Aleksandrov-Fenchel inequalities for unitary valuations of degree 2 and 3

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Abstract We extend the classical Aleksandrov-Fenchel inequality for mixed volumes to functionals arising naturally in hermitian integral geometry. As a consequence, we obtain Brunn-Minkowski and isoperimetric inequalities for hermitian quermassintegrals.

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1 Introduction

The Aleksandrov-Fenchel inequality for mixed volumes states that

\[ V(K_1, K_2, K_3, \ldots, K_n)^2 \geq V(K_1, K_1, K_3, \ldots K_n) V(K_2, K_2, K_3, \ldots, K_n) \]  

(1)

for all convex bodies \( K_1, K_2, \ldots, K_n \) in \( \mathbb{R}^n \) (\( n \geq 2 \)). A whole series of important inequalities between mixed volumes of convex bodies, including the Brunn-Minkowski and isoperimetric inequalities for quermassintegrals, can be deduced from (1) and hence the Aleksandrov-Fenchel inequality can be regarded as the main inequality in the Brunn-Minkowski theory of convex bodies. Special cases of (1) have been extended to non-convex domains, see [20, 25, 40]. For applications of the Aleksandrov-Fenchel inequality to the geometry of convex...
bodies and other fields such as combinatorics, geometric analysis and mathematical physics, we refer the reader to [11, 19, 33, 37, 38, 45, 46] and the references therein.

Several different proofs of the Aleksandrov-Fenchel inequality are known. In $\mathbb{R}^3$, the first proof of (1) was discovered by Minkowski [41] in 1903. In the 1930s, Aleksandrov [2, 3] gave two different proofs of his inequality, one based on strongly isomorphic polytopes and another, building on ideas of Hilbert [29, Chapter 19], based on elliptic operator theory. Around the same time, also Fenchel [21] sketched a proof of the inequality (1). In the 1970s, Khovanski˘ı [18, Section 27] and Teissier [50] independently discovered that the Aleksandrov-Fenchel inequality can be deduced from the Hodge index theorem from algebraic geometry. More recently, special cases of (1) have been proved using optimal mass transport and curvature flow techniques, see [8, 20, 25]. For a more complete account of the history of the Aleksandrov-Fenchel inequality, we refer the reader to [46, p. 398].

In this work we extend the Aleksandrov-Fenchel inequality for mixed volumes to functionals arising naturally in hermitian integral geometry [5, 16]. One way to describe these functionals is as follows: It is a well-known fact that for $1 < k < 2n - 1$ the action of the unitary group $U(n)$ decomposes the Grassmannian $Gr_k = Gr_k(\mathbb{C}^n)$ of $k$-dimensional, real subspaces of $\mathbb{C}^n$ into infinitely many orbits. For $k = 2, 3$ and $n \geq k$ the orbits of $Gr_k(\mathbb{C}^n)$ can be described by a single real parameter, known as the Kähler angle $\theta \in [0, \pi/2]$. For example, isotropic (with respect to the standard Kähler form on $\mathbb{C}^n$) subspaces have Kähler angle $\pi/2$ and complex subspaces have Kähler angle 0. For each Kähler angle we define two functionals on $K(\mathbb{C}^n)$, the space of convex bodies, i.e. non-empty, compact convex sets, of $\mathbb{C}^n$,

$$\varphi_\theta(K) = \int_{Gr_2(\theta)} \text{vol}_2(K|E) \, dE$$

and, for $n \geq 3$,

$$\psi_\theta(K) = \int_{Gr_3(\theta)} \text{vol}_3(K|E) \, dE.$$

Here $Gr_k(\theta)$ ($k = 2, 3$) denotes the orbit of $Gr_k(\mathbb{C}^n)$ corresponding to the Kähler angle $\theta$, $\text{vol}_k(K|E)$ is the $k$-dimensional volume of the orthogonal projection of the convex body $K$ on the $k$-dimensional subspace $E$, and $dE$ denotes the $U(n)$-invariant probability measure on the orbit.

Any linear combination $\mu$ of the functionals $\varphi_\theta$ (respectively, $\psi_\theta$) is called a unitary valuation. Observe that $\mu(tK) = t^2 \mu(K)$ (respectively, $\mu(tK) = t^3 \mu(K)$) for $t > 0$. If $\mu$ is homogeneous of degree $k$, then

$$\mu(K_1, K_2, \ldots, K_k) = \frac{1}{k!} \frac{\partial^k}{\partial t_1 \partial t_2 \cdots \partial t_k} \bigg|_{t_0=0} \mu(t_1 K_1 + t_2 K_2 + \cdots + t_k K_k)$$

is called the polarization of $\mu$. Here $K + L$ denotes the Minkowski sum of convex bodies. Note that $\mu(K, K, \ldots, K) = \mu(K)$.

Our main result is as follows:

**Theorem 1.1** If $\mu$ belongs to the convex cone generated by $\psi_\theta$ with

$$0 \leq \cos^2 \theta \leq \frac{3(n + 1)}{5n - 1},$$

then

$$\mu(K, L, M)^2 \geq \mu(K, K, M)\mu(L, L, M)$$

(3)
for all convex bodies $K, L, M$. Moreover, if $\mu$ belongs to the convex cone generated by $\varphi_\theta$ with
\[
0 \leq \cos^2 \theta \leq \frac{n + 1}{2n},
\]
then
\[
\mu(K, L)^2 \geq \mu(K, K)\mu(L, L).
\]
(5)

If $\mu = \varphi_\theta$ with $\frac{n+1}{2n} < \cos^2 \theta$, then there exist convex bodies for which (5) does not hold.

We remark that as in the classical Aleksandrov-Fenchel inequality (1) our inequalities also hold if the first convex body is replaced by the difference of support functions of two convex bodies. Moreover, since $\theta$ can be chosen such that $\psi_\theta(K)$ is proportional to
\[
V(K, K, K, \ldots, B),
\]
where $B$ denotes the unit ball in $\mathbb{C}^n$, see Lemma 2.1 below, the inequality (3) contains the Aleksandrov-Fenchel inequality with $K_4 = \cdots = K_{2n} = B$ as a special case.

Aleksandrov’s second proof [3] of (1), which we follow closely, makes critical use of Aleksandrov’s inequality for mixed discriminants. To prove (3), we first use Gårding’s theory of hyperbolic polynomials [24] to establish a hermitian analog of this fundamental determinant inequality (see Proposition 3.3 below) and then associate to each $\mu$ an elliptic differential operator.

A complete characterization of the equality cases in the Aleksandrov-Fenchel inequality (1) is not known. However, in various special cases such a characterization exists, see [46, Section 7.6]. We say that a convex body $K$ is $C^{2,\alpha}_+$ if its support function $h_K$ lies in the Hölder space $C^{2,\alpha}(S^{n-1})$ and $\det(\nabla^2 h_K + h_K \mathbb{I}) > 0$, where $\mathbb{I}$ denotes the standard Riemannian metric on the unit sphere $S^{n-1}$ and $\nabla$ denotes the covariant derivative with respect to this metric. We denote by $\mathcal{H}_1$ the space of spherical harmonics of degree one, i.e. restrictions of linear functionals to the unit sphere, and by $\mathcal{H}_{1,1} \subset \mathcal{H}_2$ the subspace of spherical harmonics of degree 2 which are invariant under the canonical circle action on the odd-dimensional sphere $S^{2n-1} \subset \mathbb{C}^n$. We establish the following description of equality cases in the inequalities (3) and (5).

**Theorem 1.2** Suppose $\mu$ belongs to the convex cone generated by $\varphi_\theta$ with
\[
0 < \cos^2 \theta < \frac{3(n + 1)}{5n - 1},
\]
and $M \in C^{2,\alpha}_+$. If $\mu(L, L, M) > 0$, then equality holds in the inequality
\[
\mu(K, L, M)^2 \geq \mu(K, K, M)\mu(L, L, M)
\]
if and only if $K$ and $L$ are homothetic. If $M$ is a ball, then the above characterization extends to $\cos^2 \theta = 0$ and $\frac{3(n+1)}{5n-1}$.

If $\mu$ belongs to the convex cone generated by $\varphi_\theta$ with
\[
0 \leq \cos^2 \theta < \frac{n + 1}{2n}
\]
and $\mu(L, L) > 0$, then equality holds in the inequality
\[
\mu(K, L)^2 \geq \mu(K, K)\mu(L, L)
\]
if and only if $K$ and $L$ are homothetic. If $\mu = \varphi_\theta$ with $\cos^2 \theta = \frac{n+1}{2n}$, then equality holds if and only if there exists a constant $\alpha$ such that $h_K$ and $\alpha h_L$ differ by an element of $\mathcal{H}_1 \oplus \mathcal{H}_{1,1}$.
We remark that we obtain the above characterization of equality cases in the more general situation where $K$ is replaced by the difference of support functions of two convex bodies.

As a consequence of Theorems 1.1 and 1.2, we obtain among several other inequalities a hermitian extension of the Brunn-Minkowski inequality (see Theorem 6.1 below) and the following isoperimetric inequalities for hermitian quermassintegrals.

**Theorem 1.3** Let $\theta \in [\pi/4, \pi/2]$ and choose $\theta' \in [0, \pi/2]$ such that $3 \cos^2 \theta' = \cos^2 \theta$. Then

$$\left( \int_{\text{Gr}_1} \text{vol}_1(K|E) \, dE \right)^2 \geq \frac{4}{\pi} \int_{\text{Gr}_2(\theta)} \text{vol}_2(K|E) \, dE$$

and

$$\left( \int_{\text{Gr}_2(\theta')} \text{vol}_2(K|E) \, dE \right)^3 \geq \frac{9\pi}{16} \left( \int_{\text{Gr}_3(\theta)} \text{vol}_3(K|E) \, dE \right)^2$$

for all convex bodies in $\mathbb{C}^n$. Equality holds if and only if $K$ is a ball.

The above inequalities hold in particular for averages over isotropic (resp. Lagrangian) subspaces ($\theta = \pi/2$), but the case of complex subspaces ($\theta = 0$) is not covered by the theorem. In fact, although the inequalities hold for a slightly larger range of $\theta$ than stated in Theorem 1.3, we show in Proposition 5.4 that the first inequality fails for $\theta < \pi/4$ when $n$ is sufficiently large.

2 Valuations and area measures

Valuations are a classical notion from convex geometry. A function $\mu : \mathcal{K}(V) \to \mathbb{R}$ on the set of non-empty, convex, compact subsets of a finite-dimensional vector space is called a valuation if

$$\mu(K \cup L) = \mu(K) + \mu(L) - \mu(K \cap L)$$

whenever the union of $K$ and $L$ is again convex. The space of continuous (with respect to the Hausdorff metric) and translation-invariant valuations is denoted by $\text{Val} = \text{Val}(V)$. A valuation $\mu$ is called homogeneous of degree $k$ if $\mu(\lambda K) = \lambda^k \mu(K)$ for every $\lambda > 0$ and $\text{Val}_k \subset \text{Val}$ denotes the subspace of $k$-homogeneous valuations. By a fundamental result of McMullen [39], every continuous and translation-invariant valuation is the sum of homogeneous valuations

$$\text{Val} = \bigoplus_{k=0}^n \text{Val}_k.$$ 

As a consequence, one can associate to each $\mu \in \text{Val}_k$ a unique function on the $k$-fold product $\mathcal{K}(V) \times \cdots \times \mathcal{K}(V)$, which is again denoted by $\mu$ and called the polarization of $\mu$, such that (i) $\mu(K, \ldots, K) = \mu(K)$; (ii) $\mu$ is symmetric in its arguments; and (iii) for every $K, L, K_2, \ldots, K_k \in \mathcal{K}(V)$ and $s, t > 0$

$$\mu(sK + tL, K_2 \ldots, K_k) = s \mu(K, K_2 \ldots, K_k) + t \mu(L, K_2 \ldots, K_k).$$

If $P$ is just a point, then, by the translation-invariance of $\mu$,

$$\mu(P, K_2, \ldots, K_k) = 0. \quad (8)$$
From now on let $V$ be a finite-dimensional, euclidean vector space. The support function of $K \in \mathcal{K}(V)$ is the function on the unit sphere of $V$ defined by $h_K(u) = \sup_{x \in K} \langle u, x \rangle$, where $\langle u, x \rangle$ denotes the inner product on $V$. If $f$ is the difference of two support functions, say $f = h_K - h_L$, then one defines
\[
\mu(f, K_2, \ldots, K_k) = \mu(K, K_2, \ldots, K_k) - \mu(L, K_2, \ldots, K_k).
\]
Similarly, $\mu(f_1, f_2, K_3, \ldots, K_k)$, where $f_1$ and $f_2$ are differences of support functions, is defined. In the following we will make frequent use of the fact that every $C^2$ function on the sphere is the difference of two support functions, see, e.g., [46, Lemma 1.7.8].

We denote by $\text{Gr}_k = \text{Gr}_k(V)$ the Grassmannian of real $k$-dimensional subspaces of $V$. If $\mu \in \text{Val}_k$ is even, that is $\mu(-K) = \mu(K)$, then, by a theorem of Hadwiger (see below), the restriction of $\mu$ to $E \in \text{Gr}_k$ is a multiple of the $k$-dimensional Lebesgue measure on $E$, and the corresponding factor is denoted by $\text{Kl}_\mu(E)$. The function $\text{Kl}_\mu : \text{Gr}_k \to \mathbb{R}$ is called the Klain function of $\mu$ and, by a theorem of Klain [31], it determines $\mu$ uniquely.

A celebrated theorem of Hadwiger characterizes linear combinations of the intrinsic volumes, which are continuous and isometry invariant. In particular, this result shows that the space of continuous and isometry invariant valuations on $\mathbb{R}^n$ is finite-dimensional.

In Alesker [5] proved the following hermitian extension of Hadwiger’s theorem: the space of valuations on $\mathbb{C}^n$ which are continuous and invariant under affine unitary transformations is finite-dimensional. The space of these valuations is denoted by $\text{Val}^U(n)$ and its elements are called unitary valuations. Here $U(n)$ denotes the group of unitary transformations, i.e. those $\mathbb{C}$-linear maps $A : \mathbb{C}^n \to \mathbb{C}^n$ which preserve the standard Kähler form $\omega = \sum_{i=1}^n dx_i \wedge dy_i$.

For every integer $0 \leq k \leq 2n$, we denote by $\text{Val}^U_k(n)$ the subspace of $k$-homogeneous valuations. While $\text{Val}_k^{O(n)}$ is one-dimensional and spanned by the intrinsic volume $\mu_k$, Alesker proved in [5] that
\[
\dim \text{Val}^U_k(n) = 1 + \min \left\{ \left\lfloor \frac{k}{2} \right\rfloor, \frac{2n-k}{2} \right\}. \tag{9}
\]

For more information on valuation theory, see [4,7,9,26,27,34–36,43,44,47,48] and the references therein. For recent applications to integral geometry, we refer the reader to [1,6,13–17,23,52].

In this article we establish inequalities for unitary valuations of degree 2 and 3. Let us describe the spaces $\text{Val}^U_k(n)$ for $k = 2, 3$ and $n \geq k$ explicitly. For $k = 2, 3$, the action of $U(n)$ decomposes $\text{Gr}_k(\mathbb{C}^n)$ into infinitely many orbits parametrized by the Kähler angle $\theta \in [0, \pi/2]$. Given $E \in \text{Gr}_k$, the Kähler angle $\theta = \theta(E)$ is defined by
\[
\cos^2 \theta = |\omega_E|^2,
\]
where $\omega_E$ denotes the restriction of the Kähler form $\omega$ to $E$ and $| \cdot |$ denotes the induced euclidean norm on $\wedge^2 E$. The Kähler angle of $E \in \text{Gr}_{2n-k}(\mathbb{C}^n)$ is, by definition, the Kähler angle of $E^\perp$. In a similar way, the $U(n)$-orbits of $\text{Gr}_k(\mathbb{C}^n)$ for $3 < k < 2n - 3$ can be described by multiple Kähler angles, see [49].
Since every $\mu \in \text{Val}_k^{U(n)}$ is even, it is uniquely determined by its Klain function $\text{Kl}_\mu$, which, by the $U(n)$-invariance of $\mu$, is constant on every $U(n)$-orbit. For $k = 2, 3$ and $n \geq k$, the space $\text{Val}_k^{U(n)}$ is 2-dimensional and spanned by two special valuations $\mu_{k,0}$ and $\mu_{k,1}$. In terms of Klain functions, they are given by

$$\text{Kl}_{\mu_{k,0}} = 1 - \cos^2 \theta \quad \text{and} \quad \text{Kl}_{\mu_{k,1}} = \cos^2 \theta$$

where $\theta$ denotes the Kähler angle, see [16, Corollary 3.8]. Moreover, the spaces $\text{Val}_{2n-k}^{U(n)}$ are also 2-dimensional and spanned by two valuations $\mu_{2n-k,n-k}$ and $\mu_{2n-k,n-k+1}$ satisfying

$$\text{Kl}_{\mu_{2n-k,n-k}} = 1 - \cos^2 \theta \quad \text{and} \quad \text{Kl}_{\mu_{2n-k,n-k+1}} = \cos^2 \theta.$$

The following lemma expresses the valuations $\varphi_\theta$ and $\psi_\theta$ defined in the introduction in terms of $\mu_{k,0}$ and $\mu_{k,1}$.

**Lemma 2.1**

$$\varphi_\theta = \frac{1}{4n(n-1)} \left( (2n - 1 - \cos^2 \theta) \mu_{2,0} + 2(n - 1) (1 + \cos^2 \theta) \mu_{2,1} \right)$$

and

$$\psi_\theta = \frac{2^{n-2}(n - 3)!}{n\pi(2n - 3)!} \left( 1 + \frac{1}{3} \cos^2 \theta \right) \mu_{3,1}.$$

**Proof** Fix $E \in \text{Gr}_k(\theta)$ and let $B_{E^\perp}$ be the unit ball in $E^\perp$. Observe that

$$\text{vol}_{2n}(K + r B_{E^\perp}) = \omega_{2n-k} \text{vol}_k(K|E) r^{2n-k} + O(r^{2n-k-1})$$

and hence

$$\int_{\text{Gr}_k(\theta)} \text{vol}_k(K|E) \, dE = \frac{1}{\omega_{2n-k}} \lim_{r \to \infty} \frac{1}{r^{2n-k}} \int_{U(n)} \text{vol}_{2n}(K + g r B_{E^\perp}) \, dg$$

$$= \frac{1}{\omega_{2n-k}} \lim_{r \to \infty} \frac{1}{r^{2n-k}} \int_{U(n) \times \mathbb{C}^n} \chi(K \cap g' r B_{E^\perp}) \, dg',$$

where $\chi$ is the Euler characteristic. The integral on the right-hand side can be evaluated using the principal kinematic formula for the unitary group established by Bernig and Fu [16]. Since an even valuation is uniquely determined by its Klain function, it suffices to check the formula for $\varphi_\theta$ only for 2-dimensional convex bodies $K$. The $(2, 2n - 2)$ bi-degree part of the principal kinematic formula is given by

$$\frac{1}{4n(n-1)} \left[ (2n - 1)\mu_{2,0} \otimes \mu_{2n-2,n-2} + 2(n - 1)\mu_{2,0} \otimes \mu_{2n-2,n-1} 
+ 2(n - 1)\mu_{2,1} \otimes \mu_{2n-2,n-2} + 4(n - 1)\mu_{2,1} \otimes \mu_{2n-2,n-1} \right],$$

see [16, p. 941]. Since $\mu_{2n-2,n-2}(B_{E^\perp}) = \omega_{2n-2}(1 - \cos^2 \theta)$ and $\mu_{2n-2,n-1}(B_{E^\perp}) = \omega_{2n-2} \cos^2 \theta$, the claim follows.
The formula for $\psi_\theta$ is proved in the same way using that the $(3, 2n - 3)$ bi-degree part of the principal kinematic formula is given by
\[
\frac{2^{n-2}(n-3)!}{n\pi(2n-3)!!} \left[ (2n-3)\mu_{3,0} \otimes \mu_{2n-2,n-3} + 2(n-2)\mu_{3,0} \otimes \mu_{2n-3,n-2} + 2(n-2)\mu_{3,1} \otimes \mu_{2n-3,n-3} + \frac{8(n-2)}{3}\mu_{3,1} \otimes \mu_{2n-3,n-2} \right].
\]

\[ \square \]

**Corollary 2.2** The valuation $\mu = c_0\mu_{2,0} + c_1\mu_{2,1}$ belongs to the convex cone generated by $\varphi_\theta$ with $\theta$ satisfying (4) if and only if
\[
2(n-1)c_0 \leq (2n-1)c_1 \quad \text{and} \quad (4n+1)c_1 \leq 2(3n+1)c_0.
\]
The valuation $\mu = c_0\mu_{3,0} + c_1\mu_{3,1}$ belongs to the convex cone generated by $\psi_\theta$ with $\theta$ satisfying (2) if and only if
\[
2(n-2)c_0 \leq (2n-3)c_1 \quad \text{and} \quad 5c_1 \leq 6c_0.
\]

For every $\mu \in \operatorname{Val}_k$ which is given by integration with respect to the normal cycle (see, e.g., [10] for this notion) and every convex body $K$ there exists a signed Borel measure $S_\mu(K)$ on the unit sphere of $V$, called the area measure associated to $\mu$, such that
\[
\mu(K, \ldots, K, L) = \frac{1}{k} \left. \frac{d}{dt} \mu(K + tL) \right|_{t=0} = \frac{1}{k} \int h_L \, dS_\mu(K),
\]
for every convex body $L$. Explicitly, if $\mu = \int_{N(K)} \omega$, then
\[
S_\mu(K) = \pi_2^*(N(K)_\ast(T \cup D\omega)),
\]
where $\pi_2 : V \times V \to V, \pi_2((u,v)) = v, N(K)$ is the normal cycle of $K, T$ denotes the Reeb vector field on the sphere bundle of $V$, and $D$ is the Rumin differential, see Proposition 2.2 of [51].

Observe that $K \mapsto S_\mu(K)$ is a translation-invariant, $(k-1)$-homogeneous valuation with values in the space of signed Borel measures on the unit sphere, which is continuous: If $K_i \to K$ with respect to the Hausdorff metric, then $S_\mu(K_i) \to S_\mu(K)$ with respect to the weak-* topology (see Lemma 2.4 of [52]). Hence, by a result of McMullen [39, Theorem 14], there exists a polarization of $S_\mu$, which, for the sake of simplicity, we denote again by $S_\mu$. More precisely, there exists a unique map $S_\mu$ from the $(k-1)$-fold product $\mathcal{K}(V) \times \cdots \times \mathcal{K}(V)$ to the space of signed Borel measures on the unit sphere such that (i) $S_\mu(K, \ldots, K) = S_\mu(K)$; (ii) $S_\mu$ is symmetric in its arguments; (iii) for every $K, L, K_2, \ldots, K_{k-1}$ and $s, t > 0$
\[
S_\mu(sK + tL, K_2, \ldots, K_{k-1}) = sS_\mu(K, K_2, \ldots, K_{k-1}) + tS_\mu(L, K_2, \ldots, K_{k-1});
\]
and (iv) for all convex bodies $K_1, \ldots, K_{k-1}, L$
\[
\mu(K_1, \ldots, K_{k-1}, L) = \frac{1}{k} \int h_L \, dS_\mu(K_1, \ldots, K_{k-1}).
\]
Moreover, since $\mu(L, K_1, K_2, \ldots, K_{k-1}) = \mu(K_1, L, K_2, \ldots, K_{k-1})$, property (iv) implies
\[
\int h_L \, dS_\mu(K_1, K_2, \ldots, K_{k-1}) = \int h_{K_1} \, dS_\mu(L, K_2, \ldots, K_{k-1}).
\]
We call the polarization of $S_\mu$ the mixed area measure associated to $\mu$. 

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3 Elliptic differential operators associated to unitary valuations

In the following we always assume that \( \mu \in \text{Val}^U_k \) for \( k = 2, 3 \) and \( n \geq k \). In a first step, we associate to every valuation \( \mu = c_0 \mu_{k,0} + c_1 \mu_{k,1} \) a polynomial function \( p_\mu \) on \( \text{Sym}^2(\mathbb{R} \oplus \mathbb{C}^{n-1}) \), the space of symmetric bilinear forms on \( \mathbb{R} \oplus \mathbb{C}^{n-1} \). To this end we choose an orthonormal basis \( \{ e_\mathcal{T}, e_2, e_\mathcal{T}, \ldots, e_n, e_\mathcal{T} \} \) of \( \mathbb{R} \oplus \mathbb{C}^{n-1} \) such that \( e_\mathcal{T} \) is an element of the first summand and \( Je_i = e_\mathcal{T} \). Here and in the following \( J \) denotes the standard complex structure on \( \mathbb{C}^n \). With respect to this basis a bilinear form \( A \in \text{Sym}^2(\mathbb{R} \oplus \mathbb{C}^{n-1}) \) is represented by a matrix \( (A^i_j) \). We denote by \( A_{i_1 \ldots i_k}^j \) the determinant of the submatrix of \( A \) obtained from the rows \( i_1, \ldots, i_k \) and columns \( j_1, \ldots, j_k \). For \( \mu = c_0 \mu_{2,0} + c_1 \mu_{2,1} \), we define the polynomial \( p_\mu \) by

\[
p_\mu(A) = \frac{1}{\omega_{2n-2}}((2n-1)c_1 - 2(n-1)c_0)A^\mathcal{T} + (2c_0 - c_1) \sum_{i=2}^n (A^i_i + A^{\mathcal{T}}_i)\]

and, for \( \mu = c_0 \mu_{3,0} + c_1 \mu_{3,1} \), by

\[
p_\mu(A) = \frac{1}{\omega_{2n-3}}((2n-3)c_1 - 2(n-2)c_0) \sum_{i=2}^n (A^{\mathcal{T}i}_{\mathcal{T}i} + A^{\mathcal{T}i}_{\mathcal{T}i}) + (3c_0 - 2c_1) \sum_{2 \leq i < j \leq n} (A^{ij}_{ij} + A^{\mathcal{T}ij}_{ij} + A^{\mathcal{T}ij}_{ij} + A^{\mathcal{T}ij}_{ij} - 2A^{\mathcal{T}ij}_{ij}) + c_1 \sum_{2 \leq i, j \leq n} A^{ij}_{ij}.\]

Note that the definition of \( p_\mu \) does not depend on the particular choice of the orthonormal basis \( \{ e_\mathcal{T}, e_2, e_\mathcal{T}, \ldots, e_n, e_\mathcal{T} \} \) with the above properties.

For every \( u \in S^{2n-1} \), choose an orthonormal basis \( \{ e_\mathcal{T}, e_2, e_\mathcal{T}, \ldots, e_n, e_\mathcal{T} \} \) of \( T_u S^{2n-1} \) such that \( Ju = e_\mathcal{T} \) and \( Je_i = e_\mathcal{T} \). If \( f \) is a \( C^2 \) function on the unit sphere, we define

\[
D_\mu(f) = p_\mu(\nabla^2 f + f \tilde{g}),
\]

where \( \tilde{g} \) denotes the canonical metric on the unit sphere and \( \nabla \) the covariant derivative with respect to this metric. If \( K \) is a convex body with \( C^2 \) support function, then we also write \( D_\mu(K) \) instead of \( D_\mu(h_K) \). In the case \( \mu = c_0 \mu_{3,0} + c_1 \mu_{3,1} \) we consider also the polarization of the 2-homogeneous polynomial \( p_\mu \), again denoted by \( p_\mu \), and define

\[
D_\mu(f_1, f_2) = p_\mu(\nabla^2 f_1 + f_1 \tilde{g}, \nabla^2 f_2 + f_2 \tilde{g}),
\]

for \( C^2 \) functions \( f_1, f_2 \) on the unit sphere. Note that \( D_\mu(f, f) = D_\mu(f) \). If \( K, L \) are convex bodies with \( C^2 \) support functions, we write \( D_\mu(K, L) \) instead of \( D_\mu(h_K, h_L) \).

**Proposition 3.1** If \( K \) is a convex body with support function in \( C^2(S^{2n-1}) \), then

\[
dS_\mu(K) = D_\mu(K) \ du,
\]

where \( du \) denotes the Riemannian measure on the sphere.

For the proof of (13) we have to introduce more notation. Choose an orthonormal basis \( \{ e_1, e_\mathcal{T}, e_2, e_\mathcal{T}, \ldots, e_n, e_\mathcal{T} \} \) of \( \mathbb{C}^n \) such that \( Je_i = e_\mathcal{T} \) and denote by

\[
(x_1, y_1, \ldots, x_n, y_n, \xi_1, \eta_1, \ldots, \xi_n, \eta_n)
\]
the corresponding coordinates on $\mathbb{C}^n \oplus \mathbb{C}^n$. The 1-forms

$$\alpha = \sum_{i=1}^{n} \xi_i dx_i + \eta_i dy_i,$$

$$\beta = \sum_{i=1}^{n} \xi_i dy_i - \eta_i dx_i,$$

$$\gamma = \sum_{i=1}^{n} \xi_i d\eta_i - \eta_i d\xi_i,$$

and the 2-forms

$$\theta_0 = \sum_{i=1}^{n} d\xi_i \wedge d\eta_i,$$

$$\theta_1 = \sum_{i=1}^{n} dx_i \wedge d\eta_i - dy_i \wedge d\xi_i,$$

$$\theta_2 = \sum_{i=1}^{n} dx_i \wedge dy_i,$$

are $U(n)$-invariant and hence do not depend on the choice of basis used for their definition.

The restriction of these forms to $\mathbb{C}^n \times S^{2n-1}$ together with the Kähler form on $\mathbb{C}^n$ generate the algebra of translation- and $U(n)$-invariant forms on the sphere bundle $\mathbb{C}^n \times S^{2n-1}$, see [16].

For non-negative integers $k, q$ with $\max\{0, k-2n\} \leq q \leq \frac{k}{2} < n$ Bernig and Fu [16] define the $(2n-1)$-forms

$$\beta_{k,q} = c_{n,k,q} \beta \wedge \theta_0^{n-k+q} \wedge \theta_1^{k-2q-1} \wedge \theta_2^q, \quad q < \frac{k}{2},$$

$$\gamma_{k,q} = \frac{c_{n,k,q}}{2} \gamma \wedge \theta_0^{n-k+q-1} \wedge \theta_1^{k-2q} \wedge \theta_2^q, \quad k-n < q,$$

where

$$c_{n,k,q} = \frac{1}{q!(n-k+q)!(k-2q)!(n-2k)}.$$

In terms of integration over the normal cycle,

$$\mu_{k,q}(K) = \int_{N(K)} \beta_{k,q} = \int_{N(K)} \gamma_{k,q}.$$

Let $K$ be a convex body with $C^1$ boundary. We denote by $\nu : \partial K \to S^{2n-1}$ the Gauss map and by $\overline{v} : \partial K \to \mathbb{C}^n \times S^{2n-1}$, $\overline{v}(x) = (x, \nu(x))$, the graphing map. If $K \in C^2_+$, which we assume in the following, then the Gauss map is a $C^1$-diffeomorphism. Fix now a point $u \in S^{2n-1}$ and put $x = \nu^{-1}(u)$. By $U(n)$-invariance, we may assume that $u = e_1$. Under this assumption we have at the point $u$,
\( (\nu \circ \nu^{-1})^* dx_i = r_i^T dy_1 + \sum_{j=2}^{n} (r_j^i dx_j + r_j^j dy_j), \quad 1 < i \leq n, \)

\( (\nu \circ \nu^{-1})^* dy_i = \tilde{r}_i^T dy_1 + \sum_{j=2}^{n} (\tilde{r}_j^i dx_j + \tilde{r}_j^j dy_j), \quad 1 \leq i \leq n, \)

where \( (r_j^j) \) is the matrix representing the bilinear form

\[ \{d_u \nu^{-1}(X), Y\} = \nabla^2 h_K(X, Y) + h_K \langle X, Y \rangle \]

with \( X \in T_u S^{2n-1} \) and \( Y \in T_x \partial K \cong T_u S^{2n-1}. \) Moreover,

\( (\nu \circ \nu^{-1})^* \alpha = 0, \quad (\nu \circ \nu^{-1})^* \beta = (\nu \circ \nu^{-1})^* dy_1, \quad (\nu \circ \nu^{-1})^* \gamma = (\nu \circ \nu^{-1})^* d\eta_1 = d\eta_1, \)

and

\( (\nu \circ \nu^{-1})^* d\xi_1 = 0, \quad (\nu \circ \nu^{-1})^* d\xi_i = dx_i, \quad (\nu \circ \nu^{-1})^* d\eta_i = dy_i \)

for \( 1 < i \leq n. \)

**Lemma 3.2** Suppose \( K \in C_{+}^2. \) Then

\( (\nu \circ \nu^{-1})^* \beta_{1,0} = \frac{1}{\omega_{2n-1}} r_{i1}^T du, \)

\( (\nu \circ \nu^{-1})^* \gamma_{1,0} = \frac{1}{2(n-1)\omega_{2n-1}} \sum_{i=2}^{n} \left( r_i^1 + r_i^2 \right) du, \)

\( (\nu \circ \nu^{-1})^* \beta_{2,0} = \frac{1}{2\omega_{2n-2}} \sum_{i=2}^{n} \left( \tilde{r}_{i1} + \tilde{r}_{i1} \right) du, \)

\( (\nu \circ \nu^{-1})^* \gamma_{2,0} = \frac{1}{2(n-2)\omega_{2n-2}} \sum_{2 \leq i < j \leq n} \left( r_{ij} + r_{ij}^T + \tilde{r}_{ij} + \tilde{r}_{ij}^T - 2r_{ij}^T \right) du, \)

\( (\nu \circ \nu^{-1})^* \gamma_{2,1} = \frac{1}{2(n-1)\omega_{2n-2}} \sum_{2 \leq i, j \leq n} r_{ij}^T du, \)

where \( du \) denotes the Riemannian volume form.

**Proof** Using the above relations, the proof is a straightforward computation. \( \square \)

**Proof of Proposition 3.1** If \( \mu = \int_{N(\cdot)} \omega, \) then, by Eq. (10),

\[ \int_{S^{2n-1}} f \ dS_\mu(K) = \int_{N(K)} \pi_2^* f \ omega', \]

where \( \omega' = T_{\omega} D\omega \) and \( D\omega \) denotes the Rumin differential of \( \omega. \) Bernig and Fu have computed \( T_{\omega} D\omega \) for each of the invariant forms \( \beta_{k,q} \) and \( \gamma_{k,q}, \) see Propositions 3.4 and 4.6 of [16]. Using this, we obtain

\[ T_{\omega} D\omega = \frac{\omega_{2n-1}}{\omega_{2n-2}} ((2n-1)c_1 - 2(n-1)c_0) \beta_{1,0} + 2(n-1)(2c_0 - c_1) \gamma_{1,0} \]
for \( \mu = c_0 \mu_{2,0} + c_1 \mu_{2,1} \) and
\[
T_3 D\omega = \frac{2\omega_{2n-2}}{\omega_{2n-3}}\left( (2n-3)c_1 - 2(n-2)c_0 )c_{2,0} + (n-2)(3c_0 - 2c_1)\gamma_{2,0} + (n-1)c_1\gamma_{2,1} \right)
\]
(14)
for \( \mu = c_0 \mu_{3,0} + c_1 \mu_{3,1} \). Hence, if \( K \in C_+^2 \), then (13) follows from
\[
\int_{\omega_{2n-1}} \pi_2^* f \omega' = \int_{\omega_{2n-1}} f (\nu \circ \nu^{-1})^* \omega',
\]
and Lemma 3.2.

If \( K \) is a convex body whose support function is merely \( C^2 \), then for every \( \varepsilon > 0 \) the Minkowski sum \( K_\varepsilon = K + \varepsilon B \) is in \( C_+^2 \). Therefore, \( S_\mu(K_\varepsilon) = D_\mu(K_\varepsilon) \, du \). Since \( S_\mu(K_\varepsilon) \) and \( D_\mu(K_\varepsilon) \) are polynomial in \( \varepsilon > 0 \) by (11) and \( h_{K_\varepsilon} = h_K + \varepsilon \), letting \( \varepsilon \to 0 \) concludes the proof.

For later use we note that (12) and (13) imply
\[
\int f_1 D_\mu(f_2, f_3) \, du = \int f_2 D_\mu(f_1, f_3) \, du
\]
(15)
for all \( C^2 \) functions \( f_1, f_2, \) and \( f_3 \).

A homogeneous polynomial \( P \) of degree \( m \) defined on \( \mathbb{R}^n \) is called hyperbolic in direction \( a \in \mathbb{R}^n \) if \( P(a) > 0 \) and for every \( x \in \mathbb{R}^n \) the univariate polynomial
\[
t \mapsto P(ta + x)
\]
has exactly \( m \) real roots (counted with multiplicities). If \( P \) is hyperbolic in direction \( a \), then \( \Gamma = \Gamma(P, a) \) denotes the connected component of the set \( \{ P > 0 \} \) containing \( a \) and is called the hyperbolicity cone of \( P \). It was shown by Gårding [24] that \( \Gamma \) is a convex cone and that \( P \) is hyperbolic in direction \( b \) for every \( b \in \Gamma \).

For example, \( x \mapsto x_1 \cdots x_n \) is a homogeneous polynomial on \( \mathbb{R}^n \) which is hyperbolic in direction \( (1, \ldots, 1) \). Since every symmetric \( n \times n \) matrix \( A \) has \( n \) real eigenvalues, the determinant \( A \mapsto \det A \) is hyperbolic in direction of the identity matrix.

**Proposition 3.3** Suppose \( \mu = c_0 \mu_{3,0} + c_1 \mu_{3,1} \). Then
\[
2(n-2)c_0 < (2n-3)c_1 \text{ and } 5c_1 < 6c_0.
\]
(16)
if and only if for every \( A, X \in \text{Sym}^2(\mathbb{R} \oplus \mathbb{C}^{n-1}) \) with \( A \) positive definite
\[
p_\mu(A, X) = 0 \implies p_\mu(X, X) \leq 0
\]
(17)
and equality holds if and only if \( X = 0 \).

**Proof** We show first that (16) implies
\[
p_\mu(I, X) = 0 \implies p_\mu(X, X) \leq 0
\]
(18)
for every \( X \in \text{Sym}^2(\mathbb{R} \oplus \mathbb{C}^{n-1}) \), where \( I \) denotes the bilinear form corresponding to the identity matrix \( (\delta^i_j) \). Note that since \( p_\mu(I) > 0 \) and
\[
p_\mu(tI + X) = p_\mu(X, X) + 2tp_\mu(I, X) + t^2p_\mu(I, I),
\]
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the claim (18) is equivalent to the statement that $p_\mu$ is hyperbolic in direction $I$. Indeed, if $p_\mu$ is hyperbolic in direction $I$, then (18) holds. Conversely, given any $X \in \text{Sym}^2(\mathbb{R} \oplus \mathbb{C}^{n-1})$, put $X' = X - \lambda I$ with $\lambda = p_\mu(I, X)/p_\mu(I, I)$. Thus, $p_\mu(I, X') = 0$ and hence, by (18),

$$p_\mu(X', X') = p_\mu(X, X) - \frac{p_\mu(I, X)^2}{p_\mu(I, I)} \leq 0. \quad (19)$$

This shows that $p_\mu$ is hyperbolic in direction $I$.

By the $\{1\} \times U(n-1)$-invariance of (18), we may assume that $X = (X^i_j)$ satisfies

$$\begin{cases}
X^2_2 = 0, \\
X^2_3 = X^2_3, X^2_3 = 0, \\
\vdots \\
X^2_\pi = X^2_\pi = \cdots = X^{n-1}_\pi = X^{n-1}_\pi = X^n_\pi = 0.
\end{cases}$$

In this case, the minors $X^{i_j}_{j_j}$ vanish for $i \neq j$ and hence

$$p_\mu(X) \leq p_\mu(\tilde{X}),$$

where $(\tilde{X}^i_j)$ has the same diagonal entries as $(X^i_j)$, but all off-diagonal entries are 0.

The condition $p_\mu(I, X) = 0$ means explicitly that

$$p_\mu(I, X) = \frac{1}{\omega_{2n-3}} \left[ (n-1)((2n-3)c_1 - 2(n-2)c_0)X^T_I \\
+ (2(n-2)c_0 - (n-3)c_1) \sum_{i=2}^n (X^i_i + X^T_i) \right] = 0$$

and, hence,

$$X^T_I = -\frac{2(n-2)c_0 - (n-3)c_1}{(n-1)((2n-3)c_1 - 2(n-2)c_0)} \sum_{i=2}^n (X^i_i + X^T_i).$$

Thus $p_\mu(\tilde{X})$ is in fact a homogeneous polynomial of degree 2 in the variables $X^2_2, X^2_3, \ldots, X^n_\pi, X^n_\pi$,

$$\omega_{2n-3} p_\mu(\tilde{X}) = \frac{a}{2} \sum_{i=2}^n (X^i_i)^2 + (X^T_i)^2 + b \sum_{i=2}^n X^i_i X^T_i \\
+ c \sum_{2 \leq i < j \leq n} (X^i_i X^j_j + X^i_i X^T_j + X^T_i X^j_j + X^T_i X^T_j) \\
= q(X^2_2, X^2_3, \ldots, X^n_\pi, X^n_\pi)$$

with

$$a = -\frac{2(2n-2)c_0 - (n-3)c_1}{(n-1)}, \quad b = c_1 + a, \quad c = (3c_0 - 2c_1) + a.$$
In order to show \( q \leq 0 \), it will be sufficient to compute the eigenvalues of the Hessian of \( q \). Since

\[
\text{Hess } q = \begin{pmatrix}
  a & b & c & c \\
  b & a & c & c \\
  c & c & a & b \\
  c & c & b & a \\
  \ddots & \ddots & \ddots & \ddots \\
  c & a & b & c \\
  b & a & c & c \\
  c & c & a & b \\
  c & c & b & a
\end{pmatrix},
\]

we conclude that Hess \( q \) has the eigenvalues

\[
\begin{align*}
  a - b &= -c_1, \\
  a + b - 2c &= 5c_1 - 6c_0, \\
  a + b + 2(n-1)c &= -\frac{2(n+1)(n-3)}{n-1}c_0 - \frac{3n+5}{n-1}c_1,
\end{align*}
\]

with multiplicities \( n - 1, n - 2, \) and 1. By assumption (16), all eigenvalues are negative and hence \( q \leq 0 \).

Next, we claim that \( p_\mu(A) > 0 \) if \( A \) is positive definite. Again by \( \{1\} \times U(n-1) \)-invariance, we may assume that \( A_{ij}^{kl} = 0 \) for \( i \neq j \). Since \( (2n-3)c_1 - 2(n-2)c_0 > 0 \) and \( 3c_1 - 2c_0 > 0 \), we conclude that \( p_\mu(A) > 0 \). Thus every positive definite bilinear form \( A \) is contained in the hyperbolicity cone \( \Gamma(p_\mu, I) \) and hence \( p_\mu \) is hyperbolic in direction \( A \). This implies (17).

Consider now the problem of maximizing \( p_\mu(X) \) subject to the condition \( g(X) := p_\mu(A, X) = 0 \). By the method of Lagrange multipliers, if \( X \) maximizes \( p_\mu \), then there exists some number \( \lambda \) such that

\[
\nabla p_\mu(X) = \lambda \nabla g(X) \quad \text{and} \quad g(X) = 0.
\]

A straightforward computation shows that \( \nabla p_\mu(X) = \lambda \nabla g(X) \) is equivalent to

\[
2X + \lambda A = 0.
\]

Since \( p_\mu(A, A) > 0 \), \( g(X) = 0 \) implies \( \lambda = 0 \) and hence \( X = 0 \).

Conversely, to see that (17) implies (16), choose \( A = I \) and plug \( X \) diagonal or of rank at most 2 into (17). \( \square \)

For later use we remark that (17) is equivalent to the statement that for every \( A, X \in \text{Sym}^2(\mathbb{R} \oplus \mathbb{C}^{n-1}) \) with \( A \) positive definite

\[
p_\mu(A, X)^2 \geq p_\mu(A)p_\mu(X)
\]

and equality holds if and only if there exists \( \lambda \in \mathbb{R} \) such that \( X = \lambda A \). Indeed, the proof of the equivalence of (18) and (19) with \( I \) replaced by \( A \) yields the equivalence of (17) and (20).

Let \( M \) be a smooth manifold. A linear map \( D : C^2(M) \to C(M) \) is called linear differential operator of order at most 2 if for every coordinate neighborhood \( U \) in \( M \) with local
coordinates \((x^1, \ldots, x^n)\) there exist continuous functions \(a^{ij} = a^{ji}, b^i, c\) such that given any \(f \in C^2(M)\) the restriction \(Df\mid_U\) to \(U\) is given by

\[
Df\mid_U = \sum_{i,j=1}^{n} a^{ij} \frac{\partial^2 f}{\partial x^i \partial x^j} + \sum_{i=1}^{n} b^i \frac{\partial f}{\partial x^i} + cf.
\]

(21)

The operator \(D\) is called elliptic if

\[
d^{ij} \xi_i \xi_j \neq 0
\]

for every \(\xi \in \mathbb{R}^n\) with \(\xi \neq 0\), see [Aubin, p. 125]. The principal symbol of \(D\) is the contravariant, symmetric tensor \(\sigma(D)^{ij} = d^{ij}\).

**Corollary 3.4** Suppose \(\mu = c_0\mu_{3,0} + c_1\mu_{3,1}\),

\[
2(n - 2)c_0 < (2n - 3)c_1 \quad \text{and} \quad 5c_1 < 6c_0,
\]

and \(M \in C^2_+\). Then the operator \(f \mapsto D_{\mu,M}f := D_{\mu}(M, f)\) is a formally self-adjoint, elliptic linear differential operator of order at most 2. Moreover,

\[
D_{\mu}(M, f) = 0 \; \Rightarrow \; D_{\mu}(f, f) \leq 0
\]

and equality holds if and only if \(f\) is the restriction of a linear function to the unit sphere.

**Proof** The symmetry of \(\mu(K, L, M)\) implies that the operator \(D_{\mu,M}\) is formally self-adjoint. Indeed, every \(C^2\) function can be expressed as the difference of two \(C^2\) support functions, and hence, for \(f = h_{K_1} - h_{K_2}\) and \(g = h_{L_1} - h_{L_2}\)

\[
(f, D_{\mu}(M, g))_{L^2} = \mu(M, K_1, L_1) - \mu(M, K_1, L_2) - \mu(M, K_2, L_1) + \mu(M, K_2, L_2)
\]

\[
= (D_{\mu}(M, f), g)_{L^2}.
\]

In order to prove ellipticity, fix a point \(p \in S^{2n-1}\) and choose normal coordinates \(x^1, \ldots, x^{2n-1}\) for \(p\) such that \(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^{2n-1}}\) is a basis of the form \(\{e_T, e_2, e_2, \ldots, e_n, e_n\}\) for \(T_pS^{2n-1}\). At the point \(p\) we have

\[
\sigma(D_{\mu,M})^{ij} \xi_i \xi_j = p_\mu(\nabla^2 h_M + h_M \nabla^2 \xi, \xi^* \xi)
\]

and the last expression is, by Proposition 3.3, zero if and only if \(\xi = 0\). Hence \(D_{\mu,M}\) is elliptic.

Since every linear functional is the support function of some point \(P\), (8) yields

\[
D_{\mu}(M, f) = D_{\mu}(f, f) = 0.
\]

Conversely, if \(D_{\mu}(M, f) = D_{\mu}(f, f) = 0\) then \(\nabla^2 f + f \nabla^2 = 0\) by Proposition 3.3. In particular, \(\text{tr}_\Sigma(\nabla^2 f + f \nabla^2) = \Delta f + (2n - 1) f = 0\), that is, \(f\) is an eigenfunction of the Laplace-Beltrami operator on the sphere with eigenvalue \(-2n + 1\). As is well known, see (24), this is possible if and only if \(f\) is the restriction of a linear functional to the unit sphere. \(\square\)

In the following \(JN\) will denote the canonical vector field on \(S^{2n-1} \subset \mathbb{C}^n\) given by \(JN(u) = Ju\). Since the trajectories of the vector field \(JN\) are geodesics, \(\nabla_{JN} JN = 0\) and, hence, \(\nabla^2 f(JN, JN) = JN(JN f)\). Consequently, we have for \(\mu = c_0\mu_{2,0} + c_1\mu_{2,1}\),

\[
D_{\mu} = \frac{1}{\omega_{2n-2}} \left[ 2n(c_1 - c_0) JN(JN) + (2c_0 - c_1) \Delta + (2(n - 1)c_0 + c_1) \right].
\]

(22)

Similarly, if \(M = B\) is the unit ball in \(\mathbb{C}^n\), we have
with
\[
a = (n - 1)((2n - 3)c_1 - 2(n - 2)c_0) \quad \text{and} \quad b = 2(n - 2)c_0 - (n - 3)c_1.
\]

We denote by \( \mathcal{H}_m = \mathcal{H}_m(S^{2n-1}), n \geq 2 \), the space of spherical harmonics of degree \( m \), i.e. the space of restrictions of harmonic, \( m \)-homogeneous polynomials \( P \in \mathbb{C}[x_1, y_1, \ldots, x_n, y_n] \) to the unit sphere. It is well known that
\[
\Delta f = -m(m + 2n - 2)f \quad \text{for} \quad f \in \mathcal{H}_m(S^{2n-1}).
\]

For non-negative integers \( k, l \) we denote by \( \mathcal{H}_{k,l} \) the space of harmonic polynomials \( P \in \mathbb{C}[x_1, y_1, \ldots, x_n, y_n] = \mathbb{C}[z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n] \) restricted to the unit sphere for which
\[
P(\lambda, z) = \lambda^{k-l} P(z) \quad \text{for} \quad \lambda \in \mathbb{C}.
\]

Clearly, \( \mathcal{H}_{k,l} \subset \mathcal{H}_{k+1,l} \). The space \( \mathcal{H}_{k,l} \) is called the space of spherical harmonics of bi-degree \((k, l)\). Under the canonical action of the unitary group \( U(n) \) on \( L^2(S^{2n-1}) \), the spaces \( \mathcal{H}_{k,l} \) are invariant and irreducible. In particular, we have the decompositions
\[
\mathcal{H}_m = \bigoplus_{k+l=m} \mathcal{H}_{k,l} \quad \text{and} \quad L^2(S^{2n-1}) = \bigoplus_{k,l} \mathcal{H}_{k,l}
\]
into pairwise orthogonal, irreducible subspaces.

Fix some point \( e \in S^{2n-1} \). A function \( P \) is called a spherical function with respect to \( U(n-1) \) if \( P \) is contained in some \( \mathcal{H}_{k,l} \), \( P \) is \( U(n-1) \)-invariant, and \( P(e) = 1 \). The existence of a unique spherical function in every \( \mathcal{H}_{k,l} \) follows from Frobenius reciprocity and the fact that irreducible \( U(n) \)-representations decompose with multiplicity 1 under \( U(n-1) \), see [32, p. 569]. One can show that the unique spherical function in \( \mathcal{H}_{k,l} \), denoted by \( P_{k,l}(w, e) \), is given by
\[
P_{k,l}(r e^{i\theta}) = (r e^{i\theta})^{k-l} Q_l(k-l, n-2, r^2) \quad \text{if} \quad k \geq l; \quad \text{and} \quad P_{k,l} = \overline{P_{l,k}} \quad \text{if} \quad l > k.
\]

Here \( \{ Q_l(a, b, t) : l = 0, 1, 2, \ldots \} \) is the complete set of polynomials in \( t \) \((Q_l \text{ has degree } l)\) orthogonal on \([0, 1]\) with weight \( t^a (1-t)^b \, dt \) and satisfying \( Q_l(a, b, 1) = 1 \), \( a > -1 \), \( b > -1 \).

The above description of spherical functions is essentially due to Johnson and Wallach [30, Theorem 3.1 (3)]; see also [42] and the references therein for more information on these spherical functions.

**Lemma 3.5** For \( f \in \mathcal{H}_{k,l}(S^{2n-1}) \),
\[
JN(JNf) = -(k-l)^2 f.
\]

**Proof** Fix \( e = e_1 \). Since \( f \mapsto JNf \) is a \( U(n) \)-intertwining operator, it will be sufficient to compute \( JNf \) for \( f = P_{k,l}(\cdot, e) \). Let \( a, b \in \mathbb{C} \) be such that \( |a|^2 + |b|^2 = 1 \) and \( 0 < |a| < 1 \), and choose \( z \in S^{2n-1} \) such that \( e \perp z \). Put \( w = ae + bz \) and let \( \gamma : \mathbb{R} \to S^{2n-1} \) be the curve \( \gamma(t) = \cos(t)w + \sin(t)Jw \). Then \( (\gamma(t), e) = (\cos t + i \sin t)a = ae^{it}, \gamma(0) = w, \gamma'(0) = Jw = JNw, \) and
\[
JNf(w) = \frac{d}{dt} \bigg|_{t=0} P_{k,l}(\gamma(t), e)) = i(k-l) f(w).
\]

\[\square\]
For the proof of Theorems 1.1 and 1.2, we need the following description of the spectrum of the differential operators (22) and (23).

**Proposition 3.6** Let $D : C^2(S^{2n-1}) \to C(S^{2n-1})$ be the differential operator defined in (22) or (23) and denote by $D^C$ its extension to $\mathbb{C}$-valued functions. If

$$2(n-1)c_0 \leq (2n-1)c_1 \quad \text{and} \quad (4n+1)c_1 < 2(3n+1)c_0$$

or

$$2(n-2)c_0 \leq (2n-3)c_1 \quad \text{and} \quad (4n^2 - 9n - 3)c_1 < 2(3n^2 - 5n - 2)c_0,$$

respectively, then $D^C$ has precisely one positive eigenvalue, which corresponds to the 1-dimensional space of constant functions, and the kernel of $D^C$ consists of the restriction of linear functionals to the unit sphere.

Moreover, if $0 < (4n+1)c_1 = 2(3n+1)c_0$, then the kernel of $D^C$ is $\mathcal{H}_{1,0} \oplus \mathcal{H}_{0,1} \oplus \mathcal{H}_{1,1}$.

**Proof** To prove the statement for the operator defined in (22), it suffices, by (24) and Lemma 3.5, to show that in the specified range for $c_0$ and $c_1$,

$$-2n(c_1 - c_0)(k-1)^2 - (2c_0 - c_1)(k+1)(k + l + 2n - 2) + (2n - 1)c_0 + c_1$$

is negative if $k + l > 1$. To this end put $\alpha = 2n(c_1 - c_0)$, $\beta = 2c_0 - c_1$, $\gamma = 2(n - 1)c_0 + c_1$, $k + l = m$, $j = |k - l|$, and observe that $\beta > 0$ and

$$\frac{\alpha}{\beta} + 2n - 1 = \frac{\gamma}{\beta}.$$

Thus (25) becomes

$$\frac{\alpha}{\beta}(1 - j^2) - ((m + n - 1)^2 - n^2),$$

which is negative for $1 < m$ and $0 \leq j \leq m$ if and only if $-\beta \leq \alpha < (2n + 1)\beta$.

Finally, defining $\alpha = a - b$, $\beta = b$, $\gamma = a + 2(n - 1)b$, and using that $\frac{\alpha}{\beta} + 2n - 1 = \frac{\gamma}{\beta}$, we conclude as before that $-\beta \leq \alpha < (2n + 1)\beta$. \qed

### 4 Proof of the inequalities

In this section we prove that if $\mu$ belongs to the convex cone generated by the valuations $\psi_\theta$ with $\theta$ satisfying (2), then

$$\mu(f, L, M)^2 \geq \mu(f, f, M)\mu(L, L, M)$$

(26)

for all convex bodies $L, M$ and all differences of support functions $f$. Moreover, we show that

$$\mu(f, L)^2 \geq \mu(f, f)\mu(L, L)$$

(27)

whenever $\mu$ belongs to the convex cone generated by the valuations $\varphi_\theta$ with $\theta$ satisfying (4).

Since every convex body can be approximated in the Hausdorff metric by convex bodies with non-empty interior, $C^\infty$ boundary, and $C^\infty$ support function, it will suffice to prove (26) and (27) for such convex bodies and smooth functions $f$. 

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Proposition 4.1 Let $L$, $M$ be convex bodies with non-empty interior, $C^\infty$ boundary, and $C^\infty$ support function, and $f$ a $C^\infty$ function on the unit sphere. Suppose $\mu$ belongs to the convex cone generated by the valuations $\psi_\theta$ with $\theta$ satisfying (6). Then the condition

$$\mu(f, L, M) = 0$$

implies

$$\mu(f, f, M) \leq 0$$

and equality holds if and only if $f$ is the restriction of a linear functional to the unit sphere.

If $\mu$ belongs to the convex cone generated by the valuations $\varphi_\theta$ with $0 < \cos^2 \theta < \frac{n+1}{2n}$, then the condition

$$\mu(f, L) = 0$$

implies

$$\mu(f, f) \leq 0$$

and equality holds if and only if $f$ is the restriction of a linear functional to the unit sphere.

The inequalities (26) and (27) are readily implied by Proposition 4.1. Indeed, if $L$, $M$ are convex bodies with non-empty interior, $C^\infty$ boundary, and $C^\infty$ support function, then, by inequality (20),

$$\mu(L, L, M) > 0$$

and hence there exists a real number $\lambda$ such that

$$\mu(f, L, M) - \lambda \mu(L, L, M) = 0.$$ 

Put $f' = f - \lambda h_L$. Then $\mu(f', L, M) = 0$ by linearity, and hence

$$0 \geq \mu(f', f', M) = \mu(f, f, M) - \frac{\mu(f, L, M)^2}{\mu(L, L, M)}.$$

We follow Hilbert [29, Chapter 19] and Aleksandrov [3] to prove Proposition 4.1. By translation-invariance, we may assume that $L$ and $M$ contain the origin in their interior and hence $h_L, h_M > 0$. Moreover, $D_{\mu, M}(L) > 0$ by (20). Consider the eigenvalue problem

$$D_{\mu, M}(f) + \lambda \frac{D_{\mu, M}(L)}{h_L} f = 0.$$ 

Since $\mu$ belongs to the convex cone generated by $\psi_\theta$ with $\theta$ satisfying (6), $D_{\mu, M}$ is, by Corollary 3.4, a formally self-adjoint, elliptic linear differential operator. Replacing $D_{\mu, M}$ by the formally self-adjoint, elliptic linear differential operator

$$\tilde{D}_{\mu, M}(f) = \left( \frac{h_L}{D_{\mu, M}(L)} \right)^{\frac{1}{2}} D_{\mu, M} \left( \left( \frac{h_L}{D_{\mu, M}(L)} \right)^{\frac{1}{2}} f \right),$$

the general theory of such operators implies (see, e.g., [12, p. 125]) that there exists an orthonormal basis $\{f_k\}_{k=1}^\infty$ of $L^2(S^{2n-1}, \frac{D_{\mu, M}(L)}{h_L} du)$ such that $f_k$ is $C^\infty$ and a solution of (29).
The set of eigenvalues of (29) is countable and discrete and the corresponding eigenspaces are finite-dimensional. Moreover, there are only finitely many negative eigenvalues.

We investigate the set of eigenvalues of (29) more closely.

**Proposition 4.2** If \( \mu \) belongs to the convex cone generated by \( \psi_0 \) with \( \theta \) satisfying (6), then 0 and \(-1\) are eigenvalues of (29) and the corresponding eigenspaces are spanned by the restriction of linear functionals to the unit sphere and by \( h_L \), respectively. All other eigenvalues of (29) are positive.

**Proof** Suppose \( \lambda = 0 \) and that \( f \) is a solution of (29). Then, \( D_\mu(M, f) = 0 \), and Corollary 3.4 yields \( D_\mu(f, f) \leq 0 \). From (15) we have
\[
0 = \int f D_\mu(M, f) = \int h_M D_\mu(f, f) \leq 0,
\]
which implies \( D_\mu(f, f) = 0 \). By Corollary 3.4, this is possible if and only if \( f \) is the restriction of a linear functional to the unit sphere.

If \( \lambda = -1 \), then it is clear that \( f = h_L \) is a solution of (29). We show now that every other solution of (29) with \( \lambda = -1 \) must be a multiple of \( h_L \) and that there are no other negative eigenvalues.

We prove this statement first for \( L = M = B \), where \( B \) denotes the unit ball in \( \mathbb{C}^n \). In this case the eigenvalue problem in (29) reduces to
\[
D_{\mu, B}(f) + \lambda \frac{n-1}{\omega_{2n-3}} (2(n-2)c_0 + 3c_1) f = 0,
\]
where \( D_{\mu, B} \) is given explicitly by Eq. (23) and the constants \( c_0, c_1 \) arise from \( \mu = c_0 \mu_{3,0} + c_1 \mu_{3,1} \). Since \( 2(n-2)c_0 + 3c_1 > 0 \) and \( D_{\mu, B} \) and its complexification have the same spectrum, the desired statement follows directly from Proposition 3.6.

Let \( L, M \) now be general convex bodies with \( C^\infty \) boundary and \( C^\infty \) support function containing the origin in the interior. Since \( L, M \) have all principal curvatures strictly positive, see, e.g., [46, p. 115],
\[
L_t = (1-t)B + tL \quad \text{and} \quad M_t = (1-t)B + tM, \quad t \in [0, 1],
\]
are convex bodies with \( C^\infty \) boundary and \( C^\infty \) support function containing the origin in the interior. Hence \( \{D_{\mu, M_t} : t \in [0, 1]\} \) is a family of uniformly elliptic, self-adjoint, linear differential operators, i.e.
\[
\sigma(D_{\mu, M_t})^{ij}\xi_i\xi_j \geq c|g|^{ij}\xi_i\xi_j \quad \text{for} \; \xi \in \mathbb{R}^{2n-1}
\]
with some constant \( c > 0 \) independent of \( t \). We denote by
\[
\lambda_1(t) \leq \lambda_2(t) \leq \lambda_3(t) \leq \cdots
\]
the eigenvalues of
\[
D_{\mu, M_t}(f) + \lambda f = 0,
\]
ordered and repeated according to their multiplicity. Since the family \( \{D_{\mu, M_t} : t \in [0, 1]\} \) is uniformly elliptic, Theorem 2.3.3 of [28] (which is stated only for bounded domains of \( \mathbb{R}^n \), but the proof works also for compact manifolds) guarantees the continuous dependence of \( \lambda_k(t) \) on \( t \).

Suppose there exists some \( t \in [0, 1] \) such that \( \lambda_2(t) < 0 \). Put
\[
t_0 = \inf \{t \in [0, 1] : \lambda_2(t) < 0\}.
\]
By continuity, \( \lambda_2(t_0) = 0 \). Moreover, since for every \( t \in [0, 1] \) the eigenvalue 0 has multiplicity \( 2n \), \( \lambda_{2n+2}(t_0) > 0 \). If \( t_0 < 1 \), then for \( t > t_0 \) sufficiently close to \( t_0 \) we have \( \lambda_{2n+2}(t) > 0 \) and hence \( \lambda_2(t) = 0 \). This contradicts the definition of \( t_0 \). We conclude that \( \lambda_2(t) = 0 \) for \( t \in [0, 1] \).

To conclude the proof of Proposition 4.1 suppose that

\[
\mu(f, L, M) = 0.
\]

Let \( f = \sum_{k=1}^{\infty} f_k \) be the expansion of \( f \) into eigenfunctions of (29). Here we stipulate that every \( f_k \) corresponds to a different eigenvalue \( \lambda_k \), ordered by their size. In particular, we have \( \lambda_1 = -1 \) and \( \lambda_2 = 0 \). Since \( f_k \) and \( f_l \) for \( k \neq l \) are orthogonal with respect to the \( L^2 \) inner product with weight \( D \mu, M(L)/h_L \ du \) and \( h_L \) spans the eigenspace corresponding to \( \lambda_1 = -1 \), we conclude that

\[
0 = \mu(f, L, M) = \int h_L D\mu, M(f) \ du = -\sum_{k=1}^{\infty} \lambda_k \int h_L f_k D\mu, M(L) h_L \ du = \int f_1 D\mu, M(L) h_L \ du.
\]

Since \( f_1 \) is a multiple of \( h_L \), this implies \( f_1 = 0 \). Hence

\[
\mu(f, f, M) = \int f D\mu, M(f) \ du = -\sum_{k=3}^{\infty} \lambda_k \int f f_k D\mu, M(L) h_L \ du = -\sum_{k=3}^{\infty} \lambda_k \int f_k^2 D\mu, M(L) h_L \ du \leq 0.
\]

Equality holds if and only if \( f_k = 0 \) for \( k \geq 3 \). Hence \( \mu(f, f, M) = 0 \) if and only if \( f \) is the restriction of a linear functional to the unit sphere.

The case that \( \mu \) belongs to the convex cone generated by \( \varphi_\theta \) with \( \theta \) satisfying (28) is proved along the same lines, the only change is that instead of the eigenvalue problem (29), one has to consider now the eigenvalue problem

\[
D \mu(f) + \lambda D \mu(L) h_L f = 0.
\]

### 5 Equality cases

We say that the unitary valuation \( \mu \in \text{Val}_k^{U(n)} \) satisfies the Aleksandrov-Fenchel inequality if

\[
\mu(f, L, M_1, \ldots, M_{k-2})^2 \geq \mu(f, f, M_1, \ldots, M_{k-2}) \mu(L, L, M_1, \ldots, M_{k-2})
\]

for all convex bodies \( L, M_1, \ldots, M_{k-2} \), and all differences of support functions \( f \). In the following we will use the abbreviations \( M = (M_1, \ldots, M_{k-2}) \) and \( \mu(f, L, M) = \mu(f, L, M_1, \ldots, M_{k-2}) \).

**Lemma 5.1** Suppose \( \mu \) satisfies the Aleksandrov-Fenchel inequality. Let \( L \) and \( M_1, \ldots, M_{k-2} \) be convex bodies, \( f \) the difference of two support functions and assume

\[
\mu(L, L, M) > 0.
\]
Then equality holds in the inequality
\[ \mu(f, L, \mathcal{M})^2 \geq \mu(f, f, \mathcal{M}) \mu(L, L, \mathcal{M}) \]
if and only if
\[ S_\mu(f, \mathcal{M}) = \alpha S_\mu(L, \mathcal{M}) \]
for some constant \( \alpha \).

**Proof** Since \( \mu(L, L, \mathcal{M}) > 0 \) and \( \mu \) satisfies the Aleksandrov-Fenchel inequality, we immediately obtain that for every \( f \) the condition
\[ \mu(f, L, \mathcal{M}) = 0 \]
implies
\[ \mu(f, f, \mathcal{M}) \leq 0. \]
Assume \( \mu(f, L, \mathcal{M})^2 = \mu(f, f, \mathcal{M}) \mu(L, L, \mathcal{M}) \) for some \( f \). Then \( f' = f - \lambda h_L \) with \( \lambda = \mu(f, L, \mathcal{M}) / \mu(L, L, \mathcal{M}) \) satisfies
\[ \mu(f', L, \mathcal{M}) = 0 \quad \text{and} \quad \mu(f', f', \mathcal{M}) = 0. \]
Consequently, \( f' \) maximizes \( \mu(f, f, \mathcal{M}) \) under the constraint \( \mu(f, L, \mathcal{M}) = 0 \) and therefore there exists a constant \( \alpha \) such that
\[ \alpha \mu(Z, L, \mathcal{M}) = \mu(Z, f', \mathcal{M}) \]
for every difference of support functions \( Z \). Hence, by the definition of the mixed area measure \( S_\mu \),
\[ \alpha \int Z \, dS_\mu(L, \mathcal{M}) = \int Z \, dS_\mu(f', \mathcal{M}) \]
for every \( Z \) and as such \( \alpha S_\mu(L, \mathcal{M}) = S_\mu(f', \mathcal{M}) \).

If \( S_\mu(f, \mathcal{M}) = \alpha S_\mu(L, \mathcal{M}) \) for some constant \( \alpha \), then multiplying this identity by \( f \) and \( h_L \) and integrating, yields \( \alpha \mu(f, L, \mathcal{M}) = \mu(f, f, \mathcal{M}) \) and \( \alpha \mu(L, L, \mathcal{M}) = \mu(L, f, \mathcal{M}) \).
Thus equality holds in the Aleksandrov-Fenchel inequality. \( \square \)

On a smooth manifold \( M \), the Hölder space \( C^{k, \alpha} \), \( 0 < \alpha < 1 \), is defined as the subspace of \( C^k(M) \) such that for every coordinate neighborhood \( U \) of \( M \) the \( k \)-th order derivatives of the restriction \( f|_U \) are locally Hölder continuous with exponent \( 0 < \alpha < 1 \). We say that a convex body \( K \) in \( \mathbb{R}^n \) is \( C^{2, \alpha} \) if the support function of \( K \) is \( C^{2, \alpha}(S^{n-1}) \) and
\[ \det(\nabla^2 h_K + h_K \mathbf{g}) > 0. \]
In particular, \( K \) has a \( C^2 \) boundary and all its principal curvatures strictly positive.

**Lemma 5.2** Suppose \( \mu \) belongs to the convex cone generated by \( \psi_\theta \) with \( \theta \) satisfying (6) and \( M \in C^{2, \alpha}_+ \). Then \( D_{\mu, M} : C^{2, \alpha}(S^{2n-1}) \to C^{0, \alpha}(S^{2n-1}) \) satisfies
\[ C^{0, \alpha}(S^{2n-1}) = \ker D_{\mu, M} \oplus \operatorname{im} D_{\mu, M}, \]
where the summands are orthogonal with respect to the standard \( L^2 \) inner product and \( \ker D_{\mu, M} \) consists precisely of the restriction of linear functionals to the unit sphere.
Proof The assertion that \( \ker D_{\mu,M} \) consists precisely of the restriction of linear functionals to the unit sphere can be proved as in Proposition 4.2. If the support function of \( M \) is \( C^\infty \) and \( D_{\mu,M} : C^\infty \to C^\infty \), then the decomposition

\[
C^\infty (S^{2n-1}) = \ker D_{\mu,M} \oplus \text{im } D_{\mu,M}
\]

follows from the general theory of self-adjoint, elliptic linear differential operators, see, e.g., [53, Theorem 4.12]. Now we may proceed exactly as in [54, Lemma 6.1], approximating \( M \) by smooth convex bodies and using the Schauder interior estimates, to obtain the corresponding decomposition if \( M \) is only \( C^{2,\alpha} \).

\[\square\]

**Theorem 5.3** Suppose \( \mu \) belongs to the convex cone generated by \( \psi_\theta \) with \( \theta \) satisfying (6) and \( M \in C^{2,\alpha}_+ \). If

\[\mu(L, L, M) > 0\]

then equality holds in the inequality

\[\mu(f, L, M)^2 \geq \mu(f, f, M) \mu(L, L, M)\]

if and only if there exists a constant \( \alpha \) such that \( \alpha h_L \) and \( f \) differ by the restriction of a linear functional to the unit sphere. If \( M \) is a ball, then the above characterization extends to \( \cos^2 \theta = 0 \) and \( \frac{3(n+1)}{2n-1} \).

If \( \mu \) belongs to the convex cone generated by \( \varphi_\theta \) with \( \theta \) satisfying (7) and

\[\mu(L, L) > 0\]

then equality holds in the inequality

\[\mu(f, L)^2 \geq \mu(f, f) \mu(L, L)\]

if and only if there exists a constant \( \alpha \) such that \( \alpha h_L \) and \( f \) differ by the restriction of a linear functional to the unit sphere. If \( \mu = \varphi_\theta \) with \( \cos^2 \theta = \frac{n+1}{2n} \), then equality holds if and only if there exists a constant \( \alpha \) such that \( \alpha h_L \) and \( f \) differ by an element of \( \mathcal{H}_{1,0} \oplus \mathcal{H}_{0,1} \oplus \mathcal{H}_{1,1} \).

Proof Let \( Z \) be a convex body. Multiplying the equality \( S_\mu(f, M) = \alpha S_\mu(L, M) \) by the support function of \( Z \), integrating and using (11) and (12), we obtain

\[\int (f - \alpha h_L) \, dS_\mu(M, Z) = 0.\]

Consequently,

\[\int (f - \alpha h_L) D_\mu(M, g) = 0\]

for every \( g \in C^{2,\alpha} \) and hence, by Lemma 5.2, \( f \) and \( \alpha h_L \) differ only by the restriction of a linear functional to the unit sphere.

Using Proposition 3.6 instead of Lemma 5.2, the remaining cases can be proved. \(\square\)

Now we show that the bound (4) is optimal.

**Proposition 5.4** If \( \mu = \varphi_\theta \) with

\[\frac{n+1}{2n} < \cos^2 \theta,\]

then there exist convex bodies \( K, L \) such that

\[\mu(K, L)^2 < \mu(K, K) \mu(L, L).\]
Proof Let $L = B$ be the unit ball in $\mathbb{C}^n$ and $f \in \mathcal{H}_{1,1}$ be real-valued and non-zero (e.g., $f(z) = \text{Re}(z_1 \overline{z_2})$). For $\varepsilon$ sufficiently small $1 + \varepsilon f$ is the support function of a convex body $K$. Since $1$ and $f$ are eigenfunctions of (22), we obtain

$$D_\mu(K) = \frac{1}{\omega_{2n-2}} ((2(n-1)c_0 + c_1) + ((4n+1)c_1 - 2(3n+1)c_0)\varepsilon f).$$

Since $1$ and $f$ are orthogonal with respect to the standard $L^2$ inner product, we have

$$\mu(K, L) = \mu(L, L) = 2\pi (2(n-1)c_0 + c_1),$$

$$\mu(K, K) = 2\pi \left( 2(n-1)c_0 + c_1 + \frac{(4n+1)c_1 - 2(3n+1)c_0}{2n\omega_{2n}} \int f^2 \, du \right).$$

Since $(4n+1)c_1 - 2(3n+1)c_0 > 0$, we obtain

$$\mu(K, L)^2 < \mu(K, K) \mu(L, L).$$

### 6 Brunn-Minkowski and isoperimetric inequalities

A straightforward consequence of the Aleksandrov-Fenchel inequality (1) is the following generalization of the Brunn-Minkowski inequality: For $m \in \{2, \ldots, n\}$ and all convex bodies $K_0, K_1, K_{m+1}, \ldots, K_n$ in $\mathbb{R}^n$,

$$V(K_0 + K_1[m], K_{m+1}, \ldots, K_n)^{\frac{1}{m}} \geq V(K_0[m], K_{m+1}, \ldots, K_n)^{\frac{1}{m}} + V(K_1[m], K_{m+1}, \ldots, K_n)^{\frac{1}{m}},$$

(32)

where here and in the following we use the shorthand

$$(K[m], K_{m+1}, \ldots, K_n) = \underbrace{(K, \ldots, K)}_{m \text{ times}},$$

Proofs of (32) were first published by Fenchel [22] and Aleksandrov [2]. In the case $m = n$ the inequality (32) reduces to the classical Brunn-Minkowski inequality.

**Theorem 6.1** Suppose $\mu$ belongs to the convex cone generated by the valuations $\psi_\theta$ with $\theta$ satisfying (2) and $m \in \{2, 3\}$. Then

$$\mu(K_0 + K_1[m], K_{m+1}, \ldots, K_3)^{\frac{1}{m}} \geq \mu(K_0[m], K_{m+1}, \ldots, K_3)^{\frac{1}{m}} + \mu(K_1[m], K_{m+1}, \ldots, K_3)^{\frac{1}{m}},$$

(33)

for all convex bodies $K_0, K_1, K_{m+1}, \ldots, K_3$ in $\mathbb{C}^n$. If $\theta$ satisfies (6), $K_3$ (or $K_1$ if $m = 3$) is of class $C^2_\alpha$, and

$$\mu(K_1[m], K_{m+1}, \ldots, K_3) > 0,$$

(34)

then equality holds in the inequality if and only if $K_0$ and $K_1$ are homothetic.

A corresponding inequality with $m = 2$ holds if $\mu$ belongs to the convex cone generated by the valuations $\phi_\theta$ with $\theta$ satisfying (4). If $\theta$ satisfies (7) and $\mu(K_1, K_1) > 0$, then equality holds if and only if $K_0$ and $K_1$ are homothetic.
Proof In order to deduce (33) from the Aleksandrov-Fenchel inequality (3) one may proceed exactly as in the case of (32), see, e.g., [46, Theorem 7.4.5]. Turning to the equality cases, first note that (33) implies the concavity of the function

\[ f(\lambda) = \mu((1 - \lambda)K_0 + \lambda K_1[m], K_{m+1}, \ldots, K_3) \frac{1}{\pi}, \quad \lambda \in [0, 1] \]

and that \( f \) is \( C^\infty \) on \( (0, 1) \) by (34). If equality holds in (33), then

\[ f(\lambda) - (1 - \lambda) f(0) - \lambda f(1) \geq 0 \]

attains a global minimum at \( \lambda = 1/2 \). Since \( f \) is also concave, we obtain

\[ 0 = f''(1/2) = 4(m - 1)\mu_{(0)}^{1/2} \left( \mu_{(0)}\mu_{(1)} - \mu_{(1)}^2 \right), \]

where

\[ \mu_{(i)} = \mu(2^{-k}(K_0 + K_1)[i], K_1[m - i], K_{m+1}, \ldots, K_3) \]

for \( i = 0, 1, 2 \). From Theorem 1.2 we deduce that \( K_0 \) and \( K_1 \) are homothetic. \( \square \)

The term “quermassintegral” is derived from the German “Quermaß”, which can be the measure of either a cross-section or a projection. The classical isoperimetric inequalities for quermassintegrals \( k = 1, \ldots, n - 1 \),

\[ \left( \int_{Gr_{k-1}} \text{vol}_{k-1}(K|E) \, dE \right)^k \geq \frac{\omega_{k-1}^k}{\omega_{k-1}^k} \left( \int_{Gr_k} \text{vol}_{k}(K|E) \, dE \right)^{k-1} \]  

are a direct consequence of the Aleksandrov-Fenchel inequality (1). Applying (3) iteratively, yields, as in the euclidean case, the inequalities

\[ \mu(K, L, L)^3 \geq \mu(K)\mu(L)^2 \]

and

\[ \mu(K, L, M)^3 \geq \mu(K)\mu(L)\mu(M). \]

In particular, letting \( K \) or \( L \) be the unit ball of \( \mathbb{C}^n \), we obtain

\[ \mu(K, B, B)^3 \geq \mu(B)^2\mu(K) \quad \text{and} \quad \mu(K, K, B)^3 \geq \mu(B)\mu(K)^2. \]  

In both inequalities, as a consequence of Theorem 1.2, equality holds if and only if \( B \) is a ball.

Lemma 6.2 Let \( \theta, \theta' \in [0, \pi/2] \). If \( \mu = \psi_\theta \), then

\[ \mu(K, K, B) = \frac{4}{3} \psi_{\theta'} \]

with \( 3 \cos^2 \theta' = \cos^2 \theta \).

Proof From the definition of \( S_\mu \) and (14), we have for \( \mu = c_0 \mu_{3,0} + c_1 \mu_{3,1} \)

\[ \mu(K, K, B) = \frac{2\omega_{2n-2}}{3\omega_{2n-3}} ((c_1 + (n - 2)c_0)\mu_{2,0} + (n - 1)c_1 \mu_{2,1}) \].

This and Lemma 2.1 imply that \( \mu(K, K, B) \) is a multiple of \( \psi_\theta \) if and only if

\[ \frac{3 + \cos^2 \theta}{\cos^2 \theta - 3(2n - 1)} = \frac{1 + \cos^2 \theta'}{\cos^2 \theta' - 2n + 1} \]

which is the case if and only if \( 3 \cos^2 \theta' = \cos^2 \theta \). \( \square \)
Combining (37) and Lemma 6.2 yields the following.

**Theorem 6.3** Let \( \theta \) satisfy (2) and choose \( \theta' \in [0, \pi/2] \) such that \( 3 \cos^2 \theta' = \cos^2 \theta \). Then

\[
\left( \int_{\text{Gr}_2(\theta')} \text{vol}_2(K|E) \ dE \right)^3 \geq \frac{9\pi}{16} \left( \int_{\text{Gr}_3(\theta)} \text{vol}_3(K|E) \ dE \right)^2
\]

for all convex bodies in \( \mathbb{C}^n \). Equality holds if and only if \( K \) is a ball.

By (9), the space \( \text{Val}_{U(n)}^1 \) is 1-dimensional and as such spanned by the first intrinsic volume or mean width which is defined by

\[
\int_{\text{Gr}_1} \text{vol}_1(K|E) \ dE.
\]

In particular, \( \mu(K, B) \) is a constant multiple of the mean width of \( K \). Hence Theorems 1.1 and 1.2 imply the following.

**Theorem 6.4** If \( \theta \) satisfies (4), then

\[
\left( \int_{\text{Gr}_1} \text{vol}_1(K|E) \ dE \right)^2 \geq \frac{4}{\pi} \int_{\text{Gr}_2(\theta)} \text{vol}_2(K|E) \ dE
\]

for all convex bodies in \( \mathbb{C}^n \). Equality holds if and only if \( K \) is a ball.

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