ENTIRE SELF-SIMILAR SOLUTIONS TO LAGRANGIAN MEAN CURVATURE FLOW

ALBERT CHAU, JINGYI CHEN, AND WEIYONG HE

Abstract. We consider self-similar solutions to mean curvature evolution of entire Lagrangian graphs. When the Hessian of the potential function $u$ has eigenvalues strictly uniformly between $-1$ and $1$, we show that on the potential level all the shrinking solitons are quadratic polynomials while the expanding solitons are in one-to-one correspondence to functions of homogenous of degree 2 with the Hessian bound. We also show that if the initial potential function is cone-like at infinity then the scaled flow converges to an expanding soliton as time goes to infinity.

1. Introduction

Let $u : \mathbb{R}^n \to \mathbb{R}$ be a smooth function. Then $L_u = \{(x, Du(x)) : x \in \mathbb{R}^n\}$ defines a graphical Lagrangian submanifold in $\mathbb{R}^{2n} = \mathbb{R}^n \oplus \sqrt{-1} \mathbb{R}^n = \mathbb{C}^n$. On the other hand, any entire graphical Lagrangian submanifold in $\mathbb{R}^{2n}$ can be obtained by such a potential $u$, up to addition of constants. Let

$$G(D^2 u) = \frac{1}{\sqrt{-1}} \log \frac{\det(I_n + \sqrt{-1} D^2 u)}{\sqrt{\det(I_n + (D^2 u)^2)}},$$

where $I_n$ is the $n$ dimensional identity matrix. The operator $G(D^2 u)$ is strictly elliptic and arises in the equation for minimal Lagrangian graphs. Namely, $L_u$ is a minimal submanifold in $\mathbb{R}^{2n}$ exactly when $G(D^2 u)$ is equal to some constant $\Theta$. In this note, we will consider the following two equations for $u(x)$:

(1) \hspace{1cm} G(D^2 u) - u + \frac{1}{2} \nabla u \cdot x = 0,

(2) \hspace{1cm} G(D^2 u) + u - \frac{1}{2} \nabla u \cdot x = 0.

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Our motivation is that (1) and (2) are the defining equations for self-expanding and self-shrinking solutions to the Lagrangian mean curvature flow of entire graphs in $\mathbb{R}^{2n}$. We describe this in more detail in the next section. We will refer to an entire solution to (1) as a self-expanding soliton and an entire solution to (2) as a self-shrinking soliton.

**Definition 1.1.** We say that $u_0$ satisfies Condition A if $D^2 u_0 \in L^\infty(\mathbb{R}^n)$ and

$$-(1 - \delta)I_n \leq D^2 u_0 \leq (1 - \delta)I_n$$

for some $\delta \in (0, 1)$ where $I_n$ is the $n$ dimensional identity matrix. We say $u_0$ satisfies Condition B if $D^2 u_0 \in L^\infty(\mathbb{R}^n)$ and

$$u_0(x) = \frac{1}{\lambda^2} u_0(\lambda x)$$

for any $\lambda > 0$.

We now state our main result

**Theorem 1.1.** If $u$ is a smooth self-shrinking soliton satisfying Condition A, then $u$ is a quadratic. There exists a one-to-one correspondence between smooth self-expanding solitons satisfying Condition A and functions which satisfy Condition A and Condition B. In particular, there are infinitely many nontrivial smooth self-expanding solitons satisfying Condition A.

We also obtain a convergence result for the following fully nonlinear parabolic equation which arises from the Lagrangian mean curvature flow (see §2):

\[
\begin{cases}
\frac{du}{dt} = G(D^2u) \\
u(x, 0) = u_0(x).
\end{cases}
\]

(3)

**Theorem 1.2.** Let $u_0$ satisfy Condition A and suppose that

$$\lim_{\lambda \to \infty} \lambda^{-2} u_0(\lambda x) \to U_0(x)$$

for some $U_0(x)$ as $\lambda \to \infty$. Let $u(x, t)$ and $U(x, t)$ be solutions to (3) with initial data $u_0(x)$ and $U_0(x)$ respectively. Then $t^{-1} u(\sqrt{t}x, t)$ converges to $U(x, 1)$ uniformly and smoothly in compact subsets of $\mathbb{R}^n$ when $t \to \infty$, where $U(x, 1)$ is a self-expanding soliton.

If we define $v(x, s) = t^{-1} u(\sqrt{t}x, t)$ where $s = \log t$ and $t \geq 1$. Then $v(x, s)$ satisfies the following equation

$$\frac{\partial v}{\partial s} = G(D^2 v) - v + \frac{1}{2} \nabla v \cdot x.$$
Theorem 1.2 implies that \( v(x, s) \) converges to \( U(x, 1) \) when \( s \to \infty \). For mean curvature flow of entire graphic hypersurface, if the initial surface is cone-like at infinity, in the sense that the normal component of the graph decays to zero at infinity at a certain rate, then the scaled mean curvature flow converges to a self-expanding soliton [4], where the monotonicity formula plays an important role. Our method is different from [4] and is based on the existence result and the estimates in [2] and the uniqueness result in [3].

Some interesting examples of non-graphical self-similar solutions to Lagrangian mean curvature flow in \( \mathbb{C}^n \) have recently been constructed [5, 6]; also see references therein.

2. Preliminaries

We now describe the connections among the elliptic equations (1) and (2), the parabolic equation (3) and the Lagrangian mean curvature flow.

Let \( F_0 : M^n \to \mathbb{R}^{n+m} \) be an embedding of a manifold \( M^n \) into \( \mathbb{R}^{n+m} \). Then the mean curvature flow with initial condition \( F_0 \) is the equation

\[
\begin{align*}
\frac{dF}{dt} &= H \\
F(x, 0) &= F_0(x)
\end{align*}
\]

where \( H(x, t) \) is the mean curvature vector of the submanifold \( F(\cdot, t)(M) \) of \( \mathbb{R}^n \) at \( F(x, t) \). The solution \( F(\cdot, t) : M^n \to \mathbb{R}^{n+m} \) is called self-expanding if it is defined for all \( t > 0 \) and \( F(\cdot, t) \) has the form

\[
M_t = \sqrt{t} M_1, \quad \text{for all } t > 0
\]

where \( M_t = F(\cdot, t)(M) \) and \( \sqrt{t} \) represents the homothety \( p \mapsto \sqrt{t}p, \ p \in \mathbb{R}^{n+m} \). Similarly, \( F(\cdot, t) : M^n \to \mathbb{R}^{n+m} \) is called self-shrinking if it is defined for all \( t < 0 \) and \( F(x, t) \) has the form

\[
M_t = \sqrt{-t} M_{-1}, \quad \text{for all } t < 0.
\]

Now when \( F_0(x) = (x, Du_0(x)) \) for \( x \in \mathbb{R}^n \) defines an entire Lagrangian graph for some potential \( u_0 : \mathbb{R}^n \to \mathbb{R} \), then (5) is equivalent to the following fully nonlinear parabolic equation (3):

\[
\begin{align*}
\frac{du}{dt} &= G(D^2u) \\
u(x, 0) &= u_0(x).
\end{align*}
\]
In particular, if \( u \) is a solution to (3) then there exists a family of diffeomorphisms \( \{ r_t : \mathbb{R}^n \to \mathbb{R}^n \} \) such that \( F(x, t) = (r_t(x), Du(r_t(x), t)) \) solves (5). Now suppose the family \( F(x, t) \) satisfies (6). Then we have

\[
(8) \quad D \left( u(x, t) - tu \left( \frac{x}{\sqrt{t}}, 1 \right) \right) = 0, \quad \forall \ t > 0.
\]

Thus by letting \( t = 1 \), we have

\[
(9) \quad u(x, t) = tu \left( \frac{x}{\sqrt{t}}, 1 \right), \quad \forall \ t > 0.
\]

Using (3) and (9) we verify directly that \( u(x, 1) \) satisfies (11), in other words, \( u(x, 1) \) is a self-expanding soliton. Similarly, we can derive that if \( F(x, t) \) is a self-shrinking solution to (5) with corresponding solution \( u(x, t) \) to (3), then \( u(x, t) \) satisfies

\[
(10) \quad u(x, t) = -tu \left( \frac{x}{\sqrt{-t}}, -1 \right), \quad \forall \ t > 0,
\]

and \( u(x, -1) \) satisfies (2). So \( u(x, 1) \) is a self-shrinking soliton.

Conversely, it is not hard to show that if \( u(x) \) solves either (11) or (2), then using (9) or (10) respectively we can generate a solution \( F(x, t) \) to (5) which is either shrinking or expanding. We illustrate this in detail in the expanding case. The case for shrinking solutions is similar. Suppose \( u(x) \) solves (11) and define \( u(x, t) := tu(x/\sqrt{t}) \). Then as above, the family \( M_t := \{ (x, Du(x, t)) : x \in \mathbb{R}^n \} \) is easily checked to satisfy (6). On the other hand, we also have

\[
\frac{du}{dt}(x, t) = u \left( \frac{x}{\sqrt{t}} \right) - \frac{1}{2} \nabla u(x/\sqrt{t}) \cdot \frac{x}{\sqrt{t}}
\]

\[
= G \left( D^2 u \left( \frac{x}{\sqrt{t}} \right) \right)
\]

\[
= G \left( D^2 u(x, t) \right)
\]

where \( y = x/\sqrt{t} \). In other words, \( u(x, t) \) solves (3) and by the discussion above there exists a family \( r_t \) such that \( F(x, t) = (r_t(x), Du(r_t(x), t)) \) is solution to (5) which is self-expanding.

3. Proofs of Theorems

The proof of Theorem 1.1 is to consider the parabolic equation (3) with Lipschitz initial data. Let us recall the main theorem in [2]:
Theorem 3.1. Let $u_0 : \mathbb{R}^n \to \mathbb{R}$ satisfy Condition A. Then (3) has a longtime smooth solution $u(x, t)$ for all $t > 0$ with initial condition $u_0$ such that the following estimates hold:

(i) $-(1 - \delta) I_n \leq D^2 u \leq (1 - \delta) I_n$ for all $t > 0$,

(ii) $\sup_{x \in \mathbb{R}^n} |D^l u(x, t)|^2 \leq C(l, \delta)/t^{l-2}$, for $l \geq 3$ and some $C(l, \delta)$.

(iii) $u(x, t) \in C^\infty(\mathbb{R}^n \times (0, \infty))$ and $u(x, t)$ converges to $u_0(x)$ in the Lipschitz norm when $t \to 0$.

We will apply Theorem 3.1 and the uniqueness result in [3] for the parabolic equation (3) to prove Theorem 1.1. The proof consists of the following two lemmas.

Lemma 3.1. Let $u_0 : \mathbb{R}^n \to \mathbb{R}$ satisfy Condition A and Condition B. Then (3) has a unique longtime smooth solution $u(x, t)$ for all $t > 0$ with initial condition $u_0$ such that $u(x, t) = tu(x/\sqrt{t}, 1)$. In particular $u(x, 1)$ is a smooth self-expanding soliton satisfying (1). Conversely, if $v$ is smooth a self-expanding soliton satisfying Condition A, then there exists $u_0$ satisfying Condition A and Condition B such that $u(x, t)$ is the unique solution of (3) with initial condition $u_0$ and $v = u(x, 1)$.

Proof. If $u_0$ satisfies Condition A, then by Theorem 3.1 there exists a smooth solution $u(x, t)$ to (3) for all $t > 0$ with initial data $u_0$. It is clear that

$$u_\lambda(x, t) := \lambda^{-2} u(\lambda x, \lambda^2 t)$$

is also a solution to (3) with initial data

$$u_\lambda(x, 0) = \lambda^{-2} u_0(\lambda x) = u_0(x)$$

where we have used that $u_0$ satisfies Condition B. Since $u_\lambda(x, 0) = u_0$, the uniqueness result in [3] implies

$$u(x, t) = u_\lambda(x, t).$$

for any $\lambda > 0$. Therefore $u(x, t)$ satisfies (3), and hence $u(x, 1)$ solves (1). In other words, $u(x, 1)$ is a smooth self-expanding soliton.

Now suppose that $v$ is a smooth solution to (1) satisfying Condition A (in fact, it suffices to assume $D^2 v$ is bounded). Define $u(x, t)$ for $t > 0$ by

$$u(x, t) = tv \left( \frac{x}{\sqrt{t}} \right).$$

It is clear that $u(x, t)$ satisfies the evolution equation in (3) since $v$ satisfies (1). Now we claim that the limit of $u(x, t)$ exists when $t$ goes to zero. To see this, begin by noting $u(0, t) = tv(0)$ for $t > 0$ and so

$$\lim_{t \to 0} u(0, t) = 0.$$
Moreover, it is clear that
\[ Du(0, t) = \sqrt{t}Dv(0) \]
and that for any \( t > 0 \)
\[ -(1 - \delta)I_n \leq D^2u(x, t) \leq (1 - \delta)I_n \]
since
\[ D^2u(x, t) = D^2 \left( tv \left( \frac{x}{\sqrt{t}} \right) \right) = D^2v \left( \frac{x}{\sqrt{t}} \right). \]
We may then conclude that for any sequence \( t_i \to 0 \) there is a subsequence \( t_{k_i} \) such that \( u(x, t_{k_i}) \) converges in \( C^{1,\alpha} \) uniformly in compact subsets of \( \mathbb{R}^n \) for any \( 0 < \alpha < 1 \). This limit is in fact independent of the sequence \( t_i \). Indeed, let \( u_1 \) and \( u_2 \) be two such limits along sequences \( t_i \) and \( t_i' \) respectively. Since \( u(x, t) \) is a solution to (3), \( \partial u/\partial t \) is also uniformly bounded for any \( t > 0, x \in \mathbb{R}^n \). Thus for any \( x \in \mathbb{R}^n \) we may have
\[ |u(x, t_i) - u(x, t_i')| \leq C|t_i - t_i'| \]
for some \( C \) independent of \( i \). Letting \( i \to \infty \), we conclude that \( u_1(x) = u_2(x) \). So for different sequences \( t_i \to 0 \), the limit is unique. Let
\[ u_0(x) = \lim_{t \to 0} u(x, t). \]
Then \( D^2u_0 \in L^\infty \) and we have
\[ -(1 - \delta)I_n \leq D^2u_0 \leq (1 - \delta)I_n. \]
Further,
\[
\frac{1}{\lambda^2} u_0(\lambda x) = \frac{1}{\lambda^2} \lim_{t \to 0} tv \left( \frac{\lambda x}{\sqrt{t}} \right) = \lim_{t \to 0} \left( \frac{\sqrt{t}}{\lambda} \right)^2 v \left( \frac{\lambda x}{\sqrt{t}} \right) = u_0(x).
\]
Therefore \( u_0 \) satisfies Condition B. \( \square \)

Remark 3.1. Lemma 3.1 provides many examples of entire graphical Lagrangian self-expanding solitons. For instance, we can choose \( u_0 \) such that
\[
u_0(x) = \left\{ \begin{array}{ll}
a_1 x_1^2 + \sum_{i=2}^n a_i x_i^2, & \text{if } x_1 \geq 0 \\
-a_1 x_1^2 + \sum_{i=2}^n a_i x_i^2, & \text{if } x_1 < 0. \end{array} \right.
\]
If \( a_i \in (-1 + \delta, 1 - \delta) \) for any fixed \( \delta \in (0, 1) \), then \( u_0 \) satisfies Condition A and Condition B and \( u(x, 1) \) is a self-expanding soliton which satisfies (1). As \( u_0 \) is not smooth, \( u(x, 1) \) cannot be a quadratic polynomial.

Next, we show that smooth self-shrinking solitons are trivial if Condition A holds.

**Lemma 3.2.** If \( v \) is a smooth solution of (2) such that \(-(1 - \delta)I_n \leq D^2v \leq (1 - \delta)I_n\), then \( v \) is a quadratic polynomial.

**Proof.** If \( v \) is a smooth solution to (2), then \( u(x, t) = (1 - t)v\left(\frac{x}{\sqrt{1 - t}}\right) \) is a solution to (3) for \( t \in (0, 1) \) and \( u(x, 0) = v(x) \), hence Theorem 3.1 applies to \( u(x, t) \) since this solution is unique, and in particular \( |D^3u(x, t)| \leq C \) for some constant \( C \) when \( t \geq 1/2 \) for any \( x \). But one checks directly that
\[
|D^3v(x)| = \left|D^3v\left(\frac{x\sqrt{1 - t}}{\sqrt{1 - t}}\right)\right| \leq C\sqrt{1 - t}
\]
for any \( x \). It follows that \( D^3v(x) = 0 \), thus \( v \) is quadratic. \( \square \)

Now we prove Theorem 1.2.

**Proof.** Let \( u(x, t) \) be the solution to (3) with initial data \( u_0 \) satisfying Condition A. It is clear that for any \( \lambda \),
\[
u_\lambda(x, t) = \lambda^{-2}u(\lambda x, \lambda^2 t)
\]
is a solution of (3) with initial data \( u_\lambda(x, 0) = \lambda^{-2}u_0(\lambda x) \) satisfying Condition A. By the uniqueness result in 3, we may then apply the estimates in Theorem 3.1 to \( u_\lambda(x, t) \). For any sequence \( \lambda_i \to \infty \), consider the solutions \( u_{\lambda_i}(x, t) \). For \( t > 0 \), it is clear that
\[
D^2u_{\lambda_i}(x, t) = D^2u(\lambda_i x, \lambda_i^2 t)
\]
and so by Theorem 3.1,
\[
-(1 - \delta)I_n \leq D^2u_{\lambda_i}(x, t) \leq (1 - \delta)I_n
\]
for all \( x \) and \( t \geq 0 \), and for \( t > 0 \) and \( l \geq 3 \)
\[
|D^l u_{\lambda_i}(x, t)| \leq C(l, \delta)\sqrt{t^{2-l}}.
\]
Thus by (3) we can also get the following estimates that, for any \( m \geq 1, l \geq 0 \), there is a constant \( C(m, l, \delta) \) such that
\[
\left| \frac{\partial^m}{\partial t^m} D^l u_{\lambda_i} \right| \leq C(m, l, \delta) \sqrt{t^{2-l-2m}}.
\]
In particular, there are constants \( C \) and \( C(\delta) \) such that
\[
\left| \frac{\partial u_{\lambda_i}}{\partial t} \right| \leq C, \quad \left| \frac{\partial D u_{\lambda_i}}{\partial t} \right| \leq \frac{C(\delta)}{\sqrt{t}}
\]
for all \( t > 0 \). We observe that \( u_{\lambda_i}(0,0) = \lambda_i^{-2} u_0(0) \) and \( D u_{\lambda_i}(0,0) = \lambda_i^{-1} D u_0(0) \) are both bounded, thus \( u_{\lambda_i}(0,t) \) and \( D u_{\lambda_i}(0,t) \) are uniformly bounded in \( \lambda_i \) for any fixed \( t \). By Arzelà–Ascoli theorem, there exists a subsequence \( \lambda_{k_i} \) such that \( u_{\lambda_{k_i}}(x,t) \) converges smoothly and uniformly in compact subsets of \( \mathbb{R}^n \times (0,\infty) \) to a solution \( U(x,t) \) of (3). Moreover, \( U(x,t) \) satisfies the estimates in Theorem 3.1. Since \( \partial U/\partial t \) is uniformly bounded for any \( t > 0 \), \( U(x,t) \) converges to some function \( U(x,0) \) when \( t \to 0 \). In particular we have
\[
U(x,0) = \lim_{\lambda_{k_i} \to \infty} \lambda_{k_i}^{-2} u_0(\lambda_{k_i}x).
\]
It is clear that \( U(x,0) \) satisfies Condition A. Similarly as in the proof of Lemma 2.1, the hypothesis (4) implies that \( U(x,0) \) satisfies Condition B. Thus by Theorem 1.1 we know that \( U(x,1) \) is a self-expanding soliton, and \( U(x,t) \) satisfies (9).

Moreover, notice that \( U(x,0) = U_0(x) \) by (4) and \( U_0(x) \) does not depend on the sequence \( \lambda_i \), it then follows from the uniqueness result in [3] that the solution \( U(x,t) \) is also independent of the sequence \( \lambda_i \) and we may write
\[
u_{\lambda}(x,t) \to U(x,t).
\]
as \( \lambda \to \infty \). In particular, letting \( \lambda = \sqrt{t} \) we have \( t^{-1} u(\sqrt{t}x,t) \) converges to \( U(x,1) \) smoothly and uniformly in compact subsets of \( \mathbb{R}^n \) when \( t \to \infty \).

Remark 3.2. The family of Lagrangian graphs \( (x + at, Du_0(x) + bt) \) is a translating solution to (5) precisely when \( u \) satisfies
\[
\sum_i \arctan \lambda_i + \sum_i a_i \frac{\partial u_0}{\partial x_i} - \sum_i b_i x_i = c,
\]
(12)
for some constant $c$. In [2] the authors proved that if $u$ is a smooth solution to (12) which satisfies condition A, then $u$ is a quadratic function. We observe here that this result can also be obtained from the uniqueness result in [3] together with Theorem 3.1. Namely, if $u_0(x)$ solves (12) then $v(x, t) = u_0(x - at) + b \cdot x t + ct$ solves (3) with initial condition $v(x, 0) = u_0(x)$. On the other hand, Theorem 3.1 guarantees a longtime solution $u(x, t)$ to (3) with initial condition $u_0$ for which $\sup_{x \in \mathbb{R}^n} |D^3 u(x, t)| \rightarrow 0$ as $t \rightarrow \infty$. Then as $u(x, t) = v(x, t)$ by the uniqueness result in [3], $\sup_{x \in \mathbb{R}^n} |D^3 u_0(x - at)| = \sup_{x \in \mathbb{R}^n} |D^3 v(x, t)| \rightarrow 0$ as $t \rightarrow \infty$ and we conclude that $\sup_{x \in \mathbb{R}^n} |D^3 u_0(x)| = 0$, in other words $u_0$ must be quadratic.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BRITISH COLUMBIA, VANCOUVER, B.C., V6T 1Z2, CANADA

E-mail address: chau@math.ubc.ca

E-mail address: jychen@math.ubc.ca

E-mail address: whe@math.ubc.ca