Representation of Artinian Partially Ordered Sets over Semiartinian Von Neumann Regular Algebras

Giuseppe Baccella

Dedicated to the memory of Adalberto Orsatti (Il Maestro) and of Dimitri Tyukavkin

Abstract. If \( R \) is a semiartinian Von Neumann regular ring, then the set \( \text{Prim}_R \) of primitive ideals of \( R \), ordered by inclusion, is an artinian poset in which all maximal chains have a greatest element. Moreover, if \( \text{Prim}_R \) has no infinite antichains, then the lattice \( L_2(R) \) of all ideals of \( R \) is anti-isomorphic to the lattice of all upper subsets of \( \text{Prim}_R \). Since the assignment \( U \mapsto r_R(U) \) defines a bijection from any set \( \text{Simp}_R \) of representatives of simple right \( R \)-modules to \( \text{Prim}_R \), a natural partial order is induced in \( \text{Simp}_R \), under which the maximal elements are precisely those simple right \( R \)-modules which are finite dimensional over the respective endomorphism division rings; these are always \( R \)-injective. Given any artinian poset \( I \) with at least two elements and having a finite cofinal subset, a lower subset \( I' \subset I \) and a field \( D \), we present a construction which produces a semiartinian and unit-regular \( D \)-algebra \( D_I \) having the following features: (a) \( \text{Simp}_{D_I} \) is order isomorphic to \( I \); (b) the assignment \( H \mapsto \text{Simp}_{D_I/H} \) realizes an anti-isomorphism from the lattice \( L_2(D_I) \) to the lattice of all upper subsets of \( \text{Simp}_{D_I} \); (c) a non-maximal element of \( \text{Simp}_{D_I} \) is injective if and only if it corresponds to an element of \( I' \), thus \( D_I \) is a right \( V \)-ring if and only if \( I' = I \); (d) \( D_I \) is a right and left \( V \)-ring if and only if \( I \) is an antichain; (e) if \( I \) has finite dual Krull length, then \( D_I \) is (right and left) hereditary; (f) if \( I \) is at most countable and \( I' = \emptyset \), then \( D_I \) is a countably dimensional \( D \)-algebra.

0. Introduction

For a given right semiartinian ring \( R \) we introduced in [11] what we called the natural preorder “\( \preceq \)” in the class of all simple right modules over \( R \). The idea was to define, for every simple module \( U_R \), a particular \( U \)-peak ideal \( I(U) \) (in the sense that the right socle of \( R/I(U) \) is essential, projective and \( U \)-homogeneous) and, given another simple module \( V_R \), to declare that \( U \preceq V \) in case \( I(U) \subset I(V) \). It turns out that the natural preorder is a Morita invariant; moreover \( U \simeq V \) if and only if both \( U \preceq V \) and \( V \preceq U \), so that “\( \preceq \)” induces the natural partial order in any set \( \text{Simp}_R \) of representatives of simple right \( R \)-modules. With respect to the natural partial order, \( \text{Simp}_R \) is an artinian poset in which every maximal chain has a maximum.

Date: March 10, 2009.

1991 Mathematics Subject Classification. Primary 16E50; Secondary 16D60, 16S50, 16E60, 06A06.

Key words and phrases. Von Neumann regular ring; Artinian poset; well founded poset; semiartinian ring; \( V \)-ring; simple module.
It is worth to observe that, since the class of right semiartinian rings is closed by factor rings, for every $U \in \text{Simp}_R$ the primitive ring $R/\tau_R(U)$ has nonzero socle. This implies that $U$ is the unique (up to an isomorphism) simple and faithful right $R/\tau_R(U)$-module and the assignment $U \mapsto \tau_R(U)$ defines a bijection from $\text{Simp}_R$ to the set $\text{Prim}_R$ of (right) primitive ideals of $R$. In view of this fact it would appear quite natural to declare $U \preceq V$ in case $\tau_R(U) \subset \tau_R(V)$; moreover we must recall that Camillo and Fuller already remarked in \cite{15} that the set $\text{Prim}_R$, ordered by inclusion, is always artinian when $R$ is right semiartinian. The point is that in many interesting cases $\text{Prim}_R$ is just the set of all maximal (two-sided) ideals and the above partial order becomes the trivial one, giving thus no insight into the structure of $R$; for example, this is the case when if $R$ is left perfect, in particular when $R$ is right artinian. The situation changes dramatically when $R$ is a regular ring; in this case it turns out that $I(U) = \tau_R(U)$ for all $U \in \text{Simp}_R$, therefore $U \preceq V$ if and only if $\tau_R(U) \subset \tau_R(V)$; moreover $U$ is a maximal element of $\text{Simp}_R$ if and only if $U$ is finite dimensional as a vector space over the division ring $\text{End}(U_R)$ and, if it is the case, then $U_R$ is injective. By the regularity, every ideal of $R$ is the intersection of all right primitive ideals containing it, therefore the order structure of $\text{Simp}_R$, or equivalently of $\text{Prim}_R$, strictly affects the order structure of the lattice $\text{L}_2(R)$ of all ideals of $R$; for instance, if $\text{Simp}_R$ has no infinite antichains, then $\text{L}_2(R)$ is anti-isomorphic to the lattice of all upper subsets of $\text{Simp}_R$, therefore $\text{L}_2(R)$ is artinian (see \cite{11} Corollary 4.8 and Theorem 4.5).

The main subject of the present work is to investigate which artinian partially ordered sets can be realized as $\text{Simp}_R$, or equivalently as $\text{Prim}_R$, for some semiartinian and regular ring $R$. This problem appears as a special instance of the more general problem of determining those complete lattices which are isomorphic to $\text{L}_2(R)$ for some regular ring $R$. A rather general answer to this problem was given by Bergman in \cite{12}, by showing that if $L$ is a complete and distributive lattice, which has a compact greatest element and each element of which is the supremum of compact join-irreducible elements, then there exists a unital, regular and locally matricial algebra $R$ over any given field $F$ such that $\text{L}_2(R)$ is isomorphic to $L$. Our main result is that if $I$ is an artinian poset and $D$ is a division ring, then there exists a unit-regular and semiartinian ring $D_I$, having $D$ as subring, such that $\text{Simp}_{D_I}$ is isomorphic to $I$ provided $I$ has a finite cofinal subset, otherwise $\text{Simp}_{D_I}$ is isomorphic to the poset obtained from $I$ by adding a suitable maximal element.

As it was proved in \cite{10}, if $R$ is a semiartinian and unit-regular ring, then the abelian group $K_0(R)$ is free of rank $|\text{Simp}_R|$; however, in the same paper the order structure of $K_0(R)$ was investigated only in the case in which $R$ satisfies the so called restricted comparability axiom (see in Section 4 below for the definition). In a forthcoming paper we will resume that investigation, precisely we will characterize those partially ordered abelian groups which can be realized as $K_0(R)$ for some semiartinian and unit-regular ring $R$. In particular we will see that if $I$ is an artinian poset having a finite cofinal subset, then $K_0(D_I)$ is isomorphic to the free abelian group generated by $I$, together with the submonoid generated by the elements $i - j$ for $j < i$ in $I$ as the positive cone.

Now $K_0(R)$ is the Grothendieck group of the abelian monoid $\mathcal{V}(R)$ of isomorphism classes of finitely generated projective right $R$-modules. When $R$ is a regular ring, then $\mathcal{V}(R)$ enjoys some fundamental and well known properties. The inverse
problem of deciding whether, given an abelian monoid $M$ having the same properties, there exists a regular ring $R$ such that $V(R)$ is isomorphic to $M$, is known as the Realization Problem for Von Neumann Regular Rings; Ara recently wrote a nice survey on it (see [5]). Only after the present work was complete we became aware of the recent important works by Ara and Brustenga [4] and by Ara [6] on this problem. Precisely, given a field $K$, in the first one a regular $K$-algebra $Q(E)$ is associated to a column-finite quiver $E$, via the Leavitt path algebra $L(E)$ of $E$ (see [1]), in such a way that $V(Q(E))$ is isomorphic to $V(L(E))$; in the second one a regular $K$-algebra $Q(P)$ is functorially associated to each finite poset $P$, in such a way that $V(Q(P))$ is the abelian monoid generated by $P$ with the only relations given by $p = p + q$ if and only if $q < p$ in $P$. To some extent our present research parallels the above works. Our construction of the ring $D_I$ is far from being functorial on $I$, exactly as the map which assigns to a set $X$ the ring $\text{CFM}_X(D)$ of all column-finite $X \times X$-matrices with entries in a given ring $D$ is not a functor on $X$.

Nonetheless, if $I$ and $J$ are isomorphic artinian posets, then the rings $D_I$ and $D_J$ turn out to be isomorphic and we can list several nice ring and module theoretical features of $D_I$. It would be interesting to find relationships, if any, between the algebra $Q(P)$ of Ara and our algebra $D_P$ when $P$ is a finite poset.

Our work is divided into nine sections. In section 1 we examine some basic features of artinian posets needed when dealing with semiartinian and regular rings. In particular, given an artinian poset $I$, for every ordinal $\alpha$ we consider the $(\alpha + 1)$-th layer $I^*_{\alpha+1}$ of $I$, namely: $I^*_0$ is the set of all minimal elements of $I$ and, for every ordinal $\alpha > 1$ one defines recursively $I^*_\alpha$ as the set of all minimal elements of the set $I \setminus \left( \bigcup_{\beta < \alpha} I^*_\beta \right)$. The set of all layers is a partition of $I$ and we define the canonical length function $\lambda_I: I \to \text{Ord}$ as the function which assigns to every $i \in I$ the (unique) successor ordinal $\lambda_I(i)$ such that $i$ belongs to the $\lambda_I(i)$-th layer of $I$ (recall that a length function on an artinian poset $I$ is any strictly increasing map from $I$ to the well ordered class $\text{Ord}$ of all ordinals).

The second, third and fourth sections are devoted to the study of the natural partial order of $\text{Simp}_R$, when $R$ is a semiartinian and regular ring. We recall that if $R$ is any right semiartinian ring and $M$ is any right $R$-module, then we define the ordinal $h(M) = \min\{ \alpha \mid M \cdot \text{Soc}_\alpha(R) = M \}$; if $M$ is finitely generated, then $h(M)$ is a successor ordinal if. If $U_R$ is simple and $h(U) = \alpha + 1$, then $U_R/\text{Soc}_\alpha(R_R)$ is projective and $\alpha$ is the largest ordinal such that $\text{Hom}_R(U, R/\text{Soc}_\alpha(R_R)) \neq 0$ (see [4] Theorem 1.3) while, if $R$ is regular, $\alpha$ is the unique ordinal with this property. Now $h$ defines a length function on the artinian poset $\text{Simp}_R$ and if $\lambda$ denotes the canonical length function on $\text{Simp}_R$, then it turns out that $\lambda(U) \leq h(U)$ for every $U \in \text{Simp}_R$. We concentrate our attention on two special classes of semiartinian and regular rings. A ring $R$ belongs to the first one if and only if the two length functions $\lambda$ and $h$ coincide, while it belongs to the second one if and only if the assignment $H \mapsto \text{Simp}_R/H$ realizes an anti-isomorphism from the lattice $L_2(R)$ to the lattice of all upper subsets of $\text{Simp}_R$. We say that $R$ is well behaved in the first case and very well behaved in the second. Of course, if $R$ is very well behaved then $R$ is well behaved and, in addition, $\text{Simp}_R$ has only finitely many maximal elements. We illustrate with examples that these latter two conditions are actually independent and, together, do not imply that $R$ is very well behaved. Next, for any semiartinian and regular ring $R$, we pass to establish which properties of the poset $\text{Simp}_R$ are connected with the various comparability axioms on $R$. 
We start with section 5 our construction of semiartinian unit-regular rings. The scenario of the whole drama is the ring $Q = \text{CFM}_X(D)$ of all column-finite matrices with entries in a given ring $D$, where $X$ is a suitable transfinite ordinal, together with the ideal $\mathbb{F}R_X(D)$ of all matrices with only finitely many nonzero rows. It is well known that if $R$ is any ring and $\varphi, \psi : Q \to R$ are two ring isomorphisms, then $\varphi(\mathbb{F}R_X(D)) = \psi(\mathbb{F}R_X(D))$; let’s say that the elements of this latter ideal are the finite-ranked elements of $R$. Thus, the first main step is to associate to every ordinal $\xi \leq X$ a family $(Q_\alpha)_{\alpha \leq \xi}$ of unital subrings of $Q$ having the following features: (a) if $\alpha < \xi$, then $Q_\alpha$ is isomorphic to $Q$, (b) by denoting with $F_\alpha$ the ideal of $Q_\alpha$ of all finite-ranked elements when $\alpha < \xi$, then $Q_\beta \cap F_\alpha = 0$ whenever $\alpha < \beta \leq \xi$. Actually, we already gave in [3] a construction which aimed to the same objective. Unfortunately the proof of Proposition 4.2 in that paper contains a gap. Filling that gap - if ever possible, would have required a considerable work and the result would have not been suitable for our present purposes either. Thus we decided to completely reorganize the construction by using a totally different approach, in which we rely mainly on ordinal arithmetic. With the new construction we have at disposal a total control of the parametrization of the entries of the matrices we deal with, as it is needed in order to accomplish the subsequent main construction.

With section 6 artinian posets enter the scene. First, we define a polarized (artinian) poset as an ordered pair $(I, I')$, where $I$ is an artinian poset and $I'$ is a lower subset of $I$. Starting from a polarized artinian poset $(I, I')$, a ring $D$ and an appropriately sized transfinite ordinal $X$, to each element $i \in I$ we associate a (not necessarily unital) subring $H_i$ of $Q = \text{CFM}_X(D)$, in such a way that $\mathcal{H} = \{H_i \mid i \in I\}$ is an independent set of $(D, D)$-submodules of $Q$ with the following features: (a) if $i$ is a maximal element of $I$, then $H_i$ is isomorphic to $D$; (b) if $i$ is not maximal and belongs to $I'$ (resp. to $I \setminus I'$), then $H_i$ is isomorphic to $\mathbb{F}R_X(D)$ (resp. to the left ideal $\mathbb{F}M_X(D)$ of $Q$ whose elements are all matrices with only finitely many nonzero entries); moreover $H_iH_j = 0$ if and only if $i, j$ are not comparable, while both $H_iH_j$ and $H_jH_i$ are nonzero and are contained in $H_i$ if $i \leq j$. This enables us to consider the (not necessary unital) subring $H_I = \bigoplus_{i \in I} H_i$ and the unital subring $D_I = H_I + 1_QD$ of $Q$ and we show that $H_I = D_I$ if and only if $I$ has a finite cofinal subset. The study of this subring, together with the strict relationship between upper subsets of $I$ and ideals of $D_I$, is the subject of sections 7 and 8.

In section 9, finally, we take $D$ as a division ring and show that, given a polarized artinian poset $(I, I')$, the ring $D_I$ has the following features: (a) $D_I$ is a unit-regular and semiartinian ring, which is also (right and left) hereditary in case $I$ has finite dual Krull length; (b) there is a map $i \mapsto U_I$ from $I$ to $\text{Simp}_{D_I}$ which is an order isomorphism in case $I$ has a finite cofinal subset, otherwise $\text{Simp}_{D_I}$ contains $D_I/H_I$ as an additional maximal element; (c) a non-maximal element $U_i$ of $\text{Simp}_{D_I}$ is injective if and only if $i \in I'$, thus $D_I$ is a right $V$-ring if and only if $I' = I$; (d) $D_I$ is a right and left $V$-ring if and only if $I$ is an antichain; (e) if $I$ has a finite cofinal subset, then the assignment $H \mapsto \text{Simp}_{D_I/H}$ realizes an anti-isomorphism from the lattice $L_2(D_I)$ to the lattice of all upper subsets of $\text{Simp}_{D_I}$; (f) if $I$ is at most countable and $I' = \emptyset$, then $D_I$ is countably dimensional over $D$.

We conclude this introduction with a few remarks about terminology and notations. In several instances we deal with rings without multiplicative identity and subrings which are not unital subrings but, often, they have their own multiplicative
identities. However, in order to avoid any ambiguity, if not otherwise stated the word “ring” means “ring with multiplicative identity”, while “subring” means “unital subring” (that is, if we state that a ring $R$ is a subring of some ring $T$ we mean that $R$ shares the same multiplicative identity of $T$) and all ring homomorphisms preserve multiplicative identity.

Given a ring $R$, we shall denote with $\text{Simp}_R$ a chosen irredundant set of representatives of all simple right $R$-modules, while $\text{Prosimp}_R$ will be the subset of $\text{Simp}_R$ of representatives of all simple and projective right $R$-modules. If any given set $U$ of simple right $R$-modules turns out to be an irredundant set of representatives of all simple right $R$ modules, we shall summarize this fact by writing $U = \text{Simp}_R$.

Recall that the Loewy chain (or lower Loewy chain, according to some authors) of a right $R$-module $M$ is the non-decreasing chain of submodules $(\text{Soc}_\alpha(M))_{\alpha \geq 0}$, parametrized over the ordinals, defined by the following rules: set $\text{Soc}_0(M) = 0$ and, recursively, define $\text{Soc}_{\alpha+1}(M)$ in such a way that $\text{Soc}_{\alpha+1}(M)/\text{Soc}_\alpha(M) = \text{Soc}(M/\text{Soc}_\alpha(M))$ (we denote by $\text{Soc}(M)$ the socle of $M$) for each ordinal $\alpha$ and $\text{Soc}_\alpha(M) = \bigcup_{\beta < \alpha} \text{Soc}_\beta(M)$ if $\alpha$ is a limit ordinal. The module $M/\text{Soc}_\alpha(M)$ is called the $\alpha$-th Loewy factor of $M$, the first ordinal $\xi$ such that $\text{Soc}_\xi(M) = \text{Soc}_{\xi+1}(M)$ is called the Loewy length of $M$ (denoted by $\text{L}(M)$) and one says that $M$ is semiartinian or a Loewy module if $\text{Soc}_\xi(M) = M$. The ring $R$ is right semiartinian if the module $R_R$ is semiartinian or, equivalently, if every non-zero right $R$-module contains a simple submodule; if it is the case, then each $\text{Soc}_\alpha(R_R)$ is an ideal.

If $R$ is a right semiartinian ring and $M$ is some right $R$-module, we define the ordinal $h(M) = \min\{\alpha \mid M/\text{Soc}_\alpha(R_R) = M\}$; clearly, when $M$ is finitely generated $h(M)$ is not a limit ordinal if. If $U_R$ is simple and $h(U) = \alpha + 1$, then $U_R/\text{Soc}_\alpha(R_R)$ is projective and $\alpha$ is the largest ordinal such that $\text{Hom}_R(U, R/\text{Soc}_\alpha(R_R)) \neq 0$ (see [9 Theorem 1.3]) while, if $R$ is regular, $\alpha$ is the unique ordinal with this property.

1. Some preliminary notions on artinian partially ordered sets.

Let $I$ be a given partially ordered set. For every subset $J \subset I$ define
\[
\{ \leq J \} := \{ k \in I \mid k \leq j \text{ for all } j \in J \},
\{ J \leq \} := \{ k \in I \mid j \leq k \text{ for all } j \in J \};
\]
thus the notations $\{ \leq i \}$ and $\{ i \leq \}$ have an obvious meaning for every element $i \in I$. A lower subset (resp. upper subset) of a poset $I$ is a subset $J \subset I$ such that if $j \in J$, then $\{ \leq j \} \subset J$ (resp. $\{ J \leq \} \subset J$). In particular $\{ \leq K \}$ and $\{ K \leq \}$ are respectively the smallest lower subset and the smallest upper subset of $I$ which contain a given subset $K \subset I$. We denote by $\uparrow I$ (resp. $\downarrow I$) the set of all upper subsets (resp. lower subsets) of $I$; both $\uparrow I$ and $\downarrow I$ are complete and distributive lattices and the map $J \mapsto I \setminus J$ is an anti-isomorphism from $\uparrow I$ to $\downarrow I$.

For every subset $J$ of $I$ let us denote by $J_1$ the set of all minimal elements of $J$. We recall that the dual classical Krull filtration of a poset $I$ is the ascending chain $(I_\alpha)_{0 \leq \alpha}$ of subsets of $I$ defined as follows (see [3]):
\[
I_0 := \emptyset,
I_{\alpha+1} := I_\alpha \cup (I \setminus I_\alpha) \quad \text{for all } \alpha,
I_\alpha := \bigcup_{\beta < \alpha} I_\beta \quad \text{if } \alpha \text{ is a limit ordinal}.
\]
Clearly there exists a smallest ordinal \( \xi \) such that \( I_{\xi+1} = I_\xi \); moreover \( I \) is artinian (i.e. it satisfies the DCC or, equivalently, every chain of \( I \) is well ordered) if and only if \( I = I_\xi \) and, in this case, the ordinal \( \xi \) is called the dual classical Krull dimension of \( I \). In the sequel we shall make use of the following further notations: for every ordinal \( \alpha \)

\[ I_\alpha^{**} : = I \setminus I_\alpha, \quad I_{\alpha+1}^* : = (I \setminus I_\alpha)_1. \]

Observe that \( I_\alpha \) is a lower subset, while \( I_\alpha^{**} \) is an upper subset. If \( I \) is artinian, then it is clear that \( \{ I_{\alpha+1}^* \mid \alpha < \xi \} \) is a partition of \( I \) and

\[ I_\alpha = \bigcup_{\beta < \alpha} I_{\beta+1}^* \]

for all \( \alpha < \xi \); we will often call \( I_{\alpha+1}^* \) the \((\alpha + 1)\)-th layer of \( I \). A similar notion is introduced in E. Harzheim book [18] where, given a finite poset \( I \), for every positive integer \( n \) the \( n \)-level \( L_n \) of \( I \) is defined exactly as our \( n \)-th layer. Of course, every subset of an artinian poset is artinian with respect to the induced partial order.

**Proposition 1.1.** If \( J \) is a lower subset of an artinian poset \( I \), then

\[ J_\alpha = J \cap I_\alpha \quad \text{for every ordinal } \alpha. \]

**Proof.** It is obvious that \( J_0 = J \cap I_0 \). Take any ordinal \( \alpha > 0 \) and assume inductively that \( J_\beta = J \cap I_\beta \) for every \( \beta < \alpha \). If \( \alpha \) is a limit ordinal, then one immediately infers that \( J_\alpha = J \cap I_\alpha \). Suppose that \( \alpha = \beta + 1 \) for some \( \beta \). From the inductive hypothesis it follows easily that \( J \setminus J_\beta = J \cap (I \setminus I_\beta) \) and then \( (J \setminus J_\beta)_1 \subset J \cap (I \setminus I_\beta)_1 \), because \( J \) is a lower subset of \( I \). As a result we obtain:

\[ J_{\beta+1} = J_\beta \cup (J \setminus J_\beta)_1 = (J \cap I_\beta) \cup [J \cap (I \setminus I_\beta)_1] = J \cap [I_\beta \cup (I \setminus I_\beta)_1] = J \cap I_{\beta+1}, \]

as wanted. \( \square \)

If \( I \) is any partially ordered class, Gary Brookfield defines in [13] the minimum length function \( \lambda_I : I \to \text{Ord} \) as follows: for every \( i \in I \)

\[ \lambda_I(i) := \min \{ \lambda(j) : \lambda \text{ is a length function on } I \}, \]

where a length function on \( I \) is any strictly increasing function from \( I \) to \( \text{Ord} \). If it exists, \( \lambda_I \) itself is a length function. It turns out that if \( I \) is an artinian poset, then \( I \) admits a length function and \( \lambda_I \) can be defined recursively as follows: for every \( i \in I \)

\[ \lambda_I(i) = \begin{cases} 0 & \text{if } i \text{ is a minimal element of } I, \\ \sup \{ \lambda_I(j) + 1 : j < i \} & \text{otherwise} \end{cases} \]

(see [13] Proposition 3.9).

**Proposition 1.2.** Let \( I \) be an artinian poset, whose dual classical Krull dimension is \( \xi \), and let \( i \in I \). Then for every ordinal \( \alpha \) we have

\[ \lambda_I(i) = \alpha \quad \text{if and only if } i \in I_{\alpha+1}^*. \]

Consequently \( \lambda_I(I) \) is an ordinal and

\[ \lambda_I(I) = \xi. \]
Proof. Denoting by $P(\alpha)$ the statement \((1.2)\), we see that $P(0)$ is obviously true. Given an ordinal $\alpha > 0$, assume that $P(\beta)$ is true whenever $\beta < \alpha$. Suppose that $\lambda_I(i) = \alpha$ and let $\gamma$ be the unique ordinal such that $i \in I^\bullet_{\alpha+1}$. Necessarily $\alpha \leq \gamma$ by the inductive hypothesis, therefore $i \in I^\bullet_{\alpha+1}$. Assume that $i \not\in I^\bullet_{\alpha+1}$, that is, $i$ is not a minimal element of $I^\bullet_{\alpha+1}$. Then there would be some $j \in I^\bullet_{\alpha+1}$ such that $j < i$ and hence $\lambda_I(j) < \lambda_I(i) = \alpha$. Using the inductive hypothesis we would get $j \in I^\bullet_{\lambda_I(j)+1} \cap I^\bullet_{\alpha+1} = \emptyset$: a contradiction. Hence $i \in I^\bullet_{\alpha+1}$. Conversely, suppose that the latter condition holds. If $j < i$, then $j \in I_\alpha$ and so there is some $\beta < \alpha$ such that $j \in I^\bullet_{\beta+1}$. As a consequence it follows from the inductive hypothesis that $\lambda_I(j) = \beta < \alpha$ and hence $\lambda_I(j) + 1 \leq \alpha$, showing that $\lambda_I(i) \leq \alpha$. It is not the case that $\lambda_I(i) < \alpha$ otherwise, again from the inductive hypothesis we would get $i \in I^\bullet_{\alpha+1} \cap I^\bullet_{\lambda_I(i)+1} = \emptyset$. We conclude that $\lambda_I(i) = \alpha$, namely that $P(\alpha)$ holds and this shows the first part of the proposition.

Now, by the assumption we have that
\[
I = \bigcup_{\alpha < \xi} I^\bullet_{\alpha+1}.
\]
If $\alpha < \xi$, namely $\alpha \in \xi$, then $I^\bullet_{\alpha+1}$ is not empty and, by the above, $\lambda_I(i) = \alpha$ for every $i \in I^\bullet_{\alpha+1}$. Thus $\xi \subseteq \lambda_I(I)$. Conversely, if $\alpha \in \lambda_I(I)$, that is $\alpha = \lambda_I(i)$ for some $i \in I$, again by the above we must have that $i \in I^\bullet_{\alpha+1}$, therefore $\alpha < \xi$. As a result $\lambda_I(I) \subseteq \xi$, which proves the equality \((1.3)\). \(\square\)

Notation 1.3. If $I$ is an artinian poset and $i \in I$, we shall denote by $\lambda(i)$ the ordinal $\lambda_I(i)+1$; in other words $\lambda(i)$ will be the unique successor ordinal such that $i \in I^\bullet_{\lambda(i)}$. Of course, the map $i \mapsto \lambda(i)$ defines a particular length function $\lambda: I \to \text{Ord}$; we call it the canonical length function, since it suits our future purposes better than the minimal length function.

According to \([13]\) Corollary 3.5, if $I$ is an artinian poset and $i \in I$, then $\lambda_I(j) = \lambda_{\{i\}}(j)$ for every $j \in \{i\}$; thus, combining Proposition\([12]\) with \([13]\) Proposition 3.6 we obtain the following result.

Corollary 1.4. Let $I$ be an artinian poset and let $i \in I$. Then for every ordinal $\alpha < \lambda(i)$ there exists an element $j \in I^\bullet_{\alpha+1}$ such that $j < i$.

Remark 1.5. It is quite natural that sometimes authors working in different areas of Mathematics concentrate the interest on the same object. As often happened, and continues to happen, according tho the specific area in which it is considered that object gets different names. This is the case for posets which satisfy DCC: ring theorists call them artinian posets, as we do, while set theorists, in particular those who investigate partially ordered sets, call them well-founded posets and call well quasi-ordered, or partially well-ordered the well-founded posets without infinite antichains (see \([18]\), for instance).

2. The natural partial order of \(\text{Simp}_R\) when \(R\) is a semiartinian regular ring.

We recall that if $R$ is any regular ring, then \(\text{Soc}(R_R) = \text{Soc}(R_R);\) in fact, every minimal right (or left) ideal of $R$ is generated by an idempotent and, for every idempotent $e \in R$, we have that $eR_R$ is simple if and only if $eRe$ is simple. By a straightforward induction it follows also that $\text{Soc}_\alpha(R_R) = \text{Soc}_\alpha(R_R)$ for every
ordinal $\alpha$. Thus, when dealing with a regular ring $R$, there will be no ambiguity in using the notations $\text{Soc}(R)$ and $\text{Soc}_\alpha(R)$.

Throughout this section, if not otherwise specified, $R$ will be a given semiartinian and regular ring with Loewy length $\xi$ and we set

$$L_\alpha := \text{Soc}_\alpha(R)$$

for every ordinal $\alpha$. As a first consequence it is easy to infer that if $x \in R$, then

$$h(xR) = \min\{ \alpha \leq \xi \mid x \in L_\alpha \}$$

and we write $h(x)$ for $h(xR)$ (see the introduction for the definition of the length function $h$). As we anticipated in the introduction, by the regularity of $R$ the correspondence $U \mapsto r_R(U)$ defines an order isomorphism from the set $\text{Simp}_R$, equipped with the natural partial order introduced in \[11\], and the set $\text{Prim}_R$ of all primitive ideals ordered by inclusion; this latter is then an artinian poset in which every maximal chain has a maximum. The hypothesis of regularity of $R$ allows to give the following characterizations of the natural partial order of $\text{Simp}_R$, in addition to those we gave in \[11\, \text{Theorem 2.2}\].

**Theorem 2.1.** Let $R$ be a semiartinian and regular ring and let $U, V$ be simple right $R$-modules such that $\alpha + 1 = h(U) < \beta + 1 = h(V)$. Then the following conditions are equivalent:

1. $U \prec V$.
2. If $y \in L_{\beta+1} \setminus L_{\beta}$ is such that $(yR + L_{\beta+1})/L_{\beta} \simeq V$, then
   $$U^n \lesssim (yR + L_{\alpha+1})/L_{\alpha} \quad \text{for every positive integer } n.$$
3. If $y \in L_{\beta+1} \setminus L_{\beta}$ is such that $(yR + L_{\beta+1})/L_{\beta} \simeq V$, then for every positive integer $n$ there is $x \in L_{\alpha+1} \setminus L_\alpha$ such that $(xR + L_{\alpha+1})/L_{\alpha} \simeq U$ and
   $$(xR)^n \lesssim yR \quad \text{(here } (xR)^n \text{ stands for the direct sum of } n \text{ copies of } xR).$$
4. If $y \in L_{\beta+1} \setminus L_{\beta}$ is such that $(yR + L_{\beta+1})/L_{\beta} \simeq V$, then there is $x \in L_{\alpha+1} \setminus L_\alpha$ such that $(xR + L_{\alpha+1})/L_{\alpha} \simeq U$ and
   $$RxR \subset RyR.$$

The above elements $x, y$ can be chosen to be idempotent.

**Proof.** (1)$\Rightarrow$(2) Assume (1), take $y \in L_{\beta+1} \setminus L_{\beta}$ with $(yR + L_{\beta+1})/L_{\beta} \simeq V$, set $A = (yR + L_{\alpha+1})/L_\alpha$ and note that $U \lesssim A$ by \[11\, \text{Theorem 2.2}\]. Let $B = A \cap \text{Tr}_R/L_\alpha(U)$ and suppose that $B$ is finitely generated. Then $A = B \oplus C$ for some $C \leq A$ and there is an idempotent $z \in R$ such that $C = (zR + L_{\alpha+1})/L_\alpha$. Observing that $B = BL_\beta$, we infer that $V \simeq A/AL_\beta \simeq (B/BL_\beta) \oplus (C/CL_\beta) \simeq (C/CL_\beta) \simeq (zR + L_{\beta+1})/L_{\beta}$; on the other hand $\text{Hom}_R(U, (zR + L_{\alpha+1})/L_\alpha) = 0$ by the above and this leads to a contradiction with (1), taking \[11\, \text{Theorem 2.2}\] into account. Thus (2) holds.

(2)$\Rightarrow$(3) Suppose (2), let $y$ be as in (3) and choose $u \in L_{\alpha+1} \setminus L_\alpha$ with $uR/uL_\alpha \simeq (uR + L_{\alpha+1})/L_\alpha \simeq U$. As $U \lesssim (yR + L_{\alpha+1})/L_\alpha \simeq yR/yL_\alpha$, it follows from \[21\, \text{Proposition 2.20}\] that $uR = xR \oplus x'R$, where $xR \lesssim yR$ and $x'R \subset L_\alpha$. Thus $xR/xL_\alpha \simeq U$ and (3) is true with $n = 1$. Let $n \geq 1$ and assume that $(uR)^n \lesssim yR$ for some $u \in R$ such that $uR/uL_\alpha \simeq U$. Then $yR = y'R \oplus y''R$, where $y'R \simeq (uR)^n \subset L_\beta$ and therefore $y''R/y''L_\beta \simeq V$. By the inductive hypothesis
modules are always minimal elements of $\text{Simp}$ every ordinal minimal simple modules (see also Example 2.8, Section 3). primitive factors artinian (see [17] and [8]); in this case every element of Proposition and Definition 2.2. general result.

with the above notations, the following conditions are equivalent:

$$(2.1) \quad \text{Simp}_R$$

where $U 
\text{Simp}_R$ and

(2.1) \quad \lambda(U) \leq h(U) \quad \text{for all } U \in \text{Simp}_R,$

where $U \mapsto \lambda(U)$ is the canonical length function on $\text{Simp}_R$ (Notation [13]). The inequality in (2.1) may be strict. For example, given any successor ordinal $\xi$, there exists a regular and semiartinian ring $R$ with Loewy length $\xi$ and having all primitive factors artinian (see [17] and [8]); in this case every element of $\text{Simp}_R$ is maximal (see [11, Corollary 4.8]), that is $\text{Simp}_R$ is an antichain and, if $\xi > 1$, for every ordinal $\alpha$ such that $1 \leq \alpha < \xi$ there are infinitely many $U \in \text{Simp}_R$ with $h(U) = \alpha$, while $\lambda(U) = 1$ for every $U \in \text{Simp}_R$. Thus, while simple projective modules are always minimal elements of $\text{Simp}_R$, there may exist non-projective minimal simple modules (see also Example [2.3, Section 3].

We now investigate when the inequality (2.1) is actually an equality. First a general result.

**Proposition and Definition 2.2.** If $R$ is a regular and semiartinian ring $R$ then, with the above notations, the following conditions are equivalent:

1. $\lambda(U) = h(U)$ for every $U \in \text{Simp}_R$.
2. For every ordinal $\alpha$ the following equality holds:

$$\text{Simp}_{R/L_\alpha} = \{ U \in \text{Simp}_R \mid U\lambda_\alpha = U \}.$$

3. For every ordinal $\alpha$ the following equality holds:

$$\text{Prosimp}_{R/L_\alpha} = \left( \text{Simp}_{R/L_\alpha} \right)_1.$$

If any, and hence all of the above conditions holds, then we say that $R$ is well behaved.

**Proof.** First, observe that for every ordinal $\alpha$ we have the equalities

$$\text{Simp}_{R/L_\alpha} = \{ U \in \text{Simp}_R \mid \lambda(U) \leq \alpha \},$$

$$\{ U \in \text{Simp}_R \mid U\lambda_\alpha = U \} = \{ U \in \text{Simp}_R \mid h(U) \leq \alpha \},$$

the first of which follows from Proposition [2.2]. Thus, since $\lambda(U) \leq h(U)$ for every $U \in \text{Simp}_R$, the equivalence between (1) and (2) easily follows.
Given any ordinal \( \alpha \), it follows from (2) that
\[
\text{Prosimp}_{R/L_\alpha} = \{ U \in \text{Simp}_R \mid h(U) = \alpha + 1 \} = (\text{Simp}_R)^{\alpha+1} = (\text{Simp}_R \setminus (\text{Simp}_R)_\alpha) \setminus_1 = \left( \text{Simp}_{R/L_\alpha} \right)_1,
\]
hence the equality \((2.3)\) holds.

Assume (1), let \( P(\alpha) \) denote the property
\[
\text{Simp}_R = \{ U \in \text{Simp}_R \mid UL_\alpha = U \} = (\text{Simp}_R)_\alpha
\]
and let us prove that \( P(\alpha) \) is true for every ordinal \( \alpha \).
If \( \alpha = 0 \), then \( P(\alpha) \) is merely \( \emptyset = \emptyset \).
Given an ordinal \( \alpha > 0 \), assume that \( P(\beta) \) holds for every \( \beta < \alpha \).
If \( \alpha \) is a limit ordinal, then \( P(\alpha) \) follows from the fact that \( L_\alpha = \bigcup_{\beta < \alpha} L_\beta \).
Assume that \( \alpha = \beta + 1 \) for some \( \beta \).
Then we have
\[
(\text{Simp}_R)^{\beta+1} = (\text{Simp}_R)^{\beta} \cup (\text{Simp}_R \setminus (\text{Simp}_R)^{\beta}) \setminus_1 = \{ U \in \text{Simp}_R \mid UL_\beta = U \} \cup (\text{Simp}_R/L_\beta) \setminus_1 = \{ U \in \text{Simp}_R \mid UL_{\beta+1} = U \},
\]
proving the equality \((2.2)\). \( \square \)

There are at least three interesting situations in which a regular and semiartinian ring \( R \) turns out to be well behaved. The first two are certain finiteness conditions on the poset \( \text{Simp}_R \) and are the subject of the remaining part of the present section; the third one is connected with a comparability condition and will be discussed in Section 4.

**Lemma 2.3.** Let \( R \) be a regular and semiartinian ring and let \( U, V \in \text{Simp}_R \) be such that \( h(U) < h(V) \). If \( U, V \) are not comparable and \( x \) is an idempotent such that \( (xR + L_{h(V)-1})/L_{h(V)-1} \simeq V \), then there is a nonnegative integer \( n \) and two orthogonal idempotents \( y, z \) such that \( xR = yR \oplus zR \) and satisfying the following conditions:

\[
(2.4) \quad yR + L_{h(V)-1}/L_{h(V)-1} \simeq V;
(2.5) \quad zR + L_{h(h(V))-1}/L_{h(V)-1} \simeq U^n;
(2.6) \quad U \not\preceq (yR + L_{h(V)-1})/L_{h(V)-1}.
\]

**Proof.** According to Theorem \(\ref{thm:regularity} \) we may consider the largest nonnegative integer \( n \) such that \( U^n \) imbeds, necessarily as a direct summand, into \( (xR + L_{h(V)-1})/L_{h(V)-1} \).
By the regularity of \( R \), there are orthogonal idempotents \( y, z \) such that \( xR = yR \oplus zR \) and \( (2.5) \) holds. Now \( (2.4) \) follows since \( z \in L_{h(V)-1} \) and the choice of \( n \) guarantees that \( (2.6) \) holds as well. \( \square \)

**Proposition 2.4.** Let \( R \) be a regular and semiartinian ring. If the layer \( (\text{Simp}_R)^{\alpha} \) is finite for every \( \alpha \), then \( R \) is well behaved.
Proof. Given an ordinal \( \alpha \), let \( P(\alpha) \) denote the following property:

\[
\text{if } U \in \text{Simp}_R \text{ and } h(U) = \alpha + 1, \text{ then } \lambda(U) = h(U).
\]

Our task is to show that \( P(\alpha) \) is true for every \( \alpha \). Without the regularity hypothesis on \( R \), we already know that \( P(0) \) holds. Thus, given an ordinal \( \alpha > 0 \), suppose inductively that \( P(\beta) \) holds whenever \( \beta < \alpha \), take \( U \in \text{Simp}_R \) such that \( h(U) = \alpha + 1 \) and assume that \( \lambda(U) = \beta + 1 < \alpha + 1 \). It follows from the inductive assumption that \( \text{Prosimp}_{R/L_\beta} \) is contained in the \( \beta + 1 \)-th layer \( (\text{Simp}_R)^{\beta+1} \) to which \( U \) belongs, consequently \( V \not\in U \) for all \( V \in \text{Prosimp}_{R/L_\beta} \). On the other hand, by the hypothesis \( (\text{Simp}_R)^{\beta+1} \) is finite, therefore, by applying finite induction and Lemma 2.3, we obtain that there exists an idempotent \( y \in R \) such that \( (yR + L_\alpha)/L_\alpha \simeq U \) and \( V \not\subset (yR + L_\beta)/L_\beta \) for every \( V \in \text{Prosimp}_{R/L_\beta} \). Inasmuch as the trace of \( \text{Prosimp}_{R/L_\beta} \) in \( R/L_\beta \) equals the socle and, whence, is essential, we infer that \( (yR + L_\beta)/L_\beta = 0 \) and so \( y \in L_\beta \). This contradicts the assumption that \( h(U) = \alpha + 1 > \beta \). We conclude that \( \lambda(U) = \alpha + 1 \) and this shows that \( P(\alpha) \) is true.

There is a natural way to link the ideal structure of a regular and semiartinian ring \( R \) and the order structure of \( \text{Simp}_R \). Indeed, observe that if \( H \) is an ideal of \( R \), then \( \text{Simp}_{R/H} \) is an upper subset of \( \text{Simp}_R \), so that we may consider the decreasing map

\[
\Phi: L_2(R) \longrightarrow \uparrow\text{Simp}_R
\]

defined by \( \Phi(H) = \text{Simp}_{R/H} \). This map is injective and has as a left inverse the map

\[
\Psi: \uparrow\text{Simp}_R \longrightarrow L_2(R)
\]

defined by \( \Psi(S) = \bigcap \{ r(R(U)) \mid U \in S \} \). In fact, it is clear that \( \Phi(H) \supset \Phi(K) \) whenever \( H \subset K \). Inasmuch as \( R \) is regular, then every ideal of \( R \) is the intersection of all primitive ideals containing it. Thus, given \( H \in L_2(R) \), we have

\[
\Psi(\Phi(H)) = \Psi \left( \text{Simp}_{R/H} \right) = \bigcap \{ r(R(U)) \mid U \in \text{Simp}_{R/H} \}
\]

\[
= \bigcap \{ r(R(U)) \mid U \in \text{Simp}_R \text{ and } UH = 0 \} = H.
\]

Definition 2.5. We say that \( R \) is very well behaved in case \( \Phi \) and \( \Psi \) are anti-isomorphisms each inverse of the other.

If \( \text{Simp}_R \) has no infinite antichains, then \( R \) is very well behaved; this is a particular case of \( \text{[11, Theorem 4.5]} \), because all ideals of a regular ring are left pure. In general, as we are going to see the property of being \( R \) very well behaved entails a finiteness condition on the poset \( \text{Simp}_R \). We can see it at first in case \( R \) has all primitive factor rings artinian.

Proposition 2.6. If \( R \) is a semiartinian and regular ring with all right primitive factor rings artinian, then \( R \) is very well behaved if and only if \( R \) is semisimple.

Proof. Assume that \( R \) is not semisimple. Then \( \text{Simp}_R \) is an infinite antichain and \( \text{Prosimp}_R \) is a proper upper subset of \( \text{Simp}_R \). Since we have

\[
\Psi(\text{Prosimp}_R) = 0 = \Psi(\text{Simp}_R),
\]

it follows that \( \Phi \) is not an anti-isomorphism. \( \square \)
Proposition 2.7. Let $R$ be a regular and semiartinian ring. If $R$ is very well behaved, then the following properties hold:

1. Every factor ring of $R$ is very well behaved.
2. $R$ is well behaved and $\text{Simp}_R$ has finitely many maximal elements.

Proof. (1) Let $H$ be an ideal of $R$, let $S$ be an upper subset of $\text{Simp}_{R/H}$ and let $U \in \text{Simp}_R$, $V \in S$ be such that $V \preceq U$. Then $UH = 0$, therefore $U \in \text{Simp}_{R/H}$ and hence $U \in S$. We infer that $\uparrow \text{Simp}_{R/H} \subseteq \uparrow \text{Simp}_R$. As a consequence, the restrictions of $\Phi$ and $\Psi$ to $\{H \subset \}$ and $\uparrow \text{Simp}_{R/H}$, respectively, define an anti-isomorphism from $\{H \subset \}$ to $\uparrow \text{Simp}_{R/H}$. As a result, the assignment $K/H \mapsto \text{Simp}_R/K$ is an anti-isomorphism from $L_2(R/H)$ to $\uparrow \text{Simp}_{R/H}$, meaning that $R/H$ is very well behaved.

(2) We claim that if $R$ is very well behaved, then $\text{ProSimp}_R = (\text{Simp}_R)_1$. Indeed, by setting $S = \{\text{ProSimp}_R \preceq \}$, we have that $\Psi(S) = 0$ and consequently $S = \Phi(\Psi(S)) = \Phi(0) = \text{Simp}_R$.

As a result, for every $U \in \text{Simp}_R$ we have that $\lambda(U) = 1$ implies $h(U) = 1$, proving our claim. Given any ordinal $\alpha$, according to (1) the ring $R/L_\alpha$ is very well behaved and we infer from the above that $\text{ProSimp}_{R/L_\alpha} = (\text{Simp}_{R/L_\alpha})_1$. Thus $R$ is well behaved.

Finally, if $M$ is the set of all maximal elements of $\text{Simp}_R$ and $H = \Psi(M)$, then $R/H$ is very well behaved and has all primitive factor rings artinian. Thus $R/H$ is semisimple by Proposition 2.6 and so $M$ is finite. □

The two conditions in property (2) of the previous proposition are actually independent and, even together, do not imply that $R$ is very well behaved; moreover a factor ring of a well behaved ring need not be well behaved. We illustrate all this with the next example, which also shows that the reverse of Proposition 2.4 does not hold; however we have to wait till the last section (see Theorem 9.5, properties (7) and (8)) in order to see that there exists a regular and semiartinian ring $R$ such that each layer $(\text{Simp}_R)_\alpha$ is finite for every $\alpha$, but $\text{Simp}_R$ has infinitely many maximal elements, so that $R$ is well behaved but is not very well behaved.

Example 2.8. There exists an indecomposable, semiartinian and regular ring $R$, together with a semiartinian and regular subring $S$, satisfying the following conditions:

1. Both $\text{Simp}_R$ and $\text{Simp}_S$ have finitely many maximal elements.
2. $R$ is well behaved but not very well behaved.
3. $S$ is not well behaved and is isomorphic to a factor ring of $R$.

Proof. Given a field $F$, let us consider the ring $Q = \mathbb{CFM}_R(F)$ and remember that $\text{Soc}(Q) = F \mathbb{R}_N(F)$ consists of all matrices with finitely many nonzero rows. By setting $X = \{2, 4, 6, \ldots \}$ and $Y = \{1, 3, 5, \ldots \}$, for the purposes of the example we want to build it is convenient to view the elements of $Q$ as blocked matrices of the form $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $A \in \mathbb{CFM}_X(F)$, $B \in \mathbb{CFM}_{X,Y}(F)$, $C \in \mathbb{CFM}_{Y,X}(F)$ and $D \in \mathbb{CFM}_Y(F)$. Set $T = \prod_{n>0} T_n$, where $T_n = Q$ for all $n > 0$, and let us consider
the idempotents $v, w \in T$ defined by

$$
v_n = \begin{pmatrix}
1 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots \\
0 & 1 & 0 & \ldots & 0 & 0 & 0 & \ldots \\
0 & 0 & 1 & \ldots & 0 & 0 & 0 & \ldots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\
0 & 0 & 0 & \ldots & 1 & 0 & 0 & \ldots \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\
\end{pmatrix},
$$

and

$$
w_n = \begin{pmatrix}
0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\
\end{pmatrix},
$$

for all $n > 0$; note that $v, w$ are orthogonal and $v + w = 1_T$. Given $A \in \text{CMF}_X(F)$, let us define the element $x_A \in T$ by setting $(x_A)_n = \left( \frac{A}{0} \frac{0}{1} \right)$ for all $n > 0$ and set

$$K := \{x_A \mid A \in \text{CMF}_X(F)\}.$$

Next, for every $n > 0$ let $L_n$ be the subset of $T$ of those elements $x$ such that $x_m = 0$ if $m \neq n$ and $x_n = \left( \frac{0}{0} \frac{1}{D} \right)$ for some $D \in \text{CMF}_V(F)$. Now it is immediate to check that $vF, wF, K$ and $L := \bigoplus_{n>0} L_n$ are independent $F$-subspaces of $T$ and

$$R := vF \oplus wF \oplus K \oplus L$$

is a regular subring of $T$. It can be seen easily that $K, L_1, L_2, \ldots$ are minimal ideals of $R$ which are the traces of pairwise non isomorphic simple projective right $R$-modules $U_0, U_1, U_2, \ldots$ respectively; moreover

$$\text{Soc}(R) = K \oplus L \quad \text{and} \quad R/\text{Soc}(R) \simeq F \times F,$$

therefore $R$ is semiartinian with Loewy length 2. Easy computations show that

$$vR + \text{Soc}(R) = vF \oplus \text{Soc}(R), \quad wR + \text{Soc}(R) = wF \oplus \text{Soc}(R)$$

are ideals of $R$ and

$$V := (vF \oplus \text{Soc}(R))/\text{Soc}(R) \simeq R/(wF \oplus \text{Soc}(R)), \quad W := (wF \oplus \text{Soc}(R))/\text{Soc}(R) \simeq R/(vF \oplus \text{Soc}(R))$$

are non isomorphic simple right $R$-modules, which are the maximal elements of $\text{Simp}_R$. Now observe that, given $n > 0$, $a, b \in F$, $k \in K$ and $l \in L$, the element $x = va + wb + k + l$ annihilates $U_n$ if and only if $b = 0$ and $l_n = -(va)_n$. We infer that $r_R(U_n) \subset r_R(V)$ but $r_R(U_n) \not\subset r_R(V)$. On the other side $x$ annihilates $U_0$ if and only if $a = 0$ and $k = 0$, so that $r_R(U_0) \subset r_R(V)$ but $r_R(U_0) \not\subset r_R(V)$. This shows that the Hasse diagram of $\text{Simp}_R$ is

$$V \quad W \quad \ldots \quad \ldots$$

and $R$ is well behaved. If we take $S = \{W, U_1, U_2, \ldots\}$, then $S$ is an upper subset of $\text{Simp}_R$ and $r_R(S) = K$. However $\text{Simp}^R_{R/K} = \{V, W, U_1, U_2, \ldots\} \nsubseteq S$, therefore $R$ is not very well behaved.
Next, let us consider the subring
\[ S := vF \oplus wF \oplus L \]
of \( R \), which is clearly isomorphic to the factor ring \( R/K \). We see easily that in the poset \( \text{Simp}_S \) we have \( \lambda(V) = 1 \), but \( h(V) = 2 \). Thus \( S \) is not well behaved, yet \( \text{Simp}_S \) has finitely many maximal elements. Finally, both \( R \) and \( S \) are indecomposable rings, because 0 and 1 are the only central idempotents of \( R \).

3. Connected components of \( \text{Simp}_R \).

Let \( I \) be a poset. Given \( i, j \in I \), let us write \( i \bowtie j \) to mean that either \( i \leq j \), or \( i \geq j \), and write \( i \sim j \) to mean that there are \( k_0, k_1, \ldots, k_n \in I \) such that
\[ i = k_0 \bowtie k_1 \bowtie \cdots \bowtie k_n = j. \]
Then \( \sim \) is the smallest equivalence relation in \( I \) containing the partial order of \( I \). The elements of \( I/\sim \) are called the connected components of \( I \); let us call the canonical partition of \( \text{Simp}_R \) the factor set \( \text{Simp}_R/\sim \). There is a natural link between the connected components of \( \text{Simp}_R \) and central idempotents of \( R \). First note that, without any assumption on the ring \( R \), for every complete set \( \{e_1, \ldots, e_n\} \) of pairwise orthogonal and central idempotents of \( R \) the set
\[ \{\text{Simp}_{e_1}, \ldots, \text{Simp}_{e_n}\} \]
is a partition of \( \text{Simp}_R \); in our present context, in which \( R \) is semiartinian and regular, this partition is always coarser or equal to the canonical partition. To see this, it is sufficient to note that if \( U \in \text{Simp}_{e_i} \) and \( V \in \text{Simp}_{e_j} \) with \( i \neq j \), then \( r_R(U) \not\subset r_R(V) \), meaning that \( U \not\bowtie V \) is false and therefore \( U \sim V \) is false too. In particular, if \( \text{Simp}_R \) consists of a single connected component, then \( R \) is indecomposable as ring, while the converse may fail; in fact Example 2.8 displays two indecomposable semiartinian and regular rings \( R \) and \( S \) for which both \( \text{Simp}_R \) and \( \text{Simp}_S \) consist of two connected components.

As we are going to see, if \( \text{Prosimp}_R \) is finite, then there is a complete set \( \{e_1, \ldots, e_n\} \) of pairwise orthogonal and central idempotents of \( R \) such that (3.1) coincides with the canonical partition.

**Proposition 3.1.** Let \( R \) be a semiartinian and regular ring. Then \( \text{Simp}_R \) has finitely many minimal elements if and only if \( \text{Prosimp}_R \) is finite. If it is the case, then \( \text{Prosimp}_R \) coincides with the set of all minimal elements of \( \text{Simp}_R \) and there is a complete set \( \{e_1, \ldots, e_n\} \) of pairwise orthogonal and central idempotents such that (3.1) coincides with the canonical partition; in particular each \( e_iR \) is an indecomposable ring.

**Proof.** We already know that \( \text{Prosimp}_R \) is always contained in the set of minimal elements of \( \text{Simp}_R \), thus the “only if” part is obvious. Suppose that \( \text{Prosimp}_R \) is finite, let \( U \) be a minimal element of \( \text{Simp}_R \) and suppose that \( U \) is not projective. Then \( h(U) = \alpha + 1 \) for some \( \alpha > 0 \) and, by applying finite induction and Lemma 2.3, we infer that there is some \( y \in L_{\alpha+1} \) such that \( yR/yL_\alpha \simeq U \) and \( \text{Hom}_R(P, yR) = 0 \) for every \( P \in \text{Prosimp}_R \). But this means that \( yR \cap \text{Soc}(R) = 0 \), which is a contradiction since \( \text{Soc}(R) \) is essential as a right ideal and \( y \neq 0 \).

Assume now that \( \text{Prosimp}_R \) is finite and let \( \{S_1, \ldots, S_n\} \) be the canonical partition of \( \text{Simp}_R \). For every \( i \in \{1, \ldots, n\} \) and \( U \in S_i \), by applying again finite
induction and Lemma 2.3 we can choose an idempotent $y_U \in L_{h(U)}$ which satisfies the following conditions:

$$y_U R/y_U L_{h(U)}^{-1} \simeq U,$$

$$\text{Hom}_R (P, y_U R) = 0 \quad \text{for all } P \in \text{Prosimp}_R \quad \text{such that } P \not\in S_i.$$

We may then consider the ideal $R_i = \sum \{ R y_U R \mid U \in S_i \}$ and it is clear that $U = U(R y_U R) = UR_i$. We claim that $R$ decomposes as

\begin{equation}
R = R_1 \oplus \cdots \oplus R_n.
\end{equation}

First, since $R$ is regular, in order to prove that the sum $R_1 + \cdots + R_n$ is direct it is sufficient to show that if $i \neq j$, then $R_i R_j = 0$. Thus, take $U \in S_i$ and $V \in S_j$ with $i \neq j$. If $K$ is a simple right ideal contained in $y_U R$, then $K \simeq P$ for a unique $P \in \text{Prosimp}_R$. Necessarily $P \in S_i$ and therefore $\text{Hom}_R (P, y_U R) = 0$. By using the fact that $\text{Soc}(R)$ is projective, we infer that

$$\text{Soc}(R y_U R) \text{Soc}(R y_U R) = \text{Soc}(R y_U R) \cap \text{Soc}(R y_U R) = 0$$

and hence $(R y_U R) (R y_U R) = (R y_U R) \cap (R y_U R) = 0$ by the essentiality of the socle. Finally, since $U = U(R_1 \oplus \cdots \oplus R_n)$ for every simple module $U_R$, we conclude that the equality (3.2) holds. There is a complete set $\{ e_1, \ldots, e_n \}$ of pairwise orthogonal and central idempotents such that $e_i R = R_i$ for all $i$ and it follows from the above that $\text{Simp}_{e_i R} = S_i$ for all $i$. \hfill \Box

**Remark 3.2.** It is worth of note that the assumption of regularity of the ring $R$ cannot be dropped in Proposition 3.1. Indeed, with [11] Example 4.8 we presented an indecomposable Artinian algebra $R$ for which $\text{Simp}_R$ consists of two connected components; yet, $\text{Simp}_R$ is finite.

### 4. Comparability.

We keep the same setting and notations of the previous section. In the literature on regular rings we find two conditions involving comparability between principal right ideals which play a central role in the structure theory of these rings. Precisely, a regular ring $R$ satisfies the *comparability* axiom if, given $x, y \in R$, one has that either $x R \preceq y R$ or $y R \preceq x R$, while $R$ satisfies the *general comparability* axiom if, given $x, y \in R$, there exists some central idempotent $e$ such that $e x R \preceq e y R$ and $(1 - e) y R \preceq (1 - e) x R$ (see [21]). An additional axiom, which makes sense when $R$ is semiartinian and regular, was introduced in [10]: $R$ satisfies the *restricted comparability* axiom if, given $x, y \in R$, the condition $h(x) < h(y)$ implies that $x R \preceq y R$. Comparability implies general comparability. If $R$ is a regular and semiartinian ring satisfying comparability, then it satisfies also restricted comparability. Indeed, if $x, y \in R$ with $h(x) < h(y)$, it is not the case that $y R \preceq x R$ otherwise, since $x \in L_{h(x)}$, it would follow that $y \in L_{h(x)}$ too, that is $h(y) \leq h(x)$. Thus $x R \preceq y R$. As we know from Theorem 2.1 the natural partial order of $\text{Simp}_R$ can be expressed in terms of the existence of an imbedding between certain principal right ideals; thus, it appears quite natural to ask if, given a semiartinian and regular ring $R$, there is any relationship between the above axioms and properties of the poset $\text{Simp}_R$. The results which follow give some answer to this question.
Proposition 4.1. Let $R$ be a semiartinian and regular ring. Then $R$ satisfies the restricted comparability axiom if and only if the following condition holds:

\[(4.1) \quad \text{for every } U, V \in \text{Simp}_R, \text{ if } h(U) < h(V), \text{ then } U \prec V.\]

In particular $R$ satisfies the comparability axiom if and only if $\text{Simp}_R$ is a chain. If $R$ satisfies the restricted comparability axiom, then $R$ is well behaved.

Proof. The “only if” part follows immediately from Theorem 2.1. In order to prove the “if” part, we first observe that, for every ordinal $\alpha$, the Loewy chain of the ring $R/L_\alpha$ is $(L_\alpha/L_\alpha)_{\alpha \leq \gamma}$ and each primitive ideal of $R/L_\alpha$ has the form $P/L_\alpha$ for a unique primitive ideal $P$ of $R$. Consequently, if $R$ satisfies (4.1), then the same holds for $R/L_\alpha$. Given an ordinal $\alpha$, let $P(\alpha)$ denote the sentence

\[\text{“If } x, y \in R \text{ and } \alpha + 1 = h(x) < h(y), \text{ then } xR \preceq yR.”\]

Then the proof of the first part of the proposition will be complete once we have shown that $P(\alpha)$ is true for every ordinal $\alpha$. Let $y \in R$ be such that $h(y) = \beta + 1$. Then there is a decomposition $yR = y_1R \oplus \cdots \oplus y_nR$, where each $y_iR/y_iL_\beta$ is simple and $h(y_iR/y_iL_\beta) = \beta + 1$. If $x \in L_\lambda = \text{Soc}(R)$, namely $h(x) = 1$, then $xR = P_1 \oplus \cdots \oplus P_m$, where each $P_j$ is simple with $h(P_j) = 1$. Thus, given $j \in \{1, \ldots, m\}$ and $i \in \{1, \ldots, n\}$, it follows from the assumption that $P_j \prec y_iR/y_iL_\beta$ and we infer from Theorem 2.1 that $P_j \preceq y_iR$ for every positive integer $k$. This is enough to infer that $xR \preceq yR$ and so the statement $P(0)$ is true. Next, given an ordinal $\alpha > 0$, assume that $P(\beta)$ is true for every $\beta < \alpha$ and take $x, y \in R$ such that $\alpha + 1 = h(x) < h(y)$. Then $0 \neq x + L_\alpha \in L_{\alpha+1}/L_\alpha = \text{Soc}(R/L_\alpha)$, while $y + L_\alpha \notin \text{Soc}(R/L_\alpha)$. Since the ring $R/L_\alpha$ satisfies (4.1), we can apply the above argument and infer that $xR/xL_\alpha \preceq yR/yL_\alpha$. It follows from Theorem 2.1 Proposition 2.20 that there are $x', x'' \in xR$, $y', y'' \in yR$ and decompositions

\[xR = x'R \oplus x''R, \quad yR = y'R \oplus y''R,\]

where $x'R \simeq y'R$ and $x'' \in L_\alpha$. Necessarily $h(y'') = h(y)$ and, since $h(x'') \leq \alpha$, it follows that $h(x'') < h(x) < h(y) = h(y'')$. From the inductive hypothesis we infer that $x''R \preceq y'R$ and therefore $xR \preceq yR$. We conclude that $P(\alpha)$ is true.

If $R$ satisfies the comparability axiom, then $L_2(R)$ is a chain by [21] Proposition 8.5]. Consequently $\text{Prim}_R$ is a chain as well and so is $\text{Simp}_R$. Conversely, if this latter condition holds, then $L_2(R)$ is a chain because every ideal of $R$ is the intersection of primitive ideals. The proof that, consequently, $R$ satisfies the comparability axiom is identical to the proof of [10] Proposition 4].

Assume that $R$ satisfies the restricted comparability axiom. If $U \in \text{Simp}_R$ and $h(U) = 1$, then $U$ is minimal and so $\lambda(U) = 1$. Given a successor ordinal $\alpha + 1$, assume that $\lambda(U) = h(U)$ whenever $h(U) < \alpha + 1$, let $U \in \text{Simp}_R$ such that $h(U) = \alpha + 1$ and suppose that $\lambda(U) < h(U)$. Inasmuch as $\lambda(U)$ is a successor ordinal less than the Loewy length of $R$, there exists $V \in \text{Simp}_R$ such that $h(V) = \lambda(U)$ and, from the inductive hypothesis, we have that $\lambda(V) = h(V) = \lambda(U)$. Thus $U$ and $V$ are not comparable. On the other hand, let $x \in L_{\alpha+1} \setminus L_\alpha$ be such that $xR/xL_{\alpha+1} \simeq U$ and chose $y \in L_{h(V)} \setminus L_{h(V)-1}$ such that $yR/yL_{h(V)-1} \simeq V$. Since $h(V) < \alpha + 1$, by the hypothesis $yR \preceq xR$ and we infer that $\text{Hom}_R(V, xR/xL_{h(V)-1}) \neq 0$. It follows from Theorem 2.1 that $V \prec U$: a contradiction. We conclude that $\lambda(U) = h(U)$ and the proof is complete. \qed
Proposition 4.2. Let $R$ be a semiartinian and regular ring. If $R$ satisfies the general comparability axiom, then $\text{Simp}_R$ is the union of pairwise disjoint maximal chains. Conversely, if $\text{Simp}_R$ is the union of finitely many pairwise disjoint maximal chains, then $R$ satisfies the general comparability axiom and is well behaved.

Proof. Inasmuch as $\text{Simp}_R$ is artinian, it is sufficient to show that $\{U \preceq \} U$ is a chain whenever $U$ is a minimal element of $\text{Simp}_R$. However this follows from [21, Theorem 8.20], combined with Proposition 4.1, since $r_R(U)$ is a prime ideal for every $U \in \text{Simp}_R$.

Conversely, assume that $\text{Simp}_R$ is the union of finitely many pairwise disjoint maximal chains $\{S_1, \ldots, S_n\}$, which are necessarily the connected components of $\text{Simp}_R$. Then $\text{Prosimp}_R$ is finite and, according to Proposition 3.1, $R$ decomposes as in (3.2), where every $R_i$ is a semiartinian and regular ring such that $\text{Simp}_{R_i}$ is a chain. By Proposition 4.1, every $R_i$ satisfies the comparability axiom, therefore $R$ satisfies the general comparability axiom. Now, observe that if $U \in \text{Simp}_R$, then $U = UR_i$ for a unique $i$, while $UR_j = 0$ if $j \neq i$. Consequently, since each ring $R_i$ well behaved by Proposition 4.1 and $\text{Soc}_\alpha(R) = \text{Soc}_\alpha(R_1) \oplus \cdots \oplus \text{Soc}_\alpha(R_n)$ for every ordinal $\alpha$, it is an easy matter to conclude that $R$ is well behaved. \qed

The following example shows that it is not possible to remove the finiteness condition from Proposition 4.2.

Example 4.3. There exists a semiartinian and regular ring $R$, with Loewy length 2 and all primitive factors artinian (hence $\text{Simp}_R$ is the union of pairwise disjoint maximal chains), which does not satisfy the general comparability axiom.

Proof. Given a field $F$, set $R_n = M_2(F)$ for every positive integer $n$ and consider the following regular subring of the direct product $T = \prod_{n>0} R_n$:

$$R = K \oplus L \oplus \left( \bigoplus_{n>0} R_n \right),$$

where

$$K = \{ k \in T \mid \text{there is } a \in F \text{ such that } k_n = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \text{ for all } n > 0 \},$$

$$L = \{ l \in T \mid \text{there is } a \in F \text{ such that } l_n = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \text{ for all } n > 0 \}.$$

We observe that

$$\text{Soc}(R) = \bigoplus_{n>0} R_n \quad \text{and} \quad R/\text{Soc}(R) \cong F \times F,$$

therefore $R$ is semiartinian with Loewy length 2 and has all primitive factors artinian. If we set

$$u = \left( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ldots \right), \quad v = \left( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ldots \right),$$

then

$$U = (uR + \text{Soc}(R))/\text{Soc}(R) \quad \text{and} \quad V = (vR + \text{Soc}(R))/\text{Soc}(R)$$

are non-isomorphic simple $R$-modules and $\text{Simp}_{R/\text{Soc}(R)} = \{U, V\}$. Now an idempotent $e \in \text{Soc}(R)$ is central if and only if all its nonzero coordinates equal $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, while all remaining central idempotents of $R$ are of the form $1 - e$, where $e$ is a
central idempotent of \( \text{Soc}(R) \). If \( e \) is a central idempotent of \( \text{Soc}(R) \), then it is clear that \( euR \simeq evR \), but if \( (1 - e)vR \) were subisomorphic to \( (1 - e)uR \), since \( (1 - e)v \) and \( (1 - e)u \) do not belong to \( \text{Soc}(R) \), we would get

\[
V = ((1 - e)vR + \text{Soc}(R)) / \text{Soc}(R) \lesssim ((1 - e)uR + \text{Soc}(R)) / \text{Soc}(R) = U,
\]

hence a contradiction; similarly, \( (1 - e)uR \) is not subisomorphic to \( (1 - e)vR \). We conclude that \( R \) does not satisfy the general comparability axiom. \( \square \)

5. A very special well ordered chain of subrings of \( \text{CFM}_X(D) \).

With this section we begin the setup which will bring us to the construction of regular and semiartinian rings, starting from an artinian poset. We set the scenario by taking a ring \( D \) (although our final concern will be the case in which \( D \) is a division ring, unless otherwise stated we do not assume anything about \( D \), apart associativity and presence of a multiplicative identity), a transfinite ordinal \( X \) and the ring \( Q = \text{CFM}_X(D) \) of all \( X \times X \)-matrices with entries in \( D \) whose columns have finite support.

**Notations** 5.1. With the above setting, we adopt the following notations:

- We denote by \( 0 \) and \( 1 \) the zero and the unital matrices respectively.
- If \( a \in Q \) and \( x, y \in X \), we use the symbol \( a(x, y) \) to denote the entry at the intersection of the \( x \)-th row with the \( y \)-th column of \( a \) (i.e. the \( (x, y) \)-entry of \( a \)), instead of the more traditional symbol \( a_{xy} \); since we often use more complex arrays, other than single letters, in order to designate the position of the entries of the matrices we deal with, our choice should guarantee a better readability. If \( Y, Z \subseteq X \), then \( a(Y, Z) \) is the \((Y, Z)\)-block of \( a \), that is the submatrix \((a(x, y))_{y \in Y, z \in Z} \) of \( a \).
- For every \( Y \subseteq X \), we denote with \( e_Y \) the idempotent diagonal matrix such that \( e_Y(x, x) = 1 \) if \( x \in Y \) and is 0 otherwise. If \( x, y \in X \), we write \( e_x \) instead of \( e_{\{x\}} \), while \( e_{x,y} \) stands for the matrix whose \((x, y)\)-entry is 1 and all others are zero; so, in particular \( e_x = e_{x,x} \).
- \( \text{FR}_X(D) \) and \( \text{FM}_X(D) \) denote respectively the subset of \( Q \) of all matrices having only finitely many nonzero rows and the subset of \( Q \) of all matrices having only finitely many nonzero entries.

\( \text{FR}_X(D) \) is an ideal of \( Q \) which is of a special interest for us; as a right ideal, it is generated by the set \( \{ e_x \mid x \in X \} \) of pairwise orthogonal idempotents and we have the equalities

\[
(5.1) \quad e_x Q = e_x \text{FR}_X(D),
\]

\[
(5.2) \quad \text{FR}_X(D) = \bigoplus \{ e_y Q \mid y \in X \} = \text{FR}_X(D)e_x \text{FR}_X(D).
\]

Moreover \( \text{FR}_X(D) \) is fully invariant; this follows from a more general result of Del Rio and Simón (see [10], Lemma, 7) although, for the case \( X = \omega \), it was a byproduct of a theorem of Camillo (see [13] and [2]). As a consequence, if \( R \) is any ring and \( \varphi, \psi : Q \to R \) are two ring isomorphisms, then \( \varphi(\text{FR}_X(D)) = \psi(\text{FR}_X(D)) \); let’s say that the elements of this latter ideal are the *finite-ranked* elements of \( R \).

If we consider a free module \( M_B \) with a basis of cardinality \(|X|\), the map which assigns to each endomorphism of \( M \) its associated matrix with respect to \( B \) is a ring isomorphism from \( \text{End}(M_B) \) to \( Q \), which restricts to an isomorphism from the ideal of finite rank endomorphisms to the ideal \( \text{FR}_X(D) \). If \( D \) is a division ring
(thus \(M_D\) is a vector space), then it is well known that \(Q\) is regular, left selfinjective and \(\mathbb{FR}_X(D) = \text{Soc}(Q)\). We shall consider \(D\) as a subring of \(Q\) by identifying each element of \(D\) with the corresponding scalar matrix in \(Q\). We call \(D\)-subring of \(Q\) every (not necessarily unital) subring \(S\) which is closed with respect to both right and left multiplication by elements of \(D\), namely it is a \((D,D)\)-submodule of \(Q\). Of course, if \(S\) is a \(D\)-subring of \(Q\), then \(S\) is a unital subring if and only if \(D \subset S\); moreover every ideal of \(Q\) is a \(D\)-subring, while not every subring (unital or not) is a \(D\)-subring. As far as \(\mathbb{FM}_X(D)\) is concerned, it is a left ideal of \(Q\), which is not a right ideal, and for every \(x \in X\) the following hold:

\[
Qe_x = \mathbb{FM}_X(D)e_x, \tag{5.3}
\]

\[
\mathbb{FM}_X(D) = \bigoplus \{Qe_y \mid y \in X\} = \mathbb{FM}_X(D)e_x \mathbb{FM}_X(D). \tag{5.4}
\]

In the sequel it will be useful to bear in mind the obvious observation that every matrix in \(\mathbb{FM}_X(D)\) is a finite sum of matrices of the form \(de_{x,y} = e_{x,y}d\) for \(d \in D\) and \(x,y \in X\).

Finally we observe that both \(\mathbb{FR}_X(D)\) and \(\mathbb{FM}_X(D)\) are pure as left ideals of \(Q\); indeed, if \(0 \neq a \in \mathbb{FR}_X(D)\) and \(Y\) is the subset of \(X\) of those \(x\) such that the \(x\)-th row of \(a\) is not zero, then \(e_Y \in \mathbb{FM}_X(D)\) and \(a = e_Y a\).

Given any ordinal \(\xi \leq X\), our program in this section is to define a family \((Q_\alpha)_{\alpha \leq \xi}\) of unital subrings of \(Q\) having the following features: (a) if \(\alpha < \xi\), then \(Q_\alpha\) is isomorphic to \(Q\); (b) by denoting with \(F_\alpha\) the ideal of \(Q_\alpha\) of all finite-ranked elements when \(\alpha < \xi\), then \(Q_\beta \cap F_\alpha = 0\) whenever \(\alpha < \beta \leq \xi\). Our construction heavily bears on ordinal arithmetic; however, since ordinal arithmetic is not so frequently used in ring theory, we think useful to list here some of the basic facts we shall use, omitting their proof (see [19] or [20], for example).

First recall that every ordinal \(\alpha\) is just the set whose elements are all ordinals \(\beta\) such that \(\beta < \alpha\); in particular \(\alpha \not\in \alpha\), while \(\beta < \alpha\) exactly means \(\beta \in \alpha\). An initial ordinal, that is an ordinal \(\aleph\) such that \(|\alpha| < |\aleph|\) for every ordinal \(\alpha < \aleph\), is called a cardinal number; for every set \(X\) there is a unique cardinal \(\aleph\) such that \(|X| = |\aleph|\) and one writes \(|X| = \aleph\).

Ordinal addition, multiplication and exponentiation are defined as follows: given an ordinal \(\alpha\),

\[
\alpha + 0 = \alpha, \quad \alpha + 1 = \alpha \cup \{\alpha\},
\]

\[
\alpha + (\beta + 1) = (\alpha + \beta) + 1 \quad \text{for every ordinal } \beta,
\]

\[
\alpha + \beta = \sup\{\alpha + \gamma \mid \gamma < \beta\} \quad \text{for every limit ordinal } \beta \neq 0;
\]

\[
\alpha \cdot 0 = 0,
\]

\[
\alpha \cdot (\beta + 1) = (\alpha \cdot \beta) + \alpha \quad \text{for every ordinal } \beta,
\]

\[
\alpha \cdot \beta = \sup\{\alpha \cdot \gamma \mid \gamma < \beta\} \quad \text{for every limit ordinal } \beta \neq 0;
\]

\[
\alpha^0 = 1,
\]

\[
\alpha^{\beta+1} = \alpha^\beta \cdot \alpha \quad \text{for every ordinal } \beta,
\]

\[
\alpha^\beta = \sup\{\alpha^\gamma \mid \gamma < \beta\} \quad \text{for every limit ordinal } \beta \neq 0.
\]

Ordinal arithmetic differs deeply from arithmetic of cardinals. For example, if \(\omega = \aleph_0\), as ordinal exponential we have that \(2^\omega = \omega\), while \(2^\omega\) is uncountable if we consider cardinal exponentiation. Since in our work we always use ordinal
exponentiation, there will be no conflict with notations. Note that $\alpha \cdot \beta$ is isomorphic, as a well ordered set, to the direct product $\alpha \times \beta$ with the antilexicographic ordering. If $\alpha$ and $\beta$ are ordinals such that $\alpha < \beta$, then there exists a unique ordinal $\beta - \alpha$ such that $\beta = \alpha + (\beta - \alpha)$. It follows that if $\alpha < \beta < \gamma$, then $(\beta - \alpha) + (\gamma - \beta) = \gamma - \alpha$. Addition and multiplication are both associative but are not commutative; multiplication is distributive on the left with respect to addition, but not on the right. All ordinals (resp. all nonzero ordinals) are left cancellable with respect to addition (resp. multiplication), but need not be right cancellable. If $\alpha, \beta, \gamma$ are ordinals, then $\alpha < \beta$ if and only if $\gamma + \alpha < \gamma + \beta$; if, in addition, $\gamma \neq 0$, then $\alpha < \beta$ if and only if $\gamma \cdot \alpha < \gamma \cdot \beta$.

Using the definitions and induction it is easy to show that:

$$0 \cdot \alpha = 0 = \alpha \cdot 0 \quad \text{and} \quad 1 \cdot \alpha = \alpha = \alpha \cdot 1$$

for every ordinal $\alpha$; moreover, if $1 < \alpha$ and $1 < \beta$, then $\alpha < \alpha \cdot \beta$ and $\beta < \alpha \cdot \beta$.

**Proposition 5.2.** If $\alpha$, $\beta$, $\gamma$ are ordinals with $1 < \gamma$, then $\alpha < \beta$ if and only if $\gamma^\alpha < \gamma^\beta$.

It is immediate from the definition that $\beta$ is a limit ordinal if and only if $\alpha + \beta$ is limit for every $\alpha$.

**Proposition 5.3.** Given two ordinals $\alpha$ and $\beta > 0$, then both $\alpha \cdot \beta$ and $\alpha^\beta$ are limit ordinals in case either $\alpha$ or $\beta$ is limit.

**Proposition 5.4.** Given three ordinals $\alpha$, $\beta$, $\gamma \neq 0$, the following equality holds:

$$\gamma^{\alpha + \beta} = \gamma^\alpha \cdot \gamma^\beta.$$  

Division with unique quotient and remainder between ordinals is possible “on the left”, as stated in the proposition which follows. This possibility is actually the key of our construction; we will make an extensive use of it without an explicit mention.

**Proposition 5.5.** Given two ordinals $\alpha$, $\beta_1$ with $\beta_1 \neq 0$, there are unique ordinals $\gamma$, $\alpha_1$ (called respectively the quotient and the remainder of the division of $\alpha$ by $\beta_1$) such that

$$\alpha = \beta_1 \cdot \gamma + \alpha_1 \quad \text{and} \quad \alpha_1 < \beta_1.$$  

**Remark 5.6.** Let $\beta_1, \beta_2$ be nonzero ordinals. Given an ordinal $\alpha < \beta_1 \cdot \beta_2$, it follows from Proposition 5.5 that there is a unique ordinal $\gamma$ such that $\alpha$ belongs to the right open interval

$$[\beta_1 \cdot \gamma, \beta_1 \cdot \gamma + \beta_1) = \{\beta_1 \cdot \gamma + \alpha_1 \mid \alpha_1 < \beta_1\};$$

necessarily $\gamma < \beta_2$, for if $\gamma \geq \beta_2$, then $\alpha < \beta_1 \cdot \beta_2 \leq \beta_1 \cdot \beta_2 + \alpha_1 \leq \beta_1 \cdot \gamma + \alpha_1 = \alpha$ and hence a contradiction. Thus the set $[[\beta_1 \cdot \gamma, \beta_1 \cdot \gamma + \beta_1) \mid \gamma < \beta_2]$ is a partition of $\beta_1 \cdot \beta_2$. Also note that, for every $\gamma < \beta_2$, the assignment $\alpha_1 \mapsto \beta_1 \cdot \gamma + \alpha_1$ defines a bijection from $\beta_1$ to $[\beta_1 \cdot \gamma, \beta_1 \cdot \gamma + \beta_1)$. These observations will be crucial for the construction which is the objective of our work.

Another feature we shall rely on is the following $n$-th iterate of Proposition 5.5.

**Proposition 5.7.** Let $\beta_1, \ldots, \beta_n$ be nonzero ordinals. For every ordinal $\alpha$ there are unique ordinals $\gamma$ and $\alpha_k < \beta_{n-k+1}$ for $k = 1, \ldots, n$ such that

$$\alpha = \beta_1 \cdot \cdots \cdot \beta_n \cdot \gamma + \beta_1 \cdot \cdots \cdot \beta_{n-1} \cdot \alpha_1 + \cdots + \beta_1 \cdot \alpha_{n-1} + \alpha_n.$$
and γ is the quotient of the division of α by β₁ • · · · • βₙ. If β₊₁ is another ordinal such that α < β₁ • · · · • βₙ • β₊₁, then γ < β₊₁.

Proof. If n = 1, the first statement is merely Proposition 5.3 together with Remark 5.6. Suppose inductively that the statement is true for some n ≥ 1 and consider n + 1 ordinals β₁, . . . , β₊₁. Given α, by Proposition 5.3 there are unique γ and δ < α such that

\[ \alpha = \beta_1 \cdot \cdots \cdot \beta_n \cdot \gamma + \delta. \]  

(5.8)

By the inductive hypothesis, there are α₁ < β₊₁, α₂ < βₙ, . . . , α₊₁ < β₁ such that

\[ \delta = \beta_1 \cdot \cdots \cdot \beta_n \cdot \alpha_1 + \beta_1 \cdot \cdots \cdot \beta_{n-1} \cdot \alpha_2 + \cdots + \beta_1 \cdot \alpha_n + \alpha_{n+1}. \]

As a result

\[ \alpha = \beta_1 \cdot \cdots \cdot \beta_n \cdot \alpha_1 + \cdots + \beta_1 \cdot \alpha_n + \alpha_{n+1}. \]

Suppose that also

\[ \alpha = \beta_1 \cdot \cdots \cdot \beta_{n+1} \cdot \gamma' + \beta_1 \cdot \cdots \cdot \beta_n \cdot \alpha'_1 + \cdots + \beta_1 \cdot \alpha'_n + \alpha'_{n+1}, \]

where α'₁ < β₊₁, . . . , α'₊₁ < β₁. Using the left distributivity of multiplication with respect to the addition, we infer from uniqueness of the quotient and remainder of the division of α by β₁ that α₊₁ = α'₊₁ and

\[ \beta_2 \cdot \cdots \cdot \beta_{n+1} \cdot \gamma + \beta_2 \cdot \cdots \cdot \beta_n \cdot \alpha_1 + \cdots + \alpha_n \]

\[ = \beta_2 \cdot \cdots \cdot \beta_{n+1} \cdot \gamma' + \beta_2 \cdot \cdots \cdot \beta_n \cdot \alpha'_1 + \cdots + \alpha'_n. \]

Again from the inductive hypothesis it follows that γ = γ' and αₖ = α'ₖ for 1 ≤ k ≤ n. Concerning the last statement, if β₊₁ is another ordinal such that α < β₁ • · · · • βₙ • β₊₁, then it follows from Proposition 5.5 and Remark 5.6 that γ < β₊₁ and the proof is complete. □

Proposition 5.8. Given an ordinal ξ and an infinite cardinal \( \aleph \), such that ξ ≤ \( \aleph \), if 0 < α ≤ ξ then, as ordinal exponential,

\[ |\aleph^\alpha| = \aleph. \]

Proof. The equality being obvious if α = 1, suppose that 1 < α ≤ ξ and |\aleph^β| = \( \aleph \) for all nonzero β < α. If α = β + 1 for some β, then

\[ |\aleph^\alpha| = |\aleph^{\beta+1}| = |\aleph^\beta \cdot \aleph| = |\aleph^\beta \times \aleph| = |\aleph \times \aleph| = \aleph. \]

If α is limit, then \( \aleph^\alpha = \sup\{\aleph^\beta \mid \beta < \alpha\} = \bigcup\{\aleph^\beta \mid \beta < \alpha\} \), therefore

\[ |\aleph^\alpha| = \sup\{|\aleph^\beta| \mid \beta < \alpha\} \cup \{\lfloor \alpha \rfloor\} = \aleph. \]

□

In order to obtain results which are general enough to be readily used in the subsequent sections, throughout the remaining part of this section we assume that

\[ X = \aleph^\xi \cdot \beth, \]

where \( \aleph \) is a given infinite cardinal, \( \beth \) is a second nonzero cardinal such that \( \beth \leq \aleph \) and \( \xi \) is an ordinal such that \( \xi \leq \aleph \). We want to stress that we are using ordinal exponentiation and multiplication. It is clear from Proposition 5.8 that |X| = \( \aleph \).

We say that a partition \( P \) of X is an \( \aleph \)-partition if |Y| = \( \aleph \) for all Y ∈ P; we denote by \( \mathcal{P}_\aleph(X) \) the set of all such partitions. Given a cardinal \( \aleph' \leq \aleph \), we say that a partition \( Q \) of X is \( \aleph' \)-coarser than a partition \( P \in \mathcal{P}_\aleph(X) \) if each element of Q is
the union of $\aleph'$ elements of $\mathcal{P}$; if it is the case, then it is clear that $Q \in \mathbb{P}_\aleph(X)$ and $\aleph' \leq |\mathcal{P}|$. Using the natural ordering and the arithmetical properties of ordinals we can define a sequence of partitions $\{\mathcal{P}_\alpha \mid 0 < \alpha \leq \xi\}$ of the set $X$, in such a way that each $\mathcal{P}_\alpha$ is an $\aleph$-partition and $\mathcal{P}_\beta$ is $\aleph$-coarser than $\mathcal{P}_\alpha$ whenever $0 < \alpha < \beta$.

Precisely, for every $\alpha \leq \xi$ and $\lambda < \aleph^{\xi-\alpha} \cdot \aleph$ let us consider in $X$ the right open interval

$$X_{\alpha,\lambda} := \{\aleph^\alpha \cdot \lambda, \aleph^\alpha \cdot \lambda + \aleph^\alpha \} = \{\aleph^\alpha \cdot \lambda + \rho \mid \rho < \aleph^\alpha\}.$$  

Observing that $X = \aleph^\alpha \cdot \left(\aleph^{\xi-\alpha} \cdot \aleph\right)$, according to Remark 5.6 the set

$$\mathcal{P}_\alpha := \{X_{\alpha,\lambda} \mid \lambda < \aleph^{\xi-\alpha} \cdot \aleph\}$$

is a partition of $X$ and $|X_{\alpha,\lambda}| = |\aleph^\alpha|$, hence $|X_{\alpha,\lambda}| = \aleph$ by Proposition 5.8. Thus $\mathcal{P}_\alpha$ is an $\aleph$-partition of $X$. An element $x \in X$ belongs to $X_{\alpha,\lambda}$ if and only if $\lambda$ is the quotient of the division (on the left) of $x$ by $\aleph^\alpha$. We can extend the definition of the partition $\mathcal{P}_\alpha$ to the case $\alpha = 0$ by observing that $X_{0,\lambda} = \{\lambda\}$ for every $\lambda < \aleph^\xi \cdot \aleph$. Thus $\mathcal{P}_0$ is just the trivial partition of $X$ in which each member is a singleton.

**Lemma 5.9.** If $\alpha < \beta \leq \xi$, then $\mathcal{P}_\beta$ is $\aleph$-coarser than $\mathcal{P}_\alpha$; specifically

$$(5.10) \quad X_{\beta,\lambda} = \bigcup\{X_{\alpha,\aleph^{\beta-\alpha} \cdot \lambda + \mu} \mid \mu < \aleph^{\beta-\alpha}\}$$

for every $\lambda < \aleph^{\xi-\beta} \cdot \aleph$.

**Proof.** Given $\lambda < \aleph^{\xi-\beta} \cdot \aleph$, suppose that $x \in X_{\beta,\lambda}$, namely $x = \aleph^\beta \cdot \lambda + \rho$ for some $\rho < \aleph^\beta$. Then it follows from Proposition 5.7 that there are unique $\mu < \aleph^{\beta-\alpha}$ and $\sigma < \aleph^\alpha$ such that

$$x = \aleph^\alpha \cdot \aleph^{\beta-\alpha} \cdot \lambda + \aleph^\alpha \cdot \mu + \sigma \in X_{\alpha,\aleph^{\beta-\alpha} \cdot \lambda + \mu}. $$

Conversely, take any $\mu < \aleph^{\beta-\alpha}$ and observe that

$$X_{\alpha,\aleph^{\beta-\alpha} \cdot \lambda + \mu} = \{\aleph^\beta \cdot \lambda + \aleph^\alpha \cdot \mu, \aleph^\beta \cdot \lambda + \aleph^\alpha \cdot \mu + \aleph^\alpha\}. $$

Obviously $\aleph^\beta \cdot \lambda \leq \aleph^\beta \cdot \lambda + \aleph^\alpha \cdot \mu$; on the other hand, since $\mu < \aleph^{\beta-\alpha}$ and $\aleph^{\beta-\alpha}$ is a limit ordinal by Proposition 5.8 then $\mu + 1 < \aleph^{\beta-\alpha}$ and consequently

$$\aleph^\beta \cdot \lambda + \aleph^\alpha \cdot \mu + \aleph^\alpha < \aleph^\beta \cdot \lambda + \aleph^\alpha \cdot \mu + \aleph^\alpha = \aleph^\beta \cdot \lambda + \aleph^\alpha \cdot (\mu + 1) = \aleph^\beta \cdot \lambda + \aleph^\alpha \cdot \aleph^{\beta-\alpha}.$$ 

This shows that $X_{\alpha,\aleph^{\beta-\alpha} \cdot \lambda + \mu} \subset X_{\beta,\lambda}$, as wanted. \qed

**Notation 5.10.** Given $x \in X$ and $\alpha \leq \xi$, we shall denote by $x_{\alpha,q}$ and $x_{\alpha,r}$ respectively the quotient and the remainder of the (left) division of $x$ by $\aleph^\alpha$, namely the unique ordinals such that $x_{\alpha,r} < \aleph^\alpha$ and

$$(5.11) \quad x = \aleph^\alpha \cdot x_{\alpha,q} + x_{\alpha,r}. $$

Note that $x_{\alpha,q} < \aleph^{\xi-\alpha} \cdot \aleph$ by Proposition 5.7.

Let us consider the ring $Q = \mathbb{CFM}(D)$ and, for every $\alpha \leq \xi$, let us consider the subset $Q_\alpha$ of $Q$ consisting of those matrices $a$ satisfying the following condition:

$$a(x, y) = \delta(x_{\alpha,r}, y_{\alpha,r}) a(\aleph^\alpha \cdot x_{\alpha,q}, \aleph^\alpha \cdot y_{\alpha,q}) \quad \text{for all } x, y \in X $$

(here and in the sequel $\delta$ stands for the “Kronecker delta” function). Thus $Q_\alpha$ consists of those matrices $a \in Q$ such that, for every $\lambda, \mu < \aleph^{\xi-\alpha} \cdot \aleph$, the block
\( a(X_{\alpha,\lambda}, X_{\alpha,\mu}) \) is a scalar \( \mathbb{R}^\alpha \times \mathbb{R}^\alpha \)-matrix. It is clear that \( Q_0 = Q \) and \( D \subset Q_\alpha \) for all \( \alpha \).

**Theorem 5.11.** Given an ordinal \( \xi > 0 \), a cardinal \( \beth \) > 0 and a ring \( D \), let \( \mathbb{R} \) be the first infinite cardinal such that \( \sup(\{\xi, \beth\}) \leq \mathbb{R} \), set \( X = \mathbb{R}^\xi \cdot \beth \) and consider the ring \( Q = \text{CFM}_X(D) \). Then, with the above notations, the following properties hold:

1. For every \( \alpha \leq \xi \) there is a unital monomorphism \( \varphi_\alpha : \text{CFM}_{\mathbb{R}^\xi \cdot \beth} (D) \to Q \) of rings such that \( \text{Im}(\varphi_\alpha) = Q_\alpha \); in particular, if \( \alpha < \xi \), then \( Q_\alpha \) is a unital \( D \)-subring of \( Q \) isomorphic to \( Q \).

2. Given \( \alpha \leq \xi \), let us consider the \( D \)-subrings \( F_\alpha = \varphi_\alpha(\text{FR}_{\mathbb{R}^\xi \cdot \beth} (D)) \) of \( Q_\alpha \) and \( G_\alpha = \varphi_\alpha(\text{FM}_{\mathbb{R}^\xi \cdot \beth} (D)) \) of \( Q_\alpha \). Then a matrix \( b \in Q_\alpha \) belongs to \( F_\alpha \) if and only if it satisfies the following condition:

(\*) there are \( \lambda_1, \ldots, \lambda_n \in \mathbb{R}^\xi \cdot \beth \) such that if the \( x \)-th row of \( b \) is not zero, then \( x \in X_{\alpha,\lambda_1} \cup \cdots \cup X_{\alpha,\lambda_n} \),

while \( b \) belongs to \( G_\alpha \) if and only if it satisfies the following condition:

(\**) there are \( \lambda_1, \ldots, \lambda_n \in \mathbb{R}^\xi \cdot \beth \) such that if the entry \( b(x,y) \) of \( b \) is not zero, then \( x, y \in X_{\alpha,\lambda_1} \cup \cdots \cup X_{\alpha,\lambda_n} \).

3. If \( \alpha < \beta \leq \xi \), then \( Q_\beta \subset Q_\alpha \).

4. If \( \alpha_1 < \ldots < \alpha_n < \beta \leq \xi \), then

\[
[F_{\alpha_1} + \cdots + F_{\alpha_n}] \cap Q_\beta = 0;
\]

consequently the set \( \{F_\alpha \mid \alpha \leq \xi \} \) of \( (D, D) \)-submodules of \( Q \) is independent and so is, in turn, the set \( \{G_\alpha \mid \alpha \leq \xi \} \).

**Proof.** (1) Given \( \alpha \leq \xi \), let us define the map \( \varphi_\alpha : \text{CFM}_{\mathbb{R}^\xi \cdot \beth} (D) \to Q \) as follows: for all \( x, y \in X \)

\[
\varphi_\alpha(a)(x,y) = \delta(x_{\alpha,r}, y_{\alpha,r}) a(x_{\alpha,q}, y_{\alpha,q}).
\]

Then, given \( a \in \text{CFM}_{\mathbb{R}^\xi \cdot \beth} (D) \) and \( x, y \in X \), we have that

\[
\varphi_\alpha(a)(x,y) = \delta(x_{\alpha,r}, y_{\alpha,r}) \delta(0,0) a(x_{\alpha,q}, y_{\alpha,q})
\]

\[
= \delta(x_{\alpha,r}, y_{\alpha,r}) \varphi_\alpha(a)(\mathbb{N}^\alpha \cdot x_{\alpha,q}, \mathbb{N}^\alpha \cdot y_{\alpha,q}),
\]

therefore \( \text{Im}(\varphi_\alpha) \subset Q_\alpha \). Conversely, given \( b \in Q_\alpha \), let \( a \in \text{CFM}_{\mathbb{R}^\xi \cdot \beth} (D) \) be the matrix defined by \( a(\lambda, \mu) = b(\mathbb{N}^\alpha \cdot \lambda, \mathbb{N}^\alpha \cdot \mu) \) for all \( \lambda, \mu < \mathbb{R}^\xi \cdot \beth \). Then for every \( x, y \in X \) we have

\[
\varphi_\alpha(a)(x,y) = \delta(x_{\alpha,r}, y_{\alpha,r}) a(x_{\alpha,q}, y_{\alpha,q})
\]

\[
= \delta(x_{\alpha,r}, y_{\alpha,r}) b(\mathbb{N}^\alpha \cdot x_{\alpha,q}, \mathbb{N}^\alpha \cdot y_{\alpha,q})
\]

\[
= b(x,y);
\]

consequently \( b = \varphi_\alpha(a) \) and hence \( Q_\alpha = \text{Im}(\varphi_\alpha) \). It is clear that \( \varphi_\alpha(1) = 1 \) and \( \varphi_\alpha \) is a homomorphism of additive groups. Given \( a, b \in \text{CFM}_{\mathbb{R}^\xi \cdot \beth} (D) \), for all
\[ x, y \in X \text{ we have that} \]
\[
\varphi_\alpha(ab)(x, y) = \delta(x_{a,r}, y_{a,r}) (ab)(x_{a,q}, y_{a,q})
\]
\[
= \sum_{\mu < \mathbb{N}^{\beta - \alpha} \cdot \Box} \delta(x_{a,r}, y_{a,r}) a(x_{a,q}, \mu) b(\mu, y_{a,q})
\]
\[
= \sum_{\rho < \mathbb{N}^{\beta} \cdot \Box} \delta(x_{a,r}, \rho) \delta(\rho, y_{a,r}) a(x_{a,q}, \mu) b(\mu, y_{a,q})
\]
\[
= \sum_{z \in X} \delta(x_{a,r}, z_{a,r}) \delta(z_{a,r}, y_{a,r}) a(x_{a,q}, z_{a,q}) b(z_{a,q}, y_{a,q})
\]
\[
= (\varphi_\alpha(a) \varphi_\alpha(b))(x, y),
\]

hence \( \varphi_\alpha \) is a ring homomorphism. Finally, if \( a \in \mathbb{CFM}_{\mathbb{S}^{\alpha} \cdot \Box}(D) \) and \( a(\lambda, \mu) \neq 0 \) for some \( \lambda, \mu < \mathbb{N}^{\beta - \alpha} \cdot \Box \), then \( (\varphi_\alpha(a))(x, y) \neq 0 \) whenever \( x = \mathbb{N}^\alpha \cdot \lambda + \rho \) and \( y = \mathbb{N}^\alpha \cdot \mu + \rho \) for some \( \rho < \mathbb{N}^\alpha \); this shows that \( \varphi_\alpha \) is injective. As a result, if \( \alpha < \xi \), since \( |\mathbb{N}^{\beta - \alpha} \cdot \Box| = \mathbb{N} \) by Proposition 6.3, we have that \( Q_\alpha \cong \mathbb{CFM}_{\mathbb{S}^{\alpha} \cdot \Box}(D) \cong \mathbb{Q} \).

(2) Let \( b \in Q_\alpha \) and take \( a \in \mathbb{CFM}_{\mathbb{S}^{\alpha} \cdot \Box}(D) \) such that \( b = \varphi_\alpha(a) \). If \( b \in F_\alpha \), that is \( a \in \mathbb{CFM}_{\mathbb{S}^{\alpha} \cdot \Box}(D) \), then there are \( \lambda_1, \ldots, \lambda_n \in \mathbb{N}^{\beta - \alpha} \cdot \Box \) such that the \( \lambda \)-th row of \( a \) is not zero only if \( \lambda = \lambda_i \) for some \( i \). Consequently, if \( x, y \in X \), by \( \text{(5.13)} \) we see that \( b(x, y) \neq 0 \) only if \( a(x_{a,q}, y_{a,q}) \neq 0 \), only if \( x_{a,q} = \lambda_i \) for some \( i \), only if \( x \in X_{a, \lambda_i} \cup \cdots \cup X_{a, \lambda_n} \). Similarly, if \( b \in G_\alpha \), namely \( a \in \mathbb{CFM}_{\mathbb{S}^{\alpha} \cdot \Box}(D) \), then there are \( \lambda_1, \ldots, \lambda_n \in \mathbb{N}^{\beta - \alpha} \cdot \Box \) such that the \( (\lambda, \mu) \)-entry of \( a \) is not zero only if \( \mu = \lambda_j \) and \( \mu = \lambda_j \) for some \( i, j \). Consequently, if \( x, y \in X \), again from \( \text{(5.13)} \) we see that \( b(x, y) \neq 0 \) only if \( a(x_{a,q}, y_{a,q}) \neq 0 \), only if \( x_{a,q} = \lambda_i \) and \( y_{a,q} = \lambda_j \) for some \( i, j \), only if \( x \in X_{a, \lambda_i} \cup \cdots \cup X_{a, \lambda_n} \). Conversely, assume that \( b \) satisfies \( (*) \) and let \( \lambda, \mu < \mathbb{N}^{\beta - \alpha} \cdot \Box \) be such that \( a(\lambda, \mu) \neq 0 \). By taking \( x = \mathbb{N}^\alpha \cdot \lambda \) and \( y = \mathbb{N}^\alpha \cdot \mu \), we infer from \( \text{(5.13)} \) that \( b(x, y) = \delta(0, 0) a(\lambda, \mu) = a(\lambda, \mu) \neq 0 \), therefore \( x \in X_{a, \lambda_i} \) for some \( i \) and hence \( \lambda = x_{a,q} = \lambda_i \). Thus \( a \) has only a finite number of nonzero rows and so \( b \in F_\alpha \). A similar argument shows that if \( b \) satisfies \( (***) \), then \( a \) has only a finite number of nonzero entries and so \( b \in G_\alpha \).

(3) Suppose that \( \alpha < \beta \leq \xi \) and let \( a \in Q_\beta \). Since \( X = \mathbb{N}^\alpha \cdot \mathbb{N}^{\beta - \alpha} \cdot (\mathbb{N}^{\beta - \beta} \cdot \Box) \), it follows from Proposition 5.7 that for every \( x \in X \) there is a unique \( x' < \mathbb{N}^{\beta - \alpha} \) such that
\[
x = \mathbb{N}^\beta \cdot x_{\beta,q} + \mathbb{N}^\alpha \cdot x' + x_{a,r},
\]
from which
\[
x_{a,q} = \mathbb{N}^{\beta - \alpha} \cdot x_{\beta,q} + x' \quad \text{and} \quad x_{\beta,r} = \mathbb{N}^\alpha \cdot x' + x_{a,r}.
\]
As a result, since \( a \in Q_\beta \), for every \( x, y \in X \) we have the following equalities:
\[
a(x, y) = \delta(x_{\beta,r}, y_{\beta,r}) a(\mathbb{N}^\beta \cdot x_{\beta,q}, \mathbb{N}^\beta \cdot y_{\beta,q})
\]
\[
= \delta(x', y') \delta(x_{a,r}, y_{a,r}) a(\mathbb{N}^\beta \cdot x_{\beta,q}, \mathbb{N}^\beta \cdot y_{\beta,q})
\]
\[
= \delta(x_{a,r}, y_{a,r}) \delta(\mathbb{N}^\alpha \cdot x', \mathbb{N}^\alpha \cdot y') a(\mathbb{N}^\beta \cdot x_{\beta,q}, \mathbb{N}^\beta \cdot y_{\beta,q})
\]
\[
= \delta(x_{a,r}, y_{a,r}) a(\mathbb{N}^\beta \cdot x_{\beta,q} + \mathbb{N}^\alpha \cdot x', \mathbb{N}^\beta \cdot y_{\beta,q} + \mathbb{N}^\alpha \cdot y')
\]
\[
= \delta(x_{a,r}, y_{a,r}) a(\mathbb{N}^\beta \cdot x_{\beta,q} + \mathbb{N}^\alpha \cdot y_{a,q}).
\]
Thus \( \text{(5.12)} \) holds and hence \( a \in Q_\alpha \).
(4) Assume that $\alpha_1 < \ldots < \alpha_n < \beta \leq \xi$ and that there are non-zero elements $a_1 \in F_{\alpha_1}, \ldots, a_n \in F_{\alpha_n}, b \in Q_\beta$ such that

$$a_1 + \cdots + a_n = b.$$ 

If $x_0, y_0 \in X$ are such that $b(x_0, y_0) \neq 0$, then there are $Y, Z \in P_\beta$ such that $x_0 \in Y$, $y_0 \in Z$ and all the rows of the block $b(Y, Z)$ are nonzero. Note that $Y$ is the union of a subset $\mathcal{Y}$ of $P_{\alpha_n}$ of cardinality $\aleph_0$, because $P_\beta$ is $\aleph_0$-coarser than $P_{\alpha_n}$; thus, by the above, for each $U \in \mathcal{Y}$ and each $x \in U$ the $x$-th row of $b$ is not zero. On the other hand the assumptions on $a_1, \ldots, a_n$, together with the previously shown property (2) and the fact that $P_{\alpha_{i+1}}$ is $\aleph_0$-coarser than $P_{\alpha_i}$ for $1 \leq i < n$, imply that there are $Y_1, \ldots, Y_k \in P_{\alpha_n}$ such that the $x$-th row of any $a_i$ is not zero only if $x \in Y_1 \cup \cdots \cup Y_k$. As a result the $x$-th row of $b$ is not zero only if $x \in Y_1 \cup \cdots \cup Y_k$: a contradiction since $|\mathcal{Y}| = \aleph_0$ is infinite.

Remark 5.12. Given $\alpha < \xi$, we have the set $\{e_\lambda \mid \lambda \in \mathbf{N}^{\xi-\alpha} \cdot \mathbb{1}\}$ of pairwise orthogonal idempotents which generates $\mathbb{F}_{\mathbf{N}^{\xi-\alpha} \cdot \mathbb{1}}(D)$ as a right ideal of $\mathbb{CFM}_{\mathbf{N}^{\xi-\alpha} \cdot \mathbb{1}}(D)$; each $e_\lambda$ generates $\mathbb{F}_{\mathbf{N}^{\xi-\alpha} \cdot \mathbb{1}}(D)$ as a (two-sided) ideal. As a result, because of the embedding $\varphi_\alpha$, we have the set

$$\{e_x \cdot e_\lambda \mid \lambda \in \mathbf{N}^{\xi-\alpha} \cdot \mathbb{1}\} = \{e_Y \mid Y \in P_\alpha\}$$

of pairwise orthogonal idempotents of the ring $Q_\alpha$. For every $Y \in P_\alpha$ we have the equalities

$$e_Y Q_\alpha = e_Y F_\alpha,$$

$$F_\alpha = \bigoplus \{e_Z Q_\alpha \mid Z \in P_\alpha\} = F_\alpha e_Y F_\alpha$$

and, similarly,

$$Q_\alpha e_Y = G_\alpha e_Y,$$

$$G_\alpha = \bigoplus \{Q_\alpha e_Z \mid Z \in P_\alpha\} = G_\alpha e_Y G_\alpha$$

(see (5.1), (5.2), (5.3) and (5.4)).

Remark and Notation 5.13. For every $\alpha < \xi$, given $Y, Z \in P_\alpha$ we shall denote by $e_{Y,Z}$ the matrix such that the $(Y, Z)$-block is the unital $\mathbf{N}^{\alpha} \times \mathbf{N}^{\alpha}$-matrix, while all other entries are zero. As $X = X_{\alpha,\lambda}$ and $Y = X_{\alpha,\mu}$ for unique $\lambda, \mu \in \mathbf{N}^{\xi-\alpha} \cdot \mathbb{1}$, then $e_{Y,Z} = \varphi_\alpha(e_{\lambda,\mu})$; more explicitly: for every $x, y \in X$

$$e_{Y,Z}(x, y) = \begin{cases} 1, & \text{if } x = \mathbf{N}^{\alpha} \cdot \lambda + \rho, \; x = \mathbf{N}^{\alpha} \cdot \mu + \rho \text{ for some } \rho < \mathbf{N}^{\alpha}; \\ 0, & \text{otherwise.} \end{cases}$$

Each matrix in $G_\alpha$ is a finite sum of matrices of the form $de_{Y,Z} = e_{Y,Z}d$, for $d \in D$ and $Y, Z \in P_\alpha$.

6. Representing artinian partially ordered sets over $\mathbb{CFM}_X(D)$.

Let us call a polarized (artinian) poset an ordered pair $(I, I')$, where $I$ is an artinian poset and $I'$ is a lower subset of $I$. However, in order to simplify notation, from now on we shall use the single letter $I$ in order to designate a polarized artinian poset, while the symbol $I'$ will denote the prescribed lower subset of $I$. Starting from a polarized artinian poset $I$, a ring $D$ and an appropriately sized transfinite ordinal $X$, our main objective in the present section is to associate to each element $i \in I$ a (not necessarily unital) $D$-subring $H_i$ of $Q = \mathbb{CFM}_X(D)$, in such a way that
\( \mathcal{H} = \{ H_i \mid i \in I \} \) is independent as a set of \( (D, D) \)-submodules of \( Q \) and present the following features: if \( i \) is a maximal element of \( I \), then \( H_i \) is isomorphic to \( D \); if \( i \) is not maximal and belongs to \( I' \) (resp. to \( I \setminus I' \)), then \( H_i \) is isomorphic to \( \mathbb{F} \mathbb{R}_X(D) \) (resp. to \( \mathbb{F} \mathbb{M}_X(D) \)); moreover \( H_i H_j = 0 \) if and only if \( i, j \) are not comparable, while both \( H_i H_j \) and \( H_j H_i \) are nonzero and are contained in \( H_i \) if \( i \leq j \).

In order to reach this goal we need a preliminary setup, in which Theorem 5.11 will play a central role. This setup will concern just artinian posets; polarized artinian posets will enter the scene only after the setup is ready, so that the above rings \( H_i \) can be introduced and we are able to prove that they have the above outlined behavior.

**Notations** 6.1. In what follows \( I \) is a given artinian poset and, by keeping the notations introduced in the previous sections, we set the following data and further notations:

- \( \xi \) is the dual classical Krull dimension of \( I \).
- \( \mathcal{M} \) is the set of all maximal chains of \( I \): we consider the cardinal \( \mathfrak{a} := |\mathcal{M}| \) and we choose a bijection \( \chi \mapsto A_\chi \) from \( \mathfrak{a} \) to \( \mathcal{M} \).
- For every \( i \in I \), \( \mathcal{M}_i \) is the set of all maximal chains of \( I \) which include \( i \):
  \[ \mathcal{M}_i := \{ A \in \mathcal{M} \mid i \in A \} \]
- Given \( i \in I \), the binary relation \( \sim_i \) in \( \mathcal{M}_i \) defined by
  \[ A \sim_i B \iff A \cap \{ \leq i \} = B \cap \{ \leq i \} \]
  is clearly an equivalence; set \( \mathcal{D}_i = \mathcal{M}_i / \sim \) and note that there is an obvious one to one correspondence between the elements of \( \mathcal{D}_i \) and the maximal chains of \( \{ \leq i \} \).
- Denoting by \( \aleph \) the first infinite cardinal such that \( \aleph \geq \sup \{ |I|, \mathfrak{a} \} \), we consider the ordinal
  \[ X := \aleph^{\aleph+1} \cdot \mathfrak{a} \]
  Note that \( |X| = \aleph \) by Proposition 5.8.
- \( \mathcal{P}_\alpha \) is the partition of \( X \) defined by \( \{ \alpha \} \), for all \( \alpha \leq \xi + 1 \).
- Given \( \chi \in \mathfrak{a} \), \( i \in I \), \( A \in \mathcal{D}_i \), we set
  \[ X_\chi := X_{\xi+1, \chi}, \quad \text{so that} \quad \{ X_\chi \mid \chi \in \mathfrak{a} \} = \mathcal{P}_{\xi+1}; \]
  \[ \mathfrak{a}_A := \{ \chi \mid A_\chi \in A \}; \]
  \[ X_A := \bigcup \{ X_\chi \mid \chi \in \mathfrak{a}_A \} = \{ \aleph^{\xi+1} \cdot \chi + \tau \mid \chi \in \mathfrak{a}_A, \tau < \aleph^{\xi+1} \}; \]
  \[ X_i := \bigcup \{ X_A \mid A \in \mathcal{D}_i \} = \bigcup \{ X_\chi \mid A_\chi \in \mathcal{M}_i \}. \]

Note that, since \( \lambda(i) < \xi + 1 \), every \( X_\chi \) is a disjoint union of \( \aleph \) members of \( \mathcal{P}_{\lambda(i)} \), each of which has the form \( X_{\lambda(i), \lambda} \) for a unique \( \lambda \in \aleph^{\xi+1-\lambda(i)} \cdot \mathfrak{a} \) (see Lemma 5.9). Set

\[ A_A := \{ \lambda < \aleph^{\xi+1-\lambda(i)} \cdot \mathfrak{a} \mid X_{\lambda(i), \lambda} \subset X_A \}; \]
\[ \mathcal{Q}_A := \{ X_{\lambda(i), \lambda} \mid \lambda \in A_A \} = \{ Y \in \mathcal{P}_{\lambda(i)} \mid Y \subset X_A \}. \]

As a consequence \( |A_A| = |\mathcal{Q}_A| = \aleph \) and so \( \mathcal{Q}_A \in \mathbb{P}(X_A) \); moreover

\[ (6.1) \quad X_A = \{ \aleph^{\lambda(i)} \cdot \lambda + \rho \mid \lambda \in A_A, \rho < \aleph^{\lambda(i)} \} = \bigcup \mathcal{Q}_A. \]
Lemma 6.2. Given $i \in I$ and $\mathcal{A} \in \mathcal{D}_i$, with the above notations we have

$$(6.2) \quad A_\mathcal{A} = \{ \mathcal{N}^{\xi+1-\lambda(i)} \cdot \chi + \sigma \mid \chi \in \mathcal{D}_\mathcal{A}; \sigma < \mathcal{N}^{\xi+1-\lambda(i)} \}. $$

Proof. Let $\lambda = \mathcal{N}^{\xi+1-\lambda(i)} \cdot \chi + \sigma$ for some $\chi < \mathcal{D}$ and $\sigma < \mathcal{N}^{\xi+1-\lambda(i)}$. Then it follows from (6.10) that $X_{\lambda(i),\lambda} \subset X_{\xi+1,\chi}$. Consequently $X_{\lambda(i),\lambda} \subset X_\mathcal{A}$ if and only if $X_\chi \subset X_\mathcal{A}$, namely $\lambda \in A_\mathcal{A}$ if and only if $\chi \in \mathcal{D}_\mathcal{A}$. 

Proposition 6.3. Two elements $i, j \in I$ are comparable if and only if $X_i \cap X_j \neq \emptyset$. Consequently, if every maximal chain of $I$ is bounded by a maximal element and $M(I)$ denotes the set of all maximal elements of $I$, then the set $\{ X_m \mid m \in M(I) \}$ is a partition of $X$.

Proof. First note that $X_i \cap X_j \neq \emptyset$ if and only if there is $\chi \in \mathcal{D}$ such that $X_\chi \subset X_i \cap X_j$, and if and only if there is $\chi \in \mathcal{D}$ such that $A_\chi \in M_i \cap M_j$. By the Hausdorff Maximal Principle the latter condition holds if and only if $i$ and $j$ are comparable. Assume now that every maximal chain of $I$ is bounded by a maximal element. Given $\chi \in \mathcal{D}$, there is $m \in M(I)$ such that $m \in A_\chi$; hence $A_\chi \in M_m$ and so $X_\chi \subset X_m$. Since the sets $X_m$ are the members of the partition $\mathcal{P}_{\xi+1}$ of $X$ and each $X_m$ is a union of such sets, the last statement of the proposition follows from the above proven first statement. 

Given $i \in I$ and $\mathcal{A}, \mathcal{A}' \in \mathcal{D}_i$, let us choose $\mathcal{A}' \in \mathcal{A}'$ and let us consider the map

$$f_{\mathcal{A}',\mathcal{A}}: \mathcal{A} \rightarrow \mathcal{A}'$$

defined by

$$f_{\mathcal{A}',\mathcal{A}}(A) = (A' \cap \{ \leq i \}) \cup (A \cap \{ i \leq \}).$$

Since $A' \cap \{ \leq i \} = A' \cap \{ \leq i \}$ for all $A', A'' \in \mathcal{A}'$, we see that $f_{\mathcal{A}',\mathcal{A}}$ does not depend on the choice of the chain $A' \in \mathcal{A}'$. Straightforward computations show that

$$(6.3) \quad \text{for all } \mathcal{A}, \mathcal{A}', \mathcal{A}'' \in \mathcal{D}_i, \quad f_{\mathcal{A},\mathcal{A}} = 1_\mathcal{A} \quad \text{and} \quad f_{\mathcal{A}',\mathcal{A}'} f_{\mathcal{A}',\mathcal{A}} = f_{\mathcal{A}',\mathcal{A}};$$

in particular each $f_{\mathcal{A}',\mathcal{A}}$ is a bijection. Observe that $f_{\mathcal{A}',\mathcal{A}}$ induces the bijection

$$g_{\mathcal{A}',\mathcal{A}}: \mathcal{D}_\mathcal{A} \rightarrow \mathcal{D}_{\mathcal{A}'}$$

defined as follows: if $\chi \in \mathcal{D}_\mathcal{A}$, then $g_{\mathcal{A}',\mathcal{A}}(\chi)$ is the unique element of $\mathcal{D}_{\mathcal{A}'}$ such that $A_{g_{\mathcal{A}',\mathcal{A}}(\chi)} = f_{\mathcal{A}',\mathcal{A}}(A_\chi)$. It follows immediately from (6.3) that

$$(6.4) \quad \text{for all } \mathcal{A}, \mathcal{A}', \mathcal{A}'' \in \mathcal{D}_i \quad g_{\mathcal{A},\mathcal{A}} = 1_{\mathcal{D}_\mathcal{A}} \quad \text{and} \quad g_{\mathcal{A}',\mathcal{A}'} g_{\mathcal{A}',\mathcal{A}} = g_{\mathcal{A}',\mathcal{A}''}.$$
Proof. (1) and (2). Let \( A \in \mathcal{A} \cap \mathcal{B} \). Then \( i, j \in A \), say \( i \leq j \). If \( B \in \mathcal{B} \), that is \( B \sim_{i} A \), then necessarily \( i \in B \) and \( B \sim_{i} A \). This shows that \( \mathcal{B} \subseteq \mathcal{A} \). Similarly \( j \leq i \) implies \( A \subseteq \mathcal{B} \).

(3) Suppose that \( i < j \) and let \( \mathcal{A} \in \mathcal{D}_i \). Given \( A \in \mathcal{A} \), by the Hausdorff's Maximal Principle there is some \( B \in \mathcal{M} \) such that \( (A \cap \{ \leq i \}) \cup \{ j \} \subseteq B \); since \( B \sim_{i} A \), then \( B \in \mathcal{A} \). If \( B \) is the unique element of \( \mathcal{D}_j \) such that \( B \in \mathcal{B} \), then \( B \subseteq \mathcal{A} \). Next, let \( B' \in \mathcal{D}_j \) and assume that there is some \( A \in \mathcal{A} \cap \mathcal{B}' \). Then for every \( B \in \mathcal{B} \) we have that \( B \sim_{i} A \) and, since \( i \in \mathcal{A} \), we infer that \( B \sim_{i} A \) as well and therefore \( B \in \mathcal{A} \), proving that \( \mathcal{B}' \subseteq \mathcal{A} \).

(4) Suppose that \( i < j \) and let \( \mathcal{A}, \mathcal{A}' \in \mathcal{D}_i \), \( \mathcal{B} \in \mathcal{D}_j \) be such that \( \mathcal{B} \subseteq \mathcal{A} \). By the definition of \( f_{\mathcal{A} \mathcal{A}} \) it is clear that \( f_{\mathcal{A} \mathcal{A}}(B) \subseteq \mathcal{B}' \) for some \( \mathcal{B}' \in \mathcal{D}_j \) and, according to (1), we must have \( \mathcal{B}' \subseteq \mathcal{A}' \). Similarly, there is \( \mathcal{B}'' \in \mathcal{D}_j \) such that \( f_{\mathcal{A} \mathcal{A}}(\mathcal{B}'') \subseteq \mathcal{B}'' \subseteq \mathcal{A} \). On the other hand we have

\[
B = f_{\mathcal{A} \mathcal{A}}(f_{\mathcal{A} \mathcal{A}}(B)) \subseteq f_{\mathcal{A} \mathcal{A}}(B') \subseteq \mathcal{B}'',
\]

this forces \( B = \mathcal{B}'' \) and consequently \( f_{\mathcal{A} \mathcal{A}}(B) = \mathcal{B}' \). Finally, choose any \( \mathcal{B}' \in \mathcal{B} \). If \( B \in \mathcal{B} \), it follows from the above that \( B \cap \{ i, j \} = B' \cap \{ i, j \} \), therefore

\[
f_{\mathcal{A} \mathcal{A}'}(B) = (B' \cap \{ \leq j \}) \cup (B \cap \{ i \leq j \}) = (B' \cap \{ \leq i \}) \cup (B \cap \{ i \leq j \}) = f_{\mathcal{A} \mathcal{A}}(B),
\]

as wanted. \( \square \)

Remark 6.5. Let \( i, j \in I \) be such that \( i < j \) and, according to Lemma 6.4, take \( \mathcal{A} \in \mathcal{D}_i \), \( \mathcal{B} \in \mathcal{D}_j \) such that \( \mathcal{B} \subseteq \mathcal{A} \). If \( Y = \mathcal{Q}_\mathcal{B} \), then \( Y \) is the union of \( \mathcal{R} \) elements of \( \mathcal{Q}_\mathcal{A} \). In fact, since \( \lambda(i) < \lambda(j) \) and \( \mathcal{Q}_\mathcal{B} \subseteq \mathcal{P}_\lambda(j) \), then \( Y \) is the union of \( \mathcal{R} \) elements of \( \mathcal{P}_\lambda(j) \). But if \( Z \in \mathcal{P}_\lambda(i) \) and \( Z \subseteq Y \), then \( Z \subseteq Y \subseteq X_B \subseteq X_A \) and so \( Z \in \mathcal{Q}_A \).

The next step toward our construction is to define, for every \( i \in I \), appropriate families of bijections

\[
(t_{\mathcal{A} \mathcal{A}}: X_A \to X_{\mathcal{A}'}, \mathcal{A}', \mathcal{A} \in \mathcal{D}_i) \quad \text{and} \quad (t_A: X_A \to X), \mathcal{A} \in \mathcal{D}_i
\]
such that

\[
t_{\mathcal{A} \mathcal{A}'} = t_{\mathcal{A} \mathcal{A}}; t_{\mathcal{A} \mathcal{A}}; t_{\mathcal{A} \mathcal{A}} = 1_{X_A} \quad \text{and} \quad t_A = t_{\mathcal{A} \mathcal{A}'} t_{\mathcal{A} \mathcal{A}'},
\]

for all \( \mathcal{A}, \mathcal{A}', \mathcal{A}'' \in \mathcal{D}_i \). First observe that, for any \( \mathcal{A} \in \mathcal{D}_i \), by the definition of \( X_A \) we have \( x \in X_A \) if and only if \( x_{t+1, q} \in X_A \) (see Notations 5.10). Thus, given \( \mathcal{A}, \mathcal{A}' \in \mathcal{D}_i \), for every \( x \in X_A \) we can define

\[
t_A(x) = \mathcal{R}^{t+1} \bullet g_{\mathcal{A} \mathcal{A}'}(x_{t+1, q}) + x_{t+1, r},
\]

noting that the second member actually belongs to \( X_{\mathcal{A}'} \). Straightforward computations with the use of (6.5) show that the first two equalities of (6.5) hold for every \( \mathcal{A}, \mathcal{A}', \mathcal{A}'' \in \mathcal{D}_i \) and so each \( t_{\mathcal{A} \mathcal{A}} \) is a bijection. It is clear that \( t_{\mathcal{A} \mathcal{A}} \) restricts to a bijection from \( X_X \) to \( X_{\mathcal{A} \mathcal{A}'}(x) \) for all \( x \in \mathcal{A} \); moreover from (4) of Lemma 6.4 we obtain the following corollary.

Corollary 6.6. Assume that \( i < j \). If \( \mathcal{A}, \mathcal{A}' \in \mathcal{D}_i \), \( \mathcal{B} \in \mathcal{D}_j \) and \( \mathcal{B} \subseteq \mathcal{A} \), by setting \( \mathcal{B}' = f_{\mathcal{A} \cdot \mathcal{A}'}(\mathcal{B}) \), for every \( x \in X_B \) we have

\[
t_{\mathcal{A} \mathcal{A}'}(x) = t_{\mathcal{B} \mathcal{B}'}(x).
\]

Next, given \( i \in I \) and \( \mathcal{A}, \mathcal{A}' \in \mathcal{D}_i \), let us consider the bijection

\[
k_{\mathcal{A} \mathcal{A}'}: \mathcal{A} \to \mathcal{A}';
\]
defined by
\[ k_{\mathcal{A},A}(N^{\xi+1-\lambda(i)} \cdot \chi + \sigma) = N^{\xi+1-\lambda(i)} \cdot g_{\mathcal{A},A}(\chi) + \sigma \]
for all \( \chi \in \mathcal{A} \) and \( \sigma < N^{\xi+1-\lambda(i)} \) (see Lemma 6.2). Again from (6.4) we infer that (6.6) for all \( \mathcal{A}, \mathcal{A}', \mathcal{A}'' \in \mathbb{D}_i \) \( k_{\mathcal{A},A} = 1_{\mathcal{A},A} \) and \( k_{\mathcal{A}',A''} k_{\mathcal{A},A} = k_{\mathcal{A}',A''} \).

Now, let us choose an equivalence class \( \mathcal{A}_i \in \mathbb{D}_i \) and a bijection
\[ k_{\mathcal{A}_i} : \mathcal{A}_i \rightarrow N^{\xi+1-\lambda(i)} \cdot \mathbb{N} \]
(this can be done since both \( \mathcal{A}_i \) and \( N^{\xi+1-\lambda(i)} \cdot \mathbb{N} \) have cardinality \( \aleph \) by Lemma 6.2 and Proposition 5.8), and, for each \( \mathcal{A} \in \mathbb{D}_i \), let us consider the bijection
\[ k_{\mathcal{A}} : = k_{\mathcal{A}_i} k_{\mathcal{A}} : \mathcal{A} \rightarrow N^{\xi+1-\lambda(i)} \cdot \mathbb{N} ; \]

namely
\[ k_{\mathcal{A}}(N^{\xi+1-\lambda(i)} \cdot \chi + \sigma) = k_{\mathcal{A}_i} (N^{\xi+1-\lambda(i)} \cdot g_{\mathcal{A}}(\chi) + \sigma) \]
for all \( \chi \in \mathcal{A} \) and \( \sigma < N^{\xi+1-\lambda(i)} \) (see again Lemma 6.2). Finally, let us define the map \( t_{\mathcal{A}} : X_\mathcal{A} \rightarrow X \) by setting
\[ t_{\mathcal{A}}(N^{\lambda(i)} \cdot \lambda + \rho) = N^{\lambda(i)} \cdot k_{\mathcal{A}}(\lambda) + \rho \]
for every \( \lambda \in \mathcal{A} \) and \( \rho < N^{\lambda(i)} \) (see (6.1)). Using Proposition 5.7 and the fact that \( k_{\mathcal{A}} \) is a bijection it is easy to see that \( t_{\mathcal{A}} \) is a bijection. We claim that
\[ t_{\mathcal{A}} = t_{\mathcal{A}',t_{\mathcal{A}'} \mathcal{A}}. \]

Indeed, taking (6.1) and Lemma 6.2 into account, let \( \chi \in \mathcal{A}, \sigma < N^{\xi+1-\lambda(i)}, \rho < N^{\lambda(i)} \) and consider \( \lambda = N^{\xi+1-\lambda(i)} \cdot \chi + \sigma \). Then we have:
\[ t_{\mathcal{A}} \left( N^{\lambda(i)} \cdot \lambda + \rho \right) = N^{\lambda(i)} \cdot k_{\mathcal{A}}(\lambda) + \rho \]
\[ = N^{\lambda(i)} \cdot k_{\mathcal{A}_i} \left( N^{\xi+1-\lambda(i)} \cdot g_{\mathcal{A}}(\chi) + \sigma \right) + \rho \]
\[ = t_{\mathcal{A}_i} \left( \left( N^{\lambda(i)} \cdot \left( N^{\xi+1-\lambda(i)} \cdot g_{\mathcal{A}}(\chi) + \sigma \right) + \rho \right) \right) \]
\[ = t_{\mathcal{A}_i} \cdot t_{\mathcal{A}_i} \cdot N^{\xi+1} \cdot g_{\mathcal{A}}(\chi) + N^{\lambda(i)} \cdot \sigma + \rho \]
\[ = t_{\mathcal{A}_i} \cdot t_{\mathcal{A}_i} \cdot \left( N^{\lambda(i)} \cdot \chi + N^{\lambda(i)} \cdot \sigma + \rho \right) \]
\[ = t_{\mathcal{A}_i} \cdot t_{\mathcal{A}_i} \cdot \left( N^{\lambda(i)} \cdot \lambda + \rho \right), \]
proving our claim. Now, let \( \mathcal{A}, \mathcal{A}' \in \mathbb{D}_i \). Since the first two equalities of (6.5) hold for every \( \mathcal{A}, \mathcal{A}', \mathcal{A}'' \in \mathbb{D}_i \), from \( t_{\mathcal{A}'} = t_{\mathcal{A}_i} \cdot t_{\mathcal{A}_i} \cdot \mathcal{A} \) we infer that \( t_{\mathcal{A}_i} = t_{\mathcal{A}_i} \cdot t_{\mathcal{A}_i} \cdot \mathcal{A} \);

consequently
\[ t_{\mathcal{A}} = t_{\mathcal{A}_i} \cdot t_{\mathcal{A}_i} \cdot \mathcal{A} = t_{\mathcal{A}} \cdot t_{\mathcal{A}_i} \cdot \mathcal{A} = t_{\mathcal{A}_i} \cdot t_{\mathcal{A}_i} \cdot \mathcal{A} \]
and therefore the third equality of (6.5) holds for all \( \mathcal{A}, \mathcal{A}' \in \mathbb{D}_i \).

Remark 6.7. Because of the definition of \( t_{\mathcal{A}} \), the assignment
\[ X_{\lambda(i),\lambda} \mapsto t_{\mathcal{A}}(X_{\lambda(i),\lambda}) = X_{\lambda(i),k_{\mathcal{A}}(\lambda)} \]
for \( \lambda \in \mathcal{A} \) defines a bijection from \( Q_{\mathcal{A}} \) to \( P_{\lambda(i)} = \{ X_{\lambda(i),\lambda} \mid \lambda < N^{\xi+1-\lambda(i)} \cdot \mathbb{N} \} \).
Consequently, by (6.5) the assignment
\[ X_{\lambda(i),\lambda} \mapsto t_{\mathcal{A}',A}(X_{\lambda(i),\lambda}) \]
gives a bijection from $Q_A$ to $Q_{A'}$.

As in Section 1, for a given ring $D$ let us consider the ring $Q = \mathbb{CFM}_X(D)$ and, for each $\alpha < \xi + 1$, let $Q_\alpha$ be the subring of $Q$ consisting of those matrices $a$ satisfying (6.12). For each $i \in I$ let us denote by $S_i$ the subset of $Q$ of those matrices $a$ such that

$$e_{X,A}a = e_{X,A}ae_{X,A} = ae_{X,A} \quad \text{for all } A \in D_i$$

and, for each $A, A' \in D_i$, then

$$a(x, y) = a(t_{A',A}(x), t_{A',A}(y))$$

for all $x, y \in X_A$. Roughly speaking, $S_i$ consists of those matrices which have zero entries outside the $(X_A, X_A)$-blocks for $A \in D_i$ (which are mutually disjoint) and, if $A, A' \in D_i$, the $(X_A, X_A')$-block coincides with the $(X_A, X_A)$-block “up to the bijection $t_{A',A}$”. As we are going to see, if we consider the idempotent diagonal matrix $e_{X,i}$, then $S_i$ is actually a unital $D$-subring of $e_{X,Q}e_{X,i}$ isomorphic to $Q$.

**Proposition 6.8.** With the above notations, for every $i \in I$ there is a unital $D$-linear ring monomorphism $\psi_i: Q \rightarrow e_X Q e_{X_i}$, such that

$$\psi_i(Q) = S_i$$

and

$$\psi_i(Q_\alpha) \subset S_i \cap Q_\alpha \quad \text{for all } \alpha \leq \lambda(i).$$

Moreover, for every $i, j \in I$ the following properties hold:

1. $S_i S_j = 0$ if and only if $i, j$ are not comparable.
2. If $i \leq j$, then $S_i S_j \cup S_j S_i \subset S_i$.

**Proof.** Given $i \in I$, let us define the map

$$\psi_i: Q \longrightarrow e_X Q e_{X_i}$$

as follows: given $a \in Q$, for every $x, y \in X$

$$\psi_i(a)(x, y) = \begin{cases} a(t_{A}(x), t_{A}(y)) & \text{if } x, y \in X_A \text{ for some } A \in D_i, \\ 0 & \text{otherwise}. \end{cases}$$

It is clear that $\psi_i$ is an homomorphism of $(D, D)$-bimodules and, by using (6.5), we see easily that $\psi_i(Q) \subset S_i$. Given $a \in Q$ and assume that $a(u, v) \neq 0$ for some $u, v \in X$. Given $A \in D_i$, we have $\psi_i(a)(x, y) \neq 0$ for $x = t_A^{-1}(u)$ and $y = t_A^{-1}(v)$; this shows that $\psi_i$ is a monomorphism. Next, let $a, b \in Q$ and $x, y \in X$. If $x, y \in X_A$ for some $A \in D_i$, using the fact that $t_A$ is a bijection and recalling that the subsets $X_A$ are mutually disjoint for $A$ ranging in $D_i$ we get the following:

$$\psi_i(ab)(x, y) = (ab)(t_{A}(x), t_{A}(y)) = \sum_{u \in X} [(a)(t_{A}(x), u)][(b)(u, t_{A}(y))]$$

$$= \sum_{z \in X_A} [(a)(t_{A}(x), t_{A}(z))][(b)(t_{A}(z), t_{A}(y))]$$

$$= \sum_{z \in X_A} [\psi_i(a)(x, z)][\psi_i(b)(z, y)]$$

$$= \sum_{z \in X} [\psi_i(a)(x, z)][\psi_i(b)(z, y)]$$

$$= (\psi_i(a) \psi_i(b))(x, y).$$
If there is no $A \in \mathbb{D}_i$ such that $x, y \in X_A$, through the same guidelines we obtain that
\[
(\psi_i(a) \psi_i(b))(x, y) = \sum_{z \in X} [\psi_i(a)(x, z)] [\psi_i(b)(z, y)] = 0 = \psi_i(ab)(x, y).
\]
Since $\psi_i(1) = e_{X_i}$, we conclude that $\psi_i$ is a unital ring homomorphism. Finally, let $c \in S_i$ and define the matrix $a \in Q$ as follows: choose any $A \in \mathbb{D}_i$ and, for every $u, v \in X$, set
\[
a(u, v) = c(t_A^{-1}(u), t_A^{-1}(v)).
\]
Using again (6.5) it is immediate to check that $\psi_i$ is a unital ring homomorphism. Finally, let $c \in S_i$ and define the matrix $a \in Q$ as follows: choose any $A \in \mathbb{D}_i$ and, for every $u, v \in X$, set
\[
a(u, v) = c(t_A^{-1}(u), t_A^{-1}(v)).
\]
In order to establish (6.9), given any $\alpha \leq \lambda(i)$ and $b \in Q_{\alpha}$, we must show that the matrix $a = \psi_i(b)$ satisfies (5.12). First observe that, given any $x \in X$, both $x$ and $\mathbb{N}^\alpha \cdot x_{\alpha,q}$ belong to the same member $X_{\alpha,x_{\alpha,q}}$ of the partition $P_\alpha$; on the other hand, given $A \in \mathbb{D}_i$, since $P_{\alpha+1}$ is coarser than $P_\alpha$ and $X_A$ is a union of members of $P_{\alpha+1}$, we have that either $X_{\alpha,x_{\alpha,q}} \subset X_A$ or $X_{\alpha,x_{\alpha,q}} \cap X_A = \emptyset$. We infer that $x \in X_A$ if and only if $\mathbb{N}^\alpha \cdot x_{\alpha,q} \in X_A$. Accordingly, given $x, y \in X$, if there is no $A \in \mathbb{D}_i$ such that $x, y \in X_A$, then both members of the equality in (5.12) are zero. Assume that $x, y \in X_A$ for some $A \in \mathbb{D}_i$ and note that, according to Proposition 5.7 we have the decompositions
\[
x = \mathbb{N}^\alpha \cdot \mathbb{N}^{\lambda(i)-\alpha} \cdot x_1 + \mathbb{N}^\alpha \cdot \mathbb{N}^{\lambda(i)-\alpha} \cdot x_2 + \mathbb{N}^\alpha \cdot x_3 + x_4
\]
for unique $x_1 < x_2 < \mathbb{N}^{\lambda(i)-\alpha} \cdot x_3 < \mathbb{N}^{\lambda(i)-\alpha} \cdot x_4 < \mathbb{N}^\alpha$. By setting $x_5 = \mathbb{N}^{\lambda(i)-\alpha} \cdot x_1 + x_2$ and comparing with the decomposition (5.11) we see that
\[
x_{\alpha,q} = \mathbb{N}^{\lambda(i)-\alpha} \cdot x_5 + x_3 \quad \text{and} \quad x_{\alpha,r} = x_4.
\]
We observe that $x \in X_{x_1} = X_{\alpha+1,x_1}$, therefore $X_{x_1} \subset X_A$ and so $x_1 \in X_A$. Consequently, it follows from Lemma 6.2 that $x_5 \in A_A$ and then we may consider the ordinal
\[
x_6 = \mathbb{N}^{\lambda(i)-\alpha} \cdot k_A(x_5) + x_3.
\]
We now obtain that
\[
t_A(x) = t_A(\mathbb{N}^{\lambda(i)} \cdot x_5 + \mathbb{N}^\alpha \cdot x_3 + x_{\alpha,r})
\]
\[
= \mathbb{N}^{\lambda(i)} \cdot k_A(x_5) + \mathbb{N}^\alpha \cdot x_3 + x_{\alpha,r}
\]
\[
= \mathbb{N}^\alpha \cdot x_6 + x_{\alpha,r}
\]
and a similar computation shows that
\[
t_A(\mathbb{N}^\alpha \cdot x_{\alpha,q}) = \mathbb{N}^\alpha \cdot x_6.
\]
After processing $y$ in the same way, from all above we infer finally:
\[
a(x, y) = b(t_A(x), t_A(y))
\]
\[
= b(\mathbb{N}^\alpha \cdot x_6 + x_{\alpha,r}, \mathbb{N}^\alpha \cdot y_6 + y_{\alpha,r})
\]
\[
= \delta(x_{\alpha,r}, y_{\alpha,r}) \cdot y_{\alpha,r})
\]
\[
= \delta(x_{\alpha,r}, y_{\alpha,r}) \cdot b(t_A(\mathbb{N}^\alpha \cdot x_{\alpha,q}), t_A(\mathbb{N}^\alpha \cdot y_{\alpha,q}))
\]
\[
= \delta(x_{\alpha,r}, y_{\alpha,r}) \cdot a(\mathbb{N}^\alpha \cdot x_{\alpha,q}, \mathbb{N}^\alpha \cdot y_{\alpha,q}).
\]
This proves that $a \in Q_\alpha$. 


(1) Let \( i, j \in I \) and assume that \( i, j \) are not comparable. Then, given \( \mathcal{A} \in \mathcal{D}_i \) and \( \mathcal{B} \in \mathcal{D}_j \), we have \( \mathcal{A} \cap \mathcal{B} = \emptyset \) by (3) of Lemma 6.4, therefore \( X_i \cap X_j = \emptyset \). As a consequence, if \( \mathbf{a} \in S_i \) and \( \mathbf{b} \in S_j \), then \( \mathbf{a} \mathbf{b} = \mathbf{e}_X \mathbf{e}_X, \mathbf{e}_Y \mathbf{b} e_{X_j} = 0 \). If, on the contrary, \( i \leq j \) and \( \mathcal{A}_i \) is any maximal chain such that \( i, j \in \mathcal{A}_i \), then \( \mathcal{X}_{X_i} \subseteq X_i \cap X_j \) and hence \( X_i \cap X_j \neq \emptyset \). Consequently \( \emptyset \neq \mathbf{e}_X, \mathbf{e}_X = \mathbf{e}_X, \mathbf{e}_X \in S_i, S_j \cap S_j S_i \).

(2) Suppose that \( i < j \), let \( \mathbf{a} \in S_i \), \( \mathbf{b} \in S_j \) and assume that \( 0 \neq (\mathbf{a} \mathbf{b})(x, y) = \sum_{x \in X} \mathbf{a}(x, z) \mathbf{b}(z, y) \) for some \( x, y \in X \). Then \( \mathbf{a}(x, z) \neq 0 \neq \mathbf{b}(z, y) \) for some \( z \in X \) and therefore \( x, z \in X_A, z, y \in X_B \) for some \( \mathcal{A} \in \mathcal{D}_i, \mathcal{B} \in \mathcal{D}_j \); necessarily \( B \subseteq A \) in view of property (2) of Lemma 6.4 and this shows that the matrix \( \mathbf{ab} \) has zero entries outside the \( (X_A, X_A) \)-blocks for \( A \in \mathcal{D}_i \). Suppose that \( \mathcal{A}, \mathcal{A}' \in \mathcal{D}_i \) and let us prove that

\[
(6.10) \quad (\mathbf{a} \mathbf{b})(x, y) = (\mathbf{a} \mathbf{b})(t_{\mathcal{A}' \mathcal{A}}(x), t_{\mathcal{A}' \mathcal{A}}(y))
\]

for all \( x, y \in X_A \). By using (6.5) and (4) of Lemma 6.4, we see that there is no \( \mathcal{B} \in \mathcal{D}_j \) such that \( y \in X_B \) if and only if there is no \( \mathcal{B}' \in \mathcal{D}_j \) such that \( t_{\mathcal{A}' \mathcal{A}}(y) \in X_B' \); if it is the case, since \( \mathbf{b} \in S_j \), both members of (6.10) are zero. Otherwise there is \( \mathcal{B} \in \mathcal{D}_j \) such that \( y \in X_B \); necessarily \( B \subseteq A \) by Lemma 6.4 and, by setting \( B' = f_{\mathcal{A}' \mathcal{A}}(\mathcal{B}) \) and using Corollary 6.6 we may compute as follows:

\[
(\mathbf{a} \mathbf{b})(x, y) = \sum_{z \in X_A} \mathbf{a}(x, z) \mathbf{b}(z, y) = \sum_{z \in X_B} \mathbf{a}(x, z) \mathbf{b}(z, y)
\]

\[
= \sum_{z \in X_B} [\mathbf{a}(t_{\mathcal{A}' \mathcal{A}}(x), t_{\mathcal{A}' \mathcal{A}}(z))] [\mathbf{b}(t_{\mathcal{B}' \mathcal{B}}(z), t_{\mathcal{B}' \mathcal{B}}(y))] = \sum_{u \in X_{B'}} [\mathbf{a}(t_{\mathcal{A}' \mathcal{A}}(x), u)] [\mathbf{b}(u, t_{\mathcal{B}' \mathcal{B}}(y))] = \sum_{u \in X_{B'}} [\mathbf{a}(t_{\mathcal{A}' \mathcal{A}}(x), u)] [\mathbf{b}(u, t_{\mathcal{A}' \mathcal{A}}(y))] = (\mathbf{a} \mathbf{b})(t_{\mathcal{A}' \mathcal{A}}(x), t_{\mathcal{A}' \mathcal{A}}(y)).
\]

Thus (6.10) holds for all \( x, y \in X_A \), showing that \( \mathbf{ab} \in S_i \). The proof that \( \mathbf{ba} \in S_i \) is similar.

We are now in a position to associate to a given polarized artinian poset \( I \) the set \( \mathcal{H} = \{ H_i \mid i \in I \} \) of (possibly non-unital) subrings of \( Q \), satisfying the conditions we outlined at the beginning of the present section. For every \( i \in I \) let us define the \( D \)-subring \( H_i \) of \( Q \) as follows:

\[
H_i = \begin{cases} 
\psi_i \left( F_{\lambda(i)} \right), & \text{if } i \text{ is not a maximal element of } I \text{ and } i \in I'; \\
\psi_i \left( G_{\lambda(i)} \right), & \text{if } i \text{ is not a maximal element of } I \text{ and } i \notin I'; \\
\psi_i(D) = \mathbf{e}_X D, & \text{if } i \text{ is a maximal element of } I.
\end{cases}
\]

(for each ordinal \( \alpha \leq \xi \), the non-unital \( D \)-subrings \( F_{\alpha} \) and \( G_{\alpha} \) of \( Q \) are defined in (2) of Theorem 5.11). Of course \( H_i \neq H_j \) if \( i \neq j \); also note that, apart from the trivial case in which \( I \) is a singleton, \( H_i \) is not a unital subring of \( Q \). It is clear that \( H_i \) has a multiplicative identity, given by \( \mathbf{e}_X \), if and only if \( i \) is a maximal element of \( I \).

Given \( i \in I \), we know that \( G_{\lambda(i)} \) contains the set \( \{ \mathbf{e}_Y \mid Y \in \mathcal{P}_{\lambda(i)} \} \) of pairwise orthogonal idempotents which generate \( F_{\lambda(i)} \) as a right ideal and \( G_{\lambda(i)} \) as a left ideal of \( Q_{\lambda(i)} \) (Remark 5.12), the images of these idempotents, under the action of the imbedding \( \psi_i \), will be relevant in order to analyze the features of the subrings \( H_i \).
and the way they interact each other. Firstly we need to introduce two additional notations.

Notations 6.9. Given \( i \in I, A \in \mathbb{D}_i, V \in \mathcal{Q}_A \) and \( Y \in \mathcal{P}_{\lambda(i)} \), we define the following subsets of \( X_i \) (see Remark 6.7):

\[
\nabla := \bigcup \{ t_{A',A}(V) \mid A' \in \mathbb{D}_i \},
\]

\[
Y(i) := \bigcup \{ t_{A'}^{-1}(Y) \mid A' \in \mathbb{D}_i \}.
\]

Clearly \( V = t_{A,A}(V) \subset \nabla \); moreover it follows from (6.10) that

\[
(6.11) \quad V = t_{A,A}(V) = (t_A(V))(i) \quad \text{for all } A, A' \in \mathbb{D}_i \text{ and } V \in \mathcal{Q}_A,
\]

while

\[
(6.12) \quad Y(i) = t_{A'}^{-1}(Y) = t_{A'}^{-1}(Y) \quad \text{for all } A, A' \in \mathbb{D}_i \text{ and } Y \in \mathcal{P}_{\lambda(i)}.
\]

As a consequence we have the equalities

\[
(6.13) \quad \{ Y(i) \mid Y \in \mathcal{P}_{\lambda(i)} \} = \{ V \mid V \in \mathcal{Q}_A \} = \{ W \mid W \in \mathcal{Q}_B \}
\]

for all \( A, B \in \mathbb{D}_i \).

Due to the definition of \( \psi_i \), for every \( Y \in \mathcal{P}_{\lambda(i)} \) we have

\[
\psi_i(e_Y) = e_{Y(i)}.
\]

Lemma 6.10. With the above notations, \( \{ \psi_i(e_Y) = e_{Y(i)} \mid Y \in \mathcal{P}_{\lambda(i)} \} \) is a set of pairwise orthogonal idempotents of \( H_i \) and

\[
\{ e_{Y(i)} \mid Y \in \mathcal{P}_{\lambda(i)} \} = \{ e_V \mid V \in \mathcal{Q}_A \} = \{ e_W \mid W \in \mathcal{Q}_B \}
\]

for every \( A, B \in \mathbb{D}_i \). Moreover, given \( j \in I \), for every \( Y \in \mathcal{P}_{\lambda(i)} \) and \( Z \in \mathcal{P}_{\lambda(j)} \) the following hold:

1. If \( i, j \) are not comparable, then \( Y(i) \cap Z(j) = \emptyset \).
2. If \( i < j \) and \( Y(i) \cap Z(j) \neq \emptyset \), then \( Y(i) \subset Z(j) \).

Proof. The first statement is a consequence of (6.13) and the fact that \( \psi_i \) is injective. Given \( j \in I \), assume that \( Y(i) \cap Z(j) = \emptyset \). Then there are \( A \in \mathbb{D}_i, B \in \mathbb{D}_j \) such that \( t_A^{-1}(Y) \cap t_B^{-1}(Z) = \emptyset \). This implies that \( X_A \cap X_B \neq \emptyset \) and hence \( A \cap B \neq \emptyset \). As a result \( i \) and \( j \) are comparable by (2) of Lemma 6.4, say \( i < j \). Thus \( \mathcal{P}_{\lambda(i)} \) is coarser than \( \mathcal{P}_{\lambda(i)} \) and, since \( t_A^{-1}(Y) \in \mathcal{P}_{\lambda(i)} \) and \( t_B^{-1}(Z) \in \mathcal{P}_{\lambda(i)} \), we infer that \( t_A^{-1}(Y) \subset t_B^{-1}(Z) \). Given any \( A' \in \mathbb{D}_i \), by setting \( B' = f_{A,A'}(B) \), we have that \( B' \in \mathbb{D}_j \) by (4) of Lemma 6.4. Thus, by using Corollary 6.6 we obtain

\[
t_A^{-1}(Y) = t_{A',A}(t_A^{-1}(Y)) = t_{B,B'}(t_A^{-1}(Y)) \subset t_{B,B'}(t_B^{-1}(Z)) = t_B^{-1}(Z) \subset Z(j).
\]

We conclude that \( Y(i) \subset Z(j) \). \( \square \)

Lemma 6.11. Assume that \( i \) is not a maximal element of \( I \). Then

\[
(6.14) \quad H_i = \bigoplus \{ e_Y(i) H_i \mid Y \in \mathcal{P}_{\lambda(i)} \}, \quad \text{if } i \in I';
\]
and

\[
(6.15) \quad H_i = H_i e_{Y(i)} H_i \quad \text{if } i \notin I';
\]

for every \( Y \in \mathcal{P}_{\lambda(i)} \). Moreover, given \( a \in S_i \), if \( i \in I' \), then the following conditions are equivalent:
(1) \( a \in H_i \).
(2) \( e_{X_A} a e_{X_A} \in F_{\lambda(i)} \) for all \( A \in D_i \).
(3) There exist \( Y_1, \ldots, Y_n \in \mathcal{P}_{\lambda(i)} \) such that the \( x \)-th row of \( a \) is not zero only if \( x \in Y_1(i) \cup \cdots \cup Y_n(i) \); equivalently
\[
a = (e_{Y_1(i)} + \cdots + e_{Y_n(i)}) a.
\]

If, on the contrary, \( i \not\in I' \), then the above conditions (1), (2), in which \( F_{\lambda(i)} \) is replaced by \( G_{\lambda(i)} \), are equivalent to the following one:

(4) There exist \( Y_1, \ldots, Y_n \in \mathcal{P}_{\lambda(i)} \) such that the entry \( a(x, y) \) of \( a \) is not zero only if \( x, y \in Y_1(i) \cup \cdots \cup Y_n(i) \); equivalently
\[
a = (e_{Y_1(i)} + \cdots + e_{Y_n(i)}) a = a(e_{Y_1(i)} + \cdots + e_{Y_n(i)}) .
\]

**Proof.** The first statement follows from Remark 6.12 while the equivalence (1) \( \iff \) (3) is clear from (6.14). Next, suppose that \( i \in I' \), assume (3) and let \( A \in D_i \). Using (6.12) we see that
\[
[Y_1(i) \cup \cdots \cup Y_n(i)] \cap X_A = \left[t_A^{-1}(Y_1) \cup \cdots \cup t_A^{-1}(Y_n)\right] \cap X_A \\
= t_A^{-1}(Y_1) \cup \cdots \cup t_A^{-1}(Y_n).
\]

Since \( t_A^{-1}(Y_r) \in \mathcal{Q}_A \subseteq \mathcal{P}_A(i) \) for all \( r \in \{1, \ldots, n\} \), it follows from (2) of Theorem 5.11 that \( e_{X_A} a e_{X_A} \in F_{\lambda(i)} \). Conversely, suppose (2) and let \( x, y \in X \) be such that \( a(x, y) \neq 0 \). Then \( x, y \in X_A \) for some \( A \in D_i \) and, by the assumption and (2) of Theorem 5.11 there are \( V_1, \ldots, V_n \in \mathcal{P}_A(i) \) such that \( x \in V_1 \cup \cdots \cup V_n \); necessarily \( V_1, \ldots, V_n \in \mathcal{Q}_A(i) \), because \( x \in X_A \). By setting \( Y_r = t_A(V_r) \in \mathcal{P}_{\lambda(i)} \) for \( r \in \{1, \ldots, n\} \), we conclude that \( x \in Y_1(i) \cup \cdots \cup Y_n(i) \), taking (6.12) into account.

The proof of the equivalence (2) \( \iff \) (3) is similar, by taking again Theorem 5.11 into account.

**Remark 6.12.** Observe that, in general, we have \( e_{X_A} H_i e_{X_A} \not\subseteq H_i \), unless \( D_i = \{ A \} \).

If \( i \) is not a maximal element of \( I \), then \( H_i \subset F_{\lambda(i)} \) if and only if \( D_i \) is finite, that is, if and only if \( i \leq i_0 \) has finitely many maximal chains.

**Lemma 6.13.** Let \( j_1 < \cdots < j_n \) be a finite chain of \( I \) with \( n > 1 \), let \( a_1 \in H_{j_1} \), \( \ldots, a_n \in H_{j_n} \), and choose \( A_1 \in D_{j_1}, \ldots, A_n \in D_{j_n} \) such that \( A_1 \supset \cdots \supset A_n \) (see (2) of Lemma 6.10). If \( a_n \neq 0 \), then the \( (X_{A_1} \times X_{A_n}) \)-block of \( a = a_1 + \cdots + a_n \) is not zero; in particular there is some \( x \in X_{A_n} \) such that the \( x \)-th row of \( a \) is not zero and coincides with the \( x \)-th row of \( a_n \).

**Proof.** For each \( r \in \{1, \ldots, n\} \) let us denote by \( Y_r \) the subset of those \( u \in X_{A_r} \) such that the \( u \)-th row of \( a_r \) is not zero. For \( r < n \) the element \( j_r \) is not maximal, therefore it follows from (2) of Theorem 5.11 and Lemma 6.11 that \( Y_r \) is the (disjoint) union of finitely many elements of \( \mathcal{Q}_A \subseteq \mathcal{P}_{\lambda(j_r)} \). Assume that \( a_n \neq 0 \). Then for every \( A \in D_{j_n} \) the \( (X_{A} \times X_{A}) \)-block of \( a_n \) is not zero and hence, in particular, \( Y_n \neq \emptyset \). Since \( \mathcal{P}_{\lambda(j_r)} \) is \( \mathfrak{R} \)-coarser than \( \mathcal{P}_{\lambda(j_r)} \) when \( r < s \leq n \), in particular \( Y_n \) is the (disjoint) union of \( \mathfrak{R} \) elements of \( \mathcal{P}_{\lambda(j_{n-1})} \); on the other hand, by the above, \( Y_1 \cup \cdots \cup Y_{n-1} \) is contained in the (disjoint) union of finitely many elements of \( \mathcal{P}_{\lambda(j_{n-1})} \). Consequently
\[
Y_n \setminus (Y_1 \cup \cdots \cup Y_{n-1}) \neq \emptyset
\]
and therefore, if \( x \in Y_n \setminus (Y_1 \cup \cdots \cup Y_{n-1}) \), the \( x \)-th row of \( a \) coincides with the \( x \)-th row of \( a_n \), which is not zero.
Lemma 6.14. Let $J$ be a finite subset of $I$, let $\mathbf{a} = \sum_{j \in J} a_j$, where $a_j \in H_j$ for $j \in J$, let $j_1 < \cdots < j_n$ be a maximal chain of $J$ and let $A_1 \in \mathcal{D}_{j_1}, \ldots, A_n \in \mathcal{D}_{j_n}$ be as in Lemma 6.13. Then the $(X_{A_n} \times X_{A_n})$-blocks of $\mathbf{a}$ and $a_{j_1} + \cdots + a_{j_n}$ coincide.

Proof. First note that, given $j \in J$, if there is some $B \in \mathcal{D}_j$ such that $X_{A_n} \cap X_B \neq 0$, namely $A_n \cap B \neq \emptyset$, then $A_r \cap B \neq \emptyset$ for all $r \in \{1, \ldots, n\}$ and therefore it follows from Lemma 6.4 that $j$ is comparable with every $j_r$. As a result $j \in \{j_1, \ldots, j_n\}$, because this latter is a maximal chain of $J$. This implies that if $j \in J \setminus \{j_1, \ldots, j_n\}$, then the $(X_{A_n} \times X_{A_n})$-block of every matrix in $H_j$ is zero and, consequently, the $(X_{A_n} \times X_{A_n})$-blocks of $\mathbf{a}$ and $a_{j_1} + \cdots + a_{j_n}$ coincide. □

Theorem 6.15. Let $I$ be a polarized artinian poset having at least two elements.

With the above notations, $\mathcal{H} = \{H_i \mid i \in I\}$ is an independent set of $(D, D)$-submodules of $Q$ which satisfy the following conditions:

1. Every $H_i$ is a non-unital subring of $Q$; it has an identity, given by $e_{X_i}$, if and only if $i$ is a maximal element of $I$.
2. $H_i H_j = 0$ if $i, j$ are not comparable.
3. Given $i \in I$, if $J \subseteq \{i \leq j\}$ and $0 \neq a \in \bigoplus_{j \in J} H_j$, then $0 \neq H_i a \subset H_i$ and $0 \neq a H_i \subset H_i$.

moreover there are $Y, Z \in P_{\lambda(i)}$ such that $0 \neq e_Y a \in H_i$ and $0 \neq a e_Z \in H_i$.

Proof. Assume that $J$ is a finite subset of $I$, suppose that $\mathbf{a} = \sum_{j \in J} a_j$, where $0 \neq a_j \in H_j$ for $j \in J$, let us choose a maximal chain $j_1 < \cdots < j_n$ of $J$ and let $A_1 \in \mathcal{D}_{j_1}, \ldots, A_n \in \mathcal{D}_{j_n}$ be such that $A_1 \supset \cdots \supset A_n$. Then by Lemma 6.14 the $(X_{A_n} \times X_{A_n})$-blocks of $\mathbf{a}$ and $\mathbf{a}' = a_{j_1} + \cdots + a_{j_n}$ coincide and, on the other hand, the $(X_{A_n} \times X_{A_n})$-block of $\mathbf{a}'$ is not zero by Lemma 6.13. As a consequence $\mathbf{a} \neq 0$ and this proves the independence of $\mathcal{H}$.

1. If $i \in I$ and $I$ is not a maximal element, then $H_i \simeq F_{\lambda(i)} \simeq \mathbb{F} \mathbb{R}_X(D)$ or $H_i \simeq G_{\lambda(i)} \simeq \mathbb{F} \mathbb{M}_X(D)$ as rings, depending on the fact that $i$ is in $I'$ or not, therefore $H_i$ is a ring without an identity. If, on the contrary, $i$ is a maximal element, then $H_i = \psi_i(D) \simeq D$ and $e_{X_i} = \psi_i(1)$ is an identity for $H_i$. Now $X_i \neq X$, because $I$ has at least two elements, consequently $H_i$ is not an unital subring of $Q$.

2. follows from the property (1) of Proposition 6.8 since $H_i \subset S_i$ for all $i \in I$.

3. It is clearly sufficient to take $i, j \in I$ with $i < j$, two nonzero elements $a \in H_j$, $b \in H_b$ and show that $ab$ and $ba$ are both in $H_i$. First, according to Proposition 6.8 we have that $ab \in S_i$ and $ba \in S_i$. Given $A \in \mathcal{D}_i$, we have from 6.7 that

$$(6.16) e_{X_A}(ab)e_{X_A} = e_{X_A}ae_{X_A}be_{X_A}. $$

By Lemma 6.11 we have that either $e_{X_A}be_{X_A} \in F_{\lambda(i)}$, or $e_{X_A}be_{X_A} \in G_{\lambda(i)}$, according to the fact that $i \in I'$ or not. Since $e_{X_A} \in Q_{\lambda(i)}$ and $a \in Q_{\lambda(i)} \subset Q_{\lambda(i)}$ and both $F_{\lambda(i)}$ and $G_{\lambda(i)}$ are left ideals of $Q_{\lambda(i)}$, we infer that the first member of (6.16) belongs to $F_{\lambda(i)}$, or to $G_{\lambda(i)}$ respectively. As a result $ab \in H_i$, again by Lemma 6.11. If $i \in I'$, since $F_{\lambda(i)}$ is also a right ideal of $Q_{\lambda(i)}$, the same argument as above shows that $ba \in H_i$. Assume that $i \notin I'$, so that $j \notin I'$ as well. In order to show that $ba \in H_i$, also in this case, it is sufficient to consider the case in which $b = e_{V(i)}$ and $a = \psi_j(\mathbf{e}_{V,W})$ for some $Y \in P_{\lambda(i)}$ and $V, W \in P_{\lambda(j)}$ (see Remark and Notation 6.13). Since $\psi_j(\mathbf{e}_{V,W}) = \psi_j(\mathbf{e}_V \mathbf{e}_{W}) = \psi_j(\mathbf{e}_V)\psi_j(\mathbf{e}_W) = \psi_j(\mathbf{e}_W) = \psi_j(\mathbf{e}_V) = \psi_j(\mathbf{e}_V) = $
\(e_{V(j)} \psi_j (e_{V,W})\), if \(Y(i) \cap V(j) = \emptyset\), then \(e_{Y(i)} \psi_j (e_{V,W}) = 0\). Otherwise, according to (2) of Lemma 6.11, we have that \(Y(i) \subset V(j)\). Given \(A \in D_i\), we claim that

\[(6.17)\]
\[e_{X_A} e_{Y(i)} \psi_j (e_{V,W}) e_{X_A} = e_{t_A^{-1}(Y)_Z} \in G_{\lambda(i)}\]

for a suitable \(Z \in P_{\lambda(i)}\); it will follow from Lemma 6.11 that \(ba = e_{Y(i)} \psi_j (e_{V,W}) \in H_i\). Since \(Y(i) \cap X_A = t_A^{-1}(Y)\), it follows from \(Y(i) \subset V(j)\) that \(t_A^{-1}(Y) \subset t_B^{-1}(V)\) for a necessarily unique \(B \in D_j\) and, given \(x, y \in X\), we have that

\[\{e_{Y(i)} \psi_j (e_{V,W}) e_{X_A}\}(x, y) \neq 0 \text{ only if } x \in t_A^{-1}(Y)\].

There are \(\lambda \prec \mathbb{N}^{\ell+1-\lambda(i)} \cdot \mathbb{N} \) and \(\lambda, \mu \prec \mathbb{N}^{\ell+1-\lambda(j)} \cdot \mathbb{N} \) such that \(Y = X_{\lambda(i), \lambda}, V = X_{\lambda(j), \lambda}\) and \(W = X_{\lambda(j), \mu}\). Let \(x \in t_A^{-1}(Y)\), so that \(x = \mathbb{N}(\lambda(i)) \cdot k_B^{-1}(\lambda) + \tau\) for a unique \(\tau \prec \mathbb{N}(\lambda(i))\). Since \(x \in t_B^{-1}(V)\) as well, there is a unique \(\sigma \prec \mathbb{N}(\lambda(j))\) such that \(x = \mathbb{N}(\lambda(j)) \cdot k_B^{-1}(\lambda) + \sigma\). Also, \(\sigma = \mathbb{N}(\lambda(i)) \cdot \sigma' + \tau\) for a unique \(\sigma' \prec \mathbb{N}(\lambda(j)) - \lambda(i)\) (see Remark 5.6) and so

\[x = \mathbb{N}(\lambda(i)) \cdot \left(\mathbb{N}(\lambda(j)) - \mathbb{N}(\lambda(i)) \cdot k_B^{-1}(\lambda) + \sigma'\right) + \tau.
\]

Consequently, for every \(y \in X_A\) we have

\[\{e_{t_A^{-1}(Y)} \psi_j (e_{V,W}) e_{X_A}\}(x, y) = \psi_j (e_{V,W})(x, y) = \begin{cases} 1, & \text{if } y = \mathbb{N}(\lambda(j)) \cdot k_B^{-1}(\mu) + \sigma; \\ 0, & \text{otherwise}. \end{cases}
\]

\[= \begin{cases} 1, & y = \mathbb{N}(\lambda(i)) \cdot \left(\mathbb{N}(\lambda(j)) - \mathbb{N}(\lambda(i)) \cdot k_B^{-1}(\mu) + \sigma'\right) + \tau; \\ 0, & \text{otherwise}. \end{cases}
\]

If we take \(Z = X_{\lambda(i), \nu}\), where \(\nu = \mathbb{N}(\lambda(j)) - \mathbb{N}(\lambda(i)) \cdot k_B^{-1}(\mu) + \sigma'\), we conclude that \((6.17)\) holds.

As far as the last statement is concerned, suppose again that \(a = \sum_{j \in J} a_j\), where \(J\) is a finite subset of \(I\) and \(0 \neq a_j \in H_j\) for \(j \in J\), let us consider a maximal chain \(j_1 < \cdots < j_n\) of \(J\) and take \(A \in D_i, A_1 \in D_{j_1}, \ldots, A_n \in D_{j_n}\) such that \(A \supset A_1 \supset \cdots \supset A_n\). As seen in the first part of the present proof, the \((X_{A_n} \times X_{A_1})\)-block of \(a\) is not zero. Let \(x, y \in X_{A_n}\) be such that \(a_{xy} \neq 0\). Since \(X_{A_n} \subset X_{A_1}\), there are (necessarily unique) \(V, W \in Q_A\) such that \(x \in V\) and \(y \in W\). If \(Y, Z\) are the unique elements of \(P_{\lambda(i)}\) such that \(Y(i) = V\) and \(Z(i) = W\) (see (6.11)), then \(e_{Y(i)} \land e_{Z(i)} \in H_i\); both \(e_{Y(i)} \land a\) and \(ae_{Z(i)}\) are nonzero and belong to \(H_i\).

\[\square\]

7. The ring \(D_J\).

As in the second half of the previous section, we assume that a polarized artinian poset \(I\) is given. The \(D\)-subrings \(H_i\) (for \(i \in I\)) of \(Q = CFM_X(D)\) we have introduced in the previous section can be used in a natural way as building blocks to construct further \(D\)-subrings of \(Q\), this time starting from subsets of \(I\). Indeed, given a subset \(J \subset I\), if we consider the \((D, D)\)-submodule \(H_J\) of \(Q\) defined by

\[H_J := \bigoplus_{j \in J} H_j,\]
then it follows from Theorem [6.13] that \( H_J \) is a \( D \)-subring; it may fail to be a unitary subring of \( Q \) and it may even lack multiplicative identity. Of course we set \( H_\emptyset = 0 \). If we define the subset \( X_J \) by setting

\[
X_J : = \bigcup \{ X_x \mid i \in J \} = \bigcup \{ X_x \mid A_x \cap J \neq \emptyset \},
\]

then \( X_J \) is the smallest subset of \( X \) such that every matrix in \( H_J \) has zero entries outside the \((X_J \times X_J)\)-block. We observe that if a matrix \( u \in Q \) acts as a multiplicative identity on \( H_J \), then the following equalities hold as well:

\[
ue_{X_J} = e_{X_J} = e_{X_J}^{}u.
\]

In fact, given \( x \in X_J \), there are \( j \in J, A \in \mathbb{D}_J \) and \( Y \in \mathbb{Q}_A \) such that \( x \in Y \) (see (6.1)). Inasmuch as \( e_{x} \in H_J \subseteq H_J \) by Lemma [6.10] we have that \( e_{x}^{}u = e_{x}^{} = u e_{x}^{} \) and, since \( x \in Y \), we infer that \( u(x, y) = \delta(x, y) = u(y, x) \) for every \( y \in X \), which proves (7.1). As a result, \( H_J + e_Z D \) is the smallest \( D \)-subring of \( Q \) which has a multiplicative identity (given by \( e_Z \)) and contains \( H_J \) as an ideal; we denote it by \( D_{I,J} \):

\[
D_{I,J} := H_J + e_{X_J} D.
\]

In case, \( J = I \), we simply write \( D_I \) instead of \( D_{I,J} \). With the next result, we give necessary and sufficient conditions under which \( H_J = D_{I,J} \). As we shall see, in this context a relevant role is played by the set \( J^* \) defined by

\[
J^* := M(I) \cap \{ J \leq J \} = M(\{ J \leq \}),
\]

(recall that \( M(I) \) denotes the set of all maximal elements of \( I \), namely the set of those maximal elements of \( J \) which follow some element of \( J \). Of course it may happen that \( J \nsubseteq \{ \leq J^* \} \), in particular that \( J^* = \emptyset \). If every element of \( I \) is bounded by a maximal element or, equivalently, all maximal chains of \( I \) have a greatest element, then it is clear that \( X_J \subset X_J^* \); this inclusion is an equality if and only if, given \( m \in J^* \), every maximal chain of \( I \) which is bounded by above by \( m \) contains an element of \( J \). Obviously this is the case if \( J^* \subset J \), in particular when \( J \) is an upper subset of \( I \); in this latter case it is clear that \( J^* = M(J) \).

We say that a subset \( J \) of \( I \) is finitely sheltered in \( I \) if the following three conditions hold:

\[
J^* \text{ is finite, } J \subset \{ \leq J^* \} \quad \text{and} \quad J^* \subset J.
\]

If \( J \) is an upper subset of \( I \), then \( J \) is finitely sheltered in \( I \) if and only if \( J \) has a finite cofinal subset; in particular \( I \) is finitely sheltered in \( I \) exactly when \( I \) has a finite cofinal subset and, if it is the case, then every subset of \( I \) is finitely sheltered in \( I \).

**Proposition 7.1.** If \( \emptyset \neq J \subset I \), then the following conditions are equivalent:

1. \( H_J \) has a multiplicative identity.
2. \( D_{I,J} = H_J \).
3. \( H_J \cap (e_{X_J} D) \neq 0 \).
4. \( J \) is finitely sheltered in \( I \).

If any (and hence all) of these conditions holds and \( J^* = \{ m_1, \ldots, m_r \} \), then

\[
e_{X_J} = e_{X_J^*} = e_{X_m} + \cdots + e_{X_m}.
\]

Consequently, either \( D_{I,J} = H_J \), or the sum in (7.3) is direct.
Proof. The equivalence between (1) and (2) follows from the previous observation, while the implication (2)⇒(3) is obvious.

(3)⇒(4). Suppose that there is a finite subset \( F \subset J \) and nonzero matrices \( d_i \in H_i \), for \( i \in F \), such that
\[
0 \neq d = \sum_{i \in F} d_i \in e_{X_J}D
\]
and let us prove first that \( X_F = X_J \). Clearly \( F \subset J \) implies that \( X_F \subset X_J \). On the other hand, given \( x \in X_J \), since the \( x \)-th row of \( d \) is not zero then, for some \( i \in F \), the \( x \)-th row of \( d_i \) is not zero. This means that there exists \( A \in \mathbb{D}_i \) such that \( x \in X_A \subset X_J \). Hence \( X_F \subset X_J \). Next, given any maximal chain \( i_1 < \ldots < i_r \) of \( F \), we claim that \( i_r \) must be a maximal element of \( I \), so that \( F^* \subset J \). Indeed, if \( A_1 \in \mathbb{D}_{i_1}, \ldots , A_r \in \mathbb{D}_{i_r} \) are such that \( A_1 \supset \cdots \supset A_r \) (see (2) of Lemma 6.14), then it follows from Lemma 6.14 that the \((X_{A_1} \times X_{A_r})\)-blocks of \( d \) and \( d_{i_1} + \cdots + d_{i_r} \) coincide. If \( i_r \) is not maximal then, with the help of (2) of Lemma 6.14 and (2) of Theorem 6.11 we see that there are \( Y_1, \ldots , Y_s \in \mathcal{Q}_{A_r} \subset \mathcal{P}_{\lambda(i_r)} \) such that, given \( x \in X_{A_r} \), the \( x \)-th row of \( d \) is not zero only if \( x \in Y_1 \cup \cdots \cup Y_s \). Since the \((X_{A_1} \times X_{A_r})\)-block of \( d \) is a nonzero scalar matrix and \( X_{A_r} \) is the union of \( \mathbb{R} \) elements of \( \mathcal{P}_{\lambda(i_r)} \), we have a contradiction and our claim is proved.

Now, let \( j \in J \) and take any \( x \in X_J \). Then there is a unique \( \chi \in \Xi \) such that \( x \in X_\chi \) and \( j \in A_\chi \). As \( X_J = X_F \), we infer that \( X_\chi \subset X_F \) and so \( A_\chi \cap F \neq \emptyset \). Thus \( A_\chi \cap F \) is a maximal chain of \( F \) which is bounded from above by an element \( m \in F^* \), as we have seen previously. As a result \( A_\chi \) itself is bounded from above by \( m \) and this proves that \( J \subset \{ \leq F^* \} \subset \{ \leq J^* \} \).

Finally, let us show that \( J^* \subset F \), from which it will follow that \( J^* = F^* \) and so \( J^* \) is finite. Assume, on the contrary, that there is some maximal element \( m \) of \( I \) such that \( m \notin F \) but \( j < m \) for some \( j \in J \). As a consequence, according to Lemma 6.3 there is some \( B \in \mathbb{D}_m \) which is contained in some \( A \in \mathbb{D}_j \) and hence \( X_B \subset X_A \subset X_j \subset X_J \). We observe that if \( i_1, \ldots , i_r \) are those elements of \( F \) such that the \((X_B \times X_{A_i})\)-blocks of \( d_{i_1}, \ldots , d_{i_r} \) are not zero, then \( r \geq 1 \) and \( i_1, \ldots , i_r \in \{ < m \} \). Indeed, if \( t \in \{ 1, \ldots , r \} \), then there is some \( C \in \mathbb{D}_{i_t} \) such that \( C \cap B \neq \emptyset \) and hence \( i_t \) and \( m \) are related by Lemma 6.4 since \( m \) is maximal in \( I \), then necessarily \( i_t < m \). We have now
\[
0 \neq d_{X_B}X_B = (d_{i_1})_{X_B}X_B + \cdots + (d_{i_r})_{X_B}X_B
\]
and, inasmuch as \( i_1, \ldots , i_r \) are not maximal elements of \( I \) and \( \lambda(i_1), \ldots , \lambda(i_r) < \lambda(m) \), we infer from (2) of Lemma 6.11 and (2) of Theorem 5.11 that there are \( Y_1, \ldots , Y_s \in \mathcal{Q}_B \subset \mathcal{P}_{\lambda(m)} \) such that, given \( x \in X_B \), the \( x \)-th row of \( d_{X_B}X_B \) is not zero only if \( x \in Y_1 \cup \cdots \cup Y_s \). This leads to a contradiction; in fact, since \( d \in e_{X_J}D \) and \( X_B \subset X_J \), for every \( x \in X_B \) the \( x \)-th row of \( d \) is not zero and \( X_B \) is the union of \( \mathbb{R} \) members of \( \mathcal{P}_{\lambda(m)} \). This shows that \( J^* \subset F \), as wanted.

(4)⇒(1) Assume (4) and set \( J^* = \{ m_1, \ldots , m_r \} \). Then it follows from Proposition 6.3 that \( \{ X_{m_1}, \ldots , X_{m_r} \} \) is a partition of \( X_J \). Thus (7.3) holds and, since \( e_{X_{m_t}} \in H_{m_t} \) for all \( t \in \{ 1, \ldots , r \} \), it follows that \( e_{X_J} \in H_J \) and therefore \( D_{I,J} = H_J \).

Formally speaking, the assignment \( I \mapsto D_I \) cannot be considered as a map from the class of all pairs \( (I, I') \), where \( I \) is an artinian posets and \( I' \) is a lower subset of \( I \), to the class of \( D \)-rings. In fact, the construction that leads us to the ring \( D_I \)
bears first on the choice of a bijection $\Xi: \chi \mapsto A_\chi$ from the cardinal $\beth = |\mathcal{M}|$ to the set $\mathcal{M}$ of all maximal chains of $I$, next on the choice of a family $\mathcal{F} = (A_i)_{i \in I}$, where each $A_i$ is an equivalence class modulo $\sim_i$ and finally on the choice of a family of bijections $\mathcal{K} = (k_{A_i}: A_i \mapsto \mathbb{N}^{\ell+1-\lambda(i)} \bullet \beth)_{i \in I}$. Thus the ring $D_I$ strictly depends on the ordered quintuple $(I, I', \Xi, \mathcal{F}, \mathcal{K})$, so that our construction realizes actually a function from the class of all such quintuples. As one might expect, if we take a second quintuple $(J, J', \Pi, \mathcal{G}, \mathcal{L})$ from this class, every order isomorphism $f: I \rightarrow J$ such that $J' = f(I')$ induces a canonical $D$-ring isomorphism from $D_I$ to $D_J$. This is not immediately obvious and, in what follows, we show how it works. Let $\mathcal{N}$ be the set of all maximal chains of $J$, so that $|\mathcal{M}| = \beth = |\mathcal{N}|$, and write let $B_\chi = \Pi(\chi)$ for every $\chi \in \beth$. As we did with Notations 6\textsuperscript{6} for every $j \in J$ we may consider the equivalence relation $\sim_j$ in the set $\mathcal{N}_j$ of all maximal chains of $J$ which contain $j$ and we get the corresponding quotient set $E_j$. Note that $f$ induces an obvious bijection $\overline{f}: \mathcal{D}_i \rightarrow \mathcal{E}_{f(i)}$. For every $B \in E_j$ we have the set $\mathcal{D}_B = \{ \chi \in \beth | B_\chi \in B \}$ and we can define the subsets $X'_B$, $X'_j$ of $X$, as well as the sets $X'_B$, $X'_j$ and the map $t_{B, B'}: X'_B \rightarrow X'_B$ for every $B, B' \in E_j$ exactly as we did with $X_A$, $X_i$, $\Lambda_A$, $Q_A$ and $t_{A, \Lambda}: X_A \rightarrow X'_{\Lambda}$ for every $i \in I$ and $A, \Lambda \in \mathcal{D}_i$. Next, for every $j \in J$ let us choose $B_j \in E_j$ and a bijection $k_{B_j}: A'_{B_j} \rightarrow \mathbb{N}^{\ell+1-\lambda(j)} \bullet \beth$ and, for every $B, B' \in E_j$, let us consider the bijections $t'_B: X'_B \rightarrow X$, $t'_{B, B'}: X'_B \rightarrow X'_B$, defined in the same fashion as $t_A$ and $t_{A, \Lambda}$. Thus, for every $j \in J$ we can define the $D$-ring $S'_j$, analogous to $S_i$ for $i \in I$, and the $D$-ring monomorphism $\psi'_j: Q \rightarrow e_{X'_j}Qe_{X'_j}$ analogous to $\psi$, as in Proposition 6\textsuperscript{1} so that $S'_j = \psi'_j(Q)$. Through a slight notational transgression, we may view $\psi$ and $\psi'$ as isomorphisms from $Q$ to $S_i$ and $S'_j$ respectively. For each $i \in I$ let us consider the $D$-ring isomorphism

$$\alpha_i: S_i \rightarrow S'_{f(i)}$$

defined by

$$\alpha_i = \psi'_{f(i)} \psi_i^{-1}.$$  

Given $A \in \mathcal{D}_i$, let us consider the bijection $s_A: X_A \rightarrow X'_{\mathcal{J}(A)}$ defined by $s_A = t_{f(A)}^{-1} t_A$. Let $a \in S_i$ and $b \in \mathcal{E}_{f(i)}$. Then $B = \overline{f}(A)$ for a unique $A \in \mathcal{D}_i$ and for every $x, y \in X'_B$ we have:

$$\alpha_i(a)(x, y) = ([\psi'_{f(i)} \psi_i^{-1}](a))((x, y) = \psi_i^{-1}(a)(t_B(x), t_B(y))$$

$$= \psi_i^{-1}(a)(t_A(s_A^{-1}(x)), t_A(s_A^{-1}(y))) = a(s_A^{-1}(x), s_A^{-1}(y)).$$

We claim that if $i, j \in I$ and $i < j$, given $a \in S_i$, $b \in S_j$ the equality

$$\alpha_i(a) \alpha_j(b) = \alpha_j(ab)$$

holds, that is,

$$[\alpha_i(a) \alpha_j(b)](x, y) = \alpha_i(ab)(x, y)$$
for every \( x, y \in X \). Indeed, first note that both members of (7.6) belong to \( S'_{\mathcal{I}(i)} \) by Proposition 6.8 (2); thus, given \( x, y \in X \), if there is no \( \mathcal{B} \in \mathcal{E}_{\mathcal{I}(i)} \) such that \( x, y \in X'_{\mathcal{B}} \), then both members of (7.7) are zero. Assume, on the contrary, that \( x, y \in X'_{\mathcal{B}} \) for some \( \mathcal{B} \in \mathcal{E}_{\mathcal{I}(i)} \) and let \( \mathcal{A} \in \mathcal{D} \) be such that \( \mathcal{B} = f(\mathcal{A}) \). Then, by using (7.8) we have:

\[
[\alpha_i(a)\alpha_j(b)](x, y) = \sum_{z \in X} \alpha_i(a)(x, z) \alpha_j(b)(z, y) \\
= \sum_{z \in X'_{\mathcal{B}}} \alpha_i(a)(x, z) \alpha_j(b)(z, y) \\
= \sum_{z \in X'_{\mathcal{B}}} a(s_{\mathcal{A}}^{-1}(x), s_{\mathcal{A}}^{-1}(z)) b(s_{\mathcal{A}}^{-1}(z), s_{\mathcal{A}}^{-1}(y)) \\
= \sum_{w \in X_{\mathcal{B}}} a(s_{\mathcal{A}}^{-1}(x), w) b(w, s_{\mathcal{A}}^{-1}(y)) \\
= [\alpha_i(ab)](x, y).
\]

Thus the equality (7.6) is proven.

Assume that both \( I, J \) are polarized in such a way that \( J' = f(I') \) and, for every \( j \in J \), define \( H'_j \) as follows:

\[
H'_j = \begin{cases} 
\psi'_j(F_{\lambda(j)}), & \text{if } j \text{ is not a maximal element of } J \text{ and } j \in J'; \\
\psi'_j(G_{\lambda(j)}), & \text{if } j \text{ is not a maximal element of } J \text{ and } j \notin J'; \\
\psi'_j(D), & \text{if } j \text{ is a maximal element of } J.
\end{cases}
\]

Then, for any \( K \subset J \), we can define \( H'_K = \bigoplus_{j \in K} H'_j \). For every \( i \in I \) it is clear that \( a \in H_i \) if and only if \( \alpha_i(a) \in H'_{\mathcal{I}(i)} \), therefore we can define the \( D \)-module isomorphism

\[
\bigoplus_{i \in I} \alpha_i : H_I \rightarrow H'_J,
\]

which extends to a \( D \)-module isomorphism

\[
\alpha : D_I \rightarrow D_J.
\]

This is obvious if \( I \) has a finite cofinal subset, since in this case we have that \( D_I = H_I \) and \( D_J = H'_J \) by Proposition 7.1; otherwise \( D_I = H_I \oplus e_X D \) and \( D_J = H'_J \oplus e_X D \), therefore \( \alpha = (\bigoplus_{i \in I} \alpha_i) \oplus 1_{e_X D} \). Now it follows from (7.6) that \( \alpha \) is a \( D \)-ring isomorphism.

**Remark 7.2.** If \( I \) is any artinian poset and \( J \subset I \), then the two rings \( D_{I,J} \) and \( D_J = D_{J,J} \) can be different and may even be non-isomorphic. This latter case occurs if, for example, \( \sup(\vert I \vert, \vert J \vert) > \aleph_0 \) and \( \sup(\vert J \vert, \vert N \vert) < \sup(\vert I \vert, \vert M \vert) \), where \( M \) and \( N \) are the sets of all maximal chains of \( I \) and \( J \) respectively.

### 8. Upper subsets of \( I \) versus ideals of the ring \( D_I \).

If \( K \subset J \subset I \), then it is clear that \( H_K \) and \( H_{J,K} \) are complementary direct summands of \( H_J \) as \( (D, D) \)-submodules, but they need not be ideals of \( D_{I,J} \). In this connection the case in which \( K \) is an upper subset of \( J \) is of a particular interest, mainly due to the following result.
Proposition 8.1. Assume that $\emptyset \neq K \subset J \subset I$. Then the following properties hold:

1. $H_{J \setminus K}$ is an ideal of $D_{I,J}$ if and only if $K$ is an upper subset of $J$.
2. If $K$ is an upper subset of $J$, then there is a unique (unital) surjective $D$-linear ring homomorphism

\[ \varphi_{K,J} : D_{I,J} \to D_{I,K} \]

such that

\[ \varphi_{K,J}(a' + a'' + e_{XJ}d) = a' + e_{XK}d \quad \text{for all } a', a'' \in H_K, e \in H_{J\setminus K}, d \in D; \]

moreover

\[ \varphi_{K,J} \text{ induces an isomorphism of } D\text{-rings} \]

\[ D_{I,K} \simeq \begin{cases} D_{I,J} & \text{if } K \text{ is finitely sheltered in } I, \\ D_{I,J} / H_{J \setminus K} & \text{if } K \text{ is not finitely sheltered in } I. \end{cases} \]

Proof. (1) Assume that $H_{J \setminus K}$ is an ideal of $D_{I,J}$ and take $j \in J, k \in K$ with $k \leq j$. If $j \notin K$, then $H_j \subset H_{J \setminus K}$ and it follows from Theorem 6.15 that $0 \neq H_j H_k \subset H_{J \setminus K} = 0$, hence a contradiction. Thus necessarily $j \in K$. Conversely, suppose that $K$ is an upper subset of $J$ and let $j \in J, k \in J \setminus K$. Then exactly one of the following possibilities occurs: a) $j \notin K$, b) $j \in K$ and $j, k$ are unrelated, c) $j \in K$ and $j > k$. In all cases it follows from Theorem 6.15 that $H_j H_k \cup H_k H_j \subset H_{J \setminus K}$. Since $H_{J \setminus K}$ is already a $(D, D)$-submodule of $D_{I,J}$, this is sufficient to conclude that it is an ideal of $D_{I,J}$.

(2) Assume now that $K$ is an upper subset of $J$. If $J$ is finitely sheltered in $I$, then $D_{I,J} = H_J = H_K \oplus H_{J \setminus K}$. Since $K$ is an upper subset of $J$, then $K$ is finitely sheltered in $I$ as well and hence $D_{I,K} = H_K$. Set $J^\star = \{m_1, \ldots, m_r, m_{r+1}, \ldots, m_{r+s}\}$, where $\{m_1, \ldots, m_r\} = K^\star$. It is clear that every maximal chain of $J$ is bounded by an element of $J^\star$, thus Proposition 6.15 tells us that both $\{X_{m_1}, \ldots, X_{m_{r+s}}\}$ and $\{X_{m_1}, \ldots, X_{m_r}\}$ are partitions of $X_J$ and $X_K$, respectively. It follows that $e_{X_{m_1}}, \ldots, e_{X_{m_r}}, e_{X_{m_{r+1}}}, \ldots, e_{X_{m_{r+s}}}$ are pairwise orthogonal idempotents and we have that

\[ e_{X_K} = e_{X_{m_1}} + \cdots + e_{X_{m_r}} \in H_K, \quad e_{X_{J\setminus K}} = e_{X_{m_{r+1}}} + \cdots + e_{X_{m_{r+s}}} \in H_{J \setminus K}, \]

hence $e_{X_J} = e_{X_K} + e_{X_{J\setminus K}}$. As a result, given $a' \in H_K, a'' \in H_{J \setminus K}$ and $d \in D$, we may write

\[ a' + a'' + e_{X_J}d = a' + e_{X_K}d + a'' + e_{X_{J\setminus K}}d \]

and, since $a' + e_{X_K}d \in H_K$ and $a'' + e_{X_{J\setminus K}}d \in H_{J \setminus K}$, we infer that (8.1) defines $\varphi_{K,J}$ as the projection of $D_{I,J}$ onto $H_K = D_{I,K}$ parallel to $H_{J \setminus K}$. If $J$ is not finitely sheltered in $I$, then

\[ D_{I,J} = H_K \oplus H_{J \setminus K} \oplus e_{X_J}D \]

by Proposition 7.1 and there exists a unique $D$-linear map $\varphi_{K,J} : D_{I,J} \to D_{I,K}$ satisfying (8.1). In any case, we have that $\varphi_{K,J}$ is well defined by mean of (8.1) and, by using the fact that $H_{J \setminus K}$ is an ideal of $D_{I,J}$, it is an easy matter to show that $\varphi_{K,J}$ is a ring homomorphism.
Finally, if $K$ is finitely sheltered in $I$, then $e_{X_{K^*}} = e_{X_K} \in H_K$ by Proposition 7.1 and so $e_{X_J} - e_{X_K} \in D_{1,J}$; consequently $\text{Ker}(\varphi_{K,J}) = H_{J,K} + (e_{X_J} - e_{X_K})D$.

If, on the contrary, $K$ is not finitely sheltered in $I$, then $J$ is not finitely sheltered in $I$ as well and we have the decomposition (8.3); in this case it is clear that $\text{Ker}(\varphi_{K,J}) = H_{J,K}$.

Given any subset $J$ of $I$ and any ordinal $\alpha < \xi$, the $\alpha$-th layer $J_\alpha$ is a lower subset of $J$, therefore it follows from Proposition 8.1 that $H_{J_\alpha}$ is an ideal of the ring $D_{1,J}$. By considering the set $J_1$ of all minimal elements of $J$, the corresponding ideal $H_{J_1}$ will play a special role, mainly due to the next result.

**Proposition 8.2.** If $\emptyset \neq J \subset I$, then $H_{J_1}$ is essential as a right ideal and is pure as a left ideal of the ring $D_{1,J}$.

**Proof.** Suppose that $0 \neq a \in D_{1,J}$ and assume first that $a \in H_{J_1}$. Then there is a smallest finite nonempty subset $K \subset J$ such that $a \in H_K$. According to Corollary 1.4 we can choose some $j \in J_1$ such that $K' = K \cap \{ j \leq \} \neq \emptyset$ and so $a = a' + a''$ for unique nonzero elements $a' \in H_K$ and $a'' \in H_{K \setminus K'}$. Consequently it follows from Theorem 6.1.14 that there is an idempotent $e \in H_j$ such that $0 \neq a' e \in H_j \subset H_{J_1}$, while $a'' e = 0$. As a result $0 \neq a e \in H_{J_1}$.

If $D_{1,J} = H_{J_1}$, the above argument shows that $H_{J_1}$ is essential as a right ideal of $D_{1,J}$. Otherwise, according to Proposition 7.1 we have that $D_{1,J} = H_{J} + e_{X_J}D$. If $0 \neq a \in e_{X_J}D$, given any $j \in J_1$ and $Y \in P_{\lambda(j)}$, we have that $e_{Y(j)} \in H_j \subset H_{J_1}$ (see Lemma 6.11) and so $0 \neq a e_{Y(j)} \in H_{J_1}$, because $Y(j) \subset X_J$. In order to complete the proof it remains to consider the case in which $a = b + d$, where $0 \neq b \in H_J$ and $0 \neq d \in e_{X_J}D$. Let $K$ be the smallest finite subset of $J$ such that $b \in H_K$. Again from Proposition 7.1 we have that either $J^* \subset J$ or $J \not\subset \{ J^* \}$, or $J^* \not\subset J$. In the first and third cases it is clear that $J^* \setminus K \neq \emptyset$.

In the second case, $J$ contains at least an infinite chain. Thus, in all cases we can choose an element $m \in J$ such that $m \neq k$ for every $k \in K$ and Corollary 1.4 allows us to take some $j \in J_1$ in such a way that $j \leq m$. By Lemma 6.1 there are $B \in D_m$ and $C \in D_j$ such that $B \subset C$. We claim that there is some $V \in Q_S$ such that the $x$-th row of $b$ is zero when $x \in V$. This is clear if $B \cap A = \emptyset$ for all $A \in D_k$ with $k \in K$. Otherwise, let $k_1, \ldots, k_r$ be those elements of $K$ such that $B \cap A_t \neq \emptyset$ for some $A_t \in D_t$, where $t \in \{ 1, \ldots, r \}$. Because of the choice of $m$, it follows from Lemma 6.4 that $k_1, \ldots, k_r$ and $m$ are pairwise comparable and we may assume that $k_1 < \cdots < k_r < m$. Let us consider the unique decomposition $b = b' + b''$, where $b' \in H_{\{ k_1, \ldots, k_r \}}$ and $b'' \in H_{K \setminus \{ k_1, \ldots, k_r \}}$. Noting that $A \cap B = \emptyset$ whenever $A \in D_k$ for some $k \in K \setminus \{ k_1, \ldots, k_r \}$, we see that the $(X_B \times X_B)$-blocks of $b$ and $b'$ coincide. Inasmuch as $k_1, \ldots, k_r$ are not maximal, by applying Lemma 6.11 to the single components of $b'$ in $H_{k_1}, \ldots, H_{k_r}$ we see that there are $V_1, \ldots, V_s \in Q_S \subset P_{\lambda(m)}$ such that if $x \in X_B$ and the $x$-th row of $b'$ is not zero, then $x \in V_1 \cup \cdots \cup V_s$. Thus, since $Q_S$ contains $8$ elements of $P_{\lambda(m)}$, there is $V \in Q_S$ such that the $x$-th row of $b$ is zero when $x \in V$ and our claim is established.

Now, pick any $W \in Q_C$ such that $W \subset V$ (see Remark 6.5) and consider the idempotent $e_W \in H_j$. By the above, $b e_W$ has zero $x$-th row for $x \in W$, while $d e_W$ has nonzero $x$-th row for all $x \in W$, because $W \subset X_B$ and $W \subset X_J$. This shows that $a e_W = (b + d) e_W$ is not zero and belongs to $H_{J_1}$, completing the proof that $H_{J_1}$ is essential as a right ideal of $D_{1,J}$.
Finally, let $j \in J_1$ and note that $H_j$ is an ideal of $D_{I,J}$ by (1) of Proposition 8.1. We have that $H_j = \psi_j(F_{\lambda(j)})$ and $\psi_j$ is a ring monomorphism; consequently, since $F_{\lambda(j)}$ is left pure, we infer that $H_j$ is left pure as well; in particular, for every $a \in H_j$ there is an idempotent $e \in H_J$ such that $a = ea$. Assume that $a \in H_{J_1} = \bigoplus_{j \in J_1} H_j$, that is, $a = a_1 + \cdots + a_n$ for some $a_1 \in H_{j_1}, \ldots, a_n \in H_{j_n}$ with $j_1, \ldots, j_n \in J_1$. Then there are appropriate idempotents $e_1 \in H_{j_1}, \ldots, e_n \in H_{j_n}$ such that $a_\alpha = e_\alpha a, a_\alpha$ for all $r = 1, \ldots, n$ and it follows from Theorem 6.13 that these idempotents are pairwise orthogonal. As a result $e = e_1 + \cdots + e_n$ is an idempotent of $H_{J_1}$ such that $a = ea$, hence $H_{J_1}$ is left pure.

\[ \square \]

9. Semiartinian unit-regular rings are coming, finally!

The setup we need is now complete for use. In this final section, starting from a given nonempty polarized artinian poset $I$, we only have to specialize the ring $D$ and check that the corresponding ring $D_I$, as defined in the previous section, is a semiartinian and regular ring which satisfies the conditions we had announced. Thus, by keeping the same data, assumptions and notations so far introduced, from now on we assume that $D$ is a division ring. Now the ring $Q = CFM_X(D)$, and hence $Q_\alpha$ for every $\alpha \leq \xi_i$ is regular, prime and right selfinjective; moreover $F_\alpha = Soc(Q_\alpha)$, so that each $H_i$ is a simple and semisimple ring, no matter if $i \in I'$ or not; it has a multiplicative identity if and only if $i$ is a maximal element of $I$, in which case $H_i$ is isomorphic to $D$.

**Lemma 9.1.** Assume that $\emptyset \neq J \subset I$. Then $D_{I,J}$ is a regular ring and

\[ H_{i_1} = Soc(D_{I,J}). \]

**Proof.** Since $J \setminus J_1$ is an upper subset of $J$, then $H_{J_1}$ is an ideal of $D_{I,J}$ by Proposition 8.1. Moreover $H_J$ is Von Neumann regular, because it is a direct sum of simple and semisimple rings. If $D_{I,J} \neq H_J$, then $D_{I,J}/H_J \cong D$ is regular, thus $D_{I,J}$ is itself regular (see [21] Lemma 1.3]). Let $i \in J_1$. Since $\{i\}$ is a lower subset of $J$, it follows from Proposition 8.1 that $H_i$ is an ideal of $D_{I,J}$. On the other hand, in view of our assumptions $H_i$ is a semisimple ring (possibly without identity), thus we infer that $H_{J_1} = \bigoplus_{j \in J_1} H_j \subset Soc(D_{I,J})$. Since $H_{J_1}$ is an essential right ideal of $D_{I,J}$ by Proposition 8.2, the opposite inclusion holds and the equality follows. \[ \square \]

As we have seen in Lemma 6.11 if $i \in I \setminus I^\star$, then $H_i$ contains the set $E_i = \{ e_{Y(i)} \mid Y \in P_{\lambda(i)} \}$ of pairwise orthogonal idempotents, each of which generates $H_i$ as an ideal of itself (see \( 6.15 \)). This time, having chosen $D$ as a division ring, each $e_{Y(i)}$ is primitive. To every element $i \in I$ we associate an idempotent $u_i \in D_I$ and a right $D_I$-module $U_i$ with the following rules: if $i \in I \setminus I^\star$, we choose $u_i \in E_i$, while if $i \in I^\star$, then we set $u_i : = e_{X_i}$. Next, set

\[ U_i : = (u_i D_I + H_{\lambda(i)-1})/H_{\lambda(i)-1}. \]

**Proposition 9.2.** For every $i \in I$ the right $D_I$-module $U_i$ is simple and

\[ r_{D_I}(U_i) = \begin{cases} H_{\lambda(\{i\})} + (1 - e_{X_i}) & \text{if } \{i\} \text{ is finitely sheltered in } I; \\ H_{\lambda(\{i\})}, & \text{if } \{i\} \text{ is not finitely sheltered in } I. \end{cases} \]

If $I$ has a finite cofinal subset, then $r_{D_I}(U_i) = H_{\lambda(\{i\})}$ for every $i \in I$.  

\[ \square \]
Proof. Firstly, it follows from (6.15) that
\begin{equation}
U_i = (u_i H_i + H_{1,i(i-1)}) / H_{1,i(i-1)} = U_i H_i.
\end{equation}
Suppose that \(0 \neq x \in U_i\). Then \(x = u_i a + H_{1,i(i-1)}\) for some nonzero \(a \in H_i\).
Inasmuch as \(u_i H_i\) is a minimal right ideal of the ring \(H_i\), then \(u_i a H_i = u_i H_i\) and consequently \(x D_I = U_i\) by (9.2), proving that \(U_i\) is a simple \(D_I\)-module. Next, taking into account that \(\{ i \leq \}\) is an upper subset of \(I\), if we specialize Proposition 8.1 by setting \(J = I\) and \(K = \{ i \leq \}\), we see that the second member of the equality (9.1) is precisely the kernel of the ring epimorphism \(\varphi = \varphi_{K,J} : D_I \to D_{I,\{ i \leq \}} = H_{i \leq \} + e_{X_{i \leq \}} D\) defined by the rule (8.1). By (9.2) and Theorem 6.15 we have that \(U_i H_j = 0\) when \(j \in I \setminus \{ i \leq \}\), therefore \(H_{I,\{ i \leq \}} \subset \rho \rho_{D_i}(U_i)\). If \(\{ i \leq \}\) is finitely sheltered in \(I\), then \(e_{X_{i \leq \}}\) is the multiplicative identity of the ring \(H_{i \leq \}\) according to Proposition 7.1 in particular \(H_i = H_i e_{X_{i \leq \}}\) and so \((1 - e_{X_{i \leq \}}) D \subset \rho \rho_{D_i}(U_i)\) by (9.2). This shows that \(\text{Ker}(\varphi) \subset \rho \rho_{D_i}(U_i)\) and hence \(U_i\) is canonically a simple right \(D_{I,\{ i \leq \}}\)-module. Now it follows from Proposition 8.1 that \(D_{I,\{ i \leq \}}\) has essential socle given by \(H_i\); since this latter is homogeneous and regular, we infer that the ring \(D_{I,\{ i \leq \}}\) is primitive. Accordingly, since \(U_i = U_i H_i\) we conclude that \(U_i\) is faithful as a simple right \(D_{I,\{ i \leq \}}\)-module and this establishes the equality (9.1).

The last statement follows directly from the equality (9.1), because if \(I\) has a finite cofinal subset, then every subset of \(I\) is finitely sheltered in \(I\) and it follows from Proposition 7.1 that \(1 - e_{X_{i \leq \}} = e_{X_{i \setminus \{ i \leq \}}} \in H_{I,\{ i \leq \}}\).

We are now in a position to analyze the main features of the regular ring \(D_I\), the first of which is that it is semiartinian. Observe that \(I^{\bullet \bullet}_\xi = \emptyset\) is finitely sheltered in \(I\); thus we may consider the ordinal \(\xi_0\) defined by
\[
\xi_0 := \min \{ \alpha \mid I^{\bullet \bullet}_\alpha \text{ is finitely sheltered in } I \} \leq \xi.
\]
As we shall see, \(\xi_0\) will be critical when determining the Loewy chain of \(D_I\). If \(\xi_0 < \xi\) and \(\{ I^{\bullet \bullet}_\xi \}^* = \{ k_1, \ldots, k_n \}\), we shall consider the idempotent
\[
f := e_{X_{k_1}} + \cdots + e_{X_{k_n}}.
\]
Remember that \(f\) is the multiplicative identity of \(H^{\bullet \bullet}_\xi = D_{I,\{ i \leq \}}\) by Proposition 7.1.

If \(\xi_0 < \xi\), then \(I^{\bullet \bullet}_\xi\) has a finite cofinal subset and therefore \(\xi\) must be a successor ordinal. In particular, it is clear that \(\xi_0 = 0\) if and only if \(I\) has a finite cofinal subset, in which case \(f = 1\) and \(D_I = H_I\) by Proposition 7.1.

We shall need a couple of lemmas, the second of which is a direct consequence of [7, Proposition 3].

**Lemma 9.3.** Let \(R\) be a ring with projective and essential right socle \(L\) and assume that \(R\) has a subring \(S\) such that \(R = S + L\) and \(R\) is left \(S\)-flat. If \(S\) is right hereditary, then \(R\) is right hereditary as well.

**Proof.** In order to prove that \(R\) is right hereditary it is sufficient to show that if \(E\) is an injective right \(R\)-module with an essential submodule \(M\), then \(E/M\) is \(R\)-injective. Firstly, by the flatness of \(S\), the injectivity of \(E_R\) implies that of \(E_S\). Now it is readily seen that the canonical right \(R/L\)-module structure on \(E/M\), arising from the fact that \(E L \subset M\), and the original structure of a factor \(R\)-module restrict to the same \(S\)-module structure. As \(S\) is right hereditary, it follows that \(E/M\) is \(S\)-injective and hence \(R/L\)-injective. Finally, since \(L\) is left pure in \(R\), we conclude that \(E/M\) is \(R\)-injective. \(\square\)
Lemma 9.4. Let $R$ be a ring with a faithful simple and projective right $R$-module $S$ and let $Q = \text{BiEnd}(S_R)$, so that $R$ can be identified with a dense subring of $Q$ and $\text{Soc}(R) = R \cap \text{Soc}(Q)$ is the trace of $S$ in $R$. Then $S_R$ is injective if and only if $\text{Soc}(R) = \text{Soc}(Q)$.

We recall that a ring $R$ is unit-regular if for every $x \in R$ there exists a unit $u \in R$ such that $x = xu$. It is well known that a regular ring $R$ is unit regular if and only if, given three finitely generated projective right $R$-modules $A, B, C$, the condition $A \oplus C \cong B \oplus C$ implies $A \cong B$. Another equivalent condition is that $R$ has stable range 1, meaning that if $a, b \in R$ and $aR + bR = R$, then there is some $c \in R$ such that $a + bc$ is a unit (see [21, Chapter 4]).

Theorem 9.5. With the above settings and notations, the ring $D_I$ satisfies the following properties:

1. For every ordinal $\alpha \leq \xi$

\[
\text{(9.3)} \quad \text{Soc}_{\alpha}(D_I) = \begin{cases} 
H_{I_{\alpha}}, & \text{if } I \text{ has a finite cofinal subset or } \alpha \leq \xi_0 \\
H_{I_{\alpha}} \oplus (1 - f)D, & \text{if } 0 < \xi_0 < \alpha.
\end{cases}
\]

Thus the ring $D_I$ is semiartinian and its Loewy length is $\xi$ (resp. $\xi + 1$) if $\xi_0 < \xi$ (resp. $\xi_0 = \xi$).

2. If $i, j \in I$, then $U_i \leq U_j$ if and only if $i \leq j$ and we have

\[
\text{(9.4)} \quad \text{Simp}_{D_I} = \begin{cases} 
\{U_i \mid i \in I\}, & \text{if } I \text{ has a finite cofinal subset}, \\
\{U_i \mid i \in I\} \cup \{D_I/H_I\}, & \text{otherwise}.
\end{cases}
\]

Thus $I$ and $\text{Simp}_{D_I}$ are isomorphic posets if $I$ has a finite cofinal subset, otherwise the additional simple module $D_I/H_I$ is a maximal element of $\text{Simp}_{D_I}$, such that

\[h(D_I/H_I) = \xi_0 + 1\]

and, for every $i \in I$,

\[
\text{(9.5)} \quad U_i \prec D_I/H_I \text{ if and only if } \{i \leq\} \text{ is not finitely sheltered in } I.
\]

3. $D_I$ is unit regular.

4. If $U \in \text{Simp}_{D_I}$, then $U D_I$ is injective if and only if $U$ is either a maximal element, or $U = U_i$ for some $i \in I'$. Consequently $D_I$ is a right V-ring if and only if $I' = I$. Moreover, $D_I$ is a right and left V-ring if and only if $\xi = 1$, if and only if all primitive factor rings of $D_I$ are artinian.

5. If $\xi$ is a natural number, in particular if $I$ is finite, then $D_I$ is (right and left) hereditary.

6. If $I' = \emptyset$ and $I$ is at most countable, then the dimension of $D_I$ as a right and a left vector space over $D$ is countable.

7. $D_I$ is well behaved (Proposition and Definition [22]) if and only if, for every $\alpha < \xi_0$, there is some $i \in I'_{\alpha+1}$ such that $\{i \leq\}$ is not finitely sheltered.

8. $D_I$ is very well behaved (Definition [22]) if and only if $I$, or equivalently $\text{Simp}_{D_I}$, has finitely many maximal elements.

Proof. (1) Obviously [9.3] holds if $\alpha = 0$, while if $\alpha = 1$, then [9.3] follows directly from Lemma 9.1. Suppose that $\alpha > 1$, assume that either $I$ has a finite cofinal
subset, or \( \alpha \leq \xi_0 \), and assume inductively that \( \text{Soc}_\beta(D_I) = H_{I_\beta} \) for every ordinal \( \beta < \alpha \). If \( \alpha \) is a limit ordinal, since \( I_\alpha := \bigcup_{\beta < \alpha} I_\beta \), then we have

\[
\text{Soc}_\alpha(D_I) = \bigcup_{\beta < \alpha} \text{Soc}_\beta(D_I) = \bigcup_{\beta < \alpha} H_{I_\beta} = H_{I_\alpha}.
\]

Suppose that \( \alpha = \beta + 1 \) for some \( \beta \) and set \( J = I_\beta^\bullet \). Since \( J \) is an upper subset of \( I \), we can consider the surjective ring homomorphism \( \varphi_{J,I} : D_I \to D_{I,J} \) as in Proposition \( \text{8.1} \). If \( \alpha \leq \xi_0 \), then \( J \) is not finitely sheltered in \( I \) and it follows from Proposition \( \text{8.1} \) that

\[
(9.6) \quad \text{Ker}(\varphi_{J,I}) = H_{I_{\beta+1}} = H_{I_\beta} = \text{Soc}_\beta(D_I).
\]

If \( I \) has a finite cofinal subset, then \( J \) is finitely sheltered in \( I \) and again Proposition \( \text{8.1} \) tells us that

\[
\text{Ker}(\varphi_{J,I}) = H_{I_{\beta+1}} + (1 - e_{X_J})D = H_{I_\beta} + (1 - e_{X_J})D;
\]

moreover we have that \( 1 - e_{X_J} \in H_{I_\beta} \), therefore \( 9.6 \) again holds. Thus, in both cases \( \varphi_{J,I} \) induces an isomorphism \( D_I/\text{Soc}_\beta(D_I) \simeq D_{I,J} \), which in turn restricts to the canonical isomorphism \( (H_{I_{\beta+1}} + \text{Soc}_\beta(D_I))/\text{Soc}_\beta(D_I) \simeq H_{I_{\beta+1}} \). As a result we obtain

\[
H_{I_{\beta+1}}/H_{I_\beta} = \left( H_{I_{\beta+1}} + H_{I_\beta} \right)/H_{I_\beta} \simeq H_{I_{\beta+1}} = H_{(I_{\beta+1})} = \text{Soc}(D_{I,J}),
\]

where the last equality comes from Lemma \( \text{9.1} \). This shows that

\[
H_{I_{\beta+1}}/H_{I_\beta} = \text{Soc}(D_I/\text{Soc}_\beta(D_I))
\]

and therefore \( \text{Soc}_\alpha(D_I) = H_{I_\alpha} \).

Next, let us consider the case in which \( 0 < \xi_0 < \alpha \). Inasmuch as \( \xi_0 \leq \beta \), then \( J^\bullet \) is finite and so we may consider the (orthogonal) idempotents \( g = \sum \{ e_{X_k} \mid k \in J^\bullet \} \) and \( h = \sum \{ e_{X_h} \mid k \in (I_{\xi_0})^\bullet \setminus J^\bullet \} \). Moreover \( g, h \in H_J = R_J \) by Proposition \( \text{7.1} \) and \( h \in H_{I_\beta} \), because \( (I_{\xi_0})^\bullet \setminus J^\bullet \subset I_\beta \). As a result, since \( I \) is not finitely sheltered in \( I \), by using again Proposition \( \text{7.1} \) and noting that \( f = g + h \) we see that

\[
(9.7) \quad D_I = H_I \oplus (1 - f)D = H_{I_\beta} \oplus D_{I,J} \oplus (1 - g - h)D
\]

and hence \( D_I/H_{I_\beta} \simeq D_{I,J} \oplus (1 - g)D \). Now, since \( g \) is the multiplicative identity of \( H_J \), it is immediately checked that \( (1 - g) + H_{I_\beta} \) is a central idempotent of \( D_I/H_{I_\beta} \). Consequently

\[
D_I/H_{I_\beta} \simeq D_{I,J} \times D
\]

as rings. We are then in a position to compute \( \text{Soc}_{\xi_0+1}(D_I) \), by putting \( \beta = \xi_0 \) in the above. Since \( \text{Soc}_{\xi_0}(D_I) = H_{I_{\xi_0}} \), as it follows from the first part of the proof, then we have that

\[
D_I/\text{Soc}_{\xi_0}(D_I) = D_I/H_{I_{\xi_0}} \simeq D_{I,J_{\xi_0}} \times D
\]

as rings. Inasmuch as \( \text{Soc}(D_{I,J_{\xi_0}}) = H_{I_{\xi_0+1}} \) by Lemma \( \text{9.1} \) and \( h = 0 \) when \( \beta = \xi_0 \), it follows from \( 9.7 \) that

\[
\text{Soc}_{\xi_0+1}(D_I) = H_{I_{\xi_0}} \oplus H_{I_{\xi_0+1}} \oplus (1 - f)D = H_{I_{\xi_0+1}} \oplus (1 - f)D.
\]

Now, assume that \( \alpha > \xi_0 + 1 \) and suppose, inductively, that

\[
(9.8) \quad \text{Soc}_\beta(D_I) = H_{I_\beta} \oplus (1 - f)D
\]
whenever $\xi_0 < \beta < \alpha$. If $\alpha = \beta + 1$ for some $\beta > \xi_0$, then it follows from (9.7) and (9.8) that $D_I/\Soc(\alpha)D_I \simeq D_{I,J}$. Since $\Soc(D_{I,J}) = H_{I}\bullet 1 + 1$ by Lemma 9.1, we infer that

$$\Soc_\alpha(D_I) = H_{I}\bowtie H_{I}\bullet 1 + 1 \oplus (1 - f)D = H_{I}\bowtie (1 - f)D,$$

as wanted. Finally, if $\alpha$ is a limit ordinal, then we have that $H_{I\alpha} = \bigcup_{\beta<\alpha} H_{I\beta}$ and hence

$$H_{I\alpha} \cap (1 - f)D = \left( \bigcup_{\beta<\alpha} H_{I\beta} \right) \cap (1 - f)D = \bigcup_{\beta<\alpha} (H_{I\beta} \cap (1 - f)D) = 0.$$ 

It follows that

$$\Soc_\alpha(D_I) = \bigcup_{\beta<\alpha} \Soc_\beta(D_I) = \bigcup_{\beta<\alpha} (H_{I\beta} \oplus (1 - f)D) = \left( \bigcup_{\beta<\alpha} H_{I\beta} \right) \oplus (1 - f)D = H_{I\alpha} \oplus (1 - f)D$$

and we are done.

As far as the Loewy length of $D_I$ is concerned, if $I$ has a finite cofinal subset, that is $\xi_0 = 0$, then $D_I = H_I = H_{I\alpha}$ and so it follows from (9.3) that $D_I$ has Loewy length $\xi$. If $0 < \xi_0 < \xi$, then by (9.3) we have that

$$\Soc_\xi(D_I) = H_{I\alpha} \oplus (1 - f)D = H_I \oplus 1D = D_I$$

and therefore $D_I$ has again Loewy length $\xi$. If $\xi_0 = \xi$, then (9.3) imply that

$$D_I/\Soc_\xi(D_I) = D_I/H_{I\alpha} = D_I/H_I \cong D_I$$

and $D_I$ has Loewy length $\xi + 1$.

(2) Let $i,j \in I$ and assume that $i \leq j$. Then $\{i \leq \} \sqsubset \{j \leq \}$, that is $I \setminus \{i \leq \} \subset I \setminus \{j \leq \}$ and therefore $H_I \setminus \{i \leq \} \subset H_I \setminus \{j \leq \}$. As a result, if $\{i \leq \}$ is not finitely sheltered in $I$, then it follows from Proposition 9.2 that $r_{D_I}(U_i) \subset r_{D_I}(U_j)$.

Suppose that $\{i \leq \}$, and hence $\{j \leq \}$, is finitely sheltered in $I$ and set $\{i \leq \} = \{m_1, \ldots, m_r, m_{r+1}, \ldots, m_s\}$, where $\{m_{r+1}, \ldots, m_s\} = \{j \leq \}$. By Proposition 6.3, $\{X_{m_1}, \ldots, X_{m_r}, X_{m_{r+1}}, \ldots, X_{m_s}\}$ and $\{X_{m_1}, \ldots, X_{m_r}\}$ are partitions of $X_{\leq \}$ and $X_{\leq \}$, respectively; therefore, we can consider the corresponding pairwise orthogonal idempotents $e_{X_{m_1}}, \ldots, e_{X_{m_r}}, e_{X_{m_{r+1}}} \ldots, e_{X_{m_s}}$. We observe that $e_{X_{m_1}}, \ldots, e_{X_{m_r}} \in H_I \setminus \{j \leq \}$. Consequently, by taking again Proposition 9.2 into account, we see that

$$1 - e_{X_{\leq \}} = 1 - e_{X_{m_1}} - \cdots - e_{X_{m_r}} - e_{X_{m_{r+1}}} - \cdots - e_{X_{m_s}}$$

Thus we have again that $r_{D_I}(U_i) \subset r_{D_I}(U_j)$, proving that $i \leq j$ implies $U_i \leq U_j$. At this point, in order to show that the reverse implication holds, it is sufficient to prove that if $U_i \leq U_j$, then $i$ and $j$ are comparable. However, if $r_{D_I}(U_i) \subset r_{D_I}(U_j)$, then necessarily $U_i H_j \neq 0$, otherwise we would get $U_j = U_j H_j = 0$. Since $U_i = U_i H_i$, then $H_i H_j \neq 0$ and so $i$ and $j$ are comparable by Theorem 6.15.

Let $V$ be any simple right $D_I$-module. If $V H_I = 0$, then $H_I \neq D_I$ and it is clear that $V \simeq D_I/H_I$. Otherwise $V H_I = V$ and we may consider the smallest ordinal $\alpha$ such that $V H_i \neq 0$ for some $i \in I^\bullet_{\alpha+1}$. According to Lemma 6.11 we have that

$$0 \neq VH_i = VH_i u_i H_i \subset VH_i.$$
Let $x \in V$ be such that $xu_i \neq 0$. If $a \in D_I$ and $u, a \in H_i$, then $xu_i a = 0$ by the choice of $\alpha$. Thus the assignment $u, a + H_i \mapsto xu_i a$ defines a nonzero $D_I$-linear map from $U_i$ to $V$ and consequently $V \simeq U_i$.

Finally, if $i \in I$ and $\{ i \}$ is not finitely sheltered in $I$, it follows from Proposition 9.2 that $U_i \nsubseteq D_I/H_I$. If, on the contrary, $\{ i \}$ is finitely sheltered in $I$, then $e_{X_1(i)} \in H_{\{ i \}}$ by Proposition 7.1. As a result, by using again Proposition 9.2 we see that

$$1 = (1 - ex_{i(\{ i \})}) + ex_{i(\{ i \})} \in \cap D_I(U_i) + H_I$$

and hence $r_{D_I(U_i)} + H_I = D_I$, proving that $r_{D_I(U_i)} \nsubseteq H_I$, namely $U_i \nsubseteq D_I/H_I$.

(3) Let us prove first that the ring $D_{I,J}$ is unit-regular whenever $\{ J \} = n$ and take $J \subseteq I$ such that $|J| = n$. Let us consider the surjective ring homomorphism

$$\varphi_{J\setminus J_1 : D_{I,J} \rightarrow D_{I,J \setminus J_1}}$$

(see Proposition 8.1) and note that $D_{I,J \setminus J_1}$ is unit-regular by the inductive hypothesis. In order to prove unit-regularity of $D_{I,J}$, according to Vasershtein criterion (see Proposition 4.12, or 3.5) for a ready-to-use version) it will be sufficient to prove that every unit of $D_{I,J \setminus J_1}$ has the form $\varphi_{J\setminus J_1}(a)$ for some unit $a$ of $D_{I,J}$ and $u_{D_{I,J}}$ is unit-regular for every idempotent $u \in \ker(\varphi_{J\setminus J_1})$.

By denoting with $K$ the set of those elements of $J$ which are isolated in $J$, i.e. are minimal and maximal in $J$, we have from Proposition 6.9 that $K = \{ j \in J \mid X_j \cap X_{J \setminus \{ j \}} = \emptyset \}$; since $K \subseteq J_1$, we infer that

$$X_{J \setminus J_1} \cap X_K = \emptyset \quad \text{and} \quad X_{J \setminus J_1} \cup X_K = X_J.$$

Let us write

$$e = e_{X_J}, \quad e' = e_{X_{J \setminus J_1}} \quad \text{and} \quad e'' = e_{X_K},$$

so that $e', e''$ are orthogonal idempotents and $e' + e'' = e$, and suppose that $b$ is a unit of $R_{J \setminus J_1}$. If $D_{I,J \setminus J_1} = H_{J \setminus J_1}$, then it follows from Proposition 7.1 that $e'$ is the multiplicative identity of $H_{J \setminus J_1}$, and $J \setminus J_1$ is finitely sheltered in $I$; consequently $e'' = e - e' \in \ker(\varphi_{J\setminus J_1})$ by 8.2. If $b'$ is the inverse of $b$ in $D_{I,J \setminus J_1}$, using the fact that $b$ and $b'$ belong to $e'D_I, e'$ it is immediately seen that $b' + e''$ is an inverse for $b + e''$ in $D_{I,J}$ and $\varphi_{J\setminus J_1}(b + e'') = b$. Assume that $D_{I,J \setminus J_1} \neq H_{J \setminus J_1}$. Then $D_{I,J \setminus J_1} = H_{J \setminus J_1} \oplus e'D$ by Proposition 7.1 and so $b = c + e'd$ for unique $c \in H_{J \setminus J_1}$ and $d \in D$. Necessarily $d \neq 0$ and if $c' + e'd'$ is the inverse of $b$ in $D_{I,J \setminus J_1}$, where $c' \in H_{J \setminus J_1}$ and $d' \in D$, then $d' = d^{-1}$. Noting that $e''H_{J \setminus J_1} = 0 = H_{J \setminus J_1}e''$, we infer that

$$(c + ed)(c' + ed') = (c + e'd + e''d)(c' + e'd' + e''d') = (c + e'd)(c' + e'd') + e''dd' = e' + e'' = e.$$

Similarly $(c' + ed')(c + ed) = e'$, hence $c + ed$ is a unit in $D_{I,J}$ and $\varphi_{J\setminus J_1}(c + ed) = c + e'd = b$.

Next, let $u$ be an idempotent of $\ker(\varphi_{J\setminus J_1})$. If $J \setminus J_1$ is not finitely sheltered in $I$, then it follows from 8.2 and Lemma 8.1 that $\ker(\varphi_{J\setminus J_1}) = H_{J_1} = \soc(D_I)$; thus $u_{D_{I,J}}$ is a semisimple ring and so it is unit-regular. If, on the contrary, $J \setminus J_1$ is finitely sheltered in $I$, then 8.2 tells us that $\ker(\varphi_{J\setminus J_1}) = H_{J_1} + e''D$ and hence, observing that $e''H_{J_1 \setminus K}e'' = 0$, we get

$$\ker(\varphi_{J\setminus J_1}) = H_{J_1 \setminus K} \oplus (H_K + e''D) = H_{J_1 \setminus K} \oplus D_{I,K}.$$
As a consequence \( u = u' + u'' \) for unique orthogonal idempotents \( u' \in H_{J_1 \setminus K} \) and \( u'' \in D_{I_1, K} \). Inasmuch as \( H_{J_1 \setminus K} \) is semisimple and \( D_{I_1, K} \) is unit-regular by the inductive hypothesis, we conclude that \( uD_{I_1, J}u = u'H_{J_1 \setminus K}u' \oplus u''D_{I_1, K}u'' \) is unit-regular and we are done.

Finally, given any \( a \in D_1 \), there is a finite subset \( J \subset I \) and two elements \( b \in H_J \), \( d \in D \) such that \( a = b + 1d = b + e_{X_J}d + (1 - e_{X_J})d \). Thus \( a \) belongs to the unital subring \( S = D_{I_1, J} + (1 - e_{X_J})D \) of \( D_I \), in which \( e_{X_J} \) is a central idempotent. If \( X_J = X \), then \( e_{X_J} = 1 \) and \( S = D_{I_1, J} \). If \( X_J \neq X \), then \( S \simeq D_{I_1, J} \times D \) as rings. In both cases \( S \) is unit-regular and hence \( a = aba \) for a unit \( b \in S \subset D_I \), as wanted.

(4) If \( U \) is a maximal element of \( \text{Simp}_{D_I} \), that is \( r_{D_I}(U) \) is a maximal right ideal, then \( U_{D_I} \) is injective by [11, Corollary 4.8]. Otherwise, according to (9.3), \( U = U_i \) for some non-maximal element \( i \in I \). Set \( J = \{ i \leq \} \) and note that \( D_{I_1, J} \simeq D_I/r_{D_I}(U) \) is a primitive ring which has \( U_i \) as the unique (up to an isomorphism) faithful simple right module. Let us consider the ring \( S_i \) introduced immediately before Proposition 6.8 and the \( D \)-linear map \( \theta: D_{I_1, J} \rightarrow S_i \) defined by \( \theta(a) = e_{X_i}ae_{X_i} \). Note that if \( J > i \) and \( a \in H_J \), then both \( e_{X_i}a \) and \( e_{X_i}b \) belong to \( H_i \) by Theorem 6.15 (3) and therefore, since \( H_i \simeq e_{X_i}H_{J_i}e_{X_i} \), for every \( a, b \in H_J \) we have that

\[ e_{X_i}(ab)e_{X_i} = e_{X_i}ae_{X_i}b e_{X_i} \]

From this we infer immediately that \( \theta \) is a unital ring homomorphism. Observe that \( \theta \) restricts to the identity on \( H_i \subset \psi_i(F_{\lambda(i)}) = \text{Soc}(S_i) \); moreover \( H_i = \text{Soc}(D_{I_1, J}) \) is essential as a right ideal of \( D_{I_1, J} \) (see Lemma 9.1), therefore \( \theta \) is a monomorphism. As a result \( D_{I_1, J} \) can be identified with a subring of \( S_i \) and \( \text{Soc}(D_{I_1, J}) = H_i = D_{I_1, J} \cap \text{Soc}(S_i) \). Since in turn \( S_i \simeq Q_i \), it follows from Lemma 9.4 that \( U = U_i \) is \( D_{I_1, J} \)-injective if and only if \( \text{Soc}(D_{I_1, J}) = \text{Soc}(S_i) \), if and only if \( H_i = \psi_i(F_{\lambda(i)}) \), if and only if \( i \in I' \). Inasmuch as \( D_I \) is regular, then \( r_{D_I}(U) \) is pure as a left ideal and so \( U_{D_I} \) is injective. Finally, according to [8, Theorem 2.7] \( D_I \) is a right and left \( V \)-ring if and only if all primitive factor rings of \( D_I \) are artinian, that is, if and only if all primitive ideals of \( D_I \) are maximal as right ideals; in view of property (2) this happens if and only if \( i \) is an antichain, that is \( \xi = 1 \).

(5) Suppose that \( \xi \) is a natural number. We will prove that \( D_{I_1, J} \) is hereditary for every subset \( J \subset I \) by applying induction on the has dual classical Krull length \( \xi(J) \) of \( J \). It will follows that, in particular, \( D_I = D_{I_1, I} \) is hereditary. If \( \xi(J) = 0 \), that is \( J = \emptyset \), then and \( D_{I_1, \emptyset} = 0 \) is trivially hereditary. Given a positive integer \( n \leq \xi \), suppose that \( D_{I_1, J} \) is hereditary whenever \( \xi(J) < n \) and let \( J \) be any subset of \( I \) such that \( \xi(J) = n \). Since \( \xi(J^{**}) = n - 1 \), then \( D_{I_1, J^{**}} \) is hereditary by the inductive hypothesis. Assume that \( J \) is finitely sheltered in \( I \) and let \( K \) be the set of all isolated elements of \( I \) which belong to \( J \). Then we have

\[ D_{I_1, J} = H_J = H_J \oplus H_J^{**} = H_K \oplus H_{J_1 \setminus K} \oplus D_{I_1, J^{**} = H_K \oplus D_{I_1, J^{**} \setminus K}} \]

taking Proposition 7.1 into account. Since \( K \) is finite, then \( H_K \simeq D_K \) is a semisimple ring and hence is hereditary; thus, in order to prove that \( D_{I_1, J} \) is hereditary we may assume that \( K = \emptyset \). Consequently \( X_J = X_I \) and therefore \( e_{X_J}e_{X_J} = e_{X_J} \), so that \( H_J^{**} = D_{I_1, J^{**} \oplus e_{X_J}D \oplus H_J^{**} \oplus e_{X_J}D \) and, by using Proposition 8.1
we infer that \( H_{J} \oplus e_{X}, D \cong D_{I,J}/H_{I} \cong D_{I,J} \). Thus \( H_{J} \oplus e_{X}, D \) is a regular and hereditary ring and Lemma 9.3 applies again, proving that \( D_{I,J} \) is right and left hereditary.

(6) If \( I \) is at most countable, then (see Notations 6.1) \(|X| = N = \aleph_{0} \) and therefore \( \mathbb{FM}_{X}(D) \) has countable dimension over \( D \). If \( I' = \emptyset \) then \( D_{I,J} \) is a maximal element of \( I \), otherwise \( H_{I} \cong \mathbb{FM}_{X}(D) \). As a result \( H_{I} = \bigoplus_{i \in I} H_{i} \) has countable dimension over \( D \) and the same occurs for \( D_{I,J} = H_{I} + e_{X}D \).

(7) If \( i \in I \), then it follows from (9.3) and property (2) that \( h(U_{i}) = \lambda(U_{i}). \) Thus we only have to check the behavior of the simple module \( V = D_{I}/H_{I} \), in case \( 0 < \xi_{0} \), that is \( I \) has not a finite cofinal subset. Set \( \alpha + 1 = \lambda(V) \) and assume that \( \alpha + 1 < h(V) = \xi_{0} + 1 \), namely \( \alpha + 1 < \xi_{0} \). Since \( \xi_{0} \leq \xi \), there is some \( i \in I \) such that \( \lambda(i) = \alpha + 1 \). We have that \( \lambda(U_{i}) = \alpha + 1 \), therefore \( U_{i} \) and \( V \) are not comparable and consequently \( \{i \leq \} \) is finitely sheltered by (9.5). Suppose, on the contrary, that \( \lambda(V) = \xi_{0} + 1 \). Given any \( \alpha < \xi_{0} \), we have from Corollary 13.3 that there is some \( U \in \mathbf{Simp}_{D_{I}} \) such that \( \lambda(U) = \alpha + 1 \) and \( U \lesssim V \). Necessarily \( U = U_{i} \) for a unique \( i \in I \) with \( \lambda(i) = \alpha + 1 \) and \( \{i \leq \} \) is not finitely sheltered by (9.5).

(8) According to Proposition 2.7 we only have to show the “if” part. Firstly, it is clear from (9.4) that \( \mathbf{Simp}_{D_{I}} \) has finitely many maximal elements if and only if \( I \) satisfies the same condition. Thus, assume that this condition holds, let \( S \) be an upper subset of \( \mathbf{Simp}_{D_{I}} \), set \( J = \{j \in I \mid U_{j} \in S\} \) and note that \( J \) is an upper subset of \( I \) by property (2). For every \( i \in I \), it follows from (9.1) that \( H_{I,J} \) annihilates \( U_{i} \) if and only if \( i \in J \). If either \( I \) has a finite cofinal subset or \( D_{I}/H_{I} \in S \), by property (2) this is enough to conclude that \( \Phi(\Psi(S)) = S \). Assume that \( I \) has not a finite cofinal subset and \( D_{I}/H_{I} \not\in S \). Then it follows from property (2) that \( J \) is order isomorphic to \( S \) and, since every element of \( \mathbf{Simp}_{D_{I}} \) is bounded by a maximal element and \( S \) is an upper subset, then \( J \) is a finitely sheltered upper subset of \( I \). Let \( m_{1}, \ldots, m_{r} \) be the maximal elements of \( J \), so that \( \{X_{m_{1}}, \ldots, X_{m_{r}}\} \) is a partition of \( X_{I} \) by Proposition 6.3. If we set \( H = H_{I,J} + (1 - e_{X_{m_{1}}} + \cdots + e_{X_{m_{r}}})D \), then we have from Propositions 7.1 and 8.1 that \( H \) is an ideal of \( D_{I} \) and \( D_{I}/H \cong D_{I,J} = H_{J} \). From this, with the help of Proposition 9.2 we infer that \( H \) annihilates \( U_{i} \) if and only if \( i \in J \). It is not the case that \( H \subset H_{I} \) otherwise, since \( e_{X_{m_{1}}}, \ldots, e_{X_{m_{r}}}, \in H_{I} \), it would follow that \( 1 \in H_{I} \supseteq D_{I} \). Again, we can conclude that \( \Phi(\Psi(S)) = S \), proving that \( D_{I} \) is very well behaved.

Concerning hereditariness, we are presently unable to exhibit an example of non-hereditary, regular and semiartinian ring; nonetheless we have the following easy result.

**Proposition 9.6.** If \( R \) is a right semiartinian ring with Loewy length at most 2 and projective right socle, then \( R \) is right hereditary.

**Proof.** If the Loewy length of \( R \) is 1, then \( R \) is semisimple and so is hereditary. Assume that \( R \) has Loewy length 2. In order to prove that \( R \) is right hereditary, it is sufficient to show that if \( E \) is an injective right \( R \)-module with an essential submodule \( M \), then \( E/M \) is an injective \( R \)-module. Set \( K = \text{Soc}(R_{M}) \) and note that \( EK \subset M \), so that \( E/M \) is canonically a right \( R/K \)-module. Since \( R/K \) is a semisimple ring, then \( E/M \) is \( R/K \)-injective and the left purity of \( K \) implies the \( R \)-injectivity of \( E/M \). \( \square \)
A couple of final remarks are in order. First, on the basis of properties (7) and (8) of Theorem 9.5, a suitable choice of the artinian poset \( I \) produces a semiartinian and regular ring \( D_I \) such that \( \text{Simp}_{D_I} \) is finite for every \( \alpha \), but \( \text{Simp}_{D_I} \) has infinitely many maximal elements, so that \( D_I \) is well behaved but not very well behaved (see Propositions 2.4 and 2.7). The second remark concerns the distribution of non-maximal, injective members of \( \text{Simp}_R \), where \( R \) is a regular and semiartinian ring. On the basis of property (4) of the previous theorem one might wonder whether the subset of these modules is always a lower subset of \( \text{Simp}_R \). However this is not the case, as shown by the following example.

**Example 9.7.** There exists a semiartinian, hereditary and unit-regular ring \( R \) such that \( \text{Simp}_R \) is a chain \( \{ U < V < W \} \), where \( V \) and \( W \) are injective but \( U \) is not injective.

**Proof.** Let \( \aleph \) and \( \beth \) be infinite cardinals with \( \beth < \aleph \) and set \( X = \aleph \cdot \beth \). With the notations of Section 3 let us consider the partition \( \mathcal{P}_1 = \{ X_{1,\lambda} \mid \lambda < \beth \} \) of \( X \), where \( X_{1,\lambda} = [\aleph \cdot \beth + \rho \mid \rho < \aleph] \) for all \( \lambda < \beth \) and note that \( |X_{1,\lambda}| = \aleph \) and \( |\mathcal{P}_1| = \beth \). Given a division ring \( D \), let us consider the ring \( Q = \text{CFM}_X(D) \) and let \( T \) be the subset of \( Q \) of all matrices whose rows have support of cardinality not exceeding \( \beth \):

\[
T = \{ A \in Q \mid |\text{Supp}(a(x, -))| \leq \beth \text{ for all } x \in X \}.
\]

Then \( T \) is a (unital) subring of \( Q \). Indeed, let \( A, B \in T \), let \( x \in X \) and set

\[
Y = \text{Supp}(a(x, -)), \quad Z = \text{Supp}(b(x, -)), \quad U = \bigcup \{ \text{Supp}(b(z, -)) \mid z \in Z \}.
\]

Then

\[
|\text{Supp}((a - b)(x, -))| = |\text{Supp}(a(x, -) - b(x, -))| \leq |Y \cup Z| \leq \beth,
\]

showing that \( T \) is an additive subgroup of \( Q \). Moreover, if \( y \in X \setminus U \), then

\[
(ab)(x, y) = \sum_{z \in X} a(x, z) b(z, y) = \sum_{z \in Y} a(x, z) b(z, y) = 0.
\]

It follows that \( ab \in T \), because \( |U| \leq \beth \). Set \( H = \text{FR}_X(D) \cap T \) and note that \( H \) is a semisimple and regular ideal of \( T \). With the notations of Theorem 5.11 we have that \( F_1 = \varphi_1(\text{FR}_X(D)) \subset T \). Finally, let us consider the ring

\[
R = H \oplus F_1 \oplus 1_Q D.
\]

Now it is easy to check that \( R \) is a regular and semiartinian ring, where \( \text{Soc} R = H \) is homogeneous, \( \text{Soc}_2(R) = H \oplus F_1 \) and \( \text{Soc}_2(R)/\text{Soc}(R) \simeq F_1 \) is homogeneous and \( R/\text{Soc}_2(R) \simeq D \). A straightforward application of Lemma 9.3 and Vasershtein criterion shows that \( R \) is hereditary and unit-regular. It is clear that \( \text{Prim}_R = \{ \{0\}, \text{Soc} R, \text{Soc}_1 R \} \), thus \( \text{Simp}_R \) is a chain \( \{ U < V < W \} \), where \( \text{Soc} R \) is the trace of \( U \) in \( R \), \( \text{Soc}_1(R)/\text{Soc} R \) is the trace of \( V \) in \( R/\text{Soc}(R) \) and \( W \simeq D \) is injective because it is a maximal element. According to Lemma 9.4 \( V \) is injective but \( U \) is not injective. \( \square \)

**References**

[1] G. Abrams and G. Aranda Pino, The Leavitt path algebra of a graph, *J. Algebra*, 293 (2005), 319-334.
[2] G. Abrams and J. J. Simón, Isomorphisms between infinite matrix rings, a survey, *Algebra and its applications* (Athens, OH, 1999), 1–12, Contemp. Math., 259, *Amer. Math. Soc.*, Providence, RI, 2000.

[3] T. Albu, Sur la dimension de Gabriel des modules, *Algebra Berichte*, Seminar Kasch-Pareigis, Verlag Uni-Druck, München, Bericht Nr. 21, 1974.

[4] P. Ara and M. Brustenga, The regular algebra of a quiver, *J. Algebra*, 309 (2007), 207-235.

[5] P. Ara, The realization problem for Von Neumann regular rings, *Preprint*, arXiv:0802.1872v1 [math.RA].

[6] P. Ara, The regular algebra of a poset, *Preprint*, arXiv:0805.2963v1 [math.RA].

[7] G. Baccella, Semiprime $ℵ_0$-QF-3 rings, *Pacific J. Math.*, 120, (2) (1985), 269-278.

[8] G. Baccella, Semiartinian $V$-rings and semiartinian Von Neumann regular rings, *J. Algebra*, 173 (1995), 587-612.

[9] G. Baccella and G. Di Campli, Semiartinian Rings whose Loewy Factors are Nonsingular, *Comm. Algebra*, 25 (1997), 2743–2764.

[10] G. Baccella, A. Ciampella, $K_0$ of semi-Artinian unit-regular rings, *Abelian groups, module theory, and topology* (Padua, 1997), 69–78. Lecture Notes in Pure and Appl. Math., 201, Dekker, New York, 1998.

[11] G. Baccella, Exchange property and the natural preorder between simple modules over semiartinian rings, *J. Algebra*, 253 (2002), 133-166.

[12] G. M. Bergman, Von Neumann regular rings with tailor-made ideal lattices. *Unpublished note*, 1986.

[13] G. Brookfield, Monoids and Categories of Noetherian Modules, Ph. D. dissertation, University of California, Santa Barbara, 1997.

[14] V. C. Camillo, Morita equivalence and infinite matrix rings, *Proc. Amer. Math. Soc.*, 90, (1984), 186-188.

[15] V. C. Camillo and K. R. Fuller, A note on Loewy rings and chain conditions on primitive ideals, *Lecture Notes in Math.*, 700, (1979), Springer-Verlag Berlin, Heidelberg and New York, 75-85.

[16] A. Del Río and J. J. Simón, Intermediate rings between matrix rings and Ornstein dual pairs, *Arch. Math.*, 75, (2000), 256-263.

[17] N. V. Dung and P. F. Smith, On Semiartinian $V$-modules, *J. Pure Appl. Algebra*, 82, no. 1, (1992), 27-37.

[18] E. Harzheim, Ordered Sets, *Advances in Mathematics*, Vol. 82, Springer, New York, 2005. xii+386 pp. MR2127991 (2006e:06001)

[19] T. Jech, “Set theory”, Second edition. Perspectives in Mathematical Logic, *Springer-Verlag, Berlin*, 1997. xiv+634 pp. MR 99k:03061

[20] A. Levy, “Basic set theory”, *Springer-Verlag, Berlin-New York*, 1979. xiv+391 pp. MR 80k:04001

[21] K. R. Goodearl, “Von Neumann Regular Rings”, Monographs and Textbooks in Mathematics, *Pitman, London*, 1979, Second edition, Krieger Publ. Co., Malabar, Florida, 1991.

[22] K. R. Goodearl, Torsion in $K_0$ of unit-regular rings, *Proc. Edinburgh Math. Soc., Series II*, 38 (1995), 331–341. MR 97c:16010

[23] L. N. Vasershtein, Stable ranks of rings and dimensionality of topological spaces, *Funct. Anal. Appl.*, 5 (1971), 17-27; translation: 102-110.

Dipartimento di Matematica Pura ed Applicata, Università di L’Aquila, L’Aquila, 67100 Italy

E-mail address: baccella@univaq.it