AN EXAMPLE OF A STABLE BUT FIBREWISE NONSTABLE
BUNDLE ON THE TWISTOR SPACE OF
A HYPERKÄHLER MANIFOLD

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Abstract. We construct an explicit example of a stable bundle on the twistor space Tw(M) of a hyperkähler manifold M whose restrictions to all the fibres of the natural twistor projection \( \pi : \text{Tw}(M) \to \mathbb{CP}^1 \) are nonstable. We also describe the relationship between bundles on Tw(M) that do not have subsheaves of strictly lower rank and bundles that stably restrict to the fibres of \( \pi \), and announce a result whose proof will appear in a forthcoming paper.

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1. Introduction

A hyperkähler manifold is a smooth manifold \( M \) together with a triple of integrable almost complex structures \( I, J, K : TM \to TM \) satisfying the quaternionic relations \( I^2 = J^2 = K^2 = -1 \), \( IJ = -JI = K \), and a Riemannian metric \( g \) which is Kähler with respect to the structures \( I, J, K \). An example of a hyperkähler manifold is a \( K3 \) surface, that is, a compact simply connected complex surface with trivial canonical bundle. It admits a hyperkähler metric as a consequence of the Calabi-Yau theorem [Y].

It’s not hard to see that a hyperkähler manifold \( M \) admits a whole family of induced complex structures, which topologically looks like a 2-sphere:

\[
S^2 = \{ aI + bJ + cK : a^2 + b^2 + c^2 = 1 \}.
\]

Identifying \( S^2 \) with \( \mathbb{CP}^1 \), we define the twistor space of \( M \) as the topological Cartesian product \( \text{Tw}(M) := M \times S^2 = M \times \mathbb{CP}^1 \). We think of \( \text{Tw}(M) \) as parametrizing the induced complex structures at points of \( M \). In the context of the twistor space, the initial structures \( I, J, K \) don’t play any vital role, and henceforth we will denote by \( I \in \mathbb{CP}^1 \) an arbitrary induced complex structure, while \( M_I \) will denote the corresponding complex manifold. Note that for any \( I \in \mathbb{CP}^1 \), \( g \) is a Kähler metric on \( M_I \).

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The twistor space $\text{Tw}(M)$ admits a natural integrable almost complex structure $\mathcal{S}$. With respect to this structure, the projection onto the second coordinate $\pi : \text{Tw}(M) \to \mathbb{C}P^1$ is holomorphic, and the fibres of $\pi$ over points $I \in \mathbb{C}P^1$ correspond to the complex manifolds $M_I$. We observe that the projection onto the first coordinate of the twistor space $\sigma : \text{Tw}(M) \to M$ is not holomorphic with respect to any of the induced complex structures on $M$, since $\text{Tw}(M)$ is not a product of $M$ and $\mathbb{C}P^1$ as complex manifolds, but only as topological manifolds. $\text{Tw}(M)$ also admits a natural Hermitian metric satisfying the balancedness condition $d(\omega^{n-1}) = 0$, where $\omega$ is the Hermitian form of this metric and $n$ is the complex dimension of $\text{Tw}(M)$ [KV].

Let $E$ be a holomorphic vector bundle on the twistor space $\text{Tw}(M)$ of a compact hyperkähler manifold $M$. We define the degree of $E$ by

\[(1) \quad \text{deg}(E) = \int_{\text{Tw}(M)} c_1(E) \wedge \omega^{n-1},\]

where by $c_1(E)$ we denote any representative of the first Chern class of $E$ in $H^2(M, \mathbb{R})$. We say that $E$ is stable if for any subsheaf $\mathcal{F} \subset E$ satisfying $0 < \text{rk}(\mathcal{F}) < \text{rk}(E)$, we have strict inequality

\[\frac{\text{deg}(\mathcal{F})}{\text{rk}(\mathcal{F})} < \frac{\text{deg}(E)}{\text{rk}(E)},\]

where the degree of $\mathcal{F}$ is defined as the degree of its determinant line bundle: $\text{deg}(\mathcal{F}) := \text{deg}(\text{det} \mathcal{F})$. The bundle $E$ is called irreducible if it does not have any nonzero subsheaves of lower rank.

Observe that the value of the integral (1) in the definition of degree does not depend on the choice of the representative of the Chern class $c_1(E)$, since the Hermitian form $\omega$ on $\text{Tw}(M)$ satisfies the balancedness condition $d(\omega^{n-1}) = 0$, as we noted earlier. For every $I \in \mathbb{C}P^1$, we can similarly define the degree of bundles on the Kähler manifold $M_I$, since any Kähler metric is a priori balanced. In this way, the notion of degree makes sense both for bundles $E$ on the twistor space $\text{Tw}(M)$ and their restrictions $E_I$ to the fibres $\pi^{-1}(I) = M_I$ of the twistor projection $\pi : \text{Tw}(M) \to \mathbb{C}P^1$.

In the paper [KV], Kaledin and Verbitsky prove the following result, among other things.

**Proposition 1.** Let $M$ be a compact hyperkähler manifold and $E$ a holomorphic vector bundle on the twistor space $\text{Tw}(M)$. If $E$ stably restricts to the generic fibre of the holomorphic twistor projection $\pi : \text{Tw}(M) \to \mathbb{C}P^1$ (in the sense of the Zariski topology on $\mathbb{C}P^1$), then it is stable as a bundle on $\text{Tw}(M)$ as well.

In the present short note, we will show that the converse statement does not not hold in general. More precisely, we will construct an explicit example of a stable holomorphic bundle $E$ of rank 2 on the twistor space $\text{Tw}(M)$ of a K3 surface $M$, all of whose restrictions to the fibres of the projection $\pi : \text{Tw}(M) \to \mathbb{C}P^1$ are nonstable. We will also formulate a stronger version of the above result (Theorem I) and briefly discuss the converse to this stronger statement.
2. An example of a stable but fibrewise nonstable bundle on $\text{Tw}(M)$

Let $M$ be an algebraic K3 surface with Picard number $\rho(M) \geq 2$ (for basic properties of K3 surfaces, as well as terminology and important theorems in complex geometry that we use below, see e.g. [GH]). The degree of any bundle on $M$ is an integer, hence for line bundles we have a homomorphism of groups $\text{deg} : \text{Pic}(M) \to \mathbb{Z}$ with a nonzero kernel. We can choose an element $L$ of this kernel in such a way that the inequality $h^1(M, L^*) \neq 0$ holds. Indeed, by the Riemann-Roch formula for K3 surfaces,

$$h^0(M, L) - h^1(M, L) + h^2(M, L) = \frac{c_1(L)^2}{2} + 2.$$ 

Let $L$ be a nontrivial holomorphic line bundle of degree zero. $L$ does not have any nonzero sections, since such a section would give an effective divisor of a strictly positive degree, by the Poincaré-Lelong formula. Hence $h^0(M, L) = 0$, and similarly $h^2(M, L) = h^0(M, L^*) = 0$, where we use Serre duality. Thus,

$$h^1(M, L) = -\frac{c_1(L)^2}{2} - 2,$$

and by the Hodge index theorem, $c_1(L)^2 < 0$. Replacing $L$ by its multiple, if necessary, we have $h^1(M, L) \neq 0$, and thus $h^1(M, L^*) \neq 0$.

Since $\text{deg} L = 0$, one can show (see [V1], Theorem 2.4) that $L$ is hyperholomorphic, that is, admits a Hermitian metric with Chern connection $\nabla$ whose curvature is a $(1,1)$-form with respect to every induced complex structure on $M$. Clearly, this means that for every $I \in \mathbb{CP}^1$, $\nabla$ endows $L$ with the structure of a holomorphic line bundle over $M_I$, which we will denote by $L_I$. Moreover, taking the pullback of $(L, \nabla)$ along the projection onto the first coordinate of the twistor space $\sigma : \text{Tw}(M) \to M$, we get a line bundle $\sigma^* L$ on $\text{Tw}(M)$ with holomorphic structure $(\sigma^* \nabla)^{0,1}$. The restriction of the holomorphic line bundle $\sigma^* L$ to the fibre $\pi^{-1}(I) = M_I$ of the twistor projection $\pi : \text{Tw}(M) \to \mathbb{CP}^1$ is precisely $L_I$. We will denote the initial complex manifold structure on our K3 surface (which corresponds to one of the $I \in \mathbb{CP}^1$) simply by $M$, while the initial holomorphic structure on our line bundle will be denoted by $L$, rather than $L_I$; this should not cause any confusion.

The higher direct images of $\sigma^* L^*$ with respect to the projection $\pi : \text{Tw}(M) \to \mathbb{CP}^1$ are as follows (see [V2], Proposition 6.3):

$$R^i \pi_* (\sigma^* L^*) \cong \mathcal{O}_{\mathbb{CP}^1}(i) \otimes_{\mathbb{C}} H^i(M, L^*).$$

Let us denote $\mathcal{O}_{\text{Tw}(M)}(-1) := \pi^* (\mathcal{O}_{\mathbb{CP}^1}(-1))$. Applying the projection formula and the above,

$$R^1 \pi_* [\sigma^* L^* \otimes \mathcal{O}_{\text{Tw}(M)}(-1)] \cong [R^1 \pi_* (\sigma^* L^*)] \otimes \mathcal{O}_{\mathbb{CP}^1}(-1) \cong \mathcal{O}_{\mathbb{CP}^1} \otimes_{\mathbb{C}} H^1(M, L^*).$$

Thus,

$$\text{Ext}^1(\sigma^* L, \mathcal{O}_{\text{Tw}(M)}(-1)) \cong H^1(\sigma^* L^* \otimes \mathcal{O}_{\text{Tw}(M)}(-1)) = \Gamma(\mathbb{CP}^1, R^1 \pi_* [\sigma^* L^* \otimes \mathcal{O}_{\text{Tw}(M)}(-1)]) \cong H^1(M, L^*).$$

This is nonzero by construction, so we can choose a nonzero element in $H^1(M, L^*) = \text{Ext}^1(\sigma^* L, \mathcal{O}_{\text{Tw}(M)}(-1))$ which corresponds to some extension

$$0 \to \mathcal{O}_{\text{Tw}(M)}(-1) \to E \to \sigma^* L \to 0.$$
Observe that the restriction of this short exact sequence to any fibre $\pi^{-1}(I) = M_I$ of the holomorphic twistor projection $\pi : \Tw(M) \to \mathbb{CP}^1$ is $0 \to \mathcal{O}_{M_I} \to E_I \to L_I \to 0$. Since $\mathcal{O}_{M_I}$ and $L_I$ both have degree zero, $\deg E_I = \deg \mathcal{O}_{M_I} + \deg L_I$ is zero as well. Therefore, the morphism $\mathcal{O}_{M_I} \to E_I$ gives a destabilizing subsheaf of $E_I$, proving that $E_I$ is nonstable as a bundle on $M_I$.

We will now show that $E$ is stable as a bundle on $\Tw(M)$. One can show (see [KV], Lemma 6.2) that the degree of any bundle on $\Tw(M)$ is equal to the degree of its restriction to any horizontal twistor line $\{m\} \times \mathbb{CP}^1 \subset \Tw(M)$, where $m \in M$. Clearly, the restriction of the exact sequence (2) to any such line has the form

$$0 \to \mathcal{O}_{\mathbb{CP}^1}(-1) \to E|_{\{m\} \times \mathbb{CP}^1} \to \mathcal{O}_{\mathbb{CP}^1} \to 0.$$  

This implies that $\deg E = \deg \mathcal{O}_{\mathbb{CP}^1}(-1) + \deg \mathcal{O}_{\mathbb{CP}^1} = -1 + 0 = -1$. Moreover, since $\text{Ext}^1(\mathcal{O}_{\mathbb{CP}^1}, \mathcal{O}_{\mathbb{CP}^1}(-1)) = H^1(\mathcal{O}_{\mathbb{CP}^1}(-1)) = 0$, we have $E|_{\{m\} \times \mathbb{CP}^1} \cong \mathcal{O}_{\mathbb{CP}^1} \oplus \mathcal{O}_{\mathbb{CP}^1}(-1)$. This means that any potential destabilizing line subsheaf $L \hookrightarrow E$, that is, one which satisfies the inequality

$$\deg(\tilde{L}) = \frac{\deg(L)}{\text{rk}(L)} \geq \frac{\deg(E)}{\text{rk}(E)} = \frac{1}{2},$$

should have degree 0. Let $\tilde{L} \hookrightarrow E$ be such a subsheaf. In the diagram

$$\begin{array}{c}
\begin{array}{ccc}
\mathcal{O}_{\Tw(M)}(-1) & \to & \mathcal{O}_{\mathbb{CP}^1} \\
\downarrow & & \downarrow \\
E & \to & \sigma^* L \\
\end{array}
\end{array}$$

(3)

if the morphism $\theta : \tilde{L} \to \sigma^* L$ is zero, then by the exactness of the bottom row, there exists a lifting of the sheaf monomorphism $\tilde{L} \hookrightarrow E$ to a monomorphism $\tilde{L} \hookrightarrow \mathcal{O}_{\Tw(M)}(-1)$.

However, such a monomorphism cannot exist, since restricting any morphism $\tilde{L} \to \mathcal{O}_{\Tw(M)}(-1)$ to any horizontal twistor line $\{m\} \times \mathbb{CP}^1$, we get $\mathcal{O}_{\mathbb{CP}^1} \to \mathcal{O}_{\mathbb{CP}^1}(-1)$, which is zero. On the other hand, if the morphism $\theta : \tilde{L} \to \sigma^* L$ is nonzero, it must be an isomorphism, since its restriction to any horizontal twistor line $\{m\} \times \mathbb{CP}^1$ has the form $\mathcal{O}_{\mathbb{CP}^1} \to \mathcal{O}_{\mathbb{CP}^1}$, and any such nonzero morphism is an isomorphism. But if $\theta$ is an isomorphism, the diagram (3) gives a splitting of the short exact sequence (2), which contradicts our choice of $E$. We have proved that such a destabilizing subsheaf $L \hookrightarrow E$ cannot exist, hence $E$ is stable.

3. Irreducible bundles and fibrewise stability

The bundle $E$ on $\Tw(M)$ that we constructed in the previous section gives a counterexample to the converse of Proposition [11]. However, looking at the proof of Lemma 7.3 in [KV], it’s not hard to see that Proposition [11] can be made stronger in the following way.

**Theorem 1.** Let $M$ be a compact hyperkähler manifold and $E$ a holomorphic vector bundle on the twistor space $\Tw(M)$. If $E$ stably restricts to the generic fibre of the holomorphic twistor projection $\pi : \Tw(M) \to \mathbb{CP}^1$, then it is irreducible as a bundle on $\Tw(M)$. 

We believe that the converse to this stronger version of the statement is in fact true. At the present time, the following partial result is known.

**Theorem 2.** Let $M$ be a compact simply connected hyperkähler manifold and $E$ a holomorphic vector bundle on the twistor space $Tw(M)$. If $E$ is irreducible, then it stably restricts to the generic fibre of the holomorphic twistor projection $\pi : Tw(M) \to \mathbb{CP}^1$, provided that the rank of $E$ is equal to 2 or 3, or if its restriction $E_I$ to the generic fibre $\pi^{-1}(I) = M_I$ of $\pi$ is a simple bundle, in the sense that $\text{Hom}(E_I, E_I) = \mathbb{C}$.

The proof of this result will be given in a forthcoming paper. In the present short note, we will content ourselves with only a brief survey of the proof. We will make use of the following construction. For an arbitrary holomorphic vector bundle $E$ on $Tw(M)$ and any $0 < s < \text{rk}(E)$, we define the cone of exterior monomials $C^s(E) \subseteq \Lambda^s E$ in the following way: at a point $x \in Tw(M)$, $C^s(E)_x$ consists of the elements of $\Lambda^s E_x$ of the form $v_1 \wedge \ldots \wedge v_s$, where $v_1, \ldots, v_s \in E_x$. If $\mathcal{F} \to E$ is a subsheaf of rank $s$, it’s not hard to verify that the image of $L = \det(\mathcal{F}) = (\Lambda^s \mathcal{F})^{**} \to (\Lambda^s E)^{**} = \Lambda^s E$ lies in $C^s(E)$. At points where $\mathcal{F}$ is a subbundle of $E$, the line $L_x \subseteq \Lambda^s E_x$ is obtained from $\mathcal{F}_x \subseteq E_x$ by virtue of the Plücker embedding. On the other hand, one can show (see [1], subsection 2.2) that starting from a line subsheaf $L \subseteq \Lambda^s E$ with image in $C^s(E)$, one can recover a subsheaf $\mathcal{F} \subseteq E$ of rank $s$. Obviously, the above also holds for bundles on the fibres $\pi^{-1}(I) = M_I$ of the projection $\pi : Tw(M) \to \mathbb{CP}^1$, and more generally on any complex manifold.

Let $M$ be a compact simply connected hyperkähler manifold. It can be shown that for a bundle $E$ on $Tw(M)$, viewed as a family of bundles on the fibres of the projection $\pi : Tw(M) \to \mathbb{CP}^1$, stability is a Zariski open condition on $\mathbb{CP}^1$. In other words, the set of $I \in \mathbb{CP}^1$ for which the restriction $E_I$ is stable is Zariski open in $\mathbb{CP}^1$. The proof of this statement is essentially an adaptation of the argument from the proof of Theorem 1.3 in [1], where an analogous statement is shown for a projection $X \times Y \to X$, where $X \times Y$ is the product of complex manifolds $X$ and $Y$ satisfying certain properties.

Let $E$ be an irreducible bundle of rank $r$ on $Tw(M)$. Arguing by contradiction, we assume that $E_I$ is nonstable as a bundle on $\pi^{-1}(I) = M_I$ for infinitely many $I \in \mathbb{CP}^1$. By the previous paragraph, it follows from this that $E_I$ is nonstable for all $I$. There exists $0 < s < r$ such that for every $I \in \mathbb{CP}^1$ there are destabilizing subsheaves $\mathcal{F}_I \to E_I$ of rank $s$, which correspond to line subsheaves $L_I \to \Lambda^s E_I$ with image in $C^s(E_I)$. Moreover, one can show that for some choice of $\mathcal{F}_I$, all the line bundles $L_I$ on $\pi^{-1}(I) = M_I$ are restrictions of a single line bundle $L$ on $Tw(M)$.

Our goal consists in "gluing" these subsheaves $L_I \to \Lambda^s E_I$ over $M_I$ into a global subsheaf of $\Lambda^s E$ over $Tw(M)$ with image in $C^s(E)$, from which we can recover a subsheaf of $E$ of rank $s$ over $Tw(M)$ and get a contradiction to the irreducibility of $E$. Consider the vector bundle $\text{Hom}(L, \Lambda^s E)$ on $Tw(M)$. Its direct image $\pi_*(\text{Hom}(L, \Lambda^s E))$ along the twistor projection $\pi : Tw(M) \to \mathbb{CP}^1$ is a vector bundle on $\mathbb{CP}^1$, and by Grauert’s theorem (see [GR], Theorem 10.5.5), $\pi_*(\text{Hom}(\sigma^* L, \Lambda^s E))_I = \text{Hom}(L_I, \Lambda^s E_I)$ at all points $I \in \mathbb{CP}^1$, except perhaps finitely many. The existence of a subsheaf of $\Lambda^s E$ over $Tw(M)$ with image in $C^s(E)$ will follow from the existence of the following algebraic morphism...
over $\mathbb{CP}^1$:

$$Y = \{[L_I \hookrightarrow C_s(E_I) \in \Lambda^s E_I]\}_{I \in \mathbb{CP}^1} \to \mathbb{P}(\pi_*(\text{Hom}(L, \Lambda^s E)))$$

If $r = 2$ or $3$, such a section always exists, since in this case $C^s(E) = \Lambda^s E$ for any $s$. If $r > 3$, the existence of a section of the morphism $Y \to \mathbb{CP}^1$ is not guaranteed, but such a section always exists over some ramified cover $f : X \to \mathbb{CP}^1$. Taking the fibred product

$$Z \xrightarrow{\varphi} \text{Tw}(M)$$

$$\rho \downarrow \quad \downarrow \pi$$

$$X \xrightarrow{f} \mathbb{CP}^1,$$

one can then proceed to construct a subsheaf $\mathcal{F} \subseteq \varphi^* E$ of rank $s$ over $Z$. If we assume that the restriction of $E$ to the generic fibre of $\pi : \text{Tw}(M) \to \mathbb{CP}^1$ is a simple bundle, then after some work it can be shown that the irreducibility of $E$ on $\text{Tw}(M)$ implies that $\varphi^* E$ is irreducible on $Z$, which leads to a contradiction.

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