Functional integral with $\phi^4$ term in the action beyond standard perturbative methods

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We propose the another, in principle nonperturbative, method of the evaluation of the Wiener functional integral for $\phi^4$ term in the action. All infinite summations in the results are proven to be convergent. We find the "generalized" Gelfand-Yaglom differential equation implying the functional integral in the continuum limit.

Introduction

The conventional functional integral calculations rely on the Gaussian method of the integration. This means, that physical problems having the second order term in the action are calculable precisely, those problems with higher orders terms in the action must be calculated perturbatively. The perturbation methods are applicable in plenty of the problems of the contemporary physics, however, more frequently than ever recently we are confronted with problems, where conventional perturbation methods are not sufficient. The non-perturbative numerical methods are successfully applied to answer the questions in the statistical physics, quantum field theory, if we mentions only the well known examples. The numerical methods give the answers "yes" or "no" on the quantitative questions, the deal of the answer, "why" is still missing.

We are going to discuss the analytical evaluation of the simplest case of the beyond Gaussian functional integral calculations, where the action possesses $\phi^4$ term in the exponent. By the convention, we call the term connected with $\phi^4$ the "coupling constant". In the conventional perturbation calculations for $\phi^4$ theory the results of the functional integral calculations are obtained in the form of the power series of the coupling constant (see, i.e., [1]). These series are asymptotic, divergent, but by sophisticated re-summation procedures one can obtain the reasonable results, allowing to take the path integral more seriously.

In this article we propose the calculation the Wiener functional integral with term of the fourth order in the exponent by the another method as in the conventional perturbation approach. In contrast to the conventional perturbation theory, we expand into the power series the term linear in the integration variable in the exponent. In such case we can profit from the representation of the functional integral by the parabolic cylinder functions. We show, that in such case the expansions into the series are uniformly convergent and we find the recurrence relations for the Wiener functional integral in the $N$-dimensional approximation. We find the continuum limit of this finite dimensional integral by procedure proposed by Gelfand-Yaglom [2] for continuum limit of the functional integral for the harmonic oscillator.

The article is organized as follows. In part 2 on the example of the one dimensional integral we remember the calculation of the integrals with the fourth order term in exponent by the parabolic cylinder functions. In the part 3 we calculate the $N$ dimensional integral as the approximation of the functional integral. We will prove the uniform convergence of the result. In the part 4 we will show on the example of the "Independent Value Model" in the sense introduced by Klauder [7], that this field theory model we can solve non-perturbatively and we discuss the problem of "triviality" of this model. In part 5 and in the appendices 1 and 2 we will study the result of the part 3. In the appendix 3 the "generalized" Gelfand-Yaglom [2] equation determining the continuum limit of the functional integral is proven. In the appendix 4 we present the same result as in part 5 for unconditional Wiener measure functional integral.

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We will define the functional integral as the limit of the finite dimensional integral. This definition has not problems with the idea of the integral measure, because for the finite dimensional integrals the integral measure is defined correctly. We explain our method of calculation on the example of the one dimensional integral. Calculating the finite dimensional integrals we must solve the problem of the calculation of the integral

\[ I_1 = \int_{-\infty}^{+\infty} dx \exp\{-(ax^4 + bx^2 + cx)\} \]  

(1)

where Re \( a > 0 \).

We are interested also to the problems, when the fourth order term is not small, and therefore it is not possible to treat it as a perturbation comparing to the rest of the action term. The generally accepted perturbative approach rely on the Taylor’s decomposition of the fourth order term with consecutive replacements of the integration and summation order:

\[ I_1 = \sum_{n=0}^{\infty} \frac{(-a)^n}{n!} \int_{-\infty}^{+\infty} dx x^{4n} \exp\{-(bx^2 + cx)\} \]  

(2)

The integrals in above relation can be calculated, but sum is divergent.

However, \( I_1 = I_1(a, b, c) \) is an entire function for any complex values of \( b \) and \( c \), since there exist all integrals

\[ \partial^m \partial^n I_1(a, b, c) = (-1)^{n+m} \int_{-\infty}^{+\infty} dx x^{2m+n} \exp\{-(ax^4 + bx^2 + cx)\} \]

Consequently, the power expansions of \( I_1 = I_1(a, b, c) \) in \( c \) and/or \( b \) has an infinite radius of convergence (and in particular they are uniformly convergent on any compact set of values of \( c \) and/or \( b \)). Let us now consider the power expansion in \( c \) which we shall frequently use:

\[ I_1 = \sum_{n=0}^{\infty} \frac{(-c)^n}{n!} \int_{-\infty}^{+\infty} dx x^n \exp\{-(ax^4 + bx^2)\} \]  

(3)

The integrals here appearing can be expressed in terms of the parabolic cylinder function \( D_\nu(z) \), \( \nu = -m - 1/2 \), (see, for instance, [4]). For \( n \) odd, due to symmetry of the integrand the integrals are zero, for \( n \) even, \( n = 2m \) we have:

\[ D_{-m-1/2}(z) = \frac{e^{-z^2/4}}{\Gamma(m+1/2)} \int_0^{+\infty} dx x^{m-1/2} \exp\{ -\frac{1}{2} x^2 -zx \} \]

\[ = \frac{e^{-z^2/4}}{\Gamma(m+1/2)} \int_{-\infty}^{+\infty} dy y^{2m} \exp\{ -\frac{1}{2} y^2 -zy^2 \} = \frac{(-1)^m}{\Gamma(m+1/2)} e^{-z^2/4} \partial^m z e^{z^2/4} D_{-1/2}(z) . \]  

(4)

Explicitly, for the Eq. (3) we have:

\[ I_1 = \frac{\Gamma(1/2)}{(2a)^{1/4}} \sum_{m=0}^{\infty} \frac{(\xi)^m}{m!} e^{z^2/4} D_{-m-1/2}(z) , \quad \xi = \frac{c^2}{4\sqrt{2a}} , \quad z = \frac{b}{\sqrt{2a}} \]  

(5)

This sum is convergent for any values of \( c, b \) and \( a \) positive.

The convergence of the infinite series [6] can be shown as follows. For \( |z| \) finite, \( |z| < \sqrt{\nu} \) and \( |\arg(-\nu)| \leq \pi/2 \), if \( \nu \to \infty \), the following asymptotic relation is valid [4]:

\[ D_\nu(z) = \frac{1}{\sqrt{2}} \exp \left[ \nu \left( \frac{\nu}{2} \ln \frac{\nu}{\nu} - 1 - \sqrt{-\nu} z \right) \right] \left[ 1 + O \left( \frac{1}{\sqrt{\nu}} \right) \right] \]  

(6)

The \( m - \text{th} \) term of the sum [5] possesses the asymptotic

\[ \frac{1}{m!} \exp \left[ \frac{m}{2} (\ln m - 1) - \sqrt{m} z + z^2/4 + m \ln \xi \right] \]

(7)

This means, following Bolzano-Cauchy’s criteria, that the sum [5] is not only absolutely, but uniformly convergent also for the finite values of the constants of the integral [1].
The Calculation of the Functional Integral

In this section we will calculate the functional integral defined as the continuum limit of the finite dimensional integral. The continuum limit means the limit $N \to \infty$, where $N$ is the number of the time slices in the integral. The advantage of this method rely in the well defined measure of the finite dimensional integral. Taking the continuum limit of the result of the $N$ dimensional integration, we bypass the problem of the continuum integral measure.

We are going to calculate Wiener functional integral for the continuum action defined by:

$$L = \int_0^\beta d\tau \left[ c/2 \left( \frac{\partial \varphi(\tau)}{\partial \tau} \right)^2 + b\varphi(\tau)^2 + a\varphi(\tau)^4 \right].$$

(8)

Following the standard procedure, we divide the integration interval to the $N$ equal slices, and we define the $N$–th approximation of the continuum action:

$$L_N = \sum_{i=1}^N \Delta \left[ c/2 \left( \frac{\varphi_i - \varphi_{i-1}}{\Delta} \right)^2 + b\varphi_i^2 + a\varphi_i^4 \right],$$

where $\Delta = \beta/N$. By definition of the integrals we have:

$$S = \lim_{N \to \infty} S_N.$$

We define the $N$– dimensional integral by the relation [3]:

$$Z_N = \int_{-\infty}^{+\infty} \prod_{i=1}^N \left( \frac{d\varphi_i}{\sqrt{2\pi\Delta}} \right) \exp \left\{ -\sum_{i=1}^N \Delta \left[ c/2 \left( \frac{\varphi_i - \varphi_{i-1}}{\Delta} \right)^2 + b\varphi_i^2 + a\varphi_i^4 \right] \right\}.$$

(9)

The Wiener unconditional measure functional integral is defined by formal limit:

$$Z = \lim_{N \to \infty} Z_N.$$

We use the formal notation for the continuum Wiener functional integral:

$$Z = \int [D\varphi(x)] \exp \left\{ -\int_0^\beta d\tau \left[ c/2 \left( \frac{\partial \varphi(\tau)}{\partial \tau} \right)^2 + b\varphi(\tau)^2 + a\varphi(\tau)^4 \right] \right\}.$$

The most important problem is to calculate the $N$–dimensional integral [11]. We rewrite the action in the convenient form for the consecutive integrations:

$$L_N = \Delta a\varphi_1^4 + \frac{c}{\Delta} (1 + \frac{b\Delta^2}{c}) \varphi_1^2 + \frac{c}{\Delta} \varphi_1 \varphi_2 + \cdots + \Delta a\varphi_i^4 + \frac{c}{\Delta} (1 + \frac{b\Delta^2}{c}) \varphi_i^2 + \frac{c}{\Delta} \varphi_i \varphi_{i+1} + \cdots + \Delta a\varphi_N^4 + \frac{c}{\Delta} (1/2 + \frac{b\Delta^2}{c}) \varphi_N^2.$$

(10)

In the integration over variable $\varphi_i$ one find the terms resulting from the preceding integration over variable $\varphi_{i-1}$, as one create the terms for integration over variable $\varphi_{i+1}$. Exploiting the formula [11]:

$$\int_0^\infty x^{\alpha-1} \exp(-px^2 - qx) \, dx = \Gamma(\alpha)(2p)^{-\alpha/2} \exp \left( \frac{q^2}{8p} \right) D_{-\alpha} \left( \frac{q}{\sqrt{2p}} \right),$$

(11)

for integration over the variable $\varphi_1$ first, we find:
\[ Z_1 = \int_{-\infty}^{+\infty} \left( \frac{d\varphi_1}{\sqrt{2\pi\Delta}} \right) \exp \left\{ -\Delta a\varphi_1^4 - \frac{c}{\Delta} (1 + \frac{b\Delta^2}{c})\varphi_1^2 + \frac{c}{\Delta} \varphi_1 \varphi_2 \right\} . \]  

(12)

By Taylor’s expansions of the term

\[ \exp \left( \frac{c}{\Delta} \varphi_1 \varphi_2 \right), \]

taking into account that terms of the odd powers in \( \varphi_1 \) disappears due to the symmetry of integral, we simplify the integral by substitution:

\[ \varphi_1^2 = \omega. \]

We reads:

\[ Z_1 = \sum_{k_1=0}^{\infty} \frac{(c\varphi_2/\Delta)^{2k_1}}{(2k_1)!} \int_{0}^{+\infty} \left( \frac{d\omega}{\sqrt{2\pi\Delta}} \right) (\omega)^{k_1-1/2} \exp \left\{ -\Delta \omega \varphi_2^2 - \frac{c}{\Delta} (1 + \frac{b\Delta^2}{c}) \omega \right\} . \]

(13)

Using formula (11) we find:

\[ Z_1 = \sum_{k_1=0}^{\infty} \frac{(c\varphi_2/\Delta)^{2k_1}}{(2k_1)!} \frac{1}{\sqrt{2\pi\Delta}} \Gamma(k_1 + 1/2) \left( \frac{1}{\sqrt{2\Delta a}} \right)^{k_1+1/2} \exp \left( \frac{z^2}{4} \right) D_{-k_1-1/2}(z), \]

(14)

where

\[ z = \frac{c(1 + b\Delta^2/c)}{\sqrt{2a\Delta^3}}. \]

Following definition of \( z \), we can use the factor

\[ \left( \frac{c}{\Delta} (1 + b\Delta^2/c) \right)^{k_1+1/2} \]

to simplify further this relation to the form:

\[ Z_1 = \frac{1}{\sqrt{2\pi(1 + b\Delta^2/c)}} \sum_{k_1=0}^{\infty} \frac{(\varphi_2)^{2k_1}}{(2k_1)!} \left( \frac{c/\Delta}{(1 + b\Delta^2/c)} \right)^{k_1} \Gamma(k_1 + 1/2) \frac{z^{k_1+1/2}}{D_{-k_1-1/2}(z)}, \]

(15)

or, equivalently:

\[ Z_1 = \frac{1}{\sqrt{2\pi(1 + b\Delta^2/c)}} \sum_{k_1=0}^{\infty} \frac{(\Delta c/\Delta^{2/3})}{(2k_1)!} (\varphi_2)^{2k_1} \Gamma(k_1 + 1/2) D_{-k_1-1/2}(z), \]

(16)

where the notation was used:

\[ D_{-k_1-1/2}(z) = z^{k_1+1/2} e^{\frac{z^2}{4}} D_{-k_1-1/2}(z). \]

(17)

From above result, the term \((\varphi_2)^{2k_1}\) will be involved into the second integration over the variable \( \varphi_2 \). This will results to the additional \( k_1 \) dependent contribution to the result of the first integration and to the introduction of the link between both steps of integration. Explicitly, for the second integration, we have:

\[ Z_2^{loc} = \int_{-\infty}^{+\infty} \frac{d\varphi_2}{\sqrt{2\pi\Delta}} (\varphi_2)^{2k_1} \exp \left\{ -\Delta a\varphi_2^4 - \frac{c}{\Delta} (1 + \frac{b\Delta^2}{c}) \varphi_2^2 + \frac{c}{\Delta} \varphi_2 \varphi_3 \right\} . \]

(18)
By Taylor’s expansion of the term of the action linear in variable \( \varphi_2 \), by the interchange of the order of the integration and summation, taking into account that the odd powers of \( \varphi_2 \) variable disappear due to symmetry of integral, and by variable substitution \( \varphi_2^2 = \omega \) we find:

\[
\mathcal{Z}_2^{\text{loc}} = \sum_{k_2=0}^{\infty} \frac{(c \varphi_3/\Delta)^{2k_2}}{(2k_2)!} \int_0^{+\infty} \left( \frac{d\omega}{\sqrt{2\pi c}} \right) \omega^{k_2+k_1-1/2} \exp \left\{ -\Delta a \omega^2 - \frac{c}{\Delta}(1 + \frac{b \Delta^2}{c})\omega \right\}.
\]

By the same evaluation as for \( Z_1 \) we have:

\[
\mathcal{Z}_2^{\text{loc}} = \frac{(c(1 + b \Delta^2/c)/\Delta)^{-k_1}}{\sqrt{2\pi(1 + b \Delta^2/c)}} \sum_{k_2=0}^{\infty} \frac{(\varphi_3)^{2k_2}}{(2k_2)!} \Gamma(k_2 + k_1 + 1/2) D_{-k_2 - k_1 - 1/2}(z).
\]

Taking both integration steps together we have:

\[
\mathcal{Z}_2 = \left( \frac{1}{\sqrt{2\pi(1 + b \Delta^2/c)}} \right)^2 \sum_{k_1, k_2=0}^{\infty} \xi^{2k_1} \frac{\varepsilon^{2k_2}}{(2k_1)!(2k_2)!} \Gamma(k_1 + 1/2) D_{-k_1 - 1/2}(z) \Gamma(k_1 + k_2 + 1/2) D_{-k_1 - k_2 - 1/2}(z),
\]

where the new symbol was introduced:

\[
\xi = \frac{1}{(1 + b \Delta^2/c)}.
\]

We can repeats this procedure for the integration variables \( \varphi_3, \cdots, \varphi_{N-1} \). For the integration over variable \( \varphi_N \), there one have no linear term in exponent, therefore we don’t expand anything and this last integration don’t add the summation to the final formula. We have:

\[
\mathcal{Z}_N^{\text{loc}} = \int_{-\infty}^{+\infty} \frac{d\varphi_N}{\sqrt{2\pi c}} (\varphi_N)^{2N-1} \exp \left\{ -\Delta a \varphi_N^4 - \frac{c}{\Delta}(1 + \frac{b \Delta^2}{c})\varphi_N^2 \right\}.
\]

and the result is:

\[
\mathcal{Z}_N^{\text{loc}} = \frac{(c(1/2 + b \Delta^2/c)/\Delta)^{k_{N-1}}}{\sqrt{2\pi(1/2 + b \Delta^2/c)}} \Gamma(k_{N-1} + 1/2) D_{-k_{N-1} - 1/2}(z_N).
\]

Repeating by recurrence this procedure, we obtain finally the non-perturbative, exact result:

\[
\mathcal{Z}_N = \left[ 2\pi(1 + b \Delta^2/c) \right]^{-1/2} \left[ 2\pi(1/2 + b \Delta^2/c) \right]^{-1/2} \sum_{k_1, \cdots, k_{N-1}=0}^{\infty} \prod_{i=1}^{N} \left[ \frac{(\xi_i)^{2k_i}}{(2k_i)!} \Gamma(k_i + k_i + 1/2) D_{-k_i - k_i - 1/2}(z_i) \right],
\]

where \( k_0 \equiv k_N \equiv 0, \) and \( \xi_1 = \xi_2 = \cdots = \xi_{N-2} = \xi, \xi_N = 0 \) also \( z_1 = z_2 = \cdots = z_{N-1} = z. \) The modifications appear due to the last integration step and we have

\[
z_N = \frac{c(1/2 + b \Delta^2/c)}{\sqrt{2a \Delta^3}}, \quad \xi_{N-1} = \frac{1}{\sqrt{(1 + b \Delta^2/c)}}, \quad \frac{1}{\sqrt{(1/2 + b \Delta^2/c)}}.
\]

For the conditional measure functional integral, the variable \( \varphi_N \) will be fixed, in time-slicing procedure the last integral over variable \( \varphi_N \) is absent. This means, that we’ll have the same values for all \( \xi_i \) as well as for \( z_i \), and upper limit for product formula in (24) should be \( N - 1 \).

In the noninteracting harmonic oscillator limit, \( (b \text{ positive}, a = 0) \), the Eq. (24) is reduced to the \( N \)-dimensional approximation to the unconditional measure Wiener functional integral for the harmonic oscillator. For \( a \neq 0 \) we can
derive from [23] by Gelfand-Yaglom procedure the differential equation of the second order. The dependence on the coupling constant is in the part characterized by multiple summations over indices $k_i$. This part is now the object of our discussion.

We are going to show that relation [23] for the $N$-dimensional integral is uniformly convergent for each of the summation index. For the summation over one index $k_i$ we have (we omit the index $i$ in $\xi_i$ and $z_i$):

$$ \sum_{k_i=0}^{\infty} \left[ \frac{(\xi)^{2k_i}}{(2k_i)!} \Gamma(k_i-1 + k_i + 1/2) \Delta^{-k_i-1/2} \right] \left[ \Gamma(k_i + k_{i+1} + 1/2) \Delta^{-k_i-1+1/2} \right]. $$

(25)

For $k_i \to \infty$ we apply for the $D$ functions the asymptotic relation [6]. We have for the $k_i$ dependent part of [25] the relation:

$$ a_{k_i} = \frac{(\xi)^{2k_i} \Gamma(k_i + k_{i+1} + 1/2) \Gamma(k_i-1 + k_i + 1/2)}{(2k_i)!} \exp(z^2/2) $$

(26)

$$ \times z^{k_i+k_{i+1}+1/2} \exp \left[ -\frac{k_i+k_{i+1}}{2}(\ln(k_i+k_{i+1}) - 1) - \sqrt{k_i+k_{i+1}}z \right] $$

$$ \times z^{k_{i-1}+k_i+1/2} \exp \left[ -\frac{k_{i-1}+k_i}{2}(\ln(k_{i-1}+k_i) - 1) - \sqrt{k_{i-1}+k_i}z \right] $$

We convert the above relation by help of the identity:

$$ \Gamma(p + k + 1/2) = \exp [\ln \Gamma(p + k + 1/2)] $$

to the form convenient for the proof of the convergence.

For the logarithm of the relation [26] we have:

$$ \ln a_{k_i} = \ln \Gamma(k_i + k_{i+1} + 1/2) + \ln \Gamma(k_{i-1} + k_i + 1/2) - \ln \Gamma(2k_i + 1) $$

$$ - \frac{k_{i-1}+k_i}{2}(\ln(k_{i-1}+k_i) - 1) - \frac{k_i+k_{i+1}}{2}(\ln(k_i+k_{i+1}) - 1) + 2k_i \ln(\xi z) + (k_{i-1} + k_{i+1} + 1) \ln z $$

$$ - (\sqrt{k_i+k_{i+1}} + \sqrt{k_{i-1}+k_i}) z + z^2/2 + o(k_i^{-1/2}) $$

(27)

For the gamma functions with argument $u \to \infty$ we can use the asymptotic relation [5]:

$$ \ln \Gamma(u) = (u - 1/2) \ln(u) - u + 1/2 \ln(2\pi) + O(u^{-1}) $$

(28)

For $k_{i-1}$ and $k_{i+1}$ fixed and finite and if $k_{i \pm 1} < k_i$, we find the following asymptotic for $a_{k_i}$:

$$ \ln a_{k_i} = -\left( k_i - \frac{k_{i-1}+k_{i+1}-1}{2} \right) \ln k_i + 2k_i(\ln(\xi z) - \ln 2 + 1/2) - 2\sqrt{k_i} z + (k_{i-1} + k_{i+1} + 1) \ln z $$

$$ + 1/2 \ln \pi + z^2/2 + o(k_i^{-1/2}) $$

(29)

The leading term of the above relation is

$$ -k_i \ln(k_i), $$

and $a_{k_i}$ is going to zero in the asymptotic region as

$$ \frac{k_i^\alpha \beta^{k_i}}{k_i! \exp(\sqrt{k_i})}, $$

where $\alpha$ and $\beta$ are finite numbers. The asymptotic feature of $a_{k_i}$ is sufficient for proof of the uniform convergence of the series [25] and therefore for the finiteness of the sum [25].

By the same method we can prove the uniform convergence of the summation over two, three, ..., $N-1$ summation indices $k_i$ in the equation [23]. This important conclusion indicate, that the $N$ dimensional approach to the the continuum functional integral can be summed up.
Independent Value Model

The another example of the use of the calculation of the functional integral by help of the parabolic cylinder function is the problem of the independent value model\[^7\]. The independent value models (ivm) are covariant models without gradient terms. The formal functional integral for generating functional of such models reads:

\[
Z_{\text{ivm}}[J] = \mathcal{N} \int D\phi \exp \left\{ \int d^3x \left[ iJ(x)\phi(x) - \frac{1}{2}m^2\phi(x)^2 - g\phi(x)^4 \right] \right\},
\]

where \(\phi(x)\) is a real field, \(J(x)\) is the source term and \(g\) is the interaction constant. The normalization constant \(\mathcal{N}\) guarantees that \(Z_{\text{ivm}}[0] = 1\). It should be noted, that physically the model is trivial (the field equation reduces to \(\phi(x) = 0\)), corresponding to independent fluctuations for different \(x\)'s. However, the functional integrals of this type have a well-defined mathematical meaning, and its investigations can serve as a tool for a better understanding of various approximative schemes of calculations.

The generating functional \(Z_{\text{ivm}}[J]\) of the ivm can be calculated as the continuum limit of the discretized finite dimensional integral approximation to the continuum functional integral

\[
\prod_{k=1}^{N} Z_{k}^{\text{ivm}}[J] = \mathcal{N}_0 \prod_{k=1}^{N} \int d\phi_k \exp \left\{ \sum_{k=1}^{N} (iJ_k \phi_k \Delta - 1/2m_0^2\phi_k^2\Delta - g_0\phi_k^4\Delta) \right\},
\]

where \(\Delta\) is the volume of the discretized space cell, and \(\mathcal{N}_0\) is the normalization constant. The values \(m_0\) and \(g_0\) are supposed to be both positive and \(\Delta\) dependent. In this model the \(\Delta\) dependence is chosen so that in the continuum limit \(\Delta \to 0\) the physical observables in the model take reasonable values. Only when this dependence is fixed the model is defined. The \(\ln Z[J]_{\text{ivm}}\) is the generating functional for the connected correlation functions and all even order connected correlations functions are nonnegative, as follows from the standard canonical form of the generating functional for the ivm model. Performing a discretized calculation Lebowitz\[^8\] proved that the connected correlation functions of the fourth order are non-positive, when a free measure of the functional integral was used. The two inequalities for the connected four point correlation function are simultaneously valid only if such function vanish. This means, that the discretized approach to the functional integral leads to the gaussian generating functional of a free, noninteracting model, with no dependence on the coupling constant in the continuum limit. Such behavior is known as "triviality".

We show that we can obtain a nontrivial result for the generating functional of ivm model non-perturbatively by taking a particular \(N \to \infty\) limit in (31) – the continuum limit of the discretized approximation. In evaluations of finite dimensional integrals we follow the parabolic cylinder functions representation of the integrals with \(x^4\) in the exponent of the integrand:

\[
\int_{-\infty}^{+\infty} x^{2m} dx \exp\{-(ax^4 + bx^2)\}
\]

\[
= \frac{1}{(2a)^{m/2+1/4}} \Gamma(m+1/2) \exp \left( \frac{b^2}{8a} \right) D_{-m-1/2} \left( \frac{b}{\sqrt{2a}} \right),
\]

where \(Re a > 0\), and \(D_{\nu}(z)\) is the parabolic cylinder function of index \(\nu\) and argument \(z\).

Applying (32) to the finite dimensional approximation of the functional integral (31), we find for the \(k-th\) factor in (31) the result:

\[
Z_{k}^{\text{ivm}}[J] = (\mathcal{N}_0)^{1/N} \sum_{i=0}^{\infty} \frac{(-\rho_k)^i}{i!} \left( z \right)^{i+1/2} \exp \left( \frac{z^2}{4} \right) D_{-i-1/2} \left( z \right),
\]

where \(\rho_k = (J_k^2 \Delta)/(2m_0^2)\) and \(z = (m_0^2 \sqrt{\Delta})/\sqrt{g_0}\). The sum in Eq.(33) is uniformly convergent with respect to \(z\) on any compact domain as follows from the asymptotic expression for parabolic cylinder functions when the index \(|\nu| \to \infty\).

The definition of the discretized approach is accomplished when the dependence of "bare" parameters in (31) on \(\Delta\) (or equivalently on \(N\)) is fixed in such way that continuum limit \(N \to \infty\) is well-defined. The requirement
\[ Z^{ivm}[J] = \lim_{N \to \infty} \left( \prod_{k=1}^{N} \left( 1 - \rho_k z^2 \frac{D_{-3/2}(z)}{D_{-1/2}(z)} + \frac{1}{2} \rho_k z \frac{D_{-5/2}(z)}{D_{-1/2}(z)} - \ldots \right) \right), \]  

where we have taken into account the proper normalization \( Z^{ivm}[0] = 1 \). In the continuum limit, \( \sum \Delta \) is replaced by \( \int dx^n \). We see immediately that the continuum limit survive only terms with the number of the summations equal to the power of \( \Delta \). This restricts the contributions to terms up to the first order in \( \rho_k \). In the limit \( N \to \infty \) (i.e., \( \Delta \to 0 \)) we obtain the nontrivial contribution to the generating functional \( Z^{ivm}[J] \) when \( m_0 \) and \( z^{-2} \sim g_0 N \) are kept fixed:

\[ Z^{ivm}[J] = \exp \left\{ -\kappa(z) \int d^nx \frac{J^2(x)}{2m_0^2} \right\}, \quad \kappa(z) = 1 - \frac{3}{2} \frac{D_{-5/2}(z)}{D_{-1/2}(z)}, \]  

The result \( Z^{ivm}[J] \) represents a free theory result, due to the quadratic \( J^2 \) dependence in the exponent. However, it possesses a nontrivial dependence on interaction term which survives in the continuum limit and is hidden in \( z \sim \sqrt{(\Delta/g_0)} \) dependence of the factor \( \kappa(z) \) multiplying the integral in the exponent. Introducing a new parameter \( m_0^2(z, m_0^2) = \frac{1}{\kappa(z)} m_0^2 \) all non-trivial dependence on \( z \) and \( m_0^2 \) is included into mass renormalization. This is a new feature, because within perturbative approach was found no dependence on the interaction constant in the continuum limit.

**Summation over \( k_i \)**

The result of the \( N \)-dimensional integration \( Z^{ivm}[J] \) is the exact one, calculated without any approximation. The multiple summations suppress this advantage somewhat, therefore we shall attempt to provide the \( k_i \) summations in the formula \( Z^{ivm}[J] \). The details of these calculations are in the Appendices 1 and 2, here we sketch the method and results only.

To obtain the formula for subsequent evaluation of the functional integral, we must to solve the problem of summations over indexes \( k_i \) in Eq. \( Z^{ivm}[J] \). For each summation index \( k_i \) we meet the sum of the series with product of two \( D \) functions. We are dealing with the sum of the series:

\[ a_{k_i} = \frac{\xi^{2k_i}}{(2k_i)!} \Gamma(k_{i-1} + k_i + 1/2)D_{-k_{i-1} - k_i - 1/2} \frac{(z)\Gamma(k_i + k_{i+1} + 1/2)D_{-k_i - k_{i+1} - 1/2}}{(z)} \]  

(36)

Of course, parabolic cylinder functions belong to the representation of the upper-triangle matrices, therefore we believe to the simplification of the product of two such functions. What follows, we propose the our approach to provide the summation over indices \( k_i \). At first, we represent one of \( D \) functions by Poincaré - type expansion \( \xi \), valid for real index and positive argument of the function, if the inequality \( a < z \) is valid:

\[ e^{z^2/4} z^{a+1/2} D_{-a-1/2}(z) = D_{-a-1/2}(z) = \sum_{j=0}^{\infty} (-1)^j \frac{(a + 1/2)^{2j}}{j! (2z^2)^j} + \epsilon_J(a, z), \]  

(37)

where \( \epsilon_J(a, z) \) is the remainder of the Poincaré - type expansion of the \( D \) function.

Inserting this relation to the sum, we divide the sum over \( k_i \) into two parts. One, over finite \( k_i \), where Poincaré - type expansion is correct and the second, the remainder over infinite \( k_i \) indexes, but small comparing to first part due to uniform convergence:

\[ \sum_{j=0}^{\infty} \frac{(-1)^j}{j! (2z^2)^j} \sum_{k_i=0}^{\infty} \frac{\xi^{2k_i}}{(2k_i)!} \Gamma(k_{i-1} + k_i + 1/2)\Gamma(k_i + k_{i+1} + 2j + 1/2)D_{-k_{i-1} - k_i - 1/2} \frac{(z)}{D_{-k_i - k_{i+1} - 1/2} \frac{(z)}{}} + \]  

(38)

\[ + \sum_{k_i=0}^{\infty} \frac{\xi^{2k_i}}{(2k_i)!} \Gamma(k_{i-1} + k_i + 1/2)\Gamma(k_i + k_{i+1} + 1/2)D_{-k_{i-1} - k_i - 1/2} \frac{(z)}{D_{-k_i - k_{i+1} - 1/2} \frac{(z)}{}} + \epsilon_J(a, z). \]
where \( N_0 < z \). In the first term of the above relation we add the terms expanding the summation over index \( k_i \) to infinity and in the asymptotic region of \( k_i > N_0 \) we expand the function \( D \) by double asymptotic properties expansions proposed by Temme \( \[11\] \):

\[
D_{-a-1/2}(z) = \frac{\exp \left( -A z^2 \right)}{(1 + 4\lambda)^{1/4}} \left[ \sum_{k=0}^{n-1} f_k(\lambda) + \frac{1}{z^{2k}} R_n(a, z) \right]
\] (39)

where the following quantities were introduced:

\[
\lambda = \frac{a}{z^2}, \quad w_0 = \frac{1}{2} \left( \sqrt{1 + 4\lambda} - 1 \right), \quad A = \frac{1}{2} w_0^2 + w_0 - \lambda \ln w_0 + \lambda \ln \lambda.
\]

The functions \( f_k(\lambda) \) are calculated in \( \[11\] \). For sum of the series (36) we find:

\[
\sum_{j=0}^{\infty} \frac{(-1)^j}{j! (2z^2)^j} \sum_{k=0}^{\infty} \frac{(\xi)^{2k_j}}{(2k)!} \Gamma(k_i - 1 + k_i + 1/2) \Gamma(k_i + k_i + 1 + 2j + 1/2) D_{-k_i - k_i - 1/2} (z) +
\]

\[
+ \sum_{k_i=0}^{N_0} \frac{(\xi)^{2k_i}}{(2k)!} \Gamma(k_i - 1 + k_i + 1/2) \Gamma(k_i + k_i + 1 + 1/2) D_{-k_i - k_i - 1/2} (z) + \epsilon_j(a, z) +
\]

\[
+ \sum_{k_i=N_0+1}^{\infty} \frac{(\xi)^{2k_i}}{(2k)!} \Gamma(k_i - 1 + k_i + 1/2) \Gamma(k_i + k_i + 1 + 1/2) D_{-k_i - k_i - 1/2} (z) \exp \left( -A z^2 \right) \frac{1}{(1 + 4\lambda)^{1/4}} \left[ \sum_{k=0}^{n-1} f_k(\lambda) \right] +
\]

\[
- \sum_{j=0}^{\infty} \frac{(-1)^j}{j! (2z^2)^j} \sum_{k_i=N_0+1}^{\infty} \frac{(\xi)^{2k_i}}{(2k)!} \Gamma(k_i - 1 + k_i + 1/2) \Gamma(k_i + k_i + 1 + 2j + 1/2) D_{-k_i - k_i - 1/2} (z)
\] (41)

The sum over index \( k_i \) in the first term of the above relation will be provided by help of the formula \( \[4\] \):

\[
e^{x^2/4} \sum_{k=0}^{\infty} \frac{\mu_k}{k!} t^k D_{-\nu-k}(x) = e^{(x-t)^2/4} D_{-\nu}(x-t).
\] (42)

Method of this sum is described in Appendix 1, we are going to evaluate the three terms of remainder now. First of all, we must to develop the criteria for the remainder. Schematically, we must evaluate the multiple sums:

\[
\sum_{k_1, \ldots, k_N=0}^{\infty} a_{k_1,k_2} a_{k_3,k_4} \cdots a_{k_{N-1},k_N} a_{k_N,k_{N+1}}
\] (43)

If each individual sum exist and it is finite, we divide the sum over index \( k_i \) to finite, principal sum and the remainder, which can be done as small as possible, comparing to the principal part of the sum. Let us define the principal sum object, where sum over \( m \) indexes \( k_i \) was not provided, but for indexes \( k_{m+1}, \ldots, k_N \) the finite sum was done:

\[
\Sigma_{m,N-m} = \sum_{k_1, \ldots, k_m=0}^{\infty} a_{k_1,k_2} a_{k_3,k_4} \cdots a_{k_{m-1},k_m} b_{k_m},
\] (44)

where

\[
b_{k_m} = \sum_{k_{m+1}, \ldots, k_N=0}^{N_0} a_{k_m,k_{m+1}} a_{k_{m+2},k_{m+3}} \cdots a_{k_N,k_{N+1}}
\] (45)

The next sum over \( k_m \) we divide into the finite, principal sum and the remainder, the infinite sum, giving minor contribution. We have the identity:

\[
\Sigma_{m,M-m} = \Sigma_{m-1,M-m+1} + \Sigma_{m-1,M-m} \epsilon_m
\] (46)
where

$$\epsilon_m = \sum_{k_m = N_0 + 1}^{\infty} a_{k_{m-1}, k_m} b_{k_m}.$$  

Let \( \epsilon = \max(\epsilon_1, \epsilon_2, \ldots, \epsilon_N) \), then we find:

$$\Sigma_{N,0} \leq \Sigma_{0,N} + \left( \frac{N}{1} \right) \Sigma_{0,N-1} \epsilon + \cdots + \left( \frac{N}{i} \right) \Sigma_{0,N-i} \epsilon^i + \cdots + \epsilon^N \quad (47)$$

If \( \Sigma_{0,k} \) are the members of non-decreasing series, the following inequality is valid:

$$\Sigma_{0,N} \leq \Sigma_{N,0} \leq \Sigma_{0,N}(1 + \epsilon)^N \quad (48)$$

In the limit \( N \to \infty \) we obtain \( \Sigma_{0,\infty} = \Sigma_{\infty,0} \), if \( \epsilon \simeq N^{-1-\varepsilon} \) where \( \varepsilon > 0 \). Let us check if the terms in the remainder corresponds to this demand.

For Poincaré - type expansion the upper bound of remainder was calculated by Olver [9]. We use the improved upper bound evaluated by Temme [10]. The upper bound for remainder in definition (37) can be read:

$$| \epsilon_\mathcal{J}(a, z) | \leq \frac{2z^2}{z^2 - 2a} \frac{(a + 1/2)\mathcal{J}}{(2a)\mathcal{J}} \left( 1 - \frac{a^2}{z^2} \right) \exp \left( \frac{4\delta}{z^2 - 2a} F_2(1/2, 1/2; 3/2; 1 - \frac{a^2}{z^2}) \right) \quad (49)$$

This relation is valid for \( 2\sqrt{a} \leq z \). The following quantity was introduced:

$$\delta = \left| a^2 \cdot \frac{3}{16} + \frac{2a}{z^2} \left( 1 + \frac{a}{2z^2} \right) \frac{z^2}{(z^2 - 2a)^2} \right|$$

The sum in remainder:

$$\sum_{k_i = 0}^{N_0} \frac{(\xi)^{2k_i}}{(2k_i)!} \Gamma(k_i + k_i + 1/2) \Gamma(k_i + k_i + 1/2) D_{-k_i - k_i - 1/2} \left( z \right) \epsilon_\mathcal{J}(k_i + k_i + 1, z) \quad (50)$$

is bounded by the relation:

$$\frac{\mathcal{M}}{(\mathcal{J} - 1)! (2\xi)^\mathcal{J}} \sum_{k_i = 0}^{N_0} \frac{(\xi)^{2k_i}}{(2k_i)!} \Gamma(k_i + k_i + 1/2) \Gamma(k_i + k_i + 1/2) D_{-k_i - k_i - 1/2} \left( z \right) \quad (51)$$

where

$$\mathcal{M} = \max \left( \frac{2z^2}{z^2 - 2a} \right) F_2(1/2, 1/2; 3/2; 1 - \frac{a^2}{z^2} \exp \left( \frac{4\delta}{z^2 - 2a} F_2(1/2, 1/2; 3/2; 1 - \frac{a^2}{z^2}) \right) \right), \quad (52)$$

and \( a = k_i + k_i + 1, k_i = 1, 2, \ldots, N_0 \). We have the freedom to choose the parameter \( \mathcal{J} \), therefore the upper bound on this contribution to the remainder can be done as small as necessary.

For the remainder in the double asymptotic property expansion [11] we find the definition [10]:

$$R_n(a, z) = \frac{z^{2a + 1/2}}{\Gamma(2a + 1/2)} \int_0^\infty s^a e^{-z^2 s} f_n(s) \frac{ds}{\sqrt{s}} \quad (53)$$

It can be shown by some algebra, that upper bound on remainder is:

$$R_n(a, z) \leq M_n,$$

where \( M_n \) is upper bound to the function \( f_n(s) \).

We are now going to estimate the upper bound of the following contribution to the remainder in (11):

$$\sum_{k_i = N_0 + 1}^{\infty} \frac{(\xi)^{2k_i}}{(2k_i)!} \Gamma(k_i + k_i + 1/2) \Gamma(k_i + k_i + 1/2) D_{-k_i - k_i - 1/2} \left( z \right) \exp \left( -Az^2 \right) \left( 1 + 4\lambda \right)^{-1/4} \frac{1}{z^n} R_n(a, z) \bigg|_{a=k_i+k_i+1} \quad (54)$$
In the asymptotic region of $k_i$ we use the Stirling formula for the gamma functions:

$$\ln \Gamma(u) = (u - 1/2) \ln u - u + 1/2 \ln (2\pi) + O(u^{-1})$$

We find:

$$\frac{(\xi)^{2k_i}}{(2k_i)!} \Gamma(k_{i-1} + k_i + 1/2) \Gamma(k_i + k_{i+1} + 1/2) = \left(\frac{\xi}{2}\right)^{2k_i} (k_i)^{k_{i+1} + k_{i-1} - 1/2} \exp\left(\frac{k_{i+1}(k_{i+1} + 1/2) + k_{i-1}(k_{i-1} + 1/2)}{k_i}\right)$$

Moreover, in the asymptotic region of $k_i$ we replace the function $\mathcal{D}_{-k_{i-1}-k_{i-1}/2}(z)$ by the principal part of the double asymptotic expansion:

$$\mathcal{D}_{-k_{i-1}-k_{i-1}/2}(z) = \exp\left(-A\frac{z^2}{2}\right) \left(-1 + \frac{2}{1 + \sqrt{1 + 4\lambda}}\right) + \lambda z^2 \ln \left(\frac{2}{1 + \sqrt{1 + 4\lambda}}\right)$$

and that for $\lambda \geq 0$ the following inequalities are valid:

$$1 - \frac{2}{1 + \sqrt{1 + 4\lambda}} \leq 1, \quad \frac{2}{1 + \sqrt{1 + 4\lambda}} \leq \frac{1}{\sqrt{\lambda}}$$

we find the upper bound to remainder (54):

$$M_n \sum_{k_i = N_0 + 1}^{\infty} \left(\frac{\xi}{2}\right)^{2k_i} (k_i)^{k_{i+1} + k_{i-1} - 1/2} \exp\left(\frac{k_{i+1}(k_{i+1} + 1/2) + k_{i-1}(k_{i-1} + 1/2)}{k_i}\right) \exp\left(k_i + \frac{k_{i+1} + k_{i-1}}{2}\right)$$

For $k_i$ big, we use the identity:

$$\left(\frac{1}{\sqrt{1 + \frac{k_{i+1}}{k_i}}}\right)^{k_{i+1} + k_i} \sim \left(1 - \frac{k_{i+1}}{k_i}\right)^{k_{i+1} + k_i} \sim \exp\left(-\frac{k_{i+1}}{2}\right)$$

Finally, we have:

$$M_n \sum_{k_i = N_0 + 1}^{\infty} \frac{1}{k_i!} \left(\frac{\xi z^2}{2}\right)^{2k_i} (k_i)^{k_{i+1} + k_{i-1} - 1/2} \exp\left(\frac{k_{i+1}(k_{i+1} + 1/2) + k_{i-1}(k_{i-1} + 1/2)}{k_i}\right)$$

The function

$$\exp\left(\frac{k_{i+1}(k_{i+1} + 1/2) + k_{i-1}(k_{i-1} + 1/2)}{k_i}\right)$$

is decreasing for $k_i \geq N_0$, therefore following mean value lemma we read:

$$M_n \exp(\rho_k) \sum_{k_i = N_0 + 1}^{\infty} \frac{1}{k_i!} \left(\frac{\xi z^2}{4}\right)^{2k_i} (k_i)^{k_{i+1} + k_{i-1} - 1/2}$$

where $\rho_k = (k_{i+1}(k_{i+1} + 1/2) + k_{i-1}(k_{i-1} + 1/2))/2$; $k \in (N_0, \infty)$. The series

$$\sum_{k_i = N_0 + 1}^{\infty} \frac{1}{k_i!} \left(\frac{\xi z^2}{4}\right)^{k} (k)^{(k_{i+1} + k_{i-1} - 1)/2}$$
is the remainder of the series uniformly convergent to the function
\[ f(x) = (x \partial_x)^m e^x, \]
where \( x = \frac{x^2}{2}, \) and \( m \geq (k_{i+1} + k_{i-1} - 1)/2. \) By choosing the parameter \( n, \) which corresponds to number of the decomposition terms taken into account of double asymptotic expansion of the function \( D, \) this contribution to the remainder can be done as small as we need.

Now we are going to estimate the last part of the remainder, corresponding to the difference of the series:
\[
\sum_{k_i=N_0+1}^{\infty} \frac{(\xi)^{2k_i}}{(2k_i)!} \Gamma(k_{i-1} + k_i + 1/2) \Gamma(k_i + k_{i+1} + 1/2) D_{k_{i-1} - k_{i-1} - 1/2} (z) e^{-(A z^2)} \frac{\sum_{k=0}^{n-1} f_k(\lambda)}{(1 + 4\lambda)^{1/4}} z^{2k} - \\
- \sum_{j=0}^{J} \frac{(-1)^{j}}{j! (2z^2)^{j}} \sum_{k_i=N_0+1}^{\infty} \frac{(\xi)^{2k_i}}{(2k_i)!} \Gamma(k_{i-1} + k_i + 1/2) \Gamma(k_i + k_{i+1} + 2j + 1/2) D_{k_{i-1} - k_{i-1} - 1/2} (z) \]

Following the identity (56) we can recognize in the sums over \( k_i \) the remainders of the uniformly convergent series. These remainders are small, if \( \lambda = k_i^2 / z \) is the large quantity. Due to the uniform convergence, we can change the order of the summations and we read:
\[
\sum_{k_i=N_0+1}^{\infty} \frac{(\xi)^{2k_i}}{(2k_i)!} \Gamma(k_{i-1} + k_i + 1/2) \Gamma(k_i + k_{i+1} + 1/2) D_{k_{i-1} - k_{i-1} - 1/2} (z) \left\{ \exp\left(\frac{(-A z^2)}{1 + 4\lambda}\right) \frac{\sum_{k=0}^{n-1} f_k(\lambda)}{(1 + 4\lambda)^{1/4}} z^{2k} - \sum_{j=0}^{J} \frac{(-1)^{j}(k_i + k_i+1 + 1/2)}{j! (2z^2)^{j}} z^{2j} \right\} \]

The double asymptotic expansion reduces to Poincaré-type expansion (10) when \( a \) is fixed after expanding the quantities in expansion that depend on \( \lambda = a/z^2 \) for small values of this parameter. In the difference
\[
\left\{ \exp\left(\frac{(-A z^2)}{1 + 4\lambda}\right) \frac{\sum_{k=0}^{n-1} f_k(\lambda)}{(1 + 4\lambda)^{1/4}} z^{2k} - \sum_{j=0}^{J} \frac{(-1)^{j}(k_i + k_i+1 + 1/2)}{j! (2z^2)^{j}} z^{2j} \right\}
\]
all terms where \( k, j \leq \min(n, J) \) cancel one another and the rest terms are proportional or smaller then \( z^{-2\min(n,J)}. \) As in the case of previous contributions to the remainder, this means, that this part of the remainder can be done as small as we need.

We shown, that we can evaluate \( N-\) dimensional integral (24) by method of Poincaré decomposition of one of the parabolic cylinder functions. We are able to accomplish the sum over the summation index of the Taylor’s decomposition of the linear exponential function. We shown, that in the continuum limit the \( N-\) dimensional integral evaluated by this method have the same value as the corresponding continuum functional integral.

The result of the recurrence summation procedure of the principal contribution is (see Appendix 1):
\[
Z_N = \left\{ \prod_{i=0}^{N} \left[ 2(1 + b \Delta^2 / c) \omega_i \right] \right\}^{-1/2} \sum_{\mu=0}^{\infty} \frac{(-1)^{\mu}}{\mu! (2z^2)^{\mu}} (N)_{2\mu, 0}
\]
Symbol \((N)_{2\mu, i}\) is defined by the recurrence relation:
\[
(\alpha + 1)_{2\mu, p} = \sum_{\lambda=0}^{\mu} \binom{\mu}{\lambda} \omega_{\lambda}^{-2(\mu - \lambda)} \sum_{i=\max(0, p-2(\mu - \lambda))]}^{2\lambda} \left( \frac{A^2}{\omega_{\lambda - 1} \omega_{\lambda}} \right)^i (\alpha)_{2\lambda, i} a_p^{2(\mu - \lambda) + i}
\]
\[(\Lambda = 1)_{2\mu, p} = \frac{a_p^{2\mu}}{\omega_{2\mu}^2}
\]
where
\[
\omega_i = 1 - \frac{A^2}{\omega_{i-1}}.
\]
\[ A = \frac{1}{2(1 + b\Delta^2/c)}, \]
\[ \omega_0 = \frac{1}{(1/2 + b\Delta^2/c)}. \] (70)

\( Z_N \) in the relation (68) is the \( N \)th approximation of the functional integral. The continuum limit of \( Z_N \) we calculate by the procedure proposed by Gelfand- Yaglom for the calculation of the functional integral for the harmonic oscillator [2]. The method and the solutions are discussed in the Appendix 3, here we present the result only.

The functional integral in the continuum limit is defined by the relation

\[ \lim_{N \to \infty} Z_N = \frac{1}{\sqrt{F(\beta)}} \]

where the function \( F(\beta) \) is the solution of the differential equation taken in the point \( \beta \):

\[ \frac{\partial^2}{\partial \tau^2} F(\tau) + 4 \frac{\partial}{\partial \tau} F(\tau) \frac{\partial}{\partial \tau} \ln S(\tau) = F(\tau) \left( \frac{2b}{c} - 2 \frac{\partial^2}{\partial \tau^2} \ln S(\tau) - 4 \left( \frac{\partial}{\partial \tau} \ln S(\tau) \right)^2 \right) \] (71)

The initial conditions are:

\[ F(0) = 1, \]
\[ \frac{\partial}{\partial \tau} F(0) = 0. \] (72)

The function \( S(\tau) \) is defined as:

\[ S(\tau) = \lim_{\Delta \to 0} \sum_{\mu=0}^{\infty} \frac{(-1)^\mu}{\mu! (2z^2\Delta^4)^\mu} (\Delta^3(N)_{2\mu, 0}) \] (73)

The equation (71) can be simplified by substitution

\[ F(\tau) = \frac{y(\tau)}{(S(\tau))^2} \]

For \( y(\tau) \) we find the equation:

\[ \frac{\partial^2}{\partial \tau^2} y(\tau) = y(\tau) \frac{2b}{c} \] (74)

In the present time the calculation of the function \( S(\tau) \) is not finished, therefore we present the calculation of the lowest order term in the power expansion in the coupling constant in the Appendix 2. For the first two terms of such expansion in the continuum we find:

\[ \lim_{\Delta \to 0} \sum_{\mu=0}^{1} \frac{(-1)^\mu}{\mu! (2z^2\Delta^4)^\mu} (N)_{2\mu, 0} = 1 - \frac{(1/2)^2}{c^2\gamma^3} \left\{ \tanh(\tau\gamma) + \tau\gamma \left[ 3 \tanh^2(\tau\gamma) - 1 \right] \right\} \]

where

\[ \gamma = \sqrt{\frac{2b}{c}}. \]

**Conclusions**

In the present article we presented the new method of the calculation of the Wiener unconditional measure functional integral for the action with the fourth order term. This method can be extended to the case with \( \varphi^{2n} \), \( n = 3, 4, ... \) term in the action, but we did not discuss these possibilities here. The main results are the analytic formula for the \( N \) dimensional integral and the generalized Gelfand-Yaglom equation implying the functional integral in the continuum limit.

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Appendix 1

In this appendix we propose and on pedagogical level to evaluate the $k_i$ summations by the recurrence method. Let we start with summation over the index $k_{N-1}$ of the Eq.(24). The sum that must be done is:

$$Z_1 = \sum_{k_{N-1}=0}^{\infty} \left[ \frac{1}{\sqrt{2\pi(1+b\Delta^2/c)}} \frac{(z/\Delta)^{k_{N-1}}}{(2k_{N-1})!} \Gamma(k_{N-2}+k_{N-1}+1/2)D_{-k_{N-2}-k_{N-1}-1/2}(z) \right]$$

(75)

Let us remind the dependence of the function $D$ on the parabolic cylinder function $D$:

$$D_{-m-1/2}(z) = z^{m+1/2} e^{z^2/4} D_{-m-1/2}(z)$$

and, also the definitions of the variables $\xi$, $\omega_0$, $z$ and $z_0$:

$$z = \frac{c(1+b\Delta^2/c)}{\sqrt{2\pi\Delta^3}}$$
$$z_0 = \frac{c(1/2+b\Delta^2/c)}{\sqrt{2\pi\Delta^3}}$$
$$\xi = \frac{1}{1+b\Delta^2/c}$$
$$\omega_0 = \frac{1/2+b\Delta^2/c}{1+b\Delta^2/c}.$$  

The sum (75) is uniformly convergent, therefore for an arbitrary small positive number $\varepsilon$ exist the number $N_0$ such that the following inequality is true:

$$| \sum_{k_{N-1}=N_0+1}^{\infty} | < N^{-1-\varepsilon}$$

(76)

When we replace the true sum by the truncated one, we introduce into our calculation error up to the order $N^{-1-\varepsilon}$. All recurrence procedure consists of the $N$ steps, therefore we require that in the continuum limit the influence of the reminders of the sums disappears. For the truncated sum we demand that

$$N_0 < z \approx N^{3/2}$$

Therefore we are permitted to use the asymptotic Poincaré type expansion of the parabolic cylinder function, which for $D$ means:

$$D_{-k_{N-1}-1/2}(z_0) \equiv z_0^{k_{N-1}+1/2} e^{z_0^2/4} D_{-k_{N-1}-1/2}(z_0) = \sum_{j=0}^{\mathcal{J}} (-1)^j \frac{(k_{N-1}+1/2)_j}{j! (2z_0^2)^j}$$

(77)

In the last relation, $\mathcal{J}$ means the number of the terms of the asymptotic expansions convenient to take into account. We apply asymptotic expansion for the function $D_{-k_{N-1}-1/2}(z_0)$ in Eq.(75). In truncated sum we interchange the order of the finite summations over indices $k_{N-1}$ and $j$. We replace $D$ by $D$, therefore a corresponding power of the variable $z$ play important role. We obtained the relation:

$$Z_{1}^{cut} = \frac{\Gamma(1/2)}{\sqrt{2\pi(1+b\Delta^2/c)\omega_0}} \sum_{j=0}^{\mathcal{J}} \frac{(-1)^j}{j! (2z_0^2)^j} \frac{\exp(z^2/4)}{\sqrt{2\pi(1+b\Delta^2/c)}} z^{k_{N-2}+1/2}$$

$$\sum_{k_{N-1}=0}^{N_0} \frac{(z/\Delta)^{k_{N-1}}}{(k_{N-1})!} \Gamma(k_{N-1}+k_{N-2}+1/2) (k_{N-1}+1/2)_{2j} D_{-k_{N-1}-k_{N-2}-1/2}(z)$$

(78)
Therefore we can extend the summation up to infinity with the error of the order $k^{-1}$

We see, that this object is a polynomial in variable $k_{N-1}$

This sum is uniformly convergent, as we could prove by the same procedure as in section "Summation over $k_i$". Therefore we can extend the summation up to infinity with the error of the order $N^{-1-\varepsilon}$. To be able provide the sum over index $k_{N-1}$, we must modify term

$$(k_{N-1} + 1/2)_{2j}.$$ 

We see, that this object is a polynomial in variable $k_{N-1}$ of the $2j-th$ order. We rewrite the polynomial in the another form:

$$(k_{N-1} + 1/2)_{2j} = \min (2j,k_{N-1}) a^2_j \frac{(k_{N-1})!}{(k_{N-1} - i)!}$$

The coefficients $a^2_j$ can be calculated by recurrence procedure from the relation:

$$\sum_{k_{N-1}=0}^{N_{0}} \frac{(k_{N-1} + 1/2)_{2j}}{(k_{N-1})!} f(k_{N-1}) = \sum_{i=0}^{2j} a^2_j \sum_{k_{N-1}=i}^{N_{0}} \frac{1}{(k_{N-1} - i)!} f(k_{N-1})$$

From the above definition, we find the recurrence equation:

$$a^k_i = (k - 1/2 + i) a^k_{i+1} + a^{k-1}_{i-1}$$

when the initial conditions are:

$$a^0_j = 1, a^0_0 = (1/2), a^0_{j+1} = 0$$

The solution of this recurrence equation is:

$$a^j_i = \binom{j}{i} (1/2)_i$$

Inserting all these replacements into Eq. (79), with help of the identity

$$\Gamma(k_{N-2} + k_{N-1} + 1/2) = \Gamma(k_{N-2} + i + 1/2) (k_{N-2} + i + 1/2)_{k_{N-1}-i}$$

after some algebra, by introducing the new summation index $k = k_{N-1} - i$ we obtain the formula:

$$Z_{\text{cut}}^{i} = \frac{z^{k_{N-2}+1/2}}{\sqrt{2(1+b\Delta^2/c)\omega_0}} \sum_{j=0}^{j} (-1)^j \frac{(i/2)^j}{j! (2\pi)^j} \frac{e^{z^2/4}}{\sqrt{2\pi(1+b\Delta^2/c)}} \sum_{i=0}^{2j} a^2_j (k_{N-2} + i + 1/2)_{k_{N-2}-i-1/2-k} (z)$$

In the above relation we extend the summation over the index $k$ up to infinity, because the added reminder disappear in the continuum limit. The sum over $k$ is now prepared for application of the identity:

$$e^{x^2/4} \sum_{k=0}^{\infty} \frac{(\nu)_k}{k!} t^k D_{-\nu-k} (x) = e^{(x-t)^2/4} D_{-\nu} (x-t)$$
The result of the first recurrence step, replacing $D$ by $D$, can be read:

$$Z_1^{\text{cut}} = \frac{1}{\sqrt{2(1+b\Delta^2/c)\omega_0}} \sqrt{2\pi(1+b\Delta^2/c)\omega_1} \sum_{j=0}^{\infty} \frac{(-1)^j}{j! (2z)^j} \sum_{i=0}^{2j} \left( \frac{\xi^2}{4\omega_0\omega_1} \right)^i \Gamma(kN-2+i+1/2)D_{D-1-i/2} (z_1)$$

(85)

where

$$z_1 = z \left(1 - \frac{\xi^2}{(4\omega_0)}\right), \quad \omega_1 = \frac{z_1}{z} = 1 - \frac{\xi^2}{(4\omega_0)}.$$ 

For the second recurrence step, we have the summation over the index $kN-2$ in the form:

$$Z_2 = \sum_{kN-2=0}^{\infty} \left[ \frac{1}{\sqrt{2(1+b\Delta^2/c)\omega_0}} \sqrt{2\pi(1+b\Delta^2/c)\omega_1} \left( \frac{\xi^2}{\omega_1} \right)^{kN-2} \Gamma(kN-3+kN-2+1/2)D_{D-1-i/2} (z_1) \right] Z_1^{\text{cut}}$$

(86)

Before replacing $Z_1^{\text{cut}}$ in the last equation, let us define the first step of the new recurrence relation by:

$$(1)_i^{2j} = \frac{1}{\omega_0^{2j}} a_i^{2j}$$

Then the above equation, taking into account the identity $z_0 = z\omega_0$ in detail can be read:

$$Z_2 = \frac{1}{\sqrt{2(1+b\Delta^2/c)\omega_0}} \sqrt{2\pi(1+b\Delta^2/c)\omega_1} \Gamma(kN-3+kN-2+1/2)D_{D-1-i/2} (z_1)$$

(87)

Now, due to the uniform convergence of the sum over index $kN-2$ we will work with truncated sum of the complete $Z_2$. In the truncated sum, we use the asymptotic Poincaré type expansion of the parabolic cylinder function $D_{D-1-i/2} (z_1)$. We have then in the relation for $Z_2^{\text{cut}}$ the finite summations only, so we rearrange them, in order that summation over $kN-2$ will be provided first. Remember, that in an intermediate step we replace $D_{D-1-i/2} (z_1)$ by $D_{D-1-i/2} (z)$ and we must take into account the corresponding power of variable $z$. We have:

$$Z_2^{\text{cut}} = \frac{1}{\sqrt{2(1+b\Delta^2/c)\omega_0}} \sqrt{2\pi(1+b\Delta^2/c)\omega_1} \frac{1}{\omega_1^{2l}} e^{z^2/4} z^{kN-3+1/2}$$

(88)

$$\times \sum_{kN-2=0}^{N_0} \left( \frac{\xi^2}{\omega_1} \right)^{kN-2} \Gamma(kN-3+kN-2+1/2) D_{D-1-i/2} (z)$$ 

if the identity

$$A = \frac{\xi}{2}$$

was introduced. Summing over index $kN-2$ as in the first recurrence step, we have:

$$Z_2^{\text{cut}} = \frac{1}{\sqrt{2(1+b\Delta^2/c)\omega_0}} \sqrt{2\pi(1+b\Delta^2/c)\omega_1} \frac{1}{\omega_1^{2l}} e^{z^2/4} z^{kN-3+1/2}$$

(89)

$$\times \sum_{j=0}^{\infty} \frac{(-1)^j}{j! (2z^2)^j} \sum_{i=0}^{2j} \left( \frac{A^2}{\omega_0\omega_1} \right)^i \sum_{p=0}^{2j+i} a_{p+i}^{2j+i} \left( \frac{A^2}{\omega_1\omega_2} \right)^p \Gamma(kN-3+p+1/2)D_{D-1-i/2} (z_2),$$
where one postulate the identities:

\[ z_2 = z\omega_2, \omega_2 = 1 - \frac{A^2}{\omega_1} \]

By adjustment in the summations:

\[ \sum_{j=0}^{J} \sum_{l=0}^{L} \frac{(-1)^{j+l}}{j!l!(2z)^{j+l}} (\cdot \cdot \cdot j, l \cdot \cdot \cdot) = \sum_{j=0}^{J} \sum_{l=0}^{L} \frac{(-1)^{\mu}}{\mu!(2z)^{\mu}} \sum_{j=0}^{\mu} \binom{\mu}{j} (\cdot \cdot \cdot j, l = \mu - j \cdot \cdot \cdot) \]

and interchange of the order of the summations:

\[ \sum_{i=0}^{2j} \sum_{p=0}^{2\mu-2j+i} \rightarrow \sum_{p=0}^{\mu} \sum_{i=\max[0, p-2\mu+2j]}^{2j} \]

we find the result of the second recurrence step:

\[ Z_{2}^{\text{cut}} = \frac{1}{\sqrt{2(1 + b\Delta^2/c)\omega_0}} \frac{1}{\sqrt{2(1 + b\Delta^2/c)\omega_1}} \frac{(z/z_2)^{k_{N-3}}}{\sqrt{2(1 + b\Delta^2/c)\omega_2}} \times \sum_{\mu=0}^{J+L} \frac{(-1)^{\mu}}{\mu!(2z)^{\mu}} \sum_{p=0}^{2\mu} (2)^{2\mu} p \left( \frac{A^2}{\omega_1\omega_2} \right)^p \Gamma(k_{N-3} + p + 1/2)D_{-k_{N-3}-p-1/2}(z_2) \]  

Where the second recurrence step of the function is defined:

\[ (2)^{2\mu} p = \sum_{j=0}^{\mu} \binom{\mu}{j} \frac{1}{\omega_1^{2\mu-2j}} \sum_{i=\max[0, p-2\mu+2j]}^{2j} \left( \frac{A^2}{\omega_1\omega_2} \right)^i (1)^{2j} a_p^{2\mu-2j+i} \]  

We can see, that after the \( \Lambda \) recurrence steps, the result of the \( \Lambda \) summations over the indices \( k_i \) can be read:

\[ Z_{\Lambda}^{\text{cut}} = \left\{ \prod_{i=0}^{\Lambda} \frac{1}{\sqrt{2(1 + b\Delta^2/c)\omega_i}} \right\} (\omega_\Lambda)^{-k_{N-1}-\Lambda} \times \sum_{\mu=0}^{J_1 + \cdots + J_\Lambda} \frac{(-1)^{\mu}}{\mu!(2z)^{\mu}} \sum_{p=0}^{2\mu} (\Lambda)^{2\mu} p \left( \frac{A^2}{\omega_{\Lambda-1}\omega_\Lambda} \right)^p \Gamma(k_{N-\Lambda-1} + p + 1/2)D_{-k_{N-\Lambda-1}-p-1/2}(z_\Lambda) . \]

We have postulated the recurrence relations:

\[ z_\Lambda = z\omega_\Lambda, \omega_\Lambda = 1 - \frac{A^2}{\omega_{\Lambda-1}}, \omega_0 = \frac{1/2 + b\Delta^2/c}{1 + b\Delta^2/c} , \]

and evaluated the recurrence definition for the function \((\Lambda)^{2\mu}\):  

\[ (\Lambda)^{2\mu} = \sum_{j=0}^{\mu} \binom{\mu}{j} \frac{1}{\omega_{\Lambda-1}^{2\mu-2j}} \sum_{i=\max[0, p-2\mu+2j]}^{2j} \left( \frac{A^2}{\omega_{\Lambda-2}\omega_{\Lambda-1}} \right)^i (\Lambda - 1)^{2j} a_p^{2\mu-2j+i} \]  

when the recurrence procedure begin from:

\[ (1)^{2j} = \frac{1}{\omega_0^{2j}} a_0^{2j} \]

After the last recurrence step, for \( \Lambda = N - 1 \), we are left with the relation of the form \( Z_{N-1}^{\text{cut}} \) where \( k_{N-\Lambda-1} \equiv 0 \) and the index of the \( D \) function is only \(-p - 1/2\). We expand the \( D_{-p-1/2}(z_{N-1}) \) as in all previous recurrence steps and we find the relation:

\[ Z_{N-1}^{\text{cut}} = \left\{ \prod_{i=0}^{N-1} \frac{1}{\sqrt{2(1 + b\Delta^2/c)\omega_i}} \right\} J_1 + \cdots + J_N \sum_{\mu=0}^{J_1 + \cdots + J_N} \frac{(-1)^{\mu}}{\mu!(2z)^{\mu}} \sum_{i=0}^{2i} \left( \frac{A^2}{\omega_{N-2}\omega_{N-1}} \right)^i (N - 1)^{2j} (1/2)_{2i+j} \]
Following the definition of the \( a_{i}^{J} \) symbols, we have:

\[
(1/2)_{2l+i} = a_{0}^{2l+i}
\]

Then in the last part of the preceding equation we read:

\[
\sum_{i=0}^{\mu} \left( \begin{array}{c} \mu \\ l \end{array} \right) \frac{1}{\omega_{N-1}^{\mu}} \sum_{i=0}^{2l} \left( \frac{A^{2}}{\omega_{N-2}\omega_{N-1}} \right)^{i} (N-1)^{2j} a_{0}^{2l+i} = (N)^{2\mu}
\]

(95)

Following the calculations done in this Appendix, we conclude, that it is possible to provide the summations in the precise formula for the \( N \) dimensional integral at least by help of the asymptotic expansions of the parabolic cylinder functions. It is possible to provide the continuum limit of the our result and there are no additional terms contributing to the result in the continuum limit. The result can be read

\[
\mathcal{Z}^{\text{cut}}_{N-1} = \left\{ \prod_{i=0}^{N-1} \frac{1}{\sqrt{2(1+b\Delta^{2}/c)\omega_{i}}} \right\}^{j_{1}+\cdots+j_{N}} \sum_{\mu=0}^{\infty} (\frac{-1}{\mu!(2\pi)^{\mu}}) (N)^{2\mu}
\]

(96)

This expression is sufficient for calculation of the continuum Wiener unconditional measure functional integral by Gelfand-Yaglom procedure leading to the differential equation of the second order. The second part of relation (96) represent the expansion of an unknown function in present time.

**APPENDIX 2**

In this appendix we will present the idea of the decomposition of the principal result by the summations over the indices \( k_{i} \) by slightly different method as was presented in the Appendix 1. Our goal is to study the unknown function from Appendix 1 in the power expansion in the coupling constant. We start as in Appendix 1 from the precise relations for the \( N \) dimensional functional integral, and we proceed by the recurrence procedure without introducing the function \( (\Lambda)^{2j} \). We start, as in Appendix 1, by summation over the index \( k_{N-1} \):

\[
\mathcal{Z}_{1} = \sum_{k_{N-1}=0}^{N-1} \left[ \frac{1}{\sqrt{2\pi(1+b\Delta^{2}/c)\omega_{0}}} \frac{\left( \frac{\xi^{2}}{z_{0}} \right)^{k_{N-1}}}{(2k_{N-1})!} \Gamma(k_{N-2} + k_{N-1} + 1/2)D_{-k_{N-2}-k_{N-1}-1/2}(z) \right]
\]

\[
\times \left[ \frac{1}{\sqrt{2\pi c z_{0}}} \frac{\Gamma(k_{N-1} + 1/2)D_{-k_{N-1}-1/2}(z_{0})}{z_{0}} \right]
\]

(97)

Repeating the same calculations as before, we will have after the second recurrence step the relations:

\[
\mathcal{Z}_{2}^{\text{cut}} = \frac{1}{\sqrt{2(1+b\Delta^{2}/c)\omega_{0}}} \frac{1}{\sqrt{2(1+b\Delta^{2}/c)\omega_{1}}} \frac{(z/z_{2})^{k_{N-3}+1/2}}{\sqrt{2(1+b\Delta^{2}/c)}} \frac{\Gamma(k_{N-3} + 1/2)}{\Gamma(k_{N-3} + 2)} \sum_{j_{1}=0}^{J_{1}} \sum_{j_{2}=0}^{J_{2}} \frac{(\frac{-1}{\mu!(2\pi)^{\mu}})}{\omega_{c}^{2j_{1}} \omega_{1}^{2j_{2}}} \Gamma(k_{N-3} + i_{2} + 1/2)D_{-k_{N-3}-i_{2}-1/2}(z_{2})
\]

\times \sum_{i_{1}=0}^{2j_{1}} a_{i_{1}}^{2j_{1}} \left( \frac{\xi^{2}}{4z_{0}z_{1}} \right)^{i_{1}} \sum_{i_{2}=0}^{2j_{2}+i_{1}} a_{i_{2}}^{2j_{2}+i_{1}} \left( \frac{\xi^{2}}{4z_{1}z_{2}} \right)^{i_{2}} \Gamma(k_{N-3} + i_{2} + 1/2)D_{-k_{N-3}-i_{2}-1/2}(z_{2})
\]

(98)

In contrary to the method in the Appendix 1, we don’t combine the summations over the asymptotic expansion indices. We see, that by this procedure we replace the summations over indices \( k_{i} \) by summations over the asymptotic expansions indices \( j_{i} \). Taking thoroughly into account the last summation term, we find:

\[
\mathcal{Z}_{N}^{\text{cut}} = \left\{ \prod_{i=1}^{N} \frac{1}{\sqrt{2(1+b\Delta^{2}/c)\omega_{i}}} \right\}^{-1/2} \sum_{j_{1},\cdots,j_{N}} \left[ \prod_{i=1}^{N} \frac{(\frac{-1}{\mu!(2\pi)^{\mu}})}{\omega_{c}^{2j_{i}} \omega_{i+1}} \right] \times \sum_{i_{1}=0}^{2j_{1}} \left( \frac{2j_{1}}{i_{1}} \right) (1/2)^{2j_{1}} \frac{A^{2}}{\omega_{0}^{2j_{1}}} \frac{1}{\omega_{1}^{i_{1}}}
\]

\times \cdots
\]
\[ \times \sum_{i_\mu=0}^{2j_\mu+i_\mu-1} \left( 2j_\mu + i_\mu - 1 \right) \left( 1/2 \right)_{2j_\mu+i_\mu-1} \left( \frac{A^2}{\omega_{i_\mu-1} \omega_{i_\mu}} \right)^{i_\mu} \]

\[ \times \cdots \times \sum_{i_{N-1}=0}^{2j_{N-1}+i_{N-2}} \left( 2j_{N-1} + i_{N-2} \right) \left( 1/2 \right)_{2j_{N-1}+i_{N-2}} \left( \frac{A^2}{\omega_{N-2} \omega_{N-1}} \right)^{i_{N-1}} \times \left( 1/2 \right)_{2j_{N-1}} \left( 1/2 + i_{N-1} \right)_{2j_N} \]

(99)

In the last equation the identities were used:

\[ \xi^2 = \frac{A^2}{\omega_{i_\mu-1} \omega_{i_\mu}} \]

where

\[ A = \frac{\xi}{2 \varepsilon} = \frac{1}{2(1 + b \Delta^2 / c)} \]

and

\[ a_i^k = \binom{k}{i} \frac{1/2}{i} \]

By the useful identity

\[ \left( 1/2 \right)_{2j_\mu+i_\mu-1} = \left( 1/2 \right)_{i_\mu-1} \left( 1/2 + i_\mu \right)_{2j_\mu} \]

we convert our result to the more reliable form for the consecutive calculations:

\[ Z_N^{cut} = \left\{ \prod_{i=1}^{N} [2(1 + b \Delta^2 / c) \omega_i] \right\}^{-1/2} \sum_{j_1, \cdots, j_N} \left[ \prod_{i=1}^{N} \left( \frac{(-1)^{j_i}}{(j_i)! (2 \varepsilon^2)^{j_i} \omega_i^{2j_i}} \right) \right] \left( 1/2 \right)_{2j_1} \]

\[ \times \sum_{i_1=0}^{2j_1} \left( \frac{2j_1}{i_1} \right) \left( 1/2 + i_1 \right)_{2j_2} \left( \frac{A^2}{\omega_{i_1} \omega_1} \right)^{i_1} \]

\[ \times \cdots \times \sum_{i_\mu=0}^{2j_\mu+i_\mu-1} \left( 2j_\mu + i_\mu - 1 \right) \left( 1/2 + i_\mu \right)_{2j_{\mu+1}} \left( \frac{A^2}{\omega_{i_\mu-1} \omega_{i_\mu}} \right)^{i_\mu} \]

\[ \times \cdots \times \sum_{i_{N-1}=0}^{2j_{N-1}+i_{N-2}} \left( 2j_{N-1} + i_{N-2} \right) \left( 1/2 + i_{N-1} \right)_{2j_N} \left( \frac{A^2}{\omega_{N-2} \omega_{N-1}} \right)^{i_{N-1}} \]

(100)

In the calculation as follows the key role plays the objects \( \omega_i \) defined by recurrence relation

\[ \omega_i = 1 - \frac{A^2}{\omega_{i-1}} \]

with the first term

\[ \omega_0 = 1/2 + B \]

where

\[ B = \frac{b \Delta^2 / c}{2(1 + b \Delta^2 / c)} \]

The \( \omega_i \) such defined are represented by the continued fraction. The continued fraction can be represented by the simpler relation as the solution of the \( n - \text{th} \) convergent problem of the continued fraction [12]. Let us shortly explain this procedure.
Let us have a continued fraction of the form:
\[
\omega_n = a_1 + \frac{b_1}{a_2 + \frac{b_2}{a_3 + \cdots}}
\]
The \(n\)–th convergent is defined as
\[
\omega_n = \frac{p_n}{q_n}
\]
where \(p_n\) and \(q_n\) are defined by equations:
\[
\begin{align*}
p_n &= a_n p_{n-1} + b_n p_{n-2} \\
q_n &= a_n q_{n-1} + b_n q_{n-2}
\end{align*}
\]
(102)
The solutions of these recurrence equations have the form:
\[
\begin{align*}
p_n &= \tilde{w}_1 \rho_1^n + \tilde{w}_2 \rho_2^n \\
q_n &= w_1 \rho_1^n + w_2 \rho_2^n
\end{align*}
\]
(103)
where \(\rho_{1,2}\) are the solutions of the characteristic equation, in our case homogenous one:
\[
\rho^2 - a_n \rho - b_n = 0
\]
For the continued fraction we have:
\[
\begin{align*}
a_n &= 1 \\
b_n &= -A^2
\end{align*}
\]
and the solution of the characteristic equation would be:
\[
\rho_{1,2} = \frac{1}{2} \left(1 \pm \sqrt{1 - 4A^2}\right)
\]
The constants \(\tilde{w}_1\) and \(w_1\) are fixed by \(\omega_0\) and \(\omega_1\) terms, that adjust the initial conditions:
\[
\begin{align*}
p_0 &= 1 + 2B \\
p_1 &= 1 + 2B - A^2 \\
q_0 &= 2 \\
q_1 &= 1 + 2B
\end{align*}
\]
The \(n\)–th convergent method solution is completed by the relations:
\[
\begin{align*}
\tilde{w}_{1,2} &= \frac{1}{2} \left[(1 + 2B) \pm \left(\sqrt{1 - 4A^2} + \frac{2B}{\sqrt{1 - 4A^2}}\right)\right] \\
w_{1,2} &= 1 \pm \frac{2B}{\sqrt{1 - 4A^2}}
\end{align*}
\]
The very important characteristic follows from the above solution:
\[
p_n = q_{n+1}
\]
which simplify our calculation significantly.
We are seeking for an expansion of the functional determinant. It look reasonable to arrange the relation following the condition
\[
\sum j_i = K = \text{constant}
\]
where by the value of this constant is labeled the term composed by the sum of all the terms of Eq. (101) suitable satisfying to the above condition. We can prove immediately, that if this constant, let us choose $K$, is $K = 0$ the contributing term is only one, it is the relation where all the $j_i = 0$ and value of this contribution is 1. Let us study the nontrivial case, when $K = 1$. The contributions will be all the members of the multiple sum (101) where only one index $j_i$ is nonzero and equal to one. By some algebra, we can express all the contributions as the sum:

$$Z_N^1 = \left\{ \prod_{i=1}^N [2(1 + b\Delta^2/c)\omega_i] \right\}^{-1/2} S_N$$

(104)

where

$$S_N \ (j_1 + j_2 + \cdots + j_N = 1) = 1 + \frac{-1}{1! 2 \delta_{2,2}} (1/2)^2$$

(105)

In this relation, the term $(2 - \delta_{m,l})$ is due to the binomial coefficients in Eq. (101). Following the result for $\omega_i$ and the functions $p_i$ and $q_i$ we can calculate the product:

$$\prod_{\alpha=k}^{m-1} \left( \frac{A^2}{\omega_{\alpha-1} \omega_{\alpha}} \right) = \prod_{\alpha=k}^{m-1} \left( \frac{A^2}{\omega_{\alpha-1} \omega_{\alpha}} \right) = \frac{Q_{k-1} Q_k}{Q_{m-1} Q_m}$$

(106)

where we have introduced the more convenient variables:

$$Q_i = \frac{q_i}{A^i} = w_1 \left( \frac{p_1}{A} \right)^i + w_2 \left( \frac{p_2}{A} \right)^i$$

and also we performed the replacement:

$$\frac{A^2}{\omega_{k-1}} = \frac{Q_{k-1}^2}{Q_k^2}$$

Inserting the last results into Eq. (105) we find:

$$S_N(j_1 + j_2 + \cdots + j_N = 1) = 1 - \frac{(1/2)^2}{2 A^2 \delta_{2,2}} \sum_{k=1}^{N-1} \left[ 2 \sum_{m=k}^{N-1} \sum_{l=m}^{N-1} \frac{Q_{k-1}^4}{Q_{m-1} Q_{l-1} Q_{Q_{l}^4}} - \sum_{l=k}^{N-1} \frac{Q_{k-1}^4}{Q_{l-1}^4} \right]$$

(107)

In the above equation we have the object symmetric in the indices $m$ and $l$

$$S_{m,l} = \frac{1}{Q_{m-1} Q_{l-1} Q_l}$$

This symmetry significantly simplifies the calculation of the double summation, because we can proceed as:

$$\sum_{m=k}^{N-1} \sum_{l=m}^{N-1} S_{m,l} = 1/2 \left\{ \sum_{m=k}^{N-1} \sum_{l=m}^{N-1} S_{m,l} + \sum_{l=k}^{N-1} \sum_{m=l}^{N-1} S_{l,m} \right\}$$

In the second term we interchange the order of the summations by identity:

$$\sum_{l=k}^{N-1} \sum_{m=l}^{N-1} S_{l,m} = \sum_{m=k}^{N-1} \sum_{l=m}^{N-1} S_{l,m}$$

Finally, we find:

$$\sum_{m=k}^{N-1} \sum_{l=m}^{N-1} S_{l,m} = 1/2 \left\{ \sum_{m=k}^{N-1} \sum_{l=m}^{N-1} S_{m,l} - \sum_{m=k}^{N-1} \sum_{l=m}^{N-1} S_{m,m} \right\}$$

(108)
where the identity can be used:

\[
\sum_{m=k}^{N-1} \sum_{l=k}^{N-1} S_{m,l} = \left(\sum_{l=k}^{N-1} \frac{1}{Q_{l-1}Q_{l}}\right)^2
\]

To calculate the sum, we introduce the relation:

\[
\tilde{Q}_i = w_1x^i - w_2y^i
\]

Let us remind, that

\[
Q_i = w_1x^i + w_2y^i
\]

where

\[
x, y = \frac{\rho_{1,2}}{A} = \frac{1}{2A} \pm \sqrt{\frac{1}{4A^2} - 1}
\]

Using the identity:

\[
\frac{\tilde{Q}_i}{Q_i} - \frac{\tilde{Q}_{i-1}}{Q_{i-1}} = \frac{2w_1w_2(x - y)}{Q_{i-1}Q_i}
\]

we are able to calculate the sum:

\[
\sum_{l=k}^{N-1} \frac{1}{Q_{l-1}Q_l} = \frac{1}{2w_1w_2(x - y)} \left(\frac{\tilde{Q}_{N-1}}{Q_{N-1}} - \frac{\tilde{Q}_{k-1}}{Q_{k-1}}\right)
\]

Inserting these intermediate results into Eq. (107), we find:

\[
S_N(j_1 + j_2 + \cdots + j_N = 1) = 1 - \frac{(1/2)2}{2A^2} \sum_{k=1}^{N-1} Q_{k-1}^2 \left[\left(\sum_{l=k}^{N-1} \frac{1}{Q_{l-1}Q_l}\right)^2 - 2 \sum_{l=k}^{N-1} \frac{1}{Q_{l-1}Q_l^2}\right]
\]

The sum over \(k\) is the problem of the sum of the finite power series. For the finite \(N\) we found the result:

\[
S_N (j_1 + j_2 + \cdots + j_N = 1) = 1 - \frac{(1/2)2}{2A^2} \frac{1}{(x - y)^3}
\]

\[
\times \left\{\frac{x^{2N-2} - y^{2N-2}}{(x + y)} \left[\frac{1}{x^2(x^{N-1} + \frac{w_1}{w_2}y^{N-1})} + \frac{1}{y^2(y^{N-1} + \frac{w_1}{w_2}x^{N-1})}\right]
\]

\[
+ \frac{2}{x(x^{N-1} + \frac{w_1}{w_2}y^{N-1})^2} - \frac{(1 - y^{2N-2})}{y(y^{N-1} + \frac{w_1}{w_2}x^{N-1})^2}\right]\]

\[
+ \frac{(x - y)(N - 1)}{2} \left\{\frac{(x^{N-1} - \frac{w_1}{w_2}y^{N-1})^2}{(x^{N-1} + \frac{w_1}{w_2}y^{N-1})^2} - 1\right\}
\]

In the above relation we have all the symbols defined in the previous text. The continuum limit means to take the limit \(N \to \infty\) in all formulas obtained for finite \(N\). Doing this, we find for symbols appeared in (110):

\[
\frac{w_2}{w_1} \to 1
\]

\[
x - y \to 2\Delta\gamma
\]

\[
x, y \to 1
\]

\[
x^N \to e^{\beta\gamma}
\]

\[
y^N \to e^{-\beta\gamma}
\]

\[
\gamma \to \sqrt{\frac{2b}{c}}
\]

\[
z^{-2} \to 2a\Delta^3/c^2
\]

\[
\Delta \to \frac{\beta}{N}
\]
Inserting into the Eq. (110) we find finally for the first nontrivial term in the continuum limit the result:

\[ S_N(j_1 + j_2 + \cdots + j_N = 1) = 1 - \frac{1}{2} \left( 1 + \frac{a}{e^{\beta \gamma}} \right) \left\{ \frac{e^{2\beta \gamma} - e^{-2\beta \gamma}}{(e^{\beta \gamma} + e^{-\beta \gamma})^2} + \beta \gamma \left[ \frac{(e^{\beta \gamma} - e^{-\beta \gamma})^2}{(e^{\beta \gamma} + e^{-\beta \gamma})^2} - 1 \right] \right\} \] (111)

In derivation of the above relation no connection between the variables \( a \) and \( b \) has been supposed. The relation is finite in the whole range of the values of \( b \), positive, zero or negative. For large \( b \) and moderate \( a \) our result corresponds to the results of the conventional perturbative method of calculation.

**APPENDIX 3: THE EVALUATION OF THE FUNCTIONAL INTEGRAL BY THE GELFAND-YAGLOM PROCEDURE**

The well-known method of the evaluation of the functional integral for the harmonic oscillator in the continuum limit was proposed by Gelfand-Yaglom [2]. We apply the same idea for the evaluation of the functional integral with the fourth order term in the potential.

In the \( N\)th approximation for such functional integral we find in the Appendix 1 the result (106) and in the Appendix 2 the approximation of this result (104), (109). The common form of the both relations for \( N\)th approximation can be expressed as:

\[ Z_N = \frac{S_{N-1}(\Delta)}{\sqrt{\prod_{i=0}^{N-1} 2(1 + b\Delta^2/c)\omega_i}} \]

The value of the functional integral in the continuum limit is defined as

\[ Z = \lim_{N \to \infty} Z_N \]

Let us define the function

\[ F_k = \frac{\prod_{i=0}^{k} 2(1 + b\Delta^2/c)\omega_i}{S_k^2(\Delta)} \] (112)

The function \( S_k(\Delta) \) is the result after first \( k \) steps od the calculation of the function \( S_N(\Delta) \), as it was demonstrated in the Appendices 1 and 2. Let us stress that

\[ Z_N = \frac{1}{\sqrt{F_N}} \]

The aim of the Gelfand-Yaglom construction is to find in the continuum limit the differential equation, such that its solution is connected to the continuum functional integral by:

\[ Z = \frac{1}{\sqrt{F(\beta)}} \]

In the sense of the Gelfand-Yaglom construction we are going to express the relation for \( F_{k+1} \) by help of \( F_k \) and \( F_{k-1} \). We have:

\[ F_{k+1} = \frac{2(1 + b\Delta^2/c) \left[ \prod_{i=0}^{k} 2(1 + b\Delta^2/c)\omega_i - \prod_{i=0}^{k-1} 2(1 + b\Delta^2/c)\omega_i \right]}{S_{k+1}^2(\Delta)} \] (113)

To obtain the above equation we used (112) and the identities:

\[ \omega_i = 1 - A^2/\omega_{i-1}, \omega_0 = 1/2 + b\Delta^2/c, A = \frac{1}{2(1 + b\Delta^2/c)} \].
To introduce $F_k$ and $F_{k-1}$, we must to express $S_{k+1}^2(\triangle)$ by help of $S_k^2(\triangle)$ and $S_{k-1}^2(\triangle)$ in the corresponding terms. This means, that we use:

\[
S_{k+1}(\triangle) = S_k(\triangle) + (S_{k+1}(\triangle) - S_k(\triangle)) \\
S_{k-1}(\triangle) = S_{k-1}(\triangle) + (S_{k+1}(\triangle) - S_{k-1}(\triangle))
\] (114)

In the case of the calculation as in the Appendix 2 the functions $S_k(\triangle)$ can be calculated without approximations. The explicit calculations shown, that $(S_{k+1}(\triangle) - S_k(\triangle))/S_k(\triangle) \approx \triangle$ also for the non-perturbative anzatz of the Appendix 1. Therefore we expands the denominators in (113) up to the terms of the second power in $\triangle$ and we obtain:

\[
\frac{1}{S_{k+1}(\triangle)^2} = \frac{1}{S_k(\triangle)^2} \left[ 1 - 2 \frac{S_{k+1}(\triangle) - S_k(\triangle)}{S_k(\triangle)} + 3 \left( \frac{S_{k+1}(\triangle) - S_k(\triangle)}{S_k(\triangle)} \right)^2 + \ldots \right]
\] (115)

\[
\frac{1}{S_{k-1}(\triangle)^2} = \frac{1}{S_k(\triangle)^2} \left[ 1 - 2 \frac{S_{k+1}(\triangle) - S_{k-1}(\triangle)}{S_{k-1}(\triangle)} + 3 \left( \frac{S_{k+1}(\triangle) - S_{k-1}(\triangle)}{S_{k-1}(\triangle)} \right)^2 + \ldots \right]
\]

After some algebra we we find for the Eq.(113) the difference equation:

\[
\frac{F_{k+1} - 2F_k + F_{k-1}}{\triangle^2} + 4 \frac{F_k - F_{k-1}}{\triangle} \frac{S_{k+1} - S_k}{\triangle S_k} = F_k \left[ 2b/c - 2 \frac{S_{k+1} - S_k}{\triangle S_k} - 2 \left( \frac{S_{k+1} - S_k}{\triangle S_k} \right)^2 \right]
\] (116)

In the continuum limit $\triangle \to 0$ we use the notation

\[
k\triangle \to \tau
\]

and we finally find the differential equation:

\[
\frac{\partial^2}{\partial \tau^2} F(\tau) + 4 \frac{\partial}{\partial \tau} F(\tau) \frac{\partial}{\partial \tau} \ln S(\tau) = F(\tau) \left( \frac{2b}{c} - 2 \frac{\partial^2}{\partial \tau^2} \ln S(\tau) - 4 \frac{\partial}{\partial \tau} \ln S(\tau)^2 \right)
\] (117)

The initial conditions are:

\[
F(0) = 1, \\
\frac{\partial}{\partial \tau} F(0) = 0.
\] (118)

By substitution

\[
F(\tau) = \frac{y(\tau)}{S^2(\tau)}
\]

the differential equation can be read in more convenient form:

\[
\frac{\partial^2}{\partial \tau^2} y(\tau) = \frac{2b}{c} y(\tau)
\] (119)

The importance of the thorough evaluation of the function $S(\tau)$ emerged in this construction of generalized Gelfand-Yaglom equation.

**APPENDIX 4: CONDITIONAL WIENER MEASURE**

The conditional measure functional integrals define the propagators in the theory, contrary to the unconditional measure functional integrals, which define the partition function. The conditional Wiener measure functional integral is defined as the continuous limit of the finite dimensional integral with fixed endpoints:

\[
I = \lim_{N \to \infty} \int_{-\infty}^{+\infty} \prod_{i=1}^{N} \left( \frac{d\varphi_i}{2\pi c} \right) \exp \left\{ - \sum_{i=1}^{N} \triangle \left[ c/2 \left( \frac{\varphi_i - \varphi_{i-1}}{\triangle} \right)^2 + b\varphi_i^2 + a\varphi_i^4 \right] \right\}
\] (120)
The only difference between the unconditional and the conditional definition inhere in the dimension of the finite dimensional integral, whereas the actions are the same. For conditional case, both the endpoints of \( \varphi \) in the time variable are fixed. This correspond to calculation of the propagator of the model, whereas the unconditional measure integral correspond to calculations of the partition functions. The result of the finite dimensional integration in Eq.\[120\], up to index \( k_{N-2} \) is the same as for the conditional measure case. For the last time slice integration, for \( k_{N-1} \) we have:

\[
\sum_{k_{N-1}=0}^{\infty} \frac{(c/\pi^2N)^{2k_{N-1}}}{(2k_{N-1})!} \int_{-\infty}^{\infty} \frac{d\varphi_{N-1}}{2\pi\Delta} (\varphi_{N-1})^{2k_{N-2}+2k_{N-1}} \exp\{-a\Delta\varphi_{N-1}^4 + c/\Delta(1 + b\Delta^2/c)\varphi_{N-1}^2\} \tag{121}
\]

The conditional measure result we derive from the result for the unconditional case, when the term depending on the summation index \( k_{N-1} \) we replace by the term:

\[
\sum_{k_{N-1}=0}^{\infty} \frac{(c\varphi_{N}^2/2\pi(1+b\Delta/c))^k_{N-1}}{(2k_{N-1})!} \Gamma(k_{N-2} + k_{N-1} + 1/2)D_{-k_{N-2}-k_{N-1}-1/2} (\xi) e^{-\{a\Delta\varphi_{N}^4 + \xi(1/2+b\Delta^2/c)\varphi_{N}^2\}} \tag{122}
\]

In this appendix we do not follow by discussion for the arbitrary value of the \( \varphi_{N} \), we are going to discuss the simplest case of the \( \varphi_{N} = 0 \), corresponding to the periodic boundary conditions in the imaginary time variable. The result of the \( N-1 \) dimensional integration can be read:

\[
\sum_{k_1,\ldots,k_{N-1}}^{\infty} \prod_{i=1}^{N-1} \left[ \frac{1}{\sqrt{2\pi(1+b\Delta/c)}} \frac{(\xi/z)^{2k_i}}{(2k_i)!} \Gamma(k_{i-1} + k_i + 1/2)D_{-k_{i-1}-k_i-1/2} (\xi) e^{-\{a\Delta\varphi_{N}^4 + \xi(1/2+b\Delta^2/c)\varphi_{N}^2\}} \right] \tag{123}
\]

In the last equation the identity \( k_0 = 0 \) is required. The main difference from the unconditional measure calculation consists in the fact, that the arguments of the parabolic cylinder functions in the Eq.\[123\] are the same in contrary to unconditional measure case, where the argument of the last function differs from the all others arguments of the \( D \) functions. We are going to repeat the calculation as in the Appendix 2, to show the differences of the unconditional measure calculations from conditional measure ones for the first two terms of the decomposition of the corresponding functional integrals. We again introduce the objects as for unconditional measure case, but corresponding to the conditional measure specifications:

\[
\omega_i = 1 - \frac{A^2}{\omega_{i-1}}
\]

with

\[
\omega_0 = 1.
\]

Following the unconditional measure calculation, by the n-th convergent method we find: If \( \omega_n \) is the continued fraction defined by:

\[
\omega_n = a_1 + \frac{b_1}{a_2 + \frac{b_2}{a_3 + \cdots}}
\]

then we can find the \( \omega_n \) in the form of the n-th convergent:

\[
\omega_n = \frac{p_n}{q_n}
\]

where \( p_n \) and \( q_n \) are the solutions of the homogeneous difference equations with the constant coefficients:

\[
p_n = a_n p_{n-1} + b_n p_{n-2}
\]

\[
q_n = a_n q_{n-1} + b_n q_{n-2}
\]

\[\tag{124}\]
The solutions have the form:
\begin{align*}
p_n &= \tilde{w}_1 \rho_1^n + \tilde{w}_2 \rho_2^n \\
q_n &= w_1 \rho_1^n + w_2 \rho_2^n
\end{align*}
(125)

where \(\rho_{1,2}\) are the solutions of the characteristic equation:
\[\rho^2 - a_n \rho - b_n = 0\]

where
\[a_n = 1\]
\[b_n = -A^2.\]

The boundary conditions for \(p_n, q_n\) are fixed by the first two relations for \(\omega_n\):
\[\omega_0 = 1\]
\[\omega_1 = 1 - A^2\]

and we have:
\begin{align*}
p_0 &= 1 \quad (126) \\
p_1 &= 1 - A^2 \\
q_0 &= 1 \quad (128) \\
q_1 &= 1 \quad (129)
\end{align*}

The \(n\) -th convergent solution for \(\omega_n\) is thus given by the relations:
\begin{align*}
\rho_{1,2} &= \frac{1}{2} \left( 1 \pm \sqrt{1 - 4A^2} \right) \quad (130) \\
\tilde{w}_{1,2} &= \frac{1}{2} \left( 1 \pm \frac{1 - 2A^2}{\sqrt{1 - 4A^2}} \right) \quad (131) \\
w_{1,2} &= \frac{1}{2} \left( 1 \pm \frac{1}{\sqrt{1 - 4A^2}} \right) \quad (132)
\end{align*}

We see from the last relations the very powerful characteristic of the solution represented by the identity:
\[p_n = q_{n+1}, \ n = 1, 2, \ldots\]

Expanding the \(N - 1\) dimensional integral for the conditional measure into the asymptotic series, we find the relation:
\[S_N = \left\{ \prod_{i=1}^{N-1} \left[ 2(1 + b \Delta^2 / c) \omega_i \right] \right\}^{-1/2} Z_N \quad (133)\]

The first part of this relation in the continuum limit is the conditional measure functional integral determinant for the harmonic oscillator type action. The second part, marked \(Z_N\), is the asymptotic expansion correction. As for the unconditional measure integral we calculated the first two corrections terms and we find:
\[1 - \frac{(1/2)^2}{2z^2} \frac{1}{(x - y)^3} \quad (134)\]

\[\times \left\{ \frac{x^{2N-2} - y^{2N-2}}{(x + y)} \left[ \frac{1}{x^2(x^{N-1} + \frac{w_2}{w_1} y^{N-1})^2} + \frac{1}{y^2(y^{N-1} + \frac{w_1}{w_2} x^{N-1})^2} \right] \right. \]
\[\left. + 2 \left[ \frac{1 + \frac{w_2}{w_1} x^{2N-2} - \frac{w_1}{w_2} y^{2N-2}}{x(x^{N-1} + \frac{w_2}{w_1} y^{N-1})^2} - \frac{1 + \frac{w_1}{w_2} y^{2N-2} - \frac{w_2}{w_1} x^{2N-2}}{y(y^{N-1} + \frac{w_1}{w_2} x^{N-1})^2} \right] \right. \]
\[\left. + \frac{(x - y)(N - 1)}{2} \left[ \frac{3(y^{N-1} - \frac{w_2}{w_1} y^{N-1})^2}{(x^{N-1} + \frac{w_2}{w_1} y^{N-1})^2} - 1 \right] \right\} \]
Above relation possesses the well defined continuum limit. We find, in the limit \( N \to \infty \) the following relations:

\[
\frac{w_2}{w_1} \to -1 \\
x - y \to 2\Delta \gamma \\
x, y \to 1 \\
x^N \to e^{\beta \gamma} \\
y^N \to e^{-\beta \gamma} \\
\gamma \to \frac{2b}{c} \\
z^{-2} \to 2a\Delta^3/c^2 \\
\Delta \to \frac{\beta}{N}
\]

then, finally, in the continuum limit the first nontrivial correction to \( Z_N \) is:

\[
1 - \left( \frac{1}{2} \right)^{\frac{1}{2}} \frac{\beta}{c^2} \frac{1}{8\gamma^3 (e^{\gamma \beta} - e^{-\gamma \beta})^2} \left\{ -3(e^{2\gamma \beta} - e^{-2\gamma \beta}) + 2\gamma \beta \left[ e^{2\gamma \beta} + e^{-2\gamma \beta} + 4 \right] \right\} 
\]

or,

\[
1 - \frac{3a}{32c^2 \gamma^3} \left\{ -3 \coth(\gamma \beta) + 2\gamma \beta \left[ \coth^2(\gamma \beta) + \frac{1}{2 \sinh^2(\gamma \beta)} \right] \right\} 
\]

We can stress, that our result is valid for finite values of the parameter "\( b \)" positive or negative, so we describe the case of the anharmonic oscillator and the Higgs model as well.

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