Central elements and the Gaeta topos: An algebraic and functorial overview on coextensive varieties

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Abstract

In this paper we show that within the context of coextensive varieties, the functor of central elements is representable. In addition, we use the theory of central elements to establish a criterion for fp-coextensive varieties that allows to decide whether the Gaeta Topos classifies indecomposable objects in terms of the indecomposability of the free algebra on one generator.

1 Introduction

Extensive categories were introduced in [24] as categories $C$ with finite coproducts and pullbacks in which the canonical functor $+: C/X \times C/Y \to C/(X + Y)$ is an equivalence, for every pair of objects $X$ and $Y$ of $C$. A category is said to be coextensive if its opposite is extensive. Coextensive varieties (as algebraic categories) are of interest because according to [25] and more recently [29], they bring an appropriate setting to develop algebraic geometry. In [36] it was shown that the theory central elements ([3], [31], [34]) can be taken as an accurate tool to study coextensive varieties. This perspective comes from the intuition that varieties with well behaved products can be algebraically described by analyzing the peculiarities on the theory that describes the behavior of the elements which concentrate the information about finite product decompositions of its algebras. Under certain circumstances, it turns out that central elements can be treated functorially. As far as we know, this approach has not yet been exploited. Most of all, because the theory of central elements was originally constrained into the realm of universal algebra.

Small extensive categories admit a particular subcanonical topology called the Gaeta topology (seemingly named in honor of the spanish algebraic geometer Federico Gaeta). This topology has to do with all the possible decompositions of objects into finite coproducts. In concrete examples ([35], [29]), it has been proved that the Gaeta topos is the classifying topos of the theory of connected objects, which can be considered as the ones who does not admit non-trivial
binary coproduct decompositions. Naturally, when considering coextensive categories, the Gaeta topology and the Gaeta topos are related with decompositions into finite products and indecomposable objects. Nevertheless, in the practice it seems not quite easy to provide an axiomatization of the theory of indecomposable objects when regarding varieties in a more general setting. Considering the characterization given in \[36\], it seems natural to wander if from the tools provided by universal algebra and the theory of central elements it is possible to determine, given a coextensive variety, whether the Gaeta topos classifies indecomposable objects.

This paper is organized as follows. Section 2 presents the most part of the definitions and basic results required for reading this work. Section 3 is devoted to the functorial treating of central elements in coextensive varieties. Section 4 deals with the characterization of those fp-coextensive varieties such that the Gaeta topos classifies indecomposable objects. The paper comes to an end with an application that connects coBoolean varieties with the Beth (definability) property for algebraizable logics. The reader is assumed to be familiar with some standard topos theory as presented in \[20\] and \[21\]. For standard notions in universal algebra the reader may consult \[28\].

2 Preliminaries

2.1 Notation and basic results

Let \( A \) be a set and \( k \) be a natural number. We write \( \vec{a} \) for an element \((a_1, \ldots, a_k) \in A^k\). If \( f : A \to B \) is a function and \( \vec{a} \in A^k \), then we write \( f(\vec{a}) \) for the element \((f(a_1), \ldots, f(a_k)) \in B^k\). If \( X \subseteq A \) we write \( f|_X \) for the restriction of \( f \) to \( X \), \( \mathcal{P}(X) \) for the power set of \( X \) and \( f[X] \) for the image of \( X \) through \( f \). If \( \vec{a} \in A^k \) and \( \vec{b} \in B^k \), we write \([\vec{a}, \vec{b}]\) for the k-tuple \(((a_1, b_1), \ldots, (a_k, b_k)) \in (A \times B)^k\). If \( g : A \times B \to C \) is a function and \([\vec{a}, \vec{b}]\in (A \times B)^k\) then we write \( g(\vec{a}, \vec{b}) \) for the element \((g(a_1, b_1), \ldots, g(a_k, b_k)) \in C^k\). If \( A \) is an algebra of a given type we denote its universe by \( A \) and its congruence lattice by \( \text{Con}(A) \). If \( \theta \in \text{Con}(A) \), and \( \vec{a} \in A^k \) we write \( \vec{a}/\theta \) for the k-tuple \( (a_1/\theta, \ldots, a_k/\theta) \in (A/\theta)^k \). The universal congruence on \( A \) and identity congruence on \( A \) are denoted by \( \nabla^A \) and \( \Delta^A \), respectively. If \( S \subseteq A \times A \), we write \( \text{Cg}^A(S) \) for the congruence generated by \( S \). We also write \( \text{Cg}^A(\vec{a}, \vec{b}) \), for the congruence generated by all pairs \((a_1, b_1), \ldots, (a_k, b_k)\) where \( \vec{a}, \vec{b} \in A^k \). We say that a congruence \( \theta \) on \( A \) is finitely generated if \( \theta = \text{Cg}^A(F) \) for some finite set \( F \subseteq A \times A \). We use \( \text{FC}(A) \) to denote
Lemma 2. For its proof, the reader may consult [33].

A variety \( \mathcal{V} \) has the Fraser-Horn property \[13] if for every \( \mathbf{A}_1, \mathbf{A}_2 \in \mathcal{V} \), it is the case that every congruence \( \theta \) in \( \mathbf{A}_1 \times \mathbf{A}_2 \) is the product congruence \( \theta_1 \times \theta_2 \) for some congruences \( \theta_1 \) of \( \mathbf{A}_1 \) and \( \theta_2 \) of \( \mathbf{A}_2 \). If \( \theta, \lambda \in \text{Con}(\mathbf{A}) \) and \( \theta \subseteq \lambda \), we write \( \lambda/\theta \) for the set of pairs \( (x/\theta, y/\theta) \) of \( \mathbf{A}/\theta \) such that \( (x, y) \in \lambda \). If \( g : \mathbf{A} \rightarrow \mathbf{B} \) is a homomorphism, we write \( \text{Ker}(g) \) for the kernel of \( g \). I.e. the congruence on \( \mathbf{A} \) defined by the set \( \{(a, b) \in A^2 : g(a) = g(b)\} \). If \( \mathbf{A} \) is an algebra of type \( \mathcal{F} = \{f_1, \ldots, f_m\} \), when required, we will write its type as an \( n \)-tuple \( (a_1, \ldots, a_m) \), where \( a_j \) denotes the arity of \( f_j \), with \( 1 \leq j \leq m \).

The following result is probably folklore but, since we have not found it in the literature, we give some of the details of its proof. It provides a description of the factor congruences of the quotients of an algebra of a given type.

**Lemma 1.** Let \( \mathbf{A} \) be an algebra of a given type and let \( \theta \in \text{Con}(\mathbf{A}) \). Consider the sets

\[
P_\theta = \{ (\lambda, \mu) \mid \theta \subseteq \lambda, \mu; \lambda \cap \mu = \theta, \lambda \circ \mu = \nabla^A \}
\]

and

\[
Z_\theta = \{ (\alpha, \beta) \in \text{FC}(\mathbf{A}/\theta)^2 : \alpha \circ \beta \},
\]

Then, the assignment \( (\lambda, \mu) \mapsto (\lambda/\theta, \mu/\theta) \) defines a bijection between \( P_\theta \) and \( Z_\theta \).

**Proof.** It is a straightforward consequence of Theorems 7.5, 6.15 and 6.20 of [10]. \(\square\)

Given a variety \( \mathcal{V} \) and a set \( X \) of variables we use \( \mathbf{T}_\mathcal{V}(X) \) for the term algebra of \( \mathcal{V} \) over \( X \) and \( \mathbf{F}_\mathcal{V}(X) \) for the free algebra of \( \mathcal{V} \) freely generated by \( X \). In particular, if \( X = \{x_1, \ldots, x_m\} \) with \( m \) a non-negative integer and if no clarification is needed, then we write \( \mathbf{T}_\mathcal{V}(m) \) and \( \mathbf{F}_\mathcal{V}(m) \) instead of \( \mathbf{T}_\mathcal{V}(\{x_1, \ldots, x_m\}) \) and \( \mathbf{F}_\mathcal{V}(\{x_1, \ldots, x_m\}) \), respectively. We recall that an algebra \( \mathbf{A} \) in \( \mathcal{V} \) is a *finitely generated free algebra* if it is isomorphic to \( \mathbf{F}_\mathcal{V}(m) \) for some finite \( m \), and *finitely presented* if it is isomorphic to an algebra of the form \( \mathbf{F}_\mathcal{V}(k)/\theta \), for some \( k \) finite and \( \theta \) finitely generated congruence on \( \mathbf{F}_\mathcal{V}(k) \).

The following Lemma is a key result that we will employ repeatedly along Section 4. For its proof, the reader may consult [33].

**Lemma 2.** Let \( \mathcal{V} \) be a variety and let \( X \) be a set of variables. Let \( r, r_1, \ldots, r_m, s, s_1, \ldots, s_m \in \mathbf{T}_\mathcal{V}(X) \). Then, the following are equivalent:

1. \( (r, s) \in \text{CG}^{\mathbf{F}_\mathcal{V}(X)}(\vec{r}, \vec{s}) \);
2. \( \mathcal{V} \models \vec{r} = \vec{s} \implies r = s \).

**Lemma 3.** Let \( \mathcal{V} \) be a variety and let \( p_i(\vec{x}, \vec{y}), q_i(\vec{x}, \vec{y}) \), \( 1 \leq i \leq n \) be terms in the language of \( \mathcal{V} \). Let 

\[ \theta = \bigvee_{i=1}^{n} \text{CG}^{\mathbf{F}_\mathcal{V}(X)}(p_i(\vec{x}, \vec{y}), q_i(\vec{x}, \vec{y})) \]
and \( H = \mathbf{F}_V(X)/\theta \). Let \( A \in V \) and suppose that \( p_i^A(a, \tilde{b}) = q_i^A(a, \tilde{b}) \), for \( 1 \leq i \leq n \). Then, for every \( \tilde{c} \in A^N \) there exists a unique \( \Omega : H \to A \) such that \( \Omega(\tilde{x}/\theta) = a \), \( \Omega(\tilde{y}/\theta) = b \) and \( \Omega(\tilde{z}/\theta) = \tilde{c} \).

Proof. It is straightforward. \( \square \)

Let \( L \) be a first order language. If a \( L \)-formula \( \varphi(\tilde{x}) \) has the form

\[
\bigwedge_{j=1}^{n} p_j(\tilde{x}) = q_j(\tilde{x}),
\]

for some positive number \( n \) and terms \( p_j(\tilde{x}) \) and \( q_j(\tilde{x}) \) in \( L \), then we say that \( \varphi(\tilde{x}) \) is a \( (\bigwedge p = \varnothing) \)-formula. If \( K \) is a class of \( L \)-structures and \( R \in L \) is a \( n \)-ary relation symbol, we say that a formula \( \varphi(x_1, \ldots, x_n) \) defines \( R \) in \( K \) if

\[
K \models \varphi(\tilde{x}) \iff R(\tilde{x}).
\]

In particular, if a \( (\bigwedge p = \varnothing) \)-formula defines \( R \), we say that \( R \) is equationally definable.

Finally, we stress that all varieties considered along this paper always will be assumed as varieties with at least a constant symbol.

### 2.2 Central Elements

By a variety with \( \tilde{0} \) and \( \tilde{1} \) we understand a variety \( V \) in which there are \( 0 \)-ary terms \( 0_1, \ldots, 0_N, 1_1, \ldots, 1_N \) such that \( V \models \tilde{0} \approx \tilde{1} \iff x \approx y \), where \( \tilde{0} = (0_1, \ldots, 0_N) \) and \( \tilde{1} = (1_1, \ldots, 1_N) \). If \( A \in V \) then we say that \( \tilde{e} = (e_1, \ldots, e_N) \in A^N \) is a central element of \( A \) if there exists an isomorphism \( \tau : A \to A_1 \times A_2 \), such that \( \tau(\tilde{e}) = (\tilde{0}_A, \tilde{1}_A) \). Also, we say that \( \tilde{e} \) and \( \tilde{f} \) are a pair of complementary central elements of \( A \) if there exists an isomorphism \( \tau : A \to A_1 \times A_2 \) such that \( \tau(\tilde{e}) = (\tilde{0}_A, \tilde{1}_A) \) and \( \tau(\tilde{f}) = (\tilde{1}_A, \tilde{0}_A) \). We write \( Z(A) \) to denote the set of central elements of \( A \) and \( \varphi_\circ_A \tilde{f} \) to denote that \( \tilde{e} \) and \( \tilde{f} \) are complementary central elements of \( A \). We say that a variety \( V \) with \( \tilde{0} \) and \( \tilde{1} \) has Boolean Factor Congruences (BFC) if the set of factor congruences of any algebra of \( V \) is a Boolean sublattice of its congruence lattice. Let \( V \) be a variety with BFC and \( A \in V \). If \( \tilde{e} \in Z(A) \), we write \( \theta^A_{\tilde{0}, \tilde{e}} \) and \( \theta^A_{\tilde{1}, \tilde{e}} \) for the unique pair of complementary factor congruences satisfying \( [\tilde{e}, \tilde{0}] \in \theta^A_{\tilde{0}, \tilde{e}} \) and \( [\tilde{e}, \tilde{1}] \in \theta^A_{\tilde{1}, \tilde{e}} \). In Theorem 1 of [31] it was proved that the assignment which sends \( \tilde{e} \) into \( \theta^A_{\tilde{0}, \tilde{e}} \) establishes a bijection between \( Z(A) \) and \( FC(A) \). Such a bijection, allows to define some operations in \( Z(A) \) as follows: given \( \tilde{e} \in Z(A) \), the complement \( \tilde{e}^*A \) of \( \tilde{e} \), is the only solution to the equations \( [\tilde{z}, \tilde{1}] \in \theta^A_{\tilde{0}, \tilde{e}} \) and \( [\tilde{z}, \tilde{0}] \in \theta^A_{\tilde{1}, \tilde{e}} \). Given \( \tilde{e}, \tilde{f} \in Z(A) \), the infimum \( \tilde{e} \wedge_A \tilde{f} \) is the only solution to the equations \( [\tilde{z}, \tilde{0}] \in \theta^A_{\tilde{0}, \tilde{e}} \cap \theta^A_{\tilde{0}, \tilde{f}} \) and \( [\tilde{z}, \tilde{1}] \in \theta^A_{\tilde{1}, \tilde{e}} \vee \theta^A_{\tilde{1}, \tilde{f}} \). Finally, the supremum \( \tilde{e} \vee_A \tilde{f} \) is the only solution to the equations \( [\tilde{z}, \tilde{0}] \in \theta^A_{\tilde{0}, \tilde{e}} \vee \theta^A_{\tilde{0}, \tilde{f}} \) and \( [\tilde{z}, \tilde{1}] \in \theta^A_{\tilde{1}, \tilde{e}} \cap \theta^A_{\tilde{1}, \tilde{f}} \). Observe
that these operations makes $Z(A) = (Z(A), \wedge_A, \vee_A, c^A, \bar{a}^A, \overline{1}^A)$ a Boolean algebra which is isomorphic to $(\text{FC}(A), \vee, \cap, *, \Delta^A, \nabla^A)$.

The following result was proved in Lemma 2.1.1. of [2] for the case $N = 1$. Nevertheless, since the arguments used for the case of an arbitrary $N$ does not change the essence of the proof, we omit the details.

**Lemma 4.** Let $\mathcal{V}$ be a variety with BFC and $A \in \mathcal{V}$. For every $\vec{e}, \vec{f} \in Z(A)$, the following holds:

1. $\vec{a} = \vec{e} \wedge_A \vec{f}$ if and only if $[\vec{0}, \vec{a}] \in \theta_{0, \vec{e}}^A$ and $[\vec{a}, \vec{f}] \in \theta_{1, \vec{e}}^A$.

2. $\vec{a} = \vec{e} \vee_A \vec{f}$ if and only if $[\vec{1}, \vec{a}] \in \theta_{1, \vec{e}}^A$ and $[\vec{a}, \vec{f}] \in \theta_{0, \vec{e}}^A$.

Let $\mathcal{V}$ be a variety with BFC. If $A, B \in \mathcal{V}$ and $f : A \rightarrow B$ is a homomorphism, we say that $f$ preserves central elements if the map $f : Z(A) \rightarrow Z(B)$ is well defined; that is to say, for every $\vec{e} \in Z(A)$, it follows that $f(\vec{e}) \in Z(B)$. We say that $f$ preserves complementary central elements if it preserves central elements and for every $\vec{e}_1, \vec{e}_2 \in Z(A)$,

$\vec{e}_1 \circ_A \vec{e}_2 \Rightarrow f(\vec{e}_1) \circ_B f(\vec{e}_2)$.

We say that a variety with BFC is center stable if every homomorphism preserves central elements and we say that it is stable by complements if every homomorphism preserves complementary central elements. In [30] it was shown that these notions are not equivalent.

### 2.3 Algebraizable Logics

The terminology and definitions of this section are based on those of [7], [6], [30] and all the references therein. Let $\mathcal{L}$ be a language of algebras and let $X$ be a countable-infinite set. We write $T_\mathcal{L}(X)$ for the set of terms in $\mathcal{L}$. A logic over $X$ is a pair $\mathcal{L} = \langle \mathcal{L}, \vdash_\mathcal{L} \rangle$ where $\vdash_\mathcal{L} \subseteq T_\mathcal{L}(X) \times \mathcal{P}(T_\mathcal{L}(X))$ is a substitution invariant consequence relation. I.e. $\vdash_\mathcal{L}$ satisfies:

\[
\begin{align*}
\varphi & \in \Gamma \\
\Gamma \vdash_\mathcal{L} \varphi & \text{ and } \Gamma \models \Delta \\
\Gamma \vdash_\mathcal{L} \varphi & \text{ for every } \psi \in \Gamma \\
\Gamma \vdash_\mathcal{L} \varphi & \Rightarrow \sigma[\Gamma] \vdash_\mathcal{L} \sigma(\varphi).
\end{align*}
\]

for every endomorphism $\sigma$ of $T_\mathcal{L}(X)$. In this context $X$ is usually referred as the set of variables of $\mathcal{L}$. The set $T_\mathcal{L}(X) \times T_\mathcal{L}(X)$ is called the set of equations of $\mathcal{L}$ and we denote it by $Eq_\mathcal{L}$. The elements $(\varphi, \psi)$ of $Eq_\mathcal{L}$ are noted as $\varphi \approx \psi$. A transformer from formulas to equations is a function $\tau : T_\mathcal{L}(X) \rightarrow \mathcal{P}(Eq_\mathcal{L})$. A transformer from equations to formulas is a function $\rho : Eq_\mathcal{L} \rightarrow \mathcal{P}(T_\mathcal{L}(X))$. In this case, $\tau$ is said to be structural if for any endomorphism $\sigma$ of $T_\mathcal{L}(X)$ and every $\varphi \in T_\mathcal{L}(X)$ we have $\tau(\sigma(\varphi)) = \sigma[\tau(\varphi)]$. On the other hand, $\rho$ is said to be structural if there is a set of formulas $\Delta(x, y)$ in at most two variables
such that for any \( \varphi, \psi \in T_L(X) \) the condition \( \rho(\varphi \approx \psi) = \Delta(\varphi, \psi) \) holds. A logic \( L \) is algebraizable (in the sense of Blok and Pigozzi) with equivalent variety semantics \( V \) if there are structural transformers \( \tau: T_L(X) \rightarrow \mathcal{P}(Eq_L) \) and \( \rho: Eq_L \rightarrow \mathcal{P}(T_L(X)) \) such that for all \( \Gamma \cup \{ \varphi \} \subseteq T_L(X) \) and \( \Theta \cup \{ \epsilon \approx \delta \} \subseteq Eq_L \), we have:

1. \( \Gamma \vdash_L \varphi \iff \tau \Gamma \models_V \tau \varphi \), and
2. \( \epsilon \approx \delta \iff \models_V \tau \rho(\epsilon \approx \delta) \).

Let \( L \) be a logic and let \( V \) and \( W \) be a disjoint pair of sets of variables such that \( T_L(V) \neq \emptyset \). Let \( \Gamma \subseteq T_L(V \cup W) \). We say that \( \Gamma \) defines \( W \) implicitly in terms of \( V \) in \( L \) if for every set of variables \( Y \), each \( z \in W \) and homomorphism \( h: T_L(V \cup W) \rightarrow T_L(Y) \), such that \( h(x) = x \) for all \( x \in V \), it follows that \( \Gamma \cup h[\Gamma] \models_{\text{Mod}^*([-1])} z \approx h(z) \). We say that \( \Gamma \) defines \( W \) explicitly in terms of \( V \) in \( L \) if for each \( z \in W \), there exists \( \varphi_z \in T_L(V) \) such that \( \Gamma \models_{\text{Mod}^*([-1])} z \approx \varphi_z \). Here \( \text{Mod}^*([-1]) \) denotes the class of all reduced matrix models of \( \vdash_L \) (i.e., \( \Gamma \vdash_L \varphi \) iff \( \models_{\text{Mod}^*([-1])} \varphi \)). We say that a logic \( L \) has the Beth (definability) property if for every disjoint pair of sets of variables \( V \) and \( W \) and \( \Gamma \subseteq T_L(V \cup W) \), if \( \Gamma \) defines \( W \) implicitly in terms of \( V \) in \( L \) then \( \Gamma \) defines \( W \) explicitly in terms of \( V \) in \( L \).

If \( L \) is an algebraizable logic with equivalent variety semantics \( V \), the underlying intuition of Beth’s (definability) property is that epimorphisms in the (algebraic) category \( V \) correspond to implicit definitions in \( L \) and surjections in \( V \) correspond to explicit definitions in \( L \). The following result, originally proved in [20], establishes that such an intuition is in fact an equivalence.

**Theorem 1.** Let \( L \) be an algebraizable logic with equivalent variety semantics \( V \). Then \( L \) has the Beth (definability) property if and only if all the epimorphisms of \( V \) are surjective.

### 3 Representability of the functor \( Z \)

We recall that a category with finite products \( C \) is called coextensive if for each pair of objects \( X \) and \( Y \) of \( C \) the canonical functor \( \times: C/X \times C/Y \rightarrow C/(X \times Y) \) is an equivalence. Classical examples of coextensive categories are the categories \textbf{Ring} and \textbf{dLat} of commutative rings with unit and bounded distributive lattices, respectively. If \( V \) is a coextensive variety, the associated algebraic category will be also denoted by \( V \). In what follows, we write \( 0 \) and \( 1 \) for the initial and terminal algebras of \( V \), respectively. If \( A \in V \) we write \( 1_A: 0 \rightarrow A \) for the unique morphism from \( 0 \) to \( A \) in \( V \). If \( \bar{e} \in Z(A) \) we write \( i_{\bar{e},0} \) and \( i_{\bar{e},1} \) for the unique morphisms from \( 0 \) to \( A/Cg^A(\bar{e},0) \) and from \( 0 \) to \( A/Cg^A(\bar{e},1) \), respectively. Finally, we recall that due to \( V \) is assumed with at least a constant symbol then \( 0 \) is isomorphic to \( F_V(\emptyset) \).
Given a variety \( \mathcal{V} \) with \( \vec{0} \) and \( \vec{1} \), BFC and center stable, it is the case that for every \( A \in \mathcal{V} \) the assignment \( A \mapsto Z(A) \) defines a functor \( Z : \mathcal{V} \to \text{Set} \) in an obvious way. In this section we prove that when \( \mathcal{V} \) is coextensive, such a functor is in fact representable by the algebra \( 0 \times 0 \). This result leads us to show that the functor \( Z \) can also be extended to a functor from \( \mathcal{V} \) to the category \( \text{Boole} \) of Boolean algebras. Moreover, a characterization of coextensivity by means of the functors \( Z \) and \( \hat{\times} \) is provided. The section concludes with an application on Beth (definability) property for logics that has coBoolean varieties as algebraic semantics.

We will begin by recalling some facts about coextensive varieties which will be essential for proving the results of this section. It is well known that every variety (as an algebraic category) has all limits. Therefore, as a restricted dual of Propositions 2.2 and 4.1 of [11] we obtain the following result.

**Proposition 1.** A variety \( \mathcal{V} \) is coextensive if and only if it has pushouts along projections and every commutative diagram

\[
\begin{array}{cccc}
0 & \xrightarrow{\pi_0} & 0 \times 0 & \xrightarrow{\pi_1} & 0 \\
\downarrow{\pi_0} & \downarrow{f} & \downarrow{\pi_1} & \downarrow{\pi_1} & \downarrow{\pi_1} \\
A_0 & \xrightarrow{g_0} & A_0 \times A_1 & \xrightarrow{g_1} & A_1
\end{array}
\]

comprises a pair of pushout squares in \( \mathcal{V} \) just when the bottom row is a product diagram in \( \mathcal{V} \).

We recall that a variety \( \mathcal{V} \) is a Pierce variety [32] if there exist a positive natural number \( N \), 0-ary terms \( 0_1, \ldots, 0_N, 1_1, \ldots, 1_N \) and a term \( U(x, y, z, w) \) such that the identities

\[
U(x, y, \vec{0}, \vec{1}) = x \quad \text{and} \quad U(x, y, \vec{1}, \vec{0}) = y
\]

hold in \( \mathcal{V} \). It is worth mentioning that in a Pierce variety \( \mathcal{V} \), it is also true that \( \theta^A_{0, \vec{e}} = \text{Cg}^A(\vec{0}, \vec{e}) \), for every \( A \in \mathcal{V} \) and every \( \vec{e} \in A \) (see [1] for details).

In [30] the following characterization of coextensive varieties by means of Pierce varieties, the equational definability of the relation “\( \vec{e} \) and \( \vec{f} \) are complementary central elements” and the stability by complements, was provided. It is a key result on which we will constantly rely for carrying on the goals of this section.

**Theorem 2.** Let \( \mathcal{V} \) be a variety. Then, the following are equivalent:

1. \( \mathcal{V} \) is coextensive.
2. \( \mathcal{V} \) is a Pierce variety in which the relation \( \vec{e} \circ_A \vec{f} \) is equationally definable.
3. \( \mathcal{V} \) is a Pierce variety stable by complements.
Let $V$ be a coextensive variety, $A, B \in V$ and $f : A \to B$ be a homomorphism. Observe that from Theorem 2 (3) the assignments

\[
A \mapsto Z(A) \\
f \mapsto f|_{Z(A)}
\]

determine a functor $Z : V \to \text{Set}$. In the following result we prove that such a functor is in fact, representable.

**Theorem 3.** Let $V$ be a coextensive variety. Then, for every $A \in V$ there is a bijection between $Z(A)$ and $V(0 \times 0, A)$. Moreover, the functor $Z : V \to \text{Set}$ is representable by $0 \times 0$.

**Proof.** Let $A \in V$ and consider the assignments $\varphi_A : V(0 \times 0, A) \to Z(A)$ and $\mu_A : Z(A) \to V(0 \times 0, A)$, defined by $\varphi_A(g) = g[\bar{0}, \bar{1}]$ and $\mu_A(\bar{e}) = i_{\bar{0},e} \times i_{\bar{1},e}$, respectively. We claim that $\varphi$ and $\mu$ are natural transformations which are inverse of each other.

We start by showing that $\varphi_A$ and $\mu_A$ are well defined. Since $[\bar{0}, \bar{1}] \in Z(0 \times 0)$, from Theorem 2 (3), $g[\bar{0}, \bar{1}] \in Z(A)$ for every $g \in V(0 \times 0, A)$ so $\varphi_A$ is well defined. Similarly, due to every $\bar{e} \in Z(A)$ induces a product decomposition of $A$, then from the coextensivity of $V$, we get that $i_{\bar{0},e} \times i_{\bar{1},e}$ is the unique morphism in $V(0 \times 0, A)$ making pushouts both squares of the diagram of above, so $\mu_A$ is also well defined.

Now we prove that $\varphi_A$ and $\mu_A$ are mutually inverse. To do so, let $\bar{e} \in Z(A)$, $h = i_{\bar{0},e} \times i_{\bar{1},e}$ and let us consider $\varphi_A(\mu_A(\bar{e})) = h[\bar{0}, \bar{1}]$.[4]

\[
\begin{array}{c}
0 \xrightarrow{\pi_0} 0 \xrightarrow{\pi_1} 0 \\
\downarrow i_{\bar{0},e} \times i_{\bar{1},e} \downarrow i_{\bar{0},e} \times i_{\bar{1},e} \\
A/Cg^A(\bar{0}, \bar{e}) \xrightarrow{p_0} A \xrightarrow{p_1} A/Cg^A(\bar{1}, \bar{e})
\end{array}
\]

If $P_k$ denote the pushouts of $\pi_k$ along $h$, with $1 \leq k \leq 2$, from Lemma 2.3 of [39], it is the case that:

\[
\begin{align*}
P_0 &\cong A/Cg^A(\bar{0}, \varphi_A(\mu_A(\bar{e}))) \cong A/Cg^A(\bar{0}, \bar{e}) \\
P_1 &\cong A/Cg^A(\bar{1}, \varphi_A(\mu_A(\bar{e}))) \cong A/Cg^A(\bar{1}, \bar{e}).
\end{align*}
\]

Therefore, for general reasons we get:

\[
\begin{align*}
Cg^A(\bar{0}, \varphi_A(\mu_A(\bar{e}))) &= Cg^A(\bar{0}, \bar{e}) \\
Cg^A(\bar{1}, \varphi_A(\mu_A(\bar{e}))) &= Cg^A(\bar{1}, \bar{e}).
\end{align*}
\]
Hence, from Corollary 4 of [34] it follows that \( \varphi_A(\mu_A(\vec{e})) = \vec{e} \). On the other hand, let \( g \in \mathcal{V}(0 \times 0, A) \) and consider \( \mu_A(\varphi_A(g)) = i_{\varphi_A(g)} \). Then we have \( A \cong A / \text{CoExt}(\bar{0}, \varphi_A(g)) \times A / \text{CoExt}(\bar{1}, \varphi_A(g)) \). Since \( \mathcal{V} \) is coextensive, there exist unique \( u : 0 \to A / \text{CoExt}(\bar{0}, \varphi_A(g)) \) and \( v : 0 \to A / \text{CoExt}(\bar{1}, \varphi_A(g)) \) such that \( g = u \times v \). Observe that due to \( 0 \) is initial in \( \mathcal{V} \), it must be the case that \( u = i_{\varphi_A(g)} \) and \( v = i_{\varphi_A(g)} \). So \( g = \mu_A(\varphi_A(g)) \), as desired.

The proof of the naturality of \( \varphi \) and \( \mu \) is straightforward.

As an immediate application of Theorem 3 and Corollary 9.33 of [31], we obtain the following result.

**Corollary 1.** If \( \mathcal{V} \) is a coextensive variety, the functor \( Z : \mathcal{V} \to \text{Set} \) preserves all limits. Therefore \( Z \) has a left adjoint.

Something more can be said about the functor \( Z \).

**Lemma 5.** Let \( \mathcal{V} \) be a coextensive variety. If \( A, B \in \mathcal{V} \) and \( f : A \to B \) is a homomorphism, then \( f|_{Z(A)} : A \to B \) is a homomorphism of Boolean algebras.

**Proof.** We start by recalling that from Theorem 3 and Lemma 4.3 of [36], for every \( A \in \mathcal{V} \) and every \( \vec{e} \in Z(A) \) we have \( \theta^A_{\vec{e}} = \text{CoExt}(\bar{0}, \vec{e}) \). Since \( f \) is a homomorphism, it is clear that \( f|_{Z(A)} \) preserves \( \bar{0} \) and \( \bar{1} \). Now, if \( \vec{e}_1, \vec{e}_2 \in Z(A) \) and \( \vec{a} = \vec{e}_1 \wedge A \vec{e}_2 \), then from Lemma 3, \( [0, \vec{a}] \in \text{CoExt}(\bar{0}, \vec{e}_1) \) and \( [\vec{a}, \vec{e}_2] \in \text{CoExt}(\bar{1}, \vec{e}_1) \). From Theorem 2(3), we get \( [0, f(\vec{a})] \in \text{CoExt}(\bar{0}, f(\vec{e}_1)) \) and \( [f(\vec{a}), f(\vec{e}_2)] \in \text{CoExt}(\bar{1}, f(\vec{e}_1)) \). Therefore, again by Lemma 2 we conclude that \( f|_{Z(A)} \) preserves the meet of \( Z(A) \). The proof that \( f|_{Z(A)} \) preserves the join of \( Z(A) \) is analogous. This concludes the proof.

Observe that as result of Lemma 5, it is the case that the functor \( Z \) can be extended to a new functor \( Z : \mathcal{V} \to \text{Boole} \). We can take advantage of this fact in order to extend the representable \( \mathcal{V}(0 \times 0, -) \) to a functor from \( \mathcal{V} \) to \( \text{Boole} \) which we will denote by \( H \). Indeed, if \( A \in \mathcal{V} \) we can endow \( \mathcal{V}(0 \times 0, A) \) with a Boolean algebra structure in such a way that the algebra obtained be isomorphic to \( Z(A) \). If \( f, g \in \mathcal{V}(0 \times 0, A) \), by using Theorem 3 we define:

\[
0 := i_{\bar{0}} \times i_A \\
1 := i_{\bar{1}} \times i_A \\
g^e := i_{\varphi_A(g)} \times i_{\varphi_A(g)} \\
g \wedge h := i_{\varphi_A(g) \wedge A \varphi_A(h)} \times i_{\varphi_A(g) \wedge A \varphi_A(h)} \\
g \vee h := i_{\varphi_A(g) \vee A \varphi_A(h)} \times i_{\varphi_A(g) \vee A \varphi_A(h)}.
\]

Notice that due to the natural isomorphism between \( Z \) and the representable \( \mathcal{V}(0 \times 0, -) \) the functoriality of \( H \) is granted.

**Corollary 2.** Let \( \mathcal{V} \) be a coextensive variety and consider the functors \( Z \) and \( H \) from \( \mathcal{V} \) to \( \text{Boole} \). Then \( Z \) and \( H \) are naturally isomorphic.

**Proof.** Immediate from Theorem 3 and Lemma 5.
At this stage, one may be wandering if a characterization of coextensive varieties in terms of the representability of the functor $Z$ can be established. We baer that in [36] it was shown that not every variety with BFC and $\vec{0}$ and $\vec{1}$ is center stable and even if it is, it may be the case that it may not be coextensive. We claim that the next result provides an effective answer to this question.

**Theorem 4.** Let $\mathcal{V}$ be a variety with BFC, $\vec{0}$ and $\vec{1}$ and center stable. Then, the following are equivalent:

1. $\mathcal{V}$ is coextensive.
2. The following conditions hold:
   
   (i) The functor $Z : \mathcal{V} \to \text{Set}$ is representable by $0 \times 0$.
   
   (ii) The functor $\hat{Z} : \mathcal{V}/0 \times \mathcal{V}/0 \to \mathcal{V}/(0 \times 0)$ is full and faithful.

**Proof.** We only prove (2) $\Rightarrow$ (1) because the converse follows from Theorem 3 and the dual of Lemma 1 of [12]. Let us assume (2). We start by noticing that from Theorem 3.4.5 of [8], $\mathcal{V}$ is cocomplete so in particular, it has pushouts along projections. In addition by (i), there exists a natural isomorphism $\phi$ from $\mathcal{V}(0 \times 0, -)$ to $Z$. If we write $\mu$ for the inverse natural transformation of $\phi$, then for every $A, B \in \mathcal{V}$, $g \in \mathcal{V}(0 \times 0, A)$, $\vec{e} \in Z(A)$ and every homomorphism $f : A \to B$ the following identities hold:

$$f(\phi_A(g)) = \phi_B(fg)$$  \hspace{1cm} (1)

$$f(\mu_A(\vec{e})) = \mu_B(f(\vec{e}))$$  \hspace{1cm} (2)

Moreover, for every $B_0, B_1 \in \mathcal{V}$, $\phi_{B_0 \times B_1}(i_{B_0} \times i_{B_1}) = [\vec{0}_{B_0}, \vec{1}_{B_1}]$.

Since $\mathcal{V}$ is a variety with $\vec{0}$ and $\vec{1}$, then it is a variety admitting constant symbols. Then from Proposition 2.1 of [9], products are codisjoint. Now, let $A, A_0, A_1 \in \mathcal{V}$, $A_0 \xrightarrow{p_0} A \xrightarrow{p_1} A_1$ be a span, and let $g \in \mathcal{V}(0 \times 0, A)$. Consider the following diagram in which the upper left and right squares are pushouts:

We will prove that the aforementioned span is a product diagram. Since $\phi_A(g) \in Z(A)$, then we have an isomorphism $h : A \to B_0 \times B_1$. Now we prove that $hg = i_{B_0} \times i_{B_1}$. To do so, we need to check that $q_0(hg) = i_{B_0}$ and $q_1(hg) = i_{B_1}$.
and \( q_1(hg) = i_{B_1}\pi_1 \). We only check the first condition because the proof of the second one is similar. Observe that:

\[
q_0(hg) = q_0(h(\varphi_A(g))), \quad \text{since } \mu_A(\varphi_A(g)) = g.
\]

\[
= \mu_{B_0}(q_0h(\varphi_A(g))), \quad \text{from (2) applied to } q_0h.
\]

\[
= \mu_{B_0}(q_0[0_{B_0}, 1_{B_1}]), \quad \text{since } h(\varphi_A(g)) = [0_{B_0}, 1_{B_1}].
\]

\[
= \mu_{B_0}(0_{B_0}).
\]

On the other hand,

\[
\varphi_{B_0}(i_{B_0}\pi_0) = \varphi_{B_0}(q_0(i_{B_0} \times i_{B_1}))
\]

\[
= q_0(\varphi_{B_0} \times 1_{B_1}(i_{B_0} \times i_{B_1})), \quad \text{from (1)}.
\]

\[
= q_0[0_{B_0}, 1_{B_1}]
\]

\[
= 0_{B_0}.
\]

Therefore, from the following calculation we obtain:

\[
i_{B_0}\pi_0 = \mu_{B_0}(\varphi_{B_0}(i_{B_0}\pi_0)) = \mu_{B_0}(0_{B_0}) = q_0(hg).
\]

So \( hg = i_{B_0} \times i_{B_1} \) as claimed. Recall that the latter implies that \( q_0(hg) = i_{B_0}\pi_0 \) and \( q_1(hg) = i_{B_1}\pi_1 \). Thus, due to each of the upper squares of the diagram of above are pushouts by assumption, there exist unique \( a_i : A_j \rightarrow B_j \) with \( j = 1, 2 \), such that each of the lower squares of the diagram commute. From (ii) and the dual of Lemma 1 of \cite{12}, the outer left and right squares of the diagram are pushouts. So, each of the lower squares of the diagram are pushouts. Since \( q_0 \) and \( q_1 \) are epi and \( h \) is an iso, then \( a_0 \) and \( a_1 \) must be iso too. Therefore, the span \( A_0 \xymatrix{ \rightarrow \ar[r]^{p_0} & A \ar[r]^{p_1} & A_1 } \) is a product, as desired. Hence from Proposition\cite{11} the result follows.

We conclude this part by introducing a particular class of coextensive varieties. It is motivated by the intimate relation they present with a concrete property of the functor \( Z \). Such a class will be related with some results in Section\cite{8}.

**Definition 1.** A coextensive variety \( \mathcal{V} \) is said to be center presentable if \( 0 \times 0 \) is finitely presentable.

**Lemma 6.** Let \( \mathcal{V} \) be a coextensive variety. Then \( \mathcal{V} \) is center presentable if and only if the functor \( Z : \mathcal{V} \rightarrow \text{Set} \) preserves filtering colimits.

**Proof.** From Theorem\cite{3} the functor \( Z \) is representable by \( 0 \times 0 \). The result follows from Proposition 3.8.14 of \cite{8}.

**3.1 coBoolean varieties and the Beth property**

Let \( C \) be a category with finite limits. We recall that \( C \) has a subobject classifier if there is a mono \( \top : 1 \rightarrow \Omega \) in \( C \) such that for every object \( X \) and mono
m : S → X in C, there exists a unique \( \chi_m : X \to \Omega \) such that the following diagram

\[
\begin{array}{ccc}
S & \rightarrow & 1 \\
m \downarrow & & \uparrow \\
X & \chi_m \rightarrow & \Omega
\end{array}
\]

is a pullback.

**Definition 2.** A category with finite colimits \( \mathcal{D} \) has a quotient coclassifier if \( \mathcal{D}^{op} \) has a subobject classifier.

The following definition is an adaptation of Definition 4.2 of [11].

**Definition 3.** A coextensive variety \( \mathcal{V} \) is said to be coBoolean if the first projection \( \pi : 0 \times 0 \to 0 \) is a quotient coclassifier.

Recall that in the case of a variety, quotients are completely determined by congruences. Therefore, if \( \mathcal{V} \) is coextensive and cooBoolean, by Theorem 3 it is the case that \( \text{Con}(A) \cong Z(A) \), for every \( A \in \mathcal{V} \). This observation motivates the following definition.

**Definition 4.** Let \( \mathcal{V} \) be a variety with BFC. We say that \( \mathcal{V} \) is congruence-factor if \( \text{Con}(A) \cong \text{FC}(A) \), for every \( A \in \mathcal{V} \).

**Remark 1.** We say that a variety \( \mathcal{V} \) has Boolean congruences if \( \text{Con}(A) \) is a Boolean algebra, for every \( A \in \mathcal{V} \). If in particular \( \mathcal{V} \) is congruence-distributive, then from Theorem 4 of [22] it follows that \( \mathcal{V} \) is semisimple. Therefore, it is immediate from Definition 4 that congruence-factor varieties are semisimple and arithmetical.

Let \( \mathcal{C} \) be a category with binary products. We say that a morphism \( X \to Y \) of \( \mathcal{C} \) is the projection of a product if there exist \( X \to Z \) in \( \mathcal{C} \) such that the span \( Y \leftarrow X \to Z \) is a product. Observe that when \( \mathcal{C} \) is a variety, the projections of a product are unique up to isomorphism in the following way: if \( f : A \to B \) is the projection of a product and \( g : A \to C \) and \( g' : A \to C' \) are such that \( A \cong B \times C \cong B \times C' \), thus \( \text{Ker}(f) \circ \text{Ker}(g) \) and \( \text{Ker}(f) \circ \text{Ker}(g') \) so \( \text{Ker}(g) = \text{Ker}(g') \). Then, by general reasons (Lemma 3.3 of [5]), there exist a unique isomorphism \( i : C \to C' \) such that \( ig = g' \).

The following result provides a characterization of coBoolean varieties by means of congruence-factor varieties.

**Lemma 7.** Let \( \mathcal{V} \) be a coextensive variety. Then, the following are equivalent:

1. \( \mathcal{V} \) is coBoolean.
2. Every epimorphism in \( \mathcal{V} \) is the projection of a product.
3. \( \mathcal{V} \) is congruence-factor and every epimorphism is surjective.
Proof. (1) \(\iff\) (2). This is a particular case of the dual of Proposition 4.4 of [11].

(2) \(\Rightarrow\) (3). In order to check that \(\mathcal{V}\) is congruence-factor, let \(A \in \mathcal{V}\) and \(\theta \in \text{Con}(A)\). By (2), the quotient map \(A \rightarrow A/\theta\) is the projection of a product, so there exist \(B \in \mathcal{V}\) and \(q : A \rightarrow B\) such that \(A/\text{Ker}(q) \cong B\) and \(A \cong A/\theta \times B\). Hence, \(\theta\) and \(\text{Ker}(q)\) are complementary factor congruences, as desired. Finally, if \(e : A \rightarrow B\) is an epimorphism, by (2) \(e\) is a projection of a product. So in particular, \(e\) is surjective.

(3) \(\Rightarrow\) (2). If \(e : A \rightarrow B\) is an epimorphism, by (3) \(e\) is surjective so \(B \cong A/\text{Ker}(e)\). Since \(\mathcal{V}\) is congruence-factor by assumption, \(\text{Ker}(e)\) has a complementary factor congruence \(\theta\). Thus we obtain that \(e\) coincides with a projection of \(B \times A/\theta\), as claimed.

We conclude this section with an application of Lemma 7 concerning algebraizable logics in the sense of Blok and Pigozzi and the Beth (definability) property. The following result reveals that cooBoolean varieties can be useful to decide whether an algebraizable logic has the Beth (definability) property.

Theorem 5. Let \(\mathbb{L}\) be an algebraizable logic with equivalent variety semantics \(\mathcal{V}\) and assume that \(\mathcal{V}\) is coextensive. Then the following hold:

1. If \(\mathcal{V}\) is cooBoolean then \(\mathbb{L}\) has the Beth (definability) property.
2. If \(\mathcal{V}\) is congruence-factor and \(\mathbb{L}\) has the Beth (definability) property, then \(\mathcal{V}\) is cooBoolean.

Proof. The result follows by a straightforward application of Lemma 7 (3), and Theorem 1.

4 The Gaeta topos and fp-coextensive varieties

In this section we show that given a coextensive variety \(\mathcal{V}\), the characterization of coextensive varieties obtained in [36] brings a suitable axiomatization of the theory of \(\mathcal{V}\)-indecomposable objects. Thereafter, we restrict our study to fp-coextensive varieties. In this setting, will provide a criterion which allows us to decide whether a cooBoolean \(\mathcal{V}\), the Gaeta topos classifies \(\mathcal{V}\)-indecomposable objects. Finally, with the aim of furnishing some examples, we apply our results to some particular coextensive varieties of interest in general algebra and algebraic logic.

We start by proving some technical results on coextensive varieties which will be used along this section.

Lemma 8. Let \(\mathcal{V}\) be a coextensive variety. Then, for every \(n\)-ary term \(p(\vec{c})\) and constant symbols \(c_1,\ldots,c_n\) in the language of \(\mathcal{V}\), there exists a \(2n\)-ary term \(q(\vec{x},\vec{y})\) such that

\[
\mathcal{V} \models p(\vec{c}) = q(\vec{0}, \vec{1}).
\]
Proof. Since $V$ is coextensive, by Theorem 2 (2), $V$ is a Pierce variety in which the relation $\vec{e} \circ A \vec{f}$ is equationally definable. So in particular, $V$ is a variety with $\vec{0}$ and $\vec{1}$. Let
\[
\sigma(\vec{x}, \vec{y}) = \bigwedge_{i=1}^{n} p_i(\vec{x}, \vec{y}) = q_i(\vec{x}, \vec{y})
\]
define the relation $\vec{e} \circ A \vec{f}$ in $V$. Because $\vec{0}$ and $\vec{1}$ are complementary central elements in $Z(0)$, then for every $\vec{A} \in V$ and $1 \leq i \leq n$,
\[
p_i^A(\vec{0}^A, \vec{1}^A) = q_i^A(\vec{0}^A, \vec{1}^A).
\]
(3)
Now we consider $X = \{\vec{x}, \vec{y}, \vec{z}\}$ and $\theta = \bigvee_{i=1}^{n} Cg^{F_{\nu}(\vec{x}, \vec{y})}(p_i(\vec{x}, \vec{y}), q_i(\vec{x}, \vec{y}))$. Let $H = F_{\nu}(X)/\theta$. Observe that since $p_i^H(\vec{x}/\theta, \vec{y}/\theta) = q_i^H(\vec{x}/\theta, \vec{y}/\theta)$ for every $1 \leq i \leq n$, then $\vec{e}/\theta \circ H \vec{f}/\theta$. Therefore, due to
\[
(p(\vec{z})/\theta, p(\vec{z})/\theta) \in \nabla^H = Cg^H(\vec{x}/\theta, \vec{0}) \circ Cg^H(\vec{y}/\theta, \vec{0}),
\]
there exists a term $t(\vec{x}, \vec{y}, \vec{z})$ such that
\[
(p^H(\vec{z})/\theta, t^H(\vec{x}/\theta, \vec{y}/\theta, \vec{z}/\theta)) \in Cg^H(\vec{x}/\theta, \vec{0})
\]
(4) and
\[
(p^H(\vec{z})/\theta, t^H(\vec{x}/\theta, \vec{y}/\theta, \vec{z}/\theta)) \in Cg^H(\vec{y}/\theta, \vec{0}).
\]
Let $\vec{A} \in V$. Then, from 3 and Lemma 3 there exists a unique $\Omega : H \rightarrow \vec{A}$ such that $\Omega(\vec{x}/\theta) = \vec{A}^0$, $\Omega(\vec{y}/\theta) = \vec{A}^1$ and $\Omega(\vec{z}/\theta) = \vec{c}^A$. Thus from 4 we obtain
\[
p^A(\vec{c}^A) = t^A(\vec{0}^A, \vec{1}^A, \vec{c}^A).
\]
Hence $V \models p(\vec{c}) = t(\vec{0}, \vec{1}, \vec{c})$. Finally, if we define $q(\vec{x}, \vec{y}) = t(\vec{x}, \vec{y}, \vec{c})$, then from the latter it is the case that $V \models p(\vec{c}) = q(\vec{0}, \vec{1})$, as required.

Let $V$ be a coextensive variety. Recall that again from Theorem 2 the relation $\vec{e} \circ A \vec{f}$ in $V$ is equationally definable. Let
\[
\sigma(\vec{x}, \vec{y}) = \bigwedge_{i=1}^{n} p_i(\vec{x}, \vec{y}) = q_i(\vec{x}, \vec{y})
\]
define the relation $\vec{e} \circ A \vec{f}$ in $V$. Now for the rest of this section we consider:
\[
\theta = \bigvee_{i=1}^{n} Cg^{F_{\nu}(\vec{x}, \vec{y})}(p_i(\vec{x}, \vec{y}), q_i(\vec{x}, \vec{y}))
\]
\[
\mu = Cg^{F_{\nu}(\vec{x}, \vec{y})}(\vec{x}, \vec{0}) \lor Cg^{F_{\nu}(\vec{x}, \vec{y})}(\vec{y}, \vec{0})
\]
\[
\lambda = Cg^{F_{\nu}(\vec{x}, \vec{y})}(\vec{x}, \vec{1}) \lor Cg^{F_{\nu}(\vec{x}, \vec{y})}(\vec{y}, \vec{1})
\]

Lemma 9. Let $V$ be a coextensive variety. Then, the following hold:
(1) $\theta \subseteq \mu, \lambda$;
(2) \( \mu \circ \lambda = \lambda \circ \mu = \nabla^{F_V(\vec{x}, \vec{y})} \);

(3) \( F_V(\vec{x}, \vec{y})/\mu \cong F_V(\vec{x}, \vec{y})/\lambda \cong 0. \)

(4) \( F_V(\vec{x}, \vec{y})/(\mu \land \lambda) \cong 0 \times 0. \)

Proof. (1) Since \( V \) is coextensive, by Theorem 2 (2), \( V \) is a Pierce variety in which the relation \( \vec{e} \circ_A \vec{f} \) is equationally definable. So in particular, by Lemma 4.3 (1) of [36], \( V \) is a variety with \( \vec{0} \) and \( \vec{1} \). Since \( \vec{0}^A, \vec{1}^A \in Z(A) \) for every \( A \in V \), it is the case that

\[ V \models \vec{x} = 0 \land \vec{y} = 1 \implies \sigma(\vec{x}, \vec{y}) \]

so in particular

\[ V \models \vec{x} = 0 \land \vec{y} = 1 \implies p_i(\vec{x}, \vec{y}) = q_i(\vec{x}, \vec{y}) \]

for every \( 1 \leq i \leq n \). Then from Lemma 2

\[ C_{g^{F_V(\vec{x}, \vec{y})}}(p_i(\vec{x}, \vec{y}), q_i(\vec{x}, \vec{y})) \subseteq \mu \]

for every \( 1 \leq i \leq n \). Therefore, \( \theta \subseteq \mu \) as required. The proof of \( \theta \subseteq \lambda \) is analogue.

(2) Let \( (s(\vec{x}, \vec{y}), t(\vec{x}, \vec{y})) \in \nabla^{F_V(\vec{x}, \vec{y})} \). By Theorem 2 (2) \( V \) is a Pierce variety so there exists a term \( U(x, y, \vec{x}, \vec{y}) \) in the language of \( V \) such that

\[ U(x, y, \vec{0}, \vec{1}) = x \quad \text{and} \quad U(x, y, \vec{1}, \vec{0}) = y. \]

Let us consider \( p(\vec{x}, \vec{y}) = U(s(\vec{x}, \vec{y}), t(\vec{x}, \vec{y}), \vec{x}, \vec{y}). \) Observe that \( p(\vec{0}, \vec{1}) = s(\vec{0}, \vec{1}) \) and \( p(\vec{1}, \vec{0}) = t(\vec{1}, \vec{0}). \) Thus, it follows that

\[ V \models (\vec{x} = 0 \land \vec{y} = 1) \implies p(\vec{x}, \vec{y}) = s(\vec{x}, \vec{y}) \]

and

\[ V \models (\vec{x} = 1 \land \vec{y} = 0) \implies s(\vec{x}, \vec{y}) = t(\vec{x}, \vec{y}). \]

Thus by Lemma 2 we get

\[ (s(\vec{x}, \vec{y}), p(\vec{x}, \vec{y})) \in \mu \quad \text{and} \quad (p(\vec{x}, \vec{y}), t(\vec{x}, \vec{y})) \in \lambda. \]

Hence \( \mu \circ \lambda = \nabla^{F_V(\vec{x}, \vec{y})} \). Finally, we stress that if we take

\[ q(\vec{x}, \vec{y}) = U(s(\vec{x}, \vec{y}), t(\vec{x}, \vec{y}), \vec{y}, \vec{x}) \]

it is no hard to see that \( q(\vec{0}, \vec{1}) = t(\vec{0}, \vec{1}) \) and \( q(\vec{1}, \vec{0}) = s(\vec{1}, \vec{0}) \). Therefore, by the same argument we employed before together with Lemma 2 we get \( \lambda \circ \mu = \nabla^{F_V(\vec{x}, \vec{y})} \), as claimed.

(3) Let \( H = F_V(\vec{x}, \vec{y})/\mu \) and \( H' = F_V(\vec{x}, \vec{y})/\lambda \). We will prove that \( H \) and \( H' \) are both isomorphic to \( 0 \). We will only exhibit the details about \( H \cong 0 \) because the proof of the last part is similar. We start by considering the map \( h : H \to F_V(\vec{x}, \vec{y}) \) defined as \( h(q(\vec{x}, \vec{y})/\mu) = q(\vec{0}, \vec{1}) \). We will show that \( h \) is an
isomorphism. In order to see that \( h \) is well defined, let \( q(\vec{x}, \vec{y}), q'(\vec{x}, \vec{y}) \in F_\mathcal{V}(\vec{x}, \vec{y}) \) and suppose that \( q(\vec{x}, \vec{y})/\mu = q'(\vec{x}, \vec{y})/\mu \). Thus \( (q(\vec{x}, \vec{y}), q'(\vec{x}, \vec{y})) \in \mu \). Then, from Lemma 2 we get
\[
\mathcal{V} \models (\vec{x} = \vec{0} \land \vec{y} = \vec{1}) \implies q(\vec{x}, \vec{y}) = q'(\vec{x}, \vec{y}),
\]
so, in particular \( q^{F_\mathcal{V}(\vec{z})}(\vec{0}, \vec{1}) = q^{F_\mathcal{V}(\vec{z})}(\vec{0}, \vec{1}) \) as claimed. Moreover, notice that the same argument applied in the reverse direction allows us to prove that \( h \) is injective. The surjectivity of \( h \) follows from Lemma 8. Finally, it is clear that \( h \) is a homomorphism. Hence \( H \Rightarrow 0 \Rightarrow H' \) as desired.

(4) Immediate from (2), (3) and Lemma 10.

4.1 \( \mathcal{V} \)-indecomposable objects in a topos

Let \( \mathcal{V} \) be a coextensive variety. An algebra \( A \) of \( \mathcal{V} \) is said to be \( \mathcal{V} \)-indecomposable if it is indecomposable by binary products; i.e. if \( A \cong B \times C \), then \( B \cong 1 \) or \( C \cong 1 \). The following result allows will show that the theory of central elements brings axiomatization for the theory of \( \mathcal{V} \)-indecomposable objects.

Lemma 10. The class of \( \mathcal{V} \)-indecomposable objects is axiomatizable by a first order formula.

Proof. From Theorem 2 (2) the relation \( \vec{e} \circ_A \vec{f} \) is equationally definable in \( \mathcal{V} \). So we can take \( \sigma(\vec{x}, \vec{y}) \) as an equation defining such a relation. It is immediate that \( A \in \mathcal{V} \) is \( \mathcal{V} \)-indecomposable if and only if in \( A \) the following sentence holds
\[
\vec{0} \neq \vec{1} \text{ and } (\forall_{\vec{x}, \vec{y}} \sigma(\vec{x}, \vec{y}) \Rightarrow ((\vec{e} = \vec{0} \land \vec{f} = \vec{1}) \lor (\vec{e} = \vec{1} \land \vec{f} = \vec{0}))).
\]

Observe that from Lemma 10 it follows that an algebra \( A \) in \( \mathcal{V} \) is \( \mathcal{V} \)-indecomposable if and only if the sequents
\[
0 = 1 \vdash
\sigma(\vec{x}, \vec{y}) \vdash_{\vec{x}, \vec{y}} (\vec{x} = \vec{0} \land \vec{y} = \vec{1}) \lor (\vec{x} = \vec{1} \land \vec{y} = \vec{0})
\]
hold in \( A \).

Let \( \mathcal{V} \) be a coextensive variety, \( f \) be a symbol in the language of \( \mathcal{V} \), \( E \) be a topos and \( M \) be an object of \( E \). If we write \( a_f \) for the arity of \( f \) (which is a natural number), recall (see D1.2.1 of [21]) that the interpretation of \( f \) in \( M \) is a morphism \( f_M : M^{a_f} \to M \). Thus a \( \mathcal{V} \)-model in \( E \) is an object \( M \) of \( E \) equipped with morphisms \( f_M : M^{a_f} \to M \) for every symbol \( f \) in the language of \( \mathcal{V} \) for which the defining identities of \( \mathcal{V} \) (expressed by diagrams in \( E \)) hold. Moreover, a homomorphism between \( \mathcal{V} \)-models \( M \) and \( R \) in \( E \) is an arrow \( h : M \to R \) in \( E \) making the diagram

\[
\begin{align*}
  M^{a_f} \xrightarrow{h^{a_f}} & R^{a_f} \\
  f_M \downarrow & \downarrow f_R \\
  M \xrightarrow{h} & R
\end{align*}
\]
commutes for every symbol $f$ in the language of $V$ (here $h^{a_f}$ denotes the product morphism of $h$ $a_f$-times). This information defines the category of $V$-models in $E$. Notice that in particular, when regarding the topos $\text{Set}$, the category of $V$-models coincides with $V$. In order to illustrate the latter, let us consider the variety $\mathcal{DL}_{01}$ of bounded distributive lattices. Then a $\mathcal{DL}_{01}$-model in $E$ is an object $L$ of $E$ endowed with arrows

$$
\begin{array}{c}
1 \\
\downarrow \odot_L \\
L \\
\uparrow \rtimes_L \\
L \times L
\end{array}
$$

such that the equations defining $\mathcal{DL}_{01}$ hold. For instance, the commutativity of the meet can be expressed by the commutativity of the following diagram in $E$:

$$
\begin{array}{c}
L \times L \\
\downarrow \langle \pi_1, \pi_2 \rangle \\
L \times L \\
\downarrow \langle \pi_2, \pi_1 \rangle \\
L \times L \\
\downarrow \rtimes_L \\
L
\end{array}
$$

Now, motivated by the observation made right after Lemma 10 we introduce the following:

**Definition 5.** Let $E$ be a topos. A $V$-model $M$ of $E$ is $V$-indecomposable if the sequents

$$
\begin{align*}
0 = 1 & \vdash \bot \\
\sigma(\bar{x}, \bar{y}) & \vdash \bar{x} = \bar{0} \wedge \bar{y} = \bar{1} \vee (\bar{x} = \bar{1} \wedge \bar{y} = \bar{0})
\end{align*}
$$

hold in the internal logic of $E$.

We stress that Definition 5 can be driven to categorical terms. To do so, let $M$ be a $V$-model in a topos $E$ and let $\hat{0}_M, \hat{1}_M$ and $[\sigma(\bar{x}, \bar{y})]_M$ be the interpretations in $M$ of the constants $\bar{0}, \bar{1}$ and a equation $\sigma(\bar{x}, \bar{y})$ defining the relation $\sigma : A \rightarrow f$ in $V$, respectively (for details, see D1.2.6 of [21]). Now, let us consider the elements $\langle \bar{0}_M, \bar{1}_M \rangle : 1 \rightarrow M^N \times M^N$ and $\langle \bar{1}_M, \bar{0}_M \rangle : 1 \rightarrow M^N \times M^N$. If $\alpha = [\langle \bar{0}_M, \bar{1}_M \rangle, \langle \bar{1}_M, \bar{0}_M \rangle]_M$ denotes the morphism from $1 + 1$ to $[\sigma(\bar{x}, \bar{y})]_M$ induced by the coproduct, then a basic exercise in the internal logic of toposes shows the following:

**Lemma 11.** Let $E$ be a topos and let $M$ be a $V$-model in $E$. The following are equivalent:

1. $M$ is $V$-indecomposable,
2. The diagram below

$$
\begin{array}{c}
0 \\
\downarrow \iota \\
1 \\
\downarrow \odot_M \\
M \times M
\end{array}
$$

is an equalizer in $E$, and the morphism $\alpha : 1 + 1 \rightarrow [\sigma(\bar{x}, \bar{y})]_M$ is an isomorphism.
(3) In the internal logic of $E$, the following sequents hold:

\[
\begin{align*}
(C1) & \quad \emptyset = \emptyset \vdash \bot, \\
(C2) & \quad \sigma(x, y) \vdash_{x, y} (x = \emptyset \land y = \emptyset) \lor (x = \emptyset \land y = \emptyset).
\end{align*}
\]

4.2 The characterization

Let $C$ be a small extensive category. For every object $X$ of $C$, we say that \{fi : Xi \to X | i \in I\} \in K_G(X) if and only if $I$ is finite and the induced arrow $\Sigma X_i \to X$ is an isomorphism. From the extensivity condition it follows that $K_G$ is a basis for a Grothendieck topology over $C$ (see III.2.1 of [26]). The topology $J_G$ generated by such a basis is called the Gaeta Topology and the Gaeta topos $G(C)$, is the topos of sheaves on the site $(C, J_G)$. As observed in [12], $G(C)$ is equivalent to the category $\text{Lex}(C^{op}, \text{Set})$ of product preserving functors to $\text{Set}$ from the category $C^{op}$ with finite products. This fact implies that $J_G$ is subcanonical. If $C$ has a terminal object 1, from Proposition 4.1 of [11], it follows that $C$ is extensive if and only if the canonical functors $1 \to C/0$ and $C/(1 + 1) \to C \times C$ are equivalences.

Remark 2. Let $C$ be an extensive category with a terminal object 1 and let $E$ be a topos. Notice that a finite limit preserving functor $G : C \to E$ is continuous (see VII.7 of [26]) with respect to the Gaeta topology over $C$ if and only if $G(0) \cong 0$ and $G(1 + 1) \cong 1 + 1$; i.e. it preserves binary coproducts.

Let $V$ be a coextensive variety. We write $\text{Mod}_p(V)$ for the full subcategory of finitely presented algebras of $V$. Let $E$ be a topos. Due to Lawvere’s duality [23], it is known that the category of $V$-models in $E$ is equivalent to the category of limit preserving functors $\text{Lex}(\text{Mod}_p(V)^{op}, E)$. So, for every $V$-model $M$ in $E$, there exists an essentially unique limit preserving functor $\phi_M : \text{Mod}_p(V)^{op} \to E$, such that $\phi_M(\text{F}_V(x)) \cong M$. In what follows we will refer to $\phi_M$ as the representative of $M$. It is worth mentioning that in the case of $E = \text{Set}$, the representative of $\text{F}_V(x)$ reflects isomorphisms.

Theorem 6. Let $V$ be a coextensive variety. Then $\text{Mod}_p(V)$ is coextensive if and only if binary products of finitely generated free algebras of $V$ are finitely presented.

Proof. Let $\text{F}_V(n)/\theta$ and $\text{F}_V(m)/\delta$ be finitely presented algebras of $V$. Then, $\theta$ and $\delta$ are finitely generated congruences on $\text{F}_V(n)$ and $\text{F}_V(m)$, respectively. From Lemma 4.3 (4) of [36], $V$ has the Fraser Horn property, thus

\[
\text{F}_V(n)/\theta \times \text{F}_V(m)/\delta \cong (\text{F}_V(n) \times \text{F}_V(m))/(\theta \times \delta).
\]  

(5)

So, due to Theorem 3 (6) of [18], $\theta \times \delta$ is finitely generated. Since $\text{F}_V(n) \times \text{F}_V(m)$ is finitely presented by assumption, there exist variables $x_1, \ldots, x_k$ and a finitely generated congruence $\epsilon$ on $\text{F}_V(k)$, such that $\text{F}_V(n) \times \text{F}_V(m) \cong \text{F}_V(k)/\epsilon$. Thus, from Theorem 6.20 of [10], there exists a compact congruence $\gamma$ on $\text{F}_V(k)$, with
\[ \epsilon \subseteq \gamma, \text{ such that } \theta \times \delta = \gamma/\epsilon. \] Therefore, from Theorem 6.15 of [10] and (15) we conclude that \( F_\mathcal{V}(n)/\theta \times F_\mathcal{V}(m)/\delta \) is finitely presented, so \( \text{Mod}_{fp}(\mathcal{V}) \) has finite products and consequently, it is coextensive. On the other hand, if \( \text{Mod}_{fp}(\mathcal{V}) \) is coextensive, it has finite products. Observe that since \( \mathcal{V} \) is coextensive, from Lemma 4.3 of [36], \( \Delta = Cg^A(\bar{u}, \bar{v}) \) for every \( A \in \mathcal{V} \). In particular, this implies that every finitely generated free algebra of \( \mathcal{V} \) is finitely presented so, binary products between them must be finitely presented. This concludes the proof. \( \square \)

A coextensive variety \( \mathcal{V} \) is said to be \emph{fp-coextensive} if it satisfies any of the equivalent conditions of Theorem 6. The following result, immediately establishes a relation between fp-coextensive varieties and center presentable varieties (see Definition 4).

**Corollary 3.** Every fp-coextensive variety is center presentable.

At this stage one may be wondering if the finiteness of the type of coextensive varieties plays any rôle to decide fp-coextensivity. The next result provides an answer to this question.

**Proposition 2.** Let \( \mathcal{V} \) be a coextensive variety of finite type. If \( \mathcal{V} \) is locally finite then it is fp-coextensive. So, in particular, the functor \( Z \) preserves filtering colimits.

**Proof.** Let us assume that \( \mathcal{V} \) is of finite type, coextensive and locally finite. Let \( F_\mathcal{V}(n) \) and \( F_\mathcal{V}(m) \) be finitely generated free algebras of \( \mathcal{V} \). From Theorem 10.15 of [10], the set \( X = F_\mathcal{V}(n) \times F_\mathcal{V}(n) \) is finite. We stress that \( F_\mathcal{V}(n) \times F_\mathcal{V}(m) \) is finitely presented because from Corollary II.10.11 of [10], such an algebra is in fact isomorphic to \( F_\mathcal{V}(X) \) quotiented by the finitely many conditions which describe the operations in \( F_\mathcal{V}(n) \times F_\mathcal{V}(m) \). The last part follows from Corollary 3. \( \square \)

Let \( \mathcal{V} \) a fp-coextensive variety. Let \( x_1, \ldots, x_k \) be a finite set of variables and let \( p_1, \ldots, p_k, q_1, \ldots, q_k \) be terms in the language of \( \mathcal{V} \) with variables \( y_1, \ldots, y_l \). If \( \delta \) denotes the congruence \( \bigwedge_{i=1}^k Cg^{F_\mathcal{V}(\bar{y})}(p_i(\bar{y}), q_i(\bar{y})) \) and \( A \) denotes the algebra \( F_\mathcal{V}(\bar{y})/\delta \), observe that the representative of \( M \) sends the finitely presentable algebra \( A \) to the following equalizer in \( E \):

\[
\phi_M(A) \xrightarrow{M^l} \frac{M^k}{\langle p_{M_1}, \ldots, p_{M_k} \rangle} \xrightarrow{\langle q_{M_1}, \ldots, q_{M_k} \rangle} M^k
\]

where \( p_{M_i} \) and \( q_{M_i} \) denote the interpretation in \( E \) of the terms \( p_i \) and \( q_i \) in \( M \), respectively, with \( 1 \leq i \leq k \). I.e. the image of \( A \) by \( \phi_M \) essentially coincides in \( E \) with the interpretation in \( M \) of the formula

\[
\varepsilon(\bar{y}) = \bigwedge_{i=1}^k p_i(\bar{y}) = q_i(\bar{y}).
\]

In what follows, we write \( G(\mathcal{V}) \) for the Gaeta topos determined by the extensive category \( \text{Mod}_{fp}(\mathcal{V})^{op} \). We recall that from VII.7.4 of [26] there is an
equivalence between the category Geo(E, G(V)) of geometric morphisms from E to G(V) and the category LexCon(\text{Mod}_p(V)^\text{op}, E) of limit preserving functors from \text{Mod}_p(V)^\text{op} to E which are continuous with respect to the Gaeta topology over \text{Mod}_p(V)^\text{op}.

As a result of the above discussion, now we can restate Lemma 11 by means of the representative of a \mathcal{V}-model in a topos \mathcal{E}.

**Lemma 12.** Let \mathcal{V} be a fp-coextensive variety, \mathcal{E} be a topos and let \text{M} be a \mathcal{V}-model in \mathcal{E}. Let \phi_M be the representative of \text{M}. Then, the following are equivalent:

1. M is \mathcal{V}-indecomposable in \mathcal{E}.
2. \phi_M(\mathcal{F}_{\mathcal{V}}(\vec{x}, \vec{y})/\theta) \cong 1 + 1 and \phi_M(1) \cong 0.

Moreover, if \mathcal{E} = \text{Set} and M = \mathcal{F}_{\mathcal{V}}(x), any of the above conditions is equivalent to \phi_{\mathcal{F}_{\mathcal{V}}(x)} preserves finite coproducts.

**Proof.** Let \text{M} be a \mathcal{V}-model in a topos \mathcal{E}. Notice that it is the case that

\[ [\sigma(\vec{x}, \vec{y})]_M \cong \phi_M(\mathcal{F}_{\mathcal{V}}(\vec{x}, \vec{y})/\theta)\]

and

\[ [\vec{0} = \vec{1}]_M \cong \phi_M(0/\text{Cg}_0(\vec{0}, \vec{1})).\]

Since \mathcal{V} is a variety with \vec{0} and \vec{1}, then

\[ 0/\text{Cg}_0(\vec{0}, \vec{1}) \cong 1.\]

Hence from Lemma 11 and Remark 2 it is immediate that a \mathcal{V}-model \text{M} in \mathcal{E} is \mathcal{V}-indecomposable in such a topos if and only if (2) holds.

For the moreover part, we start by noticing that from Lemma 9 there exist arrows \text{f} and \text{g} from \mathcal{F}_{\mathcal{V}}(\vec{x}, \vec{y})/\theta to 0. Now consider the following diagram in \mathcal{V}, in which the outer vertical arrows denote the identity of 0 and the middle vertical arrow is the arrow induced by the product.

\[
\begin{array}{ccc}
0 & \xrightarrow{\text{f}} & \mathcal{F}_{\mathcal{V}}(\vec{x}, \vec{y})/\theta & \xrightarrow{\text{g}} & 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & \xleftarrow{\text{0}} & 0 \times 0 & \xrightarrow{\text{0}} & 0
\end{array}
\]

We stress that Lemma 9 (1), (3) and (4) allows us to say that each of the squares of such a diagram is a pushout. Thus, if \mathcal{F}_{\mathcal{V}}(x) is \mathcal{V}-indecomposable in \text{Set}, by condition (2), we obtain that \phi_{\mathcal{F}_{\mathcal{V}}(x)}(\mathcal{F}_{\mathcal{V}}(\vec{x}, \vec{y})/\theta) is isomorphic to 1 + 1 and \phi_{\mathcal{F}_{\mathcal{V}}(x)}(1) is isomorphic to \emptyset. Thus, since \phi_{\mathcal{F}_{\mathcal{V}}(x)} preserves finite limits, in order to proof that such a functor preserves finite coproducts, we only need to show that \phi_{\mathcal{F}_{\mathcal{V}}(x)}(0 \times 0) is isomorphic to 1 + 1. To do so, notice that from
Lemma condition (1) and because $\phi_{FV(x)}$ preserves pullbacks, the diagram of above turns into the following diagram

\[
\begin{array}{ccc}
1 & \rightarrow & \phi_{FV(x)}(0 \times 0) & \leftarrow & 1 \\
\downarrow & & \downarrow & & \downarrow \\
1 & \rightarrow & 1 + 1 & \leftarrow & 1
\end{array}
\]

in which both squares are pullbacks in $\text{Set}$. Since $\text{Set}$ is extensive, then we conclude that $\phi_{FV(x)}(0 \times 0)$ must be isomorphic to $1 + 1$, as desired. On the other hand, if $\phi_{FV(x)}$ preserves finite coproducts, then $\phi_{FV(x)}(1) \cong \emptyset$. Thus, because both of the squares of diagram [1] are pushouts and again by the extensivity of $\text{Set}$, we get

\[
\phi_{FV(x)}(0 \times 0) \cong 1 + 1 \cong \phi_{FV(x)}(FV(x)[\bar{f}, \bar{y}] / \theta),
\]

as claimed. \hfill \Box

Now, we are ready to show the main result of this section.

**Theorem 7.** Let $\mathcal{V}$ be a fp-coextensive variety. Then, the following are equivalent:

1. $G(V)$ is a classifying topos for $\mathcal{V}$-indecomposable objects.
2. $FV(x)$ is $\mathcal{V}$-indecomposable in $\text{Set}$.

**Proof.** For the sake of readability of the following proofs, we start by fixing some notation. Let $E$ be a topos. We write $\text{Mod}(\mathcal{V}, E)$ for the category of $\mathcal{V}$-indecomposable objects in $E$ and we will denote by $A$ the category $\text{Mod}_{\text{fp}}(\mathcal{V})$.

(1) $\Rightarrow$ (2) Let us assume that $G(V)$ classifies $\mathcal{V}$-indecomposable objects. Then, for every topos $E$, the categories $\text{Mod}(\mathcal{V}, E)$ and $\text{Geo}(E, G^A)$ are equivalent. Thus, in particular, $\text{Mod}(\mathcal{V}, \text{Set}) \cong \text{Geo}(\text{Set}, G^A)$. On the other hand, from VII.7.2 of [26], there is an equivalence between the category $\text{Lex}(A^{\text{op}}, \text{Set})$ and the category $\text{Geo}(\text{Set}, G^A)$, so let $g : \text{Set} \rightarrow G^A$ be the geometric morphism corresponding to $\phi_{FV(x)}$ from this equivalence. We will prove that $\phi_{FV(x)}$ is continuous with respect to $J_G$. Because $J_G$ is subcanonical, $g$ factors through the inclusion from $G(V)$ to $G^A$, therefore by VII.7.3 of [26] we get $g^* \circ y \cong \phi_{FV(x)}$ is continuous with respect to $J_G$, as claimed. Whence,

\[
(g^* \circ y)(FV(x)) \cong \phi_{FV(x)}(FV(x)) \cong FV(x)
\]

is indecomposable in $\text{Set}$, as desired.

(2) $\Rightarrow$ (1) If $FV(x)$ is $\mathcal{V}$-indecomposable in $\text{Set}$, then, from Lemma [12] and Remark [2], $\phi_{FV(x)}$ is continuous with respect to $J_G$. So, in order to prove our claim, we need to show that there is an equivalence between $\text{Mod}(\mathcal{V}, E)$ and $\text{Geo}(E, G(V))$, for every topos $E$. Since $\text{LexCon}(A^{\text{op}}, E)$ and $\text{Geo}(E, G(V))$ are
equivalent from VII.7.4 of [26], we only need to prove that $\text{Mod}(\mathcal{V}, E)$ and $\text{LexCon}(A^{\text{op}}, E)$ are equivalent for every topos $E$. To do so, let $E$ be a topos and let $H : A^{\text{op}} \to E$ be a finite limit preserving functor continuous with respect to $J_G$. From Lawvere’s duality, it follows that $M = H(F_V(x))$ is a $\mathcal{V}$-model in $E$ and also that $\phi_M \simeq H$. Then from the following calculation

$$[\bar{0} = \bar{1}]_M \cong \phi_M(0/C^{0}(\bar{0}, \bar{1})) \cong \phi_M(1) \cong H(1) \cong 0,$$

we obtain that in the internal logic of $E$, the sequent (C1) holds. Now, observe that from (2), and the moreover part of Lemma 12 it is the case that $\phi_{F_V(x)}(F_V(\bar{x}, \bar{y})/\theta) \cong \phi_{F_V(x)}(0 \times 0)$. So since $\phi_{F_V(x)}$ reflects isomorphisms, we get $F_V(\bar{x}, \bar{y})/\theta \cong 0 \times 0$. Therefore, from the following calculation

$$[\sigma(\bar{x}, \bar{y})]_M \cong \phi_M(F_V(\bar{x}, \bar{y})/\theta) \cong H(0 \times 0) \cong 1 + 1,$$

we get that the sequent (C2) also holds in the internal logic of $E$. Hence, by Lemma 11 (3), we conclude that $M$ is indecomposable in $E$. The proof that a $\mathcal{V}$-indecomposable model $M$ in $E$ determines a functor in $\text{LexCon}(A^{\text{op}}, E)$ is similar. Hence, from D3.1.9 in [21], the functor $\text{LexCon}(A^{\text{op}}, E) \to \text{Mod}(\mathcal{V}, E)$ which sends $H$ to $H(F_V(x))$ is an equivalence of categories.

\[\square\]

4.3 Applications

4.3.1 Bounded distributive lattices

Let $\mathcal{DL}_{01}$ be the variety of bounded distributive lattices. Straightforward calculations show that the term $U(x, y, z, w) = (x \lor z) \land (y \lor w)$, together with the constants 0 and 1, makes $\mathcal{DL}_{01}$ a Pierce variety. In addition it is also well known that the relation $e \circ_A f$ in $\mathcal{DL}_{01}$ is defined by the equations $e \land f = 0$ and $e \lor f = 1$. Hence, by Theorem 2 $\mathcal{DL}_{01}$ is coextensive. Moreover, the only subdirectly irreducible member of $\mathcal{DL}$ is the two element distributive lattice $\mathbb{2}$, then from Theorem 10.16 of [10] it follows that $\mathcal{DL}_{01}$ is locally finite. Thus, by Proposition 2 $\mathcal{DL}_{01}$ is fp-coextensive. Finally, since $F_{\mathcal{DL}_{01}}(x)$ is indecomposable (it is the three element chain) from Corollary 3 and Theorem 7 we can conclude:

**Proposition 3.** The variety $\mathcal{DL}_{01}$ is fp-coextensive and $\mathcal{G}(\mathcal{DL}_{01})$ classifies $\mathcal{DL}_{01}$-indecomposable objects. In particular, the functor $Z : \mathcal{DL}_{01} \to \text{Set}$ preserves all limits and filtering colimits.

It is worth mentioning that the first part of Proposition 3 was stated without a proof in Section 8 of [25] and later on, in [35] a detailed proof was provided.

4.3.2 Integral rigs

A rig is an algebra $\mathbf{A} = (A, +, \cdot, 0, 1)$ of type $(2, 2, 0, 0)$ such that the structures $(A, \cdot, 1)$ and $(A, +, 0)$ are commutative monoids such that “product distributes
over addition” in the sense that \( x \cdot 0 = 0 \) and \( x \cdot (y + z) = (x \cdot y) + (x \cdot z) \) for every \( x, y, z \in A \). One may think such structures as “(commutative) rings (with unit) without negatives”. A rig is said to be integral if the equation \( 1 + x = 1 \) holds, for every \( x \in A \). It is immediate from the latter that the class of integral rigs is a variety. We denote such a variety by \( \mathcal{RN} \). Observe that the term \( U(x, y, z, w) = (x + z) \cdot (y + w) \), together with the constants 0 and 1 makes \( \mathcal{RN} \) a Pierce variety. In [14], it was proved that the relation \( e \circ_a f \) in \( \mathcal{RN} \) is defined by the equations \( e \cdot f = 0 \) and \( e + f = 1 \). Thus by Theorem 2 \( \mathcal{RN} \) is coextensive. Moreover, in Corollary 8.1 of [29] it was proved that \( \mathcal{RN} \) is fp-coextensive.

Proposition 4. The variety \( \mathcal{RN} \) is fp-coextensive and \( \mathcal{G}(\mathcal{RN}) \) classifies \( \mathcal{RN} \)-indecomposable objects. In particular, the functor \( Z : \mathcal{RN} \rightarrow \text{Set} \) preserves all limits and filtering colimits.

For a different proof of the fact that \( \mathcal{G}(\mathcal{RN}) \) classifies \( \mathcal{RN} \)-indecomposable objects, the reader may consult [29].

4.3.3 Commutative rings with unit

Let \( \mathcal{R} \) be the variety of commutative rings with unit. It is fairly known that \( \mathcal{A} \in \mathcal{R} \) is directly indecomposable if and only if the only idempotents in \( \mathcal{A} \) are the trivial ones. Observe that this is equivalent to say that the formula

\[
\sigma(x, y) = (xy = 0) \land (x + y = 1)
\]

defines the relation \( e \circ_a f \) in \( \mathcal{R} \). Moreover, we can employ the same term used for integral rigs and the constants 0 and 1 in order to make \( \mathcal{R} \) a Pierce variety. So from Theorem 2 we get \( \mathcal{R} \) is coextensive. It is also well known that \( \mathcal{R} \) is fp-coextensive. Recall that the free commutative rig with unit on one generator \( x \) can be identified with the ring \( \mathbb{Z}[x] \) of polynomials in the variable \( x \) with coefficients in \( \mathbb{Z} \) endowed with the usual sum and product of polynomials. It is straightforward to see that \( \mathbb{Z}[x] \) is \( \mathcal{R} \)-indecomposable. Hence, due to Corollary 3 and Theorem 7 we obtain the following:

Proposition 5. The variety \( \mathcal{R} \) is fp-coextensive and \( \mathcal{G}(\mathcal{R}) \) classifies \( \mathcal{R} \)-indecomposable objects. In particular, the functor \( Z : \mathcal{R} \rightarrow \text{Set} \) preserves all limits and filtering colimits.

4.3.4 Heyting algebras

A Heyting algebra is an algebra \( \mathcal{A} = (A, \wedge, \vee, \rightarrow, 0, 1) \) of type \( (2, 2, 2, 0, 0) \) such that the structure \( (A, \wedge, \vee, 0, 1) \) is a bounded distributive lattice satisfying \( x \wedge y \leq z \) if and only if \( x \leq y \rightarrow z \). As usual, we denote \( x \rightarrow 0 \) by \( \neg x \). Let \( \mathcal{H} \) be the class of Heyting algebras. It is fairly known that \( \mathcal{H} \) is a variety and
also that this class provide an algebraic semantics for intuitionistic logic. It is no hard to see that the constants 0 and 1 and the term

\[ U(x, y, z, w) = (z \land y) \lor (\lnot z \land x) \]

makes \( H \) a Pierce variety and moreover, that the formula

\[ \sigma(x, y) = (x \land y = 0) \land (x \lor y = 1) \]

defines the relation \( e \circ_A f \) in \( H \). Then Theorem 2 yields that \( H \) is coextensive. Further, by the dual of Proposition 5.5 of [19], \( H \) is fp-coextensive and by Theorem 3.2 [13], all free algebras of \( H \) are \( H \)-indecomposable. Therefore, we can conclude:

**Proposition 6.** The variety \( H \) is fp-coextensive and \( G(H) \) classifies \( H \)-indecomposable objects. In particular, the functor \( Z : H \to \text{Set} \) preserves all limits and filtering colimits.

### 4.3.5 Gödel algebras

A Gödel algebra is an algebra \( A = (A, \land, \lor, \to, 0, 1) \) of type \((2, 2, 2, 0, 0)\) such that the structure \((A, \land, \lor, \to, 0, 1)\) is a Heyting algebra satisfying the prelinearity condition. I.e. the equation \((x \to y) \lor (y \to x) = 1\), holds for every \( x, y \in A \). Gödel algebras provide an algebraic semantics for Gödel logic, which can be defined as the schematic extension of the intuitionistic propositional calculus by the prelinearity axiom \((\alpha \to \beta) \lor (\beta \to \alpha)\). We write \( PH \) for the variety of Gödel algebras. Observe that we can apply the same arguments used for \( H \) in order to prove that \( PH \) is coextensive. Furthermore, due to \( PH \) is a locally finite variety and its type is finite, from Lemma 2, \( PH \) is fp-coextensive. In [17] it was shown that \( F_{PH}(x) \) can be identified with the lattice displayed in Fig. 1. Nonetheless, it is the case that \( \lnot x \) and \( \lnot \lnot x \) are non-trivial complementary central elements of \( F_{PH}(x) \). We have proved:

**Proposition 7.** The variety \( PH \) is fp-coextensive so in particular the functor \( Z : PH \to \text{Set} \) preserves all limits and filtering colimits. Nevertheless \( G(PH) \) does not classifies \( PH \)-indecomposable objects.
4.3.6 MV-algebras

An MV-algebra is an algebra $(A, \oplus, \neg, 0)$ of type $(2, 1, 0)$ such that $(A, \oplus, 0)$ is a commutative monoid such that the following equations hold:

1. $\neg\neg x = x$,
2. $x \oplus 0 = 0$,
3. $\neg(x \oplus y) \oplus y = \neg(y \oplus x) \oplus x$.

We write $\mathcal{MV}$ for the variety of MV-algebras. If we define the following operations

$$x + y = \neg(\neg x \oplus y) \oplus y; \quad 1 = \neg 0; \quad x \cdot y = \neg(x \oplus \neg y),$$

straightforward calculations allows to see that the constants 0 and 1 and the term

$$U(x, y, z, w) = (x + z) \cdot (y + w)$$

makes $\mathcal{MV}$ a Pierce variety. It is known that $\mathcal{MV}$ provides an algebraic semantics for Lukasiewicz logic [15]. From Definition 1.5.2 and Theorem 1.5.3 of (op.cit.) it can be proved that the formula

$$\sigma(x, y) = (x + y = 0) \land (x \cdot y = 1)$$

defines the relation $e \circ f$ in $\mathcal{MV}$. So, by Theorem [2] $\mathcal{MV}$ is coextensive. As a consequence of Theorem 3.4 of [27], it follows that $\mathcal{MV}$ is fp-coextensive. Finally, in [16] it was proved that every free MV-algebra is semisimple and directly indecomposable. Hence we get:

**Proposition 8.** The variety $\mathcal{MV}$ is fp-coextensive and $\mathcal{G}(\mathcal{MV})$ classifies $\mathcal{MV}$-indecomposable objects. In particular, the functor $Z : \mathcal{MV} \to \text{Set}$ preserves all limits and filtering colimits.

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