NONHOLOMORPHIC CORRECTIONS TO THE ONE-LOOP $N = 2$ SUPER YANG-MILLS ACTION

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Abstract

In addition to the familiar contribution from a holomorphic function $F$, the Kähler potential of the scalars in the nonabelian $N = 2$ vector multiplet receives contributions from a real function $H$. We determine the latter at the one-loop level, taking into account both supersymmetric matter and gauge loops. The function $H$ characterizes the four-point coupling of the $N = 2$ vector-multiplet vectors for constant values of their scalar superpartners. We discuss the consequences of our results.

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Low-energy effective actions reveal much of the interesting information about a theory; for string theory and supersymmetric gauge theories, they encode non-perturbative information about the mass spectrum and the static couplings. Recently, considerable attention has been given to the study of effective actions for \( N = 2 \) supersymmetric Yang-Mills theories. This has followed Seiberg and Witten's \([1]\) construction of the exact low-energy effective action for the \( N = 2 \) \( SU(2) \) theory (which breaks down to an abelian phase). The work by Seiberg and Witten and others is based on the description of the \( N = 2 \) superspace effective action by means of a chiral integral of a holomorphic function \( F(W) \), where \( W \) is the \( N = 2 \) gauge superfield strength of the unbroken \( U(1) \) \([2]\). In terms of \( N = 1 \) superfields, this has the form

\[
S = \frac{1}{16\pi^2} \text{Im} \left[ \int d^4x \, d^2\theta \, F_{\phi\phi}(\phi) \left( \frac{1}{2} W^\alpha W_\alpha \right) + \int d^4x \, d^2\theta \, d^2\bar{\theta} \, \bar{\phi} \mathcal{F}_{\phi}(\phi) \right].
\]

Here \( \phi \) denotes the chiral superfield\(^4\) that is the lowest \( N = 1 \) superspace component of the \( N = 2 \) (unbroken) \( U(1) \) gauge multiplet, and \( W_\alpha \) is the \( N = 1 \) gauge superfield strength. This effective action is consistent with (the rigid version of) special geometry \([2]\). In particular, the Kähler potential – the chiral superfield Lagrangian – is given by

\[
K(\phi, \bar{\phi}) = \frac{1}{32\pi i} (\bar{\phi} \mathcal{F}_{\phi} - \phi \bar{\mathcal{F}}_{\phi}).
\]

Classically,

\[
\mathcal{F}(\phi) = \frac{4\pi i}{g^2} \frac{\phi^2}{2}.
\]

In this paper we examine the form of the nonabelian low-energy effective action. The kinetic terms for the component fields of a chiral superfield \( \phi \) come from an \( N = 1 \) superspace Lagrangian, the Kähler potential \( K \). Since \( K \) is a function of \( \phi \) and \( \bar{\phi} \) that does not depend on their (spinor) derivatives, we can calculate its loop corrections in much the same way as one calculates the effective potential in component theories. We compute one-loop contributions to \( K(\phi, \bar{\phi}) \) induced by both \( N = 2 \) vector multiplets and hypermultiplets. Somewhat unexpectedly, in the nonabelian case the Kähler potential cannot be written in the form \( K(\phi, \bar{\phi}) \propto \text{Im} \left[ \bar{\phi} \mathcal{F}_{\phi} \right] \) (cf. \([2]\)), so that it is not determined solely by a holomorphic function \( \mathcal{F}(\phi) \). The additional terms originate from a real function \( \mathcal{H}(W, \bar{W}) \) of the \( N = 2 \) Yang-Mills superfield strength \( W \), which is integrated with the full \( N = 2 \) superspace measure\(^5\). We deduce the one-loop contribution to \( \mathcal{H} \) from our computation of \( K(\phi, \bar{\phi}) \).

\(^4\)Except when discussing Feynman rules, we work with covariantly chiral and antichiral superfields to avoid writing explicit factors of \( e^V \).

\(^5\)Such a function has recently been discussed by Henningson \([3]\) in the abelian limit, where it does not contribute to the effect that we study.
We begin with the classical action for \( N = 2 \) supersymmetric systems written in \( N = 1 \) superspace:

\[
S = \frac{1}{4g^2} \left[ \int d^4x d^2\theta \Tr(\frac{1}{2}W^\alpha W_\alpha) + \int d^4x d^4\theta \Tr(\bar{\phi} e^{-V} \phi e^V) \right] \\
+ \int d^4x d^4\theta (Q e^V Q + \bar{Q} e^{-V} \bar{Q}) + \left( i \int d^4x d^2\theta \bar{Q}_\phi Q + h.c. \right),
\]

where \( \phi \) and \( \bar{\phi} \) are Lie-algebra valued, \( \phi = \phi^A T_A, \bar{\phi} = \bar{\phi}^A T_A \), with \( [T_A, T_B] = i f_{AB}^C T_C \), while \( Q \) and \( \bar{Q} \) are in mutually conjugate representations \( R \) and \( \bar{R} \). The superfields \( W_\alpha \) and \( \phi \) are \( N = 1 \) superfield components of the reduced chiral \( N = 2 \) superfield \( W \) which describes the \( N = 2 \) vector multiplet (see below for details), and \( W_\alpha \equiv i \bar{D}^2(e^{-V} D_\alpha e^V) \) depends on \( V \) in the usual manner. The chiral \( N = 1 \) superfields \( Q \) and \( \bar{Q} \) together describe \( N = 2 \) hypermultiplets. We use the conventions of \textit{Superspace} [4].

\[
\begin{align*}
\text{(a)} & \quad \phi \\
\text{(b)} & \quad \bar{\phi}
\end{align*}
\]

\textbf{Fig. 1. One-loop supergraphs:}

(a) Hypermultiplet contribution; (b) Vector multiplet contributions.

At the one-loop level we consider all diagrams with external \( \phi, \bar{\phi} \) lines, but when doing \( D \)-algebra, drop terms with spinor or space-time derivatives on the external lines. The contributions to the Kähler potential for the chiral superfields \( \phi \) come from internal lines corresponding to the \( (Q, \bar{Q}) \) hypermultiplets and the \( N = 2 \) vector multiplet itself. The chiral multiplets \( Q \) and \( \bar{Q} \) give rise to (with a \( d^2\theta d\bar{\theta} \) integration)

\[
\Tr_R \int \frac{d^4p}{(2\pi)^4} \frac{-1}{p^2} \sum_{n=1}^\infty \frac{1}{n} \left( \frac{\bar{\phi} \phi}{-p^2} \right)^n = \frac{1}{(4\pi)^2} \Tr_R \int_0^\infty dp^2 \ln \left( 1 + \frac{\bar{\phi} \phi}{p^2} \right),
\]

where we summed over diagrams with \( n \) \( \phi \) and \( n \) \( \bar{\phi} \) external lines, \( 2n \) propagators \(-1/\Box\), and \( n \) factors of \( D^2, \bar{D}^2 \) (from the chiral superfield vertices), \( n - 1 \) of which
cancel propagators in the course of $D$-algebra. Integrals are evaluated in Euclidean space. The momentum integral is divergent and needs to be regularized; all cut-off dependence contributes only to renormalizations of the original action (4), and the interesting terms come only from the lower limit of the integral:

$$K_Q(\phi, \bar{\phi}) = -\frac{1}{(4\pi)^2} \text{Tr}_R \left[ \bar{\phi} \phi \ln \frac{\bar{\phi} \phi}{\Lambda^2} \right].$$ (6)

where $\Lambda$ is a renormalization scale.

For the $N = 2$ vector multiplet itself, in a general gauge with a supersymmetric gauge-fixing term $\alpha^{-1}(D^2 V)(\bar{D}^2 V)$, both $V$ and $\phi$ contribute to the Kähler potential. However, it is simplest to calculate in supersymmetric Landau gauge ($\alpha = 0$), where only the $N = 1$ vector multiplet $V$ enters in the loop: the $V$-propagator carries a factor $D^a \bar{D}^2 D_\alpha$, and factors of $D^2$ or $\bar{D}^2$ from internal $\phi$, $\bar{\phi}$ lines annihilate the mixed-loop contributions stemming from vertices $\text{Tr}(\bar{\phi}[V, \phi])$ (recall that we drop terms with derivatives on external lines). Thus, only the vertices $\text{Tr}(\bar{\phi}[V, V, \phi])$ contribute. The result is independent of $\alpha$: the mixed-loop contribution cancels the $\alpha$ dependence of the $V$-propagator in the pure $V$ loops. Summing over diagrams with $n \text{Tr}(\bar{\phi}[V, [V, \phi]])$ vertices and $n V$-propagators $-D^a \bar{D}^2 D_\alpha/\Box^2$, we obtain

$$\text{Tr}_{(ad)} \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2} \sum_n \frac{1}{n} \left( \frac{\phi \bar{\phi} + \bar{\phi} \phi}{-2p^2} \right)^n = -\frac{1}{(4\pi)^2} \text{Tr}_{(ad)} \int_0^\infty dp^2 \ln \left( 1 + \frac{\bar{\phi} \phi + \bar{\phi} \phi}{2p^2} \right).$$ (7)

Doing the integral and renormalizing as before, we obtain

$$K_V(\phi, \bar{\phi}) = \frac{1}{(4\pi)^2} \text{Tr}_{(ad)} \left[ \frac{\phi \bar{\phi} + \bar{\phi} \phi}{2} \ln \frac{\phi \bar{\phi} + \bar{\phi} \phi}{2\Lambda^2} \right].$$ (8)

We can check that our renormalizations of (6) and of (8) are consistent by considering the $N = 4$ supersymmetric case. This is the theory with precisely one hypermultiplet in the adjoint representation, and we find that the dependence on the renormalization scale cancels, as it should.

We observe that the results in (6, 8) are in conflict with special geometry except in the abelian case when $\phi$ and $\bar{\phi}$ commute; in that case, it is easy to recover the following expression for the holomorphic function $\mathcal{F}$,

$$\mathcal{F}(\phi) = -\frac{i}{2\pi} \left( \text{Tr}_R \left[ \phi^2 \ln \frac{\phi^2}{\Lambda^2} \right] - \text{Tr}_{(ad)} \left[ \phi^2 \ln \frac{\phi^2}{\Lambda^2} \right] \right),$$ (9)

where $\Lambda$ has been suitably rescaled.

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\*\*For the $N = 1$ supersymmetric gauge fixing term that we use, there is no coupling of the ghosts to the external fields $\phi$.\*\*
We explicitly evaluate the Kähler potential for $SU(2)$, with $Q, \tilde{Q}$ in the fundamental and adjoint representations. In the first case we find

$$K_{Q}^{fund} = \frac{-1}{(8\pi)^2} \left( \bar{\phi} \cdot \phi \ln \frac{\phi^2 \bar{\phi}^2}{16\Lambda^4} + i|\bar{\phi} \times \phi| \ln \frac{\bar{\phi} \cdot \phi + i|\bar{\phi} \times \phi|}{\bar{\phi} \cdot \phi - i|\bar{\phi} \times \phi|} \right),$$

(10)

where $|\bar{\phi} \times \phi| = \sqrt{\phi^2 \bar{\phi}^2 - (\bar{\phi} \cdot \phi)^2}$, $\phi^2 = \phi^A \phi^A$, and $\bar{\phi} \cdot \phi = \bar{\phi}^A \phi^A$. For the adjoint representation we find

$$K_{\tilde{Q}}^{adj} = \frac{-1}{(4\pi)^2} \left( \bar{\phi} \cdot \phi \ln \frac{\phi^2 \bar{\phi}^2}{\Lambda^4} + 2\bar{\phi} \cdot \phi \ln \frac{\bar{\phi} \cdot \phi}{\sqrt{\phi^2 \bar{\phi}^2}} \right).$$

(11)

The contribution from the vector multiplet itself is

$$K_{V} = \frac{1}{(4\pi)^2} \left[ \bar{\phi} \cdot \phi \ln \frac{\phi^2 \bar{\phi}^2}{\Lambda^4} + \bar{\phi} \cdot \phi \ln \frac{\bar{\phi} \cdot \phi}{\sqrt{\phi^2 \bar{\phi}^2}} \right.$$

$$+ \frac{1}{2} \left( \bar{\phi} \cdot \phi + \sqrt{\phi^2 \bar{\phi}^2} \right) \ln \frac{\bar{\phi} \cdot \phi + \sqrt{\phi^2 \bar{\phi}^2}}{2\sqrt{\phi^2 \bar{\phi}^2}}$$

$$+ \frac{1}{2} \left( \bar{\phi} \cdot \phi - \sqrt{\phi^2 \bar{\phi}^2} \right) \ln \frac{\bar{\phi} \cdot \phi - \sqrt{\phi^2 \bar{\phi}^2}}{2\sqrt{\phi^2 \bar{\phi}^2}} \left] \right).$$

(12)

In the abelian case, only the first term in (10), (11) and (12) survives.

Thus, for the general nonabelian case, the one-loop contributions to the Kähler potential do not have the naively anticipated form. However, by considering higher-dimension local terms in the $N = 2$ superspace effective action, we will show that there is no contradiction with $N = 2$ supersymmetry.

To determine the $N = 2$ superspace terms that give rise to the contributions above, we briefly review $N = 2$ super Yang-Mills theory in superspace. In our notation, the $N = 2$ spinor coordinates form a doublet $\{\theta^{a\alpha}\} = \{\theta^{1\alpha}, \theta^{2\alpha}\}$ which transforms under a rigid $SU(2)$ group unrelated to the gauge group. We also have their complex conjugates $\{\bar{\theta}^{\dot{a}\dot{\alpha}}\}$, and spinor derivatives corresponding to all the coordinates. Chiral integrands are evaluated with the chiral measure $d^4\theta = (d^2\theta^1)(d^2\theta^2)$; the full measure is denoted by $d^4\theta d^4\bar{\theta}$. The rigid $SU(2)$ indices are raised and lowered by the antisymmetric invariant tensor $C_{ab}$, with $C_{12} = C^{12} = 1$. In this context, $N = 2$ super Yang-Mills theory is described by gauge-covariant spinor derivatives satisfying the constraints

$$\{\nabla_{a\alpha}, \nabla_{b\beta}\} = iC_{ab}C_{a\alpha} \bar{\nabla}_\beta,$$

$$\{\nabla_{\dot{a}\dot{\alpha}}, \nabla_{\dot{b}\dot{\beta}}\} = iC^{ab}C_{a\dot{\alpha}} \nabla_{\dot{\beta}},$$

$$\{\nabla_{a\alpha}, \nabla^{b}_{\beta}\} = i\delta^{b}_{a} \nabla_{a\dot{\beta}}.$$

(13)
where $W, \bar{W}$ are chiral (antichiral) scalar superfield strengths, respectively. The Bianchi identities imply the important relation

$$\nabla^a \nabla_{bc} W = C_{ac} C_{bd} \nabla^{d\dot{a}} \nabla_{\dot{a}} \bar{W}.$$  \hspace{1cm} (14)

The procedure for reducing $N = 2$ superfields and action to $N = 1$ form is well known: one defines $N = 1$ superfield components as

$$\phi \equiv W|, \quad W_\alpha \equiv -\nabla \nabla_\alpha W|,$$  \hspace{1cm} (15)

where the bar denotes setting $\theta^{2\alpha} = \bar{\theta}^{2\dot{\alpha}} = 0$. One identifies $\nabla_1$ with the $N = 1$ covariant spinor derivative, and rewrites the $N = 1$ integration measures in terms of $N = 1$ measures with the replacement $d^2 \theta^2 \to (\nabla_2)^2 \equiv \frac{1}{2} \nabla^2 \nabla_{2\alpha}$, etc.

For a chiral $N = 2$ integrand $\mathcal{F}(W)$ one finds the standard result

$$S_{\mathcal{F}} = \int d^4 x d^4 \theta \mathcal{F}(W) = \int d^4 x d^2 \theta^1 \left[ \frac{1}{2} \mathcal{F}_{AB}(\nabla_2^\alpha W^A)(\nabla_{2\alpha} W^B) + \mathcal{F}_A(\nabla_2)^2 W^A \right] | \nabla_1 = \int d^4 x d^2 \theta d^2 \bar{\theta} \mathcal{F}_A(\phi) \bar{\phi}^A,$$  \hspace{1cm} (16)

where we have used the Bianchi identity (14), dropped the superscript on $\theta^{1\alpha}$, and, in the last term of the second line, replaced $\nabla_2^2$ by $d^2 \bar{\theta}$, and we define $\mathcal{F}_A \equiv \partial \mathcal{F}/\partial \phi^A$, etc. Thus, the holomorphic contributions to the effective action that we find at the one-loop level come from $N = 2$ chiral integrands (15) (with $\phi \to W$).

We turn now to the nonholomorphic contributions. Consider the reduction to $N = 1$ of the $N = 2$ superspace integral

$$S_{\mathcal{H}} = \int d^4 \theta d^4 \bar{\theta} \mathcal{H}(W, \bar{W}) .$$  \hspace{1cm} (17)

Explicitly, we write

$$\int d^4 \theta d^4 \bar{\theta} \mathcal{H} = \int d^2 \theta_1 d^2 \bar{\theta}_1 (\nabla_2)^2 (\nabla_2^\alpha) \mathcal{H} | .$$  \hspace{1cm} (18)

We assume that $\mathcal{H}$ is gauge invariant up to total derivatives, which implies

$$f_{AC}^C \mathcal{H}_C W^B + f_{AB}^C \mathcal{H}_C \bar{W}^B = \eta_A(W) + \bar{\eta}_A(W) ,$$  \hspace{1cm} (19)

where $\eta_A(W)$ are holomorphic functions. We introduce a real function $\mu_A$, the moment map (or Killing potential), by

$$f_{AC}^C \mathcal{H}_C W^B = \eta_A(W) + i \mu_A(W, \bar{W}) .$$  \hspace{1cm} (20)

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Note that this decomposition of the $N = 2$ measure is real only modulo total derivatives.

There is a similar issue for $\mathcal{F}$, which is gauge invariant modulo a quadratic polynomial with real coefficients.
For the moment, we will set $\eta = 0$, and assume $\mathcal{H}$ is actually gauge invariant (the calculation with $\eta$ is analogous; see §3, §4).

We expand (13) and use the constraints (13) and the Bianchi identities (14), which imply, in particular,

$$
\nabla^2_{\alpha}(\nabla^2)^2 \bar{W}^A| = -i \nabla^2_{\alpha} \bar{W}^A + f^A_{BC} \bar{\phi}^B \nabla_{\alpha} \phi^C ,
$$

$$
(\nabla^2)^2 \bar{\nabla}^2_{\alpha} \bar{W}^A| = -f^A_{BC} \bar{\phi}^B \nabla_{\alpha} \bar{\phi}^C ,
$$

$$
(\nabla^2)^2(\nabla^2)^2 \bar{W}^A| = \frac{1}{2} \nabla^2_{\alpha} \nabla_{\alpha} \bar{\phi}^A + \frac{1}{2} f^A_{BC} \bar{\phi}^B \nabla_{\alpha} \phi^C 
- f^A_{BC} \bar{\phi}^B \nabla^2_{\alpha} \phi^C - f^A_{BC} f^C_{DE} \bar{\phi}^B \bar{\phi}^D \phi^E . 
$$

We find

$$
S_{\mathcal{H}} = \int d^4 x \, d^2 \theta \, d^2 \bar{\theta} \left[ \mathcal{H}_{ABCD} \left( \frac{1}{4} W^{Da} W^C \bar{W}^B \bar{W}^A \right) + \mathcal{H}_{ABC} \left( \frac{1}{2} W^{Ca} W^B \bar{W}^2 \phi^A \right) 
+ \mathcal{H}_{ABC} \left( \frac{1}{2} W^{B\bar{D}} \bar{W}^A \nabla^2 \phi^C + i W^{Ca} \bar{W}^A \phi^C \bar{\phi}^B \right) 
+ \mathcal{H}_{AB} \left( \nabla^2 \phi^B \nabla^2 \phi^A - f^A_{CD} \phi^B \nabla_{\alpha} \phi^D + i \nabla_{\alpha} \phi^C \bar{\phi} \bar{W}_{\bar{\alpha}} \right) 
+ \mathcal{H}_{\bar{A}B} \left( f^A_{CD} \nabla^2 \phi^B \nabla_{\alpha} \phi^D + \frac{1}{2} \nabla^2 \phi^B \nabla_{\alpha} \phi^D \right) 
+ \mathcal{H}_{\bar{A}} \left( \frac{1}{2} \nabla^2 \phi^A \nabla_{\alpha} \phi^B + \frac{1}{2} f^A_{BC} \phi^B \nabla_{\alpha} \phi^C - f^A_{BC} \bar{\phi}^C \phi^B 
- f^A_{BC} f^C_{DE} \bar{\phi}^B \bar{\phi}^D \phi^E \right) \right],
$$

where $\mathcal{H}$ and its derivatives are evaluated at $\theta^2 = \bar{\theta}^2 = 0$, and hence are functions of the $N = 1$ chiral superfields $\phi$, $\bar{\phi}$.

Just as for $N = 1$ supersymmetric sigma models, this effective action has a natural Kähler geometry: Since the action doesn’t change if we shift $\mathcal{H}$ by a chiral or antichiral function, $\mathcal{H}$ is a Kähler potential defined modulo the real part of a holomorphic function, i.e., a Kähler transformation. We introduce the metric, connection, and curvature:

$$
g_{AB} = \mathcal{H}_{AB} , \quad \Gamma^A_{BC} = g^{AD} \mathcal{H}_{BCD} , \quad R_{ABCD} = \mathcal{H}_{ACBD} - g_{EF} \Gamma^E_{AC} \Gamma^F_{BD} .
$$

Integrating by parts and using (20) and its derivatives, we can rewrite the action (22) in a simpler form:

$$
S_{\mathcal{H}} = \int d^4 x \, d^2 \theta \, d^2 \bar{\theta} \left[ g_{AB} \left[ -\frac{1}{2} \nabla^2 \phi^A \nabla_{\alpha} \phi^B + i \nabla_{\alpha} \phi^A \nabla_{\alpha} \phi^B \right] + i \nabla^2 \phi^B + \frac{1}{2} \Gamma^B_{CD} \nabla_{\alpha} \phi^D \left( \nabla^2 \phi^A + \frac{1}{2} \Gamma^A_{EF} W_{\phi^E} \right) 
+ \frac{1}{4} R_{ABCD} \left( W_{\phi^A} W_{\phi^B} \bar{W}^D \bar{W}^A \right) + i \mu_A \left( \frac{1}{2} \nabla^2 \phi^A + f_{BC} \phi^B \phi^C \right) \right] .
$$
Because we have rewritten the last term in (22) in terms of the moment map \( \mu \), this expression is valid even when \( \eta \) in (21) is nonvanishing.

In \( N = 1 \) language, (24) has contributions to the effective action of vector and scalar multiplets. None of these terms modifies component kinetic terms, except the last one, which is a contribution to the Kähler potential. (Clearly, it vanishes when the superfields \( \phi, \bar{\phi} \) are restricted to lie in an abelian subalgebra.) By comparing this last term to the one-loop contributions we computed above, we can attempt to reconstruct \( \mathcal{H} \). One can not read off all of \( \mathcal{H} \) from just a knowledge of this last term, as it is only determined up to an arbitrary function \( f(\phi^2, \bar{\phi}^2) \). However, we know that the \( \beta \)-function and the axial anomaly come entirely from \( F \), the holomorphic contribution to the effective action; this imposes the further constraint that under an arbitrary complex rescaling of \( \phi, \mathcal{H} \) changes at most by a Kähler transformation. Up to a Kähler transformation, this implies

\[
\mathcal{H} = \mathcal{H}^0 + c \left( \ln \frac{\phi^2}{\Lambda^2} + g^0(\phi) \right) \left( \ln \frac{\bar{\phi}^2}{\Lambda^2} + \bar{g}^0(\bar{\phi}) \right),
\]

(25)

where \( \mathcal{H}^0 \), the holomorphic function \( g^0 \), and its conjugate \( \bar{g}^0 \) are all homogeneous functions and independent of any scale:

\[
\phi^A \mathcal{H}^0_A = \bar{\phi}^A \mathcal{H}^0_{\bar{A}} = 0,
\]

(26)

and \( c \) is a constant. The one-loop contributions we have computed determine \( \mathcal{H}^0 \) uniquely. The ambiguity represented by \( c \) and \( g^0 \) is rather minor: the complex manifold with Kähler potential \( \mathcal{H} \) and coordinates \( \phi \) is a product of a manifold with Kähler potential \( \mathcal{H}^0 \) and homogeneous coordinates, and a flat space with a complex coordinate \( Z = \ln(\phi^2/\Lambda^2) + g^0(\phi) \).

In the case of \( SU(2) \), we can determine \( \mathcal{H}^0 \) and \( g^0 \) from these conditions. In particular, for \( SU(2) \), there is no holomorphic homogeneous gauge invariant function, and so \( g^0 = 0 \); thus the residual ambiguity in \( \mathcal{H} \) is just the constant \( c \) in (25).

We consider first the contribution \( K_Q(\phi, \bar{\phi}) \) in (1). The first term is a holomorphic contribution and we consider it no further. For the second term we introduce the variable

\[
s = i \frac{|\phi \times \bar{\phi}|}{\phi \cdot \bar{\phi}} = i \sqrt{\frac{\phi^2 \bar{\phi}^2}{(\phi \cdot \bar{\phi})^2} - 1},
\]

(27)

and assume that \( \mathcal{H}^0 = \mathcal{H}^0(s) \) so that \( \mathcal{H}^0_A = \mathcal{H}^0_A(s) \bar{s}_{\bar{A}} \) with

\[
s_{\bar{A}} = \frac{\partial s}{\partial \phi^A} = -\frac{1}{s} \left[ \frac{\phi^2 \bar{s}_{\bar{A}}}{(\phi \cdot \bar{\phi})^2} - \frac{\phi^2 \bar{\phi}^2 \phi^A}{(\phi \cdot \bar{\phi})^3} \right].
\]

(28)

Comparing (1) and (22), we obtain

\[
\frac{d\mathcal{H}^0}{ds} = \frac{1}{(8\pi)^2} \frac{1}{1 - s^2} \ln \frac{1 + s}{1 - s},
\]

(29)
which (rewriting the result in terms of $N = 2$ superfields) integrates to
\[ H^0_{\text{fund}}(W, \bar{W}) = \frac{1}{(16\pi^2)^2} \left[ \ln \frac{\bar{W} \cdot W + i|\bar{W} \times W|}{\bar{W} \cdot W - i|\bar{W} \times W|} \right]^2 \]  
(30)

For the other contributions in (11) and (12) it is more convenient to introduce
\[ t = \frac{\bar{\phi} \cdot \phi}{\sqrt{\phi^2 \bar{\phi}^2}} = \frac{1}{\sqrt{1 - s^2}}. \]  
(31)

We find
\[ H^0_{\text{adj}}(W, \bar{W}) = \frac{1}{2(4\pi^2)^2} \int_{T^2} du \ln \frac{u - 1}{u - 1}, \quad u \equiv t^2 \]  
(32)

and
\[ H^0_V(W, \bar{W}) = -\frac{1}{(8\pi^2)^2} \left[ 2 \left( \ln \frac{T + 1}{2} \right) \left( \ln \frac{T - 1}{2} \right) + \int_{T^2} du \ln \frac{u - 1}{u - 1} \right], \]  
(33)

with
\[ T = \frac{\bar{W} \cdot W}{\sqrt{W^2 \bar{W}^2}}. \]  
(34)

We now discuss our results. We see that the calculation of the one-loop Kähler potential, when combined with the restrictions imposed by $N = 2$ supersymmetry, gives us significant information: we obtain both the one-loop holomorphic function $F$, which determines the leading terms in the low-energy effective action (terms at most quadratic in external momenta), as well as (most of) the real function $H$, which contains the next order terms (those at most quartic in momenta).

The contributions to the one-loop Kähler potential (cf. (6) and (8)) can be interpreted in two ways: When integrating over the loop momentum from zero to some ultraviolet cut-off, we obtain a finite result because $\phi$ acts as an infrared cut-off for the nonabelian theory, precisely as described in [8]. This is equivalent to calculating with some low-energy scale $M$, and integrating out momentum modes à la Wilson between $M$ and the ultraviolet cut-off, provided that $\phi \gg M$. The physics at momenta below the scale $M$ is described by an abelian $N = 2$ supersymmetric gauge theory, and is thus encoded in the holomorphic function $F$ as in (1). For an asymptotically free theory, this is the semiclassical domain where the effective gauge coupling constant becomes small. In this case, the low-energy scale $M$ is replaced everywhere by $\phi$, and the nonholomorphic contributions to the Kähler potential do not require inverse powers of $M$ to make them dimensionally correct; instead, extra derivatives are compensated for by negative powers of $\phi$ and $\bar{\phi}$, as is clearly shown by the explicit results given above.

Nevertheless, a puzzle seems to remain. When $\phi$ is comparable in size to the low-energy cut-off $M$, we cannot reconcile our results with $N = 2$ supersymmetry. In
that case, the Kähler potential contains terms proportional to $[M^2 + \phi \bar{\phi}] \ln[M^2 + \phi \bar{\phi}]$, which cannot come from a holomorphic function (as required by special geometry) even when $\phi$ and $\bar{\phi}$ commute. We believe this breaking of $N = 2$ supersymmetry arises as follows: It is well known that the effective action is gauge invariant only if one can shift loop momenta; an infrared cutoff thus violates gauge invariance. Our quantization scheme is only $N = 1$ supersymmetric, which means that we have $N = 2$ supersymmetry only modulo gauge transformations. Therefore, there is no reason to expect that the effective action with a finite cut-off $M$, which breaks gauge-invariance, should be $N = 2$ supersymmetric. We stress that $N = 2$ supersymmetry is lost only when $\phi$ is comparable in size to the low-momentum cut-off $M$; when $\phi$ is either much smaller or much bigger than $M$, $N = 2$ supersymmetry is manifestly realized, and the Kähler potential can be computed from the holomorphic function $\mathcal{F}$ and the real function $\mathcal{H}$.

Our work may be compared to an earlier one [9] where a one-loop superspace calculation of the effective potential in $N = 1$ supersymmetric theories was presented. Superficially our calculation is very similar but the emphasis and the contributions we have considered are different. In ref. [8] supergraphs with external chiral or Yang-Mills superfields were evaluated, but only terms with a maximal number of spinor derivatives (and no spacetime derivatives) outside the loop were kept, thus obtaining a contribution which is higher order in the auxiliary fields of the chiral multiplets and/or vector multiplets, and zeroth order in derivatives of the physical scalars. Here instead, by keeping only terms with a maximal number of spinor derivatives in the loop, we obtained a contribution to the Kähler potential, which at the component level is quadratic or lower order in the auxiliary fields and involves space-time derivatives of the physical fields.

Our results appear to cast doubt on the analysis of [10]. In those works, the authors analyzed the holomorphic contributions to the effective action and concluded that at certain points in moduli space, some field destabilized because the corresponding terms in the effective action changed sign. They did not consider possible modifications due to terms such as those found above. As the results of [10] are compatible with arguments based on nonperturbative formulae for particle masses (BPS bounds), it would be interesting to explore this issue further.

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