Isogenies between K3 Surfaces over $\bar{\mathbb{F}}_p$

Ziquan Yang

Abstract

We generalize Mukai and Shafarevich’s definition of isogenies between K3 surfaces over $\mathbb{C}$ to an arbitrary perfect field and describe how to construct isogenous K3 surfaces over $\bar{\mathbb{F}}_p$ by prescribing linear algebraic data when $p$ is large. The main step is to show that isogenies between Kuga-Satake abelian varieties induce isogenies between K3 surfaces, in the context of integral models of Shimura varieties. As a byproduct, we show that every K3 surface of finite height admits a CM lifting under a mild assumption on $p$.

1 Introduction

Efforts to define the notion of isogeny between K3 surfaces have a long history. We refer the reader to [30] for a summary. Shafarevich defined an isogeny between two complex algebraic K3 surfaces $X, X'$ to be a Hodge isometry $H^2(X', \mathbb{Q}) \sim H^2(X, \mathbb{Q})$ (c.f. [38]). Mukai reserved the term for those Hodge isometries that are induced by correspondences (i.e. algebraic cycles on $X \times X'$) (c.f. [31]). By a recent result of Buskin [8], these two definitions coincide. Therefore, we make the following definition, which generalizes Mukai and Shafarevich’s:

**Definition 1.1.** Let $k$ be a perfect field with algebraic closure $\bar{k}$ and $X, X'$ be two K3 surfaces over $k$ with a correspondence $f : X \rightsquigarrow X'$.

- If char $k = 0$, we say $f$ is an *isogeny* if it induces an isometry $H^2_{\text{et}}(X'_k, \mathbb{Q}_\ell) \sim H^2_{\text{et}}(X_k, \mathbb{Q}_\ell)$ for every prime $\ell$.
- If char $k = p$, we say $f$ is an *isogeny* if it induces an isometry $H^2_{\text{et}}(X'_k, \mathbb{Q}_\ell) \sim H^2_{\text{et}}(X_k, \mathbb{Q}_\ell)$ for every prime $\ell \neq p$ and an isometry $H^2_{\text{cris}}(X'/W(k))[1/p] \sim H^2_{\text{cris}}(X/W(k))[1/p]$.

One could easily construct isogenous complex algebraic K3’s by prescribing $\mathbb{Z}$-lattices. More precisely, using the surjectivity of the period map, one easily deduces the following:

**Theorem 1.2.** Let $X$ be a complex algebraic K3 surface. Let $\Lambda$ be a quadratic lattice isomorphic to $H^2(X, \mathbb{Z})$. Then for each isometric embedding $\Lambda \subset H^2(X, \mathbb{Q})$, there exists another complex algebraic K3 surface $X'$ together with an isogeny $f : X \rightsquigarrow X'$ such that $f^*H^2(X', \mathbb{Z}) = \Lambda$.

\[\text{One could choose to distinguish correspondences only up to their actions on these cohomology groups to get an exact generalization of Mukai and Shafarevich’s definition.}\]
Surjectivity of the period map gives us a complex analytic K3 $X'$ which corresponds to the induced Hodge structure on $\Lambda$. Then one can use [3, IV Theorem 6.2] to see that $X'$ has to be algebraic.

We give an analogue of Theorem 1.2 for quasi-polarized K3 surfaces over $\mathbb{F}_p$, using K3 crystals and $\hat{\mathbb{Z}}_p$-lattices to replace $\mathbb{Z}$-lattices. K3 crystals were originally defined by Ogus to characterize the $F$-crystals that arise from K3 surfaces (c.f. [34, Definition 3.1]):

**Definition 1.3.** Let $k$ be a perfect field of characteristic $p > 0$ and let $\sigma$ be the lift of Frobenius on $W(k)$. A K3 crystal over $k$ is a finitely generated free $W(k)$-module $D$ equipped with a $\sigma$-linear injection $\varphi : D \to D$ and a perfect symmetric bilinear pairing $\langle -,- \rangle$ such that $\langle \varphi(x), \varphi(y) \rangle = p^2 \sigma(\langle x,y \rangle)$, $p^2 D \subset \varphi(D)$, and rank $\varphi \otimes k = 1$.

Recall that a quasi-polarized K3 surface is a pair $(X,\xi)$ where $X$ is a K3 surface and $\xi$ is a primitive big and nef line bundle on $X$. We call the self-intersection number of $\xi$ the degree of $(X,\xi)$. Customarily, we write $P^2_\xi(X) := \langle ch_*(\xi) \rangle^\perp \subset H^2(X)$ for the primitive cohomology of $(X,\xi)$ ($* = dR, \text{cris}, \text{ét}$, etc). Our main theorem states:

**Theorem 1.4.** Let $(X,\xi)$ be a quasi-polarized K3 surface over $\mathbb{F}_p$ of degree $2d$. Assume that $p > 18d + 4$. Let $(\Lambda_p,\varphi,\langle -,- \rangle)$ be a K3 crystal over $\mathbb{F}_p$ such that as a quadratic lattice $\Lambda_p \cong P^2_{\text{cris}}(X/W(\mathbb{F}_p))$ and let $\Lambda^p$ be a quadratic lattice isomorphic to $P^2_{\text{ét}}(X,\hat{\mathbb{Z}}_p)$. Then for each pair of

1. isometric embedding $\Lambda_p \subset P^2_{\text{cris}}(X/W(\mathbb{F}_p))[1/p]$ such that $\varphi$ agrees with the Frobenius action

2. isometric embedding of $\Lambda^p \subset P^2_{\text{ét}}(X,\mathbb{A}^p_f)$

there exists another quasi-polarized K3 surface $(X',\xi')$ of degree $2d$ over $\mathbb{F}_p$ with an isogeny $f : X \sim X'$ such that $f^* ch_*(\xi') = ch_*(\xi)$ for $* = \text{cris, ét}$, $f^* P^2_{\text{cris}}(X'/W(\mathbb{F}_p)) = \Lambda_p$ and $f^* P^2_{\text{ét}}(X',\hat{\mathbb{Z}}_p) = \Lambda^p$.

The above theorem is obtained by exploring the implications of Kisin’s work ([18]) on the Langlands-Rapoport conjecture for K3 surfaces, via the Kuga-Satake construction. We summarize the construction in the following diagram:

$$
\begin{array}{ccc}
\mathcal{S}(\text{CSpin}(L_d),\Omega) & \longrightarrow & \text{A Siegel modular variety} \\
\downarrow & & \\
\tilde{M}_{2d,\mathbb{Z}(p)} & \longrightarrow & \mathcal{S}(\text{SO}(L_d),\Omega)
\end{array}
$$

In this diagram, $L_d$ is a certain quadratic lattice; $\mathcal{S}(\text{CSpin}(L_d),\Omega)$ and $\mathcal{S}(\text{SO}(L_d),\Omega)$ are the canonical integral models of the Shimura varieties associated to algebraic groups $\text{CSpin}(L_d)$, $\text{SO}(L_d)$ and a certain period domain $\Omega$; $\tilde{M}_{2d,\mathbb{Z}(p)}$ is a moduli stack of quasi-polarized K3 surfaces of degree $2d$ with some additional structures. We have temporarily suppressed level structures, but we will introduce these objects in detail in Section 3. Kisin’s work clarifies the notion of isogeny classes on $\mathcal{S}(\text{CSpin}(L_d),\Omega)(\mathbb{F}_p)$. We could prove that, as one would expect, K3 surfaces with isogenous Kuga-Satake abelian varieties are themselves isogenous.
Proposition 1.5. (rough form) Assume $p \nmid d$ and $p \geq 5$. Let $t, t' \in \widetilde{M}_{2d, \mathbb{Z}_p}((\overline{\mathbb{F}}_p))$ correspond to quasi-polarized K3 surfaces $(X_t, \xi_t)$ and $(X_{t'}, \xi_{t'})$ of degree $2d$. If the images of $t, t'$ in $\mathcal{S}(\text{SO}(L_d), \Omega)(\overline{\mathbb{F}}_p)$ lift to points $s, s' \in \mathcal{S}(\text{CSpin}(L_d), \Omega)(\overline{\mathbb{F}}_p)$ in the same isogeny class, then there exists an isogeny $X_t \sim X_{t'}$ such that $f^*\text{ch}_*(\xi_{t'}) = \text{ch}_*(\xi_t)$ for $* = \text{cris}, \text{ét}$. The precise form is stated in Proposition 5.1. Once we have Proposition 5.1, we could prove Theorem 1.4 by some group-theoretic computation. The condition on $p$ arises when we apply the surjectivity of the period map.

As a key intermediate step in [18], Kisin proved that in each isogeny class of mod $p$ points of a Shimura variety of Hodge type, there exists a point which lifts to a special point. One could, of course, use this and Theorem 1.5 to furnish a Honda-Tate theorem for K3 surfaces. It turns out that for K3 surfaces of finite height, one could propagate the property of having CM lifting within an isogeny class, so there is no need to pass to isogeny. While preparing for this paper, the author noticed that K. Ito, T. Ito and T. Koshikawa released a new preprint [16] which contains the same result with the condition on $p$ relaxed.

Theorem 1.6. Let $X/\overline{\mathbb{F}}_p$ be a K3 surface of finite height. Suppose $X$ admits a quasi-polarization of degree $p \nmid d$ and $p \geq 5$. Then there exists a finite extension $V/W(\overline{\mathbb{F}}_p)$, a lift $X_V$ of $X$ to $V$ such that for any $V \hookrightarrow \mathbb{C}$, the complex K3 surface $X_V \times \mathbb{C}$ has commutative Mumford-Tate group. Moreover, the natural map $\text{Pic}(X_V) \to \text{Pic}(X)$ is an isomorphism.

Finally, we remark that isogenies between K3 surfaces (in our sense) can also be given by constructing moduli of twisted sheaves. The moduli theory of twisted sheaves for complex K3’s was initiated by S. Mukai and its generalization to positive characteristic has been studied by Lieblich, Maulik, Olsson and Snowden (c.f. [20], [21], [22]). Huybrechts has shown that in fact, over $\mathbb{C}$, every isogeny can be realized by a sequence of such operations [13, Theorem 0.1]. It will be interesting to figure out whether this is true over $\overline{\mathbb{F}}_p$. We also refer the reader to Liendke’s beautiful paper [23] which takes another point of view on isogenies. We remark that it is interesting that supersingular K3 surfaces are all isogenous to each other under either point of view (c.f. Remark 5.3).

Outline/Strategy of the Paper In section 2 and 3, we review the Kuga-Satake construction and the relevant Shimura varieties, mostly following [24] and [25]. In addition, we emphasize that an isogeny between Kuga-Satake abelian varieties should not be a bare isogeny between abelian varieties, but one who respects certain tensors. We call such isogenies “CSpin-isogenies”. This is an analogue of André’s notion of an isomorphism between Kuga-Satake packages (c.f. [1, Definition 4.5.1, 4.7.1]). We also discuss special endomorphisms of Kuga-Satake abelian varieties, which correspond to line bundles on K3 surfaces. Special endomorphisms are preserved by CSpin-isogenies.

In Sections 4, we prove a lifting lemma, which we use in Section 5 to prove the theorems of the paper. The main idea is that a K3 surface of finite height deforms like an elliptic curve: Such a K3 surface is also naturally associated with an one-dimensional formal group of finite height, namely its formal Brauer group $\widehat{\text{Br}}_X$. There is a natural map of deformation functors $\text{Def}_X \to \text{Def}_{\widehat{\text{Br}}_X}$. Although this map is not an isomorphism, Nygaard and Ogus constructed a natural section to this map. We make use of this section to show that away from the
supersingular locus, CSpin isogenies always lift to characteristic zero. This treats the finite height case for Proposition 1.5. We treat the supersingular case separately.

Notations In this paper, $\mathbb{S}$ denotes the Deligne torus, $\hat{\mathbb{Z}}$ the profinite completion of $\hat{\mathbb{Z}}$, $\mathbb{Z}^p$ the prime-to-$p$ part of $\hat{\mathbb{Z}}$. We write $\mathbb{A}_f$ for finite adeles and $\mathbb{A}_f^p$ its prime-to-$p$ part. If $G$ is an algebraic group over $\mathbb{Q}$ and $K \subseteq G(\mathbb{A}_f)$ is a compact open subgroup, we write $K_p \subseteq G(\mathbb{Q}_p)$ for its $p$-component and $K^p \subseteq G(\mathbb{A}_f^p)$ for its prime-to-$p$ component. We write $W$ for $W(\overline{\mathbb{F}}_p)$ and $K$ for its fraction field. The lift of Frobenius on $W$ is denoted by $\sigma$. If $H \subseteq G$ are groups, we denote the centralizer of $H$ in $G$ by $C_G(H)$. For any $p$-divisible group $\mathcal{G}$, we let $\mathbb{D}(\mathcal{G})$ denote its contravariant Dieudonné module and $\mathcal{G}^*$ its Serre dual. For a ring $R$ and a finite free $R$-module $M$, we denote by $M^{\oplus}$ the direct sum of the $R$-modules obtained from $M$ by taking duals, tensor products, symmetric and exterior powers.

2 Hodge Theoretic Preparations

Hodge Structures of K3 Surfaces We collect some definitions and facts about Hodge structures of K3 surfaces. Let $X$ be a complex projective K3 surface. Recall that by Lefschetz (1,1)-theorem we have that $\text{Pic}(X) \cong H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$. The transcendental lattice of $X$, denoted by $T(X)$, is $\text{Pic}(X)^\perp \subseteq H^2(X, \mathbb{Z})$. In the following theorem, (a) and (b) are due to Zarhin [43]. Part (c) is due to Borcea [7]. In fact, these results can be phrased purely in terms of Hodge structures of K3 type (c.f. [14] Chapter 3). We denote by $\text{MT}(\cdot)$ the Mumford-Tate group associated to a Hodge structure.

Theorem 2.1. a. $E := \text{End}_{\text{Hdg}}(T(X)_\mathbb{Q})$ is either a totally real field or a CM field.

b. $\text{MT}(T(X)_\mathbb{Q})$ is commutative if and only if $\dim_E T(X)_\mathbb{Q} = 1$. In this case, $E$ is a CM field.

c. When $E$ is a CM field, there exists a Hodge isometry $\eta$ such that $E = \mathbb{Q}(\eta)$.

Remark 2.2. $X$ is said to have CM if $\text{MT}(T(X)_\mathbb{Q})$ is commutative. Some authors will say that $X$ has CM if $E$ is a CM field. We emphasize that $E$ being a CM field is not enough to ensure that $\text{MT}(T(X)_\mathbb{Q})$ is commutative. Here is an example: Take two elliptic curves $E_1$ and $E_2$. Suppose $E_1$ has CM but $E_2$ does not. Apply the Kummer construction to the abelian surface $E_1 \times E_2$. We obtain a K3 surface $X$ with $\dim T(X)_\mathbb{Q} = 4$ but $E := \text{End}_{\text{Hdg}}(T(X)_\mathbb{Q})$ is a CM field of dimension 2. The real points of the Hodge group $\text{Hdg}(T(X)_\mathbb{Q})$ form a unitary group of signature $(1+, 1-)$ and hence $\text{MT}(T(X)_\mathbb{Q})$ cannot be commutative.

Lemma 2.3. Let $X$ be a K3 surface over $\mathbb{C}$, $\alpha : \mathbb{C} \to \mathbb{C}$ be an automorphism of $\mathbb{C}$ and $X^\alpha \to X$ be the pullback of $\alpha$. Then $X$ has CM if and only if its pullback $X^\alpha$ does.

Proof. Suppose $X$ has CM. By Theorem 2.1 one could choose a Hodge isometry which generates $\text{End}(T(X)_\mathbb{Q})$. By Buskin’s result, Hodge isometries are algebraic, and hence absolutely Hodge. Then we know that $X'$ has CM because $\dim Q \text{End}_{\text{Hdg}}(T(X)_\mathbb{Q}) = \dim Q \text{End}_{\text{Hdg}}(T(X')_\mathbb{Q})$ and $\dim Q T(X) = \dim Q T(X')$. \qed
A Review of Clifford Algebra  Let $L$ be a self-dual lattice over a commutative ring $R$. Let $q$ denote the quadratic form on $L$. Assume that 2 is invertible in $R$. We could form the Clifford algebra
\[ \text{Cl}(L) := (\bigoplus_{n \geq 0} L \otimes_n)/(v \otimes v - q(v)) \]
\[ \text{Cl}(L) \text{ has a natural } \mathbb{Z}/2\mathbb{Z}-\text{grading and we denote by } \text{Cl}^+(L) \text{ (resp. } \text{Cl}^-(L)) \text{ the even (resp. odd) part. We define the group } \text{CSpin}(L) \text{ by} \]
\[ \text{CSpin}(L) = \{ v \in \text{Cl}^+(L)^\times : vLv^{-1} = L \} \]

We could give $\text{CSpin}(L)$ the structure of an algebraic group over $R$ by defining $\text{CSpin}(L)(R') = \text{CSpin}(L_{R'})$ for any $R$-algebra $R'$. There is a norm map $\text{Nm} : \text{CSpin}(L) \to \mathbb{G}_m$ given by $v \mapsto v^* \cdot v$, where $v^*$ is the natural anti-involution of $v$. By letting $\text{CSpin}(L)$ act on $L$ by conjugation, we get the adjoint representation of $\text{CSpin}(L)$, which fits into an exact sequence (of algebraic groups)
\[ 1 \to \mathbb{G}_m \to \text{CSpin}(L) \to \text{SO}(L) \to 1 \quad (2.4) \]

Let $H$ be a free $\text{Cl}(L)$ bi-module of rank 1. Note that $H$ has a natural $\mathbb{Z}/2\mathbb{Z}$-grading. Left multiplication of $\text{Cl}(L)$ on $H$ gives us a spin representation $\text{sp} : \text{CSpin}(L) \to \text{GL}(H)$ and an embedding
\[ L \hookrightarrow \text{Cl}(L) \hookrightarrow \text{End}H \quad (2.5) \]

Equip $\text{End}H$ with the metric $(\alpha, \beta) := 2^{-\text{rank}L \cdot \text{tr}(\alpha \circ \beta)}$. We recall two simple facts, which could also be found in [25, Section 1.3]:

**Lemma 2.6.**

a. The embedding $\tilde{h}$ is an isometry with respect to $q$ and $(-,-)$.

b. There exists a unique orthogonal projection $\pi : \text{End}H \to L \subset \text{End}H$.

c. $\text{CSpin}(L)$ is the stabilizer of $\pi \in H^{(2,2)}$ among all automorphisms of $H$ as a $\mathbb{Z}/2\mathbb{Z}$-graded right $\text{Cl}(L)$-module.

**The Kuga-Satake Construction**  We assume that $(L,q)$ has signature $(n+,2-)$ and take $R = \mathbb{Q}$. The Kuga-Satake construction associates a Hodge structure $\tilde{h}$ on $H$ of type $\{(1,0),(0,1)\}$ to each Hodge structure $h$ on $L$ of type $\{(-1,1),(0,0),(1,-1)\}$ which respects $q$. It could be packed in the following diagram

\[ \text{CSpin}(L_{\mathbb{R}}) \xrightarrow{\text{sp}} \text{GL}(H_{\mathbb{R}}) \]
\[ \tilde{h} \xrightarrow{\text{ad}} \]
\[ \mathbb{S} \xrightarrow{h} \text{SO}(L_{\mathbb{R}}) \]

**Figure 1:** Kuga-Satake construction for a Hodge structure of K3 type

For each $h$, there is a unique lift $\tilde{h}$ such that $\text{Nm} \circ \tilde{h}$ gives the norm map $\mathbb{S} \to \mathbb{G}_{m,\mathbb{R}}$. The composition $\text{sp} \circ \tilde{h}$ gives the desired Hodge structure of weight 1 on $H$. In fact, the induced Hodge structure on $H$ has a simple description by that on $L$:

\[ F^1H_{\mathbb{C}} = \ker(F^1L_{\mathbb{C}}) \]
Here we are viewing $L$ as a subspace of $\text{End} H$ via left multiplication, so it makes sense to talk about $\ker F^1L_C$.

The following simple observation makes the Kuga-Satake construction naturally suitable for studying complex multiplication:

**Lemma 2.7.** MT($\tilde{h}$) is commutative if and only if MT($h$) is commutative.

**Proof.** We have the following exact sequence

$$1 \to \mathbb{G}_m, \mathbb{Q} \to \text{MT}(\tilde{h}) \to \text{MT}(h) \to 1$$

by pulling back (2.4) along the inclusion $\text{MT}(h) \hookrightarrow \text{SO}(L)$.

**Lemma 2.8.** Every isometry $g \in \text{SO}(L)(\mathbb{Q})$ preserving the Hodge structure $h$ lifts to $\tilde{g} \in \text{CSpin}(L)(\mathbb{Q})$ preserving $\tilde{h}$. In other words, the natural map

$$\text{CSpin}(L)(\mathbb{Q})(\text{MT}(\tilde{h})(\mathbb{Q})) \to \text{SO}(L)(\mathbb{Q})(\text{MT}(h)(\mathbb{Q}))$$

is surjective.

**Proof.** Let $\tilde{g} \in \text{CSpin}(L)(\mathbb{Q})$ be any lift of $g \in \text{SO}(L)(\mathbb{Q})$. In fact, since the kernel of $\text{ad}$ lies in the center, one lift works if and only if any other one does. Let $\omega$ be any generator of $F^1L_C$. Since $g$ preserves the Hodge structure on $L$ and $F^1L_C$ is one-dimensional, we have

$$\tilde{g}\omega\tilde{g}^{-1} = \lambda \omega \Rightarrow \tilde{g}\omega = \lambda \omega \tilde{g}$$

for some $\lambda \neq 0 \in \mathbb{C}$

Now if $a \in F^1H_C$, then $\omega a = \lambda^{-1} \tilde{g} \omega a = 0$, which means that $\tilde{g} a \in F^1H_C$.

### 3 Kuga-Satake via Integral Canonical Models

Let $\Lambda = \mathbb{U}^{\oplus 2} \oplus \mathbb{E}_8$ be the K3 lattice. As is customary in the study of K3 surfaces, we equip the second cohomology of K3 surfaces with the *negative* Poincaré pairing. Fix a prime $p > 2$ and $d \in \mathbb{Z}_{>0}$ which is prime to $p$. Let $e, f$ be a basis of the first copy of the hyperbolic plane $\mathbb{U}$ such that $\langle e, e \rangle = \langle f, f \rangle = 0, \langle e, f \rangle = 1$ and set $L_d = \langle e - df \rangle^\perp$. $L_d$ is an integral lattice of discriminant $2d$ and is abstractly isomorphic to $P^2(X_C, \mathbb{Z})$ for any quasi-polarized K3 surface $(X_C, \xi_C)$ of degree $2d$ over $\mathbb{C}$. Set $G = \text{CSpin}(L_d, \mathbb{Q})$ and $G^{\text{ad}} = \text{SO}(L_d, \mathbb{Q})$ to be the associated algebraic groups over $\mathbb{Q}$. In this paper, we always assume that $p \nmid 2d$ so that $L_{\mathbb{Z}(p)}$ is self-dual and $G$ and $G^{\text{ad}}$ admit natural extensions to $\mathbb{Z}(p)$, which we denote by the same symbols. Set $H$ to be $\text{Cl}(L_d)$, viewed as a $\text{Cl}(L_d)$-bimodule.

For any subgroup $K \subset G(\mathbb{A}_f)$, we denote its image in $G^{\text{ad}}(\mathbb{A}_f)$ by $K^{\text{ad}}$. Let $\mathbb{K} \subset G(\mathbb{A}_f)$ be the compact open subgroup $G(\mathbb{A}_f) \cap \text{Cl}(L_d, \mathbb{Z})^\times$. We call $\mathbb{K}^{\text{ad}} \subset G^{\text{ad}}(\mathbb{A}_f)$ the *discriminant kernel*. A compact open subgroup $K' \subset G^{\text{ad}}(\mathbb{A}_f)$ is called *admissible* if $K' \subset \mathbb{K}^{\text{ad}}$. 
3.1 Moduli stack of K3’s

A K3 surface over a scheme $S$ is a proper and smooth algebraic space $f : X \to S$ whose geometric fibers are K3 surfaces. A polarization (resp. quasi-polarization) of a K3 surface $X \to S$ is a section $\xi \in \text{Pic}(X/S)(S)$ whose fiber at each geometric fiber is an ample (resp. big and nef). A section $\xi$ is called primitive if for all geometric points $s \to S$, $\xi(s)$ is primitive, i.e. $\xi(s)$ is not a non-trivial multiple of another line bundle. $\deg \xi$ is set to be the self-intersection number of $\xi$.

Let $M_{2d}$ be the moduli problem over $\mathbb{Z}[1/2]$ which sends each $\mathbb{Z}[1/2]$-scheme $S$ to the groupoid of tuples $(f : X \to S, \xi)$, where $f : X \to S$ is a K3 surface and $\xi$ is a primitive quasi-polarization (resp. polarization) of $X$ with $\deg(\xi) = 2d$. $M_{2d}$ and $M_{2d}$ are Deligne-Mumford stacks of finite type over $\mathbb{Z}$, $M_{2d}$ is separated, and the natural map $M_{2d} \to M_{2d}$ is an open immersion (c.f. [35], [27], [24]).

Let $(f : \mathcal{X} \to \mathcal{M}_{2d, \mathbb{Z}(p)}; \xi)$ be the universal object over $\mathcal{M}_{2d, \mathbb{Z}(p)}$. For each scheme $T \to \mathcal{M}_{2d, \mathbb{Z}(p)}$, we denote by $(X_T, \xi_T)$ the pullback of $(\mathcal{X}, \xi)$. We have the relative second relative étale cohomology $H^2_{\text{ét}}$ of $\mathcal{X}$. Set $P^2_\ell$ to be the orthogonal complement of the $\ell$-adic Chern class $c_2^\ell(\xi)$ of $\xi$. Similarly we could define relative de Rham cohomology $H^2_{\text{dR}}$, which comes with a natural filtration $\text{Fil}^\bullet$, and its primitive part $P^2_{\text{dR}}$. For each scheme $T \to \mathcal{M}_{2d, \mathbb{F}_p}$, we could additionally consider the crystal $P^2_{\text{cris}, T}$ which is a vector bundle over $(T/\mathbb{Z}_p)_{\text{cris}}$. Let $\widetilde{M}_{2d} \to \mathcal{M}_{2d}$ be the étale 2-cover corresponding to the choice of an isometry of sheaves $\det(L_\ell) \otimes \mathbb{Z}_2 \sim \det(P^2_\ell)$. We call it the orientation cover.

In order to construct the period map, we will need to consider moduli of K3 surfaces with level structures. We put together the relative $\ell$-adic cohomology sheaves $H^2_\ell$ to form $H^2_{\text{ét}} := \prod_{\ell \neq p} H^2_\ell$ and similarly we put together Chern classes to form $c^\ell_{2,p}(\xi)$ in $H^2_{\text{ét}}$. Let $I^p$ be the étale sheaf over $M_{2d, \mathbb{Z}(p)}$ which associates to each morphism $S \to M_{2d, \mathbb{Z}(p)}$ the set

$$\{ \text{isometries } \eta : \Lambda_{\mathbb{Z}^p} \otimes \mathbb{Z}_2 \to H^2_{\text{ét}} \text{ such that } \eta(e - df) = c^\ell_{2,p}(\xi) \}$$

It comes equipped with a natural action by the constant sheaf of groups $K'$, for each admissible $K' \subset G^\text{ad}(\mathbb{A}_f)$. We let $M_{2d, K', \mathbb{Z}(p)}$ denote the relative moduli problem over $M_{2d, \mathbb{Z}(p)}$ which attaches to each morphism $S \to M_{2d, \mathbb{Z}(p)}$ the set $H^0(S, I^p/K'^p)$. A section $[\eta] \in H^0(S, I^p/K'^p)$ is called a $K'^p$-level structure on $S$. Denote by $M_{2d, K', \mathbb{Z}(p)}$ the corresponding pullback of $\widetilde{M}_{2d, \mathbb{Z}(p)}$. When $K'_p$ is fixed, denote by $\mathcal{M}_{2d, K'_p, \mathbb{Z}(p)}$ the inverse limit over $K'^p$.

3.2 Shimura Varieties

Let $\Omega$ be the space of oriented negative definite planes in $L_{d, \mathbb{R}}$. $\Omega$ can be seen as a $G(\mathbb{R})$—(resp. $G^\text{ad}(\mathbb{R})$)—conjugacy class of Hodge structures on $H_{\Omega}$ (resp. $L_{d, \mathbb{Q}}$) (c.f. [25] 3.1). Then $(G, \Omega)$ and $(G^\text{ad}, \Omega)$ are Shimura data with reflex field $\mathbb{Q}$ and the adjoint map $G \to G^\text{ad}$ induces a natural map $(G, \Omega) \to (G^\text{ad}, \Omega)$.

Let $K \subset G(\mathbb{A}_f)$ be a compact open subgroup with $K_p = G(\mathbb{Z}_p)$. Then the induced map
\[ \Delta(K) = \frac{\mathbb{A}_f^x}{\mathbb{Q}^x(K \cap \mathbb{A}_f^x)} = \frac{K_p^{p,x}}{\mathbb{Z}_{(p)}^{p,x}}(K_p \cap K_p^{p,x}) \] (3.1)

One may choose a non-degenerate, alternating pairing \( \psi \) on \( H = \text{Cl}(L_d) \) such that the representation \( \text{sp} : G \to \text{GL}(H) \) factors through \( \text{GSp}(H, \psi) \). By an explicit computation in [27], \( \psi \) can be chosen such that \( H_{\mathbb{Z}_{(p)}} \) is self-dual. Let \( \mathcal{H} \) be the union of Siegel half-spaces attached to \( (H, \psi) \). Then we have an embedding of Shimura data \( (G, \Omega) \to (\text{GSp}, \mathcal{H}) \) (c.f. [25 3.5]). For each compact open \( K \subset K_\psi \), we obtain a map of Shimura varieties over \( \mathbb{Q} \)

\[ i : \text{Sh}_K(G, \Omega) \to \text{Sh}_{K_\psi}(\text{GSp}, \mathcal{H}) \]

Using the same argument as in [25 3.4, 3.10], we see that when \( K \subset \mathbb{K} \), the spin (resp. adjoint) representation of \( G \) gives rise to a variational \( \mathbb{Z} \)-Hodge structure \( (H_B, \text{Fil}^* \text{H}_{dR,\mathbb{C}}) \) (resp. \( (L_B, \text{Fil}^* \text{L}_{dR,\mathbb{C}}) \)) on \( \text{Sh}_K(G, \Omega) \). \( (H_B, \text{Fil}^* \text{H}_{dR,\mathbb{C}}) \) carries in addition a \( \mathbb{Z}/2\mathbb{Z} \)-grading, right \( \text{Cl}(L_d) \)-action and a tensor \( \pi_B \in H^0(\text{Sh}_K(G, \Omega)_C, H^0_B \otimes \mathbb{Q}) \) such that \( (L_B \otimes \mathbb{Q}, \text{Fil}^* \text{L}_{dR,\mathbb{C}}) \) can be identified with the image \( \pi_B(H_B \otimes \mathbb{Q}, \text{Fil}^* \text{H}_{dR,\mathbb{C}}) \) and \( L_B^+ \subseteq L_B \otimes \mathbb{Q} \) is identified with \( \pi(H_B) \) (c.f. [25 1.5]). Similarly, the canonical representation of \( G^{\text{ad}} \) gives rise to a variational \( \mathbb{Z} \)-Hodge structure on \( \text{Sh}_{K^{\text{ad}}}(G^{\text{ad}}, \Omega)_C \), which we also denote by \( (L_B, \text{Fil}^* \text{L}_{dR,\mathbb{C}}) \), as it can be viewed as the descent of the one on \( \text{Sh}_K(G, \Omega)_C \).

Let \( K_\psi \subset \mathbb{K}_\psi \subset \text{GSp}(\mathbb{A}_f) \) be the stabilizer of \( \text{Cl}(L_d, \mathbb{Z}) \). For every compact open subgroup \( K_\psi \subset K_\psi \), we could equip \( \text{Sh}_{K_\psi}(\text{GSp}, \mathcal{H}) \) a universal abelian scheme. If we denote by \( h^{\text{an}} : A \to \text{Sh}_{K_\psi}(G, \Omega) \) the pullback family, then we could identify \( (H_B, \text{Fil}^* \text{H}_{dR,\mathbb{C}}) \) with \( (R^1h^{\text{an}}\mathbb{Z}, R^1h^{\text{an}}\Omega^*_A) \). In particular, this implies that \( A \) carries a \( \mathbb{Z}/2\mathbb{Z} \)-grading and a left \( \text{Cl}(L_d) \)-action.

**Integral Models** From now on we fix \( K_p = G(\mathbb{Z}_p) \) and \( K_p^{\text{ad}} = G^{\text{ad}}(\mathbb{Z}_p) \). Consider the limits

\[ \text{Sh}_{K_p}(G, \Omega) = \lim_{K_p \subset G(\mathbb{A}_f)} \text{Sh}_{K_p}(G, \Omega), \quad \text{Sh}_{K_p^{\text{ad}}}(G^{\text{ad}}, \Omega) = \lim_{K_p^{\text{ad}} \subset G^{\text{ad}}(\mathbb{A}_f)} \text{Sh}_{K_p^{\text{ad}}}(G^{\text{ad}}, \Omega) \]

By [17 Main Theorem], there exist canonical integral models \( \mathcal{I}_{K_p}(G, \Omega) \) and \( \mathcal{I}_{K_p^{\text{ad}}}(G^{\text{ad}}, \Omega) \) of \( \text{Sh}_{K_p}(G, \Omega) \) and \( \text{Sh}_{K_p^{\text{ad}}}(G^{\text{ad}}, \Omega) \) over \( \mathbb{Z}_{(p)} \). By the extension property of canonical integral models, the map \( \text{Sh}_{K_p}(G, \Omega) \to \text{Sh}_{K_p^{\text{ad}}}(G^{\text{ad}}, \Omega) \) extends to a pro-étale map

\[ \mathcal{I}_{K_p}(G, \Omega) \to \mathcal{I}_{K_p^{\text{ad}}}(G^{\text{ad}}, \Omega) \]

We first work with finite levels when \( K_p \) is sufficiently small. The idea of the construction is simple: Let \( K_{\psi, p} \) be the stabilizer of \( H_{\mathbb{Z}_p} \) so that \( K_p = K_{\psi, p} \cap G(\mathbb{Q}_p) \). One may choose \( K_{\psi} = K_{\psi, p} K_p^{\text{ad}} \subset \text{GSp}(\mathbb{A}_f) \) containing \( K \) such that the induced map \( \text{Sh}_K(G, \Omega) \to \text{Sh}_{K_{\psi}}(\text{GSp}, \mathcal{H}) \) is an embedding. Using the modular interpretation of \( \text{Sh}_{K_{\psi}}(\text{GSp}, \mathcal{H}) \) one could extend it to \( \mathbb{Z}_{(p)} \). The integral model \( \mathcal{I}_K(G, \Omega) \) is constructed simply by taking the normalization of

---

2We have communicated to Madapusi Pera and confirmed that in [25 (3.2)] there should be no \( "> 0" \) sign in the formula of \( \Delta(K) \).
the closure of its generic fiber inside \( \mathcal{S}_K \). The main point of [17] is to prove that this normalization is smooth and has the desired extension property. The integral model \( \mathcal{S}_K^{ad}(G^{ad}, \Omega) \) is constructed out of \( \mathcal{S}_K^{ad}(G, \Omega) \). By taking limits, one obtain the canonical models \( \mathcal{S}_K^{ad}(G, \Omega) \) and \( \mathcal{S}_K^{ad}(G^{ad}, \Omega) \). Canonical models at finite levels can be recovered by taking quotients.

Let \( h : A \to \mathcal{S}_K^{ad}(G, \Omega) \) be the pullback of the universal abelian scheme over \( \mathcal{S}_K^{ad}(G, \Omega) \). For each scheme \( T \to \mathcal{S}_K^{ad}(G, \Omega) \), we denote by \( A_T \) the pullback of \( A \). Set \( H_{dR} = R^1h_*\Omega^1_{\mathcal{S}_K^{ad}(G, \Omega)} \) to be the first relative de Rham cohomology. Clearly the previously discussed \( H_{dR, \mathcal{C}} \) is its fiber over \( \mathcal{C} \). Using the de Rham isomorphism, we could obtain a parallel section \( \pi_{dR, \mathcal{C}} \) of \( \text{Fil}^0H_{dR}^{(2,2)} \). We could further extend \( \pi_{dR, \mathcal{C}} \) to a (necessarily unique) section \( \pi_{dR} \) defined over \( \mathbb{Z}_p \).

Next we introduce the étale realizations of \( \pi \). Let \( H_{\ell} = R^1h_{\ell\text{-}et}^*\mathbb{Z}_\ell \) if \( \ell \neq p \) and \( H_p = R^1h_{\eta, \text{ ét}, *\mathbb{Z}_p} \), where \( h_\eta \) denotes the generic fiber of \( h \). The section \( \pi_B \) first gives rise to a section \( \pi_\ell \) of \( H_{\ell} \otimes \mathbb{Q}_\ell \) over \( \mathbb{C} \), which descends to \( \mathbb{Q} \) and for \( \ell \neq p \) extends to \( \mathbb{Z}_p \).

Finally we need to introduce the crystalline realization of \( \pi \), which is the most important one for our proof. Let \( H_{\text{cris}} \) be the first crystalline cohomology of \( A \) over \( \mathcal{S}_K^{ad}(G, \Omega)_{\mathbb{F}_p} \). By the de Rham-crystalline comparison isomorphism we have an identification

\[
H_{dR}|_{\mathcal{S}_K^{ad}(G, \Omega)_{\mathbb{Z}_p}} \cong \lim_{\longrightarrow} H_{\text{cris}}|_{\mathcal{S}_K^{ad}(G, \Omega)_{\mathbb{Z}_p}}
\]

where \( \mathcal{S}_K^{ad}(G, \Omega)_{\mathbb{Z}_p} \) denotes the completion of \( \mathcal{S}_K^{ad}(G, \Omega)_{\mathbb{Z}_p} \) along the special fiber. \( H_{dR} \) comes equipped with a connection \( \nabla \), which determines the \( F \)-crystal \( H_{\text{cris}} \). In particular

\[
H^0((\mathcal{S}_K^{ad}(G, \Omega)/\mathbb{Z}_p)_{\text{cris}}, H_{\text{cris}}) \cong H^0(\mathcal{S}_K^{ad}(G, \Omega)_{\mathbb{Z}_p}, H_{dR}^{(0,0), \nabla = 0})
\]

\( \pi_{dR} \) is horizontal with respect to \( \nabla \), so we obtain a global section \( \pi_{\text{cris}} \) of the crystal \( H_{\text{cris}} \). Let \( k \subset \mathbb{F}_p \) be a subfield, and \( s \in \mathcal{S}_K^{ad}(G, \Omega)(k) \) be a point. The fiber \( \pi_{\text{cris}, s} \) is naturally viewed as a tensor in \( \text{Fil}^0H_{\text{cris}}^1(A_s/W(k))^{(2,2)} \). \( \pi_{\text{cris}, s} \) has the following property (c.f. [13, Proposition 1.3.7, 1.3.10], see also [25, Proposition 4.7]): Let \( K' \) be a finite, totally ramified extension of \( W(k)[1/p] \). If \( \bar{s} \in \text{Sh}_K^{ad}(G, X)(K') \) is a point specializing to \( s \), then under the \( p \)-adic comparison isomorphism

\[
H^1_{\text{ét}}(A_{\bar{s} \otimes K'}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{\text{cris}} \cong H^1_{\text{cris}}(A_s/W(k)) \otimes_{W(k)} B_{\text{cris}}
\]

the tensor \( \pi_{p, \bar{s}} \) is carried to \( \pi_{\text{cris}, s} \). Moreover, \( \pi_{\text{cris}, s} \) is Frobenius invariant.

We denote the dual to the image of \( H_{B}^{(1,1)}, H_{dR}^{(1,1)}, H_{\ell}^{(1,1)}, H_{\text{cris}}^{(1,1)} \) under various realizations of \( \pi \) by \( L_B, L_{dR}, L_\ell, L_{\text{cris}} \) respectively. As the action of \( G \) on \( L_d \) factors through \( G^{ad} \) and the kernel of \( G \to G^{ad} \) is simply \( \mathbb{G}_m \), the sheaves \( L_B, L_{dR}, L_\ell, L_{\text{cris}} \) just constructed descend to \( \mathcal{S}_K^{ad}(G^{ad}, \Omega) \) (c.f. [25, 5.24], [21, 4.1]). We denote these sheaves by the same letters.

### 3.3 Isogeny Classes

**Definition 3.2.** Let \( k \) be a perfect field and \( s, s' \in \mathcal{S}_K^{ad}(G, \Omega)(k) \). Let \( f : A_s \to A_{s'} \) be a quasi-isogeny which respects the \( \mathbb{Z}/2\mathbb{Z} \)-grading and \( \text{Cl}(L_d) \)-action on \( A_s, A_{s'} \). Fix an algebraic closure \( k \subset k \).

- If \( \text{char } k = 0 \), we say \( f \) is a \textit{CSpin-isogeny} if it sends \( \pi_{\ell, s, \otimes \bar{k}} \) to \( \pi_{\ell, s', \otimes \bar{k}} \) for every prime \( \ell \).
• If char \( k = p \), we say \( f \) is a \textit{CSpin-isogeny} if it sends \( \pi_{\ell,s \otimes k} \) to \( \pi_{\ell,s' \otimes k} \) for every prime \( \ell \neq p \) and \( \pi_{\text{cris},s} \) to \( \pi_{\text{cris},s'} \).

Clearly, if \( k = \mathbb{C} \), it suffices to check whether \( f \) sends \( \pi_{B,s} \) to \( \pi_{B,s'} \).

Now set \( k = \mathbb{F}_p \) and let \( s \in \mathcal{K}_{K_p}(G,\Omega)(\mathbb{F}_p) \). \cite{18} tells us how to give a group-theoretic description of the isogeny class of \( s \), i.e.

\[
\{s' \in \mathcal{K}_{K_p}(G,\Omega)(\mathbb{F}_p) : \text{there exists a CSpin-isogeny } A_s \to A_{s'} \}
\]

We fix an isomorphism of \( \mathbb{Z}/2\mathbb{Z} \)-graded right \( \text{Cl}(L_d) \)-modules \( H^1_{\text{cris}}(A_s/W) \equiv H \otimes W \) which sends \( \pi_{\text{cris},s} \) to \( \pi \) (cf. \cite{25} Proposition 4.7]). Then the Frobenius action on \( H^1_{\text{cris}}(A_s/W) \) takes the form \( b\sigma \) for some \( b \in G(W) \). By \cite{18} Lemma 1.1.12], there exists a \( G_W \)-valued cocharacter \( \nu \) such that

\[
b \in G(W)\nu(p)G(W)
\]

and \( \sigma^{-1}(\nu) \) gives the filtration on \( H^1_{\text{dR}}(A_s) \). Now define

\[
X_p := \{g_p \in G(K)/G(W) : g_p^{-1}b\sigma(g_p) \in G(W)\nu(p)G(W)\}
\]

Let \( \mathcal{G}_s \) be the \( p \)-divisible group of \( A_s \). If \( g_p \in X_p \), then \( g_p \cdot \mathcal{D}(\mathcal{G}_s) \) is stable under Frobenius and satisfies the axioms of a Dieudonné module. Hence \( g_p \cdot \mathcal{D}(\mathcal{G}_s) \) corresponds to a \( p \)-divisible group \( \mathcal{G}_{g_p,s} \) naturally equipped with a quasi-isogeny \( \mathcal{G}_s \to \mathcal{G}_{g_p,s} \). We denote by \( A_{g_p,s} \) the corresponding abelian variety, which is naturally isogenous to \( A_s \). We could equip \( A_{g_p,s} \) with a polarization and a level structure using those on \( A_s \). Thus we obtain a map \( X_p \to \mathcal{K}_{K_p}(\text{GSp},\mathcal{H}) \). By \cite{18} Proposition 1.4.4, this map lifts uniquely to a map \( X_p \to \mathcal{K}_{K_p}(G,\Omega)(\mathbb{F}_p) \). Now let \( X^p = G(\mathbb{A}_f^p) \). We could define a natural map \( X_p \to \mathcal{K}_{K_p}(G,\Omega)(\mathbb{F}_p) \) by \( g^p \in X^p \mapsto g^p \cdot s \) using the \( G(\mathbb{A}_f^p) \)-action on \( \mathcal{K}_{K_p}(G,\Omega) \). Let \( \iota_s : X_p \times X^p \to \mathcal{K}_{K_p}(G,\Omega)(\mathbb{F}_p) \) be the product of these two maps. In our context, \cite{18} Proposition 1.4.15} translates to the following:

\begin{lemma}
The image of \( \iota_s \) is precisely the isogeny class of \( s \).
\end{lemma}

\begin{remark}
Kisin considered \( \mathcal{K}_{K_p}(\text{GSp},\mathcal{H}) \) as a moduli of abelian schemes up to prime-to-\( p \) isogeny together with weak polarizations and level structures. As we have chosen a \( \mathbb{Z} \)-lattice \( H \subset H_{\mathbb{Z}(p)} \), we could consider \( \mathcal{K}_{K_p}(\text{GSp},\mathcal{H}) \) as carrying an actual family of abelian schemes. The reader may refer to \cite{17} 3.4.2 for how to switch between the two points of view. In particular, to each point \( s \in \mathcal{K}_{K_p}(G,\Omega)(\mathbb{F}_p) \), one could attach a unique triple \( (A_s,\lambda_s,\xi_s^p) \) where \( \lambda_s \) is a polarization on \( A_s \) and \( \xi_s^p \) is an isomorphism of \( \mathbb{Z}/2\mathbb{Z} \)-graded right \( \text{Cl}(L_d) \)-modules \( H \otimes \mathbb{Z}^p \cong H^1_{\text{et}}(A_s,\mathbb{Z}^p) \) which sends \( \pi \) to \( \pi_{A^p,s} \). Let \( \gamma \in G(\mathbb{A}_f^p) \) and denote the image of \( s \) under the action of \( \gamma \) by \( g^p \). If \( (A_{g^p,s},\lambda_{g^p,s},\xi_{g^p,s}^p) \) is the corresponding triple for \( g^p \), then there exists a CSpin-isogeny \( f_{g^p} : A_s \to A_{g^p,s} \) such that \( f_{g^p}^*H^1_{\text{et}}(A_{g^p,s},\mathbb{Z}^p) = \xi_{g^p,s}^p(g^p \cdot H \otimes \mathbb{Z}^p) \).
\end{remark}

\subsection{The Integral Period Map}

Let \( K \subset \mathbb{K} \) be an admissible compact open subgroup with \( K_p = G(\mathbb{Z}_p) \). We start with a period map over \( \mathbb{C} \)

\[
\rho_{K_{\text{d},C}} : \tilde{M}_{2d,K_{\text{d},C}} \to \text{Sh}_{K_{\text{d},C}}(G_{\text{d},C},\Omega)
\]

10
(c.f. [24, Proposition 5.2]) which simply sends a K3 surface to its Hodge structure (up to twist). It was first proved by Rizov in [36] using CM points that $\rho_{K,\mathbb{C}}$ descends to $\mathbb{Q}$, so that we obtain

$$\rho_{K,\mathbb{Q}} : \tilde{M}_{2d,K,\mathbb{Q}} \to \text{Sh}_{Kad}(G^{ad}, \Omega)$$

We refer the reader to [24] for another proof. The extension property of the canonical integral model allows us to extend the map further to

$$\rho_{K,\mathbb{Z}(p)} : \tilde{M}_{2d,K,\mathbb{Z}(p)} \to \mathcal{S}_{Kad}(G^{ad}, \Omega)$$

A major component of Madapusi Pera’s paper [24] is to show that $\rho_{K,\mathbb{Z}(p)}$ still records the cohomological data of K3 surfaces, just like $\rho_{K,\mathbb{C}}$. This is done by comparing sheaves on $\tilde{M}_{2d,K,\mathbb{Z}(p)}$ with those on $\mathcal{S}_{Kad}(G^{ad}, \Omega)$ through $\rho_{K,\mathbb{Z}(p)}$. As in loc. cit., we denote the pullback of sheaves with respect to natural Grothendieck topologies on $\mathcal{S}_{Kad}(G^{ad}, \Omega)$ to $\tilde{M}_{2d,K,\mathbb{Z}(p)}$ by the same letters. By construction, we have isometries $\alpha_B : L_B \sim P_B^2(1)$ and $\alpha_{dr} : L_{dr} \sim P_{dr}^2(1)$ over $\tilde{M}_{2d,K,\mathbb{C}}$. We summarize some properties which we shall need in the following proposition:

**Proposition 3.5.**

a. $\alpha_B$ extends to an isometry of $\mathbb{Z}_\ell$ étale local systems $\alpha_\ell : L_\ell \sim P_\ell^2(1)$ over $\tilde{M}_{2d,K,\mathbb{Z}(p)}$ for every $\ell \neq p$ and an isometry of $\mathbb{Z}_p$ étale local systems $\alpha_p : L_p \sim P_p^2(1)$ over $\tilde{M}_{2d,K,\mathbb{Q}}$.

b. $\alpha_{dr,\mathbb{C}}$ extends to an isometry of filtered vector bundles with flat connection $\alpha_{dr} : L_{dr} \sim P_{dr}^2(1)$ which sends $\text{Fil}^1 L_{dr}$ to $\text{Fil}^1 P_{dr}^2(1)$ over $\tilde{M}_{2d,K,\mathbb{Z}(p)}$.

c. Let $T \to \tilde{M}_{2d,K,\mathbb{F}_p}$ be an étale map. $\alpha_{dr}$ induces a canonical isomorphism $L_{cris,T} \sim P_{cris,T}^2(1)$ of $F$-crystals.

**Proof.** Combine the results in [24, Section 5].

### 3.5 Special Endomorphisms

**Definition 3.6.** Let $k$ be a perfect field, $s \in \mathcal{S}_K(G, \Omega)(k)$ and $f \in \text{End}(A_s)$. Fix some algebraic closure $k \subset \bar{k}$.

- If $\text{char } k = 0$, we say $f$ is a *special endomorphism* if its $\ell$-adic realization lies in $L_{\ell,s \otimes \bar{k}} \otimes \mathbb{Q}_\ell \subset \text{End}H_{\ell,s \otimes \bar{k}} \otimes \mathbb{Q}_\ell$ for every prime $\ell$.

- If $\text{char } k = p$, we say $f$ is a *special endomorphism* if its $\ell$-adic realization lies in $L_{\ell,s \otimes \bar{k}} \otimes \mathbb{Q}_\ell \subset \text{End}H_{\ell,s \otimes \bar{k}} \otimes \mathbb{Q}_\ell$ and its crystalline realization lies in $L_{cris,s}[1/p] \subset \text{End}H_{cris,s}[1/p]$.

In either case, the special endomorphisms form a subspace $L(A_s) \subset \text{End}(A_s)$. If $f \in L(A_s)$, then $f \circ f$ is a scalar. Therefore, $L(A_s)$ has the structure of a quadratic lattice given by $f \mapsto f \circ f$. The following proposition in [24] gives a correspondence between line bundles on K3 surfaces and special endomorphisms of their Kuga-Satake abelian varieties, generalizing the natural correspondence over $\mathbb{C}$ given by Hodge theory.

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3This definition is equivalent to Madapusi Pera’s in view of [25, 5.22]. We modified the definition for it to be more parallel to Definition 1.1 and Definition 3.2.
Proposition 3.7. Let \( t \in \widehat{M}_{2d,K^{ad},\mathbb{Z}(p)}(\overline{\mathbb{F}}_p) \) be a point corresponding to a quasi-polarized K3 surface \((X_t, \xi_t)\). Let \( s \in \mathcal{A}_K(G, \Omega)(\overline{\mathbb{F}}_p) \) be a lift of \( \rho_{K^{ad},\mathbb{Z}(p)}(t) \). Then there is an isomorphism of quadratic lattices over \( \mathbb{Q} \)
\[
L(A_s)_{\mathbb{Q}} \cong \langle \xi_t \rangle^\perp \subset \text{Pic} (X_t)_{\mathbb{Q}}
\]
whose \( \ell \)-adic and crystalline realizations agree with the natural identifications \( L_{\ell,s} = L_{\ell,t} \) and \( L_{\text{cris},s} = L_{\text{cris},t} \).

Proof. This is a coarser form of [24, Theorem 5.17(4)], except that we allow quasi-polarized K3 surfaces in addition. It is ok because to establish the isomorphism of \( \mathbb{Q} \)-lattices it suffices to have the inclusion \( L(A_s) \hookrightarrow \langle \xi_t \rangle^\perp \subset \text{Pic} (X_t) \), as we know a fortiori by the Tate conjecture they have the same rank. To construct this inclusion one only need to use the fact that the deformation space of a special endomorphism is flat over \( \mathbb{Z} \) (c.f. [25, Proposition 5.21]). 

Let \( V \) be an extension of \( W \). Denote the fraction field of \( V \) by \( K' \) and fix an isomorphism \( \overline{K}' \cong \mathbb{C} \). Let \( X_V \) over \( V \) be a lift of a K3 surface \( X \) over \( \overline{\mathbb{F}}_p \) such that \( X_V \otimes \mathbb{C} \) has CM. One can infer from [15, Theorem 1.1] that if \( X \) has finite height, then the specialization map \( \text{Pic} (X_V \otimes \mathbb{C})_{\mathbb{Q}} \to \text{Pic} (X)_{\mathbb{Q}} \) is an isomorphism. In view of Proposition 3.7 there should be a corresponding statement for special endomorphisms:

Proposition 3.8. Let \( s \in \mathcal{A}_K(G, \Omega)(\overline{\mathbb{F}}_p) \) be a point such that \( A_s \) is non-supersingular. Suppose there exists a lift \( s_V \) of \( s \) over \( V \) such that \( s_V \otimes \mathbb{C} \) is a special point on \( \text{Sh}_K(G, \Omega) \). Then the specialization map
\[
L(A_{s_V})_{\mathbb{Q}} \to L(A_s)_{\mathbb{Q}}
\]
is an isomorphism.

Proof. This can be seen as a reinterpretation of [15, Theorem 1.1]. To explain the idea, we first assume that there exists \( t_V \in \widehat{M}_{2d,K^{ad},\mathbb{Z}(p)}(V) \) such that \( \rho_{K^{ad},\mathbb{Z}(p)}(t_V) \) is the image of \( s_V \). Set \( t = t_V \otimes \overline{\mathbb{F}}_p \). Since \( A_{s_V} \otimes \mathbb{C} \) has CM, so does \( X_{t_V} \otimes \mathbb{C} \) (c.f. Lemma 2.7). Then we could conclude by Proposition 3.7. Now we sketch how to interpret the computation in [15, Section 4] without appealing to K3 surfaces.

Step 1: Recall that \( L_{B,s_V} \) carries a Hodge structure of K3 type. Set \( T(L_{B,s_V}) := (L_{B,s_V}^0)^\perp \subset L_{B,s_V} \otimes \mathbb{Q} \). By Theorem 2.1 and Lemma 2.7 \( \text{End}_{\text{Hdg}}(T(L_{B,s_V})) = E \) for some CM field \( E \) such that \( T(L_{B,s_V}) \) is one-dimensional over \( E \). Let \( E_0 \) be the totally real subfield of \( E \). Zarhin tells us that (c.f. [23])
\[
\text{MT}(L_{B,s_V}) = \text{MT}(T(L_{B,s_V})) = \ker(\text{Nm} : \text{Res}_{E/\mathbb{Q}} \mathbb{G}_m \to \text{Res}_{E_0/\mathbb{Q}} \mathbb{G}_m)
\]
Denote this group by \( G_0 \).

Step 2: By extending \( V \) if necessary, we could find a number field \( F \) with \( E \subset F \subset K' \) such that \( s_K \) arises from a \( F \)-valued point \( s_F \) on \( \text{Sh}_K(G, \Omega) \). Let \( v \) be the finite place above \( p \).

\[\text{MT}(L_{B,s_V}) = \text{MT}(T(L_{B,s_V})) = \ker(\text{Nm} : \text{Res}_{E/\mathbb{Q}} \mathbb{G}_m \to \text{Res}_{E_0/\mathbb{Q}} \mathbb{G}_m)\]

\[\text{Denote this group by } G_0.\]

\[\text{Step 2: By extending } V \text{ if necessary, we could find a number field } F \text{ with } E \subset F \subset K' \text{ such that } s_K \text{ arises from a } F \text{-valued point } s_F \text{ on } \text{Sh}_K(G, \Omega). \text{ Let } v \text{ be the finite place above } p.\]
given by the inclusion $F \subset K'$ and let $q$ be the place of $E$ below $v$. Let $F_v$ be the completion of $F$ at $v$ and $\mathcal{O}_{F_v}$ be the ring of integers of $F_v$ with residue field $k(v)$. Choose an isomorphism $\bar{F}_v \cong \bar{K}'$ compatible with $F \subset K'$ so that we have $k(v) \subset \bar{\mathbb{F}}_p$. Then $s_{F_v} := s_F \otimes F_v$ extends to a $\mathcal{O}_{F_v}$-valued point $s_{\mathcal{O}_{F_v}}$ on $\mathcal{I}_K(G, \Omega)$ such that $s_{k(v)} := s_{\mathcal{O}_{F_v}} \otimes k(v)$ gives the point $s$ when base changed to $\bar{\mathbb{F}}_p$. To sum up, we have a commutative diagram

$$
\begin{array}{ccc}
\text{Spec } \bar{\mathbb{F}}_p & \longrightarrow & \text{Spec } V \\
\downarrow & & \downarrow \text{s}_F \\
\text{Spec } k(v) & \longrightarrow & \text{Spec } \mathcal{O}_{F_v}
\end{array}
$$

\[ \mathcal{I}_K(G, \Omega) \]

The descent of $s$ to $s_{k(v)}$ clearly endows $L_{\ell,s}$ with an action by the geometric Frobenius $\text{Fr} \in \text{Gal}(\bar{\mathbb{F}}_p/k(v))$.

**Step 3:** Note that we have a natural isomorphism $L(A_{s_F \otimes \bar{F}}) \cong L(A_{s_E})$. Therefore, if we define $T(L(k_1, s_F \otimes \bar{F})) := (\text{Im}(L(A_{s_F \otimes \bar{F}})))^\perp \subset L(k_1, s_F \otimes \bar{F})$, then $L_B, s_E \otimes \mathbb{A}_f \cong L(k_1, s_F \otimes \bar{F})$ restricts to an isomorphism $T(L_B, s_E) \otimes \mathbb{A}_f \cong T(L(k_1, s_F \otimes \bar{F}))$.

The $\text{Gal}(\bar{E}/F)$-action on $L(k_1, s_F \otimes \bar{F})$ restricts to one on $T(L(k_1, s_F \otimes \bar{F}))$. The theory of canonical models tells us that the induced action of $\text{Gal}(\bar{E}/F)$ on $T(L_B, s_E) \otimes \mathbb{A}_f$ is given by a homorphism $\rho : \text{Gal}(\bar{E}/F) \rightarrow G_0(\mathbb{A}_f)$. Moreover, we obtain the following commutative diagram

$$
\begin{array}{ccc}
\mathbb{A}_E^\times & \longrightarrow & \mathbb{A}_E^\times \\
\downarrow \text{art}_F & & \downarrow \text{proj} \\
\text{Gal}(F_{ab}/F) & \longrightarrow & \mathbb{A}_{E,f}^\times \\
\downarrow \rho & & \downarrow y \rightarrow c(y) y^{-1} \\
G_0(\mathbb{A}_f) & \longrightarrow & G_0(\mathbb{A}_f)/G_0(\mathbb{Q})
\end{array}
$$

where $\text{art}_F$ denotes the Artin reciprocity map, $F_{ab}$ the maximal abelian extension of $F$ and $c$ the complex conjugation on $E \subset \mathbb{C}$ (c.f. [15, Theorem 2.1], [36, Corollary 3.9.2], [37, Theorem 12]).

**Step 4:** Let $\tilde{\sigma} \in \text{Gal}(F_{ab}/F)$ be a lift of $\text{Fr} \in \text{Gal}(\bar{F}_p/k(v))$. The diagram allows us to analyze the characteristic polynomial $f_{\tilde{\sigma}}$ of $\tilde{\sigma}_{|_{F_{ab}}}$ acting on $T(L_B, s_E) \otimes \mathbb{Q}_\ell$ for $\ell \neq p$. More precisely, let $\pi_v$ be a uniformizer of $F_v$ and $x \in \mathbb{A}_E^\times$ be the element with $\pi_v$ at 0 and 1 at other places, then $x$ be a lift of $\tilde{\sigma}$. Let $z$ be the image of $x$ under the composition

$$
\mathbb{A}_E^\times \xrightarrow{\text{Nm}_{E/F}} \mathbb{A}_E^\times \xrightarrow{\text{pr}} \mathbb{A}_{E,f}^\times \xrightarrow{y \rightarrow c(y) y^{-1}} G_0(\mathbb{A}_f) \subset (E \otimes \mathbb{A}_f)^\times
$$

\[ ^5 \text{In these references, the above diagram is stated in terms of K3 surfaces. However, Rizov's theorem [36, Corollary 3.9.2] is a formal consequence of the rationality of the period map } \rho_{K,C}. \text{ The diagram itself comes from the canonical model } \text{Sh}_{K_{\text{ad}}}(G_{\text{ad}}, \Omega) \text{ of } \text{Sh}_{K_{\text{ad}}}(G_{\text{ad}}, \Omega)_C. \]
The commutativity of the diagram tells us that there exists \( \eta \in G_0(\mathbb{Q}) \) such that \( \rho(\sigma \mid_{F^{ad}}) = z\eta \).

Then we have that
\[
f_{\bar{\sigma}} = f_{\eta}^{[E: \mathbb{Q}(0)]}
\]
where \( f_\eta \) is the minimal polynomial of \( \eta \in G_0(\mathbb{Q}) \subset E^\times \).

**Step 5:** Consider the composition of maps
\[
L(A_{sc}) \to L(A_s) \hookrightarrow L_{\ell,s} \otimes \mathbb{Q}_\ell
\]
where the first map is given by specialization. \( f_{\bar{\sigma}} \) can be alternatively viewed as the characteristic polynomial of \( Fr \) acting on \( \text{Im} L(A_{sc}) \perp \subset L_{\ell,s} \). By [15, Lemma 4.1], if \( q \) does not split in \( E \), then the eigenvalues of \( f_\sigma \) are roots of unity. This implies that a power of \( Fr \) acts trivially on \( L_{\ell,s} \subset \text{End}H^1(A_s, \mathbb{Q}_\ell) \). By [9, Proposition 21], \( A_s \) must be supersingular, which contradicts our assumption. Therefore, \( q \) splits in \( E \), and by [15, Lemma 4.1] again, none of the eigenvalues of \( f_{\bar{\sigma}} \) can be a root of unity. This implies that the composition
\[
L(A_{sc}) \otimes \mathbb{Q}_\ell \to L(A_s) \otimes \mathbb{Q}_\ell \to \lim_{m} (L_{\ell,s} \otimes \mathbb{Q}_\ell)^{Fr^m = 1}
\]
is an isomorphism. Therefore, \( L(A_{sc})_\mathbb{Q} \to L(A_s)_\mathbb{Q} \) is an isomorphism. \( \square \)

## 4 Formal Groups and Cohomology

In this section, we review Nygaard-Ogus theory and prove an important lifting lemma which we shall use in Section 5.

### 4.1 Formal Groups

**Lubin-Tate Theory** We first review the basic definition in Lubin-Tate theory:

**Definition 4.1.** Let \( A, R \) be commutative rings and \( i : A \to R \) be a morphism. A formal \( A \)-module over \( R \) is a commutative formal group law \( F(X, Y) = X + Y + \cdots \in R[[X, Y]] \) together with a ring homomorphism \( \gamma : A \to \text{End}_R(F) \) such that the induced map \( A \to \text{End}_R(\text{Lie} F) \cong R \) is equal to \( i \).

The purpose of recalling this notion is to prove the following lemma:

**Lemma 4.2.** Let \( \mathcal{G} \) be a one dimensional formal group of height \( h \) over \( \mathbb{F}_p \) and \( \alpha : \mathcal{G} \to \mathcal{G} \) be an isogeny. There exists a finite extension \( V/W \) and a lift \( \mathcal{G}_V \) of \( \mathcal{G} \) to \( V \) such that \( \alpha \) lifts to an isogeny \( \alpha_V : \mathcal{G}_V \to \mathcal{G}_V \).

**Proof.** View \( \mathcal{G} \) as a \( \mathbb{Z}_p \)-module. Set \( M = \mathbb{Q}_p(\alpha) \) and let \( \mathcal{O}_M \) be its ring of integers. Let \( V := \hat{\mathcal{O}}_M^{ur} \) be the completion of the maximal unramified extension of \( \mathcal{O}_M \). Let \( e := [M : \mathbb{Q}_p] \).

Then as an \( \mathcal{O}_M \)-module \( \mathcal{G} \) has height \( h/e \). Note that \( h/e \) is always a natural number as \( \text{End}(\mathcal{G}) \) is isomorphic to the maximal order of a division algebra of invariant \( 1/h \) (c.f. [40, Theorem 1.1]). The universal deformation space of an \( \mathcal{O}_M \)-module of height \( h/e \) is isomorphic to \( \hat{\mathcal{O}}_M^{ur}[t_1, \cdots, t_{h/e-1}] \). Hence we may obtain a lift \( \mathcal{G}_V \) of \( \mathcal{O}_M \)-module \( \mathcal{G} \) to \( \hat{\mathcal{O}}_M^{ur} \) simply by setting \( t_1 = \cdots = t_{h/e-1} = 0 \). Clearly \( \mathcal{G}_V \) carries an action of \( \alpha \) and its special fiber isomorphic to \( \mathcal{G} \) by construction. \( \square \)
**Formal Brauer Groups**  An abelian variety's crystalline cohomology can be recovered from its $p$-divisible group via Dieudonné modules. There is a similar story for K3 surfaces, for which the role of $p$-torsion is taken by the formal Brauer group. Let $X$ be a K3 surface over an algebraically closed field $k$ with char $k = p$.

Let $\text{Art}_{R,k}$ be the category of Artinian local $R$-algebra with residue field $k$. Let $X_R$ be a proper smooth lifting of $X$, where $(R, \mathfrak{m}) \in \text{Art}_{R,k}$. Define a functor $\Psi_{X_R}: \text{Art}_{R,k} \to \text{Ab}$ by

$$\Psi_{X_R}(A) = H^2_{\text{fl}}(X \otimes_R A, \mu_{p^\infty})$$

Set $D = H^2_{\text{fl}}(X, \mu_{p^\infty})$. By [33] Lemma 3.1, $D$ is an abstract $p$-divisible group.

**Proposition 4.3.** $\Psi_{X_R}$ defines a $p$-divisible group over $\text{Spec} \, R$ with connected component $\Psi^0_{X_R} = \hat{\text{Br}}_{X_R}$ and étale part $\Psi^\text{et}_{X_R} = D$, where $D$ denotes the unique $p$-divisible group over $\text{Spec} \, R$ lifting $D/\text{Spec} \, k$.

*Proof.* This is a corollary to [2, IV Proposition 1.8]. See also [33, Proposition 3.2]. For the definition of the formal Brauer group $\hat{\text{Br}}_{X_R}$, we refer the reader to [2, I 1.4] (take $q = 2$).

Recall that Tate established an equivalence between the category of divisible connected formal Lie groups and that of connected $p$-divisible groups over a complete Noetherian local ring with residue field of characteristic $p$ (c.f. [39, Proposition 1]). We will freely make use of this equivalence of categories. As a formal group, $\hat{\text{Br}}_X$ has dimension 1.

**Nygaard-Ogus’ description of $H^2_{\text{cris}}(X/W)$**  From now on we assume $p \geq 5$. Nygaard and Ogus constructed an $F$-crystal $\mathbb{K}(\hat{\text{Br}}_X)$ by

$$\mathbb{K}(\hat{\text{Br}}_X) := \mathbb{D}(\hat{\text{Br}}_X^\circ) \oplus \mathbb{D}(D^*) \oplus \mathbb{D}(\hat{\text{Br}}_X)(-1)$$

and showed that there is a natural identification

$$\mathbb{K}(\hat{\text{Br}}_X) = H^2_{\text{cris}}(X/W)$$

The construction of $\mathbb{K}(\hat{\text{Br}}_X)$ explains the slope filtration on $H^2_{\text{cris}}(X/W)$: $H^2_{\text{cris}}(X/W)_{<1} = \mathbb{D}(\hat{\text{Br}}_X^\circ), H^2_{\text{cris}}(X/W)_{=1} = \mathbb{D}(D^*)$ and $H^2_{\text{cris}}(X/W)_{>1} = \mathbb{D}(\hat{\text{Br}}_X)(-1)$.

### 4.2 Constructing Liftings

Suppose $\mathcal{G}_V$ is a lifting of $\hat{\text{Br}}_X$ to a totally ramified extension $V$ of $W(k)$. Let $K'$ be the fraction field of $V$. Then $\mathcal{G}_V$ determines a filtration $\text{Fil}_{\mathcal{G}_V} \subset \mathbb{D}(\hat{\text{Br}}_X) \otimes_W K'$ with $\dim_{K'} \text{Fil}_{\mathcal{G}_V} = 1$. We could use $\text{Fil}_{\mathcal{G}_V}$ to give $\mathbb{K}(\hat{\text{Br}}_X) \otimes_W K'$ a two-step filtration $\text{Fil}_{\mathcal{G}_V}^*$:

$$\text{Fil}_{\mathcal{G}_V}^0 \subset \text{Fil}_{\mathcal{G}_V}^1 \subset \text{Fil}_{\mathcal{G}_V}^2 = \mathbb{K}(\hat{\text{Br}}_X) \otimes_W K'$$

where $\text{Fil}_{\mathcal{G}_V}^0 = \text{Fil}_{\mathcal{G}_V}(-1) \subset \mathbb{D}(\hat{\text{Br}}_X)(-1) \otimes_W K'$ and $\text{Fil}_{\mathcal{G}_V}^1 = (\text{Fil}_{\mathcal{G}_V}^2)_\perp$ with respect to the bilinear pairing.

The following result of Nygaard-Ogus allows us to choose a lift a K3 surface by choosing one for its formal Brauer group.
Proposition 4.4. For each $\mathcal{G}_V$ lifting $\widehat{Br}_X$ as above, there exists a K3 surface $X_V$ over $V$ lifting $X$ such that

$$
\Psi_{X_V} \cong \mathcal{G}_V \oplus \mathcal{D}_V \text{ and } (H^2_{dR}(X/V) \otimes K', \text{Fil}^*_\text{Hdg}) \sim (\mathbb{K}(\widehat{Br}_X) \otimes K', \text{Fil}^*_\text{Gr}_V)
$$

where $\mathcal{D}_V$ is the canonical lifting of $D$ to $V$. Moreover, the natural map $\text{Pic}(X_V) \to \text{Pic}(X)$ is an isomorphism.

Proof. This is [33, Proposition 5.5], except that we do not tensor with $\overline{K}'$ and let the crystalline-Weil group act on both sides. The requirement that $\Psi_{X_V}$ is isomorphic to the slope filtration (c.f. [42, Lemma 1.9]).

Remark 4.5. An ample line bundle on $X$ lifts to an ample line bundle on $X_{K'}$, because being ample is an open condition (c.f. [19, Theorem 1.2.17]). Hence we know that the ample cone of $X_{K'}$ lies inside that of $X$. Kleiman has shown the nef cone is the closure of the ample cone (c.f. [19, Theorem 1.4.23]), so the lift of a nef line bundle on $X$ lifts is still nef on $X_{K'}$. Finally, a nef line bundle is big if and only if its top self-intersection is strictly positive (c.f. [19, Theorem 2.2.16]), so the lift of a big and nef divisor on $X$ to $X_{K'}$ is also big and nef.

The results in this section combine to give the following lemma, which we shall use in Section 5.

Lemma 4.6. Let $(X, \xi)$ and $(X', \xi')$ be two quasi-polarized K3 surfaces of degree $2d$ over $\overline{\mathbb{F}}_p$. For any isometry of $F$-isocrystals

$$
\phi : P^2_{\text{cris}}(X/W) \otimes K \sim P^2_{\text{cris}}(X'/W) \otimes K
$$

we could find a finite extension $V/W$ with fraction field $K'$ and lifts $(X_V, \xi_V)$ and $(X'_V, \xi'_V)$ such that $\text{Pic}(X_V) \to \text{Pic}(X)$ and $\text{Pic}(X'_V) \to \text{Pic}(X')$ are isomorphisms, and $\phi \otimes K'$ respects the Hodge filtrations induced by $(X_V, \xi_V)$ and $(X'_V, \xi'_V)$ on $P^2_{\text{cris}}(X/W) \otimes K'$ and $P^2_{\text{cris}}(X'/W) \otimes K'$ via the crystalline-de Rham comparison isomorphism. When $(X, \xi) = (X', \xi')$, we may take $(X_V, \xi_V) = (X'_V, \xi'_V)$.

Proof. $p^m\phi$ restricts to a map $P^2_{\text{cris}}(X/W) \to P^2_{\text{cris}}(X'/W)$ for some $m$. Since $\phi$ respects the Frobenius action, $p^m\phi$ sends $\mathbb{D}(\widehat{Br}_X)(-1)$ into $\mathbb{D}(\widehat{Br}_X')(-1)$ and induces a nonzero map $\widehat{Br}_X' \to \widehat{Br}_X$. Therefore, $\widehat{Br}_X'$ and $\widehat{Br}_X$ must have the same height, and hence they are abstractly isomorphic. By lemma 4.2, we could find a finite extension $V/W$ and lifts $\mathcal{G}_V$ and $\mathcal{G}_V'$ of $\widehat{Br}_X$ and $\widehat{Br}_X'$ such that $\widehat{Br}_X' \to \widehat{Br}_X$ lifts to $\mathcal{G}_V' \to \mathcal{G}_V$. Let $(X_V, \xi_V)$ and $(X'_V, \xi'_V)$ be the corresponding lifts given by Proposition 4.3. When $(X, \xi) = (X', \xi')$, we could identify $\widehat{Br}_X$ with $\widehat{Br}_X'$ and lift $p^m\phi$ as an endomorphism.

Remark 4.7. Not surprisingly, for ordinary K3 surfaces, canonical liftings will always work. The reason is that the canonical lifting of an ordinary K3 surface induces a filtration which coincides with the slope filtration (c.f. [32, Lemma 1.9]).

5 Proofs of Theorems

Convention Let $R$ be a ring. For simplicity, we denote the image of a point $t \in \overline{\mathcal{M}}_{2d,K_p,\mathcal{Z}(p)}(R)$ in $\mathcal{S}_{K_p,\mathcal{Z}(p)}(G^{\text{ad}},\Omega)(R)$ by the same letter.
5.1 Proof of Proposition 1.5

Proposition 1.5 is a direct consequence of the following more precise statement:

**Proposition 5.1.** Assume \( p \nmid d \) and \( p \geq 5 \). Let \( t, t' \in \overline{M}_{2d,K_p^a,Z_{(p)}}(\overline{F}_p) \) be two points corresponding to quasi-polarized \( K3 \) surfaces \((X_t, \xi_t)\) and \((X_{t'}, \xi_{t'})\). Suppose \( t, t' \) lift to \( s, s' \in \mathcal{X}_{K_p}(G, \Omega)(\overline{F}_p) \) respectively. For each CSpin-isogeny \( \psi : A_s \to A_{s'} \), there exists an isogeny \( \phi : X_t \to X_{t'} \) which sends \( \text{ch}_{*}(\xi_t) \) to \( \text{ch}_{*}(\xi_{t'}) \) for \(* = \text{cris, ét}\) such that the following diagrams commute

\[
\begin{array}{ccc}
L_{\text{cris},t'} & \xrightarrow{\text{conj. by } \psi^*_{\text{cris}}} & L_{\text{cris},t} \\
\downarrow & & \downarrow \\
P^2_{\text{cris}}(X_{t'}/W)(1) & \xrightarrow{\phi^*_{\text{cris}}} & P^2_{\text{cris}}(X_t/W)(1) \\
\downarrow & & \downarrow \\
P^2_{\text{ ét}}(X_{t'}, A_f^p)(1) & \xrightarrow{\phi^*_{\text{ ét}}} & P^2_{\text{ ét}}(X_t, A_f^p)(1)
\end{array}
\]

**Proof.** We treat the supersingular case and the finite height case separately.

**Supersingular case:** The map \( \psi \), as a CSpin-isogeny, clearly preserves special endomorphisms, so it induces an isometry \( L(A_{s'})_Q \xrightarrow{\sim} L(A_s)_Q \). By Theorem 3.7, we obtain an isometry \( (\xi_{t'})^\perp \subset \text{Pic}(X_{t'})_Q \xrightarrow{\sim} (\xi_t)^\perp \subset \text{Pic}(X_t)_Q \). We may extend it to an isometry \( \text{Pic}(X_{t'})_Q \xrightarrow{\sim} \text{Pic}(X_t)_Q \) by sending \( \xi_{t'} \) to \( \xi_t \) and clearly it is given by a correspondence \( \phi : X_t \to X_{t'} \). On the other hand, \( (\xi_{t'})^\perp \) and \( (\xi_t)^\perp \) span all of \( P_2^\text{cris}(X_t, A_f^p) \) and \( P_2^\text{cris}(X_{t'}, A_f^p) \), the induced map \( \phi^*_{\text{cris}} : P_2^\text{cris}(X_{t'}, A_f^p) \to P_2^\text{cris}(X_t, A_f^p) \) is completely determined and has to agree the map induced by \( \psi \). The argument for crystalline cohomology is the same.

**Finite height case:** By Remark 1.5 and Lemma 4.6 we may choose lifts \( t_V, t'_{V'} : \text{Spec} \text{V} \to \overline{M}_{2d,K_p^a,Z_{(p)}} \) of \( t, t' \) such that

\[
\text{conj. by } \psi^*_{\text{cris}} : P_{\text{cris}}^2(X_{t'}/W) \otimes W K' \to P_{\text{cris}}^2(X_t/W) \otimes W K' \quad (5.2)
\]

preserves the Hodge filtration induced by \( X_{t'}, X_{t'_{V'}} \). Again, via the period map, we interpret \( t_V, t'_{V'} \) as points on \( \mathcal{X}_{K_p}(G^{\text{ad}}, \Omega) \).

Since the map \( \mathcal{X}_{K_p}(G, \Omega) \to \mathcal{X}_{K_p}(G^{\text{ad}}, \Omega) \) is pro-étale, we could lift the \( V \)-valued points \( t_V, t'_{V'} \) to \( s_V, s'_{V'} \) on \( \mathcal{X}(G, \Omega) \). Note that under the inclusion \( P_{dR}^2(X_{t_V}(1)) \subset \text{End} H_{dR}^1(A_{s_V}) \), the Hodge filtration on \( H_{dR}^1(A_{s_V}) \) is given by

\[
\text{Fil} H_{dR}^1(A_{s_V}) = \ker \text{Fil} P_{dR}^2(X_{t_V})(1)
\]

Therefore, \( \psi^*_{\text{cris}} \otimes K' : H^1_{\text{cris}}(A_{s'}/W) \otimes W K' \xrightarrow{\sim} H^1_{\text{cris}}(A_s/W) \otimes W K' \) preserves the Hodge filtrations induced by \( A_{s_V} \) and \( A_{s'_{V'}} \). By [6 Theorem 3.15], \( p^m \psi \) lifts to a morphism \( \psi_V : A_{s_V} \to A_{s'_{V'}} \) for some \( m \). The reader may easily check that \( \psi_V \otimes \mathbb{C} \) is a CSpin-isogeny. In particular, it induces by conjugation a Hodge isometry

\[
P^2(X_{tv} \otimes \mathbb{C}, \mathbb{Q}) \xrightarrow{\sim} P^2(X_{t'_{V'}} \otimes \mathbb{C}, \mathbb{Q})
\]

Extend the above map to full Hodge structures by sending the class of \( \xi_{t'} \) to that of \( \xi_t \). Buskin’s result tells us that this Hodge isometry is given by an isogeny \( X_{t_V} \otimes \mathbb{C} \to X_{t'_{V'}} \otimes \mathbb{C} \). Now we could
complete the proof by specializing this correspondence to \( \phi : X_t \rightarrow X_{t'} \). By the compatibility between specialization and cycle class maps, the diagrams in the statement of the proposition commute.

**Remark 5.3.** We remark that in fact any two supersingular K3 surfaces \( X, X' \) over \( \mathbb{F}_p \) for \( p > 2 \) are isogenous according to our definition, as Pic \((X)\mathbb{Q} \cong \text{Pic}(X')\mathbb{Q} \) as \( \mathbb{Q} \)-quadratic lattices. By works of Artin and Radukov-Shafarevich, for any supersingular \( X \) over \( \mathbb{F}_p \), Pic \((X) \cong \mathbb{N}_{p,\sigma} \) for some \( 1 \leq \sigma \leq 10 \), where \( \mathbb{N}_{p,\sigma} \) is the unique even, non-degenerate \( \mathbb{Z} \)-lattice with signature \((1,21)\) and discriminant group \((\mathbb{Z}/p\mathbb{Z})^{2\sigma}\). One may check that \( \mathbb{N}_{p,\sigma} \otimes \mathbb{Q} \cong \mathbb{N}_{p,\sigma'} \otimes \mathbb{Q} \) for any two different \( \sigma, \sigma' \) by the Hasse principle: \( \mathbb{N}_{p,\sigma} \otimes \mathbb{R} \cong \mathbb{N}_{p,\sigma'} \otimes \mathbb{R} \) as a real quadratic form is determined by its signature. For any \( \ell \neq p \), we see that Pic \((X) \otimes \mathbb{Q}_\ell = \Lambda \otimes \mathbb{Q}_\ell \) by a lifting argument. Finally, \( \mathbb{N}_{p,\sigma} \otimes \mathbb{Q}_p \cong \mathbb{N}_{p,\sigma'} \otimes \mathbb{Q}_p \) as they have the same discriminant and Hasse invariant (c.f. \cite{34}).

### 5.2 Proof of Theorem 1.4

Given Proposition 5.1, Theorem 1.4 follows from a computation of the images of isogeny classes on \( \mathcal{S}_{K_p}(G, \Omega)(\mathbb{F}_p) \). Let \( s \in \mathcal{S}_{K_p}(G, \Omega)(\mathbb{F}_p) \) and let \( X_p, X_p^{\sigma} \) be defined as in section 3.3. Let \( X_p^{\text{ad}} \) and \( X_p^{\text{ad},p} \) be the quotients of \( X_p \) and \( X_p^{\sigma} \) under the action of \( \mathbb{G}_m(\mathbb{Q}_p) \) and \( \mathbb{G}_m(A_p^{\sigma}) \) respectively.

**Lemma 5.4.** \( \iota_s \) descends to a map \( \iota_s^{\text{ad}} \) which fits into a commutative diagram

\[
\begin{array}{ccc}
X_p \times X_p^{\sigma} & \xrightarrow{\iota_s} & \mathcal{S}_{K_p}(G, \Omega)(\mathbb{F}_p) \\
\downarrow & & \downarrow \\
X_p^{\text{ad}} \times X_p^{\text{ad},p} & \xrightarrow{\iota_s^{\text{ad}}} & \mathcal{S}_{K_p}^{\text{ad}}(G, \Omega)(\mathbb{F}_p)
\end{array}
\]

**Proof.** \( \iota_s \) is equivariant under the action of \( \mathbb{G}_m(\mathbb{Q}_p) \times \mathbb{G}_m(A_p^{\sigma}) \subset Z_G(\mathbb{Q}_p) \times G(A_p^{\sigma}) \) (c.f. \cite[Corollary 1.4.13]{18}). By \( \mathfrak{3}_{\text{1.4}} \), \( \mathcal{S}_{K_p}(G, \Omega) \rightarrow \mathcal{S}_{K_p}^{\text{ad}}(G, \Omega) \) is a pro-étale Galois cover with Galois group \( \Delta = \mathbb{G}_m(A_p^{\sigma})/\mathbb{G}_m(\mathbb{Z}_p) \). One may easily check, for example by examining the generic fiber, that the action of \( \mathbb{G}_m(\mathbb{Q}_p) \times \mathbb{G}_m(A_p^{\sigma}) \) factors through \( \Delta \) via the quotient map

\[
\mathbb{G}_m(\mathbb{Q}_p) \times \mathbb{G}_m(A_p^{\sigma}) \rightarrow [\mathbb{G}_m(\mathbb{Q}_p)/\mathbb{G}_m(\mathbb{Z}_p) \times \mathbb{G}_m(A_p^{\sigma})]/\mathbb{G}_m(\mathbb{Q}) \simeq \mathbb{G}_m(A_p^{\sigma})/\mathbb{G}_m(\mathbb{Z}_p) = \Delta
\]

Recall that in section 3.3 in order to define the map \( \iota_s \), we have fixed an isomorphism of \( \mathbb{Z}/2\mathbb{Z} \)-grading right \( \text{Cl}(L_d) \)-module \( H \otimes Z_\nu(W) \cong H_{\text{cris}}(A_s/W) \) which sends \( \pi \) to \( \pi_{\text{cris},s} \), so that the Frobenius action on \( H_{\text{cris}}(A_s/W) \) takes the form \( b\sigma \) for some \( b \in G(K) \). This isomorphism also gives us an isometry

\[
L_d \otimes W \simeq L_{\text{cris},s} = L_{\text{cris},t}
\]

(5.5)

Let \( \nu \) be the cocharacter in section 3.3. Set \( \nu^{\text{ad}} := \text{ad} \circ \nu \) and \( b^{\text{ad}} := \text{ad}(b) \).

**Lemma 5.6.** \( X_p^{\text{ad}} = \{ g_p^{\text{ad}} \in G^{\text{ad}}(K)/G^{\text{ad}}(W) : (g_p^{\text{ad}})^{-1} b^{\text{ad}}(g^{\text{ad}}) \in G^{\text{ad}}(W) \nu^{\text{ad}}(p) G^{\text{ad}}(W) \} \)
Proof. Let $g_p^{ad} \in G^{ad}(K)$ be any representative of an element of $X_p^{ad}$ and let $g_p \in G(K)$ be any lift of $g_p^{ad}$. We want to show that $g_p^{-1}b\sigma(g_p) \in G(W)\nu(p)G(W)$.

Consider the central isogeny

$$\text{Nm} \times \text{ad} : G \to G' \,:= \,\mathbb{G}_m \times G^{ad}$$

Let $T \subset G_{\mathbb{Z}_p}$ (resp. $T' \subset G_{\mathbb{Z}_p}'$) be the centralizer of a maximal split torus and let $\Omega_G$ (resp. $\Omega_{G'}$) be the associated Weyl group. We could arrange that $T = T' \times G_{\mathbb{Z}_p}' G_{\mathbb{Z}_p}$ and $\nu \in X_*(T)$. The Cartan decomposition gives us an isomorphism

$$X_*(T)/\Omega_G \sim \to G(W)\backslash G(K)/G(W)$$

given by $\mu \to \mu(p)$ and an analogous one for $T' \subset G_{\mathbb{Z}_p}'$.

The element $g_p^{-1}b\sigma(g_p) \in G(W)\nu'(p)G(W)$ for some cocharacter $\nu' \in X_*(T)$. As the central isogeny $\text{Nm} \times \text{ad}$ induces an injection $X_*(T)/\Omega_G \to X_*(T')/\Omega_{G'}$, it suffices to check that $(\text{Nm} \times \text{ad}) \circ \nu'$ and $(\text{Nm} \times \text{ad}) \circ \nu'$ lie in one $\Omega_{G'}$-orbit. By assumption, $G^{ad}(W)((\text{ad} \circ \nu')(p)G(W) = G^{ad}(W)\nu'^{ad}(p)G^{ad}(W)$, so it remains to check that $G_m(W)((\text{Nm} \circ \nu')(p)G_m(W) = G_m(W)((\text{Nm} \circ \nu')(p)G_m(W)$: One easily verifies $\text{val}_p(\text{Nm}(g^{-1}b\sigma(g))) = \text{val}_p(\text{Nm}(b))$ for any $p$-adic valuation $\text{val}_p$. \hfill \Box

**Lemma 5.7.** $X^{ad,p} = G^{ad}(K^p)$

Proof. For all but finitely many $\ell$, $L_\ell \otimes \mathbb{Z}_\ell$ is self-dual. For each such $\ell$, we have that $G(\mathbb{Z}_\ell) \to G^{ad}(\mathbb{Z}_\ell)$ is surjective: $G$ and $G^{ad}$ have models over $\mathbb{Z}_\ell$. On the special fiber $G(\mathbb{F}_\ell) \to G^{ad}(\mathbb{F}_\ell)$ and by the smoothness of the kernel $G_m$, a $\mathbb{Z}_\ell$-valued point of $G^{ad}$ lifts to $G$. \hfill \Box

**Lemma 5.8.** Let $k$ be a field, $M$ be a vector space over $k$ and $G \subset GL(M)$ be a closed reductive subgroup. Let $\mu, \mu' : G \to G$ be two cocharacters. If $\mu$ and $\mu'$ induce the same filtrations on $V$, then they are conjugate over $G(k)$. \hfill \Box

Proof. This can be extracted from the proof of [18, Lemma 1.1.9]. We present the argument for readers’ convenience. Let $\text{Fil}^\bullet M$ be the filtration on $M$ induced by $\mu, \mu'$ and let $P \subset G$ denote the parabolic subgroup which respects $\text{Fil}^\bullet$. Let $U \subset P$ be the stabilizer of the associated graded vector space $\text{gr}^\bullet M$, so that $U$ is the unipotent radical of $P$ (c.f. [17, Lemma 1.1.1]). Since $\mu, \mu'$ induce the same filtrations, they induce the same grading on $\text{gr}^\bullet M$, which means that their compositions $G_m^{\mu,\mu'} : P \to P/U$ are equal. Let $L, L'$ be the centralizers of $\mu, \mu'$ in $G$. Then $L, L'$ are Levi subgroups of $P$, so $L = uL'u^{-1}$ for some $u \in U(k)$. The cocharacter $\mu'' := u\mu'u^{-1} : G_m \to L$ and $\mu$ induce the same cocharacter under the projection $L \to P/U$. Hence $\mu'' = \mu$. \hfill \Box

Next we give a lemma on K3 crystals, which is an analogue of [18, Lemma 1.1.12]. Let $k$ be a perfect field of characteristic $p$ and let $(D, \varphi, \langle \cdot, \cdot \rangle)$ be a K3 crystal over $k$. Recall that on the Frobenius action $\varphi$ gives an abstract Hodge filtration $\text{Fil}^\varphi$ on $D/pD$:

$$\text{Fil}^\varphi(D/pD) := \varphi^{-1}(p^iD) \mod p$$
By a famous result of Mazur (c.f. [6] Section 8), if \((D, \varphi, (-, -)) = H^2_{\cris}(X/W(k))\) for some K3 surface \(X\) over \(k\), then \(\Fil^*\varphi\) agrees with the Hodge filtration on
\[
H^2_{\cris}(X/W(k))/pH^2_{\cris}(X/W(k)) \cong H^2_{\text{dR}}(X)
\]
Now let \((D, \varphi, (-, -))\) be the K3 crystal of Theorem 1.4. Fix an isomorphism \((D, (-, -)) \cong L_d \otimes W\). \(\varphi\) takes that form \(pb\sigma\) with \(b \in G^{\text{ad}}(K)\).

**Lemma 5.9.** There exists a cocharacter \(\tilde{\nu} : G_m \to G^{\text{ad}}_W\) such that \(\tilde{b} \in G^{\text{ad}}(W)\tilde{\nu}(p)G^{\text{ad}}(W)\) and \(\sigma^{-1}(\tilde{\nu})\) gives the filtration \(\Fil^*\varphi(1)\).

**Proof.** By the Cartan decomposition, there exists a \(G^{\text{ad}}_W\)-valued cocharacter \(\tilde{\nu}'\) and \(h_1, h_2 \in G^{\text{ad}}(W)\) such that \(\tilde{b} = h_1\tilde{\nu}'(p)h_2 = h_1h_2\tilde{\nu}(p)\) for \(\tilde{\nu} = h_2^{-1}\tilde{\nu}'h_2\). We check that \(\sigma^{-1}(\tilde{\nu})\) gives the filtration \(\Fil^*_\varphi(1)\). The condition that \(p^2D \subset \varphi(D)\) tells us that the decomposition defined by \(\sigma^{-1}(\tilde{\nu})\) takes the form \(D = D_1 \oplus D_0 \oplus D_{-1}\) where \(D_i = \{d \in D : \sigma^{-1}(\tilde{\nu})(z) \cdot d = z^i d\}\). Now we verify that \(D_1 = \varphi^{-1}(p^2D)\). That \(D_1 \subset \varphi^{-1}(p^2D)\) is clear. To check the converse inclusion, let \(d = d_1 + d_0 + d_{-1}\) be the decomposition of an element \(d \in D\). If \(\varphi(d) = pb\sigma(d) = ph_1h_2\tilde{\nu}(p)\sigma(d) \in p^2D\), then \(d_0 \equiv d_{-1} \equiv 0 \mod p\). Hence \((d \mod p) \in D_1/pD_1\). We could check that \((D_1 \oplus D_0) \mod p = \Fil^*_\varphi(D/pD)\) similarly. 

**Proof of Theorem 1.6:** Let \((X, \xi)\) be the quasi-polarized K3 surface of Theorem 1.4. Clearly it suffices to prove the case when \(\xi\) is primitive. Let \(t \in M_{2d, K_p^{\text{ad}}, \mathbb{Z}_p}(\bar{\mathbb{F}}_p)\) be a point giving \((X, \xi)\). Recall that we have an identification \(P^2_{\cris}(X/W)(1) \simeq L_{\text{cris}, t}\) and in (5.5) we have fixed an isomorphism \(L_{\text{cris}, t} \cong L_d \otimes W\). Let \((\Lambda_p, \varphi, (-, -)) \subset P^2_{\cris}(X/W) \otimes K\) be an isometric embedding such that \(\varphi\) agrees with Frobenius action. There exists \(g_p^\text{ad} \in G^{\text{ad}}(K)\) such that \(g_p^\text{ad} \cdot P^2_{\cris}(X/W) = \Lambda_p \subset P^2_{\cris}(X/W) \otimes K\). We want to show that \(g_p^\text{ad} \in G^{\text{ad}}(W)\tilde{\nu}(p)G^{\text{ad}}(W)\). This follows from the observation that \(\nu^\text{ad}\) and \(\tilde{\nu}\) of Lemma 5.9 are \(G(W)\)-conjugate: Note that \(\sigma^{-1}(\nu^\text{ad})\) and \(\sigma^{-1}(\tilde{\nu})\) both define filtrations on \(L_d \otimes \bar{\mathbb{F}}_p\) of the form \(\Fil^1 \subset \Fil^0 = (\Fil^1)^{+} \subset \Fil^{-1} = L \otimes \bar{\mathbb{F}}_p\) with \(\Fil^1\) being isotropic of dimension 1. By the theory of quadratic forms we can find an element in \(G^{\text{ad}}(\bar{\mathbb{F}}_p)\) which sends one to the other. Then Lemma 5.8 tells us that \(\sigma^{-1}(\nu^\text{ad})\) and \(\sigma^{-1}(\tilde{\nu})\) induce conjugate \(G^{\text{ad}} \otimes \bar{\mathbb{F}}_p\) cocharacters. By [10] IX 3.3, \(\sigma^{-1}(\nu^\text{ad})\) and \(\sigma^{-1}(\tilde{\nu})\), and hence \(\nu^\text{ad}\) and \(\tilde{\nu}\), are conjugate by element of \(G^{\text{ad}}(W)\).

Similarly, for every isometric embedding \(\Lambda_p \subset P^2_{\text{et}}(X, K_p)\), we could pick \(g^\text{ad, p} \in X_p^\text{ad}\) such that \(g^{\text{ad, p}} \cdot P^2(X, \hat{\mathbb{Z}}^p) = \Lambda_p\). Under the assumption \(p > 18d + 4\), the period map \(\iota\) is known to be surjective (c.f. [26] Theorem 4.1]). Any point in the preimage of \(\iota^{\text{ad}}((g_p^\text{ad}, g^{\text{ad, p}}))\) will give us the desired \((X', \xi')\).

### 5.3 Proof of Theorem 1.6

Let \((X, \xi)\) be the quasi-polarized surface of Theorem 1.6. Clearly we could assume that \(\xi\) is primitive. Let \(t \in M_{2d, K_p^{\text{ad}}, \mathbb{Z}_p}(\bar{\mathbb{F}}_p)\) be a point giving \((X, \xi)\). We also write \((X_t, \xi_t)\) for \((X, \xi)\). Let \(s \in \mathcal{X}_{K_p}(G, \Omega)(\bar{\mathbb{F}}_p)\) be a lift of \(t\). By [18] Theorem (0.4)], the isogeny class of \(s\) contains a point \(s'\) which lifts to a special point \(s'_{\mathcal{C}}\) on \(\text{Sh}_{K_p}(G, \Omega)\). Let \(t'\) and \(t'_{\mathcal{C}}\) denote the image of \(s'\) and \(s'_{\mathcal{C}}\) respectively.
By the surjectivity of the period map over $\mathbb{C}$, $t'_c$ comes from a complex quasi-polarized K3 surface $(X'_{t'_c}, \xi_{t'_c})$. \footnote{It is not strictly necessary to know that $t'_c$ comes from a K3 surface here. We could alternatively argue purely in terms of polarizable Hodge structures of K3 type.} By Theorem 2.1 $E := \text{End}_{\text{Hdg}} T(X'_{t'_c}, Q)$ is a CM field generated by some Hodge isometry $\eta'$ as a $Q$-algebra, and we have $\text{End}_{E} T(X'_{t'_c}, Q) = 1$. We extend $\eta'$ to $\tilde{\eta'} \in \text{End}_{\text{Hdg}} (P^2(X'_{t'_c}, Q))$ by letting it act trivially on Hodge classes.

By lemma 2.8 there exists a CSpin-isogeny $\psi' : A_{s'_c} \to A_{s'_c}$ which induces the $\tilde{\eta}'$ via the identifications

$$P^2(X'_{t'_c}, Q(1)) \sim L_{B,s'_c} \otimes \mathbb{Q} = L_{B,s'_c} \otimes \mathbb{Q} \hookrightarrow \text{End} H_{B,s'_c} \otimes \mathbb{Q} = \text{End} H^1(A_{s'_c}, \mathbb{Q})$$

Now specialize $\psi'_{s'_c}$ to a quasi-isogeny $\psi' : A_s \to A_{s'}$. Let $\psi : A_s \to A_{s'}$ be a CSpin-isogeny connecting $s$ and $s'$. Set $\psi_s := \psi^{-1} \circ \psi_{s'} \circ \psi$. $\psi_s$ induces an isometry of $F$-isocrystals $\eta_{\text{cris}} : P^2_{\text{cris}}(X_t/W) \otimes K \to P^2_{\text{cris}}(X_t/W) \otimes K$. By Lemma 4.6 we can find a lift $(X_{t'V}, \xi_{t'V})$ for some finite extension $V/W$ with fraction field $K'$ such that the induced filtration on $P^2_{\text{cris}}(X_t/W) \otimes K'$ is respected by $\eta_{\text{cris}} \otimes K'$. Let $t'_{V}$ be a point on $\bar{M}_{d,K'_{\text{cris}}(V)}(\mathcal{Z}_{(p)})$ corresponding to the pair $(X_{t'V}, \xi_{t'V})$. We will show that $X_{t'_{V} \otimes V} \otimes C$ has CM for any $V \hookrightarrow \mathbb{C}$.

Let $s_V \in \mathcal{Z}_{\mathcal{K}_p}(G, \Omega)(V)$ be a lift of $t'_{V}$ such that $s = s_V \otimes \bar{\mathbb{F}}_p$. As in the proof of Theorem 4.5 $\psi_s$ respects the Hodge filtration induced on $H^1_{\text{cris}}(A_s/W) \otimes K'$ by $A_{s_V}$. By [3, Theorem 3.15] again, up to a multiple, we may lift $\psi_s$ to an isogeny $\psi_{s_V} : A_{s_V} \to A_{s_V}$. For any embedding $V \hookrightarrow C$, the CSpin-isogeny $\psi_{s_V} : A_{s_V} \otimes C : A_{s_V} \otimes C \to A_{s_V} \otimes C$ induces a Hodge isometry $\tilde{\eta} : P^2(X_{t'_{V} \otimes C}, Q) \to P^2(X_{t'_{V} \otimes C}, Q)$. One may easily check from the construction that $\tilde{\eta} \otimes \mathbb{Q}_\ell$ is sent precisely to $\tilde{\eta}' \otimes \mathbb{Q}_\ell$ via the isomorphisms

$$P^2(X_{t'_{V} \otimes C}, Q) \otimes \mathbb{Q}_\ell \cong P^2_{\ell}(X_t, \mathbb{Q}_\ell) \sim L_{\ell, t} \overset{\text{conj. by } (\psi^*_{\ell})^{-1}}{\rightarrow} L_{\ell, t'} \cong P^2(X'_{t'_c}, Q) \otimes \mathbb{Q}_\ell$$

(5.10)

Therefore, we have an isomorphism of $\mathbb{Q}$-algebras $\mathbb{Q}(\tilde{\eta}) \cong \mathbb{Q}(\tilde{\eta}')$. Moreover, we have a commutative diagram

$$\begin{align*}
\langle \xi_{t_c} \rangle^\perp \subset \text{Pic} (X_{t_c}) & \quad \longrightarrow \quad \langle \xi_t \rangle^\perp \subset \text{Pic} (X_t) \\
\downarrow & \quad \downarrow \\
L(A_{s'_c}) & \longrightarrow \quad L(A_s) \quad \overset{\text{conj. by } \psi}{\longrightarrow} \quad L(A_{s'}) \quad \longleftrightarrow \quad L(A_{s'_c})
\end{align*}$$

All arrows in the diagram are isomorphisms of quadratic lattices: the top horizontal arrow is an isomorphism because the construction in Lemma 4.6 lifts the entire Picard group, the vertical arrows are isomorphisms given by Proposition 3.7 and the specialization map $L(A_{s'_c}) \to L(A_{s'})$ is an isomorphism by Proposition 3.8. The diagram gives us an isometry

$$\langle \xi_{t_c} \rangle^\perp \subset \text{Pic} (X_{t_c}) \sim \langle \xi'_{t'_c} \rangle^\perp \subset \text{Pic} (X_{t'_c})$$

which is clearly compatible with (5.10). Therefore, (5.10) restricts to an isometry

$$T(X_{t_c}) \otimes \mathbb{Q}_\ell \cong T(X'_{t'_c}) \otimes \mathbb{Q}_\ell$$

Let $\eta$ be the restriction of $\tilde{\eta}$ to $T(X_{t'_{V} \otimes C}, Q)$. Then the above isomorphism tells us that $\mathbb{Q}(\tilde{\eta}) \cong \mathbb{Q}(\tilde{\eta}')$ restricts to $\mathbb{Q}(\eta) \cong \mathbb{Q}(\eta')$. Hence $\dim_{\mathbb{Q}(\eta)} \text{End}_{\text{Hdg}} T(X_{t_c}, Q) = 1$. By Theorem 2.1 $X_{t'_{V} \otimes C}$ has CM.
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24