Generating spaces of modular forms with $\eta$-quotients

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Abstract

In this note we consider a question of Ono, concerning which spaces of classical modular forms can be generated by sums of $\eta$-quotients. We give some new examples of spaces of modular forms which can be generated as sums of $\eta$-quotients, and show that we can write all modular forms of level $\Gamma_0(N)$ as rational functions of $\eta$-products.

1 Introduction

Let $z$ be an element of the Poincaré upper half plane $\mathcal{H}$, and let $q := \exp(2\pi iz)$. We recall the definition of the Dedekind $\eta$-function:

$$\eta(q) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).$$

It is well-known that this function plays an important role in the theory of modular forms; for instance, the unique normalised cusp form of level 1 and weight 12, the $\Delta$ function, can be written as $\Delta(q) = \eta(q)^{24}$.

Another classical application of the $\eta$-function is to the theory of partitions; a partition of $n$ is a way of writing $n$ as a sum of positive integers; for instance, $4 = 2 + 2$ is a partition of 4. The number of partitions of $n$, $p(n)$, grows quickly with $n$. For instance, in the early years of the 20th century, MacMahon computed that $p(200) = 397299029388$. The reciprocal of the $\eta$-function (with the $q^{1/24}$ removed) gives the following generating function for the $p(n)$:

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{m=1}^{\infty} \left(\frac{1}{1 - q^m}\right).$$

It can be shown that $\eta(q)^{24}$ is a modular form for $\text{SL}_2(\mathbb{Z})$. In [1], an explicit transformation formula for $\eta(q)$ under the action of $\text{SL}_2(\mathbb{Z})$ is given; it is of the form

$$\eta\left(\frac{az + b}{cz + d}\right) = \varepsilon \cdot (cz + d)^{1/2} \eta(z) \text{ for } \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in \text{SL}_2(\mathbb{Z}),$$

(1)
where $\varepsilon^{24} = 1$. (The explicit definition of $\varepsilon$, which depends on $a, b, c, d$, is complicated, but is given in Theorem 3.4 of [1]).

We see from (1) that $\eta(q)$ satisfies a “weight 1/2” transformation formula, which is consistent with the statement that $\Delta$ is a weight 12 modular form.

**Definition 1.** Let $N$ be a positive integer, let $\{r_\delta\}$ be a set of integers, and let $f$ be a meromorphic function from the Poincaré upper half plane to $\mathbb{C}$ of the form

$$f(z) = \prod_{0 < \delta | N} \eta(q^\delta)^{r_\delta},$$

where $q := \exp(2\pi i z)$.

We call $f$ an $\eta$-quotient, and if all of the $r_\delta$ are non-negative we say that $f$ is an $\eta$-product.

We see that $\Delta$ is an $\eta$-product; we will see some examples of $\eta$-quotients which are modular forms below.

There are some general theorems which tell us when an $\eta$-quotient is a classical modular form, by telling us how it transforms under elements of certain congruence subgroups.

**Theorem 2** (Ligozat [11], quoted in [13]). Suppose that $f(z)$ is an $\eta$-quotient which satisfies the following properties:

$$\sum_{0 < \delta | N} \delta \cdot r_\delta \equiv 0 \mod 24$$

(2)

and

$$\sum_{0 < \delta | N} \frac{N}{\delta} \cdot r_\delta \equiv 0 \mod 24.$$  

(3)

Then $f(z)$ satisfies

$$f\left(\frac{az + b}{cz + d}\right) = \chi(d)(cz + d)^k f(z)$$

(4)

for every $(a b c d) \in \Gamma_0(N)$, where $k := \frac{1}{2} \sum_{0 < \delta | N} r_\delta$, and

$$\chi(d) := \left(\frac{-1}{d}\right)^k s,$$

where $s := \prod_{0 < \delta | N} \delta^{r_\delta}.$

(5)

In [15], the following question is posed:

**Question 3** (Ono [15], Problem 1.68). Which spaces of classical modular forms can be generated by sums of $\eta$-quotients?

It is well-known (see for instance [16], Corollary 2 to Theorem VII.4) that every modular form of level 1 is a polynomial in the Eisenstein series $E_4$ and $E_6,$
so if we can write $E_4$ and $E_6$ in terms of $\eta$-quotients then we can write every modular form of level 1 in terms of $\eta$-quotients. In [15], we see that

$$E_4(z) = \frac{\eta(q)^{16}}{\eta(q^2)^8} + 2^8 \cdot \frac{\eta(q^2)^{16}}{\eta(q)^8}$$

and

$$E_6(z) = \frac{\eta(q)^{24}}{\eta(q^2)^{12}} - 2^5 \cdot 5 \cdot \eta(q^2)^{12} - 2^9 \cdot 3 \cdot 11 \cdot \frac{\eta(q^2)^{12} \eta(q^4)^8}{\eta(q)^8} + 2^{13} \cdot \frac{\eta(q^4)^{24}}{\eta(q^2)^{12}}.$$

One can check that these are identities of modular forms by verifying that each of these $\eta$-quotients is a modular form on $\Gamma_0(4)$ and that the $q$-expansions of both sides agree up to a suitable bound, such as the Sturm bound [10] (also known as the Hecke bound); if a modular form has a zero at the cusp $\infty$ of large enough degree, then it must be the zero modular form. In this case, we find that the dimension of the space $M_4(\Gamma_0(4))$ is 3, and the Sturm bound here is 2. We can use a computer algebra package to check that these really are equalities.

In this paper we will give further examples of this form, and prove a theorem which says that we can write every modular form for a congruence subgroup $\Gamma_0(N)$ in terms of $\eta$-quotients of level at most $\Gamma_0(4N)$.

\section{Writing eigenforms as $\eta$-products or $\eta$-quotients}

We will briefly consider $\eta$-products, and show that there are only finitely many spaces of modular forms which can be generated by $\eta$-products. These do exist: for instance, $S_2(\Gamma_0(11)) = \mathbb{C} \cdot \eta(q)^2 \eta(q^{11})^2$, and $S_1(\Gamma_1(23)) = \mathbb{C} \cdot \eta(q) \eta(q^{23})$. However, it is clear that only finitely many spaces of modular forms can be completely generated by $\eta$-products, as there are only finitely many $\eta$-products with $q$-expansion of the form $q^a + O(q^{a+1})$ for any $a$; this follows from equation (2), because as the level increases the sum increases also.

Questions of this sort have been asked before; in [2], the question of when an $\eta$ product is also a simultaneous eigenform for the Hecke operators $T_p$ is addressed. In [5], a complete list of the 30 different $\eta$-products which are classical modular cuspidal eigenforms of integer weight with no zeroes outside the cusps is given (two of these have half-integral weight). Another proof of this classification was given by [9].

Similarly, in [12], a complete list of all $\eta$-quotients which are holomorphic new eigenforms of integral weight for the congruence subgroups $\Gamma_1(N)$ is derived. There are 74 of these, with levels ranging from $\text{SL}_2(\mathbb{Z})$ (the $\Delta$-function) up to $\Gamma_1(576)$ (the modular form $\eta(q^{12})^{-2} \cdot \eta(q^{24})^6 \cdot \eta(q^{48})^{-2} \in S_1(\Gamma_1(576))$). There have been other investigations; in [8], a list of the 65 $\eta$-products which have weight 1 and level $\Gamma^*(36)$ is given; the congruence subgroup $\Gamma^*(N)$ is generated by $\Gamma_0(36)$ and the Fricke involution \begin{pmatrix} 0 & 1/\sqrt{N} \\ -1/\sqrt{N} & 0 \end{pmatrix}. Another angle of inquiry about $\eta$-products is given in [6], which considers $\eta$-products which are lacunary modular forms; in other words, they have many zero Fourier coefficients.
3 Sums of $\eta$-products and $\eta$-quotients

We see that, from the results we have mentioned in the previous section, if we are to generate all spaces of modular forms for a given congruence subgroup $\Gamma$ we will need to expand from $\eta$-products and $\eta$-quotients to consider more general objects. We will consider sums of $\eta$-products and $\eta$-quotients in this section.

From equation (4), we see that the only characters we can have are the Legendre symbols, so if we are considering congruence subgroups containing $\Gamma_1(N)$ we are restricted to the cases of $\Gamma_0(N)$ and $\Gamma_0(N)$ plus a character of order 2.

We will be using the standard formulae for the dimension of spaces of modular forms in this section; these can be found in many books on modular forms; one reference is Theorem 3.5.1 and Figure 3.3 of [4]. Let $g = \lfloor \frac{(p + 1)}{12} \rfloor$ if $p \not\equiv 1 \mod 12$ and $\lfloor \frac{(p + 1)}{12} \rfloor - 1$ otherwise (this is the genus of $X_0(p)$).

The formulae (if $N = p$ is prime) are (for $k \geq 4$ and $N \geq 5$):

$$\dim S_k(\Gamma_0(p)) = (k - 1) \cdot (g - 1) + \left\lfloor \frac{k}{4} \right\rfloor \cdot \left(1 + \left(\frac{-1}{p}\right)\right) + \left\lfloor \frac{k}{3} \right\rfloor \cdot \left(1 + \left(\frac{-3}{p}\right)\right) + k - 2.$$

The dimension of $S_2(\Gamma_0(p))$ is $g$, and the dimension of the space of Eisenstein series is 1 if $k = 2$ and 2 if $k \geq 4$ (the Eisenstein series of weight $k \geq 4$ are the images of $E_k$ under the two degeneracy maps from level 1 to level $p$, and the Eisenstein series of weight 2 is defined in terms of the Eisenstein series of weight 2 which is not a modular form). There are similar formulae for $N$ composite and for other congruence subgroups.

We will first consider some more examples of spaces of modular forms which can be generated by $\eta$-quotients: for example, we can write every modular form of level $\Gamma_0(2)$ as a sum of $\eta$-quotients. It will suffice to show that $M_2(\Gamma_0(2))$, $M_4(\Gamma_0(2))$, $M_6(\Gamma_0(2))$ and $M_8(\Gamma_0(2))$ can be generated by sums of $\eta$-quotients, because the following relation holds:

$$S_{k+8}(\Gamma_0(2)) = f \cdot M_k(\Gamma_0(2)),$$

where $f = \lambda \cdot (\eta(z) \cdot \eta(2z))^8 \in S_8(\Gamma_0(2))$. This can be derived from the dimension formulae that we have given above.

Because we can write level 1 modular forms as sums of $\eta$-quotients, and we know (for instance, from Proposition III.3.19 of [7]) that the unique normalised cusp-form $f$ of weight 8 and level $\Gamma_0(2)$ is an $\eta$-quotient, we need only to consider $M_2(\Gamma_0(2))$, which is 1-dimensional. Using Magma [3], and the fact that the Hecke bound for the space $M_2(\Gamma_0(8))$ is 3, we see that

$$\frac{\eta(q^2)^{20}}{\eta(q)^8 \cdot \eta(q^2)^8} + 16 \cdot \frac{\eta(q^8)}{\eta(q^2)^4} = E_{2,2} \in M_2(\Gamma_0(2)),$$

so every modular form of level $\Gamma_0(2)$ can be written as a sum of $\eta$-quotients. We need to consider the space of level $\Gamma_0(8)$ because the $\eta$-quotients have level $\Gamma_0(8)$. 

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Similarly, to show that every modular form for \( \Gamma_0(3) \) is a sum of \( \eta \)-quotients, we need to check that \( E_{3,2} \in M_2(\Gamma_0(3)) \) is a sum of \( \eta \)-quotients. After some calculation, we find (using the fact that the Hecke bound for \( M_2(\Gamma_0(27)) \) is 6) that

\[
36\frac{\eta(q^9)}{\eta(q^3)^2} + 12\frac{\eta(q^3)^2}{\eta(q^9)} + \frac{\eta(q^3)^{10}}{\eta(q^3)^3\eta(q^{27})^3} + 108\frac{\eta(q^9)^2\eta(q^{27})^3}{\eta(q^3)^2} + 9\frac{\eta(q^3)^3\eta(q^{27})^3}{\eta(q^3)^2} + 9\frac{\eta(q^{27})^6}{\eta(q^9)^2}
\]

is equal to \( E_{3,2} \); this, together with the knowledge that \( S_6(\Gamma_0(3)) = C\cdot\eta(q)^6\eta(q^3)^6 \) and that the Eisenstein series of weights 4 and 6 and level \( \Gamma_0 \) are oldforms and therefore the image of forms of level 1 which can be written as \( \eta \)-quotients, shows us that we can write modular forms of level \( \Gamma_0(3) \) as sums of \( \eta \)-quotients of level at most 4 \cdot 27.

In fact, we can write \( E_4 \) and \( E_6 \) as sums of \( \eta \)-quotients of level 27, so we can actually write every modular form for \( \Gamma_0(3) \) in terms of \( \eta \)-quotients of level 27.

In a similar fashion, we can write the elements of \( M_2(\Gamma_1(3)) \) as a linear combination of \( \eta(q^3)^3/\eta(q)^3 \) and \( \eta(q^9)/\eta(q^3)^3 \); these, together with the Eisenstein series of level 3 and weights 2, 4, and 6 and the cusp form of level 3 and weight 6 generate all of the modular forms of level 3 and odd weight, so we can write every modular form of level \( \Gamma_1(3) \) as a sum of \( \eta \)-quotients.

We can prove similar results for level \( \Gamma_1(4) \) as well. Using Exercise III.3.17 of Koblitz’s book on modular forms [7], we see that every \( f \in M_k(\Gamma_1(4)) \) can be written as a polynomial in the modular forms \( F := \eta(q^4)^6/\eta(q^2)^8 \), which has weight 2, and \( \theta^2 := \eta(q^2)^{10}/\eta(q)^4\eta(q^4)^4 \), which has weight 1. As both of these are \( \eta \)-quotients, we are done; every modular form of level \( \Gamma_1(4) \) can be written as a sum of \( \eta \)-quotients of level at most 4.

A rather tedious MAGMA calculation shows us that the unique Eisenstein series \( E_{5,2} \) of level \( \Gamma_0(5) \) and weight 2 can be written as the following sum of \( \eta \)-quotients of level \( \Gamma_0(20) \):

\[
E_{5,2}(q) = \frac{5 \cdot (\eta(q^2)^3\eta(q^4)^4\eta(q^5)^3\eta(q^{10})^5)}{\eta(q)^7\eta(q^{20})^4} - 2 \cdot \frac{\eta(q^2)^7\eta(q^4)^2\eta(q^5)^3\eta(q^{10})}{\eta(q)^7\eta(q^{20})^2}
\]

\[
- \frac{4 \cdot (\eta(q)^2)^7\eta(q^4)^7\eta(q^{10})^5}{\eta(q)^7\eta(q^{20})^3} + \frac{\eta(q^4)^4\eta(q^5)^6\eta(q^{10})^4}{\eta(q)^8\eta(q^{20})^4}
\]

\[
+ 2 \cdot \frac{\eta(q^2)^4\eta(q^4)^2\eta(q^5)^6}{\eta(q)^9\eta(q^{20})^2} - \frac{\eta(q^2)^4\eta(q^4)^7\eta(q^{10})^4}{\eta(q)^6\eta(q^5)^2\eta(q^{20})^3}.
\]

This follows because these six \( \eta \)-quotients generate \( M_2(\Gamma_0(20)) \), where \( E_{5,2} \) appears as an oldform. We can find the coefficients of these using linear algebra.

In a very similar way, we can write both elements of \( M_2(\Gamma_0(5), (\chi)) \) as \( \eta \)-quotients by considering them as oldforms of level 20 and finding a basis of \( \eta \)-quotients for this space of level 20 forms. This suffices to show that all modular forms with character \( (\chi) \) can be written as sums of \( \eta \)-quotients.

At this point, we notice that if \( p \geq 5 \) is a prime then we cannot write every modular form for \( \Gamma_0(p) \) as a sum of \( \eta \)-quotients of level \( p^r \). We use the dimension
that the right hand side can vanish modulo $\eta$ this by finding a basis of $M$ sum even entries (because the requirement that the character be trivial forces the we see that the right hand side is $-\frac{1}{4}$ of the $\eta$

considering the level 7 spaces as oldforms in these spaces. Here we find that $\eta$

We now reduce (8) modulo $\eta$ puting these together, we find that $\eta$

Therefore, we need to solve the following two simultaneous equations:

$$a + p \cdot b + p^2 \cdot c + \cdots + p^r \cdot z = 0 \quad (6)$$

$$a + b + c + \cdots + z = 4; \quad (7)$$

puting these together, we find that

$$(p - 1) \cdot b + (p^2 - 1) \cdot c + \cdots + (p^r - 1) \cdot z = -4. \quad (8)$$

We now reduce (5) modulo $p - 1$; the left hand side vanishes, and the only way that the right hand side can vanish modulo $p - 1$ is if $p \in \{2, 3, 5\}$.

Now we consider the case when $p = 5$. We use the formula for the character of the $\eta$-quotient given in (5) to work on (8); after dividing both sides by 4, we see that the right hand side is $-1$, whereas the left hand side has only even entries (because the requirement that the character be trivial forces the sum $b + d + f + \cdots$ to be even). This is a contradiction.

We can tackle $\Gamma_0(7)$ in a similar fashion; in this case, we have to check that we can generate $M_2(\Gamma_0(7))$ and $M_4(\Gamma_0(7))$ by $\eta$-quotients, and again we prove this by finding a basis of $\eta$-quotients for $M_2(\Gamma_0(28))$ and $M_4(\Gamma_0(28))$ and then considering the level 7 spaces as oldforms in these spaces. Here we find that

$$E_{7,2} = 16 \cdot \frac{\eta(q^2)^6 \eta(q^4)^6 \eta(q^{14})^2}{\eta(q)^8 \eta(q^{28})^2} - 32 \cdot \frac{\eta(q^2)^4 \eta(q^4)^3 \eta(q^{14})^6}{\eta(q)^6 \eta(q^{14})^2 \eta(q^{28})^2}$$

$$+ \frac{49}{2} \cdot \frac{\eta(q^2) \eta(q^7)^8 \eta(q^{14})^2}{\eta(q)^4 \eta(q^{28})^3} - 10 \cdot \frac{\eta(q^2)^4 \eta(q^4)^6 \eta(q^{14})^4}{\eta(q)^4 \eta(q^{14})^2 \eta(q^{28})^2}$$

$$- \frac{16}{7} \cdot \frac{\eta(q^2)^5 \eta(q^7)^5}{\eta(q)^3 \eta(q^{14})^2} + 6 \cdot \frac{\eta(q^2)^7 \eta(q^4)^2 \eta(q^{28})^2}{\eta(q) \eta(q^7) \eta(q^{14})^5}$$

$$- \frac{17}{14} \cdot \frac{\eta(q) \eta(q^3)^3 \eta(q^7)^5 \eta(q^{14})^5}{\eta(q^4)^7 \eta(q^{28})^3};$$

the equations for the three cuspforms in $S_4(\Gamma_0(7))$ are similar.

| Group          | Level of $\eta$-products needed |
|----------------|---------------------------------|
| $SL_2(\mathbb{Z})$ | 4 or 27                         |
| $\Gamma_1(2)$   | 8                               |
| $\Gamma_1(3)$   | 27                              |
| $\Gamma_1(4)$   | 4                               |
| $\Gamma_0(5)$   | 20                              |
| $\Gamma_0(7)$   | 28                              |

Table 1: Levels of $\eta$-quotients needed to generate spaces of modular forms
4 Generating spaces of modular forms with rational functions of $\eta$-quotients

We now consider a larger set of functions than in the previous section; we will now consider rational functions of $\eta$-quotients and $\eta$-products. We will now prove the following theorem on the generation of spaces of modular forms by rational functions of $\eta$-quotients (and in fact by rational functions of $\eta$-products):

Theorem 4. Let $N$ be a positive integer, let $k$ be a positive even integer, and let $f \in M_k(\Gamma_0(N))$ be a modular form. Then we can write $f$ as a rational function of $\eta$-quotients.

Proof. We first note that the classical $j$-invariant is an $\eta$-quotient, because we can write it as $j := \frac{E_4}{\Delta}$, where $E_4$ is an $\eta$-quotient and $\Delta = \eta(q)^{24}$. This means that the modular function $j(q^N)$ is also an $\eta$-quotient, for any $N \in \mathbb{N}$.

We now quote a theorem about the field of modular functions of weight 0 for the congruence subgroup $\Gamma_0(N)$; this is proved in the online notes of Milne [14] on modular forms and modular functions:

Theorem 5 (Milne [14], Theorem 6.1). The field $\mathbb{C}(X_0(N))$ of modular functions for $\Gamma_0(N)$ is generated over $\mathbb{C}$ by $j(q)$ and $j(q^N)$. In particular, every modular function for $\Gamma_0(N)$ is a rational function of $j(q)$ and $j(q^N)$.

By standard properties of modular forms, the quotient $f/g$ of two modular forms of weight $k$ for a congruence subgroup $\Gamma$ is a modular function of weight 0 for that congruence subgroup, with possible poles at the zeroes of $g$.

We therefore take $f \in M_k(\Gamma_0(N))$ and consider the modular function

$$g := \frac{f \cdot E_4^a \cdot E_6^b}{\Delta^c} \in \mathcal{M}_0(\Gamma_0(N)),$$

where we choose $a \in \{0, 1, 2\}$, $b \in \{0, 1\}$ and $k \in \mathbb{N}$ such that $k + 4a + 6b - 12c = 0$. This has weight 0 by definition, so by Theorem 4 we can write it as a rational function of $j(q)$ and $j(q^N)$, which are both rational functions of $\eta$-products. From before, we recall that $E_4$, $E_6$ and $\Delta$ are all rational functions of $\eta$-products, so we can write $f$ as a rational function of $\eta$-products, which is what we set out to do.

It is interesting to note that we only needed to use $\eta$-products of level $4N$ here (the $N$ coming from the level of our form $f$, and the 4 coming from the $\eta$-quotient representations of $E_4$ and $E_6$). This corresponds to the figures in Table 1, where we needed to raise the level by a factor of at least 4, corresponding to the level 4 $\eta$-quotients that we used for $E_4$ and $E_6$. 


We see also that we have to have either the square of a prime or two distinct primes dividing the level, because we always need to write $E_4$ and $E_6$ in terms of $\eta$-products, and we need to find at least two different solutions to the set of simultaneous equations in the powers of the $\eta^i$ (one for $E_4$ and one for $E_6$). This means that we need to have at least 3 $i$, so in particular we cannot write $E_4$ and $E_6$ just in terms of $\eta$-quotients of level $p$ for any prime $p$.

If we are willing to raise the level then we can use $g := \eta(q)^2 \eta(q^{11})^2 \in S_2(\Gamma_0(11))$ as a multiplier in $\mathfrak{M}$, so the only thing stopping our rational function being a polynomial is the numerator of the modular function of weight 0. However, we cannot assume that $g$ only has a zero at $\infty$, because while the modular form $\Delta(q) \in S_{12}(\text{SL}_2(\mathbb{Z}))$ only has a simple zero at $\infty$ and no zeroes elsewhere, the modular form $\Delta(q) \in S_{12}(\Gamma_0(N))$ (viewed as an oldform) has many zeroes; it has one at each cusp of $\Gamma_0(N)$.

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References

[1] Tom M. Apostol. Modular functions and Dirichlet series in number theory. Springer-Verlag, New York, 1976. Graduate Texts in Mathematics, No. 41.

[2] Anthony J. F. Biagioli. Eta-products which are simultaneous eigenforms of Hecke operators. Glasgow Math. J., 35(3):307–323, 1993.

[3] W. Bosma, J. Cannon, and C. Playoust. The Magma algebra system I: The user language. J. Symb. Comp., 24(3–4):235–265, 1997. See http://magma.maths.usyd.edu.au.

[4] Fred Diamond and Jerry Shurman. A first course in modular forms, volume 228 of Graduate Texts in Mathematics. Springer-Verlag, New York, 2005.

[5] D. Dummit, H. Kisilevsky, and J. McKay. Multiplicative products of $\eta$-functions. In Finite groups—coming of age (Montreal, Que., 1982), volume 45 of Contemp. Math., pages 89–98. Amer. Math. Soc., Providence, RI, 1985.

[6] Basil Gordon and Sinai Robins. Lacunarity of Dedekind $\eta$-products. Glasgow Math. J., 37(1):1–14, 1995.

[7] N. Koblitz. Introduction to Elliptic Curves and Modular Forms. Springer, New York, 1993.
[8] Günter Köhler. Eta products of weight 1 and level 36. *Arch. Math. (Basel)*, 76(3):202–214, 2001.

[9] Masao Koike. On McKay’s conjecture. *Nagoya Math. J.*, 95:85–89, 1984.

[10] Joan-C. Lario and René Schoof. Some computations with Hecke rings and deformation rings. *Experiment. Math.*, 11(2):303–311, 2002. With an appendix by Amod Agashe and William Stein.

[11] Gérard Ligozat. *Courbes modulaires de genre 1*. Société Mathématique de France, Paris, 1975. Bull. Soc. Math. France, Mém. 43, Supplément au Bull. Soc. Math. France Tome 103, no. 3.

[12] Yves Martin. Multiplicative \(\eta\)-quotients. *Trans. Amer. Math. Soc.*, 348(12):4825–4856, 1996.

[13] Ken McMurdy. *A Splitting Criterion for Galois Representations Associated to Exceptional Modular Forms*. PhD thesis, U.C. Berkeley, 2001. Available at [http://www.rose-hulman.edu/~mcmurdy/research/](http://www.rose-hulman.edu/~mcmurdy/research/).

[14] J. Milne. Modular functions and modular forms. Available at [http://www.jmilne.org/math/CourseNotes/math678.html](http://www.jmilne.org/math/CourseNotes/math678.html) 1997.

[15] Ken Ono. *The web of modularity: arithmetic of the coefficients of modular forms and \(q\)-series*, volume 102 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2004.

[16] J.-P. Serre. *A course in arithmetic*. Springer-Verlag, New York, 1973. Translated from the French, Graduate Texts in Mathematics, No. 7.