The Friedmann universe of dust by Regge Calculus: study of its ending point

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Abstract. We develop an evolution scheme, based on Sorkin algorithm, to evolve the most complex regular tridimensional polytope, the 600-cell. This application has been already studied before and all authors found a stop point for the evolution of the spatial section. In our opinion a clear and satisfactory meaning to this behaviour has not been given. In this paper we propose a reason why the evolution of the 600-cell stops when its volume is still far from 0. We find that the 600-cell meets a causality-breaking singularity of space–time. We study the nature of this singularity by embedding the 600-cell into a five-dimensional Lorentzian manifold. We fit 600-cell’s evolution with a continuous metric and study it as a solution of Einstein equations.

PACS number(s): 04.20.-q, 04.25.Dm

Submitted to Class. Quantum Grav.

Date: 24 March 2022

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1. Introduction

A key test of Regge Calculus which has been studied many times in literature is the Friedmann universe of dust.

The first papers on the Friedmann universe did not use simplicial complexes. The authors made use of non-simplicial blocks for which they had to give some other information besides the edge lengths. This information was deduced by symmetry. Collins and Williams (1973) used three tridimensional regular polytopes to describe the spatial sections with topology $S^3$ of the Friedmann universe of dust. Then Brewin (1987) wrote in the discrete formalism of Regge Calculus the expression of the action. Following Barrett et al (1997), for a regular 3-polytope having $n$ vertices, matter is placed on the timelike edges joining a vertex $i$ of the polytope with its evolute $i'$ of the next spatial section, finding:

$$S = \frac{1}{8\pi} \sum_b \varepsilon_b A_b - \frac{M}{n} \sum_i l_{ii'},$$  \hspace{1cm} (1)$$

where the first sum is over the bones $b$ of the manifold and $\varepsilon_b$ is the defect of the bone $b$ with area $A_b$. The second sum is done over every vertex $i$ of the first spatial section. Since a regular polytope having $n = 120$ has been chosen as spatial section, a mass $M/120$ is placed in each vertex $i$. Besides the regular polytopes, Brewin also used other non regular tridimensional polytopes. The evolution of all his polytopes, using Regge equations, stopped before the volume of spatial 3-section became zero: a point was reached where the equations had no solution. He saw that this point approaches the limit value (spatial section of null radius) as the polytope approaches the 3-sphere of the continuum solution. To overcome the obstacle, Brewin continued the evolution reverting to differential equations.

The first authors who used a simplicial complex for this metric were Barrett et al (1997). They evolved the Friedmann universe of dust using the Regge equations found by extremizing the action (1):

$$\sum_k \varepsilon_{ijk} \frac{\partial A_{ijk}}{\partial l_{ij}} = \frac{\pi}{15} M \delta_{ij},$$

where the sum is done over all vertices $k$ joined to the edge $[ij]$ of length $l_{ij}$. The bone $[ijk]$ has $\varepsilon_{ijk}$ as defect and $A_{ijk}$ as area. They made use of the evolutive scheme found by Sorkin (1975), but in our opinion their simplicial complex is not built correctly (see subsubsection 3.2.1). However they too found the point of stop and saw that its position is independent of the timelike interval between two consecutive spatial sections. The authors advanced the idea that the evolution should be continued by making spacelike the edges of the lattice which were timelike before the stop. However they did not pursue the idea.

In the present paper we propose to solve the problem of the stop of the evolution and to understand the meaning of the ending point. The paper is organized as follows.
In section 2 we embed the Robertson–Walker metric in a five-dimensional Lorentzian space. We will find it useful in order to understand the problems related to the evolution of our lattice. Then we describe our lattice and carry out the first evolution in order to create the initial simplicial sandwich (section 3). Section 4 contains the evolution of the initial sandwich and the results obtained. The description of a correct evolution algorithm for our simpicial complex based on Regge equations and the Sorkin evolutive scheme is also presented here. The nature of the singularities belonging to a general closed Robertson–Walker metric is discussed in section 5. In section 6 we propose a deeper reason why our simplicial approximation of the Friedmann universe ceases to exist before the volume of the spatial 3-section vanishes. We find that the stop condition is caused by a novel singularity of the metric. We fit the simplicial solution with a continuous metric and study its stress-energy tensor (section 7). Finally, our conclusions are reported in section 8.

2. Embedding into a 5-dimensional space

We shall find convenient to embed a space–time with closed Robertson–Walker metric into a 5-dimensional Lorentz space. This will be of help for understanding the behaviour of the Regge equations when applied to the evolution of a 600-cell space section.

The metric in consideration can be written in this way:

\[ dx^\alpha dx_\alpha = -dt^2 + R^2 \left[ d\chi^2 + \sin^2 \chi \left( d\vartheta^2 + \sin^2 \vartheta d\varphi^2 \right) \right], \]

(2)

where \( R \) depends only on the universal time \( t \).

The variety with metric (2) can be embedded into a 5-dimensional space endowed with the following metric:

\[ dx^\alpha dx_\alpha = -dv^2 + dw^2 + dx^2 + dy^2 + dz^2 \]

by putting

\[
\begin{align*}
  w &= R \cos \chi \\
  x &= R \sin \chi \sin \vartheta \cos \varphi \\
  y &= R \sin \chi \sin \vartheta \sin \varphi \\
  z &= R \sin \chi \cos \vartheta.
\end{align*}
\]

(3)

Equations (3) imply

\[
 dx^\alpha dx_\alpha = -dv^2 + dR^2 + R^2 \left[ d\chi^2 + \sin^2 \chi \left( d\vartheta^2 + \sin^2 \vartheta d\varphi^2 \right) \right].
\]

(4)

In order that metrics (2) and (4) may coincide it is necessary that

\[ dt^2 = dv^2 - dR^2. \]

(5)
If we impose that $R$ is a function only of the time variable $v$ then we have

$$dR = \frac{dR}{dv} dv,$$

so

$$dt^2 = dv^2 - \left(\frac{dR}{dv}\right)^2 dv^2 = (1 - \dot{R}^2) dv^2,$$

from which it follows that $\dot{R}^2 < 1$. In this case

$$dt = (1 - \dot{R}^2)^{1/2} dv.$$

(6)

So the metric becomes:

$$dx^\alpha dx_\alpha = - (1 - \dot{R}^2) dv^2 + R^2 \left[ d\chi^2 + \sin^2 \chi (d\vartheta^2 + \sin^2 \vartheta d\varphi^2) \right].$$

(7)

In the following we will refer to $v$ as “outer time.” We will be mainly interested in the case where $R(v)$ is an even function, so that $v = 0$ is an instant of time symmetry.

3. The 600-cell and the construction of the initial sandwich

3.1. The spatial section

We will approximate the hypersphere with the last tridimensional regular polytope, the 600-cell. It is a simplicial complex constituted by 120 vertices and 600 tetrahedra. The Friedmann metric has an instant of time symmetry where we can put a spatial section with a null extrinsic curvature. In Regge Calculus, for a spatial section whose extrinsic curvature vanishes, the following relation must hold:

$$\sum_j \varepsilon_{ij} l_{ij} = 16\pi \frac{M}{120},$$

where the sum is done over all vertices $j$ joined to vertex $i$ by an edge $[ij]$ of length $l_{ij}$ and defect $\varepsilon_{ij}$. There are 5 regular tetrahedra around a single edge, and the defect of each edge is:

$$\varepsilon = 2\pi - 5 \arccos\left(\frac{1}{3}\right).$$

Since every vertex is joined to 12 vertices which make an icosahedron, and the 600-cell is regular, calling $l_0$ its edge, we have

$$M = \frac{90}{\pi} \varepsilon l_0.$$

(8)

In order to have a criterion to compare the results of Regge Calculus with those of General Relativity, we choose that the 600-cell should approximate a 3-sphere having
the same volume. Thus the radius $R_0$ of the 3-sphere, in the instant of time symmetry, will obey the relation

$$2\pi^2 R_0^3 = 600 \frac{l_0^3}{6\sqrt{2}}$$

or

$$l_0 = \zeta R_0,$$

where

$$\zeta = \left(\frac{\pi^2 \sqrt{2}}{50}\right)^{1/3}.$$

3.2. Creation of the initial sandwich

We call “initial sandwich” the space–time between the initial spacelike section ($\Sigma_0$) and the next one ($\Sigma_1$).

3.2.1. The classes of vertices: $\alpha, \beta, \gamma, \delta, \varepsilon$. Tuckey (1993) and Barrett et al (1997) say the 120 vertices of a 600-cell can be subdivided into 4 classes. In each class there should be 30 vertices not joined to one another by an edge. So all vertices of a class could be evolved simultaneously.

But this cannot be true: it can be shown as follows. Let us consider a vertex $V$: we know it is surrounded by 12 vertices $W_i$ making an icosahedron. These vertices cannot be of the same class of $V$ because every vertex $W_i$ is joined to $V$ by an edge. But in an icosahedron it is not possible to choose more than 3 vertices not joined to one another, so that these 12 vertices must belong to at least 4 distinct classes.

To overcome this obstacle let us reason as follows. Let us consider $V$ and let it be a vertex of class $\alpha$. Let us try to subdivide the vertices $W_i$ into 4 groups of 3 vertices not joined to one another. We will name these groups with the letters $\beta, \gamma, \delta, \varepsilon$. This can be done by looking at figure 1.

Let us try to subdivide the vertices of the 600-cell into 5 classes of 24 vertices each. Every class must consist of vertices not joined together. Using the classification of the vertices of the 600-cell given by Coxeter (1973), we find that

$$\alpha = \{A_0, A_{10}, A_{20}, A_{30}, A_{40}, A_{50}, B_7, B_{17}, B_{27}, B_{37}, B_{47}, B_{57},$$

$$C_1, C_{11}, C_{21}, C_{31}, C_{41}, C_{51}, D_8, D_{18}, D_{28}, D_{38}, D_{48}, D_{58}\}.$$ 

The other classes are obtained by adding 2, 4, 6, 8 to the subscripts. It is also possible to show that the 24 vertices of each class make a 24-cell regular polytope inscribed in the 600-cell. Furthermore we can see that every vertex is actually surrounded by 12 vertices equally subdivided among the other classes. So an $\alpha$-vertex is joined to 3 $\beta$-vertices, 3 $\gamma$-vertices, 3 $\delta$-vertices and 3 $\varepsilon$-vertices. It should be noted
that the 3 vertices of the same class joined to a same vertex are not equivalent. For example the 3 \( \beta \)-vertices around the vertex \( \alpha \) in figure 1 have different neighbouring vertices. In fact they are surrounded by three sets (of 5 vertices each), which are not equal to one another. By the edge \([\alpha \beta_0]\) we will call the edge \([\alpha \beta]\) surrounded by the set of the following vertices \(\{\gamma, \gamma, \delta, \delta, \varepsilon\}\). The edge \([\alpha \beta_1]\) is surrounded by the set \(\{\gamma, \gamma, \delta, \varepsilon, \varepsilon\}\) and \([\alpha \beta_2]\) by \(\{\gamma, \delta, \delta, \varepsilon, \varepsilon\}\). As we shall see later, this inequality among the three edges \([\alpha \beta_i]\) \((i = 0, \ldots, 2)\) will give birth to three different Regge equations when an \(\alpha\)-vertex is going to be evolved.

Because of all these facts the evolution of a 600-cell consists of five steps: the evolution of each class of vertices \((\alpha, \beta, \gamma, \delta, \varepsilon)\).

### 3.2.2. The initial sandwich and the instant of time symmetry

Equation (8) has been found by choosing a spatial section having a null extrinsic curvature. In the continuum this means that the orthogonal timelike geodesics, which go out from this section, are locally parallel. In a lattice too, this characteristic should remain. So we can say that the spatial section, found by putting \(l_0\) equal to the value given by equation (8), cannot be a section of time symmetry. In fact if this were true, the edge lengths in the next section would be smaller: \(l' = l_1 < l_0\). Then the timelike edges which are the geodesics of the masses could not be parallel.

To understand better what happens in this situation let us embed the 600-cell into the 5-dimensional lorentzian space described in section 2. Since the 600-cell can be inscribed in a 3-sphere, we can look at its evolution as a sequence of 3-spheres each having a radius \(R(v)\), where \(v\) is the outer time of the 5-space. Then we put

\[
\alpha = (v, R \vec{u}_\alpha)
\]

where \(\vec{u}_\alpha\) is a unit spacelike 4-vector,

\[
\vec{u}_\alpha = (\cos \chi_\alpha, \sin \chi_\alpha \sin \vartheta_\alpha \cos \varphi_\alpha, \sin \chi_\alpha \sin \vartheta_\alpha \sin \varphi_\alpha, \sin \chi_\alpha \cos \vartheta_\alpha),
\]

pointing to the direction of \(\alpha\). There will be one such vector \(\vec{u}\) for each vertex of all 600-cell spatial sections. Let us call \(\Sigma_0\) the section corresponding to the instant of time symmetry \(v = 0\), and \(\Sigma_1\) the next one, at time \(v_1\), then

\[
\alpha = (0, R_0 \vec{u}_\alpha) \quad \beta = (0, R_0 \vec{u}_\beta) \\
\alpha' = (v_1, R_1 \vec{u}_\alpha) \quad \beta' = (v_1, R_1 \vec{u}_\beta).
\]

So we have that \([\alpha \alpha']\) is not orthogonal to \([\alpha \beta]\), unlike what happens in the continuum. In fact the straight lines

\[
\mathbf{x} = \alpha + p [\alpha \alpha'] \\
\mathbf{x} = \beta + q [\beta \beta']
\]

are never parallel and cross for \(p = q = R_0/(R_0 - R_1)\).
Nevertheless we can make an attempt in order that our lattice may have a behaviour similar to that of the continuum. Let us now put $\Sigma_0$ and $\Sigma_1$ symmetric about the instant of time symmetry $v = 0$, then $v_0 = -v_1$. Now we have

$$\alpha = (-v_1, R_1 \bar{u}_\alpha) \quad \beta = (-v_1, R_1 \bar{u}_\beta).$$

$$\alpha' = (v_1, R_1 \bar{u}_\alpha) \quad \beta' = (v_1, R_1 \bar{u}_\beta).$$

Thus we find that

$$[\alpha' \beta'] = [\alpha \beta],$$

$$[\alpha \alpha'] = [\beta \beta'] = (2v_1, 0, 0, 0, 0),$$

and $[\alpha \alpha']$ is orthogonal to $[\alpha \beta]$. Therefore we obtain

$$[\alpha' \beta] \cdot [\alpha' \beta'] = [\alpha \alpha'] \cdot [\alpha \alpha'] + [\alpha \beta] \cdot [\alpha \beta] = -4v_1^2 + l_0^2 \approx -\tau_0^2 + l_0^2,$$

where $\tau_0$ is the interval of proper time between $\Sigma_0$ and $\Sigma_1$. In fact we have also $\tau_0 \approx \Delta v$ because $\dot{R} = 0$ for $v = 0$.

Now all timelike edges (geodesics of the mass points) are orthogonal to the spatial section and parallel to one another, as we would expect if the extrinsic curvature vanished. Equation (8) does not give the edge of the time symmetric spatial section, but the edge of two consecutive 600-cells equal and symmetric about $v = 0$. All other spatial sections will also be pairwise symmetric about $v = 0$.

So we know the squared lengths of all edges of the initial sandwich and are able to build it at once without using any Regge equation.

### 3.2.3. The initial sandwich using Regge equations.

To test our assumptions we evolved the section $\Sigma_0$ making use of the Regge equations for the edges lying between $\Sigma_0$ and $\Sigma_1$, according to the Sorkin evolutive scheme. In order to allow an easier comparison between our results and those of Barrett et al (1997) we have chosen a lattice having

$$M = 10.202 \ldots \quad \text{(in arbitrary units)},$$

$$l_0 = 2.774 \ldots ,$$

$$\tau_0 = 0.0102.$$

In fact the authors considered the Friedmann universe having maximum radius $R_0$ and mass $M_\ast$ defined by

$$M_\ast \overset{\text{def}}{=} \int \varrho(t)\,dV = 2\pi^2 \varrho R_0^3 = \frac{3\pi}{4} R_0.$$

Choosing $M_\ast = 10$ they found

$$l_0 = \zeta R_0 = \frac{4\zeta}{3\pi} M_\ast = 2.774 \ldots$$

† We call “squared length” of an edge $[ij]$ the real quantity $s_{ij} = [ij] \cdot [ij]$, whereas its length is the non-negative number $l_{ij} = |s_{ij}|^{1/2}$.
In the end they found, using equation (8), the value 10.202... for $M$.

We have already remarked that the evolution scheme does not preserve the exact symmetry of the initial 600-cell. We expect however that, if the evolution is properly done, the symmetry is approximately respected. In order to test this, we examined those variables which are bound to remain equal if the symmetry holds and computed their averages and standard deviations. These variables are: the squared length $d^2$ of the oblique edge joining $\Sigma_0$ and $\Sigma_1$ and the squared length $l'^2$ of the edge of the new 600-cell forming the $\Sigma_1$ section.

We obtained

\[
\begin{align*}
l'^2 &= 7.693\,799\,901\,382\,97 \\
\sigma_{l'^2} &= 5.767 \cdot 10^{-15}
\end{align*}
\]

So, as we expected, the same value $l_0$ occurs on the first two consecutive surfaces. Furthermore we see that the relation \( d^2 = l'^2 - \tau_0^2 \) holds up to the 14th significant figure.

As we already remarked, the other surfaces, to be obtained later, are symmetric about these two. This last result has been found by Barrett et al (1997) too. The authors also found a second time-symmetric solution, in which the maximum spatial length is reached on a single hypersurface and the other hypersurfaces are symmetric in pairs around it. However this new maximum is slightly larger than that given by equation (8). These are all acceptable models and are indistinguishable, on the scale of time we used, from ours.

4. The evolution of the initial sandwich by Sorkin evolutive scheme

Given the squared lengths of all the edges lying between and on two spatial sections $\Sigma_{k-1}$ and $\Sigma_k$, in order to create a new simplicial sandwich, i.e. the space–time between $\Sigma_k$ and $\Sigma_{k+1}$, we will follow five steps.

1. To evolve the vertices of class $\alpha'$ of $\Sigma_k$. We put a timelike edge (of negative squared length $s_{\alpha''\alpha'} = -\tau_k^2$) joining $\alpha'$ to its evolute $\alpha''$ of $\Sigma_{k+1}$. We link $\alpha''$ to the 12 neighbouring vertices of $\alpha'$’s entourage in $\Sigma_k$ through spacelike edges (to ensure this, $|s_{\alpha''\alpha'}|$ should be small enough with respect to the other edges). There is one Regge equation for each “internal” edge. Then we find Regge equations around $[\alpha'\beta_i']$, $[\alpha'\gamma_i']$, $[\alpha'\delta_i']$, $[\alpha'\varepsilon_i']$ ($i = 0 ... 2$) and around $[\alpha''\alpha']$. We have as many equations as unknowns. The problem looks completely determined. Things are not so easy, however.

As widely shown in literature, on one side we must save the “lapse and shift” freedom; on the other side we expect Regge equations to be not independent
because of the contracted Bianchi identities. Therefore we must ignore 4 equations and fix 4 of the unknowns by proper lapse and shift conditions. As condition of lapse we will choose at our convenience
\[ s_{\alpha''\alpha'} = -\tau_k^2, \]
and as conditions of shift the three constraints
\[ s_{\alpha''\beta'_0} = s_{\alpha''\beta'_1}, \]
\[ s_{\alpha''\gamma'_0} = s_{\alpha''\beta'_2}, \]
\[ s_{\alpha''\gamma'_1} = s_{\alpha''\gamma'_2}. \]
We then cancel the 4 equations for \([\alpha''\alpha'], [\alpha'\beta'_1], [\alpha'\beta'_2], [\alpha'\gamma'_1]\).

As already remarked in subsubsection 3.2.1, equations over edges of the same class, e.g. \([\alpha'\delta'_0]\) are different. In fact it could be seen that \([\alpha'\delta'_0]\) is linked to 8 vertices (one \(\delta\), one \(\varepsilon\), two \(\beta'\), two \(\gamma'\), one \(\varepsilon'\) and one \(\alpha''\)) whereas \([\alpha'\delta'_1]\) is linked to 9 vertices (one \(\delta\), two \(\varepsilon\), two \(\beta'\), one \(\gamma'\), two \(\varepsilon'\) and one \(\alpha''\)).

There are still the following 9 unknowns:
\[ s_{\alpha''\beta'_0}, s_{\alpha''\gamma'_0}, s_{\alpha''\gamma'_2}, s_{\alpha''\gamma'_1}, s_{\alpha''\delta'_2}, s_{\alpha''\varepsilon'_0}. \]

Nevertheless, we can say that the 12 slanting edges belong to 4 different classes, corresponding to the class of the second vertex. It is natural to require, in order to save the symmetry, that edges belonging to the same class be of equal length.

So the unknowns reduce to 4:
\[ s_{\alpha''\beta'_1}, s_{\alpha''\gamma'_1}, s_{\alpha''\delta'_1}, s_{\alpha''\varepsilon'_1}, \]

after putting equal all connections \([\alpha''\delta']\), all connections \([\alpha''\varepsilon']\) and \(s_{\alpha''\gamma'_0} = s_{\alpha''\gamma'_2}\).

So the equations must reduce to four, for instance:
\[ [\alpha'\beta'_0], [\alpha'\delta'_0], [\alpha'\gamma'_0], [\alpha'\gamma'_0]. \]

In the following we shall verify that the other equations we have not used are satisfied anyhow.

2. To evolve the vertices of class \(\beta'\). We join \(\beta'\) to \(\beta''\) and link \(\beta''\) with the 3 vertices of class \(\alpha''\) of its entourage in \(\Sigma_{k+1}\) and with the 9 vertices (of classes \(\gamma', \delta'\) and \(\varepsilon'\)) of \(\beta'\)'s entourage in \(\Sigma_k\). By symmetry, we may reduce the new edge lengths to the following:
\[ s_{\beta''\gamma'}, s_{\beta''\delta'}, s_{\beta''\varepsilon'}, s_{\alpha''\beta''}, \]

since we put as lapse \(s_{\beta''\beta'} = -\tau_k^2\). The equations are
\[ [\beta'\gamma'], [\beta'\delta'], [\beta'\varepsilon'], [\alpha''\beta'], \]
neglecting the equation for $[\beta''\beta']$ because of Bianchi identities.

3. To evolve the vertices of class $\gamma'$. We join $\gamma'$ to $\gamma''$ and link $\gamma''$ with the 6 vertices (of classes $\alpha''$ and $\beta''$) of its entourage in $\Sigma_{k+1}$ and with the 6 vertices (of classes $\delta'$ and $\varepsilon'$) of $\gamma'$'s entourage in $\Sigma_k$. Now we have five kinds of internal edges: $[\gamma''\gamma']$, $[\alpha''\gamma']$, $[\beta''\gamma']$, $[\gamma''\delta']$ and $[\gamma'\varepsilon']$. These give rise to five distinct Regge equations. But because of the Bianchi identity only four of these can be used to find four lengths (neglecting the one for the timelike edge), whereas $s_{\gamma''\gamma'}$ must be freely given, like $s_{\alpha''\alpha'}$ and put equal to $-\tau_k^2$. Therefore we take as unknowns

$$s_{\gamma''\gamma'}, s_{\gamma''\varepsilon'}, s_{\alpha''\gamma'}, s_{\beta''\gamma'}.$$

4. To evolve the vertices of class $\delta'$. We join $\delta'$ to $\delta''$ and link $\delta''$ with the 9 vertices (of classes $\alpha''$, $\beta''$ and $\gamma''$) of its entourage in $\Sigma_{k+1}$ and with the 3 vertices of class $\varepsilon'$ of $\delta'$'s entourage in $\Sigma_k$. By symmetry, we impose that the new edge lengths are:

$$s_{\delta''\varepsilon'}, s_{\alpha''\delta''}, s_{\beta''\delta''}, s_{\gamma''\delta''}, s_{\delta''\delta'}.$$

We have only five distinct Regge equations, around $[\delta''\delta']$, $[\alpha''\delta']$, $[\beta''\delta']$, $[\gamma''\delta']$ and $[\delta'\varepsilon']$. One of these, that for the timelike edge, will not be utilized because of the Bianchi identity. So the remaining four can be used to find four lengths, whereas $s_{\delta''\delta'}$ is given, like $s_{\alpha''\alpha'}$ and equal to $-\tau_k^2$.

5. To evolve the vertices of class $\varepsilon'$. We join $\varepsilon'$ to $\varepsilon''$ and link $\varepsilon''$ with the 12 vertices (of classes $\alpha''$, $\beta''$, $\gamma''$ and $\delta''$) of its entourage in $\Sigma_{k+1}$. Now we have five kinds of internal edges: $[\varepsilon''\varepsilon']$, $[\alpha''\varepsilon']$, $[\beta''\varepsilon']$, $[\gamma''\varepsilon']$ and $[\delta''\varepsilon']$. These give rise to five distinct Regge equations. But because of the Bianchi identity only four of these can be used to find four lengths (neglecting that for the timelike edge), whereas $s_{\varepsilon''\varepsilon'}$ must be freely given, like $s_{\alpha''\alpha'}$ and put equal to $-\tau_k^2$. Therefore we take as unknowns

$$s_{\alpha''\varepsilon''}, s_{\beta''\varepsilon''}, s_{\delta''\varepsilon''}, s_{\gamma''\varepsilon''}.$$

It should be noted that every evolution consists in solving five systems of 4 equations in 4 unknowns each. Furthermore the simplicial sandwich does not have the same grade of symmetry as the initial spatial section, where all vertices were equivalent. The evolutive algorithm has broken the symmetry among vertices $\alpha$, $\beta$, $\gamma$, $\delta$, $\varepsilon$.

The evolution of $\Sigma_k$ is now finished: the new sandwich has been built up.

4.1. Results of the evolution

We evolved the initial 600-cell by imposing that the edges of the same type, like $[\alpha'\beta']$, were be all equal. In this way only four unknowns and four equations remained when
each vertex was evolved. Actually the different variables should be only two, since symmetry requires that all edges \([w'w']\) and \([w''w''']\) are equal as well. This was done by Barrett et al (1997). By allowing the edges joining vertices belonging to two different classes to be possibly different one from another, we introduce new degrees of freedom in the lattice. We have verified that this enhancement of freedom causes no problem in the evolution.

4.1.1. Behaviour of \(R_k\) as a function of the universal time \(t\). Using the evolutive algorithm we evaluated, for each section, the edge (or better the mean edge) of the 600-cell, \(l_k\). Then we calculated the quantity

\[
R_k = \frac{l_k}{\zeta},
\]

where \(R_k\) represents the radius of the 3-sphere equivalent to the 600-cell of the \(k\)-th spatial section \((\Sigma_k)\) generated by the \(k\)-th evolution. We have that \(l_k = l_k(t)\), where

\[
t = \sum_{i=0}^{k-1} \tau_i,
\]

\(\tau_i\) is the interval of proper time between \(\Sigma_i\) and \(\Sigma_{i+1}\), and \(\tau_0 = 0.0102\). As better explained in the appendix, we did not choose equal proper time intervals. Instead we took \(\tau_i = \rho l_i\), with \(\rho \approx 3.68 \cdot 10^{-3}\).

The results of this evolution can be seen in figure 2. Figure 2 shows that the evolution stops at a finite value of the radius \((R_m \approx 1.2)\) which is approximately \(\frac{1}{4}\) of the initial value \((R_0 = 4.244\ldots)\). For \(R < R_m\) Regge equations give no solution.

As it can be seen in the appendix our evolution steps were done with a constant interval of conformal time: equation (A1) says that \(\Delta \eta \approx 2.4 \cdot 10^{-3}\). Actually the critical point occurs after \(k_m \approx 750\) evolutions, or \(\eta_m \approx 1.8\). But, since \(\eta\) should reach \(\pi\), we should have done a number

\[
k_M = \frac{\pi}{\Delta \eta} \approx 1300
\]

of evolutions. This behaviour of our “platonic” polytope was also found by Barrett et al (1997) and Brewin (1987) who also studied not-simplicial complexes that, compared to the 600-cell, better approximated the 3-sphere. For these polyhedra the graph of \(R(t)\) is nearer to the cycloidal one of the Friedmann metric and the point of stop is placed in \(\eta > \eta_m\).

4.1.2. Behaviour of the two \(\sigma\). During the evolution, the two \(\sigma\) \((\sigma_{d},\sigma_{d'})\) remain below \(10^{-13}\). In proximity of the critical point these variables grow up quickly. This means that the lattice is twisting more and more.
4.1.3. Bianchi identity. We calculated, for each evolution, the residual \( f_{\alpha''\alpha'} \) of the Regge equation for the timelike edge \([\alpha''\alpha']\),

\[
f_{\alpha''\alpha'} = \sum_k \varepsilon_{\alpha''\alpha'k} \frac{\partial A_{\alpha''\alpha'k}}{\partial l_{\alpha''\alpha'}} - \frac{\pi}{15} M.
\]

This equation had not been utilised to build up the spatial section \( \Sigma_{k+1} \) because of the simplicial version of the contracted Bianchi identity, i.e. the energy conservation in the form of the Kirchhoff-like law. The smaller \( f_{\alpha''\alpha'} \), the better this identity is respected. It should be noted that the Kirchhoff-like law always exactly holds because the current of matter along the timelike edges remains the same after each evolution. Instead, Regge equations represent the Einstein tensor only approximately (Barrett (1986)).

Our results can be seen in figure 3: we see that \( f_{\alpha''\alpha'} \) is going to “explode” in proximity of the critical point.

The other equations, which are not utilized during the evolution, are all over slanting and horizontal edges. We found that they are satisfied with greater accuracy, since their residuals are all less than \(2 \cdot 10^{-15}\).

5. Analysis of the critical point

What is the nature of the critical point? Brewin (1987) gives a partial answer by noting that the following relation occurs:

\[
\frac{\Delta l}{\tau} \approx 1,
\]

i.e. the collapse becomes so fast that the vertical edges are to become spacelike. Unfortunately in our model the above fraction reaches a not easily interpretable value (approximately 2.6). So what is the true meaning of this condition, and how can we continue the iterations (if possible)?

To answer these questions, it is useful to study more deeply the characteristics of the homogeneous and isotropic metric with topology (and geometry) \(S^3\). The 5-dimensional embedding described in section 2 will be of use here.

5.1. Singularities

In order to study the singularities of metric (7) we can compute the quadratic Riemann invariant:

\[
Q = R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} = \frac{12}{R^2(1 - \dot{R}^2)^2} \left[ \frac{\ddot{R}^2}{(1 - \dot{R}^2)^2} + \frac{1}{\dot{R}^2} \right].
\] (9)

In fact a necessary and sufficient condition for a singularity is that \( Q \) becomes infinite. A glance to equation (9) shows that the singular points are those values of \( v \) where \(|\dot{R}| = 1\) or \(R = 0\) or \(|\ddot{R}| = +\infty\). Let us discuss the possible cases.
1. **The points** $R = 0$. These coincide with the infinite contraction of the universe when the volume of the spatial section goes to 0. In these points, as will be shown later, density reaches infinite values. By a volume-vanishing (VV) singularity we will mean one of these points.

2. **The points** $\dot{R} = \pm 1$. If we are not in the first case we can say that the radius of the space section increases or decreases with speed tending to that of light. If one tries to extend the manifold beyond these points, and if $|\dot{R}|$ increases further, the metric becomes positively defined. So it can no longer represent a manifold of General Relativity. This is tantamount to saying that the spacelike sections are not causally connected: the points of two different spatial sections can be joined only by spacelike geodesics. So these points will be said to be causality-breaking (CB) singularities. Actually, in a rigorous sense a CB singularity is a border of the manifold, and the extension is meaningless.

3. **The points** $\ddot{R} = \pm \infty$. If we are not in the previous two cases the singular behaviour is shown by the geodesics deviation, which becomes infinite. We will refer to these points as geodesics-deviation (GD) singularities.

These singularities can also occur together in the same point: an example is the hypothetical universe whose radius has the form $R(v) = v - A v^{3/2}$ (the critical point $v = 0$ is a VV-CB-GD singularity).

5.2. **Study of the stress-energy tensor**

From Einstein equations for the closed Robertson–Walker metric we have for the diagonal components of $T_{\alpha\beta}$ in an orthonormal basis:

$$
\frac{8\pi}{3} \rho = \frac{1}{R^2} \left( \frac{dR}{dt} \right)^2 + \frac{1}{R^2} \\
-4\pi p = \frac{1}{R} \frac{d^2R}{dt^2} + \frac{1}{2R^2} \left( \frac{dR}{dt} \right)^2 + \frac{1}{2R^2}.
$$

By expressing the time derivatives in terms of the outer time we find

$$
\rho = \frac{3}{8\pi R^2 (1 - \dot{R}^2)} \\
p = -\frac{1}{8\pi} \frac{2R\ddot{R} + 1 - \dot{R}^2}{R^2 (1 - \dot{R}^2)^2}.
$$

(10)

5.2.1. **Range of validity of the stress-energy tensor.** It is well known that the stress-energy tensor must satisfy

$$\rho \geq |p|.$$
In our case
\[-2 \leq \frac{R \ddot{R}}{1 - \dot{R}^2} \leq 1.\] (11)

In general it can happen that condition (11) be not satisfied for values of \(v\) for which the metric still has a “geometrical sense.” In fact a CB or GD or CB-GD singularity does not satisfy equation (11). Instead a VV point presents no problem. It should be noted that we cannot become aware of these new “physical” matter-limiting (ML) points, wherein \(\rho = |p|\), by looking only at the expression of the metric.

5.3. Friedmann metric
Since \(dt = R \, d\eta\), from equation (5) we obtain
\[v = \int [R^2 - (dR/d\eta)^2]^{1/2} \, d\eta.\]

For the Friedmann metric we have:
\[v = 2R_0 \sin \frac{\eta}{2}\]
and
\[R = R_0 \cos^2 \frac{\eta}{2} = R_0 - \frac{v^2}{4R_0}.\]

Metric (7) takes the following form:
\[dx^\alpha \, dx_\alpha = \frac{R}{R_0} \, dv^2 + R^2 \left[ d\chi^2 + \sin^2 \chi \left( d\vartheta^2 + \sin^2 \vartheta \, d\varphi^2 \right) \right].\]

Calling \(v_c\) the value of \(v\) where \(\dot{R} = -1\) and \(v_\Omega\) the positive value of \(v\) where \(R = 0\), for the Friedmann metric these two singularities occur at the same time, i.e. \(v_\Omega = v_c = 2R_0\). So, in the 5-space, in correspondence of the big crunch the speed of contraction equals the speed of light, i.e. the big crunch is a VV-CB point. This behaviour is shared by all matter satisfying \(p = w \rho\), where \(w\) is a constant, \(-\frac{1}{3} < w \leq 1\), since then \(\dot{R}^2 = 1 - A^2 R^{1+3w}\), \(A\) a constant. Seeing that \(\dot{R} = -\frac{1}{2} A^2 (1 + 3w) R^{3w}\), in Zel’dovich’s interval, \(0 \leq w \leq 1\), no GD singularity is present beside the other two.

6. The problem of the 600-cell
Let us now reconsider equation (5). For each evolutive step we know \(\Delta t = \tau\) and \(\Delta R = \Delta l/\zeta\). Then we can find \(\Delta v\) by calculating \(\Delta v \approx (\Delta t^2 + \Delta R^2)^{1/2}\). So we determined \(R\) as a function of \(v\) (figure 4).

We have also made a picture of \(\Delta R/\Delta v \approx dR/dv\) as a function of \(R\) in figure 5.

We can see that this derivative in the last iterations tends, in absolute value, to 1. In this sense the 600-cell meets a CB singularity before its volume vanishes. So we expect that for the 600-cell \(v_c < v_\Omega\). This leads us to believe that there is no point in trying to continue the evolution beyond this point, perhaps by making all edges spacelike as suggested by Barrett et al (1997).
7. Study of a “new” class of solutions of Einstein equations

We have just seen that the behaviour of the evolution for the 600-cell looks somewhat different from the Friedmann solution, in its reaching a CB singularity before the vanishing of $R$. So it appears expedient to study a generalisation of Friedmann metric, also having that property. For the 600-cell an empirical fit of $R(v)$ shows that a good fit is given by

$$R(v) = R_0 - \frac{a^2 v^2}{4R_0}, \quad \text{with} \quad a^2 \approx 1.128,$$

whereas $a = 1$ is the Friedmann metric.

It is useful to re-define

$$v_c \stackrel{\text{def}}{=} \frac{2R_0}{a^2},$$
$$v_\Omega \stackrel{\text{def}}{=} \frac{2R_0}{a} = a v_c,$$

so

$$R = \frac{1}{2v_c} (v_\Omega^2 - v^2)$$
$$\dot{R} = -\frac{v}{v_c}$$
$$\ddot{R} = -\frac{1}{v_c}.$$

- For $0 < a < 1$ we have that $-v_\Omega < v < v_c$ and the singularity is of VV-type.
- For $a > 1$ we have that $-v_c < v < v_\Omega$ and the singularity is of CB-type.
- For $a = 1$ the solution is the Friedmann metric. So this case is a watershed between two different behaviours of the general metric.

It is not difficult to find the expression of $t$ as a function of $v$:

$$t = \frac{1}{2} v_c \arcsin(v/v_c) + \frac{1}{2} v \left[1 - (v/v_c)^2\right]^{1/2}$$

and to show that the graph of $R(t)$ is still a cycloid, like for Friedmann metric, but scaled in $R$ by a factor $1/a^2$ and translated upwards by $\frac{1}{2} v_c (a^2 - 1)$.

7.1. Behaviour of the metric

Condition (11) implies that the following inequality must be satisfied:

$$v^2 \leq \tilde{v}^2 \stackrel{\text{def}}{=} v_c^2 - \frac{1}{3} v_c^2 |a^2 - 1|.$$
If \( a < 1 \) this condition does not carry any problem because \( \tilde{v}^2 > v_\Omega^2 \). If \( a > 1 \) the ML point \( \tilde{v} \) is the first one the metric meets in its evolution. In fact

\[
\tilde{v}^2 = v_c^2 - \frac{1}{3} v_c^2 (a^2 - 1) < v_c^2.
\]

Note that in order to have \( \tilde{v}^2 \geq 0 \) the value of \( a^2 \) must not exceed 4. So the conditions for the stress-energy tensor reduce the interval of definition of \( a \) to \( 0 < a \leq 2 \). If \( a < 1 \), the metric ends in a VV singularity reached the more softly the smaller the value of \( a \). If \( a > 1 \), the metric reaches a ML point before the CB one. It should be also pointed out that the Friedmann metric \((a = 1)\) is more “pathological” because its VV singularity is also a CB point, since \( v_\Omega = v_c \).

For this metric it is also possible in principle to write down an equation of state of matter, but of course we do not attach a physical meaning to this result.

8. Conclusions

We have seen that the evolution of the 600-cell does not describe the Friedmann universe well. Instead we can think of it as the evolution of a different type of matter. But where does such a difference come from? In our opinion, one should consider the following two points as playing an important role in answering this question.

First, the 600-cell is a good approximation to a 3-sphere, but it is still too “rough.” For every approximating method, Regge Calculus included, the smaller the step interval the better the fit. When this interval reduces, the two solutions (numerical and analytical) tend to coincide. In Regge Calculus, in order to get a better approximation, one should increase the number of tetrahedra of each space section, i.e. the 600-cell should be substituted by another non-regular polytope nearer to a 3-sphere (Brewin (1987)).

The second aspect is deeper since it is closely related to the grounds of the Regge Calculus itself. We have seen that Regge equations give rise to a different evolution of the universe, ending in a CB point. This happens because they are only an approximation of Einstein equations. The point has been discussed by Brewin and Gentle. It can be said that Regge equations are a good approximation to differential equations rather different from Einstein’s. It is wholly possible that such equations have solutions which behave in a qualitatively different way from those we are looking for: for instance, the approximate solution may exhibit singularities not belonging to the correct solution. Nevertheless, it remains true that the smaller the truncation error, the closer the behaviour of the numerically approximated equation to that of the exact solution.

Thus an idea comes to mind: could we modify Regge equations in order to have a smaller truncation error? It has been shown (see Brewin and Gentle, Gentle and Miller (1998)) that the solutions of Regge Calculus are, in general, expected to be second order accurate approximations to the corresponding continuum solutions, i.e. the truncation error for lattices obtained by Regge Calculus is \( O(l^3) \). Thus a research
program could be: to find a modification of Regge equations giving a truncation error $O(t^4)$ or better.

**Appendix. Numerical topics**

We did not use a fixed $\tau_k$ for all evolution steps. Instead we imposed a constant ratio between $\tau_k$ and $l_k$. This in order to ensure that the Courant criterion continues to be respected also when $l_k$ becomes very small. We remember that the 600-cell should approximate a Friedmann universe having the same volume. So the radius $R_k$ of the $k$-th 3-sphere is linked to the edge $l_k$ of the 600-cell of the spatial section $\Sigma_k$ of the $k$-th evolutive step by the relation $l_k = \zeta R_k$ ($\zeta$ defined in subsection 3.1). Then

$$\frac{\tau_k}{l_k} = \rho \overset{\text{def}}{=} \frac{\tau_0}{l_0} \approx 3.68 \cdot 10^{-3}.$$  

This implies that

$$\Delta \eta \approx \frac{\Delta t}{R} = \frac{\tau_k}{R_k} = \zeta \rho \approx 2.4 \cdot 10^{-3}. \quad (A1)$$

To solve Regge equations, a system of non linear equations, we use the Newton–Raphson method. To begin the iterations we have to give values to the unknowns in order to start the iterations. We call them “trial values” and they are marked by a tilde.

To give a trial value to the variables $(k)s_{\alpha''\nu'}$ (the squared lengths of the oblique edges that join $\alpha''$ to the vertices $\nu' \neq \alpha'$ of $\Sigma_k$) we can use the following argument. Keeping the notations of subsubsection 3.2.2 on, we can write

$$^{(k)}s_{\alpha''\nu'} = [\alpha'\alpha''] \cdot [\alpha'\alpha''] + [\alpha'\beta'] + [\alpha'\beta'] - 2 [\alpha'\alpha''] \cdot [\alpha'\beta'] =$$

$$= -\tau_k^2 + l_k^2 - 2 [\alpha'\alpha''] \cdot [\alpha'\beta'],$$

where

$$[\alpha'\alpha''] \cdot [\alpha'\beta'] = (\Delta v_k, (R_{k+1} - R_k) \vec{u}_\alpha) \cdot (0, R_k (\vec{u}_\beta - \vec{u}_\alpha)) =$$

$$= R_k (R_{k+1} - R_k) (\vec{u}_\alpha \cdot \vec{u}_\beta - 1) \approx -\frac{dR}{dt} (t_k) \tau_k R_k \xi,$$

being $\tau_k$ the interval of universal time between the sections $\Sigma_k$ and $\Sigma_{k+1}$, $R_k$ the radius of the 3-sphere circumscribed to the 600-cell and

$$\xi \overset{\text{def}}{=} 1 - \vec{u}_\alpha \cdot \vec{u}_\beta = \frac{3 - \sqrt{5}}{4},$$

found by calculating the angle between two vectors pointing to any two neighbouring vertices of the 600-cell (Coxeter (1973)). Furthermore we can see that $l_k = \phi R_k$, where $\phi$ is the golden ratio, $\frac{1}{2}(\sqrt{5} - 1)$. So we are able to write

$$^{(k)}s_{\alpha''\nu'} = -\tau_k^2 + l_k^2 - 2 \frac{\xi}{\phi} \tau_k l_k (l_0/l_k - 1)^{1/2}. $$
Once the vertices $\alpha'$ have been evolved we put the quantity $(k)s_{\alpha''\beta'}$ as trial value for all other oblique edges, $(k)\tilde{s}_{\beta''\beta'}$, $(k)\tilde{s}_{\gamma''\beta'}$, $(k)\tilde{s}_{\delta''\beta'}$. At the moment of the evolution of the vertex $\beta'$ we have to give another trial value to the unknown $(k)s_{\alpha''\beta'}$.

In the Friedmann metric we would have

$$R(\eta) = R_0 \cos^2(\eta/2)$$

so, since $l = \zeta R$, expanding to the second order in $\tau$, we can calculate an approximate value for the edge of the 600-cell of the section $\Sigma_{k+1}$. In fact we have

$$\tilde{l}_{k+1} = l_k - \tau_k \zeta \left( l_0/l_k - 1 \right)^{1/2} - \frac{1}{4} \zeta^2 l_0 \left( \tau_k/l_k \right)^2.$$ 

After the evolution of $\beta'$-vertices the quantity $(k)s_{\alpha''\beta'}$ is determined and we use it to give the trial values for all variables representing the edge of the 600-cell of the section $\Sigma_{k+1}$, i.e. $(k)\tilde{s}_{\alpha''\beta''}$, $(k)\tilde{s}_{\gamma''\beta''}$, $(k)\tilde{s}_{\delta''\beta''}$, $(k)\tilde{s}_{\epsilon''\beta''}$.

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Figure 1. Vertices around an $\alpha$-vertex

Figure 2. Evolution of the radius $R$ as a function of the universal time $t$ according to Regge Calculus and General Relativity
Figure 3. Residual $f_{\alpha'\alpha''}$ vs. $t$

Figure 4. Behaviour of $R$ as a function of $v$. 
Figure 5. Behaviour of $|\Delta R/\Delta v| \approx |\dot{R}|$ as a function of $v$ for the 600-cell