NORMAL FORMS FOR TENSOR RANK DECOMPOSITION

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Abstract. We propose a new algorithm for computing the tensor rank decomposition or canonical polyadic decomposition of higher-order tensors subject to a rank and genericity constraint. Reformulating this as a system of polynomial equations allows us to leverage recent numerical linear algebra tools from computational algebraic geometry. We describe the complexity of our algorithm in terms of the multigraded regularity of a multihomogeneous ideal. We prove effective bounds for many formats and ranks and conjecture a general formula. These bounds are of independent interest for overconstrained polynomial system solving. Our experiments show that our algorithm can outperform state-of-the-art algebraic algorithms by an order of magnitude in terms of accuracy, computation time, and memory consumption.

AMS subject classifications — 15A69, 65H10, 65H14

Keywords — Tensor rank decomposition, canonical polyadic decomposition, polynomial systems, normal form algorithms

1. Introduction

We introduce an original direct numerical algorithm for tensor rank decomposition or canonical polyadic decomposition (CPD) in the low-rank regime. By “direct” we mean an algorithm that does not rely on numerical optimization or other iteratively refined approximations with a data-dependent number of iterations.

Consider the vector space $\mathbb{C}^{(n_1+1) \times \cdots \times (n_d+1)}$ whose elements represent order-$d$ tensors in coordinates relative to some basis. We say that a tensor in such a space is of rank 1 or elementary if it is of the form

$$\alpha^1 \otimes \alpha^2 \otimes \cdots \otimes \alpha^d := (\alpha^k_{j_1} \alpha^2_{j_2} \cdots \alpha^d_{j_d})_{0 \leq j_k \leq n_k}, \text{ where } \alpha^k = (\alpha^k_{j_k})_{0 \leq j_k \leq n_k} \in \mathbb{C}^{n_k+1}.$$ 

Every tensor $A$ can be expressed as a linear combination of rank-1 tensors:

$$(\text{CPD}) \quad A = \sum_{i=1}^{r} \alpha^1_i \otimes \cdots \otimes \alpha^d_i, \quad \text{with } \alpha^k_i = (\alpha^k_{i,j_k})_{0 \leq j_k \leq n_k} \in \mathbb{C}^{n_k+1}.$$ 

If $r$ is minimal among all such expressions of $A$, then $r$ is called the rank of the tensor [26] and eq. (CPD) is called a CPD. The problem of computing a CPD of $A$, i.e., determining a set of rank-1 tensors summing to $A$, has many applications in science and engineering, see for instance [27,37].

The strategy we propose for computing (CPD) relies on the fact that the problem is equivalent to solving a certain system of polynomial equations. Under suitable assumptions, these equations can be obtained from the nullspace of a flattening $A(1) \in \mathbb{C}^{(n_1+1) \times \prod_{k=2}^{d}(n_k+1)}$ of the tensor $A$, as in [29]. Once we have obtained these polynomial equations, whose solutions correspond to the rank-1 terms in (CPD), we use recent numerical normal form algorithms.
techniques from \cite{2,38,39} to solve them. The following example, which is used as a running example throughout the paper, illustrates how this works.

**Example 1** (Motivating example). Consider the $4 \times 3 \times 3$ tensor $A$ with flattening

$$A_{(1)} = \begin{bmatrix}
\partial_{00} & \partial_{01} & \partial_{02} & \partial_{10} & \partial_{11} & \partial_{12} & \partial_{20} & \partial_{21} & \partial_{22} \\
1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 2 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 & 1 & 2 & 1 & 2 \\
1 & 1 & 1 & 1 & 1 & 2 & 1 & 2 & 1 & 2
\end{bmatrix}. $$

The column of this matrix indexed by $\partial_{kl}$ contains the entries $A_{jkl}$, $j = 0, \ldots, 3$. The reason for this indexing will become clear in section 2. The kernel of $A_{(1)}$ is the transpose of

$$ (1.1) \quad R_I(1,1)^\top = \begin{bmatrix}
x_0y_0 & x_0y_1 & x_0y_2 & x_1y_0 & x_1y_1 & x_1y_2 & x_2y_0 & x_2y_1 & x_2y_2 \\
f_1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\
f_2 & 0 & 0 & -1 & -1 & 0 & 1 & 0 & 0 \\
f_3 & -2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
f_4 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\
f_5 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}. $$

The column of $R_I(1,1)^\top$ corresponding to column $\partial_{kl}$ of $A_{(1)}$ is now indexed by $x_ky_l$: we interpret the rows as polynomials

$$f_1 = -x_1y_0 + x_1y_1, \quad f_2 = -x_0y_2 - x_1y_0 + x_1y_2, \quad f_3 = -2x_0y_0 + x_2y_0, \quad f_4 = -x_0y_1 + x_2y_1, \quad f_5 = -2x_0y_2 + x_2y_2. $$

These are bilinear forms in $S = \mathbb{C}[x_0, x_1, x_2, y_0, y_1, y_2]$. As explained in section 2, the common zeros of $f_1, \ldots, f_5$ form a subvariety of $\mathbb{P}^2 \times \mathbb{P}^2$ consisting of the four points

$$\zeta_1 = ((1 : 0 : 2), (1 : 0 : 0)), \quad \zeta_2 = ((1 : 0 : 1), (0 : 1 : 0)), \quad \zeta_3 = ((1 : 1 : 2), (0 : 0 : 1)), \quad \zeta_4 = ((0 : 1 : 0), (1 : 1 : 1)). $$

These points correspond to the last two factors of the rank-1 terms in

$$\begin{bmatrix}
1 \\
1 \\
1 \\
\zeta_1
\end{bmatrix} \otimes \begin{bmatrix}
1 \\
0 \\
0 \\
\zeta_2
\end{bmatrix} + \begin{bmatrix}
0 \\
1 \\
1 \\
\zeta_3
\end{bmatrix} \otimes \begin{bmatrix}
1 \\
0 \\
1 \\
\zeta_4
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
1 \\
\zeta_1
\end{bmatrix} \otimes \begin{bmatrix}
0 \\
0 \\
1 \\
\zeta_2
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
1 \\
\zeta_3
\end{bmatrix} \otimes \begin{bmatrix}
0 \\
0 \\
1 \\
\zeta_4
\end{bmatrix};$$

this is the decomposition (CPD) of $A$. △

**Contributions.** We formulate our results for the case $d = 3$, as the general case can be handled using a standard reshaping trick. Our main contribution is a new linear algebra based algorithm for computing the rank-$r$ CPD (CPD) for a third order tensor $A$, under the assumption that $A$ is $r$-identifiable and a flattening of $A$ has rank $r$, see assumption 1. We call this algorithm \texttt{cpd.hnf}. It is based on a well-known reformulation of the problem as a system of polynomial equations, which we solve using state-of-the-art methods from computational algebraic geometry. This results in algorithm 2. We show that this algorithm generalizes so-called pencil-based algorithms \cite{1,17,31,32,35} for very low ranks (theorems 3.1 and 3.2).

We give a new, explicit description of the complexity of the tensor rank decomposition problem in terms of the multigraded regularity of a non-saturated ideal in the homogeneous coordinate ring of $\mathbb{P}^m \times \mathbb{P}^n$ (proposition 3.1). We characterize the regularity in terms of the rank of a structured matrix obtained directly from the rank-one terms in (CPD), see example 6. These new insights allow us to formulate a conjecture regarding the regularity (conjecture 1). We prove this conjecture for many formats, see theorem 4.2. These results are
of independent interest for the field of polynomial system solving. They have the following consequence related to the complexity of our algorithm.

**Proposition 1.1.** Consider the tensor space $\mathbb{C}^{\ell+1} \otimes \mathbb{C}^{m+1} \otimes \mathbb{C}^{n+1}$ of dimension $M = (\ell + 1)(m + 1)(n + 1)$ with $\ell \geq m \geq n$. If conjecture 1 holds and $A$ is a generic tensor of rank $r \leq \phi mn$ with $\phi \in [0,1)$ a fixed constant, then $\text{cpd}_n \text{hnf}$ runs in polynomial time $O(M^{5/2} \lceil 1^2 - \phi \rceil + 1)$.

Our experiments in section 6 show that the proposed algorithm is highly efficient and gives accurate results (fig. 1). For instance, we can compute the decomposition of a $7 \times 7 \times 7 \times 7 \times 6 \times 6 \times 5 \times 5$ tensor of rank 1000 in double-precision arithmetic with an accuracy of order $10^{-15}$ in 441 seconds. Moreover, $\text{cpd}_n \text{hnf}$ also behaves well in the presence of noise.

**Related work.** The idea of computing tensor decompositions via polynomial root finding is central in *apolarity*-based approaches such as [3,4,6]. The *Hankel operators* play the role of normal forms in this context. These operators can be obtained partially from the tensor $A$. In most cases, a homotopy continuation procedure is needed in order to complete the Hankel operators. For instance, this is step (2) in [6, alg. 5.1]. Although our method works only under certain assumptions on the rank of $A$, in contrast to apolarity-based methods it requires *only linear algebra computations*, and it operates in polynomial rings with fewer variables. Moreover, we eliminate the choice of a so called *connected-to-one basis* [4, sec. 4.3 and 7.2].

In [29], an affine system of equations is obtained as in example 1, but it is solved using homotopy continuation methods. However, this approach is infeasible for formats that are handled without any problems by our algorithm. For instance, for the aforementioned eighth-order tensor of rank 1000, the method of [29] would need to track over $40 \cdot 10^9$ homotopy paths. We argue that the eigenvalue methods proposed in this paper are more natural to use in an overdetermined setting.

The state-of-the-art algorithms for tensor rank decomposition using only linear algebra computations were proposed in [20,21]. Although these methods work under slightly milder conditions, [21, Algorithm 1] often requires the construction of larger matrices than those in our algorithm. There is no explicit connection with polynomial equations. The algorithm and its complexity depend on a parameter $l$, which is chosen incrementally by trial and error for each format. In an analogous way, the complexity of our algorithm is governed by the choice of a parameter. However, the analysis in section 4 tells us *a priori* which parameter value should be used, circumventing a trial-and-error approach. In section 6 we demonstrate that $\text{cpd}_h \text{hnf}$ improves on [20,21] in terms of computational complexity and accuracy.

**Outline.** In section 2 we state our assumptions and show how a system of polynomial equations is obtained from the tensor $A$. In section 3 we make the connection with normal-form methods explicit. That is, we describe how a *pre-normal form* can be computed directly from the tensor $A$ and how this allows us to reduce to an eigenvalue problem. We explain how the approach generalizes so-called *pencil-based algorithms* for very small ranks in section 3.3. This leads to a complete description of our algorithm $\text{cpd}_h \text{hnf}$ in section 3.4. In section 4, we study the regularity of the ideal associated to our polynomial system. The results are the starting point for our analysis of the complexity of $\text{cpd}_h \text{hnf}$ in section 5. In section 6, we demonstrate the efficiency and accuracy of $\text{cpd}_h \text{hnf}$ through several numerical examples. The paper is completed in section 7 with some final conclusions.

## 2. From tensor decomposition to polynomial equations

### 2.1. Identifiability and flattenings.

The algorithm we propose for efficiently solving the tensor rank decomposition problem for a low-rank tensor $A$ is subject to two main restrictions:

1. $A$ is *r*-identifiable, and
(2) a (standard) flattening of the tensor $A$ is of the same rank as $A$.

The first restriction is usually very weak and only of a technical nature. Recall that the set of all rank-1 tensors forms an algebraic variety, i.e., the solution set of a system of polynomial equations, called the Segre variety $S$. A rank-$r$ CPD of a tensor $A$ is a set of $r$ rank-1 tensors whose sum is $A$. The set of all such rank-$r$ CPDs is denoted by $S^{(r)} = \{ X \subset \mathbb{C}^d | |X| = r \}$. Tensor rank decomposition consists of computing an element of the fiber of

$$f : S^{(r)} \to \mathbb{C}^{n_1+1} \times \cdots \times \mathbb{C}^{n_d+1}, \quad \{ A_1, \ldots, A_r \} \mapsto A_1 + \cdots + A_r$$

at a rank-$r$ tensor $A$. For brevity, we denote the image of $f$ by $S_r = f(S^{(r)})$. The tensor $A$ is called $r$-identifiable if the fiber $f^{-1}(A)$ contains exactly one element. That is, there is a unique set in $S^{(r)}$ whose sum is $A$. Generally, $r$-identifiability fails only on a strict closed subvariety of (the Zariski closure of) $S_r$ [5, 11−13]. This property is called the generic $r$-identifiability of $S$. These results entail that $r$-identifiability fails only on a subset of $S_r$ of Lebesgue measure zero if the rank $r$ and dimensions $(n_1 + 1, \ldots, n_d + 1)$ satisfy some very weak conditions; see [12, Section 1] and [13, Section 3] for more details. If $S$ is generically $r$-identifiable, then there is a Zariski-open subset of the closure of $S_r$, so that $f^{-1}$ is a holomorphic, bijective tensor decomposition function. We aspire to solve the tensor decomposition problem only in this well-behaved setting.

The second restriction is more limited, but allows us to tackle the tensor decomposition problem using only efficient linear algebra. Recall that the standard $k$-flattening of $A \in \mathbb{C}^{n_1+1} \times \cdots \times \mathbb{C}^{n_d+1}$ consists of interpreting $A$ as the matrix $A(k) \in \mathbb{C}^{(n_k+1) \times \prod_{k=2}^{d} (n_k+1)}$. For a rank-1 tensor $A = \alpha^1 \otimes \cdots \otimes \alpha^d$ this identification is defined by

$$A(k) = \alpha^k (\alpha^1 \otimes \cdots \otimes \alpha^{k-1} \otimes \alpha^{k+1} \otimes \cdots \otimes \alpha^d)^	op,$$

and the general case follows by linearity. The tensor product\footnote{The tensor product is defined uniquely by the linear space into which it maps by universality [23], so we do not make a distinction in notation.} in the foregoing expression is also called the reverse-order Kronecker product

$$\otimes : \mathbb{C}^{n_1+1} \times \cdots \times \mathbb{C}^{n_d+1} \to \mathbb{C}^{\prod_{k=1}^{d} (n_k+1)}, \quad (\alpha^1, \ldots, \alpha^d) \mapsto [\alpha^1_{i_1} \cdots \alpha^d_{i_d}]_{(i_1, \ldots, i_d)},$$

where the standard bases were assumed for these Euclidean spaces. Note that the 1-flattening is easy to compute when $A$ is given in coordinates relative to the standard tensor product basis. In that case it suffices to reshape the coordinate array to an $(n_1 + 1) \times \prod_{k=2}^{d} (n_k + 1)$ array (e.g., as in Matlab’s reshape function).

For dealing with higher-order tensors with $d > 3$, we rely on the well-known reshaping trick. It consists of interpreting a higher-order tensor as a third-order tensor. The approach described in the remainder of the paper can be applied directly to higher-order tensors as well, but the range of ranks $r$ in which this version operates is more restrictive than the range obtained from reshaping. Consequently, we will henceforth describe our approach only for $d = 3$.

The cpd_hnf algorithm works under the following assumption.

**Assumption 1.** The tensor $A$ is $r$-identifiable and its flattening $A_{(1)}$ has rank $r$.

Note that specific algorithms exist for verifying that a tensor is $r$-identifiable [13, 19, 21, 28, 36]. However, all of them are a posteriori certifications of $r$-identifiability that require a rank-$r$ decomposition as input. In practice, this entails that only the assumption about the rank of the flattening can be verified beforehand. The non-identifiability of the tensor $A$ will be detected as part of the proposed algorithm, hence testing it is not a prerequisite for applying cpd_hnf.

Fortunately, the following standard result by Bocci, Chiantini, and Ottaviani [5, Proposition 8.1] ensures that both restrictions are generically satisfied if the rank is small. Hence, it describes the practical working range of cpd_hnf.
Lemma 2.1. Consider the space \( \mathbb{C}^{\ell+1} \otimes \mathbb{C}^{m+1} \otimes \mathbb{C}^{n+1} \) with \( \ell \geq m \geq n > 0 \). Then, for all \( r \leq \min \{ \ell + 1, mn \} \), every tensor \( A \) outside of a strict closed subvariety of (the Zariski-closure of) \( S_r \) is both \( r \)-identifiable and its flattening \( A_{(1)} \) has rank \( r \).

2.2. Polynomial systems defined by flattenings. Having delineated the range Equation (R) in which the proposed cpd_hnf algorithm will work, we continue by describing it. Our strategy to compute the tensor rank decomposition of

\[
A = \sum_{i=1}^{r} \alpha_i \otimes \beta_i \otimes \gamma_i \in \mathbb{C}^{\ell+1} \otimes \mathbb{C}^{m+1} \otimes \mathbb{C}^{n+1}
\]

is to compute the points

\[
\zeta_i = (\beta_i, \gamma_i) \in \mathbb{C}^{n+1} \times \mathbb{C}^{n+1}, \quad i = 1, \ldots, r,
\]

first as the solution of a system of (low-degree) polynomial equations. Thereafter we compute the \( \alpha_i \)'s with efficient linear algebra by plugging \( \beta_i \) and \( \gamma_i \) into (A) and solving the resulting linear system of equations. Indeed, since

\[
A_{(1)} = \begin{bmatrix} \alpha_1 & \cdots & \alpha_r \end{bmatrix} \begin{bmatrix} \beta_1 \otimes \gamma_1 & \cdots & \beta_r \otimes \gamma_r \end{bmatrix}^T
\]

and lemma 2.1 guarantees that the second \( r \times (m+1)(n+1) \) matrix is of full rank \( r \) (so it has a right inverse), applying the latter to \( A_{(1)} \) yields the corresponding \( \alpha_i \)'s. Note that it suffices to compute the points \( \beta_i \) and \( \gamma_i \) up to a nonzero scaling factor. Therefore, it is natural to consider our problem in complex projective space.

Recall that the \( k \)-dimensional complex projective space \( \mathbb{P}^k \) is the space of equivalence classes \( \{ x \mid \lambda x \in \mathbb{C} \setminus \{0\} \} \) for \( x \in \mathbb{C}^{k+1} \setminus \{0\} \). The entries of the vector \( x = (x_0, \ldots, x_k) \in \mathbb{C}^{k+1} \setminus \{0\} \) are called homogeneous coordinates of \( x \). With a slight abuse of notation we will write \( x = (x_0 : \ldots : x_k) \in \mathbb{P}^k \) for both the equivalence class \( [x] \) and a set of homogeneous coordinates \( x \in \mathbb{C}^{k+1} \setminus \{0\} \).

The proposed cpd_hnf algorithm exploits the fact that the points \( \zeta = (\beta_i, \gamma_i) \in \mathbb{P}^m \times \mathbb{P}^n \) are defined by algebraic relations on \( X = \mathbb{P}^m \times \mathbb{P}^n \) that can be computed directly from the tensor \( A \). Such algebraic relations are homogeneous polynomials in a graded polynomial ring.

In this context, it is natural to think of degrees as 2-tuples \( (d, e) \in \mathbb{N}^2 \), where \( d \) is the degree in the variables corresponding to \( \mathbb{P}^m \) and \( e \) is the degree in the variables corresponding to \( \mathbb{P}^n \). The flexibility provided by this bi-graded setting will help us reduce the complexity of our algorithm. Concretely, we work in the \( \mathbb{N}^2 \)-graded polynomial ring

\[
S = \mathbb{C}[x_0, \ldots, x_m, y_0, \ldots, y_n] = \bigoplus_{(d,e)\in\mathbb{N}^2} S_{(d,e)}, \quad \text{where} \quad S_{(d,e)} = \bigoplus_{|a|=d,|b|=e} \mathbb{C} \cdot x^a y^b.
\]

Here we have used the shorthand notation \( x^a = x_0^{a_0} \cdots x_m^{a_m} \) and \( |a| = a_0 + \cdots + a_m \) for \( a = (a_0, \ldots, a_m) \in \mathbb{N}^{m+1} \), and analogously for \( b \). Note that the graded pieces \( S_{(d,e)} \) are \( \mathbb{C} \)-vector spaces. The variables \( x \) correspond to homogeneous coordinates on the factor \( \mathbb{P}^m \) and in \( S_{(d,e)} \), \( d \) is “the degree in the \( x \)-variables.” Analogously, elements of \( S_{(d,e)} \) have degree \( e \) in the \( y \)-variables, which are related to \( \mathbb{P}^n \).

An element \( f \in S \) is called homogeneous if \( f \in S_{(d,e)} \) for some \( (d,e) \in \mathbb{N}^2 \). The key reason that the ring \( S \), with its grading (2.2), is naturally associated to \( X \) is that homogeneous elements \( f \in S_{(d,e)} \) have well-defined zero sets on \( X \). By this we mean that for \( f \in S_{(d,e)} \) and for any \( \zeta = (x, y) \in X \), \( f(x, y) = 0 \) is independent of the choice of homogeneous coordinates. Indeed, this follows from

\[
f(\lambda x, \mu y) = \lambda^d \mu^e f(x, y) \quad \text{for} \quad \lambda, \mu \in \mathbb{C} \setminus \{0\}.
\]

Therefore, whenever \( f \) is homogeneous it makes sense to write \( f(\zeta) = 0 \) if \( f \) vanishes on some set of homogeneous coordinates for \( \zeta \), and to define the subvariety of \( X \) defined by \( f \) as \( V_X(f) = \{ \zeta \in X \mid f(\zeta) = 0 \} \). For a homogeneous ideal \( I \subset S \) (i.e., \( I \) is generated by...
homogeneous elements with respect to the grading \((2.2)\), we denote the subvariety of \(X\) corresponding to \(I\) by

\[ V_X(I) = \{ \zeta \in X \mid f(\zeta) = 0, \text{ for all homogeneous } f \in I \}. \]

That is \(V_X(I)\) contains the solutions of the polynomial system defined by the simultaneous vanishing of all homogeneous equations \(f \in I\).

**Example 2.** The polynomials \(f_1, \ldots, f_5\) in example 1 are homogeneous elements of degree \((1, 1)\) in \(S\), i.e. \(f_i \in S_{(1,1)}\). They generate the homogeneous ideal \(I\), whose corresponding subvariety is \(V_X(I) = \{ \zeta_1, \ldots, \zeta_4 \} \subset X = \mathbb{P}^2 \times \mathbb{P}^2\). \(\triangle\)

With this notation in place, we turn back to the tensor \(A\) and show that, under suitable assumptions, \(V_X(I) = \{ \zeta_1, \ldots, \zeta_r \}\) for some homogeneous ideal \(I\) defined by \(A\). This will generalize the procedure in example 1. Consider the flattening \(A_{(1)}\) from eq. \((2.1)\). It also has a natural interpretation as a \(\mathbb{C}\)-linear map

\[ A_{(1)} : S_{(1,1)} \rightarrow \mathbb{C}^{\ell+1}. \]

Indeed, we can identify the space \(\mathbb{C}^{m+1} \otimes \mathbb{C}^{n+1}\) with the graded piece \(S_{(1,1)}\) of degree \((1, 1)\) by sending

\[(2.4) e_k \otimes e_l \mapsto x_k y_l, \quad 0 \leq k \leq m, \ 0 \leq l \leq n,\]

where \(e_k\) represents the \((k+1)\)-st standard basis vector of \(\mathbb{C}^{m+1}\), and analogously for \(e_l\) and \(\mathbb{C}^{n+1}\). For brevity, we also write \(A_{(1)}\) for a matrix representation of the map \(A_{(1)}\), where the standard basis of monomials \((2.4)\) is used for \(S_{(1,1)}\). The \(\ell + 1\) rows of \(A_{(1)}\) are elements of \((S_{(1,1)})^\vee\) represented in its dual basis

\[(2.5) \left\{ \frac{\partial^2}{\partial x_k \partial y_l}, \ 0 \leq k \leq m, \ 0 \leq l \leq n \right\} . \]

From this vantage point, the rows of the second factor in eq. \((2.1)\) are

\[ (\beta_i \otimes \gamma_i)^\top = \sum_{0<k\leq m, 0\leq l\leq n} \beta_{i,k} \gamma_{i,l} \partial_{kl} \in S_{(1,1)}^\vee. \]

It is clear that for any \(f \in S_{(1,1)}\), we have \((v \otimes w)^\top (f) = f(v,w)\).

Let \(f_1, \ldots, f_s \in S_{(1,1)}\) be a \(\mathbb{C}\)-basis for the kernel \(\ker A_{(1)}\). The \(f_i\) generate a homogeneous ideal \(I = \langle \ker A_{(1)} \rangle = \langle f_1, \ldots, f_s \rangle \subset S\). The following lemma states that under assumption 1, the ideal \(I\) is a nice algebraic representation of the point set \(\{ \zeta_1, \ldots, \zeta_r \} \subset X\).

**Lemma 2.2.** Assumption 1 entails that the ideal \(I = \langle \ker A_{(1)} \rangle \subset S\) is such that \(V_X(I)\) consists of the points \(\{ \zeta_1, \ldots, \zeta_r \}\), and these points have multiplicity one.

**Proof.** By construction, the variety \(V_X(I)\) contains at least the \(\zeta_i\). Suppose that \(V_X(I)\) contains an additional point \(\zeta = (\beta, \gamma) \notin \{ \zeta_1, \ldots, \zeta_r \}\). Since the rank of \(A_{(1)}\) is \(r\) and, hence, \(\zeta_1, \ldots, \zeta_r\) (under the Segre embedding) forms a basis of the row space of \(A_{(1)}\), \(\beta \otimes \gamma\) must depend linearly on the \(\beta_i \otimes \gamma_i\). But then one of these, say \(\beta_1 \otimes \gamma_1\), can be expressed as \(\beta_1 \otimes \gamma_1 = a_0 \beta \otimes \gamma + \sum_{i=2}^r a_i \beta_i \otimes \gamma_i\) for some \(a, a_i \in \mathbb{C}\). It follows from the multilinearity of the tensor product that

\[ A = a_0 \alpha_1 \otimes \beta \otimes \gamma + \sum_{i=2}^r (a_i \alpha_1 + \alpha_i) \otimes \beta_i \otimes \gamma_i \]

is an alternative decomposition of \(A\), contradicting its \(r\)-identifiability.

So far, we showed \(V_X(I) = \{ \zeta_1, \ldots, \zeta_r \}\). The rest of the proof, regarding the multiplicities of these points, is more technical and is deferred to appendix A. \(\square\)
The fact that the points of $V_X(I) = \{\zeta_1, \ldots, \zeta_r\}$ occur with multiplicity one is a technicality we will need in section 3 in order to invoke the results from [39, Section 5.5]. For the reader who is familiar with algebraic geometry, we included a consequence of lemma 2.2 relating $I$ to the vanishing ideal of $V_X(I)$ in corollary A.1.

2.3. The high-level algorithm. We conclude the section by presenting a high-level pseudo-algorithm for computing the tensor rank decomposition $(A)$ of $A$. This is presented as algorithm 1. Its steps summarize the above discussion.

**Algorithm 1** Compute the tensor rank decomposition of $A \in \mathbb{C}^{\ell+1} \otimes \mathbb{C}^{m+1} \otimes \mathbb{C}^{n+1}$

**Require:** $A$ satisfies assumption 1.

1. $A(1) \leftarrow$ the $(\ell + 1) \times (m + 1)(n + 1)$ 1-flattening of $A$
2. $\{f_1, \ldots, f_s\} \leftarrow$ a $\mathbb{C}$-basis for $\ker A(1)$
3. $I \leftarrow$ the ideal $\langle f_1, \ldots, f_s \rangle \subset S$
4. $\{(\beta_i, \gamma_i)\}_{i=1,\ldots,r} \leftarrow$ homogeneous coordinates for $\{\zeta_i\}_{i=1,\ldots,r} = V_X(I) \subset X$
5. $(\alpha_1, \ldots, \alpha_r) \leftarrow$ solve the linear system defined by eq. (2.1)
6. return $\{(\alpha_i, \beta_i, \gamma_i)\}_{i=1,\ldots,r}$

The main focus of this paper is dealing with step 4 of algorithm 1. We will employ a state-of-the-art method for solving polynomial equations on $X$, based on homogeneous normal forms. This strategy is described in the next section.

3. From polynomial equations to eigenvalues

In this section, we employ tools from computational algebraic geometry for computing points in a product of projective spaces in step 4 of algorithm 1. The next subsection summarizes the relevant results in the present setting. We show in section 3.1 that the solutions can be obtained from an eigenvalue problem defined by preprocessing normal forms. How to compute the latter is explained in section 3.2. Thereafter, in section 3.3, we demonstrate that so-called pencil-based algorithms for decomposing tensors of very low rank are closely related to our **cpd_hnf** algorithm. The full algorithm for performing step 4 of algorithm 1 is presented in section 3.4. Finally, we conclude this section with some examples.

3.1. The eigenvalue theorem. Our algorithm is built on a multi-homogeneous version of the eigenvalue theorem (theorem 3.1 below), which allows us to find solutions of systems of polynomial equations via eigenvalue computations. Behind this is the theory of homogeneous normal forms. In our context, these are $\mathbb{N}^2$-graded versions of truncated normal forms, introduced in [40], and special cases of the more general toric homogeneous normal forms used in [2, 38] and formally introduced in [39]. For our purpose, it suffices to work with slightly simpler objects, called preprocessing normal forms, so we use homogeneous normal forms only implicitly. For full proofs and more details we refer to [39, Section 5.5.4].

Consider the space $X = \mathbb{P}^n \times \mathbb{P}^n$ and its associated ring $S$, which is the $\mathbb{N}^2$-graded polynomial ring from (2.2). Let $I \subset S$ be a homogeneous ideal. The ideal $I$ inherits the grading on $S$:

$$I = \bigoplus_{(d,e) \in \mathbb{N}^2} I_{(d,e)}, \quad \text{where} \quad I_{(d,e)} = I \cap S_{(d,e)}$$

is a $\mathbb{C}$-subspace of $S_{(d,e)}$, and so does the quotient ring

$$S/I = \bigoplus_{(d,e) \in \mathbb{N}^2} (S/I)_{(d,e)} = \bigoplus_{(d,e) \in \mathbb{N}^2} S_{(d,e)}/I_{(d,e)}$$

(these are quotients of vector spaces). Our objective is to compute a set of homogeneous coordinates of the points in $V_X(I)$ by using linear algebra computations. In order to do this,
it is necessary to work in graded pieces of \(S\), \(I\), and \(S/I\). Let \(M\) denote either of the latter. The \textit{(multi-graded) Hilbert function} \(HF_M : \mathbb{N}^2 \to \mathbb{N}\) is given by
\[
HF_M(d,e) = \dim_{\mathbb{C}} M_{(d,e)}
\]
and keeps track of the dimension of the vector space \(M\). Note that for a homogeneous ideal \(I \subset S\) and \((d,e) \in \mathbb{N}^2\), we have \(HF_{S/I}(d,e) = HF_S(d,e) - HF_I(d,e)\).

**Example 3.** The Hilbert function of the ring \(S\) is given explicitly by
\[
HF_S(d,e) = \dim_{\mathbb{C}} S_{(d,e)} = \left(\frac{m+d}{d}\right) \left(\frac{n+e}{e}\right) \quad \text{with} \quad (d,e) \in \mathbb{N}^2. \quad \triangle
\]

Before stating the main result of this section, theorem 3.1, we present three auxiliary lemmas. Lemmas 3.1 and 3.2 are well-known in commutative algebra. For lack of a precise reference and completeness, we included short proofs in appendix A.

**Lemma 3.1.** Let \(I \subset S\) be a homogeneous ideal such that \(V_X(I)\) consists of \(r < \infty\) points \(\{ζ_1, \ldots, ζ_r\}\). For each \((d', e') \in \mathbb{N}^2\), there exists a homogeneous polynomial \(h_0 \in S_{(d',e')}\) such that \(V_X(h_0) \cap V_X(I) = \emptyset\). Equivalently, we can find \(h_0 \in S_{(d',e')}\) such that \(h_0(ζ_i) ≠ 0, \ i = 1, \ldots, r\).

**Lemma 3.2.** Let \(I \subset S\) be a homogeneous ideal such that \(V_X(I)\) consists of \(r < \infty\) points, each with multiplicity one. There exists \((d,e) \in \mathbb{N}^2\) with \((d,e) ≠ (1,1)\) and \((d−1,e−1) ≥ (0,0)\) (entry-wise) such that \(HF_{S/I}(d,e) = r\).

Hereafter, we will use the following notation.
- Let \(I = \langle \ker A_{(1)} \rangle = \langle f_1, \ldots, f_s \rangle\) as in section 2.2, where \(f_1, \ldots, f_s\) \(∈ S_{(1,1)}\) form a basis for \(\ker A_{(1)}\). We assume that \(A\) satisfies assumption 1, so that \(V_X(I)\) consists of the \(r\) points \(ζ_1, \ldots, ζ_r\) with multiplicity one by lemma 2.2.
- The tuple \((d,e) \in \mathbb{N}^2\) is such that \((d,e) ≥ (1,1)\) and \(HF_{S/I}(d,e) = r\). Note that this is satisfied for \((d,e) = (1,1)\) by construction of \(I\).
- We write \((d',e') = (d−1,e−1)\), and \(h_0 \in S_{(d',e')}\) is such that \(h_0(ζ_i) ≠ 0, i = 1, \ldots, r\). This makes sense by lemma 3.1.

**Lemma 3.3.** Let \(I, (d,e)\) and \(h_0\) be as above. If a \(\mathbb{C}\)-linear map \(N : S_{(d,e)} \to \mathbb{C}^r\) is such that
\[
(3.1) \quad \text{rank}(N) = r \quad \text{and} \quad \ker N = I_{(d,e)},
\]
then the induced linear map \(N_{h_0} : S_{(1,1)} \to \mathbb{C}^r, f ↦ N(h_0 f)\) has rank \(r\).

**Proof.** Note that \(HF_{S/I}(1,1) = r\) by construction. By [39, Proposition 5.5.7], the fact that \(HF_{S/I}(d,e) = r\) implies that \((1,1)\) and \((d',e')\) form a so-called \textit{regularity pair}. Surjectivity of the map \(N_{h_0}\) then follows directly from [39, Lemma 5.5.3]. \(\square\)

Linear maps \(N : S_{(d,e)} \to \mathbb{C}^r\) satisfying (3.1) will play an important role in translating our polynomial root finding problem into an eigenvalue problem. We will therefore give them a name.

**Definition 3.1 (Pre-normal forms).** A \(\mathbb{C}\)-linear map \(N : S_{(d,e)} \to \mathbb{C}^r\) satisfying (3.1) is called a \textit{pre-normal form} on \(S_{(d,e)}\).

Note that if \(r = ℓ + 1\), assumption 1 implies that \(A_{(1)} : S_{(1,1)} \to \mathbb{C}^r\) is a pre-normal form on \(S_{(1,1)}\).

Normal forms are closely related to a classical result in computational algebraic geometry called the \textit{eigenvalue theorem} [14, §2.4], which is used for computing isolated solutions to polynomial systems. The pre-normal forms introduced above will be useful for formulating a multi-homogeneous version of this theorem, namely theorem 3.1 below. Before stating it, we need to fix some additional notation.

\(^{2}\)Often we will require \((d,e) ≠ (1,1)\) and \((d−1,e−1) ≥ (0,0)\). This makes sense by lemma 3.2. The problem of how to find such a tuple \((d,e)\) will be the topic of section 4.
In lemma 3.3, \( N_{h_0} \) is the composition of \( N \) with the linear map that represents multiplication by \( h_0 \in S_{(d,e)'} \). More generally, for \( g \in S_{(d',e')} \) we define \( N_g : S_{(1,1)} \to S_{(d,e)} \) as \( N_g(f) = N(gf) \). In theorem 3.1, we will restrict the map \( N_{h_0} \) to an \( r \)-dimensional subspace \( B \subset S_{(1,1)} \) such that the resulting map is invertible. We denote this restriction by \( (N_{h_0})_B \). In practice, it suffices to select \( r \) columns of the matrix of \( N_{h_0} \) so that the resulting \( r \times r \) submatrix \( (N_{h_0})_B \) is invertible.

In what follows, we will write \( w_i^\top = (\beta_i \otimes \gamma_i)^\top_B \in B^\vee \) for the linear functional representing evaluation at \( \zeta_i = (\beta_i, \gamma_i) \). This is the restriction of the functional \( (\beta_i \otimes \gamma_i)^\top \in S^\vee_{(1,1)} \) to the vector space \( B \). Concretely, we set

\[
(3.2) \quad w_i^\top (b) = (\beta_i \otimes \gamma_i)^\top_B (b) = b(\beta_i, \gamma_i), \quad \text{with} \ b(x, y) \in B \subset S_{(1,1)}.
\]

After fixing a basis for \( B^\vee \), \( w_i^\top \) can be represented as a row vector.

If \( g, h \in S_{(d',e')} \setminus \{0\} \) are two homogeneous polynomials of the same degree, then the fraction \( g/h \) is a well-defined function on \( X \setminus V_X(h) \). Indeed, the evaluation of this function does not depend on the choice of homogeneous coordinates. Therefore, we may write \( (g/h)(\zeta) \) for the evaluation of this rational function at \( \zeta \in X \setminus V_X(h) \).

**Theorem 3.1** (Eigenvalue theorem). Let \( I, (d, e) \) and \( h_0 \) be as above and let \( N \) be a pre-normal form. Let \( B \subset S_{(1,1)} \) be any \( r \)-dimensional subspace such that the restriction \( (N_{h_0})_B : B \to \mathbb{C}^r \) is invertible. For any \( g \in S_{(d',e')} \) we have

\[
(3.3) \quad w_i^\top \circ M_{g/h_0} = g_{h_0}(\zeta_i) \cdot w_i^\top, \quad i = 1, \ldots, r,
\]

where \( M_{g/h_0} : B \to B \) is the composition \( (N_{h_0})^{-1}_B \circ (N_g)_B \).

**Proof.** This follows from [39, Theorem 5.5.3, Propositions 5.5.4 and 5.5.5] and the fact that \((1,1)\) and \((d',e')\) form a regularity pair (see the proof of Lemma 3.3). \( \square \)

After fixing a basis for \( B \) and representing \( w_i \) in the dual basis for \( B^\vee \), (3.3) is a standard matrix eigenproblem: \( w_i^\top M_{g/h_0} = \lambda_i w_i^\top \). That is, \((\lambda_i, w_i)\) is a left eigenpair of the \( r \times r \) matrix \( M_{g/h_0} \). Note that theorem 3.1 implies that all maps of the form \( M_{g/h_0} \) share a set of eigenvectors.

We now sketch one way of using theorem 3.1 to retrieve the coordinates of \( \zeta_i = (\beta_i, \gamma_i) \), assuming a pre-normal form \( N : S_{(d,e)} \to \mathbb{C}^r \) is given. The problem of computing a pre-normal form is addressed in the next subsection. We assume \( d \geq 2 \).\(^3\) Let \( h \in S_{(d'-1,e')} \) and \( h_0 \in S_{(d',e')} \) be homogeneous polynomials that do not vanish at any of the points \( \zeta_i \). These can be chosen generically. Set \( g_j = x_j h \in S_{(d',e')}, j = 0, \ldots, m \). Choose \( B \subset S_{(1,1)} \) of dimension \( r \) such that \( (N_{h_0})_B \) is invertible and compute the matrices \( M_j = M_{g_j/h_0} = (N_{h_0})_B^{-1} \circ (N_g)_B \).

By theorem 3.1, the eigenvalues of \( M_j \) are given by \( \lambda_{ji} = (g_j/h_0)(\zeta_i) \). Writing \( \beta_{ij} \) for the \( j \)-th coordinate of \( \beta_i \), we have

\[
(\lambda_0 : \cdots : \lambda_m) = \left( \frac{\beta_0 h(\zeta_i)}{h_0(\zeta_i)} : \cdots : \frac{\beta_m h(\zeta_i)}{h_0(\zeta_i)} \right) = (\beta_{0i} : \cdots : \beta_{mi}) = (\beta_i).
\]

Note that if \((d,e) = (2,1)\), we can take \( h = 1 \).

Subsequently, we compute \( \gamma_i \) by solving the linear system of equations

\[
(3.4) \quad f_1(\beta_i, y) = \cdots = f_s(\beta_i, y) = 0.
\]

The foregoing approach requires that \((d,e) \neq (1,1)\). Otherwise \( S_{(d',e')} = S_{(0,0)} = \mathbb{C} \), and we can only evaluate constant functions using theorem 3.1.

\(^3\)If \( d = 1 \) and \( e \geq 2 \), the roles of \( d \) and \( e \) can be swapped so that the approach still works.
3.2. Computing pre-normal forms. As illustrated in the previous subsection, once we have computed a pre-normal form, we are only some linear algebra computations removed from recovering the points \( \zeta_i \). A natural next issue to address is how to compute a pre-normal form.

Our starting point is a basis \( f_1, \ldots, f_s \in S_{(1,1)} \) of \( \ker A_{(1)} \), generating our ideal \( I = \langle \ker A_{(1)} \rangle = \langle f_1, \ldots, f_s \rangle \). For any tuple \((d,e) \in \mathbb{N}^2\) such that \((d,e) \geq (1,1)\), the degree \((d,e)\)-part \( I_{(d,e)} \) of \( I \) is the \( \mathbb{C} \)-vector space spanned by

\[
\{ x^{a'} y^b f_i \mid (|a'|, |b|) = (d', e'), \ i = 1, \ldots, s \} \subset S_{(d,e)},
\]

where \((d', e') = (d-1, e-1)\). If \( d = 0 \) or \( e = 0 \), we have \( I_{(d,e)} = \{ 0 \} \). In analogy with (1.1), we construct a matrix whose rows are indexed by the monomial basis elements of \( S_{(d,e)} \) (i.e., the monomials \( \{ x^{a'} y^b \mid |a| = d, |b| = e \} \)), and whose columns are the polynomials (3.5) expanded in this basis. We denote this matrix by

\[
R_I (d,e) \in \mathbb{C}^{\text{HF}_{S}(d,e)} \times s \text{HF}_{S}(d',e').
\]

Such matrices represent graded resultant maps in the terminology of [39, Section 5.5.4]. They are multihomogeneous versions of the classical Macaulay matrices [33]. We present an explicit example below in example 4.

Observe that the Hilbert function \( \text{HF}_I (d,e) \) is given by the rank of \( R_I (d,e) \) and \( \text{HF}_{S/I}(d,e) \) by its corank. This follows immediately from the observation that the columns of \( R_I (d,e) \) span \( I_{(d,e)} \). A left nullspace matrix \( N \) of \( R_I (d,e) \) represents a map \( S_{(d,e)} \to S_{(d,e)}/I_{(d,e)} \cong \mathbb{C}^{\text{HF}_{S/I}(d,e)} \). This has the following consequence.

**Proposition 3.1.** If \((d,e) \in \mathbb{N}^2\) is such that \( \text{HF}_{S/I}(d,e) = r \), then any left nullspace matrix \( N \) of \( R_I (d,e) \) represents a pre-normal form.

We conclude that a pre-normal form \( N \) can be computed, for instance, from a full singular value decomposition (SVD) of \( R_I (d,e) \), where \( \text{HF}_{S/I}(d,e) = r \). This solves the problem of computing a pre-normal form, assuming that we know a degree \((d,e) \in \mathbb{N}^2\) for which \( \text{HF}_{S/I}(d,e) = r \). The problem of finding such degrees is addressed in section 4.

3.3. Relation to pencil-based algorithms. In order to obtain the homogeneous coordinates for \( \zeta_1, \ldots, \zeta_r \) as eigenvalues of the matrices \( M_{g/h_0} \), we usually have to work with pre-normal forms on \( S_{(d,e)} \), where \((d,e) \neq (1,1)\) and \((d',e') \geq (0,0)\). An exception is the case where \( r \leq m+1 \leq \ell + 1 \). For these tensors of very low rank, a pre-normal form \( N : S_{(1,1)} \to \mathbb{C}^r \) will suffice under the mild condition that

\[
[\beta_1 \cdots \beta_r] \in \mathbb{C}^{(m+1) \times r} \text{ has rank } r.
\]

The underlying reason is that vanishing at \( \{ \zeta_1, \ldots, \zeta_r \} \) gives \( r \) linearly independent conditions on \( S_{(1,0)} \). The proof of the following theorem is another consequence of the theory of homogeneous normal forms and is deferred to appendix A.

**Theorem 3.2** (Eigenvalue theorem for low ranks). Let \( I = \langle \ker A_{(1)} \rangle \) where \( A \) has rank \( r \leq m+1 \leq n+1 \) and satisfies both assumption 1 and (3.7). Let \( h_0 \in S_{(0,1)} \) be such that \( h_0 (\zeta_i) \neq 0 \) for \( i = 1, \ldots, r \) and let \( N : S_{(1,1)} \to \mathbb{C}^r \) be a pre-normal form on \( S_{(1,1)} \). For \( g \in S_{(0,1)} \) we define

\[
\tilde{N}_g : S_{(1,0)} \to \mathbb{C}^r \text{ given by } \tilde{N}_g (f) = N (gf).
\]

We have that \( \tilde{N}_{h_0} \) has rank \( r \). For any \( r \)-dimensional subspace \( B \subset S_{(1,0)} \) such that the restriction \( (\tilde{N}_{h_0})_B : B \to \mathbb{C}^r \) is invertible, the eigenvalues of \( M_{y_j/h_0} = (\tilde{N}_{h_0})_B^{-1} \circ (\tilde{N}_{y_j})_B \) are \( \{ \gamma_{ij} / h_0 (\zeta_i) \}_{i=1,\ldots, r} \).
This theorem can be exploited to compute \( \zeta_i = (\beta_i, \gamma_i) \) efficiently as follows. If \( A \) satisfies assumption 1 and (3.7), we take a basis of the \( r \)-dimensional row span of \( A(1) \) in theorem 3.2 as our pre-normal form. In practice, this can be obtained from an SVD of \( A(1) \). Once the \( \gamma_i \) are computed from the eigenvalues of the \( M_{gj}/h_0 \), the \( \beta_i \) can be obtained as in (3.4). Alternatively, one can use the shared eigenvectors of these commuting matrices \( M_{gj}/h_0 \) for \( j = 1, \ldots, r \) [39, Theorem 5.5.3].

Theorem 3.2 is intimately related to so-called pencil-based algorithms [1], such as [31, 32, 35], for solving eq. (CPD) when the rank satisfies \( r \leq m + 1 \leq \ell + 1 \). Recall that pencil-based algorithms assume that \( A \in \mathbb{C}^{r \times r \times (n + 1)} \) is a rank-\( r \) tensor.\(^4\) In addition, they assume that the \( \alpha_i \) form a linearly independent set, and likewise for the \( \beta_i \)'s. Then, we have that the tensor contraction of \( A \) with \( h_0^\top \in (\mathbb{C}^{n+1})^\vee \), i.e.,

\[
\tag{3.8}
h_0^\top \cdot 3 A = \sum_{i=1}^{r} (\alpha_i^\top \otimes \beta_i^\top) \cdot (h_0^\top \gamma_i^\top) = ADh_0B^\top,
\]

is an invertible \( r \times r \) matrix insofar as \( h_0^\top \gamma_i^\top \neq 0 \). Herein, \( A \in \mathbb{C}^{r \times r} \) (respectively \( B \in \mathbb{C}^{r \times r} \)) has the \( \alpha_i \)'s (respectively \( \beta_i \)'s) as columns, and \( D_{h_0} = \text{diag}(h_0^\top \gamma_1, \ldots, h_0^\top \gamma_r) \). Let \( \tilde{N}_{h_0} = h_0^\top \cdot 3 A \) and \( \tilde{N}_g = g^\top \cdot 3 A \) for \( g, h_0 \in (\mathbb{C}^{n+1})^\vee \). Then, we have

\[
M_{g/h_0} = \tilde{N}_{h_0}^{-1} \tilde{N}_g = B^{-1}D_{h_0}^{-1}D_gB^\top,
\]

so that the points \( \beta_i \) can be recovered uniquely from the matrix of eigenvectors \( B^\top \) provided that \( h_0^\top \gamma_i \neq 0 \) for all \( i = 1, \ldots, r \). The \( \alpha_i \)'s and \( \gamma_i \)'s can then be recovered from the 2-flattening; see [1, 17, 31, 32, 35] for more details. With the foregoing suggestive notation, it is easy to see that the matrix of \( \tilde{N}_{h_0} : f \mapsto N(fh_0) \) with respect to the standard bases is precisely eq. (3.8). Indeed, note that since we can take \( N = A(1) \), we have \( fh_0 \approx f \otimes h_0 \) and so \( N(fh_0) = A(1)(f \otimes h_0) \).

Pencil-based algorithms may thus be interpreted as a special case of the proposed \( \text{cpd.hnf} \) algorithm based on homogeneous normal forms when \( r \leq m + 1 \leq \ell + 1 \). Note that because of the numerical instabilities analyzed in [1], we prefer to extract the \( \zeta_i = (\beta_i, \gamma_i) \) in a different way. We compute \( \beta_i \) from the eigenvalues of \( M_{x_i/h_0} \) and the corresponding \( \gamma_i \) from the linear system eq. (3.4).

### 3.4. The algorithm

The discussion so far is distilled into algorithm 2. This algorithm implements step 4 of algorithm 1. Note that we dropped the tilde on top of the \( N_i \)’s in lines 2–7 to streamline the presentation of the algorithm.

The first phase of the algorithm, up to line 21, constructs the pre-normal form \( N \) and chooses an \( N_{h_0} \). This phase depends on whether we can invoke the more efficient theorem 3.2 (\( r \leq m + 1 \)) or we need the full power of theorem 3.1. In the former case, we can take \( N = A(1) \), while in the latter case we need to take \( N \) equal to the left null space of \( R_I(d,e) \). How we choose the degree \((d,e)\) in line 9 is explained in section 4. The matrix \( R_I(d,e) \in \mathbb{C}^{\text{HF}_S(d,e) \times \text{HF}_S(d',e')} \) can be constructed efficiently column-by-column without polynomial multiplication. Indeed, by eq. (3.5) it suffices to copy the coefficients of \( f_i \) relative to the monomial basis of \( S_{(1,1)} \) into the correct rows; see also example 4 below. The left null space \( N \) can be extracted from the last \( r \) columns of the \( U \)-factor in the SVD \( R_I(d,e) = USV^T \). In our implementation, the matrix \( N_{h_0} \in \mathbb{C}^{r \times \text{HF}_S(1,1)} \) is chosen by sampling the coefficients of \( h_0 \in S_{(d,e')} \) identically and independently distributed from a Gaussian distribution. With probability 1, such an \( h_0 \) satisfies \( h_0(\gamma_i) \neq 0 \) for all \( i \); hence, this is a valid choice of \( h_0 \).

The next phase of the algorithm, in lines 21–22, consists of choosing a basis \( B \). Although in theory theorem 3.1 enables us to choose any \( B \) such that \( (N_{h_0})^{-1}_{B} \) is invertible, it was shown

\(^4\)It is well known that the decomposition problem for a rank-\( r \) tensor in \( \mathbb{C}^{(r+1) \times (m+1) \times (n+1)} \) with \( r \leq m + 1 \leq \ell + 1 \) can always be reduced to this so-called concise case [24, 30] by computing an orthogonal Tucker decomposition [41] followed by a rank-\( r \) decomposition of the core tensor.
in [40] that for reasons of numerical stability it is crucial to choose $B$ such that $(N_{h_0})_B$ is well-conditioned. In practice, such a subspace $B$ can be found using a QR decomposition with optimal column pivoting or by using the SVD [39, Chapter 4]. We stated the QR approach in algorithm 2.

The multiplication matrices are constructed straightforwardly as the formula suggests in lines 23 to 25. Note that the upper triangular matrix $(N_{h_0})_B$ does not need to be inverted explicitly, since the system can be solved by backsubstitution.

In line 26, the matrices $M_{(hx_k)/h_0}$ need to be simultaneously diagonalized, as we have that $M_{(hx_k)/h_0} = V^{-1} \text{diag}(\beta_k)V$. We compute $V$ from a random linear combination of the $M_{(hx_k)/h_0}$s, and then extract $\beta_k$ as the diagonal entries from the (approximately) diagonalized matrix $V^{-1}M_{(hx_k)/h_0}V$.

The system in line 28 is solved efficiently by noting that the coefficients of $f_j$ can be arranged in a matrix $F_j$ of size $(m + 1) \times (n + 1)$, so that $f_j(x, y) = x^T F_j y$. Hence, (3.4) boils down to computing the kernel of $A_{y} = 0$ where the rows of $A \in \mathbb{C}^{x \times (n+1)}$ are the row vectors $\beta_i^T f_j$. Note that by assumption 1, ker $A$ is spanned by $\gamma_i$.

3.5. Some examples. We now present two illustrative examples. The first one shows how to use the techniques explained above on the tensor $A$ in example 1.
Example 4 (Example 1, continued). The Hilbert function of $S/I$, where $I$ is generated by the five $f_i$'s from example 1, is shown in table 1 for small degrees. From HF$_{S/I}(1,1) + (d', e') = r = 4$ for $(d', e') \in \mathbb{N}^2$ we see that every degree $(d, e) \geq (1, 1)$ leads to a pre-normal form. Using $(d', e') = (1, 0)$, we obtain the pre-normal form $N$ as the cokernel of $R_I(2,1) \in \mathbb{R}^{18 \times 15}$, whose transpose is

\[
\begin{pmatrix}
    x_0f_1 & -1 & 1 & -1 & 1 \\
    x_1f_1 & -1 & -1 & 1 & -1 \\
    x_2f_1 & -1 & -1 & -1 & 1 \\
    x_0f_2 & -2 & 1 & -1 & 1 \\
    x_1f_2 & -2 & -2 & 1 & 1 \\
    x_2f_2 & -1 & -1 & -1 & 1 \\
    x_0f_3 & -1 & -1 & 1 & 1 \\
    x_1f_3 & -1 & -1 & -1 & 1 \\
    x_2f_3 & -2 & -2 & -2 & 1 \\
    x_0f_4 & -2 & -2 & -2 & -2 \\
    x_1f_4 & -2 & -2 & -2 & -2 \\
    x_2f_4 & -2 & -2 & -2 & -2 \\
    x_0f_5 & -2 & -2 & -2 & -2 \\
    x_1f_5 & -2 & -2 & -2 & -2 \\
    x_2f_5 & -2 & -2 & -2 & -2 
\end{pmatrix}
\]

The missing entries represent zeros. The row indexed by $x_2f_3$ has entry $-2$ in the column indexed by $x_0x_2y_0$ and 1 in the column indexed by $x_2^2y_0$. This comes from $x_2f_3 = -2x_0x_2y_0 + x_2^2y_0$. The cokernel of $R_I(2,1)$ can be obtained, for instance, from the full SVD. We set $h_0 = x_0 + x_1 + x_2$ and use the subspace $B$ spanned by $B = \{x_0y_0, x_0y_1, x_0y_2, x_1y_0\}$. The numerical approximations of the eigenvalues of $M_{x_0/h_0}, M_{x_1/h_0}$ and $M_{x_2/h_0}$, found using Julia, are the rows of

\[
\begin{pmatrix}
    -1.03745e-16 & 0.25 & 0.333333 & 0.5 \\
    1.0 & 0.25 & -2.48091e-16 & -3.16351e-16 \\
    -1.64372e-16 & 0.5 & 0.666667 & 0.5 
\end{pmatrix}
\]

These approximate the evaluations of $x_i/h_0$ at $\zeta_4, \zeta_3, \zeta_1, \zeta_2$ (in that order, from left to right). Consequently, the columns in the display above are homogeneous coordinates for $\beta_4, \beta_3, \beta_1, \beta_2$. The $\gamma_i$'s can then be obtained by solving the linear system (3.4) of 5 equations in 3 unknowns. The left eigenvectors of the matrices $M_{x_j/h_0}$ are the columns of

\[
\begin{pmatrix}
    1.10585e-17 & 1.58104e-15 & 1.0 & -9.8273e-16 \\
    1.69823e-17 & -2.36986e-15 & 2.02579e-15 & -1.0 \\
    -1.42128e-16 & -1.0 & 1.1186e-15 & -3.681e-16 \\
    1.0 & 3.81185e-17 & -6.61522e-16 & 6.55477e-16 
\end{pmatrix}
\]

corresponding to evaluation (up to scale) of $B$ at $\zeta_4, \zeta_3, \zeta_1, \zeta_2$. △

The ideal $I$ in the previous example has the property that HF$_{S/I}(1 + d', 1 + e') = r$ for all $(d', e') \in \mathbb{N}^2$. Our next example shows that this is not the case in general.

| $i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|-----|---|---|---|---|---|---|---|---|---|---|
| 0   | 1 | 3 | 6 | 10| ...| | | | | |
| 1   | 3 | 4 | 4 | 4 | ...| | | | | |
| 2   | 6 | 4 | 4 | 4 | ...| | | | | |
| 3   | 10| 4 | 4 | 4 | ...| | | | | |
| ... | ...| ...| ...| ...| ...| | | | | |

Table 1. Hilbert functions HF$_{S/I}(i,j)$ from Example 4 (left) and Example 5 (right) for small values of $i, j$. 

\[
\begin{array}{ccccccc}
  & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
  0 & 1 & 3 & 6 & 10 & 15 & 21 \\
  1 & 3 & 4 & 4 & 4 & 15 & 16 \\
  2 & 6 & 4 & 4 & 4 & 12 & 12 \\
  3 & 10 & 4 & 4 & 4 & 12 & 12 \\
  ...
\end{array}
\]
Example 5 (A format for which $HF_{S/I}(2, 1) \neq r$). In example 4 we could take any $(d', e') \in \mathbb{N}^2$ to compute a pre-normal form. However, it may be necessary to take bigger leaps in $\mathbb{N}^2$ such that $HF_{S/I}((1, 1) + (d', e')) = r$. As a concrete example, consider a rank-12 tensor $\mathcal{A} \in \mathbb{C}^{12} \otimes \mathbb{C}^7 \otimes \mathbb{C}^3$ with decomposition

$$\mathcal{A} = \sum_{i=1}^{12} \alpha_i \otimes \beta_i \otimes \gamma_i,$$

where $\beta_1, \ldots, \beta_{12}$ are the columns of a generic $7 \times 12$ matrix, $\gamma_1, \ldots, \gamma_{12}$ are the columns of a generic $3 \times 12$ matrix and $\alpha_1, \ldots, \alpha_{12}$ are the columns of any invertible $12 \times 12$ matrix. The Hilbert function of $S/I$ where $I = \langle \ker \mathcal{A}_{(1)} \rangle$ is shown, for small degrees, in table 1. By proposition 3.1, a possible choice for $(d', e')$ is $(2, 0)$; this is underlined in the left part of table 1. Other examples are $(2, 1), (2, 2), (1, 2), (0, 4)$. Some noteworthy non-examples are $(1, 0), (1, 1), (0, 1)$. In section 4, we investigate choices of $(d', e')$ of the form $(d', 0)$. Our results will explain why, in this example, $d' = 1$ does not work, but $d' = 2$ does. \(\triangle\)

4. Regularity

One key step of the proposed cpd_hnf algorithm has not been investigated. As explained in the previous section (see also line 9 of algorithm 2), we should determine a correct degree $(d, e)$. This choice has a major impact on the computational complexity of the proposed algorithm. Indeed, it determines the dimensions of the graded resultant matrix $R_I(d, e)$ from eq. (3.6) whose left nullspace is required. The goal of this section is determining which degree $(d, e)$ is required for theorem 3.1 to apply. From this we can then deduce our algorithm’s computational complexity.

Let $A$ be a tensor as in eq. (A) satisfying assumption 1, and let $S = \mathbb{C}[x_0, \ldots, x_m, y_0, \ldots, y_n]$ be the $\mathbb{N}^2$-graded ring from (2.2). We assume that $r > m + 1$, for otherwise theorem 3.2 applies and no choice of $(d, e)$ is required.

In order to compute $\beta_i$ and $\gamma_i$ via theorem 3.1, we need to compute a pre-normal form on $S_{(d,e)}$ for $(d, e) \neq (1, 1)$ and $(d', e') \geq (0, 0)$. Proposition 3.1 tells us that we must find such a tuple $(d, e)$ for which additionally $HF_{S/I}(d, e) = r$, where $I$ is the ideal $\langle \ker \mathcal{A}_{(1)} \rangle$. Motivated by this, we make the following definition.

Definition 4.1. For a homogeneous ideal $J \subset S$ such that $J = \langle J_{(1, 1)} \rangle$ and $V_X(J)$ consists of $r$ points with multiplicity one, we define the regularity of $J$ to be the set

$$\text{Reg}(J) = \{(d, e) \in \mathbb{N}^2 \mid (d, e) \geq (1, 1) \text{ and } HF_{S/I}(d, e) = r\}.$$ 

Hence, our task is to find a tuple $(d, e) \in \text{Reg}(I) \setminus \{(1, 1)\}$. Recall that such a tuple exists by lemma 3.2. In this section, for given $\ell, m, n$ and $r$ satisfying (R), we conjecture an explicit formula for $d$ and $e$ so that $(d, 1), (1, e) \in \text{Reg}(I) \setminus \{(1, 1)\}$ for generic tensors of this format and rank. We prove it in many practical cases.

Because the results in this section are of independent interest for solving structured, overdetermined systems of polynomial equations, we formulate them in a slightly more general context. The first statement of this section, proposition 4.1, is concerned with homogeneous ideals $J$ of $S$ that are generated by elements of degree $(1, 1)$. After that, we specialize to a particular type of such $(1, 1)$-generated ideals. More precisely, to a tuple $Z = (\zeta_1, \ldots, \zeta_r) \in X^r$ we associate an ideal $J(Z)$ which is generated by elements of degree $(1, 1)$, and we investigate its Hilbert function (corollary 4.1). In our tensor setting, we will have $J(Z) = I = \langle \ker \mathcal{A}_{(1)} \rangle$. After pointing out in lemma 4.1 that for $r \leq mn$, most configurations $Z = (\zeta_1, \ldots, \zeta_r) \in X^r$ lead to an ideal $J(Z)$ such that $V_X(J(Z)) = \{\zeta_1, \ldots, \zeta_r\}$, where each of the $\zeta_i$ occurs with multiplicity one, we use corollary 4.1 to characterize $\text{Reg}(J(Z))$ in theorems 4.1 and 4.2 and conjecture 1.
For a = (a₀, ..., aₘ) ∈ ℤᵐ⁺¹, b = (b₀, ..., bₙ) ∈ ℤⁿ⁺¹ we write ∂ₐ,ₜ : S → S for the differential operator

\[ \partialₐ,ₜ = \frac{1}{a₀! \cdots aₘ! b₀! \cdots bₙ!} \partial^{a₀} \partial x_{a₀} \cdots \partial^{aₘ} \partial x_{aₘ} \partial^{b₀} \partial y₁ \cdots \partial^{bₙ} \partial yₙ . \]

such that the basis of \( S_{(d,e)}^\vee \) dual to \( \{ x^a y^b \mid |a| = d, |b| = e \} \) is given by \( \{ \partialₐ,ₜ \mid |a| = d, |b| = e \} \).

We write \( e_k \) for the standard basis vector \( (0, \ldots, 1, \ldots, 0) \) with a 1 in the \((k+1)\)-st position, such that

\[ \partial_{eₖ,ₑₗ} = \frac{\partial^2}{\partial xₖ \partial yₗ} , \quad 0 ≤ k ≤ m, 0 ≤ ℓ ≤ n . \]

Note that this differs from the shorter notation \( \partialᵣ \) used in previous sections to have a general notation for derivatives of arbitrary order. For \((d, e) ∈ ℤ²\) and \( J ⊂ S \) a homogeneous ideal, the space \( J_{(d,e)}^+ ⊂ S_{(d,e)}^\vee \) is defined as

\[ J_{(d,e)}^+ = \{ v ∈ S_{(d,e)}^\vee \mid v(f) = 0 \text{ for all } f ∈ I \} . \]

It follows from basic linear algebra that \( J_{(d,e)}^+ ≃ (S/J)_{(d,e)}^\vee \), such that \( \text{HF}_{S/J}(d, e) = \dim_ℂ J_{(d,e)}^+ \).

As before, we denote \((d', e') = (d - 1, e - 1)\). In order to simplify the notation in this section, we will use the abbreviations

\[ \sum_{a,b} = \sum_{|a|=d, |b|=e} , \quad \sum_{a', b'} = \sum_{|a'|=d', |b'|=e'} , \quad \sum_{k,l} = \sum_{0 ≤ k ≤ m, 0 ≤ ℓ ≤ n} . \]

We focus on ideals \( J \) that are generated by elements of degree \((1, 1)\). In this case, a functional belongs to \( J_{(d,e)}^+ \) if and only if it induces functionals in \( J_{(1,1)}^+ \).

**Proposition 4.1.** Let \( J ⊂ S \) be a homogeneous ideal such that \( J = \langle J_{(1,1)} \rangle \). An element \( v = \sum_{a,b} cₐ,ₜ aₜ \partialₐ,ₜ ∈ S_{(d,e)}^\vee \) is contained in \( J_{(d,e)}^+ \) if and only if

\[ \sum_{k,l} c_{a'+eₖ,b'+ₑₗ} \partial_{ₑₖ,ₑₗ} ∈ J_{(1,1)}^+ \quad \text{for all } (a', b') \text{ such that } |a'| = d', |b'| = e' . \]

**Proof.** Since \( J = \langle J_{(1,1)} \rangle \), an element \( v = \sum_{a,b} cₐ,ₜ aₜ \partialₐ,ₜ ∈ S_{(d,e)}^\vee \) is contained in \( J_{(d,e)}^+ \) if and only if \( v(hf) = 0 \) for all \( f ∈ J_{(1,1)} \) and \( h ∈ S_{(d', e')} \). Using Leibniz’ rule, we find

\[ 0 = v(hf) = \sum_{a,b} cₐ,ₜ aₜ h ∂ₐ,ₜ (hf) = \sum_{a,b} cₐ,ₜ \sum_{k,l} ∂ₖ,ₗ aₕ,ₑₗ (h) ∂ₑₖ,ₑₗ (f) , \]

with the convention that \( ∂ₐ,ₜ = 0 \) whenever \( \min(a) < 0 \) or \( \min(b) < 0 \). Regrouping the terms in this expression gives

\[ 0 = \sum_{a', b'} ∂ₐ',ₑₗ (hf) \sum_{k,l} c_{a'+eₖ,b'+ₑₗ} \partialₑₖ,ₑₗ (f) \quad \text{for all } h ∈ S_{(d', e')}, f ∈ J_{(1,1)} . \]

This proves the statement. \( \square \)

In our tensor rank decomposition setting, we are mainly interested in investigating the Hilbert function for \((1, 1)\)-generated ideals defined by point configurations in \( X = ℙ^m × ℙ^n \). To that end, fix \( r \) points \( Z = (ζ₁, ..., ζᵣ) ∈ X^r \) and let \( ζᵢ = (βᵢ, γᵢ) \). We denote \( wᵢ = (βᵢ ⊗ γᵢ) \) for \((ζᵢ) ∈ S_{(1,1)} \) such that \( wᵢ(f) = f(βᵢ, γᵢ) \) for \( f ∈ S_{(1,1)} \). In coordinates, the \( wᵢ \) are \( wᵢ = \sum_{k,l} βᵢk γᵢℓ ∂ₑₖ,ₑₗ \). To the point configuration \( Z \) we associate an ideal \( J(Z) ⊂ S \) by setting

\[ J(Z)_{(1,1)}^+ = \text{span}_ℂ(w₁, ..., wᵣ) \quad \text{and} \quad J(Z) = \langle J(Z)_{(1,1)} \rangle . \]

\( ^{5}\text{The notation is similar to section 2. Here we omit the restriction to the subspace } B \text{ (or, equivalently, we take } B = S_{(1,1)}) \). We also drop the transpose on \( wᵢ \) as we will think of them as column vectors instead of row vectors in this section.
Note that the ideal $I = \langle \ker A \rangle$ from previous sections arises in this way.\footnote{For the reader who is familiar with algebraic geometry, we note that this is our motivation for associating the ideal $J(Z)$ to $Z$, instead of the usual vanishing ideal of the points in $Z$. These are different ideals, as $J(Z)$ is usually not saturated with respect to the irrelevant ideal of $S$.} We denote $W \subset X^r$ for the Zariski-open subset in which $w_1, \ldots, w_r$ are $C$-linearly independent. If $r \leq \text{HF}_S(1,1)$, $W \subset X^r$ is non-empty, and therefore dense in the Euclidean topology.\footnote{This follows from the fact that the Segre variety is not contained in a hyperplane.} We have the following consequences of proposition 4.1.

**Corollary 4.1.** Let $Z = (z_1, \ldots, z_r) \in X^r$ and let $w_1, \ldots, w_r$ and $J(Z)$ be as above. For the maps $\iota : S_{(d,e)}^{\vee} \to S_{(1,1)}^{\vee} \otimes S_{(d',e')}^{\vee}$ and $M : (S_{(d',e')}^{\vee})^r \to S_{(1,1)}^{\vee} \otimes S_{(d',e')}^{\vee}$ with

$$
\iota \left( \sum_{a,b} c_{a,b} \partial_{a,b} \right) = \sum_{a',b'} \sum_{k,l} c_{a'+k,b'+l} \partial_{a+k,b+l} \otimes \partial_{a',b'}, \quad M(v_1, \ldots, v_r) = \sum_{i=1}^{r} w_i \otimes v_i,
$$

we have the equality $\iota \left( J(Z)^{\perp}_{(d,e)} \right) = \im \iota \cap \im M$. Moreover, if $Z \in W$, we have $\text{HF}_{S/J(Z)}(d,e) = \dim_C \left( M^{-1}(\im \iota) \right)$, where

$$
M^{-1}(\im \iota) = \left\{ (v_1, \ldots, v_r) \in (S_{(d',e')}^{\vee})^r \mid M(v_1, \ldots, v_r) \in \im \iota \right\}.
$$

**Proof.** By proposition 4.1, $v = \sum_{a,b} c_{a,b} \partial_{a,b}$ is an element of $J(Z)^{\perp}_{(d,e)}$ if and only if, for all $a',b'$ such that $|a'| = d', |b'| = e'$, there exist $v_{i,a',b'} \in C$ such that

$$
\sum_{k,l} c_{a'+k,b'+l} \partial_{a+k,b+l} = \sum_{i=1}^{r} v_{i,a',b'} w_i.
$$

Writing $v_i = \sum_{a',b'} v_{i,a',b'} \partial_{a',b'} \in S_{(d',e')}^{\vee}$ and writing (4.1) in matrix format, we see that (4.1) is equivalent to the following equality in $S_{(1,1)}^{\vee} \otimes S_{(d',e')}^{\vee}$:

$$
\sum_{i=1}^{r} w_i \otimes v_i = \sum_{a',b'} \sum_{k,l} c_{a'+k,b'+l} \partial_{a+k,b+l} \otimes \partial_{a',b'}.
$$

Linear independence of $w_1, \ldots, w_r$ implies that $M$ is injective. We have that injectivity of $\iota$ and $M$ implies along with $\text{HF}_{S/J(Z)}(d,e) = \dim_C J(Z)^{\perp}_{(d,e)}$ that

$$
\dim_C J(Z)^{\perp}_{(d,e)} = \dim_C \iota \left( J(Z)^{\perp}_{(d,e)} \right) = \dim_C (\im \iota \cap \im M) = \dim_C \left( M^{-1}(\im \iota) \right).
$$

This concludes the proof. \hfill \Box

**Corollary 4.2.** Let $Z = (z_1, \ldots, z_r) \in W \subset X^r$ and $(d,e) \geq (1,1)$. We have $\text{HF}_{S/J(Z)}(d,e) \geq r$.

**Proof.** The statement follows from [39, Lemma 5.5.7]. Nevertheless, we give an instructive proof. Let $w_i^f \in S_{(d',e')}^{\vee}, w_i^g \in S_{(d,e)}^{\vee}$ be given by $w_i^f(f) = f(\beta_i, \gamma_i), w_i^g(g) = g(\beta_i, \gamma_i)$ for $f \in S_{(d',e')}, g \in S_{(d,e)}$. Then $\iota(w_i^g) = w_i \otimes w_i^f$ and thus

$$
\mathcal{M}(w_i^f, 0, \ldots, 0), \mathcal{M}(0, w_i^g, 0, \ldots, 0), \ldots, \mathcal{M}(0, 0, \ldots, w_i^f)
$$

are all contained in $\im \iota$. Therefore $M^{-1}(\im \iota)$ contains at least $r$ linearly independent elements, so by corollary 4.1 we have $\text{HF}_{S/J(Z)}(d,e) \geq r$. \hfill \Box

The space $S_{(1,1)}^{\vee} \otimes S_{(d',e')}^{\vee}$ is identified with the space of matrices of size $\text{HF}_S(1,1) \times \text{HF}_S(d',e')$, where the rows are indexed by $\partial_{a+k,1}$ for $0 \leq k \leq m$, $0 \leq l \leq n$ and columns are indexed by $\partial_{a',b'}$ where $(a',b') \in \mathbb{N}^{m+1} \times \mathbb{N}^{n+1}$ with $|a'| = d', |b'| = e'$. For such a matrix to be contained in $\im \iota$, a collection of partial symmetry conditions needs to be satisfied. For instance, if $(a' + e_k, b' + e_l) = (a'' + e_{k'}, b'' + e_{l'})$, then the entry in the row indexed by $\partial_{a+k,1}$
and column indexed by $\partial_{\nu',\nu''}$ should be equal to the entry in the row indexed by $\partial_{\nu',\nu''}$ and column indexed by $\partial_{\nu',\nu'''}$.

By corollary 4.1 we can compute the Hilbert function of $S/J(Z)$ via a rank computation of a matrix whose entries are monomials evaluated at the points $(\beta_i, \gamma_i)$. This is most easily explained by means of an example.

**Example 6.** Let $(m, n) = (3, 2)$, $(d, e) = (2, 1)$ and $r = 6$. We consider the ideal $J(Z)$ defined by the tuple $Z = (\zeta_1, \ldots, \zeta_6) \in X^6 = (\mathbb{P}^3 \times \mathbb{P}^2)^6$ in the ring $S = \mathbb{C}[x_0, x_1, x_2, x_3, y_0, y_1, y_2]$, where $\zeta_i = (\beta_i, \gamma_i) = ((\beta_1 : \gamma_1) : \cdots : \beta_3, \gamma_3 : \gamma_2)$. As explained above, we can identify $S(1,1) \otimes S(1,0)$ with matrices of size $12 \times 4$. The image of $(v_1, \ldots, v_6) \in (S(1,0))^6$, with $v_i = \sum_{q=0}^{3} v_{iq} \partial_{\nu_q,0}$, under $\mathcal{M}$ is given by

$$
(4.3) \quad \begin{bmatrix} \beta_1 \otimes \gamma_1 & \cdots & \beta_6 \otimes \gamma_6 \end{bmatrix} \begin{bmatrix} v_1^T \\ \vdots \\ v_6^T \end{bmatrix} = \partial_{\nu_{1},0} \cdots \partial_{\nu_{r},0}.
$$

On the other hand, the image of $\sum_{|a|=2,|b|=1} c_{a,b} \partial_{a,b}$ under $\iota$ is the matrix

$$
(4.4) \quad \iota \left( \sum_{|a|=2,|b|=1} c_{a,b} \partial_{a,b} \right) = \begin{bmatrix} \partial_{\nu_{0},0} & \partial_{\nu_{1},0} & \partial_{\nu_{2},0} & \partial_{\nu_{3},0} \\ \partial_{\nu_{0},e_0} & \partial_{\nu_{0},e_1} & \partial_{\nu_{0},e_2} & \partial_{\nu_{0},e_3} \\ \partial_{\nu_{0},e_1} & \partial_{\nu_{1},e_0} & \partial_{\nu_{1},e_1} & \partial_{\nu_{1},e_2} \\ \partial_{\nu_{0},e_2} & \partial_{\nu_{1},e_2} & \partial_{\nu_{2},e_1} & \partial_{\nu_{2},e_2} \\ \partial_{\nu_{0},e_3} & \partial_{\nu_{1},e_3} & \partial_{\nu_{2},e_3} & \partial_{\nu_{3},e_3} \end{bmatrix}.
$$

In order for $\mathcal{M}(v_1, \ldots, v_6)$, i.e., (4.3), to be contained in $\text{im } \iota$, it must be such that the $(\partial_{\nu_{0},e_0}, \partial_{\nu_{0},e_0})$-entry is equal to the $(\partial_{\nu_{1},e_0}, \partial_{\nu_{1},e_0})$-entry. This gives a linear condition on the $v_{iq}$. There are 18 such conditions. Let $v_{i,q} = (v_{1,q}, v_{2,q}, v_{3,q}, v_{4,q}, v_{5,q}, v_{6,q})^T$ be the column of the second matrix in (4.3) indexed by $\partial_{\nu_{0},0}$ and let $\Gamma = [\gamma_{ij}]$ be the matrix that has the homogeneous coordinates $\gamma_{ij}$, $j = 0, \ldots, 2$, of $\gamma_i$, $i = 1, \ldots, 6$, as columns. We also let $H_q = \text{diag}(\beta_{1q}, \ldots, \beta_{6q})$. The 18 symmetry conditions are

$$
\begin{bmatrix} \Gamma H_1 & -\Gamma H_0 \\ \Gamma H_2 & -\Gamma H_0 \\ \Gamma H_3 & -\Gamma H_0 \\ \Gamma H_2 & -\Gamma H_1 \\ \Gamma H_3 & -\Gamma H_1 \\ \Gamma H_3 & -\Gamma H_2 \end{bmatrix} \begin{bmatrix} v_{i,0} \\ v_{i,1} \\ v_{i,2} \\ v_{i,3} \end{bmatrix} = 0,
$$

where the colours in the block rows of the coefficient matrix $A(Z)$ correspond to the entries in (4.4) on which they impose relations. The vector space $\{ (v_1, \ldots, v_6) \mid \mathcal{M}(v_1, \ldots, v_6) \in \text{im } \iota \}$ from corollary 4.1 is the kernel of $A(Z)$. Hence, $\text{HF}_{S/J(Z)}(2,1)$ is the corank of $A(Z)$. It is at least 6, since

$$
A(Z) \begin{bmatrix} H_0 & H_1 & H_2 & H_3 \end{bmatrix}^T = 0.
$$
These null vectors correspond to the $w'_i$ in the proof of corollary 4.2. For generic $Z \in X^r$, the corank of $A(Z)$ is exactly 6, so $HF_{S/J(Z)}(2,1) = 6$. △

From the discussion in example 6, we would like to conclude that, generically, $(2,1) \in \text{Reg}(J(Z))$. For this to make sense, i.e., to apply definition 4.1, we need to show that for most configurations $Z$, $J(Z)$ defines $r$ points with multiplicity one. By an argument analogous to lemma 2.2, this happens for small enough ranks.

**Lemma 4.1.** Suppose that $r \leq mn$ (this is (R) for $\ell = \infty$). There is a Zariski open, dense subset $U \subset X^r$ such that for all $Z = (\zeta_1, \ldots, \zeta_r) \in U$, $w_1, \ldots, w_r$ are $C$-linearly independent and $V_X(J(Z)) = \{\zeta_1, \ldots, \zeta_r\}$ consists of $r$ points with multiplicity one.

**Proof.** As before, let $\zeta_i = (\beta_i, \gamma_i)$, $i = 1, \ldots, r$ and $w_i = (\beta_i \otimes \gamma_i)^T \in S_{(1,1)}^{r'}$. By [10, Theorem 2.5], there is an open dense subset $U' \subset X^r$ such that for $Z = (\zeta_1, \ldots, \zeta_r) \in U'$, $\text{span}_C(w_1, \ldots, w_r)$ contains no points of the form $(\beta \otimes \gamma)^T$, other than the $w_i$. We set $U = U' \cap W$, which is open and dense in $X^r$ since both $U'$ and $W$ are. The rest of the proof is identical to that of lemma 2.2. □

**Proposition 4.2.** Suppose $r \leq HF_S(1,1)$ such that $W \neq \emptyset$. For fixed $(d,e) \geq (1,1)$, the Hilbert function $HF_{S/J(Z)}(d,e)$, as a function of $Z$, is upper semicontinuous on $W$. That is, for any $r^* \in \mathbb{N}$, the set

$$V_{r^*} = \{Z \in W \mid HF_{S/J(Z)}(d,e) > r^*\}$$

is Zariski closed in $W$. Consequently, either $V_{r^*} = W$ or $V_{r^*} \subset W$ is a strict closed subvariety.

**Proof.** For any $Z \in W$, the functionals $w_1, \ldots, w_r$ are linearly independent, so the map $\mathcal{M}$ from corollary 4.1 is injective. Define $\theta : S_{(d',e')}^{r'} \otimes S_{(d',e')}^{r'} \to \mathbb{C}^n$ to be any cokernel map of $\iota$. That is, $\ker \theta = \text{im} \iota$ and $q = HF_S(1,1)HF_S(d',e') - HF_S(d,e)$. We claim that $HF_{S/J(Z)}(d,e)$ is equal to $\text{corank}(\theta \circ \mathcal{M})$. Equivalently, $HF_{S/J(Z)}(d,e) = \dim_C(\ker(\theta \circ \mathcal{M}))$. To see this, note that

$$(\theta \circ \mathcal{M})(v_1, \ldots, v_r) = 0 \iff \mathcal{M}(v_1, \ldots, v_r) \in \text{im} \iota \iff (v_1, \ldots, v_r) \in \mathcal{M}^{-1}(\text{im} \iota).$$

Hence, $\ker(\theta \circ \mathcal{M}) = \mathcal{M}^{-1}(\text{im} \iota)$ and by corollary 4.1, $HF_{S/J(Z)}(d,e) = \dim_C(\ker(\theta \circ \mathcal{M}))$. The map $\theta \circ \mathcal{M}$ can be represented by a matrix whose entries are polynomials in the coordinates of $\beta_i$ and $\gamma_i$. For instance, in example 6, $A(Z) = \theta \circ \mathcal{M}$. The set $V_{r^*}$ is the vanishing locus of all $(rHF_S(d',e') - r^*)$-minors of this matrix, which is clearly Zariski closed in $X^r$, and therefore also in $W$. □

**Corollary 4.3.** Suppose that $r \leq mn$ and $(d,e) \geq (1,1)$. Let $U \subset X^r$ be the dense open subset from lemma 4.1. If for some element $Z^* \in W$ we have $HF_{S/J(Z^*)}(d,e) = r$, then there is a dense, Zariski open subset $U^0$ of $U$ such that for all $Z \in U^0$, we have $(d,e) \in \text{Reg}(J(Z))$.

**Proof.** By corollary 4.2, we know $V_{r-1} = W$. Since $Z^* \in W \setminus V_r$, proposition 4.2 implies that $V_r \subset W$ is a strict subvariety. We set $U^0 = U \setminus V_r$. Clearly, if $Z = (\zeta_1, \ldots, \zeta_r) \in U^0$, then $J(Z)$ is such that $V_X(J(Z)) = \{\zeta_1, \ldots, \zeta_r\}$, where these points occur with multiplicity one (this uses $U^0 \subset U$), and $HF_{S/J(Z)} = r$ (since $U^0 \subset V_{r-1} \setminus V_r$). □

We can use corollary 4.3 to prove that $(d,e) \in \text{Reg}(J(Z))$ for generic $Z \in X^r$ by finding one particular instance $Z^* \in W$ for which $HF_{S/J(Z)}(d,e) = r$. This will be particularly useful for proving theorem 4.2 below. First, we investigate which combinations of $m, n, d, e, r$ are possible.

For $(d,e) \geq (1,1)$ and $m, n \in \mathbb{N}_0$ we define

$$\mathcal{R}(m,n,(d,e)) = \frac{HF_S(1,1)HF_S(d',e') - HF_S(d,e)}{HF_S(d',e') - 1}.$$
If \((d, e) = (1, 1)\), we set \(\mathcal{R}(m, n, (d, e)) = \infty\). For given \(m, n\) and \((d, e) \geq (1, 1)\), the following result shows that \(\mathcal{R}(m, n, (d, e))\) bounds the rank \(r\) for which we could possibly have \(\text{HF}_{S/J}(Z)(d, e) = r\) for \(Z \in W \subset X^r\).

**Theorem 4.1.** Let \((d, e) \geq (1, 1)\) and \(Z \in W \subset X^r = (\mathbb{P}^m \times \mathbb{P}^n)^r\) with

\[
\mathcal{R}(m, n, (d, e)) < r \leq mn.
\]  

We have \(\text{HF}_{S/J}(Z)(d, e) > r\). In particular, for \(r\) in the range \((4.5)\) and \(Z \in U\), where \(U \subset W\) is the open subset from lemma 4.1, we have \((d, e) \notin \text{Reg}(J(Z))\).

**Proof.** By corollary 4.1, we have

\[
\text{HF}_{S/J}(Z)(d, e) = \dim_{\mathbb{C}}(\ker \cap \ker \mathcal{M}) 
\geq \dim_{\mathbb{C}}(\ker \mathcal{I}) + \dim_{\mathbb{C}}(\ker \mathcal{M}) - \dim_{\mathbb{C}} \left( S_{1(1)}^V \otimes S_{d'(e')}^V \right).
\]

Since \(\mathcal{I}\) and \(\mathcal{M}\) are injective for \(Z \in W\), this implies

\[
\text{HF}_{S/J}(Z)(d, e) \geq \text{HF}_{S}(d, e) + r\text{HF}_{S}(d', e') - \text{HF}_{S}(1, 1)\text{HF}_{S}(d', e').
\]

Solving \(\text{HF}_{S}(d, e) + r\text{HF}_{S}(d', e') - \text{HF}_{S}(1, 1)\text{HF}_{S}(d', e') > r\) gives exactly the first inequality in \((4.5)\).

Note that if \((d, e) = (1, 1)\), the range \((4.5)\) is empty. This agrees with the fact that \((1, 1) \in \text{Reg}(J(Z))\) for all \(Z \in W\). From corollary 4.2 we know that \(\text{HF}_{S/J}(Z)(d, e) \geq r\) for \((d, e) \geq (1, 1)\) and \(Z \in W\). Combining this with a dimension argument as in the proof of theorem 4.1, we see that

\[
\text{HF}_{S/J}(Z)(d, e) \geq \max(r, \text{HF}_{S}(d, e) + r\text{HF}_{S}(d', e') - \text{HF}_{S}(1, 1)\text{HF}_{S}(d', e')).
\]

For \(e' = 0\) or \(d' = 0\), we observe experimentally that equality holds for generic configurations \(Z = (\zeta_1, \ldots, \zeta_r) \in X^r\). In this case, it suffices to check for which \(r\) the maximum equals \(r\). This leads us to the following conjecture.

**Conjecture 1.** Let \(d \geq 1\) and let \(m, n, r\) be such that

\[
r \leq \min \{ \mathcal{R}(m, n, (d, 1)), mn \}.
\]

There is a Zariski dense, open subset \(U^o \subset U \subset X^r = (\mathbb{P}^m \times \mathbb{P}^n)^r\), were \(U\) is the open subset from lemma 4.1, such that for all \(Z \in U^o\), \((d, 1) \in \text{Reg}(J(Z))\).

**Theorem 4.2.** Conjecture 1 holds in the following cases:

1. \(m \in \mathbb{N}_0, n \in \mathbb{N}_0\) and \(d = 1\),
2. \(m \in \{1, 2\}, n \in \mathbb{N}_0\) and \(d = 2\),
3. \(2 \leq m + 1, n + 1 \leq 50\) and \(d = 2\),
4. \(2 \leq m + 1, n + 1 \leq 9\) and \(3 \leq d \leq 5\),
5. \(2 \leq m + 1, n + 1 \leq 6\) and \(6 \leq d \leq 10\).

**Proof.** (1) In this case, clearly \(U^o = U\).

(2) Following example 6, we can compute \(\text{HF}_{S/J}(Z)(2, 1)\) as the corank of the stacked matrix \(A(Z) = [g_1(Z)^\top \cdots g_m(Z)^\top]^\top\), where

\[
G_k(Z) = \begin{bmatrix}
0_{(n+1) \times (k-1)r} & \Gamma H_k & -\Gamma H_{k-1} \\
0_{(n+1) \times (k-1)r} & \Gamma H_{k+1} & -\Gamma H_k \\
\vdots & \vdots & \ddots \\
0_{(n+1) \times (k-1)r} & \Gamma H_m & -\Gamma H_{k-1}
\end{bmatrix},
\]

the matrix \(\Gamma\) contains homogeneous coordinates of the points \(\gamma_i\) in its columns, and \(H_q\) is the diagonal matrix \(\text{diag}(\beta_{q(1)}, \ldots, \beta_{q(n)})\).

By corollary 4.3, we must find one \(Z^* \in W\) such that \(\text{HF}_{S/J}(Z^*)(2, 1) = \text{corank}(A(Z^*)) = r\) for each \(r\) satisfying \((4.7)\). In fact, since \(A(Z) (H_0 \cdots H_m)^\top = 0\), we have \(\text{corank}(A(Z)) \geq r\) for any \(Z \in X^r\), and by upper semicontinuity of corank, if \(\text{corank}(A(Z^*)) = r\) for some
$Z^* \in X^r$, then $\text{corank}(A(Z)) = r$ for all $Z$ in a dense, Zariski open subset of $X^r$. For $m = 1$, (4.7) entails $r \leq n + 1$, which means that the matrix $\Gamma$ has more rows than columns. On a dense, Zariski open subset of $X^r$, $\Gamma$ has rank $r$ and, since $(\beta_0, \beta_1) \neq (0, 0)$, the matrix $A = [\Gamma H_1 - \Gamma H_0]$ has rank at least $r$. The proof for $m = 2$ is more technical. Since we could not generalize it for higher $m$, we have deferred it to appendix A.

(3)–(5) These cases consists of a computer-assisted proof.\(^8\)

For proving (3), we generated random $Z^*$ and confirmed that $A(Z^*)$ has corank $r$ as follows. We generate random $\beta_i \in \mathbb{Z}^{m+1}$ and $\gamma_i \in \mathbb{Z}^{n+1}$. Then $A(Z)$ is a matrix over $\mathbb{Z}$. We can upper bound its corank by computing the corank, via a reduction to row echelon form using Gaussian elimination, over the finite field $\mathbb{Z}_p$ for some prime number $p$. We used $p = 8191$. The corank of $A(Z^*)$ over the finite field $\mathbb{Z}_p$ is then an upper bound of the corank of $A(Z^*)$ over $\mathbb{Z}$ (and so also its algebraic closure). We implemented this approach in C++ and certified for all listed cases that there exists a $Z^*$ (with integer coefficients) such that $A(Z)$ has corank $r$.

For proving (4) and (5), we apply corollary 4.3 to verify conjecture 1 by computing the Hilbert function in Macaulay2 [22] for all listed cases as follows:

\[
F = \mathbb{Z}/8191; \\
HF = (x, y) \rightarrow \text{binomial}(m+x, x) \ast \text{binomial}(n+y, y); \\
deags = \text{apply}(m+1, i->\{0,1\}) \mid \text{apply}(n+1, i->\{0,1\}); \\
S = \text{F}_{x_0..x_m, y_0..y_n}, \text{Degrees} \Rightarrow \text{deags}; \\
X = \text{matrix} \text{apply}(m+1, i->\text{apply}(r, j->\text{random}(\text{F}))); \\
Y = \text{matrix} \text{apply}(n+1, i->\text{apply}(r, j->\text{random}(\text{F}))); \\
N = \text{transpose} \text{matrix} \text{apply}(r, i->(X_i**Y_i)); \\
I = \text{ideal} \text{basis}(\{1,1\}, S) \ast \text{gens} \text{ker} N; \\
rk = \text{rank} N; -- \text{check that this is r, so Z* is in W} \\
hf = \text{hilbertFunction}(\{d,1\}, I); -- \text{check that this equals r}
\]

where $m, n, d,$ and $r$ are respectively $m, n, d$, and $r$ from the theorem.

Note that the role of $m$ and $n$ in conjecture 1 can be interchanged, implying the analogous statement that for $e \geq 1$ and generic $Z \in W \subset X^r = (\mathbb{P}^m \times \mathbb{P}^n)^r$ with $r \leq \min\{R(m,n,(1,e)), mn\}$, we have $(1,e) \in \text{Reg}(J(Z))$.

Let $A \in C^{n+1} \otimes C^{m+1} \otimes C^{n+1}$ be a general tensor whose rank satisfies (R) and let $I = \langle \ker A_{(1)} \rangle = J(Z)$, where $Z = (\beta_i, \gamma_i)_{i=1,\ldots,r}$. An immediate consequence of conjecture 1 would be that $(d,1) \in \text{Reg}(I)$, for any $d$ such that $R(m,n,(d,1)) \geq r$ and $(1,e) \in \text{Reg}(I)$ for any $e$ such that $R(m,n,(1,e)) \geq r$.

**Example 7** (Example 5, continued). In example 5, we have $m = 6, n = 2$. We have $R(6,2,(2,1)) = 21/2 < r = 12$, which explains why $(2,1) \notin \text{Reg}(I)$. Also, $R(6,2,(3,1)) = 112/9 \geq 12$ implies, since $A$ is generic, that $(3,1) \in \text{Reg}(I)$. Similarly, the smallest $e > 1$ for which $R(6,2,(1,e)) \geq 12$ is $e = 5$, so that $(1,5) \in \text{Reg}(I)$, but $(1,e) \notin \text{Reg}(I)$ for $1 < e < 5$. These computations are confirmed by Table 1.

One can check that

\[
(4.8) \quad R(m,n,(d,1)) = \frac{(m+d-1)(d-1)}{d} \cdot \left( (m+1)(n+1) - \frac{n+1}{d} (m+d) \right).
\]

The first factor is always greater than 1 and the second factor is at least $mn$ if $d \geq n + 1$. If conjecture 1 holds, then for all tensors $A$ of rank $r$ in the range (R), we have $(n+1,1) \in \text{Reg}(I)$. In that case, $\text{cpd}_n \text{hnf}$ can take $(d,e) = (n+1,1)$ to treat all identifiable rank-$r$ tensors in the unbalanced regime. That is, when $\ell > mn$. For such shapes, generic $r$-identifiability holds up to $r \leq mn$ [5].

\(^8\)The code and certificates can be obtained at https://gitlab.kuleuven.be/u0072863/homogeneous-normal-form-cpd.
Conjecture 1 gives a way of finding degrees of the form \((d, 1) \in \text{Reg}(I) \setminus \{(1, 1)\}\). We note that there might exist other tuples of the form \((d, e) \in \text{Reg}(I) \setminus \{(1, 1)\}\), which could lead to a lower computational cost. For such degrees, \((4.6)\) may be a strict inequality, which means that other tools are needed. We leave the further exploration of effective bounds for the regularity for future research.

5. Computational complexity

One motivation for studying the regularity of the ideal \(I = \langle \ker A(1) \rangle\) is to understand the computational complexity of algorithm 1.

Assume that we are given an \(r\)-identifiable tensor \(A\) in \(\mathbb{C}^{\ell+1} \otimes \mathbb{C}^{m+1} \otimes \mathbb{C}^{n+1}\), where \(\ell \geq m \geq n\). We additionally assume that \(\ell + 1 \leq (m+1)(n+1)\). The size of the input will be denoted by \(M = (\ell+1)(m+1)(n+1)\). Because of the constraint \((m+1)(n+1) \geq \ell+1 \geq m+1 \geq n+1\), we have

\[
M^{\frac{1}{2}} \leq m + 1 \leq M^{\frac{3}{2}} \quad \text{and} \quad 2 \leq n + 1 \leq M^{\frac{1}{2}}.
\]

Since \texttt{cpd.hnf} applies only if the rank \(r\) satisfies eq. (R), we have \(r = O(M^{\frac{5}{2}})\).

Without going into details, it can be verified that a crude upper bound for the computational complexity of the steps in algorithm 1, excluding step 4, is \(O(M^{3})\).

The key contribution to the time complexity originates from algorithm 2. Its complexity is ultimately determined by the choice of \((d, e)\) in line 9 of the algorithm. We analyze what happens when the degree is selected as \((d, 1)\). In this case, bounding the size of the matrix \(R_I(d, e)\) in algorithm 2 is critical. Assuming conjecture 1 holds, we have

\[
\left(\frac{3}{2}\right)^{d+1} \leq \text{HF}_S(d, 1) = \left(\frac{m + d}{d}\right) \frac{n + 1}{(n + 1) M^{\frac{5}{2}}} \leq M^{\frac{5}{2}} M^{\frac{1}{2}}.
\]

The upper bound follows from \((m+d) \leq (m + d)^{d} \leq (2m+1)^{d} \leq ((m+1)(n+1))^{d}\). For establishing the lower bound, note that \((1 + \frac{m}{d})^{d} \leq \left(\frac{m + d}{d}\right)\) and use the facts that \(d \leq n + 1 \leq m + 1\) (see the discussion below \((4.8)\)), so that \(n \geq 1\) implies that \(\frac{m}{d} \geq \frac{1}{2}\). Computing a basis for the left null space of \(R_I(d, 1)\) requires at most \(O(\text{HF}_S(d, 1))\) and at least \(r \cdot \text{HF}_S(d, 1)\) operations, as we know \((d, 1) \in \text{Reg}(I)\) so that the dimension of the left null space is \(r\). Consequently, line 11 has a time complexity that is at least exponential in \(d\).

**Proposition 5.1.** Consider the concise tensor space \(\mathbb{C}^{M^{\frac{1}{2}}} \otimes \mathbb{C}^{M^{\frac{1}{2}}} \otimes \mathbb{C}^{M^{\frac{1}{2}}}\). If conjecture 1 holds, then for a generic tensor of rank \(r = M^\frac{1}{2} - 2M^\frac{1}{2} - 1\) in this space, the asymptotic time complexity of algorithm 1 is at least exponential in the input size \(M = M^\frac{1}{2} M^\frac{1}{2} M^\frac{1}{2}\), if the degrees are restricted to \((d, 1)\) or \((1, e)\).

**Proof.** It follows from eq. \((4.8)\) and theorem 4.1 that for sufficiently large \(M\), the degree \(d\) should be at least \(\frac{1}{2} M^{\frac{1}{2}}\). Combining this with the above discussion about the size of \(R_I(d, 1)\) concludes the proof.

One might conclude from this result that \texttt{cpd.hnf} is not an effective algorithm for tensor rank decomposition. However, we recall that computing the rank of a tensor in general is an NP-complete problem [25]. Hence, a polynomial-time algorithm that applies to all inputs for this problem is not anticipated. A typical instance of an NP-complete problem can often be solved more efficiently than the worst-case instance. Proposition 1.1, which we are now ready to prove, is in this spirit.

---

\(^{9}\)A tensor \(A\) not satisfying this constraint on \(\ell\) can always be represented in new bases by a coordinate array \(B\) in a concise \([8,30]\) tensor space \(\mathbb{C}^{\ell'+1} \otimes \mathbb{C}^{m'+1} \otimes \mathbb{C}^{n'+1}\) with \(\ell' \leq \ell, m' \leq m,\) and \(n' \leq n\). After permutation of the factors, a concise tensor space \(\mathbb{C}^{k_1} \otimes \mathbb{C}^{k_2} \otimes \mathbb{C}^{k_3}\) always satisfies \(k_3 \geq k_1 \geq k_2 \geq k_3\); see, e.g., [9]. In practice, \(B\) can be obtained as the core tensor of the higher-order singular value decomposition [18].
Proof of proposition 1.1. It follows from the upper bound in (5.1) that the time complexity of algorithm 1 is $O((HF_S(d, 1))^3) = O(M^{5d+1})$.

We determine the smallest degree $d$ under conjecture 1 such that $(d, 1) \in \text{Reg}(I)$, where $I = (\ker \mathcal{A}_1)$. From theorem 4.1 we know that a necessary condition for this is that $\mathcal{R}(m, n, (d, 1)) \geq r = \phi mn$. The last inequality is implied by $(m+1)(n+1) - \frac{n+1}{d}(m+d) \geq \phi mn$ because of eq. (4.8). This is equivalent to

$$(1 - \phi)mn + m + n + 1 \geq \frac{1}{d}(m+d)(n+1) = \frac{1}{d}m(n+1) + n + 1.$$ 

Hence, $\mathcal{R}(m, n, (d, 1)) \geq \phi mn$ is implied by

$$d \geq \frac{1}{1 - \phi} > (n+1) \frac{1}{(1 - \phi)n + 1} = \frac{m(n+1)}{(1 - \phi)mn + m}.$$ 

In other words, provided conjecture 1 holds, it suffices to take $d = \lceil \frac{1}{1 - \phi} \rceil$ to be able to cover all ranks up $\phi mn$. As $\phi \neq 1$ is a constant, this proves the result. □

We can observe that the formula $d \geq \frac{1}{1 - \phi}$ is asymptotically sharp in the sense that as $\phi \to 1$ the exponent in $O(M^*)$ needs to blow up to infinity and no polynomial in the input size can control the growth. Indeed, proposition 5.1 shows that exactly when $\phi = 1$ there exist cases that require at least an exponential growth. Note that by the discussion below (4.8), assuming conjecture 1 we can use $d = n+1$ for any $\phi \in [0, 1]$. This gives the universal, exponential complexity bound $O(M^{\frac{5}{2}n^{2.5}})$.

6. Numerical Experiments

We present several numerical results demonstrating the effectiveness of cpd_hnf. All experiments were performed on KU Leuven/UHasselt’s Tier-2 Genius cluster of the Vlaams Supercomputer Centrum (VSC). Specifically, the supercomputer’s scheduling software allocated standard skylake nodes containing two Xeon® Gold 6140 CPUs (18 physical cores, 2.3GHz clock speed, 24.75MB L3 cache) with 192GB of main memory, as well as standard cascadelake nodes which are equipped with two Xeon® Gold 6240 CPUs (18 physical cores, 2.6GHz clock speed, 24.75MB L3 cache) with 192GB of main memory for our experiments.

In all experiments, we allowed all algorithms to use up to 18 physical cores.

The proposed algorithm was implemented in Julia v1.4.0, relying on the non-base packages Arpack.jl, DynamicPolynomials.jl, GenericSchur.jl, and MultivariatePolynomials.jl.10 Our implementation follows the pseudocode in algorithms 1 and 2 and the detailed discussion in section 3.4 quite closely. The Julia code of cpd_hnf, including driver routines to reproduce cpd_hnf’s experimental results can be found at https://gitlab.kuleuven.be/u0072863/homogeneous-normal-form-cpd.

6.1. Implementation details. The algorithm is implemented both for real and complex input tensors. In the former case, all computations are performed over the reals with the exception of the computation of the left null space of $R_I(d, e)$. In the real case, the algorithm continues with the real part of the output of this step.

At the start of the algorithm, we compress the $(\ell+1) \times (m+1) \times (n+1)$ input tensor, which is to be decomposed into $r$ rank-1 terms, to a $\min\{\ell+1,r\} \times \min\{m+1,r\} \times \min\{n+1,r\}$ tensor. For this we apply ST-HOSVD compression [42] with truncation rank $\min\{\ell+1,r\}, \min\{m+1,r\}, \min\{n+1,r\}$. Most of the experiments below are chosen so that this step performs no computations.

The kernel of $\mathcal{A}_1$ in algorithm 1 is computed by an SVD. The linear system at the end of algorithm 1 is solved by computing the Khatri–Rao product $K = [\beta_i \otimes \gamma_i]_{i=1}^d$ and then

\[10\] Additional packages are used to support our experimental setup, but these are not required for the main algorithm.
solving the overdetermined linear system $KA = A_T^{(1)}$ for $A$. The rows of $A$ then correspond to the $\alpha_i$'s.

In algorithm 2, the degree $(d,e)$ is determined automatically by assuming conjecture 1 is true and selecting a valid $(d,1)$ or $(1,e)$ based on a heuristic that takes into account the estimated computational cost and numerical considerations.\(^{11}\)

The key computational bottleneck is the computation of the left nullspace of $R_I(d,e)$ in line 11 of algorithm 2. When the number of entries of $R_I(d,e)$ is smaller than 10,000, we use the standard SVD-based kernel computation. In larger cases, for efficiency, we propose to employ Arpack.jl’s eigs function to extract the left null space from the Hermitian Gram matrix $G = R_I(d,e)(R_I(d,e))^\ast$, where $^\ast$ denotes the Hermitian transpose. The eigs function (with parameters tol = 1e-6 and maxiter = 25) extracts the $r$ eigenvalues of smallest modulus and their corresponding eigenvectors. Note that Arpack.jl employs a Bunch–Kaufman factorization [7] of the positive semidefinite input matrix $G$ to perform its Lanczos iterations.\(^{12}\)

The final step of algorithm 2 is implemented as discussed at the end of section 3.4. The kernel is computed with an SVD and $\gamma_i$ vanish exactly for all $j = 1,\ldots,s$. We propose to refine the approximate roots $(\beta_i, \gamma_i)$ by applying three Newton iterations on $f_i = \cdots = f_s = 0$. This adds a computational overhead of $r(m + 1)(n + 1)(m + n)^2$. We observed in practice that this improves the final accuracy of cpd_hnf by a few (1–3) digits.

6.2. Comparison with the state of the art. In the first experiment, we compare cpd_hnf with the current state of the art in direct numerical methods for tensor rank decomposition in terms of accuracy and computational performance. The algorithm developed in [20, 21] by Domanov and De Lathauwer is an advanced direct numerical method that shares several high-level characteristics with cpd_hnf:

(i) Both methods assume the target rank $r$ is supplied to the decomposition algorithm. They operate in a similar regime of ranks and are able to treat the full range of generically $r$-identifiable tensors in unbalanced tensor spaces. The method by Domanov and De Lathauwer can even deal with some ranks $r > \ell + 1 \geq m + 1 \geq n + 1$, while our algorithm cannot.

(ii) Like our method, Domanov and De Lathauwer rely on a simultaneous diagonalization procedure to extract one or two of the factor matrices.

(iii) To deal with high ranks, both methods rely on the construction of an auxiliary matrix whose size is parameterized by an integer. By increasing this integer, the range of ranks that is covered by the algorithms is broadened at the cost of a substantially increased computational complexity. In both algorithms, the asymptotic computational complexity is determined by the cost of computing the kernel of this auxiliary matrix. Contrary to [20, 21], we are able to give a precise connection between this integer and the range of ranks we can cover (subject to conjecture 1).

(iv) When $\ell + 1 \geq m + 1 \geq r$, both approaches can be considered pencil-based algorithms. For this reason, our comparison will focus on the case where $\ell + 1 \geq r \geq m + 1 \geq n + 1$.

A Matlab implementation of the algorithm in [20, 21] was kindly provided by Domanov and De Lathauwer. We refer to it as cpd_ddl henceforth. cpd_ddl has an accurate and a fast option for computing the kernel of the auxiliary matrix. We found in our experiments that the dimension of this kernel is often misjudged by the fast implementation. Consequently, it usually increases the size of the auxiliary matrix and causes it to exceed the 16.2GB memory limit we put on the size of that matrix. As a concrete statistic, in the configuration with

\(^{11}\)For more details, see the function minimumMultiDegree in NormalFormCPD.jl.

\(^{12}\)After extensive experimentation, we concluded that using eigs does not lead to a significant degradation of the numerical accuracy. On the other hand, its total execution time was up to 33% faster than the SVD-based approach for large problems.
We generate random rank-$r$ tensors $A = \sum_{i=1}^{r} \alpha_i \otimes \beta_i \otimes \gamma_i$ by randomly sampling the elements of $\alpha_i \in \mathbb{R}^{d_i}$, $\beta_i \in \mathbb{R}^{m_i}$, and $\gamma_i \in \mathbb{R}^{n_i}$ identically and independently distributed (i.i.d.) from a standard normal distribution. We apply both cpd_hnf and cpd_dd1 to these random tensors for all of the following configurations:\footnote{Both algorithms were applied to the same rank-$r$ tensors. To deal with the different programming languages and to limit storage demands, we generated a buffer of $10^7$ reals sampled i.i.d. from a standard normal distribution. The $\alpha_i, \beta_i$, and $\gamma_i$ of the true decomposition were then generated from this buffer. This entails there is a statistical correlation between the various tensors that are decomposed. However, we judged that this does not affect the validity of our conclusions. For each line in the set of configurations, a different random buffer was generated.}

\[
\begin{align*}
40 \geq m + 1 & \geq n + 1 \geq 2 \quad \text{for } d = 2, \\
20 \geq m + 1 & \geq n + 1 \geq 2 \quad \text{for } d = 3, \\
15 \geq m + 1 & \geq n + 1 \geq 2 \quad \text{for } d = 4,
\end{align*}
\]

and in all cases we take $\ell + 1 = r = \min\{R(m, n, (d, 1)), mn\}$. We do not provide our algorithm with the value of $d$. For each input tensor, cpd_hnf determines the degree $(d, 1)$ or $(1, e)$ automatically as explained in the previous subsection. For each algorithm, we record the relative backward error, total execution time, and the size of the auxiliary matrix whose kernel is computed. As the latter is the dominant operation in both algorithms, its time and memory complexity gives a good estimate of the overall complexity. The relative backward error is $\|A - \sum_{i=1}^{r} \alpha'_i \otimes \beta'_i \otimes \gamma'_i\|_F / \|A\|_F$, where $\alpha'_i \approx \alpha_i$, $\beta'_i \approx \beta_i$, and $\gamma'_i \approx \gamma_i$ are the estimates obtained by the algorithm.

The accuracy of cpd_hnf and cpd_dd1 in this set of experiments is shown in fig. 1. The newly proposed algorithm is consistently several orders of magnitude more accurate in terms of the relative backward error than the state of the art.

The results in the left panel of fig. 1(b) for $n > 13$ are incomplete because we interrupted the computation after three days. To put this in perspective, we note that the computation after three days. To put this in perspective, we note that the computation for the left panel of fig. 1(a) took a little over 17 hours. The other missing values inside of the triangles in fig. 1(b) indicate that cpd_dd1 wanted to allocate an auxiliary matrix that would require more than 16.2GB of memory. The same constraint was also imposed on cpd_hnf, but here only the largest case with $d = 4$ and $m = n = 14$ could not be treated. The red pixel for $d = 2, m = 35, n = 13$ in fig. 1(a) indicates that cpd_hnf gave inaccurate results. This is the only case where eigs failed to find a sufficiently accurate nullspace.

The timings of our Julia implementation are shown in fig. 2. As cpd_dd1 is implemented in a different programming language, we believe a direct comparison in timings is not opportune. Nevertheless, in both algorithms computing the kernel of the auxiliary matrix has the highest asymptotic time complexity. In cpd_dd1 it is an $L \times L$ square matrix, and in our algorithm, depending on the choice of degree, $R_f(d, e)$ is an almost-square $M \times N$ matrix with $M \approx N$. Therefore, we decided to plot the ratio between the number of elements in the auxiliary matrix of cpd_dd1 and cpd_hnf. Figure 3 visualizes this factor $\mu = L^2/(MN)$ in a logarithmic scale. This number indicates the fraction of memory that cpd_dd1 requires relative to cpd_hnf. Raising it to the power $\frac{3}{2}$ gives an estimate of the speedup factor in execution time of cpd_hnf relative to cpd_dd1. Based on this estimate and the execution times we logged, it is accurate to state that the newly proposed algorithm outperforms the state of the art by up to two orders of magnitude for larger tensors.

6.3. Robustness in the noisy case. The next experiment illustrates that cpd_hnf can successfully decompose tensors even in the presence of noise. The setup is as follows. We generate a random rank-$r$ tensor $A$ of size $150 \times 25 \times 10$ by randomly sampling the $\alpha_i, \beta_i$, and $\gamma_i$ as before. Then, we add white Gaussian noise of relative magnitude $10^e$ for $e = -1, \ldots, -15$;
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Figure 1. A comparison of the \( \log_{10} \) of the relative backward error of the proposed \cpd_hnf and the state-of-the-art \cpd_ddl from [20, 21] on random rank-\( r \) tensors in \( \mathbb{R}^{\ell+1} \otimes \mathbb{R}^{m+1} \otimes \mathbb{R}^{n+1} \) with \( \ell \geq m \geq n \). The largest dimension and the rank satisfy \( \ell + 1 = r = \min\{R(m, n, (d, 1)), mn\} \), where the function \( R \) is as in eq. (4.8). The outcomes for \( d = 2, 3, 4 \) are shown respectively in the left, middle, and right plots. The color scale is the same in all plots.

Figure 2. The \( \log_{10} \) of the total execution time (seconds) of \cpd_hnf in the setup from fig. 1.

that is, we compute \( \mathcal{A}' = \mathcal{A} + 10^e \frac{\|\mathcal{A}\|}{\|\mathcal{E}\|} \mathcal{E} \). We provide \( \mathcal{A}' \) as input to our algorithm and request a rank-\( r \) decomposition.

The relative backward error between \( \mathcal{A}' \) and the computed rank-\( r \) decomposition is shown in a logarithmic scale in fig. 4. Because of our setup, the rank-\( r \) CPD of \( \mathcal{A} \) has relative backward error \( 10^e \). A good tensor decomposition algorithm should thus return a CPD with a backward error of at most \( 10^e \). Remarkably, for tensors with random rank-\( r \) CPDs, the proposed algorithm consistently manages to reach this benchmark when \( e \leq -5 \). For ranks up to about half the maximum range (from \( r = 1 \) to 70), it even consistently manages to reach this benchmark for white noise of magnitude at most \( 10^{-2} \). Based on these results, we anticipate that \cpd_hnf could be employed as a rank-\( r \) approximation algorithm in the high signal-to-noise regime. We believe this observation warrants further research.
6.4. An example of higher-order tensors. The last experiment illustrates that the reshaping trick combined with a decomposition algorithm that works well in the unbalanced regime (such as \texttt{cpd hnfh} and \texttt{cpd ddli} \cite{20,21}) is a powerful technique for decomposing higher-order, high-rank tensors, even with a balanced shape.

As an example, we generated an eighth-order real tensor of size $7 \times 7 \times 7 \times 6 \times 6 \times 5 \times 5 \times 6$ and rank $r = 1000$ with factor matrices whose entries are sampled i.i.d. from a standard normal distribution. Here is what happens:

1. Grouping $[[4, 6, 7, 8], [1, 2, 3], [5]]$ and reshaped to $(1050, 343, 6)$ tensor in $0.810859903$ s
2. Performed ST-HOSVD compression to $(1000, 343, 6)$ in $0.44888179$ s
3. Swapped factors 2 and 3, so the tensor has size $(1000, 6, 343)$
4. Selected degree increment $d_0 = [1, 0]$ in $22.333814068$ s
5. Constructed kernel of $A_1$ of size $(1000, 2058)$ in $72.176802896$ s
6. Constructed resultant map of size $(7203, 6348)$ in $1.266772414$ s
7. Constructed res res’ in $108.332858902$ s
8. Diagonalized multiplication matrices in $2.037837294$ s
9. Computed cokernel of size $(1000, 7203)$ in $78.097176017$ s
10. Diagonalized multiplication matrices and extracted solution in $151.170096114$ s
11. Refined factor matrices $Y$ and $Z$ in $0.186951757$ s
12. Recovered factor matrix $X$ in $0.939620275$ s
13. Recovered the full factor matrices in $440.457582263$ s

Relative backward error $= 3.873171296624731 \times 10^{-15}$

Relative forward error $= 7.303893102189592 \times 10^{-14}$

To our knowledge, this computation represents the first time any tensor decomposition algorithm of any type (i.e., alternating least squares, optimization-based methods, direct
algebraic methods, homotopy-based methods, or heuristic methods) successfully decomposes a high-order rank-1000 tensor that cannot be reshaped to an order-3 tensor whose CPD can be computed with a pencil-based algorithm.

7. Conclusions

The cpd_hnf algorithm proposed in this paper computes the CPD of tensors satisfying assumption 1 using numerical linear algebra techniques for solving systems of polynomial equations. Its complexity is governed (propositions 1.1 and 5.1) by the regularity of a homogeneous, \( N \)-graded ideal obtained from a flattening. We have given a formula for degrees \((d,1)\) in the regularity for generic tensors of many formats (theorem 4.2) and proposed conjecture 1 for the general case. Our experiments show that cpd_hnf produces backward errors that are almost always of the order of the machine precision. This improves upon the previous state of the art by several orders of magnitude (see fig. 1). In the high signal-to-noise-ratio regime, it seems the algorithm can be used to approximate noisy rank-\( r \) tensors.

Possible directions for future research include a further analysis of the regularity, a relaxation of the conditions in assumption 1, generalizations for (semi-)symmetric tensors, and a theoretical analysis of cpd_hnf in the noisy setting.

Acknowledgements. We are grateful to Alessandra Bernardi for fruitful discussions. We thank Ignat Domanov and Lieven de Lathauwer for kindly sharing a Matlab implementation of cpd_ddl1.

Appendix A. Proofs of the technical results

Proof of lemma 2.2. Let \( I = (f_1, \ldots, f_s) \) with \( f_i \in S_{(1,1)} \). To prove that all points in \( V_X(I) = \{ \zeta_1, \ldots, \zeta_r \} \) have multiplicity one, consider the Jacobian matrix

\[
\mathcal{J} = \left( \begin{array}{cccc}
\frac{\partial f_j}{\partial x_0} & \cdots & \frac{\partial f_j}{\partial x_m} \\
\frac{\partial f_j}{\partial y_0} & \cdots & \frac{\partial f_j}{\partial y_n}
\end{array} \right)_{1 \leq j \leq s}.
\]

The point \( \zeta_i \) has multiplicity one as a point of \( V_X(I) \) if and only if, when evaluated at \( \zeta_i \), \( \mathcal{J} \) has rank \( m + n \). Note that for \( v \in \mathbb{C}^{m+1} \), \( w \in \mathbb{C}^{n+1} \), \( \mathcal{J}(v, w) = (f_j(v, y) + f_j(x, w))_{1 \leq j \leq s} \).

Setting \((x, y) = (\beta_i, \gamma_i)\) we obtain the map

\[
\mathcal{J}_i : \mathbb{C}^{m+1} \times \mathbb{C}^{n+1} \to \mathbb{C}^s,
\]

given by \( \mathcal{J}_i(v, w) = A_1(v) + A_2(w), \)

where \( A_1 : \mathbb{C}^{m+1} \to \mathbb{C}^s \) sends \( v \mapsto (f_j(v, \gamma_i))_{1 \leq j \leq s} \), and analogously \( A_2(w) = (f_j(\beta_i, w))_{1 \leq j \leq s} \).

We need to show that \( \mathcal{J}_i \) has rank \( m + n \). Since

\[
\text{rank}(\mathcal{J}_i) = \dim_{\mathbb{C}}(\text{im} \mathcal{J}_i) = \dim_{\mathbb{C}}(\text{im} A_1 + \text{im} A_2)
= \text{rank}(A_1) + \text{rank}(A_2) - \text{dim}_{\mathbb{C}}(\text{im} A_1 \cap \text{im} A_2),
\]

it suffices to show that \( A_1 \) has rank \( m + n \), \( A_2 \) has rank \( n \) and \( \text{im} A_1 \cap \text{im} A_2 = \{0\} \).

To see that \( \text{rank}(A_1) = m \), note that \( v \in \ker A_1 \) if and only if \((v, \gamma_i) \in V_X(I) \). It is clear that \( \beta_i \in \ker A_1 \). Suppose \( v = \beta_i \in \ker A_1 \) for some \( i \neq i' \). Then

\[
(\beta_i \otimes \gamma_i)^\top(f_j) = (\beta_i' \otimes \gamma_i)^\top(f_j) = 0, \quad j = 1, \ldots, s.
\]

But by bilinearity of the tensor product this would imply that \((a\beta_i + b\beta_i') \otimes \gamma_i)^\top(f_j) = 0 \) for all \( a, b \in \mathbb{C} \) and for \( j = 1, \ldots, s \), which contradicts that \( V_X(I) \) consists of finitely many points, as we have previously shown. Therefore, \( \ker A_1 = \text{span}_{\mathbb{C}}(\beta_i) \) has dimension 1 and \( \text{rank}(A_1) = m \). Analogously, \( \text{rank}(A_2) = n \).

The next step is to show that \( \text{im} A_1 \cap \text{im} A_2 = \{0\} \). If \( A_1(v) = A_2(w) \), we need to show that \( v \in \ker A_1 \) and \( w \in \ker A_2 \). We have

\[
(v \otimes \gamma_i)^\top(f_j) - (\beta_i \otimes w)^\top(f_j) = 0, \quad j = 1, \ldots, s.
\]

\[\text{This can only make sense if rank}(A_1) + \text{rank}(A_2) = m + n \leq s. \text{ Note that, having established that } V_X(I) \text{ consists of finitely many points, we indeed have this inequality.}\]
As an immediate consequence, there are unique coefficients $a_1, \ldots, a_r \in \mathbb{C}$ such that
\begin{equation}
(v \otimes \gamma_i)^\top - (\beta_i \otimes w)^\top = a_1(\beta_1 \otimes \gamma_1)^\top + \cdots + a_r(\beta_r \otimes \gamma_r)^\top.
\end{equation}
If $a_1 = \cdots = a_r = 0$, then we must have $v \in \text{span}_\mathbb{C}(\beta_i) = \ker A_1$ and $w \in \text{span}_\mathbb{C}(\gamma_i) = \ker A_2$.
If the $a_i$ are not all zero, we can assume that $v$ and $w$ are scaled such that $|a_1| + \cdots + |a_r| = 1$.
From (A.3) we obtain
\begin{equation}
(v \otimes \gamma_i)^\top = \sum_{k \neq i} a_k(\beta_k \otimes \gamma_k)^\top + a_i(\beta_i \otimes (\gamma_i + w))^\top.
\end{equation}
Applying both sides to $f_j$ and using (A.2), we find that
\begin{equation}
(v \otimes \gamma_i)^\top (f_j) = a_i(\beta_i \otimes (\gamma_i + w))^\top (f_j) = (\beta_i \otimes w)^\top (f_j), \quad j = 1, \ldots, s.
\end{equation}
If $a_i = 1$, then $a_k = 0$ for $k \neq i$ (by our fixed scaling), and (A.4) implies $v \in \text{span}_\mathbb{C}(\beta_i) = \ker A_1$ and $w \in \text{span}_\mathbb{C}(\gamma_i) = \ker A_2$.
If $a_i \neq 1$, then by (A.5) we have
\begin{equation*}
(\beta_i \otimes (a_i \gamma_i + (a_i - 1)w))^\top (f_j) = 0, \quad j = 1, \ldots, s,
\end{equation*}
such that $(a_i \gamma_i + (a_i - 1)w) \in \ker A_2 = \text{span}_\mathbb{C}(\gamma_i)$, and hence $w \in \text{span}_\mathbb{C}(\gamma_i) = \ker A_2$.
By (A.2), we also have $v \in \ker A_1$.
Hence, $A_1(v) = A_2(w)$ if and only if $v \in \ker A_1$ and $w \in \ker A_2$, implying $\text{im} A_1 \cap \text{im} A_2 = \{0\}$.

For two ideals $I, K \subset S$, we write $(I : K^\infty) = \{f \in S \mid K^k f \subset I \text{ for some } k \in \mathbb{N}\}$. The next result follows from lemma 2.2.

**Corollary A.1.** Under assumption 1, we have the identity $(I : K^\infty) = J$, where $I = \langle \ker A(\gamma) \rangle$, $J$ is the vanishing ideal
\begin{equation*}
J = \langle f \in S \mid f \text{ is homogeneous and } f(\zeta_i) = 0, i = 1, \ldots, r \rangle,
\end{equation*}
and $K = \langle x_iy_j \mid 0 \leq i \leq m, 0 \leq j \leq n \rangle$ is the irrelevant ideal of $S$.

**Proof.** By Lemma 2.2, the modules $S/I$ and $S/J$ define the same subscheme of $X$ consisting of $r$ simple points.
It follows that $K^k(J/I) = 0$ for large enough $k$. See for instance [16, Proposition 5.3.10].

**Proof of lemma 3.1.** This follows from the fact that all $h \in S(d', e')$ satisfying $h(\zeta_i) = 0$ lie on a hyperplane $H_i$ through the origin in the $\mathbb{C}$-vector space $S(d', e')$. Any $h_0$ in the complement of $H_1 \cup \cdots \cup H_r$ satisfies $h_0(\zeta_i) \neq 0, i = 1, \ldots, r$.

**Proof of lemma 3.2.** Let $J = (I : K^\infty)$ be the saturation of $I$ with respect to the irrelevant ideal $K$ (see Corollary A.1).
First, it follows from [34, Propositions 4.4 and 6.7] that there exists $(\delta_1, \epsilon_1) \in \mathbb{N}^2$ such that $\text{HF}_{S/J}(\delta_1 + \delta', \epsilon_1 + \epsilon') = r$ for all $(\delta', \epsilon') \in \mathbb{N}^2$.
Secondly, we show that there exist $(\delta_2, \epsilon_2) \in \mathbb{N}^2$ such that $I_{(\delta_2 + \delta', \epsilon_2 + \epsilon')} = J_{(\delta_2 + \delta', \epsilon_2 + \epsilon')}$ for all $(\delta', \epsilon') \in \mathbb{N}^n$.
To see this, note that $K^kJ \subset I$ for some $k \in \mathbb{N}$ (see e.g. [15, Ch. 4, §4, Proposition 9]) and $K^k = \langle S(k,k) \rangle$.
For a finite set of homogeneous generators $g_1, \ldots, g_m$ of $J$, choose $(\delta_2, \epsilon_2)$ such that entry-wise, $(\delta_2, \epsilon_2)$ is at least $(k, k + \deg(g_i))$, $i = 1, \ldots, m$. Take $(d, e) \in (\max(\delta_1, \delta_2), \max(\epsilon_1, \epsilon_2)) + \mathbb{N}^2$ such that $(d, e) \neq (1, 1)$ and $(d - 1, e - 1) \geq (0, 0)$.

**Proof of theorem 3.2.** Under the assumptions of the theorem, $((0, 1), (1, 0))$ is a regularity pair for the ideal $J = (I : K^\infty)$, defining the same set of points $V_X(J) = V_X(I)$ (see Corollary A.1).
The statement now follows from [39, Theorem 5.5.3, Propositions 5.5.4 and 5.5.5].

**Proof of item (3) in theorem 4.2.** We prove the case $m = 2$ in item (3) of the theorem.
It suffices to show that for some configuration $Z$, the matrix
\begin{equation*}
A' = \begin{bmatrix}
-\Gamma H_0 \\
\Gamma H_2 & -\Gamma H_1
\end{bmatrix}
\end{equation*}
has full rank. We set $\beta_0 = 1, i = 1, \ldots, r$, such that $H_0$ is the identity matrix. The condition (4.7) ensures that $A'$ has more rows than columns, so it suffices to show that $\ker A' = \{0\}$. Suppose $A'v = 0$, then $v$ can be split into $v_1, v_2 \in \mathbb{C}^r$ such that $\Gamma v_1 = \Gamma v_2 = 0$. If $r \leq n + 1$, it is clear that this implies $v_1 = v_2 = 0$ for generic $Z$. Therefore, we assume $r > n + 1$ and make the following choice for $\Gamma$:

$$
\Gamma = \begin{bmatrix}
1 & \gamma_{n+2,0} & \cdots & \gamma_{r0} \\
1 & \gamma_{n+2,1} & \cdots & \gamma_{r1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & \gamma_{n+2,n} & \cdots & \gamma_{rn}
\end{bmatrix} = \left[\text{id}_{n+1} \, \hat{\Gamma}\right] \in \mathbb{C}^{(n+1) \times r},
$$

where $\hat{\Gamma} \in \mathbb{C}^{(n+1) \times \kappa}$ is the submatrix of $\Gamma$ consisting of its last $\kappa = r - (n + 1)$ columns. We have that $\Gamma v_i = 0$ implies $v_i = \Gamma^{-1} w_i$ for some $w_i \in \mathbb{C}^\kappa$ and $\Gamma^{-1} = \left[\text{id}_n \, \hat{\Gamma}\right]$. Hence $Av = 0$ is equivalent to $[\Gamma H_2 \Gamma^{-1} - \Gamma H_1 \Gamma^{-1}] [v_1, v_2] = 0$. The condition (4.7) implies $2\kappa \leq n + 1$, so that the coefficient matrix in this equation has more rows than columns. Upon closer inspection, we see that

$$
\Gamma H_1 \Gamma^{-1} = \begin{bmatrix}
(\beta_{n+2,i} - \beta_{1i}) \gamma_{n+2,0} & \cdots & (\beta_{ri} - \beta_{1i}) \gamma_{r0} \\
(\beta_{n+2,i} - \beta_{1i}) \gamma_{n+2,1} & \cdots & (\beta_{ri} - \beta_{1i}) \gamma_{r1} \\
\vdots & \vdots & \vdots \\
(\beta_{n+2,i} - \beta_{1i}) \gamma_{n+2,\kappa} & \cdots & (\beta_{ri} - \beta_{1i}) \gamma_{r\kappa}
\end{bmatrix}.
$$

In order to make rows $\kappa + 1, \ldots, 2\kappa$ equal to zero in $\Gamma H_2 \Gamma^{-1}$, we set $\beta_{\kappa+1,2} = \beta_{\kappa+2,2} = \cdots = \beta_{2\kappa,2} = \beta_{n+2,2} = \beta_{n+3,2} = \cdots = \beta_{r,2} = 1$. All other $\beta$-coordinates are chosen at random, such that all entries of $[\Gamma H_2 \Gamma^{-1} - \Gamma H_1 \Gamma^{-1}]$, except those in the rows $\kappa + 1, \ldots, 2\kappa$ of $\Gamma H_2 \Gamma^{-1}$, are of the form $\gamma_{ij}$, with $\gamma_{ij}$ some non-zero complex number. Then,

$$
[\Gamma H_2 \Gamma^{-1} - \Gamma H_1 \Gamma^{-1}] = \begin{bmatrix}
C_{11} & C_{12} \\
0 & C_{22}
\end{bmatrix},
$$

where $C_{ij} \in \mathbb{C}^{\kappa \times \kappa}$ and $D_i \in \mathbb{C}^{(n+1-2\kappa) \times \kappa}$. The minor corresponding to the first $2\kappa$ rows is $\det(C_{11}) \det(C_{22})$. This is a product of two nonzero polynomials in the parameters $\gamma_{ij}$, $i = n + 2, \ldots, r, j = 0, \ldots, n$. For generic choices of the parameters, this minor is non-zero, hence $w_1 = w_2 = 0$, and thus $v = 0$ and $\ker A' = \{0\}$. \hfill \Box

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