Anomalous scaling of passive scalar in turbulence and in equilibrium

Gregory Falkovich and Alexander Fouxon

Physics of Complex Systems, Weizmann Institute of Science, Rehovot 76100 Israel

Abstract

We analyze multi-point correlation functions of a tracer in an incompressible flow at scales far exceeding the scale $L$ at which fluctuations are generated (quasi-equilibrium domain) and compare them with the correlation functions at scales smaller than $L$ (turbulence domain). We demonstrate that the scale invariance can be broken in the equilibrium domain and trace this breakdown to the statistical integrals of motion (zero modes) as has been done before for turbulence. Employing Kraichnan model of short-correlated velocity we identify the new type of zero modes, which break scale invariance and determine an anomalously slow decay of correlations at large scales.
When the scale $L$ of an external source of fluctuations far exceeds the scale at which fluctuations are dissipated, turbulent cascade appears between those scales. One of the most interesting fundamental aspects of turbulence is the existence of anomalies: symmetries remain broken even when symmetry-breaking factors tend to zero. For example, time-reversibility and scale invariance of the statistics are not restored even when pumping scale goes to infinity and dissipation scale to zero. The mechanism of dissipative anomaly (responsible for irreversibility [1]) has been identified by Onsager as due to non-smoothness of the velocity field in the inviscid limit [2]. It is very much similar to the axial anomaly in quantum field theory [3]. Breakdown of scale invariance has been identified recently as related to the statistical integrals of motion; in particular, such integrals have been found as zero modes of the (multi-point) operator of turbulent diffusion in the Kraichnan model of short-correlated velocity [4, 5, 6, 7, 8].

Here we consider passive scalar in a random incompressible flow and ask whether the symmetries are restored at the scales far exceeding the pumping scale. It is straightforward to see that time reversibility holds (to put it simply, everything which is pumped is dissipated at smaller scales and there is no flux towards larger scales) [8]. At the level of the second moment or spectral density of the scalar, an equipartition takes place at wave-numbers smaller than $L^{-1}$ like in other systems with a direct cascade [8, 9]. Here we find four-point and higher moments at large scales and discover that the scale invariance may be broken. We employ the Kraichnan model and demonstrate that the breakdown is also due to statistical integrals of motion (even though very much different from the zero modes breaking scale invariance in turbulence). We thus find a possible link between an anomalous scaling in equilibrium systems (e.g. in critical phenomena) and in turbulence.

Consider the passive scalar $\theta(r, t)$ carried by the velocity $v(r, t)$ and pumped by $\phi(r, t)$:

$$\partial_t \theta + (v \cdot \nabla)\theta = \phi.$$  \hspace{1cm} (1)

The characteristics of (1) are called Lagrangian trajectories and defined by $\partial_t R(r, t) = v(R, t)$ and $R(r, 0) = r$. Integrating (1) by characteristics we get $\theta(r, 0) = \theta(R(r, -t), -t) + \int_{-t}^{0} \phi(R(r, t_1), t_1) dt_1$. Correlation functions, $F_n = \langle \theta_1 \ldots \theta_n \rangle$, are obtained by averaging over velocity and pumping denoted by brackets and over-bar respectively:

$$F_n = \langle \theta(R_1(-t), -t) \ldots \theta(R_n(-t), -t) \rangle + \int_{-t}^{0} dt_1 \ldots \int_{-t}^{0} dt_n \langle \phi(R_1(t_1), t_1) \ldots \phi(R_n(t_n), t_n) \rangle. \hspace{1cm} (2)$$
Here $\mathbf{R}_i(t) \equiv \mathbf{R}(\mathbf{r}_i, t)$ and $\theta_i = \theta(\mathbf{r}_i, 0)$. We assume that the pumping is white Gaussian with a zero mean and the variance $\overline{\phi(\mathbf{r}_1, t_1)\phi(\mathbf{r}_2, t_2)} = \chi(\mathbf{r}_{12})\delta(t_2 - t_1)$, $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$.

One can express the correlation functions via the multi-particle propagators. For example, assuming zero conditions at the distant past and space homogeneity, one gets

$$F_2(\mathbf{r}) = \int_0^0 dt \int P(\mathbf{R}, \mathbf{r}, t) \chi(\mathbf{R})d\mathbf{R},$$

where $P(\mathbf{R}, \mathbf{r}, t)$ is the probability density function (pdf) of $\mathbf{R}_{12}(t)$ provided $\mathbf{R}_{12}(0) = \mathbf{r}$, $\mathbf{R}_{ij}(t) = \mathbf{R}_i(t) - \mathbf{R}_j(t)$.

We use the Kraichnan model [8, 10] where velocity is Gaussian with the zero mean and the variance

$$\langle v_\alpha(\mathbf{r}_1, t)v_\beta(\mathbf{r}_2, 0) \rangle = \delta(t)\left[K_0\delta_{\alpha\beta} - K_{\alpha\beta}(\mathbf{r}_{12})\right],$$

$$K_{\alpha\beta}(\mathbf{r}) = \frac{Dr^{2-\gamma}}{d - 1} \left[(d + 1 - \gamma)\delta_{\alpha\beta} - (2 - \gamma)\frac{\mathbf{r}_\alpha\mathbf{r}_\beta}{r^2}\right].$$

Translation-invariant propagators satisfy closed differential equations in the Kraichnan model [8, 10]:

$$(\partial_t + \mathcal{L}_n)P(\mathbf{R}, \mathbf{r}, t) = 0, \quad P(\mathbf{R}, \mathbf{r}, 0) = \delta(\mathbf{R} - \mathbf{r}),$$

where $\mathbf{R} = (\mathbf{R}_1 \ldots \mathbf{R}_n)$ and $2\mathcal{L}_n \equiv \sum K_{\alpha\beta}(\mathbf{r}_{ij})\nabla_{\alpha\alpha}\nabla_{\beta\beta}$. In particular, the pdf of the two-particle separation $[P(\mathbf{R}, \mathbf{r}, t)$ integrated over angles] is expressed via the modified Bessel function ($\eta_0 = d/\gamma - 1$)

$$P_0(\mathbf{R}, \mathbf{r}, t) = \frac{(Rr)^{-\eta_0/2}}{D\gamma|t|} \exp\left[-\frac{R^\gamma + r^\gamma}{\gamma^2|t|D}\right]I_{\eta_0}\left(2\frac{(Rr)^{\gamma/2}}{\gamma^2|t|D}\right),$$

giving the Richardson law of separation $R^\gamma \sim D|t|$ [10].

We consider $\gamma > 0$ (the case $\gamma = 0$ has been described in [11]), which corresponds to Holder exponent of the velocity smaller than unity [8]. In this case, the characteristics are non-unique which is seen, for instance, from $P(\mathbf{R}, 0, t) \neq \delta(\mathbf{R})$. Respective loss of information leads to scalar dissipation, which balances pumping by $\phi$ and provides for a statistical steady state of the scalar [8]. Spatially non-smooth velocities are produced by fluid turbulence so that all scales considered in this paper are assumed to be less than the outer scale of turbulence that is the correlation scale of velocity. The equation (5) implies corresponding equation on $F_n$:

$$\mathcal{L}_n F_n = \sum_{i > j, k \neq i} \chi(\mathbf{r}_{ij}) F_{n-2}(\{\mathbf{r}_{kj}\}).$$
The simplest steady state described by (6) is the thermal equilibrium with Gaussian probability density functional \( \propto \exp \left[-\int \theta^2 dr/(2T)\right] \). Such state has \( F_2(r) = T\delta(r) \) which requires also zero correlation length \( L \) of \( \phi \):

\[
TK_{\alpha\beta}(r) \nabla_\alpha \nabla_\beta \delta(r) = -\chi(r). \tag{7}
\]

That is if the limit \( L \to 0 \) is taken in such a way that \( F_2 \to T\delta(r) \), the scalar statistics becomes Gaussian and scale-invariant. The Gaussian anzatz, \( F^G_{2n} = \sum F_2(r_{ij}) F^G_{2n-2}/n \) solves (6),

\[
\sum \nabla_\alpha^i \nabla_\beta^k [K_{\alpha\beta}(r_{ik}) - K_{\alpha\beta}(r_{jk})] F_2(r_{ij}) F^G_{2n-2} = 0,
\]

in two cases: i) \( F_2(r) = T\delta(r) \) (in the compressible case as well) and ii) \( \gamma = 2 \) where \( K_{\alpha\beta}(r) \) is \( r \)-independent.

Of course, delta-function is an idealization so let us open a Pandora box of anomalies by allowing \( \phi \) to have a finite correlation scale \( L \). We assume that at \( r \gg L \) the function \( \chi(r) \) decays faster than any power (say, exponentially). We consider velocity field correlated at different points (i.e. \( \gamma < 2 \)). Then, the scalar statistics is no longer Gaussian and very different at the scales smaller and larger than \( L \).

Let us first remind the properties of the turbulent state realized at \( r < L \). A nonzero flux \( \chi(0) \) makes the pair correlation function non-smooth at zero: \( F(0) - F(r) \propto r^\gamma \chi(0) \).

The scaling properties are characterized by the structure functions \( S_n(r) = \langle (\theta(r) - \theta(0))^n \rangle \) which are given by the zero modes of the operator \( \mathcal{L}_n \) [5, 6, 12, 13]. Zero modes are functions \( Z\{R_{ij}(t)\} \) of particles coordinates conserved on average [8, 14]. The structure functions are determined by the so-called irreducible zero modes (that involve coordinates of all the points), their exponents being \( d\ln S_n/d\ln r = n\gamma/2 - \Delta_n \) with the anomalous exponents given by \( \Delta_n = n(n-2)(2-\gamma)/2(d+2) \) in the perturbative domain, \( \Delta_n \ll n\gamma \).

One can express all the zero modes via the distance, \( R = \sqrt{R_{12}^2 + \ldots + R_{1n}^2} \), and the angles, \( e_i = R_{1i}/R \), in \( d(n-1) \)-dimensional space: \( Z\{R_{ij}\} = R^\sigma f(\hat{e}) \). The conservation, \( \mathcal{L}Z = 0 \), in a Lagrangian language means that the growth of the radial factor is compensated by the decay of the angular function. The zero modes of the hermitian scale-invariant operator \( \mathcal{L}_n \) come in pairs satisfying the duality relation: for every \( Z_n = R^{\sigma n} f_n(\hat{e}) \) there exists \( Z'_n = R^{\gamma-d(n-1)-\sigma n} \tilde{f}_n(\hat{e}) \) [14]. There is the difference between the modes with positive \( \sigma \)-s (that contribute at small scales) and with negative \( \sigma \)-s (expected at large scales): \( \mathcal{L}_n Z'_n \) produces delta-function at the origin or its derivatives (contact terms) rather than zero. A physical
reason is a nonzero probability of initially distant particles to come to the same point. For example, $\mathcal{L}_2 R^{\gamma-d} \propto \delta(R)$ at $\gamma = 2$.

After discovering that the zero modes $Z_n$ enter the solution of (6) at small scales, it is straightforward to ask what zero modes contribute at large scales where $\hat{L}_n F_n = 0$. Natural candidates to contribute $F_n$ at $r > L$ are $Z'_n$ which decay with the distances. And indeed this is true for $F_2$ for which the following representation holds

$$F_2 = \frac{q_0 r^{\gamma-d}}{D(d-\gamma)} + \int_r^\infty \frac{\chi(y)y^{d-1}(r^{\gamma-d} - y^{\gamma-d})dy}{D(d-\gamma)}.$$

Provided $q_0 = \int \chi(r) dr \neq 0$, zero mode $r^{\gamma-d}$ dual to the constant dominates $F_2$ at $r > L$. Such duality between $r > L$ and $r < L$ does not work for higher correlation functions. Considering, for instance, $F_4$ one observes that the source appearing in (6) is a linear combination of functions independent of one of three $r_{ij}$ on which $F_4$ depends. Since the action of $\mathcal{L}_n$ on $Z'_n$ produces contact terms vanishing where any $r_{ij}$ is non-zero, then $Z'_n$ do not satisfy the correct boundary conditions at $r \sim L$ and thus cannot contribute $F_n$ at large scales. One can show that $Z'_n$ occur in the solution only if irreducible terms are present in the pumping correlation functions. That means that a Gaussian pumping provides for $Z'_n$ only at small scales. Here we show that the reducible terms in the pumping correlation functions spawn a new type of zero modes at large scales, which correspond to an axial (rather than spherical) symmetry in $R$-space and to the different duality relation, $\sigma'_n = \gamma - nd/2 - \sigma_k$.

Nonzero $q_0$ means infinite $\int F_2(r) dr$ i.e an overall heating. We also call $q_0$ charge to invoke an analogy with electrostatics (literal at $\gamma = 2$ when $\mathcal{L}$ is a Laplacian). When $q_0 \neq 0$, $F_2(r)$ decays by a power law at $r \gg L$ and high-order functions obey normal scaling though statistics is non-Gaussian. We consider now a pumping with $q_0 = 0$, like in (7), when $F_2(r)$ decays faster than any power. On a particle language, the distance $R(t)$ in (3) explores the sphere $R < L$ in a completely isotropic manner with the result being proportional to $\int \chi(r) dr$. Such a symmetry cancellation makes pair-correlation small at large distances. We shall show now that mutual correlation between different pairs of distances makes for the power-law decays of multi-point correlation functions (interpreted as dipole and quadrupole contributions).

The identity (2) allows us to obtain two important conclusions when all $r_{ij} \gg L$. Consider first $t \ll [\min r_{ij}]^\gamma/D$, then the last term can be neglected since the correlation function of $\phi$ is exponentially small. Averaging over velocity at times earlier than $-t$ of which $R_i(-t)$
are independent by zero correlation time of $v$, one derives that the correlation functions at large scales are statistical invariants of Lagrangian dynamics:

$$F_n(r_1, \ldots, r_n) \approx F_n(R_1(-t), \ldots, R_n(-t))$$

(8)

This is equivalent to the above statement that $F_n$ is a zero mode of $\mathcal{L}_n$ at large distances. On the other hand, the steady state values of the correlation functions can be obtained by taking in the limit $t \to \infty$ when the first term vanishes while the second one saturates. For Gaussian statistics of $\phi$ this can only occur due to the rare events where pairs of particles traced backwards in time come within the distance $L$ from each other. The distance between the pairs remains large with overwhelming probability. Now one can choose intermediate times such that still holds but $R_i(-t)$ are already such that the inter-pair distances is much smaller than the distance between two pairs. This suggests that the main features of, say, $F_4$ can be captured by considering the special geometry of two distant pairs. In particular, this geometry will determine the scaling of a coarse-grained field, see below. The overall scaling $\sigma_n$ of $F_n$ can also be inferred from this particular case.

We consider $F_4(x, y, z)$, where $x = r_{12}$, $y = r_{34}$ and $z = (r_{13} + r_{24})/2$ at $x, y \ll z$. It is expressed via the pdf $P(X, Y, Z, x, y, z, t)$ of $X(t) = R_{12}(t)$, $Y(t) = R_{34}(t)$ and $Z(t) = (R_{13}(t) + R_{24}(t))/2$ conditioned at $X, Y, Z(t = 0) = x, y, z$:

$$F_4 \approx \int_0^\infty dt \int P(X, Y, Z, x, y, z, t)H(X, Y)dXdYdZ.$$  

(9)

Here $H(X, Y) \equiv \chi(X)F_2(Y) + \chi(Y)F_2(X)$ decays where $max[X, Y] > L$ because $q_0 = 0$. The integral is determined by $t \approx [max\{x, y, L\}]^\gamma/D \ll z^\gamma/D$, that is the distance $Z$ does not vary much. For such times, one can assume $X \ll Z, Y \ll Z$ so that $X$ and $Y$ evolve approximately independently. The lowest order approximation of the propagator is via two-particle pdfs,

$$\int dZP(X, Y, Z, x, y, z, t) \approx P(X, x, t)P(Y, y, t) \equiv P_0,$$

and it produces the reducible part of $F_4$ that is $F_2(x)f_2(y)$, which is exponentially small if $\max\{x, y\} > L$. The next-order corrections decay as powers of $x/z$ and $y/z$ and thus dominate the correlation function if $x > L$ or $y > L$. One can find that correction writing in $\mathcal{L}_4 = \hat{L}_0 + \hat{L}_1$, $\hat{L}_0 = K_{\alpha\beta}(X)\nabla x_\alpha \nabla x_\beta + K_{\alpha\beta}(Y)\nabla y_\alpha \nabla y_\beta$, see Appendix A. In the first order in $\hat{L}_1$ one derives

$$P_1 = \int_0^t dt'dx'dy'dz'P(X, x', t')P(Y, y', t')\delta(z - z')\hat{L}_1 P(x', x, t)P(y', y, t)\delta(z' - z).$$

6
We now plug that into $F_1$, use $\chi = \mathcal{L}_2 F_2$, integrate by parts and obtain the first-order term in $(x/z)^\gamma$, $(y/z)^\gamma$:

$$F_4^1 = \int_0^\infty dt dx' dy' x'_\alpha y'_\beta F_2'(y') P(x', x, t') P(y', y, t') [x'_\alpha y'_\beta \partial^2 K_{\alpha\beta}(z)/\partial z_\gamma \partial z_\delta]$$

The integration over the directions of $x'$ and $y'$ gives $P_2$ which is $P(R, r, t)$ integrated over angles with the second angular harmonics. It was found in $[\text{6}]$ that $P_2$ has the same form as $P_0$ with $\eta_0$ replaced by $\eta_2 = \gamma^{-1}\sqrt{(d-\gamma)^2 + 8d(d+1-\gamma)/(d-1)}$. Time integration also can be done explicitly and we obtain (see Appendix $A$ for the details of calculations)

$$F_4^1 = (xy)^{\gamma/4-d/2} 2\pi \gamma D \hat{x}_\alpha \hat{y}_\beta \partial^2 K_{\alpha\beta}(z) \int_0^{\infty} dx' F_2'(x') \int_0^{\infty} dy' F_2'(y') (x'y')^{\gamma/4+d/2} Q_{\eta_2-1/2}(w),$$

where $Q_{\eta_2-1/2}$ is the Legendre function of the second kind and $8w = (xyx'y')^{-\gamma/2}[(x^\gamma + y^\gamma + x'^\gamma + y'^\gamma)^2 - 4(xx')^\gamma - 4(yy')^\gamma]$. The function $F_4^1$ approximates $F_4 - F_2(x) F_2(y)$ in the whole region $z \gg L$, $x, y \ll z$ irrespectively of $L$. At $x, y \gg L$ the correlation function given by a zero mode of $\mathcal{L}_4$ reduces to the zero mode of $\hat{L}_0$:

$$F_4^1 \propto (xy)^{\delta/2} \partial^2 K_{\alpha\beta}(z) \hat{x}_\alpha \hat{y}_\beta \partial^2 K_{\alpha\beta}(z),$$

where $2\delta = \gamma \eta_2 + \gamma - d$ and $q_2 \equiv \int x^{\gamma+\delta} \chi(x) dx$. We thus have shown that, despite the fast decay of $F_2$ at large scales, $F_4$ decays as a power-law there and found the scaling exponent: $\sigma_4 = 2d + 2\delta$. The dual zero mode of $\hat{L}_0$ with the scaling exponent $\sigma'_4 = \gamma - 2d - \sigma_4$ (first found in $[\text{6}]$) appears at $x, y \ll L$:

$$Z^1 \propto (xy)^{\delta/2} \hat{x}_\alpha \hat{y}_\beta \partial^2 K_{\alpha\beta}(z).$$

One can easily verify for any vector $v$ that $x^{\delta-2}[d(x \cdot v)^2 - x^2 v^2]$ is an anisotropic zero mode of the Lagrangian evolution operator in the lowest order in the distance $x$ between two particles. The above derivation makes explicit the physical origin of the anomalous scaling thus found: The main contribution into $F_4$ at large scales comes from Lagrangian trajectories that were in the past separated into two distant pairs. If the other pair would not be there at all, the average pair prehistory would be completely isotropic that is determined by $q_0 = \int \chi(x) dx$ which vanishes. However, the presence of the other pair leads to a preferred direction in space allowing each pair to exploit the quadrupole term $q_2 = \int \chi(x) x^{\gamma+\delta} dx$. Since the correlations between the pairs decay only as a power law, the quadrupole contribution
dominates the correlation function. The possibility for more than two particles to explore non-isotropic configurations (which show themselves via non-isotropic zero mode) exists for higher correlation functions as well. To give an example of power-law correlations in high moments, let us derive the explicit expression for $n$—point correlation function in the limit of small $\xi = 2 - \gamma$. Besides reproducing the correct scaling it shows how non-trivial the angular structure of the correlation functions is at large scales. Since the statistics of $\theta$ is Gaussian at $\xi = 0$ we analyze the cumulant $\Gamma_{2n}^1 = \lim_{\xi \to 0} \Gamma_{2n}/\xi$ (see Appendix B):

$$\sum_k \nabla^2 \Gamma_{2n}^1 = F(n) \sum_{\{i_j\}} \nabla^\alpha \nabla^\beta \left[ J_{\alpha\beta}(r_{i_1i_3}) + J_{\alpha\beta}(r_{i_2i_4}) - J_{\alpha\beta}(r_{i_2i_3}) - J_{\alpha\beta}(r_{i_1i_4}) \right] \prod_{k=1}^n F_2 \left( \{r_{i_2k-1i_2k}\}, \xi = 0 \right),$$

(12)

where the sum is over all permutations of indices, $F(n)$ is an $n$—dependent constant and

$$J_{\alpha\beta}(r) = D \ln r \delta_{\alpha\beta} - r \alpha r \beta r^{-2} D/(d - 1).$$

The solution at $r_{ij} \gg L$ is as follows:

$$\Gamma_{2n}^1 \approx (-1)^{n+1} \frac{4 \Gamma(nd/2 + 4)F(n)q_1^n d^{n-2}}{(\pi)^{nd/2}D^{n-1}(d - 1)(d + 2)^3(2d)^n} \times \sum_{\{i_j\}} \frac{f(r_{i_1}, r_{i_2}, \ldots r_{i_n})}{c_{\{i_j\}}^2},$$

(14)

where $q_1 = \int x^\gamma \chi(x) \, dx$ and a homogeneous function of zero degree $f(r_{i_1}, \ldots r_{i_2n})$ is expressed via the hypergeometric functions denoted $F_{ijk} = F(nd/2 + 4; i, j + 1, i, d/2 + 2 + k, u)$ of the argument $u = -2z^2/c^2$ ($c^2 \equiv c^2_{(1,2,\ldots,2n)}$) and the scalar products $a_1 = (r_{ij} \cdot z)/r_{ij} z, a_2 = (r_{ij} \cdot z)/r_{ij} z$:

$$f = \frac{r_{ij}^2 r_{34}^2}{2c^4} \left\{ F_{000} + 8a_2^2 a_3^2 \left[ \frac{u(nd+8)}{d+4} F_{121} - F_{010} + F_{000} \right] - a_1^2 \left[ 2F_{010} + (d - 2)F_{000} \right] + 2 \left( a_2^2 + a_3^2 \right) (F_{010} - F_{000}) - 2a_1a_2a_3 \left[ 4 \frac{u(nd+8)}{d+4} F_{121} + (d - 4)(F_{010} - F_{000}) \right] \right\}.$$

(15)

One can check that in the limit of large $z$ (15) reproduces (11) at $\xi \to 0$ (where $\delta \to 2$). Let us stress that the cumulants $\Gamma_n$ are nonzero only when the balance equation (7) does not hold. If to replace $\delta(r)$ in (7) by a function with a finite $L$ then $q_0$ remains zero but dipole and quadrupole moments, $q_1$ and $q_2$, are generally nonzero and (11) (13) (15) appear. The proof that $\sigma_n = nd/2 + 2\delta$ for finite $\xi$ will be published elsewhere (15).
To study how statistics changes with the scale one defines the field coarse-grained over the scale $r$: $\tilde{\theta}_\ell = \int_{r' < \ell} \theta(r') \, dr' / \ell^{d/2}$. A fast decay of the pair correlation function together with power-law correlations of higher functions, mean that the statistics of $\tilde{\theta}_\ell$ approaches Gaussian anomalously slow as $\ell/L \to \infty$. Indeed, while the second moment $\langle \tilde{\theta}^2_\ell \rangle$ tends to a constant (exponentially) fast, the fourth cumulant may decay slower than $\ell^{-d}$ (that one would have for fields with a finite correlation radius):

$$\langle \langle \tilde{\theta}^4_\ell \rangle \rangle \propto \ell^{-d} \int c^{2d-1} dc \int F_4 z^{d-1} dz \propto (L/\ell)^{2\gamma}. \quad (16)$$

Here, the region $c \simeq L \ll z \simeq \ell$ gives the main contribution if $2\gamma < d$ (see [15] for details and also [18]). Similarly one can derive $\langle \langle \theta^2(0)\theta^2(r) \rangle \rangle \propto r^{-2\gamma}$. Note that $F_4^1$ gives zero contribution and one must account for $F_4^2 \propto z^{-2\gamma}$.

Since $\sigma_n/n$ depends on $n$ then the statistics is not scale-invariant at $r > L$ when $q_0 = 0$. If $q_0 \neq 0$ then the zero modes found here provide for an anomalous scaling of sub-leading corrections, more similar to what is generally observed in critical phenomena. What we believe is of importance here is that we trace this anomalous scaling to zero modes. That shows that at least in passive scalar problem, the mechanism of an anomalous scaling is common for turbulence and thermal equilibrium and raises an intriguing possibility that in other problems in quantum field theory and statistical physics one can relate anomalous scaling to statistical integrals of motion.

Our work benefited from helpful remarks of K. Gawędzki, V. Lebedev and A. Zamolodchikov. We are grateful to the participants of the Eilat workshop (October 2004) for the discussion of the results. After the workshop, the numerical simulations were undertaken [16], which seem to support (16) — we are grateful to A. Celani and A. Seminara for sharing their results prior to publication. This research has been supported by the Minerva grant 8464 and the EU Network "Fluid mechanical stirring and mixing: the lagrangian approach".

[1] K.R. Sreenivasan, Phys. Fluids 27, 1048-1051 (1984)
[2] L. Onsager, Nuovo Cimento, Suppl. 6, 279 (1949).
[3] A. Polyakov, Nuclear Phys. B396, 367–385, (1993).
[4] B. Shraiman and E. Siggia, C. R. Acad. Sci. 321, 279 (1995).
[5] K. Gawędzki and A. Kupiainen, Phys. Rev. Lett. 75, 3834 (1995).
The expression for the four-point correlation function $F_4(r_1, r_2, r_3, r_4)$ follows from (2)

$$F_4 = \int_{-\infty}^{\infty} dt_1 dt_2 \left\langle \left[ \chi[R_{12}(t_1)] \chi[R_{34}(t_2)] + \right. \\
+ \chi[R_{13}(t_1)] \chi[R_{24}(t_2)] + \chi[R_{14}(t_1)] \chi[R_{23}(t_2)] \right\rangle. \quad (A1)$$

Here we consider calculation of $F_4$ for the case where the absolute values of the inter-pair distances $x = r_1 - r_2$ and $y = r_3 - r_4$ are much smaller than the absolute value of the distance $z = [r_1 + r_2 - r_3 - r_4]/2$ between the centers of mass of the pairs. We shall also assume that $z \gg L$. It is easy to verify that the first summand dominates (A1) in this case. Indeed, the other two terms are contributed by the events on which $z$ shrinks to the distance of order $L$. These events have smaller probability than similar events involving $x$ and $y$ justifying the above statement. Finally, using zero-correlation time and time-reversibility of Kraichnan model and employing $H(x, y) = \chi(x)F_2(y) + \chi(y)F_2(x)$ introduced in the main text we obtain

$$F_4 \approx \int_{0}^{\infty} dt \langle H(R_{12}(t), R_{34}(t)) \rangle = \int_{0}^{\infty} dt H(X, Y)$$
\[ \times P(X, Y, Z, x, y, z, t) dX dY dZ, \]  

where we introduced the joint probability density function \( P(X, Y, Z, x, y, z, t) \) of \( X(t) = R_1(t) - R_2(t), Y(t) = R_3(t) - R_4(t) \) and \( Z = [R_1(t) + R_2(t) - R_3(t) - R_4(t)]/2. \) In Kraichnan model this function satisfies a closed evolution equation

\[
\partial_t P = [\hat{L}_0 + \hat{L}_1] P, \quad P(t = 0) = \delta(X - x) \delta(Y - y) \\
\delta(Z - z), \quad \hat{L}_0 = K_{\alpha\beta}(X) \frac{\partial^2}{\partial X_{\alpha} \partial X_{\beta}} + K_{\alpha\beta}(Y) \frac{\partial^2}{\partial Y_{\alpha} \partial Y_{\beta}}, \\
\hat{L}_1 = \frac{1}{4} \left[ K_{\alpha\beta} \left( Z + \frac{X - Y}{2} \right) + K_{\alpha\beta} \left( Z + \frac{X + Y}{2} \right) \\
+ K_{\alpha\beta} \left( Z - \frac{X + Y}{2} \right) + K_{\alpha\beta} \left( Z + \frac{Y - X}{2} \right) \\
- K_{\alpha\beta}(Y) - K_{\alpha\beta}(X) \right] \frac{\partial^2}{\partial Z_{\alpha} \partial Z_{\beta}} + \frac{1}{2} \left[ K_{\alpha\beta} \left( Z + \frac{X + Y}{2} \right) \\
+ K_{\alpha\beta} \left( Z - \frac{X + Y}{2} \right) - K_{\alpha\beta} \left( Z + \frac{X - Y}{2} \right) \\
- K_{\alpha\beta} \left( Z + \frac{Y - X}{2} \right) \right] \frac{\partial^2}{\partial X_{\alpha} \partial Y_{\beta}} + \frac{1}{2} \left[ K_{\alpha\beta} \left( Z + \frac{X + Y}{2} \right) \\
- K_{\alpha\beta} \left( Z - \frac{X + Y}{2} \right) + K_{\alpha\beta} \left( Z + \frac{X - Y}{2} \right) \\
- K_{\alpha\beta} \left( Z + \frac{Y - X}{2} \right) \right] \frac{\partial^2}{\partial Y_{\alpha} \partial Z_{\beta}}.
\]

Next we observe that the integral (9) is determined by times of the order \([\max x, y, L]^{\gamma}/D\) (we assume \(q_0 = 0\) here so that \(F_2\) is small outside \(L\)). During these times which are much smaller \(z^{\gamma}/D\) the distance \(Z\) does not vary much while \(X\) and \(Y\) remain much smaller than \(Z\). The main dynamics at these times is just independent evolution of ”fast” degrees of freedom \(X, Y\) governed by the operator \(\hat{L}_0\). Considering \(\hat{L}_1\) as a perturbation we construct \(P\) as a series

\[
P = \sum_{n=0}^{\infty} P_n, \quad \frac{\partial P_n}{\partial t} = \hat{L}_0 P_n + \hat{L}_1 P_{n-1}, \quad (A3)
\]
where \( P_0 \) is given by \( P_0(X, x, t)P_0(Y, y, t)\delta(Z - z) \). Here \( P_0(X, x, t) \) satisfies the initial condition \( P_0(t = 0) = \delta(X - x) \) and the evolution equation
\[
\frac{\partial P_0(X, x, t)}{\partial t} = K_{\alpha\beta}(X) \frac{\partial^2 P_0}{\partial X_\alpha \partial X_\beta}.
\] (A4)

The explicit solution for \( P_n \) with \( n > 0 \) reads
\[
P_n = \int_0^t dt' dx'dy'dz' P_0(X, x', t - t') P_0(Y, y', t - t')
\times \delta(Z - z') \left( \hat{L}_1 \right)_{x', y', z'} P_{n-1}(x', y', z', t')
\]

Above the subscript of Hermitian operator \( \hat{L}_1 \) signifies that it acts on variables \( x', y', z' \). The asymptotic series for \( P \) integrated according to \( \int \) produces asymptotic series \( F_4 = \sum F^n \), where \( F^{n+1}/F^n \) tends to zero as \( z \to \infty \). To write this series most concisely we note that
\[
\int dXdY P_0(X, x', t') P_0(Y, y', t') \left[ \chi(X)F_2(Y) + \chi(Y) \right]
\times F_2(X) = -\int dXdY P_0(X, x', t') P_0(Y, y', t') \left[ F_2(Y) \right]
\times K_{\alpha\beta}(X) \frac{\partial^2 F_2(X)}{\partial X_\alpha \partial X_\beta} + F_2(X) K_{\alpha\beta}(Y) \frac{\partial^2 F_2(Y)}{\partial Y_\alpha \partial Y_\beta}
\]
\[
= -\frac{\partial}{\partial t} \int dXdY P_0(X, x', t') P_0(Y, y', t') F_2(X) F_2(Y),
\]

where we used the equation satisfied by \( F_2 \). Using the above we find the following expression for \( F^n \)
\[
F^n = \int_0^\infty dt \int P_n(X, Y, Z, x, y, z, t) H(X, Y) dXdY dZ
= \int_0^\infty dt dx'dy'dz' F_2(x') F_2(y') \left( \hat{L}_1 \right)_{x', y', z'} P_{n-1}(x', y', z', t)
\] (A5)

Using the expression for \( P_n \) we find the expression for \( F^n \) with \( n > 2 \) in terms of \( F^1 \):
\[
F^n = \int_0^\infty dt dx'dy'dz' F^1 \hat{L}_1 P_{n-2}(x', y', z', t).
\] (A6)

In particular, for \( F^2 \) we obtain
\[
F^2 = \int_0^\infty dt dx'dy'dz' F^1 \hat{L}_1 F^1 P_{n-2}(x', y', z, t).
\]

We pass to study \( F^1 \) which can be written as
\[
F^1 = \int_0^\infty dt dx'dy' P_0(x, x', t') P_0(y, y', t') \tilde{x}_\alpha y_\beta F_2'(x')
\times F_2(y') \left[ K_{\alpha\beta} \left( z - \frac{x' + y'}{2} \right) + K_{\alpha\beta} \left( z - \frac{x' + y'}{2} \right) \right]
\times -K_{\alpha\beta} \left( z + \frac{x' - y'}{2} \right) - K_{\alpha\beta} \left( z + \frac{y' - x'}{2} \right).
\] (A7)
Expanding the difference in brackets at large $z$ we obtain
\[
F^1 = \int_0^\infty \! dt \! dx' \! dy' \! P_0(x, x', t') P_0(y, y', t') \hat{x}_\alpha \hat{y}_\beta \beta F_2(x') \times F'_2(y') \left[ x'_\gamma y'_\delta \frac{\partial^2 K_{\alpha \beta}(z)}{\partial z_\gamma \partial z_\delta} + O \left( z^{\gamma-2} \right) \right]
\]
It is convenient to perform angular integration by substituting the integrand by its value averaged independently over the directions of $x'$ and $y'$. We introduce functions averaged over the directions of $x'$:
\[
\langle \hat{x}_\alpha \hat{x}_\gamma P_0(x, x', t) \rangle_{\text{angle}} = \delta_{\alpha \gamma} A + \hat{x}_\alpha \hat{x}_\gamma B.
\] (A8)

Taking the trace of the above equation and multiplying it with $\hat{x}_\alpha \hat{x}_\gamma$ leads to
\[
\langle P_0(x', x, t) \rangle_{\text{angle}} = dA + B,
\]
\[
\langle (\hat{x}' \cdot \hat{x})^2 P_0(x', x, t) \rangle_{\text{angle}} = A + B.
\] (A9)

Isotropy of velocity statistics implies that the LHS of the above equations are independent of $\hat{x}$ so that $A = A(x, x', t)$, $B = B(x, x', t)$. It follows from the above relations that
\[
(1 - d)B = \langle \left[ 1 - d (\hat{x}' \cdot \hat{x})^2 \right] P_0(x, x', t) \rangle_{\text{angle}}.
\] (A10)

Using incompressibility we can write the answer in terms of the function $B$ solely
\[
F^1 = \hat{x}_\alpha \hat{x}_\gamma y_\beta \delta_{\gamma \delta} \frac{\partial^2 K_{\alpha \beta}(z)}{\partial z_\gamma \partial z_\delta} \int_0^\infty \! dt \! dx' \! dy' \! B(x, x', t)
\]
\[
B(y, y', t) \hat{x}' \hat{y}' \beta F_2(x') \beta F'_2(y').
\] (A11)

The function $B(x, x', t)$ can be found by expanding $P_0(x, x', t)$ in Jacobi polynomials in $x \cdot x'/(xx')$, see [6, 15]. One finds that $B$ is the coefficient near the Jacobi polynomial of the second-order and it is given by [6]
\[
B(r, r', t) = \frac{(rr')^{d/2 + \gamma/2}}{S^{d-1}D(d-1)\gamma t} \exp \left[ -\frac{r^\gamma + r'^\gamma}{\gamma^2(d-1)tD} \right]
\]
\[
I_{\eta_2} \left( \frac{2(rr')^{\gamma/2}}{\gamma^2(d-1)tD} \right), \quad \eta_2 = \sqrt{\left( \frac{d}{\gamma} - 1 \right)^2 + \frac{8d(d-1-\gamma)}{\gamma^2(d-1)}}.
\]

Up to a proportionality factor the above coincides with $R_1$ from [6]. The difference due to the misprint in [6] is discussed in more detail in [15]. The integral over time is given by [17]
\[
\int_0^\infty \! B(x', x, t)B(y', y, t)dt = \frac{(xyx' y')^{\gamma/4-d/2}}{(2\pi)(S^{d-1})^2(d-1)D}
\]
\[
Q_{\eta_2-1/2} \left( \frac{(x^\gamma + y^\gamma + x'^\gamma + y'^\gamma)^2 - 4(xx')^\gamma - 4(yy')^\gamma}{8(xyx' y')^{\gamma/2}} \right).
\]
The above leads to the result (10) from the main text. The analysis of the expression proceeds along the following lines. We first consider $F^1$ where $x \gg L$ and $y \gg L$. In this situation one can assume $x' \gg x$ and $y' \gg y$ and use

$$Q_{\eta_2 - 1/2} \left( \frac{(x^\gamma + y^\gamma + x'^\gamma + y'^\gamma)^2 - 4(xx')^\gamma - 4(yy')^\gamma}{8(xyxy')^{\gamma/2}} \right)$$

$$\approx \frac{\Gamma(\eta_2 + 1/2)\Gamma(1/2)}{2^{\eta_2 + 1/2}\Gamma(\eta_2 + 1)} \left( \frac{8(xyxy')^{\gamma/2}}{(x^\gamma + y^\gamma)^2} \right)^{\eta_2 + 1/2}, \quad (A12)$$

where we used the large argument asymptotic form of the Legendre function of the second kind. Substituting the above into the expression for $F^1$ we find

$$F^1 = \frac{(xy)^{\delta} \hat{x}_\alpha \hat{x}_\beta \hat{y}_\gamma \hat{y}_\delta}{(x^\gamma + y^\gamma)^{(4\delta + 2d - \gamma)/\gamma}} \frac{\partial^2 K_{\alpha \beta}(z)}{\partial z_\gamma \partial z_\delta} \frac{2^{2\eta_2}}{\pi \gamma d \Gamma(\eta_2 + 1)} \left( \int_0^\infty F_2(x') x'^{\delta + d} dx' \right)^2 \frac{\Gamma(\eta_2 + 1/2)\Gamma(1/2)}{\Gamma(\eta_2 + 1)}.$$

In the opposite limit $x \ll L$ and $y \ll L$ one finds zero mode dual to the above. Indeed, in this situation one can use the inverse inequalities $x \ll x'$, $y \ll y'$ and

$$Q_{\eta_2 - 1/2} \left( \frac{(x^\gamma + y^\gamma + x'^\gamma + y'^\gamma)^2 - 4(xx')^\gamma - 4(yy')^\gamma}{8(xyxy')^{\gamma/2}} \right)$$

$$\approx \frac{\Gamma(\eta_2 + 1/2)\Gamma(1/2)}{2^{\eta_2 + 1/2}\Gamma(\eta_2 + 1)} \left( \frac{8(xyxy')^{\gamma/2}}{(x^\gamma + y^\gamma)^2} \right)^{\eta_2 + 1/2}. \quad (A14)$$

One obtains

$$F^1 = \frac{2^{2\eta_2} \Gamma(\eta_2 + 1/2)\Gamma(1/2)}{\pi \gamma d \Gamma(\eta_2 + 1)} \frac{(xy)^{\delta} \hat{x}_\alpha \hat{x}_\beta \hat{y}_\gamma \hat{y}_\delta}{\partial z_\gamma \partial z_\delta} \frac{\partial^2 K_{\alpha \beta}(z)}{\partial z_\gamma \partial z_\delta} \times \int_0^\infty dx' \int_0^\infty dy' F_2(x') F_2(y') \frac{(x'y')^{\delta}}{(x'^\gamma + y'^\gamma)^{(4\delta + 2d - \gamma)/\gamma}}.$$

Note that now the dependencies on $x'$ and $y'$ in the integral cannot be separated.

It is easy to see that $F^1$ vanishes if $x$ or $y$ or both are zero. Indeed, if $x = 0$ then by isotropy $P_0(x', 0, t) = P_0(x', t)$. The angular integration then produces zero by identities like

$$\int_{x = R} \hat{x}_\alpha K_{\alpha \beta} \left( z + \frac{x + y}{2} \right) dS = 0. \quad (A15)$$

The same argument shows that $F^1$ averaged over the directions of $x$ or $y$ or both vanishes as well. As a result such objects as $\langle \langle \theta^2(0) \theta^2(r) \rangle \rangle$ and $\langle \langle \theta^4 \rangle \rangle$ are determined by $F^2$. These are calculated in [15].
We remind that passive scalar statistics is Gaussian at $\xi = 2 - \gamma = 0$. Thus it is convenient to discuss the limit of small $\xi$ in terms of irreducible correlation functions that are proportional to $\xi$ in this limit.

The analysis proceeds as follows. We first derive equations satisfied by the irreducible correlation functions $\Gamma_n$ in the limit of small $\xi$. We show that $\Gamma_n$ solves the potential problem (electrostatics) in the space of $nd$ dimensions with the source concentrated within a generalized cylinder. After the integration over the directions parallel to the cylinder axis we express $\Gamma_n$ as an integral of a kernel with localized source. The large-scale asymptotic expansion of $\Gamma_n$ is then derived exactly in the same way as the multi-pole expansion in electrostatics with the coefficients given by two quadrupole and $n/2 - 2$ dipole moments of the pumping correlation function. We proceed to the calculation.

1. Equations on $\Gamma_n$ in the limit of small $\xi$

We stressed above that the statistics of $\theta$ is nearly Gaussian at small $\xi$ so that $\Gamma_n$ are a small correction to the reducible part of the correlation function which is expressible via $F_2$. Thus the source appearing in the equation on the leading order term in $\Gamma_n$ should be expressible in terms of $F_2$. It is simplest to show this for $\Gamma_4 = F_4 - F_4^G = F_4 - F_2(r_{12})F_2(r_{34}) - F_2(r_{13})F_2(r_{24}) - F_2(r_{14})F_2(r_{23})$. Using the equation on $F_4$ it is easy to show that

$$-\sum_{i>j} K_{\alpha\beta}(r_i - r_j) \nabla_i \nabla_j \Gamma_4 = \frac{1}{4} \sum_{ijkl} \left[ K_{\alpha\beta}(r_i - r_j) - K_{\alpha\beta}(r_k - r_j) \right] \nabla_i \nabla_j F_2(r_i - r_k) F_2(r_j - r_l),$$

(B1)

where the sum over $ijkl$ signifies the sum over all 24 permutations of 1234. The boundary conditions on $\Gamma_4$ are vanishing with the difference of any of its arguments $|r_i - r_j|$ tending to infinity. Equations on the irreducible correlation functions of order higher than 4 involve lower order correlation functions. Let us consider the equation on $\Gamma_6$ defined by

$$F_6 = F_6^G + \sum_{i>j} F_2(r_i - r_j) \Gamma_4(\{r_k \neq i,j\}) + \Gamma_6.$$

(B2)
Using the equation on $\Gamma_4$ one finds for $\Gamma_6$

$$L_6 \Gamma_6 = \frac{1}{3} \sum_{i>j,k\neq i,j} [K_{i\beta}(r_i - r_k) - K_{i\beta}(r_j - r_k)] \nabla_{i\alpha}$$

$$\nabla_{k\beta} F_2(r_i - r_j) F_4^G(\{r_{k\neq i,j}\}) + \sum_{i>j,k\neq i,j} [K_{i\beta}(r_i - r_k)$$

$$- K_{i\beta}(r_j - r_k)] \nabla_{i\alpha} \nabla_{k\beta} F_2(r_i - r_j) \Gamma_4(\{r_{k\neq i,j}\}). \quad (B3)$$

One observes that the term that involves $\Gamma_4$ is of the second order in non-Gaussianity that is at small $\xi$ it vanishes as $\xi^2$. Thus this term is negligible to linear order in $\xi$ and in this order pair-correlation function fully determines the source in the equation. This holds for $\Gamma_n$ with $n > 6$ as well. Indeed, to linear order in $\xi$ one can neglect all products of $\Gamma-$s that enter the definition of $\Gamma_n$ (say, in the definition of $\Gamma_8$ one can neglect $\Gamma_4 \times \Gamma_4$ term). To linear order in $\xi$ the definition of $\Gamma_n$ reads

$$F_n = F_n^G + \Gamma_n + \sum_{k=2}^{n/2-1} \sum_{\{i_j\}} \frac{\Gamma_{2k}(\{r_{i_1,2\ldots,2k}\})}{(2k)!(n-2k)!} \times F_{n-2k}^G(\{r_{2k+1\ldots,n}\}). \quad (B4)$$

where $\sum_{\{i_j\}}$ runs over all permutations of $1, 2, \ldots, 2n$ and the factor $[(2k)!(2n-2k)!]^{-1}$ lifts multiple counting of terms. Using the equation on $F_{2n}^G$ it is easy to see that the equation on $\Gamma_{2n}$ to linear order in $\xi$ coincides with

$$L_{2n} \Gamma_{2n} + \sum_{i>j,k\neq i,j} \nabla_{i\alpha} \nabla_{k\beta} \left[ K_{i\beta}(r_i - r_k)$$

$$- K_{i\beta}(r_j - r_k) \right] F_2(r_i - r_j) F_{2n-2}^G(\{r_{k\neq i,j}\})$$

$$+ \sum_{k=2}^{n-1} \sum_{\{i_j\}} \frac{1}{(2k)!(2n-2k)!} L_{2k} \Gamma_{2k}(\{r_{i_1,2\ldots,2k}\})$$

$$F_{2n-2k}^G(\{r_{2k+1\ldots,2n}\}) = 0. \quad (B5)$$

Thus the equation satisfied by $\Gamma_{2n}$ can be written as follows

$$L_{2n} \Gamma_{2n} + C(n) \sum_{\{i_j\}} \nabla_{i\alpha} \nabla_{i\beta} \left[ K_{i\beta}(r_{i_1} - r_{i_3})$$

$$- K_{i\beta}(r_{i_2} - r_{i_3}) \right] \prod_{k=1}^{n} F_2(r_{i_{2k-1}} - r_{i_{2k}}) = 0.$$ 

The value of the constant $C(n)$ is not essential for our purposes. To derive the equation in the first order in $\xi$ term $\Gamma_{2n}^1 \equiv \lim_{\xi \to 0} \Gamma_{2n}/\xi$ we substitute $L_{2n}$ by its value at $\xi = 0$. Using
the fact that \( \sum_{i>j} \nabla_i \nabla_j = -\sum \nabla_i^2 / 2 \) when acting on translation-invariant functions we find

\[
\sum_n \nabla_2^2 \Gamma_{2n}^1 = F(n) \sum_{\{i,j\}} \nabla_{i\alpha} \nabla_{j\beta} \left[ J_{\alpha\beta}(r_i - r_3) 
\right.
\]

\[
+J_{\alpha\beta}(r_{i2} - r_{i4}) - J_{\alpha\beta}(r_{i2} - r_{i3}) - J_{\alpha\beta}(r_{i1} - r_{i4}) \n\]

\[
\times \prod_{k=1}^n H \left( r_{i2k-1} - r_{i2k} \right), \quad (B6)
\]

where \( F(n) \) is an \( n \)-dependent constant, \( H = F_2(\xi = 0) \) and \( J_{\alpha\beta} = \partial_\xi K_{\alpha\beta}(r, \xi)|_{\xi=0} \). Since constant part of \( J_{\alpha\beta} \) cancels in \( (B6) \) we will use the equivalent expression

\[
J_{\alpha\beta}(r) = D(d - 1) \ln r\delta_{\alpha\beta} - D \frac{r_\alpha r_\beta}{r^2}. \quad (B7)
\]

2. Solution of the equation on \( \Gamma_{2n}^1 \) and its large-scale asymptotic expansion

The solution can be expressed with the help of an auxiliary tensor \( T_{\alpha\beta} \) defined by

\[
\sum_{n=1}^{2n} \nabla_2^2 T_{\alpha\beta}(r_1, r_2, \ldots, r_{2n}, x_1, \ldots, x_k) = J_{\alpha\beta}(r_1 - r_3) 
\]

\[
\times \prod_{k=1}^n \delta(r_{2k-1} - r_{2k} - x_k). \quad (B8)
\]

One has

\[
\Gamma_{2n}^1 = F(n) \sum_{\{i,j\}} \nabla_{i\alpha} \nabla_{j\beta} \int \prod_{k=1}^n H(x_k) \prod_{k=1}^n dx_k 
\]

\[
\left[ T_{\alpha\beta}(r_{i1}, r_{i2}, r_{i3}, r_{i4}, r_{i5}, \ldots, r_{i2n}, x_1, x_2, x_3, \ldots, x_k) 
\right.
\]

\[
+T_{\alpha\beta}(r_{i2}, r_{i1}, r_{i3}, r_{i4}, x_1, x_2, x_3, \ldots, x_k) 
\]

\[
-T_{\alpha\beta}(r_{i2}, r_{i1}, r_{i3}, x_1, x_2, x_3, \ldots, x_k) 
\]

\[
-T_{\alpha\beta}(r_{i1}, r_{i2}, r_{i3}, r_{i4}, x_1, x_2, x_3, \ldots, x_k) \n\]

Using

\[
\nabla^2 \frac{\Gamma(d/2 - 1)}{4\pi^{d/2}|r - r'|^{d-2}} = -\delta(r - r'). \quad (B9)
\]
we find

\[-T_{\alpha\beta} = \int \frac{J_{\alpha\beta}(\mathbf{r}'_1 - \mathbf{r}'_3) \prod_{k=1}^{n} \delta (\mathbf{r}'_{2k-1} - \mathbf{r}'_{2k} - \mathbf{x}_k)}{\left[ \sum_{i=1}^{2n} (\mathbf{r}_i - \mathbf{r}'_i)^2 \right]^{(n-d)/2}} \times \frac{(nd - 2)!}{4\pi^{nd}} \prod_{i=1}^{2n} d\mathbf{r}'_i = \frac{\Gamma((n+2)d/2 - 1)}{2^{2 + (n-2)d/2} \pi^{(n+2)d/2}} \int \prod_{i=1}^{4} d\mathbf{r}'_i \]

\[
\frac{J_{\alpha\beta}(\mathbf{r}'_1 - \mathbf{r}'_3) \delta (\mathbf{r}'_1 - \mathbf{r}'_2 - \mathbf{x}_1) \delta (\mathbf{r}'_3 - \mathbf{r}'_4 - \mathbf{x}_2)}{\left[ \sum_{i=1}^{4} (\mathbf{r}_i - \mathbf{r}'_i)^2 + \sum_{k=3}^{n} (\mathbf{r}_{2k-1} - \mathbf{r}_{2k} - \mathbf{x}_k)^2 \right]^{(n+2)d/2 - 1}}.
\]

where we used

\[
\frac{\Gamma(n/2 - 1)}{4\pi^{n/2}} \int \prod_{i=1}^{m} d\mathbf{x}'_i \frac{\prod_{i=1}^{m} d\mathbf{x}_i}{\left[ \sum_{i=1}^{m} (\mathbf{x}_i - \mathbf{x}'_i)^2 \right]^{n/2 - 1}} = \frac{\Gamma((n - m)/2 - 1)}{4\pi^{(n-m)/2} \left[ \sum_{i=m+1}^{n} (\mathbf{x}_i - \mathbf{x}'_i)^2 \right]^{(n-m)/2 - 1}},
\]

which is readily proved as a relation between Green’s functions of a laplacian in different dimensions. Introducing auxiliary vectors \( \mathbf{z}_1 = \mathbf{r}_1, \mathbf{z}_2 = \mathbf{r}_2 + \mathbf{x}_1, \mathbf{z}_3 = \mathbf{r}_3 - \mathbf{r}, \mathbf{z}_4 = \mathbf{r}_4 + \mathbf{x}_2 - \mathbf{r} \), where \( \mathbf{r} = \mathbf{r}'_1 - \mathbf{r}'_3 \) we have

\[-T_{\alpha\beta} = \frac{\Gamma((n + 2)d/2 - 1)}{2^{2 + (n-2)d/2} \pi^{(n+2)d/2}} \int d\mathbf{r} J_{\alpha\beta}(\mathbf{r}) \int d\mathbf{r}'_1 \]

\[
\left[ \left( 2\mathbf{r}'_1 - \sum \mathbf{z}_i/2 \right)^2 + \sum \mathbf{z}_i^2 - \left( \sum \mathbf{z}_i \right)^2 / 4
\]

\[
+ \sum_{k=3}^{n} (\mathbf{r}_{2k-1} - \mathbf{r}_{2k} - \mathbf{x}_k)^2 \right]^{1-(n+2)d/2}.
\]

The integral over \( \mathbf{r}'_1 \) is easily calculated and we find

\[-T_{\alpha\beta} = \frac{\Gamma((n + 1)d/2 - 1)}{2^{3 - d/2} \pi^{(n+1)d/2}} \int d\mathbf{r} J_{\alpha\beta}(\mathbf{r}) \times \left[ (\mathbf{r} - \mathbf{b})^2 + c^2 \right]^{1-(n+1)d/2},
\]

where we introduced \( \mathbf{b} = \mathbf{r}_3 + \mathbf{r}_4 + \mathbf{x}_2 - \mathbf{r}_1 - \mathbf{r}_2 - \mathbf{x}_1 \) and \( c = \sum_{k=1}^{n} (\mathbf{r}_{2k-1} - \mathbf{r}_{2k} - \mathbf{x}_k)^2 \). It is convenient to introduce

\[
F_{\alpha\beta}(\mathbf{b}, c^2, \alpha) = \frac{\Gamma(a)}{\left[ (\mathbf{r} - \mathbf{b})^2 + c^2 \right]^\alpha} \int d\mathbf{r} \frac{J_{\alpha\beta}(\mathbf{r})}{\left[ \mathbf{r}^2 + c^2 \right]^\alpha}
\]

\[-\Gamma(a) \int d\mathbf{r} \frac{J_{\alpha\beta}(\mathbf{r})}{\left[ \mathbf{r}^2 + c^2 \right]^\alpha}.
\]

(B12)
In terms of this tensor the answer takes the form
\[
\Gamma_{2n}^1 = \sum_{\{ij\}} \frac{F(n)}{2^{d/2} \pi^{(n+1)d/2}} \int \prod_{k=1}^{n} H(x_k) dx_k \\
\times \left[ F_{\alpha\beta} \left( \frac{z_{i_1,i_2,i_3,i_4} + x_1 + x_2}{2}, c_{\{ij\}}^2, \frac{(n + 1)d}{2} - 1 \right) \\
+ F_{\alpha\beta} \left( \frac{z_{i_1,i_2,i_3,i_4} - x_1 - x_2}{2}, c_{\{ij\}}^2, \frac{(n + 1)d}{2} - 1 \right) \\
- F_{\alpha\beta} \left( \frac{z_{i_1,i_2,i_3,i_4} + x_1 - x_2}{2}, c_{\{ij\}}^2, \frac{(n + 1)d}{2} - 1 \right) \\
- F_{\alpha\beta} \left( \frac{z_{i_1,i_2,i_3,i_4} - x_1 + x_2}{2}, c_{\{ij\}}^2, \frac{(n + 1)d}{2} - 1 \right) \right],
\]
where we introduced \( z_{i_1,i_2,i_3,i_4} \equiv r_{i_1} + r_{i_2} - r_{i_3} - r_{i_4} \) and \( 2c_{\{ij\}}^2 \equiv \sum_{k=1}^{n} (r_{i_{2k-1}} - r_{i_{2k}} - x_k)^2 \).

Using
\[
\frac{\partial F_{\alpha\beta}(b, c^2, a)}{\partial c^2} = -F_{\alpha\beta}(b, c^2, a + 1), \quad \frac{\partial F_{\alpha\beta}}{\partial b_\beta} = 0
\]
we may rewrite the answer as
\[
\Gamma_{2n}^1 = \frac{F(n)}{2^{d/2} \pi^{(n+1)d/2}} \sum_{\{ij\}} \int \prod_{k=1}^{n} H(x_k) dx_k \\
\times \left( r_{i_1} - r_{i_2} - x_1 \right)_\alpha \left( r_{i_3} - r_{i_4} - x_2 \right)_\beta \left[ F_{\alpha\beta} \left( \frac{z_{i_1,i_2,i_3,i_4} + x_1 + x_2}{2}, c_{\{ij\}}^2, \frac{(n + 1)d}{2} + 1 \right) \\
+ F_{\alpha\beta} \left( \frac{z_{i_1,i_2,i_3,i_4} - x_1 - x_2}{2}, c_{\{ij\}}^2, \frac{(n + 1)d}{2} + 1 \right) \\
- F_{\alpha\beta} \left( \frac{z_{i_1,i_2,i_3,i_4} + x_1 - x_2}{2}, c_{\{ij\}}^2, \frac{(n + 1)d}{2} + 1 \right) \\
- F_{\alpha\beta} \left( \frac{z_{i_1,i_2,i_3,i_4} - x_1 + x_2}{2}, c_{\{ij\}}^2, \frac{(n + 1)d}{2} + 1 \right) \right].
\]

Let us now derive the asymptotic form of \( \Gamma_{2n}^1 \) at large distances for the case \( q_0 = 0 \). In this case \( H(x) \) decays fast at \( |x| > L \) and the expansion is derived in exactly the same way as the multi-pole expansion of the potential in electrostatics. We find
\[
\Gamma_{2n}^1 \approx \frac{F(n)}{2^{d/2} \pi^{(n+1)d/2}} \left( \int H(x) dx \right)^{n-2} \\
\sum_{\{ij\}} (r_{i_1} - r_{i_2})_\alpha (r_{i_1} - r_{i_2})_\gamma (r_{i_3} - r_{i_4})_\beta (r_{i_3} - r_{i_4})_\delta \\times \frac{\partial^2 F_{\alpha\beta}}{\partial b_\gamma \partial b_\delta} \left( \frac{z_{i_1,i_2,i_3,i_4}}{2}, c_{\{ij\}}^2, \frac{(n + 1)d}{2} + 3 \right).
\]
note that
\[ \int H(\mathbf{x})x^k d\mathbf{x} = -[(k + 2)(k + d)D]^{-1} \int \chi(\mathbf{x})x^{k+2} d\mathbf{x}. \]

To complete the calculation we pass to study \( F_{\alpha\beta} \).

3. Computation of \( F_{\alpha\beta} \)

We now concentrate on \( F_{\alpha\beta} \) defined by Eq. (B12). By isotropy and incompressibility \( F_{\alpha\beta} \) satisfies

\[
F_{\alpha\beta}(\mathbf{b}) = A(b)\delta_{\alpha\beta} + B(b)\hat{\mathbf{b}}_{\alpha}\hat{\mathbf{b}}_{\beta},
\]

\[
\frac{\partial F_{\alpha\beta}}{\partial b_\beta} = 0 = A' + B' + \frac{B(d - 1)}{b}. \tag{B13}
\]

We consider the part of \( F_{\alpha\beta} \) that enters \( \Gamma_{1/2n} \). Differentiating the above equation we find that for any two vectors \( \mathbf{v}, \mathbf{w} \) we have

\[
v_\alpha v_\gamma w_\beta w_\delta \frac{\partial^2 F_{\alpha\beta}(\mathbf{b}, c^2, a)}{\partial b_\gamma \partial b_\delta} = \frac{Bv^2w^2}{b^2} - (\mathbf{v} \cdot \mathbf{w})^2
\]

\[
\times \left( \frac{B'}{b} + \frac{(d - 2)B}{b^2} \right) + \left( \frac{B'}{b^2} - \frac{2B}{b^4} \right) \left[ v^2 (\mathbf{w} \cdot \mathbf{b})^2
\right.
\]

\[
+ w^2 (\mathbf{v} \cdot \mathbf{b})^2 \right]
\]

\[
\left. \times \left( \frac{B'}{b} \right)' - \frac{4B'}{b} + \frac{8B}{b^2} \right) \frac{(\mathbf{v} \cdot \mathbf{b})^2}{b^4}
\]

\[
\times \left( \frac{\mathbf{v} \cdot \mathbf{w}}{b^2} \right) (\mathbf{v} \cdot \mathbf{b}) (\mathbf{w} \cdot \mathbf{b}) \right]. \tag{B14}
\]

Thus our purpose is to find \( B \). Taking the trace we find the additional equation that connects \( A \) and \( B \):

\[
dA + B = Dd\Gamma(a) \left[ I(b) - I(b = 0) \right],
\]

\[
I(b) \equiv \int d\mathbf{r} \frac{\ln r}{[(r - b)^2 + c^2]^{\sigma}}.
\]

To perform angular integration conveniently we note that \( I(b) \) is a convolution of two spherically symmetric functions and its Fourier image is a product of spherically symmetric functions. To use Fourier transform techniques with logarithmic function we write \( \ln r = -\partial r^{-\xi}/\partial \xi (\xi = 0) \). Using

\[
\int d\mathbf{b} \exp[i\mathbf{b} \cdot \mathbf{q}] = \frac{(2\pi)^{d/2}(cq)^{d/2-a} K_{d/2-a}(cq)}{2^{2a-1}\Gamma(a)q^{d-2a}},
\]

\[
20
\]
\[
\int d\mathbf{b} e^{\mathbf{b} \cdot \mathbf{q} - \xi} = \frac{(2\pi)^{d/2} 2^{d/2 - \xi} \Gamma((d - \xi)/2)}{q^{d-\xi} \Gamma(\xi/2)},
\]

where \( K_\nu \) is a modified Bessel function of order \( \nu = a - d/2 \), we find

\[
I(b) = \left( -\frac{\partial}{\partial \xi} \right)_{\xi=0} (2\pi)^{d/2} c^{-\nu} \Gamma((d - \xi)/2) \frac{b^{d/2 - 1} 2^{\nu + \xi - 1} \Gamma(\xi/2) \Gamma(a)}{\Gamma(\nu + \xi/2) \Gamma((d - \xi)/2) \pi^{d/2}}
\]

\[
\int_0^\infty q^{a+\xi-d} J_{d/2-1}(qb) K_\nu(cq) dq = \left( -\frac{\partial}{\partial \xi} \right)_{\xi=0} \Gamma(\nu + \xi/2) \Gamma((d - \xi)/2) \pi^{d/2} c^{-\nu} \Gamma((d/2) \Gamma(a) - F(\nu + \xi/2, \xi, d/2, -b^2/c^2),
\]

where \( J_{d/2-1} \) is Bessel function of order \( d/2 - 1 \) and \( F \) is a hypergeometric function. Using \( I(b) \) and \( I(0) \) we find

\[
dA + B = \frac{Dd\Gamma(\nu) \pi^{d/2}}{2c^{2\nu}} \left( -\frac{\partial}{\partial \xi} \right)_{\xi=0} F(\nu, \xi, d/2, -b^2/c^2),
\]

where we used properties of hypergeometric function. One easily finds that \( B \) satisfies

\[
\frac{1}{b^4} \frac{\partial b^d B}{\partial b} = \frac{Dd\Gamma(\nu) \pi^{d/2}}{2c^{2\nu}} \left( \frac{\partial}{\partial b} \right) \left( \frac{\partial}{\partial \xi} \right)_{\xi=0} F(\nu, \xi, d/2, -b^2/c^2), \quad (B15)
\]

The result of the differentiation over \( \xi \) can be written as a series using

\[
\left( \frac{\partial(\xi)k}{\partial \xi} \right)_{\xi=0} = (k - 1)!,
\]

so that

\[
\left( \frac{\partial}{\partial \xi} \right)_{\xi=0} F(\nu, \xi, d/2, z) = \sum_{k=1}^\infty \frac{(\nu)k^k}{k(d/2)_k}. \quad (B16)
\]

We find for \( B \)

\[
B = \frac{Dd\Gamma(\nu) \pi^{d/2}}{2c^{2\nu}(d - 1)} \sum_{k=1}^\infty \frac{(\nu)k}{(d/2)_k(d/2 + k)} \left( \frac{-b^2}{c^2} \right)^k
\]

\[
= -\frac{2D\Gamma(\nu + 1) \pi^{d/2} b^2}{(d - 1)(d + 2)c^{2\nu + 2}} F(\nu + 1, 1, d/2 + 2, -b^2/c^2).
\]

Next we find

\[
\frac{B'}{b} = -\frac{4D\Gamma(\nu + 1) \pi^{d/2}}{(d - 1)(d + 2)c^{2\nu + 2}} F(\nu + 1, 2, d/2 + 2, -b^2/c^2).
\]
It follows using
\[ \frac{\partial F(\alpha, \beta, \gamma, z)}{\partial z} = \frac{\alpha \beta}{\gamma} F(\alpha + 1, \beta + 1, \gamma + 1, z) \]
that
\[ b \left( \frac{B'}{b} \right)' = \frac{32 D \Gamma(\nu + 2) \pi^{d/2}}{(d - 1)(d + 2)(d + 4) c^{2\nu + 2}} \left( \frac{b^2}{c^2} \right) \]
\[ \times F \left( \nu + 2, 3, \frac{d}{2} + 3, -\frac{b^2}{c^2} \right). \]  
(B17)

Combining the results of the last two subsections one finds the answer for $\Gamma_{2n}^2$ given in the main text.