Finite-time Singularities in Swampland-related Dark Energy Models

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In this work we shall investigate the singularity structure of the phase space corresponding to an exponential quintessence dark energy model recently related to swampland models. The dynamical system corresponding to the cosmological system is an autonomous polynomial dynamical system, and by using a mathematical theorem we shall investigate whether finite-time singularities can occur in the dynamical system variables. As we demonstrate, the solutions of the dynamical system are non-singular for all cosmic times and this result is general, meaning that the initial conditions corresponding to the regular solutions, belong to a general set of initial conditions and not to a limited set of initial conditions. As we explain, a dynamical system singularity is not directly related to a physical finite-time singularity. Then, by assuming that the Hubble rate with functional form $H(t) = f_1(t) + f_2(t)(t - t_s)^{\alpha}$, is a solution of the dynamical system, we investigate the implications of the absence of finite-time singularities in the dynamical system variables. As we demonstrate, Big Rip and a Type IV singularities can always occur if $\alpha < -1$ and $\alpha > 2$ respectively. However, Type II and Type III singularities cannot occur in the cosmological system, if the Hubble rate we quoted is considered a solution of the cosmological system.

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I. INTRODUCTION

The consistent explanation of the late-time acceleration observed in the late 90’s [1], dubbed dark energy, is a theoretical challenge to modern cosmologists. Up to date, there are various theoretical contexts that can explain this dark energy era, for example modified gravity [2–5], quintessence models [8–18] and so on, and the challenge is to find a consistent with the observational data description. Recently, the quintessence models of dark energy were considerably constrained by the string theory swampland criteria, firstly derived in Ref. [19, 20]. Many works appeared after the papers [19, 20], addressing the swampland criteria in the context of quintessence models of dark energy, see for example [21–26]. Basically, the swampland criteria constrain the field theories by ruling out those which are incompatible with quantum gravity. The implications of the swampland criteria on scalar models of dark energy are also very serious, see for example [27] for a recent account on these works. Due to the importance of the quintessence models for the dark energy explanation, in this paper we shall study the singularity structure of the phase space of exponential quintessence models. The latter serve as a characteristic class of quintessence models, and have been seriously constrained by the swampland criteria [23]. The singularity structure of the phase space can provide insights for the occurrence of physical singularities in the quintessence theory, as we shall demonstrate. Indeed, a singularity in a phase space variable does not necessarily imply that an actual physical singularity occurs, see [22] for an example of this sort in the context of $f(R)$ gravity. In this work, we shall adopt the autonomous dynamical system approach [27–31], and by using the dominant balances technique [31], we shall reveal the finite-time singularity structure of the phase space. For a recent review on dynamical systems applications in cosmology see [32]. Accordingly, we shall investigate the implications of our findings on the singularity structure of the physical theory. As we shall demonstrate based on mathematical criteria, the phase space of the exponential quintessence models is free of finite-time singularities, so in view of this result, we question the possibility of having physical finite-time singularities. By following the classification of Ref. [33], our results indicate that Type II and Type III physical finite-time singularities cannot occur in exponential quintessence models, however Big Rip and Type IV singularities can occur. With our approach we provide hints that Big Rip and Type IV are formally allowed to occur, for a certain class of Hubble rate functional forms, however we provide no actual proof that these types of singularities do actually occur. This indicates that string cosmology may not be free of future singularities like other quintessential dark energy, if eventually a specific class of cosmological evolutions is a solution to the cosmological equations. However we need to note that in power-law potentials, geodesically complete future singularities might actually occur [34].

This paper is organized as follows: In section II we briefly review the essential features of quintessence models and we present the dynamical system corresponding to exponential quintessence models. In section III we investigate the
singularity structure of the phase space of exponential quintessence models and we discuss the implications of our results on the physical singularity structure of the quintessence models under study. Finally, the conclusions follow in the end of the paper.

Before we start, we discuss in brief the geometric conventions which we shall use in this work. Particularly, we shall assume that the metric is a flat Friedmann-Robertson-Walker (FRW) of the form,

$$ds^2 = -dt^2 + a(t)^2 \sum_{i=1,2,3} (dx^i)^2,$$

where $a(t)$ is the scale factor of the Universe. In addition, we shall use a physical units system in which $\hbar = c = 1$.

II. ESSENTIAL FEATURES OF THE EXPONENTIAL QUINTESSENCE DARK ENERGY MODELS AND THEIR DYNAMICAL SYSTEM

The action of general scalar field quintessence models in vacuum for a general metric $g_{\mu\nu}$ has the following form,

$$S = \int d^4x \sqrt{-g} \left( \frac{1}{\kappa^2} R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right),$$

where $\kappa^2 = \frac{1}{M_p}$ and $M_p$ is the Planck mass scale. The equations of motion for this scalar theory in the FRW background of Eq. (1) read,

$$\ddot{\phi} + 3H \dot{\phi} + V'(\phi) = 0,$$

$$\frac{3H^2}{\kappa^2} = \frac{\dot{\phi}^2}{2} + V(\phi),$$

where the prime denotes differentiation with respect to the scalar field $\phi$. We shall focus on exponential quintessence models, so the scalar potential $V(\phi)$ has the form $V(\phi) = e^{-\lambda \phi M_p}$, in which case the equations of motion (3) can be cast in the form of an autonomous dynamical system. This can be achieved by using the following variables,

$$x = \frac{\dot{\phi}}{\sqrt{6} H M_p}, \quad y = \frac{\sqrt{V}}{\sqrt{3} H M_p},$$

where the “dot” indicates differentiation with respect to the cosmic-time $t$. By using the e-foldings number $N$ as a dynamical variable, we can construct an autonomous dynamical system by combining Eqs. (3) and (4) which has the following form,

$$\frac{dx_1}{dN} = -3x + \frac{\sqrt{6}}{2} \lambda y^2 + \frac{3}{2} x (x^2 - y^2),$$

$$\frac{dx_2}{dN} = -\frac{\sqrt{6}}{2} \lambda xy + \frac{3}{2} y (x^2 - y^2).$$

Also due to the second equation in Eq. (3), the variables $x$ and $y$ satisfy the Friedmann constraint,

$$x^2 + y^2 = 1,$$

which must be respected even at a finite-time singularity. This is a well-known feature in dynamical system analysis, see for example Ref. [27], where the singular solutions satisfy the Friedmann constraint due to the fact that the singularities cancel. The above dynamical system is an autonomous dynamical system of polynomial form. We shall not focus on the phase space structure of this dynamical system, since the fixed points and their stability were studied in Ref. [22], but we mainly focus on the singularity structure of this dynamical system. This is the subject of the next section.

III. SINGULARITY OCCURRENCE IN SWAMPLAND DARK ENERGY MODELS: DOMINANT BALANCE ANALYSIS AND PHYSICAL INTERPRETATION

The occurrence or not of finite-time singularities in polynomial autonomous dynamical systems was studied some time ago in Ref. [31]. The authors of Ref. [31] developed an asymptotic analysis technique which we shall call
dominant balance analysis hereafter, which can determine whether general singular or non-singular solutions exist in the dynamical system. This analysis was applied in cosmological systems in Ref. [35], see also Refs. [27, 29] for some recent applications. In order to maintain the article self-contained and coherent we shall briefly review this dominant balance technique, and we directly apply this for the autonomous dynamical system of Eq. (6).

Consider the following dynamical system,

\[ \dot{x} = f(x), \]  

(7)

where the \( n \)-dimensional vector \( x = (x_1, x_2, ..., x_n) \), and \( f(x) \) is a \( n \)-dimensional vector of the form \( f(x) = (f_1(x), f_2(x), ..., f_n(x)) \). We assume that the functions \( f_i(x) \) are polynomials of the dynamical system variables \( (x_1, x_2, ..., x_n) \). At a finite-time singularity \( t = t_c \), some variables of the dynamical system will have the form \((t - t_c)^{-p}\) with \( p \) being some positive number (or equivalently in terms of \( N, (N - N_c)^{-q} \), with \( q \neq p \) in general). Having the above assumptions in mind, the method of the dominant balance analysis has the following steps:

- Determine all the possible truncations of the vector function \( f(x) \) appearing in Eq. (7). One of these truncations will dominate the evolution near a finite-time singularity, so we denote this truncation as \( \hat{f}(x) \), and hence the dynamical system near the singularity takes the form,

\[ \dot{x} = \hat{f}(x). \]  

(8)

After that, we write

\[ x_1(\tau) = a_1\tau^{p_1}, \quad x_2(\tau) = a_2\tau^{p_2}, \quad ..., \quad x_n(\tau) = a_n\tau^{p_n}, \]  

(9)

where \( \tau = t - t_c \), hence we made the assumption that the vector \( x \) is written in \( \psi \)-series near the singularity. By substituting Eq. (9) in Eq. (8), and equating the exponents we get the numbers \( p_i, \ i = 1, 2, ..., n \) which must be real fractional or integer numbers. With the \( p_i \)'s at hand, we form the vector \( \vec{p} = (p_1, p_2, ..., p_n) \), and by substituting these in Eq. (9), by equating the polynomials, we finally determine the numbers \( a_i \) appearing in Eq. (9), and we form the vector \( \vec{a} = (a_1, a_2, a_3, ..., a_n) \). The vectors \( (\vec{a}, \vec{p}) \neq 0 \) constitute a dominant balance in the terminology of Ref. [31].

- The next steps of the theorem-method of dominant balances is easy to comprehend. If the vector \( \vec{a} = (a_1, a_2, a_3, ..., a_n) \) has complex entries, then the theorem of Ref. [31] clearly states that no finite-time singularities occur in the dynamical system (7), but if it has real entries, then the dynamical system has finite-time singularities, and some of its variables \( x_i \), blow up.

- At the next step, one must ensure that the solutions found in the previous step are general or non-general. Let us clarify that a general solution corresponds to a general set of initial conditions, while a non-general solution corresponds to a limited set of initial conditions. To this end, we construct the Kovalevskaya matrix \( R \), which is,

\[ R = \begin{pmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n}
\end{pmatrix}
\]

(10)

Next we calculate the Kovalevskaya matrix \( R \) at a non-zero balance found at a previous step, and we calculate the eigenvalues. If the method is applied correctly, a consistent resulting Kovalevskaya matrix \( R \) will have eigenvalues of the form \((-1, r_2, r_3, ..., r_n)\).

- If in the previous step we find \( r_i > 0, \ i = 2, 3, ..., n \), then the solutions we found are general, but if one of the above eigenvalues is negative, then the solutions we found at a previous step correspond to a limited set of initial conditions and are thus non-general.

Let us apply the method we described above, for the dynamical system (5), subject to the Friedmann constraint (6). To this end, we rewrite the dynamical system (5), in the form,

\[ \frac{d\vec{x}}{dN} = f(\vec{x}), \]  

(11)
where $\vec{x} = (x, y)$, and also the vector-valued function $f(\vec{x})$ is equal to,

$$f(x, y) = \left( \frac{f_1(x, y)}{f_2(x, y)} \right),$$  \hspace{1cm} (12)

where the $f_i$'s in Eq. (12) are equal to,

$$f_1(x, y) = -3x + \frac{\sqrt{6}}{2} \lambda y^2 + \frac{3}{2} x (x^2 - y^2),$$  \hspace{1cm} (13)

$$f_2(x, y) = -\frac{\sqrt{6}}{2} \lambda xy + \frac{3}{2} y (x^2 - y^2).$$

There exist several truncations of the vector-valued function $f(\vec{x})$, but a consistent truncation is the following,

$$\hat{f}(x, y) = \left( -\frac{3xy^2}{2} \right).$$  \hspace{1cm} (14)

By using the method we presented previously, we easily find that the corresponding vector $\vec{p}$ has the form,

$$\vec{p} = \left( -\frac{3}{2}, -\frac{1}{2} \right),$$  \hspace{1cm} (15)

and in the same way, the following non-zero vector solutions $\vec{a}$ are obtained,

$$\vec{a}_1 = \left( -\frac{i}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right),$$  \hspace{1cm} (16)

$$\vec{a}_2 = \left( -\frac{i}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right),$$

$$\vec{a}_3 = \left( \frac{i}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right),$$

$$\vec{a}_4 = \left( \frac{i}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right).$$

As it is obvious, all the solutions $\vec{a}_i$, $i = 1, 2, 3, 4$ are complex, and also it can be easily checked that Friedman constraint (6) can be satisfied for all the solutions $\vec{a}_i$ and for the vector $\vec{p}$ being chosen as in Eq. (15). This can easily be seen, since the expression $x^2 + y^2$ appearing in the Friedmann constraint, for all the $\vec{a}_i$ reads at leading order (only the leading order terms are shown),

$$\left( \pm \frac{i}{\sqrt{3}} \right)^2 \tau^{-\frac{3}{2}} + \left( \pm \frac{1}{\sqrt{3}} \right)^2 \tau^{-\frac{3}{2}},$$  \hspace{1cm} (17)

which is equal to zero, where $\tau = N - N_c$. Having verified that the Friedmann constraint can never become singular, let us proceed to the problem at hand, and with regard to the question whether singular solutions exist, the answer is no, since the solutions $\vec{a}_i$ are complex. Now what remains is to check whether these solutions are general or not. To this end, we shall calculate the corresponding Kovalevskaya matrix $R$, which in our case is,

$$R = \left( \begin{array}{cc} \frac{1}{2} - \frac{3y^2}{2} & -3xy \\ 3xy & \frac{3x^2}{2} + \frac{1}{2} \end{array} \right).$$  \hspace{1cm} (18)

By evaluating the Kovalevskaya matrix $R$ on the solution $(x, y) = \vec{a}_1$, we obtain the matrix,

$$R(\vec{a}) = \left( \begin{array}{cc} 0 & -i \\ i & 0 \end{array} \right),$$  \hspace{1cm} (19)

and by calculating the corresponding eigenvalues we get,

$$(r_1, r_2) = (-1, 1).$$  \hspace{1cm} (20)

The same eigenvalues as above can be found for all the solutions $\vec{a}_i$, $i = 2, 3, 4$. Hence, in view of the theorem we presented above, the dynamical system (5) never develops finite-time singularities, and these non-singular solutions
are general, which means that these correspond to a general set of initial conditions, and not to a limited set of initial conditions. We must note that there is another non-trivial consistent truncation, however the form of the resulting $\tilde{p}$ makes the Friedmann constraint to leading order peculiarities, so this case is excluded from our analysis.

Let us now investigate what the results we found indicate for the physical finite-time singularity structure of the cosmological system at hand. Basically, the variables $x$ and $y$ can never become singular, so this constraint restricts the allowed types of singularities that can occur. Let us recall in brief the classification of finite-time singularities according to Ref. [33], so there are four types of singularities at $t = t_s$:

- **Type I ("Big Rip")**: This is the most severe type of singularity, from a phenomenological point of view, and in this case, as $t \to t_s$, $a \to \infty$, $\rho_{\text{eff}} \to \infty$ and $|p_{\text{eff}}| \to \infty$.
- **Type II ("sudden")**: This is known as pressure singularity, and in this case, as $t \to t_s$, $a \to a_s$, $\rho_{\text{eff}} \to \rho_s$ and $|p_{\text{eff}}| \to \infty$.
- **Type III**: In this case as $t \to t_s$, $a \to a_s$, $\rho_{\text{eff}} \to \infty$ and $|p_{\text{eff}}| \to \infty$.
- **Type IV**: This is a soft type of singularity, and in this case, as $t \to t_s$, $a \to a_s$, $\rho_{\text{eff}} \to \rho_s$, $|p_{\text{eff}}| \to p_s$ and the higher derivatives ($n > 2$) of $H$ diverge.

In the above classification, $\rho_{\text{eff}}$ and $p_{\text{eff}}$ stand for the total energy density and the total pressure of the cosmological system. The most general form of a Hubble rate developing some of the above listed singularities is the following,

$$H(t) = f_1(t) + f_2(t)(t-t_s)^\alpha,$$

where for consistency we require the parameter $\alpha$ to be $\alpha = \frac{2n}{m+n}$, where $m, n$ positive integers. The functions $f_1(t)$ and $f_2(t)$ are required to be regular at the singularity time instance $t = t_s$ and also $f_1(t_s) \neq 0$, $f_2(t_s) \neq 0$. Also their derivatives up to second order are assumed to satisfy the same constraints. The Hubble rate (21) is not necessarily a solution to the field equations, but we will discuss the case that indeed it is a solution, and we will investigate the implications of such a solution on the singularity structure, in view of the results we found for the dynamical system.

Particularly, the variables $x \sim \frac{\dot{\phi}}{H}$ and $y \sim \frac{\sqrt{V(\phi)}}{H}$ must be non-singular for all the values of the cosmic time. The various singularities that the Hubble rate (21) can generate, depending on the values of the parameter $\alpha$ are listed below,

- If $\alpha < -1$ a Big Rip singularity occurs.
- If $-1 < \alpha < 0$ a Type III singularity occurs.
- If $0 < \alpha < 1$ a Type II singularity occurs.
- If $\alpha > 1$ a Type IV singularity occurs.

So let us express the terms $\dot{\phi}$ and $V(\phi)$ in terms of the Hubble rate, so by combining Eqs. (3), we get,

$$\frac{2}{\kappa^2} \dot{H} = -\dot{\phi}^2,$$

so the above can be solved in terms of $\dot{\phi}$ and the variable $x \sim \frac{\dot{\phi}}{H}$ can be expressed in terms of the Hubble rate in closed form. Also, by using the fact that $V'(\phi) = -\lambda V(\phi)$ for the exponential potential under study, we obtain the potential as a function of the Hubble rate, by combining Eqs. (3) and (22), so we finally have,

$$V(\phi) = \frac{-\dot{H} + 6\dot{H}}{\kappa^2 \lambda \phi},$$

and in effect, the potential $V(\phi)$ is also expressed in closed form as a function of the Hubble rate, in view also of Eq. (22). Without presenting the resulting expressions, which are quite lengthy, let us investigate which singularities can occur, in view of the fact that the variables $x$ and $y$ are always regular. By studying the resulting expressions, for the Type IV singularity case, the only case that this singularity cannot occur is when $1 < \alpha < 2$, due to the presence of a term $\sim (t-t_s)^{\alpha-2}$ in the final expression of the variable $y$. However, when $\alpha > 2$, the variables $x$, $y$ are always regular, so a Type IV singularity can always occur for $\alpha > 2$. For the Type II case, which occurs for $0 < \alpha < 1$, the variable $x$ is always singular due to the presence of a term $\sim (t-t_s)^{\alpha-1}$, so it cannot occur at all. With regard to the Type III case, it cannot occur due to the presence of a term $\sim (t-t_s)^{-\frac{2\alpha}{\alpha-1}}$, which is always singular for $0 > \alpha > -1$. Finally, the Big Rip singularity can always occur, since for $\alpha < -1$ the variables $x$ and $y$ are always regular. We gathered the results of our investigation in Table I. In conclusion, if the Hubble rate (21) is a solution of the field equations, the only Types of singularities that can never occur are the Type III, and the Type II. The other two types of singularities, namely the Type IV and Type I can always occur when $\alpha > 2$ for the Type IV, and for any $\alpha < -1$ for the Big Rip case.
TABLE I: Allowed Singularity Types for the Hubble rate \( H(t) = f_1(t) + f_2(t)(t - t_s)^\alpha \)

| Type | Condition |
|------|-----------|
| I (Big Rip) | Always occurs when \( \alpha < -1 \). |
| II   | Cannot occur |
| III  | Cannot occur |
| IV   | Cannot occur for \( 1 < \alpha < 2 \), but can occur for \( \alpha > 2 \). |

IV. CONCLUSIONS

In this paper we studied the singularity structure of the phase space corresponding to an exponential quintessential model of dark energy, which is constrained by the swampland criteria. We focused on the question whether the variables of the dynamical system can become singular at some finite-time instance, and due to the fact that the dynamical system is polynomial, we applied a mathematical theorem in order to see whether finite-time singularities can occur in the dynamical system variables. As we demonstrated, the variables of the dynamical system can never become singular, so this imposes restrictions on the physical finite-time singularities that the cosmological system can develop. The result indicates actually that the solutions of the dynamical system are always regular and these correspond to a general set of initial conditions and not to a limited set of initial conditions. We also clarified the difference between a finite-time singularity in a dynamical system variable and a physical finite-time singularity. As we demonstrated, if the general Hubble rate \( H(t) = f_1(t) + f_2(t)(t - t_s)^\alpha \) is a solution of the cosmological equations, then only Type IV and Big Rip singularities can always occur for \( \alpha > 2 \) and \( \alpha < -1 \) respectively. The Type II and Type III singularities cannot occur for the cosmological system, if the Hubble rate we quoted above is considered a solution of the system. What we did not study is the effect of matter and radiation perfect fluids in the dynamical system. In principle the singularity structure might change, possibly the generality of solutions, however one should check this in detail, and we hope to address this issue in a future work.

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