Measuring second Chern number from non-adiabatic effects

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The geometry and topology of quantum systems have deep connections to quantum dynamics. In this paper, I show how to measure the non-Abelian Berry curvature and its related topological invariant, the second Chern number, using dynamical techniques. The second Chern number is the defining topological characteristic of the four-dimensional generalization of the quantum Hall effect and has relevance in systems from three-dimensional topological insulators to Yang-Mills field theory. I illustrate its measurement using the simple example of a spin-3/2 particle in an electric quadrupole field. I show how one can dynamically measure diagonal components of the Berry curvature in an over-complete basis of the degenerate ground state space and use this to extract the full non-Abelian Berry curvature. I also show that one can accomplish the same ideas by stochastically averaging over random initial states in the degenerate ground state manifold. Finally I show how this system can be manufactured and the topological invariant measured in a variety of realistic systems, from superconducting qubits to trapped ions and cold atoms.

Topological invariants such as the first Chern number have become relevant in condensed matter physics due to their robustness in describing novel states of matter [1][3]. While naturally defined in the solid state Brillouin zone, these geometry concepts and the Berry phase on which they are based occur in a wide variety of systems. In particular, these ideas have been recently applied to engineer and measure topological properties of designed systems, such as many-body cold atomic systems [6][8] and few-body systems of qubits or random walkers [9][11].

It was noted in the early days of topological physics [12][13] that higher topological invariants could be defined, and particularly that a non-trivial second Chern number characterizes systems with time-reversal symmetry. More recently, this has been connected the four-dimensional generalization of the quantum Hall effect [14] and three-dimensional topological insulators [15]. It is also intricately related to the axion electrodynamics used to define 3D topological insulators and to non-perturbative instanton effects in Yang-Mills field theory.

This higher topological invariant has never been measured experimentally. Here I propose how the second Chern number may be measured using non-adiabatic effects similar to the methods used in Refs. [10] and [11] to measure the first Chern number. The proposal relies on time-reversal invariant Hamiltonians to enforce a doubly-degenerate ground state and thus the previous proposal must be extended to account for these degeneracies. This involves measuring a fundamentally non-Abelian topological object. We show two ways to account for this - one by deterministically sampling over degenerate ground states and another by stochastic sampling. Each method has its pluses and minuses that may be relevant for different experimental systems, and we close by discussing how to access this physics in current experiments.

Non-adiabatic corrections with degeneracies - Consider a Hamiltonian \( H(\lambda) \) that depends on some parameters \( \lambda \). If one starts in the non-degenerate ground state \( |\psi_0(\lambda_i)\rangle \) at \( \lambda_i \) and then ramps \( \lambda \) slowly with time, at zeroth order the system simply remains in it ground state and picks up both a dynamical and Berry phase. If the ground state remains non-degenerate during the course of the ramp, the leading non-adiabatic correction leading to population in the excited states can be calculated through a technique known as adiabatic perturbation theory [10][17]. This can be understood by translating the problem to a moving frame \( |\psi\rangle \to U(\lambda)^\dagger |\psi\rangle \), where \( U(\lambda) \) diagonalizes the Hamiltonian \( (H_d = U^\dagger H U) \). In the moving frame, the Hamiltonian becomes

\[
H_m = H_d - i\hat{\lambda}_\mu U^\dagger \partial_\mu U \equiv H_d - \hat{\lambda}_\mu A^m_\mu,
\]

where \( \partial_\mu \equiv \partial/\partial\lambda_\mu \) and repeated indices are summed.

Figure 1. Measuring second Chern number dynamically. (a) General setup where measurement is possible. \( N \) degenerate ground states are separated by a non-zero gap from the excited states. The ground states must remain degenerate and gapped from the excited states, but no there are no restrictions on the excited states. (b) Illustration of the ramping protocol to find one component \( P^{\mu\nu}_j \) of the non-Abelian Berry curvature at measurement point \( \phi \). The procedure must be iterated over \( \mu, \nu \in \{1, 2, 3, 4\} \) and \( j \in \{1, 2, \ldots, N^2\} \) at each measurement point.
over. We refer to the second term $A_m = U A_m^\dagger U$ as the Berry connection operator as its matrix elements in the energy eigenbasis $|\psi_\nu\rangle$ are $\langle \psi_m | A_\mu | \psi_\nu \rangle = i\langle \psi_m | \partial_\mu \psi_\nu \rangle$. Diagonal terms in $H_m$ give rise to the dynamical and Berry phases, since $A_\mu = \langle \psi_0 | A_\mu | \psi_0 \rangle$ is the ground state Berry connection. Off-diagonal terms in this operator give rise to non-adiabatic occupation in the excited states, which at leading order can be seen by applying static perturbation theory to $H_m$ to find $c_{n\neq0} \approx \lambda_n \langle \psi_n | A_\mu | \psi_0 \rangle / (E_n - E_0)$. Calculating the “generalized force” $M_\mu \equiv -\langle \partial_\mu H \rangle$ in this state gives a leading order correction proportional to the ground state Berry curvature, $M_\mu \approx M_\mu^0 - \lambda_\mu F_{\mu\nu}$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$, namely $M_\nu \approx M_{\nu\nu}^0 - \lambda_\nu F_{\nu\mu}^AA$. Note that the adiabatic value $M_{\nu\nu}^0$ depends on the state $|\psi_0\rangle$. Thus our results for the physical observables are clearly similar to the non-degenerate case, but with the important caveat that they depend on the history of the protocol. This is because $|\psi_0\rangle$ depends on the path taken, so two different paths that give the same value of $\lambda$ and $\Lambda$ at time $t$ will not necessarily give the same Berry curvature correction to the generalized force.

Measuring second Chern number - The question then becomes what to make of the non-Abelian Berry curvature measurement if one can not easily predict the adiabatically-connected state. We are left searching for quantities that are invariant to the choice of basis. We find such quantities in the topologically invariant Chern numbers. The simplest example is the first Chern number, defined for a closed two-dimensional manifold $M_2$ in parameter space as $C_1 = (2\pi)^{-1} \int_{M_2} d\lambda_\mu \wedge d\lambda_\sigma \text{Tr}(F_{\mu\nu})$, where $\wedge$ denotes the wedge product. A novel topological invariant that appears for the four-dimensional manifold $M_4$ is the second Chern number

$$C_2 = \int_{M_4} \omega_{\mu\nu\rho\sigma} d\lambda_\mu \wedge d\lambda_\nu \wedge d\lambda_\rho \wedge d\lambda_\sigma \quad (2)$$

$$\omega_{\mu\nu\rho\sigma} = \frac{\text{Tr}(F_{\mu\nu} F_{\rho\sigma}) - \text{Tr}(F_{\mu\nu}) \text{Tr}(F_{\rho\sigma})}{2\pi^2},$$

where and $\omega_2$ is the second Chern form. The trace is taken over the ground state (upper) indices, i.e., $\text{Tr}(F_{\mu\nu} F_{\rho\sigma}) = F_{\mu\nu}^{11} F_{\rho\sigma}^{11}$, rendering $C_2$ basis invariant. But one clearly requires knowledge of the off-diagonal elements of $F$ to take this trace, while our non-adiabatic scheme only yields diagonal elements. I will now discuss two schemes to fill in this gap, which may be suitable for different systems.

First, let us see how we can deterministically reconstruct the matrix $F$ by measuring its diagonal elements in an over-complete basis. For concreteness, assume there are $N = 2$ degenerate ground states, denoted $|\psi_0A\rangle$ and $|\psi_0B\rangle$. $F_{\mu\nu}^{11}$ is anti-symmetric w.r.t. exchange of the lower (parameter) indices and Hermitian w.r.t. the upper (ground state) ones. Thus each matrix $F_{\mu\nu}$ is determined by $N^2$ real numbers. If we measure the diagonal components in the four states $|\psi_1\rangle = |\psi_0A\rangle$, $|\psi_2\rangle = |\psi_0B\rangle$, $|\psi_3\rangle = (|\psi_0A\rangle + |\psi_0B\rangle) / \sqrt{2}$, and $|\psi_4\rangle = (|\psi_0A\rangle + i|\psi_0B\rangle) / \sqrt{2}$, then with a bit of algebra

$$F_{\mu\nu} = \begin{pmatrix} F_{\mu\nu}^{AA} & F_{\mu\nu}^{AB} \\ F_{\mu\nu}^{BA} & F_{\mu\nu}^{BB} \end{pmatrix},$$

$$= \begin{pmatrix} F_{\mu\nu}^{11} & iF_{\mu\nu}^{12} \\ F_{\mu\nu}^{21} & F_{\mu\nu}^{22} \end{pmatrix}.$$  

For the non-degenerate case, but with the important caveat that they depend on the history of the protocol. This is because $|\psi_0\rangle$ depends on the path taken, so two different paths that give the same value of $\lambda$ at time $t$ will not necessarily give the same Berry curvature correction to the generalized force.
from which evaluating the second Chern number integral is just math. This method is well-suited to controllable quantum systems such as qubits, ions, or ultracold atoms where one has the ability to prepare arbitrary initial states. It trivially generalizes to arbitrary $N$.

If one does not have such a degree of control, a similar result may be achieved stochastically. The central idea is that the object $\langle \psi | F_{\mu \nu} | \psi \rangle$ averaged over states $| \psi \rangle$ drawn uniformly from the ground state subspace contains information about the second Chern form. In particular, one can show that if $\Gamma$ denotes this state average, then

$$
\frac{\langle \psi | F_{\mu \nu} | \psi \rangle}{\langle \psi | F_{\mu \nu} | \psi \rangle} = \frac{\text{Tr}(F_{\mu \nu} F_{\rho \sigma}) + \text{Tr}(F_{\mu \nu} F_{\rho \sigma})}{N(N + 1)}
$$

This can be readily seen for $N = 2$, for which $\langle \psi | F_{\mu \nu} | \psi \rangle = \text{Tr}(\text{cos}(\theta/2) F_{\mu \nu}^{AA} + \text{cos}(\theta/2) F_{\mu \nu}^{BB}) + 2 N(N + 1)$. Averaging over the Bloch angles $\theta$ and $\phi$, phases $e^{i \phi \nu}$ vanish and one readily reproduces Eq. 4. The general formula is derived in Appendix A. This method of measurement is natural if instead of deterministically preparing the desired states, nature gives one access to random snapshots of the system but allows multiple non-destructive measurements of the same state, such that all components $F_{12}, F_{13}, \ldots$ may be measured. This may therefore be more natural in the solid state context.

**Spin-3/2 in electric quadrupole field** - I now demonstrate the applicability of the above measurement techniques on the quintessential example of system with non-trivial second Chern number: the quantum spin-3/2 in an electric quadrupole field. This model was proposed by Avron et al. [12][13] as containing the simplest “quaternionic singularity” in much the same way that the monopole singularity of the Berry curvature in a qubit yields a non-trivial first Chern number. The Hamiltonian may be written as $H = -\mathbf{\lambda} \cdot \mathbf{H}$, where $\mathbf{H} = (H_0, H_1, \ldots, H_4)$ denotes an orthonormal basis of spin-3/2 quadrupole operators and $\mathbf{\lambda}$ denotes vector of coupling parameters. In particular, we choose the basis described in Ref. [13] $H_0 = (-J_x^2 - J_y^2 + J_z^2)/3, H_1 = (J_x J_y + J_y J_z)/\sqrt{3}, H_2 = (J_y J_z + J_x J_z)/\sqrt{3}, H_3 = (J_x J_x + J_y J_y + J_z J_z)/\sqrt{3}$. These Hamiltonians are invariant under time reversal, thus the eigenvalues come in two degenerate pairs. By construction, the energy eigenvalues of each are $± 1$ due to orthonormality, this is also true for arbitrary unit 5-vector $\mathbf{\lambda}$.

It is clear from the above discussion that the only way for all four eigenvalues to be degenerate is to have $\mathbf{\lambda} = \mathbf{0}$; this is the “quaternionic monopole” that gives a non-zero second Chern number. Avron et al. showed that for a 4-sphere surrounding this degeneracy, the second Chern number is equal to 1. Furthermore, due to time reversal symmetry this system has a vanishing first Chern number, so $C_2$ is its defining topological invariant. I will now show how the above ideas can be used to measure $C_2$ directly, focusing on the deterministic method for concreteness.

Let us begin by fixing the magnitude $|\mathbf{\lambda}| = 1$ and re-parameterizing the problem in terms of the spherical angles $\phi = (\phi_1, \phi_2, \phi_3, \phi_4)$, where $\phi_4 \in [0, 2\pi), \phi_{1-3} \in [0, \pi]$. $\lambda_0 = \cos \phi_1, \lambda_1 = \sin \phi_1 \cos \phi_2, \ldots, \lambda_4 = \sin \phi_4$. To obtain the Chern form at some point $\phi$, we begin with one of the states $|\psi_{1-4}\rangle$ described earlier for the value $|\phi = \mathbf{0}\rangle$ (the North pole). Here the Hamiltonian has the simple form $H = \frac{5}{4} - J_x^2$, so that ground states are just the $m_z = 0, 3/2$ eigenstates. Starting from one of these states, say $|\psi_1\rangle$, we ramp along some arbitrary path $\phi(s)$ to the measurement point $\phi$. Then to measure the component $F_{\mu \nu}^{11}$, we ramp the parameter $\phi_\mu$ according to $\phi_\mu = \phi_\mu^{0} + u(t - t_m)^2/t_{m}$, where $\phi_\mu^{0}$ is its value at the point to be measured. This ramp is chosen such that the ramp starts smoothly (\phi_\mu(0) = 0) at $\phi_\mu(0) = \phi_\mu^{0}$ and returns to $\phi_\mu^{0}$ at time $t_m$ with velocity $u$. Repeating this ramp multiple times with velocity $u$ at $u/2$ and measuring the expectation values $M_\mu(u)$ and $M_\nu(u/2)$, the Berry curvature is $F_{\mu \nu}^{11} \approx 2(M_\mu(u/2) - M_\mu(u))/u$. This protocol, illustrated in Fig. 1 must be repeated for all pairs $(\mu, \nu)$ and all initial states $|\psi_i\rangle$ to obtain the second Chern form at the point $\phi$ via Eq. 4. Crucially, for a given point $\phi$, the same path $\phi$ must be taken for each component of the tensor to ensure that the appropriate phase relations between the $|\psi_i\rangle$’s remain once ramped to $\phi$. From the second Chern form, the second Chern number may be obtained by the integral in Eq. 2. Such higher-dimensional integrals are numerically tricky; here I do it by Monte Carlo sampling $\phi$ uniformly from its domain. Carrying out the above procedure yields $C_2 = 0.9926 \pm 0.0073$, consistent with the exact value of 1.

To demonstrate the robustness of this topological invariant, we may induce a topological transition by adding a constant offset $\Lambda_0 H_0$ to the previous Hamiltonian. This shifts the unit sphere by an amount $\Lambda_0$, and for $|\Lambda_0| > 1$ the sphere fails to surround the degeneracy at the origin. Therefore, the second Chern number jumps to being trivial. This topological transition is seen in the simulations in Fig. 2; the transition appears broadened for a finite velocity $u$ due to higher-order non-adiabatic corrections near the gapless transition point.

**Experiments** - The above procedure naturally lends itself to controllable quantum systems such as superconducting qubits, ultracold atoms, ions, and solid state defects. For such systems, more detailed topological and geometric properties such as the Wilson loop may be measured via full tomography of adiabatic protocols; the dynamic second Chern number measurement serves to supplement this natural list of tools. However, the dynamical
measurement trivially generalizes to more complicated systems where full tomography is not possible, requiring neither strict adiabaticity nor tomographic measurements that scale exponentially with system size.

An important practical concern in realizing these ideas experimentally is to use the symmetry of the sphere to reduce the number of measurements that must be made. As the simplest example of this, consider the case \( \Lambda_0 = 0 \) where the problem has full spherical symmetry. Then, since all points on the sphere are identical, the Chern number can be obtained by measuring the Berry curvature at a single point. If we start in the ground state at the North pole as before, then there are four orthonormal tangent vectors to the surface: \( \lambda_1, \lambda_2, \lambda_3, \) and \( \lambda_4 \).

We can measure the response to these parameters in the same way as we did with \( \phi \)’s. For instance, we obtain \( F_{\lambda_1, \lambda_4} \) by ramping \( \lambda_4 \) then measuring \( \lambda_1 \), (\( H_\phi \)). By symmetry, \( \text{Tr} (F_{\lambda_1, \lambda_2, \lambda_3, \lambda_4} - F_{\lambda_1, \lambda_3, \lambda_2, \lambda_4} + F_{\lambda_1, \lambda_4, \lambda_2, \lambda_3}) = 3\text{Tr} (F_{\lambda_1, \lambda_2, \lambda_3}) \) and furthermore, this value will be constant in the basis of tangent vectors at any point on the sphere. So we simply multiply by the surface area of the unit 4-sphere to get [23]

\[
C_2 = \frac{3A_{S^3}}{4\pi^2} \text{Tr}(F_{\lambda_1, \lambda_2, \lambda_3, \lambda_4}) = 2\text{Tr}(F_{\lambda_1, \lambda_2, \lambda_3, \lambda_4}). \tag{5}
\]

We thus expect that \( \text{Tr}(F_{\lambda_1, \lambda_2, \lambda_3, \lambda_4}) = 1/2 \), which is readily confirmed numerically.

The argument must be slightly modified in the presence of an offset \( \Lambda_0 \), but symmetry still significantly reduces the number of measurements required. \( \lambda_0 \) is now distinct from the other axes, which translates into a \( \phi \)-dependence of the second Chern form. The axes tangent to a point \( \phi = (\phi_1, 0, 0, 0) \) are now \( \phi_1, \lambda_2, \lambda_3 \), and \( \lambda_4 \), so the non-trivial terms \( F_{\phi_1, \lambda_2} \) and \( F_{\lambda_1, \lambda_4} \) are obtained by ramping \( \phi_1 \) and \( \lambda_3 \) and measuring \( \langle H_2 \rangle \) and \( \langle H_4 \rangle \) respectively. For a given value of \( \phi_1 \), the remaining parameters trace out a 3-sphere of radius \( \sin \phi_1 \), which has surface area \( A_{S^3} = 2\pi^2 \sin^3 \phi_1 \).

By the same logic as Eq. [5] one may then compute \( C_2 \) via

\[
C_2 = \frac{3}{4\pi^2} \int_0^\pi d\phi_1 A_{S^3}(\phi_1) \text{Tr}[F_{\phi_1, \lambda_2}(\phi_1)F_{\lambda_1, \lambda_4}(\phi_1)]
= \frac{3}{2} \int_0^\pi d\phi_1 \sin^3 \phi_1 \text{Tr}[F_{\phi_1, \lambda_2}(\phi_1)F_{\lambda_1, \lambda_4}(\phi_1)]. \tag{6}
\]

The resulting Chern number is shown in Fig. [21].

In addition to reducing the number of measurements, Eq. [6] reduces the number of control axes required; one need only ramp \( \phi_1 \) (i.e. \( \lambda_1 \) and \( \lambda_2 \)) and \( \lambda_3 \), although given the symmetry of the problem any three \( \lambda \)s will do. This is reduced further to only two \( \lambda \)s in the fully symmetric case. While one may in principle realize arbitrary 4 \times 4 Hamiltonians given four levels full-connected by drives [24], a more natural situation is partially-connected levels like the ladder system illustrated in Fig. [2] [25-27]. For such couplings, not all of the terms can be easily realized, but fortunately a sufficient number can be realized to allow the measurement using symmetry. This can be seen from the representations of \( H_i \) in the \( J_z \) basis:

\[
H_0 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad H_1 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0
\end{pmatrix}
\]

If we think of the four states as realizing a single particle in a 4-site chain, then \( H_0 \) represents on-site chemical potentials, while \( H_1 \) and \( H_2 \) represent (phased) hopping between sites 1 and 2 as well as 3 and 4. This is well within the capacity of the driven system, and even could be realized on a physical 4-site lattice or 4-site supercell within a larger lattice via lattice shaking schemes analogous to those used to generate artificial gauge fields [6-7].

Instead of imprinting the Hamiltonian structure on Hilbert space by hand, we might instead be given a system with natural structure of its own, say a set of coupled spins-1/2 [28-30]. In this case, there are two representations that may be useful. First, note that we can write the Hamiltonians in the form \( H_0 = \sigma_1^z \sigma_2^z \), \( H_1 = \sigma_1^y \sigma_2^y \), \( H_2 = \sigma_1^y \sigma_2^z \), \( H_3 = \sigma_2^z \), and \( H_4 = \sigma_2^y \). Of these operators, \( H_0, H_3 \) and \( H_4 \) are naturally realized for chains of...
two spin-1/2’s, in both trapped ions \[^{30}\] and transmon qubits \[^{31}\]. Similarly, we can imagine directly obtaining \(J = 3/2\) by fusing three spin-1/2’s. In this language, \(H_0\) is proportional to an Ising-like interaction \(\sum_{<ij>} \sigma_i^z \sigma_j^z\) in a fully-coupled three-spin ring. The other interactions seem less natural: \(H_1\) maps to \(\sum_{<ij>} \sigma_i^x \sigma_j^x\), similarly for \(H_2\) and \(H_4\), while \(H_3\) gives interactions of the form \(\sigma_i^x \sigma_j^y - \sigma_i^y \sigma_j^x\). With developments in the field, this three-qubit realization may be possible in the future, while the two-qubit version is readily available with current technology.

**Discussion** - Most of this paper has worked in the language of controllable quantum systems, for which it is natural to discuss some set of control parameters \(\lambda\). In solid state physics, a natural parameter space is the momentum \(k\) in the Brillouin zone, or more generally twists of the boundary condition \[^{32}\,^{33}\]. For four-dimensional crystals, the insulator analogous to our spin-3/2 example is the second Chern insulator, which is the natural four-dimensional generalization of the quantum Hall effect \[^{14}\]; recent work has proposed realizing this with cold atoms in an artificial 4D lattice \[^{34}\]. One obtains a quantized non-linear electromagnetic response in these systems. An interesting open question is how to relate this to our method for determining the second Chern number, which is obtained through linear response, following by classical post-processing. The stochastic method for extracting the second Chern form bears greater resemblance to the 4D Hall response, and is seemingly more natural in the solid state context. However, it remains a product of linear responses, and its relationship to non-linear fluctuations remains unclear at present.

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**Appendix A: Derivation of stochastic Chern form**

In this appendix, we will derive Eq. \[^{4}\] for generic \(N\). We start by noting that an \(N \times N\) Hilbert space may be uniformly sampled via picking a random unit vector \(\hat{n}\) from \(S^{2N}\) (the \(2N\)-sphere) and constructing the state \(|\hat{n}\rangle = (n_0 + in_1)|0\rangle + (n_2 + in_3)|1\rangle + \cdots + (n_{2N-2} + in_{2N-1})|N-1\rangle \equiv \sum_{j=0}^{N-1} \eta_j|j\rangle\). For an arbitrary Hermitian operator \(A\), the expectation value in \(|\hat{n}\rangle\) is

\[
\langle \hat{n}|A|\hat{n}\rangle = |\eta_0|^2 A_{00} + \eta_0 \eta_1 A_{01} + \cdots
\]

Averaging over the \(2N\)-sphere, only the terms without phase factors survive. Then \(\bar{|\eta_0|^2} = |\bar{\eta}_j|^2 = n_0^2 + n_1^2 = 2n_0^2\). Expressing \(\hat{n}\) in spherical coordinates \((\phi_1, \phi_2, \ldots, \phi_{2N-1})\), where \(\phi_{2N-1} \in [0, 2\pi)\) and \(\phi_j \neq 2N-1 \in [0, \pi)\), we have \(n_0 = \cos \phi_1, n_1 = \sin \phi_1 \cos \phi_2, \) and area element \(da = \sin^{N-2} \phi_1 \cdots \sin \phi_{2N-2} \sin \phi_{2N-3} d\phi_1 \cdots d\phi_{2N-1}\). Thus

\[
\bar{|\eta_0|^2} = \int \cos^2 \phi_1 da = \int_0^{\pi} \cos^2 \phi_1 (\sin \phi_1)^{2N-2} d\phi_1 = \frac{3}{4N(N+1)}
\]

Similarly, averaging the quantity \(\langle \hat{n}|A|\hat{n}\rangle \langle \hat{n}|B|\hat{n}\rangle\) for \(A\) and \(B\) Hermitian gives

\[
\bar{\langle \hat{n}|A|\hat{n}\rangle \langle \hat{n}|B|\hat{n}\rangle} = \bar{|\eta_0|^2} A_{00} B_{00} + \bar{|\eta_1|^2} A_{11} B_{11} + \cdots + \bar{|\eta_0|^2 |\eta_1|^2} (A_{00} B_{11} + A_{11} B_{00} + A_{01} B_{10} + A_{10} B_{01}) + \cdots
\]

where as before we have used the symmetry of the sphere to replace everything by \(\eta_0\) and \(\eta_1\). Then

\[
\bar{|\eta_0|^2} = (n_0^2 + n_1^2) = n_0^2 + 2n_0^2 n_1^2 + n_1^4 = 2 \left( n_0^2 + n_0^2 n_1^2 \right)
\]

\[
\bar{|\eta_0|^2 |\eta_1|^2} = (n_0^2 + n_1^2)^2 = 4n_0^2 n_1^2
\]

\[
\bar{n_0^2} = \int \cos^4 \phi_1 da = \frac{3}{4N(N+1)}
\]

\[
\bar{n_0^2 n_1^2} = \int \cos^2 \phi_1 \sin^2 \phi_1 \cos^2 \phi_2 da = \frac{1}{4N(N+1)}.
\]
Inserting these results into Eq. \[ A \] we find

\[
\langle \hat{n}|A|\hat{n}\rangle = \frac{2}{N(N+1)}(A_{00}B_{00} + A_{11}B_{11} + \ldots) + \frac{1}{N(N+1)}(A_{00}B_{11} + A_{11}B_{00} + A_{01}B_{10} + A_{10}B_{01} + A_{00}B_{22} + \ldots)
\]

\[
= \frac{1}{N(N+1)} \left[ (A_{00}B_{00} + A_{11}B_{11} + \ldots + A_{01}B_{10} + A_{10}B_{01} + \ldots) + (A_{00}B_{11} + A_{11}B_{00} + \ldots + A_{01}B_{10} + A_{10}B_{01} + \ldots) \right]
\]

which clearly reduces to Eq. \[ 4 \] when \( A = F_{\mu\nu} \) and \( B = F_{\mu\sigma} \).

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[22] More accurately, this derivation holds if the path-ordered integral and the matrix \( A_n \) are represented in the moving frame by just integrating the moving-frame Schrodinger equation within the degenerate subspace (the upper \( N \times N \) block).

Care must be taken in defining these anholonomies for large paths in parameter space, as a non-trivial \( C_2 \) serves as an obstruction to defining a global \( U(N) \) gauge.

[23] Note that because of time-reversal symmetry, the first Chern form vanishes: \( \text{Tr}(F_{\mu\nu}) = 0 \). Therefore the second Chern form reduces to \( \omega_2^{\mu\nu\rho\sigma} = \text{Tr}(F_{\mu\nu}F_{\rho\sigma})/32\pi^2 \).

[24] Note that the matrix elements of the arbitrary \( 4 \times 4 \) Hamiltonians may be controlled by tuning the amplitude, phase, and detuning of the drives.
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