The Cauchy Problem and Multi-peakons for the mCH-Novikov-CH Equation with Quadratic and Cubic Nonlinearities

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Abstract
This paper investigates the Cauchy problem of a generalized Camassa-Holm equation with quadratic and cubic nonlinearities (alias the mCH-Novikov-CH equation), which is a generalization of some special equations such as the Camassa-Holm (CH) equation, the modified CH (mCH) equation (alias the Fokas-Olver-Rosenau-Qiao equation), the Novikov equation, the CH-mCH equation, the mCH-Novikov equation, and the CH-Novikov equation. We first show the local well-posedness for the strong solutions of the mCH-Novikov-CH equation in Besov spaces by means of the Littlewood-Paley theory and the transport equations theory. Then, the Hölder continuity of the data-to-solution map to this equation are exhibited in some Sobolev spaces. After providing the blow-up criterion and the precise blow-up quantity in light of the Moser-type estimate in the Sobolev spaces, we then trace a portion and the whole of the precise blow-up quantity, respectively, along the characteristics associated with this equation, and obtain two kinds of sufficient conditions on the gradient of the initial data to guarantee the occurrence of the wave-breaking phenomenon. Finally, the non-periodic and periodic peakon and multi-peakon solutions for this equation are also explored.

Keywords mCH-Novikov-CH equation · Wave breaking · Local well-posedness · Hölder continuity · Non-periodic and periodic peakon and multi-peakon solutions

Mathematics Subject Classification 35B30 · 35G25 · 35A01 · 35B44 · 35Q53 · 35Q35
1 Introduction

We are concerned with the Cauchy problem and multi-peakons of the following nonlinear dispersive equation with quadratic and cubic nonlinearities, which is a combination of the modified Camassa-Holm equation, the Novikov equation, and the Camassa-Holm equation (alias the mCH-Novikov-CH equation)

$$\begin{align*}
\frac{m_t + k_1([u^2 - u^2]_x)}{m} + k_2(u^2m_x + 3uu_xm) + k_3(um_x + 2uxm) = 0, \quad x \in \mathbb{R}, \quad t > 0, \\
u(0, x) = u_0(x), \quad x \in \mathbb{R},
\end{align*}$$

(1.1)

where $k_1, k_2$ and $k_3$ are all real-valued parameters, and $m = u - u_{xx}$.

Eq. (1.1) with $k_{1,2} = 0$ and $k_3 = 1$ reduces to the celebrated Camassa-Holm (CH) equation

$$\frac{m_t + um_x + 2uxm = 0, \quad m = u - u_{xx},}
\text{(1.2)}$$

where $u(t, x)$ denotes the fluid velocity and $m(t, x)$ stands for the corresponding potential density, which can describe the unidirectional propagation of shallow water waves over a flat bottom [9,35,59] and the propagation of axially symmetric waves in hyperelastic rods [30,31]. Equation (1.2) can be deduced from the well-known Korteweg-de Vries (KdV) equation by means of the tri-Hamiltonian duality [73] and shares some alike properties with the KdV equation. For instance, it is completely integrable and has infinitely many conservation laws [9,36,38].

It can be written as the following bi-Hamilton structure [9]

$$m_t = J_1 \frac{\delta H_{CH,1}}{\delta m} = K_1 \frac{\delta H_{CH,2}}{\delta m}, \quad J_1 = \frac{1}{2}(m \partial + \partial m), \quad K_1 = \partial^3 - \partial$$

where

$$H_{CH,1} = \int_{\mathbb{R}} mu \, dx, \quad H_{CH,2} = \frac{1}{2} \int_{\mathbb{R}} \left(u^3 + uu_x^2\right) dx,$$

and the sum of the two Hamiltonian operators $J_1$ and $K_1$ is still a Hamiltonian operators. It can be derived from the compatibility condition of the following system for $\psi$ [9]:

$$\psi_{xx} = \left[\frac{1}{4} - \frac{m(x, t)}{2\lambda}\right] \psi, \quad \psi_t = -(\lambda + u)\psi_x + \frac{1}{2} u_x \psi.$$  

(1.3)

System (1.3) enables one to solve the Cauchy problem of the CH equation by the inverse scattering transform (IST) [24]. Also, the action angle variables of (1.2) can be constructed by using the IST [3,5,24,27]. Despite these similarities, the CH equation (1.2) has some different properties from the KdV equation. For example, there is an interesting phenomenon called wave-breaking in nature, namely, the wave itself remains bounded, while its slope becomes unbounded in finite time [4,20,83]. This kind of singularities can not be derived from the KdV equation. However, it can be modelled by Eq. (1.2). This is a significant difference between the CH and KdV equations. Now that the wave-breaking phenomenon can occur in the solution of (1.2), a natural problem is how the solution develops after the wave-breaking? This kind of problem has been considered in [6,54] and [7,55], where the authors construct the global conservative solutions and global dissipative solutions, respectively. On the other hand, the CH equation (1.2) also admits a kind of solutions called the peakons [4,9,18,22,23,29,64] whose shapes are stable under small perturbations [29,63]. The CH equation (1.2) is also related to geometry. Firstly, it represents the geodesic flows. In the aperiodic case and when some asymptotic conditions are satisfied at infinity, equation (1.2)
can be viewed as the geodesic flow on a manifold of diffeomorphism of the line [17]. In the periodic case, it can represent the geodesic flow on the diffeomorphism group of the circle [26,60]. Secondly, it can represent the families of pseudo-spherical surfaces [43,47,78]. Thirdly, it appears from a non-stretching invariant planar curve flow in the centro-affine geometry [16,47]. The geometric illustration of equation (1.2) yields the least action principle. Other mathematical facts about equation (1.2) include the existence of a recursion operator [9,10,37–40], a generalized Fourier transform via squared eigenfunctions [25] and so on. The local-in-time well-posedness for the initial value problem of (1.2) has been achieved [9,10,37–40], a generalized Fourier transform via squared eigenfunctions [25] and so on. The local-wellposedness, blow-up strong solutions [17,19–21], as well as global weak solutions [28,86,87].

Equation (1.2) has the quadratic nonlinearity, while, there do exist other CH-type equations with cubic nonlinearity. The first one of such equations is the modified CH (mCH) equation (alias the Fokas-olver-Rosenau-Qiao (FORQ) equation) [37,40,73,74]

\[ m_t + [(u^2 - u_x^2)m]_x = 0, \quad m = u - u_{xx}, \]  

which corresponds to Eq. (1.1) with \( k_{2,3} = 0 \) and \( k_1 = 1 \). Eq. (1.4) was first derived by applying the tri-Hamiltonian duality to the bi-Hamiltonian representation of the modified KdV equation [40,73]. Later, it was rededuced from the two-dimensional Euler equations [74]. Similar to the CH equation (1.2), the mCH equation (1.4) is also completely integrable. Its bi-Hamiltonian structure reads [47,73,75]

\[ m_t = J_2 \frac{\delta H_{mCH,1}}{\delta m} = K_2 \frac{\delta H_{mCH,2}}{\delta m}, \quad J_2 = -\partial_x m \partial_x^{-1} m \partial_x, \quad K_2 = \partial_x^3 - \partial_x, \]

where the Hamiltonians are

\[ H_{mCH,1} = \int_\mathbb{R} m u dx, \quad H_{mCH,2} = \frac{1}{4} \int_\mathbb{R} (u^4 + 2u^2u_x^2 - \frac{1}{3}u_x^4) dx. \]

One can solve Eq. (1.4) using the inverse scattering transform since it admits the following Lax pair [47,75]: \( \psi_t = U \psi, \quad \psi_x = V \psi \) with \( \psi = (\psi_1, \psi_2)^T \) and with

\[ U = \frac{1}{2} \begin{bmatrix} -1 & \lambda m \\ -\lambda m & 1 \end{bmatrix}, \quad V = \begin{bmatrix} \lambda^{-2} + \frac{1}{2} (u^2 - u_x^2) & -\lambda^{-1} (u - u_x) - \frac{1}{2} \lambda (u^2 - u_x^2) \lambda^{-1} (u - u_x) - \frac{1}{2} \lambda (u^2 - u_x^2) \\ \lambda^{-1} (u + u_x) + \frac{1}{2} \lambda (u^2 - u_x^2) m & -\lambda^{-2} - \frac{1}{2} \lambda (u^2 - u_x^2) \end{bmatrix}. \]

From the viewpoint of geometry, Eq. (1.4) arises from an intrinsic (arc-length preserving) invariant planar curve flow in Euclidean geometry [47]. The local-wellposedness, blow-up and wave breaking problems of Eq. (1.4) were considered in [14,41,47,68]. The explicit form of the multipeakons of Eq. (1.4) has been addressed in [11]. The peakon solution of Eq. (1.4) is orbital stability [67,77]. The Hölder continuity of Eq. (1.4) has been discussed in [51].

Another celebrated CH-type equation with cubic nonlinearity is the Novikov equation [71]

\[ m_t + u^2m_x + 3uu_xm = 0, \quad m = u - u_{xx}, \]  

which was discovered by Novikov when classifying the nonlocal partial differential equations in the light of symmetry [71]. Like the CH and mCH equations, the Novikov equation (1.5) is also completely integrable, and its Lax pair reads [71]

\[
\begin{align*}
\psi_{txx} &= \psi_x + \lambda m^2 \psi x + 2m^x \psi_{xx} + \frac{mm_{xx} - 2m^2}{m^2} \psi_x, \\
\psi_t &= \frac{1}{\lambda} \frac{u}{m} \psi_x - \frac{1}{\lambda} \frac{(mu)_x}{m^2} \psi_x - u^2 \psi_x.
\end{align*}
\]
or [56] $\Psi_x = A\Psi, \quad \Psi_t = B\Psi$, where $\Psi = (\Psi_1, \Psi_2, \Psi_3)^T$ and the two matrices $A$ and $B$ are defined as

$$
A = \begin{bmatrix}
0 & m_\lambda & 1 \\
0 & 0 & m_\lambda \\
1 & 0 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
\frac{1}{3\lambda^2} - uu_x & u^2m_\lambda & -u^2m_\lambda \\
\frac{u}{\lambda} & -2\frac{u^2}{\lambda^2} & -u^2m_\lambda \\
\frac{u}{\lambda} & 1 & \frac{1}{3\lambda^2} + uu_x
\end{bmatrix}.
$$

The bi-Hamiltonian structure of equation (1.5) was provided by [56]:

$$
m_{\kappa} = J_3 \frac{\delta H_{k+6}}{\delta m} = K_3 \frac{\delta H_k}{\delta m},
$$

where the Hamiltonian operator pairs are

$$
J_3 = -2(3m_\partial_x + 2m_x)(4\partial_x - \partial_x^3)^{-1}(3m_\partial_x + m_x), \quad K_3 = (1 - \partial_x^2)m^{-1}\partial_x m^{-1}(1 - \partial_x^2)
$$

and the first few Hamiltonians are given by

$$
H_1 = \int_\mathbb{R} \frac{1}{8}(u^4 + 2u^2u_x^2 - \frac{u^4}{3}) \, dx, \quad H_5 = \int_\mathbb{R} m^{2/3} \, dx, \quad H_7 = \int_\mathbb{R} \frac{1}{3}(m^{-8/3}m_x^2 + 9m^{-2/3}) \, dx
$$

with $k = (\pm 1 \mod 6)$. The periodic initial value problem for equation (1.5) in $H^s$ with $s > 5/2$ was considered in [81] based on Arnold’s geometric framework. The index $s > 5/2$ was improved to $s > 3/2$ in [49] based on a Galerkin-type approximation. Moreover, the continuous dependence on the initial data is showed to be optimal in [49]. The initial value problem on the line was proved to be local well-posedness in $B^s_{2,r}$ in [72] in light of Kato’s semigroup theory. [88] also considered the initial value problem in the Besov circumstances. The explicit form of multipeakons for (1.5) was computed in [57]. The wave breaking phenomenon for (1.5) was discussed in [15,58,89] under the sign-changing or non-sign-changing conditions. (1.5) admits global weak solution [61,62,84]. The measure of momentum support with upper and lower bounds after time $t$ was researched in [48]. Concerning the CH type equations with analytic initial data, Barostichi et al [8] established an Ovsyannikov type theorem for an autonomous abstract Cauchy problem. Then they apply it to some CH type equations (e.g., CH, mCH, DP, and the Novikov equations) with initial data in spaces of analytic functions to obtain the analytic lifespan of the solution as well as the continuity of the data-to-solution map in spaces of analytic functions.

Equation (1.1) with $k_2 = 0$ becomes the following mCH-CH equation with quadratic and cubic nonlinearities [37]

$$
m_t + k_1[(u^2 - u_x^2)m]_t + k_3(um_x + 2u_xm) = 0, \quad m = u - u_{xx},
$$

which is completely integrable and admits the Lax pair and bi-Hamiltonian structure [76,85], which is a linear combination of those of the CH and mCH equations, that is,

$$
m_t = J_4 \frac{\delta H_{mCH-CH,1}}{\delta m} = K_4 \frac{\delta H_{mCH-CH,2}}{\delta m}, \quad J_4 = k_1 J_2 + k_1 J_1, \quad K_4 = K_2 = K_1,
$$

where the Hamiltonians are

$$
H_{mCH-CH,1} = H_{mCH,1} = H_{CH,1}, \quad H_{mCH-CH,2} = k_1 H_{mCH,2} + k_2 H_{CH,2}.
$$

The Cauchy problem and blow-up of Eq. (1.6) were studied in [15,65,66].
Equation (1.1) with \( k_3 = 0 \) reduces to the mCH-Novikov equation with cubic nonlinearity derived recently in [13],

\[
    m_t + k_1((u^2 - u_x^2)m)_x + k_2(u^2m_x + 3uu_xm) = 0, \quad m = u - u_{xx}. \tag{1.7}
\]

It admits two conserved quantities

\[
    H_1(u) = \int_{\mathbb{R}} m u \, dx, \quad H_2(u) = \int_{\mathbb{R}} \left( u^4 + 2u^2u_x^2 - \frac{1}{3}u_x^4 \right) \, dx
\]

and a Hamiltonian structure

\[
    m_t = \frac{\delta H_1}{\delta m}, \quad J = c_1J_2 + c_2J_3.
\]

Recently, Mi et al. [70] obtained the local well-posedness of Eq. (1.7) in noncritical or critical Besov spaces and provided the blow-up criterion in Sobolev spaces as well as the wave breaking condition, the persistence property and the analyticity of the solution.

As far as we know, the initial value problem (1.1) has not been investigated yet. Employing the Littlewood-Paley theory and the transport equations theory, we will first show the local well-posedness for the strong solutions of Eq. (1.1) in Besov spaces following the spirit of [32–34]. Then, the Hölder continuity of the data-to-solution map of this equation will be shown. A Moser-type estimate in the Sobolev spaces will help us to establish a blow-up criterion and the precise blow-up quantity for (1.1). We will also provide two sufficient conditions on the initial data to ensure the occurrence of the wave breaking phenomenon.

Note that Eq. (1.1) admits a conserved quantity \( H_1 = \int_{\mathbb{R}} m u \, dx \) and a Hamiltonian structure

\[
    m_t = J' \frac{\delta H_1}{\delta m}, \quad J' = c_1J_2 + c_2J_3 + c_3J_1. \tag{1.8}
\]

For convenience, the solution spaces \( E_{p,r}^s(T) \) are defined as follows

\[
    E_{p,r}^s(T) \triangleq \begin{cases} 
        \left( C \left( [0,T); B^{s,p}_{p,r} \right) \cap C^1 \left( [0,T); B^{s-1,p}_{p,r} \right) \right), & \text{if } r < \infty, \\
        C_w \left( [0,T); B^{s,p}_{p,\infty} \right) \cap C^{0,1} \left( [0,T); B^{s-1,p}_{p,\infty} \right), & \text{if } r = \infty.
    \end{cases}
\]

We next introduce some notations to be used in this paper.

**Notation.** Let \( p(x) = \frac{1}{2}e^{-|x|}, \ x \in \mathbb{R} \) be the fundamental solution of \( 1 - \partial_x^2 \) on \( \mathbb{R} \), and two convolution operators \( p_{\pm} \) be [15]

\[
    p_{\pm} * f(x) = \frac{1}{2} e^{\mp x} \int_{-\infty}^{\pm x} e^y f(\pm y) \, dy, \tag{1.9}
\]

where the star denotes the spatial convolution. Note that \( p \) and \( p_{\pm} \) satisfy

\[
    p = p_+ + p_-, \quad p_x = p_- - p_+, \quad p_{xx} * f = p * f - f. \tag{1.10}
\]

Let \( S \) stands for the Schwartz space and \( S' \) represents the spaces of temperate distributions. Let \( L^p(\mathbb{R}) \) be the Lebesgue space equipped with the norm \( \| \cdot \|_{L^p} \) for \( 1 \leq p \leq \infty \) and \( H^s(\mathbb{R}) \) be the Sobolev space equipped with the norm \( \| \cdot \|_{H^s} \) for \( s \in \mathbb{R} \).

Our first result is the local well-posedness of the Cauchy problem (1.1) in \( E_{p,r}^s(T) \).
Theorem 1.1 Suppose \( u_0 \in B^s_{p,r} \) with \( s > \max\{5/2, 2 + 1/p\} \) and \( p, r \in [1, \infty] \). Then there exists a time \( T > 0 \) and a unique solution \( u \in E^s_{p,r}(T) \) of the Cauchy problem (1.1) such that the data-to-solution map \( u_0 \mapsto u : B^s_{p,r} \mapsto C([0, T]; B^s_{p,r}) \cap C^1([0, T]; B^{s-1}_{p,r}) \) is continuous for every \( s' < s \) as \( r = \infty \). \( \frac{1}{2} \). Let \( T^* > 0 \) be the maximal existence time of the solution \( m \) to the Cauchy problem (1.1). Then

\[
T^* < \infty \quad \Rightarrow \quad \int_0^{T^*} \|m(t)\|_{C^1}^2 dt = \infty. \tag{1.13}
\]

Remark 1.1 Suppose \( m_0 \in H^s \) with \( s > \frac{1}{2} \). Let \( m_t \) be the maximal \( H^s \) solution corresponding to \( m_0 \) with lifespan \( T_s \). Let \( m_{s'} \) be the maximal \( H^{s'} \) solution corresponding to the same initial data with lifespan \( T_{s'} \). Let \( s' < s \). Since \( H^s \hookrightarrow H^{s'} \), the uniqueness ensures that \( m_s = m_{s'} \) on \([0, T_s] \). If \( T_s < T_{s'} \), then we can conclude \( m_{s'} \in C([0, T_s]; H^s) \)
and consequently $m_s \in L^2([0, T_*]; L^\infty)$, contradicting the above blow up criterion (1.13). That is to say, the lifespan $T^*$ does not depend on the regularity index $s$ of the initial data $m_0$.

**Remark 1.2** In fact, Theorems 1.1–1.3 also hold true for the mCH-Novikov-CH equation with the linear dispersive term

$$
\begin{cases}
m_t + \kappa u_x + k_1[(u^2 - u_x^2)m]_x + k_2(u^2m_x + 3uu_xm) + k_3(um_x + 2uxm) = 0, & x \in \mathbb{R}, \ t > 0, \\
u(0, x) = u_0(x), & x \in \mathbb{R},
\end{cases}
$$

(1.14)

where $\kappa$, $k_1$, $k_2$ and $k_3$ are all real-valued parameters, and $m = u - u_{xx}$.

The following theorem shows the precise blow-up scenario for sufficiently regular solutions to the Cauchy problem (1.1).

**Theorem 1.4** Suppose $u_0 \in H^s(\mathbb{R})$ with $s > \frac{5}{2}$. Let $T^* > 0$ be the maximal existence time of the solution $u$ to the Cauchy problem (1.1). Then the solution $u$ blows up in finite time if and only if

$$
\lim_{t \to T^*} \inf_{x \in \mathbb{R}} (2k_1u(m(t, x) + 3k_2u^2(t, x) + 2k_3u_x(t, x))) = -\infty.
$$

(1.15)

From Theorem 1.4, we find that the solution blows up in finite time if and only if the quantity $M(t, x) = 2k_1u(m(t, x) + 3k_2u^2(t, x) + 2k_3u_x(t, x))$ is unbounded below. We next prove that this quantity admits an uniform upper bound as long as $m(t, x)$ exists under the assumption of Theorem 1.4.

**Theorem 1.5** Let the assumption of Theorem 1.4 be satisfied. Suppose also that $m_0(x) = (1 - \partial_x^2)^2 u_0 \geq 0$ for all $x \in \mathbb{R}$, and $m_0(x) > 0$ at some point $x_0 \in \mathbb{R}$. Then there holds

$$
\sup_{x \in \mathbb{R}} M(t, x) \leq 2k_1\|u_0\|_{H^1} \sup_{x \in \mathbb{R}} m_0(x) + 3k_2\|u_0\|_{H^1}^2 + 2k_3\|u_0\|_{H^1}
$$

(1.16)

for all $t \in [0, T^*)$.

With the precise blow-up quantity (1.15) at hand, we next display some sufficient conditions for the occurrence of the wave-breaking phenomenon to the Cauchy problem (1.1) in the case of a non-sign-changing momentum. The result reads:

**Theorem 1.6** Suppose $k_1 > 0$, $k_2$, $k_3 \geq 0$, $u_0(x) \in H^s(\mathbb{R})$ with $s > \frac{5}{2}$ and $m_0(x) \geq 0$. Let one of the following four cases be satisfied:

Case 1: For $k_1$, $k_2$, $k_3 > 0$, there exists some $x_0 \in \mathbb{R}$ and $\alpha > 2 + \frac{4k_1}{3k_2}$ such that $m_0(x_0) > 0$ and

$$
u_0(x_0) < -\frac{\alpha}{\sqrt{2}} \sqrt{\frac{2k_1 + 3k_2}{2k_1}} \left[ u(x_0) + \frac{3k_2\gamma + 3k_3}{2(2k_1 + 3k_2)} \right],$$

(1.17)

where

$$
\gamma = \frac{(3k_2k_3 + 4k_1k_3) \pm 2k_3 \sqrt{4k_1^2 + 6k_1k_2}}{3k_2^2}.
$$
Case 2: For $k_1, k_3 > 0, k_2 = 0$, there exists some $x_0 \in \mathbb{R}$ and $m_0(x_0) > 0$ such that [15]

$$u_{0,x}(x_0) < -\frac{\alpha}{\sqrt{2}} \left[ u(x_0) + \frac{3k_3}{4k_1} \right]. \tag{1.18}$$

Case 3: For $k_1, k_2 > 0, k_3 = 0$, there exists some $x_0 \in \mathbb{R}$ and $m_0(x_0) > 0$ such that

$$u_{0,x}(x_0) < -\frac{1}{\sqrt{2}} \sqrt{\frac{2k_1 + 3k_2}{2k_1}} u(x_0). \tag{1.19}$$

Case 4: For $k_1 > 0, k_2 = k_3 = 0$, there exists some $x_0 \in \mathbb{R}$ and $m_0(x_0) > 0$ such that [15]

$$u_{0,x}(x_0) < -\frac{1}{\sqrt{2}} u(x_0). \tag{1.20}$$

Then the solution $u(t, x)$ to the Cauchy problem (1.1) will blow up at time $T^*$ with

$$T^* \leq -\frac{1}{2k_1 m_0(x_0) u_{0,x}(x_0)}.$$

Remark 1.3 Similar to the proof of the wave-breaking of the generalized mCH equation considered in [15], for Case 1: $k_1, k_2, k_3 > 0$, we trace the dynamics of the ratio $\hat{u}_x/(\hat{u} + \gamma)$ along the characteristics with some nonnegative parameter $\gamma$ to be settled to make the constant term in the brackets of Eq. (7.7) to be zero in the proof of Theorem 1.6. Having determined the parameter $\gamma$ as in Eq. (7.9), we need the two conditions (7.11) and (7.12) to be satisfied simultaneously. In [15], the initial state of the corresponding condition (7.11) can ensure the corresponding condition (7.12) exactly. However, we have to multiply some factor $\alpha$ on the condition of the initial gradient to guarantee the condition (7.12) in the present paper, as we see on the right hand side of (1.17).

Theorem 1.6 has provided some sufficient conditions on the initial data for the breaking of waves, however, it contains no information of the blow-up rate. Using a different method, one can establish another wave-breaking Theorem which includes the estimation of the blow-up rate. The result reads

**Theorem 1.7** Suppose $k_1, k_2, k_3 > 0, s > \frac{5}{3}$, $u_0 \in H^s(\mathbb{R})$ and $m_0 \geq 0$. There exists some $x_1 \in \mathbb{R}$ and $C_2 > 0$ such that if $m_0(x_1) > 0$, $u_0(x_1) + 1 \leq C_2 m_0(x_1)$ and

$$\left[ 2k_1 + \frac{3k_2 u_0(x_1) + 2k_3}{m_0(x_1)} \right] \partial_x u_0(x_1) < -\left[ \frac{(2k_1 + (3k_2 + 2k_3)C_2)(\frac{38}{3}k_1 + 23k_2 + \frac{25}{3}k_3)(\|u_0\|_{H^1}^2 + \|u_0\|_{H^1}^3)}{m_0(x_1)} \right]^{1/2},$$

then, the solution $u(t, x)$ to the Cauchy problem (1.1) will blow up at time $T_*$ with

$$T_* \leq T_- := \frac{C_0}{2C_3} - \frac{1}{2} \sqrt{\frac{(C_0)^2}{C_3^2} - \frac{2}{C_3 m_0(x_1)}},$$

where $C_0, C_1$ and $C_3$ are given by

$$C_0 = -\partial_x u_0(x_1) \left( 2k_1 + \frac{3k_2 u_0(x_1) + 2k_3}{m_0(x_1)} \right),$$

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with the velocity \( c \) satisfying

\[
C_1 = \left( \frac{38}{3} k_1 + 23 k_2 + \frac{25}{2} k_3 \right) \left( \| u_0 \|_{H^1}^2 + \| u_0 \|_{H^1}^3 \right),
\]

and \( C_3 = \frac{1}{2} [2k_1 + (3k_2 + 2k_3)C_2]C_1 \), respectively.

Furthermore, when \( T_* = t_* \), one has the following estimation of the blow-up rate

\[
\lim \inf_{t \to T_*^-} \left( (T_* - t) \inf_{x \in \mathbb{R}} M(t, x) \right) \leq -\frac{1}{2}, \quad (1.22)
\]

where \( M(t, x) \) is the blow-up quantity defined in (8.1).

**Remark 1.4** The result obtained in Theorem 1.7 is inspired by [47,70]. However, we need the additional condition \( u_0(x_1) + 1 \leq C_2m_0(x_1) \) as stated in Theorem 1.7. This condition essentially comes from the terms associated with the coefficients \( k_2 \) and \( k_3 \) in Eq. (1.1) and is used to make the differential inequality (8.11) to be integrated not that hard.

According to the relation \( u = (1 - \partial_x^2)^{-1} m = p * m \) with \( p \) defined in the **Notation** part, Eq. (1.1) can be recast as the weak form

\[
u_t + \frac{k_1 + k_2}{3} u_x^3 - \frac{k_1}{3} u_x^3 + \frac{k_3}{2} u_x^2 + p_x \left[ \left( \frac{2k_1}{3} + k_2 \right) u_x^3 + (k_1 + \frac{3k_2}{2}) u_x^2 \right] + p \left[ \left( \frac{k_1}{3} + \frac{k_2}{2} \right) u_x^3 \right] = 0, \quad (1.23)
\]

which allows us to define the weak solution for (1.1) as follows:

**Definition 1.1** For \( u_0 \in W^{1,3}(\mathbb{R}) \), the function \( u(t, x) \in L^\infty_{loc}(0, T), W^{1,3}_{loc}(\mathbb{R}) \) is said to be a weak solution to the Cauchy problem (1.1) if it satisfies the following integral equality

\[
\int_0^T \int_\mathbb{R} \left\{ u \phi_t + \frac{k_1 + k_2}{3} u_x^3 \phi_x + \frac{k_1}{3} u_x^3 \phi + \frac{k_3}{2} u_x^2 \phi_x - p \left[ \left( \frac{k_1}{3} + \frac{k_2}{2} \right) u_x^3 \right] \phi \
+ p \left[ \left( \frac{2k_1}{3} + k_2 \right) u_x^3 + (k_1 + \frac{3k_2}{2}) u_x^2 \right] \phi \right\} \, dx \, dt
+ \int_\mathbb{R} u_0(x) \phi(0, x) \, dx = 0 \quad (1.24)
\]

for any smooth test function \( \phi(t, x) \in C_c^\infty([0, T] \times \mathbb{R}) \). Moreover, \( u(t, x) \) is called a global weak solution if \( T \) can be taken arbitrarily large.

In the following four propositions we would like to give the non-periodic and periodic peakon and multi-peakon solutions of Eq. (1.1).

**Proposition 1.1** For \( a \neq 0 \) and \( x_0 \in \mathbb{R} \), Eq. (1.1) admits a global weak peakon solution, in the sense of Definition 1.1, of the form

\[
u_a(t, x) = ae^{-|x - ce^{-t} - x_0|}, \quad (1.25)
\]

with the velocity \( c \) satisfying

\[
c = \left( \frac{2k_1}{3} + k_2 \right) a^2 + k_3 a. \quad (1.26)
\]
Remark 1.5 For real parameters \(a, c\) satisfying Eq. (1.26), the peakon solution (1.25) is a bright right-going travelling wave solution for \(a > 0\) and \(c > 0\), a dark right-going travelling wave solution for \(a < 0\) and \(c > 0\), a bright left-going travelling wave solution for \(a > 0\) and \(c < 0\), and a dark left-going travelling wave solution for \(a < 0\) and \(c < 0\).

Remark 1.6 In particular, one can reduce the peakon solution given by Eqs. (1.25)–(1.26) to those of the CH equation for \(k_1 = k_2 = 0, k_3 = 1\), where \(a = c [9]\); the mCH equation for \(k_2 = k_3 = 0, k_1 = 1\), where \(a = ±\sqrt{3c/2} [47]\); the Novikov equation for \(k_1 = k_3 = 0, k_2 = 1\), where \(a = ±\sqrt{c} [56]\); the mCH-CH equation for \(k_2 = 0, k_1k_3 \neq 0\), where \(a = (-3k_3 \pm \sqrt{9k_3^2 + 24ck_1})/(4k_1) [66,76]\); the mCH-Novikov equation for \(k_1k_2 \neq 0, k_3 = 0\), where \(a = ±\sqrt{3c/(2k_1 + 3k_2)} [70]\) (Notice that we here correct it); or the Novikov-CH equation for \(k_1 = 0, k_2k_3 \neq 0\), where \(a = (-k_3 \pm \sqrt{k_3^2 + 4ck_2})/(2k_2)\).

Remark 1.7 For the given real velocity \(c\) and for \(k_1k_2k_3 \neq 0\), one finds from Eq. (1.26) that
\[
a = \begin{cases} 
-3k_3 \pm \sqrt{9k_3^2 + 12c(2k_1 + 3k_2)} & , \text{as } 2k_1 + 3k_2 \neq 0, \\
c/k_3, & \text{as } 2k_1 + 3k_2 = 0, k_1k_2k_3 \neq 0. 
\end{cases}
\] (1.27)

Therefore, (i) if \(2k_1 + 3k_2 \neq 0, k_1k_2k_3 \neq 0\), and \(9k_3^2 + 12c(2k_1 + 3k_2) < 0\), then \(a\) is complex such that the solution given by Eqs. (1.25)–(1.26) is a complex peakon solution; (ii) If \(2k_1 + 3k_2 = 0, k_1k_2k_3 \neq 0\), then the peakon solution of Eq. (1.1) is of the form \(u(t, x) = q_k e^{-|x - ct - x_0|}\), which is independent of \(k_1, k_2\).

Similarly, the non-periodic multi-peakon solutions of Eq. (1.1) in the sense of Definition 1.1 is

**Proposition 1.2** The multi-peakon solutions of Eq. (1.1) in the sense of Definition 1.1 are given by
\[
u_{npm}(t, x) = \sum_{i=1}^{N} p_i(t)e^{-|\xi_i|}, \quad \xi_i = \xi_i(t) = x - q_i(t),
\] (1.28)
where the time-dependent position functions \(q_i(t)\) (without loss of generality \(q_1(t) < q_2(t) < \cdots < q_N(t)\) is required) and amplitudes \(p_i(t)\) satisfy the dynamical system
\[
\begin{aligned}
\dot{p}_i(t) &= p_i(t) \sum_{j=1}^{N} p_j(t) \text{sgn}(q_i - q_j)e^{-|q_i - q_j|} \left[ k_2 \left( \sum_{k=1}^{N} p_k(t)e^{-|q_i - q_k|} \right) + k_3 \right], \\
\dot{q}_i(t) &= -\frac{k_1}{3} p_i^2(t) - k_1 \left( \sum_{j=1}^{N} p_j(t) \text{sgn}(q_i - q_j)e^{-|q_i - q_j|} \right)^2 + (k_1 + k_2) \left( \sum_{j=1}^{N} p_j(t)e^{-|q_i - q_j|} \right)^2 + k_3 \sum_{j=1}^{N} p_j(t)e^{-|q_i - q_j|}
\end{aligned}
\] (1.29)
for \(i = 1, 2, \cdots, N\).

Remark 1.8 When \(N = 1\), the multi-peakon solution (1.28)–(1.29) reduces to the single non-periodic peakon solution given in Proposition 1.1.
Proposition 1.3 For any $a$, by using the distribution theory \[42\], the dynamical system for the mCH-Novikov equation \[70\]. System (1.29) was also given in the dynamical system for the mCH-CH equation \[85\]; when to the dynamical system for the mCH equation \[47\], when to the dynamical system for the Novikov equation \[56\]; when to the CH equation \[9\]; when to the mCH equation \[1\] for \( q(t) \neq 0 \), system (1.29) reduces to the dynamical system for \( \tanh \), whose explicit solution may be hard to be find, but can be numerically studied.

Remark 1.10 When \( k_1 = k_2 = 0 \) and \( k_3 = 1 \), system (1.29) becomes the dynamical system for the CH equation \[9\]; when \( k_1 = k_3 = 0 \) and \( k_2 = 1 \), system (1.29) becomes the dynamical system for the Novikov equation \[56\]; when \( k_2 = k_3 = 0 \) and \( k_1 = 1 \), system (1.29) becomes the dynamical system for the mCH equation \[47\], when \( k_2 = 0 \), system (1.29) becomes the dynamical system for the mCH-CH equation \[85\]; when \( k_3 = 0 \), system (1.29) becomes the dynamical system for the mCH-Novikov equation \[70\]. System (1.29) was also given in \[1\] by using the distribution theory \[42\].

Proposition 1.1 provides the explicit peakon formula for equation (1.1) on the line. Our next Proposition gives its explicit peakon formula in the periodic case.

Proposition 1.3 For any \( a \neq 0 \), the periodic function of the form

\[
 u_c(t, x) = a \cosh(\xi), \quad \xi = \xi(x, t) = 1/2 - (x - ct) + [x - ct], \tag{1.31}
\]

where the notation \([\cdot]\) denotes the floor function or the greatest integer function and

\[
 c = \left[ \frac{k_1}{3} + \cosh^2(1/2) \left( \frac{2k_1}{3} + k_2 \right) \right] a^2 + \cosh(1/2)k_3a, \tag{1.32}
\]

is a global weak periodic peakon solution to Eq. (1.1) in the sense of Definition 1.1 of the periodic case.

Remark 1.11 The definition of weak solution for the periodic case is similar to Definition 1.1, while \( \mathbb{R} \) and \( p(x) \) should be replaced by \( \mathbb{S} = [0, 1) \) and \( G(x) = \frac{1}{2} \text{csch}(1/2) \cosh (x - 1/2) \), respectively, there.

Remark 1.12 We also have a similar remark about the periodic peakon solutions as Remark 1.5. In particular, one can reduce the periodic peakon solution given by Eqs. (1.31)- (1.32) to those of the CH equation for \( k_1 = k_2 = 0 \), \( k_3 = 1 \) \[45,77\], where \( a = \text{sech}(1/2)c \); the mCH equation for \( k_2 = k_3 = 0 \), \( k_1 = 1 \) \[77\], where \( a = \pm \sqrt{3c/(2 + \cosh(1))} \); the Novikov equation for \( k_1 = k_3 = 0 \), \( k_2 = 1 \) \[44,45\], where \( a = \pm \text{sech}(1/2)\sqrt{c} \); the mCH-CH equation for \( k_2 = 0 \), \( k_1k_3 \neq 0 \) \[82\], where \( a = [-3 \cosh(1/2)k_3 \pm \sqrt{12ck_1 + 3 \cosh^2(1/2)(3k_3^2 + 8ck_1)}]/[(4 + 2 \cosh(1))k_1] \); the mCH-Novikov equation for \( k_1k_2 \neq 0 \), \( k_3 = 0 \) \[82\], where \( a = \pm \sqrt{3c/[k_1 + \cosh^2(1/2)](2k_1 + 3k_2)]} \); or the Novikov-CH equation for \( k_1 = 0 \), \( k_2k_3 \neq 0 \), where \( a = \text{sech}(1/2)(-k_3 \pm \sqrt{k_3^2 + 4ck_2})/(2k_2) \).
Remark 1.13 For the given real velocity \( c \) and for \( k_1k_2k_3 \neq 0 \), one finds from Eq. (1.32) that \( a = \left\{ \begin{array}{ll}
-3 \cosh(1/2) k_3 \pm \sqrt{9 \cosh^2(1/2) k_3^2 + 12cf(k_1, k_2)} \\
\frac{2f(k_1, k_2)}{k_3}
\end{array} \right., \quad \text{as } f(k_1, k_2) \neq 0,
\]
\[ \frac{2f(k_1, k_2)}{k_3}, \quad \text{as } f(k_1, k_2) = 0, \quad k_1k_2k_3 \neq 0,
\]
where \( f(k_1, k_2) = [2 + \cosh(1)]k_1 + 3 \cosh^2(1/2)k_2 \). Therefore, (i) if \( f(k_1, k_2) \neq 0 \), \( k_1k_2k_3 \neq 0 \) and \( 9 \cosh^2(1/2)k_3^2 + 12cf(k_1, k_2) < 0 \), then \( a \) is complex so that the solution given by Eqs. (1.25)–(1.26) is a complex peakon solution; (ii) If \( f(k_1, k_2) = 0 \), \( k_1k_2k_3 \neq 0 \), then the periodic peakon solution of Eq. (1.1) is of the form \( u_a(t, x) = \frac{\sech(1/2)c}{k_3} \cosh(x) \), which is independent of \( k_1, k_2 \).

Similarly, the periodic multi-peakon solutions of Eq. (1.1) in the sense of Definition 1.1 for the periodic case are presented as follows.

**Proposition 1.4** The periodic multi-peakon solutions of Eq. (1.1) in the sense of Definition 1.1 for the periodic case are given by
\[ u_{pm}(t, x) = \sum_{i=1}^{N} p_i(t) \cosh \left( \frac{1}{2} - (x - q_i(t)) + \lfloor x - q_i(t) \rfloor \right), \quad (1.34) \]
where the notation \( \lfloor - \rfloor \) denotes the floor function or the greatest integer function, the time-dependent position functions \( q_i(t) \) (without loss of generality, the condition, \( 0 < q_1(t) < q_2(t) < \cdots < q_N(t) < 1 \), is required) and amplitudes \( p_i(t) \) satisfy the dynamical system \( (\xi_{jm} := q_j - q_m) \)
\[ \dot{p}_m = \frac{k_2 p_m}{2 \sinh(1/2)} \left\{ \frac{1}{2} \sum_{j \neq m} p_j \text{sgn}(\xi_{jm}) \left[ p_m \left( \cosh(1/2 + |\xi_{jm}|) - \cosh(3/2 - |\xi_{jm}|) \right) \right. \right.
\]
\[ + \left. p \left( \cosh(1/2 - 2|\xi_{jm}|) - \cosh(3/2 - 2|\xi_{jm}|) \right) \right] \right) + 2 \sinh(1/2) \sum_{j < m < k} p_j p_k \sinh(\xi_{jm} + \xi_{km})
\]
\[ + \left( \sum_{m < j < k} - \sum_{j < m < k} \right) p_j p_k \left( \cosh(1/2 - |\xi_{jm}| - |\xi_{km}|) - \cosh(3/2 - |\xi_{jm}| - |\xi_{km}|) \right) \right]
\[ + \frac{k_3 p_m}{2 \sinh(1/2)} \sum_{j=1}^{N} p_j \text{sgn}(\xi_{jm}) \left( \cosh(|\xi_{jm}|) - \cosh(1 - |\xi_{jm}|) \right). \]
\[ \dot{q}_m = \left[ \frac{2k_1}{3} + k_2 \right] \sinh^2(1/2)
\]
\[ + \frac{k_2}{2} p_m^2 + \frac{(k_1 + k_2) p_m}{2 \sinh(1/2)} \sum_{j \neq m} p_j \left( \sinh(1/2 + |\xi_{jm}|) + \sinh(3/2 - |\xi_{jm}|) \right)
\]
\[ + \frac{2k_1 + k_2}{2 \sinh(1/2)} \left\{ \sum_{j < m < k} p_j p_k \left( \sinh(3/2 + \xi_{jm} - \xi_{km}) - \sinh(1/2 + \xi_{jm} - \xi_{km}) \right) \right.
\]
\[ + \sinh(1/2) \left[ \sum_{i=1}^{N} p_i^2 + 2 \left( \sum_{m < j < k} \sum_{j < m} p_j p_k \cosh(\xi_{jm} - \xi_{km}) \right) \right] \right.
\[ + \left. \frac{k_2}{2 \sinh(1/2)} \left( \sum_{j \neq m} p_j^2 \left( \sinh(3/2 - 2|\xi_{jm}|) - \sinh(1/2 - 2|\xi_{jm}|) \right) \right) \right]. \]
\[ +2 \sinh(1/2) \sum_{j < m < k} \cosh(\xi_{jm} + \xi_{km}) \]
\[ + \left( \sum_{m < j < k} + \sum_{j < k < m} \right) p_j p_k \left( \sinh(3/2 - |\xi_{km}| - |\xi_{jm}|) - \sinh(1/2 - |\xi_{km}| - |\xi_{jm}|) \right) \]
\[ + \frac{k_3}{2 \sinh(1/2)} \sum_{j=1}^{N} p_j \left( \sinh(|\xi_{jm}|) + \sinh(1 - |\xi_{jm}|) \right) \]
(1.35)

for \( m = 1, 2, \ldots, N \).

**Remark 1.14** When \( N = 1 \), the periodic multi-peakon solution (1.34) with (1.35) reduces to the single periodic peakon solution given in Proposition 1.3.

**Remark 1.15** We have investigated some properties of the Cauchy problem (1.1), including the local well-posedness of its strong solution, the Hölder continuity of its data-to-solution map, the blow-up criterion, the precise blow-up quantity and the wave-breaking phenomenon and so on. Some other properties associated with the Cauchy problem (1.1) can also be concerned with, for instance, the analyticity the persistence properties of its solution, which can be finished using similar methods as that of [52,53,70,72]. However, we will not consider these problems in the current paper in detail.

The rest of this paper is arranged as follows. The proof of Theorem 1.1 will be presented in Sect. 2, where the uniqueness and continuous dependence parts will be proved first and then the existence part. Section 3 will deal with the Hölder continuity of the data-to-solution map and prove Theorem 1.2. In Sect. 4, we will invoke the Moser-type estimates in Sobolev spaces to prove the blow-up criterion, i.e., Theorem 1.3. A global conservative property will be displayed in Sect. 5, where the precise blow-up quantity will also be established. The uniform upper bound of the blow-up quantity (1.16) stated in Theorem 1.5 will be proved in Sect. 6. Section 7 will provide the proof of the wave-breaking Theorem 1.6 by only considering a portion of the blow-up quantity. Section 8 will give the proof of Theorem 1.7 by inspecting the whole blow-up quantity. The non-periodic peakon solution stated in Proposition 1.1 and multi-peakon solutions of this equation in Proposition 1.2 will be considered in Sects. 9 and 10, respectively. The periodic peakon solution stated in Proposition 1.3 and multi-peakon in Proposition 1.4 will be considered in Sects. 11 and 12, respectively. In Appendixes A and B, some useful facts and Lemmas are given for the proofs of our main results, including the properties of Besov spaces and some estimates about the transport equation theory.

### 2 Proof of Theorem 1.1

Since the local well-posedness for the Cauchy problem (1.1) will be established in Besov-type spaces, we firstly recall some basic properties about the Littlewood-Paley theory (see Appendix A and [2,12] for more details) and some useful lemmas of the transport equation theory (see Appendix B and [2,32] for more details).

For the proof of Theorem 1.1, we first prove the uniqueness and the continuous dependent on the initial data parts. Suppose \( u, v \in B^s_{p,r} \) are two solutions of Eq. (1.23), the equivalent form of Eq. (1.1). Let \( w = u - v \). Then \( w \) satisfies the following transport equation
\[ w_t + \left[ (k_1 + k_2) v^2 - \frac{k_1}{3} (u_x^2 + u_x v_x + v_x^2) + k_3 v \right] w_x \\
= -[(k_1 + k_2) (u + v) + k_3] w u_x \\
- \left( \frac{k_1}{3} + \frac{k_2}{2} \right) p [w_x (u_x^2 + u_x v_x + v_x^2)] - \left( \frac{2k_1}{3} + k_2 \right) p_x [w(u^2 + u v + v^2)] \\
- \left( k_1 + \frac{3k_2}{2} \right) p_x [w u_x (u + v) + v^2 w_x] - k_3 p_x \left[ w(u + v) + \frac{1}{2} w_x (u_x + v_x) \right] \\
\equiv f(w, u, v) \tag{2.1} \]

with the initial condition \( w(0, x) = w_0(x) = u_0(x) - v_0(x) \).

Applying (B.2) in Lemma 12.7 to Eq. (2.1) yields

\[
\begin{align*}
\left\| w \right\|_L^{t, p, r}_{B^{1^{-1}}_{p, r}} & \leq C \int_0^t \left\| (k_1 + k_2) v^2 - \frac{k_1}{3} (u_x^2 + u_x v_x + v_x^2) + k_3 v \right\|_{B^{1^{-1}}_{p, r}} w(\tau) \left\| w(\tau) \right\|_L^{t, p, r}_{B^{1^{-1}}_{p, r}} d\tau \\
& \quad + \int_0^t \left\| f(w, u, v) \right\|_L^{t, p, r}_{B^{1^{-1}}_{p, r}} d\tau + \left\| w(0) \right\|_L^{t, p, r}_{B^{1^{-1}}_{p, r}}. \tag{2.2}
\end{align*}
\]

Thanks to the product law in the Besov spaces stated in Lemma 12.3 and the embedding relation in Lemma 12.1, one obtains

\[
\begin{align*}
\left\| (k_1 + k_2) v^2 - \frac{k_1}{3} (u_x^2 + u_x v_x + v_x^2) + k_3 v \right\|_{B^{1^{-1}}_{p, r}} & \leq C \left( \left\| v \right\|^2_{B^{2}_{p, r}} + \left\| u \right\|^2_{B^{2}_{p, r}} + \left\| v \right\|_{B^{2}_{p, r}} \right) \\
& \leq C \left( \left\| v \right\|^2_{B^{2}_{p, r}} + \left\| u \right\|^2_{B^{2}_{p, r}} + 1 \right). \tag{2.3}
\end{align*}
\]

A similar procedure yields

\[
\begin{align*}
\left\| -(k_1 + k_2) w u_x (u + v) \right\|_{B^{1^{-1}}_{p, r}} & \leq C \left\| w \right\|_{B^{1^{-1}}_{p, r}} \left\| u \right\|_{B^{1^{-1}}_{p, r}} \left\| u_x \right\|_{B^{1^{-1}}_{p, r}} \\
& \quad + C \left\| w \right\|_{B^{1^{-1}}_{p, r}} \left\| v \right\|_{B^{1^{-1}}_{p, r}} \left\| u_x \right\|_{B^{1^{-1}}_{p, r}} \\
& \leq C \left\| w \right\|_{B^{1^{-1}}_{p, r}} \left( \left\| u \right\|^2_{B^{2}_{p, r}} + \left\| v \right\|^2_{B^{2}_{p, r}} \right) \tag{2.4}
\end{align*}
\]

and

\[
\begin{align*}
\left\| -k_3 w u_x \right\|_{B^{1^{-1}}_{p, r}} & \leq C \left\| w \right\|_{B^{1^{-1}}_{p, r}} \left\| u \right\|_{B^{1^{-1}}_{p, r}}^2 \leq C \left\| w \right\|_{B^{1^{-1}}_{p, r}} \left( \left\| u \right\|^2_{B^{2}_{p, r}} + 1 \right). \tag{2.5}
\end{align*}
\]

Using the Moser-type estimates in Lemma 12.4, one finds

\[
\begin{align*}
\left\| -(k_1/3 + k_2/2) p [w_x (u_x^2 + u_x v_x + v_x^2)] \right\|_{B^{1^{-1}}_{p, r}} & \leq C \left\| w_x (u_x^2 + u_x v_x + v_x^2) \right\|_{B^{1^{-1}}_{p, r}} \\
& \leq C \left\| w_x \right\|_{B^{2}_{p, r}} \left\| u_x^2 + u_x v_x + v_x^2 \right\|_{B^{1^{-1}}_{p, r}}^2 \leq C \left\| w \right\|_{B^{2}_{p, r}} \left( \left\| u \right\|^2_{B^{2}_{p, r}} + \left\| v \right\|^2_{B^{2}_{p, r}} \right), \tag{2.6}
\end{align*}
\]
\[\|-(2k_1/3 + k_2) p_x \star [w(u^2 + u v + v^2)]\|_{B^{s-1}_{p,r}} \leq C \|w(u^2 + u v + v^2)\|_{B^{s}_{p,r}}\]
\[\leq C \|w\|_{B^{s}_{p,r}} \left\| u^2 + u v + v^2 \right\|_{B^{s}_{p,r}} \leq C \|w\|_{B^{s}_{p,r}} \left( \|u\|^2_{B^{s}_{p,r}} + \|v\|^2_{B^{s}_{p,r}} \right),\]
\[\|-(k_1 + 3k_2/2) p_x \star [wux (u + v) + v^2 w_x]\|_{B^{s-1}_{p,r}} \leq C \|wuux + wux + v^2 w_x\|_{B^{s-1}_{p,r}} \leq C \left( \|w\|_{B^{s}_{p,r}} \left( \|u\|^2_{B^{s}_{p,r}} + \|v\|^2_{B^{s}_{p,r}} \right) \right),\]
\[-k_3 p_x \star \left[ w(u + v) \right] \|_{B^{s-1}_{p,r}} \leq C \|w\|_{B^{s}_{p,r}} \left( \|u\|^2_{B^{s}_{p,r}} + \|v\|^2_{B^{s}_{p,r}} + 1 \right)\]

\[\|w\|_{B^{s-1}_{p,r}} \leq \|w(0)\|_{B^{s}_{p,r}} + C \int_0^t (\|u\|^2_{B^{s}_{p,r}} + \|v\|^2_{B^{s}_{p,r}} + 1) \|w\|_{B^{s}_{p,r}} \, dt.\]

The Gronwall’s inequality then yields
\[\|w\|_{B^{s-1}_{p,r}} \leq \|w(0)\|_{B^{s}_{p,r}} \exp \left\{ C \int_0^t \left( \|u\|^2_{B^{s}_{p,r}} + \|v\|^2_{B^{s}_{p,r}} + 1 \right) \, dt \right\},\]
which is sufficient to illustrate the uniqueness and the continuous dependence parts of Theorem 1.1.

Motivated by the proofs of local existence of the CH equation [32] and mCH-Novikov equation [70], we next prove the local existence part of Theorem 1.1. Let us start with the construction of approximate solutions for Eq. (1.1) in the spirit of Friedrichs. Let \( m^{(0)} = 0 \), and \( m^{(n+1)}(n = 0, 1, \ldots) \) being the solutions of the following linear transport equations
\[\begin{align*}
\partial_t m^{(n+1)} &+ \left( (k_1 + k_2)(m^{(n)})^2 - k_1(u_x^{(n)})^2 + k_3 u_x^{(n)} \right) \partial_x m^{(n+1)} \\
&= -u_x^{(n)} m^{(n)} \left( 2k_1 m^{(n)} + 3k_2 u_x^{(n)} + 2k_3 \right), \\
m^{(n+1)}(0, x) &= S^{n+1} u_0(x).
\end{align*}\]

Suppose \( m^{(n)} \in L^\infty(0, T; B^{s-2}_{p,r}) \), \( s - 2 > \max\{ \frac{1}{p}, \frac{1}{2} \} \). The condition on \( s \) ensures that \( B^{s-2}_{p,r} \) is an algebra. So the right hand side of Eq. (2.11) is in \( L^\infty \left( 0, T; B^{s-2}_{p,r} \right) \). Hence, by Lemma
12.8 and the high regularity of $u$, Eq. (2.11) has a global solution $m^{(n+1)} \in E_{p,r}^{x-2}$ for all positive $T$.

Applying (B.3) in Lemma 12.7 to Eq. (2.11) leads to

$$
\left\| m^{(n+1)} \right\|_{B_{p,r}^{x-2}} \leq \left\| S_{n+1} u_0 \right\|_{B_{p,r}^{x-2}} \exp \left[ C \int_0^t \left( (k_1 + k_2) (u^{(n)})^2 - k_1 (u_x^{(n)})^2 + k_3 u^{(n)} \right) d\tau \right]
+ C \int_0^t \exp \left[ C \int_\tau^t \left( (k_1 + k_2) (u^{(n)})^2 - k_1 (u_x^{(n)})^2 + k_3 u^{(n)} \right) d\tau' \right]
\times \left\| -2k_1 u^{(n)} m^{(n)} + 3k_2 u^{(n)} u_x^{(n)} m^{(n)} - 2k_3 u_x^{(n)} m^{(n)} \right\|_{B_{p,r}^{x-2}} d\tau
$$

for $n = 0, 1, 2, \cdots$.

Lemma 12.3 yields

$$
\left\| (k_1 + k_2) (u^{(n)})^2 - k_1 (u_x^{(n)})^2 + k_3 u^{(n)} \right\|_{B_{p,r}^{x-1}} \leq C \left( \left\| u^{(n)} \right\|_{B_{p,r}^{x+1}}^2 + \left\| u^{(n)} \right\|_{B_{p,r}^{x+1}} \right)
\leq C \left( \left\| u^{(n)} \right\|_{B_{p,r}^{x+1}}^2 + 1 \right),
\quad (2.13)
$$

$$
\left\| -2k_1 u_x^{(n)} m^{(n)} + 3k_2 u^{(n)} u_x^{(n)} m^{(n)} - 2k_3 u_x^{(n)} m^{(n)} \right\|_{B_{p,r}^{x-2}} \leq C \left( \left\| u^{(n)} \right\|_{B_{p,r}^{x+1}}^3 + \left\| u^{(n)} \right\|_{B_{p,r}^{x+1}}^2 \right)
\leq C \left( \left\| u^{(n)} \right\|_{B_{p,r}^{x+1}}^3 + \left\| u^{(n)} \right\|_{B_{p,r}^{x+1}}^2 \right),
\quad (2.14)
$$

Substituting (2.13)–(2.14) into (2.12), one finds

$$
\left\| u^{(n+1)} \right\|_{B_{p,r}^{x-2}} = \left\| m^{(n+1)} \right\|_{B_{p,r}^{x-2}} \leq \left\| S_{n+1} u_0 \right\|_{B_{p,r}^{x-2}} \exp \left[ C \int_0^t \left( \left\| u^{(n)} \right\|_{B_{p,r}^{x+1}}^2 + 1 \right) d\tau \right]
+ C \int_0^t \exp \left[ C \int_\tau^t \left( \left\| u^{(n)} \right\|_{B_{p,r}^{x+1}}^2 + 1 \right) d\tau' \right]
\left( \left\| u^{(n)} \right\|_{B_{p,r}^{x+1}}^3 + \left\| u^{(n)} \right\|_{B_{p,r}^{x+1}}^2 \right) d\tau.
\quad (2.15)
$$

To obtain the uniform upper bound of the solution sequence $\{u^{(n)}\}$, we assume $\left\| u^{(n)} \right\|_{B_{p,r}^{x-2}} \leq a(t)$ with $a(t)$ some unknown function of $t$. This combined with (2.15) yields

$$
\left\| u^{(n+1)} \right\|_{B_{p,r}^{x-2}} \leq \left\| u_0 \right\|_{B_{p,r}^{x-2}} \exp \left[ C \int_0^t \left( a^2(\tau) + 1 \right) d\tau \right]
+ C \int_0^t \exp \left[ C \int_\tau^t \left( a^2(\tau') + 1 \right) d\tau' \right]
\left( a^3(\tau) + a(\tau) \right) d\tau.
\quad (2.16)
$$

Let the right hand side of (2.16) be equal to $a(t)$. Then $a(t)$ satisfies the following ordinary differential equation

$$
\begin{cases}
\dot{a}(t) = 2C[a^3(t) + a(t)], \\
a(0) = \left\| u_0 \right\|_{B_{p,r}^{x-2}}.
\end{cases}
\quad (2.17)
$$
Solving (2.17), we deduce that a time $T$ can be chosen to satisfy

$$0 < T < \frac{1}{4C} \ln \left( 1 + \frac{1}{a^2(0)} \right)$$

so that the solution sequence $\{u^{(n)}\}$ of the linear transport Eq. (2.11) has the uniform bound

$$a(t) = \frac{a(0)e^{2Ct}}{\sqrt{a^2(0)(1 - e^{4Ct})} + 1}$$

for $t \in [0, T]$. Therefore, $\{u^{(n)}\}$ is uniformly bounded in $L^\infty([0, T]; B^s_{p,r})$.

In the following, we will prove that the solution sequence $\{u^{(n)}\}$ is Cauchy in $L^\infty([0, T]; B^{s-1}_{p,r})$, or equivalently $\{m^{(n)}\}$ is Cauchy in $L^\infty([0, T]; B^{s-3}_{p,r})$. Direct computation leads to

\[
\partial_t (m^{(n+1)} - m^{(n)}) + [(k_1 + k_2)(u^{(n+1)})^2 - k_1(u_x^{(n+1)})^2 + k_3u^{(n+1)}] \partial_x (m^{(n+1)} - m^{(n)}) \\
+ k_3(u^n - u^{(n+1)}) \partial_x (m^{(n+1)} - m^{(n)}) \\
- 2k_1u_x^{(n+1)}(u^{(n+1)})^2 - u_x^{(n)}(m^{(n)})^2 - 3k_2u^{(n+1)}u_x^{(n)}m^{(n)} - u^{(n)}u_x^{(n)}m^{(n)} \\
- 2k_1u_x^{(n+1)}m^{(n+1)} - u_x^{(n)}m^{(n)} = g, \quad n, l = 0, 1, \ldots.
\]

The right-hand side of (2.19) can be rewritten as

$$g = \left[ (u^{(n)} - u^{(n+1)})[(k_1 + k_2)(u^{(n)} + u^{(n+1)}) + k_3] \\
- k_1(u_x^{(n)} - u_x^{(n+1)})(u^{(n)} + u^{(n+1)}) \right] \partial_x m^{(n+1)} \\
- 2k_1[(u_x^{(n+1)} - u_x^{(n)})(m^{(n+1)})^2 + u_x^{(n)}(m^{(n)} - m^{(n+1)}) + m^{(n+1)}] \\
- 3k_2 \left[ (u^{(n+1)} - u^{(n)})u_x^{(n+1)}m^{(n+1)} + u^{(n)}u_x^{(n+1)} - u_x^{(n)}m^{(n+1)} \\
+ u^{(n)}u_x^{(n)}(m^{(n+1)} - m^{(n)}) \right] \\
- 2k_3[u_x^{(n+1)} - u_x^{(n)}m^{(n+1)} + u^{(n)}(m^{(n+1)} - m^{(n)})].$$

According to Lemma 12.7, one finds

$$\left\| m^{(n+1)} - m^{(n)} \right\|_{B^s_{p,r}} \leq \exp \left\{ C \int_0^t \| (k_1 + k_2)(u^{(n+1)})^2 - k_1(u_x^{(n+1)})^2 + k_3u^{(n+1)} \|_{B^{s-3}_{p,r}}(\tau) \mathrm{d}\tau \right\} \left\| m_0^{(n+1)} - m_0^{(n+1)} \right\|_{B^{s-3}_{p,r}}$$

$$+ \int_0^t \exp \left( -C \int_0^\tau \| (k_1 + k_2)(u^{(n+1)})^2 - k_1(u_x^{(n+1)})^2 \\
+ k_3u^{(n+1)} \|_{B^{s-3}_{p,r}}(\tau') \mathrm{d}\tau' \right) \left\| g \right\|_{B^{s-3}_{p,r}} \mathrm{d}\tau.$$
Invoking Lemma 12.3, we obtain
\[
\begin{align*}
&\left\|\{(k_1 + k_2)(u^n + u^{n+1}) + k_3(u^n - u^{n+1})\} - k_1(u_x^n - u_x^{n+1})(u_x^n + u_x^{n+1})\right\|_{B^{s-3}_{p,r}} \\
&\leq C \left\| \partial_x m^{(n+1)} \right\|_{B^{s-3}_{p,r}} \left\| u^n - u^{(n+1)} \right\|_{B^{s-1}_{p,r}} \left\| u^n + u^{(n+1)} \right\|_{B^{s-1}_{p,r}} \\
&\leq C \left\| m^{(n)} - m^{(n+1)} \right\|_{B^{s-3}_{p,r}} \left( \left\| u^n \right\|^2_{B^{s}_{p,r}} + \left\| u^{(n+1)} \right\|^2_{B^{s}_{p,r}} + \left\| u^{(n+1)} \right\|^2_{B^{s}_{p,r}} \right), \\
&\quad \leq C \left\| m^{(n)} - m^{(n+1)} \right\|^2_{B^{s-2}_{p,r}} \left\| u^n \right\|_{B^{s}_{p,r}} \left\| u^{(n+1)} \right\|_{B^{s}_{p,r}} \\
&\quad + \left\| m^{(n)} - m^{(n+1)} \right\|_{B^{s-2}_{p,r}} \left\| u_x^n \right\|_{B^{s}_{p,r}} \left\| m^{(n)} + m^{(n+1)} \right\|_{B^{s}_{p,r}} \\
&\quad \leq C \left\| m^{(n)} - m^{(n+1)} \right\|^2_{B^{s-2}_{p,r}} \left( \left\| u^n \right\|^2_{B^{s}_{p,r}} + \left\| u^{(n+1)} \right\|^2_{B^{s}_{p,r}} \right),
\end{align*}
\]
\[
\begin{align*}
&\leq C \left\| m^{(n)} - m^{(n+1)} \right\|_{B^{s-3}_{p,r}} \left( \left\| u^n \right\|^2_{B^{s}_{p,r}} + \left\| u^{(n+1)} \right\|^2_{B^{s}_{p,r}} \right) \left\| u_x^n \right\|_{B^{s}_{p,r}} \left\| m^{(n)} + m^{(n+1)} \right\|_{B^{s}_{p,r}} \\
&\leq C \left\| u^n - u^{(n+1)} \right\|_{B^{s-3}_{p,r}} \left\| u_x^n \right\|_{B^{s}_{p,r}} \left\| m^{(n)} \right\|_{B^{s-2}_{p,r}} + C \left\| u^n \right\|_{B^{s-2}_{p,r}} \left\| u_x^n \right\|_{B^{s}_{p,r}} \left\| u^{(n+1)} \right\|_{B^{s}_{p,r}} \\
&\leq C \left\| m^{(n)} - m^{(n+1)} \right\|_{B^{s-2}_{p,r}} \left( \left\| u^n \right\|^2_{B^{s}_{p,r}} + \left\| u^{(n+1)} \right\|^2_{B^{s}_{p,r}} + 1 \right).
\end{align*}
\]
and
\[
\begin{align*}
&\left\| -2k_3 \left[ (u_x^{n+1})^2 - (u_x^n)^2 \right] m^{(n+1)} + u_x^n \right\|_{B^{s-3}_{p,r}} \\
&\leq C \left\| m^{(n+1)} \right\|_{B^{s-3}_{p,r}} \left\| u_x^n - u_x^{n+1} \right\|_{B^{s-2}_{p,r}} + C \left\| m^{(n)} - m^{(n+1)} \right\|_{B^{s-3}_{p,r}} \left\| u_x^n \right\|_{B^{s-2}_{p,r}} \\
&\leq C \left\| m^{(n)} - m^{(n+1)} \right\|_{B^{s-2}_{p,r}} \left( \left\| u^n \right\|^2_{B^{s}_{p,r}} + \left\| u^{(n+1)} \right\|^2_{B^{s}_{p,r}} + 1 \right).
\end{align*}
\]
The substitution of (2.21)–(2.24) into (2.20) derives
\[
\left\| m^{(n+1)+} - m^{(n+1)} \right\|_{B^{s-3}_{p,r}} \\
\leq \exp \left\{ C \int_0^t \left( (k_1 + k_2)(u^n_{(n+1)})^2 - k_1(u_x^n)^2 + k_3u^n_{(n+1)} \right)_{B^{s-3}_{p,r}} (\tau) d\tau \right\}
\]
Then from induction one concludes this results from the following statement:

\[
\left[\left\| m^{(n+1+1)}_0 - m^{(n+1)}_0 \right\|_{B^s_{p,r}}\right] \\
+ \int_0^\tau \exp \left( -C \int_0^\tau \left( k_1 + k_2 \right)(u^{(n+1)})^2 - k_1(u^{(n+1)})^2 + k_3u^{(n+1)} \right) \left\| (\tau')^{\theta} \right\|_{B^s_{p,r}} d\tau'
\times \left\| m^{(n)} - m^{(n+1)} \right\|_{B^s_{p,r}} \left( \left\| u^{(n+1)} \right\|_{B^s_{p,r}}^2 + \left\| u^{(n+1)} \right\|_{B^s_{p,r}}^2 + \left\| u^{(n+1)} \right\|_{B^s_{p,r}}^2 + 1 \right) d\tau.
\]

(2.25)

Note that

\[
\left\| m^{(n+1+1)}_0 - m^{(n+1)}_0 \right\|_{B^s_{p,r}} = \left\| S_{n+1+1}m_0 - S_{n+1}m_0 \right\|_{B^s_{p,r}} = \left\| \sum_{q=n+1}^{n+1} \Delta_q m_0 \right\|_{B^s_{p,r}} \leq C2^{-n} \left\| m_0 \right\|_{B^s_{p,r}}
\]

(2.26)

and \( \{u^{(n)}\} \) is bounded in \( L^\infty([0, T); B^s_{p,r}) \), we thus deduce

\[
\left\| m^{(n+1+1)} - m^{(n+1)} \right\|_{B^s_{p,r}} \leq C_T \left( 2^{-n} + \int_0^\tau \left\| m^{(n+1)} - m^{(n)} \right\|_{B^s_{p,r}} d\tau \right.
\]

Then from induction one concludes

\[
\left\| m^{(n+1+1)} - m^{(n+1)} \right\|_{L^\infty(0,T; B^s_{p,r})} \leq \frac{C_T}{2n} \sum_{k=0}^n (\frac{2TCT}{k!})^k \\
+ \frac{(TCT)^{n+1}}{(n+1)!} \left\| m^{(l)} - m^{(0)} \right\|_{L^\infty(0,T; B^s_{p,r})}.
\]

(2.28)

Since \( \{m^{(n)}\} \) is uniformly bounded in \( L^\infty(0, T; B^{s-3}_{p,r}) \), one can find a new constant \( C'_T \) so that

\[
\left\| m^{(n+1+1)} - m^{(n+1)} \right\|_{L^\infty(0,T; B^s_{p,r})} \leq \frac{C'_T}{2n}.
\]

Accordingly, \( \{m^{(n)}\} \) is Cauchy in \( L^\infty(0, T; B^{s-3}_{p,r}) \) and converges to some limit function \( m \in L^\infty(0, T; B^{s-3}_{p,r}) \).

We are now in a position to prove the existence of the solution to equation (1.1). We will prove the limit function \( m \) satisfies equation (1.1) in the sense of distribution and belongs to \( E^s_{p,r} \).

Firstly, the Fatou property for the Besov spaces in Lemma 12.6(iii) and the uniform boundedness of \( \{m^{(n)}\} \) in \( L^\infty(0, T; B^{s-2}_{p,r}) \) ensure that \( m \in L^\infty(0, T; B^{s-2}_{p,r}) \).

Secondly, we claim that \( \{m^{(n)}\} \) converges to \( m \) in \( L^\infty(0, T; B^{s'}_{p,r}) \) for all \( s' < s - 2 \). In fact, this results from the following statement: \( \left\| m_n - m \right\|_{B^{s'}_{p,r}} \leq C \left\| m_n - m \right\|_{B^{s-2}_{p,r}} \) when \( s' \leq s - 3 \) and \( \left\| m_n - m \right\|_{B^{s'}_{p,r}} \leq C \left\| m_n - m \right\|_{B^{s-2}_{p,r}} \) when \( s - 3 < s' \leq s - 2 \) on account of the interpolation between Besov spaces stated in Lemma 12.2. This claim enables us to take limit in equation (2.11) to conclude that the limit function \( m \) does satisfy equation (1.1).

Note that equation (1.1) can be rewritten as a transport equation

\[
\partial_t m + \left[ k_1(u^2 - u_x^2) + k_2u^2 + k_3u \right] \partial_x m = -2k_1u_x m^2 - 3k_2uu_x m - 2k_3u_x m.
\]

(2.29)
Since $m \in L^\infty(0, T; B^{s-2}_{p,r})$, it is easy to deduce that the right hand side of the above equation also belongs to $L^\infty(0, T; B^{s-2}_{p,r})$ in view of the product law in Besov spaces and the Sobolev embedding. Consequently, Lemma 12.8 implies $m \in C([0, T); B^{s-2}_{p,r})$ when $r < \infty$ or $m \in C_w([0, T); B^{s-2}_{p,r})$ when $r = \infty$. On the other hand, from the Moser-type estimates in Lemma 12.4, we infer that $[k_1 u^2 - k_1 u_x^2 + k_2 u^2 + k_3 u] \partial_x m$ is bounded in $L^\infty(0, T; B^{s-3}_{p,r})$. Therefore, one knows $\partial_t m \in C((0, T); B^{s-3}_{p,r})$ when $r < \infty$ according to the high regularity of $u$ and equation (1.1) and thus $m \in E^{s-2}_{p,r}$. We thus complete the proof of Theorem 1.1.

### 3 Proof of Theorem 1.2

In this section, we will prove Theorem 1.2 in the spirit of [51] for the mCH equation. We first of all prove the Lipschitz continuity, namely, $\beta = 1$ in the region $A_1$. Applying $\partial_x$ to the equivalent, nonlocal form (1.23) of Eq. (1.1) and using (1.10) produce

$$\partial_t u_x = k_1 u_x^2 u_{xx} - \left( k_1 + \frac{k_2}{2} \right) u u_x^2 - (k_1 + k_2) u^2 u_{xx} + \left( \frac{2k_1}{3} + k_2 \right) u^3 - \frac{k_3}{2} u_x^2 - k_3 u u_{xx} + k_3 u^2 - p_x \left[ \left( \frac{k_1}{3} + \frac{k_2}{2} \right) u_x^3 \right] - p \left[ \left( \frac{2k_1}{3} + k_2 \right) u^3 \right] + \left( k_1 + \frac{3k_2}{2} \right) u u_x^2 + k_3 u^2 + \frac{k_3}{2} u_x^2.$$

(3.1)

Let $w = u_x$. Then Eqs. (1.23) and (3.1) lead to

$$\begin{cases}
  u_t = \frac{k_1}{3} w^3 - (k_1 + k_2) u^2 w - k_3 u w - F(u, w), \\
  w_t = k_1 u^2 w_x - \left( k_1 + \frac{k_2}{2} \right) u w^2 - (k_1 + k_2) u^2 w_x + \left( \frac{2k_1}{3} + k_2 \right) u^3 - \frac{k_3}{2} u^2 w_x + k_3 u^2 - G(u, w), \\
  u(0, x) = u_0(x), \quad w(0, x) = \partial_x u_0(x) = w_0(x),
\end{cases}$$

(3.2)

where the nonlocal terms $F$ and $G$ are given by

$$F(u, w) = p \left[ \left( \frac{k_1}{3} + \frac{k_2}{2} \right) w^3 \right] + p_x \left[ \left( \frac{2k_1}{3} + k_2 \right) u^3 + \left( k_1 + \frac{3k_2}{2} \right) u w^2 + k_3 u^2 + \frac{k_3}{2} w^2 \right],$$

(3.3)

$$G(u, w) = p_x \left[ \left( \frac{k_1}{3} + \frac{k_2}{2} \right) w^3 \right] + p \left[ \left( \frac{2k_1}{3} + k_2 \right) u^3 + \left( k_1 + \frac{3k_2}{2} \right) u w^2 + k_3 u^2 + \frac{k_3}{2} w^2 \right].$$

(3.4)

Using similar method as in [50], one can show that for $u_0(x) \in H^s (s > 5/2)$ the solution of system (3.2) corresponding to initial data $(u_0(x), w_0(x))$ satisfies $(u, w) \in C([0, T); H^{s-1})$ and the following size estimates in the lifespan of the solution

$$\|(u, w)\|_{H^{s-1}} \leq C \|u_0\|_{H^s}, \quad (s > 5/2)$$

(3.5)
with $C$ a generic constant.

Let $(v, z) \in C([0, T]; H^{l-1})$ be another solution to system (3.2) corresponding to initial data $(v_0(x), z_0(x))$. Set $\varphi = u - v$, $\psi = w - z$. Then $\varphi$ and $\psi$ satisfy

$$
\partial_t \varphi = \left[ \frac{k_1}{3} g - (k_1 + k_2) u \right] \varphi - (k_1 + k_2)(u + v) z \varphi - k_3 u \varphi - k_3 \varphi z - F(u, w) + F(v, z)
$$

(3.6)

and

$$
\partial_t \psi = \left[ k_1 u^2 - (k_1 + k_2) u^2 - k_3 u \right] \partial_x \psi
+ \left[ k_1 (w + z) \partial_x z - \left( k_1 + \frac{k_2}{2} \right) u (w + z) - \frac{k_3}{2} (w + z) \right] \varphi
$$

$$
+ \left[ \left( \frac{2k_1}{3} + k_2 \right) f - \left( k_1 + \frac{k_2}{2} \right) z^2 - (k_1 + k_2) (u + v) \partial_x z - k_3 \partial_x z + k_3 (u + v) \right] \varphi
- G(u, w) + G(v, z)
$$

(3.7)

respectively, where $f = u^2 + u v + v^2$ and $g = w^2 + w z + z^2$.

Applying the energy method to Eqs. (3.6)-(3.7) gives

$$
\frac{1}{2} \frac{d}{dt} \| \varphi \|_{H^l}^2 = \int_{\mathbb{R}} D^r \left[ \left( \frac{k_1}{3} g - (k_1 + k_2) u \right) \varphi \right] \cdot D^r \varphi dx
$$

(3.8a)

$$
- \int_{\mathbb{R}} D^r [(k_1 + k_2)(u + v) z \varphi] \cdot D^r \varphi dx
$$

(3.8b)

$$
- \int_{\mathbb{R}} D^r [k_3 u \varphi] \cdot D^r \varphi dx
$$

(3.8c)

$$
- \int_{\mathbb{R}} D^r [k_3 \varphi z] \cdot D^r \varphi dx
$$

(3.8d)

$$
- \int_{\mathbb{R}} D^r [F(u, w) - F(v, z)] \cdot D^r \varphi dx
$$

(3.8e)

and

$$
\frac{1}{2} \frac{d}{dt} \| \psi \|_{H^l}^2 = \int_{\mathbb{R}} D^r \left[ (k_1 u^2 - (k_1 + k_2) u^2 - k_3 u) \partial_x \psi \right] \cdot D^r \psi dx
$$

(3.9a)

$$
+ \int_{\mathbb{R}} D^r \left[ k_1 (w + z) \partial_x z - \left( k_1 + \frac{k_2}{2} \right) u (w + z) - \frac{k_3}{2} (w + z) \right] \varphi \right] \cdot D^r \varphi dx
$$

(3.9b)

$$
+ \int_{\mathbb{R}} D^r \left[ \left( \frac{2k_1}{3} + k_2 \right) f - \left( k_1 + \frac{k_2}{2} \right) z^2 - [(k_1 + k_2) (u + v) + k_3] \partial_x z
$$

$$
+ k_3 (u + v) \right] \varphi \right] \cdot D^r \varphi dx
$$

(3.9c)

$$
- \int_{\mathbb{R}} D^r [G(u, w) - G(v, z)] \cdot D^r \varphi dx.
$$

(3.9d)
We first estimate the terms in (3.8). As $1/2 < r \leq s - 1$, one can invoke the algebra property of $H^r$ and (3.5) to deduce

$$|\text{(3.8a)}| \leq C \left\| \left( \frac{k_1}{3} g - (k_1 + k_2) u^2 \right) \right\|_{H^r} \| \psi \|_{H^r} \leq C \left( \| g \|_{H^r} + \| u \|_{H^r}^2 \right) \| \psi \|_{H^r} \| \psi \|_{H^r}$$

$$\leq C \left( \left\| w \right\|_{H^{r-1}}^2 + \| w \|_{H^{r-1}} \left\| z \right\|_{H^{r-1}} + \| z \|_{H^{r-1}}^2 + \| u \|_{H^{r-1}}^2 \right) \| \psi \|_{H^r} \| \psi \|_{H^r}$$

$$\leq C \left( \left\| u_0 \right\|_{H^r}^2 + \left\| u_0 \right\|_{H^r} \left\| v_0 \right\|_{H^r} + \left\| v_0 \right\|_{H^r}^2 + \| u \|_{H^{r-1}}^2 \right) \| \psi \|_{H^r} \| \psi \|_{H^r}$$

$$\leq C \rho^2 \left\| \psi \right\|_{H^r} \| \psi \|_{H^r} \quad (3.10)$$

recalling $\| u_0 \|_{H^r} \leq \rho$ and $\| v_0 \|_{H^r} \leq \rho$ in the assumption of Theorem 1.2.

As $-1/2 < r \leq 1/2$ and $r \leq s - 2$, using Lemma 12.4 and (3.5) yields

$$|\text{(3.8a)}| \leq C \left( \| g \|_{H^{r+1}} + \| u^2 \|_{H^{r+1}} \right) \| \psi \|_{H^r} \| \psi \|_{H^r}$$

$$\leq C \left( \left\| w \right\|_{H^{r+1}}^2 + \| w \|_{H^{r+1}} \left\| z \right\|_{H^{r+1}} + \| z \|_{H^{r+1}}^2 + \| u \|_{H^{r+1}}^2 \right) \| \psi \|_{H^r} \| \psi \|_{H^r}$$

$$\leq C \rho^2 \| \psi \|_{H^r} \| \psi \|_{H^r}. \quad (3.11)$$

When $-1 \leq r \leq -1/2$, employing the inequality

$$\| fg \|_{H^r} \leq C \| f \|_{H^{r-1}} \| g \|_{H^r} \quad (-1 \leq r \leq 0, s > 3/2) \quad (3.12)$$

provided by Lemma 2 in [51], one finds

$$|\text{(3.8a)}| \leq C \left( \| g \|_{H^{r-1}} + \| u^2 \|_{H^{r-1}} \right) \| \psi \|_{H^r} \| \psi \|_{H^r}$$

$$\leq C \left( \left\| w \right\|_{H^{r-1}}^2 + \| w \|_{H^{r-1}} \left\| z \right\|_{H^{r-1}} + \| z \|_{H^{r-1}}^2 + \| u \|_{H^{r-1}}^2 \right) \| \psi \|_{H^r} \| \psi \|_{H^r}$$

$$\leq C \rho^2 \| \psi \|_{H^r} \| \psi \|_{H^r}. \quad (3.13)$$

Using similar procedure as above, we can estimate (3.8b)-(3.8d) as

$$|\text{(3.8b)}| \leq C \rho^2 \| \psi \|_{H^r}^2, \quad |\text{(3.8c)}| \leq C \rho \| \psi \|_{H^r} \| \psi \|_{H^r}, \quad |\text{(3.8d)}| \leq C \rho \| \psi \|_{H^r} \| \psi \|_{H^r} \quad (3.14)$$

for $r \in \{-1 \leq r \leq -1/2 \} \cup \{-1/2 < r \leq 1/2, r \leq s - 2 \} \cup \{1/2 < r \leq s - 1 \}$.

For the nonlocal terms (3.8e), a direct calculation yields

$$|\text{(3.8e)}| \leq \left| \int_R D^{r-2} \left[ \left( \frac{k_1}{3} + \frac{k_2}{2} \right) g \right] \cdot D^r \psi \ dx \right| \quad (3.15a)$$

$$+ \left| \int_R D^{r-2} \partial_x \left[ \left( \frac{2k_1}{3} + \frac{k_2}{2} \right) f + \left( k_1 + \frac{3k_2}{2} \right) z^2 + k_3 (u + v) \right] \right| \cdot D^r \psi \ dx \right| \quad (3.15b)$$

Processing similarly as the local terms, one has

$$|\text{(3.15a)}| \leq C \| g \|_{H^r} \| \psi \|_{H^r} \| \varphi \|_{H^r}$$

$$\leq \begin{cases} C \| g \|_{H^r} \| \psi \|_{H^r} \| \varphi \|_{H^r} \quad (1/2 < r \leq s - 1); \\ C \| g \|_{H^{r+1}} \| \psi \|_{H^r} \| \varphi \|_{H^r} \quad (-1/2 < r \leq 1/2, r \leq s - 2); \\ C \| g \|_{H^{r-1}} \| \psi \|_{H^r} \| \varphi \|_{H^r} \quad (-1 \leq r \leq -1/2) \end{cases}$$
Combining (3.8) and (3.10)-(3.17), one has arrived at

\[
\frac{\text{d}}{\text{d}t} \| \varphi \|_{H^r} \leq C (\rho^2 + \rho) (\| \varphi \|_{H^r} + \| \psi \|_{H^r})
\]

for \( r \in \{-1 \leq r \leq -1/2\} \cup \{-1/2 < r \leq 1/2, r \leq s - 2\} \cup \{1/2 < r \leq s - 1\} \).

We are now in a position to estimate (3.9). The following Calderon-Coifman-Meyer commutator estimate [51,80]

\[
\| [D^r \partial_x, f] g \|_{L^2} \leq C \| f \|_{H^{s-1}} \| g \|_{H^r}, (0 \leq r + 1 \leq s - 1; s - 1 > 3/2)
\]

will be used, where \([A, B] = AB - BA\) represents the commutator.

To deal with (3.9a), we first recast it as

\[
(3.9a) = \int_{\mathbb{R}} \left[ D^r \partial_x, (k_1 w^2 - (k_1 + k_2) u^2 - k_3 u) \right] \psi \cdot D^r \psi \, dx
\]

\[
+ \int_{\mathbb{R}} (k_1 w^2 - (k_1 + k_2) u^2 - k_3 u) D^r \partial_x \psi \cdot D^r \psi \, dx
\]

\[
- \int_{\mathbb{R}} D^r \left[ \psi \partial_x \left( k_1 w^2 - (k_1 + k_2) u^2 - k_3 u \right) \right] \cdot D^r \psi \, dx.
\]

Invoking (3.19), one finds

\[
| (3.20a) | \leq C \| [D^r \partial_x, (k_1 w^2 - (k_1 + k_2) u^2 - k_3 u)] \psi \|_{L^2} \| \psi \|_{H^r}
\]

\[
\leq C \| k_1 w^2 - (k_1 + k_2) u^2 - k_3 u \|_{H^{s-1}} \| \psi \|_{H^r}^2 \leq C (\rho^2 + \rho) \| \psi \|_{H^r}^2
\]

for \( r \in \{-1 \leq r \leq -1/2, r + s \geq 2\} \cup \{-1/2 < r \leq 1/2, r \leq s - 3\} \cup \{r > 1/2, r \leq s - 2\} \).

Integration by parts and Sobolev embedding inequality yields

\[
| (3.20b) | \leq C \| \partial_x [k_1 w^2 - (k_1 + k_2) u^2 - k_3 u] \|_{L^2} \| \psi \|_{H^r}^2
\]

\[
\leq C \| k_1 w^2 - (k_1 + k_2) u^2 - k_3 u \|_{L^\infty} \| \psi \|_{H^r}^2 \leq C (\rho^2 + \rho) \| \psi \|_{H^r}^2
\]

(3.20c) can be estimated as

\[
| (3.20c) | \leq C \| \psi \partial_x (k_1 w^2 - (k_1 + k_2) u^2 - k_3 u) \|_{H^r} \| \psi \|_{H^r}
\]

\[
\leq \left\{ \begin{array}{ll}
C \| \partial_x (k_1 w^2 - (k_1 + k_2) u^2 - k_3 u) \|_{H^r} \| \psi \|_{H^r}^2, & (1/2 < r \leq s - 2), \\
C \| \partial_x (k_1 w^2 - (k_1 + k_2) u^2 - k_3 u) \|_{H^{s-1}} \| \psi \|_{H^r}^2, & (-1/2 < r \leq 1/2, r \leq s - 3), \\
C \| \partial_x (k_1 w^2 - (k_1 + k_2) u^2 - k_3 u) \|_{H^{s-2}} \| \psi \|_{H^r}^2, & (-1 \leq r \leq -1/2, r + s \geq 2)
\end{array} \right.
\]
\[ \leq C\|k_2w^2 - (k_1 + k_2)u^2 - k_3u\|_{H^{r-1}}\|\psi\|_{H^r}^2 \leq C(\rho^2 + \rho)\|\psi\|_{H^r}^2, \]  
(3.23)

where the inequality \[ 51 \]
\[ \|fg\|_{H^r} \leq c_{r,s}\|f\|_{H^{r-2}}\|g\|_{H^s} \quad (-1 \leq r \leq 0, \quad r + s \geq 2, \quad s > 5/2) \]  
(3.24)

has been used to handle the third case in the brace.

Combining (3.21)-(3.23) produces
\[ |(3.9a)| \leq C(\rho^2 + \rho)\|\psi\|_{H^r}^2 \]  
(3.25)

for \( r \in \{-1 \leq r \leq -1/2, \quad r + s \geq 2\} \cup \{-1/2 < r \leq 1/2, \quad r \leq s - 3\} \cup \{r > 1/2, \quad r \leq s - 2\}.

We next estimate (3.9b). In fact, we can derive
\[ |(3.9b)| \leq C\left\| k_1(w + z)\overline{\partial_z}z - \left( k_1 + \frac{k_2}{2}\right)u(w + z) - \frac{k_3}{2}(w + z)\right\|_{H^r}\|\psi\|_{H^r} \]  
(3.26)

Using similar method as estimating (3.9b), one deduces
\[ |(3.9c)| \leq C(\rho^2 + \rho)\|\psi\|_{H^r}^2, \quad |(3.9d)| \leq C(\rho^2 + \rho)\|\psi\|_{H^r}^2 \]  
(3.27)

for \( r \in \{-1 \leq r \leq -1/2, \quad r + s \geq 2\} \cup \{-1/2 < r \leq 1/2, \quad r \leq s - 3\} \cup \{r > 1/2, \quad r \leq s - 2\}.

Finally, the estimation of the nonlocal term (3.9e) is similar to (3.8e) and we thus obtain
\[ |(3.9e)| \leq C(\rho^2 + \rho)(\|\psi\|_{H^r}^2 + \|\psi\|_{H^r}^2) \]  
(3.28)

for \( r \in \{-1 \leq r \leq -1/2\} \cup \{-1/2 < r \leq 1/2, \quad r \leq s - 2\} \cup \{1/2 < r \leq s - 1\}.

Gathering (3.20)-(3.28) into (3.9), one finds
\[ \frac{d}{dr}\|\varphi\|_{H^r} \leq C(\rho^2 + \rho)(\|\varphi\|_{H^r} + \|\psi\|_{H^r}) \]  
(3.29)

for \( r \in \{-1 \leq r \leq -1/2, \quad r + s \geq 2\} \cup \{-1/2 < r \leq 1/2, \quad r \leq s - 3\} \cup \{1/2 < r \leq s - 2\}.

Combining (3.18) and (3.29) yields
\[ \frac{d}{dr}(\|\varphi\|_{H^r} + \|\psi\|_{H^r}) \leq C(\rho^2 + \rho)(\|\varphi\|_{H^r} + \|\psi\|_{H^r}) \]  
(3.30)

or
\[ \|\varphi\|_{H^r} + \|\psi\|_{H^r} \leq C(\|\varphi(0)\|_{H^r} + \|\psi(0)\|_{H^r})e^{(\rho^2 + \rho)t}, \quad (t \in [0, T]) \]  
(3.31)

for \( r \in \{-1 \leq r \leq -1/2, \quad r + s \geq 2\} \cup \{-1/2 < r \leq 1/2, \quad r \leq s - 3\} \cup \{r > 1/2, \quad r \leq s - 2\}.

\( Springer \)
Therefore, we deduce
\[
\|u - v\|_{H^{r+1}} \leq C \|u_0 - v_0\|_{H^{r+1}} e^{(\rho^2 + \rho)T}.
\] (3.32)
Replacing \(r + 1\) with \(r\) in (3.32) leads to
\[
\|u - v\|_{H^r} \leq C \|u_0 - v_0\|_{H^r} e^{(\rho^2 + \rho)T}
\] (3.33)
for \(r \in \{0 \leq r \leq 1/2, \ r + s \geq 3\} \cup \{1/2 < r \leq 3/2, \ r \leq s - 2\} \cup \{3/2 < r \leq s - 1\}. Thus
the Lipschitz continuity in the region \(D_1\) has been established.

We next show the Hölder continuity in the region \(D_2 \cup D_3 \cup D_4\). The method is interpolation based on the Lipschitz continuity proved previously. The inequality [51]
\[
\|f\|_{H^\sigma} \leq \|f\|_{H^{\sigma_1}} \|f\|_{H^{\sigma_2}}, \quad (\sigma_1 < \sigma < \sigma_2)
\] (3.34)
will be used frequently in the remaining parts of this section.

As \((s, r) \in D_2\), we have
\[
\|u - v\|_{H^r} \leq \|u - v\|_{H^{3-s}}
\]
\[
\begin{align*}
\leq & C \|u_0 - v_0\|_{H^{3-s}} e^{(\rho^2 + \rho)T} \quad \text{by (3.33))} \\
\leq & C \|u_0 - v_0\|_{H^{3-s}} \|u_0 - v_0\|_{H^{3-s-r}} e^{(\rho^2 + \rho)T} \quad \text{by (3.34))} \\
\leq & C \rho^{3-s-r} \|u_0 - v_0\|_{H^r} e^{(\rho^2 + \rho)T}.
\end{align*}
\]

As \((s, r) \in D_3\), one finds \(s - 2 \leq r < s\), consequently
\[
\begin{align*}
\|u - v\|_{H^r} \leq & \|u - v\|_{H^{r-s}} \|u - v\|_{H^{s}} \quad \text{by (3.34))} \\
\leq & C \rho^{r-s} \|u_0 - v_0\|_{H^{s}} e^{(\rho^2 + \rho)T} \quad \text{by (3.33))} \\
\leq & C \rho^{r-s} \|u_0 - v_0\|_{H^r} e^{(\rho^2 + \rho)T}.
\end{align*}
\]

For \((s, r) \in D_4\), there holds
\[
\begin{align*}
\|u - v\|_{H^r} \leq & \|u - v\|_{H^{r-s+1}} \|u - v\|_{H^{r-s+1}} \quad \text{by (3.34))} \\
\leq & C \rho^{r-s+1} \|u_0 - v_0\|_{H^{r-s+1}} e^{(\rho^2 + \rho)T} \quad \text{by (3.33))} \\
\leq & C \rho^{r-s+1} \|u_0 - v_0\|_{H^r} e^{(\rho^2 + \rho)T}.
\end{align*}
\]

We thus complete the proof of Theorem 1.2.

4 Proof of Theorem 1.3

We would like to prove Theorem 1.3 by an induction argument on \(s > 1/2\). Let us first of all consider the case \(s \in (1/2, 1)\). Applying Lemma 12.9 to Eq. (2.29) yields
\[
\|m\|_{H^r} \leq \|m_0\|_{H^r} + C \int_0^t U'(\tau) \|m(\tau)\|_{H^{r}} d\tau + C \int_0^t 2k_1 u_x m^2 + 3k_2 u u_x m + 2k_3 u_x m \|m\|_{H^{r}} d\tau,
\] (4.1)
where \( U(t) = \int_0^t \| \partial_x (k_1(u^2 - u_x^2) + k_2u^2 + k_3u) \|_{L^\infty} \, dt \). Using \( u = (1 - \partial_x^2)^{-1}m = p \ast m \) and \( \| p \|_{L^1} = \| \partial_x p \|_{L^1} = 1 \), one finds after employing the Young inequality that for \( s \in \mathbb{R} \)
\[
\begin{align*}
\| u \|_{L^\infty} + \| u_x \|_{L^\infty} + \| u_{xx} \|_{L^\infty} & \leq C \| m \|_{L^\infty}, \\
\| u \|_{H^s} + \| u_x \|_{H^s} + \| u_{xx} \|_{H^s} & \leq C \| m \|_{H^s}.
\end{align*}
\]
(4.2)

Invoking (4.2), we derive
\[
U'(t) = \| \partial_x (k_1(u^2 - u_x^2) + k_2u^2 + k_3u) \|_{L^\infty} \leq C(\| u_x \|_{L^\infty} + \| u^2 \|_{L^\infty} + \| u \|_{L^\infty}) \leq C(\| m \|_{L^\infty} + \| m \|_{L^\infty}^2)
\]
(4.3)

and
\[
\begin{align*}
\| 2k_1u_x m^2 + 3k_2u_x m + 2k_3u_x m \|_{H^s} & \leq C(\| u_x \|_{L^\infty} \| m \|_{H^s} + \| u_x \|_{H^s} \| m \|_{L^\infty}) + C(\| u_x \|_{L^\infty} \| u \|_{H^s} + \| u \|_{H^s} \| u \|_{L^\infty}) \\
& \leq C(\| u_x \|_{L^\infty} \| m \|_{L^\infty} \| m \|_{H^s} + \| u_x \|_{H^s} \| m \|_{L^\infty}) + C(\| u_x \|_{L^\infty} \| u \|_{H^s} + \| u \|_{H^s} \| m \|_{L^\infty} \| u \|_{L^\infty}) \\
& \leq C(\| m \|_{H^s} \| m \|_{L^\infty} + \| m \|_{L^\infty}).
\end{align*}
\]
(4.4)

Substituting Eqs. (4.3) and (4.4) into Eq. (4.1) leads to
\[
\| m \|_{H^s} \leq \| m_0 \|_{H^s} + C \int_0^t \| m \|_{H^s} (\| m \|_{L^\infty}^2 + \| m \|_{L^\infty}) \, dt,
\]
which further generates
\[
\begin{align*}
\| m \|_{H^s} & \leq \| m_0 \|_{H^s} \exp \left\{ C \int_0^t (\| m \|_{L^\infty}^2 + \| m \|_{L^\infty}) \, dt \right\}
\end{align*}
\]
(4.5)
by means of the Gronwall inequality.

Accordingly, if \( \int_0^{T^*} \| m(\tau) \|_{L^\infty}^2 \, d\tau < \infty \) for the maximal existence time \( T^* < \infty \), then the inequality (4.5) indicates that \( \lim \sup \| m(t) \|_{H^s} < \infty \), which contradicts our assumption on \( T^* \). Thus we complete the proof of Theorem 1.3 for the case \( s \in (1/2, 1) \).

We are next concerned with the case \( s \in [1, 2) \). Applying \( \partial_x \) to Eq. (2.29) yields
\[
\begin{align*}
\partial_t m_x & + [k_1(u^2 - u_x^2)] + k_2u^2 + k_3u] \partial_x m_x \\
& = -\partial_x [k_1(u^2 - u_x^2) + k_2u^2 + k_3u] \partial_x m - 2k_1u_x m^2 - 2k_1u_x mm_x \\
& - 3k_2u_x^2 m - 3k_2u_x m - 3k_2u_x m - 2k_3u_x m - 2k_3u_x m_x,
\end{align*}
\]
(4.6)
which combined with Lemma 12.9 enables us to conclude
\[
\begin{align*}
\| \partial_t m \|_{H^{s-1}} & \leq \| \partial_x m_0 \|_{H^{s-1}} + C \int_0^t U'(\tau) \| \partial_x m(\tau) \|_{H^{s-1}} \, d\tau \\
& + C \int_0^t \| - \partial_x [k_1(u^2 - u_x^2) + k_2u^2 + k_3u] \partial_x m \|_{H^{s-1}} \, d\tau \\
& + C \int_0^t \| 2k_1u_x m^2 + 4k_1u_x mm_x + 3k_2u_x^2 m + 3k_2u_x m + 3k_2u_x m_x \|_{H^{s-1}} \, d\tau \\
& + C \int_0^t \| - 2k_3u_x m - 2k_3u_x m_x \|_{H^{s-1}} \, d\tau.
\end{align*}
\]
(4.7)
For the integrand of the second integral on the right hand side of (4.7), one finds
\[ || - \partial_x[k_1(u^2 - u_x^2) + k_2u^2 + k_3u]\partial_xm||_{H^{s-1}} \leq C||\partial_x(u^2 - u_x^2)\partial_xm||_{H^{s-1}} + C||uu_x\partial_xm||_{H^{s-1}} + C||u_x\partial_x m||_{H^{s-1}}. \]  \hspace{1cm} (4.8)

Employing Lemma 12.5 and (4.2), there holds
\[ || u_x\partial_x m||_{H^{s-1}} \leq ||u_x m||_{H^{s}} + ||u_x m||_{H^{s-1}} \leq ||u_x||_{L^\infty}||m||_{H^{s}} + ||m||_{L^\infty}||u_x||_{H^{s}} + ||u_x||_{L^\infty}||m||_{H^{s}} + ||m||_{L^\infty}||u_x||_{H^{s-1}} \leq ||m||_{H^{s}}||m||_{L^\infty}. \]  \hspace{1cm} (4.9)

Similar argument leads to
\[ ||uu_x\partial_x m||_{H^{s-1}} \leq ||uu_x m||_{H^{s}} + ||u_x^2 m||_{H^{s-1}} + ||uu_{xx} m||_{H^{s-1}} \leq ||u||_{L^\infty}||u_x m||_{H^{s}} + ||u_x m||_{L^\infty}||u||_{H^{s}} + ||u_x^2 m||_{L^\infty} \leq ||u||_{L^\infty}||u_x||_{L^\infty}||m||_{H^{s}} + ||u||_{L^\infty}||u_x||_{H^{s-1}} + 2||m||_{L^\infty}||m||_{H^{s}} + ||m||_{L^\infty}||u_x||_{L^\infty}||u_x||_{H^{s-1}} + ||u_x||_{L^\infty}||m||_{H^{s-1}} + ||u_x||_{L^\infty}||u||_{H^{s-1}} \leq C||m||_{H^{s}}||m||_{L^\infty} \]  \hspace{1cm} (4.10)

and
\[ ||\partial_x(u^2 - u_x^2)m||_{H^{s-1}} \leq ||u_x m||_{H^{s}} + ||u_x^2 m||_{H^{s-1}} \leq C||u||_{L^\infty}||u_x||_{H^{s}} + ||u_x||_{L^\infty}||m||_{H^{s}} + C||u_x||_{H^{s-1}} ||m||_{L^\infty} \leq C||m||_{H^{s}}||m||_{L^\infty}. \]  \hspace{1cm} (4.11)

Substituting Eqs. (4.9)-(4.11) into Eq. (4.8), we derive
\[ || - \partial_x[k_1(u^2 - u_x^2) + k_2u^2 + k_3u]\partial_xm||_{H^{s-1}} \leq C||m||_{H^{s}}||m||_{L^\infty}. \]  \hspace{1cm} (4.12)

On the other hand, the procedure of handling Eqs. (4.9)-(4.11) enables one to easily deal with the integrands of the third and fourth integrals on the right hand side of Eq. (4.7) and obtain the corresponding results in analogy with Eq. (4.12).

Plugging these estimates into Eq. (4.7) yields
\[ ||\partial_x m||_{H^{s-1}} \leq ||\partial_x m_0||_{H^{s-1}} + C\int_0^t ||m||_{H^{s}}(||m||_{L^\infty}^2 + ||m||_{L^\infty})d\tau. \]

Then arguing similarly as the case for \( s \in (1/2, 1) \), one can prove Theorem 1.3 in the case of \( s \in [1, 2) \).

We finally consider the case \( s \geq 2 \). Assume \( 2 \leq k \in \mathbb{N} \). Suppose (1.13) holds when \( k - 1 \leq s < k \). Using induction, we should prove the validity of (1.13) for \( k \leq s < k + 1 \). Applying \( \partial_x^2 \) to Eq. (2.29) yields
\[
\partial_t \partial_x^k m + [k_1 (u^2 - u_x^2) + k_2 u^2 + k_3 u] \partial_x \partial_x^k m \\
= - \sum_{l=0}^{k-1} C_k^l \partial_x^{k-l} [k_1 (u^2 - u_x^2) + k_2 u^2 + k_3 u] \partial_x^{l+1} m - 2k_1 \partial_x^k (u_x m^2) \\
- 3k_2 \partial_x^k (uu_x m) - 2k_3 \partial_x^k (u_x m). \tag{4.13}
\]

Applying Lemma 12.9 again yields

\[
\| \partial_x^k m \|_{H^{r-k}} \leq \| \partial_x^k m_0 \|_{H^{r-k}} + C \int_0^T U'(\tau) \| \partial_x^k m(\tau) \|_{H^{r-k}} d\tau \\
+ C \int_0^T \left\| \sum_{l=0}^{k-1} C_k^l \partial_x^{k-l} [k_1 (u^2 - u_x^2) + k_2 u^2 + k_3 u] \partial_x^{l+1} m \right\|_{H^{r-k}} d\tau \\
+ C \int_0^T \| - 2k_1 \partial_x^k (u_x m^2) - 3k_2 \partial_x^k (uu_x m) - 2k_3 \partial_x^k (u_x m) \|_{H^{r-k}} d\tau.
\tag{4.14}
\]

Invoking Lemma 12.5 and the Sobolev embedding inequality produces

\[
\left\| \sum_{l=0}^{k-1} C_k^l \partial_x^{k-l} [k_1 (u^2 - u_x^2) + k_2 u^2 + k_3 u] \partial_x^{l+1} m \right\|_{H^{r-k}} \\
\leq \sum_{l=0}^{k-1} C_k^l \| \partial_x^{k-l} [k_1 (u^2 - u_x^2) + k_2 u^2 + k_3 u] \partial_x^{l+1} m \|_{H^{r-k}} \\
\leq \sum_{l=0}^{k-1} C_k^l \left( \| k_1 (u^2 - u_x^2) + k_2 u^2 + k_3 u \|_{H^{r-l+1}} \| \partial_x^l m \|_{L^\infty} \\
+ \| \partial_x^{k-l} [k_1 (u^2 - u_x^2) + k_2 u^2 + k_3 u] \|_{L^\infty} \| m \|_{H^{r-k+l+1}} \right) \\
\leq \sum_{l=0}^{k-1} C_k^l \left[ (\| u \|_{L^\infty} + \| u_x \|_{L^\infty} + 1)(\| u \|_{H^{r-l+1}} + \| u_x \|_{H^{r-l+1}}) \right] \| m \|_{H^{r-k+l+1}} \\
+ (\| u \|_{H^{r-l+1}}^2 + \| u_x \|_{H^{r-l+1}}^2 + 1) \| m \|_{H^{r-k+l+1}} \\
\leq C \| m \|_{H^{r-k+\frac{1}{2}+\epsilon_0}}^2 \| m \|_{H^{r}} + C \| m \|_{H^{r-k+\frac{1}{2}+\epsilon_0}} \| m \|_{H^{r}} \\
\leq C \left( \| m \|_{H^{r-k+\frac{1}{2}+\epsilon_0}}^2 + 1 \right) \| m \|_{H^{r}}, \tag{4.15}
\]

where \( \epsilon_0 \in (0, 1/8) \) so that \( H^{r-k+\frac{1}{2}+\epsilon_0}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R}) \).

Utilizing similar method as the estimation of (4.15), we have

\[
\left\| \partial_x^k (uu_x m^2) \right\|_{H^{r-k}} \leq \left\| uu_x m^2 \right\|_{H^{r}} \leq \left\| m^2 \right\|_{H^{r}} \left\| u \right\|_{L^\infty} + \left\| u_x \right\|_{H^{r}} \left\| m \right\|_{L^\infty} \\
\leq \left\| m \right\|_{L^\infty}^2 \left\| m \right\|_{H^{r}}, \tag{4.16}
\]

\[
\left\| \partial_x^k (uu_x m) \right\|_{H^{r-k}} \leq \left\| uu_x m \right\|_{H^{r}} \leq \left\| u \right\|_{L^\infty} \left\| u_x m \right\|_{H^{r}} + \left\| u \right\|_{H^{r}} \left\| u_x m \right\|_{L^\infty}
\]
\begin{align}
&\leq \| u \|_{L^\infty} \left( \| u_x \|_{L^\infty} \| m \|_{H^s} + \| u_x \|_{H^s} \| m \|_{L^\infty} \right) \\
&\quad + \| u \|_{H^s} \| u_x \|_{L^\infty} \| m \|_{L^\infty} \\
&\leq C \| m \|_{L^\infty} \| m \|_{H^s},
\end{align}
(4.17)

and
\begin{align}
\left\| \partial_x^k (u_x m) \right\|_{H^{s-k}} &\leq \left\| u_x m \right\|_{H^s} \\
&\leq \left\| m \right\|_{H^s} \left\| u_x \right\|_{L^\infty} + \left\| u_x \right\|_{H^s} \| m \|_{L^\infty} \\
&\leq C \| m \|_{L^\infty} \| m \|_{H^s}.
\end{align}
(4.18)

Combining Eqs. (4.14)–(4.18) deduces
\begin{equation}
\| m \|_{H^s} \leq \left\| m_0 \right\|_{H^s} + C \int_0^t \left( \| m \|_{H^{s-\frac{1}{2}+\epsilon_0}}^2 + 1 \| m \|_{H^s}(\tau) \right) d\tau.
\end{equation}
(4.19)

Then the Gronwall inequality implies
\begin{equation}
\left\| m(t) \right\|_{H^s} \leq \left\| m_0 \right\|_{H^s} \exp \left\{ C \int_0^t \left( \| m(\tau) \|_{H^{s-\frac{1}{2}+\epsilon_0}}^2 + 1 \right) d\tau \right\}.
\end{equation}
(4.20)

Therefore, if the maximal existence time $T^* < \infty$ satisfies $\int_0^{T^*} \| m(\tau) \|_{L^\infty}^2 d\tau < \infty$, then the uniqueness of the solution provided by Theorem 1.1 enables us to conclude the uniform boundedness of $\| m(t) \|_{H^{s-\frac{1}{2}+\epsilon_0}}$ in $t \in (0, T^*)$ taking into account the induction assumption, which along with (4.20) indicates $\limsup_{t \to T^*} \| m(t) \|_{H^s} < \infty$, a contradiction. We thus complete the proof of Theorem 1.3.

5 Proof of Theorem 1.4

To prove Theorem 1.4, we will first of all establish a kind of global conservative property associated to Eq. (1.1). Note that the corresponding trajectory equation emanating from $x$ reads
\begin{equation}
\begin{cases}
\frac{d}{dt} q(t, x) = [k_1(u^2 - u_x^2) + k_2 u^2 + k_3 u](t, q(t, x)), & (t, x) \in (0, T^*) \times \mathbb{R}, \\
q(0, x) = x, & x \in \mathbb{R},
\end{cases}
\end{equation}
(5.1)

for the flow generated by $k_1(u^2 - u_x^2) + k_2 u^2 + k_3 u$.

The following lemma shows that, the momentum $m(t, x)$ will not change sign for all $t \in [0, T^*)$ as long as the initial value $m_0 = (1 - \partial_x^2) u_0$ not changing sign. This kind of conservative property of the momentum $m$ is significant in proving the wave-breaking Theorem 1.6 in the non-sign-changing case.
Lemma 5.1 Suppose $u_0 \in H^s(\mathbb{R})$ with $s > \frac{5}{2}$. Let $T^* > 0$ be the maximal existence time of the solution $u$ to Eq. (1.1). Then there exists a unique solution $q \in C^1([0, T^*) \times \mathbb{R}; \mathbb{R})$ to Eq. (5.1) satisfying
\begin{equation}
q_x(t, x) = \exp \left( \int_0^t [(2k_1m + 2k_2u + k_3)u_x](\tau, q(\tau, x))d\tau \right) > 0,
\end{equation}

namely, $q(t, \cdot)$ is an increasing diffeomorphism of $\mathbb{R}$. Moreover, the expression of $m(t, q(t, x))$ is
\begin{equation}
m(t, q(t, x)) = m_0(x)\exp \left( - \int_0^t [(2k_1m + 3k_2u + 2k_3)u_x](\tau, q(\tau, x))d\tau \right),
\end{equation}

$(t, x) \in [0, T^*) \times \mathbb{R}$.

Proof Applying $\partial_x$ to Eq. (5.1) leads to
\begin{equation}
\begin{aligned}
&\frac{d}{dt} q_x(t, x) = [(2k_1m + 2k_2u + k_3)u_x](t, q(t, x))q_x(t, x), \\
&q_x(0, x) = 1.
\end{aligned}
\end{equation}

Solving (5.4) produces the solution given by (5.2). The Sobolev embedding inequality gives for $T' < T^*$
\[ \sup_{(s, x) \in [0, T') \times \mathbb{R}} |(2k_1m + 2k_2u + k_3)u_x(s, x)| < \infty, \]

which along with (5.2) yields $q_x(t, x) \geq \exp(-Ct)$, $(t, x) \in [0, T^*) \times \mathbb{R}$ for some $C > 0$, which states clearly that $q(t, \cdot)$ is an increasing diffeomorphism of $\mathbb{R}$ before the blow-up time, that is, (5.2) holds.

It follows from Eqs. (1.1) and (5.1) that
\begin{equation}
\begin{aligned}
\frac{d}{dt} m(t, q(t, x)) &= m_t(t, q(t, x)) + m_x(t, q(t, x))q_t(t, x) \\
&= m_t(t, q(t, x)) + m_x(t, q(t, x))[k_1(u^2 - u_x^2) + k_2u^2 + k_3u](t, q(t, x)) \\
&= [m_t + m_x(k_1(u^2 - u_x^2) + k_2u^2 + k_3u)](t, q(t, x)) \\
&= [(-2k_1m - 3k_2u - 2k_3)u_x](t, q(t, x))m(t, q(t, x)).
\end{aligned}
\end{equation}

Solving Eq. (5.5) yields Eq. (5.3). We thus complete the proof of Lemma 5.1. \hfill $\Box$

Proof of Theorem 1.4 From the expression of $m(t, q(t, x))$ in Eq. (5.3), we conclude that if there exists a positive constant $K_1$ such that
\[ \inf_{x \in \mathbb{R}} (2k_1u_xm(t, x) + 3k_2u_xu_x(t, x) + 2k_3u_x(t, x)) \geq -K_1, \quad 0 \leq t \leq T^*, \]

then
\[ \|m(t)\|_{L^\infty} = \|m(t, q(t, x))\|_{L^\infty} \]
\[ \leq e^{K_1T^*}\|m_0\|_{L^\infty}, \]

which combined with Theorem 1.3 implies that $m(t, x)$ will not blow up in a finite time.

However, if (1.15) holds true, then Theorem 1.1 and the Sobolev embedding ensure that $m(t, x)$ will blow up in a finite time. We thus complete the proof of Theorem 1.4. \hfill $\Box$
6 Proof of Theorem 1.5

For the proof of Theorem 1.5, we need to introduce the following Lemma [20]:

**Lemma 6.1** Let \( T > 0 \) and \( w(t, x) \in C^1([0, T]; H^2(\mathbb{R})) \). Then for every \( t \in [0, T) \) there exists at least one point \( x = \xi(t) \in \mathbb{R} \) such that \( W(t) := \inf_{x \in \mathbb{R}} [w_x(t, x)] = w_x(t, \xi(t)) \). and the function \( W(t) \) is almost everywhere differentiable on \((0, T)\) with

\[
\frac{dW(t)}{dt} = w_{tx}(t, \xi(t)), \quad \text{a.e. on } (0, T).
\]

We next prove (1.16). We only need to consider the case \( s = 3 \), and the general case follows by the density argument. Let \( M(t, x) = 2k_1u_x m(t, x) + 3k_2uu_x(t, x) + 2k_3u_x(t, x) \). Firstly, Corollary 1.1 shows that \( M \in C([0, T); H^3) \cap C^1([0, T); H^{3-1}) \) with \( T \) representing the maximal existence time of the corresponding solution. Given \( t \in [0, T) \), Lemma 6.1 implies there exists some point \( x_0(t) \in \mathbb{R} \) such that

\[
M(t, x_0(t)) = \sup_{x \in \mathbb{R}} M(t, x), \quad \text{i.e., } M_x(t, x_0(t)) = 0, \quad \text{a.e. on } (0, T) \quad (6.1)
\]

The condition \( s > \frac{1}{2} \) indicates \( H^s(\mathbb{R}) \hookrightarrow C_0(\mathbb{R}) \), the space of all continuous functions on \( \mathbb{R} \) vanishing as \( |x| \to \infty \). We deduce in light of Corollary 1.1 that

\[
M(t, x_0(t)) \geq 0 \quad \text{for all } t \in [0, T) \quad (6.2)
\]

Recall that the map \( q(t, \cdot) \) is an increasing diffeomorphism of \( \mathbb{R} \) in Lemma 5.1, therefore, there exists some \( \eta_0 = \eta_0(t) \in \mathbb{R} \) satisfying \( q(t, \eta_0(t)) = x_0(t) \). At the point \( (t, q(t, \eta_0)) = (t, x_0(t)) \), one has

\[
\frac{d}{dt} m(t, x_0(t)) = -(mM)(t, x_0(t)) \quad (6.3)
\]

after substituting \( x = \eta_0(t) \) into Eq. (5.5).

Integrating (6.3) and using (6.2) enable us to conclude

\[
m(t, x_0(t)) = m_0(x_0(0)) \exp \left( -\int_0^t M(\tau, x_0(\tau))d\tau \right) \leq m_0(x_0(0)) \leq \sup_{x \in \mathbb{R}} m_0(x) \quad (6.4)
\]

Next, let’s recall the relations between \( u \) and \( m \):

\[
u(t, x) = p \ast m(t, x), \quad u_x(t, x) = p_x \ast m(t, x) \quad (6.5)
\]

On the other hand, the conditions \( m_0 \geq 0 \) and \( m_0(x_0) > 0 \) imply \( m(t, x) \geq 0 \) and \( m(t, q(t, x_0)) > 0 \) in view of the expression of \( m(t, q(t, x)) \) in Eq. (5.3). So one derives from (6.5) that \( u(t, x) \geq 0 \) and \( u(t, q(t, x_0)) > 0 \). Furthermore, it follows from (1.10) and (6.5) that

\[
u \pm u_x = 2p_x \ast m \geq 0 \quad (6.6)
\]

which indicates that \( |u_x(t, x)| \leq u(t, x) \).

From (6.4) and the inequality \( |u_x| \leq u \), one obtains

\[
M(t, x_0(t)) = 2k_1u_x m(t, x_0(t)) + 3k_2uu_x(t, x_0(t)) + 2k_3u_x(t, x_0(t)) \\
\leq 2k_1 \|u\|_{L^\infty} \sup_{x \in \mathbb{R}} m_0(x) + 3k_2 \|u\|_{L^\infty}^2 \|u\|_{L^\infty} + 2k_3 \|u\|_{L^\infty} \\
\leq 2k_1 \|u_0\|_{H^1} \sup_{x \in \mathbb{R}} m_0(x) + 3k_2 \|u_0\|_{H^1}^2 + 2k_3 \|u_0\|_{H^1} \quad (6.7)
\]

which gives (1.16). We thus complete the proof of Theorem 1.5.
7 Proof of Theorem 1.6

The main idea of the proof of Theorem 1.6 is due to Ref. [15], where the wave-breaking problem of the generalized mCH-CH equation was considered. We need to compute the dynamics of some important quantities along the characteristic $q(t, x)$ defined in (5.1).

We first calculate the dynamics of $u$ along the characteristic. For this purpose, applying $(1 - \partial_x^2)$ to $u_t + (k_1(u^2 - u_x^2) + k_2u^2 + k_3u)u_x$ and employing equation (1.1) lead to

\[
(1 - \partial_x^2)[u_t + (k_1(u^2 - u_x^2) + k_2u^2 + k_3u)u_x]
= m_t + (1 - \partial_x^2)[(k_1(u^2 - u_x^2) + k_2u^2 + k_3u)u_x]
= k_1(2u_x^2u_{xxx} + 4u_xu_x^2 - 2uu_xu_{xx} - 2u^2u_x - 2u_x^3)
+ k_2(-3u^2u_x - 3uu_xu_{xx} - 2u_x^3) + k_3(-2uu_x - u_xu_{xx}).
\]

Using the definitions of $p_\pm$ defined in (1.9), one derives from (7.1) that

\[
\begin{align*}
  u_t + (k_1(u^2 - u_x^2) + k_2u^2 + k_3u)u_x \\
  = -\frac{2}{3}k_1u_x^3 + \left(k_1 + \frac{k_2}{2}\right) \left[p_+ (u - u_x)^3 - p_- (u + u_x)^3\right] \\
  - k_3p_+ \left(u^2 + \frac{1}{2}u_x^2\right),
\end{align*}
\]

which is just the dynamics of $u$ along the characteristic.

We next operate $\partial_x$ to (7.2) to deduce

\[
\begin{align*}
  u_{xt} + (k_1(u^2 - u_x^2) + k_2u^2 + k_3u)u_{xx} \\
  = k_1 \left(\frac{1}{3}u^3 - uu_x^2\right) + k_2 \left(\frac{1}{2}u^3 - uu_x^2\right) + k_3 \left(u^2 - \frac{1}{2}u_x^2\right) \\
  - \left(k_1 + \frac{k_2}{2}\right) \left[p_+ (u - u_x)^3 + p_- (u + u_x)^3\right] - k_3p_+ \left(u^2 + \frac{1}{2}u_x^2\right),
\end{align*}
\]

which is the dynamics of $u_x$ along the characteristic.

The inequality $|u_x(t, x)| \leq u(t, x)$ and the conservative relation $\|u\|_{H^1} = \|u_0\|_{H^1}$ implies $u$ will not blow up. Consequently, it is enough to consider the quantity $N = mu_x$ on account of Theorem 1.4.

Define

\[
\begin{align*}
  \hat{u}(t) &= u(t, q(t, x_0)), \\  \hat{u}_x(t) &= u_x(t, q(t, x_0)), \\  \hat{m}(t) &= m(t, q(t, x_0)), \\  \hat{N}(t) &= (mu_x)(t, q(t, x_0)).
\end{align*}
\]

We will trace the dynamics of $N(t)$ along the characteristics emanating from $x_0$. It follows from Eqs. (7.3) and (2.29) that

\[
\begin{align*}
  \hat{N'}(t) &= \hat{m}'\hat{u}_x + \hat{m}\hat{u}_x' \\
  &= -(2k_1\hat{m}^2 + 2k_2\hat{m}\hat{u}_x^2 + 2k_3\hat{m})\hat{u}_x^2 \\
  &\quad + k_1\hat{m} \left(\frac{1}{3}u_x^3 - \hat{u}_x^2\hat{u}_x\right) + \frac{k_2}{2}\hat{m} \left(u_x^3 - \hat{u}_x^2\hat{u}_x\right) + k_3\hat{m} \left(u_x^2 - \frac{1}{2}u_x^2\right) \\
  &\quad - \left(k_1 + \frac{k_2}{2}\right)\hat{m} \left[p_+ (u - u_x)^3 + p_- (u + u_x)^3\right](t, q(t, x_0)) \\
  &\quad - k_3\hat{m}p_+ \left(u^2 + \frac{1}{2}u_x^2\right)(t, q(t, x_0)).
\end{align*}
\]
where “r” represents the derivative $\partial_t + [k_1(u^2-u_x^2)+k_2u^2+k_3u] \partial_x$ along the characteristic.

Since $p*(u^2+\frac{1}{4}u_x^2) \geq \frac{1}{4}u^2$, Eq. (7.4) can be controlled by

$$\tilde{N}'(t) \leq -2k_1\tilde{N}^2(t) + \frac{k_1}{3} \tilde{m}(\tilde{u}^2 - 3\tilde{u}_x^2) + \frac{k_2}{2} \tilde{m}(\tilde{u}^2 - 7\tilde{u}_x^2) + \frac{k_3}{2} \tilde{m}(\tilde{u}^2 - 5\tilde{u}_x^2) \quad (7.5)$$

in view of Eq. (6.6).

We want every of the last three terms on the right hand side of (7.5) to be no greater than zero to establish the Riccati-like inequality $\tilde{N}'(t) \leq C_0\tilde{N}^2(t)$ with $C_0$ being negative, so we consider the dynamics of the ratio $\tilde{u}_x/\tilde{u} + \gamma$ along the characteristics with some parameter $\gamma \geq 0$ to be settled later.

Direct computation leads to

$$\left( \frac{\tilde{u}_x}{\tilde{u} + \gamma} \right)' = \frac{1}{(\tilde{u} + \gamma)^2} \left[ \frac{k_1}{3} (\tilde{u}^2 - \tilde{u}_x^2)(\tilde{u}^2 - 2\tilde{u}_x^2) + \frac{\gamma k_1}{3} \tilde{u}^2(\tilde{u}^2 - 3\tilde{u}_x^2) 
+ \frac{k_2}{2} \tilde{u}^2(\tilde{u}^2 - \tilde{u}_x^2) + \frac{k_2}{2} \gamma \tilde{u}^2(\tilde{u}^2 - \tilde{u}_x^2) + k_3 \left( \tilde{u}^3 - \frac{1}{2} \tilde{u}^2 \tilde{u}_x \right) + \frac{\gamma k_3}{2} (2\tilde{u}^2 - \tilde{u}_x^2) 
- (\tilde{u} - \tilde{u}_x + \gamma) p_+ * (u - u_x)^3(t, q(t, x_0)) 
- (\tilde{u} - \tilde{u}_x + \gamma) p_- * (u - u_x)^3(t, q(t, x_0)) 
- k_3[(\tilde{u} - \tilde{u}_x + \gamma) p_+ + (\tilde{u} - \tilde{u}_x + \gamma) p_-] * \left( u^2 + \frac{1}{2} u_x^2 \right)(t, q(t, x_0)) \right]. \quad (7.6)$$

Employing the inequality $p_\pm * (u^2 + \frac{1}{4}u_x^2) \geq \frac{1}{4}u^2$, one can dominate Eq. (7.6) as

$$\left( \frac{\tilde{u}_x}{\tilde{u} + \gamma} \right)' \leq \frac{1}{(\tilde{u} + \gamma)^2} \left[ \frac{k_1}{3} (\tilde{u}^2 - \tilde{u}_x^2)(\tilde{u}^2 - 2\tilde{u}_x^2) + \frac{\gamma k_1}{3} \tilde{u}^2(\tilde{u}^2 - 3\tilde{u}_x^2) 
+ \frac{k_2}{2} \tilde{u}^2(\tilde{u}^2 - \tilde{u}_x^2) + \frac{k_2}{2} \gamma \tilde{u}^2(\tilde{u}^2 - \tilde{u}_x^2) + k_3 \left( \tilde{u}^3 - \frac{1}{2} \tilde{u}^2 \tilde{u}_x \right) + \frac{\gamma k_3}{2} (2\tilde{u}^2 - \tilde{u}_x^2) \right] 
= \frac{k_1}{3(\tilde{u} + \gamma)^2} \left[ (\tilde{u}^2 - \tilde{u}_x^2) \left( 2k_1 + 3k_2 \left( \tilde{u} + \frac{3}{2}k_2 \gamma + k_3 \right)^2 \right) 
+ \frac{3}{2k_1} \left( k_3 \gamma - 3(k_2 \gamma + k_3)^2 \right) - 2\tilde{u}_x^2 \right] + \gamma \tilde{u}^2(\tilde{u}^2 - 3\tilde{u}_x^2), \quad (7.7)$$

where we want the constant term in the brackets to be zero, so the parameter $\gamma$ should satisfy

$$k_3 \gamma = \frac{3(k_2 \gamma + k_3)^2}{4(2k_1 + 3k_2)}. \quad (7.8)$$

In accordance with Eq. (7.8), one should discuss the values of $k_1$, $k_2$ and $k_3$ in the following four cases (Note that the truly blow-up quantity is $mu_x$, so $k_1$ should be always greater than zero):

**Case 1:** For the case $k_1$, $k_2$, $k_3 > 0$, we obtain from Eq. (7.8) that the parameter $\gamma$ is

$$\gamma = \frac{k_3}{3k_2} \left( 4k_1 + 3k_2 + 2\sqrt{4k_1^2 + 6k_1k_2} \right) > 0, \quad (7.9)$$
from which (7.7) then becomes
\[
\left( \frac{\hat{u}_x}{\hat{u} + \gamma} \right)'(t) \leq \frac{k_1}{3(\hat{u} + \gamma)^2} \left\{ \hat{u}^2 - \hat{u}_x^2 \right\} \left[ \frac{2k_1 + 3k_2}{2k_1} \left( \hat{u} + \frac{3}{2} \cdot \frac{k_2 \gamma + k_3}{2k_1 + 3k_2} \right)^2 - 2\hat{u}_x^2 \right] \\
+ \gamma \hat{u} (\hat{u}^2 - 3\hat{u}_x^2) \right\}. 
\]

(7.10)

We need the terms in the brackets in Eq. (7.10) be negative to ensure the non-increasing of \( \hat{u}_x/(\hat{u} + \gamma) \), namely
\[
\sqrt{\frac{2k_1 + 3k_2}{2k_1}} \left| \hat{u} + \frac{3}{2} \cdot \frac{k_2 \gamma + k_3}{2k_1 + 3k_2} \right| < \sqrt{2}\hat{u}_x |
\]

(7.11)
as well as
\[
\hat{u}_x(0) < -\frac{1}{\sqrt{2}} (\hat{u}(0) + \gamma) 
\]

(7.12)
to be used to ensure the non-positivity of the last three terms on the right hand side of (7.5).

In view of Eq. (7.11), it seems the initial condition should be
\[
\hat{u}_x(0) < -\frac{1}{\sqrt{2}} \sqrt{\frac{2k_1 + 3k_2}{2k_1}} \left[ \hat{u}(0) + \frac{3}{2} \cdot \frac{k_2 \gamma + k_3}{2k_1 + 3k_2} \right].
\]

(7.13)

However, this condition cannot satisfy Eq. (7.12).

Therefore, one can consider multiply a factor \( \alpha > 1 \) to be determined later on the right of Eq. (7.13). Then simple computation yields
\[
\hat{u}_x(0) < -\alpha \sqrt{\frac{2k_1 + 3k_2}{2k_1}} \left[ \hat{u}(0) + \frac{3}{2} \cdot \frac{k_2 \gamma + k_3}{2k_1 + 3k_2} \right]
\]

(7.14)

If the factor \( \alpha \) satisfy
\[
\frac{\alpha}{2} \cdot \frac{3k_2}{2k_1 + 3k_2} > 1, \quad \text{or} \quad \alpha > 2 + \frac{4k_1}{3k_2},
\]

then (7.14) implies
\[
\frac{\hat{u}_x}{\hat{u} + \gamma}(t) < \frac{\hat{u}_x}{\hat{u} + \gamma}(0) < -\frac{1}{\sqrt{2}}. 
\]

Case 2: For the case \( k_1 > 0, k_2 = 0, k_3 > 0 \), Eq. (1.1) becomes the mCH-CH equation, and one can find from Eq. (7.8) that [15]
\[
\gamma = \frac{3k_3}{8k_1}. 
\]

(7.15)
and the inequality (7.11) then becomes
\[ |\hat{u} + \frac{3k_3}{4k_1}| < \sqrt{2}\hat{u}_x|, \]
according to which the initial condition should be
\[ \hat{u}_x(0) < -\frac{1}{\sqrt{2}} \left[ \hat{u}(0) + \frac{3k_3}{4k_1} \right]. \]
This condition can satisfy (7.12) with \( \gamma \) defined by (7.15).

**Case 3**: For the case \( k_1, k_2 > 0, k_3 = 0 \), Eq. (1.1) becomes the mCH-Novikov equation, and it follows from (7.8) that \( \gamma = 0 \). Then reasoning similarly as the above two cases, we derive the initial condition should be
\[ \hat{u}_x(0) < -\frac{1}{\sqrt{2}} \sqrt{\frac{2k_1 + 3k_2}{2k_1}} \hat{u}(0). \]

**Case 4**: For the case \( k_1 > 0, k_2 = k_3 = 0 \), Eq. (1.1) becomes the mCH equation, and one deduces [15]
\[ \gamma = 0, \quad \hat{u}_x(0) < -\frac{1}{\sqrt{2}} \hat{u}(0). \]

In each of the above four cases, one always has
\[ \hat{u} + \gamma + \sqrt{2}\hat{u}_x < 0 \]
and
\[ \hat{u}(t) + \sqrt{3}\hat{u}_x(t) < 0, \quad \hat{u}(t) + \sqrt{5}\hat{u}_x(t) < 0, \quad \hat{u}(t) + \sqrt{7}\hat{u}_x(t) < 0. \]  
(7.16)
On the other hand,
\[ \hat{u}(t) - \sqrt{3}\hat{u}_x(t) > 0, \quad \hat{u}(t) - \sqrt{5}\hat{u}_x(t) > 0, \quad \hat{u}(t) - \sqrt{7}\hat{u}_x(t) > 0. \]  
(7.17)
Substituting (7.16) and (7.17) into (7.5) leads to
\[ \hat{N}'(t) \leq -2k_1\hat{N}^2(t). \]  
(7.18)
Hence, \( \hat{N}(t) \) will blow up in finite time \( T^* \) with
\[ T^* \leq -\frac{1}{2k_1m_0(x_4)u_{0,x}(x_4)}, \]
which is obtained after integrating (7.18). We thus complete the proof of Theorem 1.6.

### 8 Proof of Theorem 1.7

Theorem 1.6 just considers a portion of the blow-up quantity, however we would like to trace the whole blow-up quantity defined as
\[ M(t, x) = (2k_1m + 3k_2u + 2k_3)u_x \]
along the characteristics (Here the main idea is similar to [47,70]) in this section.
Direct computation leads to
\[ M_t + [k_1(u^2 - u_x^2) + k_2u^2 + k_3u]M_x = 2k_1u_x(m_x + [k_1(u^2 - u_x^2) + k_2u^2 + k_3u]u_x + (2k_1m + 3k_2u + 2k_3)[u_t + [k_1(u^2 - u_x^2) + k_2u^2 + k_3u]u_{xx}] \]
\[ = A_1 + A_2, \tag{8.2} \]
where \(A_1\) is defined as the following local terms
\[ A_1 = -2k_1k_2u_x^4 - 4k_1^2m^2u_x^2 - 6k_1k_2mu u_x^2 - 4k_1k_3mu u_x^2 + (2k_1m + 3k_2u + 2k_3) \left[ k_1 \left( \frac{1}{3}u^3 - uu_x^2 \right) + k_2 \left( \frac{1}{2}u^3 - \frac{1}{2}uu_x^2 \right) + k_3 \left( u^2 - \frac{1}{2}u_x^2 \right) \right] \]
and \(A_2\) is the remaining nonlocal terms.

Recall that \(m \geq 0, u \geq 0\) and \(|u_x| \leq u\), so we have
\[
A_1 + M^2 = (2k_1m + 3k_2u + 2k_3) \left[ k_1 \left( \frac{1}{3}u^3 - uu_x^2 \right) + \frac{k_2}{2} \left( u^3 - uu_x^2 \right) + k_3 \left( u^2 - \frac{1}{2}u_x^2 \right) \right] + 4k_1k_3mu_x^2 + 6k_1k_2mu u_x^2 + 9k_2^2u_x^4 + 4k_3^2u_x^2 + 12k_2k_3u_x^2 - 2k_1k_2u_x^4 \leq (2k_1m + 3k_2u + 2k_3) \left[ \left( \frac{4}{3}k_1 + k_2 \right) ||u||^3_{L^\infty} + \frac{3}{2}k_3 ||u||^2_{L^\infty} \right] + 2k_1m \cdot 2k_3 ||u||^2_{L^\infty} + 2k_1m \cdot 3k_2u \cdot ||u||^3_{L^\infty} + 2k_3 \cdot 2k_3 ||u||^2_{L^\infty} + 3k_2u \cdot 2k_3 ||u||^3_{L^\infty} + 3k_2u \cdot \frac{2}{3}k_1 ||u||^3_{L^\infty} \leq (2k_1m + 3k_2u + 2k_3) \left[ (2k_1 + 7k_2) ||u||^3_{L^\infty} + \frac{19}{2}k_3 ||u||^2_{L^\infty} \right]. \tag{8.3} \]
We can similarly deal with the nonlocal terms \(A_2\) by noticing \(||p_{\pm}||_{L^1} \leq 1\) after using the convolution-type Young inequality
\[
A_2 \leq 3k_2u \left( \frac{16k_1}{3} ||u||^3_{L^\infty} + 8k_2 ||u||^2_{L^\infty} + \frac{3k_3}{2} ||u||_{L^\infty} \right) + (2k_1m + 3k_2u + 2k_3) \left[ \frac{16k_1}{3} ||u||^3_{L^\infty} + 8k_2 ||u||^2_{L^\infty} + \frac{3k_3}{2} ||u||_{L^\infty} \right] \leq (2k_1m + 3k_2u + 2k_3) \left[ \left( \frac{32}{3}k_1 + 16k_2 \right) ||u||^3_{L^\infty} + 3k_3 ||u||^2_{L^\infty} \right]. \tag{8.4} \]

Substituting (8.3)-(8.4) into (8.2) leads to
\[
M_t + [k_1(u^2 - u_x^2) + k_2u^2 + k_3u]M_x \leq -M^2 + C_1(2k_1m + 3k_2u + 2k_3) \tag{8.5} \]
with
\[
C_1 = \left( \frac{38}{3}k_1 + 23k_2 + \frac{25}{2}k_3 \right) (||u_0||^2_{H^1} + ||u_0||^3_{H^1}). \tag{8.6} \]

It follows from (8.5) that
\[
\frac{d}{dt} M(t, q(t, x_1)) \leq -M^2(t, q(t, x_1)) + C_1(2k_1m + 3k_2u + 2k_3)(t, q(t, x_1)). \tag{8.7} \]
Invoking Eq. (2.29), one derives
\[
\frac{d}{dt} m(t, q(t, x_1)) = -(mM)(t, q(t, x_1)). \tag{8.8} \]
Define $\bar{M}(t) := M(t, q(t, x_1)), \bar{m}(t) := m(t, q(t, x_1))$ and $\bar{u}(t) := u(t, q(t, x_1))$, then (8.7) and (8.8) can be respectively recast as
\begin{equation}
\frac{d}{dt} \bar{M}(t) \leq -\bar{M}(t)^2 + C_1(2k_1\bar{m} + 3k_2\bar{u} + 2k_3) \quad (8.9)
\end{equation}
and
\begin{equation}
\frac{d}{dt} \bar{m}(t) = -\bar{m}(t)\bar{M}(t). \quad (8.10)
\end{equation}

Consequently, one derives
\begin{align*}
\frac{d}{dt} \left( \frac{1}{\bar{m}^2(t)} \frac{d}{dt} \bar{m}(t) \right) &= \frac{d}{dt} \left( -\frac{1}{\bar{m}(t)} \bar{M}(t) \right) \\
&= \frac{1}{\bar{m}^2(t)} \left( -\bar{m}(t) \frac{d}{dt} \bar{M}(t) + \bar{M}(t) \frac{d}{dt} \bar{m}(t) \right) \\
&\geq \frac{1}{\bar{m}^2(t)} \left[ \bar{m}(t) \left( \bar{M}^2(t) - C_1(2k_1\bar{m}(t) + 3k_2\bar{u}(t) + 2k_3) \right) - \bar{m}(t)\bar{M}^2(t) \right] \\
&= -2k_1C_1 - 3k_2C_1 \frac{\bar{u}(t)}{\bar{m}(t)} - 2k_3C_1 \frac{1}{\bar{m}(t)} \\
&\geq -2k_1C_1 - (3k_2 + 2k_3)C_1 \frac{\bar{u}(t)}{\bar{m}(t)} + \frac{1}{\bar{m}(t)}. \quad (8.11)
\end{align*}

If $\frac{\bar{u}(t)}{\bar{m}(t)} \leq C_2$ holds initially, then it follows from (8.11) that
\begin{equation}
\frac{d}{dt} \left( \frac{1}{\bar{m}^2(t)} \frac{d}{dt} \bar{m}(t) \right) \geq -[2k_1 + (3k_2 + 2k_3)C_2]C_1. \quad (8.12)
\end{equation}

Integrating (8.12) yields
\begin{equation}
\frac{1}{\bar{m}^2(t)} \frac{d}{dt} \bar{m}(t) \geq C_0 - [2k_1 + (3k_2 + 2k_3)C_2]C_1 t = C_0 - 2C_3 t, \quad (8.13)
\end{equation}
where
\begin{align*}
C_0 &= \bar{m}'(0) \bar{m}^{-2}(0) = -\bar{M}'(0) \bar{m}^{-2}(0) = -\partial_x u_0(x_1) \left( 2k_1 + \frac{3k_2u_0(x_1) + 2k_3}{m_0(x_1)} \right), \\
C_3 &= [k_1 + (3k_2/2 + k_3)C_2]C_1. \quad (8.14)
\end{align*}

From (8.10) and (8.13), one obtains
\begin{equation}
\bar{M}(t) = -\frac{1}{\bar{m}(t)} \frac{d}{dt} \bar{m}(t) \leq -\bar{m}(t)(C_0 - 2C_3 t). \quad (8.15)
\end{equation}

Integrating (8.13) from 0 to $t$ yields
\begin{equation}
\frac{1}{\bar{m}(t)} - \frac{1}{\bar{m}(0)} \leq C_3 t^2 - C_0 t, \quad (8.16)
\end{equation}
which leads to
\begin{equation}
\frac{1}{\bar{m}(t)} \leq C_3 \left( t^2 - \frac{C_0}{C_3} t + \frac{1}{C_3\bar{m}(0)} \right) = C_3 \left( t^2 - \frac{C_0}{C_3} t + \frac{1}{C_3m_0(x_1)} \right). \quad (8.17)
\end{equation}

Solving the algebraic equation
\begin{equation}
\frac{C_0}{C_3} t + \frac{1}{C_3m_0(x_1)} = 0,
\end{equation}

leads to
\[ t_- := \frac{C_0}{2C_3} - \frac{1}{2} \sqrt{\left(\frac{C_0}{C_3}\right)^2 - \frac{2}{C_3m_0(x_1)}}, \quad t_+ := \frac{C_0}{2C_3} + \frac{1}{2} \sqrt{\left(\frac{C_0}{C_3}\right)^2 - \frac{2}{C_3m_0(x_1)}}. \] (8.18)

According to the condition (1.21), one derives
\[ \left(\frac{C_0}{C_3}\right)^2 > \frac{2}{C_3m_0(x_1)}, \quad \frac{2}{C_3m_0(x_1)} < t_- < \frac{C_0}{2C_3} < t_+ \] (8.19)
and consequently
\[ 0 \leq \frac{1}{m(t)} \leq C_3 (t - t_-) (t - t_+), \] (8.20)
from which we can find a time \(0 < T_* \leq t_-\) such that
\[ m(t) \rightarrow +\infty, \quad M(t) \rightarrow -\infty, \quad \text{as} \quad t \rightarrow T_* \leq t_- . \]

Thus one has
\[ \inf_{x \in \mathbb{R}} M(t, x) \leq M(t) \rightarrow -\infty, \quad \text{as} \quad t \rightarrow T_* \leq t_- , \]
implying the blow-up of \(m(t, x)\) at time \(T_*\) since \(u\) and \(u_x\) are both bounded.

We finally prove the blow-up rate (1.22). Combining (8.15) and (8.20) leads to
\[ 2 (T_* - t) \inf_{x \in \mathbb{R}} M(t, x) \leq 2 (T_* - t) M(t) \leq (T_* - t) \overline{m}(t) (C_0 - 2C_3 t) \leq 2 (T_* - t) \frac{2}{(t - t_-) (t - t_+)} \left( t - \frac{C_0}{2C_3} \right) , \]
which gives (1.22) when \(T_* = t_-\). We thus complete the proof of Theorem 1.7.

### 9 Proof of Proposition 1.1

Without loss of generality, one can take \(x_0 = 0\) in the peakon solution (1.25). Let \(c\) be a unknown real parameter, and the initial value be \(u_{0,c}(x) = u_a(0, x), \ x \in \mathbb{R}\). We first recall that [47] for all \(t \geq 0\)
\[ \partial_x u_a(t, x) = -\text{sgn}(x - ct)u_a(t, x), \quad \partial_t u_a(t, x) = c \text{sgn}(x - ct)u_a(t, x) \in L^\infty(\mathbb{R}) \] (9.1)
in the sense of distribution \(S'(\mathbb{R})\), and
\[ \lim_{t \to 0^+} \| u_a(t, \cdot) - u_{0,c}(\cdot) \|_{W^{1,\infty}} = 0 \] (9.2)
In light of Eqs. (9.1)-(9.2), one obtains for any test function \(\varphi(t, x) \in C_c^\infty([0, +\infty) \times \mathbb{R})\) that
\[
\int_0^{+\infty} \int_{\mathbb{R}} \left( u_a \partial_x \varphi + \frac{k_1 + k_2}{3} u_a^3 \varphi + \frac{k_1}{3} (\partial_x u_a)^3 \varphi + \frac{k_3}{2} u_a^2 \partial_x \varphi \right) dx dt + \int_{\mathbb{R}} u_a(0, x) \varphi(0, x) dx \\
= - \int_0^{+\infty} \int_{\mathbb{R}} \varphi (\partial_t u_a + (k_1 + k_2) u_a^2 \partial_x u_a + k_3 u_a \partial_x u_a - \frac{k_1}{3} (\partial_x u_a)^3) dx dt \\
= - \int_0^{+\infty} \int_{\mathbb{R}} \varphi \text{sgn}(x - ct) u_a \left[ c - \left( \frac{2k_1}{3} + k_2 \right) u_a^2 - k_3 u_a \right] dx dt. \tag{9.3}
\]

We next deal with the nonlocal terms in Eq. (1.24) with \( u_a(t, x) \) instead of \( u(t, x) \).

\[
\int_0^{+\infty} \int_{\mathbb{R}} \left\{ p_x \left[ \left( \frac{2k_1}{3} + k_2 \right) u_a^3 + \left( k_1 + \frac{3k_2}{2} \right) u_a (\partial_x u_a)^2 + k_3 u_a^2 + k_3 (\partial_x u_a)^2 \right] \right\} dx dt \\
= - \int_0^{+\infty} \int_{\mathbb{R}} \left\{ \varphi p_x \left[ \left( \frac{k_1}{3} + \frac{k_2}{2} \right) (\partial_x u_a)^3 \varphi \right] \right\} dx dt \\
= - \int_0^{+\infty} \int_{\mathbb{R}} \left\{ \varphi p_x \left[ k_3 u_a^2 + \frac{k_3}{2} (\partial_x u_a)^2 \right] \right\} dx dt, \tag{9.4}
\]

where the equality \( \text{sgn}(x - ct) u_a^3 = - \frac{1}{2} \partial_x (u_a^3) \) has been used to deduce

\[
(2k_1 + 3k_2) u_a^2 \partial_x u_a + \left( \frac{k_1}{3} + \frac{k_2}{2} \right) (\partial_x u_a)^3 = \left( \frac{7k_1}{9} + \frac{7k_2}{6} \right) \partial_x (u_a^3). \tag{9.5}
\]

Invoking \( \partial_x p(x) = - \frac{1}{2} \text{sgn}(x) e^{-|x|} \) leads to

\[
p_x \left[ \left( \frac{k_1}{3} + \frac{k_2}{2} \right) u_a (\partial_x u_a)^2 + \left( \frac{7k_1}{9} + \frac{7k_2}{6} \right) u_a^3 + k_3 u_a^2 + \frac{k_3}{2} (\partial_x u_a)^2 \right] \\
= - \frac{1}{2} \int_{-\infty}^{+\infty} \text{sgn}(x - y) e^{-|x-y|} \left\{ a^3 \left[ \left( \frac{7k_1}{9} + \frac{7k_2}{6} \right) + \left( k_1 + \frac{3k_2}{2} \right) \text{sgn}^2 (y - ct) \right] e^{-3|y-ct|} \right\} dy \\
+ a^2 \left[ k_3 + \frac{k_3}{2} \text{sgn}^2 (y - ct) \right] e^{-2|y-ct|} \right\} dy. \tag{9.6}
\]

For the case \( x > ct \), the right hand side of Eq. (9.6) can be separated into three parts:

\[
p_x \left[ \left( k_1 + \frac{3k_2}{2} \right) u_a (\partial_x u_a)^2 + \left( \frac{7k_1}{9} + \frac{7k_2}{6} \right) u_a^3 + k_3 u_a^2 + \frac{k_3}{2} (\partial_x u_a)^2 \right] \\
= - \frac{1}{2} \left\{ \left( \int_{-\infty}^{ct} + \int_{ct}^{x} + \int_{x}^{+\infty} \right) \text{sgn}(x - y) e^{-|x-y|} \right\} \\
\times \left\{ a^3 \left[ \left( \frac{7k_1}{9} + \frac{7k_2}{6} \right) + \left( k_1 + \frac{3k_2}{2} \right) \text{sgn}^2 (y - ct) \right] e^{-3|y-ct|} \right\} \\
+ a^2 \left[ k_3 + \frac{k_3}{2} \text{sgn}^2 (y - ct) \right] e^{-2|y-ct|} \right\} dy \\
= a^3 \left( \frac{2k_1}{3} + k_2 \right) (e^{3(ct-x)} - e^{ct-x}) + a^2 k_3 (e^{2(ct-x)} - e^{ct-x}). \tag{9.7}
\]
For the case $x \leq ct$, we split the right hand side of (9.6) into the following three parts:

\[
p_x \left[ \left( k_1 + \frac{3k_2}{2} \right) u_a(\partial_x u_a)^2 + \left( \frac{7k_1}{9} + \frac{7k_2}{6} \right) u_a^3 + k_3 u_a^2 + \frac{k_3}{2}(\partial_x u_a)^2 \right] \\
= -\frac{1}{2} \left( \int_{-\infty}^{x} + \int_{x}^{ct} + \int_{ct}^{+\infty} \right) \text{sgn}(x - y)e^{-|x-y|} \\
\times \left\{ a^3 \left[ \left( \frac{7k_1}{9} + \frac{7k_2}{6} \right) + \left( k_1 + \frac{3k_2}{2} \right) \text{sgn}^2(y - ct) \right] e^{-3|y-ct|} \\
+ a^2 \left[ k_3 + k_3 \text{sgn}^2(y - ct) \right] e^{-2|y-ct|} \right\} dy \\
= a^3 \left( \frac{2k_1}{3} + k_2 \right) (e^{x-ct} - e^{3(x-ct)}) + a^2 k_3 (e^{x-ct} - e^{2(x-ct)}).
\]

(9.8)

Thus, combining Eqs. (9.7) and (9.8), and according to the definition of $u_a$ yield

\[
\text{sgn}(x - ct)u_a \left[ c - \left( \frac{2}{3} k_1 + k_2 \right) u_a^2 - k_3 u_a \right] \\
+ p_x \left[ \left( k_1 + \frac{3k_2}{2} \right) u_a(\partial_x u_a)^2 + \left( \frac{7k_1}{9} + \frac{7k_2}{6} \right) u_a^3 + k_3 u_a^2 + \frac{k_3}{2}(\partial_x u_a)^2 \right] \\
= \left\{ \left[ -\left( \frac{2}{3} k_1 + k_2 \right) a^3 - k_3^2 a^2 + ac \right] e^{ct-x} \right. \text{as } x > ct, \\
\left. \left[ \left( \frac{2k_1}{3} + k_2 \right) a^3 + k_3^2 a^2 - ac \right] e^{x-ct} \right. \text{as } x \leq ct,
\]

which is equal to zero if

\[
\left( \frac{2}{3} k_1 + k_2 \right) a^3 + k_3^2 a^2 - ac = 0,
\]

(9.9)

that is, $c$ satisfies Eq. (1.26). With the condition (1.26) satisfied by $c$, we have arrived at

\[
\int_{0}^{+\infty} \int_{\mathbb{R}} \left\{ u_a \varphi_t + \frac{k_1 + k_2}{3} u_a^3 \varphi_x + \frac{k_1}{3}(\partial_x u_a)^3 \varphi + \frac{k_3}{2} u_a^2 \varphi_x \right. \\
+ p \left[ \left( \frac{2k_1}{3} + k_2 \right) u_a^3 + \left( k_1 + \frac{3k_2}{2} \right) u_a(\partial_x u_a)^2 + k_3 u_a^2 + \frac{k_3}{2}(\partial_x u_a)^2 \right] \partial_x \varphi \right.

\left. - p \left[ \left( \frac{k_1}{3} + \frac{k_2}{2} \right)(\partial_x u_a)^3 \right] \varphi \right\} dt \, dx + \int_{\mathbb{R}} u_a(0, x) \varphi(0, x) \, dx = 0
\]

for any smooth test function $\varphi(t, x) \in C^\infty_\mathbb{R}([0, +\infty) \times \mathbb{R})$. We thus complete the proof of Proposition 1.1.

10 Proof of Proposition 1.2

In the last section, we have deduced the single peakon solution of Eq. (1.1). It is found that Eq. (1.1) also admits the multi-peakon solutions. We are now in a position to derive them. Suppose the multi-peakon solutions of Eq. (1.1) are of the form (1.28). Then we have

\[
u_{n,p, t} = \sum_{i=1}^{N} \left[ \hat{p}_i(t) + p_i(t) \hat{q}_i(t) \text{sgn}(\xi_i) \right] e^{-|\xi_i|}, \quad u_{n,p, x} = -\sum_{i=1}^{N} \text{sgn}(\xi_i) p_i(t) e^{-|\xi_i|}.
\]

(10.1)
Without loss of generality, we assume that the time-dependent positions are ordered, that is, \( q_1(t) < q_2(t) < \cdots < q_{N-1}(t) < q_N(t) \). Since Eq. (1.1) has the non-periodic multi-peakon solution (1.28) in the sense of Definition 1.1, the multi-peakon solution \( u_{npm}(x, t) \) given by Eq. (1.28) satisfies \((m = u - u_{xx} \text{ and } u = u_{npm}(x, t))\) the following equation

\[
\int_0^\infty \int_\mathbb{R} \left\{ u_t(\psi - \psi_{xx}) + \frac{1}{3}(k_1 + k_2)u^2\psi_{xxx} + \frac{k_1}{3}u^3\psi_x - \left( k_1 + \frac{4}{3}k_2 \right)u^3\psi_x \right. \\
- \left( k_1 + \frac{3}{k_2} \right)u'\psi_x + \frac{1}{2}k_3u^2\psi_{xxx} - \frac{3}{2}k_3u^2\psi_x - \frac{1}{2}k_3u^2\psi_x + \frac{1}{2}k_2u^3\psi_x \right\} dxdt = 0
\]

(10.2)

after substituting \( \psi = (1 - \partial_x^2)^{-1}\varphi \) in Eq. (1.24) for \( \varphi(t, x) \in C^\infty([0, \infty) \times \mathbb{R}) \).

Note that it has been calculated in [47] that

\[
\int_0^\infty \int_\mathbb{R} u_t(\psi - \psi_{xx})dxdt = 2\int_0^\infty \sum_{j=1}^N (p_j'\psi(q_j) + p_j q_j'\psi_x(q_j))dt.
\]

We just need to deal with the remaining terms in (10.2). For the term \( \int_0^\infty \int_\mathbb{R} u^3\psi_{xxx}dxdt \), we have

\[
\int u^3\psi_{xxx}dx = \sum_{i,j,k=1}^N p_i p_j p_k \int e^{-|x-q_i|-|x-q_j|-|x-q_k|}\psi_{xxx}dx = K_1 + K_2 + K_3,
\]

where

\[
K_1 = \sum_{i=1}^N p_i^3 \int e^{-3|x-q_i|}\psi_{xxx}dx,
\]

\[
= -6\sum_{i=1}^N p_i^3\psi_x(q_i) - 27\sum_{i=1}^N p_i^3 \left( \int_{-\infty}^{q_i} \psi e^{3x-3q_i}dx - \int_{q_i}^{+\infty} \psi e^{-3x+3q_i}dx \right),
\]

\[
K_2 = 3\sum_{i=1}^N p_i^2 \sum_{k<i} p_k \int e^{-2|x-q_i|-|x-q_k|}\psi_{xxx}dx + \sum_{k<i} p_k \int e^{-2|x-q_i|-|x-q_k|}\psi_{xxx}dx
\]

\[
= 3\sum_{i=1}^N \sum_{k>i} p_i^2 p_k \left\{ 4[2\psi(q_i) - \psi_x(q_i)]e^{q_i-q_k} - 2[\psi_x(q_k) + 4\psi(q_k)]e^{2q_i-2q_k}
\right.
\]

\[
- 27\int_{-\infty}^{q_i} \psi e^{3x-2q_i-q_k}dx + \int_{q_i}^{q_k} \psi e^{-x+2q_i-q_k}dx + 27\int_{q_k}^{+\infty} \psi e^{-3x+2q_i+q_k}dx
\]

\[
+ 3\sum_{i=1}^N \sum_{k<i} p_i^2 p_k \left\{ -4[\psi_x(q_i) + 2\psi(q_i)]e^{q_i-q_k} - 2[\psi_x(q_k) - 4\psi(q_k)]e^{2q_i-2q_k}
\right.
\]

\[
- 27\int_{q_k}^{q_i} \psi e^{3x-2q_i-q_k}dx - \int_{q_k}^{q_i} \psi e^{-x-2q_i+q_k}dx + 27\int_{q_i}^{+\infty} \psi e^{-3x+2q_i+q_k}dx \right\}.
\]

\[
K_3 = 6\sum_{i<j<k} p_i p_j p_k \int e^{-|x-q_i|-|x-q_j|-|x-q_k|}\psi_{xxx}dx
\]

\[
+ \int_{q_j}^{q_k} e^{q_i+q_j-q_k-x}\psi_{xxx}dx + \int_{q_k}^{+\infty} e^{q_i+q_j-q_k-3x}\psi_{xxx}dx
\]

\[
\]
\[
= 6 \sum_{i < j < k} p_i p_j p_k \left\{ - 2 \psi_x (q_i) e^{2q_i - q_j - q_k} - 2 \psi_x (q_j) e^{q_i - q_k} - 2 \psi_x (q_k) e^{q_i + q_j - 2q_k} \\
+ 8 \psi (q_i) e^{2q_i - q_j - q_k} - 8 \psi (q_k) e^{q_i + q_j - 2q_k} \\
- 27 \int_{q_i}^{q_j} \psi e^{3x - q_i - q_j - q_k} \, dx - \int_{q_i}^{q_j} \psi e^{x + q_j - q_k} \, dx \\
+ \int_{q_j}^{q_k} \psi e^{-x + q_j + q_k} + 27 \int_{q_j}^{+\infty} \psi e^{-3x + q_j + q_k} \, dx \right\}.
\]

Applying similar method to the remaining terms in (10.2) produces

\[
\int_{\mathbb{R}} u^3 \psi_x \, dx = - \sum_{i=1}^{N} p_i^3 \left\{ - 2 \psi_x (q_i) - 9 \int_{-\infty}^{q_i} \psi e^{3x - 3q_i} \, dx + 9 \int_{q_i}^{+\infty} \psi e^{-3x + 3q_i} \, dx \right\} \\
- 3 \sum_{i=1}^{N} \sum_{k > i} p_i^2 p_k \left\{ - 2 [\psi_x (q_i) + 2 \psi (q_k)] e^{2q_i - 2q_k} + 4 \psi (q_i) e^{q_i - q_k} \\
- 9 \int_{-\infty}^{q_i} \psi e^{3x - 2q_i - q_k} \, dx - \int_{q_k}^{q_i} \psi e^{-x + 2q_i - q_k} \, dx + 9 \int_{q_k}^{+\infty} \psi e^{-3x + 2q_i + q_k} \, dx \right\} \\
- 6 \sum_{i < j < k} p_i p_j p_k \left\{ - 2 \psi_x (q_i) e^{2q_i - q_j - q_k} + 2 \psi_x (q_j) e^{q_i - q_k} - 2 \psi_x (q_k) e^{q_i + q_j - 2q_k} \\
+ 4 \psi (q_i) e^{2q_i - q_j - q_k} - 4 \psi (q_k) e^{q_i + q_j - 2q_k} - 9 \int_{-\infty}^{q_i} \psi e^{3x - q_i - q_j - q_k} \, dx \\
+ \int_{q_i}^{q_j} \psi e^{x + q_i - q_k} \, dx - \int_{q_j}^{q_k} \psi e^{-x + q_i + q_j - q_k} \, dx + 9 \int_{q_k}^{+\infty} \psi e^{-3x + q_i + q_j + q_k} \, dx \right\}. \tag{10.3}
\]

\[
\int_{\mathbb{R}} u^3 \psi \, dx = \sum_{i=1}^{N} p_i^3 \left\{ - 3 \int_{-\infty}^{q_i} \psi e^{3x - 3q_i} \, dx + 3 \int_{q_i}^{+\infty} \psi e^{-3x + 3q_i} \, dx \right\} \\
+ 3 \sum_{i=1}^{N} \sum_{k > i} p_i^2 p_k \left\{ - 3 \int_{-\infty}^{q_i} \psi e^{3x - 2q_i - q_k} \, dx \\
+ \int_{q_i}^{q_k} \psi e^{-x + 2q_i - q_k} \, dx + 3 \int_{q_k}^{+\infty} \psi e^{-3x + 2q_i + q_k} \, dx \right\} \\
+ 3 \sum_{i=1}^{N} \sum_{k < i} p_i^2 p_k \left\{ - 3 \int_{-\infty}^{q_i} \psi e^{3x - 2q_i - q_k} \, dx \\
- \int_{q_k}^{q_i} \psi e^{x - 2q_i + q_k} \, dx + 3 \int_{q_i}^{+\infty} \psi e^{-3x + 2q_i + q_k} \, dx \right\} \\
+ 6 \sum_{i < j < k} p_i p_j p_k \left\{ - 3 \int_{-\infty}^{q_i} \psi e^{3x - q_i - q_j - q_k} \, dx - \int_{q_i}^{q_j} \psi e^{x + q_i - q_j - q_k} \, dx \\
+ \int_{q_j}^{q_k} \psi e^{-x + q_i + q_j - q_k} \, dx + 3 \int_{q_k}^{+\infty} \psi e^{-3x + q_i + q_j + q_k} \, dx \right\}. \tag{10.4}
\]
\[
\int_{\mathbb{R}} u_{\psi}^2 \psi_x^2 \ dx = \sum_{i=1}^{N} p_i^2 \left\{ -3 \int_{-\infty}^{q_i} \psi^3 e^{-3q_i} \ dx + 3 \int_{q_i}^{+\infty} \psi e^{-3x} e^{2q_i} \ dx \right\} \\
+ 2 N \sum_{i=1}^{N} \sum_{k>i} p_i^2 p_k \left\{ 2 \psi(q_i) e^{q_i - q_k} - 2 \psi(q_k) e^{2q_i - 2q_k} - 3 \int_{q_i}^{q_k} \psi e^{-2q_i - q_k} \ dx \\
- \int_{q_i}^{q_k} \psi e^{-x} e^{2q_i - q_k} \ dx + 3 \int_{q_k}^{+\infty} \psi e^{-3x} e^{2q_i + q_k} \ dx \right\} \\
+ \sum_{i=1}^{N} \sum_{k>i} p_i^2 p_k \left\{ -3 \int_{-\infty}^{q_i} \psi^3 e^{-2q_i - q_k} \ dx + \int_{q_k}^{q_i} \psi e^{-x + 2q_i - q_k} \ dx + 3 \int_{q_k}^{+\infty} \psi e^{-3x + 2q_i + q_k} \ dx \right\} \\
+ 2 N \sum_{i=1}^{N} \sum_{k<i} p_i^2 p_k \left\{ -2 \psi(q_i) e^{q_k - q_i} + 2 \psi(q_k) e^{-2q_i - 2q_k} - 3 \int_{-\infty}^{q_i} \psi^3 e^{-2q_i - q_k} \ dx \\
+ \int_{q_i}^{q_k} \psi e^{-x + 2q_i + q_k} \ dx + 3 \int_{q_k}^{+\infty} \psi e^{-3x + 2q_i + q_k} \ dx \right\} \\
+ \sum_{i=1}^{N} \sum_{k<i} p_i^2 p_k \left\{ -3 \int_{-\infty}^{q_k} \psi^3 e^{-2q_i - q_k} \ dx - \int_{q_k}^{q_i} \psi e^{-2q_i + q_k} \ dx + 3 \int_{q_k}^{+\infty} \psi e^{-3x + 2q_i + q_k} \ dx \right\} \\
+ 2 \sum_{i<j<k} p_i p_j p_k \left\{ 4 \psi(q_i) e^{2q_k - q_j - q_i} - 4 \psi(q_j) e^{q_i + q_j - q_k} - 6 \int_{-\infty}^{q_i} \psi^3 e^{-q_i - q_j} \ dx \\
+ \int_{q_i}^{q_j} \psi e^{-x + q_i - q_j} \ dx + 3 \int_{q_j}^{+\infty} \psi e^{-3x + q_i + q_j} \ dx \right\}.
\]
\[
(10.5)
\]
\[
\int_{\mathbb{R}} u^2 \psi_{xxx} \ dx = \sum_{i=1}^{N} p_i^2 \left\{ -4 \psi_x(q_i) - 8 \int_{-\infty}^{q_i} \psi^2 e^{-2q_i} \ dx + 8 \int_{q_i}^{+\infty} \psi e^{-2x} e^{2q_i} \ dx \right\} \\
+ \sum_{i=1}^{N} \sum_{j<i} p_i p_j \left\{ -2 \psi_x(q_i) e^{q_i - q_j} - 2 \psi_x(q_j) e^{q_i - q_j} + 4 \psi(q_i) e^{q_i - q_j} - 4 \psi(q_j) e^{q_i - q_j} \\
- 8 \int_{-\infty}^{q_i} \psi^2 e^{-q_i - q_j} \ dx + 8 \int_{q_j}^{+\infty} \psi e^{-2x} e^{q_i + q_j} \ dx \right\} \\
+ \sum_{i=1}^{N} \sum_{j<i} p_i p_j \left\{ -2 \psi_x(q_i) e^{q_i - q_j} - 2 \psi_x(q_j) e^{q_i - q_j} - 4 \psi(q_i) e^{q_i - q_j} + 4 \psi(q_j) e^{q_i - q_j} \\
- 8 \int_{-\infty}^{q_j} \psi^2 e^{-q_i - q_j} \ dx + 8 \int_{q_j}^{+\infty} \psi e^{-2x} e^{q_i + q_j} \ dx \right\},
\]
\[
(10.6)
\]
\[
\int_{\mathbb{R}} u^2 \psi_x \ dx = 2 \sum_{i=1}^{N} p_i^2 \left\{ - \int_{-\infty}^{q_i} \psi^2 e^{-2q_i} \ dx + \int_{q_i}^{+\infty} \psi e^{-2x} e^{2q_i} \ dx \right\} \\
+ 2 \sum_{i=1}^{N} \sum_{j>i} p_i p_j \left\{ - \int_{-\infty}^{q_i} \psi^2 e^{-q_i - q_j} \ dx + \int_{q_j}^{+\infty} \psi^2 e^{q_i + q_j} \ dx \right\} \\
+ 2 \sum_{i=1}^{N} \sum_{j<i} p_i p_j \left\{ - \int_{-\infty}^{q_j} \psi^2 e^{-q_i - q_j} \ dx + \int_{q_j}^{+\infty} \psi^2 e^{q_i + q_j} \ dx \right\},
\]
\[
(10.7)
\]
\[
\int_{\mathbb{R}} u^2 \psi \ dx = 2 \sum_{i=1}^{N} p_i^2 \left\{ - \int_{-\infty}^{q_i} \psi^2 e^{-2q_i} \ dx + \int_{q_i}^{+\infty} \psi e^{-2x} e^{2q_i} \ dx \right\}
\]
\[ + 2 \sum_{i=1}^{N} \sum_{j>i} p_i p_j \left\{ [\psi(q_i) - \psi(q_j)] e^{q_i - q_j} \right\} \\
- \int_{-\infty}^{q_i} \psi e^{2x} e^{-q_i - q_j} dx + \int_{q_j}^{+\infty} \psi e^{-2x} e^{q_i + q_j} dx \]

+ 2 \sum_{i=1}^{N} \sum_{j<i} p_i p_j \left\{ [\psi(q_j) - \psi(q_i)] e^{q_i - q_j} \right\} \\
- \int_{-\infty}^{q_j} \psi e^{2x} e^{-q_i - q_j} dx + \int_{q_i}^{+\infty} \psi e^{-2x} e^{q_i + q_j} dx \]

\[ (10.8) \]

and

\[ \int_{\mathbb{R}} u_{x}^{3} \psi dx = \sum_{i=1}^{N} p_i \left\{ \int_{-\infty}^{q_i} \psi e^{3x} e^{-3q_i} dx - \int_{q_i}^{+\infty} \psi e^{-3x} e^{3q_i} dx \right\} \\
- 3 \sum_{i=1}^{N} \sum_{k>i} p_i^2 p_k \left\{ - \int_{-\infty}^{q_i} \psi e^{3x} e^{-2q_i - q_k} dx \right\} \\
- \int_{q_i}^{q_k} \psi e^{-x} e^{2q_i - q_k} dx + \int_{q_k}^{+\infty} \psi e^{-3x} e^{2q_i + q_k} dx \]

- 3 \sum_{i=1}^{N} \sum_{k<i} p_i^2 p_k \left\{ - \int_{-\infty}^{q_k} \psi e^{3x} e^{-2q_i - q_k} dx \right\} \\
+ \int_{q_k}^{q_i} \psi e^{x} e^{-2q_i + q_k} dx + \int_{q_i}^{+\infty} \psi e^{-3x} e^{2q_i + q_k} dx \]

- 6 \sum_{i<j<k} p_i p_j p_k \left\{ - \int_{-\infty}^{q_i} \psi e^{3x} e^{-q_i - q_j - q_k} dx + \int_{q_i}^{q_j} \psi e^{x} e^{q_i - q_j - q_k} dx \right\} \\
- \int_{q_j}^{q_k} \psi e^{-x} e^{q_i + q_j - q_k} dx + \int_{q_k}^{+\infty} \psi e^{-3x} e^{q_i + q_j + q_k} dx \right\} \].

\[ (10.9) \]

Substituting all the above-founded expressions into Eq. (10.2), and setting the coefficients of \( \psi(q_i) \) and \( \psi_x(q_i) \) both to be zero, we find the dynamical system (1.29) satisfied by \( p_i(t) \) and \( q_i(t) \) for \( i = 1, 2, \cdots, N \). This completes the proof of this Proposition.

### 11 Proof of Proposition 1.3

In the periodic case, we have

\[ u = (1 - \partial_x^2) m = G * m, \quad G(x) = \frac{1}{2} \text{csch}(1/2) \cosh(x - 1/2). \]

We will show the relation (1.32) between the two parameters \( a \) and \( c \). Similar to Ref. [77] for the mCH equation, let \( S = [0, 1) \), and \( u_c \) be periodic functions on \( S \) with period one. Notice that \( u_c \) is continuous on \( S \) with peak at \( x = 0 \). Furthermore, \( u_c \) is smooth on \( (0, 1) \) and for all \( t \in \mathbb{R}^+ \) there holds

\[ \partial_x u_c(x, t) = -a \sinh(\xi) \in L^\infty(S) \]

\[ (11.2) \]

\[ \sum Springe \]
in the sense of periodic distribution $\mathcal{P}'$, where $\mathcal{P}'$ is the dual space of the space of $C_c^\infty$ functions on $\mathbb{S}$.

Denote by $u_{c,0}(x) = u_c(x,0)$, $x \in \mathbb{S}$. Then we have
\[
\lim_{t \to 0^+} \|u_c(\cdot, t) - u_{c,0}(\cdot)\|_{W^{1,\infty}(\mathbb{S})} = 0. \tag{11.3}
\]

Also, one finds
\[
\partial_t u_c(x, t) = ac \sinh(\xi) \in L^\infty(\mathbb{S}), \quad (t \geq 0) \tag{11.4}
\]
and
\[
u_c^2 \partial_x u_c = -a^3[\sinh(\xi) + \sinh^3(\xi)], \quad u_c \partial_x u_c = -\frac{1}{2}a^2 \sinh(2\xi). \tag{11.5}
\]

Similar to the case of the peakon solution presented in Sect. 10, we firstly consider the local terms in Eq. (1.24) with $u_c$ instead of $u$ and derive by invoking Eqs. (11.2)–(11.5) that
\[
\int_0^{+\infty} \int_{\mathbb{S}} \left( u_c \partial_x \varphi + \frac{k_1 + k_2}{3} u_c^2 \partial_x \varphi + \frac{k_1}{3} (\partial_x u_c)^2 \varphi + \frac{k_2}{2} u_c^3 \partial_x \varphi \right) dx dt \\
+ \int_S u_c(0,x) \varphi(0,x) dx \\
= - \int_0^{+\infty} \int_{\mathbb{S}} \varphi \sinh(\xi) \left[ ac - (k_1 + k_2) a^3 \\
- \left( \frac{2k_1}{3} + k_2 \right) a^3 \sinh^2(\xi) - k_3 a^2 \cosh(\xi) \right] dx dt \tag{11.6}
\]
for $\varphi(t,x) \in C^\infty_c([0, +\infty) \times \mathbb{S})$.

We next deal with the nonlocal terms in Eq. (1.24) with $u_c$ and $G(x)$ instead of $u$ and $p(x)$, respectively. We derive
\[
\int_0^{+\infty} \int_{\mathbb{S}} \left[ G * \left( \frac{2k_1 + k_2}{3} u_c^3 \right) + \left( k_1 + \frac{3}{2} k_2 \right) u_c (\partial_x u_c)^2 + \frac{1}{2} k_3 (\partial_x u_c)^2 \right] \partial_x \varphi \\
- G * \left( \frac{1}{3} k_1 + \frac{1}{2} k_2 \right) (\partial_x u_c)^3 \varphi \right) dx dt \\
= - \int_0^{+\infty} \int_{\mathbb{S}} \varphi G_x * \left( k_1 + \frac{3}{2} k_2 \right) a^3 \cosh(\xi) \sinh^2(\xi) \right] dx dt \\
- \int_0^{+\infty} \int_{\mathbb{S}} \varphi G_x * \left[ k_3 a^2 \cosh^2(\xi) + \frac{1}{2} k_3 a^2 \sinh^2(\xi) \right] dx dt \\
+ \int_0^{+\infty} \int_{\mathbb{S}} \varphi G * \left[ (2k_1 + 3k_2)a^3 \sinh(\xi) + \left( \frac{7}{3} k_1 + \frac{7}{2} k_2 \right) a^3 \sinh^3(\xi) \right] dx dt. \tag{11.7}
\]
The use of $\partial_x G(x) = 1/2 \csc(1/2) \sinh(x - 1/2 - [x])$, $\zeta(x, y) = 0.5 - (x - y) + [x - y]$ and $\eta(y, t) = 1/2 - (y - ct) + [y - ct]$ gives
\[
(11.7) = \int_0^{+\infty} \int_{\mathbb{S}} \varphi \left[ \frac{(2k_1 + 3k_2)a^3}{4 \sinh(0.5)} \int_{\mathbb{S}} \sinh(\zeta) \cosh(\eta) \sinh^2(\eta) d\eta \right] dx dt \\
+ \int_0^{+\infty} \int_{\mathbb{S}} \varphi \left[ \frac{k_3 a^2}{8 \sinh(0.5)} \int_{\mathbb{S}} \sinh(\zeta) \left( 1 + 3 \cosh(2\eta) \right) d\eta \right] dx dt \\
+ \int_0^{+\infty} \int_{\mathbb{S}} \varphi \left\{ \frac{a^3}{2 \sinh(0.5)} \int_{\mathbb{S}} \cosh(\zeta) \left[ \frac{2k_1 + 3k_2}{8} \sinh(\eta) \right] \right\} dx dt.
\]
\[
+7 \left( \frac{k_1}{12} + \frac{k_2}{8} \right) \sinh(3\eta) \right) dy \right\} dxdt
\]

\[
= \int_0^{+\infty} \int_S \varphi \sinh(\xi) \left[ k_3 a^2 \cosh(0.5) - \cosh(\xi) \right) 
\]

\[
+ \left( \frac{2k_1}{3} + k_2 \right) a^3 \left( \sinh^2(0.5) - \sinh^2(\xi) \right) \right] dxdt.
\] (11.8)

Using (11.6) and (11.8), one derives

\[
\int_0^{+\infty} \int_S \left( u_c \partial_t \varphi + \frac{1}{3} (k_1 + k_2) u_c^3 \partial_x \varphi + \frac{1}{3} k_1 (\partial_x u_c)^3 \varphi + \frac{1}{2} k_3 u_c^2 \partial_x \varphi \right) dxdt
\]

\[
+ \int_S u_c(0, x) \varphi(0, x) dx 
\]

\[
+ \int_0^{+\infty} \int_S \left[ G * \left( \left( \frac{2}{3}k_1 + k_2 \right) u_c^3 + \left( k_1 + \frac{3}{2}k_2 \right) u_c (\partial_x u_c)^2 \right) 
+ k_3 u_c^2 + \frac{1}{2} k_3 (\partial_x u_c)^2 \right] \partial_x \varphi 
\]

\[
- G * \left[ \left( \frac{1}{3}k_1 + \frac{1}{2}k_2 \right) (\partial_x u_c)^3 \right] \varphi \right) dxdt
\]

\[
= \int_0^{+\infty} \int_S \varphi \sinh(\xi) \left[ (k_1 + k_2) a^3 - ac + k_3 a^2 \cosh(0.5) \right] 
\]

\[
+ \left( \frac{2k_1}{3} + k_2 \right) a^3 \sinh^2(0.5) \right] dxdt = 0
\] (11.9)

under the condition (1.32) satisfied by \( c \). We thus complete the proof of Proposition 1.3.

12 Proof of Proposition 1.4

The periodic multi-peakon solutions (1.34) will be considered in this subsection. In this case, \( u = u_{\text{pm}}(x, t) \) given by Eq. (1.34) should satisfy

\[
\int_0^{\infty} \int_0^1 \left\{ u_t(\psi - \psi_{xx}) + \frac{k_1 + k_2}{3} u^3 \psi_{xxx} + \frac{k_1}{3} u_x^3 \psi_{xx} - \left( k_1 + \frac{4k_2}{3} \right) u^3 \psi_x 
\]

\[
- \left( k_1 + \frac{3k_2}{2} \right) uu_x^2 \psi_x + \frac{k_3}{2} u^2 \psi_{xxx} - \frac{3k_3}{2} u^2 \psi_x - \frac{k_3}{2} u_x^2 \psi_x + \frac{k_2}{2} u_x^2 \psi \right\} dxdt = 0
\] (12.1)

for \( \psi \in C_c^\infty([0, \infty) \times (0, 1)) \).

Direct computation leads to

\[
\int_0^{\infty} \int_0^1 u_t(\psi - \psi_{xx}) dxdt = 2 \sinh(1/2) \int_0^{\infty} \sum_{j=1}^{N} \left[ \dot{p}_j \psi(q_j) + p_j \dot{q}_j \psi_x(q_j) \right] dt \] (12.2)

The remaining terms in (12.1) are calculated by using integration by parts, we list the results as follows:
\[
\int_0^1 u^3 \psi_{xxx} \, dx = \frac{3}{4} \sum_{i=1}^{N} p_i^3 \left\{ -2 \psi_x(q_i) \sinh(3/2) - 2 \psi_x(q_i) \sinh(1/2) \\
+ \int_0^{q_i} \psi [9 \sinh(-3/2 - 3x + 3q_i) + \sinh(-1/2 - x + q_i)] \, dx \\
+ \int_{q_i}^{1} \psi [9 \sinh(3/2 - 3x + 3q_i) + \sinh(1/2 - x + q_i)] \, dx \right\} \\
+ \frac{3}{4} \sum_{i=1}^{N} \sum_{k \neq i} p_i^2 p_k \left\{ -4 \psi_x(q_i) [\sinh(3/2 + q_i - q_k) - \sinh(-1/2 + q_i - q_k)] \\
+ 2 \psi_x(q_k) [\sinh(1/2 + 2q_i - 2q_k) - \sinh(3/2 + 2q_i - 2q_k)] \\
+ 8 \psi(q_i) [\cosh(3/2 + q_i - q_k) - \cosh(1/2 - q_i + q_k)] \\
+ 8 \psi(q_k) [\cosh(1/2 + 2q_i - 2q_k) - \cosh(3/2 + 2q_i - 2q_k)] \\
+ \int_0^{q_i} \psi [27 \sinh(-3/2 - 3x + 2q_i + q_k) + \sinh(-1/2 - x + 2q_i - q_k)] \, dx \\
+ \int_{q_i}^{q_k} \psi [27 \sinh(1/2 - 3x + 2q_i + q_k) + \sinh(3/2 - x + 2q_i - q_k)] \, dx \\
+ \int_{q_k}^{1} \psi [27 \sinh(3/2 - 3x + 2q_i + q_k) + \sinh(1/2 - x + 2q_i - q_k)] \, dx \right\} \\
+ \frac{3}{2} \sum_{i=1}^{N} \sum_{l \neq i} p_i^2 p_k \left\{ -2 \psi_x(q_k) \sinh(1/2) + \int_0^{q_i} \psi \sinh(-1/2 - x + q_k) \, dx \\
+ \int_{q_i}^{q_k} \psi \sinh(-1/2 - x + q_k) \, dx + \int_{q_k}^{1} \psi \sinh(1/2 - x + q_k) \, dx \right\} \\
+ \frac{3}{4} \sum_{i=1}^{N} \sum_{l \neq i} p_i^2 p_k \left\{ 4 \psi_x(q_i) [\sinh(-3/2 + q_i - q_k) + \sinh(-1/2 - q_i + q_k)] \\
- 2 \psi_x(q_k) [\sinh(-1/2 + 2q_i - 2q_k) - \sinh(-3/2 + 2q_i - 2q_k)] \\
+ 8 \psi(q_i) [\cosh(1/2 + q_i - q_k) - \cosh(3/2 - q_i + q_k)] \\
+ 8 \psi(q_k) [\cosh(-3/2 + 2q_i - 2q_k) - \cosh(-1/2 + 2q_i - 2q_k)] \\
+ \int_0^{q_i} \psi [27 \sinh(-3/2 - 3x + 2q_i + q_k) + \sinh(-1/2 - x + 2q_i - q_k)] \, dx \\
+ \int_{q_i}^{q_k} \psi [27 \sinh(-1/2 - 3x + 2q_i + q_k) + \sinh(-3/2 - x + 2q_i - q_k)] \, dx \\
+ \int_{q_k}^{1} \psi [27 \sinh(3/2 - 3x + 2q_i + q_k) + \sinh(1/2 - x + 2q_i - q_k)] \, dx \right\} \\
+ \frac{3}{2} \sum_{i=1}^{N} \sum_{l < i} p_i^2 p_k \left\{ -2 \psi_x(q_k) \sinh(1/2) + \int_0^{q_k} \psi \sinh(-1/2 - x + q_k) \, dx \\
+ \int_{q_k}^{1} \psi \sinh(1/2 - x + q_k) \, dx + \int_{q_k}^{i} \psi \sinh(1/2 - x + q_k) \, dx \right\} \\
+ \frac{3}{2} \sum_{j < k} p_i p_j p_k [2 \psi_x(q_i) [\sinh(1/2 + 2q_i - q_j - q_k) - \sinh(3/2 + 2q_i - q_j - q_k)] \\
+ \sinh(-1/2 + q_j - q_k) + \sinh(-1/2 - q_j + q_k)] \\
+ 8 \psi(q_j) [\cosh(3/2 + 2q_i - q_j - q_k) - \cosh(1/2 + 2q_i - q_j - q_k)] \\
+ 2 \psi_x(q_j) [\sinh(-1/2 + q_i - 2q_j + q_k) - \sinh(1/2 + q_i - 2q_j + q_k)] \right\}
\]
\[
\begin{align*}
+ \sinh(1/2 + q_i - q_k) - \sinh(3/2 + q_i - q_k) \\
+ 8\psi(q_j)[\cosh(-1/2 + q_i - 2q_j + q_k) - \cosh(1/2 + q_i - 2q_j + q_k)] \\
+ 2\psi_x(q_k)[\sinh(1/2 + q_i + q_j - 2q_k) - \sinh(3/2 + q_i + q_j - 2q_k)] \\
+ \sinh(-1/2 + q_i - q_j) - \sinh(1/2 + q_i - q_j)] \\
+ 8\psi(q_k)[\cosh(1/2 + q_i + q_j - 2q_k) - \cosh(3/2 + q_i + q_j - 2q_k)] \\
+ \frac{3}{4} \sum_{i < j < k} p_i p_j p_k \left\{ \int_0^{q_i} \psi[27 \sinh(-3/2 - 3x + q_i + q_j + q_k) + \sinh(-1/2 - x + q_i + q_j - q_k)] dx \\
+ \sinh(-1/2 - x + q_i - q_j + q_k) - \sinh(1/2 + x + q_i - q_j - q_k)] dx \\
+ \int_{q_i}^{q_j} \psi[27 \sinh(-1/2 - 3x + q_i + q_j + q_k) + \sinh(1/2 - x + q_i + q_j - q_k)] dx \\
+ \sinh(-1/2 - x + q_i - q_j + q_k) - \sinh(1/2 + x + q_i - q_j - q_k)] dx \\
+ \int_{q_j}^{q_k} \psi[27 \sinh(1/2 - 3x + q_i + q_j + q_k) + \sinh(3/2 - x + q_i + q_j - q_k)] dx \\
+ \sinh(-1/2 - x + q_i - q_j + q_k) - \sinh(1/2 + x + q_i - q_j - q_k)] dx \\
+ \sinh(1/2 - x + q_i - q_j + q_k) - \sinh(-1/2 - x + q_i - q_j - q_k)] dx \right\}.
\end{align*}
\]

\[
\int_0^1 u_x^2 \psi_{xx} dx = -\frac{1}{4} \sum_{i=1}^N p_i^3 \left\{ \int_0^{q_i} 2\psi_x(q_i)[3 \sinh(1/2) - \sinh(3/2)] dx \\
+ \int_0^{q_i} 3 \psi[3 \sinh(-3/2 - 3x + 3q_i) - \sinh(-1/2 - x + q_i)] dx \\
+ \int_0^{q_i} 3 \psi[3 \sinh(3/2 - 3x + 3q_i) - \sinh(1/2 - x + q_i)] dx \right\} \\
- \frac{3}{4} \sum_{i=1}^N \sum_{k > i} p_i^2 p_k \left\{ 2\psi_x(q_k)[\sinh(1/2 + 2q_i - 2q_k) - \sinh(3/2 + 2q_i - 2q_k)] \\
+ 4\psi(q_i)[\cosh(3/2 + q_i - q_k) - \cosh(1/2 - q_i + q_k)] \\
+ 4\psi(q_k)[\cosh(1/2 + 2q_i - 2q_k) - \cosh(3/2 + 2q_i - 2q_k)] \\
+ \int_0^{q_i} 9 \sinh(-3/2 - 3x + 2q_i + q_k) - \sinh(-1/2 - x + 2q_i - q_k)] dx \\
+ \int_{q_i}^{q_j} 9 \sinh(1/2 - 3x + 2q_i + q_k) - \sinh(3/2 - x + 2q_i - q_k)] dx \\
+ \int_{q_j}^{q_k} 9 \sinh(3/2 - 3x + 2q_i + q_k) - \sinh(1/2 - x + 2q_i - q_k)] dx \right\} \\
+ \frac{3}{2} \sum_{i=1}^N \sum_{k > i} p_i^3 p_k \left\{ \int_{q_i}^{q_j} -2\psi_x(q_k) \sinh(1/2) + \int_0^{q_i} \psi \sinh(-1/2 - x + q_k)] dx \\
+ \int_{q_j}^{q_k} \psi \sinh(-1/2 - x + q_k) dx + \int_{q_k}^{1} \psi \sinh(1/2 - x + q_k)] dx \right\} \\
- \frac{3}{4} \sum_{i=1}^N \sum_{k > i} p_i^2 p_k \left\{ 2\psi_x(q_k)[\sinh(-3/2 + 2q_i - 2q_k) - \sinh(-1/2 + 2q_i - 2q_k)] \\
+ 4\psi(q_i)[\cosh(1/2 + q_i - q_k) - \cosh(3/2 - q_i + q_k)] \right\}
\]
\begin{align*}
+ 4\psi(q_k)[\cosh(-3/2 + 2q_i - 2q_k) - \cosh(-1/2 + 2q_i - 2q_k)] \\
+ \int_0^{q_k} \psi[9 \sinh(-3/2 - 3x + 2q_i + q_k) - \sinh(-1/2 - x + 2q_i - q_k)]dx \\
+ \int_{q_i}^{q_k} \psi[9 \sinh(-1/2 - 3x + 2q_i + q_k) - \sinh(-3/2 - x + 2q_i - q_k)]dx \\
+ \int_{q_i}^{1} \psi[9 \sinh(-3/2 - 3x + 2q_i + q_k) - \sinh(1/2 - x + 2q_i - q_k)]dx \\
+ \sum_{i=1}^{N} \sum_{k \neq i}^{N} p_i^2 p_k \left\{ -2\psi_x(q_k) \sinh(1/2) + \int_0^{q_k} \psi \sinh(-1/2 - x + q_k)dx \right\} \\
+ \int_{q_i}^{q_k} \psi \sinh(1/2 - x + q_k)dx + \int_{q_i}^{1} \psi \sinh(1/2 - x + q_k)dx \\
- \sum_{i<j<k}^{N} p_i p_j p_k \left\{ 2\psi_x(q_i) [\sinh(1/2 + 2q_i - q_j - q_k) - \sinh(3/2 + 2q_i - q_j - q_k) \\
+ \sinh(1/2 - q_j + q_k) + \sinh(1/2 + q_j - q_k)] \\
+ 4\psi(q_i)[\cosh(3/2 + 2q_i - q_j - q_k) - \cosh(1/2 + 2q_i - q_j - q_k)] \\
+ 2\psi_x(q_j)[\sinh(-1/2 + q_i - 2q_j + q_k) - \sinh(1/2 + q_i - 2q_j + q_k)] \\
+ \sinh(3/2 + 2q_i - q_j - q_k) - \sinh(1/2 + q_i - q_k)] \\
+ 4\psi(q_j)[\cosh(-1/2 + q_i - 2q_j + q_k) - \cosh(1/2 + q_i - 2q_j + q_k)] \\
+ 2\psi_x(q_k)[\sinh(1/2 + q_i + q_j - 2q_k) - \sinh(3/2 + q_i + q_j - 2q_k) \\
+ \sinh(1/2 + q_i - q_j) - \sinh(-1/2 + q_i - q_j)] \\
+ 4\psi(q_k)[\cosh(1/2 + q_i + q_j - 2q_k) - \cosh(3/2 + q_i + q_j - 2q_k)] \right\} \\
- \sum_{i<j<k}^{N} p_i p_j p_k \left\{ \int_0^{q_i} \psi[9 \sinh(-3/2 - 3x + q_i + q_j + q_k) - \sinh(-1/2 - x + q_i + q_j - q_k) \\
- \sinh(-1/2 - x + q_i - q_j + q_k) + \sinh(1/2 + x + q_i - q_j - q_k)]dx \right\} \\
+ \int_{q_i}^{q_j} \psi[9 \sinh(-1/2 - 3x + q_i + q_j + q_k) - \sinh(1/2 - x + q_i + q_j - q_k) \\
- \sinh(-1/2 - x + q_i - q_j + q_k) + \sinh(3/2 + x + q_i - q_j - q_k)]dx \\
+ \int_{q_j}^{q_k} \psi[9 \sinh(1/2 - 3x + q_i + q_j + q_k) - \sinh(3/2 - x + q_i + q_j - q_k) \\
- \sinh(-1/2 - x + q_i - q_j + q_k) + \sinh(1/2 + x + q_i - q_j - q_k)]dx \\
- \sinh(-1/2 - x + q_i - q_j + q_k) + \sinh(1/2 - x + q_i - q_j - q_k)]dx \right\}, \quad (12.4) \\
\int_0^{1} u^3 \psi_x dx = \frac{1}{4} \sum_{i=1}^{N} p_i^3 \left\{ 3 \int_0^{q_i} \psi[\sinh(-3/2 - 3x + 3q_i) + \sinh(-1/2 - x + q_i)]dx \right\} \\
+ \sum_{i=1}^{N} \int_{q_i}^{1} \psi[\sinh(3/2 - 3x + 3q_i) + \sinh(1/2 - x + q_i)]dx \\
+ \sum_{i=1}^{N} \sum_{k > i}^{N} p_i^2 p_k \left\{ \int_0^{q_i} \psi[3 \sinh(-3/2 - 3x + 2q_i + q_k) + \sinh(-1/2 - x + 2q_i - q_k)]dx \right\}
\end{align*}
\[
\int q_i \psi(3 \sinh(1/2 - 3x + 2q_i + q_k) + \sinh(3/2 - x + 2q_i - q_k))\,dx \\
+ \int_{q_k}^{q_i} \psi(3 \sinh(3/2 - x + 2q_i - q_k) + \sinh(1/2 - x + 2q_i - q_k))\,dx \\
+ \frac{3}{2} \sum_{i=1}^{N} \sum_{k<i} p_i^2 p_k \left\{ \left( \int_0^{q_i} + \int_{q_k}^{q_i} \right) \psi \sinh(-1/2 - x + q_k)\,dx + \int_{q_k}^{q_i} \psi \sinh(1/2 - x + q_k)\,dx \right\} \\
+ \frac{3}{4} \sum_{i=1}^{N} \sum_{k<i} p_i^2 p_k \left\{ \int_0^{q_k} \psi(3 \sinh(-3/2 - 3x + 2q_i + q_k) + \sinh(-1/2 - x + 2q_i - q_k))\,dx \\
+ \int_{q_k}^{q_i} \psi(3 \sinh(-1/2 - x + 2q_i - q_k) + \sinh(-3/2 - x + 2q_i - q_k))\,dx \\
+ \int_{q_k}^{q_i} \psi(3 \sinh(3/2 - 3x + 2q_i + q_k) + \sinh(1/2 - x + 2q_i - q_k))\,dx \\
+ \frac{3}{2} \sum_{i=1}^{N} \sum_{k<i} p_i^2 p_k \left\{ \left( \int_0^{q_k} + \int_{q_i}^{q_k} \right) \psi \sinh(-1/2 - x + q_k)\,dx + \left( \int_{q_i}^{q_k} + \int_{q_k}^{q_i} \right) \psi \sinh(1/2 - x + q_k)\,dx \right\} \\
+ \frac{3}{2} \sum_{i=1}^{N} \sum_{k<i} p_i p_k p_j p_k \left\{ \int_0^{q_1} \psi(3 \sinh(-3/2 - 3x + q_i + q_j + q_k) + \sinh(-1/2 - x + q_i + q_j - q_k))\,dx \\
+ \sinh(-1/2 - x + q_i - q_j + q_k) - \sinh(1/2 + x + q_i - q_j - q_k))\,dx \\
+ \int_{q_k}^{q_i} \psi(3 \sinh(-1/2 - 3x + q_i + q_j + q_k) + \sinh(1/2 - x + q_i + q_j - q_k))\,dx \\
+ \sinh(1/2 - x + q_i - q_j + q_k) - \sinh(3/2 + x + q_i - q_j - q_k))\,dx \\
+ \int_{q_j}^{q_k} \psi(3 \sinh(1/2 - 3x + q_i + q_j + q_k) + \sinh(3/2 - x + q_i + q_j - q_k))\,dx \\
+ \sinh(-1/2 - x + q_i - q_j + q_k) - \sinh(1/2 + x + q_i - q_j - q_k))\,dx \\
+ \int_{q_k}^{q_j} \psi(3 \sinh(3/2 - 3x + q_i + q_j + q_k) + \sinh(1/2 - x + q_i + q_j - q_k))\,dx \\
+ \sinh(1/2 - x + q_i - q_j + q_k) - \sinh(-1/2 + x + q_i - q_j - q_k))\,dx \right\}, \quad (12.5)
\]

\[
\int_0^1 uu^2 \psi^2 \,dx = \frac{1}{4} \sum_{i=1}^{N} p_i^3 \left\{ \int_0^{q_i} \psi(3 \sinh(-3/2 - 3x + q_i) - \sinh(-1/2 - x + q_i))\,dx \\
+ \int_{q_i}^{q_k} \psi(3 \sinh(3/2 - 3x + q_i) - \sinh(1/2 - x + q_i))\,dx \right\} \\
+ \frac{1}{2} \sum_{i=1}^{N} \sum_{k>i} p_i^2 p_k \left\{ 2 \psi(q_i) \cosh(3/2 + q_i - q_k) - \cosh(1/2 - q_i + q_k) \right\} \\
+ 2 \psi(q_k) \cosh(1/2 + 2q_i - 2q_k) - \cosh(3/2 + 2q_i - 2q_k) \\
+ \int_{q_i}^{q_k} \psi(3 \sinh(-3/2 - 3x + 2q_i + q_k) - \sinh(-1/2 - x + 2q_i - q_k))\,dx \\
+ \int_{q_i}^{q_k} \psi(3 \sinh(1/2 - 3x + 2q_i + q_k) - \sinh(3/2 - x + 2q_i - q_k))\,dx \\
+ \int_{q_k}^{q_i} \psi(3 \sinh(3/2 - 3x + 2q_i + q_k) - \sinh(1/2 - x + 2q_i - q_k))\,dx \right\}
\]
\[
\begin{align*}
&+ \frac{1}{2} \sum_{i=1}^{N} \sum_{k < j} p_i^2 p_k \left\{ 2\psi(q_i)[\cosh(1/2 + q_i - q_k) - \cosh(3/2 - q_i + q_k)] \\
&+ 2\psi(q_k)[\cosh(-3/2 + 2q_i - 2q_k) - \cosh(-1/2 + 2q_i - 2q_k)] \\
&+ \int_0^{q_k} \psi[3\sinh(-3/2 - 3x + 2q_i + q_k) - \sinh(-1/2 - x + 2q_i - q_k)]dx \\
&+ \int_{q_k}^{q_i} \psi[3\sinh(-1/2 - 3x + 2q_i + q_k) - \sinh(-3/2 - x + 2q_i - q_k)]dx \\
&+ \int_{q_i}^{1} \psi[3\sinh(3/2 - 3x + 2q_i + q_k) - \sinh(1/2 - x + 2q_i - q_k)]dx \right\} \\
&+ \frac{1}{4} \sum_{j=1}^{N} \sum_{l < j} p_j^2 p_l \left\{ \int_0^{q_j} \psi[3\sinh(-3/2 - 3x + 2q_j + q_l) + \sinh(-1/2 - x + 2q_j - q_l)]dx \\
&+ \int_{q_l}^{q_j} \psi[3\sinh(1/2 - 3x + 2q_j + q_l) + \sinh(3/2 - x + 2q_j - q_l)]dx \\
&+ \int_{q_j}^{1} \psi[3\sinh(3/2 - 3x + 2q_j + q_l) + \sinh(1/2 - x + 2q_j - q_l)]dx \right\} \\
&- \frac{1}{2} \sum_{j=1}^{N} \sum_{l < j} p_j^2 p_l \left\{ \left( \int_0^{q_j} + \int_{q_l}^{q_j} \right) \psi \sinh(-1/2 - x + q_l)dx + \int_{q_l}^{1} \psi \sinh(1/2 - x + q_l)dx \right\} \\
&+ \frac{1}{4} \sum_{j=1}^{N} \sum_{l < j} p_j^2 p_l \left\{ \int_0^{q_j} \psi[3\sinh(-3/2 - 3x + 2q_j + q_l) + \sinh(-1/2 - x + 2q_j - q_l)]dx \\
&+ \int_{q_l}^{q_j} \psi[3\sinh(1/2 - 3x + 2q_j + q_l) + \sinh(3/2 - x + 2q_j - q_l)]dx \\
&+ \int_{q_j}^{1} \psi[3\sinh(3/2 - 3x + 2q_j + q_l) + \sinh(1/2 - x + 2q_j - q_l)]dx \right\} \\
&- \frac{1}{2} \sum_{j=1}^{N} \sum_{l < j} p_j^2 p_l \left\{ \int_0^{q_j} \psi \sinh(-1/2 - x + q_l)dx + \left( \int_{q_l}^{q_j} + \int_{q_l}^{1} \right) \psi \sinh(1/2 - x + q_l)dx \right\} \\
&+ \frac{1}{2} \sum_{i < j < k} p_i p_j p_k \left\{ 4\psi(q_i)[\cosh(-3/2 + q_j + q_k - 2q_i) - \cosh(-1/2 + q_j + q_k - 2q_i)] \\
&+ 4\psi(q_j)[\cosh(-1/2 + q_i + q_k - 2q_j) - \cosh(1/2 + q_i + q_k - 2q_j)] \\
&+ 4\psi(q_k)[\cosh(1/2 + q_i + q_k - 2q_k) - \cosh(3/2 + q_i + q_k - 2q_k)] \right\} \\
&+ \int_0^{q_i} \psi[9\sinh(-3/2 - 3x + q_i + q_j + q_k) - \sinh(-1/2 - x + q_i + q_j - q_k) \\
&- \sinh(-1/2 - x + q_i - q_j + q_k) + \sinh(1/2 + x + q_i - q_j - q_k)]dx \\
&+ \int_{q_i}^{q_j} \psi[9\sinh(-1/2 - 3x + q_i + q_j + q_k) - \sinh(-1/2 - x + q_i + q_j - q_k) \\
&- \sinh(1/2 - x + q_i - q_j + q_k) + \sinh(3/2 + x + q_i - q_j - q_k)]dx \\
&+ \int_{q_j}^{q_k} \psi[9\sinh(1/2 - 3x + q_i + q_j + q_k) - \sinh(3/2 - x + q_i + q_j - q_k) \\
&- \sinh(-1/2 - x + q_i - q_j + q_k) + \sinh(1/2 + x + q_i - q_j - q_k)]dx \\
&+ \int_{q_k}^{1} \psi[9\sinh(3/2 - 3x + q_i + q_j + q_k) - \sinh(1/2 - x + q_i + q_j - q_k)]dx
\end{align*}
\]
\[- \sinh(1/2 - x + q_i - q_j + q_k) + \sinh(-1/2 + x + q_i - q_j - q_k)]dx \right \}, \\ 
(12.6) 
\int_0^1 u^2 \psi_{xx} \, dx = 2 \sum_{i=1}^N p_i^2 \left\{ - \psi_x(q_i) \sinh(1) + 2 \int_0^{q_i} \psi \sinh(-1 - 2x + 2q_i) \, dx \\
+ 2 \int_1^{q_i} \psi \sinh(1 - 2x + 2q_i) \, dx \right\} \\
+ \sum_{i=1}^N \sum_{j>i} p_i p_j \left\{ [\psi_x(q_i) + \psi_x(q_j)] [\sinh(q_i - q_j) - \sinh(1 + q_i - q_j)] \\
+ 2[\psi(q_i) - \psi(q_j)] [\cosh(q_i - q_j) - \cosh(1 + q_i - q_j)] \\
+ 4 \int_0^{q_i} \psi \sinh(-1 - 2x + q_i + q_j) \, dx + 4 \int_{q_i}^{q_j} \psi \sinh(-2x + q_i + q_j) \, dx \\
+ 4 \int_{q_j}^1 \psi \sinh(1 - 2x + q_i + q_j) \, dx \right\}, \\
(12.7) 
\int_0^1 u^2 \psi_x \, dx = \sum_{i=1}^N p_i^2 \left\{ \int_0^{q_i} \psi \sinh(-1 - 2x + 2q_i) \, dx + \int_1^{q_i} \psi \sinh(1 - 2x + 2q_i) \, dx \right\} \\
+ \sum_{i=1}^N \sum_{j>i} p_i p_j \left\{ \int_0^{q_i} \psi \sinh(-1 - 2x + q_i + q_j) \, dx + \int_{q_i}^{q_j} \psi \sinh(-2x + q_i + q_j) \, dx \\
+ \int_{q_j}^1 \psi \sinh(1 - 2x + q_i + q_j) \, dx \right\} \\
+ \sum_{i=1}^N \sum_{j<i} p_i p_j \left\{ \int_0^{q_i} \psi \sinh(-1 - 2x + q_i + q_j) \, dx + \int_{q_i}^{q_j} \psi \sinh(-2x + q_i + q_j) \, dx \\
+ \int_{q_i}^1 \psi \sinh(1 - 2x + q_i + q_j) \, dx \right\}, \\
(12.8) 
\int_0^1 u^2 \psi_x \, dx = \sum_{i=1}^N p_i^2 \left\{ \int_0^{q_i} \psi \sinh(-1 - 2x + 2q_i) \, dx + \int_1^{q_i} \psi \sinh(1 - 2x + 2q_i) \, dx \right\} \\
+ \sum_{i=1}^N \sum_{j>i} p_i p_j \left\{ [\psi(q_i) - \psi(q_j)] [\cosh(1 + q_i - q_j) - \cosh(q_i - q_j)] \\
+ \int_0^{q_i} \psi \sinh(-1 - 2x + q_i + q_j) \, dx + \int_{q_i}^{q_j} \psi \sinh(-2x + q_i + q_j) \, dx \\
+ \int_{q_j}^1 \psi \sinh(1 - 2x + q_i + q_j) \, dx \right\} \]
\[
+ \sum_{i=1}^{N} \sum_{j<i} p_i p_j \left\{ [\psi(q_i) - \psi(q_j)] \left[ \cosh(q_i - q_j) - \cosh(1 - q_i + q_j) \right] \right.
\]
\[
+ \int_0^{q_j} \psi \sinh(-1 - 2x + q_i + q_j) dx + \int_{q_j}^{q_i} \psi \sinh(-2x + q_i + q_j) dx
\]
\[
+ \int_{q_j}^1 \psi \sinh(1 - 2x + q_i + q_j) dx \right\},
\]
\[
\int_0^1 u^i \psi(x) dx = -\frac{1}{4} \sum_{i=1}^{N} \left\{ \int_0^{q_i} \psi [\sinh(-3/2 - 3x + q_i)] - 3 \sinh(-1/2 - x + q_i)] dx
\]
\[
+ \int_{q_i}^1 \psi [\sinh(3/2 - 3x + q_i)] - 3 \sinh(1/2 - x + q_i)] dx \right\}
\]
\[
- \frac{3}{4} \sum_{i=1}^{N} \sum_{k>i} p_i^2 p_k \left\{ \int_0^{q_k} \psi [\sinh(-3/2 - 3x + 2q_i + q_k)] - \sinh(-1/2 - x + 2q_i - q_k)] dx
\]
\[
+ \int_{q_k}^{q_i} \psi [\sinh(-1/2 - 3x + 2q_i + q_k)] - \sinh(-3/2 - x + 2q_i - q_k)] dx
\]
\[
+ \int_{q_k}^1 \psi [\sinh(3/2 - 3x + 2q_i + q_k)] - \sinh(1/2 - x + 2q_i - q_k)] dx \right\}
\]
\[
+ \frac{3}{2} \sum_{i=1}^{N} \sum_{k>i} p_i^2 p_k \left\{ \left( \int_0^{q_k} + \int_{q_k}^{q_i} \right) \psi \sinh(-1/2 - x + q_k) dx + \int_{q_k}^1 \psi \sinh(1/2 - x + q_k) dx \right\}
\]
\[
- \frac{3}{4} \sum_{i=1}^{N} \sum_{k>i} p_i^2 p_k \left\{ \int_0^{q_k} \psi [\sinh(-3/2 - 3x + 2q_i + q_k)] - \sinh(-1/2 - x + 2q_i - q_k)] dx
\]
\[
+ \int_{q_k}^{q_i} \psi [\sinh(-1/2 - 3x + 2q_i + q_k)] - \sinh(-3/2 - x + 2q_i - q_k)] dx
\]
\[
+ \int_{q_k}^1 \psi [\sinh(3/2 - 3x + 2q_i + q_k)] - \sinh(1/2 - x + 2q_i - q_k)] dx \right\}
\]
\[
+ \frac{3}{2} \sum_{i=1}^{N} \sum_{k>i} p_i^2 p_k \left\{ \int_0^{q_k} \psi \sinh(-1/2 - x + q_k) dx + \left( \int_{q_k}^{q_i} + \int_{q_k}^1 \right) \psi \sinh(1/2 - x + q_k) dx \right\}
\]
\[
- \frac{3}{2} \sum_{i<j<k} p_i p_j p_k \left\{ \int_0^{q_k} \psi [\sinh(-3/2 - 3x + q_i + q_j + q_k)] - \sinh(-1/2 - x + q_i + q_j + q_k)] dx
\]
\[
- \sinh(-1/2 - x + q_i - q_j + q_k)] + \sinh(1/2 + x + q_i - q_j - q_k)] dx
\]
\[
+ \int_{q_k}^{q_j} \psi [\sinh(-1/2 - 3x + q_i + q_j + q_k)] - \sinh(1/2 - x + q_i + q_j - q_k)] dx
\]
\[
+ \int_{q_j}^{q_k} \psi [\sinh(1/2 - 3x + q_i + q_j + q_k)] - \sinh(3/2 - x + q_i + q_j - q_k)] dx
\]
\[
- \sinh(-1/2 - x + q_i - q_j + q_k)] + \sinh(1/2 + x + q_i - q_j - q_k)] dx
\]
\[
+ \int_{q_k}^1 \psi [\sinh(3/2 - 3x + q_i + q_j + q_k)] - \sinh(1/2 - x + q_i + q_j - q_k)] dx
\]
\[
- \sinh(-1/2 - x + q_i - q_j + q_k)] + \sinh(-1/2 + x + q_i - q_j - q_k)] dx \right\}.
\]
\[
(12.9)
\]
\[
(12.10)
\]
Substituting Eqs. (12.2)-(12.10) into Eq. (12.1), one finds the coefficient of the term $\psi(q_m)$ are

$$
2 \sinh(1/2) p_m + \frac{k_2}{2} \left\{ \sum_{k>m} p_{kn}^2 [\cosh(3/2 + q_m - q_k) - \cosh(1/2 - q_m + q_k)]
+ \sum_{i<m} p_{im}^2 p_m [\cosh(1/2 + 2q_i - 2q_m) - \cosh(3/2 + 2q_i - 2q_m)]
+ \sum_{k<m} p_{km}^2 p_k [\cosh(1/2 + q_m - q_k) - \cosh(3/2 - q_m + q_k)]
+ \sum_{i>m} p_{im}^2 p_m [\cosh(1/2 + 2q_i - 2q_m) - \cosh(1/2 + 2q_i - 2q_m)]
+ 2 \sum_{m<j<k} p_m p_j p_k [\cosh(3/2 + 2q_m - q_j - q_k) - \cosh(1/2 + 2q_m - q_j - q_k)]
+ 2 \sum_{i<m<j<k} p_i p_m p_k [\cosh(1/2 + q_i + q_j - 2q_m) - \cosh(3/2 + q_i + q_j - 2q_m)]
\right\}

+ \left(2 \sinh(1/2) \sum_{i<m} p_{im}^2 p_m + 2 \sum_{m<j<k} p_m p_j p_k [\cosh(1/2 + q_j - q_k) - \sinh(1/2 + q_j - q_k)]
+ 2 \sum_{i<m<j<k} p_i p_m p_k [\sinh(1/2 + q_j - q_k) - \sinh(3/2 + q_j - q_k)]
+ 2 \sum_{i<j<m} p_{ij} p_j p_m [\sinh(-1/2 - q_i - q_j) - \sinh(1/2 + q_i - q_j)]
\right)$$

and the coefficient of $\psi_x(q_m)$ are

$$
2 \sinh(1/2) p_m q_m - \left(\frac{k_1}{3} + \frac{k_2}{2}\right) p_{kn}^3 [\sinh(3/2) + \sinh(1/2)] - \frac{2k_1}{3} p_{kn}^3 \sinh(1/2)

- k_3 p_m^2 \sinh(1) + k_3 \left\{ \sum_{k>m} p_{km}^2 p_k [- \sinh(3/2 + q_m - q_k) + \sinh(-1/2 + q_m - q_k)]
- 2 \sinh(1/2) \sum_{i<m} p_{im}^2 p_m + 2 \sum_{m<j<k} p_m p_j p_k [\sinh(-1/2 - q_m + q_k) + \sinh(-3/2 + q_m - q_k)]
- 2 \sinh(1/2) \sum_{i>m} p_{im}^2 p_m + 2 \sum_{m<j<k} p_m p_j p_k [\sinh(-1/2 + q_j - q_k) - \sinh(1/2 + q_j - q_k)]
\right\}

+ 2 \sum_{i<m<j<k} p_i p_m p_k [\sinh(1/2 + q_i - q_j) - \sinh(3/2 + q_i - q_j)]

+ 2 \sum_{i<j<m} p_{ij} p_j p_m [\sinh(-1/2 + q_i - q_j) - \sinh(1/2 + q_i - q_j)]

+ k_2 \left\{ \sum_{k>m} p_{km}^2 p_k [- \sinh(3/2 + q_m - q_k) + \sinh(-1/2 + q_m - q_k)]
+ 2 \sum_{i<m} p_{im}^2 p_m [\sinh(1/2 + 2q_i - 2q_m) - \sinh(3/2 + 2q_i - 2q_m) - 2 \sinh(1/2)]
+ \sum_{k<m} p_{km}^2 p_k [\sinh(-1/2 - q_m + q_k) + \sinh(-3/2 + q_m - q_k)]
\right\}$$
\[ + \frac{1}{2} \sum_{i > m} p_i^2 p_m [\sinh(-3/2 + 2q_i - 2q_m) - \sinh(-1/2 + 2q_i - 2q_m) - 2 \sinh(1/2)] + \sum_{m < j < k} p_m p_j p_k [- \sinh(3/2 + 2q_m - q_j - q_k) + \sinh(1/2 + 2q_m - q_j - q_k) + \sinh(-1/2 + q_j - q_k) + \sinh(-1/2 - q_j + q_k)] + \sum_{i < m < k} p_i p_m p_k [\sinh(-1/2 + q_i - 2q_m + q_k) - \sinh(1/2 + q_i - 2q_m + q_k) + \sinh(1/2 + q_i - q_k) - \sinh(3/2 + q_i - q_k)] + \sum_{i < j < m} p_i p_j p_m [\sinh(1/2 + q_i + q_j - 2q_m) - \sinh(3/2 + q_i + q_j - 2q_m) + \sinh(-1/2 + q_i - q_j) - \sinh(1/2 + q_i - q_j)] + k_3 \left\{ \sum_{j > m} p_m p_j [\sinh(-1 + q_j - q_m) - \sinh(q_j - q_m)] + \sum_{j < m} p_m p_j [\sinh(q_j - q_m) - \sinh(1 + q_j - q_m)] \right\}. \] (12.12)

According to the arbitrariness of the function \( \psi(t, x) \), one derives the dynamical system (1.35) satisfied by \( p_m(t) \) and \( q_m(t) \) \((m = 1, 2, \ldots, N)\) after setting the coefficients given by Eqs. (12.11) and (12.12) both to be zero. This completes the proof of Proposition 1.4.

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### Appendix A. Basics Properties of the Littlewood-Paley Theory

Let \( B(x_0, r) \) be the open ball centered at \( x_0 \) with radius \( r \), \( C \equiv \{ \xi \in \mathbb{R}^d | 4/3 \leq |\xi| \leq 8/3 \} \), and \( \tilde{C} \equiv B(0, 2/3) + C \). Then there are two radial functions \( \chi \in \mathcal{D}(B(0, 4/3)) \) and \( \varphi \in \mathcal{D}(C) \) satisfying

\[
\left\{ \begin{array}{l}
\chi(\xi) + \sum_{q \geq 0} \varphi(2^{-q} \xi) = 1, \\
|q - q'| \geq 2 \Rightarrow \text{Supp} \varphi(2^{-q} \cdot) \cap \text{Supp} \varphi(2^{-q'} \cdot) = \emptyset, \\
q \geq 1 \Rightarrow \text{Supp} \chi(\cdot) \cap \text{Supp} \varphi(2^{-q'} \cdot) = \emptyset, \\
|q - q'| \geq 5 \Rightarrow 2^q \tilde{C} \cap 2^q C = \emptyset.
\end{array} \right.
\]

The dyadic operators \( \Delta_q \) and \( S_q \) acting on \( u(t, x) \in S'(\mathbb{R}^d) \) are defined as

\[
\Delta_q u = \begin{cases} 
0, & q \leq -2, \\
\chi(D) u = \int_{\mathbb{R}^d} \tilde{h}(y) u(x - y) dy, & q = -1, \\
\varphi(2^{-q} D) u = 2^q \int_{\mathbb{R}^d} h(2^q y) u(x - y) dy, & q \geq 0,
\end{cases}
\]

\[
S_q u = \sum_{q' \leq q - 1} \Delta_q' u,
\]

where \( h = \mathcal{F}^{-1} \varphi \) and \( \tilde{h} = \mathcal{F}^{-1} \chi \) with \( \mathcal{F}^{-1} \) denoting the inverse Fourier transform.

The Besov spaces are \( B^s_{p, r}(\mathbb{R}^d) = \left\{ u \in S' \right\} \| u \|_{B^s_{p, r}(\mathbb{R}^d)} = \left( \sum_{j \geq -1} 2^{js} \| \Delta_j u \|_{L^p(\mathbb{R}^d)}^r \right)^{1/r} \). With the above-defined Besov spaces, we next recall some of their properties.

**Lemma 12.1** (Embedding property) \([2, 12, 33]\) Suppose \( 1 \leq p_1 \leq p_2 \leq \infty, 1 \leq r_1 \leq r_2 \leq \infty \) and \( s \) be real. Then it holds that \( B^s_{p_1, r_1}(\mathbb{R}^d) \hookrightarrow B^{s-d(1/p_1-1/p_2)}_{p_2, r_2}(\mathbb{R}^d) \). If \( s > d/p \) or \( s = d/p, r = 1 \), then there holds \( B^s_{p, r}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d) \).
Lemma 12.2 (Interpolation) [2,12,33] Let $s_1$, $s_2$ be real numbers with $s_1 < s_2$ and $\theta \in (0, 1)$. Then there exists a constant $C$ such that
\[
\|u\|_{B^{s_1+(1-\theta)s_2}_{p,r}} \leq \|u\|_{B^{\theta}_{p,r}} \|u\|_{B^{s_2}_{p,r}}^{(1-\theta)},
\|u\|_{B^{s_1+(1-\theta)s_2}_{p,1}} \leq \frac{C}{s_2-s_1(1-\theta)} \|u\|_{B^{s_1}_{p,\infty}} \|u\|_{B^{s_2}_{p,\infty}}^{(1-\theta)},
\]
where $(p, r) \in [1, \infty)^2$.

Lemma 12.3 (Product law) [2,12,33] Let $(p, r) \in [1, \infty)^2$ and $s$ be real. Then $\|uv\|_{B^{s}_{p,r}(\mathbb{R}^d)} \leq C(\|u\|_{L^\infty(\mathbb{R}^d)} \|v\|_{B^{s}_{p,r}(\mathbb{R}^d)} + \|u\|_{B^{s}_{p,r}(\mathbb{R}^d)} \|v\|_{L^\infty(\mathbb{R}^d)})$, namely, the space $L^\infty(\mathbb{R}^d) \cap B^{s}_{p,r}(\mathbb{R}^d)$ is an algebra. Moreover, if $s > \frac{d}{p}$ or $s = \frac{d}{p}, r = 1$, then there exists $C$ such that $\|uv\|_{B^{s}_{p,r}(\mathbb{R}^d)} \leq C\|u\|_{B^{s}_{p,1}(\mathbb{R}^d)} \|v\|_{B^{s}_{p,r}(\mathbb{R}^d)}$.

Lemma 12.4 (Moser-type estimates) [2,32] Let $s > \max\{d/p, d/2\}$ and $(p, r) \in [1, \infty)^2$. Then, for any $a \in B^{s-1}_{p,r}(\mathbb{R}^d)$ and $b \in B^{s}_{p,r}(\mathbb{R}^d)$, there holds $\|ab\|_{B^{s-1}_{p,r}(\mathbb{R}^d)} \leq C\|a\|_{B^{s}_{p,1}(\mathbb{R}^d)} \|b\|_{B^{s}_{p,r}(\mathbb{R}^d)}$.

The following Lemma is useful for proving the blow-up criterion.

Lemma 12.5 (Moser-type estimates) [46,47] Let $s \geq 0$. Then one has
\[
\|fg\|_{H^{s}(\mathbb{R})} \leq C(\|f\|_{H^{s}(\mathbb{R})} \|g\|_{L^\infty(\mathbb{R})} + \|f\|_{L^\infty(\mathbb{R})} \|g\|_{H^{s}(\mathbb{R})}),
\|f\partial_{x}g\|_{H^{s}(\mathbb{R})} \leq C(\|f\|_{H^{s+1}(\mathbb{R})} \|g\|_{L^\infty(\mathbb{R})} + \|f\|_{L^\infty(\mathbb{R})} \|\partial_{x}g\|_{H^{s}(\mathbb{R})}),
\]
where $C$’s are constants independent of $f$ and $g$.

Lemma 12.6 [2] Let $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$. Then the Besov spaces have the following properties:

- $B^{s}_{p,r}(\mathbb{R}^d)$ is a Banach space and continuously embedding into $S^{'}(\mathbb{R}^d)$, where $S^{'}(\mathbb{R}^d)$ is the dual space of the Schwartz space $S(\mathbb{R}^d)$;
- If $p, r < \infty$, then $S(\mathbb{R}^d)$ is dense in $B^{s}_{p,r}(\mathbb{R}^d)$;
- If $u_n$ is a bounded sequence of $B^{s}_{p,r}(\mathbb{R}^d)$, then an element $u \in B^{s}_{p,r}(\mathbb{R}^d)$ and a subsequence $u_{n_k}$ exist such that $\lim_{k \to \infty} u_{n_k} = u$ in $S^{'}(\mathbb{R}^d)$ and $\|u\|_{B^{s}_{p,r}(\mathbb{R}^d)} \leq C \lim \inf_{k \to \infty} \|u_{n_k}\|_{B^{s}_{p,r}(\mathbb{R}^d)}$.

Appendix B. Some Lemmas in the Theory of the Transport Equation

We recall some a priori estimates [2,32] for the following transport equation
\[
\phi_t + v \cdot \nabla \phi = \omega, \quad \phi|_{t=0} = \phi_0.
\] (B.1)

Lemma 12.7 [2,32] Let $1 \leq p \leq p_1 \leq \infty, 1 \leq r \leq \infty$ and $s \geq -d \min(1/p_1, 1 - 1/p)$. Let $\phi_0 \in B^{s}_{p_1,r}(\mathbb{R}^d)$, $\omega \in L^1([0, T]; B^{s}_{p_1,r}(\mathbb{R}^d))$ and $\nabla v \in L^1([0, T]; B^{s}_{p_1,r}(\mathbb{R}^d)) \cap L^\infty(\mathbb{R}^d)$, then there exists a unique solution $\phi \in L^\infty([0, T]; B^{s}_{p_1,r}(\mathbb{R}^d))$ to Eq. (B.1) satisfying:
\[
\|\phi\|_{B^{s}_{p,r}(\mathbb{R}^d)} \leq \|\phi_0\|_{B^{s}_{p,r}(\mathbb{R}^d)} + \int_0^t \|\omega(t')\|_{B^{s}_{p,r}(\mathbb{R}^d)} + CU_{p_1}(t') \|\phi(t')\|_{B^{s}_{p,r}(\mathbb{R}^d)} dt',
\] (B.2)
\[
\|\phi\|_{B^{s}_{p,r}(\mathbb{R}^d)} \leq \left[\|\phi_0\|_{B^{s}_{p,r}(\mathbb{R}^d)} + \int_0^t \|\omega(t')e^{-CU_{p_1}(t')}\|_{B^{s}_{p,r}(\mathbb{R}^d)} dt'\right] e^{CU_{p_1}(t)}.
\] (B.3)
where $U_{p_1}(t) = \int_0^t \|\nabla v\|_{B_{p_1}^{d,p_1}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)} dt'$ if $s < 1 + d/p_1$, $U_{p_1}(t) = \int_0^t \|\nabla v\|_{B_{p_1}^{d-1,p_1}(\mathbb{R}^d)} dt'$ if $s > 1 + d/p_1$ or $s = 1 + d/p_1$, $r = 1$, and $C$ is a constant depending only on $s$, $p$, $p_1$, and $r$.

Lemma 12.8 \cite{2} Let $s \geq -d \min(1/p_1, 1 - 1/p)$. Let $\phi_0 \in B_{p,r}^s(\mathbb{R}^d)$, $\omega \in L^1([0, T]; B_{p,r}^s(\mathbb{R}^d))$ and $v \in L^p([0, T]; B_{-M}^{-s,\infty}(\mathbb{R}^d))$ for some $\rho > 1$ and $M > 0$ be a time-dependent vector field satisfying

$$
\nabla v \in \begin{cases} L^1([0, T]; B_{p_1/p}^{d/p}(\mathbb{R}^d)), & \text{if } s < 1 + d/p_1, \\
L^1([0, T]; B_{p_1}^{s-1}(\mathbb{R}^d)), & \text{if } s > 1 + d/p_1 \text{ or } s = 1 + d/p_1 \text{ and } r = 1.
\end{cases}
$$

Then, Eq. (B.1) has a unique solution $\phi \in C([0, T]; B_{p,r}^s(\mathbb{R}^d))$ for $r < \infty$, or $\phi \in (\bigcap_{s < 0} C([0, T]; B_{p,\infty}^s(\mathbb{R}^d))) \cap C([0, T]; B_{p,\infty}^s(\mathbb{R}^d)))$ for $r = \infty$. Furthermore, the inequalities (B.2)-(B.3) hold.

Lemma 12.9 (A priori estimate in the Sobolev spaces) \cite{2,46} Let $0 \leq \sigma < 1$. Let $\phi_0 \in H^\sigma$, $\omega \in L^1(0, T; H^\sigma)$ and $\partial_\nu v \in L^1(0, T; L^\infty)$. Then the solution $\phi$ to Eq. (B.1) belongs to $C([0, T]; H^{\sigma})$. More precisely, there is a constant $C$ depending only on $\sigma$ such that

$$
\|\phi\|_{H^\sigma} \leq \|\phi_0\|_{H^\sigma} + \int_0^t \left[ \|\omega(t)\|_{H^\sigma} + C U'(\tau)(\|\phi(\tau)\|_{H^\sigma}) \right] d\tau, \quad U(t) = \int_0^t \|\partial_\nu v(\tau)\|_{L^\infty} d\tau.
$$

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