Estimation and Selection Properties of the LAD Fused Lasso Signal Approximator

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Abstract

The fused lasso is an important method for signal processing when the hidden signals are sparse and blocky. It is often used in combination with the squared loss function. However, the squared loss is not suitable for heavy tail error distributions nor is robust against outliers which arise often in practice. The least absolute deviations (LAD) loss provides a robust alternative to the squared loss. In this paper, we study the asymptotic properties of the fused lasso estimator with the LAD loss for signal approximation. We refer to this estimator as the LAD fused lasso signal approximator, or LAD-FLSA. We investigate the estimation consistency properties of the LAD-FLSA and provide sufficient conditions under which the LAD-FLSA is sign consistent. We also construct an unbiased estimator for the degrees of freedom of the LAD-FLSA for any given tuning parameters. Both simulation studies and real data analysis are conducted to illustrate the performance of the LAD-FLSA and the effect of the unbiased estimator of the degrees of freedom.

Keywords: Estimation consistency; Jump selection consistency; Block selection consistency; Degrees of freedom; Fused lasso; Least absolute deviations; Sign consistency.

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# 1 Introduction

High-dimensional data arise in many fields including signal processing, image de-noising and genomic and genetic studies. When a model is sparse and has certain known structures, penalized methods have been widely used since they can incorporate known structures into penalty functions in a natural way and can do estimation and variable selection simultaneously. A biological example is the analysis of copy-number variations in a human genome. In this problem, we are interested in detecting the changes in copy numbers based on data from comparative genomic hybridization (CGH) arrays. For instance, Snijders et al. (2001) studied a CGH array consisting of 2400 bacterial artificial chromosome (BAC) clones, where the log base 2 intensity ratios at all clones are measured. In Figure 1, we plot a sample CGH copy number data on chromosomes 1–4 from cell line GM 13330. The data set has two characteristics: 1) there are only a small number of chromosomal locations where the copy numbers of genes change, that is, the underlying signals are sparse; 2) the adjacent markers tend to have similar observations, i.e., the signals are blocky.

In a signal approximation model with sample size $n$, the $i$th observation $y_i$ is considered to be a realization of the true signal $\mu^0_i$ plus random noise $\varepsilon_i$,

$$y_i = \mu^0_i + \varepsilon_i, \quad i = 1, \ldots, n.$$  

In many cases, the true signal vector $\mu^0 = (\mu^0_1, \ldots, \mu^0_n)'$ is both blocky and sparse, meaning that the intensities of true signals within each block are the same and most blocks consist of zero signals. The goal here is to find a solution not only to recover all the changes points, but also to identify the nonzero blocks. We can use the lasso penalty to enforce a sparse solution by penalizing the $\ell_1$ norm of the signals $\|\mu\|_1 \equiv \sum_{i=1}^{n} |\mu_i|$, and use the fusion (total variation) penalty to enforce a blocky solution by penalizing $\|\mu\|_{TV} \equiv \sum_{i=2}^{n} |\mu_i - \mu_{i-1}|$. The combination of these two penalties results in the fused lasso (FL) penalty (Tibshirani et al., 2005).

For detecting changing points in copy number variations, Tibshirani and Wang (2008) proposed to use the fused lasso with a squares loss function. We refer to this approach as the least squares fused lasso signal approximator (LS-FLSA). For $\lambda_n = (\lambda_{1n}, \lambda_{2n})$, the LS-FLSA seeks to find $\hat{\mu}_{n}^{\ell_2}(\lambda_n) = (\hat{\mu}_{1n}^{\ell_2}(\lambda_n), \ldots, \hat{\mu}_{nn}^{\ell_2}(\lambda_n))'$ that minimizes

$$\sum_{i=1}^{n} (y_i - \mu_i)^2 + \lambda_{1n} \sum_{i=1}^{n} |\mu_i| + \lambda_{2n} \sum_{i=2}^{n} |\mu_i - \mu_{i-1}|,$$  

where $\lambda_{1n}$ and $\lambda_{2n}$ are two nonnegative penalty parameters.

Recently, Rinaldo (2009) studied the selection properties of the LS-FLSA and adaptive LS-FLSA under the block partition assumption in the underlying signal. Several authors have also studied the properties of related procedures. For example, Mammen and van de Geer (1997) studied the rate of convergence in bounded variation function classes of the parameter functions; Harchaoui and Lévy-Leduc (2010) investigated the estimation properties of change points using the total variation penalty; Boysen et al. (2009) studied the asymptotic properties for jump-penalized least-squares regression aiming at approximating a regression function by piecewise-constant functions. These studies significantly advanced our understanding of LS-FLSA or fusion penalized LS methods in the context signal detection or nonparametric estimation. However, all these
results are obtained for methods with the least squares loss and/or require normality assumption on the errors. The LS-FLSA is easily affected by outliers which arise often in practice, for example, in CGH copy number variation data.

A more robust fused lasso signal approximator can be constructed by using the LAD loss, which we shall refer to as LAD-FLSA. For any given $\lambda_n = (\lambda_{1n}, \lambda_{2n})$, the LAD-FLSA is defined as

$$
\widehat{\mu}_n(\lambda_n) = \arg \min_{\mu \in \mathbb{R}^n} \left\{ \sum_{i=1}^n |y_i - \mu_i| + \lambda_{1n} \sum_{i=1}^n |\mu_i| + \lambda_{2n} \sum_{i=2}^n |\mu_i - \mu_{i-1}| \right\}.
$$

(3)

The convex minimizer $\widehat{\mu}_n$ in (3) has been applied to detect copy number variation breakpoints in Gao and Huang (2010a). However, its theoretic properties of have not been studied.

In this paper, we seek to answer the following questions about the LAD-FLSA: (1) how close $\widehat{\mu}_n$ can be to the true model $\mu^0$ asymptotically? (2) how accurately $\widehat{\mu}_n$ can recover the true nonzero blocks with a large probability when $\mu^0$ is both sparse and blocky? (3) what is the degrees of freedom of LAD-FLSA? The contributions of this paper are as follows.

- We show that the LAD-FLSA is rate consistent if the number of blocks is relatively small.
- We provide sufficient conditions under which the LAD-FLSA is able to recover the block patterns and distinguish nonzero blocks from zero ones correctly with high probability. That is, the LAD-FLSA can determine all the jumps, identify all the nonzero blocks, and also distinguish the positive signals from negative ones under some conditions.
- In terms of model complexity measures, we justify that the number of nonzero blocks generated from a LAD-FLSA estimate is an unbiased estimator of the degrees of freedom of the LAD-FLSA.
- Without the assumption of Gaussian or sub-Gaussian random error, our studies can be widely applied for signal detection in signal processing when random noises do not follow nice distributions or the signal-to-noise ratios are large.

The rest of the paper is organized as follows. We list some notations in Section 2. We study the estimation consistency and sign consistency properties of the LAD-FLSA respectively in Section 3 and 4. In Section 5 we derive an unbiased estimator of the degrees of freedom of the LAD-FLSA. In Section 6 we conduct simulation studies and real data analysis to demonstrate the performance of the LAD-FLSA. We also verify the effect of unbiased estimator of the degrees of freedom numerically in this section. Section 7 concludes the paper with some discussions. All the technical proofs are postponed to the Appendix.

## 2 Preliminaries

Suppose the true model $\mu^0 = (\mu^0_1, \cdots, \mu^0_n)'$ includes $J_0$ blocks and there exists a unique vector $\nu^0 = (\nu^0_1, \cdots, \nu^0_{J_0})$ such that

$$
\mu^0_i = \sum_{j=1}^{J_0} \nu^0_j I(i \in B^0_j),
$$

(4)
where \( \{B_0^j, \ldots, B_0^N\} \) is a mutually exclusive block partition of \( \{1, \ldots, n\} \) generated from \( \mu^0 \). Based on the block partition, we can rewrite \( 1 = i_0 < \cdots < i_{J_0-1} \leq i_{J_0} = n \) and \( \{i_{j-1}, \ldots, i_j-1\} \) gives the \( j \)-th block set \( B_j \). We denote the jump set in the true model as \( J^0 \). Then \( J^0 = \{1, \ldots, i_{J_0-1}\} \) and \( J_0 = |J^0| + 1 \), where \( |J^0| \) is the cardinality of \( J^0 \). Let \( K^0 = K(\mu^0) = \{1 \leq j \leq J_0 : \nu_j^0 \neq 0\} \) be the set of nonzero blocks of \( \mu^0 \) and the number of nonzero blocks \( K_0 = |K^0| \). We now list the following notations that will be used throughout the paper, some of which are adopted from Rinaldo (2009).

- For the true model \( \mu^0 \) defined in (4), we introduce the following notations (I–IV):
  
  (I) \( b_j^0 = |B_j^0| \), the size of the block set \( B_j^0 \) for \( 1 \leq j \leq J_0 \);
  
  (II) \( b_{\min}^0 = \min_{1 \leq j \leq J_0} b_j^0 \), the smallest block size;
  
  (III) \( a_n = \min_{i \in J^0} |\mu_i^0 - \mu_i^{0-1}| \), the smallest jump;
  
  (IV) \( \rho_n = \min_{i \in K^0} |\nu_i^0| \), the smallest nonzero signal intensity.

- Corresponding notations are analogous to a LAD-FLSA estimate \( \hat{\mu}_n \) in (3) as follows:
  
  (V) \( \hat{J} = J(\hat{\mu}_n) \), \( \hat{J} = J(\hat{\mu}_n) \), \( \hat{B}_j = B_j(\hat{\mu}_n) \) with \( \hat{b}_j = |\hat{B}_j| \) and \( \hat{\nu}_j = \nu_j(\hat{\mu}_n) \) for \( 1 \leq j \leq \hat{J} \); \( \hat{B}_j \) are the estimated jump set, number of blocks, block partitions of \( \{1, \ldots, n\} \) with corresponding block size and the associated unique vector generated from \( \hat{\mu}_n \);
  
  (VI) \( \hat{K} = K(\hat{\mu}_n) = \{1 \leq j \leq \hat{J} : \hat{\nu}_j \neq 0\} \) is the set of estimated nonzero blocks and \( \hat{K} = |\hat{K}| \).

3 Estimation consistency of LAD-FLSA estimators

In this section, we study the estimation consistency of the LAD-FLSA \( \hat{\mu}_n \). We first consider the following conditions:

(A1) Error assumption: random errors \( \varepsilon_i \)’s in model (1) are independent and identically distributed with median 0, and have a density \( f \) that is continuous and positive in a neighborhood of 0.

(A2) Block number assumption: the true block number \( J_0 < M_1 \Lambda_n \) for a constant \( M_1 > 0 \), where \( \Lambda_n = \max\{16n/(\lambda_{2n}^2 - 2n^2 \lambda_{1n}^2), n/(\lambda_{2n} - n \lambda_{1n})\} + 1 \) with \( \lambda_{2n}^2 > 2n \lambda_{1n}^2 \).

In (A1), we only require that the random errors have a density that is continuous and positive in neighborhood of zero and have median zero. This is a much weaker condition than the Gaussian random error assumption required in Harchaoui and Lévy-Leduc (2010) and Rinaldo (2009). Indeed, (A1) allows all heavy-tail distributions of the errors, including the Cauchy distribution whose moments do not exist. Condition (A2) requires that the number of blocks in the underlying model can increase with \( n \) but at a slower rate than \( O(\Lambda_n) \). As a by-product, the tuning parameter for jumps, \( \lambda_{2n} \), grows much faster that the tuning parameter for signals intensities, \( \lambda_{1n} \). It is a reasonable assumption since the true model is block-wise, that is, the number of nonzero jumps can be much smaller than the number of the nonzero signals. For example, if the number of jumps is \( O((\log(n))^{1/2}) \), then we can let \( \lambda_{2n} = n^{1/2}(\log(n))^{-1/4} \) and \( \lambda_{1n} = n^{-1/2} \).
In order to study the asymptotic properties of the LAD-FLSA estimator \( \hat{\mu}_n \) in (5), we first investigate its LS-FLSA approximation,

\[
\hat{\mu}_n(\lambda_n) = \arg \min_{\mu} \left\{ \sum_{i=1}^{n} (z_i - (f(0))^{1/2} \mu_i)^2 + \lambda_1 n \sum_{i=1}^{n} |\mu_i| + \lambda_2 n \sum_{i=2}^{n} |\mu_i - \mu_{i-1}| \right\},
\]

where \( z_i = (f(0))^{1/2} \mu_i^0 + \eta_i \) with \( \eta_i = (4f(0))^{-1/2} \text{sgn}(\varepsilon_i) \) for \( 1 \leq i \leq n \) consist of pseudo-signal data. Thus, all estimates listed in (V–VI) can be analogous to the corresponding ones generated from \( \hat{\mu}_n \). We now provide some rate upper bounds for the number of blocks generated from \( \hat{\mu}_n \), \( \tilde{\mu}_n \) in (5) and \( \tilde{\mu}_n \) in (3), respectively.

**Lemma 1** Under (A1), we have (i) \( \tilde{J} \leq 16n/(\lambda_2^2 - 2n^2 \lambda_1^2) + 1 \), provided \( \lambda_2^2 > 2n^2 \lambda_1^2 \) and (ii) \( \tilde{J} \leq n/((\lambda_2 - 2n \lambda_1)n + 1) \), provided \( \lambda_2 > n^2 \lambda_1 \). In addition, suppose (A2) holds, we also have (iii) \( \tilde{J} + \tilde{J} + J_0 < (M_1 + 2) \Lambda_n \), where both \( M_1 \) and \( \Lambda_n \) are defined in (A2).

The proof of Lemma 1 is given in the Appendix. Lemma 1 gives upper bounds for the number of blocks associated with \( \hat{\mu}_n \) and \( \tilde{\mu}_n \). We can interpret bounds in (i) and (ii) as the maximal dimension of any linear space where \( \hat{\mu}_n \) and \( \tilde{\mu}_n \) may belong, respectively. Similarly, (iii) provides us an unified rate upper bound for the dimension of any linear space to which \( \hat{\mu}_n \), \( \tilde{\mu}_n \) and \( \mu^0 \) can belong. Lemma 1 is useful in obtaining the estimation consistencies of \( \hat{\mu}_n \) and \( \tilde{\mu}_n \). Furthermore, it is important to notice that those upper bounds in Lemma 1 are mainly affected by the rate of \( \lambda_2 \), which is reasonable since the number of jumps in an FLSA model is mainly determined by \( \lambda_2 \).

Denote \( \|\mu\|_n = \sum_{i=1}^{n} \mu_i^2 / n \) and \( \|\mu\|_2 = \sum_{i=1}^{n} \mu_i^2 \). Below we present the estimation properties of \( \hat{\mu}_n \) in (5).

**Lemma 2** Suppose (A1-A2) hold. Then there exists a constant \( 0 < c < 1 \), such that

\[
P \left( \|\hat{\mu}_n - \mu^0\|_n \geq \alpha_n \right) \leq \Lambda_n \exp \left\{ \Lambda_n \log n - \left( 1 - c \right)^2 \left( f(0)/2 \right) n \alpha_n^2 \right\},
\]

where \( \Lambda_n \) is defined in (A2) and \( \alpha_n = 1/(c\sqrt{f(0)}) \left[ \lambda_1 n + 2\lambda_2 n + \left( (M_1 + 1) \Lambda_n / n \right)^{1/2} \right] \). Furthermore, if we let \( \alpha_n = (2M_2 \Lambda_n \log n/n)^{1/2} \) and choose \( \lambda_1 n + 2\lambda_2 n = c\sqrt{f(0)} \alpha_n - \left( (M_1 + 1) \Lambda_n / n \right)^{1/2} \), then

\[
P \left( \|\hat{\mu}_n - \mu^0\|_n \geq \alpha_n \right) \leq \Lambda_n n^{(1-M_2 f(0)(1-c)^2)/2} \Lambda_n.
\]

The proof of Lemma 2 is given in the Appendix. Lemma 2 gives us the estimation consistency result for a pseudo LS-FLSA \( \tilde{\mu}_n \) (using pseudo data \( z_i \)'s and bounded noises \( \eta_i \)'s). It is worthwhile to point out that even though we only report the consistency result for a pseudo LS-FLSA estimator \( \tilde{\mu}_n \) with bounded noises \( \eta_i \)'s in Lemma 2, we can obtain a similar consistency result for the regular LS-FLSA estimator (2) under the assumption of Gaussian noises without much extra work. Thus, the estimation consistency properties of the LS signal approximator with the total variation penalty in Harchaoui and Lévy-Leduc (2010) can also be obtained from Lemma 2 by taking \( \lambda_2 = 0 \) and \( \Lambda_n = K_{\max} \).

The consistency result of \( \hat{\mu}_n \) in Lemma 2 plays an important role in deriving the corresponding estimation consistency result of \( \tilde{\mu}_n \) in the following Theorem 1.
Theorem 1  Suppose (A1) and (A2) hold. Then there exists a constant $0 < c < 1$ such that
\[
P\left( \|\hat{\mu}_n - \mu^0\|_2 \geq \gamma_n \right) \leq \Lambda_n \exp\{\Lambda_n \log n - (1 - c)^2(f(0)/8)n\gamma_n^2 \} + (8/f(0))(\Lambda_n/(n\gamma_n^2))^{1/2},
\]
where $\Lambda_n$ is defined in (A2) and $\gamma_n = 2/(c\sqrt{f(0)})[\lambda_{1n} + 2\lambda_{2n} + (((M_1 + 1)/\Lambda_n)n)^{1/2}]$.

Furthermore, if we let $\gamma_n = (8M_3\Lambda_n(\log n)/n)^{1/2}$ for a constant $M_3 > 1/(f(0)(1 - c)^2)$ and choose $\lambda_{1n}$ and $\lambda_{2n}$ such that $\lambda_{1n} + 2\lambda_{2n} = (c\sqrt{f(0)/2})\gamma_n - (((M_1 + 1)/\Lambda_n)n)^{1/2}$, then
\[
P\left( \|\hat{\mu}_n - \mu^0\|_2 \geq \gamma_n \right) \leq \Lambda_n n^{-(M_3f(0)(1-c)^2-1)} + O\left(1/\sqrt{\log n}\right).
\]

The proof of Theorem 1 is given in the Appendix. Theorem 1 implies that the LAD-FLSA $\hat{\mu}_n$ can be consistent for estimating $\mu^0$ at the rate of $O(\Lambda_n(\log n)/n)^{1/2})$. Furthermore, if the number of blocks in true signals is bounded, the rate of convergence can be stated more explicitly as in the following Corollary 1.

Corollary 1  Suppose (A1) holds and there exists $J_{\max} > 0$ such that $J_0 \leq J_{\max}$. Then
\[
P\left( \max(J, \bar{J}) < J_{\max}\right) \cap \{\|\hat{\mu}_n - \mu^0\|_2 \geq \theta_n \} \leq \Lambda_n n^{-{2MJ_{\max}}} + O\left(1/\sqrt{\log n}\right)
\]
for $\theta_n = (8MJ_{\max}(\log n)/n)^{1/2}$ and $\lambda_{1n} + 2\lambda_{2n} = (c_1M J_{\max}(\log n)/n)^{1/2} - (J_{\max}/n)^{1/2}$. Here $M > 1/((1 - c)^2(f(0)))$ is a constant, $c_1M = (2Mc_2(f(0)))^{1/2}$ and $c_2M = f(0)M(1 - c)^2 - 1$.

Corollary 1 says that the $\hat{\mu}_n$ is consistent for estimating $\mu^0$ at the rate $O((J_{\max}(\log n)/n)^{1/2})$ if the numbers of both true and estimated jumps are bounded above. This convergence rate can be compared to $n^{-1/2}$, which is argued by Yao and Yu (1989) to be optimal for LS estimators of the levels of a step function. Notice that if $\lim_{n \to \infty} P(\bar{J} = J^0) = 1$, then $\sum_{i=1}^n (\hat{\mu}_i - \mu_i^0)^2 = \sum_{i=1}^{b_0^0} (\hat{\nu}_i - \nu_i^0)^2 \geq b_{\min}^0 \sum_{j=1}^{b_0^0} (\hat{\nu}_j - \nu_j^0)^2$ for large $n$ almost surely. Thus Corollary 1 implies that, for large $n$
\[
P\left( \|\hat{\nu}_n - \nu^0\|_2 \geq (8MJ_{\max}(\log n)/b_{\min}^0)^{1/2} \right) \leq J_{\max} n^{-{8M(1-c)^2-1}} \]
where the convergence rate is affected by $b_{\min}^0$. Therefore, $\hat{\nu}_n$ can converge to $\nu^0$ in $\ell_2$ norm at rate $O((J_{\max}(\log n)/b_{\min}^0)^{1/2})$. In other words, a block estimator $\hat{\nu}_n$ can converge faster to the true model $\nu_0$ with larger block size.

4  Block sign consistency of LAD-FLSA

In this section, we study the sign consistency of the LAD-FLSA. The sign consistency has been studied by Zhao and Yu (2006) and Gao and Huang (2010b) for both the LS-Lasso and LAD-Lasso in high-dimensional linear regression settings. It is a stronger result than variable selection consistency since it not only requires that variables to be selected correctly, but also their signs are estimated correctly with high probability.

In light of the block structure in the hidden signals, we consider the selection consistency and sign consistency for jumps and blocks separately.
Definition 1 \( \hat{\mu}_n \) is jump selection consistent if
\[
\lim_{n \to \infty} \mathbb{P} \left( \{ \hat{J} = J_0 \} \cap \{ \cap_{1 \leq j \leq J_0} \{ \hat{B}_j = B^0_j \} \} \right) = 1.
\]

Definition 2 \( \hat{\mu}_n \) is jump sign consistent if
\[
\lim_{n \to \infty} \mathbb{P} \left( \{ \hat{J} = J^0 \} \cap \{ \text{sgn}(\hat{\mu}_i - \hat{\mu}_{i-1}) = \text{sgn}(\mu_i^0 - \mu_{i-1}^0), \forall i \in J^0 \} \right) = 1.
\]

Definition 3 \( \hat{\mu}_n \) is block selection consistent if
\[
\lim_{n \to \infty} \mathbb{P} \left( \{ \hat{J} = J^0 \} \cap \{ \hat{K} = K^0 \} \right) = 1.
\]

Definition 4 \( \hat{\mu}_n \) is block sign consistent if
\[
\lim_{n \to \infty} \mathbb{P} \left( \{ \hat{J} = J^0 \} \cap \{ \hat{K} = K^0 \} \cap \{ \text{sgn}(\hat{\nu}_j) = \text{sgn}(\nu_j^0), \forall j \in J_0 \} \right) = 1.
\]

4.1 Jump selection consistency

For \( \lambda_{1n} = 0 \), a LAD-FLSA becomes a LAD signal approximator using only the total variation penalty (LAD-FSA), defined as
\[
\hat{\mu}^\text{FL}_n(\lambda_{2n}) = \hat{\mu}^\text{FL}_n(0, \lambda_{2n}) = \arg\min \left\{ \sum_{i=1}^{n} |y_i - \mu_i| + \lambda_{2n} \sum_{i=2}^{n} |\mu_i - \mu_{i-1}| \right\}.
\]

Suppose \( \hat{\mu}^\text{FL}_n(\lambda_{2n}) \) can do the block partition correctly. Then we expect to sort out those nonzero blocks by increasing \( \lambda_{1n} \) slowly from 0. So we first investigate the jump selection consistency of \( \hat{\mu}^\text{FL}_n(\lambda_{2n}) \). Below we list some conditions on the smallest value of true jumps and smallest size of the true blocks in model (1) and (4) for the jump sign consistency. Recall that \( b^{0}_{\text{min}} \) and \( a_n \) are defined in II and III in Section 2.

(B1) (a) \( \lambda_{2n} \to \infty \); (b) there exists a \( \delta > 0 \), such that \( \lambda_{2n} (\log(n - J_0))^{-1/2} > (1 + \delta)/2 \).

(B2) (a) \( (b^{0}_{\text{min}})^{1/2} a_n \to \infty \); (b) there exists \( \delta > 0 \), such that \( (b^{0}_{\text{min}}/\log(J_0))^{1/2} a_n > 3(1 + \delta)/(\sqrt{2} f(0)) \) for sufficiently large \( n \).

(B3) \( \lambda_{2n} < (f(0)/3) b^{0}_{\text{min}} a_n \) for sufficiently large \( n \).
Here (B1) and (B3) indicate that $\lambda_{2n}$ increases with $n$ faster than $O((\log(n - J_0))^{1/2})$ but slower than $O(b_{\min}^{0}a_n)$. (B2-a) requires that either the smallest jump or the smallest size of all blocks in the true model should be large enough so that $\{1, \ldots, n\}$ can be partitioned into different blocks correctly. (B2-b) strengthens (B1-a) by providing a lower bound. Conditions (B1-B3) provide us some helpful information in finding an optimal tuning parameter in model (7). When the above conditions are satisfied, the LAD-FSA estimator $\hat{\mu}_F^n(\lambda_{2n})$ can group all signals into different blocks correctly with a large probability.

**Theorem 2** Consider the signal approximation model (1) with the true model (4). A LAD-FSA estimator $\hat{\mu}_F^n(\lambda_{2n})$ is jump sign consistent under (A1) and (B1-B3).

The proof of Theorem 2 is postponed to the Appendix. Theorem 2 tells us that we can apply a LAD-FSA approach to recover not only the true jumps, but also their signs correctly with high probability if the true hidden signal vector is blocky and the tuning parameter $\lambda_{2n}$ is chosen appropriately.

### 4.2 Block selection consistency

We have seen that a LAD-FSA solution can be jump selection consistent to the blocky hidden signal vector under some conditions. In many cases, the true signal vector includes some zero blocks, which cannot be separated from nonzero ones using the LAD-FSA approach since the total variation penalty only shrinks adjacent differences but not signals themselves. The additional lasso penalty of FLSA can force the estimates of some block intensities to be exactly zero. We are interested in finding a LAD-FLSA solution to not only recover the true jumps, but also find the zero blocks and keep only the nonzero ones with a large probability. Eventually, we need to study how to choose tuning parameters $\lambda_{1n}$ and $\lambda_{2n}$ appropriately, such that the LAD-FLSA is block selection consistent.

When the true block model in (4) is also sparse, we need the following additional conditions to separate nonzero blocks from zero ones.

**Condition (C1):**

(a) $\lambda_{1n}(b_{\min}^{0})^{1/2} \to \infty$ when $n \to \infty$; (b) there exists $\delta > 0$, such that $\lambda_{1n}(b_{\min}^{0}/\log(J_0 - K_0))^{1/2} > 4\sqrt{2}(1 + \delta)$.

**Condition (C2):**

$\lambda_{2n}/b_{\min}^{0} < \lambda_{1n}/8$ for sufficiently large $n$.

**Condition (C3):**

(a) $\rho_n(b_{\min}^{0})^{1/2} \to \infty$ when $n \to \infty$; (b) there exists $\delta > 0$ such that $\rho_n(b_{\min}^{0}/\log(K_0))^{1/2} > 2\sqrt{2}(1 + \delta)/f(0)$.

**Condition (C4):**

$\lambda_{2n}/b_{\min}^{0} < f(0)\rho_n/3$ for sufficiently large $n$.

**Condition (C5):**

$\lambda_{1n} < f(0)\rho_n/2$ for sufficiently large $n$.

Here Condition (C1) and (C2) indicate that either $\lambda_{1n}$ or the smallest block size $b_{\min}^{0}$ should grow with $n$ with a lower bound provided in (C1-b) since $\lambda_{2n}$ grows with $n$ from (B1). Especially, if $\lambda_{1n}$ is relatively small as seen in (C5), $b_{\min}^{0}$ must be large enough. (C4) and (C5) provide us a lower bound for the smallest nonzero signal $\rho_n$ when $n$ is large. Above interpretations are consistent with (C3-a), which requires either the block size or the true nonzero signal intensities
should be large enough such that the nonzero blocks can be separated from zero ones. In other words, if \( \rho_n \) is relatively smaller, it becomes harder to separate nonzero ones from zero ones. However, it is not impossible for us to distinguish those nonzero blocks if we have larger enough block size since more observations can be used to estimate \( \rho_j^0 \) within \( j \)th block. (C3-b) provides us an upper bound of the number of nonzero blocks. It is worthwhile to point out that even though these conditions seem to be complicated, some of them can be redundant. For instance, (C2) and (C5) can be used to derive a smaller upper bound than the one in (C4). Thus, if both (C2) and (C5) are satisfied, (C4) can be redundant. One can see that (C3-a) can be also redundant if (C5) and (C1-a) hold.

**Theorem 3** Under (A1), (B1-B3) and (C1-C5), a LAD-FLSA solution is block sign consistent.

Theorem 3 tells us that the LAD-FLSA can first recover the block patterns of hidden signals by detecting all the true jumps, and then rule out those nonzero blocks. Furthermore, with a very large probability, those nonzero blocks are identified correctly to have either positive or negative signals. Thus, the LAD-FLSA is justified to be a promising approach for signal processing when the true hidden signal vector is both blocky and sparse and the observed data are contaminated by outliers. The proof of Theorem 3 is provided in the Appendix.

**Remark 1:** The block assumption of the true model in (4) is crucial in our study. If the model is grouped, but not blocky, fused lasso might be misleading since the fusion term is used to generate the block-wise solution. Some other techniques such as group lasso (Yuan and Lin, 2006) or smooth lasso (Hebiri, 2008) can be more useful to generate the corresponding group sparsity structure.

**Remark 2:** The relaxation of Gaussian or sub-Gaussian random error assumption is important since it is very common to see some contaminated data in signal processing, especially when repeated measurements are not available. Some normalization methods such as Loess have been used in preprocessing the real data in order to improve the robustness of LS-FLSA. However, those techniques may over-smooth the data and then generate some false negatives.

### 4.3 Additional remarks on asymptotic properties

We will provide two additional comments on the asymptotic results obtained in Section 3 and 4.

**Remark 3:** An LAD-FLSA may not reach the estimation consistency and sign consistency simultaneously.

The rate estimation consistency in Theorem 3 holds for \( \lambda_{1n} + 2\lambda_{2n} = O(\log(n)/n)^{1/2} \). However, from (B1-b) and (C2), we know one of the sufficient conditions for the sign consistency in Theorem 3 requiring \( \lambda_{kn} > O(\log(n))^{1/2} \) for \( k = 1, 2 \). So an LAD-FLSA may not be able to reach both the estimation consistency and sign consistency simultaneously. However, this claim is not theoretically justified since all conditions assumed in both Theorem 2 and 3 are sufficient.

**Remark 4:** The weak irrepresentable condition is not necessary for the jump point detection consistency in Theorem 2.

To understand Remark 4, we will transform the signal approximation model in (7) into a Lasso representation. Consider a linear regression model

\[
y_i = \sum_{j=1}^{p} x_{ij} \beta_j + \varepsilon_i, \quad 1 \leq i \leq n,
\]  

(8)
where \((y_i, x_{i1}, \cdots, x_{ip})\) and \(\beta = (\beta_1, \cdots, \beta_p)'\) represent the observed data and coefficients vector. A Lasso solution (Tibshirani, 1996) of \(\beta\) is

\[
\hat{\beta}(\lambda) = \arg \min \left\{ \frac{1}{2} \sum_{i=1}^{n} (y_i - \sum_{j=1}^{p} x_{ij} \beta_j)^2 + \lambda \sum_{j=1}^{p} |\beta_j| \right\}.
\]

If we further divide the coefficients vector \(\beta = (\beta_1', \beta_2')'\), where \(\beta_1\) include those nonzero coefficients and \(\beta_2\) includes zeros only, and correspondingly, we can write \(X = (X_1, X_2)\) and \(s_1 = \text{sgn}(\beta_1)\) consist of sign mappings of non-zero coefficients in the true model, then the weak irrepresentable condition of the designed matrix \(X\) means

\[
|X_2'X_1(X_1'X_1)^{-1}s_1| < 1,
\]

where \(1\) is a vector with element being 1. Naturally, we can write the LAD-FLSA in (7) into a Lasso solution of \(\hat{\nu} = (\nu_1, \cdots, \nu_n)'\),

\[
\hat{\nu}^s_n(\lambda_{2n}) = \arg \min \left\{ \|y - Z\nu\|_2 + \lambda_{2n} \sum_{i=2}^{n} |\nu_i| \right\},
\]

where \(\nu_1 = \mu_1, \nu_i = \mu_i - \mu_{i-1}\) for \(2 \leq i \leq n\) and \(Z\) is the low triangular design matrix with nonzero items being 1. Zhao and Yu (2006) proved that the weak irrepresentable condition is a necessary condition for a Lasso solution in (8) to be sign consistent under two regularity conditions. We list the result in the following Lemma 3.

**Lemma 3** (Zhao and Yu, 2006) Suppose two regularity conditions are satisfied for the designed matrix \(X\): (1) there exists a positive definite matrix \(C\) such that the covariance matrix \(X'X/n \to C\) as \(n \to \infty\), and (2) \(\max_{1 \leq i \leq n} x_i'x_i/n \to 0\) as \(n \to \infty\). Then Lasso is general sign consistent, \(\lim_{n \to \infty} P(3\lambda \geq 0, \text{sgn}(\hat{\beta}(\lambda)) = \text{sgn}(\beta_0)) = 1\), only if there exists \(N\) so that \(X\) satisfies the weak irrepresentable condition holds for \(n > N\). Here \(\beta_0\) is the true coefficient vector.

Unfortunately, it is easy for us to verify that the design matrix \(Z\) in (10) does not satisfy the weak irrepresentable condition. For example, if we consider a signal approximation data with only five observations where \(\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5\), then the first row vector of \(Z'Z_1(Z_1'Z_1)^{-1}\) is \(0, 0, 1)'\). Thus (9) is violated. However, there is no contradiction between the sign consistency result in Theorem 2 and Lemma 3 since both two regularity conditions in Lemma 3 are violated for design matrix \(Z\). Suppose \(\rho_1 \leq \cdots \rho_n\) are eigenvalues of \(Z'Z/n\). we know that (a) \(\rho_1 < 1/(3n) \to 0\) and \(\rho_n > 4n^{1/2} \to \infty\) when \(n \to \infty\), and in addition, (b) \(\max_{1 \leq i \leq n} z_i'z_i/n = 1\).

### 5 Degrees of freedom of LAD-FLSA

It is crucial to seek appropriate \(\lambda_{1n}\) and \(\lambda_{2n}\) in (3). Large \(\lambda_{1n}\) will generate all zero coefficients, while large \(\lambda_{2n}\) will generate all zero jumps. Conditions on \(\lambda_{1n}\) and \(\lambda_{2n}\) in Section 3 and 4 provide us some guidance in choosing the rates of two tuning parameters to obtain a well-behaved LAD-FLSA estimate. This section helps us to choose two optimal tuning parameters from the model selection point of view.
For given $\lambda_1$ and $\lambda_2$, a LAD-FLSA approach is a modeling procedure including both model selection and model fitting. The complexity of a modeling procedure is defined as the generalized degrees of freedom (df) and measured by the sum of the sensitivity of the predicted values. See Ye (1998) and Gao and Fang (2011) for the discussion on the df for a modeling procedure under both the $\ell_2$ and $\ell_1$ loss functions, respectively. For $1 \leq i \leq n$, let $\hat{\mu}_i(y; \lambda_1, \lambda_2)$ be a LAD-FLSA fitted value of $y_i$ for any given $\lambda_1$ and $\lambda_2$. The degrees of freedom of a LAD-FLSA approach,

$$
df(\lambda_1, \lambda_2) = \sum_{i=1}^{n} \partial E[\hat{\mu}_i(y; \lambda_1, \lambda_2)]/\partial y_i. \tag{11}
$$

In Theorem 4, we provide an unbiased estimator of $\df(\lambda_1, \lambda_2)$ in (11) for a LAD-FLSA modeling procedure.

**Theorem 4** Consider a LAD-FLSA modeling procedure defined for (1), (3) and (4). For any fixed positive $\lambda_1$ and $\lambda_2$, we have

$$
E[|\hat{K}(\lambda_1, \lambda_2)|] = \df(\lambda_1, \lambda_2). \tag{12}
$$

Theorem 4 indicates that the number of nonzero blocks, $|\hat{K}(\lambda_1, \lambda_2)|$, is an unbiased estimator of the degrees of freedom of a LAD-FLSA modeling procedure with any given $\lambda_1$ and $\lambda_2$. We provide both the numerical demonstration and theoretical proof are provided in Section 6.3 and the Appendix, respectively. In fact, such an unbiased estimator in (12) can be also observed from Theorem 2 of Li and Zhu (2008). For example, if $\lambda_1 = 0$, for any $\lambda_2 > 0$, the LAD-FLSA reduces to a LAD-LASSO solution of $w_i = \mu_i - \mu_{i-1}$ for $2 \leq i \leq n$. Then $\sum_{i=1}^{n} \partial \hat{y}_i(0, \lambda_2)/\partial y_i = |\hat{J}(0, \lambda_2)|$. Suppose $\lambda_2 > 0$ is fixed, the block partition is decided. Then for $\lambda_1 > 0$, the LAD-FLSA becomes a LASSO model of $\nu$ ($\nu$ is the block intensity vector). Therefore, $\sum_{i=1}^{n} \partial \hat{y}_i(\lambda_1, \lambda_2)/\partial y_i = |\hat{K}(\lambda, \lambda_2)|$.

Results in Theorem 4 can be used to choose two optimal tuning parameters from the model selection point of view. Let $y_i^0$’s denote new observations generated from the same mechanism generating $y_i$’s. The prediction error of $\hat{\mu}(y; \lambda_1, \lambda_2)$ is defined as

$$
E_0\{\sum_{i=1}^{n} |\hat{\mu}_i(\lambda_1, \lambda_2) - y_i^0|\}, \tag{13}
$$

where $E_0$ is taken over $y_i^0$’s. From Theorem 4, we can estimate the prediction error (13) by

$$
\sum_{i=1}^{n} |y_i - \hat{\mu}_i(\lambda_1, \lambda_2)| + |\hat{K}(\lambda_1, \lambda_2)|. \tag{14}
$$

Thus some existing model selection criteria can be modified to choose an optimal combination of tuning parameters. For instance, we can extend AICR (Ronchetti, 1985), BIC (Schwarz, 1978) and GCV (Wahba, 1990) to the LAD-FLSA as follows,

$$
\text{AICR} : \quad \sum_{i=1}^{n} |y_i - \hat{\mu}_i(\lambda_1, \lambda_2)| + |\hat{K}(\lambda_1, \lambda_2)|,
$$

$$
\text{BIC} : \quad \sum_{i=1}^{n} |y_i - \hat{\mu}_i(\lambda_1, \lambda_2)| + |\hat{K}(\lambda_1, \lambda_2)| \log(n)/2,
$$

$$
\text{GCV} : \quad \sum_{i=1}^{n} |y_i - \hat{\mu}_i(\lambda_1, \lambda_2)|/[1 - |\hat{K}(\lambda_1, \lambda_2)|/n]. \tag{14}
$$
6 Numerical studies

In this section, we first use some simulation studies and real data analysis to demonstrate the performances of the LAD-FLSA approach in recovering the true hidden signals. Then we verify Theorem 4 numerically using a sample copy number data.

6.1 Recovery of hidden signals

We illustrate the performance of the LAD-FLSA by modifying the block example studied in both Donoho and Johnstone (1995) and Harchaoui and Lévy-Leduc (2010), where the signal vector is only blocky but not sparse. We choose \( t = (1.5, -3, 4.3, -3.1, -2)' \) and \( h = (0.1, 0.2, .65, .76, .9)' \) and round \( \sum_j h_j(1 + \text{sgn}(i/n - t_j))/2 \) to the nearest integers to get \( \mu_i^0 \) for \( 1 \leq i \leq n \). Then the generated true hidden signal vector,

\[
\mu^0 = (0', 2'_p, 2'_q, 3'_p, 0'_p, 2'_{n-q})'
\]

is blocky and sparse with four nonzero blocks and two zero ones. Here \( q = p_1 + \cdots + p_5 \). The observed data are generated from model (1) by simulating \( \varepsilon_i \)'s from 1) normal distribution with mean 0 and standard deviation \( \sigma \), 2) double exponential distribution with center 0 and standard deviation \( \sigma \), and 3) standard cauchy distribution with a multiplier \( 0.1 \sigma \). Similar to Harchaoui and Lévy-Leduc (2010), we consider weak, mild and strong noises by setting \( \sigma = 0.1, 0.5 \) and 1 in all three types of distributions. In Figure 4, we plot a sample data set generated from 2), where the observed data, the true hidden signals and the LAD-FLSA estimates are plotted using gray, black and red colors, respectively.

The data is standardized as \( y_i/s(y) \) and analyzed using the LAD-FLSA approach in (3), where \( s(y) \) is the standard deviation of \( (y_1, \cdots, y_n)' \). We choose “optimal” \( \lambda_1 \) and \( \lambda_2 \) by minimizing BIC in (14) for \( 0 < \lambda_1 < 0.5 \) with increments of 0.01 and \( (n/\log(n))^{1/2} < \lambda_2 < n^{1/2} \) with increments of 0.1, respectively. To demonstrate the robust properties of the LAD-FLSA approach, we also report the simulation results from the LS-FLSA approach in (2). For each model, we illustrate the variable selection effect using CFR+6, the ratio of either recovering \( \hat{\mu}^0 \) correctly or over-fitting the model by including six additional noises over 1000 replicates. We choose “six” here since we have six blocks in the true model. We also report JUMP, the average number (with standard deviation) of jumps over 1000 replicates. A jump is counted only if the adjacent differences is at least 0.1. We demonstrate the estimation effects by computing the least absolute relative error (LARE) as follows:

\[
\text{LARE}(\hat{\mu}_n, \mu^0) = \frac{\sum_{i=1}^n |\hat{\mu}_i - \mu_i^0|}{\sum_{i=1}^n |\mu_i|}. \tag{15}
\]

The simulation results for sample size \( n = 1000 \) and 5000 are reported in Table 1, where we can see that the LAD-FLSA approach has much better performance than the LS-FLSA for both strong and mild signal noises. When the signal noises are weak, the LAD-FLSA still has some advantages over the LS-FLSA especially when the data is contaminated by Cauchy distributed noises. For example, for Cauchy error and \( \sigma = 0.1 \), the LAD-FLSA recovers the true signal vector exactly at a ratio of 87% for \( n = 1000 \) and 92% for \( n = 5000 \), while the LS-FLSA only recovers the true model exactly 49% and 78% of the time.
6.2 BAC array

In Section 1 we introduced a sample BAC CGH data, where the observation of each entry for cell line GM 13330 is the log 2 fluorescence ratios from all 23 chromosomes resulted from the BAC experiment sorted in the order of the clones locations on the genome. The purpose of the study is to detect the locations where there are some significant deletions or amplifications. As a demonstration of the effect of the LAD-FLSA applied to copy number analysis, we only analyze the data from chromosome 1–4 with 129, 67, 83 and 167 markers, respectively. Since the log 2 ratios at many markers are observed to be around 0 and the data may also have some spatial dependence properties, it is reasonable to assume the true hidden signals to be both sparse and blocky. We analyze each chromosome independently by using both the LAD-FLSA in (3) and LS-FLSA in (5). Tuning parameters are chosen the same as in Section 6.1. The final estimates from both methods on all 4 chromosomes are plotted together in Figure 1. The LAD-FLSA estimates (top panel) provides four blocks with one amplification region in chromosome 1 and one deletion region in chromosome 4. Besides the two variation regions detected by the LAD-FLSA, the LS-FLSA estimates (bottom panel) also show an amplification at a single point in chromosome 2, which is not confirmed by spectral karyotyping in Snijders et al. (2001).

6.3 Effect of unbiased estimator of the degrees of freedom

We now conduct some simulations based upon the sample BAC array studied in Section 6.2 to examine Theorem 4 numerically. To illustrate the effect of the unbiased estimator of the degrees of freedom, we only take chromosome 1 with 129 locations as an example. One can see the sample data from Figure 1.

We generate 500 Monte Carlo simulations based on the same hypothetical model

\[ y_i^0 = y_i + \varepsilon_i^0, \quad i = 1, \ldots, 129, \]

where \( y_i \)'s are observations at 129 locations and \( \varepsilon_i^0 \)'s are independent normal with center 0 and standard deviation \( 0.1 \sigma^* \), where \( \sigma^* \) is the standard deviation of \( y \). For each combined \( (\lambda_1, \lambda_2) \) with \( 0 < \lambda_1, \lambda_2 \leq 1 \), we record \( \hat{\text{df}}(\lambda_1, \lambda_2) \) from \( \hat{K}(\lambda_1, \lambda_2) \), and compute the true df(\( \lambda_1, \lambda_2 \)) defined in (11) using the Monte Carlo simulation from Algorithm 1 in Ye (1998). In Figure 2, we plot \( \hat{\text{df}}(\lambda_1, \lambda_2) \) of the LAD-FLSA estimate for every combination of \( 0 < \lambda_1, \lambda_2 \leq 1 \), with the increment of 0.05, respectively. The averages of df(\( \lambda_1, \lambda_2 \)) and \( \hat{K}(\lambda_1, \lambda_2) \) over 500 repetitions are reported in Figure 3. Those simulation results show that the number of estimated nonzero blocks \( \hat{K}(\lambda_1, \lambda_2) \) is a promising estimate to the df(\( \lambda_1, \lambda_2 \)) numerically, especially when the number of estimated nonzero blocks is not deviated from the true one seriously.

7 Concluding remarks

In this paper, we study the asymptotic properties of the LAD signal-approximation approach using the fused lasso penalty. By assuming the true model to be both blocky and sparse, we investigate both the estimation consistency and sign consistency of the LAD-FLSA estimator. In terms of estimation consistency, the consistency rate is optimal up to a logarithmic factor if the
dimension of any linear space where the true model and its estimates belong is bounded from above. In terms of sign consistency, we justify that a LAD-FLSA approach can not only recover the true block pattern but also distinguish those nonzero blocks from the zero ones correctly with high probability under reasonable conditions. In fact, those jump selection and block selection consistency results can be made stronger by matching the corresponding signs correctly with a large probability. Thus, by choosing two tuning parameters \( \lambda_1 \) and \( \lambda_2 \) properly, we can reach a well-behaved LAD-FLSA estimate to recover the true hidden signal vector under some random noises. The consistency results in this paper extend the theoretical properties of the LS-FSA in Harchaoui and Lévy-Leduc (2010) and the LS-FLSA in Rinaldo (2009) to the LAD signal approximation, which amplify the study of signal approximation using linear regression when the random error does not follow a Gaussian distribution. Furthermore, we demonstrate that the number of estimated nonzero blocks is an unbiased estimator of the degrees of freedom of the LAD-FLSA. Thus, the existing model selection criteria can be extended to the LAD-FLSA for choosing the tuning parameters.

As in many recent studies, our results are proved for penalty parameters that satisfy the conditions as stated in the theorems. It is not clear whether the penalty parameters selected using data-driven procedures satisfy those conditions. However, our numerical study shows a satisfactory finite-sample performance of the LAD-FLSA. Particularly, we note that the tuning parameters selected based on the BIC seem sufficient for our simulated data. This is an important and challenging problem that requires further investigation, but is beyond the scope of the current paper. Also, a basic assumption required in our results is that the random error terms \( \varepsilon_i \) in (1) are independent. Since the observations \( y_1, \ldots, y_n \) are in a natural order in this model, for example, copy number variation data based on genetic markers are ordered according to their chromosomal locations, it would be interesting to study the behavior of the LAD-FLSA allowing for certain dependence structure in the error terms.

**Appendix**

**Proof of Lemma** Let \( \tilde{w}_i = \tilde{w}_i(\lambda_{1n}, \lambda_{2n}) = \tilde{\mu}_i - \tilde{\mu}_{i-1} \) and \( \hat{w}_i = \hat{w}_i(\lambda_{1n}, \lambda_{2n}) = \hat{\mu}_i - \hat{\mu}_{i-1} \) be the LS-FSA and LAD-FLSA estimates of \( \text{ith jump in (5)} \) and (3). Using the Karush–Kuhn–Tucker (KKT) conditions (5), we get

\[
-2\sqrt{f(0)}(z_i - \sqrt{f(0)} \sum_{j=1}^{i} \tilde{w}_j) + \lambda_{1n} \sum_{k=i}^{n} \text{sgn}(\sum_{j=1}^{k} \tilde{w}_j) = -\lambda_{2n} \text{sgn}(\tilde{w}_i) \quad \text{if} \quad \tilde{w}_i \neq 0.
\]

Then

\[
\lambda_{2n}^{2} \tilde{J} \leq 8f(0) \sum_{i=1}^{n}(z_i - \sqrt{f(0)} \sum_{j=1}^{i} \tilde{w}_j)^2 + 2\lambda_{1n}^{2} n^2 \tilde{J}.
\]

Thus

\[
|\tilde{J}| \leq 8f(0)(\lambda_{2n}^{2} - 2n^{2}\lambda_{1n}^{2})^{-1} \sum_{i=1}^{n} z_i^2 \leq 16f(0)n/(\lambda_{2n}^{2} - 2n^{2}\lambda_{1n}^{2}).
\]
Using the KKT equations of (3), we have

\[-\text{sgn}(y_i - \sum_{j=1}^{i} \hat{w}_j) + \lambda_{1n} \sum_{k=i}^{n} \text{sgn}(\sum_{j=1}^{k} \hat{w}_j) = -\lambda_{2n} \text{sgn}(\hat{w}_i) \quad \text{if } \hat{w}_i \neq 0.\]

Then

\[|\hat{f}| \leq n/(\lambda_{2n} - \lambda_{1n} n). \quad (17)\]

Finally, combining with (16), (17) and (A2), (iii) holds, which completes the proof of Lemma 1. □

**Proof of Lemma 2**

From the definition of \(\bar{\mu}_n\) in (5), we have

\[
\sum_{i=1}^{n} (z_i - \sqrt{f(0)} \mu_i)^2 \leq \sum_{i=1}^{n} (z_i - \sqrt{f(0)} \mu_i^0)^2 + \lambda_{1n} \sum_{i=1}^{n} ||\bar{\mu}_i - \mu_i|| + \lambda_{2n} \sum_{i=2}^{n} ||\mu_i^0 - \mu_{i-1}^0|| - ||\bar{\mu}_i - \bar{\mu}_{i-1}||. \quad (18)
\]

From the triangle inequality, (18) becomes

\[
f(0) \sum_{i=1}^{n} (\bar{\mu}_i - \mu_i^0)^2 \leq 2\sqrt{f(0)} \sum_{i=1}^{n} \eta_i (\bar{\mu}_i - \mu_i^0) + \lambda_{1n} \sum_{i=1}^{n} ||\bar{\mu}_i - \mu_i^0|| + \lambda_{2n} \sum_{i=2}^{n} ||\mu_i^0 - \mu_{i-1}^0|| - ||\bar{\mu}_i - \bar{\mu}_{i-1}||. \quad (19)
\]

The rest of the proof is similar to the proof of Proposition 2 in Harchaoui and Lévy-Leduc (2010). For \(\mu \in \mathcal{R}^n\), we define

\[G(\mu) = 2\sqrt{f(0)} \sum_{i=1}^{n} \eta_i (\mu_i - \mu_i^0)/||\mu - \mu^0||_2.\]

Thus (19) becomes

\[f(0) \sum_{i=1}^{n} (\bar{\mu}_i - \mu_i^0)^2 \leq (\lambda_{1n} + 2\lambda_{2n}) \sqrt{n} ||\bar{\mu}_n - \mu^0||_2 + G(\bar{\mu}_n) ||\bar{\mu}_n - \mu^0||_2.\]

Then,

\[\sqrt{f(0)} ||\bar{\mu}_n - \mu^0||_2 \leq (\lambda_{1n} + 2\lambda_{2n}) \sqrt{n} + G(\bar{\mu}_n). \quad (20)\]

Let \(\{S_K\}\) be a collection of any \(K\)-dimensional linear space to which \(\bar{\mu}_n\) may belong. From Lemma 1, \(1 \leq K \leq \Lambda_n\). From (20), for any \(\delta_n > 0\),

\[P(||\bar{\mu}_n - \mu^0||_2 \geq \delta_n) \leq P(G(\bar{\mu}_n) \geq \sqrt{f(0)} \delta_n - (\lambda_{1n} + 2\lambda_{2n}) \sqrt{n}) \leq \sum_{k=1}^{\Lambda_n} \eta^k P\left(\sup_{\mu \in S_K} G(\mu) \geq \sqrt{f(0)} \delta_n - (\lambda_{1n} + 2\lambda_{2n}) \sqrt{n}\right). \quad (21)\]

Notice that \(E(G(\mu)) = 0\) and \(\text{Var}(G(\mu)) = 1\). As a consequence of Cirel’son, Ibragimov and Sudakov’s (1976) inequality,

\[P\left( \text{sup}_{\mu \in S_K} G(\mu) \geq E\left[ \text{sup}_{\mu \in S_K} G(\mu) \right] + z \right) \leq \exp\{-z^2/2\} \text{ for some constant } z > 0. \quad (22)\]
Consider the collection \( \{ S_K \} \). Let \( \Omega \) be the \( D \)-dimensional space to which \( \mu - \mu^0 \) belongs and \( \psi_1, \cdots, \psi_D \) be its orthogonal basis.

\[
\sup_{\mu \in S_K} G(\mu) \leq \sup_{\omega \in \Omega} \frac{2\sqrt{f(0)} \sum_{i=1}^n \eta_i \omega_i}{\sqrt{n} ||\omega||_n}
\]

Combining (21) and (25), we have
\[
= \sup_{\omega \in \mathbb{R}^D} \frac{2\sqrt{f(0)} \sum_{i=1}^n \eta_i (\sum_{j=1}^D a_j \psi_j(i))}{\sqrt{n} \sum_{j=1}^D a_j \psi_j(n)}
\]

where the last “\( \leq \)” is obtained using the Cauchy-Schwarz inequality. From (A2) and (i) in Lemma 1, there exists \( M_1 > 0 \) such that \( D < (M_1 + 1)\Lambda_n \). Then by taking expectations on both sides of (25), we have

\[
E[\sup_{\mu \in S_K} G(\mu)] \leq 2\sqrt{f(0)} E\left[ (\sum_{j=1}^D (\sum_{i=1}^n \eta_i \psi_{j,i})^2)^{1/2} \right]
\]

Combining (22) and (24), we get

\[
P\left( \sup_{\mu \in S_K} G(\mu) \geq ((M_1 + 1)\Lambda_n)^{1/2} + \epsilon \right) \leq \exp\{-\epsilon^2/2\}.
\] (25)

Let \( 0 < c < 1 \) such that

\[
c\sqrt{f(0)}\delta_n = (\lambda_{1n} + 2\lambda_{2n})\sqrt{n} + ((M_1 + 1)\Lambda_n)^{1/2}.
\]

Then we can choose a positive \( \epsilon = \sqrt{f(0)}\delta_n - (\lambda_{1n} + 2\lambda_{2n})\sqrt{n} - ((M_1 + 1)\Lambda_n)^{1/2} \) in (25). Combining (21) and (25),

\[
P(\|\tilde{\mu}_n - \mu^0\|_2 \geq \delta_n) \leq \Lambda_n \exp\{-\Lambda_n \log n - (1/2)[\sqrt{f(0)}\delta_n - (\lambda_{1n} + 2\lambda_{2n})\sqrt{n} - ((M_1 + 1)\Lambda_n)]^2\}
\]

For \( \alpha_n = \delta_n/\sqrt{n} \), we have

\[
P(\|\tilde{\mu}_n - \mu^0\|_n \geq \alpha_n) \leq \Lambda_n \exp\{-\Lambda_n \log n - (1/2)(1-c)^2f(0)\alpha_n^2\}.
\]

Thus the first part of Lemma 2 holds. Furthermore, if we also have \( \alpha_n = (2M_2\Lambda_n(\log n)/n)^{1/2} \), then

\[
P(\|\tilde{\mu}_n - \mu^0\|_n \geq \sqrt{2M_2\Lambda_n(\log n)/n}) \leq \Lambda_n n^\frac{(1-M_2f(0)(1-c)^2)\Lambda_n}{n}
\]

which completes the proof. \( \square \)

**Proof of Theorem 1**

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Define
\[
L_n(\mu) = n^{-1} \left[ \sum_{i=1}^{n} |y_i - \mu_i| - \sum_{i=1}^{n} |y_i - \mu_i^0| + \lambda_1 n \sum_{i=1}^{n} |\mu_i| + \lambda_2 n \sum_{i=2}^{n} |\mu_i - \mu_{i-1}| \right]
\]
and
\[
M_n(\mu) = n^{-1} \left[ f(0) \sum_{i=1}^{n} (\mu_i - \mu_i^0)^2 - \sum_{i=1}^{n} \text{sgn}(\varepsilon_i)(\mu_i - \mu_i^0) + \lambda_1 n \sum_{i=1}^{n} |\mu_i| + \lambda_2 n \sum_{i=2}^{n} |\mu_i - \mu_{i-1}| \right].
\]
Then \( \tilde{\mu}_n = \arg \min \{ L_n(\mu) \} \) and \( \bar{\mu}_n = \arg \min \{ M_n(\mu) \} \). Define \( R_{ni} = R_{ni}(\mu_i, \varepsilon_i) = |\varepsilon_i - (\mu_i - \mu_i^0)| - |\varepsilon_i| + \text{sgn}(\varepsilon_i)(\mu_i - \mu_i^0) \) and \( \xi_{ni} = R_{ni} - \mathbb{E}[R_{ni}] \). Following Gao and Huang (2010b), we can verify
\[
|L_n(\mu) - M_n(\mu)| = \sum_{i=1}^{n} \xi_{ni}/n + \tau_n(\mu, \mu^0),
\]
where \( \tau_n(\mu, \mu^0) = o(\|\mu - \mu^0\|_n^2) \). For any \( \delta > 0 \), we define \( S_\delta = \{ \mu : \|\mu - \tilde{\mu}_n\|_2 \leq \delta \} \), \( S_\delta^d = \{ \mu : \|\mu - \bar{\mu}_n\|_2 = \delta \} \) and \( S_\delta^0 = \{ \mu : \|\mu - \mu^0\|_2 \leq \delta \} \). We define
\[
\Delta_n(\delta) = \sup_{\mu \in S_\delta} |L_n(\mu) - M_n(\mu)|
\]
and
\[
h_n(\delta) = \inf_{\mu \in S_\delta^0} (M_n(\mu) - M_n(\tilde{\mu}_n)).
\]
We have
\[
M_n(\mu) - M_n(\tilde{\mu}_n) = f(0)\|\mu - \tilde{\mu}_n\|_n^2 + 2n^{-1} f(0) \sum_{i=1}^{n} (\mu_i - \tilde{\mu}_i)(\tilde{\mu}_i - \mu_i^0) + n^{-1} \sum_{i=1}^{n} \text{sgn}(\varepsilon_i)(\tilde{\mu}_i - \mu_i)
\]
\[
+ n^{-1} \lambda_1 n \sum_{i=1}^{n} |\mu_i| - |\tilde{\mu}_i| + n^{-1} \lambda_2 n \sum_{i=2}^{n} (|\mu_i - \mu_{i-1}| - |\tilde{\mu}_i - \tilde{\mu}_{i-1}|) \quad (28)
\]
Since \( \partial M_n(\mu)/\partial \mu_i|_{\mu_i = \tilde{\mu}_i} = 0 \),
\[
2n^{-1} f(0)(\tilde{\mu}_i - \mu_i^0) - n^{-1} \text{sgn}(\varepsilon_i) + n^{-1} \lambda_2 n [\text{sgn}(\tilde{\mu}_i - \mu_{i-1}) - \text{sgn}(\mu_{i+1} - \tilde{\mu}_i)] = 0.
\]
Multiply \( (\mu_i - \tilde{\mu}_i) \) on both sides and take sums. Then (29) becomes
\[
M_n(\mu) - M_n(\tilde{\mu}_n) = f(0)\|\mu - \tilde{\mu}_n\|_n^2 + n^{-1} \lambda_1 n \sum_{i=1}^{n} \left[ |\tilde{\mu}_i| - |\mu_i| + \text{sgn}(\mu_i) - (|\tilde{\mu}_i| - |\mu_i|) \right] 
\]
\[
+ n^{-1} \lambda_2 n \sum_{i=1}^{n} \left[ \text{sgn}(\mu_{i+1} - \tilde{\mu}_i)(\mu_i - \mu_i^0) + \text{sgn}(\tilde{\mu}_i - \mu_i)(\tilde{\mu}_i - \mu_i) \right] 
\]
\[
+ n^{-1} \lambda_2 n \sum_{i=2}^{n} (|\mu_i - \mu_{i-1}| - |\tilde{\mu}_i - \tilde{\mu}_{i-1}|) \quad (29)
\]
Then for any \( \kappa_n > 0 \), we have
\[
h_n(\kappa_n) > f(0)\kappa_n^2/n.
\]
From the Convex Minimization Theorem in Hjort and Pollard (1993), we have
\[
\mathbb{P}(\|\tilde{\mu}_n - \bar{\mu}_n\|_2 \geq \kappa_n) \leq \mathbb{P}(\Delta_n(\kappa_n)) > h_n(\kappa_n)/2 
\]
\[
\leq \mathbb{P}(\sup_{\mu \in S_{\kappa_n}} n^{-1} \sum_{i=1}^{n} |\xi_{ni}| \geq f(0)\kappa_n^2/(2n)) + \mathbb{P}(\sup_{\mu \in S_{\kappa_n}} |\tau_n(\mu, \mu^0)| \geq f(0)\kappa_n^2/(2n)).
\]
Suppose \( r_n = o(1) \) and \( \tau_n = \tau_n(\bar{\mu}_n, \mu^0) = r_n \| \bar{\mu}_n - \mu^0 \|_n^2 \). Then

\[
\lim_{n \to \infty} P \left( \sup_{\mu \in S_{\kappa_n}} |\tau_n(\mu, \mu^0)| \geq f(0)\kappa_n^2/(2n) \right) \\
\leq \lim_{n \to \infty} P \left( \sup_{\mu \in S_{\kappa_n}} 2|\mu|\|\bar{\mu}_n - \mu^0\|_n^2 \geq f(0)\kappa_n^2/(4n) \right) \\
+ \lim_{n \to \infty} P \left( \sup_{\mu \in S_{\kappa_n}} 2|r_n|\|\mu - \bar{\mu}_n\|_n^2 \geq f(0)\kappa_n^2/(4n) \right) \\
= \lim_{n \to \infty} P \left( \sup_{\mu \in S_{\kappa_n}} |r_n|\|\bar{\mu}_n - \mu^0\|_n^2 \geq f(0)\kappa_n^2/(8n) \right). \tag{30}
\]

Combining (27) and (29),

\[
P(\| \bar{\mu} - \mu \|_2 \geq \kappa_n) \leq P \left( \sup_{\mu \in S_{\kappa_n}} \sum_{i=1}^n \xi_{ni}/n > f(0)\kappa_n^2/(2n) \right) \\
+ P \left( \sup_{\mu \in S_{\kappa_n}} |r_n|\|\bar{\mu}_n - \mu^0\|_n^2 \geq f(0)\kappa_n^2/(8n) \right) + o(1) \tag{31}
\]

where \( \kappa_n' = 2\kappa_n \). Let the \( u_i \)'s be a Rademacher sequence and \( \psi_1, \cdots, \psi_D \) be the orthogonal basis of a \( D \)-dimensional space to which \( \mu - \mu^0 \) belongs. Using the Contraction Theorem in Ledoux and Talagrand (1991) and also the Cauchy-Schwarz inequality, we have

\[
E \left[ \sup_{\mu \in S_{\kappa_n}} |\sum_{i=1}^n \xi_{ni}/n| \right] = (8/n) E \left[ \sup_{\mu \in S_{\kappa_n}} |\sum_{i=1}^n u_i(\mu_i - \mu_0^i)| \right] \\
\leq (8/n) E \left[ \sup_{\mu \in R^D} \sup_{\mu \in S_{\kappa_n}} |\sum_{j=1}^D a_j \sum_{i=1}^n u_i(\psi_{j,i})| \right] \\
\leq (8/n) E \left[ \sup_{\mu \in R^D} \sup_{\mu \in S_{\kappa_n}} (\sum_{j=1}^D a_j^2)^{1/2}(\sum_{j=1}^D (\sum_{i=1}^n u_i\psi_{j,i})^2)^{1/2} \right] \\
\leq (8/n) \sup_{\mu \in S_{\kappa_n}} \sqrt{D(\sum_{j=1}^D a_j^2)^{1/2}} \\
= (8/n) \sup_{\mu \in S_{\kappa_n}} \sqrt{D\|\mu - \mu^0\|_2} \\
\leq 16\sqrt{\Lambda_n} \kappa_n/n.
\]

\[
P \left( \sup_{\mu \in S_{\kappa_n}} |\sum_{i=1}^n \xi_{ni}/n| > f(0)\kappa_n^2/(2n) \right) \leq E \left[ \sup_{\mu \in S_{\kappa_n}} |\sum_{i=1}^n \xi_{ni}/n| \right]/(f(0)\kappa_n^2/(2n)) \leq 32\sqrt{\Lambda_n}/(f(0)\kappa_n) \leq 8\sqrt{\Lambda_n}/(f(0)\kappa_n). \tag{32}
\]

Let \( \gamma_n = \kappa_n/\sqrt{n} \). Combining (31) and (32),

\[
P(\| \mu - \mu^0 \|_n \geq \gamma_n) \leq P(\| \mu - \mu^0 \|_n \geq \kappa_n/2) + P(\| \mu - \mu^0 \|_n \geq \kappa_n/2) \\
\leq P(\| \mu - \mu^0 \|_n \geq \kappa_n/2) + P(\| \mu - \mu^0 \|_n \geq \kappa_n/2) \\
\leq 32\sqrt{\Lambda_n}/(f(0)\gamma_n\sqrt{n}) + 2\Lambda_n \exp(-\Lambda_n n - (1 - c)^2 f(0) n \gamma_n^2/8).
\]

The last “\( \leq \)” is from (32) and Lemma 2 by choosing \( \gamma_n = 2(c\sqrt{f(0)})^{-1}[\Lambda_n + 2\lambda_2 + ((M_1 + 1)\Lambda_n/n)^{1/2}] \). Thus the first part of Theorem 2 holds. Furthermore, if we let \( \lambda_1 + 2\lambda_2 = [2c^2 f(0) M_3 \Lambda_n (\log n)/n]^{1/2} - [(M_1 + 1)\Lambda_n]^{1/2} \) for \( M_3 > 1/(1 - c)^2 f(0) \) and \( \gamma_n = (8M_3 \Lambda_n (\log n)/n)^{1/2} \), then \( \sqrt{\Lambda_n}/(f(0)\gamma_n\sqrt{n}) = O(1/\sqrt{\log n}) \). Thus,

\[
P(\| \mu - \mu^0 \|_n \geq \gamma_n) \leq O(1/\sqrt{\log n}) + 2\Lambda_n \exp(-\Lambda_n n (1 - M_3(1 - c)^2 f(0))).
\]
Proof of Corollary 1

If we replace the upper bound of maximal dimension of any linear space where \(\mu_n, \tilde{\mu}_n\) or \(\mu_0\) belong by \(J_{\text{max}}\) in the proof of Theorem 1, we can obtain the consistency result in Corollary 1. We do not repeat the proof here. □

Proof of Theorem 2

Suppose vector \(\mu\) has \(J\) blocks and \(\{B_1, \ldots, B_J\}\) is the corresponding unique block partition. Let \(\nu_j\) be the intensity at \(j\)th block for \(1 \leq j \leq J\). From Lemma A.1 in Rinaldo (2009), the subdifferential of the total variation penalty

\[
\partial \left( \lambda_2 n \sum_{j=2}^{J} |\nu_j - \nu_{j-1}| \right) = \begin{cases} 
-\lambda_2 n \text{sgn}(\nu_{j+1} - \nu_j), & j = 1 \\
\lambda_2 n \left( \text{sgn}(\nu_{j+1} - \nu_j) - \text{sgn}(\nu_j - \nu_{j-1}) \right), & 1 < j < J \\
\lambda_2 n \text{sgn}(\nu_j - \nu_{j-1}), & j = J
\end{cases},
\]

(33)

where \(\text{sgn}(x) = 1, 0, -1\) when \(x > 0, = 0, < 0\), respectively. We define \(c_j^0\) and \(\tilde{c}_j\) as the subdifferentials (33) at both \(\nu^0\) and \(\tilde{\nu}_n\) scaled by the corresponding block sizes. In other words, we have

\[
c_j^0 = \begin{cases} 
-\lambda_2 n \text{sgn}(\nu_{j+1}^0 - \nu_j^0) / b_j^0, & j = 1 \\
\lambda_2 n \left( \text{sgn}(\nu_{j+1}^0 - \nu_j^0) - \text{sgn}(\nu_j^0 - \nu_{j-1}^0) \right) / b_j^0, & 1 < j < J_0 \\
\lambda_2 n \text{sgn}(\nu_j^0 - \nu_{j-1}^0) / b_j^0, & j = J_0
\end{cases},
\]

(34)

and

\[
\tilde{c}_j = \begin{cases} 
-\lambda_2 n \text{sgn}(\tilde{\nu}_{j+1} - \tilde{\nu}_j) / \tilde{b}_j, & j = 1 \\
\lambda_2 n \left( \text{sgn}(\tilde{\nu}_{j+1} - \tilde{\nu}_j) - \text{sgn}(\tilde{\nu}_j - \tilde{\nu}_{j-1}) \right) / \tilde{b}_j, & 1 < j < J \\
\lambda_2 n \text{sgn}(\tilde{\nu}_j - \tilde{\nu}_{j-1}) / \tilde{b}_j, & j = J
\end{cases}.
\]

(35)

For an estimate \(\tilde{\mu}_n\), we let \(\tilde{B}_{j(i)}\) be the block estimate where \(i\) stays, that is, \(\tilde{\mu}_i\) are all the same for \(i \in \tilde{B}_{j(i)}\). Let \(\tilde{b}_{j(i)} = |\tilde{B}_{j(i)}|\) be the size of \(\tilde{B}_{j(i)}\). Then \(B_{j(i)}^0\) \((b_{j(i)}^0)\) is the corresponding block set (size). From notations (I) and (V) in Section 2, we have \(b_{j(i)}^0 = |B_{j(i)}^0|\) for \(1 \leq j \leq J_0\). From the KKT conditions, \(\tilde{\mu} F\) is a LAD-FSA solution if and only if

\[
\begin{cases} 
\sum_{k \in \tilde{B}_{j(i)}} \text{sgn}(y_k - \tilde{\mu}_i) = \tilde{b}_{j(i)} \tilde{c}_{j(i)} & \text{if } i \in \tilde{J} \\
|\sum_{k \in \tilde{B}_{j(i)}} \text{sgn}(y_k - \tilde{\mu}_i)| < 2\lambda_2 n & \text{if } i \notin \tilde{J}.
\end{cases}
\]

(36)

Let \(\tilde{\mu}_i\) and \(\mu_i^0\) satisfy

\[
\begin{cases} 
\tilde{\mu}_i = \mu_i^0 + (2f(0)b_{j(i)}^0)^{-1} \left( \sum_{k \in \tilde{B}_{j(i)}} \text{sgn}(\varepsilon_k) - b_{j(i)}^0 c_{j(i)}^0 + \tilde{h}_i \right) & \forall i \in J^0 \\
\tilde{\mu}_i = \tilde{\mu}_{i-1} & \forall i \notin J^0.
\end{cases}
\]

(37)

Here \(\tilde{h}_i\) is the remainder term with the stochastically equicontinuity, more specifically,

\[
| (b_{j(i)}^0)^{-1/2} \tilde{h}_i | = O_p(1), \forall 1 \leq i \leq n.
\]

(38)
In fact, \( \mathcal{N}_i = 2f(0)b_{j(i)}^0 (\bar{\mu}_i - \mu_i) + \sum_{k \in B_0^{j(i)}} \mathbb{E}[r_i^k] + \sum_{k \in B_0^{j(i)}} \zeta_k^i \) with \( r_i^k = \text{sgn}(\varepsilon_k - (\bar{\mu}_i - \mu_i)) - \text{sgn}(\varepsilon_k) \) and \( \zeta_k^i = r_i^k - \mathbb{E}r_i^k \) for \( k \in B_0^{j(i)} \). Define the difference vector \( w = (w_1, \ldots, w_n)' \), with \( w_1 = \mu_1 \) and \( w_i = \mu_i - \mu_{i-1} \) for \( 2 \leq i \leq n \). If \( \text{sgn}(\bar{w}_i) = \text{sgn}(w_i^0), \forall i \in \mathcal{J}^0 \), then (36) holds for \( \bar{\mu}_n \) in (37). Thus, \( \bar{\mu}_n \) is a LAD-FSA solution. Define

\[
\mathcal{R}_{\lambda_2n} \equiv \{ \bar{\mathcal{J}} = \mathcal{J}^0 \} \cap \{ \text{sgn}(\bar{w}_i^0) = \text{sgn}(w_i^0), \forall i \in \mathcal{J}^0 \}
\]

Then \( \mathcal{R}_{\lambda_2n} \) holds if

\[
\begin{align*}
\text{sgn}(\bar{w}_i) &= \text{sgn}(w_i^0) \quad \forall i \in \mathcal{J}^0 \\
\sum_{k \in B_0^{j(i)}} \text{sgn}(y_k - \bar{\mu}_i), &< 2\lambda_{2n} \quad \text{if } i \notin \mathcal{J}^0
\end{align*}
\]

It is easy to verify that \( \text{sgn}(\bar{w}_i) = \text{sgn}(w_i^0), \forall i \in \mathcal{J}^0 \) holds if

\[
|\text{sgn}(\bar{w}_i)(w_i^0 - \bar{w}_i)| < |w_i^0|, \text{ for } i \in \mathcal{J}^0
\]

Plug \( \bar{\mu}_n \) (37) into (40) and (39b), and then use the triangle inequality, we know that \( \mathcal{R}_{\lambda_2n} \) holds if

\[
\begin{align*}
\max_{i \in \mathcal{J}^0} |(b_{j(i)}^0)^{-1} \sum_{k \in B_0^{j(i)}} \text{sgn}(\varepsilon_k) - (b_{j(i-1)}^0)^{-1} \sum_{k \in B_0^{j(i-1)}} \text{sgn}(\varepsilon_k)|/w_i^0 + \\
\max_{i \in \mathcal{J}^0} |(b_{j(i)}^0)^{-1} \mathcal{N}_i - (b_{j(i-1)}^0)^{-1} \mathcal{N}_{i-1}|/w_i^0 + \max_{i \in \mathcal{J}^0} |c_{j(i)}^0 - c_{j(i-1)}^0|/w_i^0 \leq 2f(0)
\end{align*}
\]

and

\[
\max_{i \in \mathcal{J}^0} |\text{sgn}(\varepsilon_i) - \text{sgn}(\varepsilon_{i-1}) + \mathcal{N}_i - \mathcal{N}_{i-1}| < 4\lambda_{2n}.
\]

We have

\[
\mathbb{E}[\text{sgn}(\varepsilon_i) - \text{sgn}(\varepsilon_{i-1})] = 0
\]

and

\[
\text{Var}[\text{sgn}(\varepsilon_i) - \text{sgn}(\varepsilon_{i-1})] = 2 \text{ for } 2 \leq i \leq n
\]

and for \( 2 \leq i_1, i_2 \leq n \),

\[
\text{Cov}(\text{sgn}(\varepsilon_{i_1}) - \text{sgn}(\varepsilon_{i_1-1}), \text{sgn}(\varepsilon_{i_2}) - \text{sgn}(\varepsilon_{i_2-1})) = -1, 0 \text{ for } |i_1 - i_2| = 1, \text{ otherwise}.
\]

Suppose \( d_i^* \) are independent copies of \( N(0, 2) \). Then we have

\[
\mathbb{P}(I_4) \equiv \mathbb{P}(\max_{i \in \mathcal{J}^0} |\text{sgn}(\varepsilon_i) - \text{sgn}(\varepsilon_{i-1})| > 2\lambda_{2n}) \\
\leq \mathbb{P}(\max_{i \in \mathcal{J}^0} |d_i^*| > 2\lambda_{2n}) \\
\leq 2 \exp\{-4\lambda_{2n}^2 + \log|\mathcal{J}_0^0|\}
\]

where we get the first “\( \leq \)” using Slepian’s inequality, the second “\( \leq \)” using Chernoff’s bound. Then \( \mathbb{P}(I_4) = o(1) \) if conditions in (B1) hold. Define

\[
X_i = (2f(0)b_{j(i)}^0)^{-1} \sum_{k \in B_0^{j(i)}} \text{sgn}(\varepsilon_k) - (2f(0)b_{j(i-1)}^0)^{-1} \sum_{k \in B_0^{j(i-1)}} \text{sgn}(\varepsilon_k), \forall i \in \mathcal{J}^0.
\]
Then $E[X_i] = 0$ and $\max_{i \in J} \text{Var}[X_i] \leq (2f(0)b^0_{j(i)})^{-1}$. Consider independent copies $X_i^* \sim N(0, (2f(0)b^0_{j(i)})^{-1}), i \in J^0$. We have

$$P(I_1) = \mathbb{P}(\max_{i \in J^0}|(b^0_{j(i)})^{-1} \sum_{k \neq b^0_{j(i)}} \text{sgn}(\varepsilon_k) - (b^0_{j(i)-1})^{-1} \sum_{k \neq b^0_{j(i)-1}} \text{sgn}(\varepsilon_k)| > 2f(0)a_n/3) \leq \mathbb{P}(\max_{i \in J^0}|X_i^*| > a_n/3) \leq 2 \exp\{-2b^0_{\min}f^2(0)a_n^2/9 + \log|J^0|\}.$$

Thus $P(I_1) = o(1)$ if conditions in (B2) holds. Since $\max_{i \in J^0}|c^0_{j(i)} - c^0_{j(i)-1}| \leq 2\lambda_2/n^0\min$, from (B3),

$$P(I_2) = \mathbb{P}(\max_{i \in J^0}|c^0_{j(i)} - c^0_{j(i)-1}| > 2f(0)a_n/3) = 0.$$

Furthermore, we have

$$P(I_5) = \mathbb{P}(\max_{i \in J^0}|\tilde{\nu}_i - \tilde{\nu}_{i-1}| > 2\lambda_2/n) = 0.$$

From (38), we have

$$P(I_3) = \mathbb{P}(\max_{i \in J^0}|(w^0_{j(i)})^{-1/2}\tilde{\nu}_i - (w^0_{j(i-1)})^{-1/2}\tilde{\nu}_{i-1}| > 2f(0)/3) \leq \mathbb{P}(\max_{i \in J^0}|(b^0_{j(i)})^{-1/2}\tilde{\nu}_i - (b^0_{j(i-1)})^{-1/2}\tilde{\nu}_{i-1}| > 2f(0)(b^0_{\min})^{1/2}a_n/3) = o(1).$$

Then from (41) and (42), we get

$$P(R^c_{\lambda_2n}) \leq P(I_1) + P(I_2) + P(I_3) + P(I_4) + P(I_5) \to 0 \text{ when } n \to \infty.$$

□

**Proof of Theorem 3**

From Theorem 2, we know that if (A1) and (B1-B3) hold, the LAD-FLSA can choose all jumps with probability 1. Thus, we can prove the main results based on the true block partition. By the KKT, $\tilde{\nu}_n$ is a LAD-FLSA solution if and only if

$$\begin{cases}
\sum_{k \neq b^0_j} \text{sgn}(y_k - \tilde{\nu}_j) + \tilde{b}_j \hat{c}_j = \lambda_{1n} \tilde{b}_j \text{sgn}(\tilde{\nu}_j) & \text{if } \tilde{\nu}_j \neq 0 \\
\sum_{k \neq b^0_j} \text{sgn}(y_k - \tilde{\nu}_j) + \tilde{b}_j \hat{c}_j \leq \lambda_{1n} \tilde{b}_j & \text{if } \tilde{\nu}_j = 0.
\end{cases} \tag{43}$$

Let $\tilde{\nu}_j$ and $\nu^0_j$ satisfy

$$\begin{cases}
\tilde{\nu}_j = \nu^0_j + (2f(0)b^0_j)^{-1} \left(\sum_{i \neq b^0_j} \text{sgn}(\varepsilon_i) + \tilde{b}_j \nu^0_j - \lambda_{1n} b^0_j \text{sgn}(\nu^0_j) + \tilde{\nu}_j\right) & \forall \ j \in K^0 \\
\tilde{\nu}_j = 0 & \forall \ j \in K^0.
\end{cases} \tag{44}$$

Here, by abuse of notation, $\tilde{\nu}_j$ is the remainder term with the stochastically equicontinuity,

$$|(b^0_j)^{-1/2}\tilde{\nu}_j| = O_p(1), \ \forall 1 \leq i \leq n. \tag{45}$$

In fact, $\tilde{\nu}_j = 2f(0)b^0_j(\nu^0_j - \nu^0_j) + \sum_{i \neq b^0_j} \mathbb{E}[r^j_i] + \sum_{i \neq b^0_j} \varsigma^j_i$, with $r^j_i = \text{sgn}(\varepsilon_i - (\tilde{\nu}_j - \nu^0_j)) - \text{sgn}(\varepsilon_i)$ and $\varsigma^j_i = r^j_i - \mathbb{E}(r^j_i)$ for $i \in B^0$ and $1 \leq j \leq J_0 + 1$. If $\{\text{sgn}(\tilde{\nu}_j) = \text{sgn}(\nu^0_j), \forall \ j \in K^0\}$, then $\nu_n$ in (44) satisfies kkt-flsa, and therefore is a LAD-FLSA solution. Define an event

$$R_n = R(\lambda_{1n}, \lambda_{2n}) \equiv \{K = K^0\} \cap \{\text{sgn}(\tilde{\nu}_j) = \text{sgn}(\nu^0_j), \forall \ j \in K^0\}.$$
Then $\mathcal{R}_n$ holds if

\[
\left\{ \begin{array}{l}
\text{sgn}(\bar{\nu}_j) = \text{sgn}(\nu_j^0), \\
\sum_{k \in \mathcal{B}_0^j} \text{sgn}(y_k - \bar{\nu}_j) + \hat{b}_j \lambda_1 \bar{b}_j < \lambda_1 n \bar{b}_j \quad \forall \ j \notin \mathcal{K}_0
\end{array} \right. \quad \forall \ j \in \mathcal{K}_0 \tag{46}
\]

Therefore, from (44) and (46), we have

\[
\sum_{k \in \mathcal{B}_0^j} \text{sgn}(y_k - \bar{\nu}_j) + \hat{b}_j \lambda_1 \bar{b}_j < \lambda_1 n \bar{b}_j \quad \forall \ j \notin \mathcal{K}_0.
\]

We can verify that $\text{sgn}(\bar{\nu}_j) = \text{sgn}(\nu_j^0), \forall \ j \in \mathcal{K}_0$ holds if $|\text{sgn}(\bar{\nu}_j)(\nu_j^0 - \bar{\nu}_j)| < |\nu_j^0|$, for $j \in \mathcal{K}_0$. Thus we have

\[
\left\{ \begin{array}{l}
|\sum_{k \in \mathcal{B}_0^j} \text{sgn}(\varepsilon_k) + b_j^0 \nu_j^0 - \lambda_1 n b_j^0 \text{sgn}(\nu_j^0) + \hat{h}_j | < 2 f(0) b_j^0 |\nu_j^0| \\
|\sum_{k \in \mathcal{B}_0^j} \text{sgn}(\varepsilon_k) + b_j^0 c_j^0 + \hat{h}_j | < \lambda_1 n b_j^0
\end{array} \right. \quad \forall \ j \in \mathcal{K}_0 \quad \forall \ j \notin \mathcal{K}_0. \tag{47}
\]

Thus we have

\[
P(\mathcal{R}^c) \leq P(\max_{j \in \mathcal{K}_0} |\sum_{i \in \mathcal{B}_0^j} \text{sgn}(\varepsilon_i)| > 2 f(0) \min_{j \in \mathcal{K}_0} b_j^0 \min_{j \in \mathcal{K}_0} |\nu_j^0|/4) + P(\max_{j \in \mathcal{K}_0} c_j^0 > 2 f(0) \min_{j \in \mathcal{K}_0} |\nu_j^0|/4) + P(\max_{j \in \mathcal{K}_0} \lambda_1 n \text{sgn}(\nu_j^0) > 2 f(0) \min_{j \in \mathcal{K}_0} |\nu_j^0|/4)
\]

\[
+ P(\max_{j \in \mathcal{K}_0} \hat{h}_j / (b_j^0 |\nu_j^0|) > f(0)/2) + P(\max_{j \in \mathcal{K}_0} \lambda_1 n \text{sgn}(\varepsilon_i) > \lambda_1 n b_j^0 /3) + P(\max_{j \in \mathcal{K}_0} c_j^0 > \lambda_1 n /3) + P(\max_{j \in \mathcal{K}_0} \hat{h}_j / b_j^0 > \lambda_1 n /3)
\]

\[
\equiv P(S_1) + P(S_2) + P(S_3) + P(S_4) + P(S_5) + P(S_6) + P(S_7).
\]

Let $Z_j = \sum_{i \in \mathcal{B}_0^j} \text{sgn}(\varepsilon_i)/b_j^0$. Then $E[Z_j] = 0$ and $\text{Var}(Z_j) = 1/b_j^0$. Then $Z_j$s are independent sub-Gaussian. From (C3), we have

\[
P(S_1) \leq 2K_0 \exp\{-b_{\min}^0 f^2(0) \rho_n^2/8\} = o(1).
\]

We can verify $P(S_2) = o(1)$ from (C4), $P(S_3) = o(1)$ from (C5) and $P(S_6) = o(1)$ from (C2). From (C1),

\[
P(S_5) \leq 2(J_0 - K_0) \exp\{-b_{\min}^0 \lambda_1^2 n /32\} = o(1).
\]

Furthermore, we have $P(S_4) = o(1)$ and $P(S_4) = o(1)$. From (48), we have $P(\mathcal{R}) \rightarrow 1$ when $n \rightarrow \infty$, which completes the proof. □

The rest of the Appendix are presented to prove Theorem 4.

Recall that $w$ is the jump coefficients vector with $w_i = \mu_i - \mu_{i-1}$ for $2 \leq i \leq n$ and $\nu = (\nu_1, \cdots, \nu_J)^t$ is the block coefficients factor. From Proposition 3 in Rosset and Zhu (2007), we know that the following results of the LAD-FLSA solution.

**Lemma 4**  (i) For any $\lambda_1 = 0$, there exists a set of values of $\lambda_2$,

\[
0 = \lambda_{2,0} < \lambda_{2,1} < \cdots < \lambda_{2,m_2} < \lambda_{2,m_2+1} = \infty
\]

such that $\hat{\omega}(0, \lambda_{2,k})$ for $1 \leq k \leq m_2$ is not uniquely defined, the set of optimal solutions of each $\lambda_{2,k}$ is a straight line in $\mathcal{R}^n$, and for any $\lambda_2 \in (\lambda_{2,k}, \lambda_{2,k+1})$, the solution $\hat{\omega}(0, \lambda_2)$ is constant.

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(ii) For above $\lambda_{2,k}, 1 \leq k \leq m_2$, there exists a set of values of $\lambda_1$, 

$$0 = \lambda_{1,0} < \lambda_{1,1} < \cdots < \lambda_{1,m_2} < \lambda_{1,m_2+1} = \infty$$

such that $\mathcal{D}(\lambda_{1,j}, \lambda_{2,k})$ for $1 \leq j \leq m_1$ is not uniquely defined, the set of optimal solutions of each $\lambda_{1,j}$ is a straight line in $\mathbb{R}^n$, and for any $\lambda_1 \in (\lambda_{1,j}, \lambda_{1,j+1})$, the solution $\mathcal{D}(\lambda_1, \lambda_{2,k})$ is constant.

In Lemma 4 if we define $\lambda_{2,k}, 1 \leq k \leq m_2$ from (i) as the transition points for $w$ and $N_{0,\lambda_2}$ as the set of $y \in \mathbb{R}^n$ such that $\lambda_2$ is a transition point for $w$, then the jumps set $J(0, \lambda_2)$ only changes at those $\lambda_{2,k}$’s. Furthermore, Let $\lambda_2 = \lambda_{2,k}$ for some $1 \leq k \leq m_2$. If we can also define $\lambda_{1,j}, 1 \leq j \leq m_1$ from (ii) as the transition points for $\nu$ and $N_{\lambda_1,\lambda_2}$ as the set of $y \in \mathbb{R}^n$ such that $\lambda_1$ is a transition point for $\nu$, then the set of nonzero blocks, $\mathcal{K}(\lambda_1, \lambda_2)$, only changes at $\lambda_{1,j}$’s or $\lambda_{2,k}$’s, and $N_{\lambda_1,\lambda_2}$, is in a finite collection of hyperplanes in $\mathbb{R}^n$. From Lemma 4 we know that for any given $y \in \mathbb{R}^n/N_{\lambda_1,\lambda_2}$, $\mathcal{D}(\lambda_1, \lambda_2)$ is fixed and then $\{1, 2, \cdots, n\}$ is divided into two sets, $\mathcal{E}_{\nu,\lambda_1,\lambda_2}$ and $\mathcal{E}_{\nu,\lambda_1,\lambda_2}'$, where $\mathcal{E}_{\nu,\lambda_1,\lambda_2} = \{1 \leq i \leq n : y_i - \mathcal{D}(j(i) = 0, \mathcal{D}(j(i) = 0) \}$ and $j(i)$ specifies the block where $\mathcal{D}_i$ stays. Thus, we have 

$$|\mathcal{E}_{\nu,\lambda_1,\lambda_2}| = |\mathcal{K}(\lambda_1, \lambda_2)|.$$ 

(49)

**Lemma 5** For any $\lambda_1 > 0$ and $\lambda_2 > 0$, if $y \in \mathbb{R}^n/N_{\lambda_1,\lambda_2}$, $\mathcal{D}(\lambda_1, \lambda_2, y)$ is a continuous function of $y$, and then $\mathcal{E}_{\nu,\lambda_1,\lambda_2}$ is locally constant.

**Proof of Lemma 5**

Let $L(\mu, y)$ denote the function 

$$L(\mu, y) = \sum_{i=1}^{n} |y_i - \mu_i| + \lambda_1 \sum_{i=1}^{n} |\mu_i| + \lambda_2 \sum_{i=2}^{n} |\mu_i - \mu_{i-1}|.$$ 

Since $y \in \mathbb{R}^n/N_{\lambda_1,\lambda_2}$, $\mathcal{D}$ does not change from Lemma 4 For any $y_0 \in \mathbb{R}^n/N_{\lambda_1,\lambda_2}$, and for any sequence $\{y_m\}$ such that $\{y_m\} \to y_0$, we want to prove that $\mathcal{D}(\lambda_1, \lambda_2, y_m) \to \mathcal{D}(\lambda_1, \lambda_2, y_0)$. It is equivalent to prove $\mathcal{D}(\lambda_1, \lambda_2, y_m) \to \mathcal{D}(\lambda_1, \lambda_2, y_0)$. Because $\|\mathcal{D}(\lambda_1, \lambda_2, y)\|_1 \leq \|\mathcal{D}(0, 0, y)\|_1 = \|y\|_1$, $\mathcal{D}(\lambda_1, \lambda_2, y)$ is bounded. Thus we only need to check that for every converging subsequence of $\{y_m\}$, say $\{y_{m_k}\}$, we have $\mathcal{D}(\lambda_1, \lambda_2, y_{m_k}) \to \mathcal{D}(\lambda_1, \lambda_2, y_0)$. Suppose that $\mathcal{D}(\lambda_1, \lambda_2, y_{m_k}) \to \mathcal{D}(\lambda_1, \lambda_2, y_0)$ when $m_k \to \infty$. Let $\Delta(\mu, y, y') = L(\mu, y) - L(\mu, y')$. On the one hand, we have 

$$L(\mathcal{D}(\lambda_1, \lambda_2, y_0), y_0) = L(\mathcal{D}(\lambda_1, \lambda_2, y_{m_k}), y_{m_k}) + \Delta(\mathcal{D}(\lambda_1, \lambda_2, y_0), y_0, y_{m_k})$$

$$\geq L(\mathcal{D}(\lambda_1, \lambda_2, y_{m_k}), y_{m_k}) + \Delta(\mathcal{D}(\lambda_1, \lambda_2, y_0), y_0, y_{m_k})$$

$$= L(\mathcal{D}(\lambda_1, \lambda_2, y_{m_k}), y_0) + \Delta(\mathcal{D}(\lambda_1, \lambda_2, y_{m_k}), y_{m_k}, y_0) + \Delta(\mathcal{D}(\lambda_1, \lambda_2, y_0), y_0, y_{m_k}).$$

On the other hand, we have 

$$\Delta(\mathcal{D}(\lambda, y_{m_k}), y_{m_k}, y_0) + \Delta(\mathcal{D}(\lambda_1, \lambda_2, y_0), y_0, y_{m_k})$$

$$= \sum_{i=1}^{n} \left[ |y_{m_k} - \mathcal{D}(\lambda_1, \lambda_2, y_{m_k})| - |y_{m_k} - \mathcal{D}(\lambda_1, \lambda_2, y_{m_k})| \right]$$

$$+ |y_{m_k} - \mathcal{D}(\lambda_1, \lambda_2, y_0)| - |y_{m_k} - \mathcal{D}(\lambda_1, \lambda_2, y_0)| \right]$$

$$\leq 2 \sum_{i=1}^{n} |y_{m_k} - y_{i,0}| \to 0 \text{ when } k \to \infty.$$
Thus \( L(\hat{\mu}(\lambda_1, \lambda_2, y_0), y_0) \geq \lim_{k \to \infty} L(\hat{\mu}(\lambda_1, \lambda_2, y_m_k), y_0) = L(\hat{\mu}(\lambda_1, \lambda_2, y_0), y_0) \). Since \( \hat{\mu}(\lambda_1, \lambda_2, y_0) \) is the unique minimizer of \( L(\mu, y_0) \), we have \( \hat{\mu}(\lambda_1, \lambda_2, y_0) = \hat{\mu}(\lambda_1, \lambda_2, y_0) \). □

Proof of Theorem 4

From (49) and Lemma 5, there exists \( \epsilon > 0 \) such that \( y \in \text{Ball}(y, \epsilon) \), \( \mathcal{E}_{y, \lambda_1, \lambda_2} \) stays the same when neither \( \lambda_1 \) and \( \lambda_2 \) is a transitional point. Thus, \( \partial \nu_j(i)(\lambda_1, \lambda_2)/\partial y_i = 1 \) if \( i \in \mathcal{E}_{y, \lambda_1, \lambda_2} \) and \( \partial \nu_j(i)(\lambda_1, \lambda_2)/\partial y_i = 0 \) if \( i \notin \mathcal{E}_{y, \lambda_1, \lambda_2} \). Overall, we have \( \sum_{i=1}^n \partial \nu_j(i)(\lambda_1, \lambda_2)/\partial y_i \rightleftharpoons \sum_{i \in \mathcal{E}_{y, \lambda_1, \lambda_2}} \partial \nu_j(i)(\lambda_1, \lambda_2)/\partial y_i \) for \( y \in \mathcal{N}_{\lambda_1, \lambda_2} \). Since \( \mathcal{N}_{\lambda_1, \lambda_2} \) is in a collection of finite hyperplanes, we can obtain the conclusion by taking the expectation. □

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Figure 1: Copy Number data set from the GM 13330 BAC CGH array. The top and bottom panels give outputs from the LAD-FLSA and LS-FLSA, respectively. Both observed data (gray dots) and estimates (dark solid lines) from chromosome 1–4 are plotted. Data from different chromosomes are separated by gray vertical lines.
Figure 2: The estimated degrees of freedom of LAD-FLSA for every combined $\lambda_1$ and $\lambda_2$ for the chromosome 1 data from the GM 13330 BAC array.

Figure 3: Hypothetical model from 500 Monte Carlo simulations of 129 markers for chromosome 1 data from the GM 13330 BAC array. It shows that the estimated number of nonzero blocks $|\mathcal{K}(\lambda_1, \lambda_2)|$ is very close to the true $D(\mathcal{M}_{\lambda_1, \lambda_2})$ using the 45 degree line.
Figure 4: Example of observed data (grey) with true hidden signals (black) and LAD-FLSA estimates (red). There are 6 blocks with 4 nonzero ones. Random noise $\varepsilon_i$’s are generated from double exponential distributions with center 0 and scale $0.5/\sqrt{2}$.

Table 1: Simulation results for Section 6.1.

| $\varepsilon_i$ | $\sigma$ | Model   | LARE$^1$ | CFR+6$^2$ | JUMP$^3$ | LARE   | CFR+6  | JUMP   |
|-----------------|----------|---------|----------|-----------|----------|--------|--------|--------|
| Normal          | 1.0      | LAD-FLSA | 0.197    | 89% (17%) | 7.12 (1.32) | 0.173  | 82% (15%) | 7.24 (1.33) |
|                 |          | LS-FLSA  | 0.035    | 18% (3%)  | 7.82 (1.47) | 0.021  | 5% (0%)  | 7.7 (1.48)  |
|                 | 0.5      | LAD-FLSA | 0.098    | 97% (32%) | 5.59 (0.75) | 0.087  | 96% (22%) | 5.54 (0.72) |
|                 |          | LS-FLSA  | 0.016    | 48% (13%) | 5.68 (0.74) | 0.007  | 57% (7%)  | 5.61 (0.74) |
|                 | 0.1      | LAD-FLSA | 0.019    | 100% (93%) | 5.00 (0.00) | 0.017  | 100% (94%) | 5.00 (0.00) |
|                 |          | LS-FLSA  | 0.013    | 100% (93%) | 5.00 (0.00) | 0.003  | 100% (94%) | 5.00 (0.00) |
| Double Exp.     | 1.0      | LAD-FLSA | 0.154    | 88% (22%) | 7.42 (1.54) | 0.128  | 89% (25%) | 7.18 (1.35) |
|                 |          | LS-FLSA  | 0.031    | 12% (0%)  | 7.42 (1.42) | 0.021  | 3% (1%)  | 7.31 (1.43) |
|                 | 0.5      | LAD-FLSA | 0.077    | 97% (34%) | 5.95 (0.90) | 0.064  | 100% (41%) | 5.74 (0.86) |
|                 |          | LS-FLSA  | 0.016    | 57% (12%) | 5.73 (0.78) | 0.007  | 62% (19%) | 5.68 (0.87) |
|                 | 0.1      | LAD-FLSA | 0.015    | 100% (97%) | 5.00 (0.00) | 0.013  | 100% (90%) | 5.00 (0.00) |
|                 |          | LS-FLSA  | 0.013    | 100% (97%) | 5.00 (0.00) | 0.003  | 100% (89%) | 5.00 (0.00) |
| Cauchy          | 1.0      | LAD-FLSA | 0.048    | 87% (56%) | 6.12 (1.07) | 0.029  | 82% (45%) | 6.14 (0.95) |
|                 |          | LS-FLSA  | 0.239    | 17% (4%)  | 16.37 (5.38) | 0.275  | 2% (0%)  | 59.41 (14.38) |
|                 | 0.5      | LAD-FLSA | 0.028    | 99% (70%) | 5.56 (0.86) | 0.015  | 87% (66%) | 5.59 (0.78) |
|                 |          | LS-FLSA  | 0.120    | 39% (17%) | 10.67 (3.62) | 0.132  | 15% (3%)  | 32.05 (8.51) |
|                 | 0.1      | LAD-FLSA | 0.007    | 95% (92%) | 5.18 (0.46) | 0.003  | 96% (87%) | 5.17 (0.40) |
|                 |          | LS-FLSA  | 0.029    | 94% (78%) | 6.30 (1.34) | 0.023  | 87% (49%) | 10.14 (3.13) |

NOTE 1: LARE is the least absolute relative ratio defined in (15).
NOTE 2: CFR+6 is the ratio of recovering $\hat{\mu}^0$ correctly or plus at most six additional false positives (correctly fitted ratio).
NOTE 3: JUMP is the average number (standard deviation) of the number of jumps.