Area-Preserving Diffeomorphisms, $W_\infty$ and $\mathcal{U}_q(sl(2))$

in Chern-Simons theory and Quantum Hall system

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Abstract

We discuss a quantum $\mathcal{U}_q(sl(2))$ symmetry in Landau problem, which naturally arises due to the relation between the $\mathcal{U}_q(sl(2))$ and the group of magnetic translations. The last one is connected with the $W_\infty$ and area-preserving (symplectic) diffeomorphisms which are the canonical transformations in the two-dimensional phase space. We shall discuss the hidden quantum symmetry in a $2 + 1$ gauge theory with the Chern-Simons term and in a Quantum Hall system which are both connected with the Landau problem.

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1. Introduction.

Landau problem [1], i.e. quantum mechanical description of a charged particle with mass $m$ moving on the plane in a constant magnetic field $B$ normal to the plane, has important applications in different areas of theoretical physics. The spectrum of one-body problem consists of degenerate energy levels - the so called Landau level. The degeneracy of each level proportional to total magnetic flux and this system exhibits a lot of interesting properties.

This problem is the cornerstone of the quantum Hall effect (QHE) description [2]. Landau problem also arises [3] in a topologically massive gauge theory [4], i.e. 2 + 1-dimensional gauge theory with the Chern-Simons term, quantization. It was shown [3], [5] that the Hilbert space of the theory is a direct product of the massive gauge particles Hilbert space (one free massive particle in the most simple $U(1)$ case) and some quantum-mechanical Hilbert space. In the $U(1)$ case this quantum-mechanical Hilbert space is the product of the $g$ copies (for a genus $g$ Riemann surface) of the Hilbert space for the Landau problem on the torus. In the infinite mass limit all levels except the first one are decoupled as well as massive particles Hilbert space and we have only the first Landau level which becomes the Hilbert space of the topological Chern-Simons theory [6].

Landau Hamiltonian is not invariant under the translation group, however it was known for a long time that it is invariant under the group of magnetic translations [7]. Recently Wiegmann and Zabrodin [8] demonstrated that magnetic translations can be expressed through generators of the quantum algebra $\mathcal{U}_q(sl(2))$ [9], [10] and applied this representation to formulate the Bethe-Ansatz for the Ahbel-Hofstadter problem, i.e. the problem of Bloch electron in magnetic field. The Bethe-Ansatz solution was generalised in [11]. Let us note that several years ago Floratos [12] constructed the representations of the quantum group $GL_q(2)$ with $q = \exp(2\pi i/N)$ using the $N \times N$ representations of the Heisenberg group algebra, which were equivalent to the representations of the magnetic translations.

It was also discussed recently that there is a $W_\infty$ symmetry [13] in the Landau problem [14] - [16]. This symmetry was discussed in a context of a topologically massive gauge theory (TMGT) in [14] and of a quantum Hall effect (QHE) in [15], [16]. It is amusing that the $W_\infty$ algebras are connected with an algebra of area-preserving (or symplectic) diffeomorphisms of the two-dimensional manifolds, for example it is an infinite-dimensional algebra of canonical transformation in a two-dimensional phase space $(p, q)$. It is also a
symmetry of the relativistic membranes after gauge fixing - this symmetry and it connection with the $SU(\infty)$ were considered in [17], [18].

The aim of this paper is to discuss the connection between quantum symmetry, magnetic translations and area-preserving diffeomorphisms in Landau problem and to discuss the $U_q(sl(2))$ symmetry in $2+1$ gauge theories with Chern-Simons terms and quantum Hall systems. Let us note that for general $q$ the irreducible representations of $U_q(sl(2))$ are qualitatively the same as in $sl(2)$ case [19]. However in the case $q^n = 1$ there is only finite number of irreducible representations [20] and the new classification of the states in TMGT and quantum Hall systems exists.

The organization of the paper is as follows. In the next section, which bears essentially a review character, we consider the Landau problem and discuss the relations between canonical transformations, magnetic translations and the quantum algebra $U_q(sl(2))$. We shall demonstrate how after the restriction on the first Landau level the symmetry under the area-preserving diffeomorphisms will appear. We shall demonstrate how the this symmetry is connected with quantum group symmetry and will discuss the construction of quantum algebra $U_q(sl(2))$ from the group of magnetic translations. At the end of this section this construction will be generalized to the case of Fairlie-Fletcher-Zachos triginometric algebras with nontrivial central extension [18], which can not be reduced to any magnetic translations. In section 3 the canonical quantization and Landau levels picture in topologically massive gauge theory will be considered and the action of the quantum group on the Hilbert space will be obtained. The corresponding problem, as we shall show, is equivalent to the Landau problem on a torus and we shall consider several examples of different representations of $U_q(sl(2))$. It will be shown that the natural value for deformation parameter $q = \exp(4\pi i/k)$ where $k$ is the Chern-Simons coefficient. In the section 4 we shall consider the quantum group symmetry in a quantum Hall system. We shall remember how $W_\infty$ (to be more precise $W_{1+\infty}$ appears in a quantum Hall system due to the incompressibility of the ground state and then generalize our construction of $W_\infty$ and $U_q(sl(2))$ on many-body case and will present arguments in favour of deformation parameter $q = \exp(2\pi i\nu)$, where $\nu$ is a filling factor. We shall also consider the action of $U_q(sl(2))$ on ground state and lowest excitations. In conclusion the obtain results will be discussed as well as some possibilities to find the quantum group symmetry in other physical models invariant under the area-preserving diffeomorphisms and $W_\infty$, such as $c = 1$ strings and two-dimensional Yang-Mills theory.
2. $W_\infty$, magnetic translations and $\mathcal{U}_q(sl(2))$ in Landau problem

2.1. Landau problem

The action for Landau problem, i.e. for particle with mass $m$ moving on the $(x_1, x_2)$ plane in a magnetic field $B$ is

$$S_L = \frac{m}{2} \int \dot{x}_i^2 dt + \int A_i \dot{x}_i dt = \frac{m}{2} \int \dot{x}_i^2 dt + \frac{B}{2} \int (x_2 dx_1 - x_1 dx_2)$$

(2.1)

where we choose a vector potential in a symmetric gauge $A_i = (B/2) \epsilon_{ij} x_j$.

Let us consider the configuration and phase spaces of this problem. The canonical momenta:

$$p_i = \frac{\partial L}{\partial \dot{x}_i} = m \dot{x}_i + \frac{B}{2} \epsilon_{ij} x_j$$

(2.2)

with the usual commutation relations (or Poisson brackets in the classical limit $[,] \to -i\{,\}$)

$$[p_i, p_j] = [x_i, x_j] = 0; \quad [p_i, x_j] = -i \delta_{ij}$$

(2.3)

The canonical Landau Hamiltonian and the eigenvalues $E_n$ are

$$H = \frac{1}{2m} \left( p_i - \frac{B}{2} \epsilon_{ij} x_j \right)^2 = -\frac{1}{2m} \left( \frac{\partial}{\partial x_i} - i \frac{B}{2} \epsilon_{ij} x_j \right)^2$$

$$E_n = (n + 1/2) \frac{B}{m}$$

(2.4)

Let us notice that Hamiltonian (2.4) depends effectively only on two coordinates in the phase space, not four. Introducing the variables $a$ and $a^+$ depending on $(p_i - A_i)$ only

$$a^+ = \left( p_1 - \frac{B}{2} x_2 \right) + i \left( p_2 + \frac{B}{2} x_1 \right) = 2p_z + i \frac{B}{2} \bar{z}$$

$$a = \left( p_1 - \frac{B}{2} x_2 \right) - i \left( p_2 + \frac{B}{2} x_1 \right) = 2p_z - i \frac{B}{2} \bar{z}$$

(2.5)

one gets the Hamiltonian

$$H = \frac{1}{4m} (aa^+ + a^+ a), \quad [a, a^+] = 2B$$

(2.6)
Here \( z(\bar{z}) = x_1 \pm ix_2 \) and \( p_{z, \bar{z}} = -i\partial/\partial z(\bar{z}) \) are the corresponding conjugate momenta. There is another pair \( b \) and \( b^+ \), which depends on \( (p_i + A_i) \), commuting with \( a \) and \( a^+ \)

\[
\begin{align*}
\frac{b^+}{2} &= p_1 + B x_2 - i \left( p_2 - \frac{B}{2} x_1 \right) = 2p_z + i\frac{B}{2} \bar{z} \\
\frac{b}{2} &= p_1 + B x_2 + i \left( p_2 - \frac{B}{2} x_1 \right) = 2p_{\bar{z}} - i\frac{B}{2} z
\end{align*}
\]  

(2.7)

with commutation relation \([b, b^+] = 2B\). The angular momentum operator can be written as

\[ J = b^+ b - a^+ a \]  

(2.8)

One can see that \( b^+ \) and \( a \) increase and \( b \) and \( a^+ \) decrease the angular momentum.

It is easy to see that the states on the first Landau level \(|1\rangle\) are annihilated by \( a \) and has the form

\[ a|1\rangle = 0, \quad <z, \bar{z}|1\rangle = \Psi_1(z, \bar{z}) = \Phi(\bar{z}) \exp(-\frac{B}{4} z\bar{z}) \]  

(2.9)

where \( \Phi(\bar{z}) \) is an arbitrary antiholomorphic function and operators \( b \) and \( b^+ \) do not change the level number when acting on states at a given level. For a state with angular momentum \( l \) one has \( \Phi_l(\bar{z}) = \bar{z}^l \). One can consider another basis, parametrized by a momentum in \( x_1 \) (or any other) direction

\[
\begin{align*}
\Psi_1(p|x) &= \exp \left( -i\frac{B}{2} x_1 x_2 + ipx_2 \right) \exp \left( -\frac{B}{2} (x_2 - p)^2 \right) = \\
&= \exp(-\frac{B}{4} z\bar{z}) \Phi_p(\bar{z}) = \exp(-\frac{B}{4} z\bar{z}) \exp \left( -\frac{B}{2} p^2 + ip\bar{z} + \frac{B}{4} \bar{z}^2 \right)
\end{align*}
\]  

(2.10)

where momentum is defined as \( Bp \).

Let us note that one can consider the restriction on the first Landau level taking the limit \( m \to 0 \). In this limit one gets from (2.7)

\[ 2p_z = p_1 + ip_2 = -i\frac{B}{2} \bar{z}; \quad 2p_{\bar{z}} = p_1 - ip_2 = i\frac{B}{2} z \]  

(2.11)

and then

\[ a = a^+ = 0; \quad b^+ = iB\bar{z}; \quad b = -iBz \]  

(2.12)

The physical meaning of this reduction is the follows - operators \( a \) and \( a^+ \) acting on the state at the given level \( n \) shift it to \( n \mp 1 \). To be at a first level we must put these operators
to zero after which $b$ and $b^+$ play the role of the coordinate on the reduced phase space. Let us also note that the action (2.1) in the limit $m \to 0$ transforms into
\[
S_{m=0} = \frac{B}{2} \int (x_1 \dot{x}_2 - x_2 \dot{x}_1) dt = \frac{B}{2} \int (x_2 dx_1 - x_1 dx_2)
\] (2.13)
and one can easily see that $x_1$ and $x_2$ becomes the coordinates on the phase space, the action in the case of the closed trajectories proportional to the area $A = \oint (x_2 dx_1 - x_1 dx_2)$ and is invariant under the action of the area-preserving diffeomorphisms. The last one are nothing but a canonical transformation on the first Landau level. Let us also note that the connection between Chern-Simons theory and Landau problem in the limit $m \to 0$, i.e. reduction on the lowest level, where discussed in [21].

### 2.2. Canonical transformation on the first Landau level, $W_\infty$ algebra and the group of magnetic translations

Let us remember some well-known facts about canonical transformations (see, for example [22]). By definition canonical transformations are diffeomorphisms of the phase space which preserve the symplectic structure $\omega = \sum p_i dq_i \wedge dp_i$.

The canonical transformations are usually defined by the generating function depending on both old ($p$ or $q$) and new ($P$ and $Q$) phase space coordinates, for example one can consider arbitrary $F(q, Q)$ and put
\[
p_i = \partial F/\partial q_i; \quad P_i = -\partial F/\partial Q_i
\] (2.14)
It is easy to see that $P, Q$ are new canonical coordinates. There is however another representation, namely one can consider evolution with respect to some ”Hamiltonian” $W(p, q)$ (which is an arbitrary function on the phase space and has nothing common with the physical Hamiltonian). The change in quantities $p$ and $q$ during this evolution may itself be regarded as a series of canonical transformations. Let $p$ and $q$ be the values of the canonical variables at time $t$ and $P$ and $Q$ are their values at another time $t + \tau$. The latter are some function of the former, depending on $\tau$ as on parameter
\[
Q = Q(q, p; \tau), \quad P = P(q, p; \tau)
\] (2.15)
These formulae can be considered as the canonical transformation from the old coordinates $p, q$ to the new ones $P, Q$. This representation is convenient for the infinitesimal transformation, when $\tau \to 0$. In this case using Hamiltonian equations of motion with
"Hamiltonian" $W(p, q)/\tau$ one gets

$$Q_i = q_i + \dot{q}_i \tau = q_i + \{q_i, W\}; \quad P_i = p_i + \dot{p}_i \tau = p_i + \{p_i, W\}$$ (2.16)

where

$$\{A, B\} = \sum_i \frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial B}{\partial q_i} \frac{\partial A}{\partial p_i}$$ (2.17)

is the usual Poisson brackets.

The canonical transformations acting on the two-dimensional phase space $(q, p)$ are defined by

$$\delta q = \{q, W(p, q)\} = \frac{\partial W(p, q)}{\partial p}; \quad \delta p = \{p, W(p, q)\} = - \frac{\partial W(p, q)}{\partial q}$$ (2.18)

where $W(p, q)$ is an arbitrary function. The fact that these transformations preserves the area can be easily checked using the fact that the general infinitesimal area-preserving diffeomorphism takes the form

$$x_i \rightarrow x_i + \xi_i(x); \quad \partial_i \xi_i = 0$$ (2.19)

where $x_i = (q, p)$. General solution of $\partial_i \xi_i = 0$ is the sum of the two terms

$$\xi_i(x) = \epsilon_{ij} \partial_j W(x) + \sum_{a=1}^{b_1} c_a \xi_i^a$$ (2.20)

where first term describes infinite number (all possible functions $W(\xi)$) of the local coexact solutions and the second term describes the finite number (given by the first Betti number $b_1$) of the harmonic forms on two-dimensional phase space. It is easy to see that diffeomorphisms generated by the first term are nothing but canonical transformations (2.18).

Any function $f$ on the phase space is transformed under the canonical transformation generated by $W$ according to the rule $w f = \delta_W f = \{f, W\}$, where $w$ is the operator corresponding to function $W(\xi)$. Using the Jacobi identity $\{\{f, W_1\} W_2 - \{f, W_2\} W_1\} + \{\{W_1, W_2\} f\} = 0$ one can check that algebra of the area-preserving diffeomorphisms is given by the Poisson brackets

$$[w_1, w_2] f = [\delta_{W_1}, \delta_{W_2}] f = \delta_{\{W_1, W_2\} f}$$ (2.21)
Any function $W$ can be written in terms of the complete set of harmonics

$$W_{\vec{n}} = e^{x(n\vec{x})}$$  \hspace{1cm} (2.22)

where $\vec{n} = (n_1, n_2)$ with real $n_1, n_2$ in the case when the phase space is a plane. If the phase space is a torus (as it will be in the case of TMGT or QHE on torus) one has integers $n_1, n_2$ if one defines a torus phase space $T^2$ to be a square with both sides equal to $2\pi$. One gets the commutation relations for operators $w_{\vec{n}}$ computing the Poisson bracket for $W_{\vec{n}}$, \cite{17}

$$[w_{\vec{m}}, w_{\vec{n}}] = (\vec{m} \times \vec{n}) w_{\vec{m}+\vec{n}}$$  \hspace{1cm} (2.23)

where $\vec{a} \times \vec{b} = a_1 b_2 - a_2 b_1$. One can see that (2.23) is nothing but the commutation relations for the classical $w_\infty$ algebra \cite{13}.

Let us now consider canonical transformations acting on the Landau problem phase space. The general canonical transformations are acting on the whole four-dimensional phase space and after quantization they will mix different Landau levels. However there is a special subgroup of the canonical transformations acting on the two-dimensional subspace of the phase space generated by commuting with the Hamiltonian. This means that this transformations do not mix different Landau levels and thus acting on each Landau level as on two-dimensional phase space. It is evident that generators of this area-preserving (symplectic) transformations will depend on $b$ and $b^{+}$ (see (2.7)) which commute with Hamiltonian (2.6).

After quantization we get instead of (2.22) the quantum version

$$\mathcal{W}_{n,\bar{n}} = \exp\left(\frac{1}{2}(nb^{+} - \bar{n}b)\right) = \exp\left(\frac{1}{2}nb^{+}\right)\exp\left(-\frac{1}{2}\bar{n}b\right)\exp\left(-\frac{B}{4}n\bar{n}\right);$$

$$[\mathcal{W}_{n,\bar{n}}, \mathcal{W}_{m,\bar{m}}] = -2i\sin\frac{B}{2}(n_1 m_2 - n_2 m_1)\mathcal{W}_{n+m,\bar{n}+\bar{m}}$$  \hspace{1cm} (2.24)

Here $n(\bar{n}) = n_1 \pm in_2$ and the classical limit corresponds to weak magnetic field $B \to 0$ after obvious rescaling of $\mathcal{W}_{n,\bar{n}} \to Bw_{n,\bar{n}}$. For integer $n$ and $m$ the algebra (2.24) is the Fairlie-Fletcher-Zachos (FFZ) trigonometric algebra \cite{18}.

Let us note that this is precisely the algebra of magnetic translations \cite{7} which (in a
gauge $A_i = (B/2)\epsilon_{ij}x_j$ is generated by the operators\footnote{Let us note that the definition of magnetic translations is gauge dependent - $\vec{\nabla} + i\vec{A}$ commute with a Hamiltonian only in a symmetric $A_i = (B/2)\epsilon_{ij}x_j$ gauge. In general case one must add some gauge dependent terms in (2.25). One can show, however, that the algebra of magnetic translation does not depend on gauge and later we shall work only in the symmetric gauge.}

$$T_\xi = \exp \left( \xi \left( \vec{\nabla} + i\vec{A} \right) \right), \quad T_\xi T_\eta = \exp \left( -i \frac{B}{2} (\xi \times \eta) \right) T_{\xi + \eta}, \quad (2.25)$$

Substituting (2.7) into (2.24) one find that

$$W_{n,\bar{n}} = T_\xi, \quad \xi_i = \epsilon_{ij} n_j \quad (2.26)$$

and the action on the first level wave functions (2.9) is as follows:

$$T_\xi \Psi_1(\vec{x}) = \exp \left( i \frac{B}{2} (\xi \times \vec{x}) \right) \Psi_1(\vec{x} + \vec{\xi}) = \exp \left( \frac{B}{4} (\bar{\xi}z - \xi \bar{z}) \right) \Psi_1(z + \xi, \bar{z} + \bar{\xi}) \quad (2.27)$$

Later we shall use a notation $|p\rangle$ for wave function $\Psi_1(p|\vec{x})$.

The FFZ algebra (2.24) commutes with the Hamiltonian $H \sim (aa^+ + a^+a)$ and thus acts independently on each Landau level. One can construct another $W_\infty$ algebra from $a, a^+$ operators which we shall call $\tilde{W}_\infty$ and gets: $W_\infty \otimes \tilde{W}_\infty$. The first algebra acts on each Landau level, the second one (tilde) algebra mixes the level and acts in a simple form onto the coherent states $|\alpha, \bar{\alpha}\rangle \sim \exp (\alpha a^+) |0\rangle$.

One can consider another form of generators [15], [16]

$$\mathcal{L}_{n,m} = (b^+)^{n+1}b^{m+1}, \quad n, m \geq -1 \quad (2.29)$$

with commutation relations

$$[\mathcal{L}_{n,m}, \mathcal{L}_{k,l}] = 2B \left( (m + 1)(k + 1) - (n + 1)(l + 1) \right) \mathcal{L}_{n+k,m+l} + O(B^2) \quad (2.30)$$

This algebra is called $W_{1+\infty}$ in the literature (see [16] and references therein). After obvious rescaling one has classical $w_\infty$ in the limit $B \to 0$. It is easy to see that expanding generators $W_{n,\bar{n}}$ (2.24) in $n, \bar{n}$ one gets generators $\mathcal{L}_{n,m}$ as expansion coefficients.

$$W_{n,\bar{n}} = \exp (-B/4 n\bar{n}) \sum_{k,l=0}^{\infty} (-1)^l \frac{n^k}{2^k k!} \frac{\bar{n}^l}{2^l l!} \mathcal{L}_{k-1,l-1} \quad (2.31)$$
Let us note that FFZ algebra (2.24) is a Weyl-Moyal [23] deformation of the Poisson-Lie algebra with usual Poisson brackets \( \{ f, g \} = \partial_z f \partial_{\bar{z}} g - \partial_{\bar{z}} f \partial_z g \). The Moyal bracket \( \{ \cdot, \cdot \}_M \) is defined as follows:

\[
\{ f, g \}_M = \sum_{s=0}^{\infty} \frac{(-)^s B^{2s}}{(2s + 1)!} \sum_{j=0}^{2s+1} (-j) \left( \begin{array}{c} 2s + 1 \\ j \end{array} \right) \left[ \partial^j_z \partial_{2s+1-j} f \right] \left[ \partial_{2s+1-j} \partial^j_{\bar{z}} g \right]
\]  
\text{(2.32)}

It is easy to check for \( W_n, \bar{n} = \exp[1/2(nz - \bar{n}\bar{z})] \) one gets the classical \( w_\infty \) algebra (2.23) for usual Poisson brackets \( \{ \cdot, \cdot \} \) and FFZ algebra (2.24) for Moyal brackets \( \{ \cdot, \cdot \}_M \).

2.3. \( GL_q(2) \) and \( U_q(sl(2)) \) in Landau problem

There is a natural connection between the FFZ algebra (2.23) [24], [25] and quantum group \( GL_q(2) \). Let us consider the quantum plane

\[ UV = qVU \]  
\text{(2.33)}

and introduce a quantum group \( GL_q(2) \) which is defined as \( 2 \times 2 \) matrices

\[
L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]  
\text{(2.34)}

where the matrix elements \( a, b, c, d \) obey relations

\[
ab = q^{-1}ba, \quad ac = q^{-1}ca, \quad cd = q^{-1}dc, \quad bd = q^{-1}db, \quad bc = cb, \quad ad - da = (q^{-1} - q)bc
\]  
\text{(2.35)}

It was shown in [24] that \( GL_q(2) \) is the group of automorphisms of the quantum plane (2.33), i.e. \( U'V' = qV'U' \), where \( U' \) and \( V' \) are the images of the action of \( GL_q(2) \) on \( U \) and \( V \). It is easy to see that \( T_n = q^{n_1n_2} U^{n_1} V^{n_2} \) with \( q = \exp(iB) \) are generators of the FFZ algebra and thus \( GL_q(2) \) naturally acts on this algebra preserving the commutation relations (2.24).

It is amusing that one can construct also quantum algebra \( U_q(sl(2)) \) from the elements of the magnetic translations group (2.25) (see [8], [11] and references therein). The commutation relations of \( U_q(sl(2)) \) are defined as follows [8], [11]:

\[
q^{J_3} J_{\pm} q^{-J_3} = q^{\pm 1} J_{\pm}
\]

\[
[J_+, J_-] = \frac{q^{2 J_3} - q^{-2 J_3}}{q - q^{-1}}
\]  
\text{(2.36)}
where the first relation equivalent to usual commutation relation \([J_3, J_{\pm}] = \pm J_{\pm}\). This algebra is transformed into an ordinary Lie algebra \(sl(2)\) in the limit \(q \to 1\).

The construction of the \(J_{\pm}\) and \(J_3\) generators from the given magnetic translations group (2.23) is not unique. One can get arbitrary value for a deformation parameter \(q\) in a general case. However later we shall demonstrate that in physically interesting situations like quantum Hall effect or 2 + 1 dimensional gauge theory the choice of \(J_{\pm}\) and \(J_3\) will be dictated by additional physical arguments and there will be some ”natural” choice of parameters.

Let us present the following construction depending on two arbitrary noncollinear vectors \(\vec{a}\) and \(\vec{b}\) on a plane and four complex parameters \(\alpha, \beta, \gamma, \delta\). Considering the following superpositions of magnetic translation generators:

\[
J_+ = \frac{1}{q - q^{-1}} \left( \alpha \, T_\vec{a} + \beta \, T_\vec{b} \right), \quad J_- = \frac{1}{q - q^{-1}} \left( \gamma \, T_{-\vec{a}} + \delta \, T_{-\vec{b}} \right)
q^{2J_3} = T_{\vec{b} - \vec{a}}, \quad q^{-2J_3} = T_{\vec{a} - \vec{b}}
\]

with

\[
q = \exp \left( \frac{iB}{2} (\vec{a} \times \vec{b}) \right)
\]

and calculating the commutation relations for the \(J_{\pm}\) and \(J_3\) using (2.25) one can easily reproduce (2.36) if \(\alpha \delta = \beta \gamma = -1\).

In the end of this section let us note that that action of \(U_q(sl(2))\) generators \(J_{\pm}\) and \(J_3\) on the wave functions on the first Landau level (generalization on the case of arbitrary level \(n\) is straightforward) depends (even after fixing \(\alpha, \beta, \gamma, \delta\)) both on choice of a fundamental cell \((\vec{a}, \vec{b})\) and a basis of the wavefunctions. We shall present here the action of \(U_q(sl(2))\) on basic wave functions (2.10) for a generic \((\vec{a}, \vec{b})\) for \(\alpha = \gamma = 1, \beta = \delta = -1\). Using (2.28) one can find

\[
J_+|p> = \frac{\exp \left( +iBpa_1 - \frac{i}{2}Ba_1a_2 \right) |p - a_2> - \exp \left( +iBpb_1 - \frac{i}{2}Bb_1b_2 \right) |p - b_2>}{q - q^{-1}}
q^{2J_3}|p> = \exp \left( \pm iBp(b_1 - a_1) - \frac{i}{2}B(a_1 - b_1)(a_2 - b_2) \right) |p \pm (a_2 - b_2)>
\]

\[
J_-|p> = \frac{\exp \left( -iBpa_1 - \frac{i}{2}Ba_1a_2 \right) |p + a_2> - \exp \left( -iBpb_1 - \frac{i}{2}Bb_1b_2 \right) |p + b_2>}{q - q^{-1}}
q^{2J_3}|p> = \exp \left( \pm iBp(b_1 - a_1) - \frac{i}{2}B(a_1 - b_1)(a_2 - b_2) \right) |p \pm (a_2 - b_2)>
\]

Acting many times by operators \(J_{\pm}\) and \(q^{2J_3}\) on state \(|p>\) in general case when \(a_2\) and \(b_2\) are two incommensurable real numbers, one can obtain the state \(|p'>\) with
\( p' = p \pm n_1a_2 \pm n_2b_2 \) arbitrary close to \( p \) for large enough \( n_1 \) and \( n_2 \). It is possible however to choose the fundamental cell in a more restrictive way. Choosing, for example, \( a_2 = b_2 = b \) we see that \( J_{\pm} \) acting on \( |p> \) create one, not two as in general case, state \( |p \pm b> \) and \( q^{\pm J_3} \) are now diagonal. The natural choice is to take \( a_1 = -b_1 = a \), i.e. to have

\[
\vec{a} = (a, b), \quad \vec{b} = (-a, b), \quad q = \exp(iBab)
\]

In this case we get very simple representation

\[
J_+|p> = \left[ \frac{p - b}{2} \right]_q |p - b>, \quad J_-|p> = -\left[ \frac{p + b}{2} \right]_q |p + b>, \quad q^{\pm J_3}|p> = q^{\mp i\vec{b}}|p>
\]

where the quantum symbol \([x]_q\) is defined as

\[
[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}
\]

Let us calculate the value of the \( q \)-dimension of the representation which is defined as the value of the \( q \)-deformed Casimir operator

\[
C_q = J_-J_+ + \left[ J_3 + \frac{1}{2} \right]^2_q = J_+J_- + \left[ J_3 - \frac{1}{2} \right]^2_q
\]

When acting on the highest weight vector \( |j> \), such that \( J_+|j> = 0 \), \( J_3|j> = j|j> \), one gets \( C_q|j> = [j + 1/2]^2_q|j> \) which gives us the \( q \)-dimension of representation. However representation (2.41) is neither of highest nor of lowest weight and calculating its \( q \)-dimension

\[
C_q|p> = -\left[ \frac{p - b}{2} \right]_q |p> + \left[ \frac{1}{2} - \frac{p}{b} \right]_q |p> = 0
\]

we get zero.

Let us note that one gets the zero \( q \)-dimension for highest weight representation \( |j> \) with \( j = (nk - 1)/2 \) for \( q \) being root of unity \( q^k = 1 \) because \([j + 1/2]_q = [nk/2]_q = 0 \) in this case. The representation theory for \( q \) being root of unity was considered in [20], see also [26]. There are two types of representations:

- Type-I representations which have \( q \)-dimension zero and are either mixed, i.e. not highest weight representations, or irreducible highest weight representations with \( j = (nk - 1)/2 \).
• Type-II representations with nonzero $q$-dimension which are irreducible highest weight representations with $0 \leq j < k/2 - 1$.

In our case the deformation parameter $q$ is arbitrary and depends on our choice of fundamental cell, to be more precise it depends on the flux $\Phi = 2Bab$ through this cell $q = \exp(i\Phi/2)$. For $q^k = 1$ one must have $Bab = 2\pi/k$. In the case of Azbel-Hofstadter problem which was considered in [8], [11] the fundamental cell is the minimal plaquette on the lattice and the interesting case is when the flux through plaquette is rational. Here we shall consider two other problems - 2 + 1 abelian TMGT and QHE where the choice of $q$ also will be dictated by rational numerical parameters of the corresponding problem - the value of the numerical coefficient in front of Chern-Simons term or the filling factor in quantum Hall system. One can see that the special values of $j$ corresponds to $p = (nk/2 - 1/2)b$. Remembering that momentum in $x_1$ direction (see (2.10)) is $Bp = (nk/2 - 1/2)bB = (2\pi/a)(n/2 - 1/2k)$ we get discrete momenta. There is a natural appearence of discrete momenta in Landau problem on cylinder or torus and as we shall see in the next sections these are precisely the cases which we shall be interested in.

2.4. $U_q(sl(2))$ from a central extension of the FFZ trigonometric algebra

It is known that a classical algebra of the area-preserving diffeomorphisms on a torus has a central extension (which does not exist in the case of the area-preserving diffeomorphisms on a sphere) [7], [8]

\[ [w_{\bar{n}}, w_{\bar{m}}] = (\bar{n} \times \bar{m}) w_{\bar{m}+\bar{n}} + \bar{a}\bar{n}\delta_{\bar{m}+\bar{n},0} \] (2.45)

as well as a trigonometric FFZ algebra

\[ [\mathcal{W}_{\bar{n}}, \mathcal{W}_{\bar{m}}] = -2i \sin \frac{B}{2}(\bar{n} \times \bar{m})\mathcal{W}_{\bar{m}+\bar{n}} + \bar{a}\bar{n}\delta_{\bar{m}+\bar{n},0} \] (2.46)

where the central element is given by an arbitrary vector $\bar{A}$.

The algebra (2.46) can not be obtained from magnetic translations (2.25) because now $\mathcal{W}_{\bar{n}}$ and $\mathcal{W}_{\bar{m}}$ do not commute. However we still can construct the quantum algebra $U_q(sl(2))$ from trigonometric FFZ algebra with a central extension $\bar{a}$ using the same construction as before

\[
J_+ = \frac{1}{q - q^{-1}} (\alpha \mathcal{W}_{\bar{n}} + \beta \mathcal{W}_{\bar{m}}), \quad J_- = \frac{1}{q - q^{-1}} (\gamma \mathcal{W}_{\bar{n}} - \delta \mathcal{W}_{\bar{m}}) \\
q^{2J_3} = \mathcal{W}_{\bar{m} - \bar{n}}, \quad q^{-2J_3} = \mathcal{W}_{\bar{n} - \bar{m}} \quad q = \exp \left( + \frac{B}{2} (\bar{n} \times \bar{m}) \right) \] (2.47)
Calculating the commutation relations using (2.46) we get (putting as before $\alpha \delta = \beta \gamma = -1$)

$$[J_+, J_-] = \frac{q^{2J_3} - q^{-2J_3}}{q - q^{-1}} + \frac{1}{(q - q^{-1})^2} \vec{a} (\alpha \gamma \vec{n} + \beta \delta \vec{m})$$

$$[q^{-2J_3}, q^{+2J_3}] = [\mathcal{W}_{\vec{n} - \vec{m}}, \mathcal{W}_{\vec{m} - \vec{n}}] = \vec{a}(\vec{n} - \vec{m})$$

and a commutation relations between $J_\pm$ and $J_3$ are not affected by a central extension $\vec{a}$. To get the $U_q(sl(2))$ commutation relations (2.36) we must have

$$\vec{a}(\vec{n} - \vec{m}) = 0, \quad \alpha \gamma + \beta \delta = 0, \quad \alpha \delta = \beta \gamma = -1$$

i.e. the vector $\vec{a}$ must be orthogonal to the vector $\vec{n} - \vec{m}$ and there are 3 constraints (complex) for four parameters (complex) $\alpha, \beta, \gamma \delta$

$$\gamma = -\frac{1}{\beta^2}, \quad \delta = -\frac{1}{\alpha}, \quad \alpha^2 + \beta^2 = 0$$

leaving us with one-parameter family

$$\alpha = \rho e^{i \phi}, \quad \beta = \rho e^{i \phi \pm i \pi / 2}, \quad \gamma = \rho^{-1} e^{-i \phi \pm i \pi / 2}, \quad \delta = \rho^{-1} e^{-i \phi \pm i \pi}$$

We see that in a case of a nontrivial central extension one has to deal with more restrictions when constructing a quantum algebra. First of all the choice of the fundamental cell $\vec{n}, \vec{m}$ is no longer arbitrary but the basic vectors must be chosen in a way that $\vec{n} - \vec{m}$ is orthogonal to the vector $\vec{a}$. Besides this the four parameters $\alpha, \beta, \gamma \delta$ are completely determined by one (complex) parameter (for example, $\alpha$), contrary to a case $\vec{a} = 0$ with two independent parameters (for example $\alpha$ and $\beta$).

3. Quantum symmetry in $2+1$ gauge theory with Chern-Simons term

3.1. Canonical quantization of the $2+1$ TMGT

Let us consider an abelian topologically massive gauge theory [4]:

$$S_{U(1)} = -\frac{1}{4\gamma} \int \sqrt{-g} g^{\mu \alpha} g^{\nu \beta} F_{\mu \nu} F_{\alpha \beta} + \frac{k}{8\pi} \int \epsilon_{\mu \nu \lambda} A_\mu \partial_\nu A_\lambda$$

(3.1)
To perform canonical quantization let us chose a \(A_0 = 0\) gauge. Representing vector-potential on a plane as \(A_i = \partial_i \xi + \epsilon_{ij} \partial_j \chi\) and substituting this decomposition into constraint

\[
\frac{1}{\gamma} \partial_i \dot{A}_i + \frac{k}{4\pi} \epsilon_{ij} F_{ij} = 0,
\] (3.2)

one gets \(\partial^2 \dot{\xi} = (k\gamma/2\pi) \partial^2 \chi\). Neglecting all possible zero modes we put \(\dot{\xi} = (k\gamma/2\pi)\chi = (M/2)\chi\). Substituting this constraint into action (3.1) one gets

\[
S = \frac{1}{2} \int \frac{1}{2} (\partial_i \dot{\chi})^2 - (\partial^2 \chi)^2 - M^2 \chi \partial^2 \chi
\] (3.3)

which becomes a free action for the field \(\Phi = \sqrt{\partial^2 / \gamma \chi}\)

\[
S = \frac{1}{2} \int \dot{\Phi}^2 - (\partial_i \Phi)^2 - M^2 \Phi^2
\] (3.4)

describing the free particle with mass \(M = \gamma k/4\pi\). In obtaining this action we used the constraint (3.2). However there are some field configurations which are escaped from this constraint. It is easy to see that on the plane the spatial independent fields \(A_i(x, t) = A_i(t)\) are not affected by (3.2) - because both terms \(F_{ij}\) and \(\partial_i E_i\) are zero for space-independent vector potential (but not electric field \(E_i = \dot{A}_i\)). For these fields one gets the Landau Lagrangian (2.1)

\[
L = \frac{1}{2\gamma} \dot{A}_i^2 - \frac{k}{8\pi} \epsilon_{ij} A_i \dot{A}_j
\] (3.5)

which describes the particle with mass \(m = \gamma^{-1}\) on the plane \(A_1, A_2\) in a magnetic field \(B = k/4\pi\). From (2.4) the mass gap is \(M = B/m = \gamma k/4\pi\) which is precisely the mass of gauge particle.

Let us note that \(A_1\) and \(A_2\) belong to the configuration space, however if reduced to the first Landau level the configuration space is transformed into the phase space as we have discussed before. In this case a reduction to the first Landau level means \(m = 1/\gamma \to 0\), i.e. the reduction to the pure Chern-Simons theory which is an exactly solvable 2 + 1 dimensional topological field theory.

Is it possible to consider a constant gauge field as a physical, i.e. gauge invariant variable in the theory? Can one simply gauge away the constant field? To answer this question we must define the boundary conditions at infinity, i.e. to compactify our plane into a 2-dimensional Riemann surface of genus \(g\). It is well-known that any one-form \(A\) can be uniquely decomposed according to Hodge theorem as

\[
A = d\xi + \delta \chi + A, \quad dA = \delta A = 0
\] (3.6)
which generalizes the decomposition on the plane we have used before. The harmonic form $A$ equals

$$A = \sum_{p=1}^{g} (A^p \alpha_p + B^p \beta_p) \quad (3.7)$$

where $\alpha_p$ and $\beta_p$ are canonical harmonic 1-forms (1-cohomology) on a Riemann surface and there are precisely $2g$ harmonic 1-forms on genus $g$ Riemann surface (two in case of a torus which are these two constant modes we have discussed). After diagonalization one finds that there are $g$ copies of the Landau problem and the total Hilbert space $\mathcal{H}$ of the abelian topologically massive gauge theory

$$\mathcal{H} = \mathcal{H}_\Phi \otimes \prod_{i=1}^{g} \mathcal{H}_A \quad (3.8)$$

is the product of the free massive particle Hilbert space $\mathcal{H}_\Phi$ and $g$ copies of the Landau problem’s Hilbert space $\mathcal{H}_A$.

There is a dependence on the moduli of the Riemann surface due to the dependence of the $F^2$ term in (3.1) on metric $g_{\mu\nu}$. It is easy to see that for $g_{00} = 1$ and $g_{ij} = \rho(x)h_{ij}(\tau)$ the $F^2_{\Phi}$ term does not depend on conformal factor $\rho$. Let us consider dependence on the moduli $\tau$ in the most simple case of a torus where $\tau$ is a complex number and metric $h_{ij}$ can be parametrized as

$$h^{ij} = \frac{1}{(Im\tau)^2} \begin{pmatrix} 1 & Re \tau \\ Re \tau & |\tau|^2 \end{pmatrix} \\ h_{ij} = \begin{pmatrix} |\tau|^2 & -Re \tau \\ -Re \tau & 1 \end{pmatrix} \quad (3.9)$$

and $h = det h_{ij} = (Im\tau)^2$. Lagrangian takes the form

$$L = \frac{1}{2\gamma} \sqrt{h} h^{ij} \dot{A}_i \dot{A}_j - \frac{k}{8\pi} e^{ij} A_i \dot{A}_j \quad (3.10)$$

which can be transformed to diagonal form (3.3) for new fields

$$A_{(a)} = e^i_{(a)} A_i \quad (3.11)$$

where zweibein $e^i_{(a)}$ defines the metric $h^{ij} = e^i_{(a)} e^j_{(b)} \delta^{(a)(b)}$ and $\epsilon^{(a)(b)} e^i_{(a)} e^j_{(b)} \sim \epsilon^{ij}$. It is easy to find that

$$\begin{pmatrix} A_{(1)} \\ A_{(2)} \end{pmatrix} = \begin{pmatrix} 1 & Re \tau \\ 0 & Im \tau \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \quad (3.12)$$
In terms of the new variables the Lagrangian (3.10) takes the form

\[ L = \frac{1}{2\gamma Im\tau} \dot{A}_i^2 - \frac{k}{8\pi Im\tau} \epsilon^{(i)(j)} A_i \dot{A}_j \]  

(3.13)

and we see that the Chern-Simons coefficient depends on moduli: \( k \rightarrow k/Im\tau \). However the mass gap is unchanged because \( \gamma \) is also changed \( \gamma \rightarrow \gamma Im\tau \) and \( M = \gamma k/4\pi \) does not depend on \( \tau \).

Thus we get the Landau problem on the plane \( (A_{(1)}, A_{(2)}) \). However we forgot about large gauge transformations acting on the quantum-mechanical coordinate \( A_i \rightarrow A_i + 2\pi N_i \), where \( N_i \) are integers. These transformations act on gauge potential because the only gauge-invariant objects one can construct for \( A_i \) - Wilson lines

\[ W(C) = exp(i \oint_C A_\mu dx^\mu) \]  

(3.14)

are invariant under these transformations (we choose coordinate on a torus in a way that \( x^1 \sim x^1 + 1 \) and \( x^2 \sim x^2 + 1 \)) and one can consider torus \( 0 \leq A_i < 2\pi \) with the area \((2\pi)^2\). However after we consider the new variables \( A_{(i)} \) one gets the torus (see (3.12)) generated by the shifts \( 2\pi \) and \( 2\pi\tau \) with an area \( S = (2\pi)^2 Im\tau \).

Let us note that being reduced to the first Landau level this torus becomes the phase space - thus for the consistent quantization this area must be proportional to the integer (the total number of the states must be integer). It is known that the density of states \( \rho \) on Landau level equals to \( B/2\pi \), where \( B \) is a magnetic field. In our case the "magnetic field" in \( (A_{(1)}, A_{(2)}) \) plane can be easily obtained from (3.13) and equals to \( B = (k/4\pi) Im\tau \), thus the total number of states will be \( N = (1/2\pi)(k/4\pi Im\tau) \times (2\pi)^2 Im\tau = k/2 \) and does not depend on \( \tau \) but only on \( k \).

We see that it is possible to factorize over whole large gauge transformations only for even \( k \), for rational \( k = 2m/n \) one can not any longer maintain the whole large gauge transformations group and only the subgroup with shifts \( A_i \rightarrow A_i + nN_i \), are survive and so one gets the torus in a phase space \( 0 \leq A_i < 2\pi n \) with the total area \((2\pi n)^2 \) (we put \( \tau = i \) here because as we have mentioned before the number of states does not depend on moduli) and the number of states is \( N = (1/2\pi)(m/2\pi n) \times (2\pi n)^2 = mn \).

### 3.2. \( \mathcal{U}_q(sl(2)) \) in Chern-Simons theory and Landau problem on a torus

Now it is clear that to study the properties of the ground state in TMGT (or the whole Hilbert space in a topological Chern-Simons theory) one has to consider a Landau
problem on a torus and this problem was considered in [27]. Let us start from the first Landau level wavefunction on a plane \((A_1, A_2)\)

\[
\Psi(A_1, A_2) = \exp \left( -\frac{ik}{8\pi} A_1 A_2 + \frac{ik}{4\pi} p A_1 - \frac{k}{8\pi} (A_2 - p)^2 \right) = \\
\exp \left( -\frac{k}{16\pi} A \bar{A} \right) \exp \left( -\frac{k}{8\pi} p^2 + \frac{k}{16\pi} \bar{A}^2 + ikp \bar{A} \right) 
\]

which can be easily obtained from (2.10) substituting \(B = k/4\pi\) and using \(A_1, A_2\) notation instead of \(x_1, x_2\) and \(A(\bar{A})\) instead of \(z \bar{z}\).

To get the correct wave functions on a torus let us consider the simplest case \(\tau = i\). One can construct a torus first making a periodicity in \(A_1\) direction with period \(2\pi\), which leads to an evident quantization condition \(p_n = 4\pi n/k\). At the same time \(p\) is a \(A_2\) coordinate of the center of the wavepacket and for first \(k/2\) numbers \(p_n\) (for even \(k\)) this coordinate is inside an interval \(A_2 \in [0, 2\pi]\). Now to make a torus, i.e. make a periodicity in \(A_2\) direction with period \(2\pi\) (we consider now only even \(k\)) one has to sum over all \(p = 2\pi n\) and it is easy to see that for even \(k\) there are \(k/2\) different classes \(p = 4\pi r/k + 2\pi n, \ r = 1, \ldots k/2; \ n \in Z\) which gives \(k/2\) basic wave functions.

\[
\Psi_1(r|A_1, A_2) = \exp \left( -\frac{k}{16\pi} A \bar{A} \right) \exp \left( \frac{k}{16\pi} \bar{A}^2 \right) \theta \left[ \begin{array}{c} 2r/k \\ 0 \end{array} \right] \left( \frac{k\bar{A}}{4\pi} \right) \theta \left[ \begin{array}{c} \frac{ik}{2} \\ 0 \end{array} \right] = \\
\exp \left( -\frac{k}{16\pi} A \bar{A} \right) \exp \left( \frac{k}{16\pi} \bar{A}^2 \right) \sum_n \exp \left[ -\frac{\pi k}{2} (n + 2r/k)^2 + ik \frac{\bar{A}}{2} (n + 2r/k) \right] 
\]

where \(r = 1, 2, \ldots k/2\) and theta function is defined as follows

\[
\theta \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right] (z|\tau) = \sum_n \exp[i\pi \tau(n + \alpha)^2 + 2\pi i(n + \alpha)(z + \beta)] 
\]

Let us use the compact notation \(||r>|\) for the wave function \(\Psi_1(r|A_1, A_2)\) on a torus with \(\tau = i\). Using (3.16) and (2.10) we can write

\[
| |r> = \sum_{m=-\infty}^{\infty} |2\pi(m + 2r/k)>, \hspace{1cm} ||r>| = ||r + k/2> 
\]

* In a general case of rational \(k = 2m/l\) it will be first \(lm\) numbers \(p_n = 4\pi n/kl\) which gives us the coordinate of the wavepacket center in the interval \(A_2 \in [0, 2\pi]\)
Now let us consider the action of the $W_\infty$ (or magnetic translations) generators (2.24), (2.25) on the wave functions (3.18). It is easy to see that in this case one is dealing with generators $T_\vec{\xi}$, where $\vec{\xi} = \vec{n}/B = \frac{4\pi}{k}\vec{n}$ and $n_1, n_2$ are integers. Using equation (2.28) we get from (3.18)

$$T_\vec{\xi}||r > = \sum_{m=-\infty}^{\infty} T_\vec{\xi} |2\pi(m + 2r/k)| > = \exp \left( -\frac{2\pi i}{k} n_1(n_2 - 2r) \right) ||r - n_2 >$$

(3.19)

This is a demonstration that the ground state (first Landau level) wave functions in the topologically massive gauge theory (not only in the pure Chern-Simons case with infinite mass gap) form a unitary representation of a quantum $W_\infty$, i.e. FFZ algebra (2.24). Let us note that states at higher Landau levels which can be obtained from ground state wave functions $\Psi_r$ by the action of the $a^+$ operator (2.5)

$$\Psi_N(r|A_1, A_2) = \frac{(a^+)^N}{\sqrt{N!}} \Psi_1(r|A_1, A_2)$$

(3.20)

form unitary equivalent representations because generators of $W_\infty$ (magnetic translations) are built from $b$ and $b^+$ operators only and thus commute with $a$ and $a^+$.

Now let us consider the "minimal" quantum algebra $U_q(sl(2))$ with generators $J_\pm$

$$J_+ = \frac{1}{q - q^{-1}} \left( T_{(1,1)} - T_{(-1,1)} \right), \quad J_- = \frac{1}{q - q^{-1}} \left( T_{(-1,-1)} - T_{(1,-1)} \right)$$

$$q^{2J_3} = T_{(-2,0)}, \quad q^{-2J_3} = T_{(2,0)}, \quad q = \exp \left( \frac{4\pi i}{k} \right)$$

(3.21)

where notation $(n_1, n_2)$ corresponds to a vector $\vec{\xi} = (\xi_2, \xi_2) = \frac{4\pi}{k}(n_1, n_2)$. These generators act on states (3.18) in the following way:

$$J_+ ||r > = [r - 1/2]_q ||r - 1 >$$

$$J_- ||r > = [-r + 1/2]_q ||r + 1 >$$

$$q^{\pm 2J_3} ||r > = q^{\mp 2r} ||r >$$

(3.22)

We have two different types of representations in case of $k = 4n$ and $k = 4n + 2$ (don’t forget that here we are dealing only with even $k$).

\[\dagger\]This is only one possible choice and one can consider another constructions, choosing, for example, not $T_{\pm 1,\pm 1}$ but $T_{\pm n_1,\pm n_2}$. In that case the generators $J_\pm$ will shift state $||r >$ to $||r \mp n_2 >$.
In the case when $k = 4n + 2$ we have a highest and a lowest weight vectors. In this case $q = \exp\left(\frac{2n\pi i}{2n+1}\right)$ and it is easy to see that $[n+1/2]_q = 0$. This means that $||n+1 >$ is the highest and $||n >$ is the lowest weight vectors

$$
J_+||n+1 > = [n+1/2]_q||n > = 0 \\
J_-||n > = -[n+1/2]_q||n+1 > = 0
$$

and we have $2n+1$-dimensional representation

$$
||n+1 > \rightarrow \ldots \rightarrow ||2n+1 > \rightarrow ||1 > \rightarrow ||2 > \rightarrow \ldots \rightarrow ||n > \\
||n+1 < \leftarrow \ldots \leftarrow ||2n+1 < \leftarrow ||1 < \leftarrow ||2 < \leftarrow \ldots \leftarrow ||n >
$$

(3.23)

In the case $k = 4n$ there are no highest and/or lowest weight vectors and we have cyclic representation with dimension $k/2 = 2n$

$$
\ldots \rightarrow ||1 > \rightarrow ||2 > \rightarrow \ldots \rightarrow ||k/2 = 2n > \rightarrow ||1 > \rightarrow \ldots
$$

(3.24)

and the the same (with opposite directed arrows) for $J_+$. The $q$-dimension in both cases is zero.

Let us note that one can get the highest weight representation in the case $k = 4n$ if instead of usual periodical boundary conditions on a torus we shall consider the twisted boundary conditions in $A_1$ direction which leads to modified quantization condition $p_n = 4\pi(n-\alpha)/k$ where $\alpha \in [0, 1)$ defines an additional phase factor (twisting) $\exp(2\pi i\alpha)$ which arises in a wave function on cylinder (and torus) after $2\pi$ shift in $A_1$ direction. One can get this shift if there is a flux through cylinder (or torus) $\Phi = 2\pi\alpha$. In that case the $k/2 = 2n$ state vectors will be $||r-\alpha >, r = 1, \ldots, 2n$ and one can have highest and lowest weight vectors for $k = 4n$ for $\alpha = 1/2$. This corresponds to antiperiodic boundary conditions in $A_1$ direction or to a flux $\Phi = \pi$. Then it is easy to that $||n+1/2 >$ is the highest and $||n−1/2>$ is the lowest weight vectors

$$
J_+||n+1/2 > = [n]_q||n−1/2 > = 0 \\
J_-||n−1/2 > = −[n]_q||n+1/2 > = 0
$$

(3.26)

because here $q = \exp(\pi i/n)$ and we have $2n$-dimensional representation completely analogous to (3.24).

One can consider in a same way the case of rational $k = 2m/n$. Let us note that there is an ultimate connection between $2 + 1$ topological Chern-Simons theory and $1 + 1$
conformal field theory (CFT) \cite{3}. In our case the corresponding conformal field theory is \( c = 1 \) model and the states on the first Landau level are in one-to-one correspondence with the conformal blocks of a \( c = 1 \) model \cite{28}. This means that there is \( \mathcal{U}_q(sl(2)) \) symmetry in \( c = 1 \) CFT and conformal blocks are a representation of quantum algebra.

4. Quantum symmetry in a quantum Hall system

4.1. Area-preserving diffeomorphisms and incompressibility in a quantum Hall system

The area-preserving diffeomorphisms and corresponding \( W_\infty \) symmetry were discussed recently in quantum Hall systems in \cite{15}, \cite{16}. The physical reason for the very existence of this symmetry is based on a Laughlin idea \cite{29} that ground state of the quantum Hall system at rational (integer and fractional) values of a filling factor \( \nu = 2\pi \rho / B \), where \( \rho \) is an electron density, is described by an incompressible quantum fluid, i.e. there is an energy gap in a spectrum of excitations. In the case of an integer quantum Hall effect (IQHE) this gap is a mobility gap between Landau levels (taking into account the disorder and localised states) which for strong magnetic field \( B \) is much larger then the energy of Coulomb repulsion between electrons. Thus one can neglect the interaction in the IQHE effect and consider it as a system of \( \nu \) completely filled Landau levels. The fractional quantum Hall effect (FQHE) occurs in low-disorder, high-mobility samples with partially filled Landau levels. In this case there is no single-particle gap and only after taking into account many-body correlation due to the Coulomb repulsion the excitation gap appears in a spectrum as a collective effect. In the case of filling \( \nu = 1 / (2p + 1) \) the ground state wave function is described by Laughlin wave function \cite{29} (let us note that in our notation the wave function on the first Landau level depends on \( \bar{z} \), not \( z \))

\[
\Psi(z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n) = \prod_{i<j}(\bar{z}_i - \bar{z}_j)^{2p+1} \exp \left( -\frac{B}{4} \sum_i |z_i|^2 \right) \tag{4.1}
\]

In the case \( p = 0 \) this function describes a completely filled first Landau level.

Now let us consider an operator

\[
L_{n,m} = \sum_{i=1}^{N} L_{n,m}^i = \sum_{i=1}^{N} (b_i^+)^{n+1} b_i^{m+1}, \quad n, m \geq -1 \tag{4.2}
\]
which is the sum of $N$ independent operators (2.29). It is easy to see that the commutation relations for these operators are the same as for one-particle ones $L_{n,m}$ (2.30). If we are on a first Landau level the angular momentum is given by $\sum_{i=1}^{N} (b_i^+) b_i = L_{0,0}$ and one can see that $[L_{0,0}, L_{n,m}] = (n - m) L_{n,m}$, i.e. $L_{n,m}$ with $n < m$ are decreasing an angular momentum and in result compress the Hall liquid. Thus, being applied to the uncompressible completely filled level it must annihilate it

$$L_{n,m} \Psi_{\nu=1} = 0, \quad n < m \tag{4.3}$$

There exists a second-quantized representation for these generators

$$L_{n,m} = \int d^2 x \hat{\Psi}^\dagger(\vec{x}, t) (b_i^+)^{n+1} b_m^{m+1} \hat{\Psi}(\vec{x}, t) \tag{4.4}$$

where $\hat{\Psi}(\vec{x}, t)$ is the field operator for the fermions in an external magnetic field

$$\hat{\Psi}(\vec{x}, t) = \sum_{n=1}^{\infty} \sum_{k} F_k^{(n)}(\vec{x}) \exp \left( -\frac{1}{m} (n + \frac{1}{2}) \right) \tag{4.5}$$

and $\phi_k^{(n)}(\vec{x})$ are the wave functions on the $n$-th Landau level and $F_k^{(n)}(F_k^{(n)} \dagger)$ are the fermionic creation and annihilation operators. One can use this representation to obtain the expression for $L_{n,m}$ in terms of fermionic creation and annihilation operators (see details in [18]) and then it is easy to show that conditions (4.3) are valid for arbitrary integer-valued filling $\nu$ also. The case of the fractional filling was also considered in [16] and in a recent paper [30]. One can also show using the second-quantized representation and (2.31) that the Fourier-transformed second-quantized density operator $\rho(\vec{k}) = \int d^2 x \exp(ik\vec{x}) \hat{\Psi}^\dagger(x) \hat{\Psi}(x)$, being projected on the first Landau level, becomes proportional to a $W_{k,\vec{k}}$ generator with $k(\vec{k}) = k_1 \pm k_2$.

The incompressibility of the quantum Hall liquid thus naturally leads to a some $W_\infty$ symmetry (to be more precise it is called $W_{1+\infty}$ in a literature). If one considers a droplet of a quantum Hall liquid it is evident that the only possible deformations of this droplet preserving the area turns are the waves at the boundary of the droplet describing the deformation of shape - the so-called edge excitations [31].

4.2. Magnetic translations and $U_q(s\ell(2))$ in quantum Hall system

After this brief review of area-preserving diffeomorphisms and $W_\infty$ symmetries in a quantum Hall system let us consider our construction of $U_q(s\ell(2))$ and ask the obvious
question - what will be a natural value for a deformation a parameter \( q \) and how this symmetry will act on the physical states. Let us note that there are two possibilities to construct the generators of a \( W_\infty \) algebra in this case. First of all we can use the connection (2.31) between \( W_{n,\bar{n}} \) and \( L_{n,m} \) and construct \( W_{n,\bar{n}} \) generators in a complete analogy with \( L_{n,m} \), i.e. summing over all one-particle generators \( W^i_n, \bar{W}^i_n \)

\[
W_{n,\bar{n}} = \sum_{i=1}^{N} W^i_{n,\bar{n}} = \sum_{i=1}^{N} \exp \left( \frac{1}{2} (nb^+_i - \bar{n}b_i) \right) \tag{4.6}
\]

One can rewrite it in a second-quantized form as

\[
W_{n,\bar{n}} = \int d^2x \hat{\Psi}^\dagger(\vec{x}, t) \exp \left( \frac{1}{2} (nb^+ - \bar{n}b) \right) \hat{\Psi}(\vec{x}, t) = \exp \left( \frac{B}{4n\bar{n}} \right) \int d^2x \hat{\Psi}^\dagger(\vec{x}, t) \exp \left( -\frac{1}{2} \bar{n}b \right) \exp \left( \frac{1}{2} nb^+ \right) \hat{\Psi}(\vec{x}, t) \tag{4.7}
\]

It is easy to see that being projected on the first Landau level one can effectively substitute \( b^+ = iB\bar{z}, \ b = -iBz \) (see (2.12)) and thus get the Fourier-transformation of the density operator \( \rho(\vec{x}) = \hat{\Psi}^\dagger(\vec{x}, t)\hat{\Psi}(\vec{x}, t) \) projected on the first Landau level

\[
W_{\vec{\xi} \vec{\eta}} = \exp \left( \frac{B}{4} n \bar{n} \right) \int d^2x \hat{\Psi}^\dagger(\vec{x}, t) \exp \left( i \frac{B}{2} (n\bar{z} + \bar{n}z) \right) \hat{\Psi}(\vec{x}, t) = \exp \left( \frac{B}{4} n \bar{n} \right) \int d^2x \exp(iB\bar{n}\vec{x}) \rho(\vec{x}) \tag{4.8}
\]

Let us note that because of the projection on a given level the time dependence disappears from \( \rho(\vec{x}) \) because all \( \exp(-i(B/2m)t) \) factors from \( \hat{\Psi}(\vec{x}, t) \) will be cancelled by \( \exp(+i(B/2m)t) \) factors in \( \hat{\Psi}^\dagger(\vec{x}, t) \).

We can, however, define the total \( W_\infty \) generators acting on the whole quantum Hall system in another way, namely one can multiply all one-particle generators \( W^n_{\vec{\eta}} \) or equivalent magnetic translation generators \( T^i_{\vec{\eta}} \) and get

\[
T_\vec{\xi} = \prod_{i=1}^{N} T^{i}_{\vec{\xi}^i}, \quad T_\vec{\xi}T_{\vec{\eta}} = \exp \left( -i \frac{B}{2} N (\vec{\xi} \times \vec{\eta}) \right) T_{\vec{\xi}+\vec{\eta}} \tag{4.9}
\]

Before we shall discuss the quantum algebra structure let us consider how the translations (4.9) acts on the Laughlin wave function (1.1). Using (2.27) and the fact that \( T^{i}_{\vec{\xi}} \) factors act independently on \( z_i, \bar{z}_i \) arguments, we can get after simple calculations

\[
T_\vec{\xi} \Psi(z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n) = \exp \left( -\frac{B}{4} \vec{\xi} \vec{x} - \frac{B}{2} \frac{1}{2} \sum_i \vec{z}_i \right) \Psi(z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n) \tag{4.10}
\]
One can see that up to an overall phase factor depending only on the center of mass coordinate $\sum_i \bar{z}_i$ (which is absent in a center of mass frame) the Laughlin wave function is invariant under the magnetic translations.

Let us consider how the quasihole wave function

$$\Psi(u, \bar{u}; z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n) = \prod_i (\bar{u} - \bar{z}_i) \prod_{i<j} (\bar{z}_i - \bar{z}_j)^{2p+1} \exp \left( -\frac{B}{4} \sum_i |z_i|^2 \right) \quad (4.11)$$

is transformed under the action of $T_{\mathbf{\xi}}$. Repeating the same arguments one gets (let us note that $T$ acts only on $z_i$, not quasihole coordinate $u$!)

$$T_{\mathbf{\xi}} \Psi(u, \bar{u}; z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n) = \prod_i (\bar{u} - \bar{\xi} - \bar{z}_i) \ T_{\mathbf{\xi}} \Psi(u, \bar{u}; z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n) \quad (4.12)$$

and we see that $T_{\mathbf{\xi}}$ shifts a quasihole coordinate $u \rightarrow u - \xi, \bar{u} \rightarrow \bar{u} - \bar{\xi}$.

Now let us construct the $U_q(sl(2))$ generators from $T_{\mathbf{\xi}}$. Taking as usually

$$J_+ = \frac{1}{q - q^{-1}} \left( T_{(a,b)} - T_{(-a,b)} \right), \ J_- = \frac{1}{q - q^{-1}} \left( T_{(-a,-b)} - T_{(a,-b)} \right), q^{\pm 2J_3} = T_{(\mp 2a,0)} \quad (4.13)$$

we get the quantum algebra $U_q(sl(2))$ with

$$q = \exp (iBNab) \quad (4.14)$$

where $N$ is the total number of electrons. To construct the "minimal" $U_q(sl(2))$ one have to choose the minimal $a$ and $b$. It can be shown that for a system with sizes $L_1$ and $L_2$ the minimal shifts $a$ and $b$ must be choosen as (see discussion in section 3.2)

$$a = \frac{2\pi}{L_2 B}, \quad b = \frac{2\pi}{L_1 B} \quad (4.15)$$

With this choice one has

$$q = \exp (iNab)) = \exp \left( 2\pi i \frac{2\pi N}{BL_1 L_2} \right) = \exp (2\pi i \nu) \quad (4.16)$$

where we used the fact that filling factor $\nu$ is defined as the ratio of total number of particles $N$ to the total number of available states on the Landau level $BL_1 L_2/2\pi$, i.e $\nu = 2\pi N/BL_1 L_2$. We see that the ground state is a singlet (up to total shift of the center of mass) under the action of a quantum group. However the quasihole wave function is transformed under $U_q(sl(2))$ and the basis (4.11) is not the convenient one to study.
the action of quantum algebra - one can see that $J_3$ in this basis is not diagonal. One can consider the QHE on a cylinder or a torus and then using the arguments from the section 3.2 we shall get the same representations as (3.24) and (3.25). Let us remember that in a first case we had $q = \exp(2\pi i/(2p + 1))$ which is precisely the case of the fractional filling factor with an odd denominator $\nu = 1/(2n + 1)$. It is amusing that in this case we had $2p + 1$-dimensional representation with highest and lowest weight vectors. In the case $q = \exp(2\pi i/2p)$ which corresponds to the fractional filling factor with an even denominator $\nu = 1/2n$ there were no highest and lowest weight vectors. It is interesting that it is Pauli principle which prescribes the odd denominators for Laughlin wave function (4.1) to be antisymmetric under the permutations $z_i \leftrightarrow z_j$. It is amusing that our quantum algebra construction know about it in some way. Let us also remind that if it is a flux $\Phi = \pi$ through the cylinder the highest weight representation will be in the even denominator case $q = \exp(i/2p)$ as have been discussed at the end of section 3.2. However this flux corresponds to the antiperiodic boundary condition which may be treated as a statistical transmutations from fermions to bosons - the even denominators appearence in this case becomes obvious.

Let us also mention another relation between quantum algebras in Chern-Simons theory and in a quantum Hall system. For CS theory we got the deformation parameter $q_{CS} = \exp(4\pi i/k)$ and for quantum Hall system $q_{QH} = \exp(2\pi i\nu)$. One can describe the large-scale properties of a quantum Hall system by an effective Ginzburg-Landau theory with the Chern-Simons term for ”statistical” gauge field $a_\mu$ (don’t mix with electromagnetic field $A_\mu$) \cite{32}, \cite{33}. In a simplest case $\nu = 1/(2p + 1)$ one has the effective action with a Chern-Simons term

$$\frac{1}{4\pi\nu} \epsilon_{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda$$

and comparing with (3.1) we get

$$\nu = \frac{2}{k}, \quad q_{QH} = \exp(2\pi i\nu) = \exp\left(\frac{4\pi i}{k}\right) = q_{CS}$$

i.e. for both descriptions - microscopical and effective, based on Chern-Simons theory, we got the same quantum algebra $U_q(sl(2))$.

In a more general case $\nu = m/(2pm + 1)$ \cite{34} instead of one abelian Chern-Simons field we have $m$ ones \cite{33} and have to generalize our quantum group construction for multidimensional Landau problem. This interesting question will be discussed elsewhere.

* In section 3.2 we used letter $n$ instead of $p$ here
5. Conclusion

We discussed in this paper how the quantum algebra $\mathcal{U}_q(sl(2))$ can be constructed from the generators of the $W_\infty$ transformations. The physical model for this construction was the Landau problem and the $W_\infty$ generators were nothing but the magnetic translations. However as we had demonstrated in the case of the FFZ algebra with a nontrivial central extension one can construct the quantum algebra from the $W_\infty$ algebra which cannot be obtained from the group of magnetic translations. Two examples of a quantum $\mathcal{U}_q(sl(2))$ algebra were considered - with $q_{CS} = \exp(4\pi i/k)$ in an abelian 2+1 gauge theory with a Chern-Simons term and with $q_{QH} = \exp(2\pi i\nu)$, where $\nu$ is a filling factor, in a quantum Hall systems. We have demonstrated using the effective large-scale description of a quantum Hall system in terms of "statistical" Chern-Simons field that $q_{CS} = q_{QH}$ in the case of filling factor $\nu = 1/(2p+1)$. In a general case $\nu = m/(2pm+1)$ we have to consider a quantum algebra construction for a general $\prod_{i=1}^m U_i(1)$ Chern-Simons theory - which reduces to a multidimensional Landau problem.

There are a lot of other interesting questions which are still open. First of all it is interesting to generalise this construction to the nonabelian case for the Chern-Simons theory. It is unclear what will be the analog of quantum algebra in this case. Let us note that this question include not only topologically massive nonabelian Yang-Mills theory but also 2+1-dimensional gravity which can be considered as Chern-Simons gauge theory too [35].

It also will be extremely interesting to understand if there is a more general connection between area-preserving diffeomorphisms and quantum groups (algebras). One can study the geometric action on the coadjoint orbit of $w_\infty$ or $W_\infty$ as it was discussed in [36], [37]. These actions are relevant to both $w$ gravity and two-dimensional hydrodynamics (the finite dimensional analog of the trigonometric FFZ algebra in an ideal two-dimensional hydrodynamics was considered in [38]). It is unclear if there is a hidden quantum symmetry of these geometrical actions.

It is known that there is a $W_\infty$ in the $c = 1$ strings and corresponding matrix models [39]-[40]. What quantum algebra (if any) can be constructed in this model and what is the natural value of a deformation parameter $q$ in this case? Will this quantum symmetry exist in the case of deformed $c = 1$ model, for example in the case of two-dimensional black hole?

It is also known that the action for pure Yang-Mills theory in two dimensions is
invariant under the area-preserving diffeomorphisms (see, for example \[11\]). Can one find a quantum symmetry in this case?

Let us finally note that recently $W_\infty$ has been discussed in a context of a bosonization of current-current interactions \[12\]. In the framework of this approach the quantum symmetry may appear in some new condensed matter problems.

We hope to return to these and related questions in the following publications.

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**Note added**

After this paper has been submitted for publication I was aware about recent preprint \[13\] where the quantum group symmetry in Landau problem and in quantum Hall system had been also discussed.
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