Article

Coefficient Bounds for Certain Classes of Analytic Functions Associated with Faber Polynomial

Adel A. Attiya 1,2, Abdel Moneim Lashin 2,3, Ekram E. Ali 1,4 and Praveen Agarwal 5,6,7,8,*

1 Department of Mathematics, College of Science, University of Ha’il, Ha’il 81451, Saudi Arabia; aatiya@mans.edu.eg (A.A.A.); ekram_008eg@yahoo.com (E.E.A.)
2 Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt
3 Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia; alashin@mans.edu.eg
4 Department of Mathematics and Computer Science, Faculty of Science, Port Said University, Port Said 42521, Egypt
5 Department of Mathematics, Anand International College of Engineering, Jaipur 303012, India
6 International Center for Basic and Applied Sciences, Jaipur 302029, India
7 Department of Mathematics, Harish-Chandra Research Institute, Allahabad 211 019, India
8 Department of Mathematics, Netaji Subhas University of Technology, New Delhi 110078, India
* Correspondence: praveen.agarwal@anandice.ac.in or praveen2011@gmail.com

Abstract: In this paper, we introduce a family of analytic functions in the open unit disk which is bi-univalent. By the virtue of the Faber polynomial expansions, the estimation of \( n - th \) \((n \geq 3)\) Taylor–Maclaurin coefficients \(|a_n|\) is obtained. Furthermore, the bounds value of the first two coefficients of such functions is established.

Keywords: Faber polynomial; coefficient bounds; uniformly convex; uniformly starlike; univalent functions; bi-univalent functions

1. Introduction

Faber polynomials, which were introduced by Faber in 1903 [1], play an important role in the theory of functions of a complex variable and different areas of mathematics and there is a rich literature [2–7] describing their properties and their applications. Given a function \( h(z) \) of the form

\[
h(z) = z + b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots,
\]

consider the expansion

\[
\frac{\zeta h'\left(\zeta\right)}{h\left(\zeta\right) - w} = \sum_{n=0}^{\infty} \Psi_n(w) \zeta^{-n},
\]

valid for all \( \zeta \) in some neighborhood of \( \infty \). The function \( \Psi_n(w) = w^n + \sum_{k=1}^{n} a_{nk} w^{n-k} \) is a polynomial of degree \( n \), called the \( n \)-th Faber polynomial with respect to the function \( h(z) \). In particular,

\[
\Psi_0(w) = 1, \quad \Psi_1(w) = w - b_0,
\]
\[
\Psi_2(w) = w^2 - 2b_0w + (b_0^2 - 2b_1),
\]
\[
\Psi_3(w) = w^3 - 3b_0w^2 + (3b_0^2 - 3b_1)w + (b_0^3 + 3b_1b_0 - 3b_2).
\]

Let \( \Psi_n(0) = F_n(b_0, b_1, \ldots, b_n), n \geq 0, \) see ([8], p. 118). Let \( A \) denote the class of all functions of the form:

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n,
\]
which are analytic in the open unit disc \( U = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \} \) and let \( S \) be the class of all functions in \( A \) which are univalent in \( U \). By using the Faber polynomial expansion of functions of the form (1), Airault and Bouali [9], p. 184 showed that

\[
\frac{z f'(z)}{f(z)} = 1 - \sum_{j=2}^{\infty} F_{j-1}(a_2, a_3, \ldots, a_j) z^{j-1},
\]

where \( F_{j-1}(a_2, a_3, \ldots, a_j) \) is the Faber polynomial given by:

\[
F_{j-1}(a_2, a_3, \ldots, a_j) = \sum_{i_1+2i_2+\ldots+(j-1)i_{j-1}=j-1} A(i_1, i_2, \ldots, i_{j-1})(a_2^{i_1}, a_3^{i_2}, \ldots, a_j^{i_{j-1}})
\]

and

\[
A(i_1, i_2, \ldots, i_{j-1}) := (-1)^{(j-1)+2i_1+\ldots+j_{i_{j-1}}}(i_1+i_2+\ldots+i_{j-1}-1)! \frac{(j-1)!}{(i_1)!(i_2)\ldots(i_{j-1})!}.
\]

The first few terms of the Faber polynomials \( F_{j-1}, j \geq 2 \), are given by (e.g., see [10], p. 52)

\[
\begin{align*}
F_1 &= -a_2, \\
F_2 &= a_2^2 - 2a_3, \\
F_3 &= -a_2^3 + 3a_2a_3 - 3a_4, \\
F_4 &= a_2^4 - 4a_2^2a_3 + 4a_2a_4 + 2a_5 - 4a_6 \\
F_5 &= -a_2^5 + 5a_2^3a_3 + 5a_2^2a_4 - 5a_2a_5^2 - a_5 + 5a_3a_4 - 5a_6.
\end{align*}
\]

The Koebe one-quarter theorem [8], p. 31 ensures the range of every function of the class \( S \) contains the disc \( \{ w : |w| < \frac{1}{4} \} \). Thus every univalent function \( f \in S \) has an inverse \( f^{-1} \), which is defined by

\[
f^{-1}(f(z)) = z \quad (z \in U)
\]

and

\[
f(f^{-1}(\omega)) = \omega \quad (|\omega| < \frac{1}{4}).
\]

The inverse map \( g := f^{-1} \) of the function \( f \in A \) has Taylor expansion given by (see [9], p. 185)

\[
g(\omega) = f^{-1}(\omega) = w + \sum_{n=2}^{\infty} \frac{1}{n!} K_{n-1}^{-n}(a_2, a_3, \ldots, a_n) \omega^n
\]

\[
= w - a_2\omega^2 + (2a_2^2 - a_3)\omega^3 - (5a_2^3 - 5a_2a_3 + a_4)\omega^4 + \ldots,
\]
where the coefficients $K_n^p(a_2, a_3, \ldots, a_n)$ are given by

\[
K_1^p = pa_2, \quad K_2^p = \frac{1}{2}p(p - 1)a_2^2 + pa_3, \\
K_3^p = p(p - 1)a_2a_3 + pa_4 + \frac{p(p - 1)(p - 2)}{3!}a_2^3, \\
K_4^p = p(p - 1)a_2a_4 + pa_5 + \frac{p(p - 1)(p - 2)}{2}a_2^2a_3 + \frac{p(p - 1)(p - 2)(p - 3)}{24}a_2^4 + \frac{p(6p - 11)}{24}a_2^3a_4, \\
\vdots \\
K_n^p = \frac{p!}{(p - n)!n!}a_2^n + \frac{p!}{(p - n + 1)!(n - 2)!}a_2^{n-2}a_3 + \frac{p!}{(p - n + 2)!(n - 3)!}a_2^{n-3}a_4 + \frac{p!}{(p - n + 3)!(n - 4)!}a_2^{n-4}a_5 + \frac{p!}{(p - n + 4)!(n - 5)!}a_2^{n-5}a_6 + \sum_{j=6}^{n} a_2^{n-j}V_j
\]

(3)

and $V_j$ is homogeneous polynomial of degree $j$ in the variables $a_3, \ldots, a_n$, see ([11], p. 349 and [9], p. 183 and p. 205).

**Lemma 1.** (Schwarz lemma [8], p. 3) Let $\omega(z)$ be analytic in the unit disc $U$, with $\omega(0) = 0$ and $|\omega(z)| < 1$ in $U$. Then $|\omega(z)| < |z|$ and $|\omega'(0)| < 1$ in $U$.

If $f$ and $g$ are analytic functions in $U$, we say that $f$ is subordinate to $g$, written $f(z) \prec g(z)$ if there exists a Schwarz function $\omega(z)$ such that $f(z) = g(\omega(z))$. Let $\phi$ be an analytic function with positive real part in $U$, satisfying $\phi(0) = 1, \phi'(0) > 0$, and $\phi(U)$ is symmetric with respect to the real axis. Such a function has a Taylor series of the form

\[
\phi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \ldots (B_1 > 0).
\]

(4)

Using this $\phi$, Ma and Minda [12] considered the classes

\[
S(\phi) = \left\{ f \in A : \frac{zf'(z)}{f(z)} \prec \phi(z), \ z \in U \right\}
\]

and

\[
K(\phi) = \left\{ f \in A : zf'(z) \in S(\phi), \ z \in U \right\}
\]

Several well-known classes can be obtained by specializing of the function $\phi$, for instance

1. By taking $\phi(z) = \frac{1+Az}{1+Bz}, -1 < B < A \leq 1$, we obtain the classes $S[A, B]$ and $K[A, B]$ of the well-known Janowski starlike and convex functions.
2. If we set $\phi(z) = \frac{1}{1+z^{\alpha}}$, we obtain the classes $S^{\alpha}(a)$ and $K(a)$ of starlike and convex functions of order $a(0 \leq a < 1)$.
3. The class $S_{\frac{1}{2}} := S(\sqrt{1+z})$ was considered by Sokol and Stankieicz [13], consisting of functions $f$ such that $\frac{f(z)}{1+z}$ lies in the region bounded by the right half of the Bernoulli lemniscate given by $|w^2 - 1| < 1$.
4. Taking $\phi(z) = \left( \frac{1+z}{1+z^{\alpha}} \right)^{\delta} (0 < \delta \leq 1)$ yields the classes of strongly starlike and convex functions.
5. The function class $S^*_{C}: = S(z + \sqrt{1 + z^2})$ was considered by Raina and Sokol [14], consisting of normalized starlike functions $f$ satisfying the inequality
\[
\left\{ \frac{zf'(z)}{f(z)} \right\}^2 - 1 < 2 \left| \frac{zf'(z)}{f(z)} \right|.
\]

6. Kanas et al. [15] considered the family of analytic functions $S\left(\frac{1}{1 - z^2}\right)$ and $K\left(\frac{1}{1 - z^2}\right)$ with the property that $zf'(z)$ and $1 + z^2f''(z)$ lie in a domain bounded by a hyperbola $\beta = \rho(s) = (2 \cos \frac{s}{2})^{-1}(0 < s \leq 1, |\phi| < \frac{\pi}{2})$.

7. The function class $S^*_\alpha: = S(e^\alpha)$ was introduced and studied by Shams et al. [18].

8. The classes $S_\alpha(z) := S(e^\alpha)$ and the class $S \alpha = S(e^\alpha)$ are convex and hence starlike functions.

Moreover, the exponential function $\phi(z) = e^\alpha$ has positive real part in $U$, maps $U$ onto a domain $\phi(U) := \{w \in C : |log w| < 1\}$ which is symmetric about the real axis.

An interesting families of the domains that are bounded by a conic sections were introduced and studied by Shams et al. [18], they introduced the class $SD(\alpha, \beta)$ of $\beta$-uniformly starlike functions of order $\alpha$ $(0 \leq \alpha < 1)$ in $U$ consisting of functions $f \in A$ which satisfy the following inequality
\[
\Re \left\{ \frac{zf'(z)}{f(z)} - \alpha \right\} > \beta \left| \frac{zf'(z)}{f(z)} - 1 \right| \quad (\beta \geq 0; 0 \leq \alpha < 1; z \in U). \tag{5}
\]

and class $KD(\alpha, \beta)$ of $\beta$-uniformly convex of order $\alpha$ $(0 \leq \alpha < 1)$, defined by
\[
f \in KD(\alpha, \beta) \iff zf'(z) \in SD(\alpha, \beta).
\]

Since $Re(w > \alpha|w - 1| + \gamma$ if and only if $Re\{w(1 + \alpha \theta) - \alpha e^{i\theta}\} > \gamma$ (see [19]), then the condition (5) is equivalent to
\[
\Re \left\{ 1 + \beta e^{i\theta} \frac{zf'(z)}{f(z)} - \beta e^{i\theta} \right\} > \alpha.
\]

Motivated by the classes $SD(\alpha, \beta)$ and $KD(\alpha, \beta)$ we now introduce and investigate the following subclasses of $A$, and obtain some interesting results.

**Definition 1.** A function $f(z) \in A$ is said to be in the class $M(\lambda, \beta, \gamma, \phi)$ if it satisfies
\[
(1 + \beta e^{i\gamma}) \frac{zf''(z) + \lambda z f''(z)}{(1 - \lambda)f(z) + \lambda z f'(z)} - \beta e^{i\gamma} < \phi(z) \quad (z \in U),
\]

where $\beta \geq 0, 0 \leq \lambda \leq 1$ and $-\pi \leq \gamma < \pi$.

We note that:

1. The class $M(0, 0, \gamma, \phi) = S(\phi)$ and the class $M(1, 0, \gamma, \phi) = K(\phi)$.
2. The class $M(0, \beta, \gamma, \frac{1 + (1 - 2\alpha \phi)}{1 - \phi}) = SD(\alpha, \beta)$ and the class $M(1, \beta, \gamma, \frac{1 + (1 - 2\alpha \phi)}{1 - \phi}) = KD(\alpha, \beta)$.
3. The class $M(\lambda, 0, \gamma, \frac{1 + (1 - 2\alpha \phi)}{1 - \phi})$ was introduced and studied by Aouf et al. [20].
Definition 2. A function \( f(z) \in A \) is said to be in the class \( S(\lambda, \beta, \gamma, \phi) \) if it satisfies

\[
(1 + \beta e^{i\gamma}) \left[ (1 - \lambda) \frac{zf'(z)}{f(z)} + \lambda (1 + \frac{zf''(z)}{f'(z)}) \right] - \beta e^{i\gamma} \prec \phi(z) \quad (z \in U),
\]

where \( \beta \geq 0, 0 \leq \lambda \leq 1 \) and \( -\pi \leq \gamma < \pi \).

We note that:
1. The class \( S(0,0,\gamma,\phi) = S(\phi) \) and \( S(1,0,\gamma,\phi) = K(\phi) \).
2. \( M(1,\beta,\gamma, \frac{1+\lambda-2\alpha}{1-\alpha}) = SD(\alpha, \beta) \) and \( S(0,\beta,\gamma, \frac{1+\lambda-2\alpha}{1-\alpha}) = KD(\alpha, \beta) \).

A single-valued function \( f \) analytic in a domain \( D \subset \mathbb{C} \) is said to be univalent there if it never take the same value twice; that is, if \( f(z_1) \neq f(z_2) \) for all points \( z_1 \) and \( z_2 \) in \( D \) with \( z_1 \neq z_2 \) (see [8], p. 26). A function \( f \in A \) is said to be bi-univalent in \( U \) if \( f \) and its inverse map \( f^{-1} \) are univalent in \( U \). Let \( \sigma \) denote the class of bi-univalent functions in \( U \) given by (1). The class of analytic bi-univalent functions was first introduced and studied by Lewin [21] and showed that \(|a_2| < 1.51\). Recently, many authors found non-sharp estimates on the first two Taylor–Maclaurin coefficients \(|a_2|\) and \(|a_3|\) for various subclasses of bi-univalent functions, see for example, ([22–43]). For other related topics see also, ([44–47]).

Definition 3. A function \( f \in \sigma \) given by (1) is said to be in the class \( M_C(\lambda, \beta, \gamma, \phi) \) if both \( f \) and its inverse map \( g = f^{-1} \) are in \( M(\lambda, \beta, \gamma, \phi) \).

We note that:
1. The class \( M_C(0,1,\gamma,\phi) = \mathbb{H}_c(\phi) \) was introduced and studied by Darwish et al. [48].
2. The class \( M_C(0,0,\gamma, \frac{1+\lambda-2\alpha}{1-\alpha}) = S[A,B] \) was introduced and studied by Hamidi and Jahangiri [49].

Definition 4. A function \( f \in \sigma \) given by (1) is said to be in the class \( S_C(\lambda, \beta, \gamma, \phi) \) if both \( f \) and its inverse map \( g = f^{-1} \) are in \( S(\lambda, \beta, \gamma, \phi) \).

We note that:
1. The class \( S_C(0,1,\gamma,\phi) = \mathbb{H}_c(\phi) \).
2. The class \( S_C(0,0,\gamma, \frac{1+\lambda-2\alpha}{1-\alpha}) = S[A,B] \).
3. The class \( S_C(\lambda,0,\gamma,\phi) = M^*_C(\lambda, \phi) \) was introduced and studied by Goyal and Kumar [50], see also Zireh et al. [51].

In this paper, we use the Faber polynomial expansions to obtain bounds for the general coefficients \(|a_n|\) of bi-univalent functions in \( M_C(\lambda, \beta, \gamma, \phi) \) and \( S_C(\lambda, \beta, \gamma, \phi) \) as well as we provide estimates for the initial coefficients of these functions.

2. Coefficient Estimates for the Class \( M_C(p, \lambda, \tau, \phi) \)

Theorem 1. Let the function \( f \in \sigma \) given by (1) be in the class \( M_C(\lambda, \beta, \gamma, \phi) \). If \( a_k = 0 \) for \( 2 \leq k \leq n - 1 \), then

\[
|a_n| \leq \frac{B_1}{(n-1)[1+\lambda(n-1)]} \frac{1+\beta e^{i\gamma}}{|1+\beta e^{i\gamma}|} \quad n \geq 3.
\]

Proof. If we set \( F(z) := (1-\lambda)f(z) + \lambda zf'(z) = z + \sum_{n=2}^{\infty} [1+\lambda(n-1)]a_nz^n := z + \sum_{n=2}^{\infty} \delta_nz^n \),

then

\[
f \in M(\lambda, \beta, \gamma, \phi) \Leftrightarrow (1 + \beta e^{i\gamma}) \frac{zf'(z)}{f(z)} - \beta e^{i\gamma} \prec \phi(z).
\]
Since, both functions $f$ and its inverse map $g = f^{-1}$ are in $M(\lambda, \beta, \gamma, \phi)$, by the definition of subordination, there are analytic functions $u, v : U \to U$ with $u(0) = v(0) = 0$, $|u(z)| < 1$ and $|v(z)| < 1$, such that

\[
(1 + \beta e^{i\gamma}) \frac{zF(z)}{F(z)} - \beta e^{i\gamma} = \phi(u(z)) \quad (z \in U)
\]

and

\[
(1 + \beta e^{i\gamma}) \frac{wG'(w)}{G(w)} - \beta e^{i\gamma} = \phi(v(w)) \quad (z \in U),
\]

where $G(z) := (1 - \lambda)g(z) + \lambda zg'(z) = z + \sum_{n=2}^{\infty} [1 + \lambda(n - 1)]d_n z^n := z + \sum_{n=2}^{\infty} \zeta_n z^n$ and $d_n = \frac{1}{n} K_{n-1}(a_2, a_3, \ldots, a_n)$. Define the functions $u(z)$ and $v(z)$ by

\[
u(z) = \sum_{n=1}^{\infty} b_n z^n, \quad v(z) = \sum_{n=1}^{\infty} c_n z^n \quad (z \in U).
\]

It is well known that (see Duren [8], p. 265)

\[
|b_n| \leq 1, |c_n| \leq 1 \quad n = 2, 3, \ldots.
\]

By a simple calculation, we have

\[
\phi(u(z)) = 1 - B_1 \sum_{n=1}^{\infty} K_n^{-1}(b_1, b_2, \ldots, b_n, B_1, B_2, B_3, \ldots, B_n) z^n
\]

\[
= 1 + B_1 b_1 z + (B_1 b_2 + B_2 b_1^2) z^2 + \ldots \quad (z \in U),
\]

and

\[
\phi(v(\omega)) = 1 - B_1 \sum_{n=1}^{\infty} K_n^{-1}(c_1, c_2, \ldots, c_n, B_1, B_2, B_3, \ldots, B_n) w^n
\]

\[
= 1 + B_1 c_1 \omega + (B_1 c_2 + B_2 c_1^2) \omega^2 + \ldots \quad (\omega \in U),
\]

In general (see [52], p. 649), the coefficients $K_n^\mu(k_1, k_2, \ldots, k_n, B_1, B_2, B_3, \ldots, B_n)$ are given by

\[
K_n^\mu(k_1, k_2, \ldots, k_n, B_1, B_2, B_3, \ldots, B_n)
\]

\[
= \frac{p!}{(p - n)!} k_1^{n-1} k_2^{(1)n-1} B_n \frac{B_1}{B_1} + \frac{p!}{(p - n + 1)!} k_1^{n-2} k_2^{(1)n-1} B_{n-1} \frac{B_1}{B_1} - \frac{p!}{(p - n + 2)!} (\mu - 3)! B_{n-2} k_1^{n-3} k_3^{(1)n-1} B_1
\]

\[
+ \frac{p!}{(p - n + 3)!} k_1^{n-4} [k_4^{(1)n-2} B_{n-3} \frac{B_1}{B_1} + \frac{p - n + 3}{2} k_1^{n-5} (\mu - 1)^n B_{n-2} + \sum_{j=5}^{\infty} \frac{p!}{(p - n + 4)!} X_j]
\]

where $X_j$ is a homogeneous polynomial of degree $j$ in the variables $k_2, \ldots, k_n$.

Using the Faber polynomial expansion (2) yield the following identities

\[
(1 + \beta e^{i\gamma}) \frac{zF(z)}{F(z)} - \beta e^{i\gamma} = (1 + \beta e^{i\gamma}) [1 - \sum_{j=2}^{\infty} F_{j-1}(\delta_2, \delta_3, \ldots, \delta_j) z^{j-1}] - \beta e^{i\gamma},
\]
and
\[
(1 + \beta e^{i\gamma}) \frac{wG(w)}{G(w)} - \beta e^{i\gamma} = (1 + \beta e^{i\gamma}) [1 - \sum_{j=2}^{\infty} F_{j-1}(\zeta_2, \zeta_3, \ldots, \zeta_j) w^{j-1}] - \beta e^{i\gamma}.
\] (13)

Comparing the corresponding coefficients of (10) and (12) yields
\[
(1 + \beta e^{i\gamma}) F_{n-1}(\delta_2, \delta_3, \ldots, \delta_n) = B_1 K_{n-1}^{-1}(b_1, b_2, \ldots, b_{n-1}, B_1, B_2, B_3, \ldots, B_{n-1})
\] (14)
and similarly, from (11) and (13), we have
\[
(1 + \beta e^{i\gamma}) F_{n-1}(\xi_2, \xi_3, \ldots, \xi_n) = B_1 K_{n-1}^{-1}(c_1, c_2, \ldots, c_{n-1}, B_1, B_2, B_3, \ldots, B_{n-1}).
\] (15)
Since \(a_k = 0\) for \(2 \leq k \leq n - 1\), by substituting \(\delta_n = [1 + \lambda(n - 1)] a_n, \xi_n = [1 + \lambda(n - 1)] d_n\) and \(d_n = -a_n\) in (14) and (15), we have
\[
(1 + \beta e^{i\gamma})(n - 1) [1 + \lambda(n - 1)] a_n = B_1 b_{n-1}
\]
and
\[
-(1 + \beta e^{i\gamma})(n - 1) [1 + \lambda(n - 1)] a_n = B_1 c_{n-1}.
\]

By using (9), we conclude that
\[
|a_n| \leq \frac{B_1}{1 + \beta e^{i\gamma} [n(n-1)]} |1 + \lambda(n-1)|
\]
this completes the proof. \(\Box\)

To prove our next theorem, we shall need the following lemma.

**Lemma 2.** Ref. [52] Let the function \(\Phi(z) = \sum_{n=1}^{\infty} \Phi_n z^n\) be a Schwarz function with \(|\Phi(z)| < 1\), \(z \in U\). Then for \(-\infty < \rho < \infty\),
\[
|\Phi_2 + \rho \Phi_1^2| \leq \begin{cases} 
1 - (1 - \rho) |\Phi_1^2| & \rho > 0 \\
1 - (1 + \rho) |\Phi_1^2| & \rho \leq 0
\end{cases}
\]

**Theorem 2.** Let the function \(f \in \sigma\) given by (1) be in the class \(M_\rho(\lambda, \beta, \gamma, \Phi)\), then
\[
|a_2| \leq \begin{cases} 
\frac{B_1 \sqrt{B_1}}{\sqrt{1 + \beta e^{i\gamma}|(1 + 2\lambda - i^{2}) \beta_1^2 + (1 + \beta e^{i\gamma})^2(1 + \lambda)^2(B_1 + B_2)}} & (B_2 \leq 0, B_1 + B_2 \geq 0) \\
\frac{B_1 \sqrt{B_1}}{\sqrt{1 + \beta e^{i\gamma}|(1 + 2\lambda - i^{2}) \beta_1^2 + (1 + \beta e^{i\gamma})^2(1 + \lambda)^2(1 + B_1 - B_2)}} & (B_2 > 0, B_1 - B_2 \geq 0)
\end{cases}
\] (16)
and
\[
|a_3 - a_2^2| \leq \begin{cases} 
\frac{B_1}{2(1 + 2\lambda)|1 + \beta e^{i\gamma}|} & (B_1 \geq |B_2|) \\
\frac{|B_2|}{2(1 + 2\lambda)|1 + \beta e^{i\gamma}|} & (B_1 < |B_2|)
\end{cases}
\] (17)

**Proof.** Replacing \(n = 2\) and \(3\) in (14) and (15), respectively, we find that
\[
(1 + \beta e^{i\gamma})(1 + \lambda) a_2 = B_1 b_1,
\] (18)
\[
(1 + \beta e^{i\gamma})[2(1 + 2\lambda) a_3 - (1 + \lambda)^2 a_2^2] = [B_1 b_2 + B_2 b_3^2],
\] (19)
\[
-1 + \beta e^{i\gamma}(1 + \lambda) a_2 = B_1 c_1,
\] (20)
\[
(1 + \beta e^{i\gamma})[-2(1 + 2\lambda) a_3 + 4(1 + 2\lambda) - (1 + \lambda)^2 a_2^2] = [B_1 c_2 + B_2 c_3^2].
\] (21)
\[ b_1 = -c_1. \] (22)

Adding (19) to (21) implies
\[ 2(1 + \beta e^{i\gamma})|2(1 + 2\lambda) - (1 + \lambda)^2|a_2^2 = B_1(b_2 + c_2) + B_2\left(b_1^2 + c_1^2\right). \] (23)

Taking the absolute values of both sides of the above equation, we get
\[ |a_2|^2 \leq \frac{B_1}{2|1 + \beta e^{i\gamma}|(1 + 2\lambda - \lambda^2)}\left|b_2 + \frac{B_2}{B_1}b_1^2 + c_2 + \frac{B_2}{B_1}c_1^2\right|^2. \] (24)

Case 1. Let \( B_2 \leq 0. \) Applying Lemma 2 with \( \rho = \frac{B_2}{B_1} \leq 0 \) and using (22) we obtain
\[ |a_2|^2 \leq \frac{B_1}{|1 + \beta e^{i\gamma}|(1 + 2\lambda - \lambda^2)}\left(1 - |\frac{B_1 + B_2}{B_1}|b_1|^2\right). \]

If \( B_1 + B_2 \geq 0, \) then (18) yields
\[ |a_2| \leq \frac{B_1\sqrt{B_1}}{|1 + \beta e^{i\gamma}|(1 + 2\lambda - \lambda^2)B_1^2 + |1 + \beta e^{i\gamma}|^2(1 + \lambda)^2(B_1 + B_2)}. \] (25)

Case 2. Let \( B_2 > 0. \) Applying Lemma 2 with \( \rho = \frac{B_2}{B_1} > 0 \) and using (22), we obtain
\[ |a_2|^2 \leq \frac{B_1}{|1 + \beta e^{i\gamma}|[2(1 + 2\lambda) - (1 + \lambda)^2]}\left(1 - |\frac{B_1 - B_2}{B_1}|b_1|^2\right). \]

If \( B_1 - B_2 \geq 0, \) then (18) gives
\[ |a_2| \leq \frac{B_1\sqrt{B_1}}{|1 + \beta e^{i\gamma}|[1 + 2\lambda - \lambda^2]B_2^2 + |1 + \beta e^{i\gamma}|^2(1 + \lambda)^2(B_1 - B_2)}. \] (26)

From (25) and (26) we obtain the desired estimate of \( |a_2| \) given by (16). Next, from (19) and (21), we have
\[ |a_3 - a_2|^2 \leq \frac{B_1}{4(1 + 2\lambda)|1 + \beta e^{i\gamma}|}\left|b_2 + \frac{B_2}{B_1}b_1^2 + c_2 + \frac{B_2}{B_1}c_1^2\right|^2. \] (27)

Let \( B_2 \leq 0. \) Applying Lemma 2 for \( \rho = \frac{B_2}{B_1} \leq 0, \) we get
\[ |a_3 - a_2|^2 \leq \frac{B_1}{4(1 + 2\lambda)|1 + \beta e^{i\gamma}|}\left(\left|1 - \frac{B_1 + B_2}{B_1}|b_1|^2\right| + |1 - \frac{B_1 + B_2}{B_1}|c_1|^2\right). \] (28)

If \( B_1 + B_2 \geq 0, \) then (28) gives
\[ |a_3 - a_2|^2 \leq \frac{B_1}{2(1 + 2\lambda)|1 + \beta e^{i\gamma}|}. \]

If \( B_1 + B_2 < 0, \) then (9) and (28) lead to
\[ |a_3 - a_2|^2 \leq \frac{B_1}{2(1 + 2\lambda)|1 + \beta e^{i\gamma}|}\left[1 - \frac{B_1 + B_2}{B_1}\right] = -\frac{B_2}{2(1 + 2\lambda)|1 + \beta e^{i\gamma}|}. \]
Let \( B_2 > 0 \). Applying Lemma 2 for \( \rho = \frac{B_2}{M} > 0 \), (27) gives
\[
|a_3 - a_2^2| \leq \frac{B_1}{4(2\lambda)|1 + \beta e^{i\gamma}|} \left( \left| 1 - \frac{B_1 - B_2}{B_1} |b_1| \right|^2 + \left| 1 - \frac{B_1 - B_2}{B_1} |c_1| \right|^2 \right).
\]
(29)
If \( B_1 - B_2 \geq 0 \), then (29) gives
\[
|a_3 - a_2^2| \leq \frac{B_1}{2(1 + 2\lambda)|1 + \beta e^{i\gamma}|}.
\]
If \( B_1 - B_2 < 0 \), then from (9) and (29) we have
\[
|a_3 - a_2^2| \leq \frac{B_1}{2(1 + 2\lambda)|1 + \beta e^{i\gamma}|} \left( 1 - \frac{B_1 - B_2}{B_1} \right) = \frac{B_2}{2(1 + 2\lambda)|1 + \beta e^{i\gamma}|}.
\]
Which is the second part of assertion (17). This completes the proof of Theorem 2.

**Remark 1.** If we take \( \beta = 0 \) in Theorem 2 we obtain that the bounds on \( |a_3 - a_2^2| \) given by Deniz et al. [52] when \( \gamma = 1 \).

If we set
\[
\phi(z) = \frac{1 + A z}{1 + B z} = 1 + (A - B)z - B(A - B)z^2 + B^2(A - B)z^3 \ldots
\]
in Definition 3 of the bi-univalent function class \( M_\sigma(\lambda, \beta, \gamma, \phi) \), we obtain a new class \( M_\sigma(\lambda, \beta, \gamma, A, B) \) given by Definition 5 below.

**Definition 5.** A function \( f \in \sigma \) given by (1) is said to be in the class \( M_\sigma(\lambda, \beta, \gamma, A, B) \), \(-1 \leq B < A \leq 1\), if the following conditions are satisfied:
\[
(1 + \beta e^{i\gamma}) \frac{zf'(z) + \lambda z^2 f''(z)}{(1 - \lambda)f(z) + \lambda z f'(z)} - \beta e^{i\gamma} (1 + \frac{A z}{1 + B z}) (z \in U)
\]
and
\[
(1 + \beta e^{i\gamma}) \frac{zg'(\omega) + \lambda z^2 g''(\omega)}{(1 - \lambda)g(\omega) + \lambda zg'(\omega)} - \beta e^{i\gamma} (1 + \frac{A \omega}{1 + B \omega}) (\omega \in U),
\]
where \( g = f^{-1} \).

Using the parameter setting of Definition 5 in Theorems 1 and 2, respectively, we get the following corollaries.

**Corollary 1.** Let the function \( f \in M_\sigma(\lambda, \beta, \gamma, A, B) \) be given by (1). If \( a_k = 0 \) for \( 2 \leq k \leq n - 1 \), then
\[
|a_n| \leq \frac{(A - B)}{(n - 1)(1 + \lambda(n - 1))|1 + \beta e^{i\gamma}|} \quad n \geq 3.
\]

**Corollary 2.** If the function \( f \in \sigma \) given by (1) be in the class \( M_\sigma(\lambda, \beta, \gamma, A, B) \), then
\[
|a_2| \leq \left\{ \begin{array}{ll}
\left( \frac{A - B}{2(1 + 2\lambda)|1 + \beta e^{i\gamma}|} \right) & (B \geq 0) \\
\left( \frac{A - B}{\sqrt{|1 + \beta e^{i\gamma}|(1 + 2\lambda - A^2)|1 + \beta e^{i\gamma}|^2(1 - B)^2(1 - B)^2}} \right) & (1 \leq B < 0)
\end{array} \right.
\]
and
\[
|a_3 - a_2^2| \leq \frac{A - B}{2(1 + 2\lambda)|1 + \beta e^{i\gamma}|}.
\]
If we set
\[ \phi(z) = \left( \frac{1 + z}{1 - z} \right)^{\delta} = 1 + 2\delta z + 2\delta^2 z^2 + \ldots (0 < \delta \leq 1, z \in U) \]

in Definition 3 of the bi-univalent function class \( M_{\sigma}(\lambda, \beta, \gamma, \phi) \), we obtain a new class \( M_{\sigma}(\lambda, \beta, \gamma, \phi) \) given by Definition 6 below.

**Definition 6.** Let \( 0 < \delta \leq 1 \). A function \( f \in \sigma \) given by (1) is said to be in the class \( M_{\sigma}(\lambda, \beta, \gamma, \phi) \), if the following conditions are satisfied:

\[
\left| \arg \left( \frac{zf'(z) + \lambda z^2 f''(z) - \beta e^{i\gamma}}{(1 - \lambda)f(z) + \lambda zf'(z)} \right) \right| \leq \frac{\pi}{2} \delta (z \in U) \\
\]

and

\[
\left| \arg \left( \frac{zg'(\omega) + \lambda z^2 g''(\omega) - \beta e^{i\gamma}}{(1 - \lambda)g(\omega) + \lambda zg'(\omega)} \right) \right| \leq \frac{\pi}{2} \delta (\omega \in U),
\]

where \( g = f^{-1} \).

Using the parameter setting of Definition 6 in Theorems 1 and 2, respectively, we get the following corollaries.

**Corollary 3.** Let the function \( f \in M_{\sigma}(\lambda, \beta, \gamma, \phi) \) be given by (1). If \( a_k = 0 \) for \( 2 \leq k \leq n - 1 \), then

\[ |a_n| \leq \frac{2\delta}{(n - 1)[1 + \lambda(n - 1)]} |1 + \beta e^{i\gamma}|, \quad n \geq 3. \]

**Corollary 4.** Let \( 0 < \delta \leq 1 \). If the function \( f \in \sigma \) given by (1) be in the class \( M_{\sigma}(\lambda, \beta, \gamma, \phi) \), then

\[ |a_2| \leq \frac{2\delta}{\sqrt{[1 + \beta e^{i\gamma}]^2[1 + 2\lambda - 2\delta] + (1 + \beta e^{i\gamma})^2(1 + \lambda)^2(1 - \delta)}} \\
\]

and

\[ |a_3 - a_2^2| \leq \frac{\delta}{(1 + 2\lambda)[1 + \beta e^{i\gamma}]}. \]

If we set
\[ \phi(z) = \frac{1 + (1 - 2v)z}{1 - z} = 1 + 2(1 - v)z + 2(1 - v)z^2 + \ldots (0 \leq v < 1, z \in U) \]

in Definition 3 of the bi-univalent function class \( M_{\sigma}(\lambda, \beta, \gamma, \phi) \), we obtain a new class \( M_{\sigma}(\lambda, \beta, \gamma) \) given by Definition 7 below.

**Definition 7.** Let \( 0 \leq v < 1 \). A function \( f \in \sigma \) given by (1) is said to be in the class \( M_{\sigma}(\lambda, \beta, \gamma) \) if the following conditions hold true:

\[
\Re \left( \frac{zf'(z) + \lambda z^2 f''(z) - \beta e^{i\gamma}}{(1 - \lambda)f(z) + \lambda zf'(z)} \right) > v (z \in U) \\
\]

and

\[
\Re \left( \frac{zg'(\omega) + \lambda z^2 g''(\omega) - \beta e^{i\gamma}}{(1 - \lambda)g(\omega) + \lambda zg'(\omega)} \right) > v (\omega \in U),
\]

where \( g = f^{-1} \).
Using the parameter setting of Definition 7 in Theorems 1 and 2, respectively, we get the following corollaries.

**Corollary 5.** Let the function \( f \in M_0^\sigma(\lambda, \beta, \gamma) \) be given by (1). If \( a_k = 0 \) for \( 2 \leq k \leq n - 1 \), then

\[
|a_n| \leq \frac{2(1 - v)}{(n - 1)|1 + \lambda(n - 1)||1 + \beta e^{i\gamma}|}, \quad n \geq 3.
\]

**Corollary 6.** Let the function \( f \in M_0^\sigma(\lambda, \beta, \gamma) \) be given by (1). Then

\[
|a_2| \leq \sqrt{\frac{2(1 - v)}{(1 + 2\lambda - \lambda^2)|1 + \beta e^{i\gamma}|}}
\]

and

\[
|a_3 - a_2^2| \leq \frac{(1 - v)}{(1 + 2\lambda)|1 + \beta e^{i\gamma}|}.
\]

If we set

\[
\phi(z) = \sqrt{1 + z} = 1 + \frac{1}{2}z - \frac{1}{8}z^2 + \ldots (z \in U)
\]

in Definition 3 of the bi-univalent function class \( M_\sigma(\lambda, \beta, \gamma, \phi) \), we obtain a new class \( M_{L\sigma}(\lambda, \beta, \gamma) \) given by Definition 8 below.

**Definition 8.** A function \( f \in \sigma \) given by (1) is said to be in the class \( M_{L\sigma}(\lambda, \beta, \gamma) \), if the following conditions are satisfied:

\[
\left| \frac{(1 + \beta e^{i\gamma}) \frac{z f'(z) + \lambda z^2 f''(z)}{(1 - \lambda)f(z) + \lambda z f'(z)} - \beta e^{i\gamma}}{1} \right| < 1 \quad (z \in U)
\]

and

\[
\left| \frac{(1 + \beta e^{i\gamma}) \frac{z g'(\omega) + \lambda z^2 g''(\omega)}{(1 - \lambda)g(\omega) + \lambda z g'(\omega)} - \beta e^{i\gamma}}{1} \right| < 1 \quad (\omega \in U),
\]

where \( g = f^{-1} \).

Using the parameter setting of Definition 8 in Theorems 1 and 2, respectively, we get the following corollaries.

**Corollary 7.** Let the function \( f \in M_{L\sigma}(\lambda, \beta, \gamma) \) be given by (1). If \( a_k = 0 \) for \( 2 \leq k \leq n - 1 \), then

\[
|a_n| \leq \frac{1}{2(n - 1)|1 + \lambda(n - 1)||1 + \beta e^{i\gamma}|}, \quad n \geq 3.
\]

**Corollary 8.** If the function \( f \in \sigma \) given by (1) be in the class \( M_{L\sigma}(\lambda, \beta, \gamma) \), then

\[
|a_2| \leq \frac{1}{\sqrt{2}|1 + \beta e^{i\gamma}||1 + 2\lambda - \lambda^2| + 3|1 + \beta e^{i\gamma}|^2(1 + \lambda)^2}
\]

and

\[
|a_3 - a_2^2| \leq \frac{1}{4(1 + 2\lambda)|1 + \beta e^{i\gamma}|}.
\]

If we set

\[
\phi(z) = z + \sqrt{1 + z^2} = 1 + z + \frac{1}{2}z^2 - \frac{1}{8}z^4 \ldots (z \in U),
\]
in Definition 3 of the bi-univalent function class $M_{\sigma}(\lambda, \beta, \gamma, \phi)$, we obtain a new class $M_{\sigma}^{3}(\lambda, \beta, \gamma)$ given by Definition 9 below.

**Definition 9.** A function $f \in \sigma$ given by (1) is said to be in the class $M_{\sigma}^{3}(\lambda, \beta, \gamma)$ if the following conditions are satisfied:

$$\left| \frac{\left(1 + \beta e^{i\gamma}\right) zf(z) + \lambda z^2 f^\prime(z)}{(1 - \lambda)f(z) + \lambda zf(z)} - \beta e^{i\gamma} \right|^2 - 1 < 2 \left| \frac{zf(z) + \lambda z^2 f^\prime(z)}{(1 - \lambda)f(z) + \lambda zf(z)} - \beta e^{i\gamma} \right| (z \in \mathbb{U})$$

and

$$\left| \frac{\left(1 + \beta e^{i\gamma}\right) zg'(\omega) + \lambda z^2 g''(\omega)}{(1 - \lambda)g(\omega) + \lambda zg''(\omega)} - \beta e^{i\gamma} \right|^2 - 1 < 2 \left| \frac{g'(\omega) + \lambda z^2 g''(\omega)}{(1 - \lambda)g(\omega) + \lambda zg''(\omega)} - \beta e^{i\gamma} \right| (\omega \in \mathbb{U})$$

where $g = f^{-1}$.

Using the parameter setting of Definition 9 in Theorems 1 and 2, respectively, we get the following corollaries.

**Corollary 9.** Let the function $f \in M_{\sigma}^{3}(\lambda, \beta, \gamma)$ be given by (1). If $a_k = 0$ for $2 \leq k \leq n - 1$, then

$$|a_n| \leq \frac{1}{(n - 1)|1 + \lambda(n - 1)|1 + \beta e^{i\gamma}|}, \quad n \geq 3.$$  

**Corollary 10.** If the function $f \in \sigma$ given by (1) be in the class $M_{\sigma}^{3}(\lambda, \beta, \gamma)$, then

$$|a_2| \leq \sqrt{\frac{2}{2|1 + \beta e^{i\gamma}||1 + 2\lambda - \lambda^2| + |1 + \beta e^{i\gamma}|^2(1 + \lambda)^2}}$$

and

$$|a_3 - a_2^2| \leq \frac{1}{2(1 + 2\lambda)|1 + \beta e^{i\gamma}|}.$$  

If we set

$$\phi(z) = \frac{1}{(1 - z)^s} = 1 + sz + \frac{s(s + 1)}{2}z^2 + \frac{s(s + 1)(s + 2)}{6}z^3 \ldots$$

$$= 1 + \sum_{n=1}^{\infty} \frac{s(s + 1) \ldots (s + n - 1)}{n!}z^n \quad (z \in U),$$

in Definition 3 of the bi-univalent function class $M_{\sigma}(\lambda, \beta, \gamma, \phi)$, we obtain a new class $M_{\sigma}(\lambda, \beta, \gamma, s)$ given by Definition 10 below.

**Definition 10.** Let $0 < s \leq 1$. A function $f \in \sigma$ given by (1) is said to be in the class $M_{\sigma}(\lambda, \beta, \gamma, s)$, if the following conditions are satisfied:

$$\left| \frac{(1 + \beta e^{i\gamma}) zf(z) + \lambda z^2 f^\prime(z)}{(1 - \lambda)f(z) + \lambda zf(z)} - \beta e^{i\gamma} \right| < \frac{1}{(1 - z)^s} \quad (z \in \mathbb{U})$$
and

\[(1 + \beta e^{i\gamma}) \frac{zg'(\omega) + \lambda z^2 g''(\omega)}{(1 - \lambda)g(\omega) + \lambda zg'(\omega)} - \beta e^{i\gamma} \lesssim \frac{1}{(1 - \omega)^s} \quad (\omega \in U),\]

where \(g = f^{-1} \).

Using the parameter setting of Definition 10 in Theorems 1 and 2, respectively, we get the following corollaries.

**Corollary 11.** Let the function \(f \in M_\sigma(\lambda, \beta, \gamma, s)\) be given by (1). If \(a_k = 0\) for \(2 \leq k \leq n - 1\), then

\[|a_n| \leq \frac{s}{(n - 1)[1 + \lambda(n - 1)][1 + \beta e^{i\gamma}]^s} \quad n \geq 3.\]

**Corollary 12.** If the function \(f \in \sigma\) given by (1) be in the class \(M_\sigma(\lambda, \beta, \gamma, s)\), then

\[|a_2| \leq \frac{\sqrt{2s}}{\sqrt{[1 + \beta e^{i\gamma}][1 + 2\lambda - \lambda^2]2s + [1 + \beta e^{i\gamma}]^2(1 + \lambda)^2(1 - s)}},\]

and

\[|a_3 - a_2| \leq \frac{s}{2(1 + 2\lambda)[1 + \beta e^{i\gamma}]}.\]

If we set

\[\phi(z) = e^z = 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \ldots \quad (z \in U),\]

in Definition 3 of the bi-univalent function class \(M_\sigma(\lambda, \beta, \gamma, \phi)\), we obtain a new class \(M_{\sigma\sigma}(\lambda, \beta, \gamma)\) given by Definition 11 below.

**Definition 11.** A function \(f \in \sigma\) given by (1) is said to be in the class \(M_{\sigma\sigma}(\lambda, \beta, \gamma)\) if the following conditions are satisfied:

\[\left| \log \left(1 + \beta e^{i\gamma} \right) \frac{zf(z) + \lambda z^2 f''(z)}{(1 - \lambda)f(z) + \lambda zf'(z)} - \beta e^{i\gamma} \right| < 1 \quad (z \in U),\]

and

\[\left| \log \left(1 + \beta e^{i\gamma} \right) \frac{zg'(\omega) + \lambda z^2 g''(\omega)}{(1 - \lambda)g(\omega) + \lambda zg'(\omega)} - \beta e^{i\gamma} \right| < 1 \quad (\omega \in U),\]

where \(g = f^{-1} \).

Using the parameter setting of Definition 11 in Theorems 1 and 2, respectively, we get the following corollaries.

**Corollary 13.** Let the function \(f \in M_\sigma(\lambda, \beta, \gamma, s)\), be given by (1). If \(a_k = 0\) for \(2 \leq k \leq n - 1\), then

\[|a_n| \leq \frac{1}{(n - 1)[1 + \lambda(n - 1)][1 + \beta e^{i\gamma}]^s} \quad n \geq 3.\]

**Corollary 14.** If the function \(f \in \sigma\) given by (1) be in the class \(M_\sigma(\lambda, \beta, \gamma, s)\), then

\[|a_2| \leq \frac{2}{\sqrt{[1 + \beta e^{i\gamma}][1 + 2\lambda - \lambda^2] + [1 + \beta e^{i\gamma}]^2(1 + \lambda)^2}}.\]
and
\[ |a_3 - a_2| \leq \frac{1}{2(1 + 2\lambda)[1 + \beta e^{i\gamma}]} \]

3. Coefficient Estimates for the Class \( S_c(\lambda, \beta, \gamma, \phi) \)

**Theorem 3.** Let the function \( f \in \sigma \) given by (1) be in the class \( S_c(\lambda, \beta, \gamma, \phi) \). If \( a_k = 0 \) for \( 2 \leq k \leq n - 1 \), then
\[
|a_n| \leq \frac{B_1}{[1 + \beta e^{i\gamma}][n - 1][1 + (n - 1)\lambda]} \quad n \geq 3.
\]

**Proof.** Since, both functions \( f \) and its inverse map \( g = f^{-1} \) are in \( S_c(\lambda, \beta, \gamma, \phi) \), by the definition of subordination, there are analytic functions \( u, v : U \to U \) given by (8) such that
\[
(1 + \beta e^{i\gamma}) \left[ (1 - \lambda) \frac{z f'(z)}{f(z)} + \lambda \left( \frac{z f'(z)}{f(z)} \right)' \right] - \beta e^{i\gamma} = \phi(u(z))
\]
and
\[
(1 + \beta e^{i\gamma}) \left[ (1 - \lambda) \frac{w g'(w)}{g(w)} + \lambda \left( \frac{w g'(w)}{g(w)} \right)' \right] - \beta e^{i\gamma} = \phi(v(\omega)).
\]
Now, from (2), we get that
\[
(1 + \beta e^{i\gamma}) \left[ (1 - \lambda) \frac{z f'(z)}{f(z)} + \lambda \left( \frac{z f'(z)}{f(z)} \right)' \right] - \beta e^{i\gamma}
\]
\[
= 1 - (1 + \beta e^{i\gamma}) \sum_{j=2}^{\infty} [(1 - \lambda)F_{j-1}(a_2, a_3, \ldots, a_j) + \lambda F_{j-1}(2a_2, 3a_3, \ldots, ja_j)]z^{j-1}, \tag{30}
\]
and
\[
(1 + \beta e^{i\gamma}) \left[ (1 - \lambda) \frac{w g'(w)}{g(w)} + \lambda \left( \frac{w g'(w)}{g(w)} \right)' \right] - \beta e^{i\gamma}
\]
\[
= 1 - (1 + \beta e^{i\gamma}) \sum_{j=2}^{\infty} [(1 - \lambda)F_{j-1}(d_2, d_3, \ldots, d_j) + \lambda F_{j-1}(2d_2, 3d_3, \ldots, jd_j)]w^{j-1}, \tag{31}
\]
where \( d_n = \frac{1}{n}K_{n-1}^{-a_n}(a_2, a_3, \ldots, a_n) \). Now, upon comparing the corresponding coefficients in (10) and (30), we get
\[
(1 + \beta e^{i\gamma})[(1 - \lambda)F_{n-1}(a_2, a_3, \ldots, a_n) + \lambda F_{n-1}(2a_2, 3a_3, \ldots, na_n)]
\]
\[
= B_1K_{n-1}^{-1}(b_1, b_2, \ldots, b_{n-1}, B_1, B_1, B_2, B_3, \ldots, B_{n-1}) \tag{32}
\]
and similarly, from (11) and (31), we have
\[
(1 + \beta e^{i\gamma})[(1 - \lambda)F_{n-1}(d_2, d_3, \ldots, d_n) + \lambda F_{n-1}(2d_2, 3d_3, \ldots, nd_n)]
\]
\[
= B_1K_{n-1}^{-1}(c_1, c_2, \ldots, c_{n-1}, B_1, B_1, B_2, B_3, \ldots, B_{n-1}). \tag{33}
\]
Since \( a_k = 0 \) for \( 2 \leq k \leq n - 1 \), by using \( d_n = -a_n \) and \( F_n-1(a_2, a_3, \ldots, a_n) = -(n-1)a_n \), we have

\[
(1 + \beta e^{i\gamma})(n - 1)[1 + (n - 1)\lambda]a_n = B_1b_{n-1}
\]

and

\[
-(1 + \beta e^{i\gamma})(n - 1)[1 + (n - 1)\lambda]a_n = B_1c_{n-1}.
\]

By using (9), we conclude that

\[
|a_n| \leq \frac{B_1}{1 + \beta e^{i\gamma}([n - 1][1 + (n - 1)\lambda]).
\]

\[
\Box
\]

**Remark 2.** If we take \( \beta = 0 \) in Theorem 3, then we have the results which were given by Zireh et al. [51] when \( \varphi(z) = 1 \).

**Theorem 4.** If the function \( f \in \sigma \) given by (1) be in the class \( S_{\varphi}(\lambda, \beta, \gamma, \phi) \), then

\[
|a_2| \leq \begin{cases}
\frac{B_1\sqrt{B_1}}{\sqrt{1 + \beta e^{i\gamma}|(1 + \lambda)B_1^2 + [1 + \beta e^{i\gamma}]^2(1 + \lambda)^2(B_1 + B_2)}} & (B_2 \leq 0, B_1 + B_2 \geq 0) \\
\frac{B_1\sqrt{B_1}}{\sqrt{1 + \beta e^{i\gamma}|(1 + \lambda)B_1^2 + [1 + \beta e^{i\gamma}]^2(1 + \lambda)^2(B_1 - B_2)}} & (B_2 > 0, B_1 - B_2 \geq 0)
\end{cases}
\]

and

\[
|a_3 - a_2^2| \leq \begin{cases}
\frac{B_1}{2[1 + \beta e^{i\gamma}(1 + 2\lambda)]} & (B_1 \geq |B_2|) \\
\frac{B_1}{2[1 + \beta e^{i\gamma}(1 + 2\lambda)]} & (B_1 < |B_2|)
\end{cases}
\]

**Proof.** Letting \( n = 2 \) and \( 3 \) in (32) and (33), respectively, we find that

\[
(1 + \beta e^{i\gamma})(1 + \lambda)a_2 = B_1b_1,
\]

\[
(1 + \beta e^{i\gamma})[2(1 + 2\lambda)a_3 - (1 + 3\lambda)a_2^2] = B_1b_2 + B_2b_1^2,
\]

\[
-(1 + \beta e^{i\gamma})(1 + \lambda)a_2 = B_1c_1,
\]

\[
(1 + \beta e^{i\gamma})\{ -2(1 + 2\lambda)a_3 + [4(1 + 2\lambda) - (1 + 3\lambda)a_2^2 \} = B_1c_2 + B_2c_1^2.
\]

Equations (38) and (40) lead to

\[
b_1 = -c_1.
\]

Adding (39) and (41) yields

\[
2(1 + \beta e^{i\gamma})(1 + \lambda)a_2^2 = B_1(b_2 + c_2) + B_2\left(b_1^2 + c_1^2\right)
\]

or

\[
|a_2^2| \leq \frac{B_1}{2[1 + \beta e^{i\gamma}(1 + \lambda)]}\left(|b_2 + \frac{B_2}{B_1}b_1^2| + |c_2 + \frac{B_2}{B_1}c_1^2|\right).
\]

First, let \( B_2 \leq 0 \). Applying Lemma 2 with \( \rho = \frac{B_2}{B_1} \leq 0 \) and using (42), we get

\[
|a_2^2| \leq \frac{B_1}{1 + \beta e^{i\gamma}(1 + \lambda)}\left(1 - \left|\frac{B_1 + B_2}{B_1}\right|b_1^2\right)
\]
If \( B_1 + B_2 \geq 0 \), then (38) yields
\[
|a_2| \leq \frac{B_1}{\sqrt{1 + \beta e^{i\gamma}}(1 + \lambda) B_1^2 + |1 + \beta e^{i\gamma}|^2(1 + \lambda)^2(B_1 + B_2)}.
\]
(45)

Similarly, for \( B_2 > 0(\rho = \frac{B_2}{\rho_1} > 0, B_1 - B_2 \geq 0) \), we have
\[
|a_2| \leq \frac{B_1}{\sqrt{1 + \beta e^{i\gamma}}(1 + \lambda) B_1^2 + |1 + \beta e^{i\gamma}|^2(1 + \lambda)^2(B_1 - B_2)}.
\]
(46)

From (45) and (46) we obtain the desired estimate of \(|a_2|\) given by (36).

Next, in order to find the bound on \(|a_3 - a_2^2|\), by subtracting (41) from (39), we have
\[
|a_3 - a_2^2| \leq \frac{B_1}{4|1 + \beta e^{i\gamma}|(1 + 2\lambda)} \left( \left| B_2 + \frac{B_2}{B_1} |b_1|^2 \right| + \left| c_2 + \frac{B_2}{B_1} c_1^2 \right| \right).
\]
(47)

Let \( B_2 \leq 0 \). Applying Lemma 2 with \( \rho = \frac{B_2}{\rho_1} \leq 0 \), we get
\[
|a_3 - a_2^2| \leq \frac{B_1}{4|1 + \beta e^{i\gamma}|(1 + 2\lambda)} \left( 1 - \frac{B_1 + B_2}{B_1} |b_1|^2 \right) + \left| 1 - \frac{B_1 + B_2}{B_1} |c_1|^2 \right| \right).
\]
(48)

If \( B_1 + B_2 \geq 0 \), then (48) gives \(|a_3 - a_2^2| \leq \frac{B_1}{2|1 + \beta e^{i\gamma}|(1 + 2\lambda)}\).

If \( B_1 + B_2 < 0 \), then (9) and (48) give
\[
|a_3 - a_2^2| \leq \frac{B_1}{2|1 + \beta e^{i\gamma}|(1 + 2\lambda)} \left( 1 - \frac{B_1 + B_2}{B_1} \right) + \left| 1 - \frac{B_1 + B_2}{B_1} |c_1|^2 \right| \right).
\]
(49)

Let \( B_2 > 0 \). Applying Lemma 2 with \( \rho = \frac{B_2}{\rho_1} > 0 \), (47) gives
\[
|a_3 - a_2^2| \leq \frac{B_1}{4|1 + \beta e^{i\gamma}|(1 + 2\lambda)} \left( \left| B_2 + \frac{B_2}{B_1} |b_1|^2 \right| + \left| 1 - \frac{B_1 + B_2}{B_1} |c_1|^2 \right| \right).
\]
(49)

If \( B_1 - B_2 \geq 0 \), then (49) gives
\[
|a_3 - a_2^2| \leq \frac{B_1}{2|1 + \beta e^{i\gamma}|(1 + 2\lambda)}.
\]

If \( B_1 - B_2 < 0 \), then from (9) and (49) we get
\[
|a_3 - a_2^2| \leq \frac{B_1}{2|1 + \beta e^{i\gamma}|(1 + 2\lambda)} \left( 1 - \frac{B_1 - B_2}{B_1} \right) = \frac{B_2}{2|1 + \beta e^{i\gamma}|(1 + 2\lambda)}.
\]

This completes the proof of Theorem 3. \( \Box \)

**Remark 3.** If we set \( \beta = 0 \) in Theorem 4, then we obtain the results of Goyal and Kumar [50] when \( \phi(z) = 1 \).

If we set \( \phi(z) = \left( \frac{1+z}{1-z} \right)^{\delta} (0 < \delta \leq 1, z \in U) \) in Definition 4 of the bi-univalent function class \( S_c(\lambda, \beta, \gamma, \phi) \), we obtain a new class \( S_c^\delta(\lambda, \beta, \gamma) \) given by Definition 12 below.
Definition 12. Let $0 < \delta \leq 1$. A function $f \in \sigma$ given by (1) is said to be in the class $S_{\varphi}^{\delta}(\lambda, \beta, \gamma)$ if the following subordinations hold:

$$\left| \arg\left(1 + \beta e^{i\gamma}\right) \left((1 - \lambda)\frac{zf(z)}{f(z)} + \lambda(1 + \frac{zf''(z)}{f'(z)})\right) - \beta e^{i\gamma}\right| \leq \frac{\pi}{2}\delta \ (z \in U)$$

and

$$\left| \arg\left(1 + \beta e^{i\gamma}\right) \left((1 - \lambda)\frac{wg'(w)}{g(w)} + \lambda(1 + \frac{wg''(w)}{g'(w)})\right) - \beta e^{i\gamma}\right| \leq \frac{\pi}{2}\delta \ (\omega \in U),$$

where $g = f^{-1}$.

Using the parameter setting of Definition 12 in Theorems 3 and 4, respectively, we get the following corollaries.

Corollary 15. Let the function $f \in S_{\varphi}^{\delta}(\lambda, \beta, \gamma)$ be given by (1). If $a_k = 0$ for $2 \leq k \leq n - 1$, then

$$|a_n| \leq \frac{2\delta}{(n - 1)(1 + \lambda(n - 1))\left|1 + \beta e^{i\gamma}\right|}, \quad n \geq 3.$$  

Corollary 16. Let $0 < \gamma \leq 1$. If the function $f \in \sigma$ given by (1) be in the class $S_{\varphi}^{\delta}(\lambda, \beta, \gamma)$, then

$$|a_2| \leq \frac{2\delta}{\sqrt{2|1 + \beta e^{i\gamma}|(1 + \lambda)\delta + \left|1 + \beta e^{i\gamma}\right|^2(1 + \lambda)^2(1 - \delta)}$$

and

$$|a_3 - a_2^2| \leq \frac{\delta}{\left|1 + \beta e^{i\gamma}\right|(1 + 2\lambda)}.$$  

If we set $\phi(z) = \frac{1 + (1 - 2z)}{1 - z} (0 \leq v < 1, z \in U)$ in Definition 4 of the bi-univalent function class $S_{\varphi}(\lambda, \beta, \gamma, \phi)$, we obtain a new class $S_{\varphi}(\lambda, \beta, \gamma, v)$ given by Definition 13 below.

Definition 13. Let $0 \leq v < 1$. A function $f \in \sigma$ given by (1) is said to be in the class $S_{\varphi}(\lambda, \beta, \gamma, v)$, if the following conditions are satisfied:

$$\Re\left(1 + \beta e^{i\gamma}\right) \left((1 - \lambda)\frac{zf(z)}{f(z)} + \lambda(1 + \frac{zf''(z)}{f'(z)})\right) - \beta e^{i\gamma}\right) > v \ (z \in U)$$

and

$$\Re\left(1 + \beta e^{i\gamma}\right) \left((1 - \lambda)\frac{wg'(w)}{g(w)} + \lambda(1 + \frac{wg''(w)}{g'(w)})\right) - \beta e^{i\gamma}\right) > v \ (\omega \in U),$$

where $g = f^{-1}$.

Using the parameter setting of Definition 13 in Theorems 3 and 4, respectively, we get the following corollaries.

Corollary 17. Let the function $f \in S_{\varphi}(\lambda, \beta, \gamma, v)$ be given by (1). If $a_k = 0$ for $2 \leq k \leq n - 1$, then

$$|a_n| \leq \frac{2(1 - v)}{(n - 1)(1 + \lambda(n - 1))\left|1 + \beta e^{i\gamma}\right|}, \quad n \geq 3.$$
Corollary 18. Let $0 \leq \nu < 1$. If the function $f \in S_\sigma(\lambda, \beta, \gamma, \nu)$ be of the form (1), then

$$|a_2| \leq \sqrt{\frac{2(1 - \nu)}{|1 + \beta e^{i\gamma}||1 + \lambda|}}$$

and

$$|a_3 - a_2^2| \leq \frac{(1 - \nu)}{|1 + \beta e^{i\gamma}||1 + 2\lambda|}.$$

If we set $\phi(z) = \frac{1 + A_z}{1 + B_z}$ in Definition 4 of the bi-univalent function class $S_\sigma(\lambda, \beta, \gamma, \phi)$, we obtain a new class $S_\sigma(\lambda, \beta, \gamma, A, B)$ given by Definition 14 below.

Definition 14. A function $f \in \sigma$ given by (1) is said to be in the class $S_\sigma(\lambda, \beta, \gamma, A, B)$, $-1 \leq B < A \leq 1$, if the following conditions are satisfied:

$$(1 + \beta e^{i\gamma}) \left[ (1 - \lambda) \frac{zf'(z)}{f(z)} + \lambda(1 + \frac{zf''(z)}{f'(z)}) \right] - \beta e^{i\gamma} \prec \frac{1 + Az}{1 + Bz} (z \in U)$$

and

$$(1 + \beta e^{i\gamma}) \left[ (1 - \lambda) \frac{wg'(w)}{g(w)} + \lambda(1 + \frac{wg''(w)}{g'(w)}) \right] - \beta e^{i\gamma} \prec \frac{1 + A\omega}{1 + B\omega} (w \in U),$$

where $g = f^{-1}$.

Using the parameter setting of Definition 14 in Theorems 3 and 4, respectively, we get the following corollaries.

Corollary 19. Let the function $f \in S_\sigma(\lambda, \beta, \gamma, A, B)$ be given by (1). If $a_k = 0$ for $2 \leq k \leq n - 1$, then

$$|a_n| \leq \frac{(A - B)}{(n - 1)|1 + \lambda(n - 1)||1 + \beta e^{i\gamma}|} \quad n \geq 3.$$  

Corollary 20. If the function $f \in \sigma$ given by (1) be in the class $S_\sigma(\lambda, \beta, \gamma, A, B)$, then

$$|a_2| \leq \begin{cases} 
\frac{(A - B)}{\sqrt{|1 + \beta e^{i\gamma}||1 + \lambda(A - B) + |1 + \beta e^{i\gamma}||1 - B|}} & (B \geq 0) \\
\frac{(A - B)}{\sqrt{|1 + \beta e^{i\gamma}||1 + \lambda(A - B) + |1 + \beta e^{i\gamma}||1 + B|}} & (-1 \leq B < 0) 
\end{cases}$$

and

$$|a_3 - a_2^2| \leq \frac{A - B}{2(1 + 2\lambda)|1 + \beta e^{i\gamma}|}.$$  

Remark 4. If we put $\beta = \lambda = 0$ in Corollaries 19 and 20, then we obtain the results of Hamidi and Jahangiri [49].

If we set $\phi(z) = \sqrt{1 + z}$ in Definition 4 of the bi-univalent function class $S_\sigma(\lambda, \beta, \gamma, \phi)$, we obtain a new class $S_{\sigma L}(\lambda, \beta, \gamma)$ given by Definition 15 below.

Definition 15. A function $f \in \sigma$ given by (1) is said to be in the class $S_{\sigma L}(\lambda, \beta, \gamma)$ if the following conditions are satisfied:

$$\left| \left(1 + \beta e^{i\gamma}\right) \left[ (1 - \lambda) \frac{zf'(z)}{f(z)} + \lambda(1 + \frac{zf''(z)}{f'(z)}) \right] - \beta e^{i\gamma} \right|^2 \prec 1 \quad (z \in U)$$
where $g = f^{-1}$.

Using the parameter setting of Definition 15 in Theorems 3 and 4, respectively, we get the following corollaries.

**Corollary 21.** Let the function $f \in S_{Lr}(\lambda, \beta, \gamma)$, be given by (1). If $a_k = 0$ for $2 \leq k \leq n - 1$, then

$$|a_n| \leq \frac{1}{2(n-1)|1 + \lambda(n-1)||1 + \beta e^{i\eta}|}, \quad n \geq 3.$$

**Corollary 22.** If the function $f \in \sigma$ given by (1) be in the class $S_{Lr}(\lambda, \beta, \gamma)$, then

$$|a_2| \leq \frac{1}{\sqrt{2|1 + \beta e^{i\eta}|(1 + \lambda) + 3|1 + \beta e^{i\eta}|^2(1 + \lambda)^2}}$$

and

$$|a_3 - a_2^2| \leq \frac{1}{4(1 + 2\lambda)|1 + \beta e^{i\eta}|}.$$

If we set $\phi(z) = z + \sqrt{1 + z^2}$ in Definition 4 of the bi-univalent function class $S_{Lr}(\lambda, \beta, \gamma, \phi)$, we obtain a new class $S^\alpha_{Lr}(\lambda, \beta, \gamma)$ given by Definition 16 below.

**Definition 16.** A function $f \in \sigma$ given by (1) is said to be in the class $S^\alpha_{Lr}(\lambda, \beta, \gamma)$ if the following conditions are satisfied:

$$\left| \left(1 + \beta e^{i\eta} \right) \left[ (1 - \lambda) \frac{w g'(w)}{g(w)} + \lambda \left(1 + \frac{w g''(w)}{g'(w)}\right) - \beta e^{i\eta} \right] - 1 \right| < 2 \left| \left(1 + \beta e^{i\eta} \right) \left[ (1 - \lambda) \frac{w g'(w)}{g(w)} + \lambda \left(1 + \frac{w g''(w)}{g'(w)}\right) - \beta e^{i\eta} \right] \right| (z \in U)$$

and

$$\left| \left(1 + \beta e^{i\eta} \right) \left[ (1 - \lambda) \frac{w g'(w)}{g(w)} + \lambda \left(1 + \frac{w g''(w)}{g'(w)}\right) - \beta e^{i\eta} \right] - 1 \right| < 2 \left| \left(1 + \beta e^{i\eta} \right) \left[ (1 - \lambda) \frac{w g'(w)}{g(w)} + \lambda \left(1 + \frac{w g''(w)}{g'(w)}\right) - \beta e^{i\eta} \right] \right| (w \in U)$$

where $g = f^{-1}$.

Using the parameter setting of Definition 9 in Theorems 3 and 4, respectively, we get the following corollaries.

**Corollary 23.** Let the function $f \in S^\alpha_{Lr}(\lambda, \beta, \gamma)$, be given by (1). If $a_k = 0$ for $2 \leq k \leq n - 1$, then

$$|a_n| \leq \frac{1}{(n-1)|1 + \lambda(n-1)||1 + \beta e^{i\eta}|}, \quad n \geq 3.$$
Corollary 24. If the function \( f \in \sigma \) given by (1) be in the class \( S^{\phi}_{\lambda}(\lambda, \beta, \gamma) \), then
\[
|a_2| \leq \sqrt{\frac{2}{2|1 + \beta e^{i\gamma}|(1 + \lambda) + |1 + \beta e^{i\gamma}|^2(1 + \lambda)^2}}
\]
and
\[
|a_3 - a_2^2| \leq \frac{1}{2(1 + 2\lambda)|1 + \beta e^{i\gamma}|}.
\]

If we set \( \phi(z) = \frac{1}{(1 - z)^2} \) in Definition 4 of the bi-univalent function class \( S_{\phi}(\lambda, \beta, \gamma, \phi) \), we obtain a new class \( S_{\phi}(\lambda, \beta, \gamma, s) \) given by Definition 17 below.

Definition 17. Let \( 0 < s \leq 1 \). A function \( f \in \sigma \) given by (1) is said to be in the class \( S_{\phi}(\lambda, \beta, \gamma, s) \), if the following conditions are satisfied:

\[
(1 + \beta e^{i\gamma}) \left[ (1 - \lambda) \frac{z f'(z)}{f'(z)} + \lambda(1 + \frac{z f''(z)}{f'(z)}) \right] - \beta e^{i\gamma} < \frac{1}{(1 - z)^s} \quad (z \in U)
\]

and
\[
(1 + \beta e^{i\gamma}) \left[ (1 - \lambda) \frac{w g'(w)}{g'(w)} + \lambda(1 + \frac{w g''(w)}{g'(w)}) \right] - \beta e^{i\gamma} < \frac{1}{1 - (1 - \omega)^s} \quad (\omega \in U),
\]

where \( g = f^{-1} \).

Using the parameter setting of Definition 17 in Theorems 3 and 4, respectively, we get the following corollaries.

Corollary 25. Let the function \( f \in S_{\phi}(\lambda, \beta, \gamma, s) \) be given by (1). If \( a_k = 0 \) for \( 2 \leq k \leq n - 1 \), then
\[
|a_n| \leq \frac{s}{(n - 1)(1 + \lambda(n - 1))|1 + \beta e^{i\gamma}|} \quad n \geq 3.
\]

Corollary 26. If the function \( f \in \sigma \) given by (1) be in the class \( S_{\phi}(\lambda, \beta, \gamma, s) \), then
\[
|a_2| \leq \frac{\sqrt{2}s}{\sqrt{1 + \beta e^{i\gamma}|(1 + \lambda)2s + |1 + \beta e^{i\gamma}|^2(1 + \lambda)^2(1 - s)}}
\]

and
\[
|a_3 - a_2^2| \leq \frac{s}{2(1 + 2\lambda)|1 + \beta e^{i\gamma}|}.
\]

If we set \( \phi(z) = e^z \) in Definition 4 of the bi-univalent function class \( S_{\phi}(\lambda, \beta, \gamma, \phi) \), we obtain a new class \( S_{\phi}(\lambda, \beta, \gamma) \) given by Definition 18 below.

Definition 18. A function \( f \in \sigma \) given by (1) is said to be in the class \( S_{\phi}(\lambda, \beta, \gamma) \), if the following conditions are satisfied:

\[
\left| \log \left( (1 + \beta e^{i\gamma}) \left[ (1 - \lambda) \frac{z f'(z)}{f'(z)} + \lambda(1 + \frac{z f''(z)}{f'(z)}) \right] - \beta e^{i\gamma} \right) \right| < 1 \quad (z \in U)
\]

and
\[
\left| \log \left( (1 + \beta e^{i\gamma}) \left[ (1 - \lambda) \frac{w g'(w)}{g'(w)} + \lambda(1 + \frac{w g''(w)}{g'(w)}) \right] - \beta e^{i\gamma} \right) \right| < 1 \quad (\omega \in U),
\]
where \( g = f^{-1} \).

Using the parameter setting of Definition 18 in Theorems 3 and 4, respectively, we get the following corollaries.

**Corollary 27.** Let the function \( f \in S_\sigma(\lambda, \beta, \gamma, s) \), be given by (1). If \( a_k = 0 \) for \( 2 \leq k \leq n - 1 \), then
\[
|a_n| \leq \frac{1}{(n-1)[1+\lambda(n-1)][1+\beta e^{i\gamma}]}, \quad n \geq 3.
\]

**Corollary 28.** If the function \( f \in \sigma \) given by (1) be in the class \( S_\sigma(\lambda, \beta, \gamma, s) \), then
\[
|a_2| \leq \frac{2}{\sqrt{2[1+\beta e^{i\gamma}][(1+\alpha) + |1+\beta e^{i\gamma}|^2(1+\alpha)^2}}
\]
and
\[
|a_3 - a_2^2| \leq \frac{1}{2(1+2\lambda)[1+\beta e^{i\gamma}]}.\]

**Author Contributions:** All authors contributed equally to the writing of this paper. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research has been funded by Scientific Research Deanship at University of Hai‘l- Saudi Arabia through project number RG-20020.

**Institutional Review Board Statement:** Not applicable.

**Informed Consent Statement:** Not applicable.

**Data Availability Statement:** Not applicable.

**Acknowledgments:** This research has been funded by Scientific Research Deanship at University of Hai‘l- Saudi Arabia through project number RG-20020.

**Conflicts of Interest:** The authors declare no conflict of interest.

**References**

1. Faber, G. Über polynomische Entwickelungen. *Math. Ann.* 1903, 57, 389–408, ISSN 0025-5831, doi:10.1007/BF01444293.
2. Curtiss, J.H. Faber Polynomials and the Faber Series. *Am. Math. Mon. Math. Assoc. Am.* 1971, 78, 577–596, doi:10.2307/2316567, ISSN 0002-9890.
3. Faber, G. Über Tschebyscheffsche Polynome. *J. Reine Angew. Math.* 1919, 150, 79–106, ISSN 0075-4102. (In German)
4. Grunsky, H. Koeffizientenbedingungen für schlicht abbildende meromorphe Funktionen. *Math. Z.* 1939, 45, 29–61, ISSN 0025-5874, doi:10.1007/BF01580272.
5. Schur, I. On Faber polynomials. *Am. J. Math.* 1945, 67, 33–41, ISSN 0002-9327, doi:10.2307/2371913.
6. Suetin, P.K. *Series of Faber Polynomials; Analytical Methods and Special Functions*, 1; Gordon and Breach Science Publishers: New York, NY, USA, 1998; ISBN 978-90-5699-058-9.
7. Suetin, P.K. *Faber Polynomials*; Encyclopedia of Mathematics; EMS Press: Berlin, Germany: 2001.
8. Duren, P.L. *Univalent Functions*, Grundlehren Math. Wissenschaften, Band 259; Springer: New York, NY, USA; Berlin/Heidelberg, Germany; Tokyo, Japan, 1983.
9. Airault, H.; Bouali, A. Di erential calculus on the Faber polynomials. *Bull. Sci. Math.* 2006, 130, 179–222.
10. Bouali, A. Faber polynomials, Cayley-Hamilton equation and Newton symmetric functions. *Bull. Sci. Math.* 2006, 130, 49–70.
11. Airault, H.; Ren, J. An algebra of di erential operators and generating functions on the set of univalent functions. *Bull. Sci. Math.* 2002, 126, 343–367.
12. Ma, W.C.; Minda, D. A uniied treatment of some special classes of univalent functions. In Proceedings of the Conference on ComplexAnalysis, Tianjin, China, 19–23 June 1992; pp. 157–169.
13. Sokol, J.; Stankiewicz, J. Radius of convexity of some subclasses of strongly starlike functions. *Zesz. Nauk. Politech. Rzesz. Mat.* 1996, 19, 101–105.
14. Raine, R.K.; Sokol, J. Some properties related to a certain class of starlike functions. *C. R. Math. Acad. Sci. Paris* 2015, 353, 973–978.
15. Kanas, S.; Masih, V.S.; Ebadian, A. Relations of a planar domain bounded by hyperbola with family of holomorphic functions. *J. Inequalities Appl.* 2019, 246, 1–14.
16. Mendiratta, R.; Nagpal, S.; Ravichandran, V. On a subclass of strongly starlike functions associated with exponential function. *Bull. Malays. Math. Sci. Soc.* 2015, 38, 365–386.
17. Goel, P.; Kumar, S.S. Certain class of starlike functions associated with modified sigmoid function. *Bull. Malays. Math. Sci. Soc.* 2020, 43, 957–991.
18. Shams, S.; Kulkarni, S.R.; Jahangiri, J.M. Classes of uniformly starlike and convex functions. *Int. J. Math. Math. Sci.* 2004, 55, 2959–2961.
19. Hussain, S.; Rasheed, A.; Zaighum, M.A.; Darus, M. A Subclass of Analytic Functions Related to k-Uniformly Convex and Starlike Functions. *J. Funct. Spaces* 2017, 2017, 9010964.
20. Aouf, M.K.; Hossen, H.M.; Lasahin, A.Y. On certain families of analytic functions with negative coefficients. *Indian J. Pure Appl. Math.* 2000, 31, 999–1015.
21. Lewin, M. On a coefficient problem for bi-univalent functions. *Proc. Am. Math. Soc.* 1967, 18, 63–68.
22. Altınkaya, S. Bounds for a new subclass of bi-univalent functions subordinate to the Fibonacci numbers. *Turk. J. Math.* 2020, 44, 533–560.
23. Altınkaya, S.; Yalcn, S. Faber polynomial coefficient bounds for a subclass of bi-univalent functions. *Comptes Rendus Math.* 2015, 353, 1075–108.
24. Aouf, M.K.; El-Ashwah, R.M.; Abd-Eltawab, A.M. New subclasses of bi-univalent functions involving Dzio–Srivastava operator. *Int. Sch. Res. Not.* 2013, 2013, 387178.
25. Bulut, S. Coefficient estimates for new subclasses of analytic and bi-univalent functions defined by Al-Oboudi differential operator. *J. Funct. Spaces Appl.* 2013, 2013, 181932.
26. Bulut, S. Coefficient estimates for a new subclass of analytic and bi-univalent functions defined by Hadamard product. *J. Complex Anal.* 2014, 2014, 302019.
27. Caglar, M.; Orhan, H.; Yagmur, N. Coefficient bounds for new subclasses of bi-univalent functions. *Filomat* 2013, 27, 1165–1171.
28. Deniz, E. Certain subclasses of bi-univalent functions satisfying subordinate conditions. *J. Class. Anal.* 2013, 2, 49–60.
29. Hayami, T.; Owa, S. Coefficient bounds for bi-univalent functions. *Pan-Am. Math. J.* 2012, 22, 15–26.
30. Jahangiri, J.M.; Hamidi, S.G. Faber polynomial coefficient estimates for analytic bi-bazilevic functions. *Mat. Vesn.* 2015, 67, 123–129.
31. Lashin, A.Y. On certain subclasses of analytic and bi-univalent functions. *J. Egypt. Math. Soc.* 2016, 24, 220–225.
32. Lashin, A.Y. Coefficient estimates for two subclasses of analytic and bi-univalent functions. *Ukr. Math. J.* 2019, 70, 1484–1492.
33. Lashin, A.Y.; L-Emam, F.Z.E. Coefficient estimates for certain subclasses of analytic and biunivalent functions. *Turk. J. Math.* 2020, 44, 1345–1361.
34. Magesh, N.; Rosy, T.; Varma, S. Coefficient estimate problem for a new subclass of bi-univalent functions. *J. Complex Anal.* 2013, 2013, 474231.
35. Magesh, N.; Yamini, J. Coefficient bounds for certain subclasses of bi-univalent functions. *Int. Math. Forum.* 2013, 8, 1337–1344.
36. Murugusundaramoorthy, G.; Magesh, N.; Prameela, V. Coefficient bounds for certain subclasses of bi-univalent functions. *Abstr. Appl. Anal.* 2013, 2013, 573017.
37. Peng, Z.-G.; Han, Q.-Q. On the coefficients of several classes of bi-univalent functions. *Acta Math. Sci. Ser. B Engl. Ed.* 2014, 34, 228–240.
38. Porwal, S.; Darus, M. On a new subclass of bi-univalent functions. *J. Egypt. Math. Soc.* 2013, 21, 190–193.
39. Srivastava, H.M.; Bulut, S.; Caglar, M.; Yagmur, N. Coefficient estimates for a general subclass of analytic and bi-univalent functions. *Filomat* 2013, 27, 831–842.
40. Srivastava, H.M.; Murugusundaramoorthy, G.; Magesh, N. Certain subclasses of bi-univalent functions associated with the Hohlov operator. *Glob. J. Math. Anal.* 2013, 1, 67–73.
41. Srivastava, H.M.; Murugusundaramoorthy, G.; Vijaya, K. Coefficient estimates for some families of bi-Bazilevic functions of the Ma–Minda type involving the Hohlov operator. *J. Class. Anal.* 2013, 2, 167–181.
42. Srivastava, H.M.; Motamednezhad, A.; Adegani, E.A. Faber Polynomial Coefficient Estimates for bi-univalent functions defined by using differential subordina-tion and a certain fractional derivative operator. *Mathematics* 2020, 8, 172.
43. Tang, H.; Deng, G.-T.; Li, S.-H. Coefficient estimates for new subclasses of Ma–Minda bi-univalent functions. *J. Inequalities Appl.* 2013, 2013, 317.
44. Agarwal, P.; Nieto, J.J. Some fractional integral formulas for the Mittag-Leffler type function with four parameters. *Open Math.* 2015, 1, 537–546.
45. Agarwal, P.; Chand, M.; Baleanu, D.; O’Regan, D.; Jain, S. On the solutions of certain fractional kinetic equations involving k-Mittag-Leffler function. *Adv. Differ. Equ.* 2018, 2018, 249, 13.
46. Alderremy, A.A.; Saad, K.M.; Agarwal, P.; Aly, S.; Jain, S. Certain new models of the multi space-fractional Gardner equation. *Phys. A* 2020, 545, 11.
47. Saoudi, K.; Agarwal, P.; Kumam, P.; Ghanmi, A.; Thounthong, P. The Nehari manifold for a boundary value problem involving Riemann-Liouville fractional derivative. *Adv. Differ. Equ.* 2018, 263, 18.
48. Darwish, H.E.; Lashin, A.Y; El-Ashwah, R.M.; Madar, E.M. Coefficient estimates of some classes of rational functions. *Int. J. Open Probl. Complex Anal.* 2019, 11, 16–30.
49. Hamidi, S.G.; Jahangiri, J.M. Faber polynomial coefficients of bi-subordinate functions. *Comptes Rendus Math.* 2016, 354, 365–370.
50. Goyal, S.P.; Kumar, R. Coefficient estimates and quasi-subordination properties associated with certain subclasses of analytic and bi-univalent functions. *Math. Slovaca* 2015, 65, 533–544.
51. Zireh, A.; Adegani, E.A.; Bidkham, M. Faber polynomial coefficient estimates for subclass of bi-univalent functions defined by quasi-subordinate. *Math. Slovaca* 2018, 68, 369–378.
52. Deniz, E.; Jahangiri, J.M.; Hamidi, S.G.; Kina, S.K. Faber polynomial coefficients for generalized bi-subordinate functions of complex order. *J. Math. Inequalities* 2018, 12, 645–653.