Vacuum fluctuations and topological Casimir effect in Friedmann-Robertson-Walker cosmologies with compact dimensions

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Abstract

We investigate the Wightman function, the vacuum expectation values of the field squared and the energy-momentum tensor for a massless scalar field with general curvature coupling parameter in spatially flat Friedmann-Robertson-Walker universes with an arbitrary number of toroidally compactified dimensions. The topological parts in the expectation values are explicitly extracted and in this way the renormalization is reduced to that for the model with trivial topology. In the limit when the comoving lengths of the compact dimensions are very short compared to the Hubble length, the topological parts coincide with those for a conformal coupling and they are related to the corresponding quantities in the flat spacetime by standard conformal transformation. This limit corresponds to the adiabatic approximation. In the opposite limit of large comoving lengths of the compact dimensions, in dependence of the curvature coupling parameter, two regimes are realized with monotonic or oscillatory behavior of the vacuum expectation values. In the monotonic regime and for nonconformally and nonminimally coupled fields the vacuum stresses are isotropic and the equation of state for the topological parts in the energy density and pressures is of barotropic type. For conformal and minimal couplings the leading terms in the corresponding asymptotic expansions vanish and the vacuum stresses, in general, are anisotropic though the equation of state remains of barotropic type. In the oscillatory regime, the amplitude of the oscillations for the topological part in the expectation value of the field squared can be either decreasing or increasing with time, whereas for the energy-momentum tensor the oscillations are damping. The limits of validity of the adiabatic approximation are discussed.

1 Introduction

Friedmann-Robertson-Walker (FRW) spacetimes are among the most popular backgrounds in gravitational physics. There are several reasons for this. First of all current observations give a strong motivation for the adoption of the cosmological principle stating that at large scales the
Universe is homogeneous and isotropic and, hence, its large-scale structure is well described by
the FRW metric. Another motivation for the interest to FRW geometries is related to the high
symmetry of these backgrounds. Due this symmetry numerous physical problems are exactly
solvable and a better understanding of physical effects in FRW models could serve as a handle
to deal with more complicated geometries. In particular, the investigation of quantum effects
at the early stages of the cosmological expansion has been a subject of study in many research
papers (see, for example, [1, 2] and references therein). The original motivations in this research
were mainly to avoid the initial singularity and to solve the initial conditions problem for the
dynamics of the Universe. The further increase of the interest to this topic was related to the
appearance of inflationary cosmological scenarios [3]. During an inflationary epoch, quantum
fluctuations introduce inhomogeneities and may affect the transition toward the true vacuum.
These fluctuations provide a mechanism which explains the origin of the primordial density
perturbation needed to explain the formation of the large-scale structure in the universe.

In many problems we need to consider the physical model on the background of manifolds
with compact spatial dimensions. In particular, the idea of compact extra dimensions has been
extensively used in supergravity and superstring theories. From an inflationary point of view,
universes with compact dimensions, under certain conditions, should be considered a rule rather
than an exception [4]. The models of a compact universe with non-trivial topology may play
important roles by providing proper initial conditions for inflation. The quantum creation of
the universe having toroidal spatial topology is discussed in [5] and in references [6] within the
framework of various supergravity theories. In the case of non-trivial topology, the boundary
conditions imposed on quantum fields give rise to the modification of the spectrum for vacuum
fluctuations and, as a result, to the Casimir-type contributions in the vacuum expectation values
of physical observables (for the topological Casimir effect and its role in cosmology see [7, 8] and
references therein). In the Kaluza-Klein-type models, the topological Casimir effect has been
used as a stabilization mechanism for moduli fields which parametrize the size and the shape of
the extra dimensions. Recently, the relevance of the Casimir energy as a model for dark energy
needed for the explanation of the present accelerated expansion of the universe has been pointed
out (see [9] and references therein).

In most work on the topological Casimir effect in cosmological backgrounds, the results for
the corresponding static counterparts were used replacing the static length scales by comoving
lengths in the cosmological bulk. This procedure is valid in conformally invariant situations or
under the assumption of a quasi-adiabatic approximation. For non-conformal fields the calcula-
tions should be done directly within the framework of quantum field theory on time-dependent
backgrounds. In a series of recent papers we have considered the topological Casimir effect for
a massive scalar field with general curvature coupling [10] and for massive fermionic field [11]
in background of the de Sitter spacetime with toroidally compactified spatial dimensions. Due
to the high symmetry of the de Sitter spacetime in both cases the corresponding problems are
exactly soluble.

In the present paper we will consider another exactly soluble problem for the topological
Casimir effect on background of a more general class of spacetimes. Namely, we investigate the
vacuum expectation values of the field squared and the energy-momentum tensor for a massless
scalar field with arbitrary curvature coupling parameter induced by the compactness of spatial
dimensions in spatially flat FRW universes where the scale factor is a power of the comoving time.
The vacuum polarization and the particle creation in the FRW cosmological models with trivial
topology have been considered in a large number of papers (see [1, 2, 7] and references therein
for early research and [12]-[23] and references therein for later developments). In particular, the
vacuum expectation values of the field squared and the energy-momentum tensor in models with
power law scale factors have been discussed in [14, 15, 20-32] (see also [2]). The expectation
value of the field squared is the crucial quantity in discussions of the topological mass generation as a result of one-loop quantum corrections and dynamical symmetry breaking. It also is of interest in considerations of phase transitions in the early universe, in particular, related to inflationary scenarios. In addition to describing the physical structure of the quantum field at a given point, the expectation value of the energy-momentum tensor acts as the source of gravity in the Einstein equations. It therefore plays an important role in modelling a self-consistent dynamics involving the gravitational field.

The present paper is organized as follows. In the next section we evaluate the positive-frequency Wightman function in spatially flat FRW model with topology $R^p \times (S^1)^q$. By using the Abel-Plana summation formula, we decompose this function into two parts: the first one is the corresponding function in the geometry of the uncompactified FRW spacetime and the second one is induced by the compactness of the spatial dimensions. In section 3 we use the Wightman function for the evaluation of the vacuum expectation value of the field squared. The asymptotic behavior of the topological part is investigated in the early and late stages of the cosmological evolution. In section 4 we consider the vacuum expectation value of the energy-momentum tensor. The part in this expectation value corresponding to the uncompactified FRW model is well investigated in the literature and we are mainly concerned with the topological part. The special case of the model with a single compact dimension and the corresponding numerical results are discussed in section 5. The main results of the paper are summarized in Section 6.

2 Wightman function

We consider a scalar field with curvature coupling parameter $\xi$ evolving on background of the $(D+1)$-dimensional spatially flat FRW spacetime with power law scale factor. Such a field is described by the equation

$$\left(\nabla_l \nabla^l + m^2 + \xi R\right) \varphi = 0,$$

(1)

where $\nabla_l$ denotes the covariant derivative and $R$ is the curvature scalar of the background spacetime. For the most important special cases of minimally and conformally coupled fields one has the values of the curvature coupling parameter $\xi = 0$ and $\xi = \xi_D \equiv (D - 1)/4D$ respectively. The spatially flat FRW line element expressed in comoving time coordinate is

$$ds^2 = dt^2 - a^2(t) \sum_{i=1}^{D} (dz^i)^2, \quad a(t) = \alpha t^c,$$

(2)

with the curvature scalar

$$R = Dc \left[(D+1)c - 2\right] / t^2.$$

(3)

From the $(D+1)$-dimensional Einstein equations we find the energy density $\rho$ and the pressure $p$ for the source of the metric corresponding to (2):

$$\rho = \frac{D(D - 1) c^2}{16\pi G} t^2, \quad p = \frac{D - 1 c(2 - c D)}{16\pi G} t^2,$$

(4)

where $G$ is the $(D+1)$-dimensional gravitational constant. The corresponding equation of state has the form $p = (2/(c D) - 1)\rho$. In the special case

$$c = 2/(D + 1),$$

(5)

the equation of state is of the radiation type and the Ricci scalar vanishes. The case $c = 2/D$ corresponds to the dust-matter driven models. Note that the line element (2) with $c > 1$ describes
the power law \[33\] and extended \[34\] inflationary models. Such a power law expansion may be realized if a scalar field \( \phi \) with an exponential potential \( V(\phi) = e^{-\lambda \phi} \) dominates the energy density of the universe. The corresponding cosmological models admit power law inflationary solutions of the form \[2\] (see, for instance, \[35\]) with \( c = 4\lambda^{-2}/(D - 1) \). Solutions having power law scale factor with \( c = 2(n - 1)(2n - 1)/(D - 2n) \) arise in higher-order gravity theories with the Lagrangian \( R^n \) \[36\].

For the further discussion, in addition to the synchronous time coordinate \( t \) it is convenient to introduce the conformal time \( \tau \) in accordance with \( dt = a(t)d\tau \) or

\[
t = [(1 - c)\tau]^{1/(1 - c)}.
\]

(6)

Here we assume that \( c \neq 1 \). In the case \( c = 1 \) one has \( t = t_0 e^{\alpha \tau} \), with a constant \( t_0 \), and this case should be considered separately. In terms of the conformal time the line element takes the form:

\[
ds^2 = \Omega^2(\tau)[d\tau^2 - \sum_{i=1}^{D}(dz_i^2)], \quad \Omega(\tau) = \alpha[(1 - c)\tau]^{c/(1 - c)},
\]

(7)

which is manifestly conformal to the Minkowski spacetime. Note that one has \( 0 \leq \tau < \infty \) for \( 0 < c < 1 \) and \( -\infty < \tau \leq 0 \) for \( c > 1 \).

We will assume that the spatial coordinates \( z^l, l = p + 1, \ldots, D, \) are compactified to \( S^1 \), \( 0 \leq z^l \leq L_l \), and for the other coordinates we have \(-\infty \leq z^l \leq +\infty, l = 1, \ldots, p \). Hence, we consider the spatial topology \( R^p \times (S^1)^q \) with \( p + q = D \). For \( p = 0 \), as a special case we obtain the toroidally compactified FRW spacetime. Along the compact dimensions we will consider the boundary conditions

\[
\varphi(\tau, z_p, z_q + e_l L_l) = e^{2\pi i \alpha_l} \varphi(\tau, z_p, z_q),
\]

(8)

where \( z_p = (z^1, \ldots, z^p) \), \( z_q = (z^{p+1}, \ldots, z^D) \), and \( e_l, l = p + 1, \ldots, D \), is the unit vector along the direction \( z^l \). In \[S\] we have introduced phases with constants \( 0 \leq \alpha_l \leq 1 \). The special cases \( \alpha_l = 0 \) and \( \alpha_l = 1/2 \) correspond to periodic and antiperiodic boundary conditions. The corresponding fields are also called as untwisted and twisted ones.

Among the most important characteristics of the vacuum state are the vacuum expectation values (VEVs) for the field squared and the energy-momentum tensor. These VEVs are obtained from the Wightman function and its derivatives in the coincidence limit of the arguments. For the topology under consideration we will denote this function by \( G^+_{p,q}(x, x') \). The Wightman function is also important in the consideration of the response of particle detectors (see, for instance, \[H\]). Let \( \{\varphi_\sigma(x), \varphi_\sigma^*(x)\} \) be a complete set of solutions to the classical field equation satisfying the periodicity conditions \[S\] and \( \sigma \) is a set of quantum numbers specifying the solution. The Wightman function may be evaluated by using the mode-sum formula

\[
G^+_{p,q}(x, x') = \langle 0 | \varphi(x) \varphi(x') | 0 \rangle = \sum_\sigma \varphi_\sigma(x) \varphi_\sigma^*(x'),
\]

(9)

where \( |0\rangle \) is the amplitude for the vacuum state under consideration.

In accordance with the symmetry of the problem the mode solutions of the field equation are separable: \( \varphi_\sigma(x) = \phi(\tau)e^{ik_p z_p + ik_q z_q} \), where \( k_p = (k_1, k_2, \ldots, k_p) \) and \( k_q = (k_{p+1}, k_{p+2}, \ldots, k_D) \). For the components of the wave vector along the uncompacted directions we have \(-\infty < k_l < \infty, l = 1, 2, \ldots, p \). The components along the compact dimensions are quantized by the periodicity conditions \[S\] and for the corresponding eigenvalues we have

\[
k_l = 2\pi(n_l + \alpha_l)/L_l, \quad n_l = 0, \pm 1, \pm 2, \ldots, \quad l = p + 1, \ldots, D.
\]

(10)
From the field equation (11) we obtain an ordinary differential equation for the function \( \phi(\tau) \):

\[
\frac{d^2 \phi}{d\tau^2} + \frac{(D-1)c}{(1-c)\tau} \frac{d\phi}{d\tau} + k^2 \phi + \alpha^2 \left\{ \frac{\alpha(1-c)\tau}{2\xi(1-c)} \right\} \phi = 0,
\]

where

\[
k = \sqrt{k_p^2 + k_q^2}.
\]

(12)

For a massless scalar field, \( m = 0 \), the general solution of this equation is expressed in terms of a combination of Hankel functions:

\[
\phi(\tau) = \eta^b \left[ c_1 H_{\nu}^{(1)}(k\tau) + c_2 H_{\nu}^{(2)}(k\tau) \right],
\]

with the notations

\[
\eta = |\tau|, \quad b = \frac{cD - 1}{2(c - 1)},
\]

and

\[
\nu = \frac{1}{2|1-c|} \sqrt{(cD - 1)^2 - 4\xi Dc [(D+1)c - 2]].
\]

(15)

Note that \( \nu \) is either real and nonnegative or pure imaginary. For a conformally coupled field we have \( \nu = 1/2 \) and for a minimally coupled field \( \nu = |b| \). Different choices of the coefficients in (13) correspond to different vacuum states. We will consider the Bunch-Davies vacuum [26] for which \( c_1 = 0 \). With this choice, in the limit \( c \to 0 \) we recover the standard vacuum in the Minkowski spacetime associated with the modes \( k^{-1/2} \exp(ik\cdot z_p + ik\cdot z_q - i\kappa \tau) \).

The corresponding eigenfunctions satisfying the periodicity conditions take the form

\[
\varphi_{\sigma}(x) = C_{\sigma} \eta b H_{\nu}^{(2)} (k\tau) e^{i k_p \cdot z_p + i k_q \cdot z_q}.
\]

(16)

The coefficient \( C_{\sigma} \) with \( \sigma = (k_p, n_p + 1, \ldots, n_D) \) is found from the orthonormalization condition

\[
\int d^D x \sqrt{|g^{00}|} \left( \varphi_{\sigma}(x) \partial_\tau \varphi_{\sigma}^*(x) - \varphi_{\sigma}^*(x) \partial_\tau \varphi_{\sigma}(x) \right) = \nu \delta_{\sigma\sigma'},
\]

(17)

where the integration goes over the spatial hypersurface \( \tau = \text{const} \), and \( \delta_{\sigma\sigma'} \) is understood as the Kronecker delta for discrete indices and as the Dirac delta-function for continuous ones. By using the well known properties of the Hankel functions, this leads to the result

\[
|C_{\sigma}|^2 = \frac{\alpha^{(D-1)c/(c-1)}}{2^{p+3/2}\pi^{p-1}\alpha D - 1 V_q} e^{-(\nu - \nu^*) \pi i/2},
\]

(18)

where and in what follows we denote by

\[
V_q = L_{p+1} \cdots L_D
\]

(19)

the volume of the compact subspace.

In the case \( p = 0, q = D \) (topology \( (S^1)^D \)) and \( \alpha_l = 0, l = 1, \ldots, D \), there is also a normalizable zero mode with \( k_D = 0 \) which is independent of the spatial coordinates (for the discussion of zero modes in topologically nontrivial spaces see [37]). The corresponding eigenfunction has the form \( \varphi_0(\tau) = \eta^b(c_1 \eta^\nu + c_2 \eta^{-\nu}) \). From the normalization condition we have the relation \( c_1^2 c_2 - c_1 c_2^2 = i\text{sign}(\tau)/(2\alpha V_D) \) for real \( \nu \) and the relation \( |c_2|^2 - |c_1|^2 = \text{sign}(\tau)/(2\nu |\alpha V_D|) \) for imaginary \( \nu \). These formulae give a relation between the coefficients \( c_{10} \) and \( c_{20} \) and one of them remains undetermined. This arbitrariness arises because of freedom to choose the quantum state [37]. In the discussion below we will consider the contribution to the VEVs from the nonzero modes. If the zero mode is present, its contribution should be added separately.
Substituting the eigenfunctions (16) with the normalization coefficient (18) into the mode-sum formula for the Wightman function, one finds

$$G^+_{p,q}(x, x') = \frac{A(\eta \eta')^b}{2 \pi p^{\frac{b+1}{2}}} \int dk_p e^{ik_p \cdot \Delta z_p} \sum_{n_q \in \mathbb{Z}^b} e^{i k_{q-1} \cdot \Delta z_{q-1}}$$

$$\times K_{\nu}(\eta \eta e^{\text{sign}(\tau)\pi i/2}) K_{\nu}(\eta \eta e^{-\text{sign}(\tau)\pi i/2}),$$  \hspace{1cm} (20)

where $n_q = (n_{p+1}, \ldots, n_D)$, $\Delta z' = z - z'$, and

$$A = \alpha^{1-D}[\alpha |1 - c|](D-1)c/(c-1).$$  \hspace{1cm} (21)

In formula (20), for the further convenience, we have written the Hankel function in terms of the modified Bessel function. For a scalar field with periodic boundary conditions along compact dimensions ($\alpha_l = 0$) and for $\text{Re} \nu > p/2$ the integral in (20) has an infrared divergence at $k_p = 0$ coming from the modes $n_q = 0$. In this case the Bunch-Davies vacuum is well defined only if $\text{Re} \nu < p/2$. For the case $p = 0$ and $\alpha_l = 0$, in (20) $n_q = n_D \neq 0$ and the contribution of the additional zero mode should be added.

We apply to the sum over $n_{p+1}$ in (20) the Abel-Plana type summation formula

$$\sum_{n=-\infty}^{\infty} g(n + \beta)f(|n + \beta|) = \int_0^{\infty} du [g(u) + g(-u)] f(u)$$

$$+ \frac{1}{2} g(i \lambda u) \sum_{\lambda = \pm 1} e^{2\pi i (u + i \lambda \beta)} - 1.$$  \hspace{1cm} (22)

This formula is obtained by combining the summation formulæ given in [38]. After the application of this formula, the term in the expression of the Wightman function with the first integral on the right of (22) corresponds to the Wightman function in the FRW model with $p + 1$ uncompactified and $q - 1$ toroidally compactified dimensions, $G^+_{p+1,q-1}(x, x')$. As a result one finds the recurrence formula

$$G^+_{p,q}(x, x') = G^+_{p+1,q-1}(x, x') + \Delta_{p+1} G^+_{p,q}(x, x'),$$  \hspace{1cm} (23)

where the second term on the right is induced by the compactness of the $z^{b+1}$ - direction and is given by the expression

$$\Delta_{p+1} G^+_{p,q}(x, x') \approx \frac{A(\eta \eta')^b}{2 \pi p^{\frac{b+1}{2}}} \int dk_p e^{ik_p \cdot \Delta z_p} \sum_{n_q \in \mathbb{Z}^{b-1}} e^{i k_{q-1} \cdot \Delta z_{q-1}}$$

$$\times \int_0^{\infty} dy \frac{y}{\sqrt{y^2 + k_p^2 + k_{q-1}^2}} \sum_{\lambda = \pm 1} e^{L_{p+1} \sqrt{y^2 + k_p^2 + k_{q-1}^2} + 2\pi i \lambda \alpha_{p+1}}$$

$$\times \{ K_{\nu}(\eta y) [I_{-\nu}(\eta y') + I_{\nu}(\eta y')] + [I_{\nu}(\eta y) + I_{-\nu}(\eta y)] K_{\nu}(\eta y') \},$$  \hspace{1cm} (24)

where $I_{\nu}(z)$ is the modified Bessel function. Here the notations $V_{q-1} = L_{p+2}, \ldots, L_D$ and

$$n_{q-1} = (n_{p+2}, \ldots, n_D), k_{n_{q-1}}^2 = \sum_{l=p+2}^{D} (2\pi n_l / L_l)^2,$$  \hspace{1cm} (25)
where \( \langle \phi^2 \rangle_{p,q} \) is the expectation value in the FRW spacetime with the spatial topology \( R^p \times (S^1)^q \) and the second term on the right is the part due to the compactness of the

\[ \langle \phi^2 \rangle_{p+1,q-1} = \Delta_{p+1} \langle \phi^2 \rangle_{p,q}, \]

(31)

Note that we have

\[ A(\eta') \frac{(\xi - 1)!}{(\xi)!} \Gamma(D/2 + \nu) \Gamma(D/2 - \nu) \Gamma((D + 1)/2) F \left( \frac{D}{2} + \nu, \frac{D}{2} - \nu, \frac{D + 1}{2}, \frac{1 - u}{2} \right). \]

(29)

In the special case \( D = 3 \) and \( \xi = 0 \) the formula (29) coincides with the result given in [26] (with the missprint in the coefficient corrected, see also [11]). Shifting the time coordinate \( t \to t + t_0 \), with a constant \( t_0 \) and taking the limit \( c \to \infty \) for fixed \( t \) and \( t_0 \), from (2) we obtain the corresponding line element in de Sitter spacetime with the scale factor \( a(t) = e^t \). In this limit the expression (29) tends to \( t_0^{-D} \) and from (22) we recover the expression for the Wightman function in \((D + 1)\)-dimensional de Sitter spacetime.

### 3 VEV of the field squared

The VEV of the field squared in FRW model with topology \( R^p \times (S^1)^q \) is obtained from the two-point function \( G_{p,q}^{+}(x,x') \) taking the coincidence limit of the arguments. In this limit the Wightman function diverges and a renormalization procedure is necessary. The crucial is that under the toroidal compactification the local geometry is not changed and the divergences are contained in the part corresponding to the uncompactified FRW spacetime. As we have already extracted this part, the renormalization is reduced to that in the uncompactified FRW model which is well investigated in the literature (see for example [11,2] and references given therein). For the VEV of the field squared we have the recurrence formula
In this case the topological part of the VEV is given by formula (36) with the power law expansion with $z$ and $x$ arguments. The integral over $k_p$ is evaluated by making use of the formula
\[
\sum_{\lambda=\pm 1} \int dk_p \frac{(k_p^2 + a^2)^{-1/2}}{e^{k_p/\sqrt{k_p^2+a^2+2\pi i\lambda p+1}} - 1} = 4(2\pi)^{(p-1)/2} \sum_{n=1}^{\infty} \cos(2\pi n \alpha_{p+1}) (nL_{p+1})^{p-1} f_{(p-1)/2}(nL_{p+1}a), \quad (32)
\]
with the notation $f_{\nu}(x) = x^\nu K_{\nu}(x)$. Now for the topological part in (31) we find
\[
\Delta_{p+1} \langle \varphi^2 \rangle_{p,q} = \frac{4A \eta^{2b}}{(2\pi)^{(p+3)/2} V_{q-1}} \sum_{n_{l-1} \in \mathbb{Z}^{q-1}} \int_0^\infty \frac{dy}{y} [I_{-\nu}(y) + I_{\nu}(y)] K_{\nu}(y) \sum_{n=1}^{\infty} \frac{\cos(2\pi n \alpha_{p+1})}{(nL_{p+1})^{p-1} f_{(p-1)/2}(nL_{p+1}\sqrt{y^2 + k_n^2})}. \quad (33)
\]
This integral representation is valid for Re $\nu < 1$.

Similar to (26), we find the following decomposition for the VEV of the field squared:
\[
\langle \varphi^2 \rangle_{p,q} = \langle \varphi^2 \rangle_{\text{FRW}} + \langle \varphi^2 \rangle_{p,q}^{(t)}; \langle \varphi^2 \rangle_{p,q}^{(t)} = \sum_{j=\rho}^{D-1} \Delta_{j+1} \langle \varphi^2 \rangle_{j,D-j}, \quad (34)
\]
where $\langle \varphi^2 \rangle_{\text{FRW}} = \langle \varphi^2 \rangle_{D,0}$ is the VEV in the spatial topology $R^D$ and the part $\langle \varphi^2 \rangle_{p,q}^{(t)}$ is induced by the nontrivial topology. As it can be seen from formula (33), the topological part has the following general structure: $\langle \varphi^2 \rangle_{p,q}^{(t)} = t^{1-D} f(L_{p+1}/\eta, \ldots, L_D/\eta)$, where the form of the function $f(x_1, \ldots, x_q)$ is directly obtained from (33) and (31). The ratios in the arguments of this function may also be written as
\[
L_l/\eta = ([1 - c]/c) L_l^{(c)}/r_H, \quad (35)
\]
with $L_l^{(c)} = \Omega L_l$ and $r_H = t/c$ being the comoving length of the compact dimension and the Hubble length respectively.

For a conformally coupled massless scalar field one has $\nu = 1/2$ and $[I_{-\nu}(x) + I_{\nu}(x)] K_{\nu}(x) = 1/x$. The integral in (33) is explicitly evaluated and we find
\[
\Delta_{p+1} \langle \varphi^2 \rangle_{p,q} = \frac{2A \eta^{2b-1}}{(2\pi)^{p/2+1} V_{q-1}} \sum_{n_{l-1} \in \mathbb{Z}^{q-1}} \sum_{n=1}^{\infty} \frac{\cos(2\pi n \alpha_{p+1})}{(nL_{p+1})^{p/2} f_{p/2}(nL_{p+1}k_{n_{l-1}})}. \quad (36)
\]
We could obtain this result directly from the corresponding formula in the Minkowski spacetime with topology $R^p \times (S^1)^q$, by taking into account that $A \eta^{2b-1} = \Omega^{1-D}$ and by using the fact that two problems are conformally related: $\langle \varphi^2 \rangle_{p,q}^{(t)} = \Omega^{1-D} \langle \varphi^2 \rangle_{p,q}^{(t,M)}$. The expression for $\langle \varphi^2 \rangle_{p,q}^{(t,M)}$ is directly obtained from (36) and is valid for arbitrary values of the curvature coupling parameter $\xi$. Note that we have $\nu = 1/2$ in the general case of the curvature coupling parameter for the power law expansion with $c = 2/(D + 1)$ which corresponds to radiation driven models. In this case the topological part of the VEV is given by formula (36) with $b = -1/2$ and $A = \alpha^{-D-1}(D + 1)^2/(D - 1)^2$.

Now we turn to the investigation of the topological part $\Delta_{p+1} \langle \varphi^2 \rangle_{p,q}$ in the VEV of the field squared in the asymptotic regions of the ratio $L_{p+1}/\eta$. For small values of this ratio, $L_{p+1}/\eta \ll 1$, we introduce the new integration variable $y = L_{p+1}x$. By taking into account
that for large values \( x \) one has \( [I_{-\nu}(x) + I_\nu(x)] K_\nu(x) \approx 1/x \), we find that to the leading order \( \Delta_{p+1}(\varphi^2)_{p,q} \) coincides with the corresponding result for a conformally coupled massless field:

\[
\Delta_{p+1}(\varphi^2)_{p,q} \approx \frac{2^{\nu-1-D}}{(2\pi)^{p/2+1}V_q} \sum_{n_{q-1} \in \mathbb{Z}^{q-1}} \sum_{n=1}^{\infty} \frac{\cos(2\pi n \alpha_{p+1})}{(nL_{p+1})^p} f_{p/2}(nL_{p+1}k_{n_{q-1}}), \quad L_{p+1}/\eta \ll 1. \tag{37}
\]

In terms of the comoving time coordinate the topological part behaves as \( \Delta_{p+1}(\varphi^2)_{p,q} \propto t^{(1-D)c} \). The limit under consideration corresponds to early stages of the cosmological expansion \( (t \to 0) \) for \( c > 1 \) and to late stages \( (t \to \infty) \) for \( 0 < c < 1 \). Note that, by taking into account \((35)\), the condition \( L_{p+1}/\eta \ll 1 \) can also be written in the form \( L_{p+1}^{(c)} \ll r_H \). Hence, the asymptotic formula \((37)\) corresponds to small proper lengths of compact dimensions compared with the Hubble length. The expression on the right hand side of \((37)\) is obtained from the corresponding expression in Minkowski spacetime with the topology \( R^p \times (S^1)^q \) and with the lengths of the compact dimensions \( L_{p+1}, \ldots, L_D \), replacing \( L_l \) by the comoving lengths \( L_l^{(c)} \). Hence, \((37)\) corresponds to the adiabatic approximation. In this regime the topological part monotonically decreases with increasing scale factor.

In the opposite limit of large values for the ratio \( L_{p+1}/\eta \), the behavior of the topological part is qualitatively different for real and imaginary values of the parameter \( \nu \). For positive values \( \nu \), by using the asymptotic formulae for the modified Bessel functions for small values of the arguments, to the leading order we find

\[
\Delta_{p+1}(\varphi^2)_{p,q} \approx \frac{2^{\nu+1}A_{p+1}^{2b-2\nu} \Gamma(\nu)}{(2\pi)^{(p+3)/2}V_q} \sum_{n_{q-1} \in \mathbb{Z}^{q-1}} \sum_{n=1}^{\infty} \frac{\cos(2\pi n \alpha_{p+1})}{(nL_{p+1})^{p+1-2\nu}} \times f_{p+1}/2-\nu(nL_{p+1}k_{n_{q-1}}), \quad L_{p+1}/\eta \gg 1. \tag{38}
\]

In terms of the synchronous time coordinate we have \( \Delta_{p+1}(\varphi^2)_{p,q} \propto t^{(2-\nu)\nu-D+1} \). This limit corresponds to early stages of the cosmological expansion \( (t \to 0) \) for \( 0 < c < 1 \) and to late stages of the cosmological expansion \( (t \to \infty) \) when \( c > 1 \). Note that for \( 0 < c < 1/D \) and \( \xi < 0 \) one has \( b > \nu \) and the topological part in the expectation value of \( \varphi^2 \) increases with increasing comoving time \( t \). For a minimally coupled scalar field and for \( c \in (0,1/D) \cup (1, \infty) \) one has \( \nu = b \), and in the limit under consideration the VEV tends to finite nonzero value. In the case \( \nu = 0 \) the corresponding asymptotic formula is obtained from \((38)\) replacing \( \Gamma(\nu) \to 4\ln(L_{p+1}/\eta) \) and after substituting \( \nu = 0 \). The limit under consideration corresponds to large comoving length of the compact dimension compared to the Hubble length: \( L_{p+1}^{(c)} \gg r_H \).

For pure imaginary values \( \nu \) and in the same limit \( L_{p+1}/\eta \gg 1 \), we use the formula

\[
K_\nu(z) [I_{-\nu}(z) + I_\nu(z)] \approx \Re \left[ \frac{\Gamma(\nu)(z/2)^{-2\nu}}{\Gamma(1-\nu)} \right], \tag{39}
\]

valid for small values \( z \). Now we can see that in the leading order

\[
\Delta_{p+1}(\varphi^2)_{p,q} \approx 4AB\eta^{2b} \cos(2\nu\ln(L_{p+1}/\eta + \phi)}{(2\pi)^{(p+3)/2}V_q} f_{p+1}/2-\nu(nL_{p+1}k_{n_{q-1}}), \quad L_{p+1}/\eta \gg 1, \tag{40}
\]

where the constants \( B \) and \( \phi \) are defined by the relation

\[
Be^{\phi} = 2^{2\nu} \Gamma(i\nu) \sum_{n_{q-1} \in \mathbb{Z}^{q-1}} \sum_{n=1}^{\infty} \frac{\cos(2\pi n \alpha_{p+1})}{(nL_{p+1}k_{n_{q-1}})} f_{p+1}/2-\nu(nL_{p+1}k_{n_{q-1}}). \tag{41}
\]
Hence, the behaviour of the topological part in the VEV for the field squared is oscillatory in the early (late) stages of the cosmological expansion for \(0 < c < 1\) \((c > 1)\). The amplitude of these oscillations behaves like \(t^{1-cD}\) and the distance between the nearest zeros linearly increases with increasing \(t\). Note that for \(0 < c < 1/D\) the oscillation amplitude increases with time.

From the asymptotic analysis given above it follows that in the special case of a minimally coupled field the topological part \(\Delta_{p+1}(\varphi^2)_{p,q}\) tends to finite nonzero value at early stages of the cosmological expansion when \(0 < c < 1/D\). At late (early) stages and for \(c < 1\) \((c > 1)\) the topological part behaves like \(t^{(1-D)c}\). At early times and for \(1/D < c < 1\) one has \(\Delta_{p+1}(\varphi^2)_{p,q} \propto t^{2(1-cD)}\).

### 4 VEV of the energy-momentum tensor

Having the Wightman function and the VEV of the field squared, the expectation value of the energy-momentum tensor is evaluated with the help of the formula

\[
\langle T_{ik}\rangle_{p,q} = \lim_{x' \to x} \partial_i \partial'_k G_{p,q}^+(x, x') + \left[ \left( \xi - \frac{1}{4} \right) g_{ik} \nabla_l \nabla^l - \xi \nabla_i \nabla_k \right] \langle \varphi^2 \rangle_{p,q},
\]

where \(R_{ik}\) is the Ricci tensor for the spatially flat FRW spacetime with the components

\[
R_{00} = \frac{Dc^2 - 2}{c - 1}, \quad R_{ik} = -\delta_{ik} \frac{c(Dc - 1)}{(1 - c)^2} t, \quad i, k = 1, \ldots, D.
\]

In Eq. (42) we have used the expression of the metric energy-momentum tensor for a scalar field which differs from the standard one (see, for example, [1]) by the term vanishing on the solutions of the field equation [11]. The topological part in the VEV of the energy-momentum tensor is obtained substituting (23) and (31) into formula (42). In this calculation we need the covariant d’Alembertian acted on the topological part of the field squared:

\[
\nabla_l \nabla^l (\Delta_{p+1}(\varphi^2)_{p,q}) = \frac{2^{(1-p)/2} A \pi}{(p+3)/2} \int_{V_{p-1}} \sum_{n_{p-1} \in \mathbb{Z}^{p-1}} \sum_{n=1}^{\infty} \frac{\cos(2\pi n \alpha_{p+1})}{(n L_{p+1})^{p-1}} \int_0^\infty dy y^{3-2b} \times f_{(p-1)/2}(n L_{p+1} \sqrt{y^2 + k^2_{n_{p-1}}}) \left( \frac{1}{z} \right) \tilde{I}_\nu(z) \tilde{K}_\nu(z) |_{z=\eta y}.
\]

Here and in the discussion below, to simplify our notations, we have introduced the functions

\[
\tilde{K}_\nu(z) = z^b K_\nu(z), \quad \tilde{I}_\nu(z) = z^{-b} \left[ I_\nu(z) + I_{-\nu}(z) \right],
\]

and the function \(f_\nu(x)\) is defined after formula (32).

For the VEV of the energy-momentum tensor we have the following recurrence formula

\[
\langle T_{ik}^k \rangle_{p,q} = \langle T_{ik}^1 \rangle_{p+1,q-1} + \Delta_{p+1}(\varphi^2)_{p,q},
\]

where the second term on the right hand side is due to the compactness of the direction \(z^{p+1}\). For the topological part in the energy density we find

\[
\Delta_{p+1}(\varphi^2)_{p,q} = \frac{4A \pi^{-2}}{(2\pi)^{(p+3)/2} V_{q-1}} \int_0^\infty dy y^{3-2b} \times \sum_{n_{q-1} \in \mathbb{Z}^{q-1}} f_{(p-1)/2}(n L_{p+1} \sqrt{y^2 + k^2_{n_{q-1}}}) F^{(0)}(\eta y),
\]
In formula (49), the function $F$ where $\langle \text{traceless.}$

In particular, for a conformally coupled field the corresponding energy-momentum tensor is with the notation $\langle \text{energy-momentum tensor obey the trace relation}$

In a similar way, for the vacuum stresses the following representation holds (no summation over $i$)

$$
\Delta_{p+1}(T^i_{i,p,q}) = \frac{4A\Omega^{-2}}{(2\pi)^{(p+3)/2}V^{-1}_{q-1}} \sum_{n_{q-1} \in \mathbb{Z}^q} \sum_{n=1}^{\infty} \frac{\cos(2\pi n \alpha_{p+1})}{(nL_{p+1})^{p-1}} \int_0^\infty dy y^{3-2b} f_{(p-1)/2}(z) F(\eta y) - f_{(i)}(z) \frac{\bar{T}_\nu(y)\bar{K}_\nu(\eta y)}{(nL_{p+1}y)^2} \right] \right|_{y=nL_{p+1}\sqrt{y^2+k^2_{n-1}}},
$$

where we have used the notations

$$
f_{(p)}(y) = f_{(p+1)/2}(y), \quad i = 1, 2, \ldots, p,
$$

$$
f_{(p+1)}(y) = -pf_{(p+1)/2}(y) - y^2 f_{(p-1)/2}(y),
$$

$$
f_{(i)}(y) = (nL_{p+1}k_i)^2 f_{(p-1)/2}(y), \quad i = p + 2, \ldots, D.
$$

In formula (49), the function $F(z)$ is defined as

$$
F(z) = 2 \left( \xi - \frac{1}{4} \right) \bar{T}_\nu(z) \bar{K}_\nu(z) - \frac{c\xi}{z - c} (\bar{T}_\nu \bar{K}_\nu)'
$$

$$
+ 2 \left[ \xi - \frac{1}{4} - Dc\xi \frac{(Dc - 2)(\xi - \xi_D) + \xi c}{z^2(1 - c)^2} \right] \bar{T}_\nu(z) \bar{K}_\nu(z).
$$

Now it can be checked that the topological parts in the VEVs of the field squared and the energy-momentum tensor obey the trace relation

$$
\Delta_{p+1}(T^i_{i,p,q}) = D(\xi - \xi_D) \nabla_i \nabla^i (\Delta_{p+1}(\varphi^2)_{p,q}).
$$

In particular, for a conformally coupled field the corresponding energy-momentum tensor is traceless.

The recurrence relation (46) allows to present the VEV of the energy-momentum tensor as the sum

$$
\langle T^i_{i,p,q} \rangle = \langle T^i_{i,\text{FRW}} \rangle + \langle T^k_{k,1} \rangle \langle T^i_{i,p,q} \rangle = \sum_{j=1}^{D-1} \Delta_{j+1}(T^i_{i,j,D-j}),
$$

where $\langle T^i_{i,\text{FRW}} \rangle$ is the part corresponding to the uncompactified FRW spacetime and $\langle T^k_{k,1} \rangle$ is induced by the the nontrivial topology. The first term is well investigated in the literature (see [12] and references therein) and in the following we will concentrate on the topological part. Note that this part has the general structure $\langle T^k_{k,1} \rangle = t^{-D-1} \delta^k_i f^{(i)}(L_{p+1}/\eta, \ldots, L_D/\eta)$, where the form of the functions $f^{(i)}(x_1, \ldots, x_q)$ directly follows from formulae (47) and (49). Recall that the ratios in the arguments of these functions may also be written in the form (55).

The expressions for the VEV of the energy-momentum tensor are further simplified for a conformally coupled scalar field. In this case, after some calculations we find (no summation over $i$):

$$
\Delta_{p+1}(T^i_{i,p,q}) = \frac{2\Omega^{-D-1}}{(2\pi)^{(p+2)/2}V^{-1}_{q-1}} \sum_{n=1}^{\infty} \sum_{n_{q-1} \in \mathbb{Z}^q} \frac{\cos(2\pi n \alpha_{p+1})}{(nL_{p+1})^{p+2}} f^{(i)}(nL_{p+1}k_{n-1}),
$$

(54)
where

\[ f^{(i)}_{(c)p}(x) = f_{p/2+1}(x), \quad i = 0, 1, \ldots, p, \]

\[ f^{(p+1)}_{(c)p}(x) = -(p+1)f_{p/2+1}(x) - x^2 f_{p/2}(x), \quad (55) \]

\[ f^{(i)}_{(c)p}(x) = (nL_{p+1}k_i)^2 f_{p/2}(x), \quad i = p + 2, \ldots, D. \]

This result is also obtained from the corresponding formulae in the toroidally compactified Minkowski spacetime by using the standard relation for the VEVs in conformally related problems. Formula (54) for a conformally coupled field holds for any conformally flat bulk with general scale factor \( \Omega(\tau) \). As we see, for a conformally coupled field the stresses along the uncompactified dimensions are equal to the vacuum energy density.

In the special case of power law expansion with \( c = 2/(D+2) \) and for general \( \xi \) we again have \( \nu = 1/2 \). After the integration over \( y \), in this case we find

\[ \Delta_{p+1}(T^i_{1})_{p,q} = \frac{2\Omega^{-D-1}}{(2\pi)^{p/2+1}V_q-1} \sum_{n_{q-1} \in \mathbb{Z}^{q-1}} \sum_{n_1=1}^{\infty} \frac{\cos(2\pi n_{q-p+1})}{(nL_{p+1})^{p+2}} \]

\[ \times \left[ 2f_0 \frac{\xi - \xi_D}{D - 1} \left( \frac{nL_{p+1}}{\eta} \right)^2 f_{p/2}(nL_{p+1}k_{n-1}) - f^{(i)}_{(c)p}(nL_{p+1}k_{n-1}) \right], \quad (56) \]

where \( f_0 = D \) and \( f_i = 1 \) for \( i = 1, 2, \ldots, D \). Recall that this special case corresponds to the radiation driven expansion and the corresponding Ricci scalar is zero.

The formulae given above for the VEV of the energy-momentum tensor in the general case of the curvature coupling are simplified in the asymptotic regions of the ratio \( L_{p+1}/\eta \). For small values of this ratio the arguments of the modified Bessel functions in the expressions for the VEV of the energy-momentum tensor are large. In the leading order we have \( F^{(0)}(z) \approx -z^{2b-1} \) and to this order the first term in the square brackets in the expression (49) for the vacuum stresses does not contribute. The integral over \( y \) is explicitly evaluated and we find

\[ \Delta_{p+1}(T^i_{1})_{p,q} \approx -\frac{2\Omega^{-D-1}}{(2\pi)^{p/2+1}V_q-1} \sum_{n_{q-1} \in \mathbb{Z}^{q-1}} \sum_{n_1=1}^{\infty} \frac{\cos(2\pi n_{q-p+1})}{(nL_{p+1})^{p+2}} f^{(i)}_{(c)p}(nL_{p+1}k_{n-1}), \quad L_{p+1}/\eta \ll 1. \]

(57)

As we see, in the leading order the VEV coincides with the corresponding expression for a compactly coupled field. This limit corresponds to late stages of the cosmological expansion \( (t \rightarrow \infty) \) for \( 0 < c < 1 \) and to early stages \( (t \rightarrow 0) \) for \( c > 1 \). In terms of the comoving time coordinate, the topological part behaves as \( \Delta_{p+1}(T^i_{1})_{p,q} \propto t^{-(D+1)c} \). By using the relation (35), we see that the asymptotic expression (57) corresponds to small proper length of the compact dimension compared to the Hubble length.

For large values of the ratio \( L_{p+1}/\eta \) and in the case of positive \( \nu \) for the topological part in the VEV of the energy-momentum tensor one has (no summation over \( i \))

\[ \Delta_{p+1}(T^i_{1})_{p,q} \approx \frac{F^{(i)}}{(\Omega \eta)^2} \Delta_{p+1}(\varphi^2)_{p,q}, \quad L_{p+1}/\eta \gg 1, \quad (58) \]

where the asymptotic expression for the field squared is given by Eq. (35) and we have defined

\[ F^{(0)} = \frac{1}{2} (b - \nu)^2 + 2D\xi c \frac{b - \nu}{1 - c} + \frac{\xi D(D - 1)c^2}{2(1 - c)^2}, \]

\[ F^{(i)} = 2 \left( \xi - \frac{1}{4} \right) (b - \nu)^2 - 2c\xi \frac{b - \nu}{1 - c} - 2Dc\xi (Dc - 2)(\xi - \xi_D) + \xi c \frac{(Dc - 2)(\xi - \xi_D)}{(1 - c)^2}. \]
with \( l = 1, \ldots, D \). As it is seen, the leading terms in the vacuum stresses are isotropic. In terms of the comoving time coordinate, we have the behaviour \( \Delta_{p+1}(T^0_i)_{p,q} \propto t^2(c-1)\nu - cD-1 \). Note that this holds for the total topological part \( \langle T_{i}^{(t)} \rangle_{p,q} \) as well. The limit under consideration corresponds to late times of the cosmological expansion \( (t \to \infty) \) for \( c > 1 \) and to early times \( (t \to 0) \) for \( 0 < c < 1 \), or alternatively, to large comoving length of the compact dimension in units of the Hubble length: \( L_{p+1}^{(c)} / \rho_H \gg 1 \). In the case \( \nu = 0 \) the asymptotics of the topological part in the VEV of the energy-momentum tensor are still given by relation (58), where now

\[
F^{(0)} = \frac{\xi(D - 1)}{(1 - 1/c)^2}, \quad F^{(l)} = -\frac{\xi c}{2} \left[ 1 + \frac{D - 1}{(1 - 1/c)^2} \right], \quad \nu = 0.
\] (60)

Note that the coefficient in (58) does not depend on \( p \) and, hence, similar relation takes place between the total topological parts in the VEVs of the field squared and the energy-momentum tensor (given by formulae (54) and (53)). In the limit under consideration the vacuum stresses are isotropic and the equation of state for the topological parts in the VEVs of the field squared and the energy-momentum part in the VEV of the energy-momentum tensor are still given by relation (58), where now \( L \) and, as it follows from (59), \( \Delta_{p+1}(T^0_i)_{p,q} \approx -\Delta_{p+1}(T^0_i)_{p,q}, i = 1, 2, \ldots, D \) (no summation over \( i \)). This corresponds to the equation of state for stiff fluid with pressure equal to the energy density.

For a minimally coupled scalar field in the case \( c \in (0, 1/D) \cup (1, \infty) \) (in this case \( \nu = b \)) and for a conformally coupled field the functions \( F^{(i)} \) in (59) vanish. In these cases we should keep the next terms in the corresponding asymptotic expansions. For a conformally coupled field the behavior is described by the exact formula (54). In the case of a minimally coupled field and for \( c \in (0, 1/D) \cup (1, \infty) \), to the leading order we have (no summation over \( i \)):

\[
\frac{\Delta_{p+1}(T^0_i)_{p,q}}{b - 1} \approx \frac{\Delta_{p+1}(T^0_i)_{p,q}}{b - 2} \approx \frac{2^{b+1}\Gamma(b)A\Omega^{-2}}{(2\pi)^{(p+3)/2}V_q^{-1}} \sum_{n=1}^{\infty} \cos(2\pi n\alpha_{p+1}) \sum_{n_q=1}^{n_{L_{p+1}}} f_{(p+3)/2-b(nL_{p+1}k_{n_q-1})}, \quad i = 1, 2, \ldots, p,
\] (61)

and for the stresses along the compact dimensions

\[
\Delta_{p+1}(T^{(i)}_{p+1})_{p,q} \approx \frac{p + 1 - b}{b - 1} \Delta_{p+1}(T^0_{p+1})_{p,q} + \frac{2^{b+1}\Gamma(b)A\Omega^{-2}}{(2\pi)^{(p+3)/2}V_q^{-1}} \sum_{n=1}^{\infty} \cos(2\pi n\alpha_{p+1}) \sum_{n_q=1}^{n_{L_{p+1}}} f_{(p+3)/2-b(nL_{p+1}k_{n_q-1})},
\]

\[
\times \sum_{n_q=1}^{n_{L_{p+1}}} (nL_{p+1}k_{n_q-1})^2 f_{(p+1)/2-b(nL_{p+1}k_{n_q-1})}, \quad i p + 2, \ldots, D.
\] (62)

Hence, in this case the topological part behaves as \( \Delta_{p+1}(T^0_i)_{p,q} \propto t^{-2c} \) in the limit \( t \to \infty \) for \( c > 1 \) and in the limit \( t \to 0 \) for \( 0 < c < 1 / D \).

For small values of the ratio \( \eta / L_{p+1} \) and imaginary \( \nu \) we use the asymptotic formula (39). To the leading order this gives (no summation over \( l \))

\[
\Delta_{p+1}(T^0_i)_{p,q} \approx \frac{2^{(1-p)/2}B_0\Omega^{-D-1}}{\pi^{(p+3)/2}V_q^{p}L_{p+1}^{p} \eta} \cos \left[ 2\nu \left| \ln(L_{p+1}/\eta) + \phi_{l} \right| \right], \quad L_{p+1}/\eta \gg 1,
\] (63)
where the constants $B_l$ and $\phi_l$ are defined by the relation

$$B_l e^{i\phi_l} = 2^{||\nu|\Gamma(i|\nu|)}F^{(l)} \sum_{n_q \in \mathbb{Z}^{D-1}} \sum_{n=1}^{\infty} \cos(2\pi n \alpha_{p+1}) f_{(p+1)/2-i|\nu|}(n L_{p+1} k_{n_{q-1}}),$$

with $F^{(l)}$, $l = 0, 1, \ldots, D$, from \cite{59}. Hence, for imaginary $\nu$ the behaviour of the VEVs is oscillatory. The corresponding amplitude behaves as $t^{-cD-1}$ and we have damping oscillations.

Summarizing the analysis given above, for a minimally coupled scalar field in the models with $0 < c < 1$, at late stages, $t \to \infty$, the topological part in the VEV of the energy-momentum tensor behaves as $t^{-(D+1)c}$. At early stages, $t \to 0$, we have $\langle T^{(i)}_{i,p,q} \rangle \propto t^{-2c}$ for $0 < c < 1/D$ and $\langle T^{(i)}_{i,p,q} \rangle \propto t^{-2Dc}$ for $1/D < c < 1$. In the models with $c > 1$ and at late stages of the cosmological expansion the topological part decays like $t^{-2c}$. At early stages and for $c > 1$ one has $\langle T^{(i)}_{i,p,q} \rangle \propto t^{-(D+1)c}$.

5 Special case and numerical examples

As an application of the general formulae given above let us consider the special case of topology $R^{D-1} \times S^1$. For this case the expression for the topological part in the VEV of the field squared takes the form

$$\langle \varphi^2 \rangle_{D-1,1} = \frac{4(2\pi)^{-D/2-1}}{(\Omega \eta)^{D-1}} \sum_{n=1}^{\infty} \frac{\cos(2\pi n \alpha_D)}{(n L/\eta)^{D-2}} \times \int_0^{\infty} dy [I_{-\nu}(y) + I_{\nu}(y)] K_{\nu}(y) f_{D/2-1}(ny LD/\eta).$$

(65)

Note that in this formula $\Omega \eta = t/|1-c]$. For a conformally coupled field this formula is reduced to

$$\langle \varphi^2 \rangle_{D-1,1} = \frac{\Gamma((D-1)/2)}{2\pi (D+1)/2(\Omega L_D)^{D-1}} \sum_{n=1}^{\infty} \frac{\cos(2\pi n \alpha_D)}{n^{D-1}}, \xi = \xi_D.$$  

(66)

Formula (66) describes the asymptotic behavior of the VEV in the general case of $\xi$ in early stages of the cosmological expansion ($t \to 0$) for $c > 1$ and at late stages ($t \to \infty$) for $0 < c < 1$. In the opposite limit of late times ($t \to \infty$) for $c > 1$ and early times ($t \to 0$) for $0 < c < 1$, for positive values of $\nu$, the asymptotic behavior is given by the expression

$$\langle \varphi^2 \rangle_{D-1,1} \approx \frac{\Gamma((D-1)/2)}{2\pi (D+1)/2(\Omega L_D)^{D-1}} \sum_{n=1}^{\infty} \frac{\cos(2\pi n \alpha_D)}{n^{D-2\nu}}.$$  

(67)

Note that in this expression $\Omega L_D$ is the comoving length of the compact dimension. For imaginary values of $\nu$ we have the asymptotic described by the relation (63). In particular, in the models of power law inflation ($c > 1$) we have $\langle \varphi^2 \rangle_{D-1,1} \propto t^{(1-D)c}$ at early times. At late times the topological part behaves monotonically as $t^{2(1-c)/D}$ in the case of positive $\nu$ and oscillatory like $t^{1-cD} \cos[2|\nu|(c-1)ln t + \text{const}]$ for imaginary values of $\nu$. In the latter case the distance between the nearest zeros linearly increases with increasing $t$.

Note that the ratio $\langle \varphi^2 \rangle_{D-1,1}/\langle \varphi^2 \rangle_{D-1,1}^{(t,c)}$ with $\langle \varphi^2 \rangle_{D-1,1}^{(t,c)}$ being the VEV for a conformally coupled field given by (66), is a function of $L_D/\eta$ alone and depends on the parameters $\xi$ and $c$ through $\nu$. In figure 1 we have plotted this ratio as a function of $L/\eta$, with $L = L_D$ being the length of the compact dimension, for untwisted $D = 3$ scalar field ($\alpha_D = 0$) and for various values of the parameter $\nu$ (numbers near the curves). In this special case one has
\[ \langle \varphi^2 \rangle_{D-1,1} = 1/[12(\Omega L_D)^2] \]. Note that the ratio \( L/\eta \) is related to the comoving length of the compact dimension, measured in units of the Hubble length, by Eq. 35. Figure 1 clearly shows that the adiabatic approximation for the topological part is valid only for small values of the ratio \( L/(c \eta) \).

Figure 1: The ratio \( \langle \varphi^2 \rangle_{D-1,1}/\langle \varphi^2 \rangle_{D-1,1} \) versus \( L/\eta \) for different values of the parameter \( \nu \) (numbers near the curves) and for \( D = 3 \) scalar field with the periodicity condition along the compact dimension.

In the special case under consideration, for the topological part in the VEV of the energy-momentum tensor we have the following representation (no summation over \( i \))

\[
\langle T_{i}^{i} \rangle_{D-1,1} = 4(2\pi)^{-D/2 - 1} \sum_{n=1}^{\infty} \frac{\cos(2\pi n \alpha_D)}{(nL_D/\eta)^{D-2}} \int_{0}^{\infty} dy y^{3-2b} \left[ F^{(i)}(y) f_{D/2-1}(nyL/\eta) - \frac{\bar{I}_{\nu}(y)K_{\nu}(y)}{(nyL_D/\eta)^2} f_{D-1}(nyL_D/\eta) \right],
\]

where \( F^{(0)}(y) \) is defined by (58), \( f_{D-1}(x) = 0 \), and \( F^{(i)}(y) = F(y) \) for \( i = 1, \ldots, D \). The functions \( f_{D-1}^{(i)}(x) \) for \( i = 1, \ldots, D \) are given by expressions (59). For a conformally coupled field we have (no summation over \( i \))

\[
\langle T_{i}^{i} \rangle_{D-1,1} = \frac{1}{D^{2}} \langle T_{D}^{D} \rangle_{D-1,1} = -\frac{\Gamma((D+1)/2)}{\pi^{(D+1)/2}(\Omega L_D)^{D+1}} \sum_{n=1}^{\infty} \frac{\cos(2\pi n \alpha_D)}{n^{D+1}} \left. f_{i} \left( \frac{\xi - \xi D}{D - 1} \left( \frac{nL_D}{\eta} \right)^2 - g_{i} \frac{D - 1}{2} \right) \right].
\]

Formula (69) describes the asymptotic behavior of the VEV for the general case of the curvature coupling parameter in the limit \( L_D/\eta \ll 1 \). In the opposite limit the corresponding asymptotic formulae directly follow from the expressions (58) and (63) for real and imaginary \( \nu \) respectively. The formulae are simplified in the case \( c = 2/(D + 1) \) for general curvature coupling parameter (no summation over \( i \)):

\[
\langle T_{i}^{i} \rangle_{D-1,1} = \frac{\Gamma((D - 1)/2)}{\pi^{(D+1)/2}(\Omega L_D)^{D+1}} \sum_{n=1}^{\infty} \frac{\cos(2\pi n \alpha_D)}{n^{D+1}} \left[ f_{i} \left( \frac{\xi - \xi D}{D - 1} \left( \frac{nL_D}{\eta} \right)^2 - g_{i} \frac{D - 1}{2} \right) \right],
\]
where \( g_i = 1 \) for \( i = 0, 1, \ldots, D - 1 \), \( g_D = -D \) and \( f_i \) is defined after formula (76).

In order to see the relative contributions of separate terms in the VEV of the energy-momentum tensor, in figure 2 we have plotted the ratio \( \langle T_{i}^{(t)}(i)D_{-1,1,1}/\langle T_{0}^{(t)}\rangle_{\text{FRW}} \) in the radiation driven model \((c = 2/(D + 1))\) for \( D = 3 \) minimally coupled untwisted (left panel) and twisted (right panel) scalar fields versus \( \eta/L \) with \( L = L_{D} \) being the length of the compact dimension. Here, \( \langle T_{0}^{(t)}\rangle_{\text{FRW}} = 1/[480\pi^{2}(2t)^{4]} \) is the vacuum energy density for a massless minimally coupled scalar field in the corresponding FRW model with trivial topology. Note that for the special case under consideration one has \( \eta = (2/\alpha)^{1/2} \). In the same model and for a conformally coupled scalar fields we have (no summation over \( i = 1, 2 \))

\[
\langle T_{0}^{(t)}(i)\rangle_{2,1} = \langle T_{1}^{(t)}(i)\rangle_{2,1} = -\frac{1}{3}\langle T_{3}^{(t)}(i)\rangle_{2,1} = \frac{\pi^{2}C}{(2t)^{4}}\left(\frac{\eta}{L}\right)^{4},
\]

where \( C = -1/90 \) and \( C = 7/720 \) for untwisted and twisted fields respectively.

![Figure 2: The topological parts in the VEV of the energy-momentum tensor for the case of radiation driven model with spatial topology \( \mathcal{R}^{2} \times S^{1} \) as functions of the ratio \( \eta/L \). The left/right panel corresponds to untwisted/twisted scalar fields.](image)

Now let us estimate the relative contributions of different sources in the total energy density for the \( D = 3 \) radiation dominated model. Introducing the Planck mass \( M_{P}^{2} = 1/G \) and the Planck time \( t_{P} \), for the energy density being the source of the background metric one has \( \rho \sim M_{P}^{1}(t/t_{P})^{2} \). The average vacuum energy density in FRW model with trivial topology is estimated as \( \langle T_{0}^{(t)}\rangle_{\text{FRW}} \sim M_{P}^{1}(t/t_{P})^{4} \). For the topological part in the vacuum energy density in the case of the conformally coupled field we have \( \langle T_{0}^{(t)}\rangle_{\text{conf}} \sim 1 + \text{const} \cdot (L^{(c)}/t_{P})^{2} \), where \( L^{(c)} = a(t)L \) is the comoving length of the compact dimension and \( t_{P} \) is the Planck length. For a minimally coupled field one has \( \langle T_{0}^{(t)}\rangle_{\min} \sim 1 + \text{const} \cdot (L^{(c)}/t_{P})^{2} \). From these relations, for the ratios of the separate parts in the vacuum energy density to the source term we find

\[
\langle T_{0}^{(t)}\rangle_{\text{FRW}}/\rho \sim (t_{P}/t)^{2}, \quad \langle T_{0}^{(t)}\rangle_{\text{conf}}/\rho \sim \left(l_{P}r_{H}/L^{(c)2}\right)^{2},
\]

where \( r_{H} \) is the Hubble length. From these relations we conclude that the term \( \langle T_{0}^{(t)}\rangle_{\text{FRW}} \) is comparable to the source term \( \rho \) only near the Planck time. The topological part dominates the part \( \langle T_{0}^{(t)}\rangle_{\text{FRW}} \) when \( L^{(c)} < r_{H} \) and becomes of the order of the source term for \( L^{(c)2} \sim l_{P}r_{H} \). In the latter case the backreaction of quantum topological effects should be taken into account.
6 Conclusion

In the present paper we have investigated one-loop quantum effects for a scalar field with general curvature coupling, induced by toroidal compactification of spatial dimensions in spatially flat FRW cosmological models with power law scale factor. We treat gravity as a given classical background field and do not consider the backreaction of the quantum effects on the metric. General boundary conditions with arbitrary phases are considered along compact dimensions. As special cases they include periodicity and antiperiodicity conditions corresponding to untwisted and twisted fields. The boundary conditions imposed on possible field configurations change the spectrum of vacuum fluctuations and lead to the Casimir-type contributions in the VEVs of physical observables. Among the most important characteristics of the vacuum state are the expectation values of the field squared and the energy-momentum tensor. Though the corresponding operators are local, due to the global nature of the vacuum these VEVs carry an important information on the global structure of the background spacetime. As the first step we evaluate the positive-frequency Wightman function. By using the Abel-Plana summation formula, we have derived recurrence relation connecting the Wightman functions for the topologies $R^p \times (S^1)^q$ and $R^{p+1} \times (S^1)^{q-1}$. The repeating application of this relation allows to present the Wightman function as the sum of the function for topologically trivial FRW model and the topological part. The latter is finite in the coincidence limit and in this way the renormalization of the VEVs for the field squared and the energy-momentum tensor is reduced to that for the FRW universe with trivial topology. The latter problem is well investigated in the literature and we concentrate on the topological parts.

The topological parts are given by formulae (33) and (34) for the field squared and by formulae (17), (19), (33) for the energy density and the stresses. In the case of a conformally coupled scalar field the problem is conformally related to the corresponding problem in flat spacetime with topology $R^p \times (S^1)^q$ and the general formulae are simplified to (36) and (51) for the field squared and the energy-momentum tensor respectively. These formulae describe the asymptotic behavior of the VEVs in the general case of the curvature coupling in the limit $L_1/\eta \ll 1$. In this limit the comoving lengths of the compact dimensions measured in units of the Hubble length are small and it corresponds to early stages of the cosmological expansion for $0 < c < 1$ and to late stages of the cosmological expansion when $c > 1$. The topological parts in the VEVs behave like $\langle \varphi^2 \rangle_{p,q} \propto t^{c(1-D)}$ and $\langle T^k_{l,p,q} \rangle \propto t^{-c(1+D)}$ and the stresses along the uncompactified dimensions are equal to the vacuum energy density.

In the opposite limit $L_1/\eta \gg 1$, corresponding to large comoving lengths of compact dimensions compared to the Hubble length, the behaviour of the VEVs is qualitatively different for real and imaginary values of the parameter $\nu$ defined by Eq. (15). For real values, the asymptotics are given by formulae (38) and (58) and the topological parts in the VEVs behave as $\langle \varphi^2 \rangle_{p,q} \propto t^{2(c-1)\nu-cD+1}$ and $\langle T^k_{l,p,q} \rangle \propto t^{2(c-1)\nu-cD-1}$. In this limit the vacuum stresses are isotropic and the equation of state for the topological parts in the vacuum energy density and pressures is of the barotropic type. For a minimally coupled scalar field in the case $c \in (0, 1/D) \cup (1, \infty)$ and for a conformally coupled field, the leading terms in the asymptotic expansions for the VEV of the energy-momentum tensor vanish. The corresponding asymptotics are given by $\langle T^k_{l,p,q} \rangle \propto t^{-c(1+D)}$ for conformally coupled field and by $\langle T^k_{l,p,q} \rangle \propto t^{-2c}$ for a minimally coupled field in the model with $c \in (0, 1/D) \cup (1, \infty)$ (see formulae (61) and (62)). In these special cases the vacuum stresses are anisotropic though the equation of state remains of the barotropic type with the coefficients depending on the lengths of compact dimensions. In the limit $L_1/\eta \gg 1$ and for imaginary values of the parameter $\nu$ the asymptotic behaviour of the topological parts is described by the formulae (17) and (33) and this behaviour is oscillatory with the oscillating factor in the form $\cos[2\nu(c-1)\ln(t/t_0)+\psi]$. For the field squared the
amplitude of the oscillations behaves as \( t^{1-cD} \). For the energy-momentum tensor the amplitude of the oscillations decreases like \( t^{-cD-1} \) and the oscillations are damping for all values of the parameter \( c \).

The general results for the VEVs of the field squared and the energy-momentum tensor are specified in section 5 for the case of a model with a single compact dimension. By numerical examples for this case we have illustrated that the adiabatic approximation for nonconformally coupled fields is valid only in the limit when the comoving length of the compact dimension is very short compared to the Hubble length. We have also estimated the relative contributions of various sources in the total energy density. In particular, in the radiation driven model, for the comoving lengths \( L^{(c)} \lesssim l_{pl} H \) the topological part of the energy density becomes of the order of the source term and the backreaction of quantum effects should be taken into account.

In this paper we have considered the case of a massless field when the equation (11) for the mode functions is exactly integrable in the class of special functions. By using this equation, we can estimate the role of effects due to the nonzero mass of the field. The ratio of the first and second terms in figure braces of (11) is of the order \( m^2 t^2 \). Hence, for \( mt \ll 1 \) the effects related to the nonzero mass may be disregarded and the results described in the present paper are applicable for massive fields as well.

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