Collective motion in the frame of phase space moments (Nuclear Scissors)

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Abstract. We consider the phase space moments (or Wigner Function Moments (WFM)) method, which is developed to describe the collective motion. The method is generalized to take into account pair correlations. Its connection with RPA and Green’s function method is analyzed in the simple model, the Harmonic Oscillator with Quadrupole–Quadrupole (HO+QQ) interaction. Possibilities of WFM method are demonstrated on an example of the nuclear scissors mode.

Keywords: nuclear scissors, Wigner function moments.

1. Introduction

The method of phase space moments, or Wigner function moments method was suggested in the Bogoliubov Laboratory of Theoretical Physics about 30 years ago. It was successfully applied to study isoscalar and isovector giant multipole resonances and low-lying collective modes of rotating and nonrotating nuclei with various realistic forces [1]. The method is developed and modified continually. For example, the exact relation between the RPA and WFM variables and the respective dynamical equations is found. The very close connection of the WFM method with the Green’s Function (GF) one is established [2]. Last time the method is generalized to take into account pair correlations [3].

All these points will be discussed on an example of simple model HO+QQ, which allows one to obtain the majority of results in the analytical form. The object of the demonstration will be the scissors mode - very interesting and rather curious collective excitation predicted...
33 years ago. Its experimental discovery has initiated a cascade of theoretical studies. An excellent review was given by D. Zawischa [4]. Very briefly the situation can be described in the following way. All microscopic calculations with effective forces reproduce experimental data with respect to the position and the strength of the scissors mode. However, the situation is more obscure in regard to simple phenomenological models whose aim is to explain the physics of the phenomenon and to interpret it in the most simple and transparent terms. So it will be interesting to compare the possibilities, advantages and disadvantages of various methods in the description of all subtleties of this mode.

2. Scissors mode by WFM, RPA and Green’s function methods

The basis of the method is the TDHF equation for the one-body density matrix \( \rho^\tau (r_1, r_2, t) \):

\[
\text{i} \hbar \frac{\partial \rho^\tau}{\partial t} = \left[ \hat{H}^\tau, \rho^\tau \right],
\]

where \( \hat{H}^\tau \) is the one-body self-consistent mean field Hamiltonian and \( \tau \) is an isotopic spin index.

It is convenient to modify equation (1) introducing the Wigner transform of the density matrix

\[
f^\tau (r, p, t) = \int d^3 s \exp(-i p \cdot s / \hbar) \rho^\tau (r + \frac{s}{2}, r - \frac{s}{2}, t)
\]

and of the Hamiltonian \( H_W^\tau (r, p) \). Then one arrives [5] at

\[
\frac{\partial f^\tau}{\partial t} = \frac{2}{\hbar} \sin \left\{ \frac{\hbar}{2} \left[ (\nabla)H : (\nabla^p)f - (\nabla^p)H : (\nabla)f \right] \right\} H_W^\tau f^\tau,
\]

where the upper index on the bracket stands for the function on which the operator in these brackets acts. It is shown in [1], that by integrating equation (3) over the phase space \( \{p, r\} \) with the weights \( x_1, x_2, \ldots, x_k, p_{k+1}, \ldots, p_{n-1}, p_n \), where \( k \) runs from 0 to \( n \), one can obtain a closed finite set of dynamical equations for Cartesian tensors of the rank \( n \). Taking linear combinations of these equations one is able to represent them through irreducible tensors, which play the role of collective variables of the problem. However, it is more convenient to derive the dynamical equations directly for irreducible tensors using the technique of tensor products [6]. For this it is necessary to rewrite the Wigner function equation (3) in terms of cyclic variables

\[
r_{+1} = -\frac{1}{\sqrt{2}} (x_1 + ix_2), \quad r_0 = x_3, \quad r_{-1} = \frac{1}{\sqrt{2}} (x_1 - ix_2)
\]

and take its moments with various tensor products of \( r_\alpha \) and \( p_\alpha \).

The microscopic Hamiltonian of the model is

\[
H = \sum_{i=1}^{A} \left( \frac{p_i^2}{2m} + \frac{1}{2} m \omega_i^2 r_i^2 \right) + \tilde{\kappa} \sum_{\mu=-2}^{2} \sum_{i} \sum_{j} q_{2-\mu}(r_i) q_{2\mu}(r_j)
\]
where \( q_{\mu} = \sqrt{16\pi/5} r^2 Y_{2\mu} \) is a quadrupole operator and \( N, Z \) are the numbers of neutrons and protons. Substituting spherical functions by tensor products \( r^2 Y_{2\mu} = \sqrt{\frac{15}{8\pi}} r^2_{2\mu} \), where \( r^2_{2\mu} \equiv \{ r \otimes r \} \lambda\mu = \sum_{\sigma, \nu} C^{\lambda\mu}_{\sigma, \nu} r_{\sigma} r_{\nu} \) and \( C^{\lambda\mu}_{\sigma, \nu} \) is the Clebsch-Gordan coefficient, one can write the mean field Hamiltonian as

\[
\hat{H}^T = \frac{\vec{p}^2}{2m} + \frac{1}{2} m \omega^2 r^2 + 6 \sum_{\mu} (-1)^\mu Z_{2-\mu} r^2_{2\mu},
\]

where \( Z_{2\mu} = \kappa R_{2\mu}^2 + \bar{\kappa} R_{2\mu}^0 \), \( Z_{0\mu} = \kappa R_{0\mu}^0 + \bar{\kappa} R_{0\mu}^0 \), \( R_{\lambda\mu}(t) = \int dt \{ \vec{p}, \vec{r} \} r^2_{\lambda\mu} f^T(\vec{r}, \vec{p}, t) \) and \( f\{ \vec{p}, \vec{r} \} \equiv \langle 2(2\pi)^{-3} \int d^3p \int d^3r \rangle \). Integration of equation (3) with the weights \( r^2_{\lambda\mu}, (rp)_{\lambda\mu} \equiv \{ r \otimes p \} \lambda\mu \) and \( p^2_{\lambda\mu} \) yields the following set of equations [7]:

\[
\begin{align*}
\frac{d}{dt} R_{\lambda\mu}^T - \frac{2}{m} L_{\lambda\mu}^T &= 0, & \lambda &= 0, 2, \\
\frac{d}{dt} L_{\lambda\mu}^T - \frac{1}{m} P_{\lambda\mu}^T + m \omega^2 R_{\lambda\mu}^T - 12\sqrt{5} \sum_{j=0}^2 \sqrt{2j + 1} \{ \mathbf{1}_{2\lambda1}(j) \} (Z_1^j R_1^j)_{\lambda\mu} &= 0, & \lambda &= 0, 1, 2, \\
\frac{d}{dt} P_{\lambda\mu}^T + 2m \omega^2 L_{\lambda\mu}^T - 24\sqrt{5} \sum_{j=0}^2 \sqrt{2j + 1} \{ \mathbf{1}_{2\lambda1}(j) \} (Z_2^j L_2^j)_{\lambda\mu} &= 0, & \lambda &= 0, 2, \end{align*}
\]

where \( \{ \mathbf{1}_{2\lambda1}(j) \} \) is the 6j-symbol and the following notation is introduced

\[
P_{\lambda\mu}^T(t) = \int d\{ \vec{p}, \vec{r} \} r^2_{\lambda\mu} f^T(\vec{r}, \vec{p}, t), \quad L_{\lambda\mu}^T(t) = \int d\{ \vec{p}, \vec{r} \} (rp)_{\lambda\mu} f^T(\vec{r}, \vec{p}, t).
\]

By definition \( q_{\mu} = \sqrt{15/2} r_{2\mu} \), the quadrupole moment \( Q_{2\mu} = \sqrt{6} R_{2\mu}^0 \), \( R_{00} = -Q_{00}/\sqrt{3} \) with \( Q_{00} = N_\tau < r^2 > \) being the mean square radius of neutrons or protons. The tensor \( L_{1\nu} \) is connected with angular momentum by the relations \( L_{10} = \frac{1}{\sqrt{2}} I_3, \quad L_{1\pm 1} = \frac{1}{2}(I_2 \mp i I_1) \).

We rewrite equations (6) in terms of isoscalar and isovector variables \( R_{\lambda\mu} = R_{\lambda\mu}^0 + R_{\lambda\mu}^0 \), \( \bar{R}_{\lambda\mu} = R_{\lambda\mu}^0 - R_{\lambda\mu}^0 \) (and so on) with the isoscalar \( \kappa_0 = (\kappa + \bar{\kappa})/2 \) and isovector \( \kappa_1 = (\kappa - \bar{\kappa})/2 \) strength constants. Equations will be simplified to get the results in analytical form. Writing all variables as a sum of their equilibrium value plus a small deviation \( R_{\lambda\mu}(t) = R_{\lambda\mu}^0 + \bar{R}_{\lambda\mu}(t), P_{\lambda\mu}(t) = P_{\lambda\mu}^0 + \bar{P}_{\lambda\mu}(t), L_{\lambda\mu}(t) = L_{\lambda\mu}^0 + \bar{L}_{\lambda\mu}(t), \bar{R}_{\lambda\mu}(t) = \bar{R}_{\lambda\mu}^0 + \bar{\bar{R}}_{\lambda\mu}(t), \bar{P}_{\lambda\mu}(t) = \bar{P}_{\lambda\mu}^0 + \bar{\bar{P}}_{\lambda\mu}(t), \bar{L}_{\lambda\mu}(t) = \bar{L}_{\lambda\mu}^0 + \bar{\bar{L}}_{\lambda\mu}(t) \), we linearize the equations of motion in \( R_{\lambda\mu}, P_{\lambda\mu}, L_{\lambda\mu}, \bar{R}_{\lambda\mu}, \bar{P}_{\lambda\mu}, \bar{L}_{\lambda\mu} \) (small-amplitude approximation). Further, the equilibrium deformation and mean square radius of neutrons are supposed to be equal to that of protons, i.e. \( R_{2\mu}^0 = \bar{R}_{2\mu}^0 = 0 \). Due to this approximation the equations for isoscalar and isovector systems are decoupled, every set of equations being split into five independent subsets with \( \mu = 0, \pm 1, \pm 2 \).
The scissors mode is described by the isovector subset with $\mu = 1$ Supposing, as usual, $\kappa_1 = \alpha \kappa_0$, and taking for $\kappa_0$ the self-consistent value $\kappa_0 = -\frac{m\bar{\omega}^2}{4Q_{00}}$, where $\bar{\omega}^2 = \frac{\omega^2}{1 + \frac{2}{3} \delta}$ [7] with the standard definition of the deformation parameter $Q_{20} = (4/3)Q_{00}\delta$, we find the characteristic equation whose solutions are

$$
\Omega_{\pm}^2 = \bar{\omega}^2(2 - \alpha)(1 + \delta/3) \pm \sqrt{\bar{\omega}^4(2 - \alpha)^2(1 + \delta/3)^2 - 4\bar{\omega}^4(1 - \alpha)^2 \delta^2}.
$$

(8)

The high-lying solution $\Omega_+$ gives the frequency $\Omega_{iv}$ of the $\mu = 1$ branch of the IsoVector GQR (IVGQR). The low-lying solution $\Omega_-$ gives the frequency $\Omega_{sc}$ of the scissors mode.

A direct way of calculating the reduced transition probabilities is provided by the theory of the linear response of a system to a weak external field

$$
\hat{W}(t) = \hat{W} \exp(-i\Omega t) + \hat{W}^\dagger \exp(i\Omega t).
$$

(9)

For magnetic excitations $\hat{W} = \hat{W}_{\mu} = \sum_{s=1}^Z \hat{w}_{1\mu}(s)$, $\hat{w}_{1\mu} = \sqrt{\frac{3}{2\pi}} \frac{\mu N}{N} (r \hat{p})_{1\mu}$, $\mu_N = \frac{\varepsilon h}{2mc}$, and

$$
B(M1)_\nu = \frac{1 - \alpha}{8\pi} \frac{m\bar{\omega}^2}{\hbar} Q_{00} \delta^2 \frac{\Omega_{\nu}^2 - 2(1 + \delta/3)\bar{\omega}^2}{\Omega_{\nu}^2 - \bar{\omega}^2(2 - \alpha)(1 + \delta/3)} \mu_N^2.
$$

(10)

For electric excitations $\hat{W} = \hat{W}_{2\mu} = \sum_{s=1}^Z \hat{w}_{2\mu}(s)$, $\hat{w}_{2\mu} = e r^2 Y_{2\mu} = e \sqrt{\frac{3}{8\pi}} r^2 \mu_N$, and

$$
B(E2)_\nu = \frac{e^2 h}{m} \frac{5}{16\pi} Q_{00}\frac{(1 + \delta/3)\Omega_{\nu}^2 - 2(\bar{\omega}^2)^2}{\Omega_{\nu}^2 - \bar{\omega}^2(2 - \alpha)(1 + \delta/3)}.
$$

(11)

The RPA approach gives exactly the same results for energy and transition probabilities, therefore one can suspect very close relation between WFM and RPA. Really, this relation can be established by considering the linear response of the system to a weak external time-dependent field (9). In this case WFM collective variables can be written as linear combinations of corresponding Transition Matrix Elements (TME) [2], for example,

$$
\delta R_{\lambda\mu}(t) \equiv \hat{R}_{\lambda\mu}(t) = \sum_{\nu} \langle <0|\hat{R}_{\lambda\mu}|\nu > c_{\nu} - <\nu|\hat{R}_{\lambda\mu}^\dagger|0 > \tilde{c}_{\nu}\rangle e^{-i\Omega t} + c.c.
$$

(12)

where $\hat{R}_{\lambda\mu} = \sum_{s=1}^A \{ r_s \otimes r_s \} \lambda\mu$. Inserting such expressions for $\delta R_{\lambda\mu}$, $\delta L_{\lambda\mu}$ and $\delta P_{\lambda\mu}$ into (6) we can derive equations, which can be called as dynamical equations for corresponding transition matrix elements. Exactly the same dynamical equations for TME can be derived in RPA by combining RPA equations [5] in accordance with the definition of matrix elements:

$$
<0|\hat{F}^{\tau}_{\mu}|\nu > = \sum_{mi} (f^{\tau}_{im} X^{\tau,\nu}_{mi} + f^{\tau}_{mi} Y^{\tau,\nu}_{mi}).
$$
So, there exists one-to-one correspondence between the set of dynamical equations for WFM variables and the set of dynamical equations for transition matrix elements. On the other hand the TME equations are just linear combinations of the RPA equations. Therefore we can conclude that RPA and WFM approaches are equivalent in all aspects.

Solving the Vlasov equation (which is obtained by conserving on the right hand side of equation (3) only the first term of the sin-function expansion) by the Green’s function method with the mean field (5) one derives [8, 2] the integral equation for the Wigner function. Taking the proper phase space moments of this equation one reproduces the (linearized) dynamical equations (6), that indicates on the very close relationship of two methods.

3. Pairing

Pair correlations can be taken into account by working with time dependent HFB equations. The detailed form of the TDHFB equations is

\[
\begin{align*}
\hat{h} \dot{\hat{\rho}} &= \hat{\rho} \hat{h} - \hat{\rho} \hat{\Delta} \hat{\kappa}^\dagger + \hat{\kappa} \hat{\Delta}^\dagger, \\
-\hat{\rho} \dot{h}^* &= h^* \hat{\rho} - \hat{\rho} h^* - \hat{\Delta} \hat{\kappa} + \hat{\kappa} \hat{\Delta}^\dagger, \\
-\hat{\rho} \dot{\hat{\kappa}} &= \hat{\rho} \hat{\kappa} + \hat{\kappa} \hat{\rho}^* - \hat{\Delta} \hat{\Delta}^\dagger, \\
-\hat{\rho} \dot{\hat{\Delta}} &= \hat{\rho} \hat{\Delta}, \quad (13)
\end{align*}
\]

where \( \hat{\rho} \) and \( \hat{\kappa} \) are normal and abnormal density matrices respectively, \( \hat{\Delta} \) is the pairing gap. We work with the Wigner transformation [5] of these equations. For example, the first one reads:

\[
\begin{align*}
\hat{h} \dot{f} &= \hat{h} \{h, f\} - \Delta \kappa^* \kappa \Delta^* + \kappa \Delta \kappa^* \Delta^* - \frac{i\hbar}{2} \{\Delta, \kappa^*\} + \frac{i\hbar}{2} \{\kappa, \Delta^*\} \\
&\quad - \frac{\hbar^2}{8} \left(\{\kappa, \Delta^*\} - \{\Delta, \kappa^*\}\right) + \ldots, \quad (14)
\end{align*}
\]

where functions \( h, f, \Delta, \) and \( \kappa \) are the Wigner transforms of \( \hat{h}, \hat{f}, \hat{\Delta}, \) and \( \hat{\kappa}, \) respectively, \( \{f, g\} \) is the Poisson bracket of functions \( f(r, p) \) and \( g(r, p); \) the dots stand for terms proportional to higher powers of \( \hbar. \)

To study the quadrupole collective motion with \( K^\pi = 1^+ \) in axially symmetric nuclei, it is necessary to calculate moments of Eq. (14) (+ three other equations) with the weight functions

\[
W = xz, \quad pxpz, \quad zpx + xpz \equiv \hat{L}, \quad \text{and} \quad zpx - xpz \equiv \hat{I}_y.
\]

This procedure yields 16 equation for collective variables \( f\hat{d}(p, r)Wf \) and \( f\hat{d}(p, r)W\kappa. \) However, due to symmetry considerations 8 of them turn out trivial. Applying the approximation \( \delta\kappa_+(r, p) \ll \delta\kappa_-(r, p) \) one is able to reduce the problem to a set of only six dynamical equations (strictly speaking 12 ones: 6 for protons and 6 for neutrons). Making the standard approximation
to decouple isovector and isoscalar subsets, we find that the isovector subset has two integrals of motion allowing one to reduce the eigenvalue problem to a quadratic equation. Its two solutions

\[ E^2_\pm = D_\omega \pm \sqrt{D_\omega^2 - [8\Delta \Delta c^2 + 4(h\omega)^4\delta^2](1 - \alpha) + 24\alpha\kappa_0\tilde{\Delta}k_0 h^4/m^2} \]  

(15)
describe the energy \( E_+ \) of the IVGQR and the energy \( E_- \) of the scissors mode. Here \( \epsilon^2 = h^2\omega^2(1 + \frac{2}{3}) \), \( k_0 = 2\int d(p, r)\kappa(r, p) \), \( D_\omega = 2\Delta \Delta + \epsilon^2(2 - \alpha) \), \( 2\Delta = |V_0|[I^\Delta_{pp}, \]

\[ I^\Delta_{pp}(r, p) = \frac{\gamma^3}{\sqrt{\pi}h^6}e^{-\gamma p^3} \int \kappa^0(r, p') \left[ \phi_0(2\gamma pp') - 4\gamma^2 p'^3\phi_2(2\gamma pp') \right] e^{-\gamma p'^3} p'^2 dp' , \]  

(16)

\[ \gamma = r_p^2/4h^2, \quad \phi_0(x) = \frac{1}{x}sh(x), \quad \phi_2(x) = \frac{1}{x^3}[(1 + \frac{3}{x^2})sh(x) - \frac{3}{x}ch(x)], \]

\( r_p \) and \( V_0 \) are parameters of the pair interaction \( v(|p - p'|) = -|V_0|(r_p\sqrt{\pi})^3e^{-\gamma p^2} \). It is worth noting that contrary to the case without pairing [7] the energy \( E_- \) does not go to zero for deformation \( \delta = 0 \). The calculation of transition probabilities shows that this mode of a spherical nucleus can be excited by an electric field and it is not excited by a magnetic field.

The transition probabilities are calculated with the help of linear response theory:

\[ B(E2)_\nu = 2|\langle \nu|\hat{F}^p_{21}|0\rangle|^2 = \frac{\epsilon^2h^2}{m} \frac{5}{4\pi}Q^p_{00} \frac{(1 + \delta/3)(E_\nu^2 - 4\Delta \Delta) - 2(h\omega\delta)^2}{E_\nu[E_\nu^2 - D_\omega]} , \]  

(17)

\[ B(M1)_\nu = 2|\langle \nu|\hat{F}^p_{11}|0\rangle|^2 = \frac{m\omega^2}{4\pi} \frac{(1 - \alpha)Q^p_{00}\delta^2}{E_\nu[E_\nu^2 - D_\omega]} \frac{E_\nu^2 - 4\Delta \Delta - 2\epsilon^2}{\mu_N^2} . \]  

(18)

It is necessary to notice that the transition probability, as given by formula (18), has the experimentally observed quadratic deformation dependence.

In the isoscalar case we have two solutions

\[ E^2_\pm = 2\Delta \Delta + \epsilon^2 \pm \sqrt{(2\Delta \Delta + \epsilon^2)^2 + 24\alpha\kappa_0\tilde{\Delta}k_0 h^4/m^2} , \]  

(19)

\( E_+ \equiv E_{iv} \) being the energy of the isoscalar giant quadrupole resonance and \( E_- \equiv E_{ISL} \) being the energy of the IsoScalar Low-Lying Excitation (ISLLE). It is important to note that the very existence of the ISLLE relies on two factors: 1) pair correlations and 2) quantum correction.

Results of our calculations [3] for most of the nuclei where the scissors mode has been observed are presented in Figure 1. As it is seen, the inclusion of pairing considerably improves the agreement between the theory and experiment.
4. Conclusion

The connection of the WFM method with the GF one and RPA is discussed. It is shown that WFM and GF methods are very close to one another. Contrary to the RPA, both work in phase space with no need to introduce a single particle basis and yield identical sets of dynamical equations for the moments. To show the analytical equivalence between the WFM and RPA methods one needs to introduce the dynamical equations for the transition matrix elements. They can be derived either from the RPA equations for the amplitudes $X_{kq}$, $Y_{kq}$ or from the WFM dynamical equations for the moments.

**Figure 1.** Energies and transition probabilities of the scissors mode as a function of the mass number $A$. $E, B_{\text{new}}$: continuity equation is fulfilled, $E, B_{\text{old}}$: continuity equation is violated, $E, B_0$: theory without pairing.

The analytical formulae reproduce very well the experimentally observed deformation dependence of the energy and $B(M1)$ factor of scissors. The results of calculations are in reasonable agreement with experimental data, pair correlations being extremely important. The joint action of pairing and quantum corrections leads to the appearance of a low-lying isoscalar excitation.

The study of low-lying modes and scissors on the basis of normal and anomalous densities, calculated within a microscopic approach including spin degrees of freedom, will be the subject of the future work.

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