Abstract

We analyse the algebras generated by free component quantum fields together with the susy generators $Q, \bar{Q}$. Restricting to hermitian fields we first construct the scalar field algebra from which various scalar superfields can be obtained by exponentiation. Then we study the vector algebra and use it to construct the vector superfield. Surprisingly enough, the result is totally different from the vector multiplet in the literature. It contains two hermitian four-vector components instead of one and a spin-3/2 field similar to the gravitino in supergravity.
1 Introduction

In this paper we continue our study of quantized free superfields started in [1]. In paper [1] we have constructed the quantized chiral superfield and we have found that it has the same component expansion as its classical counterpart [2-5]. Surprisingly enough, if we go on to more complicated superfields, this is no longer the case!

To understand this interesting fact we can give the following explanations. First of all the infinitesimal susy transformation of a classical superfield \( \Phi(x, \theta, \bar{\theta}) \) is defined by applying the operator \( \xi Q + \bar{\xi} \bar{Q} \) where \( Q, \bar{Q} \) are represented by the usual superspace differential operators. On the other hand, the corresponding transformation of the quantum fields is given by a commutator

\[
\delta \Phi = [\xi Q + \bar{\xi} \bar{Q}, \Phi]
\]

where now \( Q, \bar{Q} \) are operators in Fock space. Secondly, the components of the quantum superfield satisfy (anti-)commutation relations. There is an interesting interplay between these relations, the free field equations and the supersymmetric algebra, not existing in the classical case. In fact, the construction of these supersymmetric free field algebras is the main work to be done. First this is carried out for the hermitian scalar superfield in the next section and then for the vector superfield in sect.3. The superfields themselves are obtained from the algebra simply by exponentiation. The vector algebra will be used to construct the vector superfield. The latter comes out totally different from the vector field in the literature [2-3]. For example, it contains a spin-3/2 component similar to the gravitino in supergravity and two hermitian four-vector components instead of one. We are obviously driven to another representation of the supersymmetric algebra, when we consider quantized free fields.

2 The quantized hermitian scalar superfield

We want to represent the supersymmetric algebra

\[
\{Q_a, \bar{Q}_b\} = 2\sigma^{\mu}_{ab} P_{\mu} = -2i\sigma^{\mu}_{ab} \partial_{\mu}, \quad (2.1)
\]

\[
\{Q_a, Q_b\} = 0 = \{\bar{Q}_a, \bar{Q}_b\} = [Q_a, P_{\mu}] \quad (2.2)
\]

by operators in Fock space. \( Q_a \) and \( \bar{Q}_b \) transform according to the \((\frac{1}{2}, 0)\) and \((0, \frac{1}{2})\) representations of the proper Lorentz group, respectively. As usual one rewrites (2.1) as a Lie-algebra commutator

\[
[\theta^a Q_a, \bar{\theta}^b \bar{Q}_b] = 2\theta^a \sigma^{ab} \bar{\theta} P = -[\theta Q, \bar{\theta} \bar{Q}] \quad (2.3)
\]

by introducing anti-commuting C-numbers \( \theta^a, \bar{\theta}^b \). It is our aim to construct a superfield \( S \) with the transformation law

\[
\delta S = \left. \frac{\partial}{\partial \xi} S_{\xi} \right|_{\xi=1} = i[\theta Q + \bar{\theta} \bar{Q}, S] \quad (2.4)
\]

under infinitesimal supersymmetric transformations. Taking as initial value the scalar component field \( C(x) \), the solution is given by the finite transformation

\[
S_{\xi}(x, \theta, \bar{\theta}) = e^{i(\theta Q + \bar{\theta} \bar{Q})} C(x) e^{-i(\theta Q + \bar{\theta} \bar{Q})}. \quad (2.5)
\]

The factor \( i \) has been inserted to get a hermitian field

\[
S_1(x, \theta, \bar{\theta})^+ = S_1(x, \theta, \bar{\theta}) \quad (2.6)
\]
by assuming $C(x)$ to be hermitian. Then $C(x)$ must be quantized as a neutral scalar field of mass $m$
\[
[C(x), C(x')] = -iD(x - x'),
\]
(2.7)
where $D(x)$ is the causal Jordan-Pauli distribution for mass $m$.

We want to calculate (2.5) by means of the Lie series. For the first order terms we need the commutators
\[
[Q_a, C(x)] = \chi_a(x), \quad [\bar{Q}_b, C(x)] = -\bar{\chi}_b(x),
\]
(2.8)
where $\chi$ and $\bar{\chi}$ are quantized Majorana fields. The second commutation relation follows from the first by taking the adjoint, because (see [1] eq.(2.21))
\[
Q^+_a = \bar{Q}_a
\]
(2.9)
and similarly for $\chi_a$. Next we need the anticommutator
\[
\{Q_a, \chi_b(x)\} = \{Q_a, [Q_b, C]\} = \{Q_b, [C, Q_a]\} - [C, \{Q_a, Q_b\}] = -\{Q_b, \chi_a(x)\},
\]
(2.10)
where the Jacobi identity has been used. Due to the antisymmetry in $a, b$, the anticommutator must be of the form
\[
\{Q_a, \chi_b(x)\} = -im\varepsilon_{ab}F(x),
\]
(2.11)
where $F(x)$ is a scalar field and the mass factor has been introduced for dimensional reasons. Since the $\varepsilon$-tensor is real, the adjoint relation is
\[
\{Q^+_{\bar{a}}, \bar{\chi}_b(x)\} = im\varepsilon_{\bar{a}b}F(x)^+.
\]
(2.12)
The Majorana field $\chi_a$ must be quantized according to
\[
\{\chi_a(x), \chi_b(x')\} = im\varepsilon_{ab}D(x - x'),
\]
\[
\{\chi_a(x), \bar{\chi}_b(x')\} = -\sigma^\mu_{ab}\partial_\mu D(x - x').
\]
(2.13)
On the other hand, again by the Jacobi identity the first anticommutator is equal to
\[
= \{\chi_a(x), [Q_b, C(x')]\} = \{Q_b, [C(x'), \chi_a(x)]\} - [C, \{\chi_a, Q_b\}] =
\]
\[
= \{Q_b, [C(x'), \chi_a(x)]\} + im\varepsilon_{ba}[C(x'), F(x)].
\]
The commutator in the first term must be assumed to be 0, hence we find
\[
[F(x), C(x')] = D(x - x').
\]
We can therefore write
\[
F(x) = M(x) + iC(x)
\]
(2.14)
with
\[
[M(x), C(x')] = 0.
\]
(2.15)
We specialize (2.11) to $a = 2, b = 1$. Using $\varepsilon_{21} = 1$ we get
\[
\{Q_2, \chi_1\} = -imF.
\]
(2.16)
This implies
\[ [Q_a, F(x)] = \frac{i}{m} [Q_a, \{ Q_2, \chi_1(x) \}] = \]
\[ = -\frac{i}{m} ([Q_2, \{ \chi_1, Q_a \}] + [\chi_1, \{ Q_a, Q_2 \}]) = -\varepsilon_{a1}[Q_2, F(x)]. \] (2.17)

For \( a = 1 \) this gives
\[ [Q_1, F(x)] = 0 \]
and for \( a = 2 \)
\[ [Q_2, F(x)] = -[Q_2, F(x)] = 0. \] (2.18)

The adjoint relations are
\[ [\bar{Q}_a, F^+(x)] = 0. \] (2.19)

Next we consider
\[ [F(x), F(x')] = \frac{i}{m} [F(x), \{ Q_2, \chi_1(x') \}] = \]
\[ = -\frac{i}{m} \{ Q_2, [\chi_1(x'), F(x)] \} = 0 = [M(x), M(x')] - [C(x), C(x')]. \]

Since the first commutator is \(-iD(x - x')\) it follows
\[ [M(x), M(x')] = -iD(x - x'), \] (2.20)
so that \( M(x) \) is also a hermitian scalar field. This can also be derived from (2.13). To determine the other (anti)commutators we raise the second spinor index in (2.12) with the \( \varepsilon \)-tensor
\[ \{ \bar{Q}_\bar{a}, \bar{\chi}^\bar{c} \} = -im\delta_{\bar{a}}^\bar{c}F^+. \] (2.21)

Taking the trace \( \bar{a} = \bar{c} = 1, 2 \), we have
\[ F^+ = \frac{i}{2m} \{ \bar{Q}_\bar{a}, \bar{\chi}^\bar{a} \}. \] (2.22)

Let us apply the Dirac operator \( i\sigma_\mu^{ab} \partial_\mu \) to (2.21) and use the Dirac equation
\[ \{ \bar{Q}_\bar{a}, i\sigma_\mu^{ab} \partial_\mu \bar{\chi}^\bar{c} \} = m \{ \bar{Q}_\bar{a}, \chi_a \} = m\sigma_\mu^{ab} \partial_\mu F^+. \]

This leads to
\[ \{ \bar{Q}_\bar{a}, \chi_b \} = \sigma_\mu^{ab} \partial_\mu F^+ \] (2.23)
and the adjoint relation
\[ \{ Q_a, \bar{\chi}_b \} = \sigma_\mu^{ab} \partial_\mu F. \] (2.24)

By means of (2.22) and (2.24) we calculate
\[ [Q_a, F^+(x)] = \frac{i}{2m} [Q_a, \{ \bar{Q}_\bar{a}, \bar{\chi}^\bar{a} \}] = \]
\[ = -\frac{i}{2m} \left( [\bar{Q}_\bar{a}, \{ \bar{\chi}^\bar{a}, Q_a \}] + [\bar{\chi}^\bar{a}, \{ Q_a, \bar{Q}_\bar{a} \}] \right) \]
\[ = \frac{i}{2m} \sigma_\mu^{ab} [\bar{Q}^b, \partial_\mu F] + \frac{1}{m} \sigma_\mu^{ab} \partial_\mu \bar{\chi}^a. \] (2.25)
The adjoint equation is

\[ [\bar{Q} b, F(x)] = -\frac{i}{2m} \sigma^\mu_{bb} (Q^b \sigma_\mu + \overline{\theta} \overline{\chi}_b \sigma^\mu_\mu). \quad \text{(2.26)} \]

Substituting this into (2.25) and using the relations

\[ \hat{\sigma}^{\mu a} = \epsilon^{ab} \sigma^\mu_{bb}, \quad \text{(2.27)} \]

\[ (\hat{\sigma}^\nu \sigma^\mu + \hat{\sigma}^\mu \sigma^\nu)_{\bar{b}} = 2\eta^{\nu \mu} \delta^\mu_\bar{b}, \quad \text{(2.28)} \]

we conclude

\[ \frac{3}{4} [\bar{Q} b, F] = -\frac{i}{2} \overline{\chi}_b - \frac{1}{m} \sigma^\mu_{ab} \partial_\mu \chi^a. \quad \text{(2.29)} \]

In the second term on the r.h.s. we can use the (adjoint) Dirac equation

\[ \sigma^\mu_{ab} \partial_\mu \chi^a = im \overline{\chi}_b. \quad \text{(2.30)} \]

This finally gives

\[ [\bar{Q} b, F(x)] = -2i \overline{\chi}_b(x), \quad \text{(2.31)} \]

and the adjoint relation

\[ [Q b, F^+] = -2i \chi_b. \quad \text{(2.32)} \]

Now we are ready to evaluate (2.5) because all multiple commutators on the r.h.s. are known. The result is

\[ S_1(x, \theta, \bar{\theta}) = C(x) + i(\theta \chi - \bar{\theta} \overline{\chi}) + \]

\[ -i \left( \frac{m}{2} (\theta \theta F - \bar{\theta} \overline{\theta} F^+) + \theta \sigma^\mu \overline{\theta} \partial_\mu M + \right. \]

\[ +i \left( \frac{m}{3} (\theta \overline{\theta} \overline{\chi} - \theta \chi \overline{\theta}) + \frac{1}{3} \theta \sigma^\mu \overline{\theta} (\partial_\mu \overline{\chi} - \theta \overline{\partial}_\mu \chi) + \right. \]

\[ \left. + \frac{m^2}{4} \theta \theta \overline{\theta} C(x) \right). \quad \text{(2.33)} \]

The third order terms can be rewritten by means of the Dirac equation and the identity

\[ \bar{\theta}^a \overline{\theta}^b = \frac{1}{2} \epsilon^{ab} \overline{\theta} \overline{\theta} \quad \text{(2.34)} \]

in the form

\[ -i \frac{m}{2} (\overline{\theta} \theta \chi - \theta \overline{\theta} \overline{\chi}). \quad \text{(2.35)} \]

Another hermitian scalar superfield \( S_0 \) is obtained as the sum of the chiral plus the anti-chiral superfield \( \Phi + \Phi^+ \) constructed in [1]. This field has a component expansion of the same form as \( S_1 \) (2.33), but is actually different as can be seen by comparing the coefficients. The difference is not surprising if one realizes the different behaviour under infinitesimal susy-transformation. Indeed, from

\[ \Phi_\xi = e^{\xi(\theta Q + \bar{\theta} \overline{Q})} A e^{-\xi(\theta Q + \bar{\theta} \overline{Q})} \]  \quad \text{(2.36)}

we find

\[ \delta_\xi (\Phi + \Phi^+) = \left. \frac{\partial}{\partial \xi} (\Phi_\xi + \Phi_\xi^+) \right|_{\xi=1} = [\theta Q + \bar{\theta} \overline{Q}, \Phi] - [\theta Q + \bar{\theta} \overline{Q}, \Phi^+], \quad \text{(2.37)} \]
in contrast to (2.4). A third scalar field $S_2$ can be constructed as $S_1$ (2.5) but starting with the spinor component

$$S_2(x, \theta, \bar{\theta}) = e^{i(\theta Q + \bar{\theta} \bar{Q})}(i\theta \chi(x) - i\bar{\theta} \bar{\chi}(x))e^{-i(\theta Q + \bar{\theta} \bar{Q})}.$$  

(2.38)

All commutators are the same as before so that we immediately find

$$S_2 = i(\theta \chi - \bar{\theta} \bar{\chi}) - im(\theta \theta F - \bar{\theta} \bar{\theta} F^+) + (\theta \sigma^a \bar{\theta})(\partial_a F + \partial_a F^+) +$$

$$+ \frac{3}{2} im(\theta \chi \bar{\theta} - \bar{\theta} \bar{\chi} \theta) + \frac{i}{2} m^2 \theta \theta \bar{\theta}\bar{\theta}(F^+ - F).$$  

(2.39)

This scalar field is the most important one because we shall need it when we consider gauge transformations.

Still this is not the whole story because supersymmetry gives further constraints on the component fields. To see this we take the commutator of (2.24) with $\bar{Q}_e$ and use (2.31)

$$\sigma^a_{ab}[\bar{Q}_e, \partial_a F] = -2i\sigma^a_{ab} \partial_a \bar{\chi}_e = [\bar{Q}_e, \{Q_a, \bar{\chi}_b\}] =$$

$$= -[Q_a, \{\bar{\chi}_b, \bar{Q}_e\}] - [\bar{\chi}_b, \{\bar{Q}_e, Q_a\}].$$

Using the relations (2.12),(2.1) and (2.32) the last two commutators can be evaluated. This leads to the following relation

$$\sigma^a_{ab} \partial_a \bar{\chi}_e - \sigma^a_{ac} \partial_a \bar{\chi}_b = -im \varepsilon_{eb} \chi_a.$$  

(2.40)

Here we must only check the nontrivial case $b = 1, c = 2$, say. Then the relation is satisfied due to the Dirac equation. The adjoint relation is

$$\sigma^a_{ba} \partial_a \chi_e - \sigma^a_{ca} \partial_a \chi_b = im \varepsilon_{eb} \bar{\chi}_a.$$  

(2.41)

3 Quantized vector superfield

We have learnt in the last section that we do not obtain a vector component if we start from a scalar field. Therefore, we now start from a Majorana field $\lambda_a(x)$ as in $S_2$ (2.36), but we do not assume that $\lambda(x)$ is obtained from a scalar field like (2.8). We want to construct the hermitian superfield

$$V(x, \theta, \bar{\theta}) = e^{i(\theta Q_a + \bar{\theta} \bar{Q}_a)}(i\theta \lambda_a(x) + \text{h.c.})e^{-i(\theta Q_a + \bar{\theta} \bar{Q}_a)}.$$  

(3.1)

If we omit the $\theta^a$ starting from $i\lambda_a$, we obtain a spinor field $W_a(x, \theta, \bar{\theta})$. To determine the necessary commutators we start with the mixed anticommutator $\{Q_a, \lambda_b(x)\}$. It must be proportional to $\sigma^a_{ab}$, because there is a one-to-one correspondence between spinors of type $(\frac{1}{2}, \frac{1}{2})$ and four-vectors. Therefore, we introduce two self-conjugate vector fields $v, w$ by requiring

$$\{Q_a, \bar{\lambda}_b(x)\} = -m\sigma^a_{ab}(v \mu + iw \mu).$$  

(3.2)

The adjoint relation is

$$\{\bar{Q}_a, \lambda_b(x)\} = -m\sigma^a_{ba}(v \mu - iw \mu).$$  

(3.3)

As in the last section, two other relations follow by means of the Dirac equation. We multiply (3.2) by $\varepsilon^a_{eb}$ and apply the Dirac operator $i\sigma^c_{be} \partial_c$. This gives

$$\{Q_a, i\sigma^c_{be} \partial_c \bar{\lambda}^b\} = m\{Q_a, \lambda_b\} = -im\sigma^a_{ab}\varepsilon^c_{eb} \sigma^c_{be} \partial_c(v \mu + iw \mu) =$$
\[ = -i m \sigma_{bc}^\nu \tilde{\sigma}^{\mu c} \varepsilon_{ac} \partial_{\nu} (v_{\mu} + i w_{\mu}), \]

or

\[ \{ Q_a, \lambda_b \} = -i (\sigma^\nu \tilde{\sigma}^{\mu \nu})_{ba} \partial_{\nu} (v_{\mu} + i w_{\mu}). \]  \hspace{1cm} (3.4)

The adjoint relation reads

\[ \{ \bar{Q}_a, \bar{\lambda}_b \} = -i (\tilde{\sigma}^{\nu} \sigma^{\mu \nu})_{b\bar{a}} \partial_{\nu} (v_{\mu} - i w_{\mu}). \]

Next we have to determine the commutators of \( v_{\mu} \) and \( w_{\mu} \). Taking the commutator of (3.2) with \( Q_b \) we get

\[ [Q_b, \{ Q_a, \bar{\lambda}_b \}] = -m \sigma_{ab}^\mu [Q_b, v_{\mu} + i w_{\mu}]. \]  \hspace{1cm} (3.5)

Since the l.h.s. is antisymmetric in \( a, b \) due to the Jacobi identity, we conclude

\[ m \sigma_{ab}^\mu [Q_b, v_{\mu} + i w_{\mu}] = \frac{1}{2} \varepsilon_{ab} \bar{\Lambda}_b \]  \hspace{1cm} (3.6)

where

\[ \bar{\Lambda}_b = [Q_c, \{ Q^c, \bar{\lambda}_b \}]. \]  \hspace{1cm} (3.7)

It is easy to see that \( \{ Q_a, \bar{\Lambda}_b \} = 0 \) follows and also the adjoint relation \( \{ \bar{Q}_a, \Lambda_b \} = 0 \). Since \( \Lambda \) satisfies the Dirac equation as \( \lambda \), we also have \( \{ Q_a, \Lambda_b \} = 0 \). Using the Jacobi identity again

\[ [\bar{Q}_a, \{ Q_a, \Lambda_b \}] = 0 = -[Q_a, \{ \Lambda_b, \bar{Q}_a \}] - [\Lambda_b, \{ \bar{Q}_a, Q_a \}] = -2i \sigma_{a\bar{b}}^{\mu \nu} \partial_{\mu} \Lambda_{b}, \]  \hspace{1cm} (3.8)

we finally conclude \( \Lambda = 0 \). Hence, by (3.6)

\[ [Q_b, v_{\mu} + i w_{\mu}] = 0 = [\bar{Q}_b, v_{\mu} - i w_{\mu}], \]  \hspace{1cm} (3.9)

or

\[ [Q_b, v_{\mu}] = -i [Q_b, w_{\mu}], \quad [\bar{Q}_b, v_{\mu}] = i [\bar{Q}_b, w_{\mu}]. \]  \hspace{1cm} (3.10)

This corresponds to the result (2.18-19) in the scalar case.

Let us now consider the commutators

\[ [Q_a, v^{\mu}] = -i [Q_a, w^{\mu}] \overset{\text{def}}{=} f_a^{\mu}, \]  \hspace{1cm} (3.11)

\[ [\bar{Q}_a, v^{\mu}] = i [\bar{Q}_a, w^{\mu}] \overset{\text{def}}{=} - f_a^{\mu}. \]  \hspace{1cm} (3.12)

From (3.3) we get

\[ [Q_c, \{ \bar{Q}_{\bar{a}}, \lambda_b (x) \}] = -m \sigma_{b\bar{a}}^\mu [Q_c, v_{\mu} (x) - i w_{\mu} (x)] = -2m \sigma_{b\bar{a}}^\mu f_{\mu c}. \]

By the Jacobi identity this is equal to

\[ = -[\bar{Q}_a, \{ \lambda_b, Q_c \}] - [\lambda_b, \{ Q_c, \bar{Q}_a \}]. \]

Using (3.4) and (2.1) we obtain

\[ = i (\sigma^\nu \tilde{\sigma}^{\mu \nu})_{bc} [\bar{Q}_a, \partial_{\nu} (v_{\mu} + i w_{\mu})] - 2i \sigma_{c\bar{a}}^{\mu \nu} \partial_{\mu} \lambda_b. \]

This gives an inhomogeneous linear field equation

\[ -i (\sigma^\nu \tilde{\sigma}^{\mu \nu})_{bc} \partial_{\nu} f_{\mu \bar{a}} + m \sigma_{b\bar{a}}^\mu f_{\mu c} = i \sigma_{c\bar{a}}^{\mu \nu} \partial_{\mu} \lambda_b. \]  \hspace{1cm} (3.13)
The adjoint equation reads
\[ -i(\dot{\sigma}^\nu \sigma^\mu)_{bc} \partial_\nu f_{\mu a} + m \sigma^\mu_{ab} \dot{f}^\nu_{bc} = -i \sigma^\mu_{ab} \partial_\mu \lambda^c. \] (3.14)

With the same technique we treat
\[ \{Q_a, [\bar{Q}_b, v^\mu + iw^\mu]\} = -2\{Q_a, \bar{f}^\mu_b\} \]
\[ = -[v^\mu + iw^\mu, \{Q_a, \bar{Q}_b\}] + \{\bar{Q}_b, [v^\mu + iw^\mu, Q_a]\} \]
\[ = -2i \sigma^\nu_{ab} \partial_\nu (v^\mu + iw^\mu). \]

This gives the anticommutator
\[ \{Q_a, \bar{f}^\mu_b\} = i \sigma^\nu_{ab} \partial_\nu (v^\mu + iw^\mu), \] (3.15)
and the adjoint relation
\[ \{\bar{Q}_b, f^\mu_a\} = -i \sigma^\nu_{ba} \partial_\nu (v^\mu - iw^\mu). \]

To determine the anticommutator of $f$ we use the differential equation (3.13). We multiply it by $\dot{\sigma}^{\alpha a}$:
\[ -i \sigma^{\nu a} (\dot{\sigma}^{\mu \alpha} f_{bc} \partial_\nu \bar{f}_{\mu a} + m (\sigma^{\mu \nu})_{bc} f_{\mu c} = i (\sigma^{\mu \nu})_{bc} \partial_\mu \lambda_b. \]

We put $a = b$ and sum over $b$. Using the trace $tr(\sigma^{\mu \nu}) = 2\eta^{\mu \nu}$, we obtain
\[ -i (\dot{\sigma}^{\nu a} \sigma^{\mu \alpha}) (\sigma^{\mu \nu})_{bc} \partial_\nu \bar{f}_{\mu a} + 2m f^\nu_c = i (\sigma^{\mu \nu})_{bc} \partial_\mu \lambda_b. \] (3.16)

From (3.14) we find in the same way
\[ -i (\sigma^{\nu a} \sigma^{\mu \alpha}) (\sigma^{\mu \nu})_{bc} \partial_\nu \bar{f}_{\mu a} + 2m f^\nu_c = -i (\dot{\sigma}^{\nu a} \sigma^{\mu \alpha}) (\sigma^{\mu \nu})_{bc} \partial_\mu \lambda_b. \] (3.17)

The two equations (3.16-17) are similar to the Dirac equation, but contain three $\sigma$-matrices instead of one. For brevity we will call the system (3.16-17) the $3\sigma$-equations.

Now we operate with the 3\sigma-operator $-i (\dot{\sigma}^{\nu a} \sigma^{\mu \alpha}) (\sigma^{\mu \nu})_{bc} \partial_\alpha$ on (3.15) and substitute (3.16):
\[ -2m \{Q_a, f^\nu_c\} + i (\sigma^{\mu \nu})_{bc} \{Q_a, \partial_\mu \lambda_b\} = \]
\[ = \sigma^{\mu \nu}_{ac} (\dot{\sigma}^{\nu a} \sigma^{\mu \alpha}) (\sigma^{\mu \nu})_{bc} \partial_\alpha \partial_\nu (v^\nu + iw^\nu). \]

Using (3.4) we get
\[ -2m \{Q_a, f^\nu_c\} = -2 (\sigma^{\mu \nu})_{bc} (\sigma^{\mu \nu})_{ba} \partial_\mu \partial_\nu (v^\nu + iw^\nu) \]
\[ + (\sigma^{\mu \nu})_{bc} \partial_\mu \partial_\nu (v^\nu + iw^\nu). \] (3.18)

The result is antisymmetric in $a, c$
\[ 2m \{Q_a, f^\nu_c\} = [ (\sigma^{\mu \nu})_{bc} (\sigma^{\mu \nu})_{ba} - (\sigma^{\mu \nu})_{bc} (\sigma^{\mu \nu})_{ba} ] \partial_\mu \partial_\nu (v^\nu + iw^\nu). \] (3.19)

This antisymmetry is essential for consistency of the algebra because it follows from $\{Q_a, [Q_c, v^\mu]\}$ by means of the Jacobi identity.
It is not hard to simplify the product of four $\sigma$-matrices. Only the antisymmetric terms $\sim \varepsilon_{ac}$ survive so that we finally obtain

$$m\{Q_a, f_c^\mu\} = \varepsilon_{ac} \left[ -\Box (v^\rho + i w^\rho) + 2 \partial^\rho \partial_\mu (v^\mu + i w^\mu) \right].$$

(3.20)

It is clear that the second derivatives on the r.h.s. must be simplified by means of the wave equations for the vector fields. As in quantum gauge theories [7] we assume the massive vector fields $v, w$ to satisfy the Klein-Gordon equation

$$\Box v_\nu = -m^2 v_\nu,$$

(3.21)

and the same is assumed for $w$. Then we arrive at the result

$$\{Q_a, f_b^\mu\} = \varepsilon_{ab} \left[ m(v^\mu + i w^\mu) + \frac{2}{m} \partial_\mu \partial_\nu (v^\nu + i w^\nu) \right].$$

(3.22)

To exhaust all consequences of supersymmetry we have to analyse the commutator

$$[Q_c, \{\bar{Q}_a, f_b^\mu\}] = -2i \sigma_\nu^a \partial_\nu f_c^\mu.$$

Using (3.22), (2.1), and the Jacobi identity we find another equation for $f$

$$i(\sigma_\nu^a \partial_\nu f_c^\mu - \sigma_\nu^b \partial_\nu f_b^\mu) = m \varepsilon_{cb} f_a^\mu + \frac{2}{m} \varepsilon_{cb} \partial_\nu \partial_\mu f_a^\nu.$$

(3.24)

This equation has a similar form as (2.41). Therefore, we simplify it in the same way as we have treated (2.40). Due to the antisymmetry in $b, c$, it is sufficient to consider the case $b = 1, c = 2$. Then the equation assumes the form

$$i(\sigma_\nu^a \partial_\nu f_1^\mu - \sigma_\nu^a \partial_\nu f_2^\mu) = m \bar{f}_1^\mu + \frac{2}{m} \bar{\partial}_\nu \partial_\mu \bar{f}_2^\nu.$$

(3.25)

This is a modified Dirac equation

$$-i \sigma_\alpha^\nu \partial_\nu f_1^\alpha = m f_1^\mu + \frac{2}{m} \bar{\partial}_\nu \partial_\mu \bar{f}_2^\nu.$$

(3.26)

But due to the second derivatives it is not a true equation of motion as the $3\sigma$-equations. The adjoint equation reads

$$i \sigma_\alpha^\nu \partial_\nu \bar{f}_2^\alpha = m f_2^\mu + \frac{2}{m} \bar{\partial}_\nu \partial_\mu \bar{f}_1^\nu.$$

(3.27)

The next step is the decoupling of the fields. For this purpose we set

$$f_a^\mu = g_a^\mu + \alpha \partial_\mu \lambda_a + \beta \sigma_\alpha^\mu \tilde{\lambda}^a,$$

(3.28)

$$\bar{f}_a^\mu = \bar{g}_a^\mu + \alpha^* \partial_\mu \lambda_a + \beta^* \sigma_\alpha^\mu \bar{\lambda}^a,$$

(3.29)

and we choose the constants $\alpha, \beta$ in such a way that the $g$'s satisfy the homogeneous $3\sigma$-equations

$$-i(\dot{\sigma}^\rho \sigma^\nu \dot{\sigma}_c^\mu) \partial_\nu g_\mu^a = 2m g_a^\rho = 0,$$

(3.30)

$$-i(\dot{\sigma}^\rho \sigma^\nu \sigma^\mu)_{ab} \partial_\nu g_\mu^a + 2m \bar{g}_c^\rho = 0.$$  

(3.31)
This implies the relation
\[ \alpha = \frac{i}{m}(1 + 2\beta^*). \]  
(3.32)

On the other hand, if we substitute the same form (3.28-29) into (3.26) we see that all \( \lambda \)-terms cancel provided \( \beta \) is real
\[ \beta = \beta^*. \]  
(3.33)

We remind the reader that all our fields are free quantum fields. Therefore, the spinor field \( \lambda(x) \) satisfies the anticommutation relations (2.13). The situation is not so clear for the vector fields because there exist different possibilities. Calculating the commutator of (3.2) with \( v_\mu + iw_\mu \), we find
\[ [(v_\mu + iw_\mu)(x), (v_\nu + iw_\nu)(x')] = 0, \]  
(3.34)

so that
\[ [v_\mu(x), v_\nu(x')] = [w_\mu(x), w_\nu(x')] \overset{\text{def}}{=} C_{\mu\nu}(x - x'), \]  
(3.35)

assuming that all different fields commute. Similarly, computing the commutators \([v_\mu(x), \{\bar{Q_a}, \lambda_b(x')\}] \) and \([v_\mu(x), \{Q_a, \lambda_b(x')\}] \), we obtain the relations
\[ \{\bar{f}_{\mu a}(x), \lambda_b(x')\} = -m\sigma_{ba}^\mu C_{\mu\alpha}(x - x'), \]  
(3.36)

\[ \{f_{\mu a}(x), \lambda_b(x')\} = -i(\sigma^a_\nu \sigma^\mu)_{ba} \partial_\nu C_{\mu\beta}(x - x'). \]  
(3.37)

Substituting (3.29) into (3.36) and assuming that \( \bar{g}_{ab}^\mu \) anticommutes with \( \lambda_b \) we find
\[ \alpha^* \sigma_{ba}^\mu \partial_\mu D(x - x') + im\beta^* \sigma_{\mu ba} D(x - x') = -m\sigma_{ba}^\mu C_{\mu\nu}(x - x'). \]

Similarly (3.37) implies
\[ im\alpha_{ab} \partial^\mu D - \beta(\sigma^\nu \partial^\mu)_{ba} \partial_\nu D = -i(\sigma^\nu \partial^\mu)_{ba} \partial_\nu C_{\mu\beta}^\mu. \]  
(3.38)

We postpone the choice of the coefficients \( \alpha, \beta \) to the next section.

Since we now have all commutators, we can write down the vector superfield \( V(3.1) \)
\[ V = i\theta \lambda + i\theta \sigma^\nu \partial^\mu \partial_\nu (v_\mu + iw_\mu) - m\theta \sigma^\mu \partial_\nu (v_\mu - iw_\mu) \]
\[ + (\theta \sigma^\nu \partial^\mu \partial_\nu (\bar{f}_\mu) - im(\theta \sigma^\mu \bar{\theta})(\theta f_\mu) \]
\[ - i\frac{m}{3} (\theta \sigma^\nu \partial^\mu \partial_\nu (v_\mu - iw_\mu) + i\frac{m}{3} (\theta \sigma^\nu \bar{\theta})(\theta \sigma^\mu \bar{\theta}) \partial_\nu (v_\mu - iw_\mu) + \text{h.c.} \]  
(3.39)

The last two terms can be rewritten in the form
\[ i\frac{m}{2} (\theta \partial^\mu (v_\mu - iw_\mu) + \text{h.c.} \]
Furthermore, since \( \theta \sigma^\nu \partial^\mu \theta = \theta \eta^{\nu\mu} \), we get the final result
\[ V = i\theta \lambda + i(\theta \partial^\mu (v_\mu + iw_\mu) - m\theta \sigma^\mu \partial_\nu (v_\mu - iw_\mu) \]
\[ + (\theta \partial^\mu \partial_\nu (\bar{f}_\mu) - im(\theta \sigma^\mu \bar{\theta})(\theta f_\mu) \]
\[ + i\frac{m}{2} (\theta \bar{\theta} \partial^\mu (v_\mu - iw_\mu) + \text{h.c.} \]  
(3.40)
We want to consider this field $V$ as a gauge superfield. If we add the scalar field $S_2$ (2.39) to $V$, then the component fields are changed consistently according to

$$m(v_\mu + iw_\mu) \to m(v_\mu + iw_\mu - \frac{1}{m} \partial_\mu F)$$

$$m\partial^\mu(v_\mu + iw_\mu) \to m\partial^\mu(v_\mu + iw_\mu) - \Box F = m\partial^\mu(v_\mu + iw_\mu) + m^2 F$$

\[ \lambda \to \lambda + \chi \]

$$f_{\mu a} \to f_{\mu a} + \frac{i}{m} \partial_\mu \chi_a. \tag{3.43}$$

in all orders. The transformation (3.39) is the usual classical gauge transformation. Therefore, the field $V$ is a good starting point for supergauge theory. We will further investigate the properties of the vector superfield $V$ in the next section by analysing the mixed field $g^\mu_{\text{a}}(x)$. The free quantum superfields are the basis of supergauge theory if considered as operator theory [6] in the spirit of a recent monograph [7].

4 Investigation of the mixed field $g^\mu_{\text{a}}$

The field $g$ must satisfy the homogeneous $3\sigma$-equations (3.30-31) as well as the modified Dirac equations (3.26-27)

$$-i\sigma^\nu_{\text{a}\text{b}} \partial_\nu g^{\text{a}\text{b}} = m\bar{g}^\text{a}_{\text{b}} + \frac{2}{m} \partial^\nu \partial_\nu \bar{g}^\text{a}_{\text{b}}, \tag{4.1}$$

$$i\sigma^\alpha_{\text{a}\text{b}} \partial_\alpha g^{\text{a}\text{b}} = mg^\text{a}_{\text{b}} + \frac{2}{m} \partial^\alpha \partial_\alpha g^\text{a}_{\text{b}}. \tag{4.2}$$

In addition we get constraints from the anticommutators with the supercharges. The anticommutator (3.15) implies

$$\{\bar{Q}_\alpha, g^\mu_{\text{a}}\} = -i\sigma^\nu_{\text{a}\text{b}} \partial_\nu(v^\mu - iw^\mu) + \alpha m\sigma^\nu_{\text{b}\text{a}} \partial^\mu(v_\nu - iw_\nu)$$

$$+ i\beta(\sigma^\mu \hat{\sigma}^\alpha \sigma^\beta)_{\text{b}\text{a}} \partial_\alpha(v_\beta - iw_\beta). \tag{4.3}$$

and from (3.22) we obtain

$$\{Q_\alpha, g^\mu_{\text{a}}\} = \varepsilon_{\alpha\beta} \left[ m(v^\mu + iw^\mu) + \frac{2}{m} \partial^\mu \partial^\nu(v_\nu + iw_\nu) \right]$$

$$+ i\alpha(\sigma^\alpha \hat{\sigma}^\beta)_{\text{b}\text{a}} \partial_\beta \partial^\mu(v_\beta + iw_\beta) - \beta m(\sigma^\mu \hat{\sigma}^\nu)_{\text{b}\text{a}} (v_\nu + iw_\nu). \tag{4.4}$$

These relations show that $g$ must be different from zero.

Since the vector index $\mu$ in $g^\mu_{\text{a}}$ corresponds to the spinor representation $(\frac{1}{2}, \frac{1}{2})$, the mixed field realizes the representation $(\frac{1}{2}, 0) \times (\frac{1}{2}, \frac{1}{2})$ of the proper Lorentz group. By the Clebsch-Gordan decomposition it splits into irreducible representations as follows

$$(\frac{1}{2}, 0) \times (\frac{1}{2}, \frac{1}{2}) = (0, \frac{1}{2}) + (1, \frac{1}{2}). \tag{4.6}$$

The dimensions of the representations are $2 \times 4 = 2 + 6$. We calculate with the most general (reducible) tensor and set

$$g^{\mu a} = \sigma^\mu_{\text{b}\text{c}} g^{\text{b}\text{c}}. \tag{4.7}$$
The l.h.s is the so-called Rarita-Schwinger representation [8] of the spinor on the r.h.s. The adjoint field is given by

\[ \bar{g}^\mu_{\bar{a}} = \sigma^\mu_{\bar{c}b} \bar{g}^{\bar{b}c}. \]  

(4.8)

If we substitute this representation into the 3\sigma-equation (3.30) we find the following Dirac equation for higher spin fields [8]

\[ i\sigma^\nu_{\bar{a}c} \partial_\nu \bar{g}^{\bar{b}c} = mg_{\bar{b}c}. \]  

(4.9)

Similarly, the other 3\sigma-equation (3.31) gives the adjoint Dirac equation

\[ i\hat{\sigma}^{\nu\bar{a}} \partial_\nu \bar{g}_{\bar{c}b} = mg_{\bar{c}b}. \]  

(4.10)

The indices \( c, \bar{c} \) are simply spectators in this Dirac equation.

Next we have to fulfill the equation (4.1). It boils down to the second order equation

\[ -i\sigma^\nu_{a\bar{d}} \partial_\nu g^a_{\bar{d}d} = m\bar{g}^{\bar{d}a} + \frac{1}{m} \sigma^\mu_{ad} \sigma^\nu_{\bar{b}c} \partial_\nu g_{\bar{b}c}. \]  

(4.11)

Here we insert (4.9) in the form

\[ \sigma^\nu_{\bar{a}c} \partial_\nu \bar{g}^{\bar{b}c} = -img_{\bar{b}c}. \]  

(4.12)

and get

\[ -i\sigma^\nu_{a\bar{d}} \partial_\nu g^a_{\bar{d}d} = m\bar{g}^{\bar{d}a} - i\sigma^\mu_{ad} \partial_\mu g_{\bar{c}a}. \]  

(4.13)

Now we decompose \( g \) into a symmetric and antisymmetric part

\[ g_{\bar{a}d} = \frac{1}{4}(g_{\bar{a}d} + g_{d\bar{a}}) + \frac{1}{4}(g_{\bar{a}d} - g_{d\bar{a}}). \]  

(4.14)

Since an antisymmetric second rank tensor has only one independent component, we can write

\[ g_{\bar{a}d} = g'_{\bar{a}d} + \varepsilon_{ad}\tilde{\psi}_d, \]  

(4.15)

\[ \bar{g}_{d\bar{a}} = \bar{g}'_{d\bar{a}} + \varepsilon_{ad}\tilde{\psi}_d. \]  

(4.16)

This is the decomposition (4.6) into irreducible Lorentz tensors. As we will see, \( \tilde{\psi} \) is a true independent spinor field, and \( g' \) is symmetric in the indices of the same kind. We use this decomposition in the second Dirac equation (4.10) and obtain

\[ i\sigma^\nu_{d\bar{a}} \partial_\nu (g_{\bar{b}c} + \varepsilon_{ab}\tilde{\psi}_c) = -m\bar{g}_{\bar{a}c}. \]  

(4.17)

Substituting this into (4.13) we get

\[ m(\tilde{g}_{\bar{a}d} - \tilde{g}^d_{\bar{a}}) = -2i\sigma^{\nu d}_{\bar{a}} \partial_\nu \tilde{\psi}_d + i\sigma^{\nu d}_{\bar{a}} \partial_\nu g_{\bar{c}a}. \]  

(4.18)

By (4.15) we can express \( \tilde{\psi} \) as a contracted tensor

\[ 2\tilde{\psi}_d = g^b_{bd}. \]  

(4.19)

Then (4.17) gives the following equation for the spinor field \( \psi \):

\[ m\varepsilon_{ad} \psi^{bd} = i(\sigma^{\nu d}_{\bar{a}} \partial_\nu \tilde{\psi}_a - \sigma^{\nu d}_{d\bar{a}} \partial_\nu \tilde{\psi}_d). \]  

(4.19)

This is just the relation (2.40) which shows that \( \psi(x) \) is a spin-1/2 field satisfying the Dirac equation. Since all steps can be inverted, the equation (4.11) is then satisfied. However, the
fields \( g' \) and \( \psi \) are not completely decoupled. In fact, substituting (4.15) into (4.10) and using (4.19) we obtain an inhomogeneous Dirac equation for \( g' \):
\[
 i \sigma^{\nu\mu} \partial_{\nu} g'_{\alpha} - m g'_{a b c} = - i \sigma^{\mu}_{c e} \partial_{\mu} \bar{\psi}_{\alpha}.
\]  
(4.20)
Consequently, \( g'_{ab c}(x) \), although being symmetric in \( a, b \), is not identical with the Rarita - Schwinger field for spin-3/2 particles [8].

Finally, there remains to check the consistency of the anticommutators with the supercharge, because this is the point where something goes wrong if there is an incorrect step in the construction. The anticommutator (4.4) gives the corresponding anticommutator for the spinor field:
\[
 \{ Q_{\alpha}, g_{bc} \} = \varepsilon_{ab} \sigma^\mu_{ce} \left[ \frac{m}{2} (v_\mu + iw_\mu) + \partial^\nu (v_\nu + iw_\nu) \frac{1}{m} \frac{i}{2} \alpha \right] 
- \beta m \varepsilon_{cb} \sigma^\nu_{ca} (v_\nu + iw_\nu) + \alpha \sigma^\mu_{ce} \sigma^\nu_{bc} \partial_{\alpha} (v_\beta + iw_\beta).
\]  
(4.21)
For any choice of \( \alpha \) and \( \beta \) the r.h.s. can be decomposed into a symmetric and antisymmetric part with respect to \( b, c \). This allows to identify the anticommutators of \( g' \) and \( \psi \) and shows the consistency of the whole construction. A preferred choice is
\[
 \alpha = 0, \quad \beta = - \frac{1}{2}.
\]  
(4.22)
because by (3.36) this leads to the propagator
\[
 C_{\mu \nu}(x - x') = \frac{i}{2} \eta_{\mu \nu} D(x - x')
\]  
(4.23)
which has a good ultraviolet behaviour. For this choice let us write down the antisymmetric part of (4.21):
\[
 \{ Q_{\alpha}, \bar{\psi}_{\beta} \} = - \frac{1}{2} \sigma^\mu_{ac} \left[ \frac{3}{2} m (v_\mu + iw_\mu) + \frac{1}{m} \partial^\nu (v_\nu + iw_\nu) \right].
\]  
(4.24)
Here we have used the simple identity
\[
 \varepsilon_{ab} \sigma^\mu_{ce} - \varepsilon_{ae} \sigma^\mu_{bc} = \varepsilon_{cb} \sigma^\mu_{ae}.
\]  
(4.25)
For later use we also calculate the anti-commutation relations of the mixed fields. By means of (3.12) we find
\[
 \{ f^\mu_a(x), \bar{f}^\nu_b(x') \} = - \{ f^\mu_a(x), [\bar{Q}_b, v^\nu(x')] \} = [v^\nu(x'), \{ f^\mu_a(x), \bar{Q}_b \}]
= i \sigma^\rho_{ab} \partial_\rho [v^\mu(x) - iw^\mu(x), v^\nu(x')]
= i \sigma^\rho_{ab} \partial_\rho C^{\mu \nu}(x - x')
= \frac{1}{2} \eta^{\mu \nu} \sigma^\rho_{ab} \partial_\rho D(x - x').
\]  
(4.26)
for the preferred choice (4.23). This implies the following anti-commutator for the mixed \( g' \)-field:
\[
 \{ g^\mu_a(x), \bar{g}^\nu_b(x') \} = \frac{1}{2} \eta^{\mu \nu} \sigma^\rho_{ab} \partial_\rho D(x - x') - \frac{1}{4} (\sigma^\mu \sigma^\nu)_{ab} \partial_\rho D(x - x').
\]  
(4.27)
In a similar way we obtain
\[
 \{ f^\mu_a(x), f^\nu_b(x') \} = - \frac{i}{2} \varepsilon_{ab} (m \eta^{\mu \nu} + \frac{2}{m} \partial^{\mu} \partial^{\nu}) D(x - x'),
\]  
(4.28)
and
\[
\{ g_\mu^a(x), g_\nu^b(x') \} = \frac{i}{4} m (\varepsilon_{ab} \eta^\mu\nu - 2i \sigma^\mu\nu_{ab}) D(x - x') - \frac{i}{m} \varepsilon_{ab} \partial^\mu \partial^\nu D(x - x'), \quad (4.29)
\]
\[
\{ g_\mu^a(x), g_\nu^b(x') \} = \frac{i}{4} m (\varepsilon_{ab} \eta^\mu\nu + 2i \sigma^\mu\nu_{ab}) D(x - x') + \frac{i}{m} \varepsilon_{ab} \partial^\mu \partial^\nu D(x - x'), \quad (4.30)
\]
\[
\{ g_\mu^a(x), g_\nu^b(x') \} = -\frac{1}{2} \eta^\mu\nu \sigma^\rho_{ab} \partial_\rho D(x - x') - \frac{1}{4} (\sigma^\nu \partial^\rho \sigma^\mu)_{ab} \partial_\rho D(x - x'). \quad (4.31)
\]

It is interesting to discuss our results from the point of view of representation theory. The massive representations of the $N = 1$ supersymmetric algebra have the following spin components:

| spin | $\Omega_0$ | $\Omega_{1/2}$ | $\Omega_1$ | $\Omega_{3/2}$ |
|------|------------|----------------|------------|----------------|
| 0    | 2          | 1              |            |                |
| 1/2  | 1          | 2              | 1          |                |
| 1    | 1          | 2              | 1          |                |
| 3/2  | 1          | 2              |            |                |
| 2    |            |                |            | 1              |

The representation $\Omega_0$ corresponds to the scalar superfield $S_1$ (2.33) because it has 2 scalar and 1 spinor component. The vector superfield $V$ (3.40) may be related to $\Omega_1$, if we consider the mixed field $f_\mu^a$ as the spin-3/2 component. But this is not entirely justified because $f_\mu^a$ contains also a spin-1/2 piece. The surprising result is that it was impossible to realize the representation $\Omega_{1/2}$ with free quantum fields. In fact, if we start with the single scalar component required by this representation, as we did in sect.2, we are driven into the representation $\Omega_0$.

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