Quantum non-linear effects in colliding light beams and interferometers

S Hacyan
Instituto de Física, Universidad Nacional Autónoma de México, Apdo. Postal 20-364, Cd. México 01000, Mexico
E-mail: hacyan@fisica.unam.mx

Keywords: nonlinear electrodynamics, interferometry, quantum vacuum, paraxial approximation

Abstract
Small corrections to the electromagnetic field in colliding light beams are evaluated taking into account the interaction of light with the quantum vacuum, as predicted by quantum electrodynamics. The nonlinear formulation of McKenna and Platzman (1963) is used. Possible implications for very energetic light beams and the radiation pressure in interferometers (e.g., gravitational waves detector) are considered. It is also shown that a head-on collision produces secondary waves along conical directions at angles $\arccos(1/3)$.

1. Introduction
In 1936, Euler and Heisenberg [1] proved that photons can interact with photons through the intermediate creation of electron-positron pairs, thus introducing nonlinear corrections to the Maxwell equations. This interaction, being of second order in the fine structure constant $\alpha$, is difficult to observe in practical conditions. Nevertheless, there have been several works devoted to the small but not entirely negligible effects predicted by quantum electrodynamics. Some interesting effects are the induction of dichroism and birefringence in the quantum vacuum [2–6], the splitting of photons [7–10], the bending of light by strong magnetic fields [11, 12], corrections in laser interferometry [13–16], and other processes related to the propagation of light [17–25].

The aim of the present paper is to study the possible effects of the quantum vacuum on interacting collimated light beams. In an interferometer, for instance, photons bounce back and forth (multiple times in a Fabri-Pérot [26]) and, if the light beam is sufficiently powerful, there might be small corrections to the radiation pressure and other physical parameters. Such a problem has been addressed previously by several authors [13–16]; here, a different (and relatively simpler) approach is proposed which is based on the paraxial approximation commonly used in many problems of optics. In particular, having in mind extremely powerful lasers as those used in gravitational waves observatories, it is worth evaluating whether or not the effects of nonlinear electrodynamics should be taken into account among the many possible sources of noise. Fortunately, as shown in the present paper, this is not yet the case, since it would be noticeable for extremely high-intensity focused laser beams. Nevertheless, it may be relatively important in the future if much more powerful light beams were available. In the meantime, some curious properties of light interacting with the quantum vacuum can be deduced theoretically.

The calculations in the present paper are based on the formulation of McKenna and Platzman [17], which we review in section 2 for the sake of completeness. In section 3, the nonlinear effects of the interaction between emitted and reflected waves in an interferometer are evaluated within the paraxial approximation. A brief discussion of the results is given in section 5.

2. Nonlinear electrodynamics
As shown in 1964 by McKenna and Platzman [17], the equations of electrodynamics that follow to order $\alpha^2$ from the Euler-Heisenberg Lagrangian are equivalent to the standard Maxwell equations with ‘sources’ arising from the nonlinear terms Explicitly, the ‘density’ and ‘current’ are defined as
\[ \rho = -\frac{\lambda}{4\pi} \nabla \cdot \delta E, \]  
(2.1)  
\[ \mathbf{J} = \frac{\lambda}{4\pi} \left[ \frac{\partial}{\partial t} \delta \mathbf{E} - \nabla \times \delta \mathbf{B} \right], \]  
(2.2)  
where the nonlinear terms are  
\[ \delta \mathbf{E} = 2(\mathbf{E}^2 - \mathbf{B}^2) \mathbf{E} + 7(\mathbf{E} \cdot \mathbf{B}) \mathbf{B}, \]  
\[ \delta \mathbf{B} = 2(\mathbf{E}^2 - \mathbf{B}^2) \mathbf{B} - 7(\mathbf{E} \cdot \mathbf{B}) \mathbf{E}. \]  
(2.3)  
Here  
\[ \lambda = \frac{\alpha^2}{45\pi m_e}, \]  
m_e is the electron mass, and naturalized Gaussian cgs units (i.e., \( \hbar = c = 1 \) and \( \alpha = e^2 \)) are used.  
The substitution of equations (2.3) in (2.1) and (2.2) leads to a set of modified Maxwell equations that can be solved by approximation methods, as shown in the following. Since only the physical fields appear in the equations, the results must be gauge-independent.  
The corrections to the electromagnetic field can be expressed in the form  
\[ \mathbf{E}_0 = \mathbf{E}_0 + \lambda \mathbf{E}_1, \quad \mathbf{B}_0 = \mathbf{B}_0 + \lambda \mathbf{B}_1, \]  
(2.4)  
where \( \mathbf{E}_0 \) and \( \mathbf{B}_0 \) are the solutions of the standard linear equations. Thus, the following first order equations are obtained [17]:  
\[ \left( \nabla^2 - \frac{\partial^2}{\partial t^2} \right) \mathbf{E}_1(\mathbf{r}, t) = 4\pi \left[ \frac{\partial}{\partial t} \mathbf{J}_0(\mathbf{r}, t) + \nabla \rho_0(\mathbf{r}, t) \right], \]  
(2.5)  
\[ \left( \nabla^2 - \frac{\partial^2}{\partial t^2} \right) \mathbf{B}_1(\mathbf{r}, t) = -4\pi \nabla \times \mathbf{J}_0(\mathbf{r}, t), \]  
(2.6)  
where it is understood that \( \rho_0 \) and \( \mathbf{J}_0 \) are given in terms of the linear field through equation (2.3).  

3. Linear and nonlinear fields  

In order to fix the ideas, let us consider the incident and reflected waves in an interferometer. Consider first a standard background field without nonlinear corrections. A stationary linearly polarized EM wave of intensity \( U(\mathbf{r}) \) interfering in vacuum with its reflection from a mirror has the form:  
\[ \mathbf{E}_0(\mathbf{r}, t) = U(\mathbf{r})(\cos f_+ + r \cos f_-) \mathbf{e}_x, \]  
(3.1)  
where  
\[ f_+ = -\omega t + kz, \quad f_- = -\omega t - kz + \varphi, \]  
r is the reflection coefficient (\( r = 1 \) for the head-on collision of two identical beams) and \( \varphi \) is a phase. Since the initial fields appear nonlinearly in the previous expressions, we use only their real values.  
In the paraxial approximation  
\[ \left| \frac{\partial U}{\partial z} \right| \gg \left| \frac{\partial^2 U}{\partial z^2} \right|, \]  
k = \omega, and we have  
\[ \mathbf{B}_0 = U(\mathbf{r})(\cos f_+ - r \cos f_-) \mathbf{e}_y \]  
(3.2)  
(\( \mathbf{e}_x \) and \( \mathbf{e}_y \) are unit vectors in the \( x \) and \( y \) directions).  
Accordingly,  
\[ \mathbf{E}_0 \cdot \mathbf{B}_0 = 0, \]  
\[ \mathbf{E}_0^2 - \mathbf{B}_0^2 = 4r U^2(\mathbf{r}) \cos f_+ \cos f_- \]  
(3.3)  
In the following, we will use the electromagnetic wave in the simple form (in cylindrical coordinates)  
\[ U(\mathbf{r}) = \mathcal{E}_0 e^{-\rho^2/w_0^2}, \]  
(3.4)  
where \( \rho = \sqrt{x^2 + y^2} \) and \( \mathcal{E}_0 \) is the electric field amplitude. This corresponds to the limiting case of a Gaussian beam with width \( w_0 \) and Rayleigh range tending to infinity.
3.1. Nonlinear corrections

Let us take (3.1) as the electric field inside the interferometer. For the nonlinear terms, we have from (2.3) and (3.3)

\[
\delta E = 8rU^3 \cos f_+ \cos f_- (\cos f_+ + r \cos f_-) \mathbf{e}_x, \tag{3.5}
\]

\[
\delta B = 8rU^3 \cos f_+ \cos f_- (\cos f_+ - r \cos f_-) \mathbf{e}_y, \tag{3.6}
\]

and therefore, with some straightforward algebra,

\[
\nabla \cdot \delta E = 0,
\]

\[
\frac{\partial}{\partial t} \delta E - \nabla \times \delta B = 4r\omega U^3 \{ \sin(-3\omega t + kz + \phi) + r \sin(-3\omega t - kz + 2\phi) + \sin(\omega t - 3kz + \phi) + 2 \sin(-\omega t - kz + \phi) + r \sin(\omega t + 3kz - 2\phi) + 2 \sin(-\omega t + kz) \}, \tag{3.7}
\]

keeping only terms of order \( \omega \), still in the paraxial approximation. Thus, equation (2.6) take the form

\[
\left( \nabla^2 - \frac{\partial^2}{\partial t^2} \right) E_1 = -3Ce^{-3\rho/\rho_0} \Re \{ e^{-3i\omega t + i\rho} [e^{i\kappa z} + re^{i(z + k\phi)}] - e^{-i\omega t - i\rho} [e^{-i\kappa z} + re^{-i(z + k\phi)}] \} \mathbf{e}_x,
\]

\[
\left( \nabla^2 - \frac{\partial^2}{\partial t^2} \right) B_1 = -Ce^{-3\rho/\rho_0} \Re \{ e^{-3i\omega t + i\rho} [e^{i\kappa z} - re^{i(z + k\phi)}] - e^{-i\omega t - i\rho} [e^{-i\kappa z} - re^{-i(z + k\phi)}] \} \mathbf{e}_y,
\]

where \( C = 4r\omega U^3 \). As shown in the appendix, the terms proportional to \( e^{-i(\omega t \pm 3kz)} \) and \( e^{-i(\omega t \pm k\phi)} \) make no relevant contribution to the solutions of the field equations. In anticipation of this result, they are not included in all the following expressions.

Defining

\[
E_1 = \Re \{ \Delta E \} \mathbf{e}_x, \quad B_1 = \Re \{ \Delta B \} \mathbf{e}_y, \tag{3.10}
\]

and using the retarded Green function, the solution of the above equations for the electric field turns out to be

\[
\Delta E = -\frac{3}{4\pi} Ce^{-3\rho/\rho_0} \int d\rho' d\rho' d\phi' e^{-3(\rho'/\rho_0)^2} \times (e^{ikz'} + re^{-i(kz' + \phi)}) \frac{e^{i\kappa |x - x'|}}{|x - x'|}, \tag{3.11}
\]

Now, if the beam is highly concentrated around the axis \( \rho' = 0 \), we can approximate

\[
|x - x'| \approx |\rho^2 + (z - z')^2|^{1/2},
\]

and the above integral reduces to

\[
\Delta E = -\frac{C}{4} w_0^2 e^{-3i\omega t + i\rho}(1 - e^{-3(\rho/\rho_0)^2})(I_+ + re^{i\phi}L_+), \tag{3.12}
\]

where

\[
I_+(\rho, z) = \int_0^L dz' e^{i\kappa z'} \frac{e^{i\kappa |\rho^2 + (z - z')^2|^{1/2}}}{|\rho^2 + (z - z')^2|^{1/2}}, \tag{3.13}
\]

\( R \) is the radius of the interferometer and \( L \) its length.

For the magnetic field \( \Delta B \), just change \( r \rightarrow -r \) in (3.12) and divide by a factor \( 3 \).

As shown in the appendix below, the integrals (3.13) can be solved approximately. Thus, using formula (A.2), the corrections to the EM field due to non-linear effects can be approximated as

\[
\Delta E = -\lambda E_0^3 \alpha \omega^2 w_0^2 (1 - e^{-3(\rho/\rho_0)^2})e^{-3i\omega t + i\rho} \frac{\pi}{\sqrt{2} k\rho} \times \left[ e^{i\kappa z} \Theta(z - \rho/\sqrt{8}) + re^{-i\kappa z + i\rho} \Theta(L - \rho/\sqrt{8} - z) \right], \tag{3.14}
\]

(\( \Theta \) is the step-function). The same formula for \( \Delta B \) applies with the only changes \( r \rightarrow -r \) in the term between square brackets in equation (3.14) and an overall factor \( 1/3 \).
The above equation (3.14) can be written in general units as
\[
\frac{|\Delta E|}{E_0} \approx \frac{\pi}{45} \left( \frac{\hbar}{2m c} \right)^2 \left( \frac{w_0}{\lambda_\ell} \right) (1 - e^{-3(\lambda R/w_0^2)}) \sqrt{\frac{\lambda_\ell}{2\sqrt{2}}} \left( \frac{\mathcal{P}}{\mathcal{P}_0} \right),
\]  
(3.15)
where \( \hbar = \alpha \hbar / m_c c \approx 2.8 \times 10^{-15} \text{ m} \) is the classical electron radius,
\( \mathcal{P}_0 \equiv m_e^2 c^4 / \hbar \approx 6.4 \times 10^7 \text{ W} \)
is the basic ‘quantum power’ associated to the electron, \( \mathcal{P} = \pi R^2 E_0^2 \) is the laser power and \( \lambda_\ell \) is its wavelength. The formula (3.15) is written as a dimensionless quantity in order to give an idea of the order of magnitude of the nonlinear effects.

Yet another way to write the above formula is in terms of the intensities as
\[
\frac{|\Delta E|}{E_0} \approx 0.12 \frac{w_0}{\sqrt{\lambda_\ell} R} \left( \frac{\mathcal{P}/w_0^2}{8.2 \times 10^{12} \text{ W/cm}^2} \right),
\]  
(3.16)
after an integration and averaging over the cross section \( 0 < \rho < R \), and where \( \mathcal{P}_\rho/w_0^2 \approx 8.2 \times 10^{12} \text{ W/cm}^2 \) can be interpreted as the ‘quantum intensity’ associated to the electron.

4. Radiation pressure

With the above results, we can now calculate the contribution of the non-linear electrodynamic effects to the intensity as given by the Poynting vector (which is entirely in the z direction):
\[
\mathcal{I}(\rho, z) = |\mathbf{E} \times \mathbf{B}|.
\]  
(4.1)
It must be noticed that to order \( \lambda \) only, the Poynting vector oscillates very rapidly with a frequency equal to \( n \omega \) \( (n = 1, 2, 3, \ldots) \); therefore, it does not contribute, in the average, to the radiation pressure. For a non-null time averaged contribution, we must go to order \( \lambda^2 \).

For the present purpose, it is enough to restrict the calculations to the end points of the interferometer and to consider \( L > \rho/8 \). Let us first consider the EM field at the end of the interferometer, \( z = L \). There, only the right moving wave contributes to the QED correction. Accordingly, the contribution to the Poynting vector from the right-moving wave turns out to be:
\[
\langle \Delta \mathcal{I}(L) \rangle = \frac{\sqrt{2}}{3} \pi^\frac{3}{4} |\mathcal{E}_0|^2 |r|^2 \omega^4 w_0^2 (1 - e^{-3(\lambda R/w_0^2)})^2 R,
\]  
(4.2)
after a time averaging and yet another averaging over the cross section of the beam (i.e., \( 0 < \rho < R \)).

Likewise,
\[
\langle \Delta \mathcal{I}(0) \rangle = |r|^2 \langle \Delta \mathcal{I}(L) \rangle.
\]  
(4.3)
Formula (4.2) can be written in general units as
\[
\langle \Delta \mathcal{I}(L) \rangle = \left( \frac{2 \mathcal{P}}{c} \right) \left[ \frac{4 \sqrt{2}}{3(45)^{\frac{3}{2}}} |r|^2 (1 - e^{-3(\lambda R/w_0^2)})^2 \left( \frac{\hbar}{2m c} \right)^2 \left( \frac{w_0}{\lambda_\ell R} \right)^4 \left( \frac{\mathcal{P}}{\mathcal{P}_0} \right)^{\frac{3}{2}} \right]
\approx \left( \frac{2 \mathcal{P}}{c} \right) \left[ 0.003 |r|^2 \left( \frac{w_0}{\lambda_\ell R} \right)^3 \left( \frac{\mathcal{P}/w_0^2}{8.2 \times 10^{12} \text{ W/cm}^2} \right)^{\frac{3}{2}} \right].
\]  
(4.4)
The additional force on the mirror at one end of the interferometer is precisely \( \langle \Delta \mathcal{I}(L) \rangle \), to be compared with the force \( 2\mathcal{P}/c \) produced by the classical radiation pressure (see e.g., [27], Sect. 9.4.2). In equation (4.4), the term in square brackets is precisely the dimensionless QED correction to the radiation pressure force.

5. Conclusions

The formulas obtained above describe the generation of waves of frequency \( 3\omega \) with a propagation vector along a conical direction \( \sqrt{8} \mathbf{e}_\phi \pm \mathbf{e}_z \), and with a opening semi-angle \( \alpha = \arccos(1/3) \approx 70^\circ \). One can interpret this result as describing the splitting along a conical direction of two beams of photons colliding head-on. However, it must be noticed that the electric and magnetic field vectors remain in the z constant plane and the Poynting vector is in the z direction: this can be interpreted as a sideways propagation of the generated waves (as in anisotropic crystals).

Formulas (3.16) and (4.4) imply (assuming \( w_0 \sim R \)) that the corrections due to non-linear EM effects are of the order \( \sqrt{w_0/\lambda_\ell} \) for the EM field and of order \( (w_0/\lambda_\ell)^3 \) for the radiation pressure in one or the other end of the
interferometer. Thus, shorter wavelengths and thicker beams would produce the more relevant non-linear effects.

In the particular case of an extremely powerful laser, of about 750 kW as reached by LIGO [28] (using a dual-recycled Fabry-Pérot-Michelson interferometer [26]), and a wave length \( \lambda_0 \approx 1000 \text{ nm} \), with \( R \approx w_0 \approx 10 \text{ cm} \), the above correction factor, as given by (4.4), turns out to be of the order of \( 10^{-34} \). Clearly, it can be safely neglected in all measurements. There remains, nevertheless, the possibility that the effects of nonlinear electrodynamics could be detected in the future with extremely energetic light beams of short wavelengths, most likely in the range of x-rays or shorter.

**Appendix**

Consider \( I_1(\rho, z) \) first. Notice that the integrand oscillates very rapidly, except around the minimum of the function

\[
w(z') = z'+ 3[\rho^2 + (z-z')^2]^{1/2}
\]

appearing in the exponent. This minimum is located at \( z'_0 = z - \rho/\sqrt{8} \) which falls inside the integration range if \( 0 < z'_0 < L \) and outside it otherwise. Accordingly, the main contribution to \( I_1(\rho, z) \) comes from the point \( z'_0 \) in the former case, while \( I_1(\rho, z) \) is negligible in the latter case. Expanding \( w(z') \) around its minimum,

\[
w(z') = z + \sqrt{8}\rho + \frac{8\sqrt{2}}{9\rho} (z' - z + \rho/\sqrt{8})^2 + ...
\]

(A.1)

and extending, without much error, the limits of integration of the above integral to infinity, one finds as a rough approximation:

\[
I_1(\rho, z) \approx \frac{\pi}{\sqrt{2} k\rho} e^{i2z+i2\pi k\rho+i\pi/4} \Theta(z - \rho/\sqrt{8}),
\]

(A.2)

where \( \Theta \) is the step-function (\( \Theta(x) = 1 \) if \( x > 0 \) and \( 0 \) is \( x < 0 \)).

An entirely similar argument leads to

\[
I_1(\rho, z) \approx \frac{\pi}{\sqrt{2} k\rho} e^{-i2z+i2\pi k\rho+i\pi/4} \Theta(L - \rho/\sqrt{8} - z).
\]

(A.3)

Now, as for the terms \( \exp\{i(\omega t \pm 3kz)\} \) and \( \exp\{i(-\omega t \pm kz)\} \) that appear in (3.8) and (3.9), they can be neglected in the integral (3.11) since they would contribute with terms such as \( \exp\{iw_1\} \) and \( \exp\{iw_2\} \), where

\[
w_1(z') = 3z' + [\rho^2 + (z-z')^2]^{1/2},
\]

\[
w_2(z') = z' + [\rho^2 + (z-z')^2]^{1/2}.
\]

However, these two functions, unlike \( w(z') \) above, have no minima, and therefore they only contribute with oscillating integrals that are negligible in comparison with the first integral with the exponent \( w(z') \).

**ORCID iDs**

S Hacyan @ https://orcid.org/0000-0002-4258-3860

**References**

[1] Heisenberg W and Euler H 1936 *Zeit. Physik* **98** 714
[2] Klein J and Nigam B P 1964 *Phys. Rev.* **135** B1279
[3] Klein J and Nigam B P 1964 *Phys. Rev.* **136** B1540
[4] Heyl J S and Hernquist L 1997 *J. Phys. A: Math. Gen.* **30** 6485
[5] Zavattini G, Gastaldi U, Pengo R, Ruoso G, Della Valle F and Milotti E 2012 *Int. J. Mod. Phys. A* **27** 1260017
[6] Cantatore G, Della Valle F, Milotti E, Dabrowski L and Rizzo C 1991 *Phys. Lett. B* **265** 418–24
[7] Aneck I and Kruglyak L 1987 *Phys. Rev. Lett.* **59** 1065
[8] Adler S L and Schubert C 1996 *Phys. Rev. Lett.* **77** 1695
[9] Bialynicka-Birula Z and Bialynicki-Birula J 1970 *Phys. Rev. D* **2** 2341
[10] Di Piazza A, Milstein A I and Keitel C H 2007 *Phys. Rev. A* **76** 032103
[11] Denisov V I and Sverdlov S I 2003 *Astron. Astrophys.* **399** L39
[12] Kim J Y and Lee T 2011 *J. Cosmology Astropart. Phys.* **11017
[13] Denisov V I and Krivchenkov I V and Kravtsov N V 2004 *Phys. Rev. D* **69** 066008
[14] Zavattini G and Calloni E 2009 *Eur. Phys. J. C* **62** 459–66
[15] Schelistedt G O, Perlick V and Lämmerzahl C 2015 *Phys. Rev. D* **92** 025039
[16] Narozhny N B and Fedotov A M 2007 *Laser Phys. 17* 350–7
[17] McKenna J and Platzman P M 1963 *Phys. Rev.* **129** 2354
[18] Gies H and Dittrich W 1998 Phys. Lett. B 431 420–9
[19] Dittrich W and Gies H 1998 Phys. Rev. 58 025004
[20] Obukhov Y N and Rubilar G F 2002 Phys. Rev. D 66 024042
[21] Mendonca J T, Marklund M, Shukla P K and Brodin G 2006 Phys. Lett. A 359 700
[22] Fedotov A M and Narozhny N B 2007 Phys. Lett. A 362 1
[23] King B and Keitelli C H 2012 New J. Phys. 14 103002
[24] Avetissian H K 2016 Relativistic nonlinear electrodynamics Springer Series on Atomic, Optical, and Plasma Physics vol 88 (Berlin: Springer) p 506
[25] Bohl P, King B and Ruhl H 2016 J. Plasma Phys. 82 655820202
[26] Danilishin S L and Khalili F Ya 2012 Living Rev. Rel. 15 5
[27] Maggiore M 2007 Gravitational Waves: Theory and Experiments vol 1 1st ed. (Oxford: Oxford University Press)
[28] Dergachev V and Papa M A 2019 Phys. Rev. Lett. 123 101101