The Heintze-Karcher inequality for sets of finite perimeter and bounded mean curvature

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Abstract

We establish the Heintze-Karcher inequality for sets of finite perimeter and bounded generalized mean curvature (in the sense of varifold's theory) and we prove that the equality case is uniquely characterized by finite unions of disjoint open balls.

1 Introduction

A beautiful integral inequality and a rigidity result for compact and embedded submanifolds of the Euclidean space is proved by Montiel and Ros in [MR91] following the ideas of [HK78]. It asserts that if Ω is a compact and connected smooth open subset of $\mathbb{R}^{n+1}$ whose mean curvature $h$ with respect to the exterior normal is everywhere positive then

$$(n+1)\mathcal{L}^{n+1}(Ω) \leq \int_{\partial Ω} \frac{n}{h} d\mathcal{H}^n,$$

and the equality holds if and only if Ω is a round sphere. The inequality is commonly known as **Heintze-Karcher inequality.** This result is also contained in [Ros87] with a different proof based on Reilly’s methods in [Rei77] and it contains as a special case the celebrated Alexandrov’s rigidity theorem on the smooth critical points of the isoperimetric problem. An explicit representation of the Heintze Karcher deficit $\int_{\partial Ω} \frac{n}{h} d\mathcal{H}^n - (n+1)\mathcal{L}^{n+1}(Ω)$ in terms of a volume integral and a defect measure is given in [GP13]. Recently Brendle established the Heintze-Karcher inequality for a large class of warped product spaces in [Bre13]; see also [QX15] for further results in Riemannian manifolds.

All results mentioned so far hold for smooth varieties. It is our aim in this paper to investigate this topic for singular varieties and, in this regard, we work with sets of finite perimeter. They appear to be the most general class that allows to study such a problem. Adopting the notion of **generalized mean curvature** developed in the theory of varifolds (see 2.4) the main result of this paper reads as follows (see section 2 for notation).
1.1 Theorem. Suppose \( E \) is a finite perimeter in \( \mathbb{R}^{n+1} \) with \( \mathcal{L}^{n+1}(E) < \infty \) and \( h \) is the generalized mean curvature of \( E \). If there exists \( 0 < c < \infty \) such that

\[
0 < h(z) \leq c \quad \text{for } \mathcal{H}^n \text{ a.e. } z \in \partial M E,
\]

then

\[
\mathcal{L}^{n+1}(E) \leq \frac{n}{n + 1} \int_{\partial M E} \frac{1}{h} d\mathcal{H}^n.
\]

The equality holds if and only if there exist finitely many disjoint open balls with radii not smaller than \( n/c \) whose union \( F \) satisfies

\[
\mathcal{L}^{n+1}((E \sim F) \cup (F \sim E)) = 0.
\]

Since a set of finite perimeter which is a critical point of the isoperimetric problem corresponds to an equality case in 1.1 (see \[DM19, Lemma 5\]), our result contains the rigidity theorem of \[DM19\]. Furthermore, we remark that an Heintze-Karcher type inequality for an open set of finite perimeter \( \Omega \) is given in \[DM19, Theorem 8\] in terms of the pointwise principal curvatures of the level sets of the distance function from \( \partial \Omega \). However in \[DM19, Theorem 8\] the authors do not investigate the connection between the aforementioned principal curvatures and the more natural notion of generalized mean curvature given in \[2.4\]. Our result (and its proof) clarifies this issue under the hypothesis of bounded generalized mean curvature. It is an open problem to establish the validity of the Heintze-Karcher inequality (1) assuming weaker assumptions on \( h \) (e.g., \( h \in L^p(\mathcal{H}^n \cup \partial M E) \) for \( 1 \leq p < \infty \)).

We now briefly describe the ideas of our proof. To obtain the inequality (1) we carefully adapt to sets of finite perimeter the integral-geometric argument employed for smooth varieties in \[MR91, Theorem 3\]. Our adaptation uses in a crucial way tools from the theory of curvature for arbitrary closed sets, see \[San17\], in combination with a key property of the generalized normal bundle of \( E \), called Lusin (N) condition, which holds for all varifolds of bounded mean curvature and arbitrary codimension, \[San19, 3.7(1)\]. To treat the equality case in (1) we cannot generalize the argument of \[DM19\], since in our case Allard’s regularity theory (see \[All72, 8.1\]) only ensures that the regular part of \( \partial M E \) is a \( C^{1,\alpha} \) hypersurface (for every \( \alpha < 1 \)) and not an analytic hypersurface, as in the case of constant mean curvature. Therefore we cannot easily deduce local rigidity of the regular part of \( \partial M E \) using classical theorems for umbilical surfaces (as in \[DM19, (3-55)\]). Here we adopt a different method, which is completely independent of Allard’s regularity theory: firstly we prove that the complementary of \( E \) (or, better said, a closed set \( C \) which is \( \mathcal{L}^n \) almost equal to the complementary of \( E \)) is a set of positive reach using \[HHL04\], then we notice that for all sufficiently small \( r > 0 \) the \( r \)-level sets of the distance function from \( C \) are \( C^{1,1} \) closed and umbilical hypersurfaces, which means that they are union of finitely many spheres by \[2.5\]. Letting \( r \to 0 \) we obtain the conclusion in the second part of Theorem 1.1.
We conclude the paper analyzing the stability of the Heintze-Karcher deficit $\int_{\partial M} \frac{H^n}{n+1} dH^n - (n+1) \mathcal{L}^{n+1}(E)$ for sequences of sets of finite perimeter; see 3.2.

Our method uses less varifold’s theory than [DM19] (in particular, our proof is independent of Schätzle’s maximum principle [Sch04]) and, instead, relies on somewhat more general argument originating from convex geometry. In a forthcoming paper [RKS] the methods and the results of this work will be extended to cover the anisotropic (non-crystalline) isoperimetric problem.

2 Preliminaries

Basic notation

Let $m$ be a non-negative integer. The symbol $U(a, r)$ denotes the open ball with centre $a$ and radius $r$; $S^m$ is the $m$ dimensional unit sphere in $\mathbb{R}^{m+1}$; $\mathcal{L}^m$ and $\mathcal{H}^m$ are the $m$ dimensional Lebesgue and Hausdorff measure ([Fed69 2.10.2]); given a measure $\mu$, we denote by $\Theta^m(\mu, \cdot)$, $\Theta^m(\mu, \cdot)$ and $\Theta^m(\mu, \cdot)$ the $m$ dimensional densities of $\mu$ ([Fed69 2.10.19]). Moreover, given a function $f$, we denote by $d_{\text{mn}} f$ and $i_{\text{m}} f$ the domain and the image of $f$. The symbol $\cdot$ denotes the standard inner product of $\mathbb{R}^m$. If $\nu \in \mathbb{R}^m \sim \{0\}$, then $\nu^\perp$ is the hyperplane orthogonal to $\nu$. If $X$ and $Y$ are sets, $Z \subseteq X \times Y$ and $S \subseteq X$, then $Z|S = Z \cap \{(x, y) : x \in S\}$.

To treat rectifiable sets we adopt the terminology introduced in [Fed69 3.2.14]. We refer to [Fed69 3.1.21] for the notions of tangent and normal cone of a set; moreover, given a measure $\mu$ and a positive integer $m$, the approximate tangent cone $\text{Tan}^m(\mu, \cdot)$ is defined as in [Fed69 3.2.16].

Sets of finite perimeter and generalized mean curvature

Here we recall few basic definitions and facts on sets of finite perimeter. Let $E \subseteq \mathbb{R}^{n+1}$ be $\mathcal{L}^{n+1}$ measurable.

2.1 Definition. If $\Omega \subseteq \mathbb{R}^{n+1}$ is open the perimeter of $E$ in $\Omega$ equals

$$\sup \left\{ \int_E \text{div} \phi d\mathcal{L}^{n+1} : \phi \in C_c(\Omega, \mathbb{R}^{n+1}), \|\phi\|_\infty = 1 \right\}.$$  

We say that $E$ has finite perimeter in $\Omega$ if the perimeter of $E$ in $\Omega$ is finite.

We define the measure theoretic boundary $\partial_M E$ of $E$ (see [Zie89 5.8.4]) as the set of $x \in \mathbb{R}^{n+1}$ such that

$$\Theta^{n+1}(\mathcal{L}^{n+1} \downarrow E, x) > 0 \quad \text{and} \quad \Theta^{n+1}(\mathcal{L}^{n+1} \downarrow \mathbb{R}^{n+1} \sim E, x) > 0.$$  

Let $b \in \mathbb{R}^{n+1}$, a vector $u \in S^n$ is the exterior measure theoretic normal of $E$ at $b$ (see [Fed69 4.5.5] or [Zie89 5.6.4]) if and only if

$$\Theta^{n+1}(\mathcal{L}^{n+1} \downarrow \{x : (x - b) \cdot u > 0\} \cap E, b) = 0 \quad \text{and} \quad \Theta^{n+1}(\mathcal{L}^{n+1} \downarrow \{x : (x - b) \cdot u < 0\} \sim E, b) = 0.$$
For each \( b \in \mathbb{R}^{n+1} \) there exists at most one exterior normal \( u \) of \( E \) at \( b \), see [Fed69 4.5.5], and we denote it by

\[
\mathbf{n}(E, b)
\]

whenever it exists. We define \( \partial^* E \) to be the domain of \( \mathbf{n}(E, \cdot) \). Evidently, \( \partial^* E \subseteq \partial_M E \). If \( b \in \partial^* E \) then \( \mathbf{n}(E, b) = -\mathbf{n}(\mathbb{R}^n \sim E, b) \) and

\[
\text{Tan}^{n+1}(\mathcal{L}^{n+1} \setminus E, b) = \{ v : v \cdot \mathbf{n}(E, b) \leq 0 \},
\]

as one may verifies from the definition of approximate tangent cone in [Fed69 3.2.16]. If \( E \) has finite perimeter in \( \mathbb{R}^{n+1} \) then it follows from [Fed69 4.5.6] (or [Zie89 5.7.3, 5.6.8, 5.9.5]) that \( \partial^* E \) is countably \((\mathcal{H}^n, n)\) rectifiable and

\[
\mathcal{H}^n(\partial_M E \sim \partial^* E) = 0;
\]

moreover it follows from [Fed69 4.5.3] that \( \text{spt}\mathcal{H}^n \setminus \partial_M E = \partial_M \mathcal{E} \).

2.2 Lemma. Let \( E \) be a set of finite perimeter in \( \mathbb{R}^{n+1} \) such that

\[
\mathcal{H}^n(\partial_M E \sim \partial_M E) = 0.
\]

Then there exists an open set \( P \subseteq \mathbb{R}^{n+1} \) such that

\[
\mathcal{L}^{n+1}((P \sim E) \cup (E \sim P)) = 0 \quad \text{and} \quad \mathcal{H}^n(\partial P \sim \partial_M P) = 0.
\]

Proof. We define

\[
P = \mathbb{R}^{n+1} \cap \{ x : \mathcal{L}^{n+1}(U(x, \rho) \sim E) = 0 \text{ for some } \rho > 0 \}
\]

\[
Q = \mathbb{R}^{n+1} \cap \{ x : \mathcal{L}^{n+1}(U(x, \rho) \cap E) = 0 \text{ for some } \rho > 0 \}
\]

and we notice that they are open subsets of \( \mathbb{R}^{n+1} \). It follows from [Fed69 4.5.3] that

\[
\text{spt}\mathcal{H}^n \setminus \partial_M E = \mathbb{R}^{n+1} \sim (P \cup Q).
\]

We apply [Fed69 2.9.11] to infer

\[
\mathcal{L}^{n+1}(P \sim E) = 0, \quad \mathcal{L}^{n+1}(E \cap Q) = 0,
\]

\[
\mathcal{L}^{n+1}(E \sim P) = \mathcal{L}^{n+1}(E \cap Q) + \mathcal{L}^{n+1}(\text{spt}\mathcal{H}^n \setminus \partial_M E) = 0.
\]

We deduce that \( \partial_M P = \partial_M E \) and, since \( \partial P \subseteq \text{spt}\mathcal{H}^n \setminus \partial_M E \) by \((2)\), we conclude

\[
\mathcal{H}^n(\partial P \sim \partial_M P) = 0.
\]

\( \square \)

2.3 Remark. If \( E \) is a set of finite perimeter such that \( \Theta^n(\mathcal{H}^n \setminus \partial_M E, x) > 0 \) for every \( x \in \partial_M E \) then \( \mathcal{H}^n(\partial_M E \sim \partial_M E) = 0 \) by [Fed69 2.10.19(4)].
2.4 Definition. Let $E$ be a set of finite perimeter in $\mathbb{R}^{n+1}$. A function $h \in L^1(\mathcal{H}^n \cup \partial M E)$ is the generalized mean curvature of $E$ if and only if

$$
\int_{\partial M E} Dg(x) \cdot n(E, x) \frac{1}{2} d\mathcal{H}^n x = -\int_{\partial M E} h(x) n(E, x) \cdot g(x) d\mathcal{H}^n
$$

for every $g \in C^1_0(\mathbb{R}^{n+1})$.

Using the terminology from the theory of varifolds [All72], we say that a function $h$ is the generalized mean curvature of $E$ if and only if the unit-density varifold $V = v(\partial_M E, 1)$ associated with $\partial_M E$ has locally bounded first variation absolutely continuous with respect to $\mathcal{H}^n \cup \partial M E$; in this case the function $h$ equals $h(V, \cdot) \cdot n(E, \cdot)$, where $h(V, \cdot) \in L^1(\mathcal{H}^n \cup \partial M E, \mathbb{R}^{n+1})$ is the mean curvature vector of $V$ defined in [All72, 4.3].

**Totally umbilical $C^{1,1}$ hypersurfaces**

A closed and connected hypersurface of class $C^2$ which is umbilical at every point must be a plane or a sphere. This result was proved by Hartman in [Har47]. A simplified proof of this result appears in [Pau08]. The same techniques can be easily adapted to cover the case of hypersurfaces of class $C^{1,1}$, which is the relevant case for the purpose of the present paper. For completeness, we provide the details here.

2.5 Theorem. Suppose $M \subset \mathbb{R}^{n+1}$ is a closed and connected $C^1$ hypersurface, suppose $\eta : M \to S^n$ is a Lipschitzian map with $\eta(x) \in \text{Nor}(M, x)$ for every $x \in M$ and suppose that for $\mathcal{H}^n$ a.e. $x \in M$ there exists $\kappa(x) \in \mathbb{R}$ such that

$$
D \eta(x)(u) = \kappa(x) u \quad \text{for every } u \in \text{Tan}(M, x).
$$

Then $M$ is an $n$ dimensional plane or an $n$ dimensional round sphere.

Proof. Claim 1: $\kappa$ is $(\mathcal{H}^n$ almost equal to) a constant function on $M$.

Since $M$ is connected, this is equivalent to prove that $\kappa$ is locally constant around each point of $M$. Since $M$ locally corresponds at each point $a \in M$ to a graph of a $C^{1,1}$ function, we exploit (3) to see that it is enough to prove the following claim: if $U_1, \ldots, U_n$ are bounded open intervals of $\mathbb{R}$, $U = U_1 \times \ldots \times U_n$ and $f : U \to \mathbb{R}$ is a $C^{1,1}$-function such that the conditions

$$
\partial_i ((1 + |\nabla f|^2)^{-1/2} \partial_j f) = 0 \quad \text{if } i \neq j
$$

$$
\partial_i ((1 + |\nabla f|^2)^{-1/2} \partial_i f) = \partial_{i+1} ((1 + |\nabla f|^2)^{-1/2} \partial_{i+1} f) \quad \text{for } i = 1, \ldots, n - 1
$$

hold on $\mathcal{L}^n$ almost all of $U$, then $\partial_i ((1 + |\nabla f|^2)^{-1/2} \partial_i f)$ is constant on $U$. It follows from (4) that for every $i = 1, \ldots, n$ there exists a Lipschitzian function $a_i : U_i \to \mathbb{R}$ such that

$$
((1 + |\nabla f|^2)^{-1/2} \partial_i f)(x) = a_i(x) \quad \text{for } x \in U;
$$

5
then we use $\xi'$ to conclude that

$$a_i'(x_i) = a_{i+1}'(x_{i+1}) \text{ for } \mathcal{L}^n \text{ a.e. } x \in U,$$

whence we deduce that for each $i$ the function $a_i'$ is constant and the conclusion follows.

It follows from $[3]$ and Claim 1 that there exists $\lambda \in \mathbb{R}$ such that

$$D\eta(x)(u) = \lambda u$$

for every $u \in \text{Tan}(M, x)$ and for $\mathcal{H}^n$ a.e. $x \in M$. If $\lambda = 0$ then $\eta$ is constant on $M$ and $M$ is a plane. If $\lambda \neq 0$ then $\eta - \lambda1_M$ is constant on $M$ and $M$ is a sphere of radius $1/|\lambda|$.

### Normal bundle and curvatures of arbitrary closed sets

Here we recall few basic facts on the notion of curvature for arbitrary closed sets.

Suppose $A \subseteq \mathbb{R}^{n+1}$ is closed. The distance function to $A$ is denoted by $\delta_A$ and $S(A, r) = \{x : \delta_A(x) = r\}$. If $U$ is the set of all $x \in \mathbb{R}^{n+1}$ such that there exists a unique $a \in A$ with $|x - a| = \delta_A(x)$, we define the nearest point projection onto $A$ as the map $\xi_A$ characterised by the requirement

$$|x - \xi_A(x)| = \delta_A(x) \text{ for } x \in U.$$ 

Let $U(A) = \text{dmm } \xi_A \sim A$. The functions $\nu_A$ and $\psi_A$ are defined by

$$\nu_A(z) = \delta_A(z)^{-1}(z - \xi_A(z)) \quad \text{and} \quad \psi_A(z) = (\xi_A(z), \nu_A(z)),$$

whenever $z \in U(A)$. We define (see $[San17]$ 3.6, 3.7) the upper semicontinuous function $\rho(A, \cdot)$ setting

$$\rho(A, x) = \sup\{t : \delta_A(\xi_A(x) + t(x - \xi_A(x))) = t\delta_A(x)\} \text{ for } x \in U(A),$$

and we say that $x \in U(A)$ is a regular point of $\xi_A$ if and only if $\xi_A$ is approximately differentiable at $x$ with symmetric approximate differential and $\text{aplim}_{y \to x} \rho(A, y) \geq \rho(A, x) > 1$. The set of regular points of $\xi_A$ is denoted by $R(A)$. It is proved in $[San17]$ 3.14 that $\mathcal{L}^{n+1}(\mathbb{R}^{n+1} \sim (A \cup R(A))) = 0$ and $\xi_A(x) + t(x - \xi_A(x)) \in R(A)$ for every $x \in R(A)$ and for every $0 < t < \rho(A, x)$.

Next we define the generalized unit normal bundle of $A$ as

$$N(A) = (A \times \mathbb{S}^n) \cap \{(a, u) : \delta_A(a + su) = s \text{ for some } s > 0\},$$

with $N(A, a) = \{v : (a, v) \in N(A)\}$ for $a \in A$. The positive boundary of $A$ is defined by

$$\partial^+ A = A \cap \{a : N(A, a) \neq \emptyset\}.$$ 

The set $N(A)$ is a countably $n$ rectifiable subset of $\mathbb{R}^{n+1} \times \mathbb{S}^n$ in the sense of $[Fed69]$ 3.2.14, see $[San17]$ 4.3; however it may not have locally finite $\mathcal{H}^n$. 

measure. If $x \in R(A)$ we call $\psi_A(x)$ regular point of $N(A)$. One may check (see [San17, 4.5]) that $\mathcal{H}^n(N(A) \sim R(N(A)) = 0$. For every $(a, u) \in R(N(A))$, if $x \in R(A)$ and $\psi_A(x) = (a, u)$, we define

$$T_A(a, u) = \text{im} \text{D} \xi_A(x).$$

and we define a symmetric bilinear form $Q(a, u) : T_A(a, u) \times T_A(a, u) \to R$ which maps $(\tau, \tau_1) \in T_A(a, u) \times T_A(a, u)$ into

$$Q_A(a, u)(\tau, \tau_1) = \tau \cdot \text{ap D} \nu_A(x)(\sigma_1),$$

where $\sigma_1 \in R^{n+1}$ is any vector such that $\text{ap D} \xi_A(x)(\sigma_1) = \tau_1$. This is a well-posed definition, see [San17, 4.6, 4.8]. We call $Q_A(a, u)$ second fundamental form of $A$ at $a$ in the direction $u$. It is not difficult to check that if $A$ is smooth submanifold, then $Q_A$ agrees with the classical notion of differential geometry. Moreover, if $(a, u) \in R(N(A))$ we define the principal curvatures of $A$ at $(a, u)$ to be the numbers

$$\kappa_{A,1}(a, u) \leq \ldots \leq \kappa_{A,m}(a, u),$$

such that $\kappa_{A,m+1}(a, u) = \infty$, $\kappa_{A,1}(a, u), \ldots, \kappa_{A,m}(a, u)$ are the eigenvalues of $Q_A(a, u)$ and $m = \dim T_A(a, u)$.

Now we study this abstract theory in the special case of sets of finite perimeter with bounded generalized mean curvature.

**2.6 Theorem.** Suppose $E$ is a set of finite perimeter in $R^{n+1}$ with generalized mean curvature $h \in L^\infty(\mathcal{H}^n \setminus \partial M E)$. Then there exists a closed set $C \subseteq R^{n+1}$ of finite perimeter in $R^{n+1}$ such that

(a) $\mathcal{L}^{n+1}(E \cap C) = 0$ and $\mathcal{L}^{n+1}(R^{n+1} \sim (E \cup C)) = 0$;

(b) $\partial M C = \partial M E$ and $n(E, \cdot) = -n(C, \cdot)$;

(c) $\mathcal{H}^n(\partial C \sim \partial M C) = 0$;

(d) $N(C, a) = \{n(C, a)\}$ for every $a \in \partial^+ C \cap \partial^* C$;

(e) $\mathcal{H}^n(N(C)|S) = 0$ whenever $S \subseteq R^{n+1}$ with $\mathcal{H}^n(S) = 0$;

(f) $T_C(a, n(C, a))$ is an $n$ dimensional plane perpendicular to $n(C, a)$ and

$$\text{trace } Q_C(a, n(C, a)) = -h(a)$$

for $\mathcal{H}^n$ a.e. $a \in \partial^* C \cap \partial^+ C$.

**Proof.** Since $\Theta^n(\mathcal{H}^n \setminus \partial M E, \cdot)$ is an upper semicontinuous function on $R^{n+1}$ by [Fed69, 8.6], $\Theta^n(\mathcal{H}^n \setminus \partial M E, x) = 1$ for $\mathcal{H}^n$ a.e. $x \in \partial M E$ by [Fed69, 3.2.19] and $\partial M E = \text{spt } \mathcal{H}^n \setminus \partial M E$, we infer that

$$\Theta^n(\mathcal{H}^n \setminus \partial M E, x) \geq 1 \quad \text{for every } x \in \partial M E.$$
Then we apply 2.3 and 2.2 to get an open subset $\Omega$ of $\mathbb{R}^{n+1}$ such that
\[ H^n(\partial \Omega \sim \partial_M \Omega) = 0 \quad \text{and} \quad \mathcal{L}^{n+1}(E \sim \Omega) \cup (\Omega \sim E) = 0. \]
Let $C = \mathbb{R}^{n+1} \sim \Omega$ and we notice that (a), (b) and (c) follow.

Since $\partial C$ is an $(n, \|h\|_\infty)$ subset of $\mathbb{R}^{n+1}$ by \cite{Whi16, 2.8}, it follows that
\[ H^n(N(\partial C)|S) = 0 \]
whenever $S \subseteq \mathbb{R}^{n+1}$ with $H^n(S) = 0$ by \cite{San19, 3.1, 3.7(1)]. Then (e) holds because $N(C) \subseteq N(\partial C)$. Additionally \cite{San17, 4.14] implies that
\[ T_C(z, \eta) = T_{\partial C}(z, \eta) \quad \text{and} \quad Q_C(z, \eta) = Q_{\partial C}(z, \eta) \]
for $H^n$ a.e. $(z, \eta) \in N(C)$. If $a \in \partial^+ C \cap \partial^* C$ then
\[ \text{Nor}^{n+1}(\mathcal{L}^{n+1} \upharpoonright C, a) = \{ m(C,a) : t \geq 0 \} \]
and (d) holds because $N(C,a) \subseteq \text{Nor}(C,a) \subseteq \text{Nor}^{n+1}(\mathcal{L}^{n+1} \upharpoonright C, a)$. It follows from \cite{San17, 4.8] and \cite{San19, 3.7(2)] that
\[ \dim T_C(z, n(C,z)) = n \quad \text{and} \quad T_C(z, n(C,z)) \subseteq \{ v : v \cdot n(C,z) = 0 \} \]
for $H^n$ a.e. $z \in \partial^+ C \cap \partial^* C$, whence we obtain the first part of (f). The second part of (f) follows from (b) and \cite{San19, 3.9].

3 Heintze Karcher inequality

3.1 Theorem. Suppose $E$ is a finite perimeter in $\mathbb{R}^{n+1}$ with $\mathcal{L}^{n+1}(E) < \infty$ and $h$ is the generalized mean curvature of $E$. If there exists $0 < c < \infty$ such that
\[ 0 < h(z) \leq c \quad \text{for } H^n \text{ a.e. } z \in \partial_M E, \]
then
\[ \mathcal{L}^{n+1}(E) \leq \frac{n}{n+1} \int_{\partial_M E} \frac{1}{h} dH^n. \]

The equality holds if and only if there exist finitely many disjoint open balls with radii not smaller than $n/c$ whose union $F$ satisfies
\[ \mathcal{L}^{n+1}(E \sim F) \cup (F \sim E) = 0. \]

Proof. Let $C \subseteq \mathbb{R}^{n+1}$ be a closed set satisfying 2.1(a) (f) let $\Omega = \mathbb{R}^{n+1} \sim C$ and define $Q$ as the set of $z \in \partial^* C \cap \partial^* C$ such that
\[ h(z) > 0, \quad \dim T_C(z, n(C,z)) = n, \]
\[ \text{trace} Q_C(z, n(C,z)) = -h(z). \]

Claim 1: if $y \in \xi_{C^{-1}}(Q) \cap \Omega$ then
\[ \nu_C(y) = n(C, \xi_C(y)) \quad \text{and} \quad -\delta_C(y)^{-1} \leq \kappa_{C,1}(\psi_C(y)) < 0. \]
In fact, \( \nu_C(y) = n(C, \xi_C(y)) \) follows from [San17, 4.8] and \(-\delta_C(y)^{-1} \leq \kappa_C,1(\psi_C(y)) \) follows from [San17, 4.8]. Moreover, 
\[
n\kappa_C,1(\psi_C(y)) \leq \text{trace} Q_C(\psi_C(y)) = -h(\xi_C(y)) < 0.
\]

Claim 2: \( L^{n+1}(\Omega \sim \xi_C^{-1}(Q)) = 0 \).

We use 2.6 to infer that \( H^n(S(C, r) \cap U(C) \sim \xi_C^{-1}(Q)) = 0 \), and \( H^n(N(C) | (\partial^+ C \sim Q)) = 0 \). Since \( \psi_C(S(C, r) \cap U(C) \sim \xi_C^{-1}(Q)) \subseteq N(C) | (\partial^+ C \sim Q) \) for every \( r > 0 \), it follows from [San17, 3.3] that 
\[
H^n(S(C, r) \cap U(C) \sim \xi_C^{-1}(Q)) = 0 \quad \text{for every } r > 0.
\]

Since \( L^{n+1}(R^{n+1}) = 0 \) (see [San17, 3.2]), it follows from Coarea formula that 
\[
L^{n+1}(\Omega \sim \phi(Z)) = 0
\]
by Claim 3. Noting that 
\[
\tan^{n+1}(N(C) \times R, (z, \eta, t)) = \tan^n(N(C), (z, \eta)) \times R
\]
for \( \mathcal{H}^n \) a.e. \( (z, \eta, t) \in N(C) \times R \) (here the approximate tangent spaces are defined as in [AFP00, 2.86]), we infer from [San17, 4.11(1)] that the \( n+1 \) dimensional approximate jacobian of \( \phi \) is given by 
\[
J_{n+1}(z, \eta, t) = J(z, \eta) \cdot \prod_{j=1}^{n} \left| 1 + t\kappa_C,j(z, \eta) \right|
\]
for \( \mathcal{H}^n \) a.e. \( (z, \eta, t) \in Z \), where 
\[
J(z, \eta) = \prod_{j=1}^{n} \frac{1}{(1 + \kappa_C,j(z, \eta)^2)^{1/2}}
\]
Then we apply [AFP00, 2.91], the classical inequality relating the arithmetic and geometric means of positive numbers and [San17, 5.4] in combination with 2.6(d) to estimate

\begin{equation}
\mathcal{L}^{n+1}(\phi(Z)) \\
\leq \int_{\phi(Z)} \mathcal{H}^0(\phi^{-1}(y)) \, d\mathcal{L}^{n+1}y \\
= \int_{\mathcal{Z}} \text{ap} \, J_{n+1}(z, \eta, t) \, d\mathcal{H}^{n+1}(z, \eta, t) \\
\leq \int_{N(C)\setminus Q} J(z, \eta) \int_0^{\kappa_{C,1}(z,\eta)^{-1}} \left(1 + \frac{t}{n} \text{trace} \, Q_C(z, \eta)\right) \, dt \, d\mathcal{H}^n(z, \eta) \\
= \int_{Q} \int_0^{\kappa_{C,1}(z,\eta)^{-1}} \left(1 - \frac{t}{n} h(z)\right)^n \, dt \, d\mathcal{H}^n z \\
\leq \int_{Q} \int_0^{n/h(z)} \left(1 - \frac{t}{n} h(z)\right)^n \, dt \, d\mathcal{H}^n z \\
= \int_{\partial^+C} \int_0^{n/h(z)} \left(1 - \frac{t}{n} h(z)\right)^n \, dt \, d\mathcal{H}^n z \\
= \frac{n}{n+1} \int_{\partial^+C} \frac{1}{h} \, d\mathcal{H}^n.
\end{equation}

Then using (8) we obtain (6).

From now on we assume now that equality holds in (6) and we observe from the previous estimate that

\begin{equation}
\mathcal{L}^{n+1}(\phi(Z) \sim \Omega) = 0,
\end{equation}

\begin{equation}
\mathcal{H}^0(\phi^{-1}(y)) = 1 \quad \text{for } \mathcal{L}^{n+1} \text{ a.e. } y \in \phi(Z),
\end{equation}

\begin{equation}
-\kappa_{C,j}(z, n(C, z))^{-1} = \frac{n}{h(z)} \quad \text{for } \mathcal{H}^n \text{ a.e. } z \in \partial^+C \text{ and } j = 1, \ldots, n.
\end{equation}

Our goal is to prove that \( \Omega \) is a finite union of disjoint open balls. This conclusion will be deduced from the following two claims.

**Claim 3:** reach \( C \geq n/c \).

Note \( h(z) \leq c \) for \( \mathcal{H}^n \) a.e. \( z \in \partial C \). Let \( 0 < \rho < n/c \) and

\[
Q_\rho = Q \cap \{ z : \rho < -\kappa_{C,1}(z, n(C, z))^{-1} \}.
\]

Since it follows from (7) and (11) that

\[
\mathcal{H}^n(\partial^+C \sim Q_\rho) = 0 \quad \text{and} \quad \mathcal{H}^n(N(C)|\partial^+C \sim Q_\rho) = 0,
\]

we argue as in Claim 2 to conclude that

\begin{equation}
\mathcal{L}^{n+1}(\Omega \sim \xi_c^{-1}(Q_\rho)) = 0.
\end{equation}
We define
\[ C_\rho = \{ z : \delta_C(z) \leq \rho \} \] and
\[ Z_\rho = (N(C)|Q_\rho) \times \{ t : 0 < t \leq \rho \} \]
and we notice that
\[ (13) \quad \xi_\rho^{-1}(Q_\rho) \cap \Omega \cap C_\rho \subseteq \phi(Z_\rho) \subseteq C_\rho. \]

Let \( f : \mathbb{R}^{n+1} \times \mathbb{S}^n \rightarrow \mathbb{R} \) be a Borel measurable function with compact support. Then we employ the generalized Area formula [AFP00, 2.91] and the Coarea formula [San17, 5.4] to compute
\[
\int_{\Omega \cap C_\rho} f(\psi_C(y))dL^{n+1}y
\]
\[
= \int_{\Omega \cap C_\rho \cap \xi_\rho^{-1}(Q_\rho)} f(\psi_C(y))dL^{n+1}y \quad \text{by (12)}
\]
\[
= \int_{\Omega \cap C_\rho \cap \xi_\rho^{-1}(Q_\rho)} \int_{\phi^{-1}(y)} f d\mathcal{H}^0 dL^{n+1}y \quad \text{by (10)}
\]
\[
= \int_{\phi(Z_\rho)} \int_{\phi^{-1}(y)} f d\mathcal{H}^0 dL^{n+1}y \quad \text{by (9), (12), (13)}
\]
\[
= \int_{Z_\rho} J_n\phi(z, \eta, t) f(z, \eta) d\mathcal{H}^{n+1}(z, \eta, t)
\]
\[
= \int_{Q_\rho} f(z, n(C, z)) \int_0^\rho \prod_{j=1}^n |1 + t\kappa_{C,j}(z, n(C, z))| dt d\mathcal{H}^n z
\]
\[
= \int_{Q_\rho} f(z, n(C, z)) \int_0^\rho \left(1 - \frac{t}{n} h(z)\right)^n dt d\mathcal{H}^n z \quad \text{by (11)}
\]
\[
= \int_{\partial^+ C} f(z, n(C, z)) \int_0^\rho \left(1 - \frac{t}{n} h(z)\right)^n dt d\mathcal{H}^n z
\]
\[
= \sum_{i=1}^{n+1} c_i(f) \rho^i,
\]
where, for \( i = 1, \ldots, n+1 \),
\[
c_i(f) = \left( -\frac{1}{n} \right) i^{-1} \frac{n!}{i!(n-i+1)!} \int_{\partial^+ C} f(z, n(C, z)) h(z)^{i-1} d\mathcal{H}^n z.
\]

Therefore reach \( C \geq n/c \) by [HHL14, Theorem 3].

Claim 4: if \( 0 < r < n/c \) then \( S(C, r) \) is a finite union of disjoint spheres.

Since reach \( C \geq n/c \) it follows from [Fed59, 4.8] that \( S(C, r) \) is a closed \( C^1 \) hypsersurface in \( \mathbb{R}^{n+1} \) and \( \nu_C|S(C, r) \) is a unit normal Lipschitzian vector field over \( S(C, r) \). We define
\[
T = \partial^+ C \cap \{ z : 0 < h(z) \leq c, \kappa_{C,j}(z, n(C, z)) = -h(z)/n \text{ for } j = 1, \ldots, n \},
\]
we notice that \( \mathcal{H}^n(\partial^+ C \sim T) = 0 \) by (11) and the Lusin (N) condition implies (arguing as in Claim 2)

\[
\mathcal{H}^n(S(C, r) \sim \xi_C^{-1}(T)) = 0.
\]

Moreover if \( x \in S(C, r) \cap \xi_C^{-1}(T) \) then we employ [San17, 4.10] to conclude

\[
\chi_{C,j}(x) = \frac{\kappa_{C,j}(\xi_C(x), \mathbf{n}(C, \xi_C(x)))}{1 + r\kappa_{C,j}(\xi_C(x), \mathbf{n}(C, \xi_C(x)))} = \frac{h(\xi_C(x))}{rh(\xi_C(x)) - n}
\]

for \( j = 1, \ldots, n \) and, noting that

\[
0 < \frac{h(\xi_C(x))}{n - rh(\xi_C(x))} \leq \frac{c}{n - rc} < \infty,
\]

we employ 2.5 to conclude that \( S(C, r) \) is a union of at most countably many spheres with radii not smaller than \( c^{-1}(n - rc) \). Since \( \mathcal{L}^{n+1}(\Omega) < \infty \), there are only finitely many spheres and Claim 4 is proved.

We are now ready to conclude the proof. We notice from [Fed59, 4.20] that

\[
\partial C = \partial^+ C = \{ x : \dim \text{Nor}(C, x) \geq 1 \}.
\]

Since \( \text{reach} \ C \geq n/c \) by Claim 3, we deduce that \( \xi_C(S(C, r)) = \partial C \) for \( 0 < r < n/c \). Since \( \nu_C|S(C, r) \) is a unit normal vector field over \( S(C, r) \) and

\[
\xi_C(x) = x - r\nu_C(x) \quad \text{for} \quad x \in S(C, r),
\]

the conclusion follows from Claim 4.

3.2 Theorem. Let \( E \) be a set of finite perimeter in \( \mathbb{R}^{n+1} \) with \( \mathcal{L}^{n+1}(E) < \infty \), let \( E_j \) be a sequence of sets of finite perimeter in \( \mathbb{R}^{n+1} \) such that \( E_j \to E \) in measure in \( \mathbb{R}^{n+1} \) (see [APP10, 3.37]) and \( \mathcal{H}^n(\partial_M E_j) \to \mathcal{H}^n(\partial_M E) \). Furthermore suppose that there exists a bounded upper-semicontinuous function \( h : \mathbb{R}^{n+1} \to [0, +\infty) \) and \( 0 < C < \infty \) such that

(a) \( h \) is continuous at \( x \) for \( \mathcal{H}^n \) a.e. \( x \in \partial_M E \);

(b) \( 0 < h(x) \leq C \) for \( \mathcal{H}^n \) a.e. \( x \in \partial_M E \),

(c) for every \( g \in C^1_c(\mathbb{R}^{n+1}, \mathbb{R}^{n+1}) \) the sequence of numbers

\[
\int_{\partial_M E_j} Dg(x) \cdot \mathbf{n}(E_j, x) \, d\mathcal{H}^n x - \int_{\partial_M E_j} h(x)\mathbf{n}(E_j, x) \cdot g(x) \, d\mathcal{H}^n x
\]

converges to 0 as \( j \to \infty \);

(d)

\[
\limsup_{j \to \infty} \left( \int_{\partial_M E_j} \frac{n}{r} \, d\mathcal{H}^n - (n + 1)\mathcal{L}^{n+1}(E_j) \right) \leq 0.
\]
Then there exist finitely many disjoint open balls with radii not smaller than \( n/C \) whose union \( F \) satisfies
\[
\mathcal{L}^{n+1}(E \sim F) = 0.
\]

**Proof.** It follows from [AFP00, 3.13-3.15] that
\[
\mathbf{n}(E_j, \cdot) \mathcal{H}^n \downarrow \partial M E_j \overset{*}{\rightharpoonup} \mathbf{n}(E, \cdot) \mathcal{H}^n \downarrow \partial M E, \quad \mathcal{H}^n \downarrow \partial M E_j \overset{*}{\rightharpoonup} \mathcal{H}^n \downarrow \partial M E.
\]
It follows from [AFP00, 1.62] that
\[
\lim_{j \to \infty} \int_{\partial M E_j} h(x) \mathbf{n}(E_j, x) \cdot g(x) \, d\mathcal{H}^n x = \int_{\partial M E} h(x) \mathbf{n}(E, x) \cdot g(x) \, d\mathcal{H}^n x
\]
for every \( g \in C_c(\mathbb{R}^{n+1}, \mathbb{R}^{n+1}) \),
\[
\lim \inf_{j \to \infty} \int_{\partial M E_j} \frac{n}{h} \, d\mathcal{H}^n \geq \int_{\partial M E} \frac{n}{h} \, d\mathcal{H}^n
\]
and we use (d) to infer
\[
(n + 1) \mathcal{L}^{n+1}(E) \geq \int_{\partial M E} \frac{n}{h} \, d\mathcal{H}^n.
\]
If \( g \in C^1_c(\mathbb{R}^{n+1}, \mathbb{R}^{n+1}) \) we define the function
\[
\Psi_g : \mathbb{R}^{n+1} \times S^n \to \mathbb{R}
\]
by \( \Psi_g(x, \nu) = D g(x) \cdot \nu^\perp \) and we apply Reshetnyak theorem [AFP00, 2.39] to conclude
\[
\lim_{j \to \infty} \int_{\partial M E_j} \Psi_g(x, \mathbf{n}(E_j, x)) \, d\mathcal{H}^n x = \int_{\partial M E} \Psi_g(x, \mathbf{n}(E, x)) \, d\mathcal{H}^n x.
\]
Then we use (14) and (c) to see that \( h \) is the generalized mean curvature of \( E \) and we use (d) to conclude
\[
(n + 1) \mathcal{L}^{n+1}(E) \leq \int_{\partial M E} \frac{n}{h} \, d\mathcal{H}^n.
\]
Therefore combining (15) and (16) we obtain the conclusion from the second part of 3.1. \( \square \)

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