On the reduction of the CSP dichotomy conjecture to digraphs

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Abstract. It is well known that the constraint satisfaction problem over general relational structures can be reduced in polynomial time to digraphs. We present a simple variant of such a reduction and use it to show that the algebraic dichotomy conjecture is equivalent to its restriction to digraphs and that the polynomial reduction can be made in logspace. We also show that our reduction preserves the bounded width property, i.e., solvability by local consistency methods. We discuss further algorithmic properties that are preserved and related open problems.

1 Introduction

A fundamental problem in constraint programming is to understand the computational complexity of constraint satisfaction problems (CSPs). While it is well known that the class of all constraint problems is NP-complete, there are many subclasses of problems for which there are efficient solving methods. One way to restrict the instances is to only allow a fixed set of constraint relations, often referred to as a constraint language [6] or fixed template. Classifying the computational complexity of fixed template CSPs has been a major focus in the theoretical study of constraint satisfaction. In particular it is of interest to know which templates produce polynomial time solvable problems to help provide more efficient solution techniques.

The study of fixed template CSPs dates back to the 1970’s with the work of Montanari [21] and Schaefer [24]. A standout result from this era is that of Schaefer who showed that the CSPs arising from constraint languages over 2-element domains satisfy a dichotomy. The decision problem for fixed template

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CSPs over finite domains belong to the class NP, and Schaefer showed that in the 2-element domain case, a constraint language is either solvable in polynomial time or NP-complete. Dichotomies cannot be expected for decision problems in general, since (under the assumption that $P \neq NP$) there are many problems in NP that are neither solvable in polynomial time, nor NP-complete [17]. Another important dichotomy was proved by Hell and Nešetřil [10]. They showed that if a fixed template is a finite simple graph (the vertices make up the domain and the edges make up the only allowable constraints), then the corresponding CSP is either polynomial time solvable or NP-complete. The decision problem for a graph constraint language can be rephrased as graph homomorphism problem (a graph homomorphism is a function from the vertices of one graph to another such that the edges are preserved). Specifically, given a fixed graph $H$ (the constraint language), an instance is a graph $G$ together with the question “Is there a graph homomorphism from $G$ to $H$?”. In this sense, 3-colorability corresponds to $H$ being the complete graph on 3 vertices. The notion of graph homomorphism problems naturally extends to directed graph (digraph) homomorphism problems and to relational structure homomorphism problems.

These early examples of dichotomies, by Schaefer, Hell and Nešetřil, form the basis of a larger project of classifying the complexity of fixed template CSPs. Of particular importance in this project is to prove the so-called CSP Dichotomy Conjecture of Feder and Vardi [9] dating back to 1993. It states that the CSPs relating to a fixed constraint language over a finite domain are either polynomial time solvable or NP-complete. To date this conjecture remains unanswered, but it has driven major advances in the study of CSPs.

One such advance is the algebraic connection revealed by Jeavons, Cohen and Gyssens [15] and later refined by Bulatov, Jeavons and Krokhin [6]. This connection associates with each finite domain constraint language $A$ a finite algebraic structure $A$. The properties of this algebraic structure are deeply linked with the computational complexity of the constraint language. In particular, for a fixed constraint language $A$, if there does not exist a particular kind of operation, known as a Taylor polymorphism, then the class of problems determined by $A$ is NP-complete. Bulatov, Jeavons and Krokhin [6] go on to conjecture that all other constraint languages over finite domains determine polynomial time CSPs (a stronger form of the CSP Dichotomy Conjecture, since it describes where the split between polynomial time and NP-completeness lies). This conjecture is often referred to as the Algebraic CSP Dichotomy Conjecture. Many important results have been built upon this algebraic connection. Bulatov [7] extended Schaefer’s [24] result on 2-element domains to prove the CSP Dichotomy Conjecture for 3-element domains. Barto, Kozik and Niven [4] extended Hell and Nešetřil’s result [10] on simple graphs to constraint languages consisting of a finite digraph with no sources and no sinks. Barto and Kozik [2] gave a complete algebraic description of the constraint languages over finite domains that are solvable by local consistency methods (these problems are said to be of bounded width) and as a consequence it is decidable to determine whether a constraint language can be solved by such methods.
In their seminal paper, Feder and Vardi [9] not only conjectured a dichotomy, they also reduced the problem of proving the dichotomy conjecture to the particular case of digraph homomorphism problems, and even to digraph homomorphism problems where the digraph is balanced (here balanced means that its vertices can be partitioned into levels). Specifically, for every template $A$ (a finite relational structure of finite type) there is a balanced digraph (digraphs are particular kinds of relational structures) $D(A)$ such that the CSP over $A$ is polynomial time equivalent to that over $D(A)$.

2 The main results

In general, fixed template CSPs can be modelled as relational structure homomorphism problems [9]. For detailed formal definitions of relational structures, homomorphisms and polymorphisms, see Section 3.

Let $A$ be a finite structure with signature $R$ (the fixed template), then the constraint satisfaction problem for $A$ is the following decision problem.

**Constraint satisfaction problem for $A$.**

| CSP($A$) |
|-----------|
| INSTANCE: A finite $R$-structure $X$. |
| QUESTION: Is there a homomorphism from $X$ to $A$? |

The dichotomy conjecture [9] can be stated as follows:

**CSP dichotomy conjecture.** Let $A$ be a finite relational structure. Then CSP($A$) is solvable in polynomial time or NP-complete.

The dichotomy conjecture is equivalent to its restriction to digraphs [9], and thus can be restated as follows:

**CSP dichotomy conjecture.** Let $H$ be a finite digraph. Then CSP($H$) is solvable in polynomial time or NP-complete.

Every finite relational structure $A$ has a unique core substructure $A'$ (see Section 3.3 for the precise definition) such that CSP($A$) and CSP($A'$) are identical problems, i.e., the “yes” and “no” instances are precisely the same. The algebraic dichotomy conjecture [6] is the following:

**Algebraic CSP dichotomy conjecture.** Let $A$ be a finite relational structure that is a core. If $A$ has a Taylor polymorphism then CSP($A$) is solvable in polynomial time, otherwise CSP($A$) is NP-complete.

Indeed, perhaps the above conjecture should be called the algebraic tractability conjecture since it is known that if a core $A$ does not possess a Taylor polymorphism, then CSP($A$) is NP-complete [6].

Feder and Vardi [9] proved that every fixed template CSP is polynomial time equivalent to a digraph CSP. This article will provide the following theorem, which replaces “polynomial time” with “logspace” and reduces the algebraic dichotomy conjecture to digraphs.
Theorem 1. Let $A$ be a finite relational structure. There is a finite digraph $D(A)$ such that

(i) $\text{CSP}(A)$ and $\text{CSP}(D(A))$ are logspace equivalent,
(ii) $A$ has a Taylor polymorphism if and only if $D(A)$ has a Taylor polymorphism, and
(iii) $A$ is a core if and only if $D(A)$ is a core.

Furthermore, if $A$ is a core, then $\text{CSP}(A)$ has bounded width if and only if $\text{CSP}(D(A))$ has bounded width.

Proof. To prove (i), one reduction follows from Lemma 3 and Lemma 1. The other reduction is Lemma 4.

To prove (ii) we employ Theorem 2; it suffices to show that $A$ has a WNU polymorphism if and only if $D(A)$ has a WNU polymorphism. The forward implication (which is the crucial part of our proof) is proved in Lemma 5 and the converse follows from Lemma 6 and Lemma 2. Item (iii) is Corollary 1.

The preservation of bounded width follows from Corollary 1, Lemma 8 and Theorem 3. □

See Remark 1 in Section 4 for the size of $D(A)$. The “Taylor polymorphism” in Theorem 1(ii) can be replaced by many other polymorphism properties, but space constraints do not allow us to elaborate here.

As a direct consequence of Theorem 1(ii) and (iii) above, we can restate the algebraic dichotomy conjecture:

Algebraic CSP dichotomy conjecture. Let $H$ be a finite digraph that is a core. If $H$ has a Taylor polymorphism then $\text{CSP}(H)$ is solvable in polynomial time, otherwise $\text{CSP}(H)$ is NP-complete.

3 Background and definitions

We approach fixed template constraint satisfaction problems from the “homomorphism problem” point of view. For background on the homomorphism approach to CSPs, see [9], and for background on the algebraic approach to CSP, see [6].

A relational signature $R$ is a (in our case finite) set of relation symbols $R_1$, each with an associated arity $k_i$. A (finite) relational structure $A$ over relational signature $R$ (called an $R$-structure) is a finite set $A$ (the domain) together with a relation $R_i \subseteq A^{k_i}$, for each relation symbol $R_i$ of arity $k_i$ in $R$. A CSP template is a fixed finite $R$-structure, for some signature $R$.

For simplicity we do not distinguish the relation with its associated relation symbol, however to avoid ambiguity, sometimes we write $R^A$ to indicate that $R$ belongs to $A$. We will often refer to the domain of relational structure $A$ simply by $A$. When referring to a fixed relational structure, we may simply specify it as $A = (A; R_1, R_2, \ldots, R_k)$. For technical reasons we require that all the relations of a relational structure are nonempty.
3.1 Notation

For a positive integer \( n \) we denote the set \( \{1, 2, \ldots, n\} \) by \([n]\). We write tuples using boldface notation, e.g. \( a = (a_1, a_2, \ldots, a_k) \in A^k \) and when ranging over tuples we use superscript notation, e.g. \((r^1, r^2, \ldots, r^i) \in R^i \subseteq (A^k)^i\), where \( r^i = (r^1_i, r^2_i, \ldots, r^k_i) \), for \( i = 1, \ldots, l \).

Let \( R_i \subseteq A^k \) be relations of arity \( k_i \), for \( i = 1, \ldots, n \). Let \( k = \sum_{i=1}^n k_i \) and \( l_i = \sum_{j<i} k_j \). We write \( R_1 \times \cdots \times R_n \) to mean the \( k \)-ary relation \[
\{(a_1, \ldots, a_k) \in A^k \mid (a_{i+1}, \ldots, a_{i+k_i}) \in R_i \text{ for } i = 1, \ldots, n\}.
\]

An \( n \)-ary operation on a set \( A \) is simply a mapping \( f : A^n \to A \); the number \( n \) is the arity of \( f \). Let \( f \) be an \( n \)-ary operation on \( A \) and let \( k > 0 \). We define \( f^{(k)} \) to be the \( n \)-ary operation obtained by applying \( f \) coordinatewise on \( A^k \).

That is, we define the \( n \)-ary operation \( f^{(k)} \) on \( A^k \) by
\[
(f^{(k)}(a^1, \ldots, a^n)) = (f(a_1^1, \ldots, a^n_1), \ldots, f(a_1^n, \ldots, a_k^n)),
\]
for \( a^1, \ldots, a^n \in A^k \).

We will be particularly interested in so-called idempotent operations. An \( n \)-ary operation \( f \) is said to be idempotent if it satisfies the equation
\[
f(x, x, \ldots, x) = x.
\]

3.2 Homomorphisms, cores and polymorphisms

We begin with the notion of a relational structure homomorphism.

**Definition 1.** Let \( \mathbb{A} \) and \( \mathbb{B} \) be relational structures in the same signature \( \mathcal{R} \). A homomorphism from \( \mathbb{A} \) to \( \mathbb{B} \) is a mapping \( \varphi \) from \( A \) to \( B \) such that for each \( n \)-ary relation symbol \( R \) in \( \mathcal{R} \) and each \( n \)-tuple \( a \in A^n \), if \( a \in R^k \), then \( \varphi(a) \in R^\mathbb{B} \), where \( \varphi \) is applied to \( a \) coordinatewise.

We write \( \varphi : \mathbb{A} \to \mathbb{B} \) to mean that \( \varphi \) is a homomorphism from \( \mathbb{A} \) to \( \mathbb{B} \), and \( \mathbb{A} \to \mathbb{B} \) to mean that there exists a homomorphism from \( \mathbb{A} \) to \( \mathbb{B} \).

An isomorphism is a bijective homomorphism \( \varphi \) such that \( \varphi^{-1} \) is a homomorphism. A homomorphism \( \mathbb{A} \to \mathbb{A} \) is called an endomorphism. An isomorphism from \( \mathbb{A} \) to \( \mathbb{A} \) is an automorphism. It is an easy fact that if \( \mathbb{A} \) is finite, then every surjective endomorphism is an automorphism.

A finite relational structure \( \mathbb{A}' \) is a core if every endomorphism \( \mathbb{A}' \to \mathbb{A}' \) is surjective (and therefore an automorphism). For every \( \mathbb{A} \) there exists a relational structure \( \mathbb{A}' \) such that \( \mathbb{A} \to \mathbb{A}' \) and \( \mathbb{A}' \to \mathbb{A} \) and \( \mathbb{A}' \) is minimal with respect to these properties. The structure \( \mathbb{A}' \) is called the core of \( \mathbb{A} \). The core of \( \mathbb{A} \) is unique (up to isomorphism) and \( \text{CSP}(\mathbb{A}) \) and \( \text{CSP}(\mathbb{A}') \) are the same decision problems. Equivalently, the core of \( \mathbb{A} \) can be defined as a minimal induced substructure that \( \mathbb{A} \) retracts onto. (See [11] for details on cores for graphs, cores for relational structures are a natural generalisation.)

The notion of polymorphism is central in the so called algebraic approach to CSP. Polymorphisms are a natural generalization of endomorphisms to higher arity operations.
Definition 2. Given an \( \mathcal{R} \)-structure \( \mathcal{A} \), an \( n \)-ary polymorphism of \( \mathcal{A} \) is an \( n \)-ary operation \( f \) on \( \mathcal{A} \) such that \( f \) preserves the relations of \( \mathcal{A} \). That is, if \( a^1, \ldots, a^n \in \mathcal{R} \), for some \( k \)-ary relation \( R \in \mathcal{R} \), then \( f^{(k)}(a^1, \ldots, a^n) \in \mathcal{R} \).

Thus, an endomorphism is a 1-ary polymorphism.

In this paper we will be interested in the following kind of polymorphisms.

Definition 3. A weak near-unanimity (WNU) polymorphism is an \( n \)-ary idempotent polymorphism \( \omega \), for some \( n \geq 3 \), that satisfies the following equations (for all \( x, y \)):

\[
\omega(x, \ldots, x, y) = \omega(x, \ldots, y, x) = \cdots = \omega(y, x, \ldots, x).
\]

We call the above WNU equations.

Note that since we assume that a WNU polymorphism \( \omega \) is idempotent it also satisfies the equation

\[
\omega(x, x, \ldots, x) = x.
\]

Of particular interest, with respect to the algebraic dichotomy conjecture, are Taylor polymorphisms. We will not need to explicitly define Taylor polymorphisms (and only need consider WNU polymorphisms) by the following theorem.

Theorem 2. [20] A finite relational structure \( \mathcal{A} \) has a Taylor polymorphism if and only if \( \mathcal{A} \) has a WNU polymorphism.

Weak near-unanimity polymorphisms can be also used to characterise CSPs of bounded width (see [2] for a detailed explanation of the bounded width algorithm).

Theorem 3. [2, 20] Let \( \mathcal{A} \) be a finite relational structure that is a core. Then CSP(\( \mathcal{A} \)) is of bounded width if and only if \( \mathcal{A} \) has WNU polymorphisms of all but finitely many arities.

3.3 Primitive positive definability

A first order formula is called primitive positive if it is an existential conjunction of atomic formulae. Since we only refer to relational signatures, a primitive positive formula is simply an existential conjunct of formulae of the form \( x = y \) or \( (x_1, x_2, \ldots, x_n) \in R \), where \( R \) is a relation symbol of arity \( n \).

For example, if we have a binary relation symbol \( E \) in our signature, then the formula

\[
\psi(x, y) = (\exists z)((x, z) \in E \land (z, y) \in E),
\]

pp-defines a binary relation in which elements \( a, b \) are related if there is a directed path of length 2 from \( a \) to \( b \) in \( E \).

Definition 4. A relational structure \( \mathcal{B} \) is primitive positive definable in \( \mathcal{A} \) (or \( \mathcal{A} \) pp-defines \( \mathcal{B} \)) if
(i) the set $B$ is a subset of $A$ and is definable by a primitive positive formula interpreted in $A$, and
(ii) each relation $R$ in the signature of $B$ is definable on the set $B$ by a primitive positive formula interpreted in $A$.

The following result relates the above definition to the complexity of CSPs.

**Lemma 1.** Let $A$ be a finite relational structure that pp-defines $B$. Then, $CSP(B)$ is polynomial time (indeed, logspace) reducible to $CSP(A)$.

It so happens that, if $A$ pp-defines $B$, then $B$ inherits the polymorphisms of $A$. See [6] for a detailed explanation.

**Lemma 2.** Let $A$ be a finite relational structure that pp-defines $B$. If $\varphi$ is a polymorphism of $A$, then its restriction to $B$ is a polymorphism of $B$.

In particular, if $A$ pp-defines $B$ and $A$ has a WNU polymorphism $\omega$, then $\omega$ restricted to $B$ is a WNU polymorphism of $B$.

**Example 1.** Let $A = (A; R_1, \ldots, R_n)$, where each $R_i$ is $k_i$-ary, and define $R = R_1 \times \cdots \times R_n$. Then the structure $A' = (A; R)$ is pp-equivalent to $A$.

Indeed, let $k = \sum_{i=1}^n k_i$ be the arity of $R$ and $l_i = \sum_{j<i} k_j$ for $i = 1, \ldots, n$. The relation $R$ is pp-definable from $R_1, \ldots, R_n$ using the formula

$$\Psi(x_1, \ldots, x_k) = \bigwedge_{i=1}^n (x_{l_i+1}, \ldots, x_{l_i+k_i}) \in R_i.$$ 

The relation $R_1$ can be defined from $R$ by the primitive positive formula

$$\Psi(x_1, \ldots, x_{k_1}) = (\exists y_{k_1+1}, \ldots, y_k)((x_1, \ldots, x_{k_1}, y_{k_1+1}, \ldots, y_k) \in R)$$

and the remaining $R_i$’s can be defined similarly.

Example [1] shows that when proving Theorem [1] we can restrict ourselves to relational structures with a single relation.

### 3.4 Digraphs

A directed graph, or digraph, is a relational structure $\mathcal{G}$ with a single binary relation symbol $E$ as its signature. We typically call the members of $G$ and $E G$ vertices and edges, respectively. We usually write $a \to b$ to mean $(a, b) \in E G$, if there is no ambiguity.

A special case of relational structure homomorphism (see Definition [1]), is that of digraph homomorphism. That is, given digraphs $\mathcal{G}$ and $H$, a function $\varphi : G \to H$ is a homomorphism if $(\varphi(a), \varphi(b)) \in E H$ whenever $(a, b) \in E G$.
Definition 5. For \( i = 1, \ldots, n \), let \( G_i = (G_i, E_i) \) be digraphs. The direct product of \( G_1, \ldots, G_n \), denoted by \( \prod_{i=1}^n G_i \), is the digraph with vertices \( \prod_{i=1}^n G_i \) (the cartesian product of the sets \( G_i \)) and edge relation

\[
\{(a, b) \in (\prod_{i=1}^n G_i)^2 | (a_i, b_i) \in E_i \text{ for } i = 1, \ldots, n\}.
\]

If \( G_1 = \cdots = G_n = G \) then we write \( G^n \) to mean \( \prod_{i=1}^n G_i \).

With the above definition in mind, an \( n \)-ary polymorphism on a digraph \( G \) is simply a digraph homomorphism from \( G^n \) to \( G \).

Definition 6. A digraph \( P \) is an oriented path if it consists of a sequence of vertices \( v_0, v_1, \ldots, v_k \) such that precisely one of \((v_{i-1}, v_i), (v_i, v_{i-1})\) is an edge, for each \( i = 1, \ldots, k \). We require oriented paths to have a direction; we denote the initial vertex \( v_0 \) and the terminal vertex \( v_k \) by \( iP \) and \( tP \), respectively.

Given a digraph \( G \) and an oriented path \( P \), we write \( a \to^P b \) to mean that we can walk in \( G \) from \( a \) following \( P \) to \( b \), i.e., there exists a homomorphism \( \varphi : P \to G \) such that \( \varphi(iP) = a \) and \( \varphi(tP) = b \). Note that for every \( P \) there exists a primitive positive formula \( \psi(x, y) \) such that \( a \to^P b \) if and only if \( \psi(a, b) \) is true in \( G \). If there exists an oriented path \( P \) such that \( a \to^P b \), we say that \( a \) and \( b \) are connected. If vertices \( a \) and \( b \) are connected, then the distance from \( a \) to \( b \) is the number of edges in the shortest oriented path connecting them. Connectedness forms an equivalence relation on \( G \); its classes are called the connected components of \( G \). We say that a digraph is connected if it consists of a single connected component.

A connected digraph \( G \) is balanced if it admits a level function \( \text{lvl} : G \to \mathbb{N} \), where \( \text{lvl}(b) = \text{lvl}(a) + 1 \) whenever \((a, b)\) is an edge, and the minimum level is \( 0 \). The maximum level is called the height of the digraph. Oriented paths are natural examples of balanced digraphs.

By a zigzag we mean the oriented path \( \bullet \to \bullet \leftrightarrow \bullet \to \bullet \) and a single edge is the path \( \bullet \to \bullet \). For oriented paths \( P \) and \( P' \), the concatenation of \( P \) and \( P' \), denoted by \( P + P' \), is the oriented path obtained by identifying \( tP \) with \( iP' \).

Our digraph reduction as described in Section 4 relies on oriented paths obtained by concatenation of zigzags and single edges. For example, the path in Figure 4 is a concatenation of a single edge followed by two zigzags and two more single edges (for clarity, we organise its vertices into levels).

4 The reduction to digraphs

In this section we take an arbitrary finite relational structure \( A \) and construct a balanced digraph \( \mathcal{D}(A) \) such that \( \text{CSP}(A) \) and \( \text{CSP}(\mathcal{D}(A)) \) are logspace equivalent.

Let \( A = (A; R_1, \ldots, R_n) \) be a finite relational structure, where \( R_i \) is of arity \( k_i \), for \( i = 1, \ldots, n \). Let \( k = \sum_{i=1}^n k_i \) and let \( R \) be the \( k \)-ary relation.
\( R_1 \times \cdots \times R_n \). For \( I \subseteq [k] \) define \( Q_{I,l} \) to be a single edge if \( l \in I \), and a zigzag if \( l \in [k] \setminus I \).

We define the oriented path \( Q_I \) (of height \( k + 2 \)) by

\[
Q_I = \bullet \rightarrow \bullet + Q_{I,1} + Q_{I,2} + \cdots + Q_{I,k} + \bullet \rightarrow \bullet
\]

Instead of \( Q_\emptyset, Q_{\emptyset,l} \) we write just \( Q, Q_l \), respectively. For example, the oriented path in Figure 1 is \( Q_I \) where \( k = 3 \) and \( I = \{3\} \). We will need the following observation.

**Observation.** Let \( I, J \subseteq [k] \). A homomorphism \( \varphi : Q_I \rightarrow Q_J \) exists, if and only if \( I \subseteq J \). In particular \( Q \rightarrow Q_I \) for all \( I \subseteq [k] \). Moreover, if \( \varphi \) exists, it is unique and surjective.

We are now ready to define the digraph \( D(A) \).

**Definition 7.** For every \( e = (a, r) \in A \times R \) we define \( P_e \) to be the path \( Q_{\{i : a = r_i\}} \).

The digraph \( D(A) \) is obtained from the digraph \( (A \cup R; A \times R) \) by replacing every \( e = (a, r) \in A \times R \) by the oriented path \( P_e \) (identifying \( \iota P_e \) with \( a \) and \( \tau P_e \) with \( r \)).

(We often write \( P_{e,l} \) to mean \( Q_{I,l} \) where \( P_e = Q_{I} \).)

**Example 2.** Consider the relational structure \( A = ([0, 1]; R) \) where \( R = \{(0, 1), (1, 0)\} \), i.e., \( A \) is the 2-cycle. Figure 2 is a visual representation of \( D(A) \).

**Remark 1.** The number of vertices in \( D(A) \) is \((3k + 1)|R||A| + (1 - 2k)|R| + |A|\) and the number of edges is \((3k + 2)|R||A| - 2k|R|\). The construction of \( D(A) \) can be performed in logspace (under any reasonable encoding).

**Proof.** The vertices of \( D(A) \) consist of the elements of \( A \cup R \), along with vertices from the connecting paths. The number of vertices lying strictly within the connecting paths would be \((3k + 1)|R||A|\) if every \( P_e \) was \( Q \). We need to deduct 2 vertices whenever there is a single edge instead of a zigzag and there are \( \sum_{(a,r) \in A \times R} |\{i : a = r_i\}| = k|R| \) such instances. The number of edges is counted very similarly. \( \square \)
Fig. 2. $D(A)$ where $A$ is the 2-cycle

Remark 2. Note that if we apply this construction to itself (that is, $D(D(A))$) then we obtain balanced digraphs of height 4. When applied to digraphs, the $D$ construction is identical to that given by Feder and Vardi [9, Theorem 13].

The following lemma, together with Lemma 1, shows that CSP($A$) reduces to CSP($D(A)$) in logspace.

Lemma 3. $A$ is pp-definable from $D(A)$.

Proof. Example 1 demonstrates that $A$ is pp-equivalent to $(A; R)$. We now show that $D(A)$ pp-defines $(A; R)$, from which it follows that $D(A)$ pp-defines $A$.

Note that $Q \rightarrow P(e)$ for all $e \in A \times R$, and $Q \{i\} \rightarrow P(a,r)$ if and only if $a = r_i$. The set $A$ is pp-definable in $D(A)$ by $A = \{x \mid (\exists y) (x \xrightarrow{Q} y)\}$ and the relation $R$ can be defined as the set $\{(x_1, \ldots, x_k) \mid (\exists y) (x_i \xrightarrow{Q \{i\}} y \text{ for all } i \in [k])\}$, which is also a primitive positive definition.

It is not, in general, possible to pp-define $D(A)$ from $A$. Nonetheless the following lemma is true.

Lemma 4. CSP($D(A)$) reduces in logspace to CSP($A$).

The proof of Lemma 4 is rather technical, though broadly follows the polynomial process described in the proof of [9, Theorem 13] (as mentioned, our construction coincides with theirs in the case of digraphs). Details of the argument will be presented in a subsequent expanded version of this article.

Lemma 3 and Lemma 4 complete the proof of part (i) of Theorem 4. As this improves the oft-mentioned polynomial time equivalence of general CSPs with digraph CSPs, we now present it as stand-alone statement.

Theorem 4. Every fixed finite template CSP is logspace equivalent to the CSP over some finite digraph.

1 Using the definition of pp-definability as described in this paper, this is true for cardinality reasons. However, a result of Kazda [16] can be used to show that the statement remains true even for more general definitions of pp-definability.
5 Preserving cores

In what follows, let $\mathcal{A}$ be a fixed finite relational structure. Without loss of generality we may assume that $\mathcal{A} = (A; R)$, where $R$ is a $k$-ary relation (see Example 1).

Lemma 5. The endomorphisms of $\mathcal{A}$ and $\mathcal{D}(\mathcal{A})$ are in one-to-one correspondence.

Proof. We first show that every endomorphism $\varphi$ of $\mathcal{A}$ can be extended to an endomorphism $\overline{\varphi}$ of $\mathcal{D}(\mathcal{A})$. Let $\overline{\varphi}(a) = \varphi(a)$ for $a \in A$, and let $\overline{\varphi}(r) = \varphi^{(k)}(r)$ for $r \in R$. Note that $\varphi^{(k)}(r) \in R$ since $\varphi$ is an endomorphism of $\mathcal{A}$.

Let $c \in \mathcal{D}(\mathcal{A}) \setminus (A \cup R)$ and let $e = (a, r)$ be such that $c \in P_e$. Define $e' = (\varphi(a), \varphi^{(k)}(r))$. If $P_{e,l}$ is a single edge for some $l \in [k]$, then $v_l = a$ and $\varphi(v_l) = \varphi(a)$, and therefore $P_{e,l}$ is a single edge. Thus there exists a (unique) homomorphism $P_e \to P_{e'}$. Define $\overline{\varphi}(c)$ to be the image of $c$ under this homomorphism, completing the definition of $\overline{\varphi}$.

We now show that every endomorphism $\Phi$ of $\mathcal{D}(\mathcal{A})$ is of the form $\overline{\varphi}$, for some endomorphism $\varphi$ of $\mathcal{A}$. Let $\Phi$ be an endomorphism of $\mathcal{D}(\mathcal{A})$. Let $\varphi$ be the restriction of $\Phi$ to $A$. By Lemma 2 and Lemma 3 $\varphi$ is an endomorphism of $\mathcal{A}$.

For every $e = (a, r)$, the endomorphism $\Phi$ maps $P_e$ onto $P_{(\varphi(a), \varphi^{(k)}(r))}$. If we set $a = \varphi_l$, then $P_{e,l}$ is a single edge. In this case it follows that $P_{(\varphi(a), \varphi^{(k)}(r))}$ is also a single edge. Thus, by the construction of $\mathcal{D}(\mathcal{A})$ the $l$th coordinate of $\Phi(r)$ is $\varphi(a) = \varphi(l)$. This proves that the restriction of $\Phi$ to $R$ is $\varphi^{(k)}$ and therefore $\Phi = \overline{\varphi}$.

The following corollary is Theorem 1 (iii).

Corollary 1. $\mathcal{A}$ is a core if and only if $\mathcal{D}(\mathcal{A})$ is a core.

Proof. To prove the corollary we need to show that an endomorphism $\varphi$ of $\mathcal{A}$ is surjective if and only if $\overline{\varphi}$ (from Lemma 5) is surjective. Clearly, if $\overline{\varphi}$ is surjective then so is $\varphi$.

Assume $\varphi$ is surjective (and therefore an automorphism of $\mathcal{A}$). It follows that $\varphi^{(k)}$ is surjective on $R$ and therefore $\overline{\varphi}$ is a bijection when restricted to the set $A \cup R$. Let $a \in A$ and $r \in R$. By definition we know that $\overline{\varphi}$ maps $P_{(a, r)}$ homomorphically onto $P_{(\varphi(a), \varphi^{(k)}(r))}$. Since $\varphi$ has an inverse $\varphi^{-1}$, it follows that $\varphi^{-1}$ maps $P_{(\varphi(a), \varphi^{(k)}(r))}$ homomorphically onto $P_{(a, r)}$. Thus $P_{(a, r)}$ and $P_{(\varphi(a), \varphi^{(k)}(r))}$ are isomorphic, completing the proof.

To complete the proof of Theorem 1 it remains to show that our reduction preserves WNU.

6 The reduction preserves WNU

For $m > 0$ let $\Delta_m$ denote the connected component of the digraph $\mathcal{D}(\mathcal{A})^m$ containing the diagonal (i.e., the set $\{(c, c, \ldots, c) \mid c \in \mathcal{D}(\mathcal{A})\}$).
Lemma 6. The elements of the diagonal are all connected in $\mathcal{D}(A)^m$. Furthermore, $A^m \subseteq \Delta_m$ and $R^m \subseteq \Delta_m$.

Proof. The first statement follows from the fact that $\mathcal{D}(A)$ is connected. To see that $A^m \subseteq \Delta_m$ and $R^m \subseteq \Delta_m$, fix $a \in A$, and so by definition, $(a, a, \ldots, a) \in \Delta_m$. Let $(r^1, \ldots, r^m) \in R^m$ and for every $i \in [m]$ let $\varphi_i : Q \to \mathbb{P}_{(a, r^i)}$. The homomorphism defined by $x \mapsto (\varphi_1(x), \ldots, \varphi_m(x))$ witnesses $(a, \ldots, a) \stackrel{Q}{\rightarrow} (r^1, \ldots, r^m)$ in $\mathcal{D}(A)^m$. Hence $R^m \subseteq \Delta_m$. A similar argument gives $A^m \subseteq \Delta_m$. $\square$

The following lemma shows that there is only one non-trivial connected component of $\mathcal{D}(A)^m$ that contains tuples (whose entries are) on the same level in $\mathcal{D}(A)$; namely $\Delta_m$.

Lemma 7. Let $m > 0$ and let $\Gamma$ be a connected component of $\mathcal{D}(A)^m$ containing an element $c$ such that $\ell(c_1) = \cdots = \ell(c_m)$. Then every element $d \in \Gamma$ is of the form $\ell(d_1) = \cdots = \ell(d_m)$ and the following hold.

(i) If $c \rightarrow d$ is an edge in $\Gamma$ such that $c \notin A^m$ and $d \notin R^m$, then there exist $e_1, \ldots, e_m \in A \times R$ and $l \in [k]$ such that $c, d \in \prod_{i=1}^m \mathbb{P}_{e_i,l}$.

(ii) Either $\Gamma = \Delta_m$ or $\Gamma$ is a one-element.

Proof. First observe that if an element $d$ is connected in $\mathcal{D}(A)^m$ to an element $c$ with $\ell(c_1) = \cdots = \ell(c_m)$, then there is an oriented path $Q'$ such that $c \stackrel{Q'}{\rightarrow} d$ from which it follows that $\ell(d_1) = \cdots = \ell(d_m)$.

To prove (i), let $c \rightarrow d$ be an edge in $\Gamma$ such that $c \notin A^m$ and $d \notin R^m$. For $i = 1, \ldots, m$ let $e_i$ be such that $c_i \in \mathbb{P}_{e_i,l}$ and let $l = \ell(c_1)$. The claim now follows immediately from the construction of $\mathcal{D}(A)$.

It remains to prove (ii). If $|\Gamma| > 1$, then there is an edge $c \rightarrow d$ in $\Gamma$. If $c \in A^m$ or $d \in R^m$, then the result follows from Lemma 6. Otherwise, from (i), there exists $l \in [k]$ and $e_i = (a_i, r^i)$ such that $c, d \in \prod_{i=1}^m \mathbb{P}_{e_i,l}$.

For every $i \in [m]$ we can walk from $c_i$ to $i\mathbb{P}_{e_i,l}$ following the path $i \rightarrow i \leftarrow i$; and so $c$ and $(i\mathbb{P}_{e_1,l}, \ldots, i\mathbb{P}_{e_m,l})$ are connected. For every $i \in [m]$ there exists a homomorphism $\varphi_i : Q \to \mathbb{P}_{e_i}$ such that $\varphi_i(i \mathbb{Q}) = a_i$ and $\varphi_i(i \mathbb{Q}) = i \mathbb{P}_{e_i,l}$. The homomorphism $Q \to \mathcal{D}(A)^m$ defined by $x \mapsto (\varphi_1(x), \ldots, \varphi_m(x))$ shows that $(a_1, \ldots, a_m)$ and $(i\mathbb{P}_{e_1,l}, \ldots, i\mathbb{P}_{e_m,l})$ are connected. By transitivity, $(a_1, \ldots, a_m)$ is connected to $c$ and therefore $(a_1, \ldots, a_m) \in \Gamma$. Using (i) we obtain $\Gamma = \Delta_m$. $\square$

We are now ready to prove the main ingredient of Theorem 1 (ii). The proof of Lemma 6 is similar in essence to the proof of Lemma 5 although more complicated.

Lemma 8. If $A$ has an $m$-ary WNU polymorphism, then $\mathcal{D}(A)$ has an $m$-ary WNU polymorphism.

Proof. Let $\omega$ be an $m$-ary WNU polymorphism of $A$. We construct an $m$-ary operation $\mathcal{D}(A)$. We split the definition into several cases and subcases. Let $c \in \mathcal{D}(A)^m$. 
Case 1. \(|\{\text{lvl}(c_1), \ldots, \text{lvl}(c_m)\}| > 1\).

1a. If there exists \(i \in [m]\) such that \(|\{\text{lvl}(c_j) \mid j \neq i\}| = 1\), we define \(\overline{\omega}(c) = c_i\).

1b. Else let \(\overline{\omega}(c) = c_1\).

Case 2. \(\text{lvl}(c_1) = \cdots = \text{lvl}(c_m)\), but \(c \notin \Delta_m\).

2a. If there exists \(i \in [m]\) such that \(|\{c_j \mid j \neq i\}| = 1\), we define \(\overline{\omega}(c) = c_i\).

2b. Else let \(\overline{\omega}(c) = c_1\).

Case 3. \(c \in \Delta_m\).

3a. If \(\{c_1, \ldots, c_m\} \subseteq A\), we define \(\overline{\omega}(c) = \omega(c)\).

3b. If \(\{c_1, \ldots, c_m\} \subseteq R\), we define \(\overline{\omega}(c) = \omega^{(k)}(c)\).

3c. Else, there exists \(d \in \Delta_m \setminus (A^m \cup R^m)\) such that \(c \rightarrow d\) or \(d \rightarrow c\) in \(\mathcal{D}(\mathcal{A})^m\).

By Lemma 7 (ii), there exist \(l \in [k]\) and \(c_j = (a_i, r^*)\) such that \(c \in \prod_{i=1}^m \mathbb{P}_{c_i,l}\).

Let \(e = (a, r)\), where \(a = \omega(a_1, \ldots, a_m)\) and \(r = \omega^{(k)}(r^1, \ldots, r^m)\). We set \(\overline{\omega}(c) = \Phi(e)\), where \(\Phi : \prod_{i=1}^m \mathbb{P}_{e_i,l} \rightarrow \mathbb{P}_{e,l}\) is defined as follows.

1. If \(\mathbb{P}_{e,l}\) is a single edge, then we set

\[
\Phi(u) = \begin{cases} 
\tau \mathbb{P}_{e,l} & \text{if lvl}(u_1) = \cdots = \text{lvl}(u_m) = \text{lvl}(\tau \mathbb{P}_{e,l}) \\
\mathbb{P}_{e,l} & \text{otherwise}.
\end{cases}
\]

2. If \(\mathbb{P}_{e,l}\) is a zigzag, then let \(I = \{i \in [m] \mid \mathbb{P}_{e_i,l}\text{ is a zigzag}\}\). For every \(i \in I\) let \(\phi_i : \mathbb{P}_{e_i,l} \rightarrow \mathbb{P}_{e,l}\) be the unique isomorphism. We define \(\Phi(u)\) to be the vertex from \(\{\phi_i(u_i) : i \in I\}\) with minimal distance from \(\tau \mathbb{P}_{e,l}\).

Let us first comment on correctness of the definition. In subcase 3b, \(\omega^{(k)}(c) \in R\) follows from the fact that \(\omega\) preserves \(R\). In subcase 3c, if \(\mathbb{P}_{e,l}\) is a zigzag, then \(I \neq \emptyset\). Indeed, if all the \(\mathbb{P}_{e_i,l}\)'s were single edges, then \(r_1 = \omega(r^1) = \omega(a_1, \ldots, a_m) = a\) and so \(\mathbb{P}_{e,l}\) would also be a single edge. The \(c_i\)'s are uniquely determined by \(c\), and the choice of \(l\) is unique as well, with one exception: if \(l < k\) and \(c_i = \tau \mathbb{P}_{e_i,l} = \mathbb{P}_{e_i,l+1}\) for every \(i \in [m]\), then we have \(d \rightarrow c\) for some \(d, d' \in \mathcal{D}(\mathcal{A})^m\) and we can choose \(l + 1\) instead of \(l\). However, it is not hard to see that the value assigned to \(\overline{\omega}(c)\) is the same in both cases, namely it is the vertex \(\tau \mathbb{P}_{e,l} = \mathbb{P}_{e,l+1}\) (see property (b) below).

We need the following properties of the mapping \(\Phi\) defined in 3c:

(a) \(\Phi\) is a homomorphism.

(b) \(\Phi(\tau \mathbb{P}_{e_1,l}, \ldots, \mathbb{P}_{e_m,l}) = \tau \mathbb{P}_{e,l}\) and \(\Phi(\tau \mathbb{P}_{e_{c,i},l}, \ldots, \tau \mathbb{P}_{e_{c,m},l}) = \tau \mathbb{P}_{e,l}\).

(c) \(\Phi\) does not depend on the ordering of the tuple \((e_1, \ldots, e_m)\).

(d) If \(e_1 = \cdots = e_m = e\), then \(\Phi : \mathbb{P}_{e,l} \rightarrow \mathbb{P}_{e,l}\) is idempotent, i.e., \(\Phi(u, \ldots, u) = u\) for all \(u \in \mathbb{P}_{e,l}\).

All of the above properties follow easily from the definition of \(\Phi\). We leave the verification to the reader. It remains to prove that \(\overline{\omega}\) is a WNU polymorphism of \(\mathcal{D}(\mathcal{A})\).

Claim. \(\overline{\omega}\) is idempotent and satisfies the WNU equations.
Let $c, d \in \mathcal{D}(\mathcal{A})$. Note that all of the tuples $(c, \ldots, c, d), (c, \ldots, d, c), \ldots, (d, c, \ldots, c)$ fall into the same subcase; the possibilities are 3a, 3b, 3c and if $c \neq d$, then also 1b or 2a. In subcases 1a and 2a the definition does not depend on the ordering of the input tuple at all; therefore the WNU equations hold (and since $c \neq d$, idempotency does not apply).

In case 3 we use the fact that $\omega$ and $\omega^{(k)}$ are idempotent and satisfy the WNU equations. In 3a and 3b the claim follows immediately. In 3c note that $e$ is the same for all of the tuples $(c, \ldots, c, d), \ldots, (d, c, \ldots, c)$, and if $c = d$, then $e_1 = \cdots = e_m = e$. The WNU equations follow from property (c) of the mapping $\Phi$, and idempotency follows from (d).

**Claim.** $\overline{\pi}$ is a polymorphism of $\mathcal{D}(\mathcal{A})$.

Let $c, d \in \mathcal{D}(\mathcal{A})^m$ be such that $c \rightarrow d$ is an edge in $\mathcal{D}(\mathcal{A})^m$, that is, $c_i \rightarrow d_i$ for all $i \in [m]$. Both the tuples $c$ and $d$ fall into the same case and, by Lemma 7(iii), it cannot be case 2. If it is case 1, then they also fall into the same subcase and it is easily seen that $\overline{\pi}(c) \rightarrow \overline{\pi}(d)$.

If $c$ falls into subcase 3a, then $d$ falls into 3c. Let $e_i$ be such that $d_i \in \mathcal{P}_{e_i}$. As $c_i \rightarrow d_i$, it follows that $e_i = (c_i, r^i)$ for some $r^i \in R$ and $d_i = \pi_{e_i, 1}$. Let us define $c = \omega(c), r = \omega^{(k)}(r^1, \ldots, r^m)$ and $e = (c, r)$. Now $\overline{\pi}(d)$ is the result of the mapping $\Phi$ applied to the tuple of initial vertices of the $\mathcal{P}_{e_i, 1}$’s, which (by property (b)) is the initial vertex of $\mathcal{P}_{e, 1}$. So $\overline{\pi}(c) = c \rightarrow \pi_{e, 1} = \overline{\pi}(d)$. The argument is similar if $d$ falls into 3b.

It remains to verify that $\overline{\pi}(c) \rightarrow \overline{\pi}(d)$ if both $c$ and $d$ fall into subcase 3c. Both $\overline{\pi}(c)$ and $\overline{\pi}(d)$ are defined using the mapping $\Phi : \prod_{i=1}^{m} \mathcal{P}_{e_i, 1} \rightarrow \mathcal{P}_{e, 1}$. Since $\Phi$ is a homomorphism, we have $\overline{\pi}(c) = \Phi(c) \rightarrow \Phi(d) = \overline{\pi}(d)$, concluding the proof.

7 Discussion

The algebraic dichotomy conjecture proposes a polymorphism characterisation of tractability for core CSPs. A number of other algorithmic properties are also either proved or conjectured to correspond to the existence of polymorphisms with special equational properties. For instance, solvability by the few subpowers algorithm (a generalization of Gaussian elimination) as described in Idziak et al. [13] has a polymorphism characterisation [5], as well as problems of bounded width (see Theorem 3). The final statement in Theorem 1 already shows that $\mathcal{A}$ has bounded width if and only if $\mathcal{D}(\mathcal{A})$ has bounded width. Kazda [16] showed that every digraph with a Maltsev polymorphism must have a majority polymorphism, which is not the case for finite relational structures in general. In a later version of the present article we will show that Theorem 1(ii) extends to include almost all commonly encountered polymorphism properties aside from Maltsev. For instance the CSP over $\mathcal{A}$ is solvable by the few subpowers algorithm if and only if the same is true for $\mathcal{D}(\mathcal{A})$. Among other conditions preserved under our reduction to digraphs is that of arc consistency (or width 1 problems [8]) and problems of bounded strict width [9]. The following example is a powerful consequence of the result.

$\Box$
Example 3. Let $A$ be the structure on $\{0, 1\}$ with a single 4-ary relation
\[
\{(0, 0, 0, 1), (0, 1, 1, 1), (1, 0, 1, 1), (1, 1, 0, 1)\}.
\]
Clearly $A$ is a core. The polymorphisms of $A$ can be shown to be the idempotent term functions of the two element group, and from this it follows that $\text{CSP}(A)$ is solvable by the few subpowers algorithm of [13], but is not bounded width. Then the CSP over the digraph $D(A)$ is also solvable by few subpowers but is not bounded width (that is, is not solvable by local consistency check).

Prior to the announcement of this example it had been temporarily conjectured by some researchers that solvability by the few subpowers algorithm implied solvability by local consistency check in the case of digraphs (this was the opening conjecture in Maróti’s keynote presentation at the Second International Conference on Order, Algebra and Logics in Krakow 2011 for example). With 78 vertices and 80 edges, Example 3 also serves as a simpler alternative to the 368-vertex, 432-edge digraph whose CSP was shown by Atserias in [1, §4.2] to be tractable but not solvable by local consistency check.

There are also conjectured polymorphism classifications of the property of solvability within nondeterministic logspace and within logspace; see Larose and Tesson [18]. The required polymorphism conditions are among those we can show are preserved under the transition from $A$ to $D(A)$. It then follows that these conjectures are true provided they can be established in the restricted case of CSPs over digraphs.

Open problems. We conclude our paper with some further research directions. It is possible to show that the logspace reduction in Lemma 4 cannot be replaced by first order reductions. Is there a different construction that translates general CSPs to digraph CSPs with first order reductions in both directions?

Feder and Vardi [9] and Atserias [1] provide polynomial time reductions of CSPs to digraph CSPs. We vigorously conjecture that their reductions preserve the properties of possessing a WNU polymorphism (and of being cores; but this is routinely verified). Do these or other constructions preserve the precise arity of WNU polymorphisms? What other polymorphism properties are preserved? Do they preserve the bounded width property? Can they preserve conservative polymorphisms (the polymorphisms related to list homomorphism problems)? The third and fourth authors with Kowalski [14] have recently shown that a minor variation of the $D$ construction in the present article preserves $k$-ary WNU polymorphisms (and can serve as an alternative to $D$ in Theorem 1), but always fails to preserve many other polymorphism properties (such as those witnessing strict width).

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A. The logspace reduction of CSP($\mathcal{D}(A)$) to CSP($A$).

In this appendix, we give the proof of Lemma 4 by showing that CSP($\mathcal{D}(A)$) reduces in logspace to CSP($A$). A sketch of a polynomial time reduction is given in the proof of [9, Theorem 13]; technically, that argument is for the special case where $A$ is itself already a digraph, but the arguments can be broadened to cover our case. To perform this process in logspace is rather technical, with many of the difficulties lying in details that are omitted in the polynomial time description in the proof of [9, Theorem 13]. We wish to thank Barnaby Martin for encouraging us to pursue Lemma 4.

A.1 Outline of the algorithm.

We first assume that CSP($A$) is itself not trivial (that is, that there is at least one no instance and one yes instance): this uninteresting restriction is necessary because CSP($\mathcal{D}(A)$) will have no instances always. Now let $G$ be an instance of CSP($\mathcal{D}(A)$). Also, let $n$ denote the height of $\mathcal{D}(A)$ and $k$ the arity of the single fundamental relation $R$ of $A$: so, $n = k + 2$. Recall that the vertices of $\mathcal{D}(A)$ include those of $A$ as well as the elements in $R$. The rough outline of the algorithm is as follows.

(Stage 1.) Some initial analysis of $G$ is performed to decide if it is broadly of the right kind of digraph to be a possible yes instance. If not, some fixed no instance of CSP($A$) is output.

(Stage 2.) It is convenient to remove any components of $G$ that are too small. These are considered directly, and in logspace we determine whether or not they are YES or NO instances of CSP($\mathcal{D}(A)$). If all are YES we ignore them. If one returns NO we reject the entire instance and return some fixed NO instance of CSP($A$).
(Stage 3.) Now it may be assumed that \( G \) is roughly similar to a digraph of the form \( D(\mathcal{B}) \), but where some vertices at level 0 have been lost, and other vertices at this level and at level \( n \) have been split into numerous copies, with each possibly containing different parts of the information in the connecting edges of \( D(\mathcal{B}) \). Essentially, the required \( \mathcal{B} \) is a kind of quotient of an object definable from \( G \), though some extra vertices must be added (this is similar to the addition of vertices to account for existentially quantified variables in a primitive positive definition of a relation: only new vertices are added, and they are essentially unconstrained beyond the specific purpose for which they are added). To construct \( \mathcal{B} \) in logspace, we work in two steps: we describe a logspace construction of some intermediate information. Then we describe a logspace reduction from strings of suitable information of this kind to \( \mathcal{B} \). The overall process is logspace because a composition of two logspace reductions is a logspace reduction.

(Stage 3A.) From \( G \) we output a list of “generalised hyperedges” consisting of \( k \)-tuples of sets of vertices, with some labelling to record how these were created. These generalised hyperedges arise from \( G \) but each position in the generalised hyperedge consists of a set of vertices of \( G \), possibly including some new vertices, instead of individual vertices.

(Step 3B.) The actual structure \( \mathcal{B} \) is constructed from the generalised hyperedges in the previous step. The input consists of generalised hyperedges. To create \( \mathcal{B} \) numerous undirected graph reachability checks are performed. The final “vertices” of \( \mathcal{B} \) are in fact sets of vertices of \( G \), so that the generalised hyperedges become actual hyperedges in the conventional sense (\( k \)-tuples of “vertices”, now consisting of sets of vertices of \( G \)). This may be reduced to an adjacency matrix description as a separate logspace process, but that is routine.

Stage 1 is described in Subsection A.3, while Stage 2 is described in Subsection A.4. The most involved part of the algorithm is stage 3A. In Subsection A.3 we give some preliminary discussion on how the process is to proceed: an elaboration on the item listed in the present subsection. In particular a number of definitions are introduced to aid the description of Stage 3A. The actual algorithm is detailed in Subsection A.6. Step 3B is described in Subsection A.7. After a brief discussion of why the algorithm is a valid reduction from CSP(\( D(\mathcal{A}) \)) to CSP(\( \mathcal{A} \)), we present an example of Stages 3A and 3B in action. This example may be a useful reference while reading Subsection A.6 and A.7.

Before we begin describing the algorithm we recall some basic logspace processes that we will use frequently.

### A.2 Subroutines

The algorithm we describe makes numerous calls on other logspace computable processes. Our algorithm may be thought of as running on an oracle machine, with several query tapes. Each query tape verifies membership in some logspace solvable problem. It is well known that \( L^L = L \), and this enables all of the query
tapes to be eliminated within logspace. For the sake of clarity, we briefly recall some basic information on logspace on an oracle machine. A oracle program with logspace query language $U$, has access to an input tape, a working tape (or tapes) and an output tape. Unlimited reading may be done from the input, but no writing. Unlimited writing may be done to the output tape, but no reading. Unlimited writing may be done to the query tape, but no reading. Once the query state is reached however, the current word written to the query tape is tested for membership in the language $U$ (at the cost of one step of computation), and a (correct) answer of either yes or no is received by the program, and the query tape is immediately erased. The space used is measured only from the working tape, where both reading and writing is allowed. If such a program runs in logspace, then it can be emulated by an actual logspace program (with no oracle), so that $L^\mathbb{O} = L$. The argument is essentially the fact that a composition of logspace reductions is a logspace reduction: each query to the oracle (of a string $w$ for instance) during the computation is treated as a fresh instance of a reduction to the membership problem of $U$, which is then composed with the logspace algorithm for $U$ (which is, as usual, done without ever writing any more than around one symbol of $w$ at a time—plus a short counter—which is why space used on the oracle tape does not matter in the oracle formulation of logspace, and why we may assume that the query tape may be erased after completion of the query).

In this subsection we describe the basic checks that are employed during an algorithm.

**Undirected reachability.** Given an undirected graph and two vertices $u, v$, there is a logspace algorithm to determine if $u$ is reachable from $v$ (Reingold [22,23]). In the case of an undirected graph we may use this to determine if two vertices are connected by some oriented path (simply treat the graph as an undirected graph, and use undirected reachability). This means, for example, that we can construct, in logspace, the smallest equivalence relation containing some input binary relation. Starting with $i = 0$ and $j = i + 1$, consider the $i^{th}$ vertex. Check to see if any earlier vertices are weakly connected using queries to the undirected reachability test. If so, then increment $i$ and $j$. If not, then initiate the output of the block of the equivalence relation containing vertex $i$. We may output $i$ (possibly “$\{i\}$ if we wish to output the blocks of the equivalence relation in set notation), and then query whether the $j^{th}$ vertex is undirected reachable from the $i^{th}$. If so, then output the $j^{th}$ vertex, otherwise ignore the $j^{th}$ vertex. Then increment $j$ and repeat these checks until all vertices have been checked (and close the brackets }). Now increment $i$ and repeat.

A second process we frequently perform is undirected reachability checks in graphs whose edges are not precisely those of the current input graph. A typical instance might be where we have some fixed vertex $u$ in consideration, and we wish to test if some vertex $v$ can be reached from $u$ by an oriented path avoiding all vertices failing some property $Q$, where $Q$ is a logspace testable property. This is basically undirected graph reachability, except as well as ignoring the edge direction, we must also ignore any vertex failing property $Q$. This is again
logspace, because it can be performed in logspace on an oracle machine running an algorithm for undirected graph reachability and whose query tape tests property $Q$.

**Component checking.** Undirected graph reachability is also fundamental to checking properties of induced subgraphs. In a typical situation we have some induced subgraph $C$ of $G$ (containing some vertex $u$, say) and we want to test if it satisfies some property $\mathcal{P}$. Membership of vertices in $C$ is itself determined by some property $Q$, testable in logspace. It is convenient to assume that the query tape for $\mathcal{P}$ expects inputs that consist of a list of directed edges (if adjacency matrix is preferred, then this involves one further nested logspace process, but the argument is routine). We may construct a list of the directed edges in the component $C$ on a logspace machine with a query tape for $\mathcal{P}$, for undirected graph reachability and for $Q$. We write $C$ to the query tape for $\mathcal{P}$ as follows.

Systematically enumerate pairs of vertices $v_1, v_2$ of $G$ (re-using some fixed portion of work tape for each pair), in each case testing for undirected reachability of both $v_1$ and $v_2$ from $u$, and also for satisfaction of property $Q$. If both are reachable, and if $(v_1, v_2)$ is an edge of $G$ then we output the edge $(v_1, v_2)$ to the query tape for $\mathcal{P}$. After the last pair has been considered, we may finally query $\mathcal{P}$.

**Testing for path satisfaction.** The basic properties we wish to test of components are usually satisfiability within some fixed finite family of directed paths. We consider the paths $Q_S$, where $S$ is some subset of $[k] = \{1, \ldots, k\}$; recall that these have zigzags in a position $i$ when $i \not\in S$. We first argue that it suffices to show that such queries are logspace in the the case of singleton $S$. Consider $Q_{[k]\setminus\{i\}}$, endowed with not just the edge relation, but also all singleton unary constraints. Because $Q_{[k]\setminus\{i\}}$ is a core digraph, it is well known that the CSP over $Q_{[k]\setminus\{i\}}$ is logspace equivalent to the CSP over $Q_{[k]\setminus\{i\}}$ with singleton unary relations added.

For a typical input digraph $H$, there will be a smallest set $S \subseteq \{1, \ldots, k\}$ such that $H$ can be satisfied in $Q_S$. This can be found by testing for satisfaction in $Q_{[k]\setminus\{i\}}$, where $i$ ranges from 1 to $k$. The set $S$ will consist of those $i$ for which $H \notin \text{CSP}(Q_{[k]\setminus\{i\}})$. A technicality: if the height of $H$ is less than $n$, then it may be satisfiable within some $Q_{[k]\setminus\{i\}}$ at different levels. In order to combine the positive $i$ into the set $S$, it is necessary that these interpretations be done at the same level. This can be done by fixing some vertex of $H$, and using a unary singleton constraint to force it to be interpreted at some fixed level; testing all $Q_{[k]\setminus\{i\}}$. In almost every case we encounter however, we will know in advance that certain vertices of $H$ must be satisfied at either the unique vertex adjacent to $iQ_S$, or to $\tau Q_S$. Thus we only look for such interpretations.

**Lemma 9.**

1. $\text{CSP}(Q_{[k]\setminus\{i\}})$ is solvable in logspace for any $i \in \{1, \ldots, k\}$, even when singleton unary relations are added.

2. For any $S \subseteq \{1, \ldots, k\}$ the problem $\text{CSP}(Q_S)$ is solvable in logspace, including when particular vertices are constrained to be satisfied at particular points.
3. If $Q_{S_1}, Q_{S_2}, \ldots, Q_{S_\ell}$ is a family of connecting paths in $D(A)$, then the CSP over the digraph formed by amalgamating the $Q_{S_i}$ at either all the initial points, or at all the terminal points is logspace solvable.

Proof. The first item is a special case of the second, however the proof of the second case will use the first case.

(1). This is crucial and not trivial ($Q_{[k]\{i\}}$ does not have a Maltsev polymorphism, so it is not covered under the most widely used polymorphism techniques for establishing logspace solvability). Note that as $Q_{[k]\{i\}}$ is a core, we have CSP($Q_{[k]\{i\}}$) logspace equivalent to the CSP over $Q_{[k]\{i\}}$ with all unary singletons added.

Given an input digraph $H$, we first test if $H$ is satisfiable in the directed path $P_n$ of height $n$ (checking that $H$ is balanced, and of sufficiently small height). Reject if NO. Otherwise we may assume that $H$ is a single component (because it suffices to satisfy each component, and component checking has been described as logspace in earlier discussion).

Fix any vertex $u$ of $H$. It is trivial that $H$ is satisfiable in $Q_{[k]\{i\}}$ if and only it is satisfied with $u$ interpreted at some height in $Q_{[k]\{i\}}$. We will fix a possible height for $u$, and successively try new heights if this fails until all are exhausted or $H$ is found to be a Yes instance. Once we have fixed a target height $j$ for $u$, this determines the height of all other vertices in $H$ (this is the only place where $H$ being one component is used). For example, to determine the height of another vertex $v$, we may successively test for satisfaction of $H$ in $Q_{[k]}$, with $u$ constrained at fixed height $j$, and $v$ constrained at height 0, then 1, then 2 and so on, until a YES is returned. There are only $n + 1$ heights, and each is just a query to a logspace oracle. Because $H$ is balanced and connected, there is at most one height at which $v$ will produce a YES. If none is produced, then $u$ cannot interpret in $Q_{[k]}$ at height $j$, so that $j$ itself must be incremented.

In this way, we may assume within logspace that we have access to the heights of all vertices during the following argument.

Now, in any satisfying interpretation of $H$ in $Q_{[k]\{i\}}$, any vertices of the same height $j \notin \{i, i + 1\}$ are identified. We need only ensure there is no path of vertices of height $i - 1, i + 1, i + 2$. So it suffices to enumerate all 4-tuples of vertices $u_1, \ldots, u_4$, check if $u_1 \to u_2 \to u_3 \to u_4$, and if so, check that the height of $u_1$ is not $i - 1$. If it is, then reject. Otherwise accept.

(2). For a typical instance $H$ of CSP($Q_{S_i}$), select some arbitrary vertex $h$ and then, for each possible interpretation of $h$ in $Q_{S_i}$, test membership in CSP($Q_{[k]\{i\}}$) for each $i \in S$, ensuring $h$ is constrained to lie a consistent location in each.

(3). Consider some instance $H$. As above, we may assume that $H$ is connected, balanced and is of sufficiently small height. If $H$ has a solution that does not involve the amalgamated vertex, then it has a solution within one of the paths $Q_{S_i}$, in which case this can be discovered by systematically applying part (1) to each of the paths $Q_{S_1}, Q_{S_2}, \ldots$. This is a fixed finite number of applications of part (1).

Now we consider solutions that do involve the amalgamated vertex, which we denote by $a$. (In typical instances where the lemma is used, $a$ will be either
an element of \( A \) or an element of \( R \) and the \( Q_{S_i} \) will be the paths emanating from \( a \). Without loss of generality, we will assume that \( a \) is of height 0 in the fan (rather than of height \( n \)). For each vertex \( h \in H \) we are going to first check that \( h \) can be interpreted at height 0 using the subroutine for checking height described earlier. We now consider the components of the induced subgraph of \( H \) that avoids all vertices at height 0 (given that \( h \) has been set to height 0). Each such component \( C \) is systematically tested for satisfaction in \( Q_{S_i} \) (with the lowest level vertices of \( C \) constrained to lie at height 1 in \( Q_{S_i} \)), then in \( Q_{S_2} \), and asking that it be constraint by the singleton relation \( \{a\} \). If each component \( C \) test positive for satisfaction in some \( Q_{S_i} \) (where \( i \) possibly varies), then \( H \) can be satisfied in the fan. Otherwise, replace \( h \) by the next vertex of \( H \), until either a positive check is identified, or all vertices are shown to fail, and \( H \) is discovered as a NO instance.

\[ A.3 \] Stage 1: Verification that \( G \) is balanced and a test for height.

Let \( P_n \) denote the directed path on vertices 0, 1, \ldots, \( n \): it is just \( Q_{\{1, \ldots, k\}} \). The CSP over the directed path \( P_n \) on vertices is solvable in logspace, so as an initial check, it is verified in logspace that \( G \) is balanced and of height at most \( n \), the height of \( D(A) \). Moreover, the precise height of \( G \) can be recorded, by testing for membership in \( \text{CSP}(P_{n-1}) \), \( \text{CSP}(P_{n-2}) \) and so on until some minimal height is recorded. Because singleton unary relations can be added to \( P_k \) without changing the logspace solvability of the CSP, a more refined test can be performed to determine the minimal height of any chosen vertex \( u \) (with heights starting at height 0): simply ask for satisfaction of \( G \) in \( \text{CSP}(P_{\text{hgt}(G)}) \) with the vertex \( u \) constrained by the relation \( \{0\} \), then \( \{1\} \), and so on until some minimal height is achieved. This subroutine will be performed frequently (but without further detailed discussion) throughout the remaining steps of the argument.

\[ A.4 \] Stage 2: Elimination of “short components”.

If \( G \) contains some component of height strictly less than \( n \), then we can test directly whether or not it is a YES or NO instance of \( \text{CSP}(D(A)) \) (this is explained in the next paragraph). If any are NO instances, then so is \( G \) and we can output some fixed NO instance of \( \text{CSP}(A) \). Otherwise (if all are YES instances), we may simply ignore these short components. If \( G \) itself has height less than \( n \), then instead of ignoring all components of \( G \) we simply output some fixed YES instance of \( \text{CSP}(A) \), completing the reduction.

The process for testing membership in \( \text{CSP}(D(A)) \) is as follows. We consider some connected component \( H \). Now in any satisfying interpretation of \( H \) in \( D(A) \), we must either interpret within some single path \( Q_{S_i} \) connecting \( A \) to \( R \) in \( D(A) \), or at some fan of such paths emanating from some vertex in \( A \) or some vertex in \( R \). There are a fixed finite number of such subgraphs of \( D(A) \), and we may use Lemma \[ B \] for each one.

For the remainder of the algorithm we will assume that all connected components have height \( n \).
A.5 Stage 3: $\text{hgt}(G) = n$ (and all components of $G$ have height $n$).

In this case we eventually output an actual structure $B$ with the property that $G$ is a YES instance of CSP($D(A)$) if and only if $B$ is a YES instance of CSP($A$).

In fact we focus on the production of a preliminary construction $B'$ that is not specifically a relational structure, but holds all the information for constructing $B$ using undirected graph reachability checks. The output $B'$ will consist of a list of “generalised hyperedges”, that will (in step 3B) eventually become the actual hyperedges of $B$. A typical generalised hyperedge will consist of a labelled $k$ tuple $[s_1, s_2, \ldots, s_k] e$, where each $s_i$ is a set of either existing vertices of $G$, or some newly constructed vertices, and where the “label” $e$ is itself a vertex from $G$ used for book-keeping and later amalgamation of the sets of vertices. (Some generalised hyperedges will not require the label $e$.)

For the remainder of the argument, an internal component of $G$ means a connected component of the induced subgraph of $G$ obtained by removing all vertices of height 0 and $n$. Note that we have already described that testing for height can be done in logspace. A base vertex for such a component $C$ is a vertex at height 0 that is adjacent to $C$, and a top vertex for $C$ is a level $n$ vertex adjacent to $C$. Note that an internal component may have none, or more than one base vertices, and similarly for top vertices. Every internal component must have at least one of a base vertex or a top vertex however, because we have already considered the case of “short” components in Stage 2.

We will frequently apply Lemma 9(1) to internal components $C$, to discover the smallest $S$ for which $C$ is satisfiable in $Q_S$. Once an internal component has a base or top (and it must have at least one), then in a satisfying interpretation of $G$ in $D(A)$, the component $C$ must either be satisfied within some single path, with any vertices adjacent to the base of $C$ (or to a top of $C$) being interpreted adjacent to the initial point of the path (or adjacent to the terminal point of the path, respectively). Thus we will tacitly assume that our applications of Lemma 9(1) involve suitably constraining $C$ to be consistent with this. Also note that any digraph of height $n$ is satisfiable $Q[k]$, so we will always find some smallest $S$ using our test.

Connecting paths in $D(A)$ encode positions of base level vertices in hyperedges, and we may use a check of Lemma 9(1) to determine which positions are being asserted as “filled” by any given internal component $C$ (recalling that $C$ must either have a base vertex or a top vertex, so that interpretations in paths $Q[k] \setminus \{i\}$ occur at fixed positions only). In this way, an internal component determines a subset $I_C$ of $[k]$ by

$$i \in I_C \text{ if and only if } C \text{ is satisfiable in } Q[k] \setminus \{i\}$$

(Recall, that if $C$ has a base vertex, then we consider only interpretations that would place the base vertex at $i Q[k] \setminus \{i\}$, while if $C$ has a top vertex, then we consider only interpretations that would place the base vertex at $\tau Q[k] \setminus \{i\}$.)
A.6 Stage 3A: Constructing the approximation $B'$ to $B$.

To begin with we do not output $B$ itself, but rather some approximation $B'$ to $B$. This piece of information consists of a list of “generalised hyperedges” plus a list of equalities. These generalised hyperedges consist of $k$-tuples of lists of vertex names: vertices in the same list will later be identified to create $B$, but this is a separate construction. Some hyperedges also encode some extra vertex of $G$ from which they were created. So a typical generalised hyperedge may look like $[V_1, V_2, \ldots, V_k]$, where $e$ is some vertex of $G$ (at height $n$) used for book-keeping purposes and the $V_i$ are lists consisting of some vertices of $G$ (from height 0) and some new vertices we create during the algorithm. Other hyperedges may not require the special book-keeping subscript.

Note that any new vertices created during the algorithm should be different each time (even though we often use $x$ to denote such a vertex): we should use some counter on a fixed spare piece of tape for the entire algorithm; this counter is incremented at each creation of a new variable, and its value recorded within the new vertex name. (There will be only polynomially many new variables created, so only logspace used to store this one counter.)

1 To output the generalised hyperedges. There are two causes for writing generalised hyperedges to the output: the first is due to vertices at height $n$, and the second is due to vertices at height 0 that are the base vertex for some internal component with no top vertices. The generalised hyperedges will be written in such a way to record some extra information that will be used for identifications.

For each vertex $e$ at height $n$ we will need to output a generalised hyperedge, however there may be many different vertices placed at a given position: these vertices will later be identified. We will also record in the encoding that the generalised hyperedge is created from vertex $e$. The following process is performed for each height $n$ vertex $e$ and in each case, we perform the following process for $i = 1$ to $k$.

1.1 Systematically search for an internal component $C$ in which $i \in \Gamma(C)$ and for which $e$ is a top vertex. These searches involve the following: we systematically search through all vertices of $G$ until some $u$ is found to be undirected reachable from $e$ amongst vertices not at height 0 or $n$. To avoid unnecessary duplication, we may also check that $u$ does not lie in the same internal component as some earlier vertex (in which case we may ignore $u$: this internal component has already been considered). Then we proceed to systematically search through all vertices of $G$ to identify the internal component $C_u$ of $G$ containing $u$. This component is then checked using Lemma 8 for whether $i \in \Gamma(C_u)$. If $i \in \Gamma(C)$ we go to substep 1.1.1. If $i \notin \Gamma(C_u)$ we increment $u$ and continue our search for an internal component $C$ with $e$ as top and with $i \in \Gamma(C)$. If no such components are encountered we proceed to substep 1.1.2.

1.1.1 We have identified an internal component $C$ with $i \in \Gamma(C)$ and for which $e$ is a top vertex. If $C$ has base vertices $b_1, \ldots, b_j$ then these will be written to the vertex set for the $i^{th}$ coordinate of the output hyperedge.
If $C$ does not have base vertices, then we will create some new vertex $x$ and write the vertex set $\{x\}$ to the $i^{th}$ coordinate.

1.1.2 No internal component $C$ is found with $i \in \Gamma(C)$ and for which $e$ is a top vertex. In this case only one vertex will appear in the vertex set for coordinate $i$ of this generalised hyperedge: a new vertex $x$.

1.2 Generalised hyperedges may also be created because of level 0 vertices. The following is performed for each level 0 vertex $b$ and for each internal component $C$ for which $b$ is the base vertex and such that $C$ has no top vertex. (If none are found there is nothing to do and no generalised hyperedge is written at step 1.2 for $b$.) We create a generalised hyperedge by performing the following checks for $i = 1, \ldots, k$.

1.2.1 If $i \in \Gamma(C)$ then $\{b\}$ is placed in position $i$ of the generalised hyperedge, 1.2.2 If $i \notin \Gamma(C)$ then a new vertex $x$ is created and $\{x\}$ is placed in position $i$ of the generalised hyperedge.

2 Finally we output information that will later be used to find certain vertices that be forced to be identified in any satisfying interpretation of $G$.

2.1 For each pair of distinct height $n$ vertices $e, f$, if $e$ and $f$ are the top vertex for the same internal component, then we write $e = f$ to the output tape.

2.2 For each pair of distinct height 0 vertices $b, c$, if $b$ and $c$ are base vertices for the same internal component we write $b = c$.

This completes the construction of $B'$. There are clearly further identifications that will be forced: for example, if $b$ appears in the list of position $i$ vertices for some generalised hyperedge $e$, and $c$ appears in the list of position $i$ vertices for some generalised hyperedge $f$, and if $e = f$ has been output, then we must have $b$ and $c$ identified. Accounting for these is stage 3B.

A.7 Stage 3B: construction of $B$.

We now need to construct $B$ from the list of generalised hyperedges and equalities. We describe this as a separate logspace process, and use the fact that a composition of logspace constructions is itself logspace. The actual vertices of $B$ will consist of sets of the vertices currently stated. If desired, this could be simplified as a later separate logspace process (perhaps by using only the earliest vertex from each set). Currently the input consists of generalised hyperedges where the entry in a given position is a set of vertices of $G$ or new vertices. To create $B$ we only need to amalgamate these sets, also taking into account the equality constraints.

In the following, a “vertex” refers to an element of some set within the position of some hyperedge. A “vertex set” consists of a set of vertices. The actual vertices of $B$ will be vertex sets, produced from those appearing within $B'$ by amalgamation.

The amalgamation process involves considering an undirected graph on the vertices, which we refer to as the equality graph. The undirected edges of the equality graph arise in several different ways. There will be an undirected edge
from a vertex \(a\) to a vertex \(b\) if \(a\) and \(b\) lie within the same vertex set somewhere in the input list, or if \(a = b\) is written as an equality constraint. For a further kind reason for an undirected edge, recall that a hyperedge created from a height \(n\) vertex \(e\) records the vertex \(e\) in its description. The role of this is just so that \(e\) acts as a place holder, and we now use this. There will be an undirected edge from \(a\) and \(b\) if they appear in vertex sets at position \(i\) of two generalised hyperedges, either with the same label \(e\), or with different labels \(e, f\) but where \(e = f\) appears in the input. There is a logspace check for these undirected edges, and using the logspace solvability of undirected graph reachability we may determine if two vertices identified within our list of generalised hyperedges are connected in the equality graph, all within logspace.

For each vertex \(u\) we first check if there is some lexicographically earlier vertex \(v\) for which \(u\) and \(v\) are connected in the equality graph. If an earlier vertex is discovered, then we ignore \(u\) and continue to the next vertex. Otherwise, if no earlier vertex is discovered, we proceed to write down the vertex set of the component of the equality graph containing \(u\). For each \(v\) lexicographically later than \(u\), we check whether \(v\) is reachable from \(u\) (in the equality graph) and if so include it in vertex set of \(u\). For the actual hyperedges of \(B\) we may simply write the existing generalised hyperedges (removing the book-keeping subscript), which can be read in the following way. A vertex set \(U\) appears in the \(i\)th position of a hyperedge \(E\) if the intersection of \(U\) with the vertices listed for position \(i\) in \(E\) is nontrivial. Some hyperedges may be repeated in this output and obviously this could also be neated by following with a totally new logspace reduction (even to an adjacency matrix).

A.8 If and only if.

Any homomorphism \(\phi\) from \(B\) (the amalgamated “vertex sets”) into \(A\), determines a function \(\Phi\) from the height 0 vertices of \(G\) to the height 0 vertices of \(\mathcal{D}(A)\). The construction of the hyperedges of \(B\) exactly reflects the satisfiability of the internal components of \(G\), so that the function \(\Phi\) extends to cover all of \(G\) (this ignores any “short components” that we considered directly in stage 2A: but they cannot have been NO instances, as otherwise \(B\) was already created in stage 2 to be some fixed NO instance).

The converse is also true. First, we only grouped vertices together in vertex sets and their later amalgamation if they were forced to be identified in any possible interpretation in \(\mathcal{D}(A)\). So any homomorphism \(f\) from \(G\) to \(\mathcal{D}(A)\) determines a function \(F\) from the “vertices” of \(B\) to the vertices of \(A\). The hyperedges of \(B\) were determined by the internal components of \(G\), which \(f\) is satisfying within the encoding (in \(\mathcal{D}(A)\)) of the hyperedges of \(A\). So the \(B\) hyperedges are preserved by \(F\).

A.9 An example

The following diagram depicts a reasonably general instance \(G\) of CSP(\(\mathcal{D}(A)\)) in the case that \(A\) itself is a digraph, so that \(k = 2\). We are considering stage 3,
so that \( G \) is a single connected digraph of height 4. The vertices at height 0 are \( b_1, \ldots, b_6 \), and the vertices at height 4 are \( e_1, \ldots, e_4 \). The shaded regions depict internal components: each is labelled by a subset of \( \{1, 2\} \), depicting \( \Gamma(C) \).

Let us examine how Stage 3A proceeds. We arrive at the first height 4 vertex \( e_1 \). For \( i = 1 \), discover no internal components with \( e_1 \) as the top, and with 1 \( \in \Gamma(C) \) (both have \( \Gamma(C) = \emptyset \), so we be in case 1.1.2) and therefore return \( \{x_1\} \) for the vertex set in the coordinate 1. For \( i = 2 \), we have the same outcome, so the edge that is generalised hyperedge that is actually written is \( \{x_1\}, \{x_2\} \).

Then we proceed to the next height 4 vertex \( e_2 \). We encounter just one internal component \( C \) with \( 1 \in C \), and its base vertices are \( \{b_2\} \) (so this is in case 1.1.1). For \( i = 2 \) we we also find just one internal component whose \( \Gamma \) value contains 2, and it has \( \{b_3\} \) as the base vertices (also case 1.1.1). The generalised hyperedge \( \{b_2\}, \{b_3\} \) is written.

For \( e_3 \) and \( i = 1 \) we encounter two internal components producing base vertices. We find \( b_2 \) as the only base vertex of the first, and \( b_4 \) for the second (case 1.1.1), so the first coordinate of the generalised hyperedge is \( \{b_2, b_4\} \). For \( i = 2 \), no internal components yield a base vertex (case 1.1.2), so we output \( \{x_3\} \). The actual generalised hyperedge written is \( \{b_2, b_4\}, \{x_3\} \).

The vertex \( e_4 \) similar results in the generalised hyperedge \( \{b_4, x_4\}, \{x_5\} \).

This completes step 1.1 and we continue with step 1.2. We discover the height 0 vertex \( b_1 \) as the base of an internal component \( C \) with no top. We find \( 1 \notin \Gamma(C) \), so \( \{x_6\} \) is written to the first coordinate of a generalised hyperedge (step 1.2.2 for \( i = 1 \)). For \( i = 2 \) we find \( 2 \in \Gamma(C) \) so return \( \{b_1\} \) for the second coordinate. The actual output written is \( \{x_6\}, \{b_1\} \) (there are no subscripts to hyperedges from step 1.2). Level 0 vertices \( b_4 \) and \( b_5 \) also lead to the creation of generalised hyperedges. The overall output after the completion of steps 1.1
and 1.2 is

\[
\begin{align*}
\{x_1, x_2\} e_1 & \quad (\text{from } e_1, \text{ step } 1.1) \\
\{b_2, b_3\} e_2 & \quad (\text{from } e_2, \text{ step } 1.1) \\
\{b_2, b_4\}, \{x_3\} e_3 & \quad (\text{from } e_3, \text{ step } 1.1) \\
\{b_4, x_4\}, \{x_5\} e_4 & \quad (\text{from } e_4, \text{ step } 1.1) \\
\{x_6\}, \{b_1\} & \quad (\text{from } b_1, \text{ step } 1.2) \\
\{x_7\}, \{x_8\} & \quad (\text{from } b_4, \text{ step } 1.2) \\
\{x_9\}, \{x_{10}\} & \quad (\text{from } b_5, \text{ step } 1.2) \\
\{b_3\}, \{x_{11}\} & \quad (\text{from } b_5, \text{ step } 1.2) \\
\{b_6\}, \{x_{12}\} & \quad (\text{from } b_6, \text{ step } 1.2)
\end{align*}
\]

For step 2 of the algorithm, we output the following equalities

\[
e_1 = e_2, \quad (\text{from } 2.1) \\
b_4 = b_5 = b_6 \quad (\text{from } 2.2)
\]

This completes Stage 3A: the list just given is \(B'\). We note that hyperedges such as \([\{x_7\}, \{x_8\}]\) will be no hinderance to satisfiability of \(B\) in \(A\), and we could word our algorithm to avoid writing these altogether.

Stage 3B then produces the digraph \(B\) with hyperedges

\[
\begin{align*}
\{b_2, b_4, b_5, b_6, x_1, x_4\}, \{b_3, x_2\} \\
\{b_2, b_4, b_5, b_6, x_1, x_4\}, \{b_3, x_2\} \\
\{b_2, b_4, b_5, b_6, x_1, x_4\}, \{x_3\} \\
\{b_2, b_4, b_5, b_6, x_1, x_4\}, \{x_3\} \\
\{x_6\}, \{b_1\} \\
\{x_7\}, \{x_8\} \\
\{x_9\}, \{x_{10}\} \\
\{b_2, b_4, b_5, b_6, x_1, x_4\}, \{x_{11}\} \\
\{b_2, b_4, b_5, b_6, x_1, x_4\}, \{x_{12}\}
\end{align*}
\]

Which is a digraph with 12 vertices (namely, the 12 different sets of vertices appearing in hyperedges).

The algorithm itself is the composite of stage 3A and stage 3B: we have used the fact that a composition of logspace reductions is logspace.

### B Preserving Maltsev conditions

Given a finite relational structure \(A\), we are interested in the following question: How similar are the algebras of polymorphisms of \(A\) and \(D(A)\)? More precisely, which equational properties (or Maltsev conditions) do they share? In this section we give a partial answer to this question.
B.1 The result

We start by an overview and statement of the main result of this section. All the new notions are introduced later, in Subsection B.2.

Since $A$ is pp-definable from the digraph $D(A)$ (see Lemma 3), it follows that $A$ and $R$ are subuniverses of $D(A)$ and for any $f \in \text{Pol}(D(A))$, the restriction $f|_A$ is a polymorphism of $A$. Consequently, for any set of identities $\Sigma$,

$$D(A) \models \Sigma \text{ implies that } A \models \Sigma.$$  

Theorem 1 (or, more precisely, Lemma 8) already shows that the above implication is in fact an equivalence, if $\Sigma$ is a Maltsev condition describing having a Taylor term, a WNU operation, or generating a congruence meet semidistributive variety. The list of such conditions, for which there is an equivalence, can be greatly expanded.

**Theorem 5.** Let $A$ be a finite relational structure. Let $\Sigma$ be a linear idempotent set of identities such that the algebra of polymorphisms of the zigzag satisfies $\Sigma$ and each identity in $\Sigma$ is either balanced or contains at most two variables. Then

$$D(A) \models \Sigma \text{ if and only if } A \models \Sigma.$$  

The following corollary lists some popular properties that can be expressed as sets of identities satisfying the above assumptions. Indeed, they include many commonly encountered Maltsev conditions.

**Corollary 2.** Let $A$ be a finite relational structure. Then each of the following hold equivalently on $A$ and $D(A)$.

1. Taylor polymorphism or equivalently weak near-unanimity (WNU) polymorphism \([20]\) or equivalently cyclic polymorphism \([9]\) (conjectured to be equivalent to being tractable if $A$ is a core \([2]\));
2. Polymorphisms witnessing $SD(\land)$ (equivalent to bounded width \([6]\));
3. (for $k \geq 4$) $k$-ary edge polymorphism (equivalent to few subpowers \([5], [13]\));
4. $k$-ary near-unanimity polymorphism (equivalent to strict width \([2]\));
5. totally symmetric idempotent (TSI) polymorphisms of all arities (equivalent to width 1 \([5], [2]\));
6. Hobby-McKenzie polymorphisms (equivalent to the corresponding variety satisfying a non-trivial congruence lattice identity);
7. Gumm polymorphisms witnessing congruence modularity;
8. Jónsson polymorphisms witnessing congruence distributivity;
9. polymorphisms witnessing $SD(\lor)$ (conjectured to be equivalent to NL \([19]\));
10. (for $n \geq 3$) polymorphisms witnessing congruence $n$-permutability (together with (9) is conjectured to be equivalent to L \([19]\)).

Note that the list includes all six conditions for omitting types in the sense of Tame Congruence Theory \([12]\).

We will prove Theorem 5 and Corollary 2 in subsection B.4.
B.2 Preliminaries

Given a finite relational structure $A$, let $\text{Pol}_A$ denote the set of all polymorphisms of $A$. The algebra of polymorphisms of $A$ is simply the algebra with the same universe whose operations are all polymorphisms of $A$. A subset $B \subseteq A$ is a subuniverse of $A$, denoted by $B \leq A$, if it is a subuniverse of the algebra of polymorphisms of $A$, i.e., it is closed under all $f \in \text{Pol}_A$.

An (operational) signature is a (possibly infinite) set of operation symbols with arities assigned to them. By an identity we mean an expression $u \approx v$ where $u$ and $v$ are terms in some signature. An identity $u \approx v$ is linear if both $u$ and $v$ involve at most one occurrence of an operation symbol (e.g. $f(x, y) \approx g(x)$, or $h(x, y, x) \approx x$); and balanced if the sets of variables occurring in $u$ and in $v$ are the same (e.g. $f(x, x, y) \approx g(y, x, x)$).

A set of identities $\Sigma$ is linear if it contains only linear identities; balanced if all the identities in $\Sigma$ are balanced; and idempotent if for each operation symbol $f$ appearing in an identity of $\Sigma$, the identity $f(x, x, \ldots, x) \approx x$ is in $\Sigma$. For example, the identities $p(y, x, x) \approx y$, $p(x, x, y) \approx y$, $p(x, x, x) \approx x$ (defining the so called Maltsev term) form a linear idempotent set of identities which is not balanced.

The strong Maltsev condition, a notion usual in universal algebra, can be defined in this context as a finite set of identities. A Maltsev condition is an increasing chain of strong Maltsev conditions, ordered by syntactical consequence. In all results from this section, “set of identities” can be replaced with “Maltsev condition”.

Let $\Sigma$ be a set of identities in a signature with operation symbols $F = \{ f_\lambda \mid \lambda \in \Lambda \}$. We say that a relational structure $A$ satisfies $\Sigma$ (and write $A \models \Sigma$), if for every $\lambda \in \Lambda$ there is a polymorphism $f_\lambda^A \in \text{Pol}_A$ such that the identities in $\Sigma$ hold universally in $A$ when for each $\lambda \in \Lambda$ the symbol $f_\lambda$ is interpreted as $f_\lambda^A$.

B.3 Polymorphisms of the zigzag

In the following, let $Z$ be a zigzag with vertices 00, 01, 10 and 11, i.e., the oriented path 00 $\rightarrow$ 01 $\leftarrow$ 10 $\rightarrow$ 11. The digraph $Z$ satisfies most of the important Maltsev conditions (an exception being the Maltsev term). We need the following.

**Lemma 10.** The digraph $Z$ satisfies any set of identities which holds in the variety of distributive lattices.

**Proof.** Define the operations $\land$ and $\lor$ in the following way. Let $x \land y$ by the vertex from $\{x, y\}$ closer to 00 and $x \lor y$ the vertex closer to 11. These two operations are polymorphisms and they form a distributive lattice; the rest follows immediately.

**Corollary 3.** The digraph $Z$ has a majority polymorphism, and it satisfies any balanced set of identities.

---

2 We can relax this condition and require the identity $f(x, x, \ldots, x) \approx x$ only to be a consequence of identities in $\Sigma$. 
Proof. The ternary operation defined by \( m(x, y, z) = (x \land y) \lor (x \land z) \lor (y \land z) \) (the \text{median}) is a majority polymorphism.

Let \( \Sigma \) be a balanced set of identities. For every operation symbol \( f \) (say \( k \)-ary) occurring in \( \Sigma \), we define \( f^\Sigma(x_1, \ldots, x_k) = \bigwedge_{i=1}^{k} x_i \). It is easy to check that \( f^\Sigma \) is a polymorphism and that such a construction satisfies any balanced identity.

**Lemma 11.** The digraph \( \mathbb{Z} \) is congruence 3-permutable.

**Proof.** The ternary polymorphisms \( p \) and \( q \) witnessing 3-permutability can be defined as follows:

\[
p_1(x, y, z) = \begin{cases} 01 & \text{if } 01 \in \{x, y, z\} \text{ and } y \neq z, \\ 10 & \text{if } 10 \in \{x, y, z\} \text{ and } 01 \notin \{x, y, z\} \text{ and } y \neq z, \\ x & \text{otherwise}, \end{cases}
\]

\[
p_2(x, y, z) = \begin{cases} 01 & \text{if } 01 \in \{x, y, z\} \text{ and } x \neq y, \\ 10 & \text{if } 10 \in \{x, y, z\} \text{ and } 01 \notin \{x, y, z\} \text{ and } x \neq y, \\ z & \text{if } x = y, \\ x & \text{otherwise} \end{cases}
\]

If we have triples \( (a_1, a_2, a_3) \) and \( (b_1, b_2, b_3) \) such that \( a_i \rightarrow b_i \), for \( i = 1, 2, 3 \), then \( \{a_1, a_2, a_3\} \subseteq \{00, 10\} \) and \( \{b_1, b_2, b_3\} \subseteq \{01, 11\} \). Clearly the result follows if \( p_1(a_1, a_2, a_3) = 10 \) since \( p_1(b_1, b_2, b_3) \in \{01, 11\} \). The only case where \( p_1(a_1, a_2, a_3) \neq 10 \) is when \( a_1 = 00 \) and \( a_2 = a_3 = 10 \) in which case \( b_1 = 01 \) and \( b_2, b_3 \in \{01, 11\} \). In any case, \( p_1(b_1, b_2, b_3) = 01 \). A similar argument works for \( p_2 \), and hence \( p_1 \) and \( p_2 \) are polymorphisms.

The identities \( p_1(x, y, y) \approx x \) and \( p_2(x, x, y) \approx y \) follow directly from the definitions of \( p_1 \) and \( p_2 \). To prove the equation \( p_1(x, x, y) \approx p_2(x, y, y) \) we can assume that \( x \neq y \). If \( 01 \) or \( 10 \) are in \( \{x, y\} \), then \( p_1 \) and \( p_2 \) agree. If not, then \( p_1(x, x, y) = p_2(x, y, y) = x \).

### B.4 Proofs

In this subsection we prove Theorem 3 and Corollary 2. Fix a finite relational structure, without loss of generality we can assume that \( \mathcal{A} = (A; R) \), where \( R \) is a \( k \)-ary relation. Fix an arbitrary linear order \( \preceq \) on the set \( E = A \times R \). We define the mapping \( \epsilon : \mathcal{D}(\mathcal{A}) \rightarrow E \) by setting \( \epsilon(x) \) to be the \( \preceq \)-minimal \( \epsilon \in E \) such that \( x \in \mathbb{P}_x \).

We will use the following linear order \( \sqsubseteq \) on the vertices of \( \mathcal{D}(\mathcal{A}) \): Put \( x \sqsubseteq y \) if either of the following is true:

- \( \text{lvl}(x) < \text{lvl}(y) \).
- \( \text{lvl}(x) = \text{lvl}(y) \) and \( \epsilon(x) < \epsilon(y) \).
- \( \text{lvl}(x) = \text{lvl}(y) \), \( \epsilon(x) = \epsilon(y) \), and \( x \) is closer to \( \epsilon x \) than \( y \).

We will need the following easy fact, we leave the verification to the reader.
Observation. Let \( C \) and \( D \) be subsets of \( \mathcal{D}(\mathcal{A}) \setminus (A \cup R) \) such that

- for every \( x \in C \) there exists \( y \in D \) such that \( x \rightarrow y \), and
- for every \( y \in D \) there exists \( x \in C \) such that \( x \rightarrow y \).

If \( c \) and \( d \) are the \( \sqsubseteq \)-minimal elements of \( C \) and \( D \), respectively, then \( c \rightarrow d \).

Note that the subset \( \{00, 10\} \) of the zigzag \( Z \) is closed under all polymorphisms of \( Z \) (as it is pp-definable using the formula \((\exists y)(x \rightarrow y)\), see Lemma \[2\]. The same holds for \( \{01, 11\} \). We use this fact in the construction below, namely in case 3.

**Proof of Theorem 5.** Let \( \Sigma \) be a set of identities in operation symbols \( \{f_\lambda : \lambda \in A\} \) satisfying the assumptions. Let \( \{f_\lambda^A \mid \lambda \in A\} \) and \( \{f_\lambda^Z \mid \lambda \in A\} \) be interpretations of the operation symbols witnessing \( \mathcal{A} \models \Sigma \) and \( Z \models \Sigma \), respectively.

We will now define polymorphisms \( \{f_\lambda^D \mid \lambda \in A\} \) witnessing that \( \mathcal{D}(\mathcal{A}) \models \Sigma \). Fix \( \lambda \in A \) and assume that \( f_\lambda \) is \( m \)-ary. We split the definition of \( f_\lambda^D \) into several cases and subcases. Let \( c \in \mathcal{D}(\mathcal{A})^m \).

**Case 1.** \( c \in A^m \cup R^m \).

1a If \( c \in A^m \), we define \( f_\lambda^D(c) = f_\lambda^A(c) \).

1b If \( c \in R^m \), we define \( f_\lambda^D(c) = (f_\lambda^A)(c) \).

**Case 2.** \( c \in \Delta_m \setminus (A^m \cup R^m) \).

Let \( e_i = e(c_i) \) for \( i \in [m] \) and \( e = (f_\lambda^A)(k+1)(e_1, \ldots, e_m) \). Let \( l \in [k] \) be minimal such that \( e_i \in P_{e,l} \) for all \( i \in [m] \). (Its existence is guaranteed by Lemma \[4\] (1).)

2a If \( P_{e,l} \) is a single edge, then we define \( f_\lambda^D(c) \) to be the vertex from \( P_{e,l} \) having the same level as all the \( c_i \)'s.

If \( P_{e,l} \) is a zigzag, then at least one of the \( P_{e,l} \)’s is a zigzag as well. (This follows from the construction of \( \mathcal{D}(\mathcal{A}) \) and the fact that \( f_\lambda^D \) preserves \( R \).) For every \( i \in [m] \) such that \( P_{e,l} \) is a zigzag let \( \Phi : P_{e,l} \rightarrow Z \) be the (unique) isomorphism. Let \( \Phi \) denote the isomorphism from \( P_{e,l} \) to \( Z \).

2b If all of the \( P_{e,l} \)’s are zigzags, then the value of \( f_\lambda^D \) is defined as follows:

\[
f_\lambda^D(c) = \Phi^{-1}(f_\lambda^Z(\Phi_1(c_1), \ldots, \Phi_m(c_m))).
\]

2c Else, we define \( f_\lambda^D(c) \) to be the \( \sqsubseteq \)-minimal element from the set

\[
\{\Phi^{-1}(\Phi_i(c_i)) \mid P_{e,l} \text{ is a zigzag}\}.
\]

**Case 3.** \( c \notin \Delta_m \).

3a If \( |\{\text{lv}(c_i) \mid i \in [m]\}| = 1 \) and \( |\{\text{e}(c_i) \mid i \in [m]\}| = 2 \), say that \( \text{e}(c_i) \in \{e, e'\} \) for all \( i \in [m] \) and \( e \prec e' \), then let \( \Phi : \{e, e'\} \rightarrow \{00, 10\} \) be the bijection mapping
e to 00 and e' to 10. We define $f_{\lambda}^{D(\lambda)}(c)$ to be the $\sqsubseteq$-minimal element from the set
\[
\{c_i : e(c_i) = \Phi^{-1}(f_{\lambda}^{Z}(\Phi(e(c_1)), \ldots, \Phi(e(c_m))))\}.
\]

3b. If $\{|\ell|(c_i) | i \in [m]| = 2$, say that $\ell(c_i) \in \{l, l'\}$ for all $i \in [m]$ and $l < l'$, then let $\Phi : \{l, l'\} \to \{00, 10\}$ be the bijection mapping $l$ to 00 and $l'$ to 10. We define $f_{\lambda}^{D(\lambda)}(c)$ to be the $\sqsubseteq$-minimal element from the set
\[
\{c_i : \ell(c_i) = \Phi^{-1}(f_{\lambda}^{Z}(\Phi(\ell(c_1)), \ldots, \Phi(\ell(c_m))))\}.
\]

3c. In all other cases we define $f_{\lambda}^{D(\lambda)}(c)$ to be the $\sqsubseteq$-minimal element from the set $\{c_1, \ldots, c_m\}$.

We need to verify that the operations we constructed are polymorphisms and that they satisfy all identities from $\Sigma$. We divide the proof into three claims.

Claim. For every $\lambda \in \Lambda$, $f_{\lambda}^{D(\lambda)}$ is a polymorphism of $D(\Lambda)$.

Proof. Let $c \to d$ be an edge in $D(\Lambda)^m$. Note that $c \in \Delta_m$ if and only if $d \in \Delta_m$. The tuple $c$ cannot fall under subcase 1b or under 3a, because these cases both prevent an outgoing edge from $c$ (see Lemma 7 (ii) for why this is true for 3a).

We first consider the situation where $c$ falls under subcase 1a of the definition. Then $d$ falls under case 2 and, moreover, $d_i = \ell P_{e,1}$ for all $i \in [m]$. It is not hard to verify that $f_{\lambda}^{D(\lambda)}(d) = \ell P_{e,1}$. (In subcase 2b we need the fact that $f_{\lambda}^{Z}$ is idempotent.) Therefore $f_{\lambda}^{D(\lambda)}(c) = \ell P_{e} \to \ell P_{e,1} = f_{\lambda}^{D(\lambda)}(d)$ and the polymorphism condition holds. The argument is similar when $d$ falls under subcase 1b (and so $c$ under case 2).

Consider now that $c$ falls under case 2. Then $d$ falls either under subcase 1b, which was handled in the above paragraph, or also under case 2. The choice of $c_1, \ldots, c_m$ and $e$ is the same for both $c$ and $d$. By Lemma 7 (i), there exists $l \in [k]$ such that $c_i, d_i \in P_{e,l}$ for all $i \in [m]$.

If the value of $l$ is also the same for both $c$ and $d$, then $f_{\lambda}^{D(\lambda)}(c) \to f_{\lambda}^{D(\lambda)}(d)$ follows easily; in subcase 2a trivially, in 2b from the fact that $f_{\lambda}^{Z}$ is a polymorphism of $Z$ and in 2c from the observation about $\sqsubseteq$.

It may be the case that this $l$ is not minimal for the tuple $c$, that is, that $c_i \in P_{e,l-1}$ for all $i \in [m]$. However, it then follows that $c_i = \tau P_{e,l-1} = \ell P_{e,l}$ and thus $f_{\lambda}^{D(\lambda)}(c) = \ell P_{e,l}$ (again, using idempotency of $f_{\lambda}^{Z}$ in subcase 2b).

Knowing this allows for the same argument as in the above paragraph to apply.

If $c$ falls under one of the subcases 3b or 3c, then $d$ falls under the same subcase. The fact that $f_{\lambda}^{D(\lambda)}(c) \to f_{\lambda}^{D(\lambda)}(d)$ follows from the above observation about $\sqsubseteq$. (In subcase 3b our construction “chooses” either the lower or the higher level, and it is easy to see that this choice is the same for both $c$ and $d$.)

Claim. The $f_{\lambda}^{D(\lambda)}$s satisfy every balanced identity from $\Sigma$. 

Proof. Let $f_\lambda(u) \approx f_\mu(v) \in \Sigma$ be a balanced identity in $s$ distinct variables $\{x_1, \ldots, x_s\}$. Let $E : \{x_1, \ldots, x_s\} \rightarrow D(A)$ be some evaluation of the variables. Let $u^E$ and $v^E$ denote the corresponding evaluation of these tuples.

Note that both $f_\lambda^{D(A)}(u^E)$ and $f_\mu^{D(A)}(v^E)$ fall under the same subcase of the definition. The subcase to be applied depends only on the set of elements occurring in the input tuple, except for case two, where the choice of $e$ matters as well. However, since the identity holds in $A$, this $e$ is the same for both $u^E$ and $v^E$. Therefore, to verify that $f_\lambda^{D(A)}(u^E) = f_\mu^{D(A)}(v^E)$, it is enough to consider the individual subcases separately.

In case 1 it follows immediately from the fact that the identity holds in $A$. In case 2 it is easily seen that both $f_\lambda^{D(A)}(u^E)$ and $f_\mu^{D(A)}(v^E)$ have the same level, and since the identity holds in $A$, they also lie on the same path $P_{c,l}$. To see that these two elements are equal, note that in subcase 2a it is trivial, in 2b it follows directly from the fact that the identity holds in $Z$, and in 2c we use the fact that the identity is balanced: they are both $\sqsubseteq$-minimal element of the same set $\{E(x_1), \ldots, E(x_s)\}$. Similar arguments can be used in case 3. In 3a both $f_\lambda^{D(A)}(u^E)$ and $f_\mu^{D(A)}(v^E)$ are chosen from the set $\{e, e'\}$; they are the same since the identity holds in $Z$. In 3b the level is the same for both of them (since the identity holds in $Z$) and they are both $\sqsubseteq$-minimal element of the set of elements from $\{E(x_1), \ldots, E(x_s)\}$ lying on that level. In 3c both are $\sqsubseteq$-minimal element of the set $\{E(x_1), \ldots, E(x_s)\}$.

Claim. The $f_\lambda^{D(A)}$'s satisfy every identity from $\Sigma$ in at most two variables.

Proof. Balanced identities fall under the scope of the previous claim. Since $\Sigma$ is idempotent, we may without loss of generality consider only identities of the form $f_\lambda(u) \approx x$, where $u \in \{x, y\}^n$. Suppose that $x$ and $y$ evaluate to $c$ and $d$ in $D_A$, respectively, and let $c \in \{c, d\}^n$ be the evaluation of $u$. We want to prove that $f_\lambda^{D(A)}(c) \approx c$.

The tuple $c$ cannot fall into subcase 3c of the definition of $f_\lambda^{D(A)}$. If it falls into case 1, the equality follows from the fact that the identity holds in $A$ while in subcases 3a and 3b we use the fact that it holds in $Z$.

In case 2 it is easily seen that $f_\lambda^{D(A)}(c)$ lies on the same path $P_{c,l}$ as $c$ (using that the identity holds in $A$) as well as on the same level of this path. In 2a it is trivial that $f_\lambda^{D(A)}(c) = c$ while in 2b it follows from the fact that the identity holds in $Z$. If $c$ falls under subcase 2c, then $P_{c,d}, l$ must be a single edge, and thus $f_\lambda^{D(A)}(c)$ is defined to be the $\sqsubseteq$-minimal element of the one element set $\{c\}$.

Proof of Corollary 2 All items are expressible by linear idempotent sets of identities. In all items except (5) they are in at most two variables, in item (5) the defining identities are balanced. It remains to check that all these conditions are satisfied in the zigzag, which follows from Lemma 11 for item (10) and from Lemma 3 for all other items.