UNIVERSAL CENTRAL EXTENSIONS AND NON-ABELIAN TENSOR PRODUCT OF HOM-LIE–RINEHART ALGEBRAS

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Abstract. In this paper, we study universal central extensions and non-abelian tensor product of hom-Lie–Rinehart algebras. We discuss universal $\alpha$-central extensions, and the lifting of automorphisms and $\alpha$-derivations to central extensions of hom-Lie–Rinehart algebras. We deduce the lifting of automorphisms and $\alpha^k$-derivations to the central extensions in the particular case of hom-Lie algebras.

1. Introduction. The study of quantum deformations of Witt and Virasoro algebras leads to new non-associative algebra structures known as hom-algebraic structures. Such a structure first appeared as the notion of hom-Lie algebra in [HLS], where J. Hartwig and coauthors used this new type of algebras to describe $q$-deformations using $\sigma$-derivations in place of usual derivations. There is a growing interest in hom-algebraic structures because of their close relationship to discrete and deformed vector fields, and differential calculus [HLS, LS1, LS2]. Subsequently, many concepts and properties have been investigated for hom-Lie algebras. In particular, a categorical interpretation of hom-Lie algebras is given in [CG11]; representations and deformations of hom-Lie algebras are studied in [AEM, MS10, Sh12]; geometric and algebraic generalizations of hom-Lie algebras are described in [CLS, GT13, MM2]; quantization of hom-Lie algebras is studied in [Y12]; and the universal enveloping algebra of a hom-Lie algebra is constructed in [GMT, Y08].

In recent years, different hom-algebra structures corresponding to classical algebras have been investigated. Recent developments on hom-Poisson structures including the quantization (with explicit formulas of the Moyal product in [O12]) of these structures demand an appropriate formulation of hom-Lie algebroids. Hom-Lie algebroids are introduced in [GT13]. The definition of hom-Lie algebroids is not easy to figure out from the defini-
tion of Lie algebroids since there is no notion of a hom-Lie groupoid. It is widely known that there is a bijective correspondence between Lie algebroid structures on a vector bundle and Gerstenhaber algebra structures on the exterior algebra of multisections of the vector bundle. This correspondence makes it natural to define hom-Lie algebroids through the notion of hom-Gerstenhaber algebra in [GT13]. The correspondence mentioned above is a simple consequence of a more general categorical result, namely the canonical adjunction between the category of Lie–Rinehart algebras and the category of Gerstenhaber algebras. This adjunction led us to define hom-Lie–Rinehart algebras (in [MM2]) as an algebraic analogue of hom-Lie algebroids. Hom-Lie–Rinehart algebras, hom-Gerstenhaber algebras, and hom-Lie algebroids are further studied in [MM1, MM2, MM3, ZHB].

The importance of central extensions of groups and Lie algebras in physics is paramount. Centrally extended Lie algebras such as Kac–Moody algebras and Virasoro algebras play a dominant role in conformal field theory, string theory, and M-theory. There is a well-developed theory of universal central extensions for Lie algebras, Leibniz algebras, Lie–Rinehart algebras (see [Ga80], [K73] and [CGM]). In the case of classical algebras such as Lie algebras, Leibniz algebras, Lie–Rinehart algebras, etc., the composition of two central extensions is a central extension. This result does not hold in the case of hom-Lie algebras, which leads to $\alpha$-central extensions in [CIP]. Thus, the classical results in the theory of universal central extensions of Lie algebras do not follow for hom-Lie algebras.

In this work, we develop a theory of universal central extensions for hom-Lie–Rinehart algebras and discuss the lifting of automorphisms and $\alpha$-derivations of hom-Lie–Rinehart algebras to central extensions. In this process, one can also deduce such a lifting of automorphisms and $\alpha^k$-derivations (defined in [Sh12]) to the central extensions of hom-Lie algebras.

A non-abelian tensor product of Lie algebras was introduced in [El91] and its relationship with universal central extensions of Lie algebras was described. Later on, this study was extended to Leibniz algebras, Lie–Rinehart algebras, hom-Lie algebras and hom-Leibniz algebras in [Gn99], [CGM], [CKP1] and [CKP2], respectively.

In this paper, we define a non-abelian tensor product for hom-Lie–Rinehart algebras and discuss its properties. We also express universal central extensions and universal $\alpha$-central extensions of hom-Lie–Rinehart algebras in terms of the non-abelian tensor product.

In Section 2, we recall some basic notions for hom-Lie–Rinehart algebras. Then in Section 3, we discuss universal central extensions of a hom-Lie–Rinehart algebra. We prove that for a perfect hom-Lie–Rinehart algebra, there exists a universal central extension. In Section 4, we consider $\alpha$-central
extensions and universal $\alpha$-central extensions. The lifting of automorphisms and $\alpha$-derivations is discussed in Section 5. In the last section, we discuss the non-abelian tensor product of hom-Lie–Rinehart algebras and describe universal ($\alpha$-) central extensions in terms of this tensor product.

2. Preliminaries on hom-Lie–Rinehart algebras. In this section, in order to fix notations, we recall basic definitions concerning hom-algebra structures from [MM2, HLS, MS08, Sh12]. Let $R$ be a commutative ring with unit. We consider all modules, algebras and their tensor products over the ring $R$ and all linear maps to be $R$-linear unless otherwise stated.

A hom-Lie algebra is a triplet $(L, [-, -], \alpha)$ where $L$ is an $R$-module equipped with a linear map $[-, -]: L \wedge L \to L$ and a linear map $\alpha: L \to L$ such that

- $\alpha[x, y] = [\alpha(x), \alpha(y)];$
- $[\alpha(x), [y, z]] + [\alpha(y), [z, x]] + [\alpha(z), [x, y]] = 0$

for all $x, y, z \in L$. If the structure map $\alpha$ is an automorphism of the $R$-module $L$, then the hom-Lie algebra $(L, [-, -], \alpha)$ is called regular.

A representation of a hom-Lie algebra $(L, [-, -], \alpha)$ on an $R$-module $V$ is a pair $(\theta, \beta)$ of $R$-linear maps $\theta: L \to gl(V)$ and $\beta: V \to V$ such that

- $\theta(\alpha(x)) \circ \beta = \beta \circ \theta(x);$
- $\theta([x, y]) \circ \beta = \theta(\alpha(x)) \circ \theta(y) - \theta(\alpha(y)) \circ \theta(x)$

for all $x, y \in L$.

**Definition 2.1.** Given an associative commutative algebra $A$, an $A$-module $M$ and an algebra endomorphism $\phi: A \to A$, we call an $R$-linear map $\delta: A \to M$ a $\phi$-derivation of $A$ into $M$ if it satisfies the following twisted derivation rule:

$$\delta(ab) = \delta(a)\delta(b) + \phi(b)\delta(a) \quad \text{for all } a, b \in A.$$ 

Let $\text{Der}_{\phi}(A)$ denote the set of all $\phi$-derivations $\delta: A \to A$.

2.1. Hom-Lie–Rinehart algebras

**Definition 2.2.** A tuple $(A, L, [-, -]_L, \phi, \alpha_L, \rho_L)$ is called a hom-Lie–Rinehart algebra over $(A, \phi)$ if $A$ is an associative commutative algebra, $L$ is an $A$-module, $[-, -]_L: L \times L \to L$ is a skew-symmetric bilinear map, $\phi: A \to A$ is an algebra homomorphism, $\alpha_L: L \to L$ is a linear map satisfying $\alpha_L([x, y]_L) = [\alpha_L(x), \alpha_L(y)]_L$, and $\rho_L: L \to \text{Der}_{\phi}(A)$ is an $R$-linear map such that the following conditions hold:

1. $(L, [-, -]_L, \alpha_L)$ is a hom-Lie algebra;
2. $\alpha_L(a.x) = \phi(a).\alpha_L(x)$ for all $a \in A$ and $x \in L$;
3. $(\rho_L, \phi)$ is a representation of $(L, [-, -]_L, \alpha_L)$ on $A$;
(4) $\rho_L(a.x) = \phi(a).\rho_L(x)$ for all $a \in A$ and $x \in L$;
(5) $[x,a.y]_L = \phi(a)[x,y]_L + \rho_L(x)(a)\alpha_L(y)$ for all $a \in A$ and $x, y \in L$.

If we take $\alpha_L = \text{id}_L$ in the above definition, then $\phi = \text{id}_A$ and the hom-Lie–Rinehart algebra $(A, L, [-, -]_L, \phi, \alpha_L, \rho_L)$ is a Lie–Rinehart algebra $L$ over $A$.

For simplicity, we denote the hom-Lie–Rinehart algebra $(A, L, [-, -]_L, \phi, \alpha_L, \rho_L)$ by $(\mathcal{L}, \alpha_L)$.

**Example 2.3.** A hom-Lie algebra $(L, [-, -]_L, \alpha_L)$ structure over an $R$-module $L$ gives the hom-Lie–Rinehart algebra $(A, L, [-, -]_L, \phi, \alpha_L, \rho_L)$ with $A = R$, the algebra morphism $\phi = \text{id}_R$ and the trivial action of $L$ on $R$.

**Example 2.4.** Any hom-Lie algebroid $(A, \phi, [-, -], \rho, \alpha)$ over a smooth manifold $M$ gives a hom-Lie–Rinehart algebra $(C^\infty(M), \Gamma A, [-, -], \phi^*, \alpha, \rho)$ over $(C^\infty(M), \phi^*)$, where $\Gamma A$ is the space of sections of the underlying vector bundle $A$ over $M$, and the algebra homomorphism $\phi^* : C^\infty(M) \to C^\infty(M)$ is induced by the smooth map $\phi : M \to M$.

**Example 2.5.** If we consider a Lie–Rinehart algebra $L$ over $A$ along with an endomorphism

$$(\phi, \alpha) : (A, L) \to (A, L)$$

in the category of Lie–Rinehart algebras then $(A, L, [-, -]_\alpha, \phi, \alpha, \rho_\phi)$ is a hom-Lie–Rinehart algebra, said to be “obtained by composition”, where

(1) $[x,y]_\alpha = \alpha[x,y]$ for $x, y \in L$;
(2) $\rho_\phi(x)(a) = \phi(\rho(x)(a))$ for $x \in L$ and $a \in A$.

**Example 2.6.** Suppose $(A, L, [-, -]_L, \phi, \alpha_L, \rho_L)$ and $(A, M, [-, -]_M, \phi, \alpha_M, \rho_M)$ are hom-Lie–Rinehart algebras over $(A, \phi)$. We consider

$L \times_{\text{Der}_\phi(A)} M = \{(l, m) \in L \times M : \rho_l(l) = \rho_M(m)\},$

where $L \times M$ denotes the Cartesian product. Then $(A, L \times_{\text{Der}_\phi(A)} M, [-, -], \phi, \alpha, \rho)$ is a hom-Lie–Rinehart algebra, where

- the bracket is given by

$[(l_1, m_1), (l_2, m_2)] = ([l_1, l_2], [m_1, m_2]);$

- the endomorphism $\alpha : L \times_{\text{Der}_\phi(A)} M \to L \times_{\text{Der}_\phi(A)} M$ is given by

$\alpha(l, m) = (\alpha_L(l), \alpha_M(m));$

- the anchor map $\rho : L \times_{\text{Der}_\phi(A)} M \to \text{Der}_\phi(A)$ is given by

$\rho(l, m)(a) = \rho_L(a) = \rho_M(a),$

for all $l, l_1, l_2 \in L$, $m, m_1, m_2 \in M$, and $a \in A$. The above structure gives the categorical product in the category $hLR^\phi_A$. 

A subalgebra of a hom-Lie–Rinehart algebra \((L,\alpha_L)\) over \((A,\phi)\) is a pair \((M,\alpha_M)\) such that the underlying \(A\)-module \(M\) is an \(A\)-submodule of \(L\), restriction of \(\alpha_L\) to \(M\) is an endomorphism of \(M\) \((\alpha_M = \alpha_L|_M)\), and \([m,n] \in M\) for \(m,n \in M\). The pair \((M,\alpha_L|_M)\) is called a quasi-ideal in \((L,\alpha_L)\) if \([m,x] \in M\) for \(m \in M\) and \(x \in L\). Moreover, if \(\rho(m) = 0\) for each \(m \in M\), then \((M,\alpha_L|_M)\) is called an ideal in \((L,\alpha_L)\).

**Definition 2.7.** Suppose \((A,L,[-,-]_L,\phi,\alpha_L,\rho_L)\) and \((B,L',[-,-]_{L'},\psi,\alpha_{L'},\rho_{L'})\) are hom-Lie–Rinehart algebras. Then a homomorphism of hom-Lie–Rinehart algebras is a pair of maps \((g,f)\), where \(g : A \to B\) is an \(R\)-algebra homomorphism and \(f : L \to L'\) is an \(R\)-linear map such that the following identities hold:

- \(f(a.x) = g(a).f(x)\) for all \(x \in L\) and \(a \in A\);
- \(f[x,y]_L = [f(x),f(y)]_{L'}\) for all \(x,y \in L\);
- \(f(\alpha_L(x)) = \alpha_{L'}(f(x))\) for all \(x \in L\);
- \(g(\phi(a)) = \psi(g(a))\) for all \(a \in A\);
- \(g(\rho_L(x)(a)) = \rho_{L'}(f(x))(g(a))\) for all \(x \in L\) and \(a \in A\).

The Hom-Lie–Rinehart algebras with homomorphisms form a category, which we denote by \(hLR\). Note that the category of Lie–Rinehart algebras is a full subcategory of \(hLR\).

**Remark 2.8.** If \(A = B\) and \(\phi = \psi\), then by taking \(g = \text{id}_A\) we get a homomorphism of hom-Lie–Rinehart algebras over \((A,\phi)\). We denote by \(hLR^\phi_A\) the category of hom-Lie–Rinehart algebras over \((A,\phi)\).

**Definition 2.9.** Let \(M\) be an \(A\)-module, and \(\beta \in \text{End}_R(M)\). Then \((M,\beta)\) is a left module over a hom-Lie–Rinehart algebra \((L,\alpha_L)\) if

- there is a map \(\theta : L \otimes M \to M\) such that \((\theta,\beta)\) is a representation of the hom-Lie algebra \((L,[-,-]_L,\alpha_L)\) on \(M\); we denote \(\theta(x,m)\) by \(\{x,m\}\) for \(x \in L\) and \(m \in M\);  
- \(\beta(a.m) = \phi(a)\beta(m)\) for all \(a \in A\) and \(m \in M\);
- \(\{a.X,m\} = \phi(a)\{X,m\}\) for all \(a \in A\), \(X \in L\), \(m \in M\);
- \(\{X,a.m\} = \phi(a)\{X,m\} + \rho_L(X)(a)\beta(m)\) for all \(X \in L\), \(a \in A\), \(m \in M\).

In particular, for \(\alpha_L = \text{id}_L\) and \(\beta = \text{id}_M\), \((L,\alpha_L)\) is a Lie–Rinehart algebra and \(M\) is a left Lie–Rinehart module over the Lie–Rinehart algebra \(L\).

**Example 2.10.** The pair \((A,\phi)\) is a canonical left \((L,\alpha_L)\)-module, where the left action of \(L\) on \(A\) is given by the anchor map.

Let \((L,\alpha_L)\) be a hom-Lie–Rinehart algebra over \((A,\phi)\) and \((M,\beta)\) be a left module over \((L,\alpha_L)\). Consider the graded \(R\)-module

\[C^*(L;M) := \bigoplus_{n \geq 1} C^n(L;M),\]
where $C^n(L; M) \subseteq \text{Hom}_R(\wedge^n_R L, M)$ consists of those $f \in \text{Hom}_R(\wedge^n_R L, M)$ which satisfy

1. $f(\alpha_L(x_1), \ldots, \alpha_L(x_n)) = \beta(f(x_1, x_2, \ldots, x_n))$;
2. $f(x_1, \ldots, a.x_i, \ldots, x_n) = \phi^{n-1}(a)f(x_1, \ldots, x_i, \ldots, x_n),$

for $x_1, \ldots, x_n \in L$ and $a \in A$. Define $R$-linear maps $\delta : C^n(L; M) \to C^{n+1}(L; M)$ by

$$\delta f(x_1, \ldots, x_{n+1}) := \sum_{i=1}^{n+1} (-1)^{i+1}\{\alpha_L^{n-1}(x_i), f(x_1, \ldots, \hat{x}_i, \ldots, x_{n+1})\}$$

$$+ \sum_{1 \leq i < j \leq n+1} f([x_i, x_j], \alpha_L(x_1), \ldots, \alpha_L(x_i), \ldots, \alpha_L(x_j), \ldots, \alpha_L(x_{n+1}))$$

for all $f \in C^n(L; M)$, $x_i \in L$, and $1 \leq i \leq n+1$. Then $(C^\ast(L; M), \delta)$ forms a cochain complex (see [MM2] for more details). The associated cohomology is the cohomology of the hom-Lie–Rinehart algebra $(L, \alpha_L)$ with coefficients in $(M, \beta)$, denoted by $H^\ast_{hLR}(L, M)$.

### 3. Universal central extensions of hom-Lie–Rinehart algebras

We recall from [MM2] some necessary definitions and results about extensions of a hom-Lie–Rinehart algebra. First note that the category $hLR^\phi_A$ does not have a zero object. Thus, by a short exact sequence in $hLR^\phi_A$,

$$(L'', \alpha_{L''}) \xrightarrow{i} (L', \alpha_{L'}) \xrightarrow{\sigma} (L, \alpha_L),$$

we mean that the map $i : (L'', \alpha_{L''}) \to (L', \alpha_{L'})$ is injective, $\sigma : (L', \alpha_{L'}) \to (L, \alpha_L)$ is surjective, and $\text{Ker}(\sigma) = \text{Im}(i)$.

**Definition 3.1.** A short exact sequence in the category $hLR^\phi_A$,

$$(L'', \alpha_{L''}) \xrightarrow{i} (L', \alpha_{L'}) \xrightarrow{\sigma} (L, \alpha_L),$$

is called an extension of the hom-Lie–Rinehart algebra $(L, \alpha)$ by the hom-Lie–Rinehart algebra $(L'', \alpha_{L''})$. Here, the anchor map of $(L'', \alpha_{L''})$ is zero, i.e., $\rho_{L''} = 0$ since $\sigma \circ i = 0$.

An extension

$$(L'', \alpha_{L''}) \xrightarrow{i} (L', \alpha_{L'}) \xrightarrow{\sigma} (L, \alpha_L)$$

is said to be $A$-split if there is an $A$-module map $\tau : (L, \alpha_L) \to (L', \alpha_{L'})$ such that

1. $\sigma \circ \tau = \text{id}_{(L, \alpha_L)}$;
2. $\tau \circ \alpha_L = \alpha_{L'} \circ \tau$.

In this case, we say $\tau$ is a section of $\sigma$. If $\tau$ is furthermore a homomorphism of hom-Lie–Rinehart algebras, then the extension is said to be split in the category of hom-Lie–Rinehart algebras.
First, observe that any hom-Lie–Rinehart algebra module \((M, \beta)\) gives a hom-Lie–Rinehart algebra \((A, M, [-, -]_M, \phi, \beta, \rho_M) \in hLR^A_\phi\) with a trivial bracket and a trivial anchor map. We denote this object in \(hLR\) by \((M, \beta)\). Now recall abelian extensions of hom-Lie–Rinehart algebras from [MM2].

**Definition 3.2.** Let \((\mathcal{L}, \alpha_L)\) be a hom-Lie–Rinehart algebra over \((A, \phi)\) and \((M, \beta)\) be a left \((\mathcal{L}, \alpha_L)\)-module. Then a short exact sequence

\[
(M, \beta) \overset{i}{\rightarrow} (\mathcal{L}', \alpha_{L'}) \overset{\sigma}{\rightarrow} (\mathcal{L}, \alpha_L)
\]

in the category \(hLR^A_\phi\) is called an abelian extension of \((\mathcal{L}, \alpha_L)\) by \((M, \beta)\) if

\[
[i(m), x]_{L'} = i((\epsilon(x)).m) \quad \text{for all } m \in M \text{ and } x \in L'.
\]

Here, \((M, \beta)\) is a hom-Lie–Rinehart algebra \((A, M, [-, -]_M, \phi, \beta, \rho_M)\) with \([-,-]_M = 0\) and \(\rho_M = 0\).

In [MM2], it is proved that the second cohomology space \(H^2_{hLR}(L, M)\) of a hom-Lie–Rinehart algebra \((\mathcal{L}, \alpha_L)\) with coefficients in \((M, \beta)\) classifies \(A\)-split abelian extensions of \((\mathcal{L}, \alpha_L)\) by \((M, \beta)\). In particular, the following result generalizes the well-known classification theorems for the classical cases of Lie algebras [H71] and Lie–Rinehart algebras [CGM].

**Theorem 3.3 (MM2).** There is a one-to-one correspondence between the equivalence classes of \(A\)-split abelian extensions of a hom-Lie–Rinehart algebra \((\mathcal{L}, \alpha)\) by \((M, \beta)\) and the cohomology classes in \(H^2_{hLR}(L, M)\).

Let us recall that the **center** of a hom-Lie–Rinehart algebra \((\mathcal{L}, \alpha_L)\) is

\[
Z_A(\mathcal{L}) = \{ x \in L : [a.x, z]_L = [a.\alpha_L(x), z]_L = 0 \text{ and } \rho_L(x)(a) = 0 \text{ for all } a \in A, z \in L \}.
\]

It is an ideal of \((\mathcal{L}, \alpha_L)\).

**Definition 3.4.** A short exact sequence of hom-Lie–Rinehart algebras

\[
(M, \beta) \overset{i}{\rightarrow} (\mathcal{L}', \alpha_{L'}) \overset{\sigma}{\rightarrow} (\mathcal{L}, \alpha_L)
\]

is called a central extension of \((\mathcal{L}, \alpha_L)\) if \(i(M) = \text{Ker}(\sigma) \subset Z_A(\mathcal{L}')\). Here, \((M, \beta)\) is a hom-Lie–Rinehart algebra \((A, M, [-, -]_M, \phi, \beta, \rho_M)\).

**Remark 3.5.** (1) Since \(\sigma \circ i = 0\), we have \(\rho_M = 0\).

(2) Note that \(i(M) = \text{Ker}(\sigma) \subset Z_A(\mathcal{L}')\) implies \((M, \beta)\) is a hom-Lie–Rinehart algebra with trivial bracket, since \(i[m, n]_M = [i(m), i(n)]' = 0\) and \(i\) is an injective map.

(3) If \((M, \beta)\) is a trivial module over \((\mathcal{L}, \alpha_L)\), then an abelian extension of \((\mathcal{L}, \alpha)\) by \((M, \beta)\) is a central extension.
Proposition 3.6 ([MM2]). There is a one-to-one correspondence between the equivalence classes of $A$-split central extensions

$$(\mathcal{M}, \beta) \xrightarrow{i} (\mathcal{L}', \alpha_{L'}) \xrightarrow{\tau} (\mathcal{L}, \alpha_L)$$

of $(\mathcal{L}, \alpha_L)$ by $(\mathcal{M}, \beta) := (A, M, [-, -], \phi, \beta, \rho_M)$ and the cohomology classes in $H^2_{hLR}(L, M)$, where $(\mathcal{M}, \beta)$ is a trivial module over the hom-Lie–Rinehart algebra $(\mathcal{L}, \alpha_L)$.

Definition 3.7. A central extension

$$(\mathcal{M}, \alpha_M) \xrightarrow{i} (\mathcal{K}, \alpha_K) \xrightarrow{\sigma} (\mathcal{L}, \alpha_L)$$

is said to be a universal central extension if for any other central extension

$$(\mathcal{M}', \alpha_{M'}) \xrightarrow{i} (\mathcal{L}', \alpha_{L'}) \xrightarrow{\tau} (\mathcal{L}, \alpha_L)$$

there exists a unique homomorphism $h : (\mathcal{K}, \alpha_K) \to (\mathcal{L}', \alpha_{L'})$ in the category $hLR^\phi_A$ such that $\tau \circ h = \sigma$.

Next, we have the following characterization of the universal central extension of a hom-Lie–Rinehart algebra.

Theorem 3.8. Let $(\mathcal{M}, \alpha_M) \xrightarrow{i} (\mathcal{K}, \alpha_K) \xrightarrow{\sigma} (\mathcal{L}, \alpha_L)$ be a central extension of a hom-Lie–Rinehart algebra $(\mathcal{L}, \alpha_L)$. If every central extension of $(\mathcal{K}, \alpha_K)$ splits uniquely in the category $hLR^\phi_A$, then $(\mathcal{M}, \alpha_M) \xrightarrow{i} (\mathcal{K}, \alpha_K) \xrightarrow{\sigma} (\mathcal{L}, \alpha_L)$ is a universal central extension.

Proof. Let $(\mathcal{M}', \alpha_{M'}) \xrightarrow{i} (\mathcal{L}', \alpha_{L'}) \xrightarrow{\tau} (\mathcal{L}, \alpha_L)$ be a central extension of $(\mathcal{L}, \alpha_L)$. We consider a hom-Lie–Rinehart algebra for the pull-back diagram corresponding to the homomorphisms $\sigma : (\mathcal{K}, \alpha_K) \to (\mathcal{L}, \alpha_L)$ and $\tau : (\mathcal{L}', \alpha_{L'}) \to (\mathcal{L}, \alpha_L)$ as follows:

- consider an $A$-module $P = \{(k, l') : K \times L' : \sigma(k) = \tau(l')\}$;
- define $\alpha_P : P \to P$ by $\alpha_P(k, l') = (\alpha_K(k), \alpha_{L'}(l'))$;
- define $\rho_P : P \to \text{Der}_A(\mathcal{L})$ by $\rho_P(k, l')(a) = \rho_L(\sigma(k))(a) = \rho_L(\tau(l'))(a)$.

Then $(\mathcal{P}, \alpha_P) = (A, P, [-, -], \phi, \alpha_P, \rho_P)$ is a hom-Lie–Rinehart algebra (defined in Example 2.6).

Let $\pi_1 : P \to K$ denote the projection map onto the first factor. Then the short exact sequence $(\text{Ker}(\pi_1), \alpha_P|_{\text{Ker}(\pi_1)}) \hookrightarrow (\mathcal{P}, \alpha_P) \xrightarrow{\pi_1} (\mathcal{K}, \alpha_K)$ is a central extension of $(\mathcal{K}, \alpha_K)$. Therefore, by assumption it splits uniquely. Let $s : (\mathcal{K}, \alpha_K) \to (\mathcal{P}, \alpha_P)$ be the unique section of $\pi_1$, i.e., $\pi_1 \circ s = \text{id}_K$. Then for any $k \in K$, we have $s(k) = (k, l')$ for some $l' \in L'$. Thus $s$ induces a map $h : K \to L'$. Since $s$ is a morphism in the category $hLR^\phi_A$, it follows that $h : (\mathcal{K}, \alpha_K) \to (\mathcal{L}', \alpha_{L'})$ is a morphism in $hLR^\phi_A$ such that $\tau \circ h = \sigma$. The uniqueness of $h$ follows from the uniqueness of the section $s$. ■
The converse of the above theorem may not be true; it does not hold even in the case of hom-Lie algebras (see [CIP]).

### 3.1. Existence of a universal central extension for a perfect hom-Lie–Rinehart algebra

**Definition 3.9.** A hom-Lie–Rinehart algebra $(\mathcal{L}, \alpha_L)$ over $(A, \phi)$ is said to be perfect if $L = \{L, L\}$, where $\{L, L\}$ is the $A$-submodule of $L$ generated by all elements of the form $[x, y]_L$ for $x, y \in L$.

In particular, if $\alpha_L = \text{id}_L$ then a perfect hom-Lie–Rinehart algebra is simply a perfect Lie–Rinehart algebra (defined in [CGM]). Moreover, if $A = R$ and $\phi = \text{id}_A$, then a perfect hom-Lie–Rinehart algebra is simply a perfect hom-Lie algebra (defined in [CIP]).

**Remark 3.10.** Let $\sigma : (K, \alpha_K) \to (\mathcal{L}, \alpha_L)$ be a surjective morphism in the category $hLR_A^\phi$ and suppose $(K, \alpha_K)$ is a perfect hom-Lie–Rinehart algebra. Then $(\mathcal{L}, \alpha_L)$ is also a perfect hom-Lie–Rinehart algebra.

**Lemma 3.11.** Let $(\mathcal{M}', \alpha_{M'}) \xrightarrow{\tilde{\rho}} (\mathcal{L}', \alpha_{L'}) \xrightarrow{\tau} (\mathcal{L}, \alpha_L)$ be a central extension of $(\mathcal{L}, \alpha_L)$ and $(K, \alpha_K)$ be a perfect hom-Lie–Rinehart algebra. If there exist hom-Lie–Rinehart algebra homomorphisms $f, g : (K, \alpha_K) \to (\mathcal{L}', \alpha_{L'})$ such that $\tau \circ f = \tau \circ g$ then $f = g$.

**Proof.** Since $(K, \alpha_K)$ is a perfect hom-Lie–Rinehart algebra, the assertion follows if we can show that $g(a[x, y]) = f(a[x, y])$ for all $a \in A$ and $x, y \in K$.

Observe that if $\tau(x_1) = \tau(x_2)$ and $\tau(y_1) = \tau(y_2)$ then $x_1 - x_2, y_1 - y_2 \in \text{Ker}(\tau)$ and since $\text{Ker}(\tau) \subset Z_A(\mathcal{L}', \alpha_{L'})$, it follows that $[x_1, y_1] = [x_2, y_2]$. Next by taking $x_1 = f(x), y_1 = f(y)$ and $x_2 = g(x), y_2 = g(y)$, we get $[f(x), f(y)] = [g(x), g(y)]$. Therefore, $g(a[x, y]) = a[g(x), g(y)] = a[f(x), f(y)] = f(a[x, y])$. ■

Next, we obtain a hom-Lie–Rinehart algebra $(A, \text{ucc}_A^\phi(L), [-, -], \phi, \tilde{\alpha}_L, \tilde{\rho}_L)$ for a given hom-Lie–Rinehart algebra $(\mathcal{L}, \alpha_L)$ over $(A, \phi)$. Consequently, this construction yields a functor

$$\text{ucc}_A^\phi : hLR_A^\phi \to hLR_A^\phi$$

associating a universal central extension to a perfect hom-Lie–Rinehart algebra.

First, we consider an $A$-submodule $M^\phi_A L$ of $A \otimes L \otimes L$, which is generated by all elements of the following forms:

1. $a \otimes x \otimes x$;
2. $a \otimes x \otimes y + a \otimes y \otimes x$;
3. $a \otimes \alpha_L(x) \otimes [y, z]_L + a \otimes \alpha_L(y) \otimes [z, x]_L + a \otimes \alpha_L(z) \otimes [x, y]_L$;
4. $\phi(a) \otimes [x, y]_L \otimes [x', y']_L + \rho([x, y]_L)(a) \otimes \alpha_L(x') \otimes \alpha_L(y') - 1 \otimes [x, y]_L \otimes a[x', y']_L$
where $x, x', y, y', z \in L$ and $a \in A$. Consider the quotient $A$-module

$$ucc^\phi_A(L) := A \otimes L \otimes L/M^\phi_A$$

and denote any coset $a \otimes x \otimes y + M^\phi_A$ simply by $(a, x, y)$. From the definition of $ucc^\phi_A(L)$ we get the identities

1. $(a, x, y) = -(a, y, x)$;
2. $(a, \alpha_L(x), [y, z]_L) + (a, \alpha_L(y), [z, x]_L) + (a, \alpha_L(z), [x, y]_L) = 0$;
3. $(1, [x, y]_L, a[x', y']_L) = (\phi(a), [x, y]_L, [x', y']_L) + (\rho([x, y]_L)(a), \alpha_L(x'), \alpha_L(y'))$,

for any $x, x', y, y', z \in L$ and $a \in A$. Define an $A$-module homomorphism

$$\Psi: A \otimes L \otimes L \to L$$

by $\Psi(a, x, y) = a[x, y]_L$. Since $\Psi$ vanishes on $M^\phi_A$ it induces an $A$-linear map $u_L: ucc^\phi_A(L) \to L$.

It follows that the tuple $(A, ucc^\phi_A(L), [-, -], \phi, \tilde{\alpha}_L, \tilde{\rho}_L)$ is a hom-Lie–Rinehart algebra over $(A, \phi)$, where

- the bracket $[-, -]$ is defined as
  $$[(a_1, x_1, y_1), (a_2, x_2, y_2)] = (\phi(a_1 a_2), [x_1, y_1]_L, [x_2, y_2]_L) + (\phi(a_1), [x_1, y_1]_L(a_2), \alpha_L(x_2), \alpha_L(y_2))$$
  $$- (\phi(a_2), [x_2, y_2]_L(a_1), \alpha_L(x_1), \alpha_L(y_1));$$
- the endomorphism $\tilde{\alpha}_L: ucc^\phi_A(L) \to ucc^\phi_A(L)$ is defined as
  $$\tilde{\alpha}_L(a, x, y) = (\phi(a), \alpha_L(x), \alpha_L(y));$$
- the anchor $\tilde{\rho}_L: ucc^\phi_A(L) \to \text{Der}_\phi(A)$ is defined as
  $$\tilde{\rho}_L(a, x, y)(b) = \phi(a)\rho_L([x, y]_L)(b),$$

for all $x_1, x_2, x, y_1, y_2, y \in L$ and $a, b, a_1, a_2 \in A$. Set

$$ucc^\phi_A(L, \alpha_L) := (A, ucc^\phi_A(L), [-, -], \phi, \tilde{\alpha}_L, \tilde{\rho}_L).$$

Then the previously defined map $u_L: ucc^\phi_A(L, \alpha_L) \to (L, \alpha_L)$ is a morphism in the category $hLR^\phi_A$.

Let $f : (L, \alpha_L) \to (M, \alpha_M)$ be an arbitrary morphism in $hLR^\phi_A$. Define a map $ucc^\phi_A(f): ucc^\phi_A(L) \to ucc^\phi_A(M)$ by

$$ucc^\phi_A(f)(a, x, y) = (a, f(x), f(y)) \quad \text{for } a \in A \text{ and } x, y \in L.$$
Then \( uce_A^\phi(f) \) is a morphism in \( hLR_A^\phi \) and we have the commutative diagram

\[
\begin{array}{ccc}
uce_A^\phi(L, \alpha_L) & \xrightarrow{uce_A^\phi(f)} & uce_A^\phi(M, \alpha_M) \\
u_L & & u_M \\
(L, \alpha_L) & \xrightarrow{f} & (M, \alpha_M)
\end{array}
\]

This in turn implies that \( uce_A^\phi : hLR_A^\phi \rightarrow hLR_A^\phi \) is a functor.

Let \( \{L, L\} \) denote the \( A \)-submodule of \( L \) generated by all elements of the form \([x, y]_L\) for \( x, y \in L \). Then we get a hom-Lie–Rinehart subalgebra \( \{L, L\}, \alpha_{\{L, L\}} \) of \( (L, \alpha_L) \) by restricting the hom-Lie bracket, the anchor map, and the endomorphism \( \alpha_L \) to the \( A \)-submodule \( \{L, L\} \) (here \( \alpha_{\{L, L\}} \) denotes the restriction of \( \alpha_L \) to \( \{L, L\} \)). Further \( u_L : uce_A^\phi(L, \alpha_L) \rightarrow (\{L, L\}, \alpha_{\{L, L\}}) \) is a surjective homomorphism and it gives a central extension of the hom-Lie–Rinehart algebra \( \{L, L\}, \alpha_{\{L, L\}} \). Moreover, we have the following result.

**Remark 3.12.** If \( (L, \alpha_L) \) is a perfect hom-Lie–Rinehart algebra, then \( u_L : uce_A^\phi(L, \alpha_L) \rightarrow (L, \alpha_L) \) is surjective. Assume that the maps \( \alpha_L \) and \( \phi \) are invertible, i.e., \( (L, \alpha_L) \) is a regular hom-Lie–Rinehart algebra over \( (A, \phi) \). We claim that

\[
u_L^{-1}(Z_A(L, \alpha_L)) = Z_A(uce_A^\phi(L, \alpha_L)).
\]

First, it is clear that \( Z_A(uce_A^\phi(L, \alpha_L)) \subseteq u_L^{-1}(Z_A(L, \alpha_L)) \). Conversely, assume that \( X \in u_L^{-1}(Z_A(L, \alpha_L)) \); then \( \tilde{\rho}_L(X)(a) = \rho_L(u_L(X))(a) = 0 \). For all \( a \in A \) and \( Z \in uce_A^\phi(L) \), we have

\[
u_L[a.X, Z] = [a.u_L(X), u_L(Z)] = 0 = u_L[a.\tilde{\alpha}_L(X), Z] = [a.\alpha_L(u_L(X)), Z],
\]

i.e., \([a.X, Z], [a.\tilde{\alpha}_L(X), Z] \in \text{Ker}(u_L) \subseteq Z_A(uce_A^\phi(L, \alpha_L)) \) for all \( a \in A \) and \( Z \in uce_A^\phi(L) \). Observe that the map \( \tilde{\alpha}_L \) is also invertible and

\([a.\tilde{\alpha}_L^{-1}(X), Z] \in \text{Ker}(u_L) \subseteq Z_A(uce_A^\phi(L, \alpha_L)) \) for all \( a \in A \) and \( Z \in uce_A^\phi(L) \).

Now, using the hom-Jacobi identity for the hom-Lie bracket \([-, -]\) on \( uce_A^\phi(L, \alpha_L) \), it follows that \( X \in Z_A(uce_A^\phi(L, \alpha_L)) \). Thus, \( u_L^{-1}(Z_A(L, \alpha_L)) = Z_A(uce_A^\phi(L, \alpha_L)) \).

If the hom-Lie–Rinehart algebra is centerless, i.e., \( Z_A(L, \alpha_L) = 0 \), then

\[\text{Ker}(u_L) = Z_A(uce_A^\phi(L, \alpha_L)).\]

**Main Theorem 3.13.** Let \( (L, \alpha_L) \) be a perfect hom-Lie–Rinehart algebra. Then the short exact sequence

\[
(P, \alpha_P) \rightarrow uce_A^\phi(L, \alpha_L) \rightarrow (L, \alpha_L)
\]

is a universal central extension of \( (L, \alpha_L) \) where the underlying \( A \)-module in \( (P, \alpha_P) \) is \( P = \text{Ker}(u_L) \) and \( \alpha_P \) is the restriction of \( \tilde{\alpha}_L \) to \( \text{Ker}(u) \).
Define a skew-symmetric bracket on 
uce
(L, αL) be a two-dimensional real vector space with basis
uce
α
symmetric bracket on 
uce
Next by Lemma 3.11, \( \tau: (uce_\alpha^\phi (L), \tilde{\alpha}_L) \rightarrow (M', \alpha_{M'}) \).

Next by Lemma 3.11 \( \tau \) is unique on the subalgebra \{uce_\alpha^\phi (L), uce_\alpha^\phi (L)\}. Since
uce
(L, αL) is a perfect hom-Lie–Rinehart algebra, the hom-Lie–Rinehart algebra
uce_\alpha^\phi (L, αL) is also perfect, i.e., \{uce_\alpha^\phi (L), uce_\alpha^\phi (L)\} = uce_\alpha^\phi (L). Therefore,
there exists a unique homomorphism \( \tau: uce_\alpha^\phi (L, αL) \rightarrow (M', \alpha_{M'}) \) such that
\( \sigma \circ \tau = uL \). This completes the proof. ■

4. Universal α-central extensions of hom-Lie–Rinehart algebras.
In this section, we assume that \( A \) is an associative commutative unital \( R \)
-algebra and \( \phi: A \rightarrow A \) is an algebra epimorphism. Now let us define an
α-central extension of a hom-Lie–Rinehart algebra.

**Definition 4.1.** An extension of a hom-Lie–Rinehart algebra \( (L, \alpha_L) \),
(\( M, \alpha_M \) \( \mapsto (L', \alpha_{L'}) \)) \( \sigma \rightarrow (L, \alpha_L) \),
is called an **α-central extension** if \( i(\alpha_M (M)) \subset Z_A (L', \alpha_{L'}) \).

**Remark 4.2.** Any central extension of a hom-Lie–Rinehart algebra is an
α-central extension, but the converse may not be true.

Consider the following example in the particular case of hom-Lie al-
gebras [CIP].

**Example 4.3.** Let \( A = \mathbb{R} \), the field of real numbers, and \( \phi = \text{id}_\mathbb{R} \). Let \( L \)
be a two-dimensional real vector space with basis \{\( x_1, x_2 \)\}. Define a skew-
symmetric bracket on \( L \) by \( [x_1, x_2]_L = x_1 \), and consider an endomorphism
\( \alpha_L = 0 \) on \( L \). Then \( (L, \alpha_L) := (\mathbb{R}, L, [-, -], \rho_L = 0, \alpha_L = 0) \) is a hom-Lie–
Rinehart algebra over \( (\mathbb{R}, \text{id}_\mathbb{R}) \).

Consider a three-dimensional real vector space \( K \) with basis \{\( y_1, y_2, y_3 \)\}. Define a skew-
symmetric bracket on \( K \) by \( [y_1, y_2] = y_1, [y_2, y_3] = y_2 \), and
otherwise zero. Also, define \( \alpha_K : K \to K \) as the zero map. Then
\[
(\mathcal{K}, \alpha_K) := (\mathbb{R}, K, [-, -]_K, \rho_K = 0, \alpha_K = 0)
\]
is a hom-Lie–Rinehart algebra over \((\mathbb{R}, \text{id}_\mathbb{R})\).

Define a surjective linear map \( \sigma : (\mathcal{K}, \alpha_K) \to (\mathcal{L}, \alpha_L) \) by \( \sigma(y_1) = 0 \), \( \sigma(y_2) = x_1 \), and \( \sigma(y_3) = x_2 \). Denote \( M := \text{Ker}(\sigma) = \langle \{y_1\} \rangle \). Then trivially the extension \((\mathcal{M}, \alpha_M) \xrightarrow{i} (\mathcal{K}, \alpha_K) \xrightarrow{\sigma} (\mathcal{L}, \alpha_L)\) is an \( \alpha \)-central extension of \((\mathcal{L}, \alpha_L)\). However, it is not a central extension since \([\text{Ker}(\sigma), K] \neq 0\).

In the case of Lie algebras, Leibniz algebras, Lie–Rinehart algebras and other classical algebras, the composition of two central extensions is again a central extension. This property does not hold for hom-Lie–Rinehart algebras (see \cite[Example 4.9]{CIP}). In the following lemma, we prove that the composition of two central extensions in \( hLR^0_A \) is an \( \alpha \)-central extension.

**Lemma 4.4.** Let \((\mathcal{M}, \alpha_M) \xrightarrow{i} (\mathcal{K}, \alpha_K) \xrightarrow{\sigma} (\mathcal{L}, \alpha_L)\) be a central extension, where \((\mathcal{K}, \alpha_K)\) is a perfect hom-Lie–Rinehart algebra. Then for a central extension of \((\mathcal{K}, \alpha_K)\) given by \((\mathcal{N}, \alpha_N) \xrightarrow{j} (\mathcal{L}', \alpha_{L'})\), \( \tau \) on \((\mathcal{N}', \alpha_{N'})\) the composition
\[
(\mathcal{N}', \alpha_{N'}) \xrightarrow{j'} (\mathcal{L}', \alpha_{L'}) \xrightarrow{\sigma \circ \tau} (\mathcal{L}, \alpha_L)
\]
is an \( \alpha \)-central extension.

**Proof.** Since \((\mathcal{K}, \alpha_K)\) is perfect, any element \( X \in K \) can be written as
\[
X = \sum_i a_i [X_{i_1}, X_{i_2}]
\]
for some \( a_i \in A \) and \( X_{i_1}, X_{i_2} \in K \). For any \( Y \in L' \), \( \tau(Y) = \sum_i a_i[k_{i_1}, k_{i_2}] \) for some \( k_{i_1}, k_{i_2} \in K \) and \( a_i \in A \). Since \( \tau \) is surjective, we have \( \tau(Y) = \sum_i a_i \tau(Y_{i_1}, Y_{i_2}) \). Thus \( Y - \sum_i a_i[Y_{i_1}, Y_{i_2}] \in \text{Ker}(\tau) \), i.e.,
\[
Y = \sum_i a_i[Y_{i_1}, Y_{i_2}] + \eta \quad \text{for some } \eta \in \text{Ker}(\tau).
\]

Now using \((4.1)\), the fact that \( \text{Ker}(\tau) \subset Z_A(\mathcal{L}', \alpha_{L'}) \) and the hom-Jacobi identity, we get
\[
[a, \alpha_n](n), Y] = [\alpha_n(a', n), Y] = 0
\]
for each \( n \in N' = \text{Ker}(\sigma \circ \tau) \) and \( a \in A \) (since \( \phi \) is surjective there exists \( a' \in A \) such that \( \phi(a') = a \)). Moreover, for each \( n \in N' \) and \( a \in A \), we have \( n(a) = 0 \). Therefore \( \alpha_{N'}(\text{Ker}(\sigma \circ \tau)) \subset Z_A(\mathcal{L}') \), i.e., the composition
\[
(\mathcal{N}', \alpha_{N'}) \xrightarrow{j'} (\mathcal{L}', \alpha_{L'}) \xrightarrow{\sigma \circ \tau} (\mathcal{L}, \alpha_L)
\]
is an \( \alpha \)-central extension. \( \blacksquare \)
A hom-Lie–Rinehart algebra \((\mathcal{L}, \alpha_L)\) over \((A, \phi)\) is called \(\alpha\)-perfect if \(L\) equals \(\{\alpha_L(L), \alpha_L(L)\}\), the \(A\)-submodule of \(L\) generated by all elements of the form \([\alpha_L(x), \alpha_L(y)]_L\) for \(x, y \in L\).

Note that any \(\alpha\)-perfect hom-Lie–Rinehart algebra \((\mathcal{L}, \alpha_L)\) is also perfect. Moreover, since \(\phi : A \to A\) is an algebra epimorphism, it is clear that the map \(\alpha_L\) is surjective.

In particular, if \(A = R\) and \(\phi = \text{id}_R\) then any \(\alpha\)-perfect hom-Lie–Rinehart algebra \((\mathcal{L}, \alpha_L)\) is just an \(\alpha\)-perfect hom-Lie algebra, defined in [CKP1].

Example 4.6 ([CKP1]). Let \(L\) be a three-dimensional real vector space with basis \(\{x_1, x_2, x_3\}\). Let us define a skew-symmetric bracket on \(L\) as follows:

\[ [x_1, x_2] = x_3, \quad [x_2, x_3] = x_1, \quad [x_3, x_1] = x_2, \quad \text{and otherwise zero.} \]

Consider the map \(\alpha_L : L \to L\) defined on basis elements as follows:

\[ \alpha_L(x_1) = \frac{\sqrt{2}}{2} x_1 + \frac{\sqrt{2}}{2} x_3, \quad \alpha_L(x_2) = -x_2, \quad \alpha_L(x_3) = \frac{\sqrt{2}}{2} x_1 - \frac{\sqrt{2}}{2} x_3. \]

Then \((\mathcal{L}, \alpha_L)\) is an \(\alpha\)-perfect hom-Lie–Rinehart algebra over \((R, \text{id}_R)\) with trivial anchor map.

Lemma 4.7. Let \((K, \alpha_K)\) be an \(\alpha\)-perfect hom-Lie–Rinehart algebra and suppose the short exact sequence

\[ (M', \alpha_{M'}) \xrightarrow{i} (\mathcal{L}', \alpha_{L'}) \xrightarrow{\tau} (\mathcal{L}, \alpha_L) \]

is an \(\alpha\)-central extension of \((\mathcal{L}, \alpha_L)\). If there exist hom-Lie–Rinehart algebra homomorphisms \(f, g : (K, \alpha_K) \to (\mathcal{L}', \alpha_{L'})\) such that \(\tau \circ f = \tau \circ g\) then \(f = g\).

Proof. The proof is similar to the proof of Lemma 3.11.

Definition 4.8. A central extension

\[ (\mathcal{M}, \alpha_M) \xrightarrow{i} (K, \alpha_K) \xrightarrow{\sigma} (\mathcal{L}, \alpha_L) \]

is called a universal \(\alpha\)-central extension if for every \(\alpha\)-central extension

\[ (\mathcal{M}', \alpha_{M'}) \xrightarrow{i} (\mathcal{L}', \alpha_{L'}) \xrightarrow{\tau} (\mathcal{L}, \alpha_L) \]

there exists a unique homomorphism \(h : (K, \alpha_K) \to (\mathcal{L}', \alpha')\) in the category \(hLR_A^\phi\) such that \(\tau \circ h = \sigma\).

Any central extension of a hom-Lie–Rinehart algebra is an \(\alpha\)-central extension and hence a universal \(\alpha\)-central extension of a hom-Lie–Rinehart algebra is also a universal central extension. However, by Remark 4.2 and Example 4.3 it is clear that a universal central extension of a hom-Lie–Rinehart algebra may not be a universal \(\alpha\)-central extension.
Theorem 4.9. Let \((M, \alpha_M) \xrightarrow{i} (K, \alpha_K) \xrightarrow{\sigma} (L, \alpha_L)\) be a central extension of a hom-Lie–Rinehart algebra \((L, \alpha_L)\). If this extension is a universal \(\alpha\)-central extension then the hom-Lie–Rinehart algebra \((K, \alpha_K)\) is perfect and every central extension of \((K, \alpha_K)\) splits uniquely in the category \(h\)\(LR_A\).

Proof. Assume that \((K, \alpha_K)\) is not perfect. Consider the \(A\)-module \(K' = K \times K' / \{K, K\}\). Then \((A, K', [-, -]_{K'}, \phi, \alpha_{K'}, \rho_{K'})\) is a hom-Lie–Rinehart algebra where

- \([x, y + \{K, K\}], (x', y' + \{K, K\})]_{K'} = ([x, x'], 0);
- \(\alpha_{K'}(x, y + \{K, K\}) = (\alpha_K(x), \alpha_K(y) + \{K, K\})
- \(\rho_{K'}(x, y + \{K, K\})(a) = \rho_K(x)(a)\).

Consider the central extension
\[
(M', \alpha_{M'}) \xrightarrow{i} (K', \alpha_{K'}) \xrightarrow{\tilde{\sigma}} (L, \alpha_L)
\]
where \(K' = K \times K / \{K, K\}, \tilde{\sigma} : K' \rightarrow L\) is given by \(\tilde{\sigma}(x, y + \{K, K\}) = \sigma(x)\) for \(x, y \in K\) and \(M' = \text{Ker}(\tilde{\sigma})\). Now define

- \(f : K \rightarrow K \times K / \{K, K\}\) by \(f(x) = (x, 0)\) for all \(x \in K\);
- \(g : K \rightarrow K \times K / \{K, K\}\) by \(g(x) = (x, x + \{K, K\})\) for all \(x \in K\).

Then \(\tilde{\sigma} \circ f = \tilde{\sigma} \circ g = \sigma\). Since \((M, \alpha_M) \xrightarrow{i} (K, \alpha_K) \xrightarrow{\sigma} (L, \alpha_L)\) is a universal \(\alpha\)-central extension, it is also a universal central extension. Therefore, by the universal property \(f = g\) and \(K / \{K, K\} = 0\). So, \((K, \alpha_K)\) is a perfect hom-Lie–Rinehart algebra, a contradiction.

Let \((N, \alpha_N) \xrightarrow{j} (L', \alpha_{L'}) \xrightarrow{\tau} (K, \alpha_K)\) be a central extension of \((K, \alpha_K)\). By Lemma 4.4, \((P, \alpha_P) \xrightarrow{k} (L', \alpha_{L'}) \xrightarrow{\sigma \circ \tau} (L, \alpha_L)\) is an \(\alpha\)-central extension where \(P = \text{Ker}(\sigma \circ \tau)\). Since \((M, \alpha_M) \xrightarrow{i} (K, \alpha_K) \xrightarrow{\sigma} (L, \alpha_L)\) is a universal \(\alpha\)-central extension, we have a unique homomorphism \(\gamma : (K, \alpha_K) \rightarrow (L', \alpha_{L'})\) such that \(\sigma \circ (\tau \circ \gamma) = \sigma\). Since \((K, \alpha_K)\) is perfect, by using Lemma 3.11 we get \(\tau \circ \gamma = \text{id}_K\). Thus every central extension of \((K, \alpha_K)\) splits uniquely in \(h\)\(LR_A\).

Now, for a given \(\alpha\)-perfect hom-Lie–Rinehart algebra \((L, \alpha_L)\) over \((A, \phi)\) we obtain a hom-Lie–Rinehart algebra \((A, uce_A^{\alpha, \phi}(L), [-, -], \phi, \hat{\alpha}_L, \hat{\rho}_L)\). Recall that in this subsection we are assuming that \(\phi : A \rightarrow A\) is an epimorphism. This, in turn, implies that \(\alpha_L\) is surjective.

First consider the \(A\)-submodule \(M_A^{\alpha, \phi} L\) of \(A \otimes \alpha_L(L) \otimes \alpha_L(L)\) generated by all elements of the following forms:

1. \(a \otimes x \otimes x\);
2. \(a \otimes x \otimes y + a \otimes y \otimes x\);
3. \(a \otimes \alpha_L(x) \otimes [y, z]_L + a \otimes \alpha_L(y) \otimes [z, x]_L + a \otimes \alpha_L(z) \otimes [x, y]_L\).
(4) $\phi(a) \otimes [x, y]_L \otimes [x', y']_L + \rho([x, y]_L)(a) \otimes \alpha_L(x') \otimes \alpha_L(y') - 1 \otimes [x, y]_L \otimes a[x', y']_L$,

where $x, x', y, y', z \in L$ and $a \in A$. Since the hom-Lie–Rinehart algebra $(\mathcal{L}, \alpha_L)$ is $\alpha$-perfect, we have $L = \{\alpha_L(L), \alpha_L(L)\} \subset \alpha_L(L)$. Thus, elements of the forms (1)–(4) are in $A \otimes \alpha_L(L) \otimes \alpha_L(L)$. Consider the quotient $A$-module

$$\text{uce}_A^{\alpha_\phi}(L) := A \otimes \alpha_L(L) \otimes \alpha_L(L) / M_A^{\alpha_\phi} L$$

and denote any coset $a \otimes \alpha_L(x) \otimes \alpha_L(y) + M_A^{\alpha_\phi} L$ simply by $(a, \alpha_L(x), \alpha_L(y))$. From this definition the following identities hold:

- $(a, \alpha_L(x), \alpha_L(y)) = -(a, \alpha_L(y), \alpha_L(x));$
- $(a, \alpha_L^2(x), [\alpha_L(y), \alpha_L(z)]_L) + (a, \alpha_L^2(y), [\alpha_L(z), \alpha_L(x)]_L) + (a, \alpha_L^2(z), [\alpha_L(x), \alpha_L(y)]_L) = 0;$
- $(1, \alpha_L([x, y]_L), a, \alpha_L([x', y']_L)) = (\phi(a), \alpha_L([x, y]_L), \alpha_L([x', y']_L)) + (\rho(\alpha_L[x, y]_L)(a), \alpha_L^2(x'), \alpha_L^2(y'))$

for any $x, x', y, y', z \in L$ and $a \in A$. Define an $A$-module homomorphism

$$\Psi : A \otimes \alpha_L(L) \otimes \alpha_L(L) \to L$$

by $\Psi(a, \alpha_L(x), \alpha_L(y)) = a[\alpha_L(x), \alpha_L(y)]_L$. Since $\Psi$ vanishes, it induces an $A$-linear map $u^\phi_A : \text{uce}_A^{\alpha_\phi}(L) \to L$.

Next, $(A, \text{uce}_A^{\alpha_\phi}(L), [-, -], \phi, \hat{\alpha}_L, \hat{\rho}_L)$ is a hom-Lie–Rinehart algebra over $(A, \phi)$, where

- the bracket $[-, -]$ is defined as
  $$[(a_1, \alpha_L(x_1), \alpha_L(y_1)), (a_2, \alpha_L(x_2), \alpha_L(y_2))] = (\phi(a_1a_2), [\alpha_L(x_1), \alpha_L(y_1)]_L, [\alpha_L(x_2), \alpha_L(y_2)]_L) + (\phi(a_1), [\alpha_L(x_1), \alpha_L(y_1)]_L(a_2), \alpha_L^2(x_2), \alpha_L^2(y_2)) - (\phi(a_2), [\alpha_L(x_2), \alpha_L(y_2)]_L(a_1), \alpha_L^2(x_1), \alpha_L^2(y_1));$$

- the endomorphism $\hat{\alpha}_L : \text{uce}_A^{\alpha_\phi}(L) \to \text{uce}_A^{\alpha_\phi}(L)$ is defined as
  $$\hat{\alpha}_L(a, \alpha_L(x), \alpha_L(y)) = (\phi(a), \alpha_L^2(x), \alpha_L^2(y));$$

- the anchor $\hat{\rho}_L : \text{uce}_A^{\alpha_\phi}(L) \to \text{Der}_\phi(A)$ is defined as
  $$\hat{\rho}_L(a, \alpha_L(x), \alpha_L(y))(b) = \phi(a)\rho_L([\alpha_L(x), \alpha_L(y)]_L)(b),$$

for all $x_1, x_2, x, y_1, y_2, y \in L$ and $a, b, a_1, a_2 \in A$. Set

$$\text{uce}_A^{\alpha_\phi}(\mathcal{L}, \alpha_L) := (A, \text{uce}_A^{\alpha_\phi}(L), [-, -], \phi, \hat{\alpha}_L, \hat{\rho}_L).$$

Let $(\mathcal{L}, \alpha_L)$ and $(\mathcal{M}, \alpha_M)$ be $\alpha$-perfect hom-Lie–Rinehart algebras. If $f : (\mathcal{L}, \alpha_L) \to (\mathcal{M}, \alpha_M)$ is an arbitrary morphism in the category $hLR^A_\phi$,
then define \( \text{uce}_{\alpha,\phi}^{A}(f) : \text{uce}_{\alpha,\phi}^{A}(L) \to \text{uce}_{\alpha,\phi}^{A}(M) \) by
\[
\text{uce}_{\alpha,\phi}^{A}(f)(a, \alpha_{L}(x), \alpha_{L}(y)) = (a, f(\alpha_{L}(x)), f(\alpha_{L}(y)))
\]
for \( a \in A \) and \( x, y \in L \). Then \( \text{uce}_{\alpha,\phi}^{A}(f) \) is a morphism in \( hLR_{A}^{\phi} \) and we have the commutative diagram
\[
\begin{array}{ccc}
\text{uce}_{\alpha,\phi}^{A}(\mathcal{L}, \alpha_{L}) & \xrightarrow{\text{uce}_{\alpha,\phi}^{A}(f)} & \text{uce}_{\alpha,\phi}^{A}(\mathcal{M}, \alpha_{M}) \\
\downarrow u_{L}^{\alpha} & & \downarrow u_{M}^{\alpha} \\
(\mathcal{L}, \alpha_{L}) & \xrightarrow{f} & (\mathcal{M}, \alpha_{M})
\end{array}
\]
Since \((\mathcal{L}, \alpha_{L})\) is an \( \alpha \)-perfect hom-Lie–Rinehart algebra, so is \( \text{uce}_{\alpha,\phi}^{A}(\mathcal{L}, \alpha_{L}) \), i.e.,
\[
\{\tilde{\alpha}_{L}(\text{uce}_{\alpha,\phi}^{A}(L)), \tilde{\alpha}_{L}(\text{uce}_{\alpha,\phi}^{A}(L))\} = \text{uce}_{\alpha,\phi}^{A}(L)
\]
and the induced map \( u_{L}^{\alpha} : \text{uce}_{\alpha,\phi}^{A}(\mathcal{L}, \alpha_{L}) \to (\mathcal{L}, \alpha_{L}) \) is a surjective homomorphism that gives a central extension.

**Main Theorem 4.10.** Let \((\mathcal{L}, \alpha_{L})\) be an \( \alpha \)-perfect hom-Lie–Rinehart algebra. Then
\[
(\mathcal{P}, \alpha_{P}) \to \text{uce}_{\alpha,\phi}^{A}(\mathcal{L}, \alpha_{L}) \xrightarrow{u_{L}^{\alpha}} (\mathcal{L}, \alpha_{L})
\]
is a universal \( \alpha \)-central extension of \((\mathcal{L}, \alpha_{L})\).

**Proof.** Note that \( \text{uce}_{\alpha,\phi}^{A}(\mathcal{L}, \alpha_{L}) \) is also \( \alpha \)-perfect. Let
\[
(\mathcal{N}, \alpha_{N}) \xrightarrow{\iota} (\mathcal{M}', \alpha_{M}') \xrightarrow{\sigma} (\mathcal{L}, \alpha_{L})
\]
be an \( \alpha \)-central extension, i.e., \( \alpha_{M}'(\ker(\sigma)) \subset Z_{A}(\mathcal{M}', \alpha_{M}') \). Take a map \( s : L \to M' \) such that \( \sigma \circ s = \text{id}_{L} \). Define \( \psi : A \times \alpha_{L}(L) \times \alpha_{L}(L) \to M' \) by
\[
\psi(a, \alpha_{L}(x), \alpha_{L}(y)) = a[\alpha_{M}'(s(x)), \alpha_{M}'(s(y))]_{M'}
\]
for all \( x, y \in L \). Following the arguments in the proof of Theorem 3.13, it is clear that \( \psi \) is well-defined and it extends to a hom-Lie–Rinehart algebra homomorphism
\[
\tau : \text{uce}_{\alpha,\phi}^{A}(\mathcal{L}, \alpha_{L}) \to (\mathcal{M}', \alpha_{M'}).\]
Finally, by using Lemma 4.7 we conclude that \( \tilde{f} \) is a unique homomorphism such that \( \sigma \circ \tau = u_{L}^{\alpha} \).

**Remark 4.11.** In the case of hom-Lie algebras, universal \( \alpha \)-central extensions are closely related to the homology of the hom-Lie algebra with trivial coefficients (see [CIP, CKP1]). However for a hom-Lie–Rinehart algebra \((\mathcal{L}, \alpha_{L})\) over \((A, \phi)\) such relations are not possible since there does not exist a canonical right \((\mathcal{L}, \alpha_{L})\)-module structure on \((A, \phi)\). Such relations are not even possible in the case of Lie–Rinehart algebras [CGM].
DEFINITION 4.12. An $\alpha$-central extension $(\mathcal{M}, \alpha) \overset{i}{\rightarrow} (\mathcal{K}, \alpha_K) \overset{\sigma}{\rightarrow} (\mathcal{L}, \alpha_L)$ of a hom-Lie–Rinehart algebra $(\mathcal{L}, \alpha_L)$ is said to be universal if for any other central extension $(\mathcal{M}', \alpha_{M'}) \overset{i'}{\rightarrow} (\mathcal{L}', \alpha_{L'}) \overset{\tau}{\rightarrow} (\mathcal{L}, \alpha_L)$ there exists a unique morphism $h : (\mathcal{K}, \alpha_K) \rightarrow (\mathcal{L}', \alpha_{L'})$ of hom-Lie–Rinehart algebras such that $\tau \circ h = \sigma$.

PROPOSITION 4.13. Let $(\mathcal{M}, \alpha_M) \overset{i}{\rightarrow} (\mathcal{K}, \alpha_K) \overset{\sigma}{\rightarrow} (\mathcal{L}, \alpha_L)$ be a central extension and $(\mathcal{N}', \alpha_{N'}) \overset{i'}{\rightarrow} (\mathcal{L}', \alpha_{L'}) \overset{\tau}{\rightarrow} (\mathcal{K}, \alpha_K)$ be a universal $\alpha$-central extension. Then the $\alpha$-central extension

$$(\mathcal{N}', \alpha_{N'}) \overset{i'}{\rightarrow} (\mathcal{L}', \alpha_{L'}) \overset{\sigma \circ \tau}{\rightarrow} (\mathcal{L}, \alpha_L)$$

is universal in the sense of Definition 4.12.

Proof. Since $(\mathcal{N}', \alpha_{N'}) \overset{i'}{\rightarrow} (\mathcal{L}', \alpha_{L'}) \overset{\tau}{\rightarrow} (\mathcal{K}, \alpha_K)$ is a universal $\alpha$-central extension, by Theorem 4.9, the hom-Lie–Rinehart algebra $(\mathcal{L}', \alpha_{L'})$ is perfect and consequently $(\mathcal{K}, \alpha_K)$ is also perfect. Next by Lemma 4.4, $(\mathcal{N}', \alpha_{N'}) \overset{i'}{\rightarrow} (\mathcal{L}', \alpha_{L'}) \overset{\sigma \circ \tau}{\rightarrow} (\mathcal{L}, \alpha_L)$ is an $\alpha$-central extension. Let

$$(\mathcal{H}, \alpha_H) \overset{k}{\rightarrow} (\mathcal{F}, \alpha_F) \overset{\gamma}{\rightarrow} (\mathcal{L}, \alpha_L)$$

be a central extension. Then similar to the proof of Theorem 3.8, we can construct a central extension of $(\mathcal{L}', \alpha_{L'})$ via the pull-back diagram with respect to homomorphisms $\sigma \circ \tau$ and $\gamma$ of $(\mathcal{L}', \alpha_{L'})$ given by

$$(\mathcal{N}'', \alpha_{N''}) \overset{k'}{\rightarrow} (\mathcal{P}, \alpha_P) \overset{p_{L'}}{\rightarrow} (\mathcal{L}', \alpha_{L'})$$

where the $A$-module $P$ is given as follows:

$$P = \{(l, f) \in L' \times_{\text{Der}_\phi(A)} F : \sigma \circ \tau(l) = \gamma(f)\}.$$

The map $\alpha_P : P \rightarrow P$ is defined by $\alpha_P(l, f) = (\alpha_{L'}(l), \alpha_F(f))$. Note that by Theorem 4.9, every central extension of $(\mathcal{L}', \alpha_{L'})$ splits uniquely. Therefore, the extension $(\mathcal{N}'', \alpha_{N''}) \overset{k'}{\rightarrow} (\mathcal{P}, \alpha_P) \overset{p_{L'}}{\rightarrow} (\mathcal{L}', \alpha_{L'})$ splits uniquely and the splitting gives a unique homomorphism $h : (\mathcal{L}', \alpha_{L'}) \rightarrow (\mathcal{F}, \alpha_F)$ such that $\gamma \circ h = \sigma \circ \tau$. Hence, the $\alpha$-central extension $(\mathcal{N}'', \alpha_{N''}) \overset{i''}{\rightarrow} (\mathcal{L}', \alpha_{L'}) \overset{\sigma \circ \tau}{\rightarrow} (\mathcal{L}, \alpha_L)$ is universal in the sense of Definition 4.12.

5. Lifting of automorphisms and derivations to central extensions. In this section, we assume that $A$ is an associative commutative algebra and $\phi : A \rightarrow A$ is an algebra epimorphism. The approach developed in this section provides results of [CGM] in the particular case $\alpha = \text{id}$, $\phi = \text{id}$.

Let $(\mathcal{N}, \alpha_N) \overset{i}{\rightarrow} (\mathcal{K}, \alpha_K) \overset{f}{\rightarrow} (\mathcal{L}, \alpha_L)$ be a central extension, where $(\mathcal{K}, \alpha_K)$ is an $\alpha$-perfect hom-Lie–Rinehart algebra. Then $(\mathcal{L}, \alpha_L)$ is also $\alpha$-perfect and...
we get the commutative diagram
\[
\begin{array}{ccc}
\uce_A^{\alpha,\phi}(K, \alpha_K) & \xrightarrow{\uce_A^{\alpha,\phi}(f)} & \uce_A^{\alpha,\phi}(L, \alpha_L) \\
u_K^{\alpha} & & \downarrow u_L^{\alpha} \\
(K, \alpha_K) & \xrightarrow{f} & (L, \alpha_L)
\end{array}
\]

Since the central extension \((N_K, \alpha_N) \xrightarrow{j} (K, \alpha_K) \xrightarrow{\tilde{\iota}} (L, \alpha_L)\) is a universal \(\alpha\)-central extension, by using Proposition 4.13, the \(\alpha\)-central extension induced by the map \(\uce_A^{\alpha,\phi}(K, \alpha_K) \xrightarrow{f \circ u_K^{\alpha}} (L, \alpha_L)\) is universal in the sense of Definition 4.12. Consequently, we get a unique homomorphism
\[h : \uce_A^{\alpha,\phi}(K, \alpha_K) \to \uce_A^{\alpha,\phi}(L, \alpha_L)\]
such that \(u_L^{\alpha} \circ h = f \circ u_K^{\alpha}\), i.e., \(h = \uce_A^{\alpha,\phi}(f)\).

Since \((N_L, \alpha_N) \xrightarrow{j} (L, \alpha_L)\) is a universal \(\alpha\)-central extension, we have a unique homomorphism \(g : \uce_A^{\alpha,\phi}(L, \alpha_L) \to \uce_A^{\alpha,\phi}(K, \alpha_K)\) such that
\[f \circ u_K^{\alpha} \circ g = u_L^{\alpha}.
\]

From Lemma 4.7, it follows that \(\uce_A^{\alpha,\phi}(f) \circ g = \text{id}_{\uce_A^{\alpha,\phi}(L, \alpha_L)}\) and \(g \circ \uce_A^{\alpha,\phi}(f) = \text{id}_{\uce_A^{\alpha,\phi}(K, \alpha_K)}\), i.e., we have an isomorphism
\[\uce_A^{\alpha,\phi}(K, \alpha_K) \cong \uce_A^{\alpha,\phi}(L, \alpha_L).
\]

It is immediate to see that we get a central extension
\[\(\mathcal{P}, \alpha_P\) \xrightarrow{k} \uce_A^{\alpha,\phi}(L, \alpha_L) \xrightarrow{u_K^{\alpha} \circ \uce_A^{\alpha,\phi}(f)^{-1}} (K, \alpha_K)\]
where the underlying \(A\)-module \(P\) equals
\[\ker(u_K^{\alpha} \circ \uce_A^{\alpha,\phi}(f)^{-1}) = \uce_A^{\alpha,\phi}(f)(\ker(u_K^{\alpha})).\]

Let \(\text{Aut}(L, \alpha_L)\) denote the set of automorphisms of a hom-Lie–Rinehart algebra \((L, \alpha_L)\) in the category \(hLR^\phi_A\). Then we have the following result on lifting of automorphisms.

**Main Theorem 5.1.** Let \((N, \alpha_N) \xrightarrow{j} (K, \alpha_K) \xrightarrow{f} (L, \alpha_L)\) be a central extension, where \((K, \alpha_K)\) is an \(\alpha\)-perfect hom-Lie–Rinehart algebra. If \(h \in \text{Aut}(L, \alpha_L)\) then there exists a unique \(\tilde{h} \in \text{Aut}(K, \alpha_K)\) such that \(h \circ f = f \circ \tilde{h}\) (i.e., there exists a lifting of \(h\)) if and only if \(\uce_A^{\alpha,\phi}(h)(P) = P\). Moreover, \(\tilde{h}(\ker(f)) = \ker(f)\) and we have a group isomorphism
\[\Phi : \{h \in \text{Aut}(L, \alpha_L) : \uce_A^{\alpha,\phi}(h)(P) = P\} \to \{g \in \text{Aut}(K, \alpha_K) : g(\ker(f)) = \ker(f)\}\]
given by \(\Phi(h) = \tilde{h}\).
Proof. First assume that there exists $\tilde{h} \in \text{Aut}(\mathcal{K}, \alpha_K)$ such that $h \circ f = f \circ \tilde{h}$. Then it is clear that

$$\text{ucc}_A^{\alpha, \phi}(h) \circ \text{ucc}_A^{\alpha, \phi}(f) = \text{ucc}_A^{\alpha, \phi}(f) \circ \text{ucc}_A^{\alpha, \phi}(\tilde{h}).$$

Therefore,

$$\text{ucc}_A^{\alpha, \phi}(h)(\text{ucc}_A^{\alpha, \phi}(f)(\text{Ker}(u_K^n))) = (\text{ucc}_A^{\alpha, \phi}(f) \circ \text{ucc}_A^{\alpha, \phi}(\tilde{h}))(\text{Ker}(u_K^n))$$

$$= \text{ucc}_A^{\alpha, \phi}(f)(\text{Ker}(u_K^n)),$$

i.e., $\text{ucc}_A^{\alpha, \phi}(h)(P) = P$.

Conversely, assume that $\text{ucc}_A^{\alpha, \phi}(h)(P) = P$. Then we consider the central extension

$$(\mathcal{P}, \alpha_P) \xrightarrow{k} \text{ucc}_A^{\alpha, \phi}(\mathcal{L}, \alpha_L) \xrightarrow{u_K^n \circ \text{ucc}_A^{\alpha, \phi}(f)^{-1}} (\mathcal{K}, \alpha_K)$$

and define a map $\tilde{h} : (\mathcal{K}, \alpha_K) \to (\mathcal{K}, \alpha_K)$ by

$$\tilde{h}(k) = u_K^n \circ \text{ucc}_A^{\alpha, \phi}(f)^{-1}(\text{ucc}_A^{\alpha, \phi}(h)k)$$

for any $k \in K$. Here, $\tilde{k} \in \text{ucc}_A^{\alpha, \phi}(L)$ is such that $u_K^n \circ \text{ucc}_A^{\alpha, \phi}(f)^{-1}(\tilde{k}) = k$.

Since $\text{ucc}_A^{\alpha, \phi}(h)(P) = P$, it is immediate to see that

$$\text{Ker}(u_K^n \circ \text{ucc}_A^{\alpha, \phi}(f)^{-1} \circ \text{ucc}_A^{\alpha, \phi}(h)) = \text{Ker}(u_K^n \circ \text{ucc}_A^{\alpha, \phi}(f)^{-1}) = P.$$

Then it is easily seen that $\tilde{h}$ is an automorphism. By using the condition $u_L^f = f \circ u_K^n \circ \text{ucc}_A^{\alpha, \phi}(f)^{-1}$ we get $f \circ \tilde{h} = h \circ f$. The uniqueness of $\tilde{h}$ follows from Lemma 3.11. Also from $f \circ h = h \circ f$ we obtain

$$\tilde{h}(\text{Ker}(f)) = \text{Ker}(f).$$

If $h \in \text{Aut}(\mathcal{L}, \alpha_L)$ and $\text{ucc}_A^{\alpha, \phi}(h)(P) = P$, then $h$ lifts to a unique $\tilde{h} \in \text{Aut}(\mathcal{K}, \alpha_K)$ such that $h \circ f = f \circ \tilde{h}$ and hence $\Phi$ is well-defined. Moreover, by Lemma 3.11 the map $\Phi$ is injective. Let $g \in \text{Aut}(\mathcal{K}, \alpha_K)$ be such that $g(\text{Ker}(f)) = \text{Ker}(f)$. Define $h : (\mathcal{L}, \alpha_L) \to (\mathcal{L}, \alpha_L)$ by

$$h(x) = f(g(\tilde{x}))$$

for any $x \in L$. Here, $\tilde{x} \in K$ such that $f(\tilde{x}) = x$. Using the property $g(\text{Ker}(f)) = \text{Ker}(f)$, we get $h \in \text{Aut}(\mathcal{L}, \alpha_L)$. It is clear that $g \in \text{Aut}(\mathcal{K}, \alpha_K)$ is a lifting of $h \in \text{Aut}(\mathcal{L}, \alpha_L)$ and since a lifting exists, we have $\text{ucc}_A^{\alpha, \phi}(h)(P) = P$. Hence, $\Phi$ is a group isomorphism. ■

**Definition 5.2.** Let $(\mathcal{L}, \alpha_L)$ be a hom-Lie–Rinehart algebra over $(A, \phi)$. Then a map $D : L \to L$ is said to be an $\alpha$-derivation of the hom-Lie–Rinehart algebra $(\mathcal{L}, \alpha_L)$ if the following conditions hold:

- $D \circ \alpha_L = \alpha_L \circ D$;
- $D([x, y]) = [D(x), \alpha_L(y)] + [\alpha_L(x), D(y)]$ for any $x, y \in L$;
• there exists a \( \phi \)-derivation \( \sigma_D \in \text{Der}_\phi(A) \) called the symbol of \( D \) such that 
\[
\sigma_D \circ \phi = \phi \circ \sigma_D \quad \text{and} \quad D(a.x) = \phi(a).D(x) + \sigma_D(a).\alpha(x) \quad \text{for any} \quad x \in L \quad \text{and} \quad a \in A;
\]
• \( \sigma_D(x(a)) = \alpha(x)(\sigma_D(a)) + D(x)(\phi(a)) \quad \text{for any} \quad x \in L \quad \text{and} \quad a \in A. \)

When \( A = R \) and \( \phi = \text{id}_R \), the hom-Lie–Rinehart algebra \( (\mathcal{L}, \alpha_L) \) becomes a hom-Lie algebra and an \( \alpha \)-derivation of \( (\mathcal{L}, \alpha_L) \) is simply an \( \alpha \)-derivation of the hom-Lie algebra \( (L, [\cdot, \cdot], \alpha_L) \) (as defined in Sh12).

**Example 5.3.** Let \( (\mathcal{L}, \alpha_L) \) be a hom-Lie–Rinehart algebra over \( (A, \phi) \) and \( x \in L \) be a fixed point of \( \alpha_L \), i.e., \( \alpha_L(x) = x \). Then define \( D : L \to L \) by \( D(y) = [x, y] \) for any \( y \in L \). Consider \( \sigma_D \in \text{Der}_\phi(A) \) defined by \( \sigma_D(a) = x(a) \). Then \( D \) is an \( \alpha \)-derivation of the hom-Lie–Rinehart algebra \( (\mathcal{L}, \alpha_L) \) with symbol \( \sigma_D \).

**Proposition 5.4.** Let \( (N, \alpha_N) \xrightarrow{j} (K, \alpha_K) \xrightarrow{\omega} (\mathcal{L}, \alpha_L) \) be a central extension of the hom-Lie–Rinehart algebra \( (\mathcal{L}, \alpha_L) \). If \( D \) and \( D' \) are \( \alpha \)-derivations of \( (K, \alpha_K) \) with the same symbol \( \sigma_D \) such that \( f \circ D = f \circ D' \) then \( D|_{\{K,K\}} = D'|_{\{K,K\}} \).

**Proof.** Since \( f \circ D = f \circ D' \), for any \( x \in K \) we have \( D(x) - D'(x) \in \text{Ker}(f) \subset Z_A(K, \alpha_K) \). Thus for any \( x, y \in K \) and \( a \in A \), we have \( a[D(x), y] = a[D'(x), y] \). Let \( x \in \{K, K\} \), i.e., \( x = \sum a_i[x_{i1}, x_{i2}] \). Then 
\[
D(x) = D\left( \sum_i a_i[x_{i1}, x_{i2}] \right) = \sum_i \phi(a_i).D([x_{i1}, x_{i2}]) + \sigma_D(a).\alpha_K([x_{i1}, x_{i2}])
\]
\[
= \sum_i \phi(a_i).([D(x_{i1}), \alpha_K(x_{i2})] + [\alpha_K(x_{i1}), D(x_{i2})])
\]
\[
+ \sigma_D(a).\alpha_K([x_{i1}, x_{i2}])
\]
\[
= D'\left( \sum_i a_i[x_{i1}, x_{i2}] \right) = D'(x),
\]
as desired. \( \blacksquare \)

**Proposition 5.5.** Let \( D \) be an \( \alpha \)-derivation of an \( \alpha \)-perfect hom-Lie–Rinehart algebra \( (\mathcal{L}, \alpha_L) \) with symbol \( \sigma_D \). Define a map \( \text{uce}^{\alpha, \phi}_A(D) : \text{uce}^{\alpha, \phi}_A(L) \to \text{uce}^{\alpha, \phi}_A(L) \) by 
\[
\text{uce}^{\alpha, \phi}_A(D)(\alpha_L(a), \alpha_L(x), \alpha_L(y))
\]
\[
= (\sigma_D(a), \alpha_L^2(x), \alpha_L^2(y)) + (\phi(a), D(\alpha_L(x)), \alpha_L^2(y)) + (\phi(a), \alpha_L^2(x), D(\alpha_L(y)))
\]
for any \( x, y \in L \). Then \( \text{uce}^{\alpha, \phi}_A(D) \) is an \( \alpha \)-derivation of the hom-Lie–Rinehart algebra \( \text{uce}^{\alpha, \phi}_A(L, \alpha_L) \) with symbol \( \sigma_D \). Moreover, 
\[
\text{uce}^{\alpha, \phi}_A(D)(\text{Ker}(u^\alpha_L)) \subset \text{Ker}(u^\alpha_L).
\]
Proof. By the definition of the hom-Lie–Rinehart algebra \( \text{ucc}_A^{\alpha, \phi}(\mathcal{L}, \alpha_L) \) and the map \( \text{ucc}_A^{\alpha, \phi}(D) \), it easily follows that \( \text{ucc}_A^{\alpha, \phi}(D) \) is an \( \alpha \)-derivation of \( \text{ucc}_A^{\alpha, \phi}(\mathcal{L}, \alpha_L) \) with symbol \( \sigma_D \).

Next, let \((a, \alpha_L(x), \alpha_L(y)) \in \text{Ker}(u_L^\alpha)\), i.e., \( D(a[\alpha_L(x), \alpha_L(y)]) = 0 \). Then
\[
\begin{align*}
  u_L^\alpha((D)(a, \alpha_L(x), \alpha_L(y))) &= u_L^\alpha((\sigma_D(a), \alpha_L^2(x), \alpha_L^2(y)) + (\phi(a), D(\alpha_L(x)), \alpha_L^2(y)) \\
  &\quad + (\phi(a), \alpha_L^2(x), D(\alpha_L(y)))) \\
  &= \sigma_D(a),[\alpha_L^2(x), \alpha_L^2(y)] + \phi(a),[D(\alpha_L(x)), \alpha_L^2(y)] + \phi(a),[\alpha_L^2(x), D(\alpha_L(y))] \\
  &= D(a,[\alpha_L(x), \alpha_L(y)]) = 0.
\end{align*}
\]
Thus \( \text{ucc}_A^{\alpha, \phi}(D)(\text{Ker}(u_L^\alpha)) := D_u^\phi(\text{Ker}(u_L^\alpha)) \subset \text{Ker}(u_L^\alpha) \).

Remark 5.6. Let \((K, \alpha_K), (\mathcal{L}, \alpha_L)\) be \( \alpha \)-perfect hom-Lie–Rinehart algebras, and \( f : (K, \alpha_K) \rightarrow (\mathcal{L}, \alpha_L)\) be a homomorphism in the category \( h\text{L}R_A^\phi \). If \( D_K \) is an \( \alpha \)-derivation of \((K, \alpha_K)\) and \( D_L \) is an \( \alpha \)-derivation of \((\mathcal{L}, \alpha_L)\) with the same symbol \( \sigma_D \in \text{Der}_\phi(A) \) such that \( f \circ D_K = D_L \circ f \) then \( \text{ucc}_A^{\alpha, \phi}(f) \circ D_u^\phi = D_u^\phi \circ \text{ucc}_A^{\alpha, \phi}(f) \).

Theorem 5.7. Let \((\mathcal{N}, \alpha_N) \rightarrow (K, \alpha_K) \rightarrow (\mathcal{L}, \alpha_L)\) be a central extension, where \((K, \alpha_K)\) is an \( \alpha \)-perfect hom-Lie–Rinehart algebra. An \( \alpha \)-derivation \( D_L \) of \((\mathcal{L}, \alpha_L)\) with symbol \( \sigma_D \in \text{Der}_\phi(A) \) lifts to a unique \( \alpha \)-derivation \( D_K \) of \((K, \alpha_K)\) with the same symbol \( \sigma_D \in \text{Der}_\phi(A) \), satisfying the condition \( D_L \circ f = f \circ D_K \) if and only if \( D_u^\phi(P) \subset P \). Moreover, \( \text{Ker}(f) \) is invariant under \( D_K \), i.e., \( D_K(\text{Ker}(f)) \subset \text{Ker}(f) \).

Proof. Assume that there exists a lift \( D_K \) of the \( \alpha \)-derivation \( D_L \) such that \( D_L \circ f = f \circ D_K \). Then by Proposition 5.4 and the perfectness of \((K, \alpha_K)\), the lift \( D_K \) is unique. Next, we have
\[
D_u^\phi(P) = D_u^\phi(\text{ucc}_A^{\alpha, \phi}(f)(\text{Ker}(u_K^\alpha))) = (D_u^\phi \circ \text{ucc}_A^{\alpha, \phi}(f))(\text{Ker}(u_K^\alpha)) \\
= (\text{ucc}_A^{\alpha, \phi}(f) \circ D_u^\phi)(\text{Ker}(u_K^\alpha)) \subset \text{ucc}_A^{\alpha, \phi}(f)(\text{Ker}(u_K^\alpha)) = P.
\]
Here we use the commutativity condition \( D_u^\phi \circ \text{ucc}_A^{\alpha, \phi}(f) = \text{ucc}_A^{\alpha, \phi}(f) \circ D_u^\phi \) from Remark 5.6 and the fact that \( D_u^\phi(\text{Ker}(u_K^\alpha)) \subset \text{Ker}(u_K^\alpha) \).

Conversely, let \( D_L \) be an \( \alpha \)-derivation of \((\mathcal{L}, \alpha_L)\) with symbol \( \sigma_D \in \text{Der}_\phi(A) \) such that \( D_u^\phi(P) \subset P \). Also, consider the central extension
\[
(\mathcal{P}, \alpha_P) \xrightarrow{k} \text{ucc}_A^{\alpha, \phi}(\mathcal{L}, \alpha_L) \xrightarrow{u_K^\alpha \circ \text{ucc}_A^{\alpha, \phi}(f)^{-1}} (K, \alpha_K)
\]
and define \( D_K : K \rightarrow K \) by
\[
D_K(k) = u_K^\alpha \circ \text{ucc}_A^{\alpha, \phi}(f)^{-1}(\text{ucc}_A^{\alpha, \phi}(D_L)(\bar{k}))
\]
for any \( k \in K \). Here, \( \bar{k} \in \mathfrak{ucc}_A^{\alpha, \phi}(L) \) is such that \( u^K_\alpha \circ \mathfrak{ucc}_A^{\alpha, \phi}(f)^{-1} \bar{k} = k \). Then by the definition of \( D_K \) and the fact that \( D_L^u(P) \subset P \), it easily follows that \( D_K \) is a well-defined \( \alpha \)-derivation of \((K, \alpha_K)\) with symbol \( \sigma_D \). If \( k \in \text{Ker}(f) \) then
\[
f(D_K(k)) = D_L(f(k)) = 0,
\]
i.e., \( D_K(\text{Ker}(f)) \subset \text{Ker}(f) \). □

**Remark 5.8.** In particular, if \( A = R \) and \( \phi = \text{id}_A \) then any hom-Lie–Rinehart algebra \((\mathcal{L}, \alpha_L)\) over \((A, \phi)\) is simply a hom-Lie algebra \((L, [-, -]_L, \alpha_L)\). Consequently, Theorems 5.1 and 5.7 yield a necessary and sufficient condition for lifting automorphisms and \( \alpha \)-derivations (defined in [Sh12]) of \( \alpha \)-perfect hom-Lie algebras to central extensions. In fact, using the same proof and techniques one can get results on lifting \( \alpha^k \)-derivations (defined in [Sh12]) of \( \alpha \)-perfect hom-Lie algebras.

**6. Non-abelian tensor product of hom-Lie–Rinehart algebras.**
Let \( A \) be an associative commutative algebra and \( \phi : A \to A \) be an algebra endomorphism. First, we define a quasi-hom-action of a hom-Lie–Rinehart algebra on another hom-Lie–Rinehart algebra in the category \( hLR^\phi_A \).

**Definition 6.1.** Let \((\mathcal{L}, \alpha_L)\) and \((\mathcal{M}, \alpha_M)\) be hom-Lie–Rinehart algebras over \((A, \phi)\). A map \( \theta : L \times M \to M \) written \( \theta(x, m) = x_m \) is called a quasi-hom-action of \((\mathcal{L}, \alpha_L)\) on \((\mathcal{M}, \alpha_M)\) if the following identities hold:

- \( x(a.m) = \phi(a).x_m + x(a).\alpha_M(m) \);
- \( [x, y]_{\alpha_M}(m) = \alpha_L(x)(y_m) - \alpha_L(y)(x_m) \);
- \( \alpha_L(x)[m, n] = [x_m, \alpha_M(n)] + [\alpha_M(m), x_n] \);
- \( x_m(\phi(a)) = \alpha_L(x)(m(a)) - \alpha_M(m)(x(a)) \);
- \( \alpha_M(x_m) = \alpha_L(x)\alpha_M(m) \)

for all \( x, y \in L \), \( m, n \in M \) and \( a \in A \).

In particular, if \( A = R \) and \( \phi = \text{id}_A \) then the above definition gives a hom-action of the hom-Lie algebra \((L, [-, -]_L, \alpha_L)\) on the hom-Lie algebra \((M, [-, -]_M, \alpha_M)\) (as defined in [CKPT]).

**Example 6.2.** Let \((\mathcal{L}, \alpha_L)\) be a hom-Lie–Rinehart algebra over \((A, \phi)\). Then the left action of \((\mathcal{L}, \alpha_L)\) on itself by the underlying hom-Lie bracket \( (x, y) \mapsto [x, y]_L \) is a quasi-hom-action of \((\mathcal{L}, \alpha_L)\) on itself.

Let \((\mathcal{L}, \alpha_L)\) and \((\mathcal{M}, \alpha_M)\) be hom-Lie–Rinehart algebras over \((A, \phi)\) which have quasi-actions on each other. Then the two quasi-actions \( L \times M \to M \) written \( (x, m) \mapsto x_m \) and \( M \times L \to L \) written \( (m, x) \mapsto m.x \) are said to be compatible if

- \( x.m(a) = -m.x(a) \).
for all \( x, y \in L, m, n \in M \) and \( a \in A \).

Let \((\mathcal{L}, \alpha_L)\) be a hom-Lie–Rinehart algebra (over \((A, \phi)\)). If \((\mathcal{N}, \alpha_N)\) and \((\mathcal{P}, \alpha_P)\) are quasi-ideals of \((\mathcal{L}, \alpha_L)\) then the quasi-hom-actions of these quasi-ideals on each other are defined by

\[
\phi \left( (x, y) \right) = \phi(a)[f(x, m), f(y, n)] - \phi(a)f(x, m)(b)\alpha_N(f(y, n)) + \phi(b)f(y, n)(a)\alpha_N(f(x, m))
\]

for all \( x, y \in L, a, b \in A \) and \( m, n \in M \). A hom-Lie–Rinehart pairing \( f : L \times M \to N \) is called a hom-Lie–Rinehart pairing if the following conditions hold:

\[
\begin{align*}
& f(x, m)(a) = (x^m)(a) \quad \text{for all} \quad a \in A; \\
& f([x, y], \alpha_M(m)) = f(\alpha_L(x), y_m) - f(\alpha_L(y), x_m) \quad \text{for all} \quad x, y \in L; \\
& f(\alpha_L(x), [m, n]) = f(n_x, \alpha_M(m)) - f(m_x, \alpha_M(n)) \quad \text{for all} \quad x, m \in L; \\
& f(\alpha_L(x), \alpha_M(m)) = \alpha_N(f(x, m)) \quad \text{for all} \quad x, m \in L; \\
& f(a, m_x, b, y_n) = -\phi(ab)[f(x, m), f(y, n)] - \phi(a)f(x, m)(b)\alpha_N(f(y, n)) + \phi(b)f(y, n)(a)\alpha_N(f(x, m))
\end{align*}
\]

for all \( x, y \in L, a, b \in A \) and \( m, n \in M \). A hom-Lie–Rinehart pairing \( f : L \times M \to N \) is called universal if for any other hom-Lie–Rinehart pairing \( g : L \times M \to N' \) there exists a unique morphism \( h : (\mathcal{N}, \alpha_N) \to (\mathcal{N}', \alpha_N') \) in \( hLR^\phi_A \) such that \( h \circ f = g \).

Example 6.4. Let \((\mathcal{L}, \alpha_L)\) be a hom-Lie–Rinehart algebra and \((\mathcal{N}, \alpha_N)\) and \((\mathcal{P}, \alpha_P)\) two quasi-ideals of \((\mathcal{L}, \alpha_L)\). Then the map \( f : N \times P \to N \cap P \) defined by \( f(n, p) = [n, p] \) is a hom-Lie–Rinehart pairing.

Let \((\mathcal{L}, \alpha_L), (\mathcal{M}, \alpha_M) \in hLR^\phi_A \) and suppose both have compatible quasi-hom-actions on each other. Define an \( A \)-module \( L \star M \) which is spanned by the symbols \( x \star m \) satisfying the following conditions:

\[
\begin{align*}
& (x + ky) \star m = x \star m + k(y \star m) \quad \text{for all} \quad x, y, m \in L; \\
& x \star (m + kn) = x \star m + k(x \star n) \quad \text{for all} \quad x, y, m, n \in L; \\
& \alpha_L(x) \star [m, n] = x \star \alpha_M(m) - m \star \alpha_M(x) \quad \text{for all} \quad x, m, n \in L; \\
& a(m_x) \star b(y_n) = -b(n_y) \star a(x_m) \quad \text{for all} \quad x, y, m, n \in L; \\
& a(m_x) \star b(y_n) = \phi(ab)[m_x \star y_n - \phi(a)x_m(b)\alpha_L(y) - \alpha_M(n) + \phi(b)y_n(a)\alpha_L(x) - \alpha_M(m)] \quad \text{for all} \quad x, y, m, n \in L, a, b \in A.
\end{align*}
\]

Proposition 6.5. The tuple

\[(\mathcal{L} \star \mathcal{M}, \alpha_{L \star M}) := (A, L \star M, \phi, [-, -]_{L \star M}, \alpha_{L \star M}, \rho_{L \star M})\]

is a hom-Lie–Rinehart algebra over \((A, \phi)\) where
• the endomorphism $\alpha_{L*M} : L * M \to L * M$ is defined by
  $\alpha_{L*M}(x * m) = \alpha_L(x) * \alpha_M(m)$;

• hom-Lie bracket $[-, -]_{L*M}$ is given by
  $[a(x * m), b(y * n)]_{L*M} = -a(m^r x) * b(y^n)$;

• the anchor map $\rho_{L*M} : L * M \to \text{Der}_\phi(A)$ is defined by
  $\rho_{L*M}(x * m)(a) = x m(a)$,

for any $x, y \in L$, $m, n \in M$ and $a \in A$.

**Definition 6.6.** Let $(\mathcal{L}, \alpha_L)$ and $(\mathcal{M}, \alpha_M)$ be hom-Lie–Rinehart algebras over $(A, \phi)$ which have compatible quasi-hom-actions on each other. Then the hom-Lie–Rinehart algebra $(\mathcal{L} \ast \mathcal{M}, \alpha_{L*M})$ given by Proposition 6.5 is called the non-abelian tensor product of $(\mathcal{L}, \alpha_L)$ and $(\mathcal{M}, \alpha_M)$.

In particular, if $A = R$ and $\phi = \text{id}_A$ then the above definition gives the non-abelian tensor product of the hom-Lie algebras $(L, [-, -]_L, \alpha_L)$ and $(M, [-, -]_M, \alpha_M)$ (as defined in [CKP1]). Moreover, if $\alpha = \text{id}$ and $\phi = \text{id}$, then one recovers the notion of non-abelian tensor product of Lie–Rinehart algebras defined in [CGM].

**Remark 6.7.** Note that the map $f : L \times M \to L \ast M$ defined by $f(x, m) = x \ast m$ is a hom-Lie–Rinehart pairing. Let $(\mathcal{N}, \alpha_N)$ be a hom-Lie–Rinehart algebra and $g : L \times M \to N$ be a hom-Lie–Rinehart pairing. Then define
  \[ \Phi : (\mathcal{L} \ast \mathcal{M}, \alpha_{L*M}) \to (\mathcal{N}, \alpha_N) \]
by $\Phi(x + m) = g(x, m)$. Then $\Phi$ is a hom-Lie–Rinehart algebra homomorphism since $g$ is a hom-Lie–Rinehart pairing. Also by the definition of $\Phi$, we have $\Phi \circ f = g$. Therefore, $f : L \times M \to L * M$ is a universal hom-Lie–Rinehart pairing.

**Remark 6.8.** Consider the hom-Lie–Rinehart algebra $(\mathcal{M} \ast \mathcal{L}, \alpha_{M*L})$. Define $g : L \times M \to M \ast L$ by $g(x, m) = m \ast x$. It easily follows that $g$ is a universal hom-Lie–Rinehart pairing. Hence, by using the universality and Remark 6.7 we have an isomorphism of hom-Lie–Rinehart algebras which implies that
  \[ (\mathcal{L} \ast \mathcal{M}, \alpha_{L*M}) \cong (\mathcal{M} \ast \mathcal{L}, \alpha_{M*L}). \]

The next result follows easily from Definition 6.6 of non-abelian tensor product.

**Proposition 6.9.** There exist hom-Lie–Rinehart algebra homomorphisms
  \[ \pi_1 : (\mathcal{L} \ast \mathcal{M}, \alpha_{L*M}) \to (\mathcal{L}, \alpha_L) \quad \text{and} \quad \pi_2 : (\mathcal{L} \ast \mathcal{M}, \alpha_{L*M}) \to (\mathcal{M}, \alpha_M) \]
defined by $\pi_1(a.(x * m)) = -a(m^r x)$ and $\pi_2(a.(x * m)) = a(x m)$ for $x \in L$, $m \in M$ and $a \in A$. Moreover, $\text{Ker}(\pi_1) \subset Z_A((\mathcal{L} \ast \mathcal{M}, \alpha_{L*M}))$ and $\text{Ker}(\pi_2) \subset Z_A((\mathcal{L} \ast \mathcal{M}, \alpha_{L*M}))$. 
Proof. First observe that for all \( a \in A, m \in M \) and \( x \in L \) we have \( \pi_1(x * m)(a) = -(^m x)(a) = (x * m)(a) \).

Moreover, by the definitions of \( L * M \), of compatible quasi-hom-actions, and of \([-,-]_{L*M}\), we obtain
\[
\pi_1([x * m, a.(y * n)]) = \pi_1\left( -\phi(a)(^m x * ^y n) - ^m x(a).\alpha_L(y) * \alpha_M(n) \right) \\
= \phi(a).[^m x, ^y n] + ^m x(a).\alpha_M(n)\alpha_L(y) \\
= [\pi_1(x * m), a.\pi_1(y * n)].
\]

By the definition of quasi-hom-action, \( \pi_1 \circ \alpha_{L*M} = \alpha_L \circ \pi_1 \). Thus,
\[
\pi_1 : (\mathcal{L} * \mathcal{M}, \alpha_{L*M}) \rightarrow (\mathcal{L}, \alpha_L)
\]
is a hom-Lie–Rinehart algebra homomorphism.

Similarly, \( \pi_2 : (\mathcal{L} * \mathcal{M}, \alpha_{L*M}) \rightarrow (\mathcal{M}, \alpha_M) \) is a hom-Lie–Rinehart algebra homomorphism. Now, let \( a.(x * m) \in \text{Ker}(\pi_1) \), i.e., \( a.(^m x) = 0 = a.(^x m) \). Then
\[
[b.(a.(x * m)), c.(y * n)] = -b.(a.(^m x)) * c.(^y n) = 0,
\]
\[
[b.\alpha_{L*M}(a.(x * m)), c.(y * n)] = -b.(\phi(a).\alpha_M(m)\alpha_L(x)) * c.(^y n) = 0
\]
for \( b \in A \) and \( c.(^y n) \in L * M \). Furthermore, for any \( b \in A \),
\[
(a.(x * m))(b) = a.(^x m)(b) = 0.
\]

Hence, \( \text{Ker}(\pi_1) \in Z_A((\mathcal{L} * \mathcal{M}, \alpha_{L*M})) \). Similarly,
\[
\text{Ker}(\pi_2) \in Z_A((\mathcal{L} * \mathcal{M}, \alpha_{L*M})). \quad \Box
\]

Let \( (\mathcal{L}, \alpha_L) \) and \( (\mathcal{M}, \alpha_M) \) be hom-Lie–Rinehart algebras over \( (A, \phi) \) with trivial quasi-hom-actions on each other. If \( \alpha_M \) and \( \alpha_L \) are surjective then by the definition of non-abelian tensor product for any \( x, y \in L \) and \( m, n \in M \), we have
\[
[x, y] * m = 0 = x * [m, n].
\]

Further, we have an isomorphism of \( A \)-modules \( \Phi : L * M \rightarrow L^{ab} \otimes M^{ab} \) defined by \( \Phi(x * m) = x \otimes m \) for any \( x \in L \) and \( m \in M \). It follows that \( \Phi \) gives an isomorphism of hom-Lie–Rinehart algebras. This implies
\[
(\mathcal{L} * \mathcal{M}, \alpha_{L*M}) \cong (\mathcal{L}^{ab} \otimes \mathcal{M}^{ab}, \alpha_{L^{ab} \otimes M^{ab}})
\]
where \( (\mathcal{L}^{ab} \otimes \mathcal{M}^{ab}, \alpha_{L^{ab} \otimes M^{ab}}) \) is a hom-Lie–Rinehart algebra with underlying \( A \)-module \( L^{ab} \otimes M^{ab} \) \( (L^{ab} = L/[L, L] \) and \( M^{ab} = M/[M, M] \) are the abelianizations of hom-Lie algebras), zero bracket, trivial anchor map, and the map \( \alpha_{L^{ab} \otimes M^{ab}} \) is induced by \( \alpha_L \) and \( \alpha_M \).

Let \( \sigma : (\mathcal{L}_1, \alpha_{L_1}) \rightarrow (\mathcal{L}_2, \alpha_{L_2}) \) and \( \tau : (\mathcal{M}_1, \alpha_{M_1}) \rightarrow (\mathcal{M}_2, \alpha_{M_2}) \) be morphisms in \( hLB^\phi_A \). Also assume that \( (\mathcal{L}_i, \alpha_{L_i}) \) and \( (\mathcal{M}_i, \alpha_{M_i}) \) (for \( i = 1, 2 \)) have compatible quasi-hom-actions on each other. We say that \( \sigma \) and \( \tau \)
preserve these quasi-hom-actions if
(6.1) \[ \sigma^{(m \cdot x)} = \tau^{(m)} \sigma(x) \] and \[ \tau^{(x \cdot m)} = \sigma(x) \tau(m) \] for \( x \in L_1 \) and \( m \in M_1 \).

Now, assuming that \( \sigma \) and \( \tau \) preserve the quasi-hom-actions, consider the \( A \)-linear map \( \sigma \ast \tau : L_1 \ast M_1 \to L_2 \ast M_2 \) with \( x \ast m \mapsto \sigma(x) \ast \tau(m) \) for \( x \in L_1 \) and \( m \in M_1 \). Then by (6.1),

\[ \sigma \ast \tau : (L_1 \ast M_1, \alpha_{L_1 \ast M_1}) \to (L_2 \ast M_2, \alpha_{L_2 \ast M_2}) \]

is a morphism in \( hLR^A \).

**Proposition 6.10.** Let \( (L_1, \alpha_{L_1}) \xrightarrow{f} (L_2, \alpha_{L_2}) \xrightarrow{g} (L_3, \alpha_{L_3}) \) be a short exact sequence in \( hLR^A \) and \( (M, \alpha_M) \) be a hom-Lie–Rinehart algebra over \( (A, \phi) \). If for each \( i \in \{1, 2, 3\} \), \( (M, \alpha_M) \) and \( (L_i, \alpha_{L_i}) \) have compatible quasi-hom-actions on each other and the morphisms \( f \) and \( g \) preserve these quasi-hom-actions, i.e.,

\[ f(m \cdot x) = m \cdot f(x), \quad x \cdot m = f(x) \cdot m, \quad g(m \cdot y) = m \cdot g(y) \] and \( y \cdot m = g(y) \cdot m \)

for all \( x \in L_1 \), \( y \in L_2 \) and \( m \in M \), then the sequence

\[ (L_1 \ast M, \alpha_{L_1 \ast M}) \xrightarrow{f \ast \text{id}_M} (L_2 \ast M, \alpha_{L_2 \ast M}) \xrightarrow{g \ast \text{id}_M} (L_3 \ast M, \alpha_{L_3 \ast M}) \]

is exact and the map \( g \ast \text{id}_M \) is surjective.

**Proof.** First, observe that the morphism \( g \ast \text{id}_M \) is surjective since \( g \) is surjective and \( \text{Im}(f \ast \text{id}_M) \subseteq \text{Ker}(g \ast \text{id}_M) \) since \( \text{Im}(f) = \text{Ker}(g) \). Next, observe that

- \( 0 = x \ast m(a) = x \cdot m(a) = f(x) \cdot m(a) = ((f \ast \text{id}_M)(x \ast m))(a) \);
- \[ [f(x) \ast m, y \cdot n] = m \cdot f(x) \cdot y \cdot n = f(m \cdot x) \cdot y \cdot n = f \ast \text{id}_M(m \cdot x \cdot y \cdot n) \]

for all \( x \in L_1 \), \( m \in M \) and \( a \in A \). Therefore, \( \text{Im}(f \ast \text{id}_M) \) is an ideal of \( (L_2 \ast M, \alpha_{L_2 \ast M}) \). Let us consider the quotient hom-Lie–Rinehart algebra \( (L_2 \ast M, \text{Im}(f \ast \text{id}_M), \alpha_{L_2 \ast M}) \) where the underlying \( A \)-module is the quotient module \( L_2 \ast M/\text{Im} g(f \ast \text{id}_M) \) and \( \alpha_{L_2 \ast M} \) is the induced linear map. Next, to show that \( \text{Ker}(g \ast \text{id}_M) \subseteq \text{Im}(f \ast \text{id}_M) \) we prove that

\[ (L_2 \ast M, \text{Im}(f \ast \text{id}_M), \alpha_{L_2 \ast M}) \cong (L_3 \ast M, \alpha_{L_3 \ast M}) \]

Since \( \text{Im}(f \ast \text{id}_M) \subseteq \text{Ker}(g \ast \text{id}_M) \), we have a natural surjective homomorphism

\[ \Psi : (L_2 \ast M, \text{Im}(f \ast \text{id}_M), \alpha_{L_2 \ast M}) \to (L_3 \ast M, \alpha_{L_3 \ast M}) \]

Define \( \varphi : L_3 \times M \to (L_2 \ast M)/\text{Im}(f \ast \text{id}_M) \) by \( \varphi(t, m) = p \ast m + \text{Im}(f \ast \text{id}_M) \) for any \( t \in L_3 \), \( p \in L_2 \) such that \( g(p) = t \) and \( m \in M \). It follows easily that \( \varphi \) is a hom-Lie–Rinehart pairing and hence by the universal property of non-abelian tensor product we have a unique homomorphism in \( hLR^A \), say

\[ \Phi : (L_3 \ast M, \alpha_{L_3 \ast M}) \to (L_2 \ast M/\text{Im}(f \ast \text{id}_M), \alpha_{L_2 \ast M}) \].
Since $\Phi$ and $\Psi$ are inverse to each other, it follows that $\text{Ker}(g \ast \text{id}_M) \subset \text{Im}(f \ast \text{id}_M)$. □

Let $(\mathcal{L}, \alpha_L)$ be a perfect hom-Lie–Rinehart algebra over $(A, \phi)$ and suppose it has a quasi-hom-action on itself by the underlying hom-Lie bracket, i.e., $\pi y = [x, y]_L$. Then from Proposition 6.9, the short exact sequence

$$(\mathcal{P}, \alpha_P) \xrightarrow{i} (\mathcal{L} \ast \mathcal{L}, \alpha_{L \ast L}) \xrightarrow{p} (\mathcal{L}, \alpha_L)$$

is a central extension, where $\pi(a(x \ast y)) = a[x, y]_L$, $P = \text{Ker}(\pi)$ and $\alpha_P = \alpha_{L \ast L}|P$.

**Main Theorem 6.11.** The extension

$$(\mathcal{P}, \alpha_P) \xrightarrow{i} (\mathcal{L} \ast \mathcal{L}, \alpha_{L \ast L}) \xrightarrow{p} (\mathcal{L}, \alpha_L)$$

is a universal central extension.

**Proof.** Let $p : (\mathcal{M}, \alpha_M) \rightarrow (\mathcal{L}, \alpha_L)$ be a central extension and $s : L \rightarrow M$ be a map with $p \circ s = \text{id}_L$. Then

- $s[x, y]_L - [s(x), s(y)]_M \in \text{Ker}(p)$;
- $s(a.x) - a.s(x) \in \text{Ker}(p)$;
- $s(\alpha_L(x)) - \alpha_M(s(x)) \in \text{Ker}(p)$;
- $s(x)(a) = p(s(x))(a) = x(a)$

for any $x, y \in L$ and $a \in A$. Now, define $q : L \times L \rightarrow M$ by

$$q(x, y) = [s(x), s(y)]_M.$$ 

Then by the above observations and since $\text{Ker}(p) \subset \text{Z}_A(\mathcal{M}, \alpha_M)$, the map $q : L \times L \rightarrow M$ is a hom-Lie–Rinehart pairing and therefore it extends to a hom-Lie–Rinehart algebra homomorphism $q : (\mathcal{L} \ast \mathcal{L}, \alpha_{L \ast L}) \rightarrow (\mathcal{M}, \alpha_M)$. Here,

$$q(\alpha_{L \ast L}(x \ast y)) = q(\alpha_L(x), \alpha_L(y)) = \alpha_M(q(x, y)) = \alpha_M(q(x \ast y)),$$

i.e., $q \circ \alpha_{L \ast L} = \alpha_M \circ q$. It easily follows from the definition of $q$ that $p \circ q = \pi$. Now, by Definition 6.6 $(\mathcal{L} \ast \mathcal{L}, \alpha_{L \ast L})$ is a perfect hom-Lie–Rinehart algebra (because $(\mathcal{L}, \alpha_L)$ is perfect). Then by Lemma 3.11 $q$ is unique. Hence, $(\mathcal{L} \ast \mathcal{L}, \alpha_{L \ast L})$ is a universal central extension of $(\mathcal{L}, \alpha_L)$. □

Moreover, if $\phi : A \rightarrow A$ is an algebra epimorphism and $(\mathcal{L}, \alpha_L)$ is an $\alpha$-perfect hom-Lie–Rinehart algebra then the central extension

$$(\mathcal{P}, \alpha_P) \xrightarrow{i} (\alpha_L(\mathcal{L}) \ast \alpha_L(\mathcal{L}), \alpha_{L \ast L}) \xrightarrow{p} (\mathcal{L}, \alpha_L)$$

is a universal $\alpha$-central extension. In fact, for an $\alpha$-central extension

$$(\mathcal{N}, \alpha_N) \xrightarrow{i} (\mathcal{M}, \alpha_M) \xrightarrow{p} (\mathcal{L}, \alpha_L),$$
consider a map $s : L \to M$ such that $p \circ s = \text{id}_L$. Then we can define $f : \alpha_L(L) \times \alpha_L(L) \to M$ by
\[
f(\alpha_L(x), \alpha_L(y)) = [\alpha_M(s(x)), \alpha_M(s(y))] \quad \text{for all } x, y \in L.
\]
Then $f$ is a hom-Lie–Rinehart pairing and therefore it extends to a hom-Lie–Rinehart algebra homomorphism $\tilde{f} : (\alpha_L(L) \star \alpha_L(L), \alpha_L) \to (M, \alpha_M)$. Since $(\mathcal{L}, \alpha_L)$ is $\alpha$-perfect, so is $(\alpha_L(L) \star \alpha_L(L), \alpha_L)$ by Definition 6.6. Next, by Lemma 4.7, the map $\tilde{f}$ is unique. Hence, the central extension
\[
(\mathcal{P}, \alpha_P) \xrightarrow{i} (\alpha_L(L) \star \alpha_L(L), \alpha_L) \xrightarrow{\pi} (\mathcal{L}, \alpha_L)
\]
is a universal $\alpha$-central extension.

**Remark.** The particular case $\phi = \text{id}$, $\alpha = \text{id}$ provides the corresponding results for Lie–Rinehart algebras in [CGM]; the case $\phi = \text{id}$, $A = R$ provides the corresponding results for hom-Lie algebras in [CKP1].

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