Statistical modeling of the fluid dual to Boulware-Deser black hole

J. L. López, a,b*, Swastik Bhattacharya, a and S. Shankaranarayanan, a‡

a School of physics, Indian Institute of Science Education and Research (IISER-TVM)
Thiruvananthapuram 695106, India
b Departamento de Física, División de ciencias e Ingenierías Campus León,
Universidad de Guanajuato, A.P. E-143, C.P. 37150, León, Guanajuato, México.

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In this work we study the statistical and thermodynamic properties of the horizon fluid corresponding to the Boulware-Deser (BD) black hole of Einstein-Gauss-Bonnet (EGB) gravity. Using mean field theory, we show explicitly that the BD fluid exhibits the coexistence of two phases; a BEC and a non-condensed phase corresponding to the Einstein term and the Gauss-Bonnet term in the gravity action, respectively. In the fluid description, the high-energy corrections associated to Gauss-Bonnet gravity are modeled as excitations of the fluid medium. We provide statistical modeling of the excited part of the fluid and explicitly show that it is characterized by a generalized dispersion relation which in \( D = 6 \) dimensions corresponds to a non-relativistic fluid. We also shed light on the ambiguity found in the literature regarding the expression of the entropy of the horizon fluid. We provide a general prescription to obtain the entropy and show that it is indeed given by Wald entropy.

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I. INTRODUCTION

Many interesting features of gravity have arisen since the formal relation between the laws of thermodynamics and the laws of black hole dynamics were found [1–3]. These relations allow the possibility of extracting information about the microscopic degrees of freedom by providing statistical mechanical description of the macroscopic properties of black hole horizons. [In this work, by degrees of freedom we mean the microscopic degrees of freedom corresponding to the black-hole entropy.] In other words, one aims to arrive at the microscopic features of black holes from their semi-classical properties as we yet lack information about the quantum degrees of freedom of gravity.

Fluid/Gravity correspondence is another approach that aims to associate fluid degrees of freedom to the horizon and, eventually, to the gravitational degrees of freedom [4–9]. This correspondence allows the connection between macroscopic and microscopic physics through the study of the statistical properties of the fluid on the horizon of the black hole. For instance, the fluid on the horizon of a Schwarzschild black hole can be modeled as a relativistic Bose gas with all its degrees of freedom in the lowest energy level, allowing the fluid to be in a condensed state [10, 11]. The collective behavior of the microscopic degrees of freedom, described within the Landau-Ginzburg mean field theory of phase transitions, lead to the Bekenstein-Hawking entropy in Einstein gravity [11, 12] (see also [13]). If the fluid-gravity correspondence is a generic feature of any horizon, it is imperative to see whether the mean field theory approach can be extended to higher derivative gravity theories that are relevant in large curvature limit where the Einstein-Hilbert action is not well suited.

Lanczos-Lovelock (LL) gravity is a generalization to Einstein gravity that is consistent with having no more than second order time derivatives in the equations of motion [14, 15], and is free from ghosts when perturbed around flat spacetime. The higher order terms in LL actions represent high-energy corrections to Einstein gravity. In particular the second order term known as the Gauss-Bonnet term, that is of interest in this work, appears in low energy effective actions in string theories [16, 17], where the Einstein-Hilbert term arises as the lowest order curvature term in the action that is relevant at low energy scales. The black hole solution related to the second order LL gravity, also known as Einstein-Gauss-Bonnet (EGB) gravity, was found by Boulware and Deser [17] and a thermodynamic analysis of general LL gravity was first done in [18] (see also [19, 20]).

One of the main questions we ask here is the following: Is there a statistical mechanical description for the fluid living on the BD event horizon? As mentioned earlier, such a description was recently provided for Einstein gravity within mean field theory, where the fluid is modeled as a Bose-Einstein condensate near the critical point [11]. In this work, we extend the statistical analysis to BD fluid [19, 22]. As BD fluid contains high energy corrections, one may expect that the mean field theory de-
scription can not be extended naturally. We construct an explicit model of the BD fluid here where two phases co-exist, a Bose-Einstein condensate and a non-condensed (normal) phase. As we shall see, the condensed phase corresponds to the Einstein-Hilbert term in the action whereas the non-condensed phase arises due to the presence of the Gauss-Bonnet term.

Before we go for the statistical analysis of BD fluid, we note that in the literature there exists an ambiguity in the definition of entropy of the BD horizon fluid. Specifically, it has been argued that the BD fluid entropy is less than the Wald entropy corresponding to the BD black hole \[20, 22\]. We identify the root cause of this ambiguity and show that this arises from assuming a linear relation between free energy and the volume of the fluid. We provide a consistent thermodynamic framework and show that the entropy of the fluid is given by the Wald entropy for the BD black hole, and forms the basis for further statistical analysis.

We list below the steps followed to obtain the macroscopic quantities associated with the horizon-fluid from which we obtain a statistical mechanical description for BD fluid:

1. Using the expressions for pressure \[2\] and temperature \[3\] together with the free energy representation of the first law of thermodynamics \[4\], we define the thermodynamic Potential (\(\Omega\)) or free energy \[12\] corresponding to the horizon-fluid.

2. From the free energy \[12\], we obtain the entropy of the horizon-fluid. As mentioned above, there is an ambiguity in the literature regarding the value of this entropy \[20, 22\]. In Sec. \([II.A]\), we explicitly show that the entropy of the fluid comes out to be identical to the Wald entropy when one takes into account that the free energy \(\Omega\) is a non-linear function of \(A\).

3. From the expression of free energy \[12\], we obtain the energy of the horizon-fluid \[14\] using an integral of the form \(\int PdV\). It is important to note that this definition of the energy of the fluid is different from the definition of Komar mass for the BD black hole although it matches with the accepted value of energy (with no cosmological constant) given in the literature \([18, 19]\).

The paper is organized as follows: In section II we resolve the existing ambiguity in the expression for the entropy of the BD fluid, which differs from the Wald entropy of the BD black hole \([10, 20, 22]\). We explicitly show how to obtain the entropy for the BD fluid that leads to a value identical to the Wald entropy in a consistent manner. In section III, we apply the mean field theory for the BD fluid and find that it shows the coexistence of two, condensate and non-condensate, phases. Under general assumptions, we provide a statistical description of the non-condensed part of the fluid up to first order in the coupling constant parameter (\(\lambda\)) in the Gauss-Bonnet term. In section IV we conclude and discuss the implications of our results. In appendix A, we generalize the procedure followed in \([11]\) for the higher dimensional Schwarzschild black hole and show that the correct value for Bekenstein-Hawking entropy is also obtained as the difference of two phases near a critical transition point. In appendix B we obtain the generalized dispersion for arbitrary dimensions. Finally in appendix C, we deduce that a dispersion relation with a linear term in momentum, namely a term related to phonon modes, can also be present but is unimportant for large values of momentum.

II. ENTROPY OF THE EINSTEIN-GAUSS-BONNET FLUID

The Lanczos-Lovelock Lagrangian, in a \(D\)-dimensional spacetime, is the sum of particular Lagrangian densities, where each term is characterized by a coupling constant \(\lambda_m\) and each one of these terms is a special contraction with products of completely antisymmetric tensors \(g_{b_1...a_n}^{\ b_1...a_n}\) with Riemann tensors \(R_{a_1...a_n}^{\ b_1...b_n}\). The first, second and third terms in the action correspond to the cosmological constant, Einstein-Hilbert and Gauss-Bonnet Lagrangian densities respectively. We will restrict ourselves to second order LL Lagrangian with zero cosmological constant and from now on, we refer \(\lambda\) to the coupling constant in the Gauss-Bonnet term, the action is given by

\[
S = \int d^Dx \sqrt{-g} \left[ \frac{1}{16\pi}(R + \lambda \mathcal{L}_{GB}) \right].
\] (1)

where \(\mathcal{L}_{GB} = R^2 - 4R\mu^\mu R_{\mu
u\rho\sigma} R_{\mu
u\rho\sigma} + G\). \(G\) is set to unity. The spherically symmetric black hole corresponding to this action was found in Ref. \([17]\) and is referred as BD black hole.

As mentioned in the Introduction, the equations of motion of Einstein gravity when projected to the surface horizon of its black hole give rise to Navier-Stokes equations \([4]\). This projection has been done for the equations of motion followed by Eq. \([11]\) and for particular background geometries like the Boulware-Deser \([21]\). We refer to the fluid in this particular background geometry as BD fluid. The transport coefficients related to pure LL terms in Eq. \([11]\) have been found in Ref. \([22]\). The projection from the equations of motion was made using a limiting process from an stretched horizon to the true horizon and a detailed discussion of this process and its validity is given in Refs. \([21, 22]\). The pressure and temperature for the \(D\)-dimensional BD fluid are given by \([21, 22]\)

\[
P = \frac{(D-3)}{16\pi a} + \frac{\tilde{\alpha}(D-5)}{16\pi a^3},
\] (2)

\[
T = \frac{(D-3)a^2 + (D-5)\tilde{\alpha}}{4\pi a^3 + 8\pi \tilde{\alpha}a},
\] (3)
where \( \alpha \) is the horizon radius of the BD black hole, and \( \bar{\alpha} = \lambda(D - 3)(D - 4) \). Now we turn our attention to the relation between the pressure and entropy of the fluid on the horizon. The prescription to calculate the entropy for a general diffeomorphism invariant theory of gravity for a Killing horizon was given by Wald \cite{24}. Following this prescription, the Wald entropy for the BD black hole is given by \cite{11, 25, 25}

\[
S_W = \frac{A}{4} + \lambda \frac{(D - 2)(D - 3)(A_{D-2})^{\frac{\bar{\alpha}^2}{2}}}{2} A^{\bar{\alpha}^{-2}},
\]

(4)

where \( A_{D-2} \) is the area of the unit \( (D - 2) \) dimensional sphere. While Wald entropy is uniquely defined, it seems that the entropy for the BD fluid is not defined uniquely \cite{20, 22}. In particular, using Eqs. (2), (3), the pressure can be expressed as

\[
P = \frac{T}{4} + \frac{\bar{\alpha}(A_{D-2})^{\frac{\bar{\alpha}^2}{2}}}{2} TA^{-\bar{\alpha}^2},
\]

(5)

where this relation can be understood as an equation of state for the pressure as a function of temperature and area. From this relation we can easily see that the pressure for the Schwarzschild fluid is recovered for \( \lambda = 0 \) \cite{11}. Eq. (5) can be rewritten to find another interesting relation between pressure and entropy:

\[
\frac{PA}{T} = \frac{A}{4} + \frac{\bar{\alpha}(A_{D-2})^{\frac{\bar{\alpha}^2}{2}}}{2} A^{\bar{\alpha}^{-2}}.
\]

(6)

Comparing with Wald entropy Eq. (4), we can write Eq. (6) in the following way

\[
\frac{PA}{T} = S_W^1 + \frac{(D - 4)}{(D - 2)} S_W^2,
\]

(7)

where the upper index designates the corresponding contribution to Wald entropy of individual LL terms in Eq. (4), i.e.

\[
S_W^1 = \frac{A}{4},
\]

\[
S_W^2 = \lambda \frac{(D - 2)(D - 3)(A_{D-2})^{\frac{\bar{\alpha}^2}{2}}}{2} A^{\bar{\alpha}^{-2}}.
\]

(8)

The relation (7), for the general LL Lagrangian was found in \cite{20, 22}, and it is the entropy of the BD fluid. This entropy is explicitly given by

\[
S_F = \frac{A}{4} + \lambda \frac{(D - 3)(D - 4)(A_{D-2})^{\frac{\bar{\alpha}^2}{2}}}{2} A^{\bar{\alpha}^{-2}},
\]

(9)

and differs from Wald entropy Eq. (4) by a numerical factor in the second term. It is important to note that \( S_F \) is always less than \( S_W \) (\( S_F < S_W \)). In the remainder of this section we first point out the reason for the mismatch and show a consistent way to relate the thermodynamic quantities of the BD fluid. In this approach, the entropy of the BD fluid coincides with the Wald entropy for the BD black hole.

\[\text{A. Thermodynamics of BD fluid}\]

Here we point out the origin of the discrepancy between the two entropies, namely the Wald entropy for the BD black hole Eq. (4), and the entropy for the BD fluid given by Eq. (9). The main observation is that Eq. (5) is an equation of state for the pressure, that should be derived from a thermodynamic potential as a fundamental relation, namely the free energy \( \Omega(A, T) \) satisfying the differential first law of thermodynamics \cite{25}.

\[
d\Omega = -SdT - PdA,
\]

(10)

\[
S = -\frac{\partial \Omega}{\partial T} \bigg|_A, \quad P = -\frac{\partial \Omega}{\partial A} \bigg|_T.
\]

(11)

At this point it is important to remark that if we attempt to construct the free energy from the expression, \( \Omega(A, T) = -PA \) with the pressure given by Eq. (5) we see that the thermodynamic equations \cite{11} give the entropy as in Eq. (9) and an incorrect expression for the pressure. The reason why the last construction fails is because the energy \( U(A, T) \) is not a homogeneous function of order one in its extensive variables and, as a consequence of this, the Euler relation \( U = TS - PA \) does not hold, which is a necessary requirement for the free energy to be given by \( \Omega = -PA \) \cite{27}. This is a particular feature of the LL thermodynamic relations. Therefore, we need to construct the free energy \( \Omega(A, T) \) from the differential relation \cite{10}. Using the input, that is the pressure \cite{9}, and imposing that the entropy at the origin vanishes we get the following free energy

\[
\Omega = -\frac{TA}{4} - \lambda \frac{(D - 2)(D - 3)(A_{D-2})^{\frac{\bar{\alpha}^2}{2}}}{2} A^{\bar{\alpha}^{-2}}.
\]

(12)

The equations of state related to this \( \Omega(A, T) \) are the correct expression for pressure Eq. \cite{5} and Wald entropy Eq. \cite{4} for the (BD) black hole \cite{18, 10, 25}. In this way, the thermodynamic relations for the BD fluid are seen to be consistent and the problem of the ambiguity in entropy disappears, i.e. the entropy of the BD fluid is the same as Wald entropy for the BD black hole of EGB gravity.

We also see that it follows directly from the equations of state that \( \frac{\partial S_W}{\partial T} = \frac{\partial S_F}{\partial T} \), and this quantity depends solely on the area \( A \) which means that
\[ dU = T dS - P dA = 0, \quad (13) \]

This relation is consistent with the corresponding BD black hole constraints as one parameter system depending on the area of its horizon \( A \) or the black hole radius \( a \). The energy corresponding to the horizon-fluid can be obtained from the free energy \( \Omega(A, T) \), i.e.,

\[ E = \int P dA = \int \left( -\frac{\partial \Omega(A, T)}{\partial A} \right) |_T dA. \quad (14) \]

Substituting Eq. (12) for the free energy in the above expression leads to

\[ E = \frac{(D - 2)(A_{D-2})^{\frac{1}{D-1}} A^{\frac{D-3}{D-2}}}{16\pi} \]

\[ + \lambda \frac{(D - 2)(D - 3)(D - 4)(A_{D-2})^{\frac{1}{D-2}} A^{\frac{D-5}{D-2}}}{16\pi}. \quad (15) \]

This is one of the key results of this work and we would like to stress the following points: First, the thermodynamic quantities are obtained from the free energy and are computed from an independent construction using the properties of the BD fluid. More importantly, the thermodynamic properties of the BD black hole calculated using the Euclidean action approach \[18\], and the procedure followed here are independent. Second, although the thermodynamic quantities are calculated using a different approach, they match with the correct accepted values for the BD black hole \[18, 19, 25, 28\]. The matching of the macroscopic quantities from geometric and thermodynamic routes reinforces the correspondence between the fluid and gravity on the horizon. Third, there are several inequivalent definitions of mass in general relativity. The mass or the "energy" of a black hole is also not free from this problem. For global conditions like staticity and asymptotically flatness (zero cosmological constant), the Komar integral gives the correct mass parameter, at least for Schwarzschild black holes in 4 dimensions \[24, 31\]. However, the definition of Komar energy for the Kerr black hole \[32, 33\] has an anomalous factor 2. The value of energy from our calculation matches the correct value of energy for the BD black hole \[18, 19, 28\], which is not based on Komar-like calculations.

### III. STATISTICAL MODELING OF THE BD HORIZON FLUID

In this section we turn our attention to the statistical modeling of the BD fluid.

Following the same procedure as in \[11\], we show in appendix A that Bekenstein-Hawking entropy arises as the difference in the entropy between two phases near a critical point for the higher dimensional Schwarzschild black hole. This means that the mean field theory is well suited to describe the Schwarzschild fluid near the critical point and all its degrees of freedom are in the ground state. This will correspond to the condensed phase of the BD fluid.

Here we note the fact that EGB gravity is understood as a gravity theory containing high energy corrections or equivalently contributions coming from the physics at a much shorter length scale compared to the scale where Einstein gravity is a good approximation \[14\]. It is well known, that mean field theory description of the BEC includes only the long wavelength physics and is independent of the physics at smaller length scales. Hence it is expected that the part of the fluid corresponding to the GB term cannot be described by a BEC. In fact, one encounters problems if one tries to model GB horizon fluid by a mean field theory. Accordingly, we shall treat the part of the horizon fluid corresponding to the GB term as being in a non-condensed or normal phase. We are interested in understanding how these high energy contributions manifest in the fluid description of the BD black hole. We can separate what we call now the total free energy \( \Omega \) into two parts \( \Omega_C \) and \( \Omega_N \) corresponding to the condensed and non-condensed parts respectively. They are given by,

\[ \Omega_C = -\frac{TA}{4}, \quad (16) \]

\[ \Omega_N = -\lambda \frac{(D - 2)(D - 3)(A_{D-2})^{\frac{1}{D-2}}}{2} TA^{\frac{D-5}{D-2}}. \]

The free energy \( \Omega_C \) is given by the mean field theory expansion (see details in appendix A),

\[ \Omega_C = \Phi_0 + a(T - T_c)\eta^2 + B\eta^4. \quad (17) \]

The following points are in order regarding the above expression: First, it tells us that the properties of the condensed part can be described by a mean field theory. This part does not explicitly depend on \( \lambda \) and it matches with the free energy of Einstein gravity. Second, the free energy part that explicitly depends on \( \lambda \) can not be described as a relativistic Bose gas. In hindsight, this is not surprising as Gauss-Bonnet gravity corresponds to high-energy corrections. In the fluid description, this translates to the non-condensed part of the fluid that comprises of the excitations of the medium. In other words, the statistical description of the non-condensate part can provide us with information about the high energy features of the gravity theory.

#### A. Thermodynamic quantities characterizing the non-condensed phase of the fluid

Having identified the free energy corresponding to the non-condensed part of the fluid, we provide the statistical mechanical description for this non-condensate phase...
characterized by $\Omega_N$ in Eq. \ref{eq:16}. To achieve that, first we obtain the thermodynamic relations for the non-condensed part of the fluid. The free energy $\Omega_N$ \ref{eq:16} satisfies the thermodynamic relation,

$$d\Omega_N = -S_NdT - P_NdA. \tag{18}$$

The thermodynamic quantities for the non-condensed fluid that follow from this expression are given by,

$$S_N = \lambda \left( \frac{(D-2)(D-3)(A_{D-2})^{\frac{2}{D-2}}}{2} \right) A^{\frac{D-4}{D-2}}, \tag{19}$$

$$P_N = \lambda \left( \frac{(D-3)(D-4)(A_{D-2})^{\frac{2}{D-2}}}{2} \right) A^{\frac{D-4}{D-2}}, \tag{20}$$

$$E_N = \lambda \left( \frac{(D-2)(D-3)(D-4)(A_{D-2})^{\frac{3}{D-2}}}{16 \pi} \right) A^{\frac{D-5}{D-2}},$$

where,

$$S = S_C + S_N, \tag{21}$$

$$P = P_C + P_N, \tag{22}$$

$$\Omega = \Omega_C + \Omega_N, \tag{23}$$

$$E = E_C + E_N. \tag{24}$$

The expressions for $S, P, \Omega$ and $E$ are given by Eqs. \ref{eq:14}, \ref{eq:15}, \ref{eq:16}, and Eq. \ref{eq:17} respectively.

**B. Statistical modeling of the non-condensed fluid**

Since we are working with the fluid description of the BD horizon, the degrees of freedom in the non-condensed phase can be viewed as the excitations of the medium (i.e. the fluid) due to the vibrational modes. To go about with the statistical modeling of the non-condensed phase of the fluid, we consider it to be a weakly interacting gas. The following three physical assumptions can then be made: (i) As we have shown in the last section, we are dealing with a Bose-like fluid with a fraction of its degrees of freedom in the condensate state, we assume a Bose-Einstein probability distribution for the vibrational modes in the fluid as a function of energy in the grand canonical ensemble with zero chemical potential. (ii) We will assume that the density of states is proportional to the volume accessible to the fluid, in this case, the area of the horizon. (iii) Finally, we will assume a power law for the density of states, i.e. the dispersion relation between energy and momentum is given by, $\epsilon_k = ck^2$, where, $\sigma$ is the spectral index \ref{eq:34} \ref{eq:35}. This relation can be recovered knowing the density of states $n(\epsilon)$,

$$\frac{d\Sigma(\epsilon)d\epsilon}{d\epsilon} = n(\epsilon)d\epsilon. \tag{27}$$

where, $\Sigma(\epsilon)$ is the number of microstates in this infinitesimal interval. After taking into account the last two assumptions, the density of states takes the form,

$$n(\epsilon)d\epsilon = aA\epsilon^{\gamma}d\epsilon, \tag{28}$$

where, $a$ and $\gamma$ are constants to be determined by the constraints of the model. Note that $\sigma$ and $\gamma$ are different. Following this, the logarithm of the partition function $\ln Q$ is given by,

$$\ln Q = -aA \int_0^\infty \epsilon^{\gamma} \ln (1 - e^{-\beta \epsilon})d\epsilon, \tag{29}$$

and the energy is given by

$$\tilde{E} = \frac{\partial}{\partial \beta}(\ln Q) = aA\Gamma(\gamma + 2)\zeta(\gamma + 2)(k_BT)^{\gamma + 2} \tag{30}$$

where, $\Gamma(x)$ and $\zeta(x)$ are the Euler gamma function and Riemann zeta function respectively. Let us focus on $D = 6$ dimensions. The thermodynamic properties are given by,

$$E_N = \frac{3\lambda(A_4)^{\frac{2}{5}} A^{\frac{4}{5}}}{2\pi}, \tag{31}$$

$$P_N = \frac{16\lambda\pi^2 T^3}{3}, \tag{32}$$

$$S_N = \frac{27\lambda(A_4)^{\frac{6}{5}}}{8\pi T^2}. \tag{33}$$

Matching the thermodynamic energy density Eq. \ref{eq:27} with the statistical energy density coming from Eq. \ref{eq:24}, $a$ and $\gamma$ are given by,

$$a = \frac{16\lambda\pi^2}{9\zeta(3)k_B}, \quad \gamma = 1. \tag{34}$$
Using the values of $a$ and $\gamma$ in Eq. (23) we get,
\[
\ln Q = \frac{16 \lambda \pi^2 A T^2}{9 k_B}.
\] (29)

The statistical equations of state for the pressure and entropy are
\[
\tilde{P} = \frac{\partial (k_B T \ln Q)}{\partial A} \bigg|_T, \quad \tilde{S} = \frac{\partial (k_B T \ln Q)}{\partial T} \bigg|_A.
\] (30)

As partial derivations are involved in finding these equations of state, it is important to note that $k_B T \ln Q$ is defined up to some functions of pressure and temperature. Making use of this freedom, we shall match the expressions for the variables from the thermodynamics and statistical mechanics. To this end, we define,
\[
\bar{\Omega}(A, T) = k_B T \ln Q + \frac{6 \lambda (A_4 \lambda^2 A_4 \lambda^2)}{\pi} - \frac{27 \lambda (A_4)}{16 \pi^2 T}.
\] (31)

An expression from which we get the statistical quantities
\[
\bar{P} = \frac{\partial \bar{\Omega}}{\partial A} \bigg|_T, \quad \bar{S} = \frac{\partial \bar{\Omega}}{\partial T} \bigg|_A.
\] (32)

Rewriting the statistical mechanical quantities obtained from (32) in terms of $T$ we get,
\[
\bar{P} = \frac{16 \lambda \pi^2 T^3}{3}, \quad \bar{S} = \frac{27 \lambda (A_4)}{8 \pi^2 T^2}.
\] (33)

It is important to note that this matches with the quantities (27) as obtained from the thermodynamics of BD fluid.

Now that we have a complete statistical picture of the non-condensed part of the fluid, we can gain more insight into the microscopic features of the BD fluid by looking at the dispersion relation satisfied by the vibrational modes. To this end, we look at the differential relation involving the density of states $n(\epsilon)$,
\[
n(\epsilon) d\epsilon = g(p) dp,
\] (34)

using (13) (with $\hbar = 1$) for the $D = 6$-dimensional space-time (4-dimensional fluid), $g(p) dp$ is given by,
\[
g(p) dp = \frac{A p^3}{8 \pi^2} dp,
\] (35)

and using Eq. (22) with the value for $a$ given in Eq. (28), the non-condensed fluid satisfies the following dispersion relation,
\[
\epsilon = \frac{3 \sqrt{\zeta(3)} k_B^{3/2}}{16 \pi^2 \sqrt{\lambda}} p^2.
\] (36)

The following points are noteworthy regarding the above results: First, the non-condensed phase satisfies the dispersion relation of a non-relativistic gas with $\lambda$ playing the role of mass. When $\lambda << 1$, which corresponds to the approximation we used, the mass of the particles is small and this dispersion relation corresponds to the excitation of low energy modes. Phonon modes are present but they are not important for large values of momentum (see appendix C). Second, at the leading order in $\lambda$, the above analysis can be extended to any dimensions. It can be seen that, for any dimension, the energy constraint can be satisfied for $\gamma = 1$ and the dispersion relation will depend on the space-time dimensions. These generalized dispersion relations arise naturally in higher dimensional Bose-Einstein condensates [34, 35] and we can see generally that the condensation property is satisfied for higher dimensional fluids like the ones encountered here.

IV. CONCLUSIONS

The paradigm that gravity is emergent is supported by quite a few independent lines of evidences. The connection of the gravitational dynamics with the Navier-Stokes equation is definitely one of the most striking among these. However, except for the Fluid-Gravity duality that emerges from the AdS-CFT correspondence, it is not known whether there is an underlying microscopic theory corresponding to the fluid description of the dynamics of horizons. Though the AdS-CFT correspondence is restricted to certain classes of black hole spacetimes, still it is possible to glean a lot of insight about the microscopic theory of gravity via Fluid-Gravity duality. It is interesting to find out what the microscopic theory underlying a fluid description of the event horizon of a black hole is in a gravity theory. For the Schwarzschild black hole spacetime, a solution of the Einstein theory, this study has already been performed. Here we have extended this approach to include black holes in Lanczos-Lovelock gravity.

In this work we modeled the horizon fluid of BD black hole as a fluid with two coexisting phases; a BEC phase corresponding to the Einstein gravity and a non-condensed phase that corresponds to the Gauss-Bonnet term. Our analysis has two parts: (i) Purely thermodynamic and (ii) Statistical mechanical modeling of the fluid.

In the thermodynamic aspect of our work, we provided a consistent way to derive the equations of state for the general $D$-dimensional horizon-fluid in Lovelock theory and showed that the entropy of this fluid coincides with the Wald entropy [24, 25]. Our prescription resolves the ambiguity existing in the literature in the definition of the entropy for the black hole solution of EGB gravity in its fluid description [24, 22], and establish a solid basis for the thermodynamics of the BD fluid.

On the statistical mechanical part, we generalized the results found in [11, 12] for the higher dimensional
Schwarzschild black hole and using the mean field theory analysis, we find that the BD fluid shows the coexistence of a condensed phase and a non-condensed (normal) phase. Finally, to have a better insight in the nature of the non-condensed fluid, we provided detailed statistical analysis. Under general assumptions, we found a consistent statistical model of the non-condensed part of the BD fluid. In the approximation used, we find that the normal fluid behaves like a non-relativistic fluid with a general dispersion relation given by Eq. (B3). The 6-dimensional model is treated in detail where the coupling constant $\lambda$ plays the role of mass for the low energy modes of the excited part. In the limit of $\lambda \to 0$, we recovered Schwarzschild fluid condensate.

Our work implies that the counting of the DOF on the black hole event horizon in Gauss-Bonnet gravity is greater than that for the Einstein theory of gravity. This is consistent with results of Brustein and Medved [36], where the authors have shown that the Lovelock theory of gravity can be effectively described as Einstein gravity coupled to a 2-form gauge field. This is reminiscent of the $f(R)$ gravity in a conformally transformed frame, where it can be described as Einstein gravity coupled to a scalar field. Also from the formula for Wald entropy for a black hole in a Lovelock theory [18, 19, 25], we see that the entropy is greater if the Gauss-Bonnet coupling term is non-zero. While the first point evidence that the Lovelock theory of gravity has extra degrees of freedom compared to the Einstein theory of gravity, the Wald entropy formula shows that black hole entropy in Gauss-Bonnet theory is greater than the entropy of the black holes in the Einstein theory of gravity (for the black holes having the same area in both theories). Even though in 4-dimensions, Gauss-Bonnet term is topological, it gives a non-vanishing constant term to the Wald entropy. In other words, in four dimensions, the entropies of the black hole in the two theories differ by a constant term. This also seems to indicate that if the Wald entropy denotes the number of microstates of a black hole, then probably that number is greater for a black hole in Lovelock gravity. This is true even if the coupling constant is small. To our understanding, the reason has not been understood completely.

At this point, it is also interesting to note that the Boulware-Deser black hole solution lies in one branch. The solution on the other branch has a naked singularity. The approach pioneered by Damour works requires event horizon, i.e. a null horizon generated by a Killing vector. Since horizon fluid cannot be defined for the other branch of solutions, hence our approach cannot be extended to provide a fluid description of such solutions.

In this work, we have considered horizon-fluid of Boulware-Deser black hole which is an asymptotically flat. Our aim is extend the analysis for asymptotically AdS space-times. In the case of Einsteinian gravity, we showed that a negative cosmological constant acts like an external magnetic field that induces order in the system leading to the appearance of a tri-critical point in the phase diagram [11]. In the case of 5-D Gauss-Bonnet gravity with (positive or negative) cosmological term, $\lambda = 1/4$ (in Geometric units) has a critical value and the symmetry enhances to the full SO(4, 2) group and the 5-D Gauss-Bonnet gravity action with cosmological term is Chern-Simons Lagrangian for the AdS group [37]. It will be interesting to obtain a fluid description for the critical system as it will be non-perturbative. We hope to address this elsewhere.

The coexistence of two phases in our model is a feature that is common to the two fluid model of Superfluidity as well. Given the context, it is natural to ask the question, whether a two fluid model can be developed that describes the dynamics of the BD fluid. If possible, we might be able to relate the physics of the BD fluid to the physics of superfluidity where the coexistence of two phases occurs naturally. The two fluid models also exhibit some other interesting properties [38] and one could check whether those features are seen in this case also.

We hope to address these questions elsewhere.

Appendix A: Higher dimensional Schwarzschild black hole entropy from criticality

Following the same procedure as in [11], we show that for general higher dimensional Schwarzschild black hole, Bekenstein-Hawking entropy arises as the entropy difference between two phases near a critical point. We start from the generalized expressions Eqs. (3), (14) with $\lambda = 0$, being the relevant ones the Komar energy and temperature appearing in the order parameter $\eta$ in the statistical field (free energy) expansion [11, 39]. The energy and temperature for $\lambda = 0$ in terms of area $A = A_{D-2}^{D-2}$ are given by

$$E = \frac{(D-2)(A_{D-2})^{\frac{1}{2}} A^{\frac{3}{2}}}{16\pi},$$  \hspace{1cm} (A1)

$$T = \frac{(D-3)(A_{D-2})^{\frac{1}{2}} A^{\frac{1}{2}}}{4\pi}.$$  \hspace{1cm} (A2)

Now, we can construct the quantity $N(A) = E/\alpha T$, we get

$$N(A) = \frac{(D-2)A}{4\alpha(D-3)}.$$  \hspace{1cm} (A3)

The form of the constraint $N(A)$ needed to construct the order parameter $\eta$ was justified from the micro canonical point of view for the Schwarzschild black hole fluid ($D = 4$) in [11, 11], and this statement can be generalized to the $D$-dimensional case in which we are interested now and the same functional form of $N(A)$ is found. The order parameter $\eta^2$ is given by $\eta^2 = \kappa N(A)$. We can also see from Eq. (14) that in this higher dimensional case, it is also satisfied that the form of the equation of state is $P = T/4$. We can use the thermodynamic potential for
In this case, Eq. (4) given by $\Omega(A,T) = -TA/4$ and the mean field theory expansion is [10, 30]

$$-\frac{TA}{4} = \Phi_0 + a(T - T_c)\eta^2 + B\eta^4,$$  
(A3)

when matching coefficients on both sides of the equation we get the value of $a$, $a = \frac{(D-3)p_0}{2(D-2)p}$, and the value of $\eta$ of the extremum of the statistical field is obtained from $\partial\Phi/\partial\eta = 0$, this is

$$\eta^2 = -\frac{a(T - T_c)}{2B},$$  
(A4)

where using $\eta^2 = \frac{\kappa(D-2)A}{4(D-3)\alpha}$ we get

$$\frac{(T - T_c)}{2B} = \frac{(D - 2)^2\kappa^2}{4(D - 3)\alpha^2} A.$$  
(A5)

Finally, we find the entropy, $\Delta S = -\partial\Phi/\partial T$

$$\Delta S = \frac{\partial\Phi}{\partial T} = -a\eta^2 = \frac{a^2(T - T_c)}{2B},$$  
(A6)

and substituting the previous values this gives

$$\Delta S = \frac{A}{4}.$$  
(A7)

We find that this difference in entropy between the two phases of the fluid near the critical point is in accordance with the Bekenstein-Hawking entropy.

Appendix B: General dispersion relation

For a general $(D - 2)$ dimensional fluid on the horizon of a $D$-dimensional black hole, the number density of the states for the vibrational modes of the fluid is given by,

$$\Sigma(p) = \frac{A}{h^{D-2}} \int d^{D-2}p = \frac{A}{h^{D-2}} \frac{\pi^{\frac{D-2}{2}} p^{D-2}}{\Gamma\left(\frac{D}{2}\right)}$$  
(B1)

and from this we get $g(p)$

$$d\Sigma(p) dp = g(p)dp = \frac{(D - 2)A\pi^{\frac{D-2}{2}} p^{D-3}}{h^{D-2}\Gamma\left(\frac{D}{2}\right)}$$  
(B2)

and finally from the relation $n(\epsilon)d\epsilon = g(p)dp$ we get the generalized dispersion relation,

$$\epsilon = \frac{p^{\frac{D-2}{2}}}{2^{\frac{D-2}{2}} \pi^{\frac{D-2}{4}} h^{\frac{D-2}{4}} \sqrt{\Gamma(D/2)/\pi a}}$$  
(B3)

where, the value of the constant $a$ depends on $\lambda$ and can be fixed using the energy constraint.

Appendix C: Dispersion relation with a linear term

Here we show that the leading term in the dispersion relation is $\epsilon_0(p) = cp^2$ [see Eq. (39)], while the other terms in the dispersion relation are subleading corrections. Since we are considering the excitations in a fluid medium, it is natural to consider an UV cut-off. Then the total energy of the non-condensed part of the fluid in terms of momentum is given by,

$$\tilde{E} = \int_0^{p_D} \frac{g(p)\epsilon(p)}{e^{\beta\epsilon(p)} - 1} dp.$$  
(C1)

where, $p_D$ is the cut-off. We now recall that the thermodynamic expression for the energy is equated with the statistical average assuming no cut-off in Eq. (24) (with $\gamma = 1$) where $\epsilon(p)$ is given by $\epsilon_0(p)$. In order to recover the average energy $\tilde{E}, \epsilon(p)$ in Eq. (C1) has to be slightly different from $\epsilon_0(p)$. We may express this as, $\epsilon(p) = \epsilon_0(p) + \delta\epsilon(p)$, where $\delta\epsilon(p)$ represents the change in the dispersion relation which will be of the order of $(1/p_D)$. So, the relation (C1) can be written as,

$$\int_0^{p_D} \frac{g(p)\epsilon(p)}{e^{\beta\epsilon(p)} - 1} dp \to \int_0^{\infty} \frac{g(p)[\epsilon(p) - \delta\epsilon(p)]}{e^{\beta\epsilon(p)} - 1} dp.$$  
(C2)

Expanding the right hand side of Eq. (C2) up to the first order in $1/p_D$ we get,

$$\tilde{E} = \int_0^{\infty} \frac{g[p(\epsilon)]\epsilon}{e^{\beta\epsilon} - 1} \left(\frac{dp}{d\epsilon}\right) d\epsilon - \frac{1}{p_D} \left[\frac{g(p')}{{e^{\beta\epsilon(p')}} - 1} - \beta g(p')\epsilon(p')e^{\beta\epsilon(p')}\right].$$  
(C3)

The explicit change in $\epsilon(p)$ is given by, $\epsilon(p) = \epsilon_0(p) + \frac{1}{p_D}[a_1p + a_2p^2 + a_3p^3 + ...]$, where, $a_1, a_2, a_3, ...$ are arbitrary constants and are constrained by a consistency condition which will be derived below. We invert the expression of $\epsilon(p)$ to express $p$ as a function of $\epsilon$ perturbatively, i.e.

$$p(\epsilon) = p_0(\epsilon) + \frac{p_1(\epsilon)}{p_D} + \frac{p_2(\epsilon)}{p_D^2} + ...$$  
(C4)

where, $p_0 = (\epsilon/c)^{1/2}$. We find $p(\epsilon)$ up to the coefficient $a_3$ and to first order in $1/p_D$. The factor $g(p)$ for the six dimensional case is given by, $g(p) = bp^3$, where, $b$ is a constant Eq. (35), in this case $g[p(\epsilon)] = bp^3(\epsilon)$. Hence, substituting $p(\epsilon)$ in Eq. (C3) up to first order in $1/p_D$, we get,
Now we show that this equation can be satisfied for all the factors on the left hand side is negative for all $p$ so the right hand side of this equation is positive, so the arbitrary constants $a_i$ can be positive or be in such a combination to make at least the first term positive. This shows that it is possible to have a dispersion relation in this case, that has a phonon like term. The contribution of these modes of excitations is present but is small in general. For sufficiently small values of momentum however, the phonon like term would become important.

This analysis can be extended for $D > 6$ in the same way. The dispersion relation in that case would be given by: $\epsilon(p) = \epsilon_0(p) + (1/p^0) [a_1 p + a_2 p^2 + ...]$, with $\epsilon_0(p)$ given as in Eq. (23) and with all other terms suppressed for a large value of momentum.

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We remind the reader that in the fluid description, the volume of the fluid is given by the area of the black hole horizon. That is the reason of using $A$ in our thermodynamic relations.