Generalized strategies in the Minority Game

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Abstract

We show analytically how the fluctuations (i.e. standard deviation $\sigma$) in the Minority Game (MG) can be made to decrease below the random coin-toss limit if the agents use more general behavioral strategies. This suppression of $\sigma$ results from a cancellation between the actions of a crowd, in which agents act collectively and make the same decision, and an anticrowd in which agents act collectively by making the opposite decision to the crowd.

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The Minority Game (MG) of Challet and Zhang \[1–3\] offers a simple paradigm for complex, adaptive systems. The MG comprises an odd number \(N\) of agents, each with \(s\) strategies and a memory size \(m\), who repeatedly compete to be in the minority. In the basic MG, where agents always use their highest scoring strategy, the size of the fluctuations (i.e. standard deviation \(\sigma\)) falls below the random, coin-toss limit as \(m\) varies \[3\]. Cavagna et al \[4\] considered a fascinating generalization of the basic MG, the ‘Thermal Minority Game’ (TMG), whereby agents choose between their strategies using an exponential probability weighting. Their numerical simulations demonstrated that \(\sigma\) could be pushed below the random coin-toss limit just by altering the relative probability weighting, which plays the role of temperature, of the strategies \[4\]. As pointed out by Marsili et al \[5\], such a probabilistic strategy weighting has a tradition in economics and encodes a particular behavioral model.

In this brief note, we show analytically that such a reduction in the standard deviation \(\sigma\) below the random, coin-toss limit can be understood in terms of crowd effects \[6,7\]. In particular, such generalized strategy rules tend to increase the cancellation between the actions of a crowd of like-minded agents, and its anti-correlated partner (anticrowd). Our theoretical approach builds on a recently proposed explanation of the basic MG in terms of such crowd effects \[7\].

The MG \[1\] comprises an odd number of agents \(N\) who choose repeatedly between option 0 (e.g. buy) and option 1 (e.g. sell). The winners are those in the minority group, e.g. sellers win if there is an excess of buyers. The outcome at each timestep represents the winning decision, 0 or 1. A common bit-string of the \(m\) most recent outcomes is made available to the agents at each timestep. The agents randomly pick \(s\) strategies at the beginning of the game, with repetitions allowed, from the pool of all possible strategies. After each turn, the agent assigns one (virtual) point to each of his strategies which would have predicted the correct outcome. In the basic MG, each agent uses the most successful strategy in his possession, i.e. the one with the most virtual points. Because of crowd effects \[7\], \(\sigma\) is large for small \(m\). Here we are interested in investigating analytically the reason why the large \(\sigma\) in this ‘crowded’ regime (i.e. small \(m\)) gets reduced below the random limit when
the strategy-picking rule is generalized, such as in Ref. [4]. We hence follow the analytic
crowd-anticrowd approach of Ref. [7] for small $m$ and $s = 2$.

Consider any two strategies $r$ and $r^*$ within the list of $2^{m+1}$ strategies in the reduced
strategy space [1,7]. At any moment in the game, the strategies can be ranked according
to their virtual points, $r = 1, 2 \ldots 2^{m+1}$ where $r = 1$ is the best strategy, $r = 2$ is second
best, etc. Note that in the small $m$ regime of interest, the virtual-point strategy ranking and
popularity ranking for strategies can be taken to be identical to a good approximation [7].

Let $p(r, r^*|r^* \geq r)$ be the probability that a given agent picks $r$ and $r^*$, where $r^* \geq r$ (i.e. $r$ is
the best among his $s = 2$ strategies). In contrast, let $p(r, r^*|r^* \leq r)$ be the probability that
a given agent picks $r$ and $r^*$, where $r^* \leq r$ (i.e. $r$ is the worst among his $s = 2$ strategies).
Let $\theta$ be the probability that the agent uses the worst of his $s = 2$ strategies, while $1 - \theta$ is
the probability that he uses the best. The probability that the agent plays $r$ is given by

$$p_r = \sum_{r^* = 1}^{2^{m+1}} \left[ \theta p(r, r^*|r^* \leq r) + (1 - \theta) p(r, r^*|r^* \geq r) \right]$$

$$= (1 - \theta) p_+(r) + \theta p_-(r) + 2^{-2(m+1)} \theta \tag{1}$$

where $p_+(r)$ is the probability that the agent has picked $r$ and that $r$ is the agent’s best (or
equal best) strategy; $p_-(r)$ is the probability that the agent has picked $r$ and that $r$ is the
agent’s worst strategy. It is straightforward to show that

$$p_+(r) = \left( \left[ 1 - \frac{(r - 1)}{2^{m+1}} \right]^2 - \left[ 1 - \frac{r}{2^{m+1}} \right]^2 \right). \tag{2}$$

Note that $p_+(r) + p_-(r) = p(r)$ where

$$p(r) = 2^{-m}(1 - 2^{-(m+2)}) \tag{3}$$

is the probability that the agent holds strategy $r$ after his $s = 2$ picks, with no condition
on whether it is best or worst. An expression for $p_-(r)$ follows from Eqs. (2) and (3). The
basic MG [1] thus corresponds to the case $\theta = 0$.

The TMG is a generalized version of the basic MG in which each agent is equipped at
each timestep with his own (biased) coin characterised by exponential probability weightings
An agent then flips this coin to decide which strategy to use. To relate the present analysis to the TMG in Ref. [4], we consider $0 \leq \theta \leq 1/2$: $\theta = 0$ roughly corresponds to ‘temperature’ $T = 0$ [4] while $\theta \to 1/2$ roughly corresponds to $T \to \infty$, although we note that this correspondence is not precise (see later discussion). Here, it is shown that we can capture the essence of such generalized strategy play without having to include the detailed stochastics for each agent. Consider the mean number of agents playing strategy $r$ which is given by

$$n_r = N p_r = N (1 - 2\theta) p_+(r) + N \theta p(r) + 2^{-2(m+1)} N \theta \quad .$$

(4)

If $n_r$ agents use the same strategy $r$, then they will act as a ‘crowd’, i.e. they will make the same decision. If $n_{\bar{r}}$ agents simultaneously use the strategy $\bar{r}$ anticorrelated to $r$, they will make the opposite (anticorrelated) decision and will hence act as an ‘anticrowd’ [7]. For small $m$, which is our regime of interest, crowds are sizeable while anticrowds tend to be small in the basic MG. For general values of $\theta$, the standard deviation $\sigma_\theta$ in the number of agents making a particular decision, say 0, is given by [7]

$$\sigma_\theta = \left[ \frac{1}{2} \sum_{r=1}^{2m+1} \left[ \frac{1}{4} |n_r - n_{\bar{r}}|^2 \right] \right]^{1/2} .$$

(5)

Substituting Eqs. (3) and (4) for $r$ and $\bar{r} = 2^{m+1} + 1 - r$ into Eq. (5) yields

$$\sigma_\theta = |1 - 2\theta| \sigma_{\theta=0}$$

(6)

where

$$\sigma_{\theta=0} = \frac{N}{\sqrt{8}} \left[ \sum_{r=1}^{2m+1} \left[ \left[ 1 - \frac{(r - 1)}{2m+1} \right]^2 - \left[ 1 - \frac{r}{2m+1} \right]^2 \right] + \left[ 1 - \frac{(2m+1 - r + 1)}{2m+1} \right]^2 - \left[ 1 - \frac{(2m+1 - r)}{2m+1} \right]^2 \right]^{1/2}$$

(7)

is the standard deviation for the basic MG (i.e. $\theta = 0$). Equation (6) explicitly shows that the standard deviation $\sigma_\theta$ decreases as $\theta$ increases (recall $0 \leq \theta \leq 1/2$): in other words, the standard deviation decreases as agents use their worst strategy with increasing probability. An increase in $\theta$ leads to a reduction in the size of the larger crowds using high-scoring
strategies, as well as an increase in the size of the smaller anticrowds using lower-scoring strategies, hence resulting in a more substantial cancellation effect between the crowd and the anticrowd. Note that Eq. (6) holds for all $N$ and $m$, and hence any value of $\sigma_{\theta=0}$ as long as the MG remains in the ‘crowded’ regime given by $2^m << N$. In Ref. [7], we showed that $\sigma_{\theta=0}$ provides a reasonable fit to the numerical data for $m$ up to $m \sim 5 - 6$ with $N = 101$. For small $m$ (e.g. $m \leq 4$ for $N = 101$) $\sigma_{\theta=0}$ lies above the random coin-toss result of $\sqrt{N}/2$. As $\theta$ increases, $\sigma_{\theta}$ will eventually drop below the random coin-toss result at $\theta = \theta_c$ where

$$\theta_c = \frac{1}{2} - \frac{\sqrt{N}}{4 \sigma_{\theta=0}}.$$  \hspace{1cm} (8)

Because of the difference in the details for the strategy-mixing in the present analytic calculation as compared to Ref. [4], the present analytic model cannot be expected to reproduce the same quantitative features as Ref. [4] for $\theta \to 1/2$ (i.e. the high temperature limit of Ref. [4]). In particular, the present analytic model does not include the stochastics of the coin-toss involved at each time-step for each agent which ultimately leads to a finite $\sigma$ as $T \to \infty$ [4]. Nor does it account for the fact that agents are discrete entities [7]: we have treated $n_r$ as a continuous quantity which is reasonable for small $m$ in the basic MG, but will become less good an approximation as the fluctuations become small (i.e. $\sigma \to 0$) or $m$ becomes large [7]. Despite these limitations, it is remarkable that our simple analytic approach invoking crowd effects can capture the main feature whereby $\sigma$ falls below the random, coin-toss limit as $\theta$ (and hence temperature [4]) increases. It also strengthens our belief that many results of the MG can be understood using simple notions of crowd-anticrowd interplay.
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