Reconstruction of a Source Term in a Parabolic Integro-Differential Equation from Final Data*

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Abstract. The identification of a source term in a parabolic integro-differential equation is considered. We study the existence of the quasi-solution to this problem, Tikhonov regularization and a related gradient method.

Keywords: Inverse problem, integro-differential equation, quasi-solution.

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1 Introduction

Heat flow processes in media with memory are governed by parabolic integro-differential equations [7]. A number of papers is devoted to inverse problems to determine kernels of these equations in different formulations making use of measurements over time (see e.g. [4, 6, 7, 8, 11, 13, 14]).

Recently some papers appeared that deal with the reconstruction of source terms or coefficients of these equations making use of final or integral over-determination [5, 12]. In particular, the authors’ paper [5] extends former existence and uniqueness results of Isakov [3] to the integro-differential case. The existence of the solutions to the inverse problems to determine unknown source terms from final over-determination of the temperature requires sufficient regularity and a certain monotonicity of a time-component of this term.

In the present paper we follow another approach. Instead of the conventional solution, we deal with the quasi-solution of the inverse problem that uses final data. Then we can build up a theory without any smoothness or monotonicity restrictions on the source. Similar results in the case of the parabolic differential equation without an integral term in the one-dimensional case were obtained by Hasanov [2]. Quasi-solutions of other integro-differential inverse problems were studied in [1, 9].

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2 Direct Problem

Let $\Omega$ be a $n$-dimensional domain with sufficiently smooth boundary $\Gamma$ and $\Gamma = \Gamma_1 \cup \Gamma_2$ where $\operatorname{meas} \Gamma_1 \cap \Gamma_2 = 0$. Assume that for any $j \in \{1; 2\}$ it holds either $\Gamma_j = \emptyset$ or $\operatorname{meas} \Gamma_j > 0$. Denote $\Omega_T = \Omega \times (0,T)$, $\Gamma_{1,T} = \Gamma_1 \times (0,T)$, $\Gamma_{2,T} = \Gamma_2 \times (0,T)$. Consider the problem (direct problem) to find $u(x,t) : \Omega_T \to \mathbb{R}$ such that

\begin{align*}
    u_t &= Au - m * Au + f + \nabla \phi \quad \text{in } \Omega_T, \quad (2.1) \\
    u &= u_0 \quad \text{in } \Omega \times \{0\}, \quad (2.2) \\
    u &= g \quad \text{in } \Gamma_{1,T}, \quad (2.3) \\
    - \nu_A \cdot \nabla u + m * \nu_A \cdot \nabla u &= \partial u + h \quad \text{in } \Gamma_{2,T} \quad (2.4)
\end{align*}

where

$$ Av = \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left( a_{ij} \frac{\partial}{\partial x_j} v \right) + av, $$

$$ \nu_A = \sum_{j=1}^n a_{ij} \nu_j, \quad \nu = (\nu_1, \ldots, \nu_n) - \text{outer normal of } \Gamma_2, $$

$a_{ij}, a, u_0 : \Omega \to \mathbb{R}$, $f : \Omega_T \to \mathbb{R}$, $\phi : \Omega_T \to \mathbb{R}^n$, $g : \Gamma_{1,T} \to \mathbb{R}$, $\vartheta : \Gamma_2 \to \mathbb{R}$, $h : \Gamma_{2,T} \to \mathbb{R}$, $m : (0,T) \to \mathbb{R}$ are given functions and

$$ m * w(t) = \int_0^t m(t-\tau)w(\tau) \, d\tau $$

denotes the time convolution. In case $\Gamma_1 = \emptyset$ ($\Gamma_2 = \emptyset$), the boundary condition (2.3) (2.4) is dropped.

The problem (2.1)-(2.4) describes the heat flow in a body $\Omega$ with the thermal memory. Concerning the physical background we refer the reader to [7].

Let us introduce some additional notation. Let $X$ be a Banach space. We denote by $C([0,T];X)$ the space of abstract continuous functions from $[0,T]$ to $X$ endowed with the usual maximum norm $\|v\|_{C([0,T];X)} := \max_{t \in [0,T]} \|v(t)\|$. Moreover, let

$$ L^2((0,T);X) := \{ v : (0,T) \to X : \|v\|_{L^2((0,T);X)} := \left[ \int_0^T \|v(t)\|^2 \, dt \right]^{1/2} < \infty \}. $$

In addition, we need spaces of fractional order and anisotropic spaces. To this end, let us first introduce the following notation for difference quotients of $x$- and $(x,t)$-dependent functions with powers:

$$ \langle v \rangle_p(x_1,x_2) := \frac{v(x_1)-v(x_2)}{|x_1-x_2|^p}, \quad \langle v \rangle_p(x_1,x_2;t) := \frac{v(x_1,t)-v(x_2,t)}{|x_1-x_2|^p}, \quad \langle v \rangle_p(x;t_1,t_2) := \frac{v(x,t_1)-v(x,t_2)}{|t_1-t_2|^p}, $$
where $|x|$ denotes the Euclidean norm of $x$ in the space $\mathbb{R}^n$. For any $l \geq 0$ we introduce the Sobolev–Slobodeckij spaces (cf. [10, 15])

$$W^l_2(\Omega) = \left\{ v: \|v\|_{W^l_2(\Omega)} := \sum_{|\alpha| \leq |l|} \left[ \int_{\Omega} |D_x^\alpha v(x)|^2 \, dx \right]^\frac{1}{2} + \Theta_l \sum_{|\alpha| = |l|} \left[ \int_{\Omega} dx_1 \int_{\Omega} |\langle D_x^\alpha v \rangle|^\frac{2}{l+|\alpha|} \, dx_2 \right]^\frac{1}{2} < \infty \right\},$$

$$W^{l+\frac{1}{2}}_2(\Omega_T) = \left\{ v: \|v\|_{W^{l+\frac{1}{2}}_2(\Omega_T)} := \sum_{2j+|\alpha| = |l|} \left[ \int_{0}^{T} dt \int_{\Omega} dx \int_{\Omega} |\langle D_t^j D_x^\alpha v \rangle|^\frac{2}{l+|\alpha|} \, dx_2 \right]^\frac{1}{2} + \Theta_{\frac{1}{2}} \sum_{l-2j = |\alpha| \in (0, 2)} \left[ \int_{\Omega} dx \int_{0}^{T} dt \int_{\Omega} |\langle D_t^j D_x^\alpha v \rangle|^\frac{2}{l+|\alpha|} \, dx_2 \right]^\frac{1}{2} < \infty \right\}.$$

Here $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_i \in \{0, 1, 2, \ldots\}$ is the multi-index, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $D^\alpha_x v = \frac{\partial^{\alpha}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$ and $D_t^j v = \frac{\partial^j}{\partial t^j} v$. Moreover, $|\alpha|$ is the greatest integer $\leq l$ and $\Theta_l = 0$ and $\Theta_l = 1$ in the cases of integer $l$ and non-integer $l$, respectively. The definition of $W^{l+\frac{1}{2}}_2$ is in a standard manner extended from the boundary components $\Gamma_{1,T}$ and $\Gamma_{2,T}$ (for details see [10]).

Now we return to the direct problem (2.1)–(2.4). Throughout the paper we assume the following basic regularity conditions on the coefficients, the kernel and the initial and boundary functions:

$$a_{ij} \in C^1(\overline{\Omega}), \quad a_{ij} = a_{ji}, \quad a \in C(\overline{\Omega}), \quad \vartheta \in C(\overline{T_2}), \quad \vartheta \geq 0, \quad (2.5)$$

$$m \in L^1(0, T), \quad g \in W^{1, \frac{1}{2}}_2(1, T), \quad h \in L^2(2, T), \quad (2.6)$$

$$u_0 \in L^2(\Omega), \quad f \in L^2(\Omega_T), \quad \phi = (\phi_1, \ldots, \phi_n) \in (L^2(\Omega_T))^n \quad (2.7)$$

and the ellipticity condition

$$\sum_{i,j=1}^{n} a_{ij} \lambda_i \lambda_j \geq \epsilon |\lambda|^2, \quad x \in \overline{\Omega}, \quad \lambda \in \mathbb{R}^n \text{ with some } \epsilon > 0. \quad (2.8)$$

The first aim is to reformulate the problem (2.1)–(2.4) in a weak form. Let us suppose that (2.1)–(2.4) has a classical solution $u \in W^{2,1}_2(\Omega_T)$ and the term $\phi$ satisfies the following additional conditions: $\frac{\partial}{\partial x_i} \phi_i \in (L^2_0(\Omega_T))^n, i = 1, \ldots, n, \phi|_{\Gamma_{2,T}} = 0$. Then, we multiply (2.1) with a test function $\eta$ from the space

$$T(\Omega_T) = \left\{ \eta \in L^2((0, T); W^1_2(\Omega)): \eta_t \in L^2((0, T); L^2(\Omega)), \right.$$

$$\eta|_{\Gamma_1} = 0 \text{ in case } \Gamma_1 \neq \emptyset \}$$

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and integrate by parts with respect to time and space variables. We obtain the following relation:

\[
0 = \int_{\Omega} [u(x,T)\eta(x,T) - u_0(x)\eta(x,0)] \, dx - \int_{\Omega_T} w_{\eta} \, dx \, dt \\
+ \int_{\Omega_T} \sum_{i,j=1}^{n} a_{ij}(u_{x_j} - m \ast u_{x_j})\eta_{x_i} - a(u - m \ast u)\eta \, dx \, dt \\
+ \int_{\Gamma_2 \times \tau} (\vartheta u + h)\eta \, d\Gamma \, dt - \int_{\Omega_T} (f - \phi \cdot \nabla \eta) \, dx \, dt. \quad (2.9)
\]

This relation makes sense also in a more general case when \( \phi \) satisfies only (2.7) and \( u \) doesn’t have regular first order time and second order spatial derivatives. We call a weak solution of the problem (2.1)–(2.4) a function from the space

\[
\mathcal{U}(\Omega_T) = C([0,T];\mathcal{L}^2(\Omega)) \cap \mathcal{L}^2((0,T);\mathcal{W}_2^2(\Omega))
\]

that satisfies the relation (2.9) for any \( \eta \in \mathcal{T}(\Omega_T) \) and in case \( \Gamma_1 \neq \emptyset \) fulfills the boundary condition (2.3).

**Theorem 1.** The problem (2.1)–(2.4) has a unique weak solution. If, in addition, \( \phi = 0, \, g \in \mathcal{W}^{\frac{1}{2}+\epsilon}_{2} (\Gamma_1,\tau), \, h \in \mathcal{W}^{\frac{1}{2}+\epsilon}_{2} (\Gamma_2,\tau), \, u_0 \in \mathcal{H}^1(\Omega) \) and \( u_0 = g \) on \( \Gamma_1 \times \{0\} \), then this solution belongs to the space \( \mathcal{W}^{2,1}_2(\Omega_T) \) and satisfies (2.1)–(2.4) in the classical sense.

**Proof.** It is well known (see e.g. [10]) that in the particular case \( m = 0 \) the solution exists, is unique and the operator \( \mathcal{H} \), that assigns to the data vector \( u_0, g, h, f, \phi \) the weak solution is Lipschitz-continuous from the space \( \mathcal{L}^2(\Omega) \times \mathcal{W}^{\frac{1}{2}+\epsilon}_{2} (\Gamma_1,\tau) \times \mathcal{L}^2(\Gamma_2,\tau) \times \mathcal{L}^2(\Omega_T)^{n+1} \) to the space \( \mathcal{U}(\Omega_T) \). Let us denote \( \mathcal{G}(f, \phi) = \mathcal{H}(0,0,0,f,\phi) \). Then, denoting by \( \hat{u} \) the solution corresponding to \( m = 0 \), the problem (2.1)–(2.4) for \( u \) in \( \mathcal{U}(\Omega_T) \) equivalent to the following operator equation for the function \( v = u - \hat{u} \):

\[
v = \mathcal{F}u + \mathcal{F}v \quad (2.10)
\]

with the linear operator \( \mathcal{F}v = \mathcal{G}(-m \ast (av), -m \ast (\sum_{j=1}^{n} a_{ij}v_{x_j})) \). We are going to estimate \( \mathcal{F} \). To this end, we make use of the following inequality that immediately follows from the estimate (19) in [5]:

\[
\|m \ast w\|_{\mathcal{L}^2(\Omega_t)} \leq \int_{0}^{t} |m(t - \tau)| \|w\|_{\mathcal{L}^2(\Omega_\tau)} \, d\tau, \quad t \in (0,T). \quad (2.11)
\]

Here \( \Omega_t = \Omega \times (0,t) \) for \( t \in (0,T) \) and \( w \) is an arbitrary element of \( \mathcal{L}^2(\Omega_T) \). Moreover, we define the cutting operator \( P_t \) by the formula

\[
P_t w = \begin{cases} w & \text{in } \Omega_t, \\ 0 & \text{in } \Omega_T \setminus \Omega_t. \end{cases}
\]
Note that it holds \( G(P_t f, P_t \phi)(x, t) = G(f, \phi)(x, t) \) for any \((x, t) \in \Omega_t\). Therefore, observing the Lipschitz-continuity of \( G \) and (2.11) we can estimate as follows:

\[
\|Fv\|_{\mathcal{U}(\Omega_t)} = \|G\left(-m \ast (av), -m \ast \left( \sum_{j=1}^{n} a_{ij}v_{x_j} \right) \right)\|_{\mathcal{U}(\Omega_t)}
\]

\[
= \|G\left(-P_t \left[ m \ast (av) \right], -P_t \left[ m \ast \left( \sum_{j=1}^{n} a_{ij}v_{x_j} \right) \right] \right)\|_{\mathcal{U}(\Omega_t)}
\]

\[
\leq \|G\left(-P_t \left[ m \ast (av) \right], -P_t \left[ m \ast \left( \sum_{j=1}^{n} a_{ij}v_{x_j} \right) \right] \right)\|_{\mathcal{U}(\Omega_T)}
\]

\[
\leq C_1 \left[ \|P_t \left[ m \ast (av) \right]\|_{L^2(\Omega_T)} + \sum_{i=1}^{n} \|P_t \left[ m \ast \sum_{j=1}^{n} a_{ij}v_{x_j} \right]\|_{L^2(\Omega_T)} \right]
\]

\[
= C_1 \left[ \|m \ast (av)\|_{L^2(\Omega_T)} + \sum_{i=1}^{n} \|m \ast \sum_{j=1}^{n} a_{ij}v_{x_j}\|_{L^2(\Omega_T)} \right]
\]

\[
\leq C_2 \int_{0}^{t} |m(t - \tau)| (\|v\|_{L^2(\Omega_T)} + \|\nabla v\|_{L^2(\Omega_T)}) \, d\tau
\]

\[
\leq C_2 \int_{0}^{t} |m(t - \tau)| \|v\|_{\mathcal{U}(\Omega_T)} \, d\tau
\]

for any \( t \in (0, T) \) with some constants \( C_1, C_2 \). Now we introduce the weighted norms in \( \mathcal{U}(\Omega_T) \): \( \|v\|_{\sigma} = \sup_{0 < t < T} e^{-\sigma t} \|v\|_{\mathcal{U}(\Omega_t)} \) where \( \sigma \geq 0 \). Using the deduced estimate for \( F \) we obtain

\[
\|Fv\|_{\sigma} \leq C_2 \sup_{0 < t < T} e^{-\sigma t} \int_{0}^{t} |m(t - \tau)| \|v\|_{\mathcal{U}(\Omega_T)} \, d\tau
\]

\[
= C_2 \sup_{0 < t < T} \int_{0}^{t} e^{-\sigma(t - \tau)} |m(t - \tau)| e^{-\sigma \tau} \|v\|_{\mathcal{U}(\Omega_T)} \, d\tau
\]

\[
\leq C_2 \int_{0}^{T} e^{-\sigma s} |m(s)| \, ds \|v\|_{\sigma}.
\]

Since \( \int_{0}^{T} e^{-\sigma s} |m(s)| \, ds \to 0 \) as \( \sigma \to \infty \), the operator \( F \) is a contraction for sufficiently large \( \sigma \). Consequently, (2.10) has a unique solution in \( \mathcal{U}(\Omega_T) \). This proves the existence of the unique weak solution of (2.1)–(2.4).

Secondly, let us prove the classical solvability assertion of the theorem. Again, we use the results in case \( m = 0 \). It is known [15] that in case \( m = 0 \) the solution belongs to \( W_2^1(\Omega_T) \) and the operator \( \mathcal{H}^1 \) that assigns to the data vector \( u_0, g, h, f \) the classical solution is Lipschitz-continuous from the space \( H^1(\Omega) \times W_2^{\frac{1}{2}, \frac{1}{2}}(I_{1,T}) \times W_2^{\frac{1}{2}, \frac{1}{2}}(I_{2,T}) \times L^2(\Omega_T) \) to the space \( W_2^{2,1}(\Omega_T) \). Define \( \mathcal{G}^1(h, f) = \mathcal{H}^1(0, 0, h, f) \). The problem for \( u \) is equivalent to the following operator equation for \( v = u - \hat{u} \):

\[
v = F^1 \hat{u} + F^1 v, \tag{2.12}
\]

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where \( \mathcal{F}^1 v = \mathcal{G}^1 (-m * \nu_A \cdot \nabla v |_{\Gamma^2,T}, -m * Av) \). This time we have to introduce a more complicated extension operator instead of \( P_t \) because the argument of \( \mathcal{F}^1 \) has traces on slices \( \Omega \times \{t\} \). Let us define

\[
\tilde{P}_t w(x, s) = \begin{cases} 
  w(x, s) & \text{for } s < t, \\
  w(x, 2t - s) & \text{for } t < s < \min\{2t; T\}, \\
  0 & \text{for } s > 2t \text{ in case } 2t < T.
\end{cases}
\]

Then, since the function \( v \) in the range of \( \mathcal{F}^1 \) satisfies \( v|_{t=0} = 0 \), it holds \( \tilde{P}_t v \in W^{2,1}_2 (\Omega_T) \) for \( t \in (0, T) \). Moreover, \( \mathcal{G}^1 (\tilde{P}_t \tilde{h}, \tilde{P}_t \tilde{f})(x, t) = \mathcal{G}^1 (\tilde{h}, \tilde{f})(x, t) \) for any \( (x, t) \in \Omega_t \) and \( \| \tilde{P}_t \tilde{h} \|_{W^{2,1}_2 (\Gamma^2_t)} \leq 2 \| \tilde{h} \|_{W^{2,1}_2 (\Gamma^2_t)} \), \( \| \tilde{P}_t \tilde{f} \|_{L^2(\Omega_T)} \leq 2 \| \tilde{f} \|_{L^2(\Omega_T)} \), where \( \tilde{h} = m * \nu_A \cdot \nabla v |_{\Gamma^2,T} \) and \( \tilde{f} = m * Av \). Consequently, in view of the Lipschitz-continuity of \( \mathcal{G}^1 \) we deduce

\[
\| \mathcal{F}^1 v \|_{W^{2,1}_2 (\Omega_t)} = \left\| \mathcal{G}^1 (-m * \nu_A \cdot \nabla v |_{\Gamma^2,T}, -m * Av) \right\|_{W^{2,1}_2 (\Omega_t)}
\]

\[
= \| \mathcal{G}^1 (-P_t [m * \nu_A \cdot \nabla v |_{\Gamma^2,T}], -P_t [m * Av]) \|_{W^{2,1}_2 (\Omega_t)}
\]

\[
\leq \| \mathcal{G}^1 (-P_t [m * \nu_A \cdot \nabla v |_{\Gamma^2,T}], -P_t [m * Av]) \|_{W^{2,1}_2 (\Omega_t)}
\]

\[
\leq C_3 \left[ \| P_t [m * \nu_A \cdot \nabla v] \|_{W^{\frac{5}{2}, \frac{3}{2}} (\Gamma^2_t)} + \| P_t [m * Av] \|_{L^2(\Omega_T)} \right]
\]

\[
\leq 2C_3 \left[ \| m * \nu_A \cdot \nabla v \|_{W^{\frac{5}{2}, \frac{3}{2}} (\Gamma^2_t)} + \| m * Av \|_{L^2(\Omega_T)} \right]
\]

for any \( t \in (0, T) \) with some constant \( C_3 \) and \( \Gamma^2_t = \Gamma^2 \times (0, t) \). Using the trace theorem for Sobolev–Slobodeckij spaces \([10]\) and the relation \( (m * v)_t = m * v_t \), that holds due to \( v|_{t=0} = 0 \), we compute

\[
\| m * \nu_A \cdot \nabla v \|_{W^{\frac{5}{2}, \frac{3}{2}} (\Gamma^2_t)} = \| \nu_A \cdot \nabla (m * v) \|_{W^{\frac{5}{2}, \frac{3}{2}} (\Gamma^2_t)} \leq C_4 \| m * v \|_{W^{2,1}_2 (\Omega_t)}
\]

\[
= C_4 \left[ \sum_{|\alpha| \leq 2} \| m * D^\alpha_x v \|_{L^2(\Omega_t)} + \| m * v_t \|_{L^2(\Omega_t)} \right]
\]

with some constant \( C_4 \). Applying this estimate in (2.13) and using (2.11) we deduce

\[
\| \mathcal{F}^1 v \|_{W^{2,1}_2 (\Omega_t)} \leq C_5 \int_0^t |m(t - \tau)||v|_{W^{2,1}_2 (\Omega_\tau)} d\tau, \quad t \in (0, T)
\]

with a constant \( C_5 \). We define the weighted norms

\[
\| v \|_\sigma^* = \sup_{0 < t < T} e^{-\sigma t} \| v \|_{W^{2,1}_2 (\Omega_t)}
\]

in the space \( W^{2,1}_2 (\Omega_T) \) and, as in the first part of the proof, show that \( \mathcal{F}^1 \) is a contraction in \( W^{2,1}_2 (\Omega_T) \) if \( \sigma \) is sufficiently large. This proves the unique solvability of (2.12) and in turn the classical solvability assertion of theorem.

\( \Box \)
3 Formulation of Inverse Problem. Existence of Quasi-Solution

Let \( \hat{F} \) be a linear closed subspace of \( L^2(\Omega_T) \). Suppose that the source term \( f \) is of the following form: \( f = f_0 + F \), where \( f_0 \in L^2(\Omega_T) \) is known. We pose an inverse problem to determine the function \( F \in \hat{F} \) making use of the final measurement

\[
 u(x, T) = u_T(x), \quad x \in \Omega.
\]

More precisely, we will search a quasi-solution of this problem. This is a solution of the following minimization problem for the cost functional: find \( \psi \) such that

\[
 \text{arg min}_{F \in \hat{F}} J(F), \quad J(F) = \|u(\cdot, T; F) - u_T\|^2_{L^2(\Omega)}, \quad (3.1)
\]

where \( F \subseteq \hat{F} \) is a subset including constraints. Here \( u(x, t; F) \) stands for the solution of the direct problem corresponding to the given \( F \).

Let us introduce some cases of \( \hat{F} \).

Case 1. Define \( \hat{F} = \{F : F(x, t) = \kappa(t)w(x), \; w \in L^2(\Omega) \} \), where \( \kappa \in L^2(0, T), \; \kappa \neq 0 \) is a prescribed function.

Case 2. Let \( \Omega \) be a cylinder: \( \Omega = S \times (0, l) \), where for any \( x = (x_1, \ldots, x_n) \in \Omega \) we have \( \bar{\pi} = (x_1, \ldots, x_{n-1}) \in \Sigma, \; x_n \in (0, l) \). Define \( \hat{F} = \{F : F(x, t) = \kappa(x_n)w(\bar{\pi}, t), \; w \in L^2(S_T) \} \), where \( \kappa \in L^2(0, l), \; \kappa \neq 0 \) is a prescribed function and \( S_T = S \times (0, T) \).

Case 3. Define \( \hat{F} = \{F : F(x, t) = \sum_{j=1}^{N} w_j \kappa_j(x, t), \; w = (w_j)_{j=1}^{N} \in \mathbb{R}^N \} \), where \( \kappa = (\kappa_j)_{j=1}^{N} \in (L^2(\Omega_T))^N \), \( \kappa \neq 0 \) is a prescribed vector-function. In practice, the component \( \kappa_j \) may be the characteristic function of a subdomain \( \Omega_j \subset \Omega \).

Now let us consider the first variation of the cost functional

\[
 \Delta J(F) = J(F + \Delta F) - J(F)
\]

\[
 = 2 \int_{\Omega} [u(x, T; F) - u_T(x)] \Delta u(x, T; F) \, dx + \int_{\Omega} [\Delta u(x, T; F)]^2 \, dx, \quad (3.2)
\]

where \( \Delta u(x, t; F) = u(x, t; F + \Delta F) - u(x, t; F) \). By Theorem 1, the function \( \Delta u \) belongs to \( W_{2,1}^{2,1}(\Omega_T) \) and solves the following problem in the classical sense:

\[
 \Delta u_t = A \Delta u - m * A \Delta u + \Delta F \quad \text{in} \; \Omega_T, \quad (3.3)
\]

\[
 \Delta u = 0 \quad \text{in} \; \Omega \times \{0\}, \quad (3.4)
\]

\[
 \Delta u = 0 \quad \text{in} \; \Gamma_{1,T}, \quad (3.5)
\]

\[
 - \nu_A \cdot \nabla \Delta u + m * \nu_A \cdot \nabla \Delta u = \psi \Delta u \quad \text{in} \; \Gamma_{2,T}. \quad (3.6)
\]

Moreover, let us introduce the following adjoint problem with the solution \( \psi(x, t; F) \):

\[
 \psi_t(x, t; F) = -A \psi(x, t; F) + \int_{\mathbb{T}} m(\tau - t) A \psi(x, \tau; F) \, d\tau \quad \text{in} \; \Omega_T, \quad (3.7)
\]

\[
 \psi(x, T; F) = 2[u(x, T; F) - u_T(x)] \quad \text{in} \; \Omega, \quad (3.8)
\]

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\[ \psi(x, t; F) = 0 \quad \text{in } \Gamma_{1,T}, \quad \text{(3.9)} \]
\[ - \nu_A \cdot \nabla \psi(x, t; F) + \int_t^T m(\tau - t) \nu_A \cdot \nabla \psi(x, \tau; F) \, d\tau = \vartheta \psi(x, t; F) \quad \text{in } \Gamma_{2,T}. \quad \text{(3.10)} \]

It is easy to see that the equivalent problem for \( \tilde{u}(x, t) = \psi(x, T - t; F) \) is of the form (2.1)-(2.4) with homogeneous differential conditions and the initial condition \( \tilde{u} = 2[u(\cdot, T; F) - u_T] \in L^2(\Omega) \) in \( \Omega \times \{0\} \).

Therefore, applying Theorem 206 K. Kasemets and J. Janno \( \psi \) is a unique weak solution. The weak problem for \( \psi(x, T - t; F) \) reads
\[ 0 = \int_{\Gamma_{1,T}} [\psi(x, 0; F) \eta(x, T) - 2[u(x, T; F) - u_T(x)] \eta(x, 0)] \, dx \]
\[ - \int_{\Gamma_{1,T}} \int_{\Gamma_{1,T}} [\psi(x, T - t; F) \eta_t(x, t) \, dx \, dt + \int_{\Gamma_{1,T}} \int_{\Gamma_{1,T}} \left[ \sum_{i,j=1}^n a_{ij}(x) (\psi_{x_j}(x, T - t; F) \right. \]
\[ - \int_0^t m(t - \tau) \psi_{x_j}(x, T - \tau; F) \, d\tau \left. \right) \eta_{x_i}(x, t) \]
\[ - a(x) \left( \psi(x, T - t; F) - \int_0^t m(t - \tau) \psi(x, T - \tau; F) \, d\tau \right) \eta(x, t) \right] \, dx \, dt \]
\[ + \int_{\Gamma_{2,T}} \vartheta \psi(x, T - t; F) \eta(x, t) \, d\Gamma \, dt \quad \forall \eta \in \mathcal{T}(\Omega_T). \quad \text{(3.11)} \]

**Lemma 1.** It holds the following formula:
\[ 2 \int_{\Omega} [u(x, T; F) - u_T(x)] \Delta u(x, T, F) \, dx = \int_{\Gamma_{1,T}} \int_{\Gamma_{1,T}} \psi(x, t; F) \Delta F(x, t) \, dx \, dt. \quad \text{(3.12)} \]

**Proof.** Since \( \Delta u \in W^{2,1}_0(\Omega_T) \) satisfies the homogeneous boundary condition on \( \Gamma_1 \), it holds \( \Delta u(x, T - t, F) \in \mathcal{T}(\Omega_T) \). Let us use the test function \( \eta(x, t) = \Delta u(x, T - t, F) \) in (3.11). This yields (changing the variable \( t \) by \( T - t \) under the integrals and observing that \( \eta(x, T) = 0 \) and omitting \( F \) in the arguments for the sake of shortness)
\[ 0 = -2 \int_{\Omega} [u(x, T) - u_T(x)] \Delta u(x, T) \, dx + \int_{\Gamma_{1,T}} \int_{\Gamma_{1,T}} \psi(x, t) \Delta u_t(x, t) \, dx \, dt \]
\[ + \int_{\Gamma_{1,T}} \int_{\Gamma_{1,T}} \left[ \sum_{i,j=1}^n a_{ij}(x) (\psi_{x_j}(x, t) - \int_0^t m(t - \tau) \psi_{x_j}(x, \tau) \, d\tau \right) \Delta u_{x_i}(x, t) \]
\[ - a(x) \left( \psi(x, t) - \int_0^t m(t - \tau) \psi(x, \tau) \, d\tau \right) \Delta u(x, t) \right] \, dx \, dt \]
\[ + \int_{\Gamma_{2,T}} \vartheta \psi(x, t) \Delta u(x, t) \, d\Gamma \, dt. \quad \text{(3.13)} \]
On the other hand, the problem (3.3)–(3.6) in the weak form reads

\[
0 = \int_\Omega \Delta u(x) \zeta(x) \, dx - \int_\Omega \Delta u \zeta_t \, dx \, dt \\
+ \int_\Gamma_2 \vartheta \Delta u \zeta \, d\Gamma - \int_\Omega \Delta F \zeta \, dx \, dt \\
\forall \zeta \in T(\Omega_T).
\]  

(3.14)

Since \( \Delta u \in W^{2,1}_2(\Omega_T) \) has the regular time derivative, we can integrate by parts the integral \( \int_\Omega \Delta u \zeta_t \, dx \, dt \) in (3.14). This results in the relation

\[
0 = \int_\Omega \Delta u_t \zeta \, dx \, dt + \int_\Omega \sum_{i,j=1}^n a_{ij} (\Delta u x_j - m * \Delta u_x) \zeta_{x_i} \, dx \, dt \\
- a(\Delta u - m * \Delta u) \zeta \, dx \, dt + \int_\Gamma_2 \vartheta \Delta u \zeta \, d\Gamma - \int_\Omega \Delta F \zeta \, dx \, dt. 
\]  

(3.15)

It is important that this relation doesn’t contain the time derivative of the test function \( \zeta \). Therefore, we can extend the set of test functions of (3.15) from \( T(\Omega_T) \) to \( U_0(\Omega_T) = \{ \zeta \in U(\Omega_T): \zeta|_{r_1,T} = 0 \text{ in case } r_2 \neq \emptyset \} \). In particular, it is possible to take the test function \( \zeta = \psi \in U_0(\Omega_T) \). Then we obtain

\[
0 = \int_\Omega \Delta u_t \psi \, dx \, dt + \int_\Omega \sum_{i,j=1}^n a_{ij} (\Delta u x_j - m * \Delta u_x) \psi_{x_i} \, dx \, dt \\
- a(\Delta u - m * \Delta u) \psi \, dx \, dt + \int_\Gamma_2 \vartheta \Delta u \psi \, d\Gamma - \int_\Omega \Delta F \psi \, dx \, dt. 
\]  

(3.16)

Subtracting (3.16) from (3.13) and changing the order of integration in convolution terms we deduce the formula (3.12). Lemma is proved. \( \square \)

**Theorem 2.** Let \( \mathcal{F} \) be a bounded, closed and convex subset of \( \hat{\mathcal{F}} \). Then the problem (3.1) has a solution in \( \mathcal{F} \). Moreover, the set of all solutions \( \mathcal{F}^* \) form a closed convex subset of \( \mathcal{F} \).

**Proof.** The assertion follows from Weierstrass existence theorem (see [16, Section 2.5]) once we have proved that \( J(\mathcal{F}) \) is weakly sequentially lower semicontinuous in \( \mathcal{F} \), i.e.

\[
J(\mathcal{F}) \leq \liminf_{n \to \infty} J(F_n) \quad \text{as } F_n \rightharpoonup F \text{ in } \mathcal{F} 
\]  

(3.17)

and convex, i.e.

\[
J(\gamma F_1 + (1 - \gamma) F_2) \leq \gamma J(F_1) + (1 - \gamma) J(F_2) \quad \forall \gamma \in [0,1], \ F_1, F_2 \in \mathcal{F}.
\]

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Let us compute:

\[ J(F) = \int_{\Omega} [u(x,T;F) - u_T(x)]^2 \, dx = \int_{\Omega} [u(x,T;F_n) - u_T(x)]^2 \, dx \]

\[ - \int_{\Omega} [u(x,T;F_n) - u(x,T;F)]^2 \, dx \]

\[ - 2 \int_{\Omega} [u(x,T;F) - u_T(x)][u(x,T;F_n) - u(x,T;F)] \, dx \]

\[ = J(F_n) - \int_{\Omega} [u(x,T;F_n) - u(x,T;F)]^2 \, dx \]

\[ - 2 \int_{\Omega} [u(x,T;F) - u_T(x)] \Delta u_n(x,T;F) \, dx \]

where \( \Delta u_n(x,t;F) = u(x,T;F_n) - u(x,T;F) \) is the change of \( u \) corresponding to the change of the free term \( \Delta F_n = F_n - F \). Thus, in view of (3.12) we have

\[ J(F) \leq J(F_n) - \int_{\Omega} \int_{T} \psi(x,t;F) \Delta F_n(x,t) \, dx \, dt. \]

Since \( \psi \in L^2(\Omega_T) \), this implies the relation (3.17). To prove the convexity, we firstly note that

\[ u(x,t;\gamma F_1 + (1 - \gamma) F_2) = \gamma u(x,t;F_1) + (1 - \gamma) u(x,t;F_2), \quad \text{for} \quad \gamma \in [0,1]. \]

Therefore, in view of the convexity of the quadratic function we obtain

\[ J(\gamma F_1 + (1 - \gamma) F_2) = \int_{0}^{T} [u(x,T,\gamma F_1 + (1 - \gamma) F_2) - u_T(x)]^2 \, dx \]

\[ = \int_{0}^{T} \left[ \gamma [u(x,T,F_1) - u_T(x)] + (1 - \gamma) [u(x,T,F_2) - u_T(x)] \right]^2 \, dx \]

\[ \leq \gamma \int_{0}^{T} [u(x,T,F_1) - u_T(x)]^2 \, dx + (1 - \gamma) \int_{0}^{T} [u(x,T,F_2) - u_T(x)]^2 \, dx \]

\[ = \gamma J(F_1) + (1 - \gamma) J(F_2) \quad \text{for} \quad \gamma \in [0,1]. \]

This shows the convexity of \( J \). Theorem is proved. \( \square \)

Remark 1. In order to prove the existence in an unbounded set \( F \) incl. \( \hat{F} \), it is sufficient to have the weak coercivity of \( J(F) \). This is a difficult problem, because monotonicity methods in general fail for problems in integro-differential PDE. However, the boundedness assumption of \( F \) seems not very restrictive, because in practice some bound for \( F \) may be available.

4 Regularized Problem

In [5] we proved that in a particular case the solution of the inverse problem under consideration continuously depends on certain derivatives of the data.
This shows the ill-posedness of the problem in case the data have noise in $L^2$ space. We can easily incorporate Tikhonov regularization in quasi-solution. In this case we minimize the stabilized cost functional: find

$$ F^* = \arg \min_{F \in \mathcal{F}} J_\alpha(F), \quad J_\alpha(F) = \alpha \|F\|^2_{L^2(\Omega_T)} + \|u(\cdot, T; F) - u_T\|^2_{L^2(\Omega)}. $$

Here $\alpha > 0$ is the regularization parameter that depends on the noise level of the data $u_T$. If we set here $\alpha = 0$, we get the original problem (3.1).

**Theorem 3.** Let $\alpha > 0$ and $\mathcal{F}$ be a closed and convex subset of $\hat{\mathcal{F}}$ (may be also $\mathcal{F} = \hat{\mathcal{F}}$). Then the problem (4.1) has a unique solution in $\mathcal{F}$.

**Proof.** Obviously the additional term $I(F) = \alpha \|F\|_{L^2(\Omega_T)}$ is strictly convex:

$$ I(\gamma F_1 + (1 - \gamma)F_2) < \gamma I(F_1) + (1 - \gamma)I(F_2) \quad \forall \gamma \in (0, 1), \ F_1, F_2 \in \mathcal{F} $$

and weakly coercive, i.e., $I(F) \to \infty$ as $\|F\|_{L^2(\Omega_T)} \to \infty$. This makes the whole functional $J_\alpha$ strictly convex and weakly coercive. Moreover, it is easy to check that $I(F)$ is weakly sequentially lower semi-continuous. Since $J(F) = \|u(\cdot, T; F) - u_T\|^2_{L^2(\Omega)}$ is also weakly lower semi-continuous (this was shown in the proof of Theorem 2), the whole functional $J_\alpha$ is weakly lower semi-continuous. Now the assertion of the theorem follows from Weierstrass existence theorem [16, Section 2.5]. $\square$

### 5 Auxiliary Estimates

**Lemma 2.** The following estimate is valid with a constant $C_0$:

$$ \|\Delta u(\cdot, T; F)\|_{L^2(\Omega)} \leq C_0 \|\Delta F\|_{L^2(\Omega_T)}. \tag{5.1} $$

**Proof.** For the sake of shortness, we omit $F$ in the list of arguments of $\Delta u$. Firstly, we prove this assertion in case $\|m\|_{L^1(0,T)}$ is small enough and the equation for $\Delta u$ (3.3) contains an additional term, namely it has the form

$$ \Delta u_t = A \Delta u - \sigma \Delta u - m * A \Delta u + \Delta F \quad \text{in } \Omega_T, \tag{5.2} $$

where $\sigma$ is a sufficiently large number such that $\sigma - a(x) \geq \epsilon$ for any $x \in \Omega$. By Theorem 1, $\Delta u$ belongs to $W^{2,1}_2(\Omega_T)$ and solves the problem (5.2), (3.4)–(3.6) in the classical sense. Let us multiply the equation (5.2) by $\Delta u$ and integrate by parts taking into account the definition of $A$ and the homogeneous boundary conditions (3.5), (3.6):

$$ 0 = \int_{\Omega_T} \left[ \Delta u_t - (A - \sigma) \Delta u + m * A \Delta u - \Delta F \right] \Delta u \, dx \, dt $$

$$ = \frac{1}{2} \int_{\Omega_T} |\Delta u|^2_t \, dx \, dt + \int_{\Omega_T} \left[ \sum_{i,j=1}^n a_{ij} \Delta u_{x_j} \Delta u_{x_i} + (\sigma - a) \Delta u^2 \right] \, dx \, dt $$

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\[
- \int\int_{\Omega_T} \left[ \sum_{i,j=1}^{n} a_{ij} (m \ast \Delta u_{x_j}) \Delta u_{x_i} - a (m \ast \Delta u) \right] dx \, dt \\
+ \int\int_{\Gamma_{\omega},T} \vartheta \Delta u^2 \, d\Gamma \, dt - \int\int_{\Omega_T} \Delta F \Delta u \, dx \, dt.
\]

In view of the homogeneous initial condition (3.4), this relation can be transformed to the form

\[
\frac{1}{2} \int_{\Omega} [\Delta u(x,T)]^2 \, dx + \int\int_{\Gamma_{\omega},T} \vartheta \Delta u^2 \, d\Gamma \, dt
\]

\[
+ \int\int_{\Omega_T} \left[ \sum_{i,j=1}^{n} a_{ij} \Delta u_{x_j} \Delta u_{x_i} + (\sigma - a) \Delta u^2 \right] dx \, dt
\]

\[
= \int\int_{\Omega_T} \left[ \sum_{i,j=1}^{n} a_{ij} (m \ast \Delta u_{x_j}) \Delta u_{x_i} - a (m \ast \Delta u) \right] dx \, dt + \int\int_{\Omega_T} \Delta F \Delta u \, dx \, dt.
\]

Due to the assumptions \( \vartheta \geq 0, (2.8) \) and \( \sigma - a \geq \epsilon \), the left hand side of (5.3) can be estimated from below:

\[
\frac{1}{2} \int_{\Omega} [\Delta u(x,T)]^2 \, dx + \epsilon \int\int_{\Omega_T} [\nabla \Delta u]^2 + \Delta u^2 \, dx \, dt =: I^2.
\]

The right-hand side of (5.3) is estimated from above by means of the Cauchy–Schwarz inequality:

\[
\int\int_{\Omega_T} \left[ \sum_{i,j=1}^{n} a_{ij} (m \ast \Delta u_{x_j}) \Delta u_{x_i} - a (m \ast \Delta u) \right] dx \, dt + \int\int_{\Omega_T} \Delta F \Delta u \, dx \, dt
\]

\[
\leq \bar{C}_1 \left[ \sum_{i,j=1}^{n} \| m \ast \Delta u_{x_j} \|_{L^2(\Omega_T)} \| \Delta u_{x_i} \|_{L^2(\Omega_T)} \right]
\]

\[
+ \| m \ast \Delta u \|_{L^2(\Omega_T)} \| \Delta u \|_{L^2(\Omega_T)} + \| \Delta F \|_{L^2(\Omega_T)} \| \Delta u \|_{L^2(\Omega_T)}
\]

(5.5)

where \( \bar{C}_1 \) is a constant depending on the coefficients \( a_{ij} \) and \( a \). For the convolution terms we apply the Young’s inequality in the space \( L^2(\Omega_T) = L^2((0,T); L^2(\Omega)) \). This yields

\[
\| m \ast \Delta u_{x_j} \|_{L^2(\Omega_T)} \leq \| m \|_{L^1(0,T)} \| \Delta u_{x_j} \|_{L^2(\Omega_T)}, \quad j = 1, \ldots, n,
\]

\[
\| m \ast \Delta u \|_{L^2(\Omega_T)} \leq \| m \|_{L^1(0,T)} \| \Delta u \|_{L^2(\Omega_T)}.
\]

(5.6)
Using (5.4)–(5.6) in (5.3) we obtain

\[ I^2 \leq C_1 \|m\|_{L^1(0,T)} \left[ \sum_{i,j=1}^{n} \|\Delta u_{x_j}\|_{L^2(\Omega_T)} \|\Delta u_{x_i}\|_{L^2(\Omega_T)} + \|\Delta u\|_{L^2(\Omega_T)}^2 \right] + \|\Delta F\|_{L^2(\Omega_T)} \|\Delta u\|_{L^2(\Omega_T)}. \]

Further, we use the inequalities

\[ \|\Delta u_{x_i}\|_{L^2(\Omega_T)} \leq \|\nabla \Delta u\|_{L^2(\Omega_T)}, \quad i = 1, \ldots, n, \]

and definition of \( I \) (see (5.4)). We have

\[ I^2 \leq C_1 \|m\|_{L^1(0,T)} \left[ n^2 \|\nabla \Delta u\|_{L^2(\Omega_T)}^2 + \|\Delta u\|_{L^2(\Omega_T)}^2 \right] + \|\Delta F\|_{L^2(\Omega_T)} \|\Delta u\|_{L^2(\Omega_T)} \leq \frac{C_1 n^2 \|m\|_{L^1(0,T)}}{\epsilon} I^2 + \frac{1}{\sqrt{\epsilon}} \|\Delta F\|_{L^2(\Omega_T)} I. \]

Therefore, in case \( m \) satisfies the smallness condition

\[ \|m\|_{L^1(0,T)} \leq \frac{\epsilon}{2C_1 n^2}, \] (5.7)

we obtain \( I^2 \leq \frac{2}{\sqrt{\epsilon}} \|\Delta F\|_{L^2(\Omega_T)} I \) that yields \( I \leq \frac{2}{\sqrt{\epsilon}} \|\Delta F\|_{L^2(\Omega_T)}. \) Observing that \( \|\Delta u(\cdot, T)\|_{L^2(\Omega)} \leq \sqrt{2I} \), from the latter inequality we deduce the estimate (5.1) with the constant \( C_0 = 2\sqrt{2}/\epsilon. \)

Now let us return to the original problem (3.3)–(3.6) without the additional \( \sigma \)-term and arbitrarily large \( m \). Define the following function: \( \Delta u_\sigma(x,t) = e^{-\sigma t} \Delta u(x,t) \) where \( \sigma \in \mathbb{R} \). It is easy to check that \( \Delta u_\sigma \) solves the following problem:

\[ \Delta u_{\sigma,t} - \sigma \Delta u_\sigma - m_\sigma \star A \Delta u_\sigma + \Delta F_\sigma = 0 \quad \text{in} \quad \Omega_T, \]
\[ \Delta u_\sigma = 0 \quad \text{in} \quad \Omega \times \{0\}, \]
\[ \Delta u_\sigma = 0 \quad \text{in} \quad \Gamma_{1,T}, \]
\[ -\nu_A \cdot \nabla \Delta u_\sigma + m_\sigma \star \nu_A \cdot \nabla \Delta u_\sigma = \partial \Delta u_\sigma \quad \text{in} \quad \Gamma_{2,T}, \]

where \( m_\sigma(t) = e^{-\sigma t} m(t) \) and \( \Delta F_\sigma(x,t) = e^{-\sigma t} \Delta F(x,t) \). Clearly, there exists a sufficiently large \( \sigma \) such that \( m_\sigma \) satisfies the condition (5.7) and the inequality \( \sigma - a(x) \geq \epsilon \) is valid for \( x \in \Omega \). Therefore, the first part of the proof applies to the function \( \Delta u_\sigma \). This means that the estimate

\[ \|\Delta u_\sigma(\cdot, T)\|_{L^2(\Omega)} \leq \frac{2\sqrt{2}}{\epsilon} \|\Delta F_\sigma\|_{L^2(\Omega_T)} \] (5.8)

is valid. Finally, in view of \( \Delta u_\sigma(x,T) = e^{-\sigma T} \Delta u(x,T) \) and \( |\Delta F_\sigma(x,t)| \leq |\Delta F(x,t)| \), from (5.8) we obtain the desired estimate (5.1) with the constant \( C_0 = 2\sqrt{2}e^{\sigma T}/\epsilon \). Lemma 2 is proved. □

Further, let us estimate the difference of solutions of the adjoint problems

\[ \Delta \psi(x,t; F) = \psi(x,t; F + \Delta F) - \psi(x,t; F). \]
Lemma 3. The following estimate is valid with a constant $C_1$:

$$
\|\Delta \psi(\cdot, \cdot; F)\|_{L^2(\Omega_T)} \leq C_1 \|\Delta F\|_{L^2(\Omega_T)}. \tag{5.9}
$$

Proof. Proof is similar to the proof of the previous lemma. Observing (3.7)–(3.10) we see that the problem for $\Delta \psi(x, t; F)$ has the following form:

$$
\Delta \psi_t(x, t; F) = -A \Delta \psi(x, t; F) + \int_t^T m(\tau - t) A \Delta \psi(x, \tau; F) \, d\tau \quad \text{in } \Omega_T, \tag{5.10}
$$

$$
\Delta \psi(x, T; F) = 2 \Delta u(x, T; F) \quad \text{in } \Omega, \tag{5.11}
$$

$$
\Delta \psi(x, t; F) = 0 \quad \text{in } \Gamma_{1,T}, \tag{5.12}
$$

$$
- \nu_A \cdot \nabla \Delta \psi(x, t; F) + \int_t^T m(\tau - t) \nu_A \cdot \nabla \Delta \psi(x, \tau; F) \, d\tau
= \partial \Delta \psi(x, t; F) \quad \text{in } \Gamma_{2,T}. \tag{5.13}
$$

We start by proving the assertion in case $\|m\|_{L^1(0,T)}$ is small enough and the equation (3.3) contains an additional term, namely it has the form

$$
\Delta \psi_t(x, t; F) = -A \Delta \psi(x, t; F) + \sigma \Delta \psi(x, t; F)
+ \int_t^T m(\tau - t) A \Delta \psi(x, \tau; F) \, d\tau \quad \text{in } \Omega_T, \tag{5.14}
$$

where $\sigma$ is again sufficiently large, i.e. $\sigma - a(x) \geq \epsilon$ for any $x \in \Omega$. Since $\Delta u \in W^{2,1}_2(\Omega_T)$, by the trace theorem it holds $\Delta u|_{t=T} \in H^1(\Omega)$. Moreover, one can immediately check that the time-inverted function $\Delta \psi(x, T - t; F)$ satisfies a problem of the form (2.1)–(2.4) with an homogeneous equation, homogeneous boundary conditions and the initial condition $2 \Delta u(x, T; F)$. Therefore, applying Theorem 1 we see that the function $\Delta \psi(x, t; F)$ belongs to $W^{2,1}_2(\Omega_T)$ and satisfies the problem (5.14), (5.11), (5.12), (5.13) in the classical sense. For the sake of shortness we omit the argument $F$ of $\Delta \psi$ and $\Delta u$ in forthcoming computations. Multiplying (5.14) by $\Delta \psi$ and integrating by parts we obtain

$$
0 = \int \int_{\Omega_T} \left[ \Delta \psi_t + (A - \sigma) \Delta \psi - \int_t^T m(\tau - t) A \Delta \psi(x, \tau) \, d\tau \right] \Delta \psi \, dx \, dt
$$

$$
= \frac{1}{2} \int \int_{\Omega_T} |\Delta \psi|^2 \, dx \, dt - \int \int_{\Omega_T} \left[ \sum_{i,j=1}^n a_{ij} \Delta \psi_{x_j} \Delta \psi_{x_i} + (\sigma - a) |\Delta \psi|^2 \right] \, dx \, dt
$$

$$
+ \int \int_{\Omega_T} \left[ \sum_{i,j=1}^n a_{ij}(x) \int_t^T m(\tau - t) \Delta \psi_{x_j}(x, \tau) \, d\tau \Delta \psi_{x_i}(x, t) \right] \, dx \, dt
$$

$$
- a(x) \int_t^T m(\tau - t) \Delta \psi(x, \tau) \, d\tau \Delta \psi(x, t) \right] \, dx \, dt - \int \int_{\Gamma_{2,T}} \partial \Delta \psi^2 \, d\Gamma \, dt.
$$

Observing the final condition (5.11) and rearranging the terms we get

$$
\frac{1}{2} \int \int_{\Omega} |\Delta \psi(x, 0)|^2 \, dx + \int \int_{\Gamma_{2,T}} \partial \Delta \psi^2 \, d\Gamma \, dt \tag{5.15}
$$
By means of (5.15) we get
\[
\mathcal{M} \leq \left\| \| \nabla \Delta \psi | x \rangle \right\|_{L^2(\Omega_T)}^2 + \left\| \| \Delta \psi | x \rangle \right\|_{L^2(\Omega_T)}^2.
\]

The left-hand side of (5.15) is estimated from below:
\[
\frac{1}{2} \int_\Omega [\Delta \psi(x,0)]^2 dx + \int_\Gamma_\Omega \partial \Delta \psi^2 d\Gamma + \frac{1}{2} \int_\Omega \left[ \sum_{i,j=1}^n a_{ij} \Delta \psi_{x_j} \Delta \psi_{x_i} + (\sigma - a) \Delta \psi^2 \right] dx + \frac{1}{2} \int \Delta u(x,T)^2 dx. \tag{5.16}
\]

For the right-hand side of (5.15) we use the Cauchy–Schwarz inequality:
\[
\leq \hat{C}_1 \left[ \sum_{i,j=1}^n \left\| \int_t^T m(\tau - t) \Delta \psi_{x_j} (x, \tau) d\tau \Delta \psi_{x_i} (x, t) \right\|_{L^2(\Omega_T)} \right] + \frac{1}{2} \left\| \Delta u(\cdot, T) \right\|_{L^2(\Omega)}^2 + \frac{1}{2} \int \Delta u(x,T)^2 dx. \tag{5.17}
\]

with some constant $\hat{C}_1$. It is easy to check by means of the change of variables of integration that
\[
\left\| \int_t^T m(\tau - t) v(x, \tau) d\tau \right\|_{L^2(\Omega_T)} = \left\| m \ast v \right\|_{L^2(\Omega_T)} \quad \text{for any} \quad v.
\]

Therefore, using the Young’s inequality we get
\[
\left\| \int_t^T m(\tau - t) \Delta \psi_{x_j} (x, \tau) d\tau \right\|_{L^2(\Omega_T)} \leq \left\| m \right\|_{L^1(0,T)} \left\| \Delta \psi_{x_j} \right\|_{L^2(\Omega_T)},
\]
\[
\left\| \int_t^T m(\tau - t) \Delta \psi (x, \tau) d\tau \right\|_{L^2(\Omega_T)} \leq \left\| m \right\|_{L^1(0,T)} \left\| \Delta \psi \right\|_{L^2(\Omega_T)}. \tag{5.19}
\]

By means of (5.17)–(5.19) from (5.17) we obtain the relation
\[
S^2 \leq \hat{C}_1 \left[ \sum_{i,j=1}^n \left\| \Delta \psi_{x_j} \right\|_{L^2(\Omega_T)} \left\| \Delta \psi_{x_i} \right\|_{L^2(\Omega_T)} + \left\| \Delta \psi \right\|_{L^2(\Omega_T)}^2 \right] + \frac{1}{2} \left\| \Delta u(\cdot, T) \right\|_{L^2(\Omega)}^2.
\]

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Like in the proof of Lemma 3 from this relation and the definition of $S$ we deduce the estimate $\|\Delta \psi\|_{L^2(\Omega_T)} \leq \frac{1}{\sqrt{\epsilon}} \|\Delta u(\cdot, T)\|_{L^2(\Omega)}$ provided $m$ satisfies the inequality

$$\|m\|_{L^1(0, T)} \leq \frac{\epsilon}{2C_1 n^2}. \quad (5.20)$$

Further, applying Lemma 2 to the obtained estimate we get (5.9) with the constant $C_1 = C_0 / \sqrt{\epsilon}$.

Finally, let us consider the original problem for $\Delta \psi$ without the additional $\sigma$-term and arbitrarily large $m$. Define $\Delta \psi_\sigma(x, t) = e^{-\sigma(T-t)} \Delta u(x, t)$ with $\sigma \in \mathbb{R}$. Then $\Delta \psi_\sigma$ solves the following problem:

$$
\Delta \psi_\sigma, t(x, t) = -A \Delta \psi_\sigma(x, t) + \int_t^T m_\sigma(\tau - t) A \Delta \psi_\sigma(x, \tau) \, d\tau \quad \text{in } \Omega_T,
$$

$$
\Delta \psi_\sigma(x, T) = 2\Delta u(x, T) \quad \text{in } \Omega, \quad \Delta \psi_\sigma(x, t) = 0 \quad \text{in } \Gamma_{1, T},
$$

$$
- \nu_A \cdot \nabla \Delta \psi_\sigma(x, t) + \int_t^T m_\sigma(\tau - t) \nu_A \cdot \nabla \Delta \psi_\sigma(x, \tau; t) \, d\tau = \partial \Delta \psi_\sigma(x, t) \quad \text{in } \Gamma_{2, T},
$$

where $m_\sigma(t) = e^{-\sigma t} m(t)$ again. There exists a sufficiently large $\sigma$ such that $m_\sigma$ satisfies the condition (5.20) and the inequality $\sigma - a(x) \geq \epsilon$ is valid for $x \in \Omega$. Thus, applying the first part of the proof to $\Delta \psi_\sigma$ we have

$$
\|\Delta \psi_\sigma\|_{L^2(\Omega_T)} \leq \frac{C_0}{\sqrt{\epsilon}} \|F\|_{L^2(\Omega_T)}.
$$

Since $\|\Delta \psi_\sigma\|_{L^2(\Omega_T)} \geq e^{-\sigma T} \|\Delta \psi\|_{L^2(\Omega_T)}$ we reach the estimate (5.9) with the constant $C_1 = C_0 e^{\sigma T} / \sqrt{\epsilon}$. Lemma 3 is proved. \qed

6 Frechet Derivative and Gradient Method

It follows from Lemma 2 with (3.2) that the functional $J$ is Frechet differentiable in $L^2(\Omega_T)$. Moreover, according to Lemma 1, $J'(F)$ is identical to the element $\psi(F) = \psi(x, t; F)$ in $L^2(\Omega_T)$, i.e. it holds

$$
J'(F) \tilde{F} = (\psi(F), \tilde{F})_{L^2(\Omega_T)} = \int_{\Omega_T} \psi(x, t; F) \tilde{F}(x, t) \, dx \, dt \quad \forall \tilde{F} \in L^2(\Omega_T).
$$

Similarly, $J_\alpha$ is Frechet differentiable in $L^2(\Omega_T)$ and

$$
J_\alpha'(F) \tilde{F} = (2\alpha F + \psi(F), \tilde{F})_{L^2(\Omega_T)} = \int_{\Omega_T} (2\alpha F(x, t) + \psi(x, t; F)) \tilde{F}(x, t) \, dx \, dt \quad \forall \tilde{F} \in L^2(\Omega_T). \quad (6.1)
$$

Therefore, gradient-type methods can be used to solve the minimization problems (3.1) and (4.1). These methods must be combined by proper projection techniques to get minimum in the subset $\mathcal{F}$. However, it is possible to simplify
the minimization procedure in case the structure of the subspace \( \hat{F} \) is simple. In particular, global optimization can be used if \( F = \hat{F} \). To this end, let us consider the cases 1–3 introduced in Section 3.

**Case 1.** We introduce the functional \( \Phi_{1,\alpha}(w) = J_\alpha(\omega w) \) with \( \alpha \geq 0 \) and the set \( \mathcal{W}_1 = \{ w \in L^2(\Omega) : \omega w \in F \} \). Then the problem (4.1) (in case \( \alpha = 0 \) the problem (3.1)) can be rewritten as follows:

\[
\text{find } w^* = \arg \min_{w \in \mathcal{W}_1} \Phi_{1,\alpha}(w).
\] (6.2)

In particular, when \( F = \hat{F} \), it holds \( \mathcal{W}_1 = L^2(\Omega) \) and we have a global minimization problem. Since \( J_\alpha \) is Frechet differentiable, \( \Phi_{1,\alpha} \) is also Frechet differentiable. Moreover, from (6.1) we deduce

\[
J'_\alpha(\omega w)\omega \tilde{w} = \int_\Omega \left[ \int_0^T [2\alpha \omega(x) \kappa(t) + \psi(x, t, \omega w)] \kappa(t) dt \right] \tilde{w}(x) dx.
\]

This shows that \( \Phi'_{1,\alpha}(w) \) is identical to the element \( \int_0^T [2\alpha \omega(x) \kappa(t) + \psi(x, t, \omega w)] \kappa(t) dt \) of \( L^2(\Omega) \), that is

\[
\Phi'_{1,\alpha}(w) \tilde{w} = \left( \int_0^T [2\alpha \omega \kappa(t) + \psi(\cdot, t, \omega w)] \kappa(t) dt, \tilde{w} \right)_{L^2(\Omega)} \quad \forall \tilde{w} \in L^2(\Omega).
\]

Using Cauchy–Schwarz inequality and Lemma 3 we estimate

\[
\| \Phi'_{1,\alpha}(w + \Delta w) - \Phi'_{1,\alpha}(w) \|_{L^2(\Omega)} \\
= \left[ \int_\Omega \left\{ \int_0^T [2\alpha \Delta \omega(x) \kappa(t) + \psi(x, t, \kappa(x + \Delta w) - \psi(x, t, \kappa(w)) \kappa(t) dt \right\}^2 dx \right]^{1/2} \\
\leq \|2\alpha \Delta \omega(x) \kappa(t) + \psi(x, t, \kappa(w + \Delta w) - \psi(x, t, \kappa(w)) \|_{L^2(\Omega_T)} \| \Delta w \|_{L^2(\Omega_T)} \\
\leq (2\alpha + C_1) \| \kappa \Delta \omega \|_{L^2(\Omega_T)} \| \kappa \|_{L^2(\Omega_T)} = (2\alpha + C_1) \| \kappa \|_{L^2(\Omega)} \| \Delta w \|_{L^2(\Omega)}.
\]

This implies that \( \Phi'_{1,\alpha} \) is uniformly Lipschitz-continuous, i.e.

\[
\| \Phi'_{1,\alpha}(w + \Delta w) - \Phi'_{1,\alpha}(w) \|_{L^2(\Omega)} \leq L_\alpha \| \Delta w \|_{L^2(\Omega)}
\] (6.3)

where \( L_\alpha = (2\alpha + C_1) \| \kappa \|_{L^2(\Omega)}^2 \).

The cases 2 and 3 can be treated in a similar manner. Let us summarize the results in these cases.

**Case 2.** Define \( \Phi'_{2,\alpha}(w) = J_\alpha(\omega w) \) with \( \alpha \geq 0 \) and the set \( \mathcal{W}_2 = \{ w \in L^2(S_T) : \omega w \in F \} \). If \( F = \hat{F} \) then \( \mathcal{W}_2 = L^2(S_T) \). The problem (4.1) can be rewritten in the form: find \( w^* = \arg \min_{w \in \mathcal{W}_2} \Phi_{2,\alpha}(w) \) the functional \( \Phi_{2,\alpha} \) is Frechet differentiable, \( \Phi'_{2,\alpha}(w) \) is identical to the element \( \int_0^T [2\alpha \omega(x, t) \kappa(x) + \psi(x, t, \omega w)] \kappa(x) dx \) of \( L^2(S_T) \) and the uniform Lipschitz-estimate

\[
\| \Phi'_{2,\alpha}(w + \Delta w) - \Phi'_{2,\alpha}(w) \|_{L^2(S_T)} \leq L_\alpha \| \Delta w \|_{L^2(S_T)}
\] (6.4)

is valid with \( L_\alpha = (2\alpha + C_1) \| \kappa \|^2_{L^2(\Omega)} \).

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Case 3. Let \( \Phi_{3,\alpha}(w) = J_\alpha(\sum_{j=1}^{N} w_j \varphi_j) \) with \( \alpha \geq 0 \) and \( \mathcal{W}_3 = \{ w \in \mathbb{R}^N : \sum_{j=1}^{N} w_j \varphi_j \in \mathcal{F} \} \). If \( \mathcal{F} = \hat{\mathcal{F}} \) then \( \mathcal{W}_2 = \mathbb{R}^N \). The problem (4.1) admits the following form: find \( w^* = \arg \min_{w \in \mathcal{W}_3} \Phi_{3,\alpha}(w) \). The functional \( \Phi_{3,\alpha} \) is Fréchet differentiable, \( \Phi'_{3,\alpha}(w) \) is identical to the element \( \left( \int_{\Omega_T} [2\alpha \sum_{j=1}^{N} w_j \varphi_j(x,t) + \psi(x,t,\sum_{j=1}^{N} w_j \varphi_j)] \varphi_j(x,t) \, dx \, dt \right)_{j=1,\ldots,N} \) of \( \mathbb{R}^N \) and the estimate

\[
\|\Phi'_{3,\alpha}(w + \Delta w) - \Phi'_{3,\alpha}(w)\|_{\mathbb{R}^N} \leq L_\alpha \|\Delta w\|_{\mathbb{R}^N} \tag{6.5}
\]

with \( L_\alpha = (2\alpha + C_1) \sum_{j=1}^{N} \| \varphi_j \|_{L^2(\Omega_T)}^2 \) is valid.

In the following, let \( \Phi_{\alpha} \) be one of the functionals \( \Phi_{j,\alpha} \), \( j = 1, 2, 3 \), defined above and \( \mathcal{W} \) be the corresponding set of admissible solutions \( \mathcal{W}_j \). Then we consider the problem

\[
\text{find } w^* = \arg \min_{w \in \mathcal{W}} \Phi_{\alpha}(w). \tag{6.6}
\]

For the sake of simplicity, we assume that \( \mathcal{F} = \hat{\mathcal{F}} \). This means that we consider the unconstrained minimization and \( \mathcal{W} \) is \( L^2(\Omega) \), \( L^2(S_T) \) and \( \mathbb{R}^N \) in the cases 1, 2 and 3, respectively. Let \( w_0 \in \mathcal{W} \) be an initial guess and compute the successive approximations by means of the gradient method

\[
w_{k+1} = w_k - c_k \Phi'_{\alpha}(w_k), \quad k = 0, 1, 2, \ldots \tag{6.7}
\]

with steps \( c_k > 0 \). Let us perform a little analysis for this iteration process following partially the example of [2].

**Lemma 4.** For any \( \alpha \geq 0 \) it holds

\[
|\Phi_{\alpha}(w_{k+1}) - \Phi_{\alpha}(w_k) - \Phi'_{\alpha}(w_k)(w_{k+1} - w_k)| \leq \frac{L_\alpha}{2} \|w_{k+1} - w_k\|^2. \tag{6.8}
\]

**Proof.** Using the relation

\[
\Phi_{\alpha}(w_{k+1}) - \Phi_{\alpha}(w_k) = \int_0^1 \Phi'_{\alpha}(w_k + \tau(w_{k+1} - w_k))(w_{k+1} - w_k) \, d\tau
\]

and the estimates (6.3)–(6.5) we deduce

\[
|\Phi_{\alpha}(w_{k+1}) - \Phi_{\alpha}(w_k) - \Phi'_{\alpha}(w_k)(w_{k+1} - w_k)|
\]

\[
= \left| \int_0^1 [\Phi'_{\alpha}(w_k + \tau(w_{k+1} - w_k)) - \Phi'_{\alpha}(w_k)](w_{k+1} - w_k) \, d\tau \right|
\]

\[
\leq L_\alpha \|w_{k+1} - w_k\|^2 \int_0^1 \tau \, d\tau = \frac{L_\alpha}{2} \|w_{k+1} - w_k\|^2.
\]

This proves (6.8). \( \square \)

**Theorem 4.** Let \( \alpha \geq 0 \) and \( \delta \leq c_k \leq 2/L_\alpha - \delta \) for any \( k = 0, 1, 2, \ldots \) where \( \delta \) is some number in the half-interval \( (0, 1/L_\alpha) \). Then the sequence \( \Phi_{\alpha}(w_k) \) is
monotonically decreasing, has a limit and the following relations are valid with 
\[ q_k = c_k - L_\alpha c_k^2 / 2 \geq \delta - L_\alpha \delta^2 / 2 > 0: \]
\[
\Phi_\alpha(w_k) - \Phi_\alpha(w_{k+1}) \geq q_k \| \Phi'_\alpha(w_k) \|^2, \quad k = 0, 1, 2, \ldots, \tag{6.9}
\]
\[
\Phi'_\alpha(w_k) \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty, \tag{6.10}
\]
\[
\| w_{k+1} - w_k \|^2 \leq \frac{c_k^2}{q_k} \left[ \Phi_\alpha(w_k) - \Phi_\alpha(w_{k+1}) \right], \quad k = 0, 1, 2, \ldots \tag{6.11}
\]

Proof. Due to (6.7) it hold \[ \| w_{k+1} - w_k \|^2 \leq c_k^2 \| \Phi'_\alpha(w_k) \|^2 \]
and
\[
\Phi'_\alpha(w_k)(w_{k+1} - w_k) = (\Phi'_\alpha(w_k), -c_k \Phi'_\alpha(w_k))_W = -c_k \| \Phi'_\alpha(w_k) \|^2.
\]
Thus, by means of (6.8) we get
\[
\Phi_\alpha(w_{k+1}) - \Phi_\alpha(w_k) + c_k \| \Phi'_\alpha(w_k) \|^2
\leq \left| \Phi_\alpha(w_{k+1}) - \Phi_\alpha(w_k) + c_k \| \Phi'_\alpha(w_k) \|^2 \right| \leq \frac{L_\alpha c_k^2}{2} \| \Phi'_\alpha(w_k) \|^2.
\]
This yields \[ \Phi_\alpha(w_k) - \Phi_\alpha(w_{k+1}) \geq (c_k - \frac{L_\alpha c_k^2}{2}) \| \Phi'_\alpha(w_k) \|^2, \text{ i.e.} \tag{6.9}\]
Due to \[ q_k > 0, \] the relation (6.9) implies that \[ \Phi_\alpha(w_k) \] is monotonically decreasing and 
since \[ \Phi_\alpha(w) \] has the lower bound 0, the sequence \[ \Phi_\alpha(w_k) \] converges. Further, 
since the sequence \[ q_k \] has the positive lower bound \[ \delta - \frac{L_\alpha \delta^2}{2} \] and the left hand 
side of (6.9) converges to zero, we obtain (6.10). Finally, estimating (6.7) we have 
\[ \| w_{k+1} - w_k \|^2 = c_k^2 \| \Phi'_\alpha(w_k) \|^2. \] Using here (6.9) we obtain (6.11). Theorem is proved. \[ \square \]

Clearly, the highest decrease rate of \[ \Phi_\alpha(w_k) \] is achieved in case \[ c_k = 1/L_\alpha \]
when \[ q_k \] has the biggest value \[ q_k = 1/2L_\alpha. \]

**Theorem 5.** Let \[ \alpha > 0 \] and \[ c_k \] be chosen as in Theorem 4. Then the sequence \[ w_k \] strongly converges to the unique solution of the minimization problem (6.6).

Proof. The existence of the unique solution for the minimization problem immediately follows from Theorem 3 and the definitions of \[ \Phi_\alpha. \] Moreover, since \[ J_\alpha \] is weakly sequentially lower semi-continuous, strictly convex and weakly coercive (see the proof of Theorem 3), the same properties are valid also for \[ \Phi_\alpha. \] It is well-known that under such properties every minimizing sequence of \[ \Phi_\alpha \] weakly converges to the minimum point \[ w^*. \] Thus, firstly, let us show that \[ w_k \] is a minimizing sequence, i.e. \[ \Phi_\alpha(w_k) \rightarrow \Phi_\alpha(w^*). \]

Note that the sequence \[ w_k \] is bounded. Indeed, otherwise there exists a subsequence \[ w_{k_i} \] such that \[ \| w_{k_i} \| \rightarrow \infty \] and by the weak coercivity it holds \[ \Phi_\alpha(w_{k_i}) \rightarrow \infty \] which contradicts to the statement of Theorem 4 that \[ \Phi_\alpha(w_k) \] is monotonically decreasing.

Since \[ \Phi_\alpha \] is convex, its Frechet derivative is monotone, i.e.
\[
[\Phi'_\alpha(\bar{w}) - \Phi'_\alpha(w)](\bar{w} - w) \geq 0 \quad \forall w, \bar{w} \in \mathcal{W}. \tag{6.12}
\]
Let us choose some $\tau \in (0, 1)$. Observing that it holds $\Phi'_\alpha(w^*) = 0$ in the global minimum point $w^*$ and applying (6.12) with $w = w^*$ and $\tilde{w} = w^* + \tau(w_k - w^*)$ we have

$$
\liminf_{k \to \infty} \Phi'_\alpha(w^* + \tau(w_k - w^*))(w_k - w^*) = \frac{1}{\tau} \liminf_{k \to \infty} \left[ \Phi'_\alpha(w^* + \tau(w_k - w^*)) - \Phi'_\alpha(w^*) \right](w^* + \tau(w_k - w^*) - w^*) \geq 0.
$$

(6.13)

On the other hand, it holds $\lim_{k \to \infty} \Phi'_\alpha(w_k)(w_k - w^*) = 0$ because of the boundedness of $w_k$ and the relation (6.10). Thus, using (6.12) with $w = w_k$ and $\tilde{w} = w^* + \tau(w_k - w^*)$ we obtain

$$
\limsup_{k \to \infty} \Phi'_\alpha(w^* + \tau(w_k - w^*))(w_k - w^*) = \frac{1}{1-\tau} \limsup_{k \to \infty} \left[ \Phi'_\alpha(w^* + \tau(w_k - w^*)) - \Phi'_\alpha(w_k) \right](w_k - w^* - \tau(w_k - w^*) \leq 0.
$$

(6.14)

The estimates (6.13) and (6.14) imply $\limsup_{k \to \infty} v_k \leq 0 \leq \liminf_{k \to \infty} v_k$ for the sequence $v_k = \Phi'_\alpha(w^* + \tau(w_k - w^*)) - \Phi'_\alpha(w_k - w^*)$. Hence,

$$
\lim_{k \to \infty} \Phi'_\alpha(w^* + \tau(w_k - w^*))(w_k - w^*) = 0.
$$

(6.15)

Further, writing

$$
\Phi_\alpha(w_k) - \Phi_\alpha(w^*) = \int_0^1 \Phi'_\alpha(w^* + \tau(w_k - w^*)) (w_k - w^*) d\tau
$$

and using (6.15) we obtain $\Phi_\alpha(w_k) - \Phi_\alpha(w^*) \to 0$. This shows that $w_k$ is a minimizing sequence. Consequently, $w_k \to w^*$.

Now let us prove the assertion of the Theorem $w_k \to w^*$. In case 3 this is evident, because $W$ is of finite dimension. Thus, let us study the cases 1 and 2. Then it holds $\Phi_\alpha(w) = \alpha \nu \|w\|^2 + \Phi_0(w)$ where $\nu$ is a positive constant ($\nu = \int_0^T \sigma^2(t) dt$ in case 1 and $\nu = \int_0^T \sigma^2(x_n) dx_n$ in case 2). Since the norm is weakly lower sequentially semicontinuous, the relation $w_k \to w^*$ implies

$$
\|w^*\|^2 \leq \liminf_{k \to \infty} \|w_k\|^2.
$$

(6.16)

On the other hand, since $\Phi_\alpha(w_k)$ converges to $\Phi_\alpha(w^*)$ and $\Phi_0(w)$ is weakly lower sequentially semicontinuous and we obtain

$$
\limsup_{k \to \infty} \|w_k\|^2 = \frac{1}{\alpha \nu} \limsup_{k \to \infty} \left[ \Phi_\alpha(w_k) - \Phi_0(w_k) \right]
$$

$$
= \frac{1}{\alpha \nu} \left\{ \lim_{k \to \infty} \Phi_\alpha(w_k) + \limsup_{k \to \infty} [-\Phi_0(w_k)] \right\}
$$

$$
= \frac{1}{\alpha \nu} \left\{ \Phi_\alpha(w^*) - \liminf_{k \to \infty} \Phi_0(w_k) \right\}
$$

$$
\leq \frac{1}{\alpha \nu} \left\{ \Phi_\alpha(w^*) - \Phi_0(w^*) \right\} = \|w^*\|^2.
$$

(6.17)

Putting together (6.16) and (6.17) we get $\limsup_{k \to \infty} \|w_k\|^2 \leq \|w^*\|^2 \leq \liminf_{k \to \infty} \|w_k\|^2$. This gives $\lim_{k \to \infty} \|w_k\|^2 = \|w^*\|^2$. Since in an Hilbert space the weak convergence and the convergence of norms implies the strong convergence, we prove $w_k \to w^*$. The proof is complete. □
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