The second homology group of the homological Goldman Lie algebra

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Abstract

We determine the second homology group of the homological Goldman Lie algebra for an oriented surface.

1 Introduction

By a surface, we mean an oriented two-dimensional smooth manifold possibly with boundary. The first homology group of a surface and its intersection form reflect the topological structure of the surface. For example, they have information about the genus and the boundary components of the surface. To study them in detail, we consider a Lie algebra coming from them. Goldman introduced the Lie algebra for study of the moduli space of $GL_1(\mathbb{R})$-flat bundles over the surface. We define the Lie algebra in more general setting.

Let $H$ be a $\mathbb{Z}$-module, which is not necessary finitely generated, and $\langle -, - \rangle : H \times H \to \mathbb{Z}$, $(x, y) \mapsto \langle x, y \rangle$, an alternating $\mathbb{Z}$-bilinear form. For example, we consider that $H$ is the first homology group of a surface and $\langle -, - \rangle$ is its intersection form. We define $\mathbb{Z}$-linear map $\mu : H \to \text{Hom}_\mathbb{Z}(H, \mathbb{Z})$ by $\mu(x)(y) = \langle x, y \rangle$ for $x, y \in H$. Denote by $\mathbb{Q}[H]$ the $\mathbb{Q}$-vector space with basis the set $H$:

$$\mathbb{Q}[H] = \left\{ \sum_{i=1}^{m} c_i [x_i] | n \in \mathbb{N}, c_i \in \mathbb{Q}, x_i \in H \right\},$$

where $[-] : H \to \mathbb{Q}[H]$ is the embedding as basis. We remark $c[x] \neq [cx] \in \mathbb{Q}[H]$ for $c \neq 1$. We define $\mathbb{Q}$-linear map $\langle -, - \rangle : \mathbb{Q}[H] \times \mathbb{Q}[H] \to \mathbb{Q}[H]$ by
$[[x], [y]] = \langle x, y \rangle [x + y]$ for $x, y \in H$. It is easy to see that this bilinear map is skew and satisfies the Jacobi identity [2] p.295-p.297. The Lie algebra $(Q[H], [-, -])$ is called the homological Goldman Lie algebra of $(H, \langle -, - \rangle)$.

Our purpose is to study the algebraic structure of the homological Goldman Lie algebra. In the previous paper [7], we determined all the ideals of the homological Goldman Lie algebra. In particular, the derived Lie subalgebra $Q[H]^{(1)} = [Q[H], Q[H]]$ of $Q[H]$ is the $Q$-vector subspace $Q[H \setminus \ker \mu]$ with the basis the set $H \setminus \ker \mu$, the center $\mathfrak{z}(Q[H])$ of $Q[H]$ is the $Q$-vector subspace $Q[\ker \mu]$ with the basis the set $\ker \mu$, and the abelianization $Q[H]^{\text{ab}} = Q[H]/Q[H]^{(1)}$ of $Q[H]$ equals the center. In other words, the first homology group $H_1(Q[H])$ of $Q[H]$ equals the center. In the present paper, we determine the second homology group [3] of the homological Goldman Lie algebra. That is, our main theorem in this paper is the following.

**Theorem 1.** If $\langle -, - \rangle \neq 0$, we have isomorphisms

$$H_2(Q[H]^{(1)}) \rightarrow \bigoplus_{z \in \ker \mu} Q \otimes (H/\mathbb{Z}z) \quad \text{and}$$

$$H_2(Q[H]) \rightarrow (\wedge^2 Q[\ker \mu]) \oplus H_2(Q[H]^{(1)}).$$

Goldman introduced a more geometric Lie algebra as follows. Let $\Sigma$ be a surface. Denote by $[S^1, \Sigma]$ the set of free homotopy classes of free loops on $\Sigma$. For two free loops $\alpha$ and $\beta$ on $\Sigma$ in general position, we define $[\alpha, \beta] = \sum_{p \in \alpha \cap \beta} \varepsilon(p; \alpha, \beta)[\alpha \cdot_p \beta]$, where $\varepsilon(p; \alpha, \beta)$ is the local intersection number of $\alpha$ and $\beta$ at $p$, and $\alpha \cdot_p \beta$ is the free homotopy class of the product in the fundamental group $\pi_1(\Sigma, p)$ with base point $p$. The bracket induces a well-defined binary operation in $Q[[S^1, \Sigma]]$, the $Q$-vector space with basis the set of homotopy classes of free loops, and the operation is skew and satisfies the Jacobi identity [2]. The Lie algebra $(Q[[S^1, \Sigma]], [-, -])$ is called the Goldman Lie algebra of $\Sigma$. Goldman used the Lie algebra $Q[[S^1, \Sigma]]$ for study of the space of representations on the fundamental group $\pi_1(\Sigma)$ of the surface $\Sigma$.

The natural projection $[S^1, \Sigma] \rightarrow H_1(\Sigma; \mathbb{Z})$ induces the surjective Lie algebra homomorphism $Q[[S^1, \Sigma]] \rightarrow Q[H_1(\Sigma; \mathbb{Z})]$. Hence we can have some information of the Goldman Lie algebra through the homomorphism $Q[[S^1, \Sigma]] \rightarrow Q[H_1(\Sigma; \mathbb{Z})]$. In fact, as will be shown in Theorem [11] we have that if $\Sigma$ is connected, then the composition $H_2(Q[[S^1, \Sigma]]) \rightarrow H_2(Q[H_1(\Sigma; \mathbb{Z})]) \rightarrow H_2(Q[H_1(\Sigma; \mathbb{Z})]^{(1)})$ is surjective, where the map $H_2(Q[H_1(\Sigma; \mathbb{Z})]) \rightarrow H_2(Q[H_1(\Sigma; \mathbb{Z})]^{(1)})$
is the induced map by the projection \( \varpi^{(1)} : \mathbb{Q}[H_1(\Sigma; \mathbb{Z})] = \mathbb{Q}[\ker \mu] \oplus \mathbb{Q}[H_1(\Sigma; \mathbb{Z}) \setminus \ker \mu] \rightarrow \mathbb{Q}[H_1(\Sigma; \mathbb{Z}) \setminus \ker \mu] \).

Define the \( \mathbb{Q} \)-linear map \( K : \mathbb{Q}[H] \rightarrow \mathbb{Q} \otimes_{\mathbb{Z}} H \) by \( K([x]) = 1 \otimes x \) for \( x \in H \). The \( \mathbb{Q} \)-vector subspace \( \mathfrak{g}_K := \ker K \) is a Lie subalgebra of \( \mathbb{Q}[H] \). In Theorem 10, we prove that if \( \langle -, - \rangle \neq 0 \), then the composition \( H_2(\mathfrak{g}_K) \rightarrow H_2(\mathbb{Q}[H]) \rightarrow H_2(\mathbb{Q}[H]^{(1)}) \) is surjective. The Lie subalgebra \( \mathfrak{g}_K \) is related to the Kontsevich’s “commutative” \([5]\) as follows. Assume that \( H \) has a symplectic basis \( \{x_i, y_i\}_{i=1}^g \). Then \( \mathbb{Q} \otimes H \) has a symplectic basis \( \{1 \otimes x_i, 1 \otimes y_i\}_{i=1}^g \). The symmetric algebra \( \prod_{m=0}^\infty \text{Sym}^m(\mathbb{Q} \otimes H) \) equips a Poisson bracket defined by

\[
[f, h] = \sum_{i=1}^g \left( \frac{\partial f}{\partial (1 \otimes x_i)} \frac{\partial h}{\partial (1 \otimes y_i)} - \frac{\partial h}{\partial (1 \otimes x_i)} \frac{\partial f}{\partial (1 \otimes y_i)} \right),
\]

where \( f, h \in \prod_{m=0}^\infty \text{Sym}^m(\mathbb{Q} \otimes H) \) is regarded as formal power series in variables \( \{1 \otimes x_i, 1 \otimes y_i\}_{i=1}^g \). The constant terms \( \mathbb{Q} \subset \prod_{m=0}^\infty \text{Sym}^m(\mathbb{Q} \otimes H) \) is included in the center of \( \prod_{m=0}^\infty \text{Sym}^m(\mathbb{Q} \otimes H) \). Hence \( \prod_{m=0}^\infty \text{Sym}^m(\mathbb{Q} \otimes H)/\mathbb{Q} \) has a natural Lie bracket. The composite of the inclusion and the quotient projection \( \prod_{m=1}^\infty \text{Sym}^m(\mathbb{Q} \otimes H) \rightarrow \prod_{m=0}^\infty \text{Sym}^m(\mathbb{Q} \otimes H)/\mathbb{Q} \) is a \( \mathbb{Q} \)-linear isomorphism. Then \( \prod_{m=1}^\infty \text{Sym}^m(\mathbb{Q} \otimes H) \) also has a Lie bracket. The \( \mathbb{Q} \)-linear map \( \mathbb{Q}[H] \rightarrow \prod_{m=0}^\infty \text{Sym}^m(\mathbb{Q} \otimes H) \), \( [x] \mapsto \exp(1 \otimes x) \), is a Lie algebra homomorphism. By composition, we have a Lie algebra homomorphism \( \mathbb{Q}[H] \rightarrow \prod_{m=1}^\infty \text{Sym}^m(\mathbb{Q} \otimes H) \). We denote \( \mathfrak{c}_g := \prod_{m=2}^\infty \text{Sym}^m(\mathbb{Q} \otimes H) \), which is a Lie subalgebra of \( \prod_{m=1}^\infty \text{Sym}^m(\mathbb{Q} \otimes H) \) and the degree completion of Kontsevich’s ”commutative”. The Lie subalgebra \( \mathfrak{g}_K \) is exactly the inverse image of \( \mathfrak{c}_g \) by the Lie algebra homomorphism stated above. The Lie algebra \( \mathfrak{c}_g \) equals the Lie algebra of formal Hamiltonian vector fields \( \text{ham}_{2g} \), which has been studied in the context of symplectic foliation \([11, 13, 16]\). It would be interesting if we could describe the relation between the second cohomology group of \( \text{ham}_{2g} \) and that of our \( \mathfrak{g}_K \).

In Theorem 13, we construct an explicit nontrivial cohomology class in \( H^3(\mathbb{Q}[H]) \) if \( \langle -, - \rangle \neq 0 \). In particular, we have \( H^3(\mathbb{Q}[H]) \neq 0 \).
2 Preparation

2.1 Derived algebra of some subalgebra

Let $S$ be a subset of $H$. Then we define $S^{(1)} := \{u + v \in H \mid u, v \in S, \langle u, v \rangle \neq 0\}$. For example, we have

$$H^{(1)} = H \setminus \ker \mu.$$ 

In fact, assume $x \in H^{(1)}$. Then there exist $u$ and $v \in H$ with $\langle u, v \rangle \neq 0$ and $x = u + v$. Since $\langle x, x \rangle = \langle u, v \rangle \neq 0$, we have $x \in H \setminus \ker \mu$. Conversely, assume $x \in H \setminus \ker \mu$. Then there exists $y \in H$ with $\langle x, y \rangle \neq 0$. Since $\langle x - y, y \rangle = \langle x, y \rangle \neq 0$ and $x = (x - y) + y$, we have $x \in H^{(1)}$.

**Proposition 2.** $\mathbb{Q}[S]$ is a subalgebra of $\mathbb{Q}[H]$ if and only if $S^{(1)} \subset S$. Then, $(\mathbb{Q}[S])^{(1)} = [\mathbb{Q}[S], \mathbb{Q}[S]] = \mathbb{Q}[S^{(1)}]$.

**Proof.** Assume $\mathbb{Q}[S]$ is a subalgebra of $\mathbb{Q}[H]$. For $x \in S^{(1)}$, there exist $u$ and $v \in S$ with $x = u + v$ and $\langle u, v \rangle \neq 0$. We have $[x] = \frac{1}{\langle u, v \rangle}[[u], [v]] \in [\mathbb{Q}[S], \mathbb{Q}[S]] \subset \mathbb{Q}[S]$. Hence, we have $x \in S$. This proves $S^{(1)} \subset S$ and $\mathbb{Q}[S^{(1)}] \subset [\mathbb{Q}[S], \mathbb{Q}[S]]$.

Assume $S^{(1)} \subset S$. It is trivial $\mathbb{Q}[S]$ is a subspace of $\mathbb{Q}[H]$ as $\mathbb{Q}$-vector space. For $u$ and $v \in S$, we have

$$[[u], [v]] = \begin{cases} 0, & \text{if } \langle u, v \rangle = 0, \\ \langle u, v \rangle [u + v], & \text{if } \langle u, v \rangle \neq 0. \end{cases}$$

In both cases we have $[[u], [v]] \in \mathbb{Q}[S^{(1)}] \subset \mathbb{Q}[S]$. Hence $\mathbb{Q}[S]$ is a subalgebra of $\mathbb{Q}[H]$. And this shows $[\mathbb{Q}[S], \mathbb{Q}[S]] \subset \mathbb{Q}[S^{(1)}]$. This completes the proof of the proposition. \qed

The center $\mathfrak{z}(\mathbb{Q}[H])$ of $\mathbb{Q}[H]$ is $\mathbb{Q}[\ker \mu]$. In fact, assume $u \in \ker \mu$. Then we have $[u] \in \mathfrak{z}(\mathbb{Q}[H])$, since $[[u], [v]] = \langle u, v \rangle [u + v] = 0$ for $v \in H$. Conversely, assume $u \in H \setminus \ker \mu = H^{(1)}$. Then there exists $v \in H$ with $\langle u, v \rangle \neq 0$. Then we have $[u] \in \mathbb{Q}[H] \setminus \mathfrak{z}(\mathbb{Q}[H])$ since $[[u], [v]] = \langle u, v \rangle [u + v] \neq 0$.

The inclusion $\iota^{(1)} : \mathbb{Q}[H^{(1)}] \to \mathbb{Q}[H]$ induces the inclusion homomorphism $C_p(\iota^{(1)}) : C_p(\mathbb{Q}[H^{(1)}]) \to C_p(\mathbb{Q}[H])$ and the restriction $C_p(\iota^{(1)}) : C_p(\mathbb{Q}[H]) \to C_p(\mathbb{Q}[H^{(1)}])$. On the other hand, we can decompose $\mathbb{Q}[H]$ into the center and the derived subalgebra, $\mathbb{Q}[H] = \mathbb{Q}[\ker \mu] \oplus \mathbb{Q}[H^{(1)}]$. The projection $\overline{\iota}^{(1)} : \mathbb{Q}[H] \to \mathbb{Q}[H^{(1)}]$ of the decomposition induces the projection $C_p(\overline{\iota}^{(1)}) : C_p(\mathbb{Q}[H]) \to C_p(\mathbb{Q}[H^{(1)}])$. 

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\[ C_p(\mathbb{Q}[H]) \to C_p(\mathbb{Q}[H^{(1)}]) \text{ and the zero extension } C^p(\mathbb{Q}^{(1)}: C^p(\mathbb{Q}[H^{(1)}]) \to C^p(\mathbb{Q}[H]). \] The chain maps \( C_*(\ell^{(1)}) \) and \( C_*(\mathbb{Q}^{(1)}) \) satisfy \( C_*(\mathbb{Q}^{(1)}) \circ C_*(\ell^{(1)}) = id \) since \( \mathbb{Q}^{(1)} \circ \ell^{(1)} = id \). By the same reason, the cochain maps \( C^*(\ell^{(1)}) \) and \( C^*(\mathbb{Q}^{(1)}) \) satisfy \( C^*(\mathbb{Q}^{(1)}) \circ C^*(\ell^{(1)}) = id \).

### 2.2 Decomposition of (co)homology

Let \( S \) be a subset of \( H \) with \( S^{(1)} \subset S \). For \( p > 0 \) and \( z \in H \), denote by \( C^p(\mathbb{Q}[S])_{(z)} \) a subspace of \( C^p(\mathbb{Q}[S]) \) generated by the set \( \{[u_1] \land \cdots \land [u_p] \mid u_1, \ldots, u_p \in S, u_1 + \cdots + u_p = z \} \). We define a subspace \( C^p(\mathbb{Q}[S])_{(z)} \) of \( C^p(\mathbb{Q}[S]) \) by

\[
C^p(\mathbb{Q}[S])_{(z)} := \left\{ \omega \in C^p(\mathbb{Q}[S]) \mid u_1, \ldots, u_p \in S, u_1 + \cdots + u_p \neq z \quad \Rightarrow \quad \omega([u_1], \ldots, [u_p]) = 0 \right\}.
\]

The subspace \( C_*(\mathbb{Q}[S])_{(z)} \) is a subcomplex of the chain complex \( C_*(\mathbb{Q}[S]) \) and the subspace \( C^*(\mathbb{Q}[S])_{(z)} \) is a subcomplex of the cochain complex \( C^*(\mathbb{Q}[S]) \), that is, we have \( \partial(C_p(\mathbb{Q}[S])_{(z)}) \subset C_{p-1}(\mathbb{Q}[S])_{(z)} \) and \( d(C^p(\mathbb{Q}[S])_{(z)}) \subset C^{p+1}(\mathbb{Q}[S])_{(z)} \) for all \( p > 0 \) because \( u_1 + \cdots + u_p \) define the degree of \( \mathbb{Q}[S] \). We denote \( Z_p(\mathbb{Q}[S])_{(z)} := Z_p(C_*(\mathbb{Q}[S])_{(z)}), B_p(\mathbb{Q}[S])_{(z)} := B_p(C_*(\mathbb{Q}[S])_{(z)}), H_p(\mathbb{Q}[S])_{(z)} := H_p(\mathbb{Q}[S])_{(z)}, Z^p(\mathbb{Q}[S])_{(z)} := Z^p(C^*(\mathbb{Q}[S])_{(z)}), B^p(\mathbb{Q}[S])_{(z)} := B^p(C^*(\mathbb{Q}[S])_{(z)}) \) and \( H^p(\mathbb{Q}[S])_{(z)} := H^p(C^*(\mathbb{Q}[S])_{(z)}) \). Then we have \( Z_p(\mathbb{Q}[S]) \cong \prod_{z \in H} Z_p(\mathbb{Q}[S])_{(z)}, B_p(\mathbb{Q}[S]) \cong \prod_{z \in H} B_p(\mathbb{Q}[S])_{(z)}, H_p(\mathbb{Q}[S]) \cong \prod_{z \in H} H_p(\mathbb{Q}[S])_{(z)}, Z^p(\mathbb{Q}[S]) \cong \prod_{z \in H} Z^p(\mathbb{Q}[S])_{(z)}, B^p(\mathbb{Q}[S]) \cong \prod_{z \in H} B^p(\mathbb{Q}[S])_{(z)} \) and \( H^p(\mathbb{Q}[S]) \cong \prod_{z \in H} H^p(\mathbb{Q}[S])_{(z)} \).

We call \( H_p(\mathbb{Q}[S])_{(z)} \) or \( H^p(\mathbb{Q}[S])_{(z)} \) an inner component if \( z \in \ker \mu \), and an outer component if \( z \in H^{(1)} \).

### 3 Inner component

Fix an element \( z \in \ker \mu \) in this section. Denote by \( \hat{I} \) the \( \mathbb{Q} \)-algebra ideal in \( \land \mathbb{Q}[H] \) generated by the set \( \{(u+v)(x)-[u] \land [x+v]-[v] \land [x+u] \mid u, v, x \in H\} \). The generator system of \( \hat{I} \) consists of homogeneous elements about degree \( p \in \{0, 1, 2, \ldots\} \). Then we have \( \hat{I} = \bigoplus_{p=0}^{\infty} \hat{I} \cap C_p(\mathbb{Q}[H]) \). The generator system of \( \hat{I} \) consists of homogeneous elements about degree \( x \in H \). Then we have \( \hat{I} \cap C_p(\mathbb{Q}[H]) = \bigoplus_{x \in H} \hat{I} \cap C_p(\mathbb{Q}[H])_{(x)} \). Set \( \hat{I}_{p,(x)} = \hat{I} \cap C_p(\mathbb{Q}[H])_{(x)} \) and \( \hat{C}_p(\mathbb{Q}[H])_{(x)} = C_p(\mathbb{Q}[H])_{(x)} / \hat{I}_{p,(x)} \).

Dually, we define the subspace \( \hat{C}^p(\mathbb{Q}[H])_{(z)} \) of \( C^p(\mathbb{Q}[H])_{(z)} \) by the following condition: \( \omega \in \hat{C}^p(\mathbb{Q}[H])_{(z)} \) if and only if the map \( H^p = H \times \cdots \times H \ni
Proposition 3. The composition (the quotient projection) $\partial_p : C_p(\mathbb{Q}[H])_{(z)} \to C_{p-1}(\mathbb{Q}[H])_{(z)} \to \hat{C}_{p-1}(\mathbb{Q}[H])_{(z)}$ is a zero map.

Proof. For $u_1, \ldots, u_p \in H$ with $u_1 + \cdots + u_p = z$, we have

\[
\begin{align*}
\partial_p([u_1] \land \cdots \land [u_p]) &= \sum_{1 \leq i < j \leq p} (-1)^{i+j} \langle u_i, u_j \rangle [u_i + u_j] \land [u_1] \land \cdots \land [u_p] \\
&+ \sum_{k=1}^{p-1} (-1)^{i+p} \langle u_k, u_p \rangle [u_k + u_p] \land [u_1] \land \cdots \land [u_p] \\
&= \sum_{1 \leq i < j \leq p} (-1)^{i-1} \langle u_i, u_j \rangle [u_1] \land \cdots \land [u_j] \land \cdots \land [u_{p-1}] \land [u_p] \\
&+ \sum_{1 \leq i < j \leq p} (-1)^{i} [u_i] \land \cdots \land [u_j] \land \cdots \land [u_{p-1}] \land [u_p] \\
&+ \sum_{1 \leq i < j \leq p} (-1)^{k-1} \langle u_k, u_j \rangle [u_1] \land \cdots \land [u_k] \land \cdots \land [u_{p-1}] \land [u_p] \\
&+ \sum_{1 \leq i < j \leq p} (-1)^{k} \langle u_i, u_k \rangle [u_1] \land \cdots \land [u_k] \land \cdots \land [u_{p-1}] \land [u_p] \\
&= 0 \mod \tilde{I}_{p-1}. 
\end{align*}
\]

This completes the proof of the proposition. \(\square\)

By Proposition 3 we can define a $\mathbb{Q}$-linear map $H_p(\mathbb{Q}[H])_{(z)} \to \hat{C}_p(\mathbb{Q}[H])_{(z)}$ and $\hat{C}^p(\mathbb{Q}[H])_{(z)} \to H^p(\mathbb{Q}[H])_{(z)}$.

Proposition 4. There is a natural $\mathbb{Q}$-linear isomorphism

\[
\hat{C}_p(\mathbb{Q}[H])_{(z)} \cong \mathbb{Q} \otimes_{\mathbb{Z}} \left( \wedge^{p-1}_{\mathbb{Z}} H/\mathbb{Z}z \right).
\]
Proof. Define a map \( f : \{(u_1, \ldots, u_p) \mid u_i \in H, u_1 + \cdots + u_p = z \} \to \mathbb{Q} \otimes (\wedge^{p-1} H/\mathbb{Z}z) \) by \( f(u_1, \ldots, u_p) = 1 \otimes (u_1 \wedge \cdots \wedge u_{p-1}) \). We can consider \( \mathbb{Q}\)-linear extension \( f([u_1], \ldots, [u_p]) = f(u_1, \ldots, u_p) \). We denote by \( S_p \) the symmetric group of degree \( p \). For \( \sigma \in S_p \) with \( \sigma(p) = p \), we have \( f(u_{\sigma(1)}, \ldots, u_{\sigma(p)}) = (\text{sgn}\sigma)f(u_1, \ldots, u_p) \) easily. For \( u_1, \ldots, u_p \in H \) with \( u_1 + \cdots + u_p = z \), we have

\[
f(u_p, u_2, \ldots, u_{p-1}, u_1) = 1 \otimes (u_p \wedge u_2 \wedge \cdots \wedge u_{p-1})
\]

\[
= 1 \otimes (z \wedge u_2 \wedge \cdots \wedge u_{p-1}) - \sum_{i=1}^{p-1} 1 \otimes (u_i \wedge \cdots \wedge u_{p-1})
\]

\[
= -1 \otimes (u_1 \wedge \cdots \wedge u_{p-1})
\]

\[
= -f(u_1, \ldots, u_p).
\]

Hence we have \( f(u_{\sigma(1)}, \ldots, u_{\sigma(p)}) = (\text{sgn}\sigma)f(u_1, \ldots, u_p) \) for \( \sigma \in S_p \). This induces the \( \mathbb{Q}\)-linear map \( f : C_p(\mathbb{Q}[H]_{(z)}) \to \mathbb{Q} \otimes (\wedge^{p-1} H/\mathbb{Z}z) \) with \( f([u_1] \wedge \cdots \wedge [u_p]) = 1 \otimes (u_1 \wedge \cdots \wedge u_{p-1}) \). For \( u_1, u_2, \ldots, u_p \in H \) with \( u_1 + v_1 + u_2 + \cdots + u_p = z \), we have

\[
f(u_1 + v_1, u_2, \ldots, u_p)
\]

\[
= 1 \otimes ((u_1 + v_1) \wedge u_2 \wedge \cdots \wedge u_{p-1})
\]

\[
= 1 \otimes (u_1 \wedge u_2 \wedge \cdots \wedge u_{p-1}) + 1 \otimes (v_1 \wedge u_2 \wedge \cdots \wedge u_{p-1})
\]

\[
= f(u_1, u_2, \ldots, u_{p-1}, u_p + v_1) + f(v_1, u_2, \ldots, u_{p-1}, u_p + v_1).
\]

This mean \( f(I_{p,(z)}) = 0 \) and \( f \) induce \( f : \hat{C}_p(\mathbb{Q}[H]_{(z)}) \to \mathbb{Q} \otimes (\wedge^{p-1} H/\mathbb{Z}z) \) with \( f([u_1] \wedge \cdots \wedge [u_p]) = 1 \otimes (u_1 \wedge \cdots \wedge u_{p-1}) \).

We construct the inverse map. Define the map \( g : \mathbb{Q} \times H \times \cdots \times H \to \hat{C}_p(\mathbb{Q}[H]_{(z)}) \) by \( g(c, u_1, \ldots, u_{p-1}) = c[u_1] \wedge \cdots \wedge [u_{p-1}] \wedge [z - u_1 - \cdots - u_{p-1}] \). It is trivial that \( g \) satisfies \( g(a + b, u_1, \ldots, u_{p-1}) = g(a, u_1, \ldots, u_{p-1}) + g(b, u_1, \ldots, u_{p-1}) \) and \( g(c, u_{\sigma(1)}, \ldots, u_{\sigma(p-1)}) = (\text{sgn}\sigma)g(c, u_1, \ldots, u_{p-1}) \) for \( \sigma \in S_{p-1} \). Moreover the map \( g \) is also \( \mathbb{Z}\)-linear. In fact, we have

\[
g(c, u_1 + v_1, u_2, \ldots, u_{p-1})
\]

\[
= c[u_1 + v_1] \wedge \cdots \wedge [u_{p-1}] \wedge [z - u_1 - v_1 - u_2 - \cdots - u_{p-1}]
\]

\[
\equiv c[u_1] \wedge \cdots \wedge [u_{p-1}] \wedge [z - u_1 - \cdots - u_{p-1}]
\]

\[
+ c[v_1] \wedge \cdots \wedge [u_{p-1}] \wedge [z - v_1 - u_2 - \cdots - u_{p-1}]
\]

\[
= g(c, u_1, \ldots, u_{p-1}) + g(c, v_1, u_2, \ldots, u_{p-1}).
\]
Moreover we have
\[
g(c, z, u_2, \ldots, u_{p-1}) = c[z] \wedge [u_2] \wedge \cdots \wedge [u_{p-1}] \wedge [z - z - u_2 - \cdots - u_{p-1}]
\]
\[
= - \sum_{i=2}^{p-1} c[z + u_i] \wedge [u_2] \wedge \cdots \wedge [u_{p-1}] \wedge [u_i]
\]
\[
= 0.
\]
Hence the map \(g\) induces the \(\mathbb{Q}\)-linear \(g : \mathbb{Q} \otimes (\wedge^{p-1}H/\mathbb{Z}z) \to \hat{C}_p(\mathbb{Q}[H])_{(z)}\)
with \(g(c \otimes (u_1 \wedge \cdots \wedge u_{p-1})) = c[u_1] \wedge \cdots \wedge [u_{p-1}] \wedge [z - u_1 - \cdots - u_{p-1}]\).

Clearly \(g\) is the inverse of \(f\). \(\square\)

Dually, we have a natural isomorphism \(C^p(\mathbb{Q}[H])_{(z)} \leftarrow \text{Hom}_\mathbb{Z}(\wedge^2H/\mathbb{Z}z, \mathbb{Q})\).

For the rest of this subsection, we confine ourselves to the only second homology group. We have \(C_2(\mathbb{Q}[H^{(1)}])_{(z)} = Z_2(\mathbb{Q}[H^{(1)}])_{(z)}\) and \(C_2(\mathbb{Q}[H])_{(z)} = Z_2(\mathbb{Q}[H])_{(z)}\). In fact, \(\partial_2([u] \wedge [z - u]) = -\langle u, z - u \rangle [z] = 0\) for \(u \in H^{(1)}\) or \(u \in H\). Set the inclusion \(i^{(1)} : \mathbb{Q}[H^{(1)}] \rightarrow \mathbb{Q}[H]\).

**Proposition 5.** The kernel \(C_2(i^{(1)})^{-1}(I_{2,(z)})\) of the composition \(Z_2(\mathbb{Q}[H^{(1)}])_{(z)} \rightarrow Z_2(\mathbb{Q}[H])_{(z)} \rightarrow \hat{C}_2(\mathbb{Q}[H])_{(z)}\) equals \(B_2(\mathbb{Q}[H^{(1)}])_{(z)}\).

**Proof.** For \(u, v \in H^{(1)}\) with \(u + v \in H^{(1)}\), we have
\[
C_2(i^{(1)}) \circ \partial_3([u] \wedge [v] \wedge [z - u - v]) = -\langle u, v \rangle ([u + v] \wedge [z - u - v] - [u] \wedge [z - u] - [v] \wedge [z - v])
\]
\[
\equiv 0.
\]
Hence we obtain \(C_2(i^{(1)})^{-1}(I_{2,(z)}) \supset B_2(\mathbb{Q}[H^{(1)}])_{(z)}\). By similar calculation, we have \(I_{2,(z)} \supset B_2(\mathbb{Q}[H])_{(z)}\).

\(C_2(i^{(1)})^{-1}(I_{2,(z)})\) is generated by the set \([u + v] \wedge [z - u - v] - [u] \wedge [z - u] - [v] \wedge [z - v] \mid u, v, u + v \in H^{(1)}\). If \(\langle u, v \rangle \neq 0\), then we have \(B_2(\mathbb{Q}[H^{(1)}])_{(z)} \ni \partial_3([u/v][u] \wedge [v] \wedge [z - u - v]) = [u + v] \wedge [z - u - v] - [u] \wedge [z - u] - [v] \wedge [z - v]\). If \(u, v, u + v \in H^{(1)}\) and \(\langle u, v \rangle = 0\), we can take \(x \in H^{(1)}\) with \(\langle u, x \rangle \neq 0, \langle v, x \rangle \neq 0\) and \(\langle u + v, x \rangle \neq 0\). Then we obtain \([u + v] \wedge [z - u - v] - [u] \wedge [z - u] - [v] \wedge [z - v] \in B_2(\mathbb{Q}[H^{(1)}])_{(z)}\) since \(-[u + v + x] \wedge [z - u - v - x] + [u + v] \wedge [z - u - v] + [x] \wedge [z - x] \in B_2(\mathbb{Q}[H^{(1)}])_{(z)}\), \([u + v + x] \wedge [z - u - v - x] - [u] \wedge [z - u] + [x + v] \wedge [z - x - v] \in B_2(\mathbb{Q}[H^{(1)}])_{(z)}\), and \([v + x] \wedge [z - v - x] - [v] \wedge [z - v] + [x] \wedge [z - x] \in B_2(\mathbb{Q}[H^{(1)}])_{(z)}\). Hence we have \(C_2(i^{(1)})^{-1}(I_{2,(z)}) \subset B_2(\mathbb{Q}[H^{(1)}])_{(z)}\). This completes the proof of the proposition. \(\square\)
By Proposition 4, the composition $H_2(\mathbb{Q}[H^{(1)}])_{(z)} \to H_2(\mathbb{Q}[H])_{(z)} \to \hat{C}_2(\mathbb{Q}[H])_{(z)} \to \mathbb{Q} \otimes H/\mathbb{Z}$ is injective.

**Proposition 6.** The composition $Z_2(\mathbb{Q}[H^{(1)}])_{(z)} \to Z_2(\mathbb{Q}[H])_{(z)} \to \hat{C}_1(\mathbb{Q}[H])_{(z)} \to \mathbb{Q} \otimes H/\mathbb{Z}$ is surjective if $\langle - , - \rangle \neq 0$.

**Proof.** We have $H^{(1)} \neq \emptyset$ since $\langle - , - \rangle \neq 0$. Take $x_0 \in H^{(1)}$. The $\mathbb{Q}$-vector space $\mathbb{Q} \otimes H/\mathbb{Z}$ is generated by the set $\{1 \otimes u \mid u \in H\}$. The set $\{1 \otimes u \mid u \in H^{(1)}\}$ also generates $\mathbb{Q} \otimes H/\mathbb{Z}$ since we have $1 \otimes u = 1 \otimes (u - x_0) + 1 \otimes x_0$. For $u \in H^{(1)}$, $[u] \wedge [z - u] \in Z^2(\mathbb{Q}[H^{(1)}])_{(z)}$ corresponds to $1 \otimes u$ by the composition. □

By Propositions 4, 5 and 6 we obtain a natural isomorphism $H_2(\mathbb{Q}[H^{(1)}])_{(z)} \to \mathbb{Q} \otimes (H/\mathbb{Z})$ if $H^{(1)} \neq \emptyset$. We have an isomorphism $H_2(\mathbb{Q}[H])_{(z)} \cong H_2(\mathbb{Q}[\ker \mu])_{(z)} \oplus H_2(\mathbb{Q}[H^{(1)}])_{(z)}$ by the decomposition $\mathbb{Q}[H] = \mathbb{Q}[\ker \mu] \oplus \mathbb{Q}[H^{(1)}]$.

## 4 Outer component

Fix an element $z \in H^{(1)}$ in this section. The inclusion $\iota^{(1)} : \mathbb{Q}[H^{(1)}] \to \mathbb{Q}[H]$ is a section of the projection $\varpi^{(1)} : \mathbb{Q}[H] = \mathbb{Q}[\ker \mu] \oplus \mathbb{Q}[H^{(1)}] \to \mathbb{Q}[H^{(1)}]$. Then the homomorphism $H_p(\varpi^{(1)})_{(z)} : H_p(\mathbb{Q}[H])_{(z)} \to H_p(\mathbb{Q}[H^{(1)})]_{(z)}$ is surjective. So we have $H_p(\mathbb{Q}[H^{(1)}])_{(z)} = 0$ if $H_p(\mathbb{Q}[H])_{(z)} = 0$.

**Proposition 7.** We have $H_2(\mathbb{Q}[H])_{(z)} = 0$.

**Proof.** We can take $y \in H$ with $\langle y , z \rangle \neq 0$. Define $\Phi_p : C_p(\mathbb{Q}[H])_{(z)} \to C_{p+1}(\mathbb{Q}[H])_{(z)}$, $p = 1, 2$, by $\Phi_1([z]) = \frac{1}{\langle y , z \rangle}[y] \wedge [z - y]$ and

$$\Phi_2([u] \wedge [v]) = \frac{-1}{\langle y , z \rangle}([y] \wedge [u - y] \wedge [v] + [y] \wedge [u] \wedge [v - y]) + \frac{1}{2 \langle y , z \rangle}([2y] \wedge [u - y] \wedge [v - y]) + \frac{\langle u - y , v - y \rangle}{2 \langle y , z \rangle^2}[y] \wedge [2y] \wedge [z - 3y].$$

We can check $\Phi_1 \circ \partial_2 + \partial_3 \circ \Phi_2 = id_{C_2(\mathbb{Q}[H])_{(z)}}$. This shows the proposition. □

Combine this proposition and the results of the section 3 we obtain the main theorem. If we consider the case when $\langle - , - \rangle$ is non-degenerate, we have the following corollary.
Corollary 8. If $\langle - , - \rangle$ is non-degenerate, we have an isomorphism
\[
H_2(\mathbb{Q}[H]) \to H_2(\mathbb{Q}[H^{(1)}]) \to \mathbb{Q} \otimes H.
\]

We remark that this corollary holds also for the case $H = 0$, that is, $H^{(1)} = \emptyset$. By the universal coefficient theorem, we have another corollary.

Corollary 9. If $\langle - , - \rangle \neq 0$, we have
\[
H^2(\mathbb{Q}[H^{(1)}])(z) \cong \text{Hom}_{\mathbb{Z}}(H/\mathbb{Z}z, \mathbb{Q}), \text{ if } z \in \ker \mu,
\]
\[
H^2(\mathbb{Q}[H^{(1)}])(z) = 0, \text{ if } z \in H^{(1)},
\]
\[
H^2(\mathbb{Q}[H])(z) = H^2(\mathbb{Q}[\ker \mu])(z) \oplus H^2(\mathbb{Q}[H^{(1)}])(z),
\]
\[
H^2(\mathbb{Q}[H]) \cong \prod_{z \in \ker \mu} \text{Hom}_{\mathbb{Z}}(H/\mathbb{Z}z, \mathbb{Q}) \text{ and }
\]
\[
H^2(\mathbb{Q}[H]) = \text{Hom}_{\mathbb{Q}}(\wedge^2 \mathbb{Q}[\ker \mu], \mathbb{Q}) \oplus H^2(\mathbb{Q}[H^{(1)})].
\]

5 Applications

Let $g_K$ be the kernel of the $\mathbb{Q}$-linear map $K : \mathbb{Q}[H] \to \mathbb{Q} \otimes H$, $[x] \mapsto 1 \otimes x$. Then $g_K$ is a Lie subalgebra of $\mathbb{Q}[H]$.

Theorem 10. If $\langle - , - \rangle \neq 0$, then the composition $H_2(g_K) \to H_2(\mathbb{Q}[H]) \to H_2(\mathbb{Q}[H^{(1)}])$ is surjective, where the map $H_2(g_K) \to H_2(\mathbb{Q}[H])$ is the inclusion homomorphism and the map $H_2(\mathbb{Q}[H]) \to H_2(\mathbb{Q}[H^{(1)}])$ is the induced map by the projection $\pi^{(1)} : \mathbb{Q}[H] = \mathbb{Q}[\ker \mu] \oplus \mathbb{Q}[H^{(1)}] \to \mathbb{Q}[H^{(1)}]$.

Proof. By Theorem[1] $H_2(\mathbb{Q}[H^{(1)}])$ is generated by the set $\{[u] \wedge [z-u] \mid z \in \ker \mu, u \in H^{(1)}\}$ since $H^{(1)} \neq \emptyset$.

We consider the chain $([2u] - 2[u]) \wedge ([u - 2u] - 2[z - u] + [z]) \in C_2(g_K)$ for $z \in \ker \mu$ and $u \in H^{(1)}$. This is a cycle since $\langle u, u \rangle = 0$ and $\langle u, z \rangle = 0$. Applying the inclusion homomorphism to the cycle, then we have $([2u] - 2[u]) \wedge ([z - 2u] - 2[z - u] + [z]) \in Z_2(\mathbb{Q}[H])$. Applying the map $H_2(\pi^{(1)})$ to the cycle, then we have $([2u] - 2[u]) \wedge ([z - 2u] - 2[z - u]) \in Z_2(\mathbb{Q}[H^{(1)}])$. The homology class of the cycle equals the homology class of $6[u] \wedge [z - u]$. In fact, we have
\[
([2u] - 2[u]) \wedge ([z - 2u] - 2[z - u])
\]
\[
= [2u] \wedge [z - 2u] - 2[2u] \wedge [z - u] - 2[u] \wedge [z - 2u] + 4[u] \wedge [z - u].
\]
By Theorem 1, we have \([u] \wedge [v] = 0 \in H_2(\mathbb{Q}[H(1)])\) if \(u + v \in H(1)\) and \([2u] \wedge [z - 2u] = 2[u] \wedge [z - u] \in H_2(\mathbb{Q}[H(1)])\). This completes the proof of the proposition.

Let \(\Sigma\) be a compact surface. Let \(g\) be the genus of \(\Sigma\) and \(r\) the number of the cardinality of the set of the connected components of the boundary of \(\Sigma\). We consider the surjection from the Goldman Lie algebra of \(\Sigma\) onto the homological Goldman Lie algebra of the first homology group of \(\Sigma\) with the intersection form.

We recall the definition of the homomorphism. We identify \(H_1(\Sigma; \mathbb{Z}) = \pi_1(\Sigma, *)^{\text{abel}} = \pi_1(\Sigma, *) / [\pi_1(\Sigma, *), \pi_1(\Sigma, *)]\). Hence we have the abelianization map \(q_a : \pi_1(\Sigma, *) \rightarrow H_1(\Sigma, \mathbb{Z})\). This is a group homomorphism. We can identify \([S^1, \Sigma] = \pi_1(\Sigma, \mathbb{Z})/\text{conj.}\) since \(\Sigma\) is connected. Hence we have the quotient map \(q_c : \pi_1(\Sigma, \mathbb{Z}) \rightarrow [S^1, \Sigma]\). This map is given by forgetting the base point. This induces the map between the second homology groups of the Lie algebras. Take a projection \(q : [S^1, \Sigma] \rightarrow H_1(\Sigma; \mathbb{Z})\) with \(q \circ q_c = q_a\). This map is independent of the choice \(* \in \Sigma\). Set \(q = \mathbb{Q}[q] : \mathbb{Q}[[S^1, \Sigma]] \rightarrow \mathbb{Q}[H_1(\Sigma; \mathbb{Z})]\) the \(\mathbb{Q}\)-linear extension of \(q\). This is the desired Lie algebra homomorphism.

**Theorem 11.** The composition \(H_2(\varpi(1)) \circ H_2(q) : H_2(\mathbb{Q}[H_1(\Sigma; \mathbb{Z})]) \rightarrow H_2(\mathbb{Q}[H_1(\Sigma; \mathbb{Z})]) \rightarrow H_2(\mathbb{Q}[H_1(\Sigma; \mathbb{Z})^{(1)}])\) is surjective if \(\Sigma\) is connected.

**Proof.** If \(H^{(1)} = \emptyset\), then we have \(\mathbb{Q}[H_1(\Sigma; \mathbb{Z})^{(1)}] = 0\). Hence the proposition holds. Assume \(H^{(1)} \neq \emptyset\). Then we have \(g \geq 1\). Fix a base point \(* \in \Sigma\). Take based oriented loops \(\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g, \gamma_1, \ldots, \gamma_r\) as follows. \((g = 2, r = 3)\)

The set \(\{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g, \gamma_1, \ldots, \gamma_r\}\) is a generator system of the fundamental group \(\pi_1(\Sigma, *)\) of \(\Sigma\). They satisfy the relation

\[
\alpha_1 \beta_1 \alpha_1^{-1} \beta_1^{-1} \cdots \alpha_g \beta_g \alpha_g^{-1} \beta_g^{-1} \gamma_1 \cdots \gamma_r = 1
\]
which is a defining relation of the group \( \pi_1(\Sigma, \ast) \). Set \( q_\alpha(\alpha_i) = A_i, q_\alpha(\beta_i) = B_i \) and \( q_\alpha(\gamma_j) = C_j \). They satisfy \( \langle A_i, A_j \rangle = \langle B_1, B_j \rangle = \langle C_i, x \rangle = 0 \) and \( \langle A_i, B_j \rangle = \delta_{i,j} \). In particular, the kernel \( \ker \mu \) is generated by \( C_1, \ldots, C_r \in H \) with the defining relation \( C_1 + \cdots + C_r = 0 \). Theorem II says that the set \( \{[u] \wedge [z - u] \mid x \in \{A_1, B_1, \ldots, A_g, B_g, C_1, \ldots, C_r\}, z \in \ker \mu \} \) generates \( H_2(\mathbb{Q}[H^{(1)}]) \) as a \( \mathbb{Q} \)-vector space.

Let \( z \in \ker \mu \). Then there exist \( m_1, \ldots, m_r \in \mathbb{Z} \) with \( z = m_1C_1 + \cdots + m_rC_r \).

Assume \( u = A_i \). Set \( \alpha = \alpha_i \) and \( \beta = \gamma_i^{m_r} \cdots \gamma_1^{m_1} \alpha_i^{-1} \). We may assume \( q_\alpha(\alpha) \cap q_\beta(\beta) = \emptyset \), that is, there exist \( A \in q_\alpha(\alpha) \) and \( B \in q_\beta(\beta) \) with \( A(S^1) \cap B(S^1) = \emptyset \). This shows that \( [q_\alpha(\alpha)] \wedge [q_\beta(\beta)] \in C_2(\mathbb{Q}[S^1, \Sigma]) \) is a cycle by the definition of the bracket in \( \mathbb{Q}[S^1, \Sigma] \). We have

\[
C_2(\varpi^{(1)}) \circ C_2(q)([q_\alpha(\alpha)] \wedge [q_\beta(\beta)]) = C_2(\varpi^{(1)})([q \circ q_\alpha(\alpha)] \wedge [q \circ q_\beta(\beta)])
\]

Hence we have the homology class of \( [A_i] \wedge [z - A_i] \) is in the image of \( H_2(\varpi^{(1)}) \circ H_2(q) \).

Assume \( u = B_i \). If we set \( \alpha = \beta_i \) and \( \beta = \gamma_i^{m_r} \cdots \gamma_1^{m_1} \beta_i^{-1} \), then we have that the homology class of \( [B_i] \wedge [z - B_i] \) is included in the image of \( H_2(\varpi^{(1)}) \circ H_2(q) \) similarly.

Assume \( u = C_j \). We remark \( C_2(\varpi^{(1)}) \circ C_2(q)([q_\alpha(\gamma_j)] \wedge [q_\beta(\gamma_j^{m_r} \cdots \gamma_1^{m_1} \gamma_j^{-1})]) = 0 \) since \( C_j \in \ker \mu \). By Theorem II we have \( [C_j] \wedge [z - C_j] = [C_j - A_g] \wedge [z - C_j + A_g] + [A_g] \wedge [z - A_g] \in H_2(\mathbb{Q}[H_1(\Sigma; \mathbb{Z})^{(1)}]) \). Hence it is enough to show that \( [C_j - A_g] \wedge [z - C_j + A_g] \) is in the image. Set \( \alpha = \gamma_j^{r-1} \cdots \gamma_1 \alpha_i^{-1} \) and \( \beta = \alpha_i \gamma_1 \cdots \gamma_i \gamma_i^{m_r} \cdots \gamma_1^{m_1} \).

We may also assume \( q_\alpha(\alpha) \cap q_\beta(\beta) = \emptyset \). For example, see the following figure \((g = 2, r = 3, j = 2, m_1 - m_j = 2, m_3 - m_j = 1)\).
We have $q_\alpha(\alpha) = C_j - A_g$ and $q_\alpha(\beta) = z - C_j + A_g$ by $C_1 + \cdots + C_r = 0$. This completes the proof of the proposition. \hfill \Box

Essentially, we have the following lemma at the proof of the Proposition 5.

Lemma 12. Let $f : H^{(1)} \to \mathbb{Z}$ be a function satisfying $f(u + v) = f(u) + f(v)$ for $u, v \in H^{(1)}$ with $\langle u, v \rangle \neq 0$. Then we have $f(u + v) = f(u) + f(v)$ for $u, v \in H^{(1)}$ with $u + v \in H^{(1)}$, and $f(nu) = nf(u)$ for $u \in H^{(1)}$ and $n \in \mathbb{Z} \setminus \{0\}$.

Proof. Take $u, v \in H^{(1)}$ with $u + v \in H^{(1)}$. Then there exists $x \in H^{(1)}$ with $\langle u, x \rangle \neq 0$, $\langle v, x \rangle \neq 0$ and $\langle u + v, x \rangle \neq 0$. If $\langle u, v \rangle \neq 0$, we have $f(u + v) = f(u) + f(v)$ by the assumption of the lemma. Assume $\langle u, v \rangle = 0$. Then we have $\langle u, v + x \rangle = \langle u, x \rangle \neq 0$. By the assumption of the lemma, we have $f(u + v) = f(u) + f(v)$ because

$$f(u + v) = f(u + v + x) - f(x) = f(u) + f(v + x) - f(x) = f(u) + f(v).$$

Take $u \in H^{(1)}$ and $n \in \mathbb{Z} \setminus \{0\}$. Then there exists $x \in H^{(1)}$ with $\langle u, x \rangle \neq 0$. By the result of the first half of the lemma, it is enough to show the case $n = -1$. We obtain $f(-u) = -f(u)$ because

$$f(-u) = -f(u + x) + f(x) = -f(u) - f(x) - f(x) = -f(u).$$

This completes the proof of the lemma. \hfill \Box

Assume $z \in \ker \mu$. Then we have $\langle -, - \rangle \in \text{Hom}_\mathbb{Z}(\wedge^2 H/\mathbb{Z}, \mathbb{Q})$. This corresponds the cocycle $\omega \in Z^3(\mathbb{Q}[H^{(1)}])_{(z)}$ or $\omega \in Z^3(\mathbb{Q}[H])_{(z)}$ with $\omega([u, v], [z - u - v]) = \langle u, v \rangle$ for $u, v \in H^{(1)}$ by Proposition 4. The projection $\varpi^{(1)} : \mathbb{Q}[H] = \mathbb{Q}[\ker \mu] \oplus \mathbb{Q}[H^{(1)}] \to \mathbb{Q}[H^{(1)}]$ induces injection $H^3(\varpi^{(1)})_{(z)} : H^3(\mathbb{Q}[H^{(1)}])_{(z)} \to H^3(\mathbb{Q}[H])_{(z)}$ since $\varpi^{(1)} \circ \iota^{(1)} = \text{id}_{\mathbb{Q}[H^{(1)}]}$, where the map $\iota^{(1)} : \mathbb{Q}[H^{(1)}] \to \mathbb{Q}[H]$ is the inclusion. Therefore $[\omega] = 0 \in H^3(\mathbb{Q}[H])_{(z)}$ if $[\omega] = 0 \in H^3(\mathbb{Q}[H^{(1)}])_{(z)}$, and $[\omega] \neq 0 \in H^3(\mathbb{Q}[H^{(1)}])_{(z)}$ if $[\omega] \neq 0 \in H^3(\mathbb{Q}[H])_{(z)}$.

Theorem 13. Assume $\langle -, - \rangle \neq 0$. If $z \in \ker \mu$ is a torsion element, i.e., there exists $n \in \mathbb{Z} \setminus \{0\}$ with $nz = 0$, then we have $[\omega] \neq 0 \in H^3(\mathbb{Q}[H])_{(z)}$. If $z \in \ker \mu$ is not a torsion element, i.e., $nz \neq 0$ for any $n \in \mathbb{Z} \setminus \{0\}$, then we have $[\omega] = 0 \in H^3(\mathbb{Q}[H^{(1)}])_{(z)}$. 13
Proof. Case 1: Assume that $z$ is a torsion element. Then there exists $n \in \mathbb{Z} \setminus \{0\}$ with $nz = 0$. Assume $[\omega] = 0 \in H^3(\mathbb{Q}[H])_{(z)}$. Then there exists $\eta \in C^2(\mathbb{Q}[H])_{(z)}$ with $d\eta = \omega$. Set the map $f : H \to \mathbb{Q}$ by $f(x) = \eta([x], [z - x]) - 1$ for $x \in H$. For $u, v \in H$ with $\langle u, v \rangle \neq 0$, we have $f(u + v) = f(u) + f(v)$ since

$$
\langle u, v \rangle = \omega([u], [v], [z - u - v])
\quad = \quad d\eta([u], [v], [z - u - v])
\quad = \quad -\langle u, v \rangle (\eta([u + v], [z - u - v]) - \eta([u], [z - u]) - \eta([v], [z - v])).
$$

Therefore we have a contradiction because

$$
n\eta([z - u], [u]) = nf(z - u) + n\]
\quad = \quad f(nz - nu) + n\]
\quad = \quad -nf(u) + n\]
\quad = \quad n\eta([u], [z - u]) + n + n\]
\quad = \quad n\eta([z - u], [u]) + 2n$$

for $u \in H^{(1)}$ by Lemma [12]. Hence we obtain $[\omega] \neq 0 \in H^3(\mathbb{Q}[H])_{(z)}$.

Case 2: Assume that $z$ is not a torsion element. Then there exists a $\mathbb{Z}$-linear map $f : H \to \mathbb{Q}$ with $f(z) = 1$. Set a map $\eta : \{(u, v) \mid u, v \in H^{(1)}, u + v = z\} \to \mathbb{Q}$ by $\eta(u, z - u) = f(u) + 1$. This induces $\eta \in C^2(\mathbb{Q}[H^{(1)}])_{(z)}$ since $\eta(z - u, u) = f(z - u) - 1 = f(\eta) - f(u) - 1 = -\eta(u, z - u)$ for $u \in H^{(1)}$. We have $\omega = d\eta$. In fact, for $u, v \in H^{(1)}$ with $u + v \in H^{(1)}$, we have

$$
d\eta([u], [v], [z - u - v])
\quad = \quad -\langle u, v \rangle (\eta([u + v], [z - u - v]) - \eta([u], [z - u]) - \eta([v], [z - v]))
\quad = \quad -\langle u, v \rangle (f(u + v) + 1 - f(u) - 1 - f(v) - 1)
\quad = \quad \langle u, v \rangle
\quad = \quad \omega([u], [v], [z - u - v]).
$$

Hence we obtain $[\omega] = 0 \in H^3(\mathbb{Q}[H^{(1)}])_{(z)}$. This completes the proof of the proposition. 

We have that the map $\text{Hom}_\mathbb{Z}(\wedge^2 H/\mathbb{Z}, \mathbb{Q}) \to H^3(\mathbb{Q}[H^{(1)}])_{(z)}$ is not injective if $\ker \mu$ is not torsion free. Moreover we have that the map $\text{Hom}_\mathbb{Z}(\wedge^2 H_1(T^2; \mathbb{Z}), \mathbb{Q}) \to H^3(\mathbb{Q}[H_1(T^2; \mathbb{Z})^{(1)}])_{(0)}$ is injective since the $\mathbb{Q}$-vector space $\text{Hom}_\mathbb{Q}(\wedge^2 H_1(T^2; \mathbb{Z}), \mathbb{Q})$ is generated by $\langle -, - \rangle$, where $T^2$ is the torus.
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