INVARIANT MEASURES ON POLARIZED
SUBMANIFOLDS IN GROUP QUANTIZATION

J. Guerrero\textsuperscript{1,2,3} and V. Aldaya\textsuperscript{2,3}

November 20, 1999

Abstract

We provide an explicit construction of quasi-invariant measures on polarized
coadjoint orbits of a Lie group $G$. The use of specific (trivial) central extensions
of $G$ by the multiplicative group $\mathbb{R}^+$ allows us to restore the strict invariance
of the measures and, accordingly, the unitarity of the quantization of coadjoint
orbits. As an example, the representations of $SL(2, \mathbb{R})$ are recovered.

1 Introduction

The aim of this paper is to proceed a bit further in search of a unified algorithm for
achieving unitary and irreducible representations (unirreps for short) of Lie groups in the
context of quantization. Our starting point here is a rather developed Group Approach to
Quantization (GAQ) (see [1, 2, 3] and references there in), which generalizes and improves
Geometric Quantization (GQ) and/or the Coadjoint-Orbit Method (COM) [4, 5] in many
respects, and particularly in the treatment of the non-Kähler orbits of the Virasoro group
[6], denominated “non-quantizable orbits” in Ref. [7].

GAQ inherited, however, the technical problem of finding an appropriate and natu-
ral integration measure on the polarized submanifold of the original symplectic coadjoint
orbits (or classical phase space). In fact, even though a symplectic manifold $(M^{2n}, \omega)$
is canonically endowed with a volume, that is, $\omega^n$, a maximally isotropic submanifold
associated with a Polarization (half a symplectic manifold, so to speak) does not neces-
sarily possess a canonical measure invariant under the action of the group generators (or
quantum operators, in physical language).

Nevertheless, the virtue of GAQ working directly on a group manifold, rather than
on a coadjoint orbit, taking advantage of the tools available on any Lie group (left-

\textsuperscript{1}Departamento de Matemática Aplicada, Facultad de Informática, Campus de Espinardo, 30100 Mur-
cia, Spain
\textsuperscript{2}Instituto de Astrofísica de Andalucía, Apartado Postal 3004, 18080 Granada, Spain
\textsuperscript{3}Instituto de Física Teórica y Computacional Carlos I, Facultad de Ciencias, Universidad de Granada,
Campus de Fuentenueva, Granada 18002, Spain
and right-invariant vector fields, Haar measure, etc.) brings out again the solution to the present problem of finding invariant measures. The precise technique of pseudo-extensions employed here was introduced in [9] on an equal footing with non-trivial central extensions, and was further elaborated in [8], emphasizing its relation with COM. Now the main trick consists in considering pseudo-extensions by the multiplicative real line $\mathbb{R}^+$ along with (pseudo)-extensions by $U(1)$. Central (even trivial) extensions by $\mathbb{R}^+$ can modify well the common factor accompanying the wave functions (the weight) with an extra non-unimodular real function, thus providing half of the correction needed to make a quasi-invariant measure strictly invariant. The resulting construction shed new light on the cryptic language of “half-forms” [10], which came to faint the beauty of the original scheme of GQ.

This paper is organized as follows. In Sec. 2 we provide a general background on pseudo-extensions and the explicit connection with the coadjoint orbits of a general simply connected Lie group. In Sec. 3 the existence and uniqueness of a quasi-invariant measure $\mu$ with Radon-Nikodym derivative $\lambda$ on a homogeneous space is translated to the group $G$ itself, providing a constructive proof of the existence of such a $\lambda$. Then, with the aid of this function, we find a specific $\mathbb{R}^+$ pseudo-extension of $G$ making $\mu$ strictly invariant. The results above are applied, as an example, to the explicit construction of the representations of $SL(2, \mathbb{R})$, including the Mock representation.

## 2 Pseudo-extensions

A pseudo-extension of a simply connected Lie group $G$ is a central extension $\tilde{G}$ of $G$ by $U(1)$ by means of a 2-cocycle\(^1\) $\xi_\lambda : G \times G \to R$, which is a coboundary and therefore defines a trivial central extension; i.e. there exists a function $\lambda : G \to R$, the generating function of the coboundary, such that $\xi_\lambda(g', g) = \lambda(g' * g) - \lambda(g') - \lambda(g)$, but with the property that the Lie derivative of $\lambda$ at the identity is different from zero for some left-invariant vector fields. In other words, the gradient of $\lambda$ at the identity, $\lambda^0 \equiv \frac{\partial \lambda(g)}{\partial g}$, with respect to a basis of local canonical coordinates $\{g^i\}$ at a neighbourhood of the identity of $G$, is not zero.

It should be emphasized that $\tilde{\lambda}^0 \equiv (\lambda^0_1, \ldots, \lambda^0_n)$ defines an element in the dual $G^*$ of the Lie algebra $G$ of $G$. Before going further into the properties of pseudo-extensions and their classification into equivalence classes (in the same way as true extensions), we must introduce some definitions.

Let $\{X^L_i\}$ be a basis of left-invariant vector fields associated with the canonical coordinates $\{g^i\}$, $i = 1, \ldots, n = \dim G$ at the identity. Let $\{\theta^L_i\}$ be the dual basis of left invariant 1-forms on $G$. They verify the relations:

$$i_{X^L_i} \theta^L_j = \delta^j_i$$

\(^1\)We shall consider, following Bargmann [11], local exponents $\xi : G \times G \to R$ such that $\omega = e^{i\xi}$ defines a 2-cocycle (or factor), $w : G \times G \to U(1)$.\n
2
\[ L_{X^L_i} \theta^L_j = C^j_{ik} \theta^L_k, \]  

(1)

where \( C^j_{ik} \) are the structure constants of the Lie algebra \( \mathcal{G} \) generated by \( \{X^L_i\} \).

Right-invariant vector fields \( \{X^R_i\} \) can also be introduced together with the dual basis of right-invariant 1-forms \( \{\theta^R(i)\} \), satisfying properties similar to (1), but changing \( C^j_{ik} \) by \(-C^j_{ik}\); since right-invariant vector fields generate an algebra isomorphic to that of left-invariant ones but with the structure constant with opposite sign. Left-invariant 1-forms have zero Lie derivative with respect to right-invariant vector fields and vice versa, as it should be. An important formula which will be extensively used in this paper is the Maurer-Cartan equations:

\[ d\theta^L_i = \frac{1}{2} C^i_{jk} \theta^L_j \wedge \theta^L_k, \]  

(2)

with analogous expression for the right-invariant counterpart, but changing the sign to the structure constants, as before. These equations state, for instance, that, for an Abelian group, all left- and right-invariant 1-forms are closed, and that left- and right-invariant 1-forms dual to vector fields that are not in the commutant of \( \mathcal{G} \) are also closed. These properties will be relevant below.

Let us consider a central extension \( \tilde{\mathcal{G}} \) of \( \mathcal{G} \) by \( U(1) \) characterized by a 2-cocycle \( \xi : \mathcal{G} \times \mathcal{G} \to \mathbb{R} \), which has to satisfy the equations:

\[ \xi(g_1, g_2) + \xi(g_1 \ast g_2, g_3) = \xi(g_1, g_2 \ast g_3) + \xi(g_2, g_3), \]

\[ \xi(e, e) = 0, \]  

(3)

for all \( g_1, g_2, g_3 \in \mathcal{G} \), in order to define a (associative) group law. This group law is given by:

\[ g'' = g' \ast g, \]

\[ \zeta'' = \zeta' \zeta e^{i \xi(g' \ast g)}, \]  

(4)

where \( \zeta, \zeta', \zeta'' \in U(1) \). Left- and right-invariant vector fields for the extended group \( \tilde{\mathcal{G}} \), denoted with a tilde, can be derived from the ones of \( \mathcal{G} \) and from the 2-cocycle as follows:

\[ \tilde{X}^L_i = X^L_i + \frac{\partial \xi(g', g)}{\partial g^i} \bigg|_{g=e, g'=g} \frac{\partial}{\partial \phi}, \]

\[ \tilde{X}^R_i = X^R_i + \frac{\partial \xi(g', g)}{\partial g'^i} \bigg|_{g'=e} \frac{\partial}{\partial \phi}, \]  

(5)

where we have introduced \( \zeta = e^{i \phi} \). Left- and right invariant 1-forms do not change, and, of course, there are new left- and right invariant vectors fields and 1-forms associated with

---

\[ \text{This is due to our choice for the left and right action of the group on functions: } R_{g'} f(g) = f(g' \ast g) \]

and \( L_{g'} f(g) = f(g' \ast g) \) instead of \( L_{g'} f(g) = f(g'^{-1} \ast g) \), as is used in other contexts.
the new variable \( \zeta \in U(1) \). These are:

\[
\begin{align*}
\tilde{X}_\zeta^L &= \frac{\partial}{\partial \phi} = 2\Re(i\zeta \frac{\partial}{\partial \zeta}) \equiv \Xi \\
\tilde{X}_\zeta^R &= \Xi \\
\theta^{L(\zeta)} &= \frac{d\zeta}{i\zeta} + \frac{\partial \xi(g', g)}{\partial g^i}|_{g'=g^{-1}} dg^i \\
\theta^{R(\zeta)} &= \frac{d\zeta}{i\zeta} + \frac{\partial \xi(g', g)}{\partial g'^i}|_{g=g^{-1}} dg^i,
\end{align*}
\]

where \( \frac{d\zeta}{i\zeta} = d\phi \). We shall call \( \Theta \equiv \theta^{L(\zeta)} \) the Quantization 1-form. This 1-form defines a connection on the fibre bundle \( U(1) \to \tilde{G} \to G \), and will play an important role in our formalism, since it contains all the information about the dynamics of the system under study. In fact, \( \Theta_{PC} = \Theta - \frac{d\zeta}{i\zeta} \) is the Poincaré-Cartan 1-form, and \( d\Theta = d\Theta_{PC} \) is a presymplectic 2-form on \( G \) which defines a symplectic 2-form once the distribution generated by its kernel is removed.

Now let us assume that we add to \( \xi \) the coboundary \( \xi_\lambda \), generated by the function \( \lambda \), \( \xi_\lambda(g', g) = \lambda(g'*g) - \lambda(g') - \lambda(g) \), with \( \lambda \) satisfying \( \lambda(e) = 0 \) for \( \xi_\lambda \) to verify (3). Then \( \xi' = \xi + \xi_\lambda \) determines a new extended group \( \tilde{G}' \), and a new Quantization 1-form \( \Theta' = \Theta + \Theta_\lambda \), with

\[
\Theta_\lambda = \lambda^0 \theta^{L,i} - d\lambda.
\]

The new presymplectic 2-form is \( d\Theta' = d\Theta + d\Theta_\lambda \), with \( d\Theta_\lambda = \frac{1}{2} \lambda^0 C^i_k \theta^{L,i} \wedge \theta^{L,k} \) (making use of the Maurer-Cartan equations). We shall use this decomposition of \( \Theta' \) and \( d\Theta' \) to split an arbitrary 2-cocycle \( \xi' \) in the form

\[
\xi' = \xi + \xi_\lambda,
\]

for some \( \lambda(g) \). The term \( \xi \) is such that, when considered on its own, it determines a pure central extension, i.e. a central extension for which the Lie algebra satisfies: If \( C^k_{ij} \neq 0 \), then \( C^k_{ij} = 0 \) \( \forall k \neq \zeta \).

The term \( \xi_\lambda \) is such that, when considered on its own, it determines a pure pseudo-extension, i.e a central extension for which the Lie algebra satisfies: \( C^k_{ij} = \lambda^0_k C^k_{ij} \), \( \forall i, j \), with \( \vec{X}^0 \) the gradient at the identity of \( \lambda(g) \).

An arbitrary central extension determined by \( \xi \) will belong to a given cohomology class \([\xi] \) constituted by all 2-cocycles \( \xi' \) differing from \( \xi \) by coboundaries with arbitrary generating functions \( \lambda: G \to R \). This is the usual definition of the 2nd cohomology group \( H^2(G, U(1)) \) (see, for instance [11]). Now we are going to introduce subclasses \([\xi] \) inside \([\xi] \), called pseudo-cohomology classes. For the sake of simplicity, we shall restrict to the trivial cohomology class \([\xi]_0 \) of 2-cocycles which admit a generating function and are therefore coboundaries. The partition of \([\xi]_0 \) into pseudo-cohomology subclasses can be
translated to any other cohomology class using the decomposition (8). The equivalence relation defining the subclasses \([\xi]\) is given by:

Two coboundaries \(\xi_\lambda\) and \(\xi_{\lambda'}\) with generating functions \(\lambda\) and \(\lambda'\), respectively, are in the same subclass \([\xi]\) if and only if their gradients at the identity verify

\[
\bar{\lambda}' = \text{Ad}_g^*(\lambda)\bar{\lambda},
\]

for some \(g \in G\).

In particular, if \(\bar{\lambda}' = \bar{\lambda}\), \(\xi_\lambda\) and \(\xi_{\lambda'}\) are in the same pseudo-cohomology class. This allows us always to choose representatives that are linear in the canonical coordinates, \(\xi_{\bar{\lambda}_0} = \lambda_0^i g^i\).

The condition \(\bar{\lambda}' = \text{Ad}_g^*(\lambda)\bar{\lambda}\) simply says that \(\bar{\lambda}'\) and \(\bar{\lambda}\) lie in the same coadjoint orbit in \(G^*\), and it is justified because \(d\Theta_{\bar{\lambda}_0}\) and \(d\Theta_{\bar{\lambda}_0}'\) are symplectomorphic, the symplectomorphism being the pull-back of the coadjoint action (see [9]).

The equivalence relation we have just introduced constitutes a partition of the trivial cohomology class \([\xi_0]\) of coboundaries (or of any cohomology class once translated by the relation (8)), but there is not a one to one correspondence between pseudo-cohomology classes and coadjoint orbits, since the coadjoint orbits must satisfy the integrality condition (see [9], and [12] for the proof) for \(\lambda\) to define a central extension. This restriction can be expressed in a different manner:

The gradient at the identity \(\bar{\lambda}_0 \in G^*\) defines a linear functional of \(G\) on \(R\). But it also defines a one-dimensional representation of the isotropy lie subalgebra \(G_{\bar{\lambda}_0}\) of the point \(\bar{\lambda}_0\) under the coadjoint action of \(G\) on \(G^*\). In particular, if \(\bar{\lambda}_0\) is invariant under the coadjoint action (i.e. it constitutes a zero dimensional coadjoint orbit), it defines a one-dimensional representation of the whole Lie algebra \(G\). The condition of integrability of the coadjoint orbit passing through \(\bar{\lambda}_0\) is nothing more than the condition for \(\bar{\lambda}_0\) to be exponentiable (integrable) to a character of the isotropy subgroup \(G_{\bar{\lambda}_0}\) (whose Lie algebra is \(G_{\bar{\lambda}_0}\)).

The introduction of a pseudo-extension generated by \(\lambda(g)\) in \(G\), defining a central extension \(\tilde{G}\), has the effect of modifying left- and right-invariant vector fields in the following way:

\[
\hat{X}_L^i = X_L^i + (X_L^i \lambda_0^i)\Xi, \quad \hat{X}_R^i = X_R^i + (X_R^i \lambda_0^i)\Xi.
\]

(9)

It also modifies the commutation relations in the Lie algebra \(G\) of \(G\) (defining the commutation relations of \(\tilde{G}\)):

\[
[\hat{X}_L^i, \hat{X}_L^j] = C^k_{ij}(\hat{X}_L^k + \lambda_0^k \Xi),
\]

(10)

where \(C^k_{ij}\) are the structure constants of the original algebra \(G\). For right-invariant vector fields, we get the same commutation relations up to a sign. Once the representations of \(\tilde{G}\) have been obtained (using a technique like GAQ, for instance), we recover the representations of \(G\) by simply redefining the operators (right-invariant vector fields) in the following manner:

\[
\hat{X}_R^i \rightarrow \tilde{X}_R^i = \hat{X}_R^i + \lambda_0^i \Xi = X_R^i + (X_R^i \lambda)\Xi.
\]

(11)

It is trivial to check that the new generators \(\tilde{X}_R^i\) satisfy the (original) commutation relations of \(G\).
Once that the pseudo-extensions have been introduced and classified according to equivalence classes, they can be treated as if they were true extensions and the ordinary quantization techniques, in particular GAQ, can be applied. We refer the reader to [9] for a detailed description of GAQ, and here we shall simply use it to arrive at the irreducible representations of $SL(2, \mathbb{R})$ in Sec. 4.

3 Quasi-invariant measures

For any Lie group $G$, there exists a measure, the Haar measure, which is invariant under the left or right action of the group on itself. However, if $M$ is a manifold on which there is a transitive action of $G$ (that is, $M$ is a homogenous space under $G$), the existence of an invariant measure on $M$ is not guaranteed, despite that $M$ is locally diffeomorphic to the quotient $G/H$ of $G$ by a certain closed subgroup $H$, which is the isotropy group of an arbitrary point $x_0 \in M$. More precisely, each point in $M$ has a different isotropy group, although all of them are conjugate to each other; in particular all are isomorphic.

It can be proven (see [13] and [14]), however, that $M$ admits quasi-invariant measures. A measure $d\mu(x)$ on $M$ is called quasi-invariant if $d\mu(gx)$ is equivalent to $d\mu(x)$ for all $g \in G$, where $gx$ denotes the action of $G$ on $M$, and the equivalence relation is defined among measures that have the same sets of measure zero. Then the Radon-Nikodym theorem asserts that there exists a positive function $\lambda$ (the Radon-Nikodym derivative) on $M$ such that $d\mu(gx)/d\mu(x) = \lambda(g, x)$.

Furthermore, it turns out that any two quasi-invariant measures are equivalent ([14, 13]). Therefore, up to equivalence, there exists a unique quasi-invariant measure $d\mu(x)$ with Radon-Nikodym derivative $\lambda(\cdot, x)$ on $M$. The function $\lambda$ can be derived from a strictly positive, locally integrable, Borel function $\rho(g)$ satisfying

$$\rho(gh) = \frac{\Delta_G(h)}{\Delta_H(h)}\rho(g),$$  \hspace{1cm} (12)

where $\Delta_G, \Delta_H$ are the modular function of $G$ and $H$, respectively (a modular function of $G$ is a non-negative functions on $G$ such that, if $\mu_G(\cdot)$ is the left-invariant Haar measure on $G$, then $\mu_G(R_g f) = \Delta(g)\mu_G(f)$, where $R_g$ means right translation by the element $g$). A modular function is a homomorphism of $G$ into the positive reals with the product as composition law). The Radon-Nikodym derivative is given by:

$$\lambda(g, x) = \frac{\rho(gg')}{\rho(g')},$$  \hspace{1cm} (13)

where $g'$ is any element whose image under the natural projection $G \to M$ is $x$. This definition makes sense since $\frac{\rho(gg')}{\rho(g')}$ depends only on $x$ and not on the particular choice of $g'$.

---

3Since we are considering the quotient space $G/H$ instead of $H\backslash G$, i.e. we are changing left by right with respect to [13, 14], modular functions get inverted.
Note that if \( \Delta_H(h) = \Delta_G(h), \forall h \in H \), then \( \rho(gh) = \rho(g) \), so that we can choose \( \rho(g) = 1 \) and \( \lambda(g, x) = 1 \) as the Radon-Nikodym derivative. Thus, in this case, \( M \) admits an invariant measure under \( G \).

Let us rewrite the above considerations in infinitesimal terms. Defining the modular constants \( k_i^{G/H} \equiv \frac{\partial \Delta_G(g)}{\partial g}|_{g=e} \), \( i = 1, \ldots, n = \dim G \), and similarly for \( k_i^H \), \( i = 1, \ldots, p = \dim H \), we can rephrase the \( \rho \)-function condition (12) as:

\[
X_i^L \rho(g) = k_i^{G/H} \rho(g),
\]

where \( k_i^{G/H} \equiv k_i^G - k_i^H \), \( i = 1, \ldots, p \). Modular constants possess properties derived from those of modular functions. Firstly, it can be proven that \( k_i^G = \sum_{j=1}^n C_{ij}^j \), and accordingly, \( k_i^{G/H} = \sum_{j=p+1}^n C_{ij}^j \), where we have assumed that the first \( p = \dim H \) elements of \( G \) belong to \( \mathcal{H} \), the Lie algebra of \( H \). In addition, \( k_i^G \), \( i = 1, \ldots, n \) define a character \( k_i^G \) of the Lie algebra \( \mathcal{G} \) of \( G \), coming from the fact that \( \Delta_G(g) \) defines a character of \( G \), in such a way that \( k_i^G(X_i^L) = k_i^G \). This property implies linearity, and also \( C_{ij}^l k_i^G = 0 \), since \( k_i^G([X_i^L, X_j^L]) = 0 \). As a result, \( k_i^G = 0 \) for \( G \) semisimple.

However, \( k_i^H \) can be non-trivial, even if \( H \) is a subgroup of a semisimple group \( G \), allowing for non-trivial \( k_i^{G/H} \), and, according to (14), for the possibility of homogeneous spaces with non-invariant, although quasi-invariant, measures.

Let us develop a constructive technique for obtaining quasi-invariant measures on homogeneous spaces. That is, a procedure for constructing \( \rho \)-functions satisfying (12) (or (14)). According to Mackey [13], such a function always exists, although the proof of his theorem is not constructive.

Consider the left-invariant Haar measure \( \Omega^L \) on \( G \). This is an \( n \)-form, with \( n = \dim G \), and can be written, up to a constant, as \( \Omega^L = \theta^{L1} \wedge \theta^{L2} \wedge \cdots \wedge \theta^{Ln} \), where \( \theta^{Li}, i = 1, \ldots, n \), is the set of left invariant 1-forms on \( G \) dual to a given basis \( \{X_i^L\} \) of left-invariant vector fields. Let us suppose that the first \( p = \dim H \) elements in these bases correspond to left-invariant 1-forms and vector fields of \( H \), respectively. Then we tentatively define a measure on \( G/H \) as:

\[
\Omega_H^L = i_{X_1^L} \cdots i_{X_p^L} \Omega^L = \theta^{Lp+1} \wedge \cdots \wedge \theta^{Ln}.
\]

In general, \( \Omega_H^L \) is not an invariant measure on \( G/H \); in fact, it is not even a measure on \( G/H \), in the sense that it does not fall down to the quotient. This can be checked by computing its invariance properties under \( X_i^L \), \( i = 1, \ldots, p \). After a few computations we get \( L_{X_i^L} \Omega_H^L = -k_i^{G/H} \Omega_H^L \). Therefore, if \( k_i^{G/H} \neq 0 \) for some \( i \), \( \Omega_H^L \) does not fall down to the quotient, and this is the same condition for \( G/H \) not to have a strictly invariant measure. Therefore, these two facts seem to be related. Indeed, if we look for a function \( \rho \) on \( G \) such that \( L_{X_i^L}(\rho \Omega_H^L) = 0, i = 1, \ldots, p \), we find that \( \rho \) has to be a \( \rho \)-function, satisfying \( X_i^L \rho = k_i^{G/H} \rho \), as in (14).
Now we have to prove that equation (14) always has non-trivial solutions. We know from Mackey [13], that equation (12) always has a solution, but we would like to provide a proof in infinitesimal terms and, moreover, we would like to construct the solutions explicitly.

Let us consider the Radical of $H$, $\text{Rad}H$ – the maximal solvable ideal of $H$. We know that $H/\text{Rad}H$ is semisimple. According to the previous considerations, the $k_i$’s vanish on this quotient. Thus, the non-trivial $k_i$’s lie only on $\text{Rad}H$, which is solvable. According to one of Lie’s theorems [15], a solvable algebra of operators always possesses a common eigenvector. We proceed to construct it as follows:

Let us consider the equation $X^L_i \rho = k^{G/H}_i \rho$, $i = 1, \ldots, p$. Let $\chi$ be the general solution of $X^L_i \chi = 0$, which always exists and which we know how to construct, according to the Frobenious theorem. Then we can write $\rho = \chi h$, where $h$ is a particular solution of $X^L_i h = k^{G/H}_i h$, with $X^L_i \in \text{Rad}H$ (the rest of the equations give zero, and since $h$ is a particular solution, we can choose so as not to depend on the corresponding variables). Then Lie’s Theorem guarantees the existence of such a function $h$, since $\text{Rad}H$ is solvable.

Once we have constructed the measure $\rho \Omega^L_H$ on $G/H$, we must check its invariance properties under the action of $G$. For this, we compute $L_{X^R_i} (\rho \Omega^L_H) = \frac{1}{\rho}(X^R_i \cdot \rho)(\rho \Omega^L_H)$, $i = 1, \ldots, n$. The result is that $\rho \Omega^L_H$ is quasi-invariant under $G$ and the divergence of the vector field $X^R_i$ is $\frac{1}{\rho}(X^R_i \cdot \rho)$. Once the divergence of all vector fields have been computed, it is very easy to modify the (infinitesimal) action of the group $G$ in order to restore the invariance of $\rho \Omega^L_H$, by defining the new vector fields:

$$\tilde{X}^R_i = X^R_i + \frac{1}{2\rho}(X^R_i \cdot \rho),$$

(16)
i.e., right-invariant vector fields are modified with the addition of a multiplicative term, half the divergence of the corresponding vector field. In the context of Sec. 3, we could think of this redefinition as coming from a pseudo-extension of $G$ by means of some pseudo-cocycle generated by a certain function $\lambda$ on $G$. In fact, this is the case, since the extra term can be written as $X^R_i (\frac{1}{2} \log \rho)$, i.e., the function $\lambda$, according to equation (14), would be $\lambda = -i \frac{1}{2} \log \rho$. Note the presence of the imaginary constant $i$ in $\lambda$ (so that $\lambda$ is a pure imaginary function) revealing that $G$ has been centrally pseudo-extended by $R^+$ instead of $U(1)$. Therefore, the invariance of a measure on a quotient space $G/H$ can be restored by means of a central extension of $G$ by $R^+$ with generating function $-i \frac{1}{2} \log \rho$, where $\rho$ is a $\rho$-function.

If we compute the commutation relations of the redefined vector fields, we get:

$$[\tilde{X}^R_i, \tilde{X}^R_j] = -C^k_{ij} \tilde{X}^R_k,$$

(17)
showing that this pseudo-extension does not modify the commutation relations. As in Sec. 2 we can compute the gradient of the generating function $\lambda$ at the identity, proving to be $\lambda_0^i = -i \frac{1}{2} k^{G/H}_i$, $i = 1, \ldots, n$. It is pure imaginary, as would be expected of a pseudo-extension by $R^+$. 

8
4 Example: Representations of $SL(2, \mathbb{R})$

Let us consider, as an example of application of the formalism developed above, the study of the unitary and irreducible representations of $G = SL(2, \mathbb{R})$. Since this group is non-simply connected, in order to apply our previous considerations, we shall consider its universal covering group $\tilde{G}$, with $p : \tilde{G} \to SL(2, \mathbb{R})$ the covering map, which is a group homomorphism. The kernel of $p$ is $Z$, the first homotopy group of $SL(2, \mathbb{R})$. It is easy to check that a unirrep $U$ of $\tilde{G}$ is also a unirrep of $SL(2, \mathbb{R})$ if and only if Ker$p$ is represented as phases, i.e $U(g) = e^{i\alpha_g}, \forall g \in \text{Ker}p$. Therefore, we shall compute the representations $U$ of $\tilde{G}$ and then retain only the ones that verify $U(g) = e^{i\alpha_g}, \alpha_g \in \mathbb{R}, \forall g \in \text{Ker}p$. For simplicity, we shall denote $\tilde{G}$ just by $\tilde{G}$, bearing in mind that at the end we wish to get the representations of $SL(2, \mathbb{R})$.

Since $SL(2, \mathbb{R})$ is semisimple, it has no non-trivial central extensions by $U(1)$; i.e. its second cohomology group $H^2(G, U(1)) = \{e\}$. However, as shown in Sec. 4 this group admits non-trivial pseudo-extensions by $U(1)$, which can be classified into pseudo-cohomology classes. These pseudo-cohomology classes are in one-to-one correspondence with the coadjoint orbits of $SL(2, \mathbb{R})$ with integral symplectic 2-form (see [9]).

Thus, we must first study the coadjoint orbits of $SL(2, \mathbb{R})$. These can be classified into three types: the 1-sheet hyperboloids, the 2-sheets hyperboloids, and the cones. The cones are really three different orbits, the upper and lower cones and the origin. The origin is the only zero-dimensional orbit, and is associated with the only one-dimensional representation (character) of $SL(2, \mathbb{R})$, the trivial one.

As we shall see below, the 1-sheet hyperboloids are associated with the Principal continuous series of unirreps of $SL(2, \mathbb{R})$, the 2-sheet hyperboloids are associated with the Principal discrete series of unirreps and the two cones are associated with the Mock representations.

4.1 The group law

The $SL(2, \mathbb{R})$ group can be parameterized by:

$$SL(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) / ad - bc = 1 \right\}. \quad (18)$$

If $a \neq 0$ (the case $a = 0$ is treated in an analogous manner, changing $a$ by $c$), we can eliminate $d$, $d = \frac{1+bc}{a}$, and we arrive at the following group law from matrix multiplication:

$$a'' = a'a + b'c$$
$$b'' = a'b + b \frac{1+bc}{a}$$
$$c'' = c'a + \frac{1+b'c'}{a'}c. \quad (19)$$
Left- and right-invariant vector fields are easily derived from the group law:

\[
\begin{align*}
X^L_a &= a \frac{\partial}{\partial a} + c \frac{\partial}{\partial c} - b \frac{\partial}{\partial b} \\
X^L_b &= a \frac{\partial}{\partial b} \\
X^L_c &= \frac{1 + bc}{a} \frac{\partial}{\partial c} + b \frac{\partial}{\partial a} \\
X^R_a &= a \frac{\partial}{\partial b} \\
X^R_b &= \frac{1 + bc}{b} \frac{\partial}{\partial b} + c \frac{\partial}{\partial a} \\
X^R_c &= a \frac{\partial}{\partial c}.
\end{align*}
\]  

(20)

The Lie algebra satisfied by the (say, left-invariant) vector fields is:

\[
\begin{align*}
[X^L_a, X^L_b] &= 2X^L_b \\
[X^L_a, X^L_c] &= -2X^L_c \\
[X^L_b, X^L_c] &= X^L_a,
\end{align*}
\]  

(21)

and the Casimir for this Lie algebra is given by \( \hat{C} = \frac{1}{2}(X^L_a)^2 + X^L_b X^L_c + X^L_c X^L_a \). The left-invariant 1-forms (dual to the set of left-invariant vector fields) are given by:

\[
\begin{align*}
\theta^{L(a)} &= \frac{1 + bc}{a} da - bdc \\
\theta^{L(b)} &= \frac{1}{a} db - \frac{b^2}{a} dc + \frac{b}{a} \frac{1 + bc}{a} da \\
\theta^{L(c)} &= adc - cda.
\end{align*}
\]  

(22)

The exterior product of all left-invariant 1-forms constitutes a (left-invariant) volume form on the whole group (Haar measure):

\[ \Omega^L = \theta^{L(a)} \wedge \theta^{L(b)} \wedge \theta^{L(c)} = \frac{1}{a} da \wedge db \wedge dc. \]  

(23)

4.2 Pseudo-extensions

The different (classes of) pseudo-extensions of \( SL(2, \mathbb{R}) \) by \( U(1) \) are classified, according to the discussion in Sec. 3, by the coadjoints orbits of the group \( SL(2, \mathbb{R}) \). Let us parameterize \( \mathcal{G}^* \) by \( \{ \alpha, \beta, \gamma \} \), a coordinate system associated with the base \( \{ X^L_a, X^L_b, X^L_c \} \) of \( \mathcal{G} \). Instead of looking for the different coadjoint orbits by direct computation, we can classify them by means of the Casimir functions. The Casimirs \( C_i \) are invariant functions under the coadjoint action of the group on \( \mathcal{G}^* \), so that the equations \( C_i = c_i \) define hypersurfaces on \( \mathcal{G}^* \) invariant under the coadjoint action. Of course, these hypersurfaces could be the union of two or more coadjoint orbits, and we shall need extra conditions to characterize them (these are called invariant relations, see [9, 16]).

The only (independent) Casimir function for \( SL(2, \mathbb{R}) \) is \( C = \frac{1}{2} \alpha^2 + \beta \gamma \). This is a quadratic function, and therefore its level sets are conic sections.

It is more appropriate for our purposes to perform the change of variables \( \alpha = \alpha', \beta = \mu + \nu, \gamma = \mu - \nu \). In terms of the new variables, the Casimir function is written \( C = \frac{1}{2} \alpha'^2 + 2 \mu^2 - 2 \nu^2 \). In this form, it is easy to identify the conics, of which there are essentially
three types, depending on whether $C > 0$, $C = 0$ or $C < 0$. The case $C > 0$ corresponds to 1-sheet hyperboloids; the case $C = 0$ corresponds to the two cones and the origin, i.e. the union of three coadjoint orbits; and finally, the case $C < 0$ corresponds to 2-sheets hyperboloids (i.e. the union of two coadjoint orbits).

Now we select a particular point $\tilde{X}^0$ in each coadjoint orbit, which will be used to define a pseudo-extension in $SL(2, \mathbb{R})$ (different choices of $\tilde{X}^0$ in the same coadjoint orbit will lead to equivalent pseudo-extensions). For the case $C > 0$, the easiest choice is $\tilde{X}^0 = (\alpha, 0, 0)$. For $C = 0$, we have $\tilde{X}^0 = (0, 0, 0)$ for the origin, and we can choose $\tilde{X}^0 = (0, 0, \gamma < 0)$ for the upper cone and $\tilde{X}^0 = (0, 0, \gamma > 0)$ for the lower cone. Finally, for the case $C < 0$, we select $\tilde{X}^0 = (0, \beta > 0, \gamma = -\beta)$ for the upper sheet and $\tilde{X}^0 = (0, \beta < 0, \gamma = -\beta)$ for the lower sheet of the 2-sheets hyperboloid.

### 4.3 Representations associated with the 1-sheet hyperboloid: Principal Continuous Series

According to the above discussion, let us choose $\tilde{X}^0 = (\alpha, 0, 0)$ as the representative point in the 1-sheet hyperboloids. We need to look for a function $\lambda$ on $SL(2, \mathbb{R})$ satisfying $\frac{d}{dg}\lambda(g)\mid_{g=e} = \lambda^0$. The easiest one would be a function linear on the coordinate $a$, but we should take into account that $a$ is not a canonical coordinate, since its composition law is multiplicative. That is, the uniparametric subgroup associated with it is $R^+$ instead of $R$ (the value of $a$ at the identity of the group is 1 instead of 0). Thus, we can select for $\lambda(g) = \alpha \log a$ or rather $\lambda(g) = \alpha (a - 1)$, since the generating function $\lambda$ must satisfy $\lambda(e) = 0$ for $\xi_\lambda$ to satisfy (3).

Let us fix $\lambda(g) = \alpha (a - 1)$, to be precise (the other choice would lead to an equivalent result). The representation achieved when applying GAQ to the resulting group will be associated with the coadjoint orbit for which the Casimir is $C = \frac{1}{2} \alpha^2 > 0$. The resulting group law for $SL(2, \mathbb{R})$ pseudo-extended by $U(1)$ by means of the two-cocycle $\xi_\lambda$ is:

$$
\begin{align*}
    a'' &= a'a + b'c \\
    b'' &= a'b + b' \frac{1 + bc}{a} \\
    c'' &= c'a + \frac{1 + b'c'}{a'}c \\
    \zeta'' &= \zeta' \zeta e^{i\alpha(a'a + b'c - a' - a + 1)}. \\
\end{align*}
$$

Left- and right-invariant vector field, obtained as usual from the group law, are:

$$
\begin{align*}
    \tilde{X}_a^L &= a \frac{\partial}{\partial a} + c \frac{\partial}{\partial c} - b \frac{\partial}{\partial b} + \alpha (a - 1) \Xi \\
    \tilde{X}_b^L &= a \frac{\partial}{\partial b} + \frac{1 + bc}{a} \frac{\partial}{\partial a} \\
    \tilde{X}_c^L &= b \frac{\partial}{\partial a} + \alpha b \Xi \\
    \tilde{X}_\zeta^L &= \frac{\partial}{\partial \phi} = 2\text{Re}(i \zeta \frac{\partial}{\partial \phi}) \equiv \Xi \\
    \tilde{X}_a^R &= a \frac{\partial}{\partial a} + b \frac{\partial}{\partial b} - c \frac{\partial}{\partial c} + \alpha (a - 1) \Xi \\
    \tilde{X}_b^R &= a \frac{\partial}{\partial b} + \frac{1 + bc}{a} \frac{\partial}{\partial a} + \alpha c \Xi \\
    \tilde{X}_c^R &= a \frac{\partial}{\partial c} \\
    \tilde{X}_\zeta^R &= \Xi.
\end{align*}
$$

(25)
Left- and right-invariant 1-forms associated with the variables of $SL(2, \mathbb{R})$ remain the same, and there are extra left- and right-invariant 1-forms associated with the variable $\zeta$. We are interested in the left-invariant one, which is:

$$\Theta \equiv \theta^L(\zeta) = \frac{d\zeta}{i\zeta} + \alpha(\theta^L(a) - da) = \frac{d\zeta}{i\zeta} + \alpha\left(\frac{1 + bc - a}{a}da - bdc\right).$$

The resulting Lie algebra is that of $SL(2, \mathbb{R})$ with one of the commutators modified:

$$\begin{align*}
[\tilde{X}_a^L, \tilde{X}_b^L] &= 2\tilde{X}_b^L \\
[\tilde{X}_a^L, \tilde{X}_c^L] &= -2\tilde{X}_c^L \\
[\tilde{X}_b^L, \tilde{X}_c^L] &= \tilde{X}_a^L + \alpha\Xi.
\end{align*}$$

The 2-form

$$d\Theta = \alpha(dc \wedge db + \frac{c}{a}db \wedge da + \frac{b}{a}dc \wedge da)$$

defines a presymplectic structure on $\tilde{G}$. The characteristic module, or more precisely, $\ker d\Theta \cap \ker \Theta$, is generated by the characteristic subalgebra, $G_C = <\tilde{X}_a^L>$. We should remember that the characteristic subalgebra is nothing more than the isotropy subalgebra $G_{\vec{\lambda}_0}$ of the point $\tilde{X}_0 \in G$.

Now we have to look for polarization subalgebras. These should contain the characteristic subalgebra $G_C$ and must be horizontal (i.e., in the kernel of $\Theta$). There are essentially two, and these lead to unitarily equivalent representations (since they are related by the adjoint action of the Lie algebra on itself, and this turns out to be a unitary transformation). We shall choose as polarization

$$P = <\tilde{X}_a^L, \tilde{X}_b^L >,$$

and this, by solving the equation $\tilde{X}_a^L \Psi = \tilde{X}_b^L \Psi = 0$, provides the wave functions $\Psi = \zeta e^{-ia(\kappa - 1)}\kappa^i\alpha\Phi(\tau)$, where $\kappa \equiv a$ and $\tau \equiv \frac{\zeta}{a}$. The action of the right-invariant vector fields on polarized wave functions is:

$$\begin{align*}
\tilde{X}_a^R \Psi &= \zeta e^{-ia(\kappa - 1)}\kappa^i\alpha\left[-2\tau \frac{d}{d\tau}\right]\Phi(\tau) \\
\tilde{X}_b^R \Psi &= \zeta e^{-ia(\kappa - 1)}\kappa^i\alpha[i\alpha\tau - \tau^2 \frac{d}{d\tau}]\Phi(\tau) \\
\tilde{X}_c^R \Psi &= \zeta e^{-ia(\kappa - 1)}\kappa^i\alpha\left[\frac{d}{d\tau}\right]\Phi(\tau).
\end{align*}$$

According to Sec. 2, the right-invariant generators should be redefined as $\tilde{X}_a^R \rightarrow \tilde{X}_a^{R_1} = \tilde{X}_a^R + \lambda_0\Xi$ in order to obtain the representations of $G$, and this affects only the generators $\tilde{X}_a^R$, which changes to $\tilde{X}_a^{R_1} = \tilde{X}_a^R + \alpha\Xi$. Its action on polarized wave functions turns out to be:

$$\tilde{X}_a^{R_1} \Psi = \zeta e^{-ia(\kappa - 1)}\kappa^i\alpha\left[i\alpha - 2\tau \frac{d}{d\tau}\right]\Phi(\tau).$$
The representation of $SL(2, \mathbb{R})$ here constructed is irreducible but not unitary. One way of viewing it (before discussing integration measures) is to consider the Casimir operator, which is the quadratic operator $\hat{C} = \frac{1}{2}(\tilde{X}^a_R)^2 + \tilde{X}^a_R \tilde{X}^b_R + \tilde{X}^c_R \tilde{X}^b_R$. After the pseudoextension and redefinition of operators ($\tilde{X}^a_R$ should be changed by $\tilde{X}^a'_R$), the resulting Casimir operator, $\hat{C}'$, acts on polarized wave functions as $\hat{C}'\Psi = (-\alpha^2/2 + i\alpha)\Psi$.

The fact that it is a number reveals that the representation is irreducible, but since it is not real, the representation cannot be unitary (the Casimir is a quadratic function of (anti-)Hermitian operators, and should therefore be a self-adjoint operator in any unitary representation).

The reason for this lack of unitarity is that the support manifold for the representation does not admit an invariant measure. Since the process of polarizing wave functions really amounts to reducing the space of functions to those defined in the quotient $G/G_P$, where $G_P$ is the group associated with the polarization subalgebra $\mathcal{P}$, the support manifold is given by $G/G_P$, which is naturally a homogeneous space under $G$. According to Sec. 3, it may well happen that $G/G_P$ does not admit an invariant measure, and in fact this is the case. However, the existence of quasi-invariant measures is granted, and this fact will allow us to restore the unitarity of the representation.

If we compute the measure on $G/G_P$, derived from the left Haar measure $\Omega^L$ on $G$, we obtain $\Omega_P^L = i \tilde{X}_b^L \tilde{X}_a^L \Omega^L = adc - cda$. When expressed in terms of the new variables $\kappa$ and $\tau$, it takes the form $\Omega_P^L = \kappa^2 d\tau$. Taking into account that $G/G_P$ is parameterized by $\tau$, now becomes clear why the representation is not unitary: the measure does not even fall down to the quotient.

A solution to this problem consists in choosing any quasi-invariant measure on $G/G_P$ and introducing the appropriate Radon-Nikodym derivative [13, 14]. Here, we propose another, yet equivalent, solution to this lack of unitarity, giving a new insight into the problem according to Sec. 3. We shall consider a pseudo-extension of $G$ by $R^+$, rather than $U(1)$. The reason is that we wish to restore the unitarity of a non-unitary representation, and for this we need a “piece” of non-unitary representation, in such a way that the resulting representation is unitary. To enable a direct comparison with the treatment of Mackey, we shall employ the equivalent technique of non-horizontal polarizations instead of that of pseudo-extensions. A non-horizontal polarization $\mathcal{P}^{n.h.}$ is a polarization in which the horizontality condition has been relaxed. The polarization equations acquire the form: $\tilde{X}^j_L \Psi = i\alpha_j \Psi, \forall \tilde{X}^j_L \in \mathcal{P}^{n.h.}$ (see [13] for a discussion on the equivalence between pseudo-extensions and non-horizontal polarizations).

The key point is to keep $\Omega_P^L$ as the measure on $G/G_P$, and to impose the polarization conditions $\tilde{X}^L_i \tilde{\Psi} = \frac{1}{2} k_i^{G/G_P} \tilde{\Psi}$, instead of $\tilde{X}^L_i \Psi = 0, \forall \tilde{X}^L_i \in \mathcal{P}$. In finite terms, this condition is written as:

$$\tilde{\Psi}(g \ast h) = \sqrt{\frac{\Delta_G(h)}{\Delta_H(h)}} \tilde{\Psi}(g).$$ (32)
We can rephrase this by saying that $\tilde{\Psi}$ is a $\frac{1}{2}\rho$-function. The purpose of this definition is to make $\tilde{\Psi}^*\tilde{\Psi}'\Omega^P_L$ a well-defined quantity on $G/G_P$ and can be integrated with respect to $\tau$. In other words, $\tilde{\Psi}^*\tilde{\Psi}'$ is a $\rho$-function necessary to make $\Omega^P_L$ a quasi-invariant measure on $G/G_P$.

To begin, we must compute the modular constants $k_{Gi/G_P} = k_G - k_{G_P}i$, $i = 1, \ldots, p$. Firstly, since $G = SL(2, \mathbb{R})$ is semi-simple, $k_G = 0$, $i = 1, \ldots, n$. Secondly, we have $k^G_{a} = 2$ and $k^G_{b} = 0$. Therefore, $k_{G/G_P} = -2$ and $k_{G/G_P} = 0$.

Accordingly, the new polarization equations we have to solve are:

$$\tilde{\mathcal{X}}^L_a\tilde{\Psi} = -\tilde{\Psi}, \quad \tilde{\mathcal{X}}^L_b\tilde{\Psi} = 0.$$ (33)

It is easy to verify that the solutions of these new polarization equations are of the form:

$$\tilde{\Psi}(g) = a^{-1}\Psi(g),$$ (34)

where $\Psi(g)$ is a solution of the previous (horizontal) polarization equations. Thus, the form of the solutions is:

$$\tilde{\Psi} = \zeta \kappa^{-1}e^{-i\alpha(\kappa-1)}\kappa^{i\alpha}\Phi(\tau).$$ (35)

Now it is clear why $\tilde{\Psi}^*\tilde{\Psi}'\Omega^P_L = \Phi(\tau)^*\Phi'(\tau)d\tau$ can be integrated in $G/G_P$; the $\kappa$ dependence has been removed.

The right-invariant vector fields, when acting on $\frac{1}{2}\rho$-functions, acquire extra terms that restore the unitarity of the representation:

$$\tilde{\mathcal{X}}^R_a\tilde{\Psi} = \kappa^{-1}\tilde{\mathcal{X}}^R_i\Psi + \kappa^{-1}(\kappa\tilde{\mathcal{X}}^R_i.\kappa^{-1})\Psi.$$ (36)

In this way, the final representation has the form, restricted to its action on $\Phi(\tau)$:

$$\tilde{\mathcal{X}}^R_a\Phi(\tau) = [-1 + i\alpha - 2\tau\frac{d}{d\tau}]\Phi(\tau)$$

$$\tilde{\mathcal{X}}^R_b\Phi(\tau) = [-\tau + i\alpha \tau - \tau^2\frac{d}{d\tau}]\Phi(\tau)$$

$$\tilde{\mathcal{X}}^R_c\Phi(\tau) = [\frac{d}{d\tau}]\Phi(\tau).$$ (37)

We can readily verify that these operators are self-adjoint with respect to the quasi-invariant measure $d\tau$ (what remains of $\Omega^P_L$ after multiplication by the factor $\kappa^{-2}$ contained in the wave functions). Even more, the Casimir operator, acting on the new wave functions, turns out to be real, revealing that the representation is now unitary:

$$\hat{C}'\Phi(\tau) = -\frac{1}{2}(1 + \alpha^2)\Phi(\tau).$$ (38)

---

4Note that, according to Sec. 3, the generating function for the pseudo-extension by $R^+$ would be $\lambda = -\frac{i}{2}\log \rho = -i \log \rho^\frac{1}{2}$, with $\lambda^0_0 = -\frac{i}{2}k^G_{G/G_P} = \alpha$.

5The difference between pseudo-extensions and non-horizontal polarizations lie in the fact that pseudo-extensions modify the left- and right-invariant vector fields and non-horizontal polarizations modify the wave functions. The extra term in the reduced operators is a consequence of their acting on modified wave functions.
4.4 Representations associated with the cones: Mock representation

In accordance with Sec. 4.2, let us choose \( \lambda_0 = (0,0,\gamma) \) as the representative point in the cone. If \( \gamma < 0 \) we are in the upper cone and if \( \gamma > 0 \) we are in the lower cone. We have to look for a function \( \lambda \) on \( SL(2,\mathbb{R}) \) satisfying \( \frac{\partial}{\partial g} \lambda(g)|_{g=e} = \lambda_0 \). The easiest one is the function linear on the coordinate \( c \), since here \( c \) is a true canonical coordinate, and therefore, we fix \( \lambda(g) = \gamma c \).

The representation obtained when applying GAQ to the resulting group will be associated with one of the coadjoint orbit for which the Casimir is \( C = 0 \). The resulting group law for \( SL(2,\mathbb{R}) \) pseudo-extended by \( U(1) \) by means of the two-cocycle \( \xi \) is:

\[
\begin{align*}
a'' &= a'a + b'c \\
b'' &= a'b + b'1 + bc/a \\
c'' &= c'a + 1 + b'/c \\
\zeta'' &= \zeta'e^{\gamma(c'(a'+1)+bc'/a-c'-c)}.
\end{align*}
\] (39)

Left- and right-invariant vector field, derived as usual from the group law, are:

\[
\begin{align*}
\tilde{X}_a^L &= a\frac{\partial}{\partial a} + c\frac{\partial}{\partial c} - b\frac{\partial}{\partial b} + \gamma c\Xi \\
\tilde{X}_b^L &= a\frac{\partial}{\partial b} \\
\tilde{X}_c^L &= \frac{1+b}{a}\frac{\partial}{\partial c} + b\frac{\partial}{\partial a} + \gamma(1+bc/a - 1)\Xi \\
\tilde{X}_\zeta^L &= \frac{\partial}{\partial \zeta} = 2\text{Re}(i\zeta\frac{\partial}{\partial \zeta}) \equiv \Xi \\
\tilde{X}_a^R &= a\frac{\partial}{\partial a} + b\frac{\partial}{\partial b} - c\frac{\partial}{\partial c} - \gamma c\Xi \\
\tilde{X}_b^R &= a\frac{\partial}{\partial b} + \gamma(a-1)\Xi \\
\tilde{X}_c^R &= a\frac{\partial}{\partial c} \\
\tilde{X}_\zeta^R &= \Xi.
\end{align*}
\] (40)

The left-invariant 1-form associated with the variable \( \zeta \) is:

\[
\Theta \equiv \theta^{(\zeta)} = \frac{d\zeta}{i\zeta} + \gamma(\theta^{(\zeta)} - dc) = \frac{d\zeta}{i\zeta} + \gamma((a-1)dc - cda). \] (41)

The resulting Lie algebra is, again, that of \( SL(2,\mathbb{R}) \) with one of the commutators modified, in this case the one giving \( \tilde{X}_c^L \) on the r.h.s.:

\[
\begin{align*}
[\tilde{X}_a^L, \tilde{X}_b^L] &= 2\tilde{X}_b^L \\
[\tilde{X}_a^L, \tilde{X}_c^L] &= -2(\tilde{X}_c^L + \gamma \Xi) \\
[\tilde{X}_b^L, \tilde{X}_c^L] &= \tilde{X}_a^L.
\end{align*}
\] (42)

The 2-form

\[
d\Theta = 2\gamma da \wedge dc
\] (43)

defines a presymplectic structure on \( \widetilde{G} \). The characteristic subalgebra is \( \mathcal{G}_C = \langle \tilde{X}_b^L \rangle \).

In this case, there is essentially one polarization, given by:

\[
\mathcal{P} = \langle \tilde{X}_b^L, \tilde{X}_a^L \rangle,
\] (44)

15
and this provides, by solving the equation \( \tilde{X}_a^L \Psi = \tilde{X}_b^L \Psi = 0 \), the wave functions \( \Psi = \zeta e^{-i\gamma \tau} \Phi(\tau) \), where again \( \tau \equiv \frac{c}{a} \). The action of right-invariant vector fields on polarized wave functions is:

\[
\begin{align*}
\tilde{X}_a^R \Psi &= \zeta e^{-i\gamma \tau} \left[ -2\tau \frac{d}{d\tau} \right] \Phi(\tau) \\
\tilde{X}_b^R \Psi &= \zeta e^{-i\gamma \tau} \left[ -\tau^2 \frac{d}{d\tau} \right] \Phi(\tau) \\
\tilde{X}_c^R \Psi &= \left[ \frac{d}{d\tau} - i\gamma \right] \Phi(\tau).
\end{align*}
\]

The redefinition of the right-invariant generators \( \tilde{X}_g^R \rightarrow \tilde{X}_g^R + \lambda^0 \Xi \) in order to obtain the representation of \( G \), affects only to the \( \tilde{X}_c^R \) generator, which changes to \( \tilde{X}_c^R' = \tilde{X}_a^R + \gamma \Xi \). Its action on polarized wave functions turns out to be:

\[
\tilde{X}_c^R' \Psi = \zeta e^{-i\gamma \tau} \left[ \frac{d}{d\tau} \right] \Phi(\tau). \quad (46)
\]

The representation of \( SL(2, \mathbb{R}) \) here constructed, as in the case of the 1-sheet hyperboloid, is irreducible but not unitary.

The reason for this lack of unitarity is the same as before, that is, the lack of an invariant measure on the support manifold \( G/G_P \). In fact, the polarization \( P \) is the same as in the case of the 1-sheet hyperboloid, only the vector fields are slightly different, since they come from different pseudo-extensions. Therefore, the wave functions are essentially the same as before, and consequently \( G/G_P \) is the same as in the case of the 1-sheet hyperboloid.

The measure on \( G/G_P \) is again \( \Omega^L_P = i \tilde{X}_b^L i \tilde{X}_a^L \Omega^L = adc - cda = \kappa^2 d\tau \), which does not fall down to the quotient.

Thus, we keep \( \Omega^L_P \) as the measure on \( G/G_P \), and we impose the polarization conditions \( \tilde{X}_g^L \Psi = \frac{i}{2} k_i^{G/G_P} \tilde{\Psi} \), instead of \( \tilde{X}_g^L \Psi = 0 \), \( \forall \tilde{X}_g^L \in P \). In other words, we impose \( \tilde{\Psi} \) to be a \( \frac{1}{2} \)-\( \rho \)-function in such a way that \( \tilde{\Psi}^\ast \tilde{\Psi}' \) is a \( \rho \)-function, \( \tilde{\Psi} \) and \( \tilde{\Psi}' \) being two \( \frac{1}{2} \)-\( \rho \)-functions. Now, \( \tilde{\Psi}^\ast \tilde{\Psi}' \Omega^L_P \) is a well-defined quantity on \( G/G_P \) and can be integrated with respect to \( \tau \).

Modular constants \( k_i^{G/G_P} = k_i^G - k_i^{G_P}, i = 1, \ldots, p \), are the same as before, since \( G_P \) is the same group. Therefore, \( k_a^{G/G_P} = -2 \) and \( k_b^{G/G_P} = 0 \).

The new polarization equations are:

\[
\begin{align*}
\tilde{X}_a^L \tilde{\Psi} &= -\tilde{\Psi}, \\
\tilde{X}_b^L \tilde{\Psi} &= 0.
\end{align*}
\]

with solutions:

\[
\tilde{\Psi}(g) = a^{-1} \Psi(g), \quad (48)
\]

where \( \Psi(g) \) is a solution of the previous (horizontal) polarization equations. Thus, the form of the solutions is:

\[
\tilde{\Psi} = \zeta \kappa^{-1} e^{-i\gamma \kappa \tau} \Phi(\tau). \quad (49)
\]
The right-invariant vector fields, when acting on $\frac{1}{2}\rho$-functions, acquire extra terms restoring the unitarity of the representation:

$$\tilde{X}_i^R \Psi = \kappa^{-1} \tilde{X}_i^R \Psi + \kappa^{-1}(\kappa \tilde{X}_i^R \kappa^{-1}) \Psi.$$ (50)

This way, the final representation restricted to its action on $\Phi(\tau)$ has the form:

$$\tilde{X}_a^R \Phi(\tau) = \left[ -1 - 2\tau \frac{d}{d\tau} \right] \Phi(\tau)$$

$$\tilde{X}_b^R \Phi(\tau) = \left[ -\tau - \tau^2 \frac{d}{d\tau} \right] \Phi(\tau)$$

$$\tilde{X}_c^R \Phi(\tau) = \left[ \frac{d}{d\tau} \right] \Phi(\tau).$$ (51)

Again, we can readily verify that these operators are self-adjoint with respect to the quasi-invariant measure $d\tau$ (what remains of $\Omega_L^\rho$ after multiplication by the factor $\kappa^{-2}$ contained in the wave functions). Therefore, the representation is now unitary.

This representation can be seen as the limit $\alpha \to 0$ of the Principal series of representations. We should stress at this point that the representation does not depend on $\gamma$, nor even on its sign. Therefore, we obtain the same representation for both cones, which are clearly equivalent. The reason for this equivalence is that the group isomorphism $(a, b, c) \to (a, b, -c)$ induces a unitary transformation between the two representations. This representation (up to equivalence) is called the Mock representation and is associated with the two cones.

4.5 Representations associated with the 2-sheets hyperboloids: Discrete Series

According to Sec. 4.2, we can choose the point $\vec{\lambda}^0 = (0, \beta > 0, \gamma = -\beta)$ in the upper sheet and $\vec{\lambda}^0 = (0, \beta < 0, \gamma = -\beta)$ in the lower sheet of the 2-sheets hyperboloid, to define the pseudo-extension of $SL(2, \mathbb{R})$ by $U(1)$. Let us consider $\vec{\lambda}^0 = (0, \beta, -\beta)$, keeping the sign of $\beta$ undetermined for the time being.

The easiest function $\lambda$ on $SL(2, \mathbb{R})$ satisfying $\frac{\partial}{\partial g} \lambda(g)|_{g=e} = \lambda^0$ is the function linear on the coordinate $(b - c)$, since here $b$ and $c$ are true canonical coordinates. Therefore, we fix $\lambda(g) = \beta(b - c)$.

The representation obtained when applying GAQ to the resulting group will be associated with one of the coadjoint orbits for which the Casimir is $C = -\beta^2 < 0$. The resulting group law for $SL(2, \mathbb{R})$, pseudo-extended by $U(1)$ by means of the two-cocycle $\xi_\lambda$, is:

$$a'' = a'a + b'c$$

$$b'' = a'b + b' \frac{1 + bc}{a}$$
\[ c'' = c'a + \frac{1 + b'c'}{d'}c \]

\[ \zeta'' = \zeta' \zeta e^{i\beta((a' - 1)b' - (a' - 1)c' + \frac{1 + bc - a}{a}b' - \frac{1 + bc - a}{a}c')} \]

Left- and right-invariant vector field are:

\[
\begin{align*}
\tilde{X}_a^L &= a \frac{\partial}{\partial a} + c \frac{\partial}{\partial c} - b \frac{\partial}{\partial b} - \beta(b + c)\Xi \\
\tilde{X}_a^R &= a \frac{\partial}{\partial a} + b \frac{\partial}{\partial b} - c \frac{\partial}{\partial c} + \beta(b + c)\Xi \\
\tilde{X}_b^L &= a \frac{\partial}{\partial a} + \beta(a - 1)\Xi \\
\tilde{X}_b^R &= \frac{1 + bc - a}{a} \frac{\partial}{\partial a} + b \frac{\partial}{\partial b} + \beta(1 + bc - a)\Xi \\
\tilde{X}_c^L &= \frac{1 + bc - a}{a} \frac{\partial}{\partial a} + b \frac{\partial}{\partial b} - \beta(1 + bc - a)\Xi \\
\tilde{X}_c^R &= a \frac{\partial}{\partial c} - \beta(a - 1)\Xi \\
\tilde{X}_\zeta^L &= \Xi \\
\tilde{X}_\zeta^R &= \Xi.
\end{align*}
\]

The left-invariant 1-form associated with the variable \( \zeta \) is:

\[
\Theta \equiv \theta^{L(c)} = \frac{d\zeta}{\zeta} + \beta(\theta^{L(b)} - db - \theta^{L(c)} + dc) = \frac{d\zeta}{i\zeta} + \beta \left[ \frac{1 - a}{a}db - (1 + a + \frac{b^2}{a})dc + \frac{b}{a}(1 + bc) - c)da \right].
\]

The resulting Lie algebra is, as in the other cases, the one of \( SL(2, \mathbb{R}) \) with some of the commutators modified, in this case those giving \( \tilde{X}_b^L \) and \( \tilde{X}_c^L \) on the r.h.s.:

\[
\begin{align*}
[\tilde{X}_a^L, \tilde{X}_b^L] &= 2(\tilde{X}_b^L + \beta\Xi) \\
[\tilde{X}_a^L, \tilde{X}_c^L] &= -2(\tilde{X}_c^L - \beta\Xi) \\
[\tilde{X}_b^L, \tilde{X}_c^L] &= \tilde{X}_a^L.
\end{align*}
\]

The 2-form defining a presymplectic structure on \( \tilde{G} \) is:

\[
d\Theta = -2\beta \left[ \frac{b}{a} db \land dc + \frac{1 + bc}{a^2} da \land db + da \land dc \right].
\]

The characteristic subalgebra turns out to be \( \mathcal{G}_C = \langle \tilde{X}_b^L - \tilde{X}_c^L \rangle \). Looking for a polarization subalgebra containing the characteristic subalgebra, we get into trouble, since there is no such real subalgebra. We are forced to complexify the algebra, and then we find (essentially) two complex polarizations:

\[
\mathcal{P} = \langle \tilde{X}_b^L - \tilde{X}_c^L, \tilde{X}_b^L + \tilde{X}_c^L \pm i\tilde{X}_a^L \rangle.
\]

Clearly, the solution to these polarization equations are complex functions defined on a complex submanifold of the complexification of \( SL(2, \mathbb{R}) \). These will be holomorphic or anti-holomorphic, depending on the choice of sign in \( (57) \). The explicit construction of the representations in the discrete series, according to the group quantization framework, was firstly given in Ref. \[17\] in connection to the quantum dynamics of a free particle on Anti-de Sitter space-time. Higher-order, real polarizations were used in Ref. \[18\] in the study of the relativistic harmonic oscillator. They have also been considered in conformal field theory as factor of \( SO(2, 2) \approx SL(2, \mathbb{R}) \otimes SL(2, \mathbb{R}) \) representations \[19\].
Acknowledgements

We thank G. Marmo for very useful comments on Sec. 3.

References

[1] Aldaya, V. and de Azcárraga, J.: J. Math. Phys. 23, 1297 (1982)
[2] Aldaya, V. Navarro-Salas, J. and Ramírez, A.: Commun. Math. Phys. 121, 541-556 (1989)
[3] Aldaya, V., Calixto, M. and Guerrero, J.: Commun. Math. Phys. 178, 399-424 (1996)
[4] Kirillov, A.A.: Elements of the Theory of Representations, Springer-Verlag (1976)
[5] Kostant, B.: Quantization and Unitary Representations, Lecture Notes in Math. 170, Springer-Verlag, Berlin (1970)
[6] Aldaya, V. and Navarro-Salas, J.: Commun. Math. Phys. 139, 433 (1991)
[7] Witten, E.: Commun. Math. Phys.: 114, 1 (1988)
[8] Aldaya, V. and de Azcárraga, J.A.: Int. J. Theo. Phys. 24, 141 (1985)
[9] Aldaya, V., Guerrero, J. and Marmo, G.: Int. J. Mod. Phys. A12, 3 (1997)
[10] Woodhouse, N.: Geometric Quantization, Oxford University Press (1980)
[11] Bargmann, V.: Ann. Math. 59, 1 (1954)
[12] Pressley A., and Segal, G.: Loop Groups, Clarendon Press, Oxford (1986)
[13] G.W. Mackey: Ann. Math. 55, 101 (1952)
[14] Barut, A.O. and Raczka, R. Theory of Group Representations and Applications, World Scientific Publishing, Singapore (1986)
[15] Humphreys, J.E.: Introduction to Lie Algebras and Representation Theory, Graduate Texts in Math. Springer-Verlag (1972)
[16] Levi-Civita, T. and Amaldi, U.: Lezioni di Meccanica Razionale, Zanichelli, Bologna (1974) (reprinted version of 1949 edition)
[17] Aldaya, V., Azcárraga, J.A. de, Bisquert, J. and Cerveró, J.M.: J. Phys. A23, 707 (1990)
[18] Aldaya, V., Bisquert, J., Guerrero, J. and Navarro-Salas, J.: Rep. Math. Phys. 37, 387 (1996)

[19] Aldaya, V., Calixto, M. and Cerveró, J.M.: Commun. Math. Phys. 200, 325 (1999)