Let $V$ be a quadratic space with a form $q$ over an arbitrary local field $F$ of characteristic different from 2. Let $W = V \oplus Fe$ with the form $Q$ extending $q$ with $Q(e) = 1$. Consider the standard embedding $O(V) \hookrightarrow O(W)$ and the two-sided action of $O(V) \times O(V)$ on $O(W)$.

In this note we show that any $O(V) \times O(V)$-invariant distribution on $O(W)$ is invariant with respect to transposition. This result was earlier proven in a bit different form in [vD] for $F = \mathbb{R}$, in [AvD] for $F = \mathbb{C}$ and in [BvD] for $p$-adic fields. Here we give a different proof.

Using results from [AGS], we show that this result on invariant distributions implies that the pair $(O(V), O(W))$ is a Gelfand pair. In the archimedean setting this means that for any irreducible admissible smooth Fréchet representation $(\pi, E)$ of $O(W)$ we have $\dim \text{Hom}_{O(V)}(E, \mathbb{C}) \leq 1$.

A stronger result for $p$-adic fields is obtained in [AGRS07].

\begin{abstract}
Let $V$ be a quadratic space with a form $q$ over an arbitrary local field $F$ of characteristic different from 2. Let $W = V \oplus Fe$ with the form $Q$ extending $q$ with $Q(e) = 1$. Consider the standard embedding $O(V) \hookrightarrow O(W)$ and the two-sided action of $O(V) \times O(V)$ on $O(W)$.

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\section{Introduction}

Let $F$ be a local field of characteristic different from 2.

Let $(W, Q)$ be a quadratic space defined over $F$ and fix $e \in W$ a unit vector. Consider the quadratic space $V = e^\perp$ with $q = Q|_V$. Define the standard imbedding $O(V) \hookrightarrow O(W)$ and consider the two-sided action of $O(V) \times O(V)$ on $O(W)$ defined by $(g_1, g_2)h := g_1hg_2^{-1}$. We also consider the anti-involution $\tau$ of $O_Q$ given by $\tau(g) = g^{-1}$. In this paper we prove the following theorem

\section{Acknowledgements}

\section{From Invariant distributions to Representation theory}

\subsection{Basic Results on Invariant distributions}

\subsection{Bruhat Filtration}

\subsection{Frobenius reciprocity}

\subsection{Bernstein's Localization principle}

\section{Proof of Theorem A}

\subsection{Proof of theorem 4.1}

\section{References}

\textit{Key words and phrases.} Multiplicity one, invariant distribution, orthogonal groups, Gelfand pairs.

\textit{2000 Mathematics Subject Classification Classification:22E45, 20G05, 20G25, 46F99.}
Theorem (A). Any $O(V) \times O(V)$ invariant distribution on $O(W)$ is invariant under $\tau$.

This theorem has the following corollary in representation theory.

Theorem (B). Let $(\pi, E)$ be an irreducible admissible representation of $O(W)$. Then

$$\dim \text{Hom}_{O(V)}(E, \mathbb{C}) \leq 1$$

Here admissible representation refers to the usual notion in the non-archimedean case and to the notion of admissible smooth Fréchet representation in the archimedean setting.

Our proof for the archimedean and non-archimedean case is uniform, except at one point where the archimedean case requires an extra analysis of a certain normal bundle (see lemma 4.2).

Remark 1.1. We note that a related result for unitary representations of $SO(V, Q)$ is proved in [BvD] (for $p$-adic fields) and in [vD] (for the real numbers). In fact, the proof given in those papers implies also theorem A. Also, an analogous theorem for unitary groups is proven in [vD2].

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2. From Invariant distributions to Representation theory

In this section we recall a technique due to Gelfand and Kazhdan which allows to deduce theorem B from theorem A.

Recall the following theorem ([AGS])

Theorem 2.1. Let $H \subset G$ be reductive groups and let $\tau$ be an involutive anti-automorphism of $G$ and assume that $\tau(H) = H$. Suppose $\tau(T) = T$ for all $H$-invariant distributions on $G$. Then for any irreducible admissible representation $(\pi, E)$ of $G$ we have

$$\dim \text{Hom}_H(E, \mathbb{C}) \cdot \dim \text{Hom}_H(\bar{E}, \mathbb{C}) \leq 1,$$

where $\bar{E}$ denotes the smooth contragredient representation.

Note that in the non-archimedean case the same result is proven in [Pra].

To finish the deduction of theorem B from theorem A we will show that

Theorem 2.2. Let $(\pi, E)$ be an irreducible admissible representation of $G = O(V)$. Then $\bar{E} \cong E$ and in particular

$$\dim \text{Hom}_H(E, \mathbb{C}) = \dim \text{Hom}_H(\bar{E}, \mathbb{C})$$

For the proof we recall proposition I.2 (chapter 4) from [MVW]:

Proposition 2.3. Let $V$ be a quadratic space and let $g \in O(V)$. Then $g$ is conjugate to $g^{-1}$.

\footnote{In fact it is enough to check this only for Schwartz distributions.}
Proof of Theorem 2.2. For non-archimedean fields this is a theorem from [MVW] page 91. For archimedean fields we use the Harish-Chandra regularity theorem and the proposition that any element in \( g \in O(V) \) is conjugate in \( O(V) \) to \( g^{-1} \). Thus, the characters of \( E \) and \( \bar{E} \) are the same and hence \( \bar{E} \cong E \).

**Remark 2.4.** A related result for the groups \( SO(V) \) can be found in [GP], proposition 5.3.

3. Basic Results on Invariant distributions

In this paper we consider distributions over \( l \)-spaces and over smooth manifolds. \( l \)-spaces are locally compact totally disconnected topological spaces (see [BZ], section 1).

For \( X \) a smooth manifold or an \( l \)-space we denote by \( \mathcal{D}(X) \) the space of distributions on \( X \). When \( X \) is an \( l \)-space this means that \( \mathcal{D}(X) = S(X)^* \) where \( S(X) \) is the space of locally constant functions with compact support on \( X \). For smooth \( X \), we let \( \mathcal{D}(X) = C_c^\infty(X)^* \).

The basic tools to study invariant distributions on a \( G \)-space \( X \) are Bruhat filtration, Frobenius reciprocity ([BZ], [Bar] and [AGS]) and the Bernstein’s localization principle ([Ber] and [AG]). Let us remind the statements.

For the simplicity of formulation we provide, for each principle, two versions: for \( l \)-spaces and for smooth manifolds.

3.1. Bruhat Filtration. Although we will not need the non-archimedean version of this principle, we formulate it for completeness. It is a simple consequence of proposition 1.8 in [BZ].

**Theorem 3.1.** Let an \( l \)-group \( G \) act on an \( l \)-space \( X \). Let \( X = \bigcup_{i=0}^l X_i \) be a \( G \)-invariant stratification of \( X \). Let \( \chi \) be a character of \( G \). Suppose that \( \mathcal{D}(X_i)^{G,\chi} = 0 \). Then \( \mathcal{D}(X)^{G,\chi} = 0 \).

To formulate the archimedean version we let \( X \) be a smooth manifold and \( Y \subset X \) a smooth submanifold. We remind the definition of the conormal bundle \( CN_X^Y \). For this denote by \( T_X \) the tangent bundle of \( X \) and by \( N_X^Y := (T_X|_Y)/T_Y \) the normal bundle to \( Y \) in \( X \). The conormal bundle is defined by \( CN_X^Y := (N_X^Y)^* \). Denote by \( Sym^k(CN_X^Y) \) the \( k \)-th symmetric power of the conormal bundle.

**Theorem 3.2.** Let a real reductive group \( G \) act on a smooth affine real algebraic variety \( X \). Let \( X = \bigcup_{i=0}^l X_i \) be a smooth \( G \)-invariant stratification of \( X \). Let \( \chi \) be an algebraic character of \( G \). Suppose that for any \( k \in \mathbb{Z}_{\geq 0} \) and any \( 0 \leq i \leq l \) we have \( \mathcal{D}(X_i, Sym^k(CN_X^Y_i))^{G,\chi} = 0 \). Then \( \mathcal{D}(X)^{G,\chi} = 0 \).

For proof see [AGS], section B.2.

3.2. Frobenius reciprocity. For \( l \)-space, the following version of Frobenius reciprocity is proven in [Ber].

**Theorem 3.3** (Frobenius reciprocity). Let a unimodular \( l \)-group \( G \) act transitively on an \( l \)-space \( Z \). Let \( \varphi : X \to Z \) be a \( G \)-equivariant continuous map. Let \( z \in Z \). Suppose that its stabilizer \( Stab_G(z) \) is unimodular. Let \( X_z \) be the fiber of \( z \). Let \( \chi \) be a character of \( G \). Then \( \mathcal{D}(X)^{G,\chi} \) is canonically isomorphic to \( \mathcal{D}(X_z)^{Stab_G(z),\chi} \).

An archimedean version is considered in [Bar]. Here is a slight generalization (see [AGS]):
Theorem 3.4 (Frobenius reciprocity). Let a unimodular Lie group $G$ act transitively on a smooth manifold $Z$. Let $\phi: X \to Z$ be a $G$-equivariant smooth map. Let $z \in Z$. Suppose that its stabilizer $\text{Stab}_G(z)$ is unimodular. Let $X_z$ be the fiber of $z$. Let $\chi$ be a character of $G$. Then $\mathcal{D}(X)^{G,z}$ is canonically isomorphic to $\mathcal{D}(X_z)^{\text{Stab}_G(z),z}$. Moreover, for any $G$-equivariant bundle $E$ on $X$, $\mathcal{D}(X,E)^{G,x}$ is canonically isomorphic to $\mathcal{D}(X_z,E|_{X_z})^{\text{Stab}_G(z),x}$.

3.3. Bernstein’s Localization principle. For $l$-spaces it is taken from [Ber]:

Theorem 3.5 (Localization principle). Let $X$ and $T$ be $l$-spaces and $\phi: X \to T$ be a continuous map. Let an $l$-group $G$ act on $X$ preserving the fibers of $\phi$. Let $\chi$ be a character of $G$. Suppose that for any $t \in T$, $\mathcal{D}(\phi^{-1}(t))^{G,x} = 0$. Then $\mathcal{D}(X)^{G,x} = 0$.

For real smooth algebraic varieties, the following theorem is proven in [AG], Corollary A.0.3:

Theorem 3.6 (Localization principle). Let a real reductive group $G$ act on a smooth affine real algebraic variety $X$. Let $Y$ be a smooth real algebraic variety and $\phi: X \to Y$ be an algebraic $G$-invariant submersion. Suppose that for any $y \in Y$ we have $\mathcal{D}(\phi^{-1}(y))^{G,x} = 0$. Then $\mathcal{D}(X)^{G,x} = 0$.

4. Proof of Theorem A

Recall the setting. $(W,Q)$ is a quadratic space over $F$, $e \in W$ with $Q(e) = 1$. Also $(V,q)$ is defined by $V = e^+$ and $q = Q|_V$.

We need some further notations.
- $O_q = O(V,q)$ is the group of isometries of the quadratic space $(V,q)$.
- $G_q = O(V,q) \times O(V,q)$.
- $\Delta: O_q \to G_q$ the diagonal. $H_q = \Delta(O_q) \subset G_q$.
- $\tau(g_1, g_2) = (g_2, g_1)$.
- $\tilde{G}_q = G_q \times \{1, \tau\}$, same for $\tilde{H}_q$.
- $\chi: \tilde{G}_q \to \{+1, -1\}$ the non trivial character with $\chi(G_q) = 1$.
- $\tilde{G}_q$ acts on $O_Q$ by $(g_1, g_2)x = g_1xg_2^{-1}$ and $\tau(x) = x^{-1}$.

Clearly Theorem A follows from the following theorem:

Theorem 4.1. $\mathcal{D}(O_Q)^{\tilde{G}_q,x} = 0$

4.1. Proof of theorem 4.1 We denote by $\Gamma = \{w \in W : Q(w) = 1\}$. Note that by Witt’s theorem $\Gamma$ is an $O_Q$ transitive set and therefore $\Gamma \times \Gamma$ is a transitive $G_Q$ set where the action of $G_Q$ is the standard action on $W \oplus W$ and $\tau$ acts by flip.

Applying Frobenius reciprocity [3.3, 3.4] to projections of $O_Q \times \Gamma \times \Gamma$ first on $\Gamma \times \Gamma$ and then on $O_Q$ we have

$$\mathcal{D}(O_Q)^{\tilde{G}_q,x} = \mathcal{D}(O_Q \times \Gamma \times \Gamma)^{\tilde{G}_q,x}$$

and also that

$$\mathcal{D}(O_Q \times \Gamma \times \Gamma)^{\tilde{G}_q,x} = \mathcal{D}(\Gamma \times \Gamma)^{\tilde{H}_q,x}$$

In what follows we will abuse notation and write $Q(u,v)$ for the bilinear form defined by $Q$. Define a map $D: \Gamma \times \Gamma \to Z$ where $Z = \{(v, u) \in W \oplus W : Q(v, u) = 0, Q(v + u) = 4\}$ by

$$D(x, y) = (x + y, x - y).$$
$D$ defines an $\tilde{G}_Q$-equivariant homeomorphism and thus we need to show that
\[
D(Z)^{\tilde{\alpha}} = 0
\]
Here, the action of $G_Q$ on $Z \subset W \oplus W$ is the restriction of its action on $W \oplus W$ while the action of $\tau$ is given by $\tau(v, u) = (v, -u)$.

Now we cover $Z = U_1 \cup U_2$ where
\[
U_1 = \{(v, u) \in Z : Q(v) \neq 0\}
\]
and
\[
U_2 = \{(v, u) \in Z : Q(u) \neq 0\}
\]
We will show $D(U_1)^{\tilde{\alpha}} = 0$, and the proof for $U_2$ is analogous. This will finish the proof.

**Lemma 4.2.** $D(U_1)^{\tilde{\alpha}} = 0$

**Proof for non-archimedean $F$.** Consider $\ell_1 : U_1 \rightarrow F - \{0\}$ defined as $\ell_1(v, u) = Q(v)$. By the localization principle, it is enough to show $D(U_1^{\alpha})^{\tilde{\alpha}} = 0$ where $U_1^{\alpha} = \ell_1^{-1}(\alpha)$, for any $\alpha \in F - \{0\}$.

Let $W^\alpha = \{w \in W : Q(w) = \alpha\}$ and let $p_1 : U_1^{\alpha} \rightarrow W^\alpha$ be given by $p_1(v, u) = v$. On $W^\alpha$ our group acts transitively. Fix a vector $v_0 \in W^\alpha$.

Denote $H(v_0) := H(Q|v_0)$ and $\tilde{H}(v_0) := \tilde{H}(Q|v_0)$.

The stabilizer in $\tilde{H}$ of $v_0$ is $\tilde{H}(v_0)$. The fiber $p_1^{-1}(v_0) = \{a \in v_0 | Q(a) = 4 - \alpha\}$. Frobenius reciprocity implies that
\[
D(U_1^{\alpha})^{\tilde{\alpha}} = D(p_1^{-1}(v_0))^{\tilde{H}(v_0), \chi}
\]
But clearly $D(p_1^{-1}(v_0))^{\tilde{H}(v_0), \chi} = 0$ as $-Id \in H(v_0)$.

**Proof for archimedean $F$.** Now let us consider the archimedean case. Define $U := \{(v, u) \in U_1 | u \neq 0\}$. Note that the map $\ell_1|_U$ is a submersion, so the same argument as in the non-archimedean case shows that $D(U)^{\tilde{\alpha}} = 0$. Let $Y := \{(v \in W : Q(v) = 4) \times \{0\}\}$ be the complement to $U$ in $U_1$. By theorem 3.2 it is enough to prove $D(Y, \text{Sym}^k(CN_{Y}^{(1)}))^{\tilde{\alpha}} = 0$.

Note that the action of $\tilde{H}$ on $Y$ is transitive, and fix a point $(v, 0) \in Y$. The stabilizer in $\tilde{H}$ of $(v, 0)$ is $\tilde{H}(v)$, and the normal space to $Y$ at $(v, 0)$ is $v^\perp$. So Frobenius reciprocity (theorem 3.4) implies that
\[
D(Y, \text{Sym}^k(CN_{Y}^{(1)}))^{\tilde{\alpha}} = \text{Sym}^k(v^\perp)^{\tilde{\alpha}}
\]
But clearly $\text{Sym}^k(v^\perp)^{\tilde{\alpha}} = 0$ as $-Id \in H(v)$.

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