Simplicial Endomorphisms

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Abstract
The monoids of simplicial endomorphisms, i.e. the monoids of endomorphisms in the simplicial category, are submonoids of monoids one finds in Temperley-Lieb algebras, and as the monoids of Temperley-Lieb algebras are linked to situations where an endofunctor is adjoint to itself, so the monoids of simplicial endomorphisms are linked to arbitrary adjoint situations. This link is established through diagrams of the kind found in Temperley-Lieb algebras. Results about these matters, which were previously prefigured up to a point, are here surveyed and reworked. A presentation of monoids of simplicial endomorphisms by generators and relations has been given a long time ago. Here a closely related presentation is given, with completeness proved in a new and self-contained manner.

Mathematics Subject Classification (2000): 55U10, 20M50, 57M99, 18G30, 18C15, 18A40

Keywords: simplicial category, endomorphisms, presentation by generators and relations, Temperley-Lieb algebras, adjunction, monads, triples

1 Introduction

The simplicial category $Δ$, whose arrows are all order-preserving functions from finite ordinals to finite ordinals, plays a central role in topology. It stands behind many notions of algebraic topology, and especially those having to do with homology (see [26], VII.5, and references therein).

For every object $n \geq 0$ of $Δ$ the endomorphisms $f : n \to n$ of $Δ$ make a monoid (i.e. semigroup with unit), which has been considered in semigroup
theory under the name $O_n$ (see [12]). Multiplication in this monoid is composition of functions, and the unit is the identity function. The goal of the first part of this paper is to axiomatize, i.e. present by generators and relations, the monoids $O_n$ for every $n$ (in particular, for every $n \geq 2$, when $O_n$ is not trivial). A related presentation of $O_n$ has been given a long time ago in [2], and following [2] a number of related monoids have been presented and investigated (see [12] and references therein). Although it is related to this older presentation, our presentation of $O_n$, and our method of proving its completeness, have some features, mentioned in the next section, that hopefully make worthwhile their publication. Matters concerning this presentation will be exposed here in a self-contained manner, so as to make the whole paper more self-contained.

The interest of presenting $O_n$ is in the following. The monoids $O_n$ are submonoids of monoids one finds in Temperley-Lieb algebras (with vector multiplication), and these algebras play an important role in knot theory and low-dimensional topology in the wake of the approach to knot and link invariants through Jones’ polynomial (see [19], [25], [8] and references therein). The monoids of Temperley-Lieb algebras are monoids of endomorphisms in categories where an endofunctor is adjoint to itself (as shown in [8]). The monoids $O_n$, on the other hand, arise in adjoint situations in general.

The paper will be organized as follows. In the first part of the paper (Sections 2-3), the monoids $O_n$ are presented, and it is proved that they are the monoids of endomorphisms of $\Delta$. This proof is based on the presentation of elements of $O_n$ in a certain normal form.

In the remaining part of the paper, the results of the first part are put into context. First (Section 4), it is shown that $\Delta$ is isomorphic to the free monad (or triple) generated by a single object, and next (Sections 5-8), it is shown how $\Delta$, and hence also $O_n$, arise in adjunction. For all of that we exploit the technique of composition elimination, analogous to the proof-theoretical technique of cut elimination (see [7]). Diagrams of the kind found in Temperley-Lieb algebras, which are bound to any adjoint situation, are introduced in Section 6 under the name friezes, and it is shown how they are linked to $\Delta$ and $O_n$. At the end of the paper (Section 9), the monoids of Temperley-Lieb algebras are presented, and it is shown that the monoids $O_n$ are submonoids of them. We also consider there the standard presentation of $\Delta$ by generators and relations, and how it is related to adjunction.

The presentation of $O_n$ by generators and relations, and the proof of its completeness, in the first part of the paper, though they differ from that of [2], cannot be counted as entirely new results, but the results concerning $O_n$ in the remaining part of the paper do not seem to have been explicitly registered before. Many of the results in this remaining, contextual, part of the paper are however not quite new, and earlier references I know about will be mentioned at appropriate places in the text. Some things have been clearly realized before, while others have been prefigured, more or less clearly. I didn’t manage, however, to find them all put together. Various authors pay attention to
various things. Here an attempt is made to bridge matters, and present them systematically, without extraneous material and, to a great extent, in a self-contained manner. What is maybe most original in this part is the exhibition of the connection between friezes and order-preserving functions on the set of natural numbers in Section 6, which serves to link adjunction to \( \Delta \) and \( \mathcal{O}_n \).

## 2 The monoids \( \mathcal{O}_n \)

The simplicial category \( \Delta \) has as objects all finite ordinals (including 0) and as arrows all order-preserving functions (see [26], VII.5); namely, for \( \varphi : n \to m \) and \( i, j \in \{0, \ldots, n-1\} \), if \( i \leq j \), then \( \varphi(i) \leq \varphi(j) \). (The empty function \( \emptyset : 0 \to m \) is also order-preserving.) The category \( \Delta \) is a strict monoidal category (see [26], VII.1) with the bifunctor \( + : \Delta \times \Delta \to \Delta \), which is addition on objects, while for \( \varphi : n \to n' \) and \( \psi : m \to m' \) we have

\[
(\varphi + \psi)(i) = \begin{cases} 
\varphi(i) & \text{if } 0 \leq i \leq n-1 \\
n' + \psi(i-n) & \text{if } n \leq i \leq n+m-1.
\end{cases}
\]

The unit object of \( \Delta \) is the ordinal 0 (which is also an initial object).

For every object \( n \) of \( \Delta \), the endomorphisms \( f : n \to n \) of \( \Delta \) make a monoid with composition and the identity function \( 1 : n \to n \). We shall now axiomatize this monoid, i.e. present it by generators and relations.

For every \( n \geq 0 \), let us designate this monoid by \( \mathcal{O}_n \). Although, in principle, we allow \( n \) here to be lesser than 2, the interesting monoids \( \mathcal{O}_n \) must have \( n \geq 2 \). The generators of \( \mathcal{O}_n \) are the right-forking terms \( p^i \) and the left-forking terms \( q^i \), for every \( i \) such that \( 0 \leq i \leq n-2 \), provided \( n \geq 2 \). If \( n < 2 \), then we don’t have any of these generators.

The right-forking term \( p^i \) stands for the right-forking endomorphism \( \sigma(p^i) : n \to n \) of \( \Delta \), which satisfies \( \sigma(p^i)(j) = j \) for every \( j \in \{0, \ldots, n-1\} \) different from \( i+1 \), while \( \sigma(p^i)(i+1) = i \). Diagrammatically, we have

\[
\begin{array}{cccccccc}
0 & \cdots & i-1 & i & i+1 & i+2 & \cdots & n-1 \\
0 & \cdots & i-1 & i & i+1 & i+2 & \cdots & n-1
\end{array}
\]

The left-forking term \( q^i \) stands for the left-forking endomorphism \( \sigma(q^i) : n \to n \) of \( \Delta \), which satisfies \( \sigma(q^i)(j) = j \) for every \( j \in \{0, \ldots, n-1\} \) different from \( i \), while \( \sigma(q^i)(i) = i+1 \). Diagrammatically, we have
The generators \( p^i \) and \( q^i \) correspond to the generators \( v_{n-i} \) and \( u_i \), respectively, of [2] (see also [12], §2). The more complicated indexing of \( v_{n-i} \) stresses the left-right duality, on which [2] relies for the proof of completeness of the presentation. When comparing the presentation below with the closely related presentation of [2] one should also bear in mind that multiplication, i.e. composition, is written there in reverse order, and that instead of the finite ordinal \( n \), which is equal to \( \{0, \ldots, n-1\} \), one has \( \{1, \ldots, n\} \). The presentation of [2] is more economical, while ours is separative, which means that one could easily obtain from it the presentations of the two submonoids generated by right-forking terms alone and left-forking term alone, should one wish to consider these submonoids. The proof of completeness for our presentation can easily be adapted to prove the completeness of the presentations of these submonoids.

The terms of \( O_n \) are obtained from these generators, together with the special unit term \( 1 \), with the help of the binary operation of composition \( \circ \). The unit term \( 1 \) stands for the identity function \( 1_n : n \to n \) of \( \Delta \). For terms of \( O_n \) we use the letters \( x, y, z, t, \ldots, x_1, \ldots \)

We assume the following equations for \( O_n \):

\[
\begin{align*}
(1) & \quad x \circ 1 = x, \quad 1 \circ x = x, \\
(2) & \quad x \circ (y \circ z) = (x \circ y) \circ z, \\
(p1) & \quad p^j \circ p^j = p^j \circ p^j, \quad \text{for } j+1 < i, \\
(p2) & \quad p^j \circ p^i = p^i, \\
(p3) & \quad p^i \circ p^{i+1} \circ p^i = p^{i+1} \circ p^i \circ p^{i+1} = p^i \circ p^{i+1}, \\
(q1) & \quad q^i \circ q^i = q^i \circ q^i, \quad \text{for } j+1 < i, \\
(q2) & \quad q^i \circ q^i = q^i, \\
(q3) & \quad q^i \circ q^{i+1} \circ q^i = q^{i+1} \circ q^i \circ q^{i+1} = q^{i+1} \circ q^i, \\
(pq) & \quad p^j \circ q^j = q^j \circ p^j, \quad \text{for } j < i \text{ or } i+1 < j, \\
(pq1) & \quad p^i \circ q^i = p^i, \\
(pq2) & \quad p^i \circ q^{i+1} = q^{i+1}, \\
(gp1) & \quad q^i \circ p^i = q^i, \\
(gp2) & \quad q^{i+1} \circ p^i = p^i.
\end{align*}
\]

For \( 0 \leq j \leq i \leq n-2 \), the \( p\)-block \( p^{[i,j]} \) is defined as \( p^i \circ p^{i-1} \circ \ldots \circ p^1 \circ p^j \); we define analogously the \( q\)-block \( q^{[i,j]} \) as \( q^i \circ q^{i-1} \circ \ldots \circ q^{i+1} \circ q^j \). The \( p\)-block
p_{[i,j]}^{[k,l]}, which is defined as $p_i^j$, will be called singular, and analogously for $q$-blocks. A block is either a $p$-block or a $q$-block.

Although the definitions $p$-blocks and $q$-blocks are quite analogous, the two notions are not symmetrical. This asymmetry will become quite clear in the next section where we consider the corresponding endomorphisms after the $p$-Points Lemma and the $q$-Points Lemma. The proof of completeness for the presentation of [2], which otherwise has a similar inspiration as ours, uses an analogue of our $q$-blocks and an order-reversed symmetrical notion involving right-forking terms instead of our $p$-blocks. Our asymmetrical proof of completeness given below hopefully sheds some new light on the matter. One could imagine a third possibility for this completeness proof, symmetrical as that of [2], but different, since it would be based on $p$-blocks and an order-reversed symmetrical notion involving left-forking terms instead of our $q$-blocks.

A term of $O_n$ is in normal form when it is either 1 or it is of the form

$$r_{[i_1,j_1]} \circ \ldots \circ r_{[i_m,j_m]}$$

where $m \geq 1$, for every $k \in \{1, \ldots, m\}$ we have that $r_k$ is either $p$ or $q$, and, in case $m \geq 2$, for every $k \in \{1, \ldots, m-1\}$

- if $r_k$ and $r_{k+1}$ are both $p$ or both $q$, then $i_k < i_{k+1}$ and $j_k < j_{k+1}$;
- if $r_k$ is $p$ and $r_{k+1}$ is $q$, then $i_k+1 < j_{k+1}$;
- if $r_k$ is $q$ and $r_{k+1}$ is $p$, then $i_k < j_{k+1}$.

(This normal form is inspired by Jones’ normal form of [17], §4.1.4, p. 14; see also [3], §1, and [8], §10. It stems ultimately from the normal form for symmetric groups suggested by [6], Note C.)

For the sake of definiteness, we require that in our normal form all parentheses are associated to the left (but another arrangement of parentheses would do as well). In the reduction to normal form below we will not bother about trivial considerations concerning parentheses. The associativity equation (2) guarantees that we can move them at will, as it dispensed us from writing parentheses in $(p3)$ and $(q3)$.

That every term of $O_n$ is equal to a term in normal form will be demonstrated with the help of an alternative formulation of $O_n$, called the block formulation, which is obtained as follows. Instead of the terms $p^i$ and $q^j$, we take the $p$-blocks and the $q$-blocks as generators, we generate terms with these generators, $r \circ s$, and to the monoid equations (1) and (2) we add the following equations:

**pp equations**

(Ipp) for $k+1 < j$,

$$p_{[i,j]}^{[k,l]} \circ p_{[k,l]}^{[i,j]} = p_{[k,l]}^{[k,l]} \circ p_{[i,j]}^{[k,l]}$$

(IIpp) for $k \leq j \leq k+1$,

$$p_{[i,j]}^{[i,j]} \circ p_{[i,j]}^{[i,j]} = p_{[i,j]}^{[i,j]}$$
The equation (\ref{eq:iq}) for \(i = j \) and \(k = l\); the equation (\ref{eq:jq}) is (\ref{eq:pp}) for \(i = j = k = l\); and the equations (\ref{eq:qs}) are obtained from (\ref{eq:pp}) with \(i = j = k - 1 = l\), and from (\ref{eq:pp}) with \(i = j + 1 = k = l\).
The equation \((q_1)\) is \((Iqq)\) for \(i = j\) and \(k = l\); the equation \((q_2)\) is \((IIqq)\) for \(i = j = k = l\); and the equations \((q_3)\) are obtained from \((IIqq)\) with \(i = j = k + 1 = l\), and from \((IIqq)\) with \(i = j + 1 = k = l\).

The equation \((pq)\) is obtained from \((Ipq)\) and \((Iqp)\) with \(i = j\) and \(k = l\). The equation \((pq_1)\) is \((II.2.1pq)\) for \(i = j = k = l\), and the equation \((pq_2)\) is \((II.4pq)\) for \(i = j = k = l\), and, finally, the equation \((qp_2)\) is \((II.3.1qp)\) for \(i = j = k + 1 = l + 1\).

We have to verify too that in the block formulation we can deduce the definitions of the blocks with the terms \(p\) and \(q\) replaced by singular blocks; namely, we have to verify
\[
p[i,j] = p[i,i] \circ p[i-1,i-1] \circ \ldots \circ p[i+1,j+1] \circ p[j,j],
\]
and analogously for \(p\) replaced by \(q\). This follows readily from \((IIpp)\) and \((IIqq)\).

Note that in all that we have not used \((III.1pp)\) and most of the equations of \((IIpp)\) and \((IIqp)\). (We have put these superfluous equations in the block formulation to facilitate the proof of the Normal Form Lemma below.)

To finish showing that the block formulation of \(O_n\) is equivalent to the old formulation, we have to verify that with blocks defined via the terms \(p^i\) and \(q^i\), we can deduce all the equations of the block formulation from the old equations. This is a lengthy, though pretty straightforward exercise. For that exercise it is useful to establish by induction on \(i-j\) that in the old formulation we have the equations
\[
q[i,j] \circ q[i,j+1] = q[i-1,j] \circ q[i,j] = q[i,j] \circ q[i,j] = q[i,j], \\
p[i,j] \circ q[i+1,j+1] = q[i+1,j+1] \circ q[i,j], \\
q[i,j] \circ p[i,j] = q[i,j].
\]

Our presentation of \(O_n\) is such as to facilitate the proof of the equations of the block formulation. This, and the wish to separate the two submonoids generated by right-forking terms alone and left-forking terms alone, make us keep redundant equations. To show that the equations \((p1)\) and \((q1)\), as well as the equations \((p2)\) and \((q2)\), which can easily be derived from \((pq1)\) and \((qp1)\), are redundant, and to remove redundancies in the equations \((p3)\) and \((q3)\), one has the derivations in [2] (§2).

Then we can prove the following lemma.

**Normal Form Lemma.** Every term of \(O_n\) is equal to a term in normal form.

**Proof.** We will present a reduction procedure that transforms every term \(t\) of \(O_n\) into a term \(t'\) in normal form such that \(t = t'\) in \(O_n\). In proof-theoretical jargon, we establish that this procedure is strongly normalizing; namely, that every sequence of reductions terminates in a term in normal form. (This proves more than what is needed for the lemma: it would be enough for us if for every term some sequence of reductions terminates in a term in normal form.)
procedure starts by translating a given term of $O_n$ into a term of the block formulation, which is achieved by replacing the terms $p^i$ and $q^i$ with singular blocks.

Take a term in the block formulation of $O_n$, and with $r$ standing for either $p$ or $q$, let subterms of this term of the following forms be called redexes:

\[(rr) \ r_{i,j}^{[i,j]} r_{k,l}^{[k,l]} \text{ for } k \leq i \text{ or } l \leq j,
\]

\[(pq) \ p_{k,l}^{[k,l]} q_{i,j}^{[i,j]} \text{ for } j \leq k + 1,
\]

\[(qp) \ q_{i,j}^{[i,j]} p_{k,l}^{[k,l]} \text{ for } l \leq i,
\]

\[(1) \ x \cdot 1, 1 \cdot x.\]

A reduction of the $rr$ sort consists in replacing a redex of the form $(rr)$ by the corresponding term on the right-hand side of one of the $pp$ equations or $qq$ equations. We define analogously reductions of the sorts $pq$, $qp$ and $1$ via the $pq$, $qp$ and $1$ equations, respectively.

Let the weight of a block $r_{i,j}^{[i,j]}$ be $i - j + 2$, and for a term $t$ in the block formulation let $m_1 \geq 0$ be the sum of the weights of all the blocks in $t$. For any subterm $s$ of a term $t$ in the block formulation let $m_2 \geq 0$ be the sum of the weights of all the blocks in $s$. The complexity $\mu(t)$ is the ordered pair $(m_1, m_2)$; such pairs are well-ordered lexicographically.

Then we check that if $t'$ is obtained from $t$ by a reduction, then $\mu(t')$ is strictly smaller than $\mu(t)$. With reductions based on $(1pp)$, $(1qq)$, $(1pq)$, $(1qp)$ and $(1)$, the number $m_2$ diminishes, while $m_1$ doesn’t change. With reductions based on all the remaining equations of the block formulation of $O_n$, except equation (2), the number $m_1$ diminishes. So, by induction on the complexity $\mu(t)$, we obtain that every term of the block formulation is equal to a term without redexes, and it is easy to see that a term of the block formulation is without redexes if and only if it is in normal form.

We will see towards the end of the next section that this normal form is unique, but we need not establish this uniqueness for proving the completeness of our presentation of $O_n$.

3 $O_n$ and $\Delta$

In this section we will show that $O_n$ is the monoid of endomorphisms of $\Delta$ in $n$.

Let $\varphi: n \rightarrow m$ be an arrow of $\Delta$, i.e. an order-preserving function from $n$ to $m$, and for every $j \in \{0, \ldots, m-1\}$ let $\varphi^{-1}(j) = \{i \mid 0 \leq i \leq n-1 \text{ and } \varphi(i) = j\}$. When $\varphi^{-1}(j)$ is empty, $j$ will be called an empty point of $\varphi$, when $\varphi^{-1}(j)$ is a
starting from $\sigma$ Then we can check easily that $\sigma$ Soundness Lemma conclude that $\phi$ Then we define different from $k = \phi$ every $l \in [\phi]$ if $i < j$ that $i$ is a multiple and $j$ an empty point of $\phi$. The e-m and m-e pairs of $\phi$ are called the critical pairs of $\phi$. The weight of a critical pair $(i, j)$ is $j - i$.

The complexity $\nu(\phi)$ is the pair $(n_1, n_2)$ where $n_1 \geq 0$ is the number of empty points of $\phi$, and $n_2 \geq 0$ is the minimal weight among the weights of the critical pairs of $\phi$; if there are no critical pairs of $\phi$, then $n_2$ is 0. The pairs $(n_1, n_2)$ are well-ordered lexicographically. Then we can prove the following lemma.

SURJECTIVITY LEMMA. Every endomorphism $\phi: n \to n$ of $\Delta$ is equal to an endomorphism from $n$ to $n$ of $\Delta$ generated from the endomorphisms $\sigma(\phi') : n \to n$ and $\sigma(\phi') : n \to n$ of $\Delta$, where $i \in \{0, \ldots, n-1\}$, together with the identity function $1_n : n \to n$, with the help of composition.

Proof. We proceed by induction on the complexity $\nu(\phi)$. If $\nu(\phi) = (0, 0)$, then $\phi = 1_n$.

Suppose now that $\nu(\phi) = (n_1, n_2) \neq (0, 0)$. This means that $n_1 \neq 0$ and $n_2 \neq 0$, and among all the critical pairs of $\phi$ take one with the minimal weight $n_2$. Let that pair be $(i, j)$.

Suppose $(i, j)$ is an e-m pair, and let $k = \min \phi^{-1}(i+1)$. Note we have either that $i + 1 = j$, or that $i + 1 < j$ with $i + 1$ being a single point of $\phi$. Then we define the endomorphism $\phi' : n \to n$ of $\Delta$ by taking that $\phi'(l) = \phi(l)$ for every $l \in \{0, \ldots, n-1\}$ different from $k$, and $\phi'(k) = i$. It is easy to check that $\phi = \sigma(\phi') \cdot \phi'$. If $i+1 = j$, then in $\nu(\phi') = (n'_1, n'_2)$ we have that $n'_1 < n_1$, while if $i+1 < j$, then $n'_1 = n_1$ but $n'_2 < n_2$. In both cases $\nu(\phi') < \nu(\phi)$, and we can apply the induction hypothesis to $\phi'$.

Suppose, on the other hand, $(i, j)$ is an m-e pair, and let $k = \max \phi^{-1}(j-1)$. Then we define $\phi' : n \to n$ by taking that $\phi'(l) = \phi(l)$ for every $l \in \{0, \ldots, n-1\}$ different from $k$, and $\phi'(k) = j$. If $i+1 < j$, we check that $\phi = \sigma(\phi^{-1}) \cdot \phi'$. Then we conclude that $\nu(\phi') < \nu(\phi)$, and we apply the induction hypothesis to $\phi'$.

Let $\sigma$ be the map from the terms of $\mathcal{O}_n$ to the arrows of $\Delta$ defined inductively, starting from $\sigma(\phi')$ and $\sigma(\phi')$, by putting

$$\sigma(1) = 1_n, \quad \sigma(x \cdot y) = \sigma(x) \cdot \sigma(y).$$

Then we can check easily that $\sigma$ defines a homomorphism from $\mathcal{O}_n$ to $\Delta$.

SOUNDNESS LEMMA. If $x = y$ in $\mathcal{O}_n$, then $\sigma(x) = \sigma(y)$ in $\Delta$.

For $\phi: n \to n$ an endomorphism of $\Delta$, an empty point $i$ of $\phi$ will be called a
bottom p-point of \( \varphi \) when \( \varphi(i) < i \). For \( i \) a multiple point of \( \varphi \), a member \( j \) of \( \varphi^{-1}(i) \) such that \( i \leq j < \max \varphi^{-1}(i) \) will be called a top p-point of \( \varphi \).

If the diagram of \( \varphi: 8 \to 8 \) is the following one:

```
0 1 2 3 4 5 6 7
0 1 2 3 4 5 6 7
```

then 3, 6 and 7 are bottom p-points and 2, 3 and 6 are top p-points of \( \varphi \). The denominations “bottom” and “top” stem from our putting the domain at the top and the codomain at the bottom of the diagram. The denomination \( p \) is explained by the following lemma.

**p-Points Lemma.** The endomorphism \( \sigma(p^{i,j}) \) has a single bottom p-point \( i+1 \) and a single top p-point \( j \).

Diagrammatically, we have for \( \sigma(p^{i,j}) \)

```
0 j-1 j j+1 j+2 i+1 i+2 n-1
... ... ... ... ...
0 j-1 j j+1 i i+1 i+2 n-1
```

and this proves the lemma.

Let a bottom q-point of \( \varphi \) be a single or multiple point \( i \) of \( \varphi \) such that \( \min \varphi^{-1}(i) < i \). If \( i \) a bottom q-point of \( \varphi \), then \( \min \varphi^{-1}(i) \) will be called a top q-point of \( \varphi \). These definitions are explained by the following lemma.

**q-Points Lemma.** The endomorphism \( \sigma(q^{i,j}) \) has a single bottom q-point \( i+1 \) and a single top q-point \( j \).

It is clear from the diagram of \( \sigma(q^{i,j}) \), which looks as follows:

```
0 j-1 j j+1 ... i i+1 i+2 n-1
... ... ...
0 j-1 j j+1 ... i i+1 i+2 n-1
```

that the lemma holds. Note that a bottom q-point is not necessarily a multiple point as in the diagram. (In the next diagram below, 7 is a bottom q-point that is not multiple.)

Consider now the following term of \( S_{15} \) in normal form
We proceed by induction on \(m\). If \(r_m\) is \(p\) and \(i_m + 1 < k\), or \(r_m\) is \(q\) and \(i_m < k\), then \(\sigma(t)(k) = k\), and if \(r_m\) is \(p\) and \(j_m < k \leq i_m + 1\), then \(\sigma(t)(k) < k\).

**Proof.** We proceed by induction on \(m\). If \(m = 1\), then the lemma is clear from the diagrams of \(\sigma(p_{[i,j]})\) and \(\sigma(q_{[i,j]})\) above. Suppose now \(m > 1\), and the lemma holds for the term \(r_1^{[i_1,j_1]} \cdots r_m^{[i_m,j_m]}\), which we call \(t'\).

If \(r_m\) is \(p\) and \(i_m + 1 < k\), then \(\sigma(r_m^{[i_m,j_m]})(k) = k\). If \(r_m - 1\) is \(p\), then \(i_m - 1 < i_m + 1 < k\), and if \(r_m - 1\) is \(q\), then \(i_m - 1 < i_m + 1 < j_m + 1 \leq i_m + 1 < k\).

If \(r_m\) is \(q\) and \(i_m < k\), then \(\sigma(r_m^{[i_m,j_m]})(k) = k\). If \(r_m - 1\) is \(p\), then \(i_m - 1 + 1 < j_m \leq i_m < k\), and if \(r_m - 1\) is \(q\), then \(i_m - 1 < i_m < k\).

So we obtain

\[
\sigma(t)(k) = \sigma(t')(\sigma(r_m^{[i_m,j_m]})(k)) \\
= \sigma(t')(k) \\
= k, \quad \text{by the induction hypothesis.}
\]

If \(r_m\) is \(p\) and \(j_m < k \leq i_m + 1\), then \(\sigma(r_m^{[i_m,j_m]})(k) = k - 1\). If \(r_m - 1\) is \(p\), then \(j_m - 1 < j_m < k - 1\), and if \(r_m - 1\) is \(q\), then \(i_m - 1 < j_m \leq k - 1\). Then we obtain
σ(t)(k) = σ(t')(σ(r_{[m,j_m]}^i))(k)
= σ(t')(k-1)
≤ k−1, by the induction hypothesis
< k. □

The bottom points of an endomorphism ϕ : n → n of Ω are the bottom p-points and the bottom q-points of ϕ, and the top points of ϕ are the top p-points and the top q-points of ϕ. Note that the identity function 1_n : n → n has neither bottom points nor top points. For terms in normal form other than 1 we have the following lemma.

**Key Lemma.** If t is the term r_{[1]_{i_j}} ... r_{[m,j_m]} of Ω_n in normal form, then i_1+1, ..., i_{m+1} are all the bottom points and j_1, ..., j_{m} are all the top points of σ(t). Moreover, i_k+1, for 1 ≤ k ≤ m, is a bottom p-point or q-point depending on whether r_k is p or q, respectively, and analogously for the top point j_k.

Before embarking on the proof of this lemma, we illustrate it with our diagram

which corresponds to the term p_{[1,0]} p_{[2,1]} p_{[3,3]} q_{[6,5]} q_{[8,6]} p_{[11,9]} of S_{15} in normal form we had above as an example. Then 1+1, 2+1, 3+1 and 11+1 are the bottom p-points of this term, 6+1 and 8+1 are the bottom q-points, 0, 1, 3 and 9 are the top p-points, while 5 and 6 are the top q-points.

**Proof of the Key Lemma.** We proceed by induction on m. If m = 1, then we apply the p-Points Lemma and the q-Points Lemma. Suppose now m > 1 and the lemma holds for the term r_{1_{[i_{j_1}]} ... r_{m-1,j_{m-1}}}^i, which we call t'.

If r_m is p, then i_{m+1} is a bottom point of σ(t), because it is an empty point of σ(r_{[m,j_m]}) and a single point of σ(t'), while σ(t')(i_{m+1}) = i_{m+1}, by the Last Block Lemma (if r_{m-1} is p, then i_{m-1}+1 < i_{m+1}, and if r_{m-1} is q, then i_{m-1} < j_m < i_{m+1}). We also have by the Last Block Lemma that σ(t)(i_{m+1}) < i_{m+1}, and hence i_{m+1} is a bottom p-point of σ(t).

If r_m is p, then σ(r_{[m,j_m]})^i(j_m) = σ(r_{[m,j_m]})(j_m+1) = j_m, and hence σ(t)(j_m) = σ(t')(j_m). So both j_m and j_m+1 belong to σ(t)^{-1}(σ(t')(j_m)) by using the Last Block Lemma (if r_{m-1} is p, then j_m−1 < j_m, and if r_{m-1} is q, then i_{m-1} < j_m). Therefore j_m is a top p-point of σ(t).
If \( r_m \) is \( q \), then \( j_m = \min \sigma (r_m^{i_m,j_m})^{-1}(i_m+1) < i_m+1 \). Since \( \sigma (t')(i_m+1) = i_m+1 \), by the Last Block Lemma, and \( i_m+1 \) is a single point of \( \sigma (t') \), we can conclude that \( j_m = \min \sigma (t)^{-1}(i_m+1) < i_m+1 \). Hence \( i_m + 1 \) is a bottom \( q \)-point and \( j_m \) is a top \( q \)-point of \( \sigma (t) \).

It remains to verify that there are no other bottom and top points in \( \sigma (t) \) greater than \( i_{m-1}+1 \) and \( j_{m-1} \), respectively, save \( i_m+1 \) and \( j_m \). This matter is rather lengthy, but pretty straightforward.

As a consequence of the Key Lemma we have the following two lemmata.

**Auxiliary Lemma.** If \( x \) and \( y \) are terms of \( \mathcal{O}_n \) in normal form and \( \sigma (x) = \sigma (y) \) in \( \Delta \), then \( x \) and \( y \) are the same term.

**Proof.** If \( x \) and \( y \) are different terms of \( \mathcal{O}_n \) in normal form, then, by the Key Lemma, the functions \( \sigma (x) \) and \( \sigma (y) \) must differ with respect to bottom and top points, which entails that they are different. \( \square \)

**Injectivity Lemma.** If \( \sigma (x) = \sigma (y) \) in \( \Delta \), then \( x = y \) in \( \mathcal{O}_n \).

**Proof.** Suppose \( \sigma (x) = \sigma (y) \) in \( \Delta \), and let \( x' \) and \( y' \) be terms of \( \mathcal{O}_n \) in normal form such that \( x = x' \) and \( y = y' \) in \( \mathcal{O}_n \). Such terms exist according to the Normal Form Lemma of the preceding section. Then, by the Soundness Lemma, \( \sigma (x) = \sigma (x') \) and \( \sigma (y) = \sigma (y') \), and hence \( \sigma (x') = \sigma (y') \). From the Auxiliary Lemma we conclude that \( x' \) is the same term as \( y' \), and so \( x = y \) in \( \mathcal{O}_n \). \( \square \)

The Soundness Lemma and the Injectivity Lemma guarantee that \( \sigma \) is a one-one map from \( \mathcal{O}_n \) to \( \Delta \), and the Surjectivity Lemma guarantees that \( \sigma \) is a map onto all the endomorphisms of \( \Delta \). So \( \mathcal{O}_n \) is isomorphic to the monoid of endomorphisms of \( \Delta \) on the object \( n \).

We can also ascertain that for every term \( x \) of \( \mathcal{O}_n \) there is a unique term \( x' \) in normal form such that \( x = x' \) in \( \mathcal{O}_n \). According to the Normal Form Lemma of the preceding section, take that for \( x' \) and \( x'' \) in normal form we have \( x = x' \) and \( x = x'' \) in \( \mathcal{O}_n \). Then we have \( x' = x'' \) in \( \mathcal{O}_n \), and hence, by the Soundness Lemma, \( \sigma (x') = \sigma (x'') \). Then, by the Auxiliary Lemma, \( x' \) and \( x'' \) are the same term.

This solves the word problem for \( \mathcal{O}_n \). To check whether \( x = y \) in \( \mathcal{O}_n \) we could reduce \( x \) and \( y \) to normal form, according to the procedure of the proof of the Normal Form Lemma, and then check whether the normal forms obtained are equal. However, to reduce a term \( x \) of \( \mathcal{O}_n \) to normal form, now that we have established that \( \sigma \) is an isomorphism, we can proceed more efficiently with the endomorphism \( \sigma (x) \). Just find the bottom and top points of \( \sigma (x) \), from which we immediately obtain \( x' \). And to check whether \( x = y \) in \( \mathcal{O}_n \) it is enough to check whether \( \sigma (x) = \sigma (y) \), which we can do without going via the normal form. But to show that \( \sigma \) is an isomorphism we relied essentially on this normal form.

The fact that \( \sigma \) is an isomorphism enables us not only to prove facts about \( \mathcal{O}_n \) by going to \( \Delta \), but also facts about \( \Delta \) by going to \( \mathcal{O}_n \). For example, we can
ascertain that in every endomorphism \( \varphi \) of \( \Delta \) the number of bottom \( p \)-points is equal to the number of top \( p \)-points, that the same holds for \( q \)-points, and that these points follow each other in a regular manner, as shown by the normal form. It is not clear how one could prove that directly in \( \Delta \). We can also ascertain that every endomorphism of \( \Delta \) is completely determined by its bottom and top points.

4 Monads and \( \Delta \)

In this section we will show that \( \Delta \) is isomorphic to the free monad (or triple) generated by a single object. This insight (which perhaps should be traced far back to the Appendice of [16]) may be found in Lawvere’s paper [24] (pp. 148ff; see also [22], p. 95, [1], p. 10, and [7], §5.9).

A monad is defined in a standard manner (see [26], VI) as a quadruple \( \langle M, T, \eta, \mu \rangle \) where \( M \) is a category, \( T \) is a functor from \( M \) to \( M \), while \( \eta \) and \( \mu \) are natural transformations, the first from the identity functor on \( M \) to \( T \) and the second from the composite functor \( TT \) to \( T \), such that the following equations hold:

\[
\mu_a \circ \eta_{Ta} = \mu_a \circ T\eta_a = 1_{Ta}, \\
\mu_a \circ T\mu_a = \mu_a \circ \mu_{Ta}.
\]

In an alternative, equivalent definition (stemming from [22] and akin to a definition of [27], §1.3, Exercise 12, p. 32; see [7], §§5.1.5, 5.1.1 and 5.7.3, for the exact relationship), a monad is a quadruple \( \langle M, T, H, M \rangle \) where

\( M \) is a category,
\( T \) is a function from the objects of \( M \) to the objects of \( M \),
\( H \) is a function that maps the arrows \( f : a \to b \) of \( M \) to the arrows \( HF : a \to Tb \) of \( M \),
\( M \) is a function that assigns to every object \( b \) of \( M \) a function \( Mb \) that maps the arrows \( f : a \to Tb \) of \( M \) to the arrows \( Mb\ f : Ta \to Tb \) of \( M \),

and the following equations hold:

\[
(H) \quad Hg \circ f = H(g \circ f), \quad \text{for } f : a \to b \text{ and } g : b \to c, \\
(M) \quad Mg \circ Mf = M(Mg \circ f), \quad \text{for } f : a \to Tb \text{ and } g : b \to Tc, \\
(HM) \quad Mg \circ Hf = g \circ f, \quad \text{for } f : a \to b \text{ and } g : b \to Tc, \\
(MH) \quad MH1_a = 1_{Ta}
\]

with indices appropriately assigned to \( M \) in (\( M \)), (\( HM \)) and (\( MH \)). The two notions of monad are equivalent with the following definitions:
\[ Hf = \text{def } \eta_b \circ f, \quad \text{for } f : a \to b, \]
\[ Mbf = \text{def } \mu_b \circ Tf, \quad \text{for } f : a \to Tb, \]
\[ Tf = \text{def } Mbf Hf, \quad \text{for } f : a \to b, \]
\[ \eta_a = \text{def } H1_a, \quad \mu_a = \text{def } M1_a T a. \]

The category \( \mathcal{M}_0 \) of the free monad generated by the set of objects \{0\} will have as objects finite ordinals. The function \( T \) on objects is defined by \( Tn = a_n n+1 \). The arrow terms of \( \mathcal{M}_0 \), which we call simply \( \text{terms} \), are defined inductively as follows:

- \( 1_n : n \to n \) is a term;
- if \( f : n \to m \) and \( g : m \to k \) are terms, then \( (g \circ f) : n \to k \) is a term;
- if \( f : n \to m \) is a term, then \( Hf : n \to m+1 \) is a term;
- if \( f : n \to m+1 \) is a term, then \( Mf : n+1 \to m+1 \) is a term.

The expression “\( f : n \to m \)” is an abbreviation for “\( f \) of type \( n \to m \)”. (The specification of the type \( n \to m \) belongs to the metalanguage; in the object language we have only the terms \( f \).) As usual, we omit the outermost parentheses in \( (g \circ f) \).

Since \( T \) is here a one-one function on objects, we don’t need to index \( M \) (its index is recovered from the type of \( f \) in \( Mf \)). We impose the following equations on terms:

\[
\begin{align*}
(\text{cat } 1) \quad f \circ 1_n &= f, \\
(\text{cat } 2) \quad h \circ (g \circ f) &= (h \circ g) \circ f,
\end{align*}
\]

and the equations \((H), (M), (HM)\) and \((MH)\). (The formal construction of free monads, in particular those generated by sets of objects, which may be conceived either as arrowless graphs or as discrete categories, and the precise sense in which these monads are free, are explained in [7], §§5.3-6. Formally, the arrows of \( \mathcal{M}_0 \) are equivalence classes of arrow terms, as the elements of \( O_n \) are equivalence classes of terms of \( O_n \).)

For the category \( \mathcal{M}_0 \) of the free monad we have just introduced we can prove the following proposition due to [22] (§1), which is inspired by Gentzen’s famous cut-elimination technique (see [15]; see also [7], §§5.7, 5.8.3).

**Composition Elimination.** For every term \( h \) there is a composition-free term \( h' \) such that \( h = h' \).

**Proof.** A term of the form \( g \circ f \) where \( f \) and \( g \) are composition-free is called a *topmost composition*. In an arbitrary term we consider reductions that consist in replacing a subterm that is a topmost composition and is of the form on
the left-hand side of one of the equations (cat1), (H), (M) and (HM) by the corresponding term on the right-hand side of the equation.

Let the length of a term be the number of the symbols 1, , H and M in this term (we don’t count parentheses). Let the composition degree of a term be the sum of the lengths of all its subterms of the form g · f. Then it is easy to check that the length of every topmost composition replaced in a reduction of the previous paragraph is greater than or equal to the length of the term by which it is replaced, and that after every reduction the composition degree of the whole term is strictly smaller. It remains only to verify that we have covered with our reductions all possible forms of topmost compositions, and proceed by induction on the composition degree.

Every composition-free term of _M_0 is of the form _X_n . . . X_11, where _n_ ≥ 0 and _X_i_ is _H_ or _M_. When _a_ is 0, then this composition-free term is said to be in normal form. Every composition-free term is reduced to a term in normal form equal to it by replacing 1, with _MH_1, according to equation (MH). We will see below that this normal form is unique, i.e. that every term of _M_0 is equal to exactly one term in normal form.

We will now define inductively a functor _G_ from the category _M_0 to the simplicial category ∆. On objects _G_ is the identity function, _G_(1) is the identity function on _n_, and _G_(g · _f_) is _G_(g) · _G_(_f_). As a set of ordered pairs _G_(H _f_) is equal to _G_(_f_), (but the codomains of these two functions are different), and for _f_ : _n_ → _m_+1 the set of ordered pairs _G_(M _f_) is _G_(_f_) ∪ (_n_, _m_).

We can easily check by induction on the length of derivation that if _f_ = _g_ in _M_0, then _G_(_f_) = _G_(_g_) in ∆. Since it is clear that _G_ preserves identities and composition, we have that _G_ is a functor from _M_0 to ∆. We will establish below that _G_ is a faithful functor. Since _G_ is identity on objects, the faithfulness of _G_ amounts to its being one-one on arrows. We will also establish that _G_ is onto on arrows, so that we can conclude that the categories _M_0 and ∆ are isomorphic.

**Surjectivity Lemma.** The functor _G_ is onto on arrows.

**Proof.** Take an arbitrary order-preserving function _ϕ_ : _n_ → _m_. We construct a term _f_ of _M_0 such that _G_(_f_) = _ϕ_ by induction on _n_.

Suppose _n_ = 0. Then _ϕ_ is the empty function from Ø to _m_. If _H_^m stands for a sequence of _m_ ≥ 0 occurrences of _H_, then _f_ is _H_^m1.

Suppose _n_ = _n_+1, for _n_+1 ≥ 0. In that case _m_ cannot be 0; otherwise, _ϕ_ would not exist. Let _ϕ_(_n_+1) = _m_+1, for _m_ = _m_+ _k_, _m_+ _k_ ≥ 0, _k_ ≥ 1, and let the set of ordered pairs of the order-preserving function _ϕ_′ : _n_+1 → _m_+1 be defined as the set of ordered pairs of _ϕ_ minus the pair (_n_+1, _m_+1). By the induction hypothesis, we have constructed a term _f_′ : _n_+1 → _m_+1 such that _G_(_f_′) = _ϕ_′, and _f_ is _H_^{k−1} _M_ _f_′.

**Auxiliary Lemma.** If _f_ , _g_ : _k_ → _l_ are terms of _M_0 in normal form and _G_(_f_) = _G_(_g_), then _f_ is the same term as _g_.

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Proof. Suppose \( f \), which is \( X_n \ldots X_1 \mathbf{1}_0 \), and \( g \), which is \( Y_m \ldots Y_1 \mathbf{1}_0 \), are different terms of \( \mathcal{M}_0 \) of the same type \( k \rightarrow l \). Suppose for some \( i \) such that \( 1 \leq i \leq n \) and \( 1 \leq i \leq m \) we have that \( X_i \) is different from \( Y_i \). Let \( j \) be the least such \( i \), and let \( X_j \) be \( H \) while \( Y_j \) is \( M \). Let \( X_{j-1} \ldots X_1 \mathbf{1}_0 \), which is equal to \( Y_{j-1} \ldots Y_1 \mathbf{1}_0 \), be of type \( r \rightarrow s \).

Since \( MX_{j-1} \ldots X_1 \mathbf{1}_0 \) is defined, we must have that \( s = s' + 1 \) for \( s' \geq 0 \). Then the pair \( (r, s') \) is in \( G(MX_{j-1} \ldots X_1 \mathbf{1}_0) \), and hence also in \( G(g) \). But \( (r, s') \) cannot belong to \( G(f) \), since \( HX_{j-1} \ldots X_1 \mathbf{1}_0 \) is of type \( r \rightarrow s' + 2 \). So \( G(f) \neq G(g) \). If for every \( i \) such that \( 1 \leq i \leq n \) we have that \( X_i \) is identical to \( Y_i \), then \( f \) and \( g \) can differ only if \( n < m \). But then \( f \) and \( g \) cannot be of the same type.

\[ \square \]

Injectivity Lemma. If \( f, g : k \rightarrow l \) are terms of \( \mathcal{M}_0 \) and \( G(f) = G(g) \), then \( f = g \) in \( \mathcal{M}_0 \).

Proof. Suppose \( f' \) and \( g' \) are the normal forms of \( f \) and \( g \) respectively. Since \( f = f' \) and \( g = g' \) in \( \mathcal{M}_0 \), and hence \( G(f) = G(f') \) and \( G(g) = G(g') \) in \( \Delta \), from \( G(f) = G(g) \) we infer \( G(f') = G(g') \). Then, by the Auxiliary Lemma, \( f' \) is the same term as \( g' \), and hence \( f = g \) in \( \mathcal{M}_0 \).

The proof of this last lemma is analogous to the proof of the Injectivity Lemma of the preceding section.

As in the preceding section, with the help of the Auxiliary Lemma and of the functoriality of \( G \) we can ascertain that the normal form of terms of \( \mathcal{M}_0 \) is unique. For suppose that \( f \) and \( g \) are normal forms of the same term. Since both \( f \) and \( g \) are equal in \( \mathcal{M}_0 \) to this term, we have \( f = g \) in \( \mathcal{M}_0 \), and hence \( G(f) = G(g) \) in \( \Delta \). Then, by the Auxiliary Lemma, \( f \) and \( g \) are the same term.

Since we know that \( \Delta \) is isomorphic to \( \mathcal{M}_0 \), we know that \( \mathfrak{O}_n \) captures the endomorphisms of \( \mathcal{M}_0 \). In \( \mathcal{M}_0 \) the endomorphisms that correspond to the right-forking and left-forking terms of \( \mathfrak{O}_n \) are defined as follows. If \( Tf \) stands for \( MHT_f \), while \( T^0 \) is the empty sequence and \( T^{n+1} \) is \( T^n T \), then we have

\[
p' =_{\text{def}} T^{n-i-2} H M H H 1_{T^{n+1}} = T^{n-i-2} M T H 1_{T^0} = T^{n-i-2} (\eta_{T T^0} \circ \mu_{T^0}),\]

\[
q' =_{\text{def}} T^{n-i-2} M M H H 1_{T^{n+1}} = T^{n-i-2} M T H 1_{T^0} = T^{n-i-2} (\eta_{T T^0} \circ \mu_{T^0}).
\]

5 Composition-elimination in adjunction

An adjunction is defined in a standard manner (see [26], IV) as a sextuple \( (\mathcal{A}, \mathcal{B}, F, G, \varphi, \gamma) \) where \( \mathcal{A} \) and \( \mathcal{B} \) are categories, \( F \) and \( G \) are functors, the first from \( \mathcal{B} \) to \( \mathcal{A} \) and the second from \( \mathcal{A} \) to \( \mathcal{B} \), while \( \varphi \) and \( \gamma \) are natural transformations, the first from the composite functor \( FG \) to the identity functor on \( \mathcal{A} \) and the second from the identity functor on \( \mathcal{B} \) to the composite functor \( GF \), such that the following triangular equations hold:

\[ \varphi_{Fb} \circ F \gamma_b = 1_{Fb}, \quad G \varphi_a \circ \gamma_G a = 1_{Ga}. \]
In an alternative, equivalent definition (see [7], §§4.1-2), an adjunction is a sextuple \((\mathcal{A}, \mathcal{B}, F, G, \varphi^*, \gamma^*)\) where \(\mathcal{A}\), \(\mathcal{B}\), \(F\) and \(G\) are as in the standard definition above, while for \(\varphi^*\) and \(\gamma^*\) we have that \(\varphi^*\) is a function that maps the arrows \(f : a_1 \to a_2\) of \(\mathcal{A}\) to the arrows \(\varphi^*f : FGA_1 \to a_2\) of \(\mathcal{A}\), and \(\gamma^*\) is a function that maps the arrows \(g : b_1 \to b_2\) of \(\mathcal{B}\) to the arrows \(\gamma^*g : b_1 \to GFb_2\) of \(\mathcal{B}\), so that the following equations hold:

\[
\begin{align*}
(\text{nat } 1) & \quad f_2 \circ \varphi^* f_1 = \varphi^*(f_2 \circ f_1), & \gamma^* g_2 \circ g_1 = \gamma^*(g_2 \circ g_1), \\
(\text{nat } 2) & \quad \varphi^* f_2 \circ FGf_1 = \varphi^*(f_2 \circ f_1), & GFg_2 \circ \gamma^* g_1 = \gamma^*(g_2 \circ g_1), \\
(\varphi^*\gamma^*) & \quad \varphi^* f \circ \gamma^* g = f \circ Gg, & G \varphi^* f \circ \gamma^* g = Gf \circ g.
\end{align*}
\]

The two notions of adjunction are equivalent with the following definitions:

\[
\begin{align*}
\varphi^* f &=_{\text{def}} f \circ \varphi a, & \gamma^* g &=_{\text{def}} \gamma b \circ g, \\
\varphi a &=_{\text{def}} \varphi 1 a, & \gamma b &=_{\text{def}} \gamma 1 b.
\end{align*}
\]

We will now describe the free adjunction \((\mathcal{A}, \mathcal{B}, F, G, \varphi^*, \gamma^*)\) generated by the pair of sets \((\mathcal{O}, \{\mathcal{O}\})\), i.e. \((0,1)\). The objects of the category \(\mathcal{A}\) are generated from \(\mathcal{O}\) and those of \(\mathcal{B}\) from \(\mathcal{O}\). The category \(\mathcal{B}\) has as objects words of the form \((GF)^n\mathcal{O}\), where \((GF)^n\) stands for a possibly empty sequence of \(n \geq 0\) occurrences of \(GF\). The objects of \(\mathcal{A}\) are the objects of \(\mathcal{B}\) with \(F\) prefixed. We use \(a, a_1, \ldots\) for the objects of \(\mathcal{A}\) and \(b, b_1, \ldots\) for the objects of \(\mathcal{B}\).

The arrow terms of \(\mathcal{A}\) and \(\mathcal{B}\), which we call simply terms, are defined inductively as follows:

\[
\begin{align*}
1_a &: a \to a \text{ is a term of } \mathcal{A}; \\
1_b &: b \to b \text{ is a term of } \mathcal{B}; \\
\text{if } f_1 &: a_1 \to a_2 \text{ and } f_2 &: a_2 \to a_3 \text{ are terms of } \mathcal{A}, \text{ then } f_2 \circ f_1 &: a_1 \to a_3 \text{ is a term of } \mathcal{A}; \\
\text{if } g_1 &: b_1 \to b_2 \text{ and } g_2 &: b_2 \to b_3 \text{ are terms of } \mathcal{B}, \text{ then } g_2 \circ g_1 &: b_1 \to b_3 \text{ is a term of } \mathcal{B}; \\
\text{if } g &: b_1 \to b_2 \text{ is a term of } \mathcal{B}, \text{ then } Fg &: Fb_1 \to Fb_2 \text{ is a term of } \mathcal{A}; \\
\text{if } f &: a_1 \to a_2 \text{ is a term of } \mathcal{A}, \text{ then } Gf &: Ga_1 \to Ga_2 \text{ is a term of } \mathcal{B}; \\
\text{if } f &: a_1 \to a_2 \text{ is a term of } \mathcal{A}, \text{ then } \varphi f &: FAGa_1 \to a_2 \text{ is a term of } \mathcal{A}; \\
\text{if } g &: b_1 \to b_2 \text{ is a term of } \mathcal{B}, \text{ then } \gamma^* g &: b_1 \to GFb_2 \text{ is a term of } \mathcal{B}.
\end{align*}
\]

We impose on terms the equations (cat 1), (cat 2),

\[
\begin{align*}
(\text{fun } 1) & \quad F1_b = 1_{Fb}, & \quad G1_a = 1_{Ga}, \\
(\text{fun } 2) & \quad Fg_2 \circ Fg_1 = F(g_2 \circ g_1), & \quad Gf_2 \circ GF1_1 = G(f_2 \circ f_1),
\end{align*}
\]

and the equations (nat 1), (nat 2) and \((\varphi^*\gamma^*)\). (The formal construction of free adjunctions, in particular those generated by two sets of objects, and the
precise sense in which these adjunctions are free, are explained in [7], §§4.2-4. Formally, the arrows in the categories $A$ and $B$ of our free adjunction are equivalence classes of arrow terms.

Our notion of free adjunction is closely related to a 2-categorical notion stemming from [1] and [29] (which has recently acquired the name “walking adjunction”; see [23], §2.3). Here however we have no need for the 2-categorical context. Our simple notion of free adjunction suffices to make the connection with the simplicial category $\Delta$. This is achieved through a result about the isomorphism of the categories $B$ and $\Delta$, towards which we work in the next three sections of the paper.

For the terms $h$ of the categories $A$ and $B$ of our free adjunction generated by $(0, 1)$ we can prove Composition Elimination as follows. The statement of this Composition Elimination is as in the preceding section.

**Proof of Composition Elimination.** The notion of topmost composition is as in the proof of Composition Elimination in the preceding section. We have reductions for terms which consist in replacing a subterm that is a topmost composition or is of the form $F1_b$ or $G1_a$, according to the equations (cat 1), (fun 1), (fun 2), (nat 1), (nat 2) and $(\varphi^\alpha \gamma^\epsilon)$, all read from left to right; i.e. the subterm replaced is on the left-hand side, and the term by which it is replaced is on the right-hand side.

The length of a term is now the number of the symbols $1$, $\circ$, $F$ (applied to arrow terms), $G$ (applied to arrow terms), $\varphi^\alpha$ and $\gamma^\epsilon$ in this term. The composition degree of a term is now defined as the sum of the lengths of all its subterms of the form $g \circ f$ plus the length of the whole term. Then we reason as in the proof of the preceding section (see [7], §§4.5, 4.6.3, for details; note that the definition of composition degree we have here is the one mentioned in parentheses on p. 118 of [7], and that the other definition mentioned there, before the parentheses, is not applicable).

Every composition-free term of the categories $A$ and $B$ of our free adjunction is of the form $X_n \ldots X_11_c$, where $n \geq 0$ and $X_i$ is one of $F$, $G$, $\varphi^\alpha$ and $\gamma^\epsilon$, so that the symbol $F$ can precede immediately only $G$ or $\gamma^\epsilon$, the symbol $G$ only $F$ or $\varphi^\alpha$, the symbol $\varphi^\alpha$ only $F$ or $\varphi^\alpha$, and the symbol $\gamma^\epsilon$ only $G$ or $\gamma^\epsilon$. When $c$ is Ø, this composition-free term is said to be in normal form. Every composition-free term is reduced to a term in normal form equal to it by replacing $1_{Fb}$ with $F1_b$ and $1_{Ga}$ by $G1_a$, according to the equations (fun 1) read from right to left. We will establish in Section 7 that this normal form is unique.

We oriented (fun 1) from left to right in the proof of Composition Elimination for the ease of the proof. Now, for the composition-free normal form, we reverse the orientation, again to make the matter easier. Our normal form is an expanded normal form (as the long $\beta\eta$ normal forms of the lambda calculus). The normal form for $M_0$ in the preceding section was also expanded.
6 Friezes and $\mathcal{O}$

Let $\mathcal{O}$ be the monoid of order-preserving endomorphisms of the set of finite ordinals, i.e. the set of natural numbers, $\mathbb{N}$. In this monoid multiplication is composition of functions (order-preserving endomorphisms of $\mathbb{N}$ are closed under composition), and the unit element is the identity function on $\mathbb{N}$.

In the present section we will consider something we will call “friezes”, which corresponds to a special kind of tangle without crossings of knot theory (see [5], p. 99, [28], Chapter 9, [18], Chapter 12). In [8] the term “frieze” is used for a different, in some respects more general notion. We could have called the friezes of the present paper “adjunctional friezes”, to distinguish them from the friezes of [8], but since, in the main body of the paper, we will have no use for other friezes save adjunctional ones, we will stick to the shorter term “frieze”. In the next section we will show how the friezes introduced here are tied to adjunction.

In this section we show that our friezes make a monoid isomorphic to $\mathcal{O}$.

For $M$ an ordered set and for $a, b \in M$ such that $a < b$, let a segment $[a, b]$ in $M$ be $\{z \in M \mid a \leq z \leq b\}$. The numbers $a$ and $b$ are the end points of $[a, b]$. We say that $[a, b]$ encloses $[c, d]$ when $a < c$ and $d < b$. A set of segments is nonoverlapping when every two distinct segments in it are either disjoint or one of these segments encloses the other. A set $D$ of segments exhausts $M$ when all the segments of $D$ are segments in $M$ and for every $a \in M$ there is a segment in $D$ one of whose end points is $a$.

A segment $[a, b]$ in $\mathbb{Z} - \{0\}$ (i.e. the set of integers without 0) is called a transversal when $a$ is negative and $b$ positive; when both of $a$ and $b$ are positive it is a cup, and when they are both negative it is a cap.

A segment in $\mathbb{Z} - \{0\}$ is, of course, completely determined by its end points, and we may as well talk of pairs of integers $(a, b)$ instead of segments $[a, b]$. We talk of segments to distinguish them from other sorts of pairs.

A frieze is a set of nonoverlapping segments exhausting $\mathbb{Z} - \{0\}$ whose cups are of the form $[2k+2, 2k+3]$ and whose caps are of the form $[-(2k+2), -(2k+1)]$, for some $k \in \mathbb{N}$.

Friezes may be represented by diagrams. What we do should be clear from the following example. We draw as follows the frieze $\{[2, 3], [4, 5], [10, 11], [−2, −1], [−8, −7], [−3, 1], [−4, 6], [−5, 7], [−6, 8], [−9, 9]\} \cup \{[−(k+10), k+12] \mid k \in \mathbb{N}\}$, which, for latter reference, we call $D_1$:

```
1 2 3 4 5 6 7 8 9 10 11 12 13
-1 -2 -3 -4 -5 -6 -7 -8 -9 -10 -11 ...
```

This diagram explains the terminology of “cups”, “caps” and “transversal” segments.
Note that in a frieze 1 must be the end point of a transversal segment. Note also that an adjunctional frieze is uniquely determined by its transversal segments: the cups and caps need not be mentioned; but we may as well identify a frieze by its cups and caps: the transversal segments need not be mentioned.

The unit frieze $1$ is $\{[-(k+1), k+1] \mid k \in \mathbb{N}\}$; in this frieze there are no cups and caps. It is not so simple to define formally the composition of two friezes, but it is easy to get the idea from the following example. If $D_1$ is the frieze we had above as an example, and $D_2$ is the frieze that corresponds to the following diagram:

\begin{align*}
1 & \quad 2 & \quad 3 & \quad 4 & \quad 5 & \quad 6 & \quad 7 & \quad 8 \\
-1 & \quad -2 & \quad -3 & \quad -4 & \quad -5 & \quad -6 & \quad -7 & \quad -8 & \quad -9 & \quad -10
\end{align*}

then their composition $D_2 \circ D_1$ corresponds to the diagram obtained by putting the diagram of $D_2$ below the diagram of $D_1$ in the following manner:

\begin{align*}
\begin{array}{c}
D_1 \\
\hline
D_2
\end{array}
\end{align*}

which yields the diagram

\begin{align*}
\begin{array}{c}
1 & \quad 2 & \quad 3 & \quad 4 & \quad 5 & \quad 6 & \quad 7 & \quad 8 & \quad 9 & \quad 10 & \quad 11 & \quad 12 & \quad 13 \\
-1 & \quad -2 & \quad -3 & \quad -4 & \quad -5 & \quad -6 & \quad -7 & \quad -8 & \quad -9 & \quad -10 & \quad -11 & \quad -12 & \quad -13
\end{array}
\end{align*}

To ascertain that friezes are closed under composition, that this composition is associative, and that the unit frieze is indeed a unit with respect to composition, i.e. to ascertain that friezes make a monoid, we need a more formal definition of composition. Formally, we may define composition of friezes either in a geometrical style (see [8]), or in a set-theoretical style, as a peculiar composition of equivalence relations (see [9] or [10]). We don’t have space here to go into these formal matters, which have already been treated elsewhere. So we take for granted that friezes make a monoid.
An important aspect of this matter is that no circles, or closed loops, made of caps from one frieze and cups from the other can arise in composition of friezes. This particular fact is ascertained easily from the distribution of even and odd numbers in the end points of cups and caps.

The adjunctional friezes of the present paper are obtained from the friezes of \[8\] by permitting infinitely many cups and caps, by forgetting about circles and by requiring that cups be of the form \([2k+2, 2k+3]\) and caps of the form \([-2(k+2), -(2k+1)]\). Without this condition on cups and caps, circles may arise in composition, and according to how we treat them we obtain in \[8\] various kinds of equivalences of friezes, and various kinds of monoids. The strictest, \(L\) kind, records in what regions of the diagram the circles are located. The \(K\) kind just counts the number of circles, and the loosest, \(J\) kind, ignores circles. Since we don’t have circles in the adjunctional friezes of this paper, the various notions of equivalence of friezes of \[8\] will coincide for them. The friezes of \[8\] are tied to self-adjoint situations, where an endofunctor is adjoint to itself, while the friezes introduced here are tied to arbitrary adjoint situations.

Something analogous to adjunctional and other friezes may be found in \[4\], \[11\], \[21\], \[32\], \[31\], \[30\], and in many other papers in knot theory following Jones’ approach to knot and link invariants, which we mentioned in the introduction and at the beginning of this section.

Still another possibility to define composition of friezes is to rely on the isomorphism with \(O\) (see the end of this section). We are now going to establish this isomorphism.

A transversal segment of a frieze is \textit{odd} when its end points are two odd integers, and analogously with “odd” replaced by “even”. Every transversal segment of a frieze is either odd or even.

Let the \textit{successor} of a transversal segment in a frieze be the next transversal segment on the right-hand side in the diagram. For example, in the frieze \(D_1\), which we had as our first example above, the successor of \([-3, 1]\) is \([-4, 6]\). The successor of an odd transversal segment \([-2k_1+1, 2k_2+1]\) is an even transversal segment \([-2k_1+2, 2k_2+2k_3+2]\), and the successor of an even transversal segment \([-2k_1, 2k_2]\) is an odd transversal segment \([-2k_1+2, 2k_2+1]\).

Let a \textit{transversal pair} in a frieze be an odd transversal segment \([-2k_1+1, 2k_2+1]\) and its successor \([-2k_1+2, 2k_2+2k_3+2]\). Let us say that \(n \in \mathbb{N}\) is \textit{covered} by this transversal pair when \(k_2 \leq n \leq k_2+k_3\), and let us say that this transversal pair \textit{assigns} \(k_1\) to \(n\). In the example with \(D_1\) above, the transversal pair made of \([-3, 1]\) and \([-4, 6]\) covers 0, 1 and 2, and it assigns 1 to these three numbers. In every frieze, every \(n \in \mathbb{N}\) is covered by exactly one transversal pair, and this transversal pair assigns to every \(n\) a single number \(k_1\). What happens should be clear from the following adaptation of the diagram of \(D_1\):
So for every frieze $D$ we can define an order-preserving function $\varphi(D) : \mathbb{N} \to \mathbb{N}$ by mapping $n$ to the number assigned to $n$ by the transversal pair of $D$ covering $n$.

Conversely, for every order-preserving function $\varphi : \mathbb{N} \to \mathbb{N}$ we can define a frieze $D(\varphi)$ whose transversal segments are obtained as follows. For every $k$ such that there is an $n$ for which $\varphi(n) = k$, we have the transversal segments $[-(2k+1), 2\min\{n \mid \varphi(n) = k\}+1]$ and $[-(2k+2), 2\max\{n \mid \varphi(n) = k\}+2]$; these two segments make a transversal pair. It is easy to check that $D(\varphi(D)) = D$ and $\varphi(D(\varphi)) = \varphi$, so that we have a bijection between friezes and order-preserving endomorphisms of $\mathbb{N}$.

It is clear that for the unit frieze $1$ we have that $\varphi(1)$ is the identity function on $\mathbb{N}$. We also have the following lemma.

**Lemma.** $\varphi(D_2 \ast D_1) = \varphi(D_2) \ast \varphi(D_1)$.

**Proof.** Let $n \in \mathbb{N}$ be covered by the transversal pair of $D_1$ whose segments are $[-(2k_1+2k_3+1), 2k_2+1]$ and $[-(2k_1+2k_3+2), 2k_2+2k_3+2]$, and let $k_4+k_5$ be covered by the transversal pair of $D_2$ whose segments are $[-(2k_1+1), 2k_4+1]$ and $[-(2k_1+2), 2k_2+2k_4+2k_5+2k_6+2]$, so that $\varphi(D_2)\varphi(D_1)(n) = k_1$. Then $n$ is covered by the transversal pair of $D_2 \ast D_1$ whose segments are $[-(2k_1+1), 2k_2-2k_7+1]$ and $[-(2k_1+2), 2k_2+2k_3+2k_8+2]$, and so $\varphi(D_2 \ast D_1)(n) = k_1$. □

Hence we have that $\varphi$ establishes an isomorphism between the monoid of friezes and the monoid $\mathcal{O}$ of order-preserving endomorphisms of $\mathbb{N}$.

We could have defined formally composition of friezes by relying on the bijectivity of $\varphi$. The composition $D_2 \ast D_1$ of the friezes $D_1$ and $D_2$ could be defined as $D(\varphi(D_2) \circ \varphi(D_1))$. With this definition it is trivial to establish that friezes make a monoid isomorphic to $\mathcal{O}$. But then it remains to establish that composition so defined is the same notion we find in other possible formal definitions of composition, which we mentioned above.

The isomorphism of the monoid of friezes with the monoid $\mathcal{O}$ of order-preserving endomorphisms of $\mathbb{N}$, which was established in this section, can be relied upon in investigating an ‘augmented’ simplicial category, whose objects are the finite ordinals together with $\mathbb{N}$ (i.e. the ordinal $\omega$), and whose arrows are the order-preserving functions. This point was raised by an anonymous referee of this paper, who asked whether there is a faithful functor from the augmented simplicial category to a category with the same objects whose arrows are instances of an appropriately modified notion of frieze. It seems likely that the
question can be answered positively, but to give a precise answer would lead us too far afield, and we leave the matter for future research.

7 Adjunction and friezes

In this section we show how friezes are tied to the free adjunction generated by \((0, 1)\), which we introduced in Section 5. The connection between friezes and adjunction is made, more or less implicitly, in [21], [32], [13], [14] and [7].

For \(n, m \in \mathbb{N}\), a frieze is said to be of type \(n, m\) when for every \(k \in \mathbb{N}^+ = \mathbb{N} - \{0\}\) we have a transversal segment \([- (m + k), n + k]\) in this frieze. For example, the frieze \(D_1\) from the preceding section is of type \((11, 9)\). Not all friezes have a type, and when they have one, they are said to be of finite type. Note that types of friezes are not unique; a frieze of type \((n, m)\) is also of type \((n + k, m + k)\). It is clear that if \(D_1\) is a frieze of type \((n, m)\) and \(D_2\) a frieze of type \((m, l)\), then \(D_2 \circ D_1\) is a frieze of type \((n, l)\).

Let \(D_A\) be the category whose objects are all odd natural numbers, and whose arrows between \(n\) and \(m\) are all friezes of type \((n, m)\) indexed by \((n, m)\). We index these friezes by \((n, m)\) to ensure that every arrow has a single source \(n\) and a single target \(m\) (as we said above, every frieze of finite type has infinitely many different types). The category \(D_B\) is defined analogously save that its objects are all even numbers (including 0). We will show that these categories are isomorphic respectively to the categories \(A\) and \(B\) of the free adjunction generated by \((0, 1)\).

We define a functor \(E_A\) from \(A\) to \(D_A\) and a functor \(E_B\) from \(B\) to \(D_B\). On objects we have

\[
E_A(F(G)^nO) = 2n + 1, \\
E_B((GF)^nO) = 2n.
\]

Next we define inductively \(E_A\) and \(E_B\) on the arrow terms of \(A\) and \(B\):

\[
E_A(1_a) \text{ is the unit frieze 1 indexed by } (E_A(a), E_A(a)); \\
E_B(1_b) \text{ is the unit frieze 1 indexed by } (E_B(b), E_B(b)); \\
E_A(f_2 \circ f_1) \text{ is } E_A(f_2) \circ E_A(f_1); \\
E_B(g_2 \circ g_1) \text{ is } E_B(g_2) \circ E_B(g_1); \\
E_A(Fg) \text{ is } E_B(g) \text{ with its index } (n, m) \text{ replaced by } (n + 1, m + 1); \\
E_B(Gf) \text{ is } E_A(f) \text{ with its index } (n, m) \text{ replaced by } (n + 1, m + 1); \\
\]

if \(E_A(f)\) is indexed by \((n, m)\), then \(E_A(\varphi^*f)\) is the frieze indexed by \((n+2, m)\) obtained from the frieze \(E_A(f)\) by replacing all the transversal segments \([- (m + k), n + k]\), for every \(k \in \mathbb{N}^+\), by the cup \([n + 1, n + 2]\) and by the transversal segments \([- (m + k), n + k + 2]\);
if $E_B(g)$ is indexed by $(n, m)$, then $E_B(\gamma^* g)$ is the frieze indexed by $(n, m+2)$ obtained from the frieze $E_B(g)$ by replacing all the transversal segments $[-(m+k), n+k]$, for every $k \in \mathbb{N}^+$, by the cap $[-(m+2), -(m+1)]$ and by the transversal segments $[-(m+k+2), n+k]$.

Let us illustrate the last two clauses, for $\phi^*$ and $\gamma^*$:

- $E_A(f)$ indexed by $(1, 3)$
- $E_B(g)$ indexed by $(0, 2)$

We can check by induction on the length of derivation that if $f_1 = f_2$ in $A$, then $E_A(f_1) = E_A(f_2)$ in $D_A$, and that if $g_1 = g_2$ in $B$, then $E_B(g_1) = E_B(g_2)$ in $D_B$. For example, for the first ($\phi^* \gamma^*$) equation we have

This shows that the triangular equations of adjunctions, the essential equations of adjunctions, to which the equations ($\phi^* \gamma^*$) correspond, are about “straightening a sinuosity” (cf. [7], §4.10.1), and this straightening is based on planar ambient isotopies of knot theory (cf. [5], §1.A).

Since it is clear that $E_A$ and $E_B$ preserve identities and composition, with $E_A$ and $E_B$ we have indeed functors from $A$ to $D_A$ and from $B$ to $D_B$ respectively. We will establish that these functors are isomorphisms. First we prove the following lemma.

**Surjectivity Lemma.** *The functors $E_A$ and $E_B$ are onto on arrows.*

**Proof.** Take an arbitrary arrow $D$ of $D_A$ or $D_B$ of type $(n, m)$. We construct a term $h$ of $A$ or $B$ such that $E_A(h) = D$ or $E_B(h) = D$, as appropriate, by induction on $n+m$. 

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Suppose \( n+m = 0 \); so \( n = m = 0 \). Then \( D \) must be the unit frieze \( 1 \) indexed by \((0,0)\), which is an arrow of \( \mathcal{D}_B \), and \( h \) is the term \( 1_{\emptyset} : \emptyset \to \emptyset \) of \( \mathcal{B} \).

Suppose \( n+m > 0 \), and suppose \( n \) and \( m \) are even. Then \( D \) is an arrow of \( \mathcal{D}_B \), and for \( D \) we either have a transversal segment \([-m, n]\), or a cap \([-m, -(m-1)]\).

In the first case, by the induction hypothesis, we have constructed a term \( h' \) of \( \mathcal{A} \) such that \( E_A(h') \) is the arrow \( D \) of \( \mathcal{D}_A \) of type \((n-1, m-1)\). Then \( h \) is \( G h' \). In the second case, by the induction hypothesis, we have constructed the term \( h' \) of \( \mathcal{B} \) such that \( E_B(h') \) is the arrow \( D' \) of \( \mathcal{D}_B \) of type \((n, m-2)\), and \( D \) is obtained from \( D' \) by replacing the all the transversal segments \([-m-2+k, n+k]\), for every \( k \in \mathbb{N}^+ \), by the cap \([-m, -(m-1)]\) and by the transversal segments \([-m+k, n+k]\). Then \( h \) is \( \gamma^* h' \). We proceed analogously, in a dual manner, when \( n \) and \( m \) are odd.

For the next lemma we rely on the normal form of terms of \( \mathcal{A} \) and \( \mathcal{B} \) defined in Section 5.

**Auxiliary Lemma.** (A) If \( f_1, f_2 : a_1 \to a_2 \) are terms of \( \mathcal{A} \) in normal form and \( E_A(f_1) = E_A(f_2) \) in \( \mathcal{A} \), then \( f_1 \) is the same term as \( f_2 \).

(B) If \( g_1, g_2 : b_1 \to b_2 \) are terms of \( \mathcal{B} \) in normal form and \( E_B(g_1) = E_B(g_2) \) in \( \mathcal{B} \), then \( g_1 \) is the same term as \( g_2 \).

**Proof.** (A) Suppose \( f_1 \), which is \( X^n \ldots X^1 \mathbf{1}_{\emptyset} \), and \( f_2 \), which is \( Y^m \ldots Y^1 \mathbf{1}_{\emptyset} \), are different terms of \( \mathcal{A} \) of the same type \( a_1 \to a_2 \). Suppose for some \( i \) such that \( 1 \leq i \leq n \) and \( 1 \leq i \leq m \) we have that \( X_i \) is different from \( Y_i \). Let \( j \) be the least such \( i \). This means either that one of \( X_j \) and \( Y_j \) is \( F \), while the other is \( \gamma^* \), or that one of \( X_j \) and \( Y_j \) is \( G \), while the other is \( \varphi^* \). In both cases we obtain that \( E_A(f_1) \) cannot be equal to \( E_A(f_2) \). If for every \( i \) such that \( 1 \leq i \leq n \) we have that \( X_i \) is identical to \( Y_i \), then \( f_1 \) and \( f_2 \) can differ only if \( n < m \). But then \( f_1 \) and \( f_2 \) cannot be of the same type. The proof of (B) is analogous.

As a consequence of the last lemma, and of the functoriality of \( E_A \) and \( E_B \), we obtain that the normal form of terms of \( \mathcal{A} \) and \( \mathcal{B} \) is unique. We reason as in Sections 3 and 4. We also obtain the following lemma, by reasoning as in the proofs of the Injectivity Lemmata of Sections 3 and 4.

**Injectivity Lemma.** If \( f_1, f_2 : a_1 \to a_2 \) are terms of \( \mathcal{A} \) and \( E_A(f_1) = E_A(f_2) \) in \( \mathcal{D}_A \), then \( f_1 = f_2 \) in \( \mathcal{A} \). If \( g_1, g_2 : b_1 \to b_2 \) are terms of \( \mathcal{B} \) and \( E_B(g_1) = E_B(g_2) \) in \( \mathcal{D}_B \), then \( g_1 = g_2 \) in \( \mathcal{B} \).

Since \( E_A \) and \( E_B \) are bijections on objects, this lemma says that these functors are one-one on arrows. So, together with the Surjectivity Lemma, this yields that they are isomorphisms.

If the free adjunction is generated by \((1, 0)\) instead of \((0, 1)\), then in \( \mathcal{A} \) we have the objects \((FG)^n\emptyset\) and in \( \mathcal{B} \) the objects \( G(FG)^n\emptyset \), so that in \( \mathcal{D}_A \) the objects are even and in \( \mathcal{D}_B \) odd natural numbers. The definition of frieze would
then have cups \([2k+1, 2k+2]\) and caps \([- (2k+3), -(2k+2)]\). These new friezes correspond to relations converse to order-preserving functions. We could again establish isomorphisms between \(A\) and \(D_A\) and between \(B\) and \(D_B\), analogous to those above. (This is done in [7], §4.10, but in a manner somewhat less detailed than here, in particular with respect to friezes; the idea is, however, the same.)

If the free adjunction is generated by \((1,1)\), then the category \(A\) is the disjoint union of the categories \(A\) generated by \((0,1)\) and \((1,0)\), and analogously for \(B\).

We have then again isomorphisms with the appropriately defined categories \(D_A\) and \(D_B\).

8 Adjunction and \(\Delta\)

We will now show that the category \(D_B\) of the preceding section is isomorphic to the simplicial category \(\Delta\). We define first a functor \(S\) from \(D_B\) to \(\Delta\). On objects, we have that \(S(2n) = n\). On arrows, we have that if \(D\) is a frieze indexed by \((2n, 2m)\), then \(S(D)\) is the order-preserving function from \(n\) to \(m\) obtained by restricting the domain of the order-preserving endomorphism \(\varphi(D) : \mathbb{N} \rightarrow \mathbb{N}\) (see Section 6) to \(n\), and the codomain to \(m\).

In a frieze \(D\) of type \((2n, 2m)\), for some \(k < 2m\) we have a transversal segment \([-k, 1]\), and, provided \(n > 0\), for some \(l \leq 2m\) we have a transversal segment \([-l, 2n]\). Let us call the transversal segments in between these two transversal segments, including these two segments, the specific transversal segments of \(D\). If \(n > 0\), then the number of specific transversal segments is an even number greater than or equal to 2, and if \(n = 0\), then \(2n\) is not the end point of a transversal segment, and we have zero specific transversal segments. Specific transversal segments are those out of which we take the transversal pairs that determine the value of \(S(D)\) for every number in \(\{0, \ldots, n-1\}\). This value is a number in \(\{0, \ldots, m-1\}\). If \(n = 0\), then there are no transversal pairs made of specific transversal segments, and \(S(D)\) is the empty function.

For the unit frieze \(1\) we have that \(S(1)\) is the identity function, and we check, as in the \(\varphi\) Lemma of Section 6, that \(S\) preserves composition of friezes. So \(S\) is indeed a functor from \(D_B\) to \(\Delta\).

To show that \(S\) is an isomorphism, we define a functor \(D\) from \(\Delta\) to \(D_B\). On objects, we have that \(D(n) = 2n\), and on arrows \(\varphi : n \rightarrow m\) we define \(D(\varphi)\) of type \((2n, 2m)\) as in Section 6. According to what we know from Section 6, it is clear that these two functors establish an isomorphism between \(D_B\) and \(\Delta\).

Note that the category \(D_A\) of the preceding section is isomorphic to the subcategory of \(D_B\) for whose arrows \(D\) of type \((2n, 2m)\) we have in \(D\) the transversal segment \([-2m, 2n]\). Since this segment is paired with \([- (2m-1), k]\), for some \(k < 2n\), to make a transversal pair, the category \(D_A\) is isomorphic to the subcategory of \(\Delta\) whose order-preserving functions are last-element-preserving, i.e. the order-preserving function \(\varphi : n \rightarrow m\) maps \(n-1\) to \(m-1\).

Since \(A\) and \(B\) are isomorphic to \(D_A\) and \(D_B\) respectively, we have that \(A\) is
isomorphic to a subcategory of $\mathcal{B}$. This is the subcategory of $\mathcal{B}$ whose objects
are of the form $(GF)^n\mathcal{O}$ for some $n \geq 1$, and whose arrows are arrows of $\mathcal{B}$ of
the form $Gf$ (cf. [7], §5.2.2).

The isomorphism of $\mathcal{D}_B$ with $\Delta$ yields the isomorphism of $\mathcal{B}$ with $\Delta$. The
existence of this last isomorphism may be extracted from [1] (Corollary 2.8; see
also [29]).

9 $\mathcal{O}_n$ and Temperley-Lieb algebras

We can now establish that the monoids $\mathcal{O}_n$ are submonoids of the monoids tied
to Temperley-Lieb algebras.

Let $\mathcal{B}_n$ be the monoid of endomorphisms $g : (GF)^n\mathcal{O} \to (GF)^n\mathcal{O}$ of $\mathcal{B}$
(this monoid should not be confused with a braid group, which is often named
similarly), and let $\mathcal{D}_{B_n}$ be the monoid of endomorphisms $D$ of type $(2n, 2n)$
of $\mathcal{D}_B$. (It is clear that such endomorphisms are closed under composition.)
According to what we have established in Section 7, the monoids $\mathcal{B}_n$ and $\mathcal{D}_{B_n}$
are isomorphic, and they are both isomorphic to the monoid $\mathcal{O}_n$. Here the
right-forking term $p^i$ of $\mathcal{O}_n$ corresponds to the frieze

\[ \begin{array}{cccccccc}
1 & 2i & 2i+1 & 2i+2 & 2i+3 & 2i+4 & 2i+5 & 2n \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & 2i & 2i+1 & 2i+2 & 2i+3 & 2i+4 & 2i+5 & 2n \\
\end{array} \]

which corresponds to the following arrow of $\mathcal{B}$:

\[(GF)^{n-i-2}\gamma^c G\varphi^c F(GF)^i 1_{\mathcal{O}} : (GF)^n\mathcal{O} \to (GF)^n\mathcal{O}, \]

and the left-forking term $q^i$ corresponds to the frieze

\[ \begin{array}{cccccccc}
1 & 2i & 2i+1 & 2i+2 & 2i+3 & 2i+4 & 2i+5 & 2n \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & 2i & 2i+1 & 2i+2 & 2i+3 & 2i+4 & 2i+5 & 2n \\
\end{array} \]

which corresponds to the following arrow of $\mathcal{B}$:

\[(GF)^{n-i-2}G\varphi^c F\gamma^c (GF)^i 1_{\mathcal{O}} : (GF)^n\mathcal{O} \to (GF)^n\mathcal{O}. \]

The monoid $\mathcal{K}_n$ has for every $i \in \{1, \ldots, n-1\}$ a generator $h^i$, and also
the generator $c$. The terms of $\mathcal{K}_n$ are obtained from these generators and $1$ by
closing under the binary operation $\circ$. The following equations are assumed for
$\mathcal{K}_n$ besides the monoid equations (1) and (2) of Section 2:
\[(h1) \quad h^i \cdot h^j = h^j \cdot h^i, \quad \text{for } j+1 < i,\]
\[(h2) \quad h^i \cdot h^{i+1} \cdot h^i = h^i,\]
\[(hc1) \quad h^i \cdot c = c \cdot h^i,\]
\[(hc2) \quad h^i \cdot h^j = c \cdot h^i.\]

The equations \((h1), (h2)\) and \((hc2)\), which may be derived from Jones’ paper [17] (p. 13), and which appear in the form above in many works of Kauffman (see [19], [20], §8, and references therein), are usually tied to the presentation of Temperley-Lieb algebras. They may, however, be found in Brauer algebras too (see [31], p. 180-181).

According to the isomorphism result for \(\mathcal{K}_n\) with respect to diagrams generalizing the friezes of this paper (see [3] or [8], and references therein; the first proof of this isomorphism is in [17], §4), we can conclude that the monoid \(\mathcal{D}_{B_n}\) is isomorphic to a submonoid of \(\mathcal{K}_{2n}\). Hence we have that \(\mathcal{O}_n\) is isomorphic to a submonoid of \(\mathcal{K}_{2n}\). The right-forking term \(p^i\) is represented in \(\mathcal{K}_{2n}\) by \(h^{2i+3}, h^{2i+2}\), and the left-forking term \(q^i\) by \(h^{2i+1}, h^{2i+2}\).

We can establish also that \(\mathcal{O}_n\) is isomorphic to submonoids of the monoids \(\mathcal{L}_{2n}\) and \(\mathcal{J}_{2n}\) of [8]. These last two monoids differ from \(\mathcal{K}_{2n}\) only with respect to the equations \((hc1)\) and \((hc2)\), which have to do with circles in diagrams. In \(\mathcal{J}_{2n}\) we have \(c = 1\), and instead of \((hc2)\) we have simply \(h^i \cdot h^j = h^j\), while in \(\mathcal{L}_{2n}\) we pay closer attention to circles than in \(\mathcal{K}_{2n}\), as we indicated in Section 6, and we have something more involved than \((hc2)\). Circles are, however, irrelevant for the friezes of this paper, and all of \(\mathcal{K}_{2n}, \mathcal{L}_{2n}\) and \(\mathcal{J}_{2n}\) have \(\mathcal{O}_n\) as a submonoid.

The monoid \(\mathcal{L}_\omega\) of [8] has for every \(k \in \mathbb{N}^+\) the generators \([k]\) and \([k]\). Besides the monoid equations (1) and (2) of Section 2, the following equations are assumed in \(\mathcal{L}_\omega\) for \(l \leq k\):

\[(cup) \quad [k] \cdot [l] = [l] \cdot [k+2],\]
\[(cap) \quad [l] \cdot [k] = [k+2] \cdot [l],\]
\[(cup-cap 1) \quad [l] \cdot [k+2] = [k] \cdot [l],\]
\[(cup-cap 2) \quad [k+2] \cdot [l] = [l] \cdot [k],\]
\[(cup-cap 3) \quad [k] \cdot [k \pm 1] = 1.\]

In the monoid \(\mathcal{K}_\omega\) we have the additional equation \([k] \cdot [k] = [l] \cdot [l]\), while in \(\mathcal{J}_\omega\) we have \([k] \cdot [k] = 1\). With \(h^i\) defined as \([i] \cdot [i]\) and \(c\) as \([i] \cdot [i]\), we can check easily that \(\mathcal{K}_n\) is a submonoid of \(\mathcal{K}_\omega\). It is shown in [8] that \(\mathcal{K}_\omega\) is isomorphic to the monoid of \(\mathcal{K}\)-equivalence classes of the friezes of [8], and analogously for \(\mathcal{L}\) and \(\mathcal{J}\) (see Section 6 for \(\mathcal{K}\)-equivalence). These friezes are tied to self-adjunctions, where an endofunctor is adjoint to itself (see [8]).

Let an order-preserving endomorphism \(\varphi\) of \(\mathbb{N}\) be of type \((n, m)\) when for every \(k \in \mathbb{N}\) we have that \(\varphi(n+k) = m+k\). Order-preserving endomorphisms of
The order-preserving endomorphisms of \( \mathbb{N} \) of finite type make a monoid with composition and the identity function on \( \mathbb{N} \). This monoid, which we call \( O_\omega \), can be presented by generators and relations in the following manner. The generators of \( O_\omega \) are for \( i \in \mathbb{N} \) the order-preserving surjective functions \( \sigma_i \) such that
\[
\sigma_i(n) = \begin{cases} 
 n & \text{if } n \leq i \\
 n-1 & \text{if } n > i 
\end{cases}
\]
and the order-preserving injective functions \( \delta_i \) such that
\[
\delta_i(n) = \begin{cases} 
 n & \text{if } n < i \\
 n+1 & \text{if } n \geq i.
\end{cases}
\]

The standard generators of the simplicial category \( \Delta \) are obtained by restricting the domains and codomains of \( \sigma_i \) and \( \delta_i \) (see [26], VII.5). Besides the monoid equations, these generators satisfy the following equations for \( i \leq j \):
\[
\begin{align*}
\sigma_j \circ \sigma_i &= \sigma_i \circ \sigma_{j+1}, \\
\delta_i \circ \delta_j &= \delta_{j+1} \circ \delta_i, \\
\sigma_i \circ \delta_{j+2} &= \delta_{j+1} \circ \sigma_i, \\
\sigma_{j+1} \circ \delta_i &= \delta_i \circ \sigma_j, \\
\sigma_i \circ \delta_i &= \sigma_i \circ \delta_{i+1} = 1. 
\end{align*}
\]

It can be shown that \( O_\omega \) is isomorphic to a submonoid of \( K_\omega \). The generator \( \sigma_i \) is represented in \( K_\omega \) by \([2i+2]\) and \( \delta_i \) by \([2i+1]\). We can then easily verify the equations above in \( K_\omega \). For the first equation we use \( \cup \), for the second \( \cap \), and for the remaining three \( \cup-cap \ 1 \), \( \cup-cap \ 2 \) and \( \cup-cap \ 3 \) respectively. We establish in the same manner that \( O_\omega \) is isomorphic to submonoids of \( L_\omega \) and \( J_\omega \).

The monoids \( O_n \) are of course submonoids of \( O_\omega \). We define \( p^i \) as \( \delta_{i+1} \circ \sigma_i \), which is equal to \( \sigma_i \circ \delta_{i+1} \), and \( q^i \) as \( \sigma_{i+1} \circ \delta_i \), which is equal to \( \delta_i \circ \sigma_i \).

\textbf{Acknowledgements.} I would like to thank Zoran Petrić for reading this paper, and an anonymous referee for informing me about Alzenstat’s paper, about which I didn’t know when five years ago I wrote the first version (available at: http://arXiv.org/math.GT/0301302). I would like to thank also Vitor Fernandes, who was extremely kind to send me a copy of Alzenstat’s paper. The writing of the present paper was financed by the Ministry of Science of Serbia (Grant 144013).
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