COMPLETE ORBIT SPACES
OF AFFINE TORUS ACTIONS

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Abstract. Given an action of an algebraic torus on a normal affine variety, we describe all open subsets admitting a complete orbit space.

Introduction

Let an algebraic torus $T$ act on a normal, algebraic variety $X$. It is an open problem in Geometric Invariant Theory to describe the collection of all $T$-invariant open subsets $U \subseteq X$ admitting a geometric quotient $U \to U/T$ with a complete orbit variety $U/T$. Several constructions are known to produce such $U \subseteq X$, e.g., Mumford’s method \[13\] yields in many cases subsets $U \subseteq X$ admitting projective orbit spaces, and there are more general approaches providing also non-projective complete orbit varieties, see \[8\]. However, only in very special cases, e.g., $X$ projective and $\dim(T) \leq 2$, or $X = \mathbb{P}^n$ or a toric variety, there are descriptions of all $T$-invariant open subsets $U \subseteq X$ with a complete orbit variety, see \[4\], \[5\], \[6\] and \[7\].

In the present paper, we solve the above problem for the case that an arbitrary torus $T$ acts on a normal, affine variety $X$. Our motivation to consider this case is twofold. Firstly, we hope it to be of use for the projective and, more generally, the divisorial case, because one can reduce these cases to the affine one via equivariant (multi-)cone constructions, compare \[12\] and the Example \[4.2\] given at the end. Our second motivation concerns the (in general) non-separated orbit space $W/T$ of the union $W \subseteq X$ of all $T$-orbits of maximal dimension. From a more algebraic point of view, $W/T$ is a multigraded analogue of a homogeneous spectrum, compare \[10\]. Its complete open subvarieties are precisely the complete orbit spaces $U/T$, and thus a description of them may be helpful for a better understanding of $W/T$.

So far, the known approaches to the affine case basically deal with diagonal torus actions on the affine space $X = \mathbb{K}^n$. There are treatments in terms of toric geometry, see e.g. \[11\], and, alternatively, there is a Gale dual approach as presented in \[9\]. In this paper, we provide a general approach, using the language of proper polyhedral divisors introduced in \[1\].

A proper polyhedral divisor (for short pp-divisor) on a normal projective variety $Y$ may be written as a linear combination of pairwise different prime divisors $D_i$ having certain polyhedra $\Delta_i$ as their coefficients, which live in a common rational vector space and have a common pointed tail cone:

$$D = \sum_{i=1}^n \Delta_i \otimes D_i.$$  

To any such pp-divisor $D$ one may associate in a canonical way a normal affine variety $X$ with an effective action of a torus $T$, and, conversely, any effective action of a torus on a normal affine variety is obtained in this way, see \[11\]. For convenience, we give the precise definitions and recall the basic constructions in Section \[11\].

1991 Mathematics Subject Classification. 14L24, 14M17, 14M25.
Given a proper polyhedral divisor $\mathcal{D}$ on a projective variety $Y$ as before, the basic concept of this paper is the notion of a $\mathcal{D}$-coherent collection: this is a collection of vertices $v_i \in \Delta_i$, where $i = 1, \ldots, r$, satisfying certain compatibility conditions, see Definition 3.1 which in the case of a curve $Y$ even turn out to be empty. The main result is the following, see Theorem 3.3:

**Theorem.** Let $\mathcal{D}$ be a proper polyhedral divisor on a normal projective variety $Y$, and let $X$ be the associated normal affine $T$-variety. Then the $\mathcal{D}$-coherent collections are in bijection with the $T$-invariant open subsets $U \subseteq X$ admitting a geometric quotient $U \to U/T$ with a complete orbit space $U/T$.

The paper is organized as follows. In the first section, we recall among other things the language of proper polyhedral divisors from [1], and we present the basic facts needed here. Section 2 is devoted to preparing investigations concerning complete orbit spaces. In Section 3, we formulate and prove the main result. Finally, in the last section, we discuss an application and examples.

### 1. The language of polyhedral divisors

In this section, we fix (most of) our notation, give some background on quotients and torus actions, and then recall the necessary concepts and results from [1]. In particular, we give the precise definition of a proper polyhedral divisor $\mathcal{D}$ on a semiprojective variety $Y$, and we describe the fibres of the map $\pi: \tilde{X} \to Y$ associated to $\mathcal{D}$.

We work over an algebraically closed field $K$ of characteristic zero. By a variety we mean a separated reduced $K$-scheme of finite type, and the word prevariety refers to the (possibly) nonseparated analogue. By a point of a (pre-)variety, we always mean a closed point.

An action $G \times X \to X$ of an algebraic group $G$ on a variety $X$ is always assumed to be morphic; in this setting, we also speak of the $G$-variety $X$. Now suppose that $G$ is reductive, for example $G$ is a torus, and let $X$ be a $G$-variety. We will have to distinguish between the following concepts of quotients:

- **A good prequotient** for the $G$-variety $X$ is an affine $G$-invariant morphism $\pi: X \to Y$ onto a (possibly nonseparated) prevariety $Y$ such that $\pi^*: O_Y \to \pi^*(O_X)^G$ is an isomorphism.
- **A geometric prequotient** for the $G$-variety $X$ is a good prequotient $\pi: X \to Y$ such that each set-theoretical fibre $\pi^{-1}(y)$, where $y \in Y$, consists of precisely one $T$-orbit.
- **A good quotient** for the $G$-variety $X$ is a good prequotient $\pi: X \to Y$ with a variety $Y$.
- **A geometric quotient** for the $G$-variety $X$ is a geometric prequotient $\pi: X \to Y$ with a variety $Y$.

If one of these quotients $\pi: X \to Y$ exists, then it has the following universal property: let $\varphi: X \to Z$ be a $G$-invariant morphism to a prevariety $Z$, then there is a unique morphism $\psi: Y \to Z$ with $\varphi = \psi \circ \pi$. This justifies the notations $Y = X//G$ for the good (pre-)quotient space, and $Y = X/G$ in the geometric case. We will also refer to $X/G$ as the orbit space.

We shall frequently use two existence statements on quotients. Firstly for any affine $G$-variety $X$, there is a good quotient $X \to X//G$ with $X//G$ being the spectrum of the invariants $\Gamma(X, O)^G$. Secondly, if $G$ is a torus, and $X$ is a $G$-variety containing only orbits of maximal dimension, then there is a geometric prequotient $X \to X/G$, see [13, Corollary 3].
Let us now recall the basic concepts for actions of algebraic tori $T$ on affine varieties $X$. There is a natural correspondence between multigraded affine algebras and such actions: given a lattice $M$ and an $M$-graded affine algebra

$$A = \bigoplus_{u \in M} A_u,$$

the torus $T := \text{Spec}(\mathbb{K}[M])$ acts on the variety $X := \text{Spec}(A)$ such that the homogeneous elements $f \in A_u$ are precisely the semi-invariants of $X$ with respect to the character $\chi_u: T \to \mathbb{K}^\times$, and any affine $T$-variety $X$ arises in this way.

To the affine $T$-variety $X$ arising from an $M$-graded affine algebra $A$, we may associate combinatorial data in terms of the dimension of an orbit $x$.

The grading of $\mathbb{K}^M$ gives rise to an effective action of the torus $T := \text{Spec}(\mathbb{K}[M])$ on $X$, and the canonical map $\pi: \tilde{X} \to Y$ is a good quotient for this action.

By [1, Theorem 3.1], the ring of global sections $A := \Gamma(X, \mathcal{O}) = \Gamma(Y, \mathcal{A})$ is finitely generated and normal, and there is a $T$-equivariant, birational, proper morphism $r: \tilde{X} \to X$ onto the normal, affine $T$-variety $X := \text{Spec}(A)$. Conversely [1]
Theorem 3.4], says that every normal, affine variety with an effective torus action arises in the above way from a pp-divisor on a semiprojective variety.

**Remark 1.1.** For the affine $T$-variety $X$ arising from a pp-divisor $\mathfrak{D}$ on a semiprojective variety $Y$, the following statements are equivalent:

(i) All $T$-orbits of $X$ have a common orbit $T \cdot x_0$ in their closures.
(ii) The weight cone $\omega(X)$ is pointed, and $A_0 = \mathbb{K}$ holds.
(iii) The semiprojective variety $Y$ is projective.

We will need parts of the description of the fibres of the map $\pi : \tilde{X} \to Y$ given in [1 Prop. 7.8 and Cor. 7.9]. First recall that, for a $\sigma$-polyhedron $\Delta$ in $N_\mathbb{Q}$, each face $F \leq \Delta$ defines a convex, polyhedral cone in $M_\mathbb{Q}$ via

$$
F \mapsto \lambda(F) := \{ u \in M_\mathbb{Q}; \langle u, v' \rangle \geq 0 \text{ for all } v, v' \in F \}.
$$

The collection $\Lambda(\Delta)$ of all these cones is called the normal quasifan of $\Delta$; it subdivides the dual cone $\omega \subseteq M_\mathbb{Q}$ of $\sigma \subseteq N_\mathbb{Q}$. Note that the normal quasifan $\Lambda(\Delta_1 + \Delta_2)$ of a Minkowski sum is the coarsest common refinement of $\Lambda(\Delta_1)$ and $\Lambda(\Delta_2)$.

Now, let $\mathfrak{D} = \sum \Delta_i \otimes D_i$ be a representation of our pp-divisor such that all $D_i$ are prime. For a point $y \in Y$, its fiber polyhedron is the Minkowski sum

$$
\Delta_y := \sum_{y \in D_i} \Delta_i \in \text{Pol}_y^\vee (N).
$$

**Theorem 1.2.** Let $y \in Y$, consider the affine $T$-variety $\pi^{-1}(y)$, and let $\Lambda_y$ denote the normal quasifan of the fiber polyhedron $\Delta_y$. Then there is a one-to-one correspondence:

$$
\{ T\text{-orbits in } \pi^{-1}(y) \} \to \Lambda_y, \quad T \cdot \tilde{x} \mapsto \omega(\tilde{x}).
$$

Secondly, we shall need parts of the description of the $T$-orbits of $X$ given in [1 Theorem 10.1]. This involves the canonical contraction maps

$$
\vartheta_u : Y \to \text{Proj} \left( \bigoplus_{n=0}^{\infty} \Gamma(Y, \mathcal{O}(\mathfrak{D}(nu))) \right), \quad \text{where } u \in \omega \cap M.
$$

**Theorem 1.3.** For any two $\tilde{x}_1, \tilde{x}_2 \in \tilde{X}$, the following statements are equivalent:

(i) The contraction morphism $r : \tilde{X} \to X$ identifies the orbits $T \cdot \tilde{x}_1$ and $T \cdot \tilde{x}_2$.
(ii) We have $\omega(\tilde{x}_1) = \omega(\tilde{x}_2)$ and $\vartheta_u(\pi(\tilde{x}_1)) = \vartheta_u(\pi(\tilde{x}_2))$ for some $u \in \omega(\tilde{x}_1)^\circ$.

2. **Preparing observations**

In this section, $X$ is the normal, affine $T$-variety arising from a pp-divisor $\mathfrak{D}$ living on a normal, semiprojective variety $Y$. As before, $r : \tilde{X} \to X$ denotes the associated $T$-equivariant birational contraction map, and $\pi : \tilde{X} \to Y$ is the associated good quotient for the $T$-action.

We show that existence of a complete orbit space $U/T$ for a subset $U \subseteq X$ is equivalent to existence of a complete orbit space $\bar{U}/T$ for $\bar{U} := r^{-1}(U)$, and we give a geometric characterization of the subsets $\bar{U} \subseteq \tilde{X}$ admitting a complete orbit space. We establish these facts in a series of Lemmas, and then gather them in Proposition 2.3.

**Lemma 2.1.** Let $\bar{U} \subseteq \tilde{X}$ be a $T$-invariant open subset. Then $Y' := \pi(\bar{U})$ is open in $Y$, and, for any $T$-invariant morphism $\varphi : \bar{U} \to Z$ to a variety $Z$, there is a unique morphism $\psi : Y' \to Z$ with $\varphi = \psi \circ \pi$. 
Proof. We first consider any affine open subset \( Y_0 \subseteq Y \). Then also \( \tilde{X}_0 := \pi^{-1}(Y_0) \) is affine, and hence \( \tilde{U}_0 := \tilde{U} \cap \tilde{X}_0 \) is a union of homogeneous localizations \( \tilde{U}_f := (\tilde{X}_0)_f \). For each of these localizations, we have a commutative diagram

\[
\begin{array}{ccc}
\tilde{U}_f & \xrightarrow{\pi_f} & \tilde{X}_0 \\
\downarrow{\pi_f \circ \mathbb{T}} & & \downarrow{\mathbb{T}} \\
\tilde{U}_f \mathbb{T} & \xrightarrow{\mathbb{T}\circ \psi_0} & Y_0
\end{array}
\]

Using e.g. Theorem 1.2, we see that the generic fiber of \( \pi : \tilde{X}_0 \rightarrow Y_0 \) is the closure of a single \( T \)-orbit. The above diagram tells us that the same must hold for the quotient map \( \pi_f : \tilde{U}_f \rightarrow \tilde{U}_f \mathbb{T} \). Consequently, we have canonical isomorphisms

\[ \mathbb{K}(\tilde{U}_f \mathbb{T}) \cong \mathbb{K}(\tilde{U}_f) = \mathbb{K}(\tilde{X}_0) \cong \mathbb{K}(Y_0). \]

Thus, \( \mathbb{T} : \tilde{U}_f \mathbb{T} \rightarrow Y_0 \) is birational. By Theorem 1.2, the fibers of \( \pi \) contain only finitely many \( T \)-orbits. Thus, \( \mathbb{T} \) has finite fibers, and hence is an open embedding. Since \( Y' \cap Y_0 \) is covered by the images \( Y_f := \mathbb{T}(\tilde{U}_f \mathbb{T}) \), it must be open in \( Y_0 \).

Given a \( T \)-invariant morphism \( \varphi_0 : \tilde{U}_0 \rightarrow Z \) to a variety \( Z \), consider any restriction \( \varphi_f : \tilde{U}_f \rightarrow Z \). The above consideration yields a unique morphism \( \psi_f : Y_f \rightarrow Z \) with \( \varphi_f = \psi_f \circ \pi \). Moreover, any two such \( \psi_f, \psi_g \) coincide on the dense subset

\[ Y_{fg} = \pi(\tilde{U}_f \cap \tilde{U}_g) \subseteq Y_f \cap Y_g. \]

Consequently, since \( Z \) is separated, we can glue together the morphisms \( \psi_f : Y_f \rightarrow Z \) to a morphism \( \psi_0 : Y' \cap Y_0 \rightarrow Z \), and obtain this way a unique factorization \( \varphi_0 = \psi_0 \circ \pi \).

To conclude the proof, cover \( Y \) by affine open subsets \( Y_i \). Then, by the preceding consideration, each \( Y'_i := Y' \cap Y_i \) is open, and hence \( Y' \subseteq Y \) is so. Moreover, given a \( T \)-invariant \( \varphi : \tilde{U} \rightarrow Z \) to a variety \( Z \), we have a factorization \( \varphi = \psi_i \circ \pi \) over each \( Y'_i \), and, by uniqueness over \( Y'_i \cap Y'_j \), the \( \psi_i \) can be patched together to the desired morphism \( \psi : Y' \rightarrow Z \).

Lemma 2.2. Let \( \tilde{U} \subseteq \tilde{X} \) be a \( T \)-invariant, open subset containing only \( T \)-orbits of maximal dimension, and set \( Y' := \pi(\tilde{U}) \). Then the following statements are equivalent:

(i) The orbit space \( \tilde{U}/T \) is separated.

(ii) \( \tilde{U} \cap \pi^{-1}(y) \) is a single \( T \)-orbit for every \( y \in Y' \).

If one of these statements holds, then the restriction \( \pi : \tilde{U} \rightarrow Y' \) is a geometric quotient for the \( T \)-action.

Proof. Recall from Lemma 2.1 that \( Y' = \pi(\tilde{U}) \) is open in \( Y \). Suppose that (i) holds. Then Lemma 2.1 and the universal property of \( \tilde{U} \rightarrow \tilde{U}/T \) yield that the canonical morphism \( \tilde{U}/T \rightarrow Y' \) is an isomorphism. In particular, \( \pi : \tilde{U} \rightarrow Y' \) is a geometric quotient, and (ii) holds.

Suppose that (ii) holds. Then it suffices to show that \( \pi : \tilde{U} \rightarrow Y' \) is a geometric quotient. First we note that \( \tilde{U} \) can be covered by \( T \)-invariant open affine subsets \( \tilde{U}_0 \subseteq \tilde{U} \), see [13 Cor. 2]. For each such \( \tilde{U}_0 \), we obtain a commutative diagram

\[
\begin{array}{ccc}
\tilde{U}_0 & \xrightarrow{\pi_0} & \tilde{U} \\
\downarrow{\pi_0 \circ \mathbb{T}} & & \downarrow{\mathbb{T}} \\
\tilde{U}_0 \mathbb{T} & \xrightarrow{\mathbb{T}\circ \psi_0} & Y'
\end{array}
\]
The induced map $i: \tilde{U}/T \to Y'$ is birational, and, by assumption, injective. Hence it is an open embedding, and $\pi(\tilde{U})$ is affine. Thus, $\pi: \tilde{U} \to Y'$ looks locally w.r. to $Y'$ like an affine geometric quotient, and hence is a geometric quotient. □

Lemma 2.3. Let $U \subseteq X$ be a $T$-invariant open subset containing only $T$-orbits of maximal dimension, and set $\tilde{U} := r^{-1}(U)$. Then the following statements are equivalent:

(i) The orbit space $U/T$ is a complete variety.

(ii) The orbit space $\tilde{U}/T$ is a complete variety.

In each of these two cases, $Y = \pi(\tilde{U})$ holds, $Y$ is projective, and $\pi: \tilde{U} \to Y$ is a geometric quotient; in particular, $\tilde{U}/T$ is then projective.

Proof. Note that $\tilde{U} = r^{-1}(U)$ contains only orbits of maximal dimension. Thus, there is a geometric prequotient $\tilde{U} \to \tilde{U}/T$, and we have a commutative diagram

\[
\begin{array}{ccc}
\tilde{U} & \xrightarrow{i} & U \\
\downarrow & & \downarrow \\
\tilde{U}/T & \xrightarrow{\pi} & U/T
\end{array}
\]

If (ii) holds, then we may apply [7, Lemma 3.2] to the (birational) surjective morphism $\tilde{U}/T \to U/T$, and obtain that $U/T$ is a complete variety.

Now suppose that (i) holds. If $\tilde{U}/T$ is not separated, then Lemma 2.2 provides two different orbits $T \cdot \tilde{x}_1$ and $T \cdot \tilde{x}_2$ in $\tilde{U}$, which lie in a common fibre $\pi^{-1}(y) \subseteq \tilde{X}$. By Lemma 2.1, their images $y_i \in \tilde{U}/T$ are identified to a point $y \in U/T$ under $i: \tilde{U}/T \to U/T$. Let $x \in U$ lie over $y \in U/T$. Then $r: \tilde{X} \to X$ maps each orbit $T \cdot \tilde{x}_i$ onto $T \cdot x$. By Theorems 1.2 and 1.3, this is impossible for two different $T$-orbits inside one fibre $\pi^{-1}(y) \subseteq \tilde{X}$. Thus, $U/T$ must be separated.

In order to see that $\tilde{U}/T$ is complete, it suffices to show that $\tilde{U}/T \to U/T$ is a proper morphism. Since $\tilde{U}/T$ is a variety, $\tilde{U}/T \to U/T$ is of finite type and separated. Universal closedness follows directly from that fact that, given any morphism $Z \to U/T$, we have a canonical commutative diagram

\[
\begin{array}{ccc}
Z \times_{U/T} \tilde{U} & \xrightarrow{\text{proper}} & Z \times_{U/T} U \\
\downarrow & & \downarrow \\
Z \times_{U/T} \tilde{U}/T & \xrightarrow{\cong} & Z \times_{U/T} U/T \\
\downarrow & & \downarrow \\
\cong & & \cong
\end{array}
\]

Knowing that $\tilde{U}/T$ is a complete variety, we can conclude that the canonical (dominant) morphism $\tilde{U}/T \to Y$ is surjective, which implies $\pi(\tilde{U}) = Y$. Lemma 2.1 then even says that $\tilde{U}/T \to U/T$ is projective and $\pi: \tilde{U} \to Y$ is an isomorphism. In particular, $\tilde{U}/T$ is projective.

Corollary 2.4. There exists an open, $T$-invariant subset $U \subseteq X$ admitting a complete orbit variety $U/T$ if and only if $Y$ is projective.
As announced before, we now gather the observations made in the preceding Lemmas. For this, we introduce the following notion.

**Definition 2.5.** We say that an open subset $\tilde{U} \subseteq \tilde{X}$ is simple if $\pi(\tilde{U}) = Y$ holds, we have $r^{-1}(r(\tilde{U})) = \tilde{U}$, and for every $y \in Y$ the set $\pi^{-1}(y) \cap \tilde{U}$ is a single $T$-orbit.

**Proposition 2.6.** Let $X$ the affine $T$-variety arising from a pp-divisor living on a projective variety $Y$. Then the assignments $\tilde{U} \mapsto r(\tilde{U})$ and $U \mapsto r^{-1}(U)$ define mutually inverse one-to-one correspondences between the simple subsets of $\tilde{U} \subseteq \tilde{X}$ and the $T$-invariant open subsets $U \subseteq X$ with a complete orbit space $U/T$.

### 3. Complete orbit spaces

In this section, we formulate and prove our main result describing the open subsets with a complete orbit space for a given normal affine variety $X$ with an effective torus action $T \times X \to X$. According to Corollary 2.3, we may assume that the $T$-variety $Y$ arises from a pp-divisor on a projective variety $Y$; characterizations of this case were given in Remark 1.1.

Here comes the precise setup of this section. By $Y$ we denote, as indicated, a normal, projective variety, $N$ is a lattice and $\sigma \subseteq N_\mathbb{Q}$ is a pointed cone. Let $\mathcal{D} \in \text{PPDiv}(Y, \sigma)$ be a pp-divisor on $Y$, given by a representation

$$\mathcal{D} = \sum_{i=1}^{r} \Delta_i \otimes D_i$$

with pairwise different prime divisors $D_i \in \text{WDiv}(Y)$ and $\sigma$-polyhedra $\Delta_i \subseteq N_\mathbb{Q}$. As before, we denote by $X$ the normal, affine $T$-variety arising from $\mathcal{D}$, by $r: \tilde{X} \to X$ the $T$-equivariant contraction map and by $\pi: \tilde{X} \to Y$ the associated good quotient.

Recall from Section 1 that for any $y \in Y$, there is an associated fiber polyhedron $\Delta_y \subseteq N_\mathbb{Q}$, and the normal quasifan $\Lambda_y$ of $\Delta_y$ subdivides the dual cone $\omega \subseteq M_\mathbb{Q}$ of $\sigma \subseteq N_\mathbb{Q}$. We have the bijection $F \mapsto \lambda(F)$ from the faces $F \subseteq \Delta_y$ to the cones of $\Lambda_y$. For a cone $\lambda \subseteq M_\mathbb{Q}$, we denote its relative interior by $\lambda^\circ$.

Let us introduce the combinatorial data for the description of the collection of all $T$-invariant, open subsets $U \subseteq X$ admitting a complete orbit variety $U/T$. The definition makes use of the canonical contraction maps mentioned in Section 1

$$\vartheta_u: Y \to \text{Proj} \left( \bigoplus_{n=0}^{\infty} \Gamma(Y, \mathcal{O}(\mathcal{D}(nu))) \right), \quad \text{where } u \in \omega \cap M.$$

**Definition 3.1.** Let $\mathcal{D} = \sum_{i=1}^{r} \Delta_i \otimes D_i$ be a pp-divisor on a normal, projective variety $Y$ as in (1), and consider vertices $v_i \in \Delta_i$, where $i = 1, \ldots, r$. We say that $v_1, \ldots, v_r$ is a $\mathcal{D}$-admissible collection if for any $y \in Y$ the point

$$v_y := \sum_{y \in D_i} v_i$$

is a vertex of $\Delta_y$. If $v_1, \ldots, v_r$ is a $\mathcal{D}$-admissible collection and $y \in Y$ is given, we write $\lambda_y := \lambda(v_y) \in \Lambda_y$ for the corresponding cone. We say that a $\mathcal{D}$-admissible collection $v_1, \ldots, v_r \in N_\mathbb{Q}$ is $\mathcal{D}$-coherent if for any two $y_1, y_2 \in Y$ we have

$$\lambda_{y_1} \in \lambda_{y_2} \quad \text{and} \quad \vartheta_u(y_2) = \vartheta_u(y_1) \quad \text{for some } u \in X_{y_2}^\circ \implies \lambda_{y_1} = \lambda_{y_2}.$$

Note that, if all divisors $\mathcal{D}(u)$ corresponding to interior points of $\omega$ are ample, then their contraction maps $\vartheta_u$ are trivial, and thus, every $\mathcal{D}$-admissible collection is $\mathcal{D}$-coherent. This holds for example if $Y$ is a curve, or if $Y$ has a free cyclic divisor class group.
Definition 3.2. Let $D = \sum_{i=1}^{r} \Delta_i \otimes D_i$ be a pp-divisor on a normal, projective variety $Y$ as in (1). To any $D$-admissible collection $v_1, \ldots, v_r$ we associate $T$-invariant subsets

$$\tilde{U}(v_1, \ldots, v_r) := \{ \tilde{x} \in \tilde{X}; \omega(\tilde{x}) = \lambda_{\pi(\tilde{x})} \} \subseteq \tilde{X},$$

$$U(v_1, \ldots, v_r) := r(\tilde{U}(v_1, \ldots, v_r)) \subseteq X.$$

Theorem 3.3. Let $D = \sum_{i=1}^{r} \Delta_i \otimes D_i$ be a pp-divisor on a normal, projective variety $Y$ as in (1), and let $X$ be the associated normal, affine $T$-variety. Then there is a bijection:

$$\begin{align*}
\{ \text{D-coherent collections} \} & \rightarrow \left\{ \begin{array}{l}
\text{T-invariant open } U \subseteq X \text{ with } \\
\text{a complete orbit space } U/T \end{array} \right\} \\
(v_1, \ldots, v_r) & \mapsto U(v_1, \ldots, v_r).
\end{align*}$$

Using the descriptions of projective orbit spaces and such embeddable into some toric variety in terms of orbit cones given in [2, Sec. 1], we can easily figure out such orbit spaces from their defining coherent collections, provided that the $T$-variety $X$ is factorial.

Remark 3.4. Let $D = \sum_{i=1}^{r} \Delta_i \otimes D_i$ be a pp-divisor on a normal, projective variety $Y$, and suppose that the associated normal, affine $T$-variety $X$ is factorial. Let $v_1, \ldots, v_r$ be a coherent collection, and denote by $U = U(v_1, \ldots, v_r) \subseteq X$ the associated open set with complete orbit space.

(i) The orbit space $U/T$ is projective if and only if the intersection of all relative interiors $\lambda^0_y$, where $y \in Y$ is nonempty.

(ii) The orbit space $U/T$ admits an embedding into a toric variety if and only if for any two $y_1, y_2 \in Y$, the intersection $\lambda^0_{y_1} \cap \lambda^0_{y_2}$ is nonempty.

We turn to the proof of Theorem 3.3. We shall make use of the following elementary observation in convex geometry, which is evident from the definition of the normal quasi-fan of a polyhedron.

Lemma 3.5. Let $\Delta_1, \ldots, \Delta_r$ be polyhedra in a common vector space, and fix vertices $v_i \in \Delta_i$. Denote by $\Lambda_i := \Lambda(\Delta_i)$ the normal quasifan, and let $\lambda_i \in \Lambda_i$ be the cone corresponding to $v_i$. Then the following statements are equivalent.

(i) The point $v := v_1 + \ldots + v_r$ is a vertex of the Minkowski sum $\Delta := \Delta_1 + \ldots + \Delta_r$.

(ii) The cone $\lambda := \lambda_1 \cap \ldots \cap \lambda_r$ is a maximal cone of the normal quasifan $\Lambda := \Lambda(\Delta)$.

If one of these statements holds, then $\lambda \in \Lambda$ is the maximal cone corresponding to the vertex $v \in \Delta$.

Proof of Theorem 3.3. According to Proposition 2.6, it suffices to show that the assignment $(v_1, \ldots, v_r) \mapsto \tilde{U}(v_1, \ldots, v_r)$ defines a bijection from the $D$-coherent collections to the simple subsets $\tilde{U} \subseteq \tilde{X}$.

Let $v_1, \ldots, v_r$ be a $D$-coherent collection. Our first task is to check that the $T$-invariant subset $\tilde{U}(v_1, \ldots, v_r) \subseteq \tilde{X}$ is indeed open. For this, let $\tilde{x}_0 \in \tilde{U}(v_1, \ldots, v_r)$, and set $y_0 := \pi(\tilde{x}_0)$. Then $y_0$ admits a canonical open neighbourhood

$$V := Y \setminus \bigcup_{y_0 \notin D_j} D_j \subseteq Y.$$
Then, for every $y \in V$, the normal quasifan $\Lambda_{y_0}$ of $\Delta_{y_0}$ refines the normal quasifan $\Lambda_y$ of $\Delta_y$, and, by Lemma 3.5, for the relative interiors $\lambda^o_{y_0} \subseteq \lambda_{y_0}$ and $\lambda^o_y \subseteq \lambda_y$ of the cones corresponding to the vertices $v_{y_0} \subseteq \Delta_{y_0}$ and $v_y \subseteq \Delta_y$ we have

$$\lambda^o_{y_0} \subseteq \lambda^o_y.$$  

Choose an affine open neighbourhood $V_0 \subseteq V$ of $y_0 \in V$, and an integral vector $u \in \lambda^o_{y_0}$ admitting a homogeneous function $f \in \Gamma(\pi^{-1}(V_0), \mathcal{O})_u$ with $f(\tilde{x}_0) \neq 0$. This gives an open neighbourhood of $\tilde{x}_0$ in $\tilde{X}$, namely

$$\pi^{-1}(V_0)_f \subseteq \{ \tilde{x} \in \pi^{-1}(V_0); u \in \omega(\tilde{x}) \}.$$  

According to (2), the whole set on right hand side is contained in $\tilde{U}(v_1, \ldots, v_r)$. This implies openness of the subset $\tilde{U}(v_1, \ldots, v_r) \subseteq \tilde{X}$.

Now have to verify the properties of a simple set for $\tilde{U}(v_1, \ldots, v_r)$. By Theorem 1.2, the image $\pi(\tilde{U}(v_1, \ldots, v_r))$ equals $Y$, and each fibre $\pi^{-1}(y)$, where $y \in Y$ contains exactly one $T$-orbit of $\tilde{U}(v_1, \ldots, v_r)$. So, we only have to show that $\tilde{U}(v_1, \ldots, v_r)$ is saturated with respect to the contraction map $r: \tilde{X} \to X$.

For this, let $\tilde{x}_1 \in \tilde{U}(v_1, \ldots, v_r)$ and $\tilde{x}_2 \in \tilde{X}$ with $r(\tilde{x}_2) = r(\tilde{x}_1)$. Set $y_1 := \pi(\tilde{x}_1)$. Then, by Theorem 1.3, we have $\omega(\tilde{x}_2) = \omega(\tilde{x}_1)$ and $\vartheta_u(y_2) = \vartheta_u(y_1)$ for some $u \in \omega(\tilde{x}_2)^o$. This implies $\tilde{x}_2 \in \tilde{U}(v_1, \ldots, v_r)$, because by $\mathcal{D}$-coherence of the collection $v_1, \ldots, v_r$, we have

$$\omega(\tilde{x}_2) = \omega(\tilde{x}_1) = \lambda_{y_1} = \lambda_{y_2}.$$  

Now, let $\tilde{U} \subseteq \tilde{X}$ be any simple subset. We have to show that $\tilde{U}$ arises from a $\mathcal{D}$-coherent collection. Recall from Lemma 2.2 that the restriction $\pi: \tilde{U} \to Y$ is a geometric quotient. Moreover, we have the prime divisors $D_i$ in $Y$, and (nonempty) locally closed subspaces

$$Y_i := D_i \setminus \bigcup_{j \neq i} D_j \subseteq Y, \quad \tilde{U}_i := \tilde{U} \cap \pi^{-1}(Y_i) \subseteq \tilde{X}.$$  

Note that $\pi: \tilde{U}_i \to Y_i$ is a geometric quotient, and hence $\tilde{U}_i$ is irreducible. Moreover, all points $y \in Y_i$ have the same fiber polyhedron $\Delta_y = \Delta_i$, and thus, since $\tilde{U}_i$ is irreducible, $\omega(\tilde{x})$ is constant along $\tilde{U}_i$. Finally, since also $\tilde{U} \cap \pi^{-1}(D_i)$ is irreducible, the closure of $\tilde{U}_i$ in $\tilde{U}$ is given by

$$E_i := \tilde{U}_i := \tilde{U} \cap \pi^{-1}(D_i).$$

For $\tilde{x} \in \tilde{U}_i$, set $\lambda_i := \omega(\tilde{x})$. Theorem 1.2 tells us that $\lambda_i$ is a maximal cone of the normal quasifan $\Lambda_i$ of $\Delta_i$. Let $v_i \in \Delta_i$ denote the vertex corresponding to $\lambda_i \in \Lambda_i$. We show that $v_1, \ldots, v_r$ is a $\mathcal{D}$-admissible collection with $\tilde{U} = \tilde{U}(v_1, \ldots, v_r)$. By Lemma 3.3 it suffices to show that for any subset $I \subseteq \{1, \ldots, r\}$ we have

$$\omega(\tilde{x}) = \lambda_I := \bigcap_{i \in I} \lambda_i \quad \text{for every } \tilde{x} \in \tilde{U}_I := \bigcap_{i \in I} E_i \cap \bigcup_{j \notin I} E_j.$$  

Since $\tilde{U}$ is a union of subsets $r^{-1}(X_f)$ with homogeneous $f \in \Gamma(X, \mathcal{O})$, and $\tilde{U}_i$ is contained in the closure of each $\tilde{U}_i$ with $i \in I$, we must have $\omega(\tilde{x}) \subseteq \lambda_i$ for all $\tilde{x} \in \tilde{U}_I$ and all $i \in I$. Moreover, in this situation, the fiber polyhedron $\Delta$ of $\pi(\tilde{x})$ is the Minkowski sum of the $\Delta_i$, where $i \in I$. The normal quasifan $\Lambda := \Lambda(\Delta)$ is the coarsest common refinement of the normal quasifans $\Lambda_i = \Lambda(\Delta_i)$. Thus, $\lambda_I$ is a maximal cone of $\Lambda$. Since $\omega(\tilde{x}) \in \Lambda$ holds and $\omega(\tilde{x})$ is of full dimension, we obtain $\omega(\tilde{x}) = \lambda_I$.

Finally, we have to verify $\mathcal{D}$-coherence of the $\mathcal{D}$-admissible collection $v_1, \ldots, v_r$. So, let $y_1, y_2 \in Y$ such that $\lambda_{y_2} \in \Lambda_{y_1}$ holds and we have $\vartheta_u(y_2) = \vartheta_u(y_1)$ for some integral $u \in \lambda_{y_2}^o$. By Theorem 1.3 the $T$-orbits $T \cdot \tilde{x}_1 \in \pi^{-1}(y_1)$ corresponding to
\[ \lambda_{y_2} \] are identified under \( r : \tilde{X} \to X \). Since \( \tilde{U} \) is saturated w.r. to \( r : \tilde{X} \to X \), we obtain \( \tilde{x}_1 \in \tilde{U} \), which implies \( \lambda_{y_1} = \lambda_{y_2} \). \( \square \)

4. Applications and Examples

In this section, we discuss a few examples and applications. The first observation concerns the limit \( Y' \) over all GIT-quotients associated to possible linearizations of the trivial bundle. The limit \( Y' \) contains a canonical component \( Y'_0 \) compare dominating all the GIT-quotients just mentioned, see e.g \([1\), Section 6\]. We obtain that the normalization \( Y \) of \( Y'_0 \) dominates moreover all complete orbit spaces, i.e., also those that do not arise from GIT:

**Corollary 4.1.** Let \( U \subseteq X \) admit a complete orbit space \( U(T) \). Then there is a surjective birational morphism \( Y \to U/T \) from the normalized canonical component \( Y \) of the limit over all GIT-quotients of \( X \).

**Proof.** The \( T \)-variety \( X \) admits a description by a pp-divisor living \( \mathcal{D} \) living on the normalized canonical component \( Y \), see \([1\), Section 6\]. Thus, the claim follows from Lemma 2.3. \( \square \)

We now use our result to treat an example of A. Białynicki-Birula and J. Święcicka of a \( K^* \)-action on the Grassmannian \( G(2, 4) \), see \([7\); to our knowledge, this the simplest example admitting complete orbit spaces that are not embeddable into any toric variety.

**Example 4.2.** We consider the cone \( X \) over the Grassmannian \( G(2, 4) \). In terms of Plücker Coordinates, \( X \) is given as

\[
X = V(K^6, z_1z_6 - z_2z_5 + z_3z_4) \subset K^6
\]

Let the two-dimensional torus \( T := K^* \times K^* \) act on \( X \) by defining the weight of the variable \( z_i \) as the \( i \)-th column of the matrix

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 2 & 3 & 3 & 4 & 5
\end{bmatrix}
\]

Note that this action lifts the action of the second factor \( K^* \) on \( G(2, 4) \) given in \( \mathbb{P}^5 \) by homogeneous Plücker Coordinates as

\[
t_2^i [z] := [t_2z_1, t_2z_2, t_2^2z_3, t_2^3z_4, t_2^4z_5, t_2^5z_6].
\]

The open \( T \)-invariant subsets \( U \subseteq X \) admitting a complete orbit space \( U/T \) are, via the tautological projection, in one-to-one correspondence with the open \( K^* \)-invariant subsets \( V \subseteq G(2, 4) \) admitting a complete orbit variety \( V/K^* \). The latter ones are well known, see \([7\); there are six of them; four having a projective orbit variety, and two having quite exotic orbit spaces, which are not even embeddable into toric varieties, see \([13\).

Let us see how to recover this picture via our method. We need a describing pp-divisor for the \( T \)-action on \( X \). According to the recipe discussed in \([1\), Section 11\], we first determine a pp-divisor for the (equivariant) ambient space \( K^6 \), using the language of toric varieties. We suppress the details of computation; all of them are standard toric geometry, one may even use, e.g., the software package \([3\) as a help.

As the underlying projective variety \( Y_{\text{ambient}} \), we take the normalized component of the GIT-limit of the \( T \)-action on \( K^6 \). Concretely \( Y_{\text{ambient}} \) is the toric variety given by the fan \( \Sigma \) in \( \mathbb{Q}^4 \) having its rays through the vectors

\[
v_1 := (1, 0, 0, 0), \quad v_2 := (0, 1, 0, 0), \quad v_3 := (2, 1, 0, 0), \quad v_4 := (3, 2, 1, 1),
\]

\[
a_1 := (-4, -3, -2, -2), \quad a_2 := (0, 0, 1, 0), \quad a_3 := (0, 0, 0, 1).
\]
where the U/T Y on a projective curve given by D prime divisors X image of the intersection Up to the splitting of D enumeration ∆ affine charts. This gives six coherent collections of vertices (listed according to the Example 4.3. Let tions of tori having generic orbits of small codimension (instead of small dimension): with prime divisors ∆ turns out that for K 6 we have we have four open subets W1, ..., W4 ⊂ K 6 with a complete (in fact projective) orbit variety W/T. These arise from the following four coherent collections (the vertices are listed according to the enumeration ∆1, ..., ∆4):

\{ (0, 0), (1, 1), (0, 0), (0, 0) \}, \{ (2, -1), (1, 1), (0, 0), (0, 0) \}, \{ (2, -1), (1, 1), (0, 0), (0, 0) \}, \{ (2, -1), (1, 1), (0, 0), (0, 0) \}.

A pp-divisor D for the T-action on X lives on the (normal) closure Y of the image of the intersection X ∩ (K*)6 in Yambient, and D can be taken as the pull back of Dambient with respect to the inclusion π: Y → Yambient. Pulling back toric prime divisors D1 gives

\[ \pi^*(D_1) = E_1, \quad \pi^*(D_2) = E_2, \quad \pi^*(D_3) = E_3^a \cup E_3^b, \quad \pi^*(D_4) = E_4, \]

with prime divisors E1, E2, E4 and E3^a, E3^b, the latter two being disjoint from each other. The pp-divisor for the T-action on X then is given by

\[ D = \Delta_1 \otimes E_1 + \Delta_2 \otimes E_2 + \Delta_3 \otimes E_3^a + \Delta_3 \otimes E_3^b + \Delta_4 \otimes E_4. \]

Up to the splitting of D4 into two disjoint components, the intersection behaviour of the pulled back divisors is as before, which can be directly checked in toric affine charts. This gives six coherent collections of vertices (listed according to the enumeration ∆1, ∆2, ∆3^a, ∆3^b, ∆4):

\{ (0, 0), (-1, 1), (0, 0), (0, 0), (0, 0) \}, \{ (2, -1), (-1, 1), (0, 0), (0, 0), (0, 0) \},

\{ (2, -1), (-1, 1), (0, 0), (0, 0), (0, 0) \}, \{ (2, -1), (-1, 1), (0, 0), (0, 0), (0, 0) \},

\{ (2, -1), (-1, 1), (3, -1), (3, -1), (0, 0) \}, \{ (2, -1), (-1, 1), (3, -1), (3, -1), (4, -1) \}.

The reader might be a little disappointed about the computational efforts needed for the preceding example. The situation turns much better, if one considers actions of tori having generic orbits of small codimension (instead of small dimension):

**Example 4.3.** Let X be a normal, affine variety with a good effective action of a torus T such that dim(T) = dim(X) - 1 holds. Then X arises from a pp-divisor D on a projective curve Y, and D is of the form

\[ D = \sum_{i=1}^{r} \Delta_i \otimes \{ y_i \}, \]

where the y_i ∈ Y are pairwise different points. Any collection v_1, ..., v_r of vertices v_i ∈ Δ_i is coherent, and hence the collection of open T-invariant U ⊆ X with complete orbit space U/T is in bijection to the set

\[ \text{vertices}(\Delta_1) \times \ldots \times \text{vertices}(\Delta_r). \]
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