On geometry of $p$-adic polynomials

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Abstract

An analogue of the Gauss-Lucas theorem for polynomials over the algebraic closure $\mathbb{C}_p$ of the field of $p$-adic numbers is considered.

1 Introduction

The pioneering works of I. V. Volovich [1, 2, 3] gave impetus to the rapid development of applications of non-Archimedean analysis to models and problems of mathematical physics. The current state and bibliography can be found, for example, in the review [4].

In this paper, we consider the geometry of non-Archimedean polynomials. The result can be helpful in the study of polynomial dynamical systems over the field of $p$-adic numbers [5, 6].

The Gauss-Lucas theorem states the following. Let $P(z) \in \mathbb{C}[z]$ is a polynomial over the field $\mathbb{C}$ of complex numbers, and $P'(z)$ is its derivative. Then, all the roots of the polynomial $P'(z)$ (that is, the critical points of the polynomial $P(z)$) lie in the convex hull of the set of roots of the polynomial $P(z)$. This statement admits the following equivalent formulation. Any disk in the complex plane containing zeros of the polynomial $P(z)$ also contains all zeros of the derivative $P'(z)$ [7].

The purpose of this paper is to formulate and prove an analogue of this result for an algebraically closed non-Archimedean field. As such a field, we will consider the field $\mathbb{C}_p$ — completion of the algebraic closure of the field $\mathbb{Q}_p$ of $p$-adic numbers. The norm in $\mathbb{C}_p$ will be denoted by $| \cdot |$.

The geometry of polynomials over non-Archimedean fields is a little-studied area. The author knows only one work on this topic [8], dedicated to the non-Archimedean analogue of the Sendov conjecture.
2 The main theorem

Consider the polynomial $P(z) \in \mathbb{C}_p[z]$ of degree $n$ over the field $\mathbb{C}_p$,

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0, \ a_j \in \mathbb{C}_p, \ j = 0, 1, \ldots, n.$$ 

Using $\lambda_1, \ldots \lambda_n$ we denote its roots,

$$P(z) = a_n (z - \lambda_1)(z - \lambda_2) \cdots (z - \lambda_n).$$

Using $\omega_1, \omega_2, \ldots, \omega_{n-1}$ we denote the roots of the derivative $P'(z)$,

$$P'(z) = na_n (z - \omega_1)(z - \omega_2) \cdots (z - \omega_{n-1}).$$

A disk of radius $r$ centered at $a \in \mathbb{C}_p$ is denoted by $D(a, r)$,

$$D(a, r) = \{ z \in \mathbb{C}_p : |z - a| \leq r \}.$$ 

The following theorem is valid.

**Theorem 1.** Let the roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the polynomial $P(z) \in \mathbb{C}_p[z]$ lie within the disk $D(a, r)$. Then each disk $D(a, r_k)$, $k = 1, 2, \ldots, n - 1$, where

$$r_k = \max \left\{ r \left| \frac{j}{n} \right|^{\frac{n-j}{n-1}}, \ j = 1, 2, \ldots, k \right\}$$

contains at least $k$ roots of the derivative $P'(z)$.

**Corollary 1.** The disk $D\left(a, r|n|^{-1/(n-1)}\right)$ contains at least one critical point of the polynomial $P(z)$.

**Remark 1.** This statement was proved by D. Choi and S. Lee [8]. Thus, the Theorem significantly strengthens their result.

**Corollary 2.** The disk $D\left(a, r|n|^{-1}\right)$ contains all critical points of the polynomial $P(z)$.

The statement follows directly from the statement of the Theorem and the apparent inequality $|j| \leq 1$, $j \in \mathbb{Z}$.

The following simple example shows that the estimate from Corollary cannot be improved. Let $p = 3$, consider the polynomial $P(z) = z^2(z - 1)$. Its roots $\lambda_1 = \lambda_2 = 0, \lambda_3 = 1$ lie within the circle $D(0, 1)$. The derivative $P'(z) = 3z(z - 2/3)$ has roots $\omega_1 = 0, \omega_2 = 2/3$ and, since $|2/3| = 3$, then the minimal circle containing the roots of the derivative of the polynomial $P(z)$ is $D(0, 3)$. 

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Corollary 3. In the case when the order $n$ of the polynomial is not divisible by $p$ (that is, $|n| = 1$), the critical points of the polynomial $P(z)$ lie within any disk containing zeros of this polynomial.

In other words, for polynomials of the order $n$, $p \nmid n$, the exact analogue of the Gauss-Lucas theorem for the field of complex numbers is valid.

3 Proof of the Theorem

Before proceeding to the proof of the Theorem 1, we will make a few remarks. First, without limiting the generality, we can put $a_n = 1$ since multiplication by a nonzero constant does not change the roots of the polynomial and its derivative. We can put $a = 0$ since the shift of the roots by $a$ leads to a shift of the derivative’s roots by $a$. Everywhere further, we will number the roots of the polynomial $P(z)$ and the roots of its derivative $P'(z)$ in the order of non-decreasing norm:

$$|\lambda_1| \leq |\lambda_2| \leq \cdots \leq |\lambda_n|, \quad |\omega_1| \leq |\omega_2| \leq \cdots \leq |\omega_{n-1}|.$$  \hfill (1)

Taking into account the comments made, the statement of the Theorem 1 is equivalent to the following estimate for the roots of the derivative $P'(z)$, $k = 1, 2, \ldots, n - 1$:

$$|\omega_k| \leq r_k = r \max \left\{ \left| \frac{j}{n} \right|^{1/n}, \; j = 1, 2, \ldots, k \right\}.$$  \hfill (2)

In the proof, we will use the following relations between the roots of the polynomial $P(z)$ and its coefficients (Vieta’s formulas):

$$(-1)^{n-k} a_k = \sum_{1 \leq i_1 < i_2 < \cdots < i_{n-k} \leq n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_{n-k}}, \; k = 0, 1, \ldots, n - 1,$$  \hfill (3)

and similar formulas for the derivative $P'(z)$:

$$(-1)^{n-k} \frac{k}{n} a_k = \sum_{1 \leq i_1 < i_2 < \cdots < i_{n-k} \leq n-1} \omega_{i_1} \omega_{i_2} \cdots \omega_{i_{n-k}}, \; k = 1, \ldots, n - 1.$$  \hfill (4)

The proof will be carried out by induction. First, we prove the statement of the Theorem 1 for $k = 1$. Indeed, a chain of relations follows from the formulas (1), (2), (3):
In the second last inequality in the chain, we used the strong triangle inequality for the norm $| \cdot |$. Thus, for the case of $k = 1$, the Theorem 1 is proved.

Now let’s assume that the theorem’s statement holds for $k = m - 1 \leq n - 2$ and prove in this assumption that the theorem holds for $k = m$ as well. Similarly to the reasoning for the case $k = 1$, using the relations (3), (4) and the strong triangle inequality, we obtain the following estimate for the sum of all products of $n - m$ of the roots of the polynomial $P'(z)$:

$$
\sum_{1 \leq i_1 < i_2 < \cdots < i_{n-m} \leq n} \omega_{i_1} \omega_{i_2} \cdots \omega_{i_{n-m}} = \frac{m}{n} | a_m | = \frac{m}{n} \left| \sum_{1 \leq i_1 < i_2 < \cdots < i_{n-m} \leq n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_{n-m}} \right| \leq \frac{m}{n} | r^{n-m} |. \quad (5)
$$

Further proof will be carried out by contradiction. That is, suppose that the statement of the Theorem 1 is not true for $k = m$. This means that the inequality (2) holds for all $k \leq m - 1$, but does not hold for $k = m$. Therefore, the inequality is valid:

$$
| \omega_m | > r_m = r \max \left\{ \left| \frac{j}{n} \right|, j = 1, 2, \ldots, m \right\}. \quad (6)
$$

Under this assumption, the following chain of inequalities holds:

$$
| \omega_1 | \leq | \omega_2 | \leq \cdots \leq | \omega_{m-1} | \leq r_{m-1} \leq \cdots \leq r_m < | \omega_m | \leq | \omega_{m+1} | \leq \cdots \leq | \omega_{n-1} |. \quad (7)
$$
Therefore, among all products of $n - m$ roots of the polynomial $P'(z)$, the product $\omega_m \omega_{m+1} \cdots \omega_{n-1}$ has a strictly maximal norm.

Further, using the following property of the non-Archimedean norm: $|a + b| = |b|$ if $|a| < |b|$, $a, b \in \mathbb{C}_p$, we get the estimate

$$\left| \sum_{1 \leq i_1 < i_2 < \cdots < i_{n-m} \leq n-1} \omega_{i_1} \omega_{i_2} \cdots \omega_{i_{n-m}} \right| = |\omega_m \omega_{m+1} \cdots \omega_{n-1}| \geq |\omega_m|^{n-m}. \quad (8)$$

The inequalities (5) and (8) directly imply the validity of the following estimate:

$$|\omega_m| \leq r \frac{m}{n}^{1/m}.$$  

The last inequality contradicts assumption (2). The resulting contradiction completes the proof of the Theorem.

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