On the Hawking effect

Ralf Schützhold
Institut für Theoretische Physik, Technische Universität Dresden
D-01062 Dresden, Germany
Electronic address: schuetz@theory.phy.tu-dresden.de
(November 3, 2018)

PACS-numbers: 04.70.Dy, 04.70.-s, 04.62.+v.

I. INTRODUCTION

One of the great challenges of theoretical physics is the quest for an underlying law that unifies quantum theory and general relativity. The investigation of quantum fields in curved space-times is expected to provide a chance of achieving some progress towards this aim. Fulling’s discovery of the non-uniqueness of the particle interpretation in curved space-times may be regarded as a basis for various fundamental effects, see e.g. [1–27] and references therein. Perhaps the most prominent example is the Hawking effect which predicts the evaporation of black holes. There are two alternatives for the interpretation of this striking effect: Originally Hawking calculated the Bogoliubov coefficients via the geometric optics approximation (backwards ray-tracing) in Ref. [1]. In contrast to this dynamical treatment Unruh imposed boundary conditions on the static regime in order to reproduce the main features of the Hawking effect. Since the state defined in this way – the Unruh state – is completely stationary, it merely describes the late-time part of the radiation. Of course, in general there exists some temporary amount of created particles that depends on the dynamics of the collapse. But according to Ref. [1] the number of these particles is finite with the result that they disperse after a finite period of time and thus do not affect the (divergent) late-time radiation.

Ergo it appears quite natural to assume that the state after the complete gravitational collapse coincides up to a finite number of particles with the Unruh state describing evaporation – independently of the particular dynamics of the collapse. The question of whether this assertion is strictly correct will be subject of the present article. For that purpose we shall calculate the number of created particles explicitely without employing the geometric optics approximation. It will turn out that the above statement is justified only for one particular branch of the rather general class of dynamics of the collapse.

This paper is organized as follows: In Section I we set up the basic properties of the quantum field under consideration. A brief introduction into the concept of Hadamard states is presented in Sec. II A. The number of created particles is calculated in Section II B. In Secs. II A and II B we deduce the eigen modes in terms of the Schwarzschild and the Painlevé-Gullstrand-Lemaître coordinates, respectively. The Bogoliubov coefficients are derived explicitely in Secs. II C and II D. In Section II E the relevant expectation values of the energy-momentum tensor are calculated. We shall close with a summary, some conclusions, a discussion, and an outlook.

Throughout this article natural units with \( G = \hbar = c = k_B = 1 \) will be used. Lowercase Greek indices such as \( \mu, \nu \) vary from 0 (time) to 3 (space) and describe space-time components (Einstein sum convention). Uppercase Roman indices \( I, J \) denote complete sets of quantum numbers.

II. GENERAL FORMALISM

We consider a minimally coupled, massless and neutral (i.e. Hermitian) scalar (spin-zero) quantum field \( \hat{\Phi} \) propagating on a globally hyperbolic space-time \( (\mathcal{M}, g_{\mu\nu}) \). Global hyperbolicity demands strong causality and completeness, cf. [28]. (Without these requirements the time-evolution of the quantum system is not well-defined and unitary.) In the Heisenberg representation the kinematics of the field \( \hat{\Phi} \) is governed by the Klein-Fock-Gordon equation

\[
\Box \hat{\Phi} = \frac{1}{\sqrt{-g}} \partial_{\mu} \left( \sqrt{-g} g^{\mu\nu} \partial_{\nu} \hat{\Phi} \right) = 0. \tag{1}
\]

Strictly speaking, the quantum field is represented by an operator-valued distribution \( \hat{\Phi} \) and hence the above equation has to be understood in this sense: \( \Box F = 0 \rightarrow \hat{\Phi}[F] = 0 \). In a globally hyperbolic space-time the wave equation (1) possesses unique advanced and retarded Green functions \( \Delta_{\text{adv}}(x, x') \) and \( \Delta_{\text{ret}}(x, x') \), respectively. Employing these distributions one may accomplish the
canonical quantization procedure via imposing the co-

\[ \[\Phi(x), \Phi'(x') = \Delta_{\text{rel}}(x, x') - \Delta_{\text{adv}}(x, x'). \] (2) \]

The solutions of the equation of motion \( (1) \) obey a sym-

\[ (F|G) = i \int d\Sigma^\mu F^* \overset{\leftrightarrow}{\partial_\mu} G, \] (3)

with \( F \overset{\leftrightarrow}{\partial_\mu} G = F \partial_\mu G - G \partial_\mu F \). With the aid of Gauss’ law one can show that the inner product \( (3) \) is independent of the particular Cauchy surface \( \Sigma \) for any two solutions of the Klein-Fock-Gordon equation \( \square F = \Box G = 0 \), cf. [28]. It should be mentioned here that the measure \( d\Sigma^\mu \) used above already contains volume factors like \( \sqrt{-g} \); and is normalized according to \( d\Sigma_\mu dx^\mu = \sqrt{-g} dx^x \).

The canonical commutation relations \( (2) \) imply

\[ \left( (F|G) \right) = (F|G). \] (4)

As a result the inner product of the field \( \Phi \) with positive \( (F_I) \) and negative \( (F_I') \) frequency solutions of the Klein-

\[ F^* \overset{\leftrightarrow}{\partial_\mu} G \]

Fock-Gordon equation, respectively, with \( (F_I|F_I') = - (F_I'|F_I) = \delta(I, J) \) and \( (F_I|F_J') = 0 \) defines creation and annihilation operators, respectively. As it is well-

\[ \delta(I, J) \]

known, these operators and thus also the associated number operators depend on the particular choice of the solutions \( F_I \). This ambiguity represents the non-uniqueness of the particle interpretation (see e.g. [28]) and may be regarded as the basis of the phenomenon of particle creation induced by the gravitational field.

Averaging the operator-valued distributions \( \Phi(x) \) with \( n \)-point test functions \( B_n \in C_0^\infty(M^n) \) via

\[ \hat{\Phi}^n[B_n] = \int d^4x_1 \cdots d^4x_n \hat{\Phi}(\vec{x}_1) \cdots \hat{\Phi}(\vec{x}_n) \times B_n(\vec{x}_1, \ldots, \vec{x}_n) \] (5)

we acquire well-defined operators \( \hat{\Phi}^n[B_n] \). The complete set of all these operators (constructed for all test functions) generates the *-algebra containing all possible ob-

\[ \hat{\Phi}(\vec{x}_1) \cdots \hat{\Phi}(\vec{x}_n) \]

servables of the quantum system (with the unit element \( 1 = \hat{\Phi}^0[1] \)).

The states \( \varrho \) of the quantum system can be introduced as linear \( \varrho(\mu \hat{X} + \nu \hat{Y}) = \mu \varrho(\hat{X}) + \nu \varrho(\hat{Y}) \) and non-negative \( \varrho(\hat{Z}^\dag \hat{Z}) \geq 0 \) functionals over the *-algebra with unit norm \( \varrho(1) = 1 \). All these states \( \varrho \) build up a convex set, i.e., for any two states \( \varrho_1 \) and \( \varrho_2 \) also the convex combination \( \varrho_3 = \lambda \varrho_1 + (1-\lambda) \varrho_2 \) with \( 0 < \lambda < 1 \) represents an allowed state. The extremal points of this convex set correspond to the pure states \( \hat{\varrho} = |\Psi_\varrho \rangle \langle \Psi_\varrho | \). Since every convex set is the convex hull of its extremal points all (mixed) states can be written as a (possibly infinite) linear combination of pure states.

In order to decide whether a state is pure or mixed in character one has to consider the complete algebra. Focusing on a sub-algebra a pure state may display properties that are usually connected with mixed states. This observation might be regarded as the basis of the thermo-

\[ \varrho(\hat{\Phi}(\vec{x}_1) \cdots \hat{\Phi}(\vec{x}_n)) \]

field formalism, see e.g. [29] and [8].

It might be interesting to illustrate these points by some examples: The space-time obeying the Schwarzschild geometry possesses a Killing vector mediating the associated (Schwarzschild) time-evolution. The ground state of the quantum field (with respect to that Killing vector) in the region outside the horizon is called the Boulware state \( \varrho_B \). It contains no particles – again with respect to the Killing vector measuring the time of an outside observer with a large and fixed spatial distance to the center of gravity. (A free-falling explorer may well detect particles in that state.) Ergo the Boulware state is a pure state with respect to the algebra of the exterior region \( \varrho_B = |\Psi_B \rangle \langle \Psi_B | \). The interior domain possesses no ground state at all, cf. [29]. (Again all assertions refer to the Killing field along the Schwarzschild time.) As further interesting states one may introduce the Kubo-

\[ \varrho_I \]

Martin-Schwinger (KMS, [31]) states \( \varrho_I \) describing thermal equilibrium at some given temperature \( T \). Obvi-

\[ \varrho_I \]

ously these states are mixed in character – at least from the exterior point of view. One important example is the Israel-Hartle-Hawking state \( \varrho_{\text{IHH}} \) which satisfies (in the exterior region) the KMS condition corresponding to the Hawking temperature. It can be shown [14] that this state is indeed a pure state with respect to an enlarged algebra. The Israel-Hartle-Hawking state \( \varrho_{\text{IHH}} \) contains the same number of ingoing and outgoing particles (thermal equilibrium). Hence the total energy flux vanishes. The phenomenon of the black hole evaporation can be described by the Unruh state \( \varrho_U \). This state is defined via two requirements: no ingoing/incoming parti-

\[ \varrho_U \]

cles/radiation at spatial infinity and thermal outgoing radiation near the horizon, see also [31]. If one considers a gravitational collapse of an object and assumes the initial state to be pure in character (e.g. the vacuum) then the final state is – of course – also a pure state. (Here the notion of a pure state refers in both cases to the complete algebra.) The question of whether the initial state indeed transforms into the Unruh state will be subject of Section [2].

A. Hadamard states

In general, the complete convex set is too large and contains more states than physical reasonable. One way to restrict to physically well-behaving states is to im-

\[ \varrho \]

pose the so-called Hadamard [31] condition. Hadamard states are states for which the symmetric part of the bi-

\[ \varrho(\hat{\Phi}(\vec{x}_1) \cdots \hat{\Phi}(\vec{x}_n)) \]

distribution

\[ \varrho_{\text{(I)}}(\vec{x}, \vec{x}') = \varrho \left\{ \hat{\Phi}(|\vec{x} \rangle \langle \vec{x}' |) \right\} = \text{Tr} \left\{ \varrho \hat{\Phi}(\vec{x}) \hat{\Phi}(\vec{x}') \right\}, \] (6)

\[ \varrho_{\text{(I)}}(\vec{x}, \vec{x}') \]
the fluctuation of W(2), which fulfills the Hadamard requirement (away from the horizon r > R, see Fig. 2). But only the KMS state corresponding to the Hawking temperature, i.e., the Israel-Hartle-Hawking state (after a suitable extension) meets the Hadamard condition at the horizon, see e.g. [23, 13]. However, the initial (approximately Minkowski) vacuum cannot transform into this state during a gravitational collapse of an object, cf. [23]. In contrast to the Unruh state the Israel-Hartle-Hawking state represents thermal equilibrium also for r \rightarrow \infty and the associated amount of particles and energy cannot be produced by a collapse, see also Secs. III D and III E below.

### III. PARTICLE CREATION

Within the Heisenberg representation the time-evolution of the quantum system is governed by the operators while the states remain unaffected. Hence the investigation of the Hawking effect goes along with the question: How many (final) Schwarzschild particles contains the initial state? In general, this number depends on the particular initial state and the initial metric as well as the dynamics of the metric during the collapse. According to the considerations in the previous Section we assume a \(C^\infty\)-metric throughout. It can be shown that the Hawking effect (i.e. the late-time radiation) is independent of the (regular) initial space-time, see Sec. III D below. Similarly any finite amount of particles being present initially does not alter the assertions concerning the Hawking effect (see the remarks at the end of Section III D below). For that reason we assume the initial state to coincide with the (initial) vacuum. In this situation the number of final particles can be calculated via the Bogoliubov \(\beta\)-coefficients, see e.g. [23].

\[ N_j = \langle 0^\text{in} | J^\text{out}_j | 0^\text{in} \rangle = \sum_f |\beta_{IJ}|^2 . \] (8)

In order to calculate these coefficients we have to derive the structure of the initial modes \(F^\text{in}_I\) after the collapse and to compare them with the out-solutions \(F^\text{out}_J\) by means of the inner product in Eq. (3).

#### A. Schwarzschild metric

The particle interpretation in quantum field theory is based on the selection of an appropriate time-like Killing vector. This choice refers to a certain class of associated observers whose time-evolution is generated by the Killing field. For the flat space-time example, the Killing vector mediating the (Minkowski) time translation symmetry accrues to a usual beholder at rest whereas special Lorentz boosts represent accelerated (Rindler) explorers.
Since, in general, different Killing vectors generate distinct particle definitions, the Rindler observer does not regard the Minkowski vacuum as empty with respect to (Rindler) particles. Instead, he/she experiences a thermal bath, a phenomenon which is called the Unruh effect.

The time-evolution of a beholder at a large and fixed spatial distance to the center of gravity is generated by the Killing vector corresponding to the Schwarzschild time $t$. The particles that are measured by such an observer can be described by positive frequency solutions – with respect to that time coordinate – of the Klein-Fock-Gordon equation. In contrast the evolution parameters of further coordinate representations of the Schwarzschild geometry (e.g. the Lemaître metric) represent different explorers (e.g. the free-falling one) in general.

In terms of the Schwarzschild coordinates $t, r, \vartheta, \varphi$ the 3+1 dimensional metric assumes the well-known form

$$ds^2 = \left(1 - \frac{R}{r}\right)dt^2 - \frac{1}{1 - \frac{R}{r}}dr^2 - r^2d\vartheta^2 - r^2\sin^2 \vartheta d\varphi^2,$$

where $R$ denotes the Schwarzschild radius and describes the position of the horizon.

Strictly speaking, there exist several definitions of a horizon, for example the event, the apparent, and the putative horizon, cf. [26] and [27]. The notion of the event horizon refers to the global structure of the space-time (asymptotical reachability) whereas the apparent horizon coincides with a coordinate singularity at the horizon and is therefore not considered, cf. [24]. The ingoing and outgoing modes are distinguished by $\xi = \pm 1$. (For the modifications occurring in 3+1 dimensions see Sec. VII.) $N^\text{out}$ symbolizes a normalization factor which may without any loss of generality chosen to be independent of $\xi$. These eigenfunctions are rapidly oscillating near the horizon which again hints that there is the most interesting region: This singular behavior of the modes corresponds to the freezing of the kinematics of the field (governed by the Klein-Fock-Gordon equation) in the vicinity of the (putative) horizon.

B. Painlevé-Gullstrand-Lemaître metric

The Schwarzschild metric is quite simple but exhibits a coordinate singularity at the horizon and is therefore not $C^\infty$ there. Hence it is impossible to express a manifestly $C^\infty$-metric in terms of the Schwarzschild coordinates. For this purpose one has to employ other coordinate systems. As one possible candidate we consider the Painlevé-Gullstrand-Lemaître [33] coordinates $t_{\text{PGL}}, r, \vartheta, \varphi$. These coordinates emerge from the Schwarzschild coordinates $t_S, r, \vartheta, \varphi$ by means of the transformation

$$dt_{\text{PGL}} = dt_S + \sigma \sqrt{\frac{R}{r}} \frac{dr}{1 - \frac{R}{r}},$$

for $r > R$ and vanish for $r < R$ due to the horizon, cf. [24]. The ingoing and outgoing modes are distinguished by $\xi = \pm 1$. (For the modifications occurring in 3+1 dimensions see Sec. VII.) $N^\text{out}$ symbolizes a normalization factor which may without any loss of generality chosen to be independent of $\xi$. These eigenfunctions are rapidly oscillating near the horizon which again hints that there is the most interesting region: This singular behavior of the modes corresponds to the freezing of the kinematics of the field (governed by the Klein-Fock-Gordon equation) in the vicinity of the (putative) horizon.

$$ds^2 = \left(1 - \frac{R}{r}\right)dt^2 - 2\sigma \sqrt{\frac{R}{r}} dr dt - dr^2 - r^2d\vartheta^2 - r^2\sin^2 \vartheta d\varphi^2.$$
The hyper-surfaces of constant PGL time $dt = 0$ are equivalent to flat Euclidean spaces; the space-time curvature is encoded in the shift vector (employing the Arnowitt-Deser-Misner (ADM) notation).

The evolution parameter, i.e. the PGL time, corresponds to a Killing vector leading to a stationary metric. This fact simplifies the particle definition via positive frequency solutions.

Asymptotically $r \uparrow \infty$ the PGL representation coincides with the Minkowski metric (similar to the Schwarzschild form). By virtue of Birkhoff’s theorem this coincidence persists during the dynamical period of the collapse.

The radial coordinate $r$ directly corresponds to the surface of the two-sphere $\{t, r = \text{const}\}$ via $4\pi r^2$.

Last but not least the effective acoustic metric of the sonic analogues of the Schwarzschild geometry equals – up to a conformal factor – the PGL form, see e.g. [38].

For further discussions of the properties of the PGL metric see e.g. Refs. [34–40] and references therein. For example Ref. [35] presents a pedagogical presentation of the PGL metric as well as its relation to the Eddington-Finkelstein coordinates.

It should be mentioned here that the PGL coordinates do not cover the complete fully extended Kruskal manifold: E.g., depending on the particular branch (i.e. the sign of $\sigma$) the PGL representation covers either the future (black hole) event horizon and the future (black hole) singularity for $\sigma = +1$ or the past (white hole) event horizon and the past (white hole) singularity for $\sigma = -1$, see Figs. 1 and 2. However, both branches possess an apparent horizon at $r = R = 2M$.

Since we shall calculate the inner product in terms of the new coordinates we have to transform the Schwarzschild eigenfunctions, i.e. the out-modes. This can be done by simply substituting the Schwarzschild time via $t_S = t_{PGL} - \sigma R \ln \chi + \mathcal{O}[\chi]$ in Eq. (14)

$$F_{\text{out}}(\chi) = \mathcal{N}_{\text{out}} e^{-i\omega t} \frac{1}{\sqrt{\omega}} \chi^{i(\sigma - \xi) + \omega R (1 + \mathcal{O}[\chi])}.$$  (17)

(Again we restrict our considerations to 1+1 dimensions.)

One observes that the modes with $\xi = \sigma$ are no longer singular (arbitrarily fast oscillating) at the horizon, only those with $\xi = -\sigma$ still exhibit this property. As it will become evident later on, merely the singular modes with $\xi = -\sigma$ will contribute to the Hawking effect.

Employing the Painlevé-Gullstrand-Lemaître coordinates it is possible to write down a manifestly $C^\infty$-metric modeling a gravitational collapse of an object and the subsequent formation of a horizon

$$ds^2 = \left(1 - f^2(t, r)\right) dt^2 - 2\sigma f(t,r)drdt - dr^2,$$  (18)

with $f \in C^\infty$. Initially $t \downarrow -\infty$ the metric describes a regular object with a (relatively) dilute distribution of matter and can be approximated (locally) by the Minkowski metric $f(t \downarrow -\infty, r) = f_{in}(r) \ll 1$. For reasons of simplicity we assume the horizon to be formed
at $t = 0$, i.e. $f(t \geq 0, r \geq R) = f_{\text{out}}(r) = \sqrt{R/r}$.
(Note, that we did not impose any conditions on the structure of $f$ in the interior region, i.e. beyond the horizon.)
Outside the (spherically symmetric) collapsing object the Birkhoff theorem demands a stationary metric $f(t, r \gg R) = \sqrt{R/r}$.

The Jacobi determinant is simply given by $\sqrt{-g} = 1$
and the metric as well as its inverse are smooth $g_{\mu\nu}, g^{\mu\nu} \in C^\infty$. Of course, this assertion holds true only if we omit the formation of the singularity at $r = 0$. But the region beyond the horizon is causally separated from the outside domain and (as it will turn out later) irrelevant for our purposes.

Considering the sonic analogues of the Schwarzschild geometry the function $f(t, r)$ directly corresponds to the time-dependent local velocity of the fluid, cf. [37–40].

It should be mentioned here that the knowledge of the above metric over a finite period of time is not sufficient for determining an event horizon – in contrast to the eternal (stationary) metric in Eq. (16). The local metric above does also not allow for the construction of a Penrose diagram (see Figs. 1-5) – this requires the extension to the complete space-time. Similarly it does not necessarily contain a space-time singularity. However, one may deduce the existence of an apparent horizon at $r = R$ for $t \geq 0$. In a purely 1+1 dimensional consideration the range of the coordinate $r$ in Eq. (18) might be chosen arbitrarily. But in order to keep contact to the 3+1 dimensional situation it should be specified according to $0 \leq r < \infty$.
The 3+1 dimensional bouncing-off effect (see also Fig. 3) at the origin $r = 0$ can be simulated in 1+1 dimensions by an appropriate boundary condition. As already stated in Ref. [3], Dirichlet or Neumann boundary conditions or every linear combination of them are suitable, see also the discussions at the end of the next Section.

![FIG. 3. Penrose diagram of the Minkowski space-time. The dashed line symbolizes the (regular) origin $r = 0$. Light rays originating from $J-$ bounce off at the origin and propagate to $J+$. The Minkowski time equals the PGL time for $f(t, r) = 0$. Again representative surfaces of constant (PGL) time are indicated.](image)

![FIG. 4. Penrose diagram of the collapse to a black hole as described by the branch $\sigma = +1$ of the PGL metric with an appropriate function $f(t, r)$. As one can infer from the indicated surfaces of constant PGL time, the formation of the black hole horizon (dotted line) can be described regularly by these coordinates (with $\sigma = +1$).](image)

![FIG. 5. Penrose diagram of the collapse to a white hole as described by the branch $\sigma = -1$ of the PGL metric with an appropriate function $f(t, r)$. After the formation of the white hole horizon (dotted line) no light ray originating from $J-$ can reach $r = 0$. The particular structure of this figure is based on the (not necessary) assumption that the singularity at $r = 0$ develops at a finite period of PGL time after the horizon has been formed. Again one may infer from the indicated surfaces of constant PGL time that the formation of the white hole horizon can be described regularly by the branch $\sigma = -1$ of the PGL metric. In contrast to Figs. 1 and 2 this diagram cannot be obtained by time-reversing the previous figure.](image)

Although the two distinct branches $\sigma = \pm 1$ of the stationary Painlevé-Gullstrand-Lemaître metric in Eq. (10) are related to each other via a simple change of the coor-
dines or the time-inversion $T : t \to -t$, the distinction between the different collapse dynamics for $\sigma = +1$ and $\sigma = -1$, respectively, in Eq. (18) cannot be removed by any transformation. (It is not possible to find a globally integrating factor for the differential form.) The two branches correspond to two non-equivalent collapse scenarios (see Figs. 1 and 2 and – as we shall see later – generate completely different final states of the quantum field. Nevertheless, in both cases the initial $t \downarrow -\infty$ metric describes a regular object whereas the final $t \uparrow \infty$ metric represents for $r > 0$ a vacuum solution of Einstein’s equations with a central mass $M = R/2$. So there exists a priori no reason to prefer one of the two branches, see also the remarks in Section VI.

C. Eikonal ansatz

In order to calculate the Bogoliubov coefficients we have to deduce some informations about the in-modes. For that reason we adopt the eikonal ansatz and divide the field into an amplitude and a phase

$$F_{\xi,\omega}^{\text{in}}(t, r) = \frac{1}{\sqrt{\omega}} A_{\xi}(t, r) \exp \{-i\omega S_{\xi}(t, r)\} \times \left(1 + O\left(\frac{1}{\omega}\right)\right).$$ (19)

Certainly this ansatz will be justified for compact space-time domains (which are not too large) with smooth metrics and high (initial) frequencies $\omega$ (see also the remarks at the end of this Section). But as it will turn out later, this is exactly the limit that is relevant for the Hawking effect. Inserting the above expression into the Klein-Fock-Gordon equation (8) the leading terms in $\omega$ govern the kinematics of the phase function via

$$(\partial_{\mu} S_{\xi}) g^{\mu\nu} (\partial_{\nu} S_{\xi}) = 0, \quad \text{i.e.,}$$

$$(\partial_{s} S_{\xi} - \sigma f \partial_{s} S_{\xi})^{2} = (\partial_{s} S_{\xi})^{2}. \quad \text{(20)}$$

This non-linear equation has four separate branches of solutions – e.g. for $f = 0$ one may identify the positive and negative frequency solutions on the one hand and the ingoing and outgoing components labeled by $\xi = \pm 1$ on the other hand

$$\partial_{s} S_{\xi} - \sigma f \partial_{s} S_{\xi} = \xi \partial_{s} S_{\xi}, \quad \text{i.e.,}$$

$$\partial_{s} S_{\xi} = (\sigma f + \xi) \partial_{s} S_{\xi}. \quad \text{(21)}$$

However, these four branches will not necessarily be separated for arbitrary space-time dependent functions $f(t, r)$. E.g., if $f(t, r)$ oscillates with a large elongation, a mode which is initially purely ingoing may turn its direction into outgoing and so on. Nevertheless, if we assume a sufficiently well-behaving dynamics of $f$, e.g. if it transforms directly and smoothly from $f_{\text{in}}$ to $f_{\text{out}}$ – where the relevant time-scales are smaller than the length scales ($R$) – the four branches remain separated: In this case the different branches cannot approach each other close enough during the time-evolution. As a limiting case we may consider a very rapid change (sudden approximation) of the metric $f(t, r) \approx f_{\text{in}}(r)\Theta(-t) + f_{\text{out}}(r)\Theta(t)$. In this situation the final phase function $S_{\xi}$ coincides (nearly) with its initial form while its time-derivative changes according to Eq. (21). The sudden approximation does not hold in contrast with the high frequency limit since we deal with the frequency-independent phase function $S_{\xi}$.

In summary, the above assumption of a rapid collapse ensures the separation of the four branches, e.g. if $\partial_{s} S_{\xi}$ is positive/negative initially then it remains positive/negative also after the collapse. The same applies to the time-derivative $\partial_{t} S_{\xi}$ – as long as $f < 1$, i.e. outside the horizon, see Eq. (21). As a consequence the division of the modes into ingoing and outgoing – labeled by $\xi = \pm 1$ – can be used throughout.

Additional complications arise in 3+1 dimensions, but the main result – the separation of the four branches – persists under appropriate assumptions: As demonstrated in Ref. [4], a regular spherically symmetric 3+1 dimensional space-time without horizon does not allow for the definition of ingoing and/or outgoing particles. The eigenmodes are standing waves, i.e. linear combinations of ingoing and outgoing components with equal weights. So the bouncing-off effect at $r = 0$ mixes the ingoing and outgoing components during static as well as during the dynamical period. In a 1+1 dimensional consideration this “reflection” may be simulated by an effective boundary condition at $r = 0$, cf. the remarks below Eq. (4). Selecting appropriate coordinates the point $r = 0$ becomes time-dependent. E.g., in terms of length and time-scales associated to an outside observer the center of the collapsing object goes to infinity (asymptotically at a null line) owing to the formation of the horizon. In terms of these particular coordinates the origin $r = 0$ corresponds to an accelerated mirror. Ref. [10] presents a derivation of the Hawking effect based on the moving mirror analogue.

In contrast, in terms of the Painlevé-Gullstrand-Lemaître coordinates the origin $r = 0$ obeys no time-dependence at all. Again we assume the collapse to occur fast enough: The metric is presumed to remain stationary until $t = -R$ (the beginning of the collapse) and according to Sec. [11] the (apparent) horizon at $r = R$ is formed at $t = 0$. Since the Schwarzschild eigenfunctions vanish for $r < R$ it is sufficient to consider the region outside the horizon to be formed at $r \geq R$. Within this limited space-time domain $\{-R \leq t \leq 0, r \geq R\}$ the ingoing and outgoing components are indeed effectively independent: It takes every ingoing light ray inside this domain at least the time duration $\Delta t = R$ (under the assumptions made for the function $f(t, r)$ above) to propagate to the origin, to bounce off (turning its direction into outgoing), and to reach the radius $r = R$ again. An analogue assumption was already imposed in Ref. [1]. In such a scenario the information about a possible ”reflection” at $r = 0$ cannot
influence the relevant region. In summary we arrive at the conclusion that (under the assumptions made) the four branches in Eq. (21) are indeed effectively independent also in 3+1 dimensions (see also the discussion at the end of Sec. IV).

It should be mentioned here that the above ansatz is not equivalent to the quasi-classical Jeffreys-Wentzel-Kramers-Brillouin (JWKB) approximation (expansion into powers of $\hbar$) – in spite of some similarities. It does also not coincide with the geometric optics approximation (backwards ray tracing) which was used in Ref. [1]. Instead the eikonal ansatz is based on a consequent expansion into inverse powers of the initial frequency $\omega$.

D. Bogoliubov coefficients

Now we are in the position to calculate the Bogoliubov coefficients and thereby the number of created particles explicitly. Unfortunately, it seems to be impossible to find a general solution for these overlap coefficients. Nevertheless, with an expansion into powers of the relative distance to the horizon $\chi$ and the inverse initial frequency $1/\omega$ it is possible to extract the leading contribution – the Hawking effect. (As it will turn out later, the sub-leading parts merely generate finite contributions and thus do not affect the late-time radiation.) Per definition the Hawking radiation is exactly that part of the radiation which persists at arbitrarily late times (if we neglect the back-reaction). Hence the number of created particles accounting for the Hawking effect has to diverge. Any finite amount of particles would disperse after a finite period of time and cannot generate late-time radiation. (This is a consequence of the spectral properties of the wave equation. It possesses a purely continuous spectrum and thus does only allow for scattering states but no bound states, see e.g. [24] and [25].) As demonstrated in Ref. [24], the divergent number of particles is necessary for the thermal behavior in an infinite volume. In order to isolate the divergent part of the number of created particles we have to consider the Bogoliubov $\beta$-coefficients (see e.g. [24])

$$\beta_{IJ} = i \int d\Sigma_{\mu} F^{{\text{in}}}_{I} \frac{\leftrightarrow}{\partial_{\mu}} F^{{\text{out}}}_{J}. \tag{22}$$

Since the Painlevé-Gullstrand-Lemaître coordinates are completely regular the measure $d\Sigma_{\mu}$ does not contain any singularities. As we have observed in the previous Sections, the modes $F^{{\text{in}}}_{I}$ and $F^{{\text{out}}}_{J}$ are bounded. In addition, the Birkhoff theorem implies that the modes at very large spatial distances to the collapsing object are not affected by the collapse. Consequently this region does not contribute to the $\beta$-coefficients and generates a $\delta(\omega - \omega')$-term for the $\alpha$’s, see also [1]. In summary we arrive at the conclusion that all (single) Bogoliubov $\beta$-coefficients are finite. As a result the divergence of the number of created particles $N^{{\text{out}}}_{I}$ must be traced back to the summation/integration over the initial quantum numbers $I = (\xi, \omega)$ in Eq. (8)

$$N^{{\text{out}}}_{J} = \int_{I} |\beta_{IJ}|^2.$$

There are two possibilities for a singularity, the IR- and the UV-divergence of the integration over the initial frequencies $\omega$. In the limit of small frequencies $\omega$ the modes become space- and time-independent and approach a constant – unaffected by the Klein-Fock-Gordon equation. (Here we regard the IR-singular normalization $1/\sqrt{\omega}$ as factorized out.) Ergo in the limiting case $\omega \downarrow 0$ the in- and out-modes coincide and thus possess a vanishing overlap with all other modes corresponding to finite frequencies. As a consequence the $\omega$-integration of the (absolute values squared of the) Bogoliubov coefficients is IR-save.

In summary the infinite amount of particles has to be caused by the UV-divergence of the integration over the initial frequencies in consistency with Ref. [1]. (The Hawking effect is dominated by large (initial) frequencies only if one considers a fundamental quantum field theory without any kind of dispersion. Introducing a cut-off, see e.g. [1], as an effective description of some underlying theory the calculations are different.)

Recalling the structure of the initial eigenfunctions in Eq. (19) we arrive at the conclusion that only singularities of the out-modes may induce a UV-divergence. The convolution of regular expressions with the for $\omega \uparrow \infty$ arbitrarily fast oscillating in-modes yields results of order $1/\omega$. Ergo the subsequent $\omega$-integration would be UV-save. Indeed, the out-modes are not regular at the horizon – the region that is naturally relevant for the Hawking effect. Thus it is adequate to consider the vicinity of the horizon and the high (initial) frequency limit in order to extract the Hawking effect. As it will become more evident later, exactly the leading contributions in $\chi$ and $1/\omega$ are sufficient for the derivation of the thermal radiation.

If we choose the Cauchy surface according to $\Sigma = \{0 < r < \infty, t = 0\}$ the surface element assumes the form $d\Sigma_{\mu} = (dr, 0)$ and the $\beta$-coefficients transforms into

$$\beta_{IJ} = i \int dr \ F^{{\text{in}}} (\frac{\leftrightarrow}{\partial_{t}} - \sigma f \frac{\leftrightarrow}{\partial_{r}}) F^{{\text{out}}}_{J}, \tag{23}$$

with the quantum numbers $I = (\xi, \omega)$ and $J = (\xi', \omega')$. Inserting the result of the previous Section $\partial_{t} S_{\xi} - \sigma f \partial_{r} S_{\xi} = \xi \partial_{t} S_{\xi}$ we arrive at

$$\beta_{IJ} = \int dr \ A_{\xi} \exp \{-i\omega S_{\xi}\} \times \frac{i\omega \xi \partial_{t} S_{\xi} - \omega' (1 + \sigma f [\sigma - \xi']/\chi)}{\sqrt{\omega \omega'}} \chi^{i(\sigma - \xi')\omega' R} \times N (1 + O(\chi)) \left(1 + O \left(\frac{1}{\omega} \right)\right). \tag{24}$$
At this stage the correct meaning of the Landau symbols $\mathcal{O}(|\chi|$ and $\mathcal{O}(1/\omega)$ should be explained: All terms which are of higher order in $\chi$ and of the same order in $\omega$ as well as all terms which are of lower order in $\omega$ and of the same order in $\chi$ – in comparison with the leading contributions in the integrand above – are neglected. Such a detailed consideration is especially necessary for the quantity $\exp\{-i\omega S_\xi\}$ which involves terms like $(\chi \omega)^n$. These contributions are not neglected – in contrast to terms like $(\chi \omega)^n$.

Accordingly, (exploiting the dominance of the vicinity of the horizon) we may Taylor expand the amplitude $A_{\xi,\omega}(t = 0, r) = A_{\xi,\omega}(t = 0, R) + \mathcal{O}|\chi|$). The zeroth-order term can be absorbed into the overall normalization factor $N$ via

$$N \to NA_{\xi,\omega}(t = 0, R),$$

and the higher order terms are omitted. (Here and in the following we do not change the symbol $N$ for the normalization factor and use the same letter also for the modified pre-factors.) A similar procedure can be performed with the phase function $S_\xi$. But owing to the pre-factor $\omega$ it is necessary to expand it up to first order

$$S_\xi(r) = S_\xi(r = R) + \partial_r S_\xi(r = R) R \chi + \mathcal{O}(\chi^2),$$

cf. the remarks after Eq. (24). Again the zeroth-order term $S_\xi(t = 0, r = R)$ may be absorbed by a redefinition of $N$

$$N \to \exp \{-i\omega S_\xi(t = 0, r = R)\}.$$

Since we have to integrate over the initial frequency $\omega$ in Eq. (8) in order to obtain the number of created particles, the remaining unknown first-order term $\partial_r S_\xi(t = 0, r = R)$ can be eliminated by a re-scaling of the initial frequency

$$\omega \to \tilde{\omega} = \omega \xi \partial_r S_\xi(t = 0, r = R).$$

Of course, such a transformation may be accomplished if and only if $\xi \partial_r S_\xi$ is positive. But according to the arguments at the end of the previous Section the sign of $\partial_r S_\xi$ does not change during the collapse – as long as it occurs fast enough and regularly. In this situation the four different branches of Eq. (24) do not mix and thus the sign of $\partial_r S_\xi$ equals its initial value, i.e. $\xi$. Again we may consider a very abrupt change of the metric (sudden approximation, see the previous Section) as an illustrative example, where the final phase function nearly coincides with its initial form. For the Minkowski example it is simply determined by $\partial_r S_\xi \approx \xi$ and no redefinition is necessary at all. For other initial metrics the redefinition of the frequency exactly corresponds to the fact that the Hawking effect is independent of the initial (regular and stationary) space-time.

The Jacobi factor arising from the change of the $\omega$-integral measure in Eq. (8) again modifies the normalization $N$ only. This undetermined normalization factor will be fixed later by virtue of the completeness relation in Eq. (34) below. After an analogous Taylor expansion of the function $f(t = 0, r) = 1 + \mathcal{O}|\chi|$ we find

$$\beta_{IJ} = \int_0^\infty d\chi \exp \{ -i\chi \omega R \chi \frac{\omega - \omega' \sigma - \xi'}{\sqrt{\omega \omega'}} \times \chi^{i(\sigma - \xi')\omega R} \mathcal{N}(1 + \mathcal{O}|\chi|) \left(1 + \mathcal{O}\left[\frac{1}{\omega}\right]\right).$$

As expected from the previous considerations, the Bogoliubov $\beta$-coefficients contribute only for $\sigma = -\xi'$ and vanish (in leading order) for $\sigma = \xi'$. In that case the out-modes are not singular (at the horizon) – only for $\sigma = -\xi'$ they display the arbitrarily fast oscillating behavior. Hence – depending on the sign $\sigma$ – either only ingoing (for $\sigma = -1$ and thus $\xi' = +1$) or only outgoing (for $\sigma = +1$ and thus $\xi' = -1$) particles are produced (in an infinite amount).

The integral in Eq. (29) involves generalized eigenfunctions which do not belong to the Hilbert space $L^2$ but are distributions, cf. [4]. Hence it cannot be interpreted as a well-defined Riemann integral. But – as demonstrated in Ref. [23] – it is possible to approximate (locally) the generalized eigenfunctions by well-defined wave-packets. One way to simulate such an approximation is to introduce a convergence factor via $\chi^\xi \exp\{-\xi \chi\}$ with $\xi \downarrow 0.$

For $\sigma = -\xi'$ the above integral can be solved in terms of $\Gamma$-functions. After insertion of the convergence factor we can make use of the formula [24]

$$\int_0^\infty dx e^{-xy} x^{z-1} = y^{-z} \Gamma(z),$$

which holds for $\Re(y) > 0$ and $\Re(z) > 0$, and – remembering $\Gamma(z + 1) = z \Gamma(z)$ – we arrive at

$$\beta_{IJ} = \mathcal{N} \delta_{\sigma,-\xi'} \delta_{\xi',\xi} \sqrt{\frac{\omega}{\omega'}} \Gamma(2i\sigma \omega' R) \left(i\xi \omega R + \varepsilon\right)^{2i\xi' \omega' R} \times \left(1 + \mathcal{O}\left[\frac{1}{\omega}\right]\right).$$

In view of Eq. (30) the higher order terms in $\chi$ – i.e. $x$ – cause increasing arguments $z$. Ergo these terms result in higher orders in $1/y$ – i.e. $1/\omega$ – consistently with our approximation and the arguments at the beginning of this Section. In order to evaluate the absolute value squared of the $\beta$-coefficient we may utilize the identity [4]

$$\Gamma(z)\Gamma(-z) = -\frac{\pi}{z \sin \pi z}$$

to obtain the final result

$$|\beta_{IJ}|^2 = \frac{\mathcal{N}}{\omega} \frac{\delta_{\sigma,-\xi'} \delta_{\xi',\xi}}{\exp\{4\pi \omega' R\} - 1} \left(1 + \mathcal{O}\left[\frac{1}{\omega}\right]\right).$$
This expression confirms the argumentation at the beginning of this Section. The remaining integration over \( \omega \) (or \( \tilde{\omega} \)) is indeed UV-divergent. In addition we observe that the terms of higher order in \( 1/\omega \) (and thus \( \chi \)) that we have neglected in our calculations are not UV-divergent and hence do not contribute to the Hawking effect. This observation provides an a posteriori justification of our expansion into powers of \( 1/\omega \) and \( \chi \) and the neglect of the sub-leading contributions.

The UV-divergence can be interpreted with the aid of the well-known completeness relation

\[
\sum_I \alpha_{IJ} \alpha_{IK}^* - \beta_{IJ} \beta_{IK}^* = \delta(J,K),
\]

where \( I \) again symbolizes the initial quantum number. This equality reflects the completeness of the initial modes. Special care is required concerning the derivation of an analogue expression involving the out-modes since some of those solutions are restricted to the region inside or outside the horizon, respectively, and these restricted modes are not complete in the full space-time.

In order to apply this relation we have to deduce the \( \alpha \)-coefficients as well. For that purpose we define slightly modified Bogoliubov coefficients via

\[
\tilde{\beta}(\omega, \omega') = \sqrt{\omega \omega'} \beta_{\omega \omega'},
\]

and in analogy the \( \alpha \)-coefficient. The modified Bogoliubov coefficients can be analytically continued into the complex \( \omega' \)-plane where the relations

\[
\tilde{F}_{\text{out}}(\omega') = \tilde{F}_{\text{out}}(-\omega')
\]

hold. This enables us to derive the Bogoliubov \( \alpha \)-coefficient for large initial frequencies \( \omega \). Substituting \( \omega' \to -\omega' \) in Eq. (33) together with the complex conjugation (the only difference between \( \alpha_{IJ} \) and \( \beta_{IJ} \)) is the sign in front of the term \( i\xi \omega R \) Dividing the absolute values of the two coefficients all other terms cancel and the convergence factor \( \varepsilon \) determines the side of the branch cut of the logarithm in the complex plane. Hence we find for large frequencies \( \omega \)

\[
|\beta_{IJ}| = \exp\{-2\pi \omega' R\} |\alpha_{IJ}| \left(1 + O\left(\frac{1}{\omega}\right)\right).
\]

Inserting Eq. (33) into the completeness relation (34) and considering the (singular) coincidence \( J = K \) it follows

\[
N_J = \langle \text{in} | N_J^{\text{out}} | \text{in} \rangle = \sum_I |\beta_{IJ}|^2
\]

\[
= \delta_\sigma, -\varepsilon \frac{\delta_-(I,I)}{\exp\{4\pi \omega' R\} - 1} + \text{finite}
\]

\[
= \delta_\sigma,-\varepsilon \frac{N_{\gamma} V}{\exp\{4\pi \omega' R\} - 1} + \text{finite}.
\]

According to the results of Ref. [24] the UV-divergence of the \( \omega \)-integration of the absolute values squared of the \( \beta \)-coefficients in Eq. (33) exactly corresponds to the singular quantity \( \delta_-(I,I) = \delta_-(\omega, \omega) \) and thus represents the near-horizon (\( r \downarrow R \), i.e. \( r < -\infty \)) part \( N_{\gamma} V \) of the infinite volume divergence \( N_{\gamma} V = N_{\gamma}^V + N_{\gamma}^+ V \) of the continuum normalization. As explained in Ref. [24], the infinitely large amount of particles is necessary for (quasi) thermal behavior in an unbounded volume.

It is also possible to calculate the Bogoliubov coefficients for regular modes (wave packets instead of plane waves), cf. [24]. In this case no divergences occur and all quantities are finite. Thus the (late-time) Hawking effect cannot be easily distinguished from the collapse-dependent or initially present finite amount of particles via isolating the divergent part in this situation.

As mentioned before, an initial state \( \varrho_{\text{in}} \) containing a finite number of particles does not change the final results concerning the Hawking effect. Inserting the Bogoliubov transformation the expectation value counting the number of Schwarzschild particles equals the Hawking term plus additional contributions in this situation

\[
\varrho_{\text{in}} \left( N_J^{\text{out}} \right) = N_J^{\text{Hawking}}
\]

\[
+ \sum_{IK} (\alpha_{IJ}^* \alpha_{JK} + \beta_{IJ}^* \beta_{JK}) \varrho_{\text{in}} \left( \hat{A}_I^* \hat{A}_K \right)
\]

\[
+ \sum_{IK} \alpha_{IJ}^* \beta_{IK} \varrho_{\text{in}} \left( \hat{A}_I \hat{A}_K^* \right)
\]

\[
+ \sum_{IK} \beta_{IJ}^* \alpha_{JK} \varrho_{\text{in}} \left( \hat{A}_I^* \hat{A}_K \right).
\]

For a state \( \varrho_{\text{in}} \) that contains a finite number of initial particles the above expectation values vanish in the high (initial) frequency limit \( \omega_I, \omega_K \uparrow \infty \). As a result the \( I \) and \( K \) summations/integrations are not UV-divergent. Hence the additional contributions are finite and do not affect the (divergent) Hawking effect. E.g., if we assume the collapsing object to be enclosed by a (arbitrarily large but finite) box with Dirichlet boundary conditions we may describe an initial thermal equilibrium state via the canonical ensemble. In view of the previous arguments we arrive at the conclusion that any initial temperature does also not affect the final (Hawking) temperature in this scenario. (Dropping the assumption of a finite box enclosing the collapsing object the situation becomes more complicated. In this case the number of particles being present initially diverges owing to the infinite volume. Hence the usual spatial infinity part of the infinite volume divergence still corresponds to the initial temperature whereas the near-horizon part obeys the Hawking temperature, cf. [24].)

With the aid of similar arguments one can show that the Hawking effect – i.e. the late-time radiation – is also independent of the initial metric (as long as it is regular). The number of particles created during the transi-
tion from one to another regular metric is finite. These particles disperse after some finite period of time and do not affect the (divergent) late-time part of the radiation in accordance with the arguments in the previous paragraph. In terms of the Bogoliubov coefficients this degree of freedom exactly corresponds to the redefinition of the initial frequency $\omega$. (We did not need to specify the initial metric $f_{in}(r)$ in Sec. III B.)

E. Energy-momentum tensor

In order to support the conclusions of the previous sections concerning the evaporation/anti-evaporation we calculate the late-time expectation value of the relevant component of the energy-momentum tensor. In contrast to the inherently non-covariant particle concept this quantity is manifestly covariant. The general relativistic energy-momentum tensor $T_{\mu \nu}$ as the source term in the Einstein equations can be obtained via variation of the action $A$ with respect to the metric. For the scalar field under consideration we obtain the well-known expression

$$T_{\mu \nu} = \frac{2}{\sqrt{-g}} \frac{\delta A}{\delta g_{\mu \nu}} = \partial_\mu \Phi \partial_\nu \Phi - \frac{g_{\mu \nu}}{2} \partial_\sigma \Phi \partial^\sigma \Phi . \quad (40)$$

The covariant divergence of this tensor vanishes

$$\nabla_\mu T^\mu_\nu = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} T^\mu_\nu) = -\frac{1}{2} \partial_\sigma g_{\alpha \beta} = 0 . \quad (41)$$

However, in general this equality does not imply any conservation law due to the exchange of energy and momentum between the scalar and the gravitational field (second term). Nevertheless, if the space-time possesses a Killing vector $\zeta_\nu$ mediating the time-translation symmetry (Noether theorem) we may define a conserved energy current

$$J^\mu = T^{\mu \nu} \zeta_\nu . \quad (42)$$

In view of the symmetry of the energy-momentum tensor $T^{\mu \nu} = T^{\nu \mu}$ and Eq. (11) together with the property of the Killing vector $\nabla_\mu \zeta_\nu + \nabla_\nu \zeta_\mu = 0$ this current is indeed conserved

$$\nabla_\mu J^\mu = 0 . \quad (43)$$

Now we may calculate the energy flux $\Xi$ out of (or into) the black (white) hole

$$\Xi = -\int d\Sigma_\mu J^\mu = \int d\Sigma_\mu \langle 0_{in} | \hat{T}^{\mu \nu} | 0_{in} \rangle \zeta_\nu , \quad (44)$$

where $\Sigma$ denotes the (cylindrical) hyper-surface enclosing the black/white hole. In a 3+1 dimensional space-time one may determine $\Sigma$ via the Killing vectors mediating the spherical symmetry. By virtue of the Gauss law the above quantity is invariant under deformations of this hyper-surface $\Sigma$. Hence we may consider a sphere with a radius which is much larger than the Schwarzschild radius – where the metric coincides asymptotically ($r \uparrow \infty$) with the Minkowski form. In this region the energy flux simplifies to

$$\Xi = -\frac{1}{2} \int dt \left( \partial_0 \Phi \frac{\partial \Phi}{\partial r} \right) \langle 0_{in} | \partial_0 | 0_{in} \rangle . \quad (45)$$

The symmetrization $\{ \cdot , \cdot \}$ is necessary in order to obtain a Hermitian observable $\hat{T}_{\mu \nu}$. The minus sign arises from $g_{11}(r \uparrow \infty) = -1$. For large radial distances (approximately Minkowski) the expansion of the field reads

$$\hat{\Phi}(t, r) = \sum_I \hat{a}_I^{out} F_I^{out}(t, r) + h.c. \quad (46)$$

Insertion of the above expansion into the bilinear form in Eq. (45) generates a sum over $\xi$ and $\xi'$ as well as an integration over $\omega$ and $\omega'$. The time-integration in Eq. (45) involves terms such as $\exp(\pm i\omega t)$ and thus generates $\delta(\omega \pm \omega')$-distributions. In view of the positivity of the frequencies only $\omega = \omega'$ contributes. Similarly the remaining spatial dependence $\exp(\pm i\omega(\xi - \xi')r)$ implies that merely $\xi = \xi'$ yields relevant contributions at large distances $r \uparrow \infty$. As a result only one $\langle \omega, \xi \rangle$-sum/summation/integration survives and the late-time radiation is related to the number of particles via

$$\Xi = -|\mathcal{N}|^2 \sum_{\omega \xi} \langle 0_{in} | \hat{N}_{\omega \xi} | 0_{in} \rangle \omega \xi . \quad (47)$$

This relation confirms the conclusions of the previous sections: The divergence of $\langle 0_{in} | \hat{N}_{\omega \xi}^{out} | 0_{in} \rangle$ exactly corresponds to the time-integration and the resulting singularity of the $\delta(\omega - \omega')$-distribution. The Bogoliubov coefficients and thus also $\langle 0_{in} | \hat{N}_{\omega \xi}^{out} | 0_{in} \rangle$ contribute (in an infinite amount) only for $\sigma = -\xi$. Hence the collapse to a black hole described by the branch $\sigma = +1$ of the PGL metric generates an outward $\langle \xi = -1 \rangle$ flux at late times whereas the collapse to a white hole corresponding to $\sigma = -1$ leads to an inward $\langle \xi = +1 \rangle$ flux at late times.

IV. SUMMARY

In terms of the Painlevé-Gullstrand-Lemaître coordinates it is possible to model a gravitational collapse of an object and the subsequent formation of an apparent horizon by means of a manifestly $C^\infty$-metric. This set of coordinates possesses two separate branches (labeled by $\sigma = \pm 1$). Depending on the particular branch (i.e. the sign of $\sigma$) either only ingoing or only outgoing particles are created in an infinite amount. This infinite amount of particles obeys a thermal spectrum corresponding to the Hawking temperature.
V. CONCLUSIONS

The theorems presented in Section IIA imply that during every collapse scenario that can be described by a $C^\infty$-metric an infinite number of particles with a thermal spectrum corresponding to the Hawking temperature is created. This statement is verified in the present article for a rather general ansatz for a $C^\infty$-metric in Eq. (18). For that purpose it is neither necessary to impose any conditions on the metric beyond the horizon nor to specify the explicit dynamics of $f(t, r)$ during the collapse – as long as it is regular, i.e. $C^\infty$, and fast enough, cf. Sec. III C.

So the Hawking effect is not the result of a spacetime singularity but a consequence of the formation of a horizon (strictly speaking, an apparent horizon). For the derivation of the Bogoliubov coefficients no assertions about the metric in the interior region $f(t \geq 0, r < R)$ are necessary at all. In addition, only the modes that are affected by the horizon (one-way membrane, cf. [27]) contribute to the late-time (Hawking) radiation, i.e. the outgoing particles for the black hole horizon and the ingoing particles for the white hole horizon, respectively.

Thus the properties of the produced particles crucially depend on the branch of the Painlevé-Gullstrand-Lemaître metric under consideration. Adopting the Schrödinger representation the two distinct branches generate completely different final states $\varrho_\sigma$. Only one state represents the phenomenon of evaporation while the other state corresponds to anti-evaporation.

VI. DISCUSSION

Perhaps the most striking outcome of the presented calculation is the fact that – depending on the particular branch $\sigma$ of the dynamics during the collapse – the final state of the quantum field does not necessarily represent evaporation but possibly also anti-evaporation. The phenomenon of anti-evaporation has already been discussed in the literature, see e.g. [21,22], but in a different context (Schwarzschild-de Sitter geometries, see also [1]). In contrast the calculation in the present article applies to asymptotically flat space-times.

For one branch the final state coincides – up to a finite number of particles – with the 1+1 dimensional analogue of the Unruh state $\varrho_U$ describing evaporation. The other branch generates the (in some sense) opposite final state – corresponding to anti-evaporation. In the following considerations we shall denote this state as the anti-Unruh state $\varrho_{aU}$ for convenience. This state $\varrho_{aU}$ can be obtained from the Unruh state $\varrho_U$ by means of the (Schwarzschild) time-inversion $T$

$$\varrho_{aU} = T \varrho_U, \quad (48)$$

if we regard the (Schwarzschild) metric of the space-time as fixed. Induced by the time-inversion $T$ all outgoing particles turn their direction into ingoing and vice versa. Since the neutral scalar field is neither affected by the charge conjugation C nor by the parity transformation P (in contrast to a pseudo-scalar field) and we consider a spherically symmetric situation, both, the Unruh as well as the anti-Unruh state are not CPT invariant: $CPT \varrho_U = \varrho_{aU} \neq \varrho_U$. These considerations are relevant for the investigation of unitarity and time-reversibility, see e.g. [12].

Searching for the physical implementations of the main result of the present article there are several possible interpretations:

From a conservative point of view one might argue that the branch causing anti-evaporation is unphysical and should be excluded. This assertion might perhaps be supported by physically reasonable constraints on the energy-momentum tensor, such as the energy conditions. The two branches of the $C^\infty$-metric in Eq. (18) – after the straightforward generalization to 3+1 dimensions – can be used to derive the associated Ricci tensor $R_{\mu\nu}$.

Owing to the smoothness of the metric the curvature tensor always exists and is $C^\infty$ as well. By virtue of Einstein’s equations the Ricci tensor reveals the corresponding energy-momentum tensor which could be compared with an appropriate model of a collapsing star or used to test the energy conditions, for example. It is well-known that appropriate energy conditions exclude the existence of certain worm-holes or time-machines, see e.g. [27]. However, one should be aware that the energy conditions may well be violated if one incorporates the back-reaction of the quantum field. Since the Hawking effect is most relevant for small objects and almost negligible at astrophysical orders of magnitude one would expect that such quantum effects have to be taken into account.

As another possible interpretation of the result of this article one may arrive at the conclusion that black holes evaporate but white holes anti-evaporate. The particle production by white holes has already been discussed in Ref. [12], but within a different context: The space-time under consideration in Ref. [12] was obtained via the time-inversion of a space-time representing the collapse of an object to a black hole, i.e. an anti-collapse. In contrast the space-time investigated in the present article corresponds to the gravitational collapse of an object, cf. Figs. 3 and 4. Furthermore the initial state in Ref. [12] is determined via a factorization assumption which might be questioned in general and does definitely not apply to the scenario of the present article. As a consequence the resulting radiation becomes singular at the retarded time of the termination of the horizon – a prediction which differs drastically from the outcome of the present article. Based on similar scenarios in Refs. [3] and [4] further instabilities and quantum effects connected with white holes are discussed – before (and independently of) Hawking’s discovery. The comparison of black and white holes is potentially interesting in view of the fundamental question of time-reversibility (unitarity and the
second law of thermodynamics) of quantum gravity, see e.g. [12].

The predicted anti-evaporation of white holes is certainly relevant for the sonic black/white hole analogues, see e.g. [17–10]. These flow profiles always possess an (effective) black hole and a white hole horizon, see e.g. Fig. 1 in Ref. [29]. If the fluid accelerates in such a way that its local velocity exceeds the speed of the sound (black hole horizon) it decelerates below the speed of the sound somewhere (white hole horizon) as well. Consequently, the derivation presented in this article implies that, if the perturbations of the flow profile of the liquid obey a quantum field theoretical description (with the resulting quantum fluctuations), then the associated (effective) vacuum fluctuations are converted into (quasi) particles leading to evaporation for the black hole horizon and to anti-evaporation for the white hole horizon.

In order to discuss the third opportunity regarding the interpretation of the outcome of the present article one may recall the fact that the Hawking effect is dominated by arbitrarily large (initial) frequencies. But at energies above the Planck scale one expects the breakdown of the treatment of quantum fields propagating in given (externally prescribed) space-times. In the Planck régime the back-reaction, for example, should become important. Assertions about the metric in this region (e.g. the Planck scale vicinity of the forming horizon) are a very delicate issue. Hence one is lead to the assumption that the outside observer cannot distinguish the two branches. As any explorer at a finite spatial distance to the collapsing object merely experiences the static Schwarzschild metric (Birkhoff theorem), the only possible way to obtain informations about the collapse is provided by the quantum radiation itself. If we now assume that the outside beholder cannot resolve the behavior of the metric in the Planck scale vicinity of the forming horizon, he/she cannot obtain any information about the particular branch of the metric a priori. Without any knowledge about the value of $\sigma$ during the collapse the most natural ansatz for the state governing the measurements of an outside observer is given by — remember the convexity of the states discussed in Sec. II

$$\varrho_0 = \varrho_U + \varrho_{\text{aU}} \over 2.$$ (49)

Again we adopt the Schrödinger representation. This ansatz complies with the superposition principle of quantum theory — if one assumes that the back-reaction of the quantum fields onto the metric yields relevant contributions.

The above introduced state describes some kind of quasi-thermal equilibrium — it contains the same (infinite) number of ingoing and outgoing particles with a thermal spectrum corresponding to the Hawking temperature. Although the state $\varrho_0$ displays in 1+1 dimensions a close similarity to the (1+1 dimensional analogue of the) Israel-Hartle-Hawking state $\varrho_{\text{IHH}}$, in 3+1 dimensions this quasi-thermal equilibrium state $\varrho_0$ differs drastically from the Israel-Hartle-Hawking state, which describes (at least with respect to the algebra of observables outside the horizon) real thermal equilibrium. The expectation value of the number of particles in the Israel-Hartle-Hawking state $\varrho_{\text{IHH}}$ exhibits the complete infinite volume divergence, i.e. the near-horizon part $r_* \downarrow -\infty$ as well as the usual spatial infinity $r_* \uparrow \infty$, cf. [24]. In contrast the analogue expectation value in the states $\varrho_0$, $\varrho_U$, and $\varrho_{\text{aU}}$ contains the near-horizon part only, see Sec. [III D]. As a consequence the renormalized expectation value of the energy density in the states $\varrho_0$, $\varrho_U$, and $\varrho_{\text{aU}}$ decreases for large distances $r$ with $1/r^2$ whereas the same quantity approaches a constant value (in view of the Stefan-Boltzmann law proportional to $T^4$) in the Israel-Hartle-Hawking state $\varrho_{\text{IHH}}$.

It might be noted here that — in contrast to the Unruh as well as the anti-Unruh state — the state $\varrho_0$ is CPT invariant: $\text{CPT} \varrho_0 = \varrho_0$. Therefore the unitarity and time-reversibility problem mentioned above in connection with the (anti) Unruh state does not necessarily apply to this state.

In Sec. [III C] we have observed that only the region near the horizon generates contributions that are relevant with respect to the Hawking effect. Exactly the leading terms in $1/\omega$ and $\chi$ give rise to the UV-divergence accounting for the Hawking effect. The notion of the vicinity of the horizon as the region that is essential for the Hawking effect may be illustrated via the following gedanken experiment: Let us imagine a very thin shell of matter with slowly decreasing radius. As long as the radius of the shell is larger than the associated Schwarzschild radius the number of created and radiated particles remains finite as a consequence of the regularity of the metric and the associated eigen modes. If the shell were to stop shrinking before it reached its Schwarzschild radius, no Hawking effect would be observed. Accordingly, the creation of particles accounting for the Hawking effect occurs exactly in the space-time region of the formation of the horizon.

In order to support the argumentation in Sec. [II C] concerning the independence and separation of the different branches (e.g. corresponding to ingoing and outgoing components) in Eq. (21) we may consider a conceptual clear scenario — where the effective boundary condition at $r = 0$ does not contribute at all — described in the following gedanken experiment: At first we suppose a small amount of highly charged matter to collapse at the center of gravity forming a tiny extreme Reissner black hole. The surface gravity of such a black hole vanishes with the result that there is no Hawking radiation (at this stage). After the formation of the small black hole the point $r = 0$ is hidden by the corresponding horizon. Consequently, there is no "reflection" at the origin $r = 0$ in this case. (It is possible to define ingoing and outgoing particles separately, cf. [24].) If we now suppose the matter (enclosing the tiny black hole) to collapse the origin cannot generate a mixing of the different branches (e.g. ingoing and outgoing).
VII. OUTLOOK

For the calculation of the Bogoliubov coefficients we have restricted our consideration to the 1+1 dimensional space-time of the (t, r)-sector. Even though the main result of this article should persist also in 3+1 dimensions (including the angular terms), some additional complications arise \[13\]. The Klein-Fock-Gordon equation \[1\] assumes for the 3+1 dimensional Schwarzschild metric a slightly modified form. After separating the angular dependence by spherical harmonics the centrifugal barrier and curvature scattering effects can be incorporated into an effective potential \( V_{\text{eff}} \), see e.g. \[17\]

\[
\left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^2} + V_{\text{eff}}(r_*, \ell) \right) \phi_{\ell,m} = 0. \tag{50}
\]

\( V_{\text{eff}} \) is strictly positive and approaches zero for \( r_* \uparrow +\infty \) and for \( r_* \downarrow -\infty \) with \( O[1/r^2] = O[1/r^2] \) and \( O[\chi] = O[\exp(r_*/R)] \), respectively. Unfortunately, in 3+1 dimensions no closed expression (in terms of well-known functions) for the eigen modes is available. The asymptotic behavior can be derived easily. For \( r_* \downarrow -\infty \) the positive frequency solutions again behave as \( \exp\{-i\omega t \pm i\omega r_*\} \) or linear combinations of them. These waves are purely ingoing or outgoing, respectively, for \( r_* \downarrow -\infty \). But every mode which is purely outgoing near the horizon contains for \( r_* \uparrow +\infty \) ingoing components as well owing to the scattering at the effective potential \( V_{\text{eff}} \) (inducing transmission and reflection coefficients) and vice versa. In order to obtain a complete and orthogonal (with respect to the inner product) set of positive frequency solutions of the Klein-Fock-Gordon equation one has to combine the different opportunities. E.g., within the notation of Ref. \[14\] (see Fig. 1 there) a ‘down’ mode is purely ingoing at infinity and (therefore) mixed at the horizon. According to the results of this article one would expect that an infinite amount of particles are created by the collapse in this mode – independently of the branch of the metric \[13\].

Furthermore the present article considers the most simple example of a quantum field theory, i.e. the neutral, massless, and minimally coupled scalar field \( \Phi \). Further investigations should be devoted to fields obeying more complicated equations of motion. For the spin-zero field example one may incorporate potential terms including masses \( M^2 \Phi^2 / 2 \) or conformal couplings \( R_\mu^\nu \Phi^2 / 6 \) and consider charged (i.e. non-Hermitian) fields. Moreover, it would be interesting to extend the examination to fields with higher spin, e.g. the electromagnetic field. Nevertheless, there is no obvious reason why the main conclusions of this article should not persist. The evaluation of the Hawking effect for interacting fields with non-linear equations of motion seems to be rather challenging.

Similarly the space-time under consideration describes the most simple example of a black hole. The Schwarzschild geometry represents an uncharged and non-rotating black hole where the Einstein tensor and thereby also the energy-momentum tensor vanish for \( r > 0 \). The extension of the results presented in this article to more general static (i.e. non-rotating) black-holes – e.g. the Reissner solution – seems to be straight-forward, see also \[2\]. In contrast the investigation of rotating (i.e. stationary, but not static) black-hole space-times – e.g. the Kerr solution – holds more difficulties.

Apart from the Painlevé-Gullstrand-Lemaître coordinates there are several other coordinate sets that describe the Schwarzschild geometry space-time by a manifestly \( C^\infty \)-metric, e.g. the Eddington-Finkelstein coordinates. It might be interesting to consider a collapse model in terms of these coordinates in analogy to Eq. \( (18) \) and to compare the results.

However, one should be aware that all of the previous considerations neglect the back-reaction of the quantum field onto the metric. So far the quantum field is treated as a test field propagating on a given (externally prescribed) space-time. If one attempts to leave this formalism several problems arise. The concept of Hadamard states as described in Eq. \( (5) \) is restricted to free fields obeying linear equations of motion. The two-point function of interacting fields possesses additional singularities in general. Consequently – if one regards the treatment of quantum fields in classical (general relativistic) space-times as a low-energy effective theory of some underlying theory – the imposition of the Hadamard condition is not obviously justified. Similarly the requirement of a smooth \( C^\infty \)-metric may be questioned from this point of view. Accordingly, it might be interesting to examine the consequences of collapse dynamics that are not \( C^\infty \) regarding the Hawking effect \[13\].

An exhaustive clarification of these problems probably requires the knowledge of an underlying law that unifies quantum field theory and general relativity.

ACKNOWLEDGMENT

The author is indebted to A. Calogeracos, K. Freder- hagen, I. B. Khriplovich, G. Plunien, G. Schaller, G. Soff, and W. G. Unruh for valuable discussions. This work was partially supported by BMBF, DFG, and GSI.

[1] S. A. Fulling, Nonuniqueness of Canonical Field Quantization in Riemannian Space-Time, Phys. Rev. D 7, 2850 (1973).
[2] Y. B. Zel’dovich, I. D. Novikov, and A. A. Starobinski, Quantum Effects in White Holes, Zh. Éksp. Teor. Fiz. 66, 1897 (1974); Sov. Phys. JETP 39, 933 (1974).
[3] D. M. Eardley, Death of White Holes in the Early Universe, Phys. Rev. Lett. 33, 442 (1974).
Hole in Thin $^3$He-$\Lambda$ Film, Pisma Zh. Eksp. Teor. Fiz. 69, 662 (1999); JETP Lett. 69, 705 (1999).

[40] U. R. Fischer and G. E. Volovik, Thermal Quasi-Equilibrium States across Landau Horizons in the Effective Gravity of Superfluids, e-preprint: gr-qc/0003017, to appear in Int. J. Mod. Phys. D; G. E. Volovik, Superfluid Analogies of Cosmological Phenomena, e-preprint: gr-qc/0005091, to appear in Phys. Rept.

[41] T. Jacobson, Black Hole Evaporation and Ultrashort Distances, Phys. Rev. D 44, 1731 (1991); Black Hole Radiation in the Presence of a Short Distance Cutoff, Phys. Rev. D 48, 728 (1993). W. G. Unruh, Sonic Analogue of Black Holes and the Effects of High Frequencies on Black Hole Evaporation, Phys. Rev. D 51, 2827 (1995). R. Brout, S. Massar, R. Parentani and P. Spindel, Hawking Radiation without TransPlanckian Frequencies, Phys. Rev. D 52, 4559 (1995).

[42] M. Abramowitz and I. Stegun, Handbook of Mathematical Functions, (Dover, New York, 1965).

[43] These considerations will be established in detail in a forthcoming publication.

[44] P. L. Chrzanowski and C. W. Misner, Geodesic Synchrotron Radiation in the Kerr Geometry by the Method of Asymptotically Factorized Green’s Function, Phys. Rev. D 10, 1701 (1974).