Ferromagnetism in QCD phase diagram

T. Tatsumi

Department of Physics, Kyoto University, Kyoto 606-8502, Japan
E-mail: tatsumi@ruby.scphys.kyoto-u.ac.jp

A possibility and properties of spontaneous magnetization in quark matter are investigated. Magnetic susceptibility is evaluated within Fermi liquid theory, taking into account of screening effect of gluons. Spin wave in the polarized quark matter, as the Nambu-Goldstone mode, is formulated by way of the coherent-state path integral.

Keywords: Ferromagnetism; Fermi liquid; Screening; Spin wave; Quark matter.

1. Introduction

The phase diagram of QCD has been elaborately studied in density-temperature plane. Here we study the magnetic properties of QCD. Since the discovery of magnetars with super-strong magnetic field of $O(10^{14-15})$G the origin of strong magnetic field observed in compact stars has roused our interest again. For the present there are three ideas about its origin: the first one reduces it to the fossil field, assuming the conservation of the magnetic flux during the evolution of stars. The second one applies the dynamo mechanism in the crust region. The third one seeks it at the core region, where hadron or quark matter develops. If spins of nucleons or quarks are aligned in some situations, they can provide the large magnetic field. Since elaborate studies about the spontaneous magnetization in nuclear matter have repeatedly shown the negative results, in the following, we explore the possibility of the spontaneous magnetization in quark matter.

In the first paper we have suggested a possibility of ferromagnetic phase in QCD, within the one-gluon-exchange (OGE) interaction. If such a phase is realized inside compact stars, we can see the magnetic field of $O(10^{15-17})$G is easily obtained. Directly evaluating the total energy of the polarized matter with a polarization parameter, we have shown that ferromagnetic phase is possibly realized at low densities, in analogy with the itinerant electrons by Bloch; the Fock exchange interaction is responsible to ferromagnetism in QCD. We have also seen that the phase transition is weakly first order and one may also apply the technique for the second order phase transition to analyze it, while it is one of the specific features of the magnetic transition in gauge theories.

In the first half we discuss the magnetic susceptibility of quark matter within the Fermi-liquid theory. It is well known that we must properly take into account the
screening effects in the gluon propagator to improve the IR behavior of the gauge interaction. For the longitudinal gluons we can see the static screening described in terms of the Debye screening mass. There is no static screening for the transverse gluons, while the dynamical screening appears instead due to the Landau damping. We will figure out these screening effects in evaluating the magnetic susceptibility. It would be interesting to observe that there appears the non-Fermi-liquid effect to give an anomalous term in the finite temperature case as in the specific heat.

In the second half we discuss how the spin wave, which is caused by the spontaneous magnetization, can be described within the coherent-state path integral. First we map the quark matter to a spin system by assuming the spatial wave function for each quark and leaving the degree of freedom of the direction of the spin vector. This method is inspired by the spiral approach taken by Herring in old days to discuss the spin wave in the electron gas. Introducing the collective variables $U(\theta, \phi) \in S^2$, and integrating over the individual variables, we have an effective action to see the classical Landau-Lifshitz equation for the spin wave is naturally derived by the effective action for the collective variables. We also note that there are some geometrical aspects in the effective action, which may further quantize the classical spin wave to give magnons. Thus we have magnons in the ferromagnetic phase of quarks, which directly gives the $T^{2/3}$ dependence for the reduction of the magnetization.

2. Magnetic susceptibility within the Fermi liquid theory

The magnetic susceptibility $\chi_M$ is defined as

$$\chi_M = \frac{\partial \langle M \rangle}{\partial B} \bigg|_{N, T},$$

with the magnetization $\langle M \rangle$. So we study the response of quark matter when a weak magnetic field is applied for a given quark number $N$ and temperature $T$. Using the Gordon identity, the interaction Lagrangian may be written for the constant magnetic field, $A = B \times r/2$:

$$\int d^4x L_{ext} = \mu_q \int d^4x \bar{q} \left[ -i \mathbf{r} \times \nabla + \Sigma \right] \cdot \mathbf{B} \mathbf{q},$$

for quarks with electric charge $e_q$, where $\Sigma = \left( \begin{array}{cc} \sigma & 0 \\ 0 & \sigma \end{array} \right)$, and $\mu_q$ denotes the Dirac magnetic moment $\mu_q = e_q/2m$. Then magnetization $\mathbf{M}$ may be written as

$$M_z = \langle q \Sigma q \rangle_z = \frac{\mu_q}{2} N_C \int \frac{d^3k}{(2\pi)^3} g_D(k)(n_{k,+} - n_{k,-}),$$

with the Fermi-Dirac distribution function $n_{k,\zeta}$ in the presence of $\mathbf{B}$. The gyromagnetic ratio $g_D$ is defined as

$$g_D(k)\zeta = 2 \text{tr} \left[ \Sigma_z \rho(k, \zeta) \right] = \left[ 1 - \frac{k_z^2}{E_k(E_k + m)} \right] \zeta,$$
in terms of the polarization density matrix $\rho(k, \zeta)$,

$$\rho(k, \zeta) = \frac{1}{2m}(k + m)P(a),$$  \hspace{1cm} (5)

with the projection operator, $P(a) = 1/2 \cdot (1 + \gamma_5 \delta)$.

The quasi-particle interaction consists of two terms,

$$f_{k, a} = f_{k, a}^s + f_{k, a}^a,$$  \hspace{1cm} (6)

where $f_{k, a}^{s(a)}$ is the spin-independent (dependent) interaction. Then we get the expression for the magnetic susceptibility at $T = 0$ written in terms of the Landau parameters:

$$\chi_M = \left( g_F^2 \frac{E}{2} \right) \left( \frac{\pi^2}{N_C} \right) \left( \frac{1}{3} f_1^s + \bar{f}^a \right)^{-1},$$  \hspace{1cm} (7)

where $f_1^s$ is a spin-averaged Landau parameter defined by

$$f_1^s = \frac{3}{4} \sum_{\zeta, \zeta'} \int \frac{dQ_{kq}}{4\pi} \cos \theta_{kq} f_{k, k, q, q},$$  \hspace{1cm} (8)

with the relative angle $\theta_{kq}$ of $k$ and $q$, and $\bar{f}^a$ the spin-dependent one,

$$\bar{f}^a = \int \frac{dQ_k}{4\pi} \int \frac{dQ_{kq}}{4\pi} f_{k, k, q, q},$$  \hspace{1cm} (9)

**3. Static and dynamic screening effects**

When we consider the color-symmetric forward scattering amplitude of the two quarks around the Fermi surface by the one gluon exchange interaction (OGE), the direct term should be vanished due to the color neutrality of quark matter and the Fock exchange term gives a leading contribution. The color-symmetric and flavor-symmetric OGE interaction of quasi-particles on the Fermi surface may be written,

$$\bar{f}_{\zeta, a} = \frac{1}{N_C^2} \sum_{a, b} \sum_{i, j} f_{\zeta, a, i, q, q', b, j} \left| \frac{m}{E_k} \right| \left| \frac{m}{E_q} \right| M_{\zeta, a, q', b} \left| \frac{m}{E_k} \right| \left| \frac{m}{E_q} \right|,$$  \hspace{1cm} (10)

with the invariant matrix element, $M_{\zeta, a, q', b}$. It has been well known that massless gluons often causes infrared (IR) divergences in the Landau parameters.\(^5\)

Since the one gluon exchange interaction is a long-range force and we consider the small energy-momentum transfer between quasi-particles, we must improve the gluon propagator by taking into account the screening effect,

$$D_{\mu\nu}(k - q) = D_{\mu\nu}^T(p) + D_{\mu\nu}^L(p) - \frac{p_{\mu} p_{\nu}}{p^4}$$  \hspace{1cm} (11)
with \( p = k - q \), where \( D_{T(L)}(p) = (p^2 - \Pi_{T(L)})^{-1} \), and the last term represents the gauge dependence with a parameter \( \xi \). \( P_{\mu\nu}^{\gamma(T(L))} \) is the standard projection operator onto the transverse (longitudinal) mode.

The self-energies for the transverse and longitudinal gluons are given as

\[
\Pi_L(p_0, p) = m^2_D + \frac{i \pi m_D^2}{2E_F} \frac{p_0}{|p|} \coth \frac{p_0}{2T} \left[ (k - q)^i (k - q)^j M_{ij} \right],
\]

\[
\Pi_T(p_0, p) = -i \frac{\pi m_D^2}{4E_F} \frac{p_0}{|p|} \coth \frac{p_0}{2T},
\]

(12)
in the limit \( p_0/|p| \to 0 \), with the Debye screening mass, \( m^2_D \equiv (N_F/2\pi^2)g^2E_Fk_F \). Thus the longitudinal gluon is screened to have the Debye mass \( m_D \), while the transverse gluon is not in the limit \( p_0/|p| \to 0 \) and \( T = 0 \). At finite temperature, however, the transverse gluons are also dynamically screened by the Landau damping. For quarks on the Fermi surface, the Lorentz invariant matrix element can be written as

\[
M_{\kappa\zeta, \kappa'\zeta'} = -\frac{N_F^2 - 1}{2N_C^2} g^2 \left[ -M^{00}D_L + M^{ij}P_T^{ij}D_T + \xi \frac{1}{|k - q|^4} (k - q)_i (k - q)_j M^{ij} \right],
\]

(13)

with the coefficients \( M^{\mu\nu} \),

\[
M^{\mu\nu} = \text{tr}[\gamma^\mu \rho(k, \zeta) \gamma^\nu \rho(q, \zeta')].
\]

(14)

First of all, the matrix element is obviously independent of the gauge parameter \( \xi \).

At temperature \( T = 0 \), there is no screening in the propagator of the transverse gluon, so that logarithmic divergence still remains in the Landau-Migdal parameters, \( f^1_1 \) or \( f^0 \). However, we can see that the divergences cancels each other to give a finite result for the susceptibility. The static screening effect gives a \( g^2 \ln g^{-2} \) contribution.

Fig. 1. Magnetic susceptibility as a function of the Fermi momentum. Divergence signals the onset of spontaneous spin polarization. The screening effect slightly shifts the critical density to lower densities.
In Fig. 1 we present an example of the magnetic susceptibility at $T = 0$. The ferromagnetic phase corresponds to the negative value of $\chi$ and the critical density is given by the divergence of $\chi$. Compared with the OGE calculation, we can see that the static screening effect shifts the critical density to the lower value.

At finite temperature, the dynamical screening gives $T^2 \ln T$ contribution through the density of states near the Fermi surface, besides the standard $T^2$ dependence. This behavior is a kind of non-Fermi liquid effects, as in the specific heat.

4. Spin wave in the polarized quark matter

When spontaneous magnetization occurs, the magnetization $\mathbf{M}$ is not vanished, so that rotation symmetry is violated in the ground state. Hence one can expect a Nambu-Goldstone mode, spin wave there. Different from the usual description of the spin wave, we must care about its realization in quark matter. Recalling a similar situation in electron gas, we take here an intuitive but correct framework called the spiral approach. Herring explicitly showed that the Bloch wall coefficient, which is closely related with the dispersion relation of the spin wave, is obtained for electron gas by the use of the spiral approach.

Consider the fully polarized case. Then all the quarks have a definite spin state specified by $\zeta$ (say, $\zeta = +1$). The single particle wave function with momentum $k$ is simply given as

$$u^{(\zeta)}(k) = \frac{k + m}{\sqrt{2m(E_k + m)}} u^{(\zeta)}(m, 0) e^{-ikx}$$

with the spinor in the rest frame,

$$u^{(\zeta=+1)}(m, 0) = e^{ib} \begin{pmatrix} e^{-i\phi_k/2} \cos \theta_k/2 \\ e^{i\phi_k/2} \sin \theta_k/2 \\ 0 \end{pmatrix} \equiv \begin{pmatrix} g_k(r) \\ 0 \end{pmatrix}$$

taking the spin quantization axis specified by the polar angles ($\theta_k, \phi_k$). Thus the quark wave function is characterized by the momentum $k$ and the polar variables $\theta_k, \phi_k$.

In the ground state, all the spins have the same direction, say $\theta_k = \bar{\theta}, \phi_k = \bar{\phi}$, so that the ground-state energy is degenerate in any value of them. In the following we allow them to be spatially dependent and introducing the small fluctuation fields, $\xi_k(x), \eta_k(x)$, s.t.

$$\theta_k(x) = \bar{\theta}(x) + \xi_k(x), \phi_k(x) = \bar{\phi}(x) + \eta_k(x),$$

where $\bar{\theta}, \bar{\phi}$ or $\mathbf{U} = (U^1(\bar{\theta}, \bar{\phi}), U^2(\bar{\theta}, \bar{\phi}), U^3(\bar{\theta}, \bar{\phi}))$ are the collective variables defined by

$$\cos \bar{\theta} \equiv \frac{1}{N_k} \sum_k \cos \theta_k, \cos \bar{\phi} \equiv \frac{1}{N_k} \sum_k \phi_k,$$
and we shall see they describe the Nambu-Goldstone mode (spin wave).

In the spiral approach, we assume a spin configuration described by

\[ g_k(r) = \exp \left[ i t_z (d\phi/dz) (\sigma_z/2) \right] g_k(r - t) \]  

(19)

with arbitrary displacement \( t \), which corresponds to a spin wave traveling along \( z \) axis with wave vector \( d\phi/dz \), so that \( \phi \) should be a linear function of \( z \). Actually the mean value of the spin operator \( \Sigma \) is proportional to \( \zeta \).

For a given Hamiltonian \( H \) we can evaluate the energy by putting (15), (16) into it,

\[ H(\theta_k, \phi_k) = \int d^3x \langle H \rangle, \]

(20)

which may be regarded as a classical spin Hamiltonian for quark matter. Thus we mapped quark matter to assembly of spins.

5. Coherent-state path-integral

One may reformulate the idea of the spiral approach in the framework of the path integral.\(^8\) Consider a matrix element of the evolution operator

\[ \langle \Omega'', t'' \mid \exp(-iT\hat{H}/\hbar) \mid \Omega', t' \rangle = \int D\Omega \exp \left( i \sum_k \int_{t'}^{t''} dt \int d^3x \left[ i \Omega_k \mid \dot{\Omega}_k \right] - H(\Omega) \right), \]

(21)

where \( |\Omega \rangle = |\Omega_1 \rangle \otimes \cdots \otimes |\Omega_{N_k} \rangle \) is the spin coherent state

\[ |\Omega_k \rangle = (\cos(\theta_k/2))^{2S} \exp \left[ \tan (\theta_k/2) e^{i\phi_k} \hat{S}_- \right] |0\rangle, \]

(22)

with \( S_z|0\rangle = S|0\rangle \).

Introducing collective variables \( \bar{\Omega} = (\bar{\theta}, \bar{\phi}) \), Eq. (21) may be rewritten as

\[ \int D\bar{\Omega} D\bar{\Omega} \delta \left( \Omega - \frac{1}{N_k} \sum_k \Omega_k \right) e^{\left\{ i \sum_k f_k'' \int dt d^3x \left[ \frac{1}{2} (1 - \cos \theta_k) \dot{\phi}_k - H(\theta_k, \phi_k) \right] \right\}} \]

\[ = \int D\bar{\Omega} e^{iS_{\text{eff}}(\bar{\Omega})}, \]

(23)

where the effective action is defined as

\[ e^{iS_{\text{eff}}(\bar{\Omega})} = \int D\xi D\eta \delta \left( \sum_k \xi_k \right) \delta \left( \sum_k \eta_k \right) e^{\left\{ i \sum_k f_k'' \int dt d^3x \left[ \frac{1}{2} (1 - \cos (\bar{\theta} + \xi_k)) (\dot{\phi} + \eta_k) - H(\bar{\theta}, \bar{\phi}, \xi_k, \eta_k) \right] \right\}}. \]

(24)

We expand the exponent with respect to \( \xi_k, \eta_k \) up to the second-order, discarding the higher-order terms in \( \dot{\phi} \) within the adiabatic approximation. Taking the stationary-phase approximation s.t.

\[ \delta H(\bar{\theta}, \bar{\phi}, \xi_k, \eta_k) \bigg|_{\xi_k = \xi_k^*} = \delta H(\bar{\theta}, \bar{\phi}, \xi_k, \eta_k) \bigg|_{\eta_k = \eta_k^*} = 0, \]

(25)
we have
\[ S_{\text{eff}} \approx \frac{i}{\hbar} \int_{t'}^{t''} dt \int d^3x \left[ \Sigma (1 - \cos \theta) \dot{\phi} - H(\bar{\theta}, \bar{\phi}, \xi_k, \eta_k) \right], \] (26)

with \( \Sigma = N_k / 2V \). One may show that
\[ H(\bar{\theta}, \bar{\phi}, \xi_k, \eta_k) = \frac{A}{2} (\nabla_r U)^2 \] (27)
as should be. Then, the Bloch wall coefficient reads
\[ A = \frac{N_k/V}{8E_F} + O(g^2). \] (28)
The classical equation of motion for \( U \) is given by
\[ \Sigma \dot{U} + 2A \Delta U \times U = 0, \] (29)
which is the classical Landau-Lifshitz equation for the spin wave. Then, the dispersion relation for the spin wave is deduced,
\[ \omega(q) = (2A/\Sigma) q^2 \]
\[ = \left( \frac{1}{2} E_F \right) q^2 + O(g^2). \] (30)

The effective action has some topological features, with which we can do the second quantization for the spin wave. The first term in the effective action can be written as the interaction with the dynamically induced vector potential \( A \):
\[ (1 - \cos \bar{\theta}) \dot{\bar{\phi}} = A(\bar{\zeta}) \cdot \dot{\bar{\zeta}} \] (31)
with
\[ A = \left( \frac{1 - \cos \bar{\theta}}{\sin \bar{\theta}} \right) \dot{\bar{\phi}}. \] (32)
Employing the Dirac quantization condition, we can see that \( \Sigma = \) integer or half-integer as should be.

Alternatively, the first term in terms of \( U \) can be rewritten as the line integral along the path on \( S^2 \), which is nothing but the Wess-Zumino term.

From the effective action (26), it is inferred that \( \cos \theta \) and \( \phi \) are canonical conjugate. Putting \( \bar{\pi} \equiv \Sigma(1 - \cos \bar{\theta}) \), one can verify
\[ \{ \Sigma U^+(x, t), \Sigma U^-(y, t) \}_{PB} = 2i\Sigma U^3(x, t) \delta(x - y) \approx 2i\Sigma \delta(x - y), \] (33)
where we have used the Kramers-Heller approximation for a large number of particles in the last step. Taking the Fourier transform, s.t.,
\[ U^+(x, t) = \sqrt{\frac{2}{\Sigma}} \sum_q a_q e^{iq\cdot x}, \] (34)
and \( U^-(x, t) = (U^+(x, t))^\dagger \), and assuming the quantum-mechanical commutation relation (second quantization),
\[ [a_q, a_{q'}^\dagger] = \delta_{q, q'}, \] (35)
we have an Hamiltonian as an assembly of magnons,

\[ \int d^3x H(\vec{\theta}, \vec{\phi}, \xi^c_k, \eta^c_k) = \sum_q \omega(q) a_q^\dagger a_q. \] (36)

For low temperature the thermodynamical properties may be described by
the excitation of the spin wave. We can easily see that magnetization is re-
duced by the excitation of the spin waves at finite temperature \((T^{3/2} \text{ Law})\),
\((M(T) - M(0))/N_k g D \mu_q = -\zeta(3/2) (\Sigma T/8\pi A)^{3/2}\), from which the Curie temper-
ature reads, \(T_c = 8\pi A/\Sigma(\rho/2\zeta(3/2))^{2/3}\). Thus we can roughly estimate the Curie
temperature of several tens MeV for \(2A/\Sigma \simeq 1/2E_F\)

6. Concluding remarks

In this paper we have discussed the critical line of the spontaneous polarization
on the density-temperature plane within the framework of the Fermi-liquid theory.
We have evaluated the magnetic susceptibility by taking into account the screening
effects for the gluon propagation, and figured out the important roles of the static
and dynamic screening; the former gives the \(g^2 \ln(1/g^2)\) contribution, while the
latter gives \(T^2 \ln T\) for finite temperature. Both effects surely reflect the specific
feature of the gauge interaction. To get more realistic values for the critical density
and temperature, we must consider some nonperturbative effects such as instantons
as well as the systematic analysis of the higher-order terms in QCD.

We have presented a framework to deal with the spin wave as a Nambu-
Goldstone mode. In the perspectives we can derive the magnon-quark coupling
vertex, which may give a novel cooling mechanism and a novel type of the Cooper
pairing.\(^{12}\) They are not only theoretically interesting, but also phenomenologically
important for the thermal evolution of compact stars bearing the ferromagnetic
phase inside. It should be interesting if magnon effects could distinguish the micro-
scopic origin from fossil field or the dynamo mechanism.

The author thanks K. Sato for his collaboration. This work has been partially
supported by the Grant-in-Aid for the 21st Century COE “Center for the Diversity
and Universality in Physics” and the Grant-in-Aid for Scientific Research Fund (C)
of the Ministry of Education, Culture, Sports, Science and Technology of Japan
(16540246).

References

1. A.K. Harding and D. Lai, Rep. Prog. Phys. 69 (2006) 2631.
2. T. Tatsumi, Phys. lett. **B489** (2000) 280.
   T. Tatsumi, E. Nakano and K. Nawa, *Dark Matter* (Nova Science Pub., New York,
   2006), 39.
3. E. Nakano, T. Maruyama and T. Tatsumi, Phys. Rev. **D68** (2003) 105001.
   T. Tatsumi, E. Nakano and T. Maruyama, Prog. Theor. Phys. Suppl. **153** (2004) 190.
   T. Tatsumi, T. Maruyama and E. Nakano, *Superdense QCD Matter and Compact Stars*
   (Springer, 2006), 241.
4. F. Bloch, Z. Phys. 57 (1929) 545.
5. G. Baym and C.J. Pethick, *Landau Fermi-Liquid Theory* (WILEY-VCH, 2004)
   P. Nozières, *Theory of Interacting Fermi Systems* (Westview Press,1997).
   A.A. Abrikosov, L.P. Gorkov and I.Ye. Dzyaloshinskii, *Quantum Field Theoretical
   Methods in Statistical Physics* (Pergamon, Oxford, 1965).
6. M. Le Bellac, *Thermal Field Theory* (Cambridge U. Press, 1996).
   T. Schäfer and F. Wilczek, Phys. Rev. D60 (1999) 114033.
7. T. Holstein, R.E. Norton and P. Pincus, Phys. Rev. B8 (1973) 2649.
   S. Chakravarty, R.E. Norton and O.F. Syljuasen, Phys. Rev. Lett. 74 (1995) 1423.
   A. Gerhold, A. Ipp and A. Rebhan, Phys. Rev. D70 (2004) 105015. A. Schafä r and K.
   Schwenzer, Phys. Rev. D70 (2004) 054007; 114037.
8. J.M. Radcliffe, J. Phys. A4 (1971) 313.
   J.R. Klauder, Phys. Rev. D19 (1979) 2349.
9. C. Herring, Phys. Rev. 85 (1952) 1003.
   C. Herring, *Exchange Interactions among Itinerant Electrons: Magnetism IV* (Academic
   press, New York, 1966).
10. G. Baym and S.A. Chin, Nucl. Phys. A262 (1976) 527.
11. T. Tatsumi and K. Sato, in preparation.
12. N. Karchev, J. Phys.:Condens. Matter 15 (2003) L385.
   D. Fay and J. Appel, Phys. Rev. B22 (1980) 3173.0