Short-Time Heat Content Asymptotics via the Wave and Eikonal Equations

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Abstract
In this short paper, we derive an alternative proof for some known (van den Berg & Gilkey 2015) short-time asymptotics of the heat content in a compact full-dimensional submanifolds $S$ with smooth boundary. This includes formulae like

$$\int_S \exp(t\Delta)(f\mathbb{1}_S)\,dV = \int_S f\,dV - \sqrt{\frac{t}{\pi}}\int_{\partial S} f\,dA + o(\sqrt{t}), \quad t \to 0^+,$$

and explicit expressions for similar expansions involving other powers of $\sqrt{t}$. By the same method, we also obtain short-time asymptotics of $\int_S \exp(t^m\Delta^m)(f\mathbb{1}_S)\,dV$, $m \in \mathbb{N}$, and more generally for one-parameter families of operators $t \mapsto k(\sqrt{-t\Delta})$ defined by an even Schwartz function $k$.

Keywords  Heat equation · Heat content · Riemannian manifolds · Geometrical optics

1 Introduction

Let $(M, g)$ be a complete, boundaryless, oriented Riemannian manifold with Laplace–Beltrami operator $\Delta$, and volume $dV$. On a codimension-1 submanifold of $M$, we write $dA$ for the induced surface (hyper)-area form. The heat semi-group $T_t := \exp(t\Delta)$ acting on $L^2(M, dV)$ is well defined ($\Delta$ is essentially self-adjoint on $C^\infty_c(M)$ [2]) and its behaviour as $t \to 0^+$ has been extensively investigated in the literature. Specifically, for a set $S \subset M$, the heat content of the form

1 We assume that $M$ has no boundary for the sake of simplicity, and the method presented here can be adapted to more general manifolds with boundary provided that $S$ is compactly contained in the interior of $M$. If this is not the case, such as in the classical heat content setting as in [13], it should be possible to obtain similar results by modifying the geometrical optics construction used.

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Let us briefly recall some known results. On \( \mathbb{R}^n \), sets \( S \) of finite perimeter \( P(S) \) are characterized by \([7, \text{Thm. 3.3}]\)

\[
\lim_{t \to 0^+} \sqrt{\frac{\pi}{t}} \left( \Omega_{S,1_M}(0) - \Omega_{S,1_M}(t) \right) = P(S).
\]
(1)

Extensions of this idea to abstract metric spaces are given in \([6]\). In the setting of compact manifolds \( M \) (or \( M = \mathbb{R}^n \)) and \( S \) a full-dimensional submanifold with smooth boundary \( \partial S \), the authors of \([12]\) show that

\[
\Omega_{S,f}(t) = \sum_{j=0}^{\infty} \beta_j t^j, \quad t \to 0^+,
\]
(2)

where the coefficients \( \beta_j \) depend on \( S \), \( f \) and the geometry of \( M \). The setting of \([12]\) is more general, amongst other things it includes \( f \) which have singularities. Some of the coefficients obtained in \([12, \text{corollary 1.7}]\) are

\[
\beta_0 = \int_S f \, dV, \quad \beta_1 = -\frac{1}{\sqrt{\pi}} \int_{\partial S} f \, dA, \quad \beta_2 = \frac{1}{2} \int_S \Delta f \, dV.
\]

Extensions to some non-compact manifolds \( M \) and certain non-compact \( S \) are in \([11]\).

Both Eqs. (1) and (2) are proven with significant technical effort, yielding strong results. For example, in \([7]\), explicit knowledge of the fundamental solution of the heat equation is used to obtain Eq. (1) for \( C^{1,1} \)-smooth \( \partial S \), after which geometric measure theory is used. Similarly, \([12]\) requires pseudo-differential calculus and invariance theory.

Our aim is to show that slightly weaker results can be obtained by considerably lower technical effort. In contrast to \([7]\), we treat only compact \( S \) with smooth boundary, and do not allow \( f \) to have singularities like \([12]\) does. On the other hand, we put no further restrictions than completeness on \( M \). The proof presented here is simple, comparatively short, and provides an alternative differential geometric/functional analytic point of view to questions regarding heat content. Moreover, this approach is readily extended to some other PDEs including the semi-group generated by \( \Delta^m \).

Observe that \( T(t) = k(\sqrt{-t} \Delta) \) with \( k(x) = \exp(-x^2) \). We allow \( k \) to be an arbitrary even Schwarz function, with \( \Omega_{S,f}(t) = \int_S k(\sqrt{-t} \Delta)(f \mathbb{1}_S) \, dV \) and will prove:

**Theorem 1** Let \( M \) be a complete Riemannian manifold with Laplace–Beltrami operator \( \Delta \), Riemannian volume \( dV \) and induced (hyper) area form \( dA \). Let \( S \subset M \) be a compact full-dimensional submanifold with smooth boundary. For \( f \in C^\infty(M) \) and \( N \in \mathbb{N} \),

\[
\Omega_{S,f}(t) = \sum_{j=0}^{N} \beta_j t^j + o(t^N), \quad t \to 0^+.
\]
for constants \((\beta_j)^N_{j=0}\) described further in the next theorem.

With the \(j\)th derivative \(k^{(j)}\) (for \(j \in \mathbb{N}_0\)), let \(r_j := (-1)^{j/2}k^{(j)}(0)\) for \(j\) even and \(r_j := (-1)^{(j-1)/2}\int_0^\infty \frac{2k^{(j)}(s)}{-s} ds\) for \(j\) odd. Let \(\varphi\) locally be the signed distance function (see also [8, Sect. 3.2.2]) to \(\partial S\) with \(S = \varphi^{-1}([0, \infty))\), and denote by \(\nabla\) and \(\cdot\) the gradient and (metric) inner product, respectively. The vector field \(\nu := -\nabla \varphi\) is outer unit normal at \(\partial S\).

**Theorem 2** The coefficients of Theorem 1 satisfy \(\beta_0 = r_0 \int_S f \, dV\) and \(\beta_1 = -\frac{1}{2}r_1 \int_{\partial S} f \, dA\). For even \(j \in \mathbb{N}_{\geq 2}\),

\[
\beta_j = \frac{r_j}{j!} \int_S \frac{1}{2} \Delta^{j/2} f \, dV.
\]

Moreover, given the Lie-derivative \(\mathcal{L}_\nu\) with respect to \(\nu\),

\[
\beta_3 = \frac{r_3}{2 \cdot 3!} \int_{\partial S} \mathcal{L}_\nu(-\mathcal{L}_\nu + \frac{1}{2} \Delta \varphi) f - \frac{1}{2} \Delta f + \frac{1}{2}(-\mathcal{L}_\nu + \frac{1}{2} \Delta \varphi)^2 f \, dA,
\]

similar expression can be found also for larger odd values of \(j\) (see Sect. 3).

The properties of the signed distance function \(\varphi\) may be used to express terms appearing in Theorem 2 using other quantities. For example, its Hessian \(\nabla^2 \varphi\) is the second fundamental form on the tangent space of \(\partial S\) [3, Chap. 3], and thus \(\frac{1}{2} \Delta \varphi\) is the mean curvature.

Our approach to prove Theorems 1 and 2 is to combine 3 well-known facts:

(A) The short-time behaviour of the heat flow is related to the short-time behaviour of the wave equation (cf. [1]).

(B) The short-time behaviour of the wave equation with discontinuous initial data is related to the short-time behaviour of the eikonal equation (cf. ‘geometrical optics’ and the progressing wave expansion [10]).

(C) The short-time behaviour of the wave and eikonal equations with initial data \(f 1_S\) is directly related to the geometry of \(M\) near \(\partial S\).

Though points (A)-(C) are well known in the literature, they have (to the best of our knowledge) not been applied to the study of heat content so far.

A significant portion of (C) will rest on an application of the Reynolds transport theorem. Here, denote by \(\Phi^s\) the time-\(s\) flow of the vector field \(\nu = -\nabla \varphi\). For small \(s\), the (half) tubular neighbourhood

\[
S^{-s} := \{ x \in M \setminus S : \text{dist}(x, \partial S) \leq s \}
\]

satisfies \(S \cup S^{-s} = \Phi^s(S)\). For \(a \in C^\infty((-\varepsilon, \varepsilon) \times M)\), by [5, Chap. V, Prop. 5.2],

\[
\frac{d}{ds} \int_{S^{-s}} a(s, \cdot) \, dV \bigg|_{s=0} = \frac{d}{ds} \left( \int_{S^{-s} \cup S} a(s, \cdot) \, dV - \int_S a(s, \cdot) \, dV \right) \bigg|_{s=0} = \int_S \mathcal{L}_\nu[a(0, \cdot) \, dV] - \int_S a(0, \cdot) \, dA.
\]
The last equation is a consequence of Cartan’s magic formula and Stokes’ theorem, where we use that $dV(\nu, \cdot) = dA(\cdot)$ on $\partial S$.

2 Proof for $\beta_0$, $\beta_1$

By Fourier theory (for non-Gaussian $k$, the formulae must be adapted),

$$k(t) = \exp(-t^2) = \int_0^\infty \hat{k}(s) \cos(ts) \, ds \quad \text{with} \quad \hat{k}(s) := \frac{1}{\sqrt{\pi}} \exp\left(-\frac{s^2}{4}\right).$$

On the operator level, this yields the well-known formula [10, Sect. 6.2]

$$T_t = \exp(t \Delta) = \int_0^\infty \hat{k}(s) \cos(s \sqrt{-t} \Delta) \, ds. \quad (5)$$

The operator $W^s := \cos(s \sqrt{-\Delta})$ is the time-$s$ solution operator for the wave equation with zero initial velocity, in particular $u(s, x) := (W^s f \mathbb{1}_S)(x)$ (weakly) satisfies $(\partial_t^2 - \Delta)u = 0$. Let $\langle \cdot, \cdot \rangle$ denote the $L^2(M, dV)$ inner product. Using Eq. (5),

$$\langle T_t f \mathbb{1}_S, \mathbb{1}_S \rangle = \int_0^\infty \hat{k}(s) \langle W_s \mathbb{1}_S, \mathbb{1}_S \rangle \, ds.$$

Similar reasoning has been used to great effect in [1] to derive heat-kernel bounds by making use of the finite propagation speed of the wave equation. As in [1], finite propagation speed yields for $s \geq 0$ that $\langle W_s f \mathbb{1}_S, \mathbb{1}_S \rangle = \langle W_s f \mathbb{1}_S, \mathbb{1}_S^- \rangle$, where $S^s := (M \setminus S)^-\text{t}$ is defined like Eq. (3). Even if $\mathbb{1}_{M \setminus S} \not\in L^2(M, dV)$, we have just seen that the inner product $\langle W_s f \mathbb{1}_S, \mathbb{1}_{M \setminus S} \rangle$ is nevertheless well defined. In [1], it is further observed that $\|W_s\| \leq 1$. Using the Cauchy–Schwarz inequality and assuming $f = \mathbb{1}_M$, Eq. (4) yields

$$h(s) := \langle W_s f \mathbb{1}_S, \mathbb{1}_{S^-} \rangle \leq \|\mathbb{1}_S\|_2 \|\mathbb{1}_{S^-}\|_2 \leq s \int_{\partial S} dA + o(s), \quad s \to 0^+. \quad (6)$$

In addition, $|\langle W_s f \mathbb{1}_S, \mathbb{1}_S \rangle| \leq \|f \mathbb{1}_S\|_2 \|\mathbb{1}_S\|_2$ for all $s \geq 0$, in particular as $s \to \infty$. We conclude with some calculations (cf. Lemma 3), that

$$\langle T_t \mathbb{1}_S, \mathbb{1}_S \rangle = \int_0^\infty \hat{k}(s) \left( \langle W_s \mathbb{1}_S, \mathbb{1}_M \rangle - \langle W_s \mathbb{1}_S, \mathbb{1}_{M \setminus S} \rangle \right) \, ds$$

$$= \langle \mathbb{1}_S, \mathbb{1}_M \rangle - \int_0^\infty \hat{k}(s) h(s \sqrt{t}) \, ds$$

$$\geq \int_S dV - 2 \sqrt{\frac{t}{\pi}} \int_{\partial S} dA + o(\sqrt{t}), \quad t \to 0^+. \quad (7)$$

This is weaker than the desired estimate, and restricts to $f = \mathbb{1}_M$. The problem is that the estimates in Eq. (6) are too crude. To improve them, we instead approximate
the solution $u$ to the wave equation with geometrical optics, using the “progressing wave” construction described in [10, Sect. 6.6], some details of which we recall here. The basic idea is that $u$ is in general discontinuous, with an outward—and an inward—moving discontinuity given by the zero level-set of functions $\varphi^+$ and $\varphi^-$, respectively. The functions $\varphi^\pm$ satisfy the eikonal equation $\partial_t \varphi^\pm = \pm |\nabla \varphi^\pm|$ with initial value $\varphi^\pm(0, \cdot) = \varphi(\cdot)$. Equivalently, using the (nonlinear) operator $Ew := (\partial_t w)^2 - |\nabla w|^2$, the functions $\varphi^\pm$ satisfy $E(\varphi^\pm) = 0$. Our analysis is greatly simplified by choosing the initial $\varphi$ to (locally) be the signed distance function to $\partial S$. The eikonal equation is then $\partial_t \varphi^\pm = \pm |\nabla \varphi | = \pm | - \nu | = \pm 1$, i.e. $\varphi^\pm(x, t) = \varphi(x) \pm t$.

The progressing wave construction further makes use of two (locally existing and smooth) solutions $a_0^\pm$ to the first-order transport equations $\pm \partial_t a_0^\pm(t, \cdot) + \nu \cdot \nabla a_0^\pm(t, x) = \frac{1}{2} a_0^\pm \Delta \varphi^\pm$. Observe that with the Heaviside function $\theta : \mathbb{R} \to \mathbb{R}$, and $\Box := \partial_t^2 - \Delta$, the expression $\Box(a_0^\pm \theta(\varphi^\pm))$ is given by

$$(\theta''(\varphi^\pm) \Box \varphi^\pm + \nabla \varphi^\pm \theta'(\varphi^\pm)) a_0^\pm + 2 \left( \partial_t a_0^\pm \partial_t \varphi^\pm - \nabla a_0^\pm \cdot \nabla \varphi^\pm \right) \theta'(\varphi^\pm) + \Box a_0^\pm \theta(\varphi^\pm).$$

The functions $\varphi^\pm$ and $a_0^\pm$ have been chosen so the above simplifies to

$$\Box(a_0^\pm \theta(\varphi^\pm)) = 2 \left( \pm \partial_t a_0^\pm + \nabla a_0^\pm \cdot \nu - \frac{1}{2} \Delta \varphi a_0^\pm \right) \theta'(\varphi^\pm) + \Box a_0^\pm \theta(\varphi^\pm)$$

$$= \Box a_0^\pm \theta(\varphi^\pm). \quad (8)$$

Thus $\Box(a_0^\pm \theta(\varphi^\pm))$ is as smooth as $\theta$ is. We use

$$\tilde{u}(t, x) := a_0^+(t, x) \theta(\varphi^+(t, x)) + a_0^-(t, x) \theta(\varphi^-(t, x))$$

as an approximation to the discontinuity of the solution $u$ to the wave equation. To maintain consistency with the initial values of $u$, the initial values of the approximation $\tilde{u}$ are chosen to coincide with those of $u$ at $t = 0$, this is achieved by setting $a_0^\pm(0, \cdot) = \frac{1}{2} f$ so that(at least formally) $\partial_t \tilde{u}(0, \cdot) = 0$ and also $\tilde{u}(0, \cdot) = \mathbb{1}_S f$.

The function $\tilde{u}$ approximates the discontinuous solution $u$ of the wave equation well enough that the function $(s, x) \mapsto u(s, x) - \tilde{u}(s, x)$ is continuous on $[-T, T] \times M$, see [10, Sect. 6.6, eq. 6.35]. By construction, $\tilde{u}(0, \cdot) = u(0, \cdot)$. Hence $|(u(s, x) - \tilde{u}(s, x))| = o(1)$ as $s \to 0^+$, which implies

$$|(u(s, \cdot), \mathbb{1}_S f) - (\tilde{u}(s, \cdot), \mathbb{1}_S f)| = o(s) \quad s \to 0^+. \quad (9)$$

As $\nabla \varphi = -\nu$, for sufficiently small $t$ the sets $\{ x \in M : \varphi^+(t, x) = 0 \}$ (resp. $\{ x : \varphi^-(t, x) = 0 \}$) are level sets of $\varphi$ on the outside (resp. inside) of $S$ (see also [10, Sect. 6.6]). By construction, $\theta(\varphi^-)$ vanishes outside of $S$ for $t > 0$. Consequently,
using Eq. (4), we see that as \( s \to 0^+ \),

\[
\langle \tilde{u}(s, \cdot), 1_{S^{-s}} \rangle = \int_{S^{-s}} a_0^+(s, x) 1_{[\varphi^+(s, x) \geq 0]} + a_0^-(s, x) 1_{[\varphi^-(s, x) \geq 0]} \, dV(x) \\
= s \int_{\partial S} a_0^+(0, x) \, dA(x) + o(s) = \frac{s}{2} \int_{\partial S} f \, dA + o(s).
\]

Combining Eqs. (9) and (10),

\[
h(s) = \langle W_s f 1_S, 1_{S^{-s}} \rangle = \langle u(s, \cdot), 1_{S^{-s}} \rangle = \frac{s}{2} \int_{\partial S} f \, dA + o(s), \quad s \to 0^+.
\]

Calculations along the lines of Lemma 3 and Eq. (7) yield

\[
\langle T_t f 1_S, 1_S \rangle = \int_S f \, dV - \sqrt{\frac{t}{\pi}} \int_{\partial S} f \, dA + o(\sqrt{t}), \quad t \to 0^+,
\]
as claimed.

**Lemma 3** Let \( j \in \mathbb{N} \) and \( \gamma : \mathbb{R}_{\geq 0} \to \mathbb{R} \). Let \( \gamma(s) = s^j + o(s^j) \) for \( s \to 0 \) and \( \gamma(s) = O(1) \) for \( s \to \infty \). Then for \( t \to 0^+ \),

\[
\int_0^\infty \gamma(s\sqrt{t}) \hat{k}(s) \, ds = t^{\frac{j}{2}} \begin{cases} (-1)^{\frac{j}{2}} k^{(j)}(0) & j \text{ even} \\ (-1)^{\frac{j-1}{2}} \pi \int_0^\infty \frac{2k^{(j)}(s)}{s} \, ds & j \text{ odd} \end{cases} + o(t^{\frac{j}{2}}).
\]

With \( k(s) = \exp(-s^2) \) and \( h(s) = c_0 + c_1 s + c_2 s^2 + o(s^2) \), this implies

\[
\int_0^\infty h(s\sqrt{t}) \hat{k}(s) \, ds = c_0 + \frac{2c_1}{\sqrt{\pi}} \sqrt{t} + 2c_2 t + o(t).
\]

**Proof** For even \( j \), we obtain Eq. (11) by the Fourier-transform formula for \( j \)th derivatives. If \( j \) is odd, we also need to multiply by the sign function in frequency space, and then use that the inverse Fourier-transform (unnormalized) of the sign function is given by the principal value p.v. \((\frac{2i}{x})\) [10, Sect. 4], see also [9, Chap. 7]. Equation 11 holds more generally, e.g. if \( k \) is an even Schwartz function. Equation 12 may also be verified directly without Eq. (11). \(\square\)

### 3 Proof for \( \beta_2, \beta_3, \ldots \)

We now turn to calculating \( \beta_j \) for \( j \geq 2 \). We use the \( N \)th order progressing wave construction with sufficiently large \( N \gg j \). For the sake of simplicity, we write \( O(t^\infty) \) for quantities that can be made \( O(t^k) \) for any \( k \in \mathbb{N} \) by choosing sufficiently large \( N \). As in the previous section, the construction is from [10, Sect. 6.6]. With

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\[ \theta_0 := \theta, \text{ and } \theta_i(t) := \int_{-\infty}^{t} \theta_{i-1}(s) \, ds \text{ we write} \]

\[ \tilde{u}^\pm(t, x) := \sum_{i=0}^{N} a_i^\pm(t, x) \theta_i(\varphi^\pm(t, x)). \]

Here the functions \( a_i^\pm \) are defined as before, and for \( i \geq 1 \) the \( i \)th order transport equations \( \pm \partial_t a_i^\pm = -\nu \cdot \nabla a_i^\pm + \frac{1}{2} \partial_t^2 a_i^\pm \Delta \varphi^\pm - \frac{1}{2} \Box a_i^\pm \) define \( a_i^\pm \) together with initial data \( a_i^\pm(0, \cdot) = -\frac{1}{8} (\partial_t a_{i-1}^+(0, \cdot) + \partial_t a_{i-1}^-(0, \cdot)) \). As in Eq. (8), one may verify that \( \Box \tilde{u}^\pm = \Box a_i \theta_N(\varphi^\pm) \). Writing \( u = \tilde{u}^+ + \tilde{u}^- \) and

\[ u(t, x) = \tilde{u}^+(t, x) + \tilde{u}^-(t, x) + R_N(t, x), \]

the remainder satisfies \( R_N \in C^{(N,1)}([-T, T] \times M) \) and \( R_N(t, \cdot) \) vanishes at \( t = 0 \), see [10, Sect. 6.6, eq. 6.35]. Moreover, \( R_N \) is supported on \( \{(x, t) : \text{dist}(x, S) \leq |t| \} \), all of this implies that, as \( t \to 0^+ \),

\[ h(t) = \int_{M \setminus S} u(t, x) \, dV(x) = \int_{M \setminus S} \tilde{u}^+(t, x) \, dV(x) + O(t^\infty) \quad (13) \]

and moreover \( h \in C^{\infty}([0, T]) \). The structure of \( R_N \) implies that \( \Box \tilde{u}^+(t, x) = O(t^\infty) \) on \( M \setminus S \), provided that this expression is interpreted in a sufficiently weak sense. Formally, therefore

\[ \partial_t^2 \int_{M \setminus S} \tilde{u}^+(\cdot, t) \, dV = \int_{M \setminus S} \Delta \tilde{u}^+(\cdot, t) \, dV + O(t^\infty) \]

\[ = -\int_{\partial S} \nabla \tilde{u}^+(\cdot, t) \cdot \nu \, dA + O(t^\infty), \quad (14) \]

where the last step is the divergence theorem. One may verify Eq. (14) rigorously by either doing the above steps in the sense of distributions, or by a (somewhat tedious) manual computation. Combining this with Eq. (13),

\[ h''(t) = -\int_{\partial S} \nabla \tilde{u}^+(\cdot, t) \cdot \nu \, dA + O(t^\infty). \]

(15)

The quantity \( h^{(j)}(0) \) may thus be seen to depend on \( \tilde{u}^+(0, \cdot) \) at \( \partial S \), which in turn depends on \( a_i^\pm \) at \( t = 0 \). Defining \( S_i := a_i^+ + a_i^- \) and \( D_i := a_i^+ - a_i^- \) for \( i = 0, 1, \ldots \), let \( L \) be the (spatial) differential operator defined for \( w \in C^{\infty}(M) \) by \( Lw := \frac{1}{2} \Delta \varphi w - \nu \cdot \nabla w \).

For \( i \in \mathbb{N}_0 \), the transport equations imply

\[ \partial_t S_0 = LD_0, \quad \partial_t D_0 = LS_0, \quad (16) \]

\[ \partial_t S_{i+1} = LD_{i+1} - \frac{1}{2} \Box D_i, \quad \partial_t D_{i+1} = LS_{i+1} - \frac{1}{2} \Box S_i \quad \text{for } i \geq 0, \quad (17) \]
with initial values satisfying
\[ a_0^+(0, \cdot) = \frac{1}{2} S_0(0, \cdot) = \frac{1}{2} f(\cdot), \quad D_0(0, \cdot) = 0, \quad (18) \]
\[ a_{i+1}^+(0, \cdot) = \frac{1}{2} D_{i+1}(0, \cdot) = -\frac{1}{2} \partial_t S_i(0, \cdot), \quad S_{i+1}(0, \cdot) = 0. \quad (19) \]

**Lemma 4** For $i, n \in \mathbb{N}_0$ it holds that $\partial_i^{2n} D_i(0, \cdot) = 0$ (note that as a consequence, also $a_{i+1}(0, \cdot), LD_i(0, \cdot)$, and $\Box^n D_i(0, \cdot)$ are zero).

**Proof** We will proceed by induction over $i$ and use the identities Eqs. (16)–(19). For

$i = 0, D_0(0, \cdot) = 0$ is trivially satisfied. Moreover, $\partial_i^{2n} D_0 = R^n D_0$, which is zero at $t = 0$. For $i = 1$, observe that $a_1^+(0, \cdot) = -\frac{1}{2} \partial_t S_0(0, \cdot) = -\frac{1}{2} L D_0(0, \cdot) = 0$, and thus $D_1(0, \cdot) = 0$. Likewise, $\partial_i^2 D_1 = \partial_t (L S_1 - \frac{1}{2} \Box S_0) = L (L D_1 - \frac{1}{2} \Box D_0) - \frac{1}{2} \Box L D_0$. As the operator $L$ commutes with $\partial_i^2$, this expression vanishes at $t = 0$. Induction over $n$ proves the remainder of the statement for $i = 1$. For the general case, we assume the induction hypothesis for $i$ and $i + 1$ and start by noting that $D_{i+2}(0, \cdot) = 2 a_{i+2}^+(0, \cdot) = -\partial_t S_{i+1}(0, \cdot) = -(L D_{i+1}(0, \cdot) - \frac{1}{2} \Box D_i(0, \cdot)) = 0$. Moreover, $\partial_i^2 D_{i+2} = \partial_t (L S_{i+2} - \frac{1}{2} \Box S_{i+1}) = L (L D_{i+2} - \frac{1}{2} \Box D_{i+1}) - \frac{1}{2} \Box (L D_{i+1} - \frac{1}{2} \Box D_i)$, which again vanishes at $t = 0$; the case $n > 1$ may again be proven by induction over $n$.  

**Corollary 5** For even $j \in \mathbb{N}_{\geq 2}$, the $j$th derivative of $h$ satisfies

\[ h^{(j)}(0) = -\frac{1}{2} \int_S \Delta^{j/2} f \, dV. \]

**Proof** Lemma 4 shows that for $i \geq 1$, $a_i^+(0, x) = 0$. Together with Eq. (15), thus $h''(0) = -\int_{\partial S} \nabla a_0^+(0, \cdot) \cdot v \, dA = -\frac{1}{2} \int_{\partial S} \nabla f \cdot v \, dA$. This is the case $j = 2$. More generally, for $j = 2k$ with $k \in \mathbb{N}_{\geq 2}$, we use that (for $x \in \partial S$), $\bar{u}^+$ satisfies $\partial_i^2 \bar{u}^+(t, x) = \Delta \bar{u}^+(t, x) + O(t^{\infty})$. Equation 15 ensures that as $t \to 0^+$,

\[ h^{(2k)}(t) = \int_{\partial S} \nabla (\Delta^{k-1} \bar{u}^+(t, \cdot)) \cdot v \, dA + O(t^{\infty}). \]

As for the case $k = 1$, it follows that $h^{(2)}(0) = -\int_{\partial S} \nabla (\Delta^{0} a_0^+) \cdot v \, dA$, the divergence theorem yields the claim. \hfill \Box

The odd coefficients are trickier, we only compute the case $j = 3$. We start with the observation that for $x \in \partial S, \varphi^+(t, x) = t$ and therefore

\[ \bar{u}^+(t, x) = \sum_{i=0}^{N} \frac{1}{i!} a_i^+(t, x) \quad \text{for } t \geq 0, \ x \in \partial S. \]
Recall that the Lie-derivative acts on functions $w \in C^{\infty}(M)$ by $\mathcal{L}_v w = \nabla w \cdot v$. Thus $\mathcal{L}_v \theta_{i+1}(\varphi^+(t, x)) = -\theta_i(\varphi^+(t, x))$, so for $x \in \partial S$,

$$\mathcal{L}_v \tilde{u}^+(t, x) = \sum_{i=0}^{N-1} \frac{t_i}{i!} (\mathcal{L}_v a_i^+(t, x) - a_{i+1}(t, x)) + O(t^{\infty}).$$

Therefore $\partial_t \mathcal{L}_v \tilde{u}^+(0, x) = \partial_t (\mathcal{L}_v a_1^+(0, x) - a_2^+(0, x)) + (\mathcal{L}_v a_1^+(0, x) - a_2^+(0, x))$, but the second term is zero as $a_1^+$ and $a_2^+$ vanish at $t = 0$ by Lemma 4. Substituting the transport equations and removing further zero terms leaves $\partial_t \mathcal{L}_v \tilde{u}^+(0, x) = \mathcal{L}_v L a_0^+(0, x) + \frac{1}{2} \Box a_0(0, x) = \frac{1}{2} (\mathcal{L}_v L f(x) - \frac{1}{2} \Delta f(x) + \frac{1}{2} L^2 f(x)).$ Thus (recall that $L = -\mathcal{L}_v + \frac{1}{2} \Delta \varphi$) directly from Eq. (15),

$$h^{(3)}(0) = -\frac{1}{2} \int_{\partial S} \mathcal{L}_v L f(x) - \frac{1}{2} \Delta f(x) + \frac{1}{2} L^2 f(x) \, dA(x).$$

The formula

$$\Omega_{S, f}(t) = \int_0^{\infty} k(s) \left( \int_S f \, dV - h(s \sqrt{t}) \right) \, ds$$

(20)

established in the previous section, together with Lemma 3, yields the asymptotic behaviour of $\Omega_{S, f}(t)$ by taking the Taylor expansion of $h$ using Corollary 5. This gives the remainder of the claims of theorem 2.

4 Discussion

The above-said is not specific to the heat equation. Taking $k(x) = \exp(-x^{2m})$, $m \in \mathbb{N}$, we may, for example, study the one-parameter operator family $\exp(-t^m \Delta^m)$. The wave equation estimates needed are the same. For $m \geq 2$, a brief calculation yields the explicit $t \to 0^+$ asymptotics

$$\langle \exp(t^m \Delta^m) f \mathbb{1}_S, \mathbb{1}_S \rangle = \int_S f \, dV - \left( \pi^{-1} \Gamma \left( \frac{2m - 1}{2m} \right) \int_{\partial S} f \, dA \right) \sqrt{t} + o(t).$$

We conclude with the observation that the generalization of this paper to weighted Riemannian manifolds (cf. [4]) is straightforward.

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