Multiple Weak Solutions for a Kind of Time-Dependent Equation Involving Singularity

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Abstract. The existence of at least three weak solutions for a kind of nonlinear time-dependent equation is studied. In fact, we consider the case that the source function has singularity at origin. To this aim, the variational methods and the well-known critical points theorem are main tools.

1. Introduction

The linear Sobolev equations have a real physical background (see [5, 35, 37]) and are studied in [11, 15]. Because of their complexity, they haven’t exact solutions (except some very especial cases [3]). There are different methods to study the solution of these problems. One of the standard methods is the fixed point theory that investigate the existence of solutions of nonlinear boundary value problems [2, 6, 12–14, 16, 29, 31, 34, 38]. The calculus of variation is another impressive technique and for using this technique, one needs to show that the given boundary value problem should possess a variational structure on some convenient spaces [1, 4, 9, 10, 17–28, 30, 32, 33, 36].

In the present paper, we study the weak solutions of

\[
\begin{aligned}
\frac{\partial u}{\partial t} - \frac{\partial (\Delta u)}{\partial t} &= \mu f(x,t,u) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega, \\
u(x,0) &= g(x) \quad x \in \Omega,
\end{aligned}
\]

where \(\Omega\) is a non-empty bounded open subset of \(\mathbb{R}^N\) with \(\partial \Omega \in C^1\), \(\mu\) is a positive parameter, \(f : \Omega \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}\) is a Carathéodory function and has a singularity at the origin with respect to the time variable and \(g : \Omega \to \mathbb{R}\) vanishes on \(\partial \Omega\).

The aim of this paper is to find an interval for \(\mu\) for which the problem (1) admits at least three distinct weak solutions.

By integrating the first equation of (1) we get

\[
\int_0^t \frac{\partial u(x,s)}{\partial s} ds - \int_0^t \frac{\partial \Delta u(x,s)}{\partial s} ds = \int_0^t \mu f(x,s,u) ds, \tag{2}
\]
or
\[-\Delta u(x, t) = \mu F(x, t, u) - u(x, t) + g(x) - \Delta g(x),\]  

(3)

where
\[F(x, t, u) = \int_0^t f(x, s, u)ds.\]  

(4)

The equation (3) is a time-dependent elliptic equation.

**Definition 1.1.** A function \(u : \Omega \to \mathbb{R}\) is called a weak solution of the problem (1) if \(u \in H^1_0\) and
\[\int_\Omega \nabla u(x, t) \cdot \nabla v(x)dx - \mu \int_\Omega F(x, t, u)v(x)dx + \int_\Omega u(x, t)v(x)dx - \int_\Omega g(x)v(x)dx + \int_\Omega \Delta g(x)v(x)dx = 0,\]  

(5)

for all \(v \in H^1_0\) and \(t \geq 0\).

**Definition 1.2.** Define the functionals \(\varphi, \delta : H^1_0 \to \mathbb{R}\) by \(\varphi(u) := \frac{1}{2}\|u\|^2\) and
\[\delta(u) := \int_\Omega \tilde{F}(x, t, u)dx - \frac{1}{2\mu} \int_\Omega (u(x, t))^2 dx + \frac{1}{p} \int_\Omega g(x)u(x, t)dx - \frac{1}{\mu} \int_\Omega \Delta g(x)u(x, t)dx,\]

respectively, where \(\tilde{F}(x, t, \eta) := \int_0^t F(x, t, s)ds\).

Notice that \(\varphi\) and \(\delta\) are well-defined and \(C^1, \varphi', \delta' \in X^*, \varphi'(u)(v) = \int_\Omega \nabla u(x) \cdot \nabla v(x)dx\) and
\[\delta'(u)(v) = \int_\Omega F(x, t, u)v(x)dx - \frac{1}{\mu} \int_\Omega u(x, t)v(x)dx + \frac{1}{p} \int_\Omega g(x)v(x)dx - \frac{1}{\mu} \int_\Omega \Delta g(x)v(x)dx.\]

**Remark 1.3.** A critical point of \(I_\mu := \varphi - \mu \delta\) is exactly a weak solution of (1).

Fix \(q \in [1, 2^*] \setminus \{1\}, \) Embedding Theorem [7] shows \(H^1_0(\Omega) \hookrightarrow L^q(\Omega), \) i.e. there exists \(c_q > 0\) such that for all \(u \in H^1_0(\Omega)\)
\[\|u\|_{L^q(\Omega)} \leq c_q \|u\|,\]  

(6)

where
\[c_q \leq \frac{\text{meas}(\Omega)^{\frac{2^*}{2^* - q}}}{\sqrt{N(N - 2)\pi}} \left(\frac{N!}{2\Gamma(N/2 + 1)}\right)^{\frac{1}{2}},\]  

(7)

\(\Gamma\) is the Gamma function, \(2^* = 2N/(N - 2)\) and \(\text{meas}(\Omega)\) denotes the Lebesgue measure of \(\Omega.\)

**2. Three weak solutions**

In this section the existence of at least three weak solutions for the problem (1) is proved. Due to do this, we apply [8, Theorem 3.6] which is given below

**Theorem 2.1.** (see [8], Theorem 3.6). Let \(X\) be a reflexive real Banach space, \(\Phi : X \to \mathbb{R}\) be a coercive, continuously Gateaux differentiable and sequentially weakly lower semicontinuous functional whose Gateaux derivative admits a continuous inverse on \(X^*, \Psi : X \to \mathbb{R}\) be a continuously Gateaux differentiable functional whose Gateaux derivative is compact such that \(\Phi(0) = \Psi(0) = 0.\) Assume that there exist \(r > 0\) and \(\hat{x} \in X,\) with \(r < \Phi(\hat{x}),\) such that:
Theorem 2.2. Let \( f : \Omega \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R} \) be a Carathéodory function and \( g : \Omega \to \mathbb{R} \) vanishes on \( \partial \Omega \). Assume

1. \( \sup_{x \in \Omega} \psi(x) < \psi(T), \)

2. for each \( \lambda \in \Lambda := \frac{1}{\psi(T)} \cdot \sup_{x \in \Omega} \frac{r}{\psi(x)} \) the functional \( \Phi - \lambda \Psi \) is coercive.

Then, for each \( \lambda \in \Lambda \), the functional \( \Phi - \lambda \Psi \) has at least three distinct critical points in \( X \).

Set

\[
D := \sup_{x \in \Omega} \text{dist}(x, \partial \Omega), \quad \kappa := \frac{D \sqrt{2 \lambda}}{\sqrt{2^{(N+1)}} \left( \frac{N(N+1)}{2(N+1)} \right) \frac{1}{2}}, \quad K_1 := \frac{2 \sqrt{2} (2^N-1)}{\lambda}, \quad K_2 := \frac{2 \pi \sqrt{2} (2^N-1)}{q \mu^2}. \tag{8}
\]

Now, we can state the main result.

**Theorem 2.2.** Let \( f : \Omega \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R} \) be a Carathéodory function and \( g : \Omega \to \mathbb{R} \) vanishes on \( \partial \Omega \). Assume

1. There exist non-negative constants \( m_1, m_2 \) and \( q \in [1, \frac{2N}{N-2}] \) such that
   \[
   F(x, t, s) \leq m_1 + m_2 |s|^{-1} + \frac{1}{\mu} (s - g(x) + \Delta g(x))
   \]
   for all \( (x, t, s) \in \Omega \times \mathbb{R}^+ \times \mathbb{R} \).

2. \( \overline{F}(x, t, \eta) \geq \frac{1}{p} \left( \frac{1}{2} \eta^2 - \eta g(x) + \eta \Delta g(x) \right) \) for every \( (x, t, \eta) \in \Omega \times \mathbb{R}^+ \times \mathbb{R} \).

3. There exist positive constants \( a \) and \( b < 2 \) such that
   \[
   \overline{F}(x, t, \eta) \leq a(1 + |\eta|^b) + \frac{1}{\mu} \left( \frac{1}{2} \eta^2 - \eta g(x) + \eta \Delta g(x) \right).
   \]

4. There exist positive constants \( \alpha, \beta \) with \( \beta > \alpha \mu \) such that
   \[
   \inf_{x \in \Omega} \left( \overline{F}(x, t, \beta) - \frac{1}{\beta} \left( \frac{1}{2} \beta^2 - \beta g(x) + \beta \Delta g(x) \right) \right) > m_1 \frac{K_1}{\alpha} + m_2 K_2 \alpha^b - 2,
   \]

where \( \kappa, K_1, K_2 \) are given by (8).

Then the problem (1) has at least three weak solutions in \( H^1_0(\Omega) \), for each parameter \( \mu \) belonging to \( \Lambda(\alpha, \beta) := \frac{2(2^N-1)}{\mu} \times (\delta_1, \delta_2) \), where

\[
\delta_1 := \inf_{x \in \Omega} \frac{1}{\inf_{x \in \Omega} \left( \overline{F}(x, t, \beta) - \frac{1}{\beta} \left( \frac{1}{2} \beta^2 - \beta g(x) + \beta \Delta g(x) \right) \right)} \quad \text{and} \quad \delta_2 := \frac{1}{m_1 \frac{K_1}{\alpha} + m_2 K_2 \alpha^b - 2}.
\]

**Proof.** Set \( X := H^1_0(\Omega) \) and define the functionals \( \varphi(u) \) and \( \delta(u) \) by Definition 1.2. Clearly, \( \delta \) and \( \varphi \) satisfy the assumptions of [8, Theorem 3.6]. By (1)

\[
\overline{F}(x, t, \eta) \leq \frac{1}{\mu} \left( \frac{1}{2} \eta^2 - \eta g(x) + \eta \Delta g(x) \right) + m_1 |\eta| + m_2 |\eta|^q \tag{9}
\]
Thus for every $(x, t, \eta) \in \Omega \times \mathbb{R}^+ \times \mathbb{R}$. Thus

$$
\mathcal{S}(u) := \int_\Omega \bar{F}(x, t, u)dx - \frac{1}{2\mu} \int_\Omega (u(x, t))^2 \, dx + \frac{1}{\mu} \int_\Omega g(x)u(x, t)dx
$$

$$
- \frac{1}{\mu} \int_\Omega \Delta g(x)u(x, t)dx
$$

$$
\leq \frac{1}{\mu} \int_\Omega \left( \frac{1}{2}(u(x, t))^2 - u(x, t)g(x) + u(x, t)\Delta g(x) \right) \, dx
$$

$$
+ \int_\Omega \left( m_1|u(x, t)| + m_2 \frac{|u(x, t)|^q}{q} \right) \, dx - \frac{1}{\mu} \int_\Omega (u(x, t))^2 \, dx
$$

$$
+ \frac{1}{\mu} \int_\Omega g(x)u(x, t)dx - \frac{1}{\mu} \int_\Omega \Delta g(x)u(x, t)dx
$$

$$
\leq m_1 \| u \|_{L^1(\Omega)} + \frac{m_2}{q} \| u \|_{L^q(\Omega)}.
$$

Let $r \in ]0, +\infty[$ such that $\varphi(u) \leq r$. By (6),

$$
\mathcal{S}(u) \leq \left\{ \begin{array}{c}
\sqrt{2} c_1 m_1 + \frac{2^4 c_1^2 m_2}{q} r^4 \\
\end{array} \right\}.
$$

Set $\chi(r) = \sup_{u \in \Omega, \| u \| \leq r} \mathcal{S}(u)$. Consequently

$$
\chi(r) \leq \left\{ \begin{array}{c}
\sqrt{2} c_1 m_1 + \frac{2^4 c_1^2 m_2}{q} r^4 \\
\end{array} \right\}.
$$

for every $r > 0$.

By (8), there is $x_0 \in \Omega$ such that $B(x_0, D) \subseteq \Omega$. Set

$$
u_\beta(x, t) := \left\{ \begin{array}{c}
0 \quad x \in \Omega \setminus B(x_0, D), \\
\frac{2^\beta}{D} (D - |x - x_0|) \quad x \in B(x_0, D) \setminus B(x_0, D/2), \\
\beta \quad x \in B(x_0, D/2).
\end{array} \right\}
$$

(11)

Thus $u_\beta \in H_0^1(\Omega)$. So

$$
\varphi(u_\beta) = \frac{1}{2} \int_\Omega |\nabla u_\beta|^2 \, dx
$$

$$
= \frac{1}{2} \int_{B(x_0, D) \setminus B(x_0, D/2)} (2\beta)^2 \frac{D^2}{D^2} \, dx
$$

$$
= \int_{B(x_0, D) \setminus B(x_0, D/2)} (\text{meas}(B(x_0, D)) - \text{meas}(B(x_0, D/2)))
$$

$$
= \frac{1}{2} \frac{(2\beta)^2}{D^2} \pi^{N/2} \left( \frac{D^N}{\Gamma(N/2 + 1)} \right). 
$$

(12)
If we force $\beta > \alpha x$, by (4), $\alpha^2 < \varphi(u_0)$ because $\alpha^2 < \frac{\|g\|}{\beta^2}$. Also by assumption (2),

$$
\varsigma(u_0) := \int_{\Omega} F(x, t, u_0) dx - \frac{1}{\mu} \int_{\Omega} (u_0(x, t))^2 dx + \frac{1}{\mu} \int_{\Omega} g(x) u_0(x, t) dx
$$

$$
- \frac{1}{\mu} \int_{\Omega} \Delta g(x) u_0(x, t) dx
$$

$$
= \int_{\Omega} \left[ F(x, t, u_0) - \frac{1}{\mu} \left( \frac{1}{2} u_0(x, t)^2 - g(x) u_0(x, t) + \Delta g(x) u_0(x, t) \right) \right] dx
$$

$$
\geq \int_{\Omega, 2 \Omega , \Omega / 2} \left[ F(x, t, u_0) - \frac{1}{\mu} \left( \frac{1}{2} u_0(x, t)^2 - g(x) u_0(x, t) + \Delta g(x) u_0(x, t) \right) \right] dx
$$

$$
\geq \inf_{z \in \Omega} \left( \bar{F}(x, t, \beta) - \frac{1}{\mu} \left( \frac{1}{2} \beta^2 - \beta g(x) + \beta \Delta g(x) \right) \right) \frac{\|v\|^2}{1/(N+1)} \frac{\|v\|}{2 N}.
$$

Next by dividing (12) on (13), we have

$$
\frac{\varsigma(u_0)}{\varphi(u_0)} \geq \frac{D^2}{2(2N - 1)} \inf_{z \in \Omega} \left( \bar{F}(x, t, \beta) - \frac{1}{\mu} \left( \frac{1}{2} \beta^2 - \beta g(x) + \beta \Delta g(x) \right) \right) \frac{\|v\|^2}{\beta^2}.
$$

Using (10), assumption (4) implies

$$
\chi(\alpha^2) \leq \left( \frac{\sqrt{2c_1 m_1}}{\alpha} + \frac{2 \zeta c_1 m_2 \alpha^{\gamma-2}}{q} \right)
$$

$$
= \frac{D^2}{2(2N - 1)} \left( m_1 K_1 + m_2 K_2 \alpha^{\gamma-2} \right)
$$

$$
< \frac{D^2}{2(2N - 1)} \inf_{z \in \Omega} \left( \bar{F}(x, t, \beta) - \Delta(x, t) - G(x) - G(x) \right)
$$

$$
\leq \frac{\varsigma(u_0)}{\varphi(u_0)}.
$$

Assuming $b < 2$ and considering $\|u\| \in L^2(\Omega)$ for all $u \in X$, Hölder’s inequality for $u \in X$ implies

$$
\int_{\Omega} |u(x, t)|^b dx \leq \|u\|_{L^2(\Omega)} \|\text{meas}(\Omega)\|^b.
$$

Therefore equation (6) shows for all $u \in X$

$$
\int_{\Omega} |u(x, t)|^b dx \leq c_b \|u\| \|\text{meas}(\Omega)\|^b,
$$

and by assumption (3),

$$
I_{\mu}(u) = \varphi(u) - \mu \varsigma(u)
$$

$$
= \frac{\|u\|^2}{2} - \mu \int_{\Omega} F(x, t, u) dx + \frac{1}{2} \int_{\Omega} (u(x, t))^2 dx
$$

$$
- \int_{\Omega} g(x) u(x, t) dx + \int_{\Omega} \Delta g(x) u(x, t) dx
$$

$$
\geq \frac{\|u\|^2}{2} - \mu \int_{\Omega} g(x) (1 + |u(x, t)|^b) dx
$$

$$
\geq \frac{\|u\|^2}{2} - \mu c_b \|\text{meas}(\Omega)\|^b \|u\| - \mu \alpha \|\text{meas}(\Omega)\|.
$$

This means for every $\mu \in \Lambda(\alpha, \beta) \equiv \left[ \frac{\varphi(u_0)}{\varphi(u_0)} \right] \frac{\|u\|^2}{\|u\|^2} \frac{\|u\|}{\|u\|^2} \frac{\varphi(u_0)}{\|u\|^2} \frac{\|u\|}{\|u\|^2} \frac{\varphi(u_0)}{\|u\|^2}$, $I_{\mu}$ is coercive. Therefore by Theorem 2.1 for each $\mu \in \Lambda(\alpha, \beta)$ the functional $I_{\mu}$ has at least three distinct critical points that they are weak solutions of the problem (1).
Theorem 2.2 is satisfied, i.e.
\[ u(x,0) = \frac{1}{1000} \left( \frac{1}{100} - \left( x_1^2 + x_2^2 + x_3^2 \right) \right) \quad x \in \Omega, \]
where
\[ \Omega := \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3, x_1^2 + x_2^2 + x_3^2 \leq 0.1 \right\}. \]

3. Numerical Experiment

Now, we present an example. Let
\[ \mu = 0.01, N = 3, D = r = 0.1, 2^r = 6, g(x) = 0.001 \left( 0.01 - \left( x_1^2 + x_2^2 + x_3^2 \right) \right), \quad \Delta g(x) = -0.006 \text{ and } f(x, t, u) = \frac{99}{100} \left( 1 + \exp(-t) \right) \left( 8 + 100u + u^2 \right). \]

Now, setting \( q = 3 \), then
\[ c_1 \leq 0.00445759, \quad c_2 \leq 0.171543, \]
\[ \kappa = 1.16798, \quad K_1 \leq 8.82557, \quad K_2 \leq 6.66307. \]

Clearly \( F(x, t, \eta) = \frac{99}{100} \left( 1 + \exp(-t) \right) \left( 8\eta + 50\eta^2 + \frac{\eta^3}{3} \right) \), suppose \( m_1 = 9 \) and \( m_2 = 1 \), then the assumption (1) of the Theorem 2.2 is satisfied, i.e.
\[ \frac{99}{100} \left( 1 + \exp(-t) \right) \left( 8 + 100s + s^2 \right) \leq 9 + s^2 + \frac{1}{100} \left( s - 0.001 \left( 0.01 - \left( x_1^2 + x_2^2 + x_3^2 \right) \right) - 0.006 \right), \]
for all \( (x, t, s) \in \Omega \times \mathbb{R}^+ \times \mathbb{R}. \)

Obviously \( F(x, t, \eta) = \frac{99}{100} \left( 1 + \exp(-t) \right) \left( 8\eta + 50\eta^2 + \frac{\eta^3}{3} \right) \), then it can be easily verified that the assumption (2) of the Theorem 2.2 holds, i.e. for all \( (x, t, s) \in \Omega \times \mathbb{R}^+ \times \mathbb{R} \)
\[ \frac{99}{100} \left( 1 + \exp(-t) \right) \left( 8\eta + 50\eta^2 + \frac{\eta^3}{3} \right) \leq \frac{1}{100} \left( \frac{1}{\kappa^2} - 0.001 \eta \left( 0.01 - \left( x_1^2 + x_2^2 + x_3^2 \right) \right) - 0.006 \eta \right). \]

Also, by choosing \( \alpha = 10 \), the assumption (3) of the Theorem 2.2 is satisfied, i.e. for all \( (x, t, s) \in \Omega \times \mathbb{R}^+ \times \mathbb{R} \)
\[ \frac{99}{100} \left( 1 + \exp(-t) \right) \left( 8\eta + 50\eta^2 + \frac{\eta^3}{3} \right) \leq 10(1 + \eta^2) + \frac{1}{100} \left( \frac{1}{\kappa^2} - 0.001 \eta \left( 0.01 - \left( x_1^2 + x_2^2 + x_3^2 \right) \right) - 0.006 \eta \right). \]

More, set \( \alpha = 1 \) and \( \beta = 500 > \alpha \kappa \) hence, for all \( t \geq 0 \), it is not difficult to see that
\[ \inf_{x \in \Omega} \left\{ \left( \frac{99}{100} \left( 1 + \exp(-t) \right) \left( 8\eta + 50\eta^2 + \frac{\eta^3}{3} \right) \right) \right\} \]
\[ \geq \frac{162.872}{\beta^2} > m_1 K_1 + m_2 K_2 = 86.0932. \]

Furthermore, it is observed that \( \mu = 0.01 \in \left( \frac{1}{162.872}, \frac{1}{86.0932} \right) \), therefore the problem (15) admits at least three week solutions in according to the Theorem 2.2.

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