Horizon mechanics and asymptotic symmetries with an Immirzi-like parameter in 2 + 1 dimensions

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Abstract
Starting with a generalized theory of 2 + 1 gravity containing an Immirzi-like parameter, we derive the modified laws of black hole mechanics using the formalism of weak isolated horizons. Definitions of the horizon mass and angular momentum emerge naturally in this framework. We further go on to analyse the asymptotic symmetries, as first discussed by Brown and Henneaux, and analyse their implications in a completely covariant phase space framework.

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1. Introduction
Gravity in 2 + 1 dimensions \cite{1, 2} has been an active research arena primarily because it is an example of an exactly solvable quantum system \cite{3}. The aim of this program also lies in gaining knowledge about quantum gravity phenomena for the difficult problem in 3 + 1 dimensions (reference \cite{4} has detailed discussions). Although the theory is fairly understood in a certain topological setup \cite{5, 6} (i.e. when the spatial foliations are compact Riemann surfaces), not everything is trivial when there is an inner boundary. Perhaps the most interesting, and hence most studied of these theories of 2 + 1 gravity, is the one with a negative cosmological constant which has been shown to admit the BTZ black hole as an excited state and the AdS\textsubscript{3} solution as its vacuum \cite{7}. Although there are a more general class of black holes in 2 + 1 topological gravity theories \cite{8}, of which BTZ is a special one. In the first-order formalism, this theory is equivalent to SO(2, 1) × SO(2, 1) Chern–Simons gauge theory. Chern–Simons is again a purely topological theory. The local Lorentz transformation and the diffeomorphisms are only local excitations, which being gauge, leave the theory devoid of any local physics. Global or topological degrees of freedom are the only ones to look for while constructing the physical dynamics of gravity in 2 + 1 dimensions. This is in contrast to the much studied topologically massive gravity (TMG) theory \cite{9} (or its recently understood ramifications \cite{10, 11}) in which one introduces local degrees of freedom, a parity violating massive graviton. It contains the
usual Einstein–Hilbert term, the gravitational Chern–Simons and a cosmological constant. It has also been suggested that a three-dimensional gravity can always be transformed to TMG through field redefinition and a consistent truncation [12]. The theory of TMG has some peculiarities—the massive excitations carry negative energy for a positive coupling constant (in this case, it is $G$) [9]. In the case of negative cosmological constant TMG, the situation is drastic. The change in the sign of the coupling constant gives excitations with positive energy but gives negative mass BTZ black hole solutions [13].

Black holes in such theories (see [13–16] and references therein) and their entropy have been studied in great detail [17, 18] and in [19] exhaustively for a large class of interactions governed by Chen–Simons theory. In three dimensions, black hole entropy calculations are majorly based upon two different routes. The most popular is the one which follows [20]; this again is based on the results of the seminal paper by Brown and Henneaux [21]. They showed that asymptotic symmetries of a solution of $2+1$ general relativity with negative cosmological constant (not necessarily the BTZ solution) are given by a pair of Witt algebras—the deformation algebra of $S^1$ instead of the expected isometry $SO(2, 2)$ of $AdS_3$. Canonical phase-space realization of these asymptotic symmetries however is given by a pair of Virasoro algebras, which are the centrally extended version of the symmetry algebra. A simple use of the Cardy formula for the central extensions gives the entropy (see [22] for discussions). On the other hand, there is another path (e.g. [17, 19]) which uses a covariant phase-space framework, following Wald [43]. Unfortunately, this approach heavily relies on the (bifurcate) Killing horizon structures, which have their own problems including restriction to non-extremal horizons only. Dynamical issues, conserved charges in similar class of theories, including the canonical realization of asymptotic symmetries, have been studied in [18, 24–29]. Entropy of the BTZ black hole in these modified topological theories and the TMG were also presented in these papers. Contrary to the Bekenstein–Hawking expectation, the entropy turns out not only to be proportional to the black hole area, but also to some extra terms, involving even the horizon angular momentum [18, 29]. In this paper, we will investigate related issues for a general class of theories in a covariant manner and show that such results are expected. Moreover, we will establish that our method is equally applicable to extremal and non-extremal black holes since it does not rely on the existence of bifurcation spheres.

Generalized versions of $2+1$ topological gravity which retains its topological nature came into prominence through [30]. In this work, we consider a special case of such generalized theory [31, 32]. More precisely, we will work with a theory having a negative cosmological constant and a parameter which imitates the Barbero–Immirzi parameter of $3+1$ gravity [33]. The possibility of this generalization was also hinted in the pioneering work [3]. The gauge group still remains $SO(2, 1) \times SO(2, 1)$. This theory is interestingly linked to TMG in a subtle way, as discussed in [6]. When one forces the torsion ($T^i$) to be zero, i.e. goes to the partial solution space of $T^i = 0$, one lands on TMG. The limit $\gamma \to 1$, of the new parameter (which behaves as a chiral parameter in the TMG literature), has interesting consequences both in the purely topological and the massive theories [6, 28, 34–37].

Our aim will be to establish the laws of black hole mechanics in this theory and to determine entropy in a fully covariant framework, and in doing so we will show how our approach fills up the gaps in the existing literature. In this respect, it becomes important to introduce conserved charges like the angular momentum and mass. We will use the formalism of isolated horizons to address these issues. The setup of isolated horizons is robust and conceptually straightforward, resulting in surprising simplicity in calculations. The details of isolated horizon formalism were given in a series of papers [38–41]. The isolated horizon formalism for general relativity in $2+1$ dimensions was developed in [42]. We will however
use a weaker set of boundary conditions than [42], extend to more general theories, study the asymptotic symmetries and eventually determine the entropy of horizons.

The basic idea is as follows. A horizon (black hole or cosmological) is a null hypersurface which can be described locally by providing the geometric description of that surface only. The black hole horizon (we are interested in such horizons here) is described in this formalism to be an internal boundary of spacetime which is expansion free and on which the field equations hold\(^3\). Unlike the Killing horizons/event horizons, we need not look outside/asymptotic or the spacetime in the vicinity of horizon to define isolated horizons; only horizon properties are enough. It is because of this generality that the isolated horizon is useful to describe even solutions where the asymptopia is still not well defined or has not developed yet. More precisely, all event horizons/Killing horizons are isolated horizons but not all isolated horizons are Killing/event horizons. As happens (and we will show this below), the boundary conditions enable us to prove the zeroth law of black hole mechanics directly. The first law of black hole mechanics and construction of conserved charges is not difficult in this formalism. Given a field theory, there exists a straightforward way which enables us to covariantly construct the space of solutions (and a symplectic structure on the covariant phase space), initially introduced in [47]. This has been applied successfully to study dynamics of spacetimes with an isolated horizon as an internal boundary. The conserved charges (like angular momentum) are precisely the *Hamiltonian functions* corresponding to the vector field generating canonical transformations or the so-called *Hamiltonian vector fields* (which in this case is related to the rotational Killing vector field on the horizon). The first law is, in this description, the necessary and sufficient condition for the null generator of the horizon to be a Hamiltonian vector field. These features, as we will show below, can be established very easily.

There is a precise definition of the asymptotic symmetry group, if we know the fall of behaviour of the geometry asymptotically. A natural question to ask is whether the action of this group on the pre-symplectic manifold, defined through the degenerate symplectic structure, is a Hamiltonian. As have been the expectation through canonical analyses made earlier, the answer is not in the affirmative; rather the algebra of symmetry generators gets centrally extended. This leads to finding the black hole entropy as discussed earlier (just as for the TMG case). This is similar to TMG where the parameters are the topological mass and the cosmological constant. (For TMG, this implies that the massive graviton introduced through the extra couplings has no effect on the entropy.)

The plan of the paper is as follows. In section 2, we will recall the definition of weak isolated horizons (WIH) and prove the zeroth law of black hole mechanics [45, 46]. The proof of the zeroth law is purely kinematical and does not require any dynamical information. In section 3.1, we will first discuss the generalized theories in 2+1 dimensions and then introduce the theory with the $\gamma$-parameter and negative cosmological constant. This section will also include a discussion of the BTZ solution as an example of a black hole solution in this theory. Since we will be interested in manifolds with inner and outer boundaries, we need to establish that the variational principle is well defined. In sections 3.2 and 3.3, we will establish that indeed the action principle is well defined even when the inner boundary is a WIH. In section 3.4, we construct the space of solutions and symplectic structure. The phase space contains all solutions (extremal as well as non-extremal black hole solutions), which satisfy the boundary conditions of WIH for the inner boundary and are asymptotically AdS at infinity. In section 3.6, we will show how the angular momentum can be extracted from the symplectic structure.

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3 This construction is more general than that of the Killing horizon. Laws of black hole mechanics were proved for this quasi-local definition too [43, 44]. But, as mentioned earlier, this formalism does not seem useful to address extremal horizons. Improvements by introducing extremal horizons in the same space of the non-extremal horizons in the isolated horizons framework were made in [45, 46].
The angular momentum will naturally arise as a Hamiltonian function (on the phase space) corresponding to the Hamiltonian vector field associated with the rotational Killing vector field on the spacetime. When the definition is applied to the BTZ solution, it will naturally arise that the angular momentum depends on the parameters \( J \) and \( M \) of the solution. In section 4, we will construct the vector fields which generate diffeomorphisms preserving the asymptotic conditions. We will construct Hamiltonian functions corresponding to these vector fields and show that in the presence of a WIH inner boundary, the Hamiltonian charges do not realize the algebra of vector fields. The difference is a central extension which gives rise to the entropy for black holes in these theories. We will also observe that the parameter \( \gamma \) shows up in all stages. We will discuss these issues in section 5.

2. Weak isolated horizon: kinematics

We now give a very brief introduction to WIH [45]. Let \( M \) be a 3-manifold equipped with a metric \( g_{ab} \) of signature \((-\,\,\,\,+\,\,\,+\,\,)\). Consider a null hypersurface \( \Delta \) in \( M \) of which \( \ell^a \) is a future directed null normal. However, if \( \ell^a \) is a future directed null normal, so is \( \xi \ell^a \), where \( \xi \) is any arbitrary positive function on \( \Delta \). Thus, \( \Delta \) naturally admits an equivalence class of null normals \([\xi \ell^a]\). The hypersurface \( \Delta \) being null, the metric induced on it by the spacetime metric \( g_{ab} \) will be degenerate. We will denote this degenerate metric by \( q_{ab} \equiv g_{ab} \) (since we are using abstract indices, we will distinguish intrinsic indices on \( q_{ab} \) by pullback and \( \equiv \) will mean that the equality holds only on \( \Delta \)). The inverse of \( q_{ab} \) will be denoted by \( q^{ab} \) such that \( q^{ab} q_{ab} \equiv q_{ab} q^{bd} \equiv q_{bd} \). The expansion \( \theta(\ell) \) of the null normal \( \ell^a \) is then defined by \( \theta(\ell) = q^{ab} \nabla_a \ell_b \), where \( \nabla_a \) is the covariant derivative compatible with \( g_{ab} \). Null surfaces are naturally equipped with many nice properties. Firstly, the null normal is hypersurface orthogonal and hence is twist free. Secondly, the \( \ell^a \) is also tangent to the surface. It is tangent to the geodesics generating \( \Delta \). Thus, any \( \ell^a \) in the class \([\xi \ell^a]\) satisfies the geodesic equation

\[
\ell^a \nabla_a \ell^b \equiv \kappa(\ell) \ell^b.
\]

We will interpret the acceleration \( \kappa(\ell) \) as the surface gravity. If the null normal to \( \Delta \) is such that \( \kappa \) vanishes, we will call it the extremal surface. Otherwise, the surface will be called non-extremal. The variation of \( \kappa \) in the null class \([\xi \ell]\) is \( \kappa(\ell) = \xi \kappa(\ell) + \xi \ell \).\footnote{Without the positive function \( \xi \), kappa cannot be set to zero unless the null hypersurface is part of a neighbourhood of a family of null hypersurfaces. This follows from the fact that not only the hypersurface normal is null but also that its derivative off the surface vanishes (see section 3.1 of [23]).}

In what follows, we will use the Newmann–Penrose (NP) basis for our calculations. In three dimensions, this will consist of two null vectors \( \ell^a \) and \( n^a \) and one space-like vector \( m^a \). They satisfy the condition \( \ell . n = -1 = -m . m \) while other scalar products vanish. This basis is particularly useful for our setup because the normal to \( \Delta \), denoted by \( \ell^a \), can be chosen to be the \( \ell^a \) of the NP basis. The space-like \( m^a \) will be taken to be tangent to \( \Delta \). In this basis, the spacetime metric will be given by \( g_{ab} = -2 \ell_a n_b + 2 m_a m_b \) whereas the pullback metric \( q_{ab} \) will simply be \( q_{ab} \equiv m_a m_b \).

2.1. Weak isolated horizon and the zeroth law

The null surface \( \Delta \) introduced above is an arbitrary null surface equipped with an equivalence class of null normals \([\xi \ell^a]\). The conditions on \( \Delta \) are too general to make it resemble a black hole horizon. To enrich \( \Delta \) with useful and interesting information, we need to impose some additional structures (the imposed conditions will be weaker than that in [42] in the sense that our equivalence class of null normals will be related by functions on \( \Delta \) rather than constants).
As we will see, the zeroth law and the first law of black hole mechanics will naturally follow from these conditions. These definitions will be local and only provide a construction of the black hole horizon and do not define a black hole spacetime which is a global object. However, if there is a global solution, like the BTZ one, then these conditions will be satisfied.

The null surface $\Delta$, equipped with an equivalence class of null normals $[\xi \ell^a]$, will be called a WIH if the following conditions hold.

(1) $\Delta$ is topologically $S^1 \times \mathbb{R}$.

(2) The expansion $\theta(\xi \ell) \equiv 0$ for any $\xi \ell^a$ in the equivalence class.

(3) The equations of motion and energy conditions hold on the surface $\Delta$ and the vector field $-\mathcal{T}_a \xi^b \equiv 0$ is future directed and causal.

(4) There exists a 1-form $\omega(\xi \ell)$ such that it is Lie-dragged along the horizon $\Delta$, $\mathcal{L}_{\ell} \omega(\xi \ell) \equiv 0$.

In the literature, $\Delta$ is called a non-expanding horizon (NEH) if it satisfies only the first three conditions. It is clear that the boundary conditions for a NEH hold true for the entire class of null normals $[\xi \ell^a]$ if it is valid for one null normal in that class. The Raychaudhuri equation implies that NEHs are also shear free. Thus, NEHs are twist free, expansion free and shear free, and this implies that the covariant derivative of $\ell^a$ on $\Delta$ is much simple. There exists a 1-form $\omega(\ell)$ (see the appendix for a NP-type discussion), such that $\nabla_a \left( \ell^a \right) \equiv \omega(\ell)$. The 1-form $\omega(\ell)$ varies in the equivalence class $[\xi \ell^a]$ as

$$\omega(\xi \ell) \equiv \omega(\ell) + d \ln \xi.$$  

A few other conclusions also follow. Firstly, from equations (2.1) and (2.3), it follows that $\kappa(\xi \ell) \equiv \xi \ell \cdot \omega(\xi \ell)$. Secondly, the null normals in the equivalence class are Killing vectors on the NEH $\mathcal{L}_{\ell} q_{ab} \equiv 2 \nabla_a \omega_b \equiv 0$. Thirdly, the volume form on $\Delta$ is Lie-dragged by the null normal which is in the equivalent class, $\mathcal{L}_{\ell} m \equiv 0$.

At this point, one should note that the acceleration $\kappa(\xi \ell)$ is in general a function on $\Delta$. If we want the NEH to obey the zeroth law of black hole mechanics, which requires constancy of the acceleration of the null normal on $\Delta$, we should restrict it further. This is done by demanding the fourth condition in the list of boundary conditions, equation (2.2). Although this is not a single condition (i.e. unlike the other three conditions, it is not guaranteed that if this condition holds for a single vector field $\ell^a$, it will also hold for all the others in the class $[\xi \ell^a]$ for any arbitrary $\xi$), one can always choose a class of functions $\xi$ on $\Delta$ [45, 46], for which this reduces to a single condition. For example, if the class of function is $\xi = F \exp(-\kappa(\ell) v) + \kappa(\ell) / \kappa(\ell)$, where $\ell^a = (\partial / \partial v)^a$ and $F$ is a function such that $\mathcal{L}_{\ell} F \equiv 0$, the condition holds for the entire equivalence class. Also note from (2.4) that $d\omega(\xi \ell)$, which is proportional to the Weyl tensor, is independent of the variation of $\xi$. Since the Weyl tensor vanishes identically in three dimensions, we have $d\omega(\xi \ell) \equiv 0$. Equation (2.2) then gives the zeroth law: $d\kappa(\xi \ell) \equiv 0$.

3. Weak isolated horizon: dynamics

In this section, we introduce the action for our theory and derive the laws of black hole mechanics. We use the first-order connection formulation. This formulation is tailor-made for

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5 This is a virtue in disguise in the sense that we can interpolate between extremal horizons, with $\kappa \equiv 0$, and non-extremal horizons, with $\kappa \neq 0$, using this $\xi$. In other words, we can use this formalism to accommodate extremal as non-extremal horizons in the same phase space.
our setup and the calculational simplicity will be enormous. In particular, the construction of the
covariant phase space and its associated symplectic structure is a straightforward application of
the notions used in higher dimensions [45, 46]. The use of forms also simplifies the calculation
of the first law and the conserved charges.

Our 3-manifold $\mathcal{M}$ will be taken to be topologically $M \times \mathbb{R}$ with boundaries. The inner
null boundary will be denoted by $\Delta$ which is taken to be topologically $S^1 \times \mathbb{R}$. The initial and
final space-like boundaries are denoted by $M^-$ and $M^+$, respectively. The boundary at infinity
will be denoted by $i_0$. In what follows, the inner boundary will be taken to be a WIH. In
particular, this implies that the surface $\Delta$ is equipped with an equivalent class of null normals
$[\xi^a]$ and follows equation (2.2).

3.1. The action in 2 + 1 dimensions

The action describing 2 + 1 gravity with the negative cosmological constant $\Lambda = -\frac{1}{l^2}$ in the
first-order formalism is (in our convention of $16\pi G = 1 = c$)

$$I = \int_{\mathcal{M}} \epsilon^I \wedge \left( 2 dA_I + \epsilon_JK A_J A_K \right) + \frac{1}{4l^2} \epsilon_IJK e_I \wedge e_J \wedge e_K, \quad (3.1)$$

where $\epsilon^I$ are the $SO(2, 1)$ orthonormal triad frame and the $A_I$ are connections (or canonically
projected local connection) of the frame bundle with the structure group $SO(2, 1)$. The above
action is well defined and differentiable in the absence of boundaries. However, as we will
show in the next subsection, with the present boundary conditions (at internal and asymptotic)
at hand, the variational problem is well defined with this action itself, without any boundary
term.

The equations of motion are expected first-order versions of the Einstein equation with
cosmological constant, $F_I$:

$$F_I := 2 dA_I + \epsilon_IJK A_J A_K = -\frac{1}{l^2} \epsilon_IJK e_I \wedge e_K \quad (3.2)$$

$$T_I := de_I + \epsilon_IJK e_I \wedge A^K = 0. \quad (3.3)$$

More general models for 2 + 1 gravity with negative cosmological constant were introduced
in [30] and later studied extensively in [27, 28, 31], which without matter fields read

$$I = a_1 I_1 + b_2 I_2 + \alpha_3 I_3 + \alpha_4 I_4, \quad (3.4)$$

where

$$I_1 = \int_{\mathcal{M}} \epsilon^I \wedge \left( 2 dA_I + \epsilon_IJK A_J A_K \right)$$

$$I_2 = \int_{\mathcal{M}} \epsilon_IJK e_I \wedge e_J \wedge e_K$$

$$I_3 = \int_{\mathcal{M}} A^I \wedge dA_I + \frac{1}{3} \epsilon_IJK A^I A^J \wedge A^K$$

$$I_4 = \int_{\mathcal{M}} \epsilon^I \wedge de_I + \epsilon_IJK A^I \wedge e^J \wedge e^K.$$

For more references on various applications of this model and its relevance with TMG and
chiral TMG, see [6] and references therein. However, one must note that this model does
not reproduce the Einstein equations (3.2) and (3.3) for arbitrary values of the parameters

6 Strictly speaking, one should add an asymptotic boundary term to this action, which may render the whole action
finite [38]. But since this has nothing to do with dynamics, i.e. does not affect the variation procedure, we omit it.
a, b, α₃ and α₄. We choose, as a special case of the above model, those values of these parameters which give the expected equations of motions

\[
a = 1 \quad b = \frac{1}{3l^2} \quad \alpha_3 = \frac{I}{\gamma} \quad \alpha_4 = \frac{1}{\gamma l},
\]

(3.5)

where \( \gamma \) is introduced as a new dimensionless parameter from 2+1 gravity perspective. Effectively (3.5) is the equation of a three-dimensional hypersurface parametrized by \( G, l \) and \( \gamma \) in the 4D parameter space of \( a, b, \alpha_3 \) and \( \alpha_4 \).

In [3], it was established that first-order 2+1 gravity can be written as Chern–Simons gauge theory, the gauge group being determined by the sign of the cosmological constant. It is now explicit that 2+1 gravity does not have any local degrees of freedom, since Chern–Simons theory is a topological one. Moreover, as a unique feature of three dimensions, all the invariances of first-order gravity, i.e. the local Lorentz (SO(2, 1)) transformations and arbitrarily large number of diffeomorphisms, are now taken care of by the finite-dimensional Chern–Simons gauge group, when viewed on-shell. Following [3, 31], one can introduce the SO(2, 1) or equivalently \( SL(2, \mathbb{R}) \) or \( SU(1, 1) \) connections for a principal bundle over the same base space of the frame bundle,

\[
A^{(±)} := \left( A' \pm \frac{\epsilon}{T} \right) J^{(±)}_I.
\]

Now, it also happens that \( J^{(±)}_I \) form two decoupled SO(2, 1) Lie algebras:

\[
[J^{(+)}, J^{(−)}] = \epsilon_{IKJ} J^{(+)} K \quad \left[ J^{(−)}, J^{(−)} \right] = \epsilon_{IKJ} J^{(−) K} \quad \left[ J^{(+)}, J^{(−)} \right] = 0.
\]

(3.6)

The metric on the Lie algebra is

\[
\langle J^{(+)I}, J^{(−)J} \rangle = \frac{1}{2} \eta^{IJ}.
\]

It is easily verifiable that the action

\[
I = l \left( I^{(+)} - I^{(−)} \right)
\]

(3.7)

is the same as (3.1) up to boundary terms which are guaranteed to vanish in our case. Here

\[
I^{(±)} = \int_M \text{tr} \left( A^{(±)} \wedge dA^{(±)} + \frac{2}{3} A^{(±)} \wedge A^{(±)} \wedge A^{(±)} \right).
\]

(3.8)

One striking feature of this formulation is that the last two terms of (3.4) can also be incorporated in terms of \( A^{(±)} \), for \( (\alpha_3 = l^2 \alpha_4) \), as

\[
I^{(±)} = \int_M \left( A' \wedge dA' + \frac{1}{2} \epsilon' \wedge d\epsilon' + \frac{1}{3} \epsilon_{IKJ} A' \wedge A' \wedge A' + \frac{1}{2} \epsilon_{IKJ} A' \wedge \epsilon' \wedge \epsilon' K \right)
\]

and the same equations of motion (3.2) and (3.3) are also found from varying this action. In this paper, we work with the following action:

\[
I = l \left( I^{(+)} - I^{(−)} \right) + \frac{l}{\gamma} \left( I^{(+) +} + I^{(−)} \right)
\]

\[
= l \left( 1/\gamma + 1 \right) I^{(+)} + \left( 1/\gamma - 1 \right) I^{(−)}
\]

(3.9)

with a dimensionless non-zero coupling \( \gamma \). This action (3.9) upon variations with respect to \( A^{(±)} \) and \( A^{(±)} \) gives equations of motion as expected from Chern–Simons theories. This implies that the connections \( A^{(±)} \) are flat:

\[
J^{(±)}_I := dA^{(±)}_I + \epsilon_{IKJ} A^{(±)} J \wedge A^{(±)} K = 0.
\]

(3.10)

It is also easy to check that the above flatness conditions of these SO(2, 1) bundles (3.10) are equivalent to the equations of motion of general relativity (3.2) and (3.3). Note that the new action is like the Holst action [33] used in 3+1 gravity. In our case, the parameter \( \gamma \)
can be thought of being the \((2 + 1)\)-dimensional counterpart of the original Barbero–Immirzi parameter. Moreover, the part \([V^{(+)} + I^{(-)}]\) of the action in this light qualifies to be at par with the topological (non-dynamical) term one adds with the usual Hilbert–Palatini action in \(3 + 1\) dimensions, since this term we added (being equal to a Chern–Simons action for spacetimes we consider) is also non-dynamical. But more importantly the contrast is in the fact that the original action, which is dynamical in the \(3 + 1\) case, is also non-dynamical here, when one considers local degrees of freedom only. However, there is a difference between the original B-I parameter and the present one. In the \(3 + 1\) dimensions, \(\gamma\) parameterizes canonical transformations in the phase space of general relativity. From the canonical pair of the \(SU(2)\) triad (time gauge fixed and on a spatial slice) and spin connection one goes on finding an infinitely large set of pairs parameterized by \(\gamma\). The connection is actually affected by this canonical transformation, and this whole set of parameterized connections is popularly known as the Barbero–Immirzi connection. The fact that this parameter induces canonical transformation can be checked by seeing that the symplectic structure remains invariant under the transformation on-shell. On the other hand for the case at hand, i.e. \(2 + 1\) gravity, as we will see in the following subsection that inclusion of finite \(\gamma\) is not a canonical transformation.

Just like in \(2 + 1\) gravity with a negative cosmological constant, the two-parameter family of BTZ black holes is a solution of this theory. In the standard coordinates, the solution is given by

\[
ds^2 = -N^2 \, dt^2 + N^{-2} \, dr^2 + r^2 (N^\theta \, dt + d\phi)^2,
\]

where the lapse and the shift variables contain the two parameters \(M\) and \(J\) and are defined by

\[
N^2 = \left( -\frac{M}{\pi} + \frac{r^2}{\ell^2} + \frac{J^2}{4\pi^2 \ell^2} \right) \quad \text{and} \quad N^\theta = -\frac{J}{2\pi \ell^2}.
\]

The horizon is defined through the zeros of the lapse function \(N^\theta\) which gives the position of the horizon to be

\[
r_{+} = l \left[ \frac{M}{2\pi} \left\{ 1 \mp (1 - (J/Ml)^2)^{1/2} \right\} \right]^{1/2}.
\]

It is not difficult to see that the outer horizon (at \(r_{+}\)) satisfies the conditions of WIH \(\Delta\). It is a null surface with null normal \(\ell^a = (\partial/\partial v)^a + N^\theta (r_{+}) (\partial/\partial \phi)^a\). A simple calculation also shows that \(\theta(\ell^a) \triangleq 0\). In what follows, we will always refer back to this solution to check if our definitions for conserved charges are consistent.

### 3.2. Computing tetrads and connection on \(\Delta\)

Before proceeding with the variation of the action and determining the equations of motion, it will be useful to have the values of the tetrad and connection on the null surface \(\Delta\). The usefulness of such calculation will be apparent soon. We will assume that it is possible to fix an internal null triad \((\ell^a, n^a, m^a)\) such that \(\ell^b n_b = -1 = -m^a m_a\) and all others zero. The internal indices will be raised and lowered with \(\eta_{ij}\). Given the internal triad basis \((\ell^a, n^a, m^a)\) and \(e^a_{\mu}\), the spacetime null basis \((\ell^\mu, n^\mu, m^\mu)\) can be constructed. We will further assume that the internal basis is annihilated by the partial derivative operator, \(\partial_a (\ell^\mu, n^\mu, m^\mu) = 0\).

Using the expression of the spacetime metric in the NP basis and the internal metric, we can write the tetrad \(e^a_{\mu}\) on WIH \(\Delta\) as

\[
e^a_{\mu} \triangleq -n_a \ell^\mu + m_a m^\mu.
\]

To calculate the expression of connection on \(\Delta\), we will use the NP coefficients which can be seen in the covariant derivatives of the NP basis. They are as follows:

\[
\nabla_{\mu} e^b \triangleq \omega_{a}^{(\ell)} \ell^b
\]
\[ \nabla_a n_b \triangleq -\omega_a^{(\ell)} n_b + U_a^{(\ell, m)} m_b \]  
\[ \nabla_a m^b \triangleq U_a^{(\ell, m)} \ell^b, \]  
where the superscripts on the 1-forms \( \omega_a^{(\ell)} \) and \( U_a^{(\ell, m)} \) indicate that they depend on the transformations of the corresponding basis vectors. The 1-forms used in equation (3.15) are the compact expressions of the NP coefficients. They are given by
\[ \omega_a^{(\ell)} \triangleq (-\epsilon n_a + \alpha m_a) \]  
\[ U_a^{(\ell, m)} \triangleq (-\pi n_a + \mu m_a). \]

We will now demonstrate how the NP coefficient \( \alpha \) is fixed to be a real number on \( \Delta_1 \) using topological arguments. Note from the previous discussion that \( d\omega_a^{(\ell)} \triangleq 0 \). From definition (A.9), we have \( dm^b \triangleq -\rho m \wedge n \). But because \( \Delta \) is expansion free and \( \ell^a \) is the generator of \( \Delta, \rho \triangleq 0 \). Hence, \( m^a \) is also closed on \( \Delta_1 \) \((\text{m should not strictly be exact since } \int_{\Delta_1} m \sim \text{area of horizon} \neq 0)\). Since the first cohomology group of \( \Delta \simeq \mathbb{R} \times S^1 \equiv \mathbb{R} \) is non-trivial, we have in general neither \( \omega_a^{(\ell)} \) nor \( m_a \) exact. Hence, there exist a smooth function \( \varsigma \) and a real number \( s \) for which
\[ \omega_a^{(\ell)} \triangleq d \varsigma + s m. \]

We now introduce a potential \( \psi_a^{(\ell)} \) for surface gravity (or the acceleration for \( \ell^a \)) \( \kappa_a^{(\ell)} \triangleq \ell_a \omega_a^{(\ell)} \triangleq \epsilon \) through
\[ \ell \psi_a^{(\ell)} \triangleq \kappa_a^{(\ell)}. \]

Since the zeroth law implies constancy of \( \kappa_a^{(\ell)} \) on \( \Delta, \psi_a^{(\ell)} \) can only be the function of \( v \) (could be treated as the affine parameter on \( \Delta \) ) only. Hence \( \ell m \psi_a^{(\ell)} \triangleq 0 \), which implies on the other hand \( d \psi_a^{(\ell)} \triangleq -\epsilon n \) and \( \omega \triangleq d\psi_a^{(\ell)} + \alpha m \). It is tempting to choose \( \varsigma = \psi_a^{(\ell)} \) by comparing with (3.20). That could only be supported if \( \Delta \) is axisymmetric. (Because even after choosing a triad set for which \( d n \triangleright 0 \), we end up with \( d\alpha \wedge m \triangleright 0 \), which renders \( d\alpha \triangleright 0 \) only if \( \alpha \) is axisymmetric.) For that case, we conclude \( \omega_a^{(\ell)} \triangleq (d\psi_a^{(\ell)} + \alpha m), \alpha \in \mathbb{R} \).

Now, to calculate the connection, we use two facts. First is that the tetrad is annihilated by the covariant derivative \( \nabla_a \ell_b^{\ell} = 0 \) and second that partial derivative annihilates the NP internal basis so that
\[ \nabla_a \ell^b \triangleq A_b^{\ell} \ell_j. \]

Using equations (3.18) and (3.21) and \( \ell_{JK} = 3! \ell_{[IJ} m_{K]} \), we obtain the following expression for pulled-back connection on \( \Delta \):
\[ A_b^{\ell} \triangleq -U_a^{(\ell, m)} \ell^b + \omega_a^{(\ell)} m^b. \]

Equation (3.22) will be used frequently in what follows.

### 3.3. Variation of the action

In subsection 3.1, we demanded that we get the first-order Einstein equations of motion even by varying the generalized action. For spacetime manifolds without boundary, this is trivial to check. The task is now to vary the action to obtain the equations of motion and also to verify that the action principle is obeyed in the presence of the boundaries. The variation will be over configurations which satisfy some conditions at infinity and at the inner boundary (see figure 1). At infinity, they satisfy some asymptotic conditions which are collected in the appendix of [42]. On the inner boundary \( \Delta \), they are subjected to the following conditions:
(a) the tetrads (e) are such that the vector field $\ell' = e_I^a \ell^a$ belongs to the equivalence class $[\xi \ell']$ and (b) $\Delta$ is a WIH. On variation, we will obtain equations of motion and some surface terms. The surface terms at infinity vanish because of the asymptotic conditions whereas, as we will show, those at WIH also vanish because of WIH boundary conditions.

The variation of the action with respect to the tetrad (e) and connection (A) leads to (for $\gamma^2 \neq 1$)

\[
\delta I = 0
\]

and

\[
\delta A_I = \frac{1}{2} \epsilon_{IJK} A_J \wedge A_K = - \frac{1}{2l^2} \epsilon_{IJK} e_I \wedge e_J \wedge e_K.
\] (3.24)

The first equation above just points out that the connection $A_I$ is a spin connection and the second equation is the Einstein equation. Let us now concentrate on the surface terms. The terms on the initial and final hypersurfaces $M_-$ and $M_+$ vanish because of the action principle. Those at the asymptotic boundary vanish because of the fall-offs at infinity. On $\Delta$, these are given by

\[
\delta I = - \int_{\Delta} (2m \wedge \delta \omega^{(i)} + \frac{1}{\gamma} \omega^{(i)} \wedge \delta \omega^{(i)} + \frac{1}{l^2} m \wedge \delta m).
\] (3.25)

Our strategy will be to show that the integral is constant on $\Delta$ and the integrand is a total derivative so that the integral goes on to the initial and the final boundaries where the variations are zero by assumption. This will then imply that the integral itself vanishes on $\Delta$. Note that in the above equation, $\delta \omega^{(i)}$ refers to the variation in $\omega^{(i)}$ among the configurations in the equivalence class $[\xi \ell']$. The relation between these is precisely given by equation (2.4). Now, we consider the Lie derivative of the integrands by $\xi \ell$. Since $d\omega^{(i)} \equiv 0$, it follows that $\xi \ell \omega^{(i)} \equiv 0$ and $\xi \ell \omega^{(i)} \equiv d(\xi \ell \ln \xi)$. Thus, in the first term, the total contribution is on the initial and final hypersurfaces $M_-$ and $M_+$ where the variations vanish. Identical arguments for the second and the third integrands also show that the corresponding integral vanishes. Thus, the integral is Lie-dragged on $\Delta$ and since the variations are fixed on the initial and final hypersurfaces, the entire integral vanishes and the action principle remains well defined.

3.4. Covariant phase space

An analysis of the dynamics of this theory has been considerably worked out in the literature [27] in the canonical framework even in the presence of the asymptotic boundary. A covariant
phase space [47] analysis for the same theory was presented in [6] although in the absence of boundaries. As we progress, we will see how apt the covariant analysis is in understanding horizon phenomena and even the conserved charges arising from asymptotic symmetries, using the general ideas of symplectic geometry. The covariant phase space is by definition the space of classical solutions (3.23) which satisfy the boundary conditions specified in the previous subsections. In other words, the covariant phase space $\Gamma$ will consist of solutions of the field equations which satisfy the boundary conditions of WIH at $\Delta$ and have fall-off conditions compatible with asymptotic conditions. In order to equip this space with a symplectic structure,$^7$ we find the symplectic potential from variation of the Lagrangian:

$$\delta L = d\Theta(\delta) + \text{terms vanishing on-shell.} \quad (3.26)$$

For the Lagrangian at hand given by (3.9), the symplectic potential is given by

$$\Theta(\delta) = -2 \left( e^I \wedge \delta A_I \right) - \frac{l}{\gamma} \left( A^I \wedge \delta A_I \right) - \frac{1}{l\gamma} \left( e^I \wedge \delta e_I \right). \quad (3.27)$$

Upon antisymmetrized second variation, it gives the symplectic current $J$ which is a phase-space 2-form. For two arbitrary vector fields $\delta_1$ and $\delta_2$ tangent to the space of solutions, the symplectic current for (3.27) is given by the following closed 2-form:

$$J(\delta_1, \delta_2) = \delta_1 \Theta(\delta_2) - \delta_2 \Theta(\delta_1) \quad = - 2 \left[ (\delta_1 e^I \wedge \delta_2 A_I - \delta_2 e^I \wedge \delta_1 A_I) + \frac{l}{\gamma} \delta_1 A^I \wedge \delta_2 A_I + \frac{1}{l\gamma} \delta_1 e^I \wedge \delta_2 e_I \right]. \quad (3.28)$$

Since the symplectic current is closed, $dJ(\delta_1, \delta_2) = 0$, we define the pre-symplectic structure on the phase space by

$$\Omega(\delta_1, \delta_2) = \int_{M_1 \cup M_2 \cup \partial M_1} J(\delta_1, \delta_2), \quad (3.29)$$

where the terms under the integral show contributions from the various boundaries (refer to figure 1). The surfaces $M_1$ and $M_2$ are partial Cauchy slices inside the spacetime which meet $\Delta$ in $S_1$ and $S_2$, respectively. To show that the symplectic structure is independent of the choice of the Cauchy surface, we again consider the function $\psi(\ell)$ such that $L_\ell\psi(\ell) = \kappa(\ell)$ and $\psi(\ell)$ vanishes on $S^1$ (where the affine parameter $\gamma = 0$). Choosing an orientation, it is not difficult to show that $J(\delta_1, \delta_2) \equiv dj(\delta_1, \delta_2)$ so that

$$\int_{S_1} - \int_{S_2} \right] J(\delta_1, \delta_2) = \left( \int_{S_1} - \int_{S_2} \right) J(\delta_1, \delta_2), \quad (3.30)$$

which establishes the independence of the symplectic structure on the choice of Cauchy surfaces. The pre-symplectic structure on the space of solutions of the theory in the presence of $\Delta$ turns out to be

$$\Omega(\delta_1, \delta_2) = - 2 \int_{M} \left[ (\delta_1 e^I \wedge \delta_2 A_I - \delta_2 e^I \wedge \delta_1 A_I) + \frac{l}{\gamma} \delta_1 A^I \wedge \delta_2 A_I + \frac{1}{l\gamma} \delta_1 e^I \wedge \delta_2 e_I \right]$$

$$- 2 \int_{S^1} \left( \delta_1 \psi_1(\ell) \delta_2 \left[ \left( \frac{l\alpha}{\gamma} + 1 \right) m \right] - \delta_2 \psi_1(\ell) \delta_1 \left[ \left( \frac{l\alpha}{\gamma} + 1 \right) m \right] \right). \quad (3.31)$$

We will use (3.31) to define conserved quantities like the angular momentum and prove the first law in the next two subsections. We will also construct the algebra of conserved charges using this symplectic structure and obtain the entropy for black holes in this theory.

$^7$ To be more precise, here we will be dealing with the pre-symplectic structure, since the theory has gauge redundancy, which appears as ‘degenerate directions’ for the symplectic 2-form approach.
3.5. Angular momentum

We will first introduce the concept of angular momentum starting from the symplectic structure, equation (3.31). Let us consider a fixed vector field $\phi^{\alpha}$ on $\Delta$ and all those spacetimes which will have $\phi^{\alpha}$ as the rotational Killing vector field on $\Delta$. The field $\phi^{\alpha}$ is assumed to satisfy certain properties. Firstly, it should Lie-drag all fields in the equivalence class $[\xi^{\ell\ell}]$ and secondly, it has closed orbits and the affine parameter $\epsilon \in [0, 2\pi)$. To be more precise, we can construct a submanifold $\Gamma_{\phi}$ of the covariant phase space $\Gamma$ the points of which are solutions of field equations which admit a WIH $(\Delta, [\xi^{\ell\ell}], \phi^{\alpha})$ with a rotational Killing vector field $\phi^{\alpha}$ such that $\xi_{\phi}^{\alpha} \equiv 0$, $\xi_{\phi}^{\alpha}(t) \equiv 0$. Now, let us choose a vector field $\phi$ in $\mathcal{M}$ for each point in $\Gamma_{\phi}$ such that it matches with $\phi^{\alpha}$ on $\Delta$.

We will now look for the phase-space realization of diffeomorphisms generated by this vector field $\phi^{\alpha}$ on spacetime. Corresponding to the diffeomorphisms on spacetime, we can associate a motion in the phase space $\Gamma_{\phi}$ which is generated by the vector field $\delta_{\phi} = \xi_{\phi}$. It is expected that the vector field $\delta_{\phi}$ will be Hamiltonian (i.e. generate canonical transformations). In that case, the Hamiltonian charge for the corresponding to the rotational Killing vector field can be called the angular momentum$\textsuperscript{8}$. In short, this implies that $\Omega(\delta, \delta_{\phi}) = \delta J^{(\phi)}$ and the angular momentum $J^{(\phi)}$ is given by

$$J^{(\phi)} = -\oint_{\Gamma_{\phi}} \left[ (\phi \cdot \omega)m + \frac{1}{2\gamma} (\phi \cdot \omega)\omega + \frac{1}{2\gamma} (\phi \cdot m)m \right]$$

$$+ \oint_{\Gamma_{\phi}} \left[ (\phi \cdot \mathcal{A})\mathcal{A} + \frac{1}{2\gamma} (\phi \cdot \mathcal{A})\mathcal{A} + \frac{1}{2\gamma} (\phi \cdot \mathcal{A})\mathcal{A} \right]$$

$$= -J_{\Delta} + J_{\infty}. \quad (3.32)$$

It is then natural to interpret $J_{\Delta}$ to be the angular momentum on $\Delta$. It is simple to check the expressions for $J_{\Delta}$ and $J_{\infty}$ for the BTZ spacetime. It follows that $J_{\Delta} = (J - M1/\gamma) = J_{\infty}$, leaving $J^{(\phi)} = 0$. (Note that for $\gamma \to \infty$, we obtain the value of the angular momentum of the BTZ black hole for general relativity in $2 + 1$ dimensions with a negative cosmological constant.) $J_{\Delta} = J_{\infty}$ is also supported by the fact that $\phi^{\alpha}$ is global Killing vector in the BTZ solution. However, if there are electromagnetic fields, the result differs. The value of the angular momentum at infinity $J_{\infty}$ also gets contribution from the electromagnetic fields and $J^{(\phi)} \neq 0 \textsuperscript{42}$.

3.6. First law

The first law is associated with energy which implies that we should look first for a time-like Killing vector field on spacetime. Let us consider a time-like vector field $t^{\alpha}$ in $\mathcal{M}$ associated with each point of the phase space (live) which gives the asymptotic time translation symmetry at infinity and becomes $t^{\alpha} \equiv \xi^{\alpha} - \Omega_{(\gamma)} \phi^{\alpha} x$ on $\Delta$, where $\Omega_{(\gamma)}$ is a constant on $\Delta$ but may well vary on the space of histories. Just like in the previous subsection, we ask if the associated vector field $\delta_{\gamma}$ on the phase space $\Gamma_{\phi}$ is a Hamiltonian vector field. The associated function will be related to the energy. In checking so, we have

$$\Omega(\delta, \delta_{\gamma}) = X^{(\gamma)}(\delta),$$

where

$$X^{(\gamma)}(\delta) = -2\kappa_{(\gamma)} \delta \left( 1 + \frac{1}{\gamma} \right) a_{\Delta} - 2\Omega_{(\gamma)} \delta J_{\Delta} + X^{(\infty)}(\delta) \quad (3.33)$$

$\textsuperscript{8}$ Since the theory we started with is background independent (has manifest diffeomorphism invariance in bulk) it is natural to expect that Hamiltonians generated by spacetime diffeomorphisms must consist of boundary terms, if any.
and $\kappa(t)$ actually the surface gravity associated with the vector field $\xi^a$. $X^{(t)}(\delta)$ involves integrals of fields at asymptotic infinity and can be evaluated using asymptotic conditions on the BTZ solution for example. A simple calculation gives

$$X^{(t)}(\delta) = \delta \left( M - \frac{J}{\gamma l} \right).$$

Now, the evolution along $t^a$ is Hamiltonian only if the right-hand side of (3.33) is exact on phase space. This implies that if the surface gravity is a function of area only and $\Omega(t)$ a function of the angular momentum only, there exists a phase-space function $E^t_\Delta$ such that the first law appears:

$$\delta E^t_\Delta = \left[ \kappa(\xi) \delta \left( (1 + \lambda a/\gamma) a_\Delta + \Omega(t) \delta I_\Delta \right) \right],$$

(3.34)

where $\bar{a}_\Delta = (1 + \lambda a/\gamma) a_\Delta$. The presence of $\kappa(\xi)$ in the first law indicates that the first law is the same for both extremal and non-extremal black holes. A mere choice of the function $\xi$ can help us interpolate between these class of solutions. We note here that modification in the symplectic structure of the theory leaves its footprint through $\gamma$ in the first law of (weak) isolated horizon mechanics. The term that plays the role of the ‘area’ term as it appears in this first law differs from the standard geometrical area of the horizon. If we restrict ourselves to the class of BTZ horizons, we have

$$\bar{a}_\Delta = 2\pi \left( r_+ - r_- / \gamma \right) = a_\Delta - \frac{l\pi J}{\gamma a_\Delta}.$$  

(3.35)

### 3.7. Admissible vector fields and horizon mass

In the previous discussion, we used the Hamiltonian evolution of the life-time vector field $t^a$ to deduce the first law. It is necessary and sufficient for the existence of the Hamiltonian function $E^t_\Delta$ as in (3.34) that the functions $\kappa(t), \Omega(t)$ should be functions of the independent horizon parameters $\bar{a}_\Delta$ and $I_\Delta$ only and following exactness condition should hold:

$$\frac{\partial \kappa(t)}{\partial I_\Delta} = \frac{\partial \Omega(t)}{\partial \bar{a}_\Delta}. $$

(3.36)

However, given any vector field, it is not guaranteed that these will be satisfied. In other words, not all vector fields are Hamiltonian. Vector fields $t^a$ for which these conditions are satisfied are admissible and there are infinite of them. We wish to find the class of admissible $t^a$s by solving (3.36). The essential point is to show the existence of a canonical live vector field. The horizon energy defined by this canonical live vector field is called the horizon mass. In order to proceed, we make the following change of variables for convenience:

$$(\bar{a}_\Delta, I_\Delta) \rightarrow (R_+, R_-)$$

with

$$R_+ = \sqrt{\frac{\gamma l}{2\pi (\gamma^2 - 1)}} \left( I_\Delta + \frac{\gamma \bar{a}_\Delta^2}{8\pi l} \right),$$

$$R_- = \gamma \sqrt{\frac{\gamma l}{2\pi (\gamma^2 - 1)}} \left( I_\Delta + \frac{\gamma \bar{a}_\Delta^2}{8\pi l} \right) - \frac{\gamma \bar{a}_\Delta}{4\pi}.$$ 

(3.37)

9 In our conventions, the double roots $r_+, r_-$ of the BTZ lapse polynomial are related to the BTZ ($\gamma \mapsto \infty$) mass ($M$) and angular momentum ($J$) as

$$M = 2\pi \frac{r_+^2 + r_-^2}{l^2} \quad \text{and} \quad J = 4\pi \frac{r_+ r_-}{l}.$$
Now for \( \kappa(t) \) we wish to start with a sufficiently smooth function \( \kappa_0 \) of the horizon parameters. In general \( \kappa(t) \neq \kappa_0 \). But we can always find a phase function \( \xi \) in \( \tau^a = \xi \ell^a - \Omega(t) \varphi \) such that \( \kappa(t) = \kappa_0 \). Again, there is a canonical choice, supplied by the known solution, the BTZ one, in which there is a unique BTZ black hole for each choice of the horizon parameters. We therefore set \( \kappa_0 = \kappa(t)_{(\text{BTZ})} \), where \( \tau^a \) is the global time translation Killing field of the BTZ spacetime and express it in terms of the newly introduced coordinates:

\[
\kappa_0 = \frac{R_+^2 - R_-^2}{R_+ l^2}.
\]

The angular velocity \( \Omega(t) \) satisfying (3.36) comes out as \( \Omega(t) = \frac{R_-}{R_+ l} \). Using this value of angular velocity and equation (3.37) in (3.34) we have

\[
\delta E^\Delta = \delta \left[ \frac{2\pi}{l^2} \left( R_+^2 + R_-^2 - 2R_+ R_- \gamma \right) \right]. \tag{3.38}
\]

Now, from equations (3.37) and (3.38), we have the horizon mass in terms of the independent horizon parameters:

\[
M_\Delta (J_\Delta, a_\Delta) = \frac{\gamma J_\Delta}{l} + \frac{\gamma^2 a_\Delta^2}{8\pi l^2} - \frac{a_\Delta}{2l^2} \sqrt{l y^2 - 1} \left( J_\Delta + \frac{\gamma^2 a_\Delta^2}{8\pi l^2} \right).
\]

It is not difficult to check that this works for the BTZ black hole. Restricting to BTZ values, this reads \( M_\Delta = (M - J/\gamma l) \). This exactly matches with the asymptotic charge \( X_{\infty}^\Delta (\delta) = \delta (M - J/\gamma l) \) associated with the asymptotic time translation vector \( \tau^a \) of BTZ spacetime as would have been expected. We must also note the deformations of the conserved charges: angular momentum and mass under the influence of the parameter \( \gamma \) are exactly same as those stated in [27–29] and at the ‘chiral point’ (\( \gamma = 1 \)) angular momentum and the mass become proportional to each other with opposite sign.

4. Covariant phase-space realization of asymptotic symmetry algebra

It has been suggested that microscopic details which explain the thermodynamics of black holes are independent of any theory of quantum gravity. If this is taken seriously, it implies that the microstates that describe black hole spacetime can be understood from a principle which is expected to govern all quantum gravity theory. It then seems natural to use the arguments of symmetry. Whatever be the theory of quantum gravity, it must at least preserve a part of the symmetries of classical theory. The study of asymptotic symmetries has been advocated to serve this purpose and has achieved striking success in reproducing the Bekenstein–Hawking formula. This issue was first addressed in the context of 2 + 1 gravity (with a negative cosmological constant) in [20].

In this issue, we note that diffeomorphisms which are gauges for any theory of gravity become physical symmetry at the boundaries of the spacetime manifold by physical requirements (boundary conditions). For example, in \( (3 + 1) \)-dimensional asymptotically flat spacetimes one naturally identifies a time-like vector field at asymptotic infinity as the unique time translation (Killing) as in Minkowski spacetime and fixes it once and for all. This fixes the diffeomorphisms partially and plays the role of a physical symmetry. Only then we can associate a Hamiltonian or Noether charge with time which is the Arnowitt–Deser–Misner mass. In [21], the authors considered diffeomorphisms generated by asymptotic vector fields which are a bit ‘relaxed Killing symmetries’ of the asymptotic metric in a \((2 + 1)\)-dimensional spacetime and showed that they form the pair of affine Witt algebra (2D conformal algebra, or deformation algebra of \( S^1 \)) as opposed to \( SO(2, 2) \), the isometry group of AdS\(_3\). We
will show that those vector fields actually generate flows in the phase space which are at least locally Hamiltonian and find the corresponding Hamiltonians (hence qualifying as physical symmetries), i.e. charges in the covariant phase-space framework. The preference for this framework is firstly due to its manifest covariant nature and secondly for its immense calculational simplicity, as compared to the canonical framework [18].

According to the suggestion mentioned above, this immediately implies that the quantum theory describing the microstates of black holes is a conformal field theory. The simple use of central charges in the Cardy formula determines the asymptotic density of quantum states of black holes which have same mass and angular momentum and approach the asymptotic central charges in the Cardy formula determines the asymptotic density of quantum states theory describing the microstates of black holes is a conformal field theory. The simple use of central charges in the Cardy formula determines the asymptotic density of quantum states of black holes which have same mass and angular momentum and approach the asymptotic central charges in the Cardy formula determines the asymptotic density of quantum states.

We now want to investigate if the algebra of the vector fields on the spacetime manifold (diffeomorphisms) corresponding to the vector fields \( \xi \) is also realized on the phase space, i.e. the Hamiltonian functions (or the generators of diffeomorphisms) corresponding to the vector fields \( \xi \) also satisfy the affine algebra. To see this, we first associate a phase-space vector field \( \delta \xi \) with each element \( \xi \) of the algebra such that \( \delta \xi \) acts as \( \xi \delta \) on dynamical variables\(^{10}\). Secondly, we need the symplectic structure which will enable us to construct the Hamiltonian functions as has been described in the previous sections (see (3.5) and (3.6)). Since we are interested in the asymptotic analysis, we will be interested in the contribution to the symplectic structure from the asymptopia or \( S_\infty \). If an internal boundary like NEH is present, we can assume that the vector fields whose asymptotic forms are as \( \xi \) above vanish on that boundary. From this point of view, for any arbitrary vector field \( \xi \) which vanishes on any internal boundary (in this section, we will reinstate 16\( \pi G \) but will choose \( c = h = 1 \)):

\[
8\pi G \Omega(\delta, \delta_\xi) = \oint_{S_\infty} \left[ (\xi \cdot \epsilon^I) \delta A_I + (\xi \cdot A^I) \delta \epsilon_I + \frac{l}{\gamma} (\xi \cdot A^I) \delta A_I + \frac{1}{l\gamma} (\xi \cdot \epsilon^I) \delta \epsilon_I \right].
\]

(4.2)

The under right arrows indicate the pull-back of the forms on \( S_\infty \). Therefore, the second and the fourth terms in the integral do not contribute. Only the internal component \( \epsilon^2 \) (as given above)

\(^{10}\) This is because vector fields on the spacetime manifold work as generators of infinitesimal diffeomorphisms.
survives under the pull-back which is given by \(-r\,d\phi\). This being a phase-space constant, the action of \(\delta\) on it vanishes. Hence, we obtain
\[
8\pi G \Omega(\delta, \delta_\xi) = \oint_{\Sigma_m} \left[ \xi_j \cdot \left( e^j + \frac{1}{\gamma} A^j \right) \right] \delta A_j \tag{4.3}
\]
for any arbitrary vector field \(\xi\). Using the above expressions of the fields asymptotically, we have
\[
8\pi G \Omega(\delta, \delta_{\xi_m}) = \delta H_n = \left( 1 - \frac{1}{\gamma} \right) \delta (lM + J) \delta_{n,0}. \tag{4.4}
\]
Hence, we observe that \(\delta_{\xi_m}\) are at least locally Hamiltonian for all \(n\) and have non-zero charge only for \(n = 0\). We also note using (4.1) that \(\delta_{\{\xi_n, \xi_m\}}\) is also a Hamiltonian vector field with \(\delta H[1, \xi_n, \xi_m]\) given by the right-hand side of the following equation:
\[
8\pi G \Omega(\delta, \delta_{\{\xi_n, \xi_m\}}) = -i(n - m) \left( 1 - \frac{1}{\gamma} \right) \delta (lM + J) \delta_{m+n,0}. \tag{4.5}
\]
We will now determine the current algebra of the Hamiltonian functions (i.e. \([H_{\xi_n}, H_{\xi_m}]\)) generated by the Hamiltonian vector fields \(\delta_{\xi_n}\) and \(\delta_{\xi_m}\) for arbitrary \(n\) and \(m\). This will be given by
\[
8\pi G \Omega(\delta_{\xi_m}, \delta_{\xi_n}) = \oint_{\Sigma_m} \left[ \xi_n \cdot \left( e^j + \frac{1}{\gamma} A^j \right) \right] \delta_{\xi_m} A_j. \tag{4.6}
\]
It is now important that we first pull back \(A_j\) and then calculate the action of \(\delta_{\xi_m}\) on it as the Lie derivative. After some lines of calculation, we find
\[
8\pi G \Omega(\delta_{\xi_m}, \delta_{\xi_n}) = -2i(n - m) \left( 1 - \frac{1}{\gamma} \right) (J + lM) \delta_{m+n,0} + i\pi n^3 \left( 1 - \frac{1}{\gamma} \right) \delta_{m+n,0} + \mathcal{O} \left( \frac{1}{\gamma^2} \right).
\]
Comparing (4.5) and (4.7), we infer that the asymptotic diffeomorphism algebra (4.1) is exactly realized at the canonical level (as a current algebra) except a ‘central term’ 
\(-i\pi n^3 (1 - \frac{1}{\gamma}) \delta_{m+n,0}\). This is not surprising, although all the vector fields \(\delta_{\xi_m}\) were Hamiltonian. The second cohomology group of the Witt algebra\(^{11}\) is not trivial. A theorem of symplectic geometry states that in this case the action of the algebra is not Hamiltonian and moment maps do not exist, which on the other hand implies that the action of the Lie algebra on phase space is not Hamiltonian\(^{48}\), i.e.
\[
\delta \Omega(\delta_{\xi_m}, \delta_{\xi_n}) \neq \Omega(\delta_{\{\xi_n, \xi_m\}}, \delta). \tag{4.8}
\]
Written in terms of the charges \(H_n\), (4.7) reads
\[
[H_m, H_n] = -i(n - m) \left( 1 - \frac{1}{\gamma} \right) H_{m+n} + i\pi n^3 \left( 1 - \frac{1}{\gamma} \right) \delta_{m+n,0}. \tag{4.9}
\]
which is isomorphic to the Virasoro algebra.

\(^{11}\) For any real Lie algebra \(G\) and its dual \(G^*\), a skew-symmetric bilinear map \(\alpha \in G^* \wedge G^*\) is said to be a cocycle if \(\alpha ([A, B], C) + \alpha ([B, C], A) + \alpha ([C, A], B) = 0\) for all \(A, B, C \in G\) and \([, ]\) is the usual product on \(G\). The elements \(\delta f (f \in G^*)\) defined via \(\delta f (A, B) = \frac{i}{2} f([A, B])\), automatically cocycles by Jacobi identity are called coboundary. Let us define an equivalence \(\sim\) as follows: two cocycles \(\alpha \sim \beta\) if \(\alpha = \beta + \delta g\) for any \(g \in G^*\). Now one defines \(H^2(G)\) as the additive group of equivalence classes found through the modulo action of the equivalence relation. All semi-simple Lie algebras have trivial second cohomology.

\(^{12}\) If \(J[\xi_\alpha]\) and \(J[\xi_\beta]\) are Hamiltonians (calculated in the canonical phase space) corresponding to the vector fields \(\xi_\alpha\) and \(\xi_\beta\), then \([J[\xi_\alpha], J[\xi_\beta]] = [J[\xi_\alpha], J[\xi_\beta]]\), where \([J[\xi_\alpha], J[\xi_\beta]] := \delta_{\xi_\beta} J[\xi_\alpha] = -\delta_{\xi_\alpha} J[\xi_\beta].\)
Now, all of this calculation was performed by choosing the right-moving vector fields. There also are a set of left-moving vector fields which preserve the asymptotic structure:

\[ \tilde{\xi}^a := \exp(in_{-}) \left[ l \left( 1 - \frac{l^2 n^2}{2r^2} \right) \partial_t - \frac{i}{n} \partial_r - \left( 1 + \frac{l^2 n^2}{2r^2} \right) \partial_\phi \right], \]

where \( x_- = (t/l - \phi) \). Proceeding along the very same route as before, we again end up with the result that the canonical realization of this asymptotic symmetries is also realized exactly up to a central term, which now becomes

\[ -il\pi n^3 \left( 1 + \frac{1}{\gamma} \right) \delta_{m+n,0}. \]

From the definition of the central charge of the Virasoro algebra, which is the centrally extended version of the Witt algebra, we arrive at the exact formulas for the central charges for the right- and left-moving algebras, respectively:

\[ c = \frac{3l^2}{2G} \left( 1 - \frac{1}{\gamma} \right) \quad \text{and} \quad \tilde{c} = \frac{3l^2}{2G} \left( 1 + \frac{1}{\gamma} \right). \]

Once we have the central charges, we can apply the Cardy formula to the BTZ solution to obtain the black hole entropy by noting from (4.4) that

\[ H_0 = \frac{1}{8\pi G} \left( lM + J \right) \]

for the right mover and from an almost identical calculation that

\[ \tilde{H}_0 = \frac{1}{8\pi G} \left( lM - J \right) \]

for the left mover:

\[ S = 2\pi \sqrt{\frac{H_0 c}{6}} + 2\pi \sqrt{\frac{\tilde{H}_0 \tilde{c}}{6}} = \frac{2\pi r_+}{4G} \frac{2\pi r_-}{4G} \left( \text{Since } M = 2\pi \frac{r_+^2 + r_-^2}{l^2}, J = 4\pi \frac{r_+ r_-}{l} \right) = \frac{\left( a_\Delta - \frac{l\pi J}{\gamma a_\Delta} \right)}{4G} = \frac{\tilde{a}}{4G}. \]

where \( r_+ \) and \( r_- \) are the radii of the outer and inner horizon, respectively. If we consider the thermodynamic analogy of the first law of black hole mechanics (3.34) (derived for general spacetimes only requiring the presence of a weakly isolated horizon only from classical symplectic geometric considerations), we observe that \( S \sim \tilde{a} \). Curiously, even in the quantum result (4.10), the entropy-modified area relation continues to hold.

5. Conclusion

Let us recollect the main findings of this paper. Firstly, we introduced the concept of WIH in 2 + 1 dimensions. The boundary conditions which have been imposed on a two-dimensional null surface are much weaker than the ones suggested in [42]. Our boundary conditions are satisfied by a equivalence class of null normals which are related by functions, \( [\xi \ell^a] \), rather than constants, \( [\ell^a] \), as was first proposed in [42]. The advantage of such generalization lies in the fact that it becomes possible to include extremal as well as non-extremal solutions in the same space of solutions. Just by choosing the function \( \xi \), one can move from a non-zero \( \kappa(\ell) \) to a vanishing \( \kappa(\xi) \) (see equation (2.1)) which essentially is like taking extremal limits in phase space. We also established that the zeroth law (for all solutions in this extended space of solutions) follows quite trivially from the boundary conditions.

Secondly, we have explicitly shown that in the presence of an internal boundary satisfying the boundary conditions of a WIH, the variational principle for the generalized (2 + 1)-dimensional theory remains well defined. This enables us to take the third step where we have constructed the covariant phase space of this theory. The covariant phase space now contains all solutions of the \( \gamma \)-dependent theory which satisfy the WIH boundary conditions at infinity.
As expected, extremal as well as non-extremal solutions form a part of this phase space. We then went on to define the angular momentum as a Hamiltonian function corresponding to the rotational Killing vector field on the horizon. It was also explicitly shown that for the BTZ solution, the angular momentum defined in this manner matches with the expected result.

Thirdly, we established the first law of black hole mechanics directly from the covariant phase space for isolated horizons. Instead of the usual horizon area term one encounters in this law, we find a modification due to the $\gamma$ factor. This is a completely new result in this family of theories. It arose that the first law is the necessary and sufficient condition for the existence of a time-like Hamiltonian vector field on the covariant phase space. However, not all time-like vector fields are Hamiltonian on phase space; there exist some which are admissible (there are in fact infinite of them). The canonical choice for these admissible vector fields is constructed too. Quite interestingly, the first law for the WIH formulation, equation (3.34), contains $\kappa(\xi \ell)$. This implies that the first law holds for all solutions, extremal as well as non-extremal. However, the thermodynamic implications of the first law can only be extracted for non-extremal solutions since for the extremal ones, the first law is trivial. But then, since all solutions are equivalent from the point of view of WIH boundary conditions, the entropy of both class of black hole solutions will be the same. This interpretation of our result is in contradiction to the results of [49] and [50] where the vanishing entropy for extremal black holes is obtained. On the other hand, they are in agreement to those of [51].

Using asymptotic analysis, we have calculated the entropy of black holes for the theory under consideration. Contrary to the usual approach, we construct the algebra of diffeomorphism generating Hamiltonian functions directly from the covariant phase space. As usual, we see that the algebra does not match with the Hamiltonian function for the commutator of the asymptotic vector fields. The difference is the central extension. In other words, the algebra of spacetime vector fields is not realized on the covariant phase space. The Cardy formula then gives the entropy directly which matches with the one expected from the first law. The entropy however not only depends on the geometrical area but also on other quantities such as the parameter of the solution $J$ and the $\gamma$-parameter of the theory (equation (4.10)). Keeping the thermodynamic analogy of laws of black hole mechanics in mind and concentrating on the BTZ black hole, one observes that there is a perfect harmony between this result and the modified first law. Also recall that our methods do not rely on the existence of bifurcation spheres and apply equally to extremal and non-extremal black holes. To our knowledge, this has not been reproduced earlier since the phase space of Killing horizons which satisfy laws of mechanics do not contain extremal solutions.

Our analysis for the computation of entropy is based on asymptotic symmetry analysis. The principle of using symmetry arguments to determine the density of states for the black hole is attractive, it does not depend on the details of quantum gravity. The asymptotic analysis has a major drawback—it seems to be equally applicable for any massive object placed in place of a black hole. Since such objects are not known to behave like black holes, it is not clear where to attribute such a large number of density of states. One must directly look at the near-horizon symmetry vector fields for further understanding [52]. However, a more interesting step would be to determine the horizon microstates as is done in $3+1$ dimensions. In this case, it arises from classical considerations that the degrees of freedom that reside on a WIH in $3+1$ dimensions is a Chern–Simons theory. Quantization of this theory gives an estimate of the states that contribute to a fixed area horizon and the entropy turns out to be proportional to the area. This still has not been reproduced in $2+1$ dimensions and will be investigated in the future in order to compliment the new findings already present in this paper.
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Appendix. The NP formalism for 2 + 1 dimensions

In order to make the paper self-contained we summarize here the analogue of the NP formalism in 2 + 1 dimensions, which was described in detail in [42]. We will use a triad consisting of two null vectors $\ell^a$ and $n^a$ and a real space-like vector $m^a$, subject to

$$\ell \cdot \ell = n \cdot n = 0, \quad m \cdot m = 1$$  \hspace{1cm} (A.1)

$$\ell \cdot m = n \cdot m = 0$$  \hspace{1cm} (A.2)

$$\ell \cdot n = -1.$$  \hspace{1cm} (A.3)

The spacetime metric $g_{ab}$ can be expressed as

$$g_{ab} = -2 \ell_a n_b + m_a m_b,$$  \hspace{1cm} (A.4)

and its inverse $g^{ab}$ is defined to satisfy

$$g^{ab} = -2 \ell^a n^b + m^a m^b.$$  \hspace{1cm} (A.5)

It is then easy to verify that the expression for the triad is just

$$e^I_a = -\ell_a n^I - n^a \ell^I + m_a m^I.$$  \hspace{1cm} (A.6)

Just as in the 3 + 1 case, we express the connection in the chosen triad basis, the connection coefficients being the new NP coefficients (the $\gamma$ defined below is not to be confused with the Barbero–Immirzi parameter):

$$\nabla_a \ell_b = -\epsilon n_a \ell_b + \kappa_{NP} n_a m_b - \gamma \ell_a \ell_b + \tau \ell_a m_b + \alpha m_a \ell_b - \rho m_a m_b,$$  \hspace{1cm} (A.7)

$$\nabla_a n_b = \epsilon n_a n_b - \pi n_a m_b + \gamma \ell_a n_b - \nu \ell_a m_b - \alpha m_a n_b + \mu m_a m_b,$$  \hspace{1cm} (A.8)

$$\nabla_a m_b = \kappa_{NP} n_a n_b - \pi n_a \ell_b + \tau \ell_a n_b - \nu \ell_a m_b - \rho m_a n_b + \mu m_a m_b.$$  \hspace{1cm} (A.9)

It then simply follows from the expressions above that $\nabla_a \ell^a = (\epsilon - \rho)$, $\nabla_a n^a = (\mu - \gamma)$ and $\nabla_a m^a = (\pi - \tau)$. Now, we wish to expand the connection 1-form $A^I_a$ in the triad basis with NP coefficients slated above as coefficients. In order to do so, we note that for an arbitrary 1-form $v_a$, which may be mapped uniquely to an $SO(2, 1)$ frame element $v_I = v_a e^a_I$, then, for $\nabla_a v_b = A^I_a v^I e_I b$, and using $A^I_a e^I = \epsilon_k e^I A^k_a$, we arrive at the expression

$$A^k_a = (\pi n_a + \nu \ell_a - \mu m_a) \ell^K + (\kappa_{NP} n_a + \tau \ell_a - \rho m_a) n^K + (-\epsilon n_a - \gamma \ell_a + \alpha m_a) m^K.$$  \hspace{1cm} (A.10)

13 All the NP coefficients appearing in 2 + 1 dimensions are therefore real unlike in 3 + 1.
References

[1] Deser S, Jackiw R and 't Hooft G 1984 Ann. Phys. 152 220
[2] Deser S and Jackiw R 1984 Ann. Phys. 153 405
[3] Witten E 1988 Nucl. Phys. B 311 46
[4] Carlip S 1998 *Quantum Gravity in 2 + 1 Dimensions* (Cambridge: Cambridge University Press)
[5] Ezawa K 1995 Class. Quantum Grav. 12 373
[6] Basu R and Paul S K 2010 Class. Quantum Grav. 27 125003
[7] Banados M, Teitelboim C and Zanelli J 1992 *Phys. Rev. Lett.* 69 1849
[8] Bose S, Dadhich N and Kar S 2000 Phys. Lett. B 477 451–6 (arXiv:hep-th/9911069)
[9] Deser S, Jackiw R and Templeton S 1982 *Phys. Rev. Lett.* 48 975
[10] Deser S, Jackiw R and Templeton S 1982 Ann. Phys. 140 372
[11] Rogishoef E A, Hohn O and Townsend P K 2009 *Phys. Rev. Lett.* 102 201301
[12] Liu Y and Sun Y W 2009 J. High Energy Phys. JHEP04(2009)106
[13] Gupta R K and Sen A 2008 J. High Energy Phys. JHEP07(2008)066
[14] Tachikawa Y 2007 Class. Quantum Grav. 24 737–44 (arXiv:hep-th/0611141)
[15] Strominger A 1998 J. High Energy Phys. JHEP02(1998)009
[16] Brown J D and Henneaux M 1986 *Commun. Math. Phys.* 104 207
[17] Carlip S 2005 *Class. Quantum Grav.* 22 R65
[18] Poisson E 2004 *A Relativist's Toolkit* (Cambridge: Cambridge University Press)
[19] Clement G 2003 *Phys. Rev.* D 68 024032
[20] Bouchareb A and Clement G 2007 Class. Quantum Grav. 24 5581
[21] Henneaux M, Martinez C and Troncoso R 2009 Phys. Rev. D 79 081502
[22] Compere G and Detournay S 2009 Class. Quantum Grav. 26 012001
[23] Blagojevic M and Vasilic M 2003 Phys. Rev. D 68 124007
[24] Blagojevic M and Vasilic M 2003 Phys. Rev. D 67 084032
[25] Blagojevic M and Cvetkovic B 2006 *J. Phys.: Conf. Ser.* 33 248 (arXiv:gr-qc/0511162)
[26] Blagojevic M and Cvetkovic B 2009 J. High Energy Phys. JHEP05(2009)073
[27] Blagojevic M and Cvetkovic B 2009 J. High Energy Phys. JHEP09(2009)006
[28] Mielke E W and Baekler P 1991 *Phys. Lett.* A 156 399
[29] Baekler P, Mielke E W and Hehl F W 1992 *Nuovo Cimento* B 107 91
[30] Cacciatori S L, Caldarelli M M, Giacomini A, Klemm D and Mansi D S 2006 *J. Geom. Phys.* 56 2523
[31] Bonzom V and Livine E R 2008 Class. Quantum Grav. 25 195024
[32] Holst S 1996 *Phys. Rev.* D 53 5966
[33] Li W, Song W and Strominger A 2008 J. High Energy Phys. JHEP04(2008)082
[34] Carlip S, Deser S, Waldron A and Wise D K 2009 Class. Quantum Grav. 26 075008
[35] Carlip S 2008 J. High Energy Phys. JHEP10(2008)078
[36] Grumiller D, Jackiw R and Johannson N 2008 arXiv:0806.4185 [hep-th]
[37] Grumiller D and Johannson N 2008 J. High Energy Phys. JHEP07(2008)134
[38] Smith A, Beetle C and Fairhurst S 1999 Class. Quantum Grav. 16 L1
[39] Smith A, Beetle C and Fairhurst S 2000 Class. Quantum Grav. 17 253
[40] Ashok A, Beetle C and Lewandowski J 2001 *Phys. Rev.* D 64 044016
Ashtekar A, Beetle C and Lewandowski J 2002 *Class. Quantum Grav.* **19** 1195
Ashtekar A, Corichi A and Sudarsky D 2003 *Class. Quantum Grav.* **20** 3413
Ashtekar A, Engle J, Pawlowski T and van der Broeck C 2004 *Class. Quantum Grav.* **21** 2549

[41] Ashtekar A and Krishnan B 2004 *Living Rev. Rel.* **7** 10 (available at [http://www.livingreviews.org/lrr-2004-10](http://www.livingreviews.org/lrr-2004-10))

[42] Ashtekar A, Wisniewski J and Dreyer O 2002 *Adv. Theor. Math. Phys.* **6** 507 (available at [http://www.intlpress.com/journals/ATMP/archive/2002/6-3-3.pdf](http://www.intlpress.com/journals/ATMP/archive/2002/6-3-3.pdf))

[43] Wald R M 1993 *Phys. Rev. D* **48** 3427

[44] Wald R M and Racz J 1996 *Class. Quantum Grav.* **13** 539

[45] Chatterjee A and Ghosh A 2006 *Class. Quantum Grav.* **23** 7521

Chatterjee A 2007 *AIP Conf. Proc.* **939** 96

[46] Chatterjee A and Ghosh A 2009 *Phys. Rev. D* **80** 064036

[47] Crnkovic C and Witten E 1986 *Three Hundred Years of Gravitation* ed S W Hawking and W Israel (Cambridge: Cambridge University Press)

Lee J and Wald R M 1990 *J. Math. Phys.* **31** 725

Ashtekar A, Bombelli L and Reula O 1991 *Analysis, Geometry and Mechanics: 200 Years After Lagrange’* ed M Francaviglia and D Holm (Amsterdam: North-Holland)

[48] Woodhouse N M J 1997 *Geometric Quantization* (Oxford: Oxford University Press)

[49] Hawking S W, Horowitz G T and Ross S F 1995 *Phys. Rev. D* **51** 4302

Teitelboim C 1995 *Phys. Rev. D* **51** 4315

Teitelboim C 1995 *Phys. Rev. D* **52** 6201 (erratum)

[50] Edery A and Constantineau B 2011 *Class. Quantum Grav.* **28** 045003

[51] Ghosh A and Mitra P 1997 *Phys. Rev. Lett.* **78** 1858

[52] Carlip S 1999 *Class. Quantum Grav.* **16** 3327

Dreyer O, Ghosh A and Wisniewski J 2001 *Class. Quantum Grav.* **18** 1929