Numerics for Stochastic Distributed Parameter Control Systems: a Finite Transposition Method

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Abstract

In this chapter, we present some recent progresses on the numerics for stochastic distributed parameter control systems, based on the finite transposition method introduced in our previous works. We first explain how to reduce the numerics of some stochastic control problems in this respect to the numerics of backward stochastic evolution equations. Then we present a method to find finite transposition solutions to such equations. At last, we give an illuminating example.

Key words: Numerics, stochastic distributed parameter control system, backward stochastic evolution equation, transposition solution, finite transposition method.

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1 Introduction

In this chapter, we study the numerics for stochastic distributed parameter control systems, including particularly controlled stochastic evolution equations in infinite dimensions. Although some pioneer works appeared in 1960s (e.g., Bensoussan, 1969; Kushner, 1968; Tzafestas and Nightingale, 1968), the theory for stochastic distributed parameter control systems is far from mature till now. Lots of interesting and important problems therein are not well studied. One of them is the numerical computation on control problems of such systems, for which one will meet many substantially difficulties:

- For stochastic control systems, numerical schemes should keep the adaptedness of states and controls with respect to the filtration. This restricts the application of many implicit schemes.
• Unlike the deterministic setting, solutions to stochastic evolution equations (even in finite dimensions) are usually non-differentiable with respect to the time variable. This leads to some serious troubles in the analysis of the convergence rate.

• In the continuous-time setting, similar to the deterministic situation, a typical way to solve many stochastic control problems is to introduce the corresponding dual systems, which are however backward stochastic evolution equations usually (see Lü and Zhang, 2020, for more details). Numerically, it is very hard to compute the correction terms in such sort of backward equations since there is no efficient method to compute the conditional expectation of a general random variable.

Due to these difficulties, there are very few works on the numerical analysis for stochastic distributed parameter control systems (e.g., Dunst and Prohl, 2016; Prohl and Wang, 2020a,b; Li and Zhou, 2020).

Because of the importance of backward stochastic evolution equations in the study of stochastic distributed parameter control systems, in this chapter we shall focus on the numerical solutions to such equations and the related theoretical analysis.

Up to now, there exist quite a number of numerical schemes for backward stochastic differential equations, i.e. backward stochastic evolution equations in finite dimensions, such as the method of four-step scheme (e.g., Douglas et al., 1996), the quantization tree method (e.g., Bally and Pagès, 2003), the forward/backward Euler method (e.g., Zhang, 2004; Gobet et al., 2005; Bender and Denk, 2007), the Malliavin calculus based method (e.g., Bouchard and Touzi, 2004; Hu et al., 2011), the fully time-space discretization method (e.g., Zhao et al., 2006, 2014), the finite transposition method (e.g., Wang and Zhang, 2011), the Wiener chaos decomposition method (e.g., Briand and Labart, 2014), the machine learning method (e.g., E et al., 2019), etc. However, numerical schemes for backward stochastic evolution equations in infinite dimensions (including typically backward stochastic partial differential equations) are quite limited (Wang, 2013, 2016; Dunst and Prohl, 2016).

Based on the stochastic transposition method introduced in Lü and Zhang (2013, 2014), in this chapter, we propose a new numerical algorithm to solve backward stochastic evolution equations, which can be used to compute the desired control of some controllability/optimal control problems for stochastic distributed parameter systems numerically.

To present the key idea in the simplest way, we do not pursue the full technical generality. Firstly, we consider only the simplest case of one dimensional standard Brownian motion (with respect to the time variable $t$). It would not be difficult to extend the results to the case of the $Q$-Brownian motion and the cylindrical Brownian motion. Secondly, we impose considerably strong regularity and boundedness assumptions on data appeared in the state equations and the cost functionals. Thirdly, we assume that the control operators are bounded linear ones, which cannot cover the case of stochastic partial differential equations with boundary/pointwise controls. All of these assumptions can be considerably relaxed. Fourthly, we only consider a null controllability and a linear quadratic optimal control problem. More general controllability and optimal control problems can be studied by the method presented in this chapter.

The rest of this chapter is organized as follows. In Section 2, we shall introduce the control problems and the corresponding backward stochastic evolution equations to be considered in this chapter. In Section 3, we firstly give an outline of the finite transposition method for solving backward stochastic evolution equations numerically, and then provide a choice of finite transposition space. In Section 4, based on the finite transposition space, we define the finite transposition solution to backward stochastic evolution equations, and
then present the existence and uniqueness of this solution as well as the convergence rate of this numerical method. Finally, in Section 5, as an application, we propose a numerical scheme for an optimal control problem, based on our finite transposition method.

2 Adjoint equations for stochastic distributed parameter control problems

We begin with some notations and notions to be used later. Let $T > 0$ and $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ (with $\mathbb{F} = \{\mathcal{F}_t\}_{t \in [0,T]}$) be a filtered probability space so that $(\Omega, \mathcal{F}, \mathbb{P})$ is complete, $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets in $\mathcal{F}$, and $\mathbb{F}$ is right continuous. Let $\{W(t)\}_{t \in [0,T]}$ be an $\mathbb{F}$-adapted 1-dimensional standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. Denote by $\mathbb{F}$ the progressive $\sigma$-field (in $[0,T] \times \Omega$) with respect to $\mathbb{F}$, by $\mathbb{E}\xi$ the (mathematical) expectation of an integrable random variable $\xi : (\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$, and by $C$ a generic positive constant, which may vary from one place to another.

For a Banach space $\mathcal{X}$, we write $\| \cdot \|_\mathcal{X}$ for its norm. If $\mathcal{X}$ is a Hilbert space, we denote its inner product by $\langle \cdot, \cdot \rangle_\mathcal{X}$ (When $\mathcal{X}$ is a Euclidean space, for simplicity, we denote its norm and inner product by $| \cdot |$ and $\langle \cdot, \cdot \rangle$, respectively). For any filtration $G = \{\mathcal{G}_t\}_{t \in [0,T]} \subseteq \mathbb{F}$, $p, q \in [1, \infty)$ and $t \in [0,T]$, let $L^p_{\mathcal{G}_t}(\Omega; \mathcal{X}) \triangleq L^p(\Omega, \mathcal{G}_t, \mathbb{P}; \mathcal{X})$, and

$$L^p_{\mathcal{G}_t}(0,T; L^p(\Omega; \mathcal{X})) \triangleq \left\{ \varphi : (0,T) \times \Omega \to \mathcal{X} \mid \varphi(\cdot) \text{ is } \mathbb{F}\text{-adapted} \right. \text{ and } \int_0^T (\mathbb{E}\|\varphi(t)\|_\mathcal{X}^p)^{\frac{1}{p}} \, dt < \infty \left\},
$$

$$L^p(\Omega; C([0,T]; \mathcal{X})) \triangleq \left\{ \varphi : [0,T] \times \Omega \to \mathcal{X} \mid \varphi(\cdot) \text{ is continuous, } \mathbb{F}\text{-adapted and } \mathbb{E}(\|\varphi(\cdot)\|_{C([0,T]; \mathcal{X})}) < \infty \right\},
$$

$$C^p([0,T]; L^p(\Omega; \mathcal{X})) \triangleq \left\{ \varphi : [0,T] \times \Omega \to \mathcal{X} \mid \varphi(\cdot) \text{ is } \mathbb{F}\text{-adapted} \right. \text{ and } \varphi(\cdot) : [0,T] \to L^p_{\mathcal{F}_T}(\Omega; \mathcal{X}) \text{ is continuous} \left\}.
$$

In a similar way, we can define $L^\infty_{\mathcal{G}_t}(\Omega; \mathcal{X})$ and $L^\infty_{\mathcal{F}_T}(0,T; L^\infty(\Omega; \mathcal{X}))$. One can show that, for $1 \leq p, q \leq \infty$, $L^p_{\mathcal{G}_t}(\Omega; \mathcal{X})$ and $L^p(0,T; L^p(\Omega; \mathcal{X}))$ are Banach spaces (with the canonical norms). Further, both $L^p_{\mathcal{F}_T}(\Omega; C([0,T]; \mathcal{X}))$ and $C^p([0,T]; L^p(\Omega; \mathcal{X}))$ are Banach spaces with norms given by $\|\varphi(\cdot)\|_{L^p_{\mathcal{F}_T}(\Omega; C([0,T]; \mathcal{X}))} = (\mathbb{E}(\|\varphi(\cdot)\|_{C([0,T]; \mathcal{X})})^{1/p})^{1/p}$ and $\|\varphi(\cdot)\|_{C^p([0,T]; L^p(\Omega; \mathcal{X}))} = \max_{t \in [0,T]} (\mathbb{E}(\|\varphi(t)\|_{C([0,T]; \mathcal{X})})^{1/p})^{1/p}$ respectively. Also, we write $D^p_{\mathcal{F}_T}([0,T]; L^p(\Omega; \mathcal{X}))$ for the Banach space of all $L^p(\Omega; \mathcal{X})$-valued, $\mathbb{F}$-adapted, càdlàg stochastic processes $\varphi(\cdot)$ such that $\max_{t \in [0,T]} (\mathbb{E}(\|\varphi(t)\|_{C([0,T]; \mathcal{X})})^{1/p})^{1/p} < \infty$, with the canonical norm. In the sequel, we shall simply denote $L^p_{\mathcal{G}_t}(0,T; L^p(\Omega; \mathcal{X}))$ by $L^p_{\mathcal{G}_t}(0,T; \mathcal{X})$; and further simply denote $L^p_{\mathcal{F}_T}(\Omega; \mathbb{R}), L^p_{\mathcal{G}_t}(0,T; \mathbb{R})$ and $L^p_{\mathcal{G}_t}(\Omega; C([0,T]; \mathbb{R}))$ by $L^p(\Omega, \mathbb{X})$, $L^p(0,T, \mathbb{X})$ and $L^p(\Omega; C([0,T]; \mathbb{R}))$, respectively.

In what follows, we assume that $H$ and $U$ are (real) separable Hilbert spaces, and that $A$ is an unbounded linear operator (with domain $D(A) \subseteq H$), which generates a $C_0$-semigroup $\{e^{At}\}_{t \geq 0}$ on $H$. Write $A^*$ for the adjoint operator of $A$. For any $y_0 \in H$, consider the following controlled linear stochastic evolution equation:

$$\begin{cases}
    dy(t) = (Ay(t) + Bu(t)) \, dt + (Cy(t) + Du(t)) \, dW(t), & t \in (0,T], \\
    y(0) = y_0.
\end{cases}
$$

(2.1)
In (2.1), \( C(\cdot) \in L^\infty_F(0, T; \mathcal{L}(H)), B(\cdot), D(\cdot) \in L^\infty_F(0, T; \mathcal{L}(U; H)), u(\cdot) \in L^2_F(0, T; U) \) is the control variable, and \( y(\cdot) = y(\cdot, y_0, u(\cdot)) \) is the state variable. By the well-posedness result for stochastic evolution equations (e.g., Lü and Zhang, 2020, Section 3.2), the system (2.1) admits a unique mild solution \( y(\cdot) \in C_F([0, T]; L^2(\Omega; H)) \).

Now, we recall the notion of null controllability of (2.1).

**Definition 2.1.** System (2.1) is called null controllable at time \( T \) if for any \( y_0 \in H \), there exists \( u(\cdot) \in L^2_F(0, T; U) \) such that the corresponding mild solution to (2.1) satisfies \( y(T) = 0 \), a.s.

Next, we introduce the following quadratic cost functional

\[
\mathcal{J}(y_0; u(\cdot)) = \frac{1}{2} \mathbb{E} \left[ \int_0^T \left( \langle Q y(t), y(t) \rangle_H + \langle R u(t), u(t) \rangle_U \right) dt + \langle G y(T), y(T) \rangle_H \right],
\]

where \( Q(\cdot) \in L^\infty_F(0, T; \mathcal{L}(H)) \) and \( R(\cdot) \in L^\infty_F(0, T; \mathcal{L}(U)) \) are self-adjoint operator-valued stochastic processes, while \( G \in L^\infty_F(\Omega; \mathcal{L}(H)) \) is a self-adjoint operator-valued random variable. Consider the linear quadratic optimal control problem (SLQ problem for short) as follows:

**Problem (SLQ):** For each \( y_0 \in H \), find a \( \bar{u}(\cdot) \in L^2_F(0, T; U) \) such that

\[
\mathcal{J}(y_0; \bar{u}(\cdot)) = \inf_{u(\cdot) \in L^2_F(0, T; U)} \mathcal{J}(y_0; u(\cdot)).
\]

Any \( \bar{u}(\cdot) \) satisfying (2.3) is called an *optimal control*, the corresponding state \( \bar{y}(\cdot) \) is called an *optimal state*, and \( (\bar{y}(\cdot), \bar{u}(\cdot)) \) is called an *optimal pair*.

As in the deterministic case, in order to solve the above null controllability and optimal control problems, one may employ the duality argument. For this purpose, people introduce respectively the following two backward stochastic evolution equations:

\[
\begin{aligned}
    dz(t) &= -\left( A^* z(t) + C^* Z(t) \right) dt + Z(t) dW(t), \quad t \in [0, T), \\
    z(T) &= z_T (\in L^2_{\mathcal{F}_T}(\Omega; H)),
\end{aligned}
\]

and

\[
\begin{aligned}
    dz(t) &= -\left( A^* z(t) - Q y(t) + C^* Z(t) \right) dt + Z(t) dW(t), \quad t \in [0, T), \\
    z(T) &= -G y(T).
\end{aligned}
\]

These two equations can be used to solve aforementioned two stochastic control problems respectively (see Theorem 2.2 and Theorem 2.3).

As far as we know, the study of backward stochastic evolution equations is stimulated by the works Bensoussan (1983); Hu and Peng (1991). Now, backward stochastic evolution equations and its variants play fundamental roles in the theory of stochastic distributed parameter control systems (e.g., Lü and Zhang, 2020).

Since neither the usual natural filtration condition nor the quasi-left continuity is assumed for the filtration \( \mathcal{F} \), one cannot apply the existing results on infinite dimensional backward stochastic evolution equations (e.g., Hu and Peng, 1991; Al-Hussein, 2009) to obtain the well-posedness of the equations (2.4) and (2.5) in the sense of mild or weak solutions. Therefore, we shall employ the stochastic transposition method, developed first in our paper Lü and Zhang (2013) for backward stochastic differential equations, then in Lü and Zhang...
(2014) for backward stochastic evolution equations, to study the well-posedness of the equations (2.4) and (2.5).

Consider the following backward stochastic evolution equation (in a general form):
\[
\begin{cases}
dz(t) = -A^*z(t)dt + F(t, z(t), Z(t))dt + Z(t)dW(t), & t \in [0, T), \\
z(T) = z_T.
\end{cases}
\]
(2.6)

Here \( z_T \in L^2_{\mathcal{F}_T}(\Omega; H) \) and \( F : [0, T] \times \Omega \times H \times H \to H \) is a given function satisfying that
\[
\begin{cases}
F(\cdot, 0, 0) \in L^1(0, T; L^2(\Omega; H)), \\
\|F(t, y_1, z_1) - F(t, y_2, z_2)\|_H \leq C(\|y_1 - y_2\|_H + \|z_1 - z_2\|_H), \\
\forall y_1, y_2, z_1, z_2 \in H, \ a.e. \ (t, \omega) \in [0, T] \times \Omega.
\end{cases}
\]
(2.7)

To define the transposition solution to (2.6), we introduce the following stochastic evolution equation:
\[
\begin{cases}
d\varphi(s) = (A\varphi(s) + v_1(s))ds + v_2(s)dW(s), & s \in (t, T], \\
\varphi(t) = \eta,
\end{cases}
\]
(2.8)
where \( t \in [0, T], \ v_1(\cdot) \in L^1(t, T; L^2(\Omega; H)), \ v_2(\cdot) \in L^2(t, T; H) \) and \( \eta \in L^2_{\mathcal{F}_t}(\Omega; H) \). By the classical well-posedness result for stochastic evolution equations (e.g., Liu and Zhang, 2020, Section 3.2), the equation (2.8) admits a unique mild solution \( \varphi(\cdot) \in C_{\mathcal{F}}([t, T]; L^2(\Omega; H)) \), and
\[
\|\varphi(\cdot)\|_{C_{\mathcal{F}}([t, T]; L^2(\Omega; H))} \leq C\left(\|\eta\|_{L^2_{\mathcal{F}_t}(\Omega; H)} + \|v_1(\cdot)\|_{L^1(t, T; L^2(\Omega; H))} + \|v_2(\cdot)\|_{L^2(t, T; H)}\right).
\]

If (2.6) admits a classical mild solution in Hu and Peng (1991); Al-Hussein (2009) (say, when \( \mathcal{F} \) is the natural filtration of the Brownian motion \( W(\cdot) \)), then by Itô’s formula, we have
\[
\mathbb{E}\langle \varphi(T), z_T \rangle_H - \mathbb{E} \int_t^T \langle \varphi(s), F(s, z(s), Z(s)) \rangle_H ds
\]
\[
= \mathbb{E}\langle \eta, z(t) \rangle_H + \mathbb{E} \int_t^T \langle v_1(s), z(s) \rangle_H ds + \mathbb{E} \int_t^T \langle v_2(s), Z(s) \rangle_H ds.
\]
(2.9)
Motivated by this, we introduce the following notion.

**Definition 2.2.** We call \((z(\cdot), Z(\cdot)) \in D_{\mathcal{F}}([0, T]; L^2(\Omega; H)) \times L^2_{\mathcal{F}}(0, T; H)\) a transposition solution to the equation (2.6) if for any \( t \in [0, T], \ v_1(\cdot) \in L^1(t, T; L^2(\Omega; H)), \ v_2(\cdot) \in L^2(t, T; H), \ \eta \in L^2_{\mathcal{F}_t}(\Omega; H) \) and the corresponding solution \( \varphi(\cdot) \in C_{\mathcal{F}}([t, T]; L^2(\Omega; H)) \) to (2.8), the equality (2.9) holds.

The space for the first component of the solution is chosen to be \( D_{\mathcal{F}}([0, T]; L^2(\Omega; H)) \) rather than \( C_{\mathcal{F}}([0, T]; L^2(\Omega; H)) \). This is quite natural because the filtration \( \mathcal{F} \) is assumed only to be right-continuous.

It is easy to see that if (2.6) has a mild solution, then such a solution is also a transposition solution to (2.6).

**Remark 2.1.** The above stochastic transposition method is stimulated by the classical transposition method to solve the non-homogeneous boundary value problems for deterministic partial differential equations (see, e.g., Lions and Magenes, 1972). This method is a variant of the standard duality method, and in some sense it provides a way to see something
which is not easy to be detected directly. Specifically, for the equation (2.6), the point of
this method is to interpret the solution to a backward stochastic evolution equation in terms
of a forward stochastic evolution equation which is well studied. The key tool to do this is
the Riesz type representation theorem for $L^p_T(0, T; L^p(\Omega; \mathcal{X}))$ ($1 \leq p, q < \infty$) (see Lü et al.,
2012, 2018). On the other hand, the equality (2.9) can be regarded as a variational for-
mulation of the equation (2.6). As pointed out in (Lü and Zhang, 2013, Remark 3.2), it
provides a way to solve the equation (2.6) numerically.

We have the following well-posedness result for the equation (2.6).

**Theorem 2.1** (Lü and Zhang (2014, Theorem 3.1)). For any $z_T \in L^2_{F_T}(\Omega; H)$, the equation
(2.6) admits a unique transposition solution $(z(\cdot), Z(\cdot)) \in D_T([0, T]; L^2(\Omega; H)) \times L^2_T(0, T; H)$.
Furthermore,

$$
\| (z(\cdot), Z(\cdot)) \|_{D_T([0, T]; L^2(\Omega; H)) \times L^2_T(0, T; H)} \leq C \left( \| z_T \|_{L^2_{F_T}(\Omega; H)} + \| F(\cdot, 0, 0) \|_{L^2_T(0, T; L^2(\Omega; H))} \right).
$$

From Theorem 2.1, it follows that both (2.4) and (2.5) are well-posed in the sense of
transposition solution. Then we can study the null controllability and optimal control
problems by (2.4) and (2.5), respectively.

We first consider the problem of finding the control which drives the state of the system
(2.1) to rest. To this end, define a functional on $L^2_{F_T}(\Omega; H)$ as follows:

$$
\mathcal{J}(z_T) = \frac{1}{2} \mathbb{E} \int_0^T \| B^* z(t) + D^* Z(t) \|^2 \, dt + \mathbb{E} \langle y_0, z(0) \rangle_H,
\forall z_T \in L^2_{F_T}(\Omega; H),
$$

(2.10)

where $(z(\cdot), Z(\cdot)) = (z(\cdot; z_T), Z(\cdot; z_T))$ is the transposition solution to the equation (2.4)
(corresponding to the final datum $z_T$).

The following result holds:

**Theorem 2.2** (Lü and Zhang (2020, Theorems 7.17 and 7.28)). If the system (2.1) is null
controllable at time $T$, then, among all controls transferring the state of (2.1) from $y_0$ to 0
at time $T$, the one given by

$$
u(t) = B^* z(t; \hat{z}_T) + D^* Z(t; \hat{z}_T), \quad \text{a.e. } (t, \omega) \in [0, T] \times \Omega
$$

(2.11)

has the minimal $L^2_T(0, T; U)$-norm, where $\hat{z}_T \in L^2_{F_T}(\Omega; H)$ is the minimizer of the functional
$\mathcal{J}(\cdot)$.

**Remark 2.2.** In this chapter, since we focus on the numerics, we will not consider problem
that when (2.1) is null controllable, which can be reduced to an observability estimate of the
equation (2.4) (e.g., Lü and Zhang, 2020, Theorem 7.17).

Theorem 2.2 provides an explicit formula of the desired control for null controllability
of (2.1). Therefore, to get such a control, one has to solve the following two problems:

(1) Finding the minimizer $\hat{z}_T$ of the functional $\mathcal{J}(\cdot)$;

(2) Solving (2.4) to obtain $B^* z(t; \hat{z}_T) + D^* Z(t; \hat{z}_T)$.

If we know how to solve the equation (2.4), then we can adopt the gradient method/Newton’s
method to find the minimizer of the functional $\mathcal{J}(\cdot)$ numerically. Hence, the key point to
compute the control (2.11) is how to solve the equation (2.4).

Next, we consider Problem (SLQ). With the aid of (2.5), we have the following result.
Theorem 2.3 (Lü and Zhang (2015, Theorem 5.2)). Let \((\bar{y}(\cdot), \bar{u}(\cdot))\) be an optimal pair of Problem (SLQ). Then, for the transposition solution \((z(\cdot), Z(\cdot))\) to (2.5) with \(y\) replaced by \(\bar{y}(\cdot)\), it holds that
\[
R(t)\bar{u}(t) - B(t)^\ast z(t) - D(t)^\ast Z(t) = 0, \quad \text{a.e. } (t, \omega) \in [0, T] \times \Omega. \tag{2.12}
\]

According to Theorem 2.3, by (2.1) and (2.5), when \(R(\cdot)^{-1} \in L_\mathcal{F}^\infty(0, T; \mathcal{L}(U))\), in order for finding the optimal pair \((\bar{y}(\cdot), \bar{u}(\cdot))\) of Problem (SLQ), it suffices to solve the following coupled forward-backward stochastic differential equation:
\[
\left\{ \begin{array}{l}
d\bar{y}(t) = (A\bar{y}(t) + BR^{-1}B^\ast z(t) + BR^{-1}D^\ast Z(t))dt \\
\quad + (C\bar{y}(t) + DR^{-1}B^\ast z(t) + DR^{-1}D^\ast Z(t))dW(t) \\
z(t) = -(A^\ast z(t) - Q\bar{y}(t) + C^\ast Z(t))dt + Z(t)dW(t)
\end{array} \right. \quad t \in (0, T], \tag{2.13}
\]

Nevertheless, a key point to solve (2.13) is to handle the backward equation (for \((z(\cdot), Z(\cdot))\)) therein.

From the above discussion, we see that it is very important to solve backward stochastic evolution equations in the study of control problems for stochastic evolution equations. Note that, for almost all backward stochastic evolution equations, it is impossible to find an explicit formula of the solutions. Hence, for practical applications, it is crucial to find numerical solutions to these equations.

3 The space of finite transposition

In the following two sections, based on the transposition method, we shall present the finite transposition method, introduced in (Wang and Zhang, 2011; Wang et al., 2020) for backward stochastic differential equations, to find numerical solutions for backward stochastic evolution equations. To present this method clearly, we only consider the following backward stochastic heat equation:
\[
\left\{ \begin{array}{l}
dz(t, x) = (-\Delta z(t, x) + F(t, x, z(t, x), Z(t, x))) dt + Z(t, x)dW(t), \quad (t, x) \in [0, T) \times D, \\
z(t, x) = 0, \quad (t, x) \in [0, T) \times \partial D, \\
z(T, x) = z_T(x), \quad x \in D, \tag{3.1}
\end{array} \right.
\]

where \(D \subset \mathbb{R}^\ell\) (for some \(\ell \in \mathbb{N}\)) is a bounded domain with a \(C^2\) boundary \(\partial D\).

Let us first introduce the following two assumptions:

(A1) \(F\) is the natural filtration generated by \(\{W(t)\}_{t \in [0, T]}\), \(z_T \in L_\mathcal{F}^2(\Omega; H_0^1(D))\), and \(F: [0, T] \times D \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}\) is \(1/2\)-Hölder continuous with respect to \(t\) and \(\bar{u}\) and \(\bar{y}\) continuous with respect to \(z\) and \(Z\). Moreover, \(F(\cdot, \cdot, 0, 0) \in L^2(0, T; H_0^1(D))\).

(A2) For the (equal time-interval) partition \(\pi: 0 = t_0 < t_1 < \cdots < t_N = T\) with \(N \in \mathbb{N}\), \(\tau = \frac{T}{N}\) and \(t_i = i\tau\) (for \(i = 0, 1, \cdots, N\)), there exists a constant \(C > 0\) such that the correction term \(Z\) in the equation (3.1) satisfies
\[
\sum_{n=0}^{N-1} \mathbb{E} \int_{t_n}^{t_{n+1}} \left( \|Z(t) - Z(t_{n+1})\|_{L^2(D)}^2 + \|Z(t) - Z(t_n)\|_{L^2(D)}^2 \right) dt \leq C\tau. \tag{3.2}
\]
Remark 3.1. (1) The filtration $F$ can be generalized to quasi-left continuous one, and $F(\cdot,\cdot,\cdot)$ can be stochastic. But we do not consider these general case to avoid technical complexity.

(2) In this chapter, we adopt the equipartition for simplicity. Our method can also be applied to quasi-uniform partition, i.e.
\[
\max_{k=0,1,\ldots,N-1} \{t_{k+1} - t_k\} \leq C \min_{k=0,1,\ldots,N-1} \{t_{k+1} - t_k\}.
\]

(3) In Zhang (2004), the estimate in the form of (3.2) is called the $L^2$-regularity of $Z(\cdot)$. Under some suitable conditions, (3.2) can be guaranteed (see, e.g., Wang, 2016, 2020). When $Z(\cdot)$ appears in the drift term, (3.2) is crucial to prove rates of convergence for temporal discretization. If $Z(\cdot)$ does not appear in the drift term, when proving the rate for $z(\cdot)$, we do not need the condition (3.2) (see Prohl and Wang, 2020b).

We shall use the following notations: For $k = 0, 1, \ldots, N - 1$, $t \in [t_k, t_{k+1}),$
\[
\Delta_{k+1}W = W(t_{k+1}) - W(t_k), \quad \nu(t) = t_k, \quad \mu(t) = t_{k+1}, \quad \pi(t) = k, \tag{3.3}
\]
and
\[
\nu(T) = \mu(T) = T, \quad \pi(T) = N. \tag{3.4}
\]

The numerical method (based on the transposition solutions to backward stochastic differential equations) given in Wang et al. (2020) (see also Lü and Zhang, 2013; Wang and Zhang, 2011) can be regarded as a Galerkin method, and hence we call it the finite transposition method. Let us propose below the outline of the finite transposition method to solve the equation (3.1):

(1) Determine a finite dimensional subspace $S$ of $L^2(D)$.

(2) Choose a finite transposition space $\mathbb{H} = \text{span}\{e_i\}$, which is a finite dimensional subspace of $L^2_{\mathbb{F}}(0,T;\mathbb{S})$ (in analogy with the finite element space of the finite element method in solving partial differential equations).

(3) Introduce the following variational equation:
\[
\mathbb{E}\langle \varphi(T), z_T \rangle_{L^2(D)} = \mathbb{E} \int_0^T \left[ \langle \varphi(t), F(\mu(t), \cdot, z(t), Z(t)) \rangle_{L^2(D)} + \langle v_1(t), z(t) \rangle_{L^2(D)} + \langle v_2(t), Z(t) \rangle_{L^2(D)} \right] dt, \tag{3.5}
\]
where
\[
\varphi(t) = \int_0^{\mu(t)} (\Delta \varphi(\nu(s)) + v_1(s)) ds + \int_0^{\nu(t)} v_2(s) dW(s),
\]
with $v_1(\cdot)$ and $v_2(\cdot)$ being stochastic processes in suitable finite dimensional subspaces $\mathbb{H}_1 \subset L^2_{\mathbb{F}}(t,T;L^2(\Omega;\mathbb{S}))$ and $\mathbb{H}_2 \subset L^2_{\mathbb{F}}(0,T;\mathbb{S})$, respectively, and $\nu(\cdot)$ and $\mu(\cdot)$ being piecewise constant functions defined in (3.4). Based on this variational equation, one can prove the existence and uniqueness of approximate solution $(z(\cdot), Z(\cdot))$, called the finite transposition solution, to the equation (3.1) in the form
\[
z(\cdot) = \sum_{i=1}^{\dim(\mathbb{H}_1)} \alpha_i e_i, \quad Z(\cdot) = \sum_{i=1}^{\dim(\mathbb{H}_2)} \beta_i e_i.
\]
Find coefficients $\alpha_i, \beta_i$ of the finite transposition solution via the variational equation (3.5), and prove the rate of convergence.

**Remark 3.2.** There is a useful method — the stochastic finite element method — to solve partial differential equations with random parameter (see, e.g., Ghanem and Spanos, 1991). A main ingredient in this method is the orthogonal expansions, such as polynomial chaos expansion, Karhunen-Loève expansion, etc. Let $\{e_i\}_{i=1}^{\infty}$ be an orthogonal basis for $L^2_{\mathcal{F}_T}(\Omega; L^2(D))$. Then for any stochastic process $X(\cdot)$, there exists an expansion

$$X(\cdot) = \sum_{i=1}^{\infty} x_i(\cdot)e_i.$$  

Since the solutions $(z(\cdot), Z(\cdot))$ to backward stochastic evolution equations are progressively measurable stochastic processes, people need to find the orthogonal basis of $L^2_{\mathcal{F}_T}(0, T; L^2(D))$. However, it seems that there is no simple explicit orthogonal basis for such a Hilbert space. Note that $L^2_{\mathcal{F}_T}(0, T; L^2(D))$ is much more complicated than $L^2_{\mathcal{F}_T}(\Omega; L^2(D))$, and hence the classical "stochastic finite element method" does not work for our problem.

Now we give details for the above outline of the finite transposition method.

Firstly, we determine a finite dimensional subspace of $L^2(D)$ based on the Galerkin method. Define

$$A : D(A) = H^1_0(D) \cap H^2(D) \to L^2(D), \quad A\varphi = \Delta \varphi, \quad \forall \varphi \in D(A). \tag{3.6}$$

Let $\{(-\lambda_i, \phi_i)\}_{i=1}^{\infty}$ be the sequence of eigenvalues and eigenfunctions of operator $A$, such that $\|\phi_i\|_{L^2(D)} = 1$ for $i \in \mathbb{N}$. Then $\{\phi_i\}_{i=1}^{\infty}$ constitutes an orthonormal basis of $L^2(D)$ and $\{\phi_i\}_{i=1}^{\infty}$ is also an orthogonal basis of $H^1_0(D)$. Take the subspace

$$\mathbb{S}_m = \text{span}\{\phi_1, \cdots, \phi_m\}, \quad m \in \mathbb{N}. \tag{3.7}$$

Secondly, we construct a finite transposition space by the Wiener chaos. To this end, we review some notations and results on Wiener chaos (see Nualart, 2006, for more details). Recall that $\mathcal{F}_T = \sigma\{W(t); 0 \leq t \leq T\}$. Define the Itô isometry $\mathbb{W} : L^2(0, T) \to L^2_{\mathcal{F}_T}(\Omega)$ by

$$\mathbb{W}(h) = \int_0^T h(t)dW(t).$$

For $x \in \mathbb{R}$, let

$$H_n(x) = \begin{cases} 
\frac{(-1)^n}{n!} e\frac{x^2}{2} \frac{d^n}{dx^n} (e^{-\frac{x^2}{2}}), & n > 0, \\
1, & n = 0
\end{cases}$$

be the $n$-th Hermite polynomial. Denote by $\mathcal{I}$ the set of all sequences $\alpha = (\alpha_1, \alpha_2, \cdots)$, $\alpha_i \in \mathbb{N}_0$, such that all the terms, except a finite number of them, vanish. For $k \in \mathbb{N}$, let $\mathcal{I}(k)$ be the set of all sequences $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_k)$, $\alpha_i \in \mathbb{N}_0$. For all $\alpha \in \mathcal{I}$ (resp. $\alpha \in \mathcal{I}(k)$), write

$$\alpha! \equiv \prod_{i=1}^{\infty} \alpha_i! \quad (\text{resp.} \prod_{i=1}^{k} \alpha_i!), \quad \text{and} \quad |\alpha| \equiv \sum_{i=1}^{\infty} \alpha_i \quad (\text{resp.} \sum_{i=1}^{k} \alpha_i),$$

and for $x = (x_1, x_2, \cdots) \in \mathbb{R}^\infty$,

$$H_{\alpha}(x) = \prod_{i=1}^{\infty} H_{\alpha_i}(x_i), \quad \alpha \in \mathcal{I} \quad \left(\text{resp.} \prod_{i=1}^{k} H_{\alpha_i}(x_i), \quad \alpha \in \mathcal{I}(k)\right),$$

where $H_{\alpha}(x) = \prod_{i=1}^{k} H_{\alpha_i}(x_i), \quad \alpha \in \mathcal{I}(k)$. 

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which is called the generalized Hermite polynomial. Also, we put
\[ g_i(\cdot) = \frac{X(t_{i-1}, t_i)(\cdot)}{\sqrt{\tau}}, \quad i = 1, 2, \ldots, N. \]
Furthermore, for any \( k = 1, 2, \ldots, N \), define the Wiener chaos of order \( n \) in \( H \) as follows:
\[ \mathcal{H}_n(k; H) \overset{\Delta}{=} \text{span} \left\{ \prod_{i=1}^{k} H_{\alpha_i}(\mathcal{W}(g_i))\phi \mid \alpha \in \mathcal{I}(k), |\alpha| = n, \phi \in H \right\}. \]
Set \( \mathcal{H}_n(0; H) = H \). Take \( \mathcal{G}_0 = \mathcal{B}(\mathbb{R}) \), and for \( k = 1, 2, \ldots, N \), define
\[ \mathcal{G}_k \overset{\Delta}{=} \sigma\{\Delta_1 W, \Delta_2 W, \ldots, \Delta_k W\}, \]
where \( \Delta_k W \) is defined in (3.3). By virtue of the Wiener chaos of order \( n \), we have the following orthogonal decomposition result.

**Theorem 3.1** (Nualart (2006, Theorem 1.1.1)). For any \( N, m \in \mathbb{N} \), and \( k = 0, 1, \ldots, N \), it holds that
\[ L^2_{\mathcal{G}_k}(\Omega; \mathbb{S}_m) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n(k; \mathbb{S}_m). \]

Based on Theorem 3.1, we set
\[ \mathcal{H}^M(k; \mathbb{S}_m) \overset{\Delta}{=} \bigoplus_{n=0}^{M} \mathcal{H}_n(k; \mathbb{S}_m), \]
\[ \mathcal{H}^M_N(k; \mathbb{S}_m) \overset{\Delta}{=} \text{span} \left\{ \chi_{[t_k, t_{k+1})}(\cdot)\xi \mid \xi \in \mathcal{H}^M(k; \mathbb{S}_m) \right\}, \]
and the desired finite transposition space as follows:
\[ \mathbb{H}_{N, M}(\mathbb{S}_m) \overset{\Delta}{=} \bigoplus_{k=0}^{N-1} \bigoplus_{n=0}^{M} \mathcal{H}^M_N(k; \mathbb{S}_m) \subset L^2_T(0, T; \mathbb{S}_m). \tag{3.8} \]
For simplicity, we denote \( \mathcal{H}_n(k; \mathbb{R}) \), \( \mathcal{H}^M(k; \mathbb{R}) \), \( \mathcal{H}^M_N(k; \mathbb{R}) \), and \( \mathbb{H}_{N, M}(\mathbb{R}) \) by \( \mathcal{H}_n(k) \), \( \mathcal{H}_n^M(k) \), \( \mathcal{H}_n^M_N(k) \) and \( \mathbb{H}_{N, M}^M(k) \), respectively. We write \( \mathcal{H}^M(k) \otimes^n \mathbb{R} \) and \( \mathbb{H}_{N, M}^M(k) \otimes^n \mathbb{R} \) for the \( n \)-copies of \( \mathcal{H}^M(k) \) and \( \mathbb{H}_{N, M}(\mathbb{R}) \), respectively.

**Remark 3.3.** In Theorem 3.1, we only list a decomposition of \( L^2_{\mathcal{G}_k}(\Omega; \mathbb{S}_m) \), which is a subspace of \( L^2_T(\Omega; L^2(D)) \). For the latter space, by letting
\[ \mathcal{H}_n(L^2(D)) \overset{\Delta}{=} \text{span} \left\{ \prod_{i=1}^{\infty} H_{\alpha_i}(\mathcal{W}(g_i))\phi_j \mid \alpha \in \mathcal{I}, |\alpha| = n, \{g_i\}_{i=1}^{\infty} \text{ is an orthonormal basis of } L^2(0, T), \{\phi_j\}_{j=1}^{\infty} \text{ is an orthonormal basis of } L^2(D) \right\}, \]
we have
\[ L^2_{\mathcal{G}_k}(\Omega; L^2(D)) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n(L^2(D)). \]
For the projection operator
\[ \Gamma_M : L^2_{\mathcal{G}_k}(\Omega; L^2(D)) \to \bigoplus_{n=0}^{M} \mathcal{H}_n(L^2(D)), \tag{3.9} \]
it is easy to check that \( \Gamma_M(L^2_{\mathcal{G}_k}(\Omega; \mathbb{S}_m)) = \mathcal{H}^M(N; \mathbb{S}_m) \). This fact is crucial in the sequel.
Remark 3.4. By the construction of the finite transposition space $\mathbb{H}_{N,M}$, it is easy to see that the orthonormal basis of $H^M(k)$ is
\[
\{h_{k,i}\}_{i=1}^{M_k} = \left\{ \sqrt{\alpha!} \prod_{i=1}^{k} H_{\alpha_i}(\mathbb{W}(g_i)) \mid \alpha \in I(k), |\alpha| \leq M \right\},
\]
which is a finite dimensional subspace of $L^2_{\text{bit}_k}(\Omega)$. Note that $M_k$ depends on $M$. For example, it is easy to check that
\[
M_k = \begin{cases} 
\frac{(k+1)(k+2)}{2}, & M = 2, \\
\frac{k^3 + 8k^2 + 19k + 6}{6}, & M = 3.
\end{cases}
\]
Hence the orthonormal basis of $\mathbb{H}_{N,M}$ is
\[
\{e_{k,i}(\cdot)\}_{k,i} = \left\{ x_{[t_k,t_{k+1})}(\cdot) \frac{1}{\sqrt{\tau}} h_{k,i} \mid 1 \leq i \leq M_k, 0 \leq k \leq N - 1 \right\},
\]
which is also a finite dimensional subspace of $L^2_\mathbb{P}(0,T)$.

By virtue of $\{h_{k,i}\}_{i=1}^{M_k}, \{e_{k,i}(\cdot)\}_{1 \leq i \leq M_k, 0 \leq k \leq N - 1}$ and $\{\phi_j\}_{j=1}^m$, we can obtain the orthonormal basis of $H^M(k;\mathbb{S}_m)$ and $\mathbb{H}_{N,M}(\mathbb{S}_m)$.

Remark 3.5. In this chapter, since we mainly focus on presenting the idea of handling the difficulties caused by the stochastic setting, we choose the finite dimensional subspace of $L^2(D)$ given by (3.7). One can also choose that space as the one in other numerical methods for solving deterministic partial differential equations, such as the finite element space.

For the infinite dimensional space of $L^2_\mathbb{P}(0,T)$, subspace constructed by characteristic functions instead of Wiener chaos is also a choice (e.g., Dai et al., 2017).

Remark 3.6. A Wiener chaos decomposition method is proposed in Briand and Labart (2014). It should be pointed out that, although in both Briand and Labart (2014) and the present chapter, some tools of Wiener chaos expansion are used, the numerical method in Briand and Labart (2014) (based on Malliavin analysis) is totally different from the finite transposition method in this chapter (based on the variational equation (3.5)).

4 Finite transposition method for backward stochastic evolution equations

Based on the finite transposition space $\mathbb{H}_{N,M}(\mathbb{S}_n)$, we can provide the definition of the finite transposition solution.

Definition 4.1. For any $N, M, n \in \mathbb{N}$, a couple $(\zeta(\cdot), Z(\cdot)) \in \mathbb{H}_{N,M}(\mathbb{S}_n) \times \mathbb{H}_{N,M-1}(\mathbb{S}_n)$ is called a finite transposition solution to the equation (3.1), if for any $v_1(\cdot) \in \mathbb{H}_{N,M}(\mathbb{S}_n)$ and $v_2(\cdot) \in \mathbb{H}_{N,M-1}(\mathbb{S}_n)$, the following variational equation holds
\[
\mathbb{E}\langle \varphi(T), z_{\mathbb{N},T}^2 \rangle_{L^2(D)} = \mathbb{E} \int_0^T \left[ \langle \varphi(t), F(\mu(t), \cdot, z(t), Z(t)) \rangle_{L^2(D)} + \langle v_1(t), z(t) \rangle_{L^2(D)} + \langle v_2(t), Z(t) \rangle_{L^2(D)} \right] dt.
\]
(4.1)
Here $z_{n,T}^k$ is an approximation of $z_T$, and $\varphi(\cdot)$ solves the following forward equation:

$$
\begin{aligned}
\varphi(t_k) &= \int_0^{t_k} (A\varphi(t) + v_1(t)) dt + \int_0^{t_k} v_2(t)dW(t), \quad k = 1, 2, \ldots, N - 1, \\
\varphi(0) &= 0, \\
\varphi(t) &= \varphi(\nu(t)), \quad t \in (0, T),
\end{aligned}
$$

(4.2)

where $A$ is defined in (3.6).

The finite transposition method for equation (3.1) is the algorithm to obtain the finite transposition solution by the variational equation (4.1).

The rest of this section is divided into two subsections. In the first one, we discretize the original equation with respect to space variables. In the second one, we discretize the obtained equation with respect to the time variable.

### 4.1 Spacial discretization of (3.1)

Denote by $\Pi_n$ the orthogonal projection from $L^2(D)$ to $S_n$ and define $A_n$ by $A_n = A|S_n$. The semi-discrete problem in space corresponding to the equation (3.1) is to find a pair $(z_n(\cdot), Z_n(\cdot)) \in L^2_2(\Omega; C([0,T];S_n)) \times L^2_2(0,T;S_n)$ solving the following equation:

$$
\begin{aligned}
dz_n(t) &= (-A_n z_n(t) + \Pi_n F(t, z_n(t), Z_n(t)))dt + Z_n(t)dW(t), \quad t \in [0,T], \\
z_n(T) &= \Pi_n z_T.
\end{aligned}
$$

(4.3)

The following result states the rate of convergence for the solution of the equation (4.3) to the one of the equation (3.1).

**Theorem 4.1** (Wang (2013, Theorem 5.4), Wang (2016, Theorem 3.1)). Suppose that (A1) holds. Let $(z(\cdot), Z(\cdot))$ and $(z_n(\cdot), Z_n(\cdot))$ be solutions to the equations (3.1) and (4.3), respectively. Then the following estimate holds

$$
\|z(\cdot) - z_n(\cdot)\|_{L^2_2(\Omega; C([0,T];L^2(D)))} + \|Z(\cdot) - Z_n(\cdot)\|_{L^2_2(0,T;L^2(D))} \\
\leq \frac{C}{\lambda_{n+1}} \left(\|z_T(\cdot)\|_{L^2_2(\Omega; H^1_0(D))}^2 + \|F(\cdot, \cdot, 0, 0)\|_{L^2_2(0,T;H^1_0(D))}^2\right).
$$

As we have explain in Remark 3.5, to figure out the main part, $S_n$ in the finite transposition space $H_{N,M}(S_n)$ is constructed by the first $n$ eigenfunctions of the operator $A$. We can also choose other finite element spaces $V_h$, and construct the finite transposition space $H_{N,M}(V_h)$. The readers are referred to Dunst and Prohl (2016); Prohl and Wang (2020b) for error estimates for finite-element based spacial discretization.

### 4.2 Temporal discretization of (4.3)

In this part, firstly we rewrite the definition of the finite transposition solution and the finite transposition method for the backward stochastic differential equation (4.3). Then we present the existence and uniqueness of the finite transposition solution. Finally, we give the rate of convergence.
Since $S_n = \text{span}(\phi_1, \ldots, \phi_n)$, and the solution $(z_n(\cdot), Z_n(\cdot))$ to the equation (4.3) is in the space $L_2^2(\Omega; C([0, T]; S_n)) \times L_2^2(0, T; S_n)$, we may take $(z_n(\cdot), Z_n(\cdot))$ to be the following form

\[
z_n(\cdot) = \sum_{j=1}^n a_{n,j}(\cdot) \phi_j, \quad Z_n(\cdot) = \sum_{j=1}^n b_{n,j}(\cdot) \phi_j,
\]

where $a_{n,j}(\cdot) \in L_2^2(\Omega; C([0, T]))$ and $b_{n,j}(\cdot) \in L_2^2(0, T)$, for $j = 1, 2, \ldots, n$. Set

\[
a_n(\cdot) = \begin{pmatrix} a_{n,1}(\cdot) \\ a_{n,2}(\cdot) \\ \vdots \\ a_{n,n}(\cdot) \end{pmatrix}, \quad b_n(\cdot) = \begin{pmatrix} b_{n,1}(\cdot) \\ b_{n,2}(\cdot) \\ \vdots \\ b_{n,n}(\cdot) \end{pmatrix}, \quad \Lambda_n = \begin{pmatrix} -\lambda_1 & 0 & \cdots & 0 \\ 0 & -\lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -\lambda_n \end{pmatrix},
\]

and

\[
F_n(\cdot, a_n(\cdot), b_n(\cdot)) = \begin{pmatrix} \langle F(\cdot, z_n(\cdot), Z_n(\cdot)), \phi_1 \rangle_{L_2^2(D)} \\ \langle F(\cdot, z_n(\cdot), Z_n(\cdot)), \phi_2 \rangle_{L_2^2(D)} \\ \vdots \\ \langle F(\cdot, z_n(\cdot), Z_n(\cdot)), \phi_n \rangle_{L_2^2(D)} \end{pmatrix}.
\]

Then $(a_n(\cdot), b_n(\cdot))$ solves the following backward stochastic differential equation:

\[
\begin{cases}
da_n(t) = ( -\Lambda_n a_n(t) + F_n(t, a_n(t), b_n(t))) dt + b_n(t) dW(t), & t \in [0, T], \\
a_n(T) = (\langle z_T, \phi_1 \rangle_{L_2^2(D)}, \langle z_T, \phi_2 \rangle_{L_2^2(D)}, \ldots, \langle z_T, \phi_n \rangle_{L_2^2(D)})\top.
\end{cases}
\]

Definition 4.2. For any $N, M, n \in \mathbb{N}$, a couple $(a_{n,N,M}(\cdot), b_{n,N,M-1}(\cdot)) \in H_{N,M}^\otimes n \times H_{N,M-1}^\otimes n$ is called a finite transposition solution to the backward stochastic differential equation (4.5) if for any $v_1(\cdot) \in H_{N,M}^\otimes n$ and $v_2(\cdot) \in H_{N,M-1}^\otimes n$, the following variational equation holds

\[
\mathbb{E}\langle x(T), a_n^\pi, t \rangle = \mathbb{E}\int_0^T \left[ \langle x(t), F_n(\mu(t), a_n,N,M(t), b_n,N,M-1(t)) \rangle + \langle v_1(t), a_n,N,M(t) \rangle + \langle v_2(t), b_n,N,M-1(t) \rangle \right] dt,
\]

where $x(\cdot)$ is given by

\[
\begin{cases}
x(t_k) = \int_0^{t_{k+1}} (\Lambda_n x(\nu(t)) + v_1(t)) dt + \int_0^{t_k} v_2(t) dW(t), & k = 1, \ldots, N - 1, \\
x(0) = 0, \\
x(t) = x(\nu(t)), & t \in (0, T],
\end{cases}
\]

and $a_n^\pi, t \in (H_{N,M}^\otimes n)$ is an approximation of $a_n(T)$.

By choosing $a_n^\pi, t = (\langle z_N^\pi, T, \phi_1 \rangle_{L_2^2(D)}, \langle z_N^\pi, T, \phi_2 \rangle_{L_2^2(D)}, \ldots, \langle z_N^\pi, T, \phi_n \rangle_{L_2^2(D)})\top$, and combining Definitions 4.1 and 4.2, we can see that the pair of stochastic processes $(\sum_{j=1}^n (a_{n,N,M}(\cdot))_j \phi_j, \sum_{j=1}^n b_{n,N,M-1}(\cdot)_j \phi_j)$ is just the finite transposition solution to (3.1), where $(a_{n,N,M}(\cdot))_j$ and $(b_{n,N,M-1}(\cdot))_j$ are the $j$-th component of $a_{n,N,M}(\cdot)$ and $b_{n,N,M-1}(\cdot)$ respectively. Based on this, from now on, we study (4.6) instead of (4.1).

By Remark 3.4, we can denote the orthonormal basis of $H_{N,M}^\otimes n$ by

\[
\left\{ e_{k,i}^\pi, \cdot \right\} \quad 1 \leq \ell \leq n, \quad 1 \leq i \leq M_k, \quad 0 \leq k \leq N - 1.
\]
In what follows, we would apply the variational equation (4.2) of the finite transposition space $H_{N,M}$. By Definition 4.1, the following explicit algorithm:

$$\text{Algorithm. We can also provide other schemes based on the finite transposition method, such as the following explicit algorithm:}$$

$$\mathbb{E}\langle x(T), a_{n,T}^n \rangle = \mathbb{E} \int_0^T \left[ \langle x(t), F_n(\mu(t), a_{n,N,M}(\mu(t)), b_{n,N,M-1}(\mu(t))) \rangle \right. $$

$$+ \langle v_1(t), a_{n,N,M}(t) \rangle + \langle v_2(t), b_{n,N,M-1}(t) \rangle \big] \, dt.$$ 

By Remark 3.4, we know that $\{e_{k,i}(\cdot) | 1 \leq i \leq M_k, 0 \leq k \leq N - 1\}$ is an orthonormal basis of the finite transposition space $H_{N,M}$. Hence, by Theorem 4.2, we can write

$$a_{n,N,M}(\cdot) = \sum_{\ell=1}^{n} \sum_{k=0}^{N-1} \sum_{i=1}^{M_k} \alpha_{k,i}^\ell e_{k,i}^\ell(\cdot),$$

$$\alpha_{k,i}^\ell \in \mathbb{R} \text{ for } i = 1, \ldots, M_k, \ k = 0, \ldots, N - 1, \ \ell = 1, \ldots, n,$$

$$b_{n,N,M-1}(\cdot) = \sum_{\ell=1}^{n} \sum_{k=0}^{N-1} \sum_{i=1}^{M_k} \beta_{k,i}^\ell e_{k,i}^\ell(\cdot),$$

$$\beta_{k,i}^\ell \in \mathbb{R} \text{ for } i = 1, \ldots, M_k, \ k = 0, \ldots, N - 1, \ \ell = 1, \ldots, n.$$ 

In what follows, we would apply the variational equation (4.6) to determine the coefficients of the finite transposition solution $(a_{n,N,M}(\cdot), b_{n,N,M-1}(\cdot))$.

By choosing $v_1(\cdot) = e_{k,i}^\ell(\cdot)$ and $v_2(\cdot) = 0$, we obtain

$$x(t) = \begin{cases} 0, & t \in [0, t_k), \\ \Lambda_0^{\pi(t)-k+1} \sqrt{\tau} h_{k,i}^\ell, & t \in [t_k, T]. \end{cases}$$

Here

$$\Lambda_0 = (I_n - \Lambda_n \tau)^{-1}.$$ 

By virtue of (4.6), we have

$$\mathbb{E}\langle \Lambda_0^{N-k} \sqrt{\tau} h_{k,i}^\ell, a_{n,T}^n \rangle$$

$$= \mathbb{E} \int_{t_k}^T \left\langle \Lambda_0^{\pi(t)-k+1} \sqrt{\tau} h_{k,i}^\ell, F_n(\mu(t), a_{n,N,M}(t), b_{n,N,M-1}(t)) \right\rangle \, dt$$

$$+ \frac{1}{\sqrt{\tau}} \mathbb{E} \int_{t_k}^{t_{k+1}} \left\langle h_{k,i}^\ell, a_{n,N,M}(t) \right\rangle \, dt.$$
\[
\begin{align*}
&= \mathbb{E} \int_{t_k}^{T} \left\langle \Lambda_0^{\pi(t)-k+1} \sqrt{\tau} h_{k,i}^{\ell}, F_n(\mu(t), a_{n,M}(t), b_{n,N,M-1}(t)) \right\rangle \, dt \\
&\quad + \frac{1}{\tau} \mathbb{E} \int_{t_k}^{T+k+1} \left\langle h_{k,i}^{\ell}, \sum_{m=1}^{n} \sum_{l=0}^{M_l} \sum_{j=1}^{N-1} \alpha_{l,j}^{m} X_{[t_l,t_{l+1}]}(t) h_{l,j}^{m} \right\rangle \, dt \\
&= \mathbb{E} \int_{t_k}^{T} \left\langle \Lambda_0^{\pi(t)-k+1} \sqrt{\tau} h_{k,i}^{\ell}, F_n(\mu(t), a_{n,M}(t), b_{n,N,M-1}(t)) \right\rangle \, dt + \alpha_{k,i}^{\ell},
\end{align*}
\]

which implies
\[
\alpha_{k,i}^{\ell} = \mathbb{E} \left\langle \Lambda_0^{N-k} \sqrt{\tau} h_{k,i}^{\ell}, a_{n,T}^{\pi} \right\rangle - \mathbb{E} \int_{t_k}^{T} \left\langle \Lambda_0^{\pi(t)-k+1} \sqrt{\tau} h_{k,i}^{\ell}, F_n(\mu(t), a_{n,M}(t), b_{n,N,M-1}(t)) \right\rangle \, dt. \tag{4.9}
\]

In the same vein, by taking \( v_1 = 0, v_2 = e_{k,i}^{\ell} \), we see that
\[
x(t) = \begin{cases} 
0, & t \in [0, t_k+1), \\
\Lambda_0^{\pi(t)-k} \Delta_{k+1} W \sqrt{\tau} h_{k,i}^{\ell}, & t \in [t_k+1, T],
\end{cases}
\]

and
\[
\mathbb{E} \left\langle \Lambda_0^{N-k} \frac{\Delta_{k+1} W \sqrt{\tau}}{\sqrt{\tau}} h_{k,i}^{\ell}, a_{n,T}^{\pi} \right\rangle
\]
\[
= \mathbb{E} \int_{t_k}^{T} \left\langle \Lambda_0^{\pi(t)-k} \frac{\Delta_{k+1} W \sqrt{\tau}}{\sqrt{\tau}} h_{k,i}^{\ell}, F_n(\mu(t), a_{n,M}(t), b_{n,N,M-1}(t)) \right\rangle \, dt \\
+ \frac{1}{\tau} \mathbb{E} \int_{t_k}^{T+k+1} \left\langle h_{k,i}^{\ell}, b_{n,N,M-1}(t) \right\rangle \, dt \\
= \mathbb{E} \int_{t_k}^{T} \left\langle \Lambda_0^{\pi(t)-k} \frac{\Delta_{k+1} W \sqrt{\tau}}{\sqrt{\tau}} h_{k,i}^{\ell}, F_n(\mu(t), a_{n,M}(t), b_{n,N,M-1}(t)) \right\rangle \, dt \\
+ \frac{1}{\tau} \mathbb{E} \int_{t_k}^{T+k+1} \left\langle h_{k,i}^{\ell}, \sum_{m=1}^{n} \sum_{l=0}^{M_l} \sum_{j=1}^{N-1} \beta_{l,j}^{m} X_{[t_l,t_{l+1}]}(t) h_{l,j}^{m} \right\rangle \, dt \\
= \mathbb{E} \int_{t_k}^{T} \left\langle \Lambda_0^{\pi(t)-k} \frac{\Delta_{k+1} W \sqrt{\tau}}{\sqrt{\tau}} h_{k,i}^{\ell}, F_n(\mu(t), a_{n,M}(t), b_{n,N,M-1}(t)) \right\rangle \, dt + \beta_{k,i}^{\ell}.
\]

It follows that
\[
\beta_{k,i}^{\ell} = \mathbb{E} \left\langle \Lambda_0^{N-k} \frac{\Delta_{k+1} W \sqrt{\tau}}{\sqrt{\tau}} h_{k,i}^{\ell}, a_{n,T}^{\pi} \right\rangle - \mathbb{E} \int_{t_k}^{T} \left\langle \Lambda_0^{\pi(t)-k} \frac{\Delta_{k+1} W \sqrt{\tau}}{\sqrt{\tau}} h_{k,i}^{\ell}, F_n(\mu(t), a_{n,M}(t), b_{n,N,M-1}(t)) \right\rangle \, dt. \tag{4.10}
\]

By virtue of (4.9) and (4.10), we can obtain the finite transposition solution \((a_{n,M}(\cdot), b_{n,N,M-1}(\cdot))\) to the backward stochastic differential equation (4.5). Applying Wiener chaos expansion, Theorem 4.3 given below proposes the convergence rate of the finite transposition method, and also shows the relationship between the Euler method and the finite transposition one. The following space is needed in the proof:
\[
D((-A)^{3/2}) = \left\{ \varphi = \sum_{i=1}^{\infty} \varphi_i \phi_i \left| \sum_{i=1}^{\infty} |\varphi_i|^2 \lambda_i^3 < \infty \right. \right\}.
\]
Theorem 4.3. Assume (A1) and (A2), and suppose that for given \( N, M, n \in \mathbb{N} \), \( (a_n(\cdot), b_n(\cdot)) \in L^2_p(\Omega; C([0, T]; \mathbb{R}^n)) \times L^2_p(0, T; \mathbb{R}^n) \) and \( (a_{nN,M}(\cdot), b_{nN,M-1}(\cdot)) \in H^m(\mathbb{R}^n) \times H^{m-1}(\mathbb{R}^n) \) are adapted solution and finite transposition solution to the backward stochastic differential equation (4.5), respectively. Then,

\[
a_{nN,M}(\cdot) = \sum_{k=0}^{N-1} \chi_{[t_k, t_{k+1})}(\cdot) \left[ E \left( \Lambda_0^{N-k} a_{\pi, t} \big| \mathcal{F}_{t_k} \right) \right]
- E \left( \int_{t_k}^T \Lambda_0^{\pi(t)-k+1+N} \Gamma_M F_n(\mu(t), a_{nN,M}(t), b_{nN,M-1}(t)) dt \bigg| \mathcal{F}_{t_k} \right), \tag{4.11}
\]

\[
b_{nN,M-1}(\cdot) = \sum_{k=0}^{N-1} \chi_{[t_k, t_{k+1})}(\cdot) E \left( \Delta \mathcal{F}_{\pi, t}^{N-k+1} a_{nN,M}(t_{k+1}) \big| \mathcal{F}_{t_k} \right), \tag{4.12}
\]

where \( \Lambda_0 = (I_n + \Lambda_0^\tau)^{-1} \). Furthermore, the following rate of convergence holds true:

\[
\sup_{0 \leq t \leq T} E|a_{nN,M}(t) - a_n(t)|^2 + E \int_0^T |b_{nN,M-1}(t) - b_n(t)|^2 dt \leq C \left[ E|a_n(T) - \bar{a}_n(T)|^2 + \lambda^2 \tau + E \int_0^T |(I_n - \Gamma_M) F_n(s, a_n(s), b_n(s))|^2 ds \right], \tag{4.13}
\]

where \( \Gamma_M \) is defined in (3.9). Moreover, if \( F(\cdot, \cdot, \cdot) \) is linear with respect to the last two components, and \( z_T \in L^2_{\mathcal{F}_T}(\Omega; D((-A)^{3/2})) \), \( F(\cdot, \cdot, 0, 0) \in L^2(0, T; H^1(\mathbb{R}^n)) \), then it holds that

\[
\sup_{0 \leq t \leq T} E|a_{nN,M}(t) - a_n(t)|^2 + E \int_0^T |b_{nN,M-1}(t) - b_n(t)|^2 dt \leq C \left[ E|a_n(T) - \bar{a}_n(T)|^2 + \tau \right]. \tag{4.14}
\]

Proof. The proof is long, and we carry out it by the following three steps.

Step 1. In this step, we prove (4.11). By (4.8) and (4.9), we know that

\[
a_{nN,M}(\cdot) = \sum_{k=0}^{n-1} \sum_{k=0}^{M_k} \sum_{i=1}^{M_k} \left[ E \left( \Lambda_0^{N-k} \sqrt{\tau} h_{k,i}^{\pi} a_{nN,M} \right) \right]
- E \left( \int_{t_k}^T \Lambda_0^{\pi(t)-k+1+N} \Gamma_M F_n(\mu(t), a_{nN,M}(t), b_{nN,M-1}(t)) dt \bigg| \mathcal{F}_{t_k} \right) \times e_k(\cdot).
\]

For the first term on the right side of the above equality, we have that

\[
E \left( \Lambda_0^{N-k} \sqrt{\tau} h_{k,i}^{\pi} a_{nN,M} \right) e_k(\cdot) = E \left( \Lambda_0^{N-k} \sqrt{\tau} h_{k,i}^{\pi} a_{nN,M} \right) \chi_{[t_k, t_{k+1})}(\cdot) \frac{1}{\sqrt{\tau}} h_{k,i}^{\pi} \tag{4.15}
\]

By the definition of \( L^2_{\mathcal{G}_T}(\Omega; \mathbb{R}^n) \), \( H^M(k; \mathbb{R}^n) \) and Remark 3.4, we can extend the orthonormal basis \( \{ h_{k,i} | 1 \leq i \leq n, 1 \leq i \leq M_k \} \) of \( (H^M(k))^{\otimes n} \) to the orthonormal basis \( \{ h_{k,i} | 1 \leq i \leq n, 1 \leq i \leq \infty \} \) of \( L^2_{\mathcal{G}_T}(\Omega; \mathbb{R}^n) \). Since for any \( \eta \in L^2_{\mathcal{G}_T}(\Omega; \mathbb{R}^n) \), \( \Gamma_M \eta \in (H^M(N))^{\otimes n} \),

\[
E(\Gamma_M \eta \big| \mathcal{F}_{t_k}) \in (H^M(N))^{\otimes n} \cap L^2_{\mathcal{G}_T}(\Omega; \mathbb{R}^n) = (H^M(k))^{\otimes n}. \tag{4.16}
\]
By (4.15), (4.16) and the fact that \( \{h_{k,i}^\ell \mid 1 \leq \ell \leq n, 1 \leq i \leq M_k\} \) is an orthonormal basis of \( (\mathcal{H}^M(k))^\otimes_n \) and \( a_{n,T}^\pi \in (\mathcal{H}^M(N))^\otimes_n \), we see that

\[
\sum_{\ell=1}^n \sum_{k=0}^{N-1} \sum_{\pi=1}^{M_k} \mathbb{E} \left\langle \Lambda_0^{N-k} \sqrt{\tau h_{k,i}^\ell}, a_{n,T}^\pi \right\rangle e_{k,i}^\ell(\cdot) = \sum_{k=0}^{N-1} \chi_{[t_k,t_{k+1}]}(\cdot) \Lambda_0^{N-k} \mathbb{E} \left( a_{n,T}^\pi \mid \mathcal{F}_{t_k} \right). \tag{4.17}
\]

Similarly,

\[
\sum_{\ell=1}^n \sum_{k=0}^{N-1} \sum_{\pi=1}^{M_k} \mathbb{E} \int_{t_k}^T \left\langle \Lambda_0^{(\pi(t)-k)1} \sqrt{\tau h_{k,i}^\ell}, F_n \right\rangle (\mu(t), a_{n,N,M}(t), b_{n,N,M-1}(t)) \right) dt e_{k,i}^\ell(\cdot)
\]

\[
= \sum_{\ell=1}^n \sum_{k=0}^{N-1} \chi_{[t_k,t_{k+1}]}(\cdot) \sum_{\pi=1}^{M_k} \mathbb{E} \int_{t_k}^T \left( h_{k,i}^\ell, \Lambda_0^{(\pi(t)-k)1} \Gamma_M F_n \right) (\mu(t), a_{n,N,M}(t), b_{n,N,M-1}(t)) dt e_{k,i}^\ell(\cdot)
\]

\[
= \sum_{k=0}^{N-1} \chi_{[t_k,t_{k+1}]}(\cdot) \left( \int_{t_k}^T \Lambda_0^{(\pi(t)-k)1} \Gamma_M F_n \right) (\mu(t), a_{n,N,M}(t), b_{n,N,M-1}(t)) dt \left| \mathcal{F}_{t_k} \right). \tag{4.18}
\]

Therefore, by (4.15), (4.17) and (4.18), we have (4.11).

**Step 2.** In this step, we prove (4.12). Noting that \( a_{n,T}^\pi \in (\mathcal{H}^M(N))^\otimes_n \), by Remark 3.4, we can get that

\[
a_{n,T}^\pi = \sum_{\ell=1}^n \sum_{m=0}^M d^{\alpha,\ell} \sqrt{\alpha!} \prod_{i=1}^N H_{\alpha_i}^\ell(\mathcal{W}(g_i)).
\]

Recalling that \( H_n(x) \) is the Hermite polynomial, we have

\[
x H_n(x) = (n + 1) H_{n+1}(x) - H_{n-1}(x).
\]

Consequently,

\[
\Delta_{k+1} W a_{n,T}^\pi = \sum_{\ell=1}^n \sum_{m=0}^M d^{\alpha,\ell} \sqrt{\alpha!} \prod_{i=1}^N H_{\alpha_i}^\ell(\mathcal{W}(g_i)) \sqrt{\tau} H_{1}(\mathcal{W}(g_{k+1}))
\]

\[
= \sqrt{\tau} \sum_{\ell=1}^n \sum_{m=0}^M d^{\alpha,\ell} \sqrt{\alpha!} \prod_{i=1}^N \left[ (\alpha_{k+1} + 1) H_{\alpha_i}^\ell(\mathcal{W}(g_i)) - H_{\alpha_i}^\ell(\mathcal{W}(g_i)) \right],
\]

where \( \tilde{\alpha} = \alpha + \gamma_{k+1} \) and \( \tilde{\alpha} = \alpha - \gamma_{k-1} \). Here the \( k \)-th component of \( \gamma_k (\in \mathcal{I}) \) is 1, and the others are 0. By the definition of \( \tilde{\alpha} \), we have \( \tilde{\alpha}_{k+1} \geq 1 \). Therefore, by noting that \( |\tilde{\alpha}| \leq M - 1 \), we arrive at

\[
\mathbb{E} (\Delta_{k+1} W a_{n,T}^\pi \mid \mathcal{F}_{t_k})
\]

\[
= \sqrt{\tau} \sum_{\ell=1}^n \sum_{m=0}^M d^{\alpha,\ell} \sqrt{\alpha!} \mathbb{E} \left( \prod_{i=1}^N \left[ (\alpha_{k+1} + 1) H_{\alpha_i}^\ell(\mathcal{W}(g_i)) - H_{\alpha_i}^\ell(\mathcal{W}(g_i)) \right] \mid \mathcal{F}_{t_k} \right)
\]

\[
= \sqrt{\tau} \sum_{\ell=1}^n \sum_{m=0}^M d^{\alpha,\ell} \sqrt{\alpha!} \mathbb{E} \left( \prod_{i=1}^k H_{\alpha_i}^\ell(\mathcal{W}(g_i)) \right) \mathbb{E} \left( \prod_{i=k+1}^N H_{\alpha_i}^\ell(\mathcal{W}(g_i)) \mid \mathcal{F}_{t_k} \right)
\]

\[
- \sqrt{\tau} \sum_{\ell=1}^n \sum_{m=0}^M d^{\alpha,\ell} \sqrt{\alpha!} \mathbb{E} \left( \prod_{i=1}^N H_{\alpha_i}^\ell(\mathcal{W}(g_i)) \right) \mathbb{E} \left( \prod_{i=k+1}^N H_{\alpha_i}^\ell(\mathcal{W}(g_i)) \mid \mathcal{F}_{t_k} \right)
\]

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= -\sqrt{\tau} \sum_{\ell=1}^{n} \sum_{m=0}^{M} \sum_{|a|=m} d^{n,\ell} \sqrt{\alpha!} \mathbb{E} \left( \prod_{i=1}^{N} H_{\alpha_i}^{\ell} (\mathbb{W}(g_i)) \mid F_{t_k} \right) \\
\in (H^{M-1}(k))^{\otimes n}.

Furthermore, we can deduce that

\[ \sum_{\ell=1}^{n} \sum_{i=1}^{M^{'}} \mathbb{E} \left( \frac{\Delta_{k+1} W}{\sqrt{\tau}} h_{k,i}^{\ell} \right) h_{k,i}^{\ell} = \left( \sum_{\ell=1}^{n} \sum_{i=1}^{M^{'}} \mathbb{E} \left( \frac{\Delta_{k+1} W}{\sqrt{\tau}} a_{n,T}^\pi \right) \right) h_{k,i}^{\ell} \]

(4.19)

With the same procedure, we can get that

\[ \sum_{\ell=1}^{n} \sum_{k=0}^{N-1} \sum_{i=1}^{M^{'}} \mathbb{E} \int_{t_{k+1}}^{T} \left( \frac{\Delta_{k+1} W}{\sqrt{\tau}} h_{k,i}^{\ell} \right) e_{k,i}^{\ell} \left( \cdot \right) \]

(4.20)

Combining (4.8), (4.10), (4.19) and (4.20), we conclude that

\[ b_{n,N,M-1}(\cdot) = \sum_{\ell=1}^{n} \sum_{k=0}^{N-1} \sum_{i=1}^{M^{'}} \mathbb{E} \left( \Delta_{0}^{N-k-1} \frac{\Delta_{k+1} W}{\sqrt{\tau}} h_{k,i}^{\ell}, a_{n,T}^\pi \right) \]

\[ - \mathbb{E} \int_{t_{k+1}}^{T} \left( \Delta_{0}^{\pi(t)-k} \frac{\Delta_{k+1} W}{\sqrt{\tau}} h_{k,i}^{\ell}, \Gamma_{M} F_{n}(\mu(t), a_{n,N,M}(t), b_{n,N,M-1}(t)) \right) dt e_{k,i}^{\ell} (\cdot) \]

(4.12)

which is (4.12).

**Step 3.** In this step, by means of (4.11) and (4.12), we prove the error estimates for the finite transposition method, which is a slight variation of (Wang, 2016, Theorem 4.2). We provide a sketch for completeness.
Set \( \bar{a}_n(t_k) = a_{n,N,M}(t_k) \in L^2_{\mathcal{F}_{t_k}}(\Omega;\mathbb{R}^n) \), for \( k = 0, 1, \ldots, N \). By martingale representation theorem, there exists a square integrable process \( \bar{b}_n(\cdot) \), such that

\[
\bar{a}_n(t_{k+1}) = \mathbb{E}(\bar{a}_n(t_{k+1})|\mathcal{F}_{t_k}) + \int_{t_k}^{t_{k+1}} \bar{b}_n(s)dW(s), \quad k = 0, 1, \ldots, N - 1.
\]

For \( t \in [t_k, t_{k+1}) \), \( k = 0, 1, \ldots, N - 1 \), define

\[
\bar{a}_n(t) = (I_n - \Lambda_n \tau) \bar{a}_n(t_k) + (t - t_k)\Gamma_MF_n(t_{k+1}, \bar{a}_n(t_k), b_{n,N,M-1}(t_k)) + \int_{t_k}^{t} \bar{b}_n(s)dW(s).
\]

Subsequently, by noting that \( b_{n,N,M-1}(t_k) = \frac{1}{\tau}\mathbb{E}(\int_{t_k}^{t_{k+1}} \bar{b}_n(s)ds|\mathcal{F}_{t_k}) \), difference between \( (a_n(\cdot), b_n(\cdot)) \) and \( (\bar{a}_n(\cdot), \bar{b}_n(\cdot)) \) can be estimated as follows:

\[
\begin{align*}
\mathbb{E}|(I_n - \Lambda_n \tau)(a_n(t_k) - \bar{a}_n(t_k))|^2 &+ \mathbb{E}\int_{t_k}^{t_{k+1}} |b_n(s) - \bar{b}_n(s)|^2ds \\
= \mathbb{E}|(a_n(t_k) - \bar{a}_n(t_k))| - \int_{t_k}^{t_{k+1}} |F_n(s, a_n(s), b_n(s)) - \Gamma_MF_n(t_{k+1}, \bar{a}_n(t_k), b_{n,N,M-1}(t_k))| ds \\
&\leq (1 + \mathcal{C}\tau)|a_n(t_{k+1}) - \bar{a}_n(t_{k+1})|^2 \\
&+ \mathcal{C}\left\{ \mathbb{E}\int_{t_k}^{t_{k+1}} \left[ |\Lambda_n(a_n(t_j) - a_n(s))|^2 + |(I_n - \Gamma_M)F_n(s, a_n(s), b_n(s))|^2 \right] ds \\
&+ \mathbb{E}\int_{t_k}^{t_{k+1}} |\Gamma_M(F_n(t_{k+1}, \bar{a}_n(t_k), b_{n,N,M-1}(t_k)) - F_n(s, a_n(s), b_n(s)))|^2 ds \right\}.
\end{align*}
\]

By Gronwall’s inequality and assumption (A2), we get that

\[
\max_{k=0,1,\ldots,N} \mathbb{E}|a_n(t_k) - a_{n,N,M}(t_k)|^2 = \max_{k=0,1,\ldots,N} \mathbb{E}|a_n(t_k) - \bar{a}_n(t_k)|^2 \\
\leq \mathcal{C}\left[ \mathbb{E}|a_n(T) - \bar{a}_n(T)|^2 + \tau + \mathbb{E}\int_{0}^{T} \lambda_n^2 \left( |\Lambda_n a_n(s)|^2 + |b_n(s)|^2 + |F_n(t, 0, 0)|^2 \right) ds \right. \\
\left. + \mathbb{E}\int_{0}^{T} |(I_n - \Gamma_M)F_n(s, a_n(s), b_n(s))|^2 ds \right] \\
\leq \mathcal{C}\left[ \mathbb{E}|a_n(T) - \bar{a}_n(T)|^2 + \lambda_n^2 \tau + \mathbb{E}\int_{0}^{T} |(I_n - \Gamma_M)F_n(s, a_n(s), b_n(s))|^2 ds \right].
\]

Summing (4.21) from \( k = 0 \) to \( N - 1 \), applying (4.22) and the fact \( b_{n,N,M-1}(t_k) = \frac{1}{\tau}\mathbb{E}(\int_{t_k}^{t_{k+1}} \bar{b}_n(s)ds|\mathcal{F}_{t_k}) \), we can derive (4.13). In the similar vein, we can prove (4.14). That completes the proof.

\[
\square
\]

**Remark 4.2.** By formulas (4.11) and (4.12), for a linear backward stochastic evolution equation, the transposition method is just the Euler method under \( \mathcal{H}^M(N; S_n) \)-valued approximations of the terminal value. Nevertheless, for nonlinear equations, the finite transposition method is different from the Euler method. For a linear equation, since the terminal value is approximated by \( \mathcal{H}^M(N; S_n) \)-valued random variables, thanks to the variational equation (4.1), there is no need to calculate conditional expectations.
5 Numerical method for optimal controls

In this section, we present an application of the finite transposition method. To avoid technical complexity, we consider a simple SLQ problem. To be specific, we consider a cost functional

\[ J(y_0; u(\cdot)) = \frac{1}{2} \mathbb{E} \left[ \int_0^T \left( \|y(t)\|^2_{L^2(D)} + \|u(t)\|^2_{L^2(D)} \right) dt + \|y(T)\|^2_{L^2(D)} \right] \]

subject to the (controlled forward) stochastic heat equation with an additive noise

\[
\begin{aligned}
    \left\{ 
    dy(t) &= (Ay(t) + u(t)) dt + \sigma dW(t), \quad t \in (0, T], \\
    y(0) &= y_0.
    \right. 
\end{aligned}
\]  

(5.1)

Here \( A \) is defined in (3.6), and \( y_0, \sigma \in H^1_0(D) \cap H^2(D) \). Let us now state a SLQ problem as follows:

**Problem (SLQ).** Search for \( \bar{u}(\cdot) \in L^2_F(0, T; L^2(D)) \), such that

\[ J(y_0; \bar{u}(\cdot)) = \inf_{u(\cdot) \in L^2_F(0, T; L^2(D))} J(y_0; u(\cdot)). \]

By Lü and Zhang (2015), the solvability of Problem (SLQ) is equivalent to the solvability of the following forward-backward stochastic evolution equation:

\[
\begin{aligned}
    d\bar{y}(t) &= (A\bar{y}(t) + \bar{u}(t)) dt + \sigma dW(t), \quad t \in (0, T], \\
    d\bar{z}(t) &= (-Az(t) + \bar{y}(t)) dt + Z(t) dW(t), \quad t \in [0, T], \\
    \bar{y}(0) &= y_0, \quad \bar{z}(T) = -\bar{y}(T),
    \end{aligned}
\]  

(5.2)

with the condition

\[ \bar{u}(t) - \bar{z}(t) = 0, \quad \text{a.e. } (t, \omega) \in [0, T] \times \Omega. \]  

(5.3)

In the following, we mainly propose the discretization of (5.2)-(5.3). To be specific, for the forward equation (5.2)\(_1\), we adopt the time-implicit Galerkin method (see, e.g., Grecksch and Kloeden, 1996), while for the backward one (5.2)\(_2\), we apply the finite transposition method proposed in Section 4; see also Algorithm 1 below.

Since (5.2)-(5.3) is coupled, how to obtain the convergence of this discretization strategy? In what follows, we adopt optimal control theory to deduce the convergence rates. To do this, we discretize Problem (SLQ) within two steps: firstly, we get the spacial semi-discretization (which is referred to as Problem (SLQ)\(_s\)); secondly, we obtain a spatio-temporal discretization (which is referred to as Problem (SLQ)\(_{ST}\)). Now, we present these two discretizations.

**Problem (SLQ)\(_s\).** For a fixed \( n \in \mathbb{N} \), minimize the following cost functional over \( L^2_F(0, T; S_n) \):

\[ J(\Pi_n y_0; u(\cdot)) = \frac{1}{2} \mathbb{E} \left[ \int_0^T \left( \|y_n(t)\|^2_{L^2(D)} + \|u_n(t)\|^2_{L^2(D)} \right) dt + \|y_n(T)\|^2_{L^2(D)} \right] \]

subject to the following \( S_n \)-valued stochastic differential equation:

\[
\begin{aligned}
    d y_n(t) &= (A_n y_n(t) + u_n(t)) dt + \Pi_n \sigma dW(t), \quad t \in [0, T], \\
    y_n(0) &= \Pi_n y_0 = \sum_{k=1}^{n} \langle y_0, \phi_k \rangle_{L^2(D)} \phi_k.
    \end{aligned}
\]  

(5.4)
By Yong and Zhou (1999), the solvability of Problem (SLQ) is equivalent to the solvability of the following forward-backward stochastic differential equation:

\[
\begin{align*}
    d\bar{y}_n(t) &= (A_n\bar{y}_n(t) + \bar{u}_n(t)) dt + \Pi_n \sigma dW(t), \\
    dz_n(t) &= (-A_n z_n(t) + \bar{y}_n(t)) dt + Z_n(t) dW(t), \quad t \in [0, T], \\
    \bar{y}_n(0) &= \Pi_n y_0, \quad z_n(T) = -\bar{y}_n(T),
\end{align*}
\]

with the condition

\[
    \bar{u}_n(t) - z_n(t) = 0, \quad \text{a.e.} \quad (t, \omega) \in [0, T] \times \Omega. \tag{5.6}
\]

The following result is on the rate of convergence for spacio-temporal discretization of Problem (SLQ).

**Theorem 5.1.** Let \((\bar{y}(\cdot), \bar{u}(\cdot))\) be the optimal pair of Problem (SLQ) and \((\bar{y}_n(\cdot), \bar{u}_n(\cdot))\) be the solution of Problem (SLQ) for \(n \in \mathbb{N}\). Then, there exists a constant \(C\) such that

\[
\begin{align*}
    (i) & \quad \mathbb{E}\left( \sup_{t \in [0, T]} \| \bar{u}(t) - \bar{u}_n(t) \|_{L^2(D)}^2 \right) \leq \frac{C}{\lambda_{n+1}}, \\
    (ii) & \quad \mathbb{E}\left( \sup_{t \in [0, T]} \| \bar{y}(t) - \bar{y}_n(t) \|_{L^2(D)}^2 \right) + \mathbb{E} \int_0^T \| \bar{y}(t) - \bar{y}_n(t) \|_{H^1_0(D)}^2 dt \leq \frac{C}{\lambda_{n+1}}.
\end{align*}
\]

**Proof.** Applying Itô’s formula to \(\| \nabla \bar{y}(\cdot) \|_{L^2(D)}^2 + \| \bar{y}(\cdot) \|_{L^2(D)}^2\), we obtain

\[
    \sup_{t \in [0, T]} \mathbb{E}\| \bar{y}(t) \|_{H^1_0(D)}^2 \leq C \left[ \| y_0 \|_{L^2(D)}^2 + \mathbb{E} \int_0^T \left( \| \bar{u}(t) \|_{L^2(D)}^2 + \| \sigma \|_{L^2(D)}^2 \right) dt \right].
\]

On the other hand, Theorem 4.1 yields that

\[
    \mathbb{E}\left( \sup_{t \in [0, T]} \| z(t) - z_n(t) \|_{L^2(D)}^2 \right) \leq \frac{C}{\lambda_{n+1}} \left( \mathbb{E}\| \bar{y}(T) \|_{H^1_0(D)}^2 + \mathbb{E} \int_0^T \| \bar{y}(t) \|_{H^1_0(D)}^2 dt \right).
\]

The maximum conditions (5.3) and (5.6) and above two inequalities lead to assertion (i). Itô’s formula to \(\| \bar{y}(\cdot) - \bar{y}_n(\cdot) \|_{L^2(D)}^2\), Gronwall’s inequality and assertion (i) lead to

\[
\begin{align*}
    \sup_{t \in [0, T]} \mathbb{E}\| \bar{y}(t) - \bar{y}_n(t) \|_{L^2(D)}^2 &+ \mathbb{E} \int_0^T \| \nabla (\bar{y}(s) - \bar{y}_n(s)) \|_{L^2(D)}^2 ds \\
    &\leq C \left[ \| y_0 - \Pi_n y_0 \|_{L^2(D)}^2 + \mathbb{E} \int_0^T \left( \| \bar{u}(s) - \bar{u}_n(s) \|_{L^2(D)}^2 + \| \sigma - \Pi_n \sigma \|_{L^2(D)}^2 \right) ds \right] \\
    &\leq \frac{C}{\lambda_{n+1}}.
\end{align*}
\]

Applying Itô’s formula to \(\| \bar{y}(\cdot) - \bar{y}_n(\cdot) \|_{L^2(D)}^2\) again, then using Burkholder-Davis-Gundy inequality and (5.7), we can derive assertion (ii). \(\Box\)

Before presenting the spatio-temporal discretization of Problem (SLQ), we introduce the following two spaces:

\[
\begin{align*}
    X_T^\Delta &\Delta = \left\{ x(\cdot) \in L^2_T(0, T; \mathbb{S}_n) \mid x(t) = x(t_k), \forall t \in [t_k, t_{k+1}), k = 0, 1, \cdots, N - 1 \right\}, \\
    U_T^\Delta &\Delta = \left\{ u(\cdot) \in L^2_T(0, T; \mathbb{S}_n) \mid u(t) = u(t_k), \forall t \in [t_k, t_{k+1}), k = 0, 1, \cdots, N - 1 \right\},
\end{align*}
\]
and for any \(x(\cdot) \in X_\tau\) and \(u(\cdot) \in U_\tau\),
\[
\|x(\cdot)\|_{X_\tau} \triangleq \left( \tau \sum_{n=1}^{N} \mathbb{E}\|x(t_n)\|_{L^2(D)} \right)^{1/2}, \quad \|u(\cdot)\|_{U_\tau} \triangleq \left( \tau \sum_{n=0}^{N-1} \mathbb{E}\|u(t_n)\|_{L^2(D)} \right)^{1/2}.
\]

**Problem (SLQ)\(_{ST}\).** For fixed \(N, n \in \mathbb{N}\), minimize the following functional over \(U_\tau\):
\[
\mathcal{J}_{ST}(\Pi_n y_0; u_n(\cdot)) = \frac{1}{2} (\|y_n(N)(\cdot)\|^2_{X_\tau} + \|u_n(N)(\cdot)\|^2_{U_\tau}) + \frac{1}{2} \mathbb{E}\|y_n(N)(T)\|^2_{L^2(D)},
\]
where \((y_n(N), u_n(N))\) satisfies
\[
\begin{cases}
y_n(t_{k+1}) - y_n(t_k) = \tau [A_n y_n(t_{k+1}) + u_n(t_k)] + \Pi_n \sigma (\kappa_{k+1} W, \quad k = 0, 1, \ldots, N - 1, \quad (5.8) 
y_n(0) = \Pi_n y_0.
\end{cases}
\]

The following result, which guarantees the solvability of Problem (SLQ)\(_{ST}\), is a variant of (Prohl and Wang, 2020b, Theorem 4.2).

**Theorem 5.2.** Problem (SLQ)\(_{ST}\) admits a unique minimizer \((\bar{y}_n(\cdot), \bar{u}_n(\cdot)) \in X_\tau \times U_\tau\), which satisfies the following coupled equation for \(0 \leq k \leq N - 1\):
\[
\begin{cases}
\bar{y}_n(t_{k+1}) = \Lambda_0 \bar{y}_n(t_k) + \tau \Lambda_0 \bar{u}_n(t_k) + \Lambda_0 \Pi_n \sigma (\bar{z}_n(t_{k+1}) - \bar{y}_n(t_{k+1})|\bar{F}_t), 
\bar{z}_n(t_k) = \Lambda_0 \mathbb{E}(z_n(t_{k+1}) - \tau \bar{y}_n(t_{k+1})|\bar{F}_t), 
\bar{y}_n(0) = y_n(0), \quad \bar{z}_n(T) = -\bar{y}_n(T),
\end{cases}
\]

\[
\text{together with the maximum condition} \quad \bar{u}_n(t_k) - z_n(t_k) = 0 \quad k = 0, 1, \ldots, N - 1, \quad \text{a.s.,}
\]

where \(\Lambda_0 = (I_n - \Lambda_n \tau)^{-1}\).

Based on Malanowski (1982), as well as the regularity of the optimal pairs \((\bar{y}_n(\cdot), \bar{u}_n(\cdot))\) and \((\bar{y}_n(\cdot), \bar{u}_n(\cdot))\), we can prove the following convergence rates (e.g., Prohl and Wang, 2020b, Theorem 4.3).

**Theorem 5.3.** For any \(n, N \in \mathbb{N}\), assume that \((\bar{y}_n(\cdot), \bar{u}_n(\cdot))\) and \((\bar{y}_n(\cdot), \bar{u}_n(\cdot))\) are the optimal pairs of Problem (SLQ)\(_{S}\) and (SLQ)\(_{ST}\), respectively. Then, it holds that
\[
\begin{align*}
(i) \quad & \sum_{k=0}^{N-1} \mathbb{E}\|\bar{u}_n(t) - \bar{u}_n(t_k)\|^2_{L^2(D)} dt \leq C \tau; \\
(ii) \quad & \max_{0 \leq k \leq N} \mathbb{E}\|\bar{y}_n(t_k) - y_n(t_k)\|^2_{L^2(D)} + \tau \mathbb{E}\sum_{k=1}^{N} \|\bar{y}_n(t_k) - y_n(t_k)\|^2_{H^1_0(D)} \leq C \tau.
\end{align*}
\]

**Remark 5.1.** The equation (5.9) is the implicit Euler scheme for the equation (5.5) and the Galerkin based implicit Euler method for (5.2). Notice that (5.2) and (5.5) are coupled forward-backward equations. Numerical methods for these coupled equations and the convergence analysis is highly nontrivial. To deduce the rates of convergence, additional conditions such as sufficiently small \(T\), or equations’ weak coupling, are needed (see, e.g., Bender and Zhang, 2008). In this section, we show an idea to solve these equations. To be specific, if a coupled equation can be transferred to an optimal control problem, one can try to utilize tools in optimal control theory.
The equation (5.9) is a coupled equation and contains conditional expectations. There are some methods to deal with these conditional expectations. But for coupled forward-backward equation, even for that in finite dimensions, numerical schemes are rare and the convergence analysis is far from complete (see, e.g., Bender and Zhang, 2008). Here we can adopt the finite transposition method and gradient method to solve (5.2).

Algorithm 1 Solving Problem (SLQ) by Galerkin based finite transposition method and the gradient method

Fix $N, M, n \in \mathbb{N}$, and $\kappa > 0$. Let $\tilde{u}(0) \in \mathbb{H}_{N,M}(\mathbb{S}_0)$. For any $\ell \in \mathbb{N}$, update $\tilde{u}^{(\ell)} \in \mathbb{H}_{N,M}(\mathbb{S}_n)$ as follows:

1. Compute $\tilde{y}^{(\ell)} \in \mathbb{H}_{N,M}(\mathbb{S}_n)$ via time-implicit Galerkin method:

$$
\begin{cases}
\tilde{y}^{(\ell)}(t_{n+1}) - \tilde{y}^{(\ell)}(t_n) = \tau A_n \tilde{y}^{(\ell)}(t_{n+1}) + \Pi_n \sigma \Delta_n W, & n = 0, 1, \cdots, N - 1, \\
\tilde{y}^{(\ell)}(0) = \Pi_n y_0.
\end{cases}
$$

2. Utilize $\tilde{y}^{(\ell)} \in \mathbb{H}_{N,M}(\mathbb{S}_n)$ in (5.2)2 to compute $\tilde{z}^{(\ell)} \in \mathbb{H}_{N,M}(\mathbb{S}_n)$ via the finite transposition method.

3. Compute the update $\tilde{u}^{(\ell+1)} \in \mathbb{H}_{N,M}(\mathbb{S}_n)$ via

$$
\tilde{u}^{(\ell+1)} = (1 - \frac{1}{\kappa}) \tilde{u}^{(\ell)} + \frac{1}{\kappa} \tilde{z}^{(\ell)}.
$$

In Problem (SLQ), $\kappa$ can be taken as $1 + T + T^2$. We can prove that $(\tilde{u}^{(k)}(\cdot), \tilde{y}^{(k)}(\cdot))$ converges to $(\bar{u}(\cdot), \bar{y}(\cdot))$ in $X_T \times U_T$ as $k \to \infty$ (e.g., Prohl and Wang, 2020b, Section 5).

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