Exponential decay properties of a mathematical model for a certain fluid-structure interaction

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Abstract
In this work, we derive a result of exponential stability for a coupled system of partial differential equations (PDEs) which governs a certain fluid-structure interaction. In particular, a three-dimensional Stokes flow interacts across a boundary interface with a two-dimensional mechanical plate equation. In the case that the PDE plate component is rotational inertia-free, one will have that solutions of this fluid-structure PDE system exhibit an exponential rate of decay. By way of proving this decay, an estimate is obtained for the resolvent of the associated semigroup generator, an estimate which is uniform for frequency domain values along \( i\mathbb{R} \). Subsequently, we proceed to discuss relevant point control and boundary control scenarios for this fluid-structure PDE model, with an ultimate view to optimal control studies on both finite and infinite horizon. (Because of said exponential stability result, optimal control of the PDE on time interval \((0, \infty)\) becomes a reasonable problem for contemplation.)

Introduction
In this work, we undertake a stability analysis of a certain partial differential equation (PDE) system, that is (1.2)-(1.3) below, which has been previously studied in [11] and [10], among other works, inasmuch as it simultaneously constitutes a mathematically interesting and physically relevant model of a fluid-structure (F-S) dynamics. This PDE model comprises a Stokes flow, evolving within a three-dimensional cavity \( O \), coupled via a boundary interface, to a two dimensional Euler-Bernoulli or Kirchhoff plate which displaces upon a bounded open set \( \Omega \), which is taken to be a portion of the cavity boundary \( \partial O \). Our main result here (Theorem 1.3 below) is the derivation of exponential decay rates for the composite fluid-structure dynamics, in the case that the Euler-Bernoulli plate PDE model – i.e. the one corresponding to “rotational inertia” parameter \( \rho = 0 \) in (1.2e) – is used to describe the mechanical displacements along \( \Omega \).

In point of fact, this stability result was originally given in [11]; the real novelty in the present work lies in the method of proof: whereas in [11] the exponential decay of the given

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*The research of G. Avalos was partially supported by the NSF Grants DMS-0908476 and DMS-1211232. The research of F. Bucci was partially supported by the Italian MIUR under the PRIN 2009KNZ5FK Project (*Metodi di viscosità, geometrici e di controllo per modelli diffusivi nonlineari*), by the GDRE (Groupement De Recherche Européen) CONEDP ( Control of PDEs), and also by the Università degli Studi di Firenze under the Project *Calcolo delle variazioni e teoria del controllo*.

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fluid-structure dynamics is obtained via a Lyapunov functional approach, with the authors of [11] operating strictly within the time domain, the present work is centered upon working instead in the frequency domain. In particular, we work to attain a uniform estimate for the resolvent operator of the generator of the associated fluid-structure semigroup, as it assumes values along the imaginary axis. With such resolvent estimate in hand, we can then appeal to a well-known resolvent criterion of Prüss (posted as Theorem 2.1 here), so as to ultimately infer exponential decay. The virtue of the frequency domain approach which is employed here, is that it can eventually be adapted so as to treat the case \( \rho > 0 \) (Kirchhoff plate).

Indeed, the frequency domain methodology outlined here is invoked and refined in [3], so as to provide rational decay rates for Stokes-Kirchhoff plate dynamics. (The higher topology for the mechanical velocity component in Kirchhoff plate equation – viz., \( H^1(\Omega) \) for the Kirchhoff plate, as opposed to \( L^2(\Omega) \) for Euler-Bernoulli – prevents the attainment of exponential decay in [3]. Hence, weaker polynomial rates of decay are established for \( \rho > 0 \).) We should also state that our estimate of said resolvent on the imaginary axis is direct and explicit, in the style of what was performed in [7]; previous exponential stability works which are geared so as to eventually invoke said resolvent criterion of Prüss, tend to obtain the requisite resolvent estimates via an argument by contradiction (see, e.g., [18]).

In addition, in the final Section of this work we offer some insight into a further analysis which is needed to pursue solvability of natural/appropriate optimal control problems (with quadratic functionals) associated with the PDE system under investigation. We note that a full understanding of the stability properties of this F-S interaction is not only of intrinsic interest, but indeed a prerequisite step in the study of optimal control problems over an infinite time horizon. In this respect, the uniform (exponential) stability result established for the composite PDE system (1.2) in the presence of an elastic equation of Euler-Bernoulli type ensures that both finite and infinite time horizon problems are equally valid objects of investigation. Instead, when the elastic equation is of Kirchhoff type, the (rational) decay rates of solutions shown in [3] will prevent us from taking into consideration optimal control problems on an infinite time interval.

We also point out here that the presence of boundary or point control actions into the model, will necessitate a careful technical analysis of the regularity properties of the kernel term \( e^{At}B \) of the (so called) “input-to-state map” for the abstract controlled equation \( y' = Ay + Bg \) corresponding to the controlled boundary value problem. This is particularly so, inasmuch as, in relevant applications the control operator \( B \) is (intrinsically) not bounded from the state space into the control space.

In turn, as is well known, sharp PDE regularity estimates for the solutions to the “free” (or uncontrolled) system should be instrumental in bringing about the sought regularity properties of the operator \( e^{At}B \). It is only by having at hand such information on the sharp regularity of the fluid-structure PDE under present study, that one can verify whether the recent results on the LQ-problem and Riccati equations for abstract dynamics inspired by and tailored for coupled PDE systems of hyperbolic/parabolic type (such as [1, 2]) are applicable, or whether novel theories need to be devised. A brief description of a couple of relevant scenarios for the placement of control functions in the model is given, along with some remarks about the technical challenges which are expected.

1 The PDE model, statement of the main result

In what follows, the same geometry which prevailed in [11] will obtain here. Namely: (fluid) domain \( \mathcal{O} \subset \mathbb{R}^3 \) will be bounded with sufficiently smooth boundary \( \partial \mathcal{O} \). Moreover, \( \partial \mathcal{O} = \overline{\partial \mathcal{O}} \cup \overline{\Omega} \).
with \( S \cap \Omega = \emptyset \), \( \Omega \) being a flat portion of \( \partial \mathcal{O} \). In particular, \( \partial \mathcal{O} \) has the following specific spatial configuration:

\[
\Omega \subset \{ x = (x_1, x_2, 0) \} \; , \; S \subset \{ x = (x_1, x_2, x_3) : x_3 \leq 0 \} \; .
\]

So if \( \nu(x) \) denotes the unit normal vector to \( \partial \mathcal{O} \), pointing outward, then

\[
\nu|_{\Omega} = [0, 0, 1] \; .
\]  

(1.1)

See, e.g., the picture below.

On such geometry, the PDE model is as as follows, with rotational inertia parameter \( \rho \geq 0 \), and in solution variables \( u(x,t) = [u^1(x,t), u^2(x,t), u^3(x,t)] \) and \([w(x,t), w_t(x,\cdot)]\):

\[
\begin{align*}
& u_t - \Delta u + \nabla p = 0 & \text{in } \mathcal{O} \times (0,T) \\
& \text{div}(u) = 0 & \text{in } \mathcal{O} \times (0,T) \\
& u = 0 & \text{on } S \\
& u = [u^1, u^2, u^3] = [0, 0, w_t] & \text{on } \Omega \\
& w_{tt} - \rho \Delta w_{tt} + \Delta^2 w = p|_{\Omega} & \text{in } \Omega \times (0,T) \\
& w = \frac{\partial w}{\partial \nu} = 0 & \text{on } \partial \Omega
\end{align*}
\]

(1.2a-1.2f)

with initial conditions

\[
[u(0), w(0), w_t(0)] = [u_0, w_0, w_1] \in H_\rho.
\]  

(1.3)

Here, the space of initial data \( H_\rho \) is defined as follows: Let the (fluid) space \( H_{\text{fluid}} \) (\( H_f \), in short) be defined by

\[
H_f := \{ f \in L^2(O) : \text{div}(f) = 0; \; f \cdot \nu|_S = 0 \} \; ,
\]

(1.4)

and let

\[
V_\rho = \begin{cases} 
L^2(O)/\mathbb{R} & \text{if } \rho = 0 \\
H^1_0(\Omega) \cap L^2(O)/\mathbb{R} & \text{if } \rho > 0.
\end{cases}
\]  

(1.5)

Therewith, we then set

\[
H_\rho = \left\{ [f, h_0, h_1] \in H_f \times [H^1_0(\Omega) \cap L^2(\Omega)/\mathbb{R}] \times V_\rho, \right. \\
\left. \text{with } f \cdot \nu|_{\Omega} = [0, 0, f^3] \cdot [0, 0, 1] = h_1 \right\}.
\]

(1.6)

In this paper, we shall focus on the case \( \rho = 0 \).
In addition: By way of constructing an abstract operator \( A_D : \mathcal{D}(A_D) \subset H_\rho \to H_\rho \) which describes the PDE dynamics \( (1.2)-(1.3) \), we denote \( A_D : L^2(\Omega) \to L^2(\Omega) \) by
\[
A_D g = -\Delta g, \quad \mathcal{D}(A_D) = H^2(\Omega) \cap H^1_0(\Omega).
\] (1.6)

If we subsequently make the denotation for all \( \rho \geq 0 \)
\[
P_\rho = I + \rho A_D, \quad \mathcal{D}(P_\rho) = \begin{cases} L^2(\Omega) & \text{if } \rho = 0 \\ \mathcal{D}(A_D) & \text{if } \rho > 0 \end{cases},
\] (1.7)
then the mechanical PDE component \( (1.2) \) of the system \( (1.2) \) can be written as
\[
P_\rho w_{tt} + \Delta^2 w = p|_\Omega \quad \text{on } (0, T).
\]

Using the fact from [13] that
\[
\mathcal{D}(P_{1/2}^{1/2}) = \begin{cases} L^2(\Omega) & \text{if } \rho = 0 \\ H^1_0(\Omega) & \text{if } \rho > 0 \end{cases},
\] (1.8)
then we can endow the Hilbert space \( H_\rho \) with the norm-inducing inner product
\[
(\mu_0, \omega_1, \omega_2), (\tilde{\mu}_0, \tilde{\omega}_1, \tilde{\omega}_2)_{H_\rho} = (\mu_0, \tilde{\mu}_0)_{\mathcal{O}} + (\Delta \omega_1, \Delta \tilde{\omega}_1)_{\Omega} + (P_{1/2}^{1/2} \omega_2, P_{1/2}^{1/2} \tilde{\omega}_2)_{\Omega},
\]
where \((\cdot, \cdot)_{\mathcal{O}}\) and \((\cdot, \cdot)_{\Omega}\) are the \( L^2 \)-inner products on their respective geometries.

Moreover, as was done in [4] and [3, Lemma 1.1], so as to eliminate the pressure term \( p \) in \( (1.2)-(1.3) \) (see also [5] for an analogous elimination for a different fluid-structure PDE model), we recognize the pressure term as the solution of the following BVP, pointwise in time:
\[
\begin{align*}
\Delta p &= 0 \quad \text{in } \mathcal{O} \\
\frac{\partial p}{\partial \nu} &= \Delta u \cdot \nu|_S \quad \text{on } S \\
\frac{\partial p}{\partial \nu} + p &= \Delta^2 w + \Delta w^3|_\Omega \quad \text{on } \Omega.
\end{align*}
\] (1.8)

To “solve” for the pressure term, we then invoke appropriate (‘Robin’) maps \( R_\rho \) and \( \tilde{R}_\rho \) defined as follows:
\[
R_\rho g = f \iff \{ \Delta f = 0 \text{ in } \mathcal{O}, \frac{\partial f}{\partial \nu} + P^{-1}_\rho f = g \text{ on } \Omega, \frac{\partial f}{\partial \nu} = 0 \text{ on } S \};
\]
\[
\tilde{R}_\rho g = f \iff \{ \Delta f = 0 \text{ in } \mathcal{O}, \frac{\partial f}{\partial \nu} + P^{-1}_\rho f = 0 \text{ on } \Omega, \frac{\partial f}{\partial \nu} = g \text{ on } S \}.
\]

Therewith, we have that for all real \( s \),
\[
R_\rho \in \mathcal{L}(H^s(\Omega), H^{s+3/2}(\mathcal{O})); \quad \tilde{R}_\rho \in \mathcal{L}(H^s(S), H^{s+3/2}(\mathcal{O})).
\] (1.9)

(We are also using implicitly the fact that \( P^{-1}_\rho \) is positive definite and self-adjoint on \( \Omega \).) Consequently, the pressure variable \( p(t) \), as necessarily the solution of \( (1.8) \) — that is an appropriate harmonic extension from the boundary of \( \mathcal{O} \) into the interior — can be written pointwise in time as
\[
p(t) = G_{\rho,1}(w(t)) + G_{\rho,2}(u(t)),
\] (1.10)
where

\[ G_{\rho,1}(w) = R_\rho(P_\rho^{-1}\Delta^2 w); \quad (1.11a) \]
\[ G_{\rho,2}(u) = R_\rho(\Delta u^3|_\Omega) + \tilde{R}_\rho(\Delta u \cdot \nu|_S). \quad (1.11b) \]

These relations suggest the following choice for the generator \( A_\rho : H_\rho \to H_\rho \). We set

\[
A_\rho \equiv \begin{bmatrix}
\Delta - \nabla G_{\rho,2} & -\nabla G_{\rho,1} & 0 \\
0 & 0 & I \\
-P_\rho^{-1}G_{\rho,2}|_\Omega & -P_\rho^{-1}\Delta^2 + P_\rho^{-1}G_{\rho,1}|_\Omega & 0
\end{bmatrix}
\]

(1.12)
with domain

\[
\mathcal{D}(A_\rho) = \{ [u, w_1, w_2] \in H_\rho : u \in H^2(\mathcal{O}); \quad w_1 \in S_\rho, w_2 \in H^2_0(\Omega), \]
\[
\text{u = 0 on } S, \quad \text{u = (0, 0, w_2) on } \Omega \},
\]

(1.13)

where the mechanical displacement space, denoted by \( S_\rho \), changes with \( \rho \) as follows:

\[
S_\rho := \begin{cases} 
H^4(\Omega) \cap H^2(\Omega) & \rho = 0 \\
H^3(\Omega) \cap H^2_0(\Omega) & \rho > 0.
\end{cases}
\]

We note also, from the definition of \( \mathcal{D}(A_\rho) \) that \([u, w_1, w_2] \in \mathcal{D}(A_\rho)\) implies \( \Delta u \in L^2(\mathcal{O}) \) and \( \text{div} \Delta u = 0 \). Consequently, from elementary Stokes Theory (see, e.g., [12, Proposition 1.4, p. 5]), we have

\[
\| \Delta u \cdot \nu\|_{H^{-1/2}(\partial \mathcal{O})} \leq C\|u\|_{H^4(\mathcal{O})} \leq C\|[u, w_1, w_2]\|_{\mathcal{D}(A_\rho)}
\]

and so associated pressure \( \pi_0 \) satisfies

\[
\pi_0 \equiv G_{\rho,1}(w_1) + G_{\rho,2}(u) \in H^1(\mathcal{O}). \quad (1.14)
\]

**Remarks 1.1. (i) (Well-posedness)** Well-posedness of the (linear) coupled system (1.2)-(1.3) when \( \rho = 0 \) — namely, when the elastic equation is of Euler-Bernoulli type, of specific concern in the present investigation — was originally established in [11], by using Galerkin approximations. An alternative proof of well-posedness which encompasses both cases \( \rho = 0 \) and \( \rho > 0 \) has been recently given in [4]. It is important to emphasize that the proof appeals to the Lumer-Phillips Theorem within classical semigroup theory, and yet also utilizes in a crucial and nontrivial way the Babuška-Brezzi Theorem (see, e.g., [14, p. 116]). The corresponding statement is given below.

**Theorem 1.2 ([4]).** The operator \( A_\rho : H_\rho \to H_\rho \) defined by (1.12)-(1.13) generates a \( C_0 \)-semigroup of contractions \( \{e^{A_\rho t}\}_{t \geq 0} \) on \( H_\rho \). Thus, for any \([u_0, w_0, w_1] \in H_\rho \), the (unique) weak solution to the initial/boundary value problem (1.2)-(1.3) is given by

\[
\begin{bmatrix}
u(t) \\
w(t)
\end{bmatrix} = e^{A_\rho t} \begin{bmatrix} u_0 \\
w_0 \\
w_1
\end{bmatrix} \in C([0, T]; H_\rho).
\]

(1.15)

(ii) (Decay rates) To the authors' knowledge, the stability properties of solutions to the linear model (1.2) (again, when \( \rho = 0 \)) have been explored in [11], along the analysis of the long-term behaviour of a nonlinear coupled dynamics, comprising a 3D linearized Navier-Stokes
system for the fluid velocity field in a bounded domain, and a *nonlinear* elastic plate equation for the transversal displacement of a flat flexible part of the boundary. Among the various results established in [11], primarily pertaining to the nonlinear model, exponential stability of the linear dynamics is attained by using Lyapunov function arguments; see [11, Section 3].

We aim here at presenting a different proof of exponential stability, based instead on a (by now classical) resolvent criterion by Prüss; see Theorem 2.1 in the next Section. The adoption of a “frequency domain” approach – in contrast with the more commonly invoked “time domain” analysis – is not only of intrinsic interest, but it also proves to be very effective in order to establish the decay rates of solutions, even when exponential stability fails. Indeed, in the case \( \rho > 0 \), the very same frequency domain approach enables us to establish that the energy of strong solutions decays at the rate of \( O(1/t) \), as \( t \to +\infty \); see [3].

The main result of the present work is stated below.

**Theorem 1.3 (Exponential decay rates).** Let the rotational inertia parameter \( \rho = 0 \) in (1.2e). Then all finite energy solutions of (1.2) - (1.3) decay at an exponential rate. Namely, there exist constants \( \omega > 0 \) and \( M \geq 1 \) such that for arbitrary initial data \( [u_0, w_0, w_1] \in H_0 \), the corresponding solutions \( [u, w, w_t] \) of (1.2) - (1.3) satisfy

\[
\| [u(t), w(t), w_t(t)] \|_{H_0} \leq Me^{-\omega t} \| [u_0, w_0, w_1] \|_{H_0}.
\]

2 Exponential Stability

To show that the semigroup defined by (1.15) is exponentially stable, we appeal to a celebrated result of semigroup theory which we recall explicitly for the reader’s convenience.

**Theorem 2.1 ([19]).** Let \( (T(t))_{t \geq 0} \) be a \( C_0 \)-semigroup on a Hilbert space \( H \) with generator \( A \), such that \( i\mathbb{R} \subset \rho(A) \). Then, the following are equivalent:

(i) \( \exists C > 0 : \| R(is; A) \| \leq C \quad \forall s \in \mathbb{R} ; \)

(ii) \( \exists \omega > 0, M \geq 1 : \| T(t) \| \leq Me^{-\omega t} \quad t \geq 0. \)

In order to invoke the above resolvent criterion, we need, as a preliminary step, to show that the imaginary axis belongs to the resolvent set of the dynamics operator \( A \). This property cannot be freely taken for granted: in the context of other fluid-structure interactions, it is known that certain geometrical configurations will give rise to eigenvalues on the imaginary axis; see, e.g., [5] and [6] (and also [7], where examples of “non-pathological geometries” are given).

2.1 Preliminary step: Spectral Analysis

Here, we limit ourselves to show that \( \lambda = 0 \) belongs to the resolvent set \( \rho(A) \); in other words, the resolvent operator is boundedly invertible on the state space \( H_\rho \). The reader is referred to [6, Section 2] for a detailed analysis and proof of the fact that the spectrum has empty intersection with the whole imaginary axis, in the more challenging case \( \rho > 0 \). The arguments used therein can be easily adapted to the case \( \rho = 0 \).

As the parameter \( \rho \) equals 0 throughout, in order to simplify the notation we set \( H_0 =: H \), as well as \( A_0 =: A \). (We note that \( P_\rho \) reduces coincides with the identity operator \( I \) throughout.)

**Proposition 2.2.** The generator \( A : D(A) \subset H \to H \) is boundedly invertible on \( H \). Namely, \( \lambda = 0 \) is in the resolvent set of \( A \).
Proof. Given data \([u^*, w_1^*, w_2^*] \in \mathbf{H}\), we look for \([u, w_1, w_2] \in \mathcal{D}(A)\) which solves

\[
A \begin{bmatrix} u \\ w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} u^* \\ w_1^* \\ w_2^* \end{bmatrix}.
\] (2.1)

To this end, we must seek for \([u, w_1, w_2] \in \mathcal{D}(A)\) and associated pressure \(\pi_0 \in H^1(\mathcal{O})\) which solve

\[
\begin{align*}
\Delta u - \nabla \pi_0 &= u^* \quad \text{in } \mathcal{O} \quad (2.2a) \\
\text{div}(u) &= 0 \quad \text{in } \mathcal{O} \quad (2.2b) \\
u &= 0 \quad \text{on } S \quad (2.2c) \\
u &= (0, 0, w_2) \quad \text{on } \Omega \quad (2.2d) \\
w_2 &= w_1^* \quad \text{in } \Omega \quad (2.2e) \\
\Delta^2 w_1 - \pi_0 |_{\Omega} &= -w_2^* \quad \text{in } \Omega \quad (2.2f) \\
w_1 &= \frac{\partial w_1}{\partial \nu} = 0 \quad \text{on } \partial \Omega. \quad (2.2g)
\end{align*}
\]

Moreover, we must justify that the pressure variable \(\pi_0\) above is given by the expression

\[
\pi_0 = G_1(w_1) + G_2(u), \quad (2.3)
\]

where we simply denoted by \(G_i, i = 1, 2,\) the operators \(G_{0,i}\) defined in (1.11) (in line with the appearance of \(A\) in (1.12)).

1. The Plate Velocity. From (2.2e), the velocity component \(w_2\) is immediately resolved.

2. The Fluid Velocity. We next consider the Stokes system (2.2a)–(2.2d). From (2.2e) and (2.2c)-(2.2d) we have \(u|_{\partial \mathcal{O}}\) satisfies

\[
\int_{\partial \mathcal{O}} u \cdot \nu \, d\sigma = \int_\Omega [0,0,u^3] \cdot \nu \, d\Omega = \int_\Omega w_2 \, d\Omega = \int_\Omega w_1^* \, d\Omega = 0, \quad (2.4)
\]

where the last equality follows by the membership \([u^*, w_1^*, w_2^*] \in \mathbf{H}\). Since this compatibility condition is satisfied and data \(\{u^*, w_1^*, w_2^*\} \in L^2(\mathcal{O}) \times H^2_0(\Omega)\), we can find a unique (fluid and pressure) pair \((u, q_0) \in [H^2(\mathcal{O}) \cap H^f] \times H^1(\mathcal{O})/\mathbb{R}\) which solve

\[
\begin{align*}
\Delta u - \nabla q_0 &= u^* \quad \text{in } \mathcal{O} \quad (2.5a) \\
\text{div}(u) &= 0 \quad \text{in } \mathcal{O} \quad (2.5b) \\
u &= 0 \quad \text{on } S \quad (2.5c) \\
u &= (0, 0, w_1^*) \quad \text{on } \Omega \quad (2.5d)
\end{align*}
\]

Moreover, one has the estimate

\[
\|u\|_{H^2(\mathcal{O}) \cap H^f} + \|q_0\|_{H^1(\mathcal{O})/\mathbb{R}} \leq C [\|u^*\|_{H^f} + \|w_1^*\|_{H^2_0(\Omega)}] \quad (2.6)
\]

(see e.g., [20, Proposition 2.3, p. 25]).

3. The Mechanical Displacement. Subsequently, we consider the plate component boundary value problem (BVP) (2.2f)-(2.2g). By ellipticity and elliptic regularity (see [17]) there exists a solution \(\hat{w}_1 \in H^4(\Omega) \cap H^2_0(\Omega)\) to the problem

\[
\begin{align*}
\Delta^2 \hat{w}_1 &= q_0 |_{\Omega} - w_2^* \quad \text{in } \Omega \\
\hat{w}_1 &= \frac{\partial \hat{w}_1}{\partial \nu} = 0 \quad \text{on } \partial \Omega
\end{align*}
\]
where $q_0$ is the pressure variable in $\omega$. Moreover, we have the estimate
\[
\|\hat{w}_1\|_{H^1(\Omega)\cap H^2_0(\Omega)} \leq C \|q_0\|_{\Omega} + w_2^* \|L^2(\Omega) \leq C \|q_0\|_{H^{1/2}(\Omega)} + \|\hat{w}_2^*\|_{L^2(\Omega)} \leq C \|w^*, w_1^*, w_2^*\|_{H},
\]
(in the last inequality we have also invoked Sobolev Trace Theory and (2.9)).

Finally, as $\nu$, – orthogonal with respect to the inner product $[\omega, \tilde{\omega}] \rightarrow (\Delta \omega, \Delta \tilde{\omega})_{\Omega}$, then one can readily show that its orthogonal complement $I - \mathbb{P}$ can be characterized as
\[
(I - \mathbb{P})H^2_0(\Omega) = \text{Span}\{\varphi\}, \quad \text{where} \\
\{\Delta^2 \varphi = 1 \text{ in } \Omega, \; \varphi = \frac{\partial \varphi}{\partial \nu} = 0 \text{ on } \partial \Omega\}. \tag{2.8}
\]
(see [11] Remark 2.1, p. 6). With these projections, we then set
\[
w_1 = \mathbb{P}\hat{w}_1, \\
\pi_0 = q_0 - \Delta^2 (I - \mathbb{P})\hat{w}_1. \tag{2.9}
\]
With this assignment of variables, then by (2.7) and $\hat{w}_1 = \mathbb{P}\hat{w}_1 + (I - \mathbb{P})\hat{w}_1$, we will have that $w_1$ solves (2.2a)–(2.2d). (And of course since $\pi_0$ and $q_0$ differ only by a constant, then the pair $(u, \pi_0)$ also solves (2.2a)–(2.2d).)

Moreover, from elliptic theory, (2.6) and (2.7), we have the estimate
\[
\|w_1\|_{H^1(\Omega)\cap H^2_0(\Omega)\cap L^2(\Omega)/\mathbb{R}} + \|\pi_0\|_{H^1(\Omega)} \leq C \left( \|\Delta^2 (I - \mathbb{P})\hat{w}_1\|_{L^2(\Omega)} + \|q_0\|_{H^{1/2}(\Omega)\cap L^2(\Omega)} + \|\hat{w}_2^*\|_{L^2(\Omega)} \right) \leq C \|w^*, w_1^*, w_2^*\|_{H},
\]
where implicitly we are also using the fact that $\Delta^2 (I - \mathbb{P}) \in \mathcal{L}(H^2_0(\Omega), \mathbb{R})$, by the Closed Graph Theorem.

4. Resolution of the Pressure. As we noted in (1.14) we have $\Delta u \cdot \nu \in H^{-1/2}(\partial \Omega)$, with the estimate
\[
\|\Delta u \cdot \nu\|_{H^{-1/2}(\partial \Omega)} \leq C \|u\|_{H^2(\Omega)} \leq C \left[ \|w^*\|_{H^2} + \|w_1^*\|_{H^2(\Omega)} \right], \tag{2.10}
\]
where for the second inequality we have also used (2.6). We will apply this estimate to the pressure variable $\pi_0$ in (2.2a) – given explicitly in (2.9) – which solves a fortiori
\[
\begin{cases}
\Delta \pi_0 = 0 & \text{in } \Omega \\
\frac{\partial \pi_0}{\partial \nu} = \Delta u \cdot \nu|_S & \text{on } S \\
\frac{\partial \pi_0}{\partial \nu} + \pi_0 = \Delta^2 w_1 + \Delta w_1^* \text{ on } \Omega
\end{cases}
\]

In fact: Applying the divergence operator to both sides of (2.2a) and using $\text{div}(\Delta u) = 0$, we obtain that $\pi_0$ is harmonic in $\Omega$. Moreover, dotting both sides of (2.2a) with respect to the normal vector, and subsequently taking the boundary trace to the portion $S$, we get the boundary condition on $S$ (implicitly we are also using $u^* \cdot \nu|_S = 0$, as $[u^*, w_1^*, w_2^*] \in \mathbb{H}$). Finally, as $u^* \cdot \nu|_\Omega = w_2^*$, and as $[u^*, w_1^*, w_2^*] \in \mathbb{H}$, we have from (2.2a)
\[
\pi_0|_\Omega = w_2^* + \Delta^2 w_1 = \Delta u \cdot \nu|_\Omega - \nabla \pi_0 \cdot \nu|_\Omega + \Delta^2 w_1,
\]

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which gives the boundary condition on $\Omega$ in (2.1). Necessarily then, the pressure term must be given by the expression

$$
\pi_0 = G_{0,1}(w_1) + G_{0,2}(u) \in H^1(\mathcal{O})
$$

(2.11)

(with the well-definition of right hand side assured by (2.10)).

Finally, we collect: (i) (2.5a)–(2.5d) and (2.6) (for the fluid variable $u, w$ of (2.13), with respect to $\pi_0$); (ii) (2.7) and (2.2e) and (2.9) (for the respective structure and pressure variables $w_1, w_2$ and $\pi_0$); (iii) (2.10) and (2.11) (for the characterization of the pressure term $\pi_0$). In this way we have obtained the solution of (2.2)–(2.3) in $\mathcal{D}(A)$. In short, $0 \in \rho(A)$, which concludes the proof. 

\[ \square \]

2.2 Proof of the Main Result Theorem 1.3

Proof of Theorem 1.3. By Theorem 2.1, the fluid structure semigroup $\{e^{At}\}_{t \geq 0}$ will be uniformly stable provided its associated resolvent operator $R(\lambda; A)$ is bounded on the imaginary axis; viz.,

$$
\|R(i\beta; A)\|_H \leq C \quad \text{for all } \beta \in \mathbb{R}.
$$

(2.12)

By way of establishing (2.12), we consider the following resolvent equation, for $\beta \in \mathbb{R} \setminus \{0\}$ (recall that we have already established that $0 \in \rho(A)$): Given data $[u^*, w_1^*, w_2^*] \in H$, we look for $[w_1, w_2, u] \in \mathcal{D}(A)$ which solves

$$
(i\beta - A) \begin{bmatrix} u \\ w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} u^* \\ w_1^* \\ w_2^* \end{bmatrix}.
$$

(2.13)

(Here, we also take $|\beta| \geq 1$, as $|\beta| < 1$ is relatively straightforward.)

From (2.12) and (1.3), we see then that $[u, w_1, w_2] \in \mathcal{D}(A)$ satisfies the following PDE system

\begin{align}
\begin{aligned}
i\beta u - \Delta u + \nabla p &= u^* \in H_f \\
div(u) &= 0 & \text{in } \mathcal{O} \\
u &= 0 & \text{on } S \\
u &= (u^*, u_1^*, u_2^*) = (0, 0, i\beta w_1 - w_1^*) & \text{on } \Omega \\
i\beta w_1 - w_2 = w_1^* & \in \left[H^2_s(\Omega) \cap L^2(\Omega)/\mathbb{R}\right] \\
- \beta^2 w_1 + \Delta^2 w_1 - p|_{\Omega} &= w_2^* + i\beta w_1^* & \in L^2(\Omega)/\mathbb{R} \\
w_1|_{\partial\Omega} &= \frac{\partial w_1}{\partial n} \big|_{\partial\Omega} &= 0.
\end{aligned}
\end{align}

(2.14a–2.14g)

Step 1. (A relation for the fluid gradient) We start by taking the $H$-inner product of both sides of (2.13), with respect to $[u, w_1, w_2]$. This gives

$$
i\beta \left\| \begin{bmatrix} u \\ w_1 \\ w_2 \end{bmatrix} \right\|_H^2 - \left( A \begin{bmatrix} u \\ w_1 \\ w_2 \end{bmatrix}, \begin{bmatrix} u \\ w_1 \\ w_2 \end{bmatrix} \right)_H = \left( \begin{bmatrix} u^* \\ w_1^* \\ w_2^* \end{bmatrix}, \begin{bmatrix} u \\ w_1 \\ w_2 \end{bmatrix} \right) _H.
$$

Combining this with the readily derivable relation

$$
\left( A \begin{bmatrix} u \\ w_1 \\ w_2 \end{bmatrix}, \begin{bmatrix} u \\ w_1 \\ w_2 \end{bmatrix} \right)_H = -\|\nabla u\|_\mathcal{O}^2 - 2i \text{Im}(\Delta w_1, \Delta w_2)_\Omega
$$

(2.15)

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we then will have the following “static dissipation”:

$$\|\nabla u\|_{L^2(\Omega)}^2 = \text{Re} \left( \begin{bmatrix} u^* & u \\ w_1 & w_2 \end{bmatrix} \right) .$$

(2.16)

This gives then, for arbitrary $\epsilon > 0$,

$$\|\nabla u\|_{L^2(\Omega)} \leq \epsilon \left\| \begin{bmatrix} u \\ w_1 \\ w_2 \end{bmatrix} \right\|_H + C \epsilon \left\| \begin{bmatrix} u^* \\ w_1^* \\ w_2^* \end{bmatrix} \right\|_H .$$

(2.17)

Step 2. (Control of the mechanical velocity) This comes quickly: using the fluid Dirichlet boundary condition in (2.13) we have

$$i \beta w_1 = u^3|_{\Omega} - w_1^*. $$

We estimate this expression by invoking in sequence, the Sobolev Embedding Theorem, Poincaré Inequality and (2.17). In this way, we then obtain

$$\| \beta w_1 \|_{H^{1/2}(\Omega)} \leq \| u \|_{H^{1/2}(\Omega)} - \| w_1^* \|_{H^{1/2}(\Omega)} \leq C \left\| \frac{\nabla u}{L} \right\|_{H^1(\Omega)} + \| u \|_{H^{1/2}(\Omega)} \right\|_H \leq \epsilon C \left\| \begin{bmatrix} u \\ w_1 \\ w_2 \end{bmatrix} \right\|_H + C \epsilon \left\| \begin{bmatrix} u^* \\ w_1^* \\ w_2^* \end{bmatrix} \right\|_H .$$

(2.18)

Using subsequently the resolvent relation $w_2 = i \beta w_1 - w_1^*$ now gives

$$\| \beta w_1 \|_{H^{1/2}(\Omega)} + \| w_2 \|_{H^{1/2}(\Omega)} \leq \epsilon C \left\| \begin{bmatrix} u \\ w_1 \\ w_2 \end{bmatrix} \right\|_H + C \epsilon \left\| \begin{bmatrix} u^* \\ w_1^* \\ w_2^* \end{bmatrix} \right\|_H .$$

(2.19)

Step 3. (Control of the mechanical displacement) We multiply both sides of the mechanical equation (2.14a) by $w_1$ and integrate. This gives the relation

$$\left( \Delta^2 w_1, w_1 \right)_{L^2(\Omega)} = \left( p|_{\Omega}, w_1 \right)_{\Omega} + \beta^2 \| w_1 \|_{L^2(\Omega)}^2 + \left( w_2^* + i \beta w_1^*, w_1 \right)_{L^2(\Omega)} .$$

(2.20)

To handle the first term we use the fact that since $[w_1, w_2, u] \in H$, then in particular

$$\int_\Omega w_1 d\Omega = 0 .$$

In consequence, one has well-posedness of the following boundary value problem (see [21] Proposition 2.2)

$$\begin{align*}
-\Delta \psi + \nabla q &= 0 & \text{in } \Omega \\
\text{div} \psi &= 0 & \text{in } \Omega \\
\psi|_S &= 0 & \text{on } S \\
\psi|_\Omega = (\psi^4, \psi^2, \psi^3)|_\Omega &= (0, 0, w_1) & \text{on } \Omega
\end{align*}$$

(2.21)
Another application of Green’s formula, along the fact that the proof of exponential decay of finite energy solutions of (1.2)-(1.3) is now complete. From (2.18), we get
\[ \varepsilon > 0 \]
This yields the required uniform norm estimate (2.12), upon taking \( \varepsilon \) small enough. The use of Green’s Identities and the fluid equation in (2.23) then gives
\[
(p|\Omega, w_1)\Omega = -\left( \frac{\partial u}{\partial \nu}, \psi \right)_{L^2(\partial \Omega)} + (p \nu, \psi)_{L^2(\partial \Omega)}
\]
\[
= -\left( \frac{\partial u}{\partial \nu}, \psi \right)_{L^2(\partial \Omega)} + (p \nu, \psi)_{L^2(\partial \Omega)}
\]
Estimating the latter right hand side by means of (2.17), (2.18) and (2.22), we get for \( |\beta| > 1 \),
\[
\| (p|\Omega, w_1)\Omega \| \leq C |\beta| \| \nabla \psi \|_{L^2(\Omega)} \left( \| \nabla u \|_{L^2(\Omega)} + \| u^* \|_{L^2(\Omega)} \right)
\]
\[
\leq \varepsilon C \left\| \begin{bmatrix} u \\ w_1 \\ w_2 \end{bmatrix} \right\|_{H^1(\Omega)} + C_{\varepsilon} \left\| \begin{bmatrix} u^* \\ w_1^* \\ w_2^* \end{bmatrix} \right\|_{H^1(\Omega)}.
\] (2.24)
Applying the obtained estimate (2.24) to the right hand side of (2.20), and using once more (2.18), we get
\[
(\Delta^2 w_1, w_1)_{L^2(\Omega)} \leq C \left\| \begin{bmatrix} u \\ w_1 \\ w_2 \end{bmatrix} \right\|_{H^1(\Omega)} + C_{\varepsilon} \left\| \begin{bmatrix} u^* \\ w_1^* \\ w_2^* \end{bmatrix} \right\|_{H^1(\Omega)}.
\]
Another application of Green’s formula, along the fact that \( w_1 \) satisfies hinged boundary conditions, gives
\[
\| \Delta w_1 \|_{L^2(\Omega)}^2 \leq C \left\| \begin{bmatrix} u \\ w_1 \\ w_2 \end{bmatrix} \right\|_{H^1(\Omega)} + C_{\varepsilon} \left\| \begin{bmatrix} u^* \\ w_1^* \\ w_2^* \end{bmatrix} \right\|_{H^1(\Omega)}.
\] (2.25)
Thus, combining (2.17), (2.19) and (2.25) yields now
\[
\| [u, w_1, w_2] \|_{H^1(\Omega)}^2 \leq C \| [u, w_1, w_2] \|_{H^1(\Omega)}^2 + C_{\varepsilon} \| [u^*, w_1^*, w_2^*] \|_{H^1(\Omega)}^2.
\]
This yields the required uniform norm estimate (2.12), upon taking \( \varepsilon > 0 \) small enough. The proof of exponential decay of finite energy solutions of (1.2)-(1.3) is now complete.

\[ \square \]
3 The associated optimal control problems: relevant scenarios, expected difficulties

In this section we briefly discuss a couple of possible implementations for the placement of control actions into the PDE system (1.2)-(1.3); these are complemented with some remarks about the technical challenges which are expected in the forthcoming study of the associated optimal control problems (with quadratic functionals).

3.1 A first setup: point control on the mechanical component

A classical scenario worth studying is the case of point control exerted on the elastic wall $\Omega$. The control action may be mathematically described by

$$ Bg = \sum_{j=1}^{J} a_j g_j \delta_{\xi_j}, $$

where $\xi_j$ are points in $\Omega$, and $\delta_{\xi_j}$ denote the corresponding delta functions. The control space here is $U = \mathbb{R}^J$ and

$$ B : U \to H^{-1-\sigma}(\Omega), \quad \sigma > 0; $$

accordingly, the initial/boundary value problem (IBVP) is as follows:

$$ \begin{align*}
    u_t - \Delta u + \nabla p &= 0 \quad \text{in } O \times (0,T) \\
    \text{div } u &= 0 \quad \text{in } O \times (0,T) \\
    u &= 0 \quad \text{on } S \times (0,T) \\
    u &= (u^1, u^2, u^3) = (0, 0, w_1) \quad \text{in } \Omega \times (0,T) \\
    w_{tt} - \rho \Delta w_{tt} + \Delta^2 w - p|_{\Omega} + Bg &= 0 \quad \text{in } \Omega \times (0,T) \\
    w &= \frac{\partial w}{\partial \nu} = 0 \quad \text{on } \partial \Omega \times (0,T) \\
    u(\cdot,0) &= u_0 \quad \text{in } O \\
    w(\cdot,0) &= w_0, \quad w_{t}(\cdot,0) = w_1 \quad \text{in } \Omega.
\end{align*} \tag{3.1} $$

To deal with the IBVP problem (3.1), with the natural state space $Y$ given by $H_{\rho}$ and the control space $U$ defined above, we will appeal to the the corresponding abstract formulation

$$ \begin{align*}
    y'(t) &= Ay(t) + Bg(t), \quad 0 < t < T \\
    y(0) &= y_0 \in Y, \tag{3.2}
\end{align*} $$

where

- $A : \mathcal{D}(A) \subset Y \to Y$ is the infinitesimal generator of a $C_0$-semigroup $e^{At}$ on $Y$, $t \geq 0$;
- $B \in \mathcal{L}(U, [\mathcal{D}(A^*)]')$; equivalently, $A^{-1}B \in \mathcal{L}(U,Y)$.

The most prominent features of the controlled dynamics is that (i) the control operator $B$ is not bounded from $U$ into $Y$; (ii) the semigroup $e^{At}$ is not analytic; in fact the semigroup results from a strong coupling of analytic Stokes flow with hyperbolic plate dynamics.

To the control system (3.2) we associate a quadratic functional:

$$ J(g) = \int_0^T \left( \|Ry(t)\|_Z^2 + \|g(t)\|_U^2 \right) \, dt, \tag{3.3} $$

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where $R \in \mathcal{L}(Y,Z)$ denotes the observation operator and $Z$ the observation space; thus, the optimal control problem is formulated as follows.

**Problem 3.1** (The optimal control problem). Given $y_0 \in Y$, we seek a control function $g \in L^2(0,T;U)$ which minimizes the cost functional $\mathcal{J}(x)$, where $y(\cdot) = y(\cdot; y_0, g)$ is the solution to \( (3.2) \) corresponding to $g(\cdot)$.

As is well known, the core of the work is to pinpoint the regularity of the input-to-state behavior of the optimal control problem.

**Remark 3.2.** If the aforesaid theory applies, we will achieve well-posedness of Riccati equations with bounded gains (on a dense subset of $Y$), without requiring smoothing effects of the observation operator $R$. This has been done successfully in the case of a different F-S model (solid immersed in a fluid); see [8] and [9]. (See [15] and [16] for a study of a Bolza problem associated with the very same PDE model.)

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**3.2 A different setup: boundary control on the fluid component**

Another interesting scenario is the case of boundary control acting on some part $\Sigma$ of $S$. A tentative condition\(^1\) to be taken into consideration is $u \cdot \nu = g$ on $\Sigma$. The IBVP becomes as follows:

\[
\begin{align*}
    u_t - \Delta u + \nabla p &= 0 & \text{in } \Omega \times (0,T) \\
    \text{div } u &= 0 & \text{in } \Omega \times (0,T) \\
    u &= 0 & \text{on } S \setminus \Sigma \times (0,T) \\
    u \cdot \nu &= 0, \quad u \cdot \tau = 0, & \text{on } \Sigma \times (0,T) \\
    u &= (u^1, u^2, u^3) = (0,0, w_1) & \text{on } \Omega \times (0,T) \\
    w_{tt} - \rho \Delta w_{tt} + \Delta^2 w &= p|_\Omega & \text{in } \Omega \times (0,T) \\
    w &= \frac{\partial w}{\partial n} = 0 & \text{on } \partial \Omega \times (0,T) \\
    u(\cdot,0) &= u_0 & \text{in } \Omega \\
    w(\cdot,0) &= w_0, \; w_t(\cdot,0) = w_1 & \text{in } \Omega.
\end{align*}
\]

Notice that in the present case one has $U = L^2(\Sigma)$ and $B : g \mapsto Bg = (Bg, 0, 0)$, where $B$ is an appropriate operator which describes the particular normal component control action. The presence of boundary control acting on the parabolic component of the PDE system suggests that we investigate whether the optimal control theory of [1,2] is applicable. The analysis to be performed will require that appropriate regularity results for the boundary traces of the fluid component on $\Sigma$ are established.

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\(^1\) This condition was suggested by Giovanna Guidoboni (IUPUI; also Acting Co-Director of the School of Science Institute of Mathematical Modeling and Computational Science, Indianapolis), in connection with the modeling of ocular blood flow and specifically with the issue of reducing the ocular pressure.
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