Multicoloured Ramsey numbers
of the path of length four

Henry Liu∗ Bojan Mohar† Yongtang Shi‡
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Abstract
Let \( P_t \) denote the path on \( t \) vertices. The \( r \)-coloured Ramsey number of \( P_t \), denoted by \( R_r(P_t) \), is the minimum integer \( n \) such that whenever the complete graph on \( n \) vertices is given an \( r \)-edge-colouring, there exists a monochromatic copy of \( P_t \). In this note, we determine \( R_r(P_5) \), which is approximately \( 3r \).

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1 Introduction
All graphs in this paper are finite, undirected, and have no multiple edges or loops. For any undefined terms in graph theory, we refer to the book by Bollobás [3].

Let \( K_t, P_t \) and \( C_t \) denote the complete graph (or clique), path and cycle on \( t \) vertices. For graphs \( G \) and \( H \), we denote the graph which is the disjoint union of a copy of \( G \) and a copy of \( H \) by \( G \cup H \), and the graph with a disjoint copies of \( G \) by \( aG \). The graph union of \( G \) and \( H \) is \( G \cup H = (V(G) \cup V(H), E(G) \cup E(H)) \).

An \( r \)-edge-colouring of a graph \( G \), or \( r \)-colouring for simplicity, is a function \( f: E(G) \to \{1, \ldots, r\} \). The members of the set \( \{1, \ldots, r\} \) can be thought of as a set of \( r \) colours. The sets \( f^{-1}(i) \) for \( 1 \leq i \leq r \) are the colour classes of the \( r \)-colouring \( f \).

Given graphs \( H_1, \ldots, H_r \), the \( r \)-coloured Ramsey number \( R(H_1, \ldots, H_r) \) is the minimum integer \( n \) such that, whenever we have an \( r \)-edge-colouring of \( K_n \), there exists a monochromatic copy of \( H_i \), for some \( i \). When \( H_1 = \cdots = H_r = H \), we write \( R_r(H) \) for \( R(H, \ldots, H) \). Ramsey’s classical result [13] states that all Ramsey numbers \( R(H_1, \ldots, H_r) \) exist. When all the \( H_i \) are cliques, determining the Ramsey numbers exactly is a notoriously challenging problem, and only a few values of \( R(K_s, K_t) \), as well as \( R(K_3, K_3, K_3) \), are known. Another

∗School of Mathematics, Sun Yat-sen University, Guangzhou 510275, China. Email: liaozhx5@mail.sysu.edu.cn (corresponding author)
†Department of Mathematics, Simon Fraser University, Burnaby, BC, V5A 1S6, Canada. On leave from IMFM, Department of Mathematics, University of Ljubljana, 1000 Ljubljana, Slovenia. Email: mohar@sfu.ca
‡Center for Combinatorics and LPMC, Nankai University, Tianjin 300071, China. Email: shi@nankai.edu.cn
well studied case is when all the $H_i$ are cycles, and in this case, the values of $R(C_s, C_t)$ are completely determined, while there are many interesting results and open questions for $R(C_q, C_s, C_t)$. For more information about Ramsey numbers, we refer the reader to the survey paper of Radziszowski [12].

For the case when the $H_i$ are paths, Gerencsér and Gyárfás [6] proved that $R(P_s, P_t) = t + \lfloor \frac{s}{2} \rfloor - 1$ for $t \geq s \geq 2$. For three colours, Faudree and Schelp [4] conjectured that $R(P_t, P_s, P_1) = 2t - 2 + (t \mod 2)$ for all $t$. This conjecture has been verified for all sufficiently large $t$ by Gyárfás et al. [7]. There are many known exact values for $R(P_q, P_s, P_1)$ when $q$ and $s$ are small. For more colours, since it is well known that the edge-chromatic number of $K_n$ is $n - 1 + (n \mod 2)$, we have $R_r(P_3) = r + 1 + (r \mod 2)$. For $P_4$, we have

$$R_r(P_4) = \begin{cases} 
2r + 1 & \text{if } r \equiv 0, 2 \pmod{3}, r \neq 3, \\
2r + 2 & \text{if } r \equiv 1 \pmod{3}, \\
6 & \text{if } r = 3.
\end{cases}$$

The cases $r \equiv 1, 2 \pmod{3}$ were proved by Irving [9], and he also remarked that $R_3(P_4) = 6$. The case $r \equiv 0 \pmod{3}$ with $r$ not a power of 3 was proved by Lindström [11]. The case where $r$ is a power of 3 was proved by Bierbrauer [2].

Here, we shall determine the Ramsey numbers $R_r(P_5)$ exactly, as follows.

**Theorem 1.** Let $r \geq 1$. Then

$$R_r(P_5) = \begin{cases} 
3r + 1 & \text{if } r \equiv 0 \pmod{4}, r \neq 4, \\
3r + 2 & \text{if } r \equiv 1 \pmod{4}, \\
3r & \text{if } r \equiv 2, 3 \pmod{4}, \\
11 & \text{if } r = 4.
\end{cases}$$

We shall proceed as follows. In Section 2, we gather various results from Ramsey theory, Turán theory, and design theory. In Section 3, we prove Theorem 1.

## 2 Tools

We first observe that for $r \geq 2$ and $t \geq 4$, we have

$$R_{r-1}(P_t) < R_r(P_t). \quad (1)$$

Indeed, both terms in (1) exist. Let $R = R_{r-1}(P_t)$. Then, there exists an $(r - 1)$-colouring of $K_{R-1}$ which does not contain a monochromatic copy of $P_t$. By adding new vertex $u$ to form $K_R$ and colouring all edges incident to $u$ with a new colour, we get an $r$-colouring of $K_R$ with no monochromatic copy of $P_t$ (since $t \geq 4$). Hence, $R_r(P_t) > R = R_{r-1}(P_t)$.

Next, we recall that for a graph $H$ and $n \in \mathbb{N}$, the Turán function for $H$, denoted by $ex(n, H)$, is the maximum possible number of edges in a $H$-free graph (i.e., not containing a copy of $H$ as a subgraph) on $n$ vertices. Any $H$-free graph on $n$ vertices and attaining $ex(n, H)$ edges is said to be extremal. For any path $P_t$, the Turán function $ex(n, P_t)$, as well as the corresponding extremal graphs, were completely determined by Faudree and Schelp [4]. In order to attain $ex(n, P_t)$, we can take the graph on $n$ vertices containing as many disjoint $K_{t-1}$ cliques as possible, i.e., $\lceil \frac{n}{t-1} \rceil$ cliques, and a smaller clique on the remaining vertices. For odd $t$, this graph is the unique extremal graph, and for even $t \geq 4$ and certain values of $n$, there are other extremal graphs. Their result for $P_5$ is the following.
Theorem 2. [4] Let \( n = 4a + b \), where \( a \geq 0 \) and \( 0 \leq b \leq 3 \). We have
\[
ex(n, P_5) = 6a + \frac{b(b-1)}{2}.
\]
Moreover, the unique extremal graph is \( aK_4 \cup K_b \).

Now, we recall some notions and results from design theory. Let \( v \geq k \geq t \) and \( \lambda \) be positive integers. Let \( V \) be a set of \( v \) points, and \( B \) be a family of \( k \)-element subsets of \( V \), called blocks. The pair \((V, B)\) is a Steiner system \( S(t, k, v) \) if any \( t \) points of \( V \) are contained in exactly one block of \( B \). If “exactly one” is replaced by “at least one” and “at most one”, then \((V, B)\) is a \((t, k, v)\)-covering design and a \((t, k, v)\)-packing design, respectively. A \((t, k, v)\)-covering design (resp. \((t, k, v)\)-packing design) \((V, B)\) is optimal if \(|B|\) attains the minimum (resp. maximum) possible value. The pair \((V, B)\) is a balanced incomplete block design, or BIBD, if any two points of \( V \) are contained in exactly \( \lambda \) blocks. Thus, a Steiner system \( S(2, k, v) \) is precisely a BIBD with \( \lambda = 1 \), and we denote such a design by \( B(k, v) \).

It is well-known that a necessary condition for the existence of a \( B(k, v) \) is \( v \equiv 1 \) or \( k \equiv 1 \) (mod \( k(k-1) \)). For a \((2, k, v)\)-packing design \((V, B)\), the leave graph \( L \) is the graph with vertex set \( V(L) = V \), and edge set \( E(L) = \{xy : x, y \in V \text{ and } \{x, y\} \not\subset B \text{ for every } B \in B\} \).

For any of the above designs, a parallel class is a set of blocks of \( B \) that form a partition of \( V \). If \( B \) can be partitioned into parallel classes, then the design is resolvable. Clearly for a design to be resolvable, we must necessarily have \( v \equiv 0 \) (mod \( k \)).

Here, we are interested in designs with \( k = 4 \), \( t = 2 \) and \( \lambda = 1 \). We refer the interested reader to a survey by Reid and Rosa [14] for more information about these designs. For BIBDs, we have the following result from Hanani et al. [8].

Theorem 3. [8] For every \( v \equiv 4 \) (mod \( 12 \)), there exists a resolvable BIBD \( B(4, v) \). The number of parallel classes is \( \frac{v-1}{3} \).

For \((2, 4, v)\)-covering designs, by combining results of Lamken et al. [10], and Abel et al. [11], we have the following result.

Theorem 4. [10] For \( v \equiv 0, 8 \) (mod \( 12 \)) with \( v \neq 12 \), there exists an optimal \((2, 4, v)\)-covering design which is resolvable, except possibly for \( v \in \{108, 116, 132, 156, 204, 212\} \). The number of parallel classes is
\[
\begin{cases} 
\frac{v}{3} & \text{if } v \equiv 0 \text{ (mod } 12)\text{,} \\
\frac{v+1}{3} & \text{if } v \equiv 8 \text{ (mod } 12)\text{.}
\end{cases}
\]

For \((2, 4, v)\)-packing designs, the following result was proved by Ge et al. [5]. Several cases of the result were proved by various authors. See [5] for the references therein.

Theorem 5. [5]

(a) For \( v \equiv 0 \) (mod \( 12 \)) with \( v \neq 12 \), there exists an optimal \((2, 4, v)\)-packing design which is resolvable. The number of parallel classes is \( \frac{v^2}{3} \), and the leave graph is \( L = \frac{v}{3}K_3 \).

(b) For \( v \equiv 8 \) (mod \( 12 \)) with \( v \neq 8, 20 \), there exists an optimal \((2, 4, v)\)-packing design which is resolvable, except possibly for \( v \in \{68, 92, 104, 140, 164, 188, 200, 236, 260, 284, 356, 368, 404, 428, 476, 500, 668, 692\} \). The number of parallel classes is \( \frac{v^2}{3} \), and the leave graph is \( L = \frac{v}{2}K_2 \).
3 Ramsey numbers of the path of length four

In this section, we prove Theorem 1. First, we prove the required upper bound for $R_r(P_5)$, when $r \neq 4$.

**Lemma 6.** For $r \geq 1$, we have

$$R_r(P_5) \leq \begin{cases} 
3r + 1 & \text{if } r \equiv 0 \pmod{4}, \\
3r + 2 & \text{if } r \equiv 1 \pmod{4}, \\
3r & \text{if } r \equiv 2, 3 \pmod{4}.
\end{cases}$$

**Proof.** Let $n = 4a + b$, where $a \geq 0$ and $0 \leq b \leq 3$. By Theorem 2, we have $\text{ex}(n, P_5) = 6a + \frac{1}{2}b(b - 1) = \frac{3}{2}n + \frac{1}{2}b^2 - 2b$, and the unique extremal graph is $aK_4 \cup K_b$. Now suppose that we have an $r$-colouring of $K_n$. Then the most frequent colour, say red, has at least $\lceil \frac{n}{r} \right \rangle$ edges. If $r \equiv 0 \pmod{4}$ and $n = 3r + 1$, or $r \equiv 1 \pmod{4}$ and $n = 3r + 2$, then $b = 1$, so that $\text{ex}(n, P_5) = \frac{3}{2}n - \frac{3}{2}$. We have

$$\left\lfloor \frac{n}{r} \right\rfloor \geq \frac{3}{2}n > \text{ex}(n, P_5) + 1.$$

If $r \equiv 2 \pmod{4}$ and $n = 3r$, then $b = 2$, so that $\text{ex}(n, P_5) = \frac{3}{2}n - 2$. We have

$$\left\lfloor \frac{n}{r} \right\rfloor = \left\lfloor \frac{3(n - 1)}{2} \right\rfloor = \frac{3(n - 1) + 1}{2} = \text{ex}(n, P_5) + 1.$$

In all three cases, we have a red $P_5$.

If $r \equiv 3 \pmod{4}$ and $n = 3r$, then $b = 1$. We have

$$\left\lfloor \frac{n}{r} \right\rfloor = \frac{3(n - 1)}{2} = \text{ex}(n, P_5).$$

Suppose that there is no monochromatic copy of $P_5$. Then, every colour class has exactly $\left\lfloor \frac{n}{r} \right\rfloor = \text{ex}(n, P_5)$ edges, and moreover, must induce the unique extremal graph $aK_4 \cup K_1$. Let $u$ be the vertex of $K_n$ which corresponds to $K_1$ in the red $aK_4 \cup K_1$. Then for every other colour, the number of edges incident to $u$ is three or zero, so that there are at most $3(r - 1)$ edges at $u$. This contradicts that $u$ has degree $3r - 1$ in $K_n$. \qed

Next, we prove the matching lower bound for $R_r(P_5)$, when $r \neq 4$.

**Lemma 7.** For $r \geq 1$ with $r \neq 4$, we have

$$R_r(P_5) \geq \begin{cases} 
3r + 1 & \text{if } r \equiv 0 \pmod{4}, \\
3r + 2 & \text{if } r \equiv 1 \pmod{4}, \\
3r & \text{if } r \equiv 2, 3 \pmod{4}.
\end{cases}$$

**Proof.** We first note that the case $r \equiv 2 \pmod{4}$ follows from the case $r \equiv 1 \pmod{4}$, since by \cite{1}, we have $R_r(P_5) \geq R_{r-1}(P_5) + 1 \geq 3(r - 1) + 2 + 1 = 3r$ for $r \equiv 2 \pmod{4}$.

Now, define

$$g(r) = \begin{cases} 
3r \equiv 0 \pmod{12} & \text{if } r \equiv 0 \pmod{4}, r \neq 4, \\
3r + 1 \equiv 4 \pmod{12} & \text{if } r \equiv 1 \pmod{4}, \\
3r - 1 \equiv 8 \pmod{12} & \text{if } r \equiv 3 \pmod{4}.
\end{cases}$$

4
For \( r \equiv 1 (\text{mod} \ 4) \), by Theorem 3 there exists a resolvable BIBD \( B(4, g(r)) \). For \( r \equiv 0, 3 \) (mod 4), by Theorem 4 there exists an optimal \( (2, 4, g(r)) \)-covering design which is resolvable, except for \( g(r) \in \{108, 116, 132, 156, 204, 212\} \). In each case, we obtain an \( r \)-colouring of \( K_{g(r)} \), where an edge \( xy \) is given colour \( i \) if \( x \) and \( y \) are contained in a block in the \( i \)th parallel class. The number of parallel classes is

\[
\begin{cases}
\frac{3r}{3} = r & \text{if } r \equiv 0 \text{ (mod 4)}, r \neq 4, \\
\frac{(3r + 1) - 1}{3} = r & \text{if } r \equiv 1 \text{ (mod 4)}, \\
\frac{(3r - 1) + 1}{3} = r & \text{if } r \equiv 3 \text{ (mod 4)}.
\end{cases}
\]

We have an \( r \)-colouring of \( K_{g(r)} \) which does not contain a monochromatic component with more than four vertices, and thus does not contain a monochromatic copy of \( P_5 \).

Now let \( r \equiv 0 \) (mod 4) and \( g(r) = 3r \in \{108, 132, 156, 204\} \). By Theorem 5(a), there exists an optimal \( (2, 4, 3) \)-packing design which is resolvable. The number of parallel classes is \( \frac{3r - 3}{3} = r - 1 \), and the leave graph is \( L = rK_3 \). Similarly, for \( r \equiv 3 \) (mod 4) and \( g(r) = 3r - 1 \in \{116, 212\} \), by Theorem 5(b), there exists an optimal \( (2, 4, g(r)) \)-packing design which is resolvable. Note that 116 and 212 are not in the list of 18 exceptional values in Theorem 5(b). The number of parallel classes is \( \frac{(3r - 1) - 2}{3} = r - 1 \), and the leave graph is \( L = \frac{3r - 2}{2}K_2 \). In both cases, we obtain an \( r \)-colouring of \( K_{g(r)} \), where an edge \( xy \) is given colour \( i \) if \( x \) and \( y \) are contained in a block in the \( i \)th parallel class, for \( 1 \leq i \leq r - 1 \); and colour \( r \) if \( xy \in E(L) \). Then, all monochromatic components in this \( r \)-colouring are \( K_4 \), \( K_3 \) or \( K_2 \), so there is no monochromatic copy of \( P_5 \).

This means that for every \( r \neq 4 \), we have \( R_r(P_5) \geq g(r) + 1 \). \( \square \)

To complete the proof of Theorem 4 it remains to compute \( R_4(P_5) \).

**Lemma 8.** \( R_4(P_5) = 11 \).

**Proof.** We first obtain the lower bound \( R_4(P_5) \geq 11 \). Let \( x_1, x_2, y_1, \ldots, y_4, z_1, \ldots, z_4 \) be 10 vertices, and \( G_1, \ldots, G_4 \) to be the graphs consisting of disjoint cliques on the following sets.

\[
\begin{align*}
G_1: & \ \{x_1, x_2\}, \ \{y_1, \ldots, y_4\}, \ \{z_1, \ldots, z_4\}, \\
G_2: & \ \{x_1, y_1, y_2, z_3\}, \ \{x_2, z_1, z_2, y_3\}, \ \{y_4, z_4\}, \\
G_3: & \ \{x_1, z_1, z_2, y_4\}, \ \{x_2, y_1, y_2, z_4\}, \ \{z_3, z_3\}, \\
G_4: & \ \{x_1, y_3, z_4\}, \ \{x_2, y_4, z_3\}, \ \{y_1, y_2, z_1, z_2\}.
\end{align*}
\]

It is easy to verify that \( G_1 \cup G_2 \cup G_3 \cup G_4 = K_{10} \). We obtain a 4-colouring of \( K_{10} \) where an edge \( xy \) is given colour \( i \) if \( i \) satisfies \( xy \in E(G_i) \). This 4-colouring does not contain a monochromatic copy of \( P_5 \), and hence \( R_4(P_5) \geq 11 \).

Now we prove the matching upper bound \( R_4(P_5) \leq 11 \). Let \( K_4^- \) denote the graph obtained by deleting an edge from \( K_4 \).

**Claim 9.** If \( G \) is a \( P_5 \)-free graph with 11 vertices and 14 edges, then \( G = K_4 \cup K_4 \cup P_3 \) or \( G = K_4 \cup K_4^- \cup K_3 \).

**Proof.** Let \( F \) be a component of \( G \) with \( s \) vertices. Since \( G \) is \( P_5 \)-free, if \( F \) contains a cycle, then the longest cycle of \( F \) has length 3 or 4. If the former, then \( F \) is a \( C_3 \) with possibly some
pendent edges attached to one vertex, and \( e(F) = s \). If the latter, then \( F = C_4, K^-_4 \) or \( K_4 \), and \( e(F) = s, s+1 \) or \( s+2 \) respectively. Otherwise, \( F \) is a \( P_5 \)-free tree, and \( e(F) = s-1 \). Since \( |V(G)| = 11 \), at most two components of \( G \) can be \( K^-_4 \) or \( K_4 \). Also, since \( e(G) = |V(G)| + 3 \), this means that exactly two components are either both \( K_4 \), or one is \( K^-_4 \) and the other is \( K_4 \). We can then easily see that \( G = K_4 \cup K^-_4 \cup P_3 \) or \( G = K_4 \cup K^-_4 \cup K_3 \). \( \square \)

Suppose that we have a 4-colouring of \( K_{11} \) with vertex set \( V \), which does not contain a monochromatic copy of \( P_5 \). Let \( G_1, \ldots, G_4 \) be the four graphs on \( V \) induced by the colour classes, and assume that \( e(G_1) \geq \cdots \geq e(G_4) \). By Theorem 2 we have \( \text{ex}(11, P_5) = 15 \), with the unique extremal graph \( K^-_4 \cup K^-_4 \cup K_3 \). Hence, \( e(G_1) \leq 15 \). If \( e(G_1) = 15 \), then \( G_1 = K_4 \cup K^-_4 \cup K_3 \), and \( e(G_2) \geq \left\lceil \frac{\text{ex}(11, P_5) - 15}{3} \right\rceil = \left\lceil \frac{55-15}{3} \right\rceil = 14 \). By Claim 9, \( G_2 \) contains a copy of \( K_4 \), which is a contradiction since the complement of \( G_1 \) does not contain a copy of \( K_4 \). Otherwise, we have \( e(G_1) = e(G_2) = e(G_3) = 14 \) and \( e(G_4) = 13 \). Again by Claim 9, we have \( G_1 = K_4 \cup K^-_4 \cup P_3 \) or \( K_4 \cup K^-_4 \cup K_3 \), and \( G_2 \) and \( G_3 \) both contain a copy of \( K_4 \). We see that \( G_1 \cup G_2 \) must contain a copy of \( K_4 \cup K^-_4 \cup K_3 \), so that the complement of \( G_1 \cup G_2 \) does not contain a copy of \( K_4 \). This contradicts that \( G_3 \) contains a copy of \( K_4 \). This completes the proof of Lemma 8. \( \square \)

By Lemmas 6, 7 and 8, Theorem 1 is proved.

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