A Lower Bound for Primality of Finite Languages

Philip Sieder
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Abstract
A regular language $L$ is said to be prime, if it is not the product of two non-trivial languages. Martens et al. settled the exact complexity of deciding primality for deterministic finite automata in 2010. For finite languages, Mateescu et al. and Wieczorek suspect the NP-completeness of primality, but no actual bounds are given. Using the techniques of Martens et al., we prove the NP lower bound and give a $\Pi^P_2$ upper bound for deciding primality of finite languages given as deterministic finite automata.

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1 Introduction

Coming from number theory, the primality of regular languages is a quite natural problem. As integers have a unique prime factorisation, one could hope to decompose languages into indecomposable (and therefore possibly simpler) languages. Unfortunately the decompositions of languages do not behave as nicely as those of numbers. A language, if decomposable, can have different decompositions. Neither the number of prime factors is unique nor do different decompositions need to have common prime factors [MSY98, Section 4]. Therefore the most interesting question is, whether a language can be decomposed at all, or in other words whether a language is prime. As in number theory, the complexity of a primality test (for regular languages) was pinpointed relatively recently. Martens et al. [MNS10] showed that the problem is PSPACE-complete. For finite languages in particular, there are pursuits by Mateescu et al. [MSY98] and Wieczorek [Wie10], but, besides an NP-completeness conjecture, no actual bounds have been given. Using the ideas of Martens et al., we prove an NP lower bound and a \( \Pi^P_2 \) upper bound for the problem. So again languages behave way worse than numbers, where primality can be tested in polynomial time.

In Section 2 we establish the notation and give definitions for the general language theoretical facts we need. In Section 3 we give some insight on the necessary properties for studying primality of regular languages. Those enable the proof of the \( \Pi^P_2 \) upper bound at the end of the section. Section 4 provides the NP-hardness by establishing a chain of polynomial time reductions, similar to the one in the proof of Martens, Niewerth and Schwentick. In the final Section 5 we give a brief compilation of what is yet to be determined.

2 Preliminaries

In this section we will introduce the basic concepts and notations. We omit the facts about complexity classes and polynomial time reduction. For those concepts and definitions we refer to Papadimitriou's book [Pap94]. First let us fix some general symbols:
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Notation. \([a, b] := \{m \in \mathbb{Z} \mid a \leq m \leq b\}\) with \(a, b \in \mathbb{Z}\) integers. \(n\mathbb{Z} := \{n \cdot m \mid m \in \mathbb{Z}\}\) with \(n \in \mathbb{Z}\) an integer.

For a computational decision problem \(\text{PROBLEM}\), \(\neg\text{PROBLEM}\) describes the same problem with negated answer.

Now we will introduce the most important concepts about regular languages and finite automata we use. Since this part is mostly to fix the notation, we do not give much explanation or motivation and the definitions might have minor inaccuracies. For a more thorough understanding of those conceptions we refer to the book of Hopcroft et al. [HMRU00].

Definition 1. A (finite) alphabet is a finite set \(\Sigma\) of letters. A word \(w = a_1 \ldots a_n\) is a finite sequence of letters \(a_i \in \Sigma\) and \(|w| = |a_1 \ldots a_n| := n\) is the length of the word. The empty word (of length zero) is written as \(\varepsilon\). For two words \(v = a_1 \ldots a_m\) and \(w = b_1 \ldots b_n\), \(v \circ w := vw := a_1 \ldots a_mb_1 \ldots b_n\) describes the concatenation of the two words \(v\) and \(w\).

The Kleene closure of \(\Sigma\) is \(\Sigma^* := \bigcup_{n \geq 0} \Sigma^n\) where \(\Sigma^n\) denotes the set of all words over the alphabet \(\Sigma\) with length \(n\). Additionally \(\Sigma^+ := \bigcup_{n \geq 1} \Sigma^n\) is the set of all words with positive length. A language \(L \subseteq \Sigma^*\) is a set of words. A finite language is a language containing only finitely many words. For two languages \(L_1\) and \(L_2\) over an alphabet \(\Sigma\), the term \(L_1 \circ L_2 := L_1L_2 := \{vw \in \Sigma^* \mid v \in L_1\text{ and }w \in L_2\}\) describes the product (or concatenation) of the two languages.

Definition 2 (finite automaton). A nondeterministic finite automaton (NFA) \(M\) is a tuple \((Q, \Sigma, \delta, I, F)\) where \(Q\) is a finite set of states, \(\Sigma\) is a finite alphabet, \(\delta: Q \times \Sigma \to 2^Q\) is the transition function, \(I \subseteq Q\) is the set of initial states and \(F \subseteq Q\) is the set of accepting states. The automaton is called a deterministic finite automaton (DFA) if \(|I| = 1\) and for all \(q \in Q\) and all \(a \in \Sigma\) the inequation \(|\delta(q, a)| \leq 1\) holds.

Remark. In this thesis, if not explicitly mentioned otherwise, an “automaton” is a DFA. We allow \(\delta(q, a) = \emptyset\) for DFAs to simplify their specification. To get a model where \(\delta\) is a total function one only has to add a sink state \(g\) such that \(\delta(q, a) = \{g\}\) instead of \(\emptyset\) and \(\delta(g, a) = \{g\}\) for all \(a \in \Sigma\). When a transition function is defined in this paper, a not considered pair \((q, a) \in Q \times \Sigma\) means \(\delta(q, a) = \emptyset\). Furthermore, if \(\delta(q, a) = \{q'\}\) is a singleton, we write \(\delta(q, a) = q'\).

Notation. Let \((Q, \Sigma, \delta, I, F)\) be an NFA, \(S \subseteq Q\), \(w \in \Sigma^*\) and \(a \in \Sigma\). Then we define
An introduction to primality of regular languages

- \( \delta(S, a) := \bigcup_{q \in S} \delta(q, a) \)
- \( \delta(S, w) \) inductively as \( \delta(S, aw) := \delta(\delta(S, a), w) \)
  (the states reached from \( S \) after reading \( w \))
- \( \delta^*(S, w) \) inductively as \( \delta^*(S, aw) := \delta(S, a) \cup \delta(\delta(S, a), w) \)
  (all states visited from \( S \) by reading \( w \))

If \( S = \{ q \} \) is a singleton, we write \( \delta(q, w) \) and \( \delta^*(q, w) \).

**Definition 3.** Let \( M = (Q, \Sigma, \delta, I, F) \) be an NFA.

- The language \( L(M) := \{ w \in \Sigma^* \mid \delta(I, w) \cap F \neq \emptyset \} \) is the language defined by \( M \).
- A language \( L \subseteq \Sigma^* \) is called **regular**, if there is an NFA \( M \) such that \( L = L(M) \).

**Remark.** Every regular language \( L \) has a DFA \( M \) such that \( L = L(M) \).

**Corollary 4.** Every finite language is regular.

### 3 An introduction to primality of regular languages

In this section we give the definitions, important properties and known results about the primality of regular and finite languages. First of we start with a definition of primality.

**Definition 5** (Primality). A regular language \( L \subseteq \Sigma^* \) is called **decomposable**, if there are languages \( L_1, L_2 \subseteq \Sigma^*, L_1 \neq \{ \epsilon \} \neq L_2 \) such that \( L = L_1 \circ L_2 \). If \( L \) is not decomposable it is called **prime**.

**Remark.** As we see in Theorem 11, it makes no difference whether we require \( L_1 \) and \( L_2 \) to be regular languages.

The definition adverts the following decision problem:

**Problem 6.**

| Primality_{regular} |
|----------------------|
| Input: A regular language \( L \) over a finite alphabet \( \Sigma \) given as a DFA |
| Question: Is \( L \) prime |

The exact complexity of this problem was determined relatively recently:
3 An introduction to primality of regular languages

Theorem 7 ([MNST10, Corollary 6.10]). Primality_{regular} is PSPACE-complete.

For finite languages the exact complexity of the problem is not yet known. To the best of our knowledge, the NP-hardness, which we prove in Theorem [14] was not known before. Let us start with a definition of the problem.

Problem 8.

| Primality_{finite} |
|---------------------|
| Input: A finite language \( L \) over a finite alphabet \( \Sigma \) given as a DFA |
| Question: Is \( L \) prime |

The problem was examined before: The paper of Mateescu et al. [MSY98] establishes some notions, gives general results and treats examples. They suspect NP-completeness for Primality_{finite}, but only give a double exponential algorithm [MSY98, Theorem 3.1 and below]. A less theoretical approach takes Wieczorek [Wie10], as he offers an optimised deterministic algorithm for a finite language given as a list. If the finite language is given as a list of words, the primality problem is obviously in \text{coNP}: One guesses a partition in two parts for every word and checks whether all combinations of a first part of one and a second part of another word are again in the given language. As the description of a finite language as a list can be exponentially larger than the corresponding DFA (for instance the language of all words of a specific length), the algorithm is not useful for our problem.

To check for primality of a language \( L \), one has to consider if there are languages \( L_1 \) and \( L_2 \) that decompose \( L = L_1 L_2 \). Because we have to work with the DFA of \( L \), we should examine the states in which the words get actually split. That leads to the following definition and results:

Definition 9. Let \( L \) be a regular language, given as a DFA \( M = (Q, \Sigma, \delta, \{s\}, F) \) and \( P \subseteq Q \) a set of states. We call \( P \) a partition set and define the regular languages

\[
L_1^P := \{ w \in \Sigma^* \mid \delta(s, w) \in P \}
\]

and

\[
L_2^P := \bigcap_{p \in P} \{ w \in \Sigma^* \mid \delta(p, w) \in F \}.
\]
Remark. The languages $L_{P_1}$ and $L_{P_2}$ are regular because $(Q, \Sigma, \delta, \{s\}, P)$ is an automaton for $L_{P_1}$ and $(Q, \Sigma, \delta, \{p\}, F)$ is an automaton for $\{w \in \Sigma^* \mid \delta(p, w) \in F\}$ and an intersection of regular languages is regular again [HMRU00, Section 4.2].

**Lemma 10.** Let $L$ be a regular language given as a DFA $M = (Q, \Sigma, \delta, \{s\}, F)$ and let $P \subseteq Q$ be any subset, then $L_{P_1} \cap L_{P_2} \subseteq L$.

**Proof.** Let $w_1w_2 \in L_{P_1} \cap L_{P_2}$ with $w_1 \in L_{P_1}$, then $\delta(s, w_1) \in P$ by the definition of $L_{P_1}$ and therefore $\delta(s, w_1w_2) = \delta(\delta(s, w_1), w_2) \in F$ by the definition of $L_{P_2}$.

**Theorem 11** ([MSY98, Lemma 3.1]). Let $L$ be a regular language, given as a DFA $M = (Q, \Sigma, \delta, \{s\}, F)$, let $L = L_1 \cap L_2$ be a decomposition of $L$ and let

$$P := \{q \in Q \mid q = \delta(s, w) \text{ for some } w \in L_1 \}$$

be the set of “border”-states. Then $L_1 \subseteq L_{P_1}$, $L_2 \subseteq L_{P_2}$ and

$$L = L_{P_1} \cap L_{P_2}$$

is the decomposition of $L$ into two regular languages.

**Proof.** $L_1 \subseteq L_{P_1}$: Let $w \in L_1$, then $\delta(s, w) \in P$ and therefore $w \in L_{P_1}$.

$L_2 \subseteq L_{P_2}$: Suppose $w \in L_2 \setminus L_{P_2}$, that means $w \in L_2$ and there is a $p \in P$ such that $\delta(p, w) \notin F$. Let $v \in L_1$ such that $\delta(s, v) = p$. Then $vw \notin L$, because $\delta(s, vw) = \delta(\delta(s, v), w) = \delta(p, w) \notin F$, but at the same time $vw \in L_1L_2 = L$. That contradicts the existence of $w \in L_2 \setminus L_{P_2}$.

$L = L_{P_1} \cap L_{P_2}$: The inclusion $L \subseteq L_{P_1} \cap L_{P_2}$ follows directly from $L = L_1 \cap L_2$ and $L_i \subseteq L_{P_i}$ for $i \in \{1, 2\}$. The other inclusion was given in Lemma 10.

The theorem enables us to limit our search for decompositions to the ones that arise from this construction. The problem is, after guessing a partition set $P$, to actually check whether $L \subseteq L_{P_1} \cap L_{P_2}$. Unfortunately the intersection of $O(n)$ sets and the concatenation of two languages is not efficient, as both can lead to an exponential blow-up of the number of states.
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We do not use the following theorem from Wieczorek [Wie10], which is included for readers interested in further research. It allows to reduce the states that have to be considered for $P$, but a reduction beyond $O(n)$ is neither obvious nor likely.

**Theorem 12 ([Wie10, Theorem 3]).** Let $L$ be a decomposable finite language with a minimal DFA $M = (Q, \Sigma, \delta, \{s\}, F)$. Then there is a partition set $P$ with $L = L_1^P L_2^P$ such that for all $p \in P$ either $|\{a \in \Sigma | \delta(p, a)\}| > 1$ or $(p \in F) \land (\exists w \in \Sigma^* : \delta(p, w) \in F)$ holds.

Unfortunately we did not close the gap between the NP lower and the $\Pi_2^P$ upper bound. But let us at least provide a proof for the $\Pi_2^P$ upper bound:

**Proposition 13.** Primality$_{\text{finite}}$ is in $\Pi_2^P$.

**Proof.** The definitions for the polynomial hierarchy can be found in Papadimitriou's book [Pap94, Section 17.2]. We will argue that $\neg$Primality$_{\text{finite}}$ is in $\Sigma_2^P$ by the characterisation of [Pap94, Chapter 17, Corollary 2]:

$$\neg\text{Primality}_{\text{finite}} =$$

$$\{L = L(Q, \Sigma, \delta, I, F) | \exists P \subseteq Q \forall w \in L: (L, P, w) \in R :\iff w \in L_1^P L_2^P\}$$

Using Theorem 11 and Lemma 10, the right side is a characterisation of $\neg$Primality$_{\text{finite}}$. We have to check that the relation $R$ is polynomial-time decidable and is polynomially balanced. For a finite language $L$ let $M = (Q, \Sigma, \delta, \{s\}, F)$ be the DFA of $L$ and $n$ its size. The relation is polynomial-time decidable: One simulates $M$ on the input $w$ and stores the set $P_w := \delta^*(s, w) \cap P$ and the remaining characters of $w$ (when reaching $p \in P_w$) in $W \subseteq \Sigma^*$. If $P_w = \emptyset$, we reject. Otherwise we simulate for all $v \in W$ and all $p \in P_w$ the automaton $M_p := (Q, \Sigma, \delta, \{p\}, F)$ on $v$. If there is at least one $v$ such that $v \in L(M_p)$ for all $p \in P_w$, we accept or else we reject. So the test takes at most time $O(n + n \cdot n)$.

The relation is polynomially balanced as well since the partition set has at most $n$ elements and $w$ has length at most $n − 1$ ($M$ is acyclic since the language is finite). \qed
4 NP-\textit{hardness} of Primality\textsubscript{finite}

In this chapter we proof the following main theorem of the paper:

**Theorem 14.** Primality\textsubscript{finite} is NP-hard (for languages given as DFAs).

We will start with the NP-complete problem SquareTiling\textsubscript{edge} and build the following chain of polynomial reductions:

\[
\text{NP} \leq \text{SquareTiling\textsubscript{edge}} \leq \text{SquareTiling\textsubscript{rel}} \leq \\
\neg \text{ConcatenationEquivalence\textsubscript{finite}} \leq \text{Primality\textsubscript{finite}}
\]

The chain is actually quite similar to the one in the work of Martens et al. \cite{MNS10, Sections 5.2 and 6.2}. They reference a different form of tiling and use a special case of concatenation equivalence.

4.1 \textbf{From SquareTiling\textsubscript{edge} to SquareTiling\textsubscript{rel}}

We start with a tiling problem whose complexity is stated in the book of Garey and Johnson \cite{GJ79}. Then we will adapt the problem to a better fitting variant.

**Problem 15.**

\begin{center}
\begin{tabular}{| l |}
\hline
\textbf{SquareTiling\textsubscript{edge}} \\
\hline
\textbf{Input:} A set of colours \(C\), a set of tiles \(T \subseteq C^4\) and a natural number \\
\textbf{ } \hspace{0.5cm} \(n \leq |C|\); \\
\textbf{ } \hspace{0.5cm} A tile \(\begin{array}{c}
\alpha \\
\beta \\
\gamma \\
\delta \\
\end{array}
\begin{array}{c}
\alpha \\
\beta \\
\gamma \\
\delta \\
\end{array}
\begin{array}{c}
\alpha \\
\beta \\
\gamma \\
\delta \\
\end{array}
\begin{array}{c}
\alpha \\
\beta \\
\gamma \\
\delta \\
\end{array}
\end{array}
\) \(\in T\) has four edges with corresponding \\
\textbf{ } \hspace{0.5cm} \text{colours} \\
\hline
\textbf{Question:} Is there a tiling, i.e. an \(n \times n\) square \(A \in T^{n \times n}\) of tiles, such that \\
\textbf{ } \hspace{0.5cm} all adjacent tiles \(A(i, j) = \)
\begin{array}{c}
\alpha \\
\beta \\
\gamma \\
\delta \\
\end{array}
\begin{array}{c}
\alpha \\
\beta \\
\gamma \\
\delta \\
\end{array}
\begin{array}{c}
\alpha \\
\beta \\
\gamma \\
\delta \\
\end{array}
\begin{array}{c}
\alpha \\
\beta \\
\gamma \\
\delta \\
\end{array}
\) and \(A(i, j + 1) = \)
\begin{array}{c}
\alpha \\
\beta \\
\gamma \\
\delta \\
\end{array}
\begin{array}{c}
\alpha \\
\beta \\
\gamma \\
\delta \\
\end{array}
\begin{array}{c}
\alpha \\
\beta \\
\gamma \\
\delta \\
\end{array}
\begin{array}{c}
\alpha \\
\beta \\
\gamma \\
\delta \\
\end{array}
\) resp. \\
\textbf{ } \hspace{0.5cm} \text{fullfill } b = \delta \text{ resp. } c = \bar{a}
\hline
\end{tabular}
\end{center}
Proposition 16 ([GJ79] GP13 [1]). SquareTiling\textsubscript{edge} is NP-complete.

Problem 17.

\begin{center}
\begin{tabular}{|c|c|c|}
\hline
\textbf{SquareTiling\textsubscript{rel}} & \\
\hline
\textbf{Input:} & A set of tiles $\Theta$, relations $V, H \subseteq \Theta \times \Theta$ and a natural number $n \in \mathbb{N}$ & \\
\textbf{Question:} & Is there a tiling, i.e. an $n \times n$ square $T \in \Theta^{n \times n}$, such that adjacent tiles are in the horizontal relation $H$ resp. the vertical relation $V$: & \\
& $\forall i \forall j < n: (T(i, j), T(i, j + 1)) \in H$ & \\
& $\forall i < n \forall j: (T(i, j), T(i + 1, j)) \in V$ & \\
\hline
\end{tabular}
\end{center}

Remark. Alternatively we write $T(i \cdot n + j) := T(i, j)$ and get a list where $(T(m), T(m + 1)) \in H$ for $1 \leq m < n^2 \wedge m \notin n\mathbb{Z}$ and $(T(m), T(m + n)) \in V$ for $1 \leq m \leq n^2 - n$ has to be fulfilled.

Proposition 18. SquareTiling\textsubscript{rel} is NP-hard.

Proof. Given an input $C$, $\mathcal{T}$ and $n$ for SquareTiling\textsubscript{edge}. Let

$$H := \{(\begin{array}{c}
\alpha \\
\delta
\end{array} \begin{array}{c}
\beta \\
\gamma
\end{array}, \begin{array}{c}
\alpha \\
\delta
\end{array} \begin{array}{c}
\beta \\
\gamma
\end{array}) \in \mathcal{T} \times \mathcal{T} \mid b = \delta\},$$

$$V := \{(\begin{array}{c}
\delta \ \alpha \\
\gamma \ \beta
\end{array}, \begin{array}{c}
\delta \ \alpha \\
\gamma \ \beta
\end{array}) \in \mathcal{T} \times \mathcal{T} \mid c = \alpha\}$$

and $\Theta := T$. Then there is a tiling $T$ for SquareTiling\textsubscript{rel}(\Theta, $H$, $V$, $n$) if and only if there is one for SquareTiling\textsubscript{edge}(\Theta, $C$, $\mathcal{T}$, $n$). The construction of $\Theta$, $H$ and $V$ works obviously in polynomial time. \hfill \Box

Remark. One can translate SquareTiling\textsubscript{rel} to SquareTiling\textsubscript{edge} as well, as outlined in a paper of van Emde Boas [vEB97, p. 7].

\[1\] The source only mentions that the directed-hamilton-path problem is reduced to SquareTiling\textsubscript{edge}. To give the interested reader a basis for the proof: $n := \#\text{vertices}$, the series of vertices in the hamilton path is written on the diagonal of the $n \times n$-square and the corresponding edges are on the diagonals above and below the main diagonal.
4.2 From SquareTiling\textsubscript{rel} to ConcatenationEquivalence\textsubscript{finite}

This is the most interesting reduction in the chain. Here a truly original idea, not present in the proof of Martens et al. \cite{MNST10}, is necessary.

**Problem 19.**

| ConcatenationEquivalence\textsubscript{finite} |
|-----------------------------------------------|
| **Input:** Finite languages $L$, $L_1$ and $L_2$ over a finite alphabet $\Sigma$ given as DFAs |
| **Question:** Does $L = L_1 L_2$ hold |

Now we will reduce SquareTiling\textsubscript{rel} to $\neg$ConcatenationEquivalence\textsubscript{finite}. The most interesting point, compared to the regular language case, is that we work over the alphabet $\Theta \times [1, n^2]$ instead of just $\Theta$. This allows us, for a word in $L_1 L_2$, to detect the point where we jump from $L_1$ to $L_2$.

**Proposition 20.** ConcatenationEquivalence\textsubscript{finite} is coNP-complete.

**Proof.** It is obviously in coNP, since a word in $L \setminus L_1 L_2$ or in $L_1 L_2 \setminus L$ is a witness for $L \neq L_1 L_2$ and the longest words to consider have length $O(n)$, as the languages are finite. So we get to the coNP-hardness. Suppose we can solve ConcatenationEquivalence\textsubscript{finite}. Let $n$, $\Theta$ and $V, H \subseteq \Theta \times \Theta$ be an input for SquareTiling\textsubscript{rel}. We define

$$L_1 := \{(t_1, 1)(t_2, 2)\ldots (t_m, m) \in (\Theta \times [1, n^2])^* \mid m \leq n^2 - 2\},$$

$$L_2 := \bigcup_{1 \leq m \leq n^2, m \notin \mathbb{N}} \{(t_m, m)(t_{m+1}, m+1)\ldots (t_n, n^2) \in (\Theta \times [1, n^2])^* \mid (t_m, t_{m+1}) \notin H\} \cup \bigcup_{1 \leq m \leq n^2-n} \{(t_m, m)(t_{m+1}, m+1)\ldots (t_n, n^2) \in (\Theta \times [1, n^2])^* \mid (t_m, t_{m+n}) \notin V\}$$

and

$$L := L_1 L_2 \cup \{(t_1, 1)(t_2, 2)\ldots (t_n, n^2) \in (\Theta \times [1, n^2])^{n^2}\}.$$  

The size of the DFAs of the defined languages is polynomial in the size of the input and can be constructed in polynomial time as shown below.
The automaton for $L_1$ is pretty simple and has $n^2 - 1$ states:

![Automaton Diagram]

The automaton for $L_2$ is more complicated and depends on $V$ and $H$, but it is polynomial in size. We will give two automata $M_V$ and $M_H$ with polynomial sizes such that

$$L(M_H) = \bigcup_{1 \leq m \leq n^2, m \notin \mathbb{Z}} \{ (t_m, m)(t_{m+1}, m+1) \cdots (t_{n^2}, n^2) \in (\Theta \times [1, n^2])^* \mid (t_m, t_{m+1}) \notin H \}$$

and

$$L(M_V) = \bigcup_{1 \leq m \leq n^2} \{ (t_m, m)(t_{m+1}, m+1) \cdots (t_{n^2}, n^2) \in (\Theta \times [1, n^2])^* \mid (t_m, t_{m+n}) \notin V \}.$$ 

Obviously $L_2 = L(M_H \cup M_V)$ and the union automaton still has polynomial size [Yu97].

The automaton $M_H$ is constructed as follows: The set of states is

$$Q_H := \{ s_H \} \cup \{ \varsigma_{t,m} \mid t \in \Theta, m \in [1, n^2] \setminus n\mathbb{Z} \} \cup \{ \varsigma_i \mid i \in [2, n^2] \}.$$ 

The automaton has to check for a word $(t_1,m_1)(t_2,m_2)\cdots(t_k,m_k)$ whether $(t_1,t_2) \notin H$ and whether $m_{i+1} = m_i + 1$ for all $i$. So after reading the first letter the state has to store $t_1$ and every state has to store the most recent $m_i$. Therefore after the first character we go to the corresponding state $\varsigma_{t_1,m_1}$. If the next character fulfils both $(t_1,t_2) \notin H$ and $m_2 = m_1 + 1$, we only have to check $m_{i+1} = m_i + 1$. Hence we only store the most recent $m_i$, by going to the state $\varsigma_{m_i}$. Once we get to $\varsigma_{n^2}$ we accept. If otherwise there was any mistake we stop the run at that point.

Here a formal definition of the transition function $\delta_H$:

$$\delta_H(s_H,(t,m)) := \varsigma_{t,m} \quad \text{for } 1 \leq m < n^2 \text{ and } m \notin n\mathbb{Z}$$

$$\delta_H(\varsigma_{t,m},(t',m+1)) := \varsigma_{m+1} \quad \text{for } 1 \leq m < n^2 \text{ and } (t,t') \notin H$$

$$\delta_H(\varsigma_m,(t,m+1)) := \varsigma_{m+1} \quad \text{for } 1 < m < n^2$$
The automaton is then defined as $M_H := (Q_H, \Theta \times [1, n^2], \delta_H, \{s_H\}, \{s_{n^2}\})$ and has $1 + |\Theta| \cdot (n^2 - n) + n^2 - 1$ states.

The automaton $M_V$ is quite similar. The only difference is, that we have to check for a word $(t_1, m_1)(t_2, m_2) \cdots (t_k, m_k)$, whether $(t_1, t_{1+n}) \notin V$. Therefore we need the additional states $\sigma_{t,m,o}$, where $o$ stores how many characters away from $(t_1, m_1)$ we already are. Hence we get the following set of states

$$Q_V := \{s_V\} \cup \{\sigma_{t,m,o} \mid t \in \Theta, m \in [1, n^2 - n], o \in [0, n - 1]\} \cup \{\sigma_i \mid i \in [n + 1, n^2]\},$$

the transition function

$$\delta_V(s_H, (t, m)) := \sigma_{t,m,0} \quad \text{for } 1 \leq m \leq n^2 - n$$
$$\delta_V(\sigma_{t,m,i}, (t', m + i + 1)) := \sigma_{t,m,i+1} \quad \text{for } 0 \leq i < n - 1$$
$$\delta_V(\sigma_{t,m,n-1}, (t', m + n)) := \sigma_{m+n} \quad \text{for } 1 \leq m \leq n^2 - n \text{ and } (t, t') \notin V$$
$$\delta_V(\sigma_m, (t, m + 1)) := \sigma_{m+1} \quad \text{for } 1 \leq m < n^2$$

and finally the DFA is given as $M_V := (Q_V, \Theta \times [1, n^2], \delta_V, \{s_V\}, \{s_{n^2}\})$ with $1 + |\Theta| \cdot (n^2 - n) \cdot n + n^2 - n$ states.

So at last we have to show that $L = L_1 L_2 \cup \{(t_1, 1)(t_2, 2) \cdots (t_{n^2}, n^2) \in (\Theta \times [1, n^2])^{n^2}\}$ has a polynomial-sized automaton. Using this description, we might get an exponential blow-up from the concatenation, but (with the help from the $[1, n^2]$ part of the alphabet) the language can be characterised a bit differently. It basically contains all properly numbered tilings and additionally those with one jump and a forbidden tiling (with a fault, either vertically or horizontally, directly after the jump). An automaton for this can be constructed using a DFA $M_2 = (Q_2, \Theta \times [1, n^2], \delta_2, \{s_2\}, F_2)$ that accepts $L_2$.

As set of states we use $Q := Q_2 \setminus \{s_2\} \cup [0, n^2]$ and the transition function is as follows

$$\delta(q, (t, m)) := \begin{cases} q + 1 & m = q + 1, 0 \leq q < n^2 \quad q \in [0, n^2] \\ \delta_2(s_2, (t, m)) & m \neq q + 1 \\ \delta_2(q, (t, m)) & q \in Q_2 \setminus \{s_2\} \end{cases}$$

The idea is to check for legal numbering with the states $[0, n^2]$. If there is a leap in
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the numbering, we jump into the automaton for \( L_2 \). So the automaton is given by

\[ M := (Q, \Theta \times [1, n^2], \delta, \{0\}, \{n^2\} \cup F_2) \]

Obviously \( L(M) = L \) holds and \( M \) has polynomial size.

Thus DFAs for \( L_1 \), \( L_2 \) and \( L \) are constructed in polynomial time and have polynomial size
in the size of the tiling problem. Combining this with the following Lemma 21 yields a
reduction from SquareTiling\textsubscript{rel} to \( \neg\)ConcatenationEquivalence\textsubscript{finite}. That SquareTiling\textsubscript{rel}
is NP-hard (Proposition 18) completes the proof.

**Lemma 21.** Let \( n, \Theta, V, H \subseteq \Theta \times \Theta \) be an input for SquareTiling\textsubscript{rel} and \( L_1 \), \( L_2 \) and \( L \) constructed as above. Then \( L = L_1 L_2 \) if and only if there is no legal tiling.

**Proof.** A word in \( \{(t_1, 1)(t_2, 2)\ldots(t_{n^2}, n^2) \in (\Theta \times [1, n^2])^{n^2}\} \) can be interpreted as a tiling, where \( T(j) = t_j \). Every word \( w_1 w_2 \in \{(t_1, 1)(t_2, 2)\ldots(t_{n^2}, n^2) \in (\Theta \times [1, n^2])^{n^2}\} \cap L_1 L_2 \) with \( w_i \in L_i \) represents a tiling that violates the given relations: Let \( w_2 = (t_m, m)(t_{m+1}, m+1)\ldots(t_{n^2}, n^2) \in L_2 \), then either \( (t_m, t_{m+1}) \notin H \) or \( (t_m, t_{m+n}) \notin V \) which contradicts a legal tiling.

Let \( L = L_1 L_2 \), then \( \{(t_1, 1)(t_2, 2)\ldots(t_{n^2}, n^2) \in (\Theta \times [1, n^2])^{n^2}\} \subset L_1 L_2 \), so every possible tiling violates the relations and therefore there is no legal tiling. On the other hand, if there is no legal tiling, then every possible tiling violates a relation. Hence \( \{(t_1, 1)(t_2, 2)\ldots(t_{n^2}, n^2) \in (\Theta \times [1, n^2])^{n^2}\} \subset L_1 L_2 \) which yields \( L = L_1 L_2 \).

4.3 From ConcatenationEquivalence\textsubscript{finite} to Primality\textsubscript{finite}

**Theorem 14.** Primality\textsubscript{finite} is NP-hard.

The following proof is similar to [MNST10, Proof of Theorem 6.4]. The difference is, since they treat (non-finite) regular languages, that they reduce the problem \( L_1 L_2 \supseteq \Sigma^* \) (so for them \( L = \Sigma^* \)).

**Proof of Theorem 14.** Let \( L_1 \), \( L_2 \) and \( L \) be finite languages over the alphabet \( \Sigma \) given as DFAs. We want to construct a language \( A \), such that \( A \) is decomposable if and only if
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$L = L_1L_2$, which reduces ConcatenationEquivalence_{finite} to \neg Primality_{finite} and proves the theorem by the coNP - hardness of ConcatenationEquivalence_{finite} (Proposition 20).

If $L = \emptyset$, $L = \{\varepsilon\}$, $L_1 = \emptyset$ or $L_2 = \emptyset$, then it is easy to check whether $L = L_1L_2$. So we can assume $\emptyset \neq L \neq \{\varepsilon\}$ and $L_1 \neq \emptyset \neq L_2$.

Let $\Sigma' := \{a' \mid a \in \Sigma\}$ be a disjoint copy of the alphabet and let $\$ \notin \Sigma \cup \Sigma'$ be an additional letter. $L_1'$ and $L_2'$ are the respective languages over $\Sigma'$.

Now we define the language

$$A := L \cup L_1\$$ \cup \cup L_2 \cup L_1'\$$ \cup \cup L_2'.$$

The language’s DFA is obviously constructable in polynomial time.

**Lemma 22.** The language $A$ is either prime or its only non-trivial decomposition is $A_1 \circ A_2$ with $A_1 := L_1 \cup L_1'\$ and $A_2 := L_2 \cup \$ L_2'$.

**Remark.** The proof of Martens et al. [MNS10, Claim 6.5] in the paper’s appendix works nearly word for word. It is rather technical and adds no real value. For the sake of completeness we provide one regardless.

**Proof.** Suppose $A = A_lA_r$ is a non-trivial decomposition. We first show that $A_l \subseteq \Sigma^* \cup \Sigma^*\$ and symmetrically $A_r \subseteq \Sigma^* \cup \$ \Sigma^*$.

Suppose $A_l$ contains a word $w_l$ with two $\$-letters in it or where a symbol from $\Sigma$ precedes a $\$-sign.

In both cases, for $w_lw_r$ to be in $A$, $w_r$ has to be in $\Sigma^*$. Thus $A_r \subseteq \Sigma^*$ and, since the decomposition is non-trivial, $A_r \supseteq \{\varepsilon\}$. The language $L \subseteq \Sigma^*$ contains at least one word $v$ of length $\geq 1$ (see premises). So we can concatenate $v \in L \subseteq A_l$ (because $A_r \subseteq \Sigma^*$ and $L \subseteq A$) with a word $\varepsilon \neq w \in A_r$ and should get $vw \in A_lA_r = A$. That is a contradiction, as $A$ does not contain a word in $\Sigma^+\Sigma^*$.

The language $A_r$ includes at least one word $w$ containing a $\$$, because $A \supseteq L_1'\$$ \cup \cup L_2'$ (we assume $L_1 \neq \emptyset \neq L_2$) and any word in $A_l$ contains at most one $\$$. If any word $v \in A_l$ includes a $\$ not as its last character, we get a paradox because $vw \in A_lA_r = A$.
incorporates two $-signs that are not next to each other. That cannot happen for a word in A.

Now we know, every word in A_l contains at most one $-sign and if it contains one, the $ is the last sign and is not preceded by a letter in \( \Sigma \). Hence \( A_l \subseteq \Sigma^* \cup \Sigma^* $ . Symmetrically (one can look at the reversed languages) \( A_r \subseteq \Sigma^* \cup \Sigma^* $.

The intersection \( A \cap \Sigma^* \Sigma'^* = L_1 L_2 \) together with the structures of A_l and A_r yield \( A_l \cap \Sigma^* = L_1 \) and symmetrically \( A_r \cap \Sigma^* = L_2 \).

Similarly \( A \cap \Sigma'^* \Sigma^* = L'_1 L'_2 \) along with the structures of A_l and A_r imply \( A_l \cap \Sigma'^* = L'_1 \) and \( A_r \cap \Sigma^* = L'_2 \). Thus \( A_l = L_1 \cup L'_1 \) and \( A_r = L_2 \cup L'_2 \).

**Proposition 23.** A is decomposable if and only if \( L = L_1 L_2 \).

**Proof.** If A is decomposable, we know \( A = A_1 A_2 \) as defined in Lemma\[22\]. Since \( A \cap \Sigma^* = L \), \( A_1 \cap \Sigma^* = L_1 \), \( A_2 \cap \Sigma^* = L_2 \) and \( A = A_1 A_2 \), \( L = L_1 L_2 \) holds.

If on the other hand \( L = L_1 L_2 \), obviously \( A = A_1 A_2 \) as in Lemma\[22\].

So accordingly we get that the coNP-hard problem ConcatenationEquivalence\( \text{finite} \) is reducible to the complement of Primality\( \text{finite} \). Therefore the original problem Primality\( \text{finite} \) is NP-hard.

\]

**5 Final remarks**

There are still many open questions related to Primality\( \text{finite} \). Obviously the exact complexity has to be determined. Our attempts to find an NP-algorithm failed, so perhaps the lower bound has to be improved further. A coNP lower bound for primality with a list as input would strongly hint to a higher lower bound for Primality\( \text{finite} \). Having the input as an NFA would be yet another problem to consider. In that case basically nothing is known, since Theorem\[11\] is not applicable in its current form.
5 Final remarks

Aside from these variants for the input, the decomposition into three, four or generally into $m$ languages is a problem to consider (for all the input variants). A priori we do not know much about that. For lists we still get coNP for fixed $m$ by naively guessing the partition. And for DFAs we can check, for all possible decompositions into two languages, whether those languages are decomposable again. That approach clearly is not efficient.

Comprehensively we can say that there are still many open questions regarding the complexity of decompositions of finite languages.
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