THE TWO WAYS OF GAUGING THE POINCARÉ GROUP

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Abstract. A description of how a theory of gravity can be considered as a gauge theory (in the sense of Trautman) of the Poincaré group is given. As a result, it is shown that a gauge theory of this kind is consistent with the Equivalence Principle only if the Lagrangian and the constraints are preserved not only by the gauge transformations but also by an additional family of transformations, called pseudo-translations. Explicit expressions of pseudo-translations and of their action on gravitational gauge fields are given. They are expected to be useful for geometric interpretations of their analogues in supergravity theories.

1. Introduction

The notion of “gauge theory” is certainly very well-known, but we need to explicitly recall it in order to make clear in which sense we are going to use it. Following Trautman ([18, 19]), we call gauge theory of a Lie group \( G \) a triple formed by:

a) a class of principal \( G \)-bundles \( \pi : P \to M \) over a manifold \( M \) and endowed with connections; the manifold \( M \) represents the physical empty space-time, while the connections on \( P \) represent (potentials for) physical fields, called “gauge fields”;

b) a class of bundles \( \pi^E : E \to M \), associated to the previous \( G \)-bundles, whose sections represent other physical fields, called “particles coupled with the gauge fields”;

c) a set of partial differential equations (usually Euler-Lagrange equations determined by a Lagrangian) on the connections in (a) and the sections in (b), which is invariant under the action of local automorphisms of the \( G \)-bundles \( P \), the so-called “gauge group \( \text{Gau}(P) \)”.

Given a Lagrangian or differential equations invariant under a group \( G \), we use the expression “gauging the group \( G \)” to indicate the construction of a gauge theory of \( G \), whose Lagrangian or equations reduce to the given ones when the gauge field is flat. However, in Physics the expression “gauging” has often a wider meaning. Basically, it indicates any process of construction of Lagrangians (or differential equations) that starts from a given...
$G$-invariant Lagrangian and ends up with a new Lagrangian, which is invariant under a class of transformations, locally identifiable with $G$-valued maps $g : \mathcal{U} \subset M \rightarrow G$ on the space-time $M$.

In many classical papers, it is shown how General Relativity and other theories of gravity can be obtained by “gauging” the Poincaré group $G_P$ in this wider sense or according to a definition of gauge theory different from the above (see the classical [8, 20] or [3, 11, 7, 6, 16] and the vast bibliography in those papers and books). But not many papers are concerned with a description of General Relativity as a gauge theory according to previous definition (see [12, 15, 9]). On the other hand, we strongly believe that a sure understanding of this latter description is a necessary step if one aims to geometric constructions of supergravity theories as gauge theories of super-extension of the Poincaré group (see [2] for the classification of these super-extensions in any dimension and signatures).

The purpose of this paper is to give a complete description of how a theory of gravity can be considered as a gauge theory of $G_P$. As a by-product we show that, if a theory of this kind is required to satisfy the Equivalence Principle, then the Lagrangian or the differential equations of the theory must be preserved by the gauge transformations plus an additional family of transformations, called pseudo-translations. Finally, we provide a presentation of such pseudo-translations, which admits immediate generalizations to the context of supermanifolds and hence, hopefully, of supergravity theories.

Our presentation of gravity as gauge theory is essentially equivalent to the one in [12], but it differs in the following aspect. Any connection form $\omega$ on a principal bundle $\pi : P \rightarrow M$ induces an horizontal distribution on any associated vector bundle $\pi^E : E \rightarrow M$ and hence a differential operator $\nabla$ of covariant derivation for the sections of $E$. In case $P$ is a $G_P$-bundle, there is always an associated bundle $\tilde{E}$, which has the structure of an affine bundle, endowed with a pseudo-Riemannian metric on the fibers. Moreover, there is a natural one-to-one correspondence between the covariant derivations of sections of $\tilde{E}$ and the connection forms on $P$ (Prop. 2.5). Having applications to supergravity in mind, we represent the gauge fields of a $G_P$-theory just as covariant derivations of an affine bundle with metric. By the previous observation, this is fully equivalent to consider gauge fields as in (a) and in [12], but has the advantage that deals with objects (covariant derivations) that have easily defined “super” analogues, in contrast with the “super” analogues of principal bundles and connection forms, which require not so straightforward mathematical definitions (see e.g. [11, 14]).

Our first main result can be summarized as follows. Let $\tilde{\pi} : \tilde{E} \rightarrow M$ be an affine bundle over $M$ modeled on a vector bundle $\pi : E \rightarrow M$, $(\nabla, \nabla)$ a covariant derivation for the sections of $\tilde{E}$ and $\psi_0 : M \rightarrow \tilde{E}$ a fixed section of $\tilde{E}$ (see §2 for all definitions). If $\psi_0$ satisfies suitable regularity conditions,
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the map

\[ L : TM \to E, \quad L(v) \overset{\text{def}}{=} \nabla_v \psi_o \in T_x \tilde{E} \cong E_x \quad v \in T_x M, \ x \in M, \]

is a bundle isomorphism that can be used to induce from \( \tilde{E} \) a pseudo-Riemannian metric \( g \) and a metric covariant derivation \( \nabla \) on \( M \). This construction is used to show that there is a natural correspondence between the pairs \( ((\tilde{\nabla}, \nabla); \psi_o) \), formed by a covariant derivation on \( \tilde{E} \) and regular sections \( \psi_o \) of \( \tilde{E} \), and the pairs \( (g, \nabla) \), formed by pseudo-Riemannian metrics and metric covariant derivations on \( M \). Since such pairs \( (g, \nabla) \) are the objects usually considered to represent gravitational fields, we conclude that any theory of a gravity, determined by a Lagrangian on pairs \( (g, \nabla) \), can be interpreted as a gauge theory, provided that:

- the role of the bundles, whose sections represent the particles coupled with the gauge group, is played by metric affine bundles \( \tilde{E} \);
- the role of the gauge fields is played by the affine covariant derivations \( (\tilde{\nabla}, \nabla) \) on \( \tilde{E} \);
- the theory includes a special field (Higgs field), coupled with the gauge field, consisting of a regular section \( \psi_o \) of \( \tilde{E} \);
- the Lagrangian and the constraints, expressed in terms of \( (\tilde{\nabla}, \nabla) \), are invariant under the family of all gauge transformations of \( \tilde{E} \), explicitly described in §2.1.3.

We leave to the reader the physical interpretation of the objects involved in this scheme. We need however to stress the following fact: any gauge transformation on a pair \( ((\tilde{\nabla}, \nabla); \psi_o) \) corresponds just to the identity transformation when expressed in terms of the pair \( (g, \nabla) \). This is due to the presence of the Higgs field \( \psi_o \): It reduces the representation of the gauge group \( \text{Gau}(P) \) to the faithful representation of a smaller class \( \mathcal{G}_{\psi_o} \subseteq \text{Gau}(P) \), which act on the vielbeins of \( g \) by pointwise dependent Lorentz transformations and hence makes absolutely no change on \( g \) and \( \nabla \) (see also [15]). This means that any choice for the Lagrangian and the constraints on the pairs \( (g, \nabla) \) corresponds to a gauge theory on the corresponding pairs \( ((\tilde{\nabla}, \nabla); \psi_o) \).

On the other hand, any sensible theory of gravity has to satisfy the so-called Equivalence Principle. This principle can be briefly stated saying that the Lagrangian of the theory must be invariant under arbitrary local diffeomorphisms of the space-time (or, equivalent, under arbitrary local changes of coordinates). A slightly weaker requirement is to ask that the Lagrangian is invariant under the diffeomorphisms generated in the flows of local vector fields. We call it infinitesimal version of Equivalence Principle and, for simplicity, we consider only this weaker condition.

We show that it corresponds to require that the Lagrangian on the pairs \( ((\tilde{\nabla}, \nabla); \psi_o) \) must be invariant under certain bundle transformations of the metric affine bundle \( (\tilde{E}, E) \), which we call pseudo-translations and of
which we determine the explicit action on the gauge field. These pseudo-
translations correspond to the transformations on \((g, \nabla)\) determined by the
flows of vector fields on \(M\) and their expressions in coordinates resembles
closely certain gauge transformations. This is probably at the origin of the
widespread idea that the transformations, given by flows of vector fields of
the space-time, are the outcome of a “gauging process” of the transfor-
mations of the Poincarè group.

The notion of pseudo-translations bring to our third result: theories of
gravity given by Lagrangians on pairs of the form \((g, \nabla)\) and satisfying the
infinitesimal version of the Equivalence Principle are in natural correspon-
dence with the gauge theories of the Poincarè group \(G^P\), whose Lagrangians
is not only invariant under the gauge group \(\text{Gau}(P)\) but also under the family
of all pseudo-translations. In other words, there are two classes of trans-
formations to be considered when a gravity theory is presented as a gauge
theory\footnote{See also \cite{9}, where the authors adopt a notion of gauge theory, according to which a
covariance under two (and not one) groups of transformations is required.}. In a future paper, we will consider the analogous picture in case
of super-extensions of the Poincarè group, providing a new interpretation of
the “gauging” processes of the super Poincarè group in the construction of
theories of supergravity.

The structure of the paper is as follows. In §2, we give the first proper-
erties of the covariant derivations on pseudo-Riemannian affine bundles and
of groups of gauge transformations of \(G^P\)-bundles. In §3 we discuss the
theories of gravity as gauge theories of \(G^P\). In §4, we consider the Equiv-
alence Principle, we introduce the notion of pseudo-translations and prove
the stated correspondence between the infinitesimal Equivalence Principle
and a principle of covariance under pseudo-translations.

Notation. The class of all smooth real functions on a manifold \(M\) is denoted
by \(\mathfrak{X}(M)\), while the class of vector fields is denoted by \(\mathfrak{X}(M)\). For any bundle
\(\pi : E \to M\), we denote by \(\Sigma(E)\) the family of all global sections of \(E\) and
by \(\Sigma_{\text{loc}}(E)\) the class of local sections defined on open subsets \(U \subset M\).

2. Preliminaries

2.1. Gauge transformations of metric affine bundles.

2.1.1. Affine bundles and their covariant derivations.

Definition 2.1. \cite{5} Let \(M\) be a manifold of dimension \(n\). An affine bundle
over \(M\) modeled on a vector bundle \(\pi : E \to M\) is a fiber bundle \(\tilde{E} \to M\)
equipped with a bundle morphism \(+ : \tilde{E} \times_M E \to \tilde{E}\) so that, for each \(x \in M\),
the induced map

\[
+ \mid_x : E_x \times E_x \to \tilde{E}_x, \quad (\tilde{e}, e) \mapsto \tilde{e} + e
\]  

(2.1)
determines a structure of affine space on \( \tilde{E}_x \), modeled on the vector space \( E_x \). The rank of \( E \) is called rank of the affine bundle.

In the following, we often denote an affine bundle simply by the pair \((\tilde{E}, E)\). Notice that the bundle map \((2.1)\) induces the map \(+ : \Sigma(\tilde{E}) \times \Sigma(E) \to \Sigma(\tilde{E})\), defined by \((\tilde{\sigma} + \sigma)_x \overset{\text{def}}{=} \tilde{\sigma}_x + \sigma_x\), which makes \(\Sigma(\tilde{E})\) an affine space modeled on \(\Sigma(E)\).

**Definition 2.2.** A covariant derivation on an affine bundle \((\tilde{E}, E)\) is a pair \((\tilde{\nabla}, \nabla)\), where \(\nabla\) is a linear connection on \(E\) and \(\tilde{\nabla}\) is an operator \(\tilde{\nabla} : \mathfrak{X}(M) \times \Sigma(\tilde{E}) \to \Sigma(E)\) that associates to any vector field \(X \in \mathfrak{X}(M)\) and any \(\psi \in \Sigma(E)\) a section \(\tilde{\nabla}_X \psi \in \Sigma(E)\) that satisfies

i) \(\tilde{\nabla}_{fX + gY} \psi = f \tilde{\nabla}_X \psi + g \tilde{\nabla}_Y \psi\);

ii) \(\tilde{\nabla}_X (\psi + \varphi) = \tilde{\nabla}_X \psi + \tilde{\nabla}_X \varphi\)

for any \(\psi \in \Sigma(E)\), \(\varphi \in \Sigma(E)\), \(X, Y \in \mathfrak{X}(M)\) and \(f, g \in \mathfrak{F}(M)\).

This definition is motivated by the following facts.

Let us call affine frame of \(\tilde{E}\) at \(x\) any pair \(u = (\tilde{e}_0; (e_1, \ldots, e_p))\), formed by a point \(\tilde{e}_0\) of the fiber \(\tilde{E}_x\) and a basis \((e_1, \ldots, e_p)\) of the vector space \(E_x\). The family \(A(\tilde{E})\) of affine frames of \(\tilde{E}\) has a natural structure of principal bundle over \(M\), with structure group \(\text{Aff}(\mathbb{R}^p) = \text{GL}_p(\mathbb{R}) \times \mathbb{R}^p\), and \(\tilde{E}\) and \(E\) are naturally isomorphic to the associated bundles

\[
\tilde{E} \simeq A(\tilde{E}) \times_{\text{Aff}(\mathbb{R}^p), \tilde{\rho}} \mathbb{R}^p, \quad E \simeq A(\tilde{E}) \times_{\text{Aff}(\mathbb{R}^p), \rho} \mathbb{R}^p \tag{2.2}
\]

where \(\tilde{\rho}\) is the standard representation of \(\text{Aff}(\mathbb{R}^p)\) as group of affine transformations of \(\mathbb{R}^p\) and \(\rho\) is the surjective homomorphism from \(\text{Aff}(\mathbb{R}^p)\) onto \(\text{GL}_p(\mathbb{R}) = \text{GL}_p(\mathbb{R}) \times \mathbb{R}^p / \mathbb{R}^p\).

Any connection form \(\omega\) on \(A(\tilde{E})\) determines a parallel transport between fibers on \(A(\tilde{E})\), \(\tilde{E}\) and \(E\). In particular it defines a pair of covariant derivations \(\tilde{\nabla}\) and \(\nabla\) for the sections of \(\tilde{E}\) and \(E\), respectively. It can be checked that \(\nabla\) is always a linear connection, while \(\tilde{\nabla}\) is an operator that satisfies (i) and (ii) of Definition 2.2. Conversely, for any pair \((\tilde{\nabla}, \nabla)\) that satisfies the conditions of Definition 2.2, there exists a connection form on \(A(\tilde{E})\), whose parallel transport determines \(\tilde{\nabla}\) and \(\nabla\) as associated covariant derivations. In other words, the covariant derivations \((\tilde{\nabla}, \nabla)\) considered in Definition 2.2 coincide with the covariant derivations determined by the connection forms \(\omega\) on the principal bundle \(A(\tilde{E})\) (see e.g. [17], §2.2).

Let \((\tilde{\nabla}, \nabla)\) be a covariant derivation on \((\tilde{E}, E)\) and fix a global section \(\psi_o \in \Sigma(\tilde{E})\) (by well-known properties of fiber bundles, \(\Sigma(\tilde{E})\) is not empty). Let \(L : \mathfrak{X}(M) \to \Sigma(E)\) be the linear map

\[
L(X) = \tilde{\nabla}_X \psi_o . \tag{2.3}
\]
From (i) and (ii) of Definition 2.2, one can check that \( L : \mathfrak{X}(M) \to \Sigma(E) \) can be identified with a section \( L \) of \( T^*M \otimes_M E \). Given \( L \) and \( \nabla \), the operator \( \tilde{\nabla} \) can be recovered by means of the identity
\[
\tilde{\nabla}_X \psi = L(X) + \nabla_X (\psi - \psi_o) , \quad \psi \in \Sigma(\tilde{E}) , \quad X \in \mathfrak{X}(M).
\] (2.4)
Conversely, given a section \( L \in \Sigma(T^*M \otimes_M E) \) and a linear connection \( \nabla \) on \( E \), one can check that the operator \( \tilde{\nabla} : \mathfrak{X}(M) \times \Sigma(\tilde{E}) \to \Sigma(E) \), defined by (2.4), satisfies conditions (i) and (ii) of Definition 2.2 and that the correspondence
\[
(\tilde{\nabla}, \nabla) \mapsto (L \overset{\text{def}}{=} \tilde{\nabla}\psi_o, \nabla)
\]
is one to one. For this reason, once \( \psi_o \) is given, we will often indicate a covariant derivation \( (\tilde{\nabla}, \nabla) \) by the corresponding pair \( (L, \nabla) \).

2.1.2. Pseudo-Riemannian affine bundles and metric connections.

We now introduce the concept of pseudo-Riemannian metrics on affine bundles. Let \( \pi : Q \to M \) be a fiber bundle and denote by \( T^vQ \subset TQ \) the vertical distribution. We call vertical tensor field of type \((r, s)\) on \( Q \) any section of the bundle \( \pi : \bigotimes^r T^vQ \otimes \bigotimes^s T^vQ \to Q \). A vertical tensor field \( \tilde{\alpha} \) of type \((r, s)\) of an affine bundle \( \tilde{E} \to M \) modeled on \( E \) will be called affine if, for any \( x \in M \), there exists a tensor \( \alpha_x \in \bigotimes^r E_x \otimes \bigotimes^s E_x \) so that for any \( u \in E_x \) the tensor \( \tilde{\alpha}_u \) is identifiable with \( \alpha_x \) under the natural isomorphism \( T^v_x \tilde{E} \simeq E_x \). Therefore any affine tensor field \( \tilde{\alpha} \) on \( \tilde{E} \) is uniquely determined by a corresponding section \( \alpha \) of the tensor bundle \( \pi : \bigotimes^r E \otimes_M \bigotimes^s E^* \to M \).

Definition 2.3. A pseudo-Riemannian metric on an affine bundle \((\tilde{E}, E)\) is an affine vertical tensor field \( \tilde{g} \) of type \((0, 2)\), determined by a section \( g \) of \( \bigotimes^2 E^* \), so that \( g_x \) is an inner product on \( E_x \) of constant signature for any \( x \in M \). The triple \((\tilde{E}, E, g)\) is called pseudo-Riemannian affine bundle modeled on \( E \).

We also call pseudo-Riemannian metric on a vector bundle \( E \) any vertical tensor field of type \((0, 2)\) on \( E \) that satisfies the above conditions, taking \( E \) as affine bundle modeled on itself.

Given a pseudo-Riemannian affine bundle \((\tilde{E}, E, g)\) of signature \((p, q)\), any affine frames \( u = (\bar{e}_0, (e_1, \ldots, e_p)) \) with \((e_1, \ldots, e_p)\) orthonormal w.r.t. the inner product \( g_x, x = \pi(u) \), is called orthonormal frame bundle and the class \( O_g(\tilde{E}) \) of all orthonormal affine frames is a \( O_{p,q} \times \mathbb{R}^{p+q} \)-reduction of \( \text{Aff}(\tilde{E}) \). Conversely, for any \( O_{p,q} \times \mathbb{R}^{p+q} \)-reduction \( P \subset \text{Aff}(\tilde{E}) \), there exists a unique pseudo-Riemannian metric \( \tilde{g} \) on the associated bundle \( \tilde{E} = O_g(\tilde{E}) \times_{O_{p,q} \times \mathbb{R}^{p+q}} \mathbb{R}^{p+q} \) for which \( P \) is the corresponding orthonormal affine bundle \( P = O_g(M) \).
Definition 2.4. Let $(\tilde{E}, E, g)$ be a pseudo-Riemannian affine bundle and $(\tilde{\nabla}, \nabla)$ an affine covariant derivation of $(\tilde{E}, E)$. We say that $(\tilde{\nabla}, \nabla)$ is metric if the linear connection $\nabla$ is so that $\nabla_X g = 0$ for any $X \in \mathfrak{X}(M)$.

The following proposition can be obtained with standard arguments on reductions. For a detailed proof, see e.g. [17], §3.2.2.

Proposition 2.5. Any connection form $\omega$ on the orthonormal affine frame bundle $O_g(\tilde{E})$ of a pseudo-Riemannian affine bundle $(\tilde{E}, E, g)$ induces a pair covariant derivation $(\tilde{\nabla}, \nabla)$ on the associated bundles $\tilde{E} = O_g(\tilde{E}) \times O_{p,q} \rtimes \mathbb{R}^{p+q}$ and $E = O_g(E) \times O_{p,q} \rtimes \mathbb{R}^{p+q}$, which is a metric affine covariant derivation. Conversely, any metric affine covariant derivation $(\tilde{\nabla}, \nabla)$ is uniquely determined by a connection form $\omega$ on $O_g(\tilde{E})$.

2.1.3. Affine gauge transformations.

We recall that a gauge transformation of a principal bundle $\pi : P \to M$ is a bundle automorphism $f : P \to P$ which induces the identity on $M$. The gauge transformations of the associated bundles are the bundle automorphisms induced by the gauge transformations of $P$. When $P$ is the orthonormal frame bundle $P = O_g(\tilde{E})$ of a pseudo-Riemannian affine bundle $(\tilde{E}, E, g)$, one can check that the gauge transformations of $\tilde{E}$ and $E$ coincide with the maps given in the following definition.

Definition 2.6. Let $(\tilde{E}, E, g)$ be a pseudo-Riemannian affine bundle. A gauge transformation of $E$ (resp. $\tilde{E}$) is any bundle automorphism $f$ of $E$ (resp. $\tilde{E}$) which preserve the fibers and, for any $x \in M$, the map $f|_{E_x}$ (resp. $f|_{\tilde{E}_x}$) is an isometry of $(E_x, g_x)$ (resp. $(\tilde{E}_x, g_x)$).

We call infinitesimal gauge transformation of $E$ (resp. of $\tilde{E}$) any vector field $V \in \mathfrak{X}(E)$ (resp. $\mathfrak{X}(\tilde{E})$) whose flow $\Phi^V_t$ is a one-parameter group of gauge transformations.

We denote by $\text{Gau}(\tilde{E})$ and $\text{Gau}(E)$ the groups of gauge transformations of $\tilde{E}$ and $E$, respectively. We remark that there exists a natural surjective homomorphism $\alpha : \text{Gau}(E) \to \text{Gau}(E)$ defined as follows. If $f \in \text{Gau}(\tilde{E})$ we have that $f|_{\tilde{E}_x}$ is an isometry of $(\tilde{E}_x, g_x)$ and hence it is an affine transformation for any $x \in M$. Let $h_x : E_x \to E_x$ be the linear map associated to $f|_{\tilde{E}_x}$ so that

$$f(\tilde{e}) - f(\tilde{e}) = h_x(\tilde{e} - \tilde{e})$$

for any $\tilde{e}, \tilde{e}' \in \tilde{E}_x$. The map $h = \alpha(f) \in \text{Gau}(E)$ is the gauge transformation, which induces on each fiber $E_x$ the linear map $h_x$ just described. Clearly,
\( f \in \ker \alpha \) if and only if \( f|_E \) is a translation of the affine space \( \tilde{E}_x \) for any \( x \in M \), i.e. if and only if it is a map of the form

\[
\tilde{\xi} : \tilde{E} \to \tilde{E}, \quad \tilde{\xi}(\tilde{e}) \mathrel{\overset{\text{def}}{=}} \tilde{e} + \xi|_x, \quad x = \tilde{\pi}(\tilde{e}). \tag{2.6}
\]

for some section \( \xi \in \Sigma(E) \). We call any map of this kind point-depending \textit{translation} (or simply \textit{translation}) \textit{determined by} \( \xi \) and we denote by \( \text{Gau}^T(\tilde{E}) \) the normal subgroup of \( \text{Gau}(\tilde{E}) \) of all translations. Summing up, we have the following exact sequence:

\[
0 \longrightarrow \text{Gau}^T(\tilde{E}) \overset{1}{\longrightarrow} \text{Gau}(\tilde{E}) \overset{\alpha}{\longrightarrow} \text{Gau}(E) \longrightarrow 1, \tag{2.7}
\]

which is the analogue of the corresponding sequence for \( \text{Aff}(\mathbb{R}^p) \).

Now, fix a section \( \psi_o \in \Sigma(\tilde{E}) \). A gauge transformation \( h : E \to E \) induces the following gauge transformation on \( \tilde{E} \):

\[
\tilde{h}^{\psi_o} : \tilde{E} \to \tilde{E}, \quad \tilde{h}^{\psi_o}(\tilde{e}) \mathrel{\overset{\text{def}}{=}} \psi_o(x) + h(\tilde{e} - \psi_o(x)), \quad x = \tilde{\pi}(\tilde{e}). \tag{2.8}
\]

This map will be called \textit{point-depending rotation} (or simply \textit{rotation}) \textit{around} \( \psi_o \) \textit{determined by} \( h \).

**Lemma 2.7.**

1) A gauge transformation \( f : \tilde{E} \to \tilde{E} \) is a rotation around \( \psi_o \) if and only if \( f(\psi_o(x)) = \psi_o(x) \) for any \( x \in M \).

2) Any gauge transformation \( f \) of \( \tilde{E} \) can be uniquely written as

\[
f = \tilde{\xi} \circ \tilde{h}, \tag{2.9}
\]

where \( \tilde{\xi} \) is the translation by the section \( \xi(x) \mathrel{\overset{\text{def}}{=}} f(\psi_o(x)) - \psi_o(x) \), while \( \tilde{h} \) is the rotation around \( \psi_o \) defined by \( \tilde{h} \mathrel{\overset{\text{def}}{=}} \xi^{-1} \circ f \).

3) The homomorphism \( \beta : \text{Gau}(E) \to \text{Gau}(\tilde{E}) \) defined by \( \beta(h) \mathrel{\overset{\text{def}}{=}} \tilde{h}^{\psi_o} \) makes \( \text{(2.7)} \) a splitting exact sequence.

**Proof.** (1) The necessity is immediate. Conversely, assume that \( f \) fixes \( \psi_o \) and let \( h = \alpha(f) : E \to E \) be the gauge transformation defined by \( \text{(2.5)} \). From construction it follows that

\[
f(\tilde{e}) = f(\psi_o(x)) + h(\tilde{e} - \psi_o(x)) = \psi_o(x) + h(\tilde{e} - \psi_o(x)), \quad x = \tilde{\pi}(\tilde{e}),
\]

and hence that \( f \) coincides with the rotation around \( \psi_o \) by \( h \).

(2) We only need to check that \( \tilde{h} \) fixes \( \psi_o \). Since \( \tilde{\xi}^{-1} \) is the translation by the section \( -\xi = \psi_o - f \circ \psi_o \), the conclusion follows immediately.

(3) It is immediate to verify that \( \alpha \circ \beta = \text{Id} \). \( \square \)

Recall that any vertical tangent subspace \( T_x^v \tilde{E} \) is naturally identified with the fiber \( E_x, x = \tilde{\pi}(\tilde{e}), \) of \( E \). By this identification, any section \( \xi \in \Sigma(E) \) corresponds to a vertical vector field \( V^\xi \) on \( \tilde{E} \) whose flow \( \Phi^v_t^\xi \) consists of the translations by the sections \( t \cdot \xi \in \Sigma(E) \), \( t \in \mathbb{R} \). In other words, \( V^\xi \) is an infinitesimal gauge transformation and we call it \textit{infinitesimal translation}.
2.1.4. Actions of gauge transformations on affine covariant derivations.

**Definition 2.8.** Let $(\tilde{\nabla}, \nabla)$ be an a metric affine covariant derivation on $(\tilde{E}, E, g)$ and $f \in \text{Gau}(\tilde{E})$. Let also $h = \alpha(f)$ the corresponding gauge transformation on $E$. We call **deformation of $(\tilde{\nabla}, \nabla)$ by $f$** the covariant derivation $(f_*\tilde{\nabla}, f_*\nabla)$ defined by

$$ f_*\tilde{\nabla} \overset{\text{def}}{=} h \circ \tilde{\nabla} \circ f^{-1}, \quad f_*\nabla \overset{\text{def}}{=} h \circ \nabla \circ h^{-1}. \quad (2.10) $$

If $V \in \mathfrak{X}(\tilde{E})$ is an infinitesimal gauge transformation, we call **infinitesimal deformation of $(\tilde{\nabla}, \nabla)$ by $V$** the pair of operators $(V(\tilde{\nabla}), V(\nabla))$ respectively defined by

$$ V(\tilde{\nabla})_X e \overset{\text{def}}{=} \frac{d}{dt} \left( \Phi_t^* V \tilde{\nabla} \right)_X e \bigg|_{t=0}, \quad V(\nabla)_X e \overset{\text{def}}{=} \frac{d}{dt} \left( \Phi_t^* \nabla \right)_X e \bigg|_{t=0} $$

for any $X \in \mathfrak{X}(M), \ e \in \Sigma(\tilde{E})$ and $e \in \Sigma(E)$.

**Remark 2.9.** By Proposition 2.5, any metric affine covariant derivation on $(\tilde{E}, E)$ corresponds to a unique connection form $\omega$ and associated horizontal distribution $\mathcal{H} = \ker \omega$ on $O_Y(\tilde{E})$. On the other hand, by the remarks in §2.1.3, $f \in \text{Gau}(\tilde{E})$ corresponds to a unique gauge transformation $\hat{f} : O_Y(\tilde{E}) \to O_Y(\tilde{E})$. One can check that $(f_*\tilde{\nabla}, f_*\nabla)$ coincides with the affine covariant derivation associated with the push-forward of $\mathcal{H}$ by $\hat{f}$, which is a horizontal distribution associated with the connection form $\hat{f}^* \omega$ (see e.g. [17]). This fact motivated the previous Definition 2.8.

We recall that, by the remarks in §2.1.1, given a fixed section $\psi_o \in \Sigma(\tilde{E})$, the covariant derivation $(\tilde{\nabla}, \nabla)$ can be identified with the pair $(L, \nabla)$, with $L$ defined in (2.3). A straightforward computation shows that the pair $(f_*L, f_*\nabla)$, corresponding to the deformation $(f_*\tilde{\nabla}, f_*\nabla)$ by $f \in \text{Gau}(\tilde{E})$, is given by

$$ (f_*L)(X) = h \left( L(X) + \nabla_X \left( f^{-1}(\psi_o) - \psi_o \right) \right) \quad (2.11) $$

and of course by the second formula in (2.10). It follow immediately that, if $f$ is a translation by $\xi \in \Sigma(E)$, the deformation $(f_*L, f_*\nabla)$ is

$$ f_*L = L - \nabla \xi, \quad f_*\nabla = \nabla, \quad (2.12) $$

while, if $f$ is a rotation around $\psi_o$ by $h \in \text{Gau}(E)$, $(f_*L, f_*\nabla)$ is

$$ f_*L = h \circ L, \quad f_*\nabla = h \circ \nabla \circ h^{-1}. \quad (2.13) $$
3. Gravity theories as theories of metric affine connections

3.1. Gravitational fields and metric affine connections.

Let $M$ be a manifold of dimension $n$. We call a gravitational field on $M$ any pair $(g, \nabla)$ formed by a pseudo-Riemannian metric $g$ of signature $(p, q)$ and a metric covariant derivation $\nabla$, i.e. so that $\nabla g = 0$. This terminology is motivated by the fact that, in General Relativity, gravity is represented by a 4-dimensional space-time $M$ and a pair $(g, \nabla)$, where $g$ is a pseudo-Riemannian metric of signature $(1, 3)$ and $\nabla$ is the Levi-Civita connection of $g$ (i.e. metric and torsion free) so that the well-known Einstein equations are satisfied. In other words, we may say that in General Relativity the gravity is represented by a gravitational field $(g, \nabla)$ of signature $(1, 3)$ satisfying the conditions

$$ T = 0, \quad \text{Ric} - \frac{s}{2} g = T^g $$

where $T$, $\text{Ric}$ and $s$ are the torsion, the Ricci tensors and the scalar curvature of $\nabla$, respectively, and $T^g$ the stress-energy tensor determined by other physical fields. It is therefore natural to consider the generalizations of General Relativity as theories of gravitational fields $(g, \nabla)$, subjected to systems of equations that are extensions or modifications of (3.1).

In this section, we want to show how any gravitational field $(g, \nabla)$ can be naturally associated with a covariant derivation on a suitable metric affine bundle, i.e. to the associated bundle of a principal bundle with structure group given by the Poincaré group $G = O_{p,q} \ltimes \mathbb{R}^n$ (see also [10, 13]).

Let $(\tilde{E}, E, g_0)$ be a metric affine bundle over $M$ of rank $n$ and $\psi_0$ a section of $\tilde{E}$. A metric affine covariant derivation $(L, \nabla)$ will be called regular w.r.t. $\psi_0$ if the map $L_x : T_x M \to E_x$ is a linear isomorphism for any $x \in M$. Clearly, if a quadruple $(\tilde{E}, E, g_0, \psi_0)$ admits a regular covariant derivation, then there exists an affine bundle isomorphism between $(\tilde{E}, E)$ and $(TM, TM)$ which maps $\psi_0$ into the zero section $\tilde{0}$ of $TM$ (here we consider $TM$ as an affine bundle modeled on itself).

From now on we will always assume that $(\tilde{E}, E, g_0)$ admits some regular covariant derivation and hence that $(\tilde{E}, E, g_0, \psi_0) \simeq (TM, TM, g_0, \tilde{0})$ for some pseudo-Riemannian metric $g_0$ on $M$. However, since the (non canonical) identification $\tilde{E} \simeq TM$, $E \simeq TM$, etc. is not relevant and it might even cause confusion in certain arguments, we will avoid it in all what follows.

As observed in §2.1.1 any given $\psi_0 \in \Sigma(\tilde{E})$ brings to the identification of covariant derivations $(\nabla, \nabla)$ with corresponding pairs $(L, \nabla)$. On the other hand, any such pair associated with a regular covariant derivation determines a gravitational field $(g, \nabla^L)$ on $M$ as follows:

$$ g(X, Y) \overset{\text{def}}{=} g_0(L(X), L(Y)), \quad \nabla^L_X Y \overset{\text{def}}{=} L^{-1}(\nabla_X (L(Y))). $$

(3.2)
Notice that $\nabla^L$ is metric w.r.t. $g$, because $\nabla$ metric w.r.t. $g_o$ and hence $\nabla^L g = 0$ for any $X \in \mathfrak{X}(M)$.

Using (2.13), one can check that two regular covariant derivations $(L, \nabla)$, $(L', \nabla')$ determine the same gravitational field $(g, \nabla^L)$ if and only if they differ by a rotation around $\psi_o$.

Now, let us consider the following notation:

- $\text{Conn}(\tilde{E})$ denotes the class of all metric affine covariant derivations of $(\tilde{E}, E, g_o)$;
- $\text{Conn}(\tilde{E})^{\psi_o, \text{reg}} \subset \text{Conn}(\tilde{E})$ is the subclass of regular ones w.r.t. $\psi_o$ and $\text{Conn}(\tilde{E})^{\text{reg}} = \bigcup_{\psi_o \in \Sigma(\tilde{E})} \text{Conn}(\tilde{E})^{\psi_o, \text{reg}}$;
- $G_{\psi_o} \overset{\text{def}}{=} \text{Gau}_{\psi_o}(\tilde{E}) \subset \text{Gau}(\tilde{E})$ is the isotropy subgroup of $\text{Gau}(\tilde{E})$ at $\psi_o$ (i.e. the group of rotations around $\psi_o$);
- $\text{Grav}_{p,q}(M)$ is the class of all gravitational fields $(g, \nabla)$ on $M$ with $g$ of signature $(p, q)$ (the same of $g_o$).

Then, the correspondence described in (3.2) determines a map

$$
\iota_{\psi_o} : \text{Conn}(\tilde{E})^{\psi_o, \text{reg}} \longrightarrow \text{Grav}_{p,q}(M)
$$

(3.3)

which induces an injection from the space of $G_{\psi_o}$-orbits $\text{Conn}(\tilde{E})^{\psi_o, \text{reg}}/G_{\psi_o}$ into the family of gravitational field.

The map $\iota_{\psi_o}$ is indeed a projection. In fact, by standard facts on inner products, any pseudo-Riemannian metric $g$ of signature $(p, q)$ is of the form (3.2) for some suitable tensor field $L$ of type $(1,1)$. Moreover, if $\nabla'$ is a covariant derivation on $M$ which is metric for $g$, then the covariant derivation on $E$ defined by $\nabla = L \circ \nabla' \circ L^{-1}$ is metric for $g_o$ and the pair $(L, \nabla)$ is mapped onto $(g, \nabla')$ via (3.2). This proves the surjectivity. Summing up, we proved the following.

**Theorem 3.1.** For any given $\psi_o \in \Sigma(\tilde{E})$, the map $\iota_{\psi_o}$ defined in (3.2) induces a one to one correspondence between the orbit space $\text{Conn}(\tilde{E})^{\psi_o, \text{reg}}/G_{\psi_o}$ and gravitational fields $(g, \nabla)$ in $\text{Grav}_{p,q}(M)$.

**Expressions in coordinates.** Let $(x^1, \ldots, x^n) : \mathcal{U} \subset M \rightarrow \mathbb{R}^n$ be a system of coordinates on $M$ and fix a collection $(e_1^o, \ldots, e_n^o)$ of sections $e_i^o \in \Sigma(E)$ so that $(e_i^o|_x)$ is an orthonormal basis of $(E_x, g_o)$. Any metric affine covariant derivation $(L, \nabla)$ is of the form

$$
L = \theta_i^\mu e_i^o \otimes dx^\mu, \quad \nabla_{\frac{\partial}{\partial x^\mu}} (\varphi^i e_i^o) = \left( \frac{\partial \varphi^i}{\partial x^\mu} + \Gamma_{\mu j}^i \varphi^j \right) e_i^o,
$$

where $\Gamma_{\mu j}^i$ are the components of the derivatives $\nabla_{\frac{\partial}{\partial x^\mu}} e_i^o = \Gamma_{\mu j}^i e_j^o$. The vector fields

$$
e_i = e_i^o \frac{\partial}{\partial x^\mu} \overset{\text{def}}{=} L^{-1}(e_i^o) \in \mathfrak{X}(M)
$$
constitute an orthonormal frame field (or vielbein) for the metric \( g = g_o(L(\cdot), L(\cdot)) \), while \( \nabla^L \) is of the form

\[
\nabla^L_{\partial x^\mu}(X^i e_i) = \left( \frac{\partial X_i}{\partial x^\mu} + \Gamma^j_{\mu i} X^j \right) e_i
\]

Since \( e^\mu_i \theta^j_{\mu} = \delta^j_i \), it is clear that the application \( L \) can be completely recovered from the vielbein \( (e_i) \). Moreover,

\[
0 = g(\nabla^L_{\partial x^\mu} e_i, e_j) + g(e_i, \nabla^L_{\partial x^\mu} e_j) = \Gamma^j_{\mu i} + \Gamma^j_{\mu j}
\]

and one can easily check that any set of functions \( \Gamma^j_{\mu i} \) that satisfies (3.4) determines uniquely a metric covariant derivation \( \nabla' \) on \( M \) and hence a metric covariant derivation \( \nabla = L \circ \nabla' \circ L^{-1} \) on \( E \). Therefore, the class of covariant derivations \( (L, \nabla) \) can be locally identified with the pairs \( ((e_i), (\Gamma^j_{\mu i})) \) formed by a vielbein \( (e_i) \) and functions \( \Gamma^j_{\mu i} \) with \( \Gamma^j_{\mu i} = -\Gamma^j_{i\mu} \).

Let us now write the formulae that express the action of the gauge transformations in terms of the pairs \( ((e_i), (\Gamma^j_{\mu i})) \). Assume that \( f = \tilde{\xi} \in \text{Gau}(\tilde{E}) \) is a translation by \( \xi = \xi^i e^i_o \in \Sigma(E) \). Denoting as before by \( \theta^j_{\mu} \) the functions which give the components of \( L \) and are hence defined in terms of the vielbein \( (e_i) \) by the relations \( e^\mu_i \theta^j_{\mu} = \delta^j_i \), one can immediately obtain that

\[
((e_i), (\Gamma^j_{\mu i})) \mapsto ((e'_i), (\Gamma'^j_{\mu i}))
\]

where \( \Gamma'^j_{\mu i} = \Gamma^j_{\mu i} \) and \( e'_i = e^\mu_i \frac{\partial}{\partial x^\mu} \) is defined by the following equations

\[
e'_i \left( \theta^j_{\mu} - \frac{\partial \xi^j}{\partial x^\mu} - \Gamma^j_{\mu \ell} \xi^\ell \right) = \delta^j_i.
\]

In case \( f = \tilde{h} \in \text{Gau}(\tilde{E}) \) is a rotation around \( \psi_o \) by \( h \in \text{Gau}(E) \), \( h(e^i_o) = h^j_i e^j_o \), we have that

\[
((e_i), (\Gamma^j_{\mu i})) \mapsto \left( (h^{-1})^j_i e_j, \left( h^j_i \Gamma^k_{\mu j} (h^{-1})^k_i + h^j_i \frac{\partial (h^{-1})^j_i}{\partial x^\mu} \right) \right).
\]

Notice that Theorem 3.1 implies that locally any gravitational fields \( (g, \nabla) \) can be identified with a pair of the form \( ((e_i), (\Gamma^j_{\mu i})) \) uniquely determined up to a transformation \( (3.7) \).

### 3.2. Theories of gravity as gauge theories.

Consider the actions of the elements \( f \in \text{Gau}(\tilde{E}) \) on the section \( \psi_o \) and on the corresponding map \( \iota_{\psi_o} \). We claim that

\[
\iota_{f(\psi_o)} \circ f_* = \iota_{\psi_o}
\]

(3.8)
for any \( f \in \text{Gau} (\tilde{E}) \). To prove this, by Lemma 2.7 and previous remarks, we may assume with no loss of generality that \( f = \bar{\xi} \) is a translation determined by a section \( \xi \) of \( E \). Then
\[
 f_\ast (L, \nabla) = (L - \nabla \xi, \nabla)
\]
and
\[
 \iota_{f(\psi_o)} (f_\ast (L, \nabla)) = (g_0 (\tilde{\Lambda} (\cdot), \tilde{\Lambda} (\cdot)), \nabla), \quad \text{where} \quad \tilde{\Lambda} \defeq (f_\ast \nabla)f(\psi_o).
\]
Since \((f_\ast \nabla)f(\psi_o) = \tilde{\nabla} (\tilde{\xi}^{-1} (\xi(\psi_o))) = \tilde{\nabla} \psi_o\), we have that \( \tilde{\Lambda} = L \) and (3.8) follows. Due to (3.8) and Theorem 3.1 if we set \( \Omega \defeq \bigcup_{\psi_o \in \Sigma (\bar{E})} \text{Conn} (\bar{E})^{\psi_o \text{reg}} \times \{ \psi_o \}, \) the map
\[
 \iota : \Omega \to \text{Grav}_{p,q} (M), \quad \iota ((\tilde{\nabla}, \nabla); \psi_o) \defeq \iota_{\psi_o} (L, \nabla) = (g, \nabla_L)
\]
induces a one-to-one correspondence between \( \Omega / \text{Gau} (\tilde{E}) \) and \( \text{Grav}_{p,q} (M) \).

This identification \( \text{Grav}_{p,q} (M) \simeq \Omega / \text{Gau} (\tilde{E}) \) is not in contrast with the identification \( \text{Grav}_{p,q} (M) \simeq \text{Conn} (\tilde{E})^{\psi_o \text{reg}} / \mathcal{G}_{\psi_o} \) given in Theorem 3.1. In fact, the space \( \Omega \) is union of the \( \text{Gau}^T (\tilde{E}) \)-orbits of \( \text{Conn} (\tilde{E})^{\psi_o \text{reg}} \simeq \text{Conn} (\tilde{E})^{\psi_o \text{reg}} \times \{ \psi_o \} \) and hence \( \text{Grav}_{p,q} (M) \simeq \text{Conn} (\tilde{E})^{\psi_o \text{reg}} / \mathcal{G}_{\psi_o} \simeq \left( \frac{\Omega}{\text{Gau}^T (\tilde{E})} \right) / \mathcal{G}_{\psi_o} = \frac{\Omega}{\text{Gau} (\tilde{E})}. \)

For practical purposes, the identification \( \text{Grav}_{p,q} (M) \simeq \text{Conn} (\tilde{E})^{\psi_o \text{reg}} / \mathcal{G}_{\psi_o} \) is more efficient and it is the only one we use in the following. On the other hand, the identification \( \text{Grav}_{p,q} (M) \simeq \Omega / \text{Gau} (\tilde{E}) \) allows to state that the theories on gravity fields \( (g, \nabla) \) are in natural correspondence with theories on the triples \((\tilde{\nabla}, \nabla, \psi_o) \in \Omega \) that are invariant under the full gauge group \( \text{Gau} (\tilde{E}) \), i.e. of the gauge group of the \( O_{p,q} \times \mathbb{R}^n \)-bundle \( P = O_q (\tilde{E}) \). Moreover, it must be stressed that, via the map \( \iota \), the action of \( \text{Gau} (\tilde{E}) \) on \( \Omega \) corresponds to the trivial action on \( \text{Grav}_{p,q} (M) \). Hence, the presentation of the theories on gravity fields as gauge theories of \( \text{Gau} (\tilde{E}) \) does not carry any practical advantage for analyzing the dynamics of gravity fields \((g, \nabla)\). The main application we have in mind is to provide a solid scheme of geometric construction for gauge theories with gauge group of super-extensions of the Poincarè group \( O_{p,q} \times \mathbb{R}^n \), i.e. of theories of supergravity.

**Remark 3.2.** Since \( \text{Gau} (\tilde{E}) \) acts transitively on the sections of \( \tilde{E} \), no constraint on \( \psi_o \) might occur if one look for gauge invariant equations on \( \Omega \). On the other hand, since \( \text{Gau} (\tilde{E}) \) acts trivially on \( \text{Grav}_{p,q} (M) \), there is no effect if we break the gauge invariance by considering \( \psi_o \) as fixed (see also [15], end of §1). So, with no loss of generality, we may state that the theories on gravity fields \((g, \nabla)\) are in natural correspondence with the theories on regular metric affine connections \((\tilde{\nabla}, \nabla) = (L, \nabla)\), which are invariant under the reduced gauge group \( \mathcal{G}_{\psi_o} = \text{Gau}_{\psi_o} (\tilde{E}) \).

---

2On invariance under the transformations in \( \text{Gau} (\tilde{E}) \), see also [17] and [15], §1.
Remark 3.3. Even if the subgroup of translations $\text{Gau}^T(\tilde{E}) \subset \text{Gau}(\tilde{E})$ does act (locally) on the class of regular metric affine covariant derivations $\text{Conn}(\tilde{E})^{\psi_0,\text{reg}}$, the map $i$ cannot be used to induce any corresponding action (not even the trivial one) on the space of gravity fields $\text{Grav}_{p,q}(M) \simeq \text{Conn}(\tilde{E})^{\psi_0}/\mathcal{G}_{\psi_0}$. In fact, the isotropy gauge group $\mathcal{G}_{\psi_0} = \text{Gau}_{\psi_0}(\tilde{E})$ is not normalized by the action of $\text{Gau}^T(\tilde{E})$ and hence there is no induced action of $\text{Gau}^T(\tilde{E})$ on the quotient $\text{Conn}(\tilde{E})^{\psi_0}/\mathcal{G}_{\psi_0}$. To check this directly, consider two covariant derivations $(L, \nabla)$ and $(L', \nabla')$ in the same $\mathcal{G}$-orbit (i.e. $L' = h \circ L$ and $\nabla' = h \circ \nabla \circ h^{-1}$ for some rotation $h \in \mathcal{G}$) and let $\tilde{\xi} \in \text{Gau}^T(\tilde{E})$ determined by $\tilde{\xi} \in \Sigma(E)$. Then $(\tilde{\xi}_s L, \tilde{\xi}_s \nabla) = (L - \nabla \xi, \nabla)$ and $(\tilde{\xi}_s L', \tilde{\xi}_s \nabla') = (h \circ (L - \nabla \eta), h \circ \nabla \circ h^{-1})$, with $\eta = L^{-1}(\xi)$. It follows that, in general, $(\tilde{\xi}_s L, \tilde{\xi}_s \nabla)$ and $(\tilde{\xi}_s L', \tilde{\xi}_s \nabla')$ are not in the same $\mathcal{G}$-orbit.

4. Theories of gravity as gauge theories satisfying the Equivalence Principle

The Equivalence Principle of General Relativity (i.e. covariance under changes of coordinates) requires that all constraints and equations must be covariant under any local diffeomorphism, that is the class of their solutions is invariant under the action of local diffeomorphisms. This corresponds to an invariance property on the corresponding gauge theory on $\text{Conn}(\tilde{E})^{\psi_0,\text{reg}}$ distinct from the invariance w.r.t. to $\mathcal{G}_{\psi_0}$, described in the previous section. Therefore, possible generalizations of General Relativity that satisfy the Equivalence Principle must be searched amongst $\mathcal{G}_{\psi_0}$-invariant theories in $\text{Conn}(\tilde{E})^{\psi_0,\text{reg}}$ that are invariant under an additional pseudogroup of local transformations, namely under a pseudogroup acting on $\text{Conn}(\tilde{E})^{\psi_0,\text{reg}}$ in a way that corresponds to the action of the local diffeomorphisms on $\text{Grav}_{p,q}(M)$. In the next two sections we determine the infinitesimal transformations of such pseudogroup.

4.1. Pseudo-translations and the Equivalence Principle for a theory in $\text{Conn}(\tilde{E})^{\psi_0,\text{reg}}$.

4.1.1. Torsion and curvature of metric affine covariant derivation. Parameterizations by the torsion.

Definition 4.1. We call torsion of a regular metric affine covariant derivation $(L, \nabla)$ the section of $\Sigma(\Lambda^2 E^* \otimes E)$ defined by

$$T(s, s') \overset{\text{def}}{=} \nabla_{L^{-1}(s)} s' - \nabla_{L^{-1}(s')} s - [L^{-1}(s), L^{-1}(s')]$$
for any \( s, s' \in \Sigma(E) \). We call curvature of \((L, \nabla)\) the section of \( \Sigma(\Lambda^2 E^* \otimes \mathfrak{so}(E, \gamma_0)) \) defined by

\[
R(s, s')(s'') \defeq \nabla_{L^{-1}(s)} \nabla_{L^{-1}(s')}(s'') - \nabla_{L^{-1}(s')} \nabla_{L^{-1}(s)}(s'') - \nabla_{[L^{-1}(s), L^{-1}(s')]}(s'')
\]

for any \( s, s', s'' \in \Sigma(E) \).

Notice that the tensor fields \( T^L = L^* T \) and \( R^L = L^* R \) on \( M \) coincide with the torsion and curvature of the connection \( \nabla^L = L^{-1} \nabla \circ L \) associated with \((L, \nabla)\). This motivates our terminology.

We want now to show that the torsions can be used to completely parameterize the space of metric affine covariant derivations, in full analogy with the parameterization by torsions of the metric connections on pseudo-Riemannian manifolds. First of all, fix a metric covariant derivation \( \nabla^o \) on the vector bundle \((E, g_0)\) and for any \((L, \nabla) \in \text{Conn}^{\psi_{\text{reg}}}(E)\) let

\[
\delta \nabla : \Sigma(E) \times \Sigma(E) \longrightarrow \Sigma(E)
\]

By construction and definitions, for any \( \lambda, \mu \in \mathfrak{g}(M) \) and \( s, s', s'' \in \Sigma(E) \) we have that \( \delta \nabla(\lambda s + \mu s', s'') = \lambda \delta \nabla(s, s'') + \mu \delta \nabla(s', s'') \) and \( \delta \nabla(s, \lambda s' + \mu s'') = \lambda \delta \nabla(s, s') + \mu \delta \nabla(s, s') \). This means that \( \delta \nabla \) can be uniquely represented as a vertical tensor field with values in \( E^* \otimes_M E^* \otimes_M E \). Being \( \nabla^o \) and \( \nabla \) both metric w.r.t. \( g_0 \), from equalities

\[
g_0(\nabla^o_{L^{-1}(s)} s' - \nabla^o_{L^{-1}(s')} s'', s'') =
L^{-1}(s)g_0(s', s'') - g_0(s', \nabla^o_{L^{-1}(s)} s'') - L^{-1}(s)g_0(s', s'') + g_0(s', \nabla_{L^{-1}(s)} s'') =
g_0(s', \nabla^o_{L^{-1}(s)} s'' - \nabla_{L^{-1}(s)} s'') ,
\]

we conclude that

\[
g_0(\delta \nabla(s, s'), s'') + g_0(s', \delta \nabla(s, s'')) = 0 ,
\]

meaning that \( \delta \nabla \) is indeed a section of \( E^* \otimes \mathfrak{so}(E, g_0) \) (here \( \pi : \mathfrak{so}(E, g_0) \to M \) is the vector bundle of vertical tensor fields in \( E^* \otimes E \), skew-symmetric w.r.t. \( g_0 \), i.e. with fibers equal to \( \mathfrak{so}(E_x, g_0) \)).

Conversely, given a pair \((L, \delta \nabla)\), consisting of vertical tensor fields in \((T^* M \otimes_M E) + (E^* \otimes \mathfrak{so}(E, g_0))\) with \( L_x : T^*_x M \to E_x \) invertible for any \( x \in M \), we may consider the pair \((L, \nabla)\) with \( \nabla \) defined by

\[
\nabla_X s \defeq \nabla_X s + \delta \nabla(L(X), s).
\]

From previous remarks, \((L, \nabla)\) is a metric affine covariant derivation in \( \text{Conn}^{\psi_{\text{reg}}}(E) \) and \((4.3)\) allows to \( \text{Conn}^{\psi_{\text{reg}}}(E) \simeq \Sigma(T^* M \otimes_M E) \times \Sigma(E^* \otimes \mathfrak{so}(E, g_0)) \). In particular, we may conclude that \( \text{Conn}^{\psi_{\text{o}}}(E) \) has a structure of Frechet space with tangent spaces isomorphic to the space of sections \( \Sigma(T^* M \otimes_M E \oplus_M E^* \otimes_M \mathfrak{so}(E, g_0)) \).
Let us now consider the skew-symmetrizing map \( \partial : \Sigma(E^* \otimes \mathfrak{so}(E, g_0)) \rightarrow \Sigma(\Lambda^2 E^* \otimes E) \), called Spencer operator, defined by

\[
\partial H(s, s') = H(s, s') - H(s', s)
\]  

(4.4)

**Lemma 4.2.** The operator \( \partial \) determines an isomorphism between the space of sections \( \Sigma(E^* \otimes \mathfrak{so}(E, g_0)) \) and the space of sections \( \Sigma(\Lambda^2 E^* \otimes E) \).

**Proof.** It suffices to check that, for any \( x \in M \), the linear map \( \partial_x : E_x^* \otimes \mathfrak{so}(E_x, g_0) \rightarrow \Lambda^2 E_x^* \otimes E_x \), defined by \( \partial H_x(s, s') = H_x(s, s') - H_x(s', s) \), is an isomorphism. By dimension counting, it suffices to check that \( \ker \partial_x = 0 \).

Following a very classical argument, this is proved noticing that if \( \partial H_x = 0 \), then for any \( s, s', s'' \) one has

\[
g_o(H_x(s, s'), s'') = -g_o(H_x(s, s''), s') = -g_o(H_x(s'', s), s') =
\]

\[
= g_o(H_x(s'', s'), s) = g_o(H_x(s', s''), s) = -g_o(H_x(s', s), s'') =
\]

\[
- g_o(H_x(s, s'), s'')
\]

and hence that \( H_x = 0 \) by nondegeneracy of \( g_o \).

If \( T^o \) is the torsion of \( \nabla^o \), the torsion \( T \) of any other derivation \( (L, \nabla) \), represented by the pair \((L, \delta \nabla)\), is given by

\[
T = T^o + \partial \delta \nabla.
\]  

(4.5)

From Lemma 4.2, \( \delta \nabla \) (and hence \( \nabla \)) can be completely recovered from \( T \) and \( T^o \) and the following correspondence is one-to-one:

\[
(L, T) \xrightarrow{\partial} (L, \nabla) = (L, \nabla^o + \partial^{-1}(T - T^o)(L(\cdot), \cdot))
\]  

(4.6)

**Lemma 4.3.** The correspondence (4.6) is independent of \( \nabla^o \) and \( T^o \) and gives a one-to-one correspondence between the set of sections \( (L, T) \in \Sigma(T^* M \otimes_M E) \times \Sigma(\Lambda^2 E^* \otimes E) \), with \( L \) regular, and the connections in \( \text{Conn}^{\psi_o}(E) \).

**Proof.** Let \( \nabla^o \) and \( \nabla^{o'} \) two metric covariant derivations of \((E, g_o)\), with torsions \( T^o \) and \( T^{o'} \), respectively, and \( \delta \nabla^o \overset{\text{def}}{=} \nabla^{o'} - \nabla^o \). By (4.5), \( T^{o'} = T^o + \partial \delta \nabla^o \) and the conclusion follows from

\[
\nabla^{o'} - \partial^{-1}(T - T^o)(L(\cdot), \cdot) =
\]

\[
= \nabla^o + \delta \nabla^o + \partial^{-1}(T - T^o)(L(\cdot), \cdot) - \partial^{-1}(\partial \delta \nabla^o)(L(\cdot), \cdot) =
\]

\[
= \nabla^o + \partial^{-1}(T - T^o)(L(\cdot), \cdot).
\]

(4.7)

**Remark 4.4.** By Lemma 4.2, for any given \( L \) there is a unique \( \nabla^{oL} \) with vanishing torsion \( T^o \). Inserting \( \nabla^{oL} \) in (4.6), the expression simplifies into

\[
(L, T) \xrightarrow{\partial} (L, \nabla) = (L, \nabla^{oL} + \partial^{-1}(T)(L(\cdot), \cdot))
\]  

(4.7)
Expressions in coordinates. In the notation used in 3.1 for the expressions in coordinates, the torsion $T$ and the curvature $R$ of $(L, \nabla)$ are of the form

\[
T = T^k_{ij} e^i_o \otimes e^j_o \otimes e^o_k, \quad T^k_{ij} = e^\mu_i \Gamma^k_{\mu j} - e^\mu_j \Gamma^k_{\mu i} \\
R = R^m_{ijk} e^i_o \otimes e^j_o \otimes e^o_m, \quad R^m_{ijk} = e^\mu_i e^\nu_j (\partial_\mu \Gamma^m_{\nu k} - \partial_\nu \Gamma^m_{\mu k} - \Gamma^m_{\mu \nu} \Gamma^\ell_{\mu k} + \Gamma^m_{\nu \ell} \Gamma^\ell_{\mu k}).
\]

Also the map $\partial^{-1}$ can be determined explicitly. It is equal to

\[
\partial^{-1}(T^k_{ij} e^i_o \otimes e^j_o \otimes e^o_k) = \frac{1}{2} \left( T^k_{ij} + T^k_{jk} - T^k_{ik} \right) e^i_o \otimes e^j_o \otimes e^o_k,
\]

which is the usual formula for the so-called “contorsion”.

4.1.2. Infinitesimal pseudo-translations.

In all the following, the section $\psi_o \in \Sigma(\tilde{E})$ is considered fixed and $\text{Conn}(\tilde{E})^{\psi, \text{reg}}$ is identified with the regular pairs $(L, \nabla)$. We introduce now the notion of pseudo-translations. In the next Theorem 4.6 it is shown that they correspond to the infinitesimal transformations by vector fields on $M$.

**Definition 4.5.** Let $X$ be a vector field on $M$. We call **infinitesimal pseudo-translation associated with $X$** the map

\[
\tau^{(X)} : \text{Conn}(\tilde{E})^{\psi, \text{reg}} \to \Sigma(T^* M \otimes_M E) \times \Sigma(E^* \otimes \mathfrak{so}(E, g_o))
\]

\[
\tau^{(X)}(L, \nabla) \triangleq (L(\nabla^L X + T^L_X), \partial^{-1}(\delta_X T)(L(\cdot, \cdot)))
\]

(4.8)

where $T^L$ is the torsion of $\nabla^L$ (see 3.2) and $\delta_X T \in \Sigma(E^* \otimes \mathfrak{so}(E, g_o))$ is defined by

\[
(\delta_X T)_{s, s'} = T_L(X)_{T(s, s')} + T_L((\nabla_{L^{-1}(s')} L(X))s' - T_L(\nabla_{L^{-1}(s')} L(X))s' +
- (\nabla_{L^{-1}(s')} L(X))s' L(X) + (\nabla_{L^{-1}(s')} L(X))s - R_{ss'} L(X) + R_{L(X)ss'} + R_{s'L(X)s'}
\]

Here $T$ and $R$ are the torsion and curvature of $(L, \nabla)$ and $\nabla_Y T$ is the vertical tensor defined by $(\nabla_Y T)_{ZW} \triangleq \nabla_Y (T_{ZW}) - T_{\nabla_Y ZW} - T_{Z \nabla_Y W}$.

Any infinitesimal pseudo-translation can be considered as a “vector field” on $\text{Conn}(\tilde{E})^{\psi, \text{reg}}$ with associated flow $T^{(X)}_t : \text{Conn}(\tilde{E})^{\psi, \text{reg}} \to \text{Conn}(\tilde{E})^{\psi, \text{reg}}$ defined by

\[
\left. \frac{d T^{(X)}_t (L, \nabla)}{d t} \right|_{t = t_o} = \tau^{(X)}(T^{(X)}_t (L, \nabla))
\]

We call $T^{(X)}_t$ **flow of pseudo-translations generated by $X$**.

The following theorem collects the main properties of pseudo-translations. In particular, it shows that the action of the flow of a pseudo-translation $\tau^{(X)}$ on the element $(L, \nabla) \in \text{Conn}(\tilde{E})^{\psi, \text{reg}}$ induces an action on the corresponding gravitational fields $(g, \nabla^L)$ on $M$ which coincide with the action of the flow of $X$ on $M$.

**Theorem 4.6.**
i) For any \( X \in \mathfrak{X}(M) \), the flow \( T^X_t \) commutes with \( \text{Gau}_{\psi_0}(\mathcal{E}) \).

ii) Let \( t_{\psi_0} : \text{Conn}(\mathcal{E})_{\psi_0}^{\text{reg}} \to \text{Grav}_{p,q}(M) \) the correspondence \((3,2)\), \((L, \nabla) \in \text{Conn}(\mathcal{E})_{\psi_0}^{\text{reg}} \) and \((g, \nabla^L) = t_{\psi_0}(L, \nabla) \). For any \( X \in \mathfrak{X}(M) \)

\[
\left. \frac{d}{dt} t_{\psi_0}(T^X_t(L, \nabla)) \right|_{t=0} = \left( \mathcal{L}_X g, \frac{d}{dt} \Phi^X_t(\nabla^L) \right|_{t=0} \right) .
\]

iii) The infinitesimal pseudo-translations have a natural structure of (infinite-dimensional) Lie algebra isomorphic to the Lie algebra of vector fields \( \mathfrak{X}(M) \).

**Proof.** To check (i), it is first necessary to observe that if \( h \in \text{Gau}_{\psi_0}(\mathcal{E}) \), then by definitions and \((2.13)\), the torsion and curvature of \((h_*L, h_*\nabla)\) are equal to \( h_*T = (h \circ T)(h^{-1}(\cdot), h^{-1}(\cdot)) \) and \( h_*R = (h \circ T)(h^{-1}(\cdot), h^{-1}(\cdot), h^{-1}(\cdot)) \), respectively. Using this, a straightforward computation implies the claim.

For (ii), let \((g^{(t)}, \nabla^{(t)}) = t_{\psi_0}(T^X_t(L, \nabla))\) and denote by \( T^{(t)} \) the torsion of \( \nabla^{(t)} \). Since \( g^{(0)} = g \), \( \nabla^{(0)} = \nabla^L \) and \( T^L = T^{(0)} \), we only need to show that \( \frac{d}{dt} g^{(t)} \big|_{t=0} = \mathcal{L}_X g \) and \( \frac{d}{dt} \nabla^{(t)} \big|_{t=0} = \mathcal{L}_X T^{(0)} \). These identities are consequence of the definition of pseudo-translations, standard properties of metric connections and first Bianchi identities. In fact, they can be obtained observing that, for any \( Y, Z \in \mathfrak{X}(M) \),

\[
\mathcal{L}_X g(Y, Z) = X(g(Y, Z)) - g(\mathcal{L}_X Y, Z) - g(Y, \mathcal{L}_X Z) =
\]

\[
= g(\nabla^L_Y X, Z) + g(T^L_X Y, Z) + g(Y, \nabla^L_Z X) + g(Y, T^L_X Z) =
\]

\[
= g_0(L(\nabla^L_Y X + T^L_X Y), L(Z)) + g_0(L(Y), L(\nabla^L_Z X + T^L_X Z))
\]

and

\[
\mathcal{L}_X T^L_{Y Z} = X(T^L_{Y Z}) - T^L_{L X Y} Z - T^L_{L Y Z} X = (\nabla^L_X T^L) Y Z + T^L_{Y \nabla^L_X Z} X + T^L_{T^L_X Y} Z +
\]

\[
+ T^L_{Y \nabla^L_X Z} X + T^L_{T^L_X Y} Z .
\]

Claim (iii) follows immediately from (ii) if we set \([\tau^X, \tau^X'] \overset{\text{def}}{=} \tau[X, X']\) the Lie bracket between two infinitesimal pseudo-translations. \(\square\)

By the above theorem, we get the last result, mentioned in the Introduction: an action on pairs \((L, \nabla) \in \text{Conn}(\mathcal{E})_{\psi_0}^{\text{reg}}\) corresponds to an action on gravitational fields \((g, \nabla^L)\), whose Euler-Lagrange equations satisfy the infinitesimal Equivalence Principle if and only if it is invariant under pseudo-translations “on-shell”, i.e. at the points given by solutions of the Euler-Lagrange equations.

It is also clear that if \((L, \nabla)\) is so that \( T = 0 \), by definitions and first Bianchi identities, one has

\[
\tau^X(L, \nabla) = (\nabla L(X), \partial^{-1}(0))
\]

\((4.10)\).
and hence the flow $T^X_t$ maps $(L, \nabla)$ into other metric affine connections with vanishing torsion, i.e. the condition $T = 0$ is preserved by pseudo-translations, as it should be by the correspondence (4.9).

4.1.3. A classical example of gauge-invariant Lagrangian preserved by pseudo-translations: the Palatini action.

Consider a space-time $M$ of dimension $n$, a metric affine bundle $(\tilde{E}, E, g_0)$ over $M$ with signature $(p, q)$, and assume that there exists an affine vertical tensor field $\tilde{\omega}_o$, determined by a vertical volume form $\omega_o$ on each fiber $E_x$, which is equal to 1 on suitably ordered orthonormal frames $(e^o_i)$ of $E_x$. A volume form $\omega_o$ of this kind exists if and only if the bundle $O_{g_0}(E)$ of the orthonormal frames of the fibers of $E$ admits an $SO_{p,q}(\mathbb{R})$-reduction.

Given $(L, \nabla) \in \text{Conn}(\tilde{E})^{\psi_0\text{reg}}$, we denote by $\tilde{R}$ the section in $\Sigma(\Lambda^2T^*M \otimes_M \Lambda^2E)$ determined by the relation

$$e^i_o \otimes e^j_o(\tilde{R}_{XY}) \overset{\text{def}}{=} g_0(R_{L(X)L(Y)} \cdot e^i_o, e^j_o)$$

where $R$ is the curvature defined in Definition 4.11 and $(e^j_o)$ is the coframe field dual to $(e^i_o)$. Observe that, for any vector fields $Y_i \in \mathcal{T}M$, $1 \leq i \leq n$

$$\omega_0(\tilde{R}_{Y_1 Y_2} \wedge L(Y_3) \wedge \cdots \wedge L(Y_n)) = \sum_{i=1}^n \omega_0(R_{Y_1 Y_2} \cdot e^i_o, e^i_o, L(Y_3), \ldots, L(Y_n)) \cdot$$

We also denote by $\text{Alt} : \otimes^nT^*M \to \Lambda^nT^*M$ the usual alternating map.

Now, we consider the Palatini action on pairs $(L, \nabla) \in \text{Conn}(\tilde{E})^{\psi_0\text{reg}}$

$$S_{Pal}(L, \nabla) \overset{\text{def}}{=} \int_M \text{Alt} \left( \omega_0(\tilde{R} \wedge L \wedge \cdots \wedge L) \right) =$$

$$= \int_M \sum_{i=1}^n \text{Alt} \left( \omega_0((R \cdot e^i_o) \wedge e^i_o \wedge L \wedge \cdots \wedge L) \right). \quad (4.11)$$

Using (2.13), one can check that the Lagrangian $L = \text{Alt} \left( \omega_0(\tilde{R} \wedge L \wedge \cdots \wedge L) \right)$ is invariant under any rotation in $G_{\psi_0}$. If desired, one can also extend $L$ and obtain a fully gauge-invariant action on $\Omega$ (see §3.2) by simply imposing that $L$ is constant along the $\text{Gau}^T(\tilde{E})$-orbits passing through the points of $\text{Conn}(\tilde{E})^{\psi_0\text{reg}}$.

The reader can also check that, expressed in terms of the pairs $(g^L, \nabla^L) \in \text{Grav}_{p,q}(M)$, the action $S_{Pal}$ becomes the usual Hilbert action $S_{Hilb} = \int_M \text{Scal}(g)\omega_g$, while from the Euler-Lagrange equations determined by (4.11) coincide with those obtained from the Palatini action with the Palatini method of variation (see e.g. [21]), i.e.

$$T^L = 0, \quad \text{Ric}^L = 0.$$
Remark 4.7. In a future paper, we will consider gauge theories of super-extensions of Poincarè group and the corresponding analogues of pseudo-translations. The correspondence with vector fields on super-manifolds are expected to relates the invariance under pseudo-translations to a “super” version of the Equivalence Principle.

REFERENCES

[1] A. L. Almorox, Supergauge theories in graded manifolds in “Differential Geometrical methods in Mathematical Physics” Proc. Salamanca 1985 - Eds. P. L. Garcia, A Perez-Rendón, Lectures Notes in Math. n. 1251, Springer, 1987.
[2] D. V. Alekseevsky and V. Cortes, Classification of N-(super)-extended Poincaré algebras and bilinear invariants of the Spinor representation of Spin(p,q), Comm. Math. Phys. 183 (1977), 477–510.
[3] Y. M. Cho, Gauge theory of Poincarè symmetry, Phys. Rev. D 14 n. 12 (1976), 3335–3340.
[4] J. Deligne and J. W. Morgan, Notes on supersymmetry (following Joseph Bernstein) in “Quantum Fields and Strings: A Course for Mathematicians, Vol. I, American Mathematical Society, Providence, R. I., 1999.
[5] H. Goldschmidt, Integrability criteria for systems on non-linear partial differential equations, J. Differential Geom. 1 (1967), 269-307.
[6] F. W. Hehl, Four lectures on Poincarè gauge field theory. Cosmology and gravitation (Bologna, 1979), p. 5–61 in “NATO Adv. Study Inst. Ser. B: Physics”, vol. 58, Plenum, New York-London, 1980.
[7] F. W. Hehl, J. D. McCrea, E. W. Mielke and Y. Ne’eman, Metric-affine gauge theory of gravity: field equations, Noether identities, world spinors, and breaking of dilation invariance, Phys. Rep. 258, no. 1-2 (1995), 1–171.
[8] T. W. Kibble, Lorentz invariance and the gravitational field, J. Math. Phys. 2 (1961), 212–221.
[9] E. A. Ivanov and J. Niederle, Gauge formulation of gravitation theories. I. The Poincarè, de Sitter and conformal cases, Phys. Rev. D 25 (1982), 976 – 987.
[10] B. T. McInnes, On the affine approach to Riemann-Cartan space-time geometry, Class. Quantum Grav. 1 (1984), 115–123.
[11] Y. Ne’eman, Gravity is the gauge theory of the parallel transport of the Poincarè group, in “Diff. Geom. Methods in Math. Phys.”, p. 189–216, Lect. Notes in Math. vol. 676 (Proc. Bonn 1977), K. Bleuler, H. R. Petry and A. Reetz eds., Springer - Berlin, 1979.
[12] K. A. Pilch, Geometrical meaning of the Poincarè group gauge theory, Lett. Math. Phys. 4 (1980), 49–51.
[13] D. A. Popov and L. I. Daikhin, Einstein spaces and Yang-Mills fields, Sov. Phys. Dokl. 20 (12) (1976), 818–820.
[14] T. Stavrakou, Theory of connections on graded principal bundle, Rev. Math. Phys. 10, n. 1 (1998), 47–79.
[15] S. Sternberg, The interaction of Spin and Torsion. II. The Principle of General Covariance, Ann. Phys. 162 (1985), 85–99.
[16] G. Sardanashvili and O. Zacharov, Gauge Gravitation Theory, World Scientific, Singapore, 1992.
[17] S. Tantucci, Connessioni su fibrati affini e rappresentazioni dell’interazione gravitazionale come campo di gauge, Tesi di Laurea in Fisica, Università di Camerino, Camerino, Italy, 2007.
[18] A. Trautman, *Fibre bundles associated with space-time*, Reports on Math. Phys. 1, n. 1 (1970), 29–62.
[19] A. Trautman, *The Geometry of Gauge Fields*, Czech, J. Phys. B 29 (1979), 107–116.
[20] R. Utiyama, *Invariant theoretical interpretation of interaction*, Phys. Rev. 101 (1956), 1597–1607.
[21] D. K. Wise, *MacDowell-Mansouri Gravity and Cartan Geometry*, preprint posted on ArXiv.

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