Inverse cascade and intermittency of passive scalar in 1d smooth flow

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Random advection of Lagrangian tracer scalar field \(\theta(t,x)\) by a one-dimensional, spatially smooth and short-correlated in time velocity field is considered. Scalar fluctuations are maintained by a source concentrated at the integral scale \(L\). The statistical properties of both scalar differences and the dissipation field are analytically determined, exploiting the dynamical formulation of the model. The Gaussianity known to be present at small scales for incompressible velocity fields emerges here at large scales \((x \gg L)\). These scales are shown to be excited by an inverse cascade of \(\theta^2\) and the probability distribution function (PDF) of the corresponding scalar differences to approach the Gaussian form, as larger and larger scales are considered. Small scales \((x \ll L)\) statistics is shown to be strongly non-Gaussian. Collapse of scaling exponents for scalar structure functions takes place: moments of order \(p \geq 1\) scale all linearly, independently of the order \(p\). Smooth scaling \(x^p\) is found for \(-1 < p < 1\). Tails of scalar differences PDF are exponential while, at the center, a cusped shape tends to develop when smaller and smaller ratios \(x/L\) are considered. The same tendency is present for scalar gradients PDF with respect to the inverse of the Péclet number (the pumping-to-diffusion scale ratio). The tails of the latter PDF are however much more extended, decaying as a stretched exponential of exponent \(2/3\), smaller than unity. This slower decay is physically associated with the strong fluctuations of the dynamical dissipative scale.

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1. INTRODUCTION

Small-scale statistics of a passive scalar advected by a large-scale incompressible velocity field is an old problem in turbulence theory \(^{[1,2]}\). Scalar fluctuations are maintained by a large-scale forcing, with typical scale \(L\). According to the classical picture \(^{[3]}\) of direct cascade of the scalar, the injected scalar is mainly transferred downscales to the convective interval and then to the dissipative region. For smooth velocities, the statistics of this scalar transfer can be analyzed systematically and has been characterized in much detail in \(^{[2,4,5]}\). The core of the one-point scalar probability distribution functions (PDF) is Gaussian with variance \(O(\log Pe)\), where the Péclet number \(Pe\) is very large \(^{[6]}\). Far tails of the PDF decay exponentially \(^{[1,3]}\). The physical key ingredient at the basis of these results is that material lines are stretched, i.e. the maximum Lyapunov exponent \(\lambda\) for particles separation is positive. Typical trajectories will therefore be exponentially stretched and dynamically contracted trajectories are so rare, that they can only affect the extreme tails of the statistics. On the other hand, much interest has been recently attracted by Kraichnan model \(^{[1]}\) for its intermittent scaling behavior \(^{[1,2,3]}\). The picture emerging there is that dynamically contracted Lagrangian trajectories play a crucial role for structure functions scaling exponents \(c_p\) and, thus, for intermittency. The constant asymptotic behavior of \(c_p\) for large orders \(p\) found in \(^{[1]}\) comes, for example, from the most contracting possible trajectories and the value \(c_\infty\) from nontrivial fluctuations of the degree of freedom constrained by incompressibility to still be dynamically stretched.

Previous remarks have led us to investigate scalar transport in smooth compressible flow. The motivation is that compressibility might slow down Lagrangian separations and, thus, lead to nontrivial scaling and intermittency properties. These can be then analyzed systematically, using techniques specific for smooth velocities. Positive Lyapunov exponents are indeed characteristic of isotropic, solenoidal flow \(^{[1,5]}\), but this property might be lost when compressible flow are considered. It is, for example, known that for compressible flow a substantial slow-down of large-times transport can take place (see \(^{[3]}\)). Since trapping effects are amplified when the dimensionality of space is low, we have focused our attention on the one-dimensional case. More specifically, we have considered the smooth limit of the one-dimensional \(\delta\)-correlated in time model recently introduced in \(^{[3]}\). In the absence of pumping and dissipation, any function of the tracer \(\theta(t,x)\) (say temperature) is advected along Lagrangian trajectories and globally conserved in average (provided the velocity is temporarily fast or spatially smooth). Switching on the energy (\(\theta^2\)) supply at the integral scale, one expects that a steady (or quasi-steady, as discussed in Section \(^{[2]}\)) distribution of the
scalar is attained. The main question raised here is how trapping effects due to compressibility affect the redistribution of energy among the scales and the intermittency properties of the scalar field at the stationary state. To answer this question, we exploit the dynamical formulation of the model to calculate the statistical properties of $\theta$ both up and downwards, i.e. at the scales smaller and larger than the integral scale $L$. Since the scalar is a tracer in the velocity field and the velocity is smooth, the problem reduces to studying Lagrangian separations statistics.

The major physical difference appearing with respect to the incompressible case is that the maximum Lyapunov exponent for Lagrangian separations is negative. This means that, along typical trajectories, distances are mostly contracted and the stretching process is strongly intermittent in time. As in $[18]$, the second and higher powers of the distance $R(t)$ between Lagrangian trajectories grow exponentially, while its low-order positive moments decay exponentially. This is the dynamical origin of the major results found in this paper: inverse cascade and Gaussianity at large scales and extreme intermittency at small scales.

Scalar correlations are indeed essentially governed by the statistics of the time spent by particles at distances smaller than $L$. Consider then two particles initially separated by a distance $x \gg L$. Despite of the fact that $x \gg L$ initially, the distance $R(t)$ will typically reduce to $O(L)$ by a large time $\sim \log (x/L)/|\lambda|$. The consequence is that even scales much larger than $L$ are strongly excited and this is the dynamical hint of the inverse energy cascade. Moreover, for moments of order $n \ll \log (x/L)$, relative fluctuations around the previous typical time are small and this leads to the Gaussianity, e.g. of the core of scalar differences PDF. On the contrary, the statistics at small scales $x \ll L$ is associated with the stretching process. The time to reach the integral scale $L$ strongly fluctuates and this is reflected in the intermittency of both scalar differences and gradients.

Note that the inverse cascade of the scalar taking place here differs in one important respect from other known examples of inverse cascades (say, energy cascade in 2d Navier-Stokes turbulence $[19]$ and number of waves cascade in wave turbulence $[20]$): no flux of another integral of motion (like enstrophy or wave energy) is present. The origin of the inverse cascade found here is purely dynamical and associated with trapping effects. An interesting consequence of the inverse cascade is that the equation for velocity differences PDF, derived here exactly from the dynamics, coincides with the one without dissipative anomaly (operator product expansion, which may result in the anomaly, was proposed in $[18]$ in the context of the Burgers turbulence; see also $[19]$ for possible extensions to the passive scalar turbulence). The absence of anomaly is indeed a consequence of the inverse cascade, preventing the rare trajectories emerging from the dissipative range to affect the convective range behavior.

Strong downscales intermittency emerges all over the quantities calculated in Sections IV-VI. Moments of scalar differences of order $n \geq 1$ all scale linearly with $x$, independently of the order. This collapse of exponents carries over to the dissipation field $\epsilon$, that has all its integer moments scaling with the same power of the Péclet number. Smooth scaling is observed for low-order moments of both scalar differences and dissipation. Very large fluctuations of these two quantities behave, however, quite differently. Scalar differences PDF has indeed a Lorentzian shape for values smaller than unity and exponential tails. The tails of the dissipation field PDF are, on the contrary, stretched exponentials with exponent $1/3$ (and not $1/2$). The additional probability for these strong events comes from fluctuations of the dynamical dissipative scale. Comparison of the $n$-th moment of the dissipation field with the $2n$-th moment of scalar differences establishes indeed the effective viscous scale. This appears to be a strongly fluctuating quantity, growing factorially with $n\uparrow$. This factorial dependence is the cause of the $1/3$ stretched exponential.

Even though the explicit calculations are quite lengthy, the underlying technical ideas, that make the analytical calculations doable, are simple to explain. Since the scalar is passive, its $n$-th order correlation function can be presented in terms of a matrix element of an auxiliary “Quantum” mechanics. There are then two important steps in the evaluation of such matrix elements (and thus of the scalar PDF’s). First, smoothness of the velocity field (stretchings and compressions are uniform on all the particles) allows to reduce (Section 3 and Appendix A) the multi-particle to a one-parameter problem. The fact that the Lyapunov exponent is negative clearly emerges in this procedure. This leads, after the very direct calculations of Sections 4 and 5, to a compact expression for the PDF of the scalar differences in the convective interval. Here, the temporal dynamics of the fluctuating degree of freedom is local, while the locality is lost in the dissipative range. The second step comes then into play. To describe the convection-diffusion interplay in Section 3 and Appendix B, we use a scale separation procedure. The temporarily nonlocal (diffusive) and local (convective) dynamics of the fluctuating degree of freedom are well separated by time $t_0$, $1 \ll t_0 \ll \ln[Pe]$, if the Péclet number Pe is large. Independence of the resulted average (say, the gradient’s PDF) over both local and nonlocal domains on $t_0$, and smallness of the neglected terms with respect to the inverse Péclet number, justify the

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1. An essential enhancement of the dissipative scale was observed also in $[20]$, where the tail of the scalar pdf was studied for the incompressible case by an instanton technique.
scale separation procedure.

The plan of the paper is as follows. In the following section, the one-dimensional passive scalar model is briefly recalled, its relevant time and length-scales are discussed, and the inverse cascade issue is explained from consideration of scalar pair correlation function. The dynamical formulation associated with the passive scalar equation is the subject of Section 3 and the Appendix A. The latter is based on the Martin-Siggia-Rose formalism, while the former is in terms of particles formalism. A key point arising in both procedures is that compressibility couples the dynamics with a global mode. This mode must then be taken into account in order to get the dynamical formulations. Multi-point correlation functions of scalar gradients are discussed in Section 4. Scalar differences PDF is described in Section 5, where behaviors up and downscale with respect to $L$ are considered in two different Subsections. To describe the advection-diffusion interplay, we develop a scale separation procedure in Section 6 which is devoted to the PDF of dissipation. We use the scale-separation procedure also in Appendix B to study the question, how does the steady regime for the pair correlation function of the gradients (discussed in Section 2) form? The final section is reserved for conclusions and discussion of questions which may be of a general relevance for other problems in turbulence and physics of disordered systems.

2. THE MODEL. PAIR CORRELATION FUNCTION AND INVERSE CASCADE.

Our aim here will be, first, to briefly recall the equations of the one-dimensional model introduced in $\text{[9]}$, and then, solving the equation for scalar pair correlation function, show the effects of compressibility on the redistribution of energy among the scales and how does the inverse cascade of the scalar work.

The advection-diffusion equation governing the evolution of the passive scalar $\theta(t, x)$ is

$$\partial_t \theta + u \partial_x \theta = \kappa \partial_x^2 \theta + f. \tag{1}$$

The velocity field $u$ and the force $f$ are both assumed to be Gaussian and $\delta$-correlated in time. The force has correlation function

$$\langle f(t, x) f(t', x') \rangle = \delta(t - t') \chi \left( \frac{x - x'}{L} \right), \tag{2}$$

spatially concentrated at the integral scale $L$. The velocity has zero average and correlation function

$$\langle u(t, x) u(t', x') \rangle = [V_0 - S(x - x')] \delta(t - t') \quad \text{with} \quad S(x) = |x|^{2-\gamma}. \tag{3}$$

The specific smooth case considered here corresponds to $\gamma = 0$. The scaling behavior of the structure function $S$ persists up to the infrared cut-off $L_\omega$, the largest scale in our problem. The scale independent part of the velocity correlation function $V_0$ is estimated by the infrared cutoff of the velocity field squared, $\sim L_\omega^2$. For scales larger than $L_\omega$, velocity correlations decay to zero, i.e. the structure function saturates to the constant value $V_0$. The equation of motion for the gradient field $\omega = \partial_x \theta$ immediately follows from (1)

$$\partial_t \omega + u \partial_x (u \omega) = \kappa \partial_x^2 \omega + \partial_x f. \tag{4}$$

The $\delta$-correlation in time of both the velocity and the forcing allows to derive closed equations of motion for equal-time correlation functions $\text{[9]}$. It is for example easy, e.g. using Gaussian integration by parts (see $\text{[21]}$), to derive the equation for the second-order correlations $\Omega(t, x) = \langle \omega(t, x) \omega(t, 0) \rangle$ and $F(t, x) = \langle \theta(t, x) \theta(t, 0) \rangle$

$$\partial_t \Omega = \partial_x^2 \left[ (S(x) + 2\kappa) \Omega \right] - \partial_x^2 \chi, \tag{5}$$

$$\partial_t F = \partial_x^2 \left[ (S(x) + 2\kappa) \partial_x^2 F + \chi \right]. \tag{6}$$

Note that there are two types of passive fields for compressible flow. Lagrangian tracers, like entropy or temperature (provided pressure is slowly varying in space and time), are conserved along Lagrangian trajectories in the absence of diffusion and pumping. The local maxima of the field do not grow in the absence of pumping. Concentration fields, e.g. of a pollutant, are, on the contrary, only globally conserved and their maxima can be amplified. The equations for the two types of fields differ by the position of the space-derivative in the velocity term. In our one-dimensional case, the two possibilities correspond to the $\theta$ and the $\omega$ fields, respectively.
It is convenient to consider first equation (3) and then recover the correlations of the scalar by integration. From the very definition of the correlation function, it follows that the solution of (3) is even in $x$. Looking for a stationary solution, there is another boundary condition needed to fix the integration constants. This comes from the dynamics. Let us indeed consider the situation when the system starts from rest ($\theta \equiv 0$). At any subsequent time, the solution satisfies $\int \Omega \, dx = 0$, where the integral is taken over all the space. Note that such a condition is consistent and actually dictated by the dynamics: the integral of the correlation function of the gradients is conserved in the evolution on account of the double space derivative in the r.h.s. of (3). It is then easy to find the stationary solution of (3) 

$$
\Omega(x) = \frac{\chi(x)}{2\kappa + S(x)} - \frac{C}{2\kappa + S(x)} \quad \text{where} \quad C = \frac{\int \chi/(2\kappa + S)}{\int 1/(2\kappa + S)},
$$

(7)

(see Appendix B for a dynamical derivation of (3)). The expression (4) is very illuminating for several reasons. Let us first consider the case where no infrared cutoff $L_\infty$ is present. The integral $\int 1/(2\kappa + S)$ is then convergent and, for small molecular diffusivity, varies as $1/\sqrt{\kappa}$. For the sake of concreteness, let us specifically consider here a forcing that is regular at the origin. The dominant terms for distances much smaller than the integral scale $L$ are then 

$$
\Omega(x) = \frac{|\chi''(0)|}{2(2\kappa + S)} [aL\sqrt{\kappa} - x^2],
$$

(8)

where $a$ is constant $O(1)$, dependent on the detailed form of the pumping. The first and the second term dominate respectively at scales smaller and larger than $L/\sqrt{\kappa}$, where the Péclet number $Pe \equiv L/\sqrt{\kappa}$ is supposed to be very large. Note that this scale is still much larger than the dissipative scale $L/Pe$. The most interesting aspect of (5) is that the dissipation $\kappa \Omega(0)$ vanishes as $1/Pe$, i.e. there is no direct cascade. The energy $\Omega$ is actually transferred upscale by an inverse cascade, as also emerges from the behavior of scalar correlations. Let us indeed insert into (3) the correlation $\langle \theta(t,x_1)\theta(t,x_2) \rangle = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \Omega(y_1 - y_2)$ with the $\omega$-correlation function having the expression (5). It is easily checked that the energy satisfies $\dot{\Omega}(\theta^2) = C$, i.e. it grows linearly with time. Note that the growing mode is constant in space and thus disappears when differences or gradients are considered. The effect of the advective term $u\partial_x \theta$ is therefore to transfer energy upwards in the scales. Since dissipation is quadratic in the wavenumber, the energy on the large scales cannot practically be dissipated and it is piled-up. This is at the origin of the linear growth in time of $\langle \theta^2 \rangle$. $C \neq 0$ corresponds to the inverse cascade, which therefore holds generically for any scaling exponent $\gamma < 1$ in (3) (if the source function $\chi$ is not exceptional).

Let us now introduce an infrared cutoff $L_\infty$ in the velocity field. Since $S$ saturates to a constant, it is clear now that the integral $\int 1/(2\kappa + S)$ diverges. The constant $C$ in (3) must then vanish. In the presence of a finite cutoff $L_\infty$, there will then be a very long intermediate in time asymptotic where for scales $\ll L_\infty$ the behavior without cutoff is observed. However, after a very large time $T_{L_\infty} \sim (L_\infty/\kappa)^2$ the dynamics changes: the inverse cascade stops, $\langle \theta^2 \rangle$ saturates to a finite value, the system starts dissipating a finite amount of energy in the limit of large $Pe$ and the correlation function $\Omega(x)$ tends to the solution (5) with $C = 0$. The finite contribution $-\int \chi/(2\kappa + S)$ needed to ensure the zero integral condition comes from a strip of negative values that becomes more and more extended with time and whose amplitude tends to vanish. More details on the dynamics of the pair correlation function of the scalar gradients at infinite times in the presence of an infrared cutoff $L_\infty$ may be found in Appendix B.

3. DYNAMICS IN PARTICLE (LAGRANGIAN) FORMALISM

In this Section we shall discuss the dynamical formulation of the equations of motion. The goal is the same as in (22): reduce the calculation of simultaneous scalar statistics to averaging of functionals of the random-in-time strain-vorticity matrix. This reduction is crucially based on the fact that the velocity field is smooth ($\gamma = 0$) and can be performed for any smooth flow, independently of its compressibility and the dimension of space. In the specific 1d case, no matrices are obviously involved and one is left with averaging of a single scalar field. Compressibility makes however the derivation slightly more involved and some care must be taken in the ordering of the advective term. This emerges, in particular, in the non-vanishing average of $\sigma$ in the weight (21) for the Lagrangian trajectories (22). The

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3It is important to mention that, in the absence of pumping and dissipation, the average over the velocity of any function of the scalar $\langle f(\theta) \rangle$ is conserved, although the average over all the space of $f(\theta)$ itself is not conserved in the particular realizations.
dynamical formulation is derived here using particle formalism. The equivalent derivation using Martin-Siggia-Rose field formalism is reserved for the Appendix.

Equation (9) for the passive scalar can be presented in the form

\[
\theta(T; x) = \int_0^T \mathcal{T} \exp \left[ \int_0^T \mathcal{P}(t', x) dt' \right] \phi(t; x) dt = \int dt \int dy \Psi(t, T; x, y) \phi(t; y),
\]

where it was supposed that no pumping was supplied at negative times. In (11), the operator \( \mathcal{P}(t; x) = -u(t; x) \partial_x + \kappa \partial_x^2 \), time-ordered exponential is denoted by \( \mathcal{T} \exp \) and the function \( \Psi \) can be expressed by using Lagrangian trajectories as

\[
\Psi(t, T; y, x) = \int_{\rho(t)=y} \mathcal{D} \rho \exp \left[ -\int_t^T (\dot{\rho} - u)^2 / 4 \kappa \right].
\]

This formula expresses the simple fact that Lagrangian trajectories are fixed by the velocity \( u \) and smoothed by the molecular diffusivity \( \kappa \). One can express (10) in the more convenient Hamiltonian form

\[
\Psi(t, T; x, y) = \int_{\rho_n=x}^{\rho_n=y} \prod_{n=0}^{N-1} dp_n \prod_{n=1}^{N-1} d\rho_n \exp \left[ \frac{\Delta}{\kappa} \sum_{n=1}^N \left[ \frac{1}{\Delta} p_{n-1}(\rho_n - \rho_{n-1}) - p_{n-1} u(t_n; \rho_n) + \kappa p_{n-1}^2 \right] \right],
\]

\[
\rho_n = \rho(t_n = t + n\Delta), \quad p_n = p(t + n\Delta), \quad \Delta = \frac{T - t}{N - 1} \to 0,
\]

where the momenta integrations \( (dp_k) \) run along the imaginary axis and regularizations have been specified. Using the property that both velocity and pumping are Gaussian and have correlation functions (3) and (4), we can easily perform the averages in the 2\( n \)-th simultaneous product of the scalar field \( \theta(T; x) \). We thus obtain

\[
F(T; x_1, \cdots, x_{2n}) \equiv \langle \theta(T, x_1) \cdots \theta(T, x_{2n}) \rangle
\]

\[
= \int_0^T \int \prod_{i=1}^{2n} dy_i \mathcal{R}(T - t; x_i, y_i) \left[ F(t; y_1, \cdots, y_{2n-2}) \chi(y_{2n} - y_{2n-1}) + \text{permutations} \right],
\]

where the eddy-diffusivity resolvent is defined as

\[
\mathcal{R}(T; x_i, y_i) \equiv \int_{\rho_i(0)=x_i}^{\rho_i(T)=y_i} \left\langle \Psi(T, 0; x_i, y_i) \right\rangle \mathcal{D} p_i(t) \mathcal{D} \rho_i(t) \exp \left( \int_0^T dt \left[ p_i \dot{\rho}_i - \frac{1}{2} p_i (\rho_i - \rho_j)^2 p_j + \kappa p_i^2 \right] \right).
\]

Here, one inverses the direction of time in comparison with (11) \( t \to T - t \) and thus the convective term \( (p^2 \rho^2) \) is regularized in a way such that its \( \rho \)-dependent part is retarded with respect to the \( p \)-dependent one. The Hubbard-Stratonovich transformation of the diffusive term gives

\[
\mathcal{R}(T; x_i, y_i) = \int_{\rho_i(0)=x_i}^{\rho_i(T)=y_i} \mathcal{D} \rho_i(t) \mathcal{D} p_i(t) \mathcal{D} \xi_i(t) \exp \left[ -\mathcal{S} - \mathcal{S}_\xi \right],
\]

\[
\mathcal{S} = \int_0^T dt \left[ -p_i \dot{\rho}_i + \frac{1}{2} p_i (\rho_i - \rho_j)^2 p_j - \rho_i \xi_i \right], \quad \mathcal{S}_\xi = \frac{1}{4 \kappa} \int_0^T \xi_i^2 dt.
\]
variables, say \( \rho = \sum \rho_i \) and the collective momentum \( P \equiv \sum_{i=1}^{2n} p_i/(2n) \). To get rid of the superfluous degrees of freedom, let us come from the old set of variables \( \{ \rho_1, \ldots, \rho_{2n}; p_1, \ldots, p_{2n} \} \) to the new one \( \{ \rho, \rho_2, \ldots, \rho_{2n}; P, p_2, \ldots, p_{2n} \} \). The new momenta \( \tilde{p}_i = p_i - P \) are considered in the system comoving with the “center of mass” and the positions \( \tilde{p}_i = p_i - p_1 \) are with respect to one, e.g. \( p_1 \), taken as reference. The action \( S \) can then be decomposed as \( S = S_{\text{col}} + \tilde{S} \), where

\[
S_{\text{col}} \equiv \int_0^T dt \left[ -P^2 \left( \sum_{i>1} \tilde{p}_i \right)^2 + 2nP^2 \left( \sum_{i>1} \tilde{p}_i \right)^2 + 2nP \left( \sum_{i>1} \tilde{p}_i \right) \left( \sum_{j>1} \tilde{p}_j \right) - \left( \sum_{i>1} \tilde{p}_i \right) \left( \sum_{j>1} \tilde{p}_j \right) - P \tilde{p} - \sum_{i=1}^n \xi_i \right],
\]

(17)

\[
\tilde{S} \equiv \int_0^T dt \left[ - \left( \sum_{i>1} \tilde{p}_i \tilde{p}_i \right) - \sum_{i>1} \tilde{p}_i \tilde{p}_i - \sum_{i>1} \tilde{p}_i (\xi_i - \xi_1) \right] .
\]

(18)

Let us now consider the integral \( \int \prod_{i=1}^{2n} d\tilde{y}_i \mathcal{R}(T; x_i, y_i) f(y_i) \) where, as in (13), \( f \) is a function of differences only \( f(y_i) = f(y_i - y) \). Collective degrees of freedom \( \rho \) and \( P \) are easily integrated. The principal point for this integration is the absence of a “potential” \( \rho \) dependence in the action \( S \). The integral is then reduced to \( \int \prod_{i>1} d\tilde{y}_i \mathcal{R}(T; x_i, y_i) f(y_i + y_1) \), where the effective resolvent

\[
\tilde{\mathcal{R}}(T; \tilde{\tilde{x}}, \tilde{y}) = \int_{\tilde{\tilde{p}}(0) = \tilde{\tilde{x}}} \mathcal{D} \tilde{p}(t) D\xi_i(t) D\tilde{\sigma}(t) \exp \left[ -S_\sigma - S_\xi \sum_{m,i} \delta \left( \int_0^T \left( \frac{\tilde{p} - \rho}{\Delta} - \tilde{\sigma} \right) \right) \right],
\]

(19)

and the measure of averaging \( S_\sigma = \sum_i \left[ \tilde{\sigma}^{\text{sym}} \right]^2 / 4 \). In order to derive (19), we have decomposed the quadratic over \( \tilde{p} \) term by means of Hubbard-Stratonovich trick, introducing an additional collective integration over \( \tilde{\sigma} \). The integration with respect to momentum \( \tilde{p}_i \) is already performed in the last line of (13) (the effective action appears to be linear in \( \tilde{p} \) in the result of the Hubbard-Stratonovich transformation). The continuous version of the equation under the \( \delta \)-function sign in (19) and the measure of averaging over \( \tilde{\sigma} \) are

\[
\dot{\tilde{p}}_i = \sigma \tilde{p}_i + \xi_i - \xi_1, \quad \sigma \equiv \tilde{\sigma} - 1,
\]

(20)

\[
S_\sigma = \frac{1}{4} \int_0^T dt \left| \sigma + 1 \right|^2 .
\]

(21)

To see the relation between (19) and (20) one can check, particularly, that both discrete and continuous versions give \( \langle \rho_i(t) \rangle = \rho_i(0) \). The formal solution of (20) is

\[
\tilde{p}_i(t) = W(t)(x_i - x_1) + W(t) \int_0^t dt' W^{-1}(t') (\xi_i - \xi_1), \quad W(t) \equiv \exp \left[ \int_0^t dt' \sigma(t') \right].
\]

(22)

It is finally easy to recalculate the \( 2n \)-particle correlation function of the scalar from the \( 2n \)-particle resolvent

\[
F(T; x_1, \ldots, x_{2n}) = \left< \Xi[T; \{ \sigma(t) \}; x_1 - x_2, \ldots, \Xi[T; \{ \sigma(t) \}; x_{2n-1} - x_{2n}] + \text{permutations} \right>_\sigma ,
\]

(23)

\[
\Xi[T; \{ \sigma(t) \}; x_i - x_j] \equiv \int_0^T dt \left< \chi \left[ W(t) \frac{x_i - x_j}{L} + W(t) \int_0^t dt' W^{-1}(t') \frac{\xi_i - \xi_j}{L} \right] \right>_{\xi_i, \xi_j} ,
\]

(24)

where averages over \( \sigma(t) \) and \( \xi_i(t) \) are fixed by the measures \( \exp[-S_\sigma] \) and \( \exp[-S_\xi] \) defined in (21) and (16), respectively.

By continuous version we understand, particularly, a symmetrical smearing of the temporal \( \delta \)-function in the \( \sigma \)-field correlation for a small but finite (which is still much larger than the temporal slice \( \Delta \) width.)
4. CORRELATION FUNCTIONS OF SCALAR GRADIENTS IN THE CONVECTIVE INTERVAL

Using the results of the previous Section, the dynamical expression of correlation functions of the scalar gradient \( \omega(t, x) = \partial_x \theta(t, x) \) can be simply found differentiating \((23)\) with respect to all spatial arguments. Here, we shall be interested in the behavior of these simultaneous correlation functions for distances such that molecular diffusivity can be neglected. We first derive a general formula valid for arbitrary form of forcing correlation and then treat more specifically the case of an exponential pumping. The resulting expression shows that the ratio between the irreducible and the reducible contributions to the \(2n\)-th correlation function grows as \((L/x)^{n-1}\). This evidences both the non-Gaussianity of the field and the fact that it increases going downscales in the convective interval. The same phenomenon takes place for generic forms of the pumping. The discussion of the range of scales where these considerations hold, is postponed to the end of the Section.

It follows from \((23)\) that scalar gradients correlation functions are given by

\[
\langle \omega(T, x_1) \cdots \omega(T, x_{2n}) \rangle = \frac{(-1)^n}{L^{2n}} \left[ \int_0^T dt W^2(t) \chi'' \left( W(t) \frac{x_1 - x_2}{L} \right) \right] \cdots \left[ \int_0^T dt W^2(t) \chi'' \left( W(t) \frac{x_{2n} - x_{2n-1}}{L} \right) \right] + \text{permutations},
\]

(25)

where molecular diffusion effects have been neglected. The only averaging left in \((25)\) is then with respect to the \(\sigma\) statistics defined by \((21)\). A possible way of performing this average is to introduce the auxiliary object

\[
A(s_{1,2}, \cdots, s_{2n,2n-1}) = \left\langle \exp \left[ \int_0^T W^2(t) \left( s_{1,2} \chi'' \left( W(t) \frac{x_1 - x_2}{L} \right) + \cdots + s_{2n,2n-1} \chi'' \left( W(t) \frac{x_{2n} - x_{2n-1}}{L} \right) \right) dt \right] \right\rangle.
\]

(26)

Differentiating \((26)\) over \(s\) variables, \((21)\) is clearly reproduced up to the sign and the \(L\) dependent prefactor. Inserting the weight \((21)\) into \((26)\), one can easily recognize the path-integral structure associated with the time evolution of the “quantum mechanical” Schrödinger equation, having Hamiltonian \(H = -\partial_y^2 - U(\exp \eta) \exp(2\eta)\). The “potential” \(U\) appearing in the Hamiltonian is

\[
U(y) = s_{1,2} \chi'' \left( y \frac{x_1 - x_2}{L} \right) + \cdots + s_{2n,2n-1} \chi'' \left( y \frac{x_{2n} - x_{2n-1}}{L} \right),
\]

(27)

and the “space” variable \(\eta\) is defined as \(\eta = \int_0^t dt' \sigma(t')\). Using standard notation for quantum mechanics matrix elements, it is easy to check that \((26)\) can be presented as

\[
A = \exp[-T/4] \left\langle \delta(\eta) \left| e^{-TH} \right| e^{-\eta/2} \right\rangle = \left[ e^{-T/4} \Pi(T; \eta) \right]_{\eta=0}.
\]

(28)

where the ”wave-function” \(\Pi(T; \eta)\) satisfies \((\partial_T - \hat{H})\Pi = 0\), and the initial condition is \(\Pi(0; \eta) = e^{-\eta/2}\). We can now remark that the potential part of the Hamiltonian vanishes at \(\eta \to -\infty\), while the initial condition does not. The resulting asymptotic behavior at large times \(T\) will then be

\[
\Pi(T; \eta = \ln[y]) \to \left[ e^{T/4} \frac{\Pi_0}{\sqrt{y}} \right] \left[ \partial_y^2 + U(y) \right] \Pi_0(y) = 0.
\]

(29)

The new variable \(y = \exp \eta\) has been introduced. The boundary conditions for the spatial part \(\Pi_0\) are easily derived from those for \(\Pi\): it should tend to unity for \(y \to 0\) and the ratio \(\Pi_0/\sqrt{y}\) should vanish for \(y\) tending to infinity. It follows from \((23)\) that the auxiliary object \(A\) is simply the function \(\Pi_0\) calculated at \(y = 1\).

Derivatives of \(A\) at \(s = 0\) are needed for the calculation of gradients correlations \((23)\). It is then convenient to present the solution of \((29)\) in the form of an expansion with respect to the “potential” \(U\):

\[
A(s_{1,2}, \cdots, s_{2n,2n-1}) = \sum_{k=0}^{\infty} \frac{1}{y_{1}} \int_0^\infty dy_1 \int_0^\infty dy_2 U[y_2] \int_0^\infty dy_3 \int_0^\infty dy_4 U[y_4] \cdots \int_0^\infty dy_{2k-1} \int_0^\infty dy_{2k} U[y_{2k}].
\]

(30)
Only the $n$-th order term of the expansion in $\chi$ actually contributes to the $2n$-th order scalar gradients correlation function. Its final expression reads

$$
\langle \omega_1 \cdots \omega_{2n} \rangle = \frac{(-1)^n}{L^{2n}} \sum_{k_1, \ldots, k_{2n}} \int_{y_1}^{\infty} dy_1 \int_{y_2}^{\infty} dy_2 \chi'' \left[ y_1 \frac{x_{k_1} - x_k}{L} \right] \cdot \cdots \cdot \int_{y_2n-1}^{\infty} dy_{2n-1} \int_{y_2n}^{\infty} dy_{2n} \chi'' \left[ y_{2n} \frac{x_{k_{2n}} - x_{k_{2n-1}}}{L} \right] ,
$$

(31)

where summation is performed over all the splittings of the set \{k_1, \ldots, k_{2n}\} into $n$ ordered pairs.

Formula (31) holds for any pumping $\chi$. Let us now specifically consider the case of the exponential pumping $\chi(z) = \exp\[-z\nu\]$, where the integrals appearing in (31) can be easily performed. For the second-order correlation, we obtain that the dominant contribution for small distances $x_{12}/L$ is $-1/(x_{12}L)$, in agreement with the solution (6). For the fourth-order correlation function, we find

$$
\langle \omega(x_1, t) \omega(x_2, t) \omega(x_3, t) \omega(x_4, t) \rangle = F_c(x_{12}, x_{34}) + F_c(x_{13}, x_{24}) + F_c(x_{14}, x_{23}),
$$

(32)

where $x_{ij} = |x_i - x_j|$, the subscript is intended to remind that this specific expression holds for the exponential pumping and only dominant terms in $x_{ij}/L$ have been retained. Distinguishing between reducible (Gaussian) and irreducible contributions into (22), one observes that the irreducible part is $L/x \gg 1$ times larger. More generally, if $x$ denotes the typical distance among the various particles, i.e. $x_{ij} \sim x$, then $\langle \omega_1 \cdots \omega_{2n} \rangle \sim 1/(x^{2n-1}L)$. Both the correlation functions and the degree of non-Gaussianity of the scalar gradient (ratio of the $2n$-th moment to its reducible part) are then growing with $x_{ij}$ going downscales.

Let us finally come to the range of validity of the previous convective arguments. The exponential form of the pumping is a special one, being not regular at the origin. The first term of its expansion at small distances is linear and not quadratic. This affects the dependence on $x$ of the second-order correlation. For a regular pumping, the dominant term would indeed be constant, as follows from (8). On the contrary, one can check using (31) that the dependence of correlations of order $\geq 4$ on $x$ and $L$ remains the same as for the exponential pumping. The regularity at the origin of the pumping also enters the range of scales where neglecting molecular diffusivity effects is allowed. Performing the small distance expansion, as in (3) for a regular pumping, a logarithmic correction $\sim \log Pe$ appears. The ultraviolet criterium of applicability of the previous convective considerations for the exponential pumping is then $x \gg \sqrt{x} \log Pe$. For a pumping regular at the origin, the criterium is the same as in (8), i.e. $x \gg L/\sqrt{Pe}$.

5. SCALING AND PDF OF SCALAR DIFFERENCES

Scalar structure functions $S_{2n}(x) = \langle (\theta(T, x) - \theta(T, 0))^{2n} \rangle$ can be easily expressed in terms of scalar correlation functions $F$ by taking the appropriate combinations of them. A dynamical expression for $S_{2n}$ can thus be derived directly from (24), obtained in Section 2 for the $F$'s. This is however not a very practical procedure. Each of the contributions, appearing in the sum giving the structure functions, contains indeed the constant mode. As it was discussed in Section 2 and as it also appears from the dynamical expression (24), this mode grows linearly with the observation time $T$. It is just in the whole sum that these divergent contributions are canceled, thus leaving the time-independent final result for structure functions. It is then more convenient to restore structure functions directly from scalar gradients correlations as $S_{2n}(x) = \int dx_1 \cdots \int dx_{2n} \langle \omega_1 \cdots \omega_{2n} \rangle$. An important question is whether, to calculate $S_{2n}$ in the convective interval, we can avoid to take dissipation explicitly into account or not. This means essentially asking whether it is enough to know just the convective expressions for gradients correlations or their whole behavior is needed. This point can be tested by simply taking the convective expressions for $\langle \omega_1 \cdots \omega_{2n} \rangle$ found in the previous Section and inserting them into the integral expression for $S_{2n}$. One can then check that all the integrals for any structure function are convergent on the ultraviolet and dominated by the infrared side of the convective range.

The expression for structure functions in the convective interval is then

$$
S_{2n}(x) = \frac{(2n - 1)!n!2^n}{x/L} \int_{y_1}^{\infty} dy_1 \int_{y_2}^{\infty} dy_2 \frac{\chi(0) - \chi(y_2)}{y_2^2} \cdot \cdots \cdot \int_{y_{2n-1}}^{\infty} dy_{2n-1} \int_{y_{2n}}^{\infty} dy_{2n} \frac{\chi(0) - \chi(y_{2n})}{y_{2n}^2},
$$

(33)
where we have already performed the $2n$ integrals over the $dx_i$'s. The whole set of equations (33) can be recast into the more compact equation

$$
(x^2 \partial_x^2 - \lambda^2 [\chi(0) - \chi(x/L)]) Z(x/L, \lambda) = 0,
$$

(34)

for the generating function $Z(x, \lambda) = \langle \exp (-i\lambda \delta \theta_x) \rangle$ of scalar differences $\delta \theta_x$ taken at the scale $x$. From this very definition, it follows that the function $Z$ must tend to unity for vanishing $x$ and, for the convergence of the integrals in (33), it should grow slower than linearly at infinity.

It is worth to remind that (34) was found as the result of an accurate dynamical procedure: we first calculated correlation functions of the scalar gradient for all points being separated; the resulting gradients correlation functions were then integrated to obtain structure functions and, finally, the generating function for scalar differences was reconstructed from its moments. We thus avoided to explicitly handle diffusion, paying for this the price of taking many particles into consideration. The closed differential equation (34) for scalar differences generating function emerges as the result of this procedure. On the other hand, one could generally (also for the non-smooth case, i.e. $\gamma \neq 0$ in (3)) derive the following unclosed Fokker-Planck equation for the generating function

$$
(x^2 - \gamma \partial_x^2 - \lambda^2 [\chi(0) - \chi(x/L)]) Z(x/L, \lambda) = \kappa \langle [\partial_x^2 \theta_1 - \partial_x^2 \theta_2] \exp [\lambda (\theta_1 - \theta_2)] \rangle.
$$

(35)

This equation is simply obtained averaging the equation of motion (1) for the scalar at two reference points. The smooth ($\gamma = 0$) limit of (35) differs from (34) by the r.h.s. dissipative term. There is a general expectation that this term may remain finite even in the limit $\kappa \to 0$, thus providing a nonvanishing anomaly in the terminology of Polyakov [18]. Equation (34), derived microscopically without any conjecture, shows then the absence of anomaly for the one dimensional smooth flow.

Physically, the absence of anomaly is associated with the vanishing direct flux of $\theta$ in the limit of infinite Péclet numbers. This point, already appeared in Section 2, will emerge even more clearly in the full analysis of the dissipation field in the next Section. In the two following Subsections, solutions of (35) at $x \gg L$ and $x \ll L$ will be discussed.

A. Upscale interval

Let us first consider scales larger than the integral scale $L$. The asymptotic solution for the generating function, which can also be found from (35) replacing $\chi(0) - \chi(x/L)$ by $\chi(0)$, is

$$
Z = \left( \frac{x}{L} \right)^{1/2} \exp \frac{1}{\sqrt{1+4\chi(0)\lambda^2/2}}.
$$

(36)

The inverse Fourier transform of (36), calculated in the saddle-point manner (the large parameter is $x/L$ or, equivalently, large values of $|\delta \theta_x|$), gives

$$
\mathcal{P}^{\delta \theta_x}(y) = \frac{1}{2\sqrt{\pi \chi(0) \ln(x/L)}} \begin{cases} 
\exp \left[ -y^2/(4\chi(0) \ln(x/L)) \right], & |y| \ll \ln(x/L), \\
\exp \left[ -|y|/(2\sqrt{\chi(0)}) \right], & |y| \gg \ln(x/L).
\end{cases}
$$

(37)

Thus, the arguments for Gaussianity presented in [5] for the incompressible case can indeed be reversed and applied here for the upper interval. Structure functions of orders much less than $\ln(x/L)$ scale indeed logarithmically, the core of scalar differences PDF is Gaussian and the PDF’s tail is exponential.

B. Downscale interval

We shall now obtain a general formula for the PDF at $y \ll 1$, no matter how $y$ and $x/L$ relate to each other, provided both of them are small. Indeed, replacing the $\chi$ dependent “potential” by the first term of the expansion

\footnote{We acknowledge A. Polyakov for attracting out attention to the matter discussed in the paragraph.}
over \(x/L\), and solving the resulting differential equation with the same boundary conditions as before, one gets the simple exponential form for the generating function \(Z(x/L, \lambda) = \exp[-\lambda \sqrt{\chi''[0]/2} \ x/L]\). The inverse Fourier transform produces the following Lorentzian expression for the PDF

\[
P^{(\delta \theta_x)}(y) = \frac{1}{\pi} \frac{L}{x} \frac{1}{y^2 + |\chi''[0]| x^2/(2L^2)},
\]

which is thus generally valid at \(|y| \ll 1\). It results from (38) that the PDF is smooth in a small region around the origin \(y = 0\), where it can actually be expanded in \(y^2 L^2 / x^2\). This region extends approximately up to \(x/L\), where the second behavior in \(1/y^2\) sets in. Note that the concavity of the PDF in this second region is upwards and remains upwards up to the small values \(x/L\). It is just for very small values \(|y| \ll x/L\) that the concavity is reversed downwards. An “experimental” histogram of such a PDF would then look strongly cusped at the origin.

The tail of the PDF matching the Lorentzian (38) at \(|y| \sim 1\) is exponential. To see this, and generally to obtain an explicit analytic solution for the PDF in the whole domain of \(x\) and \(\delta \theta_x\), let us consider the following specific form of pumping

\[
\chi^*(\frac{x}{L}) = \frac{1}{1 + (x/L)^2}.
\]

Making in (34) the change of variable \(\cot[\varphi] = x/L\), the solution of the resulting equation can be expressed in terms of associated Legendre functions as

\[
Z_\ast(\cot[\varphi]; \lambda) = \frac{2^{\nu-1/2} (\nu + 1) \Gamma^2 [\nu + (1/2)] P^{-\nu - 1/2} (\cos[\varphi])}{\sqrt{\pi \sin[\varphi]}},
\]

where the upper index \(\nu = -1/2 + 1/2 \sqrt{1 + 4 \lambda^2}\). The choice of the sign for the square root is such as to ensure that the generating function \(Z(x, \lambda)\) grows at infinity slower than linearly. The notation \(Z_\ast\) in (39) is intended to stress that this explicit solution was obtained for the pumping \(\nu \ll 1\). Note, that (40) is particularly applicable for the upper interval discussed in the previous Subsection. Considering (40) at \(\varphi \ll 1\) and \(\lambda \gg 1\), one indeed recovers (38). Using the integral representation (8.714) in (23) and the doubling formula for the gamma function, (40) can be presented in the integral form

\[
Z_\ast(\cot[\varphi]; \lambda) = \frac{2 \Gamma[\nu + (3)/2]}{\sqrt{\pi \sin[\varphi]} \Gamma[\nu + (1/2)/2]} \int_0^\varphi \cos[t] \left( \frac{\cos[t] - \cos[\varphi]}{\sin[\varphi]} \right)^\nu \ dt.
\]

The integral representation (41) is useful, since it clearly shows the analytic structure of \(Z_\ast\) with respect to \(\lambda\). One observes, particularly, that \(Z_\ast\) is analytic in the whole upper semi-plane, except on the line of imaginary \(\lambda\) from \(i/2\) to \(i\infty\). The path on the real axis in the inverse Fourier transform can be then deformed into the following one surrounding the cut: the left branch goes from \(i\infty - 0^+\) to \(i/2 - 0^+\), bypass of \(\lambda = i/2\) from below and the right branch goes from \(i/2 + 0^+\) to \(i\infty + 0^+\). The final result is

\[
P^{(\delta \theta_x)}(y) = \frac{1}{4\pi} \int_0^\infty \frac{qdq}{\sqrt{1 + q^2}} \exp \left[-\frac{|y|}{2} \sqrt{1 + q^2}\right] G(\cot[\varphi]; q),
\]

where the function \(-iG\) is the difference between \(Z_\ast\) on the right and the left of the cut, i.e. \(\nu \to -1/2 \pm iq/2\), respectively. The general expression for \(G\) can be derived from the integral representation (41) as

\[
G(\cot[\varphi]; q) = \frac{4}{\sqrt{\pi \sin[\varphi]}} \int_0^\varphi \frac{\cos[t] dt}{\cos[t] - \cos[\varphi]} \Re \left[ \frac{\Gamma \left( \frac{5+iq}{4} \right)}{\Gamma \left( \frac{3+iq}{4} \right)} \left( \frac{\sin[\varphi]}{\cos[t] - \cos[\varphi]} \right)^{-iq/2} \right].
\]

The calculation of the PDF of scalar differences in the convective interval is now reduced to the evaluation of the asymptotic behavior of \(G\) in (13) and then to perform the integral (12). In the convective interval, \(x/L = \cot[\varphi] \ll 1\), i.e. the angle \(\varphi\) is very close to \(\pi/2\). The dominant expression of \(G\) in this region can be obtained by simply expanding directly in (13). Inserting this expansion into (12), we obtain
\[ P^*(\delta \sigma_y) = \frac{\pi x}{8 L} \int_0^\infty dq \sqrt{1 + q^2} \sinh[\pi q/2] \exp \left[ -\frac{|y|}{2} \sqrt{1 + q^2} \right] \left[ \Gamma \left[ (1 + i q)/4 \right] \right]^4 \cosh[\pi q/2] \rightarrow \begin{cases} \frac{1}{2} \frac{1}{x} \frac{1}{\sqrt{2\pi}}, & 1 \gg |y| \gg x/L, \\ \frac{1}{2} \exp[-|y|/2], & |y| \gg 1, \end{cases} \] (44)

where, at \( y \gg 1 \), the prefactor algebraic in \( y \) has not been considered. Varying the pumping function \( \chi(x/L) \), one can change the number behind \( |y| \) in the exponential, but the exponential behavior itself will never change.

One may check that the PDF’s asymptotic for the smallest values \( |y| \ll x/L \) derived from (40, 41) is consistent with the general formula (38). Indeed, the respective large \( q \) asymptotic of \( \mathcal{G} \) is

\[ \mathcal{G}(\cot[\varphi]; q) \sim 2 \frac{q}{\pi \sin[\varphi]} \int_0^\varphi \frac{\cos[t]}{\sqrt{\cos[t] - \cos[\varphi]}} \sin \left( \frac{q}{2} \ln \left[ \frac{\sin[\varphi]}{\cos[t] - \cos[\varphi]} \right] - \frac{\pi}{4} \right) . \] (45)

Substituting (45) into (42) and performing first the integral in \( q \) and then the one in \( t \), we obtain at \( |y| \ll x/L \ll 1 \)

\[ P^*(\delta \sigma_y) (y) \sim \frac{1}{2\pi} \sqrt{\frac{y}{t}} \int_0^\varphi \frac{\cos[t] dt}{\sqrt{\cos[t] - \cos[\varphi]}} \ln^{-3/2} \left[ \frac{1}{\cos[t] - \cos[\varphi]} \right] \left( 1 - \frac{15}{8} \frac{y^2}{\ln^2 \left[ \frac{1}{\cos[t] - \cos[\varphi]} \right]} \right) \rightarrow \frac{1}{2\pi} \frac{L}{x} \left( 1 - \frac{L^2 y^2}{x^2} \right) . \] (46)

Expressions (44) and (46) are clearly in agreement with (38), valid for an arbitrary form of pumping.

Let us now derive the scaling behavior of scalar structure functions \( \langle |\theta_1 - \theta_2|^n \rangle \) at \( x \ll L \). The scaling for orders \( a > 1 \) is dominated by the behavior of the PDF at values of order unity. On the contrary, for \( a < 1 \), the region in \( 1/y^2 \) dominates. The resulting scaling behavior of structure functions is

\[ \langle |\theta_1 - \theta_2|^n \rangle \sim \begin{cases} \frac{x}{T}, & \text{for } a \geq 1; \\ \left( \frac{x}{T} \right)^a, & \text{for } -1 < a \leq 1. \end{cases} \] (47)

For \( a < -1 \) the moments do not exist at all, since the PDF is finite at the origin. Remark that (47) is exactly the same scaling as for velocity structure functions in Burgers equation. In the present model the collapse of high-order exponents is associated with the uniformity in space of stretchings and compressions.

6. PDF OF SCALAR DISSIPATION AND GRADIENTS

The \( n \)-th order moment of the dissipation field \( \epsilon = \kappa (\partial_x \sigma(t; x))^2 \) can be obtained from the dynamical expression (21) as

\[ \langle \epsilon^n \rangle = (2n - 1)!! \kappa^n \left[ \partial_x \sigma(t; x) \right]^n \left. x_i - x_j \right|_{x_i = x_j} \equiv (2n - 1)!! \left( Q[T; \langle \sigma(t) \rangle]^n \right) . \] (48)

The average with respect to the noises \( \xi_j \) is easily performed in Fourier space, thus obtaining

\[ Q = \frac{1}{\text{Pe}^2} \int_0^T dt W^2(t) G \left[ \frac{W^2(t)}{\text{Pe}^2} \int_0^t dt' W^{-2}(t') \right] \] (49)

where \( W(t) \) has been defined in (21) and \( \chi_q \) is the Fourier transform of the pumping correlation function \( \chi(x) \). Let us remind that the Péclet number is defined as the pumping-to-diffusive scale ratio \( \text{Pe} \equiv L/\sqrt{2\kappa} \) and is supposed to be large. To obtain the stationary value \( \langle \epsilon^n \rangle \), the limit \( T \rightarrow \infty \) should be considered. The average over \( \sigma \) with the Gaussian weight (21) has to be calculated in (18). This is very hard to do explicitly in the general case. We can however exploit the presence of the large parameter \( \text{Pe} \) to develop an asymptotic theory that captures the dominant terms in (18) with respect to \( \text{Pe} \). The important point is that, when \( \text{Pe} \) is large, there are two very different time scales in the dynamics. For the Lagrangian trajectories relevant to (18), particles start very close. The additive molecular noise term in the Langevin equation for Lagrangian trajectories is dominant at these distances and remains dominant as long as the particles do not separate by a distance comparable to the dissipative scale. This phase of the dynamics corresponds to the formation of the integral in the square brackets in (19) and takes place.
on times of order unity (not scaling with Pe). Once the particles have reached the dissipative scale and enter into the convective region, random multiplicative effects due to the velocity become dominant. Due to the multiplicative nature of the dynamics, the time to go from the dissipative scale to the integral scale varies as ln Pe. This phase is associated with the growth of the $W^2$ terms in $|\Phi(\sigma)|$. For large Péclet numbers, the two processes, formation of the integral in the square brackets and growth of $W^2$ terms in $|\Phi(\sigma)|$, are well separated in time. Let us then consider a time $t_0$ satisfying $1 \ll t_0 \ll \ln|\text{Pe}|$. At the dominant order in Péclet, $Q$ can be approximated as

$$Q[T; \{\sigma(t)\}] \approx \frac{\beta}{\text{Pe}^2} \int_{t_0}^{T} dt \exp \left[ 2 \int_{t_0}^{t} \sigma_{>}(t') dt' \right] G \left[ \frac{\exp \left[ 2 \int_{t_0}^{t} \sigma_{>}(t') dt' \right] - \eta}{\text{Pe}^2} \right],$$  

(50)

where $\alpha$ and $\beta$ are defined as

$$\alpha = \exp \left[ 2 \int_{0}^{t_0} \sigma(s) ds \right] \int_{0}^{t} dt' \exp \left[ -2 \int_{0}^{t'} \sigma(t'') dt'' \right], \quad \beta = \exp \left[ 2 \int_{0}^{t_0} \sigma(t') dt' \right].$$  

(51)

In order to obtain $|\Phi|$ from the original expression $|\Phi(\sigma)|$, we have made the following two steps: $Q[t_0; \{\sigma(t)\}]$ has been neglected and the upper bound $t$ in the integral over $dt'$ for $\alpha$ has been replaced by $t_0$. Both steps are motivated by the time scales separation at large Péclet numbers. More precise conditions of validity of the approximation will be discussed later in the Section. The moments $\langle Q^n \rangle$ can now be obtained by taking the $n$-th power of $|\Phi|$ and averaging. The average over $\sigma(t)$ is decomposed in a small times part $D\sigma_< \equiv \prod_{t<t_0} d\sigma(t)$ and a large times part $D\sigma_\sigma \equiv \prod_{t>t_0} d\sigma(t)$.

The corresponding weights are simply obtained decomposing the integral over $t$ in $|\Phi|$ as $S_\sigma = 1/4 \int_{0}^{t_0} [\sigma_\sigma + 1]^2$ and in $S_\sigma$ the integration runs from $t_0$ to $T$. The great advantage of $|\Phi|$ is that, in the large times averaging, both $\alpha$ and $\beta$ are just external parameters, depending neither on time $t$ nor on $\sigma_\sigma$. Once the average over $\sigma_\sigma$ is performed, we are then left with a function of $\alpha$ and $\beta$. This is, on his turn, averaged over $\sigma_\sigma$ giving the final result $\langle Q^n \rangle$.

A compact way for averaging over large times statistics is to introduce the Laplace transform of the PDF $P_s^\sigma \equiv \langle \exp[-sQ] \rangle$. It is indeed easy to recognize that its path integral coincides with a matrix element in the "Quantum mechanics" with Hamiltonian $\tilde{H} = -\partial_y^2 + s\beta \exp(2\eta) G [\alpha \exp(2\eta) / \text{Pe}^2] / \text{Pe}^2$. The "space" variable $\eta = \int_{t_0}^{t} \sigma(t') dt'$. Using standard quantum mechanical notation, the expression for $P_s^\sigma$ can be presented as

$$P_s^\sigma \equiv \langle \exp[-sQ] \rangle = \exp \left[-(T-t_0)/4 \right] \int \delta(\eta) e^{-\eta^2/2} e^{-\sigma_\sigma/4} e^{-\eta^2/2} = \left[ e^{-(T-t_0)/4} \Phi(T-t_0; \eta) \right]_{\eta=0},$$  

(52)

where $\Phi(t;\eta)$ satisfies $(\partial_t - \tilde{H}) \Phi = 0$. The initial condition at $t = 0$ for the "wave-function" $\Phi(t;\eta)$ is $\exp(-\eta^2/2)$. Noting that the potential part of the Hamiltonian vanishes at $\eta \to -\infty$ and $\Phi(0;\eta)$ does not, we obtain the asymptotic behavior of $\Phi$ at large times

$$\Phi(t, \eta = \ln[y/\text{Pe}]) \to \left. e^{t/4} \frac{\Phi_0(y)}{\sqrt{\text{Pe} y}} \right|_{t \to \infty} = \left. \left[ \partial_y^2 - s\beta G [\alpha y^2] \right] \Phi_0(y) \right|_{y=0} = 0.$$  

(53)

The new variable $y = \langle \exp[\eta] \rangle / \text{Pe}$ has been introduced. The function $\Phi_0(y)$ should tend to unity for $y \to 0$ and $\Phi_0 / \sqrt{y}$ should vanish for $y \to \infty$. It follows from (52) that $P_s^\sigma$ is simply the function $\Phi_0$ calculated at $y = 1/\text{Pe}$.

The general way to attack (34) for an arbitrary form of pumping is to proceed as we have already done for (30) in Section 4. Starting with a constant unit solution, the term with $s$ in (34) is treated perturbatively. A series in $s$ is then obtained $P_s^\sigma \equiv 1 + \sum_{n=0}^\infty c_n(y) s^n$ with the $c_n$'s having an expression similar to (31). The $n$-th moment $\langle n \rangle$ averaged over $\sigma_\sigma$ coincides with $c_n(1/\text{Pe})$, up to simple combinatorial factors. The final result is

$$\langle e^n \rangle = (2n-1)!! n! \left( \frac{\beta}{\alpha} \right)^{n-1/2} \frac{a_n \sqrt{\beta}}{\text{Pe}},$$  

(54)

where the coefficients $a_n$ are given by
Here, \( \Psi(t) \) is the appropriate to introduce the random variable \( \mu \).

It is again convenient for averaging to introduce the Laplace transform of the PDF \( \mathcal{P}^>_{\kappa} \) as

\[
\mathcal{P}^>_{\kappa} = \frac{I_0 \left( \sqrt{s \beta / \alpha} e^{-\sqrt{\pi} \kappa \beta / \alpha} \right) - 1}{I_0 \left( \sqrt{s \beta / \alpha} \right)}.
\]

The advantage with respect to the general case with arbitrary form of pumping is clearly that \( \mathcal{P}^>_{\kappa} \) is now known explicitly and this will permit us to reconstruct the PDF of the dissipation field.

Having averaged over \( \kappa \), we need now to take into account fluctuations at small times, i.e. average over \( \kappa_\kappa \).

The important remark is that, both in the general case (54) and in (56), we need to average quantities of the form \( \mathcal{P}^<_s \mathcal{P}^<_s \), we need now to take into account fluctuations at small times, i.e. average over \( \kappa_\kappa \).

It is again convenient for averaging to introduce the random variable \( \mu = \alpha / \beta \) and consider its distribution function weighted with \( \sqrt{\beta} \):

\[
\mathcal{P}^<_{\mu} = \int_0^\infty e^{-s \mu} \mathcal{P}^<_s d\mu = e^{-t_0/4} \left\{ \exp \left[ -t_0 \hat{H} \right] \right\} \left. \right|_{\eta=0}.
\]

Here, \( \Psi(t; \eta) \) satisfies \( (\partial_t - \hat{H}) \Psi = 0 \), the Hamiltonian is \( \hat{H}(\eta; s) = -\partial_t^2 + s \exp (-2\eta) \), the space variable \( \eta = \int_0^t \sigma \) and the initial condition for the “wave function” \( \Psi(t; \eta) \) is \( \exp (\eta/2) \). The asymptotic behavior at large times \( t \) can be obtained as in (53), noting that the potential part in \( \hat{H} \) decreases at infinity and \( \Psi(0; \eta) \) does not. It follows

\[
\Psi(t; \eta) \rightarrow \left. e^{t_0/4} \Psi(0; \eta) \right|_{t\rightarrow \infty} = e^{t_0/4} \exp \left[ \eta/2 - \sqrt{\eta} \exp [-\eta] \right],
\]

where \( \Psi_0(\eta) \) satisfies \( \hat{H}(1/4) \Psi_0 = 0 \) and behaves as \( \exp (\eta/2) \) for large \( \eta \)'s. Requiring \( \exp (t_0/4) \gg 1 \), we can plug the asymptotic expression (59) into (58) and obtain \( \mathcal{P}^<_{\mu} = \exp [-\sqrt{\eta}] \). The PDF of \( \mu \) and the moments relevant for \( \kappa_n \) are easily restored as

\[
\mathcal{P}^<_\mu = \frac{1}{2\pi i} \int_{0^- + i\infty}^{0^+ + i\infty} ds \mathcal{P}^<_s e^{s \mu} = \frac{1}{2\pi i} \int_{0^- - i\infty}^{0^+ - i\infty} ds \exp \left[ -s \sqrt{\eta} \right] e^{s \mu} = \frac{1}{2\sqrt{\pi} \mu^{3/2}} \exp \left[ -\frac{1}{4\mu} \right],
\]

\[
\left\{ \frac{\beta}{\alpha} \right\}^{n-1/2} \sqrt{\beta} = \int_0^\infty d\mu \mu^{-n+1/2} \mathcal{P}^<_\mu = 2^{2n-1} \frac{(n-1)!}{\sqrt{\pi}}.
\]

This expression can be used to calculate the moments appearing in (54). Note that the expression (54) derived exploiting the time scales separation, coincides for the first moment \( \langle \kappa \rangle \) with the dominant order of the known solution derived in Section 3.
\[ \langle \epsilon \rangle = \frac{1}{\pi \Pe} \int_0^\infty \frac{[\chi(0) - \chi(x)] \, dx}{x^2 + \Pe^{-2}}. \]  
(62)

One step further can be made for the specific form of pumping \( G^*[x^2] = \exp[-2x] \), allowing to obtain the explicit solution \( 5d \). From \( 5d \), we obtain indeed the Laplace transform of the PDF for \( \beta \)

\[ P_s \equiv \langle P_s^> \rangle_s \rightarrow 1 - \frac{\sqrt{s}}{\Pe} \int_0^\infty \frac{d\mu P_s^< \langle \mu \rangle I_1(\sqrt{s/\mu})}{I_0(\sqrt{s/\mu})} = 1 - \frac{1}{\sqrt{\pi \Pe}} \int_0^\infty \ln[I_0(2\sqrt{sx})]e^{-x} \, dx. \]  
(63)

The moments \( \langle \epsilon^n \rangle \) can be immediately reconstructed, according to \( 15s \), from the derivatives of \( 13s \) at \( s = 0 \). They read

\[ \langle \epsilon^n \rangle = \left( \begin{array}{c} 2n - 1 \end{array} \right)!n! \frac{\partial^n}{\partial z^n} \ln \left[ \frac{1}{J_0(\sqrt{z})} \right] \bigg|_{z=0} = i^n \frac{\Gamma(n + 1/2)[\Gamma(n + 1)]^2 2^n}{2\pi^2 \Pe} \int_C \frac{\ln[J_0(\sqrt{z})] \, dz}{(2\pi)^{n+1}}, \]  
(64)

where \( C \) is a close contour about \( z = 0 \) in the complex \( z \)-plane. The Cauchy integral representation \( 24 \) is useful for getting the large \( n \) asymptotic of \( 14d \). The integral is indeed saddle around the square of the first zero of the Bessel function \( z_0 = x_0^2 \), \( J_0(x_0) = 0 \). The saddle estimation for the integral is \( \sim x_0^2 \), thus giving

\[ \ln[\Pe \langle \epsilon^n \rangle] \rightarrow \left. n \left( 3 \ln[n] - 3 + \ln[2x_0] \right) \right|_{n \to \infty}. \]
(65)

The whole expression \( 13s \) can actually be inverted. From \( 13s \), it follows indeed that the PDF \( P^{(\epsilon)}(\epsilon) \) of the dissipation \( \epsilon \) is given by

\[ P^{(\epsilon)}(\epsilon) = \frac{1}{2\pi \Pe} \int_0^\infty \frac{dx}{2\pi} e^{-x} \int_0^{+i\infty} ds P_s \exp[\epsilon s/(2x)] \]  
(66)

where \( qF_2 \) is the generalized hypergeometric function with the \( q \) parameters in the numerator and the \( p \) parameters in the denominator. The \( \delta \)-function at the origin arising from the unit term in \( 56 \) has not been considered in \( 60 \). The reason is that, as we shall see in a moment, the range of validity of \( 60 \) is \( \epsilon \gg \Pe^{-1} \).

It is indeed time to clarify the limits of validity of the calculations performed. To be concrete, we shall specifically consider the pumping \( G^* \) leading to \( 5c \). The remainder of the expansion over \( \Pe^{-1} \) in \( 5d \) is bounded by

\[ \beta \left( I_1 \left( \sqrt{s/\alpha} \right) \right) \frac{\exp(s/\alpha) \sqrt{s}}{\sqrt{\pi}} + sI_1 \left( \sqrt{s/\beta} \right) / (2\Pe^2). \]  
(Contrary to \( 13d \), we need now to consider averages of \( \mu = \alpha/\beta \) with weight \( \beta \). The problem can again be reduced to the calculation of a quantum mechanical matrix element and it is found that the final result depends on \( t_0 \) as \( \exp(2t_0) \). The condition for the remainder to be subdominant with respect to the terms kept in \( 14d \) is therefore that \( \exp(2t_0)/\Pe \) should vanish as \( \Pe \to \infty \). On the other hand, the observation time must be much larger than unity in order to attain the stationary state, i.e. \( t_0 \gg 1 \). This condition is clearly compatible with the previous one, in the limit of large Péclet numbers, and gives the ordering \( 1 \ll t_0 \ll \ln[\Pe] \). The other delicate point is the criterion of applicability of \( 13d \) with respect to \( s \). For the expansion in \( 10d \) to be meaningful, the second term should be much smaller than unity. This shows that we should require \( s \ll \Pe^2 \). The expansion \( 10d \) fails then to describe the large \( s \) tails of \( P_s \) and thus the smallest values of \( \epsilon \) in the respective PDF (the relation between large \( s \) and small \( \epsilon \) is direct since the decay of the generating function is relatively slow).

Another simple approximation is however available for the high \( s \) limit. The trajectories contributing to \( \langle \exp(-sQ) \rangle \) at large \( s \gg \Pe^2 \) are clearly those where \( Q \) is small. From the definition \( 19d \) it follows that, for this to happen, \( W^2(t) \) should remain small all the time. The quantity \( W^2 \int W^2 \) is in this case \( O(1) \) and the argument in \( G \) is small on account of the \( 1/\Pe^2 \) factor. For the trajectories relevant at high \( s \), it is thus possible to approximate \( Q \) by \( G[0]/\Pe^2 \int W^2 \). The Laplace transform \( P_s \) reduces then to
\[ \mathcal{P}_s \equiv \exp \left[ -sG[0]/\text{Pe}^2 \int_0^T dt W^2(t) \right] = e^{-T/4} \exp \left[ -T \hat{H}(-\eta; sG[0]/\text{Pe}^2) \right] e^{-\eta/2}, \] (67)

where \( \eta(t) \equiv \int_0^t \sigma(t) \) and \( \hat{H} \) is the same as for (58). The matrix element (63) actually coincides identically with (58) when \( t_0 \) by \( T, \eta \) by \( -\eta \), and \( s \) by \( sG[0]/\text{Pe}^2 \) are replaced. Using (60), one can then easily get the final answer

\[ \text{at } s \gg \text{Pe}^2, \quad \mathcal{P}_s \to \exp \left[ -\sqrt{sG[0]/\text{Pe}} \right]. \] (68)

Inverting (88), we can obtain the PDF of \( \epsilon \) at the small values \( \epsilon \ll \text{Pe}^{-2} \). For larger values of \( \epsilon \), the PDF follows from the general formula (60). The following general behavior for the dissipation field PDF is thus obtained

\[ \mathcal{P}(\epsilon)(\epsilon) \to \begin{cases} \frac{\sqrt{2\epsilon}\text{Pe}}{2\epsilon^{1/2}} \exp \left( \frac{\epsilon}{2\epsilon^{1/2}} \right), & \text{for } \epsilon \ll \text{Pe}^{-2}; \\ \frac{\epsilon^{-1/2}}{2\epsilon^{1/2}} \exp \left( -\epsilon^{1/2} \right), & \text{for } \epsilon^{-2} \ll \epsilon \ll 1; \\ \frac{\epsilon^{-1/2}}{2\epsilon^{1/2}} \exp \left( -\epsilon \right), & \epsilon \gg 1, \end{cases} \] (69)

where algebraic prefactors have not been considered in region of exponential fall off. The PDF for scalar gradients \( \omega = \sqrt{\epsilon/\kappa} \) follows immediately from (50) as

\[ \mathcal{P}(\omega)(\omega) \to \begin{cases} \frac{\sqrt{L}}{\pi} \left( 1 - \frac{L^2(G[0])\omega^2}{\kappa/\omega^2} \right), & \text{for } \omega \ll 1/L; \\ \frac{1}{2\pi L \omega^2}, & \text{for } 1/L \ll \omega \ll 1/\sqrt{\kappa}; \\ \frac{\epsilon^{-1/2}}{2\epsilon^{1/2}} \exp \left[ -\left( \sqrt{\kappa/\epsilon} \omega \right)^{2/3} \right], & \omega \gg 1/\sqrt{\kappa}. \] (70)

Moments of the dissipation \( \langle \epsilon^n \rangle \), with \( a > 1/2 \), are all proportional to \( 1/\text{Pe} \), in agreement with (54) valid for arbitrary pumping. Moreover, we learn from (50) that the linear scaling found for scalar differences low-order moments comes down to the dissipative range. Moments \( \langle \epsilon^n \rangle \) with \(-1/2 < a \leq 1/2 \) scale indeed as \( \text{Pe}^{-2a} \). Moments with \( a < -1/2 \) do not exist. From (70), it also follows that the same tendency observed for scalar differences PDF to develop a cusped structure at the origin is present. An important difference arises for the tails. The comparison between (70) and (44) for gradients and scalar differences indicate indeed that the former decrease much more slowly. Such behavior is physically understood in terms of fluctuations of the dissipative scale as follows. Gradients can be thought of as scalar differences evaluated at the dissipative scale. This scale is however a dynamical quantity and fluctuates (strongly in a problem with inverse transfer as the one here). The statistics of these fluctuations are actually related precisely to those of the variable \( \mu \), introduced in (57). The fluctuations of \( \epsilon^n \) are therefore the product of those of scalar differences and those of the dissipative scale. The latter do not depend on the dimensional parameters of the problem, i.e., they are universal as follows from (50), but grow factorially with the order \( n \). This factorial is precisely what shifts the dominant term in (53) from \( 2n \ln n \) to \( 3n \ln n \). It follows then that the slower decay of the tails for gradients with respect to those for scalar differences is indeed due to dynamical fluctuations of the dissipative scale.

7. CONCLUSIONS AND DISCUSSION

We analyzed in the present paper statistics of the passive scalar advected by a one-dimensional, smooth and fast-correlated in time velocity field. In spite of its extreme simplicity, the dynamics presents very interesting and surprising behaviors. We expect that many of them are quite generic and it may be interesting to look for more realistic scalar (and generally turbulent) problems where the new effects discussed here could be of relevance. Thus, hereafter we recall and briefly comment the major properties of the model discussed in the paper and address, in parallel, the respective questions for future studies.

The first major property of the dynamics is the inverse cascade of the scalar. The inverse cascade is a consequence of compressibility, but in a subtle way: in [24] we consider indeed a generalized smooth \( d \)-dimensional model having
the degree of compressibility as a free parameter. A transition between inverse and direct cascades is observed there: if \( d > 4 \), the cascade is always direct, independently of the degree of compressibility; if the latter is small enough, the cascade is direct again; otherwise, it is inverse. It might also be interesting to look from the same dynamical point of view used here at the critical dimension appearing in the passive vector problem discussed in [29] and at the role of compressibility for the direction of energy transfer in other turbulent situations, e.g. 2D Navier-Stokes and MHD turbulence.

Dynamically, the inverse cascade is associated with the fact that the Lyapunov exponent for Lagrangian trajectories is negative. This implies that, in the case of inverse cascade, contraction of Lagrangian trajectories is typical and stretchings are relatively rare events. This is precisely the opposite of the canonical picture associated with the direct cascade. Rare trajectories (contracting or stretching if the Lyapunov exponent is positive or negative, respectively) are responsible for the intermittent part of the problem. This definitely confirms the picture that contractive trajectories play a crucial role for intermittency in incompressible turbulent transport.

Negative Lyapunov exponent leads to the third observation emerging from the analytical study of the model: Gaussianity of the scalar establishes at scales larger than the scale of the pumping, while strong intermittency is present at small scales. Small-scales intermittency found here is of the “Burgers” kind, i.e. all scalar differences integer moments scale linearly \( \sim r/L \). This result gives a definite ground to speculate that, generally, the stronger is the degree of compressibility (the role of contracting trajectories is then enhanced), the more intermittency downscales of the pumping is observed. Note in the respect of Gaussianity that the inverse cascade of energy in 2D Navier-Stokes turbulence is also Gaussian according to the numerics [11]. It rises yet another conjecture of a certain generality of the Gaussianity feature for a great variety of direct cascades.

The equation for scalar differences PDF, derived directly from the dynamics in Section 5, shows that no dissipative anomaly is present in the model. In the generalized one-dimensional model introduced in [1], \( \gamma = 1 \) is a natural threshold for the inverse cascade. At \( \gamma > 1 \) the cascade is direct, while it is inverse at \( \gamma < 1 \). Most probably, it means that there is no dissipative anomaly (equation for the PDF of the scalar difference could be closed) for \( \gamma < 1 \). However, it is not yet clear how to derive this consistently (as it is derived here for the smooth limit \( \gamma = 0 \)). Note that the general question of the role of the dissipative anomaly in the passive scalar physics is not yet resolved in any sense (some suggestions on this point may be found in [10,19,26]). One more unresolved question is how does the dissipative anomaly (if any) affects the anomalous scaling behavior of structure functions of integer (and generally all the) orders.

An important result for the issue of convection-diffusion interplay, is the fact that the effective dissipative scale, which sets the crossover scale between scalar differences \( 2n \)-th moment in the convective range and dissipation field \( n \)-th moment, is a strongly fluctuating quantity, growing factorially with \( n \). To make this statement, we have analytically calculated in Section 3 the dissipation field PDF exploiting a new scale-separation procedure, which is also used in Appendix B to describe the long-time dynamics of the pair correlation function of the scalar gradients. The procedure is likely to be a general tool for resolving the problem in other situations. Particularly, a slight (matrix) modification of the method should be relevant for the Kraichnan \( d \)-dimensional incompressible smooth model.

Finally, the eddy-diffusivity operator is Hermitian for Kraichnan model, where random velocity is incompressible and fast. The Hermiticity is lost, when compressible flow is considered. Generally, it might be interesting to look at scalar transport (and particularly at flow with finite correlation times) from the point of view of non-Hermiticity. This issue is the subject of a very recent interest in the field of disordered systems [32,33,34,35]. The specific object of interest for scalar transport is the resolvent \( R(t;x,y) \), describing the probability to find a Lagrangian separation equal to \( x \) at time \( t \), having initially been equal to \( y \). One may then want to study the distribution in the complex plane of the poles of its Laplace transform. A width of the poles domain along the imaginary direction might be a characteristic of the trapping degree.

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APPENDIX A: DYNAMICAL FORMULATION BY FIELD FORMALISM

Let us consider the gradient field \( \omega(t,x) \) satisfying the equation of motion \([4]\) with the velocity \( u(t,x) \) having Gaussian statistics and correlation function \( < u(t,x) u(t',x') > = V(t-t', x-x') \). The generating functional \( F(\lambda) \) for its simultaneous correlation functions can be written in the form of functional integral (see \([27,28]\)):

\[
F(\lambda) = \int DpD\omega Du \exp \left\{ \frac{1}{2} (p|\chi''|p) - \frac{1}{2} (u|V^{-1}|u) + i \int_0^T dt dx p (\partial_t \omega + \partial_x (u \omega) - \kappa \partial_x^2 \omega) + \int dx \lambda(x) \omega(T,x) \right\}
\]

(A1)

Here, \( (u|V^{-1}|u) \) denotes the diagonal matrix element of the inverse operator to \( V(t-t', x-x') \) in the Hilbert space of the functions \( u(t,x) \) and the pumping \( \chi'' \) is the second spatial derivative of the forcing correlation function \( \chi \), defined in \([2]\). The retarded regularization of the time derivative, \( (\partial_t \omega)_n = \frac{1}{\Delta} (\omega_n - \omega_{n-1}) \), does not produce any nontrivial Jacobian after derivation of (A1) from the equation of motion. \( \Delta \rightarrow 0 \) is the temporal slicing and \( \omega_n(x) \equiv \omega(n\Delta,x) \). Performing the Gaussian integration over the field \( u(t,x) \) we arrive at

\[
F(\lambda) = \int DpD\omega \exp \left\{ -\frac{1}{2} (\omega\partial_x|V|\partial_x\omega) + \frac{1}{2} (p|\chi''|p) + i \int_0^T dt dx p (\partial_t \omega - \kappa \partial_x^2 \omega) + \int dx \lambda(x) \omega(T,x) \right\}.
\]

(A2)

For the case of \( \delta \)-correlated in time velocity fields considered in this paper, we replace the general \( V(t-t', x-x') \) by specific \( \delta(t-t')V(x-x') \). The explicit version of (A1) reads then as

\[
F(\lambda) = \int DpD\omega \exp \left( -S + \int dx \lambda(x) \omega(T,x) \right)
\]

(A3)

\[
S = \frac{1}{2} \int_0^T dt \left\{ \int dx dx' (\omega \partial_x p)(t,x) V(x-x') (\omega \partial_x p)(t,x') - \int dx dx' p(t,x) \chi'' \left( \frac{x-x'}{L} \right) p(t,x') \right\}
\]

\[
-i \int_0^T dt dx p (\partial_t \omega - \kappa \partial_x^2 \omega).
\]

(A4)

Despite of the explicit presence of the \( V \)-term in (A4), the dynamical part of the action \( S \) is covariant with respect to Galilean transformations. This leads to the important Ward identity:

\[
\int DpD\omega \exp \left( -S + \int dx \lambda(x) \omega(T,x) \right) \prod_j \left( \int dx \omega(t_j,x) \partial_x p(t_j,x) \right) = 0,
\]

(A5)

where set \( \{t_j < T, j = 1, 2, ...\} \) is arbitrary. To prove it, let us consider the change of variables

\[
p(t,x) \rightarrow p \left( x + \int_t^T v(\tau) d\tau, t \right), \quad \omega(t,x) \rightarrow \omega \left( x + \int_t^T v(\tau) d\tau, t \right)
\]

(A6)

in the functional integral (A3) with \( v(\tau) \) being unconstrained. The transformation does not change the source term and it has unit Jacobian. The variation \( \delta S \) of the action \( S \) under the transformation reads as

\[
\delta S = -i \int_0^T dt v(t) \left( \int dx \omega(t,x) \partial_x p(t,x) \right).
\]

(A7)

The change of variables does not change the value of the integral. Thus, all the functional derivatives of (A3) over \( v(t) \) are equal to zero. Taking into account (A7), one finally arrives at the Ward identity (A5).

All written above is applicable for any short correlated velocity field. Now let us narrow consideration to the case of smooth velocity field \([2]\) with \( \gamma = 0 \). One gets
\[
\mathcal{F}(\lambda) = \int \mathcal{D}p \mathcal{D}\omega \exp \left( -S + \int dx \lambda(x) \omega(T,x) \right),
\]
\[\tag{A8}\]

\[
S = \frac{1}{2} \int_0^T dt \left[ - \int dx dx' (\omega \partial_x p)(t,x)(x-x')^2 (\omega \partial_x p)(t,x') \right] - i \int_0^T dt dx p(t,x) \chi'' \left( \frac{x-x'}{L} \right) p(t,x') - \frac{i}{2} \int_0^T dt dx p \left( \partial_t \omega - \kappa \partial_x^2 \omega \right),
\]
\[\tag{A9}\]

where the \(V_0\) term was dropped due to (A3). The goal is to reduce the problem to averaging of a functional of one random in time scalar field, analogously to the random matrix description used in two-dimensional \[23\] and generally \(d\)-dimensional case. However, in the present case, compressibility calls for more evaluations. Let us perform the following change of variables in (A8):

\[
\{p(t,x), \omega(t,x)\} \rightarrow \left\{ p\left(t, x + i \int_0^T d\tau \int dy y^2 \partial_y p(t,y) \omega(t,y) \right), \omega\left(t, x + i \int_0^T d\tau \int dy y^2 \partial_y p(t,y) \omega(t,y) \right) \right\},
\]
\[\tag{A11}\]

which can be considered as the field-dependent and time-dependent homogeneous spatial translation leaving the source term intact. Analytic continuation of the fields which makes the spatial arguments to be real is assumed. Action gets the following from being rewritten in the new variables

\[
S = \frac{1}{2} \int_0^T dt \left[ \left( \int dx (\omega \partial_x p)(t,x) \right)^2 - \int dx dx' (\omega \partial_x p)(t,x) \chi'' \left( \frac{x-x'}{L} \right) p(t,x') \right] - i \int_0^T dt dx p \left( \partial_t \omega - \kappa \partial_x^2 \omega \right).
\]
\[\tag{A12}\]

The transformation (A11) has non-trivial Jacobian

\[
\mathcal{D}p \mathcal{D}\omega \rightarrow \mathcal{D}p \mathcal{D}\omega \mathcal{J}(p,\omega),
\]
\[\tag{A13}\]

which depends on regularization. The regularization is fixed by the requirement for the temporal \(\delta\)– function from the correlation function of velocities to appear as a result of narrowing of an even function of temporal argument. Besides, the correctly regularized action (A12) should reproduce the respective correlation function in the limit \(T \to 0\):

\[
p_n(x) \rightarrow p_n \left( x + i \Delta \int dy y^2 \partial_y p_n(y) \omega_n(y) + i \Delta \sum_{m=n+1}^{N} \int dy y^2 \partial_y p_m(y) \omega_m(y) \right),
\]
\[\tag{A14}\]

respectively for \(\omega_n(x)\). The Jacobian \(\mathcal{J}(p,\omega)\) can be computed easily

\[
\mathcal{J}(p,\omega) = \exp \left( -i \int dt dx x \omega \partial_x p \right).
\]
\[\tag{A15}\]

The exponential of the term \(\int_0^T dt \left( \int dx x \omega \partial_x p \right)^2\) in the action (A12) multiplied by the Jacobian (A15) can be represented by means of averaging of the exponential of \(i \int_0^T dt \sigma \int dx x \omega \partial_x p\) over the auxiliary field \(\sigma(t)\). The Gaussian measure of averaging over \(\sigma\) is defined in (21). It results in the following representation for the generating functional \(\mathcal{F}(\lambda)\):

\[
\mathcal{F}(\lambda) = \int \mathcal{D}p \mathcal{D}\omega \mathcal{D}f \mathcal{D}\sigma \exp \left[ -S_{\sigma} - \frac{1}{2} (f|\chi^{-1}|f) \right]
\times \exp \left[ i \int_0^T dt dx p \left( \partial_t \omega - \sigma \partial_x(x \omega) - \kappa \partial_x^2 \omega - \partial_x f \right) + \int dx \lambda(x) \omega(T,x) \right].
\]
\[\tag{A16}\tag{A17}\]
The solution of (B2) can be written in terms of the Green function \( \Omega (t, x) \). In the present Appendix, we will find the long-time asymptotic of the pair correlation function of the scalar gradients \( \Omega (t, x) \) which obeys (B). We will model here the infrared cut-off of the velocity correlations by \( \sigma (t, x) \). Indeed, integrating (B4), one gets

\[
\Omega (x) = -\frac{1}{s} \int_0^\infty \mathcal{G}_s (x, x') \chi'' \left( x' \right) dx', \quad s\mathcal{G}_s - \partial_x^2 (S (x) + 2\kappa) \mathcal{G}_s = \delta \left( x - x' \right).
\]  

The desired long-time asymptotic corresponds to the smallest \( s \) and our aim is to find the singular structure of \( \Omega_s (x) \) at small \( s \) and \( x \gg \sqrt{\kappa} \). In what follows, we will then neglect all the logarithmic contributions like \( s \ln L_u \) and \( s \ln x \) in comparison with 1. We will also neglect the smallest \( x' (x' \ll \sqrt{\kappa}) \) contribution into the integral (B3) going to zero in the limit of small diffusivity. Therefore \( \mathcal{G}_s (x, x') \) should be studied at \( \sqrt{\kappa} \ll x' \ll L_u \) and in all the allowed domains with respect to \( x \).

Making the substitution \( \mathcal{G}_s (x, x') = g (x, x') /[S (x) + 2\kappa] \), one gets

\[
\frac{s}{S (x) + 2\kappa} g - \partial_x^2 g = \delta \left( x - x' \right).
\]  

Solution of (B4) in the domain of the largest \( x, x > L_u > x' \gg \sqrt{\kappa} \), can thus be written as

\[
g (x, x') = A \exp \left( -\frac{\sqrt{s}}{L_u} (x - L_u) \right).
\]  

Note however that only the first two terms of the (B3) expansion in \( s \) will be required for matching hereafter. If \( x \) is smaller than \( L_u \), but is still larger than a separation scale \( x_0 \), one can omit the first term on the rhs of (B4) and write the following general solution of (B4) in the intermediate domain, at \( L_u > x \gg \sqrt{\kappa} \),

\[
g (x, x') = \begin{cases} F_1 + xF_2, & x < x', \\ D_1 + xD_2, & x > x'. \end{cases}
\]  

Here, the constants \( F_1, F_2, D_1, D_2 \) should be fixed via inner matching at \( x = x' \) and outer ones at \( L_u \) and \( x_0 \). The value of \( x_0 \), restricted by \( \sqrt{\kappa} \ll x_0 \ll \sqrt{\kappa}/s \), defines the upper bound for the scales at which one cannot omit the first term of (B4) anymore. However, the point is that at \( x < x_0 < x' \) an actual dependence of \( g \) on \( x_0 \) has disappeared and we should not locate \( x_0 \) explicitly to get the asymptotic behavior of \( g(x, x') \). Indeed, integrating (B4), we get

\[
\partial_x g (x) = s \int_0^x \frac{g (z) dz}{S (x) + 2\kappa},
\]  

where the argument \( x' \) is omitted and it is already taken into account that \( g(x) \) is even: \( \partial_x g (x) |_{x=0} = 0 \). (B7) shows that the \( x \)–derivative changes sufficiently at the diffusive scale, being nonetheless small by the absolute value there. It means that the function \( g (x) \) at \( x \ll x_0 \) is a constant in the leading (zero) order in \( s \). The subleading (first order
in $s$ term is then fixed by (B7) taken at $x = x_0$. Altogether it gives the following expression for $g$ at $x < x'$ with the desirable (for the forthcoming matching) accuracy

$$g \approx g_0 \left(1 + s x + \int_{0}^{x_0} \frac{dz}{S(z) + 2\kappa} \right) \approx g_0 \left(1 + \frac{s x}{2\sqrt{2\kappa}}\right), \tag{B8}$$

where, evaluating the integral in (B8), we used that $\sqrt{\kappa} \ll x_0$. Matching (B5, B6) and (B8), we get

$$\begin{align*}
tag{B9} \text{at } & x = x', \quad D_1 + x' D_2 = F_1 + x' F_2, \quad F_2 - D_2 = 1, \\
& \text{at } x = L_u, \quad D_1 + L_u D_2 = A, \quad D_2 = -\frac{\sqrt{\kappa}}{L_u} A, \\
& \text{at } x = x_0, \quad F_1 + x_0 F_2 = g_0, \quad F_2 = s g_0 \frac{\pi}{2\sqrt{2\kappa}}.
\end{align*}$$

Keeping $s$ to be very small, while the ratios $s/\sqrt{\kappa}$ and $\sqrt{s}/L_u$ are finite, one derives from (B9)

$$\begin{align*}
F_1 &= g_0 \frac{1 + x' \sqrt{s}/L_u}{s \pi/(2r_d) + \sqrt{s}/L_u}, \\
F_2 &= \frac{s g_0 \pi}{2\sqrt{2\kappa}}, \\
D_1 &= A = 2\sqrt{2\kappa} x' \pi + \sqrt{s}/L_u, \\
D_2 &= -\frac{\sqrt{s}}{L_u} A. \tag{B10}
\end{align*}$$

It is worth noting that the explicit value of $x_0$ does not enter (B10), and the scale separation procedure is indeed justified. Substituting (B10) into (B6) ($x' < L_u$), one gets

$$\Omega_s(x) = \frac{1}{S(x) + 2\kappa} \left[\frac{\chi(x)}{s} - \frac{\chi(0)}{s + \alpha \sqrt{s}}\right], \quad \alpha = 2\sqrt{2\kappa} \pi L_u, \tag{B11}$$

and coming to the inverse Laplace-transform

$$\Omega(t, x) = \frac{1}{S(x) + 2\kappa} \left[\chi(x) - \chi(0) \frac{2}{\pi} \int_{0}^{\infty} dp \exp \left(-\alpha^2 p^2 t\right) \right]. \tag{B12}$$

The expression (B12) states that, at the times less than $T_{L_u} \sim \frac{x'^2}{\kappa}$, the correlation function coincides with the quasi-stationary limit given by (B6). At the largest times, $\Omega(t, x)$ approaches the asymptotic value

$$\frac{1}{S(x) + 2\kappa} \chi(x) \tag{B13}$$

diffusively, $\chi(x) - \Omega[S(x) + 2\kappa] = C \rightarrow \chi(0) \left(\alpha^2 \pi t\right)^{-1/2}$. 

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