A map of Ramanujan expansions

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in honor of Ramanujan expansions Masters

1. Introduction and legend of the paper.

We define a “map”, say, a paper which is neither a survey, nor a paper giving only new results; also, it is not something in which to find complete and state-of-the-art proofs. Simply, it’s a tool for people who want to have a very brief & quick look at the main properties regarding that argument: actually, this paper was born because we found ourselves in the situation to learn a great amount of, say, basic facts and shortcuts, regarding the Ramanujan expansions. So, without, of course, aiming at completeness, we embark in the not-easy-at-all, say, task of supplying a panorama in small scale: what’s a map, if not this? This, for the very interesting (and still growing) theoretical background of Ramanujan expansions. By the way, the German-style approach (very strong, but heavy, for the, say, soft mathematics applied), to these expansions, will be our assumed & quoted knowledge, but not our way to express, here, the main ideas, properties, Lemmas and so on about this argument. Last but not least, so to confirm a map is not a survey, we will also give during the exposition some small Lemmas (see our Lemmas, from 2 on, for example) of our own so to speak handmade production (not to be compared to this huge theory, we are trying to embed them in !).

Coming to good news, we will only give “quick”, say, proofs: less than two pages!

Coming to bad news, of course whenever we can’t stay in this limit, no proof, in this paper! (However, we’ll try our best to quote good literature in which to find illuminating proofs. Btw, thanks Ram for [M])

Coming to interesting news: (we cheat, as) for chosen topics, no lengthy proof at all should be required!

Prior to other definitions, properties & so on, we give(our?)current definition of Ramanujan expansion.

Given any arithmetic function $F : \mathbb{N} \to \mathbb{C}$, we say it has a Ramanujan expansion (see [CMS] Definition 2)

$$(RE) \quad F(n) = \sum_{q=1}^{\infty} \hat{F}(q) c_q(n)$$

where the Ramanujan sum is defined (as Ramanujan himself did in [R], compare [M]) as, abbreviating hereafter $(a, b) \overset{\text{def}}{=} \text{g.c.d.}(a, b)$ the greatest common divisor of $a, b \in \mathbb{Z}$, (hereafter $r \leq R$ means $1 \leq r \leq R$)

$$(RS) \quad c_q(n) \overset{\text{def}}{=} \sum_{j \leq q, (j, q) = 1} \cos(2\pi jn/q), \forall q \in \mathbb{N}, \forall n \in \mathbb{N},$$

meaning : the $q$–series in $(RE)$ converges pointwise to $F(n)$, for all fixed $n \in \mathbb{N}$, for certain $\hat{F}(q) \in \mathbb{C}$.

Notice : the definition $(RS)$ may be extended to all integers $n \in \mathbb{Z}$, giving $c_q(0) = \sum_{j \leq q, (j, q) = 1} 1 \overset{\text{def}}{=} \varphi(q)$, the Euler function and using $c_q(-n) = c_q(n)$, from parity of cosine, for negative $-n \in \mathbb{Z}$.

See: the uniqueness of these, say, Ramanujan coefficients $\hat{F}(q)$ is not guaranteed. A classic example was given by Ramanujan, after defining his sums, in [R] for the, say, constant-0-arithmetic function:

$$(C) \quad \sum_{q=1}^{\infty} \frac{1}{q} c_q(n) = 0, \forall n \in \mathbb{N},$$

thus with non-uniqueness for these $\hat{F}(q) = 1/q$, as of course also $\hat{F}(q) \equiv 0$ (here, $= 0$, $\forall q \in \mathbb{N}$)\footnote{This notation can not confuse with the congruence sign $\equiv$ notation, following.} gives the same (constant-0) function. Notice, also, that the absolute convergence can not hold, see the following, in case $\hat{F}(q) = 1/q$, while for $\hat{F}(q) \equiv 0$ it’s trivial!
Two remarks:

**Remark 1.** We say $F$ has “a”, not “the” (notice the subtlety) Ramanujan expansion . . . , because we wish to stress we are not aware (maybe it’s true, we don’t know, or we don’t say) about uniqueness of this expansion (i.e., uniqueness of $\hat{F}(q)$ coefficients).

**Remark 2.** We say that $F$ has an “absolutely convergent” Ramanujan expansion”, meaning that it has an expansion as above, that satisfies

$$\sum_{q=1}^{\infty} \left| \hat{F}(q) c_q(n) \right| < \infty, \quad \forall n \in \mathbb{N}.$$ 

These two Remarks come from the “counterexample”, (C), for the missing uniqueness of coefficients (for the constant-0-function). We wish to prove, now: (C) does not converge absolutely, namely (even more):

$$\sum_{q=1}^{\infty} \frac{1}{q} |c_q(n)| = +\infty, \quad \forall n \in \mathbb{N}.$$ 

So, we give a very short classic Lemma (compare [M]), with a “very quick”, less than half-page, proof.

**Lemma 1.** Let $q \in \mathbb{N}$. Define as usual $e_q(m) \overset{def}{=} e^{2\pi i m/q}$, $\forall m \in \mathbb{Z}$, the additive characters. Indicate with $\mathbb{Z}_q^*$ the set of reduced residue classes modulo $q$ and with $\mu$ the Möbius function [T]. Then

$$c_q(n) = \sum_{j \in \mathbb{Z}_q^*} e_q(jn) = \sum_{d|q} \sum_{d|n} \mu(q/d) \varphi(q) \frac{\mu(q/(q,n))}{\varphi(q/(q,n))}.$$ 

**Remark 3.** As $q = 1$ gives 1 everywhere and $q = 2$ everywhere $(-1)^n$, we assume $q > 2$ in the Proof.

**Proof.** First equation comes from the fact that $j \in \mathbb{Z}_q^* \Leftrightarrow -j \in \mathbb{Z}_q^*$, with $j \neq -j \mod q$ (from $q > 2$), the Euler identity:

$$e_q(jn) = \cos(2\pi jn/q) + i \sin(2\pi jn/q)$$

and the fact that sine function is odd (so imaginary part vanishes).

We write $1_q \overset{def}{=} 1$ iff (if & only if) $\varphi$ is true ($\overset{def}{=} 0$ otherwise) starting in 2nd equation proof, now.

The orthogonality of additive characters, namely 1st equation, in following (1), then g.c.d. rearranging

$$\sum_{r \leq q} e_q(rn) = q 1_{n \equiv 0 \mod q} = \sum_{d|q} \sum_{r \leq q \atop (r,q)=1} \sum_{d|q} e_q(rn) = \sum_{d|q} \sum_{d|q} c_d(jn) = \sum_{d|q} c_d(n)$$

give the, say, V.I.P. (=very important property, like we’ll abbreviate hereafter)

$$1_{q|n} = \frac{1}{q} \sum_{d|q} c_d(n), \quad \forall q \in \mathbb{N}, \quad \forall n \in \mathbb{Z}$$

and then, by Möbius inversion [T], we get 2nd equation above:

$$q 1_{q|n} = \sum_{d|q} c_d(n) \Rightarrow c_q(n) = \sum_{d|q} d 1_{d|n} \mu(q/d) = \sum_{d|q} d \mu(q/d).$$

This also proves $c_q(n)$ is a multiplicative function of $q$, so we may calculate it on prime-powers $p^K$, as Davenport [Da] does, getting last equation.

In fact, $(q, n) = 1$ implies $|c_q(n)| = \mu^2(q)$ from Lemma 1 last equation, so fixed any $n \in \mathbb{N},$

$$\sum_{q=1}^{\infty} \frac{1}{q} |c_q(n)| \geq \sum_{q=1}^{\infty} \frac{\mu^2(q)}{q} = \sum_{d|q} \mu(d) \sum_{K=1}^{\infty} \frac{\mu^2(dK)}{K} = \sum_{d|n} \mu(d) \prod_{p|d} \left(1 + \frac{1}{p}\right)^{-1} \sum_{K=1}^{\infty} \frac{\mu^2(K)}{K} = +\infty,$$
Remark 5. The main problem, here, is due to the recursion giving recursively defined coefficients, implying the a-priori dependence on variable $n$.

§

Theorem (Hildebrand). Let $F : \mathbb{N} \rightarrow \mathbb{C}$. Then, there exists a finite Ramanujan expansion

$$F(n) = \sum_q \hat{F}(q, n)c_q(n),$$

for, say, each fixed natural $n$, with Ramanujan coefficients $\hat{F}(q, n) \in \mathbb{C}$, eventually depending also on $n$.

(See [ScSp,p.167] giving recursively defined coefficients, implying the a-priori dependence on variable $n$, too.)

Remark 5. The main problem, here, is due to the $n$-dependence of coefficients. The $q$-sum is finite, but again may depend on $n$! However, we also have possible non-uniqueness of coefficients themselves (for $F(n) \equiv 0$, $\hat{F}(q) = \frac{1}{q}$ has $(C)$ above, that is clearly not a finite one, but with $\hat{F}(q) \equiv 0$ it’s trivially finite!).
Just to, say, pump up our mood, good news from 1943 (much before Hildebrand) give us (even if with possible non-uniqueness) also an explicit formula for the (at least one) sequence of coefficients, for, of course, a particular class of a.f.s.

This is the 1943 Wintner Criterion [W] of Aurel Wintner, as adapted in 1976 by Hubert Delange [De] (in this final shape [C] for which we take material from [ScSp] Schwarz-Spieler Book and [De] Delange paper).

See: [Lu] for a 4-lines proof, based on the next Lucht’s Theorem.

If \( \omega(d) \overset{def}{=} \# \{ p \in \mathbb{P} : p \mid d \} \) is the number of prime factors of \( d \), then \( 2^{\omega(d)} = \sum_{\ell \mid d} \mu^2(\ell) \) is the number of its square-free \([T]\) divisors. In the following \( F' \), the Eratosthenes transform (Wintner’s [W] terminology) of a given \( F : \mathbb{N} \rightarrow \mathbb{C} \), is, see \([T]\) for ∗, \( F' \overset{def}{=} F * \mu \); equivalently, by Möbius inversion \([T]\), \( F(n) = \sum_{d \mid n} F'(d) \).

**Theorem (Wintner-Delange Formula).** Let \( F : \mathbb{N} \rightarrow \mathbb{C} \) satisfy Delange Hypothesis, namely

\((DH)\)

\[ \sum_{d=1}^{\infty} \frac{\omega(d)}{d} |F'(d)| < \infty. \]

Then the Ramanujan expansion

\[ \sum_{q=1}^{\infty} \hat{F}(q)c_q(n) \]

converges pointwise to \( F(n), \forall n \in \mathbb{N} \), with coefficients given by the formula

\[ \hat{F}(q) = \sum_{d \equiv 0 \mod q} \frac{F'(d)}{d}, \forall q \in \mathbb{N} \]

(where the series on RHS, right hand side, converges pointwise, \( \forall q \in \mathbb{N} \)) and also by Carmichael \(^2\) formula

\[ \hat{F}(q) = \frac{1}{\varphi(q)} \lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} F(n)c_q(n), \forall q \in \mathbb{N} \]

(where the limit on RHS exists in complex numbers, \( \forall q \in \mathbb{N} \)).

**Proof.** We wish to prove that the following double series, over \( \ell, d \) summations, is absolutely convergent; so, we may write the equation expressing it in two ways (first summing over \( \ell \), then \( d \) and the vice versa):

\[ (*) \sum_{d=1}^{\infty} \sum_{\ell \mid d} \frac{F'(d)}{d} c_\ell(n) = \sum_{\ell=1}^{\infty} \sum_{d \equiv 0 \mod \ell} \frac{F'(d)}{d} c_\ell(n), \forall n \in \mathbb{N}, \]

namely, exchange sums. In fact, \((2)\) gives LHS, left-hand side,

\[ \sum_{d=1}^{\infty} \frac{F'(d)}{d} \sum_{\ell \mid d} c_\ell(n) = \sum_{d \mid n} F'(d) = F(n), \]

with, in RHS, Wintner’s coefficients

\[ (4) \sum_{d \equiv 0 \mod \ell} \frac{F'(d)}{d}, \forall \ell \in \mathbb{N}, \]

thus supplying a proof of our first formula and ensuring pointwise convergence of Ramanujan expansion:

\[ (*) \Rightarrow F(n) = \sum_{\ell=1}^{\infty} \left( \sum_{d \equiv 0 \mod \ell} \frac{F'(d)}{d} \right) c_\ell(n), \forall n \in \mathbb{N}. \]

\(^2\) The name given here is in honor of Carmichael [Ca]: maybe, compare [M, pp.26-27], it’s Wintner’s
Absolute convergence of double series comes from the fact that LHS with moduli, \( \forall d, \ell \in \mathbb{N} \), is bounded by

\[
\sum_{d=1}^{\infty} \frac{|F'(d)|}{d} \sum_{\ell|d} |c_{\ell}(n)| \leq n \sum_{d=1}^{\infty} \frac{|F'(d)|}{d} 2^{\omega(d)} < \infty, \; \forall n \in \mathbb{N},
\]
coming as we know from Delange Hypothesis, starting from the optimal bound, proved by Hubert Delange:

\[
\sum_{\ell|d} |c_{\ell}(n)| \leq n \cdot 2^{\omega(d)},
\]
see [De] (also, for comments about optimality). We prove now: *Carmichael’s coefficients* equal Wintner’s,

\[
(5) \quad \frac{1}{\varphi(q)} \lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} F(n) c_q(n) = \sum_{d \equiv 0 \mod q} \frac{F'(d)}{d},
\]
for which we plug (in LHS), for a large \( K \in \mathbb{N} \), the decomposition:

\[
F(n) = \sum_{d|n, d \leq K} F'(d) + \sum_{d|n, d > K} F'(d)
\]
rendering in the LHS the following (sums exchange is possible because \( F' \) may not depend on \( n \)):

\[
\frac{1}{x} \sum_{n \leq x} F(n) c_q(n) = \sum_{d \leq K} F'(d) \frac{1}{x} \sum_{m \leq x/d} c_q(dm) + \sum_{d > K} F'(d) \frac{1}{x} \sum_{m \leq x/d} c_q(dm),
\]
in which, now, we apply two different treatments, depending on \( d \leq K \) or \( d > K \). For low divisors \( d \), using Landau’s \( O \)--notation (see start of §6) and abbreviating \( \| \alpha \| = \min_{n \in \mathbb{Z}} |\alpha - n| \),

\[
\sum_{d \leq K} F'(d) \frac{1}{x} \sum_{m \leq x/d} c_q(dm) = \sum_{d \leq K} F'(d) \sum_{j \leq q, (j, q) = 1} \frac{1}{x} \sum_{m \leq x/d} c_q(jdm)
\]

\[
= \sum_{d \leq K} F'(d) \sum_{j \leq q, (j, q) = 1} \left( \frac{1}{d} \cdot 1_{d \equiv 0 \mod q} + O \left( \frac{1}{x} \left( 1 + \frac{1_{d \equiv 0 \mod q}}{\|jd/q\|} \right) \right) \right) = \varphi(q) \sum_{d \leq K \atop d \equiv 0 \mod q} \frac{F'(d)}{d} + O(1/x),
\]
from used-a-lot exponential sums cancellations, with a final \( O \)--constant not affecting the \( x \)--decay. While for high divisors \( d \), using Vinogradov’s \( \ll \)--notation (again §6 begin):

\[
\sum_{d > K} F'(d) \frac{1}{x} \sum_{m \leq x/d} c_q(dm) \ll \varphi(q) \sum_{d > K} \frac{|F'(d)|}{d},
\]
uniformly in \( x > 0 \), using the trivial bound \( |c_q(n)| \leq \varphi(q) \), \( \forall n \in \mathbb{Z} \). In all,

\[
\frac{1}{x} \sum_{n \leq x} F(n) c_q(n) = \varphi(q) \sum_{d \leq K \atop d \equiv 0 \mod q} \frac{F'(d)}{d} + O(1/x) + O\left( \varphi(q) \sum_{d > K} \frac{|F'(d)|}{d} \right),
\]
entailing

\[
\frac{1}{\varphi(q)} \lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} F(n) c_q(n) = \sum_{d \equiv 0 \mod q} \frac{F'(d)}{d} + O\left( \sum_{d > K} \frac{|F'(d)|}{d} \right),
\]
actually, giving the required equation, since from Delange Hypothesis the series \( \sum_{d=1}^{\infty} \frac{|F'(d)|}{d} \) converges, so errors in \( O \) are infinitesimal with \( K \), an arbitrarily large natural number (also, present LHS doesn’t depend on it!). Last but not least, we also get the convergence in RHS of these, say, \( d \leq K \)--coeff.s (as \( K \to \infty \)).
Remark 6. The non-uniqueness of coefficients is again a problem: \(F(n) \equiv 0\) satisfies Delange Hypothesis (even (WA), following), but, still, we have (see the above) both \(\hat{F}(q) = 1/q\) and \(\hat{F}(q) \equiv 0\) (by the way, so we have \(\infty^1\) choices, since any linear combination, with complex constants, gives a candidate sequence). See, in this case this last null-sequence is the one supplied by the Wintner-Delange Formula.

Remark 7. In fact, Wintner-Delange Formula above gives full concordance of Wintner’s (\(q\)-th) coefficients (4) with Carmichael’s (\(q\)-th) ones (5). These two kind of coefficients, in case series converge (\(\forall q \in \mathbb{N}\)) or limit \(\exists\) in \(\mathbb{C}\) (\(\forall q \in \mathbb{N}\)), may be defined for any a.f. \(F\) : as we do “immediately”, in next section.

We wish to close this small parade of results (only three!), with a very easy and non-technical one: we now prove it very quickly in three lines (sorry, Professor Lucht, I have to adapt to my notation!). We are quoting Theorem 3.1 [Lu].

**Theorem (Lucht).** Let \(\hat{F}: \mathbb{N} \to \mathbb{C}\) be such that

\[
d \sum_{K=1}^{\infty} \hat{F}(dK)\mu(K)
\]

converges \(\forall d \in \mathbb{N}\). Then

\[
\sum_{q=1}^{\infty} \hat{F}(q)c_q(a) = \sum_{d|a} d \sum_{K=1}^{\infty} \hat{F}(dK)\mu(K)
\]

converges \(\forall a \in \mathbb{N}\), to that function of \(a\).

**Proof.** Letting \(x \to \infty\) may assume \(x \geq a\) : apply 2nd equation of Lemma 1,

\[
\sum_{q \leq x} \hat{F}(q)c_q(a) = \sum_{d|a} d \sum_{\substack{q \leq x \mod d \leq d\ \hat{F}(q)\mu(q/d)}} = \sum_{d|a} d \sum_{\substack{K \leq x/d\ \hat{F}(dK)\mu(K)}}
\]

so the convergence of RHS (we’re assuming) implies LHS convergence. \(\square\)

Remark 8. In fact, whenever convergence problems are ignored, (but compare Lemma 3 Proof in \(\S 5\))

\[
F'(d) = d \sum_{K=1}^{\infty} \hat{F}(dK)\mu(K),
\]

so that : the Ramanujan expansion above is the one of \(\sum_{d|a} F'(d) = F(a)\) ! We’ll see again this formula in the context \(\S 4\) of finite Ramanujan expansions, for which all convergence issues, of course, may be ignored.

3. **Subtleties, shortcuts and open questions for Ramanujan expansions.**

We define *Wintner’s coefficients* (i.e. (4) above) for “any”, say, \(F: \mathbb{N} \to \mathbb{C}\), but with the property that all the following series converge \(\forall q \in \mathbb{N}\):

\[
\text{Win}_q(F) \overset{\text{def}}{=} \sum_{d=1}^{\infty} \frac{F'(d)}{d}, \quad \forall q \in \mathbb{N}.
\]

Actually, the *Wintner Assumption*

\[
(WA) \quad \sum_{d=1}^{\infty} \frac{|F'(d)|}{d} < \infty,
\]

when satisfied by \(F\), ensures at least the existence of all these coefficients and their coincidence with Carmichael’s coefficients, (5), that we now recall. This was discovered by Aurel Wintner : now it is part of the proof for above “Wintner-Delange formula”. (Last part of proof above, in fact, only assuming (WA) derives the existence and the concordance of both coefficients.)
We define \textit{Carmichael’s coefficients} (see (5) above) for “any”, say, \( F : \mathbb{N} \to \mathbb{C} \), but with the property that all the following limits exist, in \( \mathbb{C}, \forall q \in \mathbb{N} \):
\[
\text{Car}_q(F) \overset{\text{def}}{=} \frac{1}{\varphi(q)} \lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} F(n) c_q(n), \quad \forall q \in \mathbb{N}.
\]

It may seem that Wintner-Delange Formula gives a way to obtain the unique Ramanujan coefficients, but this is not true! To convince yourself of this, simply add to the Wintner’s coefficients (or Carmichael’s, which are the same in Delange Hypothesis) the ones of constant-0 function; by the way, we define it in:
\[
0(n) \overset{\text{def}}{=} 0, \forall n \in \mathbb{N} \quad \Rightarrow \quad \text{Win}_q(0) = \text{Car}_q(0) = 0(q), \quad \forall q \in \mathbb{N} \quad (\text{i.e., } \text{Win}(0) = \text{Car}(0) = 0),
\]
for which we have \( \hat{0}_{\text{Ram}}(q) \overset{\text{def}}{=} 1/q \), as we saw, as possible option. Then, given \( F \) with Carmichael’s coefficients \( \text{Car}_q(F) \), applying Wintner-Delange Formula we get (the same is true with Wintner’s coefficients of course)
\[
\hat{F}(q) = \text{Car}_q(F) + \frac{\lambda}{q}, \quad \forall q \in \mathbb{N},
\]
for all complex numbers \( \lambda \) we like!

\textbf{Remark 9.} Also L. Lucht, who proved in an elementary fashion (\( C \)) above and the divisor function \( d(a) \)--expansion [R] by his Theorem [Lu] above, in [Lu] discusses non-uniqueness of Ramanujan coefficients.

\textbf{Remark 10.} In other words, (first of subtleties) the application \( \overset{\sim}{\text{,}} \) sending \( F \) to \( \hat{F} \), is not well-defined.

Defining the \textit{Ramanujan Cloud} of a given \( F : \mathbb{N} \to \mathbb{C} \), notation \( \overline{\mathcal{F}} \), is inspired by \( F = 0 \), see (7) above :
\[
\overline{\mathcal{F}} \overset{\text{def}}{=} \left\{ \hat{F} : \mathbb{N} \to \mathbb{C} \mid \forall n, \sum_{q=1}^{\infty} \hat{F}(q)c_q(n) \text{ converges pointwise to } F(n) \right\},
\]
always non-empty (Hildebrand’s Theorem). Since Hardy in 1921 [H] found also another expansion of \( 0 \), say, \( \hat{0}_{\text{Har}}(q) \overset{\text{def}}{=} 1/\varphi(q) \in \overline{\mathcal{F}} \), this contains an \( \infty^2 \) subset: the plane \( \overline{\mathcal{F}}_{\text{plane}} \overset{\text{def}}{=} \{ \alpha \hat{0}_{\text{Ram}} + \beta \hat{0}_{\text{Har}} \mid \alpha, \beta \in \mathbb{C} \} \)

Given any \( F : \mathbb{N} \to \mathbb{C} \), a so to speak non-trivial property is that for any, fixed sequence of Ramanujan coefficients, say, \( \hat{F}_0 \overset{\text{def}}{=} \{ \hat{F}_0(q) \}_{q \in \mathbb{N}} \), the cloud of \( F \) contains an important two-dimensional subset
\[
\hat{F}_0 + \overline{\mathcal{F}}_{\text{plane}} \subseteq \overline{\mathcal{F}}
\]
as we saw before. (From Wintner-Delange, may choose \( \hat{F}_0 = \text{Car}(F) = \text{Win}(F) \).) In particular, \( \overline{\mathcal{F}}_{\text{plane}} \not\subseteq \overline{\mathcal{F}} \).

\textbf{Remark 11.} Going to open questions (this is 1st): are these sets the same? Better asked, what are the hypotheses on \( F \) to ensure these two sets coincide? (Once known \( 0 \), all clouds are, say, \( \overline{\mathcal{F}} = \hat{F}_0 + \overline{\mathcal{F}} \) !)

\textbf{Countless Remark.} In Ramanujan clouds: \textit{given any two “drops”,} \( \hat{F}_1, \hat{F}_2 \), in the same cloud, \( \overline{\mathcal{F}} \) (of a fixed \( F : \mathbb{N} \to \mathbb{C} \)), all the complex line through them, \( \{ \lambda \hat{F}_1 + (1-\lambda) \hat{F}_2 : \lambda \in \mathbb{C} \} \), is contained in the same cloud. Beautiful banner: any two drops see each other in the Ramanujan clouds!

For a different approach, based not on hypotheses on the \( F \), but on the Ramanujan expansion, we may ask: is there a sufficient condition, ensuring UNIQUENESS of Ramanujan coefficients? (We understand, from what seen above, that uniqueness, in the solely hypothesis of pointwise convergence, can’t be required)

Here of course (for me, joking) we DO HAVE an answer and is linked to the following definitions.

First of all, let’s define what’s a pure R.e.:
\[
F(n) = \sum_{q=1}^{\infty} \hat{F}(q)c_q(n) \text{ is pure } \iff \text{ all } \hat{F}(q) \text{ and their supports don’t depend on } n.
\]
For example, Hildebrand finite R.e.s are not necessarily pure, but 0 has a full plane (see above) of pure R.e.s!

From it, we build the definition we need: (hereafter, uniformly convergent ⇔ converges uniformly in \( \mathbb{N} \))

\[
F(n) = \sum_{q=1}^{\infty} \hat{F}(q) c_q(n) \text{ is a completely uniform R.e.} \quad \overset{def}{\iff} \text{it’s pure & uniformly convergent}
\]

that is a kind of strengthening the uniform convergence concept.

**Remark 12.** Taking (back to subtleties) the sum of a not-pure Hildebrand finite R.e. and a uniformly convergent R.e., we get something which is NOT completely uniform!

**Remark 13.** (Back to) Open questions (2nd & 3rd): can you prove that not-pure Hildebrand finite R.e. are all not uniformly convergent? (See that dependence on \( n \), from non-purity, of course may affect uniform convergence, that’s w.r.t. \( n \) itself!) If not, may you show a uniformly convergent f.R.e. which is not pure?

See that, luckily enough (for me, joking) you have to try to answer to open questions not answered here!

Now, by means of next result (a handmade of ours, Lemma A.4 in [CM2] Appendix) we can answer:

The completely uniform Ramanujan expansions enjoy uniqueness: have unique Ramanujan coefficients.

**Remark 14.** A kind of motto: Completely Uniform \( \implies \text{Unique!} \)

This is proved in quoted Lemma, that we reproduce here, in original formulation, from the arxiv.

**Lemma 2.** Let \( F : \mathbb{N} \to \mathbb{C} \) have an uniformly convergent Ramanujan expansion, i.e.

\[
F(h) = \sum_{q=1}^{\infty} \hat{F}(q) c_q(h), \quad \forall h \in \mathbb{N},
\]

with some coefficients \( \hat{F}(q) \in \mathbb{C} \) independent of \( h \) (even in their support). Then, these are

\[
(CF') \quad \hat{F}(\ell) = \frac{1}{\varphi(\ell)} \lim_{x \to \infty} \frac{1}{x} \sum_{h \leq x} F(h) c_\ell(h).
\]

**Remark 15.** We call the above “Carmichael’s formula”, for Ramanujan coefficients [Ca], [M].

**Proof.** Fix \( \ell \in \mathbb{N} \) and, by uniform convergence, we have \( \forall \varepsilon > 0 \exists Q = Q(\varepsilon, \ell) \), with \( Q > \ell \) and (see soon before Remark 17 for the \( d(\ell) \) definition)

\[
\left\| \sum_{q > Q} \hat{F}(q) c_q(h) \right\| < \frac{\varepsilon}{d(\ell)},
\]

entailing

\[
\frac{1}{x} \sum_{h \leq x} F(h) c_\ell(h) = \sum_{q \leq Q} \hat{F}(q) \frac{1}{x} \sum_{h \leq x} c_\ell(h) c_q(h) + \frac{1}{x} \sum_{h \leq x} c_\ell(h) \sum_{q > Q} \hat{F}(q) c_q(h)
\]

(notice purity allows sums exchange) implies (“lim”, here, abbreviating “\( \lim \)”) \( x \to \infty \)

\[
\left| \frac{1}{\varphi(\ell)} \lim_{x \to \infty} \frac{1}{x} \sum_{h \leq x} F(h) c_\ell(h) - \frac{1}{\varphi(\ell)} \sum_{q \leq Q} \hat{F}(q) \lim_{x \to \infty} \frac{1}{x} \sum_{h \leq x} c_\ell(h) c_q(h) \right| \leq \frac{\varepsilon}{\varphi(\ell) d(\ell)} \lim_{x \to \infty} \frac{1}{x} \sum_{h \leq x} (\ell, h),
\]

from \( |c_\ell(h)| \leq (\ell, h) \), see Lemma A.1 [CM2], whence the orthogonality relations (see [M])

\[
\lim_{x \to \infty} \frac{1}{x} \sum_{h \leq x} c_\ell(h) c_q(h) = \sum_{j \in \mathbb{Z}_q^*} \sum_{r \in \mathbb{Z}_q^*} \lim_{x \to \infty} \frac{1}{x} \sum_{h \leq x} e^{2\pi i (j/t - r/q)h} = \sum_{j \in \mathbb{Z}_q^*} \sum_{r \in \mathbb{Z}_q^*} 1_{q=t} 1_{r=j} 1_{q=t} \varphi(\ell)
\]
and the formula
\[
\frac{1}{x} \sum_{h \leq x} (\ell, h) = \sum_{\ell | t} \frac{t}{x} \sum_{h \leq \frac{x}{t}} 1 = \sum_{\ell | t} \frac{t}{x} \sum_{d | t} \mu(d) \left[ \frac{x}{d} \right] = \sum_{\ell | t} \sum_{d | t} \frac{\mu(d)}{d} + O \left( \frac{1}{x} \sum_{\ell | t} td(\ell/t) \right)
\]
= \sum_{\ell | t} \frac{\varphi(\ell/t)}{\ell/t} + o(1) = \sum_{d | t} \frac{\varphi(d)}{d} + o(1)

(flip to the complementary divisor \(d = \ell/t\), as \(x \to \infty\), together give
\[
\left| \frac{1}{\varphi(\ell)} \lim_{x \to \infty} \frac{1}{x} \sum_{h \leq x} F(h)c_\ell(h) - \hat{F}(\ell) \right| \leq \frac{\varepsilon}{\varphi(\ell)d(\ell)} \sum_{d | \ell} \frac{\varphi(d)}{d} \leq \frac{\varepsilon}{\varphi(\ell)} \leq \varepsilon,
\]
which, as \(\varepsilon > 0\) is arbitrary, shows \((CF)\).

**Remark 16.** (Back to) Open questions (4th & 5th) : Does complete uniformity imply \(\text{Car}(F) = \text{Win}(F)\) ? Also, what are the right hypotheses to get Lemma 2 with \(\text{Win}(F)\) instead of \(\text{Car}(F)\) ?

4. **FINITE RAMANUJAN EXPANSIONS.**

Since this is a map, we point to the “land”, where the **finite Ramanujan expansions** (f.R.e.)

\[(FRE)\]
\[
F(n) = \sum_{q \leq Q} \hat{F}(q)c_q(n), \ \forall n \in \mathbb{N}
\]

are born, i.e., \([CMS], [CM2]\), for the moment (with Ram Murty we expect to supply other papers in the series). So, all things we’ll say in this section, actually, (even if we forget) are all quoted from these(by now) two papers. (Recall that, say, Hildebrand’s f.R.e. will not be studied in this section.)

We start with a V.I.P.:

**ALL PURE f.R.e. ARE TRUNCATED DIVISOR SUMS & VICE VERSA**

and, since very easy both to state & prove, will be a “Proposition”. Recall t.d.s.=truncated divisor sum:

\[
F: \mathbb{N} \to \mathbb{C} \text{ is a t.d.s.} \iff \exists F': \mathbb{N} \to \mathbb{C} \text{ and } \exists Q \in \mathbb{N}, \text{ such that } F(n) = \sum_{d|n, d \leq Q} F'(d), \ \forall n \in \mathbb{N}
\]

where both \(Q\) and \(F'\) DO NOT depend on \(n\), of course. For example \(d(n) \overset{\text{def}}{=} \sum_{d|n} 1\), the divisor function (=number of positive divisors of \(n\), seen in Lemma 2 proof), is NOT a truncated divisor sum (not only above representation has to be \(n\)-independent, as stated, but also true \(\forall n \in \mathbb{N}: \text{otherwise, } n \leq Q \Rightarrow d \leq Q \) !)

**Remark 17.** The number \(Q \in \mathbb{N}\) is not unique, since we may choose, say, \(F'(Q) = 0\). Thus, in order to define precisely the range of a t.d.s., we have to assume \(F'(Q) \neq 0\). However, “\(F\) is of range \(Q\)” , in our jargon, may also mean: divisors are \(d \leq Q\), namely \(F'(q) = 0\), \(\forall q > Q\), and \(Q\) may not be uniquely defined !

The following Proposition will keep out of our study, actually, all the f.R.e. that are not pure : sorry for the beautiful Theorem by Hildebrand ! In fact, we know that his f.R.e. are not necessarily pure, because they are valid \(\forall F: \mathbb{N} \to \mathbb{C}\), while the t.d.s. are “a drop in this ocean”, of all a.f.s. !

**Remark 18.** Notice that above we expressed \(F(n)\) in TWO WAYS. If \(F\) is a t.d.s, through its Eratosthenes Transform \(F'\) (defined by Wintner [W] as above for all \(F\)) and through its **finite Ramanujan coefficients** \(\hat{F}\), whenever it’s a pure f.R.e. ! This duality \(F' \leftrightarrow \hat{F}\) is perfect, whenever \(F\) is AT THE SAME TIME a t.d.s. and a pure f.R.e. : this is proved by next Proposition, that renders crystal clear the duality by formulae, for the stated link !
We are ready to prove (in half page) the following.

**Proposition 1.** Take any $F : \mathbb{N} \to \mathbb{C}$. Then

$F$ is a t.d.s. $\iff$ $F$ has a pure f.R.e.

**Proof.** For $\Rightarrow$, use (2):

$$F(n) = \sum_{d|n, d \leq Q} F'(d) = \sum_{d \leq Q} \frac{F'(d)}{d} \sum_{q|d} c_q(n) = \sum_{q \leq Q} \sum_{d \leq Q \overline{d \equiv 0 \text{ mod } q}} \frac{F'(d)}{d} c_q(n) = \sum_{q \leq Q} \hat{F}(q)c_q(n),$$

for the, say, finite Ramanujan coefficients (or $Q$-truncated R.c.s)

$$\hat{F}(q) \overset{\text{def}}{=} \sum_{d \leq Q \overline{d \equiv 0 \text{ mod } q}} \frac{F'(d)}{d} \quad (\Rightarrow q \leq Q, \text{ otherwise } \hat{F}(q) = 0).$$

Vice versa, for $\Leftarrow$, use Lemma 1 second equation:

$$F(n) = \sum_{q \leq Q} \hat{F}(q)c_q(n) = \sum_{d|n} \sum_{q \leq Q \overline{q \equiv 0 \text{ mod } d}} \hat{F}(q)\mu(q/d) = \sum_{d|n} \sum_{K \leq Q/d} \hat{F}(dK)\mu(K) = \sum_{d|n, d \leq Q} F'(d),$$

for the, say, $Q$-truncated Eratosthenes transform

$$F'(d) \overset{\text{def}}{=} \sum_{K \leq Q/d} \hat{F}(dK)\mu(K) \quad (\Rightarrow d \leq Q, \text{ otherwise } F'(d) = 0).$$

Notice similarity with (6) : actually, compare Remark 8 !

**Remark 19.** From the begin to the end, the proof assumes the “$n$–independence”, for t.d.s. & for pure f.R.e. (depending on $n$, resp., ONLY in the “$d|n$” & ONLY in the Ramanujan sum $c_q(n)$.) In the following, this kind of independence will be implicit for t.d.s. & pure f.R.e. ! (Compare the “fair”, a concept in §6 !)

Very good news for uniqueness of R.e.s, in following result. (Compare the motto of Remark 14.)

**Corollary 1.** Completely uniform R.e. have unique coefficients. In particular, pure f.R.e. enjoy uniqueness.

**Remark 20.** Now we will not prove this Corollary 1, since it follows both from Lemma 2 in §3 and from previous Proposition 1 ! (The interested reader may prove its second part by Carmichael formula, i.e., following step-by-step Lemma 2 Proof in the finite part : the $q \leq Q$ sum.)

Operatively speaking, Corollary 1 tells us to apply the Carmichael Formula to pure f.R.e. (as our most powerful tool). This answers one possible question: why do we care to write t.d.s. as finite Ramanujan expansions ? (We’ll also answer “on the field”, when coming to shift-R.e. in §6.)

Now, see that equation $(\hat{F})$ in Proposition 1 Proof defines $\hat{F}$ in terms of $F'$ and vice versa, in $(F')$, $F'$ is defined in terms of $\hat{F}$ (compare [CM2] formulæ). These, say, transformation formulæ, are V.I.P. !

We give a property of finite Ramanujan coefficients: since $F'$ vanishes after $Q$, in the, say, $Q$–truncated $F_Q(n) = \sum_{d|n, d \leq Q} F'(d)$, of a given arithmetic function $F = F' * 1$, then $\hat{F}_Q = 0$ after $Q$, too and (from above $(\hat{F})$, compare [CM2, §3])

$$\left(\frac{Q}{2} < q \leq Q \implies \hat{F}_Q(q) = \frac{F'(q)}{q}.\right)$$

We quote, from [CM2], this property $(H)$, since it is a V.I.P., for finite Ramanujan expansions: it distinguishes finite R.e. from, say, classical R.e. ! In fact, the operation of truncating the Eratosthenes transform $F'$ (by previous formulæ, this is equivalent to truncating R.e.s), so to speak, reflects on Ramanujan coefficients in the “change”, say, of last R.c.s, that we’ll call “High coefficients”. (A way to say with high indices, here.)
While $F$ always has R.c.s (Hildebrand’s Theorem) and these, in general, will not vanish definitely (think about $\hat{0}(q) = 1/q$, esp.), the say $Q$–truncated R.c.s all vanish after $Q$; so they have in some sense to cope with the original expansion (of $F$, not of its $Q$–truncated counterpart) and this “forces”, so to speak, the high coefficients to rearrange and recover the final expansion (now, a finite one!), substituting all of the “missing R.c.s, i.e. the tail”, with $q > Q$. The property above is V.I.P. just for this reason, but also because a simple explicit formula is given for high R.c.s (meaning with $Q/2 < q \leq Q$ now on), in finite expansions.

Another V.I.P. for R.c.s, which is now a heuristic property, is the following:

$$(L) \quad q \leq Q_0, \text{ with } Q_0 \text{ “small”, } \text{w.r.t. } Q \implies \hat{F}_Q(q) \sim \hat{F}(q),$$

where the $F_Q(n) \overset{def}{=} \sum_{q|n,q \leq Q} F^\prime(q)$ is the $Q$–truncated counterpart of our $F = F^\prime \ast 1$ (seen above) and, as $Q \to \infty$, the new parameter $Q_0 = o(Q)$ is suitably small (esp., think about $Q_0 = \sqrt{Q}$ : say, here these “Low coefficients”, in (L), are asymptotic to classic ones (for $F$). Compare [C] for an application to $F = \Lambda$, von Mangoldt function (for primes)! Also, more in general, for truncations of a.f., see §4, [CM2].

The main question, after knowing the behavior of low & high finite R.c.s, is of course: what about intermediate ones ? This being a very difficult question (so, we don’t even list it explicitly as an o.q.) !

Address, once again, for all other f.R.e. features is in [CMS] & [CM2].

5. Partial (and short) answers to above open questions for Ramanujan expansions.

First open question (we abbreviate o.q. now on) is in Remark 11 and of course the real question is on $\hat{0}$ ! In fact, given $F$ and two sequences $\hat{F}_1 \neq \hat{F}_2$ of its Ramanujan coefficients, then, $\hat{F}_1 - \hat{F}_2 \neq 0$ of course represents $0$, so it’s in $\hat{0}$ . Can we describe this explicitly ? (I.e., give all the Ramanujan coefficients of constant 0 function ?) The answer, my friend (maybe is blowing in the wind...), I confess, I’d like to find ! (And the literature, for this, seems of little use!)

Second & third o.q.s regard (see Remark 13) not-pure Hildebrand finite R.e. and uniform convergence in $N$, recall. (Shortly, 2nd: no & 3rd: $\hat{F}(q,n) := \sin([n/q])/q^3 \Rightarrow$ total convergence from trivial $|c_q(n)| \leq \varphi(q)$.)

Recall fourth & fifth o.q.s in Remark 16. We first give answer “yes”, to 5th.

We call the following Lemma the Wintner “uniqueness” formula.

**Lemma 3.** Let $F: \mathbb{N} \to \mathbb{C}$ have a pure R.e.

$$F(a) = \sum_{q=1}^{\infty} \hat{F}(q)c_q(a), \quad \forall a \in \mathbb{N}.$$  

Then

$$F^\prime(d) = d \sum_{K=1}^{\infty} \mu(K) \hat{F}(dK), \quad \forall d \in \mathbb{N}.$$  

Furthermore, if these Ramanujan coefficients also satisfy the, say, “Dual Delange” condition

$$(8) \quad \sum_{n=1}^{\infty} 2^{\omega(n)} \left| \hat{F}(n) \right| < \infty,$$

then

$$\hat{F}(q) = \sum_{d \equiv 0 \mod q} \frac{F^\prime(d)}{d}, \quad \forall q \in \mathbb{N}.$$  

**Remark 21.** Of course, previous $d$–series converge for all $q \in \mathbb{N}$.  

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**Proof.** We start proving (6), inspired by Lucht Theorem, from

(9) \[ F(n) = \sum_{d|n} d \sum_{K=1}^{\infty} \mu(K) \hat{F}(dK), \]

thanks to Lemma 1, third eq.; then the purity, assumed, implies (6), i.e.

\[ F'(d) = d \sum_{K=1}^{\infty} \mu(K) \hat{F}(dK), \]

because: RHS of (6) inside RHS of (9) depends only on \(d\) (not on \(n\) !), so, by Möbius inversion [T], it’s \(F'(d)\).

On same lines of (Delange-style) proof of Wintner-Delange formula, we calculate Wintner’s coefficients:

\[ \text{Win}_q(F) = \sum_{d=1}^{\infty} \sum_{K=1}^{\infty} \mu(K) \hat{F}(qmK) = \sum_{K=1}^{\infty} \sum_{m=1}^{\infty} \mu(K) \hat{F}(qmK) = \sum_{K=1}^{\infty} \sum_{n=1}^{\infty} \mu(K) \hat{F}(qn), \]

exchanging \(m, K\) and, then, applying first eq. in (3), a kind of Möbius inversion, to get

(10) \[ \text{Win}_q(F) = \sum_{n=1}^{\infty} \hat{F}(qn) \sum_{K|n} \mu(K) = \hat{F}(q), \]

in which, say, we are again exchanging series; this is possible, from absolute convergence of double series (put moduli in (10) double series), we obtain using now \(\omega(qn) \geq \omega(n) \Rightarrow 2^{\omega(qn)} \geq 2^{\omega(n)}\), uniformly in \(q \in \mathbb{N}\):

\[ \sum_{n=1}^{\infty} |\hat{F}(qn)| \sum_{K|n} \mu^2(K) = \sum_{n=1}^{\infty} 2^{\omega(n)} |\hat{F}(qn)| \leq \sum_{n=1}^{\infty} 2^{\omega(m)} |\hat{F}(qn)| = \sum_{m=1}^{\infty} 2^{\omega(m)} |\hat{F}(m)| \leq \sum_{m=1}^{\infty} 2^{\omega(m)} |\hat{F}(m)|, \]

this last converging, from our hypothesis (8) above.

\[ \square \]

**Remark 22.** Everything points in the direction:

UNIQUENESS OF R.E.(F) \(\implies\) Car(F) = Win(F)

but, of course, the difficult part is to prove \(\exists\) for both Car(F) & Win(F), before!

**Remark 23.** We leave as an exercise the detailed proof of (10) under hypothesis of complete uniformity (answer to 4th o.q.).

6. **ODDS AND ENDS (FEATUREING SHIFT-RAMANUJAN EXPANSIONS).**

We begin with odds & ends, entering the “realm”, of shift-Ramanujan expansions. A realm started “upon a time” (few months) ago [CM2]. Thus genealogy is there, especially (see (CC) next) Theorem 1, Corollary 1.

First, we say \(f : \mathbb{N} \to \mathbb{C}\) satisfies the Ramanujan Conjecture or, equivalently, it’s *essentially bounded*, written \(f \ll 1\), by definition if \(f(n) \ll n^\varepsilon\), as \(n \to \infty\). Hereafter \(\ll\) is the Vinogradov notation, equivalent to Landau notation, in \(f(n) \ll g(n)\), equivalently \(f(n) = O(g(n))\), meaning that, for a certain \(n_0 \in \mathbb{N}\), \(|f(n)| \leq C g(n)\), for all \(n > n_0\) (and, of course, \(g(n) \geq 0\) at least \(\forall n > n_0\)). The constant \(C > 0\) (named the “implicit constant”) may depend on other variables, in which case they are displayed as subscripts. In our case, for \(f \ll 1\), \(C = C(\varepsilon)\) depends on \(\varepsilon > 0\), which is arbitrarily small in ANT (Analytic Number Theory). (Notice our “modified Vinogradov notation”, going back at least to Kolesnik in the 50s, doesn’t display the \(\varepsilon\)—dependence explicitly.) All of this is standard in ANT, compare [Da] & [T].

Given two arbitrary arithmetic functions \(f, g : \mathbb{N} \to \mathbb{C}\), their correlation (or shifted convolution sum) is:

\[ C_{f,g}(N, a) \overset{def}{=} \sum_{n \leq N} f(n)g(n + a), \]
which, in turn, is itself an arithmetic function of \( a \in \mathbb{N} \), called the shift. (The \( N \in \mathbb{N} \) is the length of \( C_{f,g} \).) Hence, if we expand in terms of \( a \), we get a kind of new R.e., say, the shift-Ramanujan expansion (of \( C_{f,g} \))

\[
C_{f,g}(N,a) = \sum_{\ell=1}^{\infty} C_{f,g}(N,\ell)c_\ell(a), \quad \forall a \in \mathbb{N},
\]

thanks, again, to Hildebrand’s Theorem!

Before going on, see that we may also expand the single \( f \) and \( g \) inside it. However, a big surprise came to us [CMS] when we discovered that, however we choose arbitrarily in \( (C11) \), at \( \varepsilon \)

which, in turn, is itself an arithmetic function of \( N \), the example, in [CM2] the function \( f \) arithmetic functions \( \forall \)

implying \( f \) and \( g \) (by Möbius inversion, see [T]) and observed that \( d|n, n \leq N \Rightarrow d \leq N \)!

As we see, this trick turns \( f \) and \( g \) into t.d.s., whence (see Proposition 1) we get the finite R.e.s for our arithmetic functions \( f \) and \( g \) (Proposition 1 for formulae), inside \( C_{f,g} \):

\[
C_{f,g}(N,a) = \sum_{d \leq N} \sum_{q|n+a} f'(d) \sum_{q|n+a} g'(q) = \sum_{d \leq N} \sum_{q|n+a} f'(d)g'(q) \quad \sum_{q|n+a, q \leq N+a} 1 = \sum_{d \leq N} \sum_{q|n+a, q \leq N+a} f'(d)g'(q) = \sum_{n=0}^{\infty} \sum_{n+a \equiv 0 \mod d} 1
\]

once written \( f(n) = \sum_{d|n} f'(d) \) and \( g(m) = \sum_{q|m} g'(q) \) in terms of their Eratosthenes transforms, say, \( f' \equiv f \ast \mu \) and \( g' \equiv g \ast \mu \) (by Möbius inversion, see [T])

The point, here, is that in fact these two single R.e.s may help in finding the shift-Ramanujan coefficients \( C_{\hat{f},\hat{g}}(N,\ell) \). If we may, say, exchange sums applying (CF), then orthogonality “reveals” them!

After next definition, we need for this, we see how.

We define a correlation “fair”, if the shift-dependence (the way it depends on \( a \)) is only inside the \( g \) argument, i.e., \( n + a \); there’s no other dependence on \( a \), neither in \( f \) and its support, nor in \( g \) and its support. (For example, in [CM2] the function \( f_H \) given at last in the Appendix has \( C_{f_H,f_H} \) which is not fair, since \( a \leq H \) implies \( f_H \) itself depends on \( a \), better, its support does!)

Soon after defining this, to use it, first we have to “clean up”, say, the range of \( g : \) it depends on \( a \), see the above! For this, we use the simple idea of “cutting”, with a small remainder (details in (1) of [C]):

\[
C_{f,g}(N,a) = \sum_{n=0}^{\infty} \sum_{d|n, d \leq N} f'(d) \sum_{q|n+a} g'(q) = \sum_{n=0}^{\infty} \sum_{d|n, d \leq N} f'(d) \sum_{q|n+a, q \leq N} g'(q) + O_{\varepsilon}(N^{\varepsilon}(N+a)^{\varepsilon})
\]

whenever, of course, \( f, g \ll 1 \), i.e., they satisfy Ramanujan Conjecture. Here \( g_N(m) \equiv \sum_{q|m, q \leq N} g'(q) \) is the \( N \)-truncated counterpart of \( g \) ! In our jargon, a t.d.s. of range \( N \). Now, we start using the concept of fair correlation, observing that, equivalently, \( C_{f,g} \) is fair if and only if in the formula (set \( Q = N \& F = g_N \) in (FRE), at \( \S 4 \) begin)

\[
(11) \quad C_{f,g,N}(N,a) = \sum_{q \leq N} \tilde{g}_N(q) \sum_{n \leq N} f(n)c_q(n+a), \quad \forall a \in \mathbb{N}
\]
the only, say, place in which there’s $a$–dependence is the Ramanujan sum $c_q(n + a)$. This is vital to calculate Carmichael’s coefficients (of course, also for their $\exists$), since in (11) we may exchange all the sums, when our (cut-)correlation $C_{f,g,N}(N,a)$ is fair:

\[(CC) \quad \text{Car}_\ell(C_{f,g,N}) = \sum_{q \leq N} \sum_{n \leq N} f(n) \frac{1}{\phi(\ell)} \lim_{x \rightarrow \infty} \frac{1}{x} \sum_{n \leq x} c_q(n + a)c_\ell(a) = \frac{g_N(\ell)}{\phi(\ell)} \sum_{n \leq N} f(n)c_\ell(n), \quad \forall \ell \in \mathbb{N},\]

where we’ve applied the orthogonality of Ramanujan sums (proved first time by Carmichael [Ca] himself, that’s why (CF) bears his name), see [M] for a complete proof:

\[
\lim_{x \rightarrow \infty} \frac{1}{x} \sum_{a \leq x} c_q(n + a)c_\ell(a) = 1_{q=\ell} c_\ell(n), \quad \forall \ell, n \in \mathbb{N}.
\]

There’s a way, now, to get the shift-R.e.s and passes through a remark (many vital remarks give you a vital theory!), say, Ramanujan inheritance property: “$g$ is a t.d.s” is inherited by $C_{f,g,N}$, now becoming a t.d.s. (i.e., shift’s divisors $d|a$ are truncated). Last but not least, we have to assume $(DH)$ for our $C_{f,g,N}$:

\[(***) \quad \sum_d \frac{2^{e(d)}|C_{f,g,N}(N,d)|}{d} < \infty.
\]

In [C] the Theorem and its Corollary hold assuming $(**)$, i.e., Delange Hypothesis.

See that, of course, Hardy-Littlewood Conjecture [HL] is proved conditionally in [C], under $(**).$ Is there a way to generalize further the results in [C] ? Namely, can we go from Delange to Wintner assumption, maybe making an even weaker assumption ? (Maybe in case of shift-R.e.s this is possible, but for general R.e.s maybe a shortcut to $(DH)$ is impossible)

All of the things we’ll say, after this big exposition on s.R.e., were originated (then generalized to R.e.) from our attempts (at present, not useful) to get, from $(CC)$, whence Win$_q(C_{f,g}) = 0, \forall q > N$ (this, under $(WA)$, is proved) that $C'_{f,g}(N,d) = 0, \forall d > N$ (while, instead, THIS is HARD) : from which (i.e., $(iv)$, implying $(iii)$ in Theorem 1 of [CM2]) we get next equation (compare $(iii)$ of [CM2] Th.1) called The Reef!

We wish to derive the “Ramanujan exact explicit formula”, the Reef [C], for the (cut-)correlation:

Reef:

\[C_{f,g,N}(N,a) = \sum_{q \leq N} \left( \frac{g_N(q)}{\phi(q)} \sum_{n \leq N} f(n)c_q(n) \right) c_q(a), \quad \forall a \in \mathbb{N}.
\]

This we proved above, using Delange Hypothesis $(**)$, in the style of our previous paper [C] and now we wish to obtain some weaker results, under weaker assumptions. (Namely, a weaker version of the Reef, say.)

See: $(CC)$ works under the only hypothesis of fair correlation (i.e., $C_{f,g,N}(N,a)$ is fair); the problem is that (compare the following), in general, the Carmichael coefficients can’t, so to speak, be forced to be Ramanujan coefficients (in other words, the $q$–series in $(RE)$ with Carmichael coefficients may, if convergent, not converge to $F(n)$ ! ) and this is our only problem. In fact, thanks to $(CC)$ and the vanishing of $g_N(q)$, after $N$, once given convergence to $C_{f,g,N}(N,a)$ we immediately get the Reef !

This problem (of convergence of R.e. with Carmichael coefficients) is deep and, so to speak, pervasive in the theory of R.e.! (We may say: ALL of convergence problems for R.e. in [ScSp] deal with this problem!) Compare [Lu] discussion. We’ll see, soon after this big exposition for s.R.e., illuminating, general examples.

Thus

\[C_{f,g,N}(N,a) = \sum_{d|a \leq N} C'_{f,g,N}(N,d) + \sum_{d|a > N} C'_{f,g,N}(N,d), \quad \forall a \in \mathbb{N},
\]

splitting at $N$ (also, for $g$ of range $Q$, we split at $Q$), after Möbius inversion [T], from the definition we recall:

\[C_{f,g,N}(N,d) \overset{\text{def}}{=} \sum_{t|d} C_{f,g,N}(N,t)\mu(d/t), \quad \forall d \in \mathbb{N}.
\]
From formulæ, in Proposition 1, we get
\begin{equation}
C_{f,g,N}(N,a) = \sum_{q \leq N} \hat{C}_{f,g,N}(N,N,q) c_q(a) + \sum_{d|a \atop d>N} C'_{f,g,N}(N,d), \ \forall a \in \mathbb{N},
\end{equation}
immediate from the definition:
\begin{equation}
\hat{C}_{f,g,N}(N,Q,q) \overset{\text{def}}{=} \sum_{d \in 0 \mod q \atop d \leq Q} \frac{C'_{f,g,N}(N,d)}{d}, \ \forall q \in \mathbb{N}
\end{equation}
that are, say, a kind of \(Q\)-truncated shift-Ramanujan Coefficients; recall: original ones SHOULD be the same, without truncating \(d \leq Q\), but this entails convergence problems at once! (Again, we are “playing with Wintner-Delange formulæ”.) Philosophically, \((QRC)\), as \(Q \to \infty\), should approximate original “s.R.e.”.

The advantage is clear, because now we may apply \((CF)\) to these coefficients (using Corollary 1, since fairness \(\Rightarrow \hat{C}_{f,g,N}(N,Q,q)\) are pure), getting from \((CC)\) and \((12)\)
\begin{equation}
\hat{C}_{f,g,N}(N,N,q) = \frac{\hat{g}_N(q)}{\varphi(q)} \sum_{n \leq N} f(n)c_q(n) - \frac{1}{\varphi(q)} \lim_{x \to \infty} \frac{1}{x} \sum_{m \leq x} c_q(m) \sum_{\substack{d|m \atop d>N}} C'_{f,g,N}(N,d), \ \forall q \in \mathbb{N}
\end{equation}
See that for \(q > N\) they vanish, by definition \((QRC)\) above, with \(Q = N\). Notice the presence, even in this “\(d \leq Q\)-part”, of divisors \(d > Q\) (set \(Q = N\) here)! Let us abbreviate these limits as
\begin{equation}
L(q) = L_{f,g}(q,N) \overset{\text{def}}{=} \frac{1}{\varphi(q)} \lim_{x \to \infty} \frac{1}{x} \sum_{m \leq x} c_q(m) \sum_{\substack{d|m \atop d>N}} C'_{f,g,N}(N,d), \ \forall q \in \mathbb{N}
\end{equation}
(not an \(L\)-function! \(L\) stands for limit, of course!), where \(\exists L(q) \in \mathbb{C}, \forall q \in \mathbb{N}\) and \(L(q) = 0\) after \(N\), too.

So, with the only hypothesis “\(C_{f,g,N}(N,a)\) is fair”:

**Weak Reef:** \(C_{f,g,N}(N,a) = \sum_{q \leq N} \left( \frac{\hat{g}_N(q)}{\varphi(q)} \sum_{n \leq N} f(n)c_q(n) - L(q) \right) c_q(a) + \sum_{d|a \atop d>N} C'_{f,g,N}(N,d), \ \forall a \in \mathbb{N}.\)

This is a very useful formula, when taking \(a\)-averages, say “short ones”, here \(a \leq A\) with \(A \leq N\): as a gift,
\begin{equation}
\sum_{a \leq A} C_{f,g,N}(N,a) = \sum_{q \leq N} \left( \frac{\hat{g}_N(q)}{\varphi(q)} \sum_{n \leq N} f(n)c_q(n) - L(q) \right) \sum_{a \leq A} c_q(a), \ \forall A \leq N.
\end{equation}

We address the reader, now, to other papers of ours (also future ones), for the s.R.e. and their applications to averages of correlations \([CLa]\).

We come to general R.e., no more ON s.R.e. (However, FROM s.R.e., as we say soon before the Reef.) We saw in s.R.e. the question if it’s true that Carmichael coefficients may be taken as Ramanujan coefficients (too general, a complete answer would supersede [ScSp] aim and exposition). In order to give a partial answer, we are inspired by [ScSp] in following the “classic mean value”, for a general \(F: \mathbb{N} \to \mathbb{C}\), that might even not exist: it is **simply** \(\text{Car}_1(F)\), in our notation. It influences, so to speak, all other Carmichael coefficients (as [ScSp] hints) and, of course, in case \(F \geq 0\), this is evident: in our next Lemma.

**Lemma 4.** Let \(F(n) \geq 0, \forall n \in \mathbb{N}\) and assume \(\text{Car}_1(F) = 0\). Then \(\text{Car}(F) = 0\), i.e., \(\text{Car}_q(F) = 0\), \(\forall q \in \mathbb{N}\).

**Proof.** Simply\[
\frac{1}{\varphi(q)} \frac{1}{x} \sum_{n \leq x} F(n)c_q(n) \leq \frac{1}{x} \sum_{n \leq x} F(n),
\]
from the trivial bound on Ramanujan sum and our hypothesis \(F \geq 0\); let \(x \to \infty\). \(\square\)

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See, our Lemma 4 suggests a link between 1st and subsequent Carmichael coefficients: this has been proved by Hildebrand [Hi] already in 1984, for uniformly-almost-even functions, compare [Lu, Theorem 2.5].

Remark 24. If $F$ satisfies Lemma 4 hypotheses and $F \neq 0$, then, $\text{Car}(F) = \hat{F}$ implies the absurd $F = 0$.

It may seem that $F \geq 0$, not vanishing everywhere and with classic mean-value $0$ is a too strong requirement. Actually, writing the characteristic function of sets $S$ as $1_S$, whenever the set $S \subseteq \mathbb{N}$ has a natural density, i.e. the limit defined as

$$\text{Car}_1(1_S) = \lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} 1_S(n) = \lim_{x \to \infty} \frac{\# \{ n \in S : n \leq x \}}{x} \quad \text{def} \quad \delta(S)$$

exists in $[0,1]$, from Lemma 4 (since $1_S \in \{0,1\}$ is $\geq 0$, $1_S \neq 0$ by $S \neq \emptyset$, has $\text{Car}_1(1_S) = 0$ in the following): V.I.P.: non-empty sets of naturals with natural density $0$ have characteristic function with $\text{Car} = 0$.

Thus many examples are there! (Esp., prime numbers have characteristic function with this property, even by Čebišev bound $\pi(x) \overset{\text{def}}{=} \# \{ p \in \mathbb{P} : p \leq x \} \ll x / \log x$, [T]: already $\pi(x) = o(x)$ suffices for $\text{Car}_1(1_\mathbb{P}) = 0$.)

Remark 25. A noteworthy case is the $S$ of squares: $1_S^2 = \lambda$, the Liouville function [T] and $\text{Win}(1_S^2) = 0$ by the Prime Number Theorem [Lu]. In this case, $|1_S^2| = |\lambda| = 1$, the constant-$1$ function, so the corollary: $1_S^2$ for squares doesn’t satisfy (WA). This $1_S^2 = \lambda$, completely multiplicative, inspires next Lemma 5.

We need to define the completely multiplicative arithmetic functions $f : \mathbb{N} \to \mathbb{C}$ by $f(ab) = f(a)f(b), \forall a,b \in \mathbb{N}$; in fact, next Lemma uses them: abbreviating “c.m.” for completely multiplicative, from (4),

$$F' \text{ is c.m. } \Rightarrow \text{Win}_q(F) = \sum_{d \equiv 0 \mod q} \frac{F'(d)}{d} = \frac{F'(q)}{q} \sum_{m=1}^{\infty} \frac{F'(m)}{m} = \frac{F'(q)}{q} \text{Win}_1(F), \forall q \in \mathbb{N}.$$

Hence, the following Lemma is proved.

Lemma 5. Let $F : \mathbb{N} \to \mathbb{C}$ have $F'$ completely multiplicative. Then

$$\text{Win}_1(F) = 0 \Rightarrow \text{Win}(F) = 0$$

and, assuming $\text{Win}_1(F) \neq 0$ instead,

$$\text{Win}_q(F) = 0, \forall q > Q \Rightarrow F'(q) = 0, \forall q > Q.$$

We’ll not prove following immediate Lemma.

Lemma 6. Let $F : \mathbb{N} \to \mathbb{C}$ have $F' \geq 0$. Then

$$\text{Win}_q(F) = 0, \forall q > Q \Rightarrow F'(q) = 0, \forall q > Q.$$

From these two Lemmas, we are led to formulate:

Conjecture 1. If $F : \mathbb{N} \to \mathbb{C}$ has classic mean-value $\text{Car}_1(F) = \text{Win}_1(F) \neq 0$, then

$$\text{Win}_q(F) = 0, \forall q > Q \Rightarrow F'(q) = 0, \forall q > Q.$$ 

Under this Conjecture we may: assume (WA) instead of (DH) in all the results of our previous paper [C].

In fact, (CC) above gives, assuming $C_{f,\mathbb{N}}$ is fair,

$$\text{Car}_q(C_{f,\mathbb{N}}) = 0, \forall q > N;$$

then, (WA) for it implies $\text{Car}(C_{f,\mathbb{N}}) = \text{Win}(C_{f,\mathbb{N}})$, so Conjecture 1 for $F(a) = C_{f,\mathbb{N}}(N,a)$ entails

$$\text{Win}_q(C_{f,\mathbb{N}}) = 0, \forall q > N \Rightarrow C'_{f,\mathbb{N}}(N,d) = 0, \forall d > N.$$ 

This, say finiteness condition, i.e. condition (iv) in Theorem 1 [CM2], gives the Reef (as (iii) [Th.1,CM2]).
For Ramanujan expansions, it seems that a good sign, for R.c.s, is : $\text{Car}(F) = \text{Win}(F)$.

However, we may have $\text{Car}(F) = \text{Win}(F)$ outside the Ramanujan Cloud of this $F$ (i.e., they are not $F$ Ramanujan coefficients), as we’ll see soon!

In fact, a careful analysis of our bound, linking the Carmichael and the Wintner coefficients in above Wintner-Delange proof, drives us towards our following Theorem, we call the Carmichael-Wintner Formula, for this reason; it has the noteworthy consequence of showing $\text{Car}(F) = \text{Win}(F)$, when the mean-value of $|F'(d)|$ vanishes (i.e., $\text{Car}_q(|F'|) = 0$) : see our next “Slow Decay”, say, condition on $F$. In order to prove this Formula, we need, say, a kind of approximate formula, in next Lemma (we prove in half-page).

**Lemma 7.** Let $F : \mathbb{N} \to \mathbb{C}$ be any arithmetic function. Then, as $x \to \infty$, for all fixed $q \in \mathbb{N}$, we have

\[(CW) \quad \frac{1}{\varphi(q)} \cdot \frac{1}{x} \sum_{n \leq x} F(n)c_q(n) = \sum_{d \leq x \atop d \equiv 0 \text{ mod } q} \frac{F'(d)}{d} + O_q \left( \frac{1}{x} \sum_{d \leq x} |F'(d)| \right).\]

**Remark 26.** The $O$–constant depends ONLY on $q$, here (so $x$–decay isn’t affected).

**Proof.** Let’s write $F(n) = \sum_{d|n} F'(d)$ as usual, to get

\[
\frac{1}{\varphi(q)} \cdot \frac{1}{x} \sum_{n \leq x} F(n)c_q(n) = \frac{1}{x} \sum_{d \leq x} F'(d) \frac{1}{\varphi(q)} \sum_{K \leq \frac{x}{d}} c_q(dK),
\]

where now in more detail, by exponential sums cancellation seen in Wintner-Delange Formula proof,

\[
\frac{1}{\varphi(q)} \sum_{K \leq \frac{x}{d}} c_q(dK) = \frac{1}{\varphi(q)} \sum_{j \in \mathbb{Z}_q^*} \sum_{K \leq \frac{x}{d}} e_q(jdK) = 1_{d \equiv 0 \text{ mod } q} \left[ \frac{x}{d} \right] + 1_{d \not\equiv 0 \text{ mod } q} O \left( \frac{1}{\varphi(q)} \sum_{j \in \mathbb{Z}_q^*} \frac{1}{\|jd\|} \right),
\]

where the $O(1)$ is an absolute constant (coming from : fractional parts are bounded), so this is

\[
= 1_{d \equiv 0 \text{ mod } q} \frac{x}{d} + O(1) + \sum_{\ell | d} 1_{(\ell, q) = \ell} O \left( \frac{\ell}{\varphi(q)} \sum_{j | \ell} \frac{q/\ell}{\|j\|} \right),
\]

where now the implied constant depends ONLY on $q$, getting from initial formula above

\[
\frac{1}{\varphi(q)} \cdot \frac{1}{x} \sum_{n \leq x} F(n)c_q(n) = \frac{1}{x} \sum_{d \leq x} F'(d) \left( 1_{d \equiv 0 \text{ mod } q} \frac{x}{d} + O_q(1) \right),
\]

whence we obtain the thesis. □

**Remark 27.** The name $(CW)$ is from : with infinitesimal remainders (i.e., $(SD)$ following) this, say, “approximate Carmichael-Wintner Formula” has LHS $\to \text{Car}_q(F)$ & RHS $\to \text{Win}_q(F)$!
Now, in order to conclude concordance of Carmichael & Wintner coefficients in next Theorem, we need to identify the right condition, namely to let the remainder be infinitesimal with \( x \to \infty \) : the condition of **Slow Decay** (for the Eratosthenes Transform moduli):

\[
SD) \quad \sum_{d \leq x} |F'(d)| = o(x), \quad as \ x \to \infty.
\]

This is a way to say: \( \text{Car}_1(\{F'\}) = 0 \), namely the classic Mean-Value of our \( |F'| \) vanishes.

**Remark 28.** This condition \((SD)\) is a weaker hypothesis, with respect to \((WA)\) : it can be shown (by partial summation, after splitting at \( \sqrt{x} \)) that \((WA)\) implies \((SD)\), but (as \( F'(d) = 1/\log(d+1) \) proves) the converse isn’t true! So, our next Theorem is stronger than Wintner’s Criterion (say, last part of Wintner-Delange Formula proof), even if it has a non trivial defect (compare next Remark)!

Once given this, we get the “Carmichael-Wintner Formula”: \( \text{Car}(F) = \text{Win}(F) \), in our following result (first fix \( q \in \mathbb{N} \), then \( x \to \infty \) to get \( \text{Car}_q(F) = \text{Win}_q(F) \) from \((CW)\) in Lemma 7). We say the Carmichael transform \( \text{Car}(F) \) exists, hereafter, to mean: \( \exists \text{Car}_q(F) \in \mathbb{C}, \forall q \in \mathbb{N} \) (analogously for Wintner’s Transform).

**Theorem(Carmichael-Wintner Formula).** Let \( F : \mathbb{N} \to \mathbb{C} \) satisfy \((SD)\). Then \( \forall q \in \mathbb{N} \)

\[\exists \text{Car}_q(F) \in \mathbb{C} \iff \exists \text{Win}_q(F) \in \mathbb{C} \quad \text{and, if they exist, } \text{Car}_q(F) = \text{Win}_q(F).\]

Thus the transform \( \text{Car}(F) \) exists iff the transform \( \text{Win}(F) \) exists and, in case they exist, \( \text{Car}(F) = \text{Win}(F) \).

**Remark 29.** A big defect of our result, w.r.t. the Wintner’s Criterion, is that it doesn’t get the existence of these coefficients: it has to assume it (maybe, that’s why nobody seems to have thought about it before)! However \( \exists \text{Car}(C_{f,g}) \) has, under Conjecture 1, the consequence that \((SD) \Rightarrow \text{the Reef} \), compare Remark 28.

We’ve found this result months ago (and then briefly applied to s.R.e., in the talk given on 5th September at Poznań for NTW2017). We didn’t yet know of a similar 1987 result of Delange, that we have found only days ago (thanks, Google Scholar) and, while (as you see from Lemma 7 proof) our is elementary, Delange’s 1987 Theorem in [De87] needs (even if Professor Hubert Delange didn’t say explicitly) the zero-free region, for the Riemann zeta-function at least (see His Lemma). In fact, while our previous result can’t be applied, for \( F' = \lambda \) the Liouville function (see Remark 25: this \( F' \) hasn’t \((SD)\) above), next Theorem from [De87] can be applied, as \( \lambda \neq 1 \) the characteristic function of squares (see [T] and Remark 25), is obviously bounded (and so satisfies \((i)\) next): from \( \text{Car}(1_\mathbb{S}) = 0 \) (see V.I.P. before Remark 25), we get \( \text{Win}(1_\mathbb{S}) = 0 \) (which, of course, is equivalent to PNT and requires a non-trivial arithmetic information: esp., from a zero-free region).

Next is the Theorem of Delange in [De87]. (However, expressed in our notation.)

**Theorem(Delange’87).** Let \( F : \mathbb{N} \to \mathbb{C} \) be such that, once fixed \( q \in \mathbb{N} \), the following two conditions hold:

\[(i) \quad \sum_{n \leq x} |F(n)| = O(x);\]
\[(ii) \exists \text{Car}_q(F) \in \mathbb{C}, \forall d|q.\]

Then \( \exists \text{Win}_q(F) \in \mathbb{C} \) and \( \text{Car}_q(F) = \text{Win}_q(F). \)

(Actually our \((ii)\) is Delange’s \((ii)'\), see 1.5 in [De87], which is equivalent to His condition \((ii)\), as He says.)

Summarizing the Theorem : if \( F \) is bounded on average (condition \((i)\) above), then \( \exists \text{Car}(F) \) implies \( \exists \text{Win}(F) \) and \( \text{Car}(F) = \text{Win}(F) \) ! So our V.I.P. before Remark 25 also has, consequently, \( \text{Win} = 0 \) !

If we wish to compare condition \((i)\) above with our \((SD)\) condition we might say that they look like “independent”, meaning that no one seems to imply the other ! However, considering the proofs, Delange’s result (at least, both from this point of view and from previous application to Liouville’s function) seems to be much stronger than our !

Anyway, both these results don’t say a word about R.e. of a.f.s involved ! It seems, from our proof of Wintner-Delange Formula, that the possibility to exchange sums in the double series in equation \((*)\) is the key to prove : Wintner’s coefficients of our \( F \) are in \( \mathbb{F} \) ! So we, as announced soon before the Reef above, still want to “jump”, from a definitely vanishing \( \text{Win}(C_{f,g}) \), to the Reef; but (even if our attempts and false starts produced, say, all of these odds & ends) we still haven’t found how! (A possibility is to prove Conjecture 1.)

Maybe uniqueness of R.c.s is a good indication, for our R.e.! Well, both our handmade Lemma 2 & Lemma 3 give examples of uniqueness: may you give more?
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Readers’ call: I wish to enlarge the map, so I need feedback (see email above, put subject “4theMap!”)!

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