The Absence Of Saturation Of The Level Number Variance In A
Rectangular Box

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Abstract

The variance of the number of levels in an energy interval around a level with
large quantum numbers (semiclassical quantization) is studied for a particle
in a rectangular box. Sampling involves changing the ratio of the rectangle’s
sides while keeping the area constant. For sufficiently narrow intervals, one
finds the usual linear growth with the width of the interval. For wider inter-
vals, the variance undergoes large, non-decaying oscillations around what is
expected to be the saturation value. These oscillations can be explained as a
superposition of just a few harmonics that correspond to the shortest periodic
orbits in the rectangle. The analytical and numerical results are in excellent
agreement.

I. INTRODUCTION

Two decades ago, Casati, Chirikov and Guarneri [1] and Berry [2] have undertaken
numerical and analytical studies of the level statistics in rectangular billiards for large quan-
tum numbers (semiclassical quantization). Casati et al [1] demonstrated numerically that
the distribution function of the nearest-neighbor energy level spacings is exponential (obeys
Poisson law [3]), which is generally the case for classically integrable systems and is well
understood theoretically [4]. Further, they studied the behavior of the level rigidity [3] in the spectral staircase and showed that, following the initial linear growth with the width of
the interval, the rigidity saturates to a constant value. This behavior was explained by Berry
[2] who argued that the width of the interval at which the saturation occurs corresponds to
the shortest periodic orbit in the billiard and that the saturation value of the rigidity can
be obtained as a sum over the periodic orbits. Subsequently, the results of [1] have been
reproduced with high numerical precision in Refs. [5] and [6].

In the previous work [7], we set out to interpret this result in terms of the global level
rigidity and, in particular, proposed an ansatz for the level density correlation function
that successfully describes the transition from linear to saturation behavior and that is also
similar, aside from the energy scale, to its counterparts in the Gaussian ensembles (classically
chaotic systems) [3], [4]. Furthermore, our correlation function led us to believe that the
variance should exhibit interesting, albeit decaying, oscillations on approach to its saturation
value (by the integral relationship [3] between the rigidity and the variance - see below - it
should be twice that of the rigidity).

Surprisingly, our numerical calculation indicated that the size of the oscillations of the
variance does not decay with the increase of the width of the interval. Further investigation
of the level correlation function revealed that a more accurate treatment of large energy
scales gives an analytical explanation which is in excellent agreement with the numerical
results.

It should be emphasized that the key difference of our numerical procedure relative to [5]
and [6] is that we averaged over the ratio of the rectangle’s sides, as opposed to averaging over
the spectrum keeping a fixed ratio of the sides. Indeed, both the onset of saturation and the
saturation value scale as a square root of energy, that is depend critically on the position in
the spectrum [2]. Consequently, averaging over the spectrum averages over these quantities.
Notice, for instance, that the plot of variance in [5] does show oscillations. However, because
of the averaging procedure, they appear to be decaying and of irregular pattern and did not
receive due acknowledgment.¹

II. NOTATIONS AND KNOWN RESULTS

We will consider intervals \([\varepsilon - E/2, \varepsilon + E/2]\), \(E \ll \varepsilon\), where the states with energies near \(\varepsilon\) have large quantum numbers and can be described semiclassically. We will denote \(\mathcal{N}(\varepsilon)\) the spectral staircase [4]

\[
\mathcal{N}(\varepsilon) = \sum_{k} \theta(\varepsilon - \varepsilon_k)
\]  

(1)

where \(\theta\) is unit step function and \(k\) are energy eigenstates. More precisely, we will consider the 2D-reduced [4] spectrum where the systematic energy dependence of \(\mathcal{N}(\varepsilon)\) has been eliminated². Towards this end, we make the substitution

\[
\varepsilon \rightarrow \langle \mathcal{N}(\varepsilon) \rangle
\]  

(2)

and, in particular, \(\varepsilon_k \rightarrow \langle \mathcal{N}(\varepsilon_k) \rangle\), where \(\langle \mathcal{N}(\varepsilon) \rangle\) is the ensemble average of \(\mathcal{N}(\varepsilon)\) (in the limit of an infinitely large ensemble). With such definition of energy (reduced spectrum), we have

\[
\langle \mathcal{N}(\varepsilon) \rangle \rightarrow \langle \mathcal{N}(\varepsilon) \rangle = \varepsilon = \langle \rho(\varepsilon) \rangle \varepsilon
\]  

(3)

that is \(\langle \rho(\varepsilon) \rangle = 1\), where

\[
\rho(\varepsilon) = \frac{d\mathcal{N}(\varepsilon)}{d\varepsilon} = \sum_{k} \delta(\varepsilon - \varepsilon_k)
\]  

(4)

¹We learned about Ref. [5] after completion of this work.

²For a finite domain, in particular, we must eliminate the systematic energy dependence of \(\langle \mathcal{N}(\varepsilon) \rangle\) due to boundary corrections [4], [8], [9]; for a rectangle this dependence can be obtained directly [9], [10] from the energy levels of a particle in a box, eq. (7) below, and is given by \(\langle \mathcal{N}(\varepsilon) \rangle = \varepsilon - 2\beta \sqrt{\varepsilon/\pi} + 1/4\), where \(\beta = (\alpha^{1/4} + \alpha^{-1/4})/2\), \(\alpha\) is defined in eq. (8) below and \(\varepsilon\) is measured in units of \(\Delta = 2\pi \hbar^2/mA\), eq. (15) below.
is the density of states. In dimensional units, (3) can be written as

$$\langle N(\varepsilon) \rangle = \langle \rho(\varepsilon) \rangle \varepsilon$$  \hspace{1cm} (5)$$

where

$$\langle \rho(\varepsilon) \rangle = \frac{mA}{2\pi\hbar^2}$$  \hspace{1cm} (6)$$
m/2\pi\hbar^2 being the 2D density of states in the thermodynamic limit and $A$ the system area.

For a rectangle, the energy eigenvalues are

$$\varepsilon_{nm} = \frac{\pi^2\hbar^2}{2m} \left( \frac{n^2}{a^2} + \frac{m^2}{b^2} \right)$$  \hspace{1cm} (7)$$

and averaging is performed over the “$\alpha$-ensemble,” that is by sampling the values of

$$\alpha = \frac{a^2}{b^2}$$  \hspace{1cm} (8)$$

where $a$ and $b$ are the rectangle’s sides, while keeping $A = ab$, and consequently $\langle \rho(\varepsilon) \rangle$, constant. Numerically, we use algebraic numbers for $\alpha$ to avoid accidental level degeneracies (we don’t use transcendental numbers because they are too close to rational [4]).

In this work we will not discuss the distribution of the nearest-neighbor energy level spacings, except to state that our computations [10] are congruent with those of [1], [5] and [6]. Here, we will study the correlations and fluctuations in the energy spectrum and, towards this end, the following measures will be used. First, it is the ”level rigidity” $\Delta_3$, defined [2], [3] as the best linear fit of the spectral staircase on the interval $[\varepsilon - E/2, \varepsilon + E/2]$

$$\Delta_3(\varepsilon, E) = \left\langle \min_{(A, B)} \frac{1}{E} \int_{\varepsilon - E/2}^{\varepsilon + E/2} d\varepsilon \left[ N(\varepsilon) - A - B\varepsilon \right]^2 \right\rangle$$  \hspace{1cm} (9)$$

Fig. 1 shows an example of such a fit.

For the number of levels $N$ on the interval $[\varepsilon - E/2, \varepsilon + E/2]$

$$N(\varepsilon, E) = N\left(\varepsilon + \frac{E}{2}\right) - N\left(\varepsilon - \frac{E}{2}\right)$$  \hspace{1cm} (10)$$

the variance
\[ \Sigma(\varepsilon, E) = \langle (N - \langle N \rangle)^2 \rangle \]  

is another measure of the fluctuations. Clearly,

\[ \langle N \rangle = \langle \rho \rangle E \]  

for an infinitely large ensemble.

Introducing the mean level spacing \( \langle \Delta \rangle \) as

\[ \langle \Delta \rangle = E \left( \frac{1}{\langle N \rangle} \right) \]  

we find that, due to the \( N \)-fluctuations \( \Sigma \propto E \) over the intervals \( E \) narrower than a certain energy scale, the following relationship holds [10]

\[ \langle \rho \rangle \langle \Delta \rangle = 1 + \frac{1}{\langle N \rangle} + O \left( \frac{1}{\langle N \rangle^2} \right) \]  

For classically integrable billiards the latter holds for \( E \ll \sqrt{\varepsilon \langle \rho \rangle^{-1}} \) (see below), so that we can neglect the difference between \( \langle \Delta \rangle \) and \( \langle \rho \rangle^{-1} \) only for sufficiently large intervals \( \langle N \rangle \gg 1 \), in which case we can introduce the limiting value of mean level spacing ("global mean") as

\[ \Delta = \frac{1}{\langle \rho \rangle} = \frac{2\pi \hbar^2}{m a} \]  

In what follows we will, unless mentioned otherwise, measure energies in units of \( \Delta \) (wherein \( \Delta = \langle \rho \rangle^{-1} = 1 \), as in eq. (3)).

General relationships can be established between the fluctuations measures and the correlation function of the level density [3],

\[ K(\varepsilon_1, \varepsilon_2) = \langle \delta \rho(\varepsilon_1) \delta \rho(\varepsilon_2) \rangle \]  

\[ \delta \rho(\varepsilon) = \rho(\varepsilon) - \langle \rho(\varepsilon) \rangle \]  

regardless of the specific form of \( K(\varepsilon_1, \varepsilon_2) \), for instance,

\[ \Sigma(\varepsilon, E) = \int_{\varepsilon-E/2}^{\varepsilon+E/2} \int_{\varepsilon-E/2}^{\varepsilon+E/2} K(\varepsilon_1, \varepsilon_2) d\varepsilon_1 d\varepsilon_2 \]
Using these relationships one can further show that $\Sigma$ supersedes $\Delta_3$ via an integral relationship \[3\]

\[
\Delta_3(\varepsilon, E) = \frac{2}{E^4} \int dx \left( E^3 - 2xE^2 + x^2 \right) \Sigma(\varepsilon, x)
\]

which, again, does not depend on the particulars of the level statistics and level correlations in the semiclassical energy spectrum.

For Gaussian ensembles, corresponding to classically chaotic ergodic systems, the functional form of the level correlation function $K(\varepsilon_1, \varepsilon_2)$ is well understood \[3\]. Denoting

\[
\varepsilon = \frac{\varepsilon_1 + \varepsilon_2}{2}, \quad \omega = \varepsilon_2 - \varepsilon_1
\]

we find that it can be written, in most general terms, as

\[
K(\varepsilon_1, \varepsilon_2) = K(\omega) = \Delta^{-2} \left[ \delta\left(\frac{\omega}{\Delta}\right) - K\left(\frac{\omega}{\Delta}\right) \right]
\]

where

\[
\int_{-\infty}^{\infty} K(x) \, dx = 1 \quad (22)
\]

and

\[
\int_{-\infty}^{\infty} K(\omega) \, d\omega = 0 \quad (23)
\]

The interpretation of these formulas are as follows. The $\delta$-function term in (21) describes uncorrelated levels. The $K$-function describes "level repulsion" and has a scale of $\Delta$. The integrals (22) and (23) converge to their values on the scale of $\omega \sim \Delta$ and reflect the fact that an overall "level rigidity" develops on the scale of $\Delta$.

Indeed, by definition of $\delta\rho(\varepsilon)$,

\[
\int \delta\rho(\varepsilon) \, d\varepsilon = 0 \quad (24)
\]

and thus

\[
\int K(\varepsilon_1, \varepsilon_2) \, d\varepsilon_1 = \int K(\varepsilon_1, \varepsilon_2) \, d\varepsilon_2 = 0 \quad (25)
\]
where integration is over the entire energy spectrum. The integral can be converted to the form (23) when, given a position in the spectrum $\varepsilon$, the scale on which the level rigidity (24) develops is much smaller than $\varepsilon$. Eqs. (21)-(23) show that Gaussian ensembles become rigid on the scale of $\Delta$. This fact can be also observed from the behavior of $\Sigma$

$$\Sigma (\varepsilon, E) = \frac{E}{\Delta}, \ E \lesssim \Delta$$  \hspace{1cm} (26)$$

$$\Sigma (\varepsilon, E) = C \ln \left( \frac{E}{\Delta} \right), \ E \gg \Delta$$  \hspace{1cm} (27)$$

where the constant $C$ depends on the specifics of a Gaussian ensemble [3]. The behavior of $\Delta_3$,

$$\Delta_3 (\varepsilon, E) = \frac{1}{15} \frac{E}{\Delta}, \ E \lesssim \Delta$$  \hspace{1cm} (28)$$

$$\Delta_3 (\varepsilon, E) = \frac{C}{2} \ln \left( \frac{E}{\Delta} \right), \ E \gg \Delta$$  \hspace{1cm} (29)$$

is then easily recovered from that of $\Sigma$ using (19). Notice that the linear terms in eqs. (26) and (28) originate in the $\delta$-function term in (21) (uncorrelated levels).

In the semiclassical approach, $\delta \rho (\varepsilon)$ is an oscillatory term that can be written as a sum over all periodic orbits [4]. Berry [2] used this approach to obtain the following limiting behaviors of $\Delta_3 (\varepsilon, E)$ for the rectangular box:

$$\Delta_3 (\varepsilon, E) = \frac{1}{15} \frac{E}{\Delta}, \ E \lesssim E_{\text{max}} = (\pi \alpha^{1/2} \varepsilon \Delta)^{1/2}$$  \hspace{1cm} (30)$$

$$\Delta_3 (\varepsilon, E) = \frac{1}{\pi^{5/2}} \left( \frac{\varepsilon}{\Delta} \right)^{1/2} \sum_{M_1=0}^\infty \sum_{M_2=0}^\infty \frac{\delta_M}{(M_1^2 \alpha^{1/2} + M_2^2 \alpha^{-1/2})^{3/2}}, \ E \gg E_{\text{max}}$$  \hspace{1cm} (31)$$

$$\rightarrow 0.0947 \sqrt{\frac{\varepsilon}{\Delta}}, \ \alpha \sim 1$$  \hspace{1cm} (32)$$

where the second term gives the "saturation rigidity" $\Delta_3^\infty$ and 

$$0 \quad \text{if } M_1 = M_2 = 0$$

$$\delta_M = 1/4 \quad \text{if one of } M_1 \text{ and } M_2 \text{ is zero}$$  \hspace{1cm} (33)$$

$$1 \quad \text{otherwise}$$

Here $M_1$ and $M_2$ are the "winding numbers" of the classical periodic orbits [2] and the factor $\delta_M$ differentiates between the self-retracing and non-self-retracing orbits. Notice that
both $\Delta_3^\infty$ and $E_{\text{max}}$ are the functions of the position $\varepsilon$ in the spectrum ($\propto \sqrt{\varepsilon}$), which is in stark contrast with Gaussian ensembles. The quantum scale $E_{\text{max}}$ for the onset of saturation corresponds to the time of traversal of the shortest classical periodic orbit [2],

$$E_{\text{max}} = \frac{\hbar}{T_{\text{min}}}$$

whose length is just twice the length of the rectangle’s smaller side and the winding numbers are $M_1 = 0$, $M_2 = 1$.

Notice that saturation of $\Delta_3$ to $\Delta_3^\infty$ in a rectangle, eqs.(30) and (31), is analogous to "saturation" of $\Delta_3$ to the weak logarithmic dependence for a Gaussian ensemble, eqs. (28) and (29). Consequently, one should expect the level density correlation function for a rectangular box to have the form similar to (21), except that the scale of $K$ is set by $\sim \sqrt{\varepsilon\Delta}$. Indeed, in our previous work [7] we introduced a simple ansatz that resulted in the following expression for $K$:

$$K(\varepsilon, \omega) = \Delta^{-2} \left[ \delta \left( \frac{\omega}{\Delta} \right) - \frac{\Delta}{\pi \omega} \sin \left( \frac{2\pi \omega}{E_{\text{max}}} \right) \right]$$

In this form, $K(\varepsilon, \omega)$ satisfies eq. (23) and is, thus, consistent with the level rigidity developing on the scale of $E_{\text{max}}$. Evaluation of $\Delta_3$ using eq. (35) successfully reproduces saturation to $\Delta_3^\infty$ on this scale also. For $\Sigma$, eq. (35) predicts [7] eventual saturation to $\Sigma^\infty = 2\Delta_3^\infty$, but in an oscillatory fashion

$$\frac{\Sigma - \Sigma^\infty}{\Sigma^\infty} = \frac{\sin (E/E_{\text{max}})}{E/E_{\text{max}}}$$

(The oscillatory term does not contribute to $\Delta_3^\infty$ in eq. (19)).

While successfully describing the onset of the level rigidity and the oscillations of $\Sigma$ on the scales $\sim E_{\text{max}}$, the ansatz (35) does not capture the correct behavior of $\Sigma$ on scales $> E_{\text{max}}$. The central result of this work is that the magnitude of the oscillations around $\Sigma^\infty$ is not decaying and that the onset of the level rigidity on the scale of $E_{\text{max}}$ is not, in fact, precise. This will be shown both numerically and analytically, the latter being based on a more careful analysis of the level correlation function obtained, alternatively,
either semiclassically or by the direct use of the explicit form of the level spectrum in the rectangular box.

### III. CORRELATION FUNCTION OF THE LEVEL DENSITY

The analytical expression for the level density correlation function, in the form of the infinite sum, can be obtained in two ways. First, it can be recovered from the semiclassical derivation of Berry in Ref. [2], as per eqs. (23), (38), (58), and (60) there. Indeed, taking the inverse Fourier transform in the last of these equations, we find

$$K(\varepsilon, \omega) = \frac{1}{\sqrt{(\pi \Delta)^3}} \varepsilon \sum_{M_1=0}^{\infty} \sum_{M_2=0}^{\infty} 4\delta_M \cos \left( \sqrt{\frac{4\pi}{\alpha}} \frac{(M_1^2 \alpha^{1/2} + M_2^2 \alpha^{-1/2})\omega}{\sqrt{M_1^2 \alpha^{1/2} + M_2^2 \alpha^{-1/2}}} \right)$$

(37)

The paragraph following eq. (38) in [2] explains the nature of the factor $4\delta_M$ in the above equation; our interpretation is that the factor of 4 is the intensity factor due to constructive interference between the closed orbit and its time-reversed for non-self-retracing orbits (both $M_1$ and $M_2$ are non-zero). The latter is crucial for understanding of the change in level correlations due to time-reversal symmetry breaking (for instance, by a magnetic field).

Alternatively, one can obtain (37) starting directly from eqs. (4) and (7). In Ref. [11], such a formalism was developed for $\delta\rho$ and was generalized to $K(\varepsilon, \omega)$ in Ref. [7]. It is based on the use of the Poisson summation formula, whose net effect is to convert the sum over actual levels to the sum over classical periodic orbits. Formally, it is accomplished [11], [7] by converting the sum on $(n, m)$ to the sum on $(M_1, M_2)$ of the Fourier transforms and by neglecting the rapidly oscillating terms, that is harmonic terms whose arguments depend on $\varepsilon$ and change by $\sim \sqrt{\varepsilon/\Delta} \gg 1$ when $M_{1,2}$ change by 1. As a result one recovers $^3$[7] eq.

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$^3$Notice that the correct power in the denominator of eq. (20) in Ref. [7] is 1, not 1/2. Also, we are presently investigating if the coefficient $\delta_M$ is properly recovered with this technique.
Notice that in the limit of \( \omega \ll \sqrt{\varepsilon \Delta} \), the summation over \((M_1, M_2)\) can be replaced by integration over \((x, y)\) in the entire plane, where
\[
x = M_1 \alpha^{1/4} \sqrt{\frac{4 \pi \Delta}{\varepsilon}}, \quad y = M_2 \alpha^{-1/4} \sqrt{\frac{4 \pi \Delta}{\varepsilon}}
\] 
(39)

Converting to polar coordinates, we find
\[
K(\varepsilon, \omega) \to \frac{1}{2 \pi^2 \Delta^2} \int_0^{2\pi} d\phi \int_0^\infty d\rho \cos \left( \frac{\omega}{\Delta} \rho \right) = \Delta^{-2} \delta \left( \frac{\omega}{\Delta} \right)
\] 
(40)
that is the first term in eq. (35). Notice, however, that in reality the lower limit of \(\rho\)-integration should not extend to zero since \(M_{1,2}\) are not simultaneously zero. The second (repulsion) term in eq. (35) is then obtained using the ansatz [7] in which the lower cut-off in \(\int_\rho_{\text{min}}^\infty\) is chosen at \(\rho_{\text{min}} = y_{\text{min}} = \alpha^{-1/4} \sqrt{4 \pi \Delta / \varepsilon}\).

In the opposite limit \(\omega \ll \sqrt{\varepsilon \Delta}\), the terms in the series (37) become rapidly oscillating with the amplitude rapidly decreasing with the increase of the winding numbers. Consequently, the sum will be dominated by a just few terms with the smallest \(M_{1,2}\). The latter is the reason behind the non-decaying oscillations of \(\Sigma\) that will be discussed below.

IV. LEVEL NUMBER VARIANCE

Combining eqs. (18) and (37), we find
\[
\Sigma(\varepsilon, E) = \sqrt{\frac{\varepsilon}{\pi^5 \Delta}} \sum_{M_1=0}^\infty \sum_{M_2=0}^\infty 4 \delta_M \sin^2 \left( E \sqrt{\frac{\pi \Delta}{\varepsilon}} \left( M_1^2 \alpha^{1/2} + M_2^2 \alpha^{-1/2} \right) \right) \frac{1}{(M_1^2 \alpha^{1/2} + M_2^2 \alpha^{-1/2})^{3/2}}
\] 
\[\to \sqrt{\frac{\varepsilon}{\pi^5 \Delta}} \left[ 4 \sum_{M_1 > 0} \sum_{M_2 > 0} \sin^2 \left( E \sqrt{\frac{\pi \Delta}{\varepsilon}} \left( M_1^2 + M_2^2 \right) \right) \frac{1}{(M_1^2 + M_2^2)^{3/2}} + 2 \sum_{M > 0} \sin^2 \left( E \sqrt{\frac{\pi \Delta}{\varepsilon}} M \right) \frac{1}{M^3} \right]_{\alpha \sim 1}
\] 
(41)
(42)
For narrow intervals, eq. (41) reduces to
\[
\Sigma(\varepsilon, E) = \frac{E}{\Delta}, \quad E \ll E_{\text{max}}
\] 
(43)
which follows either from eqs. (18) and (40) or directly from (41) if summation is replaced with integration using variables (39). It is also consistent with eqs. (19) and (30).
Fig. 2 shows the result of evaluation of $\Sigma$ using eq. (41) (with the upper limit of summation on $M_{1,2}$ limited to 100) versus the numerical evaluation of $\Sigma$ for the ensemble of 200 algebraic $\alpha \in [1, 2]$. We also show $\Sigma$ obtained using the ansatz (35) of Ref. [7]. Clearly, the latter adequately describes the transition from the small-scale $E \ll E_{\text{max}}$ linear behavior (43) to scales of $E \sim E_{\text{max}}$, but fails to describe the large-scale behavior, $E \gg E_{\text{max}}$. On the other hand, the agreement of numerical evaluation with the theoretical result (41) for all $E$ is quite remarkable.

Two comments are in order. First, it was necessary to have a sufficient spread of $\alpha$-values (between 1 and 2 here) to obtain reliable statistics for large quantum numbers with relatively small $\alpha$-ensembles (200 values here). However, the dependence on $\alpha$ of the relevant parameters is quite weak, through $\alpha^{\pm 1/4}$, so for $\alpha \in [1, 2]$ the difference from an ensemble whose $\alpha$-values would be close to a fixed value (1 in this particular case) is not significant. Second, for $E \gg E_{\text{max}}$ it is sufficient to limit summation on $M_{1,2}$ to 2 and 3 in the double and single sum respectively to obtain a curve which is very close to the full theoretical curve; as was already explained above this is because the amplitudes of the quickly oscillating terms in the sum rapidly fall off with the increase of $M_{1,2}$.

We now turn to the theoretical curve in Fig. 3 that represents the level number variance given by (42). It is intended to describe the $\alpha$-ensemble such that $\alpha \simeq 1$ and it can be effectively reproduced by the superposition of only 6 harmonics: 3 from the double sum and 3 (with commensurate frequencies) from the single sum. These are non-decaying oscillations whose amplitude is given by

$$\frac{\Sigma^+}{\Sigma^\infty} \approx \frac{2}{3}$$

where $\Sigma^+ = \max \{\Sigma\} - \Sigma^\infty$, $\Sigma^- = \Sigma^\infty - \min \{\Sigma\}$ and $\Sigma^\infty$ is the mean value of oscillating $\Sigma$ which is obtained by substituting $\sin^2 \rightarrow 1/2$ in eq. (41). The small asymmetry of the positive and negative swings of $\Sigma$-oscillations, $\Sigma^+ \neq \Sigma^-$, is due to the asymmetry of the Clausen function [12] $Cl_3(x)$,
\[ Cl_3(x) = \sum_{k=1}^{\infty} \frac{\cos kx}{k^3} \]

for which \( \max \{Cl_3(x)\} / \min \{Cl_3(x)\} \approx 1.3 \). This function originates in the single sum in eq. (42), which corresponds to self-retracing orbits.

It is important to point out that if one averages over the range of \( \alpha \)-values, or over the range of \( \varepsilon \)-values [5], the beats that result from a superposition of a continuous range of harmonics will present themselves, over relevant interval widths, as decaying oscillations. This is due to the corresponding continuous range of the beat frequencies represented by their envelopes.

We now turn to breaking of time reversal symmetry for charged particles due to a magnetic field. Clearly, the condition for the latter is given by

\[ BA \sim \phi_0 \] (45)

where \( \phi_0 = \hbar c/e \) is the flux quantum. For such fields, the Larmor radius is much greater than either side of the rectangle

\[ R = \frac{mcv}{eB} \sim \sqrt{A} \sqrt{\frac{\varepsilon}{\Delta}} \] (46)

where \( mv^2/2 = \varepsilon \). Therefore, the deviation from the free specular scattering will be small and we can find both the level correlation function and the level number variance via simple substitution

\[ \delta_M \rightarrow \delta_M' = \begin{cases} 0 & \text{if } M_1 = M_2 = 0 \\ 1/4 & \text{if one of } M_1 \text{ and } M_2 \text{ is zero} \\ 1/2 & \text{otherwise} \end{cases} \] (47)

in eqs. (37) and (41). For the \( \alpha \approx 1 \)-ensemble considered above, \( \Sigma' \) is shown in Fig. 4. In this case

\[ \frac{\Sigma'^\infty}{\Sigma^\infty} \approx 0.7 \] (48)
\[ \frac{\Sigma'^+}{\Sigma^\infty} \approx 0.77, \quad \frac{\Sigma'^-}{\Sigma^\infty} \approx 0.64 \] (49)
The increased asymmetry of $\Sigma$-oscillations underscores the increased relative contribution of the single sum to $\Sigma$ in eq. (42) (self-retracing orbits).

V. DISCUSSION

The central results of this work are summarized by Figs. 2-4. The first of these graphs shows that the numerical evaluation of the level number variance for a particle in a rectangular box is in excellent agreement with the theoretical result given by eq. (41). The second indicates a non-decaying oscillatory behavior of $\Sigma$. The third shows $\Sigma$ when the time-reversal symmetry is broken. Figs. 3 and 4 can be successfully reproduced with just a small number of lowest harmonics in (42).

It is remarkable that, given the center of the interval $\varepsilon$, $\Sigma(\varepsilon, E)$ exhibits large, reproducible oscillations as a function of the interval width $E$. The main implication of this result is that while the level rigidity indeed develops on the scale set by $\sim \sqrt{\varepsilon\Delta} \ll \varepsilon$, it is accurate only in an approximation where a harmonic is replaced by its average - zero. In other words, (23) converges to zero on the scale of $\omega \sim \sqrt{\varepsilon\Delta}$ only up to a sum of harmonic terms, as implied by eq. (37).

We point out that $\Sigma$-oscillations are entirely consistent with the near straight line saturation of $\Delta_3$. In fact, $\Delta_3$ also exhibit oscillatory behavior around $\Delta_3^\infty$, but it is both parametrically small in $\Delta/\varepsilon$ and decays as a function of $E$. This reduction of $\Delta_3$ can be traced, for instance, to integration in eq. (19).

In the future, we will investigate the level number variance in a variety of finite-domain and potential problems. We will also address the explicit dependence of the level correlation function on the magnetic field for a detailed description of the time-reversal breaking transition and to address the orbital magnetism of non-resonant integrable systems.

VI. ACKNOWLEDGMENTS

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VII. FIGURE CAPTIONS

1. Figure 1

Level rigidity $\Delta_3(\varepsilon, E)$ vs. the interval width $E$ for $\varepsilon/\Delta = 10^4$ and 200 algebraic $\alpha \in [1, 2]$. The straight line is $(E/15\Delta)$ and the two horizontal lines correspond to Berry’s saturation rigidity $\Delta_3^\infty$ for $\alpha = 1$ (lower line) and $\alpha = 2$. Notice that our saturation rigidity is slightly below the former, while expected to be between the two, and that the difference from $\Delta_3^\infty$ is amplified by the factor of $\sqrt{\varepsilon/\Delta}$.

2. Figure 2

Numerical evaluation of $\Sigma(\varepsilon, E)$ (red line), theoretical evaluation per eq. (41) (green line), and theoretical evaluation based on ansatz (35) (blue line) for $\varepsilon/\Delta = 10^4$ and 200 algebraic $\alpha \in [1, 2]$.

3. Figure 3

$\Sigma(\varepsilon, E)$ per eq. (42) for $\varepsilon/\Delta = 10^5$.

4. Figure 4

$\Sigma(\varepsilon, E)$ per eqs. (42) and (47) for $\varepsilon/\Delta = 10^5$ for the case of broken time reversal symmetry.
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Figure 1
Figure 3
