$L_p$ Convergence with Rates of Smooth Poisson-Cauchy Type Singular Operators

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Abstract. In this article we continue the study of smooth Poisson-Cauchy Type singular integral operators on the line regarding their convergence to the unit operator with rates in the $L_p$ norm, $p \geq 1$. The related established inequalities involve the higher order $L_p$ modulus of smoothness of the engaged function or its higher order derivative.

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1 Introduction

The rate of convergence of singular integrals has been studied in [9], [13], [14], [15], [7], [8], [4], [5], [6] and these articles motivate our work. Here we study the $L_p$, $p \geq 1$, convergence of smooth Poisson-Cauchy Type singular integral operators over $\mathbb{R}$ to the unit operator with rates over smooth functions with higher order derivatives in $L_p(\mathbb{R})$. We establish related Jackson type inequalities involving the higher $L_p$ modulus of smoothness of the engaged function or its higher order derivative. The discussed operators are not in general positive, see [10], [11]. Other motivation comes from [1], [2].

2 Results

In the next we introduce and deal with the smooth Poisson-Cauchy Type singular integral operators $M_{r,\xi}(f; x)$ defined as follows.

For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_+$ we set

$$\alpha_j = \begin{cases} (-1)^{r-j}\binom{r}{j}j^{-n}, & j = 1, \ldots, r, \\ 1 - \sum_{j=1}^{r}(-1)^{r-j}\binom{r}{j}j^{-n}, & j = 0, \end{cases}$$

that is $\sum_{j=0}^{r} \alpha_j = 1$. 

Let $f \in C^n(\mathbb{R})$ and $f^{(n)} \in L_p(\mathbb{R})$, $1 \leq p < \infty$, $\alpha \in \mathbb{N}$, $\beta > \frac{1}{2\alpha}$, we define for $x \in \mathbb{R}$, $\xi > 0$ the Lebesgue integral
\[
M_{r,\xi}(f; x) = W \int_{-\infty}^{\infty} \sum_{j=0}^{r} \alpha_j f(x + jt) \left(\frac{t^{2\alpha} + \xi^{2\alpha}}{t^{2\alpha} + \xi^{2\alpha}}\right)^{\beta} dt,
\]
where the constant is defined as
\[
W = \frac{\Gamma(\beta) \alpha \xi^{2\alpha \beta - 1}}{\Gamma(\frac{1}{2\alpha}) \Gamma(\beta - \frac{1}{2\alpha})}.
\]

**Note 1.** The operators $M_{r,\xi}$ are not, in general, positive. See [10], (18).

We notice by $W \int_{-\infty}^{\infty} \left(\frac{t^{2\alpha} + \xi^{2\alpha}}{t^{2\alpha} + \xi^{2\alpha}}\right)^{\beta} dt = 1$, that $M_{r,\xi}(c, x) = c$, $c$ constant, see also [10], [11], and
\[
M_{r,\xi}(f; x) - f(x) = W \left(\sum_{j=0}^{r} \alpha_j \int_{-\infty}^{\infty} \left[f(x + jt) - f(x)\right] \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^{\beta}} dt\right).
\]

We use also that
\[
\int_{-\infty}^{\infty} \frac{t^k}{(t^{2\alpha} + \xi^{2\alpha})^{\beta}} dt = \begin{cases} 0, & k \text{ odd}, \\ \frac{\Gamma(k+1)\Gamma(\beta-k+1)}{\Gamma(\beta)} \frac{\xi^{2\alpha \beta-k-1}}{\xi^{2\alpha-k+1}} \Gamma(\beta-k+1), & k \text{ even}, \beta > \frac{k+1}{2\alpha}, \end{cases}
\]
see [16].

We need the $r$th $L_p$-modulus of smoothness
\[
\omega_r(f^{(n)}, h)_p := \sup_{|t| \leq h} \|\Delta_r f^{(n)}(x)\|_{p,x}, \quad h > 0,
\]
where
\[
\Delta_r f^{(n)}(x) := \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} f^{(n)}(x + jw),
\]
see [12], p. 44. Here we have that $\omega_r(f^{(n)}, h)_p < \infty$, $h > 0$.

We need to introduce
\[
\delta_k := \sum_{j=1}^{r} \alpha_j j^k, \quad k = 1, \ldots, n \in \mathbb{N},
\]
and denote by $\lfloor \cdot \rfloor$ the integral part. Call
\[
\tau(w, x) := \sum_{j=0}^{r} \alpha_j j^n f^{(n)}(x + jw) - \delta_n f^{(n)}(x).
\]

Notice also that
\[
- \sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} = (-1)^r \binom{r}{0}.
\]
According to [3], p. 306, [1], we get
\[
\tau(w, x) = \Delta_{\infty} f^{(n)}(x).
\]

Thus
\[
\|\tau(w, x)\|_{p,x} \leq \omega_r(f^{(n)}, |w|)_p, \quad w \in \mathbb{R}.
\]
Using Taylor’s formula, and the appropriate change of variables, one has (see [6])
\[
\sum_{j=0}^{r} \alpha_j [f(x+jt) - f(x)] = \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k t^k + R_n(0, t, x),
\]
where
\[
R_n(0, t, x) := \int_0^t \frac{(t-w)^{n-1}}{(n-1)!} \tau(w, x) dw, \quad n \in \mathbb{N}.
\]

Using the above terminology we obtain for \( \beta > \frac{2l_0}{p} + \frac{p}{2(\alpha_n)} \) that
\[
\Delta(x) := M_{r, \xi}(f; x) - f(x) - \sum_{m=1}^{[n/2]} f^{(2m)}(x) \delta_{2m} \frac{\Gamma \left( \frac{2m+1}{2} \right) \Gamma \left( \beta - \frac{2m+1}{2} \right) \xi_{\tau}^{2m}}{\Gamma \left( \frac{1}{2} \right) \Gamma \left( \beta - \frac{1}{2} \right)} = R_n^*(x),
\]
where
\[
R_n^*(x) := W \int_{-\infty}^{\infty} R_n(0, t, x) \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^\beta} dt, \quad n \in \mathbb{N}.
\]

In \( \Delta(x) \), see (14), the sum collapses when \( n = 1 \).

We present our first result.

**Theorem 1.** Let \( p, q > 1 \) such that \( \frac{1}{p} + \frac{1}{q} = 1, \ n \in \mathbb{N}, \ \alpha \in \mathbb{N}, \ \beta > \alpha \left( \frac{1}{p} + n + r \right) \) and the rest as above. Then
\[
\|\Delta(x)\|_p \leq \frac{\left( 2\alpha \right)^{\frac{1}{p}} \Gamma \left( \beta \right) \Gamma \left( \frac{q^2}{2} - \frac{1}{2\alpha} \right)^{\frac{1}{p}} \xi_{\tau}^{n_\beta}}{\Gamma \left( \frac{q^2}{2} \right)^{\frac{1}{p}} \Gamma \left( \frac{1}{2} \right) \Gamma \left( \beta - \frac{1}{2} \right) (rp + 1)^\beta ((n-1)!)(q(n-1)+1)^{1/q}} \omega_{\tau}(f^{(n)}; \xi, p).
\]

where
\[
0 < \tau := \left[ \int_0^{\infty} (1 + u)^{rp+1} \frac{u^{n_\beta-1}}{(u^{2\alpha} + 1)^{1/2}} du - \int_0^{\infty} \frac{u^{n_\beta-1}}{(u^{2\alpha} + 1)^{3/2}} du \right]< \infty.
\]

Hence as \( \xi \to 0 \) we obtain \( \|\Delta(x)\|_p \to 0 \).

If additionally \( f^{(2m)} \in L_p(\mathbb{R}), m = 1, 2, \ldots, \left[ \frac{n}{2} \right] \) then \( \|M_{r, \xi}(f) - f\|_p \to 0, \) as \( \xi \to 0 \).

**Proof.** We observe that
\[
|\Delta(x)|^p = W^p \left[ \int_{-\infty}^{\infty} R_n(0, t, x) \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^\beta} dt \right]^p
\leq W^p \left( \int_{-\infty}^{\infty} |R_n(0, t, x)| \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^\beta} dt \right)^p
\leq W^p \left( \int_{-\infty}^{\infty} \left( \int_0^{|t|} \frac{|t-w|^{n_\beta-1}}{(n-1)!} |\tau(\text{sign}(t) \cdot w, x)| dw \right) \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^\beta} dt \right)^p.
\]

Hence we have
\[
I := \int_{-\infty}^{\infty} |\Delta(x)|^p dx \leq W^p \left( \int_{-\infty}^{\infty} \left( \int_0^{|t|} \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^\beta} dt \right)^p dx \right),
\]
where
\[
\gamma(t, x) := \int_0^{|t|} \frac{|t-w|^{n_\beta-1}}{(n-1)!} |\tau(\text{sign}(t) \cdot w, x)| dw \geq 0.
\]
Therefore by using Hölder’s inequality suitably we obtain

\[ R.H.S.(19) = W^p \left( \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \gamma(t, x) \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^{\beta/2}} dt \right)^p dx \right) \]

\[ \leq W^p \cdot \left( \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \gamma(t, x) \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^{\beta/2}} dt \right)^p \left( \int_{-\infty}^{\infty} \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^{\beta/2}} dt \right)^{\frac{p}{q}} dx \right) \]

\[ = W^p \cdot \left( \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \gamma(t, x) \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^{\beta/2}} dt \right)^p \right) \left( \int_{-\infty}^{\infty} \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^{\beta/2}} dt \right)^{\frac{p}{q}} \]

\[ = W^p \cdot \left( \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \gamma(t, x) \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^{p\beta/2}} dt \right)^p \right) \left( \int_{-\infty}^{\infty} \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^{p\beta/2}} dt \right)^{\frac{p}{q}} \]

\[ = \frac{\xi^{p\alpha\beta - 1}}{\Gamma \left( \frac{\alpha}{2} \right) \Gamma \left( \frac{\beta}{2} \right)} \left( \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \gamma(t, x) \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^{p\beta/2}} dt \right)^p \right) \left( \int_{-\infty}^{\infty} \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^{p\beta/2}} dt \right)^{\frac{p}{q}} \]

(21)

Again by Hölder’s inequality we have

\[ \gamma^p(t, x) \leq \left( \int_0^{|t|} \left| \tau(\text{sign}(t \cdot w, x)) \right|^p dw \right) \frac{|t|^{n-1}}{(q(n-1)+1)^{p/q}}. \]

(22)

Consequently we have

\[ R.H.S.(21) \leq \frac{\xi^{p\alpha\beta - 1}}{\Gamma \left( \frac{\alpha}{2} \right) \Gamma \left( \frac{\beta}{2} \right)} \left( \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \gamma(t, x) \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^{p\beta/2}} dt \right)^p \right) \frac{|t|^{n-1}}{(q(n-1)+1)^{p/q}} \]

\[ =: (\ast), \]

(calling

\[ c_1 := \frac{\xi^{p\alpha\beta - 1}}{\Gamma \left( \frac{\alpha}{2} \right) \Gamma \left( \frac{\beta}{2} \right)} \left( \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \gamma(t, x) \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^{p\beta/2}} dt \right)^p \right) \frac{|t|^{n-1}}{(q(n-1)+1)^{p/q}} \]

(23)
and

\[ (*) = c_1 \left( \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} |\tau(t, w, x)|^p dw \right) |t|^{p-1} \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^{p\beta/2}} dt \right) \]

\[ = c_1 \left( \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} |\Delta^r(t, w) f(n)(t)|^p dw \right) |t|^{p-1} \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^{p\beta/2}} dt \right) \]

\[ = c_1 \left( \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} |\Delta^r(t, w) f(n)(t)|^p dw \right) |t|^{p-1} \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^{p\beta/2}} dt \right) \]

\[ \leq c_1 \left( \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \omega_r(f(n), w)^p dw \right) |t|^{p-1} \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^{p\beta/2}} dt \right). \quad (24) \]

So far we have proved

\[ I \leq c_1 \left( \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \omega_r(f(n), w)^p dw \right) |t|^{p-1} \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^{p\beta/2}} dt \right). \quad (25) \]

By [12], p. 45 we have

\[ (R.H.S.(25)) \leq c_1 \left( \omega_r(f(n), \xi)^p \left( \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \left( 1 + \frac{w}{\xi} \right)^p \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^{p\beta/2}} dt \right) \right) =: (**) \right. \] \[ \text{But we see that} \]

\[ (***) = \left( \frac{\xi c_1}{rp + 1} \right) \left( \omega_r(f(n), \xi)^p \right) \mathcal{J}, \quad (26) \]

where

\[ \mathcal{J} = \int_{-\infty}^{\infty} \left( \left( 1 + \frac{|t|}{\xi} \right)^{p+1} - 1 \right) \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^{p\beta/2}} dt \]

\[ = 2 \int_{0}^{\infty} \left( \left( 1 + \frac{t}{\xi} \right)^{p+1} - 1 \right) \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^{p\beta/2}} dt. \quad (28) \]

Here we find

\[ \mathcal{J} = 2 \xi^{p(n-\alpha\beta)} \int_{0}^{\infty} \left( (1 + u)^{p+1} - 1 \right) \frac{1}{(u^{2\alpha} + 1)^{p\beta/2}} du \]

\[ = 2 \xi^{p(n-\alpha\beta)} \left[ \int_{0}^{\infty} \left( (1 + u)^{p+1} - 1 \right) \frac{1}{(u^{2\alpha} + 1)^{p\beta/2}} du - \int_{0}^{\infty} \left( (1 + u)^{p+1} - 1 \right) \frac{1}{(u^{2\alpha} + 1)^{p\beta/2}} du \right]. \quad (29) \]

Thus by (17) and (29) we obtain

\[ \mathcal{J} = 2 \xi^{p(n-\alpha\beta)} \mathcal{r}. \quad (30) \]
We notice that
\[
0 < \tau < \int_0^\infty \frac{(1 + u)^{p+1} u^{n-1}}{(u^{2\alpha} + 1)^{p\beta/2}} du
\]
\[
< \int_0^\infty \frac{(1 + u)^{p+1} (1 + u)^{n-1}}{(u^{2\alpha} + 1)^{p\beta/2}} du
\]
\[
= \int_0^\infty \frac{(1 + u)^{p(n+r)}}{(u^{2\alpha} + 1)^{p\beta/2}} du =: I_1.
\]

Also call
\[
K := \int_0^1 \frac{(1 + u)^{p(n+r)}}{(u^{2\alpha} + 1)^{p\beta/2}} du < \infty.
\]

Then we can write
\[
I_1 = K + \int_1^\infty \frac{(1 + u)^{p(n+r)}}{(u^{2\alpha} + 1)^{p\beta/2}} du < K + 2^p(n+r) \int_1^\infty \frac{u^{p(n+r)}}{(u^{2\alpha} + 1)^{p\beta/2}} du = K + 2^p(n+r) I_2,
\]
where \(I_2 := \int_1^\infty \frac{u^{p(n+r)}}{(u^{2\alpha} + 1)^{p\beta/2}} du.
\]

Since \(\frac{1}{1+u}\) is finite, so is \((u^{2\alpha})^{-\beta}\), for \(u \in [1, \infty).\)

So we get
\[
I_2 < \int_1^\infty u^{p(n+r-\alpha\beta)} du = \lim_{\varepsilon \to \infty} \int_1^\varepsilon u^{p(n+r-\alpha\beta)} du
\]
\[
= \lim_{\varepsilon \to \infty} \left( \frac{\varepsilon^{p(n+r-\alpha\beta)+1} - 1}{p (n + r - \alpha\beta) + 1} \right) = \frac{-1}{p (n + r - \alpha\beta) + 1},
\]

which is a positive number since \(\beta > \frac{1}{n} \left( \frac{1}{p} + n + r \right)\).

Consequently \(I_2\) is finite, so is \(I_1\), proving \(\tau < \infty\).

Using (27) and (30) we get
\[
(\ast\ast) = \left( \frac{\xi \xi_1}{rp + 1} \right) \left( \omega_r(f^{(n)}, \xi)_p \right)^p 2\xi^{p(n-\alpha\beta)} \tau
\]
\[
= \frac{2\alpha [\Gamma(\beta)]^p \Gamma\left(\frac{q\beta - 1}{2\alpha}\right)^\frac{q}{2} \tau}{(rp + 1) \Gamma\left(\frac{1}{2\alpha}\right) \Gamma\left(\beta - \frac{1}{2\alpha}\right) \Gamma\left(\frac{q\beta}{2}\right)^\frac{q}{2} ((n - 1)!)^p (q(n - 1) + 1)^{p/q}}
\]
I.e. we have established that
\[
I \leq \frac{2\alpha [\Gamma(\beta)]^p \Gamma\left(\frac{q\beta - 1}{2\alpha}\right)^\frac{q}{2} \tau}{(rp + 1) \Gamma\left(\frac{1}{2\alpha}\right) \Gamma\left(\beta - \frac{1}{2\alpha}\right) \Gamma\left(\frac{q\beta}{2}\right)^\frac{q}{2} ((n - 1)!)^p (q(n - 1) + 1)^{p/q}} \xi^{pn} \left( \omega_r(f^{(n)}, \xi)_p \right)^p.
\]

That is finishing the proof of the theorem. \(\blacksquare\)

The counterpart of Theorem 1 follows, case of \(p = 1\).

**Theorem 2.** Let \(f \in \mathcal{C}_n(\mathbb{R})\) and \(f^{(n)} \in L_1(\mathbb{R}), n \in \mathbb{N}, \alpha \in \mathbb{N}, \beta > \frac{n+r+1}{2\alpha}.\) Then
\[
\|\Delta(x)\|_1 \leq \frac{1}{(r+1)(n-1)! \Gamma\left(\frac{1}{2\alpha}\right) \Gamma\left(\beta - \frac{1}{2\alpha}\right)} \cdot \sum_{k=0}^{r+1} \binom{r+1}{k} \Gamma\left(\frac{n+k}{2\alpha}\right) \Gamma\left(\beta - \frac{n+k}{2\alpha}\right) \omega_r(f^{(n)}, \xi)_1 \xi^n.
\]
Hence as $\xi \to 0$ we obtain $\|\Delta(x)\|_1 \to 0$.
If additionally $f^{(2m)} \in L_1(\mathbb{R})$, $m = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor$ then $\|M_{r, \xi}(f) - f\|_1 \to 0$, as $\xi \to 0$.

**Proof.** It follows

$$|\Delta(x)| = W \left| \int_{-\infty}^{\infty} \mathcal{R}_n(0, t, x) \frac{1}{(t^{2\alpha} + \xi^{2\alpha})} dt \right|$$

$$\leq W \int_{-\infty}^{\infty} |\mathcal{R}_n(0, t, x)| \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^\beta} dt$$

$$\leq W \int_{-\infty}^{\infty} \left( \int_{0}^{[t]} \frac{(|t| - w)^{n-1}}{(n-1)!} |\tau(\text{sign}(t) \cdot w, x)| dw \right) \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^\beta} dt. \quad (34)$$

Thus

$$\|\Delta(x)\|_1 = \int_{-\infty}^{\infty} |\Delta(x)| dx \leq W \cdot \int_{-\infty}^{\infty} \left( \int_{0}^{[t]} \frac{(|t| - w)^{n-1}}{(n-1)!} |\tau(\text{sign}(t) \cdot w, x)| dw \right) \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^\beta} dt \cdot dx$$

$$=: (*)$$

But we see that

$$\int_{0}^{[t]} \frac{(|t| - w)^{n-1}}{(n-1)!} |\tau(\text{sign}(t) \cdot w, x)| dw \leq \frac{|t|^{n-1}}{(n-1)!} \int_{0}^{[t]} |\tau(\text{sign}(t) \cdot w, x)| dw. \quad (36)$$

Therefore it holds

$$(*) \leq W \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \frac{|t|^{n-1}}{(n-1)!} \int_{0}^{[t]} |\tau(\text{sign}(t) \cdot w, x)| dw \right) \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^\beta} dt \cdot dx$$

$$= \frac{W}{(n-1)!} \left( \int_{-\infty}^{\infty} \left( \int_{0}^{[t]} \int_{-\infty}^{\infty} |\tau(\text{sign}(t) \cdot w, x)| dx \right) dw \right) \frac{|t|^{n-1}}{(t^{2\alpha} + \xi^{2\alpha})^\beta} dt$$

$$\leq \frac{W}{(n-1)!} \left( \int_{-\infty}^{\infty} \left( \int_{0}^{[t]} \omega_r(f^{(n)}, w) dw \right) \frac{|t|^{n-1}}{(t^{2\alpha} + \xi^{2\alpha})^\beta} dt \right). \quad (37)$$

I.e. we get

$$\|\Delta(x)\| \leq \frac{W}{(n-1)!} \left( \int_{-\infty}^{\infty} \left( \int_{0}^{[t]} \omega_r(f^{(n)}, w) dw \right) \frac{|t|^{n-1}}{(t^{2\alpha} + \xi^{2\alpha})^\beta} dt \right). \quad (38)$$

Consequently we have

$$\|\Delta(x)\| \leq \frac{W \omega_r(f^{(n)}, \xi)_1}{(n-1)!} \left( \int_{-\infty}^{\infty} \left( \int_{0}^{[t]} \left( 1 + \frac{w}{\xi} \right)^r dw \right) \frac{|t|^{n-1}}{(t^{2\alpha} + \xi^{2\alpha})^\beta} dt \right)$$

$$= \frac{2\xi W \omega_r(f^{(n)}, \xi)_1}{(r+1)(n-1)!} \left( \int_{-\infty}^{\infty} \left( 1 + \frac{t}{\xi} \right)^{r+1} - 1 \right) \frac{t^{n-1}}{(t^{2\alpha} + \xi^{2\alpha})^\beta} dt$$

$$= \frac{2\Gamma(\beta) \alpha \xi^{2\alpha \beta} \omega_r(f^{(n)}, \xi)_1}{(r+1)(n-1)! \Gamma \left( \frac{1}{2\alpha} \right) \Gamma \left( \beta - \frac{1}{2\alpha} \right)} \left( \int_{0}^{\infty} \left( 1 + \frac{t}{\xi} \right)^{r+1} - 1 \right) \frac{t^{n-1}}{(t^{2\alpha} + \xi^{2\alpha})^\beta} dt. \quad (39)$$
We have gotten so far

\[
\| \Delta(x) \|_1 \leq \frac{2 \Gamma (\beta) \alpha \xi^{2 \alpha} \omega_r(f^{(\eta)}, \xi_1) \cdot \lambda}{(r + 1) (n - 1)! \Gamma \left( \frac{2 \alpha}{2 \alpha} \right) \Gamma \left( \beta - \frac{n}{2 \alpha} \right)}, \tag{40}
\]

where

\[
\lambda := \int_0^\infty \left( \left( 1 + \frac{t}{\xi} \right)^{r+1} - 1 \right) \frac{t^{n-1}}{(t^2 + \xi^{2\alpha})^\beta} dt. \tag{41}
\]

One easily finds that

\[
\lambda = \int_0^\infty \left( \sum_{k=1}^{r+1} \left( \frac{t+1}{k} \right)^k \right) \frac{t^{n-1}}{(t^2 + \xi^{2\alpha})^\beta} dt
\]

\[
= \xi^{-2\alpha} \sum_{k=1}^{r+1} \left( \frac{t+1}{k} \right)^k \int_0^\infty \frac{T^{n+k-1}}{(T^2 + 1)^\beta} dT
\]

\[
= \xi^{-2\alpha} \sum_{k=1}^{r+1} \left( \frac{t+1}{k} \right)^k K_{n+k}. \tag{42}
\]

Where

\[
K_{n+k} := \int_0^\infty \frac{T^{n+k-1}}{(T^2 + 1)^\beta} dT = \frac{\Gamma \left( \frac{n+k}{2\alpha} \right) \Gamma \left( \beta - \frac{n+k}{2\alpha} \right)}{\Gamma (\beta) 2\alpha}, \tag{43}
\]

\[
\| \Delta(x) \|_1 \leq \frac{1}{(r + 1) (n - 1)! \Gamma \left( \frac{2 \alpha}{2 \alpha} \right) \Gamma \left( \beta - \frac{n}{2 \alpha} \right)} \left[ \sum_{k=1}^{r+1} \left( \frac{t+1}{k} \right)^k \Gamma \left( \frac{n+k}{2\alpha} \right) \Gamma \left( \beta - \frac{n+k}{2\alpha} \right) \right] \omega_r(f^{(\eta)}, \xi_1)1\xi^n.
\]

We have proved (33). \hfill \blacksquare

The case \( n = 0 \) is met next.

**Proposition 1.** Let \( p, q > 1 \) such that \( \frac{1}{p} + \frac{1}{q} = 1 \), \( \alpha \in \mathbb{N} \), \( \beta > \frac{1}{\alpha} \left( r + \frac{1}{p} \right) \) and the rest as above. Then

\[
\| M_{r, \xi}(f) - f \|_p \leq \frac{(2\alpha)^\frac{1}{q} \Gamma (\beta) \left( \frac{q^2}{2} - \frac{1}{2\alpha} \right)^\frac{1}{q} \theta^\frac{1}{q}}{\Gamma \left( \frac{1}{q} \right)^\frac{1}{q} \Gamma (\beta - \frac{1}{q}) \Gamma \left( \frac{q^2}{2} \right)^\frac{1}{q}} \omega_r(f, \xi)_p, \tag{44}
\]

where

\[
0 < \theta := \int_0^\infty (1 + t)^p \frac{1}{(t^2 + 1)^{p\beta/2}} dt < \infty. \tag{45}
\]

Hence as \( \xi \to 0 \) we obtain \( M_{r, \xi} \to \) unit operator \( I \) in the \( L_p \) norm, \( p > 1 \).

**Proof.** By (3) we notice that,

\[
M_{r, \xi}(f; x) - f(x) = W \left( \sum_{j=0}^r \alpha_j \int_{-\infty}^\infty (f(x + jt) - f(x)) \frac{1}{(t^2 + \xi^{2\alpha})^\beta} dt \right)
\]

\[
= W \left( \int_{-\infty}^\infty \left( \sum_{j=0}^r \alpha_j (f(x + jt) - f(x)) \right) \frac{1}{(t^2 + \xi^{2\alpha})^\beta} dt \right)
\]

\[
= W \left( \int_{-\infty}^\infty \left( \sum_{j=1}^r \alpha_j f(x + jt) - \sum_{j=1}^r \alpha_j f(x) \right) \frac{1}{(t^2 + \xi^{2\alpha})^\beta} dt \right)
\]
\[ W \left( \int_{-\infty}^{\infty} \left( \sum_{r=1}^{\infty} \left( -1 \right)^{r-j} \binom{r}{j} j^{-n} f(x+jt) - \sum_{j=1}^{\infty} \left( -1 \right)^{r-j} \binom{r}{j} j^{-n} f(x) \right) \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^{3/2}} dt \right) \]

\[ W \left( \int_{-\infty}^{\infty} \left( \sum_{j=1}^{\infty} \left( -1 \right)^{r-j} \binom{r}{j} f(x+jt) - \sum_{j=1}^{\infty} \left( -1 \right)^{r-j} \binom{r}{j} f(x) \right) \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^{3/2}} dt \right) \]

\[ \int_{-\infty}^{\infty} (\Delta_{t}^{\alpha} f)(x) \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^{3/2}} dt \]

And then

\[ |M_{r,\xi}(f; x) - f(x)| \leq W \left( \int_{-\infty}^{\infty} |\Delta_{t}^{\alpha} f(x)| \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^{3/2}} dt \right). \]

We next estimate

\[ \int_{-\infty}^{\infty} \left| M_{r,\xi}(f; x) - f(x) \right|^p dx \leq \int_{-\infty}^{\infty} \left( W \right)^p \left( \int_{-\infty}^{\infty} \left| \Delta_{t}^{\alpha} f(x) \right| \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^{3/2}} dt \right)^p dx \]

\[ = \left( W \right)^p \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \left| \Delta_{t}^{\alpha} f(x) \right| \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^{3/2}} dt \right)^p \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^{3/2}} dx \]

\[ \leq \left( W \right)^p \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \left| \Delta_{t}^{\alpha} f(x) \right| \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^{3/2}} dt \right)^q \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^{3/2}} dx \]

\[ = \left( W \right)^p \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \left| \Delta_{t}^{\alpha} f(x) \right|^p \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^{3/2}} dt \right)^q \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^{3/2}} dx \]

\[ = \left( W \right)^p \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \left| \Delta_{t}^{\alpha} f(x) \right|^p \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^{3/2}} dt \right) \frac{\Gamma \left( \frac{q\alpha}{2} \right) \Gamma \left( \frac{q\alpha - 1}{2\alpha} \right)}{\Gamma \left( \frac{q\alpha}{2} \right) \alpha^{q\alpha-1}} dx \]

\[ = \left( W \right)^p \frac{\Gamma \left( \frac{q\alpha}{2} \right) \Gamma \left( \frac{q\alpha - 1}{2\alpha} \right)}{\Gamma \left( \frac{q\alpha}{2} \right) \alpha^{q\alpha-1}} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \left| \Delta_{t}^{\alpha} f(x) \right|^p \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^{3/2}} dt \right) dx \]
Proposition 2. \( \beta > \frac{r + 1}{2} \). It holds

\[
\| M_{r,\xi} f - f \|_1 \leq \frac{2\alpha \Gamma (\beta) (q^{\beta_2} - \frac{1}{2\alpha})}{\Gamma (\frac{1}{2\alpha}) \Gamma (\beta - \frac{1}{2\alpha})} \cdot \int_0^\infty (1 + t)^{r} \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^{3/2}} dt \| \omega_r (f, \xi) \|_1.
\]

Hence as \( \xi \to 0 \) we get \( M_{r,\xi} \to I \) in the \( L_1 \) norm.
**Proof.** By (47) we have again

\[ |M_{r,ξ}(f; x) - f(x)| \leq W \left( \int_{-\infty}^{\infty} |\Delta_r^* f(x)| \frac{1}{(t^{2\alpha} + ξ^{2\alpha})^\beta} dt \right). \]

Next we estimate

\[
\int_{-\infty}^{\infty} |M_{r,ξ}(f; x) - f(x)| \, dx \leq W \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} |\Delta_r^* f(x)| \frac{1}{(t^{2\alpha} + ξ^{2\alpha})^\beta} dt \right) \, dx \\
= W \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} |\Delta_r^* f(x)| \, dx \right) \frac{1}{(t^{2\alpha} + ξ^{2\alpha})^\beta} dt \\
\leq W \int_{-\infty}^{\infty} \omega_r(f, |t|) \frac{1}{(t^{2\alpha} + ξ^{2\alpha})^\beta} dt \\
\leq W2\omega_r(f, ξ) \int_0^{\infty} \left( 1 + \frac{t}{ξ} \right)^r \frac{1}{(t^{2\alpha} + ξ^{2\alpha})^\beta} dt \\
= \frac{\Gamma(\beta) \xi^{2\alpha\beta-1}2\alpha}{\Gamma(\frac{1}{2\alpha}) \Gamma(\beta - \frac{1}{2\alpha})} \omega_r(f, ξ) \int_0^{\infty} ξ^{1+} (1+t)^r \frac{1}{(t^{2\alpha} + 1)^\beta ξ^{2\alpha\beta}} dt \\
= \frac{\Gamma(\beta) 2\alpha}{\Gamma(\frac{1}{2\alpha}) \Gamma(\beta - \frac{1}{2\alpha})} \omega_r(f, ξ) \int_0^{\infty} (1+t)^r \frac{1}{(t^{2\alpha} + 1)^\beta} dt. \tag{50}
\]

We have proved (49).

We also notice that

\[
0 < \int_0^{\infty} (1+t)^r \frac{1}{(t^{2\alpha} + 1)^\beta} dt \\
= \int_0^{1} (1+t)^r \frac{1}{(t^{2\alpha} + 1)^\beta} dt + \int_1^{\infty} (1+t)^r \frac{1}{(t^{2\alpha} + 1)^\beta} dt \\
< \int_0^{1} (1+t)^r \frac{1}{(t^{2\alpha} + 1)^\beta} dt + 2r \int_1^{\infty} t^{-2\alpha\beta} dt \\
= \int_0^{1} (1+t)^r \frac{1}{(t^{2\alpha} + 1)^\beta} dt - \frac{2r}{(r - 2\alpha\beta + 1)},
\]

which is a positive finite constant. \[\blacksquare\]

In the next we consider \( f \in C^n(\mathbb{R}) \) and \( f^{(n)} \in L_p(\mathbb{R}), n = 0 \) or \( n \geq 2 \) even, \( 1 \leq p < \infty \) and the similar smooth singular operator of symmetric convolution type

\[ M_ξ(f; x) = W \int_{-\infty}^{\infty} f(x+y) \frac{1}{(y^{2\alpha} + ξ^{2\alpha})^\beta} dy, \quad \text{for all } x \in \mathbb{R}, \ \xi > 0. \tag{51} \]

That is

\[ M_ξ(f; x) = W \int_0^{\infty} (f(x+y) + f(x-y)) \frac{1}{(y^{2\alpha} + ξ^{2\alpha})^\beta} dy, \]

for all \( x \in \mathbb{R}, \ \xi > 0 \). Notice that \( M_{1,ξ} = M_ξ \). Let the central second order difference

\[ (\hat{Δ}_y^2 f)(x) := f(x+y) + f(x-y) - 2f(x). \tag{52} \]

Notice that

\[ (\hat{Δ}_y^2 f)(x) = (\hat{Δ}_y f)(x). \]
When \( n \geq 2 \) even using Taylor’s formula with Cauchy remainder we eventually find
\[
(\tilde{\Delta}^2_y f)(x) = 2 \sum_{\rho=1}^{n/2} \frac{f^{(2\rho)}(x)}{(2\rho)!} y^{2\rho} + R_1(x),
\]
where
\[
R_1(x) := \int_0^y (\tilde{\Delta}^2 f^{(n)})(x) \frac{(y-t)^{n-1}}{(n-1)!} \, dt.
\]

Notice that
\[
M_\xi(f;x) - f(x) = W \int_0^\infty (\tilde{\Delta}^2 f(x)) \frac{1}{(y^{2\alpha} + \xi^{2\alpha})^\beta} \, dy.
\]

Furthermore by (4), (53) and (55) we easily see that
\[
K(x) := M_\xi(f;x) - f(x) - \sum_{\rho=1}^{n/2} \frac{f^{(2\rho)}(x)}{(2\rho)!} \frac{\Gamma \left( \frac{2\rho+1}{2\alpha} \right) \Gamma \left( \frac{\beta - 2\rho+1}{2\alpha} \right)}{\Gamma \left( \frac{1}{2\alpha} \right) \Gamma \left( \frac{\beta - 1}{2\alpha} \right)} \xi^{2\rho}
\]
\[
= W \int_0^\infty \left[ \int_0^y (\tilde{\Delta}^2 f^{(n)})(x) \frac{(y-t)^{n-1}}{(n-1)!} \, dt \right] \frac{1}{(y^{2\alpha} + \xi^{2\alpha})^\beta} \, dy,
\]
where \( \beta > \frac{(n+1)}{2\alpha} \).

Therefore we have
\[
|K(x)| \leq W \int_0^\infty \left( \int_0^y |\tilde{\Delta}^2 f^{(n)}|(x) \frac{(y-t)^{n-1}}{(n-1)!} \, dt \right) \frac{1}{(y^{2\alpha} + \xi^{2\alpha})^\beta} \, dy.
\]

Here we estimate in \( L_p \) norm, \( p \geq 1 \), the error function \( K(x) \). Notice that we have \( \omega_2(f^{(n)}, h)_p < \infty \), \( h > 0 \), \( n = 0 \) or \( n \geq 2 \) even. Operators \( M_\xi \) are positive operators.

The related main \( L_p \) result here comes next.

**Theorem 3.** Let \( p, q > 1 \) such that \( \frac{1}{p} + \frac{1}{q} = 1 \), \( n \geq 2 \) even, \( \alpha \in \mathbb{N} \), \( \beta > \frac{1}{\alpha} \left( \frac{1}{p} + n + 2 \right) \) and the rest as above. Then
\[
\|K(x)\|_p \leq \frac{\tilde{\tau}^{1/p} \alpha^{1/p} \Gamma \left( \frac{q\beta}{2} - \frac{1}{2\alpha} \right)^{1/q}}{2^{1/p} \Gamma \left( \frac{1}{2\alpha} \right)^{1/p} \Gamma \left( \frac{\beta - 1}{2\alpha} \right) \Gamma \left( \frac{q\beta}{2} \right)^{1/q} (q(n-1) + 1)^{1/q} (2p+1)^{1/p} (n-1)!^\xi \omega_2(f^{(n)}, \xi)_p},
\]
where
\[
0 < \tilde{\tau} = \int_0^\infty \left( (1+u)^{2p+1} - 1 \right) u^{pn-1} \frac{1}{(1+u^{2\alpha})^{p3/2}} \, du < \infty.
\]

Hence as \( \xi \to 0 \) we get \( \|K(x)\|_p \to 0 \).

If additionally \( f^{(2m)} \in L_p(\mathbb{R}), m = 1, 2, \ldots, \frac{\beta}{\alpha} \) then \( \|M_\xi(f) - f\|_p \to 0 \), as \( \xi \to 0 \).

**Proof.** We observe that
\[
|K(x)|^p \leq W^p \left( \int_0^\infty \left( \int_0^y |\tilde{\Delta}^2 f^{(n)}|(x) \frac{(y-t)^{n-1}}{(n-1)!} \, dt \right) \frac{1}{(y^{2\alpha} + \xi^{2\alpha})^\beta} \, dy \right)^p.
\]

Call
\[
\tilde{\gamma}(y, x) := \int_0^y |\tilde{\Delta}^2 f^{(n)}|(x) \frac{(y-t)^{n-1}}{(n-1)!} \, dt \geq 0, \quad y \geq 0,
\]
then we have
\[ |K(x)|^p \leq W^p \left( \int_0^\infty \frac{1}{\left(y^{2\beta} + \xi^{2\alpha}\right)^s} \, dy \right)^p. \] (62)
Consequently
\[
\Lambda := \int_{-\infty}^{\infty} |K(x)|^p \, dx \leq W^p \int_{-\infty}^{\infty} \left( \int_0^\infty \frac{1}{\left(y^{2\beta} + \xi^{2\alpha}\right)^{s/2}} \, dy \right)^p \, dx
\]
(by Hölder’s inequality)
\[
\leq W^p \left( \int_{-\infty}^{\infty} \left( \int_0^\infty \frac{1}{\left(y^{2\beta} + \xi^{2\alpha}\right)^{s/2}} \, dy \right)^{p/q} \right)^{q/p}
\]
\[
= W^p \left( \frac{\Gamma \left( \frac{\alpha}{2\alpha} \right) \Gamma \left( \frac{\alpha}{\beta} \right)}{2 \Gamma \left( \frac{\alpha}{\beta} + 1 \right)} \right)^{p/q} \left( \int_{-\infty}^{\infty} \left( \int_0^\infty \frac{1}{\left(y^{2\beta} + \xi^{2\alpha}\right)^{s/2}} \, dy \right)^{p/q} \right)^{q/p}
\]
\[
= \frac{\Gamma (\beta)^p \alpha^{\frac{\alpha}{\beta} - 1} \Gamma \left( \frac{\alpha}{\beta} \right)^{p/q}}{2^{\frac{\alpha}{\beta}} \Gamma \left( \frac{\alpha}{\beta} + 1 \right)^{p/q}} \left( \int_{-\infty}^{\infty} \left( \int_0^\infty \frac{1}{\left(y^{2\beta} + \xi^{2\alpha}\right)^{s/2}} \, dy \right)^{p/q} \right)^{q/p}
\]
\[ = (\ast). \] (63)
By applying again Hölder’s inequality we see that
\[
\tilde{\gamma}(y, x) \leq \left( \int_0^y |\mathring{D}_y^2 f^{(n)}(x)|^p \, dt \right)^{1/p} \frac{y^{(n-1+\frac{1}{q})}}{(n-1)!} \frac{y^{(n-1+\frac{1}{q})}}{(q(n-1)+1)!}. \] (64)
Therefore it holds
\[
(\ast) \leq \frac{\Gamma (\beta)^p \alpha^{\frac{\alpha}{\beta} - 1} \Gamma \left( \frac{\alpha}{\beta} \right)^{p/q}}{2^{\frac{\alpha}{\beta}} \Gamma \left( \frac{\alpha}{\beta} + 1 \right)^{p/q}} \left( \int_{-\infty}^{\infty} \left( \int_0^y |\mathring{D}_y^2 f^{(n)}(x)|^p \, dt \right)^{p/q} \frac{1}{\left(y^{2\beta} + \xi^{2\alpha}\right)^{s/2}} \, dy \right)^{q/p}
\]
\[
= \frac{\Gamma (\beta)^p \alpha^{\frac{\alpha}{\beta} - 1} \Gamma \left( \frac{\alpha}{\beta} \right)^{p/q}}{2^{\frac{\alpha}{\beta}} \Gamma \left( \frac{\alpha}{\beta} + 1 \right)^{p/q}} \left( \int_{-\infty}^{\infty} \left( \int_0^y |\mathring{D}_y^2 f^{(n)}(x)|^p \, dt \right)^{p/q} \frac{1}{\left(y^{2\beta} + \xi^{2\alpha}\right)^{s/2}} \, dy \right)^{q/p}
\]
\[ = : (\ast\ast). \] (65)
We call
\[
c_{2} := \frac{\Gamma (\beta)^p \alpha^{\frac{\alpha}{\beta} - 1} \Gamma \left( \frac{\alpha}{\beta} \right)^{p/q}}{2^{\frac{\alpha}{\beta}} \Gamma \left( \frac{\alpha}{\beta} + 1 \right)^{p/q}} \left( \int_{-\infty}^{\infty} \left( \int_0^y |\mathring{D}_y^2 f^{(n)}(x)|^p \, dt \right)^{p/q} \frac{1}{\left(y^{2\beta} + \xi^{2\alpha}\right)^{s/2}} \, dy \right)^{q/p}. \] (66)
And hence

\[ (** ) = c_2 \left( \int_0^\infty \left( \int_{-\infty}^y \left| \Delta_+^2 f^{(n)}(x) \right|^p dx \right) \frac{1}{(y^{2\alpha} + \xi^{2\alpha})^{p\beta/2}} dy \right) \]

\[ = c_2 \left( \int_0^\infty \left( \int_{-\infty}^y \left( \int_{-\infty}^\infty |\Delta_+^2 f^{(n)}(x)|^p dx \right) dt \right) \frac{1}{(y^{2\alpha} + \xi^{2\alpha})^{p\beta/2}} dy \right) \]

\[ = c_2 \left( \int_0^\infty \left( \int_{-\infty}^y \left( \int_{-\infty}^\infty |\Delta_+^2 f^{(n)}(x-t)|^p dx \right) dt \right) \frac{1}{(y^{2\alpha} + \xi^{2\alpha})^{p\beta/2}} dy \right) \]

\[ \leq c_2 \left( \int_0^\infty \left( \int_{-\infty}^y \omega_2(f^{(n)}, t)^p dt \right) \frac{1}{(y^{2\alpha} + \xi^{2\alpha})^{p\beta/2}} dy \right) \]

\[ \leq c_2 \omega_2(f^{(n)}, \xi)_p \left( \int_0^\infty \left( \int_{0}^y \left( 1 + \frac{t}{\xi} \right)^{2p} dt \right) \frac{1}{(y^{2\alpha} + \xi^{2\alpha})^{p\beta/2}} dy \right). \tag{67} \]

I.e. so far we proved that

\[ \Lambda \leq c_2 \omega_2(f^{(n)}, \xi)_p \left( \int_0^\infty \left( \int_{0}^y \left( 1 + \frac{t}{\xi} \right)^{2p} dt \right) \frac{1}{(y^{2\alpha} + \xi^{2\alpha})^{p\beta/2}} dy \right). \tag{68} \]

But

\[ \text{R.H.S.}(68) = \frac{c_2 \xi}{2p + 1} \omega_2(f^{(n)}, \xi)_p \left( \int_0^\infty \left( \left( 1 + \frac{y}{\xi} \right)^{2p+1} - 1 \right) \frac{1}{(y^{2\alpha} + \xi^{2\alpha})^{p\beta/2}} dy \right). \tag{69} \]

Call

\[ M := \int_0^\infty \left( \left( 1 + \frac{y}{\xi} \right)^{2p+1} - 1 \right) \frac{1}{(y^{2\alpha} + \xi^{2\alpha})^{p\beta/2}} dy, \tag{70} \]

and

\[ \tilde{\tau} := \int_0^\infty \left( (1 + u)^{2p+1} - 1 \right) \frac{1}{(1 + u^{2\alpha})^{p\beta/2}} du. \tag{71} \]

That is

\[ M = \xi^{p(n-\alpha\beta)}\tilde{\tau}. \tag{72} \]

Therefore it holds

\[ \Lambda \leq \frac{\tilde{\tau} \left[ \Gamma(\beta) \right]^p \alpha \xi^{pn} \Gamma \left( \frac{q\beta}{2} - \frac{1}{2\alpha} \right)^{p/q} \omega_2(f^{(n)}, \xi)_p}{2^{\tilde{\tau}} (2p + 1) \Gamma \left( \frac{1}{2\alpha} \right) \Gamma \left( \beta - \frac{1}{2\alpha} \right)^p \Gamma \left( \frac{q\beta}{2} \right)^{p/q} \left( (n - 1) ! \right)^p (q(n - 1) + 1)^{p/q}}. \tag{73} \]

We have established (58). \( \blacksquare \)

The counterpart of Theorem 3 follows, \( p = 1 \) case.
Theorem 4. Let \( f \in C^n(\mathbb{R}) \) and \( f^{(n)} \in L_1(\mathbb{R}) \), \( n \geq 2 \) even, \( \alpha \in \mathbb{N} \), \( \beta > \frac{n+3}{2\alpha} \). Then
\[
\|K(x)\|_1 \leq \frac{1}{6\Gamma \left( \frac{1}{2\alpha} \right) \Gamma \left( \beta - \frac{1}{2\alpha} \right) (n-1)!} \left[ 3\Gamma \left( \frac{n+1}{2\alpha} \right) \Gamma \left( \beta - \frac{n+1}{2\alpha} \right) \right. \\
+ 3\Gamma \left( \frac{n+2}{2\alpha} \right) \Gamma \left( \beta - \frac{n+2}{2\alpha} \right) + \Gamma \left( \frac{n+3}{2\alpha} \right) \Gamma \left( \beta - \frac{n+3}{2\alpha} \right) \right] \omega_2(f^{(n)}, \xi)x^n.
\] (74)

Hence as \( \xi \to 0 \) we obtain \( \|K(x)\|_1 \to 0 \).

If additionally \( f^{(2m)} \in L_1(\mathbb{R}) \), \( m = 1, 2, \ldots, \frac{n}{2} \) then \( \|M_{\xi}(f) - f\|_1 \to 0 \), as \( \xi \to 0 \).

Proof. Notice that

\[
\Delta_x^2 f^{(n)}(x) = \Delta_x^2 f^{(n)}(x-t),
\] (75)

all \( x, t \in \mathbb{R} \). Also it holds

\[
\int_{-\infty}^{\infty} |\Delta_x^2 f^{(n)}(x-t)|dx = \int_{-\infty}^{\infty} |\Delta_x^2 f^{(n)}(x)|dx \leq \omega_2(f^{(n)}, t_t), \quad \text{all } t \in \mathbb{R}^+.
\] (76)

Here we obtain

\[
\|K(x)\|_1 = \int_{-\infty}^{\infty} |K(x)|dx
\]

\[
\leq W \int_{-\infty}^{\infty} \left( \int_{0}^{y} \left( \int_{0}^{y} \frac{(y-t)^{n-1}}{(n-1)!} dt \right) \frac{1}{(y^{2\alpha} + \xi^{2\alpha})^\beta} dy \right) dx
\]

\[
\leq W \int_{0}^{\infty} \left( \int_{-\infty}^{\infty} \left( \frac{y^{n-1}}{(n-1)!} \right) \frac{1}{(y^{2\alpha} + \xi^{2\alpha})^\beta} dy \right) dx
\]

\[
= W \int_{0}^{\infty} \left( \int_{-\infty}^{\infty} \left( \frac{y^{n-1}}{(n-1)!} \right) \frac{1}{(y^{2\alpha} + \xi^{2\alpha})^\beta} dy \right) dx
\]

\[
\leq W \omega_2(f^{(n)}, \xi_1) \int_{0}^{\infty} \left( \int_{0}^{y} \frac{(1+t)^2}{(n-1)!} \frac{1}{(y^{2\alpha} + \xi^{2\alpha})^\beta} dy \right) dx
\]

\[
= W \omega_2(f^{(n)}, \xi_1) \int_{0}^{\infty} \left( \int_{0}^{y} \frac{3}{(1+y)^3 - 1} \frac{1}{(y^{2\alpha} + \xi^{2\alpha})^\beta} dy \right) dx
\]

\[
= \frac{\Gamma(\beta) \alpha \xi^n}{\Gamma \left( \frac{1}{2\alpha} \right) \Gamma \left( \beta - \frac{1}{2\alpha} \right) (n-1)!} \omega_2(f^{(n)}, \xi_1) \left( \int_{0}^{\infty} \frac{1}{(Y^2 + \xi^{2\alpha})^\beta} dY \right)
\]

\[
= \frac{\Gamma(\beta) \alpha \xi^n}{\Gamma \left( \frac{1}{2\alpha} \right) \Gamma \left( \beta - \frac{1}{2\alpha} \right) (n-1)!} \omega_2(f^{(n)}, \xi_1) \left( \int_{0}^{\infty} \frac{1}{(Y^2 + \xi^{2\alpha})^\beta} dY \right)
\]
We have proved (74).

The related case here of \( n = 0 \) comes next.

**Proposition 3.** Let \( p, q > 1 \) such that \( \frac{1}{p} + \frac{1}{q} = 1 \), \( \alpha \in \mathbb{N} \), \( \beta > \frac{1}{\alpha} \left( 2 + \frac{1}{p} \right) \) and the rest as above. Then

\[
\| M_\xi(f) - f \|_p \leq \frac{p^{1/p} \Gamma(\beta) \alpha^{1/p} \Gamma \left( \frac{q\beta}{2} - \frac{1}{2\alpha} \right)}{2^{\frac{1}{2}} \Gamma \left( \frac{1}{2\alpha} \right) \Gamma \left( \frac{q\beta}{2} \right) \omega_2(f, \xi)^{1/q}} \| M_\xi(f) - f \|_p,
\]

where

\[
0 < \rho := \int_0^\infty (1 + y)^{2p} \frac{1}{(y^{2\alpha} + 1)^{\beta p/2}} dy < \infty.
\]

Hence as \( \xi \to 0 \) we obtain \( M_\xi \to I \) in the \( L_p \) norm, \( p > 1 \).

**Proof.** From (55) we get

\[
| M_\xi(f; x) - f(x) |^p \leq W_p \left( \int_0^\infty \left| \tilde{\Delta}_y^2 f(x) \right| \frac{1}{(y^{2\alpha} + \xi^{2\alpha})^{\beta/2}} dy \right)^p.
\]

We then estimate

\[
\int_{-\infty}^\infty | M_\xi(f; x) - f(x) |^p dx \leq W_p \int_{-\infty}^\infty \left( \int_0^\infty \left| \tilde{\Delta}_y^2 f(x) \right| \frac{1}{(y^{2\alpha} + \xi^{2\alpha})^{\beta/2}} dy \right)^p dx
\]

\[
= W_p \int_{-\infty}^\infty \left( \int_0^\infty \left| \tilde{\Delta}_y^2 f(x) \right| \frac{1}{(y^{2\alpha} + \xi^{2\alpha})^{\beta/2}} \frac{1}{(y^{2\alpha} + \xi^{2\alpha})^{\beta p/2}} dy \right)^p dx
\]

\[
\leq W_p \int_{-\infty}^\infty \left( \left( \int_0^\infty \left| \tilde{\Delta}_y^2 f(x) \right| \frac{1}{(y^{2\alpha} + \xi^{2\alpha})^{\beta/2}} dy \right)^p \left( \int_0^\infty \frac{1}{(y^{2\alpha} + \xi^{2\alpha})^{\beta q/2}} dy \right)^{\frac{p}{q}} \right)^{\frac{q}{p}} dx
\]

\[
= W_p \int_{-\infty}^\infty \left( \int_0^\infty \left| \tilde{\Delta}_y^2 f(x) \right|^p \frac{1}{(y^{2\alpha} + \xi^{2\alpha})^{\beta p/2}} dy \right) \left( \int_0^\infty \frac{1}{(y^{2\alpha} + \xi^{2\alpha})^{\beta q/2}} dy \right)^{\frac{p}{q}} dx
\]

\[
= W_p \left( \int_{-\infty}^\infty \left( \int_0^\infty \left| \tilde{\Delta}_y^2 f(x) \right|^p \frac{1}{(y^{2\alpha} + \xi^{2\alpha})^{\beta p/2}} dx \right) \frac{\Gamma \left( \frac{q\beta}{2} - \frac{1}{2\alpha} \right)}{2\Gamma \left( \frac{q\beta}{2} \right) \alpha\xi^{q\alpha\beta-1}} \right)^{\frac{p}{q}}
\]
Proposition 4. The proof of (78) is now completed.

Proof. Hence we get

\[
|M_\xi f - f| \leq \left[ \frac{1}{2} + \frac{\Gamma (\frac{\alpha}{2}) \Gamma (\beta - \frac{\alpha}{2})}{\Gamma (\frac{\beta}{2}) \Gamma (\beta - \frac{\beta}{2})} + \frac{\Gamma (\frac{\beta}{2}) \Gamma (\beta - \frac{\alpha}{2})}{2 \Gamma (\frac{\beta}{2}) \Gamma (\beta - \frac{\beta}{2})} \right] \omega_2(f, \xi). \tag{82}
\]

Hence as \( \xi \to 0 \) we get \( M_\xi \to I \) in the \( L_1 \) norm.

Proof. From (55) we have

\[
|M_\xi f - f(x)| \leq W \left( \int_0^\infty |\Delta_x^\alpha f(x)| \frac{1}{(y^{2\alpha} + \xi^{2\alpha})^\beta} dy \right). \tag{83}
\]

Hence we get

\[
\int_{-\infty}^\infty |M_\xi f(x) - f(x)| dx \leq W \int_{-\infty}^{\infty} \left( \int_0^{\infty} |\Delta_x^\alpha f(x)| \frac{1}{(y^{2\alpha} + \xi^{2\alpha})^\beta} dy \right) dx
\]

\[
= W \int_0^{\infty} \left( \int_{-\infty}^{\infty} |\Delta_x^\alpha f(x)| dx \right) \frac{1}{(y^{2\alpha} + \xi^{2\alpha})^\beta} dy
\]

\[
= W \int_0^{\infty} \left( \int_{-\infty}^{\infty} |\Delta_x^\alpha f(x-y)| dx \right) \frac{1}{(y^{2\alpha} + \xi^{2\alpha})^\beta} dy
\]

\[
= W \int_0^{\infty} \left( \int_{-\infty}^{\infty} |\Delta_x^\alpha f(x)| dx \right) \frac{1}{(y^{2\alpha} + \xi^{2\alpha})^\beta} dy
\]

\[
\leq W \int_0^{\infty} \omega_2(f, \xi) \frac{1}{(y^{2\alpha} + \xi^{2\alpha})^\beta} dy.
\]
\[
\begin{align*}
\leq W \omega_2(f, \xi_1) \int_0^\infty \left(1 + \frac{y}{\xi}\right)^2 \frac{1}{(y^{2\alpha} + \xi^{2\alpha})^{\beta}} dy \\
= W \omega_2(f, \xi_1) \int_0^\infty \left(1 + x\right)^2 \frac{1}{(x^{2\alpha} + 1)^{\beta} \xi^{2\alpha \beta}} dx \\
= \frac{\Gamma(\beta) \alpha}{\Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\beta - \frac{1}{2\alpha}\right)} \omega_2(f, \xi_1) \int_0^\infty \left(1 + x\right)^2 \frac{1}{(x^{2\alpha} + 1)^{\beta}} dx \\
= \left[\frac{1}{2} + \frac{\Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\beta - \frac{1}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\beta - \frac{1}{2\alpha}\right)} + \frac{\Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\beta - \frac{3}{2\alpha}\right)}{2\Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\beta - \frac{1}{2\alpha}\right)} \right] \omega_2(f, \xi_1).
\end{align*}
\]

We have established (82). ■

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