Cosmological solutions with nonlinear bulk viscosity

Luis P Chimento†, Alejandro S Jakubi†, Vicenç Méndez‡ and Roy Maartens§

† Departamento de Física, Universidad de Buenos Aires, 1428 Buenos Aires, Argentina
‡ Departament de Física, Universitat Autònoma de Barcelona, Bellaterra 08193, Spain
§ School of Computer Science and Mathematics, Portsmouth University, Portsmouth PO1 2EG, Britain

Abstract.

A recently proposed nonlinear transport equation is used to model bulk viscous cosmologies that may be far from equilibrium, as happens during viscous fluid inflation or during reheating. The asymptotic stability of the de Sitter and Friedmann solutions is investigated. The former is stable for bulk viscosity index $q < 1$ and the latter for $q > 1$. New solutions are obtained in the weakly nonlinear regime for $q = 1$. These solutions are singular and some of them represent a late-time inflationary era.

PACS numbers: 9880H, 0440N, 0570L, 9880C

Short title: Cosmological solutions with nonlinear bulk viscosity

September 21, 2018
Dissipative processes in isotropic and homogeneous Friedmann-Lemaître-Robertson-Walker (FLRW) universes are restricted to scalar dissipation by the spacetime symmetries. Such scalar dissipation may be analyzed via the relativistic theory of bulk viscosity (see [1, 2] and references cited there). Relativistic thermodynamics considers processes which remain close to equilibrium, so that the transport equation is linear in the bulk viscous stress. The original and often used theory of Eckart, and a similar theory due to Landau and Lifshitz, are non-causal (they admit superluminal signals) and their equilibrium states are unstable [3]. The theory of Israel and Stewart [4, 5], and related theories [6], overcome these pathologies, since they are causal and stable under a wide range of conditions [7].

There are dissipative processes in cosmology which are not close to equilibrium. For example, inflation driven by a viscous fluid necessarily involves a bulk viscous stress that exceeds the equilibrium pressure [1, 8]. The reheating era at the end of inflation also involves far-from-equilibrium dynamics. In [8], reheating was analyzed via causal thermodynamics. In that model, the bulk viscous stress is small, and the dominant non-equilibrium effects arise from particle creation. It is also possible to consider models with an increased contribution from the bulk viscous stress, beyond the linear regime.

In order to treat these and other dissipative processes that do not remain close to equilibrium, one requires a nonlinear generalization of the standard theories. Recently a nonlinear theory was developed in [8], via a simple phenomenological generalization of the Israel-Stewart theory. The nonlinear theory reduces to the causal theory in the linear regime, and the second law of thermodynamics is built into the theory. There is consequently an upper limit to the bulk viscous stress, as in nonlinear generalizations of the non-relativistic heat flow equation [3] and shear viscous stress equation [10].

One can define at each event a local reference equilibrium state with energy density \( \rho \), pressure \( p \), number density \( n \), specific entropy \( s \) and temperature \( T \). Then the energy-momentum tensor of the fluid for the case of scalar dissipation (i.e., no energy or particle flux and no anisotropic stress) is

\[
T_{\alpha\beta} = (\rho + p + \Pi)u_\alpha u_\beta + (p + \Pi)g_{\alpha\beta},
\]

where \( \Pi (\leq 0) \) is the bulk viscous stress, \( u^\alpha \) is the four-velocity with respect to which \( \rho, p \) and \( n \) coincide with the local equilibrium values (in FLRW spacetime, this will be the preferred four-velocity), and \( g_{\alpha\beta} \) is the metric (specialized to FLRW below). The effective non-equilibrium specific entropy is [3, 8]

\[
s_{\text{eff}} = s - \left( \frac{\tau}{2nT\zeta} \right) \Pi^2, \tag{1}
\]

where \( \tau(\rho, n) \) is the characteristic timescale for linear relaxational effects (the crucial
thermodynamic parameter in Israel-Stewart theory that ensures causality), and $\zeta(\rho,n)$ is the linear bulk viscosity. The evolution of the equilibrium and effective specific entropies is given by [8]

$$\dot{s} = -\frac{3H\Pi}{nT},$$

$$\dot{s}_{\text{eff}} = \frac{\Pi^2}{nT\zeta} \left[ 1 + \frac{\tau_*}{\zeta} \Pi \right]^{-1},$$

where $H = \frac{1}{3} \nabla_a u^a$ is the Hubble rate and $\tau_*$ is a characteristic timescale for nonlinear effects. The second law of thermodynamics imposes an upper limit on the bulk stress [8]:

$$|\Pi| \leq \frac{\zeta}{\tau_*}.$$ (4)

(Note that in the linear theory, there is no upper limit on $|\Pi|$ that is built into the theory – $|\Pi|/p \ll 1$ is implicitly imposed a priori since the theory is linear.)

The nonlinear transport equation for $\Pi$ is [8]

$$\tau \dot{\Pi} \left( 1 + \frac{\tau_*}{\zeta} \Pi \right) + \Pi (1 + 3\tau_* H)$$

$$= -3\zeta H - \frac{1}{2} \tau \Pi \left[ 3H + \frac{\dot{H}}{\tau} - \frac{\dot{\zeta}}{\zeta} - \frac{\dot{T}}{T} \right] \left( 1 + \frac{\tau_*}{\zeta} \Pi \right).$$ (5)

Israel-Stewart theory (in its full, ‘non-truncated’ form [4 [4]) is recovered when $\tau_* = 0$. If in addition $\tau = 0$, then the transport equation reduces to that of Eckart.

In a flat FLRW universe, Einstein’s equations are†

$$H^2 = \frac{1}{3}\rho,$$ (6)

$$\dot{H} + H^2 = -\frac{1}{6}(\rho + 3p + 3\Pi),$$ (7)

so that by (5) and (7), we get a second-order evolution equation for $H$ when $\tau \neq 0$. In order to find this evolution equation for nonlinear dissipative processes in a flat FLRW universe, we need to specify the thermodynamic parameters $\zeta, \tau$ and $\tau_*$, and the equations of state relating $p, \rho, T$ and $n$. These will depend on the physical situation being analyzed, and all quantities except $\tau_*$ follow from standard thermodynamics. In the nonlinear theory of [5], the microscopic interactions governing far-from-equilibrium behaviour are unknown and encoded into the phenomenological parameter $\tau_*$. As a first step towards a more complete analysis, we follow [8] and postulate here a simple ansatz for $\tau_*$, which allows us to investigate the qualitative nature of the effects of nonlinearity:

$$\tau_* = k^2 \tau,$$ (8)

† We use units with $8\pi G = 1 = c$ and $k_B = 1$. 
where \( k \) is a dimensionless constant. This assumption is a mathematical simplification, but will allow us to find some overall features of nonlinear viscosity (whose detailed physics is not known). The weakly nonlinear regime is then characterized by \( k^2 \ll 1 \), and we will linearize the evolution equation for the Hubble rate below.

The linear relaxation time is related to the bulk viscosity by the physical relation

\[
\tau = \frac{\zeta}{v^2(\rho + p)},
\]

where \( v \) is the dissipative contribution to the speed of sound \( V \) (\( \leq 1 \)), so that \( V^2 = c_s^2 + v^2 \), with \( c_s \) the adiabatic contribution. This relation follows from an analysis of the propagation of small disturbances in a dissipative fluid. A simple form for the bulk viscosity is often taken to be \( \zeta \propto \rho^{\frac{q}{2}} \) (\( q \) constant), so that by (6) we have

\[
\zeta = \alpha H^q,
\]

where \( \alpha \) (\( \geq 0 \)) is a constant. Although this is a mathematical ansatz introduced for simplicity, it does approximate the physical form of \( \zeta \) for certain fluids [12].

We also need to specify the equations of state. For the pressure, we will take the simple linear barotropic law

\[
p = (\gamma - 1)\rho,
\]

where \( \gamma \) is a constant. Typically \( \gamma \approx \frac{4}{3} \) (radiation-dominated expansion) or \( \gamma \approx 1 \) (matter-dominated expansion). (Note that the limiting case of dust, when \( p \) is exactly 0 and \( \gamma \) is exactly 1, is strictly ruled out, since the temperature is zero for dust and there can be no bulk viscous stress.) By (11), \( c_s^2 = \gamma - 1 \) is a constant, and we will assume that \( v \) is also constant, subject to the causality constraint

\[
v^2 \leq 2 - \gamma.
\]

This is a mathematical simplification, but it should not be unreasonable if (11) holds.

By equations (6) and (8)–(11), we can use the above thermodynamic assumptions to rewrite the upper limit (4) on the bulk stress, and to express \( \tau_* \) in terms of the Hubble rate:

\[
|\Pi| \leq \frac{\zeta}{\tau_*} = \frac{\gamma}{\rho} \left( \frac{v}{k} \right)^2 \leq \left[ \frac{\gamma(2 - \gamma)}{k^2} \right] \rho,
\]

\[
\tau_* = \left( \frac{\alpha k^2}{3\gamma v} \right) H^{q-2}.
\]

It follows from (14) that in the case \( q = 1 \), the nonlinear (and linear) timescales are directly determined by the expansion timescale \( H^{-1} \).
Finally, for the temperature, we will consider two cases: (a) barotropic temperature, 
\[ T = T(\rho), \]
so that
\[ T \propto \rho^{(\gamma - 1)/\gamma}, \]  
(15)
and (b) ideal-gas temperature, so that
\[ p = nT. \]  
(16)
Note that (15) and (16) can only hold simultaneously in equilibrium [2, 11, 13].

Equations (6)–(11) together with (15) and the number conservation equation
\[ \dot{n} + 3Hn = 0 \]
give the evolution equation for case (a) as
\[
\left[ 1 - \frac{k^2}{v^2} - \left( \frac{2k^2}{3\gamma v^2} \right) \frac{\dot{H}}{H^2} \right] \left\{ \ddot{H} + 3H\dot{H} + \left( \frac{1 - 2\gamma}{\gamma} \right) \frac{\dot{H}^2}{H} + \frac{9}{4} \gamma H^3 \right\} + \frac{3\gamma v^2}{2\alpha} \left[ 1 + \left( \frac{\alpha k^2}{\gamma v^2} \right) H^{q-1} \right] H^{2-q} \left( 2\dot{H} + 3\gamma H^2 \right) - \frac{9}{2} \gamma v^2 H^3 = 0. \]  
(17)
In case (b), the ideal-gas behaviour (16) leads to the new and simpler evolution equation
\[
\left( 1 - \frac{k^2}{v^2} - \frac{2k^2}{3\gamma v^2} \right) \left\{ H\ddot{H} - 2\dot{H}^2 - \frac{9}{2} \gamma v^2 H^4 \right\} + \frac{3v^2\gamma}{2\alpha} H^{3-q} (2\dot{H} + 3\gamma H^2) = 0. \]  
(18)

Considerable work has been done using the linear non-causal model of bulk viscosity (see [12, 1] for references), including the case of cosmic strings [14, 15, 16, 17]. Our results represent a generalization to the nonlinear case of previous dynamical analysis in the linear causal theory [12, 13, 18, 19, 20, 21], and also a generalization of some of the results in [8], where exact de Sitter solutions for any \( q \), and a power-law solution with \( q = 1 \) were given for equation (17). Here we find new solutions with \( q = 1 \) in the weakly nonlinear regime for both (17) and (18). We also investigate asymptotic solutions and their stability for \( q \neq 1 \). In principle our solutions and their properties could be used to model far-from-equilibrium processes, such as viscous fluid inflation and reheating. However, we will not pursue such applications here.
Models with barotropic temperature

2.1. Solution for \( q = 1 \)

When the bulk viscosity index satisfies \( q = 1 \), we have from (10) and (14) that \( \zeta \propto H \) and \( \tau, \tau_* \propto H^{-1} \). Assuming that \( k^2 \ll 1 \) and \( \dot{H}/H^2 \) is bounded, we can isolate \( \ddot{H} \) in (17) to first order in \( k^2 \):

\[
H \ddot{H} + \left( \frac{1 - 2\gamma}{\gamma} + \frac{2k^2}{\alpha} \right) \dot{H}^2 + 3 \left( 1 + \frac{\gamma v^2}{\alpha} + \frac{2\gamma k^2}{\alpha} \right) H^2 \dot{H} \\
+ \frac{9\gamma}{2} \left( \frac{1}{2} + \frac{\gamma v^2}{\alpha} - v^2 + \frac{\gamma k^2}{\alpha} \right) H^4 = 0.
\]

(19)

With the change of variables

\[
H = y^n, \quad t = \theta/(3m),
\]

where

\[
\frac{1}{n} = (1 - \gamma) \frac{2k^2}{\gamma} + \frac{2k^2}{\gamma}, \quad m = 1 \frac{\gamma v^2}{\alpha} + \frac{2\gamma k^2}{\alpha},
\]

equation (19) becomes†

\[
\frac{d^2 y}{d\theta^2} + y^n \frac{dy}{d\theta} + \frac{\beta}{n+1} y^{1+2n} = 0,
\]

(20)

where

\[
\beta = \left[ \frac{\alpha (\alpha + 2\gamma v^2 - 2\alpha v^2)}{4 (\alpha + \gamma v^2)^2} \right] + \left[ \frac{\gamma v^2 (\alpha + 2\gamma v^2 - \alpha v^2)}{(\alpha + \gamma v^2)^3} \right] k^2 + O(k^4).
\]

(21)

Then, making the nonlocal transformation \[20\]

\[
z = \frac{1}{n+1} y^{n+1}, \quad d\eta = y^n d\theta,
\]

(22)

equation (21) becomes linear:

\[
\frac{d^2 z}{d\eta^2} + \frac{dz}{d\eta} + \beta z = 0.
\]

(23)

From (21) it is easy to check that \( 1 - 4\beta > 0 \), so we obtain the general solution of (19) in parametric form

\[
H(\eta) = \left[ (n+1) \{ C_+ \exp (\lambda_+ \eta) + C_- \exp (\lambda_- \eta) \} \right]^{1/(n+1)},
\]

(24)

\[
t(\eta) = t_0 + \left( \frac{1}{3m} \right) \int \frac{d\eta}{H(\eta)},
\]

(25)

† Note that \( n = -1 \) is ruled out since \( k^2/\alpha \) is non-negative.
where \( \lambda_\pm = [-1 \pm (1 - 4\beta)^{1/2}] / 2 \) are real, and \( t_0, C_\pm \) are arbitrary integration constants. When either of \( C_\pm \) vanishes, we obtain one-parameter families of solutions, which have the form:

\[
H_\pm(t) = \frac{\nu_\pm}{(t - t_0)} \text{ if } \alpha \neq \alpha_0 = \frac{2\gamma (v^2 + k^2)}{2v^2 - 1},
\]

\[
H_- = \frac{\nu_0}{(t - t_0)}, \quad H_+ = H_0 \text{ if } \alpha = \alpha_0,
\]

where

\[
\nu_\pm = \left( \frac{2}{3\gamma} \right) \frac{m \pm \left( m^2 - [m + (\gamma/\alpha - 2) v^2] (1 + 2\gamma k^2/\alpha) \right)^{1/2}}{m + (\gamma/\alpha - 2) v^2},
\]

\[
\nu_0 = \frac{\alpha + 2\gamma k^2}{3\alpha\gamma m},
\]

and \( H_0 \) is an arbitrary positive constant giving a de Sitter solution (with \( v > 1/\sqrt{2} \)), which agrees with the linearization of the de Sitter solution given in [8], equation (41).

The parameter \( \nu_- \) is smaller than the perfect fluid value: \( 0 < \nu_- < 2/(3\gamma) < 1 \), and for small dissipative contribution to the speed of sound we get \( \nu_- \approx 2(3\gamma)^{-1}(1 - \sqrt{2}v) \). Also \( \nu_0 < (3\gamma)^{-1} < 1 \) for \( 1 < \gamma < 2 \), while \(-\infty < \nu_+ < \infty \). Thus, (27+) contains singular solutions without particle horizons.

Evaluating the local-equilibrium entropy \( s \) from (2), we find

\[
s(t) = s_0 + \gamma \left[ \frac{(3\nu_\pm^2)^{1/\gamma} a_0^3}{n_0 T_0} \right] (t - t_0)^{3\nu_\pm - 2/\gamma},
\]

for the solution given by (26), and the same for (27–), but with \( \nu_0 \) instead of \( \nu_\pm \). The effective specific entropy \( s_{\text{eff}} \) is given by (1):

\[
s_{\text{eff}}(t) = s_0 + \left[ \frac{(3\nu_\pm^2)^{1/\gamma} a_0^3}{n_0 T_0} \right] \gamma - \frac{1}{2v^2\gamma} \left( \frac{3\gamma \nu_\pm - 2}{3\nu_\pm} \right)^2 (t - t_0)^{3\nu_\pm - 2/\gamma}.
\]

Hence the requirement \( s_{\text{eff}} \geq 0 \) is fulfilled for

\[
\frac{3\nu_\pm \gamma - 2}{3\nu_\pm} < \sqrt{2}v\gamma.
\]

For the solution (27–), the restriction (32) and the solution (31) are the same, but \( \nu_\pm \) is replaced by \( \nu_0 \). Provided that \( \nu_0 > 0 \), then this restriction requires \( v > 1/\sqrt{2} \).

The second restriction, arising from positivity of the entropy production is

\[
\frac{3\nu_\pm \gamma - 2}{3\nu_\pm} < \frac{\gamma v^2}{k^2}
\]

for the solution (26), and the same but replacing \( \nu_\pm \) by \( \nu_0 \) for (27). Thermodynamics restrictions (32) and (33) are the same one for \( v = \sqrt{2}k^2 \).
We are able to find a third solution. Provided the parameters satisfy the constraint
\[
\gamma \left[ \frac{m + (\gamma/\alpha - 2) v^2}{8\alpha m^2} \right] = \frac{\alpha^2(1 - \gamma) + 2\gamma\alpha k^2}{[\alpha(2 - \gamma) + 4\gamma k^2]^2},
\]
the general solution of (19) takes an explicit form, and we get for the scale factor
\[
a(t) = a_0 \left| t - t_0 \right|^{n+1} + K \left| t - t_0 \right|^{(n+2)/3nm},
\]
where \( K \) is an integration constant. We have checked that the constraint (34) is satisfied for suitable values of the parameters.

As examples, we apply the two thermodynamical restrictions (32) and (33) and the causality constraint (12) to the solutions (26) and (27), for a radiation-dominated expansion, \( \gamma = \frac{4}{3} \). Causality demands in this case \( v^2 \leq \frac{2}{3} \).

- For the solution (26) we choose \( v^2 = \frac{1}{2} \) and \( \alpha = 1 \). Causality and the two thermodynamical restrictions are fulfilled for any \( k^2 \ll 1 \). By (28), it follows that for small \( k \) (\( k < 0.32 \)), we have \( \nu_+ > 1 \), so that these solutions are always inflationary, and \( \nu_- < 1 \), corresponding always to non-inflationary expansion. The maximum value of \( \nu_+ \) is \( \frac{1}{8}(5 + \sqrt{13}) \), and the minimum value of \( \nu_- \) is \( \frac{1}{8}(5 - \sqrt{13}) \). Thus \( a \sim t^N \) where \( 0.17 < N < 1.08 \). For small \( \alpha \) (perfect fluid limit) we get \( \nu_+ \approx 2(3\gamma)^{-1} < 1 \) and \( \nu_- \approx (2/3\gamma)(k^2/(k^2 + v^2)) \ll 1 \). Thus the solution with \( \nu_+ \) corresponds to the perfect fluid limit solution, while the other corresponds to a new solution that does not exist in this limit. On the other hand, for large \( \alpha \) we get
\[
\nu_\pm \approx \frac{2}{3\gamma} \frac{1}{1 \pm \sqrt{2}v}
\]
thus, \( \nu_\pm \) may be very large for \( v^2 \) close to \( 1/2 \) (expanding solutions require \( v^2 < 1/2 \)).

- For the solution (27–), we choose \( v^2 = \frac{2}{3} \), and by (29)
\[
\nu_0 = \frac{3}{16} \left( 1 - \frac{5}{4}k^2 \right) + O(k^4), \quad \alpha_0 = \frac{4}{3} \left( 2 + 3k^2 \right).
\]
The solution is not inflationary, and causality and thermodynamic requirements are always fulfilled.

The requirement that \( \dot{H}/H^2 \) remains bounded means that not all the two-parameter families of solutions (25) of equation (19) are also solutions of (17), but it is a consequence of the thermodynamic restriction given by positivity of the entropy production. So, from (1), (6)–(9) and (11), we find that \( \dot{H}/H^2 \) is bounded:
\[
\frac{\dot{H}}{H^2} \leq \frac{3\gamma}{2k^2}(v^2 - k^2).
\]
Solutions that satisfy this behave as follows [13]:

The evolution begins at a singularity with a Friedmann leading behaviour of the form (26–) for \( \alpha \neq \alpha_0 \) or (27–) for \( \alpha = \alpha_0 \). Then the expansion becomes
1. asymptotically Friedmann, like (26+), for $\alpha < \alpha_0$;
2. asymptotically de Sitter, like (27+), for $\alpha = \alpha_0$;
3. divergent at finite time, with leading behaviour (26+), for $\alpha > \alpha_0$.

2.2. Asymptotic solutions for $q > 1$ and stability

For $q > 1$ it is easy to check that (17) admits a solution whose leading term is $2/(3\gamma t)$ when $t \to \infty$. To study its stability, we make the change of variables

$$H^{-1} = t[u(z)]^{\gamma/(\gamma - 1)}, \ t^{q-1} = z,$$

when $\gamma > 1$. Then (17) takes the form

$$u'' + \left\{ \frac{6\gamma v^2}{\alpha R} u^{(q-2)/(\gamma - 1)} + \frac{6}{\gamma} \left( \frac{1 + k^2}{R} u^{(1-\gamma)} + q - \frac{2}{\gamma} \right) \frac{1}{z} \right\} \frac{u'}{q-1}$$

$$+ \frac{3(\gamma - 1) v^2}{(q-1)^2 \alpha R} \left[ u^{(\gamma q - 1)/(\gamma - 1)} - \frac{3\gamma}{2} u^{(\gamma q - 1)/(\gamma - 1)} \right] \frac{1}{z}$$

$$\frac{\gamma - 1}{(q-1)^2} \left[ \frac{9}{4} k^2 - 2\gamma^2 + R u^{(\gamma + 1)/(1-\gamma)} + \frac{3k^2 + R}{\gamma R} u^{(1/(1-\gamma)} - \frac{u}{\gamma^2} \right] \frac{1}{z^2},$$

(36)

where a prime denotes $d/dz$ and

$$R = 1 - \frac{k^2}{v^2} + \left( \frac{2k^2}{3\gamma v^2} \right) u^{\gamma/(\gamma - 1)} \left[ 1 + \frac{\gamma (q - 1)}{\gamma - 1} \frac{zu'}{u} \right].$$

(37)

We may rewrite (36) in the form

$$\frac{d}{dz} \left[ \frac{1}{2} u'^2 + U(u, z) \right] = D(u, u', z).$$

(38)

When $z \to \infty$, equation (36) has a constant solution $u_0 = \left( \frac{3}{2\gamma} \right)^{(\gamma-1)/\gamma}$, which is the unique minimum of $U(u, z)$. Thus, considering that $u$ lies in a neighbourhood of $u_0$, we get from (37) that

$$R \approx 1 - \frac{\gamma}{\gamma - 1} \left( \frac{2}{3\gamma} \right)^{1/\gamma} \left( \frac{k}{v} \right)^2 zu'.$$

(39)

We can see how $R \to 1$ when $z \to \infty$ through a linearization of (36):

$$u'' + 2(q - 1)\Omega u' + \Omega \frac{(u - u_0)}{z} = 0,$$

(40)

where $\Omega = 3(\gamma/\alpha)v^2(3\gamma/2)^{q-2}(q - 1)^{-2}$. By the change of variable $u = u_0 + w \exp[(1-q)\Omega z]$, equation (40) becomes

$$w'' + \left[ -(q - 1)^2 \Omega^2 + \frac{\Omega}{z} \right] w = 0,$$

(41)
which can be solved in terms of confluent hypergeometric functions. It has the following leading behaviour for $z \to \infty$:

$$w(z) \sim C_1 z^{1/2(1-q)} \exp \left[ (q-1)\Omega z \right] + C_2 z^{1/2(q-1)} \exp \left[ (1-q)\Omega z \right],$$  \hspace{1cm} (42)

where $C_{1,2}$ are arbitrary integration constants. Thus $zu' \sim z^{1/2(1-q)} \to 0$ for $z \to \infty$ and $q > 1$, and we get the expansions

$$U(u, z) = \frac{3 (\gamma - 1)^2 v^2}{(q - 1)^2 \alpha} \left[ \frac{u^{(\gamma-2)/(\gamma-1)}}{\gamma q - 2} - \frac{3\gamma u^{(\gamma-2)/\gamma}(\gamma-1)}{2 (\gamma q - 2 - \gamma)} \right] \frac{1}{z} + O \left( \frac{1}{z^2} \right),$$  \hspace{1cm} (43)

for $\gamma q \neq 2$ or $2 + \gamma$, 

$$U(u, z) = \frac{3 (\gamma - 1)^2 v^2}{(q - 1)^2 \alpha} \left[ \ln \frac{u}{u_0} + \frac{3 (\gamma - 1)}{2} u^{\gamma/(\gamma-1)} \right] \frac{1}{z} + O \left( \frac{1}{z^2} \right),$$  \hspace{1cm} (44)

for $\gamma q = 2$, 

$$U(u, z) = \frac{3 (\gamma - 1)^2 v^2}{(q - 1)^2 \alpha} \left[ \frac{\gamma - 1}{\gamma} u^{\gamma/\gamma-1} - \frac{3\gamma}{2} \ln \frac{u}{u_0} \right] \frac{1}{z} + O \left( \frac{1}{z^2} \right),$$  \hspace{1cm} (45)

for $\gamma q = 2 + \gamma$, and 

$$D(u, u', z) = -\left[ \frac{6\gamma v^2}{\alpha (q - 1)} \right] u^{(\gamma-2)/(\gamma-1)} u'^2 + O \left( \frac{1}{z} \right).$$  \hspace{1cm} (46)

For mathematical completeness, we will consider also the case $\gamma = 1$. We make the change of variables 

$$H^{-1} = t \exp \left[ -u(z) \right], \quad z = t^{q-1},$$

in (17), which then takes the form

$$u'' + \left( u'' + \frac{3u^q e^{-(q-1)u} + (q - 2) e^{-u} + 3\left( \frac{k^2}{R} + 1 \right) \frac{1}{z} }{4 (q - 1)^2 \alpha} \left[ 3e^{-(q-2)u} - 2e^{-(q-1)u} \right] \frac{1}{z} + \frac{1}{4 (q - 1)^2} \left[ 2e^{-u} + 9 \left( \frac{k^2 - v^2}{R} + \frac{1}{2} \right) e^u - 12 \left( 1 + \frac{k^2}{R} \right) \right] \frac{1}{z^2} \right) = 0.$$  \hspace{1cm} (47)

When $z \to \infty$, equation (47) has a constant solution $u_0 = \ln \frac{3}{2}$. Again, we may write (47) in the form (48), and find that $u_0$ is the unique minimum of $U(u, z)$. Thus we get 

$$U(u, z) = \frac{3v^2}{4\alpha (q - 1)^2} \left[ \frac{2}{q - 1} e^{-(q-1)u} - \frac{3}{q - 2} e^{-(q-2)u} \right] \frac{1}{z} + O \left( \frac{1}{z^2} \right),$$  \hspace{1cm} (48)

for $q \neq 2$, 

$$U(u, z) = \frac{3v^2}{4\alpha} \left( 2e^{-u} - 3u \right) \frac{1}{z} + O \left( \frac{1}{z^2} \right),$$  \hspace{1cm} (49)
for $q = 2$, and

$$D(u, u', z) = -\frac{3v^2}{2\alpha (q - 1)} e^{-(q-1)u} u'^2 + O\left(\frac{1}{z}\right). \tag{50}$$

As $U(u, z)$ has a unique minimum at $u_0$ for any $q \neq 1$, and $D(u, u', z)$ is negative definite for $q > 1$, we find that solutions with leading Friedmann behaviour $a \sim t^{2/(3\gamma)}$ when $t \to \infty$ are asymptotically stable for $q > 1$.

3. Models with ideal-gas temperature

3.1. The de Sitter solution

Using the ideal gas equation $p = nT$ instead of the power-law for the temperature one obtains (18). Setting $H = H_0$ in (18) then gives

$$H_0^{1-q} = \frac{\alpha}{v^2 \gamma} (v^2 - k^2).$$

The upper bound (13) on the bulk stress implies the limit $k \leq v$. Thus de Sitter solutions may exist. On the other hand the specific entropy may be calculated and one obtains (for $\gamma \neq 1$)

$$s(t) = s_0 + 3H_0 \left(\frac{\gamma}{\gamma - 1}\right) t, \quad s_{\text{eff}}(t) = s(t) - \frac{\gamma}{v^2 (\gamma - 1)}.$$

Linear analysis around the fixed point $(0, H_0)$ gives the following information: for $1 - \omega_1 \leq q < 1$, it is an asymptotically stable node; for $q > 1$, it is a saddle point; for $q < 1 - \omega_1$, it is an asymptotically stable focus, where

$$\omega_1 = \frac{v^6}{2\gamma (v^2 - k^2)\gamma > 0}.$$

Thus the solution is an attractor for $q < 1$ (as in the barotropic temperature case [8]). For $q = 1$, de Sitter inflation occurs for $H_0$ arbitrary, provided that $\alpha = v^2 \gamma / (v^2 - k^2)$.

3.2. Solution for $q = 1$

When $q = 1$, power-law solutions $a \sim t^N$ exist if, by (18),

$$N = \left(\frac{2}{3\gamma}\right) \frac{\alpha k^2 + v^2 \gamma}{\alpha (k^2 - v^2) + v^2 \gamma} = \frac{2}{3\gamma} \left[1 + \frac{\alpha}{\gamma} + O(\alpha^2)\right]. \tag{51}$$

For positive entropy generation we require

$$k \leq \left(\frac{3\gamma N}{3\gamma N - 2}\right)^{1/2} v, \tag{52}$$
but it is easy to show that this restriction is always fulfilled for (51) with arbitrary $v$, $k$ and $\alpha$. From (1) and (2) with $\gamma \neq 1$, we find

$$s(t) = s_0 + \left( \frac{3\gamma N - 2}{\gamma - 1} \right) \ln t, \quad s_{\text{eff}}(t) = s(t) - \frac{(3\gamma N - 2)^2}{18N^2v^2\gamma(\gamma - 1)},$$

so that positive $s$ requires $N > 2/3\gamma$ – which is always fulfilled for expanding solutions, as one can see from (51). These solutions ($N > 0$) require

$$k^2 > \left( \frac{\alpha - \gamma}{\alpha} \right) v^2.$$

In order to get an inflationary power-law solution, it is necessary that $\alpha > \alpha_{\text{min}}$ where

$$\alpha_{\text{min}} = \frac{v^2\gamma(3\gamma - 2)}{3\gamma(v^2 - k^2) + 2k^2}.$$

For $q = 1$, and assuming that $k^2/v^2 \equiv \epsilon^2 \ll 1$ and $\dot{H}/H^2$ is bounded, we can isolate $\ddot{H}$ in (18) to first order in $\epsilon^2$:

$$H\ddot{H} + \frac{3\gamma v^2}{\alpha} \left( 1 + 2\epsilon^2 \right) H^2 \dot{H} + 2 \left( \frac{k^2}{\alpha} - 1 \right) \dot{H}^2 + \frac{9}{2} v^2\gamma \left[ -1 + \frac{\gamma}{\alpha} \left( 1 + \epsilon^2 \right) \right] H^4 = 0. \quad (53)$$

With the change of variables $H = y^n, \ t = \alpha \theta/[3\gamma v^2(1 + 2\epsilon^2)],$ where $n = \alpha/(2k^2 - \alpha)$, equation (53) takes exactly the same form as (20), with

$$\beta = \frac{\epsilon^2(\gamma - \alpha + \gamma\epsilon^2)}{\gamma(1 + 2\epsilon^2)^2}. \quad (54)$$

Then, making the same nonlocal transformation (22) as before, equation (24) is again brought to the linear form (23). From (54) we find that $1 - 4\beta > 0$, so we obtain the general solution of (53) in parametric form

$$H(\eta) = \{(n + 1) \left[ C_+ \exp (\lambda_+ \eta) + C_- \exp (\lambda_- \eta) \right]\}^{\alpha/2k^2}, \quad (55)$$

$$t(\eta) = t_0 + \left[ \frac{\alpha}{3\gamma v^2(1 + 2\epsilon^2)} \right] \int \frac{d\eta}{H(\eta)}, \quad (56)$$

where $\lambda_\pm = [-1 \pm (1 - 4\beta)^{1/2}]/2$ are real, and $t_0$ and $C_\pm$ are arbitrary integration constants. When either of $C_\pm$ vanishes, we obtain one-parameter families of solutions, which have the form:

$$H_\pm(t) = \frac{\nu_\pm}{(t-t_0)} \quad \text{if} \quad \alpha \neq \alpha_0 \equiv \gamma \left( 1 + \frac{k^2}{v^2} \right), \quad (57)$$

$$H_- = \frac{\nu_0}{(t-t_0)}, \ H_+ = H_0 \quad \text{if} \quad \alpha = \alpha_0, \quad (58)$$
where

\[
\nu_{\pm} = \frac{1 + 2 \epsilon^2}{3(\gamma - \alpha + \gamma \epsilon^2)} \left\{ 1 \pm \left[ \frac{\gamma + 4 \epsilon^2 \alpha}{\gamma (1 + 2 \epsilon^2)^2} \right]^{1/2} \right\},
\]

\[
\nu_0 = \left( \frac{2}{3\gamma} \right) \frac{\epsilon^2}{1 + 2 \epsilon^2}.
\]

The specific entropy \( s \) is given by (30) and \( s_{\text{eff}} \) by (31), and the restrictions (32) and (33) also apply here, with \( \nu_{\pm} \) and \( \nu_0 \) given by (59) and (60), and satisfying similar restrictions to the barotropic temperature case. Note that \( \alpha_0 \) is the value of \( \alpha \) which arises from the requirement that the solution of (53) is de Sitter, and it can be also obtained as the first order expansion of the expression \( \alpha_0 = \gamma/(1 - \epsilon^2) \) that arises from (18).

On the other hand, provided the parameters satisfy the constraint

\[
\phi = \frac{1 - 8 \psi \epsilon^2 + 16 \psi^2 \epsilon^4}{1 + 2 \psi + \epsilon^2 - 4 \psi^2 \epsilon^2},
\]

where \( \phi \equiv \gamma/\alpha \) and \( \psi \equiv v^2/\alpha \), the general solution of (53) takes an explicit form, and we get for the scale factor

\[
a(t) = a_0 \left| t - t_0 \right|^{n+1} + K \right|^\mu,
\]

where \( K \) is arbitrary and

\[
\mu = \frac{4k^2 - \alpha}{3\gamma v^2 (1 + 2 \epsilon^2)}.
\]

We have checked that the constraint (61) is satisfied for suitable values of the parameters. For instance, if \( \epsilon < 0.1 \), then \( \phi < 1 \). This constraint also reduces, in the limit \( \epsilon \to 0 \), to the constraint on the parameters found in a previous paper for ideal-gas temperature in the linear theory [21]. It must be remarked however, that the form of the solutions is quite different. This shows that, in general, the solutions are not analytic in \( k \) for \( k \to 0 \). In other words, this limit is singular.

The requirement that \( \dot{H}/H^2 \) remains bounded means that not all the two-parameter families of solutions (56) of equation (53) are also (approximate) solutions of (18). Those that satisfy it behave as follows: The evolution begins at a singularity, with Friedmann leading behaviour as in (57–) for \( \alpha \neq \alpha_0 \), or (58–) for \( \alpha = \alpha_0 \). Then the expansion becomes

1. asymptotically Friedmann, as in (57+), for \( \alpha < \alpha_0 \);
2. asymptotically de Sitter, as in (58+), for \( \alpha = \alpha_0 \);
3. divergent at finite time, with leading behaviour (57+) for \( \alpha > \alpha_0 \). The relationships for \( s \) and \( s_{\text{eff}} \) and the the thermodynamical constraints (32) and (33) in the limit \( t \to \infty \), are the same, but with \( \nu_{\pm} \) replaced by \( \mu(n+1) \).
3.3. Asymptotic analysis for \( q > 1 \) and stability

For \( q > 1 \) it is easy to check that (18) admits a solution whose leading term is \( 2/(3\gamma t) \).

To study its stability, we make the change of variables

\[
H^{-1} = tu(z), \quad t^{q-1} = z,
\]

in (18), which then becomes

\[
u'' + \left( \frac{6\gamma v^2}{\alpha R} u^{q-2} + \frac{q}{z} \right) \frac{u'}{q-1} + \frac{3\gamma v^2}{(q-1)^2 \alpha R z} \left( u^{q-1} - \frac{3\gamma}{2} u^{q-2} \right) + \frac{9\gamma v^2}{2(q-1)^2 z^2 u} = 0 ,
\]

(64)

where

\[
R = 1 - e^2 + \frac{2e^2}{3\gamma} [u + (q-1)zu'] .
\]

We rewrite (64) in the same form as before. When \( z \to \infty \), equation (64) has a constant solution \( u_0 = \frac{3}{2}\gamma \), which is the unique minimum of \( U(u,z) \). Thus, considering that \( u \) lies in a neighbourhood of \( u_0 \), we get

\[
R \approx 1 + \frac{2e^2}{3\gamma} (q-1)zu'.
\]

We can see how \( R \to 1 \) when \( z \to \infty \) through a linearization of (64), which again leads to (10), with \( \Omega = 3(\gamma/\alpha)v^2(q-1)^{-2}(3\gamma/2)^{q-2} \). By the same change of variable \( u = u_0 + w \exp[(1-q)\Omega z] \), equation (10) is again brought to the form (11). As before, this can be solved in terms of confluent hypergeometric functions, and has the same leading behaviour for \( z \to \infty \). Thus \( zu' \sim z^{1/2(q-1)} \to 0 \) for \( z \to \infty \) and \( q > 1 \), and we get the expansions

\[
D(u,u',z) = -\frac{6\gamma v^2 u^{q-2}}{\alpha (q-1) R} u'^2 + O\left( \frac{1}{z} \right) ,
\]

\[
U(u,z) = \frac{3\gamma v^2}{(q-1)^2 \alpha} \left[ \frac{u^q}{q} - \frac{3\gamma}{2(q-1)} u^{q-1} \right] \frac{1}{z} + O\left( \frac{1}{z^2} \right) .
\]

As \( U(u,z) \) has a unique minimum at \( u_0 \) for any \( q \neq 1 \), and \( D(u,u',z) \) is negative definite for \( q > 1 \), we conclude that solutions with leading Friedmann behaviour \( a \sim t^{2/(3\gamma)} \) when \( t \to \infty \) are asymptotically stable for \( q > 1 \).
Concluding remarks

With the aim of treating dissipative processes that do not remain close to equilibrium, we have used the nonlinear generalization of the standard causal theories proposed in Ref. [8]. Barotropic and ideal-gas forms for the temperature have been used to close the dynamical equations of the universe.

The evolution of the universe is qualitatively similar in both cases, and we can summarize the main results as follows. Singular solutions are found for \( q = 1 \) with a power-law and inflationary behaviour in the limit \( t \to \infty \). The initial and final behaviour of the solutions follows closely that of the linear theory with barotropic temperature, in the ‘full’ causal as well as the ‘truncated’ causal and the noncausal cases. The logarithmic behaviour found for ideal-gas temperature in Ref. [21] no longer appears when the nonlinear transport equation is considered. Thermodynamic restrictions, arising from positivity of the entropy, have been also derived. The evolution begins at a singularity with a Friedmann leading behaviour, and the future dynamic is asymptotically Friedmannian, asymptotically de Sitter or divergent at finite time, for specific values of the bulk viscosity constant \( \alpha \).

The behaviour of particular de Sitter solutions has already been analyzed in Ref. [8], for barotropic temperature, and it is also derived here for the ideal-gas temperature. For \( q < 1 \), a stable exponential inflationary phase occurs in the far future when \( 1 \leq \gamma < 2 \). The condition \( \alpha = \alpha_0 \) is required if \( q = 1 \), and such behaviour is unstable for \( q > 1 \). Our results show that occurrence of viscosity-driven exponential inflationary behaviour depends mainly on the equation of state, rather than on the thermodynamical theory employed.

For \( q > 1 \) the perfect fluid behaviour \( a \sim t^{2/(3\gamma)} \) when \( t \to \infty \) is asymptotically stable, because the viscous pressure decays faster than the thermodynamical pressure. However, if \( q = 1 \) and \( \alpha < \alpha_0 \), both pressures decay asymptotically as \( t^{-2} \), and the exponent becomes \( \nu_+ \). The perfect fluid behaviour becomes unstable if \( q < 1 \).

In essence we have analyzed the existence of specific behaviour for a dissipative universe not close to equilibrium in terms of the thermodynamic parameters related to \( \tau, q \) and the equations of state.

A final comment concerns the robustness of our results relative to the spatial geometry and the source of the gravitational field. Firstly, the question arises as to what the effect of non-zero spatial curvature in the FLRW metric will be. In the case of non-causal bulk viscosity, where \( \Pi \) is algebraically determined by \( H \), it is relatively straightforward to investigate this question [14, 15, 16, 17]. By contrast, in the causal theory (linear as well as nonlinear), \( \Pi \) is not algebraically determined by \( H \) but satisfies an evolution equation that couples it differentially to the expansion. Curvature introduces the scale factor \( a = \exp \int H dt \) explicitly into the Friedmann equation, and
makes it more difficult to decouple the equations. Thus the effect of curvature in the
causal case will be far more difficult to determine in general. (In [18], curvature is
considered in the linear causal theory, but with highly simplified equations of state.)
This is a subject for further investigation.

Secondly, it could be interesting, but in general very difficult, to determine the effect
of introducing another fluid, without bulk viscous stress. In the non-causal theory, this
question is much less difficult to investigate [15, 16]. In the causal theory, this is another
subject for further investigation.

Acknowledgments

VM thanks the Spanish Ministry of Education for partial support under grant PB94-
0718. LPC and ASJ thank the University of Buenos Aires for partial support under
project EX-260.

References

[1] Maartens R 1995 Class. Quantum Grav. 12 1455
[2] Zimdahl W 1996 Phys. Rev. D 53 5483
[3] Lindblom L and Hiscock W A 1983 Ap. J. 267 384
[4] Israel W 1976 Ann. Phys. 100 310
[5] Israel W and Stewart J M 1979 Ann. Phys. 118 341
[6] Jou D, Casas-Vázquez J and Lebon G 1996 Extended Irreversible Thermodynamics (Berlin: Springer)
[7] Hiscock W A and Lindblom L 1983 Ann. Phys. 151 466
[8] Maartens R and Méndez V 1997 Phys. Rev. D 55 1937
[9] Zimdahl W, Pavon D and Maartens R 1997 Phys. Rev. D 55 4681
[10] Bidar H, Jou D and Criado-Sancho M 1996 Physica A 233 163
[11] Maartens R 1996 in H Rund Workshop on Relativity and Thermodynamics ed Maharaj S D (Natal University, to appear) [astro-ph/9609119]
[12] Pavon D, Bafaluy J and Jou D 1991 Class. Quantum Grav. 8 347
[13] Méndez V and Triginer J 1996 J. Math. Phys. 37 2906
[14] Barrow J D 1986 Phys. Lett. 180 235
[15] Barrow J D 1988 Nucl. Phys. B 310 743
[16] Barrow J D 1990 in The Formation and Evolution of Cosmic Strings ed Gibbons G W, Hawking S W and Vachaspati T (Cambridge: Cambridge University Press)
[17] Turok N 1988 Phys. Rev. Lett. 60 549
[18] Coley A A, van den Hoogen R J and Maartens R 1996 Phys. Rev. D 54 1393
[19] Chimento L P and Jakubi A S 1997 Class. Quantum Grav. 14 1811
[20] Chimento L P 1997 J. Math. Phys. 38 2565
[21] Chimento L P, Jakubi A S and Méndez V 1997 Int. J. Mod. Phys. D in press