Saturation Level of Turbulence in Collapsing Gas Clouds

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Abstract

We investigate the physical mechanism that determines the saturation level of the turbulence in collapsing gas clouds. We perform a suite of high-resolution numerical simulations that follow the collapse of turbulent gas clouds, with various effective polytropic exponents $\gamma_{\text{eff}}$, initial Mach numbers $M_0$, and initial turbulent seeds. Equating the energy injection rate from gravitational contraction and the dissipation rate of the turbulence, we obtain an analytic expression for the saturation level of the turbulence, and compare it with the numerical results. Consequently, the numerical results are well described by the analytic model, given that the turbulent driving scale in collapsing gas clouds is one-third the Jeans length of the collapsing core. These results indicate that the strength of the turbulence at first-core formations in the early universe/the present-day star formation process can be estimated solely by $\gamma_{\text{eff}}$.

Unified Astronomy Thesaurus concepts: Population III stars (1285); Population I stars (1282); Early universe (435); Hydrodynamics (1963); Collapsing clouds (267); Gravitational collapse (662)

1. Introduction

Astrophysical fluids are known to be turbulent and magnetized over a wide range of spatial scales, from planetary systems to galaxies. Turbulent motions and magnetic fields play important roles in the formation of astronomical objects. In particular, many earlier studies have shown that both make crucial contributions to star formation. Indeed, turbulence and magnetic fields have affected the fragmentation processes of parent gas clouds or accretion disks around protostars, in various environments, from ancient to present-day star formation sites.

Population III stars, the first generation of stars, are believed to form in high density dark matter minihalos with masses $M \sim 10^2 M_\odot$, at redshifts $z \gtrsim 10$ in the $\Lambda$CDM paradigm (Haiman et al. 1996; Tegmark et al. 1997; Nishi & Susa 1999; Fuller & Couchman 2000; Abel et al. 2002; Bromm et al. 2002; Yoshida et al. 2003). Due to the lack of heavy elements, which are efficient coolants, Population III-forming clouds are more massive and hotter than present-day clouds. As a result, the typical mass of Population III stars ($\sim 10-1000 M_\odot$) should be higher than that of present-day stars, because of the larger fragmentation scale and the higher mass accretion rate (e.g., Susa et al. 1996; Omukai & Nishi 1998; Yoshida et al. 2008). However, recent numerical studies of the mass accretion phase have shown that less massive stars ($\sim 1 M_\odot$) form through the fragmentation of accretion disks (so-called disk fragmentation; Clark et al. 2011b; Greif et al. 2011, 2012; Susa 2013, 2019; Susa et al. 2014; Stacy et al. 2016; Hirano & Bromm 2017; Inoue & Yoshida 2020; Sugimura et al. 2020; Chiaki & Yoshida 2022). Besides, other earlier studies have demonstrated that disk fragmentation is promoted by turbulence (Clark et al. 2011a; Riaz et al. 2018; Wollenberg et al. 2020) or suppressed by magnetic fields (Machida & Doi 2013; Sharda et al. 2020; Sadanari et al. 2021). Thus, the strength of the turbulence and the magnetic fields is crucial for determining the initial mass function (IMF) of Population III stars.

On the other hand, in present-day galaxies, stars form from rotating molecular cloud cores with typical temperatures of $\sim 10$ K and magnetic fields of a few tens of $\mu$G, composed of molecular hydrogen ($H_2$), carbon oxide (CO), ammonia ($NH_3$), dust grains, etc. Depending on the properties of their environments, the cores have various masses, from low ($<10 M_\odot$) to high ($>100 M_\odot$), and turbulent velocities from sub- to supersonic. Accordingly, the masses of present-day stars lie over a wide range, with a power-law distribution (e.g., Salpeter 1955; Kroupa et al. 2001; Chabrier 2005). Such a cloud core distribution could be modeled by magnetoturbulence in parent molecular clouds (e.g., Padoan & Nordlund 2002), which is reflected in the IMF, while disk fragmentation after the formation of hydrostatic cores may also modify the IMF. Thus, turbulence and magnetic fields are also crucial for the star formation process in the present-day universe.

As for the amplification of turbulence, previous studies have shown that gravitational contraction can take on that task (Vázquez-Semadeni et al. 1998; Robertson & Goldreich 2012; Murray & Chang 2015; Birnboim et al. 2018; Guerrero-Gamboa & Vázquez-Semadeni 2020; Mandal et al. 2020; Hennebelle 2021). In our most recent study, Higashi et al. (2021; hereafter, HSC21), we numerically follow the evolution of turbulence in collapsing polytropic/barotropic clouds during the collapse phase. We also obtain an analytic formula that properly describes the growth of turbulence in collapsing clouds. In these simulations, we also find that the rms of the turbulent velocity can reach a few times the sound speed, and that it eventually saturates, even when the initial seed field is feeble. Although we find that the saturation level seems to depend on the effective polytropic exponent, we do not explore detailed mechanisms.

In this paper, however, we investigate the saturation mechanisms of turbulence amplified by gravitational collapse.
in detail. For this purpose, we first perform numerical simulations that follow the cloud collapse, with various initial conditions. We describe the setup of the collapse simulations in Section 2, and we present our results in Section 3. We then theoretically estimate the saturation level of the turbulence in Section 4, and compare our numerical results and theoretical estimates in Section 5. Section 6 is devoted to discussion. We summarize the main points of this paper in Section 7.

2. Collapse Simulations

Here, we perform a suite of high-resolution 3D collapse simulations by using the N-body/adaptive mesh refinement cosmological hydrodynamics code ENZO (Bryan et al. 2014; Brummel-Smith et al. 2019). This code solves compressive hydrodynamic equations with the piecewise parabolic method in a Eulerian frame, using an HLLC Riemann solver. The basic equations are:

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \]
\[ \frac{\partial \rho \mathbf{v}}{\partial t} + \rho (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla P - \rho \nabla \phi, \]
\[ \frac{\partial E}{\partial t} + \nabla \cdot (E + P) \mathbf{v} = -\rho \mathbf{v} \cdot \nabla \phi - \Lambda + \Gamma. \]

In these equations, \( E, \rho, \mathbf{v}, \phi, \Lambda, \) and \( \Gamma \) are the total energy density, density, velocity, gravitational potential, radiative/chemical cooling rate, and radiative heating rate, respectively. The total energy density is given by \( E = e + \rho \mathbf{v}^2 / 2 \), where \( e \) is the thermal energy density. We remark that we solve Equation (3) only for a model with primordial chemistry (Section 2.2), but not for barotropic runs (Section 2.1). The gravitational potential \( \phi \) is obtained from the Poisson equation below, with a Fast Fourier Transform technique (Hockney & Eastwood 1988):

\[ \nabla^2 \phi = 4\pi G \rho. \]

We generate a gravitationally unstable Bonnor–Ebert sphere, with central density \( \rho_{\text{peak}} = 4.65 \times 10^{-26} \text{ g cm}^{-3} \), uniform temperature \( T_0 = 200 \text{ K} \), and cloud radius \( r_c = 1.5 \text{ pc} \) as the initial conditions. These are the same initial conditions as in HSC21. The computational domain has a size of 4 \( r_c \), with periodic boundary conditions. We start the calculations with a base grid of 256\(^3\) cells, and we progressively refine the cells as the cloud collapses in order to resolve the Jeans length at least by 128 cells (we refer to this minimum refinement criterion as the “Jeans parameter”). Except for the model with \( \gamma_{\text{eff}} = 1.3 \) (explained later), we terminate the simulations when the maximum cloud density reaches \( 10^{-2} \text{ g cm}^{-3} \). By the end of the simulations, the maximum refinement level reaches 27 for \( \gamma_{\text{eff}} = 1.0 \) and 20 for \( \gamma_{\text{eff}} = 1.25 \). The corresponding minimum spatial resolutions are 3.0 \( \times 10^{-3} \text{ au} \) and 0.038 au, respectively. The maximum levels in the other runs range from 20 to 27. We use the YT toolkit (Turk et al. 2011)\(^5\) to analyze the simulation data.

2.1. Barotropic Equation of State

We calculate the temperature evolution of the collapsing gas by using a simplified polytropic model, as in HSC21. The relation between the pressure \( P \) and the density \( \rho \) of the gas is expressed as

\[ P \propto \rho^{\gamma_{\text{eff}}}, \]

by using an effective polytropic exponent \( \gamma_{\text{eff}} \). In this model, we use this relation instead of solving Equation (3). We perform simulations for eight \( \gamma_{\text{eff}} = 1.0, 1.05, 1.09, 1.1, 1.15, 1.2, 1.25, \) and 1.3. The parameters \( \gamma_{\text{eff}} = 1.0, 1.09, \) and 1.2 were also employed in HSC21, which enables us to compare the results directly. Note that we terminate the simulation when the mean density within the core reaches \( 10^{-17} \text{ g cm}^{-3} \) for \( \gamma_{\text{eff}} = 1.3 \) only, because of the high computational costs. Here, the models for \( \gamma_{\text{eff}} = 1.09 \) and 1.1 mimic the primordial gas in minihalos (Omukai & Nishi 1998). In these models, we set the gaseous mean molecular weight \( \mu = 1.22 \).

2.2. Model with Primordial Chemistry

Although not explicitly calculated in the barotropic equation of state (EoS), solving the entropy production/reduction by hydrodynamic shock or radiative cooling may change the amplification/saturation level of the turbulence through the baroclinic term of the vorticity equation (e.g., Wise & Abel 2007). In order to investigate the effects of the baroclinic term on the evolution of turbulent velocities, we explicitly take into account nonequilibrium chemistry and the radiative cooling of primordial gas. For this, we solve 49 chemical reactions of 15 primordial species (e, H, H\(^+\), H\(^-\), H\(_2\), H\(_2^+\), He, He\(^+\), He\(^-\), He\(_2^+\), HeH\(^+\), D, D\(^+\), D\(^-\), HD, and HD\(^+\)) by using GRACKLE (Smith et al. 2017)\(^6\) as modified/extended by Chiaki & Wise (2019; see this paper for further details). In this model, \( \mu \) is directly calculated from the abundance of the chemical species. In the high-density region with a number density above \( 10^{17} \text{ cm}^{-3} \), the time step of the nonequilibrium chemical solver becomes prohibitively smaller than the dynamical time, resulting in huge computational costs. Hence, in this region, we calculate the abundances of hydrogen and helium with the equilibrium chemical solver using the Saha–Boltzmann equation. As the initial conditions, we assume the mass fractions of H\(_2\), H\(^+\), H, and He to be \( x_{\text{H}_2} = 7.60 \times 10^{-4}, x_{\text{H}^+} = 7.60 \times 10^{-8}, x_{\text{H}} = 0.759, \) and \( x_{\text{He}} = 0.240 \), respectively.

2.3. Initial Turbulent Flow

It is known that collapsing gas cores have turbulent velocities of some degree at the onset of runaway collapse (see the reviews of Greif 2015 for primordial clouds and McKee & Ostriker 2007 for present-day clouds).

Additionally, the power spectra of the turbulent velocities follow Larson’s law \((P(k) \equiv \langle |v_k|^2 \rangle \propto k^{-\alpha})\), for both present-day (Solomon et al. 1987) and primordial gas clouds (Prieto et al. 2011), where \( v_k \) and \( k \) are the velocity in the wavenumber space and a wavenumber, respectively. \( \langle \cdots \rangle \) denotes the average of a quantity. In order to mimic the turbulent flow of the initial cloud, we give the initial velocity field following Larson’s law, by varying the initial rms Mach number.

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\(^3\) http://enzo-project.org/
\(^4\) http://yt-project.org/
\(^5\) https://yt-project.org/
\(^6\) https://grackle.readthedocs.io/
Thus, the 1D kinetic energy spectrum $E(k) \propto k^2 P(k)$ is proportional to $k^{-2}$, which has the same power-law index as the (compressible) turbulence of Burgers (1948).\footnote{Strictly speaking, this initial velocity field does not constitute self-consistent “real” turbulence, since there is no correlation between density and velocity—see Section 6.3.}

This turbulent velocity field is composed of a natural mixture of solenoidal (divergence-free) and compressive (curl-free) modes, with a 2:1 ratio. This is a reasonable initial condition, given that transverse modes account for two-thirds of the spatial directions in a 3D system, while longitudinal modes account for the remaining one-third. Earlier studies have shown that star-forming cores have rotational energies of a few percent of the gravitational potential (e.g., Goodman et al. 1993; Yoshida et al. 2006). Therefore, we also perform a simulation for the more realistic case, where the rotational energy is 5% of the gravitational potential, to investigate the effects of initial rotation on the temporal evolution of the turbulent velocity.

To investigate the effects of the initial random seed of turbulence, we perform simulations with three different initial turbulent seeds (tagged “A,” “B,” and “C”) for $\gamma_{\text{eff}} = 1.0$, 1.05, 1.1, 1.15, and 1.2. We consider a single seed (A) for $\gamma_{\text{eff}} = 1.25$ and 1.3, because of the high computational costs. The properties of each simulation are summarized in Table 1.

In our analysis of the evolution of turbulence (Section 3), we calculate the cell volume–weighted rms turbulent velocity within a sphere with a radius of half the Jeans length $L_J$ (hereafter called the “Jeans volume”), as well as in accordance with HSC21. To define a cloud core, it is necessary to evaluate the Jeans length, because the length scale of a collapsing gas core is typically the Jeans length. However, the Jeans length depends on the mean density and sound speed within the core of a certain radius. Thus, we first need to assess the core radius accurately. We calculate the core radius iteratively, as follows:

1. First, we cut out the overdense region of $\rho > \rho_{\text{th}}$, where $\rho_{\text{th}} = \rho_{\text{peak}}/16$ in this paper. $\rho_{\text{peak}}$ is the maximum density in the computational domain.
2. Next, we calculate the center of mass $r_c$ in the region and the “tentative” Jeans length $L_J = (\pi c_s^2 / G \rho_{\text{mean}, i})^{1/2}$, using cell volume–weighted average density $\rho_{\text{mean}, i}$ and the sound speed $c_{s,i}$. We then assume this Jeans length as the tentative radius $r_0$.\footnote{This Jeans length is a first estimate.}
3. We then consider a sphere, with a radius of $r_0$ from $r_c$, and calculate the tentative Jeans length and radius $r_1$, as well as 2.
4. If $|r_1 - r_0|$ is smaller than the smallest cell width, we terminate this calculation. If not, we substitute $r_1$ for $r_0$ and go back to step 3, repeating the process until the condition is fulfilled.
5. We obtain $r_1$ as the core radius $r_1 (= L_J/2)$ after convergence.

Within the Jeans volume $V_J$, the turbulent velocity is defined as follows:

$$v_{\text{turb}}^2 \equiv \sum_{i \in V_J} \left( \frac{V_i}{V_J} (v_i - v_{\text{rad},i}) \right)^2,$$  \hspace{1cm} (6)

where $V_i$, $v_i$, and $v_{\text{rad},i}$ are the cell volume, the velocity, and the smoothed radial velocity of the $i$th cell, respectively. To calculate the turbulent velocity, we estimate the smoothed background radial velocity via the following procedure:

1. Calculate the radial velocity profile with $N_{\text{rad}}$ radial bins. $N_{\text{rad}}$ should be smaller than the Jeans parameter, so that we can remove the radial velocity fluctuations at the scale of the cell size. Here, we set $N_{\text{rad}} = 16$.
2. Linearly interpolate this profile to estimate the smoothed radial velocity $v_{\text{rad},i}$ at the position of each cell.

## 3. Results

In this section, we initially present the overall turbulence evolution in the models with the barotropic EoS, focusing on the effects of the polytropic index, the initial turbulent seed, and the initial turbulent velocity in Section 3.1. We then compare the results with/without the baroclinic term and show its effects on the evolution of the turbulence in Section 3.2.

### 3.1. Overall Turbulence Evolution in the Barotropic EoS

Figure 1 shows the evolution of the turbulent velocity as a function of the mean density $\rho_{\text{mean}}$ within the Jeans volume of each snapshot in the fiducial model (LowA). Note that we divide the results into two panels for visibility. As shown in HSC21, the turbulent velocities (solid curves) increase at a rate that depends on $\gamma_{\text{eff}}$ initially. Eventually, they saturate at $\rho_{\text{mean}} \approx 10^{-17}$ g cm$^{-3}$, and then increase with the growth rate, similar to the sound speed (dashed lines). Additionally, the ratio of the saturation velocity to the sound speed depends on $\gamma_{\text{eff}}$ and decreases as $\gamma_{\text{eff}}$ increases. The physical mechanism that determines these saturation levels is explained in detail in Sections 4 and 5.

Next, we plot the evolution of the turbulent velocities with each initial turbulent seed in Figure 2. For ease of viewing, we multiply the results by factors of $x$ for each $\gamma_{\text{eff}}$. We can see that the growth rates of the turbulent velocities before the saturation

| Name | Mach Number | Initial Rotation | $\gamma_{\text{eff}}$ | Turbulence Seed | Remarks |
|------|-------------|------------------|-----------------------|-----------------|---------|
| LowA | 0.1         | No               | All                   | A               | Fiducial |
| LowB | 0.1         | No               | All, except 1.25 and 1.3 | B               |         |
| LowC | 0.1         | No               | All, except 1.25 and 1.3 | C               |         |
| LowA w/c | 0.1 | No | With chemistry A | A |         |
| Middle | 0.5 | No | 1.09 | A |         |
| Middle w/r | 0.5 | Yes | 1.09 | A |         |
| High | 1.0 | No | 1.09 | A |         |

Note. “Low,” “Middle,” and “High” correspond to the strengths of the initial Mach numbers. “w/c” and “w/r” represent the models with chemical reactions and initial rotation, respectively. We run the simulations for $\gamma_{\text{eff}} = 1.25$ and 1.3 only with seed A, because of the high computational costs.
We can also see that the final turbulent velocities almost converge for all $\gamma_{\text{eff}}$, and that the difference in the initial seeds is a factor of 2 at most. The differences in the Mach numbers among the different seeds and the accompanying details are explained in Section 5.

Figure 3 shows the evolutions of the turbulent velocities for each model, with different initial turbulent velocities. We also plot the results of the model with rotation (Middle w/r). In all the models, the turbulent velocities saturate at $\rho_{\text{mean}} \lesssim 10^{-13} \text{ g cm}^{-3}$. The saturation levels in these models almost converge, and the differences among the different initial strengths of turbulence or the effects of rotation are about 20%, at most. This implies that the saturation level of the turbulent velocity does not depend on the initial strength of the turbulence or the presence of cloud rotation.

Compared to the turbulent component, we find that the rotational velocity is subdominant, in our particular models, over the entire course of the collapse. Thus, the resultant Mach number is not much affected by the initial rotation. However, we have to bear in mind that we must explicitly consider the rotational velocity, if it is the dominant component. In such cases, the definition of the turbulent velocity (Equation (6)) should be modified, to Equation (18) from Chiaki & Yoshida (2022).

Figure 1. The evolution of the turbulent velocity $v_{\text{turb}}$ as a function of the mean density $\rho_{\text{mean}}$ in the Jeans volume. The solid curves with different colors denote the numerical results for different $\gamma_{\text{eff}}$. The dotted lines denote the mean sound speed in the Jeans volume.

Figure 2. The same as Figure 1, but for different initial seeds of turbulence (LowA, LowB, and LowC). The differences in line style correspond to the differences between the initial seeds and the differences in color correspond to the differences in $\gamma_{\text{eff}}$. For ease of viewing, we multiply the results by factors of $x$ for each $\gamma_{\text{eff}}$ model.

Figure 3. The same as Figure 1, but for the different $M_0$ and rotation energies (LowA, Middle, Middle w/r, and High). The solid curves show the numerical results for each model. The magenta dotted line denotes the averaged sound speed in the Jeans volume for $\gamma_{\text{eff}} = 1.09$. The black dashed line is obtained by an analytic estimate (Equation (14) of HSC21).
3.2. Comparison with/without the Baroclinic Term

As mentioned in Section 2.1, \( \gamma_{\text{eff}} = 1.09 \) and 1.1 well reproduce the temperature evolutions of collapsing primordial gas clouds. Here, we compare the temporal evolutions of the turbulent velocities and the saturation levels between the models with the barotropic EoS and the model explicitly treating radiative cooling, as shown in Figure 4.

As well as the results of the barotropic EoS runs (Figure 4) in HSC21, we can see that all of the results for LowA with \( \gamma_{\text{eff}} = 1.09, 1.1 \), and w/c are in good agreement with the analytic estimate (dashed line) at \( \rho_{\text{mean}} \lesssim 10^{-13} \text{g cm}^{-3} \). We can also see that the turbulent velocities in all models saturate at \( \rho_{\text{mean}} \gtrsim 10^{-13} \text{g cm}^{-3} \) and converge. This shows that the growth and saturation of the turbulence from gravitational collapse is not greatly affected by the baroclinic term induced by shock waves and chemical reactions/radiative cooling, but depends only on \( \gamma_{\text{eff}} \).

4. Analytic Estimate of the Turbulence Saturation

We find that the growth and saturation of the turbulence depend solely on \( \gamma_{\text{eff}} \) in Section 3. In this section, extending the analytic model proposed by Robertson & Goldreich (2012), we analytically estimate the maximum saturation level of the turbulence induced by gravitational collapse from the balance between amplification and decay resulting from the dissipation of the turbulence.

In the “adiabatic heating” model in Robertson & Goldreich (2012), the temporal evolution of rms turbulent velocity on the uniformly contracting background is expressed as

\[
\frac{dv}{dt} = -Hv - \frac{v^2}{a\ell}.
\]  

Here, \( a, H(=\dot{a}/a), \eta, \) and \( \ell \) are the scale factor, the “Hubble parameter,” the dissipation coefficient of the turbulence, and the typical driving scale of the turbulence in comoving coordinates, respectively. In the uniformly changing background, the mean density fields \( \rho \) and \( a \) are given by the relation \( \rho \propto a^{-3} \).

The first term on the right-hand side in Equation (7) denotes the amplification of the turbulence due to gravitational collapse, while the second term represents the decay term resulting from dissipation, respectively. We can rewrite this equation, by using the turbulent eddy turnover frequency \( \omega = \frac{v}{a\ell} \), as

\[
\frac{dv}{da} = -\left(1 + \frac{\omega}{H}\right)\frac{v}{a}.
\]  

Robertson & Goldreich (2012) have also derived the relation between \( \omega \) and \( H \), from Equations (7) and (8):

\[
\frac{d\log(\omega/H)}{d\log(1/a)} = \frac{\omega}{H} - \frac{d\log H}{d\log(1/a)}.
\]  

Here, \( \ell \) is assumed to be a constant. If the collapse time \( t_{\text{collapse}} \sim |H|^{-1} \) is proportional to the freefall time \( (\propto 1/\sqrt{\rho}) \), the second term in Equation (9) becomes

\[
\frac{d\log(H)}{d\log(1/a)} = \frac{3}{2}.
\]  

The asymptotic relation is approached as

\[
\frac{d\log(\omega/H)}{d\log(1/a)} \rightarrow 0. \text{ We have}
\]

\[
2 + \frac{\omega}{H} - \frac{3}{2} \rightarrow 0 \quad (\text{for } \log(1/a) \rightarrow \infty).
\]  

We apply this model to the collapsing gas core in a self-similar fashion. Assuming that the driving scale in Equation (7) is a factor \( f \) times the Jeans length \( L_2 \), and re-deriving the relation corresponding to Equations (9) and (10), we obtain

\[
\frac{d\log(\omega/H)}{d\log(1/a)} = \frac{5 - 3\gamma_{\text{eff}}}{2} + \frac{\omega}{H} \quad \text{(for } a\ell \sim fL_2). \]

Here, the Jeans length is

\[
L_3 = \left(\frac{3c_s^2}{G\rho} \right)^{1/2} \propto a^{3(2-\gamma_{\text{eff}})/2},
\]  

where \( c_s \) is the sound speed of a collapsing cloud. The asymptotic value in the limit of \( \rho \rightarrow \infty \) (or \( a \rightarrow 0 \)) is

\[
\frac{\omega}{|H|} = \frac{5 - 3\gamma_{\text{eff}}}{2\eta}.
\]  

Next, we estimate \( |H|(^{=\dot{a}/a} = (1/3)(\dot{\rho}/\rho)) \) in the collapsing gas core. Since the collapse proceeds in a self-similar fashion (see Appendix A), the growth of the central density is obtained by integrating a set of ordinary differential equations (Yahil 1983; Suto & Silk 1988). From the equations, the density \( \rho(r, t) \) as a function of the radius \( r \) from the cloud center and time \( t \) is given as \( \rho(r, t) = \alpha(x)/(4\pi G t^2) \). \( \alpha(x) \) is a dimensionless quantity, as a function of a dimensionless coordinate \( x \equiv r/\sqrt{\rho}t^n \), where \( n \) and \( c_s \) are constants; for more details, see Suto & Silk (1988). At the center of the core \( (x = 0) \), the central density \( \rho_c \) is given as \( \rho_c = \tau_0/(4\pi G t^2) \), where \( \tau_0 \) is a constant depending on \( \gamma_{\text{eff}} \) (see Table 2). In a self-similarly collapsing gas core, we assume the gas density to
be uniform ($\rho_{\text{mean}} \approx \rho_c$), so we then have:

$$|H| = \left| \frac{\dot{\rho}}{3 \rho} \right| = \left| \frac{2}{3} \right| = \frac{2}{3} \sqrt{4\pi G \rho_c^4} \frac{1}{\sqrt{\alpha(0)}}.$$  \hspace{1cm} (15)

Substituting this relation into Equation (14), we obtain

$$\omega = \frac{5 - 3\gamma_{\text{eff}}}{3\eta} \frac{4\pi G \rho_{\text{mean}}}{\sqrt{\alpha(0)}}.$$  \hspace{1cm} (16)

Then, with $\omega = v/L_J$, we can derive the saturated turbulent velocity as

$$v_{\text{sat}} = \frac{5 - 3\gamma_{\text{eff}}}{3\eta} \frac{2\pi c_s}{\sqrt{\alpha(0)}} f.$$  \hspace{1cm} (17)

Finally, we obtain the maximum saturation Mach number of turbulence,

$$M_{\text{sat}} \approx \frac{5 - 3\gamma_{\text{eff}}}{3\eta} \frac{2\pi}{\sqrt{\alpha(0)}} f.$$  \hspace{1cm} (18)

This expression depends on $\gamma_{\text{eff}}$, but not on $\rho_{\text{mean}}$. This feature nicely agrees with the numerical results obtained in the previous section, as we see in Section 5.

5. Driving Scale of the Turbulence

In this section, we compare the numerical results in Section 3 with the analytic estimate in Section 4. First, we solve the ordinary differential equations—Equations (15) a and b of Suto & Silk (1988)—by using the fourth-order Runge–Kutta method to obtain $\alpha(0)$, which is necessary for calculating $M_{\text{sat}}$. We summarize the obtained $\alpha(0)$ corresponding to each $\gamma_{\text{eff}}$ in Table 2.

| $\gamma_{\text{eff}}$ | 1.0  | 1.05  | 1.09  | 1.10  | 1.15  | 1.2  | 1.25  | 1.3  |
|---------------------|------|-------|-------|-------|-------|------|-------|------|
| $\alpha(0)$         | 1.67 | 2.15  | 3.03  | 3.26  | 4.73  | 7.19 | 11.6  | 22.0 |

Besides, we also need the dissipation coefficient $\eta$ of turbulence. Mac Low (1999) performed simulations of uniform randomly driven turbulence and found that the driven turbulence saturated, resulting in a balance between energy injection and dissipation. He also found that the energy injection rate and the dissipation rate of the turbulence can be well approximated by the linear relation, and derived the value of the dissipation coefficient. However, his simulations were performed only with the isothermal EoS, so that the $\gamma_{\text{eff}}$ dependence was still unclear. Therefore, we perform another set of simulations, based on the simulations of Mac Low (1999), to confirm the value and $\gamma_{\text{eff}}$ dependence of $\eta$. Consequently, we obtain $\eta = 0.42$ for any $\gamma_{\text{eff}}$, which is the same as in Mac Low (1999); i.e., the dissipation coefficient has no $\gamma_{\text{eff}}$ dependence. Thus, we use this value to obtain the saturation Mach numbers from Equation (18). The detailed setup and the results of the simulations are summarized in Appendix B.

The upper panel of Figure 5 shows the saturation Mach numbers for each $\gamma_{\text{eff}}$ calculated from Equation (18). The red square, blue cross, and green inverse triangle symbols are obtained from our numerical results for $M_0 = 0.1$, with the initial seeds A, B, and C, respectively. The black circle symbols are the averages of the three models. As shown in Section 3, the turbulent velocities saturate at $\rho_{\text{mean}} \gtrsim 10^{-13}$ g cm$^{-3}$ in all models. Therefore, we regard the time-averaged Mach number in $\rho_{\text{mean}} \gtrsim 10^{-14}$ g cm$^{-3}$ as the saturation Mach number $M_{\text{sat}}$. The blue, orange, and green solid curves denote the $M_{\text{sat}}$ estimated from Equation (18), with turbulence driving scales of $al \sim L_j/2$, $L_j/3$, and $L_j/4$, respectively. The black dashed line in the lower panel is obtained from Equation (14).

We can see that if the turbulent driving scale is one-third of the Jeans length, the average of the three initial seeds (the black circle symbols) are in good agreement with the theoretical estimates of the saturation Mach number. This is also almost consistent with the result of Guerrero-Gamboa & Vazquez-
Semadeni (2020) for an isothermal core, who obtained a driving scale of 0.285 L_1 with the same dissipation coefficient as Mac Low (1999), but using a different theoretical model, based on virial equilibrium of a core.

The lower panel of Figure 5 shows the ratio of the eddy turnover frequency $\omega$ and the absolute value of the Hubble parameter $|H|$ as a function of $\gamma_{\text{eff}}$ in the saturation regime. The symbols indicate the results obtained with the same method as in the upper panel, and the black dashed line is obtained from Equation (14). Unlike the solid curves in the upper panel, the black dashed line in the lower panel has a weaker dependency against $\gamma_{\text{eff}}$, because Equation (14) does not contain $\alpha(0)$.

As well as the saturation Mach number, we can see that the averages of the results with the three initial seeds (the black circle symbols) are in good agreement with the theoretical estimates. These results provide clear evidence that the saturation mechanism of the turbulence involves a balance between the amplification and the decay of the turbulence.

An important point in the present study is that our theory can well describe the numerical results of various $\gamma_{\text{eff}}$. The versatility of this theory supports its significance.

6. Discussion

6.1. On Population III Formation

In this paper, we have focused on turbulence in the runaway collapse phase. As a result, given an effective polytropic index $\gamma_{\text{eff}}$, we find an analytic formula for assessing the saturation levels of the turbulence in collapsing cores. In a case where we consider primordial star formation, which is well approximated by $\gamma_{\text{eff}} \approx 1.09$, the turbulence will be supersonic, $\mathcal{M}_{\text{sat}} \sim 1.5$–2 (Figure 5 and Equation (18)), which is consistent with the results from earlier works (e.g., Greif et al. 2012). Therefore, supersonic turbulence should be a general feature of collapsing star-forming clouds in primordial environments.

As well as turbulence, magnetic fields can also have substantial impacts on the IMFs of the first stars, if they are sufficiently strong. It has been suggested that the magnetic fields were very weak in the early universe, but recent high-resolution simulations have revealed that they were amplified to certain levels by the small-scale dynamo effect resulting from turbulence (e.g., Sur et al. 2010, 2012; Federrath et al. 2011; Turk et al. 2012; Sharda et al. 2021). Thus, the magnetic energy could potentially be amplified to the level of the turbulent energy. We can estimate the maximum magnetic field strength as $B_{\text{max}} \approx 1.6 \times 10^3 \text{G}$ for $n_H = 10^{19} \text{cm}^{-3}$, assuming that the magnetic energy can reach the turbulent energy estimated from the results in Section 3, which, in turn, will affect the dynamics of the infalling gas.

The accretion phase, which is not investigated in this paper, is crucial for Population III formations. Recent numerical studies have indicated that both the turbulence and the magnetic fields play important roles in the accretion phase; for instance, the fragmentation of the accretion disk is promoted/suppressed by their effects, whereas the impact on the fragmentation depends on their strength (e.g., Sharda et al. 2020, 2021; Wollenberg et al. 2020; Prole et al. 2022). In this paper and in HSC21, we have found that amplification of the turbulence always occurs in collapsing primordial gas clouds, and that supersonic turbulence is developed. Given that the small-scale dynamo is extremely efficient at generating strong magnetic fields from turbulence, both strong turbulence and magnetic fields have likely always existed at the onset of the accretion phase, and they should be included in simulations of the accretion phase for the first star formation.

6.2. On Present-day Star Formation

In the primordial case, the relation between the gas density and the pressure is described by a single power-law index $\gamma_{\text{eff}}$. On the other hand, the temperature evolution in present-day star formation is more complicated. The temperature evolution from cloud collapse to star formation tracks four phases: (i) isothermal collapse ($\gamma_{\text{eff}} = 1.0, \rho \lesssim 10^{-4} \text{g cm}^{-3}$), (ii) first-core formation ($\gamma_{\text{eff}} = 1.4, 10^{-12} \text{g cm}^{-3} \lesssim \rho \lesssim 10^{-8} \text{g cm}^{-3}$), (iii) second collapse ($\gamma_{\text{eff}} = 1.1, 10^{-6} \text{g cm}^{-3} \lesssim \rho \lesssim 10^{-4} \text{g cm}^{-3}$), and (iv) protostar formation ($\gamma_{\text{eff}} = 1.67, 10^{-3} \text{g cm}^{-3} \lesssim \rho$) (e.g., Masunaga & Inutsuka 2000; Tomida et al. 2013). The gas is almost isothermal, until the gas density reaches $10^{-4} \text{g cm}^{-3}$, and the turbulent velocity is already transonic in molecular clouds at much lower densities. Therefore, the saturation level of the turbulence can be estimated as $\mathcal{M}_{\text{sat}} \sim 2$–3, just before the first-core formation, according to Figure 5 and Equation (12). From the first-core phase, the EoS becomes much stiffer, which may result in lower $\mathcal{M}_{\text{sat}}$. It is difficult to predict the evolution of the turbulence when $\gamma_{\text{eff}}$ changes during the collapse, because we have only investigated the single power-index $\gamma_{\text{eff}}$. We will investigate the effects of such a complex EoS in future works.

From the observational side, high-resolution instruments, such as the Atacama Large Millimeter/submillimeter Array, have begun to provide insights into the effects of turbulence on star formation, targeting dense cores in molecular clouds. For example, several authors have reported multiple systems with large separations and highly misaligned rotational axes, indicating the fragmentation due to the turbulence during protostar formation (Williams et al. 2014; Fernández-López et al. 2017; Lee et al. 2017). Besides, observations of Taurus by Tokuda et al. (2018) have indicated that intense turbulent shock heating possibly occurs in star-forming cores. Our results may indicate that strong turbulence at such small scales is driven by gravitational collapse and, in turn, that this causes the fragmentation and the shock waves. Whether such turbulence is actually driven by gravitational collapse or not will be clarified by further high-resolution observations of gravitationally unstable “starless” dense cores in the future.

6.3. A Caveat to the Initial Conditions

In this study and in HSC21, we initially give the velocity fluctuation, which has a characteristic energy spectrum $E(k) \propto k^{-\gamma_{\text{urb}}}$, Strictly speaking, this initial condition only mimics the velocity field of the turbulence in realistic gas clouds, but not the density field that should be correlated with the velocity. In order to develop this “real” turbulence, eddy turnover times that are at least twice as large will be necessary. In contrast, the eddy turnover timescale is initially longer than the collapse timescale ($\tau_{\text{urb}} < \tau_c$), as shown in Section 4.6 of HSC21. As a result, the cloud collapse proceeds before the initial field well develops into a turbulent flow. Thus, we have to bear this incompleteness of the initial field in mind. However, considering that cosmological simulations that naturally generate the initial gas distribution also obtain similar growth/saturation of the velocity field (e.g., Greif et al. 2012),
we guess that the numerical results from the “real” initial turbulence will be similar to the results in the present paper.

7. Summary

In this paper, we have studied the saturation mechanism of amplified turbulence in collapsing gas clouds.

First of all, assuming the polytropic EoS, we perform high-resolution collapse simulations with various polytropic exponents \(\gamma_{\text{eff}}\), to investigate the evolution of the turbulent velocity. We confirm that the turbulence can be amplified through gravitational collapse, and that its saturation level depends on \(\gamma_{\text{eff}}\), as found in HSC21. We also perform collapse simulations with various initial velocities, such as high Mach numbers and turbulence with initial cloud rotation, and find that there is no significant difference in the saturation levels. To investigate the effects of the baroclinic term on the time variation of the turbulent vorticity, we follow the cloud collapse until protostar formation, self-consistently solving chemical reactions in the primordial gas to explicitly introduce radiative cooling. We find that the baroclinic term resulting from shock waves/radiative cooling has quite a small effect on the evolution of the turbulence. Finally, we analytically estimate the saturation level of the turbulence by using similarity solutions and the turbulence dissipation coefficient \(\eta\).

We find that the numerical results are well described by the analytic formula with a turbulence driving scale of one-third the Jeans scale.

We emphasize that the present results show that the saturation level of the turbulence in contracting prestellar cores is given by the balance between the energy input rate from gravitational collapse and the dissipation rate of the turbulence. The former depends only on \(\gamma_{\text{eff}}\) (Equation (12)), while the latter is a constant (see Appendix B). Therefore, the strength of the turbulence at the epoch of first-core formation can be estimated solely by \(\gamma_{\text{eff}}\), both in the early and present-day universe.

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\textit{Software:} \textit{Enzo} (Bryan et al. 2014; Brummel-Smith et al. 2019), \textit{Grackle} (Smith et al. 2017), \textit{yt} (Turk et al. 2011).

Appendix A

Comparison with Similarity Solution

Assuming that the evolution of the density in a gas cloud follows the similarity solution, we have analytically estimated the saturation Mach number in Section 4. Although we have shown that this assumption is reasonable in 3D collapse simulations for \(\gamma_{\text{eff}} = 1.09\) in HSC21, we have not confirmed its validity for other \(\gamma_{\text{eff}}\). Therefore, we compare our simulation results with the similarity solution for all the \(\gamma_{\text{eff}}\) that we consider.

Figure 6 shows the evolution of the mean core density as a function of time \(-t\). The solid lines show the numerical results, which have corresponding colors, denote \(\rho_{\text{mean}} = \frac{n(0)}{4\pi\gamma G v^2}\), where \(n(0)\) are given in Table 2. Note: these values, except for that “with cooling,” \(\gamma_{\text{eff}} = 1.09\), and \(\gamma_{\text{eff}} = 1.1\), are multiplied by factors of \(x\) for ease of viewing.

![Figure 6. The evolutions of the core mean densities as a function of time \(-t\). The solid lines show the numerical results.](image)

Appendix B

Simulations of Turbulence Dissipation

To estimate \(M_{\text{sat}}\) from Equation (18), we need the dissipation coefficient \(\eta\) of turbulence. In this Appendix, we extend the simulations of Mac Low (1999), who investigated turbulent dissipation for isothermal gas. We perform simulations for various \(\gamma_{\text{eff}}\), based on the method of Mac Low (1999). Here, we remark that the dissipation coefficient \(\eta_{\text{eff}}\) defined by Mac Low (1999) is slightly different from \(\eta\) in Equation (18). They can be converted to each other by using the relationship of \(\eta_{\text{eff}} = \frac{2\pi\eta_{\text{eff}}}{\eta}\) (Equation (14) of Guerrero-Gamboa & Vazquez-Semadeni (2020)).

First, we note that each physical quantity in this simulation is described using computational code units. All runs are...
performed on a 3D uniform grid with $128^3$ cells.\footnote{Mac Low (1999) showed that if enough resolution ($128^3$ at most) is provided, the growth of the velocity in these kinds of turbulence simulations does not depend on resolution. Hence, we choose $128^3$ uniform grids. Also, the resolution is comparable to the effective resolution of the collapse simulations in Section 2.} We set the computational domain length $L = 1$, with a periodic boundary condition. The gas is uniformly distributed over the domain, with the initial density $\rho_0 = 1$ and the total mass $m = pL^3 = 1$. For comparison with the results from Section 2.1, we set $\mu = 1.22$ and $\gamma_{sd} = 1.4$, and perform simulations with the same set of $\gamma_{eff}$.

During the simulations, the turbulence is continuously driven by the injection of velocity fluctuations $\delta v$ at each time step $\delta t$. This $\delta v$ corresponds to a constant kinetic energy input rate $E_{in} = \Delta E/\Delta t$, which is set as an initial parameter. The turbulent kinetic energy spectrum at wavenumber $k$ is set to follow Larson’s law (i.e., $E(k) \propto k^{-2}$), which has a peak at $k_{\text{peak}} = 2$. Thus, the typical driving scale of the turbulence is $L/k_{\text{peak}}$. The driven velocity field fully comprises the solenoidal mode, which mimics the turbulent velocity field driven by gravitational collapse (Federrath et al. 2011; HSC21). We perform simulations with $E_{in} = 0.1, 0.3, 1, 3, 10$ for every $\gamma_{eff}$, and we terminate them at twice the sound crossing time $t_{sc}$, calculated by the initial sound speed given by each $\gamma_{eff}$.

First, as an example, we show the temporal evolution of the turbulent velocity for $E_{in} = 1$ in Figure 7. The velocity increases quickly and saturates at $\approx t_{sc}$ for all $\gamma_{eff}$. We can see that the evolution of the velocity is almost the same for every $\gamma_{eff}$, which indicates that there is no $\gamma_{eff}$ dependence during the growth of the turbulence.

Mac Low (1999) showed that when the turbulence reaches an equilibrium state, the energy dissipation rate $E_{kin}$ of the turbulence equals the energy injection rate $E_{in}$, and can be well

approximated by the linear relation

$$E_{kin} \approx -\eta vy_{rms}^3$$

(B1)

where $\eta \equiv (2\pi/L)k_{\text{peak}}$ is a “dimensionalized wavenumber.” Using this relation, we estimate the dissipation coefficient.

We plot $v_{rms}$ versus the energy dissipation rate ($=E_{in}$) after the sound crossing time from the beginning of the simulation in Figure 8. The crosses in different colors denote the results for the corresponding $\gamma_{eff}$. We can see that there are no differences among different $\gamma_{eff}$ and that the crosses almost overlap at a single point for each energy dissipation rate. From the results and from Equation (B1), we can estimate $\eta = 0.42/(2\pi)$, using the least squares method. This is fully consistent with the result for the isothermal case in Mac Low (1999). Hence, we use $\eta = 0.42$ in Section 5.

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