I. INTRODUCTION

Pseudogap signatures in the underdoped regime of the high-$T_c$ superconductors appear as intriguing features, which could contain the key to a microscopic understanding of the physics of this important region of the phase diagram. It is in this region that proximity to the Hubbard-Mott insulating phase would make one think that the underlying normal state is most profoundly different from a usual metallic Fermi liquid state. Many associated anomalous properties are observed including important violation of the c-axis optical sum rule which related the missing area under the real part of the conductivity in the superconducting state, as compared to the normal state, to the corresponding c-axis superfluid density. Above the superconducting critical temperature $T_c$, in the pseudogap regime, important changes in the plasma frequency are also observed. The preformed pair model, which envisions that electron pairs are formed at the pseudogap transition temperature ($T^*$) with $T_c$ the temperature at which phase coherence among the pairs is established, has helped understand some of these properties. In particular the pseudogap phase corresponds to phase fluctuations with pseudogap energy the same as the pairing energy. An alternative to the preformed pair model elaborated upon by Chen et al. envisions additional incoherent pair excitations of finite momentum with which goes beyond the usual BCS formulation of superconductivity. Very recently Chakravarty et al. have made a new proposal in which the pseudogap is associated with the formation of a competing d-density wave (DDW) state on a nested Fermi surface (FS). This DDW order has important consequences such as staggered currents with associated orbital magnetic moments which breaks parity and time-reversal symmetry. Since the original proposal, several works have appeared in which various properties associated with the DDW state have been elaborated upon with the aim of testing the validity of the model by comparing to experimental data. ARPES data on the pseudogap have revealed it to have $d$-wave symmetry and to exist at least in the underdoped regime. The DDW order requires a nested FS and so should be most stable at half filling in the simplest of tight binding bands, and by assumption the DDW order is taken to have $d$-wave symmetry with maximum gap at $(\pi, 0)$ and zero gap on the diagonals of the CuO$_2$ Brillouin zone. It is therefore of considerable interest to understand how much more of the known pseudogap physics such a model can explain.

Charge transport along the c-axis has long been studied in the cuprates and found to be anomalous. While for optimally doped YBa$_2$Cu$_3$O$_{7-\delta}$ (YBCO) the c-axis DC resistivity tracks fairly well the linear in temperature ($T$) in-plane resistivity, above $T_c$, in the underdoped case it shows a semiconductor-like increase with decreasing $T$ and has a maximum at $T = T_2$. Also the estimated mean free path becomes shorter than the interlayer distance indication that Bloch transport is unlikely to apply. A large decrease in the real part of the conductivity $\sigma_1(\omega)$ is also observed at small frequency $\omega$ starting at the pseudogap temperature ($T^*$). The energy scale for the pseudogap is of the same order as that for the superconducting gap in YBa$_2$Cu$_3$O$_{6.6}$. At $T = 0$, the suppression of $\sigma_1(\omega)$ for frequencies $\omega$ below the gap scale is very well developed. In fact the additional spectral weight lost under the $\sigma_1(\omega)$ curve on entering the superconducting state is found to be smaller than the superfluid density determined at $T \approx 0$ from the imaginary part of the conductivity. The c-axis sum rule is observed to be closer in value to a half than to its conventional value of one. This indicates that there must be an important change in kinetic energy on going from the pseudogap state at $T_c$ to the superconducting state at $T = 0$. The observed value of the c-axis sum rule of a half can be easily understood within a preformed pair model as can the important changes in the optical plasma frequencies which are observed in the temperature range from $T_c$ to $T^*$ in the pseudogap state. Within the preformed pair model these effects are the result of the phase fluctuations of the superconducting order parameter. In this paper we study these issues within the assumption that the pseudogap state is due instead to the formation of the DDW order.
II. FORMALISM

We begin with a phenomenological model Hamiltonian of the form: $H = -t \sum_{<i,j>\sigma} C_{i\sigma}^+ C_{j\sigma} - \mu \sum_i n_{i\uparrow} + J \sum_{<i,j>} [S_i \cdot S_j - \frac{1}{2} n_{i\uparrow} n_{j\downarrow}]$, where $C_{i\sigma}^+$ creates a spin $\sigma$ electron on the site $i$, $t$ the in-plane hopping amplitude, $\mu$ the chemical potential, and $n_i$ the occupation on the site $i$. $S_i$ is the spin and $J (> 0)$ a strength of an exchange interaction. Based on the mean field approximation, defining $\Delta = -J < C_i^+ C_{i+\delta \uparrow} >$ and making an ansatz $(-1)^i W_i = -J < C_i^+ C_{i+\delta \sigma} >$, we obtain the effective Hamiltonian for a combined superconducting state with gap $\Delta_k$ and pseudogap state (with pseudogap $W_k$)

$$H_{MF} = \sum_{k\sigma} \left[ (\epsilon_k - \mu) C_{k\sigma}^+ C_{k\sigma} + \sum_k \left[ \Delta_k C_{k\sigma}^+ C_{-k\uparrow} + h.c. \right] + \sum_{k\sigma} iW_k C_{k\sigma}^+ C_{k+Q\sigma} \right]$$

(1)

where $\epsilon_k = -2t(\cos(k_x) + \cos(k_y))$, $\Delta_k = 2\Delta_0(\cos(k_x) - \cos(k_y))$, and $W_k = 2W_0(\cos(k_x) - \cos(k_y))$. Here $\epsilon_k$ is the electron energy dispersion for a simple tight binding between two adjacent planes labeled by 1 and 2, respectively. We could equally well have taken incoherent coupling to allow for a difference in interplane hopping.

One can introduce an intersite interaction $(V/2) \sum_{<i,j>\sigma,\sigma'} \delta n_{i\sigma} n_{j\sigma'}$ to allow for a difference in interaction to exist in the d-wave superconducting (DSC) and DDW channel: $J \rightarrow J - V$ for DSC and $J \rightarrow J + V$ for DDW. Note that the site-dependent factor $(-1)^i$ in the ansatz made above ensures that nesting takes place for half filling and that $W$ is pure-imaginary.

In the $4 \times 4$ matrix form, $H_{MF} = \sum_k C_k^+ \hat{h}_k C_k$, where $\sum_k$ means a sum over half of Brillouin zone, and

$$C_k = \left( \begin{array}{cccc} C_{k\uparrow}^+, C_{-k\downarrow}, C_{k\downarrow}^+, C_{-k+Q\downarrow} \end{array} \right).$$

The $4 \times 4$ matrix $\hat{h}_k$ is

$$\hat{h}_k = \left( \begin{array}{cccc} \epsilon_k - \mu & \Delta_k & iW_k & 0 \\ -\epsilon_k - \mu & -\epsilon_k - \mu & 0 & iW_k \\ iW_k & 0 & \epsilon_k - \mu & \Delta_k + Q \\ 0 & iW_k & \Delta_k + Q & -\epsilon_k - \mu \end{array} \right).$$

(2)

This matrix defines our problem. We begin by studying c-axis properties and in particular the optical sum rule. To do this it is necessary to have some model for the c-axis charge transfer. For simplicity we will use

$$H_c = \sum_{i\sigma} t_{\perp} \left[ c_{i1\sigma}^+ c_{i2\sigma} + c_{i2\sigma}^+ c_{i1\sigma} \right],$$

(3)

where $t_{\perp}$ is the transfer integral for electron hopping between two adjacent planes labeled by 1 and 2, respectively. We could equally well have taken incoherent coupling. The c-axis conductivity at frequency $\omega$ is $\sigma_c(0, \omega) = (i/\omega) \left[ \Pi_{rel}(0, \omega) - e^2 d(H_c) \right]$, where $e$ and $d$ are

$$\sum_{i,j} \alpha_{ij} \left[ G_{ij}^2 - G_{0,i,j}^2 \right] = (G_{11}^2 - G_{0,11}^2) - (G_{13}^2 - G_{0,13}^2) - (G_{24}^2 - G_{0,24}^2) + (G_{33}^2 - G_{0,33}^2)$$

(9)

an electron charge and an interlayer spacing, respectively, and

$$\Pi(0, \omega) = 2e^2 t_{\perp}^2 dT \sum_{\omega_n} \sum_k \text{Tr} \left[ \hat{G}(k, i\omega_n) \hat{G}(k, i\omega_n + i\omega) \right],$$

(4)

and

$$\langle H_c \rangle = 2t_{\perp}^2 T \sum_{\omega_n} \sum_k \text{Tr} \left[ \hat{M} \hat{G}(k, i\omega_n) \hat{M} \hat{G}(k, i\omega_n) \right].$$

(5)

In the above equations, the $4 \times 4$ matrix Green function $\hat{G}(k, i\omega_n) = [i\omega_n - \hat{h}_k]^{-1}$ and $\hat{M} = \left( \begin{array}{cc} \hat{\tau_3} & 0 \\ 0 & \hat{\tau_3} \end{array} \right)$, where $\hat{\tau_3}$ is a Pauli matrix.

The c-axis superfluid density (or stiffness) ($\rho_s$) in the superconducting state is related to the limit as $\omega \rightarrow 0$ of the imaginary part of the conductivity. It is also related to the missing spectral weight under the real part of the conductivity on going from normal to superconducting state, which we denote by $\Delta \mathcal{N}$ and define as

$$\Delta \mathcal{N} = \int_{0^+}^{\omega_c} d\omega \left[ \sigma_c^N(\omega) - \sigma_c^S(\omega) \right].$$

(6)

Here $\omega_c$ is a cut-off frequency of order the bandwidth, and $N$ and $S$ stand for normal and superconducting state, respectively. Now, we have

$$\rho_s = \Delta \mathcal{N} - 4\pi e^2 d \left[ \langle H_c \rangle^N - \langle H_c \rangle^S \right] = 4\pi \lim_{\omega \rightarrow 0} \omega \text{Im} \sigma_c(0, \omega).$$

(7)

From the above equations we obtain a sum rule in terms of Green functions $G$ and $G_0 = G(\Delta = 0)$ as follows:

$$\frac{\Delta \mathcal{N}}{\rho_s} = \frac{1}{2} + \frac{1}{2} \sum_{\omega,k} \sum_{i,j} \alpha_{ij} \left[ G_{ij}^2 - G_{0,ij}^2 \right] \sum_{\omega,k} \sum_{i,j} \beta_{ij} G_{ij}^2,$$

(8)

where

$$\sum_{i,j} \alpha_{ij} \left[ G_{ij}^2 - G_{0,i,j}^2 \right] = (G_{11}^2 - G_{0,11}^2) - (G_{13}^2 - G_{0,13}^2) - (G_{24}^2 - G_{0,24}^2) + (G_{33}^2 - G_{0,33}^2)$$

(9)
and
\[
\sum_{i,j} \beta_{ij} G_{ij}^2 = G_{12}^2 - G_{14}^2 - G_{23}^2 + G_{34}^2. \tag{10}
\]

Note that the superfluid density \(\rho_s\) is given by
\[
\rho_s = CT \sum_{\omega_n} \sum_k \sum_{i,j} \beta_{ij} G_{ij}^2,
\]
where \(C = 32\pi e^2 t_0^2 d\). The equations above are easily generalized for incoherent interlayer coupling. As \(T \to 0\) (zero temperature limit),
\[
\rho_s = C \sum_k \left\{ \frac{\Delta_k^2}{(\epsilon_k - \mu)^2 + \Delta_k^2} + \frac{\Delta_{k+Q}^2}{(\epsilon_{k+Q} - \mu)^2 + \Delta_{k+Q}^2} \right\}^{3/2}
\]
where the summation over \(k\) is unrestricted i.e. is over the entire Brillouin zone. This is the usual expression of \(\rho_s\) with \(E_k = \sqrt{(\epsilon_k - \mu)^2 + \Delta_k^2}\) for the pure DSC case with no DDW order.

### III. C-AXIS RESPONSE

Based on experimental observations, we choose the doping dependence of the DDW gap amplitude \(W_0(x)\) to be \(W_0(x) = 0.04(1 - x/x_c)\) in unit of \(t\). Here \(x\) is doping and its critical value is \(x_c = 0.2\), where the DDW gap disappears. Also we assume \(\Delta_0(x) = 0.02\) for 0.05 < \(x\) < 0.2. A small variation of \(\Delta_0(x)\) near the optimal doping (\(x \approx 0.16\)), which is seen in the experiment, is not important because our main consideration is focused on the underdoped regime of the cuprates. [See a later discussion and an inset of Fig. 2.] In Fig. 1, we plot the calculated c-axis superfluid density vs the amplitude of the pseudogap \(W_0(x)\) at \(T = 0\) assuming a constant chemical potential. Such an assumption is of course not correct since the chemical potential also changes with the doping \(x\); however, keeping it fixed will help us to understand the physics. The solid curve is the c-axis superfluid density with the chemical potential \(\mu = -0.06\). The dashed curve is for a larger value of \(\mu = -0.2\). OD and UD at the top of Fig. 1 stand for overdoped and underdoped regime, respectively. One can see from Fig. 1 that when the absolute value of the chemical potential \(|\mu|\) is smaller than the pseudogap value of \(W(x) = 4W_0(x)\), which is its maximum for a given doping \(x\), then the superfluid density is reduced as \(W(x)\) in increased. This happens to be the case for most of the doping values \(x\) between 0.05 and 0.2 in the solid curve. On the other hand, for larger values of \(|\mu|\) the opposite holds (dashed curve). To understand the physics behind this behavior we return to the well known result embodied in Eq. (13)

which gives the superfluid density \(\rho_s\) for a pure \(d\)-wave superconductor with no DDW order i.e. \(W_0(x) = 0\). In this case assuming a cylindrical FS \(\rho_s = CN(0)/2\), where \(N(0)\) is the density of states at the FS. In this simple model, \(N(0)\) is constant throughout the band; however, if it did vary, then it would be some appropriate energy average of \(N(\omega)\) around the FS on the scale of \(\Delta_0\) that would replace it. We stress that for \(\rho_s\) the size of the zero temperature superfluid density is independent of the size of the superconducting gap \(\Delta_0\) and depends instead on normal state parameters. While Eq. (14) has an explicit factor of \(\Delta_k^2\) in the numerator, this does not translate directly into a reduced superfluid density at \(T = 0\) as \(\Delta_0(x)\) decreases.

As the pseudogap develops it competes with the superconductivity but we can understand the trends seen in Fig. 1 for \(\rho_s\) as a function of \(W_0(x)\) by considering, first, the effect of \(W_0(x)\) on the electronic density of states and then thinking of switching on the superconductivity. In the inset of Fig. 1, we show the quasiparticle density of states for the pure pseudogap state i.e. \(\Delta_0(x) = 0\). The solid curve is for \(W_0 = 0.03\) i.e. \(x = 0.05\) and the dotted curve is for \(W_0 = 0.016\) i.e. \(x = 0.12\). We will return to a detail discussion of the density of states (DOS) in a later section. For now suffice it to note that the logarithmic singularities associated with the DOS are at \(|\mu| = 4W_0/\sqrt{T^2 + W_0^2}\). For a wide band with \(t >> W_0\) they would be at \(|\mu| = 4W_0\). Now for \(\mu = -0.2\) the Fermi surface FS1 is located away from the DDW gap and the DOS around the Fermi energy increases as doping ranges from overdoped to underdoped i.e. as \(W_0(x)\) increases. Consequently, \(\rho_s\) at \(T = 0\) increase. However, when \(\mu = -0.06\) the Fermi surface FS2 sits inside the DDW gap shown for most values of \(x\) and an average of the DOS around FS2 is reduced as \(x\) becomes smaller i.e. \(W_0(x)\) becomes larger. Now the value of the superfluid density decreases as the underdoped regime is entered.
Thus the opening of the pseudogap will decrease $\rho_s$ at $T = 0$ only if $|\mu|$ is small enough in the DDW model. For realistic values of $x$, however, the chemical potential $\mu$ is not close to zero when it is calculated self-consistently for the simple tight binding band structure we have assumed and in the presence of a DDW order. While one can take empirical expression for the doping dependence of superconductor gap and pseudogap, an arbitrary choice of $\mu(x)$ would not be consistent with our model band structure and, therefore, is not allowed. We must calculate $\mu$ from the filling.

From this discussion we conclude that the opening-up of a pseudogap is not sufficient to lead to a large reduction in the $c$-axis superfluid density. On the other hand the interlayer hopping matrix element $t_\perp(x)$ in the YBCO series is known to decrease almost exponentially as the doping is decreased. This factor will dominate over effects of the DDW order in consideration of the suppression of the $c$-axis superfluid density with doping. These considerations imply that the doping dependence of the $c$-axis superfluid density is not a good quantity in which to study the role of the DDW order. A better choice is the $c$-axis optical conductivity sum rule because it is independent of the magnitude of the interlayer hopping amplitude and, therefore, of its dependence on doping. We next turn to the calculation of the $c$-axis sum rule and focus on the issue of whether or not the DDW model can describe the experimental observations in the YBCO system a conventional sum rule of one is observed for the optimally doped case and of about a half for an underdoped sample.

For the sum rule calculation we will make use of some experimental observation rather than proceeding to a complete self-consistent calculation. This is reasonable since the nature of the interaction $J - V$ and $J + V$, which determine the size of the amplitude of DSC and DDW gap respectively in the formalism, is not known. Assumptions on the variation of $\Delta_0$ and $\omega_0$ as functions of $x$ based on empirical expressions correspond to specific assumption about the unknown variation of the above interaction parameters. However, in order to take into account the band structure and doping we determine the chemical potential as a function of doping and temperature by solving the filling equation derived in the Bogoliubov-de Gennes formalism.

In practice a more useful expression than Eq. (3) to calculate the sum rule, or the normalized missing spectral weight (NMSW) $\Delta N/\rho_s$ is

$$
\frac{\Delta N}{\rho_s} = 1 + \frac{4\pi e^2 d}{\rho_s} \left[ \langle H_c \rangle - \langle H_c \rangle^n \right].
$$

Of course Eq. (8) is more direct if one wishes to understand how a sum rule of a half can be obtained. This results when the Green’s functions in the second term of the right hand side cancel between superconducting and pseudogap state as in the preform pair model. We see from Eq. (13) that the kinetic energy difference between the superconducting and pseudogap state divided by the superfluid density determines the $c$-axis conductivity sum rule. Mathematical expressions for the kinetic energy and the superfluid density are as follows:

$$
4\pi e^2 d\langle H_c \rangle = -CT \sum_{\omega_n} \sum_{\mathbf{k}} \frac{\gamma_1 + \gamma_2}{\beta^2},
$$

and

$$
\rho_s = CT \sum_{\omega_n} \sum_{\mathbf{k}} \frac{2\Delta^2 k}{\beta^2},
$$

where

$$
\gamma_1 = \left( \omega_n^2 - E_{0k}^2 + \Delta_k^2 + \mu^2 \right) \left( \omega_n^2 + E_{0k}^2 + \Delta_k^2 + \mu^2 \right) \left( \omega_n^2 + E_{0k}^2 + \Delta_k^2 + \mu^2 \right)^2 \left( \omega_n^2 + E_{0k}^2 + \Delta_k^2 + \mu^2 \right)^2
$$

$$
\gamma_2 = 4\mu^2 E_{0k}^2 \left( E_{0k}^2 + 3\omega_n^2 + 3\Delta_k^2 + \mu^2 \right)
$$

$$
\beta = \left( \omega_n^2 + E_{0k}^2 + \Delta_k^2 + \mu^2 \right)^2 - 4\mu^2 E_{0k}^2
$$

$$
\alpha = \left( \omega_n^2 + E_{0k}^2 + \Delta_k^2 + \mu^2 \right)^2 + 4\mu^2 E_{0k}^2
$$

The kinetic energy of each state should presumably be calculated at zero temperature because in some notion of the theory of interlayer coupling, the kinetic energy difference is associated with the condensation energy of superconductors. However, Basov et al. have taken $T \approx T_c$ since it is not easily to access the pseudogap state at low $T$. In the calculation of the kinetic energy we choose $T = 0.01$ for the superconducting state and $T = 0.1$ for the pseudogap state. If $T = 2000 K$, then the working temperatures are $20K$ and $200K$ in the superconducting and pseudogap state, respectively.

As we mentioned earlier, we determine self-consistently the chemical potential $\mu$ for a given doping from the band structure as well as the given temperature even though we adopt the experimental observation for $\omega_0$ and $\Delta_0$. The expression for the filling $n$, in the self-consistent formalism, is written as:

$$
n = 1 + \frac{1}{2} \sum_k \left[ \frac{E_{0k} + \mu}{E_{2k}} \operatorname{tanh} \left( \frac{E_{2k}}{2T} \right) - \frac{E_{0k} - \mu}{E_{1k}} \operatorname{tanh} \left( \frac{E_{1k}}{2T} \right) \right].
$$

In Fig. 2, we plot NMSW ($\Delta N/\rho_s$) as a function of doping ($x$). In the inset of Fig. 2 we reproduce some of the experimental phase diagram obtained recently by Talon and Loram which shows critical temperature (plus sign +), pseudogap value $\omega_0$ (solid square) and superconducting gap (solid triangle) as functions of doping for Y$_{1.5}$Ca$_{0.5}$Ba$_2$Cu$_3$O$_{7-\delta}$. Here the details are not important and we take the data as typical of the cuprates. We
approximate, in unit of $t$, $W_0(x) = 0.04(1 - x/x_c)$ and $\Delta_0(x) = 0.02$ at $T = 0.01$, and $W_0(x) = 0.025(1 - x/x_c)$ with $\Delta_0(x) = 0$ at $T = 0.1$. As shown, in the underdoped regime NMSW is less than one while it saturates to about one beyond the optimal doping. Different values of $W_0(0)$ at $T = 0.1$ do not change the overall behavior of NMSW vs $x$. However, we did find that a different choice of working temperature for the pseudogap state can change the doping dependence of NMSW and, thus, the behavior shown in Fig. 2 is not robust. It has been found that a temperature dependence of NMSW for a d-orbital shown in Fig. 2 is not robust. It has been found that the working temperature for the pseudogap state can change beyond the optimal doping. Different values of $T_{x}$ suggest the initial reduction for half filling is by a factor of 5 more than necessary to agree with experiment for YBa$_2$Cu$_3$O$_{6.6}$ shown in the inset of Fig. 4 which gives the experimental results for $T_0$. The implication that a change in $T_{x}$ is attained with temperature is lowered from $T^{*}$, and in our model as the DDW gap grows.

Within a self-consistent formalism, we calculate the kinetic energy above $T_c$ and below $T^{*}$ in the regime when DDW gap is included in the calculation, and we compare the theoretical calculations with experimental results. For simplicity, we do not consider inhomogeneity in the system. The self-consistent equations for $W_k$ and the filling $n$ are:

$$W_0 = \frac{V_{DDW}}{4} \sum_k W_0^2 \frac{\eta^2}{E_{0k}} \left[ \tanh \left( \frac{E_{0k} + \mu}{2T} \right) + \tanh \left( \frac{E_{0k} - \mu}{2T} \right) \right],$$  \hspace{1cm} (22)

and Eq. (21) with $\Delta_k = 0$, where $V_{DDW}$ is the DDW channel interaction and $\eta_k = \cos(k_x) - \cos(k_y)$. For given values of $V_{DDW}$ and $n$, $W_0(T)$ and $\mu(T)$ are self-consistently determined. In the calculation we choose the DDW channel interaction $V_{DDW} \approx 0.945$ and the values $n = 1, 0.99, 0.97$, and 0.95 for the filling. In fig. 3, we plot $W_0(T)$ versus $T$ in unit of $t$. In the inset we show $\mu(T)$ as a function of $T$ for $n = 0.95$. In this case, if we assume $t = 2000K$, then $T^{*} \approx 215K$. Since below $T_c$, which

$$\langle H_c \rangle \approx \frac{t^2}{T} \sum_k \left\{ \tanh^2 \left( \frac{E_{0k}}{T} \right) - 1 + \frac{1}{4} \left( \frac{\mu}{T} \right)^2 \left[ \tanh^2 \left( \frac{E_{0k}}{T} \right) - 1 \right] \left[ 3 \tanh^2 \left( \frac{E_{0k}}{T} \right) - 1 \right] \right\},$$  \hspace{1cm} (23)

is taken as $80K$, we would have to include an equation for $\Delta_k$ as well in the self-consistent calculation and we have not, we simply indicate $W_0(0)$ as a dashed line for $T < T_c$. Using $W_0(T)$ and $\mu(T)$ for different $n$’s, we calculate the normalized kinetic energy $\langle H_c \rangle(T)/\langle H_c \rangle(T^{*})$ for $T_c < T < T^{*}$. An analytic expression for $\langle H_c \rangle$ is achievable only if the magnitude of the chemical potential $|\mu|$ is much less than a working temperature; namely, $|\mu|/T \ll 1$. In this instance it can be shown that

The correction to the above equation is of order $(\mu/T)^4$. In Fig. 4, we plot our numerical results for the normalized kinetic energy for $T_c < T < T^{*}$. The curves are labeled by the value of filling $n$ with $n = 1.0$ half filling and $n = 0.95$ corresponding to a value of doping $x = 0.05$. It is clear from the figure that near half filling the c-axis kinetic energy and so the optical spectral weight does decrease very significantly as we go from $T^{*}$ toward $T_c$. The reduction for half filling is by a factor of 5 more than necessary to agree with experiment for YBa$_2$Cu$_3$O$_{6.6}$ shown in the inset of Fig. 4 which gives the experimental results for the square of plasma frequency as a function of $T_c$. However, as we move away from half filling, the initial reduction with decreasing temperature out of $T^{*}$ is rapidly suppressed. Also at the lower temperatures considered, the optical spectral weight begins to increase again. The tight binding band structure we have used in our calculations is at best genetic for the oxides and may not apply in a quantitative sense for a particular case. Nevertheless it is clear that the model of DDW order as an explanation of the physics of the pseudogap regime gives results for the change in c-axis kinetic energy with temperature which are rather sensitive to the assumed parameters, for example to doping.

IV. DENSITY OF STATES

Renner et al. have studied the evolution of the pseudogap features in STM tunneling (SIN) in a series of
Bi$_2$Sr$_2$CaCu$_2$O$_{8+\delta}$ (Bi2212) as a function of doping from underdoped to overdoped regime of the phase diagram. Somewhat complimentary data are given in DeWilde et al. and in Miyakawa et al where the authors concentrate more on the hump and dip feature seen around twice the gap energy in SIN and at three times in Josephson junctions. Renner et al. find a superconducting gap which is nearly temperature independent up to $T_c$ at which point it merges smoothly into a second gap like feature centered at the FS. They find the pseudogap to be present both in underdoped and overdoped samples and that its size scales in magnitude with the superconducting gap. This argues for a common origin and these authors conclude that the data is consistent with the idea of preformed pairs. Because they find that the pseudogap is tide to the FS the data do not support a conventional band structure explanation. On the other hand recent intrinsic tunneling spectroscopic results for mesa on Bi2212 single crystals (SIS) have revealed distinct features that can be associated with the superconducting gap and with the pseudogap. The temperature dependence of the superconducting peak structure in the dynamic conductance shows that it closes at $T_c$ while the hump structure which is associated with the pseudogap is unaffected by the superconducting transition. This was taken as evidence that the two phenomena are distinct and that the data do not favor a preformed pair model. Later magnetic field studies showed that the structure identified with the pseudogap is insensitive to magnetic field while the superconducting gap is strongly suppressed by the field which is taken as further confirmation of the identification made. A feature of the data which is relevant to our work is that the superconducting gap corresponds to sharp peaks which are seen to grow even sharper with decreasing temperature and fall inside the pseudogap humps. The DDW model does not relate gap and pseudogap directly and so is consistent with the above interpretation. In the self-consistent phenomenological model the two are related to $J - V$ and $J + V$ respectively and, therefore, they can be quite different in size and in variation with temperature. On the other hand, the model does not give superconducting gap falling inside a larger pseudogap structure. This behavior is generic to the model and has its origin in the fact that the superconducting gap opens up at the FS while the DDW gap is centered at the antiferromagnetic Brillouin zone boundary as described below.

Although the density of states (DOS) $N(\omega)$ is related to the in-plane dynamics, it is nevertheless relevant to our present discussion because we have already seen that the $c$-axis response reflect in-plane dynamics. Employing standard manipulations, it can be shown that the quasi-particle DOS $N(\omega)$ is:

$$\begin{align*}
N(\omega) &= -2 \sum_k \text{Im} \left[ G_R^{11} + G_R^{33} \right] \\
&= 2 \sum_k \left[ u_{1k}^2 \delta(\omega - E_{1k}) + v_{1k}^2 \delta(\omega + E_{1k}) + u_{2k}^2 \delta(\omega - E_{2k}) + v_{2k}^2 \delta(\omega + E_{2k}) \right]. \\
&= \sum_k \left\{ [1 + \xi_k/E_k] \delta(\omega - E_k) + [1 - \xi_k/E_k] \delta(\omega + E_k) + [1 + \xi_kQ/E_k + Q] \delta(\omega - E_k + Q) + [1 - \xi_kQ/E_k + Q] \delta(\omega + E_k + Q) \right\} \\
&= \sum_k \left\{ [1 + \xi_k/E_k] \delta(\omega - E_k) + [1 - \xi_k/E_k] \delta(\omega + E_k) \right\}.
\end{align*}$$

Here a factor of 2 is for the summation over spin, $G_R$ is the retarded Green’s function and the coherent factors are $u_{1k}^2 = \frac{1}{2} [1 + (E_{1k} - \mu)/E_{1k}]$ and $u_{2k}^2 = \frac{1}{2} [1 - (E_{2k} + \mu)/E_{2k}]$ with $v_{1k}^2 = v_{2k}^2 = 1$ ($i = 1, 2$). There are three limits of special interest:

i) When $W_k = 0$, one can show that $N(\omega)$ reduces to DOS for the pure DSC case as we already showed

$$
N(\omega) = \sum_k \left\{ [1 + \xi_k/E_k] \delta(\omega - E_k) + [1 - \xi_k/E_k] \delta(\omega + E_k) + [1 + \xi_kQ/E_k + Q] \delta(\omega - E_k + Q) + [1 - \xi_kQ/E_k + Q] \delta(\omega + E_k + Q) \right\},
$$

ii) When $\Delta_k = 0$, $E_{1k} = |E_{0k} - \mu|$ and $E_{2k} = |E_{0k} + \mu|$. The coherent factors are now $u_{1k}^2 = \frac{1}{2} [1 + \text{sgn}(E_{0k} - \mu)]$ and $u_{2k}^2 = \frac{1}{2} [1 - \text{sgn}(E_{0k} + \mu)]$. Simple algebra shows that the DOS for a pure DDW case is
iii) When \( \mu = 0 \), but, with nonzero DSC and DDW gap, \( E_{1 k} = E_{2 k} = \Xi_k \), where \( \Xi_k = \sqrt{\epsilon_k^2 + \Delta_k^2} \) with \( \Delta_k = \sqrt{\Delta_0^2 + W_0^2} \). In this case the DOS becomes
\[
N(\omega) = \sum_k \left[ \delta(\omega - \Xi_k) + \delta(\omega + \Xi_k) \right].
\]

Note that this DOS is nothing but the DOS of a pure DSC case or the DOS of a pure DDW case with a gap \( \Delta_k \) for \( \mu = 0 \).

For the solid curve in the top frame of Fig. 5, \( W_0 = 0.04 \) and \( \Delta_0 = 0.02 \) with the chemical potential \( \mu = -0.25 \) which is sufficiently large in absolute value that superconducting gap and pseudogap are well separated, and there is distinct and characteristic structure associated with each. The superconducting gap froms at the FS (\( \omega = 0 \)) in the figure and the DDW gap at \( \omega = |\mu| \). The position in energy \( \omega \) of the singularities corresponding to \( 4\Delta_0 \) and \( 4W_0 \) are renormalized because both gap and pseudogap are present and interfere, and because \( \mu \) is not zero. But these renormalizations are not large. For the pseudogap alone (\( \Delta_0 = 0 \) case) there should be peaks of equal heights at \( \omega \approx |\mu| \pm 4W_0 \) i.e. \( \omega \approx 0.41 \) and 0.09. While the uppermost peak is shifted upward only very slightly by the presence of \( \Delta_0 \) to 0.418, the second peak is shifted more importantly to 0.12. If the pseudogap were zero, the superconducting peaks would be close but not quite at \( \omega = \pm 0.08 \) because \( \mu \) is not zero in this example. They are at \( \omega = \pm 0.076 \). Other than this small shift the gap and pseudogap structures are quite separate in the solid curve. Note also from the general mathematical form Eq. \((24)\) for the quasiparticle DOS that structures will fall symmetrically in energy about \( \omega = 0 \) but they are of different height. In particular we note the small but clearly visible peak at \( \omega = -0.12 \) in the middle frame of Fig. 5.

The dot-dashed curve in the top frame of Fig. 5 is for comparison with the solid curve and differs from it only through a different value of \( W_0 = 0.03 \) instead of 0.04. This change clearly shifts the the two prominent pseudogap peaks leaving the superconducting gap structure much less affected. It is clear from this comparison that gap and pseudogap are pretty independent of each other although each is somewhat affected by the presence of the other. If the chemical potential is reduced to \( \mu = -0.08 \) as in the middle frame of Fig. 5, we can see that the competition between gap anf pseudogap becomes much more severe particularly in the region around the FS and the chemical potential which are now closer together. The position of the most prominent upper peak falls at \( \omega = 0.173 \) above the chemical potential (\( |\mu| = 0.08 \)). This value is still close to \( 4W_0 = 0.16 \). The third peak, which in the top frame was identified with the gap peak, falls at \( \omega = 0.036 \) which is a factor of 2 lower than \( 4\Delta_0 = 0.08 \) so that if we should identify this with the gap, we would have to conclude that it is strongly suppressed by the presence of the DDW order. This is expected since both DSC and DDW order compete for the available states in this region of energy. This interpretation finds further support in the fact that a change in the value of gap \( \Delta_0 \) affects the position of this third peak strongly while a change of the DDW gap \( W_0 \) does not. The position of the second peak however changes significantly with a change in \( \Delta_0 \) or \( W_0 \) so that this peak is a true mixture of both gap and pseudogap.

In the bottom frame of Fig. 5, we show results for \( \mu = 0 \) i.e. at half filling. In this case, as we have seen in the DOS, gap and pseudogap become a single entity \( \Delta_0 = \sqrt{\Delta_0^2 + W_0^2} = 0.045 \). A simple gap structure is obtained symmetric about \( \omega = 0 \). In this case, the singularities are located at \( \pm 4\Delta_0/\sqrt{l^2 + \Delta_0^2} \). It is clear from the trends exhibited in Fig. 5 that the DDW model does not allow for a separate well identified set of two gaps with superconducting gap falling inside a larger pseudogap about \( \omega = 0 \). This could only happen if gap and pseudogap where both pinned to the FS. In the DDW model only the superconducting gap is pinned to the FS while the DDW gap opens up at the energy of the antiferromagnetic Brillouin zone. Of course the intrinsic tunneling experiments do not measure the DOS directly so they involve SIS junctions. Our calculations can be compared more directly to the SIN data of Renner et al. \cite{20} which do not presently show the evolution with doping expected in a DDW model.

V. CONCLUSIONS

As a possible candidate model for the pseudogap state seen in the underdoped regime of the high-\( T_c \) cuprates, we have studied the effects of the formation of \( D \)-density wave (DDW) order on several properties. While it is found that the value of zero temperature superfluid density \( \rho_s \) is impacted by the opening-up of a DDW gap at the antiferromagnetic Brillouin zone, its effect can increase as well as decrease the value of \( \rho_s \). If the chemical potential is small and near zero, the effective normal state density of electronic states around the Fermi surface (FS) is on average reduced when the pseudogap is increase by going to a more underdoped case. This leads to a redution in \( \rho_s \). But if the chemical potential falls well above the pseudogap energy, the opposite holds. In any case these effects are not dominant and are masked by the near exponential reduction in interlayer transfer matrix element known to exist in \( \text{YBa}_2\text{Cu}_3\text{O}_7-\delta \) as one goes from overdoped (where \( t_{\perp} \sim 30 - 40 \text{meV} \)) to underdoped case (where \( t_{\perp} \) can be a fraction of a meV) \cite{92}.

A quantity which does not depend explicitly on \( t_{\perp} \) and its variation with doping is the \( c \)-axis optical sum rule, which is known to be anomalous in underdoped cuprates while it takes on its conventional value of one in optimally doped \( \text{YBa}_2\text{Cu}_3\text{O}_7-\delta \). In this case we find that opening a DDW gap will affect the value of the ratio of the missing area under the difference between the real part of the optical conductivity in pseudogap and superconducting state. When it is normalized to the value of the \( c \)-axis
superfluid density, which is determined from the zero frequency limit of the imaginary part of the conductivity, it is found to be close to one in the overdoped regime and decrease towards a half in the underdoped case. In the case presented, the minimum value obtained was a little less than 0.8 but the calculations show that the size of the reduction depends on details of band structure, on filling, and on size of the gap and pseudogap as well as on the reference temperature used. A reduction can occur but its size is not a robust feature of the model.

We also considered how the area under the real part of the conductivity which gives the plasma frequency (or optical spectral weight) is affected by the introduction of the DDW order. It is found that near half filling, the optical spectral weight is substantially reduced as the temperature ($T$) is decreased from the pseudogap temperature ($T^*$) towards the superconducting transition temperature ($T_c$). Agreement with experiment is possible near half filling. When the doping is increased, however, away from half filling, the plasma frequency is found to have a minimum in the region $T_c < T < T^*$. This behavior is different from that expected on the bases of the preformed pair model where the phase fluctuations are the cause of the variation in optical spectral weight which only decreases with decreasing $T$.

To achieve some physical understanding of our $c$-axis result we found it useful to introduce and consider the modifications brought to the in-plane quasiparticle density of states through the growth of DDW order. Because a generic feature of the model is that superconducting gap opens at the Fermi surface while the DDW gap opens instead at the magnetic Brillouin zone, the resulting density of states $N(\omega)$ never shows superconducting gap features distinct from pseudogap feature which are also positioned inside the pseudogap humps as reported in a recent experiment. The prediction is that when the chemical potential ($\mu$) is large enough, gap and pseudogap features are separated by $|\mu|$ and are quite distinct. When $\mu = 0$, gap and pseudogap can no longer be distinguished and the square root of the sum of the squares plays the role of a single gap instead, reminiscent of the preformed pair model.

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FIG. 1. The $c$-axis superfluid density $\rho_s$ divided by its maximum value as a function of the value of the pseudogap amplitude $W_0(x)$ which depends on the doping $x$. In the solid curve the chemical potential $\mu$ is fixed at $-0.06$ and in the dashed curve at $-0.2$ in a unit of $t$. The inset gives the density of quasiparticle states (DOS) in a pure DDW state. The solid curve is for $W_0 = 0.03$ and the dotted for 0.016. Two possible position for the Fermi surface (FS$_1$ and FS$_2$) are indicated by vertical lines. OD and UD on the top stand for overdoped regime and underdoped regime, respectively.

FIG. 2. The value of the $c$-axis sum rule $\Delta N/\rho_s$ as a function of doping $x$. The changes reflect the opening of the DDW gap as the underdoped regime is entered. The inset shows a typical phase diagram for the cuprates from Ref. 16 with the superconducting dome (plus sign +), pseudogap (solid square) and superconducting gap (solid triangle) in unit of $[K]$. Note that for both superconducting gap and pseudogap their maximum on the FS is four times $\Delta_0$ and $W_0$, respectively.

FIG. 3. The value of the DDW gap $W_0(T)$ as a function of temperature ($T$) determined self-consistently for a given value of parameters ($n = 0.95$ and $V_{DDW} \simeq 0.945$) as described in the text. The inset shows the corresponding chemical potential $\mu$ as a function of $T$.

FIG. 4. The $c$-axis kinetic divided by its value at $T^*$ as a function of temperature ($T$) in the range $T_c < T < T^*$. The curves are labeled by the values of the filling $n$ and related to the doping $x = 1 - n$. In the inset we show the square of the normalized plasma frequency as a function of temperature $T$ obtained in Ref. 14 for YBa$_2$Cu$_3$O$_{6.6}$.

FIG. 5. The density of states (DOS) $N(\omega)$ as a function of energy $\omega$ including both DDW order and superconductivity. For the solid curve in each frame, the DDW gap $W_0 = 0.04$ and the superconducting gap $\Delta_0 = 0.02$ while the dot-dashed curve in the top frame has $W_0 = 0.03$ instead. The frames differ in choice of chemical potential $\mu$. In the top frame $\mu = -0.25$, middle $\mu = -0.08$, and in the bottom $\mu = 0$. The dashed line indicates $\omega = |\mu|$ and FS stands for the Fermi surface.
Fig. 1 (Kim et al)
Fig. 2 (Kim et al)
Fig. 3 (Kim et al)
Fig. 4 (Kim et al)

![Graph showing normalized K.E. vs. T/T*](image)

Normalized K.E.

\[ \frac{\omega^2}{\omega_p^2} \]

Legend:
- 0.95
- 0.97
- 0.99
- 1.00

Inset:
- Points at T = 70 and T = 290
Fig. 5 (Kim et al)