MACDONALD DIFFERENCE OPERATORS AND
HARISH-CHANDRA SERIES

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Abstract. We analyze the centralizer of the Macdonald difference operator in an appropriate algebra of Weyl group invariant difference operators. We show that it coincides with Cherednik's commuting algebra of difference operators via an analog of the Harish-Chandra isomorphism. Analogs of Harish-Chandra series are defined and realized as solutions to the system of basic hypergeometric difference equations associated to the centralizer algebra. These Harish-Chandra series are then related to both Macdonald polynomials and Chalykh's Baker-Akhiezer functions.

1. Introduction

Important examples of eigenfunctions of the Macdonald $q$-difference operator are Macdonald polynomials and Chalykh’s Baker-Akhiezer functions. In this paper we commence with a detailed study of the general spectral analysis of the Macdonald difference operator. We construct eigenfunctions that are essentially characterized by the requirement that they behave as plane waves deep in a distinguished Weyl chamber. We call the eigenfunctions difference Harish-Chandra series and relate them to the Macdonald polynomials and to the Baker-Akhiezer functions. We work in the general set-up of Macdonald’s recent book [23], which includes Koornwinder’s [21] extension of the Macdonald theory.

The refined analysis of the spectral problem of the Macdonald difference operator involves the so-called system of basic hypergeometric difference equations. Its definition is based on Cherednik’s key observation that the affine Hecke algebra $H$ admits a realization as difference-reflection operators, in which the Macdonald difference operator arises as the difference reduction of a particular central element of $H$ (see, e.g., [6]). As such, the Macdonald difference operator is part of a commutative algebra $\mathbb{D}$ of difference operators, obtained as the difference reduction of the full center of the affine Hecke algebra. It follows from this that $\mathbb{D}$ is isomorphic to an algebra $A'_0$ of Weyl group invariant regular functions on a complex torus $T' = \text{Hom}_\mathbb{Z}(L',\mathbb{C}^\times)$, where $L'$ is either the co-weight lattice or the co-root lattice of the underlying irreducible root system $R$. A concrete realization of this isomorphism is given by a difference analog $\gamma : \mathbb{D} \to A'_0$ of the Harish-Chandra isomorphism (cf., e.g., [5 §3]), which assigns to the difference operator $D \in \mathbb{D}$ its asymptotic leading term deep in a distinguished Weyl chamber, up to a suitable twist.

The system of basic hypergeometric difference equations is then given by

$$Df = \chi(D)f, \quad \forall D \in \mathbb{D},$$

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where $\chi : D \to C$ is an algebra homomorphism. The algebra homomorphisms $\chi : D \to C$ are naturally parameterized by the orbit space $T'/W_0$, by composing the Harish-Chandra isomorphism $\gamma$ with the evaluation map $A'_0 \ni p \mapsto p(t)$ for $W_0 t \in T'/W_0$.

The system (1.1) of difference equations can be considered over various classes of (formal) trigonometric functions on the complexification $V_C$ of the ambient Euclidean space $V$ of the root system $R$ (the corresponding period lattice of the trigonometric functions is of the form $\frac{2\pi \sqrt{-1}}{\log(q)} L$ where $L$ is dual to $L'$ in a suitable sense, as we will make precise in the main text). The Macdonald polynomials are the solutions of (1.1) which are Weyl group invariant trigonometric polynomials (the corresponding $\chi$ form the so-called polynomial spectrum of the system (1.1)). The difference Harish-Chandra series we construct are formal power series solutions of (1.1).

Before giving a precise outline of the contents of the paper, we first mention three alternative contexts of the system (1.1) of basic hypergeometric difference equations which are important as guiding principle for the present work. Firstly, Cherednik [4], [5] has related the system (1.1) of basic hypergeometric difference equations to quantum affine Knizhnik-Zamolodchikov type equations using the so-called Cherednik-Matsuo correspondence (see also [25], [16] and [7, Chpt. 2]). This ties (1.1) to $q$-holonomic systems, which also arise in integrable quantum field theories and in representation theory of quantum affine algebras, see, e.g., [8] for a survey.

Secondly, for special multiplicity labels the Macdonald difference operator arises as the radial component of the quantum Casimir acting on a quantum compact symmetric space (cf., e.g., [26], [21]). In this situation the ring $D$ corresponds to the image of the center of the quantum universal enveloping algebra under the corresponding radial component map for most symmetric spaces (see, e.g., [22]), and the associated Macdonald polynomials arise as the corresponding quantum analogs of the elementary spherical functions. We expect that the present study of the system (1.1) will provide a basic step towards the understanding of harmonic analysis on quantum analogs of noncompact Riemannian symmetric spaces, cf. Harish-Chandra’s well known classical approach (see, e.g., [12] for an overview). Some initial steps towards the harmonic analysis on quantum noncompact Riemannian symmetric spaces can be found in e.g. [18], [19], [30] and [22].

Thirdly, the Macdonald difference operator is essentially the Hamiltonian of the quantum relativistic integrable system of Calogero-Moser type (cf., e.g., [28]). For generic multiplicity labels $D$ then corresponds to the associated algebra of quantum conserved integrals. The algebra of quantum conserved integrals is known to be strictly larger than $D$ for special values of the multiplicity labels, in which case one speaks of algebraic integrability. In this situation, Chalykh’s [3] Baker-Akhiezer functions arise as particular solutions of (1.1).

In the last context the classical analog of the system (1.1) has been studied in full extent by Heckman and Opdam [14], [27], [13], see also the overview in the first part of the book [15]. In this paper we develop the theory in close parallel to the theory of Heckman and Opdam.

We now proceed to give a detailed description of the content of the paper. In Section 2 we recall Cherednik’s basic representation of the affine Hecke algebra, we give the corresponding construction of the Macdonald difference operator and we
define the associated commutative ring \( D \) of difference operators, following closely Macdonald’s book [23].

In Section 3 we analyze the structure of the commutative algebra \( D \) of difference operators in detail. We consider an algebra \( \mathbb{D}_R(L')^W_0 \) of difference operators containing \( D \). It consists of \( W_0 \)-invariant difference operators with step-sizes from the lattice \( L' \) and with coefficients in a suitable \( W_0 \)-invariant algebra \( R \) of rational trigonometric functions, where \( W_0 \) is the Weyl group of the underlying root system \( R \). The functions from \( R \) satisfy the essential additional property that they converge deep in a distinguished Weyl chamber of \( V \). Using a rank reduction argument in an analogous manner to the differential theory (see, e.g., [15]) we prove that the Harish-Chandra homomorphism \( \gamma \) also defines an algebra isomorphism from the centralizer of the Macdonald difference operator in \( \mathbb{D}_R(L')^W_0 \) onto \( A_0' \). As a consequence, we obtain the result that \( D \) equals the centralizer of the Macdonald difference operator in \( \mathbb{D}_R(L')^W_0 \). We furthermore obtain a simple criterion (Corollary 3.16) when the centralizer of the Macdonald difference operator in \( \mathbb{D}_R(L') \) is strictly larger than \( D \).

In Section 4 we construct the difference analogs of the Harish-Chandra series. For root systems \( R \) of type \( A \), the Harish-Chandra series solutions have been considered before by Etingof and Kirillov Jr. [9] and Kazarnovski-Krol [17]. For generic spectral parameters we give a basis of the corresponding solution space consisting of Harish-Chandra series. We relate the Harish-Chandra series to the Macdonald polynomial when the algebra homomorphism \( \chi \) is in the polynomial spectrum. We furthermore show that the difference Harish-Chandra series reduce to Chalykh’s Baker-Akhiezer function when the system \( (1.1) \) is algebraically integrable in the sense of [3] and [10]. Finally, in Section 5 we provide a list of notations used throughout the paper.

In this paper we entirely focus on the algebraic theory of the difference Harish-Chandra series. The analytic theory, in particular the convergence of the difference Harish-Chandra series deep in a distinguished Weyl chamber, is part of research in progress of the second author with Michel van Meer. Various other natural topics, such as the basic hypergeometric function (the analog of the spherical function), connection matrices, duality, bispectrality and quantum group interpretations, are also subject to future research.

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2. Macdonald difference operators

2.1. Root system data. Macdonald polynomials, which are naturally attached to affine root systems, have been successfully analyzed through the study of Cherednik’s [6] double affine Hecke algebra. Their nonreduced extensions, the so-called Macdonald-Koornwinder polynomials (see [20]), have been incorporated in the theory using a suitable extension of the double affine Hecke algebra, see Noumi [26] and Sahi [29]. In order to capture the closely related theories all at once, we follow Macdonald’s recent book [23] and adapt its conventions and notations as much as possible throughout the paper.
The basic structure underlying the Cherednik-Macdonald theory then consists of a pair \((R, R')\) of finite, reduced irreducible crystallographic root systems in an Euclidean space \((V, \langle \cdot, \cdot \rangle)\), a pair \((L', L')\) of lattices in \(V\), and a pair \((S, S')\) of irreducible affine root systems. The definition of these pairs depends on three different cases, to which we will refer to as case \(a\), \(b\) and \(c\) throughout the paper. Before listing the pairs for each of the three cases, we first introduce some general notation.

Let \((V, \langle \cdot, \cdot \rangle)\) be a finite dimensional Euclidean space. Let \(\hat{V}\) be the space of affine linear, real functions on \(V\). Denote \(c \in \hat{V}\) for the constant function one. We identify \(\hat{V} \cong V \oplus \mathbb{R}c\) as real vector space via the scalar product \(\langle \cdot, \cdot \rangle\) by associating to \(v + sc \in V \oplus \mathbb{R}c\) \((v \in V, s \in \mathbb{R})\) the affine linear functional \(v' \mapsto \langle v, v' \rangle + s\) on \(V\). Let \(D : \hat{V} \to V\) be the gradient map, defined by \(D(v + sc) = v\) for \(v \in V\) and \(s \in \mathbb{R}\). We extend \(\langle \cdot, \cdot \rangle\) to a positive semi-definite bilinear form on \(\hat{V}\) by

\[
(f,g) := \langle Df, Dg \rangle, \quad f,g \in \hat{V}.
\]

We define for \(f \in \hat{V}\) with \(Df \neq 0\) the associated co-vector by \(f^\vee = 2f/\|f\|^2 \in \hat{V}\).

Let \(R \subset V\) be a finite, reduced irreducible crystallographic root system, with associated Weyl group \(W_0 \subset O(V)\) generated by the orthogonal reflections \(s_\alpha\) in the hyperplanes \(\alpha^\perp \subset V\) \((\alpha \in R)\). Let \(R^\vee = \{\alpha^\vee\}_{\alpha \in R}\) be the associated co-root system. Denote \(Q = Q(R)\) and \(P = P(R)\) for the root lattice and the weight lattice of \(R\) respectively, which are \(W_0\)-invariant lattices in \(V\) satisfying \(Q \subset P\). We write \(Q^\vee = Q(R^\vee)\) and \(P^\vee = P(R^\vee)\) for the co-root lattice and co-weight lattice of \(R\) in \(V\). We define the affine Weyl group \(W_{Q^\vee}\) and the extended affine Weyl group \(W_{P^\vee}\) of \(R\) as the corresponding semi-direct product groups

\[
W_{Q^\vee} = W_0 \ltimes Q^\vee, \quad W_{P^\vee} = W_0 \ltimes P^\vee.
\]

The canonical action of \(W_0\) on \(V\) extends to a faithful action of the extended affine Weyl group on \(V\) with the lattice \(P^\vee\) acting by translations,

\[
t(\lambda)(v) = v + \lambda, \quad v \in V
\]

for \(\lambda \in P^\vee\). The space \(\hat{V}\) inherits a left \(W_{P^\vee}\)-module structure by transposition of the \(W_{P^\vee}\)-action on \(V\).

The set

\[
S(R) = \{\alpha + rc \mid \alpha \in R, \, r \in \mathbb{Z}\}
\]

defines an irreducible, reduced affine root system in \(\hat{V}\) with underlying finite, gradient root system \(D(S(R)) = R\). The associated affine Weyl group, generated by the orthogonal reflections in the affine hyperplanes \(f^{-1}(0)\) \((f \in S(R))\), is isomorphic to \(W_{Q^\vee}\). The affine root system \(S(R)\) is invariant under the action of the extended affine Weyl group \(W_{P^\vee}\). Observe furthermore that \(S(R)^\vee = \{f^\vee \mid f \in S(R)\}\) is an irreducible reduced affine root system in \(\hat{V}\) with underlying gradient root system \(R^\vee\).

The affine root systems \(S(R)\) and \(S(R)^\vee\) described above are all reduced. The nonreduced irreducible affine root systems are root subsystems of the affine root system of type \(C^\vee C\), which we now proceed to describe. Denote \(\{\epsilon_i\}_{i=1}^n\) for the standard orthonormal basis of \(\mathbb{R}^n\). Write \(R_C \subset \mathbb{R}^n\) for the root system of type \(C_n\) given by the set of roots \(\pm \epsilon_i \pm \epsilon_j\) \((1 \leq i < j \leq n)\) and \(\pm 2\epsilon_m\) \((m = 1, \ldots, n)\), where all sign combinations are allowed (we suppress the dependence on \(n \in \mathbb{Z}_{>0}\) in the notations). The nonreduced affine root system of type \(C^\vee C\) can now be realized
as the set
\[ S_{nr} := \{ \pm \epsilon_i \pm \epsilon_j + rc, \pm \epsilon_m + \frac{rc}{2}, \pm 2\epsilon_m + rc \mid 1 \leq i < j \leq n, 1 \leq m \leq n, r \in \mathbb{Z} \} \]
in \( \mathbb{R}^n \) (all sign combinations are allowed). Note that the associated gradient root system \( D(S_{nr}) \) is the nonreduced root system of type \( \text{BC}_n \), containing \( R_C \) as the root subsystem of roots of squared length greater than or equal to 2 in \( D(S_{nr}) \).

We refer to the three cases \([23] (1.4.1), [23] (1.4.2) \) and \([23] (1.4.3) \) of the Cherednik-Macdonald theory as cases \( a, b \) and \( c \) respectively. We list below for each case the pairs \((R, R'), (L, L') \) and \((S, S') \).

Case \( a \). \((R, R') = (R, R'), (L, L') = (P(R), P(R'))\) and \((S, S') = (S(R), S(R'))\) with \( R \subset V \) a finite, reduced, irreducible crystallographic root system, normalized such that long roots have squared length two.

Case \( b \). \((R, R') = (R, R), (L, L') = (P(R'), P(R'))\) and \((S, S') = (S(R)^\vee, S(R)^\vee)\) with \( R \subset V \) a finite, reduced, irreducible crystallographic root system, normalized such that long roots have squared length two.

Case \( c \). \( V = \mathbb{R}^n, (R, R') = (R_C, R_C), (L, L') = (Q(R_C^\vee), Q(R_C^\vee))\) and \((S, S') = (S_{nr}, S_{nr})\).

Note that for case \( c \), long roots in \( R \) have squared norm four. Note furthermore that for case \( a \) and \( b \) we have \( L = P(R'^\vee) \) and \( L' = P(R'^\vee) \), while for case \( c \) \( L = Q(R_C^\vee) \) and \( L' = Q(R_C^\vee) \). We can thus associate to each case a pair of (extended) affine Weyl groups
\[ (W, W') = (W_{L'}, W_L), \]
which are the extended affine Weyl groups of \( R \) and \( R' \) for case \( a \) and \( b \), and the affine Weyl group of \( R \) and \( R' \) for case \( c \).

For each of the three cases, the (extended) affine Weyl group \( W \) (respectively \( W' \)) preserves \( S \) (respectively \( S' \)). In case \( a \) and \( b \) the decomposition of \( S \) into \( W \)-orbits coincides with the decomposition \( S = S_\alpha \cup S_\ell \) of \( S \) into the set \( S_\alpha \) of short roots and the set \( S_\ell \) of long roots. By the normalization of the underlying finite root system, \( S_\ell \) are the roots in \( S \) whose squared length equals two with respect to the semi-positive definite form \( \langle \cdot, \cdot \rangle \) of \( V \). In case \( c \) the affine root system \( S \) has five \( W \)-orbits, namely
\[ \mathcal{O}_1 = \{ \pm \epsilon_m + rc \mid 1 \leq m \leq n, r \in \mathbb{Z} \}, \quad \mathcal{O}_2 = 2 \mathcal{O}_1, \quad \mathcal{O}_3 = \mathcal{O}_1 + \frac{c}{2}, \quad \mathcal{O}_4 = 2 \mathcal{O}_3 \]
and \( \mathcal{O}_5 = \{ \pm \epsilon_i \pm \epsilon_j + rc \mid 1 \leq i < j \leq n, r \in \mathbb{Z} \} \).

Let \( S_1 \subset S \) be the reduced affine root subsystem of indivisible affine roots in \( S \). Then \( S_1 \) is \( W \)-invariant, \( S_1 = S \) for case \( a \) and \( b \) while \( S_1 = S(R_C)^\vee \) for case \( c \), which has three \( W \)-orbits \( \mathcal{O}_1, \mathcal{O}_3 \) and \( \mathcal{O}_5 \).

We fix a basis \( \{ \alpha_1, \ldots, \alpha_n \} \) of the root system \( R \). We associate to it a basis \( \{ a_0, a_1, \ldots, a_n \} \) of the corresponding affine root system \( S \) by
\[ \{ a_0, a_1, \ldots, a_n \} = \begin{cases} \{-\varphi + c, \alpha_1, \ldots, \alpha_n \}, & \text{case } a, \\ \{-\varphi^\vee + c, \alpha_1^\vee, \ldots, \alpha_n^\vee \}, & \text{case } b, \\ \{-\varphi^\vee + \frac{c}{2}, \alpha_1^\vee, \ldots, \alpha_n^\vee \}, & \text{case } c, \end{cases} \]
where \( \varphi \in R \) is the highest root with respect to the given basis \( \{ \alpha_1, \ldots, \alpha_n \} \) of \( R \) (which is always a long root). Note that \( \{ a_0, \ldots, a_n \} \) is also a basis of \( S_1 \), and that each affine root \( a \in S_1 \) is \( W \)-conjugate to a simple root \( a_j \) \( (j \in \{0, \ldots, n\}) \). We set \( \Delta = \{ a_1, \ldots, a_n \} \), which is a basis of the root subsystem \( R'^\vee = D(S_1) \) of indivisible
roots in the gradient root system $D(S) \subset V$ of $S$. In particular, $\Delta$ is a $\mathbb{Z}$-basis of $Q(R^\vee)$.

We denote $s_i = s_{a_i} \in W_{Q^\vee} \subset W \ (i = 0, \ldots, n)$ for the reflection associated to the simple root $a_i \in S_1 \ (i = 0, \ldots, n)$. The affine Weyl group $W_{Q^\vee}$ is a Coxeter group with respect to the simple reflections $s_i \ (i = 0, \ldots, n)$. Let $S_1^+$ (respectively $S_1^\pm$) be the positive (respectively negative) affine roots in $S_1$ with respect to the basis $\{a_i\}_{i=0}^n$ of $S_1$. The length $l(w)$ of $w \in W$ is defined by

$$l(w) = \#(S_1^+ \cap w^{-1}S_1^\pm).$$

The finite abelian subgroup $\Omega = \{w \in W \mid l(w) = 0\}$ of $W$ is isomorphic to $L'/Q(R^w)$, and we have the semi-direct product decomposition $W \simeq W_{Q^\vee} \rtimes \Omega$.

The action of $\Omega$ on $\hat{V}$ restricts to a faithful action on the finite set $\{a_i\}_{i=0}^n$ of simple roots, hence we may and will view $\Omega$ as a permutation subgroup of the index set $\{0, \ldots, n\}$. We denote $\mathbb{C}[W]$ and $\mathbb{C}[\Omega]$ for the complex group algebra of $W$ and $\Omega$, respectively.

### 2.2. The algebra of difference-reflection operators

Denote $A = \mathbb{C}[L]$ and $A' = \mathbb{C}[L']$ for the group algebras of the lattices $L$ and $L'$ over $\mathbb{C}$, respectively. We denote the canonical basis of $A$ (respectively $A'$) by $\{z^\lambda \mid \lambda \in L\}$ (respectively $\{\xi^\lambda \mid \lambda' \in L'\}$), so that

$$z^\lambda z^\mu = z^{\lambda + \mu}, \quad z^0 = 1$$

and similarly for $\xi^{\lambda'}$. We fix throughout the paper $0 < q < 1$. We extend the definition of the monomials $z^\lambda$ in $A$ (respectively $\xi^{\lambda'}$ in $A'$) by defining

$$z^{\lambda + rc} := q^r z^\lambda, \quad \xi^{\lambda' + rc} := q^r \xi^{\lambda'}$$

for $\lambda \in L, \lambda' \in L'$ and $r \in \mathbb{R}$. In particular, for $a \in S$ and $b \in S'$ we have $Da \in Q(R^\vee) \subseteq L$ and $Db \in Q(R^w) \subseteq L'$, hence $z^a \in A$ and $\xi^b \in A'$ are well defined.

Let $V_C = V \oplus \sqrt{-1} V$ be the complexification of $V$, and extend $\langle \cdot, \cdot \rangle$ to a complex bilinear form on $V_C$. We view $A$ and $A'$ as subalgebras of the algebra of complex analytic functions on $V_C$ by interpreting the canonical basis elements $z^\lambda$ and $\xi^{\lambda'}$ as complex plane waves on $V_C$,

$$z^\lambda(v) = q^{(\lambda, v)}, \quad \xi^{\lambda'}(v) = q^{(\lambda', v)}$$

for $\lambda \in L, \lambda' \in L'$ and $v \in V_C$. The (extended) affine Weyl groups $W$ and $W'$ act on $A$ and $A'$ as algebra automorphisms by transposing their respective actions on $V_C$. Concretely, for $w = vt(\lambda') \in W$ and $w' = vt(\lambda) \in W'$ with $v \in W_0$, $\lambda' \in L'$ and $\lambda \in L$ we have

$$w(z^\mu) = q^{-(\lambda', \mu)} z^{v\mu}, \quad w'(\xi^{\mu'}) = q^{-(\lambda, \mu')} \xi^{v\mu'}$$

for $\mu \in L$ and $\mu' \in L'$. Furthermore, $w(a^a) = z^{wa}$ and $w'(b^b) = \xi^{wb}$ for $a \in S$ and $b \in S'$. We write $Q$ and $Q'$ for the quotient fields of $A$ and $A'$ respectively. The $W$-action on $A$ (respectively $W'$-action on $A'$) extends uniquely to a $W$-action on $Q$ (respectively $W'$-action on $Q'$) by field automorphisms. We write $(wf)(z) \in Q$ for the rational function obtained by acting by $w \in W$ on $f(z) \in Q$.

**Definition 2.1.** Let $\mathcal{R}$ be the complex subalgebra of $Q$ generated (as an algebra) by the elements

$$\frac{1}{1 - rz^\alpha}, \quad \alpha \in R^\vee, \ r \in \mathbb{C}.$$

Note that \((1 - rz^α)^{-1} \in \mathcal{R}\) for all \(r \in \mathbb{C}\) and \(α \in S\). In particular, \(\mathcal{R}\) is a \(W\)-module subalgebra of \(\mathcal{Q}\). Furthermore,

\[
(2.2) \quad \frac{1}{1 - rz^α} = 1 - \frac{1}{1 - r^{-1}z^{-α}}, \quad α \in \mathbb{R}^\omega, \quad r \in \mathbb{C}^\times,
\]

hence \(\mathcal{R}\) is already generated by the elements \(\frac{1}{1 - rz^α}\) with \(r \in \mathbb{C}\) and with roots \(α \in \mathbb{R}^\omega_+ := \mathbb{R}^\omega \cap \mathbb{Z}_{\geq 0}\).

We associate to the \(W\)-module algebra \(\mathcal{R}\) the smash-product algebras

\[
\mathbb{D}_{\mathcal{R}}(L') := \mathcal{R}#t(L') \subset \mathcal{R}#W =: \mathbb{D}_{\mathcal{R}}(W).
\]

In other words, \(\mathbb{D}_{\mathcal{R}}(W)\) is the complex, unital, associative algebra such that

\[
\mathbb{D}_{\mathcal{R}}(W) \simeq \mathcal{R} \otimes \mathbb{C}[W]
\]

as complex vector spaces, such that the canonical linear embeddings \(\mathcal{R}, \mathbb{C}[W] \hookrightarrow \mathbb{D}_{\mathcal{R}}(W)\) are unital algebra embeddings and such that the cross relations

\[
(f_1 \otimes w_1)(f_2 \otimes w_2) = f_1w_1(f_2) \otimes w_1w_2
\]

holds for \(f_i \in \mathcal{R}\) and \(w_i \in W\) (and similarly for \(\mathbb{D}_{\mathcal{R}}(L')\)).

**Definition 2.2.** We call \(\mathbb{D}_{\mathcal{R}}(W)\) (respectively \(\mathbb{D}_{\mathcal{R}}(L')\)) the algebra of difference-reflection (respectively difference) operators with coefficients in \(\mathcal{R}\).

The terminology is justified by the canonical, faithful action of \(\mathbb{D}_{\mathcal{R}}(W)\) (respectively \(\mathbb{D}_{\mathcal{R}}(L')\)) on \(\mathcal{Q}\) as difference-reflection (respectively difference) operators by

\[
(f \otimes w)(g) := fw(g), \quad f \in \mathcal{R}, \quad w \in W, \quad g \in \mathcal{Q}.
\]

Since \(\mathcal{R}\) is a \(\mathbb{C}[t(X)]\)-submodule algebra of \(\mathcal{Q}\) for all lattices \(X \subset V\), we can similarly define the algebra \(\mathbb{D}_{\mathcal{R}}(X) = \mathcal{R}#t(X)\) of difference operators with coefficients in \(\mathcal{R}\) and step-sizes from \(X\). For lattices \(L' \subseteq X \subset V\), \(\mathbb{D}_{\mathcal{R}}(L')\) canonically embeds as a subalgebra into \(\mathbb{D}_{\mathcal{R}}(X)\).

A second extension of \(\mathbb{D}_{\mathcal{R}}(L')\) involves the coefficients of the difference operators. Recall that \(\mathcal{Q}(R^\omega) = Z\Delta \subseteq L\). We define \(\mathbb{C}[[z^{-\Delta}]]\) to be the algebra of formal power series

\[
(2.3) \quad f(z) = \sum_{x \in \mathbb{Z}_{\geq 0}\Delta} C_x z^{-x}, \quad C_x \in \mathbb{C}.
\]

A lattice \(X \subset V\) acts by algebra automorphisms on \(\mathbb{C}[[z^{-\Delta}]]\) as

\[
t(\nu)(f(z)) = \sum_{x \in \mathbb{Z}_{\geq 0}\Delta} C_x q^{(\nu,x)} z^{-x}, \quad \nu \in X,
\]

with \(f(z)\) given by \((2.3)\). The algebra of difference operators with step-sizes from \(X\) and coefficients from \(\mathbb{C}[[z^{-\Delta}]]\) is the associated smash-product algebra \(\mathbb{D}_{\mathbb{C}[[z^{-\Delta}]]}(X) := \mathbb{C}[[z^{-\Delta}]]#t(X)\). Again, for lattices \(L' \subseteq X \subset V\), \(\mathbb{D}_{\mathbb{C}[[z^{-\Delta}]]}(L')\) canonically embeds into \(\mathbb{D}_{\mathbb{C}[[z^{-\Delta}]]}(X)\).

The \(W\)-module algebra \(\mathcal{R}\) can be canonically embedded as a \(t(X)\)-module subalgebra in \(\mathbb{C}[[z^{-\Delta}]]\) for any lattice \(X \subset V\) using, for \(r \in \mathbb{C}\) and for \(α \in \mathbb{R}^\omega_+\), the formal series expansion

\[
\frac{1}{1 - rz^α} = \sum_{m=0}^{\infty} r^m z^{-mα}.
\]

Correspondingly, we have a canonical embedding

\[
\mathbb{D}_{\mathcal{R}}(X) \hookrightarrow \mathbb{D}_{\mathbb{C}[[z^{-\Delta}]]}(X).
\]
of algebras. We identify $\mathbb{D}_R(X)$ with its image under this embedding in the remainder of the paper.

We end this subsection by constructing an action of $\mathbb{D}_C[[z^{-\Delta}]](L')$ as difference operators on a space of formal linear combinations of complex plane waves on $V_C$, which will be of importance in the construction of the Harish-Chandra series in Section 4.

Let $M$ be the complex commutative subalgebra of analytic functions on $V_C$ spanned by the complex plane waves $z^u$ ($u \in V_C$), $z^u := q^{\langle u, v \rangle}$, $v \in V_C$.

The algebra $A$ naturally identifies with the subalgebra of $M$ spanned by the plane waves $z^{\lambda}$ ($\lambda \in L'$), cf. (2.1).

Denote $\mathcal{M}$ for the complex vector space of formal series $\sum_{u \in C} K_u z^u$ ($K_u \in \mathbb{C}$) for subsets $C \subset V_C$ which are contained in some finite union of sets of the form $\lambda - Z_{\geq 0}\Delta$ ($\lambda \in V_C$). In other words, an element $F(z) \in \mathcal{M}$ is a finite linear combination of formal series of the form

$$F_\lambda(z) = \sum_{x \in \mathbb{Z}_{\geq 0}\Delta} K_\lambda(x) z^{\lambda-x} \quad (K_\lambda(x) \in \mathbb{C})$$

where $\lambda \in V_C$. Note that $\mathcal{M}$ is canonically a $\mathbb{C}[[z^{-\Delta}]]$-module, and $M$ embeds in $\mathcal{M}$ as a $\mathbb{C}[[z^{-\Delta}]]$-module.

**Lemma 2.3.** The action of $\mathbb{C}[[z^{-\Delta}]]$ on $\mathcal{M}$ extends to an action of the algebra $\mathbb{D}_C[[z^{-\Delta}]](L')$ of difference operators with coefficients in $\mathbb{C}[[z^{-\Delta}]]$ on $\mathcal{M}$, with $\lambda' \in L'$ acting as

$$t(\lambda')(\sum_{u \in C} K_u z^u) = \sum_{u \in C} K_u q^{-\langle \lambda', u \rangle} z^u.$$

**Proof.** This follows from a direct computation. \qed

2.3. Cherednik’s difference-reflection operators. Cherednik’s difference-reflection operators realize the (extended) affine Hecke algebra inside $\mathbb{D}_R(W)$. We recall the construction in this subsection. The affine Hecke algebras and its realization in $\mathbb{D}_R(W)$ depend on multiplicity labels, which we now first introduce.

The complex vector space $\mathbb{C}(S)^W$ consisting of $W$-invariant functions $k : S \to \mathbb{C}$ is called the space of multiplicity labels associated to $S$. We denote $k_a = k(a)$ for $a \in S$ and $k_i = k_i(a_i)$ for $i = 0, \ldots, n$. For case $a$ and $b$ the vector space $\mathbb{C}(S)^W$ is at most two dimensional, while it is five dimensional for case $c$.

Depending on the case under consideration, we associate to a multiplicity label $k \in \mathbb{C}(S)^W$ a dual multiplicity label $k' \in \mathbb{C}(S')^{W'}$ as follows.

Case $a$. $k'(b) = k(Db')$ for all $b \in S'$.

Case $b$. $k'(b) = k(b)$ for all $b \in S'$.

Case $c$. We write $\kappa_j$ for the value of $k$ at the $W$-orbit $O_j$ of $S$ ($j = 1, \ldots, 5$). We define the dual multiplicity label $k'$ by assigning the following value $\kappa'_j$ to
the $W'$-orbit $\mathcal{O}_j$ of $S'$ $(j = 1, \ldots, 5)$,

\[
\begin{align*}
\kappa'_1 &= \frac{1}{2}(\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4), \\
\kappa'_2 &= \frac{1}{2}(\kappa_1 + \kappa_2 - \kappa_3 - \kappa_4), \\
\kappa'_3 &= \frac{1}{2}(\kappa_1 - \kappa_2 + \kappa_3 - \kappa_4), \\
\kappa'_4 &= \frac{1}{2}(\kappa_1 - \kappa_2 - \kappa_3 + \kappa_4), \\
\kappa'_5 &= \kappa_5.
\end{align*}
\]

Denote $\mathbb{C}(S_1)^W$ for the space of multiplicity labels associated to $S_1$. Associated to a multiplicity label $k \in \mathbb{C}(S)^W$ we define an invertible multiplicity label $\tau \in \mathbb{C}(S_1)^W$ and a dual invertible multiplicity label $\tau' \in \mathbb{C}(S_1)^W$ as follows. For cases $a$ and $b$, we set

\[
\tau_a = q^{\frac{1}{2}k_a} = \tau'_a, \quad a \in S_1 = S.
\]

For case $c$, we again write $\kappa_j$ for the value of $k$ on the $W'$-orbit $\mathcal{O}_j$ $(j = 1, \ldots, 5)$.

The invertible multiplicity labels $\tau$ and $\tau'$ associated to $S_1$ are then defined by

\[
\begin{align*}
\tau_a &= q^{\frac{1}{2}(\kappa_1+\kappa_2)}, \quad \tau'_a = q^{\frac{1}{2}(\kappa_1-\kappa_2)}, \quad a \in \mathcal{O}_1, \\
\tau_a &= q^{\frac{1}{2}(\kappa_3+\kappa_4)}, \quad \tau'_a = q^{\frac{1}{2}(\kappa_3-\kappa_4)}, \quad a \in \mathcal{O}_3, \\
\tau_a &= \tau'_a = q^{\frac{1}{2}\kappa_5}, \quad a \in \mathcal{O}_5.
\end{align*}
\]

We write $\tau_i = \tau_{a_i}$ and $\tau'_i = \tau'_{a_i}$ for $i = 0, \ldots, n$.

The (extended) affine Hecke algebra depends on an invertible multiplicity label $\tau \in \mathbb{C}(S_1)^W$ (hence depends at most on three continuous, complex parameters).

**Definition 2.4.** Let $\tau \in \mathbb{C}(S_1)^W$ be an invertible multiplicity label. The (extended) affine Hecke algebra $H(\tau)$ is the unital complex associative algebra generated by elements $T_i$ $(i = 0, \ldots, n)$ and $\omega$ $(\omega \in \Omega)$ satisfying

1. The linear map $\mathbb{C}[\Omega] \to H$, defined by $\omega \mapsto \omega$, is an algebra morphism.

2. The $T_i$ $(i = 0, \ldots, n)$ satisfy the braid relations

\[
T_iT_jT_i \cdots = T_jT_iT_j \cdots
\]

with $m_{ij}$ factors on both sides, for indices $i \neq j$ such that $s_is_j \in W_{Q'}$ has finite order $m_{ij}$.

3. $\omega T_i\omega^{-1} = T_{\omega(i)}$ for $i = 0, \ldots, n$ and $\omega \in \Omega$.

4. $(T_i - \tau_i)(T_i + \tau_i^{-1}) = 0$ for $i = 0, \ldots, n$.

For $w \in W$ and a reduced expression $w = \omega s_{i_1}s_{i_2} \cdots s_{i_{(w)}}$ with $\omega \in \Omega$ and $i_j \in \{0, \ldots, n\}$ we write

\[
T_w = \omega T_{i_1}T_{i_2} \cdots T_{i_{(w)}} \in H(\tau).
\]

The $T_w$ $(w \in W)$ are well defined (independent of the reduced expression) and form a linear basis of $H(\tau)$.

Fix a multiplicity label $k \in \mathbb{C}(S)^W$ and denote $\tau, \tau' \in \mathbb{C}(S_1)^W$ for the invertible multiplicity labels associated to $k$ as described above. The basic representation of $H(\tau)$ which we now proceed to define, depends also on $\tau'$ (in other words, it
under the action of the difference-reflection operators $T$, where
\begin{equation}
\tag{2.7}
c_a(z) := \frac{(1 - \tau_a \tau'_a z^a)(1 + \tau_a \tau'_a^{-1} z^a)}{\tau_a(1 - z^{2a})} \in \mathcal{R}.
\end{equation}

Note that $c_a(z) \in \mathcal{R}$ ($a \in S_1$) satisfies the elementary identity
\begin{equation}
\tag{2.6}
c_a(z) + c_{-a}(z) = \tau_a + \tau_a^{-1},
\end{equation}
and that
\[c_a(z) = \frac{(1 - \tau_a^2 z^a)}{\tau_a(1 - z^a)}\]
if $\tau_a = \tau'_a$. We define the difference-reflection operators
\[T_i(\mathbf{k}) := \tau_i + c_a(z)(s_i - 1) \in \mathcal{D}_\mathcal{R}(W), \quad i = 0, \ldots, n.
\]

The following theorem is due to Cherednik [6], and due to Noumi [26] for case e. The present uniform formulation is from [23] (4.3.10)].

**Theorem 2.5.** i) For $i = 0, \ldots, n$ the $W$-module subalgebra $A \subset \mathbb{Q}$ is invariant under the action of the difference-reflection operators $T_i(\mathbf{k}) \in \mathcal{D}_\mathcal{R}(W)$ on $\mathbb{Q}$.

ii) The assignment $T_i \mapsto T_i(\mathbf{k})|_A$ ($i = 0, \ldots, n$), together with the usual action of $\omega \in \Omega$ on the $W$-module algebra $A$, uniquely extends to a faithful representation of the (extended) affine Hecke algebra $H(\mathbf{z})$ on $A$.

Since $D \in \mathcal{D}_\mathcal{R}(W)$ is uniquely determined by its action on $A \subset \mathbb{Q}$, we obtain

**Corollary 2.6.** The assignment $\omega \mapsto \omega$ and $T_i \mapsto T_i(\mathbf{k})$ for $\omega \in \Omega$ and $i = 0, \ldots, n$ uniquely extends to an injective algebra homomorphism $\pi_{\mathbf{k}} : H(\mathbf{z}) \mapsto \mathcal{D}_\mathcal{R}(W)$.

We identify $H(\mathbf{z})$ with its image in $\mathcal{D}_\mathcal{R}(W)$ under the above algebra embedding $\pi_{\mathbf{k}}$. In particular, we write $T_i$ for $T_i(\mathbf{k}) \in \mathcal{D}_\mathcal{R}(W)$ ($i = 0, \ldots, n$) if no confusion can arise.

We now proceed to define a reduction map $\beta$ which maps difference-reflection operators to difference operators. This map is vital for constructing the Macdonald difference operators from the representation $\pi_{\mathbf{k}}$ as well as for constructing a large family of difference operators commuting with the Macdonald difference operator.

Since $\mathcal{D}_\mathcal{R}(L')$ is a $W_0$-module algebra by $W_0$-conjugation in $\mathcal{D}_\mathcal{R}(W)$,
\[D \mapsto wDw^{-1}, \quad D \in \mathcal{D}_\mathcal{R}(L'), \; w \in W_0,
\]
we have a canonical isomorphism
\begin{equation}
\tag{2.7}
\mathcal{D}_\mathcal{R}(W) \simeq \mathcal{D}_\mathcal{R}(L') \# W_0
\end{equation}
of algebras. We now define the linear map $\beta : \mathcal{D}_\mathcal{R}(W) \to \mathcal{D}_\mathcal{R}(L')$ by
\[\beta(D) = \sum_{w \in W_0} D_w, \quad D \in \mathcal{D}_\mathcal{R}(W),
\]
where $D = \sum_{w \in W_0} D_w w$ ($D_w \in \mathcal{D}_\mathcal{R}(L')$) is the unique decomposition of $D \in \mathcal{D}_\mathcal{R}(W)$ along the isomorphism (2.7). Let $H_0 = H_0(\mathbf{z}) \subset H(\mathbf{z})$ be the finite Hecke algebra, generated by $T_j$ ($j = 1, \ldots, n$). Consider the subalgebras
\[\mathcal{D}_\mathcal{R}(W)^{H_0} = \{D \in \mathcal{D}_\mathcal{R}(W) \mid [D, h] = 0 \; \forall h \in H_0\},
\]
\[\mathcal{D}_\mathcal{R}(L')^{W_0} = \{D \in \mathcal{D}_\mathcal{R}(L') \mid wDw^{-1} = D \; \forall w \in W_0\}.
\]
Note that the center $Z(H(\tau))$ of $H(\tau)$ is contained in $\mathbb{D}_R(W)^{H_0}$. The following lemma, which is the difference analog of [13] Lemma 1.2.2, gives for case $a$ a slight extension of [5] Thm. 3.3.

**Lemma 2.7.** The map $\beta$ restricts to an algebra homomorphism
\[
\beta : \mathbb{D}_R(W)^{H_0} \to \mathbb{D}_R(L')^{W_0}.
\]
In particular, the operators $\beta(D) \ (D \in \mathbb{D}_R(W)^{H_0})$ preserve the subalgebra of $W_0$-invariant elements in $Q$.

**Proof.** Let $D \in \mathbb{D}_R(W)^{H_0}$ and write $D = \sum_{w \in W_0} D_w w$ with $D_w \in \mathbb{D}_R(L')$. Fix $j \in \{1, \ldots, n\}$. Since
\[
\beta(Dc_{a_j}(z)(s_j - 1)) = 0,
\]
\[
\beta(c_{a_j}(z)(s_j - 1)D) = c_{a_j}(z)(s_j \beta(D)s_j - \beta(D)),
\]
we obtain from the fact that $[D, T_j] = 0$ in $\mathbb{D}_R(W)$,
\[
0 = \beta([D, c_{a_j}(z)(s_j - 1)]) = -c_{a_j}(z)(s_j \beta(D)s_j - \beta(D)),
\]
hence $s_j \beta(D)s_j = \beta(D)$. We conclude that $\beta(D) \in \mathbb{D}_R(L')^{W_0}$.

For $D, D' \in \mathbb{D}_R(W)^{H_0}$, written as $D = \sum_{v \in W_0} D_v v$ and $D' = \sum_{w \in W_0} D'_w w$ with $D_v, D'_w \in \mathbb{D}_R(L')$, we now have
\[
\beta(DD') = \sum_{u,v \in W_0} D_v v D'_v v^{-1} = \sum_{v \in W_0} D_v \beta(D') v^{-1} = \sum_{v \in W_0} D_v \beta(D') = \beta(D)\beta(D'),
\]
hence $\beta$, restricted to $\mathbb{D}_R(W)^{H_0}$, is an algebra homomorphism. $\square$

In the next subsection we describe Cherednik’s commuting family of difference operators $\beta(h) \in \mathbb{D}_R(L')^{W_0}$ ($h \in Z(H(\tau))$) and give the explicit expression of the Macdonald difference operators inside $\beta(Z(H(\tau)))$.

### 2.4. Cherednik-Macdonald commuting difference operators.
Let $V_+ \subset V$ be the open dominant Weyl chamber with respect to $R^+$,
\[
V_+ = \{v \in V \mid \langle v, \alpha \rangle > 0 \quad \forall \alpha \in R^+\},
\]
and write $\overline{V}_+$ for its closure in $V$. We denote
\[
L_{++} := L \cap \overline{V}_+ , \quad L'_{++} := L' \cap \overline{V}_+
\]
for the cone of dominant elements in $L$ and $L'$, respectively. Set
\[
Y^{\lambda'} := T_{t(\lambda')} \in H(\tau), \quad \lambda' \in L'_{++},
\]
and for arbitrary $\lambda' \in L'$ define
\[
Y^{\lambda'} = Y^{\mu'} (Y^{\nu'})^{-1} \in H(\tau)
\]
if $\lambda' = \mu' - \nu'$ with $\mu', \nu' \in L'_{++}$. The $Y^{\lambda'} (\lambda' \in L')$ are well defined and satisfy
\[
Y^0 = 1, \quad Y^{\lambda'} Y^{\mu'} = Y^{\lambda' + \mu'}
for \( \lambda', \mu' \in L' \). The subalgebra \( A'(Y) \) of \( H(\tau) \) spanned by the \( Y^{\lambda'} \ (\lambda' \in L') \) is isomorphic to \( A' \) by the map \( Y^{\lambda'} \mapsto \xi^{\lambda'} \) for \( \lambda' \in L' \). The multiplication map induces a linear isomorphism

\[
H(\tau) = H_0(\tau) \otimes_{\mathbb{C}} A'(Y)
\]

and

\[
Z(H(\tau)) = A'_0(Y),
\]

where \( A'_0 \) is the subalgebra of \( W_0 \)-invariant elements in \( A' \) and \( A'_0(Y) \subset H(\tau) \) is the corresponding subspace in \( A'(Y) \) via the above mentioned isomorphism \( A' \simeq A'(Y) \).

We write \( p(Y) \in A'(Y) \) for the element corresponding to \( p(\xi) \in A' \).

Similarly we write \( A_0 \) for the subalgebra of \( W_0 \)-invariant elements in \( A \). For \( \lambda \in L \) and \( \lambda' \in L' \) we define the monomial symmetric functions by

\[
m_\lambda(z) = \sum_{\mu \in W_0 \lambda} z^\mu \in A_0, \quad m_{\lambda'}(\xi) = \sum_{\mu' \in W_0 \lambda'} \xi^{\mu'} \in A'_0.
\]

The functions \( m_\lambda(z) \ (\lambda \in L_{+++}) \) (respectively \( m_{\lambda'}(\xi) \ (\lambda' \in L'_{+++}) \)) form a linear basis of \( A_0 \) (respectively \( A'_0 \)). Correspondingly, the elements \( m_{\lambda'}(Y) \ (\lambda' \in L'_{+++}) \) form a linear basis of \( Z(H(\tau)) \).

**Definition 2.8.** For \( p(\xi) \in A'_0 \) we write

\[
D_p := \beta(p(Y)) \in \mathbb{D}_R(L')^{W_0}
\]

for the corresponding \( W_0 \)-invariant difference operator. For \( p(\xi) = m_{\lambda'}(\xi) \in A'_0 \) with \( \lambda' \in L' \) we simplify the notations by writing

\[
D_{\lambda'} := D_{m_{\lambda'}} = \beta(m_{\lambda'}(Y)) \in \mathbb{D}_R(L')^{W_0}.
\]

Note that \( D_{\lambda'} \in \mathbb{D}_R(L')^{W_0} \) only depends on the \( W_0 \)-orbit \( W_0 \lambda' \) of \( \lambda' \in L' \). In practice, it will be convenient to take the antidominant representative of the orbit \( W_0 \lambda' \). The following fact (cf. [24] Thm. 3.2] for case a, and [26] for case c) is now immediate from Lemma 2.7.

**Corollary 2.9.** The \( W_0 \)-invariant difference operators \( D_p \in \mathbb{D}_R(L')^{W_0} \ (p(\xi) \in A'_0) \) pair-wise commute.

We are now in a position to define the Macdonald difference operator as a difference operator \( D_p \) for a special choice of \( p(\xi) \in A'_0 \). We call a co-weight \( \pi' \in P(R^{\tau}) \) minuscule (respectively quasi-minuscule) if it satisfies \(|\langle \pi', \alpha \rangle| \leq 1 \) for all \( \alpha \in R \) (respectively \( \pi' \in R^{\tau} \) and \(|\langle \pi', \alpha \rangle| \leq 1 \) for all \( \alpha \in R \setminus \{\pm \pi^{\tau} \} \)).

**Definition 2.10.** The operator \( D_{\pi'} \in \mathbb{D}_R(L')^{W_0} \) with \( \pi' \in L' \subset P(R^{\tau}) \) a nonzero minuscule or quasi-minuscule co-weight, is called a Macdonald difference operator.

**Remark 2.11.** The Macdonald difference operators, which can be computed explicitly (as we shall recall below), serve as the quantum Hamiltonians of relativistic versions of the quantum trigonometric Calogero-Moser systems associated to root systems in the sense of Ruijsenaars and Schneider [25]. From this viewpoint Corollary 2.9 reflects the complete quantum integrability of the corresponding quantum relativistic system, the higher quantum Hamiltonians being the difference operators \( D_p \in \mathbb{D}_R(L')^{W_0} \ (p(\xi) \in A'_0) \).
A key property of the difference operators $D_p$ ($p \in A'_0$) is their triangular action on $A_0$, which we now proceed to recall. Let

$$\rho_\omega' = \frac{1}{2} \sum_{\alpha \in R^+} k' \langle \alpha' \rangle \alpha$$

be the deformed half sum of positive roots associated to the dual multiplicity label $k'$. Let $\tilde{p}(\xi) \in A'$ for $p(\xi) \in A'$ be the associated $\rho_\omega'$-twisted trigonometric Laurent polynomial, characterized by

$$\tilde{p}(\lambda) = p(-\lambda - \rho_\omega'), \quad \forall \lambda \in V_C,$$

where we use \[2.1\] to interpret $p(\xi)$ as function on $V_C$. Concretely, if $p(\xi)$ expands as $p(\xi) = \sum_{\lambda' \in J} K(\lambda') \xi^\lambda$, then $\tilde{p}(\xi) = \sum_{\lambda' \in J} K(\lambda') q^{-\langle \rho_\omega', \lambda' \rangle} \xi^{-\lambda'}$.

By Theorem 2.5 and Lemma 2.7, $D_p|_{A_0}$ is an endomorphism of $A_0$ for all $p(\xi) \in A'_0$. In fact, by \[2.7\] they are triangular in the sense that for $p(\xi) \in A'_0$ and $\lambda \in L_{++}$,

$$D_p(m_\lambda(z)) = \tilde{p}(\lambda)m_\lambda(z) + \sum_{\mu \in L_{++}, \mu < \lambda} d_\mu(p)m_\mu(z)$$

for some $d_\mu(p) \in \mathbb{C}$, with $\leq$ the dominance order defined by $u \leq v$ for $u, v \in V_C$ if $v - u \in \mathbb{Z}_{\geq 0}\Delta$.

We end this subsection by recalling the explicit form of the Macdonald difference operators for each of the three cases $a$, $b$ and $c$. The minuscule (respectively quasi-minuscule) co-weights form a $W_0$-invariant subset of $P(R^\vee)$, so we first describe the nonzero dominant minuscule and quasi-minuscule co-weights.

Let $\pi'_i \in P(R^\vee)$ ($i = 1, \ldots, n$) be the fundamental co-weights with respect to the basis $\{\alpha_j\}_{j=1}^n$ of $R$ (so $\langle \pi'_i, \alpha_j \rangle = \delta_{i,j}$ for $i, j = 1, \ldots, n$ with $\delta_{i,j}$ the Kronecker delta function). Using the expansion

$$\varphi = \sum_{j=1}^n m_j \alpha_j$$

of the highest root $\varphi \in R$ in simple roots (in which all coefficients $m_j$ are strictly positive integers), it follows that

$$\{\pi'_j \mid j \in \{1, \ldots, n\} \text{ and } m_j = 1\}$$

is the (possibly empty) set of nonzero dominant minuscule co-weights. Macdonald difference operators of the first type are the $D_{\pi'_j} \in \mathbb{D}_R(L')^{W_0}$ with $\pi' = w_0 \pi'_j$ for some $j \in J_0$, where $w_0 \in W_0$ is the longest Weyl group element and $J_0$ is the (possibly empty) set

$$J_0 := \{j \in \{1, \ldots, n\} \mid \pi'_j \in L' \text{ and } m_j = 1\}.$$  

**Example 2.12.** i) For case $a$ and case $b$ we have $L' = P(R^\vee)$, hence $J_0$ parametrizes the nonzero dominant minuscule co-weights of $R$. Using the classification of root systems $R$ one verifies that $J_0 \neq \emptyset$ unless $R$ is of type $E_8$, $F_4$ or $G_2$.

ii) For case $c$ there is precisely one nonzero dominant minuscule co-weight, namely the fundamental co-weight $\pi'_j \in P(R^\vee)$ corresponding to the unique long simple root $\alpha_j$ from the basis $\{\alpha_1, \ldots, \alpha_n\}$ of $R$. But $\pi'_j \notin L' = Q(R^\vee)$, hence $J_0 = \emptyset$.

It is easy to show that $\varphi^\vee$ is the only nonzero dominant quasi-minuscule co-weight. Since a quasi-minuscule co-weight lies in $R^\vee$ by definition, it automatically
lies in \( L' \). The Macdonald difference operator of the second type now is \( D_{-\varphi^\vee} \in D_R(L')W_0 \), which thus exists for all the three cases \( a, b \) and \( c \) under consideration.

For \( \lambda' \in L' \) we write \( W_{0,\lambda'} \subseteq W_0 \) for the isotropy subgroup of \( \lambda' \) and \( W_{0,\lambda'}^\vee \) for a complete set of representatives of \( W_0/W_{0,\lambda'} \).

**Proposition 2.13.** Let \( \pi' \in L' \) be a nonzero minuscule or quasi-minuscule co-weight. Then

\[
D_{\pi'} = m_{\pi'}(-\rho_{\pi'}) + \sum_{w \in W_0^{\pi'}} (wf_{\pi'})(z)(t(w\pi') - 1),
\]

with \( f_{\pi'}(z) \in \mathcal{R} \) given by

\[
f_{\pi'}(z) = \prod_{a \in S_1(t(-\pi'))} c_a(z)
\]

and with \( S_1(w) = S_1^+ \cap w^{-1}S_1^- \) for \( w \in W \). If \( \pi' \) is minuscule then

\[
D_{\pi'} = \sum_{w \in W_0^{\pi'}} (wf_{\pi'})(z)t(w\pi').
\]

**Proof.** By [23, (4.4.12)] and the concluding paragraph of [23, §4.4] we have

\[
D_{\pi'} = g(z) + \sum_{w \in W_0^{\pi'}} (wf_{\pi'})(z)t(w\pi')
\]

for some \( g(z) \in \mathcal{R} \), and \( g(z) \) is zero if \( \pi' \) is minuscule. This yields the second formula for \( D_{\pi'} \).

To prove the first formula for \( D_{\pi'} \), we use (2.10) for \( \lambda = 0 \), which gives the expression

\[
g(z) = m_{\pi'}(-\rho_{\pi'}) - \sum_{w \in W_0^{\pi'}} (wf_{\pi'})(z).
\]

Combined with (2.11) we obtain the first formula for \( D_{\pi'} \). \( \square \)

Recall that for case \( a, S_1 = S(R) \) and \( R^{\vee} = R \), while for case \( b \) and \( c, S_1 = S(R)^\vee \) and \( R^{\vee} = R^\vee \). The following lemma follows now by a direct computation.

**Lemma 2.14.** i) For \( \pi' = w_0\pi_j' \) with \( j \in J_0 \) we have

\[
S_1(t(-\pi')) = \{ \alpha \in R^{\vee} \mid \langle \pi', \alpha \rangle < 0 \}.
\]

ii) For \( \pi' = -\varphi^\vee \) we have

\[
S_1(t(-\pi')) = \{ \alpha \in R^{\vee} \mid \langle \pi', \alpha \rangle < 0 \} \cup \{ (\varphi + c)^\vee \},
\]

where for case \( a \), we note that \( (\varphi + c)^\vee = \varphi + c \in S(R) = S_1 \), since \( \varphi \) has squared norm equal to two by convention.

We conclude by writing out the coefficients \( f_{\pi'}(z) \in \mathcal{R} \) explicitly. In the list below, \( \pi' \in L' \) is a nonzero anti-dominant minuscule or quasi-minuscule co-weight, so \( \pi' = w_0\pi_j' \) (\( j \in J_0 \)) or \( \pi' = -\varphi^\vee \). We use that \( z^{\varphi + c} = qz^{\varphi^\vee} \) for case \( a \), \( z^{(\varphi + c)^\vee} = qz^{\varphi^\vee} \) for case \( b \) (since \( \| \varphi \|^2 = 2 \)), and \( z^{(\varphi + c)^\vee} = q^{\frac{1}{2}}z^{\varphi^\vee} \) for case \( c \) (since \( \| \varphi \|^2 = 4 \)). We furthermore use that

\[
\{ \beta \in R \mid \langle \pi', \beta \rangle = -2 \} = \begin{cases} \emptyset, & \text{if } \pi' = w_0\pi_j' \ (j \in J_0), \\ \{ \varphi \}, & \text{if } \pi' = -\varphi^\vee, \end{cases}
\]
hence the second factor in the expression of $f_{\pi'}(z)$ for case a and case b below is one if $\pi'$ is minuscule, while it is simply the product over the singleton $\{\varphi\}$ if $\pi'$ is quasi-minuscule (cf. Chalykh’s \([3, 2.2]\) notational conventions).

Case a. We have

$$f_{\pi'}(z) = \prod_{\alpha \in R} \frac{(1 - \tau_{\alpha}z_{\alpha})}{\tau_{\alpha}(1 - z_{\alpha})} \prod_{\beta \in R} \frac{(1 - q\tau_{\beta}^{\beta}z_{\beta})}{\tau_{\beta}(1 - qz_{\beta})}.$$  

Case b. We have

$$f_{\pi'}(z) = \prod_{\alpha \in R} \frac{(1 - \tau_{\alpha}z_{\alpha}^{-\varphi})}{\tau_{\alpha}(1 - z_{\alpha}^{-\varphi})} \prod_{\beta \in R} \frac{(1 - q\tau_{\beta}^{\beta}z_{\beta}^{-\varphi})}{\tau_{\beta}(1 - qz_{\beta}^{-\varphi})}.$$  

Case c. For this case, $\pi' = -\varphi^\vee$ is quasi-minuscule as remarked before. Furthermore, $S_1(t(-\pi')) \cap O_1 = \{\varphi^\vee\}$, $S_1(t(-\pi')) \cap O_3 = \{(\varphi + \epsilon)^\vee\}$, and the remaining $2(n-1)$ elements in $S_1(t(-\pi'))$ are from the orbit $O_5$. To make the expression for $f_{\pi'}$ as explicit as possible, we choose without loss of generality the particular basis

$$\{\alpha_1, \ldots, \alpha_n\} = \{\epsilon_1 - \epsilon_2, \ldots, \epsilon_{n-1} - \epsilon_n, 2\epsilon_n\}$$

for $R = R_C$, so that the corresponding highest root is $\varphi = 2\epsilon_1$ and $\pi' = -\varphi^\vee = -\epsilon_1$. Then

$$f_{\pi'}(z) = q^{-\kappa_1(\epsilon_1-\epsilon_2)} \prod_{j=2}^{n} \frac{(1 - q^{\kappa_j} z^{\epsilon_j})}{(1 - z^{\epsilon_j})}$$

and the resulting difference operator $D_{\pi'} \in D_R(L')^{W_0}$ is Koornwinder’s \([20]\) difference operator, see also \([26]\).

### 2.5. The Harish-Chandra homomorphism.

In this subsection we describe the asymptotics of the difference operators $D_p \in D_R(L')^{W_0}$ ($p \in A_0'$) deep in the negative Weyl chamber

$$V_- := \{v \in V \mid \langle v, \alpha \rangle < 0 \quad \forall \alpha \in R^+ \}.$$  

We give an algebraic formalization in terms of a constant term map $\gamma(L)$ (see \([15, \S1.2]\) for this notion in the trigonometric differential degeneration), whose definition below is based on the fact that $z^{-\pi}(v) = q^{-\langle x, v \rangle} \to 0$ for $x \in \mathbb{Z}_{\geq 0} \Delta \setminus \{0\}$ if $\langle v, \alpha \rangle \to -\infty$ for all $\alpha \in R^+$. In this algebraic formulation we incorporate a $\partial_\lambda^\vee$-shift and we identify the algebra $\mathbb{C}[t(L')]$ of constant coefficient difference operators with $A'$ by $t(\lambda') \leftrightarrow \xi_{\lambda'}$ for $\lambda' \in L'$. For case a, the results discussed in this section are essentially due to Cherednik, see e.g. \([5, \S3]\).

**Lemma 2.15.** For

$$D = \sum_{\lambda' \in L'} \left( \sum_{x \in \mathbb{Z}_{\geq 0} \Delta} C_x(\lambda') z^{-x} \right) t(\lambda') \in D_{\mathbb{C}[z^{-\Delta}]}(L')$$

...
with \( \{C_x(\lambda')\}_{x \in \mathbb{Z}_{\geq 0}\Delta} \subset \mathbb{C} \) the zero set for all but finitely many \( \lambda' \in L' \), we define the constant term \( \gamma(k)(D) \in A' \) of \( D \) by

\[
\gamma(k)(D) = \sum_{\lambda' \in \mathcal{L}'} C_0(\lambda')q^{(\rho_{\mathfrak{L}'}, \lambda')} \xi^{\lambda'}.
\]

The resulting linear map \( \gamma(k) : D_{\mathbb{C}[z^{-\Delta}]}(L') \to A' \) is an algebra homomorphism, called the Harish-Chandra homomorphism.

**Proof.** This follows from a direct computation. \( \Box \)

Recall the \( D_{\mathbb{C}[z^{-\Delta}]}(L') \)-module \( \overline{M} \) from Lemma 2.3 and the \( \rho_{\mathfrak{L}'} \)-twist (2.9).

**Lemma 2.16.** For \( u \in V \) and \( D \in D_{\mathbb{C}[z^{-\Delta}]}(L') \) we have

\[
D(z^u) = (\gamma(k)(D))(u)z^u + \sum_{x \in \mathbb{Z}_{\geq 0}\Delta \setminus \{0\}} K_x(u)z^{u-x} \in \overline{M}
\]

for certain \( K_x(u) \in \mathbb{C} \).

**Proof.** Let \( D \in D_{\mathbb{C}[z^{-\Delta}]}(L') \), written out explicitly as

\[
D = \sum_{\lambda' \in \mathcal{L}'} \left( \sum_{x \in \mathbb{Z}_{\geq 0}\Delta} C_x(\lambda')z^{-x} \right) t(\lambda') \in D_{\mathbb{C}[z^{-\Delta}]}(L')
\]

with \( \{C_x(\lambda')\}_{x \in \mathbb{Z}_{\geq 0}\Delta} \subset \mathbb{C} \) the zero set for all but finitely many \( \lambda' \in L' \). Then

\[
D(z^u) = \sum_{x \in \mathbb{Z}_{\geq 0}\Delta} K_x(u)z^{u-x} \in \overline{M}
\]

by Lemma 2.3 with coefficients

\[
K_x(u) = \sum_{\lambda' \in \mathcal{L}'} C_x(\lambda')q^{-(\lambda', u)} , \quad x \in \mathbb{Z}_{\geq 0}\Delta.
\]

On the other hand,

\[
(\gamma(k)(D))(u) = \sum_{\lambda' \in \mathcal{L}'} C_0(\lambda')q^{-(\lambda', u)}
\]

by (2.11), (2.13) and (2.12), hence \( K_0(u) = (\gamma(k)(D))(u) \). \( \Box \)

We are now in the position to explicitly compute the constant terms \( \gamma(k)(D_p) \) for the commuting family of difference operators \( D_p \in D_{\mathbb{R}}(L')^{W_0} \subset D_{\mathbb{C}[z^{-\Delta}]}(L') \) (\( p(\xi) \in A'_0 \)).

**Proposition 2.17.** We have \( (\gamma(k)(D_p))(\xi) = p(\xi) \) for \( p(\xi) \in A'_0 \). In particular, the restriction of the algebra homomorphism

\[
\gamma(k) \circ \beta : D_{\mathbb{R}}(W)^{H_0} \to A'
\]

to \( A'_0(Y) \subset D_{\mathbb{R}}(W)^{H_0} \) is the algebra isomorphism \( A'_0(Y) \xrightarrow{\sim} A'_0 \) which maps \( p(Y) \) to \( p(\xi) \) (\( p \in A'_0 \)).

**Proof.** We embed \( A \) as subspace in \( \overline{M} \) as the complex subspace spanned by the monomials \( z^\mu \in \overline{M} \) (\( \mu \in L \)). The restriction of the action of \( D_p \in \text{End}_{\mathbb{C}}(\overline{M}) \) (\( p(\xi) \in A'_0 \)) to \( A_0 \) coincides with the action of \( D_p \) on \( A_0 \) as described in Subsections 2.3 and 2.4.
Fix \( p \in A'_0 \). For \( \lambda \in L_{++} \) and \( \mu \in W_0 \lambda \) we have \( \mu \leq \lambda \) (i.e. \( \lambda - \mu \in \mathbb{Z}_{\geq 0} \Delta \)), hence \( \gamma(D_\mu) \) and the fact that \( \mathbb{C}[[[z^{-\Delta}]]]z^u \subset \mathcal{M} \) is a \( \mathbb{D}_{\mathbb{C}[[[z^{-\Delta}]]]}(L') \)-submodule for all \( u \in V_\mathcal{C} \) imply

\[
D_p(z^\lambda) = \tilde{p}(\lambda)z^\lambda + \sum_{\mu \in L_{++} \mu < \lambda} K_\mu(\lambda)z^\mu, \quad \forall \lambda \in L_{++}
\]

for certain coefficients \( K_\mu(\lambda) \in \mathbb{C} \). Combined with Lemma 2.16 we conclude that

\[
(\gamma(D_p))(\lambda) = \tilde{p}(\lambda), \quad \forall \lambda \in L_{++}.
\]

This implies that \( (\gamma(D_p))(\xi) = \tilde{p}(\xi) \) in \( A'_0 \).

Recall the explicit form of the Macdonald difference operators \( D_{\pi'} \) from Proposition 2.13 where \( \pi' = w_0 \pi_j^i \) (\( j \in J_0 \)) or \( \pi' = \varphi^\vee \). The following technical result will be used in the next section in the analysis of the centralizer of \( D_{\pi'} \) in \( \mathbb{D}_R(L')W_0 \).

**Corollary 2.18.** For \( w \in W_0 \pi' \) we have

\[
\gamma(D_p)(wf_{\pi'}) = q^{-\langle \rho_\pi', w\pi' \rangle}.
\]

In particular,

\[
D_{\pi'} = \sum_{\mu' \in W_0 \pi'} q^{-\langle \rho_\pi', \mu' \rangle} t(\mu') + \sum_{\mu' \in W_0 \pi'} g_{\mu'}(z) (t(\mu') - 1)
\]

where for \( w \in W_0 \pi' \),

\[
g_{w\pi'}(z) := (wf_{\pi'})(z) - \gamma(D_p)(wf_{\pi'}) = \sum_{x \in \mathbb{Z}_{\geq 0} \Delta \setminus \{0\}} K_{x}(w\pi')z^{-x}
\]

for certain \( K_{x}(w\pi') \in \mathbb{C} \).

3. Centralizers of Macdonald Difference Operators

In this section we fix a nonzero anti-dominant minuscule or quasi-minuscule co-weight \( \pi' \in L' \). For the associated Macdonald difference operator \( D_{\pi'} \in \mathbb{D}_R(L')W_0 \) we analyze the centralizer algebra \( \mathbb{D}_R(L')W_0 \cdot D_{\pi'} \) (respectively \( \mathbb{D}_R(L')D_{\pi'} \)) consisting of the difference operators \( D \in \mathbb{D}_R(L')W_0 \) (respectively \( D \in \mathbb{D}_R(L') \)) which commute with \( D_{\pi'} \).

We already observed that

\[
\beta(A'_0(Y)) \subseteq \mathbb{D}_R(L')W_0 \cdot D_{\pi'} \subseteq \mathbb{D}_R(L')D_{\pi'}.
\]

We show in this section that the first inclusion is an equality. As observed in e.g. [10] and [3], the second inclusion is strict for special values of the multiplicity labels, in which case one speaks of algebraic integrability. We analyze the second inclusion using a simple symmetrization procedure for difference operators.

### 3.1. Commutativity

We start by showing that \( \mathbb{D}_R(L')D_{\pi'} \) is commutative. Denote \( \mathbb{D}_{\mathbb{C}[[[z^{-\Delta}]]]}(L')D_{\pi'} \) for the centralizer algebra of \( D_{\pi'} \) in \( \mathbb{D}_{\mathbb{C}[[[z^{-\Delta}]]]}(L') \). It contains \( \mathbb{D}_R(L')D_{\pi'} \) as a subalgebra.

**Proposition 3.1.** The Harish-Chandra homomorphism \( \gamma(D) \) restricts to an injective algebra homomorphism

\[
\gamma(D_p) : \mathbb{D}_{\mathbb{C}[[[z^{-\Delta}]]]}(L')D_{\pi'} \hookrightarrow A'_0.
\]

In particular, \( \mathbb{D}_{\mathbb{C}[[[z^{-\Delta}]]]}(L')D_{\pi'} \) and \( \mathbb{D}_R(L')D_{\pi'} \) are commutative.
Proof. We express $D \in \mathbb{D}_{\mathbb{C}[\mathfrak{z}^{-\Delta}]}(L')$ as (2.13). We can then explicitly compute the $z^{-\gamma} \mathbb{C}[t(L')]$-term of the commutant $[D, D_{\pi'}] \in \mathbb{D}_{\mathbb{C}[\mathfrak{z}^{-\Delta}]}(L')$ for all $x \in \mathbb{Z}_{\geq 0} \Delta$ using the expression from Corollary 2.11 for the Macdonald difference operator $D_{\pi'}$. It follows that $D \in \mathbb{D}_{\mathbb{C}[\mathfrak{z}^{-\Delta}]}(L')^{D_{\pi'}}$ if and only if

$$
\left( \sum_{\mu' \in W_0 \pi'} q^{-(\rho_{\mu'}, \rho_{\mu'})} (1 - q^{(\mu', x)}) t(\mu') \right) \left( \sum_{\lambda' \in L'} C_x(\lambda') t(\lambda') \right)
= \sum_{\lambda', \mu' \in W_0 \pi'} \sum_{0 < y < x} C_{\lambda' - y}(\lambda') K_y(\mu') (q^{(\mu', x-y)} - q^{(\lambda', y)}) t(\mu' + \lambda')
+ \sum_{\lambda', \mu' \in W_0 \pi'} \sum_{0 < y < x} C_{\lambda' - y}(\lambda') K_y(\mu') (q^{(\lambda', y)} - 1) t(\lambda')
$$

(3.1)
in $\mathbb{C}[t(L')]$ for all $x \in \mathbb{Z}_{\geq 0} \Delta$. Observe that the first factor

$$
\sum_{\mu' \in W_0 \pi'} q^{-(\rho_{\mu'}, \rho_{\mu'})} (1 - q^{(\mu', x)}) t(\mu')
$$
in the left hand side of (3.1) is nonzero for all $x \in \mathbb{Z}_{\geq 0} \Delta \setminus \{0\}$ since $W_0$ acts irreducibly on $V$.

Let now $D \in \mathbb{D}_{\mathbb{C}[\mathfrak{z}^{-\Delta}]}(L')^{D_{\pi'}}$ with $\gamma(k)(D) = 0$. By the definition of the Harish-Chandra homomorphism (see Lemma 2.11), we obtain

$$
C_0(\lambda') = 0, \quad \forall \lambda' \in L'.
$$

(3.2)

By induction on the height of $x \in \mathbb{Z}_{\geq 0} \Delta$ (with respect to the simple roots $\Delta$), it follows from (3.1) and (3.2) that $C_x(\lambda') = 0$ for all $\lambda' \in L'$ and all $x \in \mathbb{Z}_{\geq 0} \Delta$. Hence $D = 0$, as desired. \qed

3.2. Symmetrization of difference operators. Define a surjective linear projection $\pi : \mathbb{D}_{\mathbb{R}}(L') \to \mathbb{D}_{\mathbb{R}}(L')^{W_0}$ by

$$
\pi(D) := \frac{1}{\#W_0} \sum_{w \in W_0} w D w^{-1}, \quad D \in \mathbb{D}_{\mathbb{R}}(L').
$$

Observe that $\pi(D D') = \pi(D) D'$ and $\pi(D' D) = D' \pi(D)$ for $D \in \mathbb{D}_{\mathbb{R}}(L')$ and $D' \in \mathbb{D}_{\mathbb{R}}(L')^{W_0}$. Let $e$ be the trivial idempotent of $\mathbb{C}[W_0]$,

$$
e = \frac{1}{\#W_0} \sum_{w \in W_0} w \in \mathbb{C}[W_0] \subset \mathbb{D}_{\mathbb{R}}(W).
$$

Lemma 3.2. i) We have $e D e = \pi(D) e$ for $D \in \mathbb{D}_{\mathbb{R}}(L')$.

ii) The projection map $\pi$ restricts to a surjection

$$
\pi : \mathbb{D}_{\mathbb{R}}(L')^{D_{\pi'}} \to \mathbb{D}_{\mathbb{R}}(L')^{W_0, D_{\pi'}}. \quad \pi
$$

Proof. i) Since $w e = e$ for $w \in W_0$ we have

$$
e D e = \frac{1}{\#W_0} \sum_{w \in W_0} w D e = \frac{1}{\#W_0} \sum_{w \in W_0} w D w^{-1} e = \pi(D) e.
$$

ii) Note that $D_{\pi'} \in \beta(A_0(Y)) \subset \mathbb{D}_{\mathbb{R}}(L')^{W_0}$, hence

$$
[\pi(D), D_{\pi'}] = \pi([D, D_{\pi'}]), \quad D \in \mathbb{D}_{\mathbb{R}}(L'),
$$

which immediately leads to the desired result. \qed
Proposition 3.3. Let \( D \in \mathbb{D}_R(L')^{D_{\sigma'}} \). If \( A_0 \subset \mathcal{M} \) is invariant under the action of the difference operator \( D \) on \( \mathcal{M} \), then \( D \in \mathbb{D}_R(L')^{W_0.D_{\sigma'}} \).

Proof. Let \( D \in \mathbb{D}_R(L')^{D_{\sigma'}} \) satisfying \( D(A_0) \subseteq A_0 \), so that
\[
D(m_\lambda(z)) = \pi(D)(m_\lambda(z)), \quad \forall \lambda \in L^{++}.
\]
Equating the \( z^\lambda \)-coefficient using Lemma 2.10 we obtain
\[
(\gamma(\underline{k})(D))(\lambda) = (\gamma(\underline{k})(\pi(D)))(\lambda), \quad \forall \lambda \in L^{++},
\]
compare with the proof of Proposition 2.3. We conclude that \( \gamma(\underline{k})(D) = \gamma(\underline{k})(\pi(D)) \), hence
\[
D = \pi(D) \in \mathbb{D}_R(L')^{W_0,D_{\sigma'}}
\]
since the Harish-Chandra homomorphism \( \gamma(\underline{k}) \) is injective on \( \mathbb{D}_R(L')^{D_{\sigma'}} \).

Corollary 3.4. If \( A_0 \) is a \( \mathbb{D}_R(L')^{D_{\sigma'}} \)-submodule of \( \mathcal{M} \) then
\[
\mathbb{D}_R(L')^{D_{\sigma'}} = \mathbb{D}_R(L')^{W_0,D_{\sigma'}}.
\]

We will see in Subsection 3.4 that the converse of Corollary 3.4 is also true. Furthermore, in Subsection 4.3 we use Harish-Chandra series and the theory of Macdonald polynomials to give conditions on the multiplicity label \( \underline{k} \) for which \( A_0 \) is a \( \mathbb{D}_R(L')^{D_{\sigma'}} \)-submodule of \( \mathcal{M} \).

3.3. Rank reduction. In this subsection we compute the asymptotics of the Macdonald difference operators \( D_{\sigma'} \) along co-dimension one facets of the negative Weyl chamber. It allows us to reduce the analysis of the centralizer algebra \( \mathbb{D}_R(L')^{W_0,D_{\sigma'}} \) to the case of rank one root systems \( R \). This technique is reminiscent of the trigonometric differential case, see e.g. [15] and references therein.

For a subset \( F \subseteq \Delta \) we define the facet \( V_{-F} \) of the negative Weyl chamber \( V_- \) by
\[
V_{-F} = \{ v \in V \mid \langle v, \alpha \rangle = 0 \quad (\alpha \in F), \quad \langle v, \beta \rangle < 0 \quad (\beta \in \Delta \setminus F) \}.
\]
We write \( R_F' = R_F' \cap \mathbb{Z}F \) for the corresponding standard parabolic root subsystem of \( R_F' \), \( R_F' = \{ \alpha \in R_F' \mid \langle \alpha, \beta \rangle = 0 \} \) for the set of positive respectively negative roots in \( R_F' \) with respect to the basis \( F \) of \( R_F' \), and \( W_0,F \) for the standard parabolic subgroup of \( W_0 \) generated by the simple reflections \( s_\alpha (\alpha \in F) \). We furthermore set \( R_F = R \cap \bigoplus_{\alpha \in F} \mathbb{Q} \alpha \), \( R_F^+ = R_F^+ \cap \bigoplus_{\alpha \in F} \mathbb{Q} \alpha \) and
\[
\rho_{\underline{k},F} = \frac{1}{2} \sum_{\alpha \in R^+ \setminus R_F'} \underline{k}(\alpha) \alpha \in V_C.
\]

Note that \( \rho_{\underline{k}} = \rho_{\underline{k},F}(\emptyset) \).

We define \( \mathcal{R}_F \subseteq \mathcal{R} \) to be the subalgebra generated by
\[
1 - rz^\alpha, \quad r \in \mathbb{C}, \quad \alpha \in R_F'.
\]
The embedding \( \mathcal{R} \hookrightarrow \mathbb{C}[z^{-\Delta}] \) (see Subsection 2.2) maps \( \mathcal{R}_F \) into the subalgebra \( \mathbb{C}[z^{-F}] := \mathbb{C}[z^{-\alpha} \mid \alpha \in F] \) of \( \mathbb{C}[z^{-\Delta}] \). For a lattice \( X \subset V \) we write \( \mathbb{D}_R(X) = \mathbb{D}_R\#t(X) \) (respectively \( \mathbb{D}_C[z^{-F}])(X) = \mathbb{C}[z^{-F}]\#t(X) \), which is a subalgebra of \( \mathbb{D}_R(X) \) (respectively \( \mathbb{D}_C[z^{-\Delta}](X) \)). Note that \( \mathbb{D}_{\mathcal{R}_F}(X) \subseteq \mathbb{D}_R(X) \) is a \( W_0,F \)-invariant subalgebra if \( X \) is a \( W_0,F \)-invariant lattice in \( V \). In the special case \( F = \emptyset \) we have \( \mathcal{R}_\emptyset = \mathbb{C}[z^{-0}] = \mathbb{C} \), hence \( \mathbb{D}_{\mathcal{R}_\emptyset}(X) = \mathbb{C}[t(X)] \) is the algebra of constant coefficient difference operators with step-sizes from \( X \).
Definition 3.5. Let $F \subseteq \Delta$. The constant term map along the facet $V_{-F}$ is the algebra homomorphism $\gamma_F(\underline{k}): \mathbb{D}_{\mathbb{C}[[z^{-\Delta}]}}(L') \to \mathbb{D}_{\mathbb{C}[[z^{-F}]}}(L')$ defined by

$$\gamma_F(\underline{k})(D) = \sum_{\lambda' \in L'} \left( \sum_{x \in \mathbb{Z}_{\geq 0}^F} C_x(\lambda') z^{-x} \right) q^{(\rho_{\underline{k}'}, F, \lambda')} t(\lambda')$$

for

$$D = \sum_{\lambda' \in L'} \left( \sum_{x \in \mathbb{Z}_{\geq 0}^\Delta} C_x(\lambda') z^{-x} \right) t(\lambda') \in \mathbb{D}_{\mathbb{C}[[z^{-\Delta}]}}(L'),$$

where $\{C_x(\lambda')\}_{x \in \mathbb{Z}_{\geq 0}^\Delta} \subset \mathbb{C}$ is the zero set for all but finitely many $\lambda' \in L'$.

Lemma 3.6. The map $\gamma_F(\underline{k})$ restricts to a $W_{0,F}$-equivariant algebra homomorphism $\gamma_F(\underline{k}): \mathbb{D}_R(L') \to \mathbb{D}_{R_\pi}(L')$.

Proof. The map $\gamma_F(\underline{k})$ restricts to an algebra homomorphism $\gamma_F(\underline{k}): \mathbb{D}_R(L') \to \mathbb{D}_{R_\pi}(L')$ since for $r \in \mathbb{C}^X$,

$$\gamma_F(\underline{k}) \left( \frac{1}{1 - rz} \right) = \begin{cases} \frac{1}{1 - rz}, & \alpha \in R_{sf}^\vee, \\ 1, & \alpha \in R_{sf}^\vee \setminus R_{sf}^{\vee, -}, \\ 0, & \alpha \in R_{sf}^\vee \setminus R_{sf}^{\vee, +}. \end{cases}$$

Since $R_{sf}^\vee$ and $R_{sf}^\vee \setminus R_{sf}^{\vee, \pm}$ are $W_{0,F}$-invariant subsets of $R_{sf}^\vee$, (3.5) implies that the restriction of $\gamma_F(\underline{k})$ to $R \subset \mathbb{D}_R(L')$ is $W_{0,F}$-equivariant. Furthermore, $\rho_{\underline{k}', F} \in V_{\underline{k}}$ is $W_{0,F}$-stable since $R_{sf}^+ \setminus R_{sf}^+$ is $W_{0,F}$-invariant. It follows that $\gamma_F(\underline{k}): \mathbb{D}_R(L') \to \mathbb{D}_{R_\pi}(L')$ is $W_{0,F}$-equivariant. \(\square\)

Observe that $\gamma(\underline{k}) = \delta_F(\underline{k}) \circ \gamma_F(\underline{k})$ with $\delta_F(\underline{k}): \mathbb{D}_{\mathbb{C}[[z^{-F}]}}(L') \to A'$ the algebra homomorphism defined by

$$\delta_F(\underline{k})(D) = \sum_{\lambda' \in L'} C_0(\lambda') q^{(\rho_{\underline{k}'}, F, \lambda')} z^{\lambda'}$$

for

$$D = \sum_{\lambda' \in L'} \left( \sum_{x \in \mathbb{Z}_{\geq 0}^F} C_x(\lambda') z^{-x} \right) t(\lambda') \in \mathbb{D}_{\mathbb{C}[[z^{-F}]}}(L'),$$

where $\{C_x(\lambda')\}_{x \in \mathbb{Z}_{\geq 0}^F} \subset \mathbb{C}$ is the zero set for all but finitely many $\lambda' \in L'$.

Our aim is to compute the rank one reductions $\gamma_{\{\alpha\}}(\underline{k})(D_{\alpha})$ ($\alpha \in \Delta$) of the Macdonald difference operator $D_{\alpha}$. For $F \subseteq \Delta$, set $S_F = \{\alpha \in S \mid Da \in Z F\}$. We need several preparatory lemmas.

Lemma 3.7. Let $\pi' \in L'$ be a nonzero antidominant minuscule or quasi-minuscule co-weight. Fix $i \in \{1, \ldots, n\}$ and $w \in W_0^\pi$. 

a) If $\pi' = w_0 \pi'_j$ ($j \in J_0$), then

$$S_{\{\alpha_i\}} \cap wS_1(t(-\pi')) = \begin{cases} \{a_i\} & \text{if } (a_i, w\pi') < 0, \\ \{-a_i\} & \text{if } (a_i, w\pi') > 0, \\ 0 & \text{if } (a_i, w\pi') = 0. \end{cases}$$
b) If $\pi' = -\varphi^\vee$, then

$$S_{(a_i)} \cap wS_1(t(-\pi')) = \begin{cases} \{a_i, a_i + 2c/\|\varphi\|^2\} & \text{if } w\pi' = -a_i, \\
\{-a_i, -a_i + 2c/\|\varphi\|^2\} & \text{if } w\pi' = a_i, \\
\{a_i\} & \text{if } \langle a_i, w\pi' \rangle < 0 \text{ and } w\pi' \neq -a_i, \\
\{-a_i\} & \text{if } \langle a_i, w\pi' \rangle > 0 \text{ and } w\pi' \neq a_i, \\
\emptyset & \text{if } \langle a_i, w\pi' \rangle = 0.
\end{cases}$$

Proof. This follows directly from Lemma 2.14. □

Recall from Proposition 2.13 the rational function $f_{\pi'}(z) \in \mathcal{R}$ occurring as coefficient in the Macdonald difference operator $D_{\pi'}$.

**Lemma 3.8.** Let $i \in \{1, \ldots, n\}$, $w \in W_0^\vee$ and let $\pi' \in L'$ be a nonzero anti-dominant minuscule or quasi-minuscule co-weight. We have

$$\gamma_{\{a_i\}}(k)((w f_{\pi'})(z)) = q^{-\langle \rho\mu', (a_i), w\pi' \rangle} \prod_{a \in S(a_i) \cap wS_1(t(-\pi'))} c_a(z).$$

Proof. We define a function $\epsilon : S \to \{\pm 1\}$ by $\epsilon(a) = 1$ if $Da \in \mathbb{Z}_{\geq 0}\Delta$ and $\epsilon(a) = -1$ if $a \in \mathbb{Z}_{\leq 0}\Delta$. For $a \in S_1$ we then have

$$\gamma_{\{a_i\}}(k)((w f_{\pi'})(z)) = \begin{cases} c_a(z), & \text{if } a \in S_{(a_i)}, \\
\tau_a(\epsilon(a)), & \text{if } a \in S \setminus S_{(a_i)},
\end{cases}$$

hence

$$\gamma_{\{a_i\}}(k)((w f_{\pi'})(z)) = r_i(w) \prod_{a \in S(a_i) \cap wS_1(t(-\pi'))} c_a(z)$$

for certain $r_i(w) \in \mathbb{C}^\times$. It remains to show that $r_i(w) = q^{-\langle \rho\mu', (a_i), w\pi' \rangle}$. By Corollary 2.18, (3.6), (3.7) and the fact that $\gamma(k) = \delta_{\{a_i\}}(k) \circ \gamma_{\{a_i\}}(k)$, we have

$$r_i(w) = q^{-\langle \rho\mu', w\pi' \rangle} \prod_{a \in S(a_i) \cap wS_1(t(-\pi'))} \tau_a^{-\epsilon(a)}.$$ 

Using that $\rho\mu' = \rho\mu_{\varphi'} + \frac{1}{2}K'(\alpha_i^\vee)\alpha_i$, it suffices to show that

$$\prod_{a \in S(a_i) \cap wS_1(t(-\pi'))} \tau_a^{\epsilon(a)} = q^{-\frac{1}{2}K'(\alpha_i^\vee)\langle \alpha_i, w\pi' \rangle}.$$

This follows from straightforward computations using Lemma 3.7. As an example we discuss the proof for case $c$ in detail. For case $c$, $w\pi' = -w\varphi^\vee$ is quasi-minuscule. Suppose first that $w\pi' \neq \pm a_i$. Since $a_i = \alpha_i^\vee$, we have $(w\pi')^\vee \neq \pm \alpha_i$, hence $\langle \alpha_i, w\pi' \rangle \in \{-1, 0, 1\}$. If $\langle \alpha_i, w\pi' \rangle = 0$, then both sides of (3.8) are equal to one. Suppose now that $(a_i, w\pi') \in \{\pm 1\}$. Since $\pi'$ is anti-dominant, the left hand side of (3.8) equals $\tau_{a_i}^{\epsilon(w^{-1}(a_i))}$ by Lemma 3.7. On the other hand, the assumptions imply that $\alpha_i \in R = RC$ is a short root, hence $\|\alpha_i\|^2 = 2$ and $a_i = \alpha_i^\vee \in \mathcal{O}_a$. Consequently, $K'(\alpha_i) = K(\alpha_i) = \kappa$ and $\tau_{a_i} = q^{\frac{1}{2}K'(\alpha_i)}$. The right hand side of (3.8) thus also equals $\tau_{a_i}^{\epsilon(w^{-1}(a_i))}$. If $w\pi' \in \{\pm a_i\}$, then by Lemma 3.7 the left hand side of (3.8) equals

$$\left(\tau_{a_i}^{\tau_{a_i+c/2}}\right)^{\epsilon(w^{-1}(a_i))} = q^{\epsilon(w^{-1}(a_i))c/2},$$

with $c = 2$. If $w\pi' = -w\varphi^\vee = w(\alpha_i^\vee)^\vee$, then $w(\alpha_i^\vee)^\vee = -w(\alpha_i^\vee)^\vee = -2w(\alpha_i^\vee)^\vee$, hence $w(\alpha_i^\vee)^\vee = -2w(\alpha_i^\vee)^\vee = -2w(\alpha_i^\vee)^\vee$, and $\tau_{a_i}^{\epsilon(w^{-1}(a_i))} = q^{\epsilon(w^{-1}(a_i))c/2}$. The proof is complete. □
The lattice Lemma 3.9. The lattice side of (3.8) we use that \( \alpha \) c unless we are in case where we have used the fact that \( \alpha \) c.

\( \sum \)

For \( i \in \{1, \ldots, n\} \) we write

\[
L'_i = L' + \mathbb{Z} \frac{\alpha_i'}{2}
\]

unless we are in case c and \( a_i \in W_0 \varphi^c \), in which case we set \( L'_i = L' \). Set

\[
X_i = \{ \mu' \in L'_i \mid \langle \mu', \alpha_i \rangle = 0 \}.
\]

The lattice \( L'_i \) has the following elementary properties.

**Lemma 3.9.** The lattice \( L'_i \) is \( s_i \)-invariant. It decomposes as the orthogonal direct sum

\[
L'_i = \mathbb{Z} a_i \oplus X_i
\]

unless we are in case c and \( a_i \in W_0 \varphi^c \), in which case it decomposes as the orthogonal direct sum

\[
L'_i = \mathbb{Z} a_i \oplus X_i.
\]

**Proof.** If we are in case c and \( a_i \in W_0 \varphi^c \), then \( a_i = \alpha_i' \in Q(R^c) = L' \) with \( \alpha_i \) the unique long simple root from the basis \( \{ \alpha_1, \ldots, \alpha_n \} \) of \( R = R_C \), hence \( \langle L', \alpha_i \rangle = 2 \mathbb{Z} \) (cf. [23] (2.1.6)). Since \( \alpha_i = \alpha_i' = \alpha_i/2 \), we conclude that \( \langle L', \alpha_i \rangle \in \mathbb{Z} \). Combined with the observation that \( a_i \) has squared length one, we obtain the orthogonal decomposition (3.10) for \( L'_i = L' \).

Suppose now that we are in case a or case b, or that we are in case c and \( a_i \notin W_0 \varphi^c \). Since \( L' \subseteq P(R^c) \) and \( \langle \alpha_i'/2, \alpha_i \rangle = 1 \) we have \( \langle L'_i, \alpha_i \rangle = \mathbb{Z} \). The orthogonal decomposition (3.10) follows now immediately.

We now consider the algebra \( \mathbb{D}_{\mathcal{R}_{\{\alpha_i\}}}(L'_i) \) of difference operators with coefficients from \( \mathcal{R}_{\{\alpha_i\}} \) and with step-sizes from the enlarged lattice \( L'_i \). Since \( L'_i \) is \( s_i \)-invariant, \( \mathbb{D}_{\mathcal{R}_{\{\alpha_i\}}}(L'_i) \) is a \( W_0_{\{\alpha_i\}} \)-module algebra. Lemma 3.9 directly implies the following result.

**Corollary 3.10.** The center \( Z(\mathbb{D}_{\mathcal{R}_{\{\alpha_i\}}}(L'_i)) \) of \( \mathbb{D}_{\mathcal{R}_{\{\alpha_i\}}}(L'_i) \) is the algebra of constant coefficient difference operators with step-sizes from \( X_i \),

\[
Z(\mathbb{D}_{\mathcal{R}_{\{\alpha_i\}}}(L'_i)) = \mathbb{C}[t(X_i)].
\]

Furthermore,

\[
\mathbb{D}_{\mathcal{R}_{\{\alpha_i\}}}(L'_i) \simeq \mathbb{D}_{\mathcal{R}_{\{\alpha_i\}}}(Z a_i + \mathbb{C}[t(X_i)])
\]

as algebras unless we are in case c and \( a_i \in W_0 \varphi^c \), in which case

\[
\mathbb{D}_{\mathcal{R}_{\{\alpha_i\}}}(L'_i) \simeq \mathbb{D}_{\mathcal{R}_{\{\alpha_i\}}}(Z a_i \otimes \mathbb{C}[t(X_i)])
\]

as algebras, with the isomorphisms in (3.12) and (3.13) realized by the multiplication map.

We now decompose the operator \( \gamma_{\{\alpha_i\}}(L)(D_{\varphi^c}) \in \mathbb{D}_{\mathcal{R}_{\{\alpha_i\}}}(L'_i) \) according to the decomposition (3.12) respectively (3.13). The result can be described in terms of an explicit rank one difference operator, which we now define first (its definition depends on the case involved).
Definition 3.11. For \( i \in \{1, \ldots, n\} \) we define \( \mathcal{L}_i \in \mathbb{D}_{R_{\{a_i\}}} (\mathbb{Z}^\vee / 2)^{W_0,\{a_i\}} \) by
\[
\mathcal{L}_i = c_{a_i}(z)t(-\alpha^\vee_i/2) + c_{-a_i}(z)t(\alpha^\vee_i/2)
\]
unless we are in case \( c \) and \( a_i \in W_0\varphi^\vee \), in which case \( \mathcal{L}_i \in \mathbb{D}_{R_{\{a_i\}}} (\mathbb{Z}^\vee / 2)^{W_0,\{a_i\}} \) is defined by
\[
\mathcal{L}_i = c_{a_i}(z)c_{a_i+c/2}(z)(t(-a_i) - 1) + c_{-a_i}(z)c_{-a_i+c/2}(z)(t(a_i) - 1).
\]

Polynomial eigenfunctions of the rank one difference operator (3.14) (respectively (3.15)) are the continuous \( q \)-ultraspherical polynomials (respectively the Askey-Wilson polynomials), see [23, Chapter 6] and references therein.

We define constant coefficient difference operators \( y_i, z_i \in \mathbb{C}[X_i] \) by
\[
y_i = \sum_{w \in W_0^{\pi',}(\alpha_i, w^\pi') = 1} t\left(\frac{\alpha^\vee_i}{2} + w^\pi'\right), \quad z_i = \sum_{w \in W_0^{\pi',}(\alpha_i, w^\pi') = 0} t(w^\pi').
\]

If we are in case \( c \) and if \( a_i \in W_0\varphi^\vee \), then \( y_i \) should be read as zero. This convention is in accordance with the following slight variation of [23 (2.1.6)].

Lemma 3.12. The set
\[
\{ w \in W_0^{\pi'} \mid \langle \alpha_i, w^\pi' \rangle = -1 \}
\]
is nonempty unless \( R \) is of type \( C \), \( \pi' = -\varphi^\vee \) and \( a_i \in W_0\varphi^\vee \).

Proof. Suppose that \( \pi' \) is minuscule, or that \( \pi' = -\varphi^\vee \) and \( a_i \notin W_0\varphi^\vee \). Then \( \langle \alpha_i, w^\pi' \rangle \in \{-1, 0, 1\} \) for all \( w \in W_0^{\pi'} \). Since \( W_0 \) acts irreducibly on \( V \), there exists a \( w \in W_0^{\pi'} \) such that \( \langle \alpha_i, w^\pi' \rangle \neq 0 \), hence (3.17) is nonempty.

Suppose now that \( \pi' = -\varphi^\vee \) and \( a_i \in W_0\varphi^\vee \). Then \( \alpha_i \) is a long root in \( R \). It follows from the root system classification that \( \langle \alpha_i, \beta^\vee \rangle = -1 \) for some long root \( \beta \in R \), unless \( R \) is of type \( C \). Since long roots in \( R \) are \( W_0 \)-conjugate to \( \varphi \), we conclude that (3.17) is nonempty unless \( R \) is of type \( C \). \( \square \)

Proposition 3.13. Fix \( i \in \{1, \ldots, n\} \).

\begin{itemize}
  \item[i)] Suppose that \( \pi' = w_0\pi'_j \ (j \in J_0) \), or that \( \pi' = -\varphi^\vee \) and \( a_i \notin W_0\varphi^\vee \). Then
  \[
  \gamma_{\{a_i\}}(k)(D_{\pi'}) = y_i\mathcal{L}_i + z_i.
  \]
  \item[ii)] For cases \( a \) and \( b \) with \( \pi' = -\varphi^\vee \) and \( a_i \in W_0\varphi^\vee \) we have
  \[
  \gamma_{\{a_i\}}(k)(D_{\pi'}) = \mathcal{L}_i^2 + y_i\mathcal{L}_i + z_i - 2.
  \]
  \item[iii)] For case \( c \) with \( \pi' = -\varphi^\vee \) and \( a_i \in W_0\varphi^\vee \) we have
  \[
  \gamma_{\{a_i\}}(k)(D_{\pi'}) = \mathcal{L}_i + z_i + q^{\kappa_i} + q^{-\kappa_i}.
  \]
\end{itemize}

Proof. Note that Proposition 2.13 and Lemma 3.8 gives the initial expression
\[
\gamma_{\{a_i\}}(k)(D_{\pi'}) = m_{\pi'}(-\rho_{\vee}) + \sum_{w \in W_0^{\pi'}} g_{w,i}(z)\left(t(w^\pi') - q^{-\langle \rho_{\vee}(\omega_i), w^\pi' \rangle}\right),
\]
(3.18)
\[
g_{w,i}(z) = \prod_{a \in S(a_i) \cap \omega S_1(t(-\pi'))} c_a(z).
\]
The further computations depend on the three different cases.

i) Under the present assumptions, \( a_i = \alpha_i^\vee \in \mathcal{O}_5 \) for case c, \( \langle a_i, w \pi' \rangle < 0 \) for \( w \in W_0 \) implies \( \langle \alpha_i, w \pi' \rangle = -1 \), and

\[
q^{-\langle \rho_{w', \alpha_i}, \mu' \rangle} = q^{-\langle \rho_{w', \mu'}, \alpha_i, \mu' \rangle}, \quad \mu' \in L'_i
\]

(compare with the proof of Lemma 3.8). Combined with Lemma 3.7 and the \( s_i \)-invariance of \( \rho_{w', \alpha_i} \), the expression (3.18) becomes

\[
\gamma_{\{a_i\}}(\kappa)(D_{w' \pi}) = m_{w'}(-\rho_{w'}) + \sum_{w \in W_0' : \langle \alpha_i, w \pi' \rangle = 0} \left( t(w \pi') - q^{-\langle \rho_{w', \pi'}, \pi' \rangle} \right)
+ \sum_{w \in W_0' : \langle \alpha_i, w \pi' \rangle = -1} \left( c_{a_i}(z)t(w \pi') + c_{-a_i}(z)t(s_i w \pi') \right)
- \sum_{w \in W_0' : \langle \alpha_i, w \pi' \rangle = -1} \left( c_{a_i}(z) + c_{-a_i}(z) \right) \tau^{-1}_{a_i} q^{-\langle \rho_{w', \pi'}, \pi' \rangle}.
\]

The second line is \( y_i \mathcal{L}_i \). To show that the remaining terms sum up to \( z_i \), it suffices to note that

\[
\sum_{w \in W_0' : \langle \alpha_i, w \pi' \rangle = -1} \left( c_{a_i}(z) + c_{-a_i}(z) \right) \tau^{-1}_{a_i} q^{-\langle \rho_{w', \pi'}, \pi' \rangle} = \sum_{w \in W_0' : \langle \alpha_i, w \pi' \rangle \neq 0} q^{-\langle \rho_{w', \pi'}, \pi' \rangle}
\]

by (2.3).

ii) & iii) Under the present assumptions, \( \langle w \pi', a_i \rangle < 0 \) and \( w \pi' \not\in \{ \pm a_i \} \) imply \( \langle \alpha_i, w \pi' \rangle = -1 \). As in the proof of part i), Lemma 3.7 and (3.18) give

\[
\gamma_{\{a_i\}}(\kappa)(D_{w' \pi}) = c_{a_i}(z)c_{a_i} + w_{w \pi'}(z)\left(t(-a_i) - 1\right) + c_{-a_i}(z)c_{-a_i} + w_{w \pi'}(z)\left(t(a_i) - 1\right)
+ y_i \mathcal{L}_i + z_i + q^{\langle \rho_{w', a_i} \rangle} + q^{-\langle \rho_{w', a_i} \rangle}.
\]

For case c, \( \| \varphi \|^2 = 4 \) and \( q^{\langle \rho_{w', a_i} \rangle} = q^{\epsilon_i} \), while Lemma 3.12 implies that \( y_i = 0 \). This proves iii).

For case a and case b, the root \( a_i \) has squared length two, hence \( a_i = \alpha_i = \alpha_i^\vee \).

Since the labeling of \( S \) only depends on the gradient root system \( R^\vee \) of \( S \), it follows that

\[
t(\pm \alpha_i^\vee / 2)(c_{a_i}(z)) = c_{a_i, +c}(z), \quad t(\pm \alpha_i^\vee / 2)(c_{-a_i}(z)) = c_{-a_i, +c}(z).
\]

Combined with (2.6), we obtain

\[
\mathcal{L}_i^2 = c_{a_i}(z)c_{a_i, +c}(z)\left(t(-a_i) - 1\right) + c_{-a_i}(z)c_{-a_i, +c}(z)\left(t(a_i) - 1\right) + \left( \tau_{a_i} + \tau_{a_i}^{-1} \right)^2.
\]

Since \( \| \varphi \|^2 = 2 \) and \( q^{\langle \rho_{w', a_i} \rangle} = \tau_{a_i}^2 \), part ii) of the proposition now follows from (3.19). \( \square \)

3.4. Centralizers. Let \( \pi' \in L' \) be a nonzero anti-dominant minuscule or quasi-minuscule co-weight. For \( i \in \{ 1, \ldots, n \} \) we denote \( \mathcal{D}_i^{w'}(\kappa) \) for the centralizer of \( \gamma_{\{a_i\}}(\kappa)(D_{w' \pi'}) \) in \( \mathbb{D}_{\mathcal{R}(a_i)}(L_i^{W_0, \langle a_i \rangle}) \). Write \( \mathbb{C}[\mathcal{L}_i] \subseteq \mathbb{D}_{\mathcal{R}(a_i)}(L_i^{W_0, \langle a_i \rangle}) \) for the unital subalgebra generated by the difference operator \( \mathcal{L}_i \).
Proposition 3.14. Let \( i \in \{1, \ldots, n\} \) and let \( \pi' \in L' \) be a nonzero anti-dominant minuscule or quasi-minuscule co-weight. The restriction of the multiplication map \( D \otimes D' \rightarrow DD' \) (\( D, D' \in \mathbb{D}_{\mathcal{R}(a_i)}(L'_i) \)) to \( \mathbb{C}[\mathcal{L}_i] \otimes \mathbb{C}[X_i] \) defines an algebra isomorphism

\[
\mu_i : \mathbb{C}[\mathcal{L}_i] \otimes \mathbb{C}[X_i] \xrightarrow{\sim} \mathbb{D}_{\mathcal{R}(a_i)}(L'_i).
\]

Proof. By Corollary 3.10, the multiplication map restricts to an injective algebra homomorphism

\[
\mu_i : \mathbb{C}[\mathcal{L}_i] \otimes \mathbb{C}[X_i] \rightarrow \mathbb{D}_{\mathcal{R}(a_i)}(L'_i).
\]

A constant coefficient difference operator with step-sizes from \( X_i \) commutes with the difference operator \( \gamma_{(a_i)}(k)(D_{\pi'}) \) by Corollary 3.10, and it is \( s_i \)-invariant since \( s_i \) fixes \( X_i \) point-wise. Furthermore, by Corollary 3.10 and Proposition 3.13 we have \( \mathbb{C}[\mathcal{L}_i] \subseteq \mathbb{D}_{\mathcal{R}(a_i)}(L'_i) \). We conclude that the image of \( \mu_i \) is contained in \( \mathbb{D}_{\mathcal{R}(a_i)}(L'_i) \).

Before proving that \( \mu_i \) is a linear isomorphism, we first introduce some convenient notations and terminology. We set \( v_i = \alpha'_i/2 \) unless we are in case \( \mathbf{c} \) with \( a_i \in W_0 \varphi'_i \), in which case we set \( v_i = a_i \). For a nonzero difference operator \( D \in \mathbb{D}_{\mathcal{R}(a_i)}(L'_i) \), consider its unique expansion

\[
D = \sum_{m \in \mathbb{Z}} g_m^\pi(z) t(x + mv_i)
\]

with \( g_m^\pi(z) \in \mathcal{R}_{(a_i)} \) nonzero for at most finitely many pairs \( (m, x) \in \mathbb{Z} \times X_i \). Since \( s_i \) fixes \( X_i \) point-wise, the \( s_i \)-invariance of \( D \) implies \( s_i(g_m^\pi(z)) = g_m^\pi(z) \). We call

\[
D^{(m)} := \sum_{x \in X_i} g_m^\pi(z) t(x + mv_i)
\]

the \( m \)th order term of the difference operator \( D \). We write \( M(D) \) for the largest integer \( m \) for which \( D^{(m)} \neq 0 \). Since \( s_i(D^{(m)}) = D^{(-m)} \) we have \( M(D) \geq 0 \).

Fix now a nonzero difference operator \( D \in \mathbb{D}_{\mathcal{R}(a_i)}(L'_i) \) and set \( M = M(D) \). Consider the decomposition \( D = \sum_{x \in X_i} D_x t(x) \) with \( D_x \in \mathbb{D}_{\mathcal{R}(a_i)}(\mathbb{Z}v_i) \) the \( s_i \)-invariant difference operator

\[
D_x := \sum_{m \in \mathbb{Z}} g_m^\pi(z) t(mv_i), \quad x \in X_i.
\]

We have to show that \( D_x \in \mathbb{C}[\mathcal{L}_i] \) for all \( x \in X_i \).

By Corollary 3.10, Lemma 3.12 and Proposition 3.13 the fact that \( D \) centralizes \( \gamma_{(a_i)}(k)(D_{\pi'}) \) implies

\[
[D, \mathcal{L}_i] = 0
\]

unless we are in case \( \mathbf{a} \) or case \( \mathbf{b} \) with \( \pi' = -\varphi'_i \) and \( a_i \in W_0 \varphi'_i \), in which case it implies

\[
[D, \mathcal{L}_i]^2 + [D, \mathcal{L}_i] y_i = 0.
\]

We set \( d_i(z) = c_{-a_i}(z) \) unless we are in case \( \mathbf{c} \) with \( a_i \in W_0 \varphi'_i \), in which case we set \( d_i(z) = c_{-a_i}(z)c_{-a_i+2}(z) \). With this notation, the highest order term of \( \mathcal{L}_i \) is \( d_i(z)t(v_i) \). Considering the highest order term of the identity (3.20) and (3.21) respectively, we obtain

\[
[D^{(M)}, d_i(z)t(v_i)] = 0
\]
module algebra with action defined by $s_i$ where canonical basis denoted by $\xi_i$.

In particular, the highest order term $D(M)$ satisfies (3.22) for all the cases under consideration.

It follows from (3.22) that the coefficients $g_M^x(z) \in R_{\{a_i\}}$ ($x \in X_i$) are solutions of the difference equation

$$t(2v_i)(f(z)) = \left( \frac{t(Mv_i)(e_i(z))}{e_i(z)} \right) f(z),$$

where $e_i(z) = d_i(z)(t(v_i)(d_i(z))) \in R_{\{a_i\}}$. The space of functions $f(z) \in R_{\{a_i\}}$ satisfying (3.23) is an one-dimensional complex vector space spanned by

$$f_M(z) = \prod_{j=0}^{M-1} \left( t(jv_i)(d_i(z)) \right) \in R_{\{a_i\}},$$

hence $g_M^x(z) = K_M^x f_M(z)$ for some $K_M^x \in C \ (x \in X_i)$. In particular, $D_x \in C$ for all $x \in X_i$ if $M = 0$, which proves that $D_x \in C[L_i] \ (x \in X_i)$ for $M = 0$.

Let $M > 0$ and suppose that $D'_x \in C[L_i]$ for all $x \in X_i$ if $0 = D' \in D^x_i(k)$ and $M(D') < M$. Let $0 \neq D \in D^x_i(k)$ with $M(D) = M$. Since

$$L^M_i = \sum_{m=-M}^{M} \left( \mathcal{L}^M_i \right)^{(m)},$$

with $M$th order term given by $\left( \mathcal{L}^M_i \right)^{(M)} = f_M(z)t(Mv_i)$, it follows that

$$D' := D - \sum_{x \in X_i} K_M^x t(x) \in D^x_i(k)$$

is either zero, or it is nonzero and $M(D') < M$. By the induction hypothesis it follows that $D_x \in C[L_i]$ for all $x \in X_i$. \hfill $\square$

**Theorem 3.15.** Let $\pi' \in L'$ be a nonzero anti-dominant minuscule or quasi-minuscule co-weight.

i) The Harish-Chandra homomorphism $\gamma(k)$ restricts to an algebra isomorphism

$$\gamma(k) : D_R(L')W_0,D' \longrightarrow A'_0.$$ 

ii) The map $\beta$ restricts to an algebra isomorphism

$$\beta : A'_0(Y) \longrightarrow D_R(L')W_0,D'.$$

**Proof.** i) Let $i \in \{1, \ldots, n\}$ and write $A'_i$ for the group algebra $C[t(L'_i)]$, with canonical basis denoted by $\xi^X' \ (X' \in L'_i)$. The group algebra $A'_i$ is a $W_{0,\{a_i\}}$-module algebra with action defined by $s_i(\xi^X') := \xi^{a_iX'} \ (X' \in L'_i)$. It contains $A'$ as $W_{0,\{a_i\}}$-module algebra.

The map $\delta_{\{a_i\}}(k)$ (see (3.35)) extends to an algebra homomorphism

$$\delta_{\{a_i\}}(k) : D_{R_{\{a_i\}}}(L'_i) \rightarrow A'_i,$$

defined by the same formula (3.35) (with the finite sum over $X'$ now running over $L'_i$). By a direct computation we have

$$\delta_{\{a_i\}}(k)(\mathcal{L}_i) = \xi^{a'_i/2} + \xi^{-a'_i/2}.$$
unless we are in case \( c \) and \( a_i \in W_0 \varphi' \), in which case we have

\[
\delta(\alpha_i)(k)(L_i) = \xi^{a_i} + \xi^{-a_i} - q^{\alpha'_i} - q^{-\alpha'_i}.
\]

Combined with Proposition 3.14, we conclude that \( \delta(k) \) maps the centralizer
subalgebra \( \mathbb{D}_i^\prime(k) \) onto the subalgebra of \( W_0(\alpha_i) \)-invariant elements in \( \mathcal{A}_i' \).

Fix \( D \in \mathbb{D}_R(L')W_0,D\alpha' \). Then \( \gamma(\alpha_i)(k)(D) \in \mathbb{D}_i^\prime(k) \cap \mathbb{D}_R(\alpha_i)(L') \) by Lemma 3.6
hence the constant term

\[
\gamma(k)(D) = \delta(\alpha_i)(k)(\gamma(\alpha_i)(k)(D)) \in A'
\]
of \( D \) is \( W_0(\alpha_i) \)-invariant for all \( i \in \{1, \ldots, n\} \) by the previous paragraph. It follows
that \( \gamma(k)(D) \in A_0' \). Proposition 3.1 now completes the proof of part i).

Part ii) of the proposition follows from part i) and Proposition 2.17. \( \square \)

We now have the following stronger version of Corollary 3.4

**Corollary 3.16.** We have

\[
\mathbb{D}_R(L')^{D\alpha'} = \mathbb{D}_R(L')W_0,D\alpha'
\]
if and only if \( A_0 \) is a \( \mathbb{D}_R(L')^{D\alpha'} \)-submodule of \( \mathcal{M} \).

**Proof.** By [23] §6.4], the difference operator \( D_p|_{A_0} \) is an endomorphism of \( A_0 \) for all \( p \in A_0' \). The previous theorem thus implies that \( A_0 \) is a \( \mathbb{D}_R(L')^{W_0,D\alpha'} \)-submodule of \( \mathcal{M} \). The result follows now directly from Corollary 3.4. \( \square \)

4. **Harish-Chandra series**

4.1. **Harish-Chandra series with formal spectral parameter.** In the trigonometric differential case, Harish-Chandra series are power series solutions to the differential analogues of the commuting difference operators \( D_p \) \((p \in A_0')\), see [15] §4.2 and references therein. In this subsection we construct the natural difference analogue of the Harish-Chandra series with formal spectral parameter. For \( R \) of type \( A \), the difference analogues of the Harish-Chandra series were considered in [9] §6 in the context of weighted traces of quantum group intertwiners, see also [11] §9.

Let \( \mathcal{B}' \subseteq \mathcal{Q}' \) be an \( A' \)-submodule. We write \( \mathcal{B}'[[z^{-\Delta}]] \) for the \( A' \)-module of formal power series

\[
F(z, \xi) = \sum_{x \in \mathbb{Z}_{\geq 0} \Delta} f_x(\xi) z^{-x}, \quad f_x(\xi) \in \mathcal{B}'.
\]

The following lemma is easily checked.

**Lemma 4.1.** The canonical \( \mathbb{C}[[z^{-\Delta}]] \)-action on \( \mathcal{B}'[[z^{-\Delta}]] \) uniquely extends to an action of \( \mathbb{D}_C[z^{-\Delta}][L'] \) on the \( A' \)-module \( \mathcal{B}'[[z^{-\Delta}]] \) by

\[
t(\lambda')(F(z, \xi)) = \sum_{x \in \mathbb{Z}_{\geq 0} \Delta} f_x(\xi) \xi^{-\lambda'} q_{(x, \lambda')} z^{-x}, \quad \lambda' \in L',
\]

with \( F(z, \xi) \in \mathcal{Q}'[[z^{-\Delta}]] \) given by (4.1).

Let \( \lambda' \in L' \) be a nonzero anti-dominant minuscule or quasi-minuscule co-weight. Recall from Proposition 3.11 that the centralizer algebra \( \mathbb{D}_R(L')^{D\alpha'} \) is a commutative algebra of difference operators containing the subalgebra \( \beta(A_0'(Y)) = \mathbb{D}_R(L')W_0,D\alpha' \) of difference operators \( D_p \) \((p \in A_0')\). Recall the \( \rho_{\Delta} \)-twist (2.9).
Theorem 4.2. There exists a unique
\begin{equation} 
\Phi(z, \xi) = \sum_{x \in \mathbb{Z}_{\geq 0}\Delta} \Gamma_x(\xi)z^{-x} \in Q'[z^{-\Delta}], \quad \Gamma_x(\xi) \in Q' 
\end{equation}
normalized by \( \Gamma_0(\xi) = 1 \) and satisfying the difference equations
\( D\Phi(z, \xi) = (\gamma(\tilde{k})(D))(\xi)\Phi(z, \xi), \quad \forall D \in \mathbb{D}_{\mathbb{R}}(L')^{D_{\sigma'}} \)
with respect to the action from Lemma 4.1. In particular, \( \Phi(z, \xi) \) satisfies the difference equations
\( D_p\Phi(z, \xi) = \overline{p}(\xi)\Phi(z, \xi), \quad \forall p(\xi) \in A'_0. \)

Proof. Let \( D \in \mathbb{D}_{\mathbb{R}}(L')^{D_{\sigma'}} \subseteq \mathbb{D}_{\mathbb{C}}[z^{-\Delta}](L')^{D_{\sigma'}} \), written out explicitly as
\( D = \sum_{\lambda' \in L'} \left( \sum_{y \in \mathbb{Z}_{\geq 0}\Delta} d_{\lambda'}(y)z^{-y} \right)t(\lambda') \)
with \( d_{\lambda'}(y) \in \mathbb{C} \) and with the first sum over finitely many \( \lambda' \in L' \). We use the shorthand notation \( r_D(\xi) := (\gamma(\tilde{k})(D))(\xi) \in A' \) for the constant term of \( D \). By the definition of \( \gamma(\tilde{k}) \) (see Lemma 2.15) we then have
\( \tilde{r}_D(\xi) = \sum_{\lambda' \in L'} d_{\lambda'}(0)\xi^{-\lambda'} \).

A direct computation now shows that \( D\Phi(z, \xi) = \tilde{r}_D(\xi)\Phi(z, \xi) \), with \( \Phi(z, \xi) \in Q'[z^{-\Delta}] \) a series of the form (4.2), if and only if
\begin{equation} 
(\tilde{r}_D(\xi) - t(x)(\tilde{r}_D(\xi)))\Gamma_x(\xi) = \sum_{\lambda' \in L'} \sum_{0 \leq y < x} d_{\lambda'}(x-y)q^{(\lambda', y)}\xi^{-\lambda'}\Gamma_y(\xi) 
\end{equation}
for all \( x \in \mathbb{Z}_{\geq 0}\Delta \).

We now explore the recurrence relations (4.3) first for the Macdonald difference operator \( D_{\sigma'} \), in which case \( r_{D_{\sigma'}}(\xi) = m_{\sigma'}(\xi) \) by Proposition 2.17. We have \( t(x)(m_{\sigma'}(\xi)) \neq m_{\sigma'}(\xi) \) for all \( x \in \mathbb{Z}_{\geq 0}\Delta \setminus \{0\} \) since \( W_0 \) acts irreducibly on \( V \). Hence for \( D = D_{\sigma'} \) the recurrence relations (4.3) has a unique solution \( \Gamma_x(\xi) \in Q' \) (\( x \in \mathbb{Z}_{\geq 0}\Delta \)) normalized by \( \Gamma_0(\xi) = 1 \), and the resulting formal power series
\( \Phi(z, \xi) = \sum_{x \in \mathbb{Z}_{\geq 0}\Delta} \Gamma_x(\xi)z^{-x} \in Q'[z^{-\Delta}] \)
satisfies the recurrence relations (4.3) for \( D = D_{\sigma'} \). By a direct computation (analogous e.g. to the computations in the proof of Lemma 2.16) we furthermore have \( \Gamma_0(\xi) = 0 \), hence we conclude that \( \Gamma_x(\xi) = 0 \) for all \( x \in \mathbb{Z}_{\geq 0}\Delta \). Consequently \( D\Phi(z, \xi) = \tilde{r}_D(\xi)\Phi(z, \xi) \). Noting finally that \( r_{D_p}(\xi) = p(\xi) \) for \( p \in A'_0 \) by Proposition 2.17 we obtain the desired results. \( \square \)
Remark 4.3. It follows from (the proof of) Theorem 4.2 that the Harish-Chandra series \( \Phi(z, \xi) \in \mathcal{Q}[\mathcal{Z}^{-\Delta}] \) is uniquely characterized up to \( \mathcal{Q} \)-multiples by the single difference equation

\[
D_{\alpha'}(\Phi(z, \xi)) = \tilde{m}_{\alpha'}(\xi)\Phi(z, \xi)
\]

involving the Macdonald difference operator \( D_{\alpha'} \).

4.2. Harish-Chandra series with specialized spectral parameter. We view \( \mathcal{Q} \) as rational trigonometric functions on \( V_\mathbb{C} \) using \( \xi^\lambda'(v) = q^{(\lambda', v)} \) for \( v \in V_\mathbb{C} \) and \( \lambda' \in \mathcal{L}' \), cf. (2.1). We write \( q = e^{-2\pi\sigma} \) with \( \sigma \in \mathbb{R}_{>0} \), and

\[
Z = \{ \lambda \in \mathcal{V} \mid \langle \lambda', \lambda \rangle \in \mathbb{Z} \quad \forall \lambda' \in \mathcal{L}' \}
\]

for the lattice in \( V \) dual to \( \mathcal{L}' \). Note that \( Z = Q(R) \) for case \( a \) and \( b \), and \( Z = P(R) \) for case \( c \).

Lemma 4.4. The singularities of the coefficients \( \Gamma_x(\xi) \in \mathcal{Q} \langle x \in \mathbb{Z}_{\geq 0}\Delta \rangle \) of the Harish-Chandra series \( \Phi(z, \xi) \) are contained in the subset

\[
\mathcal{D}_{\mathbb{Z}} = \bigcup_{w \in W_0} \{ \lambda \in V_\mathbb{C} \mid \lambda - w \cdot \lambda \in \mathbb{Z}_{\geq 0}\Delta \setminus \{0\} + \sqrt{-1}\mathbb{Z}/\sigma \} \subset V_\mathbb{C},
\]

where the dot-action of \( W_0 \) on \( V_\mathbb{C} \) is defined by \( w \cdot \lambda = w(\lambda + \rho_\xi') - \rho_\xi' \) for \( w \in W_0 \) and \( \lambda \in V_\mathbb{C} \).

Proof. For \( \lambda, \mu \in V_\mathbb{C} \) we have

\[
\tilde{\rho}(\lambda) = \tilde{\rho}(\mu) \quad \forall \rho(\xi) \in A_0' \iff \lambda \in W_0 \cdot \mu + \sqrt{-1}\mathbb{Z}/\sigma.
\]

We now prove the lemma by induction to the height \( \sum_{\alpha \in \Delta} l_\alpha \in \mathbb{Z}_{\geq 0} \Delta \) of an element \( x = \sum_{\alpha \in \Delta} l_\alpha \alpha \in \mathbb{Z}_{\geq 0} \Delta \). If \( \lambda \in V_\mathbb{C} \setminus \mathcal{D}_{\mathbb{Z}} \) and \( x \in \mathbb{Z}_{\geq 0} \Delta \setminus \{0\} \), then (4.4) implies the existence of a \( p \in A_0' \) such that \( \tilde{\rho}(\lambda) \neq \tilde{\rho}(\lambda - x) \). By (4.3) applied to \( D = D_p \) the regularity of \( \Gamma_x(\xi) \) at \( \lambda \) is implied by the regularity of \( \Gamma_y(\xi) \) at \( \lambda \) for elements \( y \in \mathbb{Z}_{\geq 0} \Delta \) with height strictly smaller than \( x \).

We denote \( \mathcal{B}'_\mathbb{Z} \) for the \( A' \)-submodule of \( \mathcal{Q} \) consisting of the rational functions \( f(\xi) \in \mathcal{Q} \) with singularities contained in \( \mathcal{D}_{\mathbb{Z}} \). The previous lemma shows that \( \Phi(z, \xi) \in \mathcal{B}'_\mathbb{Z}[\mathcal{Z}^{-\Delta}] \). The following lemma will allow us to derive difference equations for the Harish-Chandra series when the spectral parameter is specialized to an element in the open and dense subset \( V_\mathbb{C} \setminus \mathcal{D}_{\mathbb{Z}} \) of \( V_\mathbb{C} \). Recall the \( \mathcal{D}_{\mathbb{C}[\mathcal{Z}^{-\Delta}]}(\mathcal{L}') \)-module \( \mathcal{M} \) from Subsection 2.2.

Lemma 4.5. For \( \lambda \in V_\mathbb{C} \setminus \mathcal{D}_{\mathbb{Z}} \) the assignment

\[
\sum_{x \in \mathbb{Z}_{\geq 0}\Delta} f_x(\xi)z^{-x} \mapsto \sum_{x \in \mathbb{Z}_{\geq 0}\Delta} f_x(\lambda)z^{\lambda - x}, \quad (f_x(\xi) \in \mathcal{B}'_\mathbb{Z}),
\]

defines a morphism \( \iota_\lambda : \mathcal{B}'_\mathbb{Z}[\mathcal{Z}^{-\Delta}] \rightarrow \mathcal{M} \) of \( \mathcal{D}_{\mathbb{C}[\mathcal{Z}^{-\Delta}]}(\mathcal{L}') \)-modules.

Proof. Direct verification. \( \square \)

Theorem 4.6. The Harish-Chandra series with spectral parameter \( \lambda \in V_\mathbb{C} \setminus \mathcal{D}_{\mathbb{Z}} \), defined by

\[
\Phi_\lambda(z) := \iota_\lambda(\Phi(z, \xi)) = \sum_{x \in \mathbb{Z}_{\geq 0}\Delta} \Gamma_x(\lambda)z^{\lambda - x} \in \mathcal{M},
\]

satisfies

\[
D(\Phi_\lambda(z)) = (\gamma(\lambda)(D))\Phi_\lambda(z), \quad \forall D \in \mathcal{D}_R(\mathcal{L}')^{\Delta'},
\]

where \( \gamma(\lambda)(D) \) acts on \( \Phi_\lambda(z) \) for \( \lambda \in V_\mathbb{C} \setminus \mathcal{D}_{\mathbb{Z}} \) by multiplication by \( \lambda \).
hence in particular
\[ D_p(\Phi_\lambda(z)) = \overline{p}(\lambda)\Phi_\lambda(z), \quad \forall p(\xi) \in A'_0. \]

The latter system of difference equations, together with the normalization \( \Gamma_0(\lambda) = 1 \), uniquely characterizes \( \Phi_\lambda(z) \) within the \( D_\C[[z^{-\Delta}]](L') \)-submodule \( \C[[z^{-\Delta}]]z^\lambda \) of \( \overline{M} \).

**Proof.** The first part follows immediately from the results of the previous section. For uniqueness, note that the coefficients \( \Gamma_x(\lambda) \) of a solution \( \sum_{x \in \Z_{\geq 0}\Delta} \Gamma_x(\lambda)z^{\lambda-x} \in \overline{M} \) of the difference equations (4.3) satisfy the homogeneous recurrence relations (4.3) specialized to \( \lambda \in V_\C \setminus D_\lambda \). With a similar argument as in the proof of Lemma 4.3 it follows that the coefficients \( \Gamma_x(\lambda) \) \( (x \in \Z_{\geq 0}\Delta) \) are determined by \( \Gamma_0(\lambda) \). \( \square \)

**Theorem 4.7.** Let \( \lambda \in V_\C \) such that
\[
\lambda - w \cdot \lambda \not\in \Z\Delta + \sqrt{-1}\Z/\sigma, \quad \forall w \in W_0 \setminus \{e\}.
\]
Then \( \{ \Phi_\mu \mid \mu \in W_0 \cdot \lambda + \sqrt{-1}\Z/\sigma \} \) is a basis of the common eigenspace
\[
\overline{M}_\lambda = \{ F(z) \in \overline{M} \mid D_p(F(z)) = \overline{p}(\lambda) F(z) \ \forall p(\xi) \in A'_0 \}.
\]

**Proof.** By (4.3), \( W_0 \cdot \lambda + \sqrt{-1}\Z/\sigma \) consists of elements from \( V_\C \setminus D_\lambda \) which are pair-wise incomparable with respect to the dominance order \( \preceq \). In particular, the Harish-Chandra series \( \Phi_\mu(z) \in \overline{M} \) \( (\mu \in W_0 \cdot \lambda + \sqrt{-1}\Z/\sigma) \) are well defined and linearly independent. Furthermore, \( \Phi_\mu(z) \in \overline{M}_\lambda \) \( (\mu \in W_0 \cdot \lambda + \sqrt{-1}\Z/\sigma) \) by (4.3).

For \( F(z) = \sum_{\nu} K_\nu z^\nu \in \overline{M} \) we write \( \Supp(F(z)) = \{ \nu \in V_\C \mid K_\nu \neq 0 \} \). We claim that if \( F(z) \in \overline{M}_\lambda \) and if \( \nu \in \Supp(F(z)) \) is a maximal element with respect to the dominance order \( \preceq \), then \( \nu \in W_0 \cdot \lambda + \sqrt{-1}\Z/\sigma \). Before proving the claim, we first show that it implies that \( F(z) \) is a finite linear combination of the Harish-Chandra series \( \Phi_\nu(z) \) \( (\mu \in W_0 \cdot \lambda + \sqrt{-1}\Z/\sigma) \).

If \( 0 \neq F(z) \in \overline{M}_\lambda \) then we can choose a maximal element \( \nu_1 \in \Supp(F(z)) \) with respect to \( \preceq \) by Zorn’s Lemma. Then \( \nu_1 \in W_0 \cdot \lambda + \sqrt{-1}\Z/\sigma \) by the claim, hence \( \nu_1 \not\in D_\lambda \) and \( \Phi_{\nu_1}(z) \in \overline{M}_\lambda \). Set \( F_1(z) = F(z) - K_{\nu_1}\Phi_{\nu_1}(z) \in \overline{M}_\lambda \). If \( F_1(z) \neq 0 \) then we choose a maximal element \( \nu_2 \) in \( \Supp(F_1(z)) \), which necessarily satisfies \( \nu_1 \neq \nu_2 \). If \( \nu_2 \in W_0 \cdot \lambda + \sqrt{-1}\Z/\sigma \). In particular, \( \nu_1 \) and \( \nu_2 \) are incomparable with respect to \( \preceq \). We proceed to define
\[
F_2(z) := F_1(z) - K_{\nu_2}\Phi_{\nu_2}(z) = F(z) - K_{\nu_1}\Phi_{\nu_1}(z) - K_{\nu_2}\Phi_{\nu_2}(z) \in \overline{M}_\lambda.
\]

Repeating the above procedure we construct \( F_m(z) \in \overline{M}_\lambda \) and \( \nu_m \in V_\C \) from \( F_{m-1}(z) \in \overline{M}_\lambda \) inductively, as long as \( F_{m-1}(z) \neq 0 \). Since the resulting spectral parameters \( \mu_1, \ldots, \mu_m \) are pair-wise incomparable with respect to \( \preceq \), we have \( F_m(z) = 0 \) for some \( m \in \Z_{\geq 0} \) in view of the definition of \( \overline{M} \). Consequently, \( F(z) \) is a linear combination of the Harish-Chandra series \( \Phi_{\nu_j}(z) \) \( (j = 1, \ldots, m) \).

It remains to prove the claim. Let \( \mu \in \Supp(F(z)) \) be a maximal element with respect to \( \preceq \) and set
\[
G(z) := F(z) - K_\mu z^\mu \in \overline{M},
\]
so that \( \Supp(G(z)) = \Supp(F(z)) \setminus \{ \mu \} \). Fix \( p \in A'_0 \). Since \( D_p(z^\nu) \in z^\nu \C[[z^{-\Delta}]] \) for \( \nu \in V_\C \), we have \( \mu \not\in \Supp(D_p(G(z))) \). Combined with Lemma 2.10 and Proposition 2.17 we conclude that the coefficient of \( z^\mu \) in
\[
D_p(F(z)) = D_p(G(z)) + K_\mu D_p(z^\mu)
\]
is $K_{\mu} \tilde{p}(\mu)$. On the other hand, the coefficient of $z^\mu$ in $D_p(F(z)) = \tilde{p}(\lambda)F(z)$ is $K_{\mu} \tilde{p}(\lambda)$. Hence $\tilde{p}(\mu) = \tilde{p}(\lambda)$ for all $p \in A'_0$, which implies that $\mu \in W_0 \cdot \lambda + \sqrt{-1}Z/\sigma$ by (4.4).

**Remark 4.8.** The Harish-Chandra series $\Phi_\lambda(z)$ for generic real spectral values $\lambda \in V$ is contained in the $D_{\mathbb{C}[z^{-\Delta}]}(L')$-submodule $M_{\lambda}' \subset M_L$ consisting of the formal series $F(z) = \sum_{a \in \mathbb{C}} K_a z^a$ in $M_L$ with $C \subset V$. Suppose that $\lambda \in V$ satisfies

$$\lambda - w \cdot \lambda \notin \mathbb{Z} \Delta, \quad \forall w \in W_0 \setminus \{e\},$$

and suppose that the multiplicity label $k'$ is real-valued, so that $\rho_{k'} \in V$ and the dot-action of $W_0$ preserves $V$. Then Theorem 4.7 implies that the common eigenspace $M_{\lambda}' := M_L' \cap M_\lambda$ is $\#W_0$-dimensional with basis $\{\Phi_{\rho \cdot \lambda}\}_{w \in W_0}$, cf. [15] Cor. 4.2.6 for the analogous statement in the trigonometric differential set-up. For type A, this result essentially is [9, Thm. 5, part 2].

**4.3. Relation to Macdonald polynomials.** The properties (2.10) and (4.4) lead to the following well-known definition of the Macdonald polynomials [23, 24, 20] (known as Koornwinder [20] polynomials for case $c$).

**Definition 4.9.** Suppose that

$$\lambda \notin W_0 \cdot \mu + \sqrt{-1}Z/\sigma \quad \forall \lambda, \mu \in L_{++} : \lambda \neq \mu. \tag{4.7}$$

The monic Macdonald polynomial $P_\lambda(z) \in A_0$ of degree $\lambda \in L_{++}$ is the unique $W_0$-invariant Laurent polynomial satisfying

$$P_\lambda(z) = m_\lambda(z) + \sum_{\mu \in L_{++} : \mu < \lambda} k_{\lambda, \mu} m_\mu(z)$$

for certain coefficients $k_{\lambda, \mu} \in \mathbb{C}$ and satisfying the system of difference equations

$$D_p(P_\lambda(z)) = \tilde{p}(\lambda)P_\lambda(z), \quad \forall p(\xi) \in A'_0. \tag{4.8}$$

Note that if the multiplicity label $k'$ is real valued then the conditions (4.7) reduce to

$$\lambda \notin W_0 \cdot \mu \quad \forall \lambda, \mu \in L_{++} : \lambda \neq \mu. \tag{4.8}$$

For instance, if $k'_a \geq 0$ for all $a \in S'$, then $\rho_{k'} \in \overline{V}_+$, hence (4.8) is satisfied.

**Proposition 4.10.** Suppose that (4.7) is satisfied. For $\lambda \in L_{++}$ not contained in $D_{\Delta}$ we have

$$\Phi_\lambda(z) = P_\lambda(z).$$

**Proof.** By the assumptions, the Harish-Chandra series $\Phi_\lambda(z)$ and the Macdonald polynomial $P_\lambda(z)$ are well defined. Furthermore, both $\Phi_\lambda(z)$ and $P_\lambda(z)$ are elements from $M_{\lambda}$ with the coefficient of $z^\lambda$ equal to one. Hence $\Phi_\lambda(z) = P_\lambda(z)$ by Theorem 4.6. □

We now return to the centralizers of Macdonald difference operators.

**Corollary 4.11.** Let $\pi' \in L'$ be a nonzero anti-dominant minuscule or quasi-minuscule co-weight. Suppose that (4.1) is satisfied and that $L_{++} \subset V_C \setminus D_{\Delta}$. Then

$$\mathbb{D}_R(L')^{D_{\pi'}} = \mathbb{D}_R(L')^{W_0 \cdot \pi'},$$

hence the centralizer of $D_{\pi'}$ in $\mathbb{D}_R(L')$ consists only of the Cherednik-Macdonald difference operators $D_p (p(\xi) \in A'_0)$.  

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Proof. The second part of the statement follows from Theorem 3.16. For the first statement it suffices to show that $A_{0} \subset M$ is a $\mathbb{D}_{R}(L')^{D_{\nu}}$-submodule in view of Corollary 4.11. Since the Macdonald polynomials $P_{\lambda}(z)$ ($\lambda \in L_{++}$) form a linear basis of $A_{0}$, it suffices to note that

$$D(P_{\lambda}(z)) = D(\Phi_{\lambda}(z))$$

$$= (\gamma(\mathbf{k})|D)(\lambda)\Phi_{\lambda}(z)$$

$$= (\gamma(\mathbf{k})|D)(\lambda)P_{\lambda}(z)$$

for $D \in \mathbb{D}_{R}(L')^{D_{\nu}}$ and $\lambda \in L_{++}$, where the first and the last equality follow from Proposition 4.10 and the second equality follows from Theorem 4.6. □

As an example, consider a multiplicity label $\mathbf{k}'$ such that

$$\rho_{\mathbf{k}'} - w(\rho_{\mathbf{k}'}) = \sum_{\alpha \in R^{+}\setminus L} \mathbf{k}'(\alpha^{\vee})\alpha \not\in L, \quad \forall w \in W_{0} \setminus \{e\}$$

(4.9) the alternative expression for $\rho_{\mathbf{k}'} - w(\rho_{\mathbf{k}'})$ follows from [23, (1.5.3)]. Then $L_{++}$ is contained in $V_{C} \setminus D_{\lambda}$ and (4.8) is satisfied, hence Corollary 4.11 implies that $\mathbb{D}_{R}(L')^{D_{\nu}} = \mathbb{D}_{R}(L')^{W_{\alpha}D_{\nu}}$. For example, for case $\mathbf{a}$ we have $L = P(R)$ so condition (4.9) implies $\mathbf{k}'(\alpha^{\vee}) \not\in \mathbb{Z}$ for all $\alpha \in R$.

Examples for which the conditions of Corollary 4.11 are violated, are discussed in the next subsection.

4.4. Relation to Baker-Akhiezer functions. For special discrete values of the multiplicity label $\mathbf{k}$, the commuting Cherednik-Macdonald difference operators $D_{p}$ ($p(\xi) \in A_{0}'$) are algebraically integrable in the sense of e.g. [3] and [10]. For such multiplicity labels, Chalykh [3] defines and studies for case $\mathbf{a}$ and case $\mathbf{c}$ eigenfunctions of the difference operators $D_{p}$ ($p(\xi) \in A_{0}'$) called Baker-Akhiezer functions (see [10] for $R$ of type $A$). In this subsection we relate the Harish-Chandra series to the Baker-Akhiezer functions for case $\mathbf{a}$. Case $\mathbf{c}$ can be treated in a similar fashion.

We assume throughout the remainder of the subsection that we are in case $\mathbf{a}$, so that $\Delta = \{\alpha_{1}, \ldots, \alpha_{n}\}$ is the basis of $R$, $\mathbb{Z}\Delta = Q(R)$ respectively $\mathbb{Z}_{\geq 0}\Delta = Q_{+}(R)$ is the root lattice respectively its cone of positive integral linear combinations of positive roots, and $(L, L') = (P(R), P(R'))$. We furthermore assume throughout the remainder of the subsection that the multiplicity label $\mathbf{k}$ satisfies

$$k_{\alpha} \in \mathbb{Z}_{\geq 0}, \quad \forall \alpha \in R.$$  

(4.10)

Note that (4.10) implies that $\rho_{\mathbf{k}'(\alpha^{\vee})} \in P(R)$ and

$$\rho_{\mathbf{k}'} := \frac{1}{2} \sum_{\alpha \in R^{+}} k_{\alpha}\alpha^{\vee} \in P(R^{\vee}).$$

With this choice of multiplicity label, the Macdonald polynomials $P_{\lambda}(z)$ are not defined for low degree $\lambda \in L_{++}$ (see [3, §5.4]), and the Harish-Chandra series $\Phi_{\lambda}(z)$ ($\lambda \in L_{++}$) are not well defined for large degree $\lambda \in L_{++}$ (specifically, for $\lambda \in L_{++}$ such that $\lambda + \rho_{\mathbf{k}'} \not\in V_{+}$). In particular, Proposition 4.10 and Corollary 4.11 are no longer valid. The theory in this set-up requires a completely different approach, which was developed by Chalykh in [3]. In this approach a key role is played by the normalized Baker-Akhiezer function $\psi_{\lambda}(z)$, whose definition we now shortly recall from [3].
Our present notations are matched with the ones from [3] as follows: the parameters \((q, \tau_v)\) correspond to \((q^2, t_\alpha)\) in [3], and the minuscule or quasi-minuscule co-weight \(\pi'\) corresponds to \(-\pi\) in [3] (with these correspondences, our Macdonald difference operator \(D_{\pi'}\) turns into the Macdonald difference operator \(D^{\pi}\) from [3] §2.2). Set

\[ N = \{ \sum_{\alpha \in R^+} l_\alpha \alpha \mid l_\alpha \in \mathbb{Z} \text{ and } 0 \leq l_\alpha \leq -k_\alpha \forall \alpha \in R \} \subset \mathbb{Q}_+(R). \]

Chalykh’s [3] Thm. 4.7 Baker-Akhiezer function \(\psi^\lambda(\lambda, z)\) associated to the co-root lattice \(R^\vee\) is now defined as follows.

**Definition 4.12** ([3]). The Baker-Akhiezer functions \(\psi_{\lambda}(z) \in M (\lambda \in V_C)\) are the unique functions of the form

\[ \psi_{\lambda}(z) = \sum_{x \in \mathcal{N}} K_x^{BA}(\lambda) z^{\lambda - \rho_{\pi'} - x} \quad (K_x^{BA}(\xi) \in A'), \]

satisfying the equalities

\[ \psi_{\lambda}(v + r \alpha^\vee / 2) = \psi_{\lambda}(v - r \alpha^\vee / 2) \quad \text{for } q^{(\alpha', \alpha)} = 1 \]

if \(\alpha \in R\) and \(r = 1, \ldots, -k_\alpha\), and normalized by

\[ K_0^{BA}(\xi) = \xi^{\rho_{\pi'}} \prod_{\alpha \in R^+} j=1 q^{-j/2} q^{-j/2} \xi_{\alpha^\vee}. \]

A key property of the Baker-Akhiezer functions \(\psi_{\lambda}(z) \in M \subset \mathcal{M}\) is the fact that

\[ D_{\pi'}(\psi_{\lambda}(z)) = m_{\pi'}(-\lambda) \psi_{\lambda}(z) \]

for a nonzero antidominant minuscule or quasi-minuscule co-weight \(\pi' \in L'\), cf. [3] Thm. 3.7(iv) & Thm. 4.7. Comparing with the properties of the Harish-Chandra series \(\Phi_{\lambda}(z) = \sum_{x \in \mathcal{Q}_+(R)} \Gamma_x(\lambda) z^{\lambda - x} \in \mathcal{M}\) \((\lambda \in V_C \setminus D_L)\) we obtain the following result.

**Proposition 4.13.** For case a and for multiplicity labels \(\kappa\) satisfying (4.10) we have

\[ \Phi_{\lambda}(z) = K_0^{BA}(\lambda + \rho_L)^{-1} \psi_{\lambda + \rho_L}(z) \]

if \(\lambda \in V_C \setminus D_L\). In particular, the rational functions \(\Gamma_x(\xi) \in Q'\) of the Harish-Chandra series \(\Phi_{\lambda}(z, \xi) = \sum_{x \in \mathcal{Q}_+(R)} \Gamma_x(\xi) z^{-x}\) with formal spectral parameter satisfy

\[ \Gamma_x(\xi) = t(-\rho_L)(K_x^{BA}(\xi) / K_0^{BA}(\xi)), \quad \forall x \in \mathcal{N}. \]

**Proof.** First note that (4.14) implies that \(\psi_{\lambda + \rho_L}(z) \in \mathcal{M}_\lambda\), by a similar argument as in the proof of Theorem 4.2 (see also Remark 4.3).

Fix now \(\lambda \in V_C \setminus D_L\). Then \(K_0^{BA}(\lambda + \rho_L) \neq 0\) and both \(\Phi_{\lambda}(z)\) and \(\psi_{\lambda + \rho_L}(z)\) are in the common eigenspace \(\mathcal{M}_\lambda\), hence \(\psi_{\lambda + \rho_L}(z) = K_0^{BA}(\lambda + \rho_L) \Phi_{\lambda}(z)\) by Theorem 4.6. \(\square\)

**Remark 4.14.** i) In the trigonometric differential degeneration, the analogue of Proposition 4.13 was established in [1] and [2] Section VI.C.

ii) Proposition 4.13 suggests that various properties of the normalized Baker-Akhiezer functions (such as duality [3] Thm. 4.7 and bispectrality [3 Cor. 4.8])
can be transferred to Harish-Chandra series for arbitrary multiplicity labels $k_i$, cf. [2] §6 for the differential set-up. We return to these issues in future work.

5. Appendix: Commonly used notation

We provide here a list of notation used throughout the paper. For each symbol, we give the subsection where it was first introduced and we provide a brief description. The reader is referred to the appropriate subsection for more information and explicit definitions.

Defined in Subsection 2.1:

- $(V,\langle \cdot,\cdot \rangle)$ finite dimensional Euclidean space
- $\hat{V}$ space of affine linear real functions on $V$
- $c \in \hat{V}$ constant function one
- $D$ the gradient map from $\hat{V}$ to $V$
- $f^\vee$ co-root $2f/\|f\|^2$
- $R$ finite reduced irreducible root system contained in $V$
- $s_\alpha$ orthogonal reflection in the hyperplane $\alpha^\perp \subset V$
- $W_0$ Weyl group for $R$ generated by the $s_\alpha, \alpha \in R$
- $Q = Q(R)$ root lattice of $R$
- $P = P(R)$ weight lattice of $R$
- $Q^\vee = Q(R^\vee)$ co-root lattice of $R$
- $P^\vee = P(R^\vee)$ co-weight lattice of $R$
- $W_{Q^\vee}$ affine Weyl group of $R$
- $W_{P^\vee}$ extended affine Weyl group of $R$
- $t(\lambda), \lambda \in P^\vee$ translation sending $v$ to $v + \lambda$, for $v \in V$
- $S(R)$ affine root system $\{\alpha + r\epsilon | \alpha \in R, \epsilon \in \mathbb{Z}\}$
- $S(R)^\vee$ dual affine root system $\{f^\vee | f \in S(R)\}$
- $S_{nr}$ roots of the nonreduced affine root system of type $C^nC_n$
- Cases $a,b,c$ Cherednik-Macdonald theory cases
- $(R,R')$ pair of root systems defined for each case
- $(L,L')$ pair of lattices defined for each case
- $(S,S')$ pair of irreducible affine root systems defined for each case
- $(W,W')$ extended Weyl groups associated to $(R,R')$
- $S_s$ set of short roots of $S$
- $S_l$ set of long roots of $S$
- $O_i, i = 1, \cdots, 5$ $W$-orbits of $S$ for case $c$
- $S_1$ reduced affine root subsystem of indivisible affine roots in $S$
- $\phi$ highest root of $R$ with respect to above basis
- $\{a_0,a_1,\ldots,a_n\}$ basis for $S$ (given in this section)
- $\Delta = \{a_1,\ldots,a_n\}$
- $R^\vee$ $D(S_1)$
- $s_i$ reflection associated to the simple root $a_i$
- $S_1^+$ positive affine roots of $S_1$ with respect to $\{a_0,\ldots,a_n\}$
- $S_1^-$ negative affine roots of $S_1$ with respect to $\{a_0,\ldots,a_n\}$
- $l(w), w \in W$ $\#(S_1^+ \cap w^{-1}S_1^-)$
- $\Omega \{w \in W | l(w) = 0\}$
- $\mathbb{C}[W]$ complex group algebra of $W$
\( \mathbb{C}[\Omega] \) complex group algebra of \( \Omega \)

Defined in Subsection 2.2

- \( A \) group algebra \( \mathbb{C}[L] \) with basis \( \{ z^\lambda \mid \lambda \in L \} \)
- \( A' \) group algebra \( \mathbb{C}[L'] \) with basis \( \{ z'^{\lambda'} \mid \lambda' \in L' \} \)
- \( V_C \) complexification of \( V \)
- \( Q \) quotient field of \( A \)
- \( Q' \) quotient field of \( A' \)
- \( \mathcal{R} \) subalgebra of \( Q \) defined by Definition 2.1
- \( \mathcal{D}_R(L') \) smash-product algebra \( \mathcal{R}\#(L') \)
- \( \mathcal{D}_R(W) \) smash-product algebra \( \mathcal{R}\#W \)
- \( X \) lattice in \( V \)
- \( \mathcal{D}_R(X) \) smash-product algebra \( \mathcal{R}\#(X) \)
- \( \mathbb{C}[[z^{-\Delta}]] \) algebra of formal power series \( \{2.3\} \)
- \( \mathcal{D}_{\mathbb{C}[[z^{-\Delta}]]}(X) \) smash-product algebra \( \mathbb{C}[[z^{-\Delta}]]\#(X) \) for lattices \( X \subset V \)
- \( M \) vector space spanned by elements of the form \( \{2.1\} \)

Defined in Subsection 2.3

- \( \mathbb{C}(S)^W \) space of multiplicity labels associated to \( S \)
- \( k \) multiplicity label associated to \( S \)
- \( k_a \) \( k \) map associated to \( a \)
- \( k_i \) \( k \) map associated to \( i \)
- \( k' \) multiplicity label associated to \( S' \) and dual to \( k \)
- \( \kappa_j \) value of \( k \) (resp. \( k' \)) at the \( W \)-orbit \( O_j \) for case \( c \)
- \( \mathbb{C}(S_i)^W \) space of multiplicity labels associated to \( S_i \)
- \( \xi, \xi' \) invertible multiplicity labels of \( S_1 \) associated to \( k \)
- \( H(\mathcal{Z}) \) extended affine Hecke algebra (Definition 2.4)
- \( T_i, i = 0, \ldots, n \) generators for \( H(\mathcal{Z}) \)
- \( T_w \) \( \omega T_{i_1} T_{i_2} \cdots T_{i_{|w|}} \) for reduced expression \( w = \omega s_{i_1} s_{i_2} \cdots s_{i_{|w|}} \)
- \( c_\mathcal{Z}(z) \) \( k \) map given by \( \{2.3\} \)
- \( \pi_k \) embedding of \( H(\mathcal{Z}) \) inside \( \mathcal{D}_R(W) \)
- \( T_i(\mathcal{Z}) \) difference reflection operator \( \pi_k(T_i) \)
- \( \beta \) linear map given by \( \beta(\sum_{w \in W_0} D_w w) = \sum_{w \in W} D_w \)
- \( H_0 = H_0(\mathcal{Z}) \) finite Hecke algebra generated by \( T_j, j = 1, \ldots, n \)
- \( Z(H(\mathcal{Z})) \) center of \( H(\mathcal{Z}) \)

Defined in Subsection 2.4

- \( V^+ \) open dominant Weyl chamber in \( V \) with respect to \( R^+ \)
- \( V_+ \) closure of \( V^+ \) in \( V \)
- \( L^++ \) \( L \cap V_+ \)
- \( L^+' \) \( L' \cap V_+ \)
- \( Y^{\lambda'}, \lambda' \in L^++ \) \( T_{i(\lambda')}^{-1} \) for \( \lambda' = \mu' - \nu' \) with \( \mu', \nu' \in L'^++ \)
- \( Y^{\lambda'}, \lambda' \in L' \) \( Y^{\mu'}(Y^{\nu'})^{-1} \) for \( \lambda' = \mu' - \nu' \) with \( \mu', \nu' \in L' \)
- \( A'(Y) \) subalgebra of \( H(\mathcal{Z}) \) spanned by the \( Y^{\lambda'}, \lambda' \in L' \)
- \( A'_0 \) algebra of \( W_0 \)-invariant elements in \( A' \)
- \( A'_0(Y) \) algebra corresponding to \( A'_0 \) via canonical isomorphism \( A' \cong A'(Y) \)
- \( m_{\lambda'}(\xi), \lambda' \in L' \) monomial symmetric function \( \sum_{\mu' \in W_0 \lambda'} \xi^{\mu'} \) in \( A'_0 \)
\[
m_\lambda(Y)
\]
\[
D_{\rho_0'}, D_{\rho_0''}
\]
\[
\rho_0', \rho_0''
\]
\[
\overline{\rho}(\xi), p(\xi) \in A'
\]
\[
\pi_{\minuscule}
\]
\[
\pi_{\trivial idempotent in L}^D
\]
\[
\pi_{\fundamental co-weights}
\]
\[
\pi_i
\]
\[
w_0
\]
\[
J_0
\]
\[
W_{0,\lambda}
\]
\[
W_{0,\lambda'}^v
\]
\[
S_i(w)
\]
\[
m_{\lambda}\left(-\rho_0''\right)
\]
\[
f_{\pi'}(z)
\]

Defined in Subsection \[2.5\]
\[
V_-
\]
\[
\gamma(\bar{k})
\]

Harish-Chandra homomorphism \[2.12\]

Defined in Subsection \[3.2\]
\[
\pi(D), D \in \mathbb{D}_R(L')
\]
\[
e
\]

trivial idempotent in \[\mathbb{C}[W_0]\]

Defined in Subsection \[3.3\]
\[
V_{-F}
\]
\[
R_{F'}^F
\]
\[
R_{F'}^{n,+} \text{ (resp. } R_{F'}^{n,-})
\]
\[
W_{0,F}
\]
\[
R_F
\]
\[
R_F^+ \text{ and } R_F^-
\]
\[
\rho_{\lambda,F}
\]
\[
S_F
\]
\[
\mathcal{R}_F
\]
\[
\mathbb{C}[z^{-F}]
\]
\[
\mathbb{C}[z^{-F}]_t(X)
\]
\[
\mathbb{C}[z^{-F}]_t(X)
\]
\[
\gamma_F(\bar{k})
\]
\[
L'_i
\]
\[
X_i
\]
\[
L_i
\]
\[
y_i, z_i
\]
\[
D_{\pi'}(\bar{k})
\]

constant term map along \[V_{-F} \text{ (Definition } 3.3\]

in cases \textbf{a}, \textbf{b}; \[L' \text{ in case } \textbf{c}\]

\[
\{\mu' \in L'_i \mid \langle \mu', \alpha_i \rangle = 0\}
\]

\[
\text{rank one difference operator defined in Definition } 3.11
\]

\[
\text{elements of } \mathbb{C}[X_i] \text{ defined by } 3.16
\]

\[
\text{centralizer of } \gamma(a_i)(\bar{k})(D_{\pi'}) \text{ in } \mathbb{D}_{\mathcal{R}_{\langle a_i \rangle}}(L'_i)^{W_{0,\langle a_i \rangle}}
\]

Defined in Subsection \[4.1\]
\[
\Phi(z, \xi)
\]

\text{difference analogue of the Harish-Chandra series (see } 4.2\)
Defined in Subsection 4.2:

\( \sigma \) positive real number such that \( q = e^{-2\pi\sigma} \)

\( Z \)

\( \{ \lambda \in V \mid \langle \lambda', \lambda \rangle \in Z \quad \forall \lambda' \in L' \} \)

\( w \cdot \lambda \)

\( w(\lambda + \rho_L') - \rho_L' \)

\( D_k \)

\( \bigcup_{w \in W_0} \{ \lambda \in V_C \mid \lambda - w \cdot \lambda \in Z_{\geq 0} \Delta \setminus \{0\} + \sqrt{-1}Z/\sigma \} \)

\( \mathcal{W}_\lambda \)

\( \lambda \in V_C \setminus D_k \) elements of \( Q' \) with singularities in \( D_k \)

\( \iota_\lambda \), \( \lambda \in V_C \setminus D_k \) specialization map (Lemma 4.5)

\( \Phi_\lambda(z) \)

\( \iota_\lambda(\Phi(z, \xi)) \in \mathcal{M} \) (see Theorem 4.6)

\( M_\lambda \)

\( \{ F(z) \in \mathcal{M} \mid D_p(F(z)) = \bar{p}(\lambda)F(z) \quad \forall p(\xi) \in A_0' \} \)

Defined in Subsection 4.3:

\( P_\lambda(z) \) Macdonald polynomial of degree \( \lambda \in L_+ \) (Definition 4.9)

Defined in Subsection 4.4:

\( \rho'_k \)

\( \frac{1}{2} \sum_{\alpha \in R^+} k_\alpha \alpha^\vee \)

\( \psi_\lambda(z) \) Baker-Akhiezer function (Definition 4.12)

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