Small $\ell$-edge-covers in $k$-connected graphs

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Abstract. Let $G = (V, E)$ be a $k$-edge-connected graph with edge costs $\{c(e) : e \in E\}$ and let $1 \leq \ell \leq k - 1$. We show by a simple and short proof, that $G$ contains an $\ell$-edge cover $I$ such that: $c(I) \leq \frac{\ell}{k} c(E)$ if $G$ is bipartite, or if $\ell|V|$ is even, or if $|E| \geq \frac{k|V|}{2} + \frac{k}{\ell}$; otherwise, $c(I) \leq \left(\frac{\ell}{k} + \frac{1}{\ell |V|}\right) c(E)$. The particular case $\ell = k - 1$ and unit costs already includes a result of Cheriyan and Thurimella [1], that $G$ contains a $(k - 1)$-edge-cover of size $|E| - \lfloor |V|/2 \rfloor$. Using our result, we slightly improve the approximation ratios for the $k$-Connected Subgraph problem (the node-connectivity version) with uniform and $\beta$-metric costs.

1 Introduction

Let $G = (V, E)$ be an undirected graph, possibly with parallel edges. For $S \subseteq V$ let $\delta(S)$ denote the set of edges in $E$ with exactly one endnode in $S$. Let $n = |V|$. An edge set $I \subseteq E$ is an $\ell$-edge-cover (of $V$) if the graph $(V, I)$ has minimum degree $\geq \ell$. For $x \in \mathbb{R}^E$ and $F \subseteq E$ let $x(F) = \sum_{e \in F} x(e)$. Let $P_{\text{cov}}^f(G, \ell)$ denote the fractional $\ell$-edge-cover polytope determined by the linear constraints

\begin{align*}
x(\delta(v)) &\geq \ell \quad v \in V \quad (1) \\
1 &\geq x_e \geq 0 \quad e \in E
\end{align*}

Let $P_{\text{cov}}(G, \ell)$ denote the integral $\ell$-edge-cover polytope, which is the convex hull of the characteristic vectors of of the $\ell$-edge-covers in $G$. It is known that if $G$ is bipartite then $P_{\text{cov}}^f(G, \ell) = P_{\text{cov}}(G, \ell)$ (see [5], (31.7) on page 340). This implies the following.

Proposition 1. Let $G = (V, E)$ be a bipartite graph and let $1 \leq \ell \leq k - 1$. Let $x \in P_{\text{cov}}(G, k)$. Then $\frac{1}{\ell} x \in P_{\text{cov}}(G, \ell)$.

Corollary 1. Let $G = (V, E)$ be a bipartite graph with edge costs $\{c(e) : e \in E\}$ and minimum degree $\geq k \geq 2$. Then for any $1 \leq \ell \leq k - 1$ $G$ contains an $\ell$-edge cover $I \subseteq E$ of cost $c(I) \leq \frac{\ell}{k} c(E)$.
Cheriyan and Thurimella [1] showed that if $G$ is bipartite and has minimum degree $\geq k$, then $G$ contains a $(k-1)$-edge-cover $I$ such that $|I| \leq |E| - n/2$. Note that this bound follows from Corollary [1] by assuming unit costs, substituting $\ell = k - 1$, and observing that $|E| \geq \frac{kn}{2}$. Unfortunately, Corollary [1] does not extend to the general (non-bipartite) case, e.g., if $G$ is a cycle of length 3, $k = 2$, and $\ell = 1$. On the positive side, it is proved in [2] that if $G$ has minimum degree $\geq k$ then $G$ contains a $(k-1)$-edge-cover $I$ of cost $c(I) \leq \frac{2k-2}{k} c(E)$. Let $\zeta(S)$ denote the set of edges in $E$ with at least one endnode in $S$. It is known that in the general case, $P_{\text{con}}(G,\ell)$ is determined by adding to the constraints of $P_{\text{fcon}}(G,\ell)$ the following inequalities (see [3], page 581, Theorem 34.13)

$$x(\zeta(S) \setminus F) \geq \frac{\ell|S|}{2} - \frac{|F| - 1}{2} \quad S \subseteq V, F \subseteq \delta(S), \ell|S| - |F| \geq 1 \text{ odd}. \quad (2)$$

A graph $G$ is $k$-edge-connected if $|\delta(S)| \geq k$ for all $\emptyset \neq S \subset V$. Cheriyan and Thurimella [1] showed that if $G$ is $k$-edge-connected, then $G$ contains a $(k-1)$-edge-cover $I$ such that $|I| \leq |E| - \lfloor n/2 \rfloor$. We present an analogue of Proposition [1] and Corollary [1] for general graphs, with simple and short proof, that also implies this bound of [1]. Let $P_{\text{fcon}}(G,k)$ denote the fractional $k$-edge-connectivity polytope, determined by

$$x(\delta(S)) \geq k \quad \emptyset \neq S \subset V$$

$$1 \geq x_e \geq 0 \quad e \in E$$

Note that $P_{\text{fcon}}(G,k) \subseteq P_{\text{con}}(G,k)$, and that if $x \in P_{\text{fcon}}(G,k)$ then $x(E) \geq \frac{kn}{2}$.

The main result of this paper is the following analogue of Proposition [1].

**Theorem 1.** Let $G = (V,E)$ be a graph, let $1 \leq \ell \leq k - 1$, and let $x \in P_{\text{fcon}}(G,k)$. Then $\frac{\ell}{k} x \in P_{\text{con}}(G,\ell)$ if $\ell |V|$ is even or if $x(E) \geq \frac{k|V|}{2} + \frac{k}{2}$; otherwise, $\frac{\ell|V|+1}{2x(E)} \cdot x \in P_{\text{con}}(G,\ell)$, and hence also $\left(\frac{\ell}{k} + \frac{1}{k|V|}\right) \cdot x \in P_{\text{con}}(G,\ell)$.

Theorem [1] immediately implies the following.

**Corollary 2.** Let $G = (V,E)$ be a $k$-edge-connected graph with edge costs $\{c(e) : e \in E\}$ and let $1 \leq \ell \leq k - 1$. Then $G$ contains an $\ell$-edge cover $I \subseteq E$ such that:

$$c(I) \leq \frac{\ell}{k} c(E) \text{ if } \ell |V| \text{ is even or if } |E| \geq \frac{k|V|}{2} + \frac{k}{2}; \text{ otherwise, } c(I) \leq \frac{\ell|V|+1}{2x(E)} c(E) \leq \left(\frac{\ell}{k} + \frac{1}{k|V|}\right) c(E).$$

Note that the bound $|I| \leq |E| - \lfloor n/2 \rfloor$ of Cheriyan and Thurimella [1] follows from Corollary [2] by assuming unit costs, substituting $\ell = k - 1$, and observing that $|E| \geq \frac{kn}{2}$. Indeed, by Corollary [2] $|E| - |I| \geq |E|/k \geq n/2$ if $(k-1)n$ is even or if $|E| \geq \frac{kn}{2} + 1$. Otherwise, $k$ is even, $n$ is odd, $|E| = \frac{kn}{2}$, and then, by Corollary [2] $|E| - |I| \geq \frac{n-1}{kn} |E| = \frac{n-1}{\frac{kn}{2}} = \lfloor n/2 \rfloor$.

We now discuss some applications of Corollaries [1] and [2] for both directed and undirected graphs, for the following classic NP-hard problem. A (simple) directed or undirected graph is $k$-connected if it contains $k$ internally disjoint paths from every node to the other.
**k-Connected Subgraph**

**Instance:** A graph $G' = (V, E')$ with edge costs and an integer $k$.

**Objective:** Find a minimum cost $k$-connected spanning subgraph $G$ of $G'$.

The case of unit costs is the Minimum Size $k$-Connected Subgraph problem. Cheriyan and Thurimella [1] suggested and analyzed the following algorithm for the Minimum Size $k$-Connected Subgraph problem, for both directed and undirected graphs; in the case of a directed graph $G = (V, E)$, we say that $I \subseteq E$ is an $\ell$-edge-cover if $(V, I)$ has minimum outdegree and minimum indegree $\geq \ell$.

**Algorithm 1**

1. Find a minimum size $(k-1)$-edge cover $I \subseteq E$.
2. Find an inclusion minimal edge set $F \subseteq E \setminus I$ such that $(V, I \cup F)$ is $k$-connected.
3. Return $I \cup F$.

They showed that this algorithm has approximation ratios

- $1 + \frac{1}{\text{opt}} \leq 1 + \frac{1}{k}$ for directed graphs;
- $1 + \frac{n}{2\text{opt}} \leq 1 + \frac{1}{k}$ for undirected graphs.

Here \text{opt} denotes the optimum solution value of a problem instance at hand. Step 1 in the algorithm can be implemented in polynomial time, c.f. [5]. Recently, the performance of this algorithm was also analyzed in [2] for so called $\beta$-metric costs, when the input graph is complete and for some $1/2 \leq \beta < 1$ the costs satisfy the $\beta$-triangle inequality $c(uv) \leq \beta[c(ua) + c(au)]$ for all $u, a, v \in V$. When $\beta = 1/2$, the costs are uniform, and we have the min-size version of the problem. If we allow the case $\beta = 1$, then the costs satisfy the ordinary triangle inequality and we have the metric version of the problem. In [2] it is shown that for undirected graphs with $\beta$-metric costs the above algorithm has ratio $1 - \frac{1}{2k-1} + \frac{2\beta}{k(1-\beta)}$. We prove the following.

**Theorem 2.** (i) For the Minimum Size $k$-Connected Subgraph problem, Algorithm 1 has approximation ratios

- $1 - \frac{1}{k} + 2n/\text{opt} \leq 1 + \frac{n}{\text{opt}}$ for directed graphs;
- $1 - \frac{1}{k} + n/\text{opt} \leq 1 + \frac{n}{2\text{opt}}$ for undirected graphs.

(ii) In the case of undirected graphs and $\beta$-metric costs, Algorithm 1 has approximation ratio $1 - \frac{1}{k} + \frac{1}{kn} + \frac{2\beta}{k(1-\beta)}$.

(iii) There exists a polynomial time algorithm that given an instance of the Minimum Size $k$-Connected Subgraph problem returns a $(k-1)$-connected spanning subgraph $G$ of $G'$ with at most $\text{opt}$ edges.

Note that in part (i) of Theorem 2 we do not improve the worse performance guarantee $1 + \frac{1}{k}$ of [1]. However, the ratio $1 + \frac{1}{k}$ applies only if $\text{opt} = kn$ in the case of directed graphs and $\text{opt} = kn/2$ in the case of undirected graphs. Otherwise, if $\text{opt}$ is larger than these minimum possible values, then both our analysis and
that of [1] give better ratios. But the ratios provided by our analysis are smaller, since \( 2n/\text{opt} - \frac{\ell}{k} \leq n/\text{opt} \) in the case of directed graphs, and \( n/\text{opt} - \frac{1}{k} \leq n/2\text{opt} \) in the case of undirected graphs. For example, in the case of directed graphs, if \( \text{opt} = \frac{4}{3}kn \) then our ratio is \( 1 + \frac{1}{3\ell} \), while that of [1] is \( 1 + \frac{2}{3\ell} \).

Part (iii) of Theorem 2 can be used to obtain a tight approximation algorithm for the Maximum Connectivity \( m \)-Edge Subgraph problem: given a graph \( G' \) and an integer \( m \), find a spanning subgraph \( G \) of \( G' \) with at most \( m \) edges and maximum connectivity \( k^* \). We can apply the algorithm in part (iii) to find the maximum integer \( k \) for which the algorithm returns a subgraph with at most \( m \) edges. Then \( k \geq k^* - 1 \), hence we obtain a polynomial time algorithm that computes a \((k^* - 1)\)-connected spanning subgraph with at most \( m \) edges. Note that this is tight, since the problem is NP-hard.

## 2 Proof of Theorem 1

Let \( x \in P_{\text{con}}^\ell(G, k) \) and let \( S \subseteq V \). It is clear that inequalities (1) are “scalable” by \( \frac{\ell}{k} \), namely, \( \frac{\ell}{k}x(\delta(S)) \geq \ell \). We show that inequalities (2) are also “scalable” by a factor of \( \mu \) defined as follows: \( \mu = \frac{\ell}{k} \) if \( \ell n \) is even or if \( x(E) \geq \frac{k\ell}{2} + \frac{k}{\ell} \), and \( \mu = \frac{\ell n + 1}{2\ell(k)} \) otherwise. Let \( F \subseteq \delta(S) \) such that \( \ell |S| - |F| \geq 1 \) is odd. We prove that then the following holds:

\[
\mu(x(\zeta(S)) - x(F)) \geq \frac{\ell|S|}{2} - \frac{|F| - 1}{2}.
\]

If \( S = V \) then \( \zeta(S) = E \) and \( F = \emptyset \). Then (3) reduces to a void condition if \( \ell |V| \) is even, and to the condition \( \mu x(E) \geq \frac{\ell n + 1}{2} \) otherwise, which holds by the definition of \( \mu \).

Henceforth assume that \( S \) is a proper subset of \( V \). We prove that then

\[
\frac{\ell}{k}(x(\zeta(S)) - x(F)) \geq \frac{\ell|S|}{2} - \frac{|F| - 1}{2}.
\]

Multiplying both sides of (4) by \( \frac{k}{\ell} \) gives

\[
x(\zeta(S)) - x(F) \geq \frac{k|S|}{2} - \frac{k}{\ell} \cdot \frac{|F| - 1}{2}.
\]

Note that \( x(\zeta(S)) \geq \frac{k|S|}{2} + \frac{x(\delta(S))}{2} \) and that \( x(F) \leq |F| \). Substituting and rearranging terms, we obtain that it is sufficient to prove that if \( x(\delta(S)) \geq k \geq \ell + 1 \geq 0 \), then

\[
x(\delta(S)) - x(F) + \frac{k - \ell}{\ell} |F| \geq \frac{k}{\ell} \quad \emptyset \neq S \subseteq V.
\]

If \( |F| \geq \frac{k}{\ell - \ell} \) then (6) holds, since \( x(\delta(S)) \geq x(F) \). Assume that \( |F| < \frac{k}{\ell - \ell} \). Then

\[
x(\delta(S)) - x(F) + \frac{k - \ell}{\ell} |F| \geq k - |F| + \frac{k - \ell}{\ell} |F| = k + \frac{k - 2\ell}{\ell} |F| \geq \frac{k}{\ell}.
\]
We explain the last inequality. If \( k \geq 2\ell \) then \( k + \frac{k-2\ell}{k-\ell}|F| \geq k \geq \frac{k}{2\ell}. \) If \( k < 2\ell \) then since \( |F| < \frac{k}{k-\ell} \)

\[
k + \frac{k-2\ell}{\ell}|F| > k + \frac{k-2\ell}{k-\ell} \cdot \frac{k}{k-\ell} = k + \frac{k}{k-\ell} - \frac{k}{k-\ell} \geq \frac{k}{2}.
\]

In both cases (3) holds, and hence the proof of Theorem 1 is complete.

3 Proof of Theorem 2

Let \( I \) and \( F \) denote the set of edges computed by Algorithm 1 at steps 1 and 2, respectively. We prove part (i), starting with the case of directed graphs. For a directed graph \( G \), the corresponding bipartite graph \( G' = (V \cup V', E') \) is obtained by adding a copy \( V' \) of \( V \) and replacing every directed edge \( uv \in E \) by the undirected edge \( uv' \), where \( v' \in V' \) is the copy of \( v \). It is not hard to verify that \( I \) is an \( \ell \)-edge-cover in \( G \) if, and only if, the set \( I' \) of edges that corresponds to \( I \) is an \( \ell \)-edge-cover in \( G' \). Thus \( |I| \leq \frac{k-1}{k-\ell} \text{opt, by Corollary 1} \)

On the other hand, by the directed Critical Cycle Theorem of Mader [4] (see [1] for details), the set of edges of \( G' \) that corresponds to \( F' \) forms a forest in \( G' \), hence \( |F| \leq 2n-1 \). Consequently, \( \frac{|I|+|F|}{\text{opt}} \leq 1 - \frac{1}{k} + \frac{n-1}{\text{opt}} \).

Let us consider undirected graphs. If \((k-1)n\) is even or if \( \text{opt} \geq \frac{kn}{2} + \frac{k}{2(k-1)} \geq \frac{kn}{2} + 1 \), then \( |I| \leq \frac{k-1}{k} \text{opt, by Corollary 2} \).

By the undirected Critical Cycle Theorem of Mader [3] (see [1] for details), \( F \) is a forest, hence \( |F| \leq n-1 \). Consequently, \( \frac{|I|+|F|}{\text{opt}} \leq 1 - \frac{1}{k} + \frac{n-1}{\text{opt}} \).

If \((k-1)n\) is odd and \( \text{opt} < \frac{kn}{2} + 1 \), then an optimal solution is \( k \)-regular and hence \( |I| \leq \frac{(k-1)n+1}{2} \leq (1 - \frac{1}{k})(\text{opt} + 1) \).

Combining we get \( \frac{|I|+|F|}{\text{opt}} \leq 1 - \frac{1}{k} + \frac{1-1/k}{\text{opt}} + \frac{n-1}{\text{opt}} < 1 - \frac{1}{k} + \frac{n}{\text{opt}} \).

Now let us consider part (ii), the case of \( \beta \)-metric costs. In [2] it is proved that \( c(F) \leq \frac{2\beta}{k(1-\beta)} \text{opt} \).

If \((k-1)n\) is even, or if there exists an optimal solution with at least \( \frac{kn}{2} + \frac{k}{2(k-1)} \leq \frac{kn}{2} + 1 \) edges, then Corollary 2 gives the bound \( c(I) \leq (1 - \frac{1}{k}) \text{opt} \). Else, Corollary 2 gives the bound \( c(I) \leq (1 - \frac{1}{k} + \frac{1}{km}) \text{opt} \), and the result follows.

We prove part (iii). We apply Algorithm 1 with \( k \) replaced by \( k-1 \), namely, \( I \subseteq E \) is a minimum size \((k-2)\)-edge cover and \( F \subseteq E \setminus I \) is an inclusion minimal edge set such that \((V,I \cup F)\) is \((k-1)\)-connected. Now we use the bounds in Corollary 2.

In the case of directed graphs we have \( |I| \leq \frac{k-2}{k} \text{opt}, |F| \leq 2n-1 \leq \frac{2}{\ell} \text{opt}, \) and the result follows. In the case of undirected graphs we have \( |I| \leq \left( \frac{k-2}{k} + \frac{1}{km} \right) \text{opt} \) and \( |F| \leq n-1 \leq \left( \frac{2}{k} - \frac{1}{km} \right) \text{opt} \), and the result follows.

The proof of Theorem 2 is complete.

References

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