ON INTEGRAL ZARISKI DECOMPOSITIONS OF PSEUDEFFECTIVE DIVISORS ON ALGEBRAIC SURFACES

B. HARBOURNE, P. POKORA, AND H. TUTAJ-GASIŃSKA

Abstract. In this note we consider the problem of integrality of Zariski decompositions for pseudoeffective integral divisors on algebraic surfaces. We show that while sometimes integrality of Zariski decompositions forces all negative curves to be $(-1)$-curves, there are examples where this is not true.

1. Introduction

In this note we work over an arbitrary algebraically closed field $K$. By a negative curve, we mean a reduced irreducible divisor $C$ with $C^2 < 0$ on a smooth projective surface. By a $(-k)$-curve, we mean a negative curve $C$ with $C^2 = -k < 0$.

In [1] the first author with Th. Bauer and D. Schmitz studied the following problem for algebraic surfaces.

Question. Let $X$ be a smooth projective surface. Does there exist an integer $d(X) \geq 1$ such that for every pseudoeffective integral divisor $D$ the denominators in the Zariski decomposition of $D$ are bounded from above by $d(X)$?

Such a question is natural when one studies Zariski decompositions [9] of pseudoeffective divisors since we have the following geometric interpretation. Given a pseudo-effective integral divisor $D$ on $X$ with Zariski decomposition $D = P + N$, then for every sufficiently divisible integer $m \geq 1$ we have the equality

$$H^0(X, \mathcal{O}_X(mD)) = H^0(X, \mathcal{O}_X(mP)),$$

which means that all sections of $\mathcal{O}_X(mD)$ come from the nef line bundle $\mathcal{O}_X(mP)$. Sufficiently divisible is required in order to clear denominators in $P$ and obtain Cartier divisors.

If such a bound $d(X)$ exists, then we say that $X$ has bounded Zariski denominators. It is an intriguing question as to whether a given smooth surface satisfies this boundedness condition. For example, it was shown in [1] that, somewhat surprisingly, boundedness of Zariski denominators on a smooth projective surface $X$ is equivalent to $X$ having bounded negativity (i.e., that there exists a number $b(X) \in \mathbb{Z}$ such that $C^2 \geq -b(X)$ for every negative curve $C$). Bounded negativity has connections to substantial open conjectures. For example, for a surface $X$ obtained by blowing up $\mathbb{P}^2$ at any finite set of generic points, the Segre-Harbourne-Gimigliano-Hirschowitz Conjecture (i.e., the SHGH Conjecture) [6] asserts that $h^1(X, \mathcal{O}_X(F)) = 0$ for every effective nef divisor $F$ and in addition that all negative curves on $X$ are $(-1)$-curves. The Bounded Negativity Conjecture (BNC) is another even older still open conjecture which asserts that smooth complex projective surfaces all have bounded negativity. Thus the equivalence of boundedness of Zariski denominators and bounded negativity provides a new perspective on these conjectures and also sheds some light on links between numerical information about divisors on a given surface $X$ and the possible negative curves on $X$.

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An interesting criterion for surfaces to have bounded Zariski denominators was given in [1], as follows, where we say that a pseudoeffective integral divisor $D$ has an integral Zariski decomposition $D = P + N$ if $P$ and $N$ are defined over the integers (i.e., all coefficients occurring in $P, N$ are integers).

**Proposition 1.1.** Let $X$ be a smooth projective surface such that for every reduced and irreducible curve $C$ one has $C^2 \geq -1$. Then all integral pseudoeffective divisors on $X$ have integral Zariski decompositions.

This raises the converse question:

**Question 1.2.** Let $X$ be a smooth projective surface having the property that

\[ (*) \quad \text{every integral pseudoeffective divisor } D \text{ has an integral Zariski decomposition.} \]

Is every negative curve then a $(-1)$-curve?

The condition $(*)$ at first glance seems to be very constraining, so it is plausible that Question 1.2 could have an affirmative answer. However, by our main result we see that the answer is negative.

**Theorem A.** There exists a smooth complex projective surface $X$ having the property that all integral pseudoeffective divisors have integral Zariski decompositions yet all negative curves on $X$ have self-intersection $-2$.

On the other hand, sometimes the answer is affirmative:

**Theorem B.** Let $X$ be a smooth projective surface such that every integral pseudoeffective divisor $D$ has an integral Zariski decomposition (i.e., $d(X) = 1$) and such that $|\Delta(X)| = 1$, where $\Delta(X)$ is the determinant of the intersection form on the Néron-Severi lattice of $X$. Then all negative curves on $X$ are $(-1)$-curves (i.e., $b(X) = 1$).

This follows from [1, Theorem 2.3], which gives the bound $b(X) \leq d(X)!|\Delta(X)|$. Thus, for example, if $X$ is a blow up of $\mathbb{P}^2$ at a finite set of points, then $|\Delta(X)| = 1$, so if Zariski decompositions are integral on $X$, then $d(X)$ is also 1, hence $b(X) = 1$. Because of the recent interest in blow ups of $\mathbb{P}^2$ at finite sets of points (see, for example, [3, 2, 4]), a direct proof in the special case of blow ups of $\mathbb{P}^2$ may be useful. We provide such a proof below.

2. Results

Before we present the main result of this note let us recall the definition of Zariski decompositions.

**Definition 2.1** (Fujita-Zariski decomposition [5, 9]). Let $X$ be a smooth projective surface and $D$ a pseudo-effective integral divisor on $X$. Then $D$ can be written uniquely as a sum $D = P + N$ of $\mathbb{Q}$-divisors such that

(i) $P$ is nef,
(ii) $N$ is effective with negative definite intersection matrix if $N \neq 0$, and
(iii) $P \cdot C = 0$ for every component $C$ of $N$.

Now we are ready to produce the surface whose existence is asserted in Theorem A.

**Theorem 2.2.** Let $X$ be a smooth projective $K3$ surface of Picard number 2 having intersection form

\[
\begin{pmatrix}
-2 & 4 \\
4 & -2
\end{pmatrix}
\]

Then all integral pseudoeffective divisors on $X$ have integral Zariski decompositions.
Proof. The existence of such a surface $X$ is a consequence of \cite{[8]} (2.9i), (2.11)] (see also \cite{[7]} Corollary 1.4). Let $C_1$ and $C_2$ be the negative curves giving a basis for the Néron-Severi group with $C_1^2 = C_2^2 = -2$ and $C_1 \cdot C_2 = 4$. Every divisor $D$, up to numerical equivalence, is of the form $mC_1 + nC_2$ for integers $m$ and $n$. If $m < 0$ and $n \leq 0$ (or vice versa), then clearly $D$ is not effective. If $m < 0$ but $n > 0$, then $D \cdot C_2 < 0$, so $D$ is effective if and only if $D - C_2$ is, and continuing this way we see that $D$ is effective if and only if $mC_1$ is, but $m < 0$, so $mC_1$ is not effective. Thus $D$ is effective if and only if $m \geq 0$ and $n \geq 0$, and hence $D$ is effective if and only if it is pseudoeffective. If $D$ were a negative curve (hence effective) but $D \neq C_1$ and $D \neq C_2$, then $D \cdot C_1 \geq 0$ and $D \cdot C_2 \geq 0$, so $D^2 \geq 0$, contrary to assumption. Thus $C_1$ and $C_2$ are the only negative curves. Thus $D$ is nef if and only if $D \cdot C_i \geq 0$ for $i = 1, 2$; i.e., if and only if $2m \geq n$ and $2n \geq m$. It is not hard to check that this holds if and only if $D$ is a nonnegative integer linear combination of $C_1 + C_2$, $2C_1 + C_2$ and $C_1 + 2C_2$.

So say $D$ is pseudoeffective and integral; i.e., $D = mC_1 + nC_2$ for $m, n \geq 0$. By symmetry, it is enough to assume that $m \geq n$. If $2n \geq m$, then $D$ is nef and the Zariski decomposition $D = P + N$ of $D$ integral since $P = D$ and $N = 0$. Now assume $m > 2n$. Take $P = n(2C_1 + C_2)$ and $N = (m - 2n)C_1$. Then $P$ is nef, $N$ clearly has negative definite intersection matrix and $P \cdot N = 0$, so $D = P + N$ is again an integral Zariski decomposition of $D$. □

Now we provide a proof of Theorem B in the special case mentioned above.

**Theorem B.** Let $\pi : X \to \mathbb{P}^2$ be the blow up of a finite set of points $p_1, \ldots, p_s$ (possibly infinitely near). Suppose that every integral pseudoeffective divisor $D$ has an integral Zariski decomposition. Then all negative curves on $X$ have self-intersection $-1$ (i.e., are $(-1)$-curves).

Proof. Denote $\pi^{-1}(p_i)$ by $E_i$ and the total transform of a line by $H$. We will consider two cases: in the first case we assume that none of the points $p_i$ is infinitely near another (so the points $p_i$ are distinct points of $\mathbb{P}^2$) and in the second case we define: $X_1$ to be the blow up of $X_0 = \mathbb{P}^2$ at any point $p_i \in X_0$; $X_2$ the blow up of $X_1$ at any point $p_2 \in X_1$; so $p_2$ can be infinitely near to $p_1$; etc.. Continuing in this way we eventually have that $X = X_s$ is the blow up of $X_{s-1}$ at any point $p_s \in X_{s-1}$. In order to avoid confusion, we indicate the exceptional curve for the blow up of $p_i \in X_{i-1}$ by $E_{i,i} \subset X_i$, and its total transform on $X_j$ for $j > i$ by $E_{i,j}$. For simplicity, we denote $E_{i,i} \subset X_i$. Thus $E_{i,i}$ is always irreducible, and $E_{i,j}$ is irreducible if and only if no point $p_i$ for $i < l < j$ is infinitely near to $p_i$.

We begin with case 1. Suppose to the contrary that $X$ has a $(-k)$-curve $C$ with $k > 1$. Since the classes of $H, E_1, \ldots, E_s$ give a basis for the divisor class group of $X$, up to linear equivalence we can write $C = dH - \sum_{j=1}^s b_{ij} E_{ij}$. Since none of the points $p_i$ is infinitely near another, the only negative curves with $d = 0$ are the $E_i$, and these are $(-1)$-curves, so we must have $d > 0$ and $b_{ij} > 0$ with $C^2 = d^2 - \sum_{j=1}^r b_{ij}^2 = -k$.

We claim that there is a big integrable divisor $D$ such that $D = P + aC$ with $a \in \mathbb{Q} \setminus \mathbb{Z}$. To see this, note that if $d' > 0$ and $a_i > 0$ are integers, then

$$A = d'H - \sum_{i=1}^s a_i E_i$$

will be an integral ample divisor. For $D$ we take $D = A + eC$ for $e \in \mathbb{Z}_{>0}$, where we choose a number $e$ such that $D \cdot C < 0$ so we have

$$0 > D \cdot C = (A + eC) \cdot C = A \cdot C - ek = d' d - \sum_{j=1}^r a_{ij} b_{ij} - ke.$$

Finding the Zariski decomposition of $D$ boils down to computing $a$. Observe that

$$a = \frac{dd' - \sum_{j=1}^r a_{ij} b_{ij} - ke}{-k} = e + \frac{\sum_{j=1}^r a_{ij} b_{ij} - dd'}{k}.$$

We just need to show that $k$ does not divide $\sum_{j} a_{ij} b_{ij} - dd'$. 

Suppose that $k$ divides $\sum_j a_j b_j - dd'$. Then we replace $A$ by $A + H$ so $d'$ becomes $d' + 1$. If $k$ does not divide $\sum_j a_j b_j - dd' - d$, then we are done. If $k$ divides this new number, then it means that $k/d$. In this case we replace $A$ instead by $A + H - E_i$. Since $H - E_i$ is nef, $A + H - E_i$ is ample, and we get $d' + 1$ in place of $d'$ and $a_i + 1$ in place of $a_i$. If $k$ does not divide the number $\sum_j a_j b_j + b_i - d(d' + 1)$, then we are done. If $k$ divides this number, then it means that $k|b_i$. We proceed along the same lines for all $j$; we are done unless $k|b_j$ for all $j$. But this is impossible because then $k^2$ would divide $d$ and each $b_j$, so $k^2$ would divide $k = -(d^2 - \sum_j b_j^2)$, where $k$ is an integer bigger than 1.

Now consider case 2. If none of the points is infinitely near another, then we are in case 1, so we may assume one the points $p_1, \ldots, p_s$ is infinitely near another. Let $p_i$ be the first such point, and let $p_j$ be the point $p_i$ is infinitely near to. Thus $p_1, \ldots, p_{j-1}$ are points of $\mathbb{P}^2$ and $p_j$ is on the exceptional locus $E_{i,j-1}$ of $p_i$ for some $i < j$. After reindexing, we may assume that $i = 1$ and $j = 2$. (The only constraint on reindexing is that if $p_v$ is infinitely near to $p_u$, then $v > u$.) On $X_2$, the curve $E_{1,2}$ has two irreducible components, $E_{1,2} = E + E_{2,2}$. Thus we have $E^2 = -2$.

Here we take $D = 3H - E_{1,2} - 2E_{2,2}$. Up to linear equivalence, we can write $2D = 6H - 2E_{1,2} - 4E_{2,2} = (6(H - E_{1,2}) + 4E) = (6H - E_{1,2}) + 3E$. Since $E$ is ample, we have $D$ is pseudo-effective (in fact it is linearly equivalent to an effective divisor). Since $H - E_{1,2}$ is nef and $(6H - E_{1,2}) + 3E$ is nef, we see that $D = P + N$ for $P = (6H - E_{1,2}) + 3E$ and $N = E_{2,2}$ is not integral. However, up to linear equivalence, $N = E = E_{1} - E_2$, and the total transforms of $E_1$ and $E_2$ under $\pi_2$ are in the span (in the divisor class group) of $E_1, \ldots, E_s$. (Just as the components of the total transform of $E_1$ under $X_2 \to X_1$ are $E_2$ and $E = E_1 - E_2$, every component of $E_i$ on $X$ is a linear integer combination of $E_1, \ldots, E_s$.)

Since $\pi_2^2(2N)$ is in the span of $E_1, \ldots, E_s$ and this span is negative definite, the intersection matrix of the components of $\pi_2^2(2N)$ is also negative definite. Since $(\pi_2^2(2N))/2 = N^2 = (E_2)^2 = -1/2$, we see that $\pi_2^2(2N)/2$ is not integral. Thus $\pi_2^2(D) = \pi_2^2(2P)/2 + \pi_2^2(2N)/2$ gives a nonintegral Zariski decomposition on $X$.

We end by posing the following problem.

**Problem 2.3.** Classify all algebraic surfaces with $d(X) = 1$.

**References**

[1] Th. Bauer, P. Pokora and D. Schmitz, On the boundedness of the denominators in the Zariski decomposition on surfaces, to appear, Journal für die reine und angewandte Mathematik.

[2] T. Bauer, S. Di Rocco, B. Harbourne, J. Huizenga, A. Lundman, P. Pokora and T. Szemberg, Bounded Negativity and Arrangements of Lines, International Math. Res. Notices (2014).

[3] T. De Fernex, Negativ curves on very general blow-ups of $\mathbb{P}^2$. Projective Varieties with Unexpected Properties, a Volume in Memory of Giuseppe Veronese, pp. 199-207, de Gruyter, Berlin, 2005.

[4] M. Dumnicki, B. Harbourne, U. Nagel, A. Seceleanu, T. Szemberg and H. Tutaj-Gasińska, Resurgences for ideals of special point configurations in $\mathbb{P}^N$ coming from hyperplane arrangements, to appear, J. Algebra.

[5] T. Fujita, On Zariski problem, Proc. Japan Acad. 55, Ser. A: 106 – 110 (1979).

[6] B. Harbourne, The geometry of rational surfaces and Hilbert functions of points in the plane, Canadian Mathematical Society Conference Proceedings 6, 95–111 (1986).

[7] S. J. Kovacs, The cone of curves of a K3 surface, Math. Ann. 300 (1994), no. 4, 681–691.

[8] D. Morrison, On K3 surfaces with large Picard number, Invent. Math. 75 (1984), no. 4, 105–121.
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[9] O. Zariski, The theorem of Riemann-Roch for high multiples of an effective divisor on an algebraic surface, *Ann. Math.* 76: 560-615 (1962)

Department of Mathematics, University of Nebraska, Lincoln, NE 68588-0130 USA  
*E-mail address*: bharbour@math.unl.edu

Institute of Mathematics, Pedagogical University, Podchorążych 2, PL-30-084, Kraków, Poland  
*E-mail address*: piotrpkr@gmail.com

Department of Mathematics and Computer Sciences, Jagiellonian University, Lojasiewicza 6, PL-30-348 Kraków  
*E-mail address*: htutaj@im.uj.edu.pl