ON MATRICES WITH DIFFERENT TROPICAL AND KAPRANOV RANKS

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Abstract. In this note, we generalize the technique developed in \cite{13} and prove that every $5 \times n$ matrix of tropical rank at most 3 has Kapranov rank at most 3, for the ground field that contains at least 4 elements. For the ground field either $\mathbb{F}_2$ or $\mathbb{F}_3$, we construct an example of a $5 \times 5$ matrix with tropical rank 3 and Kapranov rank 4.

Tropical mathematics deals with the tropical semiring, that is, the set $\mathbb{R}$ of real numbers with the operations of tropical addition and tropical multiplication defined as $a \oplus b = \min\{a, b\}$ and $a \otimes b = a + b$, for all $a, b \in \mathbb{R}$. The connection between classical and tropical mathematics can be established with Maslov dequantization \cite{1, 2, 3}. The methods of tropical mathematics are important for different applications \cite{4, 5, 6}, and are helpful for the study of algebraic geometry \cite{7, 8}. The notion of the rank is very interesting in tropical mathematics \cite{4, 9, 10}, and, in contrast with the situation of matrices over a field, there are many different rank functions for tropical matrices \cite{4, 9, 10}. This note is devoted to the concepts of the tropical and Kapranov rank functions.

We will use the symbol $\mathbb{F}$ to denote a field, and by $\mathbb{F}^*$ we will denote the set of nonzero elements of $\mathbb{F}$. By $a_{ij}$ we denote an entry of a matrix $A$, by $A^{(j)}$ the $j$th column of $A$, by $A_{(i)}$ the $i$th row, by $A^\top$ the transpose of $A$. By $A[r_1, \ldots, r_k]$ we denote the submatrix formed by the rows of $A$ with indexes $r_1, \ldots, r_k$, and by $A[r_1, \ldots, r_k|c_1, \ldots, c_l]$ the submatrix formed by the columns with indexes $c_1, \ldots, c_l$ of $A[r_1, \ldots, r_k]$.

By $\mathbb{H}_\mathbb{F}$ we denote the field \cite{11} that consists of the formal sums of the form $a(t) = \sum_{e \in \mathbb{R}} a_e t^e$, where $t$ is a variable, the coefficients $\{a_e\}$ belong to a field $\mathbb{F}$, and the support $E(a) = \{e \in \mathbb{R} : a_e \neq 0\}$ is a well-ordered subset of $\mathbb{R}$ (that is, any nonempty set of $E(a)$ has the least element). The degree of a sum $a \in \mathbb{H}_\mathbb{F}$ is the exponent of its leading term, that is, $\deg a = \min E(a)$. The element $a_0 \in \mathbb{F}$ is called the constant term of $a$. We assume the degree of the zero element from $\mathbb{H}_\mathbb{F}$ to equal $+\infty$. The matrix that is obtained from $A \in (\mathbb{H}_\mathbb{F})^{m \times n}$ by entrywise application of the mapping $\deg$ is denoted by $\deg A \in \mathbb{R}^{m \times n}$. Now we can define the notion of the Kapranov rank \cite{9} Corollary 3.4].

Definition 1. The Kapranov rank of a matrix $B \in \mathbb{R}^{m \times n}$ with respect to a ground field $\mathbb{F}$ is defined to be

$$K_\mathbb{F}(B) = \min \left\{ \text{rank}(A) \left| A \in (\mathbb{H}_\mathbb{F})^{m \times n}, \deg A = B \right. \right\},$$

where rank is the classical rank function of matrices over the field $\mathbb{H}_\mathbb{F}$.

The tropical permanent of a matrix $B \in \mathbb{R}^{n \times n}$ is defined to be

$$\text{perm}(B) = \min \left\{ b_{1, \sigma(1)} + \ldots + b_{n, \sigma(n)} \right\},$$

(1)
where $S_n$ denotes the symmetric group on \{1, \ldots, n\}. $B$ is called tropically singular if the minimum in (I) is attained at least twice. Otherwise, $B$ is called tropically non-singular.

**Definition 2.** The tropical rank, trop$(M)$, of a matrix $M \in \mathbb{R}^{p \times q}$ is the largest number $r$ such that $M$ contains a tropically non-singular $r$-by-$r$ submatrix.

The following proposition follows directly from the definitions.

**Proposition 3.** The tropical and Kapranov ranks of a matrix remain unchanged after adding a fixed number to every element of some row or some column.

For $a$ and $b$ vectors from $(\mathbb{R} \cup \{+\infty\})^m$, we denote the set of all $j$ that provide the minimum for \(\min_{j=1}^m \{a_j + b_j\}\) by $\Theta(a, b)$. The rows of a matrix $A \in \mathbb{R}^{m \times n}$ are called tropically linearly dependent (or simply tropically dependent) if there exists $\lambda \in \mathbb{R}^m$ such that $\Theta(\lambda, A^{(j)}) \geq 2$, for every $j \in \{1, \ldots, n\}$. In this case, $\lambda$ is said to realize the tropical dependence of the rows of $A$. If the rows are not tropically dependent, then they are called tropically independent. The following theorem [12, Theorem 5.11] plays an important role for our considerations.

**Theorem 4.** The tropical rank of a matrix $A \in \mathbb{R}^{m \times n}$ equals the cardinality of the largest tropically independent family of rows of $A$.

The present note is devoted to the following question, asked by Develin, Santos, and Sturmfels.

**Question 5.** [9, Section 8, Question (6)] Is there a $5 \times 5$ matrix having tropical rank $3$ but Kapranov rank $4$?

Chan, Jensen, and Rubei [13, Corollary 1.5] have shown that trop$(A) = K_C(A)$, for every matrix $A \in \mathbb{R}^{5 \times 5}$. Therefore, they answer Question 5 in the most important case, the case when the Kapranov rank function is considered with respect to a ground field $C$. On the other hand, in the paper [9], where Question 5 was proposed, the Kapranov rank was understood with respect to an arbitrary ground field [9, Definition 3.9]. In our note, we consider the problem in the case of an arbitrary field, we generalize the technique developed in [13] and give a general answer for Question 5. For a field $F$ satisfying $|F| \geq 4$ and a matrix $B \in \mathbb{R}^{5 \times 5}$ satisfying trop$(B) \leq 3$, we show that $K_F(B) \leq 3$. We provide examples of matrices $C$ with tropical rank $3$ satisfying $K_F(C) = 4$ if the field $F$ is either $\mathbb{F}_2$ or $\mathbb{F}_3$. The following lemma is helpful to prove Lemma 6 which gives a generalization for the technique developed in [13] and holds for a more general class of ground fields.

**Lemma 6.** Let $|F| \geq 4$, $S \in \mathcal{H}_F^{2 \times 2}$. Then there exists $\xi \in F^*$ such that $\deg(\xi s_{11} + s_{12}) = \min\{\deg s_{11}, \deg s_{12}\}$, for $i \in \{1, 2\}$.

**Proof.** If $s_{ij} \neq 0$, we denote the coefficient of the leading term of $s_{ij}$ by $\sigma_{ij}$. If $s_{ij} = 0$, we choose $\sigma_{ij} \in F^*$ arbitrarily. Now it remains to choose $\xi \in F \setminus \{0, -\frac{\sigma_{12}}{\sigma_{11}}, -\frac{\sigma_{22}}{\sigma_{21}}\}$. \(\square\)

**Lemma 7.** Let $|F| \geq 4$, let a matrix $A \in \mathcal{H}_F^{2 \times 2}$ be such that rank$(A) = 2$ and $\deg(a_{p1}a_{q2} - a_{q1}a_{p2}) = \min\{\deg a_{p1} + \deg a_{q2}, \deg a_{q1} + \deg a_{p2}\}$, for every different $p, q \in \{1, \ldots, 5\}$. Let also $B \in \mathbb{R}^{5 \times n}$, denote

$\Theta_{ij} = \Theta(\deg A^{(1)}, B^{(j)}), \Theta_{ij} = \Theta(\deg A^{(2)}, B^{(j)})$ for every $j \in \{1, \ldots, n\}$.

If $|\Theta_{1j}| \geq 2$, $|\Theta_{2j}| \geq 2$, $|\Theta_{1j} \cup \Theta_{2j}| \geq 3$, for every $j$, then $K_F(B) \leq 3$. 

Proof. We fix an arbitrary \( j \in \{1, \ldots, n\} \) and denote \( \theta_1 = \min_{i=1}^5 \{ \deg a_{i1} + b_{ij} \} \), \( \theta_2 = \min_{i=1}^5 \{ \deg a_{i2} + b_{ij} \} \). We assume without a loss of generality that \( 1 \in \Theta_{1j} \), \( 2 \in \Theta_{2j} \), and that both \( \Theta_1 \) and \( \Theta_2 \) have non-empty intersections with \( \{3, 4, 5\} \). These settings imply that

\[ \min_{i=3}^5 \{ \deg \det A[1, i] + b_{ij} + b_{ij} \} = \min_{i=3}^5 \{ \deg \det A[2, i] + b_{ij} + b_{ij} \} = \theta_1 + \theta_2. \]

From Lemma 6 it then follows that there exist \( \xi, \eta, \zeta \in \mathbb{F}^s \) such that

\[ \deg \left( \sum_{i=3}^5 \det A[1, i] t^{b_{ij} + b_{ij}} \xi_i \right) = \deg \left( \sum_{i=3}^5 \det A[2, i] t^{b_{ij} + b_{ij}} \xi_i \right) = \theta_1 + \theta_2. \]

Cramer’s rule then implies that the solution \((x_1, x_2)\) of

\[
\begin{cases}
 a_{11}t^{b_{1j}}x_1 + a_{21}t^{b_{2j}}x_2 = \sum_{i=3}^5 \xi_i a_{i1}t^{b_{ij}}, \\
 a_{12}t^{b_{1j}}x_1 + a_{22}t^{b_{2j}}x_2 = \sum_{i=3}^5 \xi_i a_{i2}t^{b_{ij}},
\end{cases}
\]

satisfies \( \deg x_1 = \deg x_2 = 0 \). We set \( c_{1j} = x_1t^{b_{1j}}, c_{2j} = x_2t^{b_{2j}}, c_{ij} = -\xi_t^{b_{ij}}, \) for \( \xi \in \{3, 4, 5\} \). The equations (3) imply that \( \sum_{i=3}^5 a_{1i}c_{ij} = \sum_{i=3}^5 a_{2i}c_{ij} = 0 \). Since \( j \in \{1, \ldots, n\} \) has been chosen arbitrarily, we can construct the matrix \( C \) such that \( B = \deg C \), and the rows \( \sum_{i=3}^5 a_{1i}C(i) \) and \( \sum_{i=3}^5 a_{2i}C(i) \) both consist of zero elements. From Definition it now follows that \( K_\mathbb{F}(B) \leq 3 \) \( \square \)

Lemma 8. Let the entries of a matrix \( B \in \mathbb{R}^{5 \times n} \) be nonnegative, every column of \( B \) contain at least three zeros. If \( |\mathbb{F}| \geq 4 \), then \( K_\mathbb{F}(B) \leq 3 \).

Proof. There exist different \( \eta, \zeta \in \mathbb{F} \setminus \{0, 1\} \). We set \( A = \left( \begin{array}{cc} 1 & 1 \\
 1 & \eta \end{array} \right) \in \mathbb{H}_\mathbb{F}^{2 \times 2} \). Now the result follows from Lemma 7 \( \square \)

Lemma 9. Let the entries of a matrix \( B \in \mathbb{R}^{5 \times n} \) be all nonnegative, and

\[
B = \begin{pmatrix}
 0 \ldots 0 & B_1 & B_2 & B_3 & B_4 \\
 0 \ldots 0 & B' \\
 \alpha_1 \ldots \alpha_p & \beta_1 \ldots \beta_q & \gamma_1 \ldots \gamma_r & 0 \ldots 0 \\
 v & 0 \ldots 0 & 0 \ldots 0 & 0 \ldots 0
\end{pmatrix},
\]

where \( v > 0, p > 0, q + r + s > 0 \), either \( B' \) or \( (B_1| \ldots |B_4) \) consists of positive numbers, and the numbers \( \alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q, \gamma_1, \ldots, \gamma_r \) are positive. If \( |\mathbb{F}| \geq 4 \) and \( \text{trop}(B) \leq 3 \), then \( K_\mathbb{F}(B) \leq 3 \).

Proof. The proof is by reductio ad absurdum.

1. We assume w.l.o.g. that \( B \) provides the minimal value of \( p + q + r + s \) over all matrices \( D \) of the form (4) that satisfy \( \text{trop}(D) \leq 3 \) and \( K_\mathbb{F}(D) > 3 \).

2. By \( m \) we denote the minimal element of the matrix \((B_1| \ldots |B_4)\). We add \(-m\) to every element of the first two rows of \( B \), \( m \) to every element of the first \( v \) columns. So by Proposition 8 we can assume without a loss of generality that \( B' \) consists of positive numbers and \( m = 0 \).

3. Let each of the matrices \( B_2 \) and \( B_3 \) contain a column without zeros (the numbers of these columns are denoted by \( j_1 \) and \( j_2 \)). Items 1 and 2 show that there exists \( j_3 \in \{1, \ldots, n\} \) such that either \( b_{1j_3} = 0, b_{2j_3} > 0 \) or \( b_{1j_3} > 0, b_{2j_3} = 0 \). Then we note that the matrix \( B[1, 2, 3, 4] \in \mathbb{H}_\mathbb{F}^{4 \times 2} \) is tropically non-singular. Definition 2 shows that the tropical rank of \( B \) is not less than 4, so we get a
contradiction. Thus we can assume without a loss of generality that every column of $B_3$ contains a zero element.

4. Theorem \ref{thm:main} implies that there exist $(\lambda_1, \lambda_2, \lambda_3, \lambda_5), (\mu_1, \mu_2, \mu_3, \mu_5) \in \mathbb{R}^4$ that realize the tropical dependence of the rows of $B[1, 2, 4, 5]$ and $B[1, 3, 4, 5]$, respectively. We denote $\Lambda = (\lambda_1, \lambda_2, +\infty, \lambda_4, \lambda_5)$ and $M = (\mu_1, +\infty, \mu_3, \mu_4, \mu_5)$, we then have that $\Theta(\Lambda, B^{(j)}) \geq 2$ and $\Theta(M, B^{(j)}) \geq 2$, for every $j \in \{1, \ldots, n\}$. From the equation \eqref{eq:lambda} it then follows that $\lambda_1 = \lambda_2 \leq \min\{\lambda_4, \lambda_5\}$, $\mu_3 = \mu_4 \leq \mu_5$, and $\mu_4 < \mu_1$. Now it is straightforward to check that $\Theta(\Lambda, B^{(j)}) \cup \Theta(M, B^{(j)}) \geq 3$, for every $j \in \{1, \ldots, n\}$. Finally, set $A = \left(\begin{smallmatrix} t^1 & t^2 & 0 & 0 \\ t^3 & 0 & t^4 & t^5 \\ 0 & 0 & \eta & 1 \end{smallmatrix}\right) \in \mathbb{H}_F^{5 \times 2}$, for some $\eta \in \mathbb{F} \setminus \{0, 1\}$. The application of Lemma \ref{lem:main} completes the proof. \hfill \Box

Now we can prove one of the main results of this note.

**Theorem 10.** Let $C \in \mathbb{R}^{5 \times n}$, $\text{trop}(C) \leq 3$, and $|\mathbb{F}| \geq 4$. Then $K_\mathbb{F}(C) \leq 3$.

**Proof.** 1. Theorem \ref{thm:main} implies that the rows of $C$ are tropically dependent. Applying Proposition \ref{prop:main} we assume without a loss of generality that $C$ consists of nonnegative numbers, and every column of $C$ contains at least two zeros.

2. Let the minimal element of the $i$th row of $C$ is $h_i$. For every $i \in \{1, \ldots, 5\}$, we add $(-h)$ to every entry of the $i$th row of $C$, and we denote the matrix obtained by $B$. Every row of $B$ now contains at least one zero. By item 1, the entries of $B$ are nonnegative, and every column of $B$ contains at least two zeros.

3. By Proposition \ref{prop:main} we have that $K_\mathbb{F}(B) = K_\mathbb{F}(C)$, $\text{trop}(B) \leq 3$.

4. If every column of $B$ contains at least three zeros, then Corollary \ref{cor:main} implies that $\text{trop}(B) \leq 3$. So we can further assume without a loss of generality that $b_{11} = b_{21} = 0$, and the elements $b_{31}, b_{41}, b_{51}$ are positive. The three cases are possible.

**Case 1.** Let some column of $B[3, 4, 5]$ contain exactly one zero entry. We assume without a loss of generality that $b_{32} = 0, b_{42} > 0, b_{52} > 0$. Assume $b_{i'j'} = 0, b_{i''j''} > 0$ for some $i', i'' \in \{4, 5\}, j' \in \{1, \ldots, n\}$. By item 2, there exists $j'' \in \{1, \ldots, n\}$ such that $b_{i''j''} = 0$. We note that the matrix $B[2, 3, 4, 5][1, 2, j', j'']$ is tropically non-singular, that is, $\text{trop}(B) \geq 4$, so we get a contradiction.

Thus we see that for every $j \in \{1, \ldots, n\}$, it holds that either $b_{4j} = b_{5j} = 0$ or $b_{4j}, b_{5j} > 0$. So we can see that $B$ satisfies the assumptions of Lemma \ref{lem:main} up to permutations of rows and columns.

**Case 2.** Assume that some column of $B[3, 4, 5]$ contains exactly two zero entries, and no column of $B[3, 4, 5]$ contains exactly one zero entry. In this case, $B$ satisfies the assumptions of Lemma \ref{lem:main} up to permutations of its columns.

**Case 3.** Finally, we assume that for every $j \in \{1, \ldots, n\}$, it holds that either $b_{3j} = b_{4j} = b_{5j} = 0$ or $b_{3j}, b_{4j}, b_{5j} > 0$. Let us consider the set $G$ of all $j \in \{1, \ldots, n\}$ such that the elements $b_{3j}, b_{4j}, b_{5j}$ are not all equal. If $G$ is empty, then the last three rows of $B$ coincide, so from Proposition \ref{prop:main} and Corollary \ref{cor:main} it follows that $K_\mathbb{F}(B) \leq 3$.

If $G$ is non-empty, then we denote $m = \min\{b_{3g}, b_{4g}, b_{5g}\}$. We then add $-m$ to every entry of the $j$th column of $B$ ($j$ runs over $\{1, \ldots, n\}$), we also add $m$ to every element of the first two rows of $B$. We note that the matrix obtained satisfies the conditions of Corollary \ref{cor:main}, or Case 1, or Case 2 up to permutations of its columns. By Proposition \ref{prop:main} the matrix obtained has the same tropical and Kapranov ranks as $B$. 


In any of the Cases 1–3, we see that $K_F(B) \leq 3$. The proof is complete. \hfill \Box

Now let us show that the condition $|F| \geq 4$ is necessary in the formulation of Theorem 10.

**Example 11.** Let

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$ 

Then $\text{trop}(B) = 3$, $K_{F_3}(B) = K_{F_2}(D) = 4$.

**Proof.** By Definition 2, we have straightforwardly $\text{trop}(B) = 3$. Note that if a matrix $C \in \text{H}_{F_3}^{5 \times 5}$ satisfies $D = \deg C$, then $\deg \det C[1,2,3,5][1,2,3,5] = 0$, so that $K_{F_3}(D) \geq 4$. On the other hand, the matrix $D$ contains repeating rows, thus $K_{F_2}(D) = 4$.

Assume that a matrix $A' \in \text{H}_{F_3}^{5 \times 5}$ satisfies $B = \deg A'$. Without a loss of generality it can be assumed that $a'_{ij} = t^{b_{ij}}$, for every pair $(i,j)$ satisfying $4 \in \{ i,j \}$. Note that if $\det A'[p,q,4,5][p,q,4,5] = 0$ holds for every $p,q \in \{1,2,3\}$, then the entries $a_{pq}$ have the $-1$ as their constant terms, and in this case $\deg \det A'[1,2,3,5][1,2,3,5] = 0$. Therefore, we see that $K_{F_3}(B) \geq 4$. On the other hand, we can set $a_{ij} = t^{b_{ij}}$, for every $(i,j) \in \{1,2,3,4,5\} \setminus \{(4,2),(4,3)\}$, and $a_{42} = a_{43} = 2 + 2t$, and note that the row $A_{(2)} + A_{(3)} + A_{(4)}$ is zero, and $\deg A = B$. Definition 1 shows therefore that $K_{F_3}(B) = 4$. \hfill \Box

Now we can give a general answer for Question 5.

**Theorem 12.** A $5 \times 5$ tropical matrix $B$ with tropical rank 3 and Kapranov rank 4 does exist if and only if the ground field contains at most three elements.

**Proof.** Follows directly from Theorem 10 and Example 11. \hfill \Box

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