Distillation by repeated measurements: continuous spectrum case

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Repeated measurements on a part of a bipartite system strongly affect the other part not measured, whose dynamics is regulated by an effective evolved evolution operator. When the spectrum of this operator is discrete, the latter system is driven into a pure state irrespective of the initial state, provided the spectrum satisfies certain conditions. We here show that even in the case of continuous spectrum an effective distillation can occur under rather general conditions. We confirm it by applying our formalism to a simple model.

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Distillation procedures aiming at driving quantum systems into pure states are relevant tools in the field of quantum information and computation.1 In fact, they can be exploited to control the state of a quantum system and play a crucial role to initialize quantum systems. Various purification schemes have been proposed2 and among them is a state generation strategy based on the extraction of a state through repeated measurements3. In the case of measurements projecting the total system into pure states are relevant tools in the field of quantum information and computation.1

Repeated measurements on a part of a bipartite system strongly affect the other part not measured, whose dynamics is regulated by an effective evolved evolution operator. When the spectrum of this operator is discrete, the latter system is driven into a pure state irrespective of the initial state, provided the spectrum satisfies certain conditions. We here show that even in the case of continuous spectrum an effective distillation can occur under rather general conditions. We confirm it by applying our formalism to a simple model.

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In order to discuss the possibility to obtain an effective distillation of $A$, we will compare the survival probability $P(N)$ and the purity of $A$, quantified by

$$\Pi(N) = \text{Tr}_A\{\hat{\rho}_A^2(N)\},$$

as functions of the number of measurements $N$. We endeavor to find conditions under which an effective distillation of $A$, in the sense clarified later, can be achieved and consider the cases where the right-eigenvectors are orthogonal to each other $\langle u_E | u_{E'} \rangle = \delta(E - E')$ (which is actually the case in the examples studied below). In such cases, the survival probability takes the form

$$P(N) = \int dE |\lambda_E|^{2N} \langle u_E | \hat{\rho}_A(0) | u_E \rangle,$$

and the purity of $A$ is expressed as

$$\Pi(N) = \frac{1}{P^2(N)} \int dE dE' |\lambda_E|^{2N} |\lambda_{E'}|^{2N} \times |\langle u_E | \hat{\rho}_A(0) | u_{E'} \rangle|^2.$$

Let $E_*$ be the value of $E$ at which $|\lambda_E|$ has its (unique) absolute maximum, $|\lambda_{E'}|_{E' = E_*} = 0$, and consider the Taylor expansion of $\Lambda(E) = -\ln |\lambda_E|^2$ around $E_*$, $|\lambda_E|^{2N} \approx e^{N \ln |\lambda_E|^2} \approx e^{-N\Lambda(E_*) - N\Lambda''(E_*)(E - E_*)^2/2 + \cdots}$. Notice that the higher order terms become less relevant as $N$ increases, and $|\lambda_E|^{2N}$ becomes well approximated by a Gaussian

$$|\lambda_E|^{2N} \approx f(N) \frac{1}{\sqrt{2\pi \Delta_N^2}} e^{-(E-E_*)^2/2\Delta_N^2},$$

$$f(N) = \sqrt{2\pi \Delta_N^2} e^{-N\Lambda(E_*)}, \quad \Delta_N = \frac{1}{\sqrt{N\Lambda''(E_*)}}$$

Observe here that the Gaussian in becomes narrower like $\Delta_N \propto 1/\sqrt{N}$ as $N$ increases, and a narrow band around $E_*$ is filtered in the spectrum. Putting $x = (E - E_*)/\Delta_N$, the survival probability is shown to behave for large $N$ as

$$P(N) \approx f(N) \int \frac{dx}{\sqrt{2\pi}} e^{-x^2/2} \langle u_{E_* + x\Delta_N} | \hat{\rho}_A(0) | u_{E_* + x\Delta_N} \rangle$$

$$\approx f(N) \left( g(0) + \frac{1}{2} g''(0) \Delta_N^2 \right),$$

where $g(y) \equiv \langle u_{E_* + y} | \hat{\rho}_A(0) | u_{E_* + y} \rangle$, and the purity given by becomes

$$\Pi(N) \approx \frac{f^2(N)}{P^2(N)} \int \frac{dx}{\sqrt{2\pi}} \frac{dE}{2\pi} e^{-(x^2 + x'^2)/2}$$

$$\times |\langle u_{E_* + x\Delta_N} | \hat{\rho}_A(0) | u_{E_* + x'\Delta_N} \rangle|^2$$

$$\approx 1 - \Delta_N^2 \frac{g(0) g''(0) - h_{yy}(0,0)}{g''(0)}.$$

Here we have introduced $h(y, y') = |\langle u_{E_* + y} | \hat{\rho}_A(0) \times | u_{E_* + y'} \rangle|^2$ and denoted its second partial derivative with respect to $y$ as $h_{yy}(y, y')$. One sees that, for large $N$, the purity $\Pi(N)$ approaches 1 with a rate determined by $\Delta_N^2 \propto 1/N$. System $A$ is thus asymptotically purified toward $|u_{E_*}\rangle$ by the repeated measurements on $X$.

In this process, however, one should pay attention to the behavior of the survival probability: the purity $\Pi(N)$ should reach 1 quick enough before the probability $P(N)$ decays out completely. The comparison between the rates of the decay of $P(N)$ to 0 and the approach of $\Pi(N)$ to 1 leads to the following optimization criterion in order to obtain an efficient distillation of the pure state $|u_{E_*}\rangle$. If $\Lambda(E_*) = 0$, i.e. $|\lambda_{E_*}|^2 = 1$, the exponential decay $e^{-N\Lambda(E_*)}$ in $P(N)$ disappears and the decay of $P(N)$ is ruled by $\Delta_N \propto 1/\sqrt{N}$. This decay is slower than the approach of the purity $\Pi(N)$ to 1, i.e. $\Delta_N^2 \propto 1/N$. Therefore, if the magnitude of the eigenvalue associated to the eigenstate $|u_{E_*}\rangle$ is $|\lambda_{E_*}| = 1$, system $A$ is driven toward the pure state $|u_{E_*}\rangle$ with a rate faster than the decay of the survival probability $P(N)$.

Remark that, when $\langle u_E | u_{E'} \rangle = \delta(E - E')$ is not satisfied, it can be shown that $P(N)$ and $\Pi(N)$ at best decrease and increase respectively as $\Delta_N^2 \propto 1/N$ for large $N$, so that distillation seems more difficult to achieve.

**Model.** We now apply, as an example, the above framework to a specific model. We consider a particle of mass $m$ interacting with a single cavity mode of frequency $\omega$. This system has been studied in to analyze how the repeated measurements on the particle affect the dynamics of the cavity mode. In that case, although the measured part (particle) has a continuous spectrum, the effective dynamics of the cavity mode is described by an operator $\hat{V}_\tau$ characterized by a discrete spectrum. Here, the opposite case is investigated, that is, the cavity mode is repeatedly projected onto its initial state by measurements. In this case, as we will show, the effective dynamics of the particle is described by an operator $\hat{V}_\tau$ having a continuous spectrum.

The Hamiltonian describing the system is

$$\hat{H} = \frac{\hat{p}^2}{2m} + \hbar \omega \left( \hat{a}^{\dagger} \hat{a} + \frac{1}{2} \right) + g\hat{p}(\hat{a}^{\dagger} + \hat{a}),$$

where $\hat{p}$ is the momentum operator of the particle, $\hat{a}$ and $\hat{a}^{\dagger}$ are the annihilation and creation operators of the cavity mode, respectively, satisfying the commutation rule $[\hat{a}, \hat{a}^{\dagger}] = 1$, and the real parameter $g$ is the coupling constant. The dynamics described by the Hamiltonian is exactly solvable, and the exact evolution operator at time $\tau$ in the Schrödinger picture is given by

$$\hat{U}(\tau) = e^{-i\xi g^2 \tau/2m\hbar e^{-i\omega \tau(\hat{a}^{\dagger} + 1/2)}} e^{i\hat{p}^2/2m\hbar e^{-i\omega \tau}},$$

where $\xi = 1 - (2mg^2/\hbar\omega)[1 - (\sin \omega \tau)/\omega \tau]$ and $g_\tau = g(1 - e^{-i\omega \tau})/\hbar\omega$.

The projected evolution operator $\hat{V}_\tau$ defined by strongly depends on the choice of the state $|\Phi\rangle$ on which the cavity mode is repeatedly projected at intervals $\tau$. Nevertheless, it results to be diagonal in the momentum representation for any choice of $|\Phi\rangle$. It follows that, for
any choice of the cavity state $|\Phi\rangle$, the operator $\hat{V}_r^{N}$ has a continuous spectrum of the form $\hat{V}_r^{N} = \int dp \lambda_p^{N} |p\rangle \langle p|$, where $\lambda_p$ is the eigenvalue whose explicit form depends on the choice of the cavity state $|\Phi\rangle$ to be measured. In the following, we examine how the above general framework for distillation with a continuous spectrum of $\hat{V}_r$ works, by looking at two cases with different cavity states $|\Phi\rangle$. In the first case, $|\Phi\rangle$ is assumed to be a coherent state $|\alpha\rangle$, while in the second, a number state $|n\rangle$. Both cases are of interest from an experimental point of view [5]. The first choice leads directly to a Gaussian form of $|\lambda_p|^{2N}$, while in the second case, $|\lambda_p|^{2N}$ itself is not Gaussian, but it becomes well approximated by a Gaussian, and we will see that the above general framework actually works.

As the initial state of the particle, we take a generic Gaussian state $\hat{\rho}_A(0) = \int dp dp' |p\rangle \rho_{pp'}(0) |p'\rangle$ with

$$\rho_{pp'}(0) = \frac{1}{\sqrt{2\pi(\Delta p_0)^2}} e^{-\frac{(p-p')^2}{2(\Delta p_0)^2}} e^{-i(p\cdot\tau)} \times e^{-\frac{(p-p')^2}{2(\Delta p_0)^2}},$$

where $B = \sqrt{4(\Delta p_0)^2(\Delta p_0)^2 + 2 - 1}$, $p_0$ and $x_0$, and $(\Delta p_0)^2$ and $(\Delta x_0)^2$ are the averages and variances of the momentum and the position, and $\Pi_0$ the purity of this initial state [5].

**Cavity coherent state $|\Phi\rangle = |\alpha\rangle$ ($\alpha = |\alpha|e^{i\gamma}$).** In this case, the projected evolution operator reads

$$\hat{V}_r = e^{-i\omega r/2 - 2i|\alpha|^2 \sin(\omega r/2)} e^{-i\omega r/2} \times \int dp |p\rangle \langle p| e^{-i(p\cdot\tau)/2} p^2(\Delta p_0)^2 e^{-i(p\cdot\tau)/2} \times e^{-i\omega r/2},$$

where $b = -\alpha g^*_r + e^{-i\omega r} g_r \alpha^*$. The survival probability defined in [5] takes the form

$$P(N) = \frac{\left| e^{-4N|\alpha|^2 \sin^2(\omega r/2)} \right|^2 \sin^2(\omega r/2)}{\sqrt{1 + 2g_r^2(\Delta p_0)^2N}} \times \exp\left( -\frac{(p_0 - Re b/g_r^2)^2}{2g_r^2(\Delta p_0)^2(1 + 1/2g_r^2(\Delta p_0)^2N)} \right),$$

where $Re b = -4g|\alpha|/\hbar \omega $ $\sin^2(\omega r/2)$ $\cos \gamma$. In the limit of large number of measurements, $N \gg 1/2g_r^2(\Delta p_0)^2$, $P(N)$ reduces to

$$P(N) \approx \frac{e^{-4N|\alpha|^2 \sin^2(\omega r/2)}}{\sqrt{2g_r^2(\Delta p_0)^2N}} \times \exp\left[ -\frac{1}{2g_r^2(\Delta p_0)^2N} \left( p_0 - \frac{Re b}{g_r^2} \right)^2 \right].$$

This formula corresponds to [5], without the second-order term proportional to $\Delta^2_N$, through the relationships

$$\Delta_N = \frac{1}{\sqrt{2g_r^2N}}, \quad p_* = \frac{Re b}{g_r^2},$$

$$f(N) = \sqrt{2\pi} e^{-4N|\alpha|^2 \sin^2(\omega r/2)}.$$

One can see that the selected momentum $p_*$, being proportional to $Re b$ given below (18), can be chosen by tuning the parameters properly. In particular for $\gamma = 0, \pi$, the exponential decay in the survival probability, i.e. in $f(N)$, can be suppressed, being the final momentum $p_*$ negative in the first case $\gamma = 0$ and positive in the second $\gamma = \pi$. The other parameters can be tuned in order to obtain the desired modulus of the final momentum $p_*$. Of course, the farther it is from the initial average momentum $p_0$, the smaller the exponential factor in $\langle p_*|\hat{\rho}_A(0)|p_*\rangle$ will be, since it depends on $p_* - p_0$.

Let us also look at the evolution of the purity:

$$\Pi(N) = \sqrt{\frac{1 + 2|g_r|^2(\Delta p_0)^2N}{1 + 2|g_r|^2(\Delta p_0)^2N}} \approx 1 - \frac{1}{\Pi_0^2 - \frac{1}{2g_r^2(\Delta p_0)^2N}}.$$

If the initial state of the particle is pure, $\Pi_0 = 1$, it remains pure after each measurement, while in general, the purity of the particle increases as the measurements go on. This formula is again consistent with (13).

In Fig. 1 the probability $P(N)$ in (18) and the purity $\Pi(N)$ in (21) are plotted for $\gamma = \pi$. The purity approaches 1 when the probability is yet far from 0.

**Cavity number state $|\Phi\rangle = |n\rangle$.** For $n = 1$, $\hat{V}_r$ results in

$$\hat{V}_r = \int dp |p\rangle \langle p| (-p^2|g_r|^2)^{e^{-3i\omega/2}} \times e^{-i(p\cdot\tau)/2} p^2(\Delta p_0)^2 e^{-i(p\cdot\tau)/2} \times e^{-i\omega r/2},$$

where $b = -4g|\alpha|/\hbar \omega \sin^2(\omega r/2) \cos \gamma$. In the limit of large number of measurements, $N \gg 1/2g_r^2(\Delta p_0)^2$, $P(N)$ reduces to

$$P(N) = \frac{\left| e^{-4N|\alpha|^2 \sin^2(\omega r/2)} \right|^2 \sin^2(\omega r/2)}{\sqrt{2g_r^2(\Delta p_0)^2N}} \times \exp\left[ -\frac{1}{2g_r^2(\Delta p_0)^2N} \left( p_0 - \frac{Re b}{g_r^2} \right)^2 \right].$$

This formula corresponds to [5], without the second-order term proportional to $\Delta^2_N$, through the relationships

$$\Delta_N = \frac{1}{\sqrt{2g_r^2N}}, \quad p_* = \frac{Re b}{g_r^2},$$

$$f(N) = \sqrt{2\pi} e^{-4N|\alpha|^2 \sin^2(\omega r/2)}.$$

![FIG. 1:](image-url)
On the other hand, (13) gives the asymptotic behavior of the purity $\Pi(N)$ as functions of the number of measurements $N$, when a number state $|1\rangle$ of the cavity mode is repeatedly measured. The parameters are the same as in Fig. 1 while $p_*=0$.

and the asymptotic formula $\Pi(N)$ for the survival probability $P(N)$ for a large number of measurements yields

$$P(N) \approx \frac{1}{\sqrt{6} g} e^{-\frac{\alpha^2}{2}} e^{-\frac{\alpha^2}{2N}}. \quad (24)$$

On the other hand, (13) gives the asymptotic behavior of the purity $\Pi(N)$ for a large number of measurements,

$$\Pi(N) \approx 1 - \frac{1}{\Pi_0^2} - \frac{1}{2} \frac{\Lambda}{\Delta N}, \quad (25)$$

which is the same as that obtained in the coherent state case in (21). These quantities, the survival probability $P(N)$ in (24) and the purity $\Pi(N)$ in (25), are plotted in Fig. 2 compared with the exact results computed numerically on the basis of the expression for $\hat{V}_r$ in (22). It shows that the exact results and the asymptotic formulas match already after a small number of measurements and that the purity $\Pi(N)$ approaches 1 before the survival probability $P(N)$ decays to 0.

Conclusions. We have studied the distillation process, in which one part of a bipartite system is purified by repeatedly projecting the other part onto a certain state, in the case where the dynamics is regulated by an operator $\hat{V}_r$ characterized by a continuous spectrum. When the maximum of the continuous spectrum $|\lambda_E|$ is unique at $E_*$ and the second derivative $\Lambda''(E_*) \neq 0$ exists, the spectrum $|\lambda_E|^{2N}$ becomes Gaussian around $E_*$ as the number of measurements $N$ increases, and the state is purified to $|E_*\rangle$. This purification is optimized, if it is allowed to tune the parameters so that $|\lambda_E| = 1$, by which the purity increases faster than the decrease of the survival probability. The distillation by repeated measurements has been considered to be possible only if $\hat{V}_r$ has a discrete spectrum. The present analysis reveals that this procedure can be applied to a wider class of systems.

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