Bose particles in a box III. A convergent expansion of the ground state of the Hamiltonian in the mean field limiting regime.

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Abstract

In this paper we consider an interacting Bose gas at zero temperature, constrained to a finite box and in the mean field limiting regime. The $N$ gas particles interact through a pair potential of positive type and with an ultraviolet cut-off. The (nonzero) Fourier components of the potential are assumed to be sufficiently large with respect to the corresponding kinetic energies of the modes like in the companion papers [Pi1]-[Pi2]. Using the multi-scale technique in the occupation numbers of particle states introduced in [Pi1]-[Pi2], we provide a convergent expansion of the ground state of the Hamiltonian in terms of the bare operators. In the limit $N \to \infty$ the expansion is up to any desired precision.

Summary of contents

- In Section 1, the model of an interacting Bose gas in a finite box and at zero temperature is defined along with the notation used throughout the paper. The model is analyzed at fixed total number of particles. In this context, we define the second quantized Hamiltonian and the associated particle number preserving Bogoliubov Hamiltonian (from now on Bogoliubov Hamiltonian).

- In Section 2, we review the main ideas and results of the multi-scale analysis in the occupation numbers of particle states carried out for the Bogoliubov Hamiltonian in [Pi1] and [Pi2]. The Feshbach flows implemented in [Pi1], [Pi2] are described informally in Sections 2.1 and 2.2, respectively.

- In Section 3, in the mean field limiting regime (i.e., at fixed box volume $|\Lambda|$, for a number of particles, $N$, sufficiently large, and for a coupling constant inversely proportional to the particle density) the ground state vector of the Hamiltonian is constructed as a byproduct of subsequent Feshbach flows. Each flow is associated with a couple of components, $\phi_{\pm j}$, of the Fourier expansion of the pair potential. The Fourier expansion consists of only a finite number of components because of the u.v. cut-off. This construction provides a
convergent expansion of the ground state vector of the Hamiltonian in terms of the bare operators applied to the vector with all the particles in the zero mode.

1 Interacting Bose gas in a box

We study the Hamiltonian describing a gas of (spinless) nonrelativistic Bose particles that, at zero temperature, are constrained to a $d$–dimensional box of side $L$ with $d \geq 1$. The particles interact through a pair potential with a coupling constant proportional to the inverse of the particle density $\rho$.

The rigorous description of this system has many intriguing mathematical aspects not completely clarified yet. In spite of remarkable contributions also in recent years, some important problems are still open to date, in particular in connection to the thermodynamic limit and the exact structure of the ground state vector. We shall briefly mention the results closer to our present work and give references to the reader for the details.

Some of the results have been concerned with the low energy spectrum of the Hamiltonian that in the mean field limit was predicted by Bogoliubov [Bo1], [Bo2]. Starting from the Hamiltonian of the system he defined an approximated one, the Bogoliubov Hamiltonian. For a finite box and a large class of pair potentials, upon a unitary transformation the Bogoliubov Hamiltonian describes a system of non-interacting bosons with a new energy dispersion law, which is in fact the correct description of the energy spectrum of the Bose particles system in the mean field limit.

The expression predicted by Bogoliubov for the ground state energy has been rigorously proven for certain systems in [LS1], [LS2], [ESY],[YY]. Concerning the excitation spectrum, in Bogoliubov theory it consists of elementary excitations whose energy is linear in the momentum for small momenta. After some important results restricted to one-dimensional models (see [G], [LL], [L]), this conjecture was proven by Seiringer in [Se1] (see also [GS]) for the low-energy spectrum of an interacting Bose gas in a finite box and in the mean field limiting regime, where the pair potential is of positive type. In [LNSS] it has been extended to a more general class of potentials and the limiting behavior of the low energy eigenstates has been studied. Later, the result of [Se1] has been proven to be valid in a sort of diagonal limit where the particle density and the box volume diverge according to a prescribed asymptotics; see [DN]. Recently, Bogoliubov’s prediction on the energy spectrum in the mean field limiting regime has been shown to be valid also for the high energy eigenvalues (see [NS]). These results are based on energy estimates starting from the spectrum of the corresponding Bogoliubov Hamiltonian.

A different approach to studying a gas of Bose particles is based on renormalization group. In this respect, we mention the paper by Benfatto, [Be], where he provided an order by order control of the Schwinger functions of this system in three dimensions and with an ultraviolet cut-off. His analysis holds at zero temperature in the infinite volume limit and at finite particle density. Thus, it contains a fully consistent treatment of the infrared divergences at a perturbative level. This program has been later developed in [CDPS1], [CDPS2], and, more recently, in [C] and [CG] by making use of Ward identities to deal also with two-dimensional systems where some partial control of the renormalization flow has been provided; see [C] for a detailed review of previous related results.

1In the canonical ensemble approach the diagonalization of the (particle preserving) Bogoliubov Hamiltonian is exact only in the mean field limit (see [Se1]).
Within the renormalization group approach, we also mention some results towards a rigorous construction of the functional integral for this system contained in [BFKT1], [BFKT2], and [BFKT].

In this paper we study a gas of (spinless) nonrelativistic Bose particles that, at zero temperature, are constrained to a $d$-dimensional box, $d \geq 1$, and interact through a pair potential of positive type and with an ultraviolet cut-off. We consider the number of particles fixed but we use the formalism of second quantization. We use units such that the particle mass is set equal to $\frac{1}{2}$ and $\hbar$ equal to 1. The Hamiltonian corresponding to the pair potential $\phi(x - y)$ and to the coupling constant $\lambda > 0$ is

$$
\mathcal{H} := \int (\nabla a^*)(\nabla a)(x)dx + \frac{\lambda}{2} \int \int a^*(x)a^*(y)\phi(x - y)a(x)a(y)dxdy,
$$

where reference to the integration domain $\Lambda := \{x \in \mathbb{R}^d \mid |x_i| \leq \frac{\xi}{2}, \ i = 1, 2, \ldots, d\}$ is omitted, periodic boundary conditions are assumed, and $dx$ is the Lebesgue measure in $d$ dimensions. Here the operators $a^*(x), a(x)$ are the usual operator-valued distributions on

$$
\mathcal{F} := \Gamma(L^2(\Lambda, \mathbb{C}; dx))
$$

that satisfy the canonical commutation relations

$$
[a^S(x), a^S(y)] = 0, \quad [a(x), a^*(y)] = \delta(x - y)\mathbb{1},
$$

with $a^S := a$ or $a^*$. In terms of the field modes they read

$$
a(x) = \sum_{j \in \mathbb{Z}^d} a_j e^{ik_j \cdot x}, \quad a^*(x) = \sum_{j \in \mathbb{Z}^d} a_j^* e^{-ik_j \cdot x},
$$

where $k_j := \frac{2\pi}{\xi}j, j = (j_1, \ldots, j_d), j_1, \ldots, j_d \in \mathbb{Z}$, and $|\Lambda| = \xi^d$, with CCR

$$
[a_j^S, a_{j'}^S] = 0, \quad [a_j, a_{j'}^*] = \delta_{j, j'}, \quad \text{with} \quad a_j^S = a_j \text{ or } a_j^*.
$$

The unique (up to a phase) vacuum vector of $\mathcal{F}$ is denoted by $\Omega (\|\Omega\| = 1)$.

Given any function $\varphi \in L^2(\Lambda, \mathbb{C}; dz)$, we express it in terms of its Fourier components $\varphi_j$, i.e.,

$$
\varphi(z) = \frac{1}{|\Lambda|} \sum_{j \in \mathbb{Z}^d} \varphi_j e^{ik_j \cdot z},
$$

and the Parseval identity reads

$$
\int dz|\varphi|^2(z) = \frac{1}{|\Lambda|} \sum_{j \in \mathbb{Z}^d} |\varphi_j|^2 < \infty.
$$

**Definition 1.1.** The potential $\phi(x - y)$ is a bounded, real-valued function that is periodic, i.e., $\phi(z) = \phi(z + jL)$ for $j \in \mathbb{Z}^d$, and satisfies the following conditions:

1. $\phi(z)$ is an even function, in consequence $\phi_j = \phi_{-j}$.
2. $\phi(z)$ is of positive type, i.e., the Fourier components $\phi_j$ are nonnegative.
3. The pair interaction has a fixed but arbitrarily large ultraviolet cutoff (i.e., the nonzero Fourier components $\phi_j$ form a finite set $\{\phi_0, \phi_{\pm j_1}, \ldots, \phi_{\pm j_M}\}$) with the requirement below to be satisfied:

(Strong Interaction Potential Assumption) The ratio $\epsilon_j$ between the kinetic energy of the modes $\pm j$ and the corresponding Fourier component, $\phi_j$ ($\neq 0$), of the potential, i.e.,

$$\frac{k^2}{\epsilon_j} := \epsilon_j$$

is required to be small enough to ensure the estimates used in [Pi1].

We restrict $\mathcal{H}$ to the Fock subspace $\mathcal{F}^N$ of vectors with $N$ particles

$$\mathcal{H} \upharpoonright_{\mathcal{F}^N} = \left( \int (\nabla a^*)(\nabla a)(x)dx + \frac{\lambda}{2} \int \int a^*(x)a^*(y)\phi(x-y)a(y)a(x)dxdy \right) \upharpoonright_{\mathcal{F}^N}$$

From now on, we shall study the Hamiltonian

$$H := \int (\nabla a^*)(\nabla a)(x)dx + \frac{\lambda}{2} \int \int a^*(x)a^*(y)\phi(x-y)a(y)a(x)dxdy + c_N \mathbb{1}$$

where $c_N := \frac{\partial^2}{\partial N^2} N - \frac{\partial^2}{\partial N^2} N^2$ with $\mathbf{0} = (0, \ldots, 0)$, and it is always meant to be restricted to the subspace $\mathcal{F}^N$. Notice that

$$\mathcal{H} \upharpoonright_{\mathcal{F}^N} = (H - c_N \mathbb{1}) \upharpoonright_{\mathcal{F}^N}$$

In the present paper we proceed with the study of the Hamiltonian $H$ that we started in the companion papers [Pi1], [Pi2]. Here, we deal with the complete Hamiltonian of the system in the limiting regime where the box size is fixed, the particle density is large, and the coupling constant scales like the inverse of the particle density. In this regime we provide the construction of the ground state of the Hamiltonian $H$ under the Strong Interaction Potential Assumption (see 3. in Definition 1.1) already used in the companion papers [Pi1] and [Pi2].

For this result, we have to finally control the so called “cubic” and “quartic” (in the nonzero modes) terms in the second quantized Hamiltonian of the system (see (1.21)-(1.23)), terms that are neglected in the corresponding Bogoliubov Hamiltonian.

Like in the case of the Bogoliubov Hamiltonian (see [Pi2]), with each couple of Fourier components $\{\phi_j, \phi_{-j}\}$ of the pair potential (see Definition 1.1) we associate a Feshbach flow. The new terms in the interaction are controlled thanks to a refined choice of the projections associated with the Feshbach flows. At the first step of the Feshbach flow corresponding to the couple of modes $\{j, -j\}$, a new (perpendicular) projection projects out in one single step the subspace of vectors with a number of particles in the modes $\{j, -j\}$ larger or equal to a minimum number that is chosen to be $\mathcal{O}(N^{1/4})$. In spite of this modification, the flow can be still controlled and the new terms of the interaction Hamiltonian, i.e., the “cubic” and “quartic” ones, turn out to be irrelevant. To show this we make use of the short range property of the potential in the particle states numbers by which we mean the following:

Consider a vector $\psi \in \mathcal{F}^N$ obtained as product of single particle states of the type $a^*_{j\Omega}$ containing $N_j$ particles in the modes $\pm j$. Then, it is necessary to apply the Hamiltonian to $\psi$ at least $r$ times in order to get a vector with a nonzero component in the subspace of vectors with $N_j + 2r$ or $N_j - 2r$ particles in the modes $\pm j$.

This simple but crucial property would not be enough without a refined control of a three-modes Bogoliubov Hamiltonian (see [Pi1] and [Pi2]) that will be combined with the semigroup property of the Feshbach map. In this respect, the key result is Theorem 4.1 in Section 4.
1.1 The Hamiltonian $H$ and the Hamiltonian $H^{Bog}$

Using the definitions

\[ a_+(x) := \sum_{j \in \mathbb{Z}^d \setminus \{0\}} \frac{a_j}{|\Lambda|^\frac{d}{2}} e^{ik_j \cdot x} \quad \text{and} \quad a_0(x) := \frac{a_0}{|\Lambda|^\frac{d}{2}}, \tag{1.1} \]

the Hamiltonian $H$ reads

\[
H = \sum_{j \in \mathbb{Z}^d} k_j^2 a_j^* a_j \tag{1.2}
\]
\[
+ \frac{\lambda}{2} \int \int \{a_+(x)a_+(y)\phi(x-y) + a_0(y) + h.c.\} dx dy \tag{1.3}
\]
\[
+ \lambda \int \int \{a_+(x)a_+(y)\phi(x-y) + a_0(y) + h.c.\} dx dy \tag{1.4}
\]
\[
+ \frac{\lambda}{2} \int \int \{a_0(x)a_0(y)\phi(x-y) + a_+(x)a_+(y) + h.c.\} dx dy \tag{1.5}
\]
\[
+ \lambda \int \int \{a_0(x)a_0(y)\phi(x-y) + a_+(x)a_+(y) + h.c.\} dx dy \tag{1.6}
\]
\[
+ \frac{\lambda}{2} \int \int \{a_0(x)a_0(y)\phi(x-y) + a_+(x)a_+(y) + h.c.\} dx dy \tag{1.7}
\]
\[
+ c_N \mathbb{1} \tag{1.8}
\]

Because of the implicit restriction to $\mathcal{F}^N$ and due to the choice of the constant $c_N$, it turns out that

\[
H = \sum_{j \in \mathbb{Z}^d} k_j^2 a_j^* a_j \tag{1.10}
\]
\[
+ \frac{\lambda}{2} \int \int \{a_+(x)a_+(y)\phi_{(x0)}(x-y) + a_0(y) + h.c.\} dx dy \tag{1.11}
\]
\[
+ \lambda \int \int \{a_+(x)a_+(y)\phi_{(x0)}(x-y) + a_0(y) + h.c.\} dx dy \tag{1.12}
\]
\[
+ \frac{\lambda}{2} \int \int \{a_0(x)a_0(y)\phi_{(x0)}(x-y) + a_+(x)a_+(y) + h.c.\} dx dy \tag{1.13}
\]
\[
+ \lambda \int \int \{a_0(x)a_0(y)\phi_{(x0)}(x-y) + a_+(x)a_+(y) + h.c.\} dx dy \tag{1.14}
\]

where $\phi_{(x0)}(x-y) := \phi(x-y) - \phi_{(0)}(x-y)$ with $\phi_{(0)}(x-y) := \frac{\phi_0}{|\Lambda|^\frac{d}{2}}$.

Next, we define the particle number preserving Bogoliubov Hamiltonian

\[
H^{Bog} := \sum_{j \in \mathbb{Z}^d} k_j^2 a_j^* a_j \tag{1.15}
\]
\[
+ \frac{\lambda}{2} \int \int \{a_0(x)a_0(y)\phi_{(x0)}(x-y) + a_+(x)a_+(y) + h.c.\} dx dy \tag{1.16}
\]
\[
+ \frac{\lambda}{2} \int \int \{a_0(x)a_0(y)\phi_{(x0)}(x-y) + a_+(x)a_+(y) + h.c.\} dx dy \tag{1.17}
\]
\[
+ \lambda \int \int \{a_0(x)a_0(y)\phi_{(x0)}(x-y) + a_+(x)a_+(y) + h.c.\} dx dy \tag{1.18}
\]
that we can express in terms of the field modes
\[ H^{\text{Bog}} = \sum_{j \in \mathbb{Z}^d \setminus \{0\}} \left( k_j^2 + \lambda \frac{\phi_j}{|\Lambda|} a_j^* a_j \right) + \frac{\lambda}{2} \sum_{j \in \mathbb{Z}^d \setminus \{0\}} \frac{\phi_j}{|\Lambda|} \left\{ a_j^* a_0^* a_j a_{-j} + a_j^* a_{-j}^* a_0 a_j \right\}. \]  

(1.19)

Hence, the Hamiltonian \( H \) corresponds to
\[ H = H^{\text{Bog}} + V \]  

(1.20)

with
\[ V := \lambda \int \int a_j^* (x) a_0^* (y) \phi_{(\neq 0)} (x - y) a_j (x) a_j (y) dxdy \]  

(1.21)

\[ + \lambda \int \int a_j^* (x) a_{-j}^* (y) \phi_{(\neq 0)} (x - y) a_j (x) a_0 (y) dxdy \]  

(1.22)

\[ + \frac{\lambda}{2} \int \int a_j^* (x) a_{-j}^* (y) \phi_{(\neq 0)} (x - y) a_j (x) a_j (y) dxdy. \]  

(1.23)

Following the convention of [Pi1], we set
\[ \lambda = \frac{1}{\rho}, \quad N := \rho |\Lambda| \quad \text{and even}, \]

(1.24)

where \( \rho > 0 \) is the particle density.

Assuming that
\[ \phi (z) = \frac{1}{|\Lambda|} \phi_0 + \frac{1}{|\Lambda|} \sum_{m=1}^{M} \phi_{j_m} (e^{i k_{j_m} \cdot z} + e^{-i k_{j_m} \cdot z}) \]

(1.25)

with \( M < \infty \) and \( j_m \neq 0 \), we define
\[ V_{j_1, \ldots, j_m} := \sum_{j_1=1}^{m} \ldots \sum_{j_m=1}^{m} \left( a_{j_1}^* a_0^* \phi_{j_1} a_{j_1} a_j + h.c. \right) \]  

(1.26)

\[ = \frac{1}{N} \sum_{j_1=1}^{m} \sum_{j_2=1}^{m} \ldots \sum_{j_m=1}^{m} \left( a_{j_1+j_2}^* a_0^* \phi_{j_2} a_{j_2} a_{j_1} + h.c. \right) \]  

(1.27)

\[ + \frac{1}{N} \sum_{j_1=1}^{m} \sum_{j_2=1}^{m} \ldots \sum_{j_m=1}^{m} a_{j_1+j_2}^* a_0^* \phi_{j_1} a_{j_1} a_{j_2} + h.c. \]  

(1.28)

\[ + \frac{1}{N} \sum_{j_1=1}^{m} \sum_{j_2=1}^{m} \ldots \sum_{j_m=1}^{m} \sum_{j_1'=1}^{m} a_{j_1+j_1'}^* a_{j_1+j_1'}^* \phi_{j_1} a_{j_1} a_{j_1} + h.c. \]  

(1.29)

Consequently, we can write
\[ V_{j_1, \ldots, j_m} = \frac{1}{\rho} \int \int a_j^* (x) a_0^* (y) \phi_{(\neq 0)} (x - y) a_j (x) a_j (y) dxdy \]  

(1.30)

\[ + \frac{1}{\rho} \int \int a_j^* (x) a_{-j}^* (y) \phi_{(\neq 0)} (x - y) a_j (x) a_0 (y) dxdy \]  

(1.31)

\[ + \frac{1}{2\rho} \int \int a_j^* (x) a_{-j}^* (y) \phi_{(\neq 0)} (x - y) a_j (x) a_j (y) dxdy. \]  

(1.32)

Notation
1. The symbol $\mathbb{1}$ stands for the identity operator. If helpful we specify the Hilbert space where it acts. For $c$–number operators, e.g., $z\mathbb{1}$, we may omit the symbol $\mathbb{1}$.

2. The symbol $\langle \ , \ \rangle$ stands for the scalar product in $\mathcal{F}^N$.

3. The symbol $O(\alpha)$ stands for a quantity bounded in absolute value by a constant times $\alpha$ ($\alpha > 0$). The symbol $o(\alpha)$ stands for a quantity such that $o(\alpha)/\alpha \to 0$ as $\alpha \to 0$. Throughout the paper the implicit multiplicative constants are always independent of $N$.

4. In some cases we shall use explicit constants, e.g., $C^\#_I$, if the same quantity will be used in later proofs. Unless otherwise specified or unless it is obvious from the context, the explicit constants may depend on the size of the box and on the details of the potential, in particular on the number, $M$, of couples of nonzero frequency components ($\neq 0$) in the Fourier expansion of the pair potential.

5. The symbol $|\psi\rangle\langle\psi|$, with $\|\psi\|=1$, stands for the one-dimensional projection onto the state $\psi$.

6. The word mode for the wavelength $\frac{2\pi}{L} j$ (or simply for $j$) refers to the field mode associated with it.

7. Theorems and lemmas from the companions papers [Pi1] and [Pi2] are underlined, quoted in italic, and with the numbering that they have in the corresponding paper; e.g., [Theorem 3.1 of [Pi1]].

2 Multi-scale analysis in the particle states occupation numbers for the Bogoliubov Hamiltonian: Review of results

This section serves the purpose of collecting formulae and algorithms derived in [Pi1] and [Pi2]. We shall refer to them in Section 3 where the Feshbach flow associated with the Hamiltonian $H$ is defined. For the details of the strategy the reader is encouraged to consult the companion papers [Pi1], [Pi2].

2.1 The Feshbach flow associated with a three-modes system Hamiltonian $H_{Bog}$

We started our study in [Pi1] from the three-modes Bogoliubov Hamiltonian

\[ H_{Bog} := \sum_{j \in \mathbb{Z} \setminus \{0, \pm j_\ast\}} k_j^2 a_j^* a_j + \hat{H}_{Bog} \]

(2.1)

where (see the definitions in (2.3)-(2.4))

\[ \hat{H}_{Bog} := \hat{H}^0_{Bog} + W_{j_\ast} + W^*_{j_\ast} \]

(2.2)

involves the three modes $0, \pm j_\ast$ only. Therefore, $H_{Bog}$ is the sum of:
The identity $H^\text{Bog} = \frac{1}{2} \sum_{j \in \mathbb{Z}^3 \setminus \{0, \pm j\}} \hat{H}^\text{Bog}_j$ follows from the previous definitions.

**Remark 2.1.** We stress that $H^\text{Bog}_j$ contains the kinetic energy corresponding to all the modes whereas $\hat{H}^\text{Bog}_j$ contains the kinetic energy associated with the interacting modes only.

The multi-scale analysis in the occupation numbers of particle states relies on a novel application of Feshbach map and yields a trivial effective Hamiltonian (i.e., a multiple of a one dimensional orthogonal projection) in a neighborhood of the ground state energy.

For the Hamiltonian $H^\text{Bog}_j$ applied to $\mathcal{F}^N$, we define:

- $Q^{(0,1)}_j :=$ the projection (in $\mathcal{F}^N$) onto the subspace generated by vectors with $N-0=N$ or $N-1$ particles in the modes $j$ and $-j$, i.e., the operator $a^*_j a_j + a^*_j a_{-j}$ has eigenvalues $N$ and $N-1$ when restricted to $Q_j^{(0,1)} \mathcal{F}^N$;

- $Q^{(>1)}_j :=$ the projection onto the orthogonal complement of $Q_j^{(0,1)} \mathcal{F}^N$ in $\mathcal{F}^N$.

Therefore, we have

$$Q_j^{(0,1)} + Q_j^{(>1)} = \mathbb{1}_{\mathcal{F}^N}. \tag{2.5}$$

Analogously, starting from $i = 2$ up to $i = N-2$ with $i$ even, we define $Q_j^{(i+1)}$ the projection onto the subspace of $Q_j^{(>1)} \mathcal{F}^N$ spanned by the vectors with $N-i$ or $N-i-1$ particles in the modes $j$ and $-j$. Analogously, $Q_j^{(i+1)}$ is the projection onto the orthogonal complement of $Q_j^{(i+1)} Q_j^{(>1)} \mathcal{F}^N$ in $Q_j^{(>1)} \mathcal{F}^N$, i.e.,

$$Q_j^{(i+1)} + Q_j^{(i+1)} = Q_j^{(>1)} \tag{2.5}.$$

We recall that given the (separable) Hilbert space $\mathcal{H}$, the projections $\mathcal{P}, \overline{\mathcal{P}}$ ($\mathcal{P} = \mathcal{P}^2, \overline{\mathcal{P}} = \overline{\mathcal{P}}^2$) where $\mathcal{P} + \overline{\mathcal{P}} = \mathbb{1}_\mathcal{H}$, and a closed operator $K - z$, $z$ in a subset of $\mathbb{C}$, acting on $\mathcal{H}$, the Feshbach map associated with the couple $\mathcal{P}, \overline{\mathcal{P}}$ maps $K - z$ to the operator $\mathcal{F}(K - z)$ acting on $\mathcal{P}\mathcal{H}$ where (formally)

$$\mathcal{F}(K - z) := \mathcal{P}(K - z) \mathcal{P} - \mathcal{P} K \overline{\mathcal{P}} \frac{1}{\mathcal{P}(K - z) \mathcal{P}} \overline{\mathcal{P}} K \mathcal{P}. \tag{2.6}$$
We iterate the Feshbach map starting from \( i = 0 \) up to \( i = N - 2 \) with \( i \) even, using the projections \( \mathcal{P}^{(i)} \) and \( \mathcal{P}^{(0)} \) for the \( i \)-th step of the iteration where

\[
\mathcal{P}^{(i)} := Q^{(i+1)}_{j}, \quad \mathcal{P}^{(0)} := Q^{(i+1)}_{j}.
\] (2.7)

We denote by \( \mathcal{F}^{(i)} \) the Feshbach map at the \( i \)-th step. We start applying \( \mathcal{F}^{(0)} \) to \( H^{\text{Bog}}_{j} \) \( - \) \( z \) where \( z \in \mathbb{R} \) ranges in the interval \(( -\infty, z_{\text{max}} )\) with \( z_{\text{max}} \) larger but very close to

\[
E^{\text{Bog}}_{j} := -\left[ k_{j}^{2} + \phi_{j} - \sqrt{(k_{j}^{2})^2 + 2\phi_{j}k_{j}^{2}} \right].
\] (2.8)

More precisely, for \( 0 \leq i \leq N - 2 \), we consider

\[
z \leq E^{\text{Bog}}_{j} + (\delta - 1)\phi_{j} - \sqrt{\epsilon_{j}^{2} + 2\epsilon_{j}}, \quad \epsilon_{j} := \frac{k_{j}^{2}}{\phi_{j}},
\] (2.9)

with \( \delta = 1 + \frac{1}{N} \leq \epsilon_{j} \) for some \( \nu > 1 \), and \( \epsilon_{j} \) sufficiently small; see point 3. in Definition 1.1.

As a result of the flow (for details see [Pi1] or Section 2.1 of [Pi2]), for \( i = 2, 4, 6, \ldots, N - 2 \) we obtain the Feshbach Hamiltonians

\[
\mathcal{H}^{\text{Bog}}_{j}(z) = Q^{(i+1)}_{j}(H^{\text{Bog}}_{j} - z)Q^{(i+1)}_{j}
\] (2.10)

where we have used the notation:

- \( W_{j;2,2}^{r} := Q^{(i+1)}_{j}W^{r}_{j}, W^{r}_{j} := Q^{(i+1)}_{j}Q^{(i+1)}_{j}Q^{(i+1)}_{j} \),

- \( R^{\text{Bog}}_{j;2,2}(z) := Q^{(i+1)}_{j}/Q^{(i+1)}_{j}(H^{\text{Bog}}_{j} - z)Q^{(i+1)}_{j} \);

- for \( i \geq 4 \),

\[
\Gamma^{\text{Bog}}_{j;2,2}(z) := W_{j;2,0}R^{\text{Bog}}_{j;0,0}(z)W^{r}_{j}.
\] (2.11)

For the last implementation of the Feshbach flow a new couple of projections is considered: \( \mathcal{P}_{\eta} := |\eta\rangle \langle \eta | \) (where \( \eta \) is the normalized state with all the particles in the zero mode) and \( \overline{\mathcal{P}_{\eta}} \) such that

\[
\mathcal{P}_{\eta} + \overline{\mathcal{P}_{\eta}} = \mathcal{I}_{Q^{(i+1)}_{j}}.
\] (2.14)
We notice that for \( i = N - 2 \) the projection \( Q_j^{(i+1\equiv N-1)} \) coincides with the projection onto the subspace where less than \( N - i - 1 = N - N + 1 = 1 \) particles in the modes \( j \) and \( -j \) are present, i.e., where no particle in the modes \( j \) and \( -j \) is present.

Starting from the formal expression

\[
\mathcal{K}_j^{Bog(N)}(z) := \mathcal{P}_j^{(N)}(\mathcal{K}_j^{Bog(N-2)}(z)) = \mathcal{P}_j^{(N)}(H_j^{Bog} - z) \mathcal{P}_j
\]

\[
-\mathcal{P}_j W_j R_j^{Bog}(z) \sum_{l_{N-2}=0}^{\infty} \langle f_{j_{l_{N-2}}}^{Bog}(z) R_j^{Bog}(z) \rangle_{j_{l_{N-2}}} W_j^* \mathcal{P}_j
\]

\[
-\mathcal{P}_j W_j \frac{1}{\mathcal{P}_j \mathcal{K}_j^{Bog(N-2)}(z) \mathcal{P}_j} \mathcal{P}_j W_j^* \mathcal{P}_j,
\]

the argument implemented in [Pi1] shows that the expression on the R-H-S of (2.17) is well defined for \( z \) such that

\[
z < \min \{ \frac{\Delta_0}{2} : E^{Bog}_{j} + \sqrt{\epsilon_j} \phi_j \sqrt{\epsilon_j^2 + 2\epsilon_j} \}
\]

(2.18)

where \( z_\star \) is the unique solution of \( f_j(z) = 0 \) with

\[
f_j(z) := -z - \langle \eta, W_j R_j^{Bog}(z) \rangle_{j} \sum_{l_{N-2}=0}^{\infty} \langle f_{j_{l_{N-2}}}^{Bog}(z) R_j^{Bog}(z) \rangle_{j_{l_{N-2}}} W_j^* \eta,
\]

(2.19)

Indeed, for \( z \) in the range in (2.18) the operator

\[
\mathcal{P}_j \mathcal{K}_j^{Bog(N-2)}(z) \mathcal{P}_j
\]

is bounded invertible in \( \mathcal{P}_j F^{N} \). Finally, the identities

\[
\mathcal{P}_j^{(N)}(H_j^{Bog} - z) \mathcal{P}_j = -z \mathcal{P}_j \qquad \mathcal{P}_j W_j^* \mathcal{P}_j = \mathcal{P}_j W_j \mathcal{P}_j = 0
\]

(2.21)

imply

\[
\mathcal{K}_j^{Bog(N)}(z) = f_j(z) |\eta\rangle \langle \eta|.
\]

(2.22)

The ground state energy, \( z_\star \), and the (non-normalized) ground state vector of the Hamiltonian \( H_j^{Bog} \) are then obtained exploiting Feshbach theory:

\[
\psi_j^{Bog} := \eta
\]

(2.23)

\[
-\frac{1}{Q_j^{(N-2,N-1)}(z_\star) Q_j^{(N-2,N-1)}(z_\star)} Q_j^{(N-2,N-1)} W_j^* \eta
\]

(2.24)

\[
-\sum_{j=2}^{N/2} \prod_{r=j}^{2} \left[ -\frac{1}{Q_j^{(N-2r,N-2r+1)}(z_\star) Q_j^{(N-2r,N-2r+1)}(z_\star)} W_j^* \right] \times
\]

(2.25)

\[
\times \frac{1}{Q_j^{(N-2,N-1)}(z_\star) Q_j^{(N-2,N-1)}(z_\star)} Q_j^{(N-2,N-1)} W_j^* \eta.
\]
where $\mathcal{K}_{j_1}^{\text{Bog}}(z) := H_{j_1}^{\text{Bog}} - z$. The norm of the sum in (2.25) is bounded by a multiple of
\[
\sum_{j=2}^{\infty} c_j := \sum_{j=2}^{\infty} \left\{ \prod_{l=j}^{2} \left[ 1 + \sqrt{\eta \epsilon_j} - \frac{b_j \epsilon_j}{2l - \epsilon_j} \right] \left[ 1 + a_j \epsilon_j - \frac{2b_j \epsilon_j}{2(j+1)} \right] \right\} \leq 1
\]
(2.26)
which is convergent for $\epsilon_j > 0$ because
\[
c_j = \frac{1}{c_{j-1}} \left[ 1 + \sqrt{\eta \epsilon_j} - \frac{b_j \epsilon_j}{2j - \epsilon_j} \right] \left[ 1 + a_j \epsilon_j - \frac{2b_j \epsilon_j}{2(j+1)} \right] < 1
\]
(2.27)
for $j$ sufficiently large, where $a_j, b_j, c_j$, and $0 < \Theta \leq \frac{1}{4}$ are those defined in Lemma 3.6 of [PiI]. The series in (2.26) diverges in the limit $\epsilon_j \to 0$. Hence, for any $\epsilon_j > 0$ fulfilling the Strong Interaction Potential Assumption (see point 3.) in Definition 1.1) we have derived an expansion of $\psi_j^{\text{Bog}}$ controlled by the parameter $\epsilon_j := \frac{1}{1 + \sqrt{\eta \epsilon_j} \sqrt{\epsilon_j}}$.

The ground state energy $z_*$ approaches $E_{j_1}^{\text{Bog}}$ as $N \to \infty$, more precisely (see Lemma 5.5 of [PiI]) in the mean field limiting regime the estimate $|z_* - E_{j_1}^{\text{Bog}}| \leq O(\frac{1}{\sqrt{N}})$ holds for any $0 < \beta < 1$. Starting from the formula in (2.23)-(2.25) and from the definitions in (2.11)-(2.12), in Section 4.4 of [PiI] we show how to expand the ground state $\psi_{j_1}^{\text{Bog}}$ in terms of the bare operators $\frac{1}{H_{j_1}^{\text{Bog}}} |n_E \rangle$ and $W_{j_1}$, applied to the vector $\eta$, up to any desired precision provided $N$ is sufficiently large.

**Remark 2.2.** We observe that $\psi_{j_1}^{\text{Bog}}$ is also eigenvector of $\hat{H}_{j_1}^{\text{Bog}}$ with the same eigenvalue (see the definition in (2.2)).

### 2.2 The Feshbach flows associated with the intermediate Hamiltonians $H_{j_1, \ldots, j_m}^{\text{Bog}}$

In the paper [Pi2] we have shown how the ground state of the Hamiltonian
\[
H_{j_1, \ldots, j_m}^{\text{Bog}} := \sum_{j \in \mathbb{Z}^d(\{j_1, \ldots, j_m\})} k_j^2 a_j^* a_j + \sum_{l=1}^{m} \hat{H}_{j_l}^{\text{Bog}},
\]
(2.28)
with $1 \leq m \leq M$, can be constructed by means of an inductive procedure. At each step of this procedure we exploit the Feshbach map where the (Feshbach) projections are associated with a three-modes system. In the following we shall outline the procedure.

We start from $H_{j_1}^{\text{Bog}}$ and using the results of Section 2.1 we construct
\[
\mathcal{K}_{j_1}^{\text{Bog}(N)}(z) := \mathcal{P}_{\eta}(H_{j_1}^{\text{Bog}} - z) \mathcal{P}_{\eta} - \mathcal{P}_{\eta} W_{j_1} \sum_{l_{N-2}=0}^{\infty} R_{j_1;N-2,N-2}^{\text{Bog}}(z) \left[ R_{j_1;N-2,N-2}^{\text{Bog}}(z) R_{j_1;N-2,N-2}^{\text{Bog}}(z) \right]^{l_{N-2}} W_{j_1}^* \mathcal{P}_{\eta}.
\]
(2.29)
Next, we determine the ground state energy, $z_{j_1}^{\text{Bog}}$, of $H_{j_1}^{\text{Bog}}$ by imposing
\[
z_{j_1}^{\text{Bog}} = \langle \eta, W_{j_1} \sum_{l_{N-2}=0}^{\infty} R_{j_1;N-2,N-2}^{\text{Bog}}(z_{j_1}) \left[ R_{j_1;N-2,N-2}^{\text{Bog}}(z_{j_1}) R_{j_1;N-2,N-2}^{\text{Bog}}(z_{j_1}) \right]^{l_{N-2}} W_{j_1}^* \eta \rangle.
\]
(2.30)
Hence, the (non-normalized) ground state vector, $\psi_{j_1}^{Bog}$, of $H_{j_1}^{Bog}$ is given in (2.23)-(2.25) with $j_z, z_z$ replaced with $j_1$ and $z_{j_1}^{Bog}$, respectively.

In the next step, we consider the intermediate Hamiltonian

$$H_{j_1; j_2}^{Bog} := \sum_{j \in \mathbb{Z}^d \setminus \{j_1, j_2\}} k_j^2 a_j^* a_j + \hat{H}_{j_1; j_2}^{Bog} := \sum_{j \in \mathbb{Z}^d \setminus \{j_1, j_2\}} k_j^2 a_j^* a_j + \sum_{l=1}^2 H_{j_l}^{Bog} \tag{2.31}$$

and construct the Feshbach Hamiltonians

$$\mathcal{H}_{j_1; j_2}^{Bog}(z_{j_1}^{Bog} + z) = Q_{j_2}^{(i+1)} (H_{j_1; j_2}^{Bog} - z_{j_1}^{Bog} - z) Q_{j_2}^{(i+1)} \tag{2.32}$$

$$- \sum_{l=0}^\infty Q_{j_2}^{(i+1)} W_{j_1; j_2} \Gamma_{j_1; j_2}^{Bog}(z_{j_1}^{Bog} + z) \Gamma_{j_1; j_2}^{Bog}(z_{j_1}^{Bog} + z) W_{j_1; j_2}^{\ast} Q_{j_2}^{(i+1)} \tag{2.33}$$

for $0 \leq i \leq N - 2$ and even, where we use the definitions:

* $\Gamma_{j_1; j_2; i}(z_{j_1}^{Bog} + z) := W_{j_1; j_2} R_{j_1; j_2; i}^{Bog}(z_{j_1}^{Bog} + z) \tag{2.34}$

* $\Gamma_{j_1; j_2; i, l}(z_{j_1}^{Bog} + z) := W_{j_1; j_2} R_{j_1; j_2; i, l}^{Bog}(z_{j_1}^{Bog} + z) \tag{2.35}$

and, for $i \geq 4$ and even,

$$\Gamma_{j_1; j_2; i, l}(z_{j_1}^{Bog} + z) := W_{j_1; j_2} R_{j_1; j_2; i, l-2}^{Bog}(z_{j_1}^{Bog} + z) \sum_{l_1 = 2}^\infty \left[ \Gamma_{j_1; j_2; i-2, l_1}^{Bog}(z_{j_1}^{Bog} + z) R_{j_1; j_2; i-2, l_1}^{Bog}(z_{j_1}^{Bog} + z) \right]^{l_1-2} W_{j_1; j_2}^{\ast} \tag{2.36}$$

In the last implementation of the Feshbach map we make use of the projections

$$P_{\psi_{j_1}^{Bog}} := \frac{|\psi_{j_1}^{Bog}\rangle}{||\psi_{j_1}^{Bog}\rangle||} \frac{\psi_{j_1}^{Bog}}{||\psi_{j_1}^{Bog}\rangle||} \quad \text{and} \quad P_{\psi_{j_1}^{Bog}}^\ast := \frac{1}{Q_{j_2}^{(i+1)}} - P_{\psi_{j_1}^{Bog}} \tag{2.37}$$

where $\frac{1}{Q_{j_2}^{(i+1)}}$ is the projection onto the subspace of states of $\mathcal{F}^N$ with no particles in the modes $\pm j_2$, and we define

$$\Gamma_{j_1; j_2; N,N}^{Bog}(z_{j_1}^{Bog} + z) := W_{j_1; j_2} R_{j_1; j_2; N-2, N-2}^{Bog}(z_{j_1}^{Bog} + z) \sum_{l_1 = 2}^\infty \left[ \Gamma_{j_1; j_2; N-2, l_1}^{Bog}(z_{j_1}^{Bog} + z) R_{j_1; j_2; N-2, l_1}^{Bog}(z_{j_1}^{Bog} + z) \right]^{l_1-2} W_{j_1; j_2}^{\ast} \tag{2.38}$$

For the derivation of $\mathcal{H}_{j_1; j_2}^{Bog}(N)(z_{j_1}^{Bog} + z)$, we point out that (see Remark 2.2)

$$(\hat{H}_{j_1}^{Bog} - z_{j_1}^{Bog}) P_{\psi_{j_1}^{Bog}} = 0 \quad \sum_{j \in \mathbb{Z}^d \setminus \{0, j_1\}} a_j^* a_j P_{\psi_{j_1}^{Bog}} = 0$$

and

$$P_{\psi_{j_1}^{Bog}}(H_{j_1}^{Bog} - z_{j_1}^{Bog}) P_{\psi_{j_1}^{Bog}} = P_{\psi_{j_1}^{Bog}}(\hat{H}_{j_1}^{Bog}) P_{\psi_{j_1}^{Bog}} = P_{\psi_{j_1}^{Bog}}(W_{j_1} + W_{j_2}^{\ast}) P_{\psi_{j_1}^{Bog}} = 0.$$
The identities follow from the definitions of $\mathcal{P}_{\text{Bog,1}}$, $\mathcal{P}_{\text{Bog,2}}$, and $H_{\text{Bog}}$ combined with the fact that $Q_{j_{2}^{>N-1}}$ is the projection onto the subspace of $\mathcal{F}^{N}$ of states with no particles in the modes $\pm j_{2}$. Formally, we get

$$\mathcal{P}_{\text{Bog,1}}(H_{\text{Bog}} - zj_{1} - z)\mathcal{P}_{\text{Bog,1}} = \mathcal{P}_{\text{Bog,1}}(H_{\text{Bog}} - z)\mathcal{P}_{\text{Bog,1}} = \mathcal{P}_{\text{Bog,1}}(W_{j_{1}} + W_{j_{1}}^{*})\mathcal{P}_{\text{Bog,1}} = 0.$$ (2.41)

## \[ \begin{align*}
\mathcal{P}_{\text{Bog,1}}(H_{\text{Bog}} - zj_{1} - z)\mathcal{P}_{\text{Bog,1}} &= \mathcal{P}_{\text{Bog,1}}(H_{\text{Bog}} - z)\mathcal{P}_{\text{Bog,1}} = \mathcal{P}_{\text{Bog,1}}(W_{j_{1}} + W_{j_{1}}^{*})\mathcal{P}_{\text{Bog,1}} = 0.
\end{align*} \]

These identities follow from the definitions of $\mathcal{P}_{\text{Bog,1}}$, $\mathcal{P}_{\text{Bog,2}}$, and $H_{\text{Bog}}$ combined with the fact that $Q_{j_{2}^{>N-1}}$ is the projection onto the subspace of $\mathcal{F}^{N}$ of states with no particles in the modes $\pm j_{2}$. Formally, we get

$$\mathcal{P}_{\text{Bog,1}}(H_{\text{Bog}} - zj_{1} - z)\mathcal{P}_{\text{Bog,1}} = \mathcal{P}_{\text{Bog,1}}(H_{\text{Bog}} - z)\mathcal{P}_{\text{Bog,1}} = \mathcal{P}_{\text{Bog,1}}(W_{j_{1}} + W_{j_{1}}^{*})\mathcal{P}_{\text{Bog,1}} = 0.$$ (2.41)

$$\mathcal{P}_{\text{Bog,1}}(H_{\text{Bog}} - zj_{1} - z)\mathcal{P}_{\text{Bog,1}} = \mathcal{P}_{\text{Bog,1}}(H_{\text{Bog}} - z)\mathcal{P}_{\text{Bog,1}} = \mathcal{P}_{\text{Bog,1}}(W_{j_{1}} + W_{j_{1}}^{*})\mathcal{P}_{\text{Bog,1}} = 0.$$ (2.41)

We determine the ground state energy, $\zeta^{(2)}_{j_{1},j_{2}} := \zeta^{(2)}_{j_{1}}$, of $H_{j_{1},j_{2}}$ by imposing

$$\zeta^{(2)}_{j_{1},j_{2}} = -\langle \frac{\psi_{\text{Bog}}}{||\psi_{\text{Bog}}||}, \Gamma_{j_{1},j_{2};N,N}(z^{(2)}_{j_{1}} + z^{(2)}_{j_{1}}) \frac{\psi_{\text{Bog}}}{||\psi_{\text{Bog}}||} \rangle.$$ (2.48)

$$\zeta^{(2)}_{j_{1},j_{2}} = -\langle \frac{\psi_{\text{Bog}}}{||\psi_{\text{Bog}}||}, \Gamma_{j_{1},j_{2};N,N}(z^{(2)}_{j_{1}} + z^{(2)}_{j_{1}}) \frac{\psi_{\text{Bog}}}{||\psi_{\text{Bog}}||} \rangle.$$ (2.48)

Hence, the ground state vector of $H_{j_{1},j_{2}}$ is (up to normalization)

$$\psi_{\text{Bog}}_{j_{1},j_{2}} := \psi_{\text{Bog}}_{j_{1},j_{2}}.$$ (2.51)

$$\psi_{\text{Bog}}_{j_{1},j_{2}} := \psi_{\text{Bog}}_{j_{1},j_{2}}.$$ (2.51)
where \( \mathcal{K}^{Bog}_{j_1,j_2} := H^{Bog}_{j_1,j_2} - z_{j_1,j_2} \).

At the \( m \)th step, first we define

\[
H^{Bog}_{j_1,...,j_m} := \sum_{j \in \mathbb{Z}^N \setminus \{j_1,...,j_m\}} k_j^2 \tilde{a}_{j}^* \tilde{a}_{j} + \sum_{l=1}^{m} \hat{H}^{Bog}_{j_l} . \tag{2.55}
\]

(The reader should notice that the kinetic energy of the interacting nonzero modes, \( \pm j_1, \ldots, \pm j_m \), is contained in \( \sum_{j=1}^{m} \hat{H}^{Bog}_{j_l} \)). Then, we construct

\[
\mathcal{K}^{Bog(N)}_{j_1,...,j_m} (z + z_{j_1,...,j_m-1}) \quad \tag{2.56}
\]

\[
= -z \mathcal{P}^{Bog}_{j_1,...,j_m-1} \quad \tag{2.57}
\]

\[
- \mathcal{P}^{Bog}_{j_1,...,j_m-1} \Gamma^{Bog}_{j_1,...,j_m;N,N} (z + z_{j_1,...,j_m-1}) \mathcal{P}^{Bog}_{j_1,...,j_m-1} \quad \tag{2.58}
\]

\[
- \mathcal{P}^{Bog}_{j_1,...,j_m-1} \Gamma^{Bog}_{j_1,...,j_m;N,N} (z + z_{j_1,...,j_m-1}) \mathcal{P}^{Bog}_{j_1,...,j_m-1} \times \quad \tag{2.59}
\]

\[
1 \quad \tag{2.60}
\]

with definitions analogous to (2.34)-(2.40):

\[ \mathcal{P}^{Bog}_{j_1,...,j_m-1} := \left| \frac{\tilde{a}_{j_1}^* \tilde{a}_{j_1-1}^*}{||\tilde{a}_{j_1}^* \tilde{a}_{j_1-1}^*||} \right| , \quad \mathcal{P}^{Bog}_{0} := \mathbb{I}_{Q^{(N-1)}}, \quad \mathcal{P}^{Bog}_{j_1-1} := \frac{1}{Q^{(N-1)}_{j_1}} Q^{(N-1)}_{j_1} - \mathcal{P}^{Bog}_{j_1-1} ; \tag{2.61} \]

\[ R^{Bog}_{j_1,...,j_m;i,i} (z) := \frac{1}{Q^{(i,i+1)}_{j_1,...,j_m}} \left( H^{Bog}_{j_1,...,j_m} - z_{j_1,...,j_m-1} - z \right) Q^{(i,i+1)}_{j_1,...,j_m} ; \tag{2.62} \]

\[ \Gamma^{Bog}_{j_1,...,j_m;2,2} (z) := W^{Bog}_{j_1,...,j_m;0,0} \left( \tilde{z}_{j_1,...,j_m-1} + z \right) W^{*}_{j_1,...,j_m;0,2} \tag{2.63} \]

and, for \( N = 2 \geq i \geq 4 \) and even,

\[ \Gamma^{Bog}_{j_1,...,j_m;i,i} (z) := W^{Bog}_{j_1,...,j_m;i,i} R^{Bog}_{j_1,...,j_m;i,i} (z) \times \tag{2.64} \]

\[ \times \sum_{l_{i-2} = 0}^{\infty} \left[ \Gamma^{Bog}_{j_1,...,j_m;i,i} (z) R^{Bog}_{j_1,...,j_m;i,i} (z) \right]^{l_{i-2}} W^{*}_{j_1,...,j_m;i,i} \tag{2.65} \]

and

\[ \Gamma^{Bog}_{j_1,...,j_m;N,N} (z) := W^{Bog}_{j_1,...,j_m} R^{Bog}_{j_1,...,j_m;N,N} (z) \times \tag{2.66} \]

\[ \times \sum_{l_{N-2} = 0}^{\infty} \left[ \Gamma^{Bog}_{j_1,...,j_m;N,N} (z) R^{Bog}_{j_1,...,j_m;N,N} (z) \right]^{l_{N-2}} W^{*}_{j_1,...,j_m} . \tag{2.67} \]
We compute the ground state energy, \( \psi_{Bog}^{J_1,\ldots,J_m} := \psi_{Bog}^{J_1,\ldots,J_{m-1}} + \psi^{(m)} \), of \( H_{Bog}^{J_1,\ldots,J_m} \) by solving the equation (in \( z \))

\[
J_{Bog}^{J_1,\ldots,J_m} (z + \psi_{Bog}^{J_1,\ldots,J_{m-1}}) = 0;
\]  

see (2.60). Hence, the ground state vector of \( H_{Bog}^{J_1,\ldots,J_m} \) is (up to normalization)

\[
\psi_{Bog}^{J_1,\ldots,J_m} := \psi_{Bog}^{J_1,\ldots,J_{m-1}} + \sum_{j=2}^{N/2} \left\{ \prod_{r=j}^{2} \left( \frac{1}{Q_{j_m}^{(N-2,2r-N-2r+1)}} \mathcal{K}_{Bog}^{(N-2r-2)} \psi_{Bog}^{J_1,\ldots,J_m} \right) + \mathbb{I} \right\} \times \]

\[
\times \left[ Q_{j_m}^{(N-2N-1)} \mathcal{K}_{Bog}^{(N-4)} \psi_{Bog}^{J_1,\ldots,J_m} \right] \times \]

\[
\times \left[ \mathcal{P}_{Bog}^{J_1,\ldots,J_{m-1}} \mathcal{K}_{Bog}^{(J_1,\ldots,J_m)} \psi_{Bog}^{J_1,\ldots,J_{m-1}} \right] =: \psi_{Bog}^{J_1,\ldots,J_{m-1}}
\]

where \( \mathcal{K}_{Bog}^{(J_1,\ldots,J_m)} := H_{Bog}^{J_1,\ldots,J_m} - \psi_{Bog}^{J_1,\ldots,J_m} \). Thus, we have derived the formula

\[
\psi_{Bog}^{J_1,\ldots,J_m} = T_M \cdots T_1 \eta.
\]  

Some observations are in order to understand why the procedure that we have described is not a straightforward iteration of the operations implemented for a three-modes system. Indeed, as more couples of interacting modes are considered (i.e., starting from \( H_{Bog}^{J_1,\ldots,J_m} \)) the main task is showing that the interaction terms associated with the couples of modes, \( \pm J_1, \ldots, \pm J_m \), are to some extent independent. This becomes apparent since:

- for \( m \geq 2 \) the term in (2.59) is shown to be vanishing as \( N \to \infty \);
- at later steps (i.e., starting from \( m = 2 \)) the fixed point equation in (2.69) can be written as a three-modes system fixed point equation plus a small correction that vanishes as \( N \to \infty \).

The construction implemented in [Pi2] culminates in the theorem below.

**Theorem 4.3 of [Pi2]** Let \( \max_{1 \leq m \leq M} \epsilon_{J_m} \) be sufficiently small and \( N \) sufficiently large. Then the following properties hold true for all \( 1 \leq m \leq M \):

1. The Feshbach Hamiltonian \( \mathcal{K}_{Bog}^{(J_1,\ldots,J_m)} (z + \psi_{Bog}^{J_1,\ldots,J_{m-1}}) \) in (2.56)-(2.60) is well defined for

\[
z \leq \min \left\{ z_m + \gamma \Delta_{m-1} \frac{C}{(\ln N)^{3/2}}, E_{J_m}^{Bog} + \sqrt{\epsilon_{J_m}} \phi_{J_m} \sqrt{\epsilon_{J_m}^2 + 2 \epsilon_{J_m}} \right\}, \quad \gamma = \frac{1}{2},
\]  

where:

- \( z_{J_1,\ldots,J_{m-1}} \) is the ground state energy of \( H_{J_1,\ldots,J_{m-1}}^{Bog} \) and is defined iteratively in point 2. below;
- \( z_m \) is the ground state energy of \( H_{J_m}^{Bog} \),
\[ \Delta_{m-1} \text{ (for } m \geq 1) \text{ is defined iteratively by } \Delta_0 := \min \left\{ (k_j)^2 \mid j \in \mathbb{Z}^d \setminus \{0\} \right\} \text{ and} \]

\[
\Delta_m := \gamma \Delta_{m-1} - \frac{C_{\perp}}{(\ln N)^2} - \left( \frac{2}{\gamma} \right)^m \frac{C_{III}}{(\ln N)^2} \tag{2.76}
\]

with \( C_{III} := C_I + \frac{C_{II}}{(1 - \gamma)^{\Delta_0}} \), where \( C_I, C_{II} \) are introduced in Lemma 4.3 of [Pi2].

2. For \( z \) as in (2.75), there exists a unique value \( z^{(m)} \) such that

\[
\hat{J}^{Bog}_{j_1, \ldots, j_m} (z + z^{Bog}_{j_1, \ldots, j_m}) | z^{(m)} \rangle = 0.
\]

The inequality \( |z^{(m)} - z_m| \leq \left( \frac{2}{\gamma} \right)^m \frac{C_{III}}{(\ln N)^2} \) holds true.

The Hamiltonian \( H^{Bog}_{j_1, \ldots, j_m} \) has nondegenerate ground state energy \( z^{Bog}_{j_1, \ldots, j_m} := z^{Bog}_{j_1, \ldots, j_m} + z^{(m)} \), where \( z^{Bog}_{j_1, \ldots, j_m} \mid_{m=1} \equiv 0 \). The corresponding (non-normalized) eigenvector is given in (2.70)-(2.72).

3. The spectral gap of the two operators

\[
H^{Bog}_{j_1, \ldots, j_m} , \left( \hat{H}^{Bog}_{j_1, \ldots, j_m} + \sum_{j \in \{\pm 1, \ldots, \pm j_{m+1}\}} (k_j)^2 a_j^* a_j \right) \uparrow \mathcal{F}^N_{\mathbb{R}^{j_{m+1}}}
\]

above the (common) ground state energy \( z^{Bog}_{j_1, \ldots, j_m} \) is larger or equal to \( \Delta_m \).

4. The lower bound

\[
\inf_{\mathcal{A}} \left\{ \sum_{l=1}^{m} \hat{H}^{Bog}_{a_l} \right\} - z^{Bog}_{j_1, \ldots, j_m} \geq - \frac{m}{(\ln N)^{\frac{1}{2}}} \tag{2.78}
\]

holds true.

5. For \( \tilde{\Delta}_m := \sum_{l=1}^{m} \frac{\phi_l}{\Delta_0} \) the upper bound

\[
\frac{\langle \int \hat{J}^{Bog}_{j_1, \ldots, j_m} \rangle_{\gamma}}{\| \int \hat{J}^{Bog}_{j_1, \ldots, j_m} \|} , \sum_{j \in \mathbb{Z}^d \setminus \{0\}} a_j^* a_j \| \int \hat{J}^{Bog}_{j_1, \ldots, j_m} \| \leq \tilde{\Delta}_m \tag{2.79}
\]

holds true.

Like for the three-modes system, starting from the formula in (2.70)-(2.72), in Corollary 4.5 of [Pi2] we show how to expand the ground state vector \( \varphi^{Bog}_{j_1, \ldots, j_m} \) in terms of the bare operators \( \frac{1}{H^{Bog}_{j_l}} | \varphi^{Bog}_{j_l} \rangle \) and \( W_{j_l}^* \varphi^{Bog}_{j_l} \) with \( l = 1, \ldots, m \), applied to the vector \( \eta \), up to any desired precision provided \( N \) is sufficiently large.

3  Ground state of \( H \): Outline of the construction

Due to the ultraviolet cut-off on the two-body potential \( (\phi_j \text{ in Fourier space}) \) it is convenient to write

\[
H = H_{j_1, \ldots, j_m} = H_{j_1, \ldots, j_M}^{Bog} + V_{j_1, \ldots, j_m} \tag{3.1}
\]

where, for \( 1 \leq m \leq M \), \( V_{j_1, \ldots, j_m} \) is defined in (1.27)-(1.29).

The strategy to construct the ground state of \( H \) consists in three operations:

\[ \mathcal{F}^N_{\mathbb{R}^{j_{m+1}}} \] is the subspace of vectors in \( \mathcal{F}^N \) with at least one particle in the modes \( \pm j_{m+1} \).
1. We define intermediate Hamiltonians

\[ H_{j_1, \ldots, j_m} = H_{j_1, \ldots, j_m}^{Bog} + V_{j_1, \ldots, j_m} \]  \hspace{1cm} (3.2)

obtained by adding a couple of modes, \( \{ j_m, -j_m \} \) with \( 1 \leq m \leq M \), at a time to the pair potential (see (1.27)-(1.29)) so that we obtain \( H_{j_1, \ldots, j_M} \) at the \( M - th \) step;

2. At each step, i.e., for each intermediate Hamiltonian, we use the Feshbach map flow described in Section 3.1 and associated with the new couple of modes, \( \{ j_m, -j_m \} \), that has been added to the pair potential. In comparison with the construction of the ground state of \( H_{j_1, \ldots, j_M}^{Bog} \), the new construction requires “refined” Feshbach projections. Indeed, the term \( V_{j_1, \ldots, j_m} \) cannot be controlled using the projections of Section 2.1 for all index values \( i \) of the Feshbach flow. In the new scheme, we start from \( i = \tilde{i} = N - \lceil \frac{N}{2} \rceil \) where \( \lceil \frac{N}{2} \rceil \) is assumed to be even\(^3\). With a new choice of the couple of projections \( (\mathcal{P}^{(i)}, \mathcal{Q}^{(i)}) \), in the subspace \( \mathcal{F}^N \) the number of particles in the modes \( \{ \pm j_m \} \) can range between \( N - \lfloor \frac{N}{2} \rfloor - 1 \) and \( N \).

3. We use the projection onto the ground state of an auxiliary Hamiltonian, \( H_{j_1, \ldots, j_{M-1}}^{#} \), at the \((m - 1)-th\) step as the final projection of the Feshbach map flow at the \( m-th \) step. Differently from the case of the Bogoliubov Hamiltonians, for the Hamiltonian \( H_{j_1, \ldots, j_{M-1}} \), the restriction of the pair potential to the Fourier modes associated with the set \( \{ \pm j_1, \ldots, \pm j_{M-1} \} \) does not imply that the field modes associated with \( \pm j_m \) are absent in the interaction term. The Hamiltonian \( H_{j_1, \ldots, j_{M-1}}^{#} \) will be defined starting from \( H_{j_1, \ldots, j_{M-1}} \) by omitting these terms.

The features of the new projections and the details of the Feshbach flow are described in Section 3.1. The proofs to make the construction rigorous are deferred to Section 4.

### 3.1 The Feshbach flows associated with the intermediate Hamiltonians \( H_{j_1, \ldots, j_m} \): The new projections \( \mathcal{Q}_{j_m}^{(i+1)} \) and \( \mathcal{Q}_{j_m}^{(i+1)} \)

For the derivation of the Feshbach Hamiltonians associated with the Hamiltonian \( H_{j_1, \ldots, j_m} \), we assume that the ground state of the related Hamiltonian \( H_{j_1, \ldots, j_{M-1}}^{#} \) has been already constructed. Here, we define \( H_{j_1, \ldots, j_{M-1}}^{#} \) and introduce new notation:

**Definition 3.1.**

\[ H_{j_1, \ldots, j_{m-1}} |_{m=1} := T, \quad \text{for} \quad m \geq 2 \quad H_{j_1, \ldots, j_{M-1}}^{#} := T_{j_1, \ldots, j_{M-1}} + \hat{H}_{j_1, \ldots, j_{M-1}}^{Bog} + V_{j_1, \ldots, j_{M-1}}^{#} \]  \hspace{1cm} (3.3)

with

\[ T_{j_1, \ldots, j_{M-1}} := \sum_{j \in \mathbb{Z} \setminus \{ \pm j_1, \ldots, \pm j_{M-1} \}} k_j^2 a_j^* a_j; \]  \hspace{1cm} (3.4)

\[ V_{j_1, \ldots, j_{M-1}}^{#} := \frac{1}{N} \sum_{l=1}^{m-1} \sum_{j \in \mathbb{Z} \setminus \{ \pm j_1, \ldots, \pm j_{M-1} \}} a_{j+l}^0 \phi_{j+l} a_j a_l + h.c. \]  \hspace{1cm} (3.5)

\(^3\)The exponent \( \frac{1}{16} \) is not optimal.
In this section the derivation of the Feshbach Hamiltonians is only formal.

Remark 3.3. Thus, we can write

The construction is by induction in the index \( m \) ranging from \( m = 1 \) up to \( m = M \). In this outline we present the flow associated with \( H_{j_1, \ldots, j_m} \) and defer the details of the induction scheme to the next section.

**Definition 3.2.** For the first implementation of the Feshbach map applied to the Hamiltonian \( H_{j_1, \ldots, j_m} \) we employ the couple \( \mathcal{Q}_{j_m}^{(i, j+1)}, \mathcal{Q}_{j_m}^{(i, j+1)} \), \( i = N - \lfloor N \frac{1}{2} \rfloor \) assumed to be even, that is defined here for \( N \gg 1 \):

- \( \mathcal{Q}_{j_m}^{(i, j+1)} := \sum_{j=0, \text{even}} Q_{j_m}^{(i, j+1)} \) where \( Q_{j_m}^{(i, j+1)} \) is defined in Section 2.1. Therefore, \( \mathcal{Q}_{j_m}^{(i, j+1)} \) projects onto the subspace of vectors with a number of particles in the modes \( \pm j_m \) that can range from \( N \) to \( \lfloor N \frac{1}{2} \rfloor - 1 \), i.e., the operator \( a_{\pm j_m}^* a_{\pm j_m}^* a_{\pm j_m}^* a_{\pm j_m} \) has eigenvalues \( N, N - 1, \ldots, \lfloor N \frac{1}{2} \rfloor, \lfloor N \frac{1}{2} \rfloor - 1 \) when restricted to \( \mathcal{Q}_{j_m}^{(i, j+1)} \).

- \( \mathcal{Q}_{j_m}^{(i, j+1)} \) is the projection onto the orthogonal complement of \( \mathcal{Q}_{j_m}^{(i, j+1)} \) in \( \mathcal{F}^N \).

Thus, we can write

\[
\mathcal{Q}_{j_m}^{(i, j+1)} + \mathcal{Q}_{j_m}^{(i, j+1)} = \mathbb{1}_{\mathcal{F}^N}.
\]

Starting from \( i = 1 \) up to \( i = N - 2 \) with even:

- \( \mathcal{Q}_{j_m}^{(i, j+1)} \) is the projection onto the subspace of \( \mathcal{Q}_{j_m}^{(i, j+1)} \) spanned by the vectors with \( N - i \) or \( N - i - 1 \) particles in the modes \( \pm j_m \):

- \( \mathcal{Q}_{j_m}^{(i, j+1)} \) is the projection onto the orthogonal complement of \( \mathcal{Q}_{j_m}^{(i, j+1)} \) in \( \mathcal{Q}_{j_m}^{(i, j+1)} \) in \( \mathcal{Q}_{j_m}^{(i, j+1)} \), i.e.,

\[
\mathcal{Q}_{j_m}^{(i, j+1)} + \mathcal{Q}_{j_m}^{(i, j+1)} = \mathcal{Q}_{j_m}^{(i, j+1)}. \tag{3.10}
\]

We shall iterate the Feshbach map starting from \( i = 1 \) up to \( i = N - 2 \) with even, using the projections \( p^{(i)} \) and \( \bar{p}^{(i)} \) for the \( i \)-th step of the flow where

\[
p^{(i)} := \mathcal{Q}_{j_m}^{(i, j+1)}, \quad \bar{p}^{(i)} := \mathcal{Q}_{j_m}^{(i, j+1)}. \tag{3.11}
\]

**Remark 3.3.** In this section the derivation of the Feshbach Hamiltonians is only formal. Hence, we do not specify the values of \( w \) such that the Feshbach map is well defined. Some expansions will be justified later in Section 4.
We denote by $\tilde{\mathcal{V}}^{(i)}$ the Feshbach map at the $i$-th step of the flow (with $i \geq \tilde{i}$ and even) corresponding to the couple of projections $\mathcal{P}^{(i)}$ and $\mathcal{P}^{(0)}$. We start applying $\tilde{\mathcal{V}}^{(i)}$ to $H_{1,\ldots,j_m} - w$, and we get

$$\mathcal{X}_{j_1,\ldots,j_m}^{(i)}(w) := \mathcal{X}_{j_1,\ldots,j_m}^{(i)}(H_{j_1,\ldots,j_m} - w)$$

$$= \mathop{\Sigma}_{j_m}^{(>\tilde{i}+1)}(H_{j_1,\ldots,j_m} - w)\mathop{\Sigma}_{j_m}^{(>\tilde{i}+1)}$$

$$- \mathop{\Sigma}_{j_m}^{(>\tilde{i}+1)}\tilde{W}_{j_1,\ldots,j_m}\mathop{\Sigma}_{j_m}^{(>\tilde{i}+1)}$$

(3.12)

(3.13)

(3.14)

where

$$\tilde{W}_{j_1,\ldots,j_m} \equiv H_{j_1,\ldots,j_m} = H_{j_1,\ldots,j_m-1} + V_{j_1,\ldots,j_m-1} + \tilde{H}_{j_m}^{Bog} + V_{j_m} - T_{j=j_m}$$

(3.15)

with $V_{j_1,\ldots,j_m-1} := V_{j_1,\ldots,j_m-1} - V_{j_1,\ldots,j_m-1}$ and $T_{j=j_m} := \sum_{j=j_m} k_j^2 a_j^* a_j$. Notice that for $i \geq \tilde{i} + 2$

$$\mathop{\Sigma}_{j_m}^{(i,j+1)}\tilde{W}_{j_1,\ldots,j_m}\mathop{\Sigma}_{j_m}^{(i-2,j-1)} = \mathop{\Sigma}_{j_m}^{(i,j+1)}(V_{j_1,\ldots,j_m-1} + W_{j_m} + W_{j_m}^* + V_{j_m})\mathop{\Sigma}_{j_m}^{(i-2,j-1)}.$$  

(3.16)

Next, by iteration we define

$$\mathcal{X}_{j_1,\ldots,j_m}^{(i+2)}(w) := \tilde{\mathcal{V}}^{(i+2)}(\mathcal{X}_{j_1,\ldots,j_m}^{(i)}(w)) \quad i = \tilde{i}, \ldots, N - 4, \quad \text{with } i \text{ even}.$$  

(3.17)

Using the selection rules of $\tilde{W}_{j_1,\ldots,j_m}(\equiv \tilde{W}_{j_1,\ldots,j_m})$, in Lemma 3.4 we provide the formal expression of the Feshbach Hamiltonian for $\tilde{i} + 2 \leq i \leq N - 2$.

**Lemma 3.4.** Assume that the Neumann expansion

$$\mathop{\Sigma}_{j_m}^{(i,j+1)} - \mathop{\Sigma}_{j_m}^{(i,j+1)}(H_{j_1,\ldots,j_m} - w)\mathop{\Sigma}_{j_m}^{(i,j+1)} - \mathop{\Sigma}_{j_m}^{(i,j+1)}\tilde{W}_{j_1,\ldots,j_m}\mathop{\Sigma}_{j_m}^{(i,j+1)}$$

(3.18)

$$= \sum_{l=0}^{\infty} R_{j_1,\ldots,j_m;i,l}(w)\left[\Gamma_{j_1,\ldots,j_m;i,l}(w)R_{j_1,\ldots,j_m;i,l}(w)\right]^l$$

(3.19)

$$=: (R_{j_1,\ldots,j_m;i,0}(w))^{\frac{1}{2}}\Gamma_{j_1,\ldots,j_m;i,0}(w)(R_{j_1,\ldots,j_m;i,0}(w))^{\frac{1}{2}}$$

(3.20)

holds for $\tilde{i} + 2 \leq i \leq N - 2$. Then, for $\tilde{i} + 2 \leq i \leq N - 2$ and even

$$\mathcal{X}_{j_1,\ldots,j_m}^{(i)}(w) := \tilde{\mathcal{V}}^{(i+2)}(\mathcal{X}_{j_1,\ldots,j_m}^{(i)}(w))$$

(3.21)

$$= \mathop{\Sigma}_{j_m}^{(i+1)}(H_{j_1,\ldots,j_m} - w)\mathop{\Sigma}_{j_m}^{(i+1)}$$

(3.22)

(3.23)

(3.24)

where:

- $\tilde{W}_{j_1,\ldots,j_m;i,l-2} := \mathop{\Sigma}_{j_m}^{(i+1)}\tilde{W}_{j_1,\ldots,j_m}\mathop{\Sigma}_{j_m}^{(i-2,j-1)} =: \tilde{W}_{j_1,\ldots,j_m;i,l-2}$
\[ R_{j_1, \ldots, j_m; i}(w) := \mathcal{S}_{j_1, \ldots, j_m}^{(l_{i+1})} \frac{1}{\mathcal{S}_{j_1, \ldots, j_m}^{(l_{i+1})}(H_{j_1, \ldots, j_m} - w)\mathcal{S}_{j_1, \ldots, j_m}^{(l_{i+1})}} \mathcal{S}_{j_1, \ldots, j_m}^{(l_{i+1})}, \] (3.26)

\[ \Gamma_{j_1, \ldots, j_m; i; 2, j_2+2}(w) := \mathcal{S}_{j_1, \ldots, j_m}^{(l_{i+2}, j_2+3)} \mathcal{W}_{j_1, \ldots, j_m}^{(l_{i+1})} \frac{1}{\mathcal{S}_{j_1, \ldots, j_m}^{(l_{i+1})}(H_{j_1, \ldots, j_m} - w)\mathcal{S}_{j_1, \ldots, j_m}^{(l_{i+1})}} \mathcal{W}_{j_1, \ldots, j_m}^{(l_{i+1})} \mathcal{S}_{j_1, \ldots, j_m}^{(l_{i+2}, j_2+3)}, \] (3.27)

for \( i + 4 \leq i \leq N - 2, \)

\[ \Gamma_{j_1, \ldots, j_m; i; 2, j_2+2}(w) := \mathcal{W}_{j_1, \ldots, j_m; i; 2, j_2+2}(w) \sum_{l_{i-2}}^{\infty} \left[ \Gamma_{j_1, \ldots, j_m; i-2, j_2+2}(w) \mathcal{R}_{j_1, \ldots, j_m; i-2, j_2+2}(w) \right]^{l_{i-2}} \mathcal{W}_{j_1, \ldots, j_m; i; 2, j_2+2}(w) \times \right. \]
\[ \left. \times \sum_{l_{i-2}}^{\infty} \left[ (\mathcal{R}_{j_1, \ldots, j_m; i-2, j_2+2}(w))^{l_{i-2}} \mathcal{W}_{j_1, \ldots, j_m; i-2, j_2+2}(w) (\mathcal{R}_{j_1, \ldots, j_m; i-2, j_2+2}(w))^{l_{i-2}} \right]^{l_{i-2}} \times \right. \]
\[ \left. \times (\mathcal{R}_{j_1, \ldots, j_m; i-2, j_2+2}(w))^{l_{i-2}} \mathcal{W}_{j_1, \ldots, j_m; i; 2, j_2+2}. \] (3.28)

**Proof**

We start computing \( \mathcal{X}_{j_1, \ldots, j_m}^{(i+2)} (w) \) from the operator \( \mathcal{X}_{j_1, \ldots, j_m}^{(i)} (w) \) given in (3.12)-(3.14) and from the formula

\[ \Theta_{j_1, \ldots, j_m}^{(i+2)}(\mathcal{X}_{j_1, \ldots, j_m}^{(i)}(w)) = \mathcal{X}_{j_1, \ldots, j_m}^{(i+2)}(w). \] (3.31)

We compute

\[ \mathcal{S}_{j_1, \ldots, j_m}^{(l_{i+3})} \mathcal{X}_{j_1, \ldots, j_m}^{(i)}(w) \mathcal{S}_{j_1, \ldots, j_m}^{(l_{i+3})} = \mathcal{S}_{j_1, \ldots, j_m}^{(l_{i+3})}(H_{j_1, \ldots, j_m} - w)\mathcal{S}_{j_1, \ldots, j_m}^{(l_{i+3})} \] (3.32)

\[ -\mathcal{S}_{j_1, \ldots, j_m}^{(l_{i+3})} \mathcal{W}_{j_1, \ldots, j_m}^{(l_{i+1})} \mathcal{S}_{j_1, \ldots, j_m}^{(l_{i+3})} \frac{1}{\mathcal{S}_{j_1, \ldots, j_m}^{(l_{i+1})}(H_{j_1, \ldots, j_m} - w)\mathcal{S}_{j_1, \ldots, j_m}^{(l_{i+1})}} \mathcal{W}_{j_1, \ldots, j_m}^{(l_{i+1})} \mathcal{S}_{j_1, \ldots, j_m}^{(l_{i+2}, j_2+3)} \]
\[ \mathcal{S}_{j_1, \ldots, j_m}^{(l_{i+3})} \mathcal{W}_{j_1, \ldots, j_m}^{(l_{i+1})} \mathcal{S}_{j_1, \ldots, j_m}^{(l_{i+3})} \] (3.34)

because \( \mathcal{S}_{j_1, \ldots, j_m}^{(l_{i+3})} \mathcal{W}_{j_1, \ldots, j_m}^{(l_{i+1})} = 0 \) (see (3.15)). Next, we compute

\[ \mathcal{S}_{j_1, \ldots, j_m}^{(l_{i+2}, j_2+3)} \mathcal{X}_{j_1, \ldots, j_m}^{(i)}(w) \mathcal{S}_{j_1, \ldots, j_m}^{(l_{i+2}, j_2+3)} = \mathcal{S}_{j_1, \ldots, j_m}^{(l_{i+2}, j_2+3)}(H_{j_1, \ldots, j_m} - w)\mathcal{S}_{j_1, \ldots, j_m}^{(l_{i+2}, j_2+3)} \] (3.35)

\[ -\mathcal{S}_{j_1, \ldots, j_m}^{(l_{i+2}, j_2+3)} \mathcal{W}_{j_1, \ldots, j_m}^{(l_{i+1})} \mathcal{S}_{j_1, \ldots, j_m}^{(l_{i+2}, j_2+3)} \frac{1}{\mathcal{S}_{j_1, \ldots, j_m}^{(l_{i+1})}(H_{j_1, \ldots, j_m} - w)\mathcal{S}_{j_1, \ldots, j_m}^{(l_{i+1})}} \mathcal{W}_{j_1, \ldots, j_m}^{(l_{i+1})} \mathcal{S}_{j_1, \ldots, j_m}^{(l_{i+2}, j_2+3)} \]
\[ \mathcal{S}_{j_1, \ldots, j_m}^{(l_{i+2}, j_2+3)}(H_{j_1, \ldots, j_m} - w)\mathcal{S}_{j_1, \ldots, j_m}^{(l_{i+2}, j_2+3)} - \Gamma_{j_1, \ldots, j_m; i; 2, j_2+2}(w) \]
\[ \mathcal{S}_{j_1, \ldots, j_m}^{(l_{i+2}, j_2+3)}(H_{j_1, \ldots, j_m} - w)\mathcal{S}_{j_1, \ldots, j_m}^{(l_{i+2}, j_2+3)} \] (3.37)

where we have used the definition in (3.27). Hence, using the assumption in (3.18)-(3.19) we get

\[ \mathcal{S}_{j_1, \ldots, j_m}^{(l_{i+3})} \mathcal{X}_{j_1, \ldots, j_m}^{(i)}(w) \mathcal{S}_{j_1, \ldots, j_m}^{(l_{i+3})} \] (3.38)
\[-\sum_{j_m} (^{>i+3}) \mathcal{C}^{(i)}_{j_1, \ldots, j_m} (w) \mathcal{C}^{(i+2,+3)}_{j_1,...,j_m} + \frac{1}{\sum_{j_m} (^{>i+3}) \mathcal{C}^{(i)}_{j_1, \ldots, j_m} (w) \mathcal{C}^{(i+2,+3)}_{j_1,...,j_m} \mathcal{C}^{(i)}_{j_1, \ldots, j_m} (w) \mathcal{C}^{(i+3)}_{j_1, \ldots, j_m}} \mathcal{C}^{(i+2,+3)}_{j_1,...,j_m} \mathcal{C}^{(i)}_{j_1, \ldots, j_m} (w) \mathcal{C}^{(i+3)}_{j_1, \ldots, j_m} (3.39) \]

\[= \sum_{j_m} (^{>i+3}) \mathcal{C}^{(i)}_{j_1, \ldots, j_m} (H_{j_1, \ldots, j_m} - w) \mathcal{C}^{(i+2,+3)}_{j_1,...,j_m} \mathcal{C}^{(i+3)}_{j_1, \ldots, j_m} (3.40) \]

\[-\sum_{j_m} (^{>i+3}) \mathcal{W}_{j_1, \ldots, j_m} \sum_{l=1}^{\infty} R_{j_1, \ldots, j_m ; i_l} (w) \mathcal{C}^{(i+2,+3)}_{j_1,...,j_m} \mathcal{C}^{(i+3)}_{j_1, \ldots, j_m} (3.41) \]

\[-\sum_{j_m} (^{>i+3}) \mathcal{W}_{j_1, \ldots, j_m} \mathcal{C}^{(i+2,+3)}_{j_1,...,j_m} \mathcal{C}^{(i+3)}_{j_1, \ldots, j_m} (3.42) \]

Assuming that the identity in (3.21)-(3.24) holds for \(i \geq i + 2\) we show that it is also valid for \(i + 2\). To this purpose we repeat the previous computation for \(i \geq i + 2\):

- \(\sum_{j_m} (^{>i+3}) \mathcal{C}^{(i)}_{j_1, \ldots, j_m} (w) \mathcal{C}^{(i+2,+3)}_{j_1,...,j_m} \mathcal{C}^{(i+3)}_{j_1, \ldots, j_m} (3.43) \]

\[-\sum_{j_m} (^{>i+3}) (H_{j_1, \ldots, j_m} - w) \mathcal{C}^{(i+2,+3)}_{j_1,...,j_m} \mathcal{C}^{(i+3)}_{j_1, \ldots, j_m} (3.44) \]

\[-\sum_{j_m} (^{>i+3}) \mathcal{W}_{j_1, \ldots, j_m} \sum_{l=1}^{\infty} R_{j_1, \ldots, j_m ; i_l} (w) \mathcal{C}^{(i+2,+3)}_{j_1,...,j_m} \mathcal{C}^{(i+3)}_{j_1, \ldots, j_m} (3.45) \]

\[-\sum_{j_m} (^{>i+3}) \mathcal{W}_{j_1, \ldots, j_m} \mathcal{C}^{(i+2,+3)}_{j_1,...,j_m} \mathcal{C}^{(i+3)}_{j_1, \ldots, j_m} (3.46) \]

because \(\sum_{j_m} (^{>i+3}) \mathcal{W}_{j_1, \ldots, j_m} \mathcal{C}^{(i+1)}_{j_1,...,j_m} = 0 \) (see (3.15));

- likewise

\[\sum_{j_m} (^{i+2,+3}) \mathcal{C}^{(i)}_{j_1, \ldots, j_m} (w) \mathcal{C}^{(i+2,+3)}_{j_1,...,j_m} \mathcal{C}^{(i+3)}_{j_1, \ldots, j_m} (3.47) \]

\[-\sum_{j_m} (^{i+2,+3}) (H_{j_1, \ldots, j_m} - w) \mathcal{C}^{(i+2,+3)}_{j_1,...,j_m} \mathcal{C}^{(i+3)}_{j_1, \ldots, j_m} (3.48) \]

\[-\sum_{j_m} (^{i+2,+3}) \mathcal{W}_{j_1, \ldots, j_m} \sum_{l=1}^{\infty} R_{j_1, \ldots, j_m ; i_l} (w) \mathcal{C}^{(i+2,+3)}_{j_1,...,j_m} \mathcal{C}^{(i+3)}_{j_1, \ldots, j_m} (3.49) \]

\[-\sum_{j_m} (^{i+2,+3}) \mathcal{W}_{j_1, \ldots, j_m} \mathcal{C}^{(i+2,+3)}_{j_1,...,j_m} \mathcal{C}^{(i+3)}_{j_1, \ldots, j_m} (3.50) \]

Hence, using the assumption in (3.18)-(3.19) we get

\[-\sum_{j_m} (^{>i+3}) \mathcal{C}^{(i)}_{j_1, \ldots, j_m} (w) \mathcal{C}^{(i+2,+3)}_{j_1,...,j_m} \mathcal{C}^{(i+3)}_{j_1, \ldots, j_m} (3.51) \]

\[-\sum_{j_m} (^{>i+3}) (H_{j_1, \ldots, j_m} - w) \mathcal{C}^{(i+2,+3)}_{j_1,...,j_m} \mathcal{C}^{(i+3)}_{j_1, \ldots, j_m} (3.52) \]

\[-\sum_{j_m} (^{>i+3}) \mathcal{W}_{j_1, \ldots, j_m} \sum_{l=1}^{\infty} R_{j_1, \ldots, j_m ; i_l} (w) \mathcal{C}^{(i+2,+3)}_{j_1,...,j_m} \mathcal{C}^{(i+3)}_{j_1, \ldots, j_m} (3.53) \]

\[-\sum_{j_m} (^{>i+3}) \mathcal{W}_{j_1, \ldots, j_m} \mathcal{C}^{(i+2,+3)}_{j_1,...,j_m} \mathcal{C}^{(i+3)}_{j_1, \ldots, j_m} (3.54) \]

\[= \mathcal{C}^{(i+2)}_{j_1, \ldots, j_m} (w). \]
For the last implementation (corresponding to \( i = N \)) of the Feshbach map we employ the projections

\[
\mathcal{P}^{(N)} := \mathcal{P}^{(N)} : = \left[ \frac{\psi^\#_{j_1 \ldots j_{n-1}, j_{n+1}}}{\|\psi^\#_{j_1 \ldots j_{n-1}}\|} \right] \left[ \frac{\psi^\#_{j_1 \ldots j_{n-1}}}{\|\psi^\#_{j_1 \ldots j_{n-1}}\|} \right],
\]

(3.55)

\[
\mathcal{P}^{(N)} := \mathcal{P}^{(N)} := \prod_{k=1}^{N} \left[ \frac{\psi^\#_{j_1 \ldots j_{n-1}, j_{n+1}}}{\|\psi^\#_{j_1 \ldots j_{n-1}}\|} \right] \left[ \frac{\psi^\#_{j_1 \ldots j_{n-1}}}{\|\psi^\#_{j_1 \ldots j_{n-1}}\|} \right]
\]

(3.56)

where \( \psi^\#_{j_1 \ldots j_{n-1}} = \eta \) for \( m = 1 \), and for \( m \geq 2 \) is the ground state vector (non-normalized) of the Hamiltonian \( H^\#_{j_1 \ldots j_{n-1}} \) (see (3.3)). The vector \( \psi^\#_{j_1 \ldots j_{n-1}, j_{n+1}} (\neq 0) \) and the corresponding eigenvalue \( \varepsilon^\#_{j_1 \ldots j_{n-1}} \) will be iteratively constructed in the next section.

**Remark 3.5.** The auxiliary Hamiltonian \( H^\#_{j_1 \ldots j_{n-1}} \) mediates the step from \( H_{j_1 \ldots j_{n-1}} \) to \( H^\#_{j_1 \ldots j_{n-1}} \).

In the analogous construction for the Bogoliubov Hamiltonians (see [Pi2]) the operator \( H^\#_{j_1 \ldots j_{n-1}} \) plays the role of \( H^\#_{j_1 \ldots j_{n-1}} \), thus no auxiliary Hamiltonian is necessary. Indeed, the Bogoliubov interaction associated with the Fourier modes \( \{ \pm j_1 | l = 1, \ldots, m - 1 \} \) of the pair potential contains only the field modes \( \{ 0, \pm j_1 | l = 1, \ldots, m - 1 \} \). Hence, it does not contain the operators \( a_{\pm j_1}, a^*_{\pm j_1} \).

Formally we get

\[
\mathcal{K}^{(N)}_{j_1 \ldots j_{n-1}}(w) := \mathcal{K}^{(N)}_{j_1 \ldots j_{n-1}}(w) \mathcal{P}^{(N)}_{j_1 \ldots j_{n-1}} - \mathcal{P}^{(N)}_{j_1 \ldots j_{n-1}} \Gamma_{j_1 \ldots j_{n+1}} N(w) \mathcal{P}^{(N)}_{j_1 \ldots j_{n-1}} - \mathcal{P}^{(N)}_{j_1 \ldots j_{n-1}} (V_{j_1 \ldots j_{n+1}} - \Gamma_{j_1 \ldots j_{n+1}} N(w)) \times \\
\times \mathcal{P}^{(N)}_{j_1 \ldots j_{n-1}} \mathcal{K}^{(N-2)}_{j_1 \ldots j_{n-1}}(w) \mathcal{P}^{(N-2)}_{j_1 \ldots j_{n-1}} \times (V_{j_1 \ldots j_{n+1}} - \Gamma_{j_1 \ldots j_{n+1}} N(w)) \mathcal{P}^{(N-2)}_{j_1 \ldots j_{n-1}} \mathcal{P}^{(N-2)}_{j_1 \ldots j_{n-1}}
\]

(3.57)

(3.58)

(3.59)

(3.60)

(3.61)

where

\[
\Gamma_{j_1 \ldots j_{n+1}} N(w)
\]

(3.62)

\[
\mathcal{W}_{j_1 \ldots j_{n+1}} R_{j_1 \ldots j_{n+1}} N - 2 N - 2 (w) \sum_{l=0}^{\infty} \left[ \Gamma_{j_1 \ldots j_{n+1}} N - 2 N - 2 (w) R_{j_1 \ldots j_{n+1}} N - 2 N - 2 (w) \right]^{\frac{1}{N-2}} \mathcal{W}_{j_1 \ldots j_{n+1}}
\]

(3.63)

\[
= \mathcal{W}_{j_1 \ldots j_{n+1}} (R_{j_1 \ldots j_{n+1}} N - 2 N - 2 (w))^{\frac{1}{2}} \times
\]

(3.64)

\[
\times \sum_{l=0}^{\infty} \left[ (R_{j_1 \ldots j_{n+1}} N - 2 N - 2 (w))^{\frac{1}{2}} \Gamma_{j_1 \ldots j_{n+1}} N - 2 N - 2 (w) (R_{j_1 \ldots j_{n+1}} N - 2 N - 2 (w))^{\frac{1}{2}} \right]^{\frac{1}{N-2}} \times
\]

(3.65)

Then, we set \( w = z + z_{j_1 \ldots j_{n-1}} \) and solve the fixed point equation

\[
f_{j_1 \ldots j_{n+1}} (w) = f_{j_1 \ldots j_{n+1}} (z + z_{j_1 \ldots j_{n-1}}) = 0.
\]

(3.66)
The (unique) solution, \( w \equiv z_{j_1,...,j_m} \), is the (non-degenerate) ground state energy of \( H_{j_1,...,j_m} \).

The corresponding (non-normalized) ground state vector is

\[
\psi_{j_1,...,j_m} := \begin{cases} 1 \mathcal{J}_{j_1,...,j_m}^{(N-2,N-1)} \mathcal{H}_{j_1,...,j_m}^{(N-4)}(z_{j_1,...,j_m}) \mathcal{J}_{j_1,...,j_m}^{(N-2,N-1)} \hat{W}_{j_1,...,j_m;N-2,N} \\
- \sum_{j=2}^{N-1} \prod_{r=j}^{2} \left[ - \mathcal{J}_{j_1,...,j_m}^{(N-2r,N-2r+1)} \mathcal{H}_{j_1,...,j_m}^{(N-2r-2)}(z_{j_1,...,j_m}) \mathcal{J}_{j_1,...,j_m}^{(N-2r,N-2r+1)} \hat{W}_{j_1,...,j_m;N-2r,N-2r+2} \right] \times \\
\mathcal{J}_{j_1,...,j_m}^{(N-2,N-1)} \mathcal{H}_{j_1,...,j_m}^{(N-4)}(z_{j_1,...,j_m}) \mathcal{J}_{j_1,...,j_m}^{(N-2,N-1)} \hat{W}_{j_1,...,j_m;N-2,N} \end{cases} \times
\]

\[
\times \left[ 1 - \mathcal{P}_{j_1,...,j_m-1}^{(N-2)} \mathcal{H}_{j_1,...,j_m}^{(N-4)}(z_{j_1,...,j_m}) \mathcal{P}_{j_1,...,j_m-1}^{(N-2)} \psi_{j_1,...,j_m-1}^{\#} \right] \psi_{j_1,...,j_m}^{\#}.
\]

where \( \mathcal{H}_{j_1,...,j_m}^{(i-2)}(z_{j_1,...,j_m}) = H_{j_1,...,j_m} - z_{j_1,...,j_m} \).

The ground state \( \psi_{j_1,...,j_m}^{\#} \) corresponding to the ground state energy \( z_{j_1,...,j_m} \) of the Hamiltonian \( H_{j_1,...,j_m}^{\#} \) has an analogous formula

\[
\psi_{j_1,...,j_m}^{\#} := \begin{cases} 1 \mathcal{J}_{j_1,...,j_m}^{(N-2,N-1)} \mathcal{H}_{j_1,...,j_m}^{(N-4)}(z_{j_1,...,j_m}) \mathcal{J}_{j_1,...,j_m}^{(N-2,N-1)} \hat{W}_{j_1,...,j_m;N-2,N} \\
- \sum_{j=2}^{N-1} \prod_{r=j}^{2} \left[ - \mathcal{J}_{j_1,...,j_m}^{(N-2r,N-2r+1)} \mathcal{H}_{j_1,...,j_m}^{(N-2r-2)}(z_{j_1,...,j_m}) \mathcal{J}_{j_1,...,j_m}^{(N-2r,N-2r+1)} \hat{W}_{j_1,...,j_m;N-2r,N-2r+2} \right] \times \\
\mathcal{J}_{j_1,...,j_m}^{(N-2,N-1)} \mathcal{H}_{j_1,...,j_m}^{(N-4)}(z_{j_1,...,j_m}) \mathcal{J}_{j_1,...,j_m}^{(N-2,N-1)} \hat{W}_{j_1,...,j_m;N-2,N} \end{cases} \times
\]

\[
\times \left[ 1 - \mathcal{P}_{j_1,...,j_m-1}^{(N-2)} \mathcal{H}_{j_1,...,j_m}^{(N-4)}(z_{j_1,...,j_m}) \mathcal{P}_{j_1,...,j_m-1}^{(N-2)} \psi_{j_1,...,j_m-1}^{\#} \right] \psi_{j_1,...,j_m}^{\#}.
\]

where \( \mathcal{H}_{j_1,...,j_m}^{(i-2)}(z_{j_1,...,j_m}) := H_{j_1,...,j_m}^{\#} - z_{j_1,...,j_m}^{\#} =: W_{j_1,...,j_m}^{\#} - z_{j_1,...,j_m}^{\#} \) and \( \mathcal{H}_{j_1,...,j_m}^{(i)}(z_{j_1,...,j_m}) = \mathcal{P}_{j_1,...,j_m}^{(i)}(H_{j_1,...,j_m}^{(i-2)}(z_{j_1,...,j_m})) \) with \( i \geq \tilde{i} \) and even.

The main subtleties in the construction are concerned with the very first implementation (from \( i = \tilde{i} \) to \( i = \tilde{i} + 2 \)) and the very last implementation of the Feshbach map (from \( i = N - 2 \) to \( i = N \)).

### 4 Rigorous construction of the Feshbach Hamiltonians \( \mathcal{H}_{j_1,...,j_m}^{(i)}(w) \)

The construction outlined in Section 3.1 must be implemented by induction in the index \( m \) ranging from 1 to \( M \). For the sake of clarity, first we show that for each \( m \) the Feshbach flow can be rigorously defined from \( i = \tilde{i} \) up to \( i = N - 2 \) under some conditions to be proven later in Section 4.2.
For the next estimates, it is important to isolate a term proportional to the kinetic energy operator $T := \sum_{j \in \mathbb{Z}^d} k_j^2 a_j^* a_j$ in the Hamiltonians $H_{j_1, \ldots, j_m}$ and $H_{j_1, \ldots, j_m}^\#$, by which we can dominate some of the terms of the interaction Hamiltonian. To this purpose we introduce some definitions dependent on a suitable small ($N$-dependent) positive number $\xi$.

For $m \geq 2$

$$
(H_{j_1, \ldots, j_{m-1}}^\#)_{\xi} := (1 - \xi) T_{j_1, \ldots, j_{m-1}} + (H_{j_1, \ldots, j_{m-1}}^B_{\xi}) + V_{j_1, \ldots, j_{m-1}}
$$

(4.1)

where $T_{j_1, \ldots, j_{m-1}}$ is defined in (3.4) and

$$(H_{j_1, \ldots, j_{m-1}}^B_{\xi}) := \sum_{j_1} T_{j_1}^B_{\xi}$$

(4.2)

with

$$(H_{j_1}^B_{\xi}) := \sum_{j_1} [(1 - \xi) k_j^2 + \frac{\phi_j}{N} a_j^* a_j + \frac{1}{2} \sum_{j_2} \frac{\phi_j}{N} (a_j^* a_j a_{j_2} + a_j^* a_{j_2} a_j)].$$

(4.3)

Thus we can write (see (4.1))

$$H_{j_1, \ldots, j_{m-1}} = (H_{j_1, \ldots, j_{m-1}}^\#)_{\xi} + \xi T,$$

(4.4)

and

$$H_{j_1, \ldots, j_{m}} = (H_{j_1, \ldots, j_{m-1}}^\#)_{\xi} - (1 - \xi) T_{j_1, \ldots, j_{m-1}} + V_{j_1, \ldots, j_{m-1}} + (H_{j_1, \ldots, j_{m-1}}^B_{\xi}) + \xi T$$

(4.5)

where $T_{j_1, \ldots, j_{m-1}} := \sum_{j_1} k_j^2 a_j^* a_j$ and $V_{j_1, \ldots, j_{m-1}} := V_{j_1, \ldots, j_{m-1}} - V_{j_1, \ldots, j_{m-1}}^\#$.

For $m = 1$ we set

$$(H_{j_1}^\#)_{\xi} := (1 - \xi) T.$$  

(4.6)

We recall

$$H_{j_1} = V_{j_1} + H_{j_1}^B + T_{j_1} = V_{j_1} + H_{j_1}^B + T - T_{j_1} = (1 - \xi) T - (1 - \xi) T_{j_1} + V_{j_1} + (H_{j_1}^B_{\xi}) + \xi T.$$  

(4.9)

**Lemma 4.1.** Let $M \geq m \geq 1$ and assume that $\inf \sigma[H_{j_1, \ldots, j_{m-1}}^\# - T_{j_1, \ldots, j_{m-1}}] \geq z_{j_1, \ldots, j_{m-1}}^\# - \frac{m-1}{2M}$.  

Then for $\xi = (\frac{1}{mN})^\frac{1}{4}$ and $N$ large enough the inequality

$$
(H_{j_1, \ldots, j_{m-1}}^\#)_{\xi} - (1 - \xi) T_{j_1, \ldots, j_{m-1}} \geq z_{j_1, \ldots, j_{m-1}}^\# - \frac{(m-1)\xi^\frac{1}{4}}{M}
$$

(4.10)

holds true where $z_{j_1, \ldots, j_{m-1}}^\# \equiv 0$ for $m = 1$.

**Proof**

For $m = 1$ the property is trivial. For $m \geq 2$, we notice that, due to the identity in (4.4), the assumption in the statement corresponds to

$$
\langle \psi, (H_{j_1, \ldots, j_{m-1}}^\#)_{\xi} - T_{j_1, \ldots, j_{m-1}} + \xi T \rangle \psi \geq z_{j_1, \ldots, j_{m-1}}^\# - \frac{(m-1)\xi^\frac{1}{4}}{2M}, \quad ||\psi|| = 1.
$$

(4.11)
Furthermore, we observe that

$$\langle \psi, [(H_{j_1,\ldots,j_{m-1}}^\#)^\xi - T_{j=\pm j_m} + \xi T]\psi \rangle = \langle \psi, [(H_{j_1,\ldots,j_{m-1}}^\#)^\xi - (1 - \xi)T_{j=\pm j_m} + \xi T_{j_2=\pm j_m}]\psi \rangle.$$  \hspace{1cm} (4.12)

Now, we assume that $$\langle \psi, T_{j=\pm j_m}|\psi \rangle > \frac{1}{2M^\xi}$$, so that starting from the definition in (4.1) and using a slightly modified version of Lemma 5.3 in the Appendix the inequality in (4.10) is trivially fulfilled for $$N$$ large enough. If $$\langle \psi, T_{j=\pm j_m}|\psi \rangle \leq \frac{1}{2M^\xi}$$, from (4.11) and (4.12) we readily obtain

$$\langle \psi, (H_{j_1,\ldots,j_{m-1}}^\#)^\xi \psi \rangle - (1 - \xi)\langle \psi, T_{j=\pm j_m}\psi \rangle \geq z_{j_1,\ldots,j_{m-1}}^\# - \xi \langle \psi, T_{j=\pm j_m}|\psi \rangle - \frac{(m - 1)\xi}{2M} \geq z_{j_1,\ldots,j_{m-1}}^\# - \frac{(m - 1)\xi}{M}.$$  \hspace{1cm} (4.13)

\[\square\]

In Corollary 5.1, assuming

$$(H_{j_1,\ldots,j_{m-1}}^\#)^\xi - (1 - \xi)T_{j=\pm j_m}^n - z_{j_1,\ldots,j_{m-1}}^\# \geq -\frac{(m - 1)\xi}{M}$$  \hspace{1cm} (4.14)

we show how the norm bound

$$\| [R_{j_1,\ldots,j_m}; t, -2, -2](w) ]^{1/2} \bar{W}^{n}_{j_1,\ldots,j_m}[R_{j_1,\ldots,j_m}; t, -2](w) |^{1/2} \| \leq \frac{1}{4(1 + a_{\epsilon_{j_m}} - \frac{2b_{\epsilon_{j_m}}}{N-i+2} - \frac{1-\epsilon_{j_m}}{(N-i+2)^2})}$$  \hspace{1cm} (4.15)

can be derived for $$\tilde{i} + 4 \leq i \leq N - 2$$ ($$i$$ even) and

$$w = z + z_{j_1,\ldots,j_{m-1}}^\# \leq z_{j_1,\ldots,j_{m-1}}^\# + E_{\epsilon_{j_m}}^{\text{Bog}} + (\delta - 1)\phi_{\epsilon_{j_m}} \sqrt{\epsilon_{j_m}^2 + 2\epsilon_{j_m}}, \hspace{0.5cm} \delta < 2,$$  \hspace{1cm} (4.16)

with $$\epsilon_{j_m}$$ sufficiently small and $$N$$ sufficiently large (for definitions see (5.5), (5.6) and (5.7)).

The proof of Corollary 5.1 requires some modifications with respect to Lemma 3.5 of [Pi1] and Corollary 5.1 of [Pi2]. However, the new term, $$V_{j_1,\ldots,j_m}$$, in the interaction can be controlled because it has to be evaluated only in the following expressions

$$\Sigma_{\epsilon_{j_m}}^{(i,i+1)} V_{\epsilon_{j_m}} \Sigma_{\epsilon_{j_m}}^{(i,i+1)} , \Sigma_{\epsilon_{j_m}}^{(i,i+1)} V_{\epsilon_{j_m}} \Sigma_{\epsilon_{j_m}}^{(i,i+1)} , \Sigma_{\epsilon_{j_m}}^{(i, i-1)} V_{\epsilon_{j_m}} \Sigma_{\epsilon_{j_m}}^{(i, i-1)} ,$$  \hspace{1cm} (4.17)

and, for $$i > \tilde{i}$$, $$\Sigma_{\epsilon_{j_m}}^{(i,i+1)}$$ yields a bound on the number of particles ($$\leq |N\epsilon_{j_m}| = 2$$) in the modes $$\pm j_m$$. The details are deferred to Corollary 5.1.

**Remark 4.2.** Corollary 5.1 essentially controls all the steps of the Feshbach flow from $$i = \tilde{i} + 2$$ to $$i = N - 2$$ provided that for $$w = z + z_{j_1,\ldots,j_{m-1}}^\#$$

$$\Gamma_{j_1,\ldots,j_m, \tilde{i}+2, \tilde{i}+2}(w)$$  \hspace{1cm} (4.18)

$$:= \Sigma_{\epsilon_{j_m}}^{(i, i+3)} \bar{W}_{\epsilon_{j_m}} \Sigma_{\epsilon_{j_m}}^{(i, i+3)} \Sigma_{\epsilon_{j_m}}^{(i, i+3)} (H_{j_1,\ldots,j_m} - w) \Sigma_{\epsilon_{j_m}}^{(i, i+1)} \bar{W}_{\epsilon_{j_m}} \Sigma_{\epsilon_{j_m}}^{(i, i+1)}$$  \hspace{1cm} (4.19)

is well approximated by $$\Gamma_{j_1,\ldots,j_m, \tilde{i}+2, \tilde{i}+2}(z)$$ in a sense specified in Theorem 4.1 below. For this reason, the very first step, from $$i = \tilde{i}$$ to $$i = \tilde{i} + 2$$, requires a more careful control if compared with the Feshbach flows studied in [Pi1] and [Pi2]. This is due to the new choice of the very first perpendicular projection $$\Sigma_{\epsilon_{j_m}}^{(i,i+1)}$$. In the proof of Theorem 4.1 we take advantage of the “short range property of the potential in the particle states numbers” described in the introduction (see Section 1).
Theorem 4.1. For \( M \geq m \geq 1 \) assume:
(a) 
\[
(H_{j_1,\ldots,j_m}^\#)\xi - (1 - \xi)T_{j_1,\ldots,j_m} \geq z_{j_1,\ldots,j_m}^\# - \frac{(m - 1)\xi^2}{M},
\]  
(4.20)
where \((H_{j_1,\ldots,j_m}^\#)\xi\) is defined in (4.1) for \( m \geq 2 \) and is equal to \((1 - \xi)T\) for \( m = 1 \), and where \( z_{j_1,\ldots,j_m}^\# \) is the ground state energy of \( H_{j_1,\ldots,j_m}^\# \).
(b) 
\[
w := z + z_{j_1,\ldots,j_m}^\# \leq z_{j_1,\ldots,j_m}^\# + E_{J_m}^{\text{Bog}} + (\delta - 1)\phi_{J_m} \sqrt{\epsilon_{J_m}^2 + 2\epsilon_{J_m}}
\]  
(4.21)
with \( \delta < 2 \) and \( \epsilon_{J_m} \) sufficiently small.

Then, for \( \xi = (\frac{1}{\ln N})^\frac{1}{4} \) and \( N \) sufficiently large
\[
\left\| [R_{J_1,\ldots,J_n};\hat{\iota}^2,\hat{\iota}^2](w) \right\|^\frac{1}{2} \Gamma_{J_1,\ldots,J_n;\hat{\iota}^2,\hat{\iota}^2}(w) \left[ R_{J_1,\ldots,J_n};\hat{\iota}^2,\hat{\iota}^2](w) \right]^\frac{1}{2}
\]  
\leq \frac{1}{x_{j+2}}
(4.22)
where \( x_{j+2} \) is the \( \hat{\iota} + 2 \)-term of the sequence\(^4\)
\[
x_{2j+2} := 1 - \frac{1}{4(1 + a_{J_m} - \frac{2b_{J_m}}{N - 2j - 1} - \frac{1 - c_{J_m}}{(N - 2j - 1)^2})x_{2j}}
\]  
(4.24)
starting from \( x_0 = 1 \) and defined up to \( x_{N-2} \). Here, \( a_{J_m}, b_{J_m}, \) and \( c_{J_m} \) coincide with those given in Corollary 5.1.

Proof

First, we observe that \( \Gamma_{J_1,\ldots,J_n;\hat{\iota}^2,\hat{\iota}^2}(w) \) is well defined thanks to the inequality in (5.40) of Lemma 5.3 that implies
\[
\Sigma_{j_m}^{(\hat{\iota}^2+1)}(H_{J_1,\ldots,J_n} - w)\Sigma_{j_m}^{(\hat{\iota}^2+1)} \geq C N^\frac{1}{4} \Sigma_{j_m}^{(\hat{\iota}^2+1)}
\]  
for some \( C > 0 \). Next, we recall that (see (4.5))
\[
\Sigma_{j_m}^{(\hat{\iota}^2+1)}[H_{J_1,\ldots,J_n} - w]\Sigma_{j_m}^{(\hat{\iota}^2+1)}
= \Sigma_{j_m}^{(\hat{\iota}^2+1)}[(H_{J_1,\ldots,J_n}^\#)\xi - (1 - \xi)T_{j_m\xi} - z_{j_1,\ldots,j_m-1}^\#] \Sigma_{j_m}^{(\hat{\iota}^2+1)}
+ \Sigma_{j_m}^{(\hat{\iota}^2+1)}[V_{J_m} + V_{J_1,\ldots,J_{m-1}} + (H_{J_m}^{\text{Bog}})\xi + \xi T - z] \Sigma_{j_m}^{(\hat{\iota}^2+1)}.
\]  
(4.27)

We define
\[
V_{J_m}^{(3)} := \frac{1}{N} \sum_{j \in \mathbb{Z} \backslash \{-j_m,0\}} a_{j_m}^a a_{j_m}^a \phi_{j_m} a_j a_{j_m} + \text{h.c.}
\]  
(4.28)
\[
+ \frac{1}{N} \sum_{j \in \mathbb{Z} \backslash \{j_m,0\}} a_{-j_m}^a a_{j_m}^a \phi_{j_m} a_j a_{-j_m} + \text{h.c.}
\]  
(4.29)
\[
\text{(4.30)This sequence was introduced in Lemma 3.6 of [Pi1]}
\]
where so that

Assuming the inequality in (4.14) is clear that for \(N\) sufficiently large

\[
\mathcal{S}_{\mathcal{J}_m}^{(i+2, i+3)} \mathcal{S}_{\mathcal{J}_m}^{(i+1)} > 0
\]

because

- the inequality in (4.20) has been assumed;
- the operators \(V_{\mathcal{J}_m}^{(4)}\) and \((\hat{H}_{\mathcal{J}_m}^0)\xi\) are non negative;
- \(\xi\ T\) dominates \(-\frac{1}{N} \phi_{\mathcal{J}_m} \sum_{\mathcal{J}' \in \mathbb{Z}^d \setminus \{J_m, 0\}} a_{\mathcal{J}' \mathcal{J}_m}^* a_{\mathcal{J}_m}^* a_{\mathcal{J}_m} a_{\mathcal{J}_m}^*\) for \(\xi = (\frac{1}{\ln N})^2\) and \(N\) large;
- \(-z > 0\) uniformly in \(N\).

Hence, we define the resolvent

\[
S_{\mathcal{J}_m}(z) := \frac{1}{\mathcal{S}_{\mathcal{J}_m}^{(i+2, i+3)} \mathcal{S}_{\mathcal{J}_m}^{(i+1)}} \mathcal{S}_{\mathcal{J}_m}^{(i+1)}
\]

and implement a truncated Neumann expansion so that we obtain

\[
\mathcal{S}_{\mathcal{J}_m}^{(i+2, i+3)} \mathcal{W}_{\mathcal{J}_m \cdots \mathcal{J}_m}^{(i+1)} \mathcal{S}_{\mathcal{J}_m}^{(i+1)} \frac{1}{\mathcal{S}_{\mathcal{J}_m}^{(i+1)}} \mathcal{S}_{\mathcal{J}_m}^{(i+1)} \mathcal{W}_{\mathcal{J}_m \cdots \mathcal{J}_m}^{(i+2, i+3)}
\]

\[
\mathcal{S}_{\mathcal{J}_m}^{(i+2, i+3)} \mathcal{W}_{\mathcal{J}_m \cdots \mathcal{J}_m}^{(i+1)} \mathcal{S}_{\mathcal{J}_m}^{(i+1)} \mathcal{S}_{\mathcal{J}_m}^{(i+1)} \mathcal{W}_{\mathcal{J}_m \cdots \mathcal{J}_m}^{(i+2, i+3)}
\]

\[
\mathcal{S}_{\mathcal{J}_m}^{(i+2, i+3)} \mathcal{W}_{\mathcal{J}_m \cdots \mathcal{J}_m}^{(i+1)} \mathcal{S}_{\mathcal{J}_m}^{(i+1)} \mathcal{S}_{\mathcal{J}_m}^{(i+1)} \mathcal{W}_{\mathcal{J}_m \cdots \mathcal{J}_m}^{(i+2, i+3)}
\]

\[
\mathcal{S}_{\mathcal{J}_m}^{(i+2, i+3)} \mathcal{W}_{\mathcal{J}_m \cdots \mathcal{J}_m}^{(i+1)} \mathcal{S}_{\mathcal{J}_m}^{(i+1)} \mathcal{S}_{\mathcal{J}_m}^{(i+1)} \mathcal{W}_{\mathcal{J}_m \cdots \mathcal{J}_m}^{(i+2, i+3)}
\]
Making use of the selection rules of the operator $\bar{W}_{j_1,\ldots,j_m}$ we get

\[(4.41) + (4.42)\]

\[= \sum_{\ell_0}^{i_0+3} \bar{W}_{j_1,\ldots,j_m} Q_{j_m}^{(i,j+1)} \sum_{i=0}^{n'-1} S_{j_m}(z) \left\{ \left( - (V_{j_m}^{(4)})' + V_{j_1,\ldots,j_{m-1}}^{(3)} + W_{j_m} + W_{j_m}^* \right) S_{j_m}(z) \right\} \times \]

\[\times \bar{W}_{j_1,\ldots,j_m} Q_{j_m}^{(i+2,j+3)}
\]

\[+ \sum_{\ell_0}^{i_0+3} \bar{W}_{j_1,\ldots,j_m} Q_{j_m}^{(i,j+1)} \left[ S_{j_m}(z) \left( - (V_{j_m}^{(4)})' + V_{j_1,\ldots,j_{m-1}}^{(3)} + W_{j_m} + W_{j_m}^* \right) \right]' \times \]

\[\times \sum_{\ell_0}^{i_0+3} \sum_{\ell_0}^{i_0+3} \left( H_{j_1,\ldots,j_m} - w \right) Q_{j_m}^{(i+1, j+1)}/(H_{j_1,\ldots,j_m} - w) Q_{j_m}^{(i+1, j+1)} \bar{W}_{j_1,\ldots,j_m} Q_{j_m}^{(i+2,j+3)}.
\]

**Remark 4.3.** We observe that in (4.44), (4.45), on the left of the operator \((V_{j_m}^{(4)})' + V_{j_1,\ldots,j_{m-1}}^{(3)}\), we can insert the projection \(P_m^{(N_\uparrow + 2n')}\)

where \(P_m^{(N_\uparrow + 2n')}\) projects onto the subspace of vectors with at most \([N_\uparrow + 2n']\) particles in the modes \(\pm j_m\). We also notice that

\[\left[ S_{j_m}(z), P_m^{(N_\uparrow + 2n')} \right] = 0 \]

and

\[\|Q_{j_0}^{(i,j+1)}(S_{j_m}(z))^\dagger \| (P_m^{(N_\uparrow + 2n')} (V_{j_m}^{(4)})' + V_{j_1,\ldots,j_{m-1}}^{(3)} + P_m^{(N_\uparrow + 2n')} V_{j_1,\ldots,j_{m-1}} + W_{j_m} + W_{j_m}^* ) S_{j_m}(z) \|^\dagger \| \]

\[\leq \left\{ \| (S_{j_m}(z))^\dagger \| (P_m^{(N_\uparrow + 2n')} (V_{j_m}^{(4)})' + V_{j_1,\ldots,j_{m-1}}^{(3)} + P_m^{(N_\uparrow + 2n')} V_{j_1,\ldots,j_{m-1}} ) S_{j_m}(z) \|^\dagger \| \right\} + 2 \sup_{0 \leq r \leq 2n', r \ even} \| (S_{j_m}(z))^\dagger Q_{j_m}^{(i-2,j+r-1)} W_{j_m} Q_{j_m}^{(i-2,j+r-1)} ) S_{j_m}(z) \|^\dagger \| .
\]

Next, we observe that for each summand in \((V_{j_m}^{(4)})', V_{j_1,\ldots,j_{m-1}}^{(3)}\), and \(V_{j_1,\ldots,j_{m-1}}':\]

1. at most one operator of the type \(a_0, a_0^*\) can be present;
2. at least one operator \(a_-^*\) or \(a_+^*\) is present.

Consequently, for any \(\phi \in F^N\)

\[|\phi, P_m^{(N_\uparrow + 2n')} (V_{j_m}^{(4)})' + V_{j_1,\ldots,j_{m-1}}^{(3)} | \phi \rangle| \leq C \left( \frac{N_{\uparrow}}{N} \right)^{\dagger} \| \phi, N_+ \phi \| \]

for some \(C > 0\) where \(N_+ := \sum_{j \in \mathbb{Z}_+ \setminus \{0\}} a_j^* a_j\), and we derive

\[\|P_m^{(N_\uparrow + 2n')} [S_{j_m}(z)]^\dagger \| (V_{j_m}^{(4)})' + V_{j_1,\ldots,j_{m-1}}^{(3)} || S_{j_m}(z) \|^\dagger \| \leq O \left( \frac{1}{\xi} \left( \frac{N_{\uparrow}}{N} \right)^{\dagger} \right) \]

and

\[\|P_m^{(N_\uparrow + 2n')} [S_{j_m}(z)]^\dagger \| (V_{j_m}^{(4)})' + V_{j_1,\ldots,j_{m-1}}^{(3)} || S_{j_m}(z) \|^\dagger \| \leq O \left( \frac{1}{\xi} \left( \frac{N_{\uparrow}}{N} \right)^{\dagger} \right) .\]
Next, we invoke Lemma 3.4 in [Pil] and estimate
\[
\sup_{0 \leq r \leq 2n', r \text{ even}} \| (S_{j_n}(z))^{(l-r, l-r+1)} W_{j_n} Q_{j_n}^{(l-r-2, l-r+1)} (S_{j_n}(z))^{(l-r, l-r+1)} \|^2 \leq 4(1 + a_{j_n} - \frac{2a_{j_n}}{N-i-1} - \frac{1 - c_{j_n}}{(N-i-1)^2})
\] (4.55)

where
\[
a_{j_n} := 2\epsilon_j + O(\epsilon_j^n), \quad b_{j_n} := (1 + \epsilon_{j_n}) \delta \chi(0, 2)(\delta) \sqrt{\epsilon_{j_n}^2 + 2\epsilon_{j_n}}, \quad c_{j_n} := -(1 - \delta^2 \chi(0, 2)(\delta))(\epsilon_{j_n}^2 + 2\epsilon_{j_n})
\] (4.56)

are those of Lemma 3.6 of [Pil] up to \(N\)- and \(\xi\)-dependent corrections that are hidden in the term \(O(\epsilon_j^n)\) (with \(\nu > \frac{11}{8}\)) which enters the definition of \(a_{j_n}\).

Hence, for \(N\) large enough we have derived the inequality
\[
\left\| R_{j_1, \ldots, j_n; i + 2, j + 2} (w) \right\|^2 \leq \left\| R_{j_1, \ldots, j_n; i + 2, j + 2} (w) \right\|^2 + O\left( \left( \frac{1}{1 + a_{j_n}/4} \right)^n \right)
\] (4.57)

By setting \(n' = \frac{\ln N}{\ln (1 + a_{j_n}/4)}\) and
\[
\Delta H_{j_1, \ldots, j_n} + \left( \hat{H}_{j_n}^{\text{Bog}} \right) \xi - z = \frac{(m - 1)\xi^2}{M} + \frac{\xi T}{2}
\] (4.62)

where \(\Delta H_{j_1, \ldots, j_n} \geq 0\) (recall the assumption in (5.1)), we can write
\[
\sum_{l=0}^{\infty} S_{j_n}(z) \left\{ (\xi - \frac{\xi T}{2}) \hat{H}_{j_n}^{\text{Bog}} \right\} S_{j_n}(z) \left( \hat{H}_{j_n}^{\text{Bog}} \right) \xi - z + \hat{h}_{j_1, \ldots, j_n} = \frac{(m - 1)\xi^2}{M} + \frac{\xi T}{2} + \left( \hat{H}_{j_n}^{\text{Bog}} \right) \xi - z - \frac{(m - 1)\xi^2}{M}
\] (4.63)

\[
\sum_{l=0}^{\infty} S_{j_n}(z) \left\{ (\xi - \frac{\xi T}{2}) \hat{H}_{j_n}^{\text{Bog}} \right\} S_{j_n}(z) \left( \hat{H}_{j_n}^{\text{Bog}} \right) \xi - z + \hat{h}_{j_1, \ldots, j_n} = \frac{(m - 1)\xi^2}{M} + \frac{\xi T}{2} + \left( \hat{H}_{j_n}^{\text{Bog}} \right) \xi - z - \frac{(m - 1)\xi^2}{M}
\] (4.64)

\[
\sum_{l=0}^{\infty} S_{j_n}(z) \left\{ (\xi - \frac{\xi T}{2}) \hat{H}_{j_n}^{\text{Bog}} \right\} S_{j_n}(z) \left( \hat{H}_{j_n}^{\text{Bog}} \right) \xi - z + \hat{h}_{j_1, \ldots, j_n} = \frac{(m - 1)\xi^2}{M} + \frac{\xi T}{2} + \left( \hat{H}_{j_n}^{\text{Bog}} \right) \xi - z - \frac{(m - 1)\xi^2}{M}
\] (4.65)
and estimate

\[ (4.60) + (4.61) \]

\[
\leq \left\| R_{j_1, \ldots, j_n; \xi; 2, j+2}(w) \right\|^{\frac{1}{2}} W_{j_1, \ldots, j_n} \mathcal{S}_{j_1, \ldots, j_n}^{(j+1)} W_{j_1, \ldots, j_n} \leq \left( \mathcal{S}_{j_1, \ldots, j_n}^{(j+1)} \right)^{\frac{1}{2}} W_{j_1, \ldots, j_n} \]

\[ + O\left( \frac{1}{\xi} \left\| \frac{N^\frac{1}{2}}{N} \right\|^2 \right) \]

\[ (4.67) \]

\[
\leq \left\| R_{j_1, \ldots, j_n; \xi; 2, j+2}(w) \right\|^{\frac{1}{2}} W_{j_1, \ldots, j_n} \times \mathcal{S}_{j_1, \ldots, j_n}^{(j+1)} \mathcal{S}_{j_1, \ldots, j_n}^{(j+1)} W_{j_1, \ldots, j_n} \leq \left( \mathcal{S}_{j_1, \ldots, j_n}^{(j+1)} \right)^{\frac{1}{2}} W_{j_1, \ldots, j_n} \]

\[ + O\left( \frac{1}{\xi} \left\| \frac{N^\frac{1}{2}}{N} \right\|^2 \right) \]

\[ (4.69) \]

In the step from (4.67) to (4.68) we make use of

\[ \mathcal{S}_{j_1, \ldots, j_n}^{(j+2, j+3)} \leq \mathcal{S}_{j_1, \ldots, j_n}^{(j+1)} \]

\[ (4.70) \]

and we estimate the term proportional to \( \mathcal{S}_{j_1, \ldots, j_n}^{(j+2, j+3)} \) of order \( \frac{1}{\xi} \left\| \frac{N^\frac{1}{2}}{N} \right\|^2 \); see an analogous estimate in (4.52).

We notice that because of the semigroup property of the Feshbach map the following identity holds

\[ Q_{j_1}^{(j+1)} W_{j_1} \mathcal{S}_{j_1}^{(j+1)} \leq \mathcal{S}_{j_1}^{(j+1)} W_{j_1} Q_{j_1}^{(j+1)} \]

\[ (4.72) \]

where \( R_{j_1, \ldots, j_n; \xi; 2, j+2}(z) \) and \( \mathcal{S}_{j_1, \ldots, j_n}^{(j+1)} \) have the same definition of \( R_{j_1, \ldots, j_n; \xi; 2, j+2}(z) \) and \( \mathcal{S}_{j_1, \ldots, j_n}^{(j+1)} \) but are referred to the Feshbach flow associated with \( \frac{\xi T}{2} + \mathcal{H}_{j_1}^{Bog} \leq \left\| \frac{N^\frac{1}{2}}{N} \right\|^2 \). Hence, we can write

\[ R_{j_1, \ldots, j_n; \xi; 2, j+2}(w) \]

\[ (4.74) \]

\[ (4.75) \]
Using the splitting in (4.62)-(4.63) we can estimate
\[
\|(R_{j_1, \ldots, j_m; i_1+2, i_2+2}(w))^{\frac{1}{2}}[(R_{j_1, \ldots, j_m; i_1+2, i_2+2}^{Bog}(z))_{\xi}]^{\frac{1}{2}} \|_{\mathcal{L}(\mathcal{H}_\xi)} \leq 1
\]  
(4.76)
(see a similar argument in Corollary 5.1). Finally, we use Theorem 3.1 of [Pil] to estimate
\[
\|(R_{j_1, \ldots, j_m; i_1+2, i_2+2}(z))_{\xi}\|^{\frac{1}{2}} W_{j_1, \ldots, j_m} \sum_{j=0}^{\infty} \| (R_{j_1, \ldots, j_m; i_1+2, i_2+2}^{Bog}(z))_{\xi} [(\Gamma_{j_1, \ldots, j_m; i_1+2, i_2+2}^{Bog}(z))_{\xi}]^{\frac{1}{2}} W_{j_1, \ldots, j_m} [(R_{j_1, \ldots, j_m; i_1+2, i_2+2}^{Bog}(z))_{\xi}]^{\frac{1}{2}} \|
\]  
(4.77)
and get
\[
\frac{1}{\|(R_{j_1, \ldots, j_m; i_1+2, i_2+2}(w))^{\frac{1}{2}} \|} \geq \frac{1}{\|(R_{j_1, \ldots, j_m; i_1+2, i_2+2}(z))_{\xi}\|^{\frac{1}{2}} W_{j_1, \ldots, j_m} \sum_{j=0}^{\infty} [(\Gamma_{j_1, \ldots, j_m; i_1+2, i_2+2}^{Bog}(z))_{\xi} [(R_{j_1, \ldots, j_m; i_1+2, i_2+2}^{Bog}(z))_{\xi}]^{\frac{1}{2}} W_{j_1, \ldots, j_m} [(R_{j_1, \ldots, j_m; i_1+2, i_2+2}^{Bog}(z))_{\xi}]^{\frac{1}{2}} \| + O(\frac{N}{N})\}
\]  
(4.78)
where \(x_{i_1+2}\) is the \(i_1+2\)-term of the sequence
\[
x_{i_1+2} := 1 - \frac{1}{4(1 + a_{j_m} - \frac{b_{j_m}}{N-2j_1-1} - \frac{c_{j_m}}{(N-2j_1-1)^2})x_{i_1+2}}.
\]  
(4.81)
Here, \(a_{j_m}, b_{j_m}, \) and \(c_{j_m}\) are those in Lemma 3.4 of [Pil] and in (4.56) up to \(N\) and \(\xi\)-dependent corrections that are hidden in the term \(O(e^{\nu})\) (with \(\nu > \frac{10}{16}\)) which enters the definition of \(a_{j_m}\).

We use the same notation to avoid a new symbol. Moreover, the value of \(a_{j_m}\) set here will coincide with the one in Corollary 5.1. □

Corollary 5.1 and Theorem 4.1 enable us to define the Neumann expansion in (3.18)-(3.19) rigorously, and a result analogous to Theorem 3.1 of [Pil] can be proven with the help of Lemma 3.4:

**Theorem 4.2.** Assume condition a) of Corollary 5.1. Then, for
\[
z \leq E_{j_m}^{Bog} + (\delta - 1)\phi_{j_m} \sqrt{\epsilon_j + 2\epsilon_{j_m}}
\]  
(4.82)
with \(\delta = 1 + \sqrt{\epsilon_{j_m}}, \) \(\epsilon_{j_m}\) sufficiently small and \(N\) sufficiently large, the operators \(H_{j_1, \ldots, j_m}(z + z^{#}_{j_1, \ldots, j_m}), i + 2 \leq i \leq N - 2\) and even, are well defined. For \(i = i_1 + 2, 4, 6, \ldots, N - 2\) they correspond to
\[
\mathcal{H}_{j_1, \ldots, j_m}^{(i)}(w) := \mathcal{S}_{j_m}^{(i+1)}(H_{j_1, \ldots, j_m} - w) \mathcal{S}_{j_m}^{(i+1)}
\]  
(4.83)
\[
-\mathcal{S}_{j_m}^{(i+1)} W_{j_1, \ldots, j_m} \sum_{l=0}^{\infty} R_{j_1, \ldots, j_m; i_1, l}(w) \Gamma_{j_1, \ldots, j_m; i_1, l}(w) R_{j_1, \ldots, j_m; i_1, l}(w) \times
\]  
(4.84)
\[
\times \mathcal{S}_{j_m}^{(i+1)}.
\]  
(4.85)

\(\mathcal{H}_{j_1, \ldots, j_m}^{(i)}(z + z^{#}_{j_1, \ldots, j_m})\) is self-adjoint on the domain of the Hamiltonian \(\mathcal{S}_{j_m}^{(i+1)}(H_{j_1, \ldots, j_m} - z - z^{#}_{j_1, \ldots, j_m}) \mathcal{S}_{j_m}^{(i+1)}\).
where \( w = z + z^h_{j_1,...,j_m} \). The operators \( R_{j_1,...,j_m};i,j(w) \), \( \Gamma_{j_1,...,j_m};i,j(w) \) are defined in (3.26) and (3.27)-(3.28), respectively.

The following estimates hold true for \( i + 2 \leq i \leq N - 2 \) and even:

\[
\| \hat{\Gamma}_{j_1,...,j_n};i,j(w) \| \leq \frac{1}{x_i} \tag{4.86}
\]

where

\[
\hat{\Gamma}_{j_1,...,j_n};i,j(w) := \sum_{l=0}^{\infty} \left( (R_{j_1,...,j_n};i,l(w))^{\frac{1}{2}} \Gamma_{j_1,...,j_n};i,l(w)(R_{j_1,...,j_n};i,l(w))^{\frac{1}{2}} \right)^l_i \tag{4.87}
\]

and \( x_i \) defined in (4.81) fulfills the bound (see Lemma 3.6 of [Pi1])

\[
x_i \geq \frac{1}{2} \left[ 1 + \sqrt{\eta a} - \frac{b_{\epsilon} / \sqrt{\eta a}}{N - 2j - \epsilon \Theta} \right], \quad \eta = 1 - \epsilon^{\frac{1}{4}}, \tag{4.88}
\]

for \( \epsilon \equiv \epsilon_{j_m} \) small enough and \( 0 < \Theta \leq \frac{3}{4} \).

**Proof**

The proof follows the arguments of Theorem 3.1 of [Pi1] starting from the result in Theorem 4.1. \( \Box \)

Furthermore, similarly to what seen in the companion papers [Pi1], [Pi2], Corollary 5.1 implies the expansion of the operators \( \Gamma_{j_1,...,j_m};i+2,j-2(w) \) in terms of finite sums of products of the resolvents \( R_{j_1,...,j_m};j_l(w), \bar{\Gamma}_{j_1,...,j_m};j_l(w) \), \( \bar{\Gamma}_{j_1,...,j_m};j_l(w) \), of the operator \( \hat{\Gamma}_{j_1,...,j_m};j_l(w) \), and of \( \Gamma_{j_1,...,j_m};i+2,j+2(w) \). This is the content of Proposition 4.3.

To streamline formulae, in Definition 4.4 and in Proposition 4.3 we write \( \hat{W}_{j_1,...,j_n};j_l(w) \), \( R_{j_1,...,j_n};j_l(w) \), and \( \bar{\Gamma}_{j_1,...,j_n};j_l(w) \) instead of \( \hat{W}_{j_1,...,j_n};j_l(w) \), \( \hat{W}_{j_1,...,j_n};j_l(w) \), and \( \bar{\Gamma}_{j_1,...,j_n};j_l(w) \), respectively.

**Definition 4.4.** Let \( h \in \mathbb{N}, h \geq 2, \) and

\[
w \leq z^h_{j_1,...,j_m+1} + E_{k_0}^{Bog} + (\delta - 1)\phi_{j_m} \sqrt{\epsilon_{j_m}^2 + 2\epsilon_{j_m}} \tag{4.89}
\]

with \( \delta \leq 1 \) and \( \epsilon_{j_m} \) sufficiently small. Let \( N \) be sufficiently large. We define:

1. For \( N - 2 \leq j \leq i \) and even

\[
[\Gamma_{j_1,...,j_n};j_l(w)]_{j_l,j_m}(w) = [\Gamma_{j_1,...,j_n};j_l,w)_{j_l,j_m}(w) + [\Gamma_{j_1,...,j_n};j_l,w)_{j_l,j_m}(w)] \tag{4.90}
\]

where

\[
[\Gamma_{j_1,...,j_n};j_l,w)_{j_l,j_m}(w)]_{j_l,j_m}(w) := \hat{W}_{j_1,...,j_n};j_l,w)_{j_l,j_m}(w) \hat{R}_{j_1,...,j_n};j_l,w)_{j_l,j_m}(w) \tag{4.91}
\]

\[
[\Gamma_{j_1,...,j_n};j_l,w)_{j_l,j_m}(w)] := \hat{W}_{j_1,...,j_n};j_l,w)_{j_l,j_m}(w) \hat{R}_{j_1,...,j_n};j_l,w)_{j_l,j_m}(w) \tag{4.92}
\]

\[
[\Gamma_{j_1,...,j_n};j_l,w)_{j_l,j_m}(w)]^{(0)} := \hat{W}_{j_1,...,j_n};j_l,w)_{j_l,j_m}(w) \hat{R}_{j_1,...,j_n};j_l,w)_{j_l,j_m}(w) \tag{4.93}
\]

\[
\times \sum_{l_1=1}^{b-1} \left( R_{j_1,...,j_n};j_{l_1},j_{l_1}-2 \right)^{\frac{1}{2}} \hat{W}_{j_1,...,j_n};j_{l_1},j_{l_1}-4 \right) R_{j_1,...,j_n};j_{l_1},j_{l_1}-4 \left( \hat{W}_{j_1,...,j_n};j_{l_1},j_{l_1}-2 \right)^{\frac{1}{2}} \hat{W}_{j_1,...,j_n};j_{l_1},j_{l_1}-2 \tag{4.94}
\]
with \( [\Gamma_{j+2,j+2}(w)]_{(i,h)}^{(0)} := (3.27) \):

for \( N - 2 \geq j \geq \bar{t} + 4 \) and even

\[
[\Gamma_{j,j}(w)]_{(j-2, h_\ast)} := \tilde{W}_{j,j-2} \left( R_{j-2,j-2}(w) \right)^{\frac{1}{2}} \times \\
\sum_{l_\ast = h}^{\infty} \left[ \tilde{w} \left( R_{j-2,j-2}(w) \right)^{\frac{1}{2}} [\Gamma_{j-2,j-2}(w)]_{(j-2, h_\ast)} \right]^{l_\ast} \times \\
\times (R_{j-2,j-2}(w))^{\frac{1}{2}} \tilde{W}_{j-2,j}.
\] (4.95)

2. For \( N - 2 \geq j \geq \bar{t} + 6, \bar{t} + 2 \leq l \leq j - 4 \) and even

\[
[\Gamma_{j,j}(w)]_{(l_\ast +2, h_\ast; \ldots; j-4, h_\ast; j-2, h_\ast)} := \tilde{W}_{j,j-2} \left( R_{j-2,j-2}(w) \right)^{\frac{1}{2}} \sum_{l_\ast = 1}^{h-1} \left[ \tilde{w} \left( R_{j-2,j-2}(w) \right)^{\frac{1}{2}} [\Gamma_{j-2,j-2}(w)]_{(l_\ast +2, h_\ast; \ldots; j-4, h_\ast)} \right]^{l_\ast} \times \\
\times (R_{j-2,j-2}(w))^{\frac{1}{2}} \tilde{W}_{j-2,j}.
\] (4.96)

Here, the symbol \( \sum_{l_\ast = 1}^{h-1} \) stands for a sum of terms resulting from operations \( \mathbb{A}1 \) and \( \mathbb{A}2 \) below:

\( \mathbb{A}1 \) At fixed \( 1 \leq l_\ast \leq h - 1 \) summing all the products

\[
\left[ \tilde{w} \left( R_{j-2,j-2}(w) \right)^{\frac{1}{2}} X (R_{j-2,j-2}(w))^{\frac{1}{2}} \right]^{l_\ast (j-2, h_\ast)}
\] (4.98)

that are obtained by replacing \( X \) for each factor with the operators (defined by iteration) of the type \( [\Gamma_{j-2,j-2}(w)]_{(l_\ast +2, h_\ast; \ldots; j-4, h_\ast)} \) with \( l \leq s \leq j-4 \) and even, with the constraint that if \( l \leq j-6 \) then \( X \) is replaced with \( [\Gamma_{j-2,j-2}(w)]_{(l_\ast +2, h_\ast; \ldots; j-4, h_\ast)} \) in one factor at least, whereas if \( l = j - 4 \) then \( X \) is replaced with \( [\Gamma_{j-2,j-2}(w)]_{(j-4, h_\ast)} \) in one factor at least;

\( \mathbb{A}2 \) Summing from \( l_\ast = 1 \) up to \( l_\ast = h - 1 \).

3. For \( N - 2 \geq j \geq \bar{t} + 6, \bar{t} + 2 \leq l \leq j - 4 \) and even

\[
[\Gamma_{j,j}(w)]_{(l_\ast +2, h_\ast; \ldots; j-4, h_\ast; j-2, h_\ast)} := \tilde{W}_{j,j-2} \left( R_{j-2,j-2}(w) \right)^{\frac{1}{2}} \sum_{l_\ast = 1}^{h-1} \left[ \tilde{w} \left( R_{j-2,j-2}(w) \right)^{\frac{1}{2}} [\Gamma_{j-2,j-2}(w)]_{(l_\ast +2, h_\ast; \ldots; j-4, h_\ast)} \right]^{l_\ast} \times \\
\times (R_{j-2,j-2}(w))^{\frac{1}{2}} \tilde{W}_{j-2,j}.
\] (4.100)

Here, the symbol \( \sum_{l_\ast = 1}^{h-1} \) stands for a sum of terms resulting from operations \( \mathbb{B}1 \) and \( \mathbb{B}2 \) below:

\( \mathbb{B}1 \) At fixed \( 1 \leq l_\ast \leq h - 1 \), summing all the products

\[
\left[ \tilde{w} \left( R_{j-2,j-2}(w) \right)^{\frac{1}{2}} X (R_{j-2,j-2}(w))^{\frac{1}{2}} \right]^{l_\ast (j-2, h_\ast)}
\] (4.101)

that are obtained by replacing \( X \) for each factor with the operators (defined by iteration) of the type \( [\Gamma_{j-2,j-2}(w)]_{(l_\ast +2, h_\ast; \ldots; j-4, h_\ast)} \) and \( [\Gamma_{j-2,j-2}(w)]_{(l_\ast +2, h_\ast; \ldots; j-4, h_\ast)} \) with \( l \leq s \leq j-4 \) and even, with the constraint that \( X \) is replaced with \( [\Gamma_{j-2,j-2}(w)]_{(l_\ast +2, h_\ast; \ldots; j-4, h_\ast)} \) in one factor at least.
B2) Summing from \( l_{j-2} = 1 \) up to \( h - 1 \).

**Proposition 4.3.** Let \( \epsilon_n \equiv \epsilon \) be sufficiently small and \( N \) sufficiently large. For any fixed \( 2 \leq h \in \mathbb{N} \) and for \( N - 2 \geq i \geq \tilde{i} + 4 \) and even, the splitting

\[
\Gamma_i(w) = \sum_{l = l_{i+2}, l \text{ even}}^{i-2} \left[ \Gamma_i(w) \right]_{l, i+2, i+4; \ldots; i-2, h} + \sum_{l = l_{i+2}, l \text{ even}}^{i-2} \left[ \Gamma_i(w) \right]_{l, i+2, i+4; \ldots; i-2, h} (4.102)
\]

holds true for \( w \leq z_{j_1, \ldots, j_{h-1}} + E^{Bog}_{j_m} + (\delta - 1)\phi_{j_m} \sqrt{\epsilon_j^2 + 2\epsilon_j} \) and \( \delta \leq 1 + \sqrt{\epsilon_j} \). Moreover, for \( \tilde{i} + 2 \leq l \leq i - 2 \) and even, the estimates

\[
\left\| (R_{i,l}^{Bog}(z))^{\frac{1}{2}} [R_{i,l}^{Bog}(z)]_{l, i+2, i+4; \ldots; i-2, h} (R_{i,l}^{Bog}(z))^{\frac{1}{2}} \right\| \leq \prod_{f = l+2, f \text{ even}}^{i} \frac{K_{f,\epsilon}}{(1 - Z_{f-2, \epsilon})^2}
\]

and

\[
\left\| (R_{i,l}^{Bog}(z))^{\frac{1}{2}} [R_{i,l}^{Bog}(z)]_{l, i+2, i+4; \ldots; i-2, h} (R_{i,l}^{Bog}(z))^{\frac{1}{2}} \right\| \leq (Z_{l,\epsilon})^b \prod_{f = l+2, f \text{ even}}^{i} \frac{K_{f,\epsilon}}{(1 - Z_{f-2, \epsilon})^2}
\]

hold true, where

\[
K_{i,\epsilon} := \frac{1}{4(1 + a\epsilon - \frac{b\epsilon}{N-i+1} - \frac{1-c\epsilon}{(N-i+1)^2})}, \quad Z_{l-2,\epsilon} := \frac{1}{4(1 + a\epsilon - \frac{b\epsilon}{N-i+3} - \frac{1-c\epsilon}{(N-i+3)^2})} \left[ 1 + \sqrt{\frac{1}{\epsilon_j} - \frac{b\epsilon}{N-i+4-\epsilon_j}} \right]^{\frac{2}{3}}
\]

where \( a\epsilon, b\epsilon, c\epsilon \), and \( 0 < \Theta \leq \frac{1}{2} \) are those defined in Corollary 5.1 and Lemma 3.6 of [Pi1].

**Proof**

Using the results of Theorem 4.2 the proof is like in Proposition 4.10 of [Pi1]. \( \square \)

### 4.1 Last implementation of the Feshbach map

For the last implementation of the Feshbach map, i.e., from \( i = N - 2 \) to \( i = N \), we have to make sure that

\[
\mathcal{P}_{\eta_{j_1, \ldots, j_{h-1}}}^{N} = \mathcal{P}_{\psi_{j_1, \ldots, j_{h-1}}}^{N} (z + z_{j_1, \ldots, j_{h-1}}) \mathcal{P}_{\psi_{j_1, \ldots, j_{h-1}}}^{N}
\]

is well defined in \( \mathcal{P}_{\psi_{j_1, \ldots, j_{h-1}}}^{N} \) and estimate its operator norm. This is the content of next Proposition 4.5. \( \mathcal{P}_{\psi_{j_1, \ldots, j_{h-1}}}^{N} \) is the projection onto the subspace of vectors without particles in the modes \( \pm j_m \) and orthogonal to \( \psi_{j_1, \ldots, j_{h-1}} \).

In the case \( m = 1 \) we shall use the notation

\[
\mathcal{P}_\eta := |\eta\rangle\langle\eta|, \quad \mathcal{P}_\eta^\# := \mathcal{P}_\eta^{(N-1)} - \mathcal{P}_\eta^\#
\]

(4.107)
Proposition 4.5. Let \( 1 \leq m \leq M, \varepsilon_j \) be sufficiently small and \( N \) sufficiently large such that the Feshbach flow (see (4.83)) is well defined for \( z \leq E_{j_{\text{m}}}^{\text{Bog}} + \sqrt{\varepsilon_j} \phi_j \). Let \( \Delta = \varepsilon_j \) up to \( i = N - 2 \). Assume that

1. the Hamiltonian \( H_{1,j_{\text{m}}}' \) has ground state energy \( z_{1,j_{\text{m}}}' \) with ground state vector \( \phi_{1,j_{\text{m}}}' \), \( \mathcal{F}_z \) is the subspace of vectors containing at least one particle in the modes \( \pm j_{\text{m}} \), and

\[
\text{infspec} \left\{ H_{1,j_{\text{m}}}' \right\} \geq \Delta_{m-1}
\]

for some \( \Delta_{m-1} > 0 \).

Then, there exists \( C^\perp > 0 \) such that

\[
\left| \frac{1}{(1 - \gamma \Delta_{m-1})^{\frac{1}{N}}} \right| \leq \frac{1}{(1 - \gamma \Delta_{m-1})^{\frac{1}{N}}}
\]

for \( N \) sufficiently large and

\[
z \leq \min \left\{ z_m + \gamma \Delta_{m-1} \frac{me^2}{M} \sqrt{\varepsilon_j} \phi_j \right\} ; E_{j_{\text{m}}}^{\text{Bog}} + \sqrt{\varepsilon_j} \phi_j \sqrt{e_j^2 + 2\varepsilon_j}\}
\]

where \( \bar{z} = \frac{1}{(\ln M)^\frac{1}{2}}, z_m \) is the ground state energy of \( H_{j_{\text{m}}}^{\text{Bog}}, \gamma = \frac{1}{2}, \text{ and } \Delta_{m} \equiv \Delta_0 := \min \left\{ k_j^2 \mid j \in \mathbb{Z}^d \setminus \{0\}, \right\}. \)

Proof

We set \( w \equiv z + z_{1,j_{\text{m}}}' \), and write

\[
\mathcal{K}^{(N-2)}(\xi, z_{1,j_{\text{m}}}') \leq \mathcal{K}^{(N-2)}(\xi, z_{1,j_{\text{m}}}') - \mathcal{K}^{(N-2)}(\xi, z_{1,j_{\text{m}}}' - w) - \mathcal{K}^{(N-2)}(\xi, z_{1,j_{\text{m}}}' - w) - \mathcal{K}^{(N-2)}(\xi, z_{1,j_{\text{m}}}' - w).
\]

With reference to the definition in (4.1), we proceed with the identity

\[
\mathcal{K}^{(N-2)}(\xi, z_{1,j_{\text{m}}}') = \left( \frac{1}{N} \sum_{j \in \mathbb{Z}^d \setminus \{j_{\text{m}}\}} a_j^* a_j \phi_j \right) \mathcal{K}^{(N-2)}(\xi, z_{1,j_{\text{m}}}') - \mathcal{K}^{(N-2)}(\xi, z_{1,j_{\text{m}}}' - w) - \mathcal{K}^{(N-2)}(\xi, z_{1,j_{\text{m}}}' - w).
\]

We recall that

\[
V_{j_{\text{m}}}^{(4)} = \frac{1}{N} \phi_j \sum_{j' \in \mathbb{Z}^d \setminus \{j_{\text{m}}\}} \sum_{j'' \in \mathbb{Z}^d \setminus \{j_{\text{m}}\} \cap \{0\}} a_{j''}^* a_{j'}.
\]

where \( V_{j_{\text{m}}}^{(4)} \geq 0 \) has been defined in (4.31).
For \( m = 1 \), we recall
\[
H_{j_1} - z = (1 - \xi)T - T_{j_1} + V_{j_1} + \hat{H}_{j_1}^{\text{Bog}} + \xi T - z \quad (4.119)
\]
\[
= T_{j_1} + V_{j_1} + \hat{H}_{j_1}^{\text{Bog}} - z \quad (4.120)
\]
and make use of the inequality
\[
\mathcal{P}_\eta^\# (H_{j_1} - z) \mathcal{P}_\eta^\#
\geq \mathcal{P}_\eta^\# \left[ \sum_{j \in \mathbb{Z}^d(0)} ((k_j)^2 - \frac{\phi_{j_1}}{N}) a_j^* a_j + V^{(4)}_{j_1} - z \right] \mathcal{P}_\eta^\#
\quad (4.123)
\]
that holds because \( \mathcal{P}_\eta^\# \) projects onto a subspace with no particles in the modes \( \pm j_1 \) so that
\[
\mathcal{P}_\eta^\# \mathcal{P}_\eta^\equiv \mathcal{P}_\eta^\# \hat{H}_{j_1}^{\text{Bog}} \mathcal{P}_\eta^\equiv 0.
\]
Thus, for \( N \) large enough we can estimate
\[
\mathcal{P}_\eta^\# (H_{j_1} - z) \mathcal{P}_\eta^\#
\geq (\Delta_0^\# - \frac{\xi^2}{\mathcal{M}} - z) \mathcal{P}_\eta^\# - \mathcal{P}_\eta^\# \Gamma_{j_1 : N,N}(z) \mathcal{P}_\eta^\# \quad (4.125)
\]
From Lemma 5.2 there exists a constant \( C^\#_{\perp} \) such that
\[
\mathcal{P}_\eta^\# \Gamma_{j_1 : N,N}(z) \mathcal{P}_\eta^\# \leq \frac{\phi_{j_1}}{2\epsilon_{j_1} + 2} \frac{\phi_{j_1}^2}{\phi_{j_1}} \mathcal{P}_\eta^\# \quad (4.126)
\]
where \( \mathcal{G}_{j_1 : N-2,N-2}(z) \) has been defined for a three modes system by recursion starting from
\[
\mathcal{G}_{j_1 : i,i}(z) := \left[ W_{j_1 : i,i-2}(z) W_{j_1 : i-2,i}(z) \right]_{i \leq 0}, \quad \mathcal{G}_{j_1 : 0,0}(z) = 1 \quad (4.129)
\]
with
\[
W_{j_1 : i,i-2}(z) W_{j_1 : i-2,i}(z) \quad (4.130)
\]
\[
:= \frac{1}{N^2} \frac{1}{\phi_{j_1}^2} \frac{(n_j + 1) (n_{-j} + 1)}{\phi_{j_1}^2} \left[ \frac{(n_{j_0} + 1) (n_{j_0} + 1)}{\phi_{j_1}^2} \right] \quad (4.131)
\]
\[
\times \frac{(n_{j_0})_{n_{j_0} - 2} \phi_{j_1} + k_{j_1}^2 (n_{j_1} + n_{-j_1}) + 2 (n_{j_0} - 2) \phi_{j_1} + k_{j_1}^2 - z}{\left[ \frac{(n_{j_0})_{n_{j_0} - 2}}{\phi_{j_1}^2} \phi_{j_1} + k_{j_1}^2 (n_{j_1} + n_{-j_1}) + 2 (n_{j_0} - 2) \phi_{j_1} + k_{j_1}^2 - z \right]} \quad (4.132)
\]
where
\[
n_{j_1} + n_{-j_1} = N - i \quad \text{with } i \text{ even } ; \quad n_{j_1} = n_{-j_1} ; \quad n_{j_0} = n_0 = i \quad (4.133)
\]
\( \tilde{G}_{j_i: N-2, N-2}(z) \) enters the fixed point equation for the Bogoliubov Hamiltonian associated with a three-modes system:

\[
z = -\frac{\phi_{j_i}}{2\epsilon_{j_i} + 2 - \frac{1}{\phi_{j_i}}} \tilde{G}_{j_i: N-2, N-2}(z). \tag{4.134}
\]

The additional inputs

1. \( \tilde{G}_{j_i: N-2, N-2}(z) \) is nondecreasing in \( z \) (see Remark 4.1 in [Pi1]);
2. the existence of the fixed point \( z_1 \) of (4.134) with \( |z_1 - E_{j_i}^{Bog}| = O(\frac{1}{N^\beta}) \) for any \( 0 < \beta < 1 \) (see Lemma 5.5 of [Pi1]);

imply that for \( N \) large enough and for

\[
z \leq \min \left\{ z_1 + \gamma \Delta_0^\# - \frac{\xi_1}{M} - \frac{C^\# \perp}{(\ln N)^2}; E_{j_i}^{Bog} + \sqrt{\epsilon_{j_i}} \phi_{j_i} \sqrt{\epsilon_{j_i}^2 + 2\epsilon_{j_i}} \right\} \tag{4.135}
\]

we can estimate

\[
\tilde{G}_{j_i: N-2, N-2}(z - \Delta_0^\#(1 - \frac{\phi_{j_i}(\ln N)^\perp}{N\Delta_0})) \leq \tilde{G}_{j_i: N-2, N-2}(z_1) \tag{4.136}
\]

and, consequently,

\[
\overline{\mathcal{P}}_{\eta}(H_{j_i} - z) \overline{\mathcal{P}}_{\eta}^\# - \overline{\mathcal{P}}_{\eta} \Gamma_{j_i: N,N}(z) \overline{\mathcal{P}}_{\eta}^\# \geq (\Delta_0^\# - z - \frac{\xi_1}{M} - \frac{\phi_{j_i}}{2\epsilon_{j_i} + 2 - \frac{1}{\phi_{j_i}}} \tilde{G}_{j_i: N-2, N-2}(z_1)) \overline{\mathcal{P}}_{\eta}^\# - \frac{C^\# \perp}{(\ln N)^2} \overline{\mathcal{P}}_{\eta}^\# \tag{4.137}
\]

\[
\geq (\Delta_0^\# - z - \frac{\xi_1}{M} + z_1 - \frac{C^\# \perp}{(\ln N)^2}) \overline{\mathcal{P}}_{\eta}^\# \tag{4.138}
\]

\[
\geq (1 - \gamma) \Delta_0^\# \overline{\mathcal{P}}_{\eta}^\#. \tag{4.139}
\]

For \( 2 \leq m \leq M \), we recall that \( \overline{\mathcal{P}}_{\psi_{j_1,...,j_m}}^\# \mathcal{F}^N \subset \mathcal{F}^N_{\oplus \mathcal{F}_m^{N}} \) by definition. Hence, due to the assumption in (4.108) the inequality

\[
\overline{\mathcal{P}}_{\psi_{j_1,...,j_m}}^\# \left[ (H_{j_1,...,j_m})_z - (1 - \xi) T_j z_{j_m} - z_{j_1,...,j_{m-1}} \right] \overline{\mathcal{P}}_{\psi_{j_1,...,j_m}}^\#
\]

\[
\geq (\Delta_{m-1}^\# - (m - 1)\frac{\xi_1}{M}) \overline{\mathcal{P}}_{\psi_{j_1,...,j_m}}^\#
\]

\[
\geq \left( \Delta_{m-1}^\# - \frac{(m - 1)\xi_1}{M} \right) \overline{\mathcal{P}}_{\psi_{j_1,...,j_m}}^\#
\]

(4.140)

(4.141)

can be proven with an argument similar to Lemma 4.1. Using this ingredient,

\[
\overline{\mathcal{P}}_{\psi_{j_1,...,j_m}}^\# \left[ (H_{j_1,...,j_m})_z - (1 - \xi) T_j z_{j_m} - z_{j_1,...,j_{m-1}} \right] \overline{\mathcal{P}}_{\psi_{j_1,...,j_m}}^\#
\]

\[
\geq (\Delta_{m-1}^\# - \frac{(m - 1)\xi_1}{M}) \overline{\mathcal{P}}_{\psi_{j_1,...,j_m}}^\#
\]

(4.144)

(4.145)

(4.146)

(4.147)

(4.148)
where we have exploited that $V_j^{(3)}, V_j^{(3)}$, and $(H_{j_m}^{Bog})_j$ are normal ordered and contain at least one operator $a^*_{\pm j_m}, a_{\pm j_m}$ by definition. Then, for $z$ as in (4.110) and $\frac{1}{\gamma}$ small enough

$$
(4.112) + (4.113) \\
\geq (\Delta_{m-1} - z - \frac{(m - 1)\xi^{\frac{1}{2}}}{M})\mathcal{P}_{\psi_{j_{1} \ldots j_{m-1}}} \\
- \mathcal{P}_{\psi_{j_{1} \ldots j_{m-1}}} \Gamma_{j_{1} \ldots j_{m-1}, N, N}(w) (\mathcal{P}_{\psi_{j_{1} \ldots j_{m-1}}} \\
\geq (\Delta_{m-1} - z)\mathcal{P}_{\psi_{j_{1} \ldots j_{m-1}}} + (z_m - \frac{C^\perp}{(\ln N)^\frac{1}{2}} - \frac{m\xi^{\frac{1}{2}}}{M})\mathcal{P}_{\psi_{j_{1} \ldots j_{m-1}}} \\
\geq (1 - \gamma)\Delta_{m-1} \mathcal{P}_{\psi_{j_{1} \ldots j_{m-1}}}
$$

where for the step from (4.150)-(4.151) to (4.152) we invoke (5.38) in Lemma 5.2 and repeat the argument already used for the analogous quantity in the case $m = 1$.

\[\square\]

### 4.2 Proof by induction in the index $m$

We have now all the tools to state the main result of this section contained in Theorem 4.4. This theorem concerns five properties proven by induction. For the convenience of the reader we outline the structure of the proof.

Property 1. ensures the construction of the Feshbach flow $\mathcal{F}_{j_{1}, \ldots, j_{m}}(z + z^\perp_{j_{1}, \ldots, j_{m-1}})$ (defined with Property 1.) up to the last step ($i = N$).

Property 2. provides the existence of the unique solution of the fixed point equation associated with the Feshbach Hamiltonian $\mathcal{F}_{j_{1}, \ldots, j_{m}}(z + z^\perp_{j_{1}, \ldots, j_{m-1}})$ and the construction of its ground state.

Property 3. is concerned with Property 1. and 2. but for the auxiliary Hamiltonian $H_{j_{1}, \ldots, j_{m}}^\perp$. In addition to the construction of the ground state vector, Property 3. provides the gap condition at step $m$ that must be used to get Property 1., 2., and 3. at step $m + 1$.

Property 4. provides the information on

$$\inf_{\varepsilon_j \in [\varepsilon_{j_0}, 1]} \text{spec}[H_{j_{1}, \ldots, j_{m}} - T_{j_{1} \ldots j_{m-1}}]\]$$

that is used at step $m + 1$ in Corollary 5.1. Thanks to this input, the operator norm estimate (5.3) in Corollary 5.1 can be derived as if the modes $j_1, \ldots, j_{m-1}$ were absent.

Property 5. provides a bound on the expectation value of $N^2$ in the ground state of $H_{j_{1}, \ldots, j_{m}}^\perp$. This information is needed to control the fixed point equation at step $m + 1$ both for $H_{j_{1}, \ldots, j_{m+1}}^\perp$ and for $H_{j_{1}, \ldots, j_{m}}^\perp$.

**Theorem 4.4.** Let $\max_{1 \leq m \leq M} \varepsilon_{j_m}$ be sufficiently small and $N$ sufficiently large. Then the following properties hold true for all $1 \leq m \leq M$:

1. There exists a constant $C^\perp$ such that the Feshbach Hamiltonian in (3.59)-(3.61) is well defined for

$$z \leq \min \left\{ z_m + \gamma \Delta_{m-1} - \frac{m\xi^{\frac{1}{2}}}{M} - \frac{C^\perp}{(\ln N)^{\frac{1}{2}}} ; E_{j_m}^{Bog} + \sqrt{\varepsilon_{j_m}} \phi_{j_m} \sqrt{\varepsilon_{j_m}^2 + 2\varepsilon_{j_m}} \right\}
$$

where:
• $z_m$ is the ground state energy of $H_{J_n}^{\text{Bog}}$ (see Corollary 4.6 and Theorem 4.1 of [Pi1]);

• $\gamma = \frac{1}{2}$;

• $\Delta_m^\# (\text{for } m \geq 1)$ is defined iteratively from $\Delta_0^\# = \Delta_0 := \min \{(k_j)^2 \mid j \in \mathbb{Z}^d \setminus \{0\}\}$ and for $N$ large enough

$$
\Delta_m^\# := \min \left\{ \gamma \Delta_{m-1}^\# - \frac{C^\#}{(\ln N)^{\frac{1}{2}}} ; \frac{1}{2} \sqrt{\varepsilon_j \phi_{J_m} \sqrt{\varepsilon_j^2 + 2 \delta_j}} - \frac{2m^j \xi \varepsilon_j}{\sqrt{M}} \right\} \quad (4.155)
$$

$$
= \gamma \Delta_{m-1}^\# - \frac{C^\#}{(\ln N)^{\frac{1}{2}}} - \frac{2m^j \xi \varepsilon_j}{\sqrt{M}} \quad (4.156)
$$

where $\xi = \frac{1}{(\ln N)^{\frac{1}{2}}}$.

2. For $z$ as in (4.154), there exists a unique value $z^{(m)}$ such that (see (3.66))

$$
f_{j_1, \ldots, j_m}(z + z_{j_1, \ldots, j_{m-1}}^\#)_{\mid z = z^{(m)}} = 0.
$$

The inequality

$$
|z^{(m)} - z_m| \leq \frac{2^m C^\#_{\text{III}}}{\gamma (\ln N)^{\frac{1}{2}}} + \frac{\tilde{C}^\#}{N} \quad (4.157)
$$

holds true with $C^\#_{\text{III}} := C^\#_{\gamma} + \frac{(c_{\gamma})^2}{(1-\gamma)\delta_0}$, where $\tilde{C}^\#$ is defined in point 5. below, $C^\#_{\gamma}$ and $C^\#_{\text{III}}$ are defined in Lemma 5.2.

The Hamiltonian $H_{j_1, \ldots, j_m}$ has (non-degenerate) ground state energy $z_{j_1, \ldots, j_m}^\# := z_{j_1, \ldots, j_{m-1}}^\# + z^{(m)}$, where $z_{j_1, \ldots, j_{m-1}}^\# |_{l=1} = 0$. The corresponding eigenvector, $\psi_{j_1, \ldots, j_m}$, is given in (3.67)-(3.69).

3. (a) The operator $H_{j_1, \ldots, j_m}^\#$ has (non-degenerate) ground state energy $z_{j_1, \ldots, j_m}^\#$ determined via a fixed point equation analogous to (3.66) and ground state vector $\psi_{j_1, \ldots, j_m}$ given by the formula in (3.70)-(3.72).

(b) The operator $H_{j_1, \ldots, j_m}^\# - T_j = z_{j_{m+1}}^\# \upharpoonright (\mathcal{F}^N \ominus \mathcal{F}^{N}_{\pm z_{j_{m+1}}^\#})$ has ground state vector $\psi_{j_1, \ldots, j_m}^\#$ and ground state energy $z_{j_1, \ldots, j_m}^\#$, and the gap estimate

$$
\text{infspec}(H_{j_1, \ldots, j_m}^\# - T_j = z_{j_{m+1}}^\# \upharpoonright (\mathcal{F}^N \ominus \mathcal{F}^{N}_{\pm z_{j_{m+1}}^\#}) \ominus \mathcal{C}_{\psi_{j_1, \ldots, j_m}^\#}) - z_{j_1, \ldots, j_m}^\# \geq \Delta_m^\# \quad (4.158)
$$

holds true.

4. The bound from below

$$
\text{infspec}(H_{j_1, \ldots, j_m}^\# - T_j = z_{j_{m+1}}^\# \upharpoonright (\mathcal{F}^N \ominus \mathcal{F}^{N}_{\pm z_{j_{m+1}}^\#}) \ominus \mathcal{C}_{\psi_{j_1, \ldots, j_m}^\#}) \geq z_{j_1, \ldots, j_m}^\# - \frac{m^j \xi \varepsilon_j}{\sqrt{M}}, \quad 1 \leq l \leq M - m, \quad (4.159)
$$

holds true where

$$
H_{j_1, \ldots, j_m}^{\# (l)} := T_j = z_{j_{m+1}}^\# \upharpoonright (\mathcal{F}^N \ominus \mathcal{F}^{N}_{\pm z_{j_{m+1}}^\#}) \ominus \mathcal{C}_{\psi_{j_1, \ldots, j_m}^\#}) \ominus \mathcal{C}_{\psi_{j_1, \ldots, j_m}^\#} \quad (4.146)
$$

with

• $V_{j_1, \ldots, j_m}^{\# (l)}$ corresponding to $V_{j_1, \ldots, j_m}$ (see (1.27)-(1.29)) minus all the summands containing at least one of the operators $\{a_{j_1, \ldots, j_m}^+, a_{j_1, \ldots, j_m}^- \mid l' = 1, \ldots, l\}$; consequently, $V_{j_1, \ldots, j_m}^{\# (1)} = V_{j_1, \ldots, j_m}^\#$ and $H_{j_1, \ldots, j_m}^{\# (1)} = H_{j_1, \ldots, j_m}^\#$.

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• $T_{j\in\{\pm 1,...,\pm 1\}} := \sum_{j\in\{\pm 1,...,\pm 1\}} k_j^2 a_j^* a_j$.

5. The upper bound

$$\left\langle \frac{\psi^\#_{j_1,...,j_m}}{\|\psi^\#_{j_1,...,j_m}\|}, \left( \sum_{j\in\mathbb{Z}^d\setminus\{0\}} a_j^* a_j \right)^2 \frac{\psi^\#_{j_1,...,j_m}}{\|\psi^\#_{j_1,...,j_m}\|} \right\rangle \leq O(1) \quad (4.161)$$

holds true. This implies that for some $C^\# < \infty$

$$\| \mathcal{P}_{j_1,...,j_m} V_{j_{m+1}} \mathcal{P}_{j_1,...,j_m} \| \leq \frac{C^\#}{N}. \quad (4.162)$$

Proof

Case $m = 1$

We observe that the Feshbach Hamiltonian $\mathcal{H}^{(i)}_{j_1,...,j_m}(w)$, with $m = 1$, in (3.12)-(3.14) is well defined because of the estimate in (5.40) of Lemma 5.3 that ensures the invertibility of $\mathcal{H}^{(i, j+1)}_{j_1}(H_{j_1} - w)\mathcal{H}^{(i, j+1)}_{j_2}$ in $\mathcal{S}^{(i,j+1)}_{j_1} \mathcal{F}^N$. Then, Property 1. follows from Corollary 5.1, Theorem 4.1, Theorem 4.2, and Proposition 4.5 (through Lemma 5.2). Notice that assumptions 1. and 2. in Proposition 4.5 are satisfied for $\psi^\#_{j_1,...,j_{m-1}} \big|_{m=1} \equiv \eta, \psi^\#_{j_1,...,j_{m-1}} \big|_{m=1} \equiv 0$ and $\Delta_0^\# = \min \{ (k_j)^2 \mid j \in \mathbb{Z}^d \setminus \{0\} \}$.

As far as Property 2. is concerned, at first we point out that

$$\mathcal{P}_{\eta} H_{j_1} \mathcal{P}_{\eta} = 0. \quad (4.163)$$

Similarly to the fixed point problem for the intermediate Bogoliubov Hamiltonians $H^{Bog}_{j_1,...,j_m}$ with $m \geq 2$ (see Theorem 4.3 of [Pi2]), the term in (3.60) is not zero but vanishes as $N \to \infty$.

More precisely,

- the identity

$$\mathcal{P}_{\eta} V_{j_1} \mathcal{P}_{\eta} = 0 \quad (4.164)$$

holds true because the state $\eta$ contains only particles in the zero mode, and $V_{j_1}$ is normal ordered and contains only “cubic” and “quartic” terms in the nonzero modes;

- the estimate

$$\| \mathcal{P}_{\eta} \Gamma_{j_1;N,N}(z) \mathcal{P}_{\eta} \| \leq \frac{C^\#_{II}}{(\ln N)^{\dagger}} \quad (4.165)$$

is derived in Lemma 5.2 by means of a procedure already employed for the intermediate Bogoliubov Hamiltonians (see Lemma 4.4 of [Pi2]).

Finally, we solve the fixed point equation

$$0 = -z - \langle \eta, \Gamma_{j_1;N,N}(z) \eta \rangle \quad (4.166)$$

$$-\langle \eta, \Gamma_{j_1;N,N}(z) \mathcal{P}_{\eta} \mathcal{H}^{(N-2)}_{j_1}(z) \mathcal{P}_{\eta} \Gamma_{j_1;N,N}(z)^* \eta \rangle$$

which is well defined thanks to Proposition 4.5. We claim that there is a unique solution, $z = z^{(1)} \equiv z_{j_1}$, to the equation in (4.166). Using the isospectrality of the Feshbach map, $z_{j_1}$
is the (nondegenerate) ground state energy of \( H_{j_1} \). Concerning the existence of \( z_{j_1} \), due to the estimates in (5.36), (5.37) of Lemma 5.2 and (4.109) of Proposition 4.5, the fixed point equation is equivalent to

\[
 z = -\langle \eta, \Gamma^{Bog}_{j_1;N,N}(z) \eta \rangle + \mathcal{Y}_1(z) \tag{4.167}
\]

with

\[
 |\mathcal{Y}_1(z)| \leq \frac{C^\#}{(\ln N)^{\frac{1}{2}}} + \frac{2}{\gamma} \frac{(C^\#_I)^2}{(\ln N)^{\frac{1}{2}}(1 - \gamma)\Delta_0}, \quad \gamma = \frac{1}{2}.
\]

Then, the same argument of Theorem 4.1 of [Pi1] implies that for \( N \) sufficiently large there exists a \( z^{S(1)} \) that solves the equation in (4.167). Furthermore, the inequality

\[
 |z^{S(1)} - z_1| \leq \frac{2}{\gamma} \frac{C^\#_{III}}{(\ln N)^{\frac{1}{2}}} \quad , \quad C^\#_{III} := C^\#_I + \frac{(C^\#_I)^2}{(1 - \gamma)\Delta_0}, \tag{4.168}
\]

holds true for \( z_1 \) such that \( z_1 + \langle \eta, \Gamma^{Bog}_{j_1;N,N}(z_1) \eta \rangle = 0 \) because the derivative w.r.t. \( z \) of

\[
 z + \langle \eta, \Gamma^{Bog}_{j_1;N,N}(z) \eta \rangle
\]

is not smaller than 1; see Remark 4.1 of [Pi1]. The eigenvector \( \psi_{j_1} \) of \( H_{j_1} \) corresponding to \( z^{S(1)} \) is given in expression (3.67)-(3.69) with \( m = 1 \). The uniqueness of \( z^{S(1)} \) follows from the fact that for any other value, \( (z^{S(1)})' \), that solves the fixed point problem an inequality analogous to (4.168) holds, hence

\[
 |(z^{S(1)})' - z^{S(1)}| \leq \mathcal{O}\left(\frac{1}{(\ln N)^{\frac{1}{2}}}\right). \tag{4.169}
\]

Then, using the same argument of Theorem 4.3 of [Pi2] the closeness (see (4.169)) of the two eigenvalues \( z^{#(1)} \) and \( (z^{S(1)})' \) implies

\[
 \|\psi_{j_1} - (\psi_{j_1})'\| \leq \mathcal{O}\left(\frac{1}{(\ln N)^{\frac{1}{2}}}\right). \tag{4.170}
\]

where \( (\psi_{j_1})' \) is the eigenvector corresponding to \( (z^{S(1)})' \). Thus for \( N \) large enough the two eigenvalues must coincide.

Since in the interval (4.154) \( \mathcal{X}^{(N)}_{j_1}(z) \) is well defined and \( z^{S(1)} \) is the unique fixed point of the equation in (4.166), we can conclude that (in the given interval) \( \mathcal{X}^{(N)}_{j_1}(z) \) is bounded invertible except for \( z = z^{S(1)} \) and, consequently, \( z^{S(1)} \) is the ground state energy of \( H_{j_1} \). The isospectrality of the Feshbach map implies that the eigenvalue \( z^{S(1)} \) of \( H_{j_1} \) is nondegenerate.

Regarding Property 3., as long as

\[
 z \leq \min \left\{ z_1 + \gamma \Delta_0 \frac{C^\#_{I} + \frac{1}{M} \Gamma^{Bog}_{j_1} + \sqrt{\epsilon_{j_1}} \phi_{j_1} \left( \sqrt{\epsilon_{j_1}}^2 + 2 \epsilon_{j_1} \right) \right\} \tag{4.171}
\]

the Feshbach Hamiltonian \( \mathcal{X}^{#(i)}_{j_1}(z) \) is well defined thanks to the bound in (5.39) (see Lemma 5.3). Furthermore, we can adapt Theorem 4.1 and Corollary 5.1 to the Hamiltonian \( H^\#_{j_1} \) in order to implement the Feshbach flow and define the Feshbach Hamiltonians \( \mathcal{X}^{#(i)}_{j_1}(z) \) up to \( i = N - 2 \) in the same way we proceeded for \( \mathcal{X}^{(i)}_{j_1}(z) \). To understand this, it must be noticed that
Lemma 0. Notice for example that the assumption in \((z_j)\) to any subspace \(V_j\) and this does not affect the proof. Next, we adapt Lemma 5.2 and Proposition 4.5 to the Hamiltonian \(H_j\) obtaining analogous results with the same constants. Finally, we can define \(z_j^\# := z_j^{\#(1)}\) by determining the solution, \(z_j^{(1)}\), of the fixed point equation associated with \(\mathcal{K}_j\). The eigenvalue \(z_j^{(1)}\) fulfills the bound

\[
|z_j^{(1)} - z_1| \leq \frac{2 - \zeta}{\gamma} \frac{C_{\#(1)}}{(\ln N)^2},
\]  

(4.172)

By Feshbach theory we construct the eigenvector \(\psi_j^\#\) as in (3.70)-(3.72) and similarly to our discussion on \(z_j^{(1)}\) we conclude that \(z_j^{(1)}\) is unique and is the (nondegenerate) ground state energy of \(H_j\).

We can also estimate the gap above the ground state energy. Indeed, for

\[
z \leq \min \left\{ z_1 - \frac{C_{\#}^\pm}{(\ln N)^2} - \frac{\epsilon_z^2}{M} + \gamma \Delta_0^\# ; E_j^{\text{Bog}} + \sqrt{e_j^2} \phi_j \sqrt{e_j^2 + 2e_j} \right\}, \quad \gamma = \frac{1}{2},
\]

(4.173)

the Hamiltonian \(\mathcal{K}_j^\#(N)(z)\) is bounded invertible in \(\mathcal{F}_j^\#(N)\) except for \(z \equiv z_j^{(1)}\). In addition, from Lemma 5.5 of [Pil] we know that \(|z_1 - E_j^{\text{Bog}}| = O(\frac{1}{N^\beta})\) for any \(0 < \beta < 1\). This estimate combined with (4.172) imply (for \(N\) large)

\[
\text{infspec} \left[ H_j^\# \mid_{\mathcal{F}_j^\#(N) \cap \mathcal{C}_j^\#} \right] - z_j^\# \geq \min \left\{ \gamma \Delta_0^\# - \frac{C_{\#}^\pm}{(\ln N)^2} ; \frac{1}{2} \sqrt{e_j} \phi_j \sqrt{e_j^2 + 2e_j} \right\} - \frac{2\epsilon_z^2}{M},
\]

(4.174)

\[
\geq \frac{\gamma \Delta_0^\# - \frac{C_{\#}^\pm}{(\ln N)^2} - \frac{2\epsilon_z^2}{M}}{2},
\]

(4.175)

\[
\geq \Delta_1^\#.
\]

(4.176)

As for Property 3. (b), by construction \(\psi_j^\#\) is eigenvector with eigenvalue \(z_j^\#\). From Property 3. (a) we derive

\[
\text{infspec} \left[ (H_j^\# - T_j=\pm j_{z_j}) \mid_{\mathcal{F}_j^\#(N) \cap \mathcal{C}_j^\#} \right] - z_j^\# \geq \Delta_1^\#.
\]

(4.177)

(4.178)

Concerning Property 4., we show the procedure for \(H_j^\# - T_j=\pm j_{z_j}\). (For the cases corresponding to \(2 \leq l \leq M - 1\) the proof is very similar.) We can restrict

\[
H_j^\# - T_j=\pm j_{z_j} - z - z_j^\#
\]

to any subspace \(\mathcal{F}_j\) of \(\mathcal{F}_j^\#(N)\) with fixed number of particles, \(j\), in the modes \(\pm j_{z_j}\). For simplicity, assume that \(j\) is even; the same result (Property 4.) holds if \(j\) is odd. By adapting\(^6\) Lemma 4.1, Theorem 4.1, and Corollary 5.1, the Feshbach flow can be implemented in the same way with minor modifications. More precisely:

\(^6\)Notice for example that the assumption in (5.1) in Corollary 5.1 can be replaced with \((1 - \xi)T - (1 - \xi)T_j=\pm j_{z_j} \geq 0\).
1) If \( j < N - \bar{t} = [N^\frac{3}{2}] \), we start from

\[
\mathcal{P}^{(i)}(j) := \hat{\mathcal{E}}^{(i+1)}_{j} := \mathcal{C}^{(i+1)}_{j} I_{[F^{N}]j} \quad \text{and} \quad \mathcal{P}^{(i)} := \hat{\mathcal{E}}^{(i+1)}_{j} := I_{[F^{N}]j} - \hat{\mathcal{E}}^{(i+1)}_{j},
\]

and proceed for \( i > \bar{t} \) (and even) up to the step \( i = N - 2 \) with the definitions

\[
\mathcal{P}^{(i)}(j) := \hat{\mathcal{E}}^{(i+1)}_{j} := \mathcal{C}^{(i+1)}_{j} I_{[F^{N}]j} \quad \text{and} \quad \mathcal{P}^{(i)} := \hat{\mathcal{E}}^{(i+1)}_{j} := I_{[F^{N}]j} - \hat{\mathcal{E}}^{(i+1)}_{j}.
\]

2) If \( j \geq N - \bar{t} \), we start from

\[
\mathcal{P}^{(i)}(j) := \hat{\mathcal{E}}^{(i+1)}_{j} := \mathcal{C}^{(i+1)}_{j} I_{[F^{N}]j} \quad \text{and} \quad \mathcal{P}^{(i)} := \hat{\mathcal{E}}^{(i+1)}_{j} := I_{[F^{N}]j} - \hat{\mathcal{E}}^{(i+1)}_{j},
\]

and proceed for \( i > j \) (and even) up to the step \( i = N - 2 \) with the definitions

\[
\mathcal{P}^{(i)}(j) := \hat{\mathcal{E}}^{(i+1)}_{j} := \mathcal{C}^{(i+1)}_{j} I_{[F^{N}]j} \quad \text{and} \quad \mathcal{P}^{(i)} := \hat{\mathcal{E}}^{(i+1)}_{j} := I_{[F^{N}]j} - \hat{\mathcal{E}}^{(i+1)}_{j}.
\]

We call \( \hat{\mathcal{E}}^{(N-1)}_{j} \) the Feshbach Hamiltonians so defined. One can observe that \( \hat{\mathcal{E}}^{(N-1)}_{j} \) projects onto the subspace of states with no particles in the modes \( \pm j_{1} \) and \( j \) particles in the modes \( \pm j_{2} \). We set \( w = z + z_{j}^{\#} \) and obtain

\[
\hat{\mathcal{E}}^{(N-2)}_{j}(w) = \mathcal{C}^{(N-1)}_{j}(H^{\#}_{j} - T_{j=\pm j_{2}} - w)\mathcal{C}^{(N-1)}_{j},
\]

(4.179)

\[
\hat{\mathcal{E}}^{(N-1)}_{j}(H^{\#}_{j} - T_{j=\pm j_{2}} - w)\mathcal{C}^{(N-1)}_{j},
\]

(4.180)

\[-\mathcal{C}^{(N-1)}_{j}(\hat{\mathcal{E}}^{(N-1)}_{j}) - \bar{w}_{j} \hat{R}_{j;N-2}(w) \sum_{l_{N-2}=0}^{\infty} [\Gamma_{j;N-2}(w)\hat{R}_{j;N-2}(w)]^{l_{N-2}} \bar{w}_{j} \mathcal{C}^{(N-1)}_{j},
\]

where:

1) if \( j < N - \bar{t} \) the operators \( \hat{R}_{j;N-1}(w), \hat{\Gamma}_{j;N-1}(w) \) have the same definition of \( R_{j;N-1}(w), \Gamma_{j;N-1}(w) \) but in terms of \( H^{\#}_{j} - T_{j=\pm j_{2}} \) and of the new projections;

2) if \( j \geq N - \bar{t} \) the operators \( \hat{R}_{j;N-1}(w), \hat{\Gamma}_{j;N-1}(w) \) have the same definition of \( R_{j;N-1}(w), \Gamma_{j;N-1}(w) \) but in terms of \( H^{\#}_{j} - T_{j=\pm j_{2}} \) and of the new projections, and \( \Gamma_{j;N-1}(w) \) starts from

\[
\hat{\Gamma}_{j;N-1}(w) := \mathcal{C}^{(j+2, j+3)}_{j} \bar{w}_{j} \mathcal{C}^{(j+1)}_{j} \frac{1}{\mathcal{C}^{(j+1)}_{j}(H^{\#}_{j} - T_{j=\pm j_{2}} - w)\mathcal{C}^{(j+1)}_{j}} \mathcal{C}^{(j+1)}_{j} \hat{\mathcal{E}}^{(j+2, j+3)}_{j}.
\]

We want to prove that for \( N \) large enough the operator

\[
\hat{\mathcal{E}}^{(N-2)}_{j}(w) \equiv \mathcal{C}^{(N-1)}_{j} \hat{\mathcal{E}}^{(N-2)}_{j}(w) \mathcal{C}^{(N-1)}_{j}
\]

(4.181)

is bounded invertible as long as \( z \) is less than \( -\bar{t} \). To this purpose we exploit the analogy with the estimate of a lower bound to the spectrum of

\[
\mathcal{P}_{\phi_{j}}^{(N-2)}(z + z_{j}^{\#}) \mathcal{P}_{\phi_{j}}^{(N-2)},
\]

More precisely, we replace \( \mathcal{P}_{\phi_{j}}^{(N-2)} \) with \( \hat{\mathcal{E}}^{(N-1)}_{j} \) and proceed like for \textit{Property 4.}, in \textit{Theorem 4.3} of [Pi2] by adapting\footnote{We point out that a modified Corollary 5.1 where condition a) is replaced with \((1 - \xi) T - (1 - \xi) T_{j=\pm j_{1}} \geq 0\) implies Lemma 5.2 where the Hamiltonian \( H^{\#}_{j} \) is replaced with \( H^{\#}_{j} - T_{j=\pm j_{2}} \)} (to \( H^{\#}_{j} - T_{j=\pm j_{2}} \)) estimate (5.38) of Lemma 5.2.
Finally, we conclude that \( \hat{\mathcal{X}}_{i_j}^{(N-2)}(w) \) is strictly positive if \( z \leq -\frac{\xi^2}{2M} \). Then, for \( z \) in the same interval, the operator \( H_{i_j} - T_{j=\pm j_i} - z - z_{i_j}^w \) is bounded invertible by isospectrality of the Feshbach map.

Property 5. is a straightforward consequence of (5.41) in Lemma 5.3.

**Case \( m > 1 \)**

We assume that Properties 1., 2., 3., 4., and 5. hold for \( 1 \leq m - 1 < M \) and prove that they hold for \( m \).

**Property 1.** Since Properties 3., 4., 5. hold for \( m - 1 \) we can apply Lemma 4.1, Corollary 5.1, Theorem 4.1, Theorem 4.2, Lemma 5.2 and Proposition 4.5 and get Property 1. for \( m \).

**Property 2.** We recall the definition of \( f_{j_1,...,j_m}(z + z_{j_1,...,j_{m-1}}) \) in (3.59)-(3.61) and observe that

\[
\mathcal{P}_{j_1,...,j_{m-1}}(H_{j_1,...,j_m} - z_{j_1,...,j_{m-1}}) \mathcal{P}_{j_1,...,j_{m-1}} = \mathcal{P}_{j_1,...,j_{m-1}}(H_{j_1,...,j_m} + V_{j_m} + V'_{j_1,...,j_{m-1}} + \tilde{H}^{Bog}_{j_m} - z^w_{j_1,...,j_{m-1}}) \mathcal{P}_{j_1,...,j_{m-1}}
\]

(4.182)

\[
= \mathcal{P}_{j_1,...,j_{m-1}} V_{j_m} \mathcal{P}_{j_1,...,j_{m-1}} \mathcal{P}_{j_1,...,j_{m-1}}
\]

(4.183)

\[
= \mathcal{P}_{j_1,...,j_{m-1}} V_{j_m} \mathcal{P}_{j_1,...,j_{m-1}} \mathcal{P}_{j_1,...,j_{m-1}}
\]

(4.184)

Furthermore, from Property 5. at step \( m - 1 \), we have

\[
\| \mathcal{P}_{j_1,...,j_{m-1}} V_{j_m} \mathcal{P}_{j_1,...,j_{m-1}} \| \leq \frac{C^w}{N}.
\]

The rest of the proof is analogous to the case \( m = 1 \) using (5.36) and (5.37) from Lemma 5.2 that can be applied thanks to Property 3.(a), 4., and 5. at the step \( m - 1 \). These estimates yield

\[
z = -\langle \eta, \Gamma_{j_m;N,N}(z) \eta \rangle + \mathcal{U}_m(z)
\]

(4.186)

with

\[
|\mathcal{U}_m(z)| \leq \frac{C^w}{(\ln N)^{\frac{3}{2}}} + \frac{(C^w_1)^2}{(\ln N)^{\frac{3}{2}}(1 - \gamma)\Delta_{m-1}^w} + \frac{C^w}{N}, \quad \gamma = \frac{1}{2}
\]

(4.187)

\[
= \frac{C^w}{(\ln N)^{\frac{3}{2}}} + \frac{2}{\gamma} \frac{(C^w_1)^2}{(\ln N)^{\frac{3}{2}}(1 - \gamma)\Delta_0} + \frac{C^w}{N}, \quad \gamma = \frac{1}{2}
\]

(4.188)

where we have used that \( \Delta_{m-1}^w \geq (\tilde{z})^m \Delta_0 \) for \( N \) sufficiently large. In a similar way, for \( N \) large enough one can also derive the existence of the unique solution \( z^w_{m}(m) \) with the property in (4.157). Using the isospectrality of the Feshbach map, we deduce that \( H_{j_1,...,j_m} \) has the nondegenerate eigenvalue \( z^w_{m} + z_{j_1,...,j_{m-1}} =: z_{j_1,...,j_{m}} \). The corresponding eigenvector is given in (2.70)-(2.72). We observe that in the interval

\[
z \leq \min \left\{ \frac{z_m - C^w_1}{(\ln N)^{\frac{3}{2}}} - \frac{m\xi^2}{M} + \gamma \Delta_{m-1}^w : E_{j_m}^{Bog} + \sqrt{\epsilon_{j_m}^2 + 2\epsilon_{j_m}} \right\}, \quad \gamma = \frac{1}{2}
\]

(4.189)
the Hamiltonian $\mathcal{H}_{j_1, \ldots, j_n}^{(N)} (z + z_{j_1, \ldots, j_{n-1}})$ is bounded invertible except for $z \equiv z^{(m)}$. Then, by the isospectrality of the Feshbach map we deduce that

$$z^{(m)} + z_{j_1, \ldots, j_{n-1}} =: z_{j_1, \ldots, j_n}$$

is the (nondegenerate) ground state energy of $H_{j_1, \ldots, j_n}$.

**Property 3.** Assuming Properties 3., 4., 5. at step $m - 1$ we can adapt Theorem 4.1 and Corollary 5.1 to the Hamiltonian $H_{j_1, \ldots, j_n}^\#$ in order to implement the Feshbach flow and define the Feshbach Hamiltonians $\mathcal{H}_{j_1, \ldots, j_n}^{(i)} (z + z_{j_1, \ldots, j_{n-1}})$ up to $i = N - 2$ in the same way we have proceeded for $\mathcal{H}_{j_1}^{(i)} (z)$. We can also adapt Lemma 5.2 and Proposition 4.5 to the Hamiltonian $\mathcal{H}_{j_1, \ldots, j_n}^{(N-2)} (z + z_{j_1, \ldots, j_{n-1}})$ obtaining analogous estimates with the same constants (recall that $H_{j_1, \ldots, j_n}^\#$ contains an interaction term less $(V')_{j_1, \ldots, j_n}$ with respect to the Hamiltonian $H_{j_1, \ldots, j_n}$).

Finally, we can define the eigenvalue $z_{j_1, \ldots, j_n}^\# := z_{j_1, \ldots, j_{n-1}}^{(m)} + z_{j_1, \ldots, j_{n-1}}^\#$ of $H_{j_1, \ldots, j_n}^\#$ by determining the unique solution, $z_{j_1, \ldots, j_{n-1}}^{(m)}$, of the fixed point equation corresponding to $\mathcal{H}_{j_1, \ldots, j_n}^{(N)} (z + z_{j_1, \ldots, j_{n-1}})$. This solution fulfills the bound

$$|z_{j_1, \ldots, j_{n}}^{(m)} - z_m| \leq \left(\frac{2}{\gamma}\right)^{\frac{1}{2}} \frac{C_{III}^\#}{\ln N} + \frac{C^\#}{N}. \quad (4.190)$$

By Feshbach theory we then construct the eigenvector of $H_{j_1, \ldots, j_n}^\#, \psi_{j_1, \ldots, j_n}^\#$, as in (3.70)-(3.72). Similarly to the case $m = 1$ we can conclude that $z_{j_1, \ldots, j_{n-1}}^{(m)} + z_{j_1, \ldots, j_{n-1}}$ and $\psi_{j_1, \ldots, j_n}^\#$ are the (nondegenerate) ground state energy and ground state vector, respectively.

We observe that in the interval

$$z \leq \min \left\{ z_m - \frac{C_{III}^\#}{\ln N}, \frac{m \varepsilon_{j_1}^2}{M} + \gamma \Delta_m^{-1} : E_{j_1}^{Bog} + \sqrt{\varepsilon_{j_1}^2 \gamma_{j_1} + 2 \varepsilon_{j_1}} \right\}, \quad \gamma = \frac{1}{2}, \quad (4.191)$$

the Hamiltonian $\mathcal{H}_{j_1, \ldots, j_n}^{(N)} (z + z_{j_1, \ldots, j_{n-1}})$ is bounded invertible except for $z \equiv z_{j_1, \ldots, j_{n-1}}^{(m)}$. From Corollary 4.6 of [Pi1] we know that $|z_m - E_{j_1}^{Bog}| = O\left(\frac{1}{\sqrt{N}}\right)$ for any $0 < \beta < 1$. This estimate combined with (4.190) imply

$$\inf_{\mathcal{X}} \left[H_{j_1, \ldots, j_n}^{(N)} \phi_{j_1, \ldots, j_n}^\# \right] - z_{j_1, \ldots, j_n}^\# \geq \Delta_m^\# := \gamma \Delta_m^{-1} - \frac{C_{III}^\#}{\ln N} + \frac{2 m \varepsilon_{j_1}^2}{M}, \quad (4.192)$$

and, consequently, Property 3. (b):

$$\inf_{\mathcal{X}} \left[H_{j_1, \ldots, j_{n}}^{(i)} - T_j z_{j_1, \ldots, j_{n-1}}^\# \right] \phi((\mathcal{F}^N \psi_{j_1, \ldots, j_{n-1}}^\#) \phi_{j_1, \ldots, j_n}^\#) - z_{j_1, \ldots, j_n}^\# \geq \Delta_m^\#. \quad (4.194)$$

**Property 4.** The argument is analogous to the case $m = 1$ given Properties 1.-5. at step $m - 1$. In particular, notice that in order to apply (a suitably adapted version of) Theorem 4.1 and Corollary 5.1 to

$$H_{j_1, \ldots, j_{n}}^{(i)} - T_j z_{j_1, \ldots, j_{n-1}} \equiv z_{j_1, \ldots, j_{n-1}}, \quad 1 \leq i \leq M - m,$$
Property 4. for

\[ H_{j_1,\ldots,j_M}^{(i+1)} - T_{j_1[\bar{a}_{j_0};\ldots;\bar{a}_{j_{m+i}}]} = H_{j_1,\ldots,j_{m+1}}^{(i+1)} - T_{j_1[\bar{a}_{j_{m+1}};\ldots;\bar{a}_{j_{m+i+1}}]} \]

is needed.

Property 5. This is a straightforward consequence of (5.41) in Lemma 5.3.

\[ \square \]

The very last result of this section concerns the expansion of the ground state vector \( \psi_{j_1,\ldots,j_M} \) (and of \( \psi_{j_1,\ldots,j_M}^{#} \), \( 1 \leq m \leq M - 1 \)) in terms of the bare quantities.

**Corollary 4.6.** Assume the hypotheses of Theorem 4.4. Then, for any arbitrarily small \( \zeta > 0 \), there exists \( N_\zeta < \infty \) and a vector \( (\psi_{j_1,\ldots,j_M})_\zeta \), corresponding to a \( (\xi \text{-dependent}) \) finite sum of \( (\xi \text{-dependent}) \) finite products of the interaction terms \( W_{j_1}^i + W_{j_2}^i \) and of the resolvents \( \Gamma_{j_1}^{\xi} - E_{\text{Bog}} \) (see (2.3)), \( 1 \leq l \leq M \), applied to \( \eta \), such that

\[ \| \psi_{j_1,\ldots,j_M} - (\psi_{j_1,\ldots,j_M})_\zeta \| \leq \zeta \]

for \( N > N_\zeta \).

**Proof**

The proof is very similar to the analogous result for the ground state of the Bogoliubov Hamiltonian \( H_{j_1,\ldots,j_M}^{\text{Bog}} \) derived in Corollary 4.6 of [Pi2]. However, it is important to notice that the re-expansion of the factors

\[ \frac{1}{\Sigma_{j_0}^{(N-2r,N-2r+1)} \mathcal{A}_{j_1,\ldots,j_0}^{(N-2r-2)}} (\bar{z}_{j_1,\ldots,j_0})^{2(N-2r,N-2r+1)} \]

in (3.68) produces terms containing \( \Gamma_{j_1,\ldots,j_M}^{z+2,\ldots,z+2} (\bar{z}_{j_1,\ldots,j_0}) \) that is an operator that cannot be re-expanded. This is not a problem because the norm of the sum of the contributions proportional to \( \Gamma_{j_1,\ldots,j_M}^{z+2,\ldots,z+2} (\bar{z}_{j_1,\ldots,j_0}) \) is arbitrarily small for \( N \) sufficiently large. \( \square \)

### 5 Appendix

**Corollary 5.1.** For \( M \geq m \geq 1 \) assume:

(a)

\[ (H_{j_1,\ldots,j_{m-1}}^{#})_\xi - (1 - \xi) T_{j_1[\bar{a}_{j_0};\ldots;\bar{a}_{j_m}]} \geq z_{j_1,\ldots,j_{m-1}}^{#} - \xi \frac{(m - 1)\xi}{M} \]  

where \( (H_{j_1,\ldots,j_{m-1}}^{#})_\xi \) is defined in (4.1) for \( m \geq 2 \) and is equal to \( (1 - \xi) T \) for \( m = 1 \), and where \( z_{j_1,\ldots,j_{m-1}}^{#} \) is the ground state energy of \( H_{j_1,\ldots,j_{m-1}}^{#} \).

(b)

\[ w := z + z_{j_1,\ldots,j_{m-1}}^{#} \leq z_{j_1,\ldots,j_{m-1}}^{#} + E_{j_0}^{\text{Bog}} + (\delta - 1) \psi_{j_0} \sqrt{\epsilon_{j_0}^2 + 2\epsilon_{j_0}} \]

with \( \delta < 2 \) and \( \epsilon_{j_0} \) sufficiently small.

Then, for \( \xi = (\frac{1}{\ln N})^{\frac{1}{2}} \) and \( N \) sufficiently large

\[ \| \left[ R_{j_1,\ldots,j_m}^{i,j}(w) \right] \| \leq \frac{1}{4(1 + a_{j_0} - \frac{2\epsilon_{j_0}}{M - i + 1} + 1 - \epsilon_{j_0} \frac{1}{(N - i + 1)^2})} \]  

(5.3)
holds for \(i + 4 \leq i \leq N - 2\). Here,

\[
a_{\epsilon_m} := 2\epsilon_m + O(\epsilon_m^2), \quad \nu > \frac{11}{8},
\]

\[
b_{\epsilon_m} := (1 + \epsilon_m)\delta\chi_{[0,2)}(\delta) \sqrt{\epsilon_m^2 + 2\epsilon_m},
\]

and

\[
c_{\epsilon_m} := -(1 - \delta^2\chi_{[0,2)}(\delta))(\epsilon_m^2 + 2\epsilon_m)
\]

with \(\chi_{[0,2)}(\delta)\) the characteristic function of the interval \([0,2)\).

**Proof**

For \(i + 2 \leq i \leq N - 2\), consider the operator

\[
S_{j_m} : i, (z) := \frac{1}{\Sigma_{j_m}^{(i,i+1)} \left[ V_{j_m} + V'_{j_1 \ldots j_{i-1}} + (\hat{H}_{j_m}^{Bog})_{\xi} + \xi T - \frac{(m-1)\xi}{M} - z \right] \Sigma_{j_m}^{(i,i+1)}} \Sigma_{j_m}^{(i,i+1)}.
\]

If

\[
\Sigma_{j_m}^{(i,i+1)} \left[ V_{j_m} + V'_{j_1 \ldots j_{i-1}} + (\hat{H}_{j_m}^{Bog})_{\xi} + \xi T - \frac{(m-1)\xi}{M} - z \right] \Sigma_{j_m}^{(i,i+1)} > 0
\]

then we can estimate

\[
\left\| \left[ R_{j_1 \ldots j_m : i, i+2} (w) \right]^\frac{1}{2} \tilde{W}_{j_1 \ldots j_m : i, i+2} \left[ R_{j_1 \ldots j_m : i-2, i+2} (w) \right]^\frac{1}{2} \right\|
\]

by inserting (for \(N\) sufficiently large)

\[
\mathbb{I}_{\Sigma_{j_m}^{(i,i+1)} \neq 0} = \left[ S_{j_m : i, i+2} (w) \right]^\frac{1}{2} \frac{1}{\left[ S_{j_m : i, i+2} (w) \right]^\frac{1}{2}}, \quad i - 2 \geq i + 2,
\]

and

\[
\mathbb{I}_{\Sigma_{j_m}^{(i,i+1)} \neq 0} = \frac{1}{\left[ S_{j_m : i, i+2} (w) \right]^\frac{1}{2}} \left[ S_{j_m : i, i+2} (w) \right]^\frac{1}{2}
\]

on the right and on the left of \(\tilde{W}_{j_1 \ldots j_m : i, i+2}\), respectively, i.e.,

\[
\left\| \left[ R_{j_1 \ldots j_m : i, i+2} (z) \right]^\frac{1}{2} \tilde{W}_{j_1 \ldots j_m : i, i+2} \left[ R_{j_1 \ldots j_m : i-2, i+2} (z) \right]^\frac{1}{2} \right\|
\]

\[
= \left\| \left[ R_{j_1 \ldots j_m : i, i+2} (z) \right]^\frac{1}{2} \frac{1}{\left[ S_{j_m : i, i+2} (z) \right]^\frac{1}{2}} \left[ S_{j_m : i, i+2} (z) \right]^\frac{1}{2} \tilde{W}_{j_1 \ldots j_m : i, i+2} \left[ S_{j_m : i-2, i+2} (z) \right]^\frac{1}{2} \right\| \times
\]

\[
\times \left\| \frac{1}{\left[ S_{j_m : i-2, i+2} (z) \right]^\frac{1}{2}} \left[ S_{j_m : i-2, i+2} (z) \right]^\frac{1}{2} \right\|.
\]

Next, we show that the inequality in (5.9) holds for \(\xi = \left( \frac{1}{mN^2} \right)^\frac{i}{2}\), \(N\) sufficiently large and \(z\) in the interval (5.2). To this purpose it is helpful to recall that

\[
H_{j_1 \ldots j_m} = (H_{j_1 \ldots j_m}^{Bog})_{\xi} + (1 - \xi)T_{j_1 \ldots j_m} + V_{j_1} + V'_{j_1 \ldots j_m} + (\hat{H}_{j_m}^{Bog})_{\xi} + \xi T.
\]

We point out that for \(i \geq i + 2\):
Regarding $\Sigma_{j_m}^{(i,i+1)}V_{j_m}V_{j_m}^{(i,i+1)}$, we split it into

$$
\Sigma_{j_m}^{(i,i+1)}V_{j_m}V_{j_m}^{(i,i+1)} = \sum_{j \in \mathbb{Z}^d \setminus \{-j_m,0\}} \frac{1}{N} \sum_{j \in \mathbb{Z}^d \setminus \{-j_m,0\}} a_{j,j_m}^* a_j^* a_j a_{j_m} + h.c.] \Sigma_{j_m}^{(i,i+1)}
$$

and proceed with two observations:

- For the control of (5.15)-(5.16) we point out that in each summand there is at most one of the operators $a_j$, $a_j^*$ and at least one of the operators $a_{j,j_m}, a_{j,j_m}^*$. Thus, we can exploit that the number of particles in the modes $\pm j_m$ is constrained by $\Sigma_{j_m}^{(i,i+1)}$ to the value $N - i$ or $N - i - 1$ that are smaller than $\lfloor N/2 \rfloor - 1$, and use an estimate analogous to (4.51).

- For the control of (5.17) we exploit

$$
\Sigma_{j_m}^{(i,i+1)} \geq \sum_{j \in \mathbb{Z}^d \setminus \{-j_m,0\}} \left( - \frac{\phi_{j_m}}{N} a_j^* a_j + V_{j_m}^{(d)} \right) \Sigma_{j_m}^{(i,i+1)}
$$

where

$$
V_{j_m}^{(d)} = \frac{1}{N} \sum_{j \in \mathbb{Z}^d \setminus \{-j_m,0\}} a_{j,j_m}^* a_j a_j \sum_{j' \in \mathbb{Z}^d \setminus \{j_m,0\}} a_{j'-j_m}^* a_{j'} \geq 0.
$$

Regarding $\Sigma_{j_m}^{(i,i+1)}V'_{j_m}V_{j_m}^{(i,i+1)}$, we can repeat the strategy used to control (5.15) and (5.16).

Due to the assumption in (5.1) and being $z < 0$ uniformly in $N$, we deduce that for $N$ large enough

$$
\Sigma_{j_m}^{(i,i+1)}[H_{j_1,...,j_m} - w] \geq \left( 1 - \xi T - z_{j_1,...,j_m} + \frac{(m - 1)\xi^2}{M} \right) \Sigma_{j_m}^{(i,i+1)}
$$

and

$$
\Sigma_{j_m}^{(i,i+1)}[V_{j_1,...,j_m} + V'_{j_1,...,j_m}] \geq \left( 1 - \xi T - z_{j_1,...,j_m} + \frac{(m - 1)\xi^2}{M} \right) \Sigma_{j_m}^{(i,i+1)}
$$

Consequently, we can conclude that for $i \geq \tilde{i} + 2$

$$
\left\| R_{j_1,...,j_m} \right\| \leq \frac{1}{\left\| S_{j_m}^{(i,i+1)} \right\|^2} \frac{1}{\left\| \Sigma_{j_m}^{(i,i+1)} \right\|^2} \left\| R_{j_1,...,j_m} \right\| \leq 1
$$
which implies that for $i \geq i + 4$

$$\left\| \left[ R_{j_1, \ldots, j_m; i, i(w)} \right]^\dagger \bar{W}_{j_1, \ldots, j_m; i, i-2} \left[ R_{j_1, \ldots, j_m; i-2, i-2(w)} \right]^\dagger \right\| \leq \left\| \left[ S_{j_1, \ldots, j_m; i, i(z)} \right]^\dagger \bar{W}_{j_1, \ldots, j_m; i, i-2} \left[ S_{j_1, \ldots, j_m; i-2, i-2(z)} \right]^\dagger \right\| .$$  (5.26)

The next step is showing that for $i \geq i + 4$

$$\left\| \left[ S_{j_1, \ldots, j_m; i, i(z)} \right]^\dagger V_{j_1, \ldots, j_m; i, i-2} \left[ S_{j_1, \ldots, j_m; i-2, i-2(z)} \right]^\dagger \right\| \leq O\left( \frac{(N \frac{\xi}{2})^{\frac{5}{2}}}{\xi N^{\frac{3}{2}}} \right).$$  (5.28)

where, for convenience, we recall

$$\mathcal{S}_{j_m}^{(i+1)} V_{j_1, \ldots, j_m} \mathcal{S}_{j_m}^{(i-2, i-1)}$$  (5.29)

$$= \mathcal{S}_{j_m}^{(i+1)} \left( \frac{1}{N} \sum_{l=1}^{m} \sum_{j \in \mathbb{Z}^m \setminus \{j_l, 0\}} a_{j+l,j}^* a_j^* \phi_j a_l a_{j_l} + h.c. \right) \mathcal{S}_{j_m}^{(i-2, i-1)}$$  (5.30)

$$+ \mathcal{S}_{j_m}^{(i+1)} \left( \frac{1}{N} \sum_{l=1}^{m} \sum_{j \in \mathbb{Z}^m \setminus \{j_l, 0\}} a_{j-l,j}^* a_j^* \phi_j a_j a_{j_l} = h.c. \right) \mathcal{S}_{j_m}^{(i-2, i-1)}$$  (5.31)

$$+ \mathcal{S}_{j_m}^{(i+1)} \left( \frac{1}{N} \sum_{l=1}^{m} \sum_{j \in \mathbb{Z}^m \setminus \{j_l, 0\}} \sum_{j' \in \mathbb{Z}^m \setminus \{j, 0\}} a_{j-j',j}^* a_{j'-j, j}^* \phi_j a_l a_{j' a_{j_l}} \right) \mathcal{S}_{j_m}^{(i-2, i-1)}.$$  (5.32)

Our strategy to control (5.29) and provide the estimate in (5.28) relies on the fact that in expressions (5.30)-(5.32): 1) at most one operator of the type $a_0, a_0^*$ can be present in each summand; 2) at least one operator $a_{j_m}^*$ or $a_{j_m}^*$ must be present due to the projections $\mathcal{S}_{j_m}^{(i+1)}$ and $\mathcal{S}_{j_m}^{(i-2, i-1)}$ on the left and on the right, respectively; 3) the number of particles in the modes $\pm j_m$ is constrained by $\mathcal{S}_{j_m}^{(i+1)}$ to values less than $\lfloor N \frac{\xi}{2} \rfloor - 5$ for $i \geq i + 4$.

The leading term that remains is

$$\left[ S_{j_1, \ldots, j_m; i, i(z)} \right]^\dagger W_{j_1, \ldots, j_m; i, i-2} \left[ S_{j_1, \ldots, j_m; i-2, i-2(z)} \right]^\dagger$$  (5.33)

which can be estimated like in Lemma 3.4 in [Pit]. Due to the choice of $\xi$, we arrive at the estimate in (5.4) where the corrections coming from (5.28) and the $\xi$-dependent terms in (5.8) are hidden in the term $o(\epsilon_m)$ which enters the definition of $a_{j_m}$; see (5.5). In fact these corrections vanish as $N \to \infty$. □

**Lemma 5.2.** For $M \geq m \geq 1$ assume:

(i)

$$(H^\#_{j_1, \ldots, j_{m-1}})_{j_1, \ldots, j_m} - (1 - \xi) T_{j_1, \ldots, j_m} \geq \bar{z}^\#_{j_1, \ldots, j_m} - \frac{(m - 1) \xi^{\frac{1}{2}}}{M},$$  (5.34)

where $(H^\#_{j_1, \ldots, j_{m-1}})$ is defined in (4.1) for $m \geq 2$ and is equal to $(1 - \xi) T$ for $m = 1$, and where $\bar{z}^\#_{j_1, \ldots, j_m}$ is the ground state energy of $H^\#_{j_1, \ldots, j_{m-1}}$.

(ii) The upper bound

$$\langle \psi_{j_1, \ldots, j_{m-1}}^\# | \left( \sum_{j \in \mathbb{Z}^m \setminus \{0\}} a_{j} a_{j}^* \right) \frac{\psi_{j_1, \ldots, j_{m-1}}^\#}{\| \psi_{j_1, \ldots, j_{m-1}}^\# \|} \rangle \leq O(1)$$  (5.35)
holds true where $\psi^\#_{j_1 \cdots j_{m-1}}$ is the ground state vector of $H^\#_{j_1 \cdots j_{m-1}}$.

Let $e_{j_m}$ be sufficiently small and $N$ sufficiently large. Then, for $z \leq E_{j_m}^{Bog}$ there are constants $0 < C^\#_I, C^\#_H, C^\#_\perp < \infty$ such that

$$\left| \left\langle \frac{\psi^\#_{j_1 \cdots j_{m-1}}}{\|\psi^\#_{j_1 \cdots j_{m-1}}\|}, \Gamma_{j_1 \cdots j_{m-1};N,N}(z + z^\#_{j_1 \cdots j_{m-1}}) \frac{\psi^\#_{j_1 \cdots j_{m-1}}}{\|\psi^\#_{j_1 \cdots j_{m-1}}\|} \right\rangle - \langle \eta, H_{j_m}^{Bog,N,N}(z)\eta \rangle \right| \leq \frac{C^\#_I}{(\ln N)^{\frac{1}{2}}}, \quad (5.36)$$

$$\| \mathcal{P}^\#_{j_1 \cdots j_{m-1}} (V_{j_m} - \Gamma_{j_1 \cdots j_{m-1};N,N}(z + z^\#_{j_1 \cdots j_{m-1}})) \mathcal{P}^\#_{j_1 \cdots j_{m-1}} \| \leq \frac{C^\#_H}{(\ln N)^{\frac{1}{2}}}, \quad (5.37)$$

and

$$\left\| \left( \mathcal{P}^\#_{j_1 \cdots j_{m-1}} \right)_I (\Gamma_{j_1 \cdots j_{m-1};N,N}(z + z^\#_{j_1 \cdots j_{m-1}}) \mathcal{P}^\#_{j_1 \cdots j_{m-1}}) \right\| \leq \frac{\phi_{j_m}}{2e_{j_m} + 2} \left( z - \Delta^\#_{m-1} (1 - \frac{\phi_{j_m}(\ln N)^{\frac{1}{2}}}{N\Delta^\#_{m-1}}) + \frac{C^\#_\perp}{(\ln N)^{\frac{1}{2}}} \right). \quad (5.38)$$

\textbf{Proof}

The proof is very similar to the proof of the analogous inequalities in \textit{Lemma 4.3 and Lemma 4.4} of [Pi2]. As far as (5.36) and (5.37) are concerned, the role of the Hamiltonian $H_{j_1 \cdots j_{m-1}}^{Bog}$ in the analogous estimate of \textit{Lemma 4.3} of [Pi2] is played by the operator

$$H^\#_{j_1 \cdots j_{m-1}} = T_{j = k_{j_m}} + V_{j_m} + V'_{j_1 \cdots j_{m-1}}$$

with the help of the assumption in (5.35). Likewise, in (5.37) the term proportional to $V_{j_m}$ is estimated using the assumption in (5.35). \(\Box\)

\textbf{Lemma 5.3.} Let $1 \leq m \leq M < \infty$. Then, assuming that the operators below are restricted to $\mathcal{F}^N$, the following inequalities hold true,

$$H^\#_{j_1 \cdots j_{m-1}} \geq T - \sum_{l=1}^{m-1} \phi_{j_l} \quad \text{with} \quad T := \sum_{j \in \mathbb{Z}^d} k_l^2 \alpha_j^* \alpha_j, \quad (5.39)$$

$$H_{j_1 \cdots j_{m-1}} \geq T - \sum_{l=1}^{m-1} \phi_{j_l}, \quad (5.40)$$

and

$$(H^\#_{j_1 \cdots j_{m-1}})^2 \geq C_1 N_+^2 - C_2, \quad N_+ := \sum_{j \in \mathbb{Z}^d \setminus \{0\}} \alpha_j^* \alpha_j, \quad (5.41)$$

for some $C_1, C_2 > 0$.

\textbf{Proof}
Starting from the identity
\[ H_{j_1, \ldots, j_m-1}^n = \sum_{j \in \mathbb{Z}^d} k_j^2 a_j^* a_j \] (5.42)
\[ + \sum_{l=1}^{m-1} \frac{\phi_j}{N} (a_0 a_{j_l} + a_0 a_{-j_l} + \sum_{j \in \mathbb{Z}^d \setminus \{j : j_{j_m} = \pm j_l \}} a_j^* a_j) \times \]
\[ \times (a_0 a_{-j_l}^* + a_0 a_{j_l}^*) \times \sum_{j \in \mathbb{Z}^d \setminus \{j : j_{j_m} = \pm j_l \}} a_j^* a_j \] (5.43)
\[ - \sum_{l=1}^{m-1} \frac{\phi_j}{N} \left[ a_{-j_l}^* a_{-j_l} + a_0^* a_0 + \sum_{j \in \mathbb{Z}^d \setminus \{j : j_{j_m} = \pm j_l \}} a_j^* a_j \right] \] (5.44)

it is convenient to set
\[ \mathcal{A}_l := (a_0 a_{j_l} + a_0 a_{-j_l} + \sum_{j \in \mathbb{Z}^d \setminus \{j : j_{j_m} = \pm j_l \}} a_j^* a_j), \] (5.46)
\[ \mathcal{B}_l := a_{-j_l}^* a_{-j_l} + a_0^* a_0 + \sum_{j \in \mathbb{Z}^d \setminus \{j : j_{j_m} = \pm j_l \}} a_j^* a_j. \] (5.47)

Then, the inequality in (5.39) is obvious since \( \phi_j > 0, \mathcal{A}_l^* \mathcal{A}_l \geq 0, \) and
\[ a_{-j_l}^* a_{-j_l} + a_0^* a_0 + \sum_{j \in \mathbb{Z}^d \setminus \{j : j_{j_m} = \pm j_l \}} a_j^* a_j \leq N. \] (5.48)

The proof of inequality (5.40) is essentially the same.

Regarding the third inequality, we can write
\[ (H_{j_1, \ldots, j_m-1}^n)^2 = T^2 + \left( \sum_{l=1}^{m-1} \frac{\phi_j}{N} \mathcal{A}_l^* \mathcal{A}_l \right)^2 \]
\[ + \left( \sum_{l=1}^{m-1} \frac{\phi_j}{N} \mathcal{B}_l \right)^2 \]
\[ + 2T \left( \sum_{l=1}^{m-1} \frac{\phi_j}{N} \mathcal{B}_l \right) \left( \sum_{l=1}^{m-1} \frac{\phi_j}{N} \mathcal{A}_l \right) \]
\[ + (T - \sum_{l=1}^{m-1} \frac{\phi_j}{N} \mathcal{B}_l) \left( \sum_{l=1}^{m-1} \frac{\phi_j}{N} \mathcal{A}_l \right) \]
and compute
\[ T \left( \frac{\phi_j}{N} \mathcal{A}_l^* \mathcal{A}_l \right) = \sum_{j \in \mathbb{Z}^d} k_j^2 a_j^* \left. \frac{\phi_j}{N} \mathcal{A}_l^* \mathcal{A}_l a_j + \sum_{j \in \mathbb{Z}^d} k_j^2 a_j^* \left\{ a_j, \mathcal{A}_l^* \mathcal{A}_l + \mathcal{A}_l^* \mathcal{A}_l + a_j, \mathcal{A}_l \right\} \right. \] (5.52)

We also observe that
- the expression
\[ \sum_{j \in \mathbb{Z}^d} k_j^2 a_j^* \left\{ a_j, \mathcal{A}_l^* \mathcal{A}_l \right\} = \sum_{j \in \mathbb{Z}^d} k_j^2 a_j^* \left\{ a_j, \mathcal{A}_l^* \mathcal{A}_l + \mathcal{A}_l^* \mathcal{A}_l \right\} \] (5.53)
is dominated by a constant times
\[ N_+ + \frac{\phi_j}{N} \mathcal{A}_l^* \mathcal{A}_l \] (5.54)
• the expressions
\[ \frac{\phi_{l}^n B_l}{N} \phi_{l}^m A_l, \quad -2T \frac{\phi_{l}^n B_l}{N} \quad (5.55) \]
are dominated by a constant times \( T \).
Using \( T^2 \geq \Delta^2 N^2 \) we can determine two positive constants \( C_1, C_2 \) such that
\[ (H_{\beta_{j_1, \ldots, j_{m-1}}}^\beta)^2 \geq C_1 N^2 + C_2. \quad (5.56) \]
\[ \Box \]

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