Folding, Tiling, and Multidimensional Coding

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Abstract—Folding a sequence $S$ into a multidimensional box is a method that is used to construct multidimensional codes. The well known operation of folding is generalized in a way that the sequence $S$ can be folded into various shapes. The new definition of folding is based on lattice tiling and a direction in the $D$-dimensional grid. There are potentially $2^{D-1}$ different folding operations. Necessary and sufficient conditions that a lattice combined with a direction define a folding are given. The immediate and most impressive application is some new lower bounds on the number of dots in two-dimensional synchronization patterns. This can be also generalized for multidimensional synchronization patterns. We show how folding can be used to construct multidimensional error-correcting codes and to generate multidimensional pseudo-random arrays.

I. INTRODUCTION

Multidimensional coding in general and two-dimensional coding in particular is a subject which attract lot of attention in the last three decades. It includes error-correcting codes [1], [2], [3], [4], synchronization patterns [5], [6], [7], [8], [9], perfect maps and pseudo-random arrays [10], [11], and other topics as well. But, although the related theory of the one-dimensional case is well developed, the theory for the multidimensional case is developed rather slowly. This is due that the fact the most of the one-dimensional techniques are not generalized easily to higher dimensions. Hence, specific techniques have to be developed for multidimensional coding. One technique that was used for multidimensional coding is folding [8], [9], [10]. It was used to form a two-dimensional array, in the shape of a rectangle, from a one-dimensional sequence. It is generalized for multidimensional arrays to form a multidimensional box.

In this paper we generalize the definition of folding. It is generalized in a way that we will be able to apply it to a multidimensional shape $S$ which does not necessarily has a shape of a box. We present applications of the new definition to form multidimensional synchronization patterns, error-correcting codes, and pseudo-random arrays, for many types of multidimensional shapes.

The rest of this paper is organized as follows. In Section II we define the basic concepts of folding and lattice tiling. Tilings and lattices are basic combinatorial and algebraic structures. We will consider only integer lattice tilings. We will summarize the important properties of lattices and lattice tilings. In Section III we will present the generalization of folding into multidimensional shapes. We will show that all previous known foldings are special cases of the new definition. The new definition involves a lattice tiling and a direction. We will prove necessary and sufficient conditions that a lattice with a direction define a folding. In Section IV we give a short summary on synchronization patterns and present basic theorems concerning the bounds on the number of elements in such patterns. In Section V we apply the results of the previous sections to obtain new type of synchronization patterns which are asymptotically either optimal or almost optimal. In Section VI we show how folding can be applied to construct multidimensional error-correcting codes. In section VII we generalize the construction in [10] to form pseudo-random arrays on different multidimensional shapes. We conclude in Section VIII.

II. FOLDING AND LATTICE TILING

A. Folding

Folding a rope, a ruler, or any other feasible object is a common action in every day life. Folding an one-dimensional sequence into a $D$-dimensional array is very similar, but there are a few variants. First, we will summarize three variants for folding of an one-dimensional sequence $s_0s_1\cdots s_{m-1}$ into a two-dimensional array $A$. The generalization for a $D$-dimensional array is straightforward while the description becomes more clumsy.

**F1.** $A$ is considered as a cyclic array horizontally and vertically in such a way that a walk diagonally visits all the entries of the array. The elements of the sequence are written along the diagonal of the $r \times t$ array $A$. This folding works if and only if $r$ and $t$ are relatively primes.

**F2.** The elements of the sequence are written row by row (or column by column) in $A$.

**F3.** The elements of the sequence are written diagonal by diagonal in $A$.

F1 and F2 were used by MacWilliams and Sloane [10] to form a pseudo-random arrays. F2 was used by Robinson [8] to fold a one-dimensional ruler into a two-dimensional Golomb rectangle. The generalization to higher dimensions is straight forward. F3 was used in [9] to obtain a synchronization patterns in the square grid.

B. Tiling

Tiling is one of the most basic concepts in combinatorics. We say that a $D$-dimensional shape $S$ tiles the $D$-dimensional space $\mathbb{Z}^D$ if disjoint copies of $S$ cover $\mathbb{Z}^D$. This cover of $\mathbb{Z}^D$ with disjoint copies of $S$ is called tiling of $\mathbb{Z}^D$ with $S$. For each shape $S$ we distinguish one of the points of $S$ to be the center of $S$. Each copy of $S$ in a tiling has the center in the same related point. The set $T$ of centers...
in a tiling defines the tiling, and hence the tiling is denoted by the pair \((T, S)\). Given a tiling \((T, S)\) and a grid point \((i_1, i_2, \ldots, i_D)\) we denote by \(c(i_1, i_2, \ldots, i_D)\) the center of the copy of \(S\) for which \((i_1, i_2, \ldots, i_D) \in S\). We will also assume that the origin is a center of a copy of \(S\).

**Lemma 1:** For a given tiling \((T, S)\) and a point \((i_1, i_2, \ldots, i_D)\), the point \((i_1, i_2, \ldots, i_D) - c(i_1, i_2, \ldots, i_D)\) belongs to the shape \(S\) whose center is in the origin.

One of the most common types of tiling is a lattice tiling. A lattice \(\Lambda\) is a discrete, additive subgroup of the real space \(\mathbb{R}^D\). W.l.o.g., we can assume that

\[\Lambda = \{ u_1v_1 + u_2v_2 + \cdots + u_Dv_D : u_1, \ldots, u_D \in \mathbb{Z}\} \quad (1)\]

where \({v_1, v_2, \ldots, v_D}\) is a set of linearly independent vectors in \(\mathbb{R}^D\). A lattice \(\Lambda\) defined by \((v_1, v_2, \ldots, v_D)\) is a sublattice of \(\mathbb{Z}^D\) if and only if \({v_1, v_2, \ldots, v_D}\) \(\in \mathbb{Z}^D\). We will be interested solely in sublattices of \(\mathbb{Z}^D\). The vectors \(v_1, v_2, \ldots, v_D\) are called basis for \(\Lambda \subseteq \mathbb{Z}^D\), and the \(D \times D\) matrix

\[
G = \begin{bmatrix}
v_{11} & v_{12} & \cdots & v_{1D} \\
v_{21} & v_{22} & \cdots & v_{2D} \\
\vdots & \vdots & \ddots & \vdots \\
v_{D1} & v_{D2} & \cdots & v_{DD}
\end{bmatrix}
\]

having these vectors as its rows is said to be the generator matrix for \(\Lambda\).

The volume of a lattice \(\Lambda\), denoted \(V(\Lambda)\), is inversely proportional to the number of lattice points per unit volume. More precisely, \(V(\Lambda)\) may be defined as the volume of the fundamental parallelogram \(\Pi(\Lambda)\) in \(\mathbb{R}^D\), which is given by

\[
\Pi(\Lambda) \overset{\text{def}}{=} \{ \xi_1v_1 + \xi_2v_2 + \cdots + \xi_Dv_D : 0 \leq \xi_i < 1, \quad 1 \leq i \leq D \}
\]

There is a simple expression for the volume of \(\Lambda\), namely,

\[V(\Lambda) = |\det G|\]

We say that \(\Lambda\) induces a lattice tiling of \(S\) if the lattice points can be taken as the set \(T\) to form a tiling \((T, S)\).

### III. THE GENERALIZED FOLDING METHOD

In this section we will generalize the definition of folding. All the previous three definitions are special cases of the new definition. The new definition involves a lattice tiling \((T, S)\), where \(S\) is the shape on which the tiling is performed.

A ternary vector of length \(D\), \((d_1, d_2, \ldots, d_D)\), is a word of length \(D\), where \(d_i \in \{-1, 0, +1\}\).

Let \(S\) be a \(D\)-dimensional shape and let \(\delta = (d_1, d_2, \ldots, d_D)\) be a nonzero ternary vector of length \(D\). Let \((T, S)\) be a lattice tiling defined by a \(D\)-dimensional lattice \(\Lambda\), and let \(\tilde{S}\) be the copy of \(S\) in \((T, S)\) which includes the origin. We define recursively a folded-row starting in the origin. If the point \((i_1, i_2, \ldots, i_D)\) is in \(\tilde{S}\) then the next point on its folded-row is defined as follows:

- If the point \((i_1 + d_1, i_2 + d_2, \ldots, i_D + d_D)\) is in \(\tilde{S}\) then it is the next point on the folded-row.
- If the point \((i_1 + d_1, i_2 + d_2, \ldots, i_D + d_D)\) is in \(\tilde{S}' \neq \tilde{S}\) whose center is in the point \((c_1, c_2, \ldots, c_D)\) then \((i_1 + d_1 - c_1, i_2 + d_2 - c_2, \ldots, i_D + d_D - c_D)\) is the next point on the folded-row.

The new definition of folding is based on a lattice \(\Lambda\), a shape \(S\), and a direction \(\delta\). The triple \((\Lambda, S, \delta)\) defines a folding if the definition yields a folded-row which includes all the elements of \(S\). It appears that only \(\Lambda\) and \(\delta\) determines whether the triple \((\Lambda, S, \delta)\) defines a folding. The role of \(S\) is only in the order of the elements in the folded-row; and of course \(\Lambda\) must define a lattice tiling for \(S\).

How many different folded-rows do we have? In other words, how many different folding operations can be defined? There are \(3^D - 1\) non-zero ternary vectors. If \(\Lambda\) with the ternary vector \((d_1, d_2, \ldots, d_D)\) define a folding then also \(\Lambda\) with the vector \((-d_1, -d_2, \ldots, -d_D)\) define a folding. The two folded-rows are in reverse order, and hence they will be considered to be equal. Other than these pairs of folded-rows, we don’t know whether for each \(D\), there exists a \(D\)-dimensional shape \(S\) with \(\frac{3^D-1}{2}\) different folded-rows. An example for \(D = 2\) is given next.

**Example 1:** Let \(\Lambda\) be the lattice whose generator matrix given by the matrix

\[G = \begin{bmatrix} 3 & 2 \\ 7 & 1 \end{bmatrix}\]

One can verify that shapes tiled by this lattice have different folded-rows. It can be proved that this is the lattice with the smallest volume which has this property.

The first two lemmas are an immediate consequence of the definitions and provide us a concise condition whether the triple \((\Lambda, S, \delta)\) defines a folding.

**Lemma 2:** Let \((T, S)\) be a lattice tiling defined by the \(D\)-dimensional lattice \(\Lambda\) and let \(\delta = (d_1, d_2, \ldots, d_D)\) be a nonzero ternary vector. \((\Lambda, S, \delta)\) defines a folding if and only if the set \(\{(i \cdot d_1, i \cdot d_2, \ldots, i \cdot d_D) - c(i \cdot d_1, i \cdot d_2, \ldots, i \cdot d_D) : 0 \leq i < |S|\}\) contains \(|S|\) distinct elements.

**Lemma 3:** Let \((T, S)\) be a lattice tiling defined by the \(D\)-dimensional lattice \(\Lambda\) and let \(\delta = (d_1, d_2, \ldots, d_D)\) be a nonzero ternary vector. \((\Lambda, S, \delta)\) defines a folding if and only if \(|S| \cdot d_1, \ldots, |S| \cdot d_D - c(|S| \cdot d_1, \ldots, |S| \cdot d_D) = (0, \ldots, 0)\) and for each \(i, 0 < i < |S|\) we have \((i \cdot d_1, \ldots, i \cdot d_D) - c(i \cdot d_1, \ldots, i \cdot d_D) \neq (0, \ldots, 0)\).

Before considering the general \(D\)-dimensional case we want to give a simple condition to check whether the triple \((\Lambda, S, \delta)\) defines a folding in the two-dimensional case. For each one of the four possible folding definitions we will give a necessary and sufficient condition that the triple \((\Lambda, S, \delta)\) defines a folding.

**Theorem 1:** Let \(\Lambda\) be a lattice whose generator matrix is given by

\[G = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}\]

If \(\Lambda\) defines a lattice tiling for the shape \(S\) then the triple \((\Lambda, S, \delta)\) defines a folding

- with the ternary vector \(\delta = (+1, +1)\) if and only if \(\gcd(v_{22} - v_{21}, v_{11} - v_{12}) = 1\);
- with the ternary vector \(\delta = (+1, -1)\) if and only if \(\gcd(v_{22} + v_{21}, v_{11} + v_{12}) = 1\);
- with the ternary vector \(\delta = (+1, 0)\) if and only if \(\gcd(v_{12}, v_{22}) = 1\).
• with the ternary vector \( \delta = (0,+1) \) if and only if \( \text{g.c.d.}(v_{11}, v_{21}) = 1 \).

Theorem 1 is generalized for the \( D \)-dimensional case as follows. Let \( \Lambda \) be a \( D \)-dimensional lattice tiling for the shape \( S \) with the following generator matrix.

\[
G = \begin{bmatrix}
v_{11} & v_{12} & \cdots & v_{1D} \\
v_{21} & v_{22} & \cdots & v_{2D} \\
\vdots & \vdots & \ddots & \vdots \\
v_{D1} & v_{D2} & \cdots & v_{DD}
\end{bmatrix}.
\]

Assume we have the direction vector \( \delta = (d_1, d_2, \ldots, d_D) \). W.l.o.g. we assume that the first \( \ell_1 \geq 1 \) values are +1’s, the next \( \ell_2 \) values are -1’s, and the last \( D - \ell_1 - \ell_2 \) values are 0’s. There exist integer coefficients \( \alpha_1, \alpha_2, \ldots, \alpha_D \) such that

\[
\sum_{j=1}^{D} \alpha_j (v_{j1}, v_{j2}, \ldots, v_{jD}) = (|S|, |S|, -|S|, \ldots, -|S|, 0, \ldots, 0),
\]

and there is no integer \( i, 0 < i < |S| \), and integer coefficients \( \beta_1, \beta_2, \ldots, \beta_D \) such that

\[
\sum_{j=1}^{D} \beta_j (v_{j1}, v_{j2}, \ldots, v_{jD}) = (i, \ldots, i, -i, \ldots, -i, 0, \ldots, 0).
\]

Hence we have the following \( D \) equations:

\[
\sum_{j=1}^{D} \alpha_j v_{jr} = |S|, \quad 1 \leq r \leq \ell_1,
\]

\[
\sum_{j=1}^{D} \alpha_j v_{jr} = -|S|, \quad \ell_1 + 1 \leq r \leq \ell_1 + \ell_2,
\]

\[
\sum_{j=1}^{D} \alpha_j v_{jr} = 0, \quad \ell_1 + \ell_2 + 1 \leq r \leq D,
\]

which are equivalent to the following \( D \) equations:

\[
\sum_{j=1}^{D} \alpha_j v_{j1} = |S|,
\]

\[
\sum_{j=1}^{D} \alpha_j (v_{jr} - v_{j1}) = 0, \quad 2 \leq r \leq \ell_1,
\]

\[
\sum_{j=1}^{D} \alpha_j (v_{jr} + v_{j1}) = 0, \quad \ell_1 + 1 \leq r \leq \ell_1 + \ell_2,
\]

\[
\sum_{j=1}^{D} \alpha_j v_{jr} = 0, \quad \ell_1 + \ell_2 + 1 \leq r \leq D.
\]

We define now a set of \( D(D-1) \) new coefficients \( u_{rj}, \)

\[
u_{rj} = v_{jr} - v_{j1} \quad \text{for} \quad 2 \leq r \leq \ell_1,
u_{rj} = v_{jr} \quad \text{for} \quad \ell_1 + 1 \leq r \leq \ell_1 + \ell_2,
u_{rj} = v_{jr} + v_{j1} \quad \text{for} \quad \ell_1 + \ell_2 + 1 \leq r \leq D.
\]

Consider the \((D-1) \times D\) matrix

\[
H = \begin{bmatrix}
u_{21} & u_{22} & \cdots & u_{2D} \\
u_{31} & u_{32} & \cdots & u_{3D} \\
\vdots & \vdots & \ddots & \vdots \\
u_{D1} & u_{D2} & \cdots & u_{DD}
\end{bmatrix}.
\]

Using the Cramer’s rule it is easy to verify that the following assignment is the unique solution for the \( \alpha_i \)’s.

\[
\alpha_i = (-1)^{i-1} \det H_i, \quad 1 \leq i \leq D,
\]

where \( H_i \) is the \((D-1) \times (D-1)\) matrix obtained from \( H \) by deleting column \( i \) of \( H \).

Theorem 2: If \( \Lambda \) define a lattice tiling for the shape \( S \) then the triple \( (\Lambda, S, \delta) \) defines a folding if and only if \( \text{g.c.d.}(\det H_1, \det H_2, \ldots, \det H_D) = 1 \).

One important tool that we will use to find an appropriate folding for a shape \( S' \) is to use a folding of a simpler shape \( S \) with the same volume and apply iteratively the following theorem.

Theorem 3: Let \( \Lambda \) be a lattice, \( S \) a \( D \)-dimensional shape, \( \delta = (d_1, d_2, \ldots, d_D) \) a direction, and \( (\Lambda, S, \delta) \) defines a folding. Assume the origin is a point in \( S \), \((i_1, i_2, \ldots, i_D) \in S \), \((i_1 + d_1, i_2 + d_2, \ldots, i_D + d_D) \in \tilde{S} \), \( \tilde{S} \neq S \), and the center of \( \tilde{S} \) is the point \((c_1, c_2, \ldots, c_D) \). Then for the shape \( S' = S \cup \{(i_1 + d_1, i_2 + d_2, \ldots, i_D + d_D)\} \setminus \{(i_1 + d_1 - c_1, i_2 + d_2 - c_2, \ldots, i_D + d_D - c_D)\} \) the triple \( (\Lambda, S', \delta) \) also defines a folding.

IV. BOUNDS ON SYNCHRONIZATION PATTERNS

Our motivation for the generalization of the folding operation came from the design of two dimensional synchronization patterns. Given a grid (square or hexagonal) and a shape \( S \) on the grid, we would like to find what is the largest set \( V \) of dots on grid points, \( |V| = m \), located in \( S \), such that the following property holds. All the \((m^2)\) lines between dots are distinct either in length or in slope. Such a shape \( S \) with dots is called a distinct difference configuration (DDC). If \( S \) is an \( m \times m \) array with a dot in each row and a dot in each column than \( S \) is called a Costas array [5]. If \( S \) is a \( k \times m \) array with a dot in each column then \( S \) is called a sonar sequence [5]. If \( S \) is a \( k \times n \) array then \( S \) is called a Golomb rectangle [7]. These patterns have various applications as described in [5]. A new application of these patterns to the design of key predistribution scheme for wireless sensor networks was described lately in [12]. In this application the shape \( S \) might be a Lee sphere, an hexagon, or a circle, and sometimes another regular polygon. This application requires in some cases to consider these shapes in the hexagonal grid. F3 was used for this application in [9] to form a DDC whose shape is a rectangle rotated in 45 degrees in the square grid. Henceforth, we assume that our grid is the square grid, unless stated otherwise.

Let \( S \) and \( S' \) be two-dimensional shapes in the grid. We will denote by \( \Delta(S, S') \) the largest intersection between \( S \) and \( S' \).
and \( S' \). \(|S|\) will denote the number of grid points in \( S \). Let \( m \) be a given integer. An infinite set of dots in the grid such that each given shape \( S \) on the grid is a DDC with \( m \) dots will be called an infinite \( S\)-DDC. The following theorems are generalization of similar theorem in [9].

**Theorem 4:** Assume we are given an infinite \( S\)-DDC with \( m \) dots on the grid. Let \( R \) be another shape on the grid. Then there exists a copy of \( R \) on the grid with at least \( \frac{m}{|S|R} \Delta(S, R) \) dots.

**Theorem 5:** Assume we are given an infinite \( S\)-DDC with \( m \) dots on the grid. Let \( R \) and \( U \) be another shapes on the grid. Then there exists a copy of \( U \) on the grid with at least \( \frac{m}{|S|R} \Delta(S, R) \cdot \Delta(R, U) \) dots.

In order to apply Theorem 3 and Theorem 5 we will use folding of the sequences, defined as follows, in our shape \( S \). Let \( A \) be an abelian group, and let \( E = \{a_1, a_2, \ldots, a_m\} \subseteq A \) be a sequence of \( m \) distinct elements of \( A \). We say that \( E \) is a \( B_2\)-sequence over \( A \) if all the sums \( a_{i_1} + a_{i_2} \) with \( 1 \leq i_1 \leq i_2 \leq m \) are distinct. For a survey on \( B_2\)-sequences and their generalizations the reader is referred to [13]. The following lemma is well known and can be readily verified.

**Lemma 4:** A subset \( E = \{a_1, a_2, \ldots, a_m\} \subseteq A \) is a \( B_2\)-sequence over \( A \) if and only if all the differences \( a_{i_1} - a_{i_2} \) with \( 1 \leq i_1 \neq i_2 \leq m \) are distinct in \( A \).

Note that if \( D \) is a \( B_2\)-sequence over \( \mathbb{Z}_n \) and \( a \in \mathbb{Z}_n \), then so is the shift \( a + E = \{a + e : e \in E\} \). The following theorem, due to Bose [14], shows that large \( B_2\)-sequences over \( \mathbb{Z}_n \) exist for many values of \( n \).

**Theorem 6:** Let \( q \) be a prime power. Then there exists a \( B_2\)-sequence \( a_1, a_2, \ldots, a_m \) over \( \mathbb{Z}_n \) where \( n = q^2 - 1 \) and \( m = q \).

The importance of folding a \( B_2 \) sequence \( S \) into a given shape \( S' \) is given by the following theorem.

**Theorem 7:** Let \( \Lambda \) be a lattice, \( S, n = |S| \), a \( D\)-dimensional shape, and \( \delta \) a direction. Let \( E \) be a \( B_2\)-sequence over \( \mathbb{Z}_n \). If \( (\Lambda, S, \delta) \) defines a folding then the folded-row is a \( D\)-dimensional DDC. Moreover, this DDC can be extended to infinite \( S\)-DDC.

In the sequel we will use Theorem 4, Theorem 5, and Theorem 7 to form DDCs with various given shapes with a large number of dots. To examine how good are our bounds on the number of dots we should know what is the upper bound on the number of dots in a DDC whose shape is \( S \). It was shown in [9] that for a DDC whose shape is a regular polygon or a circle, the maximal number of dots is at most \( \sqrt{s} + o(\sqrt{s}) \), when the shape contains \( s \) points of the grid.

### V. Bounds for Specific Shapes

In this section we will present some lower bounds on the number of dots in some two-dimensional DDCs with specific shapes. One of the main keys of our constructions, and the use of the given theory, is the ability to produce a DDC with a rectangle shape and any given ratio between its edges.

**Theorem 8:** For each positive number \( \gamma \) there exist two integers \( a \) and \( b \) such that \( \frac{a}{b} \approx \gamma \) and an infinite \( S\)-DDC with \( \sqrt{a} \cdot bR + o(R) \) dots whose shape is an \( \alpha \times \beta = (bR + o(R)) \times (aR + o(R)) \) rectangle, \( \alpha \beta = p^2 - 1 \) for some prime \( p \), and even \( \alpha \).

Now, we can give a few examples for specific shapes. To have some comparison, let the radius of the circle or the regular polygons be \( R \) (the radius is the distance from the center of the regular polygon to any one of its vertices).

#### A. Regular Hexagon in the Square Grid

By Theorem 8 there exists an infinite \( S\)-DDC, where \( S \) is an \( \alpha \times \beta = (aR + o(R)) \times (bR + o(R)) \) rectangle, such that \( \frac{s}{2} \approx \sqrt{s} \), \( \alpha \beta = p^2 - 1 \) for some prime \( p \), and even \( \alpha \). Let \( \Lambda \) be the lattice tiling for \( S \) with the generator matrix

\[
G = \begin{bmatrix} \beta & \frac{a}{b} + \theta \\ 0 & \alpha \end{bmatrix},
\]

where \( \theta = 1 \) if \( \alpha \equiv 0 \pmod{4} \) and \( \theta = 2 \) if \( \alpha \equiv 2 \pmod{4} \). By Theorem 1, \((\Lambda, S, \delta) = (+1, 0), \) defines a folding. Now, Theorem 3 is used iteratively to form an infinite \( S'\)-DDC, where \( S' \) is “almost” a regular hexagon with \( \sqrt{a} \cdot bR + o(R) \) dots (the six vertices of one hexagon are at \((\frac{\beta}{\sqrt{\alpha}}, 0), (0, \beta), (\frac{\beta}{\sqrt{\alpha}}, \beta), (\beta, 0), (\frac{\beta}{\sqrt{\alpha}}, 0), (0, \beta)) \). Hence, a lower bound on the number of dots in a regular hexagon with radius \( R \) is approximately \( \frac{\sqrt{3}\sqrt{\alpha}}{\sqrt{2}}bR + o(R) \).

#### B. Circle in the Square Grid

We apply Theorem 4 where \( S \) is a regular hexagon with radius \( \rho \) and \( S' \) is a circle with radius \( R \). A lower bound on the number of dots in \( S \) is approximately \( \frac{\sqrt{3}\sqrt{\alpha}}{\sqrt{2}}bR + o(\rho) \). The maximum on \( \frac{\sqrt{3}\sqrt{\alpha}}{\sqrt{2}}bR + o(\rho) \Delta(S, S') \) yields a lower bound of 1.70813\( R + o(R) \) on the number of dots in \( S' \).

#### C. Other Shapes

For most of the regular \( n \)-gons \((n \notin \{4, 6\})\) in the square grid we applied Theorem 5 with a hexagon and a circle (for \( n = 4 \) the optimum can be obtained from a Costas array). Exceptions are \( n = 3, 5, 8, 10, \) and 12, where we got a better bound by specific constructions. Table I summarizes the bounds we obtained for regular polygons and a circle in the square grid. We also consider circle in the hexagonal grid, but the main result in the hexagonal grid is a construction of an optimal DDC whose shape is a hexagon. This involves another interesting construction for a lattice tiling of another shape in the square grid and translation into the hexagonal grid. The same techniques can be used for any \( D\)-dimensional shape. Finally, we note that the problem is of interest also from discrete geometry point of view. Some similar questions can be found in [15].

### VI. Application in Error-Correction

Assume that we have a \( D\)-dimensional array of size \( n_1 \times n_2 \times \cdots \times n_D \) and we wish to correct any \( D\)-dimensional burst of length 2 (at most two adjacent positions are in error). The following construction given in [16] is based on folding the elements of a Galois field with characteristic 2 in a parity check matrix, where the order of the elements of the field is determined by a primitive element of the field.

**Construction A:** Let \( \alpha \) be a primitive element in \( GF(2^m) \) for \( 2^m - 1 \geq \prod_{i=1}^{D} n_i \). Let \( d = \log_2 D \) and \( i = (i_1, i_2, \ldots, i_D) \), where \( 0 \leq i_\ell \leq n_\ell - 1 \). Let \( A \) be a \( d \times D \)
matrix containing distinct binary \( d \)-tuples as columns. We construct the following \( n_1 \times n_2 \times \cdots \times n_D \times (m + d + 1) \) parity check matrix \( H \).

\[
H_1 = \begin{bmatrix}
\frac{A^T}{d} & 1 \\
\sum_{j=1}^{d} i_j (\prod_{i=1}^{d} n_i)
\end{bmatrix} \mod 2,
\]

for all \( i = (i_1, i_2, \ldots, i_D) \), where \( 0 \leq i_k \leq n_k - 1 \).

**Theorem 9:** The code constructed in Construction A can correct any 2-burst in an \( n_1 \times n_2 \times \cdots \times n_D \) array codeword.

**Theorem 10:** The code constructed by Construction A has redundancy which is greater by at most one from the trivial lower bound on the redundancy.

The same construction will work if instead of a \( D \)-dimensional array our codewords will have a shape \( S \) of size \( 2^m - 1 \) and there is a lattice tiling \( \Lambda \) and a direction \( \delta \) such that \((\Lambda, S, \delta)\) defines a folding. The elements of \( \text{GF}(2^m) \) will be ordered along the folded-row of \( S \). The construction can be generalized for more complicated types of multidimensional errors.

**VII. APPLICATION IN PSEUDO-RANDOM ARRAYS**

Let \( n = 2^{k_1}k_2 - 1 \) such that \( n_1 = 2^{k_1} - 1 \) and \( n_2 = \frac{n}{n_1} \) are relatively primes and greater than 1. Let \( S = s_0 s_1 \cdots s_{n-1} \) be a m-sequence (maximal length linear shift register sequence \([10],[17]\)) of length \( n \). Assume we use F1 to form an \( n_1 \times n_2 \) array \( \mathcal{A} \). \( \mathcal{A} \) has many interesting properties such as shift, recurrences, addition, auto-correlation, etc. \([10]\). It also has a \( k_1 \times k_2 \) window property, i.e., each \( k_1 \times k_2 \) possible binary matrix appears exactly once as a window in the cyclic array. These arrays were called in \([10]\) pseudo-random arrays. All these properties except for the window property are a consequence of the fact that the elements in the folded-row are consecutive elements of the m-sequence \( S \). Hence, if we use any of the folding operations to fold \( S \) into a \( D \)-dimensional shape \( S \), the shape \( S \) will have all these properties. As a consequence of Theorem 9, we have the following theorem.

**Theorem 11:** Assume \( \Lambda \) define a lattice tiling for an \( n_1 \times n_2 \) array, such that \( n_1n_2 = 2^{k_1}k_2 - 1 \), \( n_1, n_2 \) are relatively primes and greater than 1. Assume further that \( \Lambda \) defines a lattice tiling for the shape \( S \) and \((\Lambda, S, \delta)\) defines a folding for a direction \( \delta \). Then, if we fold an m-sequence \( S \) into \( S \) by the direction \( \delta \) then the resulting shape \( S \) has the \( k_1 \times k_2 \) window property if and only if the \( n_1 \times n_2 \) array \( \mathcal{A} \) has the \( k_1 \times k_2 \) window property by folding \( S \) into \( \mathcal{A} \) by the direction \( \delta \).

**VIII. CONCLUSION**

The well-known definition of folding was generalized. The generalization makes use of a lattice tiling and a direction in which the folding is performed. We demonstrated how folding in general and the new definition in particular is applied for constructions of multidimensional synchronization patterns, error-correcting codes, and pseudo-random arrays. The compressed discussion we have made raised lot of problems for further research, which can advance the research on multidimensional coding. It also raised intriguing questions which are related to discrete geometry.

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