Independence in Infinite Probabilistic Databases

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Probabilistic databases (PDBs) model uncertainty in data. The current standard is to view PDBs as finite probability spaces over relational database instances. Since many attributes in typical databases have infinite domains, such as integers, strings, or real numbers, it is often more natural to view PDBs as infinite probability spaces over database instances. In this article, we lay the mathematical foundations of infinite probabilistic databases. Our focus then is on independence assumptions. Tuple-independent PDBs play a central role in theory and practice of PDBs. Here we study infinite tuple-independent PDBs as well as related models such as infinite block-independent disjoint PDBs. While the standard model of PDBs focuses on a set-based semantics, we also study tuple-independent PDBs with a bag semantics and independence in PDBs over uncountable fact spaces.

We also propose a new approach to PDBs with an open-world assumption, addressing issues raised by Ceylan et al. (Proc. KR 2016) and generalizing their work, which is still rooted in finite tuple-independent PDBs.

Moreover, for countable PDBs we propose an approximate query answering algorithm.

CCS Concepts: • Theory of computation → Incomplete, inconsistent, and uncertain databases; • Information systems → Relational database model;

Additional Key Words and Phrases: Probabilistic databases, infinite probabilistic databases, independence assumptions, tuple-independent, block-independent disjoint

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1 INTRODUCTION

Probabilistic (relational) databases (PDBs) [76, 78] extend the relational database model by probability distributions to model uncertainty. Formally, a probabilistic database is a probability space over database instances of some schema. The database instances of a probabilistic database are then called its possible worlds.

Applications of probabilistic databases are, for example, the management of noisy sensor data [34], information extraction [40], data integration, and data cleaning [7, 33]. Detailed discussions of these and other applications of probabilistic databases can be found in References [5, 76].
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Fig. 1. A representation of a (tuple-independent) probabilistic database of orders at an online retailer. The “∼” indicates that the instance shown on the left is drawn at random from the PDB specified on the right.

1.1 Independence Assumptions

The most extensively studied class of PDBs is the class of (finite) tuple-independent probabilistic databases [27]. Therein, all facts (that is, events of the form "tuple t appears in relation R") are stochastically independent. The probabilities of these events are called marginal fact probabilities or marginals. Due to the independence, joint probabilities of facts can easily be computed by multiplying the respective marginals. Thus, the probability space is already uniquely determined by the marginal probabilities of the individual facts. In order to specify a tuple-independent PDB, it therefore suffices to give the marginal probabilities of all relevant facts.

Example 1.1 (Orders at an Online Retailer). This example is adapted from Reference [56]. Suppose we have a database with a single relation Order that stores information about the orders made at an online retailer and further suppose that the data are subject to uncertainty. We model this with a probabilistic database. For simplicity, we assume that the presence of different tuples is stochastically independent. Figure 1 depicts a database instance (on the left) drawn from such a tuple-independent PDB and the basic way to represent that PDB (on the right). The representation consists of a list of all possible facts, here Order(Joe, New York, 99), Order(Emma, Austin, 70), and Order(Dave, Atlanta, 19), together with their marginal probability, i.e., the probability of the respective fact to be present (here 0.8, 1.0, and 0.2, respectively).

Figure 1 is thus an encoding of a probabilistic database with $2^3$ possible worlds, and the probability of each world is given by the product of the probabilities of the present facts times the converse probabilities of the non-present facts. For this example, the instance shown on the left-hand side has probability $(1 - 0.8) \cdot 1.0 \cdot 0.2 = 0.04$.

The focus of theoretical work on independence assumptions, and on tuple-independence in particular has multiple reasons. First, probabilistic databases are non-trivial to represent: For finite probabilistic databases, if all facts are uncertain, then the number of possible worlds is exponential in the number of facts. Resorting to independence assumptions sacrifices expressive power to facilitate representation. Second, as a byproduct of the previous point, with independence assumptions, the probability spaces that need to be discussed have a very simple structure, and they are therefore readily accessible for theoretical investigation.

We note that the loss in expressive power is generally not as severe of an issue as it might seem and can be compensated for by adding additional mechanisms on top of the representations. For example, every finite probabilistic database can be represented as a first-order (or relational calculus) view of a tuple-independent PDB (see Reference [76]). While such a representation is infeasible for practical matters, we can often use constraints over tuple-independent PDBs to describe complex correlations more succinctly [53, 78]. In practice, some systems working with large amounts of uncertain data directly operate under the tuple-independence assumption, for example, Knowledge Vault [35], NELL [65], and DeepDive [85]. Even beyond tuple-independence, most existing PDB systems use independence assumptions at some point to span large probability spaces from
Fig. 2. A database of temperature recordings. The vertical dots are to indicate that the database contains many more (although finitely many) such entries.

independent building blocks [26], cf. References [4, 44, 76, 78]. This includes block-independent disjoint PDBs [28] but also more sophisticated representations [8, 82].

The uncertainty in probabilistic databases can come in various flavors (cf. References [76, 78]). Example 1.1 exemplifies a PDB with tuple-level uncertainty, where there is a number of possible tuples, but the presence of the individual tuples is subject to uncertainty. Under attribute-level uncertainty, there is a fixed number of present tuples, but for each of them one (or more) attribute values are subject to uncertainty (cf. the discussion in Example 1.2 below). In general, both types of uncertainty can co-occur.

1.2 From Finite to Infinite PDBs

In the literature on probabilistic databases, whenever PDBs are formally introduced, they are usually defined as a probability space over finitely many possible worlds. Then, only finitely many instances can have a non-zero probability and, in particular, only finitely many facts have a non-zero marginal probability.

This assumption is clearly not natural if facts are uncertain, but involve attributes ranging over infinite domains like the integers, real numbers, or strings. Even if implementations put restrictions on these domains (64-bit integers or floating point numbers, 256-character strings), the mathematical abstraction that we use to reason about such systems is based on the ideal infinite domains. For example, if a database records temperature measurements (or other sensor data), then the values are reals. We may want to allow for some noise in the data, naturally modeled by a normal distribution around the measured values, and this already gives us a simple example of a PDB with an infinite (even uncountable) sample space. We illustrate this situation with our next example and point out several problems that occur when we restrict ourselves to finite PDBs.

**Example 1.2 (Temperature Measurements).** Again, consider a database recording noisy temperature measurements as an example. Part of such a database instance is shown in Figure 2.

Consider the following two queries against this (non-probabilistic) database:

(Q1) *Has the temperature ever been between 20.2 °C and 20.5 °C?* Suppose that none of our finitely many facts represents a recorded temperature in this range. Then the query will return “false.”

(Q2) *Is the temperature in office 4108 always lower than the one in office 4109?* If this is supported by our finitely many facts, then the query will return “true.”

Given that the underlying data are noisy, we might want to model our database of temperature recordings as a probabilistic database, for simplicity, say, a tuple-independent one. Now reconsider (Q1) and (Q2) under the assumption that the data are modeled by such a PDB with finitely many possible facts.
(Q1) If the finitely many facts occurring in our PDB do not contain a recorded temperature between 20.2 and 20.5 °C, then the query (Q1) returns “false” with probability 1 (respectively “true” with probability 0).

(Q2) If for all temperature recordings occurring among the instances of, the values recorded in 4108 are always lower than all recordings from 4109, then the query (Q2) returns “true” with probability 1 (respectively “false” with probability 1).

However, both of these answers do not reflect what we would expect of the respective query on a stochastic model of the underlying uncertainty of the data. Instead, it seems more reasonable to assume that the events discussed above may have a high probability (after all, this is what is supported by the available data) but not exactly 1, accounting for imprecisions in measurement and potentially missing facts.

Note that even among the technically impossible events, there are distinctions in terms of plausibility. Similarly to the above, it is not reasonable to assume that the temperature in room 4108 never exceeds 23 °C with certainty. What would be reasonable instead is, for example, that a temperature slightly above 23 °C is more likely than the temperature exceeding 35 °C.

In general, conclusions drawn from the probabilistic database alone, due to the finite (closed-world) setting, diverge from what we would infer from the true underlying data, even in terms of possibility and impossibility of events.

One particular solution to the issues sketched here would be to replace the concrete temperature recordings with random variables that are distributed with a normal distribution whose mean is the original recording, together with a small variance. For example, we could replace the fact $\text{TempRec}(4108, 2021-07-01 8:00, 21.2)$ with a random fact $\text{TempRec}(4108, 2021-07-01 8:00, N(21.2, 0.1))$, where $N(21.2, 0.1)$ indicates a normally distributed random variable with mean 21.2 and variance 0.1. Note that such a PDB is basically an uncountably infinite block-independent disjoint PDB, with each block corresponding to one of the original facts.

Ceylan et al. [21] have already pointed out such issues with respect to the closed-world assumption of PDBs (that facts not mentioned by the representation have probability 0) before in the context of finite PDBs. Yet, their approach toward tackling them is still confined to a finite setting and therefore still exhibits the problems that we face in Example 1.2.

To date, there already exist a variety of practical probabilistic database systems, some of which are especially designed to support infinite domains. This includes MCDB/SimSQL [19, 52], PIP [56], Orion [71], and Trio [6, 82]. In particular, these systems can describe, and work with PDBs as the one we propose in Example 1.2. From the theoretical point of view, however, infinite probabilistic databases have lacked a general framework for a long time compared to the long history of their finite counterparts. To our knowledge, we were the first to lay such foundations in our work [46, 47].

Having a sound formal foundation for infinite PDBs is important, especially because its mathematical development proves to be far from trivial. A model of infinite PDBs needs to be consistent with our intuitions on the behavior of queries, and it should to include the usual finite model as a special case. The probability spaces we obtain may be uncountably infinite if the underlying attribute domains are. While in general the idea of query semantics is the same as in the finite setting, we then also have to pay attention to whether they are still well defined because of measurability issues. When building database systems, such foundational issues may seem remote, because in practice we are always dealing with finite approximations of the infinite space. Yet, it is desirable to have a semantics for such systems that goes beyond a specific implementation on a specific machine, and such a semantics will naturally refer to idealized infinite domains. Once we have such an idealized semantics, we can argue that a specific system adheres to it, approximately.
Our focus on tuple-independence stems from the central role of independence assumptions in theory literature for finite PDBs. Independence assumptions make PDBs easy to work with mathematically. Even more so, this is the case for infinite PDBs. This makes tuple-independent and block-independent disjoint PDBs a natural starting point for rigorous investigations of infinite PDBs.

Formally, infinite probabilistic databases are probability spaces whose sample space consists of infinitely many database instances. Even in an infinite PDB, each individual instance is finite; it is best here to think of database instances as finite sets or finite bags (or multisets) of facts. Thus, abstractly, PDBs are probability spaces over finite sets or finite bags. In probability theory, such probability spaces are known as point processes \[23, 24\].

1.3 Contributions
After carefully introducing the mathematical framework of infinite PDBs, in this article we focus on tuple-independence in infinite PDBs and on various generalizations of the tuple-independence assumption. The goal of our contribution is to broaden the understanding of independence assumptions in infinite PDBs by identifying their abstract structure and discussing it in settings of infinite set and bag PDBs.

We start by looking at countably infinite tuple-independent PDBs, which, like finite tuple-independent PDBs, can be specified by giving all the marginal fact probabilities \(P(f)\). We show that for a countable family \(F\) of facts, a tuple-independent PDB with fact probabilities \(\{P(f)\}_{f \in F}\) exists if and only if \(\sum_{f \in F} P(f) < \infty\) (Theorem 4.4). We identify independent superpositions (a notion from point process theory) as the mathematical abstraction underlying the construction of PDBs from smaller, independent PDBs. This facilitates reasoning about infinite PDBs with independence assumptions. We illustrate this by casting both countable tuple-independent and countable block-independent disjoint PDBs (PDBs made up from independent blocks of facts such that the facts within each block are mutually exclusive) as superposition constructions. This allows us to re-obtain our characterization for tuple-independent PDBs and obtain an analogous characterization of the existence of block-independent disjoint PDBs just by using the properties of superpositions.

Rather as a side note to our main story, our discussion of countable set PDBs is complemented by a few additional insights into the role of tuple-independent PDBs in terms of expressiveness and query answering. We show that in countably infinite PDBs, independence assumptions are more restrictive than their counterpart in the finite setting: While every finite probabilistic database is a first-order view over a tuple-independent one, this is not true in the countably infinite case. We also give insights into the computability of approximate query evaluation in countable tuple-independent PDBs: We show that query answering can be approximated with additive error by approximating infinite tuple-independent PDBs with finite ones. We prove that there can be no algorithm that achieves multiplicative approximations.

We use our countable tuple-independent PDBs to incorporate an open-world assumption. Extending the ideas in Reference [21], we construct potentially infinite open-world completions of a finite or infinite PDB. The key requirement is that the probability measure is faithfully extended: In an open-world completion of a PDB, the probability measure should coincide with the original one, when conditioned over the sample space of the original PDB.

Up to this point, we have treated PDBs with a set semantics, but we can extend our definitions and results to PDBs with a bag semantics. With the machinery of superpositions, it is relatively easy to prove a general existence result for tuple-independent PDBs with a bag semantics, where we simply combine the distributions of individual fact multiplicities using a superposition. When it comes to a treatment of PDBs with an uncountable sample space, a bag semantics turns out to be easier to handle. Even if one is only interested in PDBs with a set semantics, bag semantics is a useful (and to some extent necessary) intermediate step.
Note that a generalization PDBs to uncountable spaces is important, because in many applications we have real-valued attributes. The generalization of tuple-independence is not completely straightforward, because typically in an uncountable setting the individual fact probabilities will be zero. When dealing with uncountable PDBs, we build on the notion of standard PDBs introduced in References [47, 48]. Our treatment heavily draws from the mathematical theory of finite point processes [23] and the notion of completely random measures [58].

A conference version of this article, which contains the basic results for countable infinite PDBs, has been presented at the 38th ACM SIGMOD-SIGACT-SIGAI Symposium on Principles of Database Systems (PODS 2019) [46]. However, the uniform construction principle of PDBs from independent building blocks based on superpositions is new here. In a subsequent paper [47, 48], we developed a generic framework for uncountable PDBs, the so-called standard PDBs, and this framework allowed us to extend our theory of independent PDBs to the uncountable case in this article. Building on our work, Carmeli et al. [20] studied the power of tuple-independent PDBs as a representation system for countably infinite PDBs.

1.4 Related Work

The foundation of our work is the extensive literature on models for finite probabilistic databases [4, 44, 76, 78]. Dalvi et al. identified three facets of research in probabilistic data management [26]: semantics and representation, query evaluation, and user interface principles. Among these, our contribution mainly addresses semantics of probabilistic data in an infinite setting, with emphasis on independence assumptions. Concrete representation systems [44, 76], query evaluation and user interfaces are not the focus of our work but only occasionally touched upon. Note that discussions of independence assumptions are abundant in PDB literature (cf. References [44, 76, 78]). In the finite setting, such assumptions are exploited for concise representations of large probability spaces over database instances.

Incomplete databases [51, 79] are a non-probabilistic model of uncertain databases. As opposed to probabilistic databases, models of incomplete databases typically do not assume finite domains [44, 51, 79]. Essentially, a PDB augments an incomplete database with probabilities, and, thus, the problem of their representation is closely related to that of PDBs [44]. Most of the PDB literature assumes this model, referred to as the possible worlds semantics [26, 76].

There exist system-oriented approaches to PDBs that are able to handle continuous data, such as MCDB/SimSQL [19, 52], PIP [56], Orion [71], and Trio [6, 82]. Earlier, Dalvi et al. [26] noted the insufficient understanding of models for uncountable PDBs in terms of possible worlds semantics. Despite being over 10 years old, this statement is for the most part still valid today. The data model of Orion [71, 72] is worth mentioning, because it explicitly draws a connection from uncountable PDBs to the notion of possible worlds but only allows a bounded number of tuples per PDB. Measure-theoretic approaches to PDBs in terms of possible worlds semantics are scarce.

Also for probabilistic XML [3, 57], an infinite model covering continuous distributions has been introduced [1]. Yet, this approach leaves the document structure finite. On the contrary, in Reference [14], the authors propose an unbounded model of probabilistic XML that does not support continuously distributed data.

Query answering in PDBs is closely related to the problem of weighted model counting (WMC) [78], that is, to counting the models of a logical sentence in a weighted way. In recent work, the WMC problem was extended to infinite domains as well [12]. The idea of completions of probabilistic databases is introduced in OpenPDBs [21, 39] as a means to overcome problems arising from the closed-world assumption in probabilistic databases [67, 86] as lined out before. Reference [75] essentially describes a model of block-independent disjoint completions of a given incomplete data-

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base where the probabilities of the missing facts are inferred from the existing ones. In References [16, 17], ontologies are used to improve query results on PDBs.

Some related fields of research (in particular, artificial intelligence and machine learning [13, 31] and probabilistic programming [42]) have developed approaches toward infinite probabilistic data models before as modeling languages or systems such as BLOG [64] or Markov Logic Networks [69, 73, 80], as abstract data types [36], and various probabilistic programming languages as, for example, ProbLog [32, 50] and others [41, 66, 77]. On the side of programming languages, Probabilistic Programming Datalog [11, 45] has direct ties to PDBs. Programming with point processes for practical stochastic models has been recently cast into a monadic framework with category-theoretic foundations [30]. From a very abstract point of view, the basic model presented there is the model from Reference [47] that we use for uncountable PDBs in Section 5. A construction of probability spaces over sets of facts from a countably infinite product space of facts already appeared in the probabilistic programming community as distribution semantics [70].

Infinite relational structures also play a role in the discussion of limit probabilities in asymptotic combinatorics [15, 74]. For example, the classical Erdős–Rényi model \( G(n, p) \) is essentially "tuple independent". It describes a probability distribution over \( n \)-vertex graphs where every possible edge is drawn independently with probability \( p \). Usually, the focus lies on studying properties of this model as \( n \to \infty \). In such a model, properties of large graphs (or databases) dominate the observed behavior. This contrasts the infinite tuple-independence model we discuss here that is dominated by instances in the vicinity of its expected instance size (which is always finite for tuple-independent PDBs). There exists work studying limit probabilities in PDBs [25] and incomplete databases [22, 63].

1.5 Organization of the Article

After giving the necessary preliminaries in Section 2 (with additional background material being contained in the appendix), we introduce probabilistic databases (both finite and infinite) in Section 3. Section 4 contains our results on countable tuple-independent PDBs with a set semantics and generalizations such as block-independent disjoint PDBs. Specifically, in Section 4.2 we introduce countable superpositions. Section 4.8 is devoted to countable tuple-independent PDBs with a bag semantics. Finally, in Section 5 we consider tuple-independence in the general setting of (potentially uncountable) PDBs. We conclude with a few remarks and open questions in Section 6.

2 NOTATION AND MATHEMATICAL BACKGROUND

Throughout this article, \( \mathbb{N} \) denotes the set of non-negative integers and \( \mathbb{N}_{>0} \) denotes the set of positive integers. The set of real numbers is denoted by \( \mathbb{R} \). We denote open, half-open, and closed intervals of reals by \((a, b), [a, b), (a, b], \) and \([a, b] \).

We call a set or collection countable if it is finite or countably infinite. We denote the powerset of a set \( S \) (that is, the set of subsets of \( S \)) by \( \mathcal{P}(S) \). The finitary powerset of \( S \) (that is, the set of finite subsets of \( S \)) is denoted by \( \mathcal{P}_{\text{fin}}(S) \).

2.1 Bags and Sets

A bag (or multiset) is a pair \( B = (S_B, \#_B) \) where \( S_B \) is a set and \( \#_B \) is a function \( \#_B : S_B \to \mathbb{N} \cup \{ \infty \} \). We call \( B \) a bag over \( S_B \) and \( \#_B \) its multiplicity function. For \( S \subseteq S_B \), we let \( \#_B(S) := \sum_{s \in S} \#_B(s) \). The cardinality \( |B| \) of a bag \( B = (S_B, \#_B) \) is the sum of all its multiplicities, that is, \( |B| = \#_B(|B|) = \sum_{s \in S_B} \#_B(s) \in \mathbb{N} \cup \{ \infty \} \). A bag \( B \) is called finite if \( |B| < \infty \). The sets of all bags over some set \( S \) is denoted by \( \mathcal{B}(S) \). The set of all finite bags over \( S \) is denoted by \( \mathcal{B}_{\text{fin}}(S) \). We identify any bag \( B = (S_B, \#_B) \) where \( \#_B \) is \{0, 1\}-valued with the set \( \{ s \in S_B : \#_B(s) = 1 \} \).
Let $B_1 = (S_1, \#_1)$ and $B_2 = (S_2, \#_2)$ be bags. The additive union of $B_1$ and $B_2$ is the bag

$$B_1 \cup B_2 := (S_1 \cup S_2, \#_1 + \#_2)$$

with $(\#_1 + \#_2)(s) = \#_1(s) + \#_2(s)$ and the convention that $\#_i(s) = 0$ if $s \notin S_i$ for $i \in \{1, 2\}$. The additive union of bags is associative and commutative. Moreover, for every bag $B$ it holds that $B \cup \emptyset = B$ where $\emptyset$ is the empty bag. We write $\bigcup_{i=1}^n B_i$ for $B_1 \cup \cdots \cup B_n$ for all $n \in \mathbb{N}_{\geq 0}$ and bags $B_1, \ldots, B_n$. Any additive union of finitely many finite bags is finite.

### 2.2 Infinite Sums and Products

In comparison with finite probabilistic databases, the discussion of infinite probabilistic databases brings with it some new analytic challenges, as even in the most simple examples, we need to take sums and products over infinite index sets. We will therefore frequently encounter series $\sum_{i=0}^{\infty} a_i$ and infinite products $\prod_{i=0}^{\infty} a_i$ with terms $a_i \in [0, 1]$. All series as above attain well-defined values in $[0, \infty]$. If the value is finite, then the series is called convergent (and divergent otherwise). All products as above attain well-defined values in the interval $[0, 1]$. Moreover, all series and products in this article have the property that their value is independent of the order of terms. Appendix A.1 contains some more formal background on infinite series and products.

### 2.3 Probability and Measure Theory

Here in the preliminaries, we only cover the bare minimum background from probability theory. We refer to Appendix A.2 for the formal definitions and statements and pointers to textbooks.

A measure space is a tuple $(\Omega, \mathcal{A}, \mu)$ where $\Omega$ is some non-empty set, $\mathcal{A}$ is a $\sigma$-algebra$^1$ on $\Omega$, and $\mu : \mathcal{A} \to [0, \infty]$ is a function with the property that $\mu(\emptyset) = 0$ and $\mu(\bigcup_{i=0}^{\infty} A_i) = \sum_{i=0}^{\infty} \mu(A_i)$ whenever $A_1, A_2, \cdots \in \mathcal{A}$ are pairwise disjoint. This latter property of $\mu$ is called $\sigma$-additivity. The elements of $\mathcal{A}$ are called measurable sets. Measure spaces with $\mu(\Omega) = 1$ are called probability spaces. In this case, $\Omega$ is called the sample space, the sets in $\mathcal{A}$ are called events, and $\mu$ is called a probability measure. Probability measures are usually denoted by $P$.

Let $(\Omega, \mathcal{A}, P)$ be a probability space. We write

$$\Pr_{X \sim (\Omega, \mathcal{A}, P)} \{X \in A\} := P(A)$$

for the probability of the event $A \in \mathcal{A}$. If $\Phi : \Omega \to \{\text{true, false}\}$ is a measurable Boolean property (for example, given by a sentence in some logic, if $\Omega$ contains relational structures), then we write

$$\Pr_{X \sim (\Omega, \mathcal{A}, P)} \{\Phi(X)\} := \Pr_{X \sim (\Omega, \mathcal{A}, P)} \{X \text{ has property } \Phi\} = P(\{\omega \in \Omega : \Phi(\omega) = \text{true}\})$$

Events $A_0, A_1, \ldots$ in a probability space $(\Omega, \mathcal{A}, P)$ are called independent if the joint probability of any finite subset of these events is the product of their individual probabilities, that is, if $P(\bigcap_{i \in I} A_i) = \prod_{i \in I} P(A_i)$ for all finite $I \subseteq \mathbb{N}$. A family of events is independent if and only if their complements are. An important statement we need is the Borel–Cantelli Lemma [59, Theorem 2.7], which states that if $A_1, A_2, \cdots \in \mathcal{A}$ are events in a probability space $(\Omega, \mathcal{A}, P)$, then

$$\sum_{i=1}^{\infty} P(A_i) < \infty \Rightarrow P\left(\bigcap_{i=1}^{\infty} \bigcup_{j=i}^{\infty} A_j\right) = \Pr_{X \sim (\Omega, \mathcal{A}, P)} \{X \in A_i \text{ for inf. many } i\} = 0.$$

$^1$A $\sigma$-algebra on $\Omega$ is a family of subsets of $\Omega$ that contains $\Omega$ and is closed under countable unions and under taking complements.
If the $A_i$ are additionally pairwise independent, then
\[ \sum_{i=1}^{\infty} P(A_i) = \infty \Rightarrow P\left( \bigcap_{i=1}^{\infty} \bigcup_{j=i}^{\infty} A_i \right) = \Pr_{X \sim (\Omega, \mathcal{A})} \{ X \in A_i \text{ for inf. many } i \} = 1. \]

Given probability spaces $(\Omega_i, \mathcal{A}_i, P_i)$, $i = 1, 2, \ldots$, there exists a unique probability measure $P$ on $\prod_{i=1}^{\infty} \Omega_i$ (equipped with a suitable $\sigma$-algebra, that is, the product $\sigma$-algebra) such that the events $(\Omega_1 \times \cdots \times \Omega_{i-1} \times A_i \times \Omega_{i+1} \times \cdots )_{i \in \mathbb{N}_0}$ are independent, and have probability $P_i(A_i)$ where $A_i \in \mathcal{A}_i$.

### 2.4 Relational Databases

In this article, we consider the unnamed perspective on the relational model [2]. Let $\text{Rel}$ be some countably infinite set. The elements of $\text{Rel}$ are called relation symbols. We fix a function $\text{ar} : \text{Rel} \rightarrow \mathbb{N}$. Then, for all $R \in \text{Rel}$, $\text{ar}(R)$ is called the arity of $R$.

A database schema $\tau$ is a finite set of relation symbols. Let $\mathcal{U} \neq \emptyset$ be some set of arbitrary (positive) cardinality, called the universe or domain. Then a $(\tau, \mathcal{U})$-fact is an expression of the shape $R(u)$ where $u \in \text{ar}(R)$. We let $F[\tau, \mathcal{U}]$ denote the set of $(\tau, \mathcal{U})$-facts.

A database instance $D$ of schema $\tau$ over $\mathcal{U}$ (or, $(\tau, \mathcal{U})$-instance) is a finite bag of $(\tau, \mathcal{U})$-facts. We let $\text{DB}[\tau, \mathcal{U}]$ denote the set of all $(\tau, \mathcal{U})$-instances. If $D \in \text{DB}[\tau, \mathcal{U}]$ has only $\{0, 1\}$-valued fact multiplicities, then we call $D$ a set instance. Otherwise, $D$ is called a (proper) bag instance. We let $\text{DB}^{\text{set}}[\tau, \mathcal{U}]$ denote the set of all set instances of schema $\tau$ over $\mathcal{U}$. The active domain $\text{adm}(D)$ of a database instance $D$ is the restriction of $\mathcal{U}$ to the elements appearing in $D$.

### 3 PROBABILISTIC DATABASES

The following definition of possibly infinite, even uncountable probabilistic databases is the straightforward generalization of the traditional definition of PDBs as finite probability spaces. Besides the possibility of infinite probability spaces, it also allows for bag instances and treats set instances as a special case.

**Definition 3.1.** A PDB of schema $\tau$ over $\mathcal{U}$ (or, $(\tau, \mathcal{U})$-PDB) is a probability space $\mathcal{D} = (\text{DB}, \mathcal{D}, P)$ with $\text{DB} \subseteq \text{DB}[\tau, \mathcal{U}]$. The PDB $\mathcal{D}$ is called

- a set PDB, if $\text{DB} \subseteq \text{DB}^{\text{set}}[\tau, \mathcal{U}]$;
- a simple PDB, if it holds that $\Pr_{D \sim \mathcal{D}} \{ D \in \text{DB}^{\text{set}}[\tau, \mathcal{U}] \} = 1$.

A PDB $\mathcal{D} = (\text{DB}, \mathcal{D}, P)$ is called finite, countable, countably infinite, or uncountable if $\text{DB}$ has the corresponding cardinality. The class of all PDBs is denoted by $\text{PDB}$. The classes of set PDBs and simple PDBs are denoted by $\text{PDB}^{\text{set}}$ and $\text{PDB}^{\text{simple}}$, respectively. We use the subscripts "\( < \omega \)" and "\( \leq \omega \)" to refer to respective subclasses of finite and countably infinite PDBs.

If $\mathcal{D} = (\text{DB}, \mathcal{D}, P)$ is countable, then we always assume that $\mathcal{D} = \mathcal{P}(\text{DB})$ is the powerset $\sigma$-algebra on $\text{DB}$ and write $(\text{DB}, P)$ instead of $(\text{DB}, \mathcal{P}(\text{DB}), P)$. In principle, we can always restrict the sample spaces to an arbitrary subspace that carries all the probability mass. In particular, for countable PDBs, there is no need for any distinction between set and simple PDBs. In the uncountable, where measurability is not trivial anymore, it is mathematically more convenient to keep the general sample spaces. There, although happening with probability 0, an outcome drawn from a simple PDB may contain duplicates.

We note that for most applications involving uncountable PDBs, additional conditions on the $\sigma$-algebras are needed. For example, we would want that queries of typical database query language have a well-defined semantics. In order for this to be the case, the queries need to be measurable functions between PDBs, requiring the measurability of various kinds of events. While the...
Fig. 3. Representation of an (uncountably) infinite PDB.

| TempRec |
|---------|
| RoomNo | Time | Temp [°C] |
|.........|
| 4108    | 2021-07-01 8:00 | N(21.2, 0.1) |
| 4108    | 2021-07-01 14:00 | N(21.2, 0.1) |
| 4109    | 2021-07-01 8:00 | N(22.1, 0.1) |
| 4109    | 2021-07-01 14:00 | N(22.4, 0.1) |

From the point of view of probability theory, PDBs are special finite point processes. A point process is a random collection of points (usually allowing duplicates) in some measurable space \( \mathbb{X} \). A finite point process is a random finite collection of points. Point processes can be equivalently described as random integer-valued measures \( \mathbb{M} \).

### 3.1 Marginal Probabilities

Let \( D = (DB, \Xi, P) \) be a PDB. The set of facts appearing in \( DB \) is denoted by \( F(D) \). We say that \( D \) is a PDB with fact set (or, over) \( F(D) \). For every fact \( f \in F(D) \) with the property that \( \{ D \in DB : \#_D(f) > 0 \} \in \Xi \), the marginal probability of \( f \) in \( D \) is given by

\[
P(f) := \Pr_{D \sim D} \{ f \in D \} = P(\{ D \in DB : \#_D(f) > 0 \}).
\]

Potentially, we also want to include facts of marginal probability 0 in our sample spaces. As soon as we move to non-discrete distributions on uncountable spaces, this is necessary anyway. In fact, for uncountable PDBs it may happen naturally that all facts have marginal probability 0.

**Example 3.2.** Figure 3 depicts a representation of an uncountable set PDB for the temperature records example (Example 1.2). Formally, the fact space is

\[
F[\tau, \mathbb{U}] = \{\text{TempRec}(r, t, c) : r, t \in \Sigma^* \text{ and } c \in \mathbb{R}\},
\]

where \( \Sigma \) is some alphabet.

The probability measure of this PDB is given by the joint distribution of the four normally distributed random variables explicitly listed in Figure 3. Then, the marginal probability of any fact, say, for example, of the fact

\[
\text{TempRec}(4108, 2021-07-01 8:00, 21.2)
\]

is the probability of drawing a particular temperature value (here 21.2) from a normal distribution, hence, 0. Yet, with probability 1, there are exactly 3 facts in a randomly drawn instance.

Example 3.2 is closely related to PDB models based on attribute level uncertainty \([6, 72]\) that allow for continuously distributed attributes.

### 3.2 Expected Instance Size

Let \( D = (DB, \Xi, P) \) be a PDB. The **instance size function** of \( D \) is the function \( | \cdot | : DB \to \mathbb{N} \) that maps every instance \( D \in DB \) to its cardinality \( |D| \). If \( | \cdot | \) is measurable, that is, if \( \{ D \in DB : |D| = k \} \in \Xi \) for all \( k \in \mathbb{N} \), then \( | \cdot | \) is a random variable. Note that whether this is the case depends on the

---

2Point process theory is also from where we borrow the term “simple” for PDBs whose outcomes are set instances with probability 1.
σ-algebra $\mathcal{D}$ that $DB$ is equipped with in $\mathcal{D}$. If $\mathcal{D} = (DB, P)$ is a countable PDB (equipped with the powerset σ-algebra), then $|\cdot|$ is always a random variable.

If $|\cdot|$ is a random variable, then its expectation $E_\mathcal{D}(|\cdot|)$ in $\mathcal{D}$ is given as

$$E_\mathcal{D}(|\cdot|) = \sum_{n=0}^{\infty} n \cdot \Pr_{D \sim \mathcal{D}} \{ |D| = n \} = \sum_{n=0}^{\infty} n \cdot P \{ D \in DB : |D| = n \}. \quad (3.1)$$

Note that the above also holds if $\mathcal{D}$ is uncountable. (All we do is partition the range of the discrete size random variable.)

Note that for all $D \in DB$, it holds that $|D| = \#_D(F(\mathcal{D})) = \sum_{f \in F(\mathcal{D})} \#_D(f)$, where the latter sum has countably (indeed, finitely) many non-zero terms. If $F(\mathcal{D})$ is countable, then\(^7\) it holds that

$$E_\mathcal{D}(|\cdot|) = E_\mathcal{D}(\#_{\cdot}(F(\mathcal{D}))) = E_\mathcal{D} \left( \sum_{f \in F(\mathcal{D})} \#_{\cdot}(f) \right) = \sum_{f \in F(\mathcal{D})} E_\mathcal{D}(\#_{\cdot}(f)). \quad (3.2)$$

Moreover, for set PDBs, $\#_{\cdot}(f)$ is exactly the indicator random variable of the event $\{ f \in D \}$, $D \sim \mathcal{D}$. Thus, for countable set PDBs, Equation (3.2) entails that

$$E_\mathcal{D}(|\cdot|) = \sum_{f \in F(\mathcal{D})} E_\mathcal{D}(\#_{\cdot}(f)) = \sum_{f \in F(\mathcal{D})} P(f).$$

If $E_\mathcal{D}(|\cdot|) = m$ with $m \in \mathbb{R}_{\geq 0} \cup \{ \infty \}$, then we say that $\mathcal{D}$ is of expected size $m$. Although all database instances are finite, it is very easy to construct PDBs of infinite expected size. Intuitively, in such PDBs the sizes of instances grow too fast to be compensated by their probabilities.

**Example 3.3.** Let $P$ be any probability distribution on $\mathbb{N}$ with $P(n) > 0$ for infinitely many $n \in \mathbb{N}$. Suppose $\mathbb{U}$ is some countably infinite universe and $\tau = \{ \{ \} \}$ with $ar(\tau) = 1$. Let $D_0, D_1, D_2, \ldots$ be any sequence of pairwise distinct database instances over $\tau$ and $\mathbb{U}$ such that $|D_n| > \frac{1}{P(n)}$ and consider the PDB $\mathcal{D}$ with $DB = \{ D_0, D_1, D_2, \ldots \}$, $\mathcal{D} = \mathcal{P}(DB)$ and $Pr_{D \sim \mathcal{D}}(D = D_n) = P(n)$. Then

$$E_\mathcal{D}(|\cdot|) = \sum_{n=0}^{\infty} |D_n| \cdot P(n) > \sum_{n=0}^{\infty} 1 = \infty.$$ 

As a concrete example of this construction, take $P(n) = \frac{6}{\pi^2} n^2$ for all $n \in \mathbb{N}$. Note that this is indeed a probability distribution as $\sum_{n=0}^{\infty} \frac{1}{P(n)} = \frac{\pi^2}{6}$. Let $\mathbb{U} = \mathbb{N}$ and consider the instances $D_n = \{ R(1), \ldots, R(2^n) \}, n \in \mathbb{N}$. Then $E_\mathcal{D}(|\cdot|) = \sum_{n=0}^{\infty} 2^n \cdot \frac{6}{\pi^2 n^2} = \infty$.

### 3.3 Superpositions

(Independent) superposition is a standard operation of point processes [23, 24, 62] that we apply to PDBs. They provide a useful abstract tool that we can use to model how independence assumptions are cast into probabilistic databases from independent building blocks. In this section, for getting started, we give the basic idea for a superposition of two countable PDBs.

Suppose $\mathcal{D}_1 = (DB_1, P_1)$ and $\mathcal{D}_2 = (DB_2, P_2)$ are countable PDBs over $\tau$ and $\mathbb{U}$ (equipped with the powerset σ-algebras). The additive union $\cup$ of bags can be lifted to a function between PDBs in a straightforward way via $\mathcal{D}_1 \cup \mathcal{D}_2 := (DB[\tau, \mathbb{U}], P)$ where $P$ is defined by

$$P(\{D\}) := \sum_{D_1 \in DB_1, D_2 \in DB_2, D_1 \cup D_2 = D} P_1(\{D_1\}) \cdot P_2(\{D_2\}).$$

\(^7\)The expectation of a sum of countably many non-negative random variables is equal to the sum of the expectations, see Reference [59, Theorem 5.3(vi)].
This indeed defines a probability measure on $DB[\tau, \mathbb{U}]$, as

$$P(DB[\tau, \mathbb{U}]) = \sum_{D \in DB[\tau, \mathbb{U}]} P(\{D\}) = \sum_{D_1 \in DB_1} \sum_{D_2 \in DB_2} P_1(\{D_1\}) \cdot P_2(\{D_2\})$$

$$= \sum_{D_1 \in DB_1} P_1(\{D_1\}) \cdot \sum_{D_2 \in DB_2} P_2(\{D_2\}) = \sum_{D_1 \in DB_1} P_1(\{D_1\}) = 1.$$

Note that this relied on both $D_1$ and $D_2$ being discrete probability spaces. For uncountable PDBs, defining $P$ on singletons would not suffice (in fact, it could be that all singletons in both $D_1$ and $D_2$ have probability 0). The idea to extend this is to derive $P$ from the product measure of $P_1$ and $P_2$.

The PDB $D_1 \cup D_2$ is called the (independent) superposition of $D_1$ and $D_2$. The superposition of PDBs is associative. We write $\bigcup_{i=1}^{n} D_i$ for $D_1 \cup \cdots \cup D_n$.

Example 3.4. Consider the two PDBs $D_1$ and $D_2$ depicted below.

| $D$   | $\emptyset$    | $\{f\}$    | $\{f, f\}$ |
|-------|----------------|-------------|-------------|
| $P_1(\{D\})$ | $\frac{3}{4}$ | $\frac{1}{4}$ |             |
| $P_2(\{D\})$ | $\frac{3}{8}$ | $\frac{3}{8}$ | $\frac{2}{8}$ |

The following is the independent superposition of $D_1$ and $D_2$:

| $D$   | $\emptyset$    | $\{f\}$    | $\{f, f\}$ | $\{f, f'\}$ | $\{f, f, f\}$ | $\{f, f, f'\}$ |
|-------|----------------|-------------|-------------|--------------|---------------|---------------|
| $P(\{D\})$ | $\frac{9}{32}$ | $\frac{12}{32}$ | $\frac{6}{32}$ | $\frac{3}{32}$ | $\frac{4}{32}$ | $\frac{2}{32}$ |

Obviously, this is again a PDB.

Point process theory naturally considers also superpositions of countably many point processes [62] and they will naturally appear throughout this article. However, in general, the superposition "$\bigcup_{i=1}^{\infty} D_i$" of countably infinitely many PDBs $D_i$ may fail to be a PDB itself: In the "result," the multiplicity of a single fact could be infinite or there could be infinitely many different facts in a single "instance" with positive probability. This is investigated in detail in Section 4.2.

4 COUNTABLE PROBABILISTIC DATABASES

The main subject of this section is the investigation of generalizations of independence assumptions as they are known from finite PDBs [76] to countably infinite ones.

Throughout the whole section, we make the following assumptions:

(I) We only consider a fixed database schema $\tau$ and a fixed universe $\mathbb{U}$ of countable size.

(II) Whenever we consider sets $F$ of facts, then $F \subseteq F[\tau, \mathbb{U}]$. Consequentially, "facts" always means $(\tau, \mathbb{U})$-facts.

Moreover, in Sections 4.1–4.7 we make the following assumption:

(III) If not explicitly stated otherwise, all PDBs that occur in this section have sample space $P_{\text{fin}}(F(D))$ and are equipped with the powerset $\sigma$-algebra.

That is, unless explicitly stated otherwise, we use a set semantics.
4.1 Countable Tuple-Independence

With this subsection, we begin our investigation of independence assumptions in infinite PDBs by extending the well-known notion of tuple-independent PDBs toward PDBs over infinite sets of facts. We discuss the circumstances under which such PDBs exist and, if so, how to construct them by generalizing the finite construction in the natural way.

Definition 4.1 (Countable TI-PDBs). A PDB $\mathcal{D} = (DB, P)$ is called tuple-independent (or, a TI-PDB) if the events $\{D \in DB : f \in D\}$, for $f \in F(\mathcal{D})$, are independent, that is, if for all $k = 1, 2, \ldots$ and all pairwise different $f_1, \ldots, f_k \in F(\mathcal{D})$, then it holds that

$$\Pr_{D \sim \mathcal{D}} \{f_1, \ldots, f_k \in D\} = P(f_1) \cdot \ldots \cdot P(f_k). \tag{4.1}$$

The property from Equation (4.1) is referred to as independence across (or, of) facts. We denote the subclass of countable tuple-independent set PDBs of $PDB_{\leq \omega}$ by $TI_{\leq \omega}$. The class of finite tuple-independent set PDBs is denoted by $TI_{<\omega}$.

In the finite setting, a TI-PDB $\mathcal{D}$ may be specified by providing its facts $F := F(\mathcal{D})$ together with their marginal probabilities $(P(f))_{f \in F}$, and any such pair $(F, P)$ spans a unique finite TI-PDB. This is no longer the case in a countably infinite setting:

Example 4.2. Let $F \subseteq F[\tau, \mathbb{U}]$ be countably infinite and let $P : F \rightarrow [0, 1] : f \mapsto \frac{1}{2}$. Then there exists no TI-PDB with marginals according to $P$: Assume that $\mathcal{D} = (DB, P)$ is a TI-PDB with marginals $P$ and let $D \in DB$ be arbitrary. Suppose that the facts in $F$ are $f_1, f_2, \ldots$ and that $D \subseteq \{f_1, \ldots, f_k\}$ for some $k \in \mathbb{N}$. Recall that as the facts are independent, so are their complements. Then, for all $n \geq 1$, we have

$$P(\{D\}) \leq \Pr_{D \sim \mathcal{D}} (f_{k+1}, \ldots, f_{k+n} \notin D) = \prod_{i=k+1}^{k+n} (1 - P(f_i)) = \frac{1}{2}^n,$$

implying $P(\{D\}) = 0$. But then

$$1 = P(DB) = \sum_{D \in DB} P(\{D\}) = 0,$$

a contradiction.

We take the opportunity to comment on two subtleties with infinite PDBs in the light of this example:

1. In the above situation we have $E_D(| \cdot |) = \sum_{f \in F} P(f) = \infty$. We recall that this is, by itself, no contradiction to the requirement that all instances of the PDB be finite, see Example 3.3.

2. We argued that every single database instance has probability 0, and therefore, what we have is no probability space. This argument is only possible because $DB$ is countable, because then the probability of every event is given by the sum of the probabilities of the elementary events $\{D\}$ (due to $\sigma$-additivity). This does not hold for uncountable PDBs, such as the one sketched in Example 1.2.

In case of existence, however, the TI-PDB spanned by $F$ and $P$ is unique.

Proposition 4.3. Let $F \subseteq F[\tau, \mathbb{U}]$ be a countable set of facts and let $P : F \rightarrow [0, 1]$ with the property that there exists a TI-PDB $\mathcal{D}$ over $F$ with marginals $P$. Then $\mathcal{D} = (DB, P_\mathcal{D})$ is uniquely determined by $P$ and it holds that

$$P_\mathcal{D}(\{D\}) = \prod_{f \in D} P(f) \cdot \prod_{f \in F \setminus D} (1 - P(f)) \tag{4.2}$$

for all $D \in DB = \mathcal{P}_{\text{fin}}(F)$.
Note that the infinite product $\prod_{f \in F \setminus D} (1 - P(f))$ in Equation (4.2) is well defined, because all its factors are from the interval $[0, 1]$ (see Section 2.2).

**Proof.** If $F$ is finite, then the statement is clear. Let $F$ be countably infinite and let $D \in DB$. Suppose that $F = \{f_1, f_2, \ldots \}$ where $f_i \neq f_j$ for all $i \neq j$. For all $i = 1, 2, \ldots$ let $A_i$ be the set of database instances $D' \in DB$ with the property that $f_j \in D'$ if and only if $f_j \in D$ for all $j \leq i$. That is, $A_i$ is the set of instances that coincide with $D$ when restricted to $\{f_1, f_2, \ldots, f_i\}$. Then it holds that $A_1 \supseteq A_2 \supseteq \ldots$ and, moreover, that $\bigcap_{i=1}^{\infty} A_i = \{D\}$. Since $D$ is tuple-independent with marginals $P$, it holds that

$$P_D(A_i) = \prod_{(j \in \{1, \ldots, i\}: f_j \in D} P(f_j) \cdot \prod_{(j \in \{1, \ldots, i\}: f_j \notin D} (1 - P(f_j)).$$

Note that here we used, in particular, that in an independent family of events, the events are still independent when arbitrarily many of them are replaced by their complement. As $D$ is finite, it then follows from $A_1 \supseteq A_2 \supseteq \ldots$ that

$$P_D(\{D\}) = P_D(\bigcap_{i=1}^{\infty} A_i) = \prod_{j \in \mathbb{N}_{\geq 0}: f_j \in D} P(f_j) \cdot \prod_{j \in \mathbb{N}_{\geq 0}: f_j \notin D} (1 - P(f_j))$$

(see Fact A.3). That is, Equation (4.2) holds for all $D \in DB$. Due to the $\sigma$-additivity of $P_D$, and since $DB = \mathcal{P}_{\text{fin}}(F)$, Equation (4.2) implies that $P_D$ is uniquely determined. \hfill $\square$

We denote the unique $\textbf{T1}$-PDB over $F$ with marginal probabilities $P$ (in case of existence) by $\mathcal{D}_{(F, P)}$. This notation is slightly redundant as $F$ is already provided by the domain of $P$, but we deem it beneficial to make the set of facts explicit. We say that $\mathcal{D}_{(F, P)}$ is spanned by $F$ and $P$. Slightly abusing notation, we also denote the probability measure of $\mathcal{D}_{(F, P)}$ by $P$.

It remains to characterize which marginal probability assignments span countably infinite $\textbf{T1}$-PDBs.

**Theorem 4.4.** Let $F \subseteq F[\tau, \mathbb{U}]$ be a countable set of facts and let $P: F \rightarrow [0, 1]$. Then there exists a unique $\textbf{T1}$-PDB spanned by $F$ and $P$ if and only if

$$\sum_{f \in F} P(f) < \infty.$$

**Proof.** The result is trivial for finite $F$. Thus, we let $F$ be countably infinite and fix an enumeration $f_1, f_2, \ldots$ of $F$. We start with the direction from left to right. Suppose $\mathcal{D} = (DB, P)$ is the $\textbf{T1}$-PDB over $F$ with marginals $P$ and assume that $\sum_{i=1}^{\infty} P(f_i) = \infty$. Because every instance in $DB = \mathcal{P}_{\text{fin}}(F)$ is a finite set of facts, it holds that

$$\bigcap_{i=1}^{\infty} \bigcup_{j=i}^{\infty} \{D \in DB: f_j \in D\} = \emptyset.$$

But then, according to the Borel–Cantelli Lemma,

$$0 = P(\emptyset) = P\left(\bigcap_{i=1}^{\infty} \bigcup_{j=i}^{\infty} \{D \in DB: f_j \in D\}\right) = 1,$$

a contradiction. Thus, $\sum_{i=1}^{\infty} P(f_i) < \infty$. 

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For the other direction, let \( P : F \to [0,1] \) such that \( \sum_{f \in F} P(f) < \infty \). Again, let \( f_1, f_2, \ldots \) be an enumeration of \( F \). For all \( D \in DB := \mathcal{P}_{\text{fin}}(F) \) we define
\[
P(\{D\}) = \prod_{f \in D} P(f) \cdot \prod_{f \notin F \setminus D} (1 - P(f)).
\]
Then \( P(\{D\}) \in [0,1] \) is well defined. Moreover, \( P \) uniquely extends to a measure on \((DB, \mathcal{P}(DB))\) defined by \( P(D) = \sum_{D \in DB} P(\{D\}) \). Abusing notation, we denote this extended measure by \( P \) and consider the measure space \((DB, P)\).

It remains to show that \( P \) is a probability measure and has the correct marginals. Now for \( n = 0, 1, 2, \ldots \), consider \( DB_n := \mathcal{P}([f_1, \ldots, f_n]) \), i.e., the set of instances made up exclusively from the facts in \( F_n := \{f_1, \ldots, f_n\} \). Note that for all \( n \geq 0 \), \( DB_n \subseteq DB_{n+1} \) and that \( \bigcup_{n=0}^{\infty} DB_n = DB \), so in particular \( P(DB) = \lim_{n \to \infty} P(DB_n) \) (cf. Fact A.3). We have
\[
P(DB_n) = P(\{D \in DB : D \subseteq F_n\})
= \sum_{F \subseteq F_n} \prod_{f \in F} P(f) \cdot \prod_{f \notin F \setminus F_n} (1 - P(f))
= \prod_{f \notin F_n} (1 - P(f))
\]
By Equation (1.1), it follows that
\[
P(DB) = \lim_{n \to \infty} P(DB_n) = \lim_{n \to \infty} \prod_{f \notin F_n} (1 - P(f)) \geq 1 - \lim_{n \to \infty} \sum_{f \notin F_n} P(f) = 1.
\]
As \( P(DB_n) = \prod_{f \notin F_n} (1 - P(f)) \leq 1 \) for all \( n \geq 0 \), this implies \( P(DB) = 1 \).

Thus, \( P \) is a probability measure on \( DB \). It remains to show that \((DB, P)\) has the correct marginals. Recalling that \( DB = \mathcal{P}_{\text{fin}}(F) \), we see that
\[
\Pr_{D \sim D} \{f \in D\} = \sum_{D : f \in D} \prod_{g \in D} P(g) \cdot \prod_{g \notin D} (1 - P(g))
= P(f) \cdot \left( \sum_{D \in \mathcal{P}_{\text{fin}}(F(f))} \prod_{g \in D} P(g) \cdot \prod_{g \notin D} (1 - P(g)) \right) \cdot \left( P(f) + (1 - P(f)) \right)
= P(f) \cdot \sum_{D \in DB} P(\{D\})
= P(f),
\]
as desired. □

Recall from Section 3.2 that \( \sum_{f \in F(D)} P(f) \) is exactly the expected instance size \( E_D(\| \cdot \|) \) in \( D \). This immediately yields the following:

**Corollary 4.5.** If \( D = (DB, P) \) is a \( T1 \)-PDB, then
\[
E_D(\| \cdot \|) = \sum_{D \in DB} |D| \cdot P(\{D\}) < \infty.
\]
In particular, if \( D \) has infinite expected instance size, then \( D \) is not tuple-independent. We remark that Corollary 4.5 can be generalized to show that countable TI-PDBs have all moments \( E_D(|D|^k) \) of the instance size random variable finite [20].

A nice structural property of TI-PDBs is their modular setup: When conditioning a TI-PDB on a smaller set of facts, one again obtains a TI-PDB.

Let \( D = (DB, P) \) be a PDB and let \( D \subseteq DB \) such that \( P(D) > 0 \). Then \( D \mid D \) is the PDB with sample space \( D \) and probability measure \( P_D : D \rightarrow [0, 1] \) with
\[
P_D(\{D\}) = P(\{D\} | D) = \frac{P(\{D\})}{P(D)}
\]
for all \( D \in D \).

**Lemma 4.6.** Let \( D = D_{(F, P)} \) be a TI-PDB. Let \( F' \subseteq F \) such that \( P(P_{\text{fin}}(F')) > 0 \) and let \( P' \) denote the restriction of \( P \) to \( F' \).

Then it holds that
\[
D \mid P_{\text{fin}}(F') = D_{(F', P')}.
\]

In particular, \( D \mid P_{\text{fin}}(F') \) is a TI-PDB.

**Proof.** Let \( D = P_{\text{fin}}(F') \), let \( D = (DB, P) \) and let \( D \mid D = (DB_D, P_D) \). It follows from Proposition 4.3 that \( P(D) = \prod_{f \in F \setminus F'} (1 - P(f)) \) and thus, for all \( D \in D \) it holds that
\[
P_D(\{D\}) = \frac{P(\{D\})}{P(D)} = \frac{\prod_{f \in D} P(f) \cdot \prod_{f \in F \setminus D} (1 - P(f))}{\prod_{f \in F \setminus F'} (1 - P(f))} = \prod_{f \in D} P(f) \cdot \prod_{f \in F' \setminus D} (1 - P(f)) = P'(\{D\}).
\]

The claim then follows, since \( D_{(F', P')} \) is unique by Proposition 4.3. \( \square \)

The aforementioned “modularity” of TI-PDBs will be the central point of view of Section 4.3 where we investigate this property in detail.

**Remark 4.7.** In Reference [46, Definition 4.1], we introduced the following definition of tuple-independence in infinite PDBs: A countable PDB \( D \) be called tuple-independent if for all families \( F \) of pairwise disjoint (measurable) sets of facts it holds that \( \{D \cap F \neq \emptyset\} \}_{F \in \mathcal{F}} \) is independent, where \( D \sim D \). For countable PDBs, this is equivalent to Definition 4.1 (see Reference [46, Lemma 4.2]); on a high level, this easily follows from Reference [59, Theorem 2.13(iv)]. Definition 4.1 in Reference [46] only yields benefits when discussing uncountable PDBs, as there the marginal events associated to individual facts do not suffice to describe the probability spaces. We revisit the more general definition when we discuss uncountable PDBs in Section 5.

### 4.2 Countable Superpositions

In this and the following section, we present a more abstract view on the mechanisms of independence assumptions by explaining TI-PDBs via superpositions (see Section 3.3). The significance of this is that it allows us to discuss classes of PDBs that are constructed from independent parts already with the properties of superpositions at hand.

In this section in particular, we start by extending the discussion from Section 3.3 to countable superpositions with a strong focus on their application for PDBs.

Note that the results presented in this subsection are special cases of well-known propositions in point process theory [23, 24, 62]—our contribution here is to adapt these mechanisms for casting probabilistic databases into a unified framework.

Throughout this subsection, we fix a family \( (\mathcal{D}_i)_{i \in \mathbb{N} \cup 0} \) of PDBs \( \mathcal{D}_i = (DB_i, P_i) \in \text{PDB}_{\leq 0} \). Recall that by Assumptions (I) and (II) from the beginning of the section, we have that \( DB_i \subseteq DB[\tau, U] \).
for all $i = 1, 2, \ldots$, and some common database schema $\tau$ and countable universe $U$. Yet, we temporarily deviate from Assumption (III) and allow the occurring PDBs to be bag PDBs.

We now present an example that can serve as a running example throughout this section and that nicely exhibits the way we present TI-PDBs using superpositions in Section 4.3.

Example 4.8 (A Coin Flip PDB). We want to accentuate the intuition on how a tuple-independent PDB is made up from a large number of small independent events by connecting them to a coin flip experiment. For this, suppose that each fact in a TI-PDB acts as a biased coin. If the coin flips head, then it means the fact is present, and if it flips tail, then the fact is omitted.\(^4\)

We cast this intuition into a concrete TI-PDB of coin flips. Suppose we flip a countably infinite number of independent, biased coins such that the $i$th coin flips head with probability $p_i$. Let $\tau$ be the relational schema consisting of a single relation $H$ of arity 1 and let $U = \mathbb{N}$. We interpret the fact $f_i := H(i)$ as the event that the $i$th coin comes up heads. Note that $F := F[\tau, U] = \{f_i : i \in \mathbb{N}\}$. We define $P_i : F \rightarrow [0, 1]$ by $P_i(f_i) := p_i$ and $P_i(f) := 0$ for $f \neq f_i$, and we let $D_i = \tau[F, P_i]$.\(^5\)

Our goal is to express the TI-PDB $D_{(F,P)}$ with $P$ defined by $P(f_i) = P_i(f_i) = p_i$ as an independent superposition of the $D_i$. This intuitively corresponds to the observation that we obtain the same probability space, regardless of whether we flip the coins individually, or all coins together as a whole.

We write $DB^\otimes$ for the set of all sequences $\tilde{D} = (D_1, D_2, \ldots)$ where $D_i \in DB_1$ and consider the product measure space $\bigotimes_{i=1}^\infty D_i = (DB^\otimes, \mathfrak{D}^\otimes, P^\otimes)$ where $\mathfrak{D}^\otimes = \bigotimes_{i=1}^\infty P(DB_1)$. Note that this is a probability space over sequences of database instances. We now use this space to construct a measure space on database instances. Let $DB^\otimes_{\text{fin}}$ denote the set of sequences $(D_1, D_2, \ldots) \in DB^\otimes$ where all but finitely many $D_i$ are the empty instance. Note that $DB^\otimes_{\text{fin}}$ only contains countably many sequences.\(^5\) For all $(D_1, D_2, \ldots) \in DB^\otimes_{\text{fin}}$ it holds that

$$\{\langle D_1, D_2, \ldots, D_i, \ldots \rangle \} = \bigcap_{i=0}^\infty \pi_i^{-1}(\{D_i\}) \in \mathfrak{D}^\otimes,$$

where $\pi_i$ denotes the canonical projection to the $i$th component. As $DB^\otimes_{\text{fin}}$ is countable, Equation (4.3) implies $\mathfrak{D}^\otimes_{\text{fin}} := P(DB^\otimes_{\text{fin}}) \subseteq \mathfrak{D}^\otimes$. Thus, $D^\otimes_{\text{fin}} := (DB^\otimes_{\text{fin}}, P(DB^\otimes_{\text{fin}}), P^\otimes_{\text{fin}})$, where $P^\otimes_{\text{fin}}$ is the restriction of $P^\otimes$ to $\mathfrak{D}^\otimes_{\text{fin}}$ is a measure space. Note that it is not necessarily a probability space.

Example 4.9. Let us revisit the setting of Example 4.8. To define the product measure, we consider events $A$ of the form $\prod_{i=1}^\infty A_i$, where $A_i \neq DB_1 = \{\emptyset, \{f_i\}\}$ for only finitely many $i$ and define

$$P^\otimes(A) = \prod_{i \in N_{>0} : A_i = \{f_i\}} p_i \cdot \prod_{i \in N_{>0} : A_i = \emptyset} (1 - p_i).$$

The sets $A$ generate the product $\sigma$-algebra $\mathfrak{D}^\otimes$ on $DB^\otimes$, and (4.4) uniquely determines the product measure $P^\otimes$. The definition (4.4) implies that for all $\tilde{D} = (D_1, D_2, \ldots) \in DB^\otimes$ we have

$$P^\otimes(\{\tilde{D}\}) = \prod_{i \in N_{>0} : D_i = \{f_i\}} p_i \cdot \prod_{i \in N_{>0} : D_i = \emptyset} (1 - p_i).$$

\(^4\)If the number of coins is finite (as would be the analogy for finite TI-PDBs), then such experiments are called Poisson trials in probability theory literature [37, p. 218] and the distribution of the number of successes is called Poisson-binomial distribution or Poisson’s binomial distribution, cf. Reference [81]. Here we consider infinitely many coins.

\(^5\)As the $DB_1$ are countable, for every $n \in \mathbb{N}$ it holds that the set $DB^\otimes_{\text{fin}}$ contains only countably many sequences where exactly $n$ instances are non-empty. The (countable) union over all these sets of instances is $DB^\otimes_{\text{fin}}$. 

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It may well be that $P^\otimes(\{\tilde{D}\}) = 0$ for all $\tilde{D} \in DB^\otimes$. By an application of the Borel–Cantelli Lemma or by a direct calculation, it can be shown that $P^\otimes(\{\tilde{D}\}) = 0$ for all $\tilde{D} \in DB^\otimes_{\text{fin}}$ if and only if $\sum_{i=1}^\infty p_i = \infty$.

Recall that $B(S)$ denotes the set of all bags over some set $S$. Consider the additive union

$$(\cdot)^\otimes : DB^\otimes \to B\left(\bigcup_{i=1}^\infty F_i\right) : (D_1, D_2, \ldots) \mapsto (D_1, D_2, \ldots)^\otimes := \bigcup_{i=1}^\infty D_i,$$

where $F_i := F(\mathcal{D}_i) \subseteq F[\tau, \mathbb{U}]$ for all $i = 1, 2, \ldots$.

We note that the construction presented above would work exactly the same way if the PDBs $\mathcal{D}_1, \mathcal{D}_2, \ldots$ were uncountable. The only real change is that $\mathcal{B}_0^\otimes$ then becomes the product $\sigma$-algebra of the $\sigma$-algebras of $\mathcal{D}_1, \mathcal{D}_2, \ldots$ (and needing that the singleton containing the empty instance is measurable in all of them). Due to our application on countable PDBs here (and some intricacies in the uncountable with respect to set semantics), we refrain from this greater generality.

**Lemma 4.10.** Let $\tilde{D} \in DB^\otimes$.

1. The bag $\tilde{D}^\otimes$ is a database instance if and only if $\tilde{D} \in DB^\otimes_{\text{fin}}$.
2. If $\tilde{D}^\otimes$ is a database instance, then it is a set instance if and only if $\tilde{D}$ consists of pairwise disjoint set instances.

**Proof.** (1) If $\tilde{D} = (D_1, D_2, \ldots) \in DB^\otimes_{\text{fin}}$, then $D_i \neq \emptyset$ for at most finitely many $i = 1, 2, \ldots$. Thus, $\tilde{D}^\otimes$ is a database instance.

If, on the contrary, $\tilde{D} = (D_1, D_2, \ldots) \in DB^\otimes \setminus DB^\otimes_{\text{fin}}$, then $D_i \neq \emptyset$ for infinitely many $i \in \mathbb{N}_{>0}$. Thus, $|\tilde{D}^\otimes| = \infty$. In particular, $\tilde{D}^\otimes$ is not a database instance.

(2) Now suppose $\tilde{D}^\otimes$ is a database instance. Then by (1), $\tilde{D} \in DB^\otimes_{\text{fin}}$. On the one hand, if $D_1, D_2, \ldots$ are pairwise disjoint set instances among which only finitely many are non-empty, then their additive union is clearly a set instance as well. If, on the other hand, $D_1, D_2, \ldots$ contains either a proper bag instance or at least two of its components have a common fact, then $\tilde{D}^\otimes$ is a proper bag instance. □

We now define $\bigcup_{i=1}^\infty \mathcal{D}_i$ as the image measure space of $\mathcal{D}^\otimes_{\text{fin}}$ under $(\cdot)^\otimes$. That is, $\bigcup_{i=1}^\infty \mathcal{D}_i = (DB, P)$ with

$$DB = B_{\text{fin}}\left(\bigcup_{i=1}^\infty F_i\right) \quad \text{and} \quad P(\{D\}) = P^\otimes\left(\{\tilde{D} \in DB^\otimes_{\text{fin}} : \tilde{D}^\otimes = D\}\right).$$

Because the restriction of $(\cdot)^\otimes$ to $DB^\otimes_{\text{fin}}$ is measurable, this also makes $\bigcup_{i=1}^\infty \mathcal{D}_i$ a measure space. Again, however, it is not necessarily a probability space.

**Example 4.11.** We continue from Example 4.9. As in this example, for $i \neq j$ and $D_i \in DB_i$, $D_j \in DB_j$, we have $D_i \subseteq \{f_i\}$, $D_j \subseteq \{f_j\}$ and therefore $D_i \cap D_j = \emptyset$, by Lemma 4.10 all instances in $\bigcup_{i=1}^\infty \mathcal{D}_i$ are set instances, and we have $DB = P_{\text{fin}}(F)$. Note that for every instance $D \in DB$ the set $\{D | \tilde{D}^\otimes = D\}$ consists of a single sequence $\tilde{D} = (D_1, D_2, \ldots)$ with $D_i = \{f_i\}$ if $f_i \in D$ and $D_i = \emptyset$ otherwise. Thus, by Equation (4.5),

$$P(\{D\}) = \prod_{i \in \mathbb{N}_{>0} : f_i \in D} p_i \cdot \prod_{i \in \mathbb{N}_{>0} : f_i \notin D} (1 - p_i).$$
Note that this is exactly the probability \( \{D\} \) would have in a TI-PDB with fact probabilities \( P(f_i) = p_i \). Of course such a TI-PDB only exists if \( \sum_i p_i < \infty \).

**Remark 4.12.** If the measure space \( \bigcup_{i=1}^\infty D_i \) is a probability space, then it is a bag PDB. If this is the case, but additionally all \( D_i \) are set PDBs with disjoint fact sets, then we can proceed as follows to really obtain set PDBs instead of simple bag PDBs: We treat \( (\cdot)^\triangledown \) as a function into \( \mathcal{P}(\bigcup_{i=1}^\infty F_i) \) and let \( DB = \mathcal{P}_\infty(\bigcup_{i=1}^\infty F_i) \). Then \( \bigcup_{i=1}^\infty D_i \) is a set PDB.

Generalizing the previous example, our tentative goal is to express arbitrary TI-PDBs as the superposition of single fact PDBs. In particular, the superposition should have the desired independence properties, and the correct marginal probabilities. This is prepared with the following lemma.

**Lemma 4.13.** Suppose \( \mathcal{D} \coloneqq \bigcup_{i=1}^\infty D_i = (DB, P) \) is a PDB. For all \( i = 1, 2, \ldots \) let \( D_i \subseteq DB_i \) such that \( D \cap F_j = \emptyset \) for all \( D \in D_i \) and all \( j \neq i \). Let

\[
\tilde{D}_i \coloneqq DB_{\infty}^\cap \cap_{i=1}^\infty (D_i) = \left\{ \sum_{D \in DB_{\infty}^\cap \cap_{i=1}^\infty (D_i)} \mathbb{1}(D, D) \right\}. 
\]

Then

1. \( P(\tilde{D}_i) = P_i(D_i) \) for all \( i = 1, 2, \ldots \), and
2. the family \( (\tilde{D}_i)_{i \in \mathbb{N}_{>0}} \) is independent in \( \bigcup_{i=1}^\infty D_i \).

Note that the precondition of Lemma 4.13 states that the instances in \( D_i \) do not appear among the instances of the other PDBs \( D_j \), nor can they be assembled from them (apart from the empty instance). That is, the preimage of \( D_i \) under \( (\cdot)^\triangledown \) is exactly \( DB_{\infty}^\cap \cap_{i=1}^\infty (\prod_{j=1}^i DB_j \times D_i \times \prod_{j=i+1}^\infty DB_j) \).

**Proof.**

1. Let \( i \in \mathbb{N}_{>0} \). Then

\[
P(\tilde{D}_i) = P(\cap_{D \in DB_{\infty}^\cap \cap_{i=1}^\infty (D_i)} \cap_{D \in D_i} \tilde{D}_i)
= P(\cap_{D \in DB_{\infty}^\cap \cap_{i=1}^\infty (D_i)} \cap_{D \in D_i} \tilde{D}_i)
= P(\cap_{D \in DB_{\infty}^\cap \cap_{i=1}^\infty (D_i)} \cap_{D \in D_i} \tilde{D}_i) = P_i(D_i).
\]

For the second equality, we used the property discussed right before the proof. The last line uses that \( P(\cap_{D \in DB_{\infty}^\cap \cap_{i=1}^\infty (D_i)} \cap_{D \in D_i} \tilde{D}_i) = 0 \).

2. Let \( k \in \mathbb{N}_{>0} \) and let \( i_1, \ldots, i_k \in \mathbb{N}_{>0} \). Let \( \bigcup_{i=1}^\infty D_i = (DB, P) \). It holds that

\[
P(\bigcap_{j=1}^k D_{i_j}) = P(\bigcap_{j=1}^k \pi_{i_j}^{-1}(D_{i_j})) = \prod_{j=1}^k P_i(D_{i_j}),
\]

where the first equality again uses the property from before the proof, and the middle equality is due to the properties of the product measure (Fact A.5). Because the above holds for arbitrary \( k \) and \( i_1, \ldots, i_k \) it follows that \( (\tilde{D}_i)_{i \in \mathbb{N}_{>0}} \) is independent in \( \bigcup_{i=1}^\infty D_i \). \( \square \)

We close the discussion of countable superpositions with the probabilistic counterpart of Lemma 4.10.

**Lemma 4.14.**

1. The measure space \( \bigcup_{i=1}^\infty D_i \) is a PDB if and only if \( \sum_{i=1}^\infty \Pr_{D_i \sim D_i} \{D_i \neq \emptyset\} < \infty \).
2. If \( \bigcup_{i=1}^\infty D_i \) is a PDB, then it is a set PDB if and only if all \( D_i \) are set PDBs with pairwise disjoint fact sets.
Theorem be a (countable) set of facts and $\otimes_{D^i} \otimes_{1}$ for arbitrary countable index sets $I$. For all $\rightarrow [\sim D$ is a PDB.  

$N_i$ for equal $0$ and therefore $\Pr \{ \cup_{i=1}^{\infty} D_i \}$ is no PDB.  

We return to our assumptions Assumptions $\rightarrow [\sim D$ and let $\pi < \{ \emptyset \}$ denote the restriction of $\Pr$ to the full extent. Recall that for every $\sim D$ exists if and only if $\exists \otimes_{D^i} \otimes_{1}$ exists and is a partition of $DB$. $\Pr$ is a probability measure on $DB_\infty^\circ$, so $\cup_{i=1}^{\infty} D_i$ is a PDB. 

For the other direction let $\sum_{i=1}^{\infty} D_i$ are (in particular pairwise) independent by the properties of the product measure (Fact $A.6$). Thus, from Equation (4.7) and the Borel–Cantelli Lemma, we get that $P_\otimes (DB_\infty^\circ) = 0$ and therefore $\cup_{i=1}^{\infty} D_i$ is no PDB. 

(2) This is a direct consequence of the second part of Lemma 4.10.  

Remark 4.15. Theorem 4.4 can be regarded as a special case of Lemma 4.14(1) for the $D_i$ being single-fact PDBs (cf. Examples 4.8–4.11). Comparing the proofs, we find that the proof Lemma 4.14(1) is based on the more advanced machinery of product measures, which enables us to apply the Borel–Cantelli Lemma where in the proof of Theorem 4.4 we gave an ad hoc construction of the finitary part of the product space. However, at their core the two proofs are very similar.

As with finite superpositions, where it can easily be checked by hand, the independent superposition does not depend on the order of the involved spaces whatsoever, because $\otimes$ is commutative. Thus, we allow superpositions $\cup_{i \in I} D_i$ for arbitrary countable index sets $I$. Finally, note that finite superpositions could have been introduced equivalently with a product space construction instead of the explicit description in Section 3.3.

4.3 Tuple-Independence via Superpositions

Let us now carry out the argument describing $\textbf{T}I$-PDBs in terms of superpositions in detail. We return to our assumptions Assumptions (I) to (III) to the full extent. Recall that for every $F \subseteq F[\tau, U]$ and $P: F \rightarrow [0, 1]$, the (unique) $\textbf{T}I$-PDB spanned by $F$ and $P$ is denoted by $D_{\langle F, P \rangle}$, provided that it exists. We show that $\textbf{T}I$-PDBs can be decomposed into arbitrary independent components.

Theorem 4.16. Let $F \subseteq F[\tau, U]$ be a (countable) set of facts and $P: F \rightarrow [0, 1]$. Suppose that $(F_i)_{i \in I}$ is a partition of $F$ and let $P_i$ denote the restriction of $P$ to $F_i$. 

(1) The $\textbf{T}I$-PDB $D_{\langle F, P \rangle}$ exists if and only if $\cup_{i \in I} D_{\langle F_i, P_i \rangle}$ is a PDB. 

(2) If $D_{\langle F, P \rangle}$ exists, then $D_{\langle F, P \rangle} = \bigcup_{i \in I} D_{\langle F_i, P_i \rangle}$.

Proof. We first show the left-to-right direction of (1) and then handle the other direction and (2) simultaneously.
Suppose that $\mathcal{D}_{\{F_i\}}$ exists. Then by Theorem 4.4 it holds that $\sum_{f \in F} P(f) < \infty$. In particular, $\sum_{f \in F_i} P_i(f) < \infty$ for all $i \in I$, so by Theorem 4.4, all $\mathcal{D}_{\{F_i\}} =: \mathcal{D}_i$ exist. Then it holds that
\[
\sum_{i \in I} \Pr_{D \sim \mathcal{D}_i} \{ D \neq \emptyset \} = \sum_{i \in I} \Pr_{D \sim \mathcal{D}_i} \bigcup_{f \in F_i} \{ f \in D \} \\
\leq \sum_{i \in I} \sum_{f \in F_i} \Pr_{D \sim \mathcal{D}_i} \{ f \in D \} = \sum_{i \in I} \sum_{f \in F_i} P_i(f) = \sum_{f \in F} P(f) < \infty.
\]
Thus, by Lemma 4.14, $\biguplus_{i \in I} \mathcal{D}_i$ is a PDB.

If $\biguplus_{i \in I} \mathcal{D}_{\{F_i\}} =: \mathcal{D}$ is a PDB, then in particular, $\mathcal{D}_{\{F_i\}}$ is a PDB for all $i \in I$. By Lemma 4.13, the marginal probability of $f \in F_i$ in $\mathcal{D}$ coincides with $P_i(f) = P(f)$. Moreover, the events $\{ \{ f \in D \} \}_f \in F_i$ are independent in $\mathcal{D}_{\{F_i\}}$ for all $i \in I$. From Lemma 4.13, it follows that the events $\{ \{ f \in D \} \}_f \in F$ are independent in $\mathcal{D}$. Together, $\mathcal{D}$ is a $\text{T1}$-PDB with fact set $F(\mathcal{D}) = F$ and marginals according to $P$. By uniqueness (Proposition 4.3), it follows that $\mathcal{D} = \mathcal{D}_{\{F_i\}}$.

Now for a set of facts $F$, a function $P : F \to \{0, 1\}$ and $f \in F$ we let $\mathcal{D}_f = (\text{DB}_f, P_f)$ denote the single fact PDB with
\[
\text{DB}_f = \{ \emptyset, \{ f \} \} \quad \text{and} \\
P_f(\{ D \}) = \begin{cases} 
P(f) & \text{if } D = \{ f \} \text{ and} \\
1 - P(f) & \text{if } D = \emptyset
\end{cases}
\]
for all $D \in \{ \emptyset, \{ f \} \}$. Applying Theorem 4.16 to these single-fact PDBs yields the desired connection between infinite $\text{T1}$-PDBs and superpositions.

**Corollary 4.17.** Let $F$, $P$ and $\{ \mathcal{D}_f \}_{f \in F}$ as above.

1. The $\text{T1}$-PDB $\mathcal{D}_{\{F_i\}}$ exists if and only if $\biguplus_{f \in F} \mathcal{D}_f$ is a PDB.
2. If $\mathcal{D}_{\{F_i\}}$ exists, then $\mathcal{D}_{\{F_i\}} = \biguplus_{f \in F} \mathcal{D}_f$.

Explaining $\text{T1}$-PDBs by superpositions suggests natural generalizations of independence assumptions for $\text{T1}$-PDBs. In Theorem 4.16, what would happen if we were to replace the PDBs $\mathcal{D}_{\{F_i\}}$ with PDBs that are not tuple-independent? In this case, we still have the independence properties of the superposition (Lemma 4.13). One example of such PDBs are block-independent disjoint PDBs, which are the subject of the next subsection.

### 4.4 Block-Independence and Beyond

Finite probabilistic databases are referred to as being *block-independent disjoint* if the space of facts can be partitioned into stochastically independent “blocks” such that database instances almost surely contain at most one fact from each block. The following example illustrates that key constraints naturally lead to block-independent disjoint PDBs.

**Example 4.18.** We recall the setting of Example 1.1 of a database of orders in a single relation Order. Now the attribute OrderID is the key of the relation. Worlds violating the key constraint, that is, containing multiple tuples with the same key, should have probability 0. A concrete example is given in Figure 4. The grouping and the additional horizontal lines indicate the blocks of the PDB.

The positive probability outcomes of the PDB from Figure 4 have two or three facts. The first one is either the fact belonging to Joe, or the one belonging to Bob. The second fact will always...
be the one belonging to Emma. The third fact is either one of the facts belonging to Dave, Sophia, and Isabella or not present at all (with probability $1 - 0.2 - 0.6 - 0.1 = 0.1$).

Recall the coin flip interpretation of a TI-PDB. In a BID-PDB, instead of flipping a coin, we independently roll a distinguished (multi-sided, unfair) die per block. The outcome of a die roll determines the fact drawn from the block (or whether not to include a fact from the block).

**Definition 4.19 (Countable BID-PDBs).** A PDB $\mathcal{D} = (DB, P)$ is called **block-independent disjoint** (or, a BID-PDB) if there exists a partition $\mathcal{B}$ of $F(\mathcal{D})$ such that

1. for all families $(f_B)_{B \in \mathcal{B}}$ of facts from distinct blocks $B \in \mathcal{B}$, the events $f_B \in D$ are independent, that is,
   $$\Pr_{D \sim \mathcal{D}}\{f_1, \ldots, f_k \in D\} = P(f_1) \cdot \ldots \cdot P(f_k)$$
   for all $k = 1, 2, \ldots$ where $f_1, \ldots, f_k$ are from pairwise distinct blocks; and
2. for all $B \in \mathcal{B}$, the events $f \in B$ are pairwise disjoint, that is,
   $$\Pr_{D \sim \mathcal{D}}\{f, f' \in D\} = 0$$
   for all $B \in \mathcal{B}$ and all $f, f' \in B$.

Note that the blocks of a BID-PDB are unique up to facts of marginal probability 0. The **canonical block partition** of a BID-PDB is the block partition, where all facts with marginal probability 0 are grouped in a single distinguished block. We denote this partition $\mathcal{B}(\mathcal{D})$. Similarly to the TI-PDB case, the probability space of a BID-PDB is uniquely determined by the marginal probabilities. In case of existence, we denote the BID-PDB with blocks $\mathcal{B}$ and marginal probabilities $P$ by $\mathcal{D}(\mathcal{B}, P)$.

The probability measure of this PDB is as follows:

$$\Pr_{D \sim \mathcal{D}(\mathcal{B}, P)}\{\{D\}\} = \begin{cases} \prod_{B \in \mathcal{B}} P(f_B) \cdot \prod_{B \in \mathcal{B}} (1 - \sum_{f \in B} P(f)) & \text{if } |D \cap B| \leq 1 \text{ for all } B \in \mathcal{B} \\ 0 & \text{otherwise,} \end{cases} \quad (4.8)$$

where in the first case $f_B$ denotes the unique fact in $D \cap B$.

A (countable) PDB $\mathcal{D}$ is called **functional**, if $\Pr_{D \sim \mathcal{D}}\{f, f' \in D\} = 0$ for all facts $f \neq f'$.\footnote{In other words, a PDB is functional, if all possible worlds with positive probability are singletons.} Two PDBs $\mathcal{D}_1, \mathcal{D}_2$ are called **non-intersecting** if $F(\mathcal{D}_1) \cap F(\mathcal{D}_2) = \emptyset$. We say an instance $D$ intersects a block $B$ if $D$ contains a fact from $B$.\footnote{In other words, a PDB is functional, if all possible worlds with positive probability are singletons.}
**Theorem 4.20.**

(1) Every (countable) BID-PDB $\mathcal{D}$ is a superposition of non-intersecting functional PDBs, and the sum of all block probabilities is finite, that is

$$\sum_{B \in \mathcal{B}(\mathcal{D})} \Pr_{\mathcal{D} \sim \mathcal{D}} \{ D \text{ intersects } B \} < \infty.$$  

(2) For every family of functional, non-intersecting PDBs $\mathcal{D}_1, \mathcal{D}_2, \ldots$ with $\sum_{i=1}^{\infty} \Pr_{\mathcal{D} \sim \mathcal{D}_i} \{ D \neq \emptyset \} < \infty$, we can construct a BID-PDB $\mathcal{D}$ with the marginal probabilities from $\mathcal{D}_1, \mathcal{D}_2, \ldots$, that is, with

$$\Pr_{\mathcal{D} \sim \mathcal{D}_i} \{ f \in D \} = \Pr_{\mathcal{D} \sim \mathcal{D}_i} \{ f \in D \}$$

for all $f \in F(\mathcal{D}_i)$ and all $i = 1, 2, \ldots$.

**Proof.**

(1) Suppose $\mathcal{D} = \mathcal{D}(\mathcal{B}, \mathcal{P})$ where $\mathcal{B}$ is a block partition of some fact set $F \subseteq F[\mathcal{r}, \mathcal{U}]$ and $\mathcal{P}: F \to [0, 1]$ with $\sum_{f \in B} P(f) \leq 1$ for all $B \in \mathcal{B}$. For all $B \in \mathcal{B}$, recall that $\mathcal{D}_B := \mathcal{D} \mid P_{\text{fin}}(B)$ has $\Pr(\mathcal{D}_B) = B$ and

$$\Pr_{\mathcal{D} \sim \mathcal{D}_B} \{ f \in D \} = \frac{\Pr_{\mathcal{D} \sim \mathcal{D}} \{ D = \{ f \} \}}{\Pr_{\mathcal{D} \sim \mathcal{D}} \{ D \text{ does not intersect any } B^* \neq B \}} = P(f)$$

for all $f \in B$ (see Equation (4.8)). Also, the PDBs $\mathcal{D}_B$ are all functional and pairwise non-intersecting.

Because $\mathcal{D}$ is a BID-PDB, the events $\{ D \text{ intersects } B \}$ are independent in $\mathcal{D}$, and it holds that

$$\Pr_{\mathcal{D} \sim \mathcal{D}} \{ D \text{ intersects } B \text{ for infinitely many } B \in \mathcal{B} \} = 0.$$  

Thus, it follows from the Borel–Cantelli Lemma that

$$\lim_{n \to \infty} \sum_{B \in \mathcal{B}} \Pr_{\mathcal{D} \sim \mathcal{D}_B} \{ D \text{ intersects } B \text{ for infinitely many } B \in \mathcal{B} \} = 0.$$  

By Lemma 4.13, $\bigcup_{B \in \mathcal{B}} \mathcal{D}_B$ is block-independent disjoint over $F$ with blocks $B$ and marginals according to $\mathcal{P}$, so $\bigcup_{B \in \mathcal{B}} \mathcal{D}_B = \mathcal{D}(\mathcal{B}, \mathcal{P})$.

(2) Let $\mathcal{D}_1, \mathcal{D}_2, \ldots$ be a family of functional, non-intersecting PDBs with $\sum_{i=1}^{\infty} \Pr_{\mathcal{D} \sim \mathcal{D}_i} \{ D \neq \emptyset \} < \infty$. Let $F := \bigcup_{i=1}^{\infty} F(\mathcal{D}_i)$ and let $\mathcal{B}$ be the partition of $F$ into the sets $F(\mathcal{D}_i)$.

By Lemma 4.14, $\bigcup_{i=1}^{\infty} \mathcal{D}_i$ is a PDB. It follows from Lemma 4.13, that $\bigcup_{i=1}^{\infty} \mathcal{D}_i$ is block-independent disjoint with blocks $\mathcal{B}$ and such that

$$\Pr_{\mathcal{D} \sim \bigcup_{i=1}^{\infty} \mathcal{D}_i} \{ f \in D \} = \Pr_{\mathcal{D} \sim \mathcal{D}_i} \{ f \in D \}$$

for all $f \in F(\mathcal{D}_i)$ and all $i = 1, 2, \ldots$.  

We remark that analogues of Lemma 4.6 and Theorem 4.16 also hold for BID-PDBs. Regarding Lemma 4.6, one can show that the restriction of BID-PDBs to a subset of its blocks will again yield a BID-PDB. As for Theorem 4.16, it turns out that a BID-PDB can be decomposed into arbitrary smaller BID-PDBs, each of which is made up of a subset of the original blocks. In either case, the argumentation is completely parallel to the respective proof for TI-PDBs.

**Remark 4.21 (Beyond Block-Independent Disjoint PDBs.)** The concept of superposition can be used to build PDBs from arbitrary independent building blocks (provided that the condition from Lemma 4.14 is satisfied). As such, the model can be used to discuss classes of PDBs that are subject to independence assumptions but contain arbitrary correlations within the independent “blocks.”
Such a building process may even be interleaved with other standard constructions like disjoint union or convex combinations.

4.5 FO-Definability over TI-PDBs

In this section, we discuss views of tuple-independent PDBs. For (non-probabilistic) relational databases, a view is a function $V$ that maps database instances of some input schema $\tau$ to some output schema $\tau'$. If $\tau'$ consists of only a single relation symbol, say, $\tau' = \{R\}$, then $V$ is called a query. If, additionally, $R$ is 0-ary, then $V$ is called Boolean. A view is called FO-view if it is expressible as a first-order formula under the standard semantics [2] (note that we only discuss set semantics here). The following is easy to verify.

**Fact 4.22.** Let $\varphi = \varphi(x_1, \ldots, x_k)$ be an FO-formula with $k$ free variables over $\tau$, possibly mentioning constants from $U$. Let $D \in DB[\tau, U]$ such that $\varphi(D) = \{(a_1, \ldots, a_k) : D \models \varphi(a_1, \ldots, a_k)\}$ is finite. Then

$$\varphi(D) \subseteq (\text{dom}(D) \cup \text{dom}(\varphi))^k$$

where $\text{dom}(\varphi)$ denotes the set of constants from $U$ appearing in $\varphi$.

Going back to PDBs, we first clarify the semantics of queries and views. Let $D = (DB, P)$ be a PDB, and let $V$ be a view with input schema $\tau$ and output schema $\tau'$ (that is, $V : DB[\tau, U] \to DB[\tau', U]$). Let $DB' = DB[\tau', U]$. Then we let $V(D) := (DB', P')$ with

$$P'(D') := P(V^{-1}(D')) = P\{D \in DB : V(D) \in D'\} \tag{4.9}$$

for all $D' \subseteq DB'$. That is,

$$P'(\{D'\}) = \sum_{D \in DB : V(D) = D'} P(\{D\})$$

for all $D' \in DB'$. (Formally, $P'$ is the push-forward, or image measure of $P$ under $V$, cf. Appendix A.2.) We call $V(D)$ the image of $D$ under $V$.

The semantics described above lifts single instance semantics to sets of instances. In the PDB literature, this has been called possible answer sets semantics in the context of finite PDBs [76, p. 22].

**Remark 4.23.** The semantics are defined in the exact same way for uncountable PDBs, but then we have to make sure that the query or view is a measurable function of database instances (see Appendix A.2). Otherwise, (4.9) is not well defined.

Another semantics that is used for finite PDBs is the so-called possible answer semantics [76]. Therein, the result of a view is the collection of the marginal probabilities of the individual tuples. Recalling Example 3.2, this kind of representation of query results is of no use in uncountable PDBs.

We call a PDB $D$ FO-definable over a PDB $D_0$, if there exists an FO-view $V$ such that $D = V(D_0)$. If $D$ is a class of PDBs, then we let $\text{FO}(D)$ denote the class of PDBs that are FO-definable over some PDB in $D$. Moreover, for a database schema $\tau$ and universe $U$, we let $\text{FO}[\tau, U]$ denote the class of first-order formulae over $\tau$ that are allowed to use constants from $U$. One justification for tuple-independence to be a viable concept for finite PDBs is that any finite PDB is an FO-view of a finite TI-PDB [76]. This, however, does not extend to countably infinite PDBs.

**Example 4.24.** We reconsider the PDB $D$ of infinite expected size that was introduced in Example 3.3: The sample space $DB$ of $D = (DB, P)$ consists of the instances $\{D_i : i \in \mathbb{N}_{>0}\}$ where $D_n = \{R(1), \ldots, R(2^n)\}$ and the probabilities are given by $P(\{D_n\}) = \frac{6}{\pi^2 n^2}$. We claim that $D$ is not FO-definable over any TI-PDB.
To obtain a contradiction, suppose that \( \mathcal{D} = V(\mathcal{D}_0) \) for some TI-PDB \( \mathcal{D}_0 = (DB_0, P_0) \). By Corollary 4.5, \( E_{\mathcal{D}_0}(\cdot | \cdot) < \infty \) because \( \mathcal{D}_0 \) is a TI-PDB. We let \( r \) denote the maximal arity of a relation in the schema of \( \mathcal{D}_0 \). Then every \( DB_0 \)-instance \( D_0 \) satisfies
\[
|\text{adom}(D_0)| \leq r \cdot |D_0|.
\]

Since the schema of \( \mathcal{D} \) consists of a single relation symbol, \( V \) consists of a single FO-formula \( \varphi(x) \in FO[\tau_0, U] \) where \( \tau_0 \) is the schema of \( \mathcal{D}_0 \) and \( U \) is some common universe underlying \( \mathcal{D} \) and \( \mathcal{D}_0 \). Let \( c \) be the number of constants from \( U \) appearing in \( \varphi \). As \( \varphi \) has \( k \) free variables, by Fact 4.22, for every \( (\tau_0, U) \)-instance \( D_0 \) it holds that
\[
|V(D_0)| = |\varphi(D_0)| \leq (|\text{adom}(D_0)| + c)^k = |\text{adom}(D_0)| + c \leq r|D_0| + c.
\]

Thus,
\[
E_{\mathcal{D}}(\cdot | \cdot) = \sum_{D \in DB} |D| \cdot P\{\{D\}\} = \sum_{D \in DB} |D| \cdot P_0(V^{-1}(D)) = \sum_{D \in DB} |V(D_0)| \cdot P_0(\{D_0\}) \\
\leq \sum_{D_0 \in DB_0} (r \cdot |D_0| + c) \cdot P_0(\{D_0\}) = r \cdot E_{\mathcal{D}_0}(\cdot | \cdot) + c < \infty,
\]
a contradiction.

The example demonstrates the following proposition.

**Proposition 4.25.** \( FO(TI_{\leq \omega}^{\text{set}}) \subseteq PDB_{\leq \omega}^{\text{set}} \).

This means the infinite extension of TI-PDBs is in some sense not as powerful as its finite counterpart, since \( FO(TI_{\leq \omega}^{\text{set}}) = PDB_{\leq \omega}^{\text{set}} \) [76, Proposition 2.16]. Incidentally, the proof of Reference [76, Proposition 2.16] cannot be translated to the infinite setting, as it relies on an exhaustive encoding of all possible worlds into facts of marginal probability 1. This approach is not suitable for infinite PDBs as it directly causes the sum of all marginal probabilities to diverge.

### 4.6 Independent Completions

This subsection is devoted to a generalization of the idea of open-world probabilistic databases and \( \lambda \)-completions of PDBs that was introduced in Reference [21].

**Definition 4.26 (Completions).** Let \( F \) be a set of facts and let \( \mathcal{D} = (DB, P) \) be a PDB with \( DB = P_{\text{fin}}(F) \). Let \( DB = P_{\text{fin}}(F[\tau, U]) \). Then a PDB \( \overline{DB} = P_{\text{fin}}(F[\tau, U]) \) is called a completion of \( \mathcal{D} \) if \( \overline{P}(DB) > 0 \) and
\[
\overline{P}(\{D\} | DB) = P(\{D\}) \quad \text{for all } D \in DB.
\]

In other words, \( \mathcal{D}' \) is a completion of \( \mathcal{D} \) if the sample space of \( \mathcal{D}' \) is all of \( P_{\text{fin}}(F[\tau, U]) \) and \( \mathcal{D} \) is the probability space obtained by conditioning \( \mathcal{D}' \) on the sample space \( P_{\text{fin}}(F) \): Intuitively, given that we know that a database instance was one of the original instances of the uncompleted PDB, its probability stays the same. Note that there is no connection between Definition 4.26 and the notion of completion from measure theory. The homonymy is a mere coincidence.

**Proposition 4.27.** Let \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \) be two PDBs with \( F(\mathcal{D}_1) \cap F(\mathcal{D}_2) = \emptyset \) and \( F(\mathcal{D}_1) \cup F(\mathcal{D}_2) = F[\tau, U] \). Then \( \mathcal{D}_1 \cup \mathcal{D}_2 \) is a completion of \( \mathcal{D}_1 \) if and only if \( \Pr_{DB_1}(\{D\} | DB_2) > 0 \).
Proof. For $i = 1, 2$ we let $D_i = (DB_i, P_i)$ and $F_i = F(D_i)$ such that $F_1$ and $F_2$ form a partition of the set $F[\tau, U]$ of all facts.

First, suppose that $\hat{D} = D_1 \uplus D_2 = (\overline{DB}, \overline{P})$ is a completion of $D_1$. By construction,

$$0 < \hat{P}(DB_1) = P_1(DB_1) \cdot P_2(\{ \emptyset \}) = P_2(\{ \emptyset \}).$$

For the backwards direction suppose $P_2(\{ \emptyset \}) > 0$ and, again, let $D_1 \uplus D_2 = \hat{D} = (\overline{DB}, \overline{P})$. Then

$$\hat{P}(DB_1) = \frac{\hat{P}(\{ \emptyset \})}{P(DB_1)} = P_1(\{ \emptyset \}) \cdot P_2(\{ \emptyset \}) = P_1(\{ \emptyset \}).$$

That is, $\hat{D}$ is a completion of $D_1$. \qed

It is easy to see that not every completion of a PDB $D$ can be written as a superposition of $D$ with a PDB of a fact set disjoint to that of $D$. In fact, this is already the case for finite PDBs.

Example 4.28. Suppose $F[\tau, U] = \{ f_1, f_2 \}$. Let $D = (DB, P)$ with $DB = \emptyset, \{ f_1 \}$ and $P(\{ f_1 \}) = p \in (0, 1]$. Consider the PDB $\hat{D} = (DB, \hat{P})$ with $DB = \mathcal{P}(\{ f_1, f_2 \})$ and $\hat{P}$ according to the following table (where $\hat{p} \in (0, 1)$):

|          | $D$ | $\emptyset$ | $\{ f_1 \}$ | $\{ f_2 \}$ | $\{ f_1, f_2 \}$ |
|----------|-----|-------------|--------------|--------------|-------------------|
| $\hat{P}(\{ \emptyset \})$ | $(1-p)(1-\hat{p})$ | $p(1-\hat{p})$ | $0$          | $\hat{p}$         |

Observe that $\hat{P}(DB) = 1 - \hat{p}$ and $\hat{P}(\{ \emptyset \}) = P(\{ \emptyset \}) \cdot (1 - \hat{p})$ for $D \in DB$. Thus for $D \in DB$ we have

$$\hat{P}(\{ \emptyset \}) = \frac{\hat{P}(DB)}{P(DB)} = \frac{P(\{ \emptyset \}) \cdot (1 - \hat{p})}{1 - \hat{p}} = P(\{ \emptyset \}).$$

Hence $\hat{D}$ is a completion of $D$. However, for any superposition $D \uplus D' = (\overline{DB}, \overline{P})$ where $D'$ is any PDB with $F(D') = \{ f_2 \}$, it holds that $\hat{P}'(\{ f_2 \}) > 0$ whenever $\hat{P}'(\{ f_1, f_2 \}) > 0$. Thus, the completion $\hat{D}$ considered above cannot be expressed as a superposition of PDBs with disjoint fact sets.

In the light of Proposition 4.27, a straightforward recipe for building completions is to superpose a given PDB with a TI-PDB that is given by prescribed marginals for the remaining facts.

Definition 4.29 (Independent Completions). Let $D$ be a PDB. A completion $\hat{D}$ of $D$ is a TI-completion if $\hat{D} = D \uplus D'$ for some TI-PDB $D'$ with the property that for all facts $f \in F[\tau, U]$ it holds that

$$Pr_{D' \uplus D} \{ f \in D' \} > 0 \Rightarrow Pr_{D} \{ f \in D \} = 0 \quad (4.10)$$

for all facts $f \in F[\tau, U]$.

Similarly, a completion $\hat{D}$ of $D$ is a BID-completion if $\hat{D} = D \uplus D'$ for some BID-PDB satisfying (4.10).

Note that $\hat{D}$ will most likely not share the independence properties of $D'$. However, we intuitively keep the independencies from $D'$ through Lemma 4.13. For example, this applies to the independence of the new facts in the completion. Moreover, if $D$ is a TI-PDB (respectively, a BID-PDB), then $\hat{D}$ is a TI-PDB (respectively, a BID-PDB) as well.

In Reference [21], the authors consider representations of finite TI-PDBs that are given as a list of pairs $(f, P(f))_{f \in F}$ as is common practice in the treatment of finite PDBs [76, 78]. According to Reference [21], representing PDBs this way inherently comes with a closed-world assumption.
though: If $D_{(F,P)}$ is the TIPDB spanned by the list $(f, P(f))_{f \in F}$, then we do not model the uncertainty of facts $f \in F[\tau, U] \setminus F$, which for probabilistic query evaluation is equivalent to treating them as events of probability 0. As Ceylan et al. proceed to argue [21], this has several undesired practical consequences when working with such representations. They propose a model of $\lambda$-completions consisting of finite completions of a given finite TIPDB by TIPDBs modeling the probabilities of all remaining facts. In the following, we translate their construction into our framework.

**Definition 4.30 (Open-World Probabilistic Databases [21, Definitions 4 and 5].** Let $U$ be finite. An open probabilistic database is a pair $G = (D, \lambda)$ where $D = (DB, P)$ is a TI-PDB over some $F \subseteq F[\tau, U]$. A $\lambda$-completion of $D$ is a TI-PDB $\hat{D} = (\hat{DB}, \hat{P})$ with $\hat{DB} = P(F[\tau, U])$,

$$\hat{P}(f) = P(f) \quad \text{for all } f \in F \text{ and}$$

$$\hat{P}(f) \leq \lambda \quad \text{for all } f \in F[\tau, U] \setminus F.$$

Clearly, every ($\lambda$-)completion in the sense of Ceylan et al. (Definition 4.30) is a completion in the sense of Definition 4.26. That is, every $\lambda$-completion $\hat{D}$ can be written as

$$\hat{D} = D \uplus D_{(F[\tau, U] \setminus F, \hat{P})}$$

for the corresponding assignment $\hat{P}: F[\tau, U] \setminus F \to [0, \lambda]$ of marginal probabilities to the new facts.

The set of all $\lambda$-completions for some fixed $\lambda$ is the central object of study in Reference [21]. With our framework, the idea gently generalizes to infinite completions.

**Remark 4.31.** The model of independent completions we described is, to our knowledge, not (or at least not directly) expressible in the existing "infinite" PDB systems [6, 56, 71] but could be emulated in MCDB [52] as follows: First, create $n$ dummy tuples where $n$ is drawn from a Poisson distribution. For each of the $n$ dummy tuples, sample their attribute entries independently according to a common probability measure on the space of facts using MCDB’s Variable Generating functions. According to Reference [62, Proposition 3.5], this describes a Poisson process on the space of facts, so it has the desired independence condition but may contain duplicates (see also Sections 4.8 and 5). We expect though, that a considerable speedup over this approach could be achieved in an sampling based implementation that directly exploits the independence properties.

**4.7 Approximate Query Evaluation**

Query evaluation in infinite PDBs is not the main focus of this article and remains an object of future study for the most part. Nevertheless, we present two first results on query evaluation in infinite tuple-independent PDBs highlighting the bounds of possibility as well as connections to query evaluation in finite (tuple-independent) PDBs. For query evaluation in finite tuple-independent PDBs, the PDB can be given as part of the input by just specifying the list of facts together with their marginal probabilities, as seen in Example 1.1. For analyzing complexity it is typically assumed that all occurring marginal probabilities (and thus, in the finite, all instance probabilities) are rational [29, 43]. As our PDBs can be countably infinite, we need to comment on the data model. The basic assumption replacing the exhausting list of fact probabilities is that given a fact, we can determine its marginal probability in the input PDB.

The first result we give states that we can compute additive approximations of the probability of Boolean query in countably infinite TIPDBs. We first introduce this statement with respect to an oracle mechanism for accessing fact probabilities, implying that this result is independent of the concrete representation.
Let $D = (DB, P)$ be a PDB. We say an algorithm has oracle access to $D$ (cf. Reference [9, Section 3.4]), if it can query a black box

1. to obtain $P(f)$ given a fact $f$; and
2. to obtain $\sum_{f \in F(D)} P(f)$.

In employing such an oracle mechanism we avoid the discussion of representation issues at this point.

**Proposition 4.32.** There is an algorithm $A$ that, given oracle access to a $\textbf{TI}$-PDB $D$, an $\epsilon > 0$, and a Boolean query $\varphi \in \text{FO}[\tau, \mathcal{U}]$, returns a rational number $p(\varphi)$ such that

$$\Pr_{D \leftarrow D}(D \models \varphi) - \epsilon \leq p(\varphi) \leq \Pr_{D \leftarrow D}(D \models \varphi) + \epsilon.$$ 

**Remark 4.33.** We note that the proof does not actually rely on $\varphi$ being an $\text{FO}$-query. All we need is that it comes from a class of queries such that there exists an algorithm that given a finite PDB and a Boolean query from the class returns the probability of the query being true.

**Proof.** Let $F(D) = \{f_1, f_2, \ldots \}$ and let $p_i = P(f_i)$. For all $n \in \mathbb{N}_{>0}$ let $DB_n = P(\{f_1, \ldots, f_n\})$ and let $P_n := P(\{f_1, \ldots, f_n\})$. By Lemma 4.6, $D | DB_n = D(\{f_1, \ldots, f_n\})$. Because $D | DB_n$ is a finite $\textbf{TI}$-PDB (and we have access to its marginal probabilities using the oracle), the exact value of $P(\varphi | DB_n)$ can be found using the traditional techniques for query answering in finite $\textbf{TI}$-PDBs.

Since $D$ is a $\textbf{TI}$-PDB, by Theorem 4.4 it holds that $\sum_{i=1}^{\infty} p_i < \infty$, so $\lim_{i \to \infty} p_i = 0$. For all $n \in \mathbb{N}_{>0}$ we define

$$r_n := \sum_{i=n+1}^{\infty} p_i.$$ 

Then $\lim_{n \to \infty} r_n = 0$. We choose $n$ such that $r_n \leq \epsilon$. A suitable $n$ can be computed by systematically listing facts $f_1, \ldots, f_n$ until $r_n$ is small enough. The value $r_n$ itself can be calculated as $r_n = \sum_{i=n+1}^{\infty} p_i$ using the oracle access.

Let $DB_n := P(\{f_1, \ldots, f_n\})$. We let our algorithm return $A(\varphi) = p(\varphi) := P(\varphi | DB_n)$. Note that

$$P(DB_n) = \prod_{i=n+1}^{\infty} (1 - p_i)^{\text{(1.1)}} \geq 1 - \sum_{i=n+1}^{\infty} p_i = 1 - r_n \geq 1 - \epsilon.$$ 

Thus,

$$P(\varphi) = P(\varphi | DB_n) \cdot P(DB_n) + P(\varphi | (DB_n)^c) \cdot P((DB_n)^c) \leq A(\varphi) + \epsilon,$$

so $A(\varphi) \geq P(\varphi) - \epsilon$. Also, we have

$$P(\varphi) = P(\varphi | DB_n) \cdot P(DB_n) + P(\varphi | (DB_n)^c) \cdot P((DB_n)^c) \geq A(\varphi)(1 - \epsilon) \geq A(\varphi) - \epsilon,$$

so $A(\varphi) \leq P(\varphi) + \epsilon$. Together, we have

$$P(\varphi) - \epsilon \leq A(\varphi) \leq P(\varphi) + \epsilon$$

as required. \hfill \Box

In the problem discussed before, the database was fixed in the algorithm whereas the query was the input. Next we also want to consider PDBs as an input.

**Definition 4.34.** Let $D = D_\langle \tau, \Sigma, |, \epsilon, P \rangle$ be a $\textbf{TI}$-PDB where $P : F[\tau, \Sigma^*] \to [0,1] \cap \mathbb{Q}$. Let $M$ be a Turing machine with input alphabet $\Sigma \cup \tau \cup \{,\}$ and let $\xi \in \mathbb{Q}_{\geq0}$. The pair $(M, \xi)$ represents $D$ if $M$ computes the function $p_M : F[\tau, \Sigma^*] \to [0,1] \cap \mathbb{Q}$: $f \mapsto P(f)$ and $\xi = \sum_{f \in F[\tau, \Sigma]} P(f)$. 

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The general computational problem we are interested in is then the following:

| Probabilistic Query Evaluation PQE |
|------------------------------------|
| **Input** | A pair \((M_\mathcal{D}, \xi_\mathcal{D})\) representing a \textbf{TI}-PDB \(\mathcal{D}\) over \(\tau\) and \(\Sigma^*\) and a Boolean query \(\varphi \in \textbf{FO}[\tau, \Sigma^*]\) |
| **Output** | The probability of \(\varphi\) being satisfied in \(\mathcal{D} \sim \mathcal{D}\), that is, \(\Pr_{D \sim D}(D \models \varphi)\). |

We denote by \(\text{PQE}(\varphi)\) the above problem with the input query being fixed to some Boolean query \(\varphi \in \text{FO}[\tau, \Sigma^*]\).

**Corollary 4.35.** For all \(\epsilon > 0\) there exists an additive \(\epsilon\)-approximation algorithm for \(\text{PQE}\).

**Proof.** Said algorithm can be obtained by following the proof of Proposition 4.32 and replacing calls to the oracle by calculations using the representation of \(\mathcal{D}\). \(\square\)

Proposition 4.32 and Corollary 4.35 give us additive approximation results for query answering in countable \textbf{TI}-PDBs. We note that for approximation guarantee \(\epsilon\) the runtime of the algorithm used in the proof depends on the rate of convergence of the series of fact probabilities. This is because the number \(n\) of facts that are taken into consideration has to be chosen such that \(r_n\) (the sum of the remaining probabilities) is at most \(\epsilon\). In the best case, the facts are ordered in decreasing probability and \(n\) is chosen minimal. Then, for example, if the series of fact probabilities is a geometric series, then it holds that \(n = \Omega\left(\frac{1}{1-\epsilon}\right)\).\(^7\) The runtime of the complete algorithm is determined by the runtime of the method that is used for the finite query evaluation on a \textbf{TI}-PDB with \(n\) facts.

The next proposition shows that a multiplicative approximation algorithm does not exist by investigating the data complexity of a particular (and very simple) query.

**Proposition 4.36.** Let \(\Sigma = \{0, 1\}\) and \(\tau = \{R, S\}\) with \(R\) and \(S\) unary. Let \(\varphi = \exists x R(x) \in \text{FO}[\tau, \Sigma^*]\) and let \(\rho \in \mathbb{R}, \rho \geq 1\). Then there is no algorithm \(\mathcal{A}\) for \(\text{PQE}(\varphi)\) that on input a representation \((M_\mathcal{D}, \xi_\mathcal{D})\) of a \textbf{TI}-PDB \(\mathcal{D}\) satisfies

\[
\rho^{-1} \Pr_{D \sim D}(D \models \varphi) \leq \mathcal{A}(M_\mathcal{D}, \xi_\mathcal{D}) \leq \rho \Pr_{D \sim D}(D \models \varphi).
\]

**Proof.** We let

\[
\| \cdot \|_2 : \Sigma^* \to \mathbb{N}_{>0} : w_1 \ldots w_n \mapsto \sum_{i=1}^{n} w_i 2^{i-1} + 2^n
\]

be the bijection that identifies a \(\Sigma\)-string \(w\) with the positive integer whose binary representation is \(1w\). Note that \(\| \cdot \|_2\) is computable.

For a Turing machine \(M\) over alphabet \(\Sigma = \{0, 1\}\) we let \(L_M\) denote the set of strings in \(\Sigma^*\) that are accepted by \(M\). By Rice’s Theorem [68] (cf. Reference [61, Theorem 34.1]), the set \(\text{EMPTY}\), that is, the set of (encodings of) Turing machines \(M\) with \(L_M = \emptyset\), is undecidable. For every \(t \in \mathbb{N}_{>0}\), we let \(L_{M,t} = \{w \in \Sigma^* : M\) accepts \(w\) in \(\leq t\) steps\}. Then \(L_{M,t}\) is clearly decidable for all \(t \in \mathbb{N}_{>0}\) and it holds that \(L_M = \bigcup_{t \in \mathbb{N}_{>0}} L_{M,t}\).

Let \(M\) be a Turing machine over \(\Sigma\). We define a \textbf{TI}-PDB \(\mathcal{D}\) over \(F[\tau, \Sigma^*]\). Let \(<\cdot, \cdot>\) be the function from \(\mathbb{N}_{>0} \times \mathbb{N}_{>0}\) to \(\mathbb{N}_{>0}\) with

\[
<i, j> := \left( \frac{i + j - 1}{2} \right) + i = \frac{1}{2} (i + j - 1) (i + j - 2) + i
\]

\(^7\)Note though, that in general, series may converge "arbitrarily slow," see References [60, pp. 310–311].
for all \(i, j \in \mathbb{N}_{>0}\). It is well known that \(\langle \cdot, \cdot \rangle\) is a computable bijection (see, for example, Reference [61, Example J.2]). The marginal probabilities of \(\mathcal{D}\) are defined as follows for all \(w \in \Sigma^*\):

\[
P(R(w)) = \begin{cases} 2^{-|w|_2} & \text{if } |w|_2 = \langle n, t \rangle \text{ and } n \in L_{M,t} \\ 0 & \text{otherwise.} \end{cases} \tag{4.11}
\]

\[
P(S(w)) = \begin{cases} 2^{-|w|_2} & \text{if } |w|_2 = \langle n, t \rangle \text{ and } n /\in L_{M,t} \text{ and } \ \text{otherwise.} \end{cases} \tag{4.12}
\]

Note that with these definitions, it holds that \(\sum_{f \in F[\tau, \Sigma^*]} P(f) = \sum_{k \in \mathbb{N}_{>0}} 2^{-k} = 1 < \infty\). Thus, \(\mathcal{D} = \mathcal{D}(F[\tau, \Sigma^*], P)\) is a well-defined PDB. (In particular note that the tuples in \(S\) are used to bring the sum of marginal probabilities to 1.)

We observe that

\[
\Pr_{\mathcal{D} \sim \mathcal{D}} (R(w) \in D) = 0 \quad \text{for all } w \in \Sigma^* \quad \iff \quad \bigcup_{t \in \mathbb{N}_{>0}} L_{M,t} = L_M = \emptyset. \tag{4.13}
\]

Recalling that \(\varphi = \exists x R(x)\), the above equivalence entails that

\[
\Pr_{\mathcal{D} \sim \mathcal{D}} (D \models \varphi) = 0 \quad \iff \quad L_M = \emptyset. \tag{4.14}
\]

We construct a Turing machine \(\tilde{M}_D\) over the alphabet \(\tilde{\Sigma} = \Sigma \cup \tau \cup \{(,\}\) that works as follows:

- On input \(w \in \tilde{\Sigma}^*\), \(\tilde{M}_D\) checks whether \(w \in F[\tau, \Sigma^*]\). If not, then it rejects.
- Otherwise, it checks which of the cases in Equations (4.11) and (4.12) applies and outputs \(2^{-|w|_2}\) or 0 accordingly.

Then \((\tilde{M}_D, 1)\) represents \(\mathcal{D}\).

Now suppose that \(\mathcal{A}\) is a multiplicative approximation algorithm for \(\text{PQE}(\varphi)\). Then

\[
\mathcal{A}(\tilde{M}_D, 1) = 0 \quad \iff \quad \Pr_{\mathcal{D} \sim \mathcal{D}} (D \models \varphi) = 0 \quad \iff \quad L_M = \emptyset.
\]

Thus, \(\mathcal{A}\) can be used to decide \(\text{EMPTY}\). \hfill \Box

From Proposition 4.36 we immediately obtain the following corollary.

**Corollary 4.37.** Let \(\rho \geq 1\). Then there exists no multiplicative \(\rho\)-approximation algorithm for \(\text{PQE}\).

### 4.8 Tuple-Independent Bag PDBs

In this last subsection on countable tuple-independent PDBs, we study countable PDBs with a bag semantics. Still maintaining Assumptions (I) and (II), we replace Assumption (III) by the following.

\[(\text{III}’)\] If not explicitly stated otherwise, all PDBs that occur in this section have sample space \(\mathcal{B}_{\text{fin}}(F(\mathcal{D}))\) and are equipped with the powerset \(\sigma\)-algebra.

Independent PDBs with a bag semantics are interesting in their own right, but the following discussion also serves as a preparation for our treatment of uncountable PDBs in the next section. For uncountable PDBs, it is easier to work with a bag semantics, simply because the underlying probability theory of finite point processes usually has been developed for point processes where points may be repeated.

**Definition 4.38 (Countable TI-PDBs, Bag Version).** Let \(\mathcal{D}\) in \(\text{PDB}_{\text{bag}}\). Then \(\mathcal{D}\) is called **tuple-independent** (or, a TI-PDB) if the numbers of occurrences of distinct facts are mutually independent.
That is, \( \mathcal{D} \in \text{PDB}_{\leq \omega} \) is a TI-PDB if (and only if)
\[
\Pr_{\mathcal{D} \rightarrow \mathcal{D}} \{ \#_D(f_1) = n_1, \ldots, \#_D(f_k) = n_k \} = \Pr_{\mathcal{D} \rightarrow \mathcal{D}} \{ \#_D(f_1) = n_1 \} \cdots \Pr_{\mathcal{D} \rightarrow \mathcal{D}} \{ \#_D(f_k) = n_k \}
\]
for all pairwise distinct \( f_1, \ldots, f_k \in F(\mathcal{D}) \), all \( n_1, \ldots, n_k \in \mathbb{N} \), and all \( k \in \mathbb{N} \).

The class of countable tuple-independent (bag) PDBs is denoted by \( \text{TI}_{\leq \omega} \) and the subclass of finite tuple-independent (bag) PDBs by \( \text{TI}_{< \omega} \).

If we treat sets as bags with multiplicities in \( \{0, 1\} \), then Definition 4.38 is compatible with Definition 4.1. That is, \( \text{TI}_{\leq \omega}^{set} \subseteq \text{TI}_{\leq \omega} \) and, in particular, \( \text{TI}_{< \omega}^{set} \subseteq \text{TI}_{< \omega} \). Similarly to set PDBs, we can express tuple-independent bag PDBs as superpositions of single-fact PDBs.

**Corollary 4.39.**

1. Let \( \mathcal{D} \) be a countable TI-PDB. For every \( f \in F \), let \( \mathcal{D}_f \) be the bag PDB with \( F(\mathcal{D}_f) = \{f\} \), and with the probability measure defined by \( \Pr_{\mathcal{D} \rightarrow \mathcal{D}_f} (\#_D(f) = k) := \Pr_{\mathcal{D} \rightarrow \mathcal{D}} (\#_D(f) = k) \). Then
\[
\mathcal{D} = \biguplus_{f \in F(\mathcal{D})} \mathcal{D}_f.
\]
Moreover, \( \sum_{f \in F(\mathcal{D})} \Pr_{\mathcal{D} \rightarrow \mathcal{D}_f} (\#_D(f) > 0) \) is finite.

2. Let \( F \subseteq [\tau, \mathcal{U}] \), and for every \( f \in F \), let \( \mathcal{D}_f \) be a PDB with \( F(\mathcal{D}_f) = \{f\} \). Suppose that \( \sum_{f \in F} \Pr_{\mathcal{D} \rightarrow \mathcal{D}_f} (\#_D(f) > 0) < \infty \). Then \( \cup_{f \in F} \mathcal{D}_f \) is a tuple-independent PDB.

**Example 4.40.** A (countable) Poisson-PDB is a tuple-independent PDB \( \mathcal{D} \) where the fact multiplicities are Poisson distributed, that is, for every \( f \in F(\mathcal{D}) \) there is a nonnegative \( \lambda_f \in \mathbb{R} \) such that
\[
\Pr_{\mathcal{D} \rightarrow \mathcal{D}} (\#_D(f) = k) = e^{-\lambda_f} \frac{\lambda_f^k}{k!}.
\]

Our definition of tuple-independence only requires that the multiplicities of distinct facts be independent; it makes no assumptions on the distributions of the individual fact multiplicities. Intuitively, we may argue that Poisson-PDBs also make an independence assumption for these distributions. Indeed, we might think of a tuple-independent bag PDB as being generated by sampling independent identical copies of each fact. Then, if we have \( n \) identical copies of each fact, each with a probability \( p_f^{(n)} \), then the fact multiplicities will be binomially distributed. As we let the number \( n \) of copies go to infinity while keeping the expected value \( \lambda_f = np_f^{(n)} \) of the number of samples of each fact (and hence the expected size of the PDB) constant, these binomial distributions converge to a Poisson distribution with parameter \( \lambda_f \) (see Reference [37, Section 6.5]).

As we will see in the next section, Poisson-PDBs also play a special role in the theory of uncountable tuple-independent PDBs.

**Remark 4.41.** A special special case of particular interest is given when the total number of facts of a tuple-independent bag PDB is finite, but their individual multiplicities are unbounded. We call this a fact-finite PDB. Fact-finite PDBs occupy a middle ground, as although there are only finitely many different facts, the sample space can be of infinite size. Answering queries in fact-finite (bag) PDBs is studied in Reference [49].

## 5 BEYOND COUNTABLE DOMAINS

In this section, we discuss a suitable notion of tuple-independence for PDBs with uncountable sample spaces. This is an application of the theory of point processes and of random measures, and builds on some more background from measure theory and general topology that can be looked up in Appendices A.2 and A.3 whenever necessary.
We use the framework of standard PDBs from Reference [48]. From this point of view, a probabilistic database is nothing but a finite point process [23]. Essentially, a finite point process is a probability distribution over finite sets or bags of elements (“points”) in some measurable space. In the case of PDBs, these points are the facts, and the number of times a particular fact occurs in an outcome of the point process gives its multiplicity in the corresponding database instance.

In the following, let \( U \) be an uncountable universe. Following Reference [48], we require that the universe is given as a standard Borel space \((U, \mathcal{U})\). That is, \( U \) is a Polish topological space with Borel \( \sigma \)-algebra \( \mathcal{U} \). Given a database schema \( \tau \), this induces a natural \( \sigma \)-algebra \( \mathcal{G} \) on the space \( F[\tau, U] \) of \((\tau, U)\)-facts, which in turn generates a natural \( \sigma \)-algebra for probabilistic databases through the events

\[
\{ D \in DB[\tau, U] : \#_D(F_1) = n_1, \ldots, \#_D(F_k) = n_k \}
\]

for measurable sets \( F_1, \ldots, F_k \) of facts, and non-negative integers \( n_1, \ldots, n_k \). We denote this \( \sigma \)-algebra by \( \mathcal{D}[\tau, U] \). For further information on the model and the detailed constructions, we refer the reader to Reference [48].

It is worth noting that, as soon as we move to uncountable spaces of facts, the measurability of constructions, functions, and queries is a key issue that needs to be addressed. The model from Reference [48], however, has been shown to exhibit the desired properties for probabilistic databases, such as the measurability of typical database queries. That is, the model itself, and the semantics of queries are well defined.

**Definition 5.1 (Standard PDBs [48]).** Let \( \tau \) be a database schema and let \((U, \mathcal{U})\) be a standard Borel universe. A standard PDB over \( \tau \) and \( U \) is a probability space \( D = (DB, \mathcal{D}, P) \) with

- sample space \( DB = DB[\tau, U] = \mathcal{B}(F[\tau, U]) \) and
- \( \sigma \)-algebra \( \mathcal{D} = \mathcal{D}[\tau, U] \).

In probability-theoretic terms, Definition 5.1 is just the definition of a finite point process over the adequately constructed measurable space of facts \( F[\tau, U] \). In turn, finite point processes are special random measures [55], namely, random integer-valued measures.

Recall that under the tuple-independence assumption, the presence (or multiplicities) of pairwise distinct facts are independent. Given a continuum of possible facts, this definition is too weak, as it fails to capture independence between “regions” of the fact space. This was no issue in countable PDBs as there, each of the countably many possible instances can be expressed as an intersection of the marginal events from the definition of tuple-independence. Here, however, these marginal events alone do not suffice to describe the measurable structure of PDBs.

Independence has been investigated thoroughly in the general theory of random measures, though. A random measure is called completely random [58], if its values on any finite number of disjoint measurable subsets of the space are independent. Translating this to the language of PDBs gives a direct generalization of the tuple-independence assumption for PDBs over continuous spaces.

**Definition 5.2 (Tuple-Independence for Standard PDBs, cf. Reference [58]).** A standard PDB \( D = (DB, \mathcal{D}, P) \) is called tuple-independent (or, a TI-PDB) if for all \( k = 1, 2, \ldots \), all \( n_1, \ldots, n_k = 0, 1, 2, \ldots \) and all mutually disjoint \( F_1, \ldots, F_k \in \mathcal{G} \) it holds that

\[
\Pr_{D \sim \mathcal{D}} \{ \#_D(F_1) = n_1, \ldots, \#_D(F_k) = n_k \} = \prod_{i=1}^k \Pr_{D \sim \mathcal{D}} \{ \#_D(F_i) = n_i \}. \tag{5.1}
\]

As singletons \( \{ f \} \) are measurable in \( F[\tau, U] \) by the construction of Reference [48], the above is an extension of the notion of tuple-independence we introduced earlier for countable PDBs.
Independence in Infinite PDBs

Moreover, for countable PDBs, Definition 5.2 is equivalent to Definition 4.38, and for countable set PDBs, it is equivalent to Definition 4.1.

The structure of completely random measures is well understood. A classic result due to Kingman [58] presents a decomposition of completely random measures into three well-structured parts by means of superposition. Here the term superposition refers to the sum of random measures (usually implying that they are independent). This is a direct generalization of our notion of superposition for countable PDBs from Section 4.

For integer-valued completely random measures, Kingman’s decomposition simplifies as follows.

Fact 5.3 (see Reference [23, Theorem 2.4.VI]). Every integer-valued completely random measure $\mu$ is a superposition of two random measures $\mu_1$ and $\mu_2$, where $\mu_1$ is completely random with countable support, and $\mu_2$ is a random measure defined by a compound Poisson process satisfying $\Pr(\mu_2(\{x\}) > 0) = 0$ for all $x$.

To make clear the implications for PDBs with independence assumptions, let us expand a bit more on the involved terminology. A Poisson process [62] on a standard Borel space $(\Omega, \mathcal{F})$ is a point process (i.e., integer-valued random measure) that is parameterized through a measure $\lambda$ on $(\Omega, \mathcal{F})$ such that

1. the number of points in every $A \in \mathcal{F}$ is Poisson distributed with parameter $\lambda(A)$, and
2. the numbers of points in disjoint measurable sets are independent (as in Definition 5.2).

A compound Poisson process can be thought of as a generalization, specifying random locations of points by the means of a Poisson process, and for these points, prescribing separate independent, positive multiplicity distributions. For the precise statements and further details, we refer to the literature, specifically [23, Chapter 2] and [24, Chapters 9 and 10].

While the characterization from Fact 5.3 highlights a strong connection between independence assumptions and the Poisson process, Poisson processes themselves also yield a very simple model for uncountable PDBs.

Definition 5.4 (Poisson-PDBs). A PDB $D = (DB, D, P)$ is called a Poisson-PDB if there exists a finite measure $\lambda$ on $(F[\tau, U], F)$, called the parameter (measure) of $D$, such that

1. $D \in \text{StandardTI}$; and
2. for all $F \in F$ we have

\[
\Pr_{D \sim D}\{\#D(F) = k\} = \frac{\lambda(F)^k}{k!} \cdot e^{-\lambda(F)}. \tag{5.2}
\]

That is, in a Poisson-PDB, the random variable $\#_{D, \cdot}(F) : DB \rightarrow N_{> 0} : D \mapsto \#D(F)$ is Poisson distributed with parameter $\lambda(F)$. We emphasize again that the parameter $\lambda$ is not a single number but rather a function (more precisely, a measure) that maps every measurable set of facts to a nonnegative real number. We note that Poisson-PDBs over the fact space $(F[\tau, U], F)$ exist for every choice of parameter measure $\lambda$ (see Reference [62, Theorem 3.6]).

Now from Fact 5.3, we obtain the following characterization of StandardTI-PDBs.

Theorem 5.5. Let $D = (DB, D, P)$ be a StandardTI-PDB over $\tau$ and $U$. Then $D$ is a superposition of two PDBs $D_1$ and $D_2$ such that

\[\text{in particular, the diffuse deterministic component of Kingman’s characterization [58, Section 8] vanishes when going from general random measures to integer-valued ones (cf. also Reference [24, Proposition 9.1.III(i-ii)].} \]
\( \mathcal{D}_1 \) is a countable TI-PDB, and

the deduplication of \( \mathcal{D}_2 \) is a Poisson-PDB with parameter \( \lambda \) satisfying \( \lambda(\{f\}) = 0 \) for all \( f \in F[\tau, \mathbb{U}] \).

In particular, if \( \mathcal{D} \) is a StandardTI-PDB whose instances are almost surely set instances (that is, if \( \mathcal{D} \) is simple), and \( \Pr(\#\{f\} > 0) = 0 \), then it follows that \( \mathcal{D} \) is a Poisson-PDB [62, see Theorems 6.9 and 6.12].

In fact, it already follows that \( \mathcal{D} \) is a Poisson-PDB, if \( \mathcal{D} \) is any simple standard PDB, and for which there exists a diffuse, finite measure \( \lambda \) on \((F[\tau, \mathbb{U}], \mathfrak{F})\) such that \( P(\#(F) = 0) = e^{-\lambda(F)} \) for all measurable sets \( F \) of facts [62, Theorem 6.10]. Then \( \lambda \) is the parameter measure of the PDB.

Apart from their significance for the tuple-independence assumption, Poisson-PDBs have some additional nice properties (see Reference [62, Theorems 3.3 and 5.2]). For example, every superposition of two Poisson-PDBs, say, with parameters \( \lambda_1 \) and \( \lambda_2 \) is a Poisson-PDB with parameter \( \lambda_1 + \lambda_2 \). Moreover, the restriction of a Poisson-PDB to a smaller, measurable set of facts is again a Poisson-PDB.

While the above unravels the notion of tuple-independence for PDBs over uncountable fact spaces, it may not be clear where to go from here. The probabilistic tools we have touched are used in a plethora of models, for example in ecology, epidemiology and astronomy [10], and our point of view suggests that such models can be treated as probabilistic databases. A particular application we see is thus the extension of existing data by such a model (in the guise of an uncountable PDB), which could pave the way for a sophisticated variant of open-world query evaluation. Therefore, a possible future research direction is the combination of techniques from point process theory with query processing in PDBs.

6 CONCLUDING REMARKS

We introduce a formal framework of infinite probabilistic databases. Within the framework, we study tuple-independence and related independence assumptions. This adds to the theoretical foundation of existing PDB systems that support infinite domains in their data model and opens various directions for future research.

We show that countable tuple-independent PDBs exist exactly for convergent series of marginal fact probabilities. From a more abstract view, PDBs with independent components can be explained using the notion of superpositions, a known concept from point process theory. Toward this end, we investigate some general properties of superpositions of PDBs, most notably, how they preserve independence and when they indeed yield valid PDBs as a result. Following this approach, it turns out that (countable) tuple-independent PDBs can be decomposed into arbitrary smaller PDBs. The modularity of the superposition approach can also be used to reason about block-independent disjoint PDBs and possibly more general classes that are obtained by closing a subclass of PDBs under superpositions. Superpositions also enable us to define tuple-independent PDBs with a bag semantics in a natural way, leading us to the notion of Poisson-PDBs.

The vast increase in expressive power by allowing infinite probability spaces comes at a cost, though. We show that in this setting, and contrary to the finite situation, there are (countable) PDBs that cannot be expressed as a first-order view of a tuple-independent PDB. Yet, we argue that a simple tuple-independent model of completions in the sense of Reference [21] can be used to obtain more meaningful query results in (finite) PDBs. While we cannot even hope for multiplicative approximation guarantees in infinite tuple-independent PDBs, query evaluation in such PDBs can be additively approximated using the well-established methods for probabilistic query evaluation in finite PDBs.
Key problems for future research in infinite PDBs are accessible (finite) representations of infinite PDBs and query evaluation algorithms. Representations of countable PDBs as views over tuple-independent PDBs have been studied in Reference [20]. Our results about (approximate) query evaluation in infinite PDBs are only a first step, and an in-depth investigation is still open. There are some natural follow-up questions regarding our results in Section 4.7, for example, what could be said about the query evaluation problem for restricted classes of PDBs. Moreover, probabilistic query evaluation can be studied in the new setting of infinite, but fact-finite PDBs with bag semantics, such as fact-finite Poisson-PDBs. First steps into this direction were taken in Reference [49].

For putting query evaluation in infinite PDBs into practice (beyond the state of affairs that has been pointed out in the related work section), a promising approach seems to try to integrate traditional database techniques with probabilistic inference techniques for infinite domains that are used in AI, including, for example the relational languages and models BLOG [64, 84], ProbLog [32, 50] and Markov Logic [69, 73]. In general, the underlying inference problems are of high complexity, so achieving tractability is challenging.

APPENDIX

A MATHEMATICAL BACKGROUND

A.1 Series and Products

We use Reference [60] as our standard reference regarding the theory of (infinite) sums and products. For the readers convenience, this section recaps basic definitions and well-known results about series and infinite products that are used throughout the article.

Let \( (a_i)_{i \in \mathbb{N}} \) be a sequence of real numbers \( a_i \in \mathbb{R} \). The formal expression \( \sum_{i=0}^{\infty} a_i \) is called a series. If the limit \( \lim_{n \to \infty} \sum_{i=0}^{n} a_i \) exists, and is equal to \( a \in \mathbb{R} \), then \( \sum_{i=0}^{\infty} a_i \) is called convergent, \( a \) is called its value, and we write \( \sum_{i=0}^{\infty} a_i = a \). If the limit is exists and is \( \infty \), then \( \sum_{i=0}^{\infty} a_i \) is said to diverge to \( \infty \), and we write \( \sum_{i=0}^{\infty} a_i = \infty \). Every series we deal with in this article has non-negative terms only. Note that any such series is either convergent or diverges to \( \infty \). A series \( \sum_{i=0}^{\infty} a_i \) is called absolutely convergent if \( \sum_{i=0}^{\infty} |a_i| \) converges. Obviously, every convergent series with only non-negative terms is absolutely convergent.

\[ \text{Fact A.1 (}[60, \text{ch. IV, Theorem 1}]). \text{ If } \sum_{i=0}^{\infty} a_i \text{ is absolutely convergent, then } \sum_{i=0}^{\infty} a_i = \sum_{i=0}^{\infty} \tilde{a}_i \text{ for every permutation } (\tilde{a}_i)_{i \in \mathbb{N}} \text{ of } (a_i)_{i \in \mathbb{N}}. \]

Since the sequences we consider in this article will be absolutely convergent anyway, we will not have to worry about the order of summation. In particular, we sum over unordered countable index sets.

Now let \( (a_i)_{i \in \mathbb{N}} \) be a sequence of real numbers \( a_i \in \mathbb{R} \). The formal expression \( \prod_{i=0}^{\infty} a_i \) is called an infinite product. In this article, we will only deal with the case where \( a_i \in [0, 1] \) for all \( i \in \mathbb{N} \). In this situation, the limit \( \lim_{n \to \infty} \prod_{i=0}^{n} a_i \) always exists and is in the interval \([0, 1]\), because the corresponding sequence of partial products monotonically decreasing and bounded by 0 from below. If the limit is \( a \in [0, 1] \), then we write \( \prod_{i=0}^{\infty} a_i = a \). When discussing the relationship to series, it is more convenient to write infinite products in the shape \( \prod_{i=0}^{\infty} (1 - a_i) \). It is a basic fact (see Reference [60, ch. VII, Theorems 7 and 11]) that for such infinite products, if \( \sum_{i=0}^{\infty} a_i < \infty \), then \( \prod_{i=0}^{\infty} (1 - a_i) = \prod_{i=0}^{\infty} (1 - \tilde{a}_i) \) for every permutation \( (\tilde{a}_i)_{i \in \mathbb{N}} \) of \( (a_i)_{i \in \mathbb{N}} \). This justifies writing infinite products over unordered index sets.

Moreover, the following connection between series and products holds in terms of convergence.

\[ \text{Fact A.2 (see Reference [60, Theorems 125.1 and 126.4]). Let } a_i \in [0, 1] \text{ with } a_i \neq 1 \text{ for all } i \in \mathbb{N}. \text{ Then } \sum_{i=0}^{\infty} a_i < \infty \text{ if and only if } \prod_{i=0}^{\infty} (1 - a_i) > 0. \]
Furthermore, for sequences \((a_i)_{i \in \mathbb{N}}\) with \(a_i \in [0, 1]\), it holds that
\[
\prod_{i=0}^{\infty} (1 - a_i) \geq 1 - \sum_{i=0}^{\infty} a_i.
\] (1.1)
This is a variant of the Weierstrass inequalities [18, p. 104 et seq.] and can easily be shown by an induction that considers the partial products \(\prod_{i=0}^{n} (1 - a_i)\).

A.2 Probability Theory

In this subsection, we cover most of the relevant background from probability theory including some basic concepts from measure theory. We follow References [54, 59], which the reader may consult as needed for further information.

A.2.1 Measurable Spaces. Let \(\Omega \neq \emptyset\) be some set. A family \(\mathcal{A}\) of subsets of \(\Omega\) is called a \(\sigma\)-algebra on \(\Omega\) if:

1. \(\Omega \in \mathcal{A}\),
2. for all \(A \in \mathcal{A}\) it holds that \(A^c = \Omega \setminus A \in \mathcal{A}\) (closure under complement),
3. for all \(A_1, A_2, \ldots \in \mathcal{A}\) it holds that \(\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}\) (closure under countable union).

It follows from the definition that if \(\mathcal{A}\) is a \(\sigma\)-algebra on \(\Omega\), then it is also closed under countable intersection. That is, for all \(A_1, A_2, \ldots \in \mathcal{A}\) it holds that \(\bigcap_{i=1}^{\infty} A_i \in \mathcal{A}\).

A pair \((\Omega, \mathcal{A})\), where \(\Omega \neq \emptyset\) and \(\mathcal{A}\) is a \(\sigma\)-algebra on \(\Omega\), is called a measurable space. The elements of \(\mathcal{A}\) are called \((\mathcal{A}\text{-})measurable\) sets. For every non-empty set \(\Omega\), both \(\mathcal{P}(\Omega)\) and \(\{\emptyset, \Omega\}\) are \(\sigma\)-algebras on \(\Omega\).

Let \((\Omega_1, \mathcal{A}_1)\) and \((\Omega_2, \mathcal{A}_2)\) be measurable spaces. A function \(f : \Omega_1 \rightarrow \Omega_2\) is called \((\mathcal{A}_1, \mathcal{A}_2)\)-measurable if for all \(A \in \mathcal{A}_2\) it holds that \(f^{-1}(A) \in \mathcal{A}_1\), where \(f^{-1}(A) = \{\omega \in \Omega : f(\omega) \in A\}\). If \(\mathcal{A}_1\) and \(\mathcal{A}_2\) are clear from context, then the function is just called measurable.

Let \(\mathcal{G}\) be a family of subsets of \(\Omega \neq \emptyset\). Then \(\sigma(\mathcal{G})\) denotes the coarsest \(\sigma\)-algebra (that is, the smallest one with respect to set inclusion) containing \(\mathcal{G}\). Then \(\sigma(\mathcal{G})\) is indeed unique and it holds that
\[
\sigma(\mathcal{G}) = \bigcap_{\mathcal{A} \subseteq \mathcal{P}(\Omega) \text{ s.t. } \mathcal{A} \supseteq \mathcal{G} \text{ and } \mathcal{A} \text{ \(\sigma\)-algebra}} \mathcal{A}.
\]
We call \(\sigma(\mathcal{G})\) the \(\sigma\)-algebra generated by \(\mathcal{G}\).

A.2.2 Measures. Let \((\Omega, \mathcal{A})\) be a measurable space. A function \(\mu : \mathcal{A} \rightarrow [0, \infty]\) is called a measure on \((\Omega, \mathcal{A})\) (or, on \(\Omega\) if \(\mathcal{A}\) is clear from context) if

1. \(\mu(\emptyset) = 0\) and
2. for all pairwise disjoint \(A_1, A_2, \ldots \in \mathcal{A}\) it holds that \(\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)\) (\(\sigma\)-additivity).

If \(\mu\) is a measure on a measurable space \((\Omega, \mathcal{A})\), then \((\Omega, \mathcal{A}, \mu)\) is called a measure space. We call \(\mu\) finite if \(\mu(\Omega) < \infty\) and a probability measure if \(\mu(\Omega) = 1\). If \(\mu\) is a probability measure, then \((\Omega, \mathcal{A}, \mu)\) is called a probability space. In a probability space, measurable sets are also called events and for \(A \in \mathcal{A}\), \(\mu(A)\) is called the probability of \(A\). We denote probability measures by \(P\) instead of \(\mu\).

Fact A.3 (see Reference [59, Theorem 1.36]). Let \((\Omega, \mathcal{A}, \mu)\) be a measure space.

1. For all \(A_1, A_2, \ldots \in \mathcal{A}\) with \(A_1 \subseteq A_2 \subseteq \cdots\) it holds that \(\mu(\bigcup_{i=1}^{\infty} A_i) = \lim_{i \to \infty} \mu(A_i)\).
2. For all \(A_1, A_2, \ldots \in \mathcal{A}\) with \(A_1 \supseteq A_2 \supseteq \cdots\) and with \(\mu(A_i) = \infty\) and for at most finitely many \(i \in \mathbb{N}_{>0}\) it holds that \(\mu(\bigcap_{i=1}^{\infty} A_i) = \lim_{i \to \infty} \mu(A_i)\).

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If $(\Omega_1, \mathcal{F}_1, \mu)$ is a measure space, and $(\Omega_2, \mathcal{F}_2)$ a measurable space, then every measurable function $f : \Omega_1 \to \Omega_2$ induces a measure $\mu_2$ on $(\Omega_2, \mathcal{F}_2)$ via

$$\mu_2(A) = \mu_1(\{\omega \in \Omega_1 : f(\omega) \in A\}).$$

Then $\mu_2$ is called the image or push-forward measure of $\mu_1$ under $f$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ is called image measure space. If $\mu_1$ is a probability measure, then so is $\mu_2$. In this situation, $f$ is called a random variable.

### A.2.3 Stochastic Independence

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $I$ be some non-empty index set. A family of events $(A_i)_{i \in I}$ with $A_i \in \mathcal{F}$ for all $i \in I$ is called (stochastically) independent if for all $k = 1, 2, \ldots$ and all pairwise different $i_1, \ldots, i_k \in I$ it holds that

$$P\left(\bigcap_{j=1}^k A_{i_j}\right) = \prod_{j=1}^k P(A_{i_j}). \quad (1.2)$$

The family $(A_i)_{i \in I}$ is called pairwise independent if Equation (1.2) holds for $k = 2$ and $i_1, i_2 \in I$ with $i_1 \neq i_2$.

**Fact A.4** (see Reference [59, Theorem 2.5]). Let $(A_i)_{i \in I}$ and $(\bar{A}_i)_{i \in I}$ be families of events in a probability space $(\Omega, \mathcal{F}, P)$ where $\bar{A}_i \in \{A_i, A_i^C\}$ for all $i \in I$. Then $(A_i)_{i \in I}$ is independent if and only if $(\bar{A}_i)_{i \in I}$ is independent.

### A.2.4 Product Measure Spaces

Let $((\Omega_i, \mathcal{F}_i))_{i \in I}$ be a family of measurable spaces for some non-empty index set $I$. The product $\sigma$-algebra $\bigotimes_{i \in I} \mathcal{F}_i$ of the $\mathcal{F}_i$, $i \in I$ is the coarsest $\sigma$-algebra on $\Omega := \prod_{i \in I} \Omega_i$ making all the canonical projections $\pi_i : \Omega \to \Omega_i : (\omega_i)_{i \in I} \mapsto \omega_i$ measurable. That is,

$$\bigotimes_{i \in I} \mathcal{F}_i := \sigma(\{\pi_i^{-1}(A_i) : A_i \in \mathcal{F}_i\}). \quad (1.3)$$

If $I = \{1, \ldots, n\}$, then we write $\bigotimes_{i=1}^n \mathcal{F}_i$ or $\mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_n$, and if $I = \mathbb{N}_{>0}$, we write $\bigotimes_{i=1}^\infty \mathcal{F}_i$ instead of $\bigotimes_{i \in I} \mathcal{F}_i$. With (1.3), $\bigotimes_{i \in I}(\Omega_i, \mathcal{F}_i) := (\Omega, \bigotimes_{i \in I} \mathcal{F}_i)$ is a measurable space, and is called the product measurable space of the $(\Omega_i, \mathcal{F}_i)$, $i \in I$.

**Fact A.5** (see Reference [59, Corollary 14.33]). Let $(\Omega_i, \mathcal{F}_i, P_i)$ be probability spaces for $i = 1, 2, \ldots$. Moreover, let $\Omega := \prod_{i=1}^\infty \Omega_i$ and $\mathcal{F} := \bigotimes_{i=1}^\infty \mathcal{F}_i$. Then there exists a unique probability measure $P$ on $(\prod_{i=1}^\infty \Omega_i, \bigotimes_{i=1}^\infty \mathcal{F}_i)$ with the property that

$$P\left(\bigcap_{j=1}^k \pi_i^{-1}(A_{i_j})\right) = \prod_{j=1}^k P_{i_j}(A_{i_j})$$

for all $k = 1, 2, \ldots$, all pairwise distinct $i_1, \ldots, i_k \in \mathbb{N}_{>0}$ and all $A_{i_j} \in \mathcal{F}_{i_j}$.

The probability space $(\Omega, \mathcal{F}, P)$ from Fact A.5 is called the product probability space of the spaces $(\Omega_i, \mathcal{F}_i, P_i)$ and $P$ is called the associated product probability measure of the $P_i$, $i = 1, 2, \ldots$.

**Fact A.6.** Let $(\Omega, \mathcal{F}, P)$ be the product probability space from Fact A.5 and let $A_i \in \mathcal{F}_i$ for all $i = 1, 2, \ldots$. Then $(\pi_i^{-1}(A_i))_{i \in \mathbb{N}_{>0}}$ is independent in $(\Omega, \mathcal{F}, P)$.

### A.3 Some Background from Topology

This subsection contains some notions from general and metric topology that are relevant for the definitions in Section 5. For a general reference, consult Reference [83]. For standard Borel spaces and their properties see Reference [38, Section 424].
A topological space is a pair \((X, \mathcal{I})\) where \(X\) is a set and \(\mathcal{I}\) is a family of subsets of \(X\) such that

- \(\emptyset, X \in \mathcal{I}\),
- \(\mathcal{I}\) is closed under finite intersections, i.e., if \(X_i \in \mathcal{I}\) for all \(i \in I\) where \(I\) is some finite index set, then \(\bigcap_{i \in I} X_i \in \mathcal{I}\), and
- \(\mathcal{I}\) is closed under arbitrary unions, i.e., if \(X_i \in \mathcal{I}\) for all \(i \in I\) where \(I\) is an arbitrary index set, then \(\bigcup_{i \in I} X_i \in \mathcal{I}\).

The family \(\mathcal{I}\) is then called a topology on \(X\) and its elements are called the open sets of \((X, \mathcal{I})\). Complements of open sets are called closed. If \(\mathcal{I}\) is clear from context, then we just call \(X\) a topological space, referring to \((X, \mathcal{I})\).

A metric space is a pair \((X, d)\) where \(X\) is a set and \(d : X \times X \to \mathbb{R}_{\geq 0}\) is a metric on \(X\). That is,

- \(d(x, y) \geq 0\) with \(d(x, y) = 0\) if and only if \(x = y\),
- \(d(x, y) = d(y, x)\) and
- \(d(x, y) \leq d(x, z) + d(z, y)\)

for all \(x, y, z \in X\).

Let \((X, d)\) be a metric space, \(x \in X\) and \(\epsilon > 0\). Then \(B_r(x) := \{y \in X : d(x, y) < \epsilon\}\) denotes the open ball of radius \(\epsilon\) around \(x\). The metric topology \(\mathcal{I}(d)\) on \(X\) with respect to \(d\) is the topology whose open sets are exactly the sets \(Y \subseteq X\) with the property that for all \(y \in Y\) there exists \(\epsilon > 0\) such that \(B_\epsilon(y) \subseteq Y\). This is indeed a topology, and every open set is a union of open balls as defined before.

A Cauchy sequence in a metric space \((X, d)\) is a sequence of elements \(x_1, x_2, \ldots \in X\) such that \(\lim_{n \to \infty} d(x_n, x_{n+1}) = 0\). A sequence \(x_1, x_2, \ldots \in X\) converges in \((X, d)\) if there exists \(x \in X\) such that for all \(\epsilon > 0\) there exists \(n \in \mathbb{N}\) such that \(d(x_n, x) < \epsilon\). A metric \(d\) on \(X\) is called complete if all Cauchy sequences in \((X, d)\) converge.

A topological space \((X, \mathcal{I})\) is called metrizable if there exists a metric \(d\) on \(X\) such that \(\mathcal{I} = \mathcal{I}(d)\). The space is called completely metrizable if \(d\) can be chosen to be complete.

A topological space \((X, \mathcal{I})\) is called separable if it contains a countable dense subset, that is, a countable set \(D \in \mathcal{I}\) such that \(\overline{D} = \bigcap\{Y \subseteq X : Y \text{ closed and } D \subseteq Y\} = X\). (The set \(\overline{D}\) is called the closure of \(D\).)

A topological space is Polish if it is completely metrizable and separable. A measurable space \((\Omega, \mathcal{A})\) is standard Borel if \(\mathcal{A}\) is generated by the open sets of a Polish topology on \(\Omega\).

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REFERENCES

[1] Serge Abiteboul, T.-H. Hubert Chan, Evgeny Kharlamov, Werner Nutt, and Pierre Senellart. 2011. Capturing continuous data and answering aggregate queries in probabilistic XML. ACM Trans. Datab. Syst. 36, 4 (2011), 1–45. https://doi.org/10.1145/2043652.2043658

[2] Serge Abiteboul, Richard Hull, and Victor Vianu. 1995. Foundations of Databases (1st ed.). Addison-Wesley, Reading, MA.

[3] Serge Abiteboul, Benny Kimelfeld, Yehoshua Sagiv, and Pierre Senellart. 2009. On the expressiveness of probabilistic XML models. VLDB J. 18, 5 (2009), 1041–1064. https://doi.org/10.1007/s00778-009-0146-1

[4] Charu C. Aggarwal (Ed.). 2009. Managing and Mining Uncertain Data. Advances in Database Systems, Vol. 35. Springer Science+Business Media, LLC, Boston, MA. https://doi.org/10.1007/978-0-387-09690-2

[5] Charu C. Aggarwal and Philip S. Yu. 2009. A survey of uncertain data algorithms and applications. IEEE Trans. Knowl. Data Eng. 21, 5 (2009), 609–623. https://doi.org/10.1109/TKDE.2008.190

Journal of the ACM, Vol. 69, No. 5, Article 37. Publication date: October 2022.
[6] Parag Agrawal and Jennifer Widom. 2009. Continuous uncertainty in trio. In Proceedings of the 3rd VLDB Workshop on Management of Uncertain Data (MUD’09) in Conjunction with (VLDB’09), Ander de Keijzer and Maurice van Keulen (Eds.), Vol. WP09-14. Centre for Telematics and Information Technology (CTIT), University of Twente, The Netherlands, 17–32.

[7] P. Andritsos, A. Fuxman, and R.J. Miller. 2006. Clean answers over dirty databases: A probabilistic approach. In Proceedings of the 22nd International Conference on Data Engineering (ICDE’06). IEEE Computer Society, 30:1–30:18. https://doi.org/10.1109/ICDE.2006.35

[8] Lyublena Antova, Christoph Koch, and Dan Olteanu. 2009. $10^{10^{10}}$ worlds and beyond: Efficient representation and processing of incomplete information. VLDB J. 18, 5 (2009), 1021–1040. https://doi.org/10.1007/s00778-009-0149-y

[9] Sanjeev Arora and Boaz Barak. 2009. Computational Complexity: A Modern Approach. Cambridge University Press, Cambridge, UK. https://doi.org/10.1017/CBO9780511804090

[10] Adrian Baddeley. 2007. Spatial Point Processes and their Applications. Lecture Notes in Mathematics, Vol. 1892. Springer, New York, NY.

[11] Vince Bárány, Balder ten Cate, Benny Kimelfeld, Dan Olteanu, and Zografoula Vagena. 2017. Declarative probabilistic programming with datalog. ACM Trans. Datab. Syst. 42, 4 (2017), 22:1–22:35. https://doi.org/10.1145/3132700

[12] Vaishak Belle. 2017. Open-universe weighted model counting. In Proceedings of the 31st AAAI Conference on Artificial Intelligence (AAAI’17). AAAI Press, San Francisco, CA, 3701–3708.

[13] Vaishak Belle. 2020. Symbolic logic meets machine learning: A brief survey in infinite domains. In Scalable Uncertainty Management (Lecture Notes in Computer Science), Jesse Davis and Karim Tabia (Eds.). Springer International Publishing, Cham, Switzerland, 3–16. https://doi.org/10.1007/978-3-030-58449-8_1

[14] Michael Benedikt, Evgeny Kharelov, Dan Olteanu, and Pierre Senellart. 2010. Probabilistic XML via markov chains. Proc. VLDB Endow. 3, 1–2 (2010), 770–781. https://doi.org/10.14778/1920841.1920939

[15] Béla Bollobás. 2001. Random Graphs (2nd ed.). Cambridge University Press, Cambridge, UK. https://doi.org/10.1017/CBO9780511814068

[16] Stefan Borgwardt, Ismail Ilkan Ceylan, and Thomas Lukasiewicz. 2017. Ontology-mediated queries for probabilistic databases. In Proceedings of the 31st AAAI Conference on Artificial Intelligence (AAAI’17). AAAI Press, 1063–1069.

[17] Stefan Borgwardt, Ismail Ilkan Ceylan, and Thomas Lukasiewicz. 2018. Recent advances in querying probabilistic knowledge bases. In Proceedings of the 27th International Joint Conference on Artificial Intelligence (IJCAI’18). International Joint Conferences on Artificial Intelligence Organization, 5420–5426. https://doi.org/10.24963/ijcai.2018/765

[18] T. J. T. A. Bromwich. 1926. An Introduction to the Theory of Infinite Series (2nd, revised ed.). Macmillan and Company Ltd., London, UK.

[19] Zhuhua Cai, Zografoula Vagena, Luis Perez, Subramanian Arumugam, Peter J. Haas, and Christopher Jermaine. 2013. Simulation of database-valued markov chains using SimSQL. In Proceedings of the 29th ACM SIGMOD-SIGACT-SIGAI Symposium on Principles of Database Systems (PODS’20). ACM, 388–401. https://doi.org/10.1145/3452021.3458315

[20] Nofar Carmeli, Martin Grohe, Peter Lindner, and Christoph Standke. 2021. Tuple-independent representations of infinite probabilistic databases. In Proceedings of the 40th ACM SIGMOD-SIGACT-SIGART Symposium on Principles of Database Systems (PODS’21). ACM, 388–401. https://doi.org/10.1145/3452021.3458315

[21] Ismail Ilkan Ceylan, Adnan Darwiche, and Guy Van den Broeck. 2016. Open-world probabilistic databases. In Proceedings of the 15th International Conference on Principles of Knowledge Representation and Reasoning (KR’16). AAAI Press, 339–348.

[22] Marco Console, Matthias Hofer, and Leonid Libkin. 2020. Queries with arithmetic on incomplete databases. In Proceedings of the 39th ACM SIGMOD-SIGACT-SIGART Symposium on Principles of Database Systems (PODS’20). Association for Computing Machinery, New York, NY, 179–189. https://doi.org/10.1145/3375395.3387666

[23] D. J. Daley and D. Vere-Jones. 2003. An Introduction to the Theory of Point Processes, Volume I: Elementary Theory and Methods (2nd ed.). Springer, New York, NY. https://doi.org/10.1007/b97277

[24] D. J. Daley and D. Vere-Jones. 2008. An Introduction to the Theory of Point Processes, Volume II: General Theory and Structure (2nd ed.). Springer, New York, NY. https://doi.org/10.1007/978-0-387-49835-5

[25] Nilesh Dalvi, Gerome Miklau, and Dan Suciu. 2004. Asymptotic conditional probabilities for conjunctive queries. In Database Theory (ICDT’05) (Lecture Notes in Computer Science), Thomas Eiter and Leonid Libkin (Eds.), Vol. 3363. Springer-Verlag, Berlin, 289–305. https://doi.org/10.1007/978-3-540-30570-5_20

[26] Nilesh Dalvi, Christopher Ré, and Dan Suciu. 2009. Probabilistic databases: Diamonds in the dirt. Commun. ACM 52, 7 (2009), 86–94. https://doi.org/10.1145/1538788.1538810

[27] Nilesh Dalvi and Dan Suciu. 2004. Efficient query evaluation on probabilistic databases. In Proceedings of the 30th International Conference on Very Large Data Bases, Volume 30 (VLDB’04). VLDB Endowment, 864–875.

[28] Nilesh Dalvi and Dan Suciu. 2007. Management of probabilistic data: Foundations and challenges. In Proceedings of the 26th ACM SIGMOD-SIGACT-SIGART Symposium on Principles of Database Systems (PODS’07). Association for Computing Machinery, 1–12. https://doi.org/10.1145/1265530.1265531

Journal of the ACM, Vol. 69, No. 5, Article 37. Publication date: October 2022.
[29] Nilesh Dalvi and Dan Suciu. 2012. The dichotomy of probabilistic inference for unions of conjunctive queries. J. ACM 59, 6 (2012), 1–87. https://doi.org/10.1145/2395116.2395119

[30] Swaraj Dash and Sam Staton. 2021. A monad for probabilistic point processes. In Proceedings of the 3rd Annual International Applied Category Theory Conference (ACT’20) (Electronic Proceedings in Theoretical Computer Science), Vol. 333. Open Publishing Association, 19–32. https://doi.org/10.4204/EPTCS.333.2

[31] Luc De Raedt, Kristian Kersting, Sriraam Natarajan, and David Poole. 2016. Statistical Relational Artificial Intelligence: Logic, Probability, and Computation. Synthesis Lectures on Artificial Intelligence and Machine Learning, Vol. 10. Morgan & Claypool Publishers, San Rafael, CA, USA. https://doi.org/10.2200/S00692ED1V01Y201601AIM032

[32] Luc De Raedt, Angelika Kimmig, and Hannu Toivonen. 2007. ProbLog: A probabilistic prolog and its application in link discovery. In Proceedings of the 20th International Joint Conference on Artificial Intelligence (IJCAI’07). Morgan Kaufmann, San Francisco, CA, 2468–2473.

[33] Christopher De Sa, Ihab F. Ilyas, Benny Kimelfeld, Christopher Ré, and Theodoros Rekatsinas. 2019. A formal framework for probabilistic unclean databases. In Proceedings of the 22nd International Conference on Database Theory (ICDT’19) (Leibniz International Proceedings in Informatics), Pablo Barcelo and Marco Calautti (Eds.), Vol. 127. Schloss Dagstuhl–Leibniz-Zentrum für Informatik, Dagstuhl, Germany, 6:1–6:18. https://doi.org/10.4230/LIPIcs.ICDT.2019.6

[34] Amol Deshpande, Carlos Guestrin, Samuel R. Madden, Joseph M. Hellerstein, and Wei Hong. 2004. Model-driven data acquisition in sensor networks. In Proceedings of the VLDB Conference. Morgan Kaufmann, 588–599. https://doi.org/10.1016/B978-0-12088469-8.50053-X

[35] Xin Luna Dong, Evgeniy Gabrilovich, Geremy Heitz, Wilko Horn, Ni Lao, Kevin Murphy, Thomas Strohmann, Shao-hua Sun, and Wei Zhang. 2014. Knowledge vault: A web-scale approach to probabilistic knowledge fusion. In Proceedings of the 20th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining (KDD’14). ACM, New York, NY, 601–610. https://doi.org/10.1145/2623330.2623623

[36] A. Faradjiann, J. Gehrke, and P. Bonnett. 2002. GADT: A probability space ADT for representing and querying the physical world. In Proceedings of the 18th International Conference on Data Engineering (ICDE’02). IEEE, 201–211. https://doi.org/10.1109/ICDE.2002.994710

[37] William Feller. 1968. An Introduction to Probability Theory and Its Applications, Volume I (3rd ed.). John Wiley & Sons, Inc., New York, NY.

[38] David H. Fremlin. 2013. Measure Theory. Volume 4, Part 1: Topological Measure Spaces (2013 ed.). Cambridge University Press, Cambridge, UK.

[39] Tal Friedman and Guy Van den Broeck. 2013. On constrained open-world probabilistic databases. In Proceedings of the 28th International Joint Conference on Artificial Intelligence (IJCAI’19). International Joint Conferences on Artificial Intelligence Organization, 5722–5729. https://doi.org/10.24963/ijcai.2019/793

[40] Norbert Fuhr and Thomas Rölleke. 1997. A probabilistic relational algebra for the integration of information retrieval and database systems. ACM Trans. Inf. Syst. 15, 1 (1997), 32–66. https://doi.org/10.1145/239041.239045

[41] Noah D. Goodman, Vikash K. Mansinghka, Daniel Roy, Keith Bonawitz, and Joshua B. Tenenbaum. 2008. Church: A language for generative models. In Proceedings of the 24th Conference on Uncertainty in Artificial Intelligence (UAI’08). AUAI Press, 220–229.

[42] Andrew D. Gordon, Thomas A. Henzinger, Aditya V. Nori, and Sriram K. Rajamani. 2014. Probabilistic programming. In Future of Software Engineering Proceedings (FOSE’14). Association for Computing Machinery, New York, NY, 167–181. https://doi.org/10.1145/2593882.2593900

[43] Erich Grädel, Yuri Gurevich, and Colin Hirsch. 1998. The complexity of query reliability. In Proceedings of the 17th ACM SIGACT-SIGMOD-SIGART Symposium on Principles of Database Systems (PODS’98). ACM Press, 227–234. https://doi.org/10.1145/275487.295124

[44] Todd J. Green and Val Tannen. 2006. Models for incomplete and probabilistic information. In Current Trends in Database Technology: Proceedings of the International Conference on Extending Database Technology (EDBT’06), Torsten Grust, Hagen Höpner, Arantza Illarramendi, Stefan Jablonski, Marco Mesiti, Sascha Müller, Paula-Lavinia Patranjan, Kai-Uwe Sattler, Myra Spiliopoulou, and Jef Wijsen (Eds.). Springer-Verlag, Berlin, 278–296. https://doi.org/10.1007/11896548_24 Revised paper from the IIDB 2006 International Workshop on Inconsistency and Incompleteness in Databases.

[45] Martin Grohe, Benjamin Lucien Kaminski, Joost-Pieter Katoen, and Peter Lindner. 2020. Generative datalog with continuous distributions. In Proceedings of the 39th ACM SIGMOD-SIGACT-SIGAI Symposium on Principles of Database Systems (PODS’20). Association for Computing Machinery, New York, NY, 347–360. https://doi.org/10.1145/3375395.3387659

[46] Martin Grohe and Peter Lindner. 2019. Probabilistic databases with an infinite open-world assumption. In Proceedings of the 38th ACM SIGMOD-SIGACT-SIGAI Symposium on Principles of Database Systems (PODS’19). ACM, New York, NY, 17–31. https://doi.org/10.1145/3294052.3319681 Extended version available on arXiv e-prints: arXiv:1807.00607 [cs.DB].

Journal of the ACM, Vol. 69, No. 5, Article 37. Publication date: October 2022.
[47] Martin Grohe and Peter Lindner. 2020. Infinite probabilistic databases. In 23rd International Conference on Database Theory (ICDT’20) (Leibniz International Proceedings in Informatics), Carsten Lutz and Jean Christoph Jung (Eds.), Vol. 155. Schloss Dagstuhl–Leibniz-Zentrum für Informatik, Dagstuhl, Germany, 16:1–16:20. https://doi.org/10.4230/LIPIcs.ICDT.2020.16 Extended version available on arXiv e-prints: arXiv:1904.06766 [cs.DB].

[48] Martin Grohe and Peter Lindner. 2022. Infinite probabilistic databases. Logic. Methods Comput. Sci. 18, 1 (Feb 2022). https://doi.org/10.46298/lmcs-18(1:2)2022

[49] Martin Grohe, Peter Lindner, and Christoph Standke. 2022. Probabilistic query evaluation with bag semantics. arXiv:2201.11524v2. Retrieved from http://arxiv.org/abs/2201.11524.

[50] Bernd Gutmann, Manfred Jaeger, and Luc De Raedt. 2011. Extending problog with continuous distributions. In Inductive Logic Programming (ILP’10) (Lecture Notes in Artificial Intelligence), Paolo Frasconi and Francesca A. Lisi (Eds.). Springer-Verlag, Berlin, 76–91. https://doi.org/10.1007/978-3-642-21295-6_12

[51] Tomasz Imieliński and Witold Lipski. 1984. Incomplete information in relational databases. J. ACM 31, 4 (September 1984), 761–791. https://doi.org/10.1145/1634.1886

[52] Ravi Jampani, Fei Xu, Mingxi Wu, Luis Perez, Chris Jermaine, and Peter J. Haas. 2011. The monte carlo database system: Stochastic analysis close to the data. ACM Trans. Datab. Syst. 36, 3 (2011), 1–41. https://doi.org/10.1145/2000824.

[53] Abhay Jha and Dan Suciu. 2012. Probabilistic databases with markovviews. Proc. VLDB Endow. 5, 11 (2012), 1160–1171. https://doi.org/10.14778/2350229.2350236

[54] Olav Kallenberg. 2017. Random Measures, Theory and Applications. Probability Theory and Stochastic Modelling, Vol. 77. Springer International Publishing, Cham. https://doi.org/10.1007/978-3-319-41598-7

[55] Achim Klenke. 2014. Probability Theory: A Comprehensive Course. Springer London, London, UK. https://doi.org/10.1007/978-1-4471-5361-0

[56] Benny Kimelfeld and Pierre Senellart. 2013. Probabilistic XML: Models and complexity. In Advances in Probabilistic Databases for Uncertain Information Management, Zongmin Ma and Li Yan (Eds.). Studies in Fuzziness and Soft Computing, Vol. 304. Springer-Verlag, Berlin, 39–66. https://doi.org/10.1007/978-3-642-37509-5_3

[57] Benny Kimelfeld and Pierre Senellart. 2013. Probabilistic XML: Models and complexity. In Advances in Probabilistic Databases for Uncertain Information Management, Zongmin Ma and Li Yan (Eds.). Studies in Fuzziness and Soft Computing, Vol. 304. Springer-Verlag, Berlin, 39–66. https://doi.org/10.1007/978-3-642-37509-5_3

[58] J. F. C. Kingman. 1967. Completely random measures. Pac. J. Math. 21, 1 (1967), 59–78.

[59] J. F. C. Kingman. 1967. Completely random measures. Pac. J. Math. 21, 1 (1967), 59–78.

[60] Konrad Knopp. 1996. Theorie und Anwendung Der Unendlichen Reihen (6th ed.). Springer-Verlag, Berlin. https://doi.org/10.1007/978-3-642-61406-4

[61] Dexter Kozen. 1997. Automata and Computability (1st ed.). Springer-Verlag, New York, NY. https://doi.org/10.1007/978-1-4612-1844-9

[62] Günter Last and Mathew Penrose. 2017. Lectures on the Poisson Process (1st ed.). Cambridge University Press, Cambridge, UK. https://doi.org/10.1017/9781316104477

[63] Günter Last and Mathew Penrose. 2017. Lectures on the Poisson Process (1st ed.). Cambridge University Press, Cambridge, UK. https://doi.org/10.1017/9781316104477

[64] Leonid Libkin. 2018. Certain answers meet zero-one laws. In Proceedings of the 19th International Conference on Artificial Intelligence (IJCAI’05), Morgan Kaufmann, Inc., San Francisco, CA, 1352–1359.

[65] Leonid Libkin. 2018. Certain answers meet zero-one laws. In Proceedings of the 19th International Conference on Artificial Intelligence (IJCAI’05), Morgan Kaufmann, Inc., San Francisco, CA, 1352–1359.

[66] Leonid Libkin. 2018. Certain answers meet zero-one laws. In Proceedings of the 19th International Conference on Artificial Intelligence (IJCAI’05), Morgan Kaufmann, Inc., San Francisco, CA, 1352–1359.

[67] Raymond Reiter. 1981. On closed world databases. In Proceedings of the 12th International Joint Conference on Artificial Intelligence (IJCAI’81). Morgan Kaufmann, Inc., San Francisco, CA, 1352–1359.

[68] Raymond Reiter. 1981. On closed world databases. In Proceedings of the 12th International Joint Conference on Artificial Intelligence (IJCAI’81). Morgan Kaufmann, Inc., San Francisco, CA, 1352–1359.

[69] Raymond Reiter. 1981. On closed world databases. In Proceedings of the 12th International Joint Conference on Artificial Intelligence (IJCAI’81). Morgan Kaufmann, Inc., San Francisco, CA, 1352–1359.

[70] Taisuke Sato. 1995. A statistical learning method for logic programs with distribution semantics. In Proceedings of the 12th International Conference on Logic Programming (ICLP’95). MIT Press, 715–729.
[71] Sarvjeet Singh, Chris Mayfield, Sagar Mittal, Sunil Prabhakar, Susanne Hambrusch, and Rahul Shah. 2008. Orion 2.0: Native support for uncertain data. In Proceedings of the ACM SIGMOD International Conference on Management of Data. ACM, New York, NY, 1239–1242. https://doi.org/10.1145/1376616.1376744

[72] Sarvjeet Singh, Chris Mayfield, Rahul Shah, Sunil Prabhakar, Susanne Hambrusch, Jennifer Neville, and Reynold Cheng. 2008. Database support for probabilistic attributes and tuples. In Proceedings of the IEEE 24th International Conference on Data Engineering (ICDE’08). IEEE, 1053–1061. https://doi.org/10.1109/ICDE.2008.4497514

[73] Parag Singla and Pedro Domingos. 2007. Markov logic in infinite domains. In Proceedings of the 23rd Conference on Uncertainty in Artificial Intelligence (UAI’07). AUAI Press, 368–375.

[74] Joel Spencer. 2001. The Strange Logic of Random Graphs. Springer-Verlag, Berlin. https://doi.org/10.1007/978-3-662-04538-1

[75] Julia Stoyanovich, Susan Davidson, Tova Milo, and Val Tannen. 2011. Deriving probabilistic databases with inference ensembles. In Proceedings of the IEEE 27th International Conference on Data Engineering (ICDE’11). IEEE, 303–314. https://doi.org/10.1109/ICDE.2011.5767854

[76] Dan Suciu, Dan Olteanu, Christopher Ré, and Christoph Koch. 2011. Probabilistic Databases (1st ed.). Morgan & Claypool, San Rafael, CA. https://doi.org/10.2200/S00362ED1V01Y201105DTM016

[77] David Tolpin, Jan-Willem van de Meent, and Frank Wood. 2015. Probabilistic programming in anglican. In Machine Learning and Knowledge Discovery in Databases (Lecture Notes in Computer Science), Albert Bifet, Michael May, Bianca Zadrozny, Ricard Gavalda, Dino Pedreschi, Francesco Bonchi, Jaime Cardoso, and Myra Spiliopoulou (Eds.). Springer International Publishing, Cham, Switzerland, 308–311. https://doi.org/10.1007/978-3-319-23461-8_36

[78] Guy Van den Broeck and Dan Suciu. 2017. Query processing on probabilistic data: A survey. Found. Trends Datab. 7, 3–4 (2017), 197–341. https://doi.org/10.1561/1900000052

[79] Ron van der Meyden. 1998. Logical approaches to incomplete information: A survey. In Logics for Databases and Information Systems, Jan Chomicki and Gunter Saake (Eds.). Kluwer Academic Publishers, Boston, MA, 307–356.

[80] Jue Wang and Pedro Domingos. 2008. Hybrid markov logic networks. In Proceedings of the 23rd AAAI Conference on Artificial Intelligence (AAAI’08). 1106–1111.

[81] Y. H. Wang. 1993. On the number of successes in independent trials. Stat. Sin. 3, 2 (1993), 295–312.

[82] Jennifer Widom. 2009. Trio: A system for data, uncertainty, and lineage. In Managing and Mining Uncertain Data, Charu C. Aggarwal (Ed.). Advances in Database Systems, Vol. 35. Springer Science+Business, LLC, Boston, MA, 307–356. https://doi.org/10.1007/978-1-4615-5643-5_10

[83] Stephen Willard. 2004. General Topology (Dover reprint ed.). Dover Publications, Inc., Mineola, NY.

[84] Yi Wu, Siddharth Srivastava, Nicholas Hay, Simon Du, and Stuart Russell. 2018. Discrete-continuous mixtures in probabilistic programming: Generalized semantics and inference algorithms. In Proceedings of the 35th International Conference on Machine Learning (ICML ’18), Jennifer Dy and Andreas Krause (Eds.), Vol. 80. PMLR, 5343–5352.

[85] Ce Zhang. 2015. DeepDive: A Data Management System for Automatic Knowledge Base Construction. Ph.D. Dissertation. University of Wisconsin—Madison.

[86] Esteban Zimányi and Alain Pirotte. 1997. Imperfect information in relational databases. In Uncertainty Management in Information Systems (reprint of 1st ed.), Amihai Motro and Philippe Smets (Eds.). Springer Science+Business Media, 35–87. https://doi.org/10.1007/978-1-4615-6245-0_3

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