Numerical solution of fractional Volterra integral equation with piecewise continuous kernel

D Sidorov\textsuperscript{1,2}, I Muftahov\textsuperscript{1,3} and A Tynda\textsuperscript{4}

\textsuperscript{1} Melentiev Energy Systems Institute, Siberian Branch of the Russian Academy of Sciences, 130 Lermontov St., 664033 Irkutsk, Russia
\textsuperscript{2} Institute of Mathematics and Information Technologies, Irkutsk State University, 1 K. Marks St., 664003 Irkutsk, Russia
\textsuperscript{3} Main Computing Center of JSC Russian Railways, Irkutsk Department, 23 Mayakovskogo St., 664005 Irkutsk, Russia
\textsuperscript{4} Penza State University, 40 Krasnaya St., 440026 Penza, Russia

E-mail: ildar_sm@mail.ru

Abstract. Integral equations play the important role in applied mathematics, related to many areas of theory, especially applications. In this article, we consider the numerical method for solving Volterra integral equations for the fractional order of integration. The error of this method is $O(1/N)$.

1. Introduction

The Volterra integral equations [1] were introduced by Vito Volterra and then studied by Traian Lalescu in his 1908 thesis. Such weakly regular equations were introduced in [2], and in the monograph [3] the theory of such equations is generalized to the case of systems of equations and to abstract operator equations.

Fractional integration and fractional differentiation are generalizations of notions of integer-order integration and differentiation, and include $n$-th derivatives and $n$-folded integrals ($n$ denotes an integer number) as particular cases. The first mention of derivatives of non-integer order is contained in the correspondence between G. Leibniz and J. Bernoulli. There is also an interesting mention of G. Leibniz in a letter to G. Lopital about the paradox and possible useful practical application of differentials of order $1/2$. In recent years, interest in fractional calculus has been growing [4, 5, 6].

This article discusses weakly regular Volterra equations of the first kind

\[ \int_0^t K(t, s)x(s)ds = f(t), \quad 0 \leq s \leq t \leq T, \quad f(0) = 0, \]  

where the kernel is defined by the formula

\[ K(t, s) = \begin{cases} 
K_1(t, s), & t, s \in m_1, \\
\ldots & \ldots \\
K_n(t, s), & t, s \in m_n, \\
\end{cases} \quad m_i = \{ t, s \mid \alpha_{i-1}(t) < s < \alpha_i(t) \}, \]  

with

\[ \alpha_0(t) = 0, \quad \alpha_n(t) = t, \quad i = 1, n. \]
\[ \alpha_i(t), f(t) \in C^1_{[0,T]}, K_i(t,s) \text{ have continuous derivatives with respect to } t \text{ for } t \in [0,T], \ K_n(t,t) \neq 0, \ 
\alpha_i(0) = 0, \ 0 < \alpha_1(t) < \alpha_2(t) < \ldots < \alpha_{n-1}(t) < t, \ \alpha_1(t), \ldots, \alpha_{n-1}(t) \text{ grow in a small}
\] neighborhood \(0 \leq t \leq \tau, 0 < \alpha_i'(0) \leq \ldots \leq \alpha_{n-1}'(0) < 1.\)

In the case of a fractional order of integration for these equations let us now consider the
left-sided Riemann–Liouville fractional integral \([24]\) of order \(\beta \geq 0\)

\[ \int_t^\beta f(t) = \frac{1}{\Gamma(\beta)} \int_0^t \frac{f(s)}{(t-s)^\beta} ds. \quad (3) \]

For equation (1) in the case of a fractional order of integration (3), we have to solve the
following integral equation

\[ \frac{1}{\Gamma(\beta)} \int_0^t \frac{K(t,s)}{(t-s)^\beta} x(s) ds = f(t), \ 0 \leq s \leq t \leq T, \ f(0) = 0. \quad (4) \]

2. Numerical method

In this section, we show a general numerical method for weakly regular Volterra equations (1),
which is based on the use of the quadrature formula for mean rectangles. The error of the
proposed method is of the order \(O(1/N)\).

In order to construct a numerical solution to the equation (1) on the segment \([0,T]\), we
introduce a grid of nodes (not necessarily uniform)

\[ 0 = t_0 < t_1 < t_2 < \ldots < t_N = T, \ h = \max_{i=1,N} (t_i - t_{i-1}) = O(N^{-1}). \quad (5) \]

An approximate solution to the equation (1) will be sought in the form of a piecewise constant
function

\[ x_N(t) = \sum_{i=1}^N x_i \delta_i(t), \ t \in (0,T], \ \delta_i(t) = \begin{cases} 1, & \text{for } t \in \Delta_i = (t_{i-1}, t_i]; \\ 0, & \text{for } t \notin \Delta_i \end{cases} \quad (6) \]

with undefined coefficients \(x_i, i = \overline{1,N}\).

To determine the values \(x_0 = x(0)\) we differentiate both parts of each equation of (1) by \(t\)
and replace \(\frac{1}{\Gamma(\beta)} \frac{K_i(t,s)}{(t-s)^\beta}\) with \(g_i(t,s,\beta)\)

\[ f'(t) = \frac{1}{\Gamma(\beta)} \sum_{i=1}^n \left( \frac{\alpha_i(t)}{\alpha_{i-1}(t)} \int_0^t \frac{\partial g_i(t,s,\beta)}{\partial t} x(s) ds + \alpha_i'(t) g_i(t, \alpha_i(t), \beta) x(\alpha_i(t)) - \right. \]

\[ \left. \alpha_{i-1}'(t) g_i(t, \alpha_{i-1}(t), \beta) x(\alpha_{i-1}(t)) \right) \]

From the last relation we obtain

\[ x_0 = \frac{f'(0)}{\sum_{i=1}^n g_i(0,0,\beta) \left[ \alpha'_i(0) - \alpha'_{i-1}(0) \right]}. \quad (7) \]

We assume solution existence and \(\sum_{i=1}^n g_i(0,0,\beta) \cdot [\alpha'_i(0) - \alpha'_{i-1}(0)] \neq 0.\)
Next, we introduce the designation \( f_k = f(t_k), \ k = 1, \ldots, N \). In order to determine the coefficients \( x_1 \) write the original equation at the point \( t = t_1 \)

\[
\sum_{i=1}^{n} \alpha_i(t_1) \int_{\alpha_i-1(t_1)}^{\alpha_i(t_1)} g_i(t_1, s, \beta) x(s) ds = f_1. \tag{8}
\]

Since at this step the lengths of all integration segments \( \alpha_i(t_1) - \alpha_i-1(t_1) \) in (8) do not exceed the grid step \( h \), and the components of the approximate solution take the value \( x_1 \), then applying the quadrature formula of the middle rectangles, we have a system of

\[
x_1 = \frac{f_1}{\sum_{i=1}^{n} (\alpha_i(t_1) - \alpha_i-1(t_1)) g_i(t_1, \frac{\alpha_i(t_1) + \alpha_i-1(t_1)}{2}, \beta)}. \tag{9}
\]

Now we introduce the notation \( v_{ij} \) as the grid segment number (5), inside or on the right border of which the value \( \alpha_i(t_j) \) falls, i.e. \( \alpha_i(t_j) \in \Delta v_{ij} \). Obviously, \( v_{ij} < j \) for \( i = 0, n - 1 \), \( j = 1, N \), since in this case \( \alpha_i(t_j) < t_j \).

Now let the coefficients \( x_0, x_1, \ldots, x_{k-1} \) of the approximate solution be known. The equation (1) at the point \( t = t_k \), which looks like

\[
\sum_{i=1}^{n} \alpha_i(t_k) \int_{\alpha_i-1(t_k)}^{\alpha_i(t_k)} g_i(t_k, s, \beta) x(s) ds = f_k,
\]

can be rewritten as follows

\[
I_1(t_k) + I_2(t_k) + \cdots + I_n(t_k) = f_k,
\]

where

\[
I_1(t_k) = \sum_{j=1}^{\nu_1-1} \int_{t_{j-1}}^{t_j} g_1(t_k, s, \beta) x(s) ds + \int_{\nu_1-1}^{t_{\nu_1-1,k}} g_1(t_k, s, \beta) x(s) ds,
\]

\[
I_n(t_k) = \int_{\alpha_{n-1}(t_k)}^{t_{\nu_{n-1,k}}} g_n(t_k, s, \beta) x(s) ds + \sum_{j=\nu_{n-1,k}+1}^{t_k} \int_{t_j}^{t_{j-1}} g_n(t_k, s, \beta) x(s) ds.
\]

(i) If \( v_{p-1,k} \neq v_{p,k}, p = 2, \ldots, n - 1 \), then

\[
I_p(t_k) = \int_{\alpha_{p-1}(t_k)}^{t_{\nu_{p-1,k}}} g_p(t_k, s, \beta) x(s) ds + \sum_{j=\nu_{p-1,k}+1}^{t_j} \int_{t_j}^{t_{j-1}} g_p(t_k, s, \beta) x(s) ds + \int_{t_{\nu_{p,k}-1}}^{t_{p,k}} g_p(t_k, s, \beta) x(s) ds.
\]

(ii) If \( v_{p-1,k} = v_{p,k}, p = 2, \ldots, n - 1 \), then

\[
I_p(t_k) = \int_{\alpha_{p-1}(t_k)}^{t_{\nu_{p,k}}} g_p(t_k, s, \beta) x(s) ds.
\]
The number of terms in each line of the last formula depends on the array $v_{ij}$, determined by the input data: functions $\alpha_i(t)$, $i = 1, n - 1$ and fixed (for a specific $N$ value) grid.

Each integral in the last equality is now approximated by the mean rectangles formula, i.e.

$$\int_{t_{v_{p,k-1}}}^{x_p(t_k)} g_p(t_k, s, \beta)x(s)ds \approx \left(\alpha_p(t_k) - t_{v_{p,k-1}}\right) g_p\left(t_k, \frac{\alpha_p(t_k) + t_{v_{p,k-1}}}{2}, \beta\right) x_N\left(\frac{\alpha_p(t_k) + t_{v_{p,k-1}}}{2}\right).$$

In addition, on those intervals where the required function is defined, we choose $x_N(t)$ (i.e. $t_k \leq t_{k-1}$). On the remaining intervals, the unknown value of $x_k$ appears in the last few terms. We express explicitly $x_k$ and go to the next step in the loop along $k$. The number of such terms is determined by the initial data as a result of the analysis of the $v_{ij}$ array. The method error

$$\varepsilon = \max_{0 \leq i \leq N} |\bar{x}(t_i) - x^h(t_i)| \tag{10}$$

is of the order $O\left(\frac{1}{N}\right)$.

3. Numerical results

Let us the following equation

$$\frac{1}{\Gamma(\beta)} \int_{0}^{t} \frac{1}{(t-s)^\beta} x(s)ds = t^\eta, \ 0 \leq s \leq t \leq T, \ T = 1,$$

where $\beta = \frac{1}{2}, \eta = 3, x(t) = \frac{\Gamma(\eta+1)}{\Gamma(\eta+1-\beta)} t^{\eta-\beta} = \frac{\Gamma(4)}{\Gamma\left(\frac{7}{2}\right)} t^{\frac{5}{2}}$. Table 1 contains the errors of the numerical solution for various $N$.

| N   | Error     |
|-----|-----------|
| 32  | 0.124393063248813 |
| 64  | 0.099971007955374  |
| 128 | 0.076611016034783  |
| 256 | 0.057093624079822  |
| 512 | 0.041815810168616  |
| 1024| 0.030283935263120   |
| 2048| 0.021769054657969   |

Let us also show the results for the second equation

$$\frac{1}{\Gamma(\beta)} \int_{0}^{t} \frac{1}{(t-s)^\beta} x(s)ds = \frac{\Gamma(4)}{\Gamma\left(\frac{7}{2}\right)} t^{\frac{5}{2}}, \ 0 \leq s \leq t \leq T, \ T = 1,$$

where $\beta = \frac{1}{2}, x(t) = 3t^2$. Table 2 contains the errors of the numerical solution for various $N$. 




Table 2. Numerical results for $x(t) = 3t^2$

| N   | Error       |
|-----|-------------|
| 32  | 0.197401919081441 |
| 64  | 0.155472440065944 |
| 128 | 0.117764835275243 |
| 256 | 0.087139322743047 |
| 512 | 0.063532151653142 |
| 1024| 0.045875332712031 |
| 2048| 0.032912580723248 |

4. Conclusion
In this article, we have presented the numerical method that can be used to solve linear Volterra integral equations of the first kind with piecewise discontinuous kernel with fractional order of integration.

Acknowledgments
The reported study was funded by RFBR and NSFC according to the research project No. 19-58-53011/61911530132.

References
[1] Linz P 1985 Analytical and Numerical Methods for Volterra Equations (Philadelphia: SIAM)
[2] Sidorov D 2011 Volterra equations of the first kind with discontinuous kernels in the theory of evolving systems control Studia Informatica Universalis 9 135–146
[3] Sidorov D 2015 Integral dynamical models: singularities, signals and control World Scientific Series on Nonlinear Science (Series A vol 87) ed L Chua (Singapore, London: World Scientific Publishing)
[4] Atangana A and Kilicman A 2013 Analytical solutions of the space-time fractional derivative of advection dispersion equation Math. Problems in Engineering 2013 853127
[5] Baleanu D, Diethelm K, Scalas E and Trujillo J 2012 Fractional Calculus Models and Numerical Methods, Complexity, Nonlinearity and Chaos (Singapore, London: World Scientific Publishing)
[6] Kilbas A, Srivastava H and Trujillo J 2006 Theory and applications of fractional differential equations North-Holland Mathematics Studies (vol 204) (Amsterdam: Elsevier Science)