Global well-posedness of the half space problem of the Navier–Stokes equations in critical function spaces of limiting case

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Abstract
In this paper, we study the initial-boundary value problem of the Navier–Stokes equations in half-space. Let a solenoidal initial velocity be given in the function space $\dot{B}^{-1+n/p}_{p,\infty} (\mathbb{R}_n^+) \times (0, \infty)$ for $\max(1, \frac{n}{3}) < p < n$. We prove the global in time existence of weak solution $u \in L^\infty(0, \infty; \dot{B}^{-1+n/p}_{p,\infty} (\mathbb{R}_n^+))$, when the given initial velocity has small norm in function space $\dot{B}^{-1+n/p}_{p,\infty} (\mathbb{R}_n^+)$, where $\max(1, \frac{n}{3}) < p < n$. We also prove the uniqueness of solution for small initial data.

Keywords Stokes equations · Navier–Stokes equations · Homogeneous initial boundary value problem · Half-space

Mathematics Subject Classification Primary 35K61 · Secondary 76D07

1 Introduction
In this paper, we study the initial-boundary value problem of Navier–Stokes equations

\begin{equation}
\begin{aligned}
& u_t - \Delta u + \nabla p = -\text{div}(u \otimes u), \\
& \text{div} u = 0 \quad \text{in} \ \mathbb{R}^n_+ \times (0, \infty), \\
& u(x, 0) = u_0(x) \quad x \in \mathbb{R}^n_+, \\
& u(x', 0, t) = 0 \quad x' \in \mathbb{R}^{n-1}, \ t \in (0, \infty),
\end{aligned}
\end{equation}

(1.1)

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where $n \geq 2$, and $u = (u_1, \ldots, u_n)$ and $p$ are the unknown velocity and pressure, respectively, and $u_0 = (u_{01}, \ldots, u_{0n})$ is the given initial data.

Because the Navier–Stokes equations are invariant under the scaling

$$u_\lambda(x, t) = \lambda u(\lambda x, \lambda^2 t), \quad p_\lambda(x, t) = \lambda^2 p(\lambda x, \lambda^2 t), \quad u_{0\lambda}(x) = \lambda u_0(\lambda x),$$

it is important to study (1.1) in the so-called critical spaces, i.e., the function spaces with norms invariant under the scaling $u(x, t) \to \lambda u(\lambda x, \lambda^2 t)$. The homogeneous Besov space $\dot{B}_{pq}^{1-\frac{n}{p}}(\mathbb{R}^n_+)$ is one of critical spaces.

In this paper, we prove the existence of global time mild solution $u \in L^\infty(0, \infty; \dot{B}_{p\infty}^{-1+\frac{n}{p}}(\mathbb{R}^n_+))$ of (1.1) for the initial data $u_0 \in \dot{B}_{p\infty,0}^{-1+\frac{n}{p}}(\mathbb{R}^n_+)$, max$(1, \frac{n}{p}) < p < n$. See Sect. 2 for definitions of function spaces.

There are a number of papers dealing with global well-posedness for (1.1) in homogeneous Besov space $\dot{B}_{pq}^{1-\frac{n}{p}}(\mathbb{R}^n_+)$, $1 < p, q < \infty$ (see [6–9, 13, 15, 17, 25] and the references therein).

The limiting case $1 < p < \infty, q = \infty$ has been studied by M. Cannone, F. Planchon, and M. Schonbek [4] for $u_0 \in L^3(\mathbb{R}^3_+)$, by H. Amann [2] for $u_0 \in B_{p,\infty}^{-1+\frac{n}{p}}(\Omega)$, $p > \frac{n}{3}, n \geq 3, p \neq n$, where $\Omega$ is a standard domain like $\mathbb{R}^3, \mathbb{R}^3_+$ exterior or bounded domain in $\mathbb{R}^3$ and by M. Ri, P. Zhang and Z. Zhang [28] for $u_0 \in B_{0,\infty}^0(\Omega)$, where $\Omega$ is $\mathbb{R}^n, \mathbb{R}^n_+$ or bounded domain with smooth boundary, and $b_{p,\infty}^s(\Omega)$ denotes the completion of the generalized Sobolev space $H^s_p(\Omega)$ in $B_{p,\infty}^s(\Omega)$. In particular, in [4], the solution $u \in L^\infty_{\frac{1}{2}, \frac{3}{2p}}(0, \infty, L^3(\mathbb{R}^3_+))$ exists globally in time when $\|u_0\|_{\dot{B}_{p\infty,0}^{-1+\frac{n}{p}}(\mathbb{R}^n_+)} < \epsilon_*$ is small enough. Recently, H. Kozono and S. Shimizu [24] showed the existence of solution $u \in L^\infty(0, \infty; \dot{B}_{p\infty}^{-1+\frac{n}{p}}(\mathbb{R}^n_+)), \; p > n$ of (1.1) with sufficiently small norm of $u_0$ in $\dot{B}_{p\infty}^{-1+\frac{n}{p}}(\mathbb{R}^n_+)$. See also [14, 16, 19, 23, 26, 27, 30] and the references therein for the initial-boundary value problem of Navier–Stokes equations in the half space.

Our study of this paper is motivated by the results in [2, 4, 24, 28]. The following text states our main results.

**Theorem 1.1** Let max$(1, \frac{n}{p}) < p < n, n \geq 2$. Assume that $u_0 \in \dot{B}_{p\infty,0}^{-1+\frac{n}{p}}(\mathbb{R}^n_+)$ with $\text{div } u_0 = 0$. Then, there is $\epsilon_* > 0$ so that if

$$\|u_0\|_{\dot{B}_{p\infty,0}^{-1+\frac{n}{p}}(\mathbb{R}^n_+)} < \epsilon_*$$

for some $n < p_0 < \infty$, then (1.1) has a solution $u \in L^\infty(0, \infty; \dot{B}_{p\infty}^{-1+\frac{n}{p}}(\mathbb{R}^n_+)) \cap L^\infty_{\frac{2}{2-p_0}, \frac{n}{p_0}}(0, \infty; L^{p_0}(\mathbb{R}^n_+))$. Moreover, with assumption (1.2), the solution is unique in $L^\infty_{\frac{2}{2-p_0}, \frac{n}{p_0}}(0, \infty; L^{p_0}(\mathbb{R}^n_+))$.

The explanation of spaces and notations is placed in Sect. 2.
Remark 1.2 In Theorem 1.1, observe that $\mathcal{B}_{p,\infty,0}^{-1+\frac{n}{p}}(\mathbb{R}^n_+) \subset \mathcal{B}_{p_0,\infty,0}^{-1+\frac{n}{p_0}}(\mathbb{R}^n_+)$ holds (see (2.4)).

For the proof of Theorem 1.1, it is necessary to study the following initial-boundary value problem of the Stokes equations in $\mathbb{R}^n_+ \times (0, \infty)$:

\[
\begin{align*}
u_t - \Delta u + \nabla p &= f, & \text{div } u &= 0 \text{ in } \mathbb{R}^n_+ \times (0, \infty), \\
\vline_{t=0} &= u_0, & u\vline_{x_n=0} &= 0,
\end{align*}
\]

where $f = \text{div} F$.

In [18], M. Giga, Y. Giga and H. Sohr showed that if $f \in L^q(0, T; \hat{D}(A_{-\alpha}^{-1}))$ and $u_0 = 0$ then the solution $u$ of Stokes equations (1.3) satisfies that for $0 < \alpha < 1$,

\[
\int_0^T \left( \| \frac{d}{dt} u(t) \|_{L^p(\Omega)}^q + \| A_p^{-1-\alpha} u(t) \|_{L^p(\Omega)}^q \right) dt \leq c(p, q, \Omega, \alpha) \int_0^T \| A_p^{-\alpha} f(t) \|_{L^p(\Omega)} dt,
\]

where $A_p$ is Stokes operator in $\Omega$ for standard domain $\Omega$ such as bounded domain, exterior domain or half space, and $\hat{D}(T)$ is the completion of $D(T)$ in the homogeneous norm $\| T \|$. In particular, if $f = \text{div} F$ with $F \in L^q(0, T; L^p_0(\Omega))$ then

\[
\int_0^T \left( \| \frac{d}{dt} u(t) \|_{L^p(\Omega)}^q + \| \nabla u(t) \|_{L^p(\Omega)}^q \right) dt \leq c(p, q, \Omega) \int_0^T \| F(t) \|_{L^p(\Omega)} dt.
\]

H. Koch and V. A. Solonnikov [21] showed the unique local in time existence of solution $u \in L^q(0, T; L^q(\mathbb{R}^n_+))$ of (1.3) when $f = \text{div} F$, $F \in L^q(0, T; L^q(\mathbb{R}^n_+))$ and $u_0 = 0$. See also [19,22,23,30,31] and the references therein.

The following theorem states our result on the unique solvability of the Stokes equations (1.3).

Theorem 1.3 Let $1 < p_1 \leq p < \infty$ and $\alpha > 0$. Let $u_0 \in \mathcal{B}_{p,\infty,0}^{-\alpha}(\mathbb{R}^n_+)$ with div $u_0 = 0$ and $f = \text{div} F$.

1. Let $-1 + \alpha < \frac{n}{p_1} - \frac{n}{p} < 1$ and $F \in L^\infty_{\frac{1}{2}\alpha+\frac{1}{2}}(0, \infty; L^{p_1}(\mathbb{R}^n_+))$. There is a solution $u$ of (1.3) with

\[
\| u \|_{L^\infty_{\frac{1}{2}\alpha}(0, \infty; L^p(\mathbb{R}^n_+))} \leq c \left( \| u_0 \|_{\mathcal{B}_{p,\infty,0}^{-\alpha}(\mathbb{R}^n_+)} + \| F \|_{L^\infty_{\frac{1}{2}\alpha+\frac{1}{2}}(0, \infty; L^{p_1}(\mathbb{R}^n_+))} \right).
\]

2. Let $0 < \alpha < 2$ such that $1 + \frac{n-1}{p} < \frac{n}{p_1} < 1 + \frac{n}{p}$ and $F \in L^\infty_{\frac{1}{2}\alpha+\frac{n}{p_1}+\frac{n}{2p}}(0, \infty; \mathcal{B}_{p,\infty,0}^{\alpha}(\mathbb{R}^n_+))$. There is a solution $u$ of (1.3) with

\[
\| u \|_{L^\infty(0, \infty; \mathcal{B}_{p,\infty,0}^{\alpha}(\mathbb{R}^n_+))} \leq c \left( \| u_0 \|_{\mathcal{B}_{p,\infty,0}^{\alpha}(\mathbb{R}^n_+)} + \| F \|_{L^\infty_{\frac{1}{2}\alpha+\frac{n}{p_1}+\frac{n}{2p}}(0, \infty; \mathcal{B}_{p,\infty,0}^{\alpha}(\mathbb{R}^n_+))} \right).
\]

This paper is organized as follows. In Sect. 2, we introduce the function spaces and definition of the weak solutions of Stokes equations and Navier–Stokes equations. In
Sect. 3, various estimates of operators related to a Newtonian kernel and Gaussian kernel are given. In Sect. 4, we complete the proof of Theorem 1.3. In Sect. 5, we give the proof of Theorem 1.1 while applying the estimates in Theorem 1.3 to the approximate solutions.

2 Notations, function spaces and definitions of weak solutions

We denote by \( x' \) and \( x = (x', x_n) \) the points of the spaces \( \mathbb{R}^{n-1} \) and \( \mathbb{R}^n, \) \( n \geq 2, \) respectively. The multiple derivatives are denoted by \( D_x^k D_t^m = \frac{\partial^{k+m}}{\partial x^k \partial t^m} \) for multi-index \( k \) and nonnegative integers \( m. \) Throughout this paper we denote by \( c \) various generic constants.

For \( s \in \mathbb{R} \) and \( 1 \leq p, q \leq \infty, \) we denote \( \dot{H}_p^s(\mathbb{R}^n) \) and \( \dot{B}_{pq}^s(\mathbb{R}^n) \) the generalized homogeneous Sobolev spaces (space of Bessel potentials) and the homogeneous Besov spaces in \( \mathbb{R}^n, \) respectively (see [3,33] for the definition of function spaces). Note that for non-negative integer \( k, \) \( \dot{H}_p^k(\mathbb{R}^n) = \{ f \mid \sum_{|\beta| = k} \| D^\beta f \|_{L^p(\mathbb{R}^n)} < \infty \}, \) where \( \beta = (k_1, k_2, \ldots, k_n) \in (\mathbb{N} \cup \{0\})^n \) with \( |\beta| = k_1 + k_2 + \cdots + k_n. \)

From the Besov-Sobolev imbedding, for \( s - \frac{n}{p} = s_1 - \frac{n}{p_1}, \) we have

\[
\begin{align*}
\dot{B}_{pq}^s(\mathbb{R}^n) & \subset \dot{B}_{p_1 q_1}^{s_1}(\mathbb{R}^n) \quad 1 \leq p \leq p_1 < \infty, \quad 1 \leq q \leq q_1 \leq \infty \quad s, s_1 \in \mathbb{R}, \\
\dot{H}_p^s(\mathbb{R}^n) & \subset \dot{H}_{p_1}^{s_1}(\mathbb{R}^n) \quad 1 < p \leq p_1 < \infty \quad s, s_1 \in \mathbb{R}.
\end{align*}
\]

(2.1)

See Theorem 6.5.1 in [3] and its Remark.

For \( 1 \leq q \leq \infty, 0 < \theta < 1 \) and Banach spaces \( X, Y, \) we denote by \( (X, Y)_{\theta,q} \) and \( [X, Y]_\theta \) the real interpolation and complex interpolation of the Banach spaces \( X \) and \( Y. \) For \( 0 < \theta < 1, s, s_1, s_2 \in \mathbb{R} \) with \( s = \theta s_1 + (1 - \theta) s_2 \) and \( 1 \leq p, q, r \leq \infty, \)

\[
\begin{align*}
[\dot{H}_{p_1}^{s_1}(\mathbb{R}^n), \dot{H}_{p_2}^{s_2}(\mathbb{R}^n)]_\theta & = \dot{H}_p^{s}(\mathbb{R}^n), \quad s_1 \neq s_2, 1 < p_1 < p_2 < \infty, \quad \frac{1}{p} = \frac{\theta}{p_1} + \frac{1 - \theta}{p_2}, \\
(\dot{H}_p^{s_1}(\mathbb{R}^n), \dot{H}_p^{s_2}(\mathbb{R}^n))_{\theta,r} & = \dot{B}_{pr}^{s}(\mathbb{R}^n), \quad s_1 \neq s_2, 1 \leq p, r \leq \infty.
\end{align*}
\]

(2.2)

(2.3)

See Theorem 6.4.5 in [3] and its proof.

Denoted by \( \dot{H}_p^s(\mathbb{R}^n_+) \) and \( \dot{B}_{pq}^s(\mathbb{R}^n_+) \) are the restrictions of \( \dot{H}_p^s(\mathbb{R}^n) \) and \( \dot{B}_{pq}^s(\mathbb{R}^n), \) respectively, with norms

\[
\begin{align*}
\| f \|_{\dot{H}_p^s(\mathbb{R}^n_+)} & = \inf \{ \| F \|_{\dot{H}_p^s(\mathbb{R}^n)} \mid F|_{\mathbb{R}_+^n} = f, \quad F \in \dot{H}_p^s(\mathbb{R}^n) \}, \\
\| f \|_{\dot{B}_{pq}^s(\mathbb{R}^n_+)} & = \inf \{ \| F \|_{\dot{B}_{pq}^s(\mathbb{R}^n)} \mid F|_{\mathbb{R}_+^n} = f, \quad F \in \dot{B}_{pq}^s(\mathbb{R}^n) \}.
\end{align*}
\]

For a non-negative integer \( k, \) \( \dot{H}_p^k(\mathbb{R}^n_+) = \{ f \mid \sum_{|\beta| = k} \| D^\beta f \|_{L^p(\mathbb{R}^n_+)} < \infty \}. \) In particular, \( \dot{H}_p^0(\mathbb{R}^n_+) = L^p(\mathbb{R}^n_+). \)

For \( s \in \mathbb{R}, \) we denote \( \dot{B}_{pq,0}^s(\mathbb{R}^n_+), 1 \leq p, q \leq \infty \) by

\[
\dot{B}_{pq,0}^s(\mathbb{R}^n_+) = \{ f \in \dot{B}_{pq}^s(\mathbb{R}^n_+) \mid \tilde{f} \in \dot{B}_{pq}^s(\mathbb{R}^n) \},
\]

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where \( \tilde{f} \) is the zero extension of \( f \) over \( \mathbb{R}^n \). Note that \( \| \tilde{f} \|_{\dot{B}_{pq}^s(\mathbb{R}^n)} \leq c \| f \|_{\dot{B}_{pq,0}^s(\mathbb{R}^n)} \leq c \| \tilde{f} \|_{\dot{B}_{pq}^s(\mathbb{R}^n)} \). From (2.1), we have

\[
\dot{B}_{pq,0}^s(\mathbb{R}^n) \subset \dot{B}_{pq,1,0}^{s_1}(\mathbb{R}^n) \quad 1 \leq p \leq p_1 \leq \infty, \quad 1 \leq q \leq q_1 \leq \infty, \quad s, s_1 \in \mathbb{R}.
\tag{2.4}
\]

Note that for \( s \geq 0, \dot{B}_{pq,0}^{-s}(\mathbb{R}^n) \), \( 1 < p, q \leq \infty \) is the dual of \( \dot{B}_{p'q'}^{-s}(\mathbb{R}^n) \), that is, \( \dot{B}_{pq,0}^{-s}(\mathbb{R}^n) = (\dot{B}_{p'q'}^{s}(\mathbb{R}^n))^* \), where \( \frac{1}{p} + \frac{1}{p'} = 1 \) and \( \frac{1}{q} + \frac{1}{q'} = 1 \).

We define the extension operator \( E : \dot{H}_p^k(\mathbb{R}^n) \to \dot{H}_p^k(\mathbb{R}^n), k \in \mathbb{N} \cup \{ 0 \} \) by

\[
Ef(x) = \left\{ \begin{array}{ll}
 f(x), & x_n > 0, \\
 \sum_{j=1}^{k+1} \lambda_j f(x', -jx_n), & x_n < 0,
\end{array} \right.
\tag{2.5}
\]

where \( \lambda_j \) satisfies the following linear equations

\[
\sum_{j=1}^{k+1} (-j)^l \lambda_j = 1, \quad l = 0, 1, \ldots, k.
\]

(See the proof of Theorem 5.19 in [1].) Applying the extension operator \( E \) defined by (2.5) and (2.2)-(2.3) to the proofs of Proposition 2.4 and Proposition 2.17 in [20], we have that for \( 0 < \theta < 1, s, s_1, s_2 \in \mathbb{R} \) with \( s = \theta s_1 + (1 - \theta)s_2 \) and \( 1 \leq p, r \leq \infty \),

\[
[\dot{H}_{p_1}^{s_1}(\mathbb{R}^n), \dot{H}_{p_2}^{s_2}(\mathbb{R}^n)]_{\theta} = \dot{H}_p^s(\mathbb{R}^n), \quad s_1 \neq s_2, 1 < p_1 < p_2 < \infty, \quad \frac{1}{p} = \frac{\theta}{p_1} + \frac{1 - \theta}{p_2},
\tag{2.6}
\]

\[
(\dot{H}_p^{s_1}(\mathbb{R}^n), \dot{H}_p^{s_2}(\mathbb{R}^n))_{\theta,r} = \dot{B}_{pr}^s(\mathbb{R}^n), \quad s_1 \neq s_2, 1 \leq p, r \leq \infty.
\tag{2.7}
\]

For the Banach space \( X \), we denote by \( L^\infty_\beta(0, \infty; X) \), \( \beta \geq 0 \) the usual Bochner space with norm

\[
\| f \|_{L^\infty_\beta(0, \infty; X)} := \sup_{0 < t < \infty} t^\beta \| f(t) \|_X.
\]

From now, we denote \( \dot{B}_{p\infty}^\beta := \dot{B}_{p\infty}(\mathbb{R}^n_+), \dot{H}_p^\beta := \dot{H}_p^\beta(\mathbb{R}^n_+), \quad L^\infty_\alpha \dot{B}_{p\infty}^\beta := L^\infty_\alpha(0, \infty; \dot{B}_{p\infty}(\mathbb{R}^n_+)) \) and \( L^\infty_\alpha L^p := L^\infty_\alpha(0, \infty; L^p(\mathbb{R}^n_+)) \).
3 Preliminary estimates

3.1 Newtonian potential

The fundamental solution of the Laplace equation in \(\mathbb{R}^n\) is denoted by

\[
N(x) = \begin{cases} 
\frac{1}{\omega_n (2-n)|x|^{n-2}} & \text{if } n \geq 3, \\
\frac{1}{2\pi \ln |x|} & \text{if } n = 2,
\end{cases}
\]

\(\omega_n\) is the surface area of the unit sphere in \(\mathbb{R}^n\).

Lemma 3.1 For \(\alpha \geq 0\) and \(1 < p < \infty\),

\[
\|\nabla^2_x \int_{\mathbb{R}^n} N(x - y) f(y) dy \|_{\dot{B}^\alpha_p} \leq c \|f\|_{\dot{B}^\alpha_p}.
\]

The proof of Lemma 3.1 is given in “Appendix 1”.

3.2 Gaussian kernel

The fundamental solution of the heat equation in \(\mathbb{R}^n\) is denoted by

\[
\Gamma(x, t) = \begin{cases} 
\frac{1}{(2\pi t)^{n/2}} e^{-|x|^2/4t} & \text{if } t > 0, \\
0 & \text{if } t \leq 0.
\end{cases}
\]

Let \(\Gamma^*(x - y, t) := \Gamma(x' - y', x_n + y_n, t)\). For \(u_0 \in \dot{B}^\alpha_{p\infty,0}\), \(1 \leq p \leq \infty\), \(\alpha \in \mathbb{R}\), we define \(\Gamma_t * u_{0j}\) and \(\Gamma^*_t * u_{0j}\) by

\[
\Gamma_t * u_{0j}(x) := \int_{\mathbb{R}^n} \Gamma(x - y, t) u_{0j}(y) dy, \quad \Gamma^*_t * u_{0j}(x) := \int_{\mathbb{R}^n} \Gamma^*(x - y, t) u_{0j}(y) dy, \quad \alpha \geq 0,
\]

\[
\Gamma_t * u_{0j}(x) := \langle u_{0j}, \Gamma(x - \cdot, t) \rangle, \quad \Gamma^*_t * u_{0j}(x) := \langle u_{0j}, \Gamma^*(x - \cdot, t) \rangle, \quad \alpha < 0,
\]

where \(\langle \cdot, \cdot \rangle\) is dual paring between \(\dot{B}^\alpha_{p\infty}\) and \(\dot{B}^{-\alpha}_{p',1,0}\).

Lemma 3.2 (1) Let \(1 \leq p \leq \infty\) and \(\alpha > 0\). Let \(u_0 \in \dot{B}^{-\alpha}_{p\infty,0}\) with \(\text{div} \ u_0 = 0\). Then,

\[
\|\Gamma_t * u_0\|_{L^\infty} \leq c \|u_0\|_{\dot{B}^{-\alpha}_{p\infty}}. \quad (3.1)
\]

(2) Let \(1 \leq p \leq \infty\) and \(\alpha > 0\). Let \(u_0 \in \dot{B}^\alpha_{p\infty,0}\) with \(\text{div} \ u_0 = 0\). Then,

\[
\|\Gamma_t * u_0\|_{L^\infty} \leq c \|u_0\|_{\dot{B}^\alpha_{p\infty}}. \quad (3.2)
\]

Proof The proof of Lemma 3.2 is given in “Appendix 2”. 
We define $\Gamma \ast f$ and $\Gamma^* \ast f$ by

$$
\Gamma \ast f(x, t) := \int_0^t \int_{\mathbb{R}_+^n} \Gamma(x - y, t - s) f(y, s) dy ds,
$$

$$
\Gamma^* \ast f(x, t) := \int_0^t \int_{\mathbb{R}_+^n} \Gamma^*(x - y, t - s) f(y, s) dy ds.
$$

**Lemma 3.3** (1) Let $1 < p_1 \leq p < \infty$ and $\alpha > 0$ satisfying $-1 + \alpha < \frac{n}{p_1} - \frac{n}{p} < 1$. Let $f = \text{div}\mathcal{F}$ with $\mathcal{F} \in L^{\infty}_0(\mathbb{R}_+^n)$. Then,

$$
\| \Gamma \ast f \|_{L^{\infty}_2([0, \infty); L^p(\mathbb{R}_+^n))}, \quad \| \Gamma^* \ast f \|_{L^{\infty}_2([0, \infty); L^p(\mathbb{R}_+^n))}
$$

$$
\leq c \| \mathcal{F} \|_{L^{\infty}_0(\mathbb{R}_+^n)}.
$$

(2) Let $0 < \alpha < 2$ and $1 \leq p_1 \leq p$ such that $1 + \frac{n-1}{p} < \frac{n}{p_1} < 1 + \frac{n}{p}$. Let $f = \text{div}\mathcal{F}$ with $\mathcal{F} \in L^{\infty}_0(\mathbb{R}_+^n)$. Then,

$$
\| \Gamma \ast f \|_{L^{\infty}(0, \infty; \dot{B}^\alpha_{p,1}(\mathbb{R}_+^n))}, \quad \| \Gamma^* \ast f \|_{L^{\infty}(0, \infty; \dot{B}^\alpha_{p,1}(\mathbb{R}_+^n))}
$$

$$
\leq c \| \mathcal{F} \|_{L^{\infty}_0(\mathbb{R}_+^n)}, \quad \dot{B}^\alpha_{p,1}(\mathbb{R}_+^n).
$$

**Proof** The proof of Lemma 3.3 is given in “Appendix 3”.

### 3.3 Hölder type inequality

The following Hölder type inequality is a well-known result (see Lemma 2.2 in [5]).

**Lemma 3.4** Let $0 < \beta$ and $1 \leq p, q \leq \infty$. Then, for $\frac{1}{r_i} + \frac{1}{s_i} = \frac{1}{p}, i = 1, 2$,

$$
\| f_1 f_2 \|_{\dot{B}^\beta_{pq}} \leq c(\| f_1 \|_{\dot{B}^\beta_{1,q}} \| f_2 \|_{L^{r_1}} + \| f_1 \|_{L^{r_2}} \| f_2 \|_{\dot{B}^\beta_{2,q}}).
$$

### 3.4 Helmholtz projection $\mathbb{P}$

Let $\mathbb{P}$ be a Helmholtz projection operator in $\mathbb{R}_+^n$. Let $f = \text{div}\mathcal{F}, \mathcal{F} = (F_{kl})_{k,l=1}^n$, $F_{kl} = F_{ik}$, with $F_{mk}|_{x_n=0} = 0$. Then $\mathbb{P} f$ can be rewritten as $\mathbb{P} f = \text{div}\mathcal{F}'$, $\mathcal{F}' = (F'_{km})_{k,m=1}^n$, where

$$
F'_{nm} = F_{nm} - \delta_{nm} F_{nn}, \quad m = 1, \ldots, n,
$$

$$
F'_{\beta\gamma} = F_{\beta\gamma} - \delta_{\beta\gamma} F_{nn} + D_{\gamma q} \left( \sum_{q=1}^n \int_{\mathbb{R}_+^n} D_{\beta q} N^+(x, y) F_{\beta q}(y) dy \right).
$$
\[
+ \int_{\mathbb{R}^n_+} \left( D_{y_n} N^+(x, y) F_{\beta n}(y) - D_{y_\gamma} N^+(x, y) F_{\gamma n}(y) \right) dy 
\]
\[\beta, \gamma \neq n,\]
\[
F'_{\beta n} = - \sum_{\gamma=1}^{n-1} D_{x_\gamma} \int_{\mathbb{R}^n_+} D_{x_\gamma} N^+(x, y) F_{\beta \gamma}(y) dy + D_{x_\beta} \int_{\mathbb{R}^n_+} D_{x_n} N^+(x, y) F_{\beta n}(y) dy
\]
\[- 2 D_{x_\beta} (x) - 2 \sum_{\gamma=1}^{n-1} D_{x_\gamma} N^-(x, y) F_{\beta n}(y) dy \quad \beta \neq n.
\]

(See Section 3 of [12] for the details). Here \( N^+(x, y) := N(x - y) + N(x - y^*) \) and \( N^-(x, y) = N(x - y) - N(x - y^*) \). From Lemma 3.1 and real interpolation, we have
\[
\| F' \|_{\dot{H}^p_\infty} \leq c \| F \|_{\dot{H}^p_\infty} \quad 0 < \alpha, \quad 1 < p < \infty,
\]
\[
\| F' \|_{B^p_\infty} \leq c \| F \|_{B^p_\infty} \quad 0 < \alpha, \quad 1 < p < \infty.
\]

### 4 Proof of Theorem 1.3

First, we decompose the Stokes equations (1.3) as the following two equations:
\[
v_t - \Delta v + \nabla \pi = 0, \quad \text{div} \, v = 0 \quad \text{in} \, \mathbb{R}^n_+ \times (0, \infty),
\]
\[
v|_{t=0} = u_0 \quad \text{and} \quad v|_{x_n=0} = 0,
\]
and
\[
V_t - \Delta V + \nabla \Pi = \text{div} \mathcal{F}, \quad \text{div} \, V = 0 \quad \text{in} \, \mathbb{R}^n_+ \times (0, \infty),
\]
\[
V|_{t=0} = 0, \quad V|_{x_n=0} = 0.
\]

Let \( u = V + v \) and \( p = \pi + \Pi \). Then, \((u, p)\) is a solution of (1.3).

#### 4.1 Estimate of \((v, \pi)\)

The solution \((v, \pi)\) of (4.1) is represented by (see [29])
\[
v_i(x, t) = \int_{\mathbb{R}^n_+} G_{ij}(x, y, t) u_{0 j}(y) dy,
\]
\[
\pi(x, t) = \int_{\mathbb{R}^n_+} P(x, y, t) \cdot u_0(y) dy,
\]
where \( G \) and \( P \) are defined by
\[
G_{ij} = \delta_{ij}(\Gamma(x - y, t) - \Gamma(x - y^*, t))
\]
\[+ 4(1 - \delta_{jn}) \frac{\partial}{\partial x_j} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \frac{\partial N(x - z)}{\partial x_i} \Gamma(z - y^*, t) dz,
\]
\[
\odot \text{Springer}
\]
In this section, we would like to give proof of Theorem 1.1. For this purpose, we construct approximate velocities and then derive uniform convergence in $L^\infty \tilde{B}_p^{\alpha}$. 

$$P_j(x, y, t) = 4(1 - \delta_{jn}) \frac{\partial}{\partial x_j} \left[ \int_{\mathbb{R}^{n-1}} \frac{\partial N(x' - z', x_n)}{\partial x_n} \Gamma(z' - y', y_n, t) dz' \right. + \left. \int_{\mathbb{R}^{n-1}} N(x' - z', x_n) \frac{\partial \Gamma(z' - y', y_n, t)}{\partial y_n} dz' \right]. \quad (4.6)$$

From proofs of Lemma 3.1 and Lemma 3.3 in [10], we get

$$\|v(t)\|_{\tilde{H}_p^k} \leq c\left(\|\Gamma_t \ast u_0\|_{\tilde{H}_p^k} + \|\Gamma_t^{\ast} \ast u_0\|_{\tilde{H}_p^k}\right) \quad 1 < p < \infty, \; k \in \mathbb{N} \cup \{0\},$$

$$\|v(t)\|_{\tilde{B}_p^{\alpha \infty}} \leq c\left(\|\Gamma_t \ast u_0\|_{\tilde{B}_p^{\alpha \infty}} + \|\Gamma_t^{\ast} \ast u_0\|_{\tilde{B}_p^{\alpha \infty}}\right) \quad 1 < p < \infty, \; \alpha > 0. \quad (4.7)$$

### 4.2 Estimate of $(V, \Pi)$

Let $\mathbb{P} f$ be the Helmholtz projection of $f$. Note that $\text{div} \mathbb{P} f = 0$ and $(\mathbb{P} f)|_{x_n = 0} = 0$. We define $(V, \Pi_0)$ by

\begin{align*}
V_i(x, t) &= \int_0^t \int_{\mathbb{R}_+^n} G_{ij}(x, y, t - \tau)(\mathbb{P} f)_j(y, \tau) dy d\tau, \quad (4.8) \\
\Pi_0(x, t) &= \int_0^t \int_{\mathbb{R}_+^n} P(x, y, t - \tau) \cdot (\mathbb{P} f)(y, \tau) dy d\tau, \quad (4.9)
\end{align*}

where $G$ and $P$ are defined by (4.5) and (4.6). Then $(V, \Pi_0)$ satisfies

$$V_i - \Delta V + \nabla \Pi_0 = \mathbb{P} f, \quad \text{div} V = 0, \quad V|_{t=0} = 0, \quad V|_{x_n=0} = 0.$$

(See [29].) Let $\Pi = \Pi_0 + \mathbb{Q} f$. Then, $(V, \Pi)$ is the solution of (4.2).

Let $1 < p < \infty, \; 1 \leq q \leq \infty$, and $0 \leq \alpha \leq 2$. In Section 3 in [10], the authors showed that $V$ defined by (4.8) has the following estimates (using real interpolations):

$$\|V(t)\|_{\tilde{H}_p^k} \leq c\left(\|\Gamma \ast \mathbb{P} f(t)\|_{\tilde{H}_p^k} + \|\Gamma^{\ast} \ast \mathbb{P} f(t)\|_{\tilde{H}_p^k}\right) \quad 1 < p < \infty, \; k \in \mathbb{N} \cup \{0\},$$

$$\|V(t)\|_{\tilde{B}_p^{\alpha \infty}} \leq c\left(\|\Gamma \ast \mathbb{P} f(t)\|_{\tilde{B}_p^{\alpha \infty}} + \|\Gamma^{\ast} \ast \mathbb{P} f(t)\|_{\tilde{B}_p^{\alpha \infty}}\right) \quad 1 < p < \infty, \; \alpha > 0. \quad (4.10)$$

### 4.3 Estimate of $(u, p)$

Note that $(u, p)$ defined by $u = V + v$ and $p = \pi + \Pi_0 + \mathbb{Q} \text{div} \mathcal{F}$ is the solution of (1.3). From (4.7), (4.10), Lemmas 3.2 and 3.3, we obtain Theorem 1.3.

### 5 Nonlinear problem

In this section, we would like to give proof of Theorem 1.1. For this purpose, we construct approximate velocities and then derive uniform convergence in $L^\infty \tilde{B}_p^{\alpha \infty}$. 

\[ \text{Springer} \]
5.1 Approximating solutions

Let \((u^1, p^1)\) be the solution of the Stokes equations

\[
\begin{align*}
  u^1_t - \Delta u^1 + \nabla p^1 &= 0, & \text{div } u^1 &= 0, \quad \text{in } \mathbb{R}^n_+ \times (0, \infty), \\
  u^1|_{t=0} &= u_0, & u^1|_{x_n=0} &= 0.
\end{align*}
\]  

(5.1)

Let \(m \geq 1\). After obtaining \((u^1, p^1), \ldots, (u^m, p^m)\) construct \((u^{m+1}, p^{m+1})\) which satisfies the following equations

\[
\begin{align*}
  u^{m+1}_t - \Delta u^{m+1} + \nabla p^{m+1} &= f^m, & \text{div } u^{m+1} &= 0, \quad \text{in } \mathbb{R}^n_+ \times (0, \infty), \\
  u^{m+1}|_{t=0} &= u_0, & u^{m+1}|_{x_n=0} &= 0.
\end{align*}
\]  

(5.2)

where \(f^m = -\text{div}(u^m \otimes u^m)\).

5.2 Uniform boundedness in \(L^\infty \frac{n}{2} \frac{n}{p_0} L^{p_0}, n < p_0\)

From (1) of Theorem 1.3, we have

\[
\|u^1\|_{L^\infty \frac{n}{2} \frac{n}{p_0} L^{p_0}} \leq c_0\|u_0\|_{\dot{B}^{-1+n/p_0}_{p_0 \infty,0}} := N_0.
\]  

(5.3)

From (1) of Theorem 1.3, taking \(p_1 = \frac{p_0}{2}\), we have

\[
\begin{align*}
  \|u^{m+1}\|_{L^\infty \frac{n}{2} \frac{n}{p_0} L^{p_0}} &\leq c(\|u_0\|_{\dot{B}^{-1+n/p_0}_{p_0 \infty,0}} + \|u^m \otimes u^m\|_{L^\infty \frac{n}{2} \frac{n}{p_0} L^{p_0/2}}) \\
&\leq c_1(\|u_0\|_{\dot{B}^{-1+n/p_0}_{p_0 \infty,0}} + \|u^m\|_{L^\infty \frac{n}{2} \frac{n}{p_0} L^{p_0}}).
\end{align*}
\]  

(5.4)

Under the hypothesis \(\|u^m\|_{L^\infty \frac{n}{2} \frac{n}{p_0} L^{p_0}} \leq M_0\), (5.4) leads to the estimate

\[
\|u^{m+1}\|_{L^\infty \frac{n}{2} \frac{n}{p_0} L^{p_0}} \leq c_1(N_0 + M_0^2).
\]

Choose \(M_0\) and \(N_0\) so small that

\[
M_0 \leq \frac{1}{2c_1} \quad \text{and} \quad N_0 < \frac{M_0}{2c_1}.
\]  

(5.5)

By the mathematical induction argument, we conclude

\[
\|u^m\|_{L^\infty \frac{n}{2} \frac{n}{p_0} L^{p_0}} \leq M_0 \quad \text{for all } m = 1, 2, \ldots
\]  

(5.6)
5.3 Uniform boundedness in $L^\infty \dot{B}^{-1+n/p}_{p\infty}$

From (2) of Theorem 1.3, we have

$$
\|u^1\|_{L^\infty \dot{B}^{-1+n/p}_{p\infty}} \leq c_2 \|u_0\|_{\dot{B}^{-1+n/p}_{p\infty,0}} := N. 
$$

We take $1 \leq p_1 < p$ satisfying $1 + \frac{n-1}{p} < \frac{n}{p_1} < 1 + \frac{n}{p}$. From (2) of Theorem 1.3 to obtain

$$
\|u^{m+1}\|_{L^\infty \dot{B}^{-1+n/p}_{p\infty}} \leq c_1 \left( \|u_0\|_{\dot{B}^{-1+n/p}_{p\infty,0}} + \|u^m \otimes u^m\|_{L^\infty \frac{1}{2} - \frac{n}{2p_1} + \frac{n}{2p} \dot{B}^{-1+n/p}_{p1\infty}} \right). 
$$

Let $\frac{1}{p_1} = \frac{1}{p_0} + \frac{1}{p}$ such that $1 - \frac{1}{p} < \frac{n}{p_0} < 1$. By Lemma 3.4, we have

$$
\|(u^m \otimes u^m)\|_{L^\infty \frac{1}{2} - \frac{n}{2p_1} + \frac{n}{2p} \dot{B}^{-1+n/p}_{p1\infty}} \leq c_1 \|u^m\|_{L^\infty \dot{B}^{-1+n/p}_{p\infty}} \|u^m\|_{L^\infty \frac{1}{2} - \frac{n}{2p_0} + \frac{n}{2p} \dot{B}^{-1+n/p}_{p0\infty}} 
\leq c_1 \|u^m\|_{L^\infty \dot{B}^{-1+n/p}_{p\infty}} \|u^m\|_{L^\infty \frac{1}{2} - \frac{n}{2p_0} \dot{B}^{-1+n/p}_{p0\infty}}. 
$$

From (5.8)-(5.9), we have

$$
\|u^{m+1}\|_{L^\infty \dot{B}^{-1+n/p}_{p\infty}} \leq c_1 \left( N + \|u^m\|_{L^\infty \frac{1}{2} - \frac{n}{2p_0} \dot{B}^{-1+n/p}_{p0\infty}} \right). 
$$

Under the hypothesis $\|u^m\|_{L^\infty \dot{B}^{-1+n/p}_{p\infty}} \leq M$, (5.10) leads to the estimate

$$
\|u^{m+1}\|_{L^\infty \dot{B}^{-1+n/p}_{p\infty}} \leq c_1 (N + M_0 M). 
$$

Choose that $M_0$ is small and $M$ is large so that

$$
M_0 \leq \frac{1}{2c_1}, \quad 2c_1 N \leq M. 
$$

By the mathematical induction argument, we conclude

$$
\|u^m\|_{L^\infty \dot{B}^{-1+n/p}_{p\infty}} \leq M \text{ for all } m = 1, 2 \ldots. 
$$

5.4 Uniform convergence

Let $U^m = u^{m+1} - u^m$ and $P^m = p^{m+1} - p^m$. Then, $(U^m, P^m)$ satisfy the equations

$$
U^m_t - \Delta U^m + \nabla P^m = -\text{div}(u^m \otimes u^{m-1} + U^{m-1} \otimes u^{m-1}), \quad \text{div} U^m = 0, \text{ in } \mathbb{R}^N_+ \times (0, \infty), 
U^m |_{t=0} = 0, \quad U^m |_{\partial_+} = 0. 
$$
Recall the uniform estimates (5.6) and (5.12) for the approximate solutions. From Theorem 1.3, Lemma 3.4 and (5.9), we have
\[
\|U^m\|_{L_p^0} \leq c(\|u^{m-1}\|_{L_\infty^0} L_p^0 + \|u^m\|_{L_\infty^0} L_p^0) \|U^{m-1}\|_{L_p^0} \leq c_5 M_0 \|U^{m-1}\|_{L_\infty^0 (\mathcal{M}^0)} L_p^0,
\]  
(5.13)
and by (5.8), we have
\[
\|U^m\|_{L_\infty^{0 (\mathcal{M}^0)}} \leq c \|u^m \otimes U^{m-1} + U^{m-1} \otimes u^{m-1}\|_{L_\infty^{0 (\mathcal{M}^0)}}
\leq c(\|u^m\|_{L_\infty^{0 (\mathcal{M}^0)}} + \|u^{m-1}\|_{L_\infty^{0 (\mathcal{M}^0)}}) \|U^{m-1}\|_{L_p^0} + c(\|u^m\|_{L_\infty^0} L_p^0 + \|u^{m-1}\|_{L_\infty^0} L_p^0) \|U^{m-1}\|_{L_\infty^{0 (\mathcal{M}^0)}}
\leq c_6 M \|U^{m-1}\|_{L_\infty^0} L_p^0 + c_6 M_0 \|U^{m-1}\|_{L_\infty^{0 (\mathcal{M}^0)}}.
\]  
(5.14)

We take the constant $c_6$ greater than $c_5$, that is,
\[
c_6 > c_5.
\]  
(5.15)

From (5.13), if $c_5 M_0 < 1$, then \(\sum_{m=1}^\infty \|U^m\|_{L_p^0} L_p^0\) converges, that is,
\[
\sum_{m=1}^\infty U^m \text{ converges in } L_\infty^0 L_p^0.
\]

Take $A > 0$ satisfying $A(c_6 - c_5) M_0 \geq c_6 M$. Then from (5.13) and (5.14) it holds that
\[
\|U^m\|_{L_\infty^{0 (\mathcal{M}^0)}} A \|U^m\|_{L_\infty^0} L_p^0 \leq c_6 M_0 (\|U^{m-1}\|_{L_\infty^{0 (\mathcal{M}^0)}} + A \|U^{m-1}\|_{L_\infty^0} L_p^0)
\]
Again if $c_6 M_0 < 1$, then \(\sum_{m=1}^\infty (\|U^m\|_{L_\infty^{0 (\mathcal{M}^0)}} + A \|U^m\|_{L_\infty^0} L_p^0)\) converges.

This implies that \(\sum_{m=1}^\infty \|U^m\|_{L_\infty^{0 (\mathcal{M}^0)}}\) converges, that is,
\[
\sum_{m=1}^\infty U^m \text{ converges in } L_\infty^{0 (\mathcal{M}^0)}.
\]
Therefore, if $M_0$ satisfies the condition (5.11) with the additional conditions

$$M_0 < \frac{1}{c_6},$$

(5.16)

then $u^m = u^1 + \sum_{k=1}^m U^k$ converges to $u^1 + \sum_{k=1}^{\infty} U^k$ in $L^\infty \dot{B}^{-1+n/p}_{p,\infty} \cap L^\infty_{1-\frac{n}{2p_0}} L^{p_0}$.

Set $u := u^1 + \sum_{k=1}^{\infty} U^k$.

5.5 Existence

In this section, we will show that $u$ satisfies weak formulation of the Navier–Stokes equations (1.1), that is, $u$ is a weak solution of the Navier–Stokes equations (1.1) with appropriate distribution $p$.

Let $u$ be the same one constructed in the previous Section. Because $u_m \rightarrow u$ in $L^\infty_{1-\frac{n}{2p_0}} L^{p_0}$ by (5.6), we have

$$\|u\|_{L^\infty_{1-\frac{n}{2p_0}} L^{p_0}}, \quad \|u^m\|_{L^\infty_{1-\frac{n}{2p_0}} L^{p_0}} \leq M_0.$$

Let $\Phi \in C^\infty_0(\mathbb{R}_+^n \times [0, T))$ with $\text{div} \Phi = 0$ for some $T > 0$. Observe that

$$-\int_0^\infty \int \mathbb{R}_+^n u^{m+1} \cdot \Delta \Phi dx dt = \int_0^\infty \int \mathbb{R}_+^n u^{m+1} \cdot \Phi_t + (u^m \otimes u^m) : \nabla \Phi dx dt$$

$$+ \int \mathbb{R}_+^n u_0 \cdot \Phi(x, 0) dx.$$

Now, send $m$ to the infinity, then, $u^m \rightarrow u$ in $L^\infty_{1-\frac{n}{2p_0}} L^{p_0}$. Then, we have

$$\int_0^T \int \mathbb{R}_+^n (u^{m+1} - u) \cdot \Delta \Phi dx dt \leq \int_0^T \|u^{m+1} - u\|_{L^{p_0}} \|\Delta \Phi\|_{L^{p_0'}} dt$$

$$\leq \|u^{m+1} - u\|_{L^\infty_{1-\frac{n}{2p_0}} L^{p_0}} \int_0^T t^{-\frac{1}{2} + \frac{n}{2p_0}} \|\Delta \Phi\|_{L^{p_0'}} dt$$

$$\rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

Similarly, we have

$$\int_0^T \int \mathbb{R}_+^n (u^{m+1} - u) \cdot \Phi_t dx dt \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

Because $p_0 > n$, we have

$$\int_0^\infty \int \mathbb{R}_+^n (u^m \otimes (u^m - u)) : \nabla \Phi dx dt$$
\[ \| u^m \otimes (u^{m+1} - u) \|_{L_1^{\infty} - \frac{n}{p_0}} \leq \int_0^T t^{-1 + \frac{n}{p_0}} \| \Delta \Phi \|_{L^{(p_0/2)'}} \, dt \]
\[ \| u^m - u \|_{L_1^{\infty} - \frac{n}{p_0}} \leq \int_0^T t^{-1 + \frac{n}{p_0}} \| \Delta \Phi \|_{L^{(p_0/2)'}} \, dt \]
\[ \rightarrow 0 \quad \text{as} \quad m \rightarrow \infty. \]

Hence, we have the identity
\[ - \int_0^\infty \int_{\mathbb{R}_+^n} u \cdot \Delta \Phi \, dx \, dt = \int_0^\infty \int_{\mathbb{R}_+^n} u \cdot \Phi_t + (u \otimes u) : \nabla \Phi \, dx \, dt + \int_{\mathbb{R}_+^n} u_0 \cdot \Phi(x, 0) \, dx. \]

Therefore, \( u \) is a weak solution of the Navier–Stokes equations (1.1). This completes the proof of the existence part of Theorem 1.1.

### 5.6 Uniqueness in space \( L_1^{\infty} - \frac{n}{p_0} \)

Let \( u_1 \in L_1^{\infty} - \frac{n}{p_0} \) be another weak solution of the Navier–Stokes equations (1.1) with pressure \( p_1 \). Then, \( (u - u_1, p - p_1) \) satisfies the equations
\[ (u - u_1)_t - \Delta (u - u_1) + \nabla (p - p_1) = -\text{div}(u \otimes (u - u_1)) + (u - u_1) \otimes u_1 \quad \text{in} \quad \mathbb{R}_+^n \times (0, \infty), \]
\[ \text{div} (u - u_1) = 0, \quad \text{in} \quad \mathbb{R}_+^n \times (0, \infty), \]
\[ (u - u_1)|_{t=0} = 0, \quad (u - u_1)|_{x_n=0} = 0. \]

Applying the estimate of Theorem 1.3 to the above Stokes equations, we have
\[ \| u - u_1 \|_{L_1^{\infty} - \frac{n}{p_0}} \leq c \| u \otimes (u - u_1) + (u - u_1) \otimes u_1 \|_{L_1^{\infty} - \frac{n}{p_0}} \]
\[ \leq c_5 (\| u \|_{L_1^{\infty} - \frac{n}{p_0}} + \| u_1 \|_{L_1^{\infty} - \frac{n}{p_0}}) \| u - u_1 \|_{L_1^{\infty} - \frac{n}{p_0}}. \]

Taking \( M_0 > 0 \) to satisfy \( 2c_5 M_0 < 1 \), we have
\[ \| u - u_1 \|_{L_1^{\infty} - \frac{n}{p_0}} < \| u - u_1 \|_{L_1^{\infty} - \frac{n}{p_0}}. \]

This implies that \( u \equiv u_1 \) in \( \mathbb{R}_+^n \times (0, \infty) \) and so we complete the proof of the uniqueness part of Theorem 1.1.

### Appendix A. Proof of Lemma 3.1

The following lemma is well known trace theorem (see Theorem 6.6.1 in [3]).
Lemma A.1 Let \(1 < p < \infty\). If \(f \in \dot{H}^\alpha_p\) for \(\alpha > \frac{1}{p}\), then \(f \chi_{x_n=0} \in \dot{B}_{pp}^{\frac{\alpha-1}{p}}(\mathbb{R}^{n-1})\) with \(\|f \chi_{x_n=0}\|_{\dot{B}_{pp}^{\frac{\alpha-1}{p}}(\mathbb{R}^{n-1})} \leq c \|f\|_{\dot{H}^\alpha_p}^p\).

We define \(Nf\) by

\[
N f(x) = \int_{\mathbb{R}^{n-1}} N(x' - y', x_n) f(y') dy'.
\] (A.1)

Observe that \(D_{x_n} N f\) is the Poisson operator of the Laplace equation in \(\mathbb{R}^n_+\), and \(D_{x_i} N f = D_{x_n} N R'_i f\) for \(i \neq n\), where \(R' = (R_1, \ldots, R_{n-1})\) is the \((n-1)\)-dimensional Riesz operator. The Poisson operator is bounded from \(\dot{B}_{pp}^{\frac{\alpha-1}{p}}(\mathbb{R}^{n-1})\) to \(\dot{H}^\alpha_p(\mathbb{R}^n_+)\), \(\alpha \geq 0\), and \(R'\) is bounded from \(\dot{B}^s_{pp}(\mathbb{R}^{n-1})\) to \(\dot{B}^s_{pp}(\mathbb{R}^{n-1})\), \(s \in \mathbb{R}\) (see [32]). Hence the following estimates hold.

Lemma A.2 Let \(1 < p < \infty\). Then

\[
\|\nabla^3 x N f\|_{\dot{H}^\alpha_p} \leq c \|f\|_{\dot{B}_{pp}^{\frac{\alpha-1}{p}}(\mathbb{R}^{n-1})} \quad \alpha \geq 0.
\] (A.2)

According to the Calderón–Zygmund inequality

\[
\left\| \int_{\mathbb{R}^n} \nabla^2 y N(\cdot - y) f(y) dy \right\|_{L^p} \leq c \|f\|_{L^p} \quad \text{for} \quad 1 < p < \infty.
\] (A.3)

We will show that for \(k \geq 0\),

\[
\|D^k_x \nabla^2_x \int_{\mathbb{R}^n_+} N(x - y) f(y) dy(x)\|_{L^p} \leq c \|D^k_x f\|_{L^p}.
\] (A.4)

Then, by the property of complex interpolation (see (2.6)), we get Lemma 3.1.

Note that

\[
D_{x_i} \int_{\mathbb{R}^n_+} N(x - y) f(y) dy = \int_{\mathbb{R}^n_+} N(x - y) D_{y_i} f(y) dy, \quad i \neq n,
\]

\[
D_{x_n} \int_{\mathbb{R}^n_+} N(x - y) f(y) dy = \int_{\mathbb{R}^n_+} N(x - y) D_{y_n} f(y) dy
\]

\[
- \int_{\mathbb{R}^{n-1}} N(x' - y', x_n) f(y', 0) dy'.
\]

By (A.3), (A.2) and Lemma A.1, we have

\[
\|\nabla^3 x \int_{\mathbb{R}^n_+} N(x - y) f(y) dy\|_{L^p} \leq c \left( \|\nabla x f\|_{L^p} + \|f(\cdot, 0)\|_{\dot{B}_{pp}^{\frac{1}{p}}(\mathbb{R}^{n-1})} \right)
\]

\[
\leq c \|\nabla x f\|_{L^p}.
\]
By the successive argument, (A.4) can be obtained for any multiple integer \( k \geq 0 \).

**Appendix B. Proof of Lemma 3.2**

Fix a Schwartz function \( \phi \in \mathcal{S}(\mathbb{R}^n) \) satisfying \( \hat{\phi}(\xi) > 0 \) on \( \frac{1}{2} < |\xi| < 2 \), \( \hat{\phi}(\xi) = 0 \) elsewhere, and \( \sum_{j=-\infty}^{\infty} \hat{\phi}(2^{-j} \xi) = 1 \) for \( \xi \neq 0 \). Let

\[
\hat{\phi}_j(\xi) := \hat{\phi}(2^{-j} \xi) , \quad (j = 0, \pm 1, \pm 2, \ldots),
\]

where \( \hat{f} = \mathcal{F}(f) \) is a Fourier transform of \( f \). Let \( \Phi = \phi_1 + \phi_0 + \phi_1 \) and \( \Phi_j(\xi) = \Phi(2^{-j} \xi) \) such that \( \text{supp} \Phi_j \subset \{ 2^{-j-2} < |\xi| < 2^{-j+2} \} \) and \( \Phi \equiv 1 \) in \( 2^{j-1} < |\xi| < 2^{j+1} \).

**Lemma B.1** Let \( \rho_{ij}(\xi) = \Phi_j(\xi) e^{-t|\xi|^2} \) for each integer \( j \). Then \( \rho_{ij}(\xi) \) are \( L^\infty(\mathbb{R}^n) \)-multipliers with the finite norm \( M(t, j) \). Moreover for \( t > 0 \)

\[
M(t, j) \leq c e^{-\frac{1}{4} t 2^{2j}} \sum_{0 \leq i \leq n} t^i 2^{2j} \leq c e^{-\frac{1}{8} t 2^{2j}}. \tag{B.1}
\]

See Lemma 13 in [11].

Let \( \tilde{u}_0(x) = u_0(x) \) be a zero extension over \( \mathbb{R}^n \) such that \( \| \tilde{u}_0 \|_{B^p_{\infty}(\mathbb{R}^n)} \leq c \| u_0 \|_{B^p_{\infty}} \). Let

\[
v(x, t) = \int_{\mathbb{R}^n} \Gamma(x - y, t) \tilde{u}_0(y) dy.
\]

Then, we have \( \| v \|_{L^\infty B^p_{\infty}} \leq c \| v \|_{L^\infty(0, \infty; B^p_{\infty}(\mathbb{R}^n))} \).

Using the dyadic partition of unity \( \sum_{j=-\infty}^{\infty} \hat{\phi}(2^{-j} \xi) = 1 \) for \( \xi \neq 0 \), we can write

\[
\mathcal{F}(v \ast \phi_j)(\xi, t) = \hat{\phi}(2^{-j} \xi) e^{-t|\xi|^2} \hat{\tilde{u}_0}(\xi).
\]

For \( t > 0 \) we have

\[
\| v \ast \phi_j(t) \|_{L^p(\mathbb{R}^n)} \leq \left( \int_{\mathbb{R}^n} \mathcal{F}^{-1} \left( e^{-t|\xi|^2} \hat{\phi}_j(\xi) \hat{\tilde{u}_0}(\xi) \right)(x) \right)^p dx \frac{1}{p} = \left( \int_{\mathbb{R}^n} \mathcal{F}^{-1} \left( \hat{\phi}_j(\xi) e^{-t|\xi|^2} \hat{\phi}_j(\xi) \hat{\tilde{u}_0}(\xi) \right)(x) \right)^p dx \frac{1}{p}. \tag{B.2}
\]

By Lemma B.1, we have

\[
t_{\frac{1}{2} \alpha} \| v(t) \|_{L^p(\mathbb{R}^n)} \leq t_{\frac{1}{2} \alpha} \sum_{-\infty < j < \infty} \| v(t) \ast \phi_j \|_{L^p(\mathbb{R}^n)} \leq t_{\frac{1}{2} \alpha} \sum_{-\infty < j < \infty} M(t, j) \| \tilde{u}_0 \ast \phi_j \|_{L^p(\mathbb{R}^n)}.
\]
\[ \leq ct^{\frac{1}{2}} \sum_{-\infty < j < \infty} 2^{j\alpha} 2^{-12^j} 2^{-j\alpha} \| \tilde{u}_0 \ast \phi_j \|_{L^p(\mathbb{R}^n)} \]
\[ \leq ct^{\frac{1}{2}} \sum_{-\infty < j < \infty} 2^{j\alpha} 2^{-12^j} \| \tilde{u}_0 \|_{B^{-\alpha}_{p,\infty}(\mathbb{R}^n)} \]
\[ \leq c \| \tilde{u}_0 \|_{B^{-\alpha}_{p,\infty}(\mathbb{R}^n)} \]

and
\[ 2^{\alpha j} \| v(t) \ast \phi_j \|_{L^p(\mathbb{R}^n)} \leq 2^{\alpha j} M(t, j) \| \tilde{u}_0 \ast \phi_j \|_{L^p(\mathbb{R}^n)} \]
\[ \leq c 2^{\alpha j} 2^{-12^j} \| \tilde{u}_0 \ast \phi_j \|_{L^p(\mathbb{R}^n)} \]
\[ \leq c \| \tilde{u}_0 \|_{B^{-\alpha}_{p,\infty}(\mathbb{R}^n)}. \]

We complete the proof of Lemma 3.2.

**Appendix C. Proof of Lemma 3.3**

Because the proofs will be done in the same way, we prove only the case of $\Gamma_1^* \ast \mathbb{P} f$.

A crucial step in the proof of Lemma 3.3 is the following lemma, which is probably known to experts, however, we could not find it in the literature and thus, we provide its proof.

**Lemma C.1** Let $X_i$ and $Y_i$, $i = 1, 2$ be Banach spaces and $0 < t$ be fixed. Let $T : L^1(0, t; X_i) \to Y_i$, $i = 1, 2$ be linear operators such that
\[ \| Tf \|_{Y_i} \leq M_i \int_0^t (t - s)^{-\beta_i} \| f(s) \|_{X_i} ds, \quad i = 1, 2 \quad \forall f \in L^1(0, t; X_i). \]

Then, for $0 < \theta < 1$ and $1 \leq q \leq \infty$,
\[ \| Tf \|_{(Y_1, Y_2)_{\theta, q}} \leq M_1^\theta M_2^{1-\theta} \int_0^t (t - s)^{-\beta} \| f(s) \|_{(X_1, X_2)_{\theta, q}} ds, \]
where $\beta = \beta_1 \theta + \beta_2 (1 - \theta)$.

**Proof** See Lemma C.1 in [9].

**Lemma C.2** Let
\[ w(x, t) = \int_0^t \int_{\mathbb{R}^{n-1}} D_x \Gamma(x' - y', x_n, t - \tau) f(y', \tau) dy' d\tau. \]

Then, for $\beta > 0$,
\[ \| w(t) \|_{L^p} \leq c \int_0^t (t - \tau)^{\frac{1}{2p} - 1 - \frac{\beta}{2}} \| f(\tau) \|_{B^{-\beta}_{p,\infty}(\mathbb{R}^{n-1})} d\tau. \quad (C.1) \]
Proof Let $D_x = D_{x_n}$. Then,

$$
\|w(t)\|_{L^p} \leq \int_0^t \left( \int_0^\infty \frac{x_n^p}{(t - \tau)^{3p/2}} e^{-\frac{\gamma_0^2}{t - \tau} d x_n} \right) \frac{1}{p} \|\Gamma_{t - \tau} \ast' f(\tau)\|_{L^p(\mathbb{R}^{n-1})} d \tau
$$

$$
= c \int_0^t (t - \tau)^{1/2p - 1} \|\Gamma_{t - \tau} \ast' f(\tau)\|_{L^p(\mathbb{R}^{n-1})} d \tau.
$$

(C.2)

Here, $\Gamma'$ is a Gaussian kernel in $\mathbb{R}^{n-1}$ and $\ast'$ is convolution in $\mathbb{R}^{n-1}$. Then, we have

$$
\|\Gamma_{t - s} \ast' f(\tau)\|_{L^p(\mathbb{R}^{n-1})} = \left\| \sum_{-\infty < k < \infty} \Phi_k \ast' \Gamma_{t - s} \ast \phi_k \ast' f(\tau) \right\|_{L^p(\mathbb{R}^{n-1})}
$$

$$
\leq \sum_{-\infty < k < \infty} \|\Phi_k \ast' \Gamma_{t - s}\|_{M_p(\mathbb{R}^{n-1})} \|\phi_k \ast' f(\tau)\|_{L^p(\mathbb{R}^{n-1})}
$$

$$
\leq c \sum_{-\infty < k < \infty} e^{-(t - s)2^{2k}} \|\phi_k \ast' f(\tau)\|_{L^p(\mathbb{R}^{n-1})}
$$

$$
\leq c \left( \sum_{-\infty < k < \infty} 2^{-\beta p k} \|\phi_k \ast' f(\tau)\|_{L^p(\mathbb{R}^{n-1})} \right)^{\frac{1}{p}}
$$

$$
\leq c(t - s)^{-\frac{\beta}{p}} \|f(\tau)\|_{\dot{B}^{-\beta}_{p, \infty}(\mathbb{R}^{n-1})}.
$$

(C.3)

From (C.2) and (C.3), we obtain (C.1).

Let $D_x = D_{x'}$. Then,

$$
\|w(t)\|_{L^p} \leq \int_0^t \left( \int_0^\infty \frac{1}{(t - \tau)^{3p/2}} e^{-\frac{\gamma_0^2}{t - \tau} d x_n} \right) \frac{1}{p} \|D_{x'} \Gamma_{t - \tau} \ast' f(\tau)\|_{L^p(\mathbb{R}^{n-1})} d \tau
$$

$$
= c \int_0^t (t - \tau)^{1/2p - 1/2} \|D_{x'} \Gamma_{t - \tau} \ast' f(\tau)\|_{L^p(\mathbb{R}^{n-1})} d \tau.
$$

(C.4)

Here, $\Gamma'$ is a Gaussian kernel in $\mathbb{R}^{n-1}$ and $\ast'$ is convolution in $\mathbb{R}^{n-1}$. Then, we have

$$
\|D_{x_n} \Gamma_{t - s} \ast' f(\tau)\|_{L^p(\mathbb{R}^{n-1})} = \left\| \sum_{-\infty < k < \infty} D_{x'} \Phi_k \ast' \Gamma_{t - s} \ast \phi_k \ast' f(\tau) \right\|_{L^p(\mathbb{R}^{n-1})}
$$

$$
\leq \sum_{-\infty < k < \infty} \|D_{x'} \Phi_k \ast' \Gamma_{t - s}\|_{M_p(\mathbb{R}^{n-1})} \|\phi_k \ast' f(\tau)\|_{L^p(\mathbb{R}^{n-1})}
$$

$$
\leq c \sum_{-\infty < k < \infty} 2^k e^{-(t - s)2^{2k}} \|\phi_k \ast' f(\tau)\|_{L^p(\mathbb{R}^{n-1})}
$$

$$
\leq c \left( \sum_{-\infty < k < \infty} 2^{p/2} e^{-(t - s)2^{2k}} \|\phi_k \ast' f(\tau)\|_{L^p(\mathbb{R}^{n-1})} \right)^{\frac{p}{2}}
$$

$$
\leq c \left( \sum_{-\infty < k < \infty} 2^{p/2} e^{-(t - s)2^{2k}} \|\phi_k \ast' f(\tau)\|_{L^p(\mathbb{R}^{n-1})} \right)^{\frac{p}{2}}.
$$
\[
\left( \sum_{-\infty < k < \infty} 2^{-\beta p k} \| \phi_k * f(\tau) \|_{L_p(\mathbb{R}^{n-1})}^{p} \right)^{\frac{1}{p}}
\]
\[
\leq c (t - s)^{-\frac{1}{2} - \frac{n}{2p}} \| f(\tau) \|_{B^{-n}_p(\mathbb{R}^{n-1})}.
\]  
(C.5)

From (C.4) and (C.5), we obtain (C.1).

**Proof of Lemma 3.3** Recalling Helmholtz decomposition of \( f = \text{div} F \) with \( F|_{x_n=0} = 0 \) in Section 3.4, we have

\[
\Gamma^* * (\mathbb{P} f) j(x, t) = \int_0^t \int_{\mathbb{R}^n_+} \Gamma(x - y^*, t - \tau) \cdot \text{div} F_j(y, \tau) dy d\tau
\]
\[
= - \int_0^t \int_{\mathbb{R}^n_+} \nabla_y \Gamma(x - y^*, t - \tau) \cdot F_j(y, \tau) dy d\tau.
\]  
(C.6)

Using (C.6), Young’s inequality and (3.3), for \( p_1 \leq p \), we have

\[
\| \Gamma^* * (\mathbb{P} f)(t) \|_{L^p} \leq c \int_0^t (t - s)^{-\frac{1}{2} - \frac{n}{2p}(\frac{1}{p_1} - \frac{1}{p})} \sum_{j=1}^n \| F_j(s) \|_{L_{p_1}} ds
\]
\[
\leq c \int_0^t (t - s)^{-\frac{1}{2} - \frac{n}{2p}(\frac{1}{p_1} - \frac{1}{p})} \| F(s) \|_{L_{p_1}} ds.
\]  
(C.7)

Similarly, we get

\[
\| \nabla \Gamma^* * (\mathbb{P} f)(t) \|_{L^p} \leq c \int_0^t (t - s)^{-\frac{1}{2} - \frac{n}{2p}(\frac{1}{p_1} - \frac{1}{p})} \sum_{j=1}^n \| \nabla F_j(s) \|_{L_{p_1}} ds
\]
\[
\leq c \int_0^t (t - s)^{-\frac{1}{2} - \frac{n}{2p}(\frac{1}{p_1} - \frac{1}{p})} \| F(s) \|_{H^1_{p_1}} ds.
\]  
(C.8)

From (C.6), we have

\[
\nabla_x^2 \Gamma^* * (\mathbb{P} f) j(x, t) = - \int_0^t \int_{\mathbb{R}^n_+} \nabla_y \Gamma(x - y^*, t - \tau) \cdot \nabla_y^2 F_j(y, \tau) dy d\tau
\]
\[
+ \int_0^t \int_{\mathbb{R}^{n-1}} D_x \Gamma(x' - y', x_n, t - s) D_{x_n} F_j(y', 0, s) dy' d\tau.
\]  
(C.9)

Hence from Lemma C.2, for \( \beta > 0 \), we have

\[
\| D_x^2 \Gamma^* * (\mathbb{P} f)(t) \|_{L^p} \leq \int_0^t (t - \tau)^{-\frac{1}{2} - \frac{n}{2p_1} + \frac{p}{2p}} \| D_x^2 F_j(\tau) \|_{L_{p_1}(\mathbb{R}^n_+)} d\tau
\]
\[
+ \int_0^t (t - \tau)^{\frac{1}{2p} - 1 - \frac{n}{2p}} \| D_y F_j(\tau) \|_{B^{-n}_p(\mathbb{R}^{n-1})} d\tau
\]
Hence, we complete the proof of (1) of Lemma 3.3.

From (3.3), we have

\[ I_1(t) \leq c \int_0^t (t - \tau)^{-\frac{1}{2} - \frac{n}{2p_1} + \frac{n}{2p}} \| F(\tau) \|_{\dot{H}^s_{p_1}} \, d\tau. \]

Because \( 1 + \frac{n-1}{p} < \frac{n}{p_1} < 1 + \frac{n}{p} \), taking \( \beta = -1 + \frac{n}{p_1} - \frac{n-1}{p} > 0 \), we have

\[ I_2(t) \leq c \int_0^t (t - \tau)^{-\frac{1}{2p} - \frac{n}{2p_1} + \frac{n}{2p}} \| D_\xi F_j(\tau) \|_{\dot{B}^{-\frac{1}{p}}_{p_1} (\mathbb{R}^n)} \, d\tau \]

\[ \leq c \int_0^t (t - \tau)^{-\frac{1}{2p} - \frac{n}{2p_1} + \frac{n}{2p}} \| D_\xi F_j(\tau) \|_{\dot{H}^n_{p_1}(\mathbb{R}^n)} \, d\tau \]

\[ \leq c \int_0^t (t - \tau)^{-\frac{1}{2p} \left( \frac{1}{p_1} - \frac{1}{p} \right)} \| F(\tau) \|_{\dot{H}^n_{p_1}(\mathbb{R}^n)} \, d\tau. \]

Hence, we obtain

\[ \| D_x^2 \Gamma_n \ast (\mathbb{P} f)(t) \|_{L^p} \leq c \int_0^t (t - \tau)^{-\frac{1}{2} - \frac{n}{2p_1} + \frac{n}{2p}} \| F(\tau) \|_{\dot{H}^s_{p_1}} \, d\tau. \quad (C.10) \]

Then, from (C.7), we get

\[ \| \Gamma_n \ast (\mathbb{P} f)(t) \|_{L^p} \leq c \int_0^t (t - s)^{-\frac{1}{2} - \frac{n}{2} \left( \frac{1}{p_1} - \frac{1}{p} \right)} \| F(s) \|_{L^p_1} \, ds \]

\[ \leq c \sup_{0 < s < t} \left( s^{\frac{1}{2} + \frac{n}{2} \left( \frac{1}{p_1} - \frac{1}{p} \right)} \| F(s) \|_{L^p_1} \right) \]

\[ \int_0^t (t - s)^{-\frac{1}{2} - \frac{n}{2} \left( \frac{1}{p_1} - \frac{1}{p} \right)} s^{-\frac{1}{2} + \frac{n}{2} \left( \frac{1}{p_1} - \frac{1}{p} \right)} ds \]

\[ = ct^{-\frac{1}{2}} \sup_{0 < s < t} \left( s^{\frac{1}{2} + \frac{n}{2} \left( \frac{1}{p_1} - \frac{1}{p} \right)} \| F(s) \|_{L^p_1} \right). \]

Hence, we complete the proof of (1) of Lemma 3.3.

From (C.7), (C.8), Lemma C.1 and (2.7), for \( 0 < \alpha < 1 \) and \( \frac{n}{p_1} < 1 + \frac{n}{p} \), we get

\[ \| \Gamma_n \ast (\mathbb{P} f)(t) \|_{\dot{B}^\alpha_{p_1 \infty}} \leq c \int_0^t (t - s)^{-\frac{1}{2} - \frac{n}{2p_1} + \frac{n}{2p}} \| F(s) \|_{\dot{B}^\alpha_{p_1 \infty}} \, ds \]

\[ \leq c \sup_{0 < s < t} \left( s^{\frac{1}{2} - \frac{n}{2p_1} + \frac{n}{2p}} \| F(s) \|_{\dot{B}^\alpha_{p_1 \infty}} \right) \]

\[ \int_0^t (t - s)^{-\frac{1}{2} - \frac{n}{2} \left( \frac{1}{p_1} - \frac{1}{p} \right)} s^{-\frac{1}{2} + \frac{n}{2} \left( \frac{1}{p_1} - \frac{1}{p} \right)} ds \]

\[ = c \sup_{0 < s < t} \left( s^{\frac{1}{2} - \frac{n}{2p_1} + \frac{n}{2p}} \| F(s) \|_{\dot{B}^\alpha_{p_1 \infty}} \right). \quad (C.11) \]
Hence, we obtain (2) of Lemma 3.3 for $0 < \alpha < 1$.

From (C.8), (C.10), Lemma C.1 and (2.7), for $1 < \alpha < 2$ and $1 < p_1 < p$ with $1 + \frac{n-1}{p} < \frac{n}{p_1} < 1 + \frac{n}{p}$, we have

$$\|G^\ast \ast (Pf)(t)\|_{B^{\frac{\alpha}{p}}_{p\infty}} \leq \int_0^t (t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{p_1}-\frac{1}{p})} \|F(s)\|_{B^{\frac{\alpha}{p_1}\infty}} ds$$  \hspace{1cm} (C.12)

As the same estimate of (C.11), we complete the proof of (2) of Lemma 3.3.

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