Automorphisms of cotangent bundles of Lie groups

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Abstract

Let $G$ be a Lie group, $G$ its Lie algebra, $T^*G$ its cotangent bundle and $D := t^*G$ the Lie algebra of $T^*G$. We investigate the group of automorphisms of $D$. More precisely, we fully characterize the space of all derivations of $D$.

As a byproduct, we also characterize some spaces of operators on $G$ amongst which the space $J$ of bi-invariant tensors on $G$ and prove that if $G$ has a bi-invariant Riemannian or pseudo-Riemannian metric, then $J$ is isomorphic to the space of equivariant linear maps $G^* \to G$, as well as that of bi-invariant bilinear forms on $G$.

1 Introduction

Let $G$ be a Lie group whose Lie algebra $\mathcal{G} := T_\epsilon G$ is identified with its tangent space $T_\epsilon G$ at the unit $\epsilon$. As a Lie group, the cotangent bundle $T^*G$ of $G$, is seen throughout this work as the semi-direct product $G \ltimes \mathcal{G}^*$ of $G$ and the Abelian Lie group $\mathcal{G}^*$, where $G$ acts on the dual space $\mathcal{G}^*$ of $\mathcal{G}$ via the coadjoint action. Here, using the trivialization by left translations, the manifold underlying $T^* G$ has been identified with the trivial bundle $G \times \mathcal{G}^*$. The Lie algebra $\text{Lie}(T^*G) = \mathcal{G} \ltimes \mathcal{G}^*$ of $T^*G$ will be denoted by $t^* \mathcal{G}$ or simply by $D$.

It is our aim in this work to fully study the connected component of the unit of the group of automorphisms of the Lie algebra $\mathcal{D} := t^* \mathcal{G}$. Such a connected component being spanned by exponentials of derivations of $\mathcal{D}$, we will work with those derivations and the first cohomology space $H^1(\mathcal{D}, \mathcal{D})$, where $\mathcal{D}$ is seen as the $\mathcal{D}$-module for the adjoint representation.

Our motivation for this work comes from several interesting algebraic and geometric problems.

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The cotangent bundle $T^*G$ of a Lie group $G$ can exhibit very interesting and rich algebraic and geometric structures [17], [14], [11], [10], [8], [3], etc. Such structures can be better understood when one can compare, deform or classify them. This very often involves the invertible homomorphisms (automorphisms) of $T^*G$, if in particular, such structures are invariant under left or right multiplications by the elements of $T^*G$. The derivative at the unit, of automorphisms of the Lie group $T^*G$, are automorphisms of the Lie algebra $D$. Conversely, if $G$ is connected and simply connected, then so is $T^*G$ and every automorphism of the Lie algebra $D$ integrates to an automorphism of the Lie group $T^*G$. A problem involving invariant structures on a Lie group also usually transfers to one on its Lie algebra, with the Lie algebra automorphisms used as a means to compare or classify the corresponding induced structures.

In the purely algebraic point of view, finding and understanding the derivations of a given Lie algebra, is in itself an interesting problem [13], [12], [9], [15], [20], [5], etc.

On the other hand, as a Lie group, the cotangent bundle $T^*G$ is a common Drinfel’d double Lie group for all exact Poisson-Lie structures given by solutions of the Classical Yang-Baxter Equation in $G$. See e.g. [6]. Double Lie algebras (resp. groups) encode information on integrable Hamiltonian systems and Lax pairs ([2], [1], [10], [16]), Poisson homogeneous spaces of Poisson-Lie groups and the symplectic foliation of the corresponding Poisson structures ([10], [6],[16]), etc. To that extend, the complete description of the group of automorphisms of the double Lie algebra of a Poisson-Lie structure would be a big contribution towards solving very interesting and hard problems.

See Section 5 for wider discussions.

Interestingly, the space of derivations of $D$ encompasses interesting spaces of operators on $\mathcal{G}$, amongst which the derivations of $\mathcal{G}$, the second space of the invariant de Rham cohomology $H^2_{\text{inv}}(G, \mathbb{R})$ of $G$, bi-invariant endomorphisms, in particular operators giving rise to complex group structure in $G$, when they exist, ...

Throughout this work, $\text{der}(\mathcal{G})$ will stand for the Lie algebra of derivations of $\mathcal{G}$, while $J$ will denote that of linear maps $j : \mathcal{G} \to \mathcal{G}$ satisfying $j([x,y]) = [j(x),y]$, $\forall x,y \in \mathcal{G}$. We mainly work on the field $\mathbb{R}$ (or $\mathbb{C}$ in parts of Sections 4.1 and 4.2.) However, most of the results within this paper are valid for any field of characteristic zero.

We summarize some of our main results as follows.

**Theorem A.** Let $G$ be a Lie group, $\mathcal{G}$ its Lie algebra, $T^*G$ its cotangent bundle and $\mathcal{D} := \mathcal{G} \ltimes \mathcal{G}^* = \text{Lie}(T^*G)$. A derivation of $\mathcal{D}$, has the form

$$\phi(x,f) = \left( \alpha(x) + \psi(f), \beta(x) + f \circ (\alpha + j) \right), \quad \forall (x,f) \in \mathcal{D},$$

where $\alpha : \mathcal{G} \to \mathcal{G}$ is a derivation of the Lie algebra $\mathcal{G}$, the linear map $j : \mathcal{G} \to \mathcal{G}$ is in $J$, i.e. satisfies $j([x,y]) = [j(x),y]$, $\forall x,y \in \mathcal{G}$, $\beta : \mathcal{G} \to \mathcal{G}^*$ is a 1-cocycle of $\mathcal{G}$ with values in $\mathcal{G}^*$ for the coadjoint action of $\mathcal{G}$ on $\mathcal{G}^*$ and $\psi : \mathcal{G}^* \to \mathcal{G}$ is a linear map satisfying the following conditions:

$$\psi \circ \text{ad}_x^* = \text{ad}_x \circ \psi, \quad \forall x \in \mathcal{G}, \quad \text{and} \quad \text{ad}_{\psi(f)}^* g = \text{ad}_{\psi(g)}^* f, \quad \forall f,g \in \mathcal{G}^*.$$
alone, as follows,

$$\phi(x, f) = (\alpha(x) + j \circ \theta^{-1}(f), \theta \circ \alpha'(x) + f \circ (j' + \alpha)), \quad \forall (x, f) \in \mathcal{D},$$

where $\alpha, \alpha' \in \text{der}(\mathcal{G})$, $j, j' \in \mathcal{J}$, and $\theta : \mathcal{G} \to \mathcal{G}^*$ with $\theta(x)(y) := \mu(x, y), \quad \forall x, y \in \mathcal{G}$.

**Theorem B.** Let $G$ be a Lie group and $\mathcal{G}$ its Lie algebra. The group $\text{Aut}(\mathcal{D})$ of automorphisms of the Lie algebra $\mathcal{D}$ of the cotangent bundle $T^*G$ of $G$, is a symmetric group. More precisely, its Lie algebra $\text{der}(\mathcal{D})$ decomposes into a direct sum of vector spaces

$$\text{der}(\mathcal{D}) := \mathcal{G}_1 \oplus \mathcal{G}_2, \quad \text{with} \quad [\mathcal{G}_1, \mathcal{G}_1] \subset \mathcal{G}_1, \quad [\mathcal{G}_1, \mathcal{G}_2] \subset \mathcal{G}_2, \quad [\mathcal{G}_2, \mathcal{G}_2] \subset \mathcal{G}_1,$$

where $\mathcal{G}_1$ is the Lie algebra

$$\mathcal{G}_1 := \left\{ \phi : \mathcal{D} \to \mathcal{D}, \phi(x, f) = \left( \alpha(x), f \circ (\alpha + j) \right), \quad \text{with} \quad \alpha \in \text{der}(\mathcal{G}) \text{ and } j \in \mathcal{J} \right\}$$

and $\mathcal{G}_2$ is the direct sum (as a vector space) of the space $\mathcal{Q}$ of cocycles $\beta : \mathcal{G} \to \mathcal{G}^*$ and the space $\Psi$ of equivariant maps $\psi : \mathcal{G}^* \to \mathcal{G}$ with respect to the coadjoint and the adjoint representations, satisfying $\text{ad}^*_\psi(f)g = \text{ad}^*_\psi(g)f, \quad \forall f, g \in \mathcal{G}^*$. Moreover, $\mathcal{Q}$ and $\Psi$ are Abelian subalgebras of $\text{der}(\mathcal{D})$.

**Theorem C.** The first cohomology space $H^1(\mathcal{D}, \mathcal{D})$ of the (Chevalley–Eilenberg) cohomology associated with the adjoint action of $\mathcal{D}$ on itself, satisfies $H^1(\mathcal{D}, \mathcal{D}) \cong H^1(\mathcal{G}, \mathcal{G}^*) \oplus \mathcal{J}^t \oplus H^1(\mathcal{G}, \mathcal{G}^*) \oplus \Psi$, where $H^1(\mathcal{G}, \mathcal{G})$ and $H^1(\mathcal{G}, \mathcal{G}^*)$ are the first cohomology spaces associated with the adjoint and coadjoint actions of $\mathcal{G}$, respectively; and $\mathcal{J}^t := \{ j^t, j \in \mathcal{J} \}$ (space of transposes of elements of $\mathcal{J}$).

If $\mathcal{G}$ is semi-simple, then $\Psi = \{0\}$ and thus $H^1(\mathcal{D}, \mathcal{D}) \cong \mathcal{J}^t$. If $\mathcal{G}$ is a semi-simple Lie algebra over $\mathbb{C}$, then $\mathcal{J} \cong \mathbb{C}^p$, where $p$ is the number of the simple ideals $s_i$ of $\mathcal{G}$ such that $\mathcal{G} = s_1 \oplus \ldots \oplus s_p$. Hence, of course, $H^1(\mathcal{D}, \mathcal{D}) \cong \mathbb{C}^p$. In particular, if $\mathcal{G}$ is complex and simple, then $H^1(\mathcal{D}, \mathcal{D}) \cong \mathbb{C}$.

If $\mathcal{G}$ is a compact Lie algebra over $\mathbb{C}$, with centre $Z(\mathcal{G})$, we get $H^1(\mathcal{G}, \mathcal{G}) \cong \text{End}(Z(\mathcal{G})), \quad \mathcal{J} \cong \mathbb{C}^p \oplus \text{End}(Z(\mathcal{G})), \quad H^1(\mathcal{G}, \mathcal{G}^*) \cong L(Z(\mathcal{G}), Z(\mathcal{G})^*), \quad \Psi \cong L(Z(\mathcal{G})^*, Z(\mathcal{G})).$ Hence, we get $H^1(\mathcal{D}, \mathcal{D}) \cong (\text{End}(\mathbb{C}^k))^4 \oplus \mathbb{C}^p$, where $k$ is the number of the simple components of the derived ideal $[\mathcal{G}, \mathcal{G}]$ of $\mathcal{G}$. Here, if $E, F$ are vector spaces, $L(E, F)$ is the space of linear maps $E \to F$.

The paper is organized as follows. In Section 2, we explain some of the material and terminology needed to make the paper more self contained. Sections 3 and 4 are the actual core of the work where the main calculations and proofs are carried out. In Section 5, we discuss some subjects related to this work, as well as some of the possible applications.

## 2 Preliminaries

Although not central to the main purpose of the work within this paper, the following material might be useful, at least, as regards parts of the terminology used throughout this paper.
2.1 The cotangent bundle of a Lie group

Throughout this paper, given a Lie group $G$, we will always let $G \ltimes G^*$ stand for the Lie group consisting of the Cartesian product $G \times G^*$ as its underlying manifold, together with the group structure obtained by semi-direct product using the coadjoint action of $G$ on $G^*$. Recall that the trivialization by left translations, or simply the left trivialization of $T^*G$ is given by the following isomorphism $\zeta$ of vector bundles

$$\zeta : T^*G \to G \times G^*, \quad (\sigma, \nu_\sigma) \mapsto (\sigma, \nu_\sigma \circ T_\epsilon L_\sigma),$$

where $L_\sigma$ is the left multiplication $L_\sigma : G \to G, \tau \mapsto L_\sigma(\tau) := \sigma \tau$ by $\sigma$ and $T_\epsilon L_\sigma$ is the derivative of $L_\sigma$ at the unit $\epsilon$. One then endows $T^*G$ with the Lie group structure such that $\zeta$ is an isomorphism of Lie groups. The Lie algebra of $T^*G$ is then the semi-direct product $\mathcal{D} := G \ltimes G^*$. More precisely, the Lie bracket on $\mathcal{D}$ reads

$$[(x, f), (y, g)] := ([x, y], ad_x^* g - ad_y^* f), \quad \forall (x, f), (y, g) \in \mathcal{D}. \quad (1)$$

2.2 Bi-invariant metrics on Lie groups

In this work, we will refer to an object which is invariant under both left and right translations in a Lie group $G$, as a bi-invariant object. We discuss in this section, how $T^*G$ is naturally endowed with a bi-invariant pseudo-Riemannian metric.

A bi-invariant (Riemannian or pseudo-Riemannian) metric in a connected Lie group $G$, corresponds to a symmetric bilinear nondegenerate scalar form $\mu$ in its Lie algebra $G$, such that the adjoint representation of $G$ lies in the Lie algebra $O(G, \mu)$ of infinitesimal isometries of $\mu$, or equivalently

$$\mu([x, y], z) + \mu(y, [x, z]) = 0, \forall x, y, z \in G. \quad (2)$$

This means that $\mu$ is invariant under the adjoint action of $G$. In this case, we will simply say that $\mu$ is adjoint-invariant. Lie groups with bi-invariant metrics and their Lie algebras are called orthogonal or quadratic (see e.g. [18], [19].) Every orthogonal Lie group is obtained by the so-called double extension construction introduced by Medina and Revoy [18], [19]. Consider the isomorphism of vector spaces $\theta : G \to G^*$ defined by $\langle \theta(x), y \rangle := \mu(x, y)$, where $\langle \cdot, \cdot \rangle$ on the left hand side, is the duality pairing $\langle x, f \rangle = f(x)$, between elements $x$ of $G$ and $f$ of $G^*$. Then, $\theta$ is an isomorphism of $G$-modules in the sense that it is equivariant with respect to the adjoint and coadjoint actions of $G$ on $G$ and $G^*$ respectively; i.e.

$$\theta \circ ad_x = ad_{\theta(x)} \circ \theta, \quad \forall x \in G. \quad (3)$$

We also have

$$\theta^{-1} \circ ad_x = ad_{\theta^{-1}(x)} \circ \theta^{-1}, \quad \forall x \in G. \quad (4)$$

The converse is also true. More precisely, a Lie group (resp. algebra) is orthogonal if and only if its adjoint and coadjoint representations are isomorphic. See Theorem 1.4. of [19].
Semisimple Lie groups, with their Killing form, are examples of orthogonal Lie groups. Compact Lie groups, or more generally, reductive Lie groups, are orthogonal Lie groups.

The cotangent bundle of any Lie group (with its natural Lie group structure, as above) and in general any element of the larger and interesting family of the so-called Drinfel’d doubles (see Section 2.3), are orthogonal Lie groups \([10]\), as explained below.

As above, let \(D := \mathcal{G} \times \mathcal{G}^*\) be the Lie algebra of the cotangent bundle \(T^*G\) of \(G\), seen as the semi-direct product of \(G\) by \(\mathcal{G}^*\) via the coadjoint action of \(\mathcal{G}\) on \(\mathcal{G}^*\), as in (1).

Let \(\mu_0\) stand for the duality pairing \(\langle \cdot, \cdot \rangle\), that is,

\[
\mu_0((x,f),(y,g)) = f(y) + g(x), \quad \forall (x,f), (y,g) \in D.
\]

Then, \(\mu_0\) satisfies the property (2) on \(D\) and hence gives rise to a bi-invariant (pseudo-Riemannian) metric on \(T^*G\).

### 2.3 Doubles of Poisson-Lie groups and Yang-Baxter Equation

We explain in this section how cotangent bundles of Lie groups are part of the broader family of the so-called double Lie groups of Poisson-Lie groups.

A Poisson structure on a manifold \(M\) is given by a Lie bracket \(\{\cdot,\cdot\}\) on the space \(C^\infty(M, \mathbb{R})\) of smooth real-valued functions on \(M\), such that, for each \(f \in C^\infty(M, \mathbb{R})\), the linear operator \(X_f := \{f,\cdot\}\) on \(C^\infty(M, \mathbb{R})\), defined by \(g \mapsto X_f \cdot g := \{f,g\}\), is a vector field on \(M\). The bracket \(\{\cdot,\cdot\}\) defines a 2-tensor, that is, a bivector field \(\pi\) which, seen as a bilinear skew-symmetric 'form' on the space of differential 1-forms on \(M\), is given by \(\pi(df, dg) := \{f,g\}\). The Jacobi identity for \(\{\cdot,\cdot\}\) now reads \([\pi,\pi]_S = 0\), where \([\cdot,\cdot]_S\) is the so-called Schouten bracket, which is a natural extension to all multi-vector fields, of the natural Lie bracket of vector fields. Reciprocally, any bivector field \(\pi\) on \(M\) satisfying \([\pi,\pi]_S = 0\), is a Poisson tensor, i.e. defines a Poisson structure on \(M\). See e.g. \([16]\).

Recall that a Poisson-Lie structure on a Lie group \(G\), is given by a Poisson tensor \(\pi\) on \(G\), such that, when the Cartesian product \(G \times G\) is equipped with the Poisson tensor \(\pi \times \pi\), the multiplication \(m : (\sigma, \tau) \mapsto \sigma \tau\) is a Poisson map between the Poisson manifolds \((G \times G, \pi \times \pi)\) and \((G, \pi)\). In other words, the derivative \(m_*\) of \(m\) satisfies \(m_*(\pi \times \pi) = \pi\). If \(f, g\) are in \(\mathcal{G}^*\) and \(\tilde{f}, \tilde{g}\) are \(C^\infty\) functions on \(G\) with respective derivatives \(\tilde{f} = f_{*,\varepsilon}\), \(g = \tilde{g}_{*,\varepsilon}\) at the unit \(\varepsilon\) of \(G\), one defines another element \([f,g]_*\) of \(\mathcal{G}^*\) by setting \([f,g]_* := (\{\tilde{f}, \tilde{g}\})_{*,\varepsilon}\). Then \([f,g]_*\) does not depend on the choice of \(\tilde{f}\) and \(\tilde{g}\) as above, and \((\mathcal{G}^*, [\cdot,\cdot]_*)\) is a Lie algebra. Now, there is a symmetric role played by the spaces \(\mathcal{G}\) and \(\mathcal{G}^*\), dual to each other. Indeed, as well as acting on \(\mathcal{G}^*\) via the coadjoint action, \(\mathcal{G}\) is also acted on by \(\mathcal{G}^*\) using the coadjoint action of \((\mathcal{G}^*, [\cdot,\cdot]_*)\). A lot of the most interesting properties and applications of \(\pi\), are encoded in the new Lie algebra \((\mathcal{G} \oplus \mathcal{G}^*, [\cdot,\cdot]_\pi)\), where

\[
([x,f]_\pi, [y,g]_\pi) := ([x,y] + ad^*_y f - ad^*_x g, ad^*_x g - ad^*_y f + [f,g]_*),
\]

for every \(x, y, f, g \in \mathcal{G}\).

The Lie algebras \((\mathcal{G} \oplus \mathcal{G}^*, [\cdot,\cdot]_\pi)\) and \((\mathcal{G}^*, [\cdot,\cdot]_*)\) are respectively called the double and the dual Lie algebras of the Poisson-Lie group \((G, \pi)\). Endowed with the duality pairing defined in (5), the double Lie algebra of any Poisson-Lie group \((G, \pi)\), is an orthogonal Lie
algebra, such that \( \mathcal{G} \) and \( \mathcal{G}^* \) are maximal totally isotropic (Lagrangian) subalgebras. The

\[
\{ \mathcal{G} \oplus \mathcal{G}^*, \mathcal{G}^* \}
\]

is then called a Manin triple. More generally, \( \{ \mathcal{G}_1 \oplus \mathcal{G}_2, \mathcal{G}_1, \mathcal{G}_2 \} \)

is called a Manin triple and \( \{ \mathcal{G}_1, \mathcal{G}_2 \} \) a bi-algebra or a Manin pair, if \( \mathcal{G}_1 \oplus \mathcal{G}_2 \) is an orthogonal 2n-dimensional Lie algebra whose underlying adjoint-invariant pseudo-Riemannian metric is of index \((n,n)\) and \( \mathcal{G}_1, \mathcal{G}_2 \) are two Lagrangian complementary subalgebras. See \([10],[16],[6]\), for wider discussions.

Let \( r \) be an element of the wedge product \( \wedge^2 \mathcal{G} \). Denote by \( r^+ \) (resp. \( r^- \)) the left

(resp. right) invariant bivector field on \( \mathcal{G} \) with value \( r = r^+_\epsilon \) (resp. \( r = r^-\epsilon \)) at \( \epsilon \). If \( \pi_r := r^+ - r^- \) is a Poisson tensor, then it is a Poisson-Lie tensor and \( r \) is called a solution of the Yang-Baxter Equation. If, in particular, \( r^+ \) is a (left invariant) Poisson tensor on \( \mathcal{G} \), then \( r \) is called a solution of the Classical Yang-Baxter Equation (CYBE) on \( \mathcal{G} \) (resp. \( \mathcal{G}^* \)). In this latter case, the double Lie algebra \( \{ \mathcal{G} \oplus \mathcal{G}^*, [,]_{\pi_r} \} \) is isomorphic to the Lie algebra \( \mathcal{D} \) of the cotangent bundle \( T^* \mathcal{G} \) of \( \mathcal{G} \). See e.g. \([8]\). We may also consider the linear map \( \tilde{\rho} : \mathcal{G}^* \to \mathcal{G} \), where \( \tilde{\rho}(f) := r(f,\cdot) \). The linear map \( \theta_r : \{ \mathcal{G} \oplus \mathcal{G}^*, [,]_{\pi_r} \} \to \mathcal{D} \),

\[
\theta_r(x,f) := (x + \tilde{\rho}(f),f),
\]

is an isomorphism of Lie algebras, between \( \mathcal{D} \) and the double Lie algebra of any Poisson-Lie group structure on \( \mathcal{G} \), given by a solution \( r \) of the CYBE.

## 3 Group of automorphisms of \( \mathcal{D} := t^* \mathcal{G} \)

### 3.1 Derivations of \( \mathcal{D} := t^* \mathcal{G} \)

Consider a Lie group \( \mathcal{G} \) of dimension \( n \), with Lie algebra \( \mathcal{G} \). Let us also denote by \( \mathcal{D} \)

the vector space underlying the Lie algebra \( \mathcal{D} \) of the cotangent bundle \( T^* \mathcal{G} \), regarded as a \( \mathcal{D} \)-module under the adjoint action of \( \mathcal{D} \). Consider the following complex with the coboundary operator \( \partial \), where \( \partial \circ \partial = 0 \):

\[
0 \to \mathcal{D} \to \text{Hom}(\mathcal{D},\mathcal{D}) \to \text{Hom}(\Lambda^2 \mathcal{D},\mathcal{D}) \to \cdots \to \text{Hom}(\Lambda^n \mathcal{D},\mathcal{D}) \to 0.
\]

We are interested in \( \text{Hom}(\mathcal{D},\mathcal{D}) := \{ \phi : \mathcal{D} \to \mathcal{D}, \phi \text{ linear } \} \). The coboundary \( \partial \phi \) of

\( \phi \in \text{Hom}(\mathcal{D},\mathcal{D}) \) is the element of \( \text{Hom}(\Lambda^2 \mathcal{D},\mathcal{D}) \) defined by

\[
\partial \phi(u,v) := ad_u \left( \phi(v) \right) - ad_v \left( \phi(u) \right) - \phi([u,v]),
\]

for any \( u := (x,f), v := (y,g) \in \mathcal{D} \). An element \( \phi \) of \( \text{Hom}(\mathcal{D},\mathcal{D}) \) is a 1-cocycle if \( \partial \phi = 0 \), i.e.

\[
\phi([u,v]) = ad_u \left( \phi(v) \right) - ad_v \left( \phi(u) \right),
\]

\[
= [u,\phi(v)] + [\phi(u),v].
\]

In other words, 1-cocycles are the derivations of the Lie algebra \( \mathcal{D} \). In Section 3.6, we will characterize the first cohomology space \( H^1(\mathcal{D},\mathcal{D}) := \ker(\partial^2)/\text{Im}(\partial^1) \) of the associated Chevalley-Eilenberg cohomology, where for clarity, we denote by \( \partial^1 \) and \( \partial^2 \) the following restrictions \( \partial^1 : \mathcal{D} \to \text{Hom}(\mathcal{D},\mathcal{D}) \) and \( \partial^2 : \text{Hom}(\mathcal{D},\mathcal{D}) \to \text{Hom}(\Lambda^2 \mathcal{D},\mathcal{D}) \) of \( \partial \).
Theorem 3.1. Let $G$ be a Lie group, $\mathcal{G}$ its Lie algebra, $T^*G$ its cotangent bundle and $\mathcal{D} := \mathcal{G} \ltimes \mathcal{G}^*$ the Lie algebra of $T^*G$. A 1-cocycle (for the adjoint representation) hence a derivation of $\mathcal{D}$ has the following form:

$$
\phi(x, f) = \left( \alpha(x) + \psi(f), \beta(x) + \xi(f) \right),
$$

where $\alpha : \mathcal{G} \to \mathcal{G}$ is a derivation of the Lie algebra $\mathcal{G}$, $\beta : \mathcal{G} \to \mathcal{G}^*$ is a 1-cocycle of $\mathcal{G}$ with values in $\mathcal{G}^*$ for the coadjoint action of $\mathcal{G}$ on $\mathcal{G}^*$, $\xi : \mathcal{G}^* \to \mathcal{G}^*$ and $\psi : \mathcal{G}^* \to \mathcal{G}$ are linear maps satisfying the following conditions:

$$
[\xi, ad^*_x] = ad^*_{\alpha(x)}, \quad \forall x \in \mathcal{G},
$$

$$
\psi \circ ad^*_x = ad_x \circ \psi, \quad \forall x \in \mathcal{G},
$$

$$
ad^*_\psi(f)g = ad^*_\psi(g)f, \quad \forall f, g \in \mathcal{G}^*.
$$

The rest of this section is dedicated to the proof of Theorem 3.1.

Aiming to get a simpler expression for the derivations, let us write $\phi$ in terms of its components relative to the decomposition of $\mathcal{D}$ into a direct sum $\mathcal{D} = \mathcal{G} \oplus \mathcal{G}^*$ of vector spaces as follows:

$$
\phi(x, f) = \left( \phi_{11}(x) + \phi_{21}(f), \phi_{12}(x) + \phi_{22}(f) \right), \quad \forall (x, f) \in \mathcal{D},
$$

where $\phi_{11} : \mathcal{G} \to \mathcal{G}$, $\phi_{12} : \mathcal{G} \to \mathcal{G}^*$, $\phi_{21} : \mathcal{G}^* \to \mathcal{G}$ and $\phi_{22} : \mathcal{G}^* \to \mathcal{G}^*$ are all linear maps. In (14) we have made the identifications: $x = (x, 0)$, $f = (0, f)$ so that the element $(x, f)$ can also be written $x + f$. Likewise, we can write

$$
\phi(x) = (\phi_{11}(x), \phi_{12}(x)), \quad \forall x \in \mathcal{G}; \quad \phi(f) = (\phi_{21}(f), \phi_{22}(f)), \quad \forall f \in \mathcal{G}^*
$$

or simply

$$
\phi(x) = \phi_{11}(x) + \phi_{12}(x); \quad \phi(f) = \phi_{21}(f) + \phi_{22}(f).
$$

In order to find the $\phi_{ij}$’s and hence all the derivations of $\mathcal{D}$, we are now going to use the cocycle condition (9).

For $x, y \in \mathcal{G} \subset \mathcal{D}$ we have:

$$
\phi([x, y]) = \phi_{11}([x, y]) + \phi_{12}([x, y])
$$

and

$$
[\phi(x), y] + [x, \phi(y)] = \left[ \phi_{11}(x) + \phi_{12}(x), y \right] + \left[ x, \phi_{11}(y) + \phi_{12}(y) \right]
$$

$$
= \left[ \phi_{11}(x), y \right] - ad^*_x(\phi_{12}(x)) + \left[ x, \phi_{11}(y) \right] + ad^*_x(\phi_{12}(y)).
$$

Comparing (17) and (18), we first get

$$
\phi_{11}([x, y]) = \phi_{11}(x, y) + [x, \phi_{11}(y)], \quad \forall x, y \in \mathcal{G}.
$$

This means that $\phi_{11}$ is a derivation of the Lie algebra $\mathcal{G}$. 

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Secondly we have
\[ \phi_{12}([x,y]) = ad^*_x(\phi_{12}(y)) - ad^*_y(\phi_{12}(x)). \] (20)

Equation (20) means that \( \phi_{12} : G \to G^* \) is a 1-cocycle of \( G \) with values on \( G^* \) for the coadjoint action of \( G \) on \( G^* \).

Now we are going to examine the following case:
\[ \phi([x,f]) = \phi(ad^*_x f), \quad \forall x \in G, \forall f \in G^*. \] (21)

and
\[ [\phi(x,f) + x, \phi(f)] = [\phi_{11}(x) + \phi_{12}(x), f] + [x, \phi_{21}(f) + \phi_{22}(f)], \]
\[ = ad^*_{\phi_{11}(x)} f + [x, \phi_{21}(f)] + ad^*_{\phi_{22}(f)} f. \] (22)

Identifying (21) and (22) we obtain on the one hand
\[ \phi_{21}(ad^*_x f) = [x, \phi_{21}(f)], \quad \forall x \in G, \forall f \in G^*. \]

We write the above as
\[ \phi_{21} \circ ad^*_x = ad^*_x \circ \phi_{21}, \quad \forall x \in G. \]

That is, \( \phi_{21} : G^* \to G \) is equivariant (commutes) with respect to the adjoint and the coadjoint actions of \( G \) on \( G \) and \( G^* \) respectively.

We have on the other hand
\[ \phi_{22}(ad^*_x f) = ad^*_x(\phi_{22}(f)) + ad^*_{\phi_{21}(x)} f, \quad \forall x \in G, \forall f \in G^*. \] (23)

Formula (23) can be rewritten as
\[ \phi_{22} \circ ad^*_x - ad^*_x \circ \phi_{22} = ad^*_{\phi_{11}(x)}, \quad \forall x \in G. \]

i.e.
\[ [\phi_{22}, ad^*_x] = ad^*_x(\phi_{11}(x)), \quad \forall x \in G. \]

Last, for \( f, g \in G^* \) we have
\[ \phi([f, g]) = 0, \] (24)

and
\[ [\phi(f), g] + [f, \phi(g)] = [\phi_{21}(f) + \phi_{22}(f), g] + [f, \phi_{21}(g) + \phi_{22}(g)], \]
\[ = ad^*_{\phi_{21}(f)} g - ad^*_{\phi_{21}(g)} f. \] (25)

From (24) and (25) it comes that
\[ ad^*_{\phi_{21}(f)} g = ad^*_{\phi_{21}(g)} f, \quad \forall f, g \in G^*. \]

Noting \( \alpha := \phi_{11}, \beta := \phi_{12}, \psi := \phi_{21} \) and \( \xi := \phi_{22} \), we get a proof of Theorem 3.1 \( \Box \)
Remark 3.1. (Notations) From now on, if $\mathcal{G}$ is a Lie algebra, then

(1) $\mathcal{E}$ will stand for the space of linear maps $\xi : \mathcal{G}^* \to \mathcal{G}^*$ satisfying Equation (11), for some derivation $\alpha$ of $\mathcal{G}$;

(2) set $\mathcal{G}_1 := \{ \phi : \mathcal{D} \to \mathcal{D}, \phi(x, f) = (\alpha(x), \xi(f)), \text{ where } \alpha \in \text{der}(\mathcal{G}), \xi \in \mathcal{E} \text{ and } [\xi, \text{ad}_\alpha^*] = \text{ad}_\alpha^* \xi, \forall x \in \mathcal{G} \}$;

(3) we may let $\Psi$ stand for the space of 1-cocycles $\beta : \mathcal{G} \to \mathcal{G}^*$ as in (20), whereas $\Psi$ may be used for the space of equivariant linear $\psi : \mathcal{G}^* \to \mathcal{G}$ as in (12), which satisfy (13);

(4) we will denote by $\mathcal{G}_2$, the direct sum $\mathcal{G}_2 := \mathcal{Q} \oplus \Psi$ of the vector spaces $\mathcal{Q}$ and $\Psi$.

Remark 3.2. The spaces $\text{der}(\mathcal{G})$ of derivations of $\mathcal{G}$, $\mathcal{Q}$ and $\Psi$, as in Remark 3.1, are all subsets of $\text{der}(\mathcal{D})$, as follows. A derivation $\alpha$ of $\mathcal{G}$, an equivariant map $\psi \in \Psi$, a 1-cocycle $\beta \in \mathcal{Q}$ respectively seen as the elements $\phi_\alpha, \phi_\psi, \phi_\beta$ of $\text{der}(\mathcal{D})$, with

$$\phi_\alpha(x, f) := (\alpha(x), f \circ \alpha); \phi_\psi(x, f) := (\psi(f), 0); \phi_\beta(x, f) := (0, \beta(x)), \forall \,(x, f) \in \mathcal{D}.$$ 

3.2 A structure theorem for the group of automorphisms of $\mathcal{D}$

Lemma 3.1. The space $\mathcal{E}$, as in Remark 3.1, is a Lie algebra.

Namely, if $\xi_1, \xi_2 \in \mathcal{E}$ satisfy

$$[\xi_1, \text{ad}_\alpha^*] = \text{ad}_{\alpha_1(x)}^* \xi_1 \text{ and } [\xi_2, \text{ad}_\alpha^*] = \text{ad}_{\alpha_2(x)}^* \xi_2, \forall \,x \in \mathcal{G}, \tag{26}$$

then their Lie bracket $[\xi_1, \xi_2]$ is in $\mathcal{E}$ and satisfies

$$[[\xi_1, \xi_2], \text{ad}_\alpha^*] = \text{ad}_{\alpha_1 \alpha_2(x)}^* \xi_1, \forall \,x \in \mathcal{G}. \tag{27}$$

Proof. Using Jacobi identity in the Lie algebra $\mathcal{gl}(\mathcal{G}^*)$ of endomorphisms of the vector space $\mathcal{G}^*$, we get

$$[[\xi_1, \xi_2], \text{ad}_\alpha^*] = [[\xi_1, \text{ad}_\alpha^*] \xi_2] + [\xi_1, [\xi_2, \text{ad}_\alpha^*]]$$

$$= [\text{ad}_{\alpha_1(x)}^* \xi_1, \xi_2] + [\xi_1, \text{ad}_{\alpha_2(x)}^* \xi_2]$$

$$= -\text{ad}_{\alpha_2 \alpha_1(x)}^* \xi_1 + \text{ad}_{\alpha_1 \alpha_2(x)}^* \xi_2$$

$$= \text{ad}_{\alpha_1 \alpha_2(x)}^* \xi_1, \forall \,x \in \mathcal{G}. \square$$

Lemma 3.2. The space $\mathcal{G}_1$, as in Remark 3.1, is a Lie subalgebra of $\text{der}(\mathcal{D})$.

Proof. This is a consequence of Lemma 3.1. If $\phi_1 := (\alpha_1, \xi_1), \phi_2 := (\alpha_2, \xi_2) \in \mathcal{G}_1$, then $[\phi_1, \phi_2] = ([\alpha_1, \alpha_2], [\xi_1, \xi_2])$, as can easily be seen, below. For every $(x, f) \in \mathcal{D}$, we have

$$[\phi_1, \phi_2](x, f) = \phi_1(\alpha_2(x), \xi_2(f)) - \phi_2(\alpha_1(x), \xi_1(f))$$

$$= (\alpha_1 \circ \alpha_2(x), \xi_1 \circ \xi_2(f)) - (\alpha_2 \circ \alpha_1(x), \xi_2 \circ \xi_1(f))$$

$$= ([\alpha_1, \alpha_2](x), [\xi_1, \xi_2](f)). \square$$

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Lemma 3.3. Let $\beta$ and $\psi$ be elements of $Q$ and $\Psi$, respectively. Then $[\beta, \psi] = (-\psi \circ \beta, \beta \circ \psi) \in G_1$, more precisely $\beta \circ \psi \in E$, $\psi \circ \beta \in \text{der}(G)$ and $[\beta \circ \psi, ad^*_x] = -ad^*_{\psi \circ \beta(x)}, \forall x \in G$.

Proof. First, $\beta$ being a 1-cocycle is equivalent to

$$\beta \circ ad_x(y) = ad^*_x \circ \beta(y) - ad^*_y \circ \beta(x), \quad \forall x, y \in G. \quad (28)$$

Now for every $x \in G$ and $f \in G^*$, we have

$$[\beta \circ \psi, ad^*_x](f) = \beta \circ \psi \circ ad^*_x(f) - ad^*_y \circ \beta \circ \psi(f) = \beta \circ ad_x \circ \psi(f) - ad^*_y \circ \beta \circ \psi(f), \text{ now take } y = \psi(f) \text{ in } (28)$$

$$= ad_x \circ \beta \circ \psi(f) - ad^*_y \circ \beta \circ \psi(f) - ad^*_x \circ \beta \circ \psi(f),$$

$$= -ad^*_x(\beta)(x), \text{ take } g = \beta(x) \text{ in } (13)$$

$$= -ad^*_x(\beta)(x), \text{ where } \alpha = -\psi \circ \beta.$$ 

Next, the proof that $\psi \circ \beta \in \text{der}(G)$, is straightforward. Indeed, for every $x, y \in G$, we have

$$\psi \circ \beta[x, y] = \psi(ad^*_x(\beta)(y) - ad^*_y(\beta(x))$$

$$= ad_x \circ \psi \circ \beta(y) - ad_y \circ \psi \circ \beta(x))$$

$$= [x, \psi \circ \beta(y)] + [\psi \circ \beta(x), y]$$

Hence $[\beta, \psi] \in G_1$, for every $\beta \in Q$ and $\psi \in \Psi$. \hfill \qed

Lemma 3.4. Let $\phi := (\alpha, \xi) \in G_1$, $\beta : G \to G^*$ and $\psi : G^* \to G$ be respectively in $Q$ and in $\Psi$. Then both $[\phi, \beta]$ and $[\phi, \psi]$ are elements of $G_2$, more precisely $[\phi, \beta] \in Q$ and $[\phi, \psi] \in E$. Moreover, we have $[Q, Q] = 0$ and $[\Psi, \Psi] = 0$.

Proof. Let $\phi = (\alpha, \xi) \in G_1$, $\beta : G \to G^*$ a 1-cocycle and $\psi : G^* \to G$ an equivariant linear map. Using $\phi_\beta$ and $\phi_\psi$ as in Remark 3.2 we obtain

$$[\phi, \phi_\beta](x, y) = \phi(0, \beta(x)) - \phi_\beta(\alpha(x), \xi(f))$$

$$= (0, \xi \circ \beta(x)) - (0, \beta \circ \alpha(x))$$

$$= (0, (\xi \circ \beta - \beta \circ \alpha)(x)).$$

Now, let us show that $\tilde{\beta} := \xi \circ \beta - \beta \circ \alpha : G \to G^*$ is a 1-cocycle. Indeed, on the one hand we have

$$\xi \circ \beta([x, y]) = \xi(ad^*_x(\beta)(y) - ad^*_y(\beta(x))$$

$$= \left(\xi, ad^*_x \beta(y) + ad^*_y \xi \right)(\beta(x)) - \left(\xi, ad^*_x \beta(y) + ad^*_y \xi \right)(\beta(x))$$

$$= ad^*_{\alpha(x)}(\beta)(y) + ad^*_x(\xi \circ \beta(y)) - ad^*_{\alpha(y)}(\beta)(x) - ad^*_y(\xi \circ \beta(x))$$

$$= ad^*_{\alpha(x)}(\beta)(y) - ad^*_x(\xi \circ \beta(y)) + ad^*_{\alpha(y)}(\beta)(y) - ad^*_{\alpha(y)}(\beta)(x) \quad (29)$$
On the other hand, we also have
\[
\beta \circ \alpha([x, y]) = \beta([\alpha(x), y] + \beta([x, \alpha(y)])
\]
\[
= ad^*_{\alpha(x)}\beta(y) - ad^*_y(\beta \circ \alpha(x)) + ad^*_x(\beta \circ \alpha(y)) - ad^*_{\alpha(y)}\beta(x) \quad (30)
\]
Subtracting (30) from (29), we see that \(\tilde{\beta}[x, y]\) now reads
\[
\tilde{\beta}[x, y] = ad^*_x(\xi \circ \beta - \beta \circ \alpha)(y) - ad^*_y(\xi \circ \beta - \beta \circ \alpha)(x)
\]
\[
= ad^*_x\tilde{\beta}(y) - ad^*_y\tilde{\beta}(x)
\]
Hence \(\tilde{\beta} \in \mathcal{Q}\).

In the same way, we also have
\[
[\phi, \phi\psi](x, y) = \phi(\psi(f), 0) - \phi\psi(\alpha(x), \xi(f))
\]
\[
= (\alpha \circ \psi(f), 0) - (\psi \circ \xi(f), 0)
\]
\[
= ((\alpha \circ \psi - \psi \circ \xi)\psi(f), 0)
\]
The linear map \(\tilde{\psi} := \alpha \circ \psi - \psi \circ \xi : \mathcal{G}^* \to \mathcal{G}\) is equivariant, i.e. is an element of \(\Psi\). As above, this is seen by first computing, for every \(x \in \mathcal{G}\) and \(f \in \mathcal{G}^*\),
\[
\alpha \circ \psi(ad^*_x f) = \alpha([x, \psi(f)]) = [\alpha(x), \psi(f)] + [x, \alpha \circ \psi(f)] \quad (31)
\]
and
\[
\psi \circ \xi(ad^*_x f) = \psi \circ ([\xi, ad^*_x] + ad^*_x \circ \xi)(f)
\]
\[
= \psi(ad^*_\alpha(x)f) + \psi(ad^*_x \xi(f))
\]
\[
= [\alpha(x), \psi(f)] + [x, \psi \circ \xi(f)], \quad (32)
\]
then subtracting (31) and (32).

Now we have
\[
[\phi, \phi\psi'](x, f) = \phi\psi'(0, \beta'(x)) - \phi\psi'(0, \beta(x)) = 0
\]
and
\[
[\phi\psi, \phi\psi](x, f) = \phi\psi(\psi'(f), 0) - \phi\psi'(\psi(f), 0) = 0,
\]
for all \((x, f) \in \mathcal{D}\). In other words, \([\mathcal{Q}, \mathcal{Q}] = 0\) and \([\Psi, \Psi] = 0\).

We summarize all the above in the

**Theorem 3.2.** Let \(G\) be a Lie group and \(\mathcal{G}\) its Lie algebra. The group \(\text{Aut}(\mathcal{D})\) of automorphisms of the Lie algebra \(\mathcal{D}\) of the cotangent bundle \(T^*G\) of \(G\), is a symmetric group. More precisely, its Lie algebra \(\text{der}(\mathcal{D})\) decomposes into a direct sum of vector spaces
\[
\text{der}(\mathcal{D}) := \mathcal{G}_1 \oplus \mathcal{G}_2, \quad \text{with} \quad [\mathcal{G}_1, \mathcal{G}_1] \subset \mathcal{G}_1, \quad [\mathcal{G}_1, \mathcal{G}_2] \subset \mathcal{G}_2, \quad [\mathcal{G}_2, \mathcal{G}_2] \subset \mathcal{G}_1, \quad (33)
\]
where $\mathcal{G}_1$ is the Lie algebra of linear maps $\phi : \mathcal{D} \to \mathcal{D}$, $\phi(x, f) = (\alpha(x), \xi(f))$ with $\alpha$ a derivation of $\mathcal{G}$ and the linear map $\xi : \mathcal{G}^* \to \mathcal{G}^*$ satisfies,
\[
[\xi, \text{ad}_\alpha^x] = \text{ad}_{\alpha(x)}^x \forall x \in \mathcal{G};
\] (34)
and $\mathcal{G}_2$ is the direct sum (as a vector space) of the space $Q$ of 1-cocycles $\mathcal{G} \to \mathcal{G}^*$ and the space $\Psi$ of linear maps $\mathcal{G}^* \to \mathcal{G}$ which are equivariant with respect to the coadjoint and the adjoint representations and satisfy (13). Moreover, $Q$ and $\Psi$ are Abelian subalgebras of $\text{der} (\mathcal{D})$.

Remark 3.3. In Propositions 3.1 and 3.2, we will prove that every element $\xi$ of $E$ is the transpose $\xi = (\alpha + j)^t$ of the sum of a derivation $\alpha \in \text{der}(\mathcal{G})$ and an adjoint-invariant endomorphism $j \in \mathcal{J}$ of $\mathcal{G}$.

We also have the following

**Theorem 3.3.** The first cohomology space $H^1(\mathcal{D}, \mathcal{D})$ of the (Chevalley-Eilenberg) cohomology associated with the adjoint action of $\mathcal{D}$ on itself, satisfies $H^1(\mathcal{D}, \mathcal{D}) \cong H^1(\mathcal{G}, \mathcal{G}) \oplus \mathcal{J}^\perp \oplus H^1(\mathcal{G}, \mathcal{G}^*) \oplus \Psi$, where $H^1(\mathcal{G}, \mathcal{G})$ and $H^1(\mathcal{G}, \mathcal{G}^*)$ are the first cohomology spaces associated with the adjoint and coadjoint actions of $\mathcal{G}$, respectively; $\mathcal{J}^\perp$ is the space of transposes of elements of $\mathcal{J}$.

If $\mathcal{G}$ is semi-simple, then $\Psi = \{0\}$ and thus $H^1(\mathcal{D}, \mathcal{D}) \cong \mathcal{J}^\perp$. If $\mathcal{G}$ is a semi-simple Lie algebra over $\mathbb{C}$, then $\mathcal{J} \cong \mathbb{C}^p$, where $p$ is the number of the simple ideals $\mathcal{G}_i$ of $\mathcal{G}$ such that $\mathcal{G} = \mathcal{G}_1 \oplus \ldots \oplus \mathcal{G}_p$. Hence, of course, $H^1(\mathcal{D}, \mathcal{D}) \cong \mathbb{C}^p$. In particular, if $\mathcal{G}$ is complex and simple, then $H^1(\mathcal{D}, \mathcal{D}) \cong \mathbb{C}$. If $\mathcal{G}$ is a compact Lie algebra over $\mathbb{C}$, with centre $Z(\mathcal{G})$, we get $H^1(\mathcal{G}, \mathcal{G}) \cong \text{End}(Z(\mathcal{G}))$, $\mathcal{J} \cong \mathbb{C}^p \oplus \text{End}(Z(\mathcal{G}))$, $H^1(\mathcal{G}, \mathcal{G}^*) \cong L(Z(\mathcal{G}), Z(\mathcal{G}^*))$, $\Psi \cong L(Z(\mathcal{G}^*), Z(\mathcal{G}))$. Hence, we get $H^1(\mathcal{D}, \mathcal{D}) \cong (\text{End}(\mathbb{C}^k))^4 \oplus \mathbb{C}^p$, where $k$ is the dimension of the center of $\mathcal{G}$, and $p$ is the number of the simple components of the derived ideal $[\mathcal{G}, \mathcal{G}]$ of $\mathcal{G}$. Here, if $E, F$ are vector spaces, $L(E, F)$ is the set of linear maps $E \to F$.

The proof of Theorem 3.3 is given by different lemmas and propositions, discussed in Sections 3.6 and 4.

Let us now have a closer look at maps $\xi, \psi, \beta$, etc.

### 3.3 Maps $\xi$ and bi-invariant tensors of type (1,1)

#### 3.3.1 Adjoint invariant endomorphisms

Linear operators acting on vector fields of a given Lie group $G$ can be seen as fields of endomorphisms of its tangent spaces. Bi-invariant ones correspond to endomorphisms $j : \mathcal{G} \to \mathcal{G}$ of the Lie algebra $\mathcal{G}$ of $G$, satisfying the condition $j[x, y] = [jx, y]$, for all $x, y \in \mathcal{G}$. If we denote by $\nabla$ the connection on $G$ given on left invariant vector fields by $\nabla_x y := \frac{1}{2}[x, y]$, then using the covariant derivative, we have $\nabla j = 0$, (see e.g. [21]). As above, let $\mathcal{J} := \{ j : \mathcal{G} \to \mathcal{G} \mid \text{linear and } j[x, y] = [jx, y], \forall x, y \in \mathcal{G} \}$. Endowed with the bracket $[j, j'] := j \circ j' - j' \circ j$, the space $\mathcal{J}$ is a Lie algebra, and indeed a subalgebra of the Lie algebra $\mathcal{G}(\mathcal{G})$ of all endomorphisms of $\mathcal{G}$.

In the case where the dimension of $G$ is even and if in addition $j$ satisfies $j^2 = \text{identity}$, then $(G, j)$ is a complex Lie group.
### 3.3.2 Maps $\xi : G^* \rightarrow G^*$

**Proposition 3.1.** Let $G$ be a Lie algebra and $\alpha$ a derivation of $G$. A linear map $\xi' : G \rightarrow G$ satisfies $[\xi', ad_x] = ad_{\alpha(x)}$, $\forall x \in G$, if and only if there exists a linear map $j : G \rightarrow G$ satisfying

$$j([x, y]) = [j(x), y] = [x, j(y)], \quad \forall x, y \in G,$$

such that $\xi' = j + \alpha$.

**Proof.** Let $\alpha$ be a derivation and $\xi'$ an endomorphism of $G$ satisfying the hypothesis of Proposition 3.1, that is, $[\xi', ad_x] = ad_{\alpha(x)} = [\alpha, ad_x]$, $\forall x \in G$. We then have,

$$[\xi' - \alpha, ad_x] = 0, \quad \forall x \in G. \quad (36)$$

So the endomorphism $j := \xi' - \alpha$ commutes with all adjoint operators.

Now a linear map $j : G \rightarrow G$ commuting with all adjoint operators, satisfies

$$0 = [j, ad_x](y) = j([x, y]) - [x, j(y)], \quad \forall x, y \in G. \quad (37)$$

We also have,

$$0 = [j, ad_y](x) = j([y, x]) - [y, j(x)], \quad \forall x, y \in G. \quad (38)$$

From (37) and (38), we have $j([x, y]) = [j(x), y] = [x, j(y)], \forall x, y \in G$. Thus, (36) is equivalent to $\xi' = j + \alpha$, where $j$ satisfies (35).

**Remark 3.4.** The above means that the space $\mathcal{J}$ is the centralizer $Z_{\mathfrak{gl}(G)}(ad_G) := \{j : G \rightarrow G \text{ linear and } [j, ad_x] = 0, \forall x \in G\}$ in $\mathfrak{gl}(G) := \{l : G \rightarrow G \text{ linear }\}$ of the space $ad_G$ of inner derivations of $G$.

**Proposition 3.2.** Let $G$ be a nonabelian Lie algebra and $\mathcal{S}$ the space of endomorphisms $\xi' : G \rightarrow G$ such that there exists a derivation $\alpha$ of $G$ and $[\xi', ad_x] = ad_{\alpha(x)}$ for all $x \in G$. Then $\mathcal{S}$ is a Lie algebra containing $\mathcal{J}$ and $\text{der}(G)$ as subalgebras. In the case where $G$ has a trivial centre, then $\mathcal{S}$ is the semi-direct product $\mathcal{S} = \text{der}(G) \ltimes \mathcal{J}$ of $\mathcal{J}$ and $\text{der}(G)$.

A linear map $\xi : G^* \rightarrow G^*$ is an element of $\mathcal{E}$, i.e. satisfies (17), if and only if its transpose $\xi^t$ is an element of $\mathcal{S}$. The transposition $\xi \mapsto \xi^t$ of linear maps is an anti-isomorphism between the Lie algebras $\mathcal{E}$ and $\mathcal{S}$.

**Proof.** Using the same argument as in Lemma 3.1 if $[\xi'_1, ad_x] = ad_{\alpha_1(x)}$ and $[\xi'_2, ad_x] = ad_{\alpha_2(x)}$, $\forall x \in G$, then $[[\xi'_1, \xi'_2], ad_x] = ad_{[\alpha_1, \alpha_2](x)}$, $\forall x \in G$. Thus $\mathcal{S}$ is a Lie algebra. From Proposition 3.1 there exist $j_i \in \mathcal{J}$ such that $\xi'_i = \alpha_i + j_i$, $i = 1, 2$. Obviously, $\mathcal{S}$ contains $\mathcal{J}$ and $\text{der}(G)$. Thus, as a vector space, $\mathcal{S}$ decomposes as $\mathcal{S} = \text{der}(G) + \mathcal{J}$. Now, the Lie bracket in $\mathcal{S}$ reads

$$[\xi'_1, \xi'_2] = [\alpha_1 + j_1, \alpha_2 + j_2] = [\alpha_1, \alpha_2] + [\alpha_1, j_2] + [j_1, \alpha_2] + [j_1, j_2] \quad (39)$$

Of course we have $[\alpha_1, \alpha_2] \in \text{der}(G)$ and $[j_1, j_2] \in \mathcal{J}$, as $\mathcal{J}$ is a Lie algebra (Section 3.3.1). It is easy to check that

$$[\alpha, j] \in \mathcal{J}, \forall \alpha \in \text{der}(G), \forall j \in \mathcal{J}. \quad (40)$$
Indeed, the following holds
\[
\left[\alpha, j\right](\left[x, y\right]) = \alpha\left(j(x), y\right) - j\left(\left[\alpha(x), y\right] + \left[x, \alpha(y)\right]\right)
\]
\[
= \left[\alpha \circ j(x), y\right] + \left[j(x), \alpha(y)\right] - \left[j \circ \alpha(x), y\right] - \left[j(x), \alpha(y)\right]
\]
\[
= \left[\left[\alpha, j\right](x), y\right], \forall x, y \in G. \quad (41)
\]

The intersection \( \text{der}(G) \cap J \) is made of elements \( j \) of \( J \) whose image \( \text{Im}(j) \) is a subset of the centre \( Z(G) \) of \( G \). Hence if \( Z(G) = 0 \), then \( S = \text{der}(G) \oplus J \) and as a Lie algebra, \( S = \text{der}(G) \rtimes J \). Using this decomposition, we can also rewrite (39) as
\[
\left[\xi'_1, \xi'_2\right] = \left(\left[\alpha_1, j_1\right), \left(\alpha_2, j_2\right]\right) = \left(\left[\alpha_1, \alpha_2\right), \left[j_1, j_2\right] + \left[\alpha_1, j_2\right] + \left[j_1, \alpha_2\right]\right) \quad (42)
\]

Now let \( \xi \in E \), with \( \left[\xi, ad^*_\alpha\right] = ad^*_\alpha(x) \), then
\[
- ad^*_\alpha(x) = \left[\xi, ad^*_\alpha\right]^t = -\left[\xi^t, \left(ad^*_\alpha\right)^t\right] = \left[\xi^t, ad_x\right]
\]

Hence \( \xi^t \in S \), with \( ad^*_\alpha(x) = \left[\xi^t, ad_x\right], \forall x \in G \), where \( \alpha' := -\alpha \). Of course, we also know that \( \left[\xi_1, \xi_2\right]^t = -\left[\xi_1^t, \xi_2^t\right], \forall \xi_1, \xi_2 \in E \).

**Lemma 3.5.** Let \( \xi' : G \to G \) linear such that there exists \( \alpha : G \to G \) linear and \( \left[\xi', ad_x\right] = ad^*_\alpha(x), \forall x \in G \). Then \( \xi' \) preserves every ideal \( \mathcal{A} \) of \( G \) satisfying \( [\mathcal{A}, \mathcal{A}] = \mathcal{A} \). In particular, if \( G \) is semisimple and \( G = s_1 \oplus s_2 \oplus \ldots \oplus s_p \) is a decomposition of \( G \) into a sum of simple ideals \( s_1, \ldots, s_p \), then \( \xi'(s_i) \subset s_i, \forall i = 1, \ldots, p \).

**Proof.** The proof is straightforward. Indeed, every element \( x \) of an ideal \( \mathcal{A} \) satisfying the hypothesis of Lemma 3.5 is a finite sum of the form \( x = \sum \left[x_i, y_i\right] \) where \( x_i, y_i \) are all elements of \( \mathcal{A} \). But as \( \mathcal{A} \) is an ideal,
\[
\xi'(\left[x_i, y_i\right]) = \xi' \circ ad_{x_i}(y_i) = \left(\left[\xi', \left(ad_{x_i}\right)\right] + \left(ad_{x_i} \circ \xi'\right)(y_i)\right)
\]
\[
= \left(ad^*_\alpha(x_i) + ad_{x_i} \circ \xi'\right)(y_i) = \left[\alpha(x_i), y_i\right] + \left[x_i, \xi'(y_i)\right]
\]
is again an element of \( \mathcal{A} \). Hence we have \( \xi'(x) = \sum \left(\left[\alpha(x_i), y_i\right] + \left[x_i, \xi'(y_i)\right]\right) \in \mathcal{A} \).

**3.4 Equivariant maps \( \psi : G^* \to G \)**

Let \( G \) be a Lie algebra. In this section, we would like to explore properties of the space \( \Psi \) of linear maps \( \psi : G^* \to G \) which are equivariant with respect to the adjoint and the coadjoint actions of \( G \) on \( G \) and \( G^* \) respectively and satisfy Equation (13):
\[
ad^*_{\psi(f)}g = ad^*_{\psi(g)}f, \quad \forall f, g \in G^*.
\]

**Lemma 3.6.** Let \( G \) be a Lie algebra and \( \psi \in \Psi \). Then,
(a) \( \text{Im} \psi \) is an Abelian ideal of \( G \) and we have \( \psi(\left(ad^*_{\psi(g)}\right)f) = 0, \forall f, g \in G^* \);
(b) \( \psi \) sends closed forms on \( G \) in the center of \( G \);
(c) \( \left[\text{Im} \psi, G\right] \subset \ker f, \forall f \in \ker \psi \);
(d) the map \( \psi \) cannot be invertible if \( G \) is not Abelian.
Proof. (a) For every \( f \in \mathcal{G}^* \) and \( x \in \mathcal{G} \), we have,

\[
[\psi(f), x] = -(ad_x \circ \psi)(f) = -(\psi \circ ad_x^*)(f) \in \text{Im} \psi.
\]

Hence \( \text{Im}(\psi) \) is an ideal of \( \mathcal{G} \).

Now, for every \( f, g \in \mathcal{G}^* \), since \( \psi(f) \) and \( \psi(g) \) are elements of \( \mathcal{G} \), we also have \( \psi \circ ad_{\psi(f)}^* = ad_{\psi(f)} \circ \psi \) and \( \psi \circ ad_{\psi(g)}^* = ad_{\psi(g)} \circ \psi \). On the one hand,

\[
(\psi \circ ad_{\psi(f)}^*)(g) = (ad_{\psi(f)} \circ \psi)(g)
\]

and hence

\[
\psi(ad_{\psi(f)}^*)g = [\psi(f), \psi(g)]
\]

(43)

On the other hand,

\[
(\psi \circ ad_{\psi(g)}^*)(f) = (ad_{\psi(g)} \circ \psi)(f)
\]

\[
\psi(ad_{\psi(g)}^*)f = [\psi(g), \psi(f)]
\]

(44)

Using (43) and (44) we get

\[
[\psi(f), \psi(g)] = \psi(ad_{\psi(f)}^*)g = \psi(ad_{\psi(g)}^*)f = [\psi(g), \psi(f)]
\]

Equation (45) implies the following

\[
[\psi(f), \psi(g)] = \psi(ad_{\psi(g)}^*)f = 0, \quad \forall f, g \in \mathcal{G}^*.
\]

(46)

So we have proved (a).

(b) Let \( f \) be a closed form on \( \mathcal{G} \), that is, \( f \in \mathcal{G}^* \) and \( ad_x^*f = 0, \forall x \in \mathcal{G} \). The relation (43) implies that \( ad_{\psi(f)}^*g = 0, \forall g \in \mathcal{G}^* \). Thus,

\[
g([\psi(f), y]) = 0, \forall y \in \mathcal{G}, \forall g \in \mathcal{G}^*
\]

and hence \([\psi(f), y] = 0, \forall y \in \mathcal{G} \). In other words \( \psi(f) \) belongs to the center of \( \mathcal{G} \).

(c) If \( f \in \ker \psi \), then \( ad_{\psi(f)}^*f = ad_{\psi(f)}^*g = 0, \forall g \in \mathcal{G}^* \), or equivalently, for every \( x \in \mathcal{G} \) and \( g \in \mathcal{G}^* \), \( f([\psi(g), x]) = 0 \). It follows that \([\text{Im} \psi, \mathcal{G}] \subset \ker f, \forall f \in \ker \psi \).

(d) From (a), the map \( \psi \) satisfies \( \psi(ad_{\psi(g)}^*)f = 0, \forall f, g \in \mathcal{G}^* \). There are two possibilities here:

(i) either there exist \( f, g \in \mathcal{G}^* \) such that \( ad_{\psi(f)}^*f \neq 0 \), in which case \( ad_{\psi(f)}^*f \in \ker \psi \neq 0 \) and thus \( \psi \) is not invertible;

(ii) or else, suppose \( ad_{\psi(g)}^*f = 0, \forall f, g \in \mathcal{G}^* \). This implies that \( \psi(g) \) belongs to the center of \( \mathcal{G} \) for every \( g \in \mathcal{G}^* \). In other words, the center of \( \mathcal{G} \) contains \( \text{Im}(\psi) \). But since \( \mathcal{G} \) is not Abelian, the center of \( \mathcal{G} \) is different from \( \mathcal{G} \), hence \( \psi \) is not invertible.

\[\square \]

Lemma 3.7. The space of equivariant maps \( \psi : \mathcal{G}^* \to \mathcal{G} \) bijectively corresponds to that of \( \mathcal{G} \)-invariant bilinear forms on the \( \mathcal{G} \)-module \( \mathcal{G}^* \) for the coadjoint representation.

Proof. Indeed, each such \( \psi \) defines a unique coadjoint-invariant bilinear form \( \langle \cdot, \cdot \rangle_\psi \) on \( \mathcal{G}^* \) as follows,

\[
\langle f, g \rangle_\psi := \langle \psi(f), g \rangle, \quad \forall f, g \in \mathcal{G}^*,
\]

(47)
where the right hand side is the duality pairing $\langle f, x \rangle = f(x), \ x \in \mathcal{G}, \ f \in G^*$, as above. The coadjoint-invariance reads

$$\langle ad^*_x f, g \rangle_\psi + \langle f, ad^*_x g \rangle_\psi = 0, \ \forall x \in \mathcal{G}, \ \forall f, g \in G^*, \ (48)$$

and is due to the simple equalities

$$\langle ad^*_x f, g \rangle_\psi = \langle \psi(ad^*_x f), g \rangle = \langle ad_x \psi(f), g \rangle = -(\langle \psi(f), ad^*_x g \rangle = -(\langle f, ad^*_x g \rangle_\psi. $$

Conversely, every $G$-invariant bilinear form $\langle \cdot \rangle_1$ on $G^*$ gives rise to a unique linear map $\psi_1 : G^* \rightarrow \mathcal{G}$ which is equivariant with respect to the adjoint and coadjoint representations of $\mathcal{G}$, by the formula

$$\langle \psi_1(f), g \rangle := \langle f, g \rangle_1. \ (49)$$

If $\psi$ is symmetric or skew-symmetric, then so is $\langle \cdot \rangle_\psi$ and vice versa. Otherwise, $\langle \cdot \rangle_\psi$ can be decomposed into a symmetric and a skew-symmetric parts $\langle \cdot \rangle_\psi,s$ and $\langle \cdot \rangle_\psi,a$ respectively, defined by the following formulas:

$$\langle f, g \rangle_\psi,s := \frac{1}{2} [\langle f, g \rangle_\psi + \langle g, f \rangle_\psi], \ (50)$$

$$\langle f, g \rangle_\psi,a := \frac{1}{2} [\langle f, g \rangle_\psi - \langle g, f \rangle_\psi]. \ (51)$$

The symmetric and skew-symmetric parts $\langle \cdot \rangle_1,s$ and $\langle \cdot \rangle_1,a$ of a $G$-invariant bilinear form $\langle \cdot \rangle_1$, are also $G$-invariant. From a remark in p. 2297 of [19], the radical $\text{Rad} \langle \cdot \rangle_1 := \{ f \in G^*, \langle f, g \rangle_1 = 0, \forall g \in G^* \}$ of a $G$-invariant form $\langle \cdot \rangle_1$, contains the coadjoint orbits of all its points.

### 3.5 Cocycles $\mathcal{G} \to G^*$

The 1-cocycles for the coadjoint representation of a Lie algebra $\mathcal{G}$ are linear maps $\beta : \mathcal{G} \to G^*$ satisfying the cocycle condition $\beta([x, y]) = ad^*_x \beta(y) - ad^*_y \beta(x), \ \forall x, y \in \mathcal{G}$. To any given 1-cocycle $\beta$, corresponds a bilinear form $\Omega_\beta$ on $\mathcal{G}$, by the formula

$$\Omega_\beta(x, y) := \langle \beta(x), y \rangle, \ \forall x, y \in \mathcal{G}, \ (52)$$

where $\langle \cdot \rangle$ is again the duality pairing between elements of $\mathcal{G}$ and $G^*$.

The bilinear form $\Omega_\beta$ is skew-symmetric (resp. symmetric, nondegenerate) if and only if $\beta$ is skew-symmetric (resp. symmetric, invertible).

Skew-symmetric such cocycles $\beta$ are in bijective correspondence with closed 2-forms in $\mathcal{G}$, via the formula [12]. In this sense, the cohomology space $H^1(\mathcal{D}, \mathcal{D})$ contains the second cohomology space $H^2(G, \mathbb{R})$ of $\mathcal{G}$ with coefficients in $\mathbb{R}$ for the trivial action of $\mathcal{G}$ on $\mathbb{R}$. Hence, $H^1(\mathcal{D}, \mathcal{D})$ somehow contains the second space $H^2_{inv}(G, \mathbb{R})$ of invariant de Rham cohomology $H^*_{inv}(G, \mathbb{R})$ of any Lie group $G$ with Lie algebra $\mathcal{G}$. 

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Invertible skew-symmetric ones, when they exist, are those giving rise to symplectic forms or equivalently to invertible solutions of the Classical Yang-Baxter Equation. The study and classification of the solutions of the Classical Yang-Baxter Equation is a still open problem in Geometry, Theory of integrable systems, etc. In Geometry, they give rise to very interesting structures in the framework of Symplectic Geometry, Affine Geometry, Theory of Homogeneous Kähler domains, etc (see e.g. [8] and references therein).

If \( \mathcal{G} \) is semisimple, then every cocycle \( \beta \) is a coboundary, that is, there exists \( f_\beta \in \mathcal{G}^* \) such that \( \beta(x) = -ad^*_xf_\beta, \forall x \in \mathcal{G} \).

### 3.6 Cohomology space \( H^1(\mathcal{D}, \mathcal{D}) \)

Following Remarks 3.1, 3.2 we can embed \( \text{der}(\mathcal{G}) \) as a subalgebra \( \text{der}(\mathcal{G})_1 \) of \( \text{der}(\mathcal{D}) \), using the linear map \( \alpha \mapsto \phi_\alpha \), with \( \phi_\alpha(x, f) = (\alpha(x), f \circ \alpha) \). In the same way, we have constructed \( \mathcal{Q} \) and \( \Psi \) as subspaces of \( \text{der}(\mathcal{D}) \). Likewise, \( \mathcal{J}^t := \{ j^t \}, \) where \( j \in \mathcal{J} \) is seen as a subspace of \( \text{der}(\mathcal{D}) \), via the linear map \( j^t \mapsto \phi_j \), with \( \phi_j(x, f) = (0, f \circ j) \).

As a vector space, \( \text{der}(\mathcal{D}) \) is isomorphic to the direct sum \( \text{der}(\mathcal{G}) \oplus \mathcal{J}^t \oplus \mathcal{Q} \oplus \Psi \) by

\[
\Phi : \text{der}(\mathcal{G}) \oplus \mathcal{J}^t \oplus \mathcal{Q} \oplus \Psi \to \text{der}(\mathcal{D}); \quad (\alpha, j^t, \beta, \psi) \mapsto \phi_\alpha + \phi_j + \phi_\beta + \phi_\psi.
\]

In this isomorphism, we have \( \Phi(\text{der}(\mathcal{G}) \oplus \mathcal{J}^t) = \text{der}(\mathcal{G})_1 \oplus \mathcal{J}^t = \mathcal{G}_1 \) and \( \Phi(\mathcal{Q} \oplus \Psi) = \mathcal{G}_2 \).

Now an exact derivation of \( \mathcal{D} \), i.e. a 1-coboundary for the Chevalley-Eilenberg cohomology associated with the adjoint action of \( \mathcal{D} \) on \( \mathcal{D} \), is of the form \( \phi_0 = \partial v_0 = adv_0 \) for some element \( v_0 := (x_0, f_0) \) of the \( \mathcal{D} \)-module \( \mathcal{D} \). That is, \( \phi_0(x, f) = (\alpha_0(x), \beta_0(x)+\xi_0(f)) \), where \( \alpha_0(x) := [x_0, x], \beta_0(x) = -ad^*_xf_0, \xi_0(f) = ad^*_xf \). As we can see \( \phi_0 = \phi_0 + \phi_\beta = \Phi(\alpha_0, 0, \beta_0, 0) \) and

**Proposition 3.3.** The linear map \( \Phi \) in (53) induces an isomorphism \( \Phi \) in cohomology, between the spaces \( H^1(\mathcal{G}, \mathcal{G}) \oplus J^t \oplus H^1(\mathcal{G}, \mathcal{G}^*) \oplus \Psi \) and \( H^1(\mathcal{D}, \mathcal{D}) \).

**Proof.** The isomorphism in cohomology simply reads

\[
\Phi(\text{class}(\alpha), j^t, \text{class}(\beta), \psi) = \text{class}(\phi_\alpha + \phi_j + \phi_\beta + \phi_\psi).
\]

\[\square\]

### 4 Case of orthogonal Lie algebras

In this section, we prove that if a Lie algebra \( \mathcal{G} \) is orthogonal, then the Lie algebra \( \text{der}(\mathcal{G}) \) of its derivations and the Lie algebra \( \mathcal{J} \) of linear maps \( j : \mathcal{G} \rightarrow \mathcal{G} \) satisfying \( j[x, y] = [jx, y], \forall x, y \in \mathcal{G} \), completely characterize the Lie algebra \( \text{der}(\mathcal{D}) \) of derivations of \( \mathcal{D} := \mathcal{G} \ltimes \mathcal{G}^* \), and hence the group of automorphisms of the cotangent bundle of any connected Lie group with Lie algebra \( \mathcal{G} \). We also show that \( \mathcal{J} \) is isomorphic to the space of adjoint-invariant bilinear forms on \( \mathcal{G} \).

Let \( (\mathcal{G}, \mu) \) be an orthogonal Lie algebra and consider the isomorphism \( \theta : \mathcal{G} \rightarrow \mathcal{G}^* \) of \( \mathcal{G} \)-modules, given by \( \langle \theta(x), y \rangle := \mu(x, y) \), as in Section 2.2.

Of course, \( \theta^{-1} \) is an equivariant map. But if \( \mathcal{G} \) is not Abelian, invertible equivariant linear maps do not contribute to the space of derivations of \( \mathcal{D} \), as discussed in Lemma 3.6.
We pull coadjoint-invariant bilinear forms $B'$ on $\mathcal{G}^*$ back to adjoint-invariant bilinear forms on $\mathcal{G}$, as follows $B(x, y) := B'(\theta(x), \theta(y))$. Indeed, we have

$$B([x, y], z) = B'(\theta([x, y]), \theta(z)) = B'(ad_x^* \theta(y), \theta(z)) = -B'(\theta(y), ad_x^* \theta(z)) = -B(y, [x, z]).$$

**Proposition 4.1.** If a Lie algebra $\mathcal{G}$ is orthogonal, then there is an isomorphism between any two of the following vector spaces

(a) the space $\mathcal{J}$ of linear maps $\mu : \mathcal{G} \to \mathcal{G}$ satisfying $\mu([x, y]) = [\mu(x), \mu(y)]$, $\forall x, y \in \mathcal{G}$;

(b) the space of linear maps $\psi : \mathcal{G}^* \to \mathcal{G}$ which are equivariant with respect to the coadjoint and the adjoint representations of $\mathcal{G}$;

(c) the space of bilinear forms $B$ on $\mathcal{G}$ which are adjoint-invariant, i.e.

$$B([x, y], z) + B(y, [x, z]) = 0, \text{ for all } x, y, z \in \mathcal{G};$$

(d) the space of bilinear forms $B'$ on $\mathcal{G}^*$ which are coadjoint-invariant, i.e.

$$B'(ad_x^* f, g) + B'(f, ad_x^* g) = 0, \text{ for all } x \in \mathcal{G}, f, g \in \mathcal{G}^*.$$

**Proof.**

• The linear map $\psi \mapsto \psi \circ \theta$ is an isomorphism between the space of equivariant linear maps $\psi : \mathcal{G}^* \to \mathcal{G}$ and the space $\mathcal{J}$. Indeed, if $\psi$ is equivariant, we have

$$\psi \circ \theta([x, y]) = -\psi(ad_y^* \theta(x)) = -ad_y \psi(\theta(x)) = [\psi \circ \theta(x), y].$$

Hence $\psi \circ \theta \in \mathcal{J}$. Conversely, if $\theta \in \mathcal{J}$, then $\theta \circ \theta^{-1}$ is equivariant, as it satisfies

$$\theta \circ \theta^{-1} \circ ad_x^* = \theta \circ ad_x \circ \theta^{-1} = ad_x \circ \theta \circ \theta^{-1}.$$

This correspondence is obviously linear and invertible. Hence we get the isomorphism between (a) and (b).

• The isomorphism between the space $\mathcal{J}$ of adjoint-invariant endomorphisms and adjoint-invariant bilinear forms is given as follows

$$\mu : \mathcal{G}^* \to \mathcal{G}, \text{ where } B_j(x, y) := \mu(j(x), y).$$

We have

$$B_j([x, y], z) := \mu([j(x), y], z) = \mu([x, j(y)], z) = -\mu(j(y), [x, z]) = -B_j(y, [x, z]).$$

Conversely, if $B$ is an adjoint-invariant bilinear form on $\mathcal{G}$, then the endomorphism $j$, defined by

$$\mu(j(x), y) := B(x, y)$$

is an element of $\mathcal{J}$, as it satisfies

$$\mu([j(x), y], z) := B([j(x), y], z) = B([x, y], z) = B(x, [y, z]) = \mu(j(x), [y, z]) = \mu([j(x), y], z), \forall x, y, z \in \mathcal{G}.$$

• From Lemma 3.7 the space of equivariant linear maps $\psi$ bijectively corresponds to that of coadjoint-invariant bilinear forms on $\mathcal{G}^*$, via $\psi \mapsto \langle \cdot \rangle_\psi$.  \qed
Now, suppose \( \psi \) is skew-symmetric. Let \( \omega_\psi \) denote the corresponding skew-symmetric bilinear form on \( G \):

\[
\omega_\psi(x, y) := \mu(\psi \circ \theta(x), y), \quad \forall \, x, y \in G.
\] (58)

Then, \( \omega_\psi \) is adjoint-invariant. If we denote by \( \partial \) the Chevalley-Eilenberg coboundary operator, that is,

\[
(\partial \omega_\psi)(x, y, z) = -\left(\omega_\psi([x, y], z) + \omega_\psi([y, z], x) + \omega_\psi([z, x], y)\right),
\]

the following formula holds true

\[
(\partial \omega_\psi)(x, y, z) = -\omega_\psi([x, y], z).
\]

**Corollary 4.1.** The following are equivalent.

(a) \( \omega_\psi \) is closed

(b) \( \psi \circ \theta([x, y]) = 0, \quad \forall \, x, y \in G \).

(c) \( \text{Im}(\psi) \) is in the center of \( G \).

In particular, if \( \dim[G, G] \geq \dim G - 1 \), then \( \omega_\psi \) is closed if and only if \( \psi = 0 \).

**Proof.** The above equality also reads

\[
\partial \omega_\psi(x, y, z) = -\omega_\psi([x, y], z) = -\mu(\psi \circ \theta([x, y]), z), \quad \forall \, x, y, z \in G,
\] (59)

and gives the proof that (a) and (b) are equivalent. In particular, if \( G = [G, G] \) then, obviously \( \partial \omega_\psi = 0 \) if and only if \( \psi = 0 \), as \( \theta \) is invertible.

Now suppose \( \dim[G, G] = \dim G - 1 \) and set \( G = \mathbb{R}x_0 \oplus [G, G] \), for some \( x_0 \in G \). If \( \omega_\psi \) is closed, we already know that \( \psi \circ \theta \) vanishes on \([G, G]\). Below, we show that, it also does on \( \mathbb{R}x_0 \). Indeed, the formula

\[
0 = -\omega_\psi([x, y], x_0) = \omega_\psi(x_0, [x, y]) = \mu(\psi \circ \theta(x_0), [x, y]), \quad \forall \, x, y \in G,
\] (60)

obtained by taking \( z = x_0 \) in (59), coupled with the obvious equality \( 0 = \omega_\psi(x_0, x_0) = \mu(\psi \circ \theta(x_0), x_0) \), are equivalent to \( \psi \circ \theta(x_0) \) satisfying \( \mu(\psi \circ \theta(x_0), x) = 0 \) for all \( x \in G \). As \( \mu \) is nondegenerate, this means that \( \psi \circ \theta(x_0) = 0 \). Hence \( \psi \circ \theta = 0 \), or equivalently \( \psi = 0 \).

Now, as every \( f \in G^* \) is of the form \( f = \theta(y) \), for some \( y \in G \), the formula

\[
\psi \circ \theta([x, y]) = \psi \circ \text{ad}_x^\ast \theta(y) = \text{ad}_x \circ \psi \circ \theta(y) = [x, \psi \circ \theta(y)], \quad \forall x, y \in G.
\] (61)

shows that \( \psi \circ \theta([x, y]) = 0, \quad \forall x, y \in G \) if and only if \( \text{Im}(\psi) \) is a subset of the center of \( G \). Thus, (b) is equivalent to (c).

Now we pull every element \( \xi \in \mathcal{E} \) back to an endomorphism \( \xi' \) of \( G \) given by the formula \( \xi' := \theta^{-1} \circ \xi \circ \theta \).

**Proposition 4.2.** Let \((G, \mu)\) be an orthogonal Lie algebra and \( G^* \) its dual space. Define \( \theta : G \to G^* \) by \( \langle \theta(x), y \rangle := \mu(x, y) \), as in Section 2.2, and let \( \mathcal{E} \) and \( \mathcal{S} \) stand for the same Lie algebras as above. The linear map \( Q : \xi \mapsto \xi' := \theta^{-1} \circ \xi \circ \theta \) is an isomorphism of Lie algebras between \( \mathcal{E} \) and \( \mathcal{S} \).
Proof. Let \( \xi \in \mathcal{E} \), with \([\xi, ad^*_x] = ad^*_x \alpha \), \( \forall x \in \mathcal{G} \). The image \( Q(\xi) =: \xi' \) of \( \xi \), satisfies \( \forall x \in \mathcal{G} \),

\[
[\xi', ad_x] := \xi' \circ ad_x - ad_x \circ \xi' \\
:= \theta^{-1} \circ \xi \circ \theta \circ ad_x - ad_x \circ \theta^{-1} \circ \xi \circ \theta \\
= \theta^{-1} \circ \xi \circ ad^*_x \circ \theta - \theta^{-1} \circ ad^*_x \circ \xi \circ \theta \\
= \theta^{-1} \circ (\xi \circ ad^*_x - ad^*_x \circ \xi) \circ \theta \\
= \theta^{-1} \circ ad^*_\alpha(x) \circ \theta, \quad \text{since } [\xi, ad^*_x] = ad^*_\alpha(x). \\
= \theta^{-1} \circ \theta \circ ad\alpha(x), \quad \text{using } (1).
\]

Now we have \([Q(\xi_1), Q(\xi_2)]) = Q([\xi_1, \xi_2]) \) for all \( \xi_1, \xi_2 \in \mathcal{E} \), as seen below.

\[
[Q(\xi_1), Q(\xi_2)]) := Q(\xi_1)Q(\xi_2) - Q(\xi_2)Q(\xi_1) \\
:= \theta^{-1} \circ \xi_1 \circ \theta \circ \theta^{-1} \circ \xi_2 \circ \theta - \theta^{-1} \circ \xi_2 \circ \theta \circ \theta^{-1} \circ \xi_1 \circ \theta \\
= \theta^{-1} \circ [\xi_1, \xi_2] \circ \theta \\
= Q([\xi_1, \xi_2]) \). \quad (62)
\]

\[\square\]

**Proposition 4.3.** The linear map \( P : \beta \mapsto D_\beta := \theta^{-1} \circ \beta \), is an isomorphism between the space of cocycles \( \beta : \mathcal{G} \rightarrow \mathcal{G}^* \) and the space \( \text{der}(\mathcal{G}) \) of derivations of \( \mathcal{G} \).

**Proof.** The proof is straightforward. If \( \beta : \mathcal{G} \rightarrow \mathcal{G}^* \) is a cocycle, then the linear map \( D_\beta : \mathcal{G} \rightarrow \mathcal{G}, \ x \mapsto \theta^{-1}(\beta(x)) \) is a derivation of \( \mathcal{G} \), as we have

\[
D_\beta[x, y] = \theta^{-1}(ad^*_x\beta(y) - ad^*_y\beta(x)) = [x, \theta^{-1}(\beta(y))] - [y, \theta^{-1}(\beta(x))].
\]

Conversely, if \( D \) is a derivation of \( \mathcal{G} \), then the linear map \( \beta_D := P^{-1}(D) = \theta \circ D : \mathcal{G} \rightarrow \mathcal{G}^* \), is 1-cocycle. Indeed we have

\[
\beta_D[x, y] = \theta([Dx, y] + [x, Dy]) = -ad^*_y(\theta \circ D(x)) + ad^*_x(\theta \circ D(y)).
\]

\[\square\]

### 4.1 Case of semisimple Lie algebras

Suppose now \( \mathcal{G} \) is semisimple, then every derivation is inner. Thus in particular, the derivation \( \phi_{11} \) obtained in (19), is of the form

\[
\phi_{11} = ad_{x_0}, \text{ for some } x_0 \in \mathcal{G}.
\]

The semisimplicity of \( \mathcal{G} \) also implies that the 1-cocycle \( \phi_{12} \) obtained in (20) is a coboundary. That is, there exists an element \( f_0 \in \mathcal{G}^* \) such that

\[
\phi_{12}(x) = -ad^*_x f_0, \ \forall x \in \mathcal{G}.
\]

Here is a direct corollary of Lemma 3.16.
Proposition 4.4. If \( \mathcal{G} \) is a semisimple Lie algebra, then every linear map \( \psi : \mathcal{G}^* \to \mathcal{G} \) which is equivariant with respect to the adjoint and coadjoint actions of \( \mathcal{G} \) and satisfies (13), is necessarily identically equal to zero.

Proof. A Lie algebra is semisimple if and only if it contains no nonzero proper Abelian ideal. But from Lemma 3.6, \( \text{Im}(\psi) \) must be an Abelian ideal of \( \mathcal{G} \). So \( \text{Im}(\psi) = \{0\} \) and hence \( \psi = 0 \).

Remark 4.1. From Propositions 3.3 and 4.4, the cohomology space \( H^1(\mathcal{D}, \mathcal{D}) \) is completely determined by the space \( J \) of endomorphisms \( j \) with \( j([x, y]) = [j(x), y] \), \( \forall x, y \in \mathcal{G} \), or equivalently, by the space of adjoint-invariant bilinear forms on \( \mathcal{G} \).

Corollary 4.2. If \( \mathcal{G} \) is a semisimple Lie group with Lie algebra \( \mathcal{G} \), then the space of bi-invariant bilinear forms on \( \mathcal{G} \) is of dimension \( \dim H^1(\mathcal{D}, \mathcal{D}) \).

So far, we have explicitly worked with Lie algebras over the field \( \mathbb{R} \) of real numbers, although most of the results are also valid for any field of characteristic zero. Now, here is what happens for fields where the Schur lemma can be applied.

Proposition 4.5. Suppose \( \mathcal{G} \) is a simple Lie algebra over \( \mathbb{C} \). Then,

(a) every linear \( j : \mathcal{G} \to \mathcal{G} \) satisfying \( j[x, y] = [j(x), y] \), \( \forall x, y \in \mathcal{G} \), is of the form \( j(x) = \lambda x \), for some \( \lambda \in \mathbb{C} \).

(b) Every element \( \xi \) of \( \mathcal{E} \) is of the form

\[
\xi = \text{ad}^*_{x_0} + \lambda \text{Id}_{\mathcal{G}^*}, \text{ for some } x_0 \in \mathcal{G} \text{ and } \lambda \in \mathbb{C}.
\]  

(65)

Proof. The part (a) is obtained from relation (36) and the Schur’s lemma. From Propositions 3.1 and 3.2, for every \( \xi \in \mathcal{E} \), there exist \( \alpha \in \text{der}(\mathcal{G}) \) and \( j \in J \) such that \( \xi^t = \alpha + j \). As \( \mathcal{G} \) is simple and from part (a), there exist \( x_0 \in \mathcal{G} \) and \( \lambda \in \mathbb{C} \) such that \( \xi^t = \text{ad}_{x_0} + \lambda \text{Id}_{\mathcal{G}} \).

We also have the following

Proposition 4.6. Let \( G \) be a simple Lie group with Lie algebra \( \mathcal{G} \) over \( \mathbb{C} \). Let \( \mathcal{D} := \mathcal{G} \ltimes \mathcal{G}^* \) be the Lie algebra of the cotangent bundle \( T^*G \) of \( G \). Then, the first cohomology space of \( \mathcal{D} \) with coefficients in \( \mathcal{D} \) is \( H^1(\mathcal{D}, \mathcal{D}) \cong \mathbb{C} \).

Proof. Indeed, a derivation \( \phi : \mathcal{D} \to \mathcal{D} \) can be written: \( \forall (x, f) \in \mathcal{D} := \mathcal{G} \ltimes \mathcal{G}^* \),

\[
\phi(x, f) = ([x_0, x] , \text{ad}^*_{x_0} f - \text{ad}^*_{x_0} f_0 + \lambda f), \ x_0 \in \mathcal{G}, \ f_0 \in \mathcal{G}^*.
\]  

(66)

The inner derivations are those with \( \lambda = 0 \). It follows that the first cohomology space of \( \mathcal{D} \) with values in \( \mathcal{D} \) is given by

\[
H^1(\mathcal{D}, \mathcal{D}) = \{ \phi : \mathcal{D} \to \mathcal{D} : \phi(x, f) = (0, \lambda f) , \lambda \in \mathbb{C} \} = \{ \lambda(0, \text{Id}_{\mathcal{G}^*}), \lambda \in \mathbb{C} \} = \mathbb{C} \text{Id}_{\mathcal{G}^*}.
\]
As a direct consequence, we get

**Corollary 4.3.** If $\mathcal{G}$ is a semi-simple Lie algebra over $\mathbb{C}$, then $\dim H^1(\mathcal{D}, \mathcal{D}) = p$, where $p$ is the number of simple components of $\mathcal{G}$, in its decomposition into a direct sum $\mathcal{G} = \mathfrak{s}_1 \oplus \ldots \oplus \mathfrak{s}_p$ of simple ideals $\mathfrak{s}_1, \ldots, \mathfrak{s}_p$.

Consider a semisimple Lie algebra $\mathcal{G}$ over $\mathbb{C}$ and set $\mathcal{G} = \mathfrak{s}_1 \oplus \mathfrak{s}_2 \oplus \cdots \oplus \mathfrak{s}_p$, $p \in \mathbb{N}$, where $\mathfrak{s}_i$, $i = 1, \ldots, p$ are simple Lie algebras. From Lemma 3.3, $\xi'$ preserves each $\mathfrak{s}_i$. Thus from Proposition 4.7, the restriction $\xi''$ of $\xi'$ to each $\mathfrak{s}_i$, $i = 1, 2, \ldots, p$ equals $\xi''_i = ad_{x_0} + \lambda_i Id_{\mathfrak{s}_i}$, for some $x_0 \in \mathfrak{s}_i$ and $\lambda_i \in \mathbb{C}$. Hence, $\xi'' = ad_{x_0} \oplus_{i=1}^p \lambda_i Id_{\mathfrak{s}_i}$, where $x_0 = x_0 + x_0 + \cdots + x_p \in \mathfrak{s}_1 \oplus \mathfrak{s}_2 \oplus \cdots \oplus \mathfrak{s}_p$, and $\oplus_{i=1}^p \lambda_i Id_{\mathfrak{s}_i}$ acts on $\mathfrak{s}_1 \oplus \mathfrak{s}_2 \oplus \cdots \oplus \mathfrak{s}_p$ as follows: $(\oplus_{i=1}^p \lambda_i Id_{\mathfrak{s}_i})(x_1 + x_2 + \cdots + x_p) = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_p x_p$. In particular, we have proved

**Corollary 4.4.** Consider the decomposition of a semisimple complex Lie algebra $\mathcal{G}$ into a sum $\mathcal{G} = \mathfrak{s}_1 \oplus \mathfrak{s}_2 \oplus \cdots \oplus \mathfrak{s}_p$, of simple Lie algebras $\mathfrak{s}_i$, $i = 1, \ldots, p$. If a linear map $j : \mathcal{G} \to \mathcal{G}$ satisfies $j[x, y] = [x, y]$, then there exist $\lambda_1, \ldots, \lambda_p \in \mathbb{C}$ such that $j = \lambda_1 Id_{\mathfrak{s}_1} \oplus \cdots \oplus \lambda_p Id_{\mathfrak{s}_p}$. More precisely $j(x_1 + \cdots + x_p) = \lambda_1 x_1 + \cdots + \lambda_p x_p$, $x_i \in \mathfrak{s}_i$, $i = 1, \ldots, p$.

Now, we already know from Proposition 4.4 that each $\psi$ vanishes identically. So a 1-cocycle of $\mathcal{D}$ is given by:

$$\phi(x, f) = ([x_0, x], ad_{x_0} f - ad_x f_0 + \sum_{i=1}^p \lambda_i f_i)$$

for every $x \in \mathcal{G}$ and every $f := f_1 + f_2 + \cdots + f_p \in \mathfrak{s}_1 \oplus \mathfrak{s}_2 \oplus \cdots \oplus \mathfrak{s}_p = \mathcal{G}^*$, where $x_0 \in \mathcal{G}$, $f_0 \in \mathcal{G}^*$ and $\lambda_i \in \mathbb{C}$, $i = 1, \ldots, p$. We then have,

**Proposition 4.7.** Let $G$ be a semisimple Lie group with Lie algebra $\mathcal{G}$ over $\mathbb{C}$. Let $\mathcal{D} := G \ltimes \mathcal{G}^*$ be the cotangent Lie algebra of $G$. Then, the first cohomology space of $\mathcal{D}$ with coefficients in $\mathcal{D}$ is given by $H^1(\mathcal{D}, \mathcal{D}) \cong \mathbb{C}^p$, where $p$ is the number of the simple components of $\mathcal{G}$.

### 4.2 Case of compact Lie algebras

It is well known that a compact Lie algebra $\mathcal{G}$ decomposes as the direct sum $\mathcal{G} = [\mathcal{G}, \mathcal{G}] \oplus Z(\mathcal{G})$ of its derived ideal $[\mathcal{G}, \mathcal{G}]$ and its centre $Z(\mathcal{G})$, with $[\mathcal{G}, \mathcal{G}]$ semisimple and compact. This yields a decomposition $\mathcal{G}^* = [\mathcal{G}, \mathcal{G}]^* \oplus Z(\mathcal{G})^*$ of $\mathcal{G}^*$ into a direct sum of the dual spaces $[\mathcal{G}, \mathcal{G}]^*$, $Z(\mathcal{G})^*$ of $[\mathcal{G}, \mathcal{G}]$ and $Z(\mathcal{G})$ respectively, where $[\mathcal{G}, \mathcal{G}]^*$ (resp. $Z(\mathcal{G})^*$) is identified with the space of linear forms on $\mathcal{G}$ which vanish on $Z(\mathcal{G})$ (resp. $[\mathcal{G}, \mathcal{G}]$). On the other hand, $[\mathcal{G}, \mathcal{G}]$ also decomposes into as a direct sum $[\mathcal{G}, \mathcal{G}] = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_p$ of simple ideals $\mathfrak{s}_i$. From Theorem 3.1, a derivation of the Lie algebra $\mathcal{D} := G \ltimes \mathcal{G}^*$ of the cotangent bundle of $G$ has the following form

$$\phi(x, f) = \left( \alpha(x) + \psi(f), \beta(x) + \xi(f) \right),$$

with conditions listed in Theorem 3.1. Let us look at the equivariant maps $\psi : \mathcal{G}^* \to \mathcal{G}$ satisfying $ad_{\psi(g)}^* g = ad_{\psi(g)} f$, $\forall f, g \in \mathcal{G}^*$. From Lemma 3.6, $Im(\psi)$ is an Abelian ideal of $\mathcal{G}$; thus $Im(\psi) \subset Z(\mathcal{G})$. As consequence, we have $\psi(ad_x^* f) = [x, \psi(f)] = 0$, $\forall x \in \mathcal{G}$, $f \in \mathcal{G}^*$.
Lemma 4.1. Let \( \mathcal{G}, \mu \) be an orthogonal Lie algebra satisfying \( \mathcal{G} = [\mathcal{G}, \mathcal{G}] \). Then, every \( g \in \mathcal{G}^* \) is a finite sum of elements of the form \( g_i = \text{ad}^*_x \bar{g}_i \), for some \( \bar{x} \in \mathcal{G}, \bar{g}_i \in \mathcal{G}^* \).

Proof. Indeed, consider an isomorphism \( \theta : \mathcal{G} \to \mathcal{G}^* \) of \( \mathcal{G} \)-modules. For every \( g \in \mathcal{G}^* \), there exists \( x_g \in \mathcal{G} \) such that \( g = \theta(x_g) \). But as \( \mathcal{G} = [\mathcal{G}, \mathcal{G}] \), we have \( x_g = [x_1, y_1] + \ldots + [x_n, y_n] \) for some \( x_i, y_i \in \mathcal{G} \). Thus \( g = \theta([x_1, y_1]) + \ldots + \theta([x_n, y_n]) = \text{ad}^*_x \theta(y_1) + \ldots + \text{ad}^*_x \theta(y_n) = \text{ad}^*_x \bar{g}_1 + \ldots + \text{ad}^*_x \bar{g}_n \) where \( \bar{x} = x_i \) and \( \bar{g}_i = \theta(y_i) \).

A semisimple Lie algebra being orthogonal (with, e.g., its Killing form as \( \mu \)), from Lemma 4.1 and the equality \( \psi(\text{ad}^*_x f) = 0 \), for all \( x \in \mathcal{G}, f \in \mathcal{G}^* \), each \( \psi \) in \( \Psi \) vanishes on \([\mathcal{G}, \mathcal{G}]^* \). Of course, the converse is true. Every linear map \( \psi : \mathcal{G}^* \to \mathcal{G} \) with \( \text{Im}(\psi) \subset Z(\mathcal{G}) \) and \( \psi([\mathcal{G}, \mathcal{G}]^*) = 0 \), is in \( \Psi \). Hence we can make the following identification.

Lemma 4.2. Let \( \mathcal{G} \) be a compact Lie algebra, with centre \( Z(\mathcal{G}) \). Then \( \Psi \) is isomorphic to the space \( L(Z(\mathcal{G})^*, Z(\mathcal{G})) \) of linear maps \( Z(\mathcal{G})^* \to Z(\mathcal{G}) \).

The restriction of the cocycle \( \beta \) to the semisimple ideal \([\mathcal{G}, \mathcal{G}] \) is a coboundary, that is, there exists an element \( f_0 \in \mathcal{G}^* \) such that for every \( x_1 \in [\mathcal{G}, \mathcal{G}], \beta(x_1) = -\text{ad}^*_x f_0 \). Now for \( x_1 \in Z(\mathcal{G}) \), one has \( 0 = \beta[x_2, y] = -\text{ad}^*_x \beta(x_2), \forall y \in \mathcal{G}, \) since \( x_2 \in Z(\mathcal{G}) \). In other words, \( \beta(x_2)([y, z]) = 0, \forall y, z \in \mathcal{G} \). That is, \( \beta(x_2) \) vanishes on \([\mathcal{G}, \mathcal{G}] \) for every \( x_2 \in Z(\mathcal{G}) \). Hence, we write \( \beta(x) = -\text{ad}^*_x f_0 + \eta(x_2), \forall x := x_1 + x_2 \in [\mathcal{G}, \mathcal{G}] \oplus Z(\mathcal{G}) \), where \( \eta : Z(\mathcal{G}) \to Z(\mathcal{G})^* \) is a linear map. This simply means the following.

Lemma 4.3. Let \( \mathcal{G} \) be a compact Lie algebra, with centre \( Z(\mathcal{G}) \). Then the first space \( H^1(\mathcal{G}, \mathcal{G}^*) \) of the cohomology associated with the coadjoint action of \( \mathcal{G} \), is isomorphic to the space \( L(Z(\mathcal{G}), Z(\mathcal{G})^*) \).

We have already seen that \( \xi \) is such that \( \xi^j = \alpha + j \), where \( \alpha \) is a derivation of \( \mathcal{G} \) and \( j \) is an endomorphism of \( \mathcal{G} \) satisfying \( j([x, y]) = [j(x), y] = [x, j(y)] \). Both \( \alpha \) and \( j \) preserve each of \([\mathcal{G}, \mathcal{G}] \) and \( Z(\mathcal{G}) \). Thus we can write \( \alpha = \text{ad}_{x_01} \oplus \varphi \), for some \( x_01 \in [\mathcal{G}, \mathcal{G}] \), where \( \varphi \in \text{End}(Z(\mathcal{G})) \). Here \( \alpha \) acts on an element \( x := x_1 + x_2 \), where \( x_1 \in [\mathcal{G}, \mathcal{G}], x_2 \in Z(\mathcal{G}) \), as follows: \( \alpha(x) = (\text{ad}_{x_01} \oplus \varphi)(x_1 + x_2) := \text{ad}_{x_01} x_1 + \varphi(x_2) \). We summarize this as

Lemma 4.4. If \( \mathcal{G} \) is a compact Lie algebra with centre \( Z(\mathcal{G}) \), then \( H^1(\mathcal{G}, \mathcal{G}) \cong Z(\mathcal{G}) \).

Now, suppose for the rest of this section, that \( \mathcal{G} \) is a compact Lie algebra over \( \mathbb{C} \). We write \( j = \bigoplus_{i=1}^p \lambda_i \text{Id}_{[\mathcal{G}, \mathcal{G}]} \oplus \rho \), where \( \rho : Z(\mathcal{G}) \to Z(\mathcal{G}) \) is a linear map and \( j \) acts on an element \( x := x_1 + x_2 \) as follows:

\[
  j(x) = \left( \bigoplus_{i=1}^p \lambda_i \text{Id}_{[\mathcal{G}, \mathcal{G}]} \oplus \rho \right)(x_1 + x_2 + \cdots + x_1p + x_2) = \sum_{i=1}^p \lambda_i x_1i + \rho(x_2)
\]

where \( x_1 := x_{11} + x_{12} + \cdots + x_{1p} \in [\mathcal{G}, \mathcal{G}], x_2 \in Z(\mathcal{G}) \) and \( x_1i \in \mathfrak{s}_i \). Hence,

Lemma 4.5. If \( \mathcal{G} \) is a compact Lie algebra over \( \mathbb{C} \), with centre \( Z(\mathcal{G}) \), then \( J \cong \mathbb{C}^p \oplus \text{End}(Z(\mathcal{G})) \), where \( p \) the number of simple components of \([\mathcal{G}, \mathcal{G}] \).
So, the expression of $\xi$ now reads

$$\xi = \left[-ad^*_{x_0} + \left(\oplus_{i=1}^p \lambda_i Id_{[\mathcal{G},\mathcal{G}^*]}\right)\right] \oplus \varphi''$$

with $(\varphi'')^t(x_2) = \rho(x_2) + \varphi(x_2), \ \forall x_2 \in Z(\mathcal{G}).$

where $x_{01} \in [\mathcal{G},\mathcal{G}], \lambda_i \in \mathbb{C}.$

By identifying $End(Z(\mathcal{G}))$, $L(Z(\mathcal{G})^*, Z(\mathcal{G}))$ and $L(Z(\mathcal{G}), Z(\mathcal{G})^*)$ to $End(\mathbb{R}^k)$, we get

$$H^1(\mathcal{D}, \mathcal{D}) = (End(\mathbb{R}^k))^4 \oplus \mathbb{C}^p.$$  \hfill (68)

5 Some possible applications and open problems

Given two invariant structures of the same ‘nature’ (e.g. affine, symplectic, complex, Riemannian or pseudo-Riemannian, etc) on $T^*G$, one wonders whether they are equivalent, i.e. if there exists an automorphism of $T^*G$ mapping one to the other. By taking the values of those structures at the unit of $T^*G$, the problem translates to finding an automorphism of Lie algebra mapping two structures of $\mathcal{D}$. The work within this paper may also be seen as a useful tool for the study of such structures. Here are some examples of problems and framework for further extension and applications of this work. For more discussions about structures and problems on $T^*G$, one can have a look at [17], [14], [11], [10], [8], [3], etc.

5.1 Invariant Riemannian or pseudo-Riemannian metrics

As discussed in Section 2.2, the Lie group $T^*G$ possesses bi-invariant pseudo-Riemannian metrics.

Amongst others, one of the open problems in [19], is the question as to whether, given two bi-invariant pseudo-Riemannian metrics $\mu_1$ and $\mu_2$ on a Lie group $\tilde{G}$, the two pseudo-Riemannian manifolds $(\tilde{G}, \mu_1)$ and $(\tilde{G}, \mu_2)$ are homothetic via an automorphism of $\tilde{G}$. If this is the case, we say that $\mu_1$ and $\mu_2$ are isomorphic or equivalent.

When $\tilde{G} = T^*G$, the question as to how many non-isomorphic bi-invariant pseudo-Riemannian metrics can exist on $T^*G$ is still open, in the general case. For example, suppose $G$ itself has a bi-invariant Riemannian or pseudo-Riemannian metric $\mu$ and let $\mu$ stand again for the corresponding adjoint-invariant metric in the Lie algebra $\mathcal{G}$ of $G$.

Then $\mu$ induces a new adjoint-invariant metric $\langle , \rangle_\mu$ on $\mathcal{D} = \text{Lie}(T^*G)$, with

$$\langle (x, f), (y, g) \rangle_\mu := \langle (x, f), (y, g) \rangle + \mu(x, y), \ \forall x, y \in \mathcal{G}, \ f, g \in \mathcal{G}^*$$  \hfill (69)

where $\langle , \rangle$ is the duality pairing $\langle (x, f), (y, g) \rangle = f(y) + g(x).$ In some cases, the two metrics can even happen to have the same index, but are still not isomorphic via an automorphism of $\tilde{G}$. If $\tilde{\mu}$ is a bilinear symmetric form on $\mathcal{G}^*$ satisfying $\tilde{\mu}(ad^*_x f, g) = 0, \ \forall x \in \mathcal{G}, \ f, g \in \mathcal{G}^*$, then

$$\langle (x, f), (y, g) \rangle_{\tilde{\mu}} := \langle (x, f), (y, g) \rangle + \tilde{\mu}(f, g), \ \forall x, y \in \mathcal{G}, \ f, g \in \mathcal{G}^*$$  \hfill (70)

is also another adjoint-invariant metric on $\mathcal{D}$. Now in some cases, nonzero such bilinear forms $\tilde{\mu}$ may not exist, as is the case when one of the coadjoint orbits of $G$, is an open subset of $\mathcal{G}^*$, or equivalently, when $G$ has a left invariant exact symplectic structure.
More generally, these equivalence questions can also be simply extended to all left (resp. right) invariant Riemannian or pseudo-Riemannian structures on cotangent bundles of Lie groups.

5.2 Poisson-Lie structures, doubles Lie algebras, applications

The classification questions of double Lie algebras, Manin pairs, Lagrangian subalgebras, ... arising from Poisson-Lie groups, are still open problems [10], [16], etc. In [10], Lagrangian subalgebras of double Lie algebras are used as the main tool for classifying the so-called Poisson Homogeneous spaces of Poisson-Lie groups. A type of local action of those Lagrangian subalgebras is also used to describe symplectic foliations of Poisson Homogeneous spaces of Poisson-Lie groups, in [10], [6], etc.

It would be interesting to extend the results within this paper to double Lie algebras of general Poisson-Lie groups. It is hard to get a common substantial description valid for the group of automorphism of the double Lie algebras of all possible Poisson-Lie structures in a given Lie group. This is due to the diversity of Poisson-Lie structures that can coexist in the same Lie group.

Amongst other things, the description of the group of automorphisms of the double Lie algebra of a Poisson-Lie structure is a big step forward towards solving very interesting and hard problems such as:
- The classification of Manin triples [16].
- The classification of Poisson homogeneous spaces of a Lie groups. See [6], [16].
- A full description and understanding of the foliations of Poisson homogeneous spaces of Poisson Lie groups. The leaves of such foliations trap the trajectories, under a Hamiltonian flow, passing through all its points. Hence, from their knowledge, one gets a great deal of information on Hamiltonian systems.
- Etc.

5.3 Affine and complex structures on $T^*G$

In certain cases, $T^*G$ possesses left invariant affine connections, that is, left invariant zero curvature and torsion free linear connections. Here, the classification problem involves $\text{Aut}(T^*G)$ as follows. The group $\text{Aut}(T^*G)$ acts on the space of left invariant affine connections on $T^*G$, the orbit of each connection being the set of equivalent (isomorphic) ones. Recall that amongst other results in [8], the authors proved that when $G$ has an invertible solution of the Classical Yang-Baxter Equation (or equivalently a left invariant symplectic structure,) then $T^*G$ has a left invariant affine connection $\nabla$ and a complex structure $J$ such that $\nabla J = 0$.

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