ALGEBRAIC GEOMETRIC INVARIANTS OF PARAFREE GROUPS

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ABSTRACT.
Given a finitely generated (fg) group $G$, the set $R(G)$ of homomorphisms from $G$ to $SL_2\mathbb{C}$ inherits the structure of an algebraic variety known as the representation variety of $G$ in $SL_2\mathbb{C}$. This algebraic variety is an invariant of fg presentations of $G$. Call a group $G$ parafree of rank $n$ if it shares the lower central sequence with a free group of rank $n$, and if it is residually nilpotent. The deviation of a fg parafree group is the difference between the minimum possible number of generators of $G$ and the rank of $G$. So parafree groups of deviation zero are actually just free groups. Parafree groups that are not free share a host of properties with free groups. In this paper algebraic geometric invariants involving the number of maximal irreducible components (mirr) of $R(G)$, and the dimension of $R(G)$ for certain classes of parafree groups are computed. It is shown that in an infinite number of cases these invariants successfully discriminate between isomorphism types within the class of parafree groups of the same rank. This is quite surprising, since an $n$ generated group $G$ is free of rank $n$ iff $\text{Dim}(R(G)) = 3n$. In fact, a direct consequence of Theorem 1.6 in this paper is that given an arbitrary positive integer $k$, and any integer $r \geq 2$ there exist infinitely many non-isomorphic fg parafree groups of rank $r$ and deviation one with representation varieties of dimension $3r$, having more than $k \text{mirr}$ of dimension $3r$. This paper also introduces many novel and surprising dimension formulas for the representation varieties of certain one-relator groups.

General Structure of the Paper. This paper begins with an introduction where relevant ideas to what will follow are developed. It then goes on to define what a parafree group is, and how the notions that inspired G. Baumslag to give rise to such groups arose in the context of investigations conducted by W. Magnus, and a question of Hanna Neumann. The new results in this paper are Theorem 1.0, Theorem 1.1, Theorem 1.2, Corollary 1.2, Theorem 1.3, Theorem 1.5, and Theorem 1.6. In Section One several results from the author’s earlier work are introduced. In Section Two the following theorems are proven: 1.0, 1.1, and 1.6. The paper ends with a list notation.

INTRODUCTION

The methods introduced here were developed with the explicit agenda of exploring possible algebraic geometric parallels between the representation varieties of a class of fg one relator parafree groups and free groups. More precisely, the development of this approach arose as a result of a question posed to the author by G. Baumslag inquiring about the possible equality of the Krull dimension of the space of $SL_2\mathbb{C}$ representations of a rank 2 free group, and that of the group...
$G = \langle x, y, z; x^2 y^3 = z^5 \rangle$, a group that some readers may readily identify as a parafree group of rank two, and deviation one.

Let $G$ be a group generated by the finite set $X = \{x_1, x_2, \ldots, x_n\}$; then the set $R(G)$ of homomorphisms from the group $G$ to $SL_2 \mathbb{C}$ can be endowed with the structure of a complex affine variety arising as the set of solutions in $(SL_2 \mathbb{C})^n$ to the set of matrix equations obtained from the relations of a presentation of $G$ on the generators $X$. The affine variety $R(G)$ is an invariant of $fg$ presentations of the group $G$. The invariant $R(G)$ exports into the study of $fg$ groups the numerous invariants of Algebraic Geometry and Commutative Algebra associated with algebraic varieties. For a detailed account of the algebraic variety $R(G)$ the reader may consult [LM].

Given a group $G$, invariants of particular interest associated with $G$ are the dimension, and reducibility status of $R(G)$. In the sequel, $\text{Dim}(R(G))$ shall denote the dimension of $R(G)$. Fortunately, if $F_n$ is the free group of rank $n$, then using the fact that $SL(2, \mathbb{C})$ is an irreducible 3 dimensional variety and Lemma .6 it is easy to see that $R(F_n) = SL(2, \mathbb{C})^n$, and is thus an irreducible\(^1\) variety of dimension $3n$.

Denote by $\gamma_n G$ the $n$-th term of the lower central series of the group $G$. A direct consequence of W. Magnus’ 1935 paper [MW] is what in the sequel will be referred to as Magnus’ Theorem:

**Magnus’ Theorem.** A $k$-generated group $G$ with $G/\gamma_n G \cong F_k/\gamma_n F_k$ for all $n \geq 1$ is free of rank $k$, where $F_k$ denotes the free group of rank $k$.

Magnus’ result led Hanna Neumann to inquire whether it was possible for two residually nilpotent groups $G$ and $G'$ to have $G/\gamma_n G \cong G'/\gamma_n G'$ for all $n$ without $G \cong G'$. Subsequently this question prompted G. Baumslag [B1] to construct in the 1960’s a class of groups he named parafree. A group $G$ is termed parafree if:

1) $G$ is residually nilpotent.

2) There exists a free group $F$ with the property that $G/\gamma_n G \cong F/\gamma_n F$ for all $n \geq 1$.

Now denote by $\mu(G)$ the minimal number of generators of a group $G$. Define the rank of $G$, denoted here by $rk(G)$, to be $rk(G) = \mu(G/\gamma_2 G)$. Further, define the deviation of the group $G$, here denoted by $\delta(G)$, to be $\delta(G) = \mu(G) - rk(G)$. A parafree group $G$ is of rank $r$ if the free group in (2) above is also of rank $r$. Notice that by Magnus’ Theorem a parafree group of finite rank $r$ is free iff it has deviation zero.

G. Baumslag in [B1] introduced a result quite handy in building non-isomorphic parafree groups of the same rank as a previously given parafree group:

**G. Baumslag’s Theorem.** Let $r$ and $n$ be positive integers and $H$ parafree of rank $r$, and let $(x)$ be the infinite cyclic group on $x$. Further, let $W \in H$. Suppose $W$ is the $k$ power of $W'$ modulo $\gamma_2 H$, where $W'$ is itself not a power modulo $\gamma_2 H$; also, assume that $k$ and $n$ are coprime and that the generalized free product $G = \{H * (x); W = x^n\}$ is residually nilpotent. Then $G$ is parafree of rank $r$.

Now consider the groups:

\[
G_{p_1 \ldots p_n} = \langle a_1, \ldots, a_n; a_1^{p_1} a_2^{p_2} \ldots a_{n-1}^{p_{n-1}} a_n^{p_n} = 1 \rangle, 
\]

\(^1\)The product of irreducible varieties is irreducible.
where \( p_1, p_2, \ldots, p_n \) are positive integers (all \( \geq 2 \)).

In 1975, S. Meskin [MS] showed that two groups \( G \) and \( G' \) as in (1-1) are isomorphic iff there exists a permutation \( \sigma \in S(n) \) such that the sequence of exponents \( (p_1, p_2, \ldots, p_n) \) of \( G \) are sent by \( \sigma \) to the corresponding sequence of exponents of \( G' \); in particular, the two sequences must be of the same length.

**Proposition 1.** The groups \( G_{p_1\ldots p_n} \), with \( n \geq 3 \), are freely indecomposable parafree groups of rank \( (n-1) \) and deviation 1 whenever the \( p_1, p_2, \ldots, p_n \) are positive integers with all the \( p_i' \)'s \( \geq 2 \) and having no common divisor.

**Proof.** Proposition 1 is a direct result of G. Baumslag’s Theorem cited earlier, and Theorem 1 in [B6], which guarantees that the groups are residually nilpotent. That each of the \( G_{p_1\ldots p_n} \) is of deviation one follows from the fact that \( G_{p_1\ldots p_n} \) maps to a corresponding free product \( \mathbb{Z}_{p_1} \ast \cdots \ast \mathbb{Z}_{p_n} \), and an application of the Grushko-Neumann Theorem. Their free indecomposability can be deduced from Theorem 3 in [SA].

It is a well documented fact that one-relator cyclically pinched groups share a host of properties with free groups. For example, the fact that the groups in (1-1) are cyclically pinched one-relator groups which meet the conditions of a combination theorem of [BF] when \( n \geq 3 \) makes them hyperbolic, as is the case with \( f_9 \) free groups. In fact, when \( n \geq 3 \) the groups in (1-1) are also linear by a result of Shalen [SP]. These are all properties possessed by free groups. For additional properties shared by cyclically pinched one-relator groups and free groups, the reader may consult [B4], and [B5]. Incidentally, as of this writing the existence on non-linear \( f_9 \) parafree groups is unknown; also unknown is whether all finitely presented ones are hyperbolic.

In [L1] the author introduced a result permitting one to calculate the dimension and reducibility status of the representation variety of cyclically pinched one-relator groups obtained from a free group \( F_n \) by adding a new generator \( y \) and a single relation \( g = y^p \), for any non-trivial \( g \in F_n \). The use of the main result of [L1] applied to the groups \( G_{p_1\ldots p_n} \), for \( n \geq 3 \), leads to Theorem 1.1, and many of the surprising results of this paper. Before stating Theorem 1.1, note that it takes into consideration the restrictions on the exponents of the relator in each of the groups \( G_{p_1\ldots p_n} \) in (1-1), and S. Meskin’s result [MS].

**Theorem 1.1.** Let \( G_{pq^t} = \langle x_1, x_2, x_3; x_1^p x_2^q x_3^t = x_3^t \rangle \), where \( p, q \) are negative integers \( \leq -2 \), and \( t \geq 2 \) is an integer. Then,

i) \( \text{Dim} \ (R(G_{pq^t})) = 6 \),

ii) \( R(G_{pq^t}) \) is a reducible algebraic variety when at least one of the absolute values of \( p, q, t \) is strictly larger than two,

iii) There exists an infinite set \( S_2 \) of groups associated with an infinite set \( S \) of 3-tuples, \( S \subset \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \), having the property that if \( (p, q, t) \), and \( (p', q', t') \) are in \( S \), then \( R(G_{pq^t}) \not\cong R(G_{p'q't'}) \) if \( p \neq p' \), and \( q \neq q' \), and \( t \neq t' \).

In fact, the next result gives a better picture of the general situation.

**Theorem 1.2.** Let \( p_1, p_2, \ldots, p_n \) be positive integers (all \( \geq 2 \)). Then for \( n \geq 3 \), \( \text{Dim} \ (R(G_{p_1\ldots p_n})) = 3(n-1) \).
Proof. Let \( \Lambda_n \) be the \( 3(n-1) \) dimensional affine variety of Proposition 4 in Section Two. Clearly, \( R(G_{p_1...p_n}) \subset \Lambda_n \). So \( \text{Dim}(R(G_{p_1...p_n})) \leq 3(n-1) \). But, \( G_{p_1...p_n} \) has a presentation of deficiency \( n-1 \), and hence \( \text{Dim}(R(G_{p_1...p_n})) \geq 3(n-1) \), by [LM]. So \( \text{Dim}(R(G_{p_1...p_n})) = 3(n-1) \).

**Corollary 1.2.** For any integer \( r \geq 2 \) there exist an infinite number of freely indecomposable pairwise non-isomorphic parafree groups of rank \( r \) and deviation one with \( \text{Dim}(R(G)) = 3r \).

**Proof.** The proof follows from Theorem 1.2 and Proposition 1.

Notice that Corollary 1.2 gives no information about the isomorphism type of the corresponding algebraic varieties. So it is conceivable that two non-isomorphic groups could have isomorphic representation varieties. In fact, in contrast to the case \( n = 3 \), it is not known when, for fixed \( n \geq 4 \), and \( p_1, \ldots, p_n \) as in (1-1), the corresponding algebraic varieties \( R(G_{p_1, \ldots, p_n}) \) are reducible.

The next theorem, a direct consequence of Theorem 1.1, provides a rather strong illustration of the sensitivity of the invariant \( R(G) \).

**Theorem 1.3.** There exists an infinite set \( S_2 \) of freely indecomposable parafree groups of rank two and deviation one having the property that for different \( G_{pqt} \) and \( G_{p'q't'} \) in \( S_2 \) their corresponding representation varieties are reducible, six dimensional, and non-isomorphic.

**Proof.** Let \( S_2 \) be the infinite set of groups guaranteed by Theorem 1.1 part iii. By Proposition 1 each of the groups in \( S_2 \) is parafree of deviation 1, rank 2, freely indecomposable, and for different \( G_{pqt} \) and \( G_{p'q't'} \) in \( S_2 \) it is the case that \( R(G_{pqt}) \not\cong R(G_{p'q't'}) \).

Theorem 1.3 displays clear distinctions in the reducibility status of the representation variety of a rank 2 free group, and the representation varieties of an infinite number of non-isomorphic parafree groups all of rank 2 and deviation one. In fact all the groups in \( S_2 \) have non-isomorphic varieties, quite a difference from the free group of rank \( n \), where up to isomorphism there is only one representation variety. The next theorem uses an invariant of \( R(G) \) to characterize as free, or not free, an \( n \)-generated group.

**Theorem 1.4.** An \( n \)-generated group \( G \) is free of rank \( n \) iff \( \text{Dim}(R(G)) = 3n \).

**Proof.** Let \( G \) be \( n \)-generated with \( \text{Dim}(R(G)) = 3n \); and suppose \( G \) is not free. Then \( G \) is a quotient group of \( F_n \). So there exists a non-trivial word \( w \neq 1 \) in \( F_n \) such that \( w = 1 \) in \( G \). But by Theorem 1 in [L2] one gets\(^2\) that \( \text{Dim}(R(F/N(w))) < 3n \). But, \( R(G) \) injects into \( R(F/N(w)) \), and consequently \( \text{Dim}(R(G)) \leq 3n - 1 \). This is a contradiction. Conversely, suppose that \( G \) is free and generated by \( n \) elements. Then by a result of Nielsen proven also in [L2], these \( n \) elements generate \( G \) freely. Thus \( \text{Dim}(R(G)) = 3n \). The proof is complete.

Let \( V \) be an algebraic variety and let \( N_c(V) \) be the integer valued function that counts its number of maximal irreducible components (\( \text{mir} \)) of dimension exactly \( c \). Clearly \( N_c(R(F_n)) = 1 \), when \( c = 3n \), and is zero otherwise. So that the reader

\(^2 N(w) \) is the normal closure of \( w \) in \( F_n \).
gains a glimpse of the sensitivity of this invariant of algebraic varieties, and thus of \( f_g \) groups, consider the case where \( \Sigma \) is a torus knot group. In [L3] the author showed that \( N_4(R(\Sigma)) = g \), where \( g \) is the genus of the torus knot corresponding to the group \( \Sigma \). The next theorem is a consequence of ideas employed in the proof of Theorem 1.1 involving the \( \text{mirc} \) counting function \( N_c(V) \).

**Theorem 1.5.** For any integer \( r \geq 2 \) there exists an infinite set \( S_r \) of parafree groups of rank \( r \) and deviation 1 having the property that for each group \( G_i \in S_r \), \( \text{Dim}(R(G_i)) = 3r \), and such that given another \( G_j \) in \( S_r \), then \( R(G_i) \cong R(G_j) \) iff \( G_i \cong G_j \).

**Proof.** If \( r = 2 \), then let \( S_r = S_2 \), as in Theorem 1.3. If \( r \geq 3 \) let \( S_r = \{ F_{r-2} * G_i | G_i \in S_2 \} \), where \( F_{r-2} \) is the free group of rank \( r-2 \). Note that any \( G_k \in S_r \) maps onto \( F_{r-2} * \mathbb{Z}_p * \mathbb{Z}_q * \mathbb{Z}_t \), where \( p, q, t \) are all greater than 1. It follows from the Grushko-Neumann Theorem that \( G_k \) can’t be generated by fewer than \( r+1 \) generators. That \( G_k \) is parafree of rank \( r \) follows from [B2], which guarantees that the rank of a free product of parafree groups is parafree of rank the sum of the factors. The dimension condition for \( R(G_k) \) follows from Lemma .6, Theorem 1.1 and the fact that \( \text{Dim}(R(F_n)) = 3(n-2) \). The isomorphism condition for different groups in \( S_r \) follows since any two groups \( G_e, G_f \) in \( S_2 \) have the property that \( N_b(R(G_e)) \neq N_b(R(G_f)) \), unless \( G_e \) and \( G_f \) are isomorphic, (see proof of Theorem 1.1 part iii); now by virtue of Lemma .2 the same condition can be extended to the groups in \( S_r \) by employing the \( \text{mirc} \) counting function \( N_{3r}(V) \). The proof is complete.

The next theorem is perhaps one of the most surprising results so far on the representation varieties of groups in the class of \( f_g \) parafree groups of rank \( n \geq 2 \), and deviation one.

**Theorem 1.6.** Given an integer \( r \geq 2 \), and an arbitrary integer \( k \geq 1 \), there exists a parafree group \( G \) of rank \( r \) and deviation one with \( \text{Dim}(R(G)) = 3r \), and \( N_{3r}(R(G)) \geq k \). If \( r = 2 \), the parafree group can be taken to be freely indecomposable.

Compare the above result with the situation in the case of parafree groups of rank \( n \) and deviation zero. In such a case \( N_{3n}(R(G)) = 1 \), and up to isomorphism there is only one such parafree group, namely \( F_n \).

**Section One**

In this section Theorem .1 of [L1] will be introduced along with some preliminary material to be employed in the eventual proof of many of the new results of this paper.

Let \( W \neq 1 \) be a freely reduced word in \( F_n \) involving all the generators \( \{x_1, \ldots, x_n\} \) of \( F_n \). To the free group \( F_n \) add a new generator \( y \), and now consider the relation \( W = y^k \) of the one-relator group

\[
(2-1.1) \quad G = \langle x_1, \ldots, x_n, y; W = y^k \rangle,
\]

where \( k \geq 2 \) is a positive integer.
Observation 1. The relation $W = y^k$ gives rise to an equation in $SL_2 \mathbb{C}$. Solutions to this equation are $(n+1)$-tuples of $SL_2 \mathbb{C}$ matrices $(m_1, \ldots, m_{n+1})$ such that the relation $W = y^k$ is satisfied when the $(n+1)$-tuple is evaluated in $W = y^k$ under the obvious assignment $x_i \rightarrow m_i$, for all $x_i \in \{x_1, \ldots, x_n\}$, and $y \rightarrow m_{n+1}$. Further, observe that the relation can also be used to give rise to an equation $W = -y^k$ in $SL_2 \mathbb{C}$.

Notation. Denote the $SL_2 \mathbb{C}$ solutions to the equations $W = y^k$, and $W = -y^k$ of the previous observation by $S_n(W, k)$, and $S_n(-W, k)$ respectively. By $I$ shall be meant the $2 \times 2$ identity matrix.

Notice that $S_n(W, k)$ and $S_n(-W, k)$ are affine algebraic varieties, also that the representation variety $R(G)$ of the one-relator group in (2-1.1) is $S_n(W, k)$. Now introduce the following integer valued function:

\[
(2-1.2) \quad f(x) = \begin{cases} 
1 & \text{if } x \neq 2 \\
0 & \text{if } x = 2 
\end{cases}
\]

and let $P_+$ and $P_-$ be the two algebraic sub-varieties of $R(F_n)$ given by:

\[
(2-1.3) \quad P_+ = \{ \rho \mid \rho \in R(F_n), \text{and } \rho(W) = I \}, \quad P_- = \{ \rho \mid \rho \in R(F_n), \text{and } \rho(W) = -I \}.
\]

Theorem .1.

a) $\dim (S_n(W, k)) = \max \{ \dim (P_+) + 2f(k), \dim (P_-) + 2, 3n \} \leq 3n + 1$.

b) $\dim (S_n(-W, k)) = \max \{ \dim (P_-) + 2f(k), \dim (P_+) + 2, 3n \} \leq 3n + 1$.

For a proof consult [L1].

The next corollary, a consequence of the fact that an irreducible algebraic variety can have no proper sub-variety of its dimension, will be indispensable.

Corollary .1.

i) If in Theorem .1a $\max \{ \dim (P_+) + 2f(k), \dim (P_-) + 2 \} \geq 3n$, then $S_n(W, k)$ is a reducible variety.

ii) If in Theorem .1b $\max \{ \dim (P_-) + 2f(k), \dim (P_+) + 2 \} \geq 3n$, then $S_n(-W, k)$ is a reducible variety.

For a proof consult [L1].

Now a basic lemma concerning $SL_2 \mathbb{C}$ solutions of equations of the type $x^p = \pm I$ will be given, but first some notation.

Notation. Given a positive integer $p$ and $M \in SL_2 \mathbb{C}$, denote by $\Omega(p, M)$ the following set $\{ A \mid A \in SL_2 \mathbb{C}, \ A^p = M \}$. Notice that $\Omega(p, M)$ is an algebraic variety.

Lemma .1.

i) If $p = 2$, then $\dim \Omega(p, I) = 0$.

ii) If $p > 2$, then $\dim \Omega(p, I) = 2$.

iii) For $p \geq 2$, $\dim \Omega(p, -I) = 2$.

The proof of Lemma .1 can be found $^3$ directly, or by consulting [L1].

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$^3$In $SL_2 \mathbb{C}$ a matrix of finite order is conjugate to a diagonal matrix.
Lemma 2. Let $V$ be an irreducible variety with $\dim(V) = m$, and let $W$ be a reducible variety with $N_2(W) = c$. Then $N_{m+q}(V \times W) = c$.

Proof. The dimension of a product of two varieties is the sum of the dimension of each factor, and the number of $\text{miric}$ in the product is the product of the $\text{miric}$ in each factor. Now, Since there is only one $\text{miric}$ of dimension $m$ in $V$, given the stated irreducibility of the algebraic variety $V$, and since there are a number $c$ of $q$-dimensional $\text{miric}$ in $W$, it follows that the number of $(m+q)$-dimensional $\text{miric}$ of $V \times W$ is also $c$.

The first application of Lemma .1 will be the proof of the next result.

Lemma 3. By $\Sigma_1(-x_1^p, -q)$ denote the solutions over $SL_2 \mathbb{C}$ to the matrix equation $x_1^p x_2^q = -I$, where $p$ and $q$ carry the same signs and assumptions as in Theorem 1.1. Then:

i) $\dim(\Sigma_1(-x_1^p, -q)) = 4$, if $|p| > 2$ or $|q| > 2$.

ii) $\dim(\Sigma_1(-x_1^p, -q)) = 3$, if $|p| = 2$ and $|q| = 2$.

Proof i). It has been assumed that $p$ and $q$ in $x_1^p x_2^q = -I$ are negative. Consider the equation $x_1^p = -x_2^{-q}$. Let $P_- = \{ \rho | \rho \in R(F_1), \rho(x_1^p) = I \}$ and $P_+ = \{ \rho | \rho \in R(F_1), \rho(x_1^p) = -I \}$. Notice that if $|p| > 2$, then by Lemma .1 $\dim(P_-) = 2$, and by Lemma .1 above $\dim(P_+) = 2$. By Theorem .1b

$\dim(\Sigma_1(-x_1^p, -q)) = \max \{ \dim(P_-) + 2f(-q), \dim(P_+) + 2, 3 \} \leq 3 + 1$, where $f$ is the function in the statement of Theorem .1. Thus $\dim(\Sigma_1(-x_1^p, -q)) = 4$. If on the other hand $|p| = 2$, then by Lemma .1 $\dim(P_-) = 0$, and $\dim(P_+) = 2$. Thus $\dim(\Sigma_1(-x_1^p, -q)) = 4$, since $-q > 2$. This completes the proof of (i).

Proof ii). Suppose $p$ has absolute value equal to two. Notice that

$P_- = \{ \rho | \rho \in R(F_1), \rho(x_1^p) = I \}, P_+ = \{ \rho | \rho \in R(F_1), \rho(x_1^p) = -I \}$.

By Lemma .1 one gets that $\dim(P_-) = 2$, and that $\dim(P_+) = 0$. Thus by Theorem .1b it follows that $\dim(\Sigma_1(-x_1^p, -q)) = \max \{ \dim(P_-) + 2f(-q), \dim(P_+) + 2, 3 \} \leq 3 + 1$. So $\dim(\Sigma_1(-x_1^p, -q)) = 3$, since $-q = 2$. This completes the proof of (ii).

Lemma 4. Let $G$ be a fg group, and let $N$ be a normal subgroup of $G$. Let $N_2(V)$ be the $\text{miric}$ counting function in dimension $c$, and suppose that $\dim(R(G)) = c$. Then:

i) $\dim(R(G/N) \leq \dim(R(G))$.

ii) $N_2(R(G/N)) \leq N_2(R(G))$.

Proof.

i). $N$ is the normal closure of at least one non-trivial element from $G$. So $R(G/N)$ is the zero locus in $R(G)$ of at least one polynomial belonging to the coordinate algebra of $R(G)$. It follows that $\dim(R(G/N)) \leq \dim(R(G))$.

ii). By i), $\dim(R(G/N)) \leq \dim(R(G))$. Assume that $N_2(R(G/N)) > N_2(R(G))$. Then there is an irreducible component containing a component of its own dimension. This contradicts the properties of finite dimensional noetherian topological spaces.
Lemma 5. Denote by $\Sigma_2(-x_i^p x_2^q, t)$ the solutions over $SL_2 \mathbb{C}$ to the matrix equation $x_1^p x_2^q = -x_4^r$, where $p, q, r$ carry the same signs and assumptions as in Theorem 1.1. Then $\text{Dim}(\Sigma_2(-x_i^p x_2^q, t)) = 6$.

Proof. Theorem 1b will be employed. Notice that

$$P_- = \{ \rho \mid \rho \in R(F_1), -\rho(x_i^p x_2^q) = I \}, P_+ = \{ \rho \mid \rho \in R(F_1), -\rho(x_i^p x_2^q) = -I \}.$$ 

By Lemma 3, $\text{Dim}(P_-) = 4$ or $3$, depending on $|p|$ and $|q|$. Now consider $P_+$. By Theorem 1a-b and Lemma 1 follows that $\text{Dim}(P_+) = 4$. Now by Theorem 1b

$$\text{Dim}(\Sigma_2(-x_i^p x_2^q, t)) = \text{Max} \{ \text{Dim}(P_-) + 2f(t), \text{Dim}(P_+) + 2, 6 \} \leq 6 + 1$$

and thus $\text{Dim}(\Sigma_2(-x_i^p x_2^q, t)) = 6$. This completes the proof of Lemma 5.

Lemma 6. Let $G_1$ and $G_2$ be fg groups, and $V_1, V_2$ be algebraic varieties; then $R(G_1 \ast G_2) = R(G_1) \times R(G_2)$, and $\text{Dim}(V_1 \times V_2) = \text{Dim}(V_1) + \text{Dim}(V_2)$.

The proof of the above can be easily deduced from elementary facts found in [MD], and is left to the reader.

Next is introduced a result allowing computation of the dimension of certain algebraic varieties using fibers of regular maps.

Proposition 2. Let $\phi : V \to W$ be a regular map between two algebraic varieties, where $W$ is irreducible and $\text{Dim}(W) = n > 0$. Let $V_1$ and $W_1$ be two proper closed subvarieties of $V$ and $W$, respectively, such that the restricted map $\phi : V^\circ \to W^\circ$, where $V^\circ = V - V_1$ and $W^\circ = W - W_1$, is such that:

1) $\phi : V^\circ \to W^\circ$ is onto.
2) $\phi$ has zero dimensional fiber above each point of $W^\circ$.
3) $\phi^{-1}(W^\circ) = V^\circ$.

Then $\text{Dim}(\text{Cl}(W^\circ)) = \text{Dim}(\text{Cl}(V^\circ)) = n$, where $\text{Cl}(W^\circ)$ denotes the Zariski closure of $W^\circ$.

For a proof the reader may consult [L1], or use elementary facts from [MD].

Section Two

The next result, instrumental in the proof of Theorem 1.1, and Theorem 1.6, guarantees that an infinite sequence of pairs $(G_i, \mathfrak{R}_i)$ consisting of a group $G_i$ and a corresponding normal subgroup $\mathfrak{R}_i$ will have an infinite subsequence provided the mirc counting function $N_c(R(G_i/\mathfrak{R}_i))$ achieves an infinite number of values as $i$ varies. In such an instance, the corresponding subsequence can be used to obtain a well ordered infinite set of groups using the mirc counting function $N_c(V)$ on their corresponding representation varieties.

Theorem 1.0.

Let $S_{\leq i}^\infty(G_i, \mathfrak{R}_i)$ be an infinite sequence of pairs $(G_1, \mathfrak{R}_1), (G_2, \mathfrak{R}_2), \ldots$ consisting of fg groups $G_i$ and a corresponding normal subgroup $\mathfrak{R}_i$ of $G_i$. Let $N_c(V)$ be the mirc counting function in dimension $c$. Suppose that $\text{Dim}(R(G_i)) = \text{Dim}(R(G_i/\mathfrak{R}_i)) = \ldots$
c, and that the set \( \tilde{S} = \{ N_c(R(G_j/\mathfrak{N}_j)) \mid (G_j, \mathfrak{N}_j) \in S_{i=1}^\infty(G_i, \mathfrak{N}_i) \} \) contains an infinite set of integer points. Then:

i) \( S_{i=1}^\infty(G_i, \mathfrak{N}_i) \) has an infinite subsequence \( S^2 \) with the property that given two different pairs \((G_j, \mathfrak{N}_j), (G_k, \mathfrak{N}_k) \) in \( S^2 \) then \( N_c(R(G_j)) \neq N_c(R(G_k)) \).

ii) For different pairs \((G_j, \mathfrak{N}_j), (G_k, \mathfrak{N}_k) \) in \( S^2 \) then \( R(G_j) \not\approx R(G_k) \).

iii) The set of groups \( S_2 = \{ G \mid (G, \mathfrak{N}_j) \in S_{i=1}^\infty(G_i, \mathfrak{N}_i) \} \) contains an upper-bound for an infinite set of positive integers.

Proof of part i). Using the notation of Theorem .1a let \( B = \{ N_c(R(G_j)) \mid (G_j, \mathfrak{N}_j) \in S_{i=1}^\infty(G_i, \mathfrak{N}_i) \} \) that the set \( B \) is also infinite, since no finite set of positive integers can be an upper-bound for an infinite set of positive integers. Thus the set \( B^2 = \{ (N_c(R(G_j)), N_c(R(G_j/\mathfrak{N}_j))) \mid (G_j, \mathfrak{N}_j) \in S_{i=1}^\infty(G_i, \mathfrak{N}_i) \} \) is an infinite subset of the lattice \( \mathbb{Z}^2 \) with strictly positive entries. Use the set \( B^2 \) to obtain the subsequence \( S^2 \) of \( S_{i=1}^\infty(G_i, \mathfrak{N}_i) \) meeting the stipulated requirements.

Proof of part ii). Given \( G_j \) and \( G_k \) in \( S_2 \), by virtue of i, it is the case that either \( N_c(R(G_j)) < N_c(R(G_k)) \), or \( N_c(R(G_k)) < N_c(R(G_j)) \), or \( N_c(R(G_k)) = N_c(R(G_j)) \) in which case \( G_j \cong G_k \).

The next theorem is a cornerstone in the demonstration of many subsequent results.

Theorem 1.1. Let \( G_{pq} = \langle x_1, x_2, x_3; x_1^p x_2^q = x_3^t \rangle \), where \( p, q \) are negative integers \( \leq -2 \), and \( t \geq 2 \) is an integer. Then,

i) \( \dim(R(G_{pq})) = 6 \),

ii) \( R(G_{pq}) \) is a reducible algebraic variety when at least one of the absolute values of \( p, q, t \) is strictly larger than two,

iii) There exists an infinite set \( S_2 \) of groups associated with an infinite set \( S \) of \( 3 \)-tuples, \( S \subset \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \), having the property that if \( (p, q, t) \), and \( (p', q', t') \) are in \( S \), then \( R(G_{pq}) \not\approx R(G_{p'q't'}) \) if \( p \neq p' \), and \( q \neq q' \), and \( t \neq t' \).

Proof i). Using the notation of Theorem .1a let \( P_+ = \{ \rho | \rho \in R(F_2), and \rho(x_1^p x_2^q) = I \}, P_- = \{ \rho | \rho \in R(F_2), and \rho(x_1^p x_2^q) = I \} \). Now by the Theorem 1a

\[
(3-1.2) \quad \dim(R(G_{pq})) = \max \{ \dim(P_+) + 2f(t), \dim(P_-) + 2, 3n \},
\]

where \( f \) is the integer valued functions in the statement of Theorem .1. Using Theorem .1 and Lemma .1 it can be readily shown that \( \dim(P_+) = 4 \). Now it is necessary to compute \( \dim(P_-) \). This of course follows from the above lemma. Thus \( \dim(P_-) = 3 \) or \( 4 \), depending on the absolute values of \( p \) and \( q \). Thus, by Theorem 1.1a-b it follows that

\[
(3-1.3) \quad \dim(R(G_{pq})) = \max \{ \dim(P_+) + 2f(t), \dim(P_-) + 2, 6 \} \leq 7
\]

\[
= \max \{ 4 + 2f(t), 3 + 2 \) or \( 4 + 2, 6 \} = 6.
\]
Proof ii). Without loss of generality it can be assumed that \( t \) is the integer whose absolute value is greater than 2. Then in (3-1.3) it follows that since \( t \) is not two, \( \text{Dim}(P_a) + 2f(t) = 6 \). By Corollary 1 one obtains that \( R(G_{pqt}) \) is reducible.

Proof iii). This proof will make use of Theorem 1.0. Beginning with the prime 3 list the infinite progression of primes thus: 3, 5, 7, 11, ... Using the resulting list associate an infinite set \( S' \subset \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \) consisting of 3-tuples as shown: \( S' = \{ (3,5,7), (11,13,17), ... \} \). Each one of the three tuples in \( S' \) can be used to identify a group \( G_{pqt} \): example\(^{4} \), to (3,5,7) assign to the group \( < x_1, x_2, x_3; x_1^{-3}x_2^{-5} = x_3^{-7} > \). Notice also that there is a surjection from \( G_{pqt} \) to the free product of cyclicity \( \mathbb{Z}_p * \mathbb{Z}_q * \mathbb{Z}_t \), and \( G_{pqt}/N_{pq} \cong \mathbb{Z}_p * \mathbb{Z}_q * \mathbb{Z}_t \), where \( N_{pq} = N\{ x_1^p, x_2^q, x_3^t \} \); here \( N\{ x_1^1, x_2^2, x_3^3 \} \) stands for the “normal closure” in \( G_{pqt} \) of the word set \( \{ x_1^p, x_2^q, x_3^t \} \). By imitation of the above, the set \( S' \) can be used to give rise to an infinite sequence \( S_{i=1}^\infty (G_i, \mathfrak{N}_i) \) consisting of a group \( G_i \) and a corresponding normal subgroup \( \mathfrak{N}_i \), very much in the spirit of the sequence of Theorem 1.0, as follows:

\[
(3-1.4) \quad (G_1 = G_{3,5,7}, \mathfrak{N}_1 = N_{3,5,7}), (G_2 = G_{11,13,17}, \mathfrak{N}_2 = N_{11,13,17}), \ldots
\]

Now using part i which has already been proven, it follows that \( \text{Dim}(R(G_{pqt})) = 6 \) for any of the groups \( G_{pqt} \) in the sequence \( S_{i=1}^\infty (G_i, \mathfrak{N}_i) \). So as to employ Theorem 1.0, it remains to be shown that \( \text{Dim}(R(G_{pqt}/N_{pq})) = 6 \) also. But \( G_{pqt}/N_{pq} \cong \mathbb{Z}_p * \mathbb{Z}_q * \mathbb{Z}_t \). So all that needs to be shown is that \( \text{Dim}(R(\mathbb{Z}_p * \mathbb{Z}_q * \mathbb{Z}_t)) = 6 \), but this follows from Lemma 1 part ii, and Lemma 6. Now all that remains to show is that the set \( S = \{ N_{6}(R(G_j/\mathfrak{N}_j)) \mid (G_j, \mathfrak{N}_j) \in S_{i=1}^\infty (G_i, \mathfrak{N}_i) \} \) corresponding to (3-1.4) contains an infinite set of integer points. But this will be a consequence of the easily deduced fact shown in \([L1]\), or \([L2]\): (for \( p \geq 3 \) odd \( N_2(R(\mathbb{Z}_p)) = \frac{p-1}{2^3} \)). Now using Lemma 1, Lemma 2, and Lemma 6 it follows that

\[
(3-1.5) \quad N_6(R(\mathbb{Z}_p * \mathbb{Z}_q * \mathbb{Z}_t)) = \frac{(p-1)(q-1)(t-1)}{2^3}.
\]

Notice that \( N_6(R(\mathbb{Z}_p * \mathbb{Z}_q * \mathbb{Z}_t)) \) grows arbitrarily large as either \( p \), or \( q \), or \( t \) grow large. The conditions for Theorem 1.0 are met by (3-1.4). So one is guaranteed a subsequence \( S^2 \) of \( S_{i=1}^\infty (G_i, \mathfrak{N}_i) \) with the property that given two different pairs \( (G_j, \mathfrak{N}_j), (G_k, \mathfrak{N}_k) \) in \( S^2 \) then \( N_6(R(G_j)) \neq N_6(R(G_k)) \), and a sequence \( S_2 \) of groups meeting the criteria stipulated by part iii. The existence of \( S_2 \) trivially leads to the existence of \( S \). The proof is complete.

Now some additional remarks concerning the groups in (1-1) are needed.

Observation 2. As mentioned earlier, the groups \( G_{p_1 \cdots p_n} \) in (1-1) are isomorphic to the groups \( G_{-p_1 \cdots -p_n} \). Thus one can assume each \( p_i \) in (1-1) is negative as this has no consequence on the invariant \( R(G) \).

The solutions over \( SL_2 \mathbb{C} \) to the equation in \( n \)-variables obtained from the relation of (1-1) by replacing \( p_i \) with \( -p_i \):

\[
(3-1.6) \quad a_1^{-p_1}a_2^{-p_2} \cdots a_{n-1}^{-p_{n-1}}a_n^{-p_n} = I
\]

\(^{4}\)For clarity commas are inserted between \( pqt \) when actual integers are used.
give rise to an affine variety in \( \mathbb{C}^{4n} \). The equation in (3-1.6) can be thought of as the relation of a one-relator group generated by \( \{a_1, \ldots, a_n\} \). By Observation 2, the solutions to the equation (3-1.6) gives an algebraic variety isomorphic to the representation variety of a group as in (1-1). In fact, the solutions to the equation (3-1.6) are contained in the union of solutions over \( SL_2 \mathbb{C} \) to the two matrix equations in \( n \) variables obtained from

\[
(3-1.7) \quad a_1^{-p_1} a_2^{-p_2} \cdots a_{n-1}^{-p_{n-1}} = \pm a_n^{p_n}.
\]

**Notation.** Denote by \( \Lambda_n \) the union of solutions over \( SL_2 \mathbb{C} \) to the two matrix equations in the sense of Observation 1 in Section One, obtained from (3-1.7). By \( F_{n-1} \), denote the free group generated by \( \{a_1, a_2, \ldots, a_{n-1}\} \).

Recall that \( R(F_{n-1}) \) can be thought of as consisting of all \((n-1)\)-tuples of \( 2 \times 2 \) matrices from \( SL_2 \mathbb{C} \). Each one of the \((n-1)\)-tuples is a point of \( R(F_{n-1}) \) and thus can be denoted by \((m_1, m_2, \ldots, m_{n-1})\). Again, the same holds true for \( \Lambda_n \); each solution to the two matrix equations in (3-1.7) is an \( n \)-tuple, \((m_1, m_2, \ldots, m_n)\) of \( 2 \times 2 \) matrices from \( SL_2 \mathbb{C} \).

**Notation.** By \( W \) denote the word \( a_1^{-p_1} a_2^{-p_2} \cdots a_{n-1}^{-p_{n-1}} \) in \( F_{n-1} \), and also in the left side of the equation (3-1.7). By \( \Phi \) refer to the projection map

\[
(3-1.8) \quad \Phi : \Lambda_n \to R(F_{n-1})
\]

given by \( \Phi(m_1, m_2, \ldots, m_n) = (m_1, m_2, \ldots, m_{n-1}) \).

Observe that \( \Lambda_n \) is the union of the two algebraic varieties (3-1.9) and (3-2.1):

\[
(3-1.9) \quad \{(\rho(a_1), \rho(a_2), \ldots, \rho(a_{n-1}), \sigma) | \rho \in R(F_{n-1}), \sigma \in \Omega(p_n, +\rho(W))\}
\]

\[
(3-2.1) \quad \{(\rho(a_1), \rho(a_2), \ldots, \rho(a_{n-1}), \sigma) | \rho \in R(F_{n-1}), \sigma \in \Omega(p_n, -\rho(W))\}.
\]

**Proposition 3.** The map \( \Phi \) in (3-1.8) maps \( \Lambda_n \) onto \( R(F_{n-1}) \).

**Proof.** One only needs to consider the case when \( p_n \) is even, since if it is odd, then the map is onto; see [L1]. Suppose \( p_n \) is even; then if \( \rho(W) \in Orb(B) \), where by \( Orb(B) \) is meant the orbit under \( SL_2 \mathbb{C} \) conjugation of the matrix \( B = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \), one obtains in (3-1.9) that \( \Omega(p_n, \rho(W)) = \emptyset \). However, for this very same choice of \( \rho \), \( -\rho(W) \) does not lie in \( Orb(B) \). Consequently in (3-2.1) the set \( \Omega(p_n, -\rho(W)) \neq \emptyset \). Thus the map \( \Phi \) is always onto since \( \Lambda_n \) is the union of the algebraic varieties given in (3-1.9) and (3-2.1).

Next a proposition is proven whose direct corollary is Theorem 1.2.

**Proposition 4.** For \( n \geq 3 \), \( Dim(\Lambda_n) = 3(n-1) \).

**Proof.** The proof will proceed by induction on \( n \geq 3 \). This has already been established in Lemma 5 and Theorem 1.1 for \( n = 3 \). Assume the theorem true for \( n - 1 \), where \( n > 3 \). It will be then shown true for \( n \). Notice that \( \Lambda_n = \)
Clearly, given arbitrary integer $G$, Notation.

So given $r$ and an arbitrary integer $k \geq 1$, there exists a parafree group $G$ of rank $r$ and deviation one with $\text{Dim}(R(G)) = 3r$, and $N_{3r}(R(G)) \geq k$. If $r = 2$, the parafree group can be taken to be freely indecomposable.

**Proof.** The infinite set of groups $S_2$ of Theorem 1.1 part iii, or Theorem 1.3, by virtue of Theorem 1.0 can be well ordered using the \textit{micr} counting function $N_6(V)$. So given $G_i$, $G_j$ in $S_2$ then either: $N_6(R(G_i)) = N_6(R(G_j))$, or $N_6(R(G_i)) < N_6(R(G_j))$, or $N_6(R(G_i)) > N_6(R(G_j))$. Now using Lemma 2. and Lemma 6 this well ordering of $S_2$ can be extended using the \textit{micr} counting function $N_{3r}(V)$ to the infinite set of rank $r$ deviation 1 parafree groups $S_r$ in Theorem 1.5, where $r \geq 3$. Clearly, given arbitrary integer $k$, and $r \geq 3$, the well ordering on $S_r$ guarantees that there exists $G \in S_r$ such that $N_{3r}(R(G)) \geq k$; in fact, as announced, an infinite number of non-isomorphic groups meeting the demand exist. Notice that each group $G$ in $S_r$ has $\text{Dim}(R(G)) = 3r$. Finally, That the groups in $S_2$ are freely indecomposable follows from Proposition 1.

**Notation.**

$G$, a finitely generated group (fg), unless specified.

$SL_2\mathbb{C}$, group of 2 by 2 matrices of determinant one.

$R(G)$, the space of representations of $G$ in $SL_2\mathbb{C}$.

$\text{Dim}(R(G))$, dimension of the affine variety $R(G)$.

$F_n$, the free group of rank $n$.

$\gamma_n G$, the $n$-th term of the lower central series of $G$.

$G/\gamma_n G$, the $n$-th term of the lower central sequence of $G$. 

$$\{ (\rho(a_1), \rho(a_2), \ldots, \rho(a_{n-1}), \sigma) | \rho \in R(F_{n-1}), \sigma \in \Omega(p_n, \pm \rho(W)) \}.$$ Given any representation $\rho = (m_1, \ldots, m_{n-1}) \in R(F_{n-1})$ such that $m_1 p_1 m_2^{-1} \ldots m_{n-1}^{-1} = \pm I$; then it is true that $\text{Dim}(\Phi^{-1}(\rho)) = 2$, by Lemma 1. Thus the map $\Phi$ in general fails to have finite fibre; so in order to employ Proposition 2 care is needed. Let $U = \{(m_1, m_2, \ldots, m_n) | m_1, m_2, \ldots, m_n \in \Lambda_n$ and $m_1^{-1} m_2 \ldots m_{n-1}^{-1} = \pm I \}$. $U$ is a sub-variety of $\Lambda_n$. In fact, $U$ is a proper sub-variety of $\Omega = (\Lambda_n) \times \Omega(p_n, \pm I)$, with $\text{Dim}(\Omega) = \text{Dim}(U)$, where $\Lambda_n$ is the union of algebraic varieties consisting of solutions over $SL_2\mathbb{C}$ to the two equations on $(n - 1)$ variables given by $a_1^{-1} a_2^{-1} \ldots a_{n-2}^{-1} = \pm a_{n-1}^{-1}$. Notice that by the induction hypothesis

$$\text{Dim}(\Lambda_n) = 3(n - 2). \quad (3.22)$$

Consequently, by Lemma 1, $\text{Dim}(U) = 3(n - 2) + 2$. Notice that, $3(n - 2) + 2 < 3(n - 1)$. Thus $U$ is a proper sub-variety of $\Lambda_n$ since $\text{Dim}(\Lambda_n) \geq 3(n - 1)$, given that $\Lambda_n$ contains the representation variety of a one-relator group with a presentation of deficiency $(n - 1)$. Notice that $\Lambda_n - U$ is mapped by $\Phi$ onto $\{(R(F_{n-1}) - \Phi(U))$, and that $\Phi(U)$ is a sub-variety of $R(F_{n-1})$. In fact, $\Phi(U) = (\Lambda_n)$. Consequently by (3.2) $\text{Dim}(\Phi(U)) = 3(n - 2)$. Thus $\{R(F_{n-1}) - \Phi(U)\}$ is a quasi-affine variety, and since quasi-affine varieties are dense in an irreducible variety, it follows that $\text{Dim}(\{R(F_{n-1}) - \Phi(U)\}) = 3(n - 1)$. For every point $m \in \{R(F_{n-1}) - \Phi(U)\}$, the fibre $\Phi(m)^{-1}$ has zero dimension. So present are then the needed conditions for Proposition 2. Hence, $\text{Dim}(\{R(F_{n-1}) - \Phi(U)\}) = \text{Dim}(\{\Lambda_n - U\}) = 3(n - 1)$. Notice that $\Lambda_n = \{\Lambda_n - U\} \cup U$. Therefore: $\text{Dim}(\Lambda_n) = \text{Max} \{\text{Dim}(\{\Lambda_n - U\}), \text{Dim}(U)\} = \text{Max} \{(3(n - 1), 3n - 4\} = 3(n - 1).$ This completes the proof.
\( \mu(G) \), the minimal number of generators of a group \( G \).
\( \text{rk}(G) = \mu(G/\gamma_2G) \), rank of a parafree group \( G \).
\( \delta(G) = \mu(G) - \text{rk}(G) \), deviation of a parafree group \( G \).
\( W \), word in \( F_n \), where \( n \) depends on the context.
\( \Sigma_n(W,k) \), solutions to the matrix equation \( W = y^k \) in \( SL_2 \mathbb{C} \).
\( \Sigma_n(-W,k) \), solutions to the equation \( -W = y^k \) in \( SL_2 \mathbb{C} \).
\( P_+ = \{ \rho \mid \rho \in R(F_n), \text{ and } \rho(W) = I \} \).
\( P_- = \{ \rho \mid \rho \in R(F_n), \text{ and } \rho(W) = -I \} \).
\( \Omega(p, M) = \{ A \mid A \in SL_2 \mathbb{C}, A^p = M \} \), \( M \) is in \( SL_2 \mathbb{C} \).
\( N_c(V) \), number of \( c \) dimensional maximal irreducible components (mirc) of an algebraic variety \( V \).

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