Topology of spaces of smooth functions and gradient-like flows with prescribed singularities on surfaces

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Abstract

By a gradient-like flow on a closed orientable surface $M$, we mean a closed 1-form $\beta$ defined on $M$ punctured at a finite set of points (sources and sinks of $\beta$) such that there exists a Morse function $f$ on $M$, called an energy function of $\beta$, whose critical points coincide with equilibria of $\beta$, and the pair $(f, \beta)$ has a canonical form near each critical point of $f$. Let $B = B(\beta_0)$ be the space of all gradient-like flows on $M$ having the same types of local singularities as a flow $\beta_0$, and $F = F(f_0)$ the space of all Morse functions on $M$ having the same types of local singularities as an energy function $f_0$ of $\beta_0$. We prove that the spaces $F$ and $B$, equipped with $C^\infty$ topologies, are homotopy equivalent to some manifold $M_s$, moreover their decompositions into $\text{Diff}^0(M)$-orbits are given by two transversal fibrations on $M_s$. Similar results are proved for topological equivalence classes on $F$ and $B$, and for non-Morse singularities.

Key words: Morse flow, gradient-like flow, orbital topological equivalence, ADE singularities, moduli space of real-normalized meromorphic differentials

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1 Spaces of smooth functions with prescribed local singularities on surfaces

Let $M$ be a smooth orientable connected closed two-dimensional surface, and $f_0 \in C^\infty(M)$ a function all whose critical points have types $A_\mu$, $D_\mu$, $E_\mu$ (e.g. a Morse function).

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Recall that a point \( P \in M \) is critical for \( f \in C^\infty \) if \( df(P) = 0 \). A function \( f \in C^\infty(M) \) is called Morse if all its critical points are non-degenerate (of type \( A_1 \)), i.e. \( d^2f(P) \) is non-degenerate. By the Morse lemma, locally \( f = \pm x^2 \pm y^2 + f(P) \) in suitable coordinates near each critical point \( P \).

Consider the set \( \mathcal{F} = \mathcal{F}(f_0) \) of all functions \( f \in C^\infty(M) \) having the same types of local singularities as \( f_0 \). Denote by \( D^0(M) \) the identity path component of the group \( D(M) = \text{Diff}^+(M) \) of orientation-preserving diffeomorphisms endowed with \( C^\infty \) topology. The group \( D(\mathbb{R}) \times D(M) \) acts on the space \( \mathcal{F} \) by “left-right changes of coordinates”.

We want to describe the topology of the space \( \mathcal{F} \), equipped with the \( C^\infty \)-topology, and its decomposition into \( D^0(\mathbb{R}) \times D^0(M) \)- and \( D^0(M) \)-orbits. This problem was solved by the author in the cases when either \( f_0 \) is a Morse function and \( \chi(M) < 0 \) [4, 5, 7], or all critical points of \( f_0 \) have \( A_\mu \) types, \( \mu \in \mathbb{N} \) [7]. Topology of the \( D^0(M) \)-orbits was studied by S.I. Maksymenko [8] (allowing some other types of degenerate singularities) and by the author [4, 5, 7] (for \( A_\mu \)-singularities).

For any function \( f \in \mathcal{F} \), consider the set \( C_f := \{ P \in M \mid df(P) = 0 \} \) of its critical points. These critical points form five classes of topological equivalence (some classes may be empty):

\[
C_f^{\min} = \bigcup_{i \geq 1} A^{+,+}_{2i-1}(f), \quad C_f^{\max} = \bigcup_{i \geq 1} A^{+,-}_{2i-1}(f), \quad C_f^{\text{saddle}} = A^{-,-}_1(f) \cup \bigcup_{\eta = \pm} \bigcup_{i \geq 2} (A^{-,\eta}_{2i-1}(f) \cup D^\eta_{2i+1}(f)) \cup E^\eta_1(f),
\]

\[
C_f^{\text{triv}} = ( \bigcup_{i \geq 1, \eta = \pm} A^\eta_{2i}(f) ) \cup ( \bigcup_{i \geq 2} D^+_{2i}(f) ) \cup E^+_0(f) \cup E^-_0(f) \cup E^+_8(f) \cup E^-_8(f), \quad C_f^{\text{mult}} = \bigcup_{i \geq 2} D^-_{2i}(f),
\]

i.e. the critical points of local minima, local maxima, saddle points, quasi- and multysaddle points, respectively. Here \( A^{\pm,\pm}_\mu(f), D^\pm_\mu(f) \) and \( E^\pm_\mu(f) \) denote the corresponding subsets of critical points of \( A - D - E \) types. In the set \( C_f^{\text{extr}} := C_f^{\min} \cup C_f^{\max} \) of local extremum points, consider the subset \( C_f^{\text{extr}} \) of degenerate (non-Morse) critical points.

Denote \( s := \max\{0, \chi(M) + 1\} \).

**Theorem 1.** For any function \( f_0 \in C^\infty(M) \), whose all critical points have \( A - D - E \) types (e.g. a Morse function), the space \( \mathcal{F} = \mathcal{F}(f_0) \) has the homotopy type of a manifold \( M_s = M_s(f_0) \) having dimension \( \dim M_s = 2s + |C_{f_0}| + |C_{f_0}^{\text{extr}}| + |C_{f_0}^{\text{triv}}| + 2|C_{f_0}^{\text{saddle}}| + 3|C_{f_0}^{\text{mult}}| \). Moreover:

(a) There exists a surjective submersion \( \kappa : \mathcal{F} \to M_s \) and a stratification [11] (respectively, a fibration of codimension \( |C_{f_0}| \)) on \( M_s \) such that every \( D^0(\mathbb{R}) \times D^0(M) \)-orbit (resp., \( D^0(M) \)-orbit) in \( \mathcal{F} \) is the \( \kappa \)-preimage of a stratum (resp., a fiber) in \( M_s \).

(b) The map \( \kappa \) provides a homotopy equivalence between every \( D^0(M) \)-invariant subset \( I \subseteq \mathcal{F} \) and its image \( \kappa(I) \subseteq M_s \). In particular, it provides homotopy
Denote by \( \pi_*(\mathcal{F}) \) the space of all functions \( f \in \mathcal{F} \) whose all local extrema equal \( \pm F \). Then an analogue of Theorem 1 holds when \( \mathcal{F} \) and the corresponding stratum (resp., fiber) in \( \mathcal{M}_s \).

In particular, \( \pi_k(\mathcal{F}) \cong \pi_k(\mathcal{M}_s) \), \( H_k(\mathcal{F}) \cong H_k(\mathcal{M}_s) \). Thus \( H_k(\mathcal{F}) = 0 \) for all \( k > \dim \mathcal{M}_s \).

**Remark 1.** Denote by \( \mathcal{F}^1 = \mathcal{F}^1(f_0) \) the space of all functions \( f \in \mathcal{F} \) whose all local extrema equal \( \pm F \) and the sum of values at all non-extremum critical points vanishes. Then an analogue of Theorem 1 holds when \( \mathcal{F} \) and \( \mathcal{M}_s \) are replaced by \( \mathcal{F}^1 \) and a submanifold \( \mathcal{M}_s^1 \subset \mathcal{M}_s \), respectively, where \( \mathcal{M}_s^1 \) is a union of fibres of \( \mathcal{M}_s \), \( \dim \mathcal{M}_s^1 = \dim \mathcal{M}_s - |C_{f_0}^{\text{extr}}| - 1 = 2s + |C_{f_0}^{\text{extr}}| + 2|C_{f_0}^{\text{triv}}| + 3|C_{f_0}^{\text{saddle}}| + 4|C_{f_0}^{\text{mult}}| - 1 \). Actually, \( \mathcal{M}_s^1 \) is a strong deformation retract of \( \mathcal{M}_s \), so it is homotopy equivalent to \( \mathcal{M}_s \).

Our proof of Theorem 1 uses results [2, 9] about a “uniform” reduction of a smooth function to a normal form near its critical points.

## 2 Morse flows and gradient-like flows on surfaces

Suppose \( \Omega \in \Lambda^n(M) \) is a volume form on a \( n \)-manifold \( M = M^n \). Let \( \mathcal{P} \subset M \) be a finite subset. For any vector field \( \xi \) on \( M' := M \setminus \mathcal{P} \), we assign the \((n - 1)\)-form \( \beta = i_{\xi} \Omega \in \Lambda^{n-1}(M') \). Clearly, this assignment is one-to-one, and \( \xi \in \operatorname{Ker} \beta \).

Furthermore, the flow of the vector field \( \xi \) is volume-preserving if and only if \( \beta \) is a closed form. Indeed: the Lie derivative \( L_{\xi} \Omega = (i_{\xi}d + di_{\xi})\Omega = di_{\xi} \Omega = d\beta \), so the Lie derivative vanishes if and only if \( d\beta = 0 \). By abusing language, we will call the \((n - 1)\)-form \( \beta \) a flow.

Suppose now that \( n = \dim M = 2 \). Denote \( \mathcal{Z}_\beta := \{ \rho \in M' \mid \beta(\rho) = 0 \} \).

A closed 1-form \( \beta \) on \( M' = M \setminus \mathcal{P} \) will be called a Morse flow on \( M \) if, in a neighbourhood of every point \( \rho \in \mathcal{P} \cup \mathcal{Z}_\beta \), there exist local coordinates \( x, y \) such that either \( \beta = d(2xy) = d(\operatorname{Im}(z^2)) \) and \( \rho \in \mathcal{Z}_\beta \), or \( \beta = \pm(xdy - ydx)/(x^2 + y^2) = \pm d(\operatorname{Im}(\ln z)) \) and \( \rho \in \mathcal{P} \), where \( z = x + iy \). Geometrically, the set \( \mathcal{P}_\beta := \mathcal{P} \) consists of sources and sinks of the flow \( \beta \), while the set \( \mathcal{Z}_\beta \) consists of saddle points of the flow \( \beta \).

A closed 1-form \( \beta \) on \( M' = M \setminus \mathcal{P} \) will be called a gradient-like flow on \( M \) if there exists a Morse function \( f \in C^\infty(M) \), called an energy function of \( \beta \), such that

(i) the set \( \mathcal{P} \) coincides with the set of local extremum points of \( f \),

(ii) the 2-form \( df \wedge \beta|_{M \setminus \mathcal{P}} \) has no zeros and defines a positive orientation on \( M \),
(iii) in a neighbourhood of every point \( P \in C_f \), there exist local coordinates \( x, y \) such that either
\[
f = f(P) + x^2 - y^2, \quad \beta = 2xy \quad \text{and} \quad P \in Z = C_f \setminus \mathcal{P},
\]
or
\[
f = f(P) \pm (x^2 + y^2), \quad \beta = (x dy - y dx)/(x^2 + y^2) \quad \text{and} \quad P \in \mathcal{P}.
\]

Geometrically, the set \( \mathcal{P}_\beta := \mathcal{P} \) of sources and sinks of the flow \( \beta \) coincides with the set of local extremum points of the energy function \( f \), while the set \( Z_\beta := Z = C_f \setminus \mathcal{P} \) of saddle points of the flow \( \beta \) coincides with the set of saddle critical points of \( f \).

Let \( \beta_0 \) be a Morse flow on \( M \). Consider the set \( \mathcal{B} = \mathcal{B}(\beta_0) \) of gradient-like flows \( \beta \) having the same types of local singularities as \( \beta_0 \) (in particular, \(|Z_\beta| = |Z_{\beta_0}| \) and \(|\mathcal{P}_\beta| = |\mathcal{P}_{\beta_0}|\)).

First of all, let us characterize gradient-like flows among all 2D Morse flows. The following theorem is similar to a result by S. Smale characterizing gradient-like flows among all Morse-Smale flows [10].

**Theorem 2** (Characterization of 2D gradient-like flows). Let \( \beta_0 \) be a Morse flow on \( M \). Then:

(a) The space \( \mathcal{B} = \mathcal{B}(\beta_0) \) of gradient-like flows is nonempty if and only if \( \beta_0 \) has at least one sink and at least one source.

(b) A Morse flow \( \beta \) is gradient-like if and only if

(i) \( \beta \) has at least one sink and at least one source,

(ii) every separatrix of \( \beta \) has two endpoints belonging to \( Z_\beta \cup \mathcal{P}_\beta \), and

(iii) there is no an oriented closed curve \( P_1 P_2 \ldots P_{k-1} P_k \) (\( k \geq 2 \)) formed by oriented separatrices of \( \beta \), where \( P_i \in Z_\beta \) and \( P_k = P_1 \).

(c) The space \( \mathcal{B} \) of gradient-like flows is open in the space of Morse flows and is \( \mathcal{D}(M) \)-invariant. The orbit space \( \mathcal{B}_{num}/\mathcal{D}^0(M) \) is a \( 2|Z_{\beta_0}| \)-dimensional manifold, where \( \mathcal{B}_{num} \) is the space of gradient-like flows with enumerated sinks and sources.

3 Spaces of gradient-like flows on surfaces

We want to describe the topology of the space \( \mathcal{B} = \mathcal{B}(\beta_0) \), equipped with the \( C^\infty \)-topology, and its decomposition into \( \mathcal{D}^0(M) \)-orbits and into classes of (orbital) topological equivalence.
Theorem 3. For any gradient-like flow $\beta_0$ on $M$, the space $\mathcal{B} = \mathcal{B}(\beta_0)$ has the homotopy type of the manifold $\mathcal{M}^1_s = \mathcal{M}^1_s(f_0)$ from Theorem 1 and Remark 1, where $f_0$ is an energy function of $\beta_0$. Moreover:

(a) There exists a surjective submersion $\lambda: \mathcal{B} \to \mathcal{M}^1_s$, a stratification and a $(|Z_{\beta_0}| + 2s - 1)$-dimensional fibration on $\mathcal{M}^1_s$ such that every class of orbital topological equivalence (resp., $D^0(M)$-orbit) in $\mathcal{B}$ is the $\lambda$-preimage of a stratum (resp., a fibre) from $\mathcal{M}^1_s$.

(b) The map $\lambda$ provides a homotopy equivalence between every $D^0(M)$-invariant subset $I \subseteq \mathcal{B}$ and its image $\lambda(I) \subseteq \mathcal{M}^1_s$. In particular, it provides a homotopy equivalence between every class of topological equivalence (resp., $D^0(M)$-orbit) in $\mathcal{B}$ and the corresponding stratum (resp., fibre) in $\mathcal{M}^1_s$.

(c) All fibres and strata in $\mathcal{M}^1_s$ (and, thus, all classes of topological equivalence and all $D^0(M)$-orbits in $\mathcal{B}$) are homotopy equivalent either to a point, or to $T^2$, or to $SO(3)/G$ or to $S^2$, in dependence on whether $\chi(M) < 0$, or $\chi(M) = 0$, or $\chi(M) \cdot |Z_{\beta_0}| > 0$, or $\chi(M) > 0$ and $|Z_{\beta_0}| = 0$, respectively, where $G$ is a finite subgroup of $SO(3)$.

In particular, $\pi_k(\mathcal{B}) \cong \pi_k(\mathcal{M}^1_s)$, $H_k(\mathcal{B}) \cong H_k(\mathcal{M}^1_s)$. Thus $H_k(\mathcal{B}) = 0$ for all $k > \dim \mathcal{M}^1_s$.

Remark 2. The fibrations on the manifold $\mathcal{M}^1_s$ in Theorem 1 and Theorem 3 are transversal to each other, and intersections of their fibres are $2s$-dimensional submanifolds diffeomorphic to the space $M^s \setminus \Delta$ of $s$-point configurations on $M$, where $\Delta = \cup \Delta_{ij}$, $\Delta_{ij} = \{(P_1, \ldots, P_s) \in M^s \mid P_i = P_j\}$. Consider the topological space obtained from $\mathcal{M}^1_s$ by contracting each such a $2s$-dimensional submanifold to a point. This space is known as the universal moduli space of real-normalized meromorphic 1-forms on $M$ [1].

4 Describing the classifying manifolds $\mathcal{M}_s$ and $\mathcal{M}^1_s$

The manifold $\mathcal{M}_s = \mathcal{M}_s(\beta_0)$ from Theorem 1 can be constructed as follows.

Let us consider the topological spaces

$$\mathcal{F} := \{(f, \beta) \in \mathcal{F} \times \mathcal{B} \mid f \text{ is an energy function of } \beta\}, \quad \mathcal{F}^1 := \{(f, \beta) \in \mathcal{F} \mid f \in \mathcal{F}^1\}$$

endowed with $C^\infty$ topology [3].

Let us fix a $s$-point subset $N_s \subset M$, $|N_s| = s$. Consider the subgroup $D_s(M) := \{\phi \in \mathcal{D}(M) \mid N_s \subseteq \text{Fix}(\phi)\}$ of the group $\mathcal{D}(M)$ endowed with $C^\infty$ topology. Denote by
\( \mathcal{D}_s^0(M) \) the identity path component of the group \( \mathcal{D}_s(M) \). Define the moduli spaces

\[
\mathcal{M}_s = \mathcal{M}_s(f_0) := \mathbb{F}/\mathcal{D}_s^0(M), \quad \mathcal{M}_s^1 = \mathcal{M}_s^1(f_0) := \mathbb{F}^1/\mathcal{D}_s^0(M)
\]

endowed with quotient topology.

One can show that \( \mathcal{M}_s^1 \) is a strong deformation retract of \( \mathcal{M}_s \). Furthermore, by using a “uniform” reduction of a smooth function to a normal form near its critical points \([2, 9]\), one can prove that the forgetful maps

\[
\text{Forg}_1 : \mathbb{F} \to \mathcal{F}, \quad \text{Forg}_1|_{\mathbb{F}^1} : \mathbb{F}^1 \to \mathcal{F}^1, \quad \text{Forg}_2 : \mathbb{F}^1 \to \mathcal{B}
\]

are homotopy equivalences (cf. \([3]\) for \( \mathcal{F} \) and Morse singularities, for other singularity types the proof is similar).

One can show that the group \( \mathcal{D}_s^0(M) \) acts freely on \( \mathbb{F} \). Since this group is contractible, we have homeomorphisms

\[
\mathbb{F} \approx \mathcal{D}_s^0(M) \times \mathcal{M}_s, \quad \mathbb{F}^1 \approx \mathcal{D}_s^0(M) \times \mathcal{M}_s^1.
\]

Therefore, the projections

\[
\mathbb{F} \to \mathcal{M}_s, \quad \mathbb{F}^1 \to \mathcal{M}_s^1
\]  

are homotopy equivalences. It is easy to show that \( \mathcal{M}_s \) is a smooth manifold equipped with two transversal fibrations (whose fibres will be called “horizontal” and “vertical”, respectively), and \( \mathcal{M}_s^1 \) is its submanifold consisting of horizontal fibres. Namely, each horizontal fibre is the Forg\(_1\)-preimage of a \( \mathcal{D}_s^0(M) \)-orbit in \( \mathcal{F} \), while each vertical fibre is the Forg\(_2\)-preimage of a \( \mathcal{D}_s^0(M) \)-orbit in \( \mathcal{B} \).

Now, for proving Theorem 1 and Remark 1 one should consider the manifolds \( \mathcal{M}_s \) and \( \mathcal{M}_s^1 \) fibred by the horizontal fibres. For proving Theorem 3 one should consider the manifold \( \mathcal{M}_s^1 \) fibred by the vertical fibres. Then the theorems 1 and 3 follow from the homotopy equivalences (1) and (2).

Each horizontal fibre (from Theorem 1 and Remark 1) and each vertical fibre (from Theorem 3) on the manifold \( \mathcal{M}_s^1 \) are transversal to each other, and their intersection is a 2\( s \)-dimensional submanifold diffeomorphic to \( M^s \setminus \Delta \), the \( s \)-point configuration space on \( M \), where \( \Delta = \cup \Delta_{ij}, \Delta_{ij} = \{(P_1, \ldots, P_s) \in M^s \mid P_i = P_j\} \).

**Remark 3.** Let us consider the moduli spaces

\[
\mathcal{M} = \mathcal{M}(f_0) := \mathbb{F}/\mathcal{D}_s^0(M), \quad \mathcal{M}^1 = \mathcal{M}^1(f_0) := \mathbb{F}^1/\mathcal{D}_s^0(M)
\]

endowed with quotient topology. The space \( \mathcal{M}^1 \) is known as the universal moduli space of real-normalized meromorphic differentials (or meromorphic 1-forms) on \( M \). If \( \chi(M) < 0 \) then \( s = 0, \mathcal{M}_s = \mathcal{M}_0 = \mathcal{M} \) and \( \mathcal{M}_s^1 = \mathcal{M}_0^1 = \mathcal{M}^1 \), so \( \mathcal{M} \) and \( \mathcal{M}_s^1 \) are manifolds. If \( \chi(M) \geq 0 \), then \( \mathcal{M} \) and \( \mathcal{M}^1 \) are orbifolds in general, which can
be obtained from the manifolds $\mathcal{M}_s$ and $\mathcal{M}_s^1$ (resp.) by contracting the intersection of each horizontal fibre with each vertical fibre to a point.

Suppose that at least $s$ critical points of the function $f_0$ are enumerated (e.g., $\chi(M) < 0$ or the singularity type of each of these $s$ points is different from the singularity type of any other critical point of $f_0$). Then:

- The orbifolds $\mathcal{M}$ and $\mathcal{M}^1$ are in fact manifolds, and there exist homeomorphisms
  $$\mathcal{M}_s \approx (M^s \setminus \Delta) \times \mathcal{M}, \quad \mathcal{M}_s^1 \approx (M^s \setminus \Delta) \times \mathcal{M}^1$$
  (cf. [5, 6]). We remark that $M^s \setminus \Delta$ is homotopy equivalent to $D^0(M)$, which has the homotopy type of either a point or $T^2$ or $SO(3)$ in dependence on whether $\chi(M) < 0$ or $\chi(M) = 0$ or $\chi(M) > 0$.

- Each of the manifolds $\mathcal{M}$ and $\mathcal{M}^1$ (more precisely, their strong deformation retracts $\mathcal{K}$ and $\mathcal{K}^1$) is a “skew cylindric-polyhedral complex”, i.e. it can be represented as the union of “skew cylindric handles” glued to each other in a nice way [5, 6]. The skew cylindric handles of the manifold $\mathcal{M}$ (resp., $\mathcal{M}^1$) are in one-to-one correspondence with the $D^0(\mathbb{R}) \times D^0(M)$-orbits in the space $\mathcal{F}$ (resp., $\mathcal{F}^1$). The Morse index of each handle equals the codimension of the corresponding $D^0(\mathbb{R}) \times D^0(M)$-orbit in the space $\mathcal{F}$ (resp., $\mathcal{F}^1$).

- Each skew cylindric handle of the manifold $\mathcal{M}$ (resp., $\mathcal{M}^1$) is incompressible, i.e. the inclusion mapping of the handle into the manifold $\mathcal{M}$ (resp., $\mathcal{M}^1$) induces a monomorphism of the fundamental groups.

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