A POTENTIAL THEORY FOR THE $k$-CURVATURE EQUATION

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Abstract. In this paper, we introduce a potential theory for the $k$-curvature equation, which can also be seen as a PDE approach to curvature measures. We assign a measure to a bounded, upper semicontinuous function which is strictly subharmonic with respect to the $k$-curvature operator, and establish the weak continuity of the measure.

1. Introduction

The potential theory for nonlinear elliptic equations has been extensively studied. For quasilinear elliptic equations, we refer the reader to [24]. For the complex Monge-Ampère equation, the weak convergence for bounded, monotone pluri-subharmonic functions was established in [5, 6]. See [9, 10, 28, 29, 3] for further development. For the $k$-Hessian equations, analogous but stronger results were obtained in [40] (see also [42]).

In a recent paper [16], the mean curvature measure was established for upper semicontinuous functions which are subharmonic with respect to the mean curvature operator, which is a notion weaker than the generalized solution studied by Giusti [21, 22]. The purpose of this paper is to establish the $k$-curvature measures, where $1 < k < n$, for upper semicontinuous functions which are subharmonic with respect to the $k$-curvature operator, as well as the weak convergence of measures.

For a function $u \in C^{2}(\Omega)$, where $\Omega$ is a bounded domain in $\mathbb{R}^n$, the $k$-curvature of the graph of $u$ is the $k^{th}$ elementary symmetric polynomial of the principal curvatures $(\kappa_1, \cdots, \kappa_n)$ of the graph of $u$. It can also be written as

$$H_k[u] = \sigma_k\left(D\left(\frac{Du}{w}\right)\right),$$

where $w = \sqrt{1 + |Du|^2}$, and $\sigma_k$ denotes the sum of the principal minors of order $k$ of the matrix $D\left(\frac{Du}{w}\right)$. In particular, $H_1$ is the mean curvature and $H_n$ is Gauss curvature.

1991 Mathematics Subject Classification. Primary 35J60; Secondary 35D40.

Key words and phrases. Curvature measure, subharmonic functions, weak continuity.

The first author was supported by NNSFC 11271120. The second author was supported by ARC DP120102718 and ARC FL1301000118. The third author was supported by ARC DECRA and NNSFC 11101004.
We say a function $u$ in $C^2(\Omega)$ is $H_k$-subharmonic if $H_j[u] \geq 0$ for $j = 1, \cdots, k$, namely if the principal curvatures $\lambda = (\lambda_1, \cdots, \lambda_n)$ of the graph of $u$ lies in $\overline{\Gamma}_k$, the closure of the convex cone

\begin{equation}
\Gamma_k = \{ \lambda \in \mathbb{R}^n \mid \sigma_j(\lambda) > 0, j = 1, \cdots, k \}.
\end{equation}

This $H_k$-subharmonicity is equivalent to the $H_k$-admissibility in [5]. We use the terminology subharmonicity, instead of admissibility, to reflect the following extension to nonsmooth function in the viscosity sense, as in [40, 16]. Namely an upper semicontinuous function $u : \Omega \to [-\infty, \infty)$ is called subharmonic with respect to the $k$-curvature operator $H_k$, or $H_k$-subharmonic for short, if the set $\{ u = -\infty \}$ has measure zero and for any open set $\omega \subset \Omega$ and any smooth function $h \in C^2(\overline{\omega})$ with $H_k[h] \leq 0$, $h \geq u$ on $\partial \omega$, one has $h \geq u$ in $\omega$. This is equivalent to the inequality $H_k[u] \geq 0$ holding in the viscosity sense [37]. We denote the set of all $H_k$-subharmonic functions on $\Omega$ by $SH_k(\Omega)$. We say $u \in SH_k(\Omega)$ is strictly $H_k$-subharmonic if $h > u$ for any $\omega \subset \Omega$ and any continuous function $h$ with $H_k[h] = 0$ in the viscosity sense, $h \geq u$ on $\partial \omega$.

The main result of the paper is the following

**Theorem 1.1.** (i) For any bounded strictly $H_k$-subharmonic function $u$, there exists an associated Radon measure $\mu_k[u]$, such that $H_k[u]$ is the density function of $\mu_k[u]$ if $u \in C^2$.

(ii) If $\{ u_j \}$ is a sequence of bounded strictly $H_k$-subharmonic functions which converges to a bounded strictly $H_k$-subharmonic $u$ a.e., then $\mu_k[u_j] \to \mu_k[u]$ weakly.

The notion of curvature measures in Theorem 1.1 is closely related to that of Federer’s [20]. The curvature measures in Theorem 1.1 are defined in the domain $\Omega$ while Federer’s are defined in the space $\mathbb{R}^{n+1}$, supported on the graph of the function. But they are essentially the same as the curvature measures in Theorem 1.1 are projections of Federer’s from $\mathbb{R}^{n+1}$ to $\mathbb{R}^n$. When $k = n$, the $k$-curvature is the Gauss curvature and the corresponding curvature measure has been extensively studied [34].

Federer’s curvature measures apply to general sets in the Euclidean space $\mathbb{R}^{n+1}$ of positive reach. If the boundary of the set is the graph of a function, the positive reach condition means the function is semi-convex, which is a condition different from the $H_k$-subharmonicity in Theorem 1.1.

The weak convergence in part (ii) is a stability of the curvature measures $\mu_k[u]$, as the convergence is independent of the sequence $\{ u_j \}$. It also allows us to assign a measure to a bounded strictly $H_k$-subharmonic function $u$. The stability of curvature measures is useful in the numerical computation of curvature measures, see [11, 12].
For the mean curvature operator \( k = 1 \), this result was established in [16]. In this paper, we will use some ideas from [16] and [24], such as the Perron lifting in §3. The argument in §5 was also inspired by that in [16]. However, new difficulties arise in treating the cases \( 1 < k < n \). One is that the operator \( H_k \) is fully nonlinear and there is no interior regularity for the equation, as with the case of the complex Monge-Ampère equation. But the main obstacle is that the set of \( H_k \)-subharmonic functions is not convex and so we cannot use the perturbation or mollification techniques. For example, if \( u \) is \( H_k \)-subharmonic, \( u + \varepsilon \phi \) may fail to be \( H_k \)-subharmonic even if \( \phi \) is convex. An simple example is: letting \( u \) be a smooth \( H_1[u] \)-subharmonic function in \( \mathbb{R}^2 \) satisfying \( Du(0) = 0 \), \( u_{xx}(0) = -1 \), \( u_{yy}(0) = 1 \) and \( u_{xy}(0) = 0 \). Then one can check that \( H_1[u + \varepsilon y] < 0 \) at 0.

In this paper we introduce a deformation technique, by studying the Dirichlet problem of an associated obstacle problem. This deformation argument provides an approximation to a nonsmooth \( H_k \)-subharmonic function by smooth \( H_k \)-subharmonic functions of positive \( k \)-curvature. This is a key ingredient in the argument of the paper. This technique can also be used to give a new proof of the global regularization of pluri-subharmonic functions on compact Kähler manifolds. Namely a pluri-subharmonic function on a compact Kähler manifold can be approximated by smooth ones (see Appendix 2). Also because we cannot use the perturbation technique, we have to assume the strict \( H_k \)-subharmonicity condition in Theorem 1.1.

The organisation of the paper is as follows. In §2, we prove a monotonicity inequality for the \( k \)-curvature operators. The monotonicity for \( k \)-Hessian operators can be obtained by a simple integration by parts [39]. But for the \( k \)-curvature operators, the computation is much more complicated.

In §3 we introduce the Perron lifting. In §4 we prove that every \( H_k \)-subharmonic function can be approximated locally by a sequence of smooth \( H_k \)-subharmonic functions. The approximation for \( H_1 \)-subharmonic functions was obtained in [16] by Perron lifting. This technique does not work when \( k > 1 \), due to the lack of interior regularity of the equation \( H_k[u] = 0 \). In this paper we prove the smooth approximation by solving a sequence of Dirichlet problems with obstacles and introducing the notion of \( H_k \)-subharmonic envelope.

In §5 we prove Theorem 1.1, by making a perturbation for a sequence of \( H_k \)-subharmonic functions in an annulus, and using the monotonicity formula established in §2. To keep the \( H_k \)-subharmonicity under the perturbation, we need the assumption of the strict \( H_k \)-subharmonicity in Theorem 1.1.
In Appendix I we prove the existence of solutions to the Dirichlet problems with obstacle, which is needed in Section 4. As an application we prove in Appendix II that a pluri-subharmonic function on a compact Kähler manifold can be approximated by smooth ones.

2. An integral monotonicity inequality

Denote \( A = D \left( \frac{Du}{w} \right) = (a_{ij}) \), where

\[
(2.1) \quad a_{ij} = \frac{u_{ij}}{w} - \frac{u_{il}u_lu_j}{w^3}.
\]

Following [8], it can be written as

\[
(2.2) \quad a_{ij} = \frac{1}{w} u_{ip} b^{pq} b_{qj},
\]

where \( b^{ij} = \delta_{ij} - \frac{u_iu_j}{w(1+w)} \). We have [37]

\[
(2.3) \quad H_k[u] = \frac{1}{k!} \sum \left( \begin{array}{cccc}
  i_1 & \cdots & i_k \\
  j_1 & \cdots & j_k
\end{array} \right) a_{i_1j_1} \cdots a_{i_kj_k}.
\]

By proposition 4.1 in [33], \( H_k \) can be written in the divergence form

\[
(2.4) \quad H_k[u] = \frac{1}{k} \left( H^{ij} \frac{u_j}{w} \right)_i,
\]

where a subscript of a function denotes partial derivative, namely \( u_j = u_{x_j}, u_{ij} = u_{x_i x_j} \), and

\[
(2.5) \quad H^{ij} = \frac{\partial H_k}{\partial a_{ij}} = \frac{1}{(k-1)!} \sum \left( \begin{array}{cccc}
  i_1 & \cdots & i_{k-1} & i \\
  j_1 & \cdots & j_{k-1} & j
\end{array} \right) a_{i_1j_1} \cdots a_{i_{k-1}j_{k-1}}.
\]

By the divergence theorem,

\[
\int_{\Omega} H_k[u] = \int_{\partial \Omega} X_u \cdot \gamma,
\]

where \( \gamma \) is the unit outer normal of \( \partial \Omega \), and \( X_u = (H^{1j} u_j, \cdots, H^{nj} u_j) \).

**Lemma 2.1.** Suppose \( u, v \in C^2(\Omega) \). If \( u = v \) and \( Du = Dv \) on \( \partial \Omega \), then

\[
(2.6) \quad \int_{\Omega} H_k[u] = \int_{\Omega} H_k[v].
\]

**Proof.** First, we show that \( X_u \cdot \gamma \) is invariant under orthogonal transformation. For convenience, by (2.2), we rewrite \( H^{ij} \) as

\[
H^{ij} = w \frac{\partial H_k}{\partial u_{ip}} b^{pq} b_{qj},
\]
where \( b_{ij} = \delta_{ij} + \frac{u_i u_j}{1 + w} \). Let \( x = Py \) and denote \( \tilde{u}(y) = u(x) \), where \( P = (p_{ij}) \) is an orthogonal matrix. Then

\[
\tilde{w} = w, \\
\tilde{b}_{ij} = \delta_{ij} + \frac{u_i u_j}{1 + w}, \\
\frac{\partial H_k[\tilde{u}]}{\partial u_{ij}} = \frac{\partial H_k}{\partial u_{ij}} p_{ij}.
\]

Note that \( \tilde{u}_{yi} = u_{x_{\alpha} p_{\alpha i}} \) and \( \tilde{\gamma}_{\alpha} = p_{\alpha \gamma} \). Hence,

\[
X_{\tilde{u}} \cdot \tilde{\gamma} = X_{\tilde{u}} \cdot \tilde{\gamma} = \tilde{H}^{ij} \tilde{u}_{ji} \tilde{u}_{yi} \gamma_{\alpha}.
\]

Now for any given point \( p \in \partial \Omega \), by a translation and rotation of the coordinates, we may assume that \( p \) is the origin and locally \( \partial \Omega \) is given by \( x_n = \rho(x') \) such that the inner normal of \( \partial \Omega \) is \((0, \cdots, 0, 1)\) at \( p \). By \( u = v \) on \( \partial \Omega \), we have

\[
(2.7) \quad u_{ij} + u_n p_{ij} = v_{ij} + v_n p_{ij}, \quad 1 \leq i, j \leq n - 1.
\]

As \( Du = Dv \) on \( \partial \Omega \), we have \( u_{ij} = v_{ij} \) and \( u_n = v_n \) at \( p \) for \( i, j < n \). It is clear that

\[
X_{\tilde{u}} \cdot \gamma = - \sum \frac{\partial H_k}{\partial u_{n i}} (\delta_{ip} + \frac{u_i u_p}{1 + w}) \cdot (\delta_{pj} + \frac{u_j u_p}{1 + w}) \cdot \tilde{u}_j
\]

is independent of \( u_{nn} \) since \( \frac{\partial H_k}{\partial u_{n i}} \) are all independent of \( u_{nn} \). Hence \( X_{\tilde{u}} \cdot \gamma = X_{\tilde{v}} \cdot \gamma \) at \( p \). Since \( p \) is an arbitrary point on \( \partial \Omega \), the lemma follows. \( \square \)

**Lemma 2.2.** We have

\[
(2.8) \quad \frac{\partial H_k}{\partial u_{nn}} = \frac{1}{k!} \frac{\tilde{w}^{k+1}}{w^{k+2}} \sigma_{k-1}(\bar{A})
\]

\[
= \frac{1}{k!} \frac{\tilde{w}^{k+1}}{w^{k+2}} \sum \left( \begin{array}{c} i_1 \\ \vdots \\ i_{k-1} \\ j_1 \\ \vdots \\ j_{k-1} \end{array} \right) \bar{a}_{i_1 j_1} \cdots \bar{a}_{i_{k-1} j_{k-1}}.
\]
where \( A = (\bar{a}_{ij}) \),
\[
\bar{a}_{ij} = \frac{u_{ij}}{\bar{w}} - \sum_{l=1}^{n-1} u_{il} u_{lj}, \quad 1 \leq i, j \leq n - 1,
\]
\[
\bar{w} = \sqrt{1 + \sum_{i=1}^{n-1} u_{i}^{2}}.
\]

**Proof.** By definition,
\[
\frac{\partial H_{k}}{\partial u_{nn}} = \frac{\partial H_{k}}{\partial a_{nn}} \cdot \left( \frac{1}{w} - \frac{u_{n}^{2}}{w^{3}} \right) + \sum_{j=1}^{n-1} \frac{\partial H_{k}}{\partial a_{nj}} \cdot \left( -\frac{u_{n} u_{j}}{w^{3}} \right). 
\]

Since \( \frac{\partial H_{k}}{\partial a_{nn}} \) depends only on \( \{a_{il}\}_{i \neq n} \) and \( \frac{\partial H_{k}}{\partial a_{nj}} \) depends only on \( \{a_{il}\}_{i \neq n, l \neq j} \), it is easy to see that \( \frac{\partial H_{k}}{\partial a_{nn}} \) is independent of \( u_{nn} \). We will further show that \( \frac{\partial H_{k}}{\partial u_{nn}} \) is independent of \( u_{ni} \) for any \( 1 \leq i \leq n - 1 \) and \( w^{k+2} \frac{\partial H_{k}}{\partial u_{nn}} \) is independent of \( u_{n} \).

First, for any \( 1 \leq i \leq n - 1 \),
\[
\frac{\partial^{2} H_{k}}{\partial u_{nn} \partial u_{ni}} = \sum_{j=1}^{n-1} \frac{\partial^{2} H_{k}}{\partial a_{nn} \partial a_{ij}} \cdot \left( -\frac{u_{n} u_{j}}{w^{3}} \right) \cdot \left( \frac{1}{w} - \frac{u_{n}^{2}}{w^{3}} \right) 
+ \sum_{j,l=1}^{n-1} \frac{\partial^{2} H_{k}}{\partial a_{nj} \partial a_{il}} \cdot \left( -\frac{u_{n} u_{j}}{w^{3}} \right) \cdot \left( \frac{1}{w} - \frac{u_{n}^{2}}{w^{3}} \right) 
+ \sum_{j=1}^{n-1} \frac{\partial^{2} H_{k}}{\partial a_{nj} \partial a_{in}} \cdot \left( -\frac{u_{n} u_{j}}{w^{3}} \right) \cdot \left( \frac{1}{w} - \frac{u_{n}^{2}}{w^{3}} \right) 
= 0.
\]

Note that in the last equality we use the fact
\[
\frac{\partial^{2} H_{k}}{\partial a_{ip} \partial a_{jq}} = -\frac{\partial^{2} H_{k}}{\partial a_{iq} \partial a_{jp}}, \quad 1 \leq i, j, p, q \leq n.
\]

Next, we consider the dependence on \( u_{n} \). For convenience, we denote
\[
\tilde{a}_{ij} = w a_{ij} = u_{ij} - \frac{u_{i} u_{j}}{w^2}, \quad \tilde{H}_{k} = \sigma_{k} (\{\tilde{a}_{ij}\}) = w^{k} H_{k}.
\]

Then it follows
\[
\frac{\partial \tilde{a}_{ij}}{\partial u_{n}} = \begin{cases} 
0, & i = n, \\
\frac{2 u_{n}^{2} - u_{j}^{2}}{w^{3}} \sum_{l=1}^{n-1} u_{il} u_{lj}, & i < n, j = n, \\
-\frac{2 u_{n}^{2}}{w^{3}} \sum_{l=1}^{n-1} u_{il} u_{lj} u_{ln}, & i, j < n,
\end{cases}
\]
and
\[
w^{k+2} \frac{\partial H_{k}}{\partial u_{nn}} = \frac{\partial \tilde{H}_{k}}{\partial a_{nn}} w^{2} + \sum_{j=1}^{n-1} \frac{\partial \tilde{H}_{k}}{\partial a_{nj}} \cdot (-u_{n} u_{j}).
\]
Since $\frac{\partial H_k}{\partial a_{nn}}$ is independent of $u_{ni}$, we may assume $u_{ni} = 0$. Hence

$$
\frac{\partial}{\partial u_n} \left( w^{k+2} \frac{\partial H_k}{\partial u_{nn}} \right) = \sum_{i,j,l=1}^{n-1} \frac{\partial^2 \tilde{H}_k}{\partial a_{nn} \partial a_{ij}} \left( -u_j \right) + \sum_{l=1}^{n-1} \frac{\partial \tilde{H}_k}{\partial a_{nj}} \cdot \left( -u_l u_j \right) + \sum_{i,j=1}^{n-1} \frac{\partial \tilde{H}_k}{\partial a_{in}} \cdot \left( -u_l u_j \right) + \frac{n-1}{w^2} \sum_{l=1}^{n-1} u_{il} u_l u_j
$$

Again by the fact

$$
\frac{\partial^2 \tilde{H}_k}{\partial a_{ip} \partial a_{jq}} = -\frac{\partial^2 \tilde{H}_k}{\partial a_{iq} \partial a_{jp}}, \quad 1 \leq i, j, p, q \leq n,
$$

we have

$$
\frac{\partial}{\partial u_n} \left( w^{k+2} \frac{\partial H_k}{\partial u_{nn}} \right) = \sum_{j=1}^{n-1} \frac{\partial \tilde{H}_k}{\partial a_{nj}} \cdot \left( -u_j \right) + \sum_{i,j=1}^{n-1} \frac{\partial \tilde{H}_k}{\partial a_{nj} \partial a_{in}} \cdot \left( 1 - u_{il} u_l \right) \cdot \left( -u_n u_j \right) = \sum_{j=1}^{n-1} \frac{\partial \tilde{H}_k}{\partial a_{nj}} \cdot \left( -u_j \right) + \sum_{i,j=1}^{n-1} \frac{\partial \tilde{H}_k}{\partial a_{nj} \partial a_{in}} \cdot \left( -a_{in} \right) \cdot \left( -u_j \right) = 0.
$$

This implies that we may further assume $u_n = 0$ and the lemma follows by substituting $a_{ij} = \frac{w}{w} \tilde{a}_{ij}$ into $\frac{\partial H_k}{\partial u_{nn}} = \frac{\partial \tilde{H}_k}{\partial a_{nn}} w^{-1}$. $\square$

The following monotonicity integral inequality is the main result of this section. It is critical for the proof of weak continuity in Theorem 1.1. Note that in the following we do not assume the boundary $\partial \Omega$ is $(k - 1)$-convex. Due to the lack of convexity of the set of $H_k$-subharmonic functions, the computation is quite complicated.

**Lemma 2.3.** Suppose $u, v \in C^\infty(\Omega)$ are $H_k$-subharmonic functions. If $u = v$ and $u_\gamma > v_\gamma$ on $\partial \Omega$, then

$$
(2.9) \quad \int_\Omega H_k[u] \geq \int_\Omega H_k[v],
$$

where $\gamma$ is the unit outer normal of $\partial \Omega$.

**Proof.** By the divergence structure of the $k$-curvature operator $H_k$ [26, 27], the integral $\int_\Omega H_k[u]$ depends only on the value of $u$ near $\partial \Omega$. Hence by the condition $u = v$ and $u_\gamma > v_\gamma$ on $\partial \Omega$, we may also assume that $u < v$ in $\Omega$. This condition also implies that $\partial \Omega$ is smooth. Since the proof is complicated, we divide it into three steps:
For any \( p \in \partial \Omega \) and \( \delta > 0 \), denote by
\[
L_\delta(p) = \{ p + t\gamma \mid -\delta \leq t \leq 0 \},
\]
where \( \gamma \) is the unit outer normal. Assume \( \delta \) is small enough such that the set \( \{ p_\delta = p - \delta\gamma(p) \mid p \in \partial \Omega \} \) encloses a subdomain \( \Omega_\delta = \{ x \in \Omega : d(x, \partial \Omega) > \delta \} \subset \Omega \), i.e.,
\[
\Omega \setminus \Omega_\delta = \bigcup_{p \in \partial \Omega} L_\delta(p).
\]
We hope to construct a suitable function \( \tilde{u} = u + \eta \in C^2(\Omega \setminus \Omega_\delta) \), such that for some small constant \( c_\delta > 0 \),
\[
\tilde{u} = v - c_\delta, \nabla \tilde{u} = \nabla v \text{ on } \partial \Omega_\delta,
\]
\[
\tilde{u} = u, \nabla \tilde{u} = \nabla u \text{ on } \partial \Omega.
\]
Indeed we will get a sequence of functions that approximately satisfy the above condition, see (2.12), (2.13).

By the smoothness of \( u, v \), for any given \( \varepsilon > 0 \), we have
\[
|u_\gamma(x) - u_\gamma(p)|, \quad |v_\gamma(x) - v_\gamma(p)| \leq \varepsilon \quad \forall \ x \in L_\delta(p)
\]
provided \( \delta \) is small enough. Hence
\[
|v(p_\delta) - u(p_\delta) - (u_\gamma(p_\delta) - v_\gamma(p_\delta))\delta| \leq C\varepsilon\delta
\]
on \( \partial \Omega_\delta \). Let \( \lambda = \inf_{\partial \Omega} (u_\gamma - v_\gamma) > 0 \). For any \( p_\delta \in \partial \Omega_\delta \), denote
\[
(2.10) \quad a = v(p_\delta) - u(p_\delta) - \frac{\lambda \delta}{4} > 0, \quad b = u_\gamma(p_\delta) - v_\gamma(p_\delta) > 0.
\]
So we have \( v - \frac{\lambda \delta}{4} > u \) on \( \partial \Omega_\delta \) and \( v - \frac{\lambda \delta}{4} < u \) on \( \partial \Omega \). For each \( p \in \partial \Omega \), we aim to construct a function to connect \( (p_\delta, v(p_\delta) - \frac{\lambda \delta}{4}) \) and \( (p, u(p)) \), and whose first derivative coincides with \( v_\gamma(p_\delta), u_\gamma(p) \), respectively, and the second derivative is \( O(\delta^{-1}) \) as \( \delta \to 0 \) for our purpose. A function behaving like \( c_1 \frac{(-t)^{1+\alpha}}{\delta^\alpha} + c_2 \frac{t^2}{\delta} \) \( (0 < \alpha << 1) \) with \( c_1, c_2 \) to be determined would have an \( O(1) \) jump for the first derivative and an \( O(\delta^{-1}) \) jump for the second derivative.

Choose \( 0 < \alpha << 1 \) and \( \kappa << \delta \) such that
\[
(b - \frac{1}{4} \lambda)\delta < b \frac{\kappa + \delta}{1 + \alpha}.
\]
Define a function
\[
(2.11) \quad \eta = \eta_{\delta, \kappa} = s \frac{b}{(1 + \alpha)(\kappa + \delta)^\alpha} (-t + \kappa)^{1+\alpha} + (1 - s) \frac{b}{2\delta} t^2, \quad -\delta \leq t \leq 0
\]
on each line segment $L_\delta(p)$, where

$$s = \frac{a - \frac{\delta b}{2}}{\kappa + \delta} > 0.$$ 

Here the small perturbation by constant $\kappa$ ensures $\eta$ is smooth up to $\partial\Omega_\delta$. Note that $\Omega \setminus \Omega_\delta = \cup_{p \in \partial\Omega} L_\delta(p)$. We obtain a function $\bar{u} = u + \eta \in C^2(\Omega \setminus \Omega_\delta)$. Combining with (2.10), one can check that

(2.12) \hspace{1em} \bar{u} = v - \frac{\lambda\delta}{4}, D\bar{u} = Dv \hspace{1em} \text{on} \hspace{1em} \partial\Omega_\delta;

(2.13) \hspace{1em} \bar{u} = u + \frac{a - \frac{\delta b}{2}}{\kappa + \delta - \frac{(1 + \alpha)\delta}{2}} (\kappa + \delta)^{\alpha}, \bar{u}_\gamma = u_\gamma - \frac{a - \frac{\delta b}{2}}{\kappa + \delta - \frac{(1 + \alpha)\delta}{2}} \frac{(1 + \alpha)\kappa^\alpha}{(\kappa + \delta)^{\alpha}} \hspace{1em} \text{on} \hspace{1em} \partial\Omega.

By the divergence theorem,

(2.14) \hspace{1em} \int_{\partial\Omega} X\bar{u} \cdot \gamma = \int_{\partial\Omega_\delta} X\bar{u} \cdot \gamma + \int_{\Omega \setminus \Omega_\delta} H_k[\bar{u}].

By Lemma 2.1, the right hand side is

$$\int_{\Omega_\delta} H_k[v] + \int_{\Omega \setminus \Omega_\delta} H_k[\bar{u}]$$

while the left hand side converges to

$$\int_{\Omega} H_k[u]$$

when $\kappa \to 0$. It suffices to estimate the second term on the right.

(ii) In this step we estimate the derivatives of $\eta$ in $\Omega \setminus \Omega_\delta$.

For any $x_0 \in \Omega \setminus \Omega_\delta$, there exists $p$ on $\partial\Omega$ such that $x_0 \in L_\delta(p)$, i.e., $x_0 = p + t\gamma(p)$. By a translation and a rotation of the coordinates, we may assume that $p = 0$, $x_0 = (0, \ldots, 0, -\delta)$ and $\gamma(p) = (0, \ldots, 0, 1)$. Denote $\beta' = D_\gamma u(p)$, $\beta = D_\gamma v(p)$. Then near 0 we have

\[
\begin{align*}
    u &= \sum_{i=1}^{n-1} \alpha_i x_i + \beta' x_n + \sum_{i,j=1}^{n} \frac{A_{ij}}{2} x_i x_j + O(|x|^3), \\
    v &= \sum_{i=1}^{n-1} \alpha_i x_i + \beta x_n + \sum_{i,j=1}^{n} \frac{B_{ij}}{2} x_i x_j + O(|x|^3).
\end{align*}
\]

By the assumption of the lemma, $\beta' > \beta$ and near 0, by (2.7), $\partial\Omega$ is given by

(2.15) \hspace{1em} x_n = \rho(x') = \sum_{i,j=1}^{n-1} \frac{B_{ij} - A_{ij}}{2(\beta' - \beta)} x_i x_j + O(|x'|^3),
where \( x' = (x_1, \ldots, x_{n-1}) \). Then,

\[
(2.16) \quad \rho_i = \frac{B_{ij} - A_{ij}}{\beta' - \beta} x_j + O(|x'|^2).
\]

By (2.11), to estimate the derivatives of \( \eta_i \), we need to analyse \( t \) with respect to the nearby points of \( x_0 \). For any \( x = (x', x_n) \) near \( x_0 \), let \( p_x = (y', \rho(y')) \in \partial \Omega \) be the point such that \( x \in L_\delta(p_x) \). From

\[
-(\frac{D\rho}{\sqrt{1 + |D\rho|^2}} \cdot \frac{-1}{\sqrt{1 + |D\rho|^2}}) \cdot t + (y', \rho(y')) = (x', x_n),
\]

and (2.16), we have

\[
(2.17) \quad t = -(\rho(y') - x_n) \sqrt{1 + |D\rho|^2},
\]

\[
y_i = \left[ \frac{B_{ij} - A_{ij}}{\beta' - \beta} y_j + O(|y'|^2) \right] \cdot \left[ \frac{B_{ij} - A_{ij}}{2(\beta' - \beta)} y_i y_j - x_n + O(|y'|^3) \right] = x_i, 1 \leq i \leq n - 1.
\]

By (2.18),

\[
x_i = y_i + x_n \frac{B_{ij} - A_{ij}}{\beta' - \beta} x_j + O(|y'|^2).
\]

It follows

\[
(2.19) \quad y_i = x_i - x_n \frac{B_{ij} - A_{ij}}{\beta' - \beta} x_j + O(|x'|^2).
\]

Substituting it into \( t \), we have

\[
t = - \left[ \frac{B_{ij} - A_{ij}}{2(\beta' - \beta)} \left( x_i - x_n \frac{B_{ik} - A_{ik}}{\beta' - \beta} x_k \right) \left( x_j - x_n \frac{B_{jl} - A_{jl}}{\beta' - \beta} x_l \right) + O(|x'|^3) - x_n \right]
\]

\[
\cdot \left[ 1 + \left( \frac{B_{ij} - A_{ij}}{\beta' - \beta} x_j + O(|x'|^2) \right)^2 \right]^{\frac{3}{2}}
\]

\[
+ \left[ \frac{B_{ij} - A_{ij}}{2(\beta' - \beta)} x_i x_j - \left( \frac{B_{ik} - A_{ik}}{\beta' - \beta} x_k \right) \frac{B_{kj} - A_{kj}}{\beta' - \beta} x_n + O(|x_n|^2) \right] x_i x_j + O(|x'|^3) - x_n
\]

\[
\cdot \left[ 1 + \frac{1}{2} \frac{B_{ik} - A_{ik}}{\beta' - \beta} \frac{B_{jl} - A_{jl}}{\beta' - \beta} x_k x_l + O(|x'|^3) \right]
\]

\[
= x_n + \left[ \frac{B_{ij} - A_{ij}}{2(\beta' - \beta)} + \frac{3 B_{ik} - A_{ik}}{2} \frac{B_{kj} - A_{kj}}{\beta' - \beta} x_n + O(|x_n|^2) \right] x_i x_j + O(|x'|^3).
\]

At \( x_0 \), by (2.11),

\[
(2.20) \quad |\eta_k| \leq C\delta, \quad |\eta_{ni}| \leq C, \quad 1 \leq i \leq n - 1,
\]

\[
(2.21) \quad |\eta_n| \leq C, \quad \eta_{nn} = \frac{b(-t + \kappa)^{\alpha - 1}}{\delta + \kappa} + (1 - s) \frac{b}{\delta} \geq C\delta^{-1}.
\]
where $C$ is a constant depending on $u, v$ but independent of $\delta$. Since $\frac{\partial \eta}{\partial x_i} = 0$ at $x_0$ for $1 \leq i \leq n - 1$, by the expansion of $t$

\begin{equation}
(2.22) \quad \eta_{ij} = b \left[ s \frac{B_{ij} - A_{ij}}{\beta' - \beta} \cdot \left( \frac{-t + \kappa}{\delta + \kappa} \right)^\alpha - \frac{t}{\delta} \frac{B_{ij} - A_{ij}}{\beta' - \beta} (1 - s) \right] + O(\delta).
\end{equation}

Note that $b \to \beta' - \beta$ as $\delta \to 0$. By (2.22),

\begin{equation}
(2.23) \quad \tilde{u}_{ij} = u_{ij} + \eta_{ij}
\end{equation}

for $1 \leq i, j \leq n - 1$ as $\delta \to 0$.

(iii) In the final step, we estimate the integral

$$\int_{\Omega \setminus \Omega_\delta} H_k[\tilde{u}].$$

For any $x_0 \in \Omega \setminus \Omega_\delta$ and $x_0 \in L_\delta(p)$ for some $p$ on $\partial \Omega$. As in step (ii), we may assume that $p = 0$ and $\gamma(p) = (0, \cdots, 0, 1)$. We estimate $H_k[\tilde{u}]$ at $x_0$. Write

\begin{equation}
(2.24) \quad H_k[u] = \frac{\partial H_k}{\partial u_{nn}} u_{nn} + Q(\nabla u, u_{11}, \ldots, u_{ij}, \ldots, \hat{u}_{nn}).
\end{equation}

The symbol $\hat{u}_{nn}$ means $Q$ is independent of $u_{nn}$. Recall that $\frac{\partial H_k}{\partial u_{nn}}$ was given in Lemma 2.2. By (2.20), (2.21) and (2.23), the quantity $Q \geq -C_1$ for $\tilde{u}$, where $C_1 > 0$ is a constant depending on $u, v$ but independent of $\delta$.

Next we estimate $\frac{\partial H_k}{\partial u_{nn}}$ for $\tilde{u}$. For simplicity, we denote by

\begin{equation}
D_{ij} = [1 - \left( \frac{-t + \kappa}{\delta + \kappa} \right)^\alpha s - \frac{t}{\delta} (1 - s)] A_{ij} + \left[ \left( \frac{-t + \kappa}{\delta + \kappa} \right)^\alpha s + \frac{t}{\delta} (1 - s) \right] B_{ij}.
\end{equation}

and $\xi_i = u_i(p) = v_i(p), 1 \leq i \leq n - 1$. By the formula (2.8) in Lemma 2.2 and (2.23), there exists $C_\delta > 0$ such that

\begin{equation}
(2.25) \quad \frac{\partial H_k}{\partial u_{nn}} = \frac{1}{k!} \bar{w}^{k+1} \sum \left( \begin{array}{c}
i_1 \\
\vdots \\
i_{k-1} \\
\end{array} \begin{array}{c}j_1 \\
\vdots \\
j_{k-1} \\
\end{array} \right) \bar{a}_{i_1 j_1} \cdots \bar{a}_{i_{k-1} j_{k-1}} - C_\delta,
\end{equation}

for $\tilde{u}$ and $C_\delta \to 0$ as $\delta \to 0$, where

\begin{equation}
\bar{a}_{ij} = \frac{D_{ij}}{\bar{w}} - \frac{\sum_{i=1}^{n-1} D_i \xi_i_j}{\bar{w}^3}, \quad 1 \leq i, j \leq n - 1,
\end{equation}

\begin{equation}
\bar{w} = \sqrt{1 + \sum_{i=1}^{n-1} \xi^2_i},
\end{equation}

\begin{equation}
w = \sqrt{1 + \sum_{i=1}^{n} \xi^2_i}.
\end{equation}
By the $H_k$-subharmonicity of $u$, $v$, $\frac{\partial H_k}{\partial u_{nn}} \geq 0$ for $u$ and $v$. This implies the first term in (2.25) is nonnegative, i.e.,
$$\frac{\partial H_k}{\partial u_{nn}} \geq -C_\delta \to 0$$
for $\tilde{u}$.

Hence, by (2.21),
$$H_k[\tilde{u}] \geq -C_\delta \delta^{-1} - C_1.$$

at $x_0$, which implies
$$\int_{\Omega \setminus \Omega_\delta} H_k[\tilde{u}] \geq -C_\delta \frac{|\Omega \setminus \Omega_\delta|}{\delta} - C_1 |\Omega \setminus \Omega_\delta| \to 0$$
as $\delta \to 0$. Therefore sending $\delta \to 0$ in (2.14), we have
$$\int_\Omega H_k[u] \geq \int_\Omega H_k[v].$$

The following lemma is needed later to mollify a piecewise smooth $H_k$-subharmonic function. The proof is similar to that of Lemma 2.3. Note that for a smooth function defined on one side of a smooth hypersurface $\Gamma$, if it is smooth up to $\Gamma$, then one can extend it to the other side of $\Gamma$ by Taylor’s expansion in the normal bundle.

**Lemma 2.4.** Let $v, v'$ be two smooth $H_k$-subharmonic functions in $\Omega$. Let $u = \max\{v, v'\}$ and denote $\Gamma = \{x \in \Omega \mid v(x) = v'(x)\}$. Assume $\Gamma$ is a smooth hypersurface, $Dv \neq Dv'$ on $\Gamma$, and $H_k[v], H_k[v'] > 0$ near $\Gamma$. Then there is a sequence of smooth $H_k$-subharmonic functions $u_j$ which converges to $u$ and $u_j = u$ outside a small neighbourhood of $\Gamma$.

**Proof.** First, we show that $u$ can be approximated by $C^{1,1}$ smooth $H_k$-subharmonic functions. Denote $\omega = \{x \in \Omega \mid v < v'\}$. Then $\Gamma = \partial \omega$. For any $p \in \partial \omega$, denote $\beta' = D_\gamma v'(p)$, $\beta = D_\gamma v(p)$, where $\gamma$ is the unit normal of $\Gamma$. Since $Dv \neq Dv'$ on $\Gamma$, we have $\beta > \beta'$. We choose a smooth function $\eta(p)$ on $\partial \omega$ such that $0 < \eta < \beta - \beta'$ and $\eta = 0$ on $\partial \Gamma$. Now for $p \in \partial \omega$ and small $\epsilon > 0$, let
$$L(p, \epsilon) = \{p + t\gamma \mid -\epsilon \eta(p) \leq t \leq \epsilon \eta(p)\}, \quad \omega_\epsilon = \bigcup_{p \in \Gamma} L(p, \epsilon).$$

It is clear that any $x \in \omega_\epsilon$, there exists $p \in \partial \omega$ and $-\epsilon \eta \leq t \leq \epsilon \eta$ such that $x = p + t\gamma(p)$. Note that by the smoothness of $\Gamma$, $x$ corresponds to a unique $(p, t)$. Then we define
$$\varphi(x) = \frac{\beta(p) - \beta'(p)}{4\epsilon \eta(p)} (t + \epsilon \eta(p))^2$$

(2.26)
which is a smooth function in $\omega_\epsilon$, and

$$u^\epsilon(x) = \begin{cases} 
    u(x) + \varphi(x), & x \in \omega_\epsilon, -\epsilon \eta(p) \leq t \leq 0, \\
    u(x) + \varphi(x) - (\beta(p) - \beta'(p)) t, & x \in \omega_\epsilon, 0 \leq t \leq \epsilon \eta(p), \\
    u(x), & x \in \Omega \setminus \omega_\epsilon.
\end{cases}$$

(2.27)

It is obvious that $u^\epsilon \in C^{1,1}$. We show that $u^\epsilon$ is $H_k$-subharmonic in $\omega_\epsilon$ and converges to $u$ as $\epsilon \to 0$. For any $x_0 \in \omega_\epsilon$, we have $x_0 \in L(p, \epsilon)$ for some $p \in \Gamma$. As above, we assume that $p = 0$, and $\gamma(p) = (0, \cdots, 0, 1)$. Denote $A_{ij} = v_{ij}(p)$, $B_{ij} = v'_{ij}(p)$. Then near 0, $\Gamma$ is given by

$$x_n = \rho(x') = \sum_{i,j=1}^{n-1} B_{ij} - A_{ij} \frac{1}{2(\beta - \beta')} x_i x_j + O(|x'|^3).$$

(2.28)

As in the proof of Lemma 2.3, we have, at $x_0$,

$$|u_{i}^\epsilon - u_i| \leq C \epsilon, \; |u_{ni}^\epsilon - u_{ni}| \leq C, \; 1 \leq i \leq n - 1,$$

(2.29)

$$|u_{n}^\epsilon - u_n| \leq C, \; u_{nn}^\epsilon - u_{nn} \geq \frac{1}{2\epsilon'}$$

(2.30)

$$u_{ij}^\epsilon = (1 - t + \frac{\epsilon \eta}{2\epsilon'}) B_{ij} + \frac{t + \epsilon \eta}{2\epsilon'} A_{ij} + O(\epsilon), \; 1 \leq i, j \leq n - 1$$

(2.31)

for sufficiently small $\epsilon$. To estimate $H_k[u^\epsilon](x_0)$, we again use formula (2.24)

$$H_k[u] = \frac{\partial H_k}{\partial u_{nn}} u_{nn} + Q(\nabla u, u_{11}, \ldots, u_{ij}, \ldots, u_{nn}).$$

(2.32)

By (2.29)-(2.31), the quantity $Q \geq -C_1$, where $C_1$ is a positive constant independent of $\epsilon$. By the ellipticity assumption that $H_k[v], H_k[v'] > 0$ in the lemma, there exists $C_2 > 0$, such that $\frac{\partial H_k}{\partial u_{nn}} > C_2$ for $u$ and $v$. With (2.29)-(2.31), we have $\frac{\partial H_k}{\partial u_{nn}} > \frac{C_2}{2}$ for $u^\epsilon$ provided $\epsilon$ is sufficiently small. Then

$$H_k[u^\epsilon](x_0) \geq \frac{C_2}{2} \epsilon^{-1} - C_1$$

which implies $H_k[u^\epsilon](x_0) > 0$. Hence, we obtain a sequence of $C^{1,1}$ smooth $H_k$-subharmonic functions which converges to $u$.

Next, we construct the $C^{2,1}$ approximation. But the above construction, it suffices to consider $u = \max\{v, v'\}$ in the lemma with further assumption that $u \in C^1(\Omega)$. Then $\omega = \{x \in \Omega \mid v < v'\}$ and $Dv = Dv'$ on $\Gamma = \partial \omega$. For any $p \in \partial \omega$, denote $\beta = D_{\gamma\gamma} v(p)$, $\beta' = D_{\gamma\gamma} v'(p)$, where $\gamma$ is the unit normal of $\Gamma$. We have $\beta - \beta' > 0$. Choose a positive smooth function $\eta$ on $\partial \omega$ and $\eta = 0$ on $\partial \Gamma$. We use the same notations $L(p, \epsilon)$ and $\omega_\epsilon$ as
in $C^{1,1}$ approximation. Let

$$u^\epsilon(x) = \begin{cases} 
    u(x) + \varphi(x), & x \in \omega_\epsilon, -\epsilon \eta(p) \leq t \leq 0, \\
    u(x) + \psi(x), & x \in \omega_\epsilon, 0 \leq t \leq \epsilon \eta(p), \\
    u(x), & x \in \Omega \setminus \omega_\epsilon
\end{cases}$$

(2.33)

where

$$\varphi(x) = \frac{\beta - \beta'}{24 \epsilon \eta(p)}(t + \epsilon \eta(p))^3,$$

$$\psi(x) = -\frac{13(\beta - \beta')}{24 \epsilon \eta(p)}(t - \epsilon \eta(p))^3 - \frac{3(\beta - \beta')}{4(\epsilon \eta(p))^2}(t - \epsilon \eta(p))^4 - \frac{5(\beta - \beta')}{4(\epsilon \eta(p))^3}(t - \epsilon \eta(p))^5.$$ 

One can check that $u^\epsilon \in C^{2,1}$. We show that $u^\epsilon$ is $H_k$-subharmonic in $\omega_\epsilon$ and converges to $u$ as $\epsilon \to 0$. For any $x_0 \in \omega_\epsilon$, we have $x_0 \in L(p, \epsilon)$ for some $p \in \Gamma$. By a transformation, we assume that $p = 0$, and $\gamma(p) = (0, \cdots, 0, 1)$. By the assumption, we have

$$Dv(p) = Dv'(p) = Du(p),$$

$$v_{ij}(p) = v'_{ij}(p) = u_{ij}(p), i, j = 1, ..., n - 1,$$

$$v_{in}(p) = v'_{in}(p) = u_{in}(p), i = 1, ..., n - 1.$$ 

The main point is that under the construction (2.33), $Du, \{u_{ij}\}_{i,j=1}^{n-1}, \{u_{in}\}_{i,j=1}^{n-1}$ change very little, while $u_{nn}$ will always be no less than $\min\{v_{nn}, v'_{nn}\}$. First, by the above facts, $\Gamma$ is given by

$$x_n = \rho(x') = \sum_{i,j=1}^{n-1} A_{ij} x_i x_j + O(|x'|^3)$$

(2.34)

near 0 for some $A_{ij}$, $i, j = 1, ..., n - 1$. Then by similar arguments and computation as before, we have

$$|u^\epsilon_i(x_0) - v^\epsilon_i(p)| \leq C \epsilon, \quad 1 \leq i \leq n,$$

(2.35)

$$|u^\epsilon_{nn}(x_0) - v^\epsilon_{nn}(p)| \leq C \epsilon,$$

(2.36)

$$|u^\epsilon_{ij}(x_0) - v^\epsilon_{ij}(p)| \leq C \epsilon, \quad 1 \leq i, j \leq n - 1,$$

for sufficiently small $\epsilon$. Furthermore, since $\varphi_{nn} > 0$ and

$$\psi_{nn} = -\frac{13(\beta - \beta')}{4 \epsilon \eta(p)}(t - \epsilon \eta(p)) - \frac{9(\beta - \beta')}{(\epsilon \eta(p))^2}(t - \epsilon \eta(p))^2 - \frac{5(\beta - \beta')}{(\epsilon \eta(p))^3}(t - \epsilon \eta(p))^3$$

$$> - (\beta - \beta'),$$

we have

$$u^\epsilon_{nn}(x_0) \geq v^\epsilon_{nn}(p) - C \epsilon$$

(2.37)
for sufficiently small $\epsilon$. Using formula (2.24) with (2.35)-(2.37) and the ellipticity assumption that $H_k[v], H_k[v'] > 0$, we have

$$H_k[u^\epsilon](x_0) \geq H_k[v'](p) - C\epsilon > 0$$

as $\epsilon \to 0$.

By choosing the function $\varphi$ more carefully, the sequence can be made $C^{l,1}$ for any $l \geq 3$. Actually, the $C^{2,1}$ approximation is enough for the purpose of this paper. □

We point out that for $l \geq 2$, the construction of the $C^{l,1}$ approximation by small modification is complicated. If we do not request $u_j = u$ outside a small neighbourhood of $\Gamma$, then there are other ways to get the $C^\infty$ approximation from the $C^{1,1}$ approximation. One way is to mollify the $C^{1,1}$ function $u^\epsilon$ constructed in the proof of Lemma 2.4 directly by convolution. The other way is to consider the initial-boundary value problem to an associated parabolic equation and obtain a smooth solution $u(x,t)$ with initial condition $u^\epsilon$. Then $u(\cdot,t)$, as $t \to 0$, gives another smooth approximation.

3. Perron lifting

To prove Theorem 1.1, we assume that there exists two sequences of bounded $H_k$-subharmonic functions $\{u_j\}, \{v_j\}$ in $\mathcal{SH}_k(\Omega)$ which converge to an $H_k$-subharmonic function $u$ a.e. in $\Omega$. Let $B_r(x_0)$ and $B_{r+t}(x_0)$ be two balls in $\Omega$. The purpose of this section is to modify $u_j$ and $v_j$ in the annulus $B_{r+t} - B_r$ such that they are locally uniformly Lipschitz continuous and $|u_j - v_j| \to 0$ locally uniformly in the annulus.

For this purpose we use the Perron lifting for $k$-curvature equations, following the treatment in [16] for the mean curvature equation. See also [24] for quasilinear elliptic equations. As the argument is very similar, we will sketch the proof only.

However let us point out a difference, that is for the mean curvature equation, by the interior regularity one obtains a sequence of piecewise smooth functions in the annulus. For the $k$-curvature for $1 < k < n$, by the interior gradient estimate [31, 41], we only obtain a sequence of piecewise Lipchitz continuous functions in the annulus. In the next section we will show how to obtain a smooth approximation in the annulus.

Let $u$ be a $H_k$-subharmonic function in $\Omega$ and $\omega \Subset \Omega$ be a subdomain of $\Omega$. The Perron lifting of $u$, $u^\omega$, is the upper semicontinuous regularization [24] of

$$\tilde{u} = \{v \mid v \text{ is } H_k\text{-subharmonic in } \Omega \text{ and } v \leq u \text{ in } \Omega \setminus \omega\},$$

i.e.,

$$u^\omega(x) = \lim_{t \to 0} \sup_{B_t(x)} \tilde{u}. $$
It is clear that \( u^\omega \geq u \) in \( \Omega \) and \( u^\omega = u \) in \( \Omega \setminus \bar{\omega} \) but \( u^\omega = u \) on \( \partial \omega \) may not be true.

In order to study the properties of Perron lifting, we first recall the existence of solutions to the Dirichlet problem for \( k \)-curvature equations [27, 37, 38]. For any \( C^2 \) domain, we denote by \( H_k[\partial \Omega] \) the \( k \)-curvature of the boundary \( \partial \Omega \). Let us quote the following two lemmas (Theorem 4 and 5, [38]).

**Lemma 3.1.** Assume \( \partial \Omega \in C^2 \), \( \varphi \in C^0(\partial \Omega) \), \( f^k \in C^{1,1}(\bar{\Omega}) \), \( f > 0 \) in \( \Omega \). Suppose

\[
\begin{align*}
  f(x) &\leq H_k[\partial \Omega], \quad x \in \partial \Omega, \\
  \int_E f &\leq \frac{1 - \lambda}{k} H_{k-1}[\partial E]
\end{align*}
\]

for some \( \lambda > 0 \) and all subdomains \( E \subset \Omega \) with \((k-1)\)-convex boundary \( \partial E \in C^2 \). Then there exists a unique, \( H_k \)-subharmonic, viscosity solution \( u \in C^0(\bar{\Omega}) \), which is locally uniformly Lipschitz continuous in \( \Omega \) to the Dirichlet problem

\[
\begin{align*}
  H_k[u] &= f(x) \quad \text{in } \Omega, \\
  u &= \varphi \quad \text{on } \partial \Omega.
\end{align*}
\]

If \( \varphi \in C^{1,1}(\bar{\Omega}) \), then \( u \in C^{0,1}(\bar{\Omega}) \).

**Lemma 3.2.** Suppose that \( \partial \Omega \in C^{3,1} \), \( \varphi \in C^{3,1}(\bar{\Omega}) \), \( f \in C^{1,1}(\bar{\Omega}) \), \( f > 0 \) in \( \Omega \). Suppose (3.1), (3.2) hold. Then there exists a unique, \( H_k \)-subharmonic, classical solution to (3.3).

**Remark 3.3.** It is shown in [38] that (3.2) is a necessary condition for the solvability of \( k \)-curvature equations.

By choosing a sequence of \( f_i \to 0 \) and using approximation, we have

**Corollary 3.4.** Assume \( \partial \Omega \in C^2 \), \( H_k[\partial \Omega] > 0 \) and \( \varphi \in C^{1,1}(\bar{\Omega}) \). Then there exists a unique, \( H_k \)-subharmonic, viscosity solution \( u \in C^{0,1}(\bar{\Omega}) \) to the Dirichlet problem

\[
\begin{align*}
  H_k[u] &= 0 \quad \text{in } \Omega, \\
  u &= \varphi \quad \text{on } \partial \Omega.
\end{align*}
\]

We have the following existence and Lipschitz continuity for the Perron lifting.

**Lemma 3.5.** Let \( u \in S\mathcal{H}_k(\Omega) \). Then for any open set \( \omega \subset \Omega \), the Perron lifting \( u^\omega \) is \( H_k \)-subharmonic function in \( \Omega \). If we further assume \( u \in L^\infty_{\text{loc}}(\Omega) \), then \( u^\omega \) is locally Lipschitz continuous in \( \omega \) for any open set \( \omega \subset \Omega \).

**Proof.** The property that \( u^\omega \) is \( H_k \)-subharmonic function in \( \Omega \) follows by definition. It suffices to show that \( u^\omega \) locally Lipschitz continuous in \( \omega \) when \( u \in L^\infty_{\text{loc}}(\Omega) \). We outline the proof here, as it is similar to that in [16, 24]. The idea is as follows. Since \( u \) is upper...
semicontinuous, there exists a sequence \( \{ \varphi_j \} \) of smooth functions in \( \Omega \) such that \( \varphi_j \searrow u \). Assume \( B \Subset \omega \) is a ball. By Corollary 3.4, there is a solution \( u_j \in C^{0,1}(\overline{B}) \) to

\[
\begin{aligned}
H_k[u] &= 0 \quad \text{in } B, \\
u &= \varphi_j \quad \text{on } \partial B.
\end{aligned}
\]

Furthermore, since \( u \) is bounded, by the interior gradient estimate \([31, 41]\), the decreasing sequence \( u_j \) is locally uniformly Lipchitz. Hence \( u_j \) converges locally, uniformly to a locally Lipchitz function \( \hat{u} \). It is obvious that \( u_j \geq u \) and hence \( \hat{u} \geq u^\omega \geq u \). By a barrier construction, we can show that \( \hat{u} \leq u \) on \( \partial B \), in the sense that for any given \( x_0 \in \partial B \), \( \lim_{x \to x_0} \hat{u}(x) \leq u(x_0) \). Now extend \( \hat{u} \) to \( \Omega \) so that \( \hat{u} = u \) in \( \Omega \setminus B \). \( \hat{u} \) is \( H_k \)-subharmonic in \( \Omega \), which implies that \( \hat{u} = u^\omega \) by the definition of \( u^\omega \). \( \square \)

We also need the following convergence of the Perron lifting.

**Lemma 3.6.** Let \( u_j \) be a sequence of uniformly bounded \( H_k \)-subharmonic functions which converges to \( u \in \mathcal{SH}_k(\Omega) \) a.e. Let \( B_{r_0}(x_0) \subset \Omega \). Then for a.e. \( r \in (0, r_0) \), we have \( u_j^{B_r} \to u^{B_r} \) a.e. in \( \Omega \), as \( j \to \infty \).

The proof is by the monotonicity of \( u^{B_r} \) for \( r \in (0, r_0) \), i.e, \( u^{B_r} \) is increasing in \( r \) and

\[
\lim_{r \to \delta^-} u^{B_r} \leq u^{B_{\delta}} \leq \lim_{r \to \delta^+} u^{B_r}.
\]

It is similar to that in \([16]\). We refer the reader to \([16]\) for details.

4. Approximation

In this section, we prove that every \( H_k \)-subharmonic function can be approximated by a sequence of smooth \( H_k \)-subharmonic functions. The approximation for the case \( k = 1 \) was obtained in \([16]\) by the Perron lifting in small balls and using the mollification as in the proof of Lemma 2.4. However, when \( k > 1 \), the Perron lifting of an \( H_k \)-subharmonic function may fail to be smooth. In this paper we introduce a new technique to prove the approximation, by considering an obstacle problem of the equation. The regularity of the solution to the obstacle problem is proved in the Appendix 1.

For \( u \in C^0(\overline{\Omega}) \), define the \( H_k \)-subharmonic envelope

\[(4.1) \quad \tilde{u} = \sup\{v \in \mathcal{SH}_k(\Omega) \cap C^0(\Omega) \mid v \leq u \text{ in } \Omega\}.
\]

It is the greatest subsolution to an obstacle problem for the \( k \)-curvature equation.

**Theorem 4.1.** Let \( u \in \mathcal{SH}_k(\Omega) \). Then for any ball \( B_R \Subset \Omega \), there is a sequence of smooth \( H_k \)-subharmonic functions \( u_j \in C^\infty(B_R) \) converging to \( u \) in \( B_R \).
Proof. We may assume that $u$ is continuous. Indeed, by the property of Perron lifting in last section, we may use the techniques as Theorem 5.1 in [16] to construct a sequence of piecewise Lipchitz, $H_k$-subharmonic functions to approximate it.

To obtain the smooth approximation, we first choose a sequence $\{\varphi_j\}$ of smooth functions in $\Omega$ such that $\varphi_j \searrow u$ and $\varphi_j > u$. Let $\tilde{\varphi}_j$ be the $H_k$-subharmonic envelope of $\varphi_j$ in $B_R$, as defined above. Then $\varphi_j \geq \tilde{\varphi}_j \geq u$. Hence, $\tilde{\varphi}_j$ converges to $u$ as $j \to \infty$.

But the function $\tilde{\varphi}_j$ is not smooth. To obtain a smooth approximation, we choose a sequence $\delta_j > 0$, converging to 0 as $j \to \infty$, and assume that $\varphi_j \geq u + \delta_j$. Now we consider the obstacle problem

$$
(4.2) \quad \varphi_j = \sup_{v \in S_{\varphi_j}} v.
$$

where

$$
S_{\varphi_j, \delta_j} = \{v \in \mathcal{SH}_k(\Omega) \cap C^0(B_R) \mid v \leq \varphi_j \text{ in } B_R \text{ and } v \leq \varphi_j - \delta_j \text{ on } \partial B_R\}.
$$

We will show that $(\text{Theorem 6.1 in the Appendix 1})$ $\varphi_j$ is a Lipchitz $H_k$-subharmonic smooth function. Furthermore, for any $j$, there exists a sequence of smooth $H_k$-subharmonic functions $u_j$, which converges to $\tilde{\varphi}_j$ as $\epsilon \to 0$. Since $u \in S_{\varphi_j, \delta_j}$, we have $\tilde{\varphi}_j \geq \tilde{\varphi}_j \geq u$. Hence, $u_j$ converges to $u$ as $\epsilon \to 0$, $j \to \infty$. We obtain the smooth approximation. \qed

Remark 4.2. Note that the function $u$ in Theorem 4.1 satisfies $H_k[u] \geq 0$ in the viscosity sense. Namely we allow that $u$ is degenerate in the sense that $H_k[u] = 0$ at some points. But by our proof of Theorem 6.1, the smooth function $u_j$ obtained in Theorem 4.1 satisfies $H_k[u_j] > 0$ in $\Omega$. This important property enables us to assume that the sequence $\{u_j\}$ in Theorem 1.1 are smooth and $H_k[u_j] > 0$. In other words, by Theorem 4.1, it suffices to prove Theorem 1.1 (ii) for any sequence of smooth, bounded, strictly $H_k$-subharmonic functions $\{u_j\}$ satisfying $H_k[u_j] > 0$.

5. Weak continuity

In this section, we prove Theorem 1.1. First we prove

Lemma 5.1. Let $u_j$ be a sequence of $H_1$-subharmonic functions which converges to an $H_1$-subharmonic function $u$ a.e.. Then for any $\epsilon > 0, \delta > 0$, and any subset $\Omega' \subset \subset \Omega$, there exists $J > 1$ such that when $j > J$, we have

$$
u_j(x) \leq u_\delta(x) + \epsilon \quad \forall \ x \in \Omega',
$$

where

$$
u_\delta(x) = \sup \{u(y) \mid |y - x| < \delta\}.
$$
Proof. For any given \( x_0 \in \Omega' \), by subtracting a constant we assume that \( u_\delta(x_0) = 0 \). Replacing \( u_j \) by \( \max(u_j, 0) \) we may assume that \( u_j \geq 0 \) in \( B_\delta(x_0) \). By the Harnack inequality (see Corollary 2.1 in \cite{[16]}), we have

\[
u_j(x_0) \leq C\delta^{-n/p} \left( \int_{B_\delta} \nu_j^p \right)^{1/p}.
\]

But since \( u_j \) are uniformly bounded and \( u_j \to u \) a.e., the right hand side converges to 0. Hence \( u_j(x_0) \leq \varepsilon \) when \( j \) is sufficiently large. \( \Box \)

To prove Theorem 1.1, by Remark 4.2 it suffices to consider a sequence \( \{u_j\} \) which are smooth and satisfy \( H_k[u_j] > 0 \). When no confusion arises, we also use the density \( H_k[u] \) to denote the measure \( \mu_k[u] = H_k[u] \, dx \) for \( u \in SH_k \cap C^2 \).

To prove Theorem 1.1, it also suffices to prove part (ii) of it. Therefore we need only to prove

**Lemma 5.2.** Let \( u_j \in C^2(\Omega) \) be a sequence of uniformly bounded, \( H_k \)-subharmonic functions. Suppose \( u_j \) converges to a strictly \( H_k \)-subharmonic function \( u \in SH_k(\Omega) \) a.e.. Then \( H_k[u_j] \) converges to a measure \( \mu \) weakly.

**Proof.** For any ball \( B_r(x_0) \subset \subset \Omega \), let \( \phi(x) = \lambda(\|x - x_0\| - r) \), which is the representative function of a convex cone. We choose \( \lambda \) sufficiently large such that \( \phi < u \) in \( B_{r-\delta}(x_0) \) and \( \phi > u \) on \( \partial B_{r+\delta}(x_0) \), where \( \delta = \lambda^{-1} \sup_j \sup_{\Omega} |u_j| \). By Lemma 2.3, we have

\[
\int_{B_{r-\delta}(x_0)} H_k[u_j] \leq \int_{B_{r+\delta}(x_0)} H_k[\max(u_j, \phi)] = \int_{B_{r+\delta}(x_0)} H_k[\phi],
\]

which is uniformly bounded. This means that \( \int_{\Omega'} H_k[u_j] \) are uniformly bounded for \( \Omega' \subset \subset \Omega \). Hence, according to \cite{[1]} there is a subsequence of \( H_k[u_j] \) which converges to a measure \( \mu \) weakly.

To prove that the convergence is independent of the subsequences, we consider two sequences \( \{u_j\}, \{v_j\} \) in \( SH_k(\Omega) \cap C^2(\Omega) \). Suppose both of them converge to \( u \) a.e. in \( \Omega \) and

\begin{equation}
H_k[u_j] \to \mu, \quad H_k[v_j] \to \nu.
\end{equation}

Then to prove \( \mu = \nu \), it suffices to prove that for any ball \( B_r = B_r(x_0) \subset \subset \Omega \) and any \( t > 0 \),

\begin{equation}
\mu(B_r) \leq \nu(B_{r+t}), \quad \nu(B_r) \leq \mu(B_{r+t}).
\end{equation}

Our strategy of proving (5.2) is to use the Perron lifting in the annuli \( N_{t/8}(\partial B_{r+t/4}) \) and \( N_{t/8}(\partial B_{r+3t/4}) \) and use the monotonicity formula (2.9), where \( N_\delta \) denotes the \( \delta \)-neighbourhood. As can be seen in Lemma 3.1 and 3.2, the \( H_k \)-Perron lifting directly in
an annuli domain cannot guarantee the regularity. We will use the Perron lifting in a modified way by liftings in a finite sequence of finite covering balls. We also use Lemma 2.4 to get the smoothness near the boundaries of these balls, so that the Perron lifting of \( u_j \) and \( v_j \) are smooth. Details is as follows.

First we choose finitely many balls \( \hat{B}_{r_\alpha} \) of radius \( r_\alpha \approx t/8 \) and center \( x_\alpha \) on \( \partial B_{r+t/4} \), such that
\[
N_{t/16}(\partial B_{r+t/4}) \subset \bigcup_{\alpha=1}^m \hat{B}_{r_\alpha} \subset N_{t/8}(\partial B_{r+t/4}).
\]
Denote \( \hat{u}_{j,0} = u_j \). Let \( \hat{u}'_{j,1} \) be the solution to
\[
\begin{aligned}
H_k[u] &= \sigma_{j,1} \quad \text{in } \hat{B}_{r_1}, \\
u &= \hat{u}_{j,0} \quad \text{in } \Omega \setminus \hat{B}_{r_1},
\end{aligned}
\]
where \( \sigma_{j,1} \leq \inf H_k[\hat{u}_{j,0}] \) is a constant sufficiently small but positive, such that the Dirichlet problem (5.3) is solvable. Then \( \hat{u}'_{j,1} \) is smooth in \( \hat{B}_{r_1} \) and in \( B_{r+t} \setminus \hat{B}_{r_1} \), up to the boundary. By Lemma 2.4 we may modify \( \hat{u}'_{j,1} \) slightly near \( \partial B_{r_1} \) to get a smooth \( H_k \)-subharmonic function \( \hat{u}_{j,1} \) in \( \Omega \).

We define \( \hat{u}_{j,\alpha} \) inductively for \( \alpha = 2, 3, \ldots, m \), such that \( \hat{u}_{j,\alpha} \in \mathcal{SH}_k(\Omega) \) is a modification of \( \hat{u}'_{j,\alpha} \) near \( \partial \hat{B}_{r_\alpha} \), where \( \hat{u}'_{j,\alpha} \) is the solution to
\[
\begin{aligned}
H_k[u] &= \sigma_{j,\alpha} \quad \text{in } \hat{B}_{r_\alpha}, \\
u &= \hat{u}_{j,\alpha-1} \quad \text{in } \Omega \setminus \hat{B}_{r_\alpha},
\end{aligned}
\]
for some sufficiently small positive constant \( \sigma_{j,\alpha} < \inf H_k[\hat{u}_{j,\alpha-1}] \).

Denote \( \hat{u}_j = \hat{u}_{j,m} \). Similarly, we obtain \( \hat{v}_j \). Both \( \hat{u}_j \) and \( \hat{v}_j \) are smooth and \( H_k \)-subharmonic. But note that the function \( u \) is not smooth and \( H_k[u] \) may vanish at some points. So we simply let \( \hat{u}_0 = u \) and for \( \alpha = 1, 2, \ldots, m \), let \( \hat{u}_\alpha \) be the solution to
\[
\begin{aligned}
H_k[u] &= 0 \quad \text{in } \hat{B}_{r_\alpha}, \\
u &= \hat{u}_{\alpha-1} \quad \text{in } \Omega \setminus \hat{B}_{r_\alpha},
\end{aligned}
\]
and denote \( \hat{u} = \hat{u}_m \).

In conclusion, the almost everywhere convergent sequences \( \{u_j\} \), \( \{v_j\} \) become uniformly convergent sequences \( \{\hat{u}_j\} \), \( \{\hat{v}_j\} \) near \( \partial B_{r+t/4} \) after the above process. In fact, by Lemma 3.6, we may choose the radii \( r_\alpha \approx t/8 \) and choose \( \sigma_{j,\alpha} \) sufficiently small such that \( \hat{u}_j \to \hat{u} \) and \( \hat{v}_j \to \hat{u} \) uniformly in \( N_{t/16}(\partial B_{r+t/4}) \). Moreover, by the interior gradient estimate for the \( k \)-curvature equation [31, 32], \( \hat{u}_j, \hat{v}_j \) and \( \hat{u} \) are locally uniformly Lipchitz continuous in \( \bigcup_{\alpha=1}^m \hat{B}_{r_\alpha} \). Hence by subtracting a small constant we may assume that
\[
\hat{u}_j < \hat{v}_j \quad \text{on } \partial B_{r+t/4}.
\]
Next we apply the above modified Perron lifting to the functions \( \hat{u}_j \) and \( \hat{u} \) near \( \partial B_{r+3t/4} \). That is we choose finitely many balls \( \{ \hat{B}_{r,\beta} \}_{\beta=1}^{m'} \) of radius \( r_\beta \approx \frac{t}{8} \) and center \( x_\beta \) on \( \partial B_{r+3t/4} \), such that

\[
N_{t/16}(\partial B_{r+3t/4}) \subset \bigcup_{\beta=1}^{m'} \hat{B}_{r,\beta} \subset N_{t/8}(\partial B_{r+3t/4}).
\]

Let \( \tilde{u}_{j,0} = \hat{u}_j \). For \( \beta = 1, 2, 3, \ldots, m' \), let \( \tilde{u}_{j,\beta} \) be the solution to

\[
\begin{cases}
H_k[u] = \sigma_{j,\beta} & \text{in } \hat{B}_{r,\beta}, \\
u = \tilde{u}_{j,\beta-1} & \text{in } \Omega \setminus \hat{B}_{r,\beta},
\end{cases}
\]

and let \( \tilde{u}_{j,\beta} \in SH_k(\Omega) \) be a modification of \( \tilde{u}_{j,\beta}' \) near \( \partial B_{r,\beta} \) such that it is smooth and \( H_k \)-subharmonic. Denote \( \tilde{u}_j = \tilde{u}_{j,m'} \). Similarly let \( \tilde{u}_0 = \hat{u} \) and for \( \beta = 1, 2, \ldots, m', \) let \( \tilde{u}_\beta \) be the solution to

\[
\begin{cases}
H_k[u] = 0 & \hat{B}_{r,\beta}, \\
u = \tilde{u}_{\beta-1} & \Omega \setminus \hat{B}_{r,\beta},
\end{cases}
\]

and denote \( \tilde{u} = \tilde{u}_m \).

In the above we obtained by the Perron lifting the new sequences \( \tilde{u}_j, \hat{u}_j, \hat{v}_j, \) and the modified function \( \hat{u}, \tilde{u} \) with the properties

\[
\begin{align*}
\tilde{u}_j &= \hat{u}_j = u_j, \\
\tilde{u} &= \hat{u} = u, \\
\hat{v}_j &= v_j
\end{align*}
\]

near \( \partial B_r \cup \partial B_{r+t/2} \cup \partial B_{r+t} \). By assumption, \( u \) is strictly \( H_k \)-subharmonic. Hence there exists \( \varepsilon > 0 \) such that \( \tilde{u} > u + \varepsilon \) near \( \partial B_{r+3t/4} \). By Lemma 3.6, \( \tilde{u}_j \to \tilde{u} \) near \( \partial B_{r+3t/4} \).

By Lemma 5.1, we then have

\[
\tilde{u}_j > \hat{v}_j = v_j \quad \text{near } \partial B_{r+3t/4}.
\]

By (5.6) and (5.7), there exist set \( G_j \) with

\[
B_{r+t/4} \subset G_j \subset B_{r+3t/4}
\]

such that

\[
\tilde{u}_j(x) = \hat{v}_j(x)
\]

for \( x \in \partial G_j \) and

\[
\tilde{u}_j(x) < \hat{v}_j(x)
\]

for \( x \in G_j \) near \( \partial G_j \). By Sard’s lemma, we may also assume by subtracting a small constant to \( \tilde{u}_j \) that \( D\tilde{u}_j \neq D\hat{v}_j \) on \( \partial G_j \). Therefore by the monotonicity formula (2.9),

---

1Consider the set \( G_\delta = \{ \tilde{u}_j < \hat{v}_j + \delta \} \). By Sard’s lemma, the set is smooth for almost all \( \delta > 0 \). We can choose a smooth \( G_\delta \) for some small \( \delta \) and hence \( D\tilde{u}_j \neq D\hat{v}_j \) on \( \partial G_\delta \).
we have
\[ \int_{B_r} H_k[v_j] = \int_{B_r} H_k[\tilde{v}_j] \leq \int_{G_j} H_k[\tilde{v}_j] \leq \int_{G_j} H_k[\bar{u}_j] \leq \int_{B_{r+t}} H_k[\bar{u}_j] = \int_{B_{r+t}} H_k[u_j]. \]
Taking limit we obtain \( \nu(B_r) \leq \mu(B_{r+t}). \) By exchanging \( u \) and \( v \) we also have \( \mu(B_r) \leq \nu(B_{r+t}). \)

**Remark 5.3.** We point out that even if the sequence \( \{u_j\} \) are locally uniformly Lipschitz continuous, we cannot remove the strict \( H_k \)-subharmonicity condition, because we cannot modify the sequence \( u_j \) and \( v_j \) to obtain (5.6) and (5.7) simultaneously.

**Remark 5.4.** The inequalities (5.2) is built upon the monotonicity formula (2.8) and we need the modified functions of \( u_j \) and \( v_j \) satisfies (5.6) and (5.7). For the \( k \)-Hessian equation or the \( p \)-Laplace equations, to obtain an inequality like (5.7), one may simply add the function \( w_\varepsilon := \varepsilon [\max(0, |x| - r - \frac{t}{4})]^2 \) to \( u_j \). However for the \( k \)-curvature equation, \( u_j + w_\varepsilon \) is not \( H_k \)-subharmonic in general.

One may solve the Dirichlet problem

\[
\begin{cases}
H_k[u] = 0 & \text{in } B_{r+3t/4} \setminus B_{r+t/4}, \\
u = \tilde{u}_{j,\beta} + \varepsilon & \text{on } \partial B_{r+3t/4}, \\
u = \bar{u}_{j,\beta} & \text{on } \partial B_{r+t/4}.
\end{cases}
\]
and send \( \varepsilon \to 0 \). However we cannot prove the convergence of the gradient of the solution on \( \partial B_{r+t/4} \). For these reasons we have to assume that \( u \) is strictly \( H_k \)-subharmonic.

6. **Appendix 1: An obstacle problem**

Let \( \Omega \) be a smooth, bounded domain in \( \mathbb{R}^n \) with \( H_k[\partial \Omega] > 0 \) and let \( \varphi \in C^\infty(\overline{\Omega}) \). Suppose \( \delta > 0 \) is a small positive constant. In this appendix, we consider greatest viscosity subsolution to the obstacle problem

\[
\begin{cases}
H_k[u] \geq 0 & \text{in } \Omega, \\
u \leq \varphi & \text{in } \Omega, \\
u \leq \varphi - \delta & \text{on } \partial \Omega,
\end{cases}
\]
i.e.,

\[
\bar{\varphi}^\delta = \sup_{u \in S_{\varphi,\delta}} u,
\]
where
\[
S_{\varphi,\delta} = \{ u \in C(\Omega) \mid H_k[u] \geq 0, \ u \leq \varphi, \ u|_{\partial \Omega} \leq \varphi - \delta \}.
\]
It is clear that $S_{\varphi,\delta}$ is nonempty, so $\tilde{\varphi}^\delta$ is well defined.

**Theorem 6.1.** The function $\tilde{\varphi}^\delta$ is Lipchitz continuous on $\overline{\Omega}$, and satisfies

$$
\begin{cases}
H_k[\tilde{\varphi}^\delta] \geq 0 & \text{in } \Omega, \\
\tilde{\varphi}^\delta \leq \varphi & \text{in } \Omega, \\
\tilde{\varphi}^\delta = \varphi - \delta & \text{on } \partial \Omega.
\end{cases}
$$

Furthermore, $\tilde{\varphi}^\delta$ can be approximated by a sequence of smooth $H_k$-subharmonic functions $u^\epsilon$, which are solutions to the following perturbation problem:

$$
\begin{cases}
\{H_k[u]\}^\sharp = -\beta_\epsilon(\varphi - u) & \text{in } \Omega, \\
u = \varphi - \delta & \text{on } \partial \Omega,
\end{cases}
$$

where $\beta_\epsilon(t)$ is a family of smooth functions ($\epsilon > 0$) such that

- $\beta_\epsilon(0) = -1$, $\beta_\epsilon(t) < 0$,
- $\beta_\epsilon(t) \to -\infty$, $t < 0$, $\epsilon \to 0$,
- $\beta_\epsilon(t) \to 0$, $t > 0$, $\epsilon \to 0$,
- $\beta_\epsilon'(t) \geq 0$, $\beta_\epsilon''(t) \leq 0$.

An example of the function $\beta_\epsilon$ satisfying the above conditions is $\beta_\epsilon(t) = -e^{-\epsilon^{-1}t}$.

Theorem 6.1 is inspired by [32], which deals with an obstacle problem for Monge-Ampère equation.

First let us recall the a priori estimates for the curvature equation [27]

$$
\begin{cases}
H_k[u] = f(x, u) & \text{in } \Omega, \\
u = \varphi & \text{on } \partial \Omega.
\end{cases}
$$

**Theorem 6.2.** [27] Let $\Omega$ be a bounded domain in $R^n$ with $C^4$ boundary and $\varphi \in C^4(\partial \Omega)$. Assume $H_k[\partial \Omega] > 0$, $f \in C^2(\Omega \times \mathbb{R})$ and $f$ satisfies

$$
(6.6)\quad f > 0, \quad \frac{\partial f}{\partial u} \geq 0
$$

and

$$
(6.7)\quad f < H_k[\partial \Omega] \quad \text{on } \partial \Omega
$$

---

2 The condition was given by $f < \frac{n-k}{n} H_k[\partial \Omega]$ in [27] since the $k$-curvature was defined to be $\left(\frac{n}{k}\right)^{-1} \sigma_k(\kappa)$ there, where $\kappa = (\kappa_1, \cdots, \kappa_n)$ is the vector of principal curvatures.
Let $u \in C^2(\Omega)$ be a solution to (6.5). Denote $M = \sup_{\Omega} |u|$, $\nu = \inf f(x, u) > 0$. Then there exists $C > 0$ depending on $M$, $\nu$, $\|\varphi\|_{C^2(\partial\Omega)}$, $\|\partial\varphi\|_{C^4}$, $\|f\|_{C^2}$ and $H_k[\partial\Omega] - f|_{\partial\Omega}$, such that

$$
(6.8) \quad \|u\|_{C^2(\Omega)} \leq C.
$$

Applying Theorem 6.2 to equation (6.5) we have the following existence result.

**Theorem 6.3.** For any small $\epsilon > 0$, there exists a smooth $H_k$-subharmonic solution $u^\epsilon$ to Dirichlet problem (6.4).

**Proof.** Let $f(x, u) = -\beta\epsilon(\varphi - u)$. It is clear that

$$
\frac{\partial f}{\partial u} = \beta'\epsilon(\varphi - u) \geq 0.
$$

On the boundary $\partial\Omega$,

$$
f(x, u) = -\beta\epsilon(\delta) < H_k[\partial\Omega]
$$

provided $\epsilon$ is small enough. Thus conditions (6.6), (6.7) are satisfied. Hence by Theorem 6.2 and the regularity theory of Evans and Krylov [19, 30], we obtain a priori estimates in $C^{3,\alpha}(\Omega)$. By the continuity method, we obtain the existence of solutions to (6.4). \qed

**Proof of Theorem 6.1.** Let $u^\epsilon$ be the smooth solution to (6.4). In the following we show that the $C^0$ and $C^1$ bounds are independent of $\epsilon > 0$ small. First we show

$$
(6.9) \quad \sup_{\Omega} |u^\epsilon| \leq C
$$

for some constant $C > 0$ independent of $\epsilon > 0$.

Indeed, by the $H_k$-subharmonicity, $\sup_{\Omega} u^\epsilon = \sup_{\partial\Omega} \varphi$. Hence, we only need to prove $u^\epsilon$ is bounded from below. By Lemma 3.1, there exists $\sigma_0 > 0$ small enough such that the Dirichlet problem

$$
(6.10) \begin{cases}
H_k[u] = \sigma_0^k & \text{in } \Omega, \\
u = \varphi - \delta & \text{on } \partial\Omega
\end{cases}
$$

is solvable. Let $u_0$ be the unique solution to (6.10). Let $K > 0$ be a positive constant and denote

$$
E = \{ x \mid u^\epsilon(x) \leq \varphi(x) - K \}.
$$

Then $E \subset \Omega$. By the definition of $\beta\epsilon$,

$$
f = -\beta\epsilon(\varphi - u^\epsilon) \leq -\beta\epsilon(K) \leq \sigma_0
$$
when $K$ is chosen large enough. Hence by the comparison principle,
\begin{equation}
(6.11) \quad u^\epsilon \geq u_0 - C_K
\end{equation}
for some constant $C_K$ depending on $\varphi$ and $K$. Hence (6.9) holds for some constant $C > 0$ independent of $\epsilon$.

Next we show that
\begin{equation}
(6.12) \quad \sup_\Omega |\nabla u^\epsilon| \leq C
\end{equation}
for some constant $C > 0$ independent of $\epsilon$. The gradient estimate can be found in [8]. We include the proof, only to show that the upper bound is independent of $\epsilon$.

Let us first show that $f(x, u^\epsilon)$ is uniformly bounded from above. Indeed, assume that the minimum of $\beta_\epsilon(\varphi - u^\epsilon)$ is attained at a point $x_0$. If $x_0 \in \partial \Omega$, by the boundary condition, $\beta_\epsilon(\varphi - u^\epsilon)(x_0) = \beta_\epsilon(0) = -1$. If $x_0 \in \Omega$, since $\beta_\epsilon$ is monotonic, we have
\[
\nabla \varphi(x_0) = \nabla u^\epsilon(x_0), \quad D^2 \varphi(x_0) \geq D^2 u^\epsilon(x_0).
\]
Hence,
\[
f(x_0, u^\epsilon(x_0)) = \{H_k[u^\epsilon](x_0)\}^{\frac{1}{k}} \leq \{H_k[\varphi](x_0)\}^{\frac{1}{k}}.
\]
Therefore,
\[
f(x, u^\epsilon) \leq C_0,
\]
where $C_0 > 0$ is independent of $\epsilon$. This uniform bound implies
\begin{equation}
(6.13) \quad \sup_\Omega (u^\epsilon - \varphi) \to 0
\end{equation}
as $\epsilon \to 0$.

By Corollary 3.4, there exists $v_0 \in C^{0,1}(\overline{\Omega}) \cap S\mathcal{H}_k(\Omega)$,
\begin{equation}
(6.14) \quad H_k[v_0] = 0, \quad v_0|_{\partial \Omega} = \varphi - \delta.
\end{equation}
Hence by the comparison principle, we have $u^\epsilon \leq v_0$. Let $E = \{u^\epsilon < \varphi - \frac{\delta}{2}\}$ and $E_0 = \{v_0 < \varphi - \frac{\delta}{2}\}$. Then $E \supset E_0$ and $E_0$ is a neighbourhood of $\partial \Omega$, independent of $\epsilon$. Therefore, by the argument in [27] for the boundary gradient estimate, and by our construction of $\beta_\epsilon$, we obtain
\begin{equation}
(6.15) \quad \sup_{\partial \Omega} |\nabla u^\epsilon| \leq C_1,
\end{equation}
where $C_1$ is independent of $\epsilon$.

To obtain the global estimate, we denote for simplicity that $u = u^\epsilon$. Following [8] we introduce the auxiliary function
\[
G = |\nabla u| e^u.
\]
Suppose that the maximum of $G$ is attained at $x_0$. If $x_0 \in \partial \Omega$, the gradient estimates follow by (6.15). If $x_0 \in \Omega$, we may suppose that $|\nabla u| = u_1$ and $u_j = 0$ for $j \geq 2$ at $x_0$. Then at $x_0$,

\begin{align}
0 = (\log G)_i &= \frac{u_1i}{u} + u_i, \\
0 \geq (\log G)_{ij} &= \frac{u_1ij}{u} - \frac{u_1iu_1j}{u^2} + u_{ij}.
\end{align}

We now make use of the matrix

\begin{equation}
a_{ij} = \frac{1}{w} \left[ u_{ij} - \frac{u_iu_ku_{kj}}{w(1+w)} - \frac{u_ju_ku_{ki}}{w(1+w)} + \frac{u_iu_ju_ku_{kl}}{w^2(1+w)^2} \right],
\end{equation}

where $w = \sqrt{1 + |\nabla u|^2}$. Let $F_k = \sigma_k(\lambda)$, where $\lambda \in \mathbb{R}^n$ denotes the eigenvalue vector of the matrix $(a_{ij})$, and $\sigma_k$ denotes $k^{th}$ elementary symmetric function of $\lambda$. It is known [8] that

\[ F_k(a_{ij}) = H_k[u] = f(x,u) = -\beta \varepsilon (\varphi - u) \]

By (6.16), $u_{1i} = 0$ for $i \geq 2$. By a rotation of the coordinates $(x_2, \ldots, x_n)$, we may further assume that $u_{ij}$ is diagonal at $x_0$. Then $a_{ij}$ is also diagonal and

\begin{equation}
a_{11} = \frac{u_{11}}{w^3}, \quad a_{ii} = \frac{u_{ii}}{w}, \quad i > 1.
\end{equation}

Set $F^i_{ij} = \frac{\partial F_k}{\partial a_{ij}}$. $F^i_{ij}$ is diagonal at $x_0$. Assume that $u_1(x_0) > \varphi_1(x_0)$ otherwise we immediately get the gradient estimate. Differentiating the equation with respect to $x_1$, we have

\begin{equation}
F^i_{ii} \frac{\partial a_{ii}}{\partial x_1} = \beta'(\varphi - u) \cdot (u_1 - \varphi_1) \geq 0.
\end{equation}

By computation,

\begin{equation}
\frac{\partial a_{11}}{\partial x_1} = \frac{u_{111}}{w^3} - \frac{3u_{11}^2}{w^5}, \quad \frac{\partial a_{ii}}{\partial x_1} = -\frac{u_1u_{11}a_{ii}}{w^2} + \frac{u_{ii1}}{w}, \quad i > 1.
\end{equation}

Combining (6.19) - (6.21), we have

\begin{equation}
\frac{2u_1F^{11}u_{11}^2}{w^5} + \frac{u_1u_{11}}{w^2} \sum_{i=1}^n F^i_{ii} a_{ii} \leq \frac{1}{w} \sum_{i>1} F^i_{ii} u_{ii1} + \frac{1}{w^3} F^{11} u_{111}.
\end{equation}

By (6.17),

\[ \frac{1}{w^3} F^{11} u_{111} \leq \frac{F^{11}}{w^3} \left( \frac{u_{11}^2}{u_1} - u_1 u_{11} \right) = \frac{F^{11} u_{11}^2}{u_1 w^3} - u_1 F^{11} a_{11} \]

and

\[ \frac{1}{w} \sum_{i>1} F^i_{ii} u_{ii1} \leq \frac{1}{w} \sum_{i>1} F^i_{ii} u_{1i} = -u_1 \sum_{i>1} F^i_{ii} a_{ii}. \]
Substituting them into (6.22) and using (6.16), we have

\[ \frac{F_{11}u_{11}^2}{u_1w_5^5}(w^2 - 2) + \frac{u_1}{w^2} \sum_{i=1}^{n} F_{ii}a_{ii} \leq 0. \]

Since \( u \) is smooth and \( H_k \)-subharmonic, \( F_{11} > 0 \) and \( \sum_{i=1}^{n} F_{ii}a_{ii} > 0 \). Therefore, we have \( w^2 - 2 \leq 0 \). In conclusion, there exists \( C'_1 \) depending only on \( \sup_{\Omega} |u^\epsilon| \) and \( \sup_{\partial \Omega} |\nabla u^\epsilon| \) and \( \varphi \), but independent of \( \epsilon \), such that \( \sup_{\Omega} |\nabla u^\epsilon| \leq C'_1 \).

Note that the maximum bound of \( |u^\epsilon| \) implies that

\[ f(x, u^\epsilon) = -\beta_\epsilon(\varphi - u^\epsilon) \geq \nu_\epsilon > 0 \]

for some positive constant \( \nu_\epsilon \). Hence we have the second derivative estimate [27]

(6.23) \[ \sup_{\Omega} |D^2 u^\epsilon| \leq C. \]

By the global regularity theory of Evans-Krylov [19, 30], we then have

(6.24) \[ \|u^\epsilon\|_{C^{3,\alpha}(\Omega)} \leq C, \]

where \( C \) depending on \( \beta_\epsilon, \varphi, \Omega \).

To conclude the proof, we need to show that \( u^\epsilon \) converges to \( \tilde{\varphi}^\delta \) and \( \tilde{\varphi}^\delta \) is Lipchitz continuous. We have shown that \( \nabla u^\epsilon \) are uniformly bounded with respect to \( \epsilon \). Hence by choosing a subsequence, \( u^\epsilon \) converges to a Lipchitz continuous, \( H_k \)-subharmonic function \( \bar{u} \). We claim that \( \bar{u} = \tilde{\varphi}^\delta \) in \( \Omega \). Indeed, by (6.13), \( \bar{u} \leq \varphi \) in \( \Omega \), i.e, \( \bar{u} \in S_{\varphi, \delta} \). This implies \( \bar{u} \leq \tilde{\varphi}^\delta \) in \( \Omega \). Through the definition of \( \beta_\epsilon \), we also have \( H_k[\bar{u}] = 0 \) in viscosity sense in the set \( G = \{ \bar{u} < \varphi \} \subset \Omega \). If there is \( x_0 \in \Omega \) such that \( \bar{u}(x_0) < \tilde{\varphi}^\delta(x_0) \), then by the definition of \( \tilde{\varphi}^\delta \), there exists \( h \in SH_k(\Omega) \cap C^0(\Omega) \), such that \( \bar{u}(x_0) < h(x_0) \). The set \( G' = \{ \bar{u} < h \} \subset G \) is nonempty. Since

\[ H_k[h] \geq H_k[\bar{u}] = 0 \text{ in } G', \]
\[ \bar{u} = h \text{ on } \partial G', \]

by comparison, we obtain the contradiction. Hence \( \bar{u} = \tilde{\varphi}^\delta \). The theorem is proved.

Remark 6.4. We point out that the idea above can be used to prove the approximation results for other nonlinear elliptic equations. One example is the complex Monge-Ampère equation on Kähler manifolds in the next section.
7. Appendix 2: Regularization of plurisubharmonic functions on compact Kähler manifolds

In this appendix, we give a new proof for the global regularization of plurisubharmonic functions on compact Kähler manifolds following the idea in last appendix.

The local regularization of plurisubharmonic functions on domains in $\mathbb{C}^n$ is not difficult since the standard mollification keeps the plurisubharmonicity. An interesting question is whether one can globally regularize plurisubharmonic functions. The global regularization of plurisubharmonic functions on $\mathbb{C}^n$ was obtained in [15]. Since then, the problem on compact Kähler manifolds attracted much attention.

Let $(M, \omega)$ be a compact Kähler manifold. A function $u : M \to [\infty, \infty)$ is called $\omega$-plurisubharmonic (quasi-plurisubharmonic), if $u$ is an integrable, upper semicontinuous function on $M$ and $\omega_u = \omega + \partial \bar{\partial} u \geq 0$ in the distribution sense. Denote by $PSH(M, \omega)$ the set of all $\omega$-plurisubharmonic functions. The problem of global regularization refers to whether an $\omega$-plurisubharmonic function can be approximated by smooth $\omega$-plurisubharmonic functions. This approximation was first obtained in [17, 18, 23] under the assumption that $M$ admits a positive holomorphic line bundle, i.e. $\omega$ is a Hodge class. The proof used a complicated method developed by Demailly. Later on, a simpler proof was given [7], where the approximation holds in arbitrary compact Kähler case. Here we present a new proof of this theorem here by solving penalized complex Monge-Ampère equations.

**Theorem 7.1.** Let $(M, \omega)$ be a compact Kähler manifold and $u \in PSH(M, \omega)$. There exists a sequence of smooth $\omega$-plurisubharmonic functions $\{u_k\}$, such that $u_k$ converges decreasingly to $u$ a.e.

As in the previous section, the sequence $u_k$ is the solution to an associated penalty problem, namely equation (7.3) below. In Theorem 7.1, we may assume that $u$ is bounded. Otherwise, we may choose the sequence of $\omega$-plurisubharmonic functions $u_j = \max\{u, -j\}$, $j = 1, 2, 3, \ldots$, which converges to $u$ monotone decreasingly. We also assume that $\omega_u \geq \sigma^\frac{1}{n} \omega$ for some $\sigma > 0$. For if not, we can replace $\omega_u$ by $\omega + (1-\sigma^\frac{1}{n})\sqrt{-1}\partial \bar{\partial} u$. This implies that

(7.1) \[(\omega + \sqrt{-1}\partial \bar{\partial} u)^n \geq \sigma \omega^n\]

in viscosity sense.

The following theorem of Aubin and Yau [2, 43] will be needed.
Theorem 7.2. Let \((M, \omega)\) be a compact Kähler manifold. Let \(F\) be a smooth function defined on \(M \times \mathbb{R}\) with \(\frac{\partial F}{\partial t} \geq 0\). Suppose that, for some smooth function \(\psi\) defined on \(M\),
\[
\int_M e^{F(x, \psi(x))} \omega^n = \text{vol}(M).
\]
Then there exists a smooth function \(\varphi\) on \(M\) such that
\[
(\omega + \sqrt{-1} \partial \bar{\partial} \varphi)^n = e^{F(x, \varphi(x))} \omega^n,
\]
and \(\omega + \sqrt{-1} \partial \bar{\partial} \varphi\) defines a Kähler metric. Furthermore, any other smooth function satisfying the same property differs from \(\varphi\) by only a constant.

Since \(u\) is upper semicontinuous, there exist a sequence of smooth functions \(\varphi_k\) which converge decreasingly to \(u\). Furthermore, we may assume that \(\varphi_k \geq \varphi_{k+1} + \frac{1}{(k+1)^2}\). For any integer \(k\) and \(\epsilon > 0\), we consider the equation
\[
(\omega + \sqrt{-1} \partial \bar{\partial} u)^n = \beta_\epsilon(\varphi_k - u) \omega^n,
\]
where \(\beta_\epsilon(t)\) is the smooth function given in Appendix 1. It is easy to see that \(\beta_\epsilon(0) = 1\), so \(\beta_\epsilon(\varphi_k - u)\) satisfies condition (7.2) in Theorem 7.1. For any \(k\) and \(\epsilon > 0\), there exists a smooth solution \(u_{k, \epsilon} \in PSH(M, \omega)\) to the above equation (7.3).

We show that \(u_{k, \epsilon}\) converges to \(u\). First, we claim that
\[
(7.4) \sup_M \{u_{k, \epsilon} - \varphi_k, 0\} \rightarrow 0
\]
as \(\epsilon \rightarrow 0\). By the definition of \(\beta_\epsilon\), it suffices to show that \(\beta_\epsilon(\varphi_k - u_{k, \epsilon})\) is uniformly bounded with respect to \(\epsilon\). Assume that \(\beta_\epsilon(\varphi_k - u_{k, \epsilon})\) attains its maximum at \(p \in M\).

Then \(\sqrt{-1} \partial \bar{\partial} (\varphi_k - u_{k, \epsilon}) \geq 0\). It follows that
\[
(7.5) \beta_\epsilon(\varphi_k - u_{k, \epsilon}) = \frac{(\omega + \sqrt{-1} \partial \bar{\partial} u_{k, \epsilon})^n}{\omega^n} \leq \frac{(\omega + \partial \bar{\partial} \varphi_k)^n}{\omega^n} \leq C
\]
for some \(C\) independent of \(\epsilon\). Next, we consider the lower bound of \(u_{k, \epsilon}\). For any \(\delta > 0\), let \(\Omega_\delta = \{x \in M : u_{k, \epsilon} < \varphi_k - \delta\}\). It is clear that
\[
\beta_\epsilon(\varphi_k - u_{k, \epsilon}) \leq e^{-\epsilon^{-1} \delta} < \sigma
\]
as \(\epsilon \rightarrow 0\). Then by (7.1), we have
\[
(7.6) u_{k, \epsilon}(x) \geq u(x) - \delta, \quad x \in \partial \Omega_\delta
\]
\[
(\omega + \sqrt{-1} \partial \bar{\partial} u)^n > (\omega + \sqrt{-1} \partial \bar{\partial} u_{k, \epsilon})^n, \quad x \in \Omega_\delta
\]
in viscosity sense. We claim that \(u_{k, \epsilon} \geq u - \delta\) in \(\Omega_\delta\). By contradiction, suppose \(m = \min(u_{k, \epsilon} - u + \delta) < 0\) is attained at some point \(p \in \Omega_\delta\). Let \(v = u_{k, \epsilon} - m\). It follows that \(v(p) = u(p), \ v \geq u\) in \(\Omega_\delta\). Hence, as a smooth test function \(v\) satisfies
\[
(\omega + \sqrt{-1} \partial \bar{\partial} v)^n \geq \frac{29}{29} (\omega + \sqrt{-1} \partial \bar{\partial} u)^n
\]
at $p$. This is a contradiction. Letting $\delta \to 0$, we obtain $u_{k,\epsilon} \geq u$ on $M$. Along with (7.5), we have $u_{k,\epsilon}$ converges to $u$ as $\epsilon \to 0$ and $k \to \infty$. It is also clear that $\|u_{k,\epsilon}\|_{L^\infty} \leq C$ for some $C$ independent of $\epsilon$.

In order to obtain a decreasing sequence, we show that $u_{k,\epsilon}$ converges uniformly to a limit as $\epsilon \to 0$. We need the following gradient estimate.

**Lemma 7.3.** Let $-B$ be the lower bound for the bisectional curvature, where $B \geq 0$. There exists $C > 0$ depending on $\varphi_k$, $B$, and $\|u\|_{L^\infty}$, but independent of $\epsilon$, such that

$$|\nabla u_{k,\epsilon}| \leq C.$$

**Proof.** The proof is a generalisation of the gradient estimate by Blocki [4]. Denote $u = u_{k,\epsilon}$ and $f(x, u) = \beta_\epsilon(\varphi_k - u_{k,\epsilon})$. Following [4], we consider the auxiliary function

$$\alpha = \log |\nabla u|^2 + \gamma(u),$$

where $\gamma$ is a function of $u$. Assume that $\alpha$ attains the maximum at some $p \in M$. Near $p$, $\omega = \partial\bar{\partial}g$ for some smooth plurisubharmonic function $g$. Write $v = g + u$. We can choose holomorphic charts $\{z_1, ..., z_n\}$ near $p$ such that

$$(7.6) \quad g_{\bar{z}j} = \delta_{ij}, \quad g_{ij\bar{l}} = 0$$

and $(u_{ij})$ is diagonal. By the computation in [4] with the lower bound of the bisectional curvature, it follows

$$0 \geq \sum_l \frac{\alpha_{\bar{l}l}}{v_{\bar{l}l}} \geq (\gamma' - B) \sum_l \frac{1}{v_{\bar{l}l}} + \frac{2}{|\nabla u|^2} \text{Re} \left\{ \sum_j \frac{\partial \log f}{\partial z_j} u_j \right\} + \frac{1}{|\nabla u|^2} \sum_{l,j} \frac{|u_{jl}|^2}{v_{\bar{l}l}}$$

$$- [\gamma'^2 + \gamma''] \sum_l \frac{|u_l|^2}{v_{\bar{l}l}} - n\gamma'$$

$$(7.7)$$

It is shown in [4] that

$$\frac{1}{|\nabla u|^2} \sum_{l,j} \frac{|u_{jl}|^2}{v_{\bar{l}l}} \geq (\gamma')^2 \sum_l \frac{|u_l|^2}{v_{\bar{l}l}} - 2\gamma' - \frac{2}{|\nabla u|^2}.$$ 

Assume that $|\nabla u| \geq C \geq 1$ is large, depending on $\varphi_k$, so that by definition of $f$,

$$\frac{2}{|\nabla u|^2} \text{Re} \left\{ \sum_j \frac{\partial \log f}{\partial z_j} u_j \right\} = \frac{2}{|\nabla u|^2} \beta_\epsilon \text{Re} \left\{ \sum_j \epsilon^{-1} \frac{\partial (\varphi_k - u)}{\partial z_j} u_j \right\} \geq 0.$$ 

Then we have

$$0 \geq (\gamma' - B) \sum_l \frac{1}{v_{\bar{l}l}} - \gamma'' \sum_l \frac{|u_l|^2}{v_{\bar{l}l}} - (n + 2)\gamma' - 2.$$ 

$$(7.9)$$
Choose
\[ \gamma(u) = (B + 3)u - \frac{1}{\|u\|_{L^\infty}}u^2. \]
Then
\[ (7.10) \quad B + 1 \leq \gamma' = B + 3 - \frac{2}{\|u\|_{L^\infty}} u \leq B + 5, \quad \gamma'' = -\frac{2}{\|u\|_{L^\infty}} u. \]
By (7.9),
\[ \sum_{l} \frac{1}{v_l u_l} + \frac{2}{\|u\|_{L^\infty}} \sum_{l} \frac{|u_l|^2}{v_l u_l} \leq D := (n + 2)(B + 5) + 2. \]
Therefore \( \frac{1}{v_l u_l} \leq D \). Then \( v_l u_l \leq \sup f \cdot D^{n-1} \). Using (7.10) again, we have
\[ |\nabla u|^2 \leq \frac{\|u\|_{L^\infty} \sup f \cdot D^n}{2} \]
at \( p \). Combing this with the definition of \( \alpha \), we finished the proof of the Lemma. \( \square \)

By the above lemma, \( u_{k, \epsilon} \) converges uniformly to a limit \( \tilde{\varphi}_k \) as \( \epsilon \to 0 \). To get a decreasing convergent sequence, it remains to prove that \( \tilde{\varphi}_k \geq \tilde{\varphi}_{k+1} + \frac{1}{(k+1)^2} \). We claim that \( u_{k, \epsilon} \geq u_{k+1, \epsilon} + \frac{1}{(k+1)^2} \). Otherwise, \( E_{k, \epsilon} = \{ u_{k, \epsilon} < u_{k+1, \epsilon} + \frac{1}{(k+1)^2} \} \) is an open set. Then \( \varphi_k - u_{k, \epsilon} \geq \varphi_{k+1} - u_{k+1, \epsilon} \), i.e.,
\[ (\omega + \sqrt{-1} \partial \bar{\partial} u_{k, \epsilon})^n \leq (\omega + \sqrt{-1} \partial \bar{\partial} u_{k+1, \epsilon})^n \]
in \( E_k \). The contradiction follows from \( u_{k, \epsilon} = u_{k+1, \epsilon} \) on \( \partial E_{k, \epsilon} \). Letting \( \epsilon \to 0 \), we obtain \( \tilde{\varphi}_k \geq \tilde{\varphi}_{k+1} + \frac{1}{(k+1)^2} \), so the sequence \( \{ u_{k, \epsilon} \} \) is monotone decreasing.

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