Experts with Lower-Bounded Loss Feedback: A Unifying Framework

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Abstract

The most prominent feedback models for the best expert problem are the full information and bandit models. In this work we consider a simple feedback model that generalizes both, where on every round, in addition to a bandit feedback, the adversary provides a lower bound on the loss of each expert. Such lower bounds may be obtained in various scenarios, for instance, in stock trading or in assessing errors of certain measurement devices. For this model we prove optimal regret bounds (up to logarithmic factors) for modified versions of Exp3, generalizing algorithms and bounds both for the bandit and the full-information settings. Our second-order unified regret analysis simulates a two-step loss update and highlights three Hessian or Hessian-like expressions, which map to the full-information regret, bandit regret, and a hybrid of both. Our results intersect with those for bandits with graph-structured feedback, in that both settings can accommodate feedback from an arbitrary subset of experts on each round. However, our model also accommodates partial feedback at the single-expert level, by allowing non-trivial lower bounds on each loss.

Keywords: regret minimization, multi-armed bandit, best expert, feedback model, online learning

1. Introduction

The best expert setting is a classic online learning framework, where a simple game takes place between a learner and an adversary. In this game, there are $N$ available experts (choices, actions), and $T$ rounds of play, or time steps. On each round $1 \leq t \leq T$, an online algorithm $A$, the learner, picks a distribution $p_t$ over the experts and uses it to randomly select an expert $I_t$. Simultaneously, the adversary assigns the losses of the experts for that round, $l_t = (l_{1,t}, \ldots, l_{N,t}) \in \mathbb{R}^N$, and the learner incurs the loss $l_{I_t,t}$. The aim of the learner is to minimize its regret, defined as $R_{A,T} = L_{A,T} - \min_j \{L_{j,T}\}$, where $L_{A,t} = \sum_{\tau=1}^{t} l_{I_{\tau},\tau}$ is the cumulative loss of $A$ at time $t$ and $L_t = \sum_{\tau=1}^{t} l_{\tau}$ is the cumulative loss of the experts at time $t$. Importantly, a small regret should be achieved regardless of the losses chosen by the adversary. The adversary may determine its choices before the game begins (an oblivious or non-adaptive adversary) or at the time of assignment (an adaptive adversary); as has been often observed, however, the meaning of regret as a comparative benchmark is much clearer for oblivious adversaries.

In the full information version of the problem, the learner has full knowledge of the past losses of every expert. The most famous learner for this variant is the Hedge algorithm (Vovk, 1990; Littlestone and Warmuth, 1994; Freund and Schapire, 1997). For bounded single-period losses, the expected regret of Hedge has an upper bound of the form $O(\sqrt{T \log N})$. This type of bound, which depends only on the time horizon of the game, is referred to as a zeroth-order bound. A more
Algorithm 1: Hedge

**Parameters:** A learning rate $\eta > 0$ and initial weights $w_{i,1} > 0$, $1 \leq i \leq N$.

For each round $t = 1, \ldots, T$

1. Define probabilities $p_{i,t} = w_{i,t}/W_t$, where $W_t = \sum_{i=1}^{N} w_{i,t}$.

2. For each expert $i = 1, \ldots, N$, let $w_{i,t+1} = w_{i,t} e^{-\eta l_{i,t}}$.

A general so-called second-order bound of the form $O(\sqrt{q \log N})$ may also be proven given an upper bound $q$ on the relative quadratic variation of the loss sequence, defined as $\sum_{t=1}^{T} (\max_i \{l_{i,t}\} - \min_i \{l_{i,t}\})^2$ (Cesa-Bianchi et al., 2007). Both bounds are optimal for the expected regret. Other regret bounds, which depend on the cumulative loss of the best expert (first-order bounds) or more refined second-order notions of variation (Hazan and Kale, 2010; Chiang et al., 2012) have also been shown. Regret bounds that hold with a desired high probability, rather than in expectation, have also been established, for example, a zeroth-order bound of $O(\sqrt{T \log(N/\delta)})$ on the regret of Hedge, which holds with probability at least $1 - \delta$ (see, e.g., Cesa-Bianchi and Lugosi, 2006).

In contrast to the full-information setting, in the adversarial multi-armed bandit (or bandit) setting, the learner observes on each round only the loss of the expert it chooses. The Exp3 algorithm (Auer et al., 2002), which is an adaptation of Hedge for this setting, obtains a zeroth-order bound of $O(\sqrt{TN \log N})$ on the expected regret that is optimal up to logarithmic factors.\(^1\) The simpler version of Exp3 given here as Algorithm 2 (Bubeck and Cesa-Bianchi, 2012, Chapter 3), achieves the same bound for non-adaptive adversaries. A variant of Exp3, named Exp3.P, obtains a similar high-probability bound of $O(\sqrt{T N \log(N/\delta)})$.

Second-order bounds for bandits in terms of total variation were shown by Hazan and Kale (2011) and by Bubeck et al. (2018) for bounded single-period losses. The latter give a bound on the pseudo-regret (or expected regret for an oblivious adversary) of $O(\sqrt{V \log N + N \log^2 T})$, where $V$ is the total variation. For more details on the theory of bandits, see, e.g., Bubeck and Cesa-Bianchi (2012); Lattimore and Szepesvári (2020); Slivkins (2019).

### 1.1 A Generalized Model

We consider a new model for the best expert setting, where the learner receives, in addition to a bandit feedback, a lower bound on the loss of each expert. More specifically, on each round $t$, the adversary assigns the experts both losses $l_{i,t} \in \mathbb{R}^N$ and lower bounds on these losses $\lambda_{i,t} \in \mathbb{R}^N$, and the learner receives, simultaneously with its decision, the loss of that decision and all the loss lower bounds. This model is an intermediate between the bandit and full-information models, and further generalizes both. To retrieve the bandit setting, the adversary may provide trivial lower bounds, such as zero values when the losses are restricted to the range $[0, 1]$. To retrieve the full-information setting, the lower bounds may be the actual losses.

To directly motivate this model, consider a scenario of stock trading. Here the experts are stocks, and the single-period loss of an expert is minus the single-period change in the logarithm of the price of the stock (a loss that may be either positive or negative). Theoretically, any trade may be executed

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\(^1\) A different algorithm with an optimal regret bound of $O(\sqrt{TN})$ for an oblivious adversary was later presented by Audibert and Bubeck (2010).
Algorithm 2: Exp3

**Parameters:** \( \eta > 0 \).

Let \( p_1 \) be the uniform distribution over \( \{1, \ldots, N\} \), and let \( \bar{L}_0 = 0 \).

For each round \( t = 1, \ldots, T \)

1. Draw an action \( I_t \) from the probability distribution \( p_t \).

2. For each action \( i = 1, \ldots, N \), compute the estimated loss \( \bar{l}_{i,t} = \frac{l_{i,t} \mathbb{1}\{I_t = i\}}{p_{i,t}} \) and update the estimated cumulative loss \( \bar{L}_{i,t} = \bar{L}_{i,t-1} + \bar{l}_{i,t} \).

3. Compute the new distribution over actions \( p_{t+1} = (p_{1,t+1}, \ldots, p_{N,t+1}) \), where

\[
p_{i,t+1} = \frac{\exp\left(-\eta \bar{L}_{i,t}\right)}{\sum_{k=1}^{N} \exp\left(-\eta \bar{L}_{k,t}\right)}.\]

at market price. However, when trading in large volumes or in small stocks, the stock price reacts in a direction that increases the loss to the trader. Thus, real losses are lower bounded by the theoretic losses calculated from market prices.

Another scenario stems from the fact that the variance of a statistical estimator lower bounds its squared error, through the bias-variance decomposition. For example, when a sensor makes several measurements of the same real-valued quantity, the squared error of these measurements is unknown, unless the ground truth value is ascertained, possibly through the costly work of a human expert. However, the empirical variance of these measurements is always known, and may serve as a lower bound to that error. Thus, if several sensor prototypes go through a series of tests in different labs (where on each occasion a sensor makes several independent measurements, yielding variance), we would want to spend our human-expert budget increasingly on the most promising sensor, that is, the one with the lowest cumulative squared error.

Our model allows a general “soft” decomposition of the loss of each expert into a known part and a tentative part. Yet, even in a restricted dichotomous regime where each loss is either fully known or completely unknown, there are interesting hybrid scenarios. Specifically, the availability of feedback for experts may vary with time in an unexpected way. This case is handled in our model by an adversary that assigns the losses themselves as lower bounds for experts with available feedback, and trivial lower bounds otherwise.

1.2 Summary of Results

We present an algorithm, Exp3.LB, which is an adaptation of Exp3 for our model. In the limits of the full-information and bandit settings, this algorithm is equivalent to Hedge and Exp3, respectively, and thus its analysis captures both algorithms as special cases. Exp3.LB differs from Exp3 in that it updates the estimated cumulative loss of each expert by adding a sum of two elements. The first is the lower bound on the loss, like Hedge does for the true loss. The other is an estimate of the slack, or difference between the true loss and its lower bound, like Exp3 does for the loss itself.
We prove a tight second-order bound on the expected regret of Exp3.LB against non-adaptive adversaries, which has the form $O\left(\sqrt{Q \log N}\right)$, where

$$Q = \sum_{t=1}^{T} \left(\max_{i} \{\lambda_{i,t}\} - \min_{i} \{\lambda_{i,t}\}\right)^{2} + \sum_{t=1}^{T} \|l_{t} - \lambda_{t}\|^{2}.$$  

In our analysis, the estimated cumulative loss update step is broken into a bandit half-step and a full-information half-step, yielding three second-order quantities with quadratic upper bounds. The first is the relative quadratic variation of the sequence of lower bounds, which in the special full-information case translates into the usual relative quadratic variation, while the two other quantities disappear. The second quantity is the sum of all squared slacks, which in the special bandit scenario translates into the zeroth-order bandit bound while the other two factors disappear. The third is a hybrid of slacks and loss lower bounds, and is non-zero only for scenarios on the continuum between the bandit and full-information cases. It does not change the order of the bound and hence dropped from the above expression for simplicity.

We give expected regret bounds for a variable subset feedback scenario, where on each round $t$, the losses for an adversarially-chosen subset $S_{t}$ of the experts is revealed to the learner. This scenario may be modeled by the adversary setting $\lambda_{i,t} = l_{i,t}$ for every $i \in S_{t}$ and $\lambda_{i,t} = 0$ otherwise, where we assume all losses are in $[0, 1]$. Applying Exp3.LB to this scenario, we obtain an expected regret bound of $O\left(\sqrt{T + \sum_{t}(N - |S_{t}|)} \log N\right)$, which is optimal up to logarithmic factors if the subsets are identical (Alon et al., 2017).

We show that other quantities may replace the lower bounds on the losses in our model and algorithm, yielding regret bounds of a similar form. In particular, we may assume that the adversary is providing upper bounds $\upsilon_{i,t}$ on the losses. For this scenario we give an algorithm, Exp3.UB, and bound its expected regret. The algorithm and the bound are similar to those of Exp3.LB except that each occurrence of $\lambda_{i,t}$ is replaced by a quantity based on $\upsilon_{i,t}$.

Finally, we provide a variant of Exp3.LB with regret bounds that hold with high probability against an oblivious adversary. This algorithm, named Exp3.LB.P, is adapted from Exp3.LB using a biasing method unlike that of Exp3.P. We show second-order regret bounds of the form $O\left(\sqrt{Q} \log(N/\delta)\right)$, and given mild conditions, also $O\left(\sqrt{Q} \log(N/\delta)\right)$, where the bounds hold with probability at least $1 - \delta$. For the bounded single-period loss scenario, we prove the bound $O\left(\sqrt{\max\{T, Q\}} \log(N/\delta)\right)$, retrieving the zeroth-order bound types of both Exp3.P and Hedge.

### 1.3 Related Work

Several works have considered a scenario where a graph structure describes the feedback flow to the learner (Mannor and Shamir, 2011; Alon et al., 2013, 2017, 2015). Specifically, given a possibly time-dependent graph whose nodes are the experts, choosing one expert reveals the losses of neighboring experts. These works give regret bounds in terms of graph properties, such as the independence number or the size of the maximal acyclic subgraph.

The difference from our model is twofold. First, in the graph-based model, the set of additional experts providing feedback is a function of the choice made by the learner, while in our model it is not. In this sense, the graph-based model is more general. However, the feedback in our model is “soft”, rather than binary (available or not), and hence the graph-based model is more limited than...
ours. Both models can handle a scenario where on each round, the losses of a time-dependent set of experts are revealed to the learner, in addition to the loss of the expert it chose. In these cases, which clearly interpolate between the bandit and full-information settings, the regret bounds in the two models have the same form.

The work of Cesa-Bianchi and Shamir (2017) considered a setting where along with the loss of the chosen action, the learner is given an interval containing each loss. Crucially, and in contrast to our work, this interval is given to the learner on each round before it makes its choice. The purpose is to allow the learner to take advantage of easy loss sequences, in this case, where potentially only a few experts should be considered on each round. They provide regret bounds that disappear as the interval size shrinks to zero. In summation, their work is thus not truly related to ours.

Finally, we comment that contextual bands, partial monitoring, and combinatorial bandits all have a more distant connection to the topics discussed in this work. More information on these topics may be found in the bandit literature.

1.4 Outline

In Section 2 we give some useful notation. Section 3 covers our model, the Exp3.LB algorithm, and its expected regret bound. In Section 4 we give corollaries for some special scenarios of interest. Section 5 covers variants of our model and corresponding algorithms and bounds. Section 6 provides lower bounds on the regret in our model. In Section 7 we give an algorithm and regret bounds that hold with high probability in our model, and in Section 8 we conclude and discuss future directions. The appendix contains some additional claims.

2. Miscellaneous Notation

We use bold face for vectors, most often for time series of vectors in $\mathbb{R}^N$, such as $l_1, \ldots, l_T$. Their components are written as $l_t = (l_{1,t}, \ldots, l_{N,t})$. We use $\| \cdot \|$ for the $L_2$ norm, and for $x, y \in \mathbb{R}^N$, $[x, y] = \{a x + (1-a)y : 0 \leq a \leq 1\}$ denotes the line segment between $x$ and $y$. We write $\Delta_N$ for the probability simplex of $N$ elements, $\Delta_N = \{p \in \mathbb{R}^N : p_i \geq 0 \forall i = 1, \ldots, N, \sum_{i=1}^N p_i = 1\}$. For $x \in \mathbb{R}^N$, $\text{diag}(x)$ is the diagonal matrix with $x$ as its diagonal. The indicator variable of an event $E$ is denoted by $\mathbb{1}\{E\}$. We will often use the specialized notation $d(x) = \max_i \{x_i\} - \min_i \{x_i\}$ for $x \in \mathbb{R}^N$, and given a sequence of vectors $l_1, \ldots, l_T \in \mathbb{R}^N$ we will denote $q(l_1:T) = \sum_{t=1}^T d(l_t)^2 = \sum_{t=1}^T (\max_i \{l_{i,t}\} - \min_i \{l_{i,t}\})^2$, namely, the relative quadratic variation of the sequence.

3. Best Experts with Lower Bounds

We define a feedback model where on each round, after choosing an expert, the learner is given the exact loss of that expert and a lower bound on the losses of all the experts. In what follows we will denote by $\lambda_{i,t}$ the real-valued lower bound on the loss of expert $i$ at time $t$ and by $s_{i,t} = l_{i,t} - \lambda_{i,t}$ the slack between the loss and the lower bound of expert $i$ at time $t$. We observe that once the losses and lower bounds for round $t$ become known, we may subtract $\min_i \{\lambda_{i,t}\}$ from all of them, without affecting the problem. In particular, the regret of any algorithm is not affected by subtracting a constant $c_t$ from all losses $l_{1,t}, \ldots, l_{N,t}$. We will therefore assume WLOG that $\min_i \{\lambda_{i,t}\} \geq 0$ for every $t$, and thus $l_{i,t} \geq 0$ for every $i$ and $t$. 

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We propose a natural variant of Exp3, which we call Exp3.LB, to handle the lower bound information model.

**Algorithm 3: Exp3.LB**

**Parameters:** \( \eta > 0 \).

Let \( p_1 \) be the uniform distribution over \( \{1, \ldots, N\} \), and let \( \tilde{L}_0 = 0 \).

For each round \( t = 1, \ldots, T \)

1. Draw an action \( I_t \) from the probability distribution \( p_t \).

2. For each action \( i = 1, \ldots, N \), compute the estimated loss
   \[
   \tilde{l}_{i,t} = \frac{s_{i,t}}{p_{i,t}} \cdot \mathbb{I}\{I_t = i\} + \lambda_{i,t}
   \]
   and update the estimated cumulative loss \( \tilde{L}_{i,t} = \tilde{L}_{i,t-1} + \tilde{l}_{i,t} \).

3. Compute the new distribution over actions \( p_{t+1} = (p_{1,t+1}, \ldots, p_{N,t+1}) \), where
   \[
   p_{i,t+1} = \frac{\exp\left(-\eta \tilde{L}_{i,t}\right)}{\sum_{k=1}^{N} \exp\left(-\eta \tilde{L}_{k,t}\right)}.
   \]

The difference from Exp3 is in the definition of \( \tilde{l}_{i,t} \) in step 2, which now incorporates the lower bounds \( \lambda_{i,t} \). All other elements remain the same. Note that if \( \lambda_{i,t} = 0 \) for every \( i \) and \( t \) (the pure bandit case), the algorithm becomes Exp3, and if \( s_{i,t} = 0 \) for every \( i \) and \( t \) (the full-information case), it becomes Hedge.

We now prove expected regret bounds for Exp3.LB for an oblivious, or non-adaptive, adversary. Namely, we assume that all losses and lower bounds are decided by the adversary before the beginning of the game with the learner. Technically, we will bound the pseudo-regret of Exp3.LB, defined as \( \mathcal{R}_{Exp3.LB,T} = \mathbb{E}L_{Exp3.LB,T} - \min_i \mathbb{E}L_{i,t} \). For oblivious adversaries, the notions of expected regret and pseudo-regret coincide. For the rest of this paper we will assume that the adversary is oblivious.

The analysis is adapted from the work of Gofer (2014, Theorem 23) given originally for the full-information case. The main difference lies in simulating a two-part update step, which first performs the “bandit part” of the Exp3.LB update step, namely, adding \( (s_{i,t}/p_{i,t}) \cdot \mathbb{I}\{I_t = i\} \) to \( \tilde{l}_{i,t} \), and then the “full-information part”, namely, adding \( \lambda_{i,t} \).

**Theorem 1** Let \( Q \) be an upper bound on
\[
\frac{1}{2} q(\lambda_{1:T}) + 2 \sum_{i=1}^{T} \|s_i\|^2 + 4 \sum_{i=1}^{T} \max_{i} \left\{ s_{i,t} \right\} \cdot d(\lambda_t) .
\]

Then for any \( \eta > 0 \) it holds that
\[
\mathcal{R}_{Exp3.LB,T} \leq \frac{1}{\eta} \log N + \frac{1}{4} \eta Q ,
\]
and in particular for $\eta = \sqrt{4 \log N/ Q}$,
\[
    \overline{R}_{E x p B L T} \leq \sqrt{Q \log N}.
\]

**Proof** Let $\Phi(L) = -(1/\eta) \log \frac{1}{N} \sum_{j=1}^{N} e^{-\eta L_j}$ and define
\[
    \overline{L}_{i,t-\frac{1}{2}} = \overline{L}_{i,t-1} + \frac{a_i \cdot \mathbb{I}\{I_t = i\}}{p_i}.
\]
for $1 \leq t \leq T$, $1 \leq i \leq N$, recalling that $\overline{L}_0 = 0$. Noting that $\Phi(0) = 0$ we have
\[
    \Phi(\overline{L}_T) = \sum_{t=1}^{T} \Phi(\overline{L}_t) - \Phi(\overline{L}_{t-\frac{1}{2}}) + \sum_{t=1}^{T} \Phi(\overline{L}_{t-\frac{1}{2}}) - \Phi(\overline{L}_{t-1}).
\]
(1)

Now, for every $x, x' \in \mathbb{R}^N$ we have by Taylor’s expansion that
\[
    \Phi(x') - \Phi(x) = \nabla \Phi(x) \cdot (x' - x) + \frac{1}{2}(x' - x)^\top \nabla^2 \Phi(z)(x' - x),
\]
where $z \in [x, x']$. Thus, denoting
\[
    A_t = (\overline{L}_t - \overline{L}_{t-\frac{1}{2}})^\top \nabla^2 \Phi(z_t)(\overline{L}_t - \overline{L}_{t-\frac{1}{2}}),
\]
\[
    B_t = (\overline{L}_{t-\frac{1}{2}} - \overline{L}_{t-1})^\top \nabla^2 \Phi(z_{t-\frac{1}{2}})(\overline{L}_{t-\frac{1}{2}} - \overline{L}_{t-1}),
\]
\[
    C_t = (\nabla \Phi(\overline{L}_{t-\frac{1}{2}}) - \nabla \Phi(\overline{L}_{t-1})) \cdot (\overline{L}_t - \overline{L}_{t-1}),
\]
where $z_t \in [\overline{L}_{t-\frac{1}{2}}, \overline{L}_t], z_{t-\frac{1}{2}} \in [\overline{L}_{t-1}, \overline{L}_{t-\frac{1}{2}}]$ for every $t$, we have from (1) that
\[
    \Phi(\overline{L}_T) = \sum_{t=1}^{T} \nabla \Phi(\overline{L}_{t-\frac{1}{2}}) \cdot (\overline{L}_t - \overline{L}_{t-\frac{1}{2}}) + \frac{1}{2} A_t + \nabla \Phi(\overline{L}_{t-1}) \cdot (\overline{L}_{t-\frac{1}{2}} - \overline{L}_{t-1}) + \frac{1}{2} B_t
\]
\[
    = \sum_{t=1}^{T} \nabla \Phi(\overline{L}_{t-1}) \cdot (\overline{L}_t - \overline{L}_{t-1}) + \frac{1}{2} A_t + \frac{1}{2} B_t + C_t
\]
or
\[
    \Phi(\overline{L}_T) = \sum_{t=1}^{T} p_t \cdot \overline{L}_t + \frac{1}{2} A_t + \frac{1}{2} B_t + C_t.
\]
(2)

We now turn to the quadratic terms $A_t$ and $B_t$. It is a well-known fact that for every $x, z \in \mathbb{R}^N$,
\[
    x^\top \nabla^2 \Phi(z)x = -\eta Var(Y_{p,x}),
\]
where $Y_{p,x}$ is a random variable that obtains values in $\{x_1, \ldots, x_N\}$. (This claim is stated in the appendix as Lemma 15 for completeness; for its proof, see, e.g., Gofer and Mansour, 2016, Lemma 6). By Popoviciu’s inequality (Lemma 16 in the appendix) we therefore have that
\[
    A_t \geq -\frac{\eta}{4} (\max_i \lambda_{i,t} - \min_i \lambda_{i,t})^2.
\]
(3)
For $B_t$, writing $q = \nabla \Phi(z_{t-rac{1}{2}})$, we have by Lemma 15 that

$$B_t = -\eta(L_{t-rac{1}{2}} - L_{t-1})^\top (\text{diag}(q) - q q^\top)(L_{t-rac{1}{2}} - L_{t-1})$$

$$= -\eta(q_t - q_t^2) \cdot \frac{s_{t,t}^2}{p_{t,t}} ,$$

$$\geq -\eta q_t \cdot \frac{s_{t,t}^2}{p_{t,t}} ,$$

where we used the fact that only the index $I_t$ in $L_{t-rac{1}{2}} - L_{t-1}$ may be non-zero. Furthermore, since

$$z_{t-rac{1}{2}} \in \overline{[L_{t-1}, L_{t-rac{1}{2}}]} ,$$

we have that $z_{i,t-rac{1}{2}} = - \lambda_{i,t-1}$ for $i \neq I_t$ and $z_{I_t,t-1} \geq L_{I_t,t-1}$. It follows that $q_t \leq p_{I_t,t}$, and therefore

$$B_t \geq -\eta \cdot \frac{s_{t,t}^2}{p_{I_t,t}} .$$

For bounding $C_t$ we again use the fact that $p_t = \nabla \Phi(L_{t-1})$ and $p_{t+rac{1}{2}} = \nabla \Phi(L_{t-rac{1}{2}})$ are probability vectors and that $L_{t-rac{1}{2}}$ and $L_{t-1}$ may differ only by the index $I_t$, where $L_{I_t,t-\frac{1}{2}} \geq L_{I_t,t-1}$. As a result, $p_{I_t,t+\frac{1}{2}} - p_{I_t,t} \leq 0$ and for $i \neq I_t$, it holds that $p_{I_t,t+\frac{1}{2}} - p_{I_t,t} \geq 0$. Recalling that $\lambda_t = - L_t - L_{t-\frac{1}{2}}$, we have that

$$C_t = \langle p_{t+\frac{1}{2}} - p_t \rangle \cdot \lambda_t \geq \min_i \{ \lambda_{i,t} \} \cdot \sum_{i \neq I_t} (p_{I_t,t+\frac{1}{2}} - p_{I_t,t}) + \max_i \{ \lambda_{i,t} \} \cdot (p_{I_t,t+\frac{1}{2}} - p_{I_t,t})$$

$$= \min_i \{ \lambda_{i,t} \} \cdot (p_{I_t,t} - p_{I_t,t+\frac{1}{2}}) + \max_i \{ \lambda_{i,t} \} \cdot (p_{I_t,t+\frac{1}{2}} - p_{I_t,t})$$

$$= (p_{I_t,t+\frac{1}{2}} - p_{I_t,t}) \cdot d(\lambda_t) .$$

Now, by a first-order Taylor expansion of $f(x) = \frac{\partial \Phi(x)}{\partial x_{I_t}}$ we have for some $z' \in \overline{[L_{t-1}, L_{t-\frac{1}{2}}]}$

$$p_{I_t,t+\frac{1}{2}} - p_{I_t,t} = f(L_{t-\frac{1}{2}}) - f(L_{t-1}) = \nabla f(z') \cdot (L_{t-\frac{1}{2}} - L_{t-1})$$

$$= \frac{\partial^2 \Phi(z')}{\partial x_{I_t}^2} \cdot \frac{s_{I_t,t}}{p_{I_t,t}} .$$

Again by Lemma 15 for $p' = \nabla \Phi(z')$ we have

$$\frac{\partial^2 \Phi(z')}{\partial x_{I_t}^2} = \eta(p_{I_t}^2 - p_{I_t}') \geq -\eta p_{I_t}' ,$$

and again, since $z' \in \overline{[L_{t-1}, L_{t-\frac{1}{2}}]}$, we have $p_{I_t}' \leq p_{I_t,t}$. Therefore,

$$p_{I_t,t+\frac{1}{2}} - p_{I_t,t} \geq -\eta p_{I_t,t} \cdot \frac{s_{I_t,t}}{p_{I_t,t}} ,$$

yielding that

$$C_t \geq -\eta s_{I_t,t} \cdot d(\lambda_t) \geq -\eta \max_i \{ s_{i,t} \} \cdot d(\lambda_t) .$$
Finally, we observe that for every $k$,
\[
\Phi(\tilde{L}_T) - \tilde{L}_{k,T} = -\frac{1}{\eta} \log \frac{1}{N} \sum_{j=1}^{N} e^{-\eta \tilde{L}_{j,T}} + \frac{1}{\eta} \log e^{-\eta \tilde{L}_{k,T}} = \frac{1}{\eta} \log \left( \frac{N \exp(-\eta \tilde{L}_{k,T})}{\sum_{j=1}^{N} e^{-\eta \tilde{L}_{j,T}}} \right) \leq \frac{1}{\eta} \log N .
\]
Combining this with (2), (3), (4), and (5) and rearranging, we get that for every $k$,
\[
\sum_{t=1}^{T} p_t \cdot \tilde{l}_t - \tilde{L}_{k,T} \leq \frac{1}{\eta} \log N + \frac{\eta}{8} \cdot q(\lambda_{1:T}) + \frac{\eta}{2} \sum_{t=1}^{T} s_{t,t}^{2} p_{t,t} + \eta \sum_{t=1}^{T} \max_{i} \{ s_{i,t} \} \cdot d(\lambda_t) .
\] (6)

We can now take expectations on both sides, preserving the inequality. On the r.h.s., we have that
\[
E \left[ \sum_{t=1}^{T} s_{t,t}^{2} p_{t,t} \right] = \sum_{t=1}^{T} E_{I_t \sim p_t} \frac{s_{t,t}^{2}}{p_{t,t}} = \sum_{t=1}^{T} \sum_{i=1}^{N} s_{i,t}^{2} ,
\]
where we used the rule of conditional expectations. On the l.h.s., we have
\[
E \left[ \sum_{t=1}^{T} p_t \cdot \tilde{l}_t \right] = E \left[ \sum_{t=1}^{T} p_t \cdot \lambda_t + l_{t,t} - \lambda_{t,t} \right]
= E \left[ \sum_{t=1}^{T} p_t \cdot \lambda_t \right] + E \left[ \sum_{t=1}^{T} l_{t,t} \right] - E \left[ \sum_{t=1}^{T} \lambda_{t,t} \right]
= E \left[ \sum_{t=1}^{T} p_t \cdot \lambda_t \right] + E \left[ L_{\text{Exp3}, LB,T} \right] - E \left[ \sum_{t=1}^{T} p_t \cdot \lambda_t \right]
= E \left[ L_{\text{Exp3}, LB,T} \right] .
\]
In addition we have that
\[
E \left[ \tilde{L}_{k,T} \right] = \sum_{t=1}^{T} E_{I_t \sim p_t} \lambda_{k,t} = \sum_{t=1}^{T} \sum_{k=1}^{K} E_{I_t \sim p_t} \left[ \lambda_{k,t} \frac{l_{k,t} - \lambda_{k,t}}{p_{k,t}} \cdot I_t = k \right]
= \sum_{t=1}^{T} E \left[ \lambda_{k,t} + (l_{k,t} - \lambda_{k,t}) \right]
= E \left[ L_{k,T} \right] .
\]
Thus, taking expectations in (6) yields that for every $k$,
\[
E \left[ L_{\text{Exp3}, LB,T} \right] - E \left[ L_{k,T} \right] \leq \frac{1}{\eta} \log N + \frac{\eta}{8} \cdot q(\lambda_{1:T}) + \frac{\eta}{2} \sum_{t=1}^{T} \sum_{i=1}^{N} s_{i,t}^{2} + \eta \sum_{t=1}^{T} \max_{i} \{ s_{i,t} \} \cdot d(\lambda_t) .
\]
We thus have that
\[
R_{\text{Exp3}, LB,T} \leq \frac{1}{\eta} \log N + \frac{1}{4} \eta Q ,
\]
and in particular for \( \eta = \sqrt{4 \log N/Q} \),

\[
R_{\text{Exp3.LB}, T} \leq \sqrt{Q \log N},
\]

completing the proof.

The bound of Theorem 1 may be simplified without changing its order up to multiplicative constants. Note that for any \( a \geq 0 \) and \( x \in \mathbb{R}^N \) with non-negative entries, we have

\[
2 \max_i \{x_i\} \cdot a \leq 2\|x\| \cdot a \leq a^2 + \|x\|^2,
\]

and therefore

\[
\frac{1}{2}(a^2 + \|x\|^2) \leq \frac{1}{2}a^2 + 2\|x\|^2 + 4 \max_i \{x_i\} \cdot a \leq 4(a^2 + \|x\|^2).
\]

Substituting \( d(\lambda_t) \) for \( a \) and \( s_t \) for \( x \), and summing over \( t \), we get

\[
\frac{1}{2} \sum_{t=1}^{T} (d(\lambda_t)^2 + \|s_t\|^2) \leq \frac{1}{2} \sum_{t=1}^{T} d(\lambda_t)^2 + 2\|s_t\|^2 + 4 \max_i \{s_{i,t}\} \cdot d(\lambda_t) \leq 4 \sum_{t=1}^{T} (d(\lambda_t)^2 + \|s_t\|^2).
\]

Recalling that \( q(\lambda_{1:T}) = \sum_{t=1}^{T} d(\lambda_t)^2 \) by definition, we get the following.

**Corollary 2** If \( Q' \) is an upper bound on \( 4(q(\lambda_{1:T}) + \sum_{t=1}^{T} \|s_t\|^2) \), then taking \( \eta = \sqrt{4 \log N/Q'} \), it holds that \( \mathbb{E}R_{\text{Exp3.LB}, T} \leq \sqrt{Q' \log N} \).

Finally, we note that a major chunk of the proof of Theorem 1 holds in a more general scenario. Specifically, even if the algorithm used an arbitrary \( 0 \leq s'_{i,t} \) instead of \( s_{i,t} \), Equation 6 would still hold. This fact will be useful when we consider our high-probability variant, and we therefore state the following corollary:

**Corollary 3** Replacing \( s_{i,t} \) with some \( s'_{i,t} \geq 0 \) in Exp3.LB for every \( i \) and \( t \), it holds for every \( k \) that

\[
\sum_{i=1}^{T} p_t \cdot \tilde{L}_{k,T} \leq \frac{1}{\eta} \log N + \frac{\eta}{8} q(\lambda_{1:T}) + \frac{\eta}{2} \sum_{t=1}^{T} \frac{s^2_{i,t}}{p_{i,t}} + \eta \sum_{t=1}^{T} \max_i \{s'_{i,t}\} \cdot d(\lambda_t).
\]

### 3.1 Unknown Horizon

In Theorem 1 we were able to set \( \eta \) optimally, assuming foreknowledge of an upper bound \( Q \) on the quantity of interest. To remove such an assumption, it is customary to use a ‘doubling trick’, namely, to start the algorithm with a small initial guess for the upper bound, and whenever the guess is exceeded, double it and restart the algorithm. The resulting analysis typically yields a regret bound of the same general order.

In our setting, however, the slack data for unchosen actions is not observable. This is not a problem if all slacks are known to be zero (full information), but it hinders the use of a doubling trick in the most general setting. To overcome this issue to some extent, we may assume that the losses are bounded, WLOG in \( [0, 1] \), s.t. for every \( i \) and \( t \) we may replace \( s_{i,t} \) in the regret bound with its upper bound \( 1 - \lambda_{i,t} \). In this case it is straightforward to show the following.

**Corollary 4 (unknown horizon)** If \( l_{i,t} \in [0, 1] \) for every \( i \) and \( t \), then in conjunction with a doubling trick, the regret of Exp3.LB satisfies \( \mathbb{E}R_{\text{Exp3.LB}, T} = O \left( \sqrt{\max_i \{Q_uh, 1\} \log N} \right) \), where \( Q_uh = \frac{1}{2} q(\lambda_{1:T}) + 2 \sum_{t=1}^{T} \sum_{i=1}^{N} (1 - \lambda_{i,t})^2 + 4 \sum_{t=1}^{T} \max_i \{1 - \lambda_{i,t}\} \cdot d(\lambda_t) \).
4. Special Feedback Settings

The regret bound of Exp3.LB given in Theorem 1 is dominated by the quantity $Q$, which comprises three distinct terms. These will be referred to as the full information term, $q(\lambda_{1:T})$, the bandit term, $\sum_{t=1}^T \|s_t\|^2$, and the term $\sum_{t=1}^T \max_s \{s_{it}\} \cdot d(\lambda_t)$, which we will call the hybrid term.

In the full information and bandit scenarios, the bound degenerates to the appropriate single term. Specifically, in the full-information case, all slacks $s_{it}$ become zero and Exp3.LB becomes the Hedge algorithm. Theorem 1 immediately retrieves a known second-order regret bound for Hedge (Gofer, 2014, Theorem 23).

**Corollary 5 (full-information feedback)** If $q$ is an upper bound on $q(1:T)$ and $\eta = \sqrt{8 \log N/q}$ then $E_{R_{Hedge,T}} \leq \sqrt{(q/2) \log N}$.

In the bandit case, making the standard assumption that $l_{i,t} \in [0, 1]$ for every $i$ and $t$, we may take $\lambda_{1,t} = \ldots = \lambda_{N,t} = 0$, s.t. $d(\lambda_t) = 0$ for every $t$ and $q(\lambda_{1:T}) = 0$. We have for every $i$ and $t$ that $s_{it} \leq 1$, and Exp3.LB becomes Exp3. Theorem 1 then yields the zeroth-order bound for Exp3 (see, e.g., Bubeck and Cesa-Bianchi, 2012, Theorem 3.1).

**Corollary 6 (bandit feedback)** If $\eta = \sqrt{2 \log N/(NT)}$, then $E_{R_{Exp3,T}} \leq \sqrt{2TN \log N}$.

There is, however, another interesting setting highlighted by the regret bound of Exp3.LB. When on each round $t$, we have either $d(\lambda_i) = 0$ or $\max_s \{s_{it}\} = 0$, the hybrid term disappears. This happens in particular if on every round the adversary provides either a bandit feedback or a full-information feedback, possibly by adversarial choice. For this scenario, Theorem 1 gives the following zeroth-order bound.

**Corollary 7 (mixed feedback)** If there are $T_b$ bandit feedback rounds and $T_f$ full-information feedback rounds then for $\eta = \sqrt{8 \log N/(T_f + 4NT_b)}$ it holds that

$$E_{R_{Exp3,LB,T}} \leq \sqrt{(T_f/2 + 2NT_b) \log N}.$$ 

It should be noted that even without foreknowledge of $T_b$ and $T_f$, a standard doubling trick on $T_f/2 + 2NT_b$ yields the same order of bound, namely, $O(\sqrt{(T_f/2 + 2NT_b) \log N})$.

More generally, we may consider a variable subset feedback scenario, where on each round $t$, the true losses for a subset $S_t$ of the experts, chosen by the adversary, are revealed to the learner. For losses bounded in $[0, 1]$, we have that

$$\frac{1}{2} q(\lambda_{1:T}) + 2 \sum_{t=1}^T \|s_t\|^2 + 4 \sum_{t=1}^T \max_i \{s_{it}\} \cdot d(\lambda_t) \leq \frac{T}{2} + 2 \sum_{t=1}^T (N - |S_t|) + 4T \leq \frac{9}{2} T + 2 \sum_{t=1}^T (N - |S_t|),$$

and applying Exp3.LB thus yields the following.

**Corollary 8 (variable set feedback)** If for every $t$ the learner receives feedback for a subset $S_t$ of the experts, then for $\eta = \sqrt{8 \log N/(9T + 4 \sum_T (N - |S_t|))}$ it holds that

$$E_{R_{Exp3,LB,T}} \leq \sqrt{((9T/2 + 2 \sum_{t=1}^T (N - |S_t|)) \log N}.$$
Again, a doubling trick is applicable even if the sets are not known in advance, yielding the same order of bound, written more succinctly as $O(\sqrt{T + \sum_i (N - |S_i|)} \log N)$.

5. Model and Algorithm Variants

In our model, lower bounds on the losses feature as extra information given to the learner, and then play a role in the estimated losses defined by Exp3.LB, which is expected. We might ask if other quantities could feature as the extra information, and how Exp3 should be modified to accommodate them. One natural choice is upper bounds on the losses instead of lower bounds.

Let $\alpha_{i,t}$, for $1 \leq i \leq N$, $1 \leq t \leq T$, be arbitrary quantities, and assume that the adversary reveals $\alpha_{1,t}, \ldots, \alpha_{N,t}$ to the learner along with the loss of the chosen action $I_t$. We then define a variant of Exp3, denoted by Exp3.\( \alpha \), which is the same as Exp3 except that the estimated loss becomes

$$\tilde{l}_{i,t} = l_{i,t} - \alpha_{i,t} \frac{1}{p_{i,t}} I_t \{I_t = i\} + \alpha_{i,t}. $$

A careful examination of the proof of Theorem 1 reveals that $\lambda_{i,t}$ may be replaced by $\alpha_{i,t}$ (and of course, $s_{i,t}$ by $l_{i,t} - \alpha_{i,t}$), as long as $l_{i,t} - \alpha_{i,t} \geq 0$ for every $i$ and $t$. We thus obtain the following more general form of Theorem 1 (written here more succinctly):

**Theorem 9** Let $Q_\alpha$ be an upper bound on

$$\frac{1}{2} q(\alpha_{1:T}) + 2 \sum_{t=1}^{T} \sum_{i=1}^{N} (l_{i,t} - \alpha_{i,t})^2 + 4 \sum_{t=1}^{T} \max_i \{l_{i,t} - \alpha_{i,t}\} \cdot d(\alpha_t).$$

Then for $\eta = \sqrt{4 \log N/Q_\alpha}$, it holds that $\mathbb{E} R_{\text{Exp3.} \alpha, T} \leq \sqrt{Q_\alpha \log N}$.

To apply this theorem in a case where upper bounds on the losses are provided, we need an extra step. The reason is that we cannot simply take $\alpha_{i,t}$ to be the upper bound $v_{i,t}$ on $l_{i,t}$, since the requirement $l_{i,t} - \alpha_{i,t} \geq 0$ would be violated. However, assuming $M_t$ is an upper bound on $\max_i \{v_{i,t} - l_{i,t}\}$ known to the learner, then $\alpha_{i,t} = v_{i,t} - M_t$ would be a valid choice for $\alpha_{i,t}$. Denoting the resulting algorithm by Exp3.UB, we obtain the following.

**Corollary 10** For the above scenario, if $Q_\alpha$ is an upper bound on

$$\frac{1}{2} q(\alpha_{1:T}) + 2 \sum_{t=1}^{T} \sum_{i=1}^{N} (l_{i,t} - \alpha_{i,t})^2 + 4 \sum_{t=1}^{T} \max_i \{l_{i,t} - \alpha_{i,t}\} \cdot d(\alpha_t),$$

then for $\eta = \sqrt{4 \log N/Q_\alpha}$, it holds that $\mathbb{E} R_{\text{Exp3.} \text{UB}, T} \leq \sqrt{Q_\alpha \log N}$.

6. Lower Bounds

The upper bounds on the expected regret of Exp3.LB that were shown in Section 3 featured quantities of the form $\Theta(q(\lambda_{1:T}) + \sum_t ||s_t||^2)$. Given a value $Q = q(\lambda_{1:T}) + \sum_t ||s_t||^2$, we may consider either a full-information scenario, where $Q = q(1_{1:T})$ or a bandit scenario, where $Q = \sum_t ||1_t||^2$. We may then use existing lower bounds, and in both cases these bounds are of the form $\Omega(\sqrt{Q})$. 

We may also examine a more elaborate requirement, where we are prescribed both \( Q_1 = q(\lambda_{1:T}) \) and \( Q_2 = \sum_t \| s_t \|^2 \). In this case, since
\[
q(\lambda_{1:T}) + \sum_t \| s_t \|^2 = \Theta(\max\{q(\lambda_{1:T}), \sum_t \| s_t \|^2\}),
\]
we may consider a full-information scenario for \( Q_1 \) if \( Q_1 \geq Q_2 \), and a bandit scenario for \( Q_2 \), otherwise. In both cases we can then add artificial rounds to fulfill the rest of the prescription (\( Q_2 \) or \( Q_1 \)) without possibly decreasing the expected regret. Existing lower bounds then yield an \( \Omega(\sqrt{\max\{Q_1, Q_2\}}) \) or equivalently, \( \Omega(\sqrt{Q_1 + Q_2}) \), as before.

Our bounds are thus tight up to logarithmic factors for the above requirements. We comment, however, that in principle, there might be more elaborate requirements that would call for more refined bounds. In this context it is interesting to consider the bound of Corollary 8, for which results on graph-structured feedback are applicable. For the case where the set \( S_t \) is fixed over time, Alon et al. (2017) give an optimal lower bound, which is the same as our upper bound up to logarithmic factors.

7. High-Probability Regret Bounds

Like Exp3 on which it is based, Exp3.LB uses loss estimates \( \tilde{l}_{i,t} \) whose variance may behave like \( 1/p_{i,t} \). To enable regret bounds that hold with high probability, special care is required to control this variance. The authors of Exp3 introduced the algorithmic variant Exp3.P, which biases the loss estimates and mixes the probability of Exp3 with a suitable uniform distribution. Here we only bias the loss estimate of the chosen action. This allows us to define a bias that depends on the slack, which is unobservable except for the chosen action. The resulting algorithm, Exp.LB.P, is given below.

The new algorithm is identical to Exp3.LB, except for using the corrected slacks \( s_{i,t}(1 - x_{i,t}) \) instead of \( s_{i,t} \) in the estimated losses (see step 2 of the algorithm for the definition of \( x_{i,t} \)). Intuitively, for \( \beta > 0 \), the factor \( x_{i,t} \) approaches 1 for small probabilities and generally prevents extreme behavior of the estimated losses. For \( \beta = 0 \), Exp3.LB.P simply becomes Exp3.LB. Some useful properties of this correction factor are summarized in the next technical lemma.

**Lemma 11** For any \( i \) and \( t \), if \( \beta s_{i,t} \leq 1 \), then the correction factor \( x_{i,t} \) satisfies the following:

(i) \( x_{i,t} \) is well-defined, obtains values in \([0, 1]\), and \( x_{i,t} = 0 \) iff \( \beta s_{i,t} = 0 \).

(ii) \( x_{i,t} = \beta s_{i,t} \left( \frac{1 - x_{i,t}}{p_{i,t}} + 2x_{i,t} - 1 \right) \).

(iii) \( \beta s_{i,t} \left( \frac{1 - x_{i,t}}{p_{i,t}} - 1 \right) \leq 1 \).

(iv) \( p_{i,t} x_{i,t} s_{i,t} \leq \beta s_{i,t}^2 \).

(v) \( \beta s_{i,t} (1 - x_{i,t})^2 \leq 1 \).

|---|
| We note that different alternatives to the biasing mechanism of Exp3.P have also been introduced by Audibert and Bubeck (2010) and Kocák et al. (2014), the latter in the context of Exp3. |
| This behavior depends on the magnitude of \( \beta s_{i,t} \). It should also be noted that technically, we also allow \( \beta s_{i,t} = 1 \), which implies \( x_{i,t} = 1 \) for all probability values. |
Algorithm 4: Exp3.LB.P

Parameters: $\eta > 0, \beta \geq 0$.
Let $p_1$ be the uniform distribution over $\{1, \ldots, N\}$, and let $\tilde{L}_0 = 0$.
For each round $t = 1, \ldots, T$
1. Draw an action $I_t$ from the probability distribution $p_t$.
2. Define $x_{i,t} = \frac{\beta s_{i,t}(1-p_{i,t})}{p_{i,t}(1-\beta s_{i,t}) + \beta s_{i,t}(1-p_{i,t})}$ and calculate for $i = I_t$.
3. For each action $i = 1, \ldots, N$, compute the estimated loss
   \[ \tilde{l}_{i,t} = \frac{s_{i,t}(1-x_{i,t}) \mathbb{I}\{I_t = i\}}{p_{i,t}} + \lambda_{i,t} \]
   and update the estimated cumulative loss $\tilde{L}_{i,t} = \tilde{L}_{i,t-1} + \tilde{l}_{i,t}$.
4. Compute the new distribution over actions $p_{t+1} = (p_{1,t+1}, \ldots, p_{N,t+1})$, where
   \[ p_{i,t+1} = \frac{\exp(-\eta \tilde{L}_{i,t})}{\sum_{k=1}^{N} \exp(-\eta \tilde{L}_{k,t})} \]

Proof

Derived by simple arithmetic from the definition of $x_{i,t}$.

(i) Since we always have that $0 < p_{i,t} < 1$ and since $0 \leq \beta s_{i,t} \leq 1$, the claim is obvious.

(ii) If $\beta s_{i,t} = 0$ then the claim is true, so assuming $\beta s_{i,t} > 0$, we have that
   \[ x_{i,t} = \frac{\beta s_{i,t}(1-p_{i,t})}{p_{i,t} + \beta s_{i,t} - 2\beta p_{i,t} s_{i,t}} = \frac{\frac{1}{p_{i,t}} - 1}{\frac{1}{\beta s_{i,t}} + \frac{1}{p_{i,t}} - 2} , \]
   and therefore,
   \[ x_{i,t} = \frac{1}{\beta s_{i,t}} + \frac{1}{p_{i,t}} - 2 = \frac{1}{p_{i,t}} - 1 . \]

Rearranging, we get
   \[ \frac{x_{i,t}}{\beta s_{i,t}} = \frac{1 - x_{i,t}}{p_{i,t}} + 2x_{i,t} - 1 , \]
   and multiplying both sides by $\beta s_{i,t}$ yields the claim.

(iii) From (i) and (ii) we immediately have that
   \[ \beta s_{i,t} \left( \frac{1 - x_{i,t}}{p_{i,t}} - 1 \right) \leq \beta s_{i,t} \left( \frac{1 - x_{i,t}}{p_{i,t}} + 2x_{i,t} - 1 \right) = x_{i,t} \leq 1 . \]

(iv) From (ii) we have that
   \[ p_{i,t} x_{i,t} s_{i,t} = \beta s_{i,t}^2 (1 - x_{i,t} + (2x_{i,t} - 1)p_{i,t}) . \]
If \( z \in [0, 1] \), the expression \( 1 - z + (2z - 1)p_{i,t} \) attains its maximum for \( z = 0 \) or \( z = 1 \), and therefore
\[
p_{i,t}x_{i,t}s_{i,t} \leq \beta s_{i,t}^2 \max\{1 - p_{i,t}, p_{i,t}\} \leq \beta s_{i,t}^2.
\]

(v) Denote \( a = \beta s_{i,t} \). By (ii) we have that
\[
\frac{a(1 - x_{i,t})}{p_{i,t}} = x_{i,t} - a(2x_{i,t} - 1),
\]
and therefore
\[
\frac{a(1 - x_{i,t})^2}{p_{i,t}} = (1 - x_{i,t})(x_{i,t} - a(2x_{i,t} - 1)).
\]

For \( a \in [0, 1] \) it is clear that the r.h.s. of the last expression attains its maximum for \( a = 0 \) or \( a = 1 \), yielding that
\[
\frac{a(1 - x_{i,t})^2}{p_{i,t}} \leq \max\{x_{i,t}(1 - x_{i,t}), (1 - x_{i,t})^2\} \leq 1.
\]

The proof is complete.

The definition of \( x_{i,t} \) is handy in proving the following key lemma, which is modified from Lemma 3.2 in Bubeck and Cesa-Bianchi (2012).

**Lemma 12** Let \( \beta > 0 \) and \( \max_{i,t}\{\beta s_{i,t}\} \leq 1 \), and fix \( 1 \leq i \leq N \). For every \( \delta > 0 \), it holds with probability at least \( 1 - \delta \) that
\[
\overline{L}_{i,T} \leq L_{i,T} + \frac{1}{\beta} \log \frac{1}{\delta}.
\]

**Proof** Let \( \mathbb{E}_t \) be the expectation conditioned on \( I_1, \ldots, I_{t-1} \). Since \( e^z \leq 1 + z + z^2 \) for every \( z \leq 1 \), and using part (iii) of Lemma 11, we have for every \( t \) that
\[
\mathbb{E}_t \exp \left( \beta s_{i,t} \left( \frac{1 - x_{i,t}}{p_{i,t}} \right) I\{I_t = i\} - 1 \right) \leq 1 + \mathbb{E}_t \left[ \beta s_{i,t} \left( \frac{1 - x_{i,t}}{p_{i,t}} \right) I\{I_t = i\} - 1 \right] + \mathbb{E}_t \left[ \beta^2 s_{i,t}^2 \left( \frac{1 - x_{i,t}}{p_{i,t}} \right) I\{I_t = i\} - 1 \right]^2
\]
\[
= 1 - \beta x_{i,t}s_{i,t} + \beta^2 s_{i,t}^2 \left( \frac{1 - x_{i,t}}{p_{i,t}} \right)^2 + 2x_{i,t} - 1
\]
\[
= 1 + \beta s_{i,t} \left( \frac{1 - x_{i,t}}{p_{i,t}} \right)^2 + 2x_{i,t} - 1
\]
\[
= 1 + \beta s_{i,t} \left( \frac{1 - x_{i,t}}{p_{i,t}} + 2x_{i,t} - 1 \right) - x_{i,t}
\]
\[
\leq 1 + \beta s_{i,t} \left( \frac{1 - x_{i,t}}{p_{i,t}} + 2x_{i,t} - 1 \right) - x_{i,t}
\]
\[
= 1,
\]
where the last equality uses part (ii) of Lemma 11. By a further use of induction we obtain that
\[
\mathbb{E} \left[ \exp \left( \beta \sum_{t=1}^{T} \frac{s_{i,t}(1 - x_{i,t})}{p_{i,t}} I\{I_t = i\} - \beta \sum_{t=1}^{T} s_{i,t} \right) \right] \leq 1.
\]
Now, for any random variable \( X \), Markov’s inequality implies that \( \mathbb{P}(X > \log(1/\delta)) \leq \delta \mathbb{E} e^X \). Thus, with probability at least \( 1 - \delta \),

\[
\beta \sum_{t=1}^{T} \frac{s_{i,t}(1-x_{i,t})}{p_{i,t}} \mathbb{I}\{I_t = i\} - \beta \sum_{t=1}^{T} s_{i,t} \leq \log \frac{1}{\delta},
\]

or equivalently,

\[
\sum_{t=1}^{T} \frac{s_{i,t}(1-x_{i,t})}{p_{i,t}} \mathbb{I}\{I_t = i\} + \lambda_{i,t} - l_{i,t} \leq \frac{1}{\beta} \log \frac{1}{\delta},
\]

namely,

\[
\tilde{L}_{i,T} - L_{i,T} \leq \frac{1}{\beta} \log \frac{1}{\delta},
\]

completing the proof. \(\square\)

We can now bound the regret of Exp3.LB.P.

**Theorem 13** Let \( 0 < \delta < 1 \), let \( Q \) be an upper bound on

\[
\frac{1}{2} q(\lambda_{1:T}) + 2 \sum_{t=1}^{T} \|s_t\|^2 + 4 \sum_{t=1}^{T} \max_i \{s_{i,t}\} \cdot d(\lambda_t),
\]

and set \( \eta = \sqrt{\frac{1}{Q} \log N} \).

(i) If \( \beta = \sqrt{\frac{2}{Q} \log \frac{N + 3}{\delta}} \), then assuming \( \beta \max_i \{s_{i,t}\} \leq 1 \), it holds w.p. at least \( 1 - \delta \) that

\[
R_{\text{Exp3.LB.P,T}} = O\left(\sqrt{Q \log \frac{N}{\delta}}\right).
\]

(ii) If \( \beta = \sqrt{\frac{2}{Q}} \), then w.p. at least \( 1 - \delta \), \( R_{\text{Exp3.LB.P,T}} = O\left(\sqrt{Q \cdot \log(N/\delta)}\right) \).

(iii) For the scenario where \( l_{i,t} \in [0,1] \) for every \( i \) and \( t \), requiring \( Q \geq 2T \) and setting

\[
\beta = \min\left\{1, \sqrt{\frac{2}{Q} \log \frac{N + 3}{\delta}}\right\}
\]

yields that \( R_{\text{Exp3.LB.P,T}} = O\left(\sqrt{Q \log(N/\delta)}\right) \) w.p. at least \( 1 - \delta \). This bound implies the zeroth-order regret bound of Exp3.P, \( O(\sqrt{NT \log(N/\delta)}) \), for the bandit setting and of Hedge, \( O(\sqrt{T \log(N/\delta)}) \), for the full-information setting.

**Proof** Much of the analysis of Exp3.LB is also applicable to Exp3.LB.P, and this shared part is given in Corollary 3. Thus, since \( s'_{i,t} = s_{i,t}(1-x_{i,t}) \in [0,s_{i,t}] \) for every \( i \) and \( t \), we have for every \( k \) that

\[
\sum_{t=1}^{T} p_t \cdot \tilde{I}_t - \bar{L}_{k,T} \leq \frac{1}{\eta} \log N + \frac{\eta}{8} \cdot q(\lambda_{1:T}) + \frac{\eta}{2} \sum_{t=1}^{T} \frac{s_{i,t}^2}{p_{i,t}} + \eta \sum_{t=1}^{T} \max_i \{s_{i,t}\} \cdot d(\lambda_t), \quad (7)
\]
where we additionally replaced \( \max_{i} \{ s_{i,t}^\prime \} \) with \( \max_{i} \{ s_{i,t} \} \).

We next establish some high-probability bounds. First, by the Azuma-Hoeffding inequality (see, e.g., Lemma A.7 in Cesa-Bianchi and Lugosi, 2006) it holds w.p. at least \( 1 - \delta \) that

\[
\sum_{t=1}^{T} p_t \cdot \lambda_t - \lambda_{I_t,t} \geq - \left( \frac{1}{2} \log \frac{1}{\delta} \sum_{t=1}^{T} d(\lambda_t)^2 \right)^{\frac{1}{2}} = - \left( \frac{1}{2} q(\lambda_{1:T}) \log \frac{1}{\delta} \right)^{\frac{1}{2}} . \tag{8}
\]

Then, we have for every \( t \) that \( 0 \leq x_{I_t,t} s_{I_t,t} \leq \max_{i} \{ s_{i,t} \} \) and

\[
\mathbb{E}_t x_{I_t,t} s_{I_t,t} = \sum_{i=1}^{N} p_{i,t} x_{i,t} s_{i,t} \leq \sum_{i=1}^{N} \beta s_{i,t}^2 = \beta \|s_t\|^2 ,
\]

where the inequality is by part (iv) of Lemma 11. Thus, again by the Azuma-Hoeffding inequality, it holds w.p. at least \( 1 - \delta \) that

\[
\sum_{t=1}^{T} x_{I_t,t} s_{I_t,t} - \beta \sum_{t=1}^{T} \|s_t\|^2 \leq \left( \frac{1}{2} \log \frac{1}{\delta} \sum_{t=1}^{T} \max_{i} \{ s_{i,t}^2 \} \right)^{\frac{1}{2}} . \tag{9}
\]

Using (8) and (9) we obtain that

\[
\sum_{t=1}^{T} p_t \cdot \tilde{I}_t = \sum_{t=1}^{T} s_{I_t,t} (1 - x_{I_t,t}) + p_t \cdot \lambda_t = \sum_{t=1}^{T} l_{I_t,t} + \sum_{t=1}^{T} (p_t \cdot \lambda_t - \lambda_{I_t,t}) - \sum_{t=1}^{T} x_{I_t,t} s_{I_t,t} \geq \sum_{t=1}^{T} l_{I_t,t} - \left( \frac{1}{2} q(\lambda_{1:T}) \log \frac{1}{\delta} \right)^{\frac{1}{2}} - \beta \sum_{t=1}^{T} \|s_t\|^2 - \left( \frac{1}{2} \log \frac{1}{\delta} \sum_{t=1}^{T} \max_{i} \{ s_{i,t}^2 \} \right)^{\frac{1}{2}} . \tag{10}
\]

Next, we bound the term \( \sum_{t=1}^{T} \frac{s_{I_t,t}^2}{p_{I_t,t}} \) in a similar way. It holds that \( \mathbb{E}_{t} \frac{s_{I_t,t}^2}{p_{I_t,t}} \leq \|s_t\|^2 \), and in addition,

\[
\frac{s_{I_t,t}^2}{p_{I_t,t}} = \frac{s_{I_t,t}^2 (1 - x_{I_t,t})^2}{p_{I_t,t}} \leq \beta^{-1} s_{I_t,t} \leq \beta^{-1} \max_{i} \{ s_{i,t} \} ,
\]

where the first inequality is by part (v) of Lemma 11. We thus have w.p. at least \( 1 - \delta \) that

\[
\sum_{t=1}^{T} \frac{s_{I_t,t}^2}{p_{I_t,t}} - \sum_{t=1}^{T} \|s_t\|^2 \leq \left( \frac{1}{2} \beta^{-2} \log \frac{1}{\delta} \sum_{t=1}^{T} \max_{i} \{ s_{i,t}^2 \} \right)^{\frac{1}{2}} . \tag{11}
\]

Finally, by Lemma 12 we have w.h.p. that \( \tilde{L}_{k,T} \leq L_{k,T} + \frac{1}{\beta} \log \frac{1}{\delta} \). We may combine this bound with Equation (10) to yield

\[
\sum_{t=1}^{T} p_t \cdot \tilde{I}_t - \tilde{L}_{k,T} \geq \sum_{t=1}^{T} l_{I_t,t} - \left( \frac{1}{2} q(\lambda_{1:T}) \log \frac{1}{\delta} \right)^{\frac{1}{2}} - \beta \sum_{t=1}^{T} \|s_t\|^2 - \left( \frac{1}{2} \log \frac{1}{\delta} \sum_{t=1}^{T} \max_{i} \{ s_{i,t}^2 \} \right)^{\frac{1}{2}} - L_{k,T} - \frac{1}{\beta} \log \frac{1}{\delta} . \tag{12}
\]
From (7) and (11) we also have that
\[
\sum_{t=1}^{T} p_t \tilde{1}_t - \tilde{L}_{k,T} \leq \frac{1}{\eta} \log N + \frac{\eta}{8} \cdot q(\lambda_{1:T}) + \frac{\eta T}{2} \sum_{t=1}^{T} \|s_t\|^2 + \frac{\eta}{2\beta} \left( \frac{1}{\delta} \log \frac{1}{\delta} \sum_{t=1}^{T} \max_i \{s_{i,t}^2\} \right)^{\frac{1}{2}} + \eta \sum_{t=1}^{T} \max_i \{s_{i,t}\} \cdot d(\lambda_t) .
\] (13)

Combining Equations (12) and (13) and rearranging, we obtain that for every \(k\),
\[
\sum_{t=1}^{T} l_{t,t} - L_{k,T} \leq \frac{1}{\eta} \log N + \frac{\eta}{8} \cdot q(\lambda_{1:T}) + \frac{\eta T}{2} \sum_{t=1}^{T} \|s_t\|^2 + \eta \sum_{t=1}^{T} \max_i \{s_{i,t}\} \cdot d(\lambda_t) + \beta \sum_{t=1}^{T} \|s_t\|^2 + \frac{1}{\beta} \log \frac{1}{\delta'} \left( 1 + \frac{\eta}{2\beta} \right) \left( \frac{1}{\delta} \log \frac{1}{\delta} \sum_{t=1}^{T} \max_i \{s_{i,t}^2\} \right)^{\frac{1}{2}} + \left( \frac{1}{2} q(\lambda_{1:T}) \log \frac{1}{\delta} \right)^{\frac{1}{2}} .
\] (14)

We briefly comment that throughout the proof, a total of \(N + 3\) events occur w.p. at least \(1 - \delta\). As usual, we may insure that all of them occur simultaneously w.p. at least \(1 - \delta\) by using \(\delta' = \delta / (N + 3)\) instead of \(\delta\) and invoking the union bound.

Now, the first line of the r.h.s. of (14) is exactly the regret bound of Theorem 1 and is minimized similarly. The first line thus becomes simply \(\sqrt{Q \log N}\). One may also observe that \(q(\lambda_{1:T}) \leq 2Q\) and
\[
\sum_{t=1}^{T} \max_i \{s_{i,t}^2\} \leq \sum_{t=1}^{T} \|s_t\|^2 \leq \frac{1}{2} Q .
\]

Therefore, setting \(\beta = \sqrt{\frac{3}{8} \log \frac{1}{\delta'}}\), we have that
\[
\beta \sum_{t=1}^{T} \|s_t\|^2 + \frac{1}{\beta} \log \frac{1}{\delta'} \leq \sqrt{2Q \log \frac{1}{\delta'}} .
\]

Writing \(A\) for \(\text{Exp3.LB.P}\), we thus obtain that
\[
R_{A,T} \leq \sqrt{Q \log N} + \sqrt{2Q \log \frac{1}{\delta'}} + \left( 1 + \sqrt{\frac{\log N}{2 \log \frac{1}{\delta'}}} \right) \cdot \sqrt{\frac{1}{4} Q \log \frac{1}{\delta'}} + \sqrt{2Q \log \frac{1}{\delta'}}
\leq \left( 1 + \frac{1}{2\sqrt{2}} \right) \cdot \sqrt{Q \log N} + \left( \sqrt{2} + \frac{3}{2} \right) \cdot \sqrt{Q \log \frac{1}{\delta'}}
= O\left( \sqrt{Q \log \frac{N}{\delta}} \right) ,
\] (15)
proving part (i).
To avoid the extra assumption that $\beta \max_{i,t} \{s_{i,t}\} \leq 1$, we can set $\beta = \sqrt{\frac{1}{2Q}}$. We thus have that if $\max_{i,t} \{s_{i,t}\} = 0$ then $\beta \max_{i,t} \{s_{i,t}\} \leq 1$ trivially, and otherwise,

$$\beta \leq \sqrt{\frac{2}{2\sum_t \|s_t\|^2}} \leq \sqrt{\frac{1}{\max_{i,t} \{s_{i,t}\}^2}} = \frac{1}{\max_{i,t} \{s_{i,t}\}}$$

as needed. It now holds that

$$\beta \sum_{t=1}^T \|s_t\|^2 + \frac{1}{\beta} \log \frac{1}{\delta'} \leq \sqrt{\frac{1}{2Q} \cdot \log \frac{e}{\delta'}}.$$ We then have from (14) that

$$R_{A,T} \leq \sqrt{Q \log N} + \sqrt{\frac{1}{2Q} \cdot \log \frac{e}{\delta'}} + \left(1 + \sqrt{\frac{1}{2} \log N}\right) \cdot \sqrt{\frac{1}{4Q} \log \frac{1}{\delta'}} + \sqrt{Q \log \frac{1}{\delta'}}$$

$$= O \left(\sqrt{Q} \cdot \log \frac{N}{\delta'}\right),$$

yielding part (ii).

For part (iii), we first comment that if $T$ is known, we may always assume that $Q \geq 2T$ (otherwise we use $\max\{Q, 2T\}$ instead of $Q$). Next, we note that the condition $\beta \max_{i,t} \{s_{i,t}\} \leq 1$ is satisfied if $\beta \leq 1$, and in particular by setting $\beta = \min\left\{1, \sqrt{\frac{2}{Q} \log \frac{1}{\delta'}}\right\}$. If $\beta < 1$, we have by part (i) that $R_{A,T} = O \left(\sqrt{Q \log (N/\delta)}\right)$. Otherwise, $\log (1/\delta') \geq T$, and it follows trivially that $R_{A,T} \leq T \leq \sqrt{Q \log (1/\delta')}$. Therefore, in any case it holds w.p. at least $1 - \delta$ that

$$R_{A,T} = O \left(\sqrt{Q \log (N/\delta)}\right).$$

(17)

It is easy to observe that in the bandit case we may use $Q = 2NT$, yielding a regret bound of $O \left(\sqrt{NT \log (N/\delta)}\right)$, and in the full-information case we may use $Q = \max\{T/2, 2T\} = 2T$, yielding a regret bound of $O \left(\sqrt{T \log (N/\delta)}\right)$, as required.

**Remark 14** The assumption that $\beta \max_{i,t} \{s_{i,t}\} \leq 1$, which was made in part (i) of Theorem 13, is mild. It holds trivially in the full information case, namely, $\max_{i,t} \{s_{i,t}\} = 0$, and otherwise we have that

$$\beta = \sqrt{\frac{2}{Q} \log \frac{N+3}{\delta}} \leq \sqrt{\frac{\log \frac{N+3}{\delta}}{\sum_{t=1}^T \|s_t\|^2}}.$$ 

Thus, it holds if $\max_{i,t} \{s_{i,t}\} \ll \sqrt{\sum_{t=1}^T \|s_t\|^2}$, namely, if $\{s_{i,t}\}$ is not concentrated on very few indices.

Finally, we point out that the expression bounded by $Q$ is in fact of the simpler and more interpretable form $\Theta(q(\lambda_{1:T}) + \sum_{t=1}^T \|s_t\|^2)$. This argument has already been made in the context of Theorem 1 and formalized in Corollary 2.
8. Conclusion

In this work we presented an online learning model that unifies and generalizes the full-information and bandit settings. We gave algorithms and analysis for this model, thus providing a single, generalized, framework. We proved regret bounds that are optimal up to logarithmic factors and handled both the expected regret and the high-probability regret regimes.

Our generalization works by modeling partial knowledge of losses as full knowledge of their lower bounds. This is in contrast to works on graph-structured feedback, where partial knowledge is modeled as full knowledge of losses for subsets of experts. In future work it would be interesting to examine a combination of our model with graph-structured feedback.

On a more technical aspect, it appears that current methods for proving regret lower bounds are not straightforward to apply for scenarios with slightly elaborate constraints on the losses, including in our model. Lower bounds for such scenarios would either strengthen tightness results or help suggest more refined regret upper bounds.

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Appendix A. Additional Claims

Lemma 15 Let \( z \in \mathbb{R}^N \), \( p_0 \in \Delta_N \), define \( \Phi(z) = -(1/\eta) \log \sum_{j=1}^N p_j e^{-\eta z_j} \), and denote \( p = \nabla \Phi(z) \). Then \( \nabla^2 \Phi(z) = \eta \cdot (pp^T - \text{diag}(p)) \preceq 0 \). Moreover, for every \( x \in \mathbb{R}^N \), it holds that \( x^T \nabla^2 \Phi(z) x = -\eta \text{Var}(Y_{p,x}) \), where \( Y_{p,x} \) is a random variable that satisfies for every \( 1 \leq i \leq N \) that \( \mathbb{P}(Y_{p,x} = x_i) = \sum_{j: x_j = x_i} p_j \).

Lemma 16 (Popoviciu’s inequality) If \( X \) is a bounded random variable with values in \([m, M]\), then \( \text{Var}(X) \leq (M - m)^2/4 \), with equality iff \( \mathbb{P}(X = M) = \mathbb{P}(X = m) = 1/2 \).

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