Approximation algorithms on $k$– cycle covering and $k$– clique covering

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Abstract

Given a weighted graph $G(V,E)$ with weight $w : E \to \mathbb{Z}^{|E|}_+$. A $k$– cycle covering is an edge subset $A$ of $E$ such that $G - A$ has no $k$–cycle. The minimum weight of $k$–cycle covering is the weighted covering number on $k$–cycle, denoted by $\tau_k(G_w)$. In this paper, we design a $k-1/2$ approximation algorithm for the weighted covering number on $k$–cycle when $k$ is odd.

Given a weighted graph $G(V,E)$ with weight $w : E \to \mathbb{Z}^{|E|}_+$. A $k$– clique covering is an edge subset $A$ of $E$ such that $G - A$ has no $k$–clique. The minimum weight of $k$–clique covering is the weighted covering number on $k$–clique, denoted by $\hat{\tau}_k(G_w)$. In this paper, we design a $(k^2 - k - 1)/2$ approximation algorithm for the weighted covering number on $k$–clique. Last, we discuss the relationship between $k$–clique covering and $k$–clique packing in complete graph $K_n$.

Keywords: $k$– cycle covering, $k$– clique covering, $k$– clique packing

1 $k$-cycle covering

Given a weighted graph $G(V,E)$ with weight $w : E \to \mathbb{Z}^{|E|}_+$. A $k$– cycle covering is an edge subset $A$ of $E$ such that $G - A$ has no $k$–cycle. The problem of minimum weight of $k$–cycle covering can be described as follows:

$$\tau_k(G_w) = \min \{ w^T x : A x \geq 1, x \in \{0, 1\} \}$$ (1)

Where $A$ is a $k$-cycle-edge adjacent matrix and $w$ is the weight vector. $x$ represents the characteristic vector of edge. We obtain the relaxed programming of (1) as follows:

$$\min \{ w^T x : A x \geq 1, 0 \leq x \leq 1 \}$$ (2)

We compute the optimal solution $\hat{x}^*$ of (2) in polynomial time, then we transfer $\hat{x}^*$ to integral vector:

$$x_e = \begin{cases} 1 & \hat{x}_e^* \geq 1/k \\ 0 & o.w. \end{cases}$$ (3)

Obviously, $x$ is a feasible solution of ILP (1) and $w^T x \leq k w^T \hat{x}^*$, which implies a $k$-approximation algorithm of minimum $k$-cycle covering problem.

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ALGORITHM 1: Approximation algorithm of minimum $k$-cycle covering

**Input:** Weighted vector $w$, $k$-cycle-edge adjacent matrix $A$

**Output:** A feasible solution $x$ of ILP\(^1\), which reaches objective value no more than $k$ times of the optimal value.

1. Solve LP\(^2\) and get the optimal solution $\hat{x}^*$.
2. Compute $x$ by equation(3).

$2$ The $(k - \frac{1}{2})$-approximation algorithm when $k$ is odd

We take advantage of specific strategy to reach better performance when $k$ is odd.

ALGORITHM 2: Approximation algorithm of minimum $k$-cycle covering

**Input:** Weighted graph $(G, w)$

**Output:** A Edge set $E_k$, which covers every $k$-cycle in $G$.

0. Set $E_{k_1} = \emptyset$.
1. Solve LP\(^2\) and get the optimal solution $\hat{x}^*$.
2. For every $e \in E$
3.  If $\hat{x}_e^* \geq 2/(2k - 1)$ Then $E_{k_1} = E_{k_1} \cup \{e\}$.
4. Suppose $E'$ are these edges of all $k$-cycles in $G - E_{k_1}$ and let $G'$ be a subgraph of $G$ induced by the edge set $E'$.
6. Using the Greedy Algorithm or Random Algorithm, we can find an approximate solution of maximum weight bipartite graph
7. $B = (V_1, V_2, E_B)$, which satisfies $W(E_B) \geq (1/2)W(E')$. Set $E_{k_2} = E' \setminus E_B$.
8. Output $E_k = E_{k_1} \cup E_{k_2}$.

**Theorem 2.1.** The Algorithm 2 has $(k - \frac{1}{2})$-approximate ratio for the minimum $k$-cycle covering problem.

**Proof.** Suppose $\hat{x}^*$ and $x^*$ are the optimal solution of LP\(^2\) and ILP\(^1\), respectively.

Firstly, we indicate that $E_k$ is a $k$-cycle covering. Actually, for every $k$-cycle $C_k$ in $G$, if it doesn’t contain any edge in $E_{k_1}$, then it is a $k$-cycle in $G - E_{k_1}$, thus it is a $k$-cycle in $G'$. Because $G' - E_{k_2}$ is a bipartite graph, of course, $G' - E_{k_2}$ has no $k$-cycle (here $k$ is odd). Thus $C_k$ contains some edge in $E_{k_2}$. Above all, we prove that $E_k$ is a $k$-cycle covering of $G$.

Additionally, we will show the approximate ratio.

On one hand, according to the rounding regulation, we know that:

$$\sum_{e \in E_{k_1}} w_e \leq (k - \frac{1}{2}) \sum_{e \in E_{k_1}} w_e \hat{x}_e^*.$$  \(4\)

On the other hand, every $\hat{x}_e^*$ related to $e \in E'$ has the lower bound $1 - 2(k - 1)/(2k - 1) = 1/(2k - 1)$ on the grounds that there exists a $k$-cycle $C_k$ in $G'$ containing $e$, satisfying $\hat{x}_e^* \leq 2/(2k - 1)$ and $\sum_{e \in C_k} \hat{x}_e^* \geq 1$.

$$\sum_{e \in E_{k_2}} w_e \leq (1/2) \sum_{e \in E'} w_e \leq (1/2)(2k - 1) \sum_{e \in E'} w_e \hat{x}_e^* \leq (k - \frac{1}{2}) \sum_{e \in E \setminus E_{k_2}} w_e \hat{x}_e^*.$$  \(5\)

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Combine inequalities (12) and (13):

\[ \sum_{e \in E} w_e \leq \left( k - \frac{1}{2} \right) \sum_{e \in E} w_e \hat{x}_e^* \leq \left( k - \frac{1}{2} \right) \sum_{e \in E} w_e x_e^*. \]  

(6)

which completes the proof.

3 The hardness of $k$-cycle covering when $k$ is even

According to Algorithm 1, we trivially derive the $k$ approximate ratio whatever $k$ is odd or even. In previous section, we have shown $(k - \frac{1}{2})$ approximate ratio when $k$ is odd, but unfortunately we can’t improve the approximate ratio when $k$ is even by using similar techniques. The following Theorem may tell us a possible reason and the hardness of the problem when $k$ is even.

**Theorem 3.1.** (Paul Erdős, Arthur Stone, 1946[1]) The extremal function $ex(n; H)$ is defined to be the maximum number of edges in a graph of order $n$ not containing a subgraph isomorphic to $H$.

\[ ex(n; H) = \left( \frac{r - 2}{r - 1} + o(1) \right) \left( \frac{n}{2} \right) \]  

(7)

where $r$ is the color number of $H$.

It is known that, when $H$ is bipartite, $ex(n; H) = o(n^2)$. Consider the special case, $H$ is an even cycle $C_k$, $ex(n; C_k) = o(n^2)$ thus $\tau_k(K_n) = \left( \binom{n}{2} \right) - o(n^2)$. We have:

\[ \lim_{n \to \infty} \tau_k(K_n) = \left( \frac{n}{2} \right) = 1 \]  

(8)

Thus there doesn’t exist constant $0 < c < 1$ such that for every graph $G(V, E)$, $\tau_k(G) \leq cm$ holds on which is a key quality in our Algorithm 2.

4 k-clique covering

Given a weighted graph $G(V, E)$ with weight $w : E \to \mathbb{Z}^{|E|}$. A $k$–clique covering is an edge subset $A$ of $E$ such that $G - A$ has no $k$–clique. The problem of minimum weight of $k$–clique covering can be described as follows:

\[ \tau_k(G_w) = \min \{ w^T x : Ax \geq 1, x \in \{0, 1\} \} \]  

(9)

Where $A$ is a $k$-clique-edge adjacent matrix and $w$ is the weight vector. $x$ represents the characteristic vector of edge. We obtain the relaxed programming of (11) as follows:

\[ \min \{ w^T x : Ax \geq 1, 0 \leq x \leq 1 \} \]  

(10)

We compute the optimal solution $\hat{x}^*$ of (10) in polynomial time, then we transfer $\hat{x}^*$ to integral vector:

\[ x_e = \begin{cases} 
1 & \hat{x}_e^* \geq \frac{k}{2} \\
0 & \text{o.w.} 
\end{cases} \]  

(11)

Obviously, $x$ is a feasible solution of ILP (6) and $w^T x \leq \left( \binom{k}{2} \right) w^T \hat{x}^*$, which implies a $\left( \frac{k}{2} \right)$-approximation algorithm of minimum $k$-clique covering problem.
ALGORITHM 3: Approximation algorithm of minimum k-clique covering

**Input:** Weighted vector \( w \), k-clique-edge adjacent matrix \( A \)

**Output:** A feasible solution \( x \) of ILP (9), which reaches objective value no more than \( \left( \frac{k}{2} \right) \) times of the optimal value.

1. Solve LP (10) and get the optimal solution \( \hat{x}^* \).
2. Compute \( x \) by equation (11).

5 The \( (k^2 - k - 1) / 2 \)-approximation algorithm for minimum k-clique covering problem

Similarly with Algorithm 2, we have the following approximation algorithm for minimum k-clique covering problem.

ALGORITHM 4: Approximation algorithm of k-clique covering

**Input:** Weighted graph \((G, w)\)

**Output:** A Edge set \( E_k \), which covers every k-clique in \( G \).

0. Set \( E_{k1} = \emptyset \).
1. Solve LP (2) and get the optimal solution \( \hat{x}^* \).
2. For every \( e \in E \)
3. If \( \hat{x}_e^* \geq 2 / (2k^2 - 1) \)
4. Suppose \( E' \) are these edges of all k-cliques in \( G - E_{k1} \) and let \( G' \) be a subgraph of \( G \) induced by the edge set \( E' \).
5. Using the Greedy Algorithm or Random Algorithm, we can find an approximate solution of maximum weight bipartite graph \( B = (V_1, V_2, E_B) \), which satisfies \( W(E_B) \geq (1/2)W(E') \). Set \( E_{k2} = E' \setminus E_B \).
6. Output \( E_k = E_{k1} \cup E_{k2} \).

Theorem 5.1. The Algorithm 4 has \( (k^2 - k - 1) / 2 \) approximate ratio for the minimum k-clique covering problem.

**Proof.** Suppose \( \hat{x}^* \) and \( x^* \) are the optimal solution of LP (10) and ILP (9), respectively.

Firstly, we indicate that \( E_k \) is a k-clique covering. Actually, for every k-clique \( K_k \) in \( G \), if it doesn’t contain any edge in \( E_{k1} \), then it is a k-clique in \( G - E_{k1} \), thus it is a k-clique in \( G' \). Because \( G' - E_{k2} \) is a bipartite graph, of course, \( G' - E_{k2} \) has no k-clique (\( G' - E_{k2} \) has no triangle). Thus \( K_k \) contains some edge in \( E_{k2} \). Above all, we prove that \( E_k \) is a k-clique covering of \( G \).

Additionally, we will show the approximate ratio.

On one hand, according to the rounding regulation, we know that:

\[
\sum_{e \in E_{k1}} w_e \leq \left( \frac{k}{2} \right) - \frac{1}{2} \sum_{e \in E_{k1}} w_e \hat{x}_e^*.
\]  \( (12) \)

On the other hand, every \( \hat{x}_e^* \) related to \( e \in E' \) has the lower bound \( 1 - 2((k^2 - 1) / (2k^2 - 1)) \) on
the grounds that there exists a $k$-clique $K_k$ in $G'$ containing $e$, satisfying $\hat{x}^*_e \leq \frac{1}{2}(\binom{k}{2} - 1)$ and $\sum_{e \in K_k} \hat{x}^*_e \geq 1$.

$$\sum_{e \in E_{\overline{k}}} w_e \leq \frac{1}{2} \sum_{e \in E'} w_e \leq (\frac{1}{2}) (\binom{k}{2} - 1) \sum_{e \in E'} w_e \hat{x}^*_e \leq (\frac{k}{2}) - \frac{1}{2} \sum_{e \in E \setminus E_k} w_e \hat{x}^*_e. \quad (13)$$

Combine inequalities (12) and (13):

$$\sum_{e \in E_k} w_e \leq (\frac{1}{2}) \sum_{e \in E} w_e \hat{x}^*_e \leq (\frac{k}{2}) - \frac{1}{2} \sum_{e \in E} w_e x^*_e. \quad (14)$$

which completes the proof.

\[ \square \]

### 6 $k$-clique covering and $k$-clique packing in $K_n$

Given a graph $G(V, E)$, a $k$-clique packing is a set of edge-disjoint $k$-cliques in $G$. The problem of maximum number of $k$-clique packing can be described as follows:

$$\tilde{\nu}^*_k(G) = \max \{ 1^T x : A^T y \leq 1, y \in \{0, 1\} \} \quad (15)$$

It is easy to see for every graph $G$, $\tilde{\nu}^*_k(G) \leq \tilde{\tau}_k(G) \leq \binom{k}{2} \tilde{\nu}^*_k(G)$ holds on.

According to Theorem 3.1 the $k$-clique covering number of $K_n$ is $\tilde{\tau}_k(K_n) = (1/(k - 1) - o(1)) (\frac{n}{k})$.

As for the packing number, we need the classical results in Block Design Theory.

A 2-design (or BIBD, standing for balanced incomplete block design), denoted by $(v, k, \lambda)$-BIBD, is a family of $k$-element subsets of $X$, called blocks, such that any pair of distinct points $x$ and $y$ in $X$ is contained in $\lambda$ blocks. Here $v$ is number of points, number of elements of $X$, $k$ is number of points in a block, $\lambda$ is number of blocks containing any two distinct points. We have next famous theorem:

**Theorem 6.1.** (23) Given positive integers $k$ and $\lambda$, $(v, k, \lambda)$-BIBD exist for all sufficiently large integers $v$ for which the congruences $\lambda (v - 1) \equiv 0 \pmod{k - 1}$ and $\lambda v (v - 1) \equiv 0 \pmod{k(k-1)}$ are valid.

When $\lambda = 1$, it is easy to see $(n, k, 1)$-BIBD exists if and only if $K_n$ contains a perfect $k$-clique packing, which is a $k$-clique packing such that every edge belongs to a $k$-clique. Thus the above Theorem 6.1 is equivalent to the following Theorem:

**Theorem 6.2.** For all sufficiently large integers $n$ satisfying $n \equiv 1, k \pmod{k(k-1)}$, then $K_n$ contains a perfect $k$-clique packing.

For all sufficiently large integers $n$ satisfying $n \equiv 1, k \pmod{k(k-1)}$, $K_n$ contains $\frac{n(n-1)}{k(k-1)}$ edge-disjoint $k$-clique. So we have $\tilde{\nu}_k(K_n) \sim \frac{n(n-1)}{k(k-1)}$ and $k$-clique covering number over $k$-clique packing number in $K_n$ is $k/2$ when $n \to \infty$, that is:

$$\lim_{n \to \infty} \tilde{\tau}_k(K_n)/\tilde{\nu}^*_k(K_n) = \lim_{n \to \infty} (1/(k - 1) - o(1)) \left( \frac{n}{2} \right) \frac{n(n-1)}{k(k-1)} = k/2 \quad (16)$$

Recall Tuza’s Conjecture, which is related to the ratio of triangle covering number and triangle packing number:

**Conjecture 6.3.** (Tuza, 1981) $\tau(G) \leq 2 \nu(G)$ holds for every graph $G$.

For the ratio of $k$-clique covering number and $k$-clique packing number in graph $G$, the trivial upper bound is $\left( \frac{k}{2} \right)$. We guess there exists a upper bound between $k/2$ and $\left( \frac{k}{2} \right)$ for every graph $G$.  

5
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