Renormalized quark-antiquark Hamiltonian induced by a gluon mass ansatz in heavy-flavor QCD

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Abstract

In response to the growing need for theoretical tools that can be used in QCD to describe and understand the dynamics of gluons in hadrons in the Minkowski space-time, the renormalization group procedure for effective particles (RGPEP) is shown to be the simplest available context of heavy quarkonia to exhibit a welcome degree of universality in the first approximation. It yields once one assumes that beyond perturbation theory gluons obtain effective mass. Namely, in the second-order terms, the Coulomb potential with Breit-Fermi spin couplings in the effective quark-antiquark component of a heavy quarkonium, is corrected in one-flavor QCD by a spin-independent harmonic oscillator term that does not depend on the assumed effective gluon mass or the choice of the RGPEP generator. The new generator we use here is much simpler than the ones used before and has the advantage of being suitable for studies of the effective gluon dynamics at higher orders than the second and beyond the perturbative expansion.

Keywords: Hamiltonian, QCD, renormalization, heavy quarkonia, gluon mass, symmetry

1. Introduction

The growing need for understanding dynamics of gluons in QCD, comprehensively documented in \cite{1}, revives interest in Hamiltonian methods for describing heavy quarkonia in terms of wave functions in the Fock space of virtual quanta, where gluons are likely to behave in a relatively simple way because they are permanently coupled to the massive quarks that move slowly with respect to each other. Phenomenology and theory of heavy quarkonia rapidly develop, as is illustrated by many examples \cite{2-12}. The key feature regarding gluons that requires understanding is how they acquire the effective mass, so that the hadron mass spectra \cite{13} do not exhibit any small excitations such as the atomic spectra do due to massless photons. The gluon-mass generation is a subject of research from early on using continuum Dyson-Schwinger equations \cite{14-16} and it is addressed in lattice studies \cite{17,20} because its implications for theory of strong interactions would be broad, including the issue of saturation in dense gluon systems beyond a single hadron setting \cite{21}. In the canonical formulation of QCD in the front form (FF) of dynamics in the Minkowski space-time \cite{22}, the need for understanding implications of an effective gluon mass is stressed in general in \cite{24} and in the context of heavy quarkonia in \cite{24,25}. Theoretical studies of heavy quarkonia may thus be expected to increasingly focus on the Hamiltonian dynamics of their gluonic content, cf. \cite{26,27}. This article presents a first step in a program of systematic studies of dynamics of scale-dependent gluons in heavy quarkonia, starting with the simplified case of canonical FF formulation of QCD with quarks of just one heavy flavor and assuming that the effective gluons which correspond to momentum scale of quark binding mechanism in quarkonia have mass.

We calculate a renormalized, scale-dependent Hamiltonian for quarkonium constituents in the Fock space using a new formulation of the renormalization group procedure for effective particles (RGPEP) in quantum field theory, see below. In this new formulation, the key RGPEP element called its generator does not depend on the derivative of the renormalized Hamiltonian with respect to the scale parameter, in contrast to the earlier versions of the RGPEP, in which such dependence was involved. The generator dependence on the derivative of the Hamiltonian made a calculation of the latter difficult and stalled the development. With the derivative issue resolved, the improved method is now prepared for trial applications and further development. The new generator has already been verified to work beyond perturbative expansion in simple theories \cite{28,29} and it has passed the test of producing asymptotic freedom in the renormalized FF Hamiltonian of QCD \cite{30}. Here we present the new-generator calculation of a renormalized Hamiltonian in one-flavor theory including terms up to second order in powers of the strong-interaction coupling constant at the scale of heavy quarkonia.

The renormalized Hamiltonian at a suitable scale is subsequently applied to formulate the eigenvalue problem for a quarkonium in the Fock space basis that is constructed using the creation operators for effective particles of the corresponding size. We make a drastically simplifying assumption that the non-Abelian and non-perturbative effects due to components with more effective gluons than one, can be mimicked by a gluon-mass ansatz introduced in the dynamics of the compo-
ment with only one effective gluon and a quark-antiquark pair. Namely, the components with more than one effective gluon are dropped at the price of introducing the ansatz. We then express the quark-antiquark-gluon component in terms of the quark-antiquark component and calculate the effective Hamiltonian in the quark-antiquark sector. As a result, we establish the universality of effective interaction one obtains in the quark-antiquark component: the new RGPEP generator produces the same Coulomb term with Breit-Fermi spin-dependent factors and the same spin-independent harmonic oscillator term that were previously obtained with the old generator \[31, 32\]. The oscillator term is independent of the value of the mass ansatz.

The new effective term respects rotational symmetry in the quark-antiquark relative three-momentum space, in which the eigenvalue problem for quarkonium two-body component resembles a non-relativistic Shrödinger equation with a potential, except that the eigenvalue is the quarkonium mass squared instead of its energy. The effective eigenvalue function is invariant with respect to the FF kinematical Lorentz transformations. We also note that the relative momentum variables we use for heavy quarkonia resemble the variables used in the light-front holographic approach to the phenomenon of light hadrons, based on the AdS/QCD duality ideas \[33\]. The holographic potential is also of the harmonic oscillator form, but its frequency is different, which can be associated with much smaller mass of light quarks than \(\Lambda_{QCD}\) while the heavy quark mass is much greater.

Section 2 explains the preliminary nature of the gluon mass ansatz that is used to finesse the effective quark-antiquark interaction. Section 3 presents the RGPEP in application to one-flavor QCD. The renormalized eigenvalue equation for the entire quarkonium state and the effective Hamiltonian for its quark-antiquark component are introduced in Sec. 4. Section 5 shows how the effective harmonic oscillator potential emerges in the non-relativistic limit, using holographic quark-antiquark relative momentum variables. Section 6 concludes the article with comments on how our results prepare ground for renormalized Hamiltonian studies of gluon dynamics in heavy quarkonia.

2. The initial need for gluon mass ansatz

The difficulty of describing bound states in the Fock space is that one needs to deal with an \textit{a priori} infinite number of components. In the case of quarkonium, the bound state

\[
|\phi\rangle = |Q\bar{Q}\rangle + |Q\bar{Q}G\rangle + |Q\bar{Q}GG\rangle + \ldots ,
\]

is built from canonical quanta of quark and gluon fields. To deal with this issue, the RGPEP uses the concept of effective particles. They are characterized by an effective size \(s\) and are related to the bare, point-like particles of canonical theory by means of an operator transformation. The idea is that for description of observables characterized by the momentum scale \(A = 1/s\), one can use the renormalized Hamiltonian \(H_s\), so that the number of relevant Fock components in Eq. 1 is sufficiently small for carrying out computations, except that \(Q, \bar{Q}\) and \(G\) are replaced by \(Q_s, \bar{Q}_s\) and \(G_s\), respectively. Thus, infinitely many Fock components for effective particles can be neglected. The bound-state problem becomes so greatly simplified that one can attempt to seek solutions to the eigenvalue equation numerically.

The above strategy may work when the RGPEP equation for \(H_s\) is solved exactly. When one uses expansions of \(H_s\), in a series of powers of effective coupling constant, non-perturbative effects in \(H_s\) itself are not included. The eigenvalue problem for such \(H_s\) still couples the Fock components made of effective particles in significant ways. Initially, one cannot be certain that the Fock components with more than one effective gluon can be dropped from the eigenvalue problem with no consequence. Since power counting allows a mass term for gluons, perturbative RGPEP calculations imply a need for a gluon mass counterterm in the canonical Hamiltonian of one-flavor QCD, whose finite part is not known, and the phenomenology of hadrons appears to exclude massless effective gluons, one is in need to assume that neglecting components with more than one effective gluon of size \(s\) in heavy quarkonia of characteristic physical size \(s\) may only be admissible if one allows for appearance of the effective mass term for the gluon in the \(Q, \bar{Q}, G\) component, presumably produced by the non-Abelian interaction effects descendant from higher components, absent in Abelian theories.

In the variety of terms that are conceivable in renormalized Hamiltonians using FF power counting \[23\], rotational symmetry is not explicitly respected. Any ansatz one introduces to hypothetically account for the omitted terms, must satisfy the condition that the resulting effective eigenvalue problem produces the mass spectrum that exhibits degeneracy implied by rotational symmetry. Making an ansatz for the gluon mass term, rather than any other potentially allowed term, turns out to lead to the effective oscillator correction to the Coulomb terms that satisfies the rotational-symmetry condition already at the level of effective Hamiltonian itself, exhibiting the symmetry in relative momentum variables. It is unlikely that ansatz terms for quantities other than mass can easily produce such a universal result. Moreover, the mass ansatz turns out to possess the unique feature that the oscillator potential it leads to is independent of the actual value of the ansatz mass, provided it is not too small.

When the RGPEP calculation of renormalized Hamiltonian \(H_s\) is extended to higher orders than second, and when the unknowns of the effective gluon mass are relegated to the Fock sectors with more than one gluon, the preliminary gluon-mass ansatz in the dynamics of \(Q_s\bar{Q}_sG_s\) component will be replaced by true QCD terms. However, since the complexity of renormalized Hamiltonian operators requires approximations, and since their spectra are calculated numerically making further simplifications, a purely mathematical approach to identification of all important terms may turn out hard to execute. Fortunately, the spectra of Hamiltonians for heavy quarkonia in the Fock space representation with effective gluons can also be studied in comparison with experimental data. Assuming that the canonical theory one starts from is right, one can use the data in identifying scale dependence of various terms allowed by the power counting. The mass ansatz seems to be the simplest admissible term to falsify. The universality of the quarkonium effective Hamiltonian described here prepares the ground for required theoretical studies in parallel with quarkonium phenomenology.
3. RGPEP as a tool for solving bound states in QCD

Starting from the one-flavor QCD Lagrangian, we derive the renormalized Hamiltonian $H_s$ which is applied to heavy quarkonia using the effective particle basis in the Fock space. The eigenvalue problem for $H_s$ is then reduced to the effective quark-antiquark Hamiltonian $H_{s eff}$ for the quarkonium $Q, \bar{Q}$, component. A brief sketch of our method is available in [34].

3.1. Renormalized heavy-flavor Hamiltonian

Starting from the one-flavor QCD Lagrangian [13],

$$L = \bar{\psi}(i\gamma \cdot \partial - m)\psi - \frac{1}{2} \text{tr} F^{\mu \nu} F_{\mu \nu},$$  

working in the gauge $A^i = 0$ and using standard notation for all tensors exemplified by $x^i = x^0 \pm x^3, x^x = (x^1, x^2)$, we calculate the canonical FF Hamiltonian

$$\hat{H}_{QCD}^{\text{can}} = 2 \int dx^i x^i : \mathcal{H}[x = 0]:,$$

where $\mathcal{H}$ is the well-known Hamiltonian density written in terms of quantum fields and their derivatives, see e.g. [35].

The canonical Hamiltonian couples point-like particles with increasing strength when the invariant mass change it induces increases. This leads in perturbative calculations to divergent integrals over quark or gluon transverse momenta $k^\perp$, called UV divergences, and to divergent integrals over their $k^+$ momenta near zero, called small-$k^+$ divergences, since the ratio $x = k^+/P^+$, where $P^+$ denotes a hadron momentum, corresponds to the momentum fraction $x$ carried by a quark or gluon in the parton model. Therefore, our starting Hamiltonian includes the regularization described in [Appendix A]. It also includes the corresponding counterterms.

After regularization, we introduce the effective particles of size $s$ by means of a unitary transformation of field operators,

$$\psi_s = U_s \psi_0 U_s^\dagger,$$

and similarly for the gluon field $A$. The subscript 0 refers to the size $s = 0$. The Hamiltonian is to be independent of the scale $s$,

$$H_s(q_s) = H_0(q_0),$$

where $q$ denotes quark and gluon operators. This condition is satisfied if $\mathcal{H}_s = H_s(q_s)$ solves the RGPEP differential equation with respect to parameter $t = s^4$, introduced for convenience in handling dimensionful quantities. We use the subscript $t$ below as equivalent to $s$. The RGPEP equation reads

$$\mathcal{H}_t = [G_t, \mathcal{H}_t],$$

where $G_t = -U_t U_t^\dagger$ is called the RGPEP generator. The initial condition at $t = 0$ is set by the regulated canonical Hamiltonian with counterterms. The new generator we use is [36]

$$G_t = [\mathcal{H}_t, \mathcal{H}_t],$$

where the operator $\mathcal{H}_t$ is the free part of $\mathcal{H}$ and the operator $\mathcal{H}_t$ is directly related to $\mathcal{H}$ as described in [Appendix B]. The RGPEP Eq. (6) is designed according to the principles of similarity renormalization group procedure [37, 38], taking advantage of features of the double-commutator flow equations [39, 45].

We solve Eq. (7) expanding $H_s(q_s)$ in powers of the effective coupling constant, which is assumed small on the basis of asymptotic freedom in our approach [40]. Namely, we take advantage of the hierarchy of scales of the heavy quark mass $m$, inverse effective particle size $s^{-1}$ and $\Lambda_{QCD}$ in our scheme,

$$m \gtrsim s^{-1} \gg \Lambda_{QCD}.$$

3.2. Bound state problem for heavy quarkonia

Quarkonium is defined as a solution to the eigenvalue problem of the renormalized Hamiltonian, $H_s(q_s)$, denoted for brevity by $H_s$, for eigenvectors with corresponding quantum numbers,

$$H_s |\Psi⟩ = E |\Psi⟩ .$$

The state $|\Psi⟩$ is written in terms of the Fock components built using creation operators for the corresponding effective particles,

$$|\Psi⟩ = |Q, \bar{Q}_t⟩ + |Q, \bar{Q}, G_t⟩ + |Q, \bar{Q}, G_t, G_t⟩ + ... .$$

In principle, this representation has infinitely many terms. However, the interaction Hamiltonian that solves the RGPEP Eq. (6) is characterized by the vertex form factors, $f_{c,a} = e^{i x (M_c - M_a)}$, where $M_c$ and $M_a$ denote the invariant masses of particles created and annihilated in the vertex, respectively. The greater $t$ the stronger suppression of interactions that change the number of virtual particles, and one may hope that the expansion may converge, or even a few terms may be sufficient to obtain a reasonable first approximation. However, the increase of size $s$ or parameter $t$, is associated with the increase of the coupling constant in $H_s$, so that one has to study the effective dynamics in order to determine the values of $t$ that can be reliably achieved using an expansion of $H_s$ in a power series in $g$, before one solves the resulting non-perturbative eigenvalue problem for $H_s$ expecting a desired accuracy. This approach to non-perturbative bound-state eigenvalue problems is supported by numerical studies of simple models with asymptotic freedom [46, 47].

Using the RGPEP solution of the form

$$H_t = H_f + g H_{1t} + g^2 H_{2t} + ... ,$$

and keeping only terms up to second order in powers of the small coupling constant $g$, Eq. (9) can be written as

$$\begin{bmatrix} H_f + g^2 H_{2t} & g H_{1t} \\ g H_{1t} & g^2 H_{2t} \end{bmatrix} - E \begin{bmatrix} 0 \\ |Q, \bar{Q}_t⟩ \end{bmatrix} = 0,$$

where the dots represent the Fock components with more than one effective gluon and the Hamiltonian terms that involve these components. The operator $H_{2t}$ is limited in Eq. (12) to its terms that do not change the number of effective particles.
3.3. Gluon mass ansatz and effective eigenvalue problem

Introducing the gluon mass term, denoted by $\mu^2$, as a minimal price one has to pay for dropping all the infinitely many dotted terms in Eq. (12), one arrives at an approximate eigenvalue problem of the form

\[
\begin{bmatrix}
H_f + \mu^2 gH_t \\
H_t + g^2 H_{2t}
\end{bmatrix}
- E
\begin{bmatrix}
|Q_1 \bar{Q}(G_1)\rangle \\
|Q_2 \bar{Q}(G_2)\rangle
\end{bmatrix}
= 0 .
\]

We also dropped the interaction terms $H_{2t}$ in the sector with one effective gluon because in the calculation that follows they contribute to the effective Hamiltonian for the $Q, \bar{Q}$ component first in order $g^4$, while our calculation is limited here to terms order 1 and $g^2$.

The effective Hamiltonian for the $Q, \bar{Q}$ component is calculated using the projection operator on that Fock sector, $\mathcal{P}_t$, and the projector $Q_t = 1 - \mathcal{P}_t$ on the $Q, \bar{Q}(G_t)$ sector.

\[
|\Psi_P\rangle \equiv \mathcal{P}_t |\Psi\rangle = |Q, \bar{Q}\rangle , \quad |\Psi_Q\rangle \equiv Q_t |\Psi\rangle = |Q_t, \bar{Q}(G_t)\rangle .
\]

Following Ref. [48], the component $|\Psi_P\rangle$ is related to the component $|\Psi_Q\rangle$ by the operator $\mathcal{R}_t$, $|\Psi_Q\rangle = \mathcal{R}_t |\Psi_P\rangle$. The eigenvalue equation to solve is written as

\[
H_{\text{eff}} |\psi_Q(Q_t)\rangle = E |\psi_Q(Q_t)\rangle ,
\]

where $|\psi_Q(Q_t)\rangle = S_t |Q, \bar{Q}(Q_t)\rangle$, $S_t = (\mathcal{P}_t + \mathcal{R}^\dagger_t \mathcal{R}_t)^{-1/2}$ and

\[
H_{\text{eff}} = S_t (\mathcal{P}_t + \mathcal{R}^\dagger_t) H(t, \mathcal{P}_t + \mathcal{R}_t) S_t .
\]

This formula is used below to evaluate the quarkonium effective Hamiltonian $H_{\text{eff}}$, keeping terms order 1 and $g^2$ and neglecting terms order $g^3$ and smaller.

4. $Q \bar{Q}$ effective eigenvalue equation

The effective FF Hamiltonian eigenvalue equation for the heavy quarkonium $Q \bar{Q}$ component reads

\[
H_{\text{eff}} |\psi_Q(Q_t)\rangle = \frac{M^2 + \mu^2}{Q^s} |\psi_Q(Q_t)\rangle ,
\]

where $P$ is the quarkonium kinematical momentum. Its mass is denoted by $M = 2m + B$, where $B$ is traditionally called the binding energy, although in QCD it may be positive. The kinematical momentum drops out of the eigenvalue equation for the quarkonium mass squared,

\[
(P^s H_{\text{eff}} - P^s + 1/2 - M^2) |\psi_Q(Q_t)\rangle = 0 .
\]

Using the two-body relative motion wave function $\psi_{2s}(k_{2s}, x_2)$ for effective quarks of size $s = 1/2$ and momenta $k_2$ and $k_4$, where $x_2 = k_{2s}/P^s$ and $k_{2s} = x_2 k_2 - x_2 k_4$, the eigenstate is written as

\[
|\psi_{Q_t}\rangle = \sum_{k_{2s}} P^s \delta_{Q_t} \frac{1}{\sqrt{3}} \delta(P - k_2 - k_4) \psi_{2s}(k_{2s}, x_2) \psi_{1s}(k_{1s}, x_1) b_{1s} d_{2s}[0] .
\]

The sum extends over quarks spins and colors, integrations over kinematical momenta are carried out with Lorentz-invariant FF measure $[k] = dk^4 d^4k/[2k^+(2\pi)^3]$, factor 1/3 takes care of normalization of a color singlet described by the Kronecker delta in the quark color indices, and the tilde indicates that the kinematical total momentum conservation $\delta$-function is multiplied by $16\pi^3$. The mass-squared eigenvalue equation reads

\[
\begin{align*}
\left(\frac{\kappa_3^2 + M_{1,1}^2}{x_1} + \frac{\kappa_3^2 + M_{1,1}^2}{x_3} - M^2\right) \psi_{1s}(\kappa_3^2, x_1) \\
+ g^2 \int \frac{dx_2 dx_3^2 k_{2s}^2}{2(2\pi)^3 x_2 x_4} U_{\text{eff}}(13, 24) \psi_{2s}(k_{2s}, x_2) = 0 ,
\end{align*}
\]

where $M_{1,1}^2$ is the effective mass for the quark, $i = 1$, or antiquark, $i = 3$. It contains the quark mass parameter $m^2$ appearing in $H_f$ and the self-interaction contributions that include UV-finite parts of the mass counterterm in the initial QCD canonical Hamiltonian, $g^2 m^2$. Namely, as indicated in Fig. 1

\[
M_{1,1}^2 = m^2 + g^2 m_5^2 + \frac{4}{3} g^2 \int \frac{dx^2 dx^4 k_{2s}^2}{2(2\pi)^3 x(1-x) r_{G5}} \times \sum_6 \left( \frac{1}{m - M^2} - \frac{1}{m - M^2 - \mu_{G5}} \right) e^{-2(m^2 - m^2)^2},
\]

and analogously for $M_{1,1}^2$. The gluon mass appears with subscripts, such as $\mu_{G5}$ above, because the ansatz $\mu^2$ in Eq. (13) is a priori allowed to be a function of the relative motion of a gluon with respect to quark and antiquark in the meson $Q, \bar{Q}, G$, component and the subscripts indicate the three-particle state in which the ansatz function is evaluated, the gluon being always labeled by $S$ as in Figs. 1 and 2. $M^2$ denotes an invariant mass of the fermion and boson in the loop with relative momenta $x \equiv x_5/k_1^s/k_1^s$ and $k^+ = (1 - x) k_2^+ - x k_4^+$.

The interaction kernel $U_{\text{eff}}(13, 24)$ contains instantaneous FF interactions and gluon-exchange terms. There are two kinds of exchange terms. One kind comes directly from the Hamiltonian $H_t$ in Eq. (14), which is obtained from canonical one-flavor QCD with massless gluons. This kind leads to terms denoted below by $F_Z$ and $F_X$ that contribute to the Coulomb potential with Breit-Fermi spin-dependent factors. The other kind of terms comes from the exchange of the effective gluon, for which we have introduced the mass ansatz. This other kind leads to the terms denoted below by $F_R$ and $F_X$. These terms contribute the spin-independent oscillator correction to the Coulomb interaction.

In the notation of Fig. 2 $U_{\text{eff}} = H_{\text{exch}} + H_{\text{int}}$ and

\[
H_{\text{exch}} = -\frac{4}{3} \left[ \frac{\theta(x_2 - x_1)}{k_5^2} r_{25,1} r_{35,4} d_{\mu\nu}(k_5)(\bar{u}_1 y^\mu u_2)(\bar{v}_4 y^\nu v_3) F_X + \frac{\theta(x_2 - x_1)}{k_5^2} r_{15,2} r_{45,3} d_{\mu\nu}(k_5)(\bar{u}_1 y^\mu u_2)(\bar{v}_4 y^\nu v_3) F_Z \right],
\]

\[
H_{\text{int}} = -rc_{13,2} f_{13,24} \sqrt{x_1 x_3 x_2} \delta_{13} \delta_{24} \frac{4}{x_1^2 - x_2^2} ,
\]

where $d_{\mu\nu}(k_5) = -g_{\mu\nu} + (n_\mu k_5^\nu + n_\nu k_5^\mu)/k_5^2$ and

\[
F_X = f_{13,24} F_Z + \frac{f_{13,53} f_{13,52}}{R_X} R_X ,
\]

\[
F_Z = f_{13,24} F_Z + f_{13,54} f_{13,52} R_Z .
\]
Appendix A. Factors

They contain only the first-order RGPEP form factors, which are of Eq. (20): (a) corresponds to Eq. (25), (b) to Eq. (24), and (c) to Eq. (23). The heavy-quark limit of the $Q_{ar{Q}}$ interaction is also not affected by the generator change because, in comparison to the other terms that are the same as with the old generator, the new term is suppressed by the square of the ratio of relative quark-antiquark momentum to their mass. These are the two bottom-line reasons for the two different RGPEP generators to produce the same effective $Q_{ar{Q}}$ dynamics.

5. Harmonic oscillator force

The scale hierarchy of Eq. (8) allows one to consider the eigenvalue equation in the non-relativistic approximation for relative motion of quarks, because the RGPEP factors exclude invariant mass changes greater than $x^{-1}$ and the dominant relative momenta of quarks are smaller than the quark mass. We introduce the quark relative momenta $k_{ij} = (k_{ij}^+, k_{ij}^-)$, noting that analogous variables appear in light-front holography [33, 59].

$$k_{ij}^+ = \frac{1}{2} \frac{k_{ij}^+}{\sqrt{x_i x_j}}, \quad k_{ij}^- = \frac{m}{\sqrt{x_i x_j}} \left(x_i - \frac{1}{2} \right).$$

and we define the momentum transfer $\vec{q} = \vec{k}_{13} - \vec{k}_{24}$. The non-relativistic approximation is obtained in the limit $k_{ij}/m \to 0$. The eigenvalue equation in this limit reads

$$\left[ \frac{\vec{k}_{13}^2}{m} - B + \frac{\delta m^2_{13}}{2m} + \frac{\delta m^2_{34}}{2m} \right] \psi_{13}(\vec{k}_{13})$$

$$+ \int \frac{d^3q}{(2\pi)^3} \left[ V_{C,BF}(\vec{q}) + W(\vec{q}) \right] \psi_{24}(\vec{k}_{13} - \vec{q}) = 0,$$

where the Coulomb potential with Breit-Fermi spin-dependent factors is

$$V_{C,BF}(\vec{q}) = -\frac{4}{3} \frac{4\pi a}{\vec{q}^2} (1 + BF),$$

with $a = g^2/(4\pi)$, and the additional term contains

$$W(\vec{q}) = \frac{4}{3} \frac{4\pi a}{\vec{q}^2} \frac{1}{\vec{q}^2} \left( \frac{\mu^2}{\mu^2 + \vec{q}^2} - e^{-2m^2q^2/|\vec{q}|^2} \right),$$

with $\mu^2 = \theta(q_+)\mu_{35}^2 + \theta(-q_+)(\mu_{35}^2)$. The quark mass corrections are given by the same function $W$, $\delta m^2_{ij}/m = -\int [d^3q/(2\pi)^3] W(\vec{q})$, where $\mu^2 = \mu_{35}^2$ for $i = 1$ and $\mu^2 = \mu_{53}^2$ for $i = 3$, and in the mass counterterm we have chosen $m^2_q = 0$. The only dependence on the gluon mass ansatz function $\mu$ appears in the factor $\mu^2/(\mu^2 + \vec{q}^2)$, which could be replaced by one in any integral involving $W$ if the mass $\mu$ dominated the momentum $q = |\vec{q}|$ in the relevant integration range. It is shown below that such dominance appears indeed in the heavy quark limit, see Eq. (37). This implies that the mass ansatz allows us to finesse the universal oscillator result irrespective of the ansatz actual value and the choice of the RGPEP generator.
Once one renames the momentum $\hat{\kappa}_1$, as $\vec{k}$, and suppresses spin subscripts that are not important for the spin-independent correction to the Coulomb term, one obtains the same result as in Eq. (105) of [31] despite that we use the new generator and holography-motivated relative momentum variables,

$$
\left[\frac{\vec{k}^2}{m} - B\right] \psi(\vec{k}) + \int \frac{d^3q}{(2\pi)^3} V_{C, BR}(\vec{q}) \psi(\vec{k} - \vec{q})
$$
$$
+ \int \frac{d^3q}{(2\pi)^3} W(\vec{q}) \left[ \psi(\vec{k} - \vec{q}) - \psi(\vec{k}) \right] = 0 .
$$

(34)

Since the RGPEP form factors limit $\vec{q}$ to small values when $q_0$ is small, one can expand the wave function in the small region of integration over $\vec{q}$ around $\vec{k}$,

$$
\psi(\vec{k} - \vec{q}) = \psi(\vec{k}) - q_0 \frac{\partial}{\partial k_0} \psi(\vec{k}) + \frac{1}{2} q_2 q_0 \frac{\partial^2}{\partial k_0 \partial k_0} \psi(\vec{k}) + \ldots
$$

(35)

and observe that only even terms contribute, because $W$ is an even function of $\vec{q}$. The quadratic terms dominate and yield the harmonic oscillator potential, which is thus shown to not depend on the change in the generator,

$$
- \frac{4}{3} \frac{\alpha}{\pi} b^{-3} \sum_i \tau_i \frac{\partial^2}{\partial k_i^2},
$$

(36)

with $b = \sqrt{2m\mu} i$, the vector $\vec{\tau} = \int_0^1 dv \sqrt{1 - v^2} \vec{\omega}(v) \tau(v)$, and $v = q_0 / |\vec{q}|$ in $\vec{\omega} = (1 - v^2, 1 - v^2, 2v^2)$. The function $\tau(v)$,

$$
\tau(v) = \int_0^1 du \frac{u}{u^2} e^{-u^2} \left[ 1 + \frac{u^2 v^2}{2 (m s)^2 (\mu s)^2} \right]^{-1},
$$

(37)

results from the first-order RGPEP form factor and the gluon mass ansatz. The form factor is the same for the new generator as it was for the old one. It is visible that in the limit of heavy quarks the function $\tau(v)$ is a constant $\sqrt{\pi}/4$ no matter how large is the gluon mass ansatz $\mu$, and a constant $\tau$ yields a rotationally symmetric harmonic oscillator potential. The resulting effective Schrödinger equation in momentum space reads

$$
\left[ \frac{\vec{k}^2}{m} - \frac{\alpha}{\pi} \vec{k} \Delta \vec{k} - B \right] \psi(\vec{k}) + \int \frac{d^3q}{(2\pi)^3} V_{C, BR}(\vec{q}) \psi(\vec{k} - \vec{q}) = 0 ,
$$

(38)

with $\vec{k} = \alpha (m^2 \mu^2)^{-3}/(36 \sqrt{27})$. In the heavy quark limit, the harmonic oscillator frequency $\omega$ depends on the mass and size of the effective quarks but not on the value of the gluon mass ansatz. This result is formally valid as long as the gluon mass ansatz is not zero. When the gluon mass ansatz is set to zero, the harmonic oscillator potential vanishes and the Coulomb term is the same as in QED, except for the color factor $4/3$ and the strong coupling constant $g$ replacing the electric charge $e$.

Although Eq. (38) with Breit-Fermi terms in new variables of Eq. (60) has not been solved yet, estimates previously attempted [12] using the old version of the RGPEP and standard FF variables suggest that the values of size $s$ and frequency $\omega$ that could be useful in the phenomenology of charmonium and bottomonium families using our new RGPEP version, may be on

the order of $1/m$ and $200 - 300$ MeV, respectively. For such values, according to Eq. (37), a rotationally symmetric eigenvalue problem in the new variables may emerge if the gluon mass ansatz is either very large in comparison to the quark mass or it is some function of the quark motion with respect to quarks that leads to $\vec{\tau}$ with three equal components, such as, for example, $\mu^2 = v^2$. The first possibility appears unlikely when one assumes that the gluon mass should be on the order of at most few GeV for a realistic value of the coupling constant. The second option requires insight concerning the functional form implied by QCD. Therefore, the most desired course of study is to estimate the gluon mass terms in QCD using the 4th order RGPEP.

6. Conclusion

Since the effective $Q\bar{Q}$ Hamiltonian we obtain using the RGPEP is the same for different generators and does not depend on the gluon mass ansatz value, one may hope that it may serve as a reasonable initial approximation to one heavy-flavor QCD in the range of momentum scales at which the mechanism of binding of effective quarks by effective gluons is at work. The question arises if a more accurate calculation, using the solution to Eq. (3) with accuracy to terms order $g^2$ and shifting the gluon mass ansatz to the two-gluon component, also leads to the same oscillator potential. An additional reason for interest in such calculation, not addressed here, stems from the fact that the eigenvalue problem derived in a similar way for a single quark, yields an infinite eigenvalue, indicating how the renormalized effective particle approach can tackle the issue of confinement.

Our assumption that the gluon mass ansatz can mimic non-perturbative non-Abelian dynamics in higher Fock components is testable in two ways. One way is the suggested above RGPEP calculation of higher order than $g^2$ combined with a shift of the mass ansatz to components with more effective gluons than one. The other way is to include non-perturbative running of the constituent quark and quark masses with their size parameter $s$, which can be approximately described using Eq. (6).

Two qualitative arguments support our expectation that a harmonic oscillator interaction will also result from more advanced RGPEP calculations than carried out here. The first argument is that the eigenvalue of the effective FF Hamiltonian is the square of the quarkonium mass. In contrast, in the instant form (IF) of dynamics the eigenvalue is the quarkonium mass itself. Therefore, if the quark-antiquark potential commonly used to describe confinement in the IF Hamiltonians is linear in the distance between quark and antiquark, then the corresponding FF Hamiltonian should be quadratic [59]. This correspondence between linear and quadratic potentials for heavy quarkonia needs to be verified numerically. The second argument is that the type of momentum variables that we use and the quadratic nature of effective potential both appear also in the light-front holography for light hadrons. The holography claims to represent the first approximation to QCD [33]. This coincidence suggests the dynamical utility of the concept of effective particles in QCD, corresponding to the constituent quarks in light hadrons or charm and bottom quarks in heavy hadrons, both
kinds potentially describable by including effective gluons in the RGPEP.

Even the crude oscillator picture developed here is able to provide wave functions that can be applied in phenomenology of heavy quarkonia [3–12] and that can be compared with results of other approaches [50–59], including those that use the FF of Hamiltonian dynamics [24–27]. Numerical calculations are needed to determine the probability of $Q\bar{Q}, G\bar{G}$ component. The probability of two-gluons component is needed to find out if the expansion in the number of effective gluons of appropriate mass and size has a chance to converge. This requires fourth-order calculations. Such calculations are also needed for the RGPEP to complement other approaches addressing non-Abelian dynamical effects, such as Dyson-Schwinger equations [60–62], functional-renormalization group [63–65], and other Hamiltonian methods [66–68]. If the RGPEP calculations did show convergence in the number of effective gluons, the effective quantum field operators it introduces might become useful in designing optimized interpolating operators for lattice studies [69] and obtaining the Minkowski space images of hadrons.

Appendix A. Regularization

We regulate the QCD interaction terms. Let $k^+$ and $k^-$ label transverse and longitudinal momenta of a particle taking part in an interaction. The total kinematical momentum annihilated in a vertex is labeled by $P^+$ and $P^-$. The relative transverse momentum of any particle of momentum $k$ that is involved in the vertex, with respect to all other particles in that vertex, is defined by $k^\perp = k^+ - xP^+$, where $x = k^-/P^-$. Every creation or annihilation operator for a quark in every interaction vertex is supplied with a vertex factor $\exp\left[-m^2 + k^2/\left(x\Delta^2\right)\right]$, and every gluon operator is supplied with a factor $\exp\left[-(\delta^2 + k^2)/(x\Delta^2)\right]$. The parameter $\Delta \to \infty$ when the ultraviolet regularization is being removed. The parameter $\delta$ plays the role of an infinitesimal gluon mass that regulates small-$x$ divergences by the parameter $\epsilon = \delta/\Delta$. The small-$x$ regularization is being removed by $\epsilon \to 0$.

For example, in an interaction vertex where a quark with momentum $k_1$ splits into a quark with momentum $k_2$ and a gluon with momentum $k_1$, the regulating function is

$$r_{21,1} = e^{-\frac{x^2m^2}{2\Delta^2}} e^{-\frac{x^2\delta^2}{2\Delta^2}} e^{-\frac{x^2}{2\Delta^2}} = e^{-\frac{m^2 + \delta^2}{2\Delta^2}}. \quad \text{(A.1)}$$

$M_{1,1,1}^2$ denotes invariant mass of particles $k_1$ and $k_2$, in which the gluon has mass $\delta$. $M_{2,1,1}^2 = M_{1,1,1}^2 + \delta^2/\Delta^2$.

The FF canonical Hamiltonian contains also interactions with instantaneous exchange of a gluon or a quark. These terms are singular as functions of the exchanged longitudinal momentum. They are regulated as if they were made of two local interaction vertices. For example, the instantaneous interaction between a quark and an antiquark (with ingoing and outgoing momenta $k_2$ and $k_1$, respectively) is regulated as if they exchanged a gluon with momentum $k_5 = (k_1 - k_2)x/z$, where $z = x_1/x_2$. The regulating function in this case is

$$r_{C13,42} = \theta(z) r_{25,1} r_{35,4} + \theta(-z) r_{25,3} r_{15,2}. \quad \text{(A.2)}$$

Appendix B. Elements of RGPEP

The operator $H_f$, called the free Hamiltonian, is the kinetic term, which does not contain the QCD coupling constant $g$.

$$H_f = \sum_i \frac{p^2_i + m_i^2}{p_i^2} q_i q_{0i}, \quad \text{(B.1)}$$

where $i$ denotes particle species. The operator $\tilde{H}_i$ for any

$$\tilde{H}_i = \sum_{n=2}^{\infty} \sum_{i_2,\ldots,i_n} c(i_1,\ldots,i_n) q_{0i_1}^+ \cdots q_{0i_n}, \quad \text{(B.2)}$$

is defined by multiplication of each and every term in it in a square of a total + momentum involved in a term,

$$\tilde{H}_i = \sum_{n=2}^{\infty} \sum_{i_2,\ldots,i_n} c(i_1,\ldots,i_n) \left(\frac{1}{2} \sum_{k=1}^{n} p_{ik}^2\right) q_{0i_1}^+ \cdots q_{0i_n}. \quad \text{(B.3)}$$

The multiplication by this factor secures invariance of $H_i$ with respect to seven kinematical symmetries of the FF dynamics.

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