Distributed Control of Inverter-Based Lossy Microgrids for Power Sharing and Frequency Regulation Under Voltage Constraints

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Abstract

This paper presents a new distributed control framework to coordinate inverter-interfaced distributed energy resources (DERs) in island microgrids. We show that under bounded load uncertainties, the proposed control method can steer the microgrid to a desired steady state with synchronized inverter frequency across the network and proportional sharing of both active and reactive powers among the inverters. We also show that such convergence can be achieved while respecting constraints on voltage magnitude and branch angle differences. The controller is robust under various contingency scenarios, including loss of communication links and failures of DERs. The proposed controller is applicable to lossy mesh microgrids with heterogeneous R/X distribution lines and reasonable parameter variations. Simulations based on various microgrid operation scenarios are also provided to show the effectiveness of the proposed control method.

Key words: Microgrid Control, Droop Control, Frequency Synchronization, Power Sharing, Voltage Regulation

1 Introduction

Microgrids are low voltage power networks comprised of distributed generations (DGs), energy storages systems (ESSs), and loads that can operate in either grid-connected or island mode. A transition to the island mode can be triggered autonomously by faults on the main grid or taken intentionally for efficiency or reliability benefits, (see [26,27]).

Distributed energy resources (DERs) such as DGs and ESSs connect to the microgrid through DC/AC or AC/AC inverters. During the island mode, the inverters are typically operated as voltage source inverters (VSIs). These VSIs need to be controlled cooperatively to achieve desired performance and reliability properties. In AC networks, voltage magnitude and angle difference between connected buses should be regulated in some bounded ranges for system security and stability. Frequency synchronization to a nominal value is also crucial for grid connection and stability purposes. Besides frequency and voltage regulation, active and reactive power sharing are considered as important control objectives in microgrids [24,29]. They require that the power injection into the microgrid from DERs is proportional to the nominal value defined by economics or other incentives, while satisfying load demands [28]. Power sharing enables effective utilization of limited generation resources and prevent overloading [17].

To achieve the aforementioned objectives, a microgrid is typically controlled using a hierarchal structure that is similar to the one used in the traditional power systems [19], including primary, secondary, and tertiary controls [3,5,9,10]. The primary droop control of a microgrid maintains the voltage and frequency stability while balancing the generation and load with proper power sharing. The secondary controller compensates the voltage and frequency deviations from their reference values. The tertiary control establishes the optimal power sharing between inverters in both islanding and grid-connected modes.

The primary droop is generally a decentralized controller that adjusts the voltage frequency and magnitude of each inverter in response to active and reactive power deviations from their nominal values. Various droop methods are proposed to achieve proportional active and reactive load power sharing [1,15,21,24,35,36,41]. However, this is often achieved at the cost of sacrificing other control objectives such as voltage and frequency regu-
The secondary control utilizes either centralized or decentralized communication infrastructures to restore frequency and voltage deviation induced by the primary droop. Most of the existing secondary control methods require centralized communications [13, 22, 23]. On the other hand, decentralized secondary control has recently been proposed to avoid single point of failure [30]. The combined operations of the primary and secondary control require separation of time scale, resulting in slow dynamics that cannot effectively handle fast-varying loads [32]. In addition, the secondary control may destroy the proportional power sharing established in the primary control layer [34]. One possible solution is to adopt distributed or decentralized control structure for primary and secondary control layers to improve performance and support plug-and-play operation of the microgrid [5].

Many existing primary and secondary control methods rely on small signal linearization for stability analysis, which is vulnerable to parameter variations and change of operating points. Only several recent works [2, 28, 34] have rigorously analyzed the stability of microgrid with droop-controlled inverters. In particular, [34] derives a necessary and sufficient condition for the stability under primary droop control. The authors have also proposed a distributed averaging controller to fix the time scale separation issue between the primary and secondary control layers. In [28] and [2], stability conditions of lossless mesh microgrids have been provided. Despite their advantages, these nonlinear methods still suffer from several common limitations. First, all the nonlinear analyses mentioned above focus on lossless microgrids with purely inductive distribution lines. The results may not be applicable in microgrids with heterogeneous and mixed R/X ratio lines, which is common in microgrids [22]. Secondly, since only frequency droop is carefully analyzed, reactive power sharing is often not guaranteed.

To address the aforementioned limitations of the existing works, we propose a distributed control framework to coordinate VSIs in an island AC microgrid. The proposed control adjusts each inverter frequency and voltage magnitude based on the active/reactive power measurements of its neighbors. We first show that the particular control structure ensures that any equilibrium of the closed-loop system results in the desired power sharing and frequency synchronization. Secondly, conditions for power sharing, synchronized frequency at nominal value respecting voltage constraints are provided. The proposed controller can be applied to both radial and mesh microgrids with mixed R/X ratios. Furthermore, the proposed controller requires no separation of time scale and can tolerate reasonable parameter variations.

To demonstrate the robustness of the proposed distributed controller, we study the control performance under partial communication failures and the plug-and-play operations. We will show that as long as the communication network remains connected, all the desired properties including power sharing, frequency and voltage regulation still hold in these contingency scenarios. This effectively demonstrates the robustness of the proposed distributed controller. It is worth to mention that the proposed framework may require faster communications among the VSIs than the traditional secondary control. However, such communication requirement is reasonable for most microgrid control systems [11, 20, 39].

The rest of this paper is organized as follows. Section 2 formulates the microgrid control problem. Sufficient conditions for the solvability of the proportional power sharing problem respecting voltage constraints are also provided. The proposed distributed control framework is developed in Section 3. Robustness of the distributed controller under loss of communication links or failures of DERs is studied in Section 4. In Section 5, we validate the proposed controller through simulations under various microgrid operating scenarios, including abrupt changes of loads, communication delays and loss of one VSI. Some concluding remarks are given in Section 6.

**Notation** Define $\mathbb{R}_+$ and $\mathbb{R}_-$ as positive and negative real numbers, respectively. Denote $[n] := \{1, 2, ..., n\}$. Given a set $\mathcal{V}$, let $|\mathcal{V}|$ and $2^\mathcal{V}$ be its cardinality and power set, respectively. Denote the diagonal matrix of a vector $x$ as $\text{diag}(x)$. For a set of vectors $x_i, i \in \mathcal{I}$, let $\{x_i, i \in \mathcal{I}\}$ be the augmented vector of $x_i$ collecting all $i \in \mathcal{I}$. Given a polyhedron $\mathcal{B} \subseteq \mathbb{R}^n$, let $\nu(\mathcal{B})$ be the vertex set of the polyhedron. For a closed set $F \subseteq \mathbb{R}^n$, $\text{int}(F)$ and $\partial F$ are the interior and the boundary of $F$. The distance between a point $f \in \mathbb{R}^n$ and the set $F$ is denoted as $d(f, F) := \inf\{|f - f|_2 : f \in F\}$. Define $\check{1}_n \in \mathbb{R}^n$ and $\underline{0}_n \in \mathbb{R}^n$ as the vectors with all the elements being ones and zeros, respectively. For a symmetric matrix $A$, let $\lambda(A)$ and $\underline{\lambda}(A)$ be the spectrum and minimal eigenvalue of $A$, respectively. Denote $A \otimes B$ as the tensor product between matrices $A$ and $B$. Let null($A$) be the null space a matrix $A$.

## 2 Problem Formulation

In this paper, we consider a connected island microgrid network as shown in Fig. 1. Let the set of inverter buses and load buses be $\mathcal{V}_I$ and $\mathcal{V}_L$, respectively. Let $n_I = |\mathcal{V}_I|$, $n_L = |\mathcal{V}_L|$ and $n = |\mathcal{V}|$, where $\mathcal{V} = \mathcal{V}_I \cup \mathcal{V}_L$. Denote $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ as the set of distribution lines connecting the buses. The magnitude and phase angle of the bus voltage are denoted as $E_i$ and $\theta_i$, respectively. Let $x_i \triangleq [\theta_i, E_i]^T$ be the state vector at bus $i$, and let $x_I = \{x_i, i \in \mathcal{V}_I\}$ and $x_L = \{x_i, i \in \mathcal{V}_L\}$ be the inverter bus state vector.
and load bus state vector, respectively. The overall system state vector is denoted by \( x = [x_P^T, x_Q^T]^T \) and will be referred to as the system voltage profile. For each bus \( i \in V \), let \( P_i \) and \( Q_i \) be the active and reactive power injections at bus \( i \). Given the admittance matrix \( Y \in \mathbb{C}^{n \times n} \) of the microgrid, the active and reactive power injections are related to the voltage profile \( x \) by power flow equations [6]

\[
\begin{align*}
    P_i(x) &= E_i \sum_{j \in V} Y_{ij} E_j \cos(\theta_i - \theta_j - \phi_{ij}), \\
    Q_i(x) &= E_i \sum_{j \in V} Y_{ij} E_j \sin(\theta_i - \theta_j - \phi_{ij}),
\end{align*}
\]

(1)

where \( Y_{ij} = |Y_{ij}| \) and \( \phi_{ij} = \angle Y_{ij} \) are the magnitude and the phase angle of the admittance matrix element \( Y_{ij} \).

We distinguish the voltage at inverter and load buses in our formulation due to their different properties. For inverter buses, there are standard methods to control the voltage magnitude and frequency ([18], [40]). Typically, these methods can track a given inverter voltage reference almost instantaneously. Therefore, an inverter is often modeled as a controlled voltage source behind a reactance [43]. We also adopt such a model and assume \( x_I \) can be fully controlled. In contrast to \( x_I \), voltage at load buses \( x_L \) is uncertain. The voltage dynamics of the load buses is assumed to satisfy the following condition

**Assumption 1** \( \| \dot{x}_L \|_2 \leq \kappa \| \dot{x}_I \|_2 \).

where \( \kappa > 0 \) is a constant determined by load properties and microgrid topology. More explanations about this assumption are provided in Appendix A, where we show that Assumption 1 holds for various types of loads, such as constant power and constant impedance loads. Discussions on modeling various types of loads can be found in [4].

Given nominal active and reactive power injections \( P_i^* \) and \( Q_i^* \), \( i \in \mathcal{V}_I \), it is desired that the power injection of the inverters share uncertain loads proportionally to their nominal value:

**Definition 1 (Proportional Power Sharing)** The active and reactive power are proportionally shared among the buses \( j \in \mathcal{V}_I \) if

\[
\frac{P_j(x)}{P_j^*} = \frac{P_k(x)}{P_k^*}, \quad \frac{Q_j(x)}{Q_j^*} = \frac{Q_k(x)}{Q_k^*}, \quad j, k \in \mathcal{V}_I.
\]

(2)

**Remark 1** The nominal power injections \( P_i^* \) and \( Q_i^* \) are selected in tertiary control layer based on economic or environmental criteria of a microgrid (see [28] and [3]). Since we focus on controller design of the primary and secondary layers, we assume that \( P_i^* \) and \( Q_i^* \) are known constants.

The power sharing condition (2) impose a constraint on the system voltage profile. We define this constraint set as

\[
    x \in \mathcal{X}_S := \{ x | \text{Eq. (2) holds} \}.
\]

(3)

In addition to the condition (3), the control and operation of a microgrid has to respect its branch angle difference limits and voltage magnitude constraint. The branch angle difference between all connected buses is typically required to be bounded by a given constant \( \gamma \in [0, \frac{\pi}{2}] \). The upper bound \( \gamma \) is derived by maximum current allowable on each distribution line \( I_{\text{max}} \) (see [38] and [44]). In addition to branch angle difference, the voltage magnitude also needs to stay inside some secure operation range [6]. Denote \( [E_i, \bar{E}_i] \) as a given desired range of the voltage magnitude of bus \( i \). Both branch angle difference and voltage magnitude requirements impose a constraint on the system voltage profile defined below:

\[
\begin{align*}
    \mathcal{X}_\theta &= \{ x | |\theta_i - \theta_j| \leq \gamma, \ \forall (i, j) \in \mathcal{E} \}, \\
    \mathcal{X}_E &= \{ x | E_i \leq E_i \leq \bar{E}_i, \ \forall i \in \mathcal{V} \}.
\end{align*}
\]

These two constraints will be referred to as the security constraint of the microgrid

**Definition 2 (Security Constraints)** We say that a microgrid satisfies the security constraints if

\[
x \in \mathcal{X}_c := \mathcal{X}_E \cap \mathcal{X}_\theta.
\]

(4)

In addition to proportional power sharing and voltage regulation, another important microgrid control objective is known as frequency regulation. Frequency regulation is defined as synchronization without deviations from the nominal value, specifically, \( \dot{\theta}_i = \omega_0 \), \( \forall i \in \mathcal{V} \), where \( \omega_0 \) is a predefined nominal frequency of the microgrid. Note that the power flow equations (1) and the
security constraints are invariant with respect to rigid rotation of \( \theta_i \) of all buses. We can select a reference frame rotating at angular frequency \( \omega_0 \) while preserving all properties in Eq. (1)-(4). With the rotating reference, the frequency regulation condition is reduced to

\[
\dot{\theta}_i = 0, \quad \forall i \in \mathcal{V}.
\]

Conventional droop controller is widely adopted to achieve power sharing. The conventional droop controller takes the following form

\[
\dot{\theta}_i = m_{p_i}(P_i - P_i^*)
\]

\[
E_i - E_i^* = m_{q_i}(Q_i - Q_i^*), \quad \forall i \in \mathcal{V}_I,
\]

where \( m_{p_i}, m_{q_i} \in \mathbb{R}_+ \) are droop gains and \( E_i^* \in [E_i, \bar{E}_i] \) is the nominal voltage magnitude of bus \( i \). Under some assumptions, condition (3) can be met under the conventional droop controller if \( m_{p_i}, m_{q_i} \) are selected inversely proportional to \( P_i^* \) and \( Q_i^* \), respectively [42]. However, the conventional droop controller introduces frequency and voltage magnitude deviations whenever \( P_i \neq P_i^* \) and \( Q_i \neq Q_i^* \), respectively. Especially when larger values of \( m_{p_i} \) and \( m_{q_i} \) are selected to achieve better power sharing performance, the deviations become more apparent, leading to violation of conditions (4) and (5). This issue motivates us to design a new distributed control framework which does not violate conditions (4) and (5) while accomplishing the control objective (3).

As discussed in [9], the controller design to meet the requirements specified in Eq. (3)-(5) inevitably requires communication networks. In this paper, we employ a distributed communication structure similar to [29] and [32]. We assume that each inverter can communicate with its neighboring inverters to share its local measurements as shown in Fig. 1. Let \( \mathcal{G}_c = (\mathcal{V}_I, \mathcal{E}_c) \) be a connected simple graph of the communication network, where each edge \((i, j) \in \mathcal{E}_c\) represents an available communication link between buses \( i \) and \( j \). Let \( \mathcal{N}_i := \{j|(i, j) \in \mathcal{E}_c\} \cup \{i\} \) be the set of neighbors of bus \( i \), (including bus \( i \) itself). An inverter \( i \) is assumed to have access to the measurements at every inverter bus \( j \in \mathcal{N}_i \), including \( P_j, Q_j \) and \( E_j \).

Since each inverter is modeled as a VSI, the microgrid-level coordination control for each inverter \( i \) reduces to the determination of appropriate voltage frequency and magnitude setpoints. The actual frequency \( \dot{\theta}_i \) and magnitude \( E_i \) can track these setpoints almost instantaneously. The challenge here lies in that the load is uncertain and different load conditions require different voltage profile \( x_i \) in order to satisfy constraints (3)-(5). Define \( S_i = \left[P_i/P_i^*, Q_i/Q_i^*\right]^T \), \( S_{N_i} = \{S_j, j \in \mathcal{N}_i\} \) and \( E_{N_i} = \{E_j, j \in \mathcal{N}_i\} \). Our goal is thus to design a controller for each inverter \( i \) that can automatically find the desired voltage profile \( x_i \) based on local information \((S_{N_i}, E_{N_i})\). Towards this end, we propose to dynamically update \( x_i \) as follows

\[
\begin{cases}
\dot{x}_i = \mu_i(S_{N_i}(x), E_{N_i}), \quad \forall i \in \mathcal{V}_I \\
\text{subj. to } x(t) \in \mathcal{X}_c, \quad \forall t \geq 0,
\end{cases}
\]

where \( \mu_i \) is the control law of inverter \( i \) to be designed. Note that the above control structure corresponds to directly assigning frequency \( \dot{\theta}_i \), based on local information, while dynamically updating voltage magnitude \( E_i \) through simple integrator dynamics. Such structure is commonly used in the literature of microgrid control, (see [29] and [33]). The constraint \( x(t) \in \mathcal{X}_c \) is imposed to ensure that the security constraints are always satisfied.

Define \( \mathcal{X}_c := \mathcal{X}_e \cap \mathcal{X}_s \), then conditions (3) and (4) hold when \( x \in \mathcal{X}_c \). Under Assumption 1, condition (5) holds when \( \dot{x}_i = 0 \) for all \( i \in \mathcal{V}_I \). Our goal becomes to designing \( \mu_i \) such that \( \mathcal{X}_c \) forms an equilibrium set of system (6). In addition, we also want \( \mathcal{X}_c \) to be locally exponentially stable for some initial \( x(t_0) \) and uncertain \( x_L(0) \) such that \( x(0) \in \mathcal{X}_c \). If such a controller is found, it can drive the microgrid to the desired steady state where conditions (3)-(5) hold. In the rest of this paper, we will first develop the control law \( \mu_i \) such that \( \mathcal{X}_c \) forms the equilibrium set of the system (6), and then derive conditions to ensure stability of \( \mathcal{X}_c \).

3 A Distributed Microgrid Control Framework

In this section, we propose a distributed control framework to coordinate the inverters in an island AC microgrid to accomplish the control objectives (3)-(5). We first provide sufficient conditions to ensure \( \mathcal{X}_c \neq \emptyset \). A control design framework is then developed.

3.1 Existence of Solutions

A minimum requirement for the controller design is the existence of a voltage profile \( x \) satisfying all the constraints, i.e., \( \mathcal{X}_c \neq \emptyset \). Existing methods in the literature often directly assume this holds ([35], [36]). Here, we provide a brief discussion and a set of sufficient conditions to guarantee the non-emptiness of \( \mathcal{X}_c \). The existence of the voltage profile \( x \) satisfying conditions (3) and (4) involves solving nonlinear algebraic power flow equations (1). We revisit a classical result in the following.

Lemma 1 [38] Suppose that the following conditions hold

(a) The microgrid is connected,
(b) The admittance matrix \( Y \) is symmetric,
(c) \( 2E_j > E_k \) for all \( j, k \in \mathcal{V} \),
(d) \( I_{jk}^{\text{max}} \leq \frac{2}{R_{jk}} \) for all \( \{j, k\} \in \mathcal{E} \).
The microgrid with dynamics (8) is in steady state where \( \dot{x}_t = 0 \).

(b) The desired conditions (3)-(5) hold.

Proof. Since the null space of \( L \) is \( \mathbf{1}_{n_I} \), \( \text{null}(\bar{L}) = \mathcal{O} \).

The equivalency between statements (a) and (b) follows from Eq. (9).

Remark 3 Note that the proposed control structure (8) does not depend on the voltage magnitude information \( E_N \) that is also available at bus \( i \). We will show later that such a control structure is already sufficient to ensure convergence to \( \mathcal{X}_c \). In principle, the magnitude information \( E_N \) can be used to further improve the control performance, especially for voltage regulation. However, we will not study such extension in this paper.

We start our controller design from a simple property of a connected graph \( G_c \). Let \( L \in \mathbb{R}^{n_V \times n_I} \) be the Laplacian of \( G_c \). Since \( G_c \) is connected, the null space of \( L \) is \( \text{span}\{\mathbf{1}_{n_I}\} \). Observing that \( L \) has a close relation with condition (3), we design \( \mu_i(\cdot) \) as a simple linear feedback in terms of \( S_N \) in the following form

\[
\begin{aligned}
\dot{x}_i(t) &= K_i \sum_{j \in \mathcal{V}_i} L(i,j)S_i(x(t)) \\
\text{subj. to } x(t) &\in \mathcal{X}_c, \forall t \geq 0,
\end{aligned}
\]

(7)

where \( K_i \in \mathbb{R}^{2 \times 2} \) is the local control gain matrix at bus \( i \) to be designed. Denote \( S_t \triangleq \{S_i, i \in \mathcal{V}_t\} \), and \( S \triangleq \{S_i, i \in \mathcal{V}\} \). Let \( K = \text{diag}\{K_i, i \in \mathcal{V}_I\} \), and \( \bar{L} = L \otimes I_2 \). The dynamical model of the microgrid under the proposed inverter control (7) becomes

\[
\begin{aligned}
\dot{x}_i(t) &= K \bar{L}S_i(x(t)) \\
\text{subj. to } x(t) &\in \mathcal{X}_c, \forall t \geq 0,
\end{aligned}
\]

(8)

Define \( \mathcal{O} = \text{span}\{v_p, v_q\} \), where \( v_p, v_q \in \mathbb{R}^{2n_I} \), \( v_p = [1, 0, 1, \ldots, 0]^T \) and \( v_q = [0, 1, 0, \ldots, 1]^T \). The following lemma shows that under some mild conditions, every equilibrium point of system (8) satisfies the control objectives (3)-(5).

Proposition 1 If Assumption 1 holds and \( \text{null}(K) \subseteq \mathcal{O} \), the following statements are equivalent

(a) The microgrid with dynamics (8) is in steady state where \( \dot{x}_t = 0 \).

(b) The desired conditions (3)-(5) hold.

Proof. Since the null space of \( L \) is \( \mathbf{1}_{n_I} \), \( \text{null}(\bar{L}) = \mathcal{O} \).

The equivalency between statements (a) and (b) follows from Eq. (9). \( \square \)

Proposition 1 reduces the microgrid control problem with numerous requirements to the study of stability of system (8). We will therefore focus on analyzing the stability of \( \mathcal{X}_c \) of system (8).

3.3 Stability of \( \mathcal{X}_c \)

In this subsection, we derive the conditions of exponential convergence to \( \mathcal{X}_c \) where all the desired conditions (3)-(5) follow. The stability analysis of \( \mathcal{X}_c \) of system (8) is challenging in general due to the nonlinearity of the underlying system and the uncertainty of the load bus states \( x_L \). Instead of directly analyzing system (8), we apply the chain rule to obtain the dynamics of \( S_t \) under the proposed control strategy

\[
\begin{aligned}
\dot{S}_t(x(t)) &= J_{I, x(t)}K \bar{L}S_t(x(t)) + J_{L, x(t)}\dot{x}_L(t) \\
x(t) &\in \mathcal{X}_c, \forall t \geq 0,
\end{aligned}
\]

(10)

where \( J_{I, x} \) and \( J_{L, x} \) are the Jacobian matrices of \( S_t(\cdot) \) evaluated at \( x \) with respect to \( x_I \) and \( x_L \), respectively. Notice that at every time instant and for any \( \dot{x}_L \), Eq. (10) describes the dynamics of \( S_t \) when the dynamics of \( x_I \) is given by Eq. (8). According to Proposition 1, the convergence of \( S_t(x(t)) \) to \( \mathcal{O} \) implies the convergence of the state trajectory \( x(t) \) to \( \mathcal{X}_c \). The close relation between these two stability properties motivates us to focus on system (10). To simplify notation, we define \( z(t) = S_t(x(t)), B(t) = J_{I, x(t)} \), and \( \mathcal{J} = \{J_{I, x}|x \in \mathcal{X}_c\} \).
System (10) can then be written as a linear time varying (LTV) system
\[
\begin{cases}
\dot{z}(t) = B(t)KLz(t) + w(t) \\
B(t) \in \mathcal{J},
\end{cases}
\]
where \(w(t) = J_{L,x}(t)\dot{x}_L(t)\) is considered as a disturbance of system (11). With this notation, system (11) becomes a stand alone dynamic system with state variable \(z\) subject to unknown disturbance \(w(t)\). Note that in system (11), \(w\) is quadratically bounded by \(d(z,\mathcal{O})\), which is derived in the following
\[
\|w\|_2 = \|J_{L,x}\dot{x}_L\|_2 \leq \kappa \|J_{L,x}\|_2 \|\dot{z}\|_2 = \kappa \|J_{L,x}\|_2 |KLz|_2 \leq \zeta \|Lz\|_2 = \zeta \cdot d(z,\mathcal{O}),
\]
where \(\zeta \in \mathbb{R}_+\) is a constant depending on system parameters as well as control gain \(K\). Robust stability of the equilibriums of systems with bounded noise is introduced in [31], which is reviewed in the following

**Definition 3** The set \(\mathcal{O}\) is robustly stable of system (11) with degree \(\zeta\) if \(\mathcal{O}\) is globally exponentially stable for all \(w\) such that \(\|w\|_2 \leq \zeta d(z,\mathcal{O})\).

To analyze robust stability for \(\mathcal{O}\) of system (11), we apply a standard change of coordinates. Define a change of coordinate matrix equation: \(T = [v_1, v_2, \ldots, v_{2n-2}, \|v_2\|_2, \|v_{2n-2}\|_2]\) the first \(2n - 2\) vectors are arbitrary vectors such that \(T\) is an orthogonal matrix. Let \(\bar{z} = T^{-1}z\) be the state vector in the new coordinate system. The LTV system (11) becomes
\[
\begin{cases}
\dot{\bar{z}}(t) = T^{-1}B(t)KL\bar{z}(t) + T^{-1}w(t) \\
\bar{z}(t) \in \mathcal{J},
\end{cases}
\]
Since the last two coordinates of the new basis \(V\) span \(\mathcal{O}\), the last two column vectors of \(LT\) are zeros and
\[
T^{-1}B(t)KL = \begin{bmatrix} \hat{A}_{11}(t) & 0 \\ \hat{A}_{21}(t) & 0 \end{bmatrix},
\]
where \(\hat{A}_{11}(\cdot) \in \mathbb{R}^{(n_1-1)\times 2(n_1-1)}\) and \(\hat{A}_{21}(\cdot) \in \mathbb{R}^{2\times 2(n_1-1)}\). Considering that the dynamics of \(d(z(\cdot),\mathcal{O})\) of system (11) is irrelevant to the last two coordinates of the state \(\bar{z}\) of system (12), we focus on a reduced order system of (12) with state vector \(\bar{z} = [I\ 0]\bar{z} \in \mathbb{R}^{2(n_1-1)}\). Define \(\bar{\dot{w}} = [I\ 0]T^{-1}w \in \mathbb{R}^{2(n_1-1)}\) and \(G(B) = [I\ 0]T^{-1}BKL[T\ 0]^T \in \mathbb{R}^{2(n_1-1)\times 2(n_1-1)}\), we have a reduced order system of (12)
\[
\begin{cases}
\dot{\bar{z}}(t) = \hat{A}_{11}(t)\bar{z}(t) + \bar{\dot{w}}(t) \\
\hat{A}_{11}(t) \in \mathcal{A},
\end{cases}
\]
where \(\mathcal{A} := \{G(B)\ B \in \mathcal{J}\}\). Similar to \(w\) in system (11), \(\bar{w}\) is quadratically bounded by the state \(\bar{z}\) shown in the following
\[
\|\bar{w}\|_2 \leq \|w\|_2 \leq \zeta \|\bar{z}\|_2.
\]
The following lemma shows that stability properties of \(\mathcal{X}_e\) of system (8) follows if the origin of system (14) is robustly stable.

**Lemma 2** If the origin of system (14) is robustly stable with degree \(\zeta\), then there exists an non-empty \(\mathcal{X}_{e,s} \subseteq \mathcal{X}_e\) such that for all \(x(0) \in \mathcal{X}_{e,s}, x(t)\) exponentially converges to \(\mathcal{X}_e\) for system (8).

**Proof.** Since systems (14) and (12) share the same dynamics in the space \(\mathbb{R}^{2n_1}\), robust stability of the origin of system (14) implies \(\mathcal{O}\) is robustly stable of system (12) with degree \(\zeta\). In addition, robust stability of \(\mathcal{O}\) of system (11) (or system (12)) guarantees the trajectory of \(z(t)\) is bounded. Define \(\mathcal{S}_j = \{S_j(t)| x \in \mathcal{X}_e\}\). With the bounded trajectory of \(z(t) = S_j(t)(x(t))\), we can find \(\mathcal{X}_{e,s} \subseteq \mathcal{X}_e\) such that for all \(x(0) \in \mathcal{X}_{e,s}, S_j(t)(x(t)) \in \mathcal{S}_j \forall t\), which implies \(x(t) \in \mathcal{X}_e \forall t\). Therefore, for all initial \(x(0) \in \mathcal{X}_{e,s}, x(t) = S_j(t)(x(t))\) converges to \(\mathcal{O}\) with \(x(t) \in \mathcal{X}_e\) for all time in system (11) if system (14) is robustly stable. Recall that Eq. (11) describes the dynamics of \(S_j(t)\) when the dynamics of \(x_j\) is given by Eq. (8). We then conclude that for all \(x(0) \in \mathcal{X}_{e,s}, x(\cdot)\) exponentially converges to \(\mathcal{X}_e\) for system (8) if the origin of system (14) is robustly stable. \(\square\)

We now provide a set of sufficient conditions for robust stability of the origin of system (14) [31]

**Proposition 2** The origin of system (14) is robustly stable with degree \(\zeta\) if there exist \(\xi \in \mathbb{R}_+, U = U^T \succ 0\), and \(W = W^T \succ 0\) such that
\[
\begin{bmatrix}
\hat{A}_{11}^T + U \hat{A}_{11} + \zeta I & U \\
U & -I
\end{bmatrix} < -\xi W,
\]
for all \(\hat{A}_{11} \in \mathcal{A}\).

**Proof.** Conditions in Eq. (15) can be derived by quadratic Lyapunov function argument. If there exist Lyapunov function \(V(\bar{z}) = \bar{z}^T U \bar{z}\) such that for all \(\hat{A}_{11} \in \mathcal{A}, V(\bar{z}) \leq -\xi, \) then the origin of system (14) is exponentially stable. For any \(\hat{A}_{11} \in \mathcal{A}\),
\[
V(\bar{z}) = \bar{z}^T U (\hat{A}_{11} \bar{z} + \bar{w}) + (\hat{A}_{11} \bar{z} + \bar{w})^T U \bar{z} \\
\leq \left[\bar{z}^T \bar{w}\right]^T \begin{bmatrix}
\hat{A}_{11}^T + U \hat{A}_{11} + \zeta I & U \\
U & -I
\end{bmatrix} \left[\begin{bmatrix}
\bar{z} \\
\bar{w}
\end{bmatrix}\right].
\]
which leads to Eq. (15). \hfill \Box

Note that if \( A_{11} \) is polytopic, the condition (15) can be formulated into bilinear matrix inequalities (BMIs). The condition can then be checked numerically. However, \( A_{11} \) is not polytopic, so we will instead develop a way to find a convex set containing \( A_{11} \) in the next subsection.

With Lemma 2 and Proposition 2, the stability property of \( \mathcal{X}_c \) of (8) can be obtained:

**Theorem 1** If Assumption 1, hypotheses in Lemma 1 and Eq. (15) hold, then there exists an non-empty \( \mathcal{X}_{c,s} \subseteq \mathcal{X}_c \) such that for \( x(0) \in \mathcal{X}_{c,s} \), the microgrid (8) converges exponentially to the set \( \mathcal{X}_c \) where the control objectives including (3)-(5) are all satisfied.

**Proof.** By Proposition 2, we know the origin of system (14) is robustly stable. By Lemma 2, robust stability of system (14) implies existence of \( \mathcal{X}_{c,s} \subseteq \mathcal{X}_c \) such that for all \( x(0) \in \mathcal{X}_{c,s} \), the trajectories converge to \( \mathcal{X}_c \) of system (8). Since Eq. (15) ensures null(K) \( \subseteq \mathcal{O} \), we know \( \mathcal{X}_c \) is equivalent to desired control objectives (3)-(5) from Proposition 1. \hfill \Box

The results of Theorem 1 are robust with respect to small variations of system parameters. As long as the perturbations of the admittance matrix are small enough such that \( J_{I,x} \in \mathcal{J} \), the exponential convergence to \( \mathcal{X}_c \) for some \( x(0) \in \mathcal{X}_{c,s} \) still follows from Theorem 1. Different from most of the literature, the proposed controller can be applied to mixed \( R/X \) ratio distribution lines and general microgrid topology including acyclic and mesh networks. Furthermore, the controller meets all the main control objectives without the separation of time scale, which distinguishes it from the mainstream droop control methods.

### 3.4 Feedback Gain Design

In this subsection, we propose a constructive way to find a feedback gain \( K \) such that the hypotheses in Theorem 1 holds. The difficulty lies in checking the feasibility Eq. (15). As discussed in the last subsection, the robust stability condition in Eq. (15) can not be directly formulated into BMIs because \( A \) is not polytopic. We will first derive a convex hull containing \( A \) by analyzing the Jacobian of power flow equations (1) so that Eq. (15) can be checked by several BMIs. Secondly, instead of only checking the feasibility, we formulate the BMIs into an optimization problem to enhance the robustness.

Define \( C_i \subseteq \mathcal{E} \) and \( \mathcal{X}_c \) as a polyhedron replacing the constraint \( |\phi_{ij}| \leq \gamma \) in \( \mathcal{X}_c \) by \( \theta_i - \theta_j = 0 \) for all \( (i, j) \in C_i \). Let \( \Phi := \{ \phi_{ij} \mid |\phi_{ij}| + \gamma \leq \frac{n}{2} \} \). An approximated convex hull can be found by the following Proposition

**Proposition 3** If every entry of the admittance matrix \( Y \) satisfy \( \phi_{ij} \in \{ \pm \frac{\pi}{2}, \Phi \} \), the upper and lower bounds of every entry \( (i, j) \) of \( J_{I,x} \in \mathcal{J} \) are

\[
\begin{align*}
\bar{J}_{I,x}(i, j) &= \max \{ J_{I,x}(i, j), z \in Z \}, \\
\underline{J}_{I,x}(i, j) &= \min \{ J_{I,x}(i, j), z \in Z \},
\end{align*}
\]

where \( Z = \bigcup_{C_i \in 2^E} \nu(\mathcal{X}_c) \).

**Proof.** According to the power flow equation (1), the derivative of the active power injection at bus \( i \) with respect to different variants are

\[
\begin{align*}
\frac{\partial P_i(x)}{\partial \theta_i} &= -E_i \sum_{j \in V \setminus i} Y_{ij} E_j \sin(\theta_i - \theta_j - \phi_{ij}), \\
\frac{\partial P_i(x)}{\partial E_i} &= 2E_i \cos(-\phi_{ij}) + \sum_{j \in V \setminus i} Y_{ij} E_j \cos(\theta_i - \theta_j - \phi_{ij}), \\
\frac{\partial P_i(x)}{\partial \theta_j} &= E_i Y_{ij} E_j \sin(\theta_i - \theta_j - \phi_{ij}), \quad j \neq i, \\
\frac{\partial P_i(x)}{\partial E_j} &= E_i Y_{ij} \cos(\theta_i - \theta_j - \phi_{ij}), \quad j \neq i.
\end{align*}
\]

Since for all entries in the admittance matrix \( Y \), \( \phi_{ij} \in \{ \pm \frac{\pi}{2}, \Phi \} \), every summmand of the Jacobian elements in (17) has the maximal and minimal points at \( x \in Z \) if \( x \in \mathcal{X}_c \). In addition, every summmand corresponds to different set of \( E_j \) and \( \theta_i - \theta_j \) in each Jacobian element in (17), the Jacobian elements have the maximal and minimal points at \( x \in Z \) if \( x \in \mathcal{X}_c \). The same conclusion for the reactive part is reached by a similar argument. \hfill \Box

Given the upper and lower bounds of every entry of \( J_{I,x} \in \mathcal{J} \), a convex hull contains \( \mathcal{J} \) can be found. Let \( \mathcal{F} \) be a set \( \{-1, 1\} \). Define \( \theta_i \in \mathbb{F}^{2n_1 \times 2n_1} \) such that the disjoint \( \theta_i \) satisfying \( \cup_i \theta_i = \mathbb{F}^{2n_1 \times 2n_1} \). Define \( D_{\theta_i} \in \mathbb{R}^{2n_1 \times 2n_1} \) such that

\[
\begin{align*}
D_{\theta_i}(j, k) &= \bar{J}_{I,x}(j, k) \quad \text{if } \theta_i(j, k) = 1, \\
D_{\theta_i}(j, k) &= \underline{J}_{I,x}(j, k) \quad \text{if } \theta_i(j, k) = -1.
\end{align*}
\]

The following lemma is a simple consequence of Proposition 3 and provides a convex hull containing \( \mathcal{J} \).

**Lemma 3** If every entry of the admittance matrix \( Y \) satisfy \( \phi_{ij} \in \{ \pm \frac{\pi}{2}, \Phi \} \), the convex hull

\[
\mathcal{D} := CO\{ D_{\theta_1}, \ldots, D_{\theta_l} \}
\]

contains \( \mathcal{J} \), where \( l = 2^{4n_1^2} \).
Remark 5 The results of Lemma 3 allows us to replace Eq. (15) in Theorem 1 by BMIs. Instead of only finding \( K \) such that Eq. (15) is feasible, we propose to design \( K \) by solving the following optimization problem subject to BMI constraints:

\[
\begin{align*}
\text{maximize } & \zeta + \zeta \\
\text{subject to } & W = W^T > 0, \\
& M_i(\zeta) < -\xi W,
\end{align*}
\]

where \( \xi, \zeta \in \mathbb{R}_+ \) and

\[
\begin{align*}
\hat{A}_{i,1} &= [I, 0]T^{-1}D_{ij,k}L[T[I, 0]]^T, \\
M_i(\zeta) &= \begin{bmatrix} A^T_{i,1}U & U \hat{A}_{i,1} + \zeta U \zeta & U \zeta \\
U & -I \end{bmatrix}, \\
U &= U^T > 0, \ i \in [l].
\end{align*}
\]

Remark 4 The maximization of \( \xi \) is for the purpose of best convergence rate possible while the maximization of \( \zeta \) is for best toleration of different loads conditions. Notice that maximizing \( \zeta \) is only meaningful if an upper bound of \( ||K||_2 \) is imposed. More discussion on the upper bound will be included in the next section.

The following corollary is a direct consequence of BMIs (18) and Theorem 1.

**Corollary 1** If Assumption 1, BMIs (18), the hypotheses in Lemma 1 and Proposition 3 hold, then there exists an non-empty \( X_{c,s} \subset X_c \) such that for \( x(0) \in X_{c,s} \), the microgrid (8) converges exponentially to the set \( X_c \) where the control objectives including (3)-(5) are all satisfied.

We want to comment that convex optimization problem subject to BMI constraints are NP-hard to solve in general. However, efficient algorithms [12] and [16] are available if an initial feasible solution can be found. We can first substitute \( U \) by some simple positive definite matrices as an initial guess and check the feasibility. If the BMIs is feasible with an \( U \), then we apply the algorithms [12] and [16]. The BMIs (18) becomes a sequence of linear matrix inequalities (LMIs) problems and can be solved effectively using the cvx toolbox [8] [7].

Remark 5 (Trade-off between Complexity and Robustness) Let \( \mathcal{F} := CO\{J_{i,j}, z \in Z\} \). The convex hull \( \mathcal{F} \) does not necessary to contain \( \mathcal{J} \). However, there exists a compact set \( X_{c,s} \subset X_c \) such that \( \mathcal{J} \supseteq \{J_{i,j}, x \in X_{c,s}\} \). Hence, if one solves BMIs (18) by replacing \( v(\mathcal{D}) \) to \( v(\mathcal{J}) \), Corollary 1 still implies exponential convergence to a subset of \( X_c \) for some initial \( x(0) \). The benefit of replacing \( v(\mathcal{D}) \) to \( v(\mathcal{J}) \) is reduction of complexity, where \( |v(\mathcal{D})| = 2^{4n^2} \) while \( |v(\mathcal{J})| = 2^{2n_l+|E|} \). The difference becomes evident for large \( n_l \) and one may prefer to solve the BMIs with \( \mathcal{J} \) for microgrids with larger \( n_l \).

4 Flexible Operation of Microgrids

In this section, we study the plug-and-play feature of DERs under the proposed distributed control framework. The robustness under communication failure is also analyzed to solidify the proposed distributed controller.

4.1 Plug-and-Play Feature

The plug-and-play feature of the DERs refers to the property that one DER can be plugged or unplugged to a microgrid without re-engineering the entire control. We consider a whole set of inverters \( V_I \) is known, but part of inverters may disconnected to the microgrid abruptly due to some severe events. Let \( V_{I_j} \subset V_I \) be a set of normal operating inverter buses so that \( V_I \setminus V_{I_j} \) is the set of unplugged inverters. The voltage magnitude and phase angle dynamics at the buses \( i \in V_I \setminus V_{I_j} \) become unknown and are categorized as load buses. For this reason, we will partition the buses by \( V = V_{I_j} \cup V_{I_j} \), for which \( V_{I_j} \) contains all the connected inverter buses, while \( V_{I_j} \triangleq V \setminus V_{I_j} \) consists of all the other buses including load buses \( V_L \) or disconnected inverter buses \( V_I \setminus V_{I_j} \). If a feedback gain \( K \) is found such that the conditions (3)-(5) hold for possible \( V_{I_j} \), we say the DERs under the proposed distributed controller has the plug-and-play feature.

Consider the case that the communication network remains intact when some inverters are disconnected from the microgrid. The communication graph between the operating inverters is defined as \( \mathcal{G_f} := (V_{I_j}, \mathcal{E_I}) \), where \( \mathcal{E_I} = \mathcal{E} \setminus \{(i,j)|i \in V_I \setminus V_{I_j}\} \). Let \( L_f \) be the Laplacian of \( \mathcal{G_f} \). The control law of the “fault” microgrid is designed as follow:

\[
\begin{align*}
\dot{x}_i(t) &= K_i \sum_{j \in V_{I_j}} L_f(i,j)S_i(x(t)) \\
\text{subj. to } x(t) &\in X_i, \ \forall t \geq 0.
\end{align*}
\]

(19)

Notice that due to the assumption on intact communication network, the disconnected inverter can still communicate with its neighborhood so that the control law can be autonomously transformed to Eq. (19) in response to the change of microgrid operating conditions. Let \( x_{I_j} := \{x_i, i \in V_{I_j}\} \) and \( S_{I_j} := \{S_i, i \in V_{I_j}\} \). The microgrid dynamics with the controller (19) is rewritten as follow:

\[
\begin{align*}
\dot{x}_{I_j}(t) &= K_{I_j} \bar{L_f}S_{I_j}(x(t)) \\
\text{subj. to } x(t) &\in X_i, \ \forall t \geq 0.
\end{align*}
\]

(20)
where $K_I = \text{diag}(K_i, i \in \mathcal{V}_I)$, and $\bar{L}_I = L_I \otimes I_2$. Denoted $\mathcal{O}$ as the proportional power sharing space of the inverters $i \in \mathcal{V}_I$. If $G_I$ stays connected, the null space of $\bar{L}_I$ remains equivalent to $\mathcal{O}$. The “reduced” microgrid (20) can therefore be analyzed through a similar way discussed in the last section. Let $I_I = 2^{\mathcal{V}_I} \mathcal{I}_I^2$ and a convex hull $\mathcal{D} = CO(D_{\mathcal{V}_I}, \cdots, D_{\mathcal{G}_I})$ contains Jacobians of $S_I$, when $x \in \mathcal{X}$. The control objectives (3)-(5) of the reduced microgrid follow if all the hypotheses in Corollary 1 hold except that BMIs (18) are replaced by the following\footnote{Here we assume all loads stay serviceable by the remaining DERs $\mathcal{V}_I$. The voltage dynamics of buses $\mathcal{V} \setminus \mathcal{V}_I$ is bounded in the form of Assumption 1.}:

\[
\begin{align*}
\text{maximize } & \quad \zeta_f + \xi_f \\
\text{subject to } & \quad W_f = W_f^T > 0, \\
& \quad M_{i,f}(\zeta_f) = -\zeta_f W_f,
\end{align*}
\]

where $\zeta_f, \xi_f \in \mathbb{R}_+$. The orthogonal matrix $T_f$ of system (20) is defined similar to $T$ where the last two coordinates span the space of proportional power sharing. If $A$ is found such that BMIs (18) and (21) are both feasible for $\mathcal{V}_I$ and all possible $\mathcal{V}_I$, respectively, the proposed controller has the plug-and-play feature. In fact, we will show a lemma proving that the BMIs (21) are feasible if BMIs (18) are feasible and $G_f$ is connected. Before proving the lemma, we review Cauchy interlace theorem as follow

\textbf{Theorem 2 (Cauchy Interlace Theorem)} Let $A$ be a symmetric matrix of order $n$, and let $B$ be a principal submatrix of $A$ of order $n-1$. If $\lambda_n \leq \lambda_{n-1} \leq \cdots \leq \lambda_2 \leq \lambda_1$ lists the eigenvalues of $A$ and $\mu_n \leq \mu_{n-1} \leq \cdots \leq \mu_2 \leq \mu_1$ the eigenvalues of $B$, then $\lambda_n \leq \mu_n \leq \lambda_{n-1} \leq \mu_{n-1} \leq \cdots \leq \lambda_2 \leq \mu_2 \leq \lambda_1$.

The proof of the above theorem can be found in [14]. The following Lemma is derived based on the Cauchy interlace theorem

\textbf{Lemma 4} If $G_f$ is connected and $K^*, U^*, W^*$ are found such that BMIs (18) are feasible, then by setting $K_I = \text{diag}(K^*_i, i \in \mathcal{V}_I)$, BMIs (21) are reduced to LMIs and are feasible.

\textbf{Proof.} By definition of the Laplacian matrix, $\bar{L}$ and $\bar{L}_f$ are positive semidefinite. In addition, because $G_e$ and $G_f$ are connected,

\[
T^{-1}LT = \begin{bmatrix} \bar{L}_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad 0 \prec \bar{L}_1 \in \mathbb{R}^{2(n_I-1) \times 2(n_I-1)},
\]

\[
T^{-1}\bar{L}_fT_f = \begin{bmatrix} \bar{L}_{f1} & 0 \\ 0 & 0 \end{bmatrix}, \quad 0 \prec \bar{L}_{f1} \in \mathbb{R}^{2(|\mathcal{V}_f|-1) \times 2(|\mathcal{V}_f|-1)}.
\]

(23)

Since $K^*$, $U^*$, $W^*$ solve BMIs (18), the following condition holds for all $i \in [I]$

\[
[U^*, 0]T^{-1}D_{\mathcal{V}_I}K^*T[\bar{L}_1, 0]^T + \quad (24)
\]

\[
[\bar{L}_1, 0]T^{-1}D_{\mathcal{V}_I}^T(K^*)^T [U^*, 0]^T \prec -(U^*U^* + \xi W^*).
\]

The matrix inequalities are derived by applying Schur complement to BMIs (18) and symmetric property of $\bar{L}_1$. Due to Eq. (24), the eigenvalues of $H_i := D_{\mathcal{V}_I}K^* + (K^*)^T D_{\mathcal{V}_I}^T$ can be transformed into the following form by spectral decomposition

\[
\bar{H}_i = \begin{bmatrix} \bar{H}_{i1} & 0 \\ 0 & \bar{H}_{i2} \end{bmatrix},
\]

where $\lambda(\bar{H}_{i1}) \subset \mathbb{R}_{-c_i}, c_i > 0$ and $\lambda(\bar{H}_{i2}) \subset \mathbb{R}$. The matrix $\bar{H}_{i1}$ is a bijective mapping from $\mathcal{H}_i$ to $\mathcal{H}_i$, where $\mathcal{H}_i$ is some Euclidean space such that $\mathcal{H}_i \supseteq \mathbb{R}^{2n_I} \setminus \mathcal{O}$. Notice that $D_{\mathcal{V}_I}K_{I,f} + K_{I,f}^T D_{\mathcal{V}_I}^T$ is a principal submatrix of $\bar{H}_i$ and

\[
H_{r,i} := [I, 0]T_f^{-1}(D_{\mathcal{V}_I}K_{I,f} + K_{I,f}^T D_{\mathcal{V}_I}^T)T_f[I, 0]^T
\]

is a linear mapping from $\mathbb{R}^{2|\mathcal{V}_I|} \setminus \mathcal{O}$ to $\mathbb{R}^{2|\mathcal{V}_I|} \setminus \mathcal{O}_f$, where $\mathbb{R}^{2|\mathcal{V}_I|} \setminus \mathcal{O} \subset \mathcal{H}_i$. As a result, $H_{r,i}$ is a principal submatrix of $\bar{H}_{i1}$ with proper basis of $\mathcal{H}_i$. By the Cauchy interlace theorem, $\lambda(H_{r,i}) \subset \mathbb{R}_{-c}$, for all $i \in [I]$. The condition $\lambda(H_{r,i}) \subset \mathbb{R}_{-c}$ is sufficient for the feasibility of BMIs (21) by Schur compliment and the property of $\bar{L}_f$ shown in Eq. (23). \hfill \square

With Lemma 4, we can conclude stability of the “fault” microgrids by only solving BMIs (18), which reduces the computational efforts to find a proper $K$ for various microgrids operating conditions.

\subsection*{4.2 Loss of Communication links}

We consider the case of communication failures in this subsection. Let $G^* := (\mathcal{V}_I, \mathcal{E}_I)$ be a simple communication graph when communication links $\mathcal{E}_I \setminus \mathcal{E}_f$ are failed. Denote $L_f$ as the Laplacian of $G_f$. We can write down the
Consider a more general scenario when both part of the DERs and communication links are failed. Let $G_{fl} = (V_{fl}, E_{fl})$ and $L_{fl}$ be the graph and Laplacian under the operating condition, respectively. Assume the communication links 

$$\{(i,j), i \in V_{fl}, j \in V \setminus V_{fl}\} \subseteq \mathcal{E} \setminus \mathcal{E}_{fl}$$

(25)

remains intact so that the inverters can still update the Laplacian $L_{fl}$ autonomously. The desired properties in the scenario of loss of DERs and communication links tie closely to the feasibility of BMIs (21). We conclude this section by the following theorem

**Theorem 3** If the hypotheses in Corollary 1 hold, then there exists an non-empty $X_{c,s} \subset X_c$ such that for some given loads and initial $x(0) \in X_{c,s}$, the microgrid converges exponentially to the set where the control objectives including (3)-(5) are all satisfied if any one of the following operating conditions holds.

(a) Microgrid operates normally with the dynamics (8),
(b) The DERs $i \in V \setminus V_{fl}$ are disconnected with the microgrid, where $G_f$ remains connected. The microgrid dynamics is described by (20). Voltage dynamics of buses $V \setminus V_{fl}$ is bounded in the form of Assumption 1,
(c) The communication network is reduced to a connected simple graph $G_l$. The microgrid dynamics is described by Eq. (8) with $L$ replaced by $L_l$,
(d) Both parts of the DERs are unplugged and communication links are failed. The reduced communication graph $G_{fl}$ stays connected. The dynamics is described by Eq. (20) with $L_f$ replaced by $L_{fl}$. Voltage dynamics of buses $V \setminus V_{fl}$ is bounded in the form of Assumption 1.

**Proof.** Every microgrid condition listed above has its own reduced order system in the form of Eq. (14). To prove the theorem, it is sufficient to show that all the reduced order systems are robustly stable with the hypotheses of Corollary 1. Namely, the BMIs condition of robust stability of every reduced order system in the form of Eq. (18) is feasible. The operating conditions (a) and (b) have their BMI feasible due to Corollary 1 and Lemma 4. The operating condition (c) has the same dynamics to Eq. (8), only $L$ is replaced by $L_l$. Notice that the eigenspace of 0 for both $L$ and $L_l$ are the same. As a result, when $L$ is replaced by $L_l$ in BMIs (18), the new BMIs remain feasible with the gain $K$ optimizing BMI (18). The feasibility of BMIs for operating scenario (d) follows if BMIs (21) is feasible because the only difference between the two is the replacement of $L_f$ with $L_{fl}$, where a similar argument from scenario (a) to (c) applies. $\square$

**Remark 6** All the operating conditions listed in Theorem 3 require that the communication graph stays connected. Otherwise, the null space of the Laplacian matrix is bigger than the space of proportional power sharing. Despite we may still maintain the stability property of the “disconnected” microgrid, the proportional power sharing is not preserved.

## 5 Simulation Results

In this section, we validate the proposed controller by simulating an island microgrid under different operation scenarios. A typical parallel microgrid shown in Figure (2) has been implemented in MATLAB/Simulink. Since we consider the island case, the static transfer switch (STS) assumed to be opened. Each inverter is connected with one local load and the critical loads. We assume that the neighboring inverters can communicate with each other. The corresponding Laplacian matrix of the communication network is given below

$$L = \begin{bmatrix}
1 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -1 & 1
\end{bmatrix}$$

### Table 1 Microgrid Parameters

| DER 1 | DER 2 | DER 3 |
|---|---|---|
| Nom. powers | 10.55 kW | 6.85 kW | 12.64 kW |
| Q | 3.34 kVar | 3.01 kVar | 3.13 kVar |
| Phys. Capa. | 21 kW | 15 kW | 24 kW |
| Local loads | 2.08 kW | 2.23 kW | 6.57 kW |
| Nom. Freq. | 50 Hz | 50 Hz | 50 Hz |
| Nom. Volt. (rms) | 268.7 V | 268.7 V | 268.7 V |
| Line Imped. | 0.12 mΩ | 0.2 mΩ | 0.15 mΩ |
| Freq. bounds | 50.5 Hz | 50.5 Hz | 50.5 Hz |
| E bounds | 2.7 V/s | 2.7 V/s | 2.7 V/s |

Detailed parameter values of the microgrid are given in Table (1). We assume that the voltage magnitude deviation should be less than 2% of its nominal value for every bus. In addition, the branch angle difference is set...
Fig. 2. The simulated parallel microgrid configuration to be less than $\gamma = 15$ deg. We choose $U$ as an identity matrix and find LMIs reduced from BMIs (18) is feasible, where $v(D)$ is replaced by $v(J)$ to reduce the complexity as discussed in Remark 5. We use the feedback gain $K$ found in the LMIs despite a suboptimal solution of BMIs (18) may be found by implementing algorithms proposed in [12] and [16]. The calculated feedback gain $K$ is shown in Table (2). Notice that we have imposed a linear constraint on $K$ shown in the following

$$-b_m \leq K \dot{L} s \leq b_M, \quad \forall s \in v(\Xi), \quad (26)$$

where $\Xi$ represents active and reactive power capacity of inverter buses, $b_M$ and $b_m$ are the upper and lower bounds of ramp of phase angle and voltage defined in the last two rows in the Table 1. Despite a bigger $\|K\|_p$ implies faster convergence of system (14), the proper initial states region $\mathcal{X}_{c,s}$ in Theorem 1 becomes smaller. A systematic way to choose the proper bounds $b_M$ and $b_m$ for the best balance between the convergence rate and scale of $\mathcal{X}_{c,s}$ is considered as a part of future works.

Table 2

| Feedback Gains |
|----------------|
| DER 1 | DER 2 | DER 3 |
| $K_{i,(1,1)}$ | -7.2 mrad/s | -45.5 mrad/s | 0.3 mrad/s |
| $K_{i,(1,2)}$ | 20.8 mrad/s | -36.6 mrad/s | 0.4 mrad/s |
| $K_{i,(2,1)}$ | -2.65 V/s | 0 V/s | 0 V/s |
| $K_{i,(2,2)}$ | -0.043 V/s | -1.88 V/s | -0.21 V/s |

We first simulate the case when all the inverters operate normally with an abrupt change of the critical loads as shown in Fig. 3(a)- 3(e). The system initially operates without any control. At 0.5 second, the proposed controller is applied and at time $t = 5.5$ second, the load impedance changes abruptly from $Z_L = 4(\Omega)$ to $Z_L = 2.91+0.24i(\Omega)$. The corresponding power and voltage responses are shown in Fig. 3(a)- 3(e). It can be seen that the voltage magnitude always lies inside the desired range. In addition, the microgrid successfully reaches a satisfactory new steady state with proportional power sharing and synchronized frequencies.

Fig. 3. The critical load is abruptly changed at 5.5 second. The second case we consider is the network effects including the delays and package dropouts. It has been shown that the exponential stability of nonlinear time
The random communication is also considered. The last case that we consider is the failure of the inverter 1 with the network effects. From Theorem 3, $K$ solves BMIs (18) can maintain the stability and achieve the requirements (3)-(5) when inverter 1 is disconnected. As shown in Fig. 5(a)- 5(e), the inverter 2 and 3 can carry over the original power injection from the inverter 1 while the desired properties hold.

6 Conclusion

In this paper, we have presented a distributed control method to coordinate VSIIs in an island AC microgrid. Instead of using the conventional droop controller inducing steady state deviation of both frequency and voltage magnitude, our design controls the microgrid to the steady state where frequency synchronization and proportional active and reactive power sharing hold while respecting the voltage regulation constraints. The sufficient conditions of the convergence to the steady state can be approximated into solving BMIs problem. The design is robust with respect to the network delays and system parameter variations. Unlike most of the existing droop based controller, our design can be applied to the lossy mesh microgrid. Relaxation of the communication requirements and systematic ways to find optimal bounds of controller gain $K$ are considered as important future works. Extension of the controller for higher order harmonic loads, unbalanced, and other types of nonlinear loads are also among our future works.

A Assumption on $||\dot{x}_L||_2 \leq \kappa ||\dot{x}_I||_2$

We first consider the case of constant impedance loads. Given the impedance at every load bus $i$, the power injection at bus $i$ is determined by local voltage and is written as $P_i^*(x_i)$ and $Q_i^*(x_i)$, for all $i \in \mathcal{V}_L$. Applying KCL at every load bus with power flow equation (1), we have

\[
\begin{align*}
0 &= P_i^*(x_i) + P_i(x) \\
0 &= Q_i^*(x_i) + Q_i(x)
\end{align*} \quad \text{for all } i \in \mathcal{V}_L.
\]  

(A.1)

Taking time derivative of Eq. (A.1), we have

\[
0_{2n_L} = f_{I,x}\dot{x}_I + f_{L,x}\dot{x}_L,
\]

where $f_{I,x}$ and $f_{L,x}$ denote the Jacobian matrices follow from Eq. (A.1). For any $x$, $||\dot{x}_L||_2 \leq \kappa_{1x,2} ||\dot{x}_I||_2$ always follows from Eq. (A.2) if $f_{L,x}$ is invertible. In the bounded, the package loss can also be considered as a part of the delay. As shown in Fig. 4(a)- 4(e), we test the system response when the first scenario is changed by adding an uniformly distributed communication delay $\tau_{ij} \in [0, 0.5](s), \forall (i, j) \in \mathcal{E}_c$. Although the network effects induce a larger transients, the microgrid is still controlled to the same desired steady state.

Fig. 4. The critical load is abruptly changed at 5.5 second. The random communication is also considered.
f^T_{L,x}f_{L,x}, then
\[ \kappa_2,\hat{x}_{L,1} \leq \hat{x}_{L,1}^T f_{L,x}^T f_{L,x} \hat{x}_{L} = \hat{x}_{L,1}^T f_{L,x}^T f_{L,x} \hat{x}_{L} \]
\[ \Rightarrow \|\hat{x}_{L,1}\|_2 \leq \kappa_3,\|\hat{x}_{1}\|_2, \kappa_3, \in \mathbb{R}^+. \]

Since there exist a simple linear mapping from \( \hat{x}_{L,1} \) to \( \hat{x}_{L,2} \), \( \|\hat{x}_{L,2}\|_2 \leq \kappa_{4,\kappa} \|\hat{x}_{L,1}\|_2 \). We derive
\[ \|\hat{x}_{L}\|_2 = \|\hat{x}_{L,1}\|_2 = \|\hat{x}_{L,1}\|_2 + \|\hat{x}_{L,2}\|_2 \leq \kappa_{1,\kappa} \|\hat{x}_{1}\|_2, \kappa_{1,\kappa} \in \mathbb{R}^+. \]

for the case that \( f_{L,x} \) is not invertible. For \( x \in X_c \), we choose \( \kappa = \max_{x \in X_c} \kappa_{1,\kappa} \) and \( \|\hat{x}_1\|_2 \leq \kappa \|\hat{x}_1\|_2 \) follows. The derivation also applies to constant power loads, only \( P^c_i \) and \( Q^c_i \) at the load buses become constants instead of the function of local voltage profile. Consider dynamic loads with general dynamics \( \hat{x}_L = h_L(x) \). The assumption \( \|\hat{x}_L\|_2 \leq \kappa \|\hat{x}_1\|_2 \) holds only when the dynamics of \( x_1, \hat{x}_1 = h_1(x) \), is designed such that there exist a \( \kappa \in \mathbb{R}^+ \) such that \( \|h_L(x)\|_2 \leq \kappa \|h_1(x)\|_2 \) for all \( x \in X_c \).

Due to our focus is on constant impedance/power loads, the assumption has no restriction on designing the dynamics of \( x_1 \).

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