The Local-Triviality Dimension of Actions of Compact Quantum Groups

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Abstract. We define the local-triviality dimension for actions of compact quantum groups on unital C*-algebras, and say that the resulting compact quantum principal bundle is locally trivial when this dimension is finite. For commutative C*-algebras, the thus defined local triviality recovers the standard definition of local triviality of compact principal bundles. We prove that actions with finite local-triviality dimension are automatically free. Then we apply this new notion to prove the noncommutative Borsuk-Ulam-type conjecture under the assumption that a compact quantum group admits a non-trivial classical subgroup whose induced action has finite local-triviality dimension.

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1. Introduction

Noncommutative geometry (in the sense of Alain Connes [8]) was always inspired by physics, especially by gauge symmetries in field theories. The modern mathematical description of gauge theories uses the language of principal bundles and finding the right analog of a principal bundle became one of the important tasks of noncommutative geometry. The ultimate goal is to replace the space by a possibly noncommutative C*-algebra and the transformation group by a quantum group. Many approaches were introduced, which translated to the noncommutative world such notions as freeness and properness of a group action on a space [3, 11].

However, the standard definition of the principal bundle involves the notion of local triviality. From the beginning it was evident that the generalization of this
particular concept is far from straightforward as it is not clear how to define an open covering of a space in terms of the algebra of functions on that space.

In the case of compact Hausdorff spaces, piecewise triviality was introduced as a replacement of local triviality and this new definition was easily generalized to the noncommutative setting [4, 15]. Piecewise triviality uses closed covers and turned out to be more general than local triviality. Indeed, any locally trivial compact principal bundle is automatically piecewise trivial, but not the other way around. Nonetheless, the noncommutative generalization of piecewise triviality is only valid for non-simple C*-algebras.

In this paper, we introduce a definition of local triviality for actions of compact quantum groups [29] on unital C*-algebras. Our approach is based on the theory of the Rokhlin dimension [17] used in the classification of unital simple separable nuclear C*-algebras. We show that the local-triviality dimension and the Rokhlin dimension are equivalent for actions of compact Lie groups on unital commutative C*-algebras. However, even for free $\mathbb{Z}$ actions on compact metric spaces, finite Rokhlin dimension does not imply local triviality as shown in Example 4.8. We also prove that, in the noncommutative setting, local triviality does not imply piecewise triviality.

In Section 2, we recall some results from the classical theory of principal bundles and we reformulate the definition of local triviality of a compact principal bundle in a way that can be generalized to the noncommutative topology. Section 3 presents the main definition of the paper. We prove that in the classical case our definition recovers the standard notion of local triviality of a principal bundle and that locally trivial actions are automatically free. In Section 4, we examine the connection between local triviality with both the Rokhlin dimension and piecewise triviality. Section 5 concerns an application of the notion of the local-triviality dimension to prove a new Borsuk-Ulam-type theorem.

2. Locally trivial compact principal bundles

First, let us recall the definition of a locally trivial principal $G$-bundle, which is the main motivation of this work.

**Definition 2.1** (principal $G$-bundle). Let $\pi : X \to M$ be a fiber bundle of topological spaces and let $X$ be equipped with a right action of a topological group $G$. The quadruple $(X, M, \pi, G)$ is called a principal $G$-bundle if the following axioms hold:

1. For any $x \in X$ and $g \in G$, we have $\pi(xg) = \pi(x)$.
(2) For each \( m \in M \), there exists an open neighbourhood \( U \) of \( m \) in \( M \) and a fiber-preserving \( G \)-equivariant homeomorphism \( \varphi : \pi^{-1}(U) \to U \times G \), which is called a trivialization of \( \pi \) over \( U \) with a typical fibre \( G \).

Point (2) in Definition 2.1 describes the local triviality of a principal \( G \)-bundle. From (1) and (2) one can prove that the action of \( G \) on \( X \) is free. On the other hand, by a result of Gleason [14], free actions of compact Lie groups on regular topological spaces give rise to locally trivial principal \( G \)-bundles.

The above is a standard definition that can be found in textbooks, e.g. [25, 27], and should be contrasted with the definition of the Cartan principal \( G \)-bundle [4, 7], where one assumes that the action of \( G \) is free and proper and that \( M \cong X/G \) (in the original work of Cartan, it is phrased equivalently as the continuity of the translation map) instead of (2).

For our purposes, we need a slight reformulation of the notion of local triviality. This can be achieved by means of certain invariants that are always finite for locally trivial compact principal \( G \)-bundles.

**Definition 2.2** (Schwarz genus [24]). The Schwarz genus of a \( G \)-space \( X \), denoted by \( g_G(X) \), is the smallest number \( n \) such that \( X \) can be covered with open \( G \)-invariant subsets \( U_0, U_1, \ldots, U_n \) with the property that for every \( i = 0, 1, 2, \ldots, n \), there exists a \( G \)-equivariant map \( U_i \to G \). If no such \( n \) exists, we write \( g_G(X) = \infty \).

**Remark 2.3.** Now let \( X \) and \( Y \) be two \( G \)-spaces and suppose that there exists a \( G \)-map \( X \to Y \). Then, we have that \( g_G(X) \leq g_G(Y) \).

Note that if \( G \) is a compact Hausdorff group acting on a compact Hausdorff space \( X \), then \( g_G(X) < \infty \) if and only if \( \pi : X \to X/G \) is a locally trivial principal \( G \)-bundle. One can say even more, if \( g_G(X) = n \) this means that \( X/G \) can be covered with at most \( n + 1 \) trivializing open sets.

For the next definition and what follows, we introduce a concise notation for the multi-join of a topological group \( G \) with itself:

\[
E_0G := G, \quad E_nG := G \ast \ldots \ast G, \quad n > 0.
\]

Let us recall a fundamental result of Schwarz [24].

**Theorem 2.4.** Let \( G \) be a topological group. Then \( g_G(E_nG) \leq n \).

The following definition was introduced for purposes of the Borsuk-Ulam-type theorems (see for example [18]).
Definition 2.5 (G-index). Let $X$ be a $G$-space. We define the G-index of $X$ by

$$\text{ind}_G(X) := \min\{n : \exists \text{ G-map } X \to E_n G\}.$$ 

If there is no such G-map, we write $\text{ind}_G(X) = \infty$.

Remark 2.6. As for the Schwarz genus, we have that if there is an $G$-equivariant map $X \to Y$ between two $G$-spaces, then $\text{ind}_G(X) \leq \text{ind}_G(Y)$.

We conclude this section with

Theorem 2.7. Let $G$ be a compact Hausdorff group acting continuously on a compact Hausdorff $G$-space $X$. Then, $g_G(X) = \text{ind}_G(X)$.

Proof. First assume that $\text{ind}_G(X) = n$. We have a $G$-map $X \to E_n G$. The space $E_n G$ is a $G$-space with the diagonal action of $G$. Using Remark 2.3 and Theorem 2.4, we get that $g_G(X) \leq g_G(E_n G) \leq n = \text{ind}_G(X)$.

Now assume that $g_G(X) = n$. This means that $\pi : X \to X/G$ is a locally trivial principal $G$-bundle with at most $n + 1$ open sets $U_0, \ldots, U_n$ in the trivializing cover of $X/G$. Let $\{f_i\}_{i=0}^n$ and $\{\sigma_i\}_{i=0}^n$ denote a partition of unity functions and local sections subordinate to the trivializing cover $\{U_i\}_{i=0}^n$ respectively. Define a $G$-map

$$\psi : X \to E_n G : \quad x \mapsto \sum_{i=0}^n f_i(\pi(x)) \tau(\sigma_i(\pi(x)), x),$$

where $\tau : X \times X \to G$ is the translation map, i.e. $\tau(x, y) = g$ if and only if $y = xg$, and we used the simplicial notation for the multi-join. The above $G$-map is well-defined. Indeed, the condition $\sum_{i=0}^n f_i(m) = 1$, for every $m \in X/G$, assures that the image lands in $\Delta^n \times G^{n+1}$, where $\Delta^n$ is the $n$-simplex. Moreover, if $\pi(x)$ is not in a particular $U_i$, then $f_i(\pi(x)) = 0$. The map $\psi$ is continuous as each $f_i$ is continuous by definition and we know that $\tau$ is continuous as $\pi : X \to X/G$ is locally trivial. From the definition of the translation map, we see that $\tau(x, yg) = \tau(x, y)g$ and hence $\psi$ is $G$-equivariant with respect to the given action on $X$ and the diagonal action on $E_n G$. Existence of $\psi$ implies, by Remark 2.6 and Theorem 2.4, that $\text{ind}_G(X) \leq \text{ind}_G(E_n G) \leq n = g_G(X)$.

We emphasize that Theorem 2.7 shows that, for compact Hausdorff $X$ and $G$, $\text{ind}_G(X) < \infty$ if and only if $\pi : X \to X/G$ is locally trivial. Thus, we obtain a different characterization of (locally trivial) compact principal $G$-bundles.
3. Locally trivial compact quantum principal bundles

In this section, we introduce the notion of the local-triviality dimension for actions of compact quantum groups on unital C*-algebras and we prove that local triviality in that sense imply freeness.

Let us recall the definition of a compact quantum group and an action of a compact quantum group on a unital C*-algebra. All tensor products of C*-algebras in this paper are assumed to be minimal.

**Definition 3.1 (Compact quantum group [29]).** A compact quantum group is a unital C*-algebra \( C(G) \) with a unital injective *-morphism \( \Delta : C(G) \to C(G) \otimes C(G) \) that is coassociative, i.e. \( (\text{id} \otimes \Delta) \Delta = (\Delta \otimes \text{id}) \Delta \), and such that the two-sided cancellation property holds

\[
\{(a \otimes 1) \Delta(b) : a, b \in C(G)\}^{\text{cls}} = C(G) \otimes C(G) = \{(1 \otimes a) \Delta(b) : a, b \in C(G)\}^{\text{cls}},
\]

where cls denotes the closed linear span.

**Definition 3.2 (Compact quantum group action).** Let \( (C(G), \Delta) \) be a compact quantum group, let \( A \) be a unital C*-algebra and let \( \delta : A \to A \otimes C(G) \) be an injective unital *-homomorphism. We call \( \delta \) a coaction of \( C(G) \) on \( A \) (or an action of \( G \) on \( A \)) if

1. \( (\delta \otimes \text{id}) \circ \delta = (\text{id} \otimes \Delta) \circ \delta \) (coassociativity),
2. \( \{\delta(a)(1 \otimes h) : a \in A, h \in C(G)\}^{\text{cls}} = A \otimes C(G) \) (counitality).

Now we are ready to introduce the definition of the local-triviality dimension of actions of compact quantum groups.

**Definition 3.3 (Local triviality dimension).** Let \( (C(G), \Delta) \) be a compact quantum group, and let \( A \) be a unital C*-algebra equipped with a coaction \( \delta : A \to A \otimes C(G) \). For an integer \( d \geq 0 \), we say that \( \delta \) has the local-triviality dimension \( d \), written \( \dim_{\text{triv}}(\delta) = d \), if \( d \) is the minimal number such that there exist \( G \)-equivariant *-homomorphisms \( \rho_0, \ldots, \rho_d : C_c((0, 1]) \otimes C(G) \to A \) satisfying \( \sum_{j=0}^d \rho_j(t \otimes 1) = 1 \), where \( t \) is the inclusion from the half-open interval \( (0, 1] \) into \( \mathbb{C} \). We set \( \dim_{\text{triv}}(\delta) = \infty \) if no such \( d \) exists. We say that a compact quantum principal bundle \( (A, C(G), \delta) \) is locally trivial if \( \dim_{\text{triv}}(\delta) < \infty \).

First, we show that for unital commutative C*-algebras, i.e. algebras of complex-valued continuous functions on compact Hausdorff topological spaces, Definition 3.3 recovers the usual notion of local triviality.
Theorem 3.4. Let $G$ be a compact Hausdorff group, let $X$ be a compact Hausdorff space, and let $G$ act continuously on $X$. Denote by $\alpha : G \to \text{Aut}(C(X))$ the induced action. Then,

$$\text{ind}_G(X) = \dim_{\text{triv}}(\alpha).$$

Proof. Let $\dim_{\text{triv}}(\alpha) = n$. Take equivariant $*$-homomorphisms from Definition 3.3 and denote by

$$\rho_0, \ldots, \rho_n : (C_0((0, 1]) \otimes C(G))^+ \cong C(CG) \to C(X)$$

maps given by the universality of the unitalization, where $CG$ is the unreduced cone of $G$. Note that $\sum_{i=0}^n \rho_i(t \otimes 1) = 1$. We dualize the above to obtain $G$-equivariant continuous maps

$$\hat{\rho}_0, \ldots, \hat{\rho}_n : X \to CG.$$

Define a continuous $G$-map

$$\psi : X \to E_nG, \quad x \mapsto \sum_{i=0}^n \hat{\rho}_i(x),$$

where we used simplicial coordinates for $E_nG$. Every element $\hat{\rho}_i(x)$ is a class in $CG$. Let us fix an arbitrary point $x \in X$ and denote $\hat{\rho}_i(x) := [(s_i, g_i)]$, where $s_i \in [0, 1]$ and $g_i \in G$. We only need to check if $\sum_{i=0}^n s_i = 1$ for non-zero $s_i$'s:

$$1 = \sum_{i=0}^n (\rho_i(t \otimes 1))(x) = \sum_{i=0}^n (t \otimes 1)(\hat{\rho}_i(x)) = \sum_{i=0}^n (t \otimes 1)([(s_i, g_i)]) = \sum_{i=0}^n s_i.$$

Hence, $\text{ind}_G(X) \leq n = \dim_{\text{triv}}(\alpha)$.

Suppose that $\text{ind}_G(X) = n$. We have a $G$-equivariant map $\psi : X \to E_nG$. Define $G$-equivariant continuous maps

$$\hat{\rho}_i : X \to E_nG \xrightarrow{\psi} CG \xrightarrow{pr_i} CG,$$

where $i = 1, \ldots, n+1$. Again each element $\hat{\rho}_i(x)$ is a class in $CG$. Let us denote it as previously by $\hat{\rho}_i(x) := [(s_i, g_i)]$, where $x \in X$ is arbitrary. Notice that by definition $\sum_{i=1}^{n+1} s_i = 1$ and using the above calculation one shows that $\sum_{i=1}^{n+1} \rho_i(t \otimes 1) = 1$. This implies that $\dim_{\text{triv}}(\alpha) \leq n = \text{ind}_G(X)$. ■

In particular, $X \to X/G$ is trivial if and only if $\dim_{\text{triv}}(\alpha) = 0$. In the most general case, i.e. an action of a compact quantum group $G$ on a unital (possibly noncommutative) $C^*$-algebra $A$, vanishing of the local-triviality dimension is tantamount to the existence of a $G$-equivariant $*$-homomorphism from $C(G)$ to $A$. This
agrees with the approach adapted in [2], where a compact quantum principal bundle \((P, B, G)\) is called trivializable iff there exists a \(G\)-equivariant \(*\)-homomorphism \(C(G) \to P\).

Next, we show that local triviality implies freeness, in complete generality. We recall its definition first.

**Definition 3.5** (Freeness [11]). Let \(G\) be a compact quantum group, let \(A\) be a unital \(C^*\)-algebra, and let \(\alpha : A \to A \otimes C(G)\) be an action. We say that \(\alpha\) is free if

\[
\{(A \otimes 1_{C(G)})\alpha(A)\}_{\text{cls}} = A \otimes C(G).
\]

A basic example of a free action is the canonical translation action given by the coproduct \(\Delta : C(G) \to C(G) \otimes C(G)\).

**Theorem 3.6.** Let \(G\) be a compact quantum group, let \(A\) be a unital \(C^*\)-algebra, and let \(\alpha\) be an action of \(G\) on \(A\). If \(\dim_{\text{triv}}(\alpha) < \infty\), then \(\alpha\) is free.

**Proof.** Set \(B = \{(A \otimes 1_{C(G)})\alpha(A)\}_{\text{cls}}\). To show that \(B = A \otimes C(G)\), it is enough to show that \(B\) contains all simple tensors. In addition, since \(B\) is a left \((A \otimes 1_{C(G)})\)-module, it suffices to show that \(1_A \otimes x\) belongs to \(B\) for all \(x \in C(G)\).

Let \(x \in C(G)\) be fixed, and set \(d = \dim_{\text{triv}}(\alpha) < \infty\). Let \(\varepsilon > 0\). We write \(f \approx \varepsilon g\) to mean that \(\|f - g\| < \varepsilon\). Since the action \(\Delta : C(G) \to C(G) \otimes C(G)\) is free, there are \(m \in \mathbb{N}\) and \(y_1, \ldots, y_m, z_1, \ldots, z_m \in C(G)\) such that

\[
1_{C(G)} \otimes x \approx \varepsilon \sum_{k=1}^{m} (y_k \otimes 1_{C(G)})\Delta(z_k).
\]

Using the fact that \(\dim_{\text{triv}}(\alpha) = d\), find jointly unital equivariant \(*\)-morphisms \(\rho_0, \ldots, \rho_d : C_0((0, 1]] \otimes C(G) \to A\). Denote by \(b\) the function given by the inclusion of the half-open interval \((0, 1]\) into \(\mathbb{C}\). For \(j = 0, \ldots, d\), set \(\tilde{\rho}_j = \rho_j \otimes \text{Id}_{C(G)} : C_0((0, 1] \otimes C(G) \otimes C(G) \to A \otimes C(G)\), which is also an equivariant \(*\)-homomorphism. Then

\[
1_A \otimes x = \sum_{j=0}^{d} \rho_j(t \otimes 1_{C(G)}) \otimes x = \sum_{j=0}^{d} \tilde{\rho}_j(t \otimes 1_{C(G)} \otimes x)
\]

\[
\approx \varepsilon \sum_{j=0}^{d} \sum_{k=1}^{m} \tilde{\rho}_j(t \otimes (y_k \otimes 1_{C(G)})\Delta(z_k))
\]

\[
= \sum_{j=0}^{d} \sum_{k=1}^{m} \tilde{\rho}_j(\sqrt{t} \otimes (y_k \otimes 1_{C(G)})) \tilde{\rho}_j(\sqrt{t} \otimes \Delta(z_k))
\]

\[
= \sum_{j=0}^{d} \sum_{k=1}^{m} (\rho_j(\sqrt{t} \otimes y_k))\alpha(\rho_j(\sqrt{t} \otimes z_k)),
\]

where we used the equivariance of \(\rho_j\)’s.
This shows that $1_A \otimes x$ belongs to $B$, and hence $B = A \otimes C(G)$ and we conclude that $\alpha$ is free.

4. Relations with the Rokhlin dimension and piecewise triviality

In this section, we examine the connection between the local-triviality dimension and some other related notions, i.e., the Rokhlin dimension and piecewise triviality.

Let us start by recalling the definitions of a completely positive contractive order zero map and a sequence algebra, which are the basic ingredients of the definition of the Rokhlin dimension.

**Definition 4.1 (Order zero maps).** Let $A$ and $B$ be $C^*$-algebras. A completely positive contractive map $\varphi : A \to B$ is called order zero if $\varphi(a)\varphi(b) = 0$, whenever $ab = 0$, for any $a, b \in A$.

The theory of completey positive contractive order zero maps was developed by Winter and Zacharias [28] and has played a fundamental role in the recent breakthrough in the classification theory of $C^*$-algebras. The following result, based on the Stinespring theorem, will play a crucial role in exploring the connection of the Rokhlin dimension with local triviality.

**Theorem 4.2 (Winter-Zacharias [28]).** There is a bijection between completely positive contractive order zero maps $\varphi : A \to B$ and $\ast$-homomorphisms $\rho : C_0((0,1]) \otimes A \to B$.

The above is also true in the equivariant setting (see [12]).

**Definition 4.3 (Sequence algebra).** Let $A$ be a unital $C^*$-algebra, $\ell^\infty(\mathbb{N}, A)$ denote the $C^*$-algebra of all bounded sequences with elements in $A$ and $c_0(\mathbb{N}, A)$ denote the ideal consisting of sequences converging to zero in norm. The sequence algebra is defined as the quotient $A_\infty := \ell^\infty(\mathbb{N}, A)/c_0(\mathbb{N}, A)$.

We have an inclusion $A \subseteq A_\infty$, where elements of $A$ are considered as constant sequences. We write $A_\infty \cap A'$ (central sequence algebra) to denote the relative commutant of $A$ in $A_\infty$.

Definition 3.3 was inspired by the next definition.

**Definition 4.4 (Rokhlin dimension [12]).** Let $G$ be a compact Hausdorff group and let $\alpha : G \to \text{Aut}(A)$ be an action of $G$ on a unital $C^*$-algebra $A$. We say that $\alpha$ has the Rokhlin dimension $n$, written $\text{dim}_{\text{Rok}}(\alpha) = n$, if $n$ is the minimal positive integer such that there exist $G$-equivariant completely positive contractive order zero maps $\varphi_0, \ldots, \varphi_n : C(G) \to A_\infty \cap A'$. We set $\text{dim}_{\text{Rok}}(\alpha) = \infty$ if no such $n$ exists.
Using Theorem 4.2, one can compare Definitions 3.3 and 4.4. Note that the definition of the local-triviality dimension is much simpler as it does not involve the central sequence algebra. Moreover, there is a generalization of Definition 4.4 to actions of compact quantum groups, however, it is not straightforward and requires reformulations [13].

Next, using the notion of equivariant projectivity, we show that, for compact Lie group actions on unital commutative C*-algebras, the notions of local-triviality dimension and Rokhlin dimension coincide.

Let us first state the definition of equivariant projectivity [22, 23].

**Definition 4.5** (Equivariant projectivity). Let $G$ be a compact Hausdorff group and let $A$ be a $G$-C*-algebra. We say that $A$ is equivariantly projective if for any $G$-C*-algebra $B$, a $G$-invariant closed ideal $J \subseteq B$, and an equivariant *-homomorphism $\sigma : A \to B/J$, there is an equivariant *-homomorphism $\lambda : A \to B$ such that $\pi \circ \lambda = \sigma$, where $\pi : B \to B/J$ is the quotient map.

**Remark 4.6.** The above definition should be distinguished from the notion of equivariant projectivity in the category of projective modules that can be applied to C*-algebras [6].

Note that Definition 4.5 means equivariant projectivity in the category of general C*-algebras and one can restrict this definition to subcategories of unital C*-algebras, commutative C*-algebras, etc.

We recall that if we dualize Definition 4.5 for the category of commutative C*-algebras, we get that an object $C_0(X)$ is equivariantly projective if and only if $X$ is a $G$-AR ($G$-equivariant absolute retract, see [19, 20] for the definition).

**Theorem 4.7.** Let $A$ be a unital commutative C*-algebra equipped with an action $\alpha$ of a compact Lie group $G$. Then,

$$\dim_{\text{triv}}(\alpha) = \dim_{\text{Rok}}(\alpha).$$

**Proof.** First assume that $\dim_{\text{triv}}(\alpha) = d$. By Theorem 4.2, we have $G$-equivariant completely positive contractive order zero maps $\varphi_0, \ldots, \varphi_d : C(G) \to A$ such that $\sum_{i=0}^d \varphi_i(1) = 1$. Using the unital inclusion of $\iota : A \to A_\infty$ and the fact that $A$ is commutative, we obtain $G$-equivariant completely positive contractive order zero maps

$$\tilde{\varphi}_i := \iota \circ \varphi_i : C(G) \to A_\infty = A_\infty \cap A', \quad i = 0, 1, 2, \ldots, d.$$  

As $\iota$ is unital, we get that $\sum_{i=0}^d \tilde{\varphi}_i(1) = 1$. Therefore, $\dim_{\text{Rok}}(\alpha) \leq d = \dim_{\text{triv}}(\alpha)$. 

Now suppose that \( \dim \text{Rok}(\alpha) = d \) and we have equivariant completely positive contractive order zero maps \( \varphi_i : C(G) \to A_\infty \cap A' \) for \( i = 0, 1, \ldots, d \). By Theorem 4.2, we obtain equivariant \(*\)-homomorphisms
\[
\rho_i : C_0((0, 1]) \otimes C(G) \to A_\infty \cap A' = A_\infty.
\]
Since \((0, 1] \times G \) is \( G\)-AR \([1]\), \( C_0((0, 1]) \otimes C(G) \) is equivariantly projective in the category of commutative \( C^*\)-algebras. Therefore, for each \( \rho_i \), there is a \(*\)-homomorphism
\[
\lambda_i : C_0((0, 1]) \otimes C(G) \to \ell^\infty(N, A).
\]
Define \( pr_n : \ell^\infty(N, A) \to A \) as a projection on the \( n \)-th element of the sequence for some \( n \in \mathbb{N} \). Then, the maps
\[
\tilde{\rho}_{n,i} := pr_n \circ \lambda_i : C_0((0, 1]) \otimes C(G) \to A, \quad i = 0, 1, 2, \ldots, d,
\]
define equivariant \(*\)-homomorphisms.

From the fact that \( \sum_i \rho_i(t \otimes 1) = 1 \) and that \( \rho_i = \pi \circ \lambda_i \), where \( \pi : \ell^\infty(N, A) \to A_\infty \) is the quotient map, we get that
\[
\left\| pr_n \left( \sum_i \lambda_i(t \otimes 1) \right) - 1 \right\| \to 0 \quad \text{as} \quad n \to \infty.
\]

The above suggests that the Rokhlin dimension can be viewed as another noncommutative generalization of local triviality of compact principal \( G \)-bundles, where \( G \) is a compact Lie group. However, these notions differ in general, even in the classical case. The following example deals with a free action of \( \mathbb{Z} \) on \( S^1 \), but we hope that an example of that type can be also found for a compact group action.

**Example 4.8.** Let \( \theta \) be an irrational number and let \( T \) denote a rotation by angle \( 2\pi \theta \) on \( S^1 \). Powers of \( T \) define a free action of \( \mathbb{Z} \) on \( S^1 \). This action is not piecewise trivial and hence cannot be locally trivial. Indeed, take any closed subset \( V \) of the orbit space. By compactness of \( S^1 \) the preimage of \( V \) by the quotient map must be again compact, hence it cannot be homeomorphic with \( V \times \mathbb{Z} \). In contrast to the above, by a result of Szabó [26], any free action of \( \mathbb{Z} \) on a compact finite-dimensional (covering dimension) metric space \( X \) has finite Rokhlin dimension.

We conclude this section by exploring the connection of the local-triviality dimension with piecewise triviality.

**Definition 4.9** (Piecewise triviality [4]). A Cartan principal bundle \( (X, \pi, M, G) \) is called piecewise trivial, if there exist a covering of \( M \) by finitely many closed sets
Definition 4.9 was introduced in [4] along with an example (the bubble space) of a Cartan principal $G$-bundle that is piecewise trivial, but not locally trivial.

For compact Hausdorff spaces, local triviality implies piecewise triviality. Indeed, for the $V_i$’s in Definition 4.9, take the supports of functions in a partition of unity subordinate to the open trivializing cover given by local triviality.

Definition 4.9 admits a straightforward generalization to the realm of noncommutative geometry.

**Definition 4.10** (Noncommutative piecewise triviality [15]). Let $A$ be a unital $G$-$C^*$-algebra, where $G$ is a compact quantum group. An action of $G$ on $A$ is said to be piecewise trivial, if there exist $G$-invariant closed ideals $I_0, \ldots, I_n$ of $A$, such that $\bigcap_{i=0}^n I_i = 0$, and unital $G$-equivariant *-homomorphisms $\chi_i : C(G) \to A/I_i$.

Note that according to Definition 4.10, if a simple $G$-$C^*$-algebras is piecewise trivial it is in fact trivializable.

The next example shows that, in contrast with the classical case, local triviality in the sense of Definition 3.3 does not imply piecewise triviality.

**Example 4.11.** Consider the action of $\mathbb{Z}/2\mathbb{Z}$ on the algebra $\bigotimes_{n \in \mathbb{N}} M_3(\mathbb{C})$ given by $\bigotimes_{n \in \mathbb{N}} \text{Ad}(\text{diag}(1, 1, -1))$. One can verify that this action has $\dim_{\text{triv}}(\alpha) \leq 1$. It cannot be piecewise trivial, because simplicity of $M_3^\infty$ would force it to be trivial, and one can show that there is no equivariant unital homomorphism $C(\mathbb{Z}/2\mathbb{Z}) \to M_3^\infty$.

## 5. THE NONCOMMUTATIVE BORSUK-ULAM-TYPE CONJECTURE

In [2], Baum, Dąbrowski and Hajac postulated a certain noncommutative Borsuk-Ulam-type conjecture.

**Conjecture 5.1.** Let $A$ be a unital $C^*$-algebra with a free action $\alpha$ of a non-trivial compact quantum group $G$. There does not exist a $G$-equivariant *-homomorphism $A \to A \hat{\otimes} C(G)$.

Using the notion of the local-triviality dimension, we can obtain a new noncommutative Borsuk-Ulam-type result. First, we need the following lemma.

**Lemma 5.2.** Let $G$ be a compact Hausdorff group, let $A$ be a unital $C^*$-algebra, and let $\alpha$ be an action of $G$ on $A$. Let $I$ be an $G$-invariant ideal in $A$, and denote by $\overline{\alpha}$ the action induced by $\alpha$ on the quotient. Then

$$\dim_{\text{triv}}(\overline{\alpha}) \leq \dim_{\text{triv}}(\alpha).$$
The proof of the above lemma is analogous to the proof of point (2) of Theorem 3.8 in [12].

**Theorem 5.3.** Let \((C(G), \Delta)\) be a compact quantum group, and let \(A\) be unital \(C^*\)-algebra equipped with a coaction \(\delta : A \to A \otimes C(G)\). Then, if \(C(G)\) admits a non-trivial classical subgroup \(G\) whose induced action \(\alpha\) satisfies \(\dim_{triv}(\alpha) < \infty\), there is no \(G\)-equivariant \(*\)-homomorphism \(A \to A \delta \otimes C(G)\).

**Proof.** We follow the reasoning presented in [21]. Suppose that there exists a \(G\)-equivariant \(*\)-homomorphism \(A \to A \delta \otimes C(G)\). Then by [10] there exists a \(G\)-equivariant \(*\)-homomorphism \(A \to A \otimes C(G)\). Evaluation at \(e \in G\) in each point would produce a path of unital \(*\)-homomorphisms on \(A\) connecting a \(G\)-equivariant map to a one-dimensional representation. In what follows, we show that such a path cannot exists.

Denote the above-mentioned path of unital \(*\)-homomorphisms by \(\psi_t : A \to A\), where \(\psi_1\) is equivariant and \(\psi_0 : A \to C\). Existence of \(\psi_0\) implies that the \(G\)-invariant ideal \(I = \langle ab - ba : a, b \in A \rangle\) is proper and we can consider the abelianization \(A_c = A/I\). By Gelfand-Naimark, \(A_c = C(X)\) for some compact Hausdorff space \(X\). Since the action of \(G\) on \(A\) has finite local-triviality dimension, by Lemma 5.2 the induced action on \(A_c\) has finite local triviality dimension as well. In terms of spaces, this means that the principal \(G\)-bundle \(X \to X/G\) is locally trivial.

Now for every \(t \in [0, 1]\), \(\psi_t\) induce a map \(\tilde{\psi}_t : A_c \to A_c\). Such a path is dual to the homotopy of maps on the space \(X\) connecting a \(G\)-equivariant map with a constant map. This implies equivariant contractibility of \(X\), which is equivalent with the existence of a \(G\)-map \(X \ast G \to X\). However, for locally trivial actions, this map cannot exists by Edwards-Bestvina theorem [5].

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**References**

[1] S. Antonyan. Universal proper \(G\)-spaces. *Topology Appl.*, 117 (1), 23–43, 2002.
[2] P. F. Baum, L. Dąbrowski, P. M. Hajac. Noncommutative Borsuk-Ulam-type conjectures. *Banach Center Publications*, 106, 9–18, 2015.
[3] P. F. Baum, K. de Commer, P. M. Hajac. Free actions of compact quantum groups on unital $C^*$-algebras. Doc. Math., 22, 825–849, 2017.

[4] P. F. Baum, P. M. Hajac, R. Matthes, W. Szymański. Noncommutative geometry approach to principal and associated bundles. arXiv:math/0701033.

[5] M. Bestvina, R. Edwards. Unpublished, private communication.

[6] T. Brzeziński, P. M. Hajac. Galois-Type Extensions and Equivariant Projectivity. arXiv:0901.0141.

[7] H. Cartan, Séminaire Cartan. E.N.S. 1949/1950, reprinted by W.A. Benjamin, INC., New York, Amsterdam, 1967.

[8] A. Connes. Noncommutative Geometry. Academic Press, San Diego, CA, 1994.

[9] L. Dąbrowski, T. Hadfield, P. M. Hajac. Equivariant Join and Fusion of Noncommutative Algebras. SIGMA, 11, 082, 2015.

[10] L. Dąbrowski, P. M. Hajac, S. Neshveyev. Noncommutative Borsuk-Ulam-type conjectures revisited. arXiv:1611.04130.

[11] D. A. Ellwood. A new characterisation of principal actions. J. Funct. Anal., 173 (1), 49–60, 2000.

[12] E. Gardella. Rokhlin dimension for compact group actions. Indiana Univ. Math. J., 66 (2), 659–703, 2017.

[13] E. Gardella, M. Kalantar, M. Lupini. Rokhlin dimension for compact quantum group actions. arXiv:1709.00222.

[14] A. M. Gleason. Spaces with a compact Lie group of transformations. Proc. Amer. Math. Soc., 1, 35–43, 1950.

[15] P. M. Hajac, U. Krähmer, R. Matthes, B. Zieliński. Piecewise principal comodule algebras. J. Noncommut. Geom. 5 (4), 591–614, 2011.

[16] P. M. Hajac, T. Maszczyk. Pullbacks and nontriviality of associated noncommutative vector bundles. To appear in J. Noncommut. Geom.

[17] I. Hirshberg, W. Winter, J. Zacharias. Rokhlin dimension and C*-dynamics. Comm. Math. Phys., 335 (2), 637–670, 2015.

[18] J. Matoušek. Using Borsuk-Ulam theorem. Universitext, Lectures on topological methods in combinatorics and geometry, written in cooperation with Anders Björner and Günter M. Ziegler, Springer-Verlag, Berlin, 2003.

[19] M. Murayama. On the G-Homotopy Types of G-ANR’s. Publ. RIMS, Kyoto Univ., 18, 183–189, 1982.

[20] M. Murayama. On G-ANR’s and their G-homotopy types. Osaka J. Math., 20, 479–512, 1983.

[21] B. Passer. Free Actions on C*-algebra Suspensions and Joins by Finite Cyclic Groups. To appear in Indiana Univ. Math. J.

[22] N. C. Phillips. Equivariant semiprojectivity. arXiv:1112.4584.

[23] N. C. Phillips, A. P. W. Sørensen, H. Thiel. Semiprojectivity with and without a group action. J. Funct. Anal., 268 (4), 929–973, 2015.

[24] A. S. Schwarz. The genus of a fibre space. (Russian) Trudy Moskov Mat. Obšč, 10, 99–126, 1961.

[25] N. Steenrod. The topology of fibre bundles. Princeton Landmarks in Mathematics, Reprint of the 1957 edition, Princeton Paperbacks, Princeton University Press, Princeton, 1999.
[26] G. Szabó. The Rokhlin dimension of topological \( \mathbb{Z}^m \)-actions. *Proc. Lond. Math. Soc.*, 110 (3), 673–694, 2015.

[27] T. tom Dieck. *Algebraic topology*. EMS Textbooks in Mathematics, European Mathematical Society (EMS), Zürich, 2008.

[28] W. Winter, J. Zacharias. Completely positive maps of order zero. *Münster J. Math.*, 2, 311–324, 2009.

[29] S. L. Woronowicz. Compact matrix pseudogroups. *Comm. Math. Phys.*, 111, 613–665, 1987.

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