The Kepler problem in the Snyder space

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In this paper we study the Kepler problem in the non commutative Snyder scenario. We characterize the deformations in the Poisson bracket algebra under a mimic procedure from quantum standard formulations and taking into account a general recipe to build the noncommutative phase space coordinates (in the sense of Poisson brackets). We obtain an expression to the deformed potential, and then the consequences in the precession of the orbit of Mercury are calculated. This result allows us to find an estimated value for the non commutative deformation parameter introduced.

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I. INTRODUCTION

Non commutativity has became a serious fellow among the physics theories, since minimal fundamental lengths have been introduced by the leading theories of loop quantum gravity and string theory. This minimal fundamental length usually is identified as the Planck length and it is supposed that under that scale Physics is totally different, even from the standard Quantum Physics.

There are many ways to introduce non commutativity, usually the Heisenberg algebra is deformed through a matrix that encodes the lack of commutativity between the position operators. This way is incompatible with Lorentz symmetry and many difficulties arise due to the many changes that abandoning this fundamental symmetry implies. But there is a safer way. In fact, H. Snyder in the 40's \cite{1} proposed a modification of the Heisenberg algebra that implies discrete spectra of the spacetime operators. This modification is included among the $\kappa$-deformed spacetime modifications.

In fact, the noncommutative spacetime program was forgotten due the successfully renormalization program in the standard model. However there is a renewed interest due to the develop of loop quantum gravity and string theories with their discrete spacetimes.

One of the problems with the leading theories of quantum gravity today is the lack of experimental confirmations. In that direction, this paper shows a possible way to measure the implications of a non commutative spacetime, using the well known Kepler celestial mechanics; introducing a deformation parameter in the Kepler potential and forecasting deformations in the orbits of planets. There are some previous efforts dealing with this problem, but they used a non commutativity that is not compatible with Lorenz symmetry, and that is a very undesirable feature. \cite{2,3,4}.

The paper is organized as follows: in the next section, a short review of non commutative algebras is given, in the third section the Kepler problem in the Snyder spacetime is developed obtaining an advance of perihelion of a planet due to the deformed considerations and, finally conclusions are given in the last section.
II. NON COMMUTATIVE ALGEBRAS

A. General case

In a \((n + 1)\) dimensional Minkowski spacetime, we introduce the non commutativity through:

\[
[\bar{x}_\mu, \bar{x}_\nu] = lM_{\mu\nu},
\]

where \(\bar{x}\) is the non commutative coordinate and \(l\) a parameter measuring the non commutativity with dimension of squared length, usually identifying \(\sqrt{l}\) with \(l_p\), the Planck longitude and \(M_{\mu\nu}\) the rotations generator.

It is usual to demand that the Poincaré algebra is untouched, then we have the standard commutations relations

\[
[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\mu\nu}M_{\rho\sigma} - \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\sigma}M_{\mu\rho} + \eta_{\mu\sigma}M_{\nu\rho},
\]

\[
[p_\mu, p_\nu] = 0.
\]

We can obtain a general expression for the new coordinates taking

\[
\bar{x}_\mu = x_\mu \phi_1(A) + l(xp)p_\mu \phi_2(A),
\]

where \(\phi_1\) and \(\phi_2\) are two dependent functions of the quantity \(A = sp^2\), and the relation between them is

\[
\phi_2 = \frac{1 + 2\phi'_1 \phi_1}{\phi_1 - 2A \phi'_1},
\]

where \(\prime\) denotes derivative respect to \(A\).

We have freedom to take any value of \(\phi_1\) in order to obtain the realization of the non commutativity, the only restriction is the boundary condition \(\phi(0) = 1\), to retrieve the ordinary commutativity.

In the general case the commutator between coordinates and momenta is

\[
[\bar{x}_\mu, p_\nu] = i(\eta_{\mu\nu}\phi_1 + lp_\mu p_\nu \phi_2).
\]
B. Snyder case

Among the infinite possibilities choosing the value of $\phi_1$, there is a very special case: taking $\phi_1 = 1$. This choice implies that $\phi_2 = 1$, that leads to the so called Snyder space, characterized by

$$[x_\mu, x_\nu] = i l M_{\mu \nu},$$

(6)

$$[x_\mu, p_\nu] = i \delta_{\mu \nu} - il p_\mu p_\nu,$$

(7)

$$[p_\mu, p_\nu] = 0.$$  

(8)

This is a very interesting case and many works have investigated about it since the Snyder’s paper itself [1], [2], [7] and others.

III. THE KEPLER PROBLEM IN THE SNYDER NON COMMUTATIVE EUCLIDIAN SPACE

Classical euclidian $n$ dimensional Snyder Space is characterized by its non linear commutation relations (in the sense of Poisson brackets), between the variables of the phase space. They can be set following the inverse of Dirac quantization recipe

$$\{ x_i, x_j \} = l_p^2 L_{ij},$$

(9)

$$\{ x_i, p_j \} = \delta_{ij} - l_p^2 p_i p_j,$$

(10)

$$\{ p_i, p_j \} = 0,$$

(11)

where $l_p$ is the Planck longitude and measures the deformation introduced in the canonical Poisson brackets, and $L_{ij}$ is defined as a dimensionless matrix proportional to the angular momentum.

The Kepler potential $V = -\frac{\kappa}{\sqrt{x_i}}$ is implemented in the general non commutative case, taking

$$V(\bar{x}) = -\frac{\kappa}{\sqrt{\bar{x}_i \bar{x}_i}},$$

(12)
and considering the recipe from [3], we obtain at the first order in \( l \)

\[
V(x) = -\frac{\kappa}{\sqrt{x^2_1 \phi_1^2 + 2l^2(xp)^2 \phi_1 \phi_2}}.
\]

(13)

For the Snyder realization \((\phi_1 = \phi_2 = 1)\), we have that

\[
V(x) = -\frac{\kappa}{\sqrt{x^2 + 2l^2(xp)^2}},
\]

(14)

so, using polar coordinates for a plane motion,

\[
x = \rho \dot{\rho},
\]

(15)

\[
p = m(\dot{\rho} \dot{\rho} + \rho \dot{\theta} \dot{\theta}),
\]

(16)

the Lagrangian for a particle in the Snyder-Kepler potential can be written as

\[
\mathcal{L} = \frac{1}{2} m \left[ 1 - \frac{2l^2 km}{\rho} \right] \dot{\rho}^2 + \frac{1}{2} m \rho^2 \dot{\theta}^2 + \frac{k}{\rho}.
\]

(17)

We still have the angular momentum \( L = m\rho^2 \dot{\theta} \) as a constant of motion, so considering a particle with energy \( E \) we obtain for the radial equation

\[
\dot{\rho}^2 = \frac{2}{mf(\rho)} [E - V_c(\rho)],
\]

(18)

where \( f(\rho) = (1 + \frac{2l^2 m}{\rho}) \) and \( V_c(\rho) = \frac{k^2}{2m\rho^2} - \frac{k}{\rho} \), is the classical effective potential for the two-bodies problem. In this sense, our interest is to study the non-commutative correction to the confined orbit, so the constant of motion \( E \) is restricted to the values

\[
0 > E > E_c \equiv -\frac{\kappa}{2\rho_c},
\]

where \( \rho_c = (m\kappa)^{-1}L^2 \) is the radius of the circular orbit and \( E_c \) is the energy at this point. Now, we can write a dimensionless equation of motion in terms of these quantities as

\[
(-x')^2 = \left(2x - x^2 - \mathcal{E}\right) \left(1 + 2J^2(x)\right)^{-1},
\]

(19)
where \( J = (mk\ell_p)/L \), \( 1 > \mathcal{E} \equiv E/E_c > 0 \), \( x_- \geq x \equiv \rho_c/\rho \geq x_+ \) (with \( x_{\pm} \equiv \rho_c/\rho_{\pm} = 1 \mp \sqrt{1 - \mathcal{E}} \)), and \( x' = dx/d\theta \). Performing the substitution \( x = A - y \), with \( A = (4J^2 - 1)(6J^2)^{-1} \), eq. (19) becomes

\[
y'^2 = \frac{1}{8J^2} \frac{4y^3 - g_2 y - g_3}{(h - y)^2},
\]

where \( h = (1 + 2J^2)(3J^2)^{-1} \), and the invariants are given by

\[
g_2 = \frac{1 + 4J^2 + 4J^4(4 - 3\mathcal{E})}{3J^4}, \quad \text{and} \quad g_3 = \frac{(1 + 2J^2)[1 + 4J^2 - 4J^4(8 - 9\mathcal{E})]}{27J^6}.
\]

Therefore, choosing \( \theta = 0 \) at \( y = y_+ \) and integrating eq. (20), we find

\[
\frac{\theta}{\sqrt{8J^2}} = W(y_+) - W(y), \quad \text{(21)}
\]

where

\[
W(y) = (C - y)\wp(y; g_2, g_3) - \zeta(y; g_2, g_3), \quad \text{(22)}
\]

where \( \wp \) is the Weierstrass-\( p \) function, and \( \zeta \) is the Weierstrass-\( z \) function. Equation (21) represents the formal solution for the Kepler’s problem when the non-commutativity is taken into account. But we still can say something more about the deformation parameter, \( \ell_p \). To do this, we study the advance of perihelion starting from (19), expanding to order \( J^2 \), and neglecting \( x^3 \) terms. Thus, we obtain

\[
\left( -\frac{dx}{d\theta} \right)^2 \approx -\mathcal{E} + 2(1 + J^2\mathcal{E})x - (1 + 4J^2)x^2
\]

\[
= \frac{(1 + J^2\mathcal{E})^2}{(1 + 4J^2)} - \mathcal{E} - (1 + 4J^2) \left( x - \frac{(1 + J^2\mathcal{E})}{(1 + 4J^2)} \right)^2,
\]

so, it yields

\[
x \equiv \frac{\rho_c}{\rho} = C_1 + C_2 \cos(k\theta + \theta_0), \quad \text{(24)}
\]

where

\[
C_1 = \frac{1 + J^2\mathcal{E}}{1 + 4J^2}, \quad C_2 = k^{-1} \left( \frac{(1 + J^2\mathcal{E})^2}{1 + 4J^2} - \mathcal{E} \right)^{1/2}, \quad k = \sqrt{1 + 4J^2}.
\]

Therefore, the correction for the advance of perihelion is given by
\[
\Delta \theta = \frac{2\pi}{k} = 2\pi \left(1 + 4J^2\right)^{-1/2},
\]
which can be approximated as a deviation of the Newtonian orbit

\[
\Delta \theta \simeq 2\pi \left(1 - 2J^2\right) = 2\pi + \delta \theta_{nc},
\]

where \(\delta \theta_{nc} = -4\pi J^2\) is the non-commutative correction. In order to obtain the value of the deformation parameter, we can consider that the discrepancy of the observational data and the theoretical value in the specific case of Mercury (see TABLE I), could be due to the non-commutativity scenario. Obviously we choose Mercury because it is an usual natural laboratory to check deformations. This is because it is expected that any little effect can be observable in its orbit due to Mercury is the nearest planet to the sun. Therefore, we obtain \(l_p = 1.68 \times 10^{-32}\).

**TABLE I. Sources of the precession of perihelion for Mercury**

| Amount (arcsec/Julian century) | Cause                                           |
|-------------------------------|-------------------------------------------------|
| 5028.83 ± 0.04 [9]            | Coordinate (due to the precession of the equinoxes) |
| 530 [10]                      | Gravitational tugs of the other planets          |
| 0.0254                        | Oblateness of the Sun (quadrupole moment)       |
| 42.98 ± 0.04 [11]             | General Relativity                               |
| 5603.24                       | Total                                           |
| 5599.7                        | Observed                                        |
| −3.54                         | Discrepancy                                     |

**IV. FINAL REMARKS**

In this article we have described the effects of Snyder space non commutativity on Kepler problem and have studied its effect on a planetary orbit. We introduced non commutativity performing the deformations in Poisson bracket algebra under a mimic procedure from quantum standard formulations and then, using a general recipe to build the non-commutative phase space coordinates (in the sense of Poisson brackets). We found that the deformation in the central potential allows us to write a Lagrangian for a particle
in the Snyder-Kepler potential and to obtain the formal solution for the Kepler’s problem when the non-commutativity is taken into account. Our solution is given in terms of Weierstrass-p (\(\wp\)) and Weierstrass-z (\(\zeta\)) functions. Then, used our analytical results to compute the advance of perihelion of an planetary orbit. In fact, we found that is given by 
\[\Delta \theta = \frac{2\pi}{k} = 2\pi \left(1 + 4J^2\right)^{-1/2},\]
which can be approximated as a deviation of the Newtonian orbit as follow 
\[\Delta \theta \simeq 2\pi \left(1 - 2J^2\right) = 2\pi + \delta \theta_{nc}\]
where \(\delta \theta_{nc} = -4\pi J^2\) is the non-commutative correction.

Finally we applied this formula to fix the discrepancy between observational data and the theoretical value obtained from different classical sources and, under the hypothesis that the discrepancy is due to taking into account the non commutivity of the space, we obtained an estimated value for the non commutative deformation parameter given by \(l_p = 1.68 \times 10^{-32}\).
In the future we would like to see the value of deformation parameter in more general setting as the advance of perihelion in the neighbor of black hole, we left this issue for a future work.

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