GENERALIZATIONS OF THE CHOE-HOPPE HELICOID AND
CLIFFORD CONES IN EUCLIDEAN SPACE

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Abstract. By sweeping out $L$ independent Clifford cones in $\mathbb{R}^{2N+2}$
via the multi-screw motion, we construct minimal submanifolds in $\mathbb{R}^{L(2N+2)+1}$. Also, we sweep out the $L$-rays Clifford cone (introduced
in Section 2.3) in $\mathbb{R}^{L(2N+2)}$ to construct minimal submanifolds in $\mathbb{R}^{L(2N+2)+1}$. Our minimal submanifolds unify various interesting
examples: Choe-Hoppe’s helicoid of codimension one, cone over Lawson’s ruled minimal surfaces in $S^3$, Barbosa-Dajczer-Jorge’s
ruled submanifolds, and Harvey-Lawson’s volume-minimizing twisted normal cone over the Clifford torus $\frac{\mathbb{R}}{\sqrt{2}} S^N \times \frac{\mathbb{R}}{\sqrt{2}} S^N$.

Dedicated to Professor Jaigyoung Choe in honor of his 61st birthday

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1. Helicoids and minimal cones

In 1867, Riemann [21] discovered a family of complete, embedded, singly periodic minimal surfaces in Euclidean space $\mathbb{R}^3$ foliated by
circles and lines. He established that his staircases, planes, catenoids,
and helicoids are the only minimal surfaces fibered by circles or lines
in parallel planes.

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Catenoids can be generalized to higher dimensions and have been characterized in various ways. Higher dimensional catenoids in $\mathbb{R}^{K \geq 4}$ are the minimal hypersurfaces spanned by a family of coaxial $(K - 2)$-dimensional round spheres of varying radii. In 1991, Jagy [15] adopted Schoen’s argument [22] to show that if a minimal hypersurface $\Sigma^{K-1}$ in $\mathbb{R}^{K \geq 4}$ is foliated by $(K - 2)$-dimensional round spheres of varying radii in parallel $(K - 1)$-dimensional hyperplanes, then the submanifold $\Sigma^{K-1}$ should be rotationally symmetric. See also Shiffman’s Theorem [24, Theorem 1].

While there were several interesting results on higher dimensional catenoids, significant results on helicoids have been mostly in $\mathbb{R}^3$. For instance, Colding-Minicozzi’s deep description illustrates that embedded minimal disks in a ball in $\mathbb{R}^3$ are modeled on planes or helicoids [5–8]. Meeks and Rosenberg [19] proved that helicoids and planes are the only complete, properly embedded, simply connected minimal surfaces. Bernstein and Breiner [1] used the Colding-Minicozzi theory to show that a complete, properly embedded, minimal surface with finite genus and one end must be asymptotic to a helicoid.

Helicoids in $\mathbb{R}^3$ can be characterized in various ways. For instance, we can obtain helicoids by taking conjugate surfaces of catenoids. However, the notion of conjugation (for instance, see [9,16]) is not known for minimal hypersurfaces in $\mathbb{R}^{K \geq 4}$. Also, Catalan’s Theorem shows that a ruled minimal surface in $\mathbb{R}^3$ should be a helicoid $H(\lambda_0, \lambda_1) = \{ r e^{i(\lambda_1 \Theta)} \lambda_0 \Theta \in \mathbb{C} \times \mathbb{R} | \Theta, r \in \mathbb{R} \}$, $\lambda_0, \lambda_1 \in \mathbb{R}$: constants.

The helicoid $H(\lambda_0, \lambda_1)$ in $\mathbb{R}^3$ is invariant under the screw motion

$$\begin{bmatrix} x + iy \\ z \end{bmatrix} \in \mathbb{C} \times \mathbb{R} \mapsto \begin{bmatrix} e^{i\lambda_1 t} (x + iy) \\ z + \lambda_0 t \end{bmatrix} \in \mathbb{C} \times \mathbb{R}.$$ 

This geometric observation gives an insight on generalizing classical helicoids into higher dimensions as in [4].

Choe and Hoppe [4, Theorem 2] gave an explicit construction of a minimal hypersurface in $\mathbb{R}^{2N+1}$ foliated by Clifford hypercones in $\mathbb{R}^{2N}$. The Choe-Hoppe helicoid in $\mathbb{R}^{2N+1}$ is the hypersurface

$$z = f(x_1, y_1, \cdots, x_N, y_N) = \arg \left( \sqrt{(x_1 + iy_1)^2 + \cdots + (x_N + iy_N)^2} \right),$$

up to homotheties. Recently, Del Pino, Musso, and Pacard [20] produced new solutions of the Allen-Cahn equation whose zero set is the
Choe-Hoppe helicoid. See also Wei-Yang’s traveling wave solutions with vortex helix structure for Schrödinger map equation [28].

Minimal cones play an important role in solving higher dimensional Bernstein problems (for instance, see Fleming’s argument [10] and Simons’ Theorem [25 Theorem 6.2.2]) and understanding the nature of singularities of minimal varieties. Smale [26] used disjoint stable minimal hypercones in $\mathbb{R}^{k \geq 8}$ to construct many stable embedded minimal hypersurfaces with boundary, in $\mathbb{R}^{k \geq 8}$, with an arbitrary number of isolated singularities and prescribed rate of decay to their tangent cones at the singularities. Moreover, minimal cones in the unit ball become important examples of free boundary minimal varieties. See papers [11, 12] by Fraser and Schoen, and the survey [23, Example 2.10] by Schoen.

Our main goal is to generalize Choe-Hoppe’s minimal variety. By sweeping out Clifford cones or multi-rays Clifford cones (Definition 4), we explicitly construct generalized helicoids in odd dimensional Euclidean spaces (Theorem 2 and Theorem 3) and new minimal cones in even dimensional Euclidean spaces (Corollary 1, Remark 3, and Example 3). We also extend Takahashi’s classical criterion to higher codimension (Theorem 1). We shall show that our minimal submanifolds naturally unify various minimal submanifolds in Euclidean space and unit sphere. See four examples illustrated in Section 3.

2. Preliminaries

2.1. Multi-screw motions in Euclidean space. For a given angle $\theta$ and a complex vector

$$X + iY = \begin{bmatrix} x_1 + iy_1 \\ \vdots \\ x_{N+1} + iy_{N+1} \end{bmatrix} \in \mathbb{C}^{N+1} = \mathbb{R}^{N+1} + i\mathbb{R}^{N+1},$$

we adopt the notation

$$e^{i(\lambda, \theta)} (X + iY) = \begin{bmatrix} e^{i(\lambda, \theta)} (x_1 + iy_1) \\ \vdots \\ e^{i(\lambda, \theta)} (x_{N+1} + iy_{N+1}) \end{bmatrix}.$$

We also use the complex structure $J$ as the $\frac{\pi}{2}$-rotation. More explicitly,

$$J (X + iY) = i (X + iY) = \begin{bmatrix} -y_1 + ix_1 \\ \vdots \\ -y_{N+1} + ix_{N+1} \end{bmatrix}.$$

We then introduce the multi-screw motion in $\mathbb{R}^{L(2N+2)+1}$. 
Definition 1 (Multi-screw motion in Euclidean space $\mathbb{R}^{(2N+2)+1}$). Let $L \geq 1$ and $N \geq 0$ be integers. Given an $(L+1)$-tuple of real numbers 

$$\Lambda = (\lambda_0, \lambda_1, \cdots, \lambda_L),$$

we introduce the multi-screw motion $S_\Lambda$ in $\mathbb{R}^{(2N+2)+1} = C^{(N+1)L} \times \mathbb{R}$ with the pitch vector $\Lambda$. The mapping $S_\Lambda$ is defined by

$$\begin{pmatrix} X_1 + iY_1 \\ \vdots \\ X_L + iY_L \\ z \end{pmatrix} \mapsto \begin{pmatrix} e^{i\lambda_1 \theta} (X_1 + iY_1) \\ \vdots \\ e^{i(\lambda_L \theta)} (X_L + iY_L) \\ z + \lambda_0 \theta \end{pmatrix}.$$  

(2.1)

2.2. Identities on higher dimensional Clifford tori.

Definition 2 (Higher dimensional Clifford tori in sphere $\mathbb{S}^{2N+1}$). A $2N$-dimensional Clifford torus $C = \frac{1}{\sqrt{2}} S^N \times \frac{1}{\sqrt{2}} S^N$ denotes the minimal hypersurface in $\mathbb{S}^{2N+1} \subset \mathbb{R}^{2N+2}$ defined by

$$C = \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} X \\ Y \end{pmatrix} \in \mathbb{S}^{2N+1} \subset \mathbb{R}^{2N+1} \mid \|X\|_{\mathbb{R}^{N+1}} = 1, \|Y\|_{\mathbb{R}^{N+1}} = 1 \right\}.$$

Throughout this article, the symbol $\cdot$ means the standard dot product in Euclidean spaces.

Lemma 1 (Identities on Clifford tori $\frac{1}{\sqrt{2}} S^N \times \frac{1}{\sqrt{2}} S^N$). Let $C$ denote a local parameterization of the Clifford torus $\frac{1}{\sqrt{2}} S^N \times \frac{1}{\sqrt{2}} S^N$ in $\mathbb{S}^{2N+1} \subset \mathbb{R}^{2N+2}$

$$C (u_1, \ldots, u_{2N}) = \frac{1}{\sqrt{2}} \begin{pmatrix} X (u_1, \ldots, u_N) \\ Y (u_{N+1}, \ldots, u_{2N}) \end{pmatrix} \in \mathbb{R}^{2N+2},$$

where $X (u_1, \ldots, u_N)$ and $Y (u_{N+1}, \ldots, u_{2N})$ are an $\mathbb{R}^{N+1}$-valued local chart of two independent unit spheres of dimension $N$. Its unit normal vector reads

$$D (u_1, \ldots, u_{2N}) = \frac{1}{\sqrt{2}} \begin{pmatrix} X (u_1, \ldots, u_N) \\ -Y (u_{N+1}, \ldots, u_{2N}) \end{pmatrix}.$$ 

Let $\left( g^{ij} \right)_{1 \leq i, j \leq 2N}$ denote the inverse of matrix $\left( g_{ij} \right)_{1 \leq i, j \leq 2N}$ of the first fundamental form induced by the immersion $C$ in coordinates $u_1, \ldots, u_{2N}$. We also use the abbreviation $g = \det \left( g_{ij} \right)$ and introduce the $\mathbb{R}$-valued function

$$w_j = \frac{\partial C}{\partial u_j} \cdot JC, \quad j \in \{1, \cdots, 2N\}.$$

Then, we have the following identities:
(a) \[ JC = (D \cdot JC) D + \sum_{1 \leq i, j \leq 2N} g^{ij} w_i \frac{\partial C}{\partial u_i}, \]

and \[ -JD = (D \cdot JC) C + \sum_{1 \leq i, j \leq 2N} g^{ij} w_i \frac{\partial D}{\partial u_i}. \]

(b) \[ 1 - (D \cdot JC)^2 = \sum_{1 \leq i, j \leq 2N} g^{ij} w_i w_j. \]

(c) \[ \sum_{1 \leq i, j \leq 2N} \frac{\partial}{\partial u_i} \left( \sqrt{g} g^{ij} w_j \right) = 0. \]

(d) \[ \sum_{1 \leq i, j \leq 2N} g^{ij} w_j \frac{\partial}{\partial u_i} (D \cdot JC) = 0. \]

(e) \[ \sum_{1 \leq i, j \leq 2N} g^{ij} \frac{\partial}{\partial u_i} (D \cdot JC) \frac{\partial C}{\partial u_j} = -2 \left( JD + (D \cdot JC) C \right). \]

**Proof.** From the definition of the immersion \( C \), we observe that the symmetric matrix \( (g_{ij})_{1 \leq i, j \leq 2N} \) becomes a block matrix such that \( g_{ij} = g_{ji} = 0 \) holds whenever \( i \leq N \) and \( j \geq N + 1 \).

(a) Since the second identity is equivalent to the first one, we only check the first one. Fix the coordinates \( (u_1, \cdots, u_{2N}) \). Since vectors \( X, \frac{\partial X}{\partial u_1}, \cdots, \frac{\partial X}{\partial u_N} \) in \( \mathbb{R}^{N+1} \) are linearly independent, we have the linear combination

\[-Y = -(Y \cdot X) X + \sum_{i=1}^{N} \tau_i \frac{\partial X}{\partial u_i},\]

and similarly,

\[X = (X \cdot Y) Y + \sum_{i=N+1}^{2N} \tau_i \frac{\partial Y}{\partial u_i},\]

for some coefficients \( \tau_1, \cdots, \tau_{2N} \). Combining these two, we have the linear combination

\[ JC = \frac{1}{\sqrt{2}} \left[ -\frac{\sqrt{2}}{X} \right] = (D \cdot JC) D + \sum_{i=1}^{2N} \tau_i \frac{\partial C}{\partial u_i}. \]
Taking the dot product with the vector $\frac{\partial C}{\partial u_j}$ in both sides yields

$$w_j = \frac{\partial C}{\partial u_j} \cdot JC = \sum_{i=1}^{2N} g_{ij} \tau_i, \quad j \in \{1, \cdots, 2N\},$$

which implies that

$$\tau_i = \sum_{j=1}^{2N} g^{ij} w_j, \quad i \in \{1, \cdots, 2N\}.$$

(b) It follows from (a) and $C \cdot C = 1$ that

$$1 - (D \cdot JC)^2 = JC \cdot \left[ JC - (D \cdot JC) D \right] = JC \cdot \left[ \sum_{1 \leq i, j \leq 2N} g^{ij} w_i \frac{\partial C}{\partial u_j} \right] = \sum_{1 \leq i, j \leq 2N} g^{ij} w_i w_j.$$

(c) We begin with the decomposition

$$\sum_{1 \leq i, j \leq 2N} \frac{\partial}{\partial u_i} \left( \sqrt{g} \ g^{ij} w_j \right)$$

$$= \sum_{1 \leq i, j \leq 2N} \frac{\partial}{\partial u_i} \left[ \sqrt{g} \ g^{ij} \left( \frac{\partial C}{\partial u_j} \cdot JC \right) \right]$$

$$= \sum_{1 \leq i, j \leq 2N} \left( \sqrt{g} \ g^{ij} \frac{\partial C}{\partial u_j} \right) \cdot JC + \sum_{1 \leq i, j \leq 2N} \left( \sqrt{g} \ g^{ij} \frac{\partial C}{\partial u_j} \right) \cdot \frac{\partial}{\partial u_i} (JC).$$

Since the Clifford torus $\frac{1}{\sqrt{2}} S^N \times \frac{1}{\sqrt{2}} S^N$ in minimal $S^{2N+1} \subset \mathbb{R}^{2N+2}$, its mean curvature vector vanishes:

$$\Delta_{gc} C + 2NC \equiv 0,$$

which implies that the first sum vanishes:

$$\left[ \sum_{1 \leq i, j \leq 2N} \frac{\partial}{\partial u_i} \left( \sqrt{g} \ g^{ij} \frac{\partial C}{\partial u_j} \right) \right] \cdot JC = (-2NC) \cdot JC = 0.$$
By using the fact that the matrix $\left( g_{ij} \right)_{1 \leq i, j \leq 2N}$ is symmetric and by noticing that

$$\frac{\partial C}{\partial u_i} \cdot \frac{\partial}{\partial u_j} (JC) = J \frac{\partial C}{\partial u_j} \cdot J \frac{\partial}{\partial u_i} (JC) = -\frac{\partial}{\partial u_i} (JC) \cdot \frac{\partial C}{\partial u_j}$$

and

$$\frac{\partial C}{\partial u_i} \cdot \frac{\partial}{\partial u_i} (JC) = 0,$$

we see that the second sum also vanishes:

$$\sum_{1 \leq i < j \leq 2N} \sqrt{g} g^{ij} \left[ \frac{\partial C}{\partial u_j} \cdot \frac{\partial}{\partial u_i} (JC) + \frac{\partial C}{\partial u_i} \cdot \frac{\partial}{\partial u_j} (JC) \right] = 0.$$

(d) From $\frac{\partial D}{\partial u_i} \cdot JC = -\frac{\partial C}{\partial u_i} \cdot JD$ and (a), we have

$$\sum_{1 \leq i, j \leq 2N} g^{ij} w_j \frac{\partial}{\partial u_i} (D \cdot JC)$$

$$= \sum_{1 \leq i, j \leq 2N} g^{ij} w_j \left( \frac{\partial D}{\partial u_i} \cdot JC \right) + D \cdot J \left( \sum_{1 \leq i, j \leq 2N} g^{ij} w_j \frac{\partial C}{\partial u_i} \right)$$

$$= -\sum_{1 \leq i, j \leq 2N} g^{ij} w_j \left( \frac{\partial C}{\partial u_i} \cdot JD \right) - JD \cdot \left( \sum_{1 \leq i, j \leq 2N} g^{ij} w_j \frac{\partial C}{\partial u_i} \right)$$

$$= -2 \left( \sum_{1 \leq i, j \leq 2N} g^{ij} w_j \frac{\partial C}{\partial u_i} \right) \cdot JD$$

$$= -2 \left[ JC - (D \cdot JC)JD \right] \cdot JD$$

$$= -2 C \cdot JD$$

$$= 0.$$

(e) Recall that the symmetric matrix $\left( g_{ij} \right)_{1 \leq i, j \leq 2N}$ is a block matrix such that $g_{ij} = g_{ji} = 0$ for $i \leq N$ and $j \geq N + 1$, and note that

$$\frac{\partial}{\partial u_i} (D \cdot JC) = \begin{cases} 2w_i, & i \in \{1, \cdots, N\}, \\ -2w_i, & i \in \{N + 1, \cdots, 2N\}. \end{cases}$$
Using the second identity in (a), we have
\[ \sum_{1 \leq i, j \leq 2N} g^{ij} \frac{\partial}{\partial u_i} (\mathcal{D} \cdot J \mathcal{C}) \frac{\partial \mathcal{C}}{\partial u_j} \]
\[= 2 \sum_{1 \leq i, j \leq N} g^{ij} w_i \frac{\partial \mathcal{C}}{\partial u_j} - 2 \sum_{N+1 \leq i, j \leq 2N} g^{ij} w_i \frac{\partial \mathcal{C}}{\partial u_j} \]
\[= 2 \sum_{1 \leq i, j \leq 2N} g^{ij} w_i \frac{\partial \mathcal{D}}{\partial u_j} \]
\[= -2 (\mathcal{D} + (\mathcal{D} \cdot J \mathcal{C}) \mathcal{C}) \]

\[\square\]

2.3. Multi-rays cones over the submanifolds in a sphere.

**Definition 3** (L-rays cone in \(\mathbb{R}^{L(N+1)}\) over a submanifold in \(S^N \subset \mathbb{R}^{N+1}\)).
Given a submanifold \(\Sigma\) in the unit hypersphere \(S^N \subset \mathbb{R}^{N+1}\), we introduce the L-rays cone in \(\mathbb{R}^{L(N+1)}\) (possibly with a singularity at the origin)
\[C_L (\Sigma) = \left\{ \begin{array}{c} r_1 \mathcal{P} \\ \vdots \\ r_L \mathcal{P} \end{array} \right\} \in \mathbb{R}^{L(N+1)} \mid r_1, \cdots, r_L \in \mathbb{R}, \mathcal{P} \in \Sigma \}

**Theorem 1** (Takahashi type equivalence for multi-rays cones). Let \(\Sigma^n\) be a submanifold of the unit hypersphere \(S^N \subset \mathbb{R}^{N+1}\). Then the following three statements are equivalent:
(a) \(\Sigma^n\) is minimal in \(S^N\).
(b) The following submanifold \(S_L (\Sigma^n)\) is minimal in \(S^{L(N+1)-1}\).
\[S_L (\Sigma^n) = \left\{ \begin{array}{c} x_1 \mathcal{P} \\ \vdots \\ x_L \mathcal{P} \end{array} \right\} \in S^{L(N+1)-1} \subset \mathbb{R}^{L(N+1)} \mid \begin{array}{c} x_1 \\ \vdots \\ x_L \end{array} \in S^{L-1} \subset \mathbb{R}^L, \mathcal{P} \in \Sigma \}
(c) The L-rays cone \(C_L (\Sigma^n)\) is minimal in \(\mathbb{R}^{L(N+1)}\).

**Remark 1.** The case \(L = 1\) of Theorem \[\]was proved in [27] Theorem 3] and [25] Proposition 6.1.1].

**Proof.** Since \(C_L (\Sigma^n)\) is the cone over \(S_L (\Sigma^n)\), Takahashi’\'s Theorem [27] guarantees the equivalence (b) \(\Leftrightarrow\) (c). It now remains to prove the equivalence (a) \(\Leftrightarrow\) (c). Let \(\mathcal{F} (u_1, \cdots, u_n)\) be a local patch of \(\Sigma^n\) in \(S^N \subset \mathbb{R}^{N+1}\). We write
\[g = g_{\Sigma} = (g_{ij})_{1 \leq i, j \leq n} = \left( \frac{\partial \mathcal{F}}{\partial u_i} \cdot \frac{\partial \mathcal{F}}{\partial u_j} \right)_{1 \leq i, j \leq n}.\]
We observe that the induced metric of the torus. See Example 4 in Section 3 and [14, Example 3.22].

By observing that the induced metric of the $C_L$-rays cone $C_L(\Sigma^n)$ reads

$$G = dr_1^2 + \cdots + dr_L^2 + Rg, \quad R = r_1^2 + \cdots + r_L^2,$$

we are able to explicitly compute the mean curvature vector $H$ of $C_L(\Sigma^n) \subset \mathbb{R}^{L(N+1)}$ in terms of local coordinates $u_1, \ldots, u_n, r_1, \ldots, r_L$:

$$H = \Delta_C \Phi = \begin{bmatrix} \frac{r_1}{\sqrt{n}} (nF + \Delta gF) \\ \vdots \\ \frac{r_L}{\sqrt{n}} (nF + \Delta gF) \end{bmatrix}.$$ 

Therefore, we see that the mean curvature vector filed $H$ of the cone $C_L(\Sigma^n)$ vanishes if and only if $0_{\mathbb{R}^{N+1}} = nF + \Delta_g F$. \qed

The $L$-rays cone in $\mathbb{R}^{L(2N+2)}$ over the Clifford torus $\frac{1}{\sqrt{2}}S^N \times \frac{1}{\sqrt{2}}S^N$ will play an important role in Theorem 3.

**Definition 4 (L-rays Clifford cone in Euclidean space $\mathbb{R}^{L(2N+2)}$).** Let $L \geq 1$, $N \geq 0$ be integers. We introduce the $L$-rays Clifford cone in $\mathbb{R}^{L(2N+2)} = \mathbb{C}^{(N+1)L}$. Definition 3 gives an explicit expression

$$C_L \left( \frac{1}{\sqrt{2}}S^N \times \frac{1}{\sqrt{2}}S^N \right) = \left\{ \begin{bmatrix} r_1 (X + iY) \\ \vdots \\ r_L (X + iY) \end{bmatrix} : r_1, \ldots, r_L \in \mathbb{R}, \|X\|_{\mathbb{R}^{N+1}} = \|Y\|_{\mathbb{R}^{N+1}} = 1 \right\}.$$

**Corollary 1 (Minimality of multi-rays Clifford cones).** The $L$-rays Clifford cone $C_L \left( \frac{1}{\sqrt{2}}S^N \times \frac{1}{\sqrt{2}}S^N \right)$ is a minimal submanifold in $\mathbb{R}^{L(2N+2)}$.

**Remark 2.** We observe that $C_1 \left( \frac{1}{\sqrt{2}}S^N \times \frac{1}{\sqrt{2}}S^N \right)$ is the classical Clifford cone. After applying reflections in $\mathbb{R}^{4N+4}$, $C_2 \left( \frac{1}{\sqrt{2}}S^N \times \frac{1}{\sqrt{2}}S^N \right)$ is congruent to Harvey-Lawson’s twisted normal cone [14, Theorem 3.17] over the Clifford torus. See Example 4 in Section 3 and [14, Example 3.22].
3. Generalized helicoids in Euclidean space $\mathbb{R}^{L(2N+2)+1}$

For any given pair $(\lambda_0, \lambda_1)$ of real constants, the submanifold

$$\left\{ \left[ \begin{array}{c} r e^{i(\lambda_1 \Theta)} \\ \lambda_0 \Theta \end{array} \right] \in \mathbb{C} \times \mathbb{R} | \Theta, r \in \mathbb{R} \right\}$$

is minimal in $\mathbb{R}^3$. We present two generalizations and four examples.

**Theorem 2 (Sweeping out $L$ independent Clifford cones in $\mathbb{R}^{2N+2}$).** Let $L \geq 1, N \geq 0$ be integers. Given an $(L + 1)$-tuple $\Lambda = (\lambda_0, \lambda_1, \ldots, \lambda_L)$ of real numbers and a collection $\mathcal{C} = \{C_1, \ldots, C_L\}$ of $L$ independent $2N$-dimensional Clifford tori lying in the unit hypersphere $S^{2N+1} \subset \mathbb{R}^{2N+2}$, we define the generalized helicoid $\mathcal{H}^{\Lambda, \mathcal{C}} \subset \mathbb{R}^{L(2N+2)+1} = \mathbb{C}^{(N+1)L} \times \mathbb{R}$

$$\mathcal{H}^{\Lambda, \mathcal{C}} = \left\{ \left[ \begin{array}{c} r_1 e^{i(\lambda_1 \Theta)} (X_1 + iY_1) \\ \vdots \\ r_L e^{i(\lambda_L \Theta)} (X_L + iY_L) \\ \lambda_0 \Theta \end{array} \right] \in \mathbb{C} \times \mathbb{R}, r_t \in \mathbb{R}, \left[ \begin{array}{c} X_t \\ Y_t \end{array} \right] \in C_t, 1 \leq t \leq L \right\}.$$ 

Then, $\mathcal{H}^{\Lambda, \mathcal{C}}$ is a minimal submanifold in $\mathbb{R}^{L(2N+2)+1}$ and invariant under the multi-screw motion $S_\Lambda$ introduced in Definition 1.

**Remark 3.** When $\lambda_0 = 0$, $\mathcal{H}^{\Lambda, \mathcal{C}}$ becomes a minimal cone in $\mathbb{R}^{L(2N+2)}$. In the particular case when $(\lambda_0, \lambda_1, \ldots, \lambda_L) = (0, \ldots, 0)$, $\mathcal{H}^{\Lambda, \mathcal{C}}$ is the product of $L$ independent Clifford cones.

**Theorem 3 (Sweeping out the $L$-rays Clifford cone in $\mathbb{R}^{L(2N+2)}$).** Let $L \geq 1, N \geq 0$ be integers. Given two real constants $\lambda_0, \lambda$ and the $L$-rays Clifford cone $C_L \left( \frac{1}{\sqrt{2}} S^N \times \frac{1}{\sqrt{2}} S^N \right)$, we define the generalized helicoid $\mathcal{H}^{\lambda, \lambda_0, L, N}$ in Euclidean space $\mathbb{R}^{L(2N+2)+1} = \mathbb{C}^{(N+1)L} \times \mathbb{R}$

$$\mathcal{H}^{\lambda, \lambda_0, L, N} = \left\{ \left[ \begin{array}{c} e^{i(\lambda \Theta)Z} \\ \lambda_0 \Theta \end{array} \right] \in \mathbb{C}^{(N+1)L} \times \mathbb{R}, Z \in C_L \left( \frac{1}{\sqrt{2}} S^N \times \frac{1}{\sqrt{2}} S^N \right) \right\}.$$ 

More explicitly, we have

$$\mathcal{H}^{\lambda, \lambda_0, L, N} = \left\{ \left[ \begin{array}{c} r_1 e^{i(\lambda \Theta)} (X + iY) \\ \vdots \\ r_L e^{i(\lambda \Theta)} (X + iY) \\ \lambda_0 \Theta \end{array} \right] \in \mathbb{C} \times \mathbb{R}, r_t \in \mathbb{R}, \left[ \begin{array}{c} X_t \\ Y_t \end{array} \right] \in \frac{1}{\sqrt{2}} S^N \times \frac{1}{\sqrt{2}} S^N \right\}.$$ 

Then, the variety $\mathcal{H}^{\lambda, \lambda_0, L, N}$ is minimal in $\mathbb{R}^{L(2N+2)+1}$. It is invariant under the multi-screw motion $S_\Lambda = (\Lambda, \ldots, \Lambda, \lambda_0)$ introduced in Definition 1.

**Remark 4.** When $\lambda = \lambda_0 = 0$, $\mathcal{H}^{0,0,L,N}$ is a minimal cone in Corollary 1.
Example 1 (Choe-Hoppe’s minimal hypersurface [4, Theorem 2]).
Taking $L = 1$ in Theorem 2 or Theorem 3 recovers the Choe-Hoppe helicoid [4]. More explicitly, sweeping out the Clifford cone in $\mathbb{R}^{2N} \subset \mathbb{R}^{2N+1}$

$$
C^{2N-1} = \left\{ \begin{bmatrix} p_1 \\ q_1 \\ \vdots \\ p_N \\ q_N \end{bmatrix} \in \mathbb{R}^{2N} \mid p_1^2 + \cdots + p_N^2 = q_1^2 + \cdots + q_N^2 \right\}
$$

yields the minimal submanifold in $\mathbb{R}^{2N+1}$

$$
\mathcal{H}_\lambda = \left\{ \begin{bmatrix} x_1 \\ y_1 \\ \vdots \\ x_N \\ y_N \\ z \end{bmatrix} = \begin{bmatrix} p_1 \cos \Theta - q_1 \sin \Theta \\ q_1 \cos \Theta + p_1 \sin \Theta \\ \vdots \\ p_N \cos \Theta - q_N \sin \Theta \\ q_N \cos \Theta + p_N \sin \Theta \\ \lambda \Theta \end{bmatrix} \in \mathbb{R}^{2N+1} \mid \Theta \in \mathbb{R}, \lambda \in C^{2N-1} \right\},
$$

for any pitch constant $\lambda \in \mathbb{R}$.

Remark 5. Observing that $C^{2N-1}$ is a cone, one finds that $\mathcal{H}_\lambda$ is homothetic to $\mathcal{H}_1$ for any non-zero constant $\lambda \in \mathbb{R}$. Up to homotheties, the Choe-Hoppe helicoid in $\mathbb{R}^{2N+1}$ can be represented as the hypersurface

$$
z = \arg \left( \sqrt{(x_1 + iy_1)^2 + \cdots + (x_N + iy_N)^2} \right).
$$

We can also deduce its minimality by checking that the function

$$
f(x_1, y_1, \cdots, x_N, y_N) = \frac{1}{2} \arctan \left( \frac{2x_1y_1 + \cdots + 2x_Ny_N}{x_1^2 - y_1^2 + \cdots + x_N^2 - y_N^2} \right)
$$

satisfies the minimal hypersurface equation in $\mathbb{R}^{2N+1}$:

$$
0 = \sum_{k=1}^{N} \left[ \frac{\partial}{\partial x_k} \left( \frac{f_{x_k}}{W} \right) + \frac{\partial}{\partial y_k} \left( \frac{f_{y_k}}{W} \right) \right], \quad W = \sqrt{1 + \sum_{k=1}^{N} (f_{x_k}^2 + f_{y_k}^2)}.
$$

Remark 6. There are at least two geometric proofs of the minimality of the classical helicoids in $\mathbb{R}^3$, which exploits symmetries of helicoids. For instance, see [13, Section 2.2] and Karsten's lecture note [17, Section 2.2]. Interested readers may also try to give new proofs of the minimality of the Choe-Hoppe helicoid, which extend such geometric arguments.
Example 2 (Barbosa-Dajczer-Jorge’s helicoids [2]). The case \( N = 0 \) in Theorem 2 recovers Barbosa-Dajczer-Jorge’s ruled minimal submanifolds [2]. More explicitly, given an \((L + 1)\)-tuple \( \Lambda = (\lambda_0, \lambda_1, \cdots, \lambda_L) \), by sweeping out an \( L \)-dimensional plane, we have a minimal submanifold

\[
\mathcal{H}_\Lambda = \left\{ \begin{bmatrix}
    r_1 \cos (\lambda_1 \Theta) \\
    r_1 \sin (\lambda_1 \Theta) \\
    \vdots \\
    r_L \cos (\lambda_L \Theta) \\
    r_L \sin (\lambda_L \Theta) \\
    \lambda_0 \Theta
\end{bmatrix} \in \mathbb{R}^{2L+1} \mid \Theta, r_1, \cdots, r_L \in \mathbb{R} \right\}.
\]

Bryant [3] proved that it is an austere submanifold, which means that, for any normal vector, the set of eigenvalues of its induced shape operator is invariant under the multiplication by \(-1\). These submanifolds can be characterized by two uniqueness results. See [2, Theorem 3.10] and [3, Theorem 3.11]. Notice that, for \( \lambda_0 = 0 \), they become minimal cones in \( \mathbb{R}^{2L} \).

Example 3 (Minimal submanifolds in the unit sphere \( S^{L(2N+2)-1} \)). Taking \( \lambda_0 = 0 \) in Theorem 2 we obtain a minimal cone in \( \mathbb{R}^{L(2N+2)} \) given by

\[
\mathcal{H}^{\lambda_1, \cdots, \lambda_L, \mathcal{C}} = \left\{ \begin{bmatrix}
    r_t e^{i (\lambda_t \Theta)} (X_1 + i Y_1) \\
    \vdots \\
    r_L e^{i (\lambda_L \Theta)} (X_L + i Y_L)
\end{bmatrix} \mid \Theta \in \mathbb{R}, r_t \in \mathbb{R}, \begin{bmatrix} X_t \\ Y_t \end{bmatrix} \in C_t, 1 \leq t \leq L \right\},
\]

where \( \mathcal{C} = \{C_1, \cdots, C_L\} \) denotes a collection of \( L \) independent \( 2N \)-dimensional Clifford tori in \( S^{2N+1} \subset \mathbb{R}^{2N+2} \). The fact that \( \mathcal{H}^{\lambda_1, \cdots, \lambda_L, \mathcal{C}} \) is a minimal cone in \( \mathbb{R}^{L(2N+2)} \) guarantees that the intersection \( \Sigma = \mathcal{H}^{\lambda_1, \cdots, \lambda_L, \mathcal{C}} \cap S^{L(2N+2)-1} \) becomes a minimal submanifold in the unit sphere \( S^{L(2N+2)-1} \subset \mathbb{R}^{L(2N+2)} \). More explicitly, we have

\[
\Sigma = \left\{ \begin{bmatrix}
    p_t e^{i (\lambda_t \Theta)} (X_1 + i Y_1) \\
    \vdots \\
    p_L e^{i (\lambda_L \Theta)} (X_L + i Y_L)
\end{bmatrix} \mid \Theta \in \mathbb{R}, \sum_{t=1}^{L} p_t^2 = 1, \begin{bmatrix} X_t \\ Y_t \end{bmatrix} \in C_t, 1 \leq t \leq L \right\}.
\]

In the particular case when \((L, N) = (2, 0)\), we recover the family of Lawson’s ruled minimal surfaces [18, Section 7] in \( S^3 \subset \mathbb{R}^4 \):

\[
\Sigma = \left\{ \begin{bmatrix}
    \cos t \cos (\lambda_1 \Theta) \\
    \cos t \sin (\lambda_1 \Theta) \\
    \sin t \cos (\lambda_2 \Theta) \\
    \sin t \sin (\lambda_2 \Theta)
\end{bmatrix} \in \mathbb{R}^4 \mid t, \Theta \in \mathbb{R} \right\},
\]

for any pair \((\lambda_1, \lambda_2) \neq (0, 0)\) of real constants.
Example 4 (Harvey-Lawson’s volume-minimizing cone in $\mathbb{R}^{4N}$ [14]).

Corollary [7] with $L = 2$ or Theorem [3] with $\lambda_0 = \lambda = 0$ and $L = 2$ recover Harvey-Lawson’s twisted normal cone [14, Example 3.22] over the Clifford torus $\frac{1}{\sqrt{2}}S^N \times \frac{1}{\sqrt{2}}S^N \subset S^{2N+1}$. According to [14, Theorem 3.17], the austerity of the Clifford torus in $S^{2N+1}$ guarantees that the cone

$$\Sigma^{2N+2} = \left\{ \begin{array}{l} r_1X \\ r_1Y \\ r_2X \\ r_2Y \end{array} \right\} \in \mathbb{R}^{4N} \mid r_1, r_2 \in \mathbb{R}, \|X\|_{\mathbb{R}^{2N+1}} = 1, \|Y\|_{\mathbb{R}^{2N+1}} = 1 \right\}$$

is homologically volume minimizing.

4. Proof of main results

We present details of the proof of Theorem [2] which exploits five identities in Lemma [1]. Since the proof of Theorem [3] is similar, we shall omit it.

Our aim is to show that the generalized helicoid $\mathcal{H}^{\lambda, C}$ is minimal in $\mathbb{R}^{L(2N+2)+1}$. In the particular case when $(\lambda_0, \lambda_1, \cdots, \lambda_L) = (0, \cdots, 0)$, it becomes the product of $L$ independent Clifford cones. From now on, we assume that $(\lambda_0, \lambda_1, \cdots, \lambda_L) \neq (0, \cdots, 0)$.

For each index $s \in \{1, \cdots, L\}$, let $C^s(\theta) = \left( \begin{array}{c} u_1^s \\ \cdots \\ u_{2N}^s \end{array} \right)$ denote a local chart of the Clifford tori $\frac{1}{\sqrt{2}}S^N \times \frac{1}{\sqrt{2}}S^N$ in $S^{2N+1} \subset \mathbb{R}^{2N+2} = \mathbb{C}^{N+1}$. These induce a local patch $F$ of the generalized helicoid $\mathcal{H}^{\lambda, C} \subset \mathbb{R}^{L(2N+2)+1}$

$$F(u_1, \cdots, u_{2N}, \cdots, u_1^L, \cdots, u_{2N}^L, \Theta, r_1, \cdots, r_L) = \left[ \begin{array}{c} r_1 e^{i(\lambda_1 \Theta)} C_1(u_1^1, \cdots, u_{2N}^1) \\ \vdots \\ r_L e^{i(\lambda_L \Theta)} C_L(u_1^L, \cdots, u_{2N}^L) \\ \lambda_0 \Theta \end{array} \right].$$

We will show that the mean curvature vector $\Delta_{\mathcal{H}^{\lambda, C}} F$ vanishes. Here, $\Delta_{\mathcal{H}^{\lambda, C}}$ denote the Laplace-Beltrami operator on $\mathcal{H}^{\lambda, C}$ induced by the patch $F$ of the generalized helicoid $\mathcal{H}^{\lambda, C}$. More explicitly, we need to prove equalities

(a) $\Delta_{\mathcal{H}^{\lambda, C}} (\lambda_0 \Theta) \equiv 0.$

(b) $\Delta_{\mathcal{H}^{\lambda, C}} (r_t e^{i(\lambda_t \Theta)} C^t) \equiv 0, \ t \in \{1, \cdots, L\}.$
Step A. Let \((g^s_{ij})_{1 \leq i,j \leq 2N}\) denote the matrix of the first fundamental form induced by the patch \(C^s(u_1^s, \cdots, u_{2N}^s)\) of the Clifford torus \(\frac{1}{\sqrt{2}}S^N \times \frac{1}{\sqrt{2}}S^N\) lying in \(S^{2N+1} \subset \mathbb{R}^{2N+2}\). We adopt the notation
\[
g^s := \det (g^s_{ij})_{1 \leq i,j \leq 2N}.
\]
Then, the induced metric \(G_{\mathcal{H}^A}^s\) of \(\mathcal{H}^A\) in coordinates \(u_1^1, \cdots, u_{2N}^1, u_1^L, \cdots, u_{2N}^L, \Theta, r_1, \cdots, r_L\) reads
\[
G_{\mathcal{H}^A}^s = \sum_{s=1}^{L} \sum_{1 \leq i,j \leq 2N} r_s^2 g^s_{ij} du_i^s du_j^s + 2 \sum_{s=1}^{L} \sum_{i=1}^{2N} \lambda_s r_s^2 w_i^s d\Theta du_i^s + R d\Theta^2 + \sum_{s=1}^{L} dr_s^2,
\]
where we define
\[
(4.1) \quad R = \lambda_1^2 r_1^2 + \cdots + \lambda_L^2 r_L^2 + \lambda_0^2 > 0,
\]
and
\[
w_i^s = \frac{\partial C_i}{\partial u_i^s} \cdot JC_i^s, \quad s \in \{1, \cdots, L\}, \quad i \in \{1, \cdots, 2N\}.
\]
By using the cofactor expansion of determinant or the Laplace formula, we compute the determinant
\[
(4.2) \quad G := \det (G_{\mathcal{H}^A}^s) = P (r_1 \cdots r_L)^4 \prod_{s=1}^{L} g^s,
\]
where we have, by (b) of Lemma [1],
\[
(4.3) \quad P := R - \sum_{1 \leq s \leq L} \sum_{1 \leq i,j \leq 2N} \lambda_s^2 r_s^2 (g^s)_{ij} w_i^s w_j^s = \lambda_0^2 + \sum_{s=1}^{L} \lambda_s^2 r_s^2 (D^s \cdot JC^s)^2 > 0.
\]
From now on, we work on the points when \(G = \det (G_{\mathcal{H}^A}^s)\) does not vanish, or equivalently, when none of \(r_1, \cdots, r_L\) vanishes. Write
\[
(4.4) \quad d_i^s = \sum_{j=1}^{2N} \lambda_s (g^s)_{ij} w_j^s, \quad s \in \{1, \cdots, L\}, \quad i \in \{1, \cdots, 2N\}.
\]
Then, the components of \((G_{\mathcal{H}^A}^s)^{-1}\) in the local coordinates \(u_1^1, \cdots, u_{2N}^1, \cdots, u_1^L, \cdots, u_{2N}^L, \Theta, r_1, \cdots, r_L\) reads:
\[
G^w_{i'j'} = \frac{(g^s)_{ij}}{r_s^2} + \frac{d_i^s d_j^s}{P}, \quad G^w_{i\Theta} = G^{\Theta w^s} = \frac{-d_i^s}{P}, \quad G^{\Theta \Theta} = \frac{1}{P}, \quad G^{i'j'} = 1.
\]
The other components of \((G_{HΛ,C})^{-1}\) are all zero. Finally, we find the induced Laplace-Beltrami operator with respect to the metric \(G_{HΛ,C}\).

\[
\triangle G_{HΛ,C} = \frac{1}{\sqrt{G}} \sum_{s=1}^{L} \sum_{1 \leq i, j \leq 2N} \frac{\partial}{\partial u^s_i} \left( \sqrt{G} \frac{(g^e)^{ij}}{r^2} \frac{\partial}{\partial u^s_j} \right) 
+ \frac{1}{\sqrt{G}} \sum_{s=1}^{L} \sum_{1 \leq i, j \leq 2N} \frac{\partial}{\partial u^s_i} \left( \sqrt{G} \frac{d^i_{d^j}}{P} \frac{\partial}{\partial u^s_j} \right) 
+ \frac{1}{\sqrt{G}} \sum_{s=1}^{L} \sum_{i=1}^{2N} \frac{\partial}{\partial \Theta} \left( \sqrt{G} \frac{-d^i}{P} \frac{\partial}{\partial u^s_i} \right) 
+ \frac{1}{\sqrt{G}} \left( \sqrt{G} \frac{1}{P} \frac{\partial}{\partial \Theta} \right) 
+ \frac{1}{\sqrt{G}} \sum_{s=1}^{L} \frac{\partial}{\partial r^s} \left( \sqrt{G} \frac{\partial}{\partial r^s} \right).
\]

**Step B.** We next show that

\[
\triangle G_{HΛ,C} \Theta \equiv 0,
\]

which implies that the last coordinate in \(\mathbb{R}^{L(2N+2)+1}\) is harmonic on the generalized helicoid \(H^{Λ,C}\). According to the formula for \(\triangle G_{HΛ,C}\) deduced in **Step A**, it reduces to prove the equality

\[
\sum_{s=1}^{L} \sum_{i=1}^{2N} \frac{\partial}{\partial u^s_i} \left( \sqrt{G} \frac{d^i}{P} \right) = 0.
\]

We claim that, for each fixed \(s \in \{1, \cdots, L\},\)

\[
(4.5) \sum_{i=1}^{2N} \frac{\partial}{\partial u^s_i} \left( \sqrt{G} \frac{d^i}{P} \right) = 0.
\]

According to the equality

\[\sqrt{G} = \sqrt{P} (r_1 \cdots r_L)^{2N} \prod_{s=1}^{L} \sqrt{g^s},\]
and the definition (4.4)

\[ d^i = \sum_{j=1}^{2N} \lambda_s (g^s)^{ij} w^i_j, \]

it is sufficient to check the identity

\[ \sum_{1 \leq i, j \leq 2N} \frac{\partial}{\partial u^s_i} \left( \frac{1}{\sqrt{P}} \sqrt{g^s} (g^s)^{ij} w^i_j \right) = 0. \]  

or equivalently,

\[ \frac{1}{\sqrt{P}} \sum_{1 \leq i, j \leq 2N} \frac{\partial}{\partial u^s_i} \left( \sqrt{g^s} (g^s)^{ij} w^i_j \right) + \sum_{1 \leq i, j \leq 2N} \sqrt{g^s} (g^s)^{ij} w^i_j \frac{\partial}{\partial u^s_i} \left( \frac{1}{\sqrt{P}} \right) = 0. \]

The identity (c) of Lemma 1 guarantees that the first sum vanishes. To prove that the second sum vanishes, we are required to show

\[ \sum_{1 \leq i, j \leq 2N} (g^s)^{ij} w^i_j \frac{\partial P}{\partial u^s_i} = 0. \]

From the definition

\[ P = R - \sum_{1 \leq s \leq L} \sum_{1 \leq i, j \leq 2N} \lambda_s^2 r_s^2 (g^s)^{ij} w^i_j w^j_i, \]

and the identity (b) of Lemma 1, we have

\[ \frac{\partial P}{\partial u^s_i} = -\lambda^2 r^2 \frac{\partial}{\partial u^s_i} \left( \sum_{1 \leq i, j \leq 2N} (g^s)^{ij} w^i_j w^j_i \right) = -\lambda^2 r^2 \frac{\partial}{\partial u^s_i} \left( 1 - (D^s \cdot J C^s)^2 \right). \]

We thus need to prove

\[ \sum_{1 \leq i, j \leq 2N} (g^s)^{ij} w^i_j \frac{\partial}{\partial u^s_i} \left( 1 - (D^s \cdot J C^s)^2 \right) = 0. \]

So, it is enough to obtain

\[ \sum_{1 \leq i, j \leq 2N} (g^s)^{ij} w^i_j \frac{\partial}{\partial u^s_i} (D^s \cdot J C^s) = 0. \]

However, because of the identity (d) of Lemma 1, this sum vanishes.

**Step C.** It now remains to prove that, for each index \( t \in \{1, \cdots, L\}, \)

\[ \Delta_G y^L \left( r_t e^{(\Theta_C)C^t} \right) \equiv 0. \]
According to the formula for $\Delta_{G_{\eta,\lambda,c}}$ deduced in Step A, it reduces to prove the equality

$$0 = \sum_{s=1}^{L} \sum_{1 \leq i,j \leq 2N} \frac{\partial}{\partial u^s_i} \left( \sqrt{G} \frac{(g^s)^{ij}}{r_s^2} \frac{\partial}{\partial u^s_j} \left( r_i e^{i(\lambda_i(\Theta)) C} \right) \right)$$

$$+ \sum_{s=1}^{L} \sum_{1 \leq i,j \leq 2N} \frac{\partial}{\partial u^s_i} \left( \sqrt{G} \frac{d_i^s d_j^s}{\rho} \frac{\partial}{\partial u^s_j} \left( r_i e^{i(\lambda_i(\Theta)) C} \right) \right)$$

$$+ \sum_{s=1}^{L} \sum_{i=1}^{2N} \frac{\partial}{\partial r^s_i} \left( \sqrt{G} \frac{-d_s^s}{\rho} \frac{\partial}{\partial \Theta} \left( r_i e^{i(\lambda_i(\Theta)) C} \right) \right)$$

$$+ \sum_{s=1}^{L} \sum_{i=1}^{2N} \frac{\partial}{\partial u^s_i} \left( \sqrt{G} \frac{-d_s^s}{\rho} \frac{\partial}{\partial u^s_i} \left( r_i e^{i(\lambda_i(\Theta)) C} \right) \right)$$

$$+ \left( \sqrt{G} \frac{1}{\rho} \frac{\partial}{\partial \Theta} \left( r_i e^{i(\lambda_i(\Theta)) C} \right) \right)$$

$$+ \sum_{s=1}^{L} \frac{\partial}{\partial r^s} \left( \sqrt{G} \frac{\partial}{\partial r^s} \left( r_i e^{i(\lambda_i(\Theta)) C} \right) \right).$$

We express this equality as the sum

$$0 = S_1 + S_2 + S_3 + S_4 + S_5 + S_6.$$

**Step C1.** We recall that $G = \rho \left( r_1 \cdots r_L \right)^{4N} \prod_{s=1}^{L} g^s$. We introduce

$$Q_s = \sqrt{\prod_{s \in \{1, \cdots, L\}, \{a\}} \left( r_s^{4N} g^s \right)} , \quad s \in \{1, \cdots, L\}$$

to get the factorization

$$\sqrt{G} = \sqrt{\rho} r_s^{2N} \sqrt{g^s} Q_s, \quad s \in \{1, \cdots, L\}.$$
We evaluate the sum $S_1$.

$$
S_1 = \sum_{s=1}^{L} \sum_{1 \leq i, j \leq 2N} \frac{\partial}{\partial u^i_s} \left( \sqrt{G} \left( g^e \right)_{ij} \frac{\partial}{\partial u^j_s} \left( r_t e^{\left(\lambda, \Theta\right)} C^l \right) \right)
= \sum_{1 \leq i, j \leq 2N} \frac{\partial}{\partial u^i_t} \left( \sqrt{G} \left( g^t \right)_{ij} \frac{\partial}{\partial u^j_t} \left( r_t e^{\left(\lambda, \Theta\right)} C^l \right) \right)
= \frac{1}{r_t} e^{\left(\lambda, \Theta\right)} \sum_{1 \leq i, j \leq 2N} \frac{\partial}{\partial u^i_t} \left( \sqrt{P} \sqrt{g^t} \left( g^t \right)_{ij} \frac{\partial}{\partial u^j_t} \left( r_t e^{\left(\lambda, \Theta\right)} C^l \right) \right)
= r_t^{2N-1} Q_t e^{\left(\lambda, \Theta\right)} \sum_{1 \leq i, j \leq 2N} \frac{\partial}{\partial u^i_t} \left( \sqrt{g^t} \left( g^t \right)_{ij} \frac{\partial}{\partial u^j_t} \left( r_t e^{\left(\lambda, \Theta\right)} C^l \right) \right)
+ r_t^{2N-1} Q_t e^{\left(\lambda, \Theta\right)} \sum_{1 \leq i, j \leq 2N} \sqrt{g^t} \left( g^t \right)_{ij} \frac{\partial}{\partial u^i_t} \left( \sqrt{g^t} \frac{\partial}{\partial u^j_t} C^l \right).
$$

We make two observations. First, as in the proof of the identity in (c) of Lemma [1] by using the minimality of the Clifford torus $\frac{1}{\sqrt{2}} S^N \times \frac{1}{\sqrt{2}} S^N$ in the unit hypersphere $S^{2N+1} \subset \mathbb{R}^{2N+2}$, we can simplify the sum in the first term:

$$
\sum_{1 \leq i, j \leq 2N} \frac{\partial}{\partial u^i_t} \left( \sqrt{g^t} \left( g^t \right)_{ij} \frac{\partial}{\partial u^j_t} C^l \right) = \sqrt{g^t} \Delta_{g^t} C^l = -2N \sqrt{g^t} C^l,
$$

Second, from the definition (4,3) and (b) of Lemma [1] we have

$$
\frac{\partial \sqrt{P}}{\partial u^i_t} = \frac{1}{2 \sqrt{P}} \frac{\partial}{\partial u^i_t} \left( \sqrt{R - \sum_{1 \leq s \leq L} \sum_{1 \leq i, j \leq 2N} \lambda^2 r^2 \left( g^e \right)_{ij} w^i_s w^j_s} \right)
= -\frac{\lambda^2 r^2}{2 \sqrt{P}} \frac{\partial}{\partial u^i_t} \left( 1 - \left( D^i \cdot JC^l \right)^2 \right)
= \frac{\lambda^2 r^2}{\sqrt{P}} \left( D^i \cdot JC^l \right),
\frac{\partial}{\partial u^i_t} C^l = \left( D^i \cdot JC^l \right),
$$
and then, by (e) of Lemma 1

\[ \sum_{1 \leq i, j \leq 2N} \sqrt{g^i}(g^i)^{ij} \frac{\partial}{\partial u_i^t} \frac{\partial}{\partial u_j^t} = \lambda_t^2 r_t^2 \frac{\sqrt{g^i}}{\sqrt{P}} \left( D^i \cdot JC^i \right) \sum_{1 \leq i, j \leq 2N} (g^i)^{ij} \frac{\partial}{\partial u_i^t} \left( D^j \cdot JC^j \right) \frac{\partial}{\partial u_j^t}. \]

It follows that

\[ S_1 = -2N r_t^{2N-1} Q_t \sqrt{g^t} \sqrt{g^{i(\lambda_t)C^t}} - 2\lambda_t^2 r_t^{2N+1} Q_t \frac{\sqrt{g^i}}{\sqrt{P}} \left( D^i \cdot JC^i \right) e^{i(\lambda_t)\Theta} \left[ JD^i + (D^i \cdot JC^i) C^i \right]. \]

**Step C2.** We will use the factorization (4.7) obtained in **Step C1:**

\[ \sqrt{G} = \sqrt{P} r_t^{2N} \sqrt{g^t} Q_t. \]

We expand the sum \( S_2: \)

\[ S_2 = \sum_{s=1}^L \sum_{1 \leq i, j \leq 2N} \frac{\partial}{\partial u_i^s} \left( \sqrt{G} \frac{d_i^s d_j^s}{P} \frac{\partial}{\partial u_j^s} \left( r_t e^{i(\lambda_t)C^t} \right) \right) \]

\[ = \sum_{1 \leq i, j \leq 2N} \frac{\partial}{\partial u_i^t} \left( \sqrt{G} \frac{d_i^t d_j^t}{P} \frac{\partial}{\partial u_j^t} \left( r_t e^{i(\lambda_t)C^t} \right) \right) \]

\[ = r_t^{2N+1} Q_t \left\{ \sum_{i=1}^{2N} \frac{\partial}{\partial u_i^t} \left[ \sqrt{g^i} \frac{d_i^t}{\sqrt{P}} \sum_{j=1}^{2N} d_j^t \frac{\partial C^t}{\partial u_j^t} \right] \right\}. \]

From the first identity in (a) of Lemma 1 and the definition (4.4)

\[ d_j^t = \sum_{k=1}^{2N} \lambda_i (g^i)^{jk} w_k^t, \]

we compute the inner sum:

\[ \sum_{j=1}^{2N} d_j^t \frac{\partial C^t}{\partial u_j^t} = \lambda_t \sum_{1 \leq i, j \leq 2N} (g^i)^{jk} w_k^t \frac{\partial C^t}{\partial u_j^t} = \lambda_t \left[ JC^t - (D^i \cdot JC^i) D^i \right]. \]
It follows that

\[
S_2 = \lambda_t r_t^{2N+1} Q_t \sum_{i=1}^{2N} \frac{\partial}{\partial u_i} \left[ \frac{\sqrt{g_i}}{\sqrt{P}} d_i^t \left\{ J C^t - (D^t \cdot J C^t) D^t \right\} \right]
\]

\[
= \lambda_t r_t^{2N+1} Q_t \sum_{i=1}^{2N} \frac{\partial}{\partial u_i} \left( \frac{\sqrt{g_i}}{\sqrt{P}} d_i^t \right) \left\{ J C^t - (D^t \cdot J C^t) D^t \right\}
\]

\[
+ \lambda_t r_t^{2N+1} Q_t e^{i(\lambda, \Theta)} \sum_{i=1}^{2N} \frac{\sqrt{g_i}}{\sqrt{P}} d_i^t \frac{\partial}{\partial u_i} \left[ J C^t - (D^t \cdot J C^t) D^t \right].
\]

According to the identity (4.6) deduced in Step B, we notice that the sum in the first term vanishes:

\[
\sum_{i=1}^{2N} \frac{\partial}{\partial u_i} \left( \frac{\sqrt{g_i}}{\sqrt{P}} d_i^t \right) = \lambda_t \sum_{1 \leq i, j \leq 2N} \frac{\partial}{\partial u_i} \left( \frac{\sqrt{g_{ij}}}{\sqrt{P}} (g^t)^{ij} w_i^t \right) = 0.
\]

We thus obtain

\[
S_2 = \lambda_t r_t^{2N+1} Q_t e^{i(\lambda, \Theta)} \sum_{i=1}^{2N} \frac{\sqrt{g_i}}{\sqrt{P}} d_i^t \frac{\partial}{\partial u_i} \left[ J C^t - (D^t \cdot J C^t) D^t \right]
\]

\[
= \lambda_t r_t^{2N+1} Q_t \sqrt{g_i} e^{i(\lambda, \Theta)} J \left( \sum_{i=1}^{2N} d_i^t \frac{\partial C^t}{\partial u_i} \right)
\]

\[
- \lambda_t r_t^{2N+1} Q_t (D^t \cdot J C^t) \sqrt{g_i} e^{i(\lambda, \Theta)} \sum_{i=1}^{2N} d_i^t \frac{\partial D^t}{\partial u_i}
\]

\[
- \lambda_t r_t^{2N+1} Q_t \sqrt{g_i} e^{i(\lambda, \Theta)} \left[ \sum_{1 \leq i, j \leq 2N} (g^t)^{ij} w_i^t \frac{\partial C^t}{\partial u_i} \right] D^t
\]

\[
= \lambda_t r_t^{2N+1} Q_t \sqrt{g_i} e^{i(\lambda, \Theta)} J \left( \sum_{1 \leq i, j \leq 2N} (g^t)^{ij} w_i^t \frac{\partial C^t}{\partial u_i} \right)
\]

\[
- \lambda_t r_t^{2N+1} Q_t (D^t \cdot J C^t) \sqrt{g_i} e^{i(\lambda, \Theta)} \sum_{1 \leq i, j \leq 2N} (g^t)^{ij} w_i^t \frac{\partial D^t}{\partial u_i}
\]

\[
- \lambda_t r_t^{2N+1} Q_t \sqrt{g_i} e^{i(\lambda, \Theta)} \left[ \sum_{1 \leq i, j \leq 2N} (g^t)^{ij} w_i^t \frac{\partial (D^t \cdot J C^t)}{\partial u_i} \right] D^t.
\]
According to the identity (d) of Lemma 4.2.3, the third sum vanishes. By using two identities in (a) of Lemma 4.2.1 we deduce

$$S_2 = \lambda t^2 r_{t}^{2N+1} Q_t \frac{\sqrt{g}'}{\sqrt{P}} e^{i(\lambda, \Theta)} \left[ J C' - (D' \cdot J C') \ D' \right]$$

$$-\lambda t^2 r_{t}^{2N+1} Q_t \left( D' \cdot J C' \right) \frac{\sqrt{g}'}{\sqrt{P}} e^{i(\lambda, \Theta)} \left[ -J D' - (D' \cdot J C') \ C' \right]$$

$$= -\lambda t^2 r_{t}^{2N+1} Q_t \frac{\sqrt{g}'}{\sqrt{P}} \left[ 1 - (D' \cdot J C')^2 \right] e^{i(\lambda, \Theta) C'}$$

**Step C3.** The identity (4.5) and the definition (4.4) give

$$S_3 = \sum_{s=1}^{L} \sum_{i=1}^{2N} \frac{\partial}{\partial u_i} \left( \sqrt{G} \frac{-d_i}{\sqrt{P}} \frac{\partial}{\partial \Theta} \left( r_t e^{i(\lambda, \Theta) C'} \right) \right)$$

$$= -\lambda t r_t e^{i(\lambda, \Theta)} \left[ \sum_{s=1}^{L} \sum_{i=1}^{2N} \frac{\partial}{\partial u_i} \left( \sqrt{G} \frac{d_i}{\sqrt{P}} J C' \right) \right]$$

$$= -\lambda t r_t e^{i(\lambda, \Theta)} J \left[ \sum_{s=1}^{L} \sum_{i=1}^{2N} \frac{\partial}{\partial u_i} \left( \sqrt{G} \frac{d_i}{\sqrt{P}} C' \right) \right]$$

$$= -\lambda t r_t e^{i(\lambda, \Theta)} J \left[ \sum_{i=1}^{2N} \left( \sum_{s=1}^{L} \frac{\partial}{\partial u_i} \left( \sqrt{G} \frac{d_i}{\sqrt{P}} C' \right) \right) \right]$$

$$= -\lambda t r_t e^{i(\lambda, \Theta)} J \left[ \sum_{1 \leq i, j \leq 2N} \left( \sqrt{g} \right)_{ij} \frac{\partial C'}{\partial u_i} \right]$$

The first identity in (a) of Lemma 4.2.1 and the factorization (4.7) yield

$$S_3 = -\lambda t^2 r_{t}^{2N+1} Q_t \frac{\sqrt{g}'}{\sqrt{P}} e^{i(\lambda, \Theta)} \left[ J C' - (D' \cdot J C') \ D' \right]$$

$$= \lambda t^2 r_{t}^{2N+1} Q_t \frac{\sqrt{g}'}{\sqrt{P}} e^{i(\lambda, \Theta)} \left[ C' + (D' \cdot J C') \ J D' \right]$$

$$= \lambda t^2 r_{t}^{2N+1} Q_t \frac{\sqrt{g}'}{\sqrt{P}} e^{i(\lambda, \Theta)} \left[ C' + (D' \cdot J C') \ J D' \right].$$
**Step C4.** We simplify the sum $S_4$.

$$
S_4 = \sum_{s=1}^{L} \sum_{i=1}^{2N} \frac{\partial}{\partial \Theta} \left( \sqrt{G} \frac{-d_s^i}{P} \frac{\partial}{\partial u_i} \left( r_i e^{(l_i, \Theta)} C^i \right) \right)
$$

$$
= -\frac{\sqrt{G}}{P} \sum_{i=1}^{2N} \frac{\partial}{\partial \Theta} \left( \sum_{s=1}^{L} d_s^i \frac{\partial}{\partial u_i} \left( r_i e^{(l_i, \Theta)} C^i \right) \right)
$$

$$
= -\frac{\sqrt{G}}{P} \sum_{i=1}^{2N} \frac{\partial}{\partial \Theta} \left( d_s^i \frac{\partial}{\partial u_i} \left( r_i e^{(l_i, \Theta)} C^i \right) \right)
$$

$$
= -r_i \frac{\sqrt{G}}{P} \sum_{i=1}^{2N} \left( \frac{\partial}{\partial u_i} e^{(l_i, \Theta)} C^i \right)
$$

$$
= -\lambda_i r_i \frac{\sqrt{G}}{P} e^{(l_i, \Theta)} \sum_{i=1}^{2N} \frac{\partial}{\partial u_i} \left( J C^i \right)
$$

$$
= -\lambda_i r_i \frac{\sqrt{G}}{P} e^{(l_i, \Theta)} \left( \sum_{i=1}^{2N} \frac{\partial}{\partial u_i} C^i \right).
$$

From the definition $d_i^i = \sum_{j=1}^{2N} \lambda_i (g_i^i)^j w_i^j$ and the first identity in (a) of Lemma 1, we deduce

$$
S_4 = -\lambda_i r_i \frac{\sqrt{G}}{P} e^{(l_i, \Theta)} \left( \sum_{1 \leq i, j \leq 2N} (g_i^j)^i w_i^j \frac{\partial}{\partial u_i} \right)
$$

$$
= -\lambda_i r_i \frac{\sqrt{G}}{P} e^{(l_i, \Theta)} \left( J C^i \right)
$$

$$
= -\lambda_i r_i \frac{\sqrt{G}}{P} e^{(l_i, \Theta)} \left( D^i \cdot J C^i \right)
$$

By using the factorization (4.7) obtained in **Step C1**:

$$
\sqrt{G} = \sqrt{P} r_i^{2N} \left( g_i^j \right)^{1/2} Q_i,
$$

we have

$$
S_4 = \lambda_i^2 r_i^{2N+1} Q_i \frac{\sqrt{G}}{\sqrt{P}} e^{(l_i, \Theta)} \left[ C^i + \left( D^i \cdot J C^i \right) J D^i \right].
$$

**Step C5.** The term $S_5$ can be simplified to:

$$
S_5 = \lambda_i r_i \frac{\partial}{\partial \Theta} \left( \frac{\sqrt{G}}{P} e^{(l_i, \Theta)} J C^i \right) = -\lambda_i^2 r_i \frac{\sqrt{G}}{P} e^{(l_i, \Theta)} C^i.
$$
From the factorization (4.7) obtained in Step C1:
\[ \sqrt{G} = \sqrt{P} r_t^{2N} \sqrt{g_t} Q_t, \]
we have
\[ S_5 = -\lambda_t^2 r_t^{2N+1} Q_t \frac{\sqrt{g_t}}{\sqrt{P}} e^{i(\lambda_t \Theta)} C^t. \]

Step C6. We have
\[
S_6 = \sum_{s=1}^{L} \frac{\partial}{\partial r^s} \left( \sqrt{G} \frac{\partial}{\partial r^s} \left( r_t e^{i(\lambda_t \Theta)} C^t \right) \right).
\]
\[
= \frac{\partial}{\partial r^t} \left( \sqrt{G} \frac{\partial}{\partial r^t} \left( r_t e^{i(\lambda_t \Theta)} C^t \right) \right).
\]
\[
= e^{i(\lambda_t \Theta)} \frac{\partial}{\partial r^t} C^t
\]
\[
= \sqrt{g_t} Q_t e^{i(\lambda_t \Theta)} \left[ 2N r_t^{2N-1} \sqrt{P} + r_t^{2N} \frac{\partial \sqrt{P}}{\partial r^t} \right] C^t.
\]
By using (4.1), (4.3), and the first identity in (b) of Lemma 1, we deduce
\[
\frac{\partial \sqrt{P}}{\partial r_t} = \frac{1}{2 \sqrt{P}} \frac{\partial}{\partial r_t} \left( R - \sum_{1 \leq s \leq L} \sum_{1 \leq i, j \leq 2N} \lambda_s^2 r_s^{2} (g^t)^{ij} w_t^i w_t^j \right)
\]
\[
= \frac{1}{2 \sqrt{P}} \left[ 2 \lambda_t^2 r_t - 2 \lambda_t^2 r_t \sum_{1 \leq i, j \leq 2N} (g^t)^{ij} w_t^i w_t^j \right]
\]
\[
= \frac{1}{2 \sqrt{P}} \left[ 2 \lambda_t^2 r_t - 2 \lambda_t^2 r_t \left( 1 - \left( D^t \cdot JC^t \right)^2 \right) \right]
\]
\[
= \frac{\lambda_t^2 r_t}{\sqrt{P}} \left( D^t \cdot JC^t \right)^2.
\]
and meet
\[
S_6 = 2N r_t^{2N-1} Q_t \sqrt{P} \sqrt{g_t} e^{i(\lambda_t \Theta)} C^t
\]
\[
+ \lambda_t^2 r_t^{2N+1} Q_t \frac{\sqrt{g_t}}{\sqrt{P}} \left( D^t \cdot JC^t \right)^2 e^{i(\lambda_t \Theta)} C^t.
\]
Step C7. Combining the results so far, we conclude

$$S_1 + S_2 + S_3 + S_4 + S_5 + S_6$$

$$= -2N r_t^{2N-1} Q_t \sqrt{P} \sqrt{g^t} e^{i(\lambda t \Theta)} C^t$$

$$- 2\lambda_t^2 r_t^{2N+1} Q_t \sqrt{g^t} \left( D^t \cdot JC^t \right) e^{i(\lambda t \Theta)} \left[ JD^t + (D^t \cdot JC^t) C^t \right]$$

$$- \lambda_t^2 r_t^{2N+1} Q_t \sqrt{g^t} \left[ 1 - (D^t \cdot JC^t)^2 \right] e^{i(\lambda t \Theta)} C^t$$

$$+ 2 \lambda_t^2 r_t^{2N+1} Q_t \sqrt{g^t} e^{i(\lambda t \Theta)} C^t$$

$$- \lambda_t^2 r_t^{2N+1} Q_t \sqrt{g^t} e^{i(\lambda t \Theta)} C^t$$

$$+ 2N r_t^{2N-1} Q_t \sqrt{P} \sqrt{g^t} e^{i(\lambda t \Theta)} C^t$$

$$+ \lambda_t^2 r_t^{2N+1} Q_t \sqrt{g^t} \left( D^t \cdot JC^t \right)^2 e^{i(\lambda t \Theta)} C^t$$

$$= 0.$$
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