The conformal method and the conformal thin-sandwich method are the same

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Abstract
The conformal method developed in the 1970s and the more recent Lagrangian and Hamiltonian conformal thin-sandwich methods are techniques for finding solutions of the Einstein constraint equations. We show that they are manifestations of a single conformal method: there is a straightforward way to convert back and forth between the parameters for these methods so that the corresponding solutions of the Einstein constraint equations agree. The unifying idea is the need to clearly distinguish tangent and cotangent vectors to the space of conformal classes on a manifold, and we introduce a vocabulary for working with these objects without reference to a particular representative background metric. As a consequence of these conceptual advantages, we demonstrate how to strengthen previous near-CMC (constant mean curvature) existence and non-existence theorems for the original conformal method to include metrics with scalar curvatures that change sign.

Keywords: conformal method, conformal thin-sandwich, Einstein constraint equations

1. Introduction

Initial data for the (vacuum) Cauchy problem in general relativity consist of a Riemannian manifold \((M^n, g_{\alpha\beta})\) and a symmetric tensor \(K_{\alpha\beta}\) that will become the induced metric and second fundamental form of an embedding of \(M^n\) into a Ricci flat Lorentzian spacetime determined from the initial data \((g_{\alpha\beta}, K_{\alpha\beta})\). The Gauss and Codazzi relations (along with the fact that the ambient spacetime is Ricci flat) impose the following compatibility conditions on the Cauchy data:

\[
R_g - |K_g|^2 + (\text{tr}_g K)^2 = 0 \quad \text{[Hamiltonian constraint]} \tag{1a}
\]

\[
\text{div}_g K = d\tau \quad \text{[momentum constraint]} \tag{1b}
\]
where \( \tau = g^{ab}K_{ab} \) is the mean curvature. Choquet-Bruhat showed \([FB52]\) that there exists a solution of the Cauchy problem if and only if the initial data satisfy the Einstein constraint equations \((1)\), so, finding solutions of the constraint equations is a fundamental problem in general relativity.

In 1944, Lichnerowicz \([Li44]\) initiated an approach for finding solutions of the constraint equations, the so-called conformal method. Extensions of this method (due to York and his collaborators, as described below) are now the principal techniques used to construct solutions from scratch. For these methods, one starts with a Riemannian metric \( g_{ab} \), and attempts to construct a solution \( (g_{ab}, K_{ab}) \) of the constraints where \( \bar{g}_{ab} \) is conformally related to \( g_{ab} \) via some conformal factor to be determined as part of the solution. The mean curvature \( \tau = g^{ab}K_{ab} \) is also freely specified so

\[
\bar{K}_{ab} = \bar{X}_{ab} + \frac{2}{n} \tau \bar{g}_{ab}
\]

where \( \bar{X}_{ab} \) is a trace-free tensor that is determined, as part of the solution procedure, from \( \bar{g}_{ab} \), \( \tau \), and other auxiliary data specific to the particular conformal method.

Lichnerowicz’s original conformal method constructed solutions with \( \tau \equiv 0 \). In the early ’70s York proposed an extension of the method that allows one to specify \( \tau = \tau_0 \) for an arbitrary constant \( \tau_0 \) \([Yo73]\), and with Ó Murchadha described a further extension to arbitrary mean curvatures \([ÓMY74]\). In this paper, we refer to the prescription from \([Yo73]\) as the constant mean curvature (CMC) conformal method, and its extension in \([ÓMY74]\) as the 1974 conformal method. The data for this method consist of a metric \( g_{ab} \), a symmetric trace-free tensor \( \sigma_{ab} \) satisfying \( \nabla \sigma_{ab} = 0 \) \( a \), and a mean curvature \( \tau \).

Somewhat later York proposed a Lagrangian conformal thin-sandwich (CTS) approach \([Yo99]\), and subsequently with Pfeiffer described a Hamiltonian formulation of the CTS method \([PY03]\) \(^1\). The Lagrangian method starts from initial data \( (\bar{g}_{ab}, u_{ab}, \tau, N) \) where \( u_{ab} \) is an arbitrary trace-free symmetric tensor and \( N \) is a positive function related to a parameter (the lapse) that appears in the \( n + 1 \) formulation of the Cauchy problem. The CTS method has the virtue that given conformal data \( (\bar{g}_{ab}, u_{ab}, \tau, N) \) and a second conformally related metric \( \tilde{g}_{ab} \), there is a way to conformally transform the remaining data to form \( (\tilde{g}_{ab}, \tilde{u}_{ab}, \tau, \tilde{N}) \) such that the set of solutions of the constraints associated with the original data and the transformed data are the same. The property is known as conformal covariance (or sometimes conformal invariance), and is a property that is shared with the CMC conformal method but that is apparently absent for the 1974 conformal method. The ability to select a background metric within the conformal class satisfying some desired property (e.g., a metric with a scalar curvature that has constant sign) is a powerful tool, and this has occasionally lead to theorems that are stronger when using the CTS approach versus the 1974 conformal method.

The purpose of this note is to clarify the relationship between the CTS methods and the 1974 conformal method: they are the same. Specifically, there is a way to translate, in a straightforward and essentially unique way, between 1974 conformal data \( (g_{ab}, \sigma_{ab}, \tau) \) and CTS data \( (\tilde{g}_{ab}, \tilde{u}_{ab}, \tau, \tilde{N}) \) such that the corresponding solutions of the constraint equations are the same. The significance of this result arises from the fact that if \( \tau \) is not constant, then it is generally unknown how many solutions are associated with 1974 conformal data \( (g_{ab}, \sigma_{ab}, \tau) \). One hopes that there is exactly one, except perhaps for some well defined set of data where there is none. But from \([Ma11]\) we have examples showing that there can be more than one

\(^1\) We distinguish here between the CTS method and the so-called extended CTS method described at the end of \([PY03]\), which is a non-trivial modification of the CTS method and has unsatisfactory uniqueness properties \([PY05]\). We do not treat the extended CTS method.
solution, and evidence that the set for which there is no solution may be difficult to describe. Since the 1974 and CTS methods are the same, these deficiencies apply equally to both methods. Or, from a more positive perspective, we see that any result that can be proved for one method can be translated into an equivalent theorem about the other method. Questions of which data yield no solutions, or exactly one solution, or multiple solutions of the constraints can be formulated using whichever method is convenient. Moreover, since the CTS method is conformally covariant, so is the 1974 conformal method when handled correctly.

A remark that the methods are identical (along with a sketch of the equivalence) was made by the author in [Ma11], which explored conformal parameterizations of certain far-from CMC data using the CTS framework, and which asserted that the results of that paper translate to results for the 1974 conformal method. Aside from the remark in [Ma11], the fact that the methods are the same does not seem to appear in the literature (although [PY03] comes very close, but stops short and seems to have the perspective that the methods are different). Since the mathematics literature in recent years has seen progress toward understanding the 1974 conformal method for non-CMC conformal data (e.g., [HNT09] [Ma09] [DGH12]), and since physicists tend to use the CTS method, it seems useful to have a guide for how to translate results between the methods. Moreover, there are instances in both mathematics and physics publications where the methods are asserted to be different, or where results are proved for the 1974 method that are weaker than analogous results for the CTS method. The following examples illustrate how improvements can be realized by taking advantage of the equivalence.

• In [IM96] and [ACI08] the 1974 approach is used to generate near-CMC solutions of the constraint equations, but under the restriction that the scalar curvature of the metric has constant sign. These theorems admit generalizations to the CTS method, and we will see in section 8 that these can be used to establish similar near-CMC results for the 1974 method for metrics with an arbitrary scalar curvature.

• Reference [IÔM04] contains non-existence theorems for the conformal methods for certain non-CMC data. Theorem 2 (framed in the 1974 conformal framework) is weaker than theorem 3 (which uses the CTS framework) and it is asserted that the gap is related to the lack of conformal covariance of the 1974 method. However, the 1974 method is covariant and we show in section 8 that theorem 2 can be improved to be just as strong as theorem 3.

• In [DGH12], theorem 1.7 proves a variation of the non-existence result of [IÔM04] theorem 2 using the 1974 conformal method approach; it can similarly be strengthened by taking advantage of the equivalence of the 1974 and CTS methods2.

• The numerical relativity text [BS10] presents the 1974 conformal method (called there the conformal transverse traceless decomposition) and the CTS method as different techniques, with a different number of specifiable parameters for each method. We show here how to translate back and forth between the parameters of the two methods; knowing that the parameterizations are the same gives insight into both methods.

A secondary goal of this paper is to formulate the parameters of the various conformal methods in terms of objects at the level of the set $C$ of conformal classes rather than the set $M$ of metrics on $M$; i.e., we work with conformal classes, tangent and cotangent vectors to $C$, and conformal Killing operators defined in terms of conformal classes rather than representative metrics. Doing so can be thought of as a coordinate-free approach to understanding

2 We note in press that the recent preprint [GN14] provides an alternative proof that also strengthens [DGH12] theorem 1.7.
the parameters. Motivated by diffeomorphism invariance of the Einstein equations, we also give a clear interpretation of these parameters as objects associated with $C/D_0$, where $D_0$ is the connected component of the identity of the diffeomorphism group.

While a coordinate-free perspective is implicitly present in some of the physics literature, it clarifies matters to make it explicit. For example, it turns out that it is crucial to make a distinction between the tangent space $T_C$ and the cotangent space $T^*_C$ to $C$ at some conformal class $g$. Unlike the case for $M$, there is no natural way to identify tangent vectors as cotangent vectors, but there is a natural family of identifications. In the CTS method, a choice from this family is specified via the lapse. In the 1974 method the choice is specified less explicitly, and this is perhaps the reason why it is not obvious at first glance that the 1974 and CTS methods are the same. Most variations of the conformal method use the metric parameter $g_{ab}$ simply to determine the conformal class $[g_{ab}]$ of the solution metric. For the 1974 conformal method, however, the choice of $g_{ab}$ specifies both $[g_{ab}]$ and a choice of identification of $T_C$ with $T^*_C$. Changing the representative metric in the 1974 approach is equivalent to changing the lapse in the CTS approach, and working with the parameters in a coordinate-free way helps make this relationship clear.

Although the coordinate-free formulation provides insight, it also introduces an extra layer of abstraction. Readers who are already familiar with the conformal methods, and who wish to skip over this abstraction, can jump to the end of section 7 where there are concise recipes, in familiar tensorial terms, for how to convert parameters between the various conformal methods. These recipes describe mechanically how the methods are the same; we hope that the coordinate-free approach taken elsewhere in the paper illuminates why the methods are the same.

The remainder of the paper proceeds as follows. In section 2 we establish coordinate-free language for describing conformal objects, and sections 3 through 6 formulate each of the various conformal methods in terms of this language. In section 7 we establish the equivalence of all these methods, and section 8 uses this equivalence in some applications.

1.1. Notation

Throughout we assume that $M^n$ is a compact, connected, oriented $n$-manifold with $n \geq 3$. The set of smooth functions on $M$ is $C^\infty (M)$, and the positive smooth functions are denoted by $C^\infty_+ (M)$. Given a bundle $E$ over $M$, we write $C^\infty (M, E)$ for smooth sections off the bundle. The bundle of symmetric $(0, 2)$-tensors is $S_2 M$, and $M$ is the set of smooth metrics on $M$ (i.e., the open set of positive-definite elements of $C^\infty (M, S_2 M)$). The bundle of conformal classes of smooth metrics is $C$. All objects in this paper are smooth.

We use a modified form of abstract index notation. Indices are used for tensorial objects to clarify the number and type of arguments, to help with contraction operations, and so forth, but are not associated with the components of the tensor with respect to some specific coordinate system. Whenever indices might clutter notation (e.g., when the tensor is used as a subscript) we freely drop the indices. So for a metric $g_{ab}$, the name of the metric is $g$ and the indices are a helpful decoration to be used when they do not get in the way.

The Levi-Civita connection of a metric $g_{ab}$ is $\nabla$ or $\hat{\nabla}$ as needed, and its (positively-oriented) volume form is $dV$. Given a metric $g_{ab}$ and a function $\phi \in C^\infty_+ (M)$, we can form a conformally related metric

$$\tilde{g}_{ab} = \phi^{n-2} g_{ab}$$

(3)
where \( q \) is the dimensional constant

\[
q = \frac{2n}{n - 2}.
\]  

(4)

All conformal transformations in this paper will have the form (3) since the scalar curvature \( R_g \) then has the simple form

\[
R_g = \phi^{-\frac{4}{n}} \left( -2q \Delta_g \phi + R_g \phi \right),
\]  

(5)

where \( \Delta_g \) is the Laplacian of \( g \) and \( \kappa \) is the dimensional constant

\[
\kappa = \frac{n - 1}{n}.
\]  

(6)

Although the appearance of \( \kappa \) in equation (5) is somewhat awkward, it appears naturally throughout the equations connected to the conformal method, so we introduce this notation now.

The conformal class of \( g_{ab} \) is \([ g_{ab} ]\). When we do not want to emphasize some particular representative of the conformal class we use bold face instead: \( g \) denotes a conformal class as well. A boldface \( g \) and a plain \( g \) are unrelated names, so an equation such as \([ g_{ab} ] = g \) is a non-trivial statement. Tangent and cotangent vectors to \( C \) will be written with boldface as well.

Starting from a metric \( g_{ab} \) and other conformal data the conformal methods attempt to find a solution of the constraints with a metric conformally related to \( g_{ab} \). We use overbars to denote conformally transforming objects that satisfy the constraint equations, so \( \bar{g}_{ab} \) is the physical solution metric.

## 2. Conformal objects

Our goal here is to express the objects that appear in the various conformal methods intrinsically with respect to a conformal class rather than with respect to a representative metric. The set \( C \) of smooth conformal classes can be shown to be a Fréchet manifold, which provides a natural definition of tangent and cotangent vectors at some conformal class \( g \). To avoid this machinery, however, we take a more prosaic approach and define tangent and (certain) cotangent vectors to \( C \) at \( g \) as tensorial objects that transform in a certain way when changing from one representative of \( g \) to another. This is analogous to the old-fashioned approach of defining a manifold’s tangent and cotangent vectors as objects that transform in a certain way under coordinate changes. We also give a related description of tangent and cotangent vectors to \( C/D_0 \) where \( D_0 \) is the connected component of the identity of the diffeomorphism group.

### 2.1. Conformal tangent vectors

Let \( g_{ab}^0 \) be a metric and let \( g_{ab} (t) \) be a smooth path with \( g_{ab} (0) = g_{ab}^0 \). It is easy to see that if \( g_{ab} (t) \) remains in the conformal class \([ g_{ab}^0 ]\) then there is function \( \alpha \in C^\infty (M) \) such that

\[
g_{ab}^t (0) = \alpha g_{ab}^0.
\]  

(7)

Moreover, every smooth function arises this way for some path (e.g. \( g_{ab} (t) = e^{\alpha t} g_{ab}^0 \)). So we identify
as the tangent space of the conformal class \([g^0_{ab}]\) at \(g^0_{ab}\). Given an arbitrary path starting at \(g^0_{ab}\), we can uniquely decompose

\[
g^0_{ab}(0) = u_{ab} + \alpha g^0_{ab}
\]

where \(u_{ab}\) is trace-free with respect to \(g^0_{ab}\) and \(\alpha \in C^\infty(M)\). It is therefore natural to identify the trace-free tensors \(u_{ab}\) as the directions of travel through the set of conformal classes. Given a smooth function \(\beta(t)\), the paths \(g^0_{ab}(t)\) and

\[
\bar{g}_{ab}(t) = e^{\beta/2}g_{ab}(t)
\]
descend to the same path in \(\mathcal{C}\). Since \(\bar{g}^0_{ab}(0) = e^{\beta/2}u_{ab} + \left(\beta\beta + \alpha\right)g^0_{ab}\) we therefore identify \(u_{ab}\) at \(g^0_{ab}\) and \(e^\beta u_{ab}\) at \(e^\beta g^0_{ab}\) as representing the same tangent vector to \(\mathcal{C}\) at \([g^0_{ab}]\).

**Definition 1.** Let \(X\) be the set of pairs \((g^0_{ab}, u_{ab})\) where \(g^0_{ab} \in \mathcal{M}\) and where \(u_{ab} \in C^\infty(M, S^2_M)\) is trace-free with respect to \(g^0_{ab}\). A conformal tangent vector is an element of \(X/\sim\) where

\[
\left(\bar{g}^0_{ab}, \bar{u}_{ab}\right) \sim (g^0_{ab}, u_{ab})
\]

if there exists \(\phi \in C^\infty(M)\) such that

\[
g^0_{ab} = \phi^{\epsilon/2}g_{ab}, \quad \bar{u}_{ab} = \phi^{\epsilon/2}u_{ab}.
\]

We use the following notation:

- \([g^0_{ab}, u_{ab}]\) is the conformal tangent vector corresponding to \((g^0_{ab}, u_{ab})\).
- For each \(g \in \mathcal{C}\), \(T_g\mathcal{C}\) is the set of conformal tangent vectors \([g^0_{ab}, u_{ab}]\) with \(g^0_{ab} \in g\).
- \(TC = \bigcup_{g \in \mathcal{C}} T_g\mathcal{C}\).

More generally, if \(g_{ab}\) is a metric and \(S_{ab}\) is an arbitrary symmetric \((0, 2)\)-tensor field, we define

\[
[g_{ab}, S_{ab}] = [g_{ab}, u_{ab}]
\]

where \(u_{ab}\) is the trace-free part of \(S_{ab}\) (as computed with respect to \(g^0_{ab}\)). This should be thought of as the pushforward of the tangent vector \(\bar{S}_{ab}\) to the space of metrics at \(g_{ab}\) to an element of \(T_{[g^0_{ab}]}\mathcal{C}\) under the natural projection.

Suppose \(g\) is a conformal class and \(u \in T_g\mathcal{C}\). Given a representative \(g^0_{ab} \in g\) it is clear that there is a unique trace-free \(u_{ab} \in C^\infty(M, S^2_M)\) with \([g^0_{ab}, u_{ab}] = u\), which we will call the representative of \(u\) with respect to \(g^0_{ab}\). We give \(T_g\mathcal{C}\) the topology of the subspace of \(C^\infty(M, S^2_M)\) determined by this identification and note that the topology is independent of the choice of representative \(g^0_{ab}\).

### 2.2. The conformal killing operator

The conformal tangent vectors that arise by flowing a conformal class \(g\) through a path of diffeomorphisms can be described in terms of a map \(L_g\): \(C^\infty(M, TM) \to T_g\mathcal{C}\) called the conformal Killing operator.

Let \(g\) be a conformal class with representative \(g^0_{ab}\) and suppose \(\Phi\) is a path of diffeomorphisms starting at the identity with infinitesimal generator \(X^a\). The path
satisfies
\[ h_{ab}(t) = \Phi^b_{\gamma} g_{ab}, \]  
(14)
where \( \mathcal{L}_X g_{ab} \) is the Lie derivative of \( g_{ab} \) with respect to the vector field \( X^a \). We decompose \( \mathcal{L}_X g_{ab} \) into its trace and trace-free parts with respect to \( g_{ab} \) to obtain
\[ \mathcal{L}_X g_{ab} = \left( L^a \mathcal{X}_X \right)_{ab} + \frac{2 \text{div} X}{n} g_{ab} \]  
(16)
where \( \text{div}_X X = ^e V_X^n X^a \) and where
\[ \left( L^a \mathcal{X}_X \right)_{ab} = ^e V_X^n a + ^e V_X^n b - \frac{2 \text{div} X}{n} g_{ab} \]  
(17)
is the usual conformal Killing operator. The conformal tangent vector
\[ \mathbf{u} = \left[ g_{ab}, \left( L^a \mathcal{X}_X \right)_{ab} \right] \]  
(18)
does not depend on the choice of representative of \( g \). Indeed, an easy computation shows that if \( \tilde{g}_{ab} \) is another metric conformally related to \( g_{ab} \) via \( \tilde{g}_{ab} = \phi^{n-2} g_{ab} \) then
\[ L^a \mathcal{X}_X = \phi^n L^a \mathcal{X}_X \]  
(19)
and consequently
\[ \left[ \tilde{g}_{ab}, \left( L^a \mathcal{X}_X \right)_{ab} \right] = \left[ g_{ab}, \left( L^a \mathcal{X}_X \right)_{ab} \right] = \mathbf{u}. \]  
(20)
We therefore obtain a well-defined conformal Killing operator \( L^a \mathcal{X}_X : C^\infty \left( M, TM \right) \rightarrow T_m C \) given by
\[ L^a \mathcal{X}_X = \left[ g_{ab}, L^a \mathcal{X}_X \right]. \]  
(21)
for any representative \( g_{ab} \) of \( g \). Since the map \( L^a \mathcal{X}_X : C^\infty \left( M, TM \right) \rightarrow C^\infty \left( M, S_2 M \right) \) is continuous, and since the projection \( u_{ab} \rightarrow [ g_{ab}, u_{ab} ] \) is continuous, so is \( L^a \mathcal{X}_X \).

The elements of the kernel of \( L^a \mathcal{X}_X \) are called conformal Killing fields. Generically there are none [FM77].

2.3. Conformal cotangent vectors

The conformal method involves symmetric, trace-free, \((0,2)\)-tensors \( S_{ab} \) that obey the conformal transformation law
\[ \tilde{S}_{ab} = \phi^{-2} S_{ab} \]  
(22)
when \( g_{ab} \) is transformed to \( \tilde{g}_{ab} = \phi^{n-2} g_{ab} \). Such objects are not tangent vectors to \( C \) since the wrong power of \( \phi \) appears in the transformation law; rather, these encode cotangent vectors as follows.

Given the pair \( \left( \tilde{g}_{ab}, A_{ab} \right) \) we define a functional on symmetric trace free tensors \( u_{ab} \) via
\[ F_{x,a} (u) = \int_u \langle A, u \rangle_x dV_x. \]  
(23)
If we conformally transform \( u_{ab} \) as a tangent vector
\[ \tilde{u}_{ab} = \phi^{n-2} u_{ab} \]  
(24)
and we transform
\[ \tilde{A}_{ab} = \phi^{-2} A_{ab} \] (25)
then
\[ \{ \tilde{A}, \tilde{u} \} = \phi^{-2} \psi \{ \phi^{-2} A, \phi^{-2} u \} = \phi^{-1} \{ A, u \}, \] (26)

At the same time, the volume form transforms as \( dV_{\tilde{g}} = \phi^\gamma dV_g \) and therefore
\[ F_{\tilde{g}, A} (\tilde{u}) = \int_M \{ \tilde{A}, \tilde{u} \} dV_{\tilde{g}} = \int_M \phi^{-1} \{ A, u \} dV_g = \int_M \{ A, u \} dV_g = F_{g, A} (u). \] (27)
Thus we can associate with \( (g_{ab}, A_{ab}) \) a well-defined functional on \( T_g (\mathcal{M}) \) when we transform \( A_{ab} \) according to equation (25).

**Definition 2.** Let \( \mathcal{Y} \) be the set of pairs \( (g_{ab}, A_{ab}) \) where \( g_{ab} \in \mathcal{M} \) and where \( A_{ab} \in C^\infty (M, S^2 \mathcal{M}) \) is trace-free with respect to \( g_{ab} \). A (smooth) conformal cotangent vector is an equivalence class of \( \mathcal{Y}/\sim \) under the relation
\[ (\tilde{g}_{ab}, \tilde{A}_{ab}) \sim (g_{ab}, A_{ab}) \] (28)
if there is a smooth positive function \( \phi \) on \( \mathcal{M} \) such that
\[ \tilde{g}_{ab} = \phi^{\frac{1}{2}} g_{ab} \]
\[ \tilde{A}_{ab} = \phi^{-1} A_{ab} \] (29)

We use the following notation:
- \( \left[ g_{ab}, A_{ab} \right]^\# \) is the conformal cotangent vector corresponding to \( (g_{ab}, A_{ab}) \).
- For each \( g \in C^\infty \mathcal{M} \), \( T^g C \) is the set of conformal tangent vectors \( [g_{ab}, A_{ab}]^\# \) with \( g_{ab} \in g \).
- \( T^g C = \bigcup_{g_{ab} \in \mathcal{M}} T^g C_{g_{ab}} \).

More generally, if \( K_{ab} \) is an arbitrary element of \( C^\infty (M, S^2 \mathcal{M}) \) we define
\[ \left[ g_{ab}, K_{ab} \right]^\# = \left[ g_{ab}, A_{ab} \right]^\# \] (30)
where \( A_{ab} \) is the trace-free part of \( K_{ab} \) (with respect to \( g_{ab} \)).

Given a smooth conformal tangent vector \( A \in T_g ^{\#} C \) and a conformal tangent vector \( u \in T_g C \), we define
\[ \{ A, u \} = \int \{ A, u \} dV_g \] (31)
where \( \langle g_{ab}, A_{ab} \rangle \) and \( \langle g_{ab}, u_{ab} \rangle \) are any representatives of \( A \) and \( u \) with respect to the same background metric \( g_{ab} \in g \). This linear map is evidently continuous, so we identify \( T^g C \) with a subspace of \( \left( T_g C \right)^\# \).

The containment \( T^\# g C \subseteq \left( T^g C \right)^\# \) is strict since the topological dual space contains more general distributions, which motivates the modifier smooth in the previous definition. Given that we represent conformal tangent vectors using symmetric trace-free \((0, 2)\)-tensor fields, it may be more natural to represent conformal cotangent vectors using symmetric trace-free \((2, 0)\)-tensor fields. To this end, we also define
\[ \left[ g_{ab}, A^{ab} \right]^\# = \left[ g_{ab}, g_{ac} g_{bd} A_{cd} \right]^\# \] (32)
It is easy to see that if $\tilde{g}_{ab} = \phi^{-2} g_{ab}$, then
\[
\left[ \tilde{g}_{ab}, \tilde{A}^{ab} \right]^a = \left[ g_{ab}, A^{ab} \right]^a
\] (33)
if and only if $\tilde{A}^{ab} = \phi^{-2} A^{ab}$, which recovers another familiar transformation law for the conformal method. Symmetric trace-free tensors transforming according to $\tilde{A}_{ab} = \phi^{-2} A_{ab}$ or $\tilde{A}^{ab} = \phi^{-2} A^{ab}$ are both representations of conformal cotangent vectors.

The distinction between tangent and cotangent vectors is important because unlike the situation for the space $\mathcal{M}$ of metrics, we do not have a canonical identification of tangent and cotangent vectors for $\mathcal{C}$. The tangent space of $\mathcal{M}$ at a metric $g_{ab}$ is $T_g \mathcal{M} = C^\infty (M, S^2 \mathcal{M})$ and is equipped with a natural metric defined by
\[
\langle h, k \rangle = \int_M \langle h, k \rangle_d V_g.
\] (34)
The metric provides a natural identification of $T_g \mathcal{M}$ with a subspace of $(T_g \mathcal{M})^\mathbb{R}$ by taking to $h_{ab}$ to $\langle h, \cdot \rangle$. Unfortunately, this inner product does not descend to an inner product on $T_g \mathcal{C}$, and we do not have a canonical way to identify $T_g \mathcal{C}$ with a subspace of $(T_g \mathcal{C})^\mathbb{R}$. Instead, we have a family of identifications depending on the choice of a volume form on $M$.

**Proposition 2.1.** Let $\omega$ be a smooth volume form on $M$ (i.e., a non-vanishing, positively-oriented section of $\Lambda^M$). There is a unique linear map $k_{\omega}: T^*_g \mathcal{C} \to T^*_g \mathcal{C}$ satisfying the following:

- For each $g \in \mathcal{C}$, $k_{\omega}: T_g \mathcal{C} \to T^*_g \mathcal{C}$ is continuous and bijective.
- If $u, v \in T_g \mathcal{C}$, and if $u_{ab}$ and $v_{ab}$ are their representatives with respect to some common background metric $g_{ab} \in \mathcal{g}$, then
  \[
  \langle k_{\omega}(u), v \rangle = \int_M \langle u, v \rangle_g \omega
  \] (35)
- If $g$ is a conformal class and $u \in T_g \mathcal{C}$ with representative $(g_{ab}, u_{ab})$ then
  \[
  k_{\omega}(u) = \left[ g_{ab}, (\omega / dV_g) u_{ab} \right]^a.
  \] (36)

**Proof.** We define $k_{\omega}$ by equation (35) and need to show that it is well-defined and has the stated properties. To see that it is well-defined, suppose $g_{ab}$ and $\tilde{g}_{ab} = \phi^{1/2} g_{ab}$ are two representatives of a conformal class $g$ and suppose $u, v \in T_g \mathcal{C}$. Let $u_{ab}, \tilde{u}_{ab}, v_{ab}, \tilde{v}_{ab}$ be the representatives of $u$ and $v$ with respect to $g_{ab}$ and $\tilde{g}_{ab}$, so $\tilde{u}_{ab} = \phi^{1/2} u_{ab}$ and similarly with $\tilde{v}_{ab}$. Then
\[
\langle \tilde{u}, \tilde{v} \rangle_g = \phi^{1/2} \langle u, v \rangle_g = \phi^{1/2} \langle \phi^{1/2} u, \phi^{1/2} v \rangle_g = \langle u, v \rangle_g.
\] (37)
Thus
\[
\int_M \langle \tilde{u}, \tilde{v} \rangle_g \omega = \int_M \langle u, v \rangle_g \omega
\] (38)
and $k_{\omega}$ is well-defined.

It is clear that $k_{\omega}(u) \in (T_g \mathcal{C})^\mathbb{R}$. To see that it belongs to $T^*_g \mathcal{C}$, pick a representative $g_{ab} \in \mathcal{g}$, let
Then for any \( v \in T_u^C \)

\[
\langle A, v \rangle = \int_M \left( \alpha / dV_x \right)_u v_x \, dV_x = \int_M \langle u, v \rangle_x \omega = \langle \alpha_u(u), v \rangle
\]  

and hence \( \omega (u) = A \in T_u^C \).

To see that \( \omega_u \) is bijective as a map into \( T_u^* C \) we first note that \( \langle \alpha_u(u), u \rangle > 0 \) unless \( u = 0 \) and \( \omega_u \) is therefore injective. Consider an arbitrary \( A \in T_u^* C \) and write \( A = [\gamma_{ab}, A_{ab}]^* \). The previous computation shows that \( \omega_u([\gamma_{ab}, dV_x/\alpha]A_{ab}) = A \) so \( \omega_u \) is surjective as well.

The continuity of \( \omega_u \) is a straightforward consequence of the fact that the right-hand side of (35) defines a continuous map \( C^0(M, S, M) \to (C^0(M, S, M))^* \) and the continuity of the embeddings \( C^0(M, S, M) \hookrightarrow C^0(M, S, M) \) and \( (C^0(M, S, M))^* \hookrightarrow (C^0(M, S, M))^* \). □

A metric \( g_{ab} \) determines a volume form \( \omega \) and there is a one-to-one correspondence between metrics and pairs \( (g, \alpha) \) of conformal classes and volume forms. So the choice of a volume form \( \alpha \) in proposition 2.1 can be thought of, at least when working with \( T_u^* C \) for some fixed \( g \), as a choice of representative metric within the conformal class.

2.4. The divergence

We have previously defined the conformal Killing operator associated with a conformal class \( g \).

\[
L_g^*: C^\infty (M, TM) \to T_g C.
\]  

(41)

This is a continuous linear map, and hence we obtain a continuous adjoint

\[
L_g^{**}: (T_g C)^* \to (C^\infty (M, TM))^*
\]  

(42)

given by

\[
\langle L_g^*(F), X \rangle = \langle F, L_g X \rangle.
\]  

(43)

We define the divergence

\[
\text{div}_g = -\frac{1}{2} L_g^{**}
\]  

(44)

Note that if \( A = [\gamma_{ab}, A_{ab}]^* \) is a smooth cotangent vector and \( X^a \) is a smooth vector field then, using the definition of \( L_{\text{tr}} \) and integration by parts, we find

\[
\langle \text{div}_{\text{tr}}(A), X \rangle = -\frac{1}{2} \langle A, L_{\text{tr}} X \rangle
\]  

\[
= -\frac{1}{2} \left\langle \left[ g_{ab}, A_{ab} \right]^*, \left[ g_{ab}, L(X)X \right] \right\rangle
\]  

\[
= -\frac{1}{2} \int_M \langle A, L(X)X \rangle_x dV_x
\]  

\[
= \int_M \langle \text{div}(A), X^a \rangle_x dV_x.
\]  

(45)
2.5. Quotients modulo flows

The space of conformal geometries, sometimes called conformal superspace, is the quotient of $\mathcal{C}$ obtained by identifying conformal classes if there is a flow taking one to another. We will write this quotient symbolically as $C/D_0$ (here $D_0$ is the connected component of the identity of the diffeomorphism group). Because the Einstein equations are diffeomorphism invariant, the space $C/D_0$ is more fundamental than $\mathcal{C}$, and it will be important to work with tangent and cotangent vectors to this space.

Suppose we have a curve $\gamma$ of conformal classes obtained by a flow. Its tangent vector at $g = \gamma(0)$ is then $L_g X$ for some vector field $X^a$. Since $\gamma$ descends to a stationary curve in $C/D_0$, the directions $L Im g$ become null directions in $C/D_0$, which motivates the following.

**Definition 3.** Let $g \in \mathcal{C}$. The space of conformal geometric velocities at the conformal geometry represented by $g$ is the quotient space

$$T_g C/\text{Im} L_g.$$  \hfill (46)

The conformal geometric velocity represented by a conformal tangent vector $u$ is the subspace

$$[u] = u + \text{Im} L_g$$  \hfill (47)

of $T_g C$. We write $T_g (C/D_0)$ for the set of conformal velocities at the conformal geometry represented by $g$.

Note we are deliberately avoiding working with equivalence classes $[g]$ of conformal classes under flows, and that each representative of $[g]$ gives a representation $T_g (C/D_0)$ of an object that would be written as $T_{[g]} (C/D_0)$.

Every conformal tangent vector $u \in T_g C$ naturally determines the conformal geometric velocity $u + \text{Im} L_g$. Fixing a representative metric $g_{ab}$ of $g$, the conformal geometric velocities at $g$ are naturally identified with the subspaces

$$u_{ab} + \text{Im} L_g$$  \hfill (48)

of $C^\omega (\mathcal{M}, S_2 \mathcal{M})$ where $u_{ab}$ is trace-free with respect to $g_{ab}$.

Elements of the dual space $(T_g C/\text{Im} L_g)^*$ can be represented as elements of the subspace of $(T_g C)^*$ that annihilate $\text{Im} L_g$. Restricting our attention to those elements that are also smooth conformal cotangent vectors we have the following.

**Definition 4.** Let $g \in \mathcal{C}$. The space of conformal geometric momenta at the conformal geometry represented by $g$ is the subspace of $T_g^* C$ consisting of those elements that vanish on $\text{Im} L_g$. We denote this subspace by $T_g^* (C/D_0)$.

The subspace $T_g^* (C/D_0) \subseteq T_g^* C$ of conformal geometric momenta can be characterized in terms of the divergence $\text{div} \sigma$ and this leads to the notion of a transverse traceless tensor. A symmetric tensor $\sigma_{ab}$ is said to be transverse traceless (TT) with respect to a metric $g_{ab}$ if it is traceless,

$$g^{ab} \sigma_{ab} = 0,$$  \hfill (49)

and transverse,

$$\text{div} \sigma = 0.$$  \hfill (50)
Lichnerowicz observed [Li44] that TT tensors behave well with respect to conformal transformations: if \( \sigma_{ab} \) is TT with respect to \( g_{ab} \), then \( \sigma_{\phi} \sigma_{\tilde{a} \tilde{b}} = - \frac{1}{2} g_{ab} q_{\tilde{a} \tilde{b}} \) is TT with respect to \( \phi = - g_{ab} q_{ab} \). From this conformal transformation law we identify

\[
\sigma = [g_{ab}; \sigma_{ab}]^g
\]  

as a smooth conformal cotangent vector. Moreover, equation (45) implies that \( \text{div}_g \sigma = 0 \).

The following easy lemma shows that the TT tensors represent the smooth conformal cotangent vectors that annihilate the image of the conformal Killing operator; we omit the proof.

**Lemma 2.2.** For \( \sigma \in T^*_g C \) the following are equivalent.

1. \( \text{div}_g \sigma = 0 \).
2. For all smooth vector fields \( X^a \)
   \[
   \{ \sigma, L_g X \} = 0.
   \]  

3. For some \( g_{ab} \) and \( \sigma_{ab} \) with \( \sigma = [g_{ab}; \sigma_{ab}]^g \), \( \sigma_{ab} \) is TT with respect to \( g_{ab} \).
4. For all \( g_{ab} \) and \( \sigma_{ab} \) with \( \sigma = [g_{ab}; \sigma_{ab}]^g \), \( \sigma_{ab} \) is TT with respect to \( g_{ab} \).

As a consequence of lemma 2.2 we have shown

\[
T^*_g (C/\mathcal{D}_g) = \{ \sigma \in T^*_C: \text{div}_g \sigma = 0 \}.
\]  

So transverse traceless tensors are the representations, in terms of a background metric, of conformal geometric momenta.

We have seen that a conformal tangent vector \( u \) naturally defines a conformal geometric velocity \( u + \text{Im} L_g \in T^*_g (C/\mathcal{D}_g) \). On the other hand, an arbitrary conformal cotangent vector \( A \) does **not** naturally determine a conformal geometric momentum: this would require a choice of projection from \( T^*_g C \) onto the subspace \( T^*_g (C/\mathcal{D}_g) \). Our next goal is to describe a family of such projections that are closely related to the maps \( \omega_k \) from proposition 2.1. To begin, we recall the following result from [Yo73], which is a fundamental component of the 1974 conformal method.

**Proposition 2.3.** (York Splitting) Let \( g_{ab} \) be a smooth Riemannian metric on \( M \) and let \( A_{ab} \) be a smooth, trace-free, symmetric (0,2)-tensor. Then there is a smooth TT tensor field \( \sigma_{ab} \) and a smooth vector field \( X^a \) such that

\[
A_{ab} = \sigma_{ab} + \left( L_g X \right)_{ab}.
\]  

This decomposition is unique up to the addition of a conformal Killing field to \( X^a \).

Notice that the right-hand side of equation (54) does not have a natural interpretation as a conformal object: it is the sum of a representative \( \sigma_{ab} \) of a conformal cotangent vector with a representative \( \left( L_g X \right)_{ab} \) of a conformal tangent vector. Adding these together requires an identification of \( T^*_g C \) with \( T^*_g C \). We can reformulate proposition 2.3, however, in terms of conformal objects using the maps \( k_g \) defined in proposition 2.1.

**Proposition 2.4.** Let \( g \in C \) and let \( A \in T^*_g C \). Given a choice of a volume form \( \omega \), there is a conformal geometric momentum \( \sigma \) and a vector field \( X^a \) such that
\[ A = \sigma + k_\omega \left( L_\omega X \right), \]  
(55)

where \( k_\omega \) is the map defined in proposition 2.1. The decomposition is unique up to the addition of a conformal Killing field to \( X_a \).

Moreover, if \( g_{ab} \in g \) is the unique metric with \( dV_g = \omega \), and if \( A_{ab} \) and \( \sigma_{ab} \) are the representatives of \( A \) and \( \sigma \) with respect to \( g_{ab} \), then

\[ A_{ab} = \sigma_{ab} + \left( L_\omega X \right)_{ab}. \]  
(56)

**Proof.** Let \( g_{ab} \) be the unique metric in \( g \) with \( dV_g = \omega \). We wish to write equation (55) in terms of representatives with respect to \( g_{ab} \).

Proposition 2.1 equation (36) implies that for any \( \left[ g_{ab}, \psi_{ab} \right] \in T_g C \),

\[ k_\omega (\psi) = \left[ g_{ab}, \psi_{ab} \right]^g; \]  
(57)

this is the step where we use the specific choice of \( g_{ab} \). In particular, for a vector field \( X^a \),

\[ k_\omega \left( L_\omega X \right) = \left[ g_{ab}, \left( L_\omega X \right)_{ab} \right]^g. \]  
(58)

Hence equation (55) is equivalent to finding a TT tensor \( \sigma_{ab} \) and vector field \( X^a \) such that

\[ \left[ g_{ab}, A_{ab} \right]^g = \left[ g_{ab}, \sigma_{ab} \right]^g + \left[ g_{ab}, \left( L_\omega X \right)_{ab} \right]^g, \]  
(59)

where \( A_{ab} \) is the tensor field such that \( A = [g_{ab}, A_{ab}]^g \). In other words, we wish to solve

\[ A_{ab} = \sigma_{ab} + \left( L_\omega X \right)_{ab}, \]  
(60)

and the result now follows from proposition 2.3.

For each choice of volume form \( \omega \), proposition 2.4 determines a projection from the space of conformal cotangent vectors onto the subspace of conformal geometric momenta.

**Definition 5.** Let \( \omega \) be a volume form. For all \( \in C \), the projection \( P_\omega \colon T_g^\omega C \to T_g^\omega (C/D_0) \) is defined by

\[ P_\omega (A) = \sigma \]  
(61)

where \( \sigma \) is the unique conformal geometric momentum determined by equation (55).

The maps \( k_\omega \) each determine identifications of \( T_g C \) with \( T_g^\omega C \). Using the projections \( P_\omega \), we can now define related identifications \( j_\omega \colon T_g (C/D_0) \to T_g^\omega (C/D_0) \) that satisfy

\[ T_g C \xrightarrow{k_\omega^*} T_g^\omega C \]  
\[ \pi \downarrow \quad \uparrow \pi^* \]  
\[ T_g (C/D_0) \leftarrow j_\omega^* \]  
(62)

where \( \pi \) is the natural projection and \( \pi^* \) is the natural embedding.

**Definition 6.** Let \( \omega \) be a volume form. For each \( g \in C \) we define \( j_{\omega} \colon T_g (C/D_0) \to T_g^\omega (C/D_0) \) by
Its inverse is given by

\[ j^{-1}(\sigma) = k^{-1}(\sigma) + \text{Im } L_k. \]  

(64)

One needs to verify that \( j \) is well-defined, but this is an easy consequence of the uniqueness clause of proposition 2.4. Showing that \( j^{-1} \) really is the inverse of \( j \) is also an easy exercise using proposition 2.4 and is left to the reader. Note that the commutative diagram (62) is simply an alternative expression of equation (64).

### 3. The CMC conformal method

Suppose \( (\bar{g}_{ab}, \bar{K}_a) \) is a solution of the constraints such that \( \bar{g}^ab\bar{K}_a = \tau_0 \) for some constant \( \tau_0 \); we say such a solution is constant mean curvature or CMC. Letting \( \sigma_{ab} \) be the trace-free part of \( K_{ab} \) the momentum constraint (1b) then reads

\[ \text{div } \sigma = 0 \]  

and hence \( \sigma_{ab} \) is TT. So

\[ K_{ab} = \sigma_{ab} + \frac{\tau_0}{n} \sigma_{ab} \]  

(66)

for some unique TT tensor \( \sigma_{ab} \) and constant \( \tau_0 \). In this way, every CMC solution determines a unique conformal class \( g = [\bar{g}_{ab}] \), conformal geometric momentum \( \sigma = [\sigma_{ab}, \tau_0] \in T^g_0 (C/D_0) \), and constant \( \tau_0 \). We refer to a triple \( (g, \sigma, \tau_0) \) as CMC conformal data.

The CMC conformal method of [Yo73] seeks to reverse this process: starting from CMC conformal data \( (g, \sigma, \tau_0) \) we wish to construct a CMC solution \( (\bar{g}_{ab}, \bar{K}_a) \) of the constraints with

\[
\begin{bmatrix}
\bar{g}_{ab} \\
\bar{K}_a 
\end{bmatrix} = g \\
\begin{bmatrix}
\bar{g}_{ab} \\
\bar{K}_a 
\end{bmatrix}^\# = \sigma \\
\bar{g}^ab\bar{K}_a = \tau_0.
\]  

(67)

To solve this problem, let \( g_{ab} \) be an arbitrary representative of \( g \) and let \( \sigma_{ab} \) be the unique TT tensor such that

\[ [g_{ab}, \sigma_{ab}] = \sigma. \]  

(68)

If \( (\bar{g}_{ab}, \bar{K}_a) \) is a solution of the constraints satisfying (67), then there is a conformal factor \( \phi \) such that \( \bar{g}_{ab} = \phi^{-\tau} g_{ab} \) and such that \( \bar{K}_a \) satisfies equation (66) with \( \sigma_{ab} = \phi^{-1} \sigma_{ab} \). Writing the constraint equations (1) in terms of \( g_{ab} \) and \( \sigma_{ab} \), we find that the momentum constraint is automatically satisfied and (using the scalar curvature transformation law (5)) the Hamiltonian constraint is equivalent to

\[-2\kappa q \nabla_\phi + R_\phi \phi - |\nabla_\phi \phi|^{-1} + \kappa \phi^{-1} = 0, \]

(Lichnerowicz – York equation)(69)

where \( R_\phi \) is the scalar curvature of \( g_{ab} \) and \( \kappa \) and \( q \) are the dimensional constants defined by equations (4) and (6). Thus we have established the following.

---

3 Since \( \sigma \) determines \( g \) (every cotangent vector determines its base point), this description of conformal data is mildly redundant. Nevertheless, it is useful to have an explicit notation for the conformal class.
Proposition 3.1. (The CMC Conformal Method). Let \((g, \sigma, \tau)\) be CMC conformal data. Suppose \(\sigma\) is an arbitrary representative of the CMC conformal data (i.e., \([g_{ab}] = g\) and \([g_{ab}] = \sigma\)) and suppose \(\phi\) is a positive function solving the Lichnerowicz–York equation (69). Then
\[
\bar{g}_{ab} = \phi^{\frac{4-n}{2}} g_{ab},
\]
\[
\bar{K}_{ab} = \phi^{\frac{4-n}{2}} \sigma_{ab} + \frac{\tau_n}{n} \bar{g}_{ab}
\]
(70)
solve the constraint equations. Moreover,
\[
\begin{bmatrix} \bar{g}_{ab} \\ \bar{K}_{ab} \end{bmatrix} = g,
\]
\[
\begin{bmatrix} \bar{g}_{ab} \\ \bar{K}_{ab} \end{bmatrix} = \sigma,
\]
\[
\bar{g}^{ab} \bar{K}_{ab} = \tau_n.
\]
Conversely, suppose \((\bar{g}_{ab}, \bar{K}_{ab})\) is a solution of the constraint equations such that equations FOO(71) are satisfied. Let \((\bar{g}_{ab}, \sigma_{ab}, \tau_n)\) be any representative of the CMC conformal data and let \(\phi\) be the unique conformal factor such that \(\bar{g}_{ab} = \phi^{\frac{4-n}{2}} g_{ab}\). Then \(\phi\) solves the Lichnerowicz–York equation (69).

As a consequence of proposition 3.1, the set of solutions of the constraints satisfying conditions (71) is in one-to-one correspondence with the set of conformal factors \(\phi\) solving the Lichnerowicz–York equation (69) as expressed with respect to any representative of the CMC conformal data \((g, \sigma, \tau_n)\). This independence with respect to the choice of representative is known in the literature as conformal covariance.

Proposition 3.2. Suppose \((g_{ab}, \sigma_{ab}, \tau_n)\) and \((\bar{g}_{ab}, \bar{\sigma}_{ab}, \bar{\tau}_n)\) are two representatives of the same CMC conformal data, so \(\bar{g}_{ab} = \psi^{\frac{4-n}{2}} g_{ab}\) and \(\bar{\sigma}_{ab} = \psi^{-2} \sigma_{ab}\) for some conformal factor \(\psi\). Then \(\phi\) solves the Lichnerowicz–York equation (69) with respect to \((g_{ab}, \sigma_{ab}, \tau_n)\) if and only if \(\psi^{-1} \phi\) solves the Lichnerowicz–York equation (69) with respect to \((\bar{g}_{ab}, \bar{\sigma}_{ab}, \bar{\tau}_n)\), in which case the corresponding solution \((\bar{g}_{ab}, \bar{\sigma}_{ab})\) of the constraints in both cases is the same.

Proof. Suppose \(\phi\) solves the Lichnerowicz–York equation with respect to \((g_{ab}, \sigma_{ab}, \tau_n)\) and let \((\bar{g}_{ab}, \bar{K}_{ab})\) be defined by equations (70). The forward implication of proposition 3.1 implies \((\bar{g}_{ab}, \bar{K}_{ab})\) is a solution of the constraints satisfying (71).

Since \(\bar{g}_{ab} = \psi^{\frac{4-n}{2}} g_{ab}\) and \(\bar{\sigma}_{ab} = \psi^{-2} \sigma_{ab}\) it follows that \(\bar{g}_{ab} = (\phi/\psi)^{\frac{4-n}{2}} \tilde{g}_{ab}\) and hence the reverse implication of proposition 3.1 implies that \(\phi \psi^{-1}\) solves the Lichnerowicz–York equation (69) with respect to \((\bar{g}_{ab}, \bar{\sigma}_{ab}, \bar{\tau}_n)\). The solution of the constraints generated by \(\phi \psi^{-1}\) given by equation (70) is
\[
\bar{g}_{ab} = (\phi/\psi)^{\frac{4-n}{2}} \tilde{g}_{ab} = \phi^{\frac{4-n}{2}} g_{ab} = g_{ab},
\]
\[
\bar{K}_{ab} = (\phi/\psi)^{\frac{4-n}{2}} \tilde{\sigma}_{ab} + \frac{\tau_n}{n} \tilde{g}_{ab} = \phi^{-2} \sigma_{ab} + \frac{\tau_n}{n} \bar{g}_{ab} = \bar{K}_{ab}.
\]
(72)

The celebrated property of the CMC conformal method is that given representative CMC conformal data \((g_{ab}, \sigma_{ab}, \tau_n)\) there is (generically) exactly one solution of the Lichnerowicz–York equation, so there is effectively a one-to-one correspondence between CMC
conformal data and CMC solutions of the constraints. This result (accomplished over many years by several authors including York and Choquet-Bruhat, and completed and summarized by Isenberg in [Is95]) can be expressed in terms of conformal objects (independent of a choice of background metric) as follows.

**Theorem 3.3.** (CMC parameterization). Let \((g, \sigma, \tau_0)\) be CMC conformal data. Then there exists a unique solution \((g_{ab}^\text{CMC}, \underline{K}_{ab})\) of the vacuum Einstein constraint equations satisfying conditions (71) except in the following cases:

- \(g\) is Yamabe positive and \(\sigma = 0\), in which case there is no solution,
- \(g\) is Yamabe negative and \(\tau_0 = 0\), in which case there is no solution,
- \(g\) is Yamabe null and \(\sigma = 0\) or \(\tau_0 = 0\), in which case there is no solution (unless both are zero, in which case there is a one-parameter family of solutions consisting of solution metrics \(g_{ab}\) all homothetically related to a single metric with vanishing scalar curvature and with solution extrinsic curvatures \(\underline{K}_{ab}\) all vanishing identically).

### 4. The 1974 conformal method

Let \(\omega\) be a fixed volume form, and suppose \((g_{ab}^\text{CMC}, \underline{K}_{ab})\) is a solution of the constraint equations. The solution and the choice of \(\omega\) uniquely determine the following:

\[
g = \begin{bmatrix} \underline{g}_{ab} \end{bmatrix}, \tag{73a}
\]

\[
\sigma = P_\omega \left( \begin{bmatrix} \sigma_{ab}, \underline{K}_{ab} \end{bmatrix}^\omega \right), \tag{73b}
\]

\[
\tau = \underline{g}^{ab} \underline{K}_{ab}. \tag{73c}
\]

where \([\underline{g}_{ab}, \underline{K}_{ab}]^\omega\) is defined at the end of definition 2 and the projection \(P_\omega\) comes from definition 5.

We call a tuple \((g, \sigma, \tau, \omega)\) 1974 conformal data. Although it is not usually presented this way, the 1974 conformal method attempts to reverse this process: starting from conformal data \((g, \sigma, \tau, \omega)\) we seek a solution \((g_{ab}^\text{CMC}, \underline{K}_{ab})\) of the constraints satisfying conditions (73).

Suppose \((\underline{g}_{ab}, \underline{K}_{ab})\) is a solution of the constraints satisfying conditions (73) and let \(\underline{K}_{ab}\) be the trace-free part of \(\underline{K}_{ab}\), so \(\underline{K}_{ab} = A_{ab} + (\tau/n) \underline{g}_{ab}\). Equation (73b) is equivalent to the existence of a vector field \(W^a\) such that

\[
\begin{bmatrix} \underline{g}_{ab}, \underline{A}_{ab} \end{bmatrix} = \sigma + k_a (L_g W). \tag{74}
\]

Let \(g_{ab}^\text{CMC}\) be the unique element of \(g\) with \(dV' = \omega\), and let \(A_{ab}\) be the representative of \([\underline{g}_{ab}, \underline{A}_{ab}]^\omega\) with respect to \(g_{ab}^\text{CMC}\) (i.e., \(\underline{A}_{ab} = \phi^{-2} A_{ab}\)). From our specific choice of \(g_{ab}^\text{CMC}\), proposition 2.4 implies that equation (74) is equivalent to

\[
A_{ab} = \sigma_{ab} + (L_g W)_{ab} \tag{75}
\]

where \(\sigma_{ab}\) is the representative of \(\sigma\) with respect to \(g_{ab}^\text{CMC}\). We then have

\[
\underline{g}_{ab} = \phi^{n-2} g_{ab} \tag{76a}
\]

\[
\underline{K}_{ab} = \phi^{-2} \left[ \sigma_{ab} + (L_g W)_{ab} \right] + \frac{\tau}{n} \underline{g}_{ab}. \tag{76b}
\]
The preceding discussion is reversible, so we have shown that equations (73) are equivalent to
the existence a conformal factor $\phi$ and a vector field $W^a$ such that conditions (76) hold, so
long as $g_{ab}$ is the representative of $g$ with $dV = \omega$.

Substituting equations (76) into the constraint equations (1) we find that $(\tilde{g}_{ab}, \tilde{K}_{ab})$ is a
solution of the constraints if and only if $\phi$ and $W$ satisfy

\[-2\kappa q \Delta \phi + R_c \phi - \left[ \sigma + L_g W^g \right] \phi^{-1} + \kappa \tau \phi^{r-1} = 0 \quad \text{[1974 Hamiltonian constraint]} \tag{77a}\]

\[\text{div}_g L_g W = k \phi^r \sigma. \quad \text{[1974 momentum constraint]} \tag{77b}\]

These equations, which first appeared in [ÖMY74], will be called the 1974 conformally parameterized constraint equations, though we note that they have various other names in the
literature, including the LCBY equations named after Lichnerowicz, Choquet-Bruhat and
York. We have described how their solutions correspond to the solutions of the constraints
solving conditions (73), and summarize this discussion as follows.

**Proposition 4.1.** (1974 Conformal Method).

Let $(g, \sigma, \tau, \omega)$ be 1974 conformal data.

Let $g_{ab} \in g$ be the unique representative with $dV = \omega$ and let $\sigma_{ab}$ be the representative of
$\sigma$ with respect to $g_{ab}$.

Suppose $\phi$ and $W^a$ solve the 1974 conformally parameterized constraint equations (77) with respect to $g_{ab}$ and $\sigma_{ab}$. Then $(\tilde{g}_{ab}, \tilde{K}_{ab})$ defined by equations (76) satisfy the constraint equations (1) and satisfy

\[\left[ \tilde{g}_{ab} \right] = g, \quad (78a)\]

\[P_g \left( \left[ \tilde{g}_{ab}, \tilde{K}_{ab} \right] \right) = \sigma, \quad (78b)\]

\[\tilde{g}^{ab} \tilde{K}_{ab} = \tau. \quad (78c)\]

Conversely, suppose $(\tilde{g}_{ab}, \tilde{K}_{ab})$ is a solution of the constraints satisfying conditions (78). Then there exists a conformal factor $\phi$ and vector field $W^a$ (both unique up to addition of a conformal Killing field to $W^a$) such that the decomposition (76) holds and the 1974
conformally parameterized constraint equations (with respect to $g_{ab}$ and $\sigma_{ab}$) are satisfied.

Each choice of volume form $\omega$ leads to an independent parameterization of the set of solutions of the constraints in the sense that once $\omega$ is fixed, every solution $(\tilde{g}_{ab}, \tilde{K}_{ab})$ is associated with exactly one tuple $(g, \sigma, \tau, \omega)$ via equations (78). The reverse implication need not be true, however. As mentioned in the introduction, for 1974 conformal data where $\tau$ is not nearly constant it is generally unknown how many solutions of the constraints are associated with this data.

In the usual presentation of the 1974 conformal method the representative conformal data consist of a metric $g_{ab}$, a TT tensor $\sigma_{ab}$, and a mean curvature $\tau$, and we begin by writing down the corresponding 1974 conformally parameterized constraint equations. The triple $(g_{ab}, \sigma_{ab}, \tau)$ appears to be analogous to representative data for the CMC conformal method, but representative data determines conformal data $(g, \sigma, \tau, \omega)$ as follows:
\( g = [ g_{ab} ] \)
\( \sigma = [ g_{ab}, \sigma_{ab} ] \)
\( \tau = \tau \)
\( \omega = dV_c \)  \( (79) \)

Note that compared to the CMC conformal method, the choice of metric now plays two roles: it selects the conformal class of the solution metric and the choice of volume form \( \omega \) in proposition 2.4. In this second role, it determines a choice of identification of \( T_g \mathcal{C} \) with \( T_g^* \mathcal{C} \). Although the choice of volume form \( \omega \) and the choice of background metric \( g_{ab} \) used to write down the PDEs (77) are tightly connected in the 1974 conformal method, there is no particular reason why this needs to be the case. Indeed, there are good reasons to decouple these two roles. Given an arbitrary \( \omega \), one might want to work with a metric different from the one for which \( dV_c = \omega \); it may be more expedient to work with a metric with, e.g., positive scalar curvature instead. The problem of finding a solution of the constraint equations satisfying conditions (78) does not depend on the choice of a background metric. But the 1974 conformally parameterized constraint equations themselves do depend on the choice \( dV_c = \omega \). If we work with a different background metric, these equations will change, and we will see that this is the connection between the 1974 conformal method and the Hamiltonian formulation of the CTS method.

When expressed in terms of representative conformal data, the 1974 method appears to lack conformal covariance. If we start with representative data \( (g_{ab}, \sigma_{ab}, \tau) \) and conformally change to representative data \( \left( \phi^{-2} g_{ab}, \phi^{-2} \sigma_{ab}, \tau \right) \), there is no reason to expect that the corresponding solutions of the constraints will be the same. In terms of conformal objects, this transformation is equivalent to fixing \( (g, \sigma, \tau) \) but changing the choice of \( \omega \). Each choice of \( \omega \) gives a separate parameterization of the solutions of the constraint equations, and the parameterizations can be different from each other. Indeed, recent work [Ma14] shows that the 1974 conformal method parameterizes flat Kasner data in fundamentally different ways depending on the choice of volume form. Certain data \( (g, \sigma, \tau) \) generate one-parameter families of solutions for some volume forms, but generate only a single solution for others. So the choice of \( \omega \) is an important part of the parameterization. However, the task defined by the 1974 conformal method,

\[ \text{Find a solution of the constraints satisfying conditions (78).} \]

\( \text{can be expressed in terms of conformal objects and therefore is by necessity conformally} \]
\( \text{covariant; the issue is simply how to express the problem when using a representative metric} \]
\( \text{different from the one with } dV_c = \omega. \)

Finally, we observe that although the choice of \( \omega \) is important for the 1974 conformal method, if we restrict to constant mean curvature data \( \tau = \tau_0 \), then the choice of \( \omega \) is irrelevant. The 1974 conformally parameterized momentum constraint (77b) reads

\[ \text{div} L W = 0 \]  \( (80) \)

which is solved exactly by conformal Killing fields (i.e., \( (L W)_{ab} = 0 \)). So there is no longer any ambiguity about adding tangent and cotangent vectors in the expression

\[ A_{ab} = \sigma_{ab} + (L_s W)_{ab} \]  \( (81) \)

from equation (76b) and the choice of volume form is no longer needed. The 1974 conformally parameterized Hamiltonian constraint (77a) is

\[ -2 \kappa q \Delta \phi + R_s \phi - |\sigma|^2 \phi^{-1} + \kappa \tau_0^2 \phi^{-1} = 0, \]  \( (82) \)
5. The conformal thin-sandwich method

The thick-sandwich conjecture, in the vacuum setting, states that given two metrics \( g_{ab}^0 \) and \( g_{ab}^1 \) on \( M \) one can find a globally hyperbolic Ricci-flat Lorentzian spacetime, unique up to diffeomorphism, and two disjoint spacelike hypersurfaces of the spacetime, such that the induced metrics on the hypersurfaces are the given two metrics. As described in [BF93], there are reasons to doubt the validity of this conjecture. It was also shown in [BF93] that an infinitesimal variation, known as the thin-sandwich conjecture, turns out to hold under limited circumstances. The thin-sandwich conjecture asserts that given \( g_{ab} \) and its Lie derivative \( \dot{g}_{ab} = \nabla^a v^b + \nabla^b v^a \) with respect to some (to be determined) future pointing time like vector field \( v^a \) along the surface, there is a unique Ricci-flat spacetime containing a slice satisfying the initial conditions. Writing \( T = \tau \mathbf{e}^a + X^a \) where \( \tau \) is the future pointing unit normal to the surface and \( X^a \) is a vector field tangential to the surface (i.e., in terms of the lapse \( N \) and shift \( X^a \)) we have

\[
\dot{g}_{ab} = 2N \overline{K}_{ab} + L_{\alpha} \overline{g}_{ab}. \tag{83}
\]

So the goal is to find \( (g_{ab}, \overline{K}_{ab}) \) solving the constraints, along with a choice of \( N \) and \( X^a \), such that \( g_{ab} \) is the given metric and such that (83) holds. Using a perturbative technique, the authors of [BF93] exhibited an open set of data \( (\overline{g}_{ab}, \overline{K}_{ab}) \), with additional restrictions on the scalar curvature of \( g_{ab} \), for which the conjecture is true. They also conjecture, however, that the thin-sandwich conjecture is not well-posed in general.

York's CTS method [Yo99] is based on a conformal version of the thin-sandwich conjecture. Given a conformal class \( g \) and a conformal tangent vector \( u \in T_C \), we wish to find a solution \( (g_{ab}, K_{ab}) \) of the vacuum Einstein constraint equations along with a lapse \( N \) and a shift \( X^a \) such that

\[
\begin{align*}
\left[ g_{ab} \right] &= g, \quad \tag{84} \\
\left[ g_{ab}, \dot{g}_{ab} \right] &= u, \quad \tag{85}
\end{align*}
\]

where \( \dot{g}_{ab} \) is defined by equation (83) and (as noted at the end of definition 1) \( [\overline{g}_{ab}, \overline{K}_{ab}] \) should be thought of as the pushforward of the tangent vector \( g_{ab} \) to an element of \( T^*_C \). One hopes that specification of \( (g_{ab}, u) \), along with information about the trace part of \( K_{ab} \) and the coordinate freedom in \( (N, X^a) \), results in a unique solution of the constraint equations.

Let \( g \in C \) and \( u \in T_C \) be given and suppose \( (\overline{g}_{ab}, \overline{K}_{ab}, N, X^a) \) satisfies equations (84), and (85) (with \( \overline{g}_{ab} \) defined by equation (83)). Let \( \tau \) be the trace of \( K_{ab} \) so

\[
K_{ab} = \overline{K}_{ab} + \left( \tau \mathbf{e}^a \right), \tag{86}
\]

for some unique trace-free tensor \( \overline{K}_{ab} \). Decomposing equation (83) into its trace-free and trace parts we obtain

\[
\overline{g}_{ab} = \left( 2\overline{N} \overline{K}_{ab} + \left( L_\tau X \right) + \left( \overline{N} \tau + \text{div}_X \right) \right) + \left( \overline{N} \tau + \text{div}_X \right) \overline{g}_{ab}, \tag{87}
\]
so equation (85) is equivalent to

\[ 2\mathcal{N} \vec{\pi}_{ab} + \left( L_\tau X \right)_{ab} = \pi_{ab} \quad (88) \]

where \( \pi_{ab} \) is the representative of \( u \) with respect to \( g_{ab} \). Equation (88) can be solved for \( \vec{\pi}_{ab} \) to obtain

\[ \vec{\pi}_{ab} = \frac{1}{2\mathcal{N}} \left[ \vec{\pi}_{ab} - \left( L_\tau X \right)_{ab} \right] \quad (89) \]

and the constraint equations (1) can be written in terms of \( \vec{\pi}_{ab} \) and \( \tau \) to obtain

\[ R_\tau - \frac{1}{2} \frac{\kappa}{\tau} + \kappa \tau^2 = 0 \quad \text{div}_\tau \vec{A} = \kappa \text{d}r. \quad (90) \]

Substituting equation (89) into equations (90) we have

\[ R_\tau - \frac{1}{2\mathcal{N}} \left[ \vec{\pi} - \left( L_\tau X \right) \right] + \kappa \tau^2 = 0 \quad (91a) \]

\[ \text{div}_\tau \left[ \frac{1}{2\mathcal{N}} \left[ \vec{\pi} - \left( L_\tau X \right) \right] \right] = \kappa \text{d}r. \quad (91b) \]

York’s prescription for solving these equations can be described as follows. Pick an arbitrary \( g_{ab} \in \mathfrak{g} \) and let \( u_{ab} \) be the representative of \( u \) with respect to \( g_{ab} \). The solution metric \( \vec{g}_{ab} \) is then related to \( g_{ab} \) via an as-yet unknown conformal factor \( \phi \) via \( \vec{g}_{ab} = \phi^{r-2} g_{ab} \), and we set \( \pi_{ab} = \phi^{r-2} u_{ab} \) so that \( [\vec{g}_{ab}, \pi_{ab}] = [g_{ab}, u_{ab}] = u \). The shift \( X^a \) is the other unknown, and the remaining quantities \( \mathcal{N} \) and \( \tau \) are treated as parameters. The mean curvature \( \tau \) is specified directly, but the lapse obeys a non-trivial conformal transformation law: \( \mathcal{N} = \phi^r \mathcal{N} \), where \( \mathcal{N} \) is a given positive function.

Rewriting equations (91) in terms of \( \phi, X, g_{ab}, u_{ab}, \tau, \) and \( \mathcal{N} \), we obtain the CTS equations

\[ -2\kappa \phi \Delta \phi + \frac{1}{2\mathcal{N}} \left( \phi^{r-1} + \kappa r^2 \phi^{r-1} \right) \quad \text{[CTS Hamiltonian constraint]} \quad (92a) \]

\[ -\text{div} \left[ \frac{1}{2\mathcal{N}} \left( L_\tau X \right) \right] = -\text{div} \left[ \frac{1}{2\mathcal{N}} \left( u \right) \right] + \kappa \phi \text{d}r \quad \text{[CTS momentum constraint]} \quad (92b) \]

to be solved for \( \phi \) and \( X^a \).

The conformally transforming lapse is the key ingredient of the CTS method, and can be motivated by examining the term \( \tau - L_\tau X \) appearing in the momentum constraint of equation (91). This term represents a conformal tangent vector, e.g.

\[ \pi_{ab} = \left( L_\tau X \right)_{ab} \phi^{r-2} \left[ u_{ab} - \left( L_\tau X \right)_{ab} \right]. \quad (93) \]

The divergence, however, naturally acts on conformal cotangent vectors, so we should have

\[ \frac{1}{2\mathcal{N}} \left[ \vec{\pi}_{ab} - \left( L_\tau X \right)_{ab} \right] = \phi^{r-2} \frac{1}{2\mathcal{N}} \left[ u_{ab} - \left( L_\tau X \right)_{ab} \right]. \quad (94) \]

Comparing equations (93) and (94) we arrive at York’s transformation law \( \mathcal{N} = \phi^r \mathcal{N} \).
A conformally transforming lapse is a conformal object associated with a conformal class \( g \), and it will be useful to introduce the following notation.

**Definition 7.** A densitized lapse is an element of \( \left( M \times C^\infty(M) \right) / \sim \) where
\[
\left( \tilde{g}_{ab}, \tilde{N} \right) \sim \left( g_{ab}, N \right)
\]
if there is a smooth positive function \( \phi \) on \( M \) with \( \tilde{g}_{ab} = \phi^{1/2} g_{ab} \) and \( \tilde{N} = \phi^{-1} N \). We use the following notation:

- \( \left[ g_{ab}, N \right] \) is the densitized lapse determined by \( \left( g_{ab}, N \right) \).
- \( \mathcal{N}_g \) is the set of all densitized lapses \( \left[ g_{ab}, N \right] \) with \( g_{ab} \in g \).
- \( \mathcal{N} = \bigcup_{g \in \mathcal{C}} \mathcal{N}_g \).

A tuple \( (g, u, \tau, N) \) where \( g \in \mathcal{C} \), \( u \in T^*_g \mathcal{C} \) and \( N \in \mathcal{N}_g \) is called CTS data, and \( \left( g_{ab}, u_{ab}, \tau, N \right) \) is a representative if \( \left[ g_{ab} \right] = g \), \( \left[ g_{ab}, u_{ab} \right] = u \) and \( \left[ g_{ab}, N \right] = N \). With this notation, we summarize the previous discussion as follows.

**Proposition 5.1.** (The CTS Method) Let \( (g, u, \tau, N) \) be CTS data. Suppose \( \left( g_{ab}, u_{ab}, \tau, N \right) \) is a representative of the CTS data. If \( \phi \) and \( X^a \) solve the CTS equations (92) with respect to \( \left( g_{ab}, u_{ab}, \tau, N \right) \), then
\[
\tilde{g}_{ab} = \phi^{1/2} g_{ab},
\]
\[
\tilde{K}_{ab} = \phi^{-1/2} \left[ u_{ab} - (UX)_{ab} \right] + \frac{1}{2N} \tilde{g}_{ab}
\]
solve the constraint equations. Moreover, setting \( \tilde{N} = \phi^{-1} N \) and
\[
\tilde{\tau} = 2\tilde{N} \tilde{K}_{ab} + L_X \tilde{g}_{ab},
\]
we have
\[
\left[ \tilde{g} \right] = g,
\]
\[
\left[ \tilde{g}, \tilde{u} \right] = u,
\]
\[
\left[ \tilde{g}, \tilde{\tau} \right] = \tau,
\]
\[
\left[ \tilde{g}, \tilde{N} \right] = N.
\]

Conversely, suppose \( \left( \tilde{g}_{ab}, \tilde{K}_{ab} \right) \) are solutions of the constraint equations (1) and that \( \tilde{N} \) and \( X^a \) are a lapse and a shift such that conditions (98) hold. Let \( \left( g_{ab}, u_{ab}, \tau, N \right) \) be any representative CTS data for \( (g, u, \tau, N) \) and let \( \phi \) be the unique conformal factor such that \( \tilde{g}_{ab} = \phi^{1/2} g_{ab} \). Then \( \left( \phi, X^a \right) \) solve the CTS equations (92) with respect to \( \left( g_{ab}, u_{ab}, \tau, N \right) \) and equations (96) hold.

The CTS method is conformally covariant in the sense that if we change to a second background metric, and conformally transform the remaining representative conformal data to represent the same conformal objects, the resulting set of solutions of the constraint equations are the same.

**Proposition 5.2.** Let \( \left( \tilde{g}_{ab}, u_{ab}, \tau, N \right) \) and \( \left( \tilde{g}_{ab}, \tilde{u}_{ab}, \tau, \tilde{N} \right) \) be representative CTS data both corresponding to the same CTS data \( (g, u, \tau, N) \), and let \( \psi \) be the unique conformal factor such that \( \tilde{g}_{ab} = \psi^{1/2} g_{ab} \). Then \( \left( \psi, X^a \right) \) solves the CTS equations (92) with respect to \( \left( g_{ab}, u_{ab}, \tau, N \right) \) if and only if \( \left( \psi^{-1}, \phi \right) \) solves the CTS equations with respect to
(\tilde{\eta}_{ab}, \tilde{\eta}_{ab}, \tau, \tilde{N}), and the corresponding solution \((\tilde{\eta}_{ab}, \tilde{K}_{ab})\) of the Einstein constraint equations in both cases is the same.

**Proof.** The proof is analogous to that of proposition 3.2. □

Each choice of densitized lapse yields an independent parameterization of the set of solutions of the constraint equations in the sense that once the densitized lapse \(N\) is fixed, each solution of the constraints is associated with a tuple of CTS data \((\eta_{ab}, u, \tau, N)\), and this data is unique up to adding an element of \(\text{Im} \, L_{g}\) to \(u\).

**Proposition 5.3.** Let \(g\) be a conformal class and let \(N \in \mathcal{N}_{g}\). Suppose \((\eta_{ab}, K_{ab})\) is a solution of the constraints with \(\eta_{ab} \in g\). Then \((\eta_{ab}, K_{ab})\) is generated by CTS data \((\eta_{ab}, u, \tau, N)\) if and only if \(\tau = g^{-1}K_{ab}\) and \(u \in [\eta_{ab}, 2N, K_{ab}] + \text{Im} \, L_{g}\) where \(N\) is the representative of \(N\) with respect to \(g_{ab}\).

**Proof.** Let \((\eta_{ab}, K_{ab})\) be a solution of the constraints with \(\eta_{ab} \in g\), and let \(N\) be the representative of \(N\) with respect to \(g_{ab}\). From proposition 5.1 we see that \((g, u, \tau, N)\) generates \((\eta_{ab}, K_{ab})\) if and only if \(\tau = g^{-1}K_{ab}\) and there is a vector field \(X^a\) such that

\[
\dot{\eta}_{ab} = 2N \, K_{ab} + L_{N} \eta_{ab}
\]

(99)

satisfies

\[
\left[ \eta_{ab}, \dot{\eta}_{ab} \right] = u.
\]

(100)

Given a vector field \(X^a\),

\[
\left[ \eta_{ab}, 2N \, K_{ab} + L_{N} \eta_{ab} \right] = \left[ \eta_{ab}, 2N \, K_{ab} \right] + \left[ \eta_{ab}, L_{N} \eta_{ab} \right] = \left[ \eta_{ab}, 2N \, K_{ab} \right] + \left[ \eta_{ab}, L_{\tau} X \right] = \left[ \eta_{ab}, 2N \, K_{ab} \right] + L_{g} X.
\]

(101)

Thus \((g, u, \tau, N)\) generates \((\eta_{ab}, K_{ab})\) if and only if \(\tau = g^{-1}K_{ab}\) and

\[
u \in \left[ \eta_{ab}, 2N \, K_{ab} \right] + \text{Im} \, L_{g}.
\]

(102)

Proposition 5.3 shows that the true parameters for the CTS method are a conformal class \(g\), a mean curvature \(\tau\), a densitized lapse \(N\), and an element of \(T_{g}C/\text{Im} \, L_{g}\), i.e., an element of \(T_{g} \left( C/\mathcal{O}_{0} \right)\) from definition 3. After selecting a densitized lapse \(N\), a tuple

\[
(g, u + \text{Im} \, L_{g}, \tau, N)
\]

(103)

of geometric CTS data is uniquely determined by a solution \((\eta_{ab}, K_{ab})\) of the constraints, and the CTS method attempts to invert this map.

### 6. The Hamiltonian formulation of the CTS method

The CTS method was presented by York as a Lagrangian alternative to the standard (Hamiltonian) conformal method. Subsequently Pfeiffer and York demonstrated a
Hamiltonian approach [PY03] to the CTS method that will allow us to link the CTS method and to the 1974 conformal method. We will call the method described here the CTS-H method (and will call the original CTS approach the CTS-L method if emphasis on its Lagrangian nature is desired).

Although not presented this way in [PY03], the key to the CTS-H method is the introduction of a lapse-dependent way of translating between conformal tangent vectors and smooth conformal cotangent vectors defined as follows.

**Definition 8.** Let $g$ be a conformal class and let $N$ be a densitized lapse. Given a conformal velocity $u \in T^*_g C$, we wish to identify it with an element of $\delta T^*_g C$. To do this, let $g_{ab}$ be an arbitrary representative of $g$ and let $u_{ab}$ and $N$ be the representatives of $u$ and $N$ with respect to $g_{ab}$. We then define

$$ k_N(u) = \left[ g_{ab}, (1/2N) u_{ab} \right]^g. $$

(104)

To see that $k_N$ is well-defined, suppose we use a different representative metric $\tilde{g}_{ab} = \phi^{1/2} g_{ab}$. Then $\tilde{u}_{ab} = \phi^{1/2} u_{ab}$ and $\tilde{N} = \phi N$ so

$$ \frac{1}{2N} \tilde{u}_{ab} = \frac{1}{2\phi N} \phi^{1/2} u_{ab} = \phi^{-1} \frac{1}{2N} u_{ab}. $$

(105)

Hence

$$ \left[ g_{ab}, \left( 1/2N \right) \tilde{u}_{ab} \right] = \left[ \phi^{1/2} g_{ab}, \phi^{-1} (1/2N) u_{ab} \right]^g = \left[ g_{ab}, (1/2N) u_{ab} \right]^g $$

as needed.

The map $k_N$ plays the same role for the CTS-H method as $k_\omega$ does for the 1974 conformal method, and in fact there is a way to identify densitized lapses and volume forms such that the corresponding maps $k_N$ and $k_\omega$ are identical.

**Proposition 6.1.** Suppose $N \in \mathcal{N}_g$ for some conformal class $g$. Let $g_{ab}$ be an arbitrary representative of $g$, let $N$ be the representative of $N$ with respect to $g_{ab}$, and let

$$ \omega = \frac{1}{2N} dV_g. $$

(107)

Then

$$ k_N = k_\omega. $$

(108)

**Proof.** We first observe that $\omega$ defined by equation (107) does not depend on the choice of conformal representative. Indeed, if $\tilde{g}_{ab} = \phi^{1/2} g_{ab}$ for some conformal factor $\phi$ then $\tilde{N} = \phi^{1/2} N$ and $dV_{\tilde{g}} = \phi^2 dV_g$ so

$$ \frac{1}{2N} dV_g = \frac{1}{2\tilde{N}} dV_{\tilde{g}}. $$

(109)

So to establish equation (108) it suffices to work with a convenient background metric. Let $g_{ab}$ be the representative metric such that $dV_g = \omega$ (or equivalently such that $N = 1/2$).
Suppose \( u = [g_{ab}, u_{ab}] \) is a conformal tangent vector. Since \( N = 1/2 \), equation (104) then implies
\[
k_N(u) = [g_{ab}, u_{ab}]^\dagger. \tag{110}
\]
On the other hand, since \( dV_g = \alpha \), equation (36) implies
\[
k_N(u) = [g_{ab}, (dV_g/\alpha)u_{ab}]^\dagger = [g_{ab}, u_{ab}]^\dagger \tag{111}
\]
as well. Hence \( k_N(u) = k_N(u) \).

From propositions 2.1 and 6.1 it follows that each \( k_N \) is a bijection onto the space of smooth cotangent vectors and admits an inverse \( k_N^{-1} \). It then follows from equation (104) that for any smooth conformal cotangent vector \( A = [g_{ab}, A_{ab}]^\dagger \),
\[
k_N^{-1}(A) = [g_{ab}, 2NA_{ab}]^\dagger. \tag{112}
\]

From proposition 6.1 we can translate proposition 2.4 in terms of densitized lapses.

**Proposition 6.2.** Let \( g \in \mathcal{C} \) and let \( A \in T^g_+ \mathcal{C} \). Given a choice of densitized lapse \( N \) there is a conformal geometric momentum and a vector field \( W^a \) such that
\[
A = \sigma + k_N \left( L_g W \right), \tag{113}
\]

The decomposition is unique up the the addition of a conformal Killing field to \( W^a \).

Moreover, if \( g_{ab} \) is an arbitrary representative of \( g \), and if \( A_{ab}, \sigma_{ab} \) and \( N \) are the representatives of \( A, \sigma \), and \( N \) with respect to \( g_{ab} \), then
\[
A_{ab} = \sigma_{ab} + \frac{1}{2N} \left( L_g W \right)_{ab}. \tag{114}
\]

**Proof.** Equation (113) is immediate from propositions 2.4 and 6.1 and it remains to establish equation (114).

Starting from equation (113), let \( g_{ab} \) be a representative of \( g \), and let \( A_{ab}, \sigma_{ab} \) and \( N \) be the representatives of \( A, \sigma \), and \( N \) with respect to \( g_{ab} \). By definition
\[
L_g W = \left[ g_{ab}, \left( L_g W \right)_{ab} \right] \tag{115}
\]
and hence equation (104) implies
\[
k_N \left( L_g W \right) = \left[ g_{ab}, (1/2N) \left( L_g W \right)_{ab} \right]^\dagger. \tag{116}
\]
So equation (113) reads
\[
\left[ g_{ab}, A_{ab} \right]^\dagger = \left[ g_{ab}, \sigma_{ab} \right]^\dagger + \left[ g_{ab}, (2N)^{-1} \left( L_g W \right)_{ab} \right]^\dagger \tag{117}
\]
which establishes equation (114).
written with respect to an arbitrary background metric whereas equation (56) is written with respect to a single background metric (the one where d\nu = \omega).

From definition 5 we have volume-form dependent projections \( P_{\nu} \) from \( T^\nu \mathcal{C} \) to the subspace of conformal geometric momenta. We similarly define densitized-lapse-dependent projections \( P_{\Delta} \) by

\[
P_{\Delta}(A) = \sigma
\]

where \( \sigma \) is the unique conformal geometric momentum from equation (113). Following the construction of definition 6 we also have densitized-lapse-dependent identifications

\[
j_N: T^\nu \mathcal{C} \to T^\nu (C/D_0)
\]

deefined by

\[
\sigma = P_{\Delta}(k_N(\omega)),
\]

and the analogue of the commutative-diagram (62) holds as well. Indeed, all of these objects are obtained simply by replacing \( N \) with the volume form \( \omega \) defined in equation (107).

Data for the CTS-H data method consists of a conformal class \( g \), a conformal geometric momentum \( \sigma \), a mean curvature \( \tau \), and a densitized lapse \( N \) and we seek a solution \( g_{K,ab} \) of the constraints such that

\[
\begin{bmatrix} \bar{g}_{ab} \\ \bar{K}_{ab} \end{bmatrix} = g
\]

\[
P_{\Delta} \left( \begin{bmatrix} \bar{g}_{ab} \\ \bar{K}_{ab} \end{bmatrix} \right) = \sigma,
\]

\[
\bar{g}_{ab} \bar{K}_{ab} = \tau.
\]

To formulate this problem in terms of a PDE, let \( g_{ab} \) be an arbitrary representative of \( g \). Suppose \( \left( \bar{g}_{ab}, \bar{K}_{ab} \right) \) is a metric and second fundamental form with \( [\bar{g}_{ab}] = [g_{ab}] \), so \( \bar{g}_{ab} = \phi^{-2} g_{ab} \) for some conformal factor \( \phi \). Let \( \bar{K}_{ab} \) be the trace-free part of \( \bar{K}_{ab} \), and let \( A_{ab} \) and \( \sigma_{ab} \) be the representatives of \( [\bar{g}_{ab}, \bar{K}_{ab}] \) and \( \sigma \) with respect to \( g_{ab} \), so \( \bar{g}_{ab} = \phi^{-2} A_{ab} \). From the definition of \( P_{\Delta} \) and equation (113) we see that equation (120b) is equivalent to

\[
\phi \sigma_{ab} = A_{ab} + \frac{1}{2N} (L_{\phi} W)_{ab}
\]

for some vector field \( W^a \). Thus equations (120) can be written in terms of the background metric \( g_{ab} \) as

\[
\bar{g}_{ab} = \phi^{-2} g_{ab}
\]

\[
\bar{K}_{ab} = \phi^{-2} \left( \sigma_{ab} + \frac{1}{2N} (L_{\phi} W)_{ab} \right) + \frac{\tau}{n} \sigma_{ab}
\]

for some conformal factor \( \phi \) and vector field \( W^a \).

Substituting equations (122) into the constraint equations we see that \( \left( \bar{g}_{ab}, \bar{K}_{ab} \right) \) solve the constraint equations if and only if \( (\phi, W^a) \) satisfy the CTS-H equations

\[
- 2\kappa \sigma \Delta_{\phi} \phi + R_{\phi} \phi = - \sigma + \frac{1}{2N} L_{\phi} W_t \phi^{-q-1} + \kappa \tau \phi^{q+1} = 0
\]

(CTS-H Hamiltonian constraint) (123a)

\[
div_{\phi} \frac{1}{2N} L_{\phi} W = \kappa \phi \phi_t dr.
\]

(CTS-H momentum constraint) (123b)
These are equivalent to the equations that appear in [PY03], with differences appearing because we treat $\sigma_{ab}$ as the representative of a conformal geometric momentum that is freely specified whereas [PY03] treats $\sigma_{ab}$ as something to be extracted as a TT component of a freely-specified source tensor $C_{ab}$.

We summarize the previous discussion with the following proposition (noting that CTS-H data and representative data are defined analogously to their CTS-L counterparts).

**Proposition 6.3. (The CTS-H Method)** Let $(g, \sigma, \tau, N)$ be CTS-H data, and let $(g_{ab}, \sigma_{ab}, \tau, N)$ be an arbitrary representative of this data.

If $\phi$ and $W^a$ solve the CTS-H equations (123) then $(\tilde{g}_{ab}, \tilde{K}_{ab})$ defined by equations (122) satisfy the constraint equations (1) and satisfy

$$g = \left[ \tilde{g}_{ab} \right] \quad (124a)$$

$$\sigma = P_\kappa \left( \left[ \tilde{g}_{ab}, \tilde{K}_{ab} \right] \right), \quad \text{and} \quad (124b)$$

$$\tau = \tilde{g}^{ab} \tilde{K}_{ab} \quad (124c)$$

Conversely, suppose $(\tilde{g}_{ab}, \tilde{K}_{ab})$ solve the constraint equations and satisfy conditions (124). Then there exist a conformal factor $\phi$ and a vector field $W^a$, unique up to addition of a conformal Killing field to $W^a$, such that the decomposition (122) holds and the CTS-H equations (123) are satisfied.

The CTS-H method is conformally covariant; the proof is analogous to that of proposition 3.2 and is omitted.

**Proposition 6.4.** Let $(g_{ab}, \sigma_{ab}, \tau, N)$ be representative CTS-H data, let $\psi$ be a smooth positive function, and let $\tilde{g}_{ab} = \psi^{-1} g_{ab}, \tilde{\sigma}_{ab} = \phi^{-2} \sigma_{ab}, \text{and } \tilde{N} = \psi N$. Then $(\phi, W)$ solves the CTS-H equations (123) with respect to the data $(g_{ab}, \sigma_{ab}, \tau, N)$ if and only if $(\psi^{-1} \phi, W)$ solve the CTS-H equations with respect to $(\tilde{g}_{ab}, \tilde{\sigma}_{ab}, \tau, \tilde{N})$ and both yield the same solution $(\tilde{g}_{ab}, \tilde{K}_{ab})$ of the constraint equations.

### 7. Equivalence of the methods

We now show that the 1974, CTS-L, and CTS-H parameterizations are all the same by demonstrating how to translate between the parameters for these methods such that the corresponding solutions of the constraints are the same.

Starting with the CTS-L and CTS-H methods, the parameters $g$, $\tau$ and $N$ retain their roles and are fixed when moving between the two methods and we need a way to map back and forth between the velocity/momentum parameters. The momentum parameter from the CTS-H method is an element of $T^*_g(C/D_0)$, and we saw at the end of section 5 that the true velocity parameter for the CTS method is a conformal geometric velocity $u + \text{Im } L_u \in T_g(C/D_0)$. So a natural candidate for the identification is the map $\tilde{j}_N$ defined in equation (119), and this is the correct choice.

**Proposition 7.1.** The solutions of the constraint equations generated by geometric CTS-L data $(g, u + \text{Im } L_u, \tau, N)$ and the solutions generated by CTS-H data $(g, \sigma, \tau, N)$ coincide if and only if
\[ \sigma = j_N \left( \mathbf{u} + \text{Im} \mathbf{L}_g \right). \] (125)

In terms of a representative metric \( g_{ab} \in \mathfrak{g} \), representative CTS-L data \( (g_{ab}, u_{ab}, \tau, N) \) and representative CTS-H data \( (g_{ab}, \sigma_{ab}, \tau, N) \) generate the same solutions if and only if there is a vector field \( X^a \) such that
\[ u_{ab} = 2N\sigma_{ab} + \left( \mathbf{L}_g X \right)_{ab}. \] (126)

**Proof.** Suppose \( (\tilde{g}_{ab}, \tilde{K}_{ab}) \) is a solution of the constraints generated by CTS-L data \( (\mathbf{g}, \mathbf{u} + \text{Im} \mathbf{L}_g, \tau, N) \). Then there exists a vector field \( X^a \) such that conditions (98) hold. In particular,
\[ \left[ \tilde{g}_{ab}, \tilde{g}_{ab} \right] = \mathbf{u} \] (127)
where \( \tilde{g}_{ab} = 2N \tilde{K}_{ab} + \mathcal{L}_X \tilde{g}_{ab} \). Hence
\[ \left[ \tilde{g}_{ab}, 2N \tilde{K}_{ab} \right] \in \mathbf{u} + \text{Im} \mathbf{L}_g \] (128)
and therefore from equation (63) we have
\[ j_N \left( \mathbf{u} + \text{Im} \mathbf{L}_g \right) = P_N \left( k_N \left( \left[ \tilde{g}_{ab}, 2N \tilde{K}_{ab} \right] \right) \right). \] (129)

Equation (104) implies
\[ k_N \left( \left[ \tilde{g}_{ab}, 2N, \tilde{K}_{ab} \right] \right) = \left[ \tilde{g}_{ab}, \tilde{K}_{ab} \right]^g \] (130)
and thus
\[ P_N \left( \left[ \tilde{g}_{ab}, \tilde{K}_{ab} \right] \right) = j_N \left( \mathbf{u} + \text{Im} \mathbf{L}_g \right). \] (131)

Defining \( \sigma = P_N \left( \left[ \tilde{g}_{ab}, \tilde{K}_{ab} \right]^g \right) \), the solution \( (\tilde{g}_{ab}, \tilde{K}_{ab}) \) then satisfies conditions (124) and therefore is generated by CTS-H data \( (\mathbf{g}, \sigma, \tau, N) \).

The previous discussion is reversible and therefore if a solution of the constraints is generated by CTS-H data \( (\mathbf{g}, \sigma, \tau, N) \), then it is generated by geometric CTS-L data \( (\mathbf{g}, \mathbf{u} + \text{Im} \mathbf{L}_g, \tau, N) \) where
\[ \mathbf{u} + \text{Im} \mathbf{L}_g = j_N^{-1} (\sigma). \] (132)

To reformulate equation (132) in terms of a background metric \( g_{ab} \in \mathfrak{g} \), let \( u_{ab}, \sigma_{ab} \) and \( N \) be the representatives of \( \mathbf{u} \), \( \sigma \) and \( N \) with respect to \( g_{ab} \). Since
\[ j_N^{-1} (\sigma) = k_N^{-1} (\sigma) + \text{Im} \mathbf{L}_g, \] (133)
and since
\[ k_N^{-1} (\sigma) = \left[ g_{ab}, 2N\sigma_{ab} \right], \] (134)
equation (132) is equivalent to
\[ u_{ab} = 2N\sigma_{ab} + \left( \mathbf{L}_g X \right)_{ab} \] (135)
for some vector field \( X^a \), where \( u_{ab} \) is the representative of \( \mathbf{u} \) with respect to \( u_{ab} \). \( \square \)
The equivalence of the 1974 method and the CTS-H method is a consequence of the equivalences of the projections $P_n$ for the 1974 method and the projections $P^\mu_\nu$ of the CTS-H method, where we translate between volume forms and densitized lapses via equation (107).

**Proposition 7.2.** Let $g \in C$, $\sigma \in T^\mu_\nu (C/\partial_n)$ and $\tau \in C^\infty (M)$. Suppose $N \in N_g$ is a densitized lapse and $\omega$ is a volume form that satisfy equation (107). Then the set of solutions of the constraints generated by CTS-H data $(g, \sigma, \tau, N)$ is the same as the set of solutions generated by 1974 data $(g, \sigma, \tau, \omega)$.

**Proof.** Suppose $(g, \sigma, \tau, N)$ is a solution of the constraints generated by CTS-H data $(g, \sigma, \tau, N)$. Then $(\pi^{\sigma}_{\sigma ab}, K^a_{\mu \nu})$ satisfy conditions (124), and in particular \[
P_n \left( \left[ g^{ab}_\sigma, K^a_{\mu \nu} \right] \right) = \sigma.
\] But $P_n = P_\omega$ where $\omega$ is defined in equation (107), so the solution $(\pi^{\sigma}_{\sigma ab}, K^a_{\mu \nu})$ satisfies conditions (78) as well and is generated by 1974 data $(g, \sigma, \tau, \omega)$. The converse is proved similarly. \hfill $\square$

Proposition 7.2 admits the following reformulation in terms of 1974 representative data, where the volume form $\omega$ is determined implicitly by the background metric.

**Proposition 7.3.** Let $(g_{ab}, \sigma_{ab}, \tau)$ be representative 1974 data and let
\[
\begin{align*}
g &= \left[ g_{ab} \right], \\
\sigma &= \left[ g_{ab}, \sigma_{ab} \right], \\
N &= \left[ g_{ab}, 1/2 \right].
\end{align*}
\] Then the set of solutions of the constraint equations generated by the 1974 method for $(g_{ab}, \sigma_{ab}, \tau)$ is the same as the set of solutions generated by the CTS-H method for data $(g, \sigma, \tau, N)$.

Conversely, suppose $(g, \sigma, \tau, N)$ is a tuple of CTS-H data. Let $g_{ab}$ be the unique element of $g$ such that
\[
\left[ g_{ab}, 1/2 \right] = N
\] and let $\sigma_{ab}$ be the unique TT tensor such that
\[
\left[ g_{ab}, \sigma_{ab} \right] = \sigma.
\] The set of solutions of the constraint equations generated by the CTS-H method for $(g, \sigma, \tau, N)$ is the same as the set generated by 1974 conformal data $(g_{ab}, \sigma_{ab}, \tau)$.

**Proof.** Representative 1974 data $(g_{ab}, \sigma_{ab}, \tau)$ determine 1974 data $(g, \sigma, \tau, \omega)$ with $g = [g_{ab}]$, $\sigma = [g_{ab}, \sigma_{ab}]$, and $\omega = dV^\tau$, proposition 7.2 implies that the set of solutions generated by 1974 data $(g, \sigma, \tau, \omega)$ is the same as the set of solutions generated by CTS-H data $(g, \sigma, \tau, N)$ where the representative of $N$ with respect to $g_{ab}$ satisfies equation (107). Since $\omega = dV^\tau$, equation (107) implies $N = 1/2$ and hence $N = [g_{ab}, 1/2]$. This establishes the forward direction, and the converse is proved similarly. \hfill $\square$
While the ‘coordinate-free’ approach to expressing the conformal method parameters is helpful, applications frequently require working with representative data. Summarizing from propositions 7.1 and 7.3 we translate between representative data as follows.

- [1974 to CTS-H] Start with 1974 data $(g_{ab}, \sigma_{ab}, \tau)$ and adjoin a lapse $N = 1/2$. Use CTS-H data $(\tilde{g}_{ab}, \tilde{\sigma}_{ab}, \tilde{\tau}, 1/2)$, or any conformally related CTS-H data.

- [CTS-H to 1974] Start with CTS-H data $(\tilde{g}_{ab}, \tilde{\sigma}_{ab}, \tilde{\tau}, N)$ and let $\psi$ be the conformal factor satisfying $\psi^N = (1/2)$. Let $\tilde{g}_{ab} = \psi^{-1/2}g_{ab}$ and $\tilde{\sigma}_{ab} = \psi^{-1/2}\sigma_{ab}$, and use 1974 data $(\tilde{g}_{ab}, \tilde{\sigma}_{ab}, \tilde{\tau})$.

- [CTS-H to CTS-L] Start with CTS-H data $(\tilde{g}_{ab}, \tilde{\sigma}_{ab}, \tilde{\tau}, N)$ and select an arbitrary vector field $X_a$. Let $\sigma = +\left( uN X_a \right)_L^2$, and use CTS-L data $(g_{ab}, \sigma_{ab}, \tau)$ or any conformally related CTS-L data.

- [CTS-L to CTS-H] Start with CTS-L data $(g_{ab}, \sigma_{ab}, \tau)$ and let $\sigma_{ab}$ be the unique transverse traceless tensor with $\sigma = +\left( uN Y_a \right)_L^2$ for some vector field $Y_a$, as given by proposition 6.2. Use CTS-H data $(g_{ab}, \sigma_{ab}, \tau, N)$ or any conformally related CTS-H data.

8. Applications

In this section we strengthen two previous results concerning the 1974 conformal method on compact manifolds by using the correspondence between the 1974 method and the CTS-H method.

8.1. Near-CMC existence/uniqueness

The main theorem from [IM96], when restricted to smooth tensors, can be phrased as follows.

**Theorem 8.1.** Let $M^3$ be a compact 3-manifold. Suppose $g_{ab}$ is a smooth metric on $M$ that has constant scalar curvature equal to -1 and that admits only the trivial conformal Killing field, and suppose $\sigma_{ab}$ is an arbitrary transverse-traceless tensor with respect to $g_{ab}$. Then there is an open set $T_{\sigma}$ of nowhere-vanishing mean curvatures such that every non-zero constant mean curvature belongs to $T_{\sigma}$, and such that for every $\tau \in T_{\sigma}$ the 1974 conformally-parameterized constraint equations (77) for the representative 1974 data $(g_{ab}, \sigma_{ab}, \tau)$ have a unique solution. The set $T_{\sigma}$ in theorem 8.1 is defined by

$$\max \frac{|d\tau|_L}{\min |\tau|} \text{ and } |d\tau|_L$$

being sufficiently small, so theorem 8.1 is a near-CMC existence and uniqueness result. It is remarked in [IM96] that the proof of theorem 8.1 could be carried out under the more general hypothesis $R_{\sigma}(0)$ everywhere, but that the authors were unable to extend it to the most natural generalization that $g_{ab}$ is Yamabe negative. We show here that such an extension is possible.

In coordinate-free language, theorem 8.1 can be phrased as follows.
Theorem 8.2. Let $M$ be a compact 3-manifold. Suppose $\sigma$ is a Yamabe-negative conformal class on $M$ admitting only the trivial conformal Killing field, and suppose $\tau \in T^*_{\sigma}$. Let $\omega$ be the volume form of the unique representative $g_{\omega} \in \sigma$ that satisfies $R_g = -1$. Then there is an open set $T_{\sigma,\tau}$ of nowhere-vanishing mean curvatures such that for every $\tau \in T_{\sigma,\tau}$ the 1974 conformal data $(\sigma, \tau, \omega)$ determines a unique solution of the constraint equations.

In this language, the central restriction of the theorem is the choice of a single volume form $\omega$. We wish to eliminate this restriction, and we use the fact that the choice of volume form for the 1974 method corresponds to the choice of densitized lapse for the CTS-H method. So we will consider the CTS-H equations

$$-2q\Delta_{\tau}^{\phi} + R_{\phi}^{\phi} - \left[\sigma + \frac{1}{2N}L^x W\right]^{2} \phi^{-q-1} + \kappa \tau^{2} = 0$$

$$\text{div}_{\tau} \frac{1}{2N}L^x W = \phi^q \text{d}\tau$$

(143)

where $g_{\omega}$ is the unique representative with $R_g = -1$ and $N$ is an arbitrary lapse.

The equations considered by theorem 8.1 are exactly equations (143) with $N = 1/2$, so we need to consider the impact of an arbitrary choice of $N$ in equations (143) on the rather technical proof of theorem 8.1. In effect, this amounts to replacing $L^x$ with $1/(2N) L^x$ wherever it appears in the proof, and there are facts concerning the vector Laplacian $\Delta_{\tau} = \text{div}_{\tau} (2N)^{-1} L^x$ that need to be revisited for the operator

$$\Delta_{\tau,N} = \text{div}_{\tau} (2N)^{-1} L^x$$

(144)

Proposition 8.3. Let $g_{\omega}$ and $N$ be a smooth metric and positive smooth function on $M$. The operator $\Delta_{\tau,N}$ is linear, elliptic, and self-adjoint with respect to $g_{\omega}$. If $g_{\omega}$ has no conformal Killing fields then $\Delta_{\tau,N}$ has trivial kernel. Regardless of whether $g_{\omega}$ has conformal Killing fields, there is a constant $c_{\epsilon,N}$ such that if $W = \phi$ and $\eta$ satisfy

$$\Delta_{\tau,N} W = \eta$$

then

$$\left|\frac{1}{2N}L^x W\right| \leq c_{\epsilon,N} |\eta|_{\epsilon}.$$  

(145)

Proof. The fact that $\Delta_{\tau,N}$ is linear, elliptic, and self-adjoint is obvious, and an integration by parts argument shows that its kernel consists of conformal Killing fields, so it remains to establish inequality (146).

Let $\psi$ be the unique positive function with $\psi^q = (2N)^{-1}$ and let $\tilde{g}_{\omega} = \psi^{-2} g_{\omega}$. Then $L^x = \psi^{-2} L^x$ and $\text{div}_{\tilde{\tau}} = \psi^q \text{div}_{\tilde{\tau}}$. Hence

$$\Delta_{\tilde{\tau}}^{\psi} = \psi^{-q} \text{div}_{\tilde{\tau}} \psi^q L^x = \psi^{-q} \text{div}_{\tilde{\tau}} (2N)^{-1} L^x = \psi^{-q} \Delta_{\tau,N}.$$  

(147)
Now suppose $W^a$ and $\eta$ satisfy $\Delta_{g^a} W = \eta$, so
\[ \Delta_{\tilde{g}} W = \psi^2 \eta. \] (148)

From [IÓM04] Lemma 1 concerning the standard vector Laplacian we know that there is a constant $\tilde{c}$, independent of $W^a$ and $\eta$, such that
\[ \max |L_a W| \leq \tilde{c} \max (\psi^2) \max |\eta|. \] (149)

Since the norms for $g$ and $\tilde{g}$ are comparable via constants depending on $\min \psi$ and $\max \psi$, and since $L_a = \psi^{-1} L_a$, inequality (146) now follows, where the constant depends on $\tilde{c}$, $\min \psi$ and $\max \psi$ (i.e., on $g$ and $N$). \[\square\]

With proposition 8.3 in hand, the reader is now invited to walk through the proof of theorem 8.1, as presented in [IM96], to establish existence and uniqueness of equations (143). The only interesting changes occur in establishing analogues of inequalities (38) and (58) of that paper under the hypotheses that $\tau \equiv 0$ or $\tau \equiv 8$. We assert that using proposition 8.3 one can repeat the exercise just undertaken for this paper as well to prove the following.

**Theorem 8.4.** Let $M^3$ be a compact 3-manifold. Let $g$ be a Yamabe-negative conformal class on $M$ admitting no conformal Killing fields, and let $\sigma \in T^*_g C$ be arbitrary. For any choice $N$ of densitized lapse there is an open set $T_{\tau,N}$ of nowhere-vanishing mean curvatures such that every non-zero constant mean curvature belongs to $T_{\tau,N}$, and such that for every $\tau \in T_{\tau,N}$ the CTS-H data $(g, \sigma, \tau, N)$ determines a unique solution of the constraint equations.

Using proposition 7.3, theorem 8.4 then implies that theorem 8.1 holds with the restriction $R = -1$ replaced by the condition that $g_{ab}$ is Yamabe-negative.

[ACI08] contains results that are analogues of theorem 8.1 under the hypotheses that $R \equiv 0$ or $R \equiv 8$. We assert that using proposition 8.3 one can repeat the exercise just undertaken for this paper as well to prove the following.

**Theorem 8.5.** Let $M^3$ be a compact 3-manifold. Let $g$ be a Yamabe-non-negative conformal class on $M$ admitting no conformal Killing fields. For any choice $N$ of densitized lapse there is an open set $T_{\tau,N}$ of nowhere-vanishing mean curvatures such that every non-zero constant mean curvature belongs to $T_{\tau,N}$, and such that for every $\tau \in T_{\tau,N}$ the CTS-H data $(g, \sigma, \tau, N)$ determines a unique solution of the constraint equations. Hence theorem 8.1 also holds without any restriction whatsoever on the metric $g_{ab}$.

### 8.2. Near-CMC non-existence

Theorem 3.3 states that aside from some special cases, there does not exist a solution for CMC data $(g, \sigma, \tau)$ if $g$ is Yamabe non-negative and $\sigma = 0$. [IÓM04] established the following two near-CMC analogues of this fact.

**Theorem 8.6.** Let $M^3$ be a compact 3-manifold. Suppose we have 1974 representative data $(g_{ab}, \sigma_{ab}, \tau)$ with $R_{\tau} \geq 0$ and $\sigma_{ab} \equiv 0$. If $\tau = T + \rho$ for some non-zero constant $T$ and if
is sufficiently small, then the 1974 conformally parameterized constraint equations (77) do not admit a solution.

**Theorem 8.7.** Let $M^3$ be a compact 3-manifold. Suppose we have CTS-L representative data $(g_{ab}, u_{ab}, \tau, N)$ where $g$ is Yamabe non-negative and with $u_{ab} \equiv 0$. If $\tau = T + \rho$ for some non-zero constant $T$ and if

$$\left| \frac{d\rho}{g} \right| \leq T$$

(150)

is sufficiently small, then the CTS-L equations (92) do not admit a solution.

Note that theorem 8.6 only applies to metrics with everywhere non-negative scalar curvature, whereas theorem 8.7 only assumes the metric is Yamabe non-negative. We now show that theorem 8.6 can be strengthened to include the case that $g_{ab}$ is Yamabe non-negative.

Suppose $g_{ab}$ is Yamabe non-negative and that $\sigma_{ab} \equiv 0$. Following the recipes at the end of section 7, if a solution of the constraints exists for 1974 data $(g_{ab}, \sigma_{ab}, \tau)$ then it exists for CTS-H data $(g_{ab}, \sigma_{ab}, \tau, N)$ where $N = 1/2$. And if it exists for this CTS-H data, then it also exists for CTS-L data $(g_{ab}, u_{ab}, \tau, 1/2)$ where $u_{ab} \equiv 0$. Now theorem 8.7 implies that if $\tau = T + \rho$ for some non-zero constant $T$, and if $|d\rho|/T$ is sufficiently small, there is no solution for the CTS-L data $(g_{ab}, \sigma_{ab}, \tau, 1/2)$ and therefore no solution for the 1974 data $(g_{ab}, \sigma_{ab}, \tau)$.

For completeness we state the coordinate-free variation of this result and leave the proof as an exercise.

**Theorem 8.8.** Suppose $g$ is a Yamabe-non-negative conformal class.

1. If $\omega$ is a volume form, there is an open set $U_{g,\omega}$ of mean curvatures that contains the non-zero constants such that 1974 data $(g, \sigma, \tau, \omega)$ does not generate a solution of the constraints if $\alpha = 0$ and $\tau \in U_{g,\omega}$.
2. If $N$ is a densitized lapse, there is an open set $V_{g,N}$ of mean curvatures that contains the non-zero constants such that CTS-H data $(g, \sigma, \tau, N)$ does not generate a solution of the constraints if $\alpha = 0$ and $\tau \in V_{g,N}$.
3. For the same set $V_{g,N}$ as in item 2, CTS-L data $(g, u, \tau, N)$ does not generate a solution of the constraints if $u \in \text{Im} L$ and $\tau \in V_{g,N}$. Moreover, the sets $U_{g,\omega}$ and $V_{g,N}$ are the same if $\omega$ and $N$ are related via proposition 6.1.

Theorem 8.8 is not as specific as theorem 8.7 in defining the near-CMC condition because we currently have a hazy understanding of what this set is. Expression (151) is defined with respect to a particular representative metric, and the set $V_{g,N}$ can be thought of as taking a union of sets obtained from applying theorem 8.7 for each choice of background metric. The maximal set $V_{g,N}$ for which theorem 8.8 applies should be described in terms of $g$ and $N$ directly, and such a description is not yet understood.
9. Conclusion

We have demonstrated in this paper that there is really only one conformal method. The CMC conformal method is a special case of the 1974 method, and the 1974 method has equivalent formulations in terms of the CTS-L and CTS-H methods. The parameters of the conformal method are:

1. a conformal class \( g \),
2. either a conformal geometric velocity \( u + \text{Im } L_e \in \mathbb{T}_g(C/D_0) \) or a conformal geometric momentum \( \sigma \in T^*_g(C/D_0) \),
3. a mean curvature \( \tau \),
4. a choice of one member of a family of identifications of \( T_g C \) with \( T^*_g C \).

The choice in item (4) can be made alternatively by selecting a volume form \( \omega \) and using the map \( \omega_k \) from proposition 2.1, or by selecting a densitized lapse \( N \) and using the map \( kN \) from definition 8. Proposition 6.1 shows how to convert back and forth between \( kN \) and \( \omega_k \), so these are equivalent ways of expressing the same choice. The CTS-L and CTS-H methods make choice (4) explicitly via \( gN \), whereas the 1974 method makes the choice implicitly via \( \omega = dV' \).

The choice of \( kN \) determines a related identification \( jN \), given by equation (119), between conformal geometric velocities and momenta. This identification allows one to map back and forth in item (2) between velocities and momenta: the CTS-L method uses velocities, whereas the 1974 and CTS-H methods use momenta, but by using \( jN \) these are equivalent ways of expressing the same parameter.

The unifying theme of this paper is the need to clearly distinguish between tangent and cotangent vectors in the conformal method, and that there is a choice in the conformal method of how to identify these objects. This leads to the question of what this choice corresponds to (physically or otherwise). In fact, these identifications arise as Legendre transformations in the \( n+1 \) formulation of gravity when using a densitized lapse. We will return to this point and related results in forthcoming work.

Because the conformal methods are equivalent, a theorem proved for one method determines analogous theorems proved for the other methods. From a practical point of view, however, the CTS-H method seems most expedient to work with. Given a choice of CTS-H data \( (g, \sigma, \tau, N) \), the CTS-H equations (123) can be expressed with respect to any representative background metric, whereas given 1974 data \( (g, \sigma, \tau, \omega) \), the 1974 conformally parameterized constraint equations (77) are written with respect to the unique background metric with \( dV' = \omega \). This lack of flexibility led to unnecessary restrictions in the past for theorems proved for the 1974 method, and we saw in section 7 how these restrictions can be overcome. In principle one could express the 1974 method with respect to an arbitrary background metric, and doing so must lead to the CTS-H equations after converting the volume form into a densitized lapse. So there is little reason to prefer the 1974 conformally parameterized constraint equations. The only mild additional difficulty in working with the CTS-H equations comes from working with the generalized vector Laplacians \( \Delta_{g,N} = \text{div}_g (2N)^{-1}L_e \) instead of the standard vector Laplacian \( \Delta = \text{div}_gL_e \). But the operators are very closely related and proofs in the generalized case can typically be obtained by trivially modifying proofs for the standard case. Moreover, as seen in proposition 8.3, one can sometimes obtain results for the generalized operators as a corollary of a known results for the standard vector Laplacian without revisiting the steps of the original proof.

The case for using the CTS-H equations over the CTS-L equations is not especially strong, but there are some advantages. The velocity parameter \( u_{ab} \) in the CTS-L method is really a representative of the whole subspace \( u_{ab} + \text{Im } L_e \), and this makes uniqueness statements a little
more cumbersome for the CTS-L method. Moreover, the CTS-H equations (123) are a little simpler than the CTS-L equations (92), and are more familiar for researchers accustomed to working with the 1974 method: simply prepend a \(1/(2N)\) in front of every conformal Killing operator and proceed as before.

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