A Journey Through Garden Algebras

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\textbf{Summary}. The main purpose of these lectures is to give a pedagogical overview on the possibility to classify and relate off-shell linear supermultiplets in the context of supersymmetric mechanics. A special emphasis is given to a recent graphical technique that turns out to be particularly effective for describing many aspects of supersymmetric mechanics in a direct and simplifying way.
Introduction

Sometimes problems in mathematical physics go unresolved for long periods of time in mature topics of investigation. During this World Year of Physics which commemorates the pioneering efforts of Albert Einstein, it is perhaps appropriate to note the irreconciliability of the symmetry group of Maxwell Equations with that of Newton’s Equation (via his second law of motion) was one such problem. The resolution of this problem, of course, led to one of the greatest revolutions in physics. This piece of history suggests a lesson on what can be the importance of problems that large numbers of physicists regard as unimportant or unsolvable.

In light of this episode, the presentation which follows hereafter is focused on a problem in supersymmetry that has long gone unresolved and seems generally regarded as one of little importance. While there is no claim or pretension that this problem has the importance of the one resolved by the brilliant genius of Einstein, it is a problem that perhaps holds the key to a more mathematically complete understanding of the area known as “supersymmetry.”

The topic of supersymmetry is over thirty years old now. It has been vigorously researched by both mathematicians and physicists. During this entire time, this subject has been insinuated into a continuously widening array of increasingly sophisticated mathematical models. At the end of this stream of development lies the mysterious topic known as “M-theory.” Accordingly, it may be thought that all fundamental issue regarding this area have already a satisfactory resolution.

However, as surprising as it may seem, in fact very little is known about the representation theory of supersymmetry required for the classification of irreducible superfield theories in a manner that allows for quantization consistent with a manifest realization of supersymmetry.

Superspace is to supersymmetry as Minkowski space is to the Lorentz group. Superspace provides the most natural geometrical setting in which to describe supersymmetrical theories. Almost no physicist would utilize the component of Lorentz four-vectors or higher rank tensor to describe relativistic physics. Yet, the analog of this is common practice in describing supersymmetrical theory. This is so because ‘component fields’ are the predominant language by which most discussions of supersymmetry are couched.

One fact that hides this situation is that much of the language used to describe supersymmetrical theories appears to utilize the superspace formalism. However, this appearance is deceiving. Most often what appears to be
A superspace presentation is actually a component presentation in disguise. A true superspace formulation of a theory is one that uses ‘unconstrained’ superfields as their fundamental variables. This is true of an tiny subset of the discussions of supersymmetrical theories and is true of none of the most interesting such theories involving superstrings.

This has led us to the belief that possibly some important fundamental issues regarding supersymmetry have yet to be properly understood. This belief has been the cause of periodic efforts that have returned to this issue. Within the last decade this investigation has pointed toward two new tools as possibly providing a fresh point of departure for the continued study (and hopefully ultimate resolution) of this problem. One of these tools has relied on a totally new setting in which to understand the meaning of supersymmetry. This has led to the idea that the still unknown complete understanding of the representation theory of supersymmetry lies at the intersection of the study of Clifford algebras and K-theory. In particular, a certain class of Clifford algebras (to which the moniker $GR(d, N)$ have been attached) provides a key to making such a connection. Within the confines of an interdisciplinary working group that has been discussing these problems, the term “garden algebra” has been applied to the symbolic name $GR(d, N)$. It has also been shown that these Clifford algebras natural lead to a graphical representation somewhat akin to the root and weight spaces seen in the classification of compact Lie algebras. These graphs have been given the name “Adinkras.” The topic of this paper will be introducing these new tools for the study of supersymmetry representation theory.

1 $GR(d, N)$ Algebras

1.1 Geometrical interpretation of $GR(d, N)$ algebras

In a field theory, boson and fermions are to be regarded as diffeomorphisms generating two different vector spaces; the supersymmetry generators are nothing but sets of linear maps between these spaces. Following this picture we can include a supersymmetric theory in a more general geometrical framework defining the collection of diffeomorphisms

\[ \phi_i : \mathbb{R} \rightarrow \mathbb{R}^{d_L}, \quad i = 1, \ldots, d_L \]  

and

\[ \psi_\alpha : \mathbb{R} \rightarrow \mathbb{R}^{d_R}, \quad \alpha = 1, \ldots, d_R \]  

where the one dimensional dependence reminds us that we restrict our attention to mechanics. The free vector spaces generated by $\{\phi_i\}_{i = 1}^{d_L}$ and $\{\psi_\alpha\}_{\alpha = 1}^{d_R}$...
\{\psi^a_\alpha\}_{a=1}^d$ are respectively $\mathcal{V}_L$ and $\mathcal{V}_R$, isomorphic to $R^d_L$ and $R^d_R$. For matrix representations in the following, the two integers are restricted to the case $d_L = d_R = d$. Four different linear mappings can act on $\mathcal{V}_L$ and $\mathcal{V}_R$:

$$\mathcal{M}_L : \mathcal{V}_L \rightarrow \mathcal{V}_R, \quad \mathcal{M}_R : \mathcal{V}_R \rightarrow \mathcal{V}_L$$
$$\mathcal{U}_L : \mathcal{V}_L \rightarrow \mathcal{V}_L, \quad \mathcal{U}_R : \mathcal{V}_R \rightarrow \mathcal{V}_R$$

with linear maps space dimensions

$$\dim \mathcal{M}_L = \dim \mathcal{M}_R = d_R d_L = d^2$$
$$\dim \mathcal{U}_L = d_L^2 = d^2, \quad \dim \mathcal{U}_R = d_R^2 = d^2$$

as a consequence of linearity. In order to relate this construction to a general real ($\equiv \mathcal{GR}$) algebraic structure of dimension $d$ and rank $N$ denoted by $\mathcal{GR}(d,N)$, two more requirements need to be added.

1. Let us define the generators of $\mathcal{GR}(d,N)$ as the family of $N + N$ linear maps $^4$

$$L_I \in \{\mathcal{M}_L\}, \quad I = 1,\ldots,N$$
$$R_K \in \{\mathcal{M}_R\}, \quad K = 1,\ldots,N$$

such that for all $I, K = 1,\ldots,N$ we have

$$L_I \circ R_K + R_K \circ L_I = -2\delta_{IK} I_{\mathcal{V}_L},$$
$$R_I \circ L_K + R_K \circ L_I = -2\delta_{IK} I_{\mathcal{V}_R},$$

where $I_{\mathcal{V}_L}$ and $I_{\mathcal{V}_R}$ are identity maps on $\mathcal{V}_L$ and $\mathcal{V}_R$. Equations (6) will later be embedded into a Clifford algebra but one point has to be emphasized, we are working with real objects.

2. After equipping $\mathcal{V}_L$ and $\mathcal{V}_R$ with euclidean inner products $\langle \cdot, \cdot \rangle_{\mathcal{V}_L}$ and $\langle \cdot, \cdot \rangle_{\mathcal{V}_R}$ respectively, the generators satisfy the property

$$\langle \phi, R_I(\psi) \rangle_{\mathcal{V}_L} = -\langle L_I(\phi), \psi \rangle_{\mathcal{V}_R}, \quad \forall (\phi, \psi) \in \mathcal{V}_L \oplus \mathcal{V}_R.$$  

This condition relates $L_I$ to the hermitian conjugate of $R_I$, namely $R_I^\dagger$, defined as usual by

$$\langle \phi, R_I(\psi) \rangle_{\mathcal{V}_L} = \langle R_I^\dagger(\phi), \psi \rangle_{\mathcal{V}_R}$$

so that

$$R_I^\dagger = R_I^\dagger = -L_I.$$  

$^4$ Notice that in previous works on the subject $^3$ $^12$, the maps $L_I$ and $R_K$ where exchanged, so that $L_I \in \{\mathcal{M}_R\}$ and $R_K \in \{\mathcal{M}_L\}$.
The role of \( \{U_L\} \) and \( \{U_R\} \) maps is to connect different representations once a set of generators defined by conditions (6) and (7) has been chosen. Notice that \( (R_I L_J)_{ij} \in U_L \) and \( (L_I R_J)_{\hat{\alpha} \hat{\beta}} \in U_R \). Let us consider \( A \in \{U_L\} \) and \( B \in \{U_R\} \) such that

\[
\begin{align*}
A & : \phi \rightarrow \phi' = A\phi \\
B & : \psi \rightarrow \psi' = B\psi
\end{align*}
\]

then, taking the \( V_L \) sector as example, we have

\[
\langle \phi, R_I (\psi) \rangle_{V_L} \rightarrow \langle A\phi, R_I B(\psi) \rangle_{V_L} = \\
= \langle \phi, A^R R_I B(\psi) \rangle_{V_L} = \\
= \langle \phi, R_I' (\psi) \rangle_{V_L}
\]

so a change of representation transforms the generators in the following manner:

\[
R_I \rightarrow R'_I = A^R R_I B, \quad L_I \rightarrow L'_I = B^L L_I A
\]

In general, equations (6) and (7) do not identify a unique set of generators. Thus, an equivalence relation has to be defined on the space of possible sets of generators, say \( \{L_I, R_I\} \sim \{L'_I, R'_I\} \) if and only if there exist \( A \in \{U_L\} \) and \( B \in \{U_R\} \) such that \( L' = B^L L I A \) and \( R' = A^R R I B \).

Now we want to show how a supersymmetric theory arises. Algebraic derivations are defined by

\[
\begin{align*}
\delta_\epsilon \phi_i &= i\epsilon^I (R_I)_{ij} \hat{\alpha} \hat{\beta} \psi_{\hat{\alpha}} \\
\delta_\epsilon \psi_{\hat{\alpha}} &= -\epsilon^I (L_I)_{\hat{\alpha} \hat{\beta}} \partial_\tau \phi_i
\end{align*}
\]

where the real valued fields \( \{\phi_i\}^{d_L} = 1 \) and \( \{\psi_{\hat{\alpha}}\}^{d_R} = 1 \) can be interpreted as bosonic and fermionic respectively. The fermionic nature attributed to the \( V_R \) elements implies that \( M_L \) and \( M_R \) generators, together with supersymmetry transformation parameters \( \epsilon^I \), anticommute among themselves. Introducing the \( d_L + d_R \) dimensional space \( V_L \oplus V_R \) with vectors

\[
\Psi = \begin{pmatrix} \phi \\ \psi \end{pmatrix},
\]

equation (13) reads

\[
\delta_\epsilon (\Psi) = \begin{pmatrix} i\epsilon^R \psi \\ \epsilon L \partial_\tau \phi \end{pmatrix}
\]
so that

\[ [\delta_{\epsilon_1}, \delta_{\epsilon_2}] \Psi = i \epsilon_1^I \epsilon_2^J \left( \frac{R_I L_J \partial_\tau \phi}{L_IR_J \partial_\tau \psi} - i \epsilon_2^J \epsilon_1^I \left( \frac{R_J L_I \partial_\tau \phi}{L_J R_I \partial_\tau \psi} \right) \right) = -2i \epsilon_1^I \epsilon_2^J \partial_\tau \Psi , \]

\[ \text{(16)} \]

utilizing that we have classical anticommuting parameters and that equations (6) hold. It is important to stress that components of (14) can be interpreted as superfield components, so it is as if we were working with a particular superfield multiplet containing only these physical bosons and fermions. From (16) it is clear that \( \delta_\epsilon \) acts as a supersymmetry generator, so that we can set

\[ \delta_Q \Psi := \delta_\epsilon \Psi = i \epsilon^I Q_I \Psi \]

\[ \text{(17)} \]

which is equivalent to writing

\[ \delta_Q \phi_i = i \left( \epsilon^I Q_I \phi \right)_i , \]
\[ \delta_Q \psi_{\dot{\alpha}} = i \left( \epsilon^I Q_I \phi \right)_{\dot{\alpha}} , \]

\[ \text{(18)} \]

with

\[ Q_I = \begin{pmatrix} 0 & R_I \\ L_I H & 0 \end{pmatrix} , \]

\[ \text{(19)} \]

where \( H = i \partial_\tau \). As a consequence of (16) a familiar anticommutation relation appears

\[ \{ Q_I, Q_J \} = -2i \delta_{IJ} H , \]

\[ \text{(20)} \]

confirming that we are talking about genuine supersymmetry. Once the supersymmetry is recognized, we can associate to the algebraic derivations (13) the variations defining the scalar supermultiplets. However, the choice (13) is not unique: one can check that

\[ \delta_Q \xi_{\dot{\alpha}} = \epsilon^I (L_I)_{\dot{\alpha}} \hat{\psi}_i , \]
\[ \delta_Q \hat{\psi}_i = -i \epsilon^I (R_I)_{\dot{\alpha}} \partial_\tau \xi_{\dot{\alpha}} , \]

\[ \text{(21)} \]

is another proposal linked to ordinary supersymmetry as the previous one. In this case we will refer to the supermultiplet defined by (21) as the spinorial one.

### 1.2 Twisted representations

The construction outlined above suffers from an ambiguity in the definition of superfield components \((\phi_i, \psi_{\dot{\alpha}})\) and \((\xi_\alpha, A_i)\) due to the possibility of exchanging the role of \( R \) and \( L \) generators, giving rise to the new superfields \((\phi_{\dot{\alpha}}, \psi_i)\) and \((\xi_i, A_{\dot{\alpha}})\) with the same supersymmetric properties of the previous ones. The variations associated to these twisted versions are, respectively
\[
\delta_Q \phi_0 = i \epsilon^I (L_I)_0 \dot{\alpha} \psi_i,
\]
\[
\delta_Q \psi_i = - \epsilon^I (R_I)_i \dot{\alpha} \partial \phi_{\dot{\alpha}}.
\]  

and

\[
\delta_Q \xi_i = \epsilon^I (R_I)_i \dot{\alpha} F_{\dot{\alpha}},
\]
\[
\delta_Q F_{\dot{\alpha}} = - i \epsilon^I (L_I)_i \dot{\alpha} \partial \phi_i.
\]

The examples above mentioned are just some cases of a wider class of inequivalent representations, referred to as “twisted” ones. The possibility to pass from a supermultiplet to its twisted version is realized by the so-called “mirror maps”. Moreover, it is possible to define superfields in a completely different manner by parameterizing the supermultiplet using component fields which take value in the algebra vector space. We will refer to these objects as Clifford algebraic superfields. An easy way to construct this kind of representations is tensoring the superspace \( \{V_L\} \oplus \{V_R\} \) with \( \{V_L\} \) or \( \{V_R\} \). For instance, if we multiply from the right by \( \{V_L\} \) then we have

\[
(\{V_L\} \oplus \{V_R\}) \otimes \{V_L\} = \{U_L\} \oplus \{M_L\}
\]

whose fields content is

\[
\phi_i \dot{\alpha} \in \{U_L\},
\]
\[
\psi_i \dot{\alpha} \in \{M_L\},
\]

with supersymmetry transformations

\[
\delta_Q \phi_i \dot{\alpha} = - i \epsilon^I (R_I)_i \dot{\alpha} \psi_i \dot{\alpha},
\]
\[
\delta_Q \psi_i \dot{\alpha} = \epsilon^I (L_I)_i \dot{\alpha} \partial \phi_i \dot{\alpha}.
\]

still defining a scalar supermultiplet. An analogous structure can be assigned to \( \{U_R\} \oplus \{M_L\}, \{U_L\} \oplus \{M_R\} \) and \( \{U_R\} \oplus \{M_R\} \) type superspaces. Even in these cases, twisted versions can be constructed applying considerations similar to those stated above. The important difference between the Clifford algebraic superfields approach and the \( \{V_L\} \oplus \{V_R\} \) superspace one, resides in the fact that in the latter case the number of bosonic fields (which actually describe coordinates) increases with the number of supersymmetric charges, while in the first case there is a way to make this not happen, allowing for a description of arbitrary extended supersymmetric spinning particle systems, as it will be shown in the third section.

1.3 \( \mathcal{GR}(d, N) \) algebras representation theory

It is time to clarify the link with real Clifford \( \Gamma \)-matrices of Weyl type (\( \equiv \) block skew diagonal) space which is easily seen to be
In fact, due to (6), \( \Gamma \)-matrices in (27) satisfy
\[
\{ \Gamma_I, \Gamma_J \} = -2i \delta_{IJ} I, \quad \forall I, J = 1, \ldots, N,
\]
which is the definition of Clifford algebras. One further \( \Gamma \)-matrix, namely
\[
\Gamma_{N+1} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},
\]
(29)
can be added. Therefore the complete algebra obeys the relationships
\[
\{ \Gamma_A, \Gamma_B \} = -2i \eta_{AB} I, \quad \forall A, B = 1, \ldots, N+1,
\]
(30)
where
\[
\eta_{AB} = \text{diag}(1, \ldots, 1, -1).
\]
(31)
In the following we assume that \( A, B \) indices run from 1 to \( N + 1 \) while \( I, J \) run from 1 to \( N \). The generator (29), that has the interpretation of a fermionic number, allow us to construct the following projectors on bosonic and fermionic sectors:
\[
P_{\pm} = \frac{1}{2} (I \pm \Gamma_{N+1}),
\]
(32)
which are the generators of the usual projectors algebra
\[
P_a P_b = \delta_{ab} P_a.
\]
(33)
Commutation properties of \( P_{\pm} \) with \( \Gamma \)-matrices are easily seen to be
\[
P_{\pm} \Gamma_I = \Gamma_I P_{\mp},
\]
\[
P_{\pm} \Gamma_{N+1} = \pm \Gamma_{N+1} P_{\mp}.
\]
(34)
The way to go back to \( GR(d, N) \) from a real Clifford algebra is through
\[
R_I = P_+ \Gamma_I P_-, \\
L_I = P_- \Gamma_I P_+,
\]
(35)
that yield immediately the condition
\[
R_{(ILJ)} = P_+ \Gamma_{(I} P_- \Gamma_{J)} P_+ = -2 \delta_{IJ} P_+ \equiv -2 \delta_{IJ} I_+ ,
\]
(36)
\[
L_{(IRJ)} = P_- \Gamma_{(I} P_+ \Gamma_{J)} P_- = -2 \delta_{IJ} P_- \equiv -2 \delta_{IJ} I_-.
\]
(37)
In this way, we have just demonstrated that representations of \( GR(d, N) \) are in one-to-one correspondence with real valued representations of Clifford
algebras, which will be classified in the following using considerations of \[1\]. To this end, let $M$ be an arbitrary $d \times d$ real matrix and let us consider

$$S = \sum_A \Gamma_A^{-1} M \Gamma_A , \quad (38)$$

then

$$\forall \Gamma_B \in C(p, q) , \quad \Gamma_B^{-1} S \Gamma_B = \sum_A (\Gamma_B \Gamma_A)^{-1} M \Gamma_A \Gamma_B = \sum_C \Gamma_C^{-1} M \Gamma_C = S , \quad (39)$$

where we have used the property of $\Gamma$-matrices

$$\Gamma_A \Gamma_B = \epsilon_{AB} \Gamma_C + \delta_{AB} I . \quad (40)$$

Equation (39) tells us that for all $\Gamma_A \in C(p, q)$ there exists at least one $S$ such that $[\Gamma_A, S] = 0$. Thus, by Shur’s lemma, $S$ has to be invertible (if not vanishing). It follows that any set of such $M$ matrices defines a real division algebra. As a consequence of a Frobenius theorem, three possibilities exist that we are going to analyze.

1. **Normal representations (N).**
   The division algebra is generated by the identity only

$$S = \lambda I , \quad \lambda \in R . \quad (41)$$

2. **Almost complex representations (AC).**
   There exists a further division algebra real matrix $J$ such that $J^2 = -I$ and we have

$$S = \mu I + \nu J , \quad \mu, \nu \in R . \quad (42)$$

3. **Quaternionic representations (Q).**
   Three elements $E_1$, $E_2$ and $E_3$ satisfying quaternionic relations

$$E_i E_j = -\delta_{ij} E + \sum_{k=1}^3 \epsilon_{ijk} E_k , \quad i, j = 1, 2, 3 , \quad (43)$$

are present in this case. Thus it follows

$$S = \mu I + \nu E_1 + \rho E_2 + \sigma E_3 , \quad \mu, \nu, \rho, \sigma \in R . \quad (44)$$

The results about irreducible representations obtained in \[1\] for $C(p, q)$ are summarized in the table 1.

The dimensions of irreducible representations are referred to faithful ones except the $p - q = 1, 5$ cases where exist two inequivalent representations of
the same dimension, related to each other by $\tilde{\Gamma}_A = -\Gamma_A$. In order to obtain faithful representations, the dimensions of those cases should be doubled defining

$$\tilde{\Gamma}_A = \begin{pmatrix} \Gamma_A & 0 \\ 0 & -\Gamma_A \end{pmatrix}.$$ \hfill (45)

Once the faithfulness has been recovered, we can say that a periodicity theorem holds, asserting that

$$C(p + 8, 0) = C(p, 0) \otimes M_{16}(R),$$ \hfill (46)
$$C(0, q + 8) = C(0, q) \otimes M_{16}(R),$$ \hfill (47)

where $M_r(R)$ stands for the set of all $r \times r$ real matrices. Furthermore we have

$$C(p, p) = M_r(R), \quad r = 2^n.$$ \hfill (48)

The structure theorems (47) and (48) justify the restriction in table 1 to values of $p - q$ from 0 to 7. As mentioned in [2], the dimensions reported in table 1 can be expressed as functions of the signature $(p, q)$ introducing integer numbers $k, l, m$ and $n$ such that

$$q = 8k + m, \quad 0 \leq m \leq 7,$$
$$p = 8l + m + n, \quad 1 \leq n \leq 8,$$ \hfill (49)

where $n$ fix $p - q$ up to $l - k$ multiples of eight as can be seen from

$$p - q = 8(l - k) + n,$$ \hfill (50)

while $m$ encode the $p, q$ choice freedom keeping $p - q$ fixed. Obviously $k$ and $l$ take into account the periodicity properties. The expression of irreducible representation dimensionalities reads
\[ d = 2^{4k+4l+m} F(n) \]  
where \( F(n) \) is the Radon-Hurwitz function defined by

\[ F(n) = 2^r, \quad \lfloor \log_2 n \rfloor + 1 \geq r \geq \lfloor \log_2 n \rfloor, \quad r \in \mathbb{N}. \]  

Turning back to \( \mathcal{GR}(d, N) \) algebras, from (31) we deduce that we have
to deal only with \( C(N, 1) \) case which means that irreducible representation
dimensions depend only on \( N \) in the following simple manner:

\[ d = 2^{4a} F(b) \]  
where \( N = 8a + b \) with \( a \) and \( b \) integer running respectively from 1 to 8 and
from 0 to infinity. This result can be straightforwardly obtained setting \( p = 1 \)
and \( q = N \) in equations (49). Representation dimensions obtained adapting
the results of table 1 to the \( C(N, 1) \) case are summarized in table 2.

|   | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  |
|---|----|----|----|----|----|----|----|----|
| \( b = \) | \( 2^4 \) | \( 2 \cdot 2^4 \) | \( 4 \cdot 2^4 \) | \( 4 \cdot 2^4 \) | \( 8 \cdot 2^4 \) | \( 8 \cdot 2^4 \) | \( 8 \cdot 2^4 \) | \( 8 \cdot 2^4 \) |

**Table 2.** Representation dimensions for \( \mathcal{GR}(d, N) \) algebras.

In what follows we focus our attention to the explicit representations
construction. First of all we enlarge the set of linear mappings acting between
\( \mathcal{V}_L \) and \( \mathcal{V}_R \), namely \( \mathcal{M}_L \oplus \mathcal{M}_R \) ( i.e. \( \mathcal{GR}(d, N) \) ), to \( \mathcal{U}_L \oplus \mathcal{U}_R \) defining the
enveloping general real algebra

\[ \mathcal{EGR}(d, N) = \mathcal{M}_L \oplus \mathcal{M}_R \oplus \mathcal{U}_L \oplus \mathcal{U}_R. \]  

As noticed before, we have the possibility to construct elements of \( \mathcal{U}_L \) and \( \mathcal{U}_R \)
as products of alternating elements of \( \mathcal{M}_L \) and \( \mathcal{M}_R \) so that

\[ L_1 R_J, L_1 R_J L_K R_L, \ldots \in \mathcal{U}_R, \]
\[ R_1 L_J, R_1 L_J R_K L_L, \ldots \in \mathcal{U}_L, \]  
but \( L_I \) and \( R_J \) comes from \( C(N, 1) \) through (35) so all the ingredients are
present to develop explicit representation of \( \mathcal{EGR}(d, N) \) starting from Clifford
algebra.
We focus now on the building of enveloping algebras representations starting from Clifford algebras. Indeed we need to divide into three cases.

1. **Normal representations.** In this case basic definition of Clifford algebra $(56)$ suggests a way to construct a basis $\{\Gamma\}$ by wedging $\Gamma$ matrices

$$\{\Gamma\} = \{I, \Gamma^I, \Gamma^{IJ}, \Gamma^{IJK}, ..., \Gamma^{N+1}\}, \quad I < J < K, ...,$$

where $\Gamma^I, ..., \Gamma^J$ are to be intended as the antisymmetrization of $\Gamma^I \cdot \Gamma^J$ matrices otherwise denoted by $\Gamma^{[I \cdots J]}$ or $\Gamma^{[N]}$ if the product involves $N$ elements. Dividing into odd and even products of $\Gamma$ we obtain the sets

$$\{\Gamma_o\} = \{I, \Gamma^{N+1}, \Gamma^{IJK}, \Gamma^{[N]}, \Gamma^{IJK}, \Gamma^{[N+1]}, \Gamma^{[N]} \},$$

$$\{\Gamma_e\} = \{\Gamma^I, \Gamma^I \Gamma^{N+1}, \Gamma^{IJK}, \Gamma^{IJK}, \Gamma^{[N+1]} \},$$

respectively related to $\{\mathcal{M}\}$ and $\{\mathcal{U}\}$ spaces. Projectors have the key role to separate left sector from right sector. In fact, for instance, we have

$$P_+ \Gamma_{IJ} P_+ = P_+ \Gamma_{IJ} P_+ P_+ = P_+ \Gamma_{IJ} P_+ P_+ = R_{[I} L_{J]} \in \{\mathcal{U}\}.$$

and in a similar way $P_- \Gamma_{IJ} P_- = L^{[I} R^{J]} \in \{\mathcal{U}\}$. Projectors have the key role to separate left sector from right sector. In fact, for instance, we have

$$\{\mathcal{U}_R\} = \{P_-, P_-, \Gamma_{IJ} P_-, ..., P_- \Gamma^{[N]} P_-\} = \{I_{[I}, L^{[I} R^{J]}, \ldots\},$$

$$\{\mathcal{M}_R\} = \{P_+, \Gamma^{[I} P_{+, ..., P_+, \Gamma^{[N-1]} P_+\} = \{R^{[I}, R^{[I} L^{J} R^{K]}, \ldots\},$$

$$\{\mathcal{U}_L\} = \{P_+, P_+, \Gamma_{IJ} P_+, ..., P_+ \Gamma^{[N]} P_+\} = \{I_{[I}, L_{[I} R_{J]} L_{K]}, \ldots\},$$

$$\{\mathcal{M}_L\} = \{P_- \Gamma_{IJ} P_+, ..., P_- \Gamma^{[N-1]} P_+\} = \{L_{[I}, L_{[I} R_{J} L_{K}], \ldots\},$$

(59)

which we will denote as $\wedge \mathcal{GR}(d, N)$ to remember that it is constructed by wedging $L_I$ and $R_J$ generators. Clearly enough, from each $\Gamma_{[I \cdots J]}$ matrix we get two elements of $\mathcal{E} \mathcal{GR}(d, N)$ algebra as a consequence of the projection. Thus we can say that in the normal representation case, $C(N, 1)$ is in one-two correspondence with the enveloping algebra which can be identified by $\wedge \mathcal{GR}(d, N)$. By the wedging construction in $(59)$ naturally arise p-forms that is useful to denote

$$f_I = L_I, \quad \hat{f}_I = R_I,$$

$$f_{IJ} = R_{[I} L_{J]}, \quad \hat{f}_{IJ} = L^{[I} R^{J]},$$

$$f_{IJK} = L_{[I} R_{J} L_{K]}\,$$

$$\vdots \quad \vdots,$$

(60)
The superfield components for the \{U_L\} ⊕ \{M_L\} type superspace introduced in (25), can be expanded in terms of this normal basis as follows

\[\phi^j_i = \phi \delta^j_i + \phi^{IJ} (f_{IJ})^j_i + \ldots \in \{U_L\},\]
\[\psi_{\dot{\alpha}}^i = \psi^I (f_I)_{\dot{\alpha}}^i + \psi^{IJK} (f_{IJK})_{\dot{\alpha}}^i + \ldots \in \{M_L\}\]

according to the fact that \(f_{[\text{even}]}) \in \{U_L\}\) and \(f_{[\text{odd}]}) \in \{M_L\}\). We will refer to this kind of superfields as bosonic Clifford algebraic ones because of the bosonic nature of the level zero field. Similar expansion can be done for the \{U_R\} ⊕ \{M_R\} type superspace where \(\hat{f}_{[\text{even}]}) \in \{U_R\}\) and \(\hat{f}_{[\text{odd}]}) \in \{M_R\}\)

\[\phi^{\dot{\beta}}_\dot{\alpha} = \phi \delta^{\dot{\beta}}_\dot{\alpha} + \phi^{IJ} (\hat{f}_{IJ})^{\dot{\beta}}_{\dot{\alpha}} + \ldots \in \{U_R\},\]
\[\psi^i_\dot{\alpha} = \psi^I (\hat{f}_I)^i_\dot{\alpha} + \psi^{IJK} (\hat{f}_{IJK})^i_\dot{\alpha} + \ldots \in \{M_R\}\]

In the (62) case we deal with a fermionic Clifford algebraic superfield because the component \(\phi\) is a fermion. For completeness we include the remaining cases, namely \{U_R\} ⊕ \{M_L\} superspace

\[\phi^{\dot{\beta}}_\dot{\alpha} = \phi \delta^{\dot{\beta}}_\dot{\alpha} + \phi^{IJ} (\hat{f}_{IJ})^{\dot{\beta}}_{\dot{\alpha}} + \ldots \in \{U_R\},\]
\[\psi^i_\dot{\alpha} = \psi^I (\hat{f}_I)^i_\dot{\alpha} + \psi^{IJK} (\hat{f}_{IJK})^i_\dot{\alpha} + \ldots \in \{M_L\}\]

and \{U_L\} ⊕ \{M_R\} superspace

\[\phi^j_i = \phi \delta^j_i + \phi^{IJ} (f_{IJ})^j_i + \ldots \in \{U_L\},\]
\[\psi_{\dot{\alpha}}^i = \psi^I (f_I)_{\dot{\alpha}}^i + \psi^{IJK} (f_{IJK})_{\dot{\alpha}}^i + \ldots \in \{M_R\}\]

2. **Almost complex representations.** As already pointed out, those kind of representations contain one more generator \(J\) with respect to normal representations so that to span all the space, the normal part, which is generated by wedging, is doubled to form the basis for the Clifford algebra

\[\{\Gamma\} = \{I, J, I^J, I^I J, I^J J, \ldots, \Gamma^{N+1}, \Gamma^{N+1} J\}\].

Starting from (65) it is straightforward to apply considerations from (67) to (69) to end with an \(\mathcal{EGR}(d, N)\) almost complex representation in 1-2 correspondence with the previous. Concerning almost complex Clifford algebra superfields, it is important to stress that we obtain irreducible representations only restricting to the normal part.

3. **Quaternionic representations.** Three more generators \(E^\alpha\) satisfying

\[\left[ E^\alpha, E^\beta \right] = 2\epsilon^{\alpha\beta\gamma} E^\gamma\]

have to be added to the normal part to give the following quaternionic Clifford algebra basis
\[ \{ \Gamma \} = \{ I, E^\alpha, \Gamma^I, \Gamma^I E^\alpha, \Gamma^{IJ}, \Gamma^{IJ} E^\alpha, ..., \Gamma^{N+1}, \Gamma^{N+1} E^\alpha \}, \quad (67) \]

which is four times larger than the normal part. Again, repeating the projective procedure presented above, generators of Clifford algebra are quadrupled to produce the \( \mathcal{EGR}(d, N) \) quaternionic representation. Even in this case only the normal part gives irreducible representations for the Clifford algebra superfields.

Notice that from the group manifold point of view, the presence of the generator \( J \) for the almost complex case and generators \( E^\alpha \) for the quaternionic one, separate the manifold into sectors which are not connected by left or right group elements multiplication giving rise to intransitive spaces. Division algebra has the role to link those different sectors.

Finally we explain how to produce an explicit matrix representation using a recursive procedure mentioned in [3] that can be presented in the following manner for the case \( N = 8a + b \) with \( a \geq 1 \):

\[
\begin{align*}
L_1 &= i\sigma^2 \otimes I_b \otimes I_{8a} = R_1, \\
L_I &= \sigma^3 \otimes (L_b)_I \otimes I_{8a} = R_I, \quad 1 \leq I \leq b - 1, \\
L_J &= \sigma^1 \otimes I_b \otimes (L_{8a})_J = R_J, \quad 1 \leq I \leq 8a - 1, \\
L_N &= I_2 \otimes I_b \otimes I_{8a} = -R_N, \\
\end{align*}
\]

where \( I_n \) stands for the n-dimensional identity matrix while \( L_b \) and \( L_{8a} \) are referred respectively to the cases \( N = b \) and \( N = 8a \). Expressions for the cases where \( N \leq 7 \) which are the starting points to apply the algorithm in (68), can be found in appendix A of [4].

2 Relationships between different models

It turns out that apparently different supermultiplets can be related to each other using several operations.

1. leaving \( N \) and \( d \) unchanged, one can increase or decrease the number of physical bosonic degrees of freedom (while necessarily and simultaneously to decrease or increase the number of auxiliary bosonic degrees of freedom) within a supermultiplet by shifting the level of the superfield \( \theta \)-variables expansion by mean of an automorphism on the superalgebra representation space, commonly called automorphic duality (AD).
2. it is possible to reduce the number of supersymmetries maintaining fixed representation dimension (reduction).
3. the space-time coordinates can be increased preserving the supersymmetries (oxidation).
4. by a space-time compactification, supersymmetries can be eventually increased.
These powerful tools can be combined together to discover new supermultiplets or to relate the known ones. The first two points will be analyzed the following paragraphs while for the last two procedures, we remind to \[5, 6\] and references therein.

### 2.1 Automorphic duality transformations

Until now, we encountered the following two types of representation: the first one defined on \(\mathcal{V}_L\) and \(\mathcal{V}_R\) superspace complemented with the second one, Clifford algebraic superfields. In the latter case we observed that in order to obtain irreducible representations, is needed a restriction to normal representations or to their normal parts. If we consider irreducible cases of Clifford algebraic superfields then there exists the surprisingly possibility to transmute physical fields into auxiliary ones changing the supermultiplet degrees of freedom dynamical nature. The best way to proceed for an explanation of the subject is to begin with the \(N = 1\) example which came out to be the simplest. In this case only two supermultiplets are present

- the scalar supermultiplet \((X, \psi)\) respectively composed of one bosonic and one fermionic field arranged in the superfield

\[
X(\tau, \theta) = X(\tau) + i\theta \psi(\tau) ,
\]

(69)

with transformation properties

\[
\delta_Q X = i\epsilon \psi , \\
\delta_Q \psi = \epsilon \partial_\tau X ;
\]

(70)

- the spinor supermultiplet \((\xi, A)\) respectively composed of one bosonic and one fermionic field arranged in the superfield

\[
Y(\tau, \theta) = \xi(\tau) + \theta A(\tau) ,
\]

(71)

with transformation properties

\[
\delta_Q A = i\epsilon \partial_\tau \xi , \\
\delta_Q \xi = \epsilon A .
\]

(72)

The invariant Lagrangian for the scalar supermultiplet transformations

\[
\mathcal{L} = \dot{X}^2 + ig\psi \dot{\psi} ,
\]

(73)

gives to the fields \(X\) and \(\psi\) a dynamical meaning and offers the possibility to perform an automorphic duality map that at the superfield level reads

\[
Y(\tau, \theta) = -iDX(\tau, \theta) ,
\]

(74)
where $\mathcal{D} = \partial_\theta + i \partial_\tau$ is the superspace covariant derivative. At the component level, it corresponds to the map upon bosonic components

$$X(\tau) = \partial_\tau^{-1} A(\tau), \quad (75)$$

and identification of fermionic ones. The mapping is intrinsically not local but it can be implemented in a local way both in the transformations and in the Lagrangian, producing respectively equations and the Lagrangian

$$L = A^2 + ig\dot{\psi} \dot{\psi}. \quad (76)$$

As a result we get that automorphic duality transformations map $N = 1$ supermultiplets into each other in a local way, changing the physical meaning of the bosonic field $X$ from dynamical to auxiliary $A$ (not propagating) as is showed by the Lagrangian invariant for transformations. Note that the auxiliary meaning of $A$ is already encoded into transformations that enlighten on the nature of the fields and consequently of the supermultiplet.

Let us pass to the analysis of the $N = 2$ case making a link with the considerations about representation theory discussed above. At the $N = 2$ level, we deal with a $AC$ representation so, in order to implement AD transformations, we focus on the normal part, namely $\wedge GR(d, N)$, defining the Clifford algebraic bosonic superfield

$$\phi^i_j = \phi^i_j + \phi^{IJ}(f_{IJ})^i_j, \quad \psi^i_\alpha = \psi^I(f_I)^i_\alpha, \quad (77)$$

constructed with the forms. Notice that if we work in a $N$-dimensional space then the highest rank for the forms is $N$. This is the reason why, writing, we stopped at level. Some comments about transformation properties. By comparing each level of the expansion, it is straightforward to prove that superfields transform according to if the component fields transformations are recognized to be

$$\delta Q \phi^{I_1 \cdots I_{p_{even}}} = -i\epsilon^{[I_1} \phi^{I_2 \cdots I_{p_{even}] + i(p_{even} + 1)}\epsilon_{I} \phi^{I_1 \cdots I_{p_{even}} J},$$

$$\delta Q \psi^{I_1 \cdots I_{p_{odd}}} = -\epsilon^{[I_1} \dot{\phi}^{I_2 \cdots I_{p_{odd}] + i(p_{odd} + 1)}\epsilon_{I} \dot{\phi}^{I_1 \cdots I_{p_{odd}} J}. \quad (78)$$

Therefore equations for the $N = 2$ case read

$$\delta Q \phi = i\epsilon_I \psi^I, \quad \delta Q \psi^I = -\epsilon^I \dot{\phi} + 2\epsilon_J \dot{\phi}^{IJ}, \quad \delta Q \phi^{IJ} = -i\epsilon^{[I} \dot{\phi}^{J]. \quad (79)$$

Once again, transformations admit local AD maps between bosonic fields. In order to discuss this in a way that brings this discussion in line with that of, we adhere to a convention that list three numbers ($PB, PF, AB$)
where \(PB\) denotes the number of ‘propagating’ bosonic fields, \(AB\) denotes the number of ‘auxiliary’ bosonic fields and \(PF\) denotes the number of fermionic fields.

We briefly list the resulting supermultiplets arising from the dualization procedure.

- The AD map involving \(\phi\) field

\[
\phi(\tau) = \partial^{-1}_\tau A(\tau),
\]

yield a \((1, 2, 1)\) supermultiplet whose transformations properties are

\[
\delta_Q A = i\epsilon_I \partial_\tau \psi^I, \\
\delta_Q \psi^I = -\epsilon^I A + 2\epsilon_J \phi^{IJ}, \\
\delta_Q \phi^{IJ} = -i\epsilon^{IJ} \partial_\tau \psi^J.
\]

- By redefining the \(\phi^{IJ}\) field

\[
\phi^{IJ} = \partial^{-1}_\tau B^{IJ},
\]

another \((1, 2, 1)\) supermultiplet is obtained. Accordingly, we have

\[
\delta_Q \phi = i\epsilon_I \psi^I, \\
\delta_Q \psi^I = -\epsilon^I \phi + 2\epsilon_J A^{IJ}, \\
\delta_Q A^{IJ} = -i\epsilon^{IJ} \partial_\tau \psi^J.
\]

- Finally, if both redefinitions (80) and (82) are adopted, then we are left with \((0, 2, 2)\) spinor supermultiplet whose components behave as

\[
\delta_Q A = i\epsilon_I \partial_\tau \psi^I, \\
\delta_Q \psi^I = -\epsilon^I A + 2\epsilon_J A^{IJ}, \\
\delta_Q A^{IJ} = -i\epsilon^{IJ} \partial_\tau \psi^J.
\]

It is important to stress that we can make redefinitions of bosonic fields via AD maps that involve higher time derivatives. For instance, by applying \(\partial^2\) to the first equation in (79) together with the new field introduction

\[
\phi = \partial^{-2}_\tau C,
\]

transformations turn out to be free from nonlocal terms if AD for the remaining fields

\[
\psi^I = i\partial^{-3}_\tau \xi^I \\
\phi^{IJ} = \partial^{-1}_\tau B^{IJ}
\]

are enforced. Thus we end with
\[
\delta_Q C = - \epsilon_I \partial_I \xi^I \\
\delta_Q \xi^I = i \epsilon^I C - 2i \epsilon_J \bar{D}^{IJ} \\
\delta_Q D^{IJ} = \epsilon^{[I} \xi^{J]} .
\] (87)

From equations (87) one may argue that \( C \) is auxiliary while \( D \) is physical. The point is which is the meaning of the fields we started from? An invariant action from (87) is
\[
L = C^2 + ig \xi \dot{\xi} + \dot{D}^{IJ} \dot{D}^{IJ},
\] (88)
so that going backward, we can deduce the initial action
\[
L = \dddot{\phi}^2 + ig \dot{\psi} \dddot{\psi} + \dddot{\phi}_{IJ} \dddot{\phi}_{IJ} .
\] (89)

The examples above should convince any reader that Clifford superfields are a starting point to construct a wider class of representation by means of AD maps. Following this idea, one can identify each supermultiplet with a correspondent root label \((a_1, \ldots, a_k)\) where \(a_i \in \mathbb{Z}\) are defined according to
\[
(\tilde{\phi}, \tilde{\psi}^I, \tilde{\phi}^{IJ}, \ldots)_+ = (\partial^{-a_0} \phi, \partial^{a_1} \psi^I, \partial^{-a_2} \phi^{IJ}, \ldots)_+,
(\tilde{\psi}, \tilde{\phi}^I, \tilde{\psi}^{IJ}, \ldots)_- = (\partial^{a_0} \psi, \partial^{-a_1} \phi^I, \partial^{a_2} \psi^{IJ}, \ldots)_- ,
\] (90)
and \(\pm\) distinguish between Clifford superfields of bosonic and fermionic type. For instance, the last supermultiplet (87), corresponds to the case \((a_0, a_1, a_2) = (2, -1, 1)\). We name base superfield the one with all zero in the root label \((0, \ldots, 0)\), underling that in the plus (minus) case, this supermultiplet has to be intended as the one with all bosons (fermion) differentiated in the r.h.s. of variations. They are of particular interest the supermultiplets whose roots label involve only \(0\) and \(1\). All these supermultiplets form what we call root tree.

### 2.2 Reduction

It is shown in table 2 that \(N = 8, 7, 6, 5\) irreducible representations have the same dimension. The same happens for the \(N = 4, 3\) cases. This fact reflect the possibility to relate those supermultiplets via a reduction procedure. In order to explain how this method works, consider a form \( f_{I_1 \ldots I_K} \) and notice that the indices \( I_1, \ldots, I_K \) run on the number of supersymmetries: reducing this number corresponds to diminishing the components contained in the rank \( k \) form. The remaining components has to be rearranged into another form. For instance if we consider a 3-rank form for the \(N = 8\) case then the number of components is given by\(^5\) \( \binom{8}{3} = 56 \) but, reducing to the \(N = 7\) case and

\(^5\) For the construction of \(N = 8\) supersymmetric mechanics, see [9]; the nonlinear chiral multiplet has been used in this connection [10], as well as in related tasks [11].
leaving invariant the rank, we get $\binom{7}{3} = 35$ components. The remaining ones can be rearranged in a 5-rank form. This means that the maximum rank of Clifford superfield expansion is raised until the irreducible representation dimension is reached. However the right way to look at this rank enhancing is through duality. An enlightening example will be useful. By a proper counting of irreducible representation dimension for the $EGR(8,8)$, we are left with $\{\mathcal{U}_L\} \oplus \{\mathcal{M}_L\}$ type Clifford algebraic superfield

$$
\phi_{ij} = \phi \delta_{ij} + \phi^{IJ} (f_{IJ})_{ij} + \phi^{IJKL} (f_{IJKL})_{ij}
$$
$$
\psi_{\dot{i}i} = \psi^{I} (f_{I})_{\dot{i}i} + \psi^{IJ} (f_{IJK})_{\dot{i}i}
$$

where the 4-form has definite duality or, more precisely, the sign in the equation

$$
\epsilon^{IJKLMNPQ} f_{MNQP} = \pm f^{IJKLM},
$$

has been chosen, halving the number of independent components. In order to reduce to the $N = 7$ case, we need to eliminate all “8” indices and this can be done by exploiting the duality. For instance, $f_{I8}$ can disappear if transformed into

$$
\epsilon^{IJKLMNP8} f_{P8} = \pm f^{IJKLMN}.
$$

This trick adds the 6-rank to the expansion manifesting the enhancing phenomenon previously discussed. Once the method is understood, it is straightforward to prove that for the $N = 7$ case, the proper superfiels expression is

$$
\phi_{ij} = \phi \delta_{ij} + \phi^{IJ} (f_{IJ})_{ij} + \phi^{IJKL} (f_{IJKL})_{ij} + \phi^{IJKLM} (f_{IJKLM})_{ij}
$$
$$
\psi_{\dot{i}i} = \psi^{I} (f_{I})_{\dot{i}i} + \psi^{IJ} (f_{IJK})_{\dot{i}i} + \psi^{IJKL} (f_{IJKL})_{\dot{i}i} + \psi^{IJKLM} (f_{IJKLM})_{\dot{i}i}
$$

The explicit reduction procedure for $N \leq 8$ can be found in [12] and summarized in the following tables:

## 3 Applications

### 3.1 Spinning particle

Before we begin a detailed analysis of spinning particle system it is important to understand what a spinning particle is. Early models of relativistic particle with spin involving only commuting variables can be divided into the two following classes:

- vectorial models, based upon the idea of extending Minkowski space-time by vectorial internal degrees of freedom;
Table 3. $\mathcal{EGR}(4,4)$ and its reduction: algebras representation in terms of forms and division algebra. Here and in the following table, the generators $E^\mu, \epsilon_E^\mu$ are respectively the + and - projections of the quaternionic division algebra generators in the Clifford space. The same projection on complex structure originate $D, \hat{D}$.

| $\mathcal{EGR}(d,N)$ | $\mathcal{AGR}(d,N)$ basis | Division Structure |
|------------------------|-------------------------------|---------------------|
| $\mathcal{EGR}(4,4)$   | $\{U_L\} = \{I, f_{1IJ}, E^\mu, f_{1IJ}E^\mu\}$ | $E^\mu, \epsilon_E^\mu$ |
|                        | $\{M_L\} = \{f_1, f_1\hat{E}^\mu\}$ |                      |
| $\mathcal{EGR}(4,3)$   | $\{U_L\} = \{I, f_{1IJ}, E^\mu, f_{1IJ}E^\mu\}$ | $E^\mu, \epsilon_E^\mu$ |
|                        | $\{M_L\} = \{f_1, f_1\hat{E}^\mu, f_{1JK}, f_{1JK}\hat{E}^\mu\}$ |                      |

- spinorial models, characterized by the enhancing of configuration space using spinorial commuting variables.

These models lack of the following important requirement: after first quantization, they never produce relativistic Dirac equations. Moreover, in the spinorial cases, a tower of all possible spin values appear in the spectrum. Further progress in the development of spinning particle descriptions was achieved by the introduction of anticommuting variables to describe internal degrees of freedom \cite{13}. This idea stems from the classical limit ($\hbar \to 0$) formulation of Fermi systems \cite{14}, the so called 'pseudoclassical mechanics' referring to the fact that it is not a ordinary mechanical theory because of the presence of Grassmannian variables. By means of pseudoclassical approach, vectorial and spinorial models can be generalized to 'spinning particle' and 'superparticle' models, respectively. In the first case, the extension to superspace $(x_\mu, \theta_\mu, \theta_5)$ is made possible by a pseudovector $\theta_\mu$ and a pseudoscalar $\theta_5$ \cite{15}, \cite{16}. The presence of vector index associated with $\theta$-variables implies the vectorial character of the model.

In the second case, spinorial coordinates are considered, giving rise to ordinary superspace approach whose underlying symmetry is the super Poincaré group (eventually extended) \cite{14}. The superparticle is nothing but a generalization of relativistic point particle to superspace.

It turns out that after first quantization, the spinning particle model produced Dirac equations and all Grassmann variables are mapped into Clifford algebra generators. Superfields that take values on this kind of quantized superspace are precisely Clifford algebraic superfields described in the previous paragraphs. On the other side, a superspace version of Dirac equation arises...
from superparticle quantization. Moreover, θ-variables are still present in the quantized version.

To have a more precise idea, we spend a few words discussing the Barducci-Casalbuoni-Lusanna model [15] which is one of the first works on pseudoclassical model. As already mentioned, it is assumed the configuration space to be described by \((x_\mu, \theta_\mu, \theta_5)\). The Lagrangian of the system

\[
\mathcal{L}_{BCL} = -m \sqrt{\left( \frac{\dot{x}_\mu - i}{m} \theta_\mu \dot{\theta}_5 \right) \left( \dot{x}_\mu - i \theta_\mu \dot{\theta}_5 \right) - \frac{i}{2} \theta_\mu \dot{\theta}_\mu - \frac{i}{2} \theta_5 \dot{\theta}_5} \tag{95}
\]

is invariant under the transformations

\[
\begin{align*}
\delta x_\mu &= -\epsilon_\mu a \theta_5 + \epsilon_5 b \theta_\mu , \\
\delta \theta_\mu &= \epsilon_\mu , \\
\delta \theta_5 &= \epsilon_5 .
\end{align*}
\tag{96}
\]

Table 4. \(E\Gamma R(8, 8)\) and its reductions. Here the subscript \([n]\) is used in place of \(n\) anticommuting indices.
and produces the equations of motion

\[
p^2 - m^2 = 0 , \\
p_{\mu} \theta^{\mu} - m \theta_5 = 0 ,
\]  
(97)

after a canonical analysis. These equations (97) are classical limits of Klein-Gordon and Dirac equations, respectively. Moreover, the first quantization maps \( \theta \)-variables into Clifford algebra generators

\[
\theta_\mu \rightarrow \gamma_\mu \gamma_5 \ (\text{pseudovector}) , \\
\theta_5 \rightarrow \gamma_5 \ (\text{pseudoscalar}) , 
\]  
(98)

so that equations (97) exactly reproduce relativistic quantum behavior of a particle with spin. Even if it is not manifest, it is possible to find a particular direction in the \((\theta_\mu, \theta_5)\) space along which the theory is invariant under the following localized supersymmetry transformation

\[
\delta x_\mu = 2 i \frac{e_5(\tau)}{m^2} P_\mu \theta_5(\tau) - \frac{i}{m} \epsilon_5(\tau) \theta_\mu , \\
\delta \theta_\mu = \frac{1}{m} \epsilon_5(\tau) P_\mu , \\
\delta \theta_5 = \epsilon_5(\tau) ,
\]  
(99)

opening the way to supergravity. Basic concepts on the extension to minimal supergravity-coupled model can be found in [17]. Here the proposed action is a direct generalization of 1-dimensional general covariant free particle to include the spin; for the first-order formalism in the massless case we have

\[
S = \int d\tau \{ P^\mu \dot{X}_\mu - \frac{1}{2} \epsilon P^2 - \frac{i}{2} \dot{\psi}^\mu \dot{\psi}_\mu - \frac{i}{2} \chi \psi^\mu P_\mu \} ,
\]  
(100)

with the local invariances

\[
\delta \psi^\mu = \epsilon(\tau) P^\mu , \quad \delta \phi^\mu = i \epsilon(\tau) \psi^\mu , \quad \delta P^\mu = 0 , \\
\delta e = i \epsilon(\tau) \chi , \quad \delta \chi = 2 \dot{\epsilon}(\tau) ,
\]  
(101)

corresponding to pure supergravity transformations as is shown calculating the commutators

\[
[\delta_{\epsilon_1}, \delta_{\epsilon_2} ] X^\mu = \xi X^\mu + i \dot{\epsilon} \psi^\mu , \\
[\delta_{\epsilon_1}, \delta_{\epsilon_2} ] \psi^\mu = \xi \psi^\mu + i P^\mu , \\
[\delta_{\epsilon_1}, \delta_{\epsilon_2} ] e = \xi e + \dot{\xi} e + i \dot{\epsilon} \chi , \\
[\delta_{\epsilon_1}, \delta_{\epsilon_2} ] \chi = \xi \chi + \dot{\xi} \chi + 2 \dot{\epsilon} ,
\]  
(102)

where

\[
\xi = 2 ie^{-1} \epsilon_2 \epsilon_1 , \\
\dot{\epsilon} = -\frac{1}{2} \xi \chi .
\]  
(103)
In fact, the r.h.s. of (102) describes both general coordinate and local supersymmetry transformations.

In order to produce a mass-shell condition, the massive version of the above model require the presence of a cosmological term in the action

\[ S = -\frac{1}{2} \int d\tau m^2 \]  

that, in turn, imply the presence of an additional anticommuting field \( \psi_5 \), transforming through

\[ \delta \psi_5 = m \tilde{\epsilon} , \]  

(105)

to construct terms that restore the symmetries broken by (104). The complete action describing the massive spinning particle version minimally coupled to supergravity multiplet turns out to be

\[ S = \int d\tau \left[ P^\mu \dot{X}_\mu - \frac{1}{2} \epsilon (P^2 + m^2) - i \left( \psi_1^\mu \dot{\psi}_1^\mu + \psi_5 \dot{\psi}_5 \right) - i \chi (\psi^\mu \dot{P}_\mu + m \psi_5) \right] . \]  

(106)

The second-order formalism for the massless and massive model follow straightforwardly from actions (106) and (100) eliminating the \( P \) fields using their equations of motion.

An advance on this line of research yielded the on-shell N-extension [18]. However a satisfactory off-shell description with arbitrary N require the \( \mathcal{G}R(d, N) \) approach. In the paragraphs below we describe in detail how this construction is worked out.

**Second-order formalism for spinning particle with rigid N-extended supersymmetry**

The basic objects of this model are Clifford algebraic bosonic superfields valued in \( \{ \mathcal{U}_L \} \oplus \{ \mathcal{M}_L \} \) superspace with transformations [26]. One can easily check that the action

\[ S = \int d\tau \left\{ \partial_\tau (\phi_1)_i^j \partial_\tau (\phi_1)_j^i + i (\psi_1)_i^\alpha \partial_\tau (\psi_1)_i^\alpha \right\} \]  

(107)

is left unchanged by (26). The next step consists in separating the physical degrees of freedom in \( (\phi_1)_i^j, (\psi_1)_i^\alpha \) from nonphysical ones. For the bosonic superfield, valued in \( \{ \mathcal{U}_L \} \), we separate the trace from the remaining components

\[ (\phi_1)_i^j = X \delta_i^j + \tilde{\phi}_i^j , \]

\[ X = \phi_i^i , \quad \tilde{\phi}_i^i = 0 , \]  

(108)

and perform an AD transformation on tilded components.
to end with the decomposition

$$\phi^j_i = X\delta^j_i + \partial^{-1}\tau F^j_i ,$$  \hfill (109)

constrained by the equation

$$F^i_i = 0 .$$  \hfill (110)

The field component $X$ can be interpreted as the spinning particle bosonic coordinate in a background space. Nothing forbids us from considering $D$ supermultiplet of this kind that amount to add a $D$-dimensional background index $\mu$ to the superfields

$$\begin{align*}
(\phi_1)^j_i &\rightarrow (\phi_1^\mu)^j_i , \\
(\psi_1)^i_\alpha &\rightarrow (\psi_1^\mu)^i_\alpha .
\end{align*}$$  \hfill (112)

In this way the dimension of the background space has no link neither with the number of supersymmetries nor with representation dimension. However, to simplify the notation, background index will be omitted. Transformations involving the fields defined in (111) reads

$$\begin{align*}
\delta_Q X &= -\frac{1}{d}\imath\epsilon^I(R_l)^i_\alpha^\beta(\psi_1)^i_\alpha , \\
\delta_Q F^j_i &= \imath\epsilon^I(R_l)^{j}_\alpha^\beta\partial_\tau(\psi_1)^i_\alpha , \\
\delta_Q (\psi_1)^i_\alpha &= \epsilon^I(L_l)^{j}_\alpha^\beta F^j_i + \epsilon^I(L_l)^j_\alpha^\beta \partial_\tau X .
\end{align*}$$  \hfill (113)

Even in the fermionic case, we need that only the lowest component in the expansion \ref{61} has physical meaning so that the higher level components happen to be distributed in the following manner

$$\begin{align*}
(\psi_1)^i_\alpha &= \psi^I(L_l)^{j}_\alpha^\beta + \psi^{\tilde{I}}(L_l)^{j}_{\tilde{\alpha}}^\beta = \psi^I(L_l)^{j}_\alpha^\beta + \mu^j_\alpha ,
\end{align*}$$  \hfill (114)

where $\psi^I = \frac{1}{d}(R_l)^i_\alpha^\beta(\psi_1)^i_\alpha$ and the fermionic superfield $\mu^j_\alpha$ obey the constraint equation

$$(R_l)^i_\alpha^\beta\mu^j_\alpha = 0 .$$  \hfill (115)

After the substitution of the new component fields \ref{61}, transformations \ref{113} became

$$\begin{align*}
\delta_Q X &= -\frac{1}{d}\imath\epsilon^I(R_l)^i_\alpha^\beta(L_l)^j_\beta^\alpha\psi^j , \\
\delta_Q F^j_i &= \imath\epsilon^I(R_l)^{j}_\alpha^\beta(L_l)^i_\beta^\alpha\partial_\tau \psi^j - \imath\epsilon^I(R_l)^{j}_\alpha^\beta\partial_\tau\mu^j_\alpha , \\
(L_l)^i_\alpha\delta_Q \psi^j + \delta_Q \mu^j_\alpha &= \epsilon^I(L_l)^j_\alpha^\beta F^j_i + \epsilon^I(L_l)^j_\alpha^\beta \partial_\tau X .
\end{align*}$$  \hfill (118)
where we used (115) to obtain (116). Equations (116) and (117) can be simplified into
\[ \delta Q X = i \epsilon^I \psi_I, \]
\[ \delta Q F^j_i = i \epsilon^I (f_{IJ})^j_i \partial_\tau \psi^I - i \epsilon^I (R_I)_{\hat{\alpha} \hat{\beta}} \partial_\tau \mu^I_{\hat{\alpha} \hat{\beta}}, \]
(119)
if one notice that
\[ (R_I)_{\hat{\alpha} \hat{\beta}} (L_J)_{\hat{\beta} \hat{\alpha}} = -d \delta_{IJ}, \]
\[ (R_I)_{\hat{\alpha} \hat{\beta}} (L_J)_{\hat{\beta} \hat{\alpha}} = (f_{IJ})^j_i \]
(120)
while the equation (118) need more care. In order to separate the variation of \( \psi^I \) and \( \mu^I_{\hat{\alpha} \hat{\beta}} \), we multiply by \( (R_I)_{\hat{\alpha} \hat{\beta}} \) to eliminate the \( \mu^I_{\hat{\alpha} \hat{\beta}} \) contribution thanks to (115). As a result we get
\[ -d \delta Q \psi_J = \epsilon^I (f_{IJ})^j_i \mathcal{F}^j_i - d\epsilon_J \partial_\tau X \]
\[ \delta Q \psi_I = \epsilon_I \partial_\tau X - \frac{1}{d} \epsilon^I (f_{IJ})^j_i \mathcal{F}^j_i. \]
(121)
Substituting back (121) into (118) we finally have
\[ \delta Q \mu^I_{\hat{\alpha} \hat{\beta}} = -(L_I)_{\hat{\alpha} \hat{\beta}} \left[ \epsilon^I \partial_\tau X - \frac{1}{d} \epsilon^I (f_{IJ})^j_i \mathcal{F}^j_i \right] + \epsilon^I (L_I)_{\hat{\alpha} \hat{\beta}} \mathcal{F}^j_i + \epsilon^I (L_I)_{\hat{\alpha} \hat{\beta}} \partial_\tau X \]
\[ = \left[ \epsilon^I (\hat{f}_{IJ})^j_i + \frac{1}{d} \epsilon^I (\hat{f}_{IJ})^j_i (f_{IJ})^j_i \right] \mathcal{F}^j_i. \]
(122)
The supermultiplet \( (X, \mathcal{F}^j_i, \psi_I, \mu^I_{\hat{\alpha} \hat{\beta}}) \) together with transformations (119), (121) and (122), is called ‘universal spinning particle multiplet’ (USPM). Acting with the maps (111) and (113) on the action (107) we obtain the USPM invariant action
\[ S = \int d\tau \{ d(\partial_\tau X \partial_\tau X - i \psi_I \partial_\tau \psi_I) + \mathcal{F}^j_i \mathcal{F}^j_i + i \mu^I_{\hat{\alpha} \hat{\beta}} \partial_\tau \mu^I_{\hat{\alpha} \hat{\beta}} \} \]
(123)
that represent the second-order approach to the spinning particle problem with global supersymmetry. A remarkably difference between the AD presented in section (2.1) and the AD used to derive USPM resides in the fact that in the latter case we map \( \tilde{\phi}^j_i \) and \( \tilde{\psi}^I_{\hat{\alpha} \hat{\beta}} \) which are Clifford algebraic superfield while in the previous we work at the component level. Finally, it is important to keep in mind that the superfields \( \tilde{\phi}^j_i \) and \( \tilde{\psi}^I_{\hat{\alpha} \hat{\beta}} \) take values on the normal part of the enveloping algebra which is equivalent to say that they can be expanded on the basis (60).
First-order formalism for spinning particle with rigid N-extended supersymmetry

To formulate a first-order formalism, one more fermionic supermultiplet is required. This time the superfields \((\phi_2)_i^j, (\psi_2)_i^{\hat{\alpha}}\), valued in \(\{\mathcal{U}_L\} \oplus \{\mathcal{M}_R\}\) superspace, transform according to

\[
\delta_Q(\phi_2)_i^j = -ie^j(L_1)_i^j \partial_\tau (\psi_2)_i^{\hat{\alpha}}
\]
\[
\delta_Q(\psi_2)_i^{\hat{\alpha}} = e^j(R_1)_i^j (\phi_2)_i^j.
\] (124)

The expansions needed turns out to be

\[
(\phi_2)_i^j = \tilde{\phi}_i^j + G_i^j, \quad (\psi_2)_i^{\hat{\alpha}} = \tilde{\psi}_i^{\hat{\alpha}} + X_i^{\hat{\alpha}}, \quad (L_1)_i^j X_i^{\hat{\alpha}} = 0
\] (125)

that bring us to the transformations

\[
\delta_Q \tilde{\phi}_i^j = ie^j \tilde{\psi}_1^j,
\]
\[
\delta_Q G_i^j = -i \partial_\tau e^j (\tilde{f}_1)_i^j \tilde{\psi}_1^j - ie^K(L_K)_i^j X_i^{\hat{\alpha}},
\]
\[
\delta_Q \tilde{\psi}_1^j = e^j \partial_\tau X + \frac{1}{d} e^j (\tilde{f}_1)_i^j \tilde{F}_i^j,
\]
\[
\delta_Q X_i^{\hat{\alpha}} = -d^{-1} e^j (\tilde{f}_1)_i^j (\tilde{f}_1)_i^j X_i^{\hat{\alpha}} + e^j (\tilde{f}_1)_i^j G_i^j.
\] (126)

Here, the scalar supermultiplet \((\phi_1)_i^j, (\psi_1)_i^{\hat{\alpha}}\) has to be treated in the following different way: the off-trace superfield \(\mu_{\hat{\alpha}}^i\) undergoes an AD

\[
\mu_{\hat{\alpha}}^i \rightarrow \partial_\tau^{-1} A_{\hat{\alpha}}^i,
\] (127)

that slightly changes the variation \(120, 121\) and \(122\) into

\[
\delta_Q X = ie^j \tilde{\psi}_1^j,
\]
\[
\delta_Q F_i^j = ie^j (\tilde{f}_1)_i^j \partial_\tau \tilde{\psi}_1^j - ie^j (R_1)_i^j \tilde{A}_{\hat{\alpha}}^j,
\]
\[
\delta_Q \tilde{\psi}_1^j = e^j \partial_\tau X + \frac{1}{d} e^j (\tilde{f}_1)_i^j \tilde{F}_i^j,
\]
\[
\delta_Q A_{\hat{\alpha}}^j = \left[ e^j (L_1)_i^j - \frac{1}{d} e^j (\tilde{f}_1)_i^j (\tilde{f}_1)_i^j \right] \partial_\tau \tilde{F}_i^j.
\] (128)

The action can be thought as the sum of two separated pieces

\[
S_{P2} = \frac{1}{d} \int d\tau \{(\phi_2)_i^j (\phi_2)_i^j + i (\psi_2)_i^{\hat{\alpha}} \partial_\tau (\psi_2)_i^{\hat{\alpha}}\},
\]
\[
S_{PV} = \frac{1}{d} \int d\tau \{(\phi_2)_i^j \partial_\tau (\phi_1)_i^j + i (\psi_2)_i^{\hat{\alpha}} \partial_\tau (\psi_1)_i^{\hat{\alpha}}\},
\] (129)

so that, in analogy with the free particle description where the Lagrangian has the form
\[ \mathcal{L} = PV - \frac{1}{2} P^2 \]  

we consider

\[ S = S_{PV} - \frac{1}{2} S_{P^2} , \]

as the correct first-order free spinning particle model. By eliminating the fermionic supermultiplet superfields by their equations of motion, we fall into the second-order description. It is clear that the fermionic supermultiplet is nothing but the conjugated of USPM. After \( (125) \) substitution the proposed action \( (131) \) assume the final aspect

\[ S_{sp} = \int d\tau \left\{ P \partial_\tau X + \frac{1}{d} G_i^j F_j^i - i \bar{\psi}^I \partial_\tau \psi_I + \frac{i}{d} \lambda_i^\dagger \lambda_i^j \right\} , \]

where the auxiliary superfields \( P^2, F^i_j, \lambda_i^\dagger, G^i_j \) and \( X^\dagger_i \) are manifest.

Massive theory

The massive theory is obtained by adding to the previous first and second-order actions the appropriate terms where it figures an additional supermultiplet \( \left( \hat{\psi}_i^\dagger, \hat{G}_i^j \right) \), which is fermionic in nature

\[ \delta Q \hat{\psi}_i^\dagger = \epsilon^I (R_I)_j^\dagger \hat{G}_i^j , \]

\[ \delta Q \hat{G}_i^j = i \epsilon^I (L_I)_j^\dagger \partial_\tau \hat{\psi}_i^\dagger , \]

and is inserted through the action

\[ S_M = \int d\tau [i \hat{\psi}_i^\dagger \partial_\tau \hat{\psi}_i^\dagger + \hat{G}_i^j \hat{G}_i^j + M \hat{G}_i^i] . \]

Here the bosonic auxiliary trace \( G^i_i \) plays a different role with respect to the other off-trace component because it is responsible, by its equation of motion, for setting the mass equal to \( M \). It can be easily recognized the resemblance between the Sherk-Shwartz method \( \text{[19]} \) and the above way to proceed if we interpret the mass multiplet as a \( (D+1) \)-th Minkowski momentum component without coordinate analogue. We underline that the "mass multiplet" \( (133) \) is crucial if we want to insert the mass and preserve the preexisting symmetries, as it happens for the \( \psi_5 \) field \( \text{[105]} \).

First and second-order formalism for spinning particle coupled to minimal N-extended supergravity

For completeness, we include the coupling of the above models to minimal 1-dimensional supergravity. The supergravity multiplet escape from \( \mathcal{G} R(d, N) \) embedding because its off-shell fields content \( (e, \chi_I) \),
δQe = -4ie^2ε_1χ_I ,  
\delta Q\chi^I = -∂_ε e_I , \quad (135)

consists of one real boson and N real fermions. General coordinate variations are
\[
\delta_{GC} e = \dot{e}ξ - e \dot{ξ} , \\
\delta_{GC} \chi^I = \partial_\tau (\chi^I \xi) . 
\]

The gauging of supersymmetry requires the introduction of connections by means of generators \( A_{IJ} \) valued on an arbitrary Lie algebra. Local supersymmetric variations come from (119), (121) and (122) by replacing
\[
\partial_\tau \rightarrow D = e\partial_\tau + e\chi^I Q_I + \frac{1}{2} w^{JK} A_{JK} , 
\]
while the gravitino one can be written
\[
\delta Q\chi^I = -[\delta_I J \partial_\tau - \frac{1}{2} e^{-1} w^{KL}(f_{KL})_J] e^J . 
\]

One can explicitly check that the local supersymmetric invariant action for the above mentioned transformations, is
\[
S = \int d\tau \{ e^{-1}[P D_\tau X + \frac{1}{d} G^i_j F^i_j] - i\bar{\psi}D_\tau \psi_I + \frac{i}{d} \chi^I \bar{\lambda}_I \lambda_\alpha + \\
- \frac{1}{2} P^2 - \frac{1}{d} G^i_j G^j_i - \frac{i}{2} \bar{\psi}D_\tau \psi_I - \frac{i}{2d} \chi^I \bar{\lambda}_I \lambda_\alpha + \\
- i\chi_I \bar{\psi}_I P + \frac{1}{d} (f_{IJ})_j^i \bar{G}^i_j \psi_J + \frac{1}{d} (L_I)_j^i \bar{G}^i_j \chi^I \bar{\lambda}_I \lambda_\alpha + \\
\psi_I P + \frac{1}{d} (f_{IJ})_j^i \bar{G}^i_j \psi_J - \frac{1}{d} (L_I)_j^i \bar{G}^i_j \chi^I \bar{\lambda}_I \lambda_\alpha \} , 
\]

providing the first-order massless model for spinning particle minimally coupled to N-extended supergravity on the worldline. Equation of motion associated to \( P \) field reads
\[
P = D_\tau X - i\chi_I \bar{\psi}_I - i\chi_I \psi_I , \quad (140)
\]
that, substituted in (139), give us the second-order formulation
\[
S = \int d\tau \{ \frac{1}{2} D_\tau X D_\tau X + \frac{1}{d} G^i_j F^i_j - i\bar{\psi}D_\tau \psi_I + \frac{i}{d} \chi^I \bar{\lambda}_I \lambda_\alpha + \\
- \frac{1}{2d} G^i_j G^j_i - \frac{i}{2} \bar{\psi}D_\tau \psi_I - \frac{i}{2d} \chi^I \bar{\lambda}_I \lambda_\alpha + \\
- \frac{i}{d} \chi_I \bar{\psi}_I P + \frac{1}{d} (f_{IJ})_j^i \bar{G}^i_j \psi_J + (L_I)_j^i \bar{G}^i_j \chi^I \bar{\lambda}_I \lambda_\alpha + \\
(f_{IJ})_j^i \bar{G}^i_j \psi_J - (L_I)_j^i \bar{G}^i_j \chi^I \bar{\lambda}_I \lambda_\alpha + \chi \text{terms} \} . 
\]

(141)
Finally, the massive theory is obtained following the ideas of the previous paragraph. The massive supermultiplet \((133)\) is coupled to supergravity supermultiplet by the extension of \((134)\) to the local supersymmetric case

\[
S_M = \int d\tau \{ i e^{-1} \hat{\psi}_i \hat{\psi}_i \hat{\psi}_i + e^{-1} \hat{G}_j \hat{G}_j + e^{-1} M \hat{G}_i \hat{G}_i + i \chi^I (L_I)_{ij} \hat{\psi}_i \hat{\psi}_j - i M \chi^I (R_I)_i l^i \},
\]

that is exactly what we need to add to the action \((139)\) to achieve a completely off-shell massive first-order description. We close this review by noting that importance issues regarding zero-modes of the models discussed above have yet to be resolved. So we do not regard this as a completed subject yet.

### 3.2 \(N = 8\) unusual representations

There exists also some “unusual” representations in this approach to 1D supersymmetric quantum mechanics. As an illustration of these, the discussion will now treat such a case for \(\mathcal{G}R(8,8)\). It may be verified that a suitable representation is provided by the \(8 \times 8\) matrices

\[
\begin{align*}
L_1 &= i \sigma^3 \otimes \sigma^2 \otimes \sigma^1, & L_5 &= i \sigma^1 \otimes \sigma^1 \otimes \sigma^2, \\
L_2 &= i \sigma^3 \otimes \sigma^2 \otimes \sigma^3, & L_6 &= i \sigma^1 \otimes \sigma^3 \otimes \sigma^2, \\
L_3 &= i \sigma^3 \otimes \mathbb{I}_2 \otimes \sigma^2, & L_7 &= i \sigma^1 \otimes \sigma^2 \otimes \mathbb{I}_2, \\
L_4 &= -i \sigma^2 \otimes \mathbb{I}_2 \otimes \mathbb{I}_2, & L_8 &= \mathbb{I}_2 \otimes \mathbb{I}_2 \otimes \mathbb{I}_2.
\end{align*}
\]

An octet of scalar fields \(A_I\) and spinor fields \(\Psi_I\) may be introduced. The supersymmetry variation of these are defined by

\[
\delta Q A_J = i \epsilon^I (L_J)_{IK} \Psi_K, \quad \delta Q \Psi_K = - \epsilon^I (R_N)_{KI} (\partial_\tau A_N)
\]

where I, J, K, etc. now take on the values 1, 2,...,8. Proper closure of the supersymmetry algebra requires in addition to \((6)\) also the fact that

\[
(R_N)_{KJ} (L_N)_{IJ} + (R_N)_{KI} (L_N)_{JM} = -2 \delta_{IJ} \delta_{KM},
\]

which may be verified for the representation in \((143)\). This is the fact that identifies the representation in \((144)\) as being an ‘unusual’ representation. The representation in \((144)\) through the action of various AD maps generates many closely related representations. In fact, it can be seen that for any integer \(p\) (with \(0 \leq p \leq 8\)) there exist \((p, 8, 8 - p)\) representations!

Table 5 shows that there are, for example, two distinct supermultiplets that have seven propagating bosons. In order to gain some insight into how this profusion of supermultiplets comes into being, it is convenient to note that the matrices in \((143)\) can be arranged according to the identifications
Table 5. The representation in (144) has been given the name “ultra-multiplets.”
The sum of the degeneracies adds to sixteen.

| p | Degeneracy |
|---|------------|
| 8 | 1          |
| 7 | 2          |
| 6 | 1          |
| 5 | 2          |
| 4 | 4          |
| 3 | 2          |
| 2 | 1          |
| 1 | 2          |
| 0 | 1          |

Table 5. The representation in (144) has been given the name “ultra-multiplets.”
The sum of the degeneracies adds to sixteen.

\[ \alpha_{\hat{A}} = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix}, \quad \beta_{\hat{A}} = \begin{bmatrix} L_5 \\ L_6 \\ L_7 \end{bmatrix}, \]
\[ \Theta = L_4, \quad L_8 = \delta , \]  \hspace{1cm} (146)

where the quantities \( L_I \) is split into triplets of matrices \( \alpha_{\hat{A}} \) and \( \beta_{\hat{A}} \), as well as the single matrix \( \Theta \) and the identity matrix \( \delta \).

With respect to this same decomposition the eight bosonic fields may be written as

\[ A_I = \{ \mathcal{P}_{\hat{A}}, \mathcal{A}_{\hat{A}}, \mathcal{A}, \mathcal{P} \} . \]  \hspace{1cm} (147)

Now the two distinct cases where \( p = 2 \) occur from the respective AD maps

\[ \{ \mathcal{P}_{\hat{A}}, \mathcal{A}_{\hat{A}}, \mathcal{A}, \mathcal{P} \} \rightarrow \{ \mathcal{P}_{\hat{A}}, \mathcal{A}_{\hat{A}}, \partial^{-1}_{\tau} \mathcal{A}, \mathcal{P} \} \]  \hspace{1cm} (148)

and

\[ \{ \mathcal{P}_{\hat{A}}, \mathcal{A}_{\hat{A}}, \mathcal{A}, \mathcal{P} \} \rightarrow \{ \mathcal{P}_{\hat{A}}, \mathcal{A}_{\hat{A}}, \mathcal{A}, \partial^{-1}_{\tau} \mathcal{P} \} . \]  \hspace{1cm} (149)

4 Graphical supersymmetric representation technique: Adinkras

The root labels defined in (90) seems to be good candidates to classify linear representation of supersymmetry. However a more careful analysis reveals that there is not a one to one correspondence between admissible transformations and labels. For instance the \( N = 1 \) scalar supermultiplet can be identified by both \((0,0)_+ \) and \((0,1)_- \) root labels.

To fully exploit the power of the developed formalism, we need to introduce a more fundamental technique that from one side eliminate all the ambiguities and from another side reveal new structures. A useful way to encode all the informations contained in a supermultiplet, is provided by a graphical formulation where each graph is christen adinkra in honour of Asante populations.
of Ghana, West Africa, accustomed to express concepts that defy usual words, by symbols. This approach was pioneered in [20].

Basic pictures used to represent supersymmetry are circles (nodes), white for bosons and black for fermions component fields, connected by arrows that are chosen in such a way to point the higher component field which is assumed to be the one that does not appear differentiated in the r.h.s. of transformation properties. The general rule to follow in constructing variations from adinkras is

\[ \delta_Q f_i = \pm i^b \partial^a f_j, \]  

where \( f_i, f_j \) are two adjacent component fields, \( b = 1 (b = 0) \) if \( f_j \) is a fermion (boson) and \( a = 1 (a = 0) \) if \( f_j \) is the lower (higher) component field. The sign has to be the same for both the nodes connected. Its relevance became clear only for \( N > 1 \) as will be discussed in the paragraph 4.1. In the following we introduce a general procedure to classify root tree supermultiplets, that works for arbitrary \( N \).

### 4.1 \( N = 1 \) supermultiplets

It is straightforward to recognize the \( N = 1 \) scalar supermultiplet labelled by \((0,0)_+\)

\[ \delta_Q \phi = i\epsilon \psi, \quad \delta_Q \psi = \epsilon \partial \phi, \quad \Rightarrow \quad \begin{array}{c}
\end{array} \]  

and the first level dualized supermultiplet \((1,0)_+\)

\[ \delta_Q \phi = i\epsilon \partial \psi, \quad \delta_Q \psi = \epsilon \phi, \quad \Rightarrow \quad \begin{array}{c}
\end{array} \]  

that corresponds to the spinorial supermultiplet. The order of the nodes is conventionally chosen to keep contact with component fields order of the bosonic root labels (i.e. marked with a plus sign). Alternatively, starting from (151), we can dualize the second level, falling in the (152) option. The last possibility is to dualize both levels but again we go back to the scalar case. Now that we have run out all the bosonic root label possibilities, we can outline the following sequence of congruences
In this framework the AD \((153)\) is seen to be implemented by a simple change in the orientation of the arrow. We refer to this simple sequence, made of all the inequivalent root tree supermultiplets of the bosonic type, as the “base sequence”.

Beside the AD we have another kind of duality, namely the Klein flip \([7]\), which corresponds to the exchanging of bosons and fermions. If we apply the Klein flip to the previous adinkras it happens that we get what we call the “mirror sequence”\(^\rightarrow\) \(\Rightarrow\)

\[
\begin{align*}
\delta_Q \psi &= \epsilon \phi , \\
\delta_Q \phi &= i \epsilon \partial_\tau \psi ,
\end{align*}
\] (154)

\[
\begin{align*}
\delta_Q \psi &= \epsilon \partial_\tau \phi , \\
\delta_Q \phi &= i \epsilon \psi ,
\end{align*}
\] (155)

Here the KF is responsible of a changing of the supermultiplets nature from bosonic to fermionic. Accordingly, to maintain the order of fermionic root labels, we put a fermionic node on the upper position. As in the base sequence, even in the mirror one, we have congruences between root labels, precisely \((0,0)_- \simeq (1,1)_-\) are referred to \((154)\) while \((1,0)_- \simeq (0,1)_-\) to \((155)\). The power of adinkras became manifest when we try to find which supermultiplet of the base sequence is equivalent to the supermultiplets in the mirror one. It is straightforward to see that up to 180 degree rotations, only two adinkras are inequivalent

\[
\begin{align*}
(0,0)_+ &\simeq (1,1)_+ \simeq (0,1)_+ \simeq (1,0)_+ , \\
(0,0)_- &\simeq (1,1)_- \simeq (0,1)_- \simeq (1,0)_- .
\end{align*}
\] (156)

All the above results about \(N = 1\) root tree supermultiplet, can be reassumed in a compact way in the table where boxed nodes refer to auxiliary fields. Actually there exists a way to define auxiliary fields by means of adinkras without appealing to the dynamics. In the following we will denote as auxiliary all the fields whose associated bosonic (fermionic) nodes are sink (source), namely all the arrows point to (comes out from) the node.
Table 6. N=1 root tree elements.

4.2 $N = 2$ supermultiplets

Even in the $N = 2$ case we start from the scalar supermultiplet $^{13}$ whose root label is $(0,0,0)$. Choosing the representation

$$L_1 = R_1 = i\sigma_2, \quad L_2 = -R_2 = I_2,$$  \hspace{1cm} (157)

the resulting explicit transformation properties are

$$\delta_Q \phi_1 = -i\epsilon^1 \psi_2 + i\epsilon^2 \psi_1,$$
$$\delta_Q \phi_2 = i\epsilon^1 \psi_1 + i\epsilon^2 \psi_2,$$
$$\delta_Q \psi_1 = \epsilon^1 \dot{\phi}_2 + \epsilon^2 \dot{\phi}_1,$$
$$\delta_Q \psi_2 = -\epsilon^1 \dot{\phi}_1 + \epsilon^2 \dot{\phi}_2.$$  \hspace{1cm} (158)

Accordingly the adinkra associated with (158) can be drawn as

The filled arrow is inserted to take into account that appears a minus sign in the (150) involving $\phi_1$ and $\psi_2$. The $N = 2$ case furnishes new features that will be present in all higher supersymmetric extensions. One of them is the sum rule that can be stated as follows: multiplying the signs chosen in the (150) for a closed path in the adinkra, we should get a minus sign. Clearly the graph (159) satisfy this condition. Moreover it is possible to flip the sign of a field associated to a node. The net effect on the adinkra is a shift of the red arrow or the appearing of two more red arrows confirming that after this kind of flip the sum rule still holds. Another evident property is that parallel arrows correspond to the same supersymmetry. It is easy to foresee that the $N$-extended adinkras live in a $N$ dimensional space so that graphical difficulties will arise for $N \geq 4$. However suitable techniques will be developed below to treat higher dimensional cases.
The AD can be generalized to arbitrary $N$-extended cases by saying that its application to a field is equivalent to reversing \textit{all} the arrows connected to the corresponding node. However, as will be cleared in the next paragraphs, if we want to move inside the root tree, we have to implement AD level by level. This means that in (158), the AD is necessarily implemented on both $\psi_1$ and $\psi_2$ fermionic fields.

For arbitrary value of $N$, the proper way to manage the signs is to consider the scalar supermultiplet adinkra associated to (78), as the starting point to construct all the other root tree supermultiplets implementing AD, Klein flip and sign flipping of component fields. Since the scalar supermultiplet has well defined signs by construction, the resulting adinkras will be consistent with the underlying theory. This allows us to forget about the red arrows and consider equivalent all the graphs that differ from each other by a sign redefinition of a component field. Once the problem of signs is understood, let us go back to the classification problem. Following the line of the $N = 1$ case, from the scalar adinkra (159) we derive the base sequence whose inequivalent graphs, with root labels on the right, are

\begin{equation}
\rightarrow (0, 0, 0)_+ \simeq (1, 1, 1)_+ \quad (160)
\end{equation}

\begin{equation}
\rightarrow (0, 0, 1)_+ \simeq (1, 0, 0)_+ \quad (161)
\end{equation}

\begin{equation}
\rightarrow (0, 1, 0)_+ \quad (162)
\end{equation}

The KF applied to the above adinkras, provides the mirror sequence
In order to classify the $N = 2$ root tree supermultiplets, the last step is the matching of the base sequence with the mirror sequence to recognize topologically equivalent graphs. It turns out that only four of them originate different dual supermultiplets. In fact adinkra (160) is nothing but adinkra (165) rotated of 90 degrees. The same relation holds between the adinkras (162) and (163). To make the remaining adinkra relationships clear, we can arrange them in the following way

\[
\begin{align*}
K^F & \leftrightarrow \\
\uparrow AD & \leftrightarrow \\
\uparrow & AD
\end{align*}
\]

so that left column is connected by the klein flip to the right column while the upper row is the automorphic dual of the lower one.
4.3 Adinkras folding

As observed in the previous paragraph, adinkras associated to $N \geq 3$ extended supermultiplets may become problematic to draw and consequently to classify. Fortunately there exists a very simple way to reduce the dimensionality of the graphs preserving the topological structure memory. The process consists in moving the nodes and arrows into each other in a proper way. In doing this two basic rules have to be satisfied

1. only nodes of the same type can be overlapped,
2. we can make arrows lay upon each other only if they are oriented in the same way.

In the first rule, when we talk about nodes of the same type, we refer not only to the bosonic or fermionic nature but even to physical or auxiliary dynamical behavior. In order to clarify how this process works, let us graphically examine the simplest example by folding the adinkra (160)

On the left of each node it is reported its multiplicity. Thus from a 2-dim adinkra we end up to a 1-dim one, increasing the multiplicity of the nodes. We emphasize that in the example above, we have a sequence of two different folding. After the first one we are left with a partially folded adinkra while in the end we obtain a fully folded one. It is important for the following developments, to have in mind that we have various levels of folding for the same adinkra. A remarkable property of the root tree elements is that they can be always folded into a linear chain. Applying this technique to the arrangement scheme (166), the $N = 2$ root tree adinkras can be organized as follows

Table 7. Fully folded $N=2$ root tree elements.
4.4 Escheric supermultiplets

In this paragraph we want to give some hints about how to describe supermultiplets that are not in the root tree. We anticipated that implementing AD on singular nodes may bring us outside the root tree sequence. Let us examine this aspect in some detail. Starting from variations (158) we dualize

\[ \phi_1 = \partial^{-1}_\tau A , \]  

(168)

to obtain

\[ \delta_Q A = -i\epsilon^1 \dot{\psi} + i\epsilon^2 \dot{\phi}_1 , \]
\[ \delta_Q \phi_2 = i\epsilon^1 \psi_1 + i\epsilon^2 \psi_2 , \]
\[ \delta_Q \psi_1 = \epsilon^1 \dot{\psi}_2 + \epsilon^2 A , \]
\[ \delta_Q \psi_2 = -\epsilon^2 A + \epsilon^2 \dot{\phi}_2 , \]  

(169)

associated to adinkra (161). Then let the AD map act on the left fermionic node

\[ \psi_2 = i\partial \xi , \]  

(170)

to end into

\[ \delta_Q A = -i\epsilon^1 \ddot{\xi} + i\epsilon^2 \dot{\psi}_1 , \]
\[ \delta_Q \phi_2 = i\epsilon^1 \psi_1 - \epsilon^2 \dot{\xi} , \]
\[ \delta_Q \psi_1 = \epsilon^1 \dot{\phi}_2 + \epsilon^2 A , \]
\[ \delta_Q \xi = i\epsilon^1 \int^\tau d\tilde{t} A - i\epsilon^2 \dot{\phi}_2 , \]  

(171)

whose corresponding adinkra symbol is

\[ (172) \]

where the new modified arrow is used to describe the appearance of the antiderivative in the r.h.s. of \( \psi_2 \) variation. Let us notice that the usual ordering of the nodes in the adinkra (172) makes no sense because each node is upper than the previous and lower then the next one. This situation was one of the main theme of some drawings of the graphic artist Maurits Cornelis Escher (see, for instance, the lithograph “Ascending and Descending”). For this reason we will refer to these kind of supermultiplets as escheric. One of the main feature of adinkra (172) is that it can not be folded into a lower dimensional graph. This force us to introduce a new important concept which is the rank of an adinkra, namely the dimensions spanned by the fully folded
graph diminished by one. The case (172) provides an \( N = 2 \) example of a rank one adinkra while the root trees are always composed of rank zero adinkras.

A similar result can be found even in the \( N = 1 \) case. It is possible to go outside the root tree enforcing the duality

\[
\phi \rightarrow \partial_c^2 \tilde{\phi} ,
\]

(173)
on transformation properties (151), in order to obtain

\[
\begin{align*}
\delta Q \tilde{\phi} &= i \epsilon \partial_1^\tau \psi , \\
\delta Q \psi &= \epsilon \int \tilde{\tau} \, d \tilde{\tau} \, \tilde{\phi}(\tilde{\tau}) ,
\end{align*}
\]

(174)

and the new \( N = 1 \) escheric symbol

\[
\]

(175)

with equivalent root labels \((2,0)_+ \cong (0,2)_+ \cong (1,0)_- \cong (0,1)_-\).

The integral in the r.h.s. of the above transformation properties assume a particularly interesting meaning whenever the integrated boson lives in a compact manifold. If this is the case then the integral term counts the number of wrappings of the considered bosonic field.

The above discussion about escheric supermultiplet is somehow linked via AD maps to supersymmetric multiplets presented so far. However, the contact with the previous approach is completely lost by considering the adinkra

\[
\]

(176)

and associating to this graph the variations exploiting the general method of equation (150). It turns out that transformation properties referred to (176) are

\[
\begin{align*}
\delta Q \phi_1 &= - i \epsilon^1 \psi_2 + i \epsilon^2 \dot{\psi}_1 , \\
\delta Q \phi_2 &= i \epsilon^1 \psi_1 + i \epsilon^2 \dot{\psi}_2 , \\
\delta Q \psi_1 &= \epsilon^1 \dot{\phi}_2 + \epsilon^2 \phi_1 , \\
\delta Q \psi_2 &= - \epsilon^1 \dot{\phi}_1 + \epsilon^2 \phi_2 ,
\end{align*}
\]

(177)
where each field is associated to each site in the same way of adinkra and a proper minus sign has been inserted in order to accomplish the sum rule. The first thing to figure out in order to understand what kind of supersymmetric properties are hidden under this new supermultiplet, is the commutator between two variations. It is straightforward to prove that after the fields and parameters complexification

\[
\phi = \phi_1 + i\phi_2, \quad \psi = \psi_1 + i\psi_2, \quad \epsilon = \epsilon^1 + i\epsilon^2,
\]

we have\(^6\)

\[
[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)] = -2i\bar{\epsilon}_1\epsilon_2\partial^\tau - \frac{i}{2}(\epsilon_1\bar{\epsilon}_2 - \bar{\epsilon}_1\epsilon_2) \delta_Y
\]

where \(\delta_Y\) acts on the fields in the following way

\[
\delta_Y(\phi, \psi) = (\partial^2 - 1)(\phi, \psi), \quad \delta_Y(\bar{\phi}, \bar{\psi}) = -(\partial^2 - 1)(\bar{\phi}, \bar{\psi}).
\]

Expressing the variations in terms of supersymmetric charges \(Q, \bar{Q}\) and central charge \(Y\)

\[
\delta_Q(\epsilon) = \epsilon Q + \bar{\epsilon}\bar{Q}, \quad \delta_Y = \frac{Y}{4},
\]

we obtain the central extended algebra

\[
\{Q, \bar{Q}\} = H, \quad Q^2 = iY, \quad [H, Y] = 0,
\]

where \(Y\) plays the role of a purely imaginary central charge. Although real central extension of similar algebras has been studied\(^{21}\), the purely imaginary case still lacks of a completely clear interpretation.

The escheric adinkras make clear how this graphical approach can offer the possibility to describe theories that lie outside the formalism developed in the previous chapters and eventually can make arise to new non trivial features.

**4.5 Through higher \(N\)**

In principle, the techniques of the previous paragraphs are suitable even for \(N \geq 3\). Thus, for \(N = 3\) case, the scalar supermultiplet fields transformations can be written down using the representation for \(G_R(4, 3)\) given by

\(^6\) the lower indices of the supersymmetry parameters are referred to different supersymmetries while the upper one are associated to the two real charges of the same supersymmetry.
\[ L_1 = R_1 = i\sigma_1 \otimes \sigma_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \]
\[ L_2 = R_2 = i\sigma_2 \otimes I_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \]
\[ L_3 = R_3 = -i\sigma_3 \otimes \sigma_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \]

so that explicitly we have
\[ \delta \phi_1 = -i\epsilon_1 \psi_4 - i\epsilon_2 \psi_2 + i\epsilon_3 \psi_3, \]
\[ \delta \phi_2 = -i\epsilon_1 \psi_3 + i\epsilon_2 \psi_1 - i\epsilon_3 \psi_4, \]
\[ \delta \phi_3 = i\epsilon_1 \psi_2 - i\epsilon_2 \psi_4 - i\epsilon_3 \psi_1, \]
\[ \delta \phi_4 = i\epsilon_1 \psi_1 + i\epsilon_2 \psi_3 + i\epsilon_3 \psi_2, \]
\[ \delta \psi_1 = \epsilon^1 \dot{\phi}_4 + \epsilon^2 \dot{\phi}_2 - \epsilon^3 \dot{\phi}_3, \]
\[ \delta \psi_2 = \epsilon^1 \dot{\phi}_3 - \epsilon^2 \dot{\phi}_1 + \epsilon^3 \dot{\phi}_4, \]
\[ \delta \psi_3 = -\epsilon^1 \dot{\phi}_2 + \epsilon^2 \dot{\phi}_4 + \epsilon^3 \dot{\phi}_1, \]
\[ \delta \psi_4 = -\epsilon^1 \dot{\phi}_1 - \epsilon^2 \dot{\phi}_3 - \epsilon^3 \dot{\phi}_2, \]

that, translated in terms of graph, are equivalent to

Next we dualize via Klein flip to obtain
that, together with $SU(5)$, are respectively the starting point to construct all the elements of the base and mirror sequences using all possible levelwise AD. Alternatively one can derive all the base sequence from $SU(5)$ and then performs a Klein flip on each base element in order to deduce the mirror adinkras. Finally we fold all the topologically inequivalent adinkras organizing them into the table

Table 8. Fully folded $N = 3$ root tree elements.

As expected, all the fully folded root tree adinkras are one dimensional. Moreover, one can verify that each closed path without arrows within, satisfy the sum rule. The $N = 3$ case furnishes the possibility to generalize the sum rule by saying that if $a$ is the number of arrows that are circuited by the path then the sign of the sum rule turns out to be $(-1)^{a+1}$. It is of some importance to notice that by completely folding the adinkras, the levels of the nodes can be upsetted. Nevertheless if we consider the fully unfolded one dimensional adinkras obtained by folding the $N = 3$ graphs, then we can still implement AD level by level in order to get all the root tree. In other words the depth of the ADs (i.e. the minimum number of dimensions reached by the folded adinkra when the AD is applied) used to deduce the root tree is one.

If we assume that each supersymmetry corresponds to an orthogonal direction, as stated above, then the nodes are placed on the vertices of a $N$ dimensional hypercube. Consequently, at $N = 4$ we find $2^N|_{N=4} = 16$ nodes among which eight are bosonic and eight are fermionic. This is in contrast
with the irreducible representation dimension that is $4 + 4 = 8$ as reported in table 2. The problem to face is how to reduce consistently the dimension of the representation that arise from the $N = 4$ adinkras in order to obtain the irreducible representation described in the paragraph 1.3. The following two methods are effective to solve this problem: in the first one we identify consistently the nodes to obtain the proper transformation properties, while in the second one we recognize irreducible sub-adinkras making rise to gauge degrees of freedom. Let us consider the first method ( the other one will be analyzed in the next paragraph ) constructing the $N = 4$ scalar supermultiplet as example. A possible choice of $GR(\triangle, \triangle)$ generators turns out to be composed of six generators of the $N = 3$ case (183) plus the two generators

$$L_4 = - R_4 = - I_2 \otimes I_2$$

(187)

It is straightforward to figure out the following variations:

$$\begin{align*}
\delta \phi_1 &= - i \epsilon^1 \psi_4 - i \epsilon^2 \psi_2 + i \epsilon^3 \psi_3 - i \epsilon^4 \psi_1 , \\
\delta \phi_2 &= - i \epsilon^1 \psi_3 + i \epsilon^2 \psi_1 - i \epsilon^3 \psi_4 - i \epsilon^4 \psi_2 , \\
\delta \phi_3 &= i \epsilon^1 \psi_2 - i \epsilon^2 \psi_4 - i \epsilon^3 \psi_1 - i \epsilon^4 \psi_3 , \\
\delta \phi_4 &= i \epsilon^1 \psi_1 + i \epsilon^2 \psi_3 + i \epsilon^3 \psi_2 - i \epsilon^4 \psi_4 , \\
\delta \psi_1 &= \epsilon^1 \dot{\phi}_4 + \epsilon^2 \dot{\phi}_2 - \epsilon^3 \dot{\phi}_3 - \epsilon^4 \dot{\phi}_1 , \\
\delta \psi_2 &= \epsilon^1 \dot{\phi}_3 - \epsilon^2 \dot{\phi}_1 + \epsilon^3 \dot{\phi}_4 - \epsilon^4 \dot{\phi}_2 , \\
\delta \psi_3 &= - \epsilon^1 \dot{\phi}_2 + \epsilon^2 \dot{\phi}_4 + \epsilon^3 \dot{\phi}_1 - \epsilon^4 \dot{\phi}_3 , \\
\delta \psi_4 &= - \epsilon^1 \dot{\phi}_1 - \epsilon^2 \dot{\phi}_3 - \epsilon^3 \dot{\phi}_2 - \epsilon^4 \dot{\phi}_4 ,
\end{align*}$$

(188)

associated to the scalar supermultiplet. Since the number of fields are doubled by the translation into adinkras, it is conceivable that two copies of the same adinkra could fit properly to describe the supermultiplet. The $N = 3$ scalar adinkra (185) is suitable to encode the first three supesymmetries while the extra supersymmetry connect the nodes of the two $N = 3$ scalar adinkras copies. Graphically, the situation can be well depicted by the figure below.

(189)
where we omitted the arrows from $\phi_3$ to $\phi_3$ (the arrows between $\phi_2$ and $\psi_2$, and between $\phi_2$ and $\psi_2$, are not repeated in the external nodes). We can see that the fourth supersymmetry connects opposite nodes of the adinkra, so that we can render the drawing in the following more compact way:

(190)

where we agree that the dashed diagonal arrows describe the same supersymmetry even if they are not parallel. Let us notice that the second $N = 3$ adinkra in the picture is the mirrored copy of the first. In fact, the dashed line in the middle represents the mirror plane inserted to underline this property. The subtlety in this construction is hidden in the way to connect the two $N = 3$ adinkras using the fourth supersymmetry. In fact, four consistent choices of sign flips make rise to as many inequivalent supermultiplets that behave to the same conjugacy class. These four scalar supermultiplets were considered in [7]. To describe them it is useful to fold the adinkra in the following way:

(191)

where the diagonal dashed lines stand for the levels of the $N = 4$ supermultiplets. It is clear that the right side with respect of the mirror plane is redundant
since it can be deduced from the left one. Therefore it is sufficient to draw only the left side of the graph (191) in order to allow us to add the signs flips that identify each scalar supermultiplet as it follows

\begin{equation}
\begin{aligned}
+ & \quad + & \quad + & \quad + \\
- & \quad + & \quad - & \quad - \\
+ & \quad - & \quad + & \quad - \\
\end{aligned}
\end{equation}

(192)

This quaternionic structure can be neglected if we assume that each $N = 4$ adinkra in the root tree stands for a conjugacy class. If we adhere to this point of view, then it is a good exercise for the reader to derive all the fully folded root tree elements of the $N = 4$ case using the techniques described so far. The best way to proceed is to reduce the adinkra (189) to its most unfolded one dimensional version which is

\begin{equation}
\begin{aligned}
1 & \quad 4 \\
4 & \quad 6 \\
1 & \quad 4 \\
\end{aligned}
\end{equation}

(193)

obtained by identifying the nodes along the levels represented by dashed diagonal lines in the graph (191). We underline that the non trivial structure of the levels is not manifest in the drawing (189) but it becomes evident once we fold it into the linear graph (193). One can check that the root tree elements can be obtained dualizing along these levels and the resulting graphs can be arranged in the table 9. The reader is also encouraged to try to implement the AD not respecting the suggested levels. For instance, if we consider the levels of each $N=3$ sub-adinkra cube to apply AD, then it is easy to see that escheric loops may come out.

### 4.6 Gauge invariance

Before starting with the discussion of the gauge aspects of adinkras, we need to describe explicitly the $N = 4$ chiral supermultiplet. To this end, let us apply a third level AD to the adinkra (189) and fold it in the following way:
Table 9. Fully folded N=4 root tree elements.

where we disregarded the second $N = 3$ cube and the fourth supersymmetry arrows. The simplest root label associated to this supermultiplet is $(00100)_+$ and it corresponds to the shadow of the $N = 1 d = 4$ chiral supermultiplet. For this reason we refer to it as the $N = 4$ chiral supermultiplet. Analogously, the shadow of $N = 1 d = 4$ vector supermultiplet can be constructed dualizing...
the first, fourth and fifth level of the scalar supermultiplet associated to the adinkra \( \text{[189]} \). By doing this we are left with

\[
\begin{align*}
\text{(195)}
\end{align*}
\]

which is foldable to the following form

\[
\begin{align*}
\text{(196)}
\end{align*}
\]

In this particular structure is embedded the chiral supermultiplet as a sub-adinkra. It is possible to remove it from the top of the vector adinkra in order to obtain two irreducible representations

\[
\begin{align*}
\text{(197)}
\end{align*}
\]

We see that a subtraction of the nodes is performed and consequently, the topmost node of the vector adinkra assumes a negative multiplicity. Such a node acquire the meaning of a residual gauge degree of freedom. By moving the gauge node along the initial structure of adinkra \( \text{[189]} \), we fix it on the nearest remained node, as shown in the figure.
This phenomenon is nothing but the shadow of the $N = 4 \ d = 1$ Wess-Zumino gauge fixing procedure. Clearly, the method described above offers an alternative possibility to reduce the reducible supermultiplet coming out from an adinkra symbol into two reducible representations via the introduction of gauge degrees of freedom.

Conclusions

By a geometrical interpretation of supersymmetric mechanics, we reviewed a classification scheme that exploits real Clifford algebras which are in one-to-one correspondence with the geometrical framework of Garden Algebras. For supersymmetric mechanics we explicitly described the link between the number of supersymmetries and the dimension and geometry of their faithful representations. All methods used to construct the explicit representation of such algebras are reviewed in detail. Particular emphasis has been dedicated to the duality relations among different supermultiplets at fixed number of supersymmetries using Clifford algebraic superfields. The formalism developed turned out to be necessary, as well as effective, when applied to the spinning particle problem, providing, quite straightforwardly, first and second order supersymmetric actions both in the case of global and local $N$-extended supersymmetry. Another new application example has been provided by an $N = 8$ unusual representation, suggesting how to derive many related representation via automorphic duality.

The second part of these lectures concerned the translation of all the results obtained so far into a simple graphical language whose symbols are called "Adinkras". In particular, we encoded all properties of each supermultiplet into an adinkra graph in order to classify and better clarify the duality relations between supermultiplets. Using a folding procedure to reduce the dimensions of the adinkras, we succeeded in classifying, up to $N = 4$, a large class of supermultiplets (root trees) using linear graphs. Moreover, it has been demonstrated that this graphical technique offers the possibility to derive new supermultiplets through dualities, possibly with the appearance of central charges or topological charges.
Even though the attempt to formalize a method to relate adinkras to supermultiplets has been carried out successfully in these lectures, many aspects still need a proper investigation on mathematical footing. A step forward in this direction has been presented in a recent work [22]. However, the $N \geq 4$ cases still present many unresolved classification subtleties mainly due to the non trivial topology structure of the adinkras. Another line of research that may be followed deals with the implications of the duality relations between supermultiplets on higher dimensional field theories. The oxidation procedure is a nice tool that can be used to proceed in this way. Recently, exploiting the automorphic duality, it has been shown [23] that it is possible to relate not only the $N = 4$ root tree supermultiplets, but even the associated interacting sigma models.\footnote{Notice that in [23] linear and nonlinear chiral supermultiplets were obtained by the reduction of the linear supermultiplet with four bosonic and four fermionic degrees of freedom [8, 24].} Anyway, if we work outside the root tree, it is still not completely clear what kind of theories can be constructed with such supermultiplets. Especially, it should be interesting to better understand how to introduce central charges through dualities. It is our belief that the techniques reviewed here will provide new insight towards the solution of this open problem.

Acknowledgments

E.O. would like to thank the University of Maryland and S. J. Gates, Jr. for the warm hospitality during the development of this work. Furthermore, E. O. would like to express his gratitude to Lubna Rana for helpful discussions. The research of S. B. is partially supported by the European Community’s Marie Curie Research Training Network under contract MRTN-CT-2004-005104 Forces of Universe, and by INTAS-00-00254 grant. The research of S.J.G. is partially supported by the National Science Foundation Grant PHY-0354401.

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