QCD thermodynamics on the lattice and in the 
Hadron Resonance Gas model

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Abstract. We conclude our investigation on the QCD transition temperature and equation 
of state (EoS) with 2 + 1 staggered flavors and one-link stout improvement. We extend our 
previous studies by choosing even finer lattices (up to \( N_t = 16 \) for \( T_c \), up to \( N_t = 10 \) and 12 
for the EoS). These new results support our earlier findings. A Symanzik improved gauge and 
a stout-link improved staggered fermion action is taken; the light and strange quark masses are 
set to their physical values. Various observables are calculated in the temperature (\( T \)) interval 
of 100 to 1000 MeV. We compare our data to those obtained by the “hotQCD” collaboration.

1. Introduction
The study of QCD thermodynamics and that of the phase diagram are receiving increasing 
attention in recent years. A transition occurs in strongly interacting matter from a hadronic, 
confined system at small temperatures and densities to a phase dominated by colored degrees of 
freedom at large temperatures or densities. A systematic approach to determine the properties 
of this transition is through lattice QCD. Lattice simulations indicate that the transition at 
vanishing chemical potential is merely an analytic crossover [1]. One of the most interesting 
quantities that can be extracted from lattice simulations is the transition temperature \( T_c \) at 
which hadronic matter passes to a deconfined phase. We summarize here the final results on \( T_c \) 
by the Wuppertal-Budapest collaboration [2].

This field of physics is particularly appealing because the deconfined phase of QCD can be 
produced in the laboratory, in the ultrarelativistic heavy ion collision experiments at CERN SPS, 
RHIC at Brookhaven National Laboratory, ALICE at the LHC and the future FAIR at the GSI. 
The experimental results available so far show that the hot QCD matter produced experimentally 
exhibits robust collective flow phenomena, which are well and consistently described by near-
ideal relativistic hydrodynamics. These hydrodynamical models need as an input an EoS which 
relates the local thermodynamic quantities.

Most of the results on the QCD EoS have been obtained using improved staggered fermions. 
This formulation does not preserve the flavor symmetry of continuum QCD; as a consequence, 
the spectrum of low lying hadron states is distorted. Recent analyses performed by various 
collaborations [2, 3] have pointed out that this distortion can have a dramatic impact on
Figure 1. (Color online) Left: The trace anomaly on lattices with different spatial volumes: $N_s/N_t = 3$ (red band) and $N_s/N_t = 6$ (blue points). Right: $I=\epsilon-3p$ at three different $T$s as a function of $1/N_t^2$. Filled/open symbols represent results with/without tree-level improvement.

the thermodynamic quantities. To quantify this effect, one can compare the low temperature behavior of the observables obtained on the lattice, to the predictions of the Hadron Resonance Gas (HRG) model.

2. Lattice framework

Here we present our results for the following observables: subtracted chiral condensate $\Delta_{l,s} = [(\bar{\psi}\psi)_{l,T} - m_l/m_s(\bar{\psi}\psi)_{s,T}] / [(\bar{\psi}\psi)_{l,0} - m_l/m_s(\bar{\psi}\psi)_{s,0}]$ for $l=u,d$, and with $\langle \bar{\psi}\psi \rangle_q = T\partial \ln Z / (\partial m_q V)$, pressure ($p$), trace anomaly ($I=\epsilon-3p$) and speed of sound ($c_s$), for $n_f=2+1$ dynamical quarks. We improve our previous findings [4, 5, 6] by choosing finer lattices ($N_t=16$ for $T_c$, $N_t=8$, 10 and a few checkpoints at $N_t=12$ for the equation of state). We work again with physical light and strange quark masses: we fix them by reproducing the physical ratios $f_K/m_\pi$ and $f_K/m_K$ and by this procedure [4, 5, 2] we get $m_s/m_{u,d}=28.15$. We checked that there were no significant finite size effects, by performing two sets of simulations in boxes with a size of 3.5 fm and 7 fm around $T_c$. The left panel of figure 1 shows the comparison between the two volumes for the normalized trace anomaly $I/T^4$. Let us note here, that the volume independence in the transition region is an unambiguous evidence for the crossover type of the transition.

To decrease lattice artefacts, we apply tree-level improvement for our thermodynamic observables: we divide the lattice results with the appropriate improvement coefficients. These factors can be calculated analytically for our action and in case of the pressure we have the following values on different $N_t$'s: $N_t=6$ gives 1.517, $N_t=8$ gives 1.283, $N_t=10$ gives 1.159 and $N_t=12$ gives 1.099. Using thermodynamical relations one can obtain these improvement coefficients for the energy density, trace anomaly and entropy, too. The speed of sound receives no improvement factor at tree level. Note, that these improvement coefficients are exact only at tree-level, thus in the infinitely high temperature, non-interacting case. As we decrease the temperature, corrections to these improvement coefficients appear, which have the form $1 + b_2(T)/N_t^2 + \ldots$. Empirically one finds that the $b_2(T)$ coefficient, which describes the size of lattice artefacts of the tree-level improved quantities, is tiny not only at very high temperatures, but throughout the deconfined phase. The right panel of figure 1 illustrates at three temperature
Figure 2. (Color online) $\Delta_{l,s}$ as a function of $T$. The gray band is our continuum result, obtained with the stout action. Full symbols are obtained with the asqtad and p4 actions [13, 14]. The solid line is the HRG model result with physical masses. The error band corresponds to the uncertainty in the quark mass-dependence of hadron masses. The dashed lines are the HRG+$\chi$PT model result with distorted masses of the hotQCD Collaboration [13, 14] for $N_t = 8$ and $N_t = 12$.

Figure 3. (Color online) The trace anomaly normalized by $T^4$ as a function of $T$ on $N_t = 6, 8, 10$ and 12 lattices.
values ($T = 132, 167$ and $223$ MeV) the effectiveness of this improvement procedure. We show both the unimproved/improved values of the trace anomaly for $N_t=6,8,10$ and $12$ as a function of $1/N_t^2$. The lines are linear continuum extrapolations using the three smallest lattice spacings $a$. The $a\to0$ limit of both the unimproved and the improved observables converge to the same value. The figure confirms the expectations, that lattice tree-level improvement effectively reduces the cutoff effects. At all three $T$s the unimproved observables have larger cutoff effects than the improved ones. Actually, all the three values of $b_2(T)$, which indicate the remaining cutoff effects after tree-level improvement, differ from zero by less than one standard deviation.

The most popular technique to determine the EoS is the integral method. For large homogenous systems $p$ is proportional to the logarithm of the partition function. Its direct determination is difficult. Instead, one determines the partial derivatives with respect to the bare lattice parameters. Finally, $p$ is rewritten as a multidimensional integral along a path in the space of bare parameters. To obtain the EoS for various $m_\pi$, we simulate for a wide range of bare parameters on the plane of $m_{u,d}$ and $\beta$ ($m_s$ is fixed to its physical value). Having obtained this large set of data we generalize the integral method and include all possible integration paths into the analysis [7, 8].

An additive divergence is present in $p$, which is independent of $T$. One removes it by subtracting the same observables measured on a lattice, with the same bare parameters but at a different $T$ value. Here we use lattices with a large enough temporal extent, so it can be regarded as $T = 0$.

3. Hadron Resonance Gas Model

The Hadron Resonance Gas model has been widely used to study the hadronic phase of QCD in comparison with lattice data. The low temperature phase is dominated by pions. Goldstone’s theorem implies weak interactions between pions at low energies. As the temperature $T$ increases, heavier states become more relevant and need to be taken into account. The HRG model has its roots in the theorem of Ref. [9], which allows to calculate the microcanonical partition function of an interacting system, in the thermodynamic limit $V \to \infty$, to a good approximation, assuming that it is a gas of non-interacting free hadrons and resonances [10]. Staggered lattice discretization has a considerable impact on the hadron spectrum. In order to investigate these errors, we define a “lattice HRG” model, where in the hadron masses lattice discretization effects are taken into account.

In order to compare the HRG model results with our additional lattice simulations at larger-than-physical quark masses, we need the pion mass dependence (see e.g. [11] for a recent lattice result) of all hadrons and resonances included in the calculation. As we already did in [2], we assume that all resonances behave as their fundamental state hadrons as functions of the pion mass. For the fundamental hadrons, we use the pion mass dependence from Reference [12]. For larger-than-physical quark masses, the taste symmetry violation at finite $a$ has a milder impact on the pressure. This also motivates to take a pion mass of about 700 MeV as the starting point of the integration of the pressure at $T = 100$ MeV.

4. Results

As we will see different sets of data corresponding to different $N_t$ nicely agree with each other for all observables under study; thus, we expect that discretization effects are tiny.

In figure 2 we show the subtracted chiral condensate $\Delta_{l,s}$ as a function of the temperature. The gray band is our continuum result, obtained with the stout action. Full symbols are obtained with the asqtad and p4 actions [13, 14]. The solid line is the HRG model result with physical masses. The error band corresponds to the uncertainty in the quark mass-dependence of hadron masses. The dashed lines are the HRG+$\chi$PT model result with distorted masses of the hotQCD Collaboration [13, 14] for $N_t = 8$ and $N_t = 12$. As we can see, the HRG model
predictions with physical resonance spectrum nicely agree with our results. The model also nicely reproduces the data obtained by the hotQCD collaboration, once the distorted resonance spectrum is taken into account. We find that the inflection point for the subtracted chiral condensate sits at \( T = 157(3)(3) \) MeV, which can be identified as the transition temperature for chiral symmetry restoration given by this observable (since the transition is an analytic crossover, different observables can give rise to different transition temperatures).

In figure 3 we show the \( T \) dependence of \( \varepsilon \)-3p for \( n_f = 2 + 1 \). We have results at four different “\( a \)”. Results show essentially no dependence on “\( a \)”, they all lie on top of each other. Only the coarsest \( N_t = 6 \) lattice shows some deviation around \( T \sim 300 \) MeV. On the same figure, we zoom into the transition region. Here we also show the results from the HRG model: a good agreement with the lattice results is found up to \( T \sim 140 \) MeV. One characteristic temperature of the crossover transition can be defined as the inflection point of the trace anomaly. This and other characteristic features of the trace anomaly are the following: the inflection point of \( I(T)/T^4 \) is 152(4) MeV; the maximum value of \( I(T)/T^4 \) is 4.1(1), whereas \( T \) at the maximum of \( I(T)/T^4 \) is 191(5) MeV.

In figure 4 we show \( p(T) \). We have results at three different values for “\( a \)”. The \( N_t = 6 \) and \( N_t = 8 \) are in the range from 100 up to 1000 MeV. In figure 5 we present the energy density. In figure 6 \( c_s^2(T) \), the speed of sound is shown. One can also read off the characteristic points of this curve: the minimum value of \( c_s^2(T) \) is 0.133(5); \( T \) at the minimum of \( c_s^2(T) \) is 145(5) MeV; whereas \( \varepsilon \) at the minimum of \( c_s^2(T) \) is 0.20(4) GeV/fm\(^3\).

As it was already discussed in Ref. [5], there is a disagreement between the results of current large scale thermodynamical calculations. The main difference can be described by a \( \sim 20-30 \) MeV shift in the temperature. This means, that the transition temperatures are different: the temperature values that we obtain are smaller by this amount than the values of the “hotQCD” collaboration. Refs. [12] and [2] presented a possible explanation for this problem: the more severe discretization artefacts of the “asqtad” and “p4fat” actions used by the “hotQCD” collaboration lead to larger transition temperatures. This can be clearly seen in figure 2, where the results of the two collaborations for the subtracted chiral condensate are presented: the shift in temperature is evident, and can be reproduced by the HRG model once the correct resonance spectrum is taken into account.

It is interesting to look for this discrepancy in the equation of state as well. In figure 7 we compare the trace anomaly obtained in this study with the trace anomaly of the “hotQCD” collaboration. We plot the \( N_t = 8 \) data using the “p4fat” and “asqtad” actions, which we took from Refs. [13] and [15]. As it can be clearly seen, the upward going branch and the peak position are located at \( \sim 20 \) MeV higher temperatures in the simulations of the “hotQCD” group. This is the same phenomenon as the one, which was already reported for many other quantities in [5]. We also see, that the peak height is about 50% larger in the “hotQCD” case. We determined the equation of state of QCD by means of lattice simulations. Results for the \( n_f = 2 + 1 \) flavor chiral condensate, pressure, trace anomaly and for the speed of sound were presented on figures. The results were obtained by carrying out lattice simulations at several “\( a \)” values, in the \( T \) range of \( T = 100 \ldots 1000 \) MeV. In order to reduce the lattice artefacts we applied tree-level improvement for all of the thermodynamical observables. We found that there is no difference in the results at the three finest lattice spacings. This shows that the lattice discretization errors are not significant and the continuum limit can be reliably taken. The details of the present work can be found in Refs. [2, 7]

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Figure 4. (Color online) $p(T)$ normalized by $T^4$ as a function of the temperature on $N_t = 6, 8$ and 10 lattices. The Stefan-Boltzmann limit $p_{SB}(T) \approx 5.209 \cdot T^4$ is indicated. At $T = 1000$ MeV $p(T)$ is almost 20% below this limit.

Figure 5. (Color online) The energy density normalized by $T^4$ as a function of the temperature on $N_t = 6, 8$ and 10 lattices. The Stefan-Boltzmann limit $\epsilon_{SB} = 3p_{SB}$ is indicated by an arrow.
Figure 6. (Color online) The speed of sound squared as a function of the temperature on $N_t = 6, 8$ and 10 lattices. The Stefan-Boltzmann limit is $c_{s,SB}^2 = 1/3$ indicated by an arrow.

Figure 7. (Color online) The normalized trace anomaly obtained in our study is compared to recent results from the “hotQCD” collaboration [13, 15].
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