Learning Games and Rademacher Observations Losses

Richard Nock
NICTA & the Australian National University
richard.nock@nicta.com.au

February 16, 2016

Abstract

It has recently been shown that supervised learning with the popular logistic loss is equivalent to optimizing the exponential loss over sufficient statistics about the class: Rademacher observations (rados). We first show that this unexpected equivalence can actually be generalized to other example / rado losses, with necessary and sufficient conditions for the equivalence, exemplified on four losses that bear popular names in various fields: exponential (boosting), mean-variance (finance), Linear Hinge (on-line learning), ReLU (deep learning), and unhinged (statistics). Second, we show that the generalization unveils a surprising new connection to regularized learning, and in particular a sufficient condition under which regularizing the loss over examples is equivalent to regularizing the rados (with Minkowski sums) in the equivalent rado loss. This brings simple and powerful rado-based learning algorithms for sparsity-controlling regularization, that we exemplify on a boosting algorithm for the regularized exponential rado-loss, which formally boosts over four types of regularization, including the popular ridge and lasso, and the recently coined SLOPE — we obtain the first proven boosting algorithm for this last regularization. Through our first contribution on the equivalence of rado and example-based losses, Ω-R.ADABoost appears to be an efficient proxy to boost the regularized logistic loss over examples using whichever of the four regularizers (and any linear combination of them, e.g., for elastic net regularization). We are not aware of any regularized logistic loss formal boosting algorithm with such a wide spectrum of regularizers. Experiments display that regularization consistently improves performances of rado-based learning, and may challenge or beat the state of the art of example-based learning even when learning over small sets of rados. Finally, we connect regularization to ε-differential privacy, and display how tiny budgets (e.g. ε < 10⁻³) can be afforded on big domains while beating (protected) example-based learning.

1 Introduction

A recent result has shown that minimizing the popular logistic loss over examples in supervised learning is equivalent to the minimisation of the exponential loss over sufficient statistics about the class known as Rademacher observations (rados, [Nock et al., 2015]), for the same classifier. In short, we fit a classifier over data that is different from examples, and the same classifier generalizes well to new observations. It is known that sufficient statistics carry the intractability of certain processes that would otherwise be easy with data (Montanari, 2014). In the case of rados, such a computational caveat turns out to be a big advantage as privacy is becoming crucial (Enserink & Chin, 2015). Indeed, rados allow to protect data not just from a computational complexity standpoint, but also from geometric, algebraic and statistical standpoints (Nock et al., 2015), while still allowing to learn accurate classifiers.

Two key problems remain: learning from rados can compete experimentally with learning from examples, but there is a gap to reduce for rados to be not just a good material to learn from in a privacy setting, but also a serious alternative to learning from examples at large, yielding new avenues to supervised learning.
Second, theoretically speaking, it is crucial to understand if this equivalence holds only for the logistic and exponential losses, or if it can be generalised and shed new light on losses and their minimisation.

In this paper, we provide answers to these two questions, with four main contributions. Our first contribution is to show that this generalization indeed holds: other example losses admit equivalent losses in the rado world, meaning in particular that their minimiser classifier is the same, regardless of the dataset of examples. The technique we use exploits a two-player zero sum game representation of convex losses, that has been very useful to analyse boosting algorithms (Schapire, 2003; Telgarsky, 2012), with one key difference: payoffs are non-linear convex, eventually non-differentiable. These also resemble the entropic dual losses (Reid et al., 2015), with the difference that we do not enforce conjugacy over the simplex. The conditions of the game are slightly different for examples and rados. We provide necessary and sufficient conditions for the resulting losses over examples and rados to be equivalent. Informally, equivalence happens iff the convex functions of the games satisfy a symmetry relationship and the weights satisfy a linear system of equations. We give four cases of this equivalence. It turns out that the losses involved bear popular names in different communities, even when not all of them are systematically used as losses per se: exponential, logistic, square, mean-variance, ReLU, linear Hinge, and unhinged losses (Nair & Hinton, 2010; Gentile & Warmuth, 1998; Nock & Nielsen, 2008; Telgarsky, 2012; Vapnik, 1998; van Rooyen et al., 2015) (and many others).

Our second contribution came unexpectedly through this equivalence. Regularizing a loss is common in machine learning (Bach et al., 2011). We show a sufficient condition for the equivalence under which regularizing the example loss is equivalent to regularizing the rados in the rado loss, i.e. making a Minkowski sum of the rado set with a classifier-based set. This property is independent of the regularizer, and holds for all four cases of equivalence.

Third, we propose a boosting algorithm, Ω-R.ADA-BOOST, that learns a classifier from rados using the exponential regularized rado loss, with regularization choice belonging to the ridge, lasso, ℓ∞, or the recently coined SLOPE (Bogdan et al., 2015). A key property is that Ω-R.ADA-BOOST bypasses the Minkowski sums to compute regularized rados. It is therefore computationally efficient. Experiments display that Ω-R.ADA-BOOST is all the better vs ADA-BOOST (unregularized and ℓ1-regularized) as the domain gets larger, and is able to learn both accurate and sparse classifiers, making it a good contender for supervised learning at large on big domains. From a theoretical standpoint, we show that for any of these four regularizations, Ω-R.ADA-BOOST is a boosting algorithm — thus, through our first contribution, Ω-R.ADA-BOOST is an efficient proxy to boost the regularized logistic loss over examples using whichever of the four regularizers, and by extension, any linear combination of them (e.g., for elastic net regularization (Zou & Hastie, 2005)). We are not aware of any regularized logistic loss formal boosting algorithm with such a wide spectrum of regularizers.

Our fourth contribution is a direct application of our findings to ε-differential privacy (DP). We protect directly the examples, granting the property that all subsequent stages are DP as well. We show theoretically that a most popular mechanism (Dwork & Roth, 2014) used to protect examples in rados amounts to a surrogate form of regularization of the clean examples’ loss; furthermore, the amount of noise can be commensurate to the one for a direct protection of examples. In other words, since rados’ norm may be much larger than examples’ (e.g. on big domains), we can expect noise to be much less damaging if learning from protected rados, and afford tiny budgets (e.g. ε ≈ 10⁻４) at little cost in accuracy. Experiments validate this intuition.

The rest of this paper is as follows. §2, 3 and 4 respectively present the equivalence between example and rado losses, its extension to regularized learning and Ω-R.ADA-BOOST. §5, 6 and 7 respectively present differential privacy vs regularized rado losses, detail experiments, and conclude. In order not to laden the paper’s body, an appendix, starting page 15 of this draft, contains the proofs and additional theoretical and experimental results.
2 Games and equivalent example/rado losses

We first start by defining and analysing our general two players game setting. To avoid notational load, we shall not put immediately the learning setting at play, considering for the moment that the learner fits a general vector \( z \in \mathbb{R}^m \), which depends both on data (examples or rados) and classifier. Let \( [m] = \{1, 2, \ldots, m\} \) and \( \Sigma_m = \{-1, 1\}^m \), for \( m > 0 \). Let \( \varphi_e : \mathbb{R} \to \mathbb{R} \) and \( \varphi_i : \mathbb{R} \to \mathbb{R} \) two convex and lower-semicontinuous generators. We define functions \( \mathcal{L}_e : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R} \) and \( \mathcal{L}_i : \mathbb{R}^{2m} \times \mathbb{R}^m \to \mathbb{R} \):

\[
\mathcal{L}_e(p, z) = \sum_{i \in [m]} p_i z_i + \mu_e \sum_{i \in [m]} \varphi_e(p_i),
\]

\[
\mathcal{L}_i(q, z) = \sum_{J \subseteq [m]} q_J \sum_{i \in J} z_i + \mu_i \sum_{J \subseteq [m]} \varphi_i(q_J),
\]

where \( \mu_e, \mu_i > 0 \) do not depend on \( z \). For the notation to be meaningful, the coordinates in \( q \) are assumed (wlog) to be in bijection with \( 2^{[m]} \). The dependence of both problems in their respective generators is implicit and shall be clear from context. The adversary’s goal is to fit

\[
p^*(z) = \arg \min_{p \in \mathbb{R}^m} \mathcal{L}_e(p, z),
\]

\[
q^*(z) = \arg \min_{q \in \mathbb{R}^{2m}} \mathcal{L}_i(q, z),
\]

with \( \mathbb{H}^{2m} = \{ q \in \mathbb{R}^{2m} : 1^\top q = 1 \} \), so as to attain

\[
\mathcal{L}_e^*(z) = \mathcal{L}_e(p^*(z), z),
\]

\[
\mathcal{L}_i^*(z) = \mathcal{L}_i(q^*(z), z),
\]

and let \( \partial \mathcal{L}_e^*(z) \) and \( \partial \mathcal{L}_i^*(z) \) denote their subdifferentials. We view the learner’s task as the problem of maximising the corresponding problems in eq. (5) (with examples) or (6) (with rados), or equivalently minimising negative the corresponding function, then called a loss function. The question of when these two problems are equivalent from the learner’s standpoint motivates the following definition.

**Definition 1** Two generators \( \varphi_e, \varphi_i \), are said proportionate iff for any \( m > 0 \), there exists \( (\mu_e, \mu_i) \) such that

\[
\mathcal{L}_e^*(z) = \mathcal{L}_i^*(z) + b, \forall z \in \mathbb{R}^m.
\]

(\( b \) does not depend on \( z \)) \( \forall m \in \mathbb{N}, \) let

\[
G_m = \begin{bmatrix}
0_{2m-1}^\top & 1_{2m-1}^\top & G_{m-1}
\end{bmatrix}
\in \{0, 1\}^{m \times 2^n}
\]

if \( m > 1 \), and \( G_1 = [0 \ 1] \) otherwise (\( z_d \) denotes a vector in \( \mathbb{R}^d \)). Each column of \( G_m \) is the binary indicator vector for the edge vectors summed in a rado; wlog, we let these to give the bijection between \( 2^{[m]} \) and coordinates of \( q^{(*)}(z) \).

**Theorem 2** \( \varphi_e, \varphi_i \) are proportionate iff the optimal solutions \( p^*(z) \) and \( q^*(z) \) to eqs (3) and (4) satisfy

\[
p^*(z) \in \partial \mathcal{L}_e^*(z),
\]

\[
G_m q^*(z) \in \partial \mathcal{L}_i^*(z).
\]

In the case where \( \varphi_e, \varphi_i \) are differentiable, they are proportionate iff \( p^*(z) = G_m q^*(z) \).
(Proof in Appendix, Subsection 9.1) Theorem 2 gives a necessary and sufficient condition for two generators to be proportionate. It does not say how to construct one from the other, if possible. We now show that it is indeed possible and prune the search space: if \( \varphi_e \) is proportionate to some \( \varphi_r \), then it has to be a “symmetrized” version of \( \varphi_r \), according to the following definition.

**Definition 3** Let \( \varphi_r \) such that \( \text{dom}(\varphi_r) \supseteq (0,1) \). We call \( \varphi_{s(r)}(z) := \varphi_r(z) + \varphi_r(1-z) \) the symmetrisation of \( \varphi_r \).

**Lemma 4** If \( \varphi_e \) and \( \varphi_r \) are proportionate, then \( \varphi_e(z) = (\mu_e/\mu_e) \cdot \varphi_{s(r)}(z) + (b/\mu_e) \), where \( b \) appears in eq. (7).

(Proof in Appendix, Subsection 9.2) To summarize, \( \varphi_e \) and \( \varphi_r \) are proportionate iff (i) they meet the structural property that \( \varphi_e \) is (proportional to) the symmetrized version of \( \varphi_r \) (according to Definition 3), and (ii) the optimal solutions \( p^*(z) \) and \( q^*(z) \) to problems (1) and (2) satisfy the conditions of Theorem 2. Depending on the direction, we have two cases to craft proportionate generators. First, if we have \( \varphi_r \), then necessarily \( \varphi_e \propto \varphi_{s(r)} \) so we merely have to check Theorem 2. Second, if we have \( \varphi_e \), then it matches Definition 3. In this case, we have to find \( \varphi_r = f + g \) where \( g(z) = -g(1-z) \) and \( \varphi_e(z) = f(z) + f(1-z) \).

We now come back to \( L^*_e(z) \), \( L^*_r(z) \) as defined in Definition 1 and make the connection with example and rado losses. In the next definition, an \( e \)-loss \( \ell^*_e(z) \) is a function defined over the coordinates of \( z \), and a \( r \)-loss \( \ell_r(z) \) is a function defined over the subsets of sums of coordinates. Functions can depend on other parameters as well.

**Definition 5** Suppose \( e \)-loss \( \ell^*_e(z) \) and \( r \)-loss \( \ell_r(z) \) are such that there exist (i) \( f_e : \mathbb{R} \rightarrow \mathbb{R} \) and \( f_r(z) : \mathbb{R} \rightarrow \mathbb{R} \) both strictly increasing and such that \( \forall z \in \mathbb{R}^m \),

\[
- \mathbb{L}^*_e(z) = f_e(\ell^*_e(z)) \tag{11}
\]
\[
- \mathbb{L}^*_r(z) = f_r(\ell_r(z)) \tag{12}
\]

Then the couple \((\ell^*_e, \ell_r)\) is called a couple of equivalent example-rado losses.

Hereafter, we just write \( \varphi^*_e \) instead of \( \varphi_{s(r)} \).

**Lemma 6** \( \varphi_r(z) = z \log z - z \) is proportionate to \( \varphi_e = \varphi_e = z \log z + (1-z) \log (1-z) - 1 \), whenever \( \mu_e = \mu_r \).

(Proof in Appendix, Subsection 9.3)

**Corollary 7** The following example and rado losses are equivalent for any \( \mu > 0 \):

\[
\ell^*_e(z, \mu) = \sum_{i \in [m]} \log \left( 1 + \exp \left( -\frac{1}{\mu} \cdot z_i \right) \right) \tag{13}
\]
\[
\ell_r(z, \mu) = \sum_{J \subseteq [m]} \exp \left( -\frac{1}{\mu} \cdot \sum_{i \in J} z_i \right) \tag{14}
\]

(Proof in Appendix, Subsection 9.4)

**Lemma 8** \( \varphi_r(z) = (1/2) \cdot z^2 \) is proportionate to \( \varphi_e = \varphi_e = (1/2) \cdot (1-2z(1-z)) \) whenever \( \mu_e = \mu_r/2^m - 1 \).

\(^1\)Alternatively, \( -\varphi_e \) is permissible [Kearns & Mansour, 1999].
Corollary 9 The following example and rado losses are equivalent, for any \( \mu > 0 \):

\[
\ell_e(z, \mu) = \sum_{i \in [m]} \left( 1 - \frac{1}{\mu} \cdot z_i \right)^2 ,
\]

\[
\ell_r(z, \mu) = -\left( \mathbb{E}_J \left[ \frac{1}{\mu} \cdot \sum_{i \in J} z_i \right] - \mu \cdot \mathbb{V}_J \left[ \frac{1}{\mu} \cdot \sum_{i \in J} z_i \right] \right) ,
\]

where \( \mathbb{E}_J[X(J)] \) and \( \mathbb{V}_J[X(J)] \) denote the expectation and variance of \( X \) wrt uniform weights on \( J \subseteq [m] \).

(Proof in Appendix, Subsection 9.6) We now investigate cases of non differentiable proportionate generators, the first of which is self-proportionate (\( \varphi_e = \varphi_r \)). We let \( \chi_A(z) \) be the indicator function:

\[
\chi_A(z) = 0 \text{ if } z \in A \text{ (and } +\infty \text{ otherwise), convex since } A = [0, 1] \text{ is convex.}
\]

Lemma 10 \( \varphi_r(z) = \chi_{[0,1]}(z) \) is self-proportionate, \( \forall \mu_e, \mu_r \).

(Proof in Appendix, Subsection 9.7)

Corollary 11 The following example and rado losses are equivalent, for any \( \mu_e, \mu_r \):

\[
\ell_e(z, \mu_e) = \sum_{i \in [m]} \max \left\{ 0, -\frac{1}{\mu_e} \cdot z_i \right\} ,
\]

\[
\ell_r(z, \mu_r) = \max \left\{ 0, \max_{J \subseteq [m]} \left\{ -\frac{1}{\mu_r} \cdot \sum_{i \in J} z_i \right\} \right\} .
\]

(Proof in Appendix, Subsection 9.8)

Lemma 12 \( \varphi_r(z) = \chi_{[\frac{1}{2}, \frac{1}{2}]}(z) \) is proportionate to \( \varphi_e = \varphi_s = \chi_{\{\frac{1}{2}\}}(z) \), for any \( \mu_e, \mu_r \).

(Proof in Appendix, Subsection 9.9)

Corollary 13 The following example and rado losses are equivalent, for any \( \mu_e, \mu_r \):

\[
\ell_e(z, \mu_e) = \sum_i -\frac{1}{\mu_e} \cdot z_i ,
\]

\[
\ell_r(z, \mu_r) = \mathbb{E}_J \left[ -\frac{1}{\mu_r} \cdot \sum_{i \in J} z_i \right] .
\]

(Proof in Appendix, Subsection 9.10) Table 1 summarizes the four equivalent example and rado losses.

### 3 Learning with (rado) regularized losses

We now plug the learning setting. The learner is given a set of examples \( S = \{(x_i, y_i), i = 1, 2, ..., m \} \) where \( x_i \in \mathbb{R}^d, y_i \in \Sigma_1 \) (for \( i = 1, 2, ..., m \)). It returns a classifier \( h : \mathbb{R}^d \rightarrow \mathbb{R} \) from a predefined set \( \mathcal{H} \). Let \( z_i(h) = y h(x_i) \) and define \( z(h) \) as the corresponding vector in \( \mathbb{R}^m \), which we plug in the losses of Table 1 to obtain the corresponding example and rado losses. Losses simplify conveniently when \( \mathcal{H} \)
Table 1: Examples of equivalent example and rado losses. Names of the rado-losses \( \ell_r(z, \mu_r) \) are respectively the Exponential (I), Mean-variance (II), ReLU (III) and Unhinged (IV) rado loss. We use shorthands \( z^+_i = -\left(1/\mu_e\right) \cdot z_i \) and \( z^+_i = -\left(1/\mu_r\right) \cdot \sum_{i \in S} z_i \). Parameter \( \alpha_r \) appears in eq. (22). Column “\( \mu_e \) and \( \mu_r \)” gives the constraints for the equivalence to hold (see text for details).

| #  | \( \ell_e(z, \mu_e) \) | \( \ell_r(z, \mu_r) \) | \( \omega_r(z) \) | \( \mu_e \) and \( \mu_r \) | \( \alpha_r \) | Ref |
|----|----------------|----------------|-----------------|-----------------|--------|-----|
| I  | \( \sum_{i \in [m]} \log \left(1 + \exp \left(z^+_i\right)\right) \) | \( \sum_{j \leq [m]} \exp \left(z^+_j\right) \) | \( z \log z - z \) | \( \forall \mu_e = \mu_r \) | \( \mu_e \) | Cor. 7 |
| II | \( \sum_{i \in [m]} (1 + z^+_i)^2 \) | \( -(\|E_j\| - z^+_j - \mu_r \cdot \forall y \cdot z^+_j) \) | \( (1/2) \cdot z^2 \) | \( \forall \mu_e = \mu_r \) | \( \mu_e/4 \) | Cor. 9 |
| III | \( \sum_{i \in [m]} \max \{0, z^+_i\} \) | \( \max \{0, \max_{j \leq [m]} \{z^+_j\}\} \) | \( \chi_{[0,1]}(z) \) | \( \forall \mu_e, \mu_r \) | \( \mu_e \) | Cor. 11 |
| IV | \( \sum_i z^+_i \) | \( \mathbb{E}_j \{z^+_j\} \) | \( \chi_{[-1,1]}(z) \) | \( \forall \mu_e, \mu_r \) | \( \mu_e \) | Cor. 13 |

consists of linear classifiers, \( h(x) = \theta^T x \) for some \( \theta \in \Theta \subseteq \mathbb{R}^d \). In this case, the example loss can be described using edge vectors \( S_e = \{y_i \cdot x_i : i = 1, 2, ..., m\} \), since \( z_i = \theta^T (y_i \cdot x_i) \), and the rado loss can be described using rademacher observations \( \{\text{Nock et al.} 2015\} \), since \( \sum_{i \in S} z_i = \theta^T \pi\sigma \) for \( \sigma_i = y_i \) iff \( i \in S \) and \( \pi\sigma = (1/2) \cdot \sum_{i \in S} \sigma_i + y_i \cdot x_i \). Let us define \( S_e^* = \{\pi\sigma : \sigma \in S\} \) the set of all rademacher observations. We rewrite any couple of equivalent example and rado losses as \( \ell_e(S_e, \theta) \) and \( \ell_r(S_e^*, \theta) \) respectively\(^2\) omitting parameters \( \mu_e \) and \( \mu_r \), assumed to be fixed beforehand for the equivalence to hold (see Table I). Let us regularize the example loss, so that the learner’s goal is to minimize

\[
\ell_e(S_e, \theta, \Omega) = \ell_e(S_e, \theta) + \Omega(\theta),
\]

with \( \Omega \) a regularizer \( \{\text{Bach et al.} 2011\} \). The following shows that when \( f_e \) in eq. (11) is linear, there is a rado-loss equivalent to this regularized loss, regardless of \( \Omega \).

**Theorem 14** Suppose \( \mathcal{H} \) contains linear classifiers. Let \( (\ell_e(S_e, \theta), \ell_r(S_e^*, \theta)) \) be any couple of equivalent example-rado losses such that \( f_e \) in eq. (11) is linear:

\[
f_e(z) = a_e \cdot z + b_e,
\]

for some \( a_e > 0, b_e \in \mathbb{R} \). Then for any regularizer \( \Omega(\cdot) \), the regularized example loss \( \ell_e(S_e, \theta, \Omega) \) is equivalent to rado loss \( \ell_r(S_e^*, \theta, \Omega) \) computed over regularized rados:

\[
S_r^*, \Omega, \theta = S_e^* \oplus \{\hat{\Omega}(\theta) \cdot \theta\},
\]

where \( \oplus \) is Minkowski sum and \( \hat{\Omega}(\theta) = a_e \cdot \Omega(\theta)/\|\theta\|_2^2 \) if \( \theta \neq 0 \) (and 0 otherwise, assuming wlog \( \Omega(0) = 0 \)).

(Proof in Appendix, Subsection 9.11) Theorem 14 applies to all rado losses (I-IV) in Table I. The effect of regularization on rados is intuitive from the margin standpoint: assume that a “good” classifier \( \theta \) is one that ensures lowerbounded inner products \( \theta^T z \geq \tau \) for some margin threshold \( \tau \). Then any good classifier on a regularized rado \( \pi\sigma \) shall actually meet, over examples,

\[
\sum_{i : y_i = \sigma_i} \theta^T (y_i \cdot x_i) \geq \tau + a_e \cdot \Omega(\theta).
\]

Notice that ineq (24) ties an “accuracy” of \( \theta \) (edges, left hand-side) and its sparsity (right-hand side). One important question is the way the minimisation of the regularized rado loss impacts the minimisation of the regularized example loss when one *subsamples* the rados, and learns \( \theta \) from some \( S_r \subseteq S_e^* \) with eventually

\(^2\)To prevent notational overload, we blend the notions of (pointwise) loss and (samplewise) risk, as just “losses”.}
We give an answer for the log-loss (Nock et al., 2015) (row I in Table 1), and for this objective define the $\Omega$-regularized exp-rado-loss computed over \( S_r \), with \(|S_r| = n \) and \( \omega > 0 \) user-fixed:

\[
\ell_r^{\exp}(S_r, \theta, \Omega) := \frac{1}{n} \cdot \sum_{j=1}^{n} \exp \left( -\theta^\top \left( \pi_j - \omega \frac{\Omega(\theta)}{\|\theta\|^2_2} \right) \right),
\]

whenever \( \theta \neq 0 \) (otherwise, we discard the factor depending on \( \omega \) in the formula). We assume that \( \Omega \) is a norm, and let \( \ell_r^{\exp}(S_r, \theta) \) denote the unregularized loss (\( \omega = 0 \) in eq. \( 25 \)), and we let \( \ell_r^{\log}(S_r, \theta, \Omega) \) denote the \( \Omega \)-regularized log-loss. Notice that we normalize losses. We define the open ball \( B_{\Omega}(0, r) = \{ x \in \mathbb{R}^d : \Omega(x) < r \} \) and \( r^*_\Omega = (1/m) \cdot \max_{S_r} \|\pi^*_\sigma\|_\Omega \)

where \( \Omega^* \) is the dual norm of \( \Omega \). The following Theorem is a direct application of Theorem 3 in (Nock et al., 2015), and shows mild conditions on \( S_r \subseteq S^*_r \) for the minimization of \( \ell_r^{\exp}(S_r, \theta, \Omega) \) to indeed yield that of \( \ell_r^{\log}(S_r, \theta, \Omega) \).

**Theorem 15** Assume \( \Theta \subseteq \mathbb{B}_{\|\cdot\|_2}(0, r_\Theta) \), with \( r_\Theta > 0 \). Let \( g(\theta) = (\sup_{\theta' \in \Theta} \max_{\pi^* \in S^*_r} \exp(-\theta'^\top \pi^*_\sigma))/\|\theta\|^r_\exp \). Then if \( m \) is sufficiently large, \( \forall \delta > 0 \), there is probability \( \geq 1 - \delta \) over the sampling of \( S_r \) that any \( \theta \in \Theta \) satisfies:

\[
\ell_r^{\log}(S_r, \theta, \Omega) \leq \log 2 + (1/m) \cdot \log \ell_r^{\exp}(S_r, \theta, \Omega)
\]

\[+ O \left( \frac{\log(n)}{m^\beta} \cdot \sqrt{\frac{r_\Theta r^*_\Omega}{n}} + \frac{d}{nm} \log n \right),
\]

as long as \( \omega \geq \omega_m \) for some constant \( u > 0 \).

### 4 Boosting with (rado) regularized losses

\( \Omega \)-R.\textsc{AdaBoost} presents our approach to learning with rados regularized with regularizer \( \Omega \) to minimise loss \( \ell_r^{\exp}(S_r, \theta, \Omega) \) in eq. \( 25 \). Classifier \( \theta_t \) is defined as \( \theta_t \doteq \sum_{t'=1}^{t} \alpha_{t'} \cdot \ell_{t'} \), where \( \ell_{t'} \) is the \( k \)th canonical basis vector. A key property is that \( \Omega \)-R.\textsc{AdaBoost} bypasses the Minkowski sums to compute regularized rados. It is therefore computationally efficient. Frameboxes highlight the differences with \textsc{RadoBoost} (Nock et al., 2015). The expected edge \( r_t \) used to compute \( \alpha_t \) in eq. \( 27 \) is based on the following basis assignment:

\[
r_{t}(\ell) \leftarrow \frac{1}{\pi_{s_{\ell}}(t)} \sum_{j=1}^{n} w_{ij} \pi_{j_{\ell}}(t) \in [-1, 1] \right).
\]

The computation of \( r_t \) is eventually tweaked by the weak learner, as displayed in Algorithm \( \Omega\text{-}\textsc{Wl} \). We investigate four choices for \( \Omega \). For each of them, we prove the boosting ability of \( \Omega \)-R.\textsc{AdaBoost} (\( \Gamma \) is symmetric positive definite, \( S_\ell \) is the symmetric group of order \( d \), \( |\theta| \) is the vector whose coordinates are the absolute values of the coordinates of \( \theta \)):

\[
\Omega(\theta) = \begin{cases} 
\|\theta\|_1 \doteq |\theta|^\top \mathbf{1} & \text{Lasso} \\
\|\theta\|_2 \doteq \theta^\top \Gamma \theta & \text{Ridge} \\
\|\theta\|_\infty \doteq \max_k |\theta_k| & \ell_\infty \\
\|\theta\|_\Phi \doteq \max_{M \in S_d(M|\theta|)^\top} \xi & \text{SLOPE}
\end{cases}
\]

(Bach et al., 2011; Bogdan et al., 2015; Duchi & Singer, 2009; Su & Candès, 2015). The coordinates of \( \xi \) in SLOPE are \( \xi_k \doteq \Phi^{-1}(1 - kq/(2d)) \) where \( \Phi^{-1}(\cdot) \) is the quantile of the standard normal distribution and
Algorithm 1 $\Omega$-R.ADABoost

**Input** rados $S_r = \{\pi_1, \pi_2, \ldots, \pi_n\}; T \in \mathbb{N}_+; \omega \in \mathbb{R}_+; \gamma \in (0, 1);

Step 1: let $\theta_0 \leftarrow 0, w_0 \leftarrow (1/n)1$

Step 2: for $t = 1, 2, \ldots, T$

Step 2.1: call the weak learner $(\iota(t), r_t) \leftarrow \Omega$-WL$(S_r, w_t, \gamma, \omega, \theta_{t-1})$; \hfill (26)

Step 2.2: let

$$\alpha_{\iota(t)} \leftarrow \frac{1}{2\pi_{\iota(t)}} \log \frac{1 + r_t}{1 - r_t};$$

$$\delta_t \leftarrow \omega \cdot (\Omega(\theta_t) - \Omega(\theta_{t-1}));$$ \hfill (28)

Step 2.3: for $j = 1, 2, \ldots, n$

$$w_{tj} \leftarrow \frac{w_{(t-1)j}}{Z_t} \cdot \exp \left( -\alpha_{\iota(t)}\pi_{ji(t)} + \delta_t \right);$$ \hfill (29)

Return $\theta_T$;

$q \in (0, 1)$; thus, the largest coordinates (in absolute value) of $\theta$ are more penalized. We now establish the boosting ability of $\Omega$-R.ADABoost. We give no direction for Step 1 in $\Omega$-WL, which is consistent with the definition of a weak learner in the boosting theory: all we require from the weak learner is $|r_i|$ no smaller than some weak learning threshold $\gamma_{WL} > 0$.

**Definition 16** Fix any constant $\gamma_{WL} \in (0, 1)$. $\Omega$-WL is said to be a $\gamma_{WL}$-Weak Learner iff the feature $\iota(t)$ it picks at iteration $t$ satisfies $|r_{\iota(t)}| \geq \gamma_{WL}$ for any $t = 1, 2, \ldots, T$.

We also provide an optional step for the weak learner in $\Omega$-WL, which we exploit in the experimentations, which gives a total preference order on features to optimise further the convergence of $\Omega$-R.ADABoost.

**Theorem 17** (boosting with ridge). Take $\Omega(.) = \| . \|_2^2$. Fix any $0 < a < 1/5$, and suppose that $\omega$ and the number of iterations $T$ of $\Omega$-R.ADABoost are chosen so that

$$\omega < (2a \min_k \max_j \pi_{jk}^2)/(T\lambda_T),$$ \hfill (34)

where $\lambda_T > 0$ is the largest eigenvalue of $\Gamma$. Then there exists some $\gamma > 0$ (depending on $a$, and given to $\Omega$-WL) such that for any fixed $0 < \gamma_{WL} < \gamma$, if $\Omega$-WL is a $\gamma_{WL}$-Weak Learner, then $\Omega$-R.ADABoost returns at the end of the $T$ boosting iterations a classifier $\theta_T$ which meets:

$$\ell_{exp}^r(S_r, \theta_T, \| . \|_2^2) \leq \exp(-a\gamma_{WL}^2T/2).$$ \hfill (35)

Furthermore, if we fix $a = 1/7$, then we can fix $\gamma = 0.98$, and if we consider $a = 1/10$, then we can fix $\gamma = 0.999$. 8
Algorithm 2 $\Omega$-WL, for $\Omega \in \{\|\cdot\|_1, \|\cdot\|_2^2, \|\cdot\|_\infty, \|\cdot\|_\Phi\}$

**Input** set of rados $S_r = \{\pi_1, \pi_2, \ldots, \pi_n\}$; weights $w \in \triangle n$; parameters $\gamma \in (0, 1)$, $\omega \in \mathbb{R}_+$; classifier $\theta \in \mathbb{R}^d$;

Step 1: pick weak feature $\iota_* \in [d]$;

Optional — use preference order:

\[
\iota \succeq \iota' \iff |r_\iota| - \delta_\iota \geq |r_{\iota'}| - \delta_{\iota'};
\]

\[
(\delta_\iota = \omega \cdot (\Omega(\theta + \alpha_{\iota} \cdot 1) - \Omega(\theta)))
\]

// $r_\iota$ is given in (32), $\alpha_{\iota}$ is given in (27)

Step 2: if $\Omega = \|\cdot\|_2^2$ then

\[
\begin{align*}
\iota_* & \left\{ \begin{array}{ll}
r_{\iota_*} & \text{if } r_{\iota_*} \in [-\gamma, \gamma] \\
\text{sign}(r_{\iota_*}) \cdot \gamma & \text{otherwise}
\end{array} \right., \\
\end{align*}
\]

else $r_* \leftarrow r_{\iota_*}$;

Return $(\iota_*, r_*)$;

(Proof in Appendix, Subsection 9.12) Two remarks are in order. First, the cases $a = 1/7, 1/10$ show that $\Omega$-WL can still obtain large edges in eq. (32), so even a “strong” weak learner might fit in for $\Omega$-WL, without clamping edges. Second, the right-hand side of ineq. (34) may be very large if we consider that $\min_k \max_j |\pi_{jk}|$ may be proportional to $m^2$. So the constraint on $\omega$ is in fact loose, and $\omega$ may easily meet the constraint of Thm 15.

**Theorem 18 (boosting with lasso or $\ell_\infty$).** Take $\Omega(.) \in \{\|\cdot\|_1, \|\cdot\|_\infty\}$. Suppose $\Omega$-WL is a $\gamma_{WL}$-Weak Learner for some $\gamma_{WL} > 0$. Suppose $\exists 0 < a < 3/11$ s. t. $\omega$ satisfies:

\[
\omega = a\gamma_{WL} \min_k \max_j |\pi_{jk}| .
\]

Then $\Omega$-R.ADABoost returns at the end of the $T$ boosting iterations a classifier $\theta_T$ which meets:

\[
\ell_r^{\text{exp}}(S_r, \theta_T, \Omega) \leq \exp(-\tilde{T}\gamma_{WL}^2/2),
\]

where

\[
\tilde{T} \doteq \begin{cases} 
\frac{a\gamma_{WL} T}{(T - T_*) + a\gamma_{WL} \cdot T_*} & \text{if } \Omega = \|\cdot\|_1 \\
\frac{T_*}{T} & \text{if } \Omega = \|\cdot\|_\infty
\end{cases},
\]

and $T_*$ is the number of iterations where the feature computing the $\ell_\infty$ norm was updated.

(Proof in Appendix, Subsection 9.13) We finally investigate the SLOPE choice. The Theorem is proven for $\omega = 1$ in $\Omega$-R.ADABoost, for two reasons: it matches the original definition (Bogdan et al., 2015) and furthermore it unveils an interesting connection between boosting and the SLOPE properties (Su & Candès, 2015).

---

3If several features match this criterion, $T_*$ is the total number of iterations for all these features.
Algorithm 3 DP-RADOS

Input: rados \( S_r = \{\pi_1, \pi_2, \ldots, \pi_n\} \); budget \( \varepsilon > 0 \);
Step 1: let \( S_{dp}^0 \leftarrow \emptyset \);
Step 2: for \( j = 1, 2, \ldots, n \)
  Step 2.1: sample \( z_j \) as \( z_{jk} \sim \text{Lap}(z|nr_\varepsilon/\varepsilon) \), \( \forall k \);
  Step 2.2: \( S_{dp}^j \leftarrow S_{dp}^j \cup \{\pi_j + z_j\} \);
Return \( S_{dp}^\varepsilon \);

Theorem 19 (boosting with SLOPE). Take \( \Omega(.) = ||.||_\Phi \). Suppose wlog \( |\theta_{Tk}| \geq |\theta_{T(k+1)}| \), \( \forall k \), and fix \( \omega = 1 \). Let

\[
a = \min \left\{ \frac{3\gamma_{wl}}{11}, \frac{\Phi^{-1}(1-q/(2d))}{\min_k \max_j |\pi_{jk}|} \right\} . \tag{39}
\]

Suppose (i) \( \Omega\)-WL is a \( \gamma_{wl}\)-Weak Learner for some \( \gamma_{wl} > 0 \), and (ii) the \( q \)-value is chosen to meet:

\[
q \geq 2 \cdot \max_k \left\{ \left( 1 - \Phi \left( \frac{3\gamma_{wl}}{11} \cdot \max_j |\pi_{jk}| \right) \right) / \left( \frac{k}{n} \right) \right\} .
\]

Then classifier \( \theta_T \) returned by \( \Omega\)-R.ADABOOST at the end of the \( T \) boosting iterations satisfies:

\[
\ell_r^{exp}(S_r, \theta_T, ||.||_\Phi) \leq \exp(-a\gamma_{wl}^2 T/2) . \tag{40}
\]

(Proof in Appendix, Subsection 9.14) Constraint (ii) on \( q \) is interesting in the light of the properties of SLOPE (Bogdan et al. 2015; Su & Candès, 2015). Modulo some assumptions, SLOPE yields a control the false discovery rate (FDR) — that is, sparsity errors, negligible coefficients in the "true" linear model \( \theta^* \) that are actually found significant in the learned \( \theta \) —. Constraint (ii) links the "small" achievable FDR (upperbounded by \( q \)) to the "boostability" of the data: the fact that each feature \( \gamma_{wl} \) or has \( \max_j |\pi_{jk}| \) large, precisely flags potential significant features, thus reducing the risk of sparsity errors, and allowing small \( q \), which is constraint (ii). Using the second order approximation of normal quantiles (Su & Candès, 2015), a sufficient condition for (ii) is that, for some constant \( K \),

\[
\gamma_{wl} \min_j \max_j |\pi_{jk}| \geq K \cdot \sqrt{\log d + \log q^{-1}} ; \tag{41}
\]

but \( \min_j \max_j |\pi_{jk}| \) is proportional to \( m \), so ineq. (41), and thus (ii), may hold even for small samples and \( q \)-values.

We can now have a look at the regularized log-loss of \( \theta_T \) over examples, as depicted in Theorem 18 and show that it is guaranteed a monotonic decrease with \( T \), with high probability, for any applicable choice of regularization, since we get indeed that the regularized log-loss of \( \theta_T \) output by \( \Omega\)-R.ADABOOST, computed on examples, satisfies with high probability \( \ell_r^{\log}(S_\omega, \theta, \Omega) \leq \log 2 - \kappa \cdot T + \tau(m) \), with \( \tau(m) \to 0 \) when \( m \to \infty \), and \( \kappa \) does not depend on \( T \). Hence, \( \Omega\)-R.ADABOOST is an efficient proxy to boost the regularized log-loss over examples, using whichever of the ridge, lasso, \( \ell_\infty \) or SLOPE regularization, establishing the first boosting algorithm for this last choice. Notice finally that we can also choose any linear combinations of the regularizers and still keep the formal boosting property, thereby extending our results e.g., to the popular elastic nets regularization (Zou & Hastie, 2005).
Table 2: Best result of A\textsubscript{DA}Boost given the number of boosting iterations (best result over all line) / best result of regularization over We show here that the standard differential privacy (DP) mechanism (Dwork & Roth, 2014) to protect ex-
(osc) resembles the minimisation of an optimistic bound on a\textsubscript{DA}Boost’s method (in average) is indicated with “"", and the least sparse is indicated with “\textsuperscript{•}". When A\textsubscript{DA}Boost (resp. \(\ell_1\)-A\textsubscript{DA}Boost) yields the least sparse (resp. the sparsest) of all methods (including \(\Omega\)-R.A\textsubscript{DA}Boost), it is shown with “\textsuperscript{■}” (resp. “♦”). All domains but Kaggle are UCI (Bache & Lichman, 2013).

5 Regularized losses and differential privacy

We show here that the standard differential privacy (DP) mechanism (Dwork & Roth, 2014) to protect ex-
ample vectors \(e, e'\) satisfy \(\|e - e\|_1 \leq r_e\), which is ensured e.g. if all examples belong to a \(\ell_1\)-ball of diameter \(r_e\).

Theorem 20 DP-RADOS delivers \(\varepsilon\)-differential privacy. Furthermore, pick \((\Omega, \Omega^\ast)\) any couple of dual norms and assume \(S_e = S_e^\ast (|S_e| = 2^n)\). Then \(\forall \theta, \ell^{\text{exp}}(S_e^\ast, \Omega^\ast, \theta) \leq \exp \{m^{\ell^{\text{log}}(S_e, \theta, (1/m) \cdot \max_j \Omega^\ast(z_j) \cdot \Omega)}\}.

(Proof in Appendix, Subsection 9.15)

6 Experiments

We have implemented \(\Omega\)-WL using the order suggested to retrieve the topmost feature in the order. Hence, the weak learner returns the feature maximising \(|r_i| - \delta_i\). The rationale for this comes from the proofs of Theorems 17—19 showing that \(\prod_i \exp(- (|r_i|/2 - \delta_i(0)))\) is an upperbound on the exponential regularized rado-loss. We do not clamp the weak learner for \(\Omega(x) = \|w\|_p^2\), so the weak learner is restricted to the framebox in \(\Omega\)-WL\(^4\). We have tested two types of random rados, the plain random rados (Nock et al., 2015),

\(^4\)the values for \(\omega\) that we test, in \([0^{-\omega}, \omega \in \{0, 1, 2, 3, 4, 5\}\), are very small with respect to the upperbound in ineq. (34) given the number of boosting iterations \((T = 1000)\), and would yield on most domains a maximal \(\gamma = 1\).
Experiments I: (regularized) rados vs examples  

The objective of these experiments is to evaluate \( \Omega \text{-RADOS} \) as a contender for supervised learning per se. We compared \( \Omega - \text{RADOS} \) to ADABOOST (Schapire & Singer, 1999; Xi et al., 2009). All algorithms are run for a total of \( T = 1000 \) iterations, and at the end of the iterations, the classifier in the sequence that minimizes the empirical loss is kept. Notice therefore that rado-based classifiers are evaluated on the training set which computes the rados (in a privacy setting, the learner send the sequence of classifiers to the data handler, which then selects the best according to its training sample). To obtain very sparse solutions for \( \ell_1 \)-ADABOOST, we pick its \( \beta (\beta \text{ in } \{10^{-2}, 1, 10^2\}) \). The results we give, in Table [2], report only the lowest error of all of ADABOOST variants. The Appendix (Subsection [10]) details the support results, that are summarized in Table [2]. Experiments support several key observations. First, regularizing consistently reduces the test error of \( \Omega - \text{RADOS} \), by more than 15% on Magic, and 20% on Kaggle. Second, \( \Omega - \text{RADOS} \) is able to obtain both very sparse and accurate classifiers (Magic, Hardware, Marketing, Kaggle). Third, with the sole exception of domain Banknote, \( \Omega - \text{RADOS} \) competes or beats ADABOOST on all domains, and is all the better as the domain gets bigger. Fourth, it is important to have several choices of regularizers at hand. Fifth, as already remarked (Nock et al., 2015), significantly subsampling rados (e.g. Marketing, Kaggle) still yields very accurate classifiers. Finally, regularization in \( \Omega - \text{RADOS} \) successfully reduces sparsity to learn more accurate classifiers on several domains (Transfusion, Banknote, Winered, Magic, Marketing), achieving efficient adaptive sparsity control.

Experiments II: differential privacy  

We have tested DP-RADOS for a fixed number of rados of \( n = 100 \). Such a small number of rados has three advantages: (i) the privacy budget does not blow up, (ii) accurate classifiers can still be learned with a small number of rados (Nock et al., 2015), (iii) with such a small number of rados, we are within the reach of additional privacy guarantees (Nock et al., 2015). We have compared with ADABOOST, trained over a subset of \( n = 100 \) (protected) examples, randomly sampled out of the full training fold. To make sure that this does not impair the algorithm just because the sample is too small, we compute the test error for very large values of \( \varepsilon \) as a baseline. Last, for tight comparisons, we use the same set of random vectors \( z \) to protect the rados and the examples. This choice is justified and discussed at the end of the proof of Theorem 20 (Appendix, Subsection [9.15]). Yet, as we shall see, the results are exceedingly in favor of \( \Omega - \text{RADOS} \) in this case. To give a more balanced picture, we chose to compute an “approximate” example-equivalent privacy budget \( \varepsilon_a = \varepsilon_a(\varepsilon, n, m) \) for ADABOOST and \( n \) examples, which we fix to be

\[
\varepsilon_a := n \cdot \ln \left( 1 + \frac{\exp(\varepsilon/n) - 1}{m} \right).
\]

We always have \( \varepsilon_a < \varepsilon \). The “optimal” DP picture of ADABOOST shall thus be representable as a stretching of its curves in between the figures for \( \varepsilon_a \) and \( \varepsilon \). We insist on the fact that the noise for \( \varepsilon \) is conservative but always safe, while computing \( \varepsilon \) from \( \varepsilon_a \) would sometimes fail to provide \( \varepsilon_a \)-DP (Appendix, Subsection [9.15]). Table [3] presents the results obtained for three big domains (\( m \) indicated in parenthesis), in which we have run unregularized algorithms for a fixed number of \( T = 1000 \) iterations, keeping the last classifier \( \theta_{1000} \) for testing. GaussNLin is a \( d = 2 \) simulated domain, non linearly separable but for which the optimal linear classifier has error < 2%. The results are a clear advocacy in favor of using rados against examples for the straight DP protection: with plain random rados, test errors that compete with clean data can be
observed for privacy budget $\varepsilon \approx 10^{-4}$, that is, more than a hundred times smaller than most reported studies \cite{Hsu2014}. In comparison, AdaBoost’s results, even plotted against the weak protection budget $\varepsilon_a$, are very significantly worse. Finally, on UCI domains SuSy and Higgs, non-trivial protections (typically, $\varepsilon \in [0.01, 1]$) allow to beat classification on clean data, as witnessed by a 6%+ test error reduction for Higgs. In addition to “coming for free” \cite{Wang2015} in machine learning, DP may thus also be a worthwhile companion to improve learning.

7 Conclusion

We have shown that the equivalence between the log loss over examples and the exponential loss over rados, as shown in \cite{Nock2013}, can be generalized to other losses via a principled representation of a loss function in a two-player zero-sum game. Furthermore, we have shown that this equivalence extends to regularized losses, where the regularization in the rado loss is performed over the rados themselves with Minkowski sums. Because regularization with rados has such a simple form, it is relatively easy to derive efficient learning algorithms working with various forms of regularization, as exemplified with ridge, lasso, $\ell_\infty$ and SLOPE regularizations in a formal boosting algorithm that we introduce, $\Omega$-R.AdaBoost. Experiments confirm that this freedom in the choice of regularization is a clear strength of the algorithm, and that regularization dramatically improves the performances over non-regularized rado learning. $\Omega$-R.AdaBoost efficiently controls sparsity, and may be a worthy contender for supervised learning at large outside the privacy framework. Experiments also display that SLOPE regularization tends to achieve top performances, and call for an extension to rados of the formal sparsity results already known \cite{Su2015}.

8 Acknowledgments

Thanks are also due to Stephen Hardy and Giorgio Patrini for many stimulating discussions and feedback on the subject. NICTA is funded by the Australian Government through the Department of Communications and the Australian Research Council through the ICT Center of Excellence Program.

References

Bach, F., Jenatton, R., Mairal, J., and Obozinski, G. Optimization with sparsity-inducing penalties. Foundations and Trends in Machine Learning, 4:1–106, 2011.

Bache, K. and Lichman, M. UCI machine learning repository, 2013.

Bogdan, M, van den Berg, E., Sabatti, C., Su, W., and Candès, E.-J. SLOPE – adaptive variable selection via convex optimization. Annals of Applied Statistics, 2015. Also arXiv:1310.1969v2.

Duchi, J.-C. and Singer, Y. Efficient learning using forward-backward splitting. In NIPS*22, pp. 495–503, 2009.

Dwork, C. and Roth, A. The algorithmic foundations of differential privacy. Foundations and Trends in Theoretical Computer Science, 9:211–407, 2014.

Enserink, M. and Chin, G. The end of privacy. Science, 347:490–491, 2015.

Gentile, C. and Warmuth, M. Linear hinge loss and average margin. In NIPS*11, pp. 225–231, 1998.
Hsu, J., Gaboardi, M., Haeberlen, A., Khanna, S., Narayan, A., Pierce, B.-C., and Roth, A. Differential privacy: An economic method for choosing epsilon. In *Proc. of the 27th IEEE CSFS*, pp. 398–410, 2014.

Kearns, M.J. and Mansour, Y. On the boosting ability of top-down decision tree learning algorithms. *J. Comp. Syst. Sc.*, 58:109–128, 1999.

Montanari, A. Computational implications of reducing data to sufficient statistics. Technical Report 2014-12, Stanford U., 2014.

Nair, V. and Hinton, G. Rectified linear units improve restricted boltzmann machines. In *27th ICML*, pp. 807–814, 2010.

Nock, R. and Nielsen, F. On the efficient minimization of classification-calibrated surrogates. In *NIPS*21, pp. 1201–1208, 2008.

Nock, R., Patrini, G., and Friedman, A. Rademacher observations, private data, and boosting. In 32nd *ICML*, pp. 948–956, 2015.

Reid, M.-D., Frongillo, R.-M., Williamson, R.-C., and Mehta, N.-A. Generalized mixability via entropic duality. In 28th *COLT*, pp. 1501–1522, 2015.

Schapire, R.-E. The boosting approach to machine learning: An overview. In Denison, D.-D., Hansen, M.-H., Holmes, C.-C., Mallick, B., and Yu, B. (eds.), *Nonlinear Estimation and Classification*, volume 171 of *Lecture Notes in Statistics*, pp. 149–171. Springer Verlag, 2003.

Schapire, R. E. and Singer, Y. Improved boosting algorithms using confidence-rated predictions. *MLJ*, 37: 297–336, 1999.

Su, W. and Candès, E.-J. SLOPE is adaptive to unknown sparsity and asymptotically minimax. *CoRR*, abs/1503.08393, 2015.

Telgarsky, M. A primal-dual convergence analysis of boosting. *JMLR*, 13:561–606, 2012.

van Rooyen, B., Menon, A., and Williamson, R.-C. Learning with symmetric label noise: The importance of being unhinged. In *NIPS*28, 2015.

Vapnik, V. *Statistical Learning Theory*. John Wiley, 1998.

Wang, Y.-X., Fienberg, S.E., and Smola, A.-J. Privacy for free: Posterior sampling and stochastic gradient Monte Carlo. In 32nd *ICML*, pp. 2493–2502, 2015.

Xi, Y.-T., Xiang, Z.-J., Ramadge, P.-J., and Schapire, R.-E. Speed and sparsity of regularized boosting. In 12th *AISTATS*, pp. 615–622, 2009.

Zou, H. and Hastie, T. Regularization and variable selection via the elastic net. *Journal of the Royal Statistical Society B*, 67:301–321, 2005.
# Appendix — Table of contents

| Proofs | Pg |
|--------|----|
| Proof of Theorem 2 | 16 |
| Proof of Lemma 4 | 16 |
| Proof of Lemma 6 | 18 |
| Proof of Corollary 7 | 19 |
| Proof of Lemma 8 | 20 |
| Proof of Corollary 9 | 21 |
| Proof of Lemma 10 | 22 |
| Proof of Corollary 11 | 24 |
| Proof of Lemma 12 | 24 |
| Proof of Corollary 13 | 25 |
| Proof of Theorem 14 | 25 |
| Proof of Theorem 17 | 26 |
| Proof of Theorem 18 | 28 |
| Proof of Theorem 19 | 30 |
| Proof of Theorem 20 | 31 |

| Additional Experiments | Pg |
|------------------------|----|
| Supports for rados (complement to Table 2) | 33 |
| Experiments on class-wise rados | 33 |
| Test errors and supports for rados (comparison last vs best empirical classifier) | 33 |
9 Proofs

9.1 Proof of Theorem 2

We split the proof in two parts, the first concerning the case where both generators are differentiable since some of the derivations shall be used hereafter, and then the case where they are not. Remark that because of Lemma 4, we do not have to cover the case where just one of the two generators would be differentiable.

Case 1: \( \varphi_e, \varphi_r \) are differentiable. We show in this case that being proportionate is equivalent to having:

\[
p^*(z) = G_m q^*(z) .
\]  

Solving eqs. (3) and (4) bring respectively:

\[
p^*_i(z) = \varphi_e^{-1}( -\frac{1}{\mu_e} \cdot z_i ) ,
\]  

\[
q^*_j(z) = \varphi_r^{-1}( -\frac{1}{\mu_r} \cdot \sum_{i \in I} z_i + \frac{\lambda}{\mu_r} ) ,
\]

where \( \lambda \) is picked so that \( q^*(z) \in \mathbb{H}^2[m] \), that is,

\[
\sum_{J \subseteq [m]} \varphi_r^{-1}( -\frac{1}{\mu_r} \cdot \sum_{i \in J} z_i + \frac{\lambda}{\mu_r} ) = 1 .
\]

We obtain

\[
\mathcal{L}^*_e(z) = -\mu_e \sum_{i \in [m]} \varphi_e^*( -\frac{1}{\mu_e} \cdot z_i ) ,
\]

\[
\mathcal{L}^*_r(z) = \lambda - \mu_r \sum_{J \subseteq [m]} \varphi_r^*( -\frac{1}{\mu_r} \cdot \sum_{i \in J} z_i + \frac{\lambda}{\mu_r} ) ,
\]

where \( \varphi^*(z) \doteq \sup_{z'} \{zz' - \varphi(z') \} \) denotes the convex conjugate of \( \varphi \). It follows from eq. (47) that:

\[
\frac{\partial}{\partial z_i} \mathcal{L}^*_e(z) = \varphi_e^{-1}( -\frac{1}{\mu_e} \cdot z_i )
\]

\[
= \varphi_e^{-1}( -\frac{1}{\mu_e} \cdot z_i )
\]

\[
= p^*_i(z) ,
\]
where eq. (49) follows from properties of $\varphi^*$. We also have

$$\frac{\partial}{\partial z_i} L_e^*(z)$$

$$= \left(1 - \sum_{j \subseteq [m]} \varphi_j^{-1} \left(-\frac{1}{\mu_i} \cdot \sum_{j \in j} z_j + \frac{\lambda}{\mu_i}\right) \right) \cdot \frac{\partial \lambda}{\partial z_i} + \sum_{j \subseteq [m]} \left(1_{i \in j} - \frac{\partial \lambda}{\partial z_i}\right) \cdot q_i^*(z)$$

$$= \frac{\partial \lambda}{\partial z_i} \cdot \left(1 - \sum_{j \subseteq [m]} q_j^*(z)\right) + \sum_{j \subseteq [m]} 1_{i \in j} \cdot q_i^*(z)$$

$$= \sum_{j \subseteq [m]} 1_{i \in j} \cdot q_i^*(z) , \quad (51)$$

since $q^*(z) \in \mathbb{H}^m$.

Now suppose $\varphi_e$ and $\varphi_r$ proportionate. It comes that there exists $(\mu_e, \mu_r)$ such that the gradients of eq. (7) yield $\nabla L_e^*(z) = \nabla L_r^*(z)$, and from eqs. (50) and (51) we obtain $p^*(z) = G_m q^*(z)$.

Reciprocally, having $p^*(z) = G_m q^*(z)$ for some $\varphi_e, \varphi_r$ and $\mu_e, \mu_r > 0$ implies as well $\nabla L_e^*(z) = \nabla L_r^*(z)$ from eqs. (50) and (51), and therefore eq. (7) holds as well. This ends the proof of Case 1 for Theorem 2.

Case 2: $\varphi_e, \varphi_r$ are not differentiable. To simplify the statement and proofs, we assume that $\mu_e = \mu_r = 1$. We define the following problems

$$L_e(z) = \inf_{p \in \mathbb{R}^m} z^T p + \varphi_e(p) , \quad (52)$$

$$L_r(z) = \inf_{q \in \mathbb{H}^m} z^T G_m q + \varphi_r(q) , \quad (53)$$

where $\varphi_e : \mathbb{R}^m \to \mathbb{R}$ and $\varphi_r : \mathbb{H}^m \to \mathbb{R}$ are convex. Recall that $\partial L_e$ and $\partial L_r$ are their subdifferentials, and $p(z)$ and $q(z)$ the arguments of the infima, assuming without loss of generality that they are finite. We now show that being proportionate is equivalent to having, for any $z$,

$$p(z) \in \partial L_r(z) , \quad (54)$$

$$G_m q(z) \in \partial L_e(z) . \quad (55)$$

This property is an immediate consequence of the following property, which we shall in fact show:

$$p(z) \in \partial L_e(z) , \quad (56)$$

$$G_m q(z) \in \partial L_r(z) . \quad (57)$$
Granted all (54–57) hold, Eq. (43) of Theorem 2 follows whenever subgradients are singletons. To see why the statement of the Theorem follows from (54–55), if the functions are proportionate, then their subdifferentials match from Definition 1 and we immediately get (54) and (55) from (56) and (57). If, on the other hand, we have both (54) and (55), then we get from (56) and (57) that $\partial \mathcal{L}_e(z) \cap \partial \mathcal{L}_r(z) \neq \emptyset$, $\forall z$ and so $0 \in \partial (\mathcal{L}_e(z) - \mathcal{L}_r(z))$, yielding the fact that the epigraphs of $\mathcal{L}_e(z)$ and $\mathcal{L}_r(z)$ match by a translation of some $b$ that does not depend on $z$, and by extension, the fact that $\varphi_e$ and $\varphi_r$ meet Definition 1 and are proportionate.

To show (56), we first remark that $-z' \in \partial \varphi_e(p(z'))$ for any $z'$ because of the definition of $p$ in (52). So, from the definition of subdifferentials, for any $z$,

$$\varphi_e(p(z')) + (-z')^\top (p(z) - p(z')) \leq \varphi_e(p(z)) \ .$$

Reorganising and substracting $z^\top p(z)$ to both sides, we get

$$-\varphi_e(p(z')) - z^\top p(z') \geq -\varphi_e(p(z)) - z^\top p(z) + (-p(z))^\top (z' - z) \ ,$$

which shows that $-p(z) \in \partial - (\varphi_e(p(z)) + z^\top p(z))$, and so $p(z) \in \partial \mathcal{L}_e(z)$.

We then tackle (57). We show that there exists $\lambda \in \mathbb{R}$ such that $\lambda \cdot 1_{2m} - g_m z \in \partial \varphi_r(q(z))$ at the optimal $q(z)$. Suppose it is not the case. Then because of the definition of subgradients, for any $\lambda \in \mathbb{R}$, there exists $q \in \mathbb{H}^{2m}, q \neq q(z)$ such that

$$\varphi_r(q(z)) + (\lambda \cdot 1_{2m} - g_m z)^\top (q - q(z)) > \varphi_r(q) \ .$$

Reorganising and using the fact that $q, q_e \in \mathbb{H}^{2m}$, we get $\varphi_r(q(z)) + z^\top g_m q(z) > \varphi_r(q) + z^\top g_m q$, contradicting the optimality of $q(z)$. Consider any $z'$ and its corresponding optimal $q(z')$. Since $\lambda' \cdot 1_{2m} - g_m z \in \partial \varphi_r(q(z))$ for some $\lambda' \in \mathbb{R}$, we get from the definition of subgradients that

$$\varphi_r(q(z)) \geq \varphi_r(q(z')) + (\lambda' \cdot 1_{2m} - g_m z')^\top (q(z) - q(z')) \ .$$

Reorganising and using the fact that $q(z), q(z') \in \mathbb{H}^{2m}$, we get

$$-(\varphi_r(q(z')) + z'^\top g_m q(z')) \geq -(\varphi_r(q(z)) + z^\top g_m q(z)) + (g_m q(z))^\top (z' - z) \ ,$$

showing that $-g_m q(z) \in \partial - (\varphi_r(q(z)) + z^\top g_m q(z))$, and so $g_m q(z) \in \partial \mathcal{L}_r(z)$.

### 9.2 Proof of Lemma 4

Take $m = 1$, and replace $z$ by real $z_1$. We have $\mathcal{L}_e(p(z_1)) = p z_1 + \varphi_e(z_1)$ and $\mathcal{L}_r(q, z) = q_{11} z_1 + \varphi_r(q_{11}) + \varphi_r(q_0)$. Remark that we can drop the constraint $q \in \mathbb{H}^2$ since then $q_0 = 1 - q_{11}$. So we get

$$\mathcal{L}_e^*(q) = \min_{q \in \mathbb{R}} q z_1 + \mu \varphi_e(q) + \mu \varphi_r(1 - q) = \min_{q \in \mathbb{R}} q z_1 + \mu \varphi_s(q) \quad \text{if} \quad \varphi_s(q) \neq 0 \ ,$$

$$= -\mu \varphi_s^* \left( -\frac{1}{\mu} z_1 \right) \ ,$$
whereas

$$L_e^*(p) = -\mu_e \varphi_e^* \left( -\frac{1}{\mu_e} \cdot z_1 \right),$$

and since $\varphi_e$ and $\varphi_r$ are proportionate, then

$$\varphi_r^* \left( -\frac{1}{\mu_e} \cdot z_1 \right) = \frac{\mu_r}{\mu_e} \cdot \varphi_{s(r)}^* \left( -\frac{1}{\mu_r} \cdot z \right) - \frac{b}{\mu_e}. \quad (59)$$

We then make the variable change $z = -z_1/\mu_e$ and get

$$\varphi_e^*(z) = \frac{\mu_r}{\mu_e} \cdot \varphi_{s(r)}^* (z) - \frac{b}{\mu_e}, \quad (60)$$

which yields, since $\varphi_e, \varphi_r,$ and by extension $\varphi_{s(r)}$, are all convex and lower-semicontinuous,

$$\varphi_e(z) = \frac{\mu_r}{\mu_e} \cdot \varphi_{s(r)}(z) + \frac{b}{\mu_e}, \quad (61)$$

as claimed.

### 9.3 Proof of Lemma 6

We use the fact that whenever $\varphi$ is differentiable, $\varphi^*(z) = z \cdot \varphi'^{-1}(z) - \varphi(\varphi'^{-1}(z))$. We have $\varphi_r^*(z) = \log z$, $\varphi_r'^{-1}(z) = \exp z = \varphi_r^*(z)$. Therefore, the Lagrange multiplier $\lambda$ in (46) is

$$\lambda = -\mu_r \cdot \log \left( \sum_{J \subseteq [m]} \exp \left( -\frac{1}{\mu_r} \cdot \sum_{j \in J} z_j \right) \right), \quad (62)$$

which yields from (54):

$$q_j^*(z) = \frac{\exp \left( -\frac{1}{\mu_e} \cdot \sum_{i \in J} z_i \right)}{\sum_{J \subseteq [m]} \exp \left( -\frac{1}{\mu_r} \cdot \sum_{j \in J} z_j \right)}, \forall J \subseteq [m].$$

On the other hand, we also have $\varphi_e'(z) = \log(z/(1-z))$, $\varphi_e'^{-1}(z) = \exp(z)/(1 + \exp(z))$ and $\varphi_r^*(z) = 1 + \log(1 + \exp(z))$, which yields from (63):

$$p_i^*(z) = \frac{\exp \left( -\frac{1}{\mu_e} \cdot z_i \right)}{1 + \exp \left( -\frac{1}{\mu_e} \cdot z_i \right)}, \forall i \in [m].$$
We then check that for any $i \in [m]$, we indeed have
\[
\sum_{\mathcal{J} \subseteq [m]} 1_{i \in \mathcal{J}} \cdot q_i^\ast(z)
= \sum_{\mathcal{J} \subseteq [m]} 1_{i \in \mathcal{J}} \cdot \frac{\exp\left(-\frac{1}{\mu_e} \cdot \sum_{j \in \mathcal{J}} z_j\right)}{\sum_{\mathcal{J} \subseteq [m]} \exp\left(-\frac{1}{\mu_e} \cdot \sum_{j \in \mathcal{J}} z_j\right)}
= \exp\left(-\frac{1}{\mu_e} \cdot z_i\right) \cdot \frac{\sum_{\mathcal{J} \subseteq [m] \setminus \{i\}} \exp\left(-\frac{1}{\mu_e} \cdot \sum_{j \in \mathcal{J}} z_j\right)}{\sum_{\mathcal{J} \subseteq [m]} \exp\left(-\frac{1}{\mu_e} \cdot \sum_{j \in \mathcal{J}} z_j\right)}
= \exp\left(-\frac{1}{\mu_e} \cdot z_i\right) \cdot \frac{c}{1 + \exp\left(-\frac{1}{\mu_e} \cdot z_i\right)},
\]
(64)

with $c = \sum_{\mathcal{J} \subseteq [m] \setminus \{i\}} \exp\left(-\frac{1}{\mu_e} \cdot \sum_{j \in \mathcal{J}} z_j\right)$. We check that eq. (64) equals eq. (63) whenever $\mu_e = \mu_r$. Hence eq. (43) holds. We conclude that $\varphi_r$ and $\varphi_e = \varphi_s$ are proportionate whenever $\mu_e = \mu_r$.

9.4 Proof of Corollary 7

Consider $\varphi_r(z) = z \log z - z$ and $\varphi_e = \varphi_s$. We obtain from eq. (47):
\[
-L_e^\ast(z)
= f_e \left( \sum_{i \in [m]} \log \left(1 + \exp\left(-\frac{1}{\mu_e} \cdot z_i\right)\right)\right),
\]
with $f_e(z) = \mu_e \cdot z + \mu_e m$. We have also $\varphi_r^\ast(z) = \exp(z)$, and so using $\lambda$ in eq. (62) and eq. (48), we obtain
\[
-L_r^\ast (z)
= \mu_r \cdot \log \left( \sum_{\mathcal{J} \subseteq [m]} \exp\left(-\frac{1}{\mu_r} \cdot \sum_{i \in \mathcal{J}} z_i\right)\right)
+ \mu_r \cdot \exp\left(\frac{\lambda}{\mu_e}\right) \cdot \sum_{\mathcal{J} \subseteq [m]} \exp\left(-\frac{1}{\mu_e} \cdot \sum_{i \in \mathcal{J}} z_i\right)
= \mu_r \cdot \log \left( \sum_{\mathcal{J} \subseteq [m]} \exp\left(-\frac{1}{\mu_r} \cdot \sum_{i \in \mathcal{J}} z_i\right)\right)
+ \mu_r \cdot \frac{\sum_{\mathcal{J} \subseteq [m]} \exp\left(-\frac{1}{\mu_e} \cdot \sum_{i \in \mathcal{J}} z_i\right)}{\sum_{\mathcal{J} \subseteq [m]} \exp\left(-\frac{1}{\mu_r} \cdot \sum_{i \in \mathcal{J}} z_i\right)}
= f_r \left( \sum_{\mathcal{J} \subseteq [m]} \exp\left(-\frac{1}{\mu_r} \cdot \sum_{i \in \mathcal{J}} z_i\right)\right),
\]

with $f_r(z) = \mu \cdot \log z + \mu$. We get from Lemma 6 that the following example and rado risks are equivalent whenever $\mu_e = \mu_r$:

\[
\ell_e(z, \mu_e) = \sum_{i \in [m]} \log \left(1 + \exp \left(-1 \cdot \frac{\mu_e}{\mu_e} \cdot z_i\right)\right),
\]

(65)

\[
\ell_r(z, \mu_r) = \sum_{J \subseteq [m]} \exp \left(-1 \cdot \frac{\mu_r}{\mu_r} \cdot \sum_{i \in J} z_i\right),
\]

(66)

from which we get the statement of the Corollary by fixing $\mu = \mu_e = \mu_r$.

### 9.5 Proof of Lemma 8

We proceed as in the proof of Lemma 6. We have $\varphi'_r(z) = z$, $\varphi'^{-1}_r(z) = z$ and $\varphi^*_r(z) = \varphi_r(z)$. Therefore, the Lagrange multiplier $\lambda$ in (46) is

\[
\lambda = \frac{\mu_r}{2m} + \frac{1}{2m} \cdot \sum_{J \subseteq [m]} \sum_{i \in J} z_i
\]

(67)

\[
\lambda = \frac{\mu_r}{2m} + \frac{1}{2} \cdot \sum_{i \in [m]} z_i
\]

(68)

since any $i$ belongs exactly to half of the subsets of $[m]$. We obtain:

\[
q^*_r(z) = \frac{1}{2^m} - \frac{1}{\mu_r} \cdot \sum_{i \in J} z_i + \frac{1}{2\mu_r} \cdot \sum_{i \in [m]} z_i, \forall J \subseteq [m].
\]

On the other hand, we also have $\varphi'_e(z) = 2z - 1$, $\varphi'^{-1}_e(z) = (1+z)/2$ and $\varphi^*_e(z) = -(1/4) + (1/4) \cdot (1+z)^2$, which yields from (94):

\[
p^*_i(z) = \frac{1}{2} \cdot \left(1 - \frac{1}{\mu_e} \cdot z_i\right), \forall i \in [m].
\]

(69)
We then check that for any $i \in [m]$, we have

$$\sum_{J \subseteq [m]} 1_{i \in J} \cdot q_i^*(z)$$

$$= \sum_{J \subseteq [m]} 1_{i \in J} \cdot \left( \frac{1}{2^m} - \frac{1}{\mu_r} \sum_{i \in J} z_i + \frac{1}{2\mu_r} \cdot \sum_{i \in [m]} z_i \right)$$

$$= \frac{1}{2} - \frac{1}{\mu_r} \cdot \sum_{J \subseteq [m]} 1_{i \in J} \cdot \sum_{i \in J} z_i + \frac{2^{m-2}}{\mu_r} \cdot \sum_{i \in [m]} z_i$$

$$= \frac{1}{2} - \frac{2^{m-1}}{\mu_r} \cdot z_i - \frac{2^{m-2}}{\mu_r} \cdot \sum_{i \in [m] \setminus \{i\}} z_i$$

$$+ \frac{2^{m-2}}{\mu_r} \cdot \sum_{i \in [m]} z_i$$

$$= \frac{1}{2} - \frac{2^{m-1}}{\mu_r} \cdot z_i + \frac{2^{m-2}}{\mu_r} \cdot z_i$$

$$= \frac{1}{2} \left( 1 - \frac{2^{m-1}}{\mu_r} \cdot z_i \right). \quad (70)$$

We check that eq. (70) equals eq. (69) whenever $\mu_e = \mu_r / 2^{m-1}$. Hence eq. (43) holds. We conclude that $\varphi_r$ is proportionate to $\varphi_e = \varphi_s$ whenever $\mu_e = \mu_r / 2^{m-1}$.

### 9.6 Proof of Corollary 9

Consider $\varphi_r(z) \triangleq (1/2) \cdot z^2$ and $\varphi_o = \varphi_s$. We obtain from eq. (47):

$$-\mathcal{L}_e^*(z)$$

$$= f_s \left( \sum_{i \in [m]} \left( 1 - \frac{1}{\mu_e} \cdot z_i \right)^2 \right),$$
with \( f_ε(z) = (μ_ε/4) \cdot z + (μ_ε m/4) \). We have also \( ϕ^*_r(z) = (1/2) \cdot z^2 \), and so using eq. (48) and \( λ \) in eq. (67), we obtain

\[
-\mathcal{L}^*_r(z) = -\frac{μ_r}{2m} \cdot \sum_{j \subseteq [m]} \sum_{i \in j} z_i \\
+ \frac{1}{2m} \cdot \sum_{j \subseteq [m]} \left( \sum_{i \in j} z_i - \frac{μ_r}{2m} \cdot \sum_{j \subseteq [m]} \sum_{i \in j} z_i \right)^2 \\
= -\frac{μ_r}{2m} \cdot \sum_{j \subseteq [m]} \sum_{i \in j} z_i \\
+ \frac{1}{2m} \cdot \sum_{j \subseteq [m]} \left( \sum_{i \in j} z_i - \frac{μ_r}{2m} \cdot \sum_{j \subseteq [m]} \sum_{i \in j} z_i \right)^2 \\
= -\frac{μ_r}{2m+1} \cdot \sum_{j \subseteq [m]} \sum_{i \in j} z_i \\
+ \frac{2m-1}{μ_r} \cdot \sum_{j \subseteq [m]} \left( \sum_{i \in j} z_i - \frac{μ_r}{2m} \cdot \sum_{j \subseteq [m]} \sum_{i \in j} z_i \right)^2 \\
= \frac{μ_r}{2m+1} \cdot \sum_{j \subseteq [m]} \left( \mathbb{E}_{j \sim [m]} \left( \sum_{i \in j} z_i \right) + \frac{2m-1}{μ_r} \cdot \mathbb{V}_{j \sim [m]} \left( \sum_{i \in j} z_i \right) \right) \\
+ \frac{μ_r}{2m-1} \cdot \left( - \left( \frac{2m-1}{μ_r} \cdot \mathbb{E}_{j \sim [m]} \left[ \sum_{i \in j} z_i \right] - \frac{2m-1}{μ_r} \cdot \mathbb{V}_{j \sim [m]} \left[ \sum_{i \in j} z_i \right] \right) \right) \\
= f_r \left( \left( \frac{2m-1}{μ_r} \cdot \mathbb{E}_{j \sim [m]} \left[ \sum_{i \in j} z_i \right] - \frac{2m-1}{μ_r} \cdot \mathbb{V}_{j \sim [m]} \left[ \sum_{i \in j} z_i \right] \right) \right),
\]

(71)
with \( f_i(z) = (\mu_r / 2^{m-1}) \cdot z - (\mu_e / 2^{m+1}) \). Therefore, it comes from Lemma 8 that the following example and rado risks are equivalent whenever \( \mu_e = \mu_r / 2^{m-1} \):

\[
\ell_e(z, \mu_e) = \sum_{i \in [m]} \left( 1 - \frac{1}{\mu_e} \cdot z_i \right)^2 ,
\]

\[
\ell_r(z, \mu_r) = - \left( \mathbb{E}_j \left[ \frac{2^{m-1}}{\mu_r} \cdot \sum_{i \in j} z_i \right] \right) - \frac{\mu_r}{2^{m-1}} \cdot \min_J \left[ \frac{2^{m-1}}{\mu_r} \cdot \sum_{i \in j} z_i \right] .
\]

There remains to fix \( \mu = \mu_e = \mu_r / 2^{m-1} \) to obtain the statement of the Corollary.

### 9.7 Proof of Lemma 10

Define \( \Delta_d \) as the \( d \)-dimensional probability simplex. Then it comes with that choice of \( \varphi_r(q) \):

\[
\min_{q \in \mathbb{R}^m} \mathcal{L}_r(q, z) = \min_{q \in \Delta_{2^m}} \sum_{J \subseteq [m]} q_J \sum_{i \in J} z_i
\]

\[
= \left\{ \begin{array}{ll} 0 & \text{if } \sum_{i \in J} z_i > 0, \forall J \neq \emptyset , \\ \sum_{i : z_i < 0} z_i & \text{otherwise} \end{array} \right\} ,
\]

since whenever no \( z_i \) is negative, the minimum is achieved by putting all the mass (1) on \( q_0 \), and when some are negative, the minimum is achieved by putting all the mass on the smallest over all \( J \) of \( \sum_{i \in J} z_i \), which is the one which collects all the indexes of the negative coordinates in \( z \).

On the other hand, remark that fixing \( \varphi_e = \varphi_z \) still yields \( \varphi_e(z) = \chi_{[0,1]}(z) = \varphi_r(z) \), yet this time we have the following on \( \mathcal{L}_e(p, z) \):

\[
\min_{p \in \mathbb{R}^m} \mathcal{L}_e(p, z) = \min_{p \in [0,1]^m} \sum_{i \in [m]} p_i z_i
\]

\[
= -\mu_e \cdot \sum_{i \in [m]} \max \left\{ 0, -\frac{1}{\mu_e} \cdot z_i \right\} ,
\]

since the optimal choice for \( p_i^* \) is to put 1 only when \( z_i \) is negative. We obtain \( p^*(z) = G_m q^*(z) \) for any choice of \( \mu_e, \mu_r \), and so \( \varphi_i(z) \) is self-proportionate for any \( \mu_e, \mu_r \). This ends the proof of Lemma 10.

### 9.8 Proof of Corollary 11

We obtain from Lemma 10 that \( -\mathcal{L}_r^*(z) = f_i(\ell_r(z, \mu_r)) \) with \( f_i(z) = \mu_r \cdot z \) and:

\[
\ell_r(z, \mu_r) = \max \left\{ 0, \max_{J \subseteq [m]} \left\{ -\frac{1}{\mu_r} \cdot \sum_{i \in J} z_i \right\} \right\} .
\]

On the other hand, it comes from eq. (73) that \( -\mathcal{L}_e^*(z) = f_e(\ell_e(z, \mu_e)) \) with \( f_e(z) = \mu_e \cdot z \) and:

\[
\ell_e(z, \mu_e) = \sum_{i \in [m]} \max \left\{ 0, -\frac{1}{\mu_e} \cdot z_i \right\} .
\]

This concludes the proof of Corollary 11.
9.9 Proof of Lemma 12

The choice of

\[ \varphi_i(z) = X_{[\frac{z}{\mu}, \frac{1}{\mu}]}(z), \]  

(76)

under the constraint that \( q \in \mathbb{H}^m \), enforces \( q^*_j = 1/2^m, \forall j \subseteq [m] \). Furthermore, fixing \( \varphi_e = \varphi_z \) indeed yields

\[ \varphi_e = X_{[\frac{z}{\mu}, \frac{1}{\mu}]}(z) + X_{[\frac{1}{\mu}, \frac{2}{\mu}]}(1-z) = X_{[\frac{1}{\mu}, \frac{2}{\mu}]}(z), \]  

(77)

which enforces \( p^*_i = 1/2, \forall i \). Since each \( i \) belongs to exactly \( 2^m \) subsets of \( [m] \), we obtain \( p^*(z) = G_m q^*(z) \), for any \( \mu_e, \mu_r \), and so \( \varphi_i \) is proportionate to \( \varphi_e = \varphi_z \) for any \( \mu_e, \mu_r \).

9.10 Proof of Corollary 13

We obtain from Lemma 12 that \(-L^*_i(z) = f_i(\ell_i(z, \mu_r)) \) with \( f_i(z) = z \) and:

\[ \ell_i(z, \mu_r) = \mathbb{E}_j \left[ -\frac{1}{\mu_r} \cdot \sum_{i \in J} z_i \right]. \]

On the other hand, it comes from eq. (73) that \(-L^*_e(z) = f_e(\ell_e(z, \mu_e)) \) with \( f_e(z) = (1/2) \cdot z \) and:

\[ \ell_e(z, \mu_e) = \sum_i -\frac{1}{\mu_e} \cdot z_i. \]

This concludes the proof of Corollary 11.

9.11 Proof of Theorem 14

The key to the proof is the constraint \( q \in \mathbb{H}^m \) in eq. (4). Since \( f_e(z) = a_e \cdot z + b_e \), we have \( L^*_e(z) = a_e \cdot (\ell_e(z) + \omega) + b_e - a_e \cdot \omega \) for any \( \omega \in \mathbb{R} \). It follows from eq. (71) that \( a_e \cdot (\ell_e(z) + \omega) + b_e - a_e \cdot \omega = L^*_i(z) + b = \sum_{j \subseteq [m]} q_j \sum_{i \in J} z_i + \mu_r \sum_{j \subseteq [m]} \varphi_j(q_j) + b \), and so

\[ a_e \cdot (\ell_e(z) + \omega) + b_e = - \left\{ \min_{q \in \mathbb{H}^m} \left( \sum_{j \subseteq [m]} q_j \sum_{i \in J} z_i + \mu_r \sum_{j \subseteq [m]} \varphi_j(q_j) - a_e \omega \right) \right\} + b \]

\[ = - \min_{q \in \mathbb{H}^m} \left( \sum_{j \subseteq [m]} q_j \left( \sum_{i \in J} z_i - a_e \omega \right) + \mu_r \sum_{j \subseteq [m]} \varphi_j(q_j) \right) + b, \]

since \( q \in \mathbb{H}^m \) and \( a_e, \omega, a \) are not a function of \( q \). We thus get \( a_e \cdot (\ell_e(z) + \omega) + b_e = a_e \cdot f_e(\ell_e(z)) + b_e \).

where \( \ell_i(z) \) equals \( \ell_e(z) \) in which each \( \sum_{i \in J} z_i \) is replaced by \( \sum_{i \in J} z_i - a_e \omega \). For \( z_i = \theta^\top (y_i \cdot x_i) \) and \( \omega = \Omega(\theta) \), we obtain that whenever \( \theta \neq 0, \forall J \subseteq [m] \),

\[ \sum_{i \in J} z_i + a_e \omega = \theta^\top \left( \pi_{\sigma} - \frac{a_e \Omega(\theta)}{\|\theta\|_2^2} \cdot \theta \right), \]

(78)

for \( \sigma_i = y_i \) iff \( i \in J \) (and \( -y_i \) otherwise), and the statement of the Theorem follows.
9.12 Proof of Theorem 17

The proof of the Theorem contains two parts, the first of which follows ADABOOST’s exponential convergence rate proof, and the second departs from this proof to cover \( \Omega_R \text{ADABOOST} \).

We use the fact that \( \alpha_i(t) \pi_{j_i(t)} = \alpha_i(t) \cdot 1^T_i(t) \pi_j = (\theta_T - \theta_{T-1})^T \pi_j \) to unravel the weights as:

\[
\begin{align*}
\omega^T_j &= \frac{w_{(T-1)j}}{Z_T} \cdot \exp \left( -\alpha_i(T) \pi_{j_i(T)} + \delta_T \right) \\
&= \frac{w_{(T-1)j}}{Z_T} \cdot \exp \left( -(\theta_T - \theta_{T-1})^T \pi_j + \omega \cdot \|\theta_T\|^2 - \|\theta_{T-1}\|^2 \right) \\
&= \frac{w_{(T-1)j}}{Z_T} \cdot \exp \left( -\theta_T^T (\pi_j - \omega \cdot \theta_T) + \theta_{T-1}^T (\pi_j - \omega \cdot \theta_{T-1}) \right) \\
&= \frac{w_0}{\prod_{t=1}^T Z_t} \cdot \exp \left( -\theta_T^T (\pi_j - \omega \cdot \theta_T) + \theta_0^T (\pi_j - \omega \cdot \theta_0) \right) \\
&= \frac{w_0}{\prod_{t=1}^T Z_t} \cdot \exp \left( -\theta_T^T (\pi_j - \omega \cdot \theta_T) \right),
\end{align*}
\]

since the sums telescope in eq. (79) when we unravel the weight update and \( \theta_0 = 0 \). We therefore get

\[
\ell_t^{\text{exp}}(S_t, \theta, \|\cdot\|^2_2) = \prod_{t=1}^T Z_t,
\]

as in the classical ADABOOST analysis (Schapire & Singer, 1999). This time however, we have, letting \( \tilde{\pi}_{j_i(t)} \approx \pi_{j_i(t)}/\pi_{s_i(t)} \in [-1, 1] \) and \( \tilde{\alpha}_i(t) \approx \pi_{s_i(t)} \cdot \alpha_t \) for short,

\[
\begin{align*}
Z_{t+1} &= \sum_{j \in [n]} w_{tj} \cdot \exp \left( -\alpha_i(t) \pi_{j_i(t)} + \delta_t \right) \\
&= \exp(\delta_t) \cdot \sum_{j \in [n]} w_{tj} \cdot \exp \left( -\alpha_i(t) \pi_{j_i(t)} \right) \\
&= \exp(\delta_t) \cdot \sum_{j \in [n]} w_{tj} \cdot \exp \left( -\tilde{\alpha}_i(t) \tilde{\pi}_{j_i(t)} \right) \\
&\leq \exp(\delta_t) \cdot \sqrt{1 - r_t^2} \\
&= \exp \left( \omega \cdot (\|\theta_t\|^2_2 - \|\theta_{t-1}\|^2_2) - \frac{1}{2} \ln \left( \frac{1}{1 - r_t^2} \right) \right).
\end{align*}
\]

This is where our proof follows a different path from ADABOOST’s: in eq. (83), we do not upperbound the \( \sqrt{1 - r_t^2} \) term, so it can absorb more easily the new \( \exp(\delta_t) \) factor which appears because of regularization.

Ineq. (82) holds because of the convexity of \( \exp \), and eq. (83) is an equality when \( r_t < \gamma \). If \( r_t > \gamma \) is
clamped to \( r_t \leftarrow \gamma \) by the weak learner in (31), then we have instead the derivation

\[
\sum_{j \in [n]} w_t j \cdot \left( (1 + \tilde{\pi}_j(t)) \cdot \exp(-\tilde{\alpha}_j(t)) + (1 - \tilde{\pi}_j(t)) \cdot \exp(\tilde{\alpha}_j(t)) \right)
\]

\[
= (1 + r_t) \cdot \sqrt{1 - \gamma \cdot \frac{1}{1 + \gamma}} + (1 - r_t) \cdot \sqrt{1 + \gamma \cdot \frac{1}{1 - \gamma}}
\]

\[
\leq 2\sqrt{1 - \gamma^2},
\]  

(84)

since function in (84) is decreasing on \( r_t > 0 \). If \( r_t < -\gamma \) is clamped to \( r_t \leftarrow -\gamma \), we get the same conclusion as in ineq (84) because this time \( \tilde{\alpha}_j(t) = (1/2) \cdot \ln((1 - \gamma)/(1 + \gamma)) \). Summarising, whether \( r_t \) has been clamped or not by the weak learner in (31), we get

\[
Z_{t+1} \leq \exp \left( \omega \cdot \left( \|\theta_t\|_2^2 - \|\theta_{t-1}\|_2^2 \right) - \frac{1}{2} \ln \frac{1}{1 - r_t^2} \right),
\]

(85)

with the additional fact that \(|r_t| \leq \gamma\). For any feature index \( k \in [d] \), let \( \mathcal{F}_k \subseteq [T] \) the iteration indexes for which \( \iota(t) = k \). Letting \( \lambda_\Gamma (> 0) \) the largest eigenvalue of \( \Gamma \), we obtain:

\[
\prod_{t=1}^{T} Z_t \leq \exp \left( \omega \cdot \|\theta_T\|_2^2 - \sum_t \frac{1}{2} \log \frac{1}{1 - r_t^2} \right)
\]

\[
\leq \exp \left( \omega \lambda_\Gamma \cdot \|\theta_T\|_2^2 - \sum_t \frac{1}{2} \log \frac{1}{1 - r_t^2} \right)
\]

\[
= \exp \left( -\frac{1}{2} \cdot \sum_{k \in [d]} \Lambda_k \right),
\]

(86)

With

\[
\Lambda_k = \log \frac{1}{\prod_{t: \iota(t) \in \mathcal{F}_k} (1 - r_t^2)} - \frac{\omega \lambda_\Gamma}{2\pi_{\ast k}^2} \log^2 \prod_{t: \iota(t) \in \mathcal{F}_k} \left( \frac{1 + r_t}{1 - r_t} \right).
\]

(87)

Since \( \left( \sum_{t=1}^u a_t \right)^2 \leq u \sum_{t=1}^u a_t^2 \) and \( \min_k \max_j |\pi_{jk}| \leq |\pi_{\ast k}| \), \( \Lambda_k \) satisfies:

\[
\Lambda_k \geq \sum_{t: \iota(t) \in \mathcal{F}_k} \left\{ \log \frac{1}{1 - r_t^2} - \frac{T_k \omega \lambda_\Gamma}{2M^2} \log^2 \frac{1 + r_t}{1 - r_t} \right\},
\]

(88)

with \( T_k = |\mathcal{F}_k| \) and \( M = \min_k \max_j |\pi_{jk}| \). For any \( a > 0 \), let

\[
f_a(z) = \frac{1}{az^2} \cdot \left( \log \frac{1}{1 - z^2} - a \cdot \log^2 \frac{1 + z}{1 - z} \right) - 1.
\]

27
It satisfies
\[ f_a(z) \approx_0 \left( \frac{1}{a} - 5 \right) + \left( \frac{1}{2a} - \frac{8}{3} \right) \cdot z^2 + \left( \frac{1}{3a} - \frac{92}{45} \right) \cdot z^4 + o(z^4). \] (89)

Since \( f_a(z) \) is continuous for any \( a \neq 0, \forall 0 < a < 1/5, \exists z_*(a) > 0 \) such that \( f_a(z) \geq 0, \forall z \in [0, z_*] \). So, for any such \( a < 1/5 \) and any \( \omega \) satisfying \( \omega < (2aM^2)/(T_k\lambda_T) \), as long as each \( r_t \leq z_*(a) \), we shall obtain
\[ \Lambda_k \geq a \sum_{t:z(t) \in \mathcal{I}_k} r_t^2. \] (90)

There remains to tune \( \gamma \leq z_*(a) \), and remark that if we fix \( a = 1/7 \), then numerical calculations reveal that \( z_*(a) > 0.98 \), and if \( a = 1/10 \) then numerical calculations give \( z_*(a) > 0.999 \), completing the statement of Theorem \[17\].

### 9.13 Proof of Theorem \[18\]

We consider the case \( \Omega(\cdot) = \|\cdot\|_\infty \), from which we shall derive the case \( \Omega(\cdot) = \|\cdot\|_1 \). We proceed as in the proof of Theorem \[17\] with the main change that we have now \( \delta_1 = \omega \cdot (\|\theta_t\|_\infty - \|\theta_{t-1}\|_\infty) \), so in place of \( \Lambda_k \) in ineq. \[86\] we have to use, letting \( k_* \) any feature that gives the \( \ell_\infty \) norm,
\[
\Lambda_k = \begin{cases} \sum_{t:z(t) \in \mathcal{I}_k} \log \frac{1}{1-r_t} - \omega \sum_{t:z(t) \in \mathcal{I}_k} \log \frac{1+|r_t|}{1-|r_t|} & \text{if } k = k_* \, \text{.} \\
\sum_{t:z(t) \in \mathcal{I}_k} \log \frac{1}{1-r_t} & \text{otherwise}\end{cases} \tag{91}
\]

It also comes
\[
\Lambda_{k_*} \geq \sum_{t:z(t) \in \mathcal{I}_{k_*}} \left\{ \log \frac{1}{1-r_t} - \omega \frac{1+|r_t|}{1-|r_t|} \right\} \geq \sum_{t:z(t) \in \mathcal{I}_{k_*}} \left\{ \log \frac{1+|r_t|}{1-|r_t|} \right\}, \tag{92}\]

with \( M = \min_k \max_j |\pi_{jk}| \). Let us analyze \( \Lambda_{k_*} \) and define for any \( b > 0 \)
\[
g_b(z) \doteq \log \frac{1}{1-z^2} - b \cdot \log \frac{1+z}{1-z} - \left( -2bz + z^2 - \frac{2bz^3}{3} \right). \tag{93}\]

Inspecting \( g_b \) shows that \( g_b(0) = 0, g'_b(0) = 0 \) and \( g_b(z) \) is convex over \([0, 1]\) for any \( b \leq 3 \), which shows that \( g_b(z) \geq 0, \forall z \in [0,1], \forall b \leq 3 \), and so, after dividing by \( bz^2 \) and reorganising, yields in these cases:
\[
\frac{1}{bz^2} \cdot \left( \log \frac{1}{1-z^2} - b \cdot \log \frac{1+z}{1-z} \right) - 1 \geq \left( -2z + \frac{1}{b} - 1 \right) - \frac{2z}{3}. \tag{94}\]
Hence, both functions being continuous on \((0, 1)\), the function in the left-hand side zeroes before the one in the right-hand side (when this one does on \((0, 1)\)). The zeroes of the polynomial

\[
p_b(z) = -\frac{2z^2}{3} + \left(\frac{1}{b} - 1\right)z - 2
\]

exist iff \(b \leq \sqrt{3}/(4 + \sqrt{3})\), in which case any \(z \in [0, 1)\) must satisfy

\[
z \geq \frac{3}{4} \cdot \left(\frac{1}{b} - 1 - \sqrt{\left(\frac{1}{b} - 1\right)^2 - \frac{16}{3}}\right)
\]

(96)
to guarantee that \(p_b(z) \geq 0\). Whenever this happens, we shall have from (94):

\[
\log \frac{1}{1 - z^2} - b \cdot \log \frac{1 + z}{1 - z} \geq b z^2 .
\]

(97)

Since \(\Omega\)-WL is a \(\gamma_{WL}\)-weak learner, if we can guarantee that the right-hand side of ineq. (96) is no more than \(\gamma_{WL}\), then there is nothing more to require from the weak learner to have ineq. (97) — and therefore to have \(\Lambda_{k_s} \geq b \gamma_{WL}^2 \cdot |\mathcal{F}_{k_s}|\). This yields equivalently the following constraint on \(b\):

\[
b \leq \frac{8 \gamma_{WL}}{3} \frac{1}{16 \gamma_{WL}^2 + 8 \gamma_{WL} + 16} .
\]

(98)

Since \(\gamma_{WL} \leq 1\), ineq (98) ensured as long as

\[
b \leq \frac{8 \gamma_{WL}}{3} \frac{1}{16 + 8 + 16} = \frac{3 \gamma_{WL}}{11} ,
\]

(99)

which also guarantees \(b \leq \sqrt{3}/(4 + \sqrt{3})\). So, letting \(T_s = |\mathcal{F}_{k_s}|\) and recollecting

\[
b \equiv \frac{\omega}{\min_k \max_j |\pi_{jk}|}
\]

(100)

from eq. (92), we obtain from ineqs (92) and (97):

\[
\Lambda_{k_s} \geq \frac{\omega T_s \gamma_{WL}^2}{\min_k \max_j |\pi_{jk}|} .
\]

(101)

We need to ensure \(\omega \leq 3 \min_k \max_j |\pi_{jk}| \gamma_{WL}/11\) from ineq. (99), which holds if we pick it according to eq. (36). In this case, we finally obtain

\[
\Lambda_{k_s} \geq (a \gamma_{WL} T_s) \cdot \gamma_{WL}^2 .
\]

(102)

Now, since \(\log(1/(1 - x^2)) \geq x^2\), we also have for \(k \neq k_s\) in eq. (91),

\[
\Lambda_k = \sum_{t : \omega(t) \in \mathcal{F}_k} \log \frac{1}{1 - r_t^2}
\]

\[
\geq \sum_{t : \omega(t) \in \mathcal{F}_k} r_t^2
\]

\[
\geq |\mathcal{F}_k| \gamma_{WL}^2 , \forall k \neq k_s .
\]

(103)

So, we finally obtain from eq. (84) and ineq. (86),

\[
\ell_r^{\exp}(S_r, \theta, \|\cdot\|_2^2) \leq \exp \left(-\frac{T \gamma_{WL}^2}{2}\right) ,
\]

(104)

with \(T = (T - T_s) + a \gamma_{WL} \cdot T_s\), as claimed when \(\Omega(.) = \|\cdot\|_\infty\). The case \(\Omega = \|\cdot\|_1\) follows from the fact that all \(\Lambda_k\) match the bound of \(\Lambda_{k_s}\).
9.14 Proof of Theorem 19

We use the proof of Theorem 18 since when \( \Omega(.) = \| . \|_\Phi \), eq. (91) becomes

\[
\Lambda_k = \sum_{t: \iota(t) \in F_k} \log \frac{1}{1 - r_t^2} - \frac{\xi_k}{\pi_{sk}} \left| \sum_{t: \iota(t) \in F_k} \log \frac{1 + r_t}{1 - r_t} \right| \geq \sum_{t: \iota(t) \in F_k} \left\{ \log \frac{1}{1 - r_t^2} - \frac{\xi_k}{\max_j |\pi_{jk}|} \log \frac{1 + |r_t|}{1 - |r_t|} \right\},
\]

assuming without loss of generality that the classifier at iteration \( T \), \( \theta_T \), satisfies \( |\theta_{Tk}| \geq |\theta_{T(k+1)}| \) for \( k = 1, 2, ..., d - 1 \). We recall that \( \xi_k = \Phi^{-1}(1 - kq/(2d)) \) where \( \Phi^{-1}(.) \) is the quantile of the standard normal distribution and \( q \in (0, 1) \) is the user-fixed \( q \)-value. The constraint \( b \leq 3\gamma_{WL}/11 \) from ineq. (99) now has to hold with

\[
b_k = b_k = \frac{\xi_k}{\max_j |\pi_{jk}|}.
\]

Now, fix

\[
a = \min \left\{ \frac{3\gamma_{WL}}{11}, \frac{\Phi^{-1}(1 - q/(2d))}{\min_k \max_j |\pi_{jk}|} \right\}.
\]

Remark that

\[
\xi_k = \Phi^{-1} \left( 1 - \frac{kq}{2d} \right) \geq \Phi^{-1} \left( 1 - \frac{q}{2d} \right) \geq a \min_{j \neq k'} \max_j |\pi_{jk'}|.
\]

Suppose \( q \) is chosen such that

\[
\xi_k \leq \frac{3\gamma_{WL}}{11} \cdot \max_j |\pi_{jk}|, \forall k \in [d].
\]

This ensures \( b_k \leq 3\gamma_{WL}/11 (\forall k \in [d]) \) in ineq. (99), while ineq. (109) ensures

\[
\Lambda_k \geq b_k \sum_{t: \iota(t) \in F_k} r_t^2 \geq \frac{\xi_k}{\min_{k'} \max_j |\pi_{jk'}|} \cdot \sum_{t: \iota(t) \in F_k} r_t^2 \geq a|F_k|^2 \gamma_{WL}^2.
\]

Ineq. (111) holds because of ineqs (106) and (97). Ineq. (113) holds because of the weak learning assumption and ineq. (110). So, we obtain, under the weak learning assumption,

\[
\ell_{T_\exp}^\ast(S_r, \theta, \| . \|_\Phi) \leq \exp \left( -\frac{aT\gamma_{WL}^2}{2} \right).
\]
Ensuring ineq. (110) is done if, after replacing $\xi_k$ by its expression and reorganising, we can ensure

\[
q \geq 2 \cdot \max_k \frac{q_{N,k}^D}{q_{D,k}} ,
\]

with

\[
(0, 1) \ni q_{N,k} = 1 - \Phi \left( \frac{3\gamma_{wl}}{11} \cdot \max_j |\pi j_k| \right) ,
\]

\[
(0, 1) \ni q_{D,k} = \frac{k}{d} .
\]

9.15 Proof of Theorem 20

Suppose wlog that the example index on which $S_e$ and $S'_e$ differ is $m$, and let $e_m$ and $e'_m$ denote the two distinct edge vectors of the neighbouring datasets. For $n = 1$, let $\pi$ denote a rado created from first picking uniformly at random $j \in 2^m$ and then using DP-RADOS on the singleton $S_e = \{\pi_j\}$ with:

\[
\pi_j = \sum_{i \in j} y_i \cdot x_i .
\]

Let $a(S_e) = \sum_{j' \subseteq [m-1]} \mu(\pi|\pi_1 = \pi_{j'}, S = S_e)$, where $\mu(\pi_j)$ is the density of the singleton output of DP-RADOS, and $b(S_e) = \sum_{j \subseteq [m], m \in j} \mu(\pi|\pi_1 = \pi_j, S = S_e)$. We have:

\[
\frac{\mu(\pi|S_e)}{\mu(\pi|S'_e)} = \frac{a(S_e) + b(S_e)}{a(S'_e) + b(S'_e)} = \frac{a(S_e) + b(S_e)}{a(S'_e) + b(S'_e)} \leq \max \left\{ \frac{b(S_e)}{b(S'_e)}, \frac{b(S'_e)}{b(S_e)} \right\} .
\]

Eq. (120) follows from the fact that when $j' \subseteq [m-1]$, $\mu(\pi|\pi_1 = \pi_{j'}, S = S_e) = \mu(\pi|\pi_1 = \pi_{j'}, S = S'_e)$. Now, for any fixed $j' \subseteq [m]$ such that $m \in j'$, we have:

\[
\mu(\pi|\pi_1 = \pi_{j'}, S = S_e) = \frac{\varepsilon}{2r_e} \cdot \exp \left( -\frac{\varepsilon}{r_e} \cdot \|\pi_{j'} - \pi\|_1 \right) .
\]

\[
\mu(\pi|\pi_1 = \pi_{j'}, S = S'_e) = \frac{\varepsilon}{2r_e} \cdot \exp \left( -\frac{\varepsilon}{r_e} \cdot \|\pi_{j'} - \pi\|_1 \right) .
\]

\[
\mu(\pi|\pi_1 = \pi_{j'}, S = S_e) = \frac{\varepsilon}{2r_e} \cdot \exp \left( -\frac{\varepsilon}{r_e} \cdot \|\pi_{j'} - \pi\|_1 \right) .
\]

\[
\mu(\pi|\pi_1 = \pi_{j'}, S = S'_e) = \frac{\varepsilon}{2r_e} \cdot \exp \left( -\frac{\varepsilon}{r_e} \cdot \|\pi_{j'} - \pi\|_1 \right) .
\]

\[
\mu(\pi|\pi_1 = \pi_{j'}, S = S'_e) = \frac{\varepsilon}{2r_e} \cdot \exp \left( -\frac{\varepsilon}{r_e} \cdot \|\pi_{j'} - \pi\|_1 \right) .
\]

\[
\mu(\pi|\pi_1 = \pi_{j'}, S = S'_e) = \frac{\varepsilon}{2r_e} \cdot \exp \left( -\frac{\varepsilon}{r_e} \cdot \|\pi_{j'} - \pi\|_1 \right) .
\]
where \( \pi'_j = \pi_j - e_m + e'_m \) in eq. (122) is the rado that is created from the same \( j' \) but using \( S'_e \) and its potentially different example \( e'_m \). The inequality holds because of the triangle inequality. Since \( \|e_m - e'_m\|_1 \leq r_e \) by assumption, we get \( \mu(\pi | \pi_j = \pi_j, \delta = S_e) \leq \exp(\varepsilon) \cdot \mu(\pi | \pi_j = \pi_j, \delta = S'_e) \), and so, summing over all \( j' \subseteq [m] \) such that \( m \in j' \), we get \( b(S_e) / b(S'_e) \leq \exp(\varepsilon) \). Furthermore, we also have by symmetry \( b(S'_e) / b(S_e) \leq \exp(\varepsilon) \). So the delivery of one rado is \( \varepsilon \)-differentially private. The composition Theorem (Dwork & Roth, 2014) achieves the proof of the first point of Theorem 20. To prove the second point, we first define the (unregularized) log-loss,

\[
elog_e(S_e, \theta) = \frac{1}{m} \sum_i \log \left( 1 + \exp \left( -\theta^\top (y_i \cdot x_i) \right) \right)
\]

We exploit the following inequalities that hold for the log-loss and exp-rado loss:

\[
\frac{1}{m} \cdot \log \exp_e(S_e, \theta)
\]

\[
= \frac{1}{m} \cdot \log \left( \frac{1}{2m} \sum_{\sigma \in \Sigma_m} \exp \left( -\theta^\top (\pi_\sigma + z_\sigma) \right) \right)
\]

\[
\leq \frac{1}{m} \cdot \log \left( \left( \frac{1}{2m} \sum_{\sigma \in \Sigma_m} \exp \left( -\theta^\top \pi_\sigma \right) \right) \cdot \left( \sum_{\sigma \in \Sigma_m} \exp \left( -\theta^\top z_\sigma \right) \right) \right)
\]

\[
= \frac{1}{m} \cdot \log \left( \frac{1}{2m} \sum_{\sigma \in \Sigma_m} \exp \left( -\theta^\top \pi_\sigma \right) \right) + \frac{1}{m} \cdot \log \left( \sum_{\sigma \in \Sigma_m} \exp \left( -\theta^\top z_\sigma \right) \right)
\]

\[
= \log 2 + \frac{1}{m} \cdot \log \left( \frac{1}{2m} \sum_{\sigma \in \Sigma_m} \exp \left( -\theta^\top \pi_\sigma \right) \right) + \frac{1}{m} \cdot \log \left( \sum_{\sigma \in \Sigma_m} \exp \left( -\theta^\top z_\sigma \right) \right)
\]

\[
\leq \log 2 + \frac{1}{m} \cdot \log \left( \sum_{\sigma \in \Sigma_m} \exp \left( -\theta^\top \pi_\sigma \right) \right) + \frac{1}{m} \cdot \max_{\sigma} \theta^\top z_\sigma
\]

\[
\leq \log 2 + \frac{1}{m} \cdot \log \left( \sum_{\sigma \in \Sigma_m} \exp \left( -\theta^\top \pi_\sigma \right) \right) + \frac{1}{m} \cdot \max_{\sigma} \Omega^*(z_\sigma) \Omega(\theta)
\]

\[
= \ell_e(S_e, \theta) + \frac{1}{m} \cdot \max_{\sigma} \Omega^* \Omega(\theta)
\]

\[
= \ell_e(S_e, \theta, (1/m) \cdot \max_{\sigma} \Omega^*(z_\sigma) \cdot \Omega)
\]

where ineq. (124) comes from the fact that \( \sum_{a_i b_i \leq \left( \sum_{a_i} \right) \left( \sum_{b_i} \right) \text{ when all } a_i, b_i \geq 0, \) ineq. (125) is Cauchy-Schwartz and eq. (126) is Lemma 2 in (Nock et al., 2015).

**Remarks on \( \varepsilon \):** let us explain why the protection of examples using the same noise level as rados is conservative but in fact necessary in the worst case, considering for simplicity the protection of a single rado / example. The proof of Theorem 20 exploits a conservative upperbound for the likelihood ratio:

\[
\frac{\mu(\pi | S_e)}{\mu(\pi | S'_e)} = \frac{a(S_e) + b(S_e)}{a(S_e) + b(S'_e)} \leq \max \left\{ \frac{b(S_e)}{b(S'_e)}, \frac{b(S'_e)}{b(S_e)} \right\}
\]

and then upperbounds the max by \( \exp \varepsilon \) to get the DP requirement. The same strategy can be used to protect the example, but the bound is sometimes more conservative in this case. Indeed, whereas one examples participates in generating half the total number of DP rados, one example participates in only \( 1/m \) of the
generation of DP examples. For a single DP example $e$, the equality in (127) becomes

$$
\mu(e|S_a)/\mu(e|S'_a) = (a'(S_a) + b'(S_a))/(a'(S_a) + b'(S'_a))
$$

with $a'(S_a) \equiv (1 - (1/m)) \cdot \mu(e|S_a \setminus \{e_m\})$ and:

$$
b'(S_a) \equiv \frac{1}{m} \cdot \mu(e|\{e_m\}) ,
\quad b'(S'_a) \equiv \frac{1}{m} \cdot \mu(e|\{e'_m\}).
$$

(128)

Let $u \equiv \mu(e|\{e'_m\})/\mu(e|S'_a \setminus \{e'_m\}) = \mu(e|\{e'_m\})/\mu(e|S_a \setminus \{e_m\})$. If we use the same amount of protection as for one rado, then we get

$$
\frac{\mu(e|S_a)}{\mu(e|S'_a)} \leq f_u(\varepsilon),
$$

(129)

where $\varepsilon$ is the rado privacy budget and

$$
f_u(\varepsilon) \equiv \frac{(m-1) + u \exp(\varepsilon)}{m - 1 + u}.
$$

(130)

$f_u(\varepsilon)$ is always $\lt \exp(\varepsilon)$, so if we use this $\exp(\varepsilon)$ bound to pick the noise level, then we are in fact putting more protection over examples than necessary (remember that the protection is also conservative for rados, but to a lesser extent). However, this choice would not be so bad in the worst case since $\lim_{u \to \infty} f_u(\varepsilon) = \exp(\varepsilon)$. To summarise, without constraints on $u$, and to be sure to meet the DP requirements in any case, we would err on the conservative side, as we did for rados in ineq. (121), and pick $\varepsilon_a = \varepsilon$, i.e. the same amount of noise for examples. Yet, as we explain in the body of the paper, the results are exceedingly in favor of $\Omega$.R.ADABOOST in this case. To give a more balanced picture, we chose to compute an “approximate” privacy budget $\varepsilon_a = \varepsilon_a$ for $n$ examples, which we simply fix to be $\varepsilon_a \equiv n \cdot \ln(f_{u=1}(\varepsilon/n)) (< \varepsilon)$ where $\varepsilon$ is the privacy budget for $n$ rados. So, we have

$$
\varepsilon_a = n \cdot \ln \left(1 + \frac{\exp(\varepsilon/n) - 1}{m}\right).
$$

(131)

Again, when $u > 1$, fixing $\varepsilon = \varepsilon_a$ to protect examples would fail to achieve $\varepsilon$-differential privacy.

Nevertheless, one can reasonably consider that the “optimal” differentially private picture of $\text{ADABOOST}$ shall thus be representable as a stretching of its curves in between the figures for $\varepsilon_a$ and $\varepsilon$.

10 Additional experiments

10.1 Supports for rados (complement to Table 2)

Table 4 in this Appendix provides the supports used to summarize Table 2.

10.2 Experiments on class-wise rados

Tables 5 and 6 provide the test errors and supports for $\Omega$.R.ADABOOST when trained with class-wise rados, that is, rados that sum examples of the same class. The experiments do not display that class-wise rados allow for a better training of $\Omega$.R.ADABOOST, as test errors are on par with $\Omega$.R.ADABOOST trained with plain random rados (see Table 2).

10.3 Test errors and supports for rados (comparison last vs best empirical classifier)

In the paper’s main experiments, the classifier kept out of the sequence, for both $\text{ADABOOST}$ and $\Omega$.R.ADABOOST, is the best empirical classifier, that is, the classifier which minimizes the empirical risk.
out of the training sample. This setting makes sense if the objective is just the minimization of the test error without any constraint, and it is also applicable in a privacy setting where the data and the learner are distant parties (in this case, the learner sends the sequence of classifiers $\theta_1, \theta_2, ..., \theta_T$ to the party holding the data, which can then select the best in the sequence). Yet, one may wonder how the algorithms compare when the classifier returned is just the last one in the sequence, that is, $\theta_T$.

Tables 7 and 8 provide errors and supports comparing the versions of $\Omega$-R.ADABoost when the best empirical classifier is selected ($\star$), or when the last classifier in the sequence is kept ($\dagger$). They are therefore subsuming Tables 2 (for test errors) and 4 (for supports).

The intuition tells that not selecting the classifier in the sequence produced ($\dagger$) should produce either no better, or eventually worse results than when selecting the classifier to keep from the sequence $\theta_0, \theta_1, ..., \theta_T$. The results display that it is the case, for both AdaBoost and $\Omega$-R.ADABoost, and the phenomenon is more visible as the domain size increases. The degradation for $\Omega$-R.ADABoost appears to be significantly worse than that for AdaBoost on three domains, Fertility, Firmteacher and Kaggle, since not selecting the classifier using the training data incurs an increase of 8\% and more on the test error for these domains. However, for the majority of the domains, the variation in test error does not exceed 1\%, and on three domains (Winewhite, Smartphone, Eeg), the absence of selection of the classifier actually does not increase the test error at all.

Therefore, even when not marginal, the fact that the test error significantly increases only on a minority of the domains for $\Omega$-R.Adaboost calls for a rather domain-specific selection procedure of the classifier in the sequence, rather than an all-purpose selection procedure. Furthermore, on domains for which not selecting the classifier produces the worst results, such a more efficient selection procedure of the classifier might actually be bypassed by a more careful crafting of the rados, since when class-wise random rados are used (results not shown), picking the last classifier for domain Kaggle reduces the test error by approximately 10\% compared to random rados (the test error drops to 32.68$\pm$10.9 instead of 42.41$\pm$9.32 for SLOPE). Such a specific crafting of rados is an interesting and non trivial problem that deserves further attention.
Table 3: DP experiments on big domains ($\omega = 0$ in $\Omega$.R.ADABOOST). Test error as a function of privacy budgets $\varepsilon$ and $\varepsilon_a$ for ADBOOST, and as a function of $\varepsilon$ for $\Omega$.R.ADABOOST trained with plain random rados (rand) or class-wise random rados (rand+c).
| Domain      | $m$ | $d$ | $\text{supp.} \pm \sigma$ | $\text{supp.} \pm \sigma$ | $\omega = 0$ | $\Omega = ||\cdot||^2_d$ | $\Omega = ||\cdot||_1$ | $\Omega = ||\cdot||_{\infty}$ | $\Omega = ||\cdot||_{\Phi}$ |
|------------|-----|-----|--------------------------|--------------------------|-------------|---------------------|---------------------|---------------------|---------------------|
| Fertility  | 100 | 9   | $36.67 \pm 36.3$         | $14.44 \pm 5.36$         | $37.78 \pm 31.1$ | $36.67 \pm 34.8$ | $\bullet 42.22 \pm 31.0$ | $\circ 24.44 \pm 22.1$ | $32.22 \pm 17.7$ |
| Sonar      | 208 | 60  | $57.83 \pm 3.69$         | $1.83 \pm 0.52$          | $\bullet 14.17 \pm 3.62$ | $14.00 \pm 3.16$ | $13.67 \pm 3.99$ | $12.83 \pm 4.45$ | $12.67 \pm 4.17$ |
| Haberman   | 306 | 3   | $70.00 \pm 10.6$         | $33.33 \pm 0.00$         | $\bullet 66.67 \pm 22.2$ | $\bullet 66.67 \pm 15.7$ | $56.67 \pm 16.1$ | $60.00 \pm 26.3$ | $50.00 \pm 17.6$ |
| Ionosphere | 351 | 33  | $76.97 \pm 8.23$         | $3.64 \pm 1.27$          | $\bullet 13.64 \pm 3.85$ | $13.03 \pm 3.51$ | $11.82 \pm 3.63$ | $\circ 11.21 \pm 4.75$ | $\circ 11.21 \pm 4.53$ |
| Breastwisc | 699 | 9   | $90.00 \pm 3.51$         | $11.11 \pm 0.00$         | $51.11 \pm 12.0$ | $84.44 \pm 7.77$ | $48.89 \pm 9.37$ | $84.44 \pm 10.7$ | $86.67 \pm 4.68$ |
| Transfusion| 748 | 4   | $77.50 \pm 14.2$         | $25.00 \pm 0.00$         | $\circ 67.50 \pm 20.6$ | $70.00 \pm 23.0$ | $\circ 67.50 \pm 16.9$ | $\circ 67.50 \pm 23.7$ | $\bullet 72.50 \pm 14.2$ |
| Banknote   | 1372| 4   | $100.00 \pm 0.00$        | $25.00 \pm 0.00$         | $\circ 40.00 \pm 12.9$ | $50.00 \pm 0.00$ | $47.50 \pm 7.91$ | $\circ 50.00 \pm 0.00$ | $47.50 \pm 7.91$ |
| Winered    | 1599| 11  | $79.09 \pm 6.14$         | $9.09 \pm 0.00$          | $\circ 25.45 \pm 5.75$ | $27.27 \pm 6.06$ | $\bullet 27.27 \pm 7.42$ | $\circ 25.45 \pm 7.17$ | $\bullet 27.27 \pm 7.42$ |
| Abalone    | 4177| 10  | $64.00 \pm 6.99$         | $19.00 \pm 3.16$         | $\bullet 30.00 \pm 6.67$ | $10.00 \pm 0.00$ | $12.00 \pm 6.32$ | $\circ 10.00 \pm 0.00$ | $11.00 \pm 3.16$ |
| Winewhite  | 4898| 11  | $66.36 \pm 9.63$         | $9.09 \pm 0.00$          | $\bullet 28.18 \pm 2.87$ | $28.18 \pm 2.87$ | $20.91 \pm 4.39$ | $27.27 \pm 0.00$ | $18.18 \pm 0.00$ |
| Smartphone | 7352| 561 | $5.53 \pm 0.24$          | $0.36 \pm 0.00$          | $\circ 0.18 \pm 0.00$ | $71.21 \pm 20.1$ | $0.18 \pm 0.00$ | $\circ 74.72 \pm 19.7$ | $24.69 \pm 9.87$ |
| Firmteacher| 10800|16  | $48.12 \pm 30.8$         | $10.00 \pm 3.22$         | $24.38 \pm 7.48$ | $25.62 \pm 9.52$ | $21.25 \pm 4.37$ | $20.62 \pm 4.22$ | $20.62 \pm 9.34$ |
| Eeg        | 14980|14  | $14.29 \pm 3.37$         | $8.57 \pm 3.01$          | $\bullet 39.29 \pm 13.2$ | $38.57 \pm 9.04$ | $39.29 \pm 14.0$ | $38.57 \pm 13.1$ | $39.29 \pm 10.8$ |
| Magic      | 19020|10  | $45.00 \pm 7.07$         | $10.00 \pm 0.00$         | $\circ 10.00 \pm 0.00$ | $51.00 \pm 3.16$ | $10.00 \pm 0.00$ | $49.00 \pm 7.38$ | $10.00 \pm 0.00$ |
| Hardware   | 28179|96  | $11.98 \pm 7.56$         | $2.19 \pm 0.33$          | $\circ 1.04 \pm 0.00$ | $20.94 \pm 3.12$ | $1.04 \pm 0.00$ | $\circ 22.08 \pm 1.89$ | $21.25 \pm 1.49$ |
| Marketing  | 45211|27  | $65.19 \pm 5.58$         | $7.40 \pm 0.00$          | $7.41 \pm 0.00$ | $12.96 \pm 3.60$ | $\circ 3.70 \pm 0.00$ | $\bullet 13.33 \pm 4.35$ | $\circ 3.70 \pm 0.00$ |
| Kaggle     | 120269|11 | $28.18 \pm 5.16$         | $18.18 \pm 0.00$         | $\bullet 17.27 \pm 2.87$ | $9.09 \pm 0.00$ | $15.45 \pm 4.39$ | $10.00 \pm 2.87$ | $14.55 \pm 4.69$ |

Table 4: Supports of $\text{ADABOOST}$ and $\ell_1$-$\text{ADABOOST}$ vs $\Omega$-$\text{R. ADABOOST}$ for the results displayed in Table 2 ($\text{supp.}\% (\theta) = 100 \cdot ||\theta||_0 / d$). For each domain, the sparsest of $\Omega$-$\text{R. ADABOOST}$’s method (in average) is indicated with "•", and the least sparse is indicated with "○".
| domain       | m  | d  | \(\ell_1\text{-AdaBoost}\) | \(\Omega\text{-AdaBoost}\) | \(\Omega\text{-R. AdaBoost}\) |
|--------------|----|----|-----------------|-----------------|-----------------|
| Fertility    | 100| 9  | 40.00±14.1      | 48.00±16.1      | 47.00±12.5      |
| Sonar       | 208| 60 | 24.57±9.11      | 27.86±11.4      | 26.40±7.80      |
| Haberman    | 306| 3  | 25.15±6.53      | 26.76±6.92      | 25.78±6.72      |
| Ionosphere  | 351| 33 | 13.11±6.36      | 17.67±6.17      | 15.65±5.51      |
| Breastwisc  | 699| 9  | 3.00±1.96       | 3.43±1.93       | 3.57±2.15       |
| Transfusion | 748| 4  | 39.17±7.01      | 35.56±5.15      | 34.76±7.25      |
| Banknote    | 1372|4  | 2.70±1.46       | 13.70±2.30      | 13.92±3.16      |
| Winered     | 1599|11 | 26.33±2.75      | 27.64±3.16      | 27.39±2.86      |
| Abalone     | 4177|10 | 22.98±2.70      | 24.59±2.65      | 24.11±2.39      |
| Winewhite   | 4898|11 | 30.73±2.20      | 31.97±1.57      | 31.01±2.17      |
| Smartphone  | 7352|561| 0.00±0.00       | 0.67±0.25       | 0.46±0.29      |
| Firnteacher | 10800|16 | 44.44±1.43      | 40.10±4.65      | 40.86±5.16      |
| Eeg         | 14980|14 | 45.38±2.04      | 44.79±1.62      | 44.45±1.27      |
| Magic       | 19020|10 | 21.07±1.09      | 21.51±0.99      | 26.41±1.08      |
| Hardware    | 28179|96 | 16.77±0.73      | 8.85±0.68       | 9.06±3.76      |
| Marketing   | 45211|27 | 30.68±1.01      | 28.03±0.45      | 27.87±0.58      |
| Kaggle      | 120269|11 | 47.80±0.47      | 15.99±2.89      | 15.99±2.89      |

Table 5: Results of \(\text{AdaBoost}\) [Schapire & Singer, 1999] vs \(\Omega\text{-R. AdaBoost}\) (trained with random class-wise rados). Conventions follow Table 2. On each domain, the leftmost column shows a “+” when \(\Omega\text{-R. AdaBoost}\) performs better when trained with class-wise rados (instead of just plain random rados), and “-” when it performs worse.
Table 6: Supports of $\text{AdaBoost}$ vs $\Omega-\text{R.AdaBoost}$ for the results displayed in Table 5 in this Appendix. Conventions follow Table 4 in this Appendix.
| domain       | m   | d   | \(\ell_1\text{-AdaBoost}t\) \(\omega = 0\) \(\Omega = \|\cdot\|_\Phi\) | \(\Omega = \|\cdot\|_\infty\) | \(\Omega = \|\cdot\|_1\) | \(\Omega = \|\cdot\|_\Phi\) |
|--------------|-----|-----|--------------------------|--------------------------|--------------------------|--------------------------|
| Fertility†   | 100 | 9   | 48.00±15.5               | 49.00±13.7               | 49.00±14.5               | 48.00±16.9               |
| Fertility*   | 100 | 9   | 40.00±14.1               | 40.00±14.9               | 41.00±16.6               | 41.00±14.5               |
| Sonar†       | 208 | 60  | 24.02±5.42               | 29.38±8.85               | 24.50±7.53               | 24.30±10.7               |
| Sonar*       | 208 | 60  | 24.02±5.42               | 29.38±8.85               | 24.50±7.53               | 24.30±10.7               |
| Haberman†    | 306 | 3   | 50.45±12.4               | 27.19±11.2               | 26.15±9.75               | 26.49±9.21               |
| Haberman*    | 306 | 3   | 27.15±11.2               | 25.05±7.56               | 25.05±8.41               | 24.52±8.65               |
| Ionosphere†  | 351 | 33  | 10.56±4.88               | 17.68±7.38               | 15.11±6.05               | 16.54±6.46               |
| Ionosphere*  | 351 | 33  | 13.11±6.36               | 14.51±7.36               | 13.64±5.99               | 14.24±6.15               |
| Breastwic†   | 699 | 9   | 3.44±2.46                | 3.72±2.06                | 3.87±3.38                | 3.16±2.44                |
| Breastwic*   | 699 | 9   | 3.44±2.46                | 3.72±2.06                | 3.87±3.38                | 3.16±2.44                |
| Transfusion† | 474 | 4   | 39.68±6.39               | 39.02±6.60               | 38.35±7.06               | 39.01±7.63               |
| Transfusion* | 474 | 4   | 39.02±6.60               | 38.35±7.06               | 39.01±7.63               | 39.75±7.15               |
| Banknote†    | 1372| 4   | 2.70±1.09                | 15.09±3.50               | 14.07±3.80               | 14.14±3.25               |
| Banknote*    | 1372| 4   | 2.70±1.09                | 14.00±4.16               | 12.02±2.74               | 13.63±2.75               |
| Winered†     | 1599| 11  | 26.08±2.14               | 28.14±3.23               | 27.77±3.83               | 28.14±3.71               |
| Winered*     | 1599| 11  | 26.08±2.14               | 28.14±3.23               | 27.77±3.83               | 28.14±3.71               |
| Abalone†     | 4100| 17  | 2.03±2.13                | 26.48±1.75               | 25.81±1.49               | 24.18±0.94               |
| Abalone*     | 4100| 17  | 2.03±2.13                | 26.48±1.75               | 25.81±1.49               | 24.18±0.94               |
| Winewhite†   | 4898| 11  | 30.67±1.96               | 31.77±2.51               | 32.32±2.51               | 32.50±1.92               |
| Winewhite*   | 4898| 11  | 30.67±1.96               | 31.77±2.51               | 32.32±2.51               | 32.50±1.92               |
| Smartphone†  | 7352| 561| 0.00±0.00                | 0.67±0.24                | 0.20±0.13                | 0.46±0.16                |
| Smartphone*  | 7352| 561| 0.00±0.00                | 0.67±0.24                | 0.20±0.13                | 0.46±0.16                |
| Firnreacher† | 10800| 16 | 46.00±1.32               | 46.62±1.86               | 46.12±1.80               | 46.37±1.58               |
| Firnreacher* | 10800| 16 | 46.00±1.32               | 46.62±1.86               | 46.12±1.80               | 46.37±1.58               |
| Eeg†         | 14980 | 14 | 45.60±1.96               | 43.68±1.97               | 44.03±2.04               | 43.95±2.04               |
| Eeg*         | 14980 | 14 | 45.60±1.96               | 43.68±1.97               | 44.03±2.04               | 43.95±2.04               |
| Magic†       | 19020| 10 | 21.81±0.81               | 37.23±1.02               | 22.38±0.78               | 23.59±0.85               |
| Magic*       | 19020| 10 | 21.81±0.81               | 37.23±1.02               | 22.38±0.78               | 23.59±0.85               |
| Hardware†    | 28179| 96 | 16.84±0.74               | 9.41±0.55                | 11.52±0.91               | 11.51±1.20               |
| Hardware*    | 28179| 96 | 16.84±0.74               | 9.41±0.55                | 11.52±0.91               | 11.51±1.20               |
| Marketing†   | 45211| 27 | 30.67±0.61               | 28.37±1.21               | 28.06±1.04               | 28.02±0.98               |
| Marketing*   | 45211| 27 | 30.67±0.61               | 28.37±1.21               | 28.06±1.04               | 28.02±0.98               |
| Kaggel†      | 120269 | 11 | 48.30±0.67               | 44.99±1.98               | 44.04±5.80               | 44.37±3.93               |
| Kaggel*      | 120269| 11 | 48.30±0.67               | 44.99±1.98               | 44.04±5.80               | 44.37±3.93               |

Table 7: Best result of ADABoost/\(\ell_1\)-AdaBoost [Schapire & Singer (1999); Xi et al. (2009)], vs Ω-R.ADABoost (with or without regularization, trained with n = m random rados (above bold horizontal line) / n = 10000 rados (below bold horizontal line)), according to the expected true error of \(\theta_T\), when the classifier \(\theta_T\) returned is the last classifier of the sequence (\(*\): \(\theta_T = \theta_{(1000)}\)), or when it is the classifier minimizing the empirical risk in the sequence (\(*\): \(\theta_T = \theta_{\leq 1000}\)). Table shows the best result over all us, as well as the difference between the worst and best (\(\Delta\)). Shaded cells display the best result of Ω-R.ADABoost. All domains but Kaggle are UCI [Bache & Lichman (2013)].
| Domain       | m  | d  | \(\text{Adaboost}^{\omega=0}\) | \(\ell_1\text{-Adaboost}^{\omega=0}\) | \(\Omega = \| \cdot \|_2\) | \(\Omega = \| \cdot \|_1\) | \(\Omega = \| \cdot \|_{\infty}\) |
|--------------|----|----|---------------------------------|---------------------------------|-----------------|-----------------|-----------------|
| Fertility\*  | 100| 9  | 100.00 ± 0.00                   | 11.11 ± 0.00                   | 86.67 ± 10.2    | 86.67 ± 10.2    | 82.22 ± 15.0    |
| Sonar\*      | 208| 60 | 58.50 ± 4.04                   | 2.50 ± 1.41                    | 15.67 ± 3.62    | 16.50 ± 2.88    | 15.17 ± 3.09    |
| Haberman\*    | 306| 3  | 100.00 ± 0.00                   | 33.33 ± 0.00                   | 66.67 ± 22.2    | 60.00 ± 26.3    | 73.33 ± 21.1    |
| Ionosphere\*  | 351| 33 | 80.30 ± 4.34                   | 3.94 ± 2.04                    | 15.15 ± 2.47    | 14.24 ± 2.05    | 13.64 ± 3.27    |
| Breastwise\*  | 699| 9  | 100.00 ± 0.00                   | 11.11 ± 0.00                   | 75.00 ± 11.8    | 87.50 ± 13.2    | 80.00 ± 15.8    |
| Transfusion   | 748| 4  | 100.00 ± 0.00                   | 25.00 ± 0.00                   | 25.00 ± 0.00    | 25.00 ± 0.00    | 25.00 ± 0.00    |
| Banknote\*    | 1372| 4 | 100.00 ± 0.00                   | 25.00 ± 0.00                   | 37.50 ± 13.2    | 50.00 ± 0.00    | 50.00 ± 0.00    |
| Winered\*     | 1599| 11| 84.55 ± 6.14                   | 9.09 ± 0.00                    | 27.27 ± 7.42    | 27.27 ± 7.42    | 31.82 ± 6.43    |
| Abalone\*     | 4177| 10| 65.00 ± 5.27                   | 19.00 ± 3.16                   | 40.00 ± 13.3    | 83.00 ± 4.83    | 10.00 ± 0.00    |
| Winefuture\*  | 4898| 11| 65.45 ± 7.17                   | 9.09 ± 0.00                    | 27.27 ± 0.00    | 30.00 ± 6.14    | 13.64 ± 2.17    |
| Smartphone\*  | 7352| 561| 5.63 ± 0.27                    | 0.36 ± 0.00                    | 0.18 ± 0.00     | 78.81 ± 0.33    | 0.18 ± 0.00     |
| Firmeacher\*  | 10800| 16| 100.00 ± 0.00                  | 10.00 ± 3.23                   | 40.62 ± 6.07    | 39.38 ± 7.82    | 42.50 ± 7.10    |
| Eeg\*         | 14980| 14| 15.00 ± 2.26                   | 8.57 ± 3.01                    | 47.14 ± 6.90    | 47.86 ± 5.88    | 42.14 ± 7.86    |
| Magic\*       | 19020| 10| 68.00 ± 7.14                   | 10.00 ± 3.16                   | 11.00 ± 3.16    | 49.00 ± 3.16    | 10.00 ± 0.00    |
| Hardware\*    | 28179| 96| 16.35 ± 0.70                   | 2.91 ± 0.82                    | 1.04 ± 0.00     | 91.67 ± 0.00    | 1.04 ± 0.00     |
| Marketing\*   | 45211| 27| 64.07 ± 3.51                   | 7.41 ± 0.00                    | 8.89 ± 1.91     | 15.93 ± 3.51    | 3.70 ± 0.00     |
| Kaggle\*      | 120269| 11| 36.36 ± 0.00                   | 18.18 ± 0.00                   | 19.09 ± 2.87    | 33.64 ± 6.14    | 18.18 ± 0.00    |
| Kaggle\*      | 120269| 11| 28.18 ± 5.16                   | 18.18 ± 0.00                   | 17.27 ± 2.87    | 9.09 ± 0.00     | 15.45 ± 4.39    |

Table 8: Supports of Adaboost and \(\ell_1\text{-Adaboost}\) vs \(\Omega\text{-R. Adaboost}\) for the results displayed in Table2 (supp. % \(\theta\) = 100 \| \theta \|_0/d), when the classifier \(\theta_T\) returned is the last classifier of the sequence ("\*"; \(\theta_T = \theta_{1000}\)), or when it is the classifier minimizing the empirical risk in the sequence ("\*\*"; \(\theta_T = \theta_{\leq 1000}\)).