Dynamics of radiating particles in current sheets with a transverse magnetic field component

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Current sheets in the vicinity of pulsars or those predicted on upcoming multipetawatt laser facilities can feature extremely high electromagnetic fields. As a result, charged particles in such sheets may experience significant radiation losses. We present an analysis of particle motion in model fields of a relativistic neutral current sheet with two magnetic field components in the case when radiative effects must be accounted for. In the Landau-Lifshitz radiation reaction model the method of slowly changing parameters is used to quantify the influence of radiative effects. A quasiadiabatic invariant of dissipative motion is found. It is shown how the presented method can be used in a wider range of problems.

I. INTRODUCTION

Current sheets are magnetoplasma structures that both exist naturally in the Universe and can be formed in laboratory experiments. Much attention has been paid to current sheets of various structure and origin, for example, those forming as a result of solar wind interaction with planetary magnetic fields. In most of these cases the characteristic values of magnetic fields and particle energies do not require the consideration of radiation losses, and analytical solutions of equations of particle motion in such structures were obtained [1-8].

However, for current sheets formed under extreme conditions radiation losses due to abundant photon emission can play a significant role in particle dynamics. Such extreme conditions can be natural, for example those in the vicinity of pulsars [9], or be a result of solar wind interaction with planetary magnetic fields. In most of these cases the characteristic values of magnetic fields and particle energies do not require the consideration of radiation losses, and analytical solutions of equations of particle motion in such structures were obtained [1-8].

In the work [7] the dynamics of particles in a null current sheet with a fixed two-component magnetic field was studied without the consideration of relativism or radiation losses. The primary component of the magnetic field is considered to be parallel to the sheet and perpendicular to the sheet. In a recent work the influence of radiation losses was investigated theoretically and numerically for ultrarelativistic particles in a single-component fixed magnetic field [22], this quasistationary plasma-field configuration is similar to laser excited [10, 11] and space current sheets [1-2, 6, 9, 23]. The goal of the current work is to combine the results of works [7] and [22] and study the dynamics of ultrarelativistic particles experiencing weak radiation losses in a more general two-component configuration of the magnetic field.

Although current sheets are complicated self-consistent plasma-field structures and particle dynamics should ideally be considered self-consistently with the field generated by all particles, the focus of this paper is the influence of radiation losses on particle dynamics. Therefore, in this paper we study the motion of probe particles experiencing radiation losses in a fixed two-component magnetic field modeling the field structure of a current sheet.

In our study we consider the case when a relativistic particle emits photons frequently and each emitted photon carries away a negligible part of particle energy, so it is reasonable to consider radiation losses in the form of the Landau-Lifshitz (LL) force [24]. In the case of weak radiative reaction we quantify its influence on the dynamics of particles and find a quasiadiabatic invariant of dissipative motion.

The structure of this paper is as follows. In Section II we rederive some of the results of the work [7] in the relativistic case and offer a way to deal with significantly different time scales of interdependent features of particle dynamics. In Section III we consider radiation reaction in the form of a continuous force of radiative friction in the Landau-Lifshitz form and use the method of slowly changing parameters to study the dynamics of particles with this force taken into account. In Section IV we discuss how some of the assumptions we had employed can be lifted and show how this method can be applied in a wider range of problems.

II. BASE MODEL

In this section we consider without radiation losses the motion of a relativistic positron in a constant inhomogeneous magnetic field with two non-zero components

\[
\begin{align*}
B_x(x) &= kx \\
B_z &= \text{const}
\end{align*}
\] (1)

and an according vector potential \( A_z(x, y) = B_x y - kx^2/2 \). The motion of an electron is symmetrical and can be derived by replacing \( e \) with \( -e \).
Particle motion in a field of such configuration has been studied before in \([3, 4, 7]\) in the non-relativistic case. It was shown that in the case when \(B_x \ll B_{y\text{max}}\) the motion can be considered as a superposition of two motions. The "slow" quasi-closed-loop motion near the sheet plane \(x = 0\) is due to the magnetic field component \(B_x\) perpendicular to the sheet, this motion can be considered in the phase space \((x, p_x)\). The "fast" oscillations in the transverse direction are due to the \(B_y\) component, which has a null point in the center of the sheet. This motion can be considered in the phase space \((x, p_z)\) \([7}\).

We intend to show that in this problem relativity can be accounted for with rather minor adjustments. The key factor here is that, since the field is purely magnetic and in this section we are not considering radiation losses yet, the kinetic energy and the Lorentz-factor \(\gamma\) of the particle remain constant, as well as the absolute values of momentum \(p\) and velocity \(V\), any of which can be considered the first integral of motion.

Additionally, for this system a second integral of motion can be obtained: since the vector-potential \(A = (0, 0, A_z(x, y))\) does not depend on \(z\), it is evident that \(P_z = p_x + e A_z/c = \text{const}\), where \(P\) is the generalized momentum, \(c > 0\) is the positron charge, and \(c\) is the speed of light. Therefore, \(p_z = P_z - e(B_x y - k x^2/2)/c\). Using this expression, the system of equations for the motion of the particle can be written as:

\[
\begin{align*}
\dot{x} &= p_x/m\gamma \\
\dot{y} &= p_y/m\gamma \\
\dot{z} &= p_z/m\gamma \\
p_x &= -\frac{k}{\gamma} k x p_z/m\gamma \\
p_y &= \frac{e}{\gamma} B_z p_z/m\gamma \\
p_z &= P_z - \frac{e}{\gamma} (B_x y - k x^2) 
\end{align*}
\]

(2)

where \(m\) is the positron mass. Due to the conservation of total momentum the system of equations is similar to the non-relativistic one with the exception of the \(1/\gamma\) factor.

From this system we can extract the systems of equations for "fast" and "slow" motion, respectively:

\[
\begin{align*}
\dot{x} &= p_x/m\gamma \\
p_x &= -\frac{k}{\gamma} k x p_z/m\gamma \\
\dot{y} &= p_y/m\gamma \\
p_y &= \frac{e}{\gamma} B_z p_z/m\gamma \\
\dot{z} &= p_z/m\gamma \\
p_z &= P_z - \frac{e}{\gamma} (B_x y - k x^2)
\end{align*}
\]

(3)

As we can see, these systems are not completely separable, i.e. \(p_y\) depends on \(x\), and \(p_z\) depends on \(y\). However, the assumption of significantly different timescales for the "fast" and the "slow" motion allows us to separate them.

For the sake of "fast" motion \(y\) and \(p_y\) are "frozen" external parameters. Thus, the conservation of momentum during "fast" motion can be written as \(p_x^2 + p_z^2 = p^2 - p_y^2 = \text{const}\) \([3]\), since \(p_y\) does not change on the "fast" time scale. Then the parameter \(\eta\) from \([22]\) (or \(-s\) from \([7]\)) that determines the type of motion in the "fast" phase space \((x, p_x)\) can be written as

\[
\eta(y, p_y) = (P_z - e B_x y/c)/\sqrt{p^2 - p_y^2}
\]

(5)

In the work \([22]\) \(\eta\) was the second integral of motion as there was no \(y\) or \(p_0\). In this work, however, the point on the "slow" \((y, p_y)\) plane corresponds to and determines the trajectory on the "fast" \((x, p_x)\) plane.

For the sake of "slow" motion the terms depending on the "fast" variable \(x\) can be averaged over the period of "fast" motion, and the averaged values are functions of \((y, p_y)\), which can be computed with the help of the results of \([22]\). For example, as follows from \([22]\) Eq.2,

\[
x^s = \frac{2c}{ek} \left( 1 - q_y^2 \cos \varphi - \eta \right),
\]

(6)

where \(q_y \equiv p_y/p\) \([26]\) and \(\varphi\) is the "fast" motion direction of propagation (projected onto the \(xz\) plane): \(\sin \varphi = p_z/p_{xz}, \cos \varphi = p_x/p_{xz}\). The value \(\cos \varphi\) can be computed as

\[
\cos \varphi = \frac{\int_{T_x} \cos \varphi dt}{\int_{T_x} dt} = \frac{\int_{0}^{\tau_{max}} (\cos \varphi/\varphi) d\varphi}{\int_{0}^{\tau_{max}} (1/\varphi) d\varphi}.
\]

Since \(\varphi = \sqrt{2ek/m^2}\cos \varphi - \eta\) \([22]\),

\[
\cos \varphi = \frac{\int_{0}^{\tau_{max}} (\cos \varphi/\varphi) d\varphi}{\int_{0}^{\tau_{max}} (1/\varphi) d\varphi}.
\]

which with the help of the substitution \(\sin \alpha = \sin (\varphi/2)/\sqrt{(1 - \eta)/2}\) can be computed as

\[
\cos \varphi = \left\{ \begin{array}{ll}
\frac{2}{K[(1-\eta)/2]} - 1, & \eta > -1 \\
-1 + \frac{2}{K[(1+\eta)/2]}, & \eta < -1
\end{array} \right.
\]

(7)

where \(K, E\) are complete elliptic integrals of the first and second kind, respectively, and \(\eta\) is a function of \((y, p_y)\) as defined in \([3]\). The notation

\[
K[\kappa^2] = \int_{\pi/2}^\pi d\varphi/\sqrt{1 - \kappa^2 \sin^2 \varphi}
\]

\[
E[\kappa^2] = \int_{\pi/2}^\pi d\varphi/\sqrt{1 - \kappa^2 \sin^2 \varphi}
\]

is used.

In theory equations \([4]\) and \([7]\) allow excluding the term \(x^2\) from \([4]\), but one can see that even then solving \([4]\) directly would be problematic. In order to study the system \([4]\), we employ the quasidiabatic invariant \([7]\):

\[
i_x(y, q_y) = \frac{1}{2\pi} (1 - q_y^2)^{3/4} f(-\eta),
\]

(9)

where

\[
f(s) = \left\{ \begin{array}{ll}
\frac{2}{3}(K[\kappa^2](1 - s) + 2s E[\kappa^2]), & -1 \leq s \leq 1 \\
\frac{4}{3} K[\kappa^2](1 - s) + s E[\kappa^2]), & s \geq 1
\end{array} \right.
\]

(10)

and \(\kappa = \sqrt{(1 + s)/2}\). Unlike the work \([7]\), in our notation \(i_x\) is dimensionless.
Each possible value of \( i_z \) corresponds to a closed loop on the \((y, q_y)\) plane, the possible values are \( 0 < i_z < i_{z\text{max}} \equiv f_{\text{max}}/(2\pi) \approx 0.246 \), where \( f_{\text{max}} \approx 1.55 \) is the maximal value of the function \( f(s) \). \( i_z \) is well conserved along the trajectory except in the vicinity of the uncertainty curve (in red on Fig. 2) determined by \( \eta(y, q_y) = -1 \), where it experiences a quasirandom jump \([7, 22]\). Since this paper is devoted to studying the influence of radiation losses on the trajectory, and not much radiation loss is accumulated in the small vicinity of the uncertainty curve where the jump occurs, the quasirandom jumps of the quasiadiabatic invariant will be disregarded throughout this paper unless specifically stated otherwise. The green curve on Fig. 2 is the curve \( q_y = 0 \) or \( \eta(y, q_y) = \eta^* \) \([22]\), which will be important later. To the left of the green curve \( \eta^* < \eta(y, q_y) \leq 1 \) and the "fast" trajectory is of type A or B \([22]\) (meandering). Between the green and the red curves \(-1 < \eta(y, q_y) < \eta^* \) and the trajectory is of type C \([22]\). To the right of the red curve \( \eta(y, q_y) < -1 \) and the trajectory is of type D \([22]\) (non-crossing).

An interesting feature of this system is that \( \dot{z} \sim p_y \sim p_z \). As a result, with the right choice of the origin of coordinates \( z \sim p_y \), and the trajectory in the phase space \((y, p_y)\) is similar to the projection of the trajectory onto the \(yz\) plane. In the \(x\) direction the trajectory oscillates around zero if the current point on the \((y, p_y)\) plane is to the left of the red curve and around a non-zero value with a quasirandom sign \([7, 22]\) if it is to the right of the red curve. Thus, the whole trajectory resembles a split torus.

### III. IMPLEMENTING RADIATIVE RECOIL

A positron moving along a curvilinear trajectory can emit photons. Based on the preceding works \([11, 22]\) we allow the magnetic field and particle energy values to be sufficiently high in order for the particles to exhibit radiative recoil. Even though a single impact experienced by the particle as a result of photon emission can be relatively weak, particle motion can be qualitatively modified as a result of a sequence of such acts. While recoil-free motion of particles would be infinite and quasiperiodic as described in Sec. III, even recoil insignificant over one period of particle motion due to photon emission may accumulate over multiple periods and have a significant effect on the motion of particles. In the work \([11]\) current sheets are shown to be formed by ultraintense laser fields. The lifetime of these current sheets was observed to be much larger than the laser wave period, which is in turn much greater than the characteristic times of particle trajectories, so such an accumulation may indeed take place.

In this section we consider particle motion in the field structure with two non-zero components \( B_y(x) = kx \) and \( B_z = \text{const} \) with radiative recoil. In the case when particles emit photons often and they carry away a negligible part of the particle’s energy, it is reasonable to consider radiation losses in the form of a continuous Landau-Lifshitz friction force \([24]\). We also consider only the ultrarelativistic case \( p/mc \approx \gamma > 1 \), which allows us to neglect the first and second terms of the LL force \([24]\).

In our setup this translates to:

\[
\vec{F}_{\text{rad}} = \frac{2e^4V}{3m^2c^3} \gamma^2 [V \times B]^2, \tag{11}
\]

or, after expanding the vector product,

\[
\vec{F}_{\text{rad}} = \frac{2e^4V^2}{3m^2c^3} \gamma^2 [ k^2 x^2 (1 - q_y^2) + B_z^2 (1 - (1 - q_y^2) \sin^2 \varphi) - 2kx B_x q_y \sqrt{1 - q_y^2} \sin \varphi ]. \tag{12}
\]

In Sec. III \( \gamma \) was constant as there was no recoil/friction. Accounting for radiative friction forces us to treat \( \gamma \) as another parameter depending on time. Like in \([22]\), we will use the variable \( \mu = 1/\gamma \) as the main variable characterizing the current energy. Taking into account that particle energy decreases due to radiative friction given

![FIG. 1. The function f(s).](image1)

![FIG. 2. The phase space of "slow" variables \((y, q_y)\). The uncertainty curve is shown in red. The green curve represents \( p_y = 0 \). Trajectories are denoted with the corresponding value of \( i_z \).](image2)
by the LL force in Eq. (12), the equation for $\dot{\mu} = \frac{4e^2}{3mc^2}$ can be written as:

$$\dot{\mu} = \frac{2e^4}{3mc^2c^0} \left[ k^2x^2(1 - q_y^2) + B_x^2(1 - (1 - q_y^2)\sin^2\varphi) - 2kxB_yq_y\sqrt{1 - q_y^2}\sin\varphi \right]. \tag{13}$$

Thus, the main parameters characterizing the state of the particle are $\mu$ (determines the particle's energy) and $i_x$ (determines what trajectory the particle is on in the $(y,p_y)$ phase space). $(y,p_y)$ determine at which point of that trajectory that particle is at the moment, and the shape of the potential the particle oscillates in, as well as the particle's energy, in the $(x,p_x)$ phase space. It is important that $(y,p_y)$ are not independent considering a given $i_x$. And finally, $x$ or $\varphi$ determines at which point of that "fast" potential the particle is at the moment.

Our primary objective is to study the influence of weak radiative friction, so we will assume that

$$T_x \ll T_y \ll \frac{\mu}{\mu_i} i_x, \tag{14}$$

The positron’s motion at any given moment of time can be approximated by the solution for the case without radiative friction, and we treat $\mu$ and $i_x$ as slowly changing parameters. In order to account for radiative friction, we must quantify its influence on these parameters over the half-period of motion in the $(y,p_y)$ plane $T_y/2$, and find how that quantity depends on $\mu$ and $i_x$.

To quantify $\Delta \mu$ we integrate $\dot{\mu}(\mu, i_x)$ over $T_y/2$. In order to do that, we use $d\mu/dq_y = \dot{\mu}/q_y$ and integrate over $q_y$ instead. Here, from (2) and (4),

$$q_y = \frac{eB_y}{mc}\left((\gamma_x - \frac{2}{3}B_y) + \frac{(2\pi)^2}{p} \right) = \frac{eB_y}{mc\gamma_x}\left(1 - \frac{q_y^2 + \sqrt{1 - q_y^2(\cos\varphi - \eta)} - \eta} {1 - q_y\cos\varphi} \right), \tag{15}$$

However, since $\eta$ in the expression for $\dot{\mu}$ (13) depends only on $i_x$, while all dependence on $\mu$ can be factored out of the integral. As a result, $\Delta \mu(\mu, i_x)$ can be written as a product of two functions with separated variables: $\Delta \mu(\mu, i_x) = F_\mu(\mu)F_{ix}(i_x)$. In this case we can define $F_\mu(\mu) = 4e^2k/3mc^2B_x\mu^2$, then

$$F_{ix}(i_x) = \int_{i_x} (1 - q_y^2)(1 - \eta/\cos\varphi(\eta)) d\eta, \tag{19}$$

where, again, $\eta = \eta(i_x,q_y)$ according to (10) and $\cos\varphi(\eta)$ is a function of $\eta$ according to (7). We will see momentarily that the same thing applies to $\Delta i_x$, which can be written as $\Delta i_x(\mu, i_x) = G_\mu(\mu)G_{ix}(i_x)$.

Since any change in the quasidiabatic invariant $i_x$ is a direct consequence of radiative friction, $i_x$ can be calculated as $i_x = \frac{di_x}{d\mu}$, where $\frac{di_x}{d\mu}$ can be derived from (10):

$$\frac{di_x}{d\mu} = -\frac{1}{2\pi} \left(1 - q_y^2\right)^{3/4} f(-\eta) \frac{\eta}{\mu}. \tag{20}$$

Accordingly, the expression for the change of $i_x$ over the half-period $T_y/2$ can be expressed as:

$$\Delta i_x = -\frac{G_\mu(\mu)}{2\pi} \int_{i_x} (1 - q_y^2)^{7/4} f(-\eta)(1 - \eta/\cos\varphi(\eta)) d\eta, \tag{21}$$

keeping in mind (16) and (7). Here we chose $G_\mu(\mu) = -4e^2k/3mc^2B_x\mu^3$ or

$$G_\mu(\mu) = F_\mu(\mu)/\mu, \tag{22}$$
while

\[ G_{i_x}(i_x) = -\frac{1}{2\pi} \int_{i_x}^{1} (1-y^2)^{3/4} \eta f'(-\eta)(1-\eta/\cos \varphi) \, dy; \]  

(23)

\[ \mu(i_x, i_x) = \frac{F_{i_x}(i_x)}{T_{i_x}/2} \]  

and  

\[ i_x(i_x) = \frac{G_{i_x}(i_x)}{T_{i_x}/2} \]  

(24)

But in that case there exists an invariant  

\[ L(\mu, i_x) = L_{\mu}(\mu)L_{i_x}(i_x) \]  

From  \[ \frac{d}{d\mu} L(\mu, i_x) = 0 \]  

follows

\[ L''(\mu)L_{i_x}(i_x) + 2L'(\mu)L'_{i_x}(i_x) + L(\mu)L''_{i_x}(i_x) = 0 \]

or:

\[ \frac{L''(\mu)}{L(\mu)G_{i_x}(i_x)} = \frac{L'(i_x)}{L_{i_x}(i_x)G_{i_x}(i_x)} = \text{const}, \]

(25)

since the left side does not depend on \( i_x \) and the right side does not depend on \( \mu \).

In that case due to Eq. (22) it is easily found that

\[ L_{\mu} = \exp(\int \frac{G_{i_x}(i_x)}{F_{i_x}(i_x)} \, d\mu) \approx \mu. \]

(26)

\[ L_{i_x} = \exp(-\int \frac{F_{i_x}(i_x)}{G_{i_x}(i_x)} \, di_x) \approx \mu. \]

(27)

We will use Eq. (27) to find \( L_{i_x} \) for all \( i_x \) in the following subsection. But for trajectories with \( \gamma < i_{\text{max}} \) it can be estimated independently of Eq. (27) that \( L_{i_x} = i_x^2 \). Indeed, since \( p_y \sim \dot{z} \) and \( |\dot{z}| \) is very small on the near-Larmor trajectories with \( \gamma > 1 \) (to the right of the red curve on Fig. 2), a particle spends a lot more time with \( p_y > 0 \) (to the right of the green curve) than with \( p_y < 0 \) (to the left of the green curve). Additionally, its average \( x^2 \) is also larger [4,7,22], and therefore so is its exposure to radiative friction. Therefore, for a rough estimate the influence of weaker radiative friction over shorter periods of time to the left of the red curve can be neglected, and only the part of the trajectory to the right of the red curve can be considered. Finally, remembering that \( i_x \sim \int p_x \, dx / p^{3/2} \), the Larmor radius in the ultrarelativistic case \( R_L \sim \gamma \) and \( p \sim \gamma \), it can be concluded that \( i_x \sim \gamma^{1/2} = \mu^{1/2} \), thus \( i_x^2 \mu \approx \text{const} \).

A. The functions \( \Delta \mu(i_x), \Delta i_x(i_x) \) and \( L(i_x) \)

At this point the \( \mu \)-dependent parts of the discussed functions have been identified and only \( i_x \)-dependent parts remain, so we will refer to \( F_{i_x} \) as \( \mu \), to \( G_{i_x} \) as \( i_x \), and to \( L_{i_x} \) as \( L \) in order to underline their physical meaning. Although exact analytical expressions for these functions were obtained (Eqs. [19,23,27]), the elaborate nature of these expressions gives little idea of the actual form of the functions. As such, in this subsection we present the results of numerical calculations of the specified functions, as well as discuss fits for these functions in different regions, and discuss the inferences. Fits for \( i_x < i_{\text{max}} \) can usually be made on the fly, while fits for \( i_x \approx i_{\text{max}} \) have been moved into the appendix.

1. \( \Delta \mu(i_x) \)

Using asymptotic expressions for the underlying functions in the case \( i_x < i_{\text{max}} \) and \( -\eta > 1 \) a dependence \( \Delta \mu \sim i_x^{-4} \) can be derived, so we expect to see \( \mu_3 \approx \text{const} \) near \( i_x = 0 \).

The graph of \( \Delta \mu(i_x) \) is shown in Fig. 3. It is indeed evident that \( \Delta \mu_3 \approx \text{const} \) is a good approximation for \( i_x < i_{\text{max}} \) and it holds a better than 1% precision for \( i_x < 0.15 \). Note that the function has a breaking point near \( i_x = i_x^* \approx 0.212 \), which corresponds to the trajectory on Fig. 2 tangent to the red curve.

2. \( \Delta i_x(i_x) \)

Using the approximate invariant \( L \approx \mu^2 i_x \), or using asymptotic expressions for the underlying functions in the case \( i_x < i_{\text{max}} \) and \( -\eta > 1 \), a dependence \( \Delta \mu \sim i_x^{-3} \) can be derived, so we expect to see \( i_x^3 \Delta i_x \approx \text{const} \) near \( i_x = 0 \).

Note that \( \Delta i_x \) is negative, so the graph depicts the value \( -\Delta i_x \).

The graph of \( -\Delta i_x(i_x) \) is shown in Fig. 4. It is indeed evident that \( i_x^3 \Delta i_x = \text{const} \) is a good approximation for \( i_x < i_{\text{max}} \). The rough fit near \( i_x = i_{\text{max}} \) (in red) was obtained numerically and is \( -\Delta i_x = 4.4(i_{\text{max}} - i_x) - 18(i_{\text{max}} - i_x)^2 \).

3. \( \Delta \mu(i_x) / \Delta i_x(i_x) \)

The function \( \Delta \mu(i_x) / \Delta i_x(i_x) \) or \( F_{i_x}(i_x)/G_{i_x}(i_x) \) is crucial for the calculation of the invariant \( L \) according to Eq. (27). The graph describes the negative value \( -\Delta \mu(i_x)/\Delta i_x(i_x) \), or \( -\Delta \mu / \Delta i_x \). \( -\Delta \mu / \Delta i_x \approx 2/i_x \) is expected near \( i_x = 0 \).

The graph of \( -\Delta \mu(i_x) / \Delta i_x(i_x) \) is shown in Fig. 5. The expected fit \( 2/i_x \) at \( i_x \ll i_{\text{max}} \) works well, a more precise one obtained numerically and used in Fig. 5 in green is
FIG. 3. (a) $\Delta \mu(i_x)$ on a logarithmic scale. (b) The lower part of $\Delta \mu(i_x)$ ($i_x > 0.1$) on a linear scale. (c) $\Delta \mu i_x^*(i_x)$ (blue) and its fit (green), in this case a constant. (d) Accuracy of the fit on a logarithmic scale. $i_x = i_x^*$ is shown in gray.

FIG. 4. (a) $-\Delta i_x(i_x)$ on a logarithmic scale. (b) The lower part of $-\Delta i_x(i_x)$ ($i_x > 0.1$) on a linear scale (blue), its rough fit by a quadratic function (red). (c) $-i_x^3\Delta i_x$ (blue) and its fit (green), in this case a constant. (d) Accuracy of the fit on a logarithmic scale.

$2/i_x - 430i_x^{-5/2}$. Near $i_x = i_{x_{\text{max}}}$ the linear character of $-\Delta i_x$ on Fig. 4 and the roughly constant character of $\Delta \mu$ on Fig. 3 suggest a very rough fit can be made, shown in red and equal to $0.62/(i_{x_{\text{max}}} - i_x)$. On subplot (b) in the logarithmic scale a breaking point near $i_x = i_x^*$ is once again evident.
4. \( L(i_x) \)

The function \( L(i_x) \) is obtained numerically using expression (27). \( L \approx i_x^2 \) is expected at \( i_x \ll i_{x\max} \).

The graph of \( L(i_x) \) is shown in Fig. 6. The expected fit \( L \approx i_x^2 \) at \( i_x \ll i_{x\max} \) works well, a more precise one used in Fig. 6 is \( L = i_x^2 - 110 i_x^{11/2} \). As follows from the fit to \( \Delta \mu(i_x)/\Delta i_x(i_x) \), a rough fit can be made near \( i_x = i_{x\max} \), shown in red and equal to \( 5 \cdot 10^{-3} (i_{x\max} - i_x)^{0.62} \).

5. Takeaways

In the assumptions of weak radiative friction \(^{13}\) and ultrarelativity \( (\gamma \gg 1 \text{ or } \mu \ll 1) \) the motion of particles in current sheets with the field structure \(^{11}\) was studied. It is also assumed that the field component \( B_x \) contributes only to the shape of the unperturbed trajectory, making it 3D instead of 2D \(^{11,12,13,14,15}\), but does not contribute to radiative friction. Since the former is a qualitative change and the latter is a quantitative contribution, such assumptions can be justified in the case when \( B_x \) is sufficiently weak \(^{11}\).

Within these assumptions the main parameters defining the trajectory (\( \mu \) and \( i_x \)), being integrals of motion in the frictionless case, are now considered as slowly changing parameters. In order to quantify the influence of radiative friction, the change in these parameters due to radiative friction is integrated over a half-period \( T_y/2 \) of the unperturbed trajectory. The exact expressions even within our assumptions are rather intricate (See Eqs. \(^{17,19,19,19,17,18}\), mostly owing to the implicit definition of the dual-branch function \( f^{-1} \) and the piecewise properties of \( f^{-1} \) and other underlying functions. Thus, the required functions have been nondimensionalized and calculated numerically.

The analytical approximations made in the limit \( i_x \ll i_{x\max} \) work well up to at least \( i_x \approx 0.05 \), some up to \( i_x \approx 0.15 \). When working in the specified limit, these may be sufficient depending on the required precision. At higher \( i_x \) (or if precision better than \( 10^{-2} - 10^{-3} \) is required), however, the numerically obtained functions can prove to be useful as the fits no longer work very well.

We would like to comment on the surprising change of trend near \( i_x = i_x^* \approx 0.212 \) in \( \Delta \mu(i_x) \) on Fig. \(^{6,a,b}\) and the lack thereof on \( -\Delta i_x(i_x) \) in Fig. \(^{4,a,b}\). The decrease of both with the growth of \( i_x \) from 0 to \( i_x^* \) is not surprising, since the part of the trajectory to the right of the uncertainty curve is the most subject to radiation losses. We believe the slight increase in \( \Delta \mu \) at \( i_x > i_x^* \) is due to the fact that near the uncertainty curve the trajectory of the particle in the "fast" phase space is close to the separatrix, so the particle is less subject to radiation reaction \(^{22}\). or, in mathematical terms, \( x^2 \approx 0 \) in Eq. \(^{12}\) or \( \eta \approx \cos \varphi(\eta) \approx -1 \) in Eq. \(^{17}\). As a result, the increase is noticeable, but still rather slight. For \( \Delta i_x(i_x) \), however, the factor \( f'(\eta) \) in \(^{20,21}\) mitigates this, since for trajectories close to the center of the "slow" phase space (Fig. \(^{2}\)) \( i_x \approx i_{x\max} \) \( f'(\eta) \approx 0 \) as the phase space center corresponds to \( \eta = \eta^* \).
A special note should be made about the obtained form of the function \(L(i_x)\). Since it is obtained via integrating \(\Delta \mu / \Delta i_x\) \(^{27}\), and those two functions, in turn, are calculated in some number of discrete points, there is an additional risk of a numerical integration error, which in this case may accumulate over \(i_x\). The integral \(\int_{i_{xmax}}^{i_x} -\Delta \mu / \Delta i_x \, dx\) diverges to infinity on both sides. Near \(i_x = 0\) the rather precise approximation \(-\Delta \mu / \Delta i_x \approx 2 / i_x\) and the free constant of integration allow us to calculate the first few points alternatively to integration, but there is no accurate fit near \(i_x = i_{xmax}\), so the issue especially important in this region. We note that the accuracy of the fit near \(i_x = i_{xmax}\) on Fig. \(6\) (in red) represents the discrepancy between the red curve on Fig. \(6\) (rough fit) and the blue curve (numerically integrated using Eq. \(27\)), so it does not take into account the integration error described in this paragraph. While we have done our best to calculate \(L(i_x)\) accurately, an analytically derived fit near \(i_x = i_{xmax}\) would help to alleviate this problem.

**IV. DISCUSSION**

The method of slowly changing parameters was used to quantify the influence of radiative friction over a single half-period \(T_y / 2\). The obtained expressions and the performed numerical calculations can be used to evaluate the changes of key parameters \(\mu\) and \(i_x\) over \(T_y / 2\), to employ the obtained invariant \(L(\mu, i_x)\) or to construct a system of differential equations describing the system on long timescales \(\Delta t \gg T_y\). This system is much easier to analyze on the specified timescales than the original one \(^{2}\). Because, first, it is a system of only two differential equations instead of five or six, and second, one would need to resolve only the time scales \(\mu / \dot{\mu}\) and \(i_x / \dot{i_x}\) instead of \(T_x\), which is at least two orders of magnitude lower \(^{14}\).

The primary condition for the applicability of the entire model described within this paper is \(T_x / T_y \ll 1\). Using this model \(T_y\) can be expressed as \(T_y = \int dq_x / \dot{q}_x = T_y(i_x)\), and using the model from \(^{22}\) \(T_x = \int d\phi / \dot{\phi} = T_x(\eta(i_x, q_x))\) can be written. In other words, \(T_y\) depends on which trajectory of the phase plane (Fig. \(2\)) the particle is located on, while \(T_x\) also depends on which point of the trajectory it is currently at. Thus, the constructed theory allows us to obtain \(T_x / T_y\) as a function of \((i_x, q_x)\) or \((y, q_y)\). In the region to the left (and far enough from) the uncertainty curve, it was found that the condition can be reduced to \((kmc^2) / (\mu \dot{B}_x^2) \gg 1\), which is similar to the condition obtained in \(^{1}\). To the right of the uncertainty curve \(T_x \sim 1 / \sqrt{T_y}\), i.e. is much less, and \(T_y\) for such trajectories is, on the contrary, more than to the left, so if \(T_x / T_y \ll 1\) is true on the left, it will be true automatically on the right.

Additionally, while we used the method of slowly changing parameters to calculate the according functions in the assumptions described in Sec \(\text{III}\) it can be applied to a more general problem, where some of these assumptions can be completely or partially lifted.

The second and third terms in \(^{13}\) involving \(B_x\) do not have to be neglected. Finding the exact expressions for \(\Delta \mu\) and \(\Delta i_x\) would entail somewhat more intricate math, but the method applies. Interestingly, the form of
the invariant $L$ in this case remains the same. Note that the condition for weakness of $B_{\perp}$, $[7]$ is still required in order to be able to describe trajectories with the model described in Sec.[14] in the first place.

The employed assumption of ultrarelativity ($V \approx c$) allows neglecting the first two terms of the radiative friction force $[24]$. Notably, the third term is always colinear to the particle’s velocity, while the first two terms may influence the particle’s direction of propagation directly $[29]$. Such an influence would require significant modifications to the method, so in the general relativistic case the method is applicable only as long as this influence is weak. This may be useful in some field structures when the first and second terms are non-negligible in absolute value, but are colinear to the particle’s velocity; or perhaps if this influence is averaged out over either of the periods of motion. More specific instructions for method applicability in the general relativistic case require a separate investigation.

Finally, with some modifications to the method the requirement $T_y \ll \gamma/\gamma, i_x/i_x$ ($[11]$) can be lifted (but not $T_x \ll \gamma/\gamma, i_x/i_x$ or $T_x \ll T_y$). In this case differential equations for $\mu$ and $\nu$ have to be solved self-consistently alongside with the one for $q_y$ and would describe motion on time intervals within the period $T_y$.

We note that although technically the presented method can be used with some of these assumptions lifted simultaneously, it would require immense analytical and computational efforts. Therefore, we remind that one should adapt to the problem at hand and lift only those assumptions that truly require doing so.

Lastly, the effect of quasirandom jumps of the quasadiabatic invariant of nondissipative motion $i_x$ $[5]$ in the vicinity of the uncertainty curve were ignored in this work. Since they occur in a close vicinity of the uncertainty curve during time intervals $\Delta t \ll T_y$, the change to $\mu$ during this time can be neglected. Therefore, the quasadiabatic invariant of dissipative motion $L(\mu, i_x)$ will experience jumps according to the jumps in $i_x$, which have been extensively studied before.

V. CONCLUSION

The motion of an ultrarelativistic particle experiencing radiative friction in null current sheets with a two-component magnetic field (with a weak component perpendicular to the sheet) was considered. The method of slowly changing parameters was used to quantify the influence of weak radiation reaction considered in the form of the continuous Landau-Lifshitz force on key parameters defining the dynamics of the particle. The expressions quantifying this influence were found, and due to their intricacy they were nondimensionalized and computed numerically. A quasadiabatic invariant of dissipative motion was found analytically and also computed numerically in dimensionless variables. Theoretical fits were tested, their precision analyzed depending on the value of key parameters, and in some cases more precise fits were found numerically.

It was also shown how the presented method can be applied in a wider range of problems where some of the employed assumptions are completely or partially lifted.

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Appendix: Approximations in the $i_x \approx i_{x_{max}}$ limit

The limit $i_x \rightarrow i_{x_{max}}$ corresponds to the neighborhood of the center of the phase plane $(y, q_y)$, and the structure of the phase plane (Fig.2) suggests that near this point it is possible to linearize the system of differential equations (4) to $\dot{y}$ and $\dot{q}_y$. Before doing this, it is necessary to take a closer look at the function $x^2(q_y, \cos \varphi(\eta y, q_y)), \eta y, q_y)$ and the functions contained therein that need to be linearized, and their values at the point corresponding to $i_x = i_{x_{max}}$. Since the curve $q_y = 0$ passes through this point (see Fig.1), then, given (15), $\cos \varphi y = 0$. From the latter it follows that $\eta = \eta^* \approx -0.65$ $[22]$, and $x^2$ takes some non-zero value. Using the expressions (4-8,15) and leaving only the terms up to the first order of magnitude by $y - y^*$ and $q_y$, the system can be linearized and any necessary parameters of the system oscillations near the center can be obtained. Among other things, the frequency of these oscillations $\omega_0 \approx \sqrt{0.87 eB_{\perp}\mu/mc}$ was obtained, which is of interest for further research, where the coefficient is equal to the derivative of $\cos \varphi$ with respect to $\eta$: $\cos \varphi' y \approx 0.87$ which appears linearization. Thus, the half-period is $\Pi / \sqrt{\cos \varphi} y \approx 3.37$ in dimensionless units, which agrees well with the values obtained numerically near $i_x = i_{x_{max}}$.

An interesting method of calculating approximations can be used in the $i_x \rightarrow i_{x_{max}}$ limit. For this method we switch back to integrating over $dt$ as opposed to $dq_y$ as proposed withing this paper. As we will see later, many of the functions of interest to us are functions of $\eta$ and $q_y^2$. Since $\cos \varphi y \rightarrow 0$ and $q_y \rightarrow 0$ in this limit, all appearances of $\eta$ can be expressed via the Taylor series in terms of $\cos \varphi y \equiv w$. Then $\eta = \eta^* + \eta^* w(\eta - \eta^*) + \eta^* w^2(\eta - \eta^*)^2/2 + \ldots$. Here higher terms are neglected. As follows from Eq.15, any integral over a whole or halfperiod involving the first power of $\cos \varphi$ is equal to 0. As a result, we are left with terms of the order 0 (constant) and 2 (quadratic). The quadratic term can then be expressed back through $\eta$ and replaced with a Taylor expansion of Eq.16 over $\eta$. As a result, the remaining terms will be either constant or quadratic over $q_y$ with no other variable that changes within a $i_x = const$. trajectory. For quadratic terms it is well known that during harmonic oscillations
their average value is half of the max value, which can be easily expressed linearly over \((i_{x\text{max}} - i_x)\) using Eq.9. This method will allow us to calculate many integrals in this limit without actually performing integration.

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