ON THE EIGENVALUE OF $p(x)$-LAPLACE EQUATION

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Abstract

The main purpose of this paper is to show that there exists a positive number $\lambda_1$, the first eigenvalue, such that some $p(x)$-Laplace equation admits a solution if $\lambda = \lambda_1$ and that $\lambda_1$ is simple, i.e., with respect to the first eigenvalue solutions, which are not equal to zero a.e., of the $p(x)$-Laplace equation forms an one dimensional subset. Furthermore, by developing Moser method we obtained some results concerning Hölder continuity and bounded properties of the solutions. Our works are done in the setting of the Generalized-Sobolev Space. There are many perfect results about $p$-Laplace equations, but about $p(x)$-Laplace equation there are few results. The main reason is that a lot of methods which are very useful in dealing with $p$-Laplace equations are no longer valid for $p(x)$-Laplace equations. In this paper, many results are obtained by imposing some conditions on $p(x)$.

Stimulated by the development of the study of elastic mechanics, interest in variational problems and differential equations has grown in recent decades, while Laplace equations with nonstandard growth conditions share a part. The equation discussed in this paper is derived from the elastic mechanics.

Keyword: $p(x)$-Laplace equation; eigenvalue; Hölder continuity
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Chapter 1

Introduction

1.1 Background

In recent years, there has been increasing interest toward variable exponent Lebesgue and Sobolev spaces. It is clear that we cannot simply replace $p$ by $p(x)$ in the usual definition of the norm in $L^p$. However, the Lebesgue spaces can be considered as particular cases of the Orlicz spaces belonging to a larger family of so-called modular spaces. This approach enables us to define corresponding counterparts of the Luxemburg and Orlicz norms in $L^{p(x)}$.

The present line of investigation toward variable exponent Lebesgue and Sobolev spaces goes back to a paper by O.Kováčik and J.Rákosník [1] from 1991. After this paper not much happened till the late 1990’s. At this point the subject seems to have been rediscovered by several researchers independently: S. Samko [3, 4], working based on earlier Russian work (I. Sharapudinov and V. Zhikov [6]), X. Fan and collaborators drawing inspiration from the study of differential equations [40, 41, 42, 43]. The last couple of years have seen the integration of the separate lines of investigation, but much still remains to be done.

The main incentive for many of the investigators of variable exponent spaces is relaxing the coercivity conditions assumed for the solutions of a differential equation or the corresponding variational integral, while Laplace equations with nonstandard growth conditions share a part. One such application has been investigated in greater detail, electro-rheological fluids. These fluids have the interesting property that their viscosity depends on the electric field in the fluid. For some technical applications the mathematical theory was presented by some investigators: M. Růžička, E. Acerbi and G. Mingione [8, 9, 10, 23].

The remain part of this part strive to give a little more detailed an account of the mathematics of variable exponent spaces and in the next section we present the theory of Generalized Lebesgue spaces and that of Generalized Sobolev spaces. In the final section or the main section we
present a generalization of the eigenvalue problem on some $p(x)$-Laplace equation by Mitsuharu Ôtani and Toshiaki Teshima [34].

1.2 Overview of Differential Operator with Non-standard Exponent Growth

The Harnack Inequality

In a bounded domain $D$ of Euclidean space $\mathbb{R}^n$, $n \geq 2$, Yu.A.Alkhutov[30] proved the Harnack inequality and an interior a priori estimate for the Hölder norm of solutions about the equation as following:

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( |\nabla u|^{p(x)-2} \frac{\partial u}{\partial x_i} \right) = 0$$  \hspace{1cm} (1.1)

where $p(x)$ is a measurable function in $D$ and $1 < p_1 \leq p(x) \leq p_2 < \infty$, the domain $D$ is divided by a part of a Lipschitz surface into two subdomains, in each of which $p(x)$ is constant.

The Hardy-Littlewood Maximal Operator.

Assume that $1 < p^- \leq p^+ < \infty$ and there exists a constant $C > 0$ such that

$$|p(x) - p(y)| \leq \frac{C}{-\log|x - y|}$$

for every $x, y \in \mathbb{R}^n$, $|x - y| \leq \frac{1}{2}$ and

$$|p(x) - p(y)| \leq \frac{C}{-\log(e + |y|)}$$

for every $x, y \in \mathbb{R}^n$, $|y| \geq |x|$. Under these assumption on $p(x)$, Cruz-Uribe, Fiorenza and Neugebauer [13] proved that the Hardy-Littlewood maximal operator is bounded from $L^{p^+}(\mathbb{R}^n)$ to itself. This was an improvement of earlier work by Diening [19, 20, 21, 22] and Nekvinda [36]. Maximal operator have also been studied in weighted $L^{p^+}$ spaces by Kokilashvili and Samko [37, 41, 43].

Strong Maximum Principle of $p(x)$-Laplace Equation

If $\Omega \subset \mathbb{R}^N (N \geq 2)$ be an open set, $p(x) \in C^1(\overline{\Omega})$, and $p(x) > 1(x \in \Omega), q(x) \in C^0(\overline{\Omega})$, and $p(x) \leq q(x) \leq p^*(x)$ ($p^* = \frac{Np(x)}{N-p(x)}$ for $p(x) < N$; $p^* = \infty$ for $p(x) \geq N$), $d(x) \in L^\infty(\Omega), d(x) \geq 0 \text{ a.e.}$, Fan.X and Zhao.Y [45] given a strong maximum principle for super-solutions of the $p(x)$-Laplace equations

$$-\text{div}(|\nabla u|^{p(x)-2}\nabla u) + d(x)|u|^{q(x)-2}u = 0$$  \hspace{1cm} (1.2)
Fan, X and Zhao, Y proved that the nonnegative weak upper solution of (1.2) satisfies \( u \geq c, x \in K \ a.e. \), for any given nonempty compact subset \( K \subset \Omega \), where \( c > 0 \) is a constant. Furthermore if \( u \in C^1(\Omega \cup x_1) \), \( u(x_1) = 0 \), \( x_1 \in \partial \Omega \) and \( u \) satisfies the inner sphere conditions, then \( \frac{\partial u(x_1)}{\partial n} > 0 \), where \( \gamma \) is the unit inner normal vector of \( \partial \Omega \) at \( x_1 \).

**Existence of Solutions for Elliptic Systems with Nonuniform Growth**

Let \( \Omega \subset \mathbb{R}^n \) be a bounded Lipschitz domain. For the following systems:

\[
\frac{\partial A_i^\alpha}{\partial A^\alpha}(x, u(x), Du(x)) = B^i(x, u(x), Du(x)), x \in \Omega, i = 1, \ldots, N \quad (1.3)
\]

\[
u \int_G A_\alpha^i(x_0, s_0, \xi_0 + Dz(x))z_\alpha^i(x) dx \geq \nu \int_G \left| Dz(x) \right|^{p(x)} dx
\]

for each \( \xi_0 \in M^{N \times n} \), \( G \subset \mathbb{R}^n \), \( z \in C^0_0(G, \mathbb{R}^N) \) where \( \nu > 0 \) and \( \langle Du(x) \rangle^i_\alpha = \partial u^i(x)/\partial x^\alpha = u^i_\alpha(x) \). \( p : \Omega \to [1, \infty] \) is a measurable function and \( p' \) is its conjugate function. Then the Dirichlet problem (1.3), (1.4) has at least one weak solution in \( W^{1, p(\cdot)}_0(\Omega, \mathbb{R}^N) \), that is to say, there exists at least one \( u \in W^{1, p(\cdot)}_0(\Omega, \mathbb{R}^N) \) satisfying

\[
\int_\Omega \left[ A^\alpha_i(x, u, Du)z^i_\alpha(x) + B^i(x, u, Du)z^i(x) \right] dx = 0 \quad (1.5)
\]

for all \( x \in W^{1, p(\cdot)}_0(\Omega, \mathbb{R}^N) \). This generalizes the result of Acerbi and Fusco [11].

**Hölder Continuity of Minimizers of Functionals with Variable Growth Exponent**

Let \( a \in W^{1,s}(\Omega)(s > n) \), \( r \) be two nonnegative measurable functions such that \( 1 < p_0 \leq a(x) \leq q_0 \leq p_0^* \), \( 0 \leq r(x) \leq r \leq p_0^* \) and let \( f : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R} \) be a function such that
be a Carathéodory function satisfying the growth assumptions $c_1(|\xi|^{a(x)} - |u|^{r(x)} - 1) \leq f(x, u, \xi) \leq c_2(|\xi|^{a(x)} + |u|^{r(x)} + 1)$ and let $u$ be a quasiminimizer of the functional as following

$$\mathcal{F}(u) = \int_{\Omega} f(x, u, Du)dx$$

(1.6)

Valeria Chiadò Piat and Alessandra Coscia proved the the locally Hölder continuity of $u$ in $\Omega$.

**Hölder continuity of $p(x)$-Laplace equation**

Let $\Omega$ be a open set in $\mathbb{R}^N$, for the following equation:

$$-div \left( \lambda + |\nabla u|^2 \right)^{\frac{p(x)-2}{2}} \nabla u = F(x, u), x \in \Omega \subset \mathbb{R}^N$$

(1.7)

where $\lambda \geq 0$, $F \in C^0(\Omega \times \mathbb{R})$ satisfies $|F(x, u)| \leq c_1 + c_2|u|^{q(x)}$, $\forall (x, u) \in \Omega \times \mathbb{R}$ where $1 < q(x) < p^*(x)(p^* = \frac{Np(x)}{N-p(x)})$ for $p(x) < N$, $p^* = +\infty$ for $p(x) \geq N$, $p \in C^1(\Omega), p(x) > 1(\forall x \in \Omega)$. X.Fan and Zhao Dun proved the local $C^{1,\alpha}$ regularity $u$, that is to say, the weak solution of (1.7) satisfies $u \in C^{1,\alpha}_{loc}(\Omega)$.

**On the Positive Solution of $p(x)$-Laplace Equation**

Let $\Omega$ be a bounded domain in $\mathbb{R}^N (N > 1)$, for the following equation:

$$\begin{cases}
-\text{div}(|\nabla u|^{p(x)-2}\nabla u) = \lambda|u|^{\alpha(x)-2}u + |u|^{\beta(x)-2}u & x \in \Omega \\
u(x) = 0 & x \in \partial \Omega
\end{cases}$$

(1.8)

where $p, \alpha, \beta$ are the continuous functions on $\overline{\Omega}$, and $p(x) < N, \lambda > 0$. X.Fan proved that if 1) $p(x) : \overline{\Omega} \rightarrow \mathbb{R}$ is Lipschitz continuous and $p_- > 1$; 2) $1 - \alpha_- \leq \alpha^+ < p_- \leq p^+ < \beta_-, \beta(x) \leq p^*(x)$. then (1.8) has at least two positive solutions for small $\lambda$.

**Dirichlet Boundary Value Problem**

Consider a differential operator $A$ of order $2k$ in the divergence form

$$Au(x) = \sum_{|\alpha| \leq k} (-1)^{|\alpha|} D^\alpha a_\alpha(x, \delta_k u(x))$$

(1.9)

where the functions $a_\alpha(x, \delta_k u(x)) \in CAR(\Omega, m)$, $m = \left\{ \alpha \in \mathbb{N}_0^N : |\alpha| \leq k \right\}$, fulfill the growth condition $|a_\alpha(x, \xi)| \leq g(x) + c \sum_{|\alpha| \leq k} |\xi|^{p(x)-1}$ with $g \in C^{p(x)}(\Omega)$ and $c > 0$. Let $Q$ be a Banach space of functions on $\Omega$ equipped with a norm $|| \cdot ||_Q$ and such that $C^\infty(\Omega)$ is dense in $Q$ and moreover, $W_0^{k,p(x)}(\Omega) \subset Q$ A function $u \in W^{k,p(x)}(\Omega)$ is a weak solution to the
Dirichlet boundary value problem \((A, u_0, f)\) for the equation \(Au = f\) with the boundary condition given by \(u_0\), if \(u - u_0 \in W^{k,p(x)}_0(\Omega)\) and if the identify

\[
\sum_{|\alpha| \leq k} \int_{\Omega} a_\alpha(x, \delta_k u(x)) D^\alpha v(x) dx = \langle f, v \rangle 
\]  

(1.10)

holds for every \(v \in W^{k,p(x)}_0(\Omega)\). O.Kováčik and J.Rákosník \[1\] proved that if \(p(x) \in P(\Omega)\) satisfy

\[
1 < \text{ess inf} \inf_{\Omega} p(x) \leq \text{ess sup} \sup_{\Omega} p(x) < \infty
\]

the functions \(a_\alpha\) satisfy

\[
\sum_{|\alpha| \leq k} [a_\alpha(x, \xi) - a_\alpha(x, \eta)] (\xi_\alpha - \eta_\alpha) \geq 0,
\]

(1.11)

\[
\sum_{|\alpha| \leq k} a_\alpha(x, \xi) \xi_\alpha \geq c_1 \sum_{|\alpha| \leq k} |\xi_\alpha|^{p(x)} - c_2
\]

for every \(\xi, \eta \in \mathbb{R}^m\) and for a.e. \(x \in \Omega\) with some constants \(c_1, c_2 > 0\). Then the boundary value problem \((A, u_0, f)\) has at least one weak solution \(u \in W^{k,p(x)}(\Omega)\). If moreover, the inequality (1.11) is strict for \(\xi \neq \eta\) then the solution is unique.

**Existence of Positive Solution on \(p(x)\)-Laplace Equation**

For the equations as following:

\[
\begin{cases}
- \text{div}(|\nabla u|^{p(x)-2} \nabla u) = a(x)|u|^{\beta(x)}u & x \in \Omega \\
u(x) = 0 & x \in \partial \Omega
\end{cases}
\]

(1.12)

X.Fan \[4\] proved the existence of positive solution for (1.12) with \(\beta^+ < p_-\) or \(\beta^- > p^+\).

**1.3 Origin of Problem**

Our study comes from the article \[2\] written by M.Ôtani and T.Teshima. in which they study the eigenvalue of the equation as following:

\[
\begin{cases}
- \Delta_p u(x) + a(x)|u(x)|^{p^*-2} u(x) = \lambda b(x)|u(x)|^{p^*-2} u(x) & x \in \Omega \\
u(x) = 0 & x \in \partial \Omega
\end{cases}
\]

where \(\Delta_p u(x) = \text{div}(|\nabla u(x)|^{p^*-2} \nabla u(x))\), \(\lambda > 0\).

M.Ôtani and T.Teshima. proved that eigenvalue problem above has a nontrivial nonnegative solution \(u\) if and only if \(\lambda = \lambda_1\) and \(J_{\lambda_1} := A(u) - \)
\( \lambda_1 B(u) = 0 \). Furthermore, the set of all solutions consists of \( tu_1; t \in \mathbb{R}^1 \) where \( u_1 \) is a solution and \( u_1 \in C^{1,\theta}(\Omega) \) for some \( \theta \in (0,1) \). But if \( p \) is a function of \( x \in \Omega \), it’s a more difficult situation. It’s clear that we cannot simply replace \( p \) by \( p(x) \) in the equation about (43). However, we can extend the definition of \( p \)-Laplace operator by

\[
\Delta_{p(x)} u(x) = \text{Div}(p(x)|\nabla u(x)|^{p(x)-2}\nabla u(x))
\]  

(1.13)

which is named \( p(x) \)-Laplace Operator. Our work tries to give some results about the eigenvalue problem of \( p(x) \)-Laplace equation as following:

\[
-\Delta_{p(x)} u(x) + a(x)|u(x)|^{p(x)-2}u(x) = \lambda b(x)|u(x)|^{p(x)-2}u(x)
\]  

(1.14)
Chapter 2

Generalized Sobolev Space

2.1 Conceptions and Properties

For a set $\Omega \in \mathbb{R}^N$ with $|\Omega| > 0$, we define the family of all measurable functions $p : \Omega \to [1, \infty]$ by $P(\Omega)$. We put $\Omega_1 = \{x \in \Omega : p(x) = 1\}$, $\Omega_\infty = \{x \in \Omega : p(x) = \infty\}$, $\Omega_0 = \Omega \setminus (\Omega_0 \cup \Omega_\infty)$; also, we define $p^+ = \text{ess sup}_{x \in \Omega_0} p(x)$ and $p^- = \text{ess inf}_{x \in \Omega_0} p(x)$. We define the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ to consist of all measurable functions $u : \mathbb{R}^n \to \mathbb{R}$ such that

$$q_{p(\cdot)}(\lambda u) = \int_{\mathbb{R}^n} |\lambda u(x)|^{p(x)} \, dx + \text{ess sup}_{\Omega_\infty} |u(x)| < \infty$$

for some $\lambda > 0$. The function $q_{p(\cdot)} : L^{p(\cdot)}(\Omega) \to [0, \infty]$ is called the modular of the space $L^{p(\cdot)}(\Omega)$. We define a norm, the so-called Luxemburg norm, on this space by the formula

$$\|u\|_{p(\cdot)} = \inf \{\lambda > 0 : q_{p(\cdot)}(u/\lambda) \leq 1\}$$

If $p$ is a constant function, then the variable exponent Lebesgue spaces coincides with the classical Lebesgue spaces and so the notation can give rise to no confusion. The variable exponent Sobolev space $W^{k,p(\cdot)}(\Omega)$ is the subspace of functions $L^{p(\cdot)}(\Omega)$ whose distributional gradient exists almost everywhere and lies in $L^{p(\cdot)}(\Omega)$. The norm of $W^{k,p(\cdot)}(\Omega)$ defined by

$$\|u\|_{k,p} = \sum_{|\alpha| \leq k} \|D^\alpha u\|_{p(\cdot)}$$

By $W^{k,p(\cdot)}_0(\Omega)$ we denote the subspace of $W^{k,p(\cdot)}(\Omega)$ which is the closure of $C_0^\infty$ with respect to the norm (2.3).  

**Basic properties.** Variable exponent Lebesgue spaces resemble classical Lebesgue spaces in many respects $\Uparrow$ —they are Banach spaces, the Hölder inequality holds, they are reflexive if and only if $1 < p^- \leq p^+ < \infty$ and...
continuous functions are dense if \( p^+ < \infty \); The inclusion between Lebesgue spaces also generalizes naturally: if \( 0 < |\Omega| < \infty \) and \( p, q \) are variable exponents so that \( p(x) \leq q(x) \) almost everywhere in \( \Omega \) then there exists an imbedding \( L^{q(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega) \) whose norm does not exceed \( |\Omega| + 1 \); If \( p_1 \leq p^+ < \infty \) and \( (f_i) \) is a sequence of functions in \( L^{q(x)}(\Omega) \), then \( \|f_i\|_{p(x)} \to 0 \) if and only if \( q_{p(x)}(f_i) \to 0 \); The spaces \( W^{k,p(x)}(\Omega) \) and \( W_0^{k,p(x)}(\Omega) \) are Banach spaces, which are separable and reflexive if \( 1 < p^- \leq p^+ < \infty \); If \( q(x) \leq p(x) \) for a.e. \( x \in \Omega \) then \( W^{k,p(x)}(\Omega) \cap W^{k,q(x)}(\Omega) \).

2.2 Sobolev Embedding Inequalities

As we know, in dealing with some partial differential equation problems Sobolev embedding inequality is useful for us. Many good results are derived from these inequalities.

**Theorem 2.1** ([40]). Assume \( \Omega \) be a open domain in \( \mathbb{R}^N \), with cone property. Let \( p(x) \in \mathcal{P}(\Omega) \) be a Lipschitz continuous function, if \( q(x) \in \mathcal{P}(\Omega) \) satisfy \( p(x) \leq q(x) \leq P^*(x) := \frac{N p(x)}{N - k p(x)} \) a.e. \( x \in \Omega \) then there exists a continuous embedding \( W^{k,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega) \).

**Theorem 2.2** ([40]). Assume \( \Omega \) be a open domain in \( \mathbb{R}^N \), with cone property. If \( p(x) : \overline{\Omega} \to \mathbb{R} \) is uniform continuous and satisfy \( 1 < p^- \leq p^+ \leq \frac{N}{k} \) then for any measurable function \( q(x) \) defined in \( \Omega \) with \( p(x) \leq q(x), \) a.e. \( x \in \Omega \) and \( \text{essinf}_{x \in \Omega} (p^*(x) - q(x)) > 0 \) there is a continuous embedding \( W^{k,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega) \).

**Theorem 2.3** ([40]). Assume that \( \Omega \) be a open domain in \( \mathbb{R}^N \) with cone property, If \( \Omega \) is bounded, \( p(x) \in \mathcal{C}(\overline{\Omega}) \) and \( q(x) \) is the same as in theorem 2.2 then there is a continuous compact embedding \( W^{k,p(x)}(\Omega) \cap L^{q(x)}(\Omega) \).

**Theorem 2.4** ([43]). Suppose that \( p : \mathbb{R}^N \to \mathbb{R} \) is a uniformly continuous and radially symmetric function satisfying \( 1 < p^- \leq p^+ < N \) then, for any measurable function \( \alpha : \mathbb{R}^N \to \mathbb{R} \) with \( p(x) \ll \alpha(x) \ll p^*(x) \) \( \forall x \in \mathbb{R}^N \), there be a compact imbedding \( W^{1,p(x)}(\mathbb{R}^N) \cap L^{\alpha(x)}(\mathbb{R}^N) \), where \( W^{1,p(x)}(\mathbb{R}^N) := \{ u \in W^{1,p(x)}(\mathbb{R}^N) : u \text{ is radially symmetric}. \} \).

**Theorem 2.5** ([43]). If \( p : \mathbb{R}^N \to \mathbb{R} \) is a uniformly continuous and satisfies \( 1 < p^- \leq p^+ < N \) then for any measurable function \( \alpha \) with \( \overline{p} \ll \alpha \ll \overline{p}^* \), \( x \in \mathbb{R}^N \) we have the compact imbedding \( W^{1,p(x)}(\mathbb{R}^N) \cap L^{\alpha(x)}(\mathbb{R}^N) \). where \( \overline{p}(x) = \int_G p(y(x))d\mu(g), \forall x \in \mathbb{R}^N, G = O(N) \) be the orthogonal group on \( \mathbb{R}^N, \) \( \mu \) be a Haar measure on the compact group \( G \), and \( \mu(G) = 1 \).

**Theorem 2.6** ([43]). Let \( G \) be a subgroup of \( O(N) \) and \( \Omega \) be a invariant open subset in \( \mathbb{R}^N \) compatible with \( G, p : \mathbb{R}^N \to \mathbb{R} \) be \( G \)-invariant and uniformly continuous such that \( 1 < p^- \leq p^+ < N \) holds. Then for any
measurable function \( \alpha \) with \( p \ll \alpha \ll p^* \), \( x \in \Omega \), we have the compact imbedding \( W^{1,p(x)}_{0,\alpha}(\Omega) \odot L^{\alpha(x)}(\Omega) \). where \( W^{1,p(x)}_{0,\alpha}(\Omega) := \{ u \in W^{1,p(x)}_{0}(\Omega) : u \text{ is } G\text{-invariant} \} \).

### 2.3 Notations and Preliminary Results

Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^n, (n \geq 2) \). we use the standard notation for the Generalize Lebesgue and Generalize Sobolev spaces \( L^p(x)(\Omega) \) and \( W^{k,p(x)}(\Omega) \); in particular we will denote by \( \| \cdot \|_p \) and \( \| \cdot \|_{k,p} \) the corresponding norms. The Lebesgue measure of a measurable set \( A \subset \mathbb{R}^n \) will be denoted by \( |A| \), whereas a ball of radius \( R \) will be denoted by \( B_R \) and all balls mentioned in a single proposition will always be assumed to be concentric; moreover if \( u : \Omega \to \mathbb{R}, k \in \mathbb{R} \) and \( B_R \) is a ball strictly contained in \( \Omega \), we set

\[
\begin{align*}
A(k, R) &= \{ x \in B_R : u(x) > k \} \\
M(u, R) &= \sup_{B_R} u \\
m(u, R) &= \inf_{B_R} u \\
osc(u, R) &= M(u, R) - m(u, R)
\end{align*}
\]

Finally, for all \( k \in \mathbb{R} \) and \( R > 0 \) we set

\[
\begin{align*}
\int_{B_R} u \, dx &= \frac{1}{|B_R|} \int_{B_R} u \, dx \\
\int_{A(k, R)} u \, dx &= \frac{1}{|B_R|} \int_{A(k, R)} u \, dx
\end{align*}
\]

we will denote by the same letter \( C \) (or \( C(\cdots) \) to stress the dependence on some arguments) several constants, whose value may change from line to line.

In the proof of the article we shall use the following Lemma which can be found in [18], [33], [31].

**Lemma 2.7** (Moser iteration inequality). Let \( \{x_i\}_i \) be a sequence of positive real numbers such that \( x_0 \leq C^{-1/\beta}B^{-1/\beta^2}, \ x_{i+1} \leq CB^i x_i^{1+\beta} \) where \( \beta > 0, C > 0, B > 1 \). then \( x_i \to 0 \) as \( i \to +\infty \).

**Lemma 2.8** (Poincaré inequality [18]). Let \( u \) be a function in \( W^{1,p}_{0}(\mathbb{R}^n) \) then

\[
\int_{\mathbb{R}^n} |u|^p dx \leq \prod_{i=1}^{n} \left( \int_{\mathbb{R}^n} |u|^p dx \right)^{\frac{1}{p}} \leq \frac{n}{np} (2.4)
\]

holds for \( 1 \leq p \leq n \), where \( p^* = \frac{np}{n-p} \).
Lemma 2.9 (Sobolev-Poincaré inequality\[33\]). For any given bounded domain $\Omega$, if $p(x) \in L^\infty(\Omega)$, $u(x) \in W^{1,p(x)}_0(\Omega)$ then
\[
\int_{\Omega} |u(x)|^{p(x)} \, dx \leq C \int_{\Omega} |\nabla u(x)|^{p(x)} \, dx
\]
(2.5)
where $C$ is a constant depended on $\Omega$.

Lemma 2.10 (Sobolev-Poincaré inequality with variable exponent\[31\]). Assume that $a \in W^{1,s}(\Omega)$, $s > n$ satisfies $1 < p_0 < a(x) < q_0 \leq p^*_0$; then for every $M > 0$ there exists a positive radius $R_1 = R_1(M, s, n, \|a\|_{1,s})$ such that for every $\gamma > \frac{1}{n} - \frac{1}{s}$ there exist two positive constants $\chi = \chi(n, p_0, s, \gamma, \|a\|_{1,s})$ and $C = C(n, p_0, q_0)$ for which
\[
\left( \int_{B_R} \left( \frac{u}{R} \right)^{a(x)\frac{n}{n-1}} \, dx \right)^{\frac{n-1}{n}} \leq c \int_{B_R} |Du|^{a(x)} \, dx + \chi \{x \in B_R : |u| > 0\}^\gamma
\]
(2.6)
holds for every $B_R \subset \Omega$ with $0 < R < R_1$ and every $u \in W^{1,p_0}(B_R)$ such that $\int_{B_R} |Du|^{a(x)} \, dx < +\infty$, $\sup_{B_R} u = 0$ on $\partial B_R$. 

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Chapter 3

On the First Eigenvalue Problem of $p(x)$-Laplace Equation

3.1 Introduction

Suppose $\Omega$ is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial \Omega$. For certain given $p(x) \in P(\Omega)$, where $1 < p(x) \leq p^* < +\infty (p^* = \frac{np}{(n-p)})$ and $p(x)$ is continuous in $\Omega$ and the partial differential of $p(x)$, $\frac{\partial p}{\partial x_i}$ is bounded a.e. in $\Omega$. Thinking the eigenvalue problem of $p(x)$-Laplace equation $(E)_{\lambda}$ as following:

\[
\begin{cases}
-\Delta_{p(x)} u(x) + a(x)|u(x)|^{p(x)-2}u(x) = \lambda b(x)|u(x)|^{p(x)-2}u(x) & x \in \Omega \\
u(x) = 0 & x \in \partial \Omega
\end{cases}
\]

where

\begin{align*}
-\Delta_{p(x)} u(x) &= -\text{Div}(p(x)|\nabla u(x)|^{p(x)-2}\nabla u(x)) \\
a(x) &\in L^\infty_+(\Omega) = \{f \in L^\infty(\Omega) | f(x) \geq 0 \text{ a.e. } x \in \Omega\} \\
b(x) &\in L^\infty_0(\Omega) = \{f \in L^\infty(\Omega) | f^+(x) = \max\{f(x), 0\} \neq 0\}
\end{align*}

We say that $u(x)$ is the solution of the eigenvalue problem $(E)_{\lambda}$ if $u(x) \in W_0^{1,p(x)}(\Omega)$ satisfies the equation in the general sense, that is to say, for any given function $h(x) \in C_0^\infty(\Omega)$ there stands

\[
\int_{\Omega} -\text{Div}(p(x)|\nabla u(x)|^{p(x)-2}\nabla u(x))h(x)dx \\
+ \int_{\Omega} a(x)|u(x)|^{p(x)-2}u(x)h(x)dx \\
= \lambda \int_{\Omega} b(x)|u(x)|^{p(x)-2}u(x)h(x)dx
\]
In the generalized sobolev space, we study the eigenvalue of the equation. By Moser iteration, we obtain some properties about the solution of eigenvalue problem \((E)_\lambda;\) boundary, Hölder continuity, and so on.

### 3.2 On the Local Boundary of Eigenvalue Problem

#### 3.2.1 Gradient Estimate of Solution

**Lemma 3.1** (Caccioppoli Inequality on Solution). Suppose \(u(x)\) is the solution of \((E)_\lambda,\) \(p(x)\) satisfies \(1 < p \leq p(x) \leq p^* < +\infty\) in a certain open set \(\omega\) included in \(\Omega,\) then for any given spherical neighborhood \(B_R\) in \(\omega\) and every \(0 < s < t < R < 1, \) \(k > 0\) there stands

\[
\int_{A(k,s)} |\nabla u(x)|^p dx \leq C \left( \int_{A(k,t)} \frac{|u(x) - k|^q}{t - s} dx + (1 + k^q)|A(k,R)| \right) \tag{3.5}
\]

where \(C = C(p, q, \lambda)\)

**Proof:** Let \(\zeta(x)\) be a cut-off function between \(B_s\) and \(B_t\) with \(|\nabla \zeta(x)| \leq \frac{2}{t - s},\) by using a test function \(\varphi = -\zeta(u - k)^+\) in equation \((?)\), one arrives at

\[
\int_{A(k,s)} |\nabla u(x)|^{p(\varphi)} dx \leq \int_{A(k,t)} |\nabla(\zeta(x)(u(x) - k)^+)|^{p(\varphi)} dx
\]

\[
= \int_{A(k,t)} |\nabla(\zeta(x)(u(x) - k))|^{p(\varphi)} dx
\]

\[
= \int_{A(k,t)} |\zeta(x)\nabla(u(x) - k) + (u(x) - k)\nabla\zeta(x)|^{p(\varphi)} dx
\]

\[
\leq C \left\{ \int_{A(k,t)} |\zeta(x)|^{p(\varphi)} |\nabla(u(x) - k)|^{p(\varphi)} dx \right. \\
+ \int_{A(k,t)} |(u(x) - k)\nabla\zeta(x)|^{p(\varphi)} dx \right\}
\]

\[
\leq C \left\{ \int_{A(k,t)} |\nabla u(x)|^{p(\varphi)} dx + \int_{A(k,t)} |(u(x) - k)\nabla\zeta(x)|^{p(\varphi)} dx \right\}
\]

\[
\leq C(p, q, \lambda) \left\{ \int_{A(k,t)} |u(x)|^{p(\varphi)} dx + \int_{A(k,t)} |(u(x) - k)\nabla\zeta(x)|^{p(\varphi)} dx \right\}
\]

the last inequality can be deduced from \((3.4)\) in which we take \(h(x) = u(x)\) such that

\[
\int_{\Omega} -\text{Div} \left( p(x) |\nabla u(x)|^{p(x) - 2}\nabla u(x) \right) u(x) dx
\]
Moreover, by applying Younger inequality and Sobolev embedding theorem, we have

\[
\int_{\Omega} \left\{ p(x) |\nabla u(x)|^{p(x)} + a(x) |u(x)|^{p(x)} \right\} u(x) dx = \lambda \int_{\Omega} b(x) |u(x)|^{p(x)} dx
\]

or

\[
\int_{\Omega} \left\{ p(x) |\nabla u(x)|^{p(x)} + a(x) |u(x)|^{p(x)} \right\} u(x) dx = \lambda \int_{\Omega} b(x) |u(x)|^{p(x)} dx
\]

since \( a(x), b(x) \in L^\infty(\Omega), 1 < p \leq p(x) \leq q \leq p^* < +\infty \), we get the result above.

Moreover, by \( |\nabla u(x)| \leq \frac{2}{t-s} \) we obtain

\[
\int_{A(k,s)} |\nabla u(x)|^{p(x)} dx 
\]

\[
\leq C(\lambda) \int_{A(k,t)} |u(x)|^{p(x)} dx + C \int_{A(k,t)} \left| \frac{u(x) - k}{t-s} \right|^{p(x)} dx 
\]

\[
\leq C(\lambda) \int_{A(k,t)} |u(x) - k + k|^{p(x)} dx + C \int_{A(k,t)} \left| \frac{u(x) - k}{t-s} \right|^{p(x)} dx 
\]

\[
\leq C(\lambda) \int_{A(k,t)} \left( |u(x) - k|^{p(x)} + |k|^{p(x)} \right) dx + C \int_{A(k,t)} \left| \frac{u(x) - k}{t-s} \right|^{p(x)} dx 
\]

\[
\leq C(\lambda) \left\{ \int_{A(k,t)} |k|^{p(x)} dx + C \int_{A(k,t)} \left| \frac{u(x) - k}{t-s} \right|^{p(x)} dx \right\} 
\]

applying for Younger inequality and Sobolev embedding theorem, we have

\[
\int_{A(k,s)} |\nabla u(x)|^{p} dx \leq \int_{A(k,s)} \left( 1 + |\nabla u(x)|^{p(x)} \right) dx 
\]

\[
\leq C(\lambda) \left\{ \int_{A(k,t)} |k|^{p(x)} dx + C \int_{A(k,t)} \left( |u(x) - k|^{p(x)} \right) dx \right\} + |A(k,s)| 
\]

\[
\leq C(\lambda) \left\{ \int_{A(k,t)} \left( 1 + \left| \frac{u(x) - k}{t-s} \right|^{q} \right) dx + (1 + k^q)|A(k,t)| \right\} 
\]

\[
\leq C(\lambda) \left\{ \int_{A(k,t)} \left( 1 + \left| \frac{u(x) - k}{t-s} \right|^{q} \right) dx + (1 + k^q)|A(k,t)| \right\} 
\]

**Corollary 3.2.** Suppose \( u(x) \) is the solution of \((E)_{\lambda}, p(x) \) satisfies \( 1 < p \leq p(x) \leq p^* < +\infty \) in a certain open set \( \omega \) included in \( \Omega \), then for any given spherical neighborhood \( B_R \) in \( \omega \) and every \( 0 < \rho < R < 1, k > 0 \) there stands

\[
\int_{A(k,\rho)} |\nabla u(x)|^{p(x)} dx \leq C \left( \int_{A(k,R)} \left| \frac{u(x) - k}{R - \rho} \right|^{p(x)} dx + |A(k,R)| \right) 
\]
and

\[
\int_{A(k,\rho)} |\nabla u(x)|^p dx \leq C \left( \int_{A(k,R)} \left| \frac{u(x) - k}{R - \rho} \right|^q dx + |A(k,R)| \right) \tag{3.8}
\]

where \( C = C(p, q, \lambda) \), \( 1 < p^* \leq p \leq p(x) \leq p < p^* < +\infty \).

### 3.2.2 Local Bounded of Solution

**Theorem 3.3** (Local Bounded on solution). The solution of \((E)_\lambda\) is local bounded, that is, for every spherical neighborhood \(B_R \subset \Omega \) and \( 0 < R < 1 \), there exist a certain given positive number \( k > 0 \) such that \( u(x) \leq k \) where \( p(x) \) satisfies \( 1 < p \leq p(x) \leq p < p^* < +\infty \).

**Proof:** For fixed \( B_R \subset \subset \Omega \) and \( R \leq 1, k \geq k_0 > 0 \) let \( \vartheta_h = \frac{R}{2} + \frac{R}{2n+1}, \) \( \vartheta_h = \frac{\vartheta_h + \vartheta_{h+1}}{2}, k_h = k(1 - \frac{1}{2^{n+1}}), h = 0, 1, \ldots \). Obviously \( \vartheta_h \) monotonously decreases to \( \frac{R}{2} \) and \( k_h \) monotonously increases to \( k \) as \( h \to +\infty \). Define a functional \( J_h \) as

\[
J_h = \int_{A(k_h, \vartheta_h)} |u(x) - k_h|^p dx
\]

Take a function \( \xi(t) \in C^1([0, +\infty]) \) such that \( 0 \leq \xi(t) \leq 1, |\xi(t)| \leq C(\text{constant}) \) and \( \xi(t) = 1 \) when \( 0 \leq t \leq \frac{1}{2} \), \( \xi(t) = 0 \) when \( t \geq \frac{3}{4} \). From above making the cutting function \( \xi_h(x) = \xi(\frac{2k}{R}(\frac{k|x|}{R} - \frac{h}{2})) \) by the Poincaré inequality(2.4) and Caccioppoli inequality(3.5) we have

\[
J_h = \int_{A(k_h, \vartheta_h)} |u(x) - k_h|^p dx
\]

\[
\leq \int_{A(k_h, \vartheta_h)} |(u(x) - k_h)\xi_h(x)|^p dx
\]

\[
= \int_{A(k_h, \vartheta_h)} |(u(x) - k_h)^+ \xi_h(x)|^p dx
\]

\[
\leq \left\{ \prod_{i=1}^n \left( \int_{A(k_h, \vartheta_h)} \left| \frac{\partial((u(x) - k_{h+1}^+ \xi_h(x))}{\partial x_i} \right|^p dx \right) \frac{1}{i^p} \right\}^{\frac{1}{p}}
\]

\[
\leq C \left\{ \prod_{i=1}^n \left( \int_{A(k_h, \vartheta_h)} \left| \frac{\partial u(x)}{\partial x_i} \right|^p dx + 2^{hp} \int_{A(k_h, \vartheta_h)} |u(x) - k_{h+1}|^p dx \right) \right\}^{\frac{1}{p}}
\]
\[
\begin{align*}
&\leq C(\lambda) \left\{ \prod_{i=1}^{n} \left( \int_{A(k_h, \vartheta_h)} \left| u(x) - k_{h+1} \right|^p dx \right) \right\}^\frac{1}{p} \\
&\quad + 2^{hp} \int_{A(k_h, \vartheta_h)} \left| u(x) - k_{h+1} \right|^p dx \left( 1 + k_{h+1}^{p^*} |A(k_{h+1}, \vartheta_h)| \right) \right\}^\frac{1}{p} \\
&\leq C(\lambda) \left\{ \prod_{i=1}^{n} \left( 2^{hp^*} J_h + (1 + k_{h+1}^{p^*}) |A(k_{h+1}, \vartheta_h)| \right) \right\}^\frac{1}{p} \\
&\quad + 2^{hp} \int_{A(k_h, \vartheta_h)} \left| u(x) - k_{h+1} \right|^p dx \right\}^\frac{1}{p} \\
&\text{or}
\end{align*}
\]

or
\[
J_{h+1} \leq C(\lambda) \left\{ \prod_{i=1}^{n} \left( 2^{hp^*} J_h + (1 + k_{h+1}^{p^*}) |A(k_{h+1}, \vartheta_h)| \right) \right\}^\frac{1}{p} \\
\]

but
\[
\begin{align*}
(k_{h+1} - k_h)^{p^*} |A(k_{h+1}, \vartheta_h)| &= \int_{A(k_h, \vartheta_h)} |k_{h+1} - k_h|^{p^*} dx \leq \int_{A(k_h, \vartheta_h)} |u(x) - k_h|^{p^{*+\tau}} dx \leq J_h
\end{align*}
\]

or
\[
|A(k_{h+1}, \vartheta_h)| \leq \left( \frac{2^{h+2}}{k} \right)^{p^*} J_h, k^{p^*} |A(k_{h+1}, \vartheta_h)| \leq (2^{h+2})^{p^*} J_h
\]

combining (3.9) and (3.10) we get
\[
\begin{align*}
J_{h+1} &\leq C(\lambda) \left\{ \prod_{i=1}^{n} \left( 2^{hp^*} J_h + (1 + \frac{1}{k^{p^*}})(2^{h+2})^{p^*} J_h \right) \right\}^\frac{1}{p} \\
&\quad + \left\{ \left( \frac{2^{h+2}}{k} \right)^{p^*} J_h \right\} \left( \frac{p^{*+\tau}}{p^*} \frac{J_h^{p^{*+\tau}}}{J_h^{p^*}} \right) \frac{1}{p} \\
&= C(\lambda) \left\{ 2^{hp^*} J_h + (1 + \frac{1}{k^{p^*}})2^{hp^*+2p^*} J_h + 2^{(h+2)(p^*+\tau)} \left( \frac{1}{k} \right)^{p^*+\tau} J_h \right\} \frac{1}{p} \\
&\leq C(\lambda, k) \left\{ 2^{hp^*} J_h + 2^{hp^*+2p^*} J_h + 2^{(h+2)(p^*+\tau)} J_h \right\} \frac{1}{p} \\
&\leq C(\lambda, k) 2^{hp^*} \frac{J_h^{p^*+\tau}}{J_h^{p^*+\tau}} \\
&= C(\lambda, k) \left( \frac{2^{p^*}}{p} \right)^h J_h \frac{p^{*+\tau}}{p^{*+\tau}} \\
&= C(\lambda, k) \left( \frac{2^{p^*}}{p} \right)^h J_h \frac{p^{*+\tau}}{p^{*+\tau}}
\end{align*}
\]
by choosing \( k > 1 \) such that

\[
J_0 = \int_{A(k, R)} \left| u(x) - \frac{k^2}{2} \right|^{p^*} \, dx \leq C(\lambda)^\frac{n}{n-1} (2^L)^{\frac{1}{n}} 
\]  

(3.11)

where \( I = \frac{(p^*)^2}{p} \), \( \eta = \frac{(p^*)^2 - p}{p} = \frac{p}{n-p} \). Then with the help of Moser iterative inequality Lemma 2.7, we have

\[
\lim_{h \to +\infty} J_h = 0 
\]  

(3.12)

or

\[
\int_{A(k, R)} |u(x) - k|^{p^*} \, dx = 0 
\]  

(3.13)

This shows that \( u(x) \leq k, x \in B_R \). This completes the proof of theorem 3.3.

Consequently, by the similar argument to \(-u(x)\) we can prove that \( u(x) \) is local bounded. then with the compact property of \( u(x) \) we get theorem 3.4 as following:

**Theorem 3.4 (Bounded on solution).** the solution of \((E)_\lambda\) is bounded.

### 3.3 Hölder Continuity of Solution

#### 3.3.1 Sobolev-Poincaré Inequality on Solution

**Lemma 3.5.** if \( p(x) \in P(\Omega) \) satisfies \( \partial p/\partial x_i \) is bounded almost everywhere in \( \Omega \), \( \sup u(x) \leq M, x \in B_R \), then there exist \( R_1 = R_1(M, n, p(x)) \) such that for every spherical neighborhood \( B_R \subset \Omega, 0 < R < R_1 \) and \( 1 < p \leq p(x) \leq p < p^* < +\infty, (x \in B_R) \) the solution of eigenvalue problem \((E)_\lambda\) satisfies

\[
\left( \frac{\int_{B_R} u(x)^{\frac{p(x)n}{n-1}} \, dx}{R^{\frac{n-1}{n}}} \right)^{\frac{n-1}{n}} \leq C(n, p, q) \int_{B_R} |\nabla u(x)|^{p(x)} \, dx + C(n)|B_R| 
\]  

(3.14)

Proof: In lemma 2.10 let \( \gamma = 1 \), notice that \( |\{x \in B_R : |u(x)| > 0\}| \leq |B_R| \) we can get the inequality easily.

#### 3.3.2 Harnack Inequality on Solution

**Theorem 3.6.** if \( p(x) \in P(\Omega) \) satisfies \( \partial p/\partial x_i \) is bounded almost everywhere in \( \Omega \), \( 1 < p \leq p(x) \leq p < p^* < +\infty, (x \in \omega \subset \Omega) \)\), \( \sup u(x) \leq M, x \in B_R \),
then there exist $R_1 = R_1(M, n, p(x))$ such that for every spherical neighborhood $B_R \subset \omega, 0 < R < R_1$ the solution of eigenvalue problem (E)$_\lambda$ $u(x)$ satisfies

$$\sup_{B_R} u(x) \leq CR^{p/q} \left( \frac{|A(0,R)|}{R^n} \right)^\beta \int_{A(0,R)} |u/R|^{p(x)}\, dx + R^n \right)^{1/q}$$

(3.15)

where $\beta > 0, \beta(\beta + 1) = \frac{1}{n}, C = C(n, p, q, p(x), M)$

Proof: Let $h < k, \frac{4}{k} \leq \rho < \sigma \leq R < R_1 < 1$ where $R_1$ is the same as Lemma 3.5. making the cutting function on $B_\rho$ such that $|\nabla \zeta(x)| \leq 4$.

By H"older inequality, we have

$$\int_{A(k,\sigma)} \left| \frac{u-k}{\rho} \right|^{p(x)}\, dx$$

$$\leq C(n, q) \left( \int_{B_R} \left| \frac{(u-k)^+ \zeta(x)}{R} \right|^{\frac{n+1}{n}}\, dx \right)^{\frac{n-1}{n}} \left( \frac{A(k, \rho)}{R^n} \right)^\frac{1}{n}$$

$$\leq C(n, p, q) \left( \int_{B_R} \left| \nabla ((u-k)^+ \zeta(x)) \right|^{p(x)}\, dx + \left| A \left( k, \frac{\sigma + \rho}{2} \right) \right| \left( \frac{A(k, \rho)}{R^n} \right)^\frac{1}{n} \right. \right.$$ (3.16)

However, by assumed conditions for $x \in A \left( k, \frac{\sigma + \rho}{2} \right)$ we have

$$|\nabla (u(x) - k)^+ \zeta(x)| \leq |\nabla u(x)| + 4 \left| \frac{u(x) - k}{\sigma - \rho} \right|$$

(3.17)

then it follows from (3.17) and (3.16) that

$$\int_{A(k,\rho)} \left| \frac{u-k}{\rho} \right|^{p(x)}\, dx$$

$$\leq C(n, p, q) \left( \frac{|A(k, \rho)|}{R^n} \right)^\frac{1}{n} \left\{ \int_{A(k, \frac{\sigma + \rho}{2})} \left( |\nabla u(x)|^{p(x)} + \left| \frac{u-k}{\sigma - \rho} \right|^{p(x)} \right)\, dx \right.$$ (3.18)
combining (3.18) and Lemma 3.2 we have
\[
\iint_{A(k, \rho)} \left| \frac{u - k}{\rho} \right|^{p(x)} dx \leq C(n, p, q) \left( \frac{|A(k, \rho)|}{R^n} \right)^{\frac{1}{n}} \left\{ \iint_{A(k, \sigma)} \left| \frac{u - k}{\sigma - \rho} \right| dx + |A(k, \sigma)| \right\}
\]
moreover when \( h < k \)
\[
|A(k, \rho)| \leq \frac{\sigma^p}{(k - h)^q} \iint_{A(h, \sigma)} \left| \frac{u - h}{\sigma - \rho} \right| dx
\]
\[
\iint_{A(k, \sigma)} \left| \frac{u - k}{\sigma} \right|^{p(x)} dx \leq \iint_{A(h, \sigma)} \left| \frac{u - k}{\sigma} \right|^{p(x)} dx
\]
therefore, from (3.19) and (3.21)
\[
\iint_{A(k, \rho)} \left| \frac{u - k}{\rho} \right|^{p(x)} dx \leq C(n, p, q) \left( \frac{|A(h, \rho)|}{R^n} \right)^{\frac{1}{n}} \left\{ \iint_{A(h, \sigma)} \left| \frac{u - h}{\sigma - \rho} \right|^{p(x)} dx + \frac{\sigma^p}{(k - h)^q} \iint_{A(h, \sigma)} \left| \frac{u - h}{\sigma - \rho} \right|^{p(x)} dx \right\}
\]
noticing that
\[
\left( \frac{|A(k, \rho)|}{R^n} \right)^{\beta} \leq \left( C(n) \frac{\sigma^p}{(k - h)^q} \iint_{A(h, \sigma)} \left| \frac{u - h}{\sigma - \rho} \right|^{p(x)} dx \right)^{\beta}
\]
if we multiply the two sides of (3.23) by
\[
\iint_{A(k, \rho)} \left| \frac{u - k}{\rho} \right|^{p(x)} dx
\]
and
\[
C(n, p, q) \left( \frac{|A(h, \rho)|}{R^n} \right)^{\frac{1}{n}} \left\{ \iint_{A(h, \sigma)} \left| \frac{u - h}{\sigma - \rho} \right|^{p(x)} dx + \frac{\sigma^p}{(k - h)^q} \iint_{A(h, \sigma)} \left| \frac{u - h}{\sigma - \rho} \right|^{p(x)} dx \right\}
we get
\[
\left( \frac{|A(k, \rho)|}{R^n} \right)^\beta \int_{A(k, \rho)} \left| \frac{u - k}{\rho} \right|^p \, dx
\leq C(n, p, q) \left( \frac{|A(h, \rho)|}{R^n} \right)^{\frac{1}{n}} \left( \frac{\sigma^p}{(k - h)q} \right) \int_{A(h, \sigma)} \left| \frac{u - h}{\sigma} \right|^p \, dx
\]
\[
\left\{ \int_{A(h, \sigma)} \left| \frac{u - h}{\sigma - \rho} \right|^p \, dx + \frac{\sigma^p}{(k - h)q} \int_{A(h, \sigma)} \left| \frac{u - h}{\sigma - \rho} \right|^p \, dx \right\}^{1+\beta}
\]
\[
(3.24)
\]
Now we take \( \sigma = R_i = \frac{R_i}{2} + \frac{R_i}{2i+1}, \rho = R_i+1, h = k_i = dR_i^\gamma (1 - \frac{1}{2^i}), k = k_{i+1} \)
for every \( i \in \mathbb{N} \) and some \( d \in \mathbb{R} \) to be chosen later. Taking into account that
\[
k_{i+1} - k_i = \frac{dR_i^\gamma}{2i+1}, R_i - R_{i+1} = \frac{R_i}{2i+1} \] from (3.24) we have
\[
\left( \frac{|A(k_{i+1}, R_{i+1})|}{R^n} \right)^\beta \int_{A(k_{i+1}, R_{i+1})} \left| \frac{u - k_{i+1}}{R_{i+1}} \right|^p \, dx
\leq C \frac{2^{(1+\beta)q_i}}{d^q} \left( 1 + \frac{R^n}{d^q} \right) \left( \frac{|A(k_i, R_i)|}{R^n} \right)^{\frac{1}{n}} \left( \int_{A(k_i, R_i)} \left| \frac{u - k_i}{R_i} \right|^p \, dx \right)^{1+\beta}
\]
where \( C = C(n, p, q) \). let
\[
\varphi(k, \rho) = \left( \frac{|A(k, \rho)|}{R^n} \right)^\beta \int_{A(k, \rho)} \left| \frac{u - k}{\rho} \right|^p \, dx
\]
we have
\[
\varphi(k_{i+1}, R_{i+1}) \leq C(n, p, q) \frac{2^{(1+\beta)q_i}}{d^q} \left( 1 + \frac{R^n}{d^q} \right) \varphi^{1+\beta}(k_i, R_i)
\]
(3.25)
choosing \( d \) such that \( d^q \leq R^n \) and \( \varphi(k_0, R_0) = \varphi(0, R) \leq C(n, p, q)2^{(1+\beta)q_i} \) \( d^q \) from (3.24) and Lemma 2.7 we get
\[
\lim_{i \to +\infty} \varphi(k_i, R_i) = \varphi(dR_i^\gamma, \frac{R_i}{2}) = 0
\]
(3.26)
taking \( d = R^n + C(n, p, q)\varphi(0, R) \) we deduce the desired result.

**Theorem 3.7.** The weak solution of \((E)_\lambda\) is local Hölder continuous.

Proof: by the same proof with \(-u(x)\) and notice that \( \int_{B_R} |u(x)|^{p(x)} \, dx \) is bounded with \( p < p(x) < q \) we can get the estimate of \( u(x) \) on \( B_R \)
\[
\text{osc}(u(x), \frac{R}{2}) \leq C(n, p, q, M)R^\beta
\]
(3.27)
where \( M = \sup_{B_R} u(x) \).

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### 3.4 On the First Eigenvalue

#### 3.4.1 Comparison Principle

**Lemma 3.8** (Comparison Principle). Let \( F(x, u) : \Omega \times \mathbb{R}^1 \to \mathbb{R}^1 \) be measurable in \( x \) and monotone nondecreasing in \( u \), let \( u_1, u_2 \in W^{1,p(x)}(\Omega) \) satisfies

\[
-\Delta_{p(x)}u_1 + F(x, u_1) \leq -\Delta_{p(x)}u_2 + F(x, u_2)
\]

in \( W^{-1,p'(x)}(\Omega) \), \( p'(x) = p(x)/p(x) - 1 \) Then \( u_1 \leq u_2 \) on \( \partial \Omega \) implies \( u_1 \leq u_2 \) in \( \Omega \).

**Proof:** Let \( \omega(x) = \max(u_1 - u_2, 0) \) by \( u_1, u_2 \in W^{1,p(x)}(\Omega) \), \( \omega(x) \in W^{1,p(x)}(\Omega) \). Multiplying (3.28) by \( \omega \) and using the monotonicity of \( F(x, u) \), we have

\[
\int_{\Omega} (-\Delta_{p(x)}u_1(x) + F(x, u_1(x))) \omega(x)dx \\
\leq \int_{\Omega} (-\Delta_{p(x)}u_2(x) + F(x, u_2(x))) \omega(x)dx \\
\int_{\Omega} -\Delta_{p(x)}u_1(x)\omega(x)dx + \int_{\Omega} F(x, u_1(x))\omega(x)dx \\
\leq \int_{\Omega} -\Delta_{p(x)}u_2(x)\omega(x)dx + \int_{\Omega} F(x, u_2(x))\omega(x)dx
\]

or

\[
\int_{\{x \in \Omega: u_1(x) \geq u_2(x)\}} -\Delta_{p(x)}u_1(x)\omega(x)dx + \int_{\{x \in \Omega: u_1(x) \geq u_2(x)\}} F(x, u_1(x))\omega(x)dx \\
\leq \int_{\{x \in \Omega: u_1(x) \geq u_2(x)\}} -\Delta_{p(x)}u_2(x)\omega(x)dx + \int_{\{x \in \Omega: u_1(x) \geq u_2(x)\}} F(x, u_2(x))\omega(x)dx \\
\int_{\{x \in \Omega: u_1(x) \geq u_2(x)\}} -\Delta_{p(x)}u_1(x)\omega(x)dx \\
\leq \int_{\{x \in \Omega: u_1(x) \geq u_2(x)\}} -\Delta_{p(x)}u_2(x)\omega(x)dx
\]

according to the definition of \(-\Delta_{p(x)}u(x)\) it follows from above that

\[
\int_{\partial D} p(x)(|\nabla u_1(x)|^{p(x)-2}\nabla u_1(x) - |\nabla u_2(x)|^{p(x)-2}\nabla u_2(x))(\nabla u_1(x) - \nabla u_2(x))dx \leq 0
\]

(3.29)

where \( D = \{x \in \Omega : u_1(x) \geq u_2(x)\} \) but

\[
(|\nabla u_1(x)|^{p(x)-2}\nabla u_1(x) - |\nabla u_2(x)|^{p(x)-2}\nabla u_2(x))(\nabla u_1(x) - \nabla u_2(x)) \geq 0
\]

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hence,
\[ \nabla u_1(x) = \nabla u_1(x), \quad x \in \{ x \in \Omega : u_1(x) \geq u_2(x) \} \]

it’s means \( \nabla \omega(x) = 0 \) or \( u_1(x) = u_2(x) \) when \( x \in \{ x \in \Omega : u_1(x) \geq u_2(x) \} \) which implies \( u_1(x) \leq u_2(x), x \in \Omega \).

**Lemma 3.9 (Extremum Principle).** If \( u(x) \in W^{1,p(x)}(\Omega) \cap C^1(\overline{\Omega}) \) satisfies
\[
\begin{align*}
-\Delta_{p(x)} u(x) + M u^{p(x)-1}(x) &\geq 0, \quad \text{in } W^{-1,p(x)}(\Omega), M \geq 0 \\
u(x) > 0, &\quad x \in \Omega \\
u(x) = 0, &\quad x \in \partial \Omega 
\end{align*}
\]

Then the outer normal derivative \( \frac{\partial u}{\partial n} \) of \( u \) is strictly negative on \( \partial \Omega \).

Proof: for any given \( x_0 \in \partial \Omega \) and a sufficiently small \( R > 0 \), there exists \( y \in \Omega \) such that \( B_R(y) \subset \Omega \) and \( x_0 \in \partial B_{2R}(y) \cap \partial \Omega \) where \( B_R(z) =: \{ x \in R^n : |z - x| < R \} \). Set
\[ v(x) = \alpha(3R - r)^{\delta} - \alpha R^{\delta}, \quad r = |x - y| \]

for fixed \( \delta \) taking sufficiently small \( \alpha, R \) such that
\[
-\Delta_{p(x)} v(x) + M v^{p(x)-1}(x) \leq 0, \quad x \in \Omega_R \quad (3.30)
\]
\[ v(x) \leq u(x), \quad x \in \partial \Omega_R \quad (3.31)\]

where \( \Omega_R = B_{2R}(y) \setminus \overline{B_R(y)} \). Now we prove that \( (3.30) \) and \( (3.31) \) is valid. when \( x \in \partial B_{2R}(y), \) \( v(x) \leq u(x) \) is trivial. noticing that \( v(x) = (2^{\delta - 1})^{\alpha R^{\delta}}, x \in \partial B_R(y), u(x) > 0, x \in \partial B_R \), we can get \( (3.31) \) by taking sufficiently small \( \alpha, R \).

For \( (3.30) \) according to the chain rule of differential,
\[
-\operatorname{Div}(p(x)\nabla (\alpha(3R - r)^{\delta} - \alpha R^{\delta})^{p(x)-1}\nabla (\alpha(3R - r)^{\delta} - \alpha R^{\delta}))
\]
\[ = (\alpha \delta (3R - r)^{\delta-1})^{p(x)-1} \sum_{i=1}^{n} \frac{\partial p(x)}{\partial r} \frac{\partial r}{\partial x_i} \]
\[ + (\alpha \delta (3R - r)^{\delta-1})^{p(x)-1} \ln \alpha \delta (3R - r)^{\delta-1} \sum_{i=1}^{n} \frac{\partial p(x)}{\partial r} \frac{\partial r}{\partial x_i} \]
\[ - p(x)(\alpha \delta (3R - r)^{\delta-1})^{p(x)-1} \frac{p(x) - 1}{\alpha \delta (3R - r)^{\delta-1} - 1} \alpha \delta (\delta - 1)(3R - r)^{\delta-2} \]
\[ + p(x)(\alpha \delta (3R - r)^{\delta-1})^{p(x)-1} \frac{n}{r} - \frac{1}{r^2} \]
\[ \equiv H_1 + H_2 + H_3 + H_4 \]

By assumed conditions, \( \frac{\partial p(x)}{\partial r} \) and \( \frac{\partial r}{\partial x_i} \) are both bounded for every \( i = 1, \ldots, n \). hence \( H_1 \leq C_1(\alpha \delta (3R - r)^{\delta-1})^{p(x)-1} \leq C_2(\alpha \delta R^{\delta-1})^{p(x)-1} \) taking sufficient small \( R \) such that \( \ln \alpha \delta (3R - r)^{\delta-1} \leq 0, \frac{n}{r} - \frac{1}{r^2} \leq 0 \) then
we get $H_2 \leq 0, H_4 \leq 0$. because $p(x)$ is bounded in $\Omega$, we have $H_3 \leq -C_3(\alpha \delta R^\delta - 1)p(x-1)^{\delta - 1} 3R-r$. And noticing that $Mv(x)p(x-1) = M(\alpha \delta (3R - r_\xi)^{\delta - 1}(2R - r))p(x-1)$ where $r \leq r_\xi \leq 2R$ we also have $Mv(x)p(x-1) \leq M_1(\alpha \delta R^\delta - 1)p(x-1)$ stands for sufficiently small $2R - r$. All the conditions shown above imply that

\[-\Delta_{p(x)}v(x) + M_1p(x-1)(x)\]

\[\leq C_2(\alpha \delta R^\delta - 1)p(x-1) - C_3(\alpha \delta R^\delta - 1)p(x-1) \frac{\delta - 1}{3R-r} + M_1(\alpha \delta R^\delta - 1)p(x-1)\]

Let

\[C_2(\alpha \delta R^\delta - 1)p(x-1) - C_3(\alpha \delta R^\delta - 1)p(x-1) \frac{\delta - 1}{3R-r} + M_1(\alpha \delta R^\delta - 1)p(x-1) \leq 0\]

then we get

\[C_4 \frac{\delta - 1}{3R-r} \geq C_2 + M_1\]

it stands for sufficiently small $R$. For fixed $\delta$ we prove that (3.30) and (3.31) are valid. according to Lemma 3.8 we get the desired result.

### 3.4.2 The Eigenvalue Problem of Solution

We give some definitions about $(E)_\lambda$ as following

**Definition 3.10.** the first eigenvalue of $(E)_\lambda$ is

\[
\frac{1}{\lambda_1} = \sup \left\{ R(v) := \frac{B(v)}{A(v)} \left| v \in W =: W_0^{1,p(x)}(\Omega) \backslash \{0\} \right. \right\}.
\]  

(3.32)

where

\[A(v) = \int_\Omega \left\{ |\nabla v(x)|^{p(x)} + \frac{a(x)}{p(x)}|v(x)|^{p(x)} \right\} dx\]

\[B(v) = \int_\Omega \frac{b(x)}{p(x)}|v(x)|^{p(x)} dx\]

**Theorem 3.11** (Boundedness of the First Eigenvalue). For the first eigenvalue, there exists $C_1, C_2 > 0$ such that $C_1 < \lambda_1 < C_2$.

proof: suppose $B(u)$ is no positive for all $u \in W_0^{1,p(x)}(\Omega)$, then there exists a function sequence $f_n$ in $W_0^{1,p(x)}(\Omega)$ such that $f_n(x) \leq 0$ and $f_n(x) \to b^+(x) := \max\{b(x), 0\}, (n \to \infty)$ which implies $B(b^+(x)) \leq 0$ so $b^+ \equiv 0$ it contradict with the definition of $b(x)$ therefore there must be exist a function $u_0(x)$ such that

\[B(u_0(x)) > 0\]

(3.33)
or

\[ 0 < \lambda_1 < \frac{1}{R(u_0(x))} \quad (3.34) \]

On the other hand, by Lemma 2.9 and the definition of the first eigenvalue we have

\[ \frac{1}{\lambda_1} \leq C(\Omega)\|b(x)\|_{L^\infty} \quad (3.35) \]

So we have

\[ \frac{1}{C(\Omega)\|b(x)\|_{L^\infty}} \leq \lambda_1 \leq \frac{1}{R(u_0(x))} \quad (3.36) \]

**Theorem 3.12.** There exists \( u \in W_0^{1,p(x)} \) so that \( J_{\lambda_1}(u) = 0 \) implies \( u \) is the solution of eigenvalue problem \((E)_{\lambda_1}\), where

\[ J_{\lambda_1}(u) = \int_\Omega \left( |\nabla u(x)|^{p(x)} + \frac{a(x)}{p(x)} |u(x)|^{p(x)} \right) dx - \lambda_1 \int_\Omega \frac{b(x)}{p(x)} |u(x)|^{p(x)} dx \]

Proof: make the Fréchet derivation of \( J_{\lambda_1}(u) \) we have

\[
J_{\lambda_1}'(u) = \frac{d}{du} \left\{ \int_\Omega \left( |\nabla u(x)|^{p(x)} + \frac{a(x)}{p(x)} |u(x)|^{p(x)} \right) dx - \lambda_1 \int_\Omega \frac{b(x)}{p(x)} |u(x)|^{p(x)} dx \right\}
\]

\[
= \lim_{t \to 0} \frac{\int_\Omega |\nabla (u(x) + th(x))|^{p(x)} dx - \int_\Omega |\nabla u(x)|^{p(x)} dx}{th(x)} \\
+ \lim_{t \to 0} \frac{\int_\Omega \frac{a(x)}{p(x)} u(x) + th(x)|^{p(x)} dx - \int_\Omega \frac{a(x)}{p(x)} u(x)|^{p(x)} dx}{th(x)} \\
- \lambda_1 \lim_{t \to 0} \frac{\int_\Omega \frac{b(x)}{p(x)} u(x) + th(x)|^{p(x)} dx - \int_\Omega \frac{b(x)}{p(x)} u(x)|^{p(x)} dx}{th(x)}
\]

\[ = G_1 + G_2 + G_3 = 0 \]

where \( h(x) \in C^\infty_0(\Omega) \) and

\[
G_1 = \lim_{t \to \infty} \frac{\int_\Omega |\nabla (u(x) + th(x))|^{p(x)} dx - \int_\Omega |\nabla u(x)|^{p(x)} dx}{th(x)} \\
= \int_\Omega \lim_{t \to \infty} \frac{1}{h(x)} \left( |\nabla (u(x) + th(x))|^{p(x)} - |\nabla u(x)|^{p(x)} \right) dx \\
= \frac{1}{h(x)} \int_\Omega \frac{d|\nabla (u(x) + th(x))|^{p(x)}}{dt} \bigg|_{t=0} dx \\
= \frac{1}{h(x)} \int_\Omega p(x) |\nabla u(x)|^{p(x)-2} \nabla u(x) \nabla h(x) dx \\
= \frac{1}{h(x)} \int_\Omega -Div \left( p(x) |\nabla u(x)|^{p(x)-2} \nabla u(x) \right) h(x) dx
\]

(3.37)
\[ G_2 = \lim_{t \to 0} \int_{\Omega} \frac{a(x)|u(x) + th(x)|^{p(x)} dx}{th(x)} - \int_{\Omega} \frac{a(x)|u(x)|^{p(x)} dx}{th(x)} \]
\[ = \frac{1}{h(x)} \int_{\Omega} a(x)|u(x)|^{p(x)-1} h(x) dx \]  
(3.38)

\[ G_3 = \lim_{t \to 0} \int_{\Omega} \frac{b(x)|u(x) + th(x)|^{p(x)} dx}{th(x)} - \int_{\Omega} \frac{b(x)|u(x)|^{p(x)} dx}{th(x)} \]
\[ = \frac{1}{h(x)} \int_{\Omega} b(x)|u(x)|^{p(x)-1} h(x) dx \]  
(3.39)

Combining (3.37), (3.38) and (3.39) we get
\[
\int_{\Omega} - \text{Div} \left( p(x)|\nabla u(x)|^{p(x)-2} \nabla u(x) \right) h(x) dx + \int_{\Omega} a(x)|u(x)|^{p(x)-1} h(x) dx 
\]
\[ = \lambda_1 \int_{\Omega} b(x)|u(x)|^{p(x)-1} h(x) dx \]

therefore, \( u(x) \) is the solution of eigenvalue problem \((E)_{\lambda_1}\). Noticing that \( J_{\lambda_1}(u(x)) = J_{\lambda_1}(|u(x)|) \) and \( W_0^{1,p(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega) \) is a continuously compact imbedding, we obtain the existence of nonnegative solution about the eigenvalue problem \((E)_{\lambda_1}\).

**Theorem 3.13.** The first eigenvalue \( \lambda_1 \) is simple, that is to say, the set of solutions is \( \{ tu(x) : t \in \mathbb{R} \} \)

Proof: the sufficient is obvious. Let \( u_1, u_2 \) be two solutions of \((E)_{\lambda_1}\) and \( M(t,x) = \max(u_1, tu_2), \min(u_1, tu_2). \) because
\[
1 = \sup \{ R(v) := \frac{B(v)}{A(v)} : v \in W := W_0^{1,p(x)}(\Omega) \} \geq \frac{B(M)}{A(M)} 
\]
therefore
\[ J_{\lambda_1}(M) = A(M) - \lambda_1 B(M) \geq 0 \]  
(3.40)

In the same like,
\[ J_{\lambda_1}(m) = A(m) - \lambda_1 B(m) \geq 0 \]  
(3.41)

Now we show that \( M, m \) are the solutions of \((E)_{\lambda_1}\) or \( J_{\lambda_1}(M) = J_{\lambda_1}(m) = 0. \)
By the definition of $J_{\lambda_1}$ it’s easy to see that

$$J_{\lambda_1}(M) + J_{\lambda_1}(m) = A(M) - \lambda_1 B(M) + A(m) - \lambda_1 B(m)$$

$$= \int_{\Omega} \left( |\nabla M|^{p(x)} + a |M|^{p(x)} \right) dx - \lambda_1 \int_{\Omega} b |M|^{p(x)} dx$$

$$+ \int_{\Omega} \left( |\nabla m|^{p(x)} + a |m|^{p(x)} \right) dx - \lambda_1 \int_{\Omega} b |m|^{p(x)} dx$$

$$= \int_{I_{u_1}} \left( |\nabla M|^{p(x)} + a |M|^{p(x)} \right) dx - \lambda_1 \int_{I_{u_1}} b |M|^{p(x)} dx$$

$$+ \int_{I_{u_1}} \left( |\nabla m|^{p(x)} + a |m|^{p(x)} \right) dx - \lambda_1 \int_{I_{u_1}} b |m|^{p(x)} dx$$

$$+ \int_{I_{u_2}} \left( |\nabla M|^{p(x)} + a |M|^{p(x)} \right) dx - \lambda_1 \int_{I_{u_2}} b |M|^{p(x)} dx$$

$$+ \int_{I_{u_2}} \left( |\nabla m|^{p(x)} + a |m|^{p(x)} \right) dx - \lambda_1 \int_{I_{u_2}} b |m|^{p(x)} dx$$

$$= \int_{\Omega} \left( |\nabla u_1|^{p(x)} + a |u_1|^{p(x)} \right) dx - \lambda_1 \int_{\Omega} b |u_1|^{p(x)} dx$$

$$+ \int_{\Omega} \left( |\nabla u_2|^{p(x)} + a |u_2|^{p(x)} \right) dx - \lambda_1 \int_{\Omega} b |u_2|^{p(x)} dx$$

$$= J_{\lambda_1}(u_1) + J_{\lambda_1}(u_2) = 0$$

where $I_{u_1} = \{ x \in \Omega : u_1(x) \geq u_2(x) \}$, $I_{u_2} = \{ x \in \Omega : u_1(x) < u_2(x) \}$ according to (3.40) and (3.41). $u_1, u_2$ be two solutions of $(E)_{\lambda_1}$. From theorem 3.7 we have $M \in C^{1,\beta}(\Omega)$ for all $t \geq 0$.

For certain given $x_0 \in \Omega$, we take $t_0 = \frac{u_1(x_0)}{u_2(x_0)}$. As we know, for every vector $e$ there stands

$$u_1(x_0 + he) - u_1(x_0) \leq \max(u_1(x_0 + he), t_0 u_2(x_0 + he)) - u_1(x_0)$$

$$= M(t_0, x_0 + he) - M(t_0, x_0)$$

therefore, the partial derivative of $u_1(x)$ and $M(t, x)$ at $x_0$ satisfies

$$\frac{\partial u_1(x_0)}{\partial x_i} = \lim_{h \to 0^+} \frac{u_1(x_0 + he_i) - u_1(x_0)}{h}$$

$$\geq \lim_{h \to 0^+} \frac{M(x_0 + he_i) - u_1(x_0)}{h} = \frac{\partial M(x_0)}{\partial x_i}$$

$$\frac{\partial u_1(x_0)}{\partial x_i} = \lim_{h \to 0^-} \frac{u_1(x_0 + he_i) - u_1(x_0)}{h}$$

$$\leq \lim_{h \to 0^-} \frac{M(x_0 + he_i) - u_1(x_0)}{h} = \frac{\partial M(x_0)}{\partial x_i}$$

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\[
\frac{\partial u_1(x_0)}{\partial x_i} = \frac{\partial M(x_0)}{\partial x_i} \tag{3.42}
\]

where \( e_i (i = 1, 2, \cdots, n) \) are the unit normal vectors. Moreover, we have \( \nabla_x u_1(x_0) = \nabla_x M(t_0, x_0) \), where \( \nabla_x = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n} \right) \). By the same way, we obtain \( \nabla_x t_0 u_2(x_0) = \nabla_x M(t_0, x_0) \). Hence, the gradient of \( u_1 t_0 u_2 \) at \( x_0 \) is

\[
\nabla_x \left( \frac{u_1(x_0)}{t_0 u_2(x_0)} \right) = \frac{u_2(x_0) \nabla_x u_1(x_0) - u_1(x_0) \nabla_x u_2(x_0)}{u_1(x_0) - u_2(x_0)} = 0
\]

By the arbitrary of \( x_0 \), we get

\[
\frac{u_1(x)}{u_2(x)} \equiv \text{const, } x \in \Omega \tag{3.43}
\]

**Theorem 3.14.** \((E)_\lambda \) has no solution for \( \lambda > \lambda_1 \).

Proof: By theorem 3.12, we need only verify it for the positive solution. Let \( u, \upsilon \) be positive solutions of \((E)_{\lambda_1}\) and \((E)_{\lambda}\) respectively. Assume \( b(x) \geq 0 \), from above we can select some solutions \( u, \upsilon \) such that \( u \leq \upsilon \) for all \( x \in \Omega \).

Then we deduce that there must exist \( 0 < \eta < 1 \) so that

\[
-\Delta_{p(x)} u + a u^{p(x) - 1} \leq -\Delta_{p(x)} (\eta \upsilon) + a (\eta \upsilon)^{p(x) - 1} \tag{3.44}
\]

By the definition of solution and \( u \leq \upsilon \), because for all \( h \in C_0^\infty(\Omega) \) and \( h > 0 \)

\[
\int_{\Omega} -\text{Div}(p(x)|\nabla u|^{p(x) - 2} \nabla u) h dx + \int_{\Omega} a |u|^{p(x) - 2} u h dx
\]

\[
= \lambda_1 \int_{\Omega} b |u|^{p(x) - 2} u h dx
\]

\[
\leq \lambda_1 \int_{\Omega} b |v|^{p(x) - 2} v h dx
\]

\[
= \frac{\lambda_1}{\lambda} \int_{\Omega} -\text{Div}(p(x)|\nabla v|^{p(x) - 2} \nabla v) h dx + \frac{\lambda_1}{\lambda} \int_{\Omega} a |v|^{p(x) - 2} v h dx
\]

We need only to prove that

\[
\frac{\lambda_1}{\lambda} \int_{\Omega} |\nabla \upsilon|^{p(x) - 2} \nabla \upsilon h dx + \frac{\lambda_1}{\lambda} \int_{\Omega} \frac{a}{p(x)|\upsilon|^{p(x) - 2} h dx
\]

\[
\leq \int_{\Omega} |\nabla (\eta \upsilon)|^{p(x) - 2} \nabla (\eta \upsilon) h dx + \int_{\Omega} \frac{a}{p(x)|\eta \upsilon|^{p(x) - 2} h dx
\]

for certain \( 0 < \eta < 1 \). We obtain the desired result by taking \( \inf \eta^{p(x) - 1} \geq \frac{\lambda_1}{\lambda} \). Therefore applying for Lemma 3.8 \( u \leq \eta \upsilon \) in \( \Omega \). Repeating this procedure,
we deduce that \( u \leq \eta^n v \) in \( \Omega \) for all \( n \in \mathbb{N} \), which follows \( u \equiv 0 \). This is a contradiction.

For general case, let \( B^+(x) =: \max(b(x), 0) \) and \( b^-(x) =: \max(-b(x), 0) \). Then above result implies the equation

\[
-\Delta_{p(x)} \omega(x) + \{a(x) + \lambda b^-(x)\} \omega(x)^{p(x)-1} = \mu b^+(x) \omega^{p(x)-1}
\]

has a nontrivial positive solution \( \omega \) if

\[
\mu \leq \mu_1 = \lambda_1(a(x) + \lambda b^-(x), b^+(x))
\]

and

\[
I_{\mu_1}(\omega(x)) = A(\omega(x)) + \lambda \int_{\Omega} \frac{b^-(x)}{p(x)} |\omega(x)|^{p(x)} dx - \mu_1 \int_{\Omega} \frac{b^+(x)}{p(x)} |\omega(x)|^{p(x)} dx
= \min \{ I_{\mu_1}(z(x)); z(x) \in W \}
= 0
\]

Since \( v \) is a positive solution of the above equation with \( \mu_1 = \lambda \) we deduce that \( \lambda \leq \mu_1 \) and

\[
J_\lambda(v) = I_\lambda(v) \geq I_{\mu_1}(v) = \min \{ I_{\mu_1}(z(x)); z(x) \in W \} = 0
\]

However,

\[
J_\lambda(u) = J_{\lambda_1}(u) - (\lambda - \lambda_1)B(u) < 0
\]

for all \( u \in W_0^{1,p(x)} \). This is a contradiction.

**Remark**

In this chapter, we give many properties of the solutions of \((E)_\lambda\) in the sense of weak. The most important part we discussed are the boundedness and Hölder continuity of the weak solutions, which are also important to weak solution. Therefore, the eigenvalue study in the partial differential operators, in essence, includes the solution research. Finally, we show that the conditions restrict to \( p(x) \) are not necessary, it can be replaced by \( 1 < p \leq p(x) \leq q < +\infty \).
Chapter 4

Conclusion

In this paper we introduced certain $p(x)$-Laplace operator under the generalized Sobolev Space and proved some conclusions as following:

We first proved the existence, boundary and Hölder continuity of the solutions about the $p(x)$-Laplace equation which generalizes the result of M.ôtani and T.Teshima and in the same time we obtained two inequalities: the Caccioppoli inequality and the Harnack inequality about the equation with $p(x)$ exponent which are two basic conclusions in the research of equations with nonstandard exponent condition.

On the other hand, about the eigenvalue of $p(x)$-Laplace equation we showed some properties: All eigenvalues are bounded. Solutions about the first eigenvalue $\lambda_1$ is simple i.e. all the nontrivial solutions form an one dimension subset of the space $W^{1,p(x)}_0(\Omega)$. The equation has no solution with $\lambda > \lambda_1$. Furthermore, we show that the outer norm vector of the positive solutions is strictly negative at the boundary $\partial \Omega$ and got the comparison principle. Consequently, by the similar argument we generalized the result of M.ôtani and T.Teshima.

From the view of this point, how about the second eigenvale, the third, · · · and the solution about them? this is our aftetimes works.
In this article, my supervisor, YongQiang Fu, has given me so many advices that I can finish it completely. I give my best grateful thanks to him. Some relative problem about the generalized Sobolev Space has been studied in several articles written by him.
Bibliography

[1] O.Kováčik and J.Rákosník. On Spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$. Czechoslovak Math. J. 1991, 41(116): 592-618

[2] P.Maracellini. Regularity and Existence of Solutions of Elliptic Equations with p,q-growth Conditions. J. Differential Equations. 1991, 1(50): 1-30

[3] S. Samko. Convolution and Potential Typeoperators in $L^{p(x)}(R^n)$. Integral Transform. Spec. Funct. 1998, 3-4(7):261-284

[4] S. Samko. Convolution Type Operators in $L^{p(x)}$. Integral Transform. Spec. Funct. 1998, 7(1-2): 123-144

[5] I.Isharapudinov. On the Topology of the Space $L^{p(x)}[0,1]$. Matem. Zametki. 1978, 4(26): 613-632

[6] V.V.Zhikov. Averaging of Functionals of the Calculus of Variations and Elasticity theory. Math. USSR Izvestiya. 1987, 1(29): 33-66

[7] Ondrej Kovacik,Zilina and Jiri Rakosnik,Praha. on Spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$

[8] E. Acerbi and G. Mingione. Regularity Results for a class of Functionals with Non-standard Growth. Arch. Ration. Mech. Anal. 2001, (156): 121-140

[9] E. Acerbi and G. Mingione. Regularity Results for Stationary Electro-rheological Fluids. Arch. Ration. Mech. Anal. 2002, (164): 213-259

[10] E. Acerbi and G. Mingione. Regularity Results for Electro-rheological Fluids. the Stationary case C. R. Acad. Sci. Paris Ser. 2002, (334): 817-822

[11] E.Acerbi and N.Fusco. Semicontinuity problems in the calculus of variations. Arch. Rational Mech. Anal. 86(1984), 125-135

[12] A. Coscia and G. Mingione. Holder Continuity of the Gradient of $p(x)$-harmonic Mappings. C. R. Acad. Sci. Paris Ser. 1999, I(328): 363-368
[13] D. Cruz-Uribe, A. Fiorenza and C. J. Neugebauer. The Maximal Function on Variable $L^p$ Spaces. Ann. Acad. Sci. Fenn. Ser. A I Math. 2003, (28): 223-238

[14] D. E. Edmunds and J. Lang and A. Nekvinda. On $L^{p(\cdot)}$ Norms R. Soc. Lond. Proc. Ser. Amath. Phys. Eng. Sci. 1999, 1981(455): 219-225

[15] D. E. Edmunds and A. Meskhi. Potential-type Operators in $L^{p(\cdot)}$ Spaces. Z. Anal. Anwendungen. 2002, 3(21): 681-690

[16] D. E. Edmunds and J. Rakosnik. Density of Smooth Functions in $W^{k,p(\cdot)}$. Proc. Roy. Soc. London. Ser. 1992, A(437): 229-236

[17] D. E. Edmunds and J. Rakosnik. Sobolev Embedding with Variable Exponent. Studia Math. 2000, (143): 267-293

[18] D. E. Edmunds and J. Rakosnik. Sobolev Embedding with Variable Exponent, II. Math. Nachr. 2002, (246-247): 53-67

[19] L. Diening. Maximal Function on Generalized Lebesgue Spaces $L^p$. Math. Inequal. Appl. to appear

[20] L. Diening. Riesz Potential and Sobolev Embeddings of Generalized Lebesgue Spaces $L^{p(\cdot)}$ and Sobolev Spaces $W^{k,p(\cdot)}$ and Math. Nachr. to appear

[21] L. Diening and M. Ruzicka. Calderon-Zygmund Operators on Generalized Lebesgue Spaces $L^{p(\cdot)}$ and Problems related to Fluid Dynamics. preprint

[22] L. Diening and M. Ruzicka. Integral Operators on the Halfspace in Generalized Lebesgue Spaces $L^{p(\cdot)}$. preprint

[23] M. Ruzicka. Electro-rheological Fluids. Modeling and Mathematical Theory. Springer Verlag, Berlin. 2000

[24] P. Harjulehto and P. Hasto. A Capacity Approach to the Poincaré Inequality and Sobolev Embedding in Variable Exponent Sobolev Space. Rev. Mat. Comput. to appear

[25] P. Harjulehto and P. Hasto. Lebesgue Points in Variable Exponent Spaces. preprint

[26] P. Harjulehto, P. Hasto and M. Koskenoja. The Dirichlet Energy Integral on Intervals in Variable Exponent Sobolev Spaces, preprint

[27] J. Heinonen, T. Kilpelainen and O. Martio. Nonlinear Potential Theory of Degenerate Elliptic Equations. Oxford Mathematical Monographs, Oxford University Press, Oxford, 1993
[28] A. Fiorenza. A Mean Continuity Type Results for Certain Sobolev Spaces with Variable Exponent. Commun. Contemp. Math. 2002, 3(4): 587-605

[29] T.C.Halsey. Electroheological fluids. Science. 1992, (258):761-766

[30] Yu.A.Alkhutov. the Harnack Inequality and the Hölder Property of Solutions of Nonlinear Elliptic Equations with a Nonstandard Growth Conditions. Differential Equations. 1997,33(12):1653-1662

[31] Valeria ChiadòPiat and Alessandra Coscia. Hölder Continuity of Minimizers of Functionals with Variable Growth Exponent. Manuscripta Math. 1997, (93):283-299

[32] J. Rakosnik. Sobolev Inequality with Variable Exponent in Function Spaces. Differential Operators and Nonlinear Analysis (Syote, Finland). 1999: 220-228

[33] Yongqiang Fu. the Existence of Solutions for Elliptic Systems with Nonuniform Growth. Studia Mathematica. 2002, 151(3):227-246

[34] M.ótani and T.Teshima. on the First Eigenvalue of Some Quasilinear Elliptic Equations. Proc.Japan.Acad. 1998,Ser.A(64):8-10

[35] Carlo Sbordone and Nicola Fusco. some Remarks on the Regularity of Minima of Anisotropic Integrals. Commun.in Partial Differential Equations. 1993, (18):153-161

[36] A. Nekvinda: Hardy-Littlewood maximal operator on $L^{p(\cdot)}(\mathbb{R}^n)$. Math. Inequal. Appl. to appear.

[37] V. Kokilashvili and S. Samko. Maximal and fractional operators in weighted spaces $L^{p(\cdot)}(\mathbb{R}^n)$. Rev.Mat.Iberoamericana, to appear.

[38] X.Fan and Zhao Dun. the local $C^{1,\alpha}$ regularity of solution for $p(x)$-Laplace equation. Journal of Gansu Education College. 2001,15(2):1-5

[39] X. Fan. On the positive solutions of $p(x)$-Laplace equation. Journal of Gansu Education College. 2001,15(1):1-3

[40] X. Fan, J. Shen and D.Zhao. Sobolev Embedding Theorems for Spaces $W^{k,p(x)}(\Omega)$. J. Math. Anal. Appl. 2001, (262): 749-760

[41] X. Fan and D. Zhao. The Quasi-minimizer of Integral Functionals with $m(x)$ Growth Conditions. Nonlinear Anal. 2000, 39: 807-816

[42] X. Fan and D. Zhao. On the Spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$. J. Math. Anal. Appl. 2002, (263): 424-446

33
[43] X. Fan, Y. Zhao and D. Zhao. Compact Imbedding Theorems with Symmetry of Strass-Lions Type for the Space $W^{k,p(x)}(\Omega)$. J. Math. Anal. Appl. 2001, (255): 333-348

[44] X. Fan. Existence of Positive Solutions for $p(x)$-Laplace Equations. Journal of Northwest Minorities University(Natural Science). 2000,(21):1-4

[45] X.Fan. A Strong Maximum Principle for $p(x)$-Laplace equation. Chinese Journal of Contemporary Mathematics, 2003,(24):3:495-500

[46] X.L. Fan, Q.H.Zhang Existence of solutions for $p(x)$-Laplacian Dirichlet problem Nonlinear Analysis 2003,(52):1843C1852

[47] A.Adams. “Sobolev Space” (Qixiao Ye et al.,trans). People’s Education Publishing House,Beijing,1983[in Chinese].

[48] Chen.Ya.Zhe. “second order elliptic equation and equation system” science Press, Beijing, 1991[in Chinese]