ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO SCHRÖDINGER EQUATIONS NEAR AN ISOLATED SINGULARITY OF THE ELECTROMAGNETIC POTENTIAL

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Abstract. Asymptotics of solutions to Schrödinger equations with singular magnetic and electric potentials is investigated. By using an Almgren type monotonicity formula, separation of variables, and an iterative Brezis-Kato type procedure, we describe the exact behavior near the singularity of solutions to linear and semilinear (critical and subcritical) elliptic equations with an inverse square electric potential and a singular magnetic potential with a homogeneity of order $-1$.

1. Introduction

In quantum mechanics, the hamiltonian of a non-relativistic charged particle in an electromagnetic field has the form $(-i\nabla + \mathbf{A})^2 + V$, where $V : \mathbb{R}^N \to \mathbb{R}$ is the electric potential and $\mathbf{A} : \mathbb{R}^N \to \mathbb{R}^N$ is a magnetic potential associated to the magnetic field $B = \text{curl} \mathbf{A}$. For $N = 2, 3$, “curl” denotes the usual curl operator, whereas for $N > 3$ by $B = \text{curl} \mathbf{A}$ we mean the 2-form $(B_{jk})$ with $B_{jk} := \partial_j A_k - \partial_k A_j$, where $\mathbf{A} = (A_j)_{j=1,...,N}$. Linear and nonlinear elliptic equations associated to electromagnetic hamiltonians have been the object of a wide recent mathematical research; we quote, among others, [2, 7, 8, 9, 10, 17].

In this paper we are concerned with singular homogeneous electromagnetic potentials $(\mathbf{A}, V)$ which make the operator invariant by scaling, namely of the form

$$\mathbf{A}(x) = \frac{\mathbf{A}(x/|x|)}{|x|} \quad \text{and} \quad V(x) = -\frac{a(x/|x|)}{|x|^2}$$

in $\mathbb{R}^N$, where $N \geq 2$, $\mathbf{A} \in C^1(S^{N-1}, \mathbb{R}^N)$, and $a \in L^\infty(S^{N-1}, \mathbb{R})$. A prototype in dimension 2 is given by potentials associated to thin solenoids: if the radius of the solenoid tends to zero while the flux through it remains constant, then the particle is subject to a $\delta$-type magnetic field, which is called Aharonov-Bohm field. A vector potential associated to the Aharonov-Bohm magnetic field in $\mathbb{R}^2$ has the form

$$\mathbf{A}(x_1, x_2) = \alpha \left(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2}\right), \quad (x_1, x_2) \in \mathbb{R}^2,$$

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with $\alpha \in \mathbb{R}$ representing the circulation of $\mathcal{A}$ around the solenoid. We notice that the potential in (1) is singular at 0, homogeneous of degree $-1$ and satisfies the following transversality condition

$$\mathcal{A}(\theta) \cdot \theta = 0 \quad \text{for all } \theta \in \mathbb{S}^{N-1}.$$ 

We refer to [3, 15, 23] for properties of Aharonov-Bohm magnetic potentials and related Hardy inequalities. In the present paper, we consider, for $N \geq 2$, a larger class of singular vector potentials, characterized by the presence of a homogeneous isolated singularity of order $-1$ and by the transversality (or Poincaré) condition (we address the reader to [16] and [26, §8.4.2] for details about the transversal or Poincaré gauge). Such a class includes, for $N = 2$, the Aharonov-Bohm magnetic potential (1). The Aharonov-Bohm potential in dimension $N = 3$ is singular on a straight line and is not covered by the analysis performed here, which only allows treating isolated singularities. In a forthcoming paper, we will extend the present results to potentials with cylindrical singularity including the 3-dimensional Aharonov-Bohm case.

Singular homogeneous electric potentials which scale as the laplacian arise in nonrelativistic molecular physics, where the interaction between an electric charge and the dipole moment $\mathbf{D} \in \mathbb{R}^N$ of a molecule is described by an inverse square potential with an anisotropic coupling strength of the form

$$V(x) = -\frac{\lambda(x \cdot \mathbf{d})}{|x|^3}$$

in $\mathbb{R}^N$, where $\lambda > 0$ is proportional to the magnitude of the dipole moment $\mathbf{D}$ and $\mathbf{d} = \mathbf{D}/|\mathbf{D}|$ denotes the orientation of $\mathbf{D}$, see [12, 13, 21]. We notice that the above electric potential is singular at 0 and homogeneous of degree $-2$.

We aim to describe the asymptotic behavior near the singularity of solutions to equations associated to the following class of Schrödinger operators with singular homogeneous electromagnetic potentials:

$$\mathcal{L}_{\mathcal{A}, a} := \left(-i \nabla + \frac{\mathcal{A}(x/|x|)}{|x|} \right)^2 - a \frac{x}{|x|^2}.$$

We study both linear and nonlinear equations obtained as perturbations of the operator $\mathcal{L}_{\mathcal{A}, a}$ in a domain $\Omega \subset \mathbb{R}^N$ containing either the origin or a neighborhood of $\infty$. More precisely, we deal with linear equations of the type

$$\mathcal{L}_{\mathcal{A}, a} u = h(x) u, \quad \text{in } \Omega,$$

where $h \in L^\infty_{\text{loc}}(\Omega \setminus \{0\})$ is negligible with respect to the inverse square potential $|x|^{-2}$ near the singularity, and semilinear equations

$$\mathcal{L}_{\mathcal{A}, a} u(x) = f(x, u(x))$$

with $f$ having at most critical growth.

Regularity properties of solutions to Schrödinger equations with less singular magnetic and electric potentials have been studied by several authors. In particular, in [7], boundedness and decay at $\infty$ of solutions are proved in dimensions $N \geq 3$ for $L^2_{\text{loc}}$ magnetic potentials and electric potentials with $L^{N/2}$ negative part. It is also worth quoting [18] and [17], where, in dimensions $N \geq 3$, local boundedness and, respectively, a unique continuation property are established under the assumption that the electric potential and the square of the magnetic one belong to the Kato class. In [18] continuity of solutions is also obtained under restricted assumptions on the potentials.
Due to the presence of a more strong singularity which keeps potentials in $L^A_{a}$ out of the Kato class, it is natural to expect that solutions to equations (2) and (3) behave singularly at the origin: our purpose is to describe the rate and the shape of the singularity of solutions, by relating them to the eigenvalues and the eigenfunctions of a Schrödinger operator on the sphere $S^{N-1}$ corresponding to the angular part of $L_{A,a}$.

As remarked in [11, 13] for the case $A = 0$ (i.e. no magnetic vector potential), the estimate of the behavior of solutions to elliptic equations with singular potentials near the singularities has several important applications to the study of spectral properties of the associated Schrödinger operator, such as essential self-adjointness, positivity, etc. In [12], the exact asymptotic behavior near the singularity of solutions to Schrödinger equations with singular dipole-type electric potentials is established, using separation of variables combined with a comparison method. Comparison and maximum principles play a crucial role also in [24], where the existence of the limit at the singularity of any quotient of two positive solutions to Fuchsian type elliptic equations is proved. In the presence of a singular magnetic potential, comparison methods are no more available, preventing us from a direct extension of the results of [12, 24]. This difficulty is overcome by a Almgren type monotonicity formula (see [1, 14]) and blow-up methods which allow avoiding the use of comparison methods.

1.1. Assumptions and functional setting. As already mentioned, we shall deal with electromagnetic potentials $(A, V)$ in $\mathbb{R}^N$, $N \geq 2$, satisfying the following assumptions:

(A.1) $A(x) = \frac{A(\frac{x}{|x|})}{|x|}$ and $V(x) = -\frac{a(\frac{x}{|x|})}{|x|^2}$ (homogeneity)

(A.2) $A \in C^1(S^{N-1}, \mathbb{R}^N)$ and $a \in L^\infty(S^{N-1}, \mathbb{R})$ (regularity of angular coefficients)

(A.3) $A(\theta) \cdot \theta = 0$ for all $\theta \in S^{N-1}$. (transversality)

Under the transversality assumption (A.3), the operator $L_{A,a}$ acts on functions $u : \mathbb{R}^N \to \mathbb{C}$ as

$$L_{A,a}u = -\Delta u - \frac{a(\frac{x}{|x|}) - |A(\frac{x}{|x|})|^2 + i \text{div}_{S^{N-1}} A(\frac{x}{|x|})}{|x|^2} u - \frac{A(\frac{x}{|x|})}{|x|} \cdot \nabla u,$$

where $\text{div}_{S^{N-1}} A$ denotes the Riemannian divergence of $A$ on the unit sphere $S^{N-1}$ endowed with the standard metric.

The positivity properties of the Schrödinger operator $L_{A,a}$ are strongly related to the first eigenvalue of the angular component of the operator on the sphere $S^{N-1}$. More precisely, the positivity of the quadratic form associated to $L_{A,a}$ is ensured under the assumption

(A.4) $\mu_1(A,a) > -\left(\frac{N-2}{2}\right)^2$, (positive definiteness),

see Lemma [2.2] where $\mu_1(A,a)$ is the first eigenvalue of the angular component of the operator on the sphere $S^{N-1}$, i.e. of the operator

$$L_{A,a} := (-i \nabla_{S^{N-1}} + A)^2 - a.$$

When dealing with the nonlinear problem [3] we introduce the stronger condition
\( (A.5) \quad \mu_1(0, a) > -\left(\frac{N-2}{2}\right)^2. \)

From the diamagnetic inequality it follows that \( \mu_1(0, a) \leq \mu_1(A, a) \) with equality holding if and only if \( \text{curl} A = 0 \) in the sense of distributions, see Lemma \((A.2)\). In particular the assumption \((A.5)\) is in general stronger than \((A.4)\).

The spectrum of the angular operator \( L_{A, a} \) is discrete and consists in a nondecreasing sequence of eigenvalues

\[ \mu_1(A, a) \leq \mu_2(A, a) \leq \cdots \leq \mu_k(A, a) \leq \cdots \]

diverging to \( +\infty \), see Lemma \((A.5)\) in the Appendix. Condition \((A.4)\) is fundamental to introduce a proper functional setting in which to frame our analysis. Let us define \( D^{1,2}_r(\mathbb{R}^N, \mathbb{C}) \) as the completion of \( C_c^\infty(\mathbb{R}^N \setminus \{0\}, \mathbb{C}) \) with respect to the norm

\[ (4) \quad \|u\|_{D^{1,2}_r(\mathbb{R}^N, \mathbb{C})} := \left( \int_{\mathbb{R}^N} \left( |\nabla u(x)|^2 + \frac{|u(x)|^2}{|x|^2} \right) dx \right)^{1/2}. \]

It is easy to verify that

\[ D^{1,2}_r(\mathbb{R}^N, \mathbb{C}) = \left\{ u \in L^1_{\text{loc}}(\mathbb{R}^N \setminus \{0\}, \mathbb{C}) : \frac{u}{|x|} \in L^2(\mathbb{R}^N, \mathbb{C}) \text{ and } \nabla u \in L^2(\mathbb{R}^N, \mathbb{C}^N) \right\}. \]

The following lemma ensures that, under assumption \((A.4)\), the space \( D^{1,2}_r(\mathbb{R}^N, \mathbb{C}) \) coincides with the Hilbert space originated by the quadratic form \( Q_{A, a} \) associated to the operator \( L_{A, a} \)

\[ (5) \quad Q_{A, a} : D^{1,2}_r(\mathbb{R}^N, \mathbb{C}) \to \mathbb{R}, \quad Q_{A, a}(u) := \int_{\mathbb{R}^N} \left[ \left( \nabla + i \frac{A(x/|x|)}{|x|} \right) u(x) \right]^2 - \frac{a(x/|x|)}{|x|^2} |u(x)|^2 \, dx. \]

**Lemma 1.1.** Assume that \( N \geq 2 \) and \((A.2)\), \((A.3)\), \((A.4)\) hold. Then

i) \[ \inf_{u \in D^{1,2}_r(\mathbb{R}^N, \mathbb{C}) \setminus \{0\}} \frac{Q_{A, a}(u)}{\int_{\mathbb{R}^N} |x|^{-2}|u(x)|^2 \, dx} > 0 \]

ii) \[ Q_{A, a} \text{ is positive definite in } D^{1,2}_r(\mathbb{R}^N, \mathbb{C}), \text{i.e.} \quad \inf_{u \in D^{1,2}_r(\mathbb{R}^N, \mathbb{C}) \setminus \{0\}} \frac{Q_{A, a}(u)}{\|u\|_{D^{1,2}_r(\mathbb{R}^N, \mathbb{C})}^2} > 0 \]

iii) \[ D^{1,2}_r(\mathbb{R}^N, \mathbb{C}) = D^{1,2}_{A, a}(\mathbb{R}^N), \text{ where } D^{1,2}_{A, a}(\mathbb{R}^N) \text{ is the completion of } C_c^\infty(\mathbb{R}^N \setminus \{0\}, \mathbb{C}) \]

with respect to the norm

\[ \|u\|_{D^{1,2}_{A, a}(\mathbb{R}^N)} := (Q_{A, a}(u))^{1/2}. \]

Moreover the norms \( \|\cdot\|_{D^{1,2}_r(\mathbb{R}^N, \mathbb{C})} \) and \( \|\cdot\|_{D^{1,2}_{A, a}(\mathbb{R}^N)} \) are equivalent.

In any open bounded domain \( \Omega \subset \mathbb{R}^N \) containing 0, we introduce the functional space \( H^1_*(\Omega, \mathbb{C}) \) as the completion of

\[ \{ u \in H^1(\Omega, \mathbb{C}) \cap C^\infty(\Omega, \mathbb{C}) : u \text{ vanishes in a neighborhood of } 0 \}. \]
with respect to the norm
\[ \|u\|_{H^1_2(\Omega, \mathbb{C})} = \left( \|\nabla u\|_{L^2(\Omega, \mathbb{C}^N)}^2 + \|u\|_{L^2(\Omega, \mathbb{C})}^2 + \left\| \frac{u}{|x|} \right\|_{L^2(\Omega, \mathbb{C})}^2 \right)^{1/2}. \]

It is easy to verify that
\[ H^1_2(\Omega, \mathbb{C}) = \left\{ u \in H^1(\Omega, \mathbb{C}) : \frac{u}{|x|} \in L^2(\Omega, \mathbb{C}) \right\}. \]

If \( N \geq 3, H^1_2(\Omega, \mathbb{C}) = H^1(\Omega, \mathbb{C}) \) and their norms are equivalent, as one can easily deduce from the Hardy type inequality with boundary terms due to [27] (see [31]) and continuity of Sobolev trace imbeddings. On the other hand, if \( N = 2, H^1_2(\Omega, \mathbb{C}) \) is strictly smaller than \( H^1(\Omega, \mathbb{C}) \).

For any \( h \) satisfying
\[ h \in L^\infty_{\text{loc}}(\Omega \setminus \{0\}, \mathbb{C}), \quad |h(x)| = O(|x|^{-2+\varepsilon}) \quad \text{as} \quad |x| \to 0 \quad \text{for some} \quad \varepsilon > 0, \]
we introduce the notion of weak solution to (2): we say that a function \( u \in H^1_2(\Omega, \mathbb{C}) \) is a \( H^1_2(\Omega, \mathbb{C}) \)-weak solution to (2) if, for all \( w \in H^1_2(\Omega, \mathbb{C}) \) such that \( \frac{w}{|x|} \in L^2(\Omega, \mathbb{C}) \),
\[ \mathcal{Q}_{\Lambda, \sigma}^\Omega(u, w) = \int_\Omega h(x) u(x) \overline{w(x)} \, dx, \]
where \( \mathcal{Q}_{\Lambda, \sigma}^\Omega : H^1_2(\Omega, \mathbb{C}) \times H^1_2(\Omega, \mathbb{C}) \to \mathbb{C} \) is defined by
\[ \mathcal{Q}_{\Lambda, \sigma}^\Omega(u, w) := \int_\Omega \left( \nabla u(x) + i \frac{\Lambda(x)}{|x|} u(x) \right) \cdot \left( \nabla \overline{w(x)} + i \frac{\Lambda(x)}{|x|} \overline{w(x)} \right) \, dx - \int_\Omega \frac{a(x/|x|)}{|x|^2} u(x) \overline{w(x)} \, dx. \]

In an analogous way, we define the notion of weak solutions to (3) in a bounded domain for every Carathéodory function \( f : \Omega \times \mathbb{C} \to \mathbb{C} \) satisfying the growth restriction
\[ \left| \frac{f(x, z)}{z} \right| \leq \begin{cases} C_f (1 + |z|^{2^{*}-2}), & \text{if} \ N \geq 3, \\ C_f (1 + |z|^{p-2}), & \text{if} \ N = 2, \end{cases} \]
for a.e. \( x \in \Omega \) and for all \( z \in \mathbb{C} \setminus \{0\} \), where \( 2^{*} = \frac{2N}{N-2} \) is the critical Sobolev exponent and the constant \( C_f > 0 \) is independent of \( x \in \Omega \) and \( z \in \mathbb{C} \setminus \{0\} \): we say that a function \( u \in H^1_2(\Omega, \mathbb{C}) \) is a \( H^1_2(\Omega, \mathbb{C}) \)-weak solution to (3) if, for all \( w \in H^1_2(\Omega, \mathbb{C}) \) such that \( \frac{w}{|x|} \in L^2(\Omega, \mathbb{C}) \),
\[ \mathcal{Q}_{\Lambda, \sigma}^{\Omega}(u, w) = \int_\Omega f(x, u(x)) \overline{w(x)} \, dx. \]

Regularity of solutions either to (2) or to (3) outside the singularity follows from classical elliptic regularity theory, as described in the following remark.

**Remark 1.2.** If \( A \in C^1(S^{N-1}, \mathbb{R}^N), a \in L^\infty(S^{N-1}, \mathbb{R}), \) and \( h \in L^\infty_{\text{loc}}(\Omega \setminus \{0\}) \), then, from standard regularity theory and bootstrap arguments, it follows that any \( H^1_2(\Omega, \mathbb{C}) \)-weak solution \( u \) of (2) satisfies \( u \in W^{2,p}_{\text{loc}}(\Omega \setminus \{0\}) \) for any \( 1 \leq p < \infty \) and in particular \( u \in C^{1,\tau}_{\text{loc}}(\Omega \setminus \{0\}, \mathbb{C}) \) for any \( \tau \in (0, 1) \). The Brezis-Kato technique introduced in [4], standard regularity theory, and bootstrap arguments, lead to the same conclusion also for \( H^1_2(\Omega, \mathbb{C}) \)-weak solutions to (3) with \( f \) as in (7).
1.2. Statement of the main results. The following theorem provides a classification of the behavior of any solution \( u \) to \((2)\) near the singularity based on the limit as \( r \to 0^+ \) of the Almgren’s frequency function (see \([14]\))

\[
\mathcal{N}_{u,h}(r) = \frac{r \int_{B_r} \left[ \nabla u(x) + i\frac{A(x)}{|x|}u(x) \right]^2 - \frac{a(x/|x|)}{|x|^2} |u(x)|^2 - (Rh(x))|u(x)|^2 \, dx}{\int_{\partial B_r} |u(x)|^2 \, dS},
\]

where, for any \( r > 0 \), \( B_r \) denotes the ball \( \{ x \in \mathbb{R}^N : |x| < r \} \).

**Theorem 1.3.** Let \( \Omega \subset \mathbb{R}^N \), \( N \geq 2 \), be a bounded open set containing 0, \((A.1)\), \((A.2)\), \((A.3)\), \((A.4)\) hold, and \( u \) be a weak \( H^1_0(\Omega, \mathbb{C}) \)-solution to \((2)\), \( u \neq 0 \), with \( h \) satisfying \((6)\). Then, letting \( \mathcal{N}_{u,h}(r) \) as in \((3)\), there exists \( k_0 \in \mathbb{N} \), \( k_0 \geq 1 \), such that

\[
\lim_{r \to 0^+} \mathcal{N}_{u,h}(r) = -\frac{N-2}{2} + \sqrt{\left( \frac{N-2}{2} \right)^2 + \mu_{k_0}(A,a)}.
\]

Furthermore, if \( \gamma \) denotes the limit in \((9)\), \( m \geq 1 \) is the multiplicity of the eigenvalue \( \mu_{k_0}(A,a) \), and \( \{ \psi_i : j_0 \leq i \leq j_0 + m - 1 \} \) \((j_0 \leq k_0 \leq j_0 + m - 1)\) is an \( L^2(\mathbb{S}^{N-1}, \mathbb{C}) \)-orthonormal basis for the eigenspace of the operator \( L_{A,a} \) associated to \( \mu_{k_0}(A,a) \), then

\[
\lambda^{-\gamma} u(\lambda \theta) \longrightarrow \sum_{i=j_0}^{j_0+m-1} \beta_i \psi_i(\theta) \quad \text{in} \quad C^{1,\tau}(\mathbb{S}^{N-1}, \mathbb{C}) \quad \text{as} \quad \lambda \to 0^+,
\]

and

\[
\lambda^{1-\gamma} \nabla u(\lambda \theta) \longrightarrow \sum_{i=j_0}^{j_0+m-1} \beta_i \left( \gamma \psi_i(\theta) \theta + \nabla_{\mathbb{S}^{N-1}} \psi_i(\theta) \right) \quad \text{in} \quad C^{0,\tau}(\mathbb{S}^{N-1}, \mathbb{C}^N) \quad \text{as} \quad \lambda \to 0^+,
\]

for any \( \tau \in (0,1) \), where

\[
\beta_i = \int_{\mathbb{S}^{N-1}} \left[ R^{-\gamma} u(R\theta) + \int_0^R h(s \theta) u(s \theta) \left( s^{-\gamma} - \frac{s^{\gamma+N-1}}{R^{\gamma+N-2}} \right) ds \right] \psi_i(\theta) \, dS(\theta),
\]

for all \( R > 0 \) such that \( B_R = \{ x \in \mathbb{R}^N : |x| \leq R \} \subset \Omega \) and \( (\beta_{j_0}, \beta_{j_0+1}, \ldots, \beta_{j_0+m-1}) \neq (0,0,\ldots,0) \).

We notice that \((12)\) is actually a Cauchy’s integral type formula for \( u \) which allows retracing the behavior of \( u \) at the singularity from the values of \( u \) along any circle centered at 0, up to some term depending on the perturbation \( h \).

An application of Theorem 1.3 to the special case of Aharonov-Bohm magnetic fields in \( \mathbb{R}^2 \) of the form \( 0 \) is described in section 7.

Theorem 1.3 implies a strong unique continuation property as the following corollary states. Moreover, if \( \gamma > 0 \) (as e.g. it happens under assumption \((A.4)\) in dimension \( N = 2 \)) then the solutions to \((2)\) are Hölder continuous for \( 0 < \gamma < 1 \) and Lipschitz continuous for \( \gamma \geq 1 \).

**Corollary 1.4.** Suppose that all the assumptions of Theorem 1.3 hold true. Let \( \gamma \) denote the limit in \((9)\) and \( u \) be a weak \( H^1_0(\Omega, \mathbb{C}) \)-solution to \((2)\).

(i) If \( u(x) = O(|x|^k) \) as \( |x| \to 0 \) for all \( k \in \mathbb{N} \), then \( u \equiv 0 \) in \( \Omega \).

(ii) If \( 0 < \gamma < 1 \) then \( u \in C^{1,\gamma}_0(\Omega, \mathbb{C}) \).

(iii) If \( \gamma \geq 1 \) then \( u \) is locally Lipschitz continuous in \( \Omega \).
We notice that the unique continuation property proved in [17] for electromagnetic potentials in the Kato class does not contain the result stated in part (i) of Corollary [14] for singular homogeneous magnetic potentials. We also remark that the monotonicity argument used to prove Theorem [1.3] (see sections 5 and 6) actually applies when perturbing the magnetic homogeneous potential with a non singular term, namely with a magnetic potential of the form

\[
\mathcal{A}(x) = \frac{A(x)}{|x|^2} + b(x)
\]

where \(b \in C^1(\Omega \setminus \{0\}, \mathbb{C}^N)\) satisfies \(|b(x)| = O(|x|^{-1+\varepsilon})\) and \(|\nabla b(x)| = O(|x|^{-2+\varepsilon})\) as \(|x| \to 0\) for some \(\varepsilon > 0\) as \(|x| \to 0\). For sake of simplicity, we omit the details of case (13), which can be treated following closely the strategy developed in sections 5 and 6.

Due to the homogeneity of the potentials, Schrödinger operators \(L_{\mathbf{A},a}\) are invariant by the Kelvin transform,

\[
\tilde{u}(x) = |x|^{-(N-2)}u \left( \frac{x}{|x|^2} \right),
\]

which is an isomorphism of \(D_0^{1,2}(\mathbb{R}^N, \mathbb{C})\). Indeed, if \(u \in H^1(\Omega, \mathbb{C})\) weakly solves (2) in a bounded open set \(\Omega\) containing 0, then its Kelvin’s transform \(\tilde{u}\) weakly solves (2) with \(h\) replaced by \(|x|^{-2} h \left( \frac{x}{|x|^2} \right)\) in the external domain \(\Omega = \{x \in \mathbb{R}^N : x/|x|^2 \in \Omega\}\). Weak solution \(u\) of problem (2) with \(h\) satisfying

\[
h \in L^\infty(\Omega, \mathbb{C}), \quad h(x) = O(|x|^{-2-\varepsilon}) \quad \text{as} \quad |x| \to +\infty \quad \text{for some} \quad \varepsilon > 0,
\]
in an external domain \(\Omega\) (i.e. a domain \(\Omega\) such that \(\mathbb{R}^N \setminus B_{R_0} \subset \Omega \subset \mathbb{R}^N \setminus B_{R_1}\) for some \(R_0 > R_1 > 0\)), we mean a function \(u\) such that \(\frac{u}{|x|} \in L^2(\Omega, \mathbb{C})\), \(\nabla u \in L^2(\Omega, \mathbb{C}^N)\), and

\[
Q^2_{\mathbf{A},a}(u, w) = \int_{\Omega} h(x) u(x) w(x) \, dx,
\]

for any \(w \in D_0^{1,2}(\Omega, \mathbb{C})\), where \(D_0^{1,2}(\Omega, \mathbb{C})\) is the completion of \(C^\infty_0(\Omega, \mathbb{C})\) with respect to the norm

\[
||u||_{D_0^{1,2}(\Omega)} := \left( ||\nabla u||_{L^2(\Omega, \mathbb{C}^N)}^2 + ||\frac{u}{|x|}||_{L^2(\Omega, \mathbb{C})}^2 \right)^{1/2}.
\]

Theorem 1.3 and invariance by the Kelvin transform provide the following description of the behavior of solutions to (2) as \(|x| \to \infty\). The Ambregn’s frequency type function in exterior domains has the form

\[
\tilde{N}_{u,h}(r) = \frac{r \int_{\mathbb{R}^N \setminus B_r} |\nabla u(x) + i \frac{A(x)}{|x|^2} u(x)|^2 - \frac{n(x/|x|)}{|x|^2} |u(x)|^2 - (\Re h(x)) |u(x)|^2 \, dx}{\int_{\partial B_r} |u(x)|^2 \, dS},
\]

**Theorem 1.5.** Let \(\Omega \subset \mathbb{R}^N, N \geq 2,\) be an open set such that \(\mathbb{R}^N \setminus B_{R_0} \subset \Omega \subset \mathbb{R}^N \setminus B_{R_1}\) for some \(R_0 > R_1 > 0, [A.1], [A.2], [A.3], [A.4]\) hold, and \(u\) be a weak solution to (2), \(u \not\equiv 0\), with \(h\) satisfying (14). Then, letting \(\tilde{N}_{u,h}\) as in (15), there exists \(k_0 \in \mathbb{N}, k_\geq 1,\) such that

\[
\lim_{r \to +\infty} \tilde{N}_{u,h}(r) = \frac{N-2}{2} + \sqrt{\left( \frac{N-2}{2} \right)^2 + \mu_{k_0}(\mathbf{A}, a)}.
\]

Moreover, if \(\tilde{\gamma}\) denotes the limit in (16), \(m \geq 1\) is the multiplicity of the eigenvalue \(\mu_{k_0}(\mathbf{A}, a)\), and \(\{ \psi_i : j_0 \leq i \leq j_0 + m - 1 \}\) \((j_0 \leq k_0 \leq j_0 + m - 1)\) is an \(L^2(\mathbb{S}^{N-1}, \mathbb{C})\)-orthonormal basis for the
eigenspace of the operator $L_{A,a}$ associated to $\mu_{k_0}(A,a)$, then
\[ \lambda^\gamma u(\lambda) \longrightarrow \sum_{i=j_0}^{j_0+m-1} \beta_i \psi_i(\theta) \quad \text{in } C^{1,\gamma}(S^{N-1}, \mathbb{C}) \quad \text{as } \lambda \to +\infty \]
and
\[ \lambda^{\gamma+1} \nabla u(\lambda) \longrightarrow \sum_{i=j_0}^{j_0+m-1} \beta_i \left( -\tilde{\gamma} \psi_i(\theta) + \nabla_{S^{N-1}} \psi_i(\theta) \right) \quad \text{in } C^{0,\gamma}(S^{N-1}, \mathbb{C}^N) \quad \text{as } \lambda \to +\infty \]
for every $\gamma \in (0,1)$, where
\[ \tilde{\beta}_i = \int_{S^{N-1}} \left[ R^\gamma u(R\theta) + \int_{R}^{+\infty} \frac{h(s\theta)u(s\theta)}{2\gamma + N - 2} \left( s^{\gamma+1} - R^{2\gamma + N - 2}s^{-\gamma + N - 1} \right) ds \right] \psi_i(\theta) dS(\theta), \]
for all $R > 0$ such that $\mathbb{R}^N \setminus B_R \subset \Omega$ and $(\beta_{j_0}, \beta_{j_0+1}, \ldots, \beta_{j_0+m-1}) \neq (0, 0, \ldots, 0)$.

A Brezis-Kato type iteration, see [4], allows us to obtain asymptotics of solutions also for semilinear problems with at most critical growth. In order to start such an iterative procedure, we require assumption \([A.5]\) which allows transforming equation \([3]\) into a degenerate elliptic equation without singular potentials on which the Brezis-Kato method applies successfully, see Lemmas \([9.1]\) and \([10.3]\). The iteration scheme developed in sections \([9]\) and \([10]\) provides an upper bound for solutions and then reduces the semilinear problem to a linear one with enough control on the perturbing potential at the singularity to apply Theorem \([1.3]\) and to recover the exact asymptotic behavior, as stated in the following theorem.

**Theorem 1.6.** Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded open set containing 0, \([A.1]\), \([A.2]\), \([A.3]\), \([A.5]\) hold, and $u$ be a weak $H^1_0(\Omega, \mathbb{C})$-solution to \([3]\), $u \neq 0$, with $f$ being a Carathéodory function satisfying \([7]\). Then, there exists $k_0 \in \mathbb{N}$, $k_0 \geq 1$, such that
\[ \lim_{r \to 0^+} \frac{N_{u,f(\cdot, u)}(r)}{r} = \frac{N - 2}{2} + \sqrt{\left( \frac{N - 2}{2} \right)^2 + \mu_{k_0}(A,a)}. \]
Furthermore, if $\gamma$ denotes the limit in \([17]\), $m \geq 1$ is the multiplicity of the eigenvalue $\mu_{k_0}(A,a)$, and \(\{\psi_i : j_0 \leq i \leq j_0 + m - 1\} \quad (j_0 \leq k_0 \leq j_0 + m - 1)\) is an $L^2(S^{N-1}, \mathbb{C})$-orthonormal basis for the eigenspace of the operator $L_{A,a}$ associated to $\mu_{k_0}(A,a)$, then
\[ \lambda^{-\gamma} u(\lambda) \longrightarrow \sum_{i=j_0}^{j_0+m-1} \beta_i \psi_i(\theta) \quad \text{in } C^{1,\gamma}(S^{N-1}, \mathbb{C}) \quad \text{as } \lambda \to 0^+, \]
and
\[ \lambda^{1-\gamma} \nabla u(\lambda) \longrightarrow \sum_{i=j_0}^{j_0+m-1} \beta_i \left( \gamma \psi_i(\theta) + \nabla_{S^{N-1}} \psi_i(\theta) \right) \quad \text{in } C^{0,\gamma}(S^{N-1}, \mathbb{C}^N) \quad \text{as } \lambda \to 0^+, \]
for any $\gamma \in (0,1)$, where
\[ \beta_i = \int_{S^{N-1}} \left[ R^{-\gamma} u(R\theta) + \int_{R}^{+\infty} \frac{f(s\theta,u(s\theta))}{2\gamma + N - 2} \left( s^{1-\gamma} - R^{2\gamma + N - 2}s^{-\gamma + N - 1} \right) ds \right] \psi_i(\theta) dS(\theta), \]
for all $R > 0$ such that $\overline{B_R} \subset \Omega$ and $(\beta_{j_0}, \beta_{j_0+1}, \ldots, \beta_{j_0+m-1}) \neq (0, 0, \ldots, 0)$. 

Similar conclusions as those in Corollary 1.3 can be deduced from the above theorem for solutions to semilinear equations of type (3): under the same assumption as in Theorem 1.6 if \( \gamma > 0 \) then the solutions to (3) are \( \gamma \)-Hölder continuous for \( 0 < \gamma < 1 \) and Lipschitz continuous for \( \gamma \geq 1 \).

The following result is the counterpart of Theorem 1.6 in exterior domains.

**Theorem 1.7.** Let \( \Omega \subset \mathbb{R}^N, \ N \geq 2, \) be an open set such that \( \mathbb{R}^N \setminus B_{R_0} \subset \Omega \subset \mathbb{R}^N \setminus B_{R_1} \) for some \( R_0 > R_1 > 0, \) \( (A.1), (A.2), (A.3), (A.5) \) hold, and \( u \) be a weak solution to (3) in \( \Omega, \ u \neq 0, \) with \( f \) satisfying, for some \( C_f > 0, \)

\[
\left| \frac{f(x,z)}{z} \right| \leq \begin{cases} \frac{C_f(|x|^{-4} + |z|^2 - 2)}, & \text{if } N \geq 3, \\ \frac{C_f|x|^{-4(1 + |z|^p - 2)}}{2} & \text{for some } p > 2, \end{cases}
\]

for a.e. \( x \in \Omega \) and for all \( z \in \mathbb{C} \setminus \{0\} \). Then there exists \( k_0 \in \mathbb{N}, k_0 \geq 1, \) such that

\[
\lim_{r \to +\infty} \tilde{N}_{u/(\cdot , u)/u}(r) = \frac{N - 2}{2} + \sqrt{\left( \frac{N - 2}{2} \right)^2 + \mu_{k_0}(A,a)}.
\]

Moreover, if \( \tilde{\gamma} \) denotes the limit in above, \( m > 1 \) is the multiplicity of the eigenvalue \( \mu_{k_0}(A,a) \), and \( \{\psi_i : j_0 \leq i \leq j_0 + m - 1\} \) \( (j_0 \leq k_0 \leq j_0 + m - 1) \) is an \( L^2(S^{N-1}, \mathbb{C}) \)-orthonormal basis for the eigenspace of the operator \( L_{A,a} \) associated to \( \mu_{k_0}(A,a) \), then

\[
\lambda^{\tilde{\gamma}} u(\lambda^\theta) \longrightarrow \sum_{i=j_0}^{j_0+m-1} \tilde{\beta}_i \psi_i(\theta) \quad \text{in } C^{1,\tau}(S^{N-1}, \mathbb{C}) \quad \text{as } \lambda \to +\infty
\]

and

\[
\lambda^{\tilde{\gamma}+1} \nabla u(\lambda^\theta) \longrightarrow \sum_{i=j_0}^{j_0+m-1} \tilde{\beta}_i \left( -\tilde{\gamma} \psi_i(\theta) + \nabla_{S^{N-1}} \psi_i(\theta) \right) \quad \text{in } C^{0,\tau}(S^{N-1}, \mathbb{C}^N) \quad \text{as } \lambda \to +\infty
\]

for every \( \tau \in (0,1) \), where

\[
\tilde{\beta}_i = \int_{S^{N-1}} \left[ R^{\tilde{\gamma}+1} u(R^\theta \left) + \int_R^{+\infty} \frac{f(s^\theta, \psi(s^\theta))}{2\tilde{\gamma} - N + 2} \left( s^{\tilde{\gamma}+1} - R^{2\tilde{\gamma} - N + 2s^{-\tilde{\gamma}+N-1}} \right) ds \right] \psi_i(\theta) dS(\theta)
\]

for all \( R > 0 \) such that \( \mathbb{R}^N \setminus B_R \subset \Omega \) and \( \{\tilde{\beta}_{j_0}, \tilde{\beta}_{j_0+1}, \ldots, \tilde{\beta}_{j_0+m-1}\} \neq \{0, 0, \ldots, 0\} \).

The paper is organized as follows. In section 2 we prove Lemma 1.1 and discuss the relation between the positivity of the quadratic form associated to \( L_{A,a} \) and the first eigenvalue of the angular operator on the sphere \( S^{N-1} \). In section 3 we prove a Hardy type inequality with boundary terms and singular electromagnetic potential, while in section 4 we derive a Pohozaev-type identity for solutions to (2). Section 5 contains an Almgren type monotonicity formula, which is used in section 6 together with a blow-up method to prove Theorems 1.3 and 1.5. Section 7 contains an application of Theorem 1.3 to Aharonov-Bohm magnetic potentials. In section 8 we prove a Hardy-Sobolev inequality with magnetic potentials which is needed in section 9 to start a Brezis-Kato iteration procedure in order to obtain a-priori pointwise bounds for solutions to the nonlinear equation and to prove Theorems 1.6 and 1.7 in dimension \( N \geq 3 \). The proof of Theorems 1.6 and 1.7 in dimension \( N = 2 \) can be found in section 10. In a final appendix, we recall well-known results such as the diamagnetic inequality, Hardy’s inequality with boundary terms, and the description the spectrum of angular operator \( L_{A,a} \).
Notation. We list below some notation used throughout the paper.
- For all $r > 0$, $B_r$ denotes the ball $\{ x \in \mathbb{R}^N : |x| < r \}$ in $\mathbb{R}^N$ with center at 0 and radius $r$.
- For all $r > 0$, $\overline{B}_r = \{ x \in \mathbb{R}^N : |x| \leq r \}$ denotes the closure of $B_r$.
- $dS$ denotes the volume element on the spheres $\partial B_r$, $r > 0$.
- For every complex number $z \in \mathbb{C}$, $\Re z$ denotes its real part and $\Im z$ its imaginary part.
- For every complex number $z \in \mathbb{C}$, $\overline{z}$ denotes its complex conjugate.

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2. Positivity of the quadratic form

In this section, we study the quadratic form associated to the Schrödinger operator $\mathcal{L}_{\mathbf{A}, a}$ and defined in (3). To study the sign of $Q_{\mathbf{A}, a}$, we define the first eigenvalue of $Q_{\mathbf{A}, a}$ with respect to the Hardy singular weight as

$$\lambda_1(\mathbf{A}, a) := \inf_{u \in \mathcal{D}^1_2(S^{N-1}, \mathbb{C}) \setminus \{0\}} \frac{Q_{\mathbf{A}, a}(u)}{\int_{\mathbb{R}^N} \frac{|u(z)|^2}{|z|^2} \, dx},$$

and discuss the relation between $\lambda_1(\mathbf{A}, a)$ and the first eigenvalue of the angular component of the operator on the sphere $S^{N-1}$, i.e. of the operator

$$L_{\mathbf{A}, a} = (-i \nabla_{S^{N-1}} + \mathbf{A})^2 - a = -\Delta_{S^{N-1}} - (a(\theta) - |\mathbf{A}|^2 + i \text{div}_{S^{N-1}} \mathbf{A}) - 2i \mathbf{A} \cdot \nabla_{S^{N-1}}.$$

We notice that, by (A.2), $\lambda_1(\mathbf{A}, a)$ is well defined and finite. Let us introduce the Sobolev space

$$H^1_{\mathbf{A}}(S^{N-1}) := \left\{ \psi \in L^2(S^{N-1}, \mathbb{C}) : \nabla_{S^{N-1}} \psi + i \mathbf{A}(\theta) \psi \in L^2(S^{N-1}, \mathbb{C}^N) \right\},$$

endowed with the norm

$$\|\psi\|_{H^1_{\mathbf{A}}(S^{N-1})} := \left( \int_{S^{N-1}} \left[ \left( |\nabla_{S^{N-1}} \psi|^2 + |\psi(\theta)|^2 \right) dS(\theta) \right]^{1/2},$$

d$S$ denoting the volume element on the sphere $S^{N-1}$. We observe that, if $\mathbf{A} \in C^1(S^{N-1}, \mathbb{R}^N)$, then $H^1_{\mathbf{A}}(S^{N-1})$ is equal to the classical Sobolev space $H^1(S^{N-1}, \mathbb{C})$ and its norm is equivalent to the $H^1(S^{N-1}, \mathbb{C})$-norm, see Lemma [A.3] in the appendix.

Under assumption (A.2), the operator $L_{\mathbf{A}, a}$ on $S^{N-1}$ admits a diverging sequence of real eigenvalues $\mu_1(\mathbf{A}, a) \leq \mu_2(\mathbf{A}, a) \leq \cdots \leq \mu_k(\mathbf{A}, a) \leq \cdots$ the first of which can be characterized as

$$\mu_1(\mathbf{A}, a) = \min_{\psi \in H^1_{\mathbf{A}}(S^{N-1}) \setminus \{0\}} \frac{\int_{S^{N-1}} \left[ (|\nabla_{S^{N-1}} \psi|^2 + |i \mathbf{A}(\theta) \psi|^2) \right] dS(\theta)}{\int_{S^{N-1}} |\psi(\theta)|^2 dS(\theta)},$$

see Lemma [A.5] in the appendix. The relation between $\lambda_1(\mathbf{A}, a)$ and $\mu_1(\mathbf{A}, a)$ is clarified in the following lemma.

Lemma 2.1. If $N \geq 2$, (A.2) and (A.3) hold, then

$$\lambda_1(\mathbf{A}, a) = \mu_1(\mathbf{A}, a) + \left( \frac{N - 2}{2} \right)^2.$$
PROOF. Let \( \psi \in H^1_0(\mathbb{S}^{N-1}) \), \( \psi \neq 0 \), attaining \( \mu_1(A, a) \) and let \( \phi \in C^\infty_c((0, +\infty), \mathbb{R}) \) so that \( \varphi : x \mapsto \phi(|x|) \in C^\infty_c(\mathbb{R}^N \setminus \{0\}, \mathbb{R}) \). If \( u(x) = \phi(|x|)\psi(\frac{x}{|x|}) \), there holds
\[
\left( \nabla + i \frac{A(|x|)}{|x|} \right) u(x) = \phi'(|x|)\psi(\frac{x}{|x|}) \frac{x}{|x|} + \frac{1}{|x|} \phi(|x|) \nabla_{\mathbb{S}^{N-1}} \psi(\frac{x}{|x|}) + \frac{i}{|x|} A(\frac{x}{|x|}) \phi(|x|) \psi(\frac{x}{|x|})
\]
and, by assumption \( A.3 \),
\[
\left| (\nabla + i \frac{A(\frac{x}{|x|})}{|x|}) u(x) \right|^2 = |\phi'(|x|)|^2 |\psi(\frac{x}{|x|})|^2 + \frac{1}{|x|^2} |\nabla_{\mathbb{S}^{N-1}} \psi(\frac{x}{|x|}) + i A(\frac{x}{|x|}) \psi(\frac{x}{|x|})|^2.
\]
Therefore, from the definition of \( \lambda_1(A, a) \) it follows
\[
\lambda_1(A, a) \left( \int_0^{+\infty} r^{N-1} \frac{|\phi(r)|^2}{r^2} dr \right) \left( \int_{\mathbb{S}^{N-1}} |\psi(\theta)|^2 dS(\theta) \right)
\leqslant \int_{\mathbb{R}^N} \left[ \left( (\nabla + i \frac{A(|x|)}{|x|}) u(x) \right]^2 - a(\frac{x}{|x|}) \frac{|u(x)|^2}{|x|} \right] dx
\leqslant \left( \int_0^{+\infty} r^{N-1} |\phi'(r)|^2 dr \right) \left( \int_{\mathbb{S}^{N-1}} |\psi(\theta)|^2 dS(\theta) \right)
+ \left( \int_0^{+\infty} r^{N-1} |\phi(r)|^2 dr \right) \left( \int_{\mathbb{S}^{N-1}} \left[ \left( \nabla_{\mathbb{S}^{N-1}} \psi(\theta) + i A(\theta) \psi(\theta) \right)^2 - a(\theta)|\psi(\theta)|^2 \right] dS(\theta) \right)
= \left( \int_{\mathbb{S}^{N-1}} |\psi(\theta)|^2 dS(\theta) \right) \left[ \int_0^{+\infty} r^{N-1} |\phi'(r)|^2 dr + \mu_1(A, a) \int_0^{+\infty} r^{N-1} \frac{|\phi(r)|^2}{r^2} dr \right].
\]
Hence
\[
\lambda_1(A, a) - \mu_1(A, a) \leqslant \frac{\int_0^{+\infty} r^{N-1} |\phi'(r)|^2 dr}{\int_0^{+\infty} r^{N-3} |\phi(r)|^2 dr} \leqslant \frac{\int_{\mathbb{R}^N} |\nabla \bar{\varphi}(x)|^2 dx}{\int_{\mathbb{R}^N} |\bar{\varphi}(x)|^2 dx}
\]
for every radial function \( \bar{\varphi} \in C^\infty_c(\mathbb{R}^N \setminus \{0\}, \mathbb{R}) \). Hence by Schwarz symmetrization
\[
\lambda_1(A, a) - \mu_1(A, a) \leqslant \inf_{\bar{\varphi} \in C^\infty_c(\mathbb{R}^N \setminus \{0\}, \mathbb{R})} \frac{\int_{\mathbb{R}^N} |\nabla \bar{\varphi}(x)|^2 dx}{\int_{\mathbb{R}^N} |\bar{\varphi}(x)|^2 dx}
\]
\[
= \inf_{u \in C^\infty_c(\mathbb{R}^N \setminus \{0\}, \mathbb{R})} \frac{\int_{\mathbb{R}^N} |\nabla u(x)|^2 dx}{\int_{\mathbb{R}^N} |u(x)|^2 dx}
= \left( \frac{N - 2}{2} \right)^2,
\]
where the last identity is due to the optimality of the classical best Hardy constant for \( N \geqslant 3 \) and to direct calculations for \( N = 2 \). To prove the reverse inequality, let \( u \in C^\infty_c(\mathbb{R}^N \setminus \{0\}, \mathbb{C}) \). The magnetic gradient of \( u \) can be written in polar coordinates as
\[
\nabla u(x) + i \frac{A(\frac{x}{|x|})}{|x|} u(x) = (\partial_r u(r, \theta)) \theta + \frac{1}{r} \nabla_{\mathbb{S}^{N-1}} u(r, \theta) + i \frac{u(r, \theta)}{r} A(\theta), \quad r = |x|, \quad \theta = \frac{x}{|x|}.
\]
By assumption \( A.3 \), there holds
\[
\left| (\nabla + i \frac{A(\frac{x}{|x|})}{|x|}) u(x) \right|^2 = \left| \partial_r u(r, \theta) \right|^2 + \frac{1}{r^2} \left| \nabla_{\mathbb{S}^{N-1}} u(r, \theta) + i A(\theta) u(r, \theta) \right|^2.
\]
hence

\begin{equation}
Q_{\mathcal{A},\alpha}(u) = \int_{S^{N-1}} \left( \int_0^{+\infty} r^{N-1}|\partial_r u(r, \theta)|^2 \, dr \right) dS(\theta)
+ \int_0^{+\infty} r^{N-1} \left( \int_{S^{N-1}} \left[ |\nabla_{S^{N-1}} u(r, \theta) - i \mathbf{A}(\theta) u(r, \theta)|^2 - \alpha(\theta)|u(r, \theta)|^2 \right] \, dS(\theta) \right) \, dr.
\end{equation}

For all $\theta \in S^{N-1}$, let $\varphi_\theta \in C^\infty_c((0, +\infty), \mathbb{C})$ be defined by $\varphi_\theta(r) = u(r, \theta)$, and $\tilde{\varphi}_\theta \in C^\infty_c(\mathbb{R}^N \setminus \{0\}, \mathbb{C})$ be the radially symmetric function given by $\tilde{\varphi}_\theta(x) = \varphi_\theta(|x|)$. If $N \geq 3$, Hardy’s inequality yields

\begin{equation}
\int_{S^{N-1}} \left( \int_0^{+\infty} r^{N-1}|\partial_r u(r, \theta)|^2 \, dr \right) dS(\theta) = \int_{S^{N-1}} \left( \int_0^{+\infty} r^{N-1}|\varphi_\theta'(r)|^2 \, dr \right) dS(\theta)
\geq \frac{1}{\omega_{N-1}} \left( \int_{\mathbb{R}^N} |\nabla \varphi_\theta(x)|^2 \, dx \right) dS(\theta)
= \frac{1}{\omega_{N-1}} \left( \frac{N-2}{2} \right)^2 \int_{S^{N-1}} \left( \int_{\mathbb{R}^N} \frac{|\tilde{\varphi}_\theta(x)|^2}{|x|^2} \, dx \right) dS(\theta)
= \left( \frac{N-2}{2} \right)^2 \int_{S^{N-1}} \left( \int_0^{+\infty} \frac{r^{N-1}}{r^2}|u(r, \theta)|^2 \, dr \right) dS(\theta) = \left( \frac{N-2}{2} \right)^2 \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^2} \, dx,
\end{equation}

where $\omega_{N-1}$ denotes the volume of the unit sphere $S^{N-1}$, i.e. $\omega_{N-1} = \int_{S^{N-1}} dS(\theta)$. For $N = 2$ trivially holds. On the other hand, from the definition of $\mu_1(\mathbf{A}, \alpha)$ it follows that

\begin{equation}
\int_{S^{N-1}} \left[ |\nabla_{S^{N-1}} u(r, \theta) - i \mathbf{A}(\theta) u(r, \theta)|^2 - \alpha(\theta)|u(r, \theta)|^2 \right] \, dS(\theta) \geq \mu_1(\mathbf{A}, \alpha) \int_{S^{N-1}} |u(r, \theta)|^2 \, dS(\theta).
\end{equation}

From (23), (24), and (25), we deduce that

\begin{equation}
Q_{\mathcal{A},\alpha}(u) \geq \left[ \left( \frac{N-2}{2} \right)^2 + \mu_1(\mathcal{A}, \alpha) \right] \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^2} \, dx \quad \text{for all } u \in C^\infty_c(\mathbb{R}^N \setminus \{0\}, \mathbb{C}),
\end{equation}

which, by density of $C^\infty_c(\mathbb{R}^N \setminus \{0\}, \mathbb{C})$ in $D^{1,2}_{\mathcal{A}}(\mathbb{R}^N, \mathbb{C})$, implies

\[ \lambda_1(\mathbf{A}, \alpha) \geq \left( \frac{N-2}{2} \right)^2 + \mu_1(\mathbf{A}, \alpha), \]

thus completing the proof.

The relation between positivity of $Q_{\mathcal{A},\alpha}$ and the values $\mu_1(\mathbf{A}, \alpha), \lambda_1(\mathbf{A}, \alpha)$ is described in the following lemma.

**Lemma 2.2.** If $N \geq 2$, (A.2) and (A.3) hold, then the following conditions are equivalent:

\begin{enumerate}
    \item $Q_{\mathcal{A},\alpha}$ is positive definite in $D^{1,2}_{\mathcal{A}}(\mathbb{R}^N, \mathbb{C})$, i.e.
    \[ \inf_{u \in D^{1,2}_{\mathcal{A}}(\mathbb{R}^N, \mathbb{C}) \setminus \{0\}} \frac{Q_{\mathcal{A},\alpha}(u)}{\|u\|_{D^{1,2}_{\mathcal{A}}(\mathbb{R}^N, \mathbb{C})}} > 0; \]
    \item $\lambda_1(\mathbf{A}, \alpha) > 0$;
    \item $\mu_1(\mathbf{A}, \alpha) > -\left( \frac{N-2}{2} \right)^2$.
\end{enumerate}
The presence of a vector potential satisfying a suitable non-degeneracy condition, allows recovering functions in $D^{1,2}(\mathbb{R}^N, \mathbb{C})$ such that

\[ Q_{A,a}(u_\varepsilon) < \varepsilon \|u_\varepsilon\|_{D^{1,2}(\mathbb{R}^N, \mathbb{C})}^2 \leq 2(\|A\|_{L^\infty(S^{N-1}, \mathbb{R}^N)} + 1) \varepsilon \left( \int_{\mathbb{R}^N} \left| \nabla + i \frac{A(x/|x|)}{|x|} \right| u(x) \right)^2 + \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^2} \, dx \]

and hence, for $\varepsilon$ small,

\[ \lambda_1 \left( A, \frac{1}{1 - 2\varepsilon(\|A\|_{L^\infty(S^{N-1}, \mathbb{R}^N)} + 1)} \right) \leq \frac{2\varepsilon(\|A\|_{L^\infty(S^{N-1}, \mathbb{R}^N)} + 1)}{1 - 2\varepsilon(\|A\|_{L^\infty(S^{N-1}, \mathbb{R}^N)} + 1)}. \]

On the other hand, from the characterization of $\lambda_1(A,a)$ given in Lemma 2.1, we have that the map $a \mapsto \lambda_1(A,a)$ is continuous with respect to the $L^\infty$-norm and hence, letting $\varepsilon \to 0$, we obtain $\lambda_1(A,a) \leq 0$, a contradiction.

The previous lemma allows relating $D^{1,2}(\mathbb{R}^N, \mathbb{C})$ with the Hilbert space $D^{1,2}(\mathbb{R}^N)$ generated by the quadratic form $Q_{A,a}$, thus proving Lemma 1.1.

**Proof of Lemma 1.1**

i) follows from Lemma 2.1 and assumption (A.4). ii) is a direct consequence of Lemma 2.2 and (A.4). From ii) we deduce that $(Q_{A,a}(\cdot))^{1/2}$ defines a norm in $C_c^\infty(\mathbb{R}^N \setminus \{0\}, \mathbb{C})$ which is equivalent to $\|\cdot\|_{D^{1,2}(\mathbb{R}^N, \mathbb{C})}$. Hence completion of $C_c^\infty(\mathbb{R}^N \setminus \{0\}, \mathbb{C})$ with respect to the norms $(Q_{A,a}(\cdot))^{1/2}$ and $\|\cdot\|_{D^{1,2}(\mathbb{R}^N, \mathbb{C})}$ yields two coinciding spaces with equivalent norms.

By Hardy type inequalities, it is possible to compare the functional space $D^{1,2}(\mathbb{R}^N, \mathbb{C})$ with the classical Sobolev space $D^{1,2}(\mathbb{R}^N, \mathbb{C})$ defined as the completion of $C_c^\infty(\mathbb{R}^N, \mathbb{C})$ with respect to the norm

\[ \|u\|_{D^{1,2}(\mathbb{R}^N, \mathbb{C})} := \left( \int_{\mathbb{R}^N} |\nabla u(x)|^2 \, dx \right)^{1/2} \]

and with the space $D^{1,2}(\mathbb{R}^N)$ given by the completion of $C_c^\infty(\mathbb{R}^N \setminus \{0\}, \mathbb{C})$ with respect to the magnetic Dirichlet norm

\[ \|u\|_{D^{1,2}_{\text{mag}}(\mathbb{R}^N)} := \left( \int_{\mathbb{R}^N} |\nabla u(x) + i \frac{A(x/|x|)}{|x|} u(x)|^2 \, dx \right)^{1/2}. \]

The presence of a vector potential satisfying a suitable non-degeneracy condition, allows recovering a Hardy’s inequality even for $N = 2$. Indeed, if $N = 2$, (A.3) holds, and

\[ \Phi_A := \frac{1}{2\pi} \int_0^{2\pi} \alpha(t) \, dt \notin \mathbb{Z}, \quad \text{where} \quad \alpha(t) := A(\cos t, \sin t) \cdot (-\sin t, \cos t), \]

then functions in $D^{1,2}_{\text{mag}}(\mathbb{R}^2)$ satisfy the following Hardy inequality

\[ \left( \min_{k \in \mathbb{Z}} |k - \Phi_A|^2 \right) \int_{\mathbb{R}^2} \frac{|u(x)|^2}{|x|^2} \, dx \leq \int_{\mathbb{R}^2} \left| \nabla u(x) + i \frac{A(x/|x|)}{|x|} u(x) \right|^2 \, dx \]
being \( \left( \min_{k \in \mathbb{Z}} |k - \Phi_A| \right)^2 \) the best constant, as proved in [19]. It is easy to verify that, for \( N = 2 \),

\[
\mu_1(A, 0) = \min_{\psi \in H^1((0,2\pi), C)} \frac{\int_0^{2\pi} |\psi'(t) + i\alpha(t)\psi(t)|^2 dt}{\int_0^{2\pi} |\psi(t)|^2 dt},
\]

where \( \alpha(t) := A(\cos t, \sin t) \cdot (-\sin t, \cos t) \). Furthermore, \( \mu_1(A, 0) > 0 \) if and only if (26) holds. Combining Lemma 2.1 (in the case \( N = 2 \) and \( a = 0 \)) with [19], we conclude that, for \( N = 2 \),

\[
(28) \quad \mu_1(A, 0) = \left( \min_{k \in \mathbb{Z}} |k - \Phi_A| \right)^2.
\]

**Lemma 2.3.**

(i) If \( N \geq 3 \) then \( D_{1,2}^1(\mathbb{R}^N, C) = D_{1,2}^1(\mathbb{R}^N, C) \) and the norms \( \| \cdot \|_{D_{1,2}^1(\mathbb{R}^N, C)} \) and \( \| \cdot \|_{D_{1,2}^1(\mathbb{R}^N, C)} \) are equivalent.

(ii) If \( A \in C^1(S^{N-1}, \mathbb{R}^N) \) and either \( N \geq 3 \) or \( N = 2 \) and (A.3), (26) hold, then \( D_{1,2}^1(\mathbb{R}^N, C) = D_{1,2}^1(\mathbb{R}^N, \mathbb{C}) \) with equivalent norms.

**Proof.** By classical Hardy’s inequality, for \( N \geq 3 \) the norms \( \| \cdot \|_{D_{1,2}^1(\mathbb{R}^N, C)} \) and \( \| \cdot \|_{D_{1,2}^1(\mathbb{R}^N, C)} \) are equivalent over the space \( C_c^\infty(\mathbb{R}^N \setminus \{0\}, \mathbb{C}) \). The proof of i) then follows by completion after observing that, for \( N \geq 3 \), \( C_c^\infty(\mathbb{R}^N \setminus \{0\}, \mathbb{C}) \) is dense in \( D_{1,2}^1(\mathbb{R}^N, \mathbb{C}) \).

In order to prove ii), let us consider \( u \in C_c^\infty(\mathbb{R}^N \setminus \{0\}, \mathbb{C}) \). Then

\[
\|u\|_{D_{1,2}^1(\mathbb{R}^N)} = \left\| \nabla u + i \frac{A(x/|x|)}{|x|} u \right\|_{L^2(\mathbb{R}^N, C^N)} \leq \|\nabla u\|_{L^2(\mathbb{R}^N, C^N)} + \left\| \frac{A(x/|x|)}{|x|} u \right\|_{L^2(\mathbb{R}^N)}
\]

\[
\leq \|\nabla u\|_{L^2(\mathbb{R}^N, C^N)} + \sup_{S^{N-1}} |A| \left( \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx \right)^{1/2} \leq \text{const} \|u\|_{D_{1,2}^1(\mathbb{R}^N, C)}.
\]

On the other hand, by the diamagnetic inequality in Lemma A.1 classical Hardy’s inequality for \( N \geq 3 \), and (27) for \( N = 2 \), we have

\[
\|u\|_{D_{1,2}^1(\mathbb{R}^N, C)} = \|\nabla u\|_{L^2(\mathbb{R}^N, C^N)} \leq \left\| \nabla u + i \frac{A(x/|x|)}{|x|} u \right\|_{L^2(\mathbb{R}^N, C^N)} + \left\| \frac{A(x/|x|)}{|x|} u \right\|_{L^2(\mathbb{R}^N, C^N)}
\]

\[
\leq \|u\|_{D_{1,2}^1(\mathbb{R}^N)} + \sup_{S^{N-1}} |A| \left( \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx \right)^{1/2} \leq \|u\|_{D_{1,2}^1(\mathbb{R}^N)} + \text{const} \|\nabla u\|_{L^2(\mathbb{R}^N, C^N)}
\]

\[
\leq (1 + \text{const}) \|u\|_{D_{1,2}^1(\mathbb{R}^N)}.
\]

The above inequalities show that \( \| \cdot \|_{D_{1,2}^1(\mathbb{R}^N, C)} \) and \( \| \cdot \|_{D_{1,2}^1(\mathbb{R}^N, \mathbb{C})} \) are equivalent norms over the space \( C_c^\infty(\mathbb{R}^N \setminus \{0\}, \mathbb{C}) \). The proof of the lemma then follows immediately from the definition of the spaces \( D_{1,2}^1(\mathbb{R}^N, \mathbb{C}) \) and \( D_{1,2}^1(\mathbb{R}^N) \).

3. A Hardy type inequality with boundary terms

We extend to singular electromagnetic potentials the Hardy type inequality with boundary terms proved by Wang and Zhu in [27] (see Lemma A.3 in the Appendix).
Lemma 3.1. If $N \geq 2$, (A.2) and (A.3) hold, then

\begin{equation}
\int_{B_r} \left[ \left( \nabla + i \frac{A(x/|x|)}{|x|} \right) u \right]^2 - \frac{a(x/|x|)}{|x|^2} |u|^2 \, dx + \frac{N - 2}{2r} \int_{\partial B_r} |u(x)|^2 \, dS \geq \left( \mu_1(A, a) + \left( \frac{N - 2}{2} \right)^2 \right) \int_{B_r} \frac{|u(x)|^2}{|x|^2} \, dx
\end{equation}

for all $r > 0$ and $u \in H^1_s(B_r, \mathbb{C})$.

PROOF. By scaling, it is enough to prove the inequality for $r = 1$. Let $u \in C^\infty(B_1, \mathbb{C}) \cap H^1_s(B_1, \mathbb{C})$ with $0 \notin \text{supp} u$. Passing to polar coordinates and using (22), we have that

\begin{equation}
\int_{B_1} \left[ \left( \nabla + i \frac{A(x/|x|)}{|x|} \right) u \right]^2 - \frac{a(x/|x|)}{|x|^2} |u|^2 \, dx + \frac{N - 2}{2} \int_{\partial B_1} |u(x)|^2 \, dS \\
= \int_{S^{N-1}} \left( \int_0^1 r^{N-1} |\partial_r u(r, \theta)|^2 \, dr \right) dS(\theta) + \frac{N - 2}{2} \int_{S^{N-1}} |u(1, \theta)|^2 \, dS(\theta) \\
+ \int_0^1 \frac{r^{N-2}}{r^2} \left( \int_{S^{N-1}} \left| \nabla_{S^{N-1}} u(r, \theta) + i A(\theta) u(r, \theta) \right|^2 - a(\theta) |u(r, \theta)|^2 \right) \, dS(\theta) \, dr.
\end{equation}

For all $\theta \in S^{N-1}$, let $\varphi_\theta \in C^\infty_c((0, +\infty), \mathbb{C})$ be defined by $\varphi_\theta(r) = u(r, \theta)$, and $\widetilde{\varphi}_\theta \in C^\infty_c(\mathbb{R}^N \setminus \{0\}, \mathbb{C})$ be the radially symmetric function given by $\widetilde{\varphi}_\theta(x) = \varphi_\theta(|x|)$. The Hardy inequality with boundary term proved in (27) (see Lemma (A.3) in the appendix) yields, for $N \geq 3$,

\begin{equation}
\int_{S^{N-1}} \left( \int_0^1 r^{N-1} |\partial_r u(r, \theta)|^2 \, dr + \frac{N - 2}{2} |u(1, \theta)|^2 \right) \, dS(\theta) \\
= \int_{S^{N-1}} \left( \int_0^1 r^{N-1} |\varphi_\theta'(r)|^2 \, dr + \frac{N - 2}{2} |\varphi_\theta(1)|^2 \right) \, dS(\theta) \\
= \frac{1}{\omega_{N-1}} \int_{S^{N-1}} \left( \int_{B_1} |\nabla \varphi_\theta|^2 \, dx + \frac{N - 2}{2} \int_{\partial B_1} |\varphi_\theta|^2 \, dS \right) \, dS(\theta) \\
\geq \frac{1}{\omega_{N-1}} \left( \frac{N - 2}{2} \right)^2 \int_{S^{N-1}} \left( \int_{B_1} \frac{|\varphi_\theta|^2}{|x|^2} \, dx \right) \, dS(\theta) \\
= \frac{N - 2}{2} \int_{S^{N-1}} \left( \int_0^1 r^{N-1} |u(r, \theta)|^2 \, dr \right) \, dS(\theta) = \frac{N - 2}{2} \int_{B_1} \frac{|u(x)|^2}{|x|^2} \, dx.
\end{equation}

On the other hand, (31) trivially holds also for $N = 2$. From (30), (31), and (26), we deduce that

\begin{equation}
\int_{B_1} \left[ \left( \nabla + i \frac{A(x/|x|)}{|x|} \right) u \right]^2 - \frac{a(x/|x|)}{|x|^2} |u|^2 \, dx + \frac{N - 2}{2} \int_{\partial B_1} |u(x)|^2 \, dS \geq \left( \frac{N - 2}{2} \right)^2 + \mu_1(A, a) \int_{B_1} \frac{|u(x)|^2}{|x|^2} \, dx
\end{equation}

for all $u \in C^\infty(B_1, \mathbb{C}) \cap H^1_s(B_1, \mathbb{C})$ with $0 \notin \text{supp} u$,

which, by density, yields the stated inequality for all $H^1_s(B_r, \mathbb{C})$-functions for $r = 1$. \qed
Remark 3.2. In view of (28), Lemma 3.1 for \( N = 2 \) and \( \alpha \equiv 0 \) yields
\[
\int_{B_r} \left[ \left( \nabla + i \frac{A(x/|x|)}{|x|} \right) u \right]^2 dx \geq \left( \min_{k \in \mathbb{Z}} |k - \Phi A| \right)^2 \int_{B_r} |u(x)|^2 dx
\]
for all \( r > 0 \) and \( u \in H^1_0(B_r, \mathbb{C}) \).

4. A Pohozaev-type identity

Solutions to (2) satisfy the following Pohozaev-type identity.

Theorem 4.1. Let \( \Omega \subset \mathbb{R}^N, N \geq 2 \), be a bounded open set such that \( 0 \in \Omega \). Let \( a, A \) satisfy (A.2), and \( u \) be a weak \( H^1_0(\Omega, \mathbb{C}) \)-solution to (2) in \( \Omega \), with \( h \) satisfying (6). Then
\[
(32) \quad - \frac{N-2}{2} \int_{B_r} \left[ \left( \nabla + i \frac{A(x/|x|)}{|x|} \right) u \right]^2 dx - \frac{a(x/|x|)}{|x|^2} |u|^2 dx + \frac{\partial u}{\partial \nu}^2 ds = r \int_{\partial B_r} \left[ \left( \nabla + i \frac{A(x/|x|)}{|x|} \right) u \right]^2 |u|^2 dx
\]
for all \( r > 0 \) such that \( \partial B_r = \{ x \in \mathbb{R}^N : |x| \leq r \} \subset \Omega \), where \( \nu = \nu(x) \) is the unit outer normal vector \( \nu(x) = \frac{x}{|x|} \).

Proof. Let \( r > 0 \) such that \( \partial B_r \subset \Omega \). Since
\[
\int_0^r \left[ \int_{\partial B_s} \left[ \left( \nabla + i \frac{A(x/|x|)}{|x|} \right) u \right]^2 ds \right] ds + \frac{\partial u}{\partial \nu}^2 ds = \int_{B_r} \left[ \left( \nabla + i \frac{A(x/|x|)}{|x|} \right) u \right]^2 dx + \frac{\partial u}{\partial \nu}^2 dx < +\infty
\]
there exists a sequence \( \{ \delta_n \} \subset (0, r) \) such that \( \lim_{n \to +\infty} \delta_n = 0 \) and
\[
(33) \quad \delta_n \int_{\partial B_{\delta_n}} \left[ \left( \nabla + i \frac{A(x/|x|)}{|x|} \right) u \right]^2 dx + \frac{\partial u}{\partial \nu}^2 ds \to 0 \quad \text{as} \quad n \to +\infty.
\]

From classical regularity theory for elliptic equations, \( u \in W^{2,p}_{\text{loc}}(\Omega \setminus \{0\}) \) for all \( p \in (1, \infty) \) and \( u \in C^{1,\tau}_{\text{loc}}(\Omega \setminus \{0\}, \mathbb{C}) \) for any \( \tau \in (0, 1) \) (see Remark 1.2), hence we can multiply equation (2) by \( x \cdot \nabla u(x) \), integrate over \( B_r \setminus B_{\delta_n} \), and take the real part, thus obtaining
\[
(34) \quad \int_{B_r \setminus B_{\delta_n}} \mathfrak{R} \left( \nabla u(x) \cdot \nabla (x \cdot \nabla u(x)) \right) dx + \int_{B_r \setminus B_{\delta_n}} \frac{|A(x/|x|)|^2 - a(x/|x|)}{|x|^2} \mathfrak{R} (u(x) \cdot \nabla u(x)) dx + \int_{B_r \setminus B_{\delta_n}} \frac{A(x/|x|)}{|x|} \cdot \mathfrak{Im} (\nabla u(x) \cdot x) dx + \int_{B_r \setminus B_{\delta_n}} \frac{A(x/|x|)}{|x|} \cdot \mathfrak{Im} \left( \nabla u(x) \cdot x \right) dx
\]
\[
= r \int_{\partial B_r} \left| \frac{\partial u}{\partial \nu} \right|^2 ds - \delta_n \int_{\partial B_{\delta_n}} \left| \frac{\partial u}{\partial \nu} \right|^2 ds + \int_{B_r \setminus B_{\delta_n}} \mathfrak{R} (h(x) u(x) (x \cdot \nabla u(x))) dx.
\]
Integration by parts yields

\[ \int_{\partial B \setminus B_n} \nabla u(x) \cdot \nabla (x \cdot \nabla u(x)) \, dx = -(N - 1) \int_{\partial B \setminus B_n} |\nabla u(x)|^2 \, dx + r \int_{\partial B_r} |\nabla u(x)|^2 \, dS + \delta_n \int_{\partial B_n} |\nabla u(x)|^2 \, dx. \]

A further integration by parts leads to

\[ \sum_{i,j=1}^{N} \int_{\partial B \setminus B_n} x_j \frac{\partial u}{\partial x_i} \frac{\partial^2 u}{\partial x_i \partial x_j} \, dx = -N \int_{\partial B \setminus B_n} |\nabla u(x)|^2 \, dx + r \int_{\partial B_r} |\nabla u(x)|^2 \, dS + \delta_n \int_{\partial B_n} |\nabla u(x)|^2 \, dx. \]

and hence

\[ \sum_{i,j=1}^{N} \int_{\partial B \setminus B_n} \Re \left( x_j \frac{\partial u}{\partial x_i} \frac{\partial^2 u}{\partial x_i \partial x_j} \right) \, dx = -\frac{N}{2} \int_{\partial B \setminus B_n} |\nabla u(x)|^2 \, dx \]

\[ + \frac{r}{2} \int_{\partial B_r} |\nabla u(x)|^2 \, dS - \frac{\delta_n}{2} \int_{\partial B_n} |\nabla u(x)|^2 \, dS. \]

Collecting (35) and (36) we obtain

\[ \int_{\partial B \setminus B_n} \Re (\nabla u(x) \cdot \nabla (x \cdot \nabla u(x))) \, dx \]

\[ = -\frac{N - 2}{2} \int_{\partial B \setminus B_n} |\nabla u(x)|^2 \, dx + \frac{r}{2} \int_{\partial B_r} |\nabla u(x)|^2 \, dS - \frac{\delta_n}{2} \int_{\partial B_n} |\nabla u(x)|^2 \, dS. \]

Letting \( f(\theta) = |A(\theta)|^2 - a(\theta) \), we have that \( f \in L^\infty(S^{N-1}, \mathbb{R}) \) and, passing to polar coordinates \( r = \frac{x}{|x|}, \theta = \frac{x}{|x|} \), and observing that \( \partial_u(r, \theta) = \nabla u(r \theta) \cdot \theta \),

\[ \int_{\partial B \setminus B_n} \frac{f(\frac{x}{|x|})}{|x|^2} u(x) (x \cdot \nabla u(x)) \, dx = \int_{S^{N-1}} f(\theta) \left[ \int_0^r s^{N-2} u(s \theta) \partial_u u(s \theta) \, ds \right] dS(\theta) \]

\[ = \int_{S^{N-1}} f(\theta) \left[ r^{N-2} |u(r \theta)|^2 - \delta_n^{N-2} |u(\delta_n \theta)|^2 \right] \]

\[ - (N - 2) \int_{S^{N-1}} \int_0^r s^{N-3} |u(s \theta)|^2 \, ds \, dS(\theta) \]

\[ = \int_{\partial B_r} \frac{f(\frac{x}{|x|})}{|x|^2} |u(x)|^2 \, dS - \delta_n \int_{\partial B_n} \frac{f(\frac{x}{|x|})}{|x|^2} |u(x)|^2 \, dS \]

\[ - (N - 2) \int_{\partial B \setminus B_n} \int_{\partial B_n} \frac{f(\frac{x}{|x|})}{|x|^2} u(x) (x \cdot \nabla u(x)) \, dx. \]
thus leading to

\[
\int_{B_r \setminus B_{r_n}} \frac{|A(\xi)|^2 - a(\xi)|}{|x|^2} \Re (u(x) (x \cdot \nabla u(x))) \, dx
\]

\[
= -\frac{N - 2}{2} \int_{B_r \setminus B_{r_n}} \frac{|A(\xi)|^2 - a(\xi)|}{|x|^2} |u(x)|^2 \, dx + \frac{r}{2} \int_{\partial B_r} \frac{|A(\xi)|^2 - a(\xi)|}{|x|^2} |u(x)|^2 \, dS
\]

\[
- \frac{1}{2} \int_{\partial B_{r_n}} \frac{|A(\xi)|^2 - a(\xi)|}{|x|^2} |u(x)|^2 \, dS.
\]

From integration by parts it follows

\[
\int_{B_r \setminus B_{r_n}} \frac{u(x) A(\xi/|x|)}{|x|} \cdot \nabla (\nabla u(x) \cdot x) \, dx = -(N - 2) \int_{B_r \setminus B_{r_n}} \frac{u(x) A(\xi/|x|)}{|x|} \cdot \nabla u(x) \, dx
\]

\[
+ r \int_{\partial B_r} \frac{u(x) A(\xi/|x|)}{|x|} \cdot \nabla u(x) \, dS - \delta_n \int_{\partial B_{r_n}} \frac{u(x) A(\xi/|x|)}{|x|} \cdot \nabla u(x) \, dS
\]

\[
- \int_{B_r \setminus B_{r_n}} \frac{A(\xi/|x|)}{|x|} \cdot \nabla u(x) (x \cdot \nabla u(x)) \, dx
\]

and therefore

\[
\int_{B_r \setminus B_{r_n}} \frac{A(\xi/|x|)}{|x|} \cdot \Im (\nabla (\nabla u(x) \cdot x)) \, dx + \int_{B_r \setminus B_{r_n}} \frac{A(\xi/|x|)}{|x|} \cdot \Im \left((\nabla u(x) \cdot x) \nabla u(x)\right) \, dx
\]

\[
= -(N - 2) \int_{B_r \setminus B_{r_n}} \Im \left(\frac{A(\xi/|x|)}{|x|} \cdot \nabla u(x) \frac{u(x)}{|x|}\right) \, dx
\]

\[
+ r \int_{\partial B_r} \Im \left(\frac{A(\xi/|x|)}{|x|} \cdot \nabla u(x) \frac{u(x)}{|x|}\right) \, dS - \delta_n \int_{\partial B_{r_n}} \Im \left(\frac{A(\xi/|x|)}{|x|} \cdot \nabla u(x) \frac{u(x)}{|x|}\right) \, dS.
\]

Putting together (33), (37), (38), and (39) and taking into account that

\[
|\nabla + i \frac{A(\xi/|x|)}{|x|}|^2 u^2 = |\nabla u|^2 + 2 \frac{A(\xi/|x|)}{|x|} \cdot \Im (\nabla u) + |A(\xi/|x|)|^2 u^2,
\]

we obtain

\[
- \frac{N - 2}{2} \int_{B_r \setminus B_{r_n}} \left[|\nabla + i \frac{A(\xi/|x|)}{|x|}|^2 u^2 - a(\xi/|x|) |u|^2\right] \, dx
\]

\[
+ \frac{r}{2} \int_{\partial B_r} \left[|\nabla + i \frac{A(\xi/|x|)}{|x|}|^2 u^2 - a(\xi/|x|) |u|^2\right] \, dS
\]

\[
- \frac{\delta_n}{2} \int_{\partial B_{r_n}} \left[|\nabla + i \frac{A(\xi/|x|)}{|x|}|^2 u^2 - a(\xi/|x|) |u|^2\right] \, dS
\]

\[
= r \int_{\partial B_r} |\frac{\partial u}{\partial v}|^2 \, dS - \delta_n \int_{\partial B_{r_n}} |\frac{\partial u}{\partial v}|^2 \, dS + \int_{B_r \setminus B_{r_n}} \Re (h(x) u(x) (x \cdot \nabla u(x))) \, dx.
\]

Letting \( n \to +\infty \) in the above identity and using (33) we obtain (32). \( \square \)
Let $u$ be a weak $H^1_0(\Omega, \mathbb{C})$-solution to equation (2) in a bounded domain $\Omega \subset \mathbb{R}^N$ containing the origin with $h$ satisfying (4). Let $\overline{R} > 0$ be such that $\overline{B_{\overline{R}}} = \{x \in \mathbb{R}^N : |x| \leq \overline{R}\} \subset \Omega$. Thus, the following functions are well defined for every $r \in (0, \overline{R})$:

\begin{equation}
D(r) = \frac{1}{r^{N-2}} \int_{B_r} \left[ \nabla u(x) + \frac{i}{|x|} A(x/|x|) u(x) - \frac{a(x/|x|)}{|x|^2} |u(x)|^2 - (\Re h(x)) |u(x)|^2 \right] \, dx,
\end{equation}

and

\begin{equation}
H(r) = \frac{1}{r^{N-1}} \int_{\partial B_r} |u|^2 \, dS.
\end{equation}

We are going to study regularity of functions $D$ and $H$. We first differentiate $H$.

**Lemma 5.1.** Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded open set such that $0 \in \Omega$. Let $a, A$ satisfy (A.2), and $u$ be a weak $H^1_0(\Omega, \mathbb{C})$-solution to (2) in $\Omega$, with $h$ satisfying (4). If $H$ is the function defined in (41), then $H \in C^1(0, \overline{R})$ and

\begin{equation}
H'(r) = \frac{2}{r^{N-1}} \int_{\partial B_r} \Re \left( u \frac{\partial \pi}{\partial \nu} \right) \, dS \quad \text{for every } r \in (0, \overline{R}).
\end{equation}

**Proof.** Fix $r_0 \in (0, \overline{R})$ and consider the limit

\begin{equation}
\lim_{r \to r_0} \frac{H(r) - H(r_0)}{r - r_0} = \lim_{r \to r_0} \int_{\partial B_r} \frac{|u(r\theta)|^2 - |u(r_0\theta)|^2}{r - r_0} \, dS(\theta).
\end{equation}

Since $u \in C^1(B_{\overline{R}} \setminus \{0\}, \mathbb{C})$ (see Remark 1.2) then, for every $\theta \in \partial B_1$,

\begin{equation}
\lim_{r \to r_0} \frac{|u(r\theta)|^2 - |u(r_0\theta)|^2}{r - r_0} = 2\Re \left( \frac{\partial \pi}{\partial \nu} (r_0\theta) u(r_0\theta) \right).
\end{equation}

On the other hand, for any $r \in (r_0/2, \overline{R})$ and $\theta \in \partial B_1$ we have

\[ |u(r\theta)|^2 - |u(r_0\theta)|^2 | \leq 2 \sup_{B_\overline{R}} |u| \cdot \sup_{B_\overline{R} \cap B_1} |\nabla u| \]

and hence, by (43), (44), and the Dominated Convergence Theorem, we obtain that

\[ H'(r_0) = \int_{\partial B_{r_0}} 2\Re \left( \frac{\partial \pi}{\partial \nu} (r_0\theta) u(r_0\theta) \right) \, dS(\theta) = \frac{2}{r_0^{N-1}} \int_{\partial B_{r_0}} \Re \left( u \frac{\partial \pi}{\partial \nu} \right) \, dS. \]

The continuity of $H'$ on the interval $(0, \overline{R})$ follows by the representation of $H'$ given above, the fact that $u \in C^1(B_{\overline{R}} \setminus \{0\}, \mathbb{C})$, and the Dominated Convergence Theorem. \qed

In the lemma below, we study the regularity of the function $D$.

**Lemma 5.2.** Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded open set such that $0 \in \Omega$. Let $a, A$ satisfy (A.2), and $u$ be a weak $H^1_0(\Omega, \mathbb{C})$-solution to (2) in $\Omega$, with $h$ satisfying (4). If $D$ is the function defined
Then for any 

Proof

From the fact that 

We now show that 

Lemma 5.3. Let \( \Omega \subset \mathbb{R}^N \), \( N \geq 2 \), be a bounded open set such that \( 0 \in \Omega \), \( a, A \) satisfy (A.2), (A.3), (A.4), and \( u \neq 0 \) be a weak \( H^1_0(\Omega, \mathbb{C}) \)-solution to (4) in \( \Omega \), with \( h \) satisfying (A). Let \( H = H(r) \) be the function defined in (41). Then there exists \( \tau > 0 \) such that \( H(r) > 0 \) for any \( r \in (0, \tau) \).

Proof. Suppose by contradiction that there exists a sequence \( r_n \to 0^+ \) such that \( H(r_n) = 0 \). Then for any \( u, v \equiv 0 \) on \( \partial B_{r_n} \). Multiplying both sides of (2) by \( \tau \) and integrating by parts over \( B_{r_n} \) we obtain

\[
\int_{B_{r_n}} \left| \nabla u(x) + i \frac{A(x/|x|)}{|x|} u(x) \right|^2 dx - \int_{B_{r_n}} \frac{a(x/|x|)}{|x|^2} |u(x)|^2 dx \\
= \int_{B_{r_n}} h(x)|u(x)|^2 dx + \int_{\partial B_{r_n}} \frac{\partial u}{\partial n} \tau dS = \int_{B_{r_n}} h(x)|u(x)|^2 dx.
\]
Taking the real part on both sides it follows
\[
\int_{B_{r_n}} \left| \nabla u(x) + i \frac{A(x/|x|)}{|x|} u(x) \right|^2 dx - \int_{B_{r_n}} \frac{a(x/|x|)}{|x|^2} |u(x)|^2 dx = \int_{B_{r_n}} \Re(h(x)) |u(x)|^2 dx.
\]
Since \( u \equiv 0 \) on \( \partial B_{r_n} \), Lemma 3.1 and (41) yield, for some positive constant \( c_h > 0 \) depending only on \( h \),
\[
0 \geq \int_{B_{r_n}} \left| \nabla u(x) + i \frac{A(x/|x|)}{|x|} u(x) \right|^2 dx - \int_{B_{r_n}} \frac{a(x/|x|)}{|x|^2} |u(x)|^2 dx - c_h r_n^2 \int_{B_{r_n}} \frac{|u(x)|^2}{|x|^2} dx
\]
\[
\geq \left( \mu_1(A, a) + \left( \frac{N-2}{2} \right) - c_h r_n^2 \right) \int_{B_{r_n}} \frac{|u(x)|^2}{|x|^2} dx.
\]
Since \( \mu_1(A, a) + \left( \frac{N-2}{2} \right) > 0 \) and \( r_n \to 0^+ \), we conclude that \( u \equiv 0 \) in \( B_{r_n} \) for \( n \) sufficiently large. Since \( u \equiv 0 \) in a neighborhood of the origin, we may apply, away from the origin, a unique continuation principle for second order elliptic equations with locally bounded coefficients (see e.g. [28]) to conclude that \( u \equiv 0 \) in \( \Omega \), a contradiction. \( \square \)

By virtue of Lemma 5.3 the Almgren type frequency function
\[
N(r) = N_{u, h}(r) = \frac{D(r)}{H(r)}
\]
is well defined in a suitably small interval \((0, \bar{r})\). Collecting Lemmas 5.1 and 5.2 we compute the derivative of \( N \).

**Lemma 5.4.** Let \( \Omega \subset \mathbb{R}^N \), \( N \geq 2 \), be a bounded open set such that \( 0 \in \Omega \), \( a, A \) satisfy (A.2), (A.3), (A.4), and \( u \not\equiv 0 \) be a weak \( H^1_0(\Omega, \mathbb{C}) \)-solution to (2) in \( \Omega \), with \( h \) satisfying (2). Then, letting \( N \) as in (50), there holds \( N \in W^{1,1}_{\text{loc}}(0, \bar{r}) \) and
\[
N'(r) = 2r \left[ \left( \int_{B_r} \left| \frac{\partial u}{\partial r} \right|^2 dS \right) \cdot \left( \int_{B_r} |u|^2 dS \right) - \left( \int_{B_r} \Re \left( \frac{u \partial u}{\partial S} \right) dS \right)^2 \right]
\]
\[
+ \left. \int_{B_r} \Re(h(x)) u(x) (x \cdot \nabla u(x)) \right| dx + \frac{N-2}{2} \int_{B_r} \Re(h(x)) |u(x)|^2 dx - \frac{\tau}{r} \int_{\partial B_r} \Re(h(x)) |u(x)|^2 dS \right] \int_{\partial B_r} |u|^2 dS
\]
in a distributional sense and for a.e. \( r \in (0, \bar{r}) \).

**Proof.** From Lemmas 5.3, 5.1 and 5.2 it follows that \( N \in W^{1,1}_{\text{loc}}(0, \bar{r}) \). Multiplying both sides of (2) by \( \bar{u} \), integrating by parts, and taking the real part we obtain the identity
\[
\int_{B_r} \left| \nabla u(x) + i \frac{A(x/|x|)}{|x|} u(x) \right|^2 dx - \frac{a(x/|x|)}{|x|^2} |u(x)|^2 dx - \Re(h(x)) |u(x)|^2 \right] \right| dx = \int_{\partial B_r} \Re \left( \frac{\partial \bar{u}}{\partial \nu} \right) dS.
\]
Therefore, by (40) and (42) we infer
\[
D(r) = \frac{1}{2} r H'(r)
\]
for every \( r \in (0, \bar{r}) \). From (52) we have that
\[
\mathcal{N}'(r) = \frac{D'(r)H(r) - D(r)H'(r)}{(H(r))^2} = \frac{D'(r)H(r) - \frac{1}{2}r(H'(r))^2}{(H(r))^2}
\]
and, using (42) and (45), the proof of the lemma easily follows.

We now prove that \( \mathcal{N}(r) \) admits a finite limit as \( r \to 0^+ \).

**Lemma 5.5.** Under the same assumptions as in Lemma 5.4, the limit
\[
\gamma := \lim_{r \to 0^+} \mathcal{N}(r)
\]
exists and is finite.

**Proof.** We start by proving that \( \mathcal{N}(r) \) is bounded from below as \( r \to 0^+ \). By Lemma 3.4, proceeding as in (10), we arrive, for some positive constant \( c_h > 0 \) depending only on \( h \), to
\[
\int_{B_r} |\nabla u(x) + \frac{1}{|x|} A(x/|x|) u(x)|^2 \, dx - \int_{\partial B_r} a(x/|x|) |u(x)|^2 \, dx - \int_{B_r} (\Re h(x)) |u(x)|^2 \, dx
\geq -\frac{N - 2}{2r} \int_{\partial B_r} |u(x)|^2 \, dS + \mu_1(A, a) + \left( \frac{N - 2}{2} - c_h r^\varepsilon \right) \int_{B_r} |u(x)|^2 \, dx
\]
for \( r > 0 \) sufficiently small. This with (10) - (11) yields
\[
\mathcal{N}(r) > -\frac{N - 2}{2}
\]
for any \( r > 0 \) sufficiently close to zero. Thanks to (9), for some \( C_1 > 0 \), we estimate
\[
\left| \int_{B_r} \Re(h(x)u(x)(x \cdot \nabla u(x))) \, dx + \frac{N - 2}{2} \int_{B_r} (\Re h(x)) |u(x)|^2 \, dx - \frac{r}{2} \int_{\partial B_r} (\Re h(x)) |u(x)|^2 \, dS \right|
\leq C_1 r^\varepsilon \left( \int_{B_r} |\nabla u + \frac{1}{|x|} A(x/|x|) u|^2 \, dx + \int_{B_r} \frac{|u(x)|^2}{|x|^2} \, dx + r^{-N-2} H(r) \right).
\]
Together with (53), this implies that there exist \( C_2 > 0 \) and \( \tilde{r} > 0 \) such that, for any \( r \in (0, \tilde{r}) \),
\[
\left| \int_{B_r} \Re(h(x)u(x)(x \cdot \nabla u(x))) \, dx + \frac{N - 2}{2} \int_{B_r} (\Re h(x)) |u(x)|^2 \, dx - \frac{r}{2} \int_{\partial B_r} (\Re h(x)) |u(x)|^2 \, dS \right|
\leq C_2 r^{\varepsilon + N-2} [D(r) + H(r)].
\]
Therefore, for any \( r \in (0, \tilde{r}) \), we have that
\[
\int_{B_r} \Re(h(x)u(x)(x \cdot \nabla u(x))) \, dx + \frac{N - 2}{2} \int_{B_r} (\Re h(x)) |u(x)|^2 \, dx - \frac{r}{2} \int_{\partial B_r} (\Re h(x)) |u(x)|^2 \, dS
\leq C_2 r^{-1+\varepsilon} \frac{D(r) + H(r)}{H(r)} \leq C_2 r^{-1+\varepsilon} \mathcal{N}(r) + C_2 r^{-1+\varepsilon}.
\]
By Lemma 5.4 and Schwarz’s inequality, one sees that
\[
N'(r) \geq 2 \int_{B_r} \mathfrak{R}(\mathcal{H}(x)u(x)(x \cdot \nabla u(x))) \, dx + \frac{N - \int_{\partial B_r} \mathfrak{R}(\mathcal{H}(x))|u(x)|^2 \, dS}{\int_{\partial B_r} |u(x)|^2 \, dS}
\]
and hence by (55) we obtain
\[
N'(r) \geq -2C_2 r^{-1+\varepsilon} N(r) - 2C_2 r^{-1+\varepsilon}
\]
for any \( r \in (0, \tilde{r}) \). After integration it follows that, for some \( C_3 > 0 \),
\[
N(r) \leq N(\tilde{r}) e^{2C_2 (\tilde{r} - r)} + 2C_2 e^{-2C_2 \tilde{r}^2} \int_{\tilde{r}}^r s^{-1+\varepsilon} e^{2C_2 s} \, ds \leq C_3
\]
for any \( r \in (0, \tilde{r}) \). This shows that the left hand side of (55) belongs to \( L^1(0, \tilde{r}) \). In particular by Lemma 5.6 and Schwarz’s inequality we see that \( N' \) is the sum of a nonnegative function and of a \( L^1 \)-function. Therefore
\[
N(r) = N(\tilde{r}) - \int_{\tilde{r}}^r N'(s) \, ds
\]
admits a limit as \( r \to 0^+ \) which is necessarily finite in view of (55) and (57).

A first consequence of the above analysis on the Almgren’s frequency function is the following estimate of \( H(r) \).

**Lemma 5.6.** Under the same assumptions as in Lemma 5.4, let \( \gamma := \lim_{r \to 0^+} N(r) \) be as in Lemma 5.5. Then there exists a constant \( K_1 > 0 \) such that
\[
H(r) \leq K_1 r^{2\gamma}
\]
for all \( r \in (0, \tilde{r}) \). On the other hand for any \( \sigma > 0 \) there exists a constant \( K_2(\sigma) > 0 \) depending on \( \sigma \) such that
\[
H(r) \geq K_2(\sigma) r^{2\gamma + \sigma}
\]
for all \( r \in (0, \tilde{r}) \).

**Proof.** We start by proving (58). Since, by Lemma 5.5, \( N' \in L^1(0, \tilde{r}) \) and \( N' \) is bounded, then by (56), we infer that
\[
N(r) - \gamma = \int_0^r N'(s) \, ds \geq -C_4 r^{\varepsilon}
\]
for some constant \( C_4 > 0 \) and \( r \in (0, \tilde{r}) \) with \( 0 < \tilde{r} < \tilde{r} \). Therefore by (52) and (60) we deduce that for \( r \in (0, \tilde{r}) \)
\[
\frac{H'(r)}{H(r)} = \frac{2N(r)}{r} \geq \frac{2\gamma}{r} - 2C_4 r^{-1+\varepsilon}.
\]
The proof of (58) follows immediately after integration in the previous differential inequality over the interval \( (r, \tilde{r}) \) by continuity of \( H \) outside \( 0 \).

Let us prove (59). Since \( \gamma = \lim_{r \to 0^+} N(r) \), for any \( \sigma > 0 \) there exists \( r_\sigma > 0 \) such that \( N(r) < \gamma + \sigma/2 \) for any \( r \in (0, r_\sigma) \) and hence
\[
\frac{H'(r)}{H(r)} = \frac{2N(r)}{r} < \frac{2\gamma + \sigma}{r} \quad \text{for all } r \in (0, r_\sigma).
\]
Integrating over the interval \( (r, r_\sigma) \) and by continuity of \( H \) outside \( 0 \), we obtain (59) for some constant \( K_2(\sigma) \) depending on \( \sigma \). \( \square \)
6. Proofs of Theorems 1.3 and 1.5

In this section we use the monotonicity properties established in section 5 combined with a blow-up technique to deduce asymptotics of solutions near the singularity and to prove Theorems 1.3 and 1.5.

Lemma 6.1. Let \( \Omega \subset \mathbb{R}^N \), \( N \geq 2 \), be a bounded open set containing 0, \( a, A \) such that (A.2), (A.3), and (A.4) hold, and \( h \) as in (6). For \( u \in H_1^2(\Omega, \mathbb{C}) \) weakly solving (2), \( u \neq 0 \), let \( \gamma := \lim_{r \to 0^+} \mathcal{N}(r) \) as in Lemma 5.5. Then

(i) there exists \( k_0 \in \mathbb{N} \) such that \( \gamma = -\frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_{k_0}(A, a)} \);

(ii) for every sequence \( \lambda_n \to 0^+ \), there exist a subsequence \( \{\lambda_{n_k}\}_{k \in \mathbb{N}} \) and an eigenfunction \( \psi \) of the operator \( L_{A,a} \) associated to the eigenvalue \( \mu_{k_0}(A, a) \) such that \( \|\psi\|_{L_2(S^{n-1}, \mathbb{C})} = 1 \) and

\[
\frac{u(\lambda_{n_k} x)}{\sqrt{H(\lambda_{n_k})}} \to |x|^\gamma \psi \left( \frac{x}{|x|} \right)
\]

weakly in \( H^1(B_1, \mathbb{C}) \), strongly in \( H^1(B_r, \mathbb{C}) \) for every \( 0 < r < 1 \), and in \( C^{1,\tau}_{\text{loc}}(B_1 \setminus \{0\}, \mathbb{C}) \) for any \( \tau \in (0,1) \).

Proof. Let us set

\[
w^\lambda(x) = \frac{u(\lambda x)}{\sqrt{H(\lambda)}}.
\]

We notice that \( \int_{\partial B_1} |w^\lambda|^2 dS = 1 \). Moreover, by scaling and (37),

\[
(61) \quad \int_{B_1} \left| \nabla w^\lambda(x) + \frac{A\left(\frac{x}{|x|}\right)}{|x|} w^\lambda(x) \right|^2 dx - \int_{B_1} \frac{a\left(\frac{x}{|x|}\right)}{|x|^2} \left| w^\lambda(x) \right|^2 dx - \int_{B_1} \lambda^2 (\Re(h(\lambda x))) |w^\lambda(x)|^2 dx
\]

\[
= \mathcal{N}(\lambda) \leq \text{const}.
\]

Hence, by (29) and (6) there exists \( c_h > 0 \) such that

\[
\left( \mu_1(A, a) + \left(\frac{N-2}{2}\right)^2 - c_h \lambda^2 \right) \int_{B_1} \frac{|w^\lambda(x)|^2}{|x|^2} dx \leq N - \frac{2}{2} + \mathcal{N}(\lambda),
\]

and, consequently, there exist \( \bar{\lambda} > 0 \) and \( \text{const} > 0 \) such that

\[
\int_{B_1} \frac{|w^\lambda(x)|^2}{|x|^2} dx \leq \text{const} \quad \text{for every} \quad 0 < \lambda < \bar{\lambda},
\]

which, in view of (61), implies that \( \{w^\lambda\}_{\lambda \in (0,\bar{\lambda})} \) is bounded in \( H_1^2(B_1, \mathbb{C}) \).

Therefore, for any given sequence \( \lambda_n \to 0^+ \), there exists a subsequence \( \lambda_{n_k} \to 0^+ \) such that \( w^{\lambda_{n_k}} \rightharpoonup w \) weakly in \( H_1^2(B_1, \mathbb{C}) \) for some \( w \in H_1^2(B_1, \mathbb{C}) \). We notice that \( H_1^2(B_1, \mathbb{C}) \) is continuously embedded into \( H^1(B_1, \mathbb{C}) \), hence \( w^{\lambda_{n_k}} \to w \) weakly also in \( H^1(B_1, \mathbb{C}) \). Due to compactness of the trace imbedding \( H^1(B_1, \mathbb{C}) \hookrightarrow L^2(\partial B_1, \mathbb{C}) \), we obtain that \( \int_{\partial B_1} |w|^2 dS = 1 \). In particular \( w \neq 0 \). Furthermore, weak convergence allows passing to the weak limit in the equation

\[
(62) \quad L_{A,a} w^{\lambda_{n_k}}(x) = \lambda_{n_k}^2 h(\lambda_{n_k} x) w^{\lambda_{n_k}}(x)
\]
which holds in a weak sense in \( B_{\pi/\lambda_n} \supset B_1 \) (see the beginning of section 3 for the definition of \( \pi \)), thus yielding
\[
L_{\lambda, a} w(x) = 0 \quad \text{in } B_1.
\]
A bootstrap argument and classical regularity theory lead to
\[
w^{\lambda_n} \rightarrow w \quad \text{in } C^{1, \tau}_a(B_1 \setminus \{0\}, \mathbb{C})
\]
for any \( \tau \in (0, 1) \) and
\[
w^{\lambda_n} \rightarrow w \quad \text{in } H^1(B_r, \mathbb{C}) \quad \text{and} \quad H^1_0(B_r, \mathbb{C})
\]
for any \( r \in (0, 1) \). Since the functions \( w^{\lambda_n} \) solve equation (62), then for any \( r \in (0, 1) \) we may define the functions
\[
D_k(r) = \frac{1}{r^{N-2}} \int_{B_r} \left[ \nabla w^{\lambda_n}(x) + i \frac{A(x/|x|)}{|x|} w^{\lambda_n}(x) \right]^2 dx
\]
\[
- \frac{1}{r^{N-2}} \int_{B_r} \left[ a(x/|x|) |w^{\lambda_n}(x)|^2 + \lambda_n^2 (\Re h(\lambda_n x)) |w^{\lambda_n}(x)|^2 \right] dx
\]
and
\[
H_k(r) = \frac{1}{r^{N-1}} \int_{\partial B_r} |w^{\lambda_n}|^2 dS.
\]
On the other hand, since \( w \) solves (63), then we put
\[
D_w(r) = \frac{1}{r^{N-2}} \int_{B_r} \left[ \nabla w(x) + i \frac{A(x/|x|)}{|x|} w(x) \right]^2 dx
\]
\[
- \frac{1}{r^{N-2}} \int_{B_r} \left[ a(x/|x|) |w(x)|^2 - \frac{a(x/|x|)}{|x|^2} |w(x)|^2 \right] dx
\]
for all \( r \in (0, 1) \) and
\[
H_w(r) = \frac{1}{r^{N-1}} \int_{\partial B_r} |w|^2 dS \quad \text{for all } r \in (0, 1).
\]
Using a change of variables, one sees that
\[
N_k(r) := \frac{D_k(r)}{H_k(r)} = \frac{D(\lambda_n r)}{H(\lambda_n r)} = N(\lambda_n r) \quad \text{for all } r \in (0, 1).
\]
By (65) and (64), we have for any fixed \( r \in (0, 1) \)
\[
D_k(r) \rightarrow D_w(r).
\]
On the other hand, by compactness of the trace imbedding \( H^1(B_r, \mathbb{C}) \hookrightarrow L^2(\partial B_r, \mathbb{C}) \), we also have
\[
H_k(r) \rightarrow H_w(r) \quad \text{for any fixed } r \in (0, 1).
\]
From (29) it follows that \( D_w(r) > -\frac{N}{2} H_w(r) \) for all \( r \in (0, 1) \). Therefore, if, for some \( r \in (0, 1) \), \( H_w(r) = 0 \) then \( D_w(r) > 0 \), and passing to the limit in (67) should give a contradiction with Lemma 5.5. Hence \( H_w(r) > 0 \) for all \( r \in (0, 1) \). Thus the function
\[
N_w(r) := \frac{D_w(r)}{H_w(r)}
\]
is well defined for \( r \in (0, 1) \). This, together with (67), (68), (69), and Lemma 5.5 shows that
\[
N_w(r) = \lim_{k \to \infty} N(\lambda_n r) = \gamma
\]
for all $r \in (0,1)$. Therefore $N_w$ is constant in $(0,1)$ and hence $N_w'(r) = 0$ for any $r \in (0,1)$. By (63) and Lemma 5.3 with $h \equiv 0$, we obtain

\[
\left( \int_{\partial B_r} \left| \frac{\partial w}{\partial \nu} \right|^2 dS \right) \cdot \left( \int_{\partial B_r} |w|^2 dS \right) - \left( \int_{\partial B_r} \Re \left( w \frac{\partial \bar{w}}{\partial \nu} \right) dS \right)^2 = 0 \quad \text{for all } r \in (0,1),
\]

i.e.

\[
\left| \int_{\partial B_r} \Re \left( w \frac{\partial \bar{w}}{\partial \nu} \right) dS \right|^2 = \|w\|^2_{L^2(\partial B_r, \mathbb{C})} \cdot \left\| \frac{\partial w}{\partial \nu} \right\|^2_{L^2(\partial B_r, \mathbb{C})}.
\]

This shows that $w$ and $\frac{\partial w}{\partial \nu}$ have the same direction as vectors in $L^2(\partial B_r, \mathbb{C})$ and hence there exists a real valued function $\eta = \eta(r)$ such that $\frac{\partial w}{\partial \nu}(r, \theta) = \eta(r)w(r, \theta)$ for $r \in (0,1)$. After integration we obtain

\[
w(r, \theta) = e^{\int_1^r \eta(s)ds} w(1, \theta) = \varphi(r)\psi(\theta) \quad r \in (0,1), \ \theta \in S^{N-1},
\]

where we put $\varphi(r) = e^{\int_1^r \eta(s)ds}$ and $\psi(\theta) = w(1, \theta)$. Since

\[
\mathcal{L}_{A,a} w = -\frac{\partial^2 w}{\partial r^2} - \frac{N-1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} L_{A,a} w,
\]

then (71) yields

\[
\left( -\varphi''(r) - \frac{N-1}{r} \varphi'(r) \right) \psi(\theta) + \frac{\varphi(r)}{r^2} L_{A,a} \psi(\theta) = 0.
\]

Taking $r$ fixed we deduce that $\psi$ is an eigenfunction of the operator $L_{A,a}$. If $\mu_{k_0}(A,a)$ is the corresponding eigenvalue then $\varphi(r)$ solves the equation

\[
-\varphi''(r) - \frac{N-1}{r} \varphi(r) + \frac{\mu_{k_0}(A,a)}{r^2} \varphi(r) = 0
\]

and hence $\varphi(r)$ is of the form

\[
\varphi(r) = c_1 r^{\sigma_{k_0}^+} + c_2 r^{\sigma_{k_0}^-}
\]

for some $c_1, c_2 \in \mathbb{R}$, where

\[
\sigma_{k_0}^+ = -\frac{N-2}{2} + \sqrt{\left( \frac{N-2}{2} \right)^2 + \mu_{k_0}(A,a)} \quad \text{and} \quad \sigma_{k_0}^- = -\frac{N-2}{2} - \sqrt{\left( \frac{N-2}{2} \right)^2 + \mu_{k_0}(A,a)}.
\]

Since the function $\frac{1}{|x|}(x^{\sigma_{k_0}^+} \psi(\frac{|x|}{r})) \notin L^2(B_1, \mathbb{C})$ and hence $|x|^{\sigma_{k_0}^-} \psi(\frac{|x|}{r}) \notin H_1^1(B_1, \mathbb{C})$, then $c_2 = 0$ and $\varphi(r) = c_1 r^{\sigma_{k_0}^+}$. Since $\varphi(1) = 1$, we obtain that $c_1 = 1$ and then

\[
w(r, \theta) = r^{\sigma_{k_0}^+} \psi(\theta), \quad \text{for all } r \in (0,1) \text{ and } \theta \in S^{N-1}.
\]

It remains to prove part (i). Since $w$ solves (63), after integration by parts

\[
\int_{B_r} \left( \nabla w(x) + i \frac{A(x/|x|)w(x)}{|x|} \right)^2 - \frac{a(|x|/x)}{|x|^2} |w(x)|^2 \] 
\[
dx = \int_{\partial B_r} \frac{\partial w}{\partial \nu} \bar{w} dS.
\]

Therefore, by (63), (65), (70) and (72), it follows

\[
\gamma = N_w(r) = \frac{D_w(r)}{H_w(r)} = \frac{r \int_{\partial B_r} \frac{\partial w}{\partial \nu} \bar{w} dS}{\int_{\partial B_r} |w|^2 dS} = \sigma_{k_0}^+.
\]
This completes the proof of the lemma. □

A further step towards a-priori bounds for solutions to (2) relies in uniformly estimating the supremum of \(|u|\) on \(\partial B_r\) with \(H(r)\).

**Lemma 6.2.** Let \(\Omega \subset \mathbb{R}^N\), \(N \geqslant 2\), be a bounded open set containing 0, \(a, \mathbf{A}\) such that (A.2), (A.3) and (A.4) hold, and \(h\) as in (6). Then, for any weak \(H^1_0(\Omega, \mathbb{C})\)-solution \(u\) to (3) there exist \(\bar{s} > 0\) and \(C > 0\) such that

\[
\sup_{\partial B_s}|u|^2 \leqslant \frac{C}{s^{N-1}} \int_{\partial B_s} |u|^2 \, dS \quad \text{for every } 0 < s < \bar{s}.
\]

**Proof.** Let \(\gamma = \lim_{r \to 0^+} \gamma(r)\) as in Lemma 5.5 and \(k_0 \in \mathbb{N}\) such that \(\gamma = -\frac{N-2}{2} + \sqrt{(\frac{N-2}{2})^2 + \mu_{k_0}(\mathbf{A}, a)}\), see Lemma 6.1. Denote as \(\mathbf{A}_0\) the eigenspace of the operator \(L_{\mathbf{A}, a}\) associated to the eigenvalue \(\mu_{k_0}(\mathbf{A}, a)\). Since \(\dim \mathbf{A}_0\) is finite, it is easy to verify that

\[
\Lambda = \sup_{v \in \mathbf{A}_0 \setminus \{0\}} \sup_{|v| \leq 1} \int_{S^{N-1}} |v|^2 \, dS < +\infty.
\]

Let \(\tilde{C} > 2^{N-1}\Lambda\). We claim that there exists \(\bar{\lambda}\) such that

\[
(73) \quad \sup_{\partial B_{\lambda/2}} |w\lambda|^2 \leqslant \tilde{C} \int_{\partial B_{\lambda/2}} |w\lambda|^2 \, dS \quad \text{for every } \lambda \in (0, \bar{\lambda}).
\]

To prove (73), assume by contradiction that there exists a sequence \(\{\lambda_n\}_{n \in \mathbb{N}}\) such that \(\lambda_n \to 0^+\) and

\[
(74) \quad \sup_{\partial B_{\lambda_n/2}} |w\lambda_n|^2 > \tilde{C} \int_{\partial B_{\lambda_n/2}} |w\lambda_n|^2 \, dS.
\]

Lemma 6.1 implies that there exist a subsequence \(\{\lambda_{n_j}\}_{j \in \mathbb{N}}\) and an eigenfunction \(\psi \in \mathbf{A}_0\) such that \(||\psi||^2_{L^2(S^{N-1}, \mathbb{C})} = 1\) and \(w\lambda_{n_j} \to |\tau|^\gamma \psi(x)\left(\frac{\tau}{|\tau|}\right)\) weakly in \(H^1(B_1, \mathbb{C})\) and in \(C^{1,\tau}_{\text{loc}}(B_1 \setminus \{0\}, \mathbb{C})\) for any \(\tau \in (0, 1)\). Passing to limit in (74), this should imply that

\[
\sup_{|v| \leq 1} \int_{S^{N-1}} |v|^2 \, dS \geq \frac{\tilde{C}}{2^{N-1}} \int_{S^{N-1}} |\psi|^2 \, dS > \Lambda \int_{S^{N-1}} |\psi|^2 \, dS
\]

giving rise to a contradiction with the definition of \(\Lambda\). Claim (73) is thereby proved.

Estimate (73) can be written as

\[
\sup_{\partial B_{\lambda/2}} |u|^2 \leqslant \frac{\tilde{C}}{\lambda^{N-1}} \int_{\partial B_{\lambda/2}} |u|^2 \, dS \quad \text{for every } \lambda \in (0, \bar{\lambda}).
\]

Choosing \(\bar{s} = \frac{1}{2}\bar{\lambda}\) and \(C = 2^{1-N}\tilde{C}\), the conclusion follows. □

From Lemmas 5.3 and 6.2 we deduce the following pointwise estimate for solutions to (2).
Corollary 6.3. Let \( \Omega \subset \mathbb{R}^N, N \geq 2, \) be a bounded open set containing 0, a, A such that \( \text{[A.2]}, \text{[A.3]} \) and \( \text{[A.4]} \) hold, and h as in \([6]\). Then, for any weak \( H^1(\Omega, \mathbb{C}) \)-solution \( u \) to \( \text{[3]} \) there exist \( s > 0 \) and \( C > 0 \) such that
\[
|u(x)| \leq C |x|^\gamma \quad \text{for every } x \in B_s,
\]
where \( \gamma = \lim_{r \to 0^+} N(r) \) as in Lemma 5.5.

PROOF. It follows from \([58]\) and Lemma 6.2. \( \square \)

Let us now describe the behavior of \( H(r) \) as \( r \to 0^+ \).

Lemma 6.4. Under the same assumptions as in Lemma 6.3 and letting \( \gamma := \lim_{r \to 0^+} N(r) \in \mathbb{R} \) as in Lemma 5.3 the limit
\[
\lim_{r \to 0^+} r^{-2\gamma} H(r)
\]
exists and it is finite.

PROOF. In view of \([58]\) it is sufficient to prove that the limit exists. By \([11], [52], \) and Lemma 5.5 we have
\[
\frac{d}{dr} \frac{H(r)}{r^{2\gamma}} = -2\gamma r^{-2\gamma-1}H(r) + r^{-2\gamma} H'(r) = 2r^{-2\gamma-1}(D(r) - \gamma H(r)) = 2r^{-2\gamma-1} H(r) \int_0^r N'(s)ds.
\]
Denote by \( M_1(r) \) and \( M_2(r) \) respectively the first and the second term in the right hand side of \([51]\) in order to obtain, after integration over \( (r, \tilde{r}) \),
\[
(75) \quad \frac{H(\tilde{r})}{\tilde{r}^{2\gamma}} - \frac{H(r)}{r^{2\gamma}} = \int_r^{\tilde{r}} 2s^{-2\gamma-1}H(s) \left( \int_0^s M_1(t)dt \right) ds + \int_r^{\tilde{r}} 2s^{-2\gamma-1}H(s) \left( \int_0^s M_2(t)dt \right) ds.
\]
By Schwarz’s inequality we have that \( M_1(t) \geq 0 \) and hence
\[
\lim_{r \to 0^+} \int_r^{\tilde{r}} 2s^{-2\gamma-1}H(s) \left( \int_0^s M_1(t)dt \right) ds
\]
exists. On the other hand, by \([60]\) and \([58]\) we deduce that \( |M_2(r)| = O(r^{-1+\varepsilon}) \) and \( H(r) = O(r^{2\gamma}) \) as \( r \to 0^+ \). Therefore, if \( \tilde{r} \) is sufficiently small, for some const \( > 0 \) there holds
\[
\left| s^{-2\gamma-1}H(s) \left( \int_0^s M_2(t)dt \right) \right| \leq \frac{\text{const}}{\varepsilon} s^{-1+\varepsilon}
\]
for all \( r \in (0, \tilde{r}) \), which proves that \( s^{-2\gamma-1}H(s) \left( \int_0^s M_2(t)dt \right) \in L^1(0, \tilde{r}) \). We may conclude that both terms in the right hand side of \([75]\) admit a limit as \( r \to 0^+ \) thus completing the proof of the lemma. \( \square \)

The limit \( \lim_{r \to 0^+} r^{-2\gamma} H(r) \) is indeed strictly positive, as we prove in the following lemma.

Lemma 6.5. Under the same assumptions as in Lemma 6.4 and letting \( \gamma := \lim_{r \to 0^+} N(r) \in \mathbb{R} \) as in Lemma 5.3 there holds
\[
\lim_{r \to 0^+} r^{-2\gamma} H(r) > 0.
\]
Proof. Let us fix $R > 0$ such that $B_R \subset \Omega$. For any $k \in \mathbb{N} \setminus \{0\}$, let $\psi_k$ be a $L^2$-normalized eigenfunction of the operator $L_{A,a}$ on the sphere associated to the $k$-th eigenvalue $\mu_k(A,a)$, i.e. satisfying
\begin{equation}
\begin{cases}
L_{A,a} \psi_k(\theta) = \mu_k(A,a) \psi_k(\theta), & \text{in } S^{N-1}, \\
\int_{S^{N-1}} |\psi_k(\theta)|^2 dS(\theta) = 1.
\end{cases}
\end{equation}
We can choose the functions $\psi_k$ in such a way that they form an orthonormal basis of $L^2(S^{N-1}, \mathbb{C})$, hence $u$ and $hu$ can be expanded as
\begin{equation}
u(x) = u(\lambda \theta) = \sum_{k=1}^{\infty} \varphi_k(\lambda) \psi_k(\theta), \quad h(x)u(x) = h(\lambda \theta)u(\lambda \theta) = \sum_{k=1}^{\infty} \zeta_k(\lambda) \psi_k(\theta),
\end{equation}
where $\lambda = |x| \in (0, R]$, $\theta = x/|x| \in S^{N-1}$, and
\begin{equation}
\varphi_k(\lambda) = \int_{S^{N-1}} u(\lambda \theta) \psi_k(\theta) dS(\theta), \quad \zeta_k(\lambda) = \int_{S^{N-1}} h(\lambda \theta) u(\lambda \theta) \overline{\psi_k(\theta)} dS(\theta).
\end{equation}
Equations (2) and (76) imply that, for every $k$,
\begin{equation}
-\varphi_k''(\lambda) - \frac{N-1}{\lambda} \varphi_k'(\lambda) + \frac{\mu_k(A,a)}{\lambda^2} \varphi_k(\lambda) = \zeta_k(\lambda), \quad \text{in } (0, R).
\end{equation}
A direct calculation shows that, for some $c_1^k, c_2^k \in \mathbb{R}$,
\begin{equation}
\varphi_k(\lambda) = \lambda^\sigma_k^+ \left( c_1^k + \int_0^R \frac{s-\sigma_k^+}{\sigma_k^+ - \sigma_k^-} \zeta_k(s) ds \right) + \lambda^\sigma_k^- \left( c_2^k + \int_0^R \frac{s-\sigma_k^-}{\sigma_k^- - \sigma_k^+} \zeta_k(s) ds \right),
\end{equation}
where
\begin{equation}
\sigma_k^+ = -\frac{N-2}{2} + \sqrt{\left( \frac{N-2}{2} \right)^2 + \mu_k(A,a)} \quad \text{and} \quad \sigma_k^- = -\frac{N-2}{2} - \sqrt{\left( \frac{N-2}{2} \right)^2 + \mu_k(A,a)}.
\end{equation}
In view of Lemma 6.1 there exist $j_0, m \in \mathbb{N}$, $j_0, m \geq 1$ such that $m$ is the multiplicity of the eigenvalue $\mu_{j_0}(A,a) = \mu_{j_0+1}(A,a) = \cdots = \mu_{j_0+m-1}(A,a)$ and
\begin{equation}
\gamma = \lim_{r \to 0^+} N(r) = \sigma_i^+, \quad i = j_0, \ldots, j_0 + m - 1.
\end{equation}
The Parseval identity yields
\begin{equation}
H(\lambda) = \int_{S^{N-1}} |u(\lambda \theta)|^2 dS(\theta) = \sum_{k=1}^{\infty} |\varphi_k(\lambda)|^2, \quad \text{for all } 0 < \lambda \leq R.
\end{equation}
Let us assume by contradiction that $\lim_{\lambda \to 0^+} \lambda^{-2\gamma} H(\lambda) = 0$ and fix $i \in \{j_0, \ldots, j_0 + m - 1\}$. Then, (80) and (81) imply that
\begin{equation}
\lim_{\lambda \to 0^+} \lambda^{-\sigma_i^+} \varphi_i(\lambda) = 0.
\end{equation}
From (6) and Corollary 6.3 we obtain that
\begin{equation}
\zeta_i(\lambda) = O(\lambda^{-2+\epsilon+\sigma_i^+}) \quad \text{as } \lambda \to 0^+,
\end{equation}
and, consequently, the functions
\[ s \mapsto \frac{s^{-\sigma_i^+ + 1}}{\sigma_i^+ - \sigma_i^-} \zeta_i(s) \quad \text{and} \quad s \mapsto \frac{s^{-\sigma_i^- + 1}}{\sigma_i^- - \sigma_i^+} \zeta_i(s) \]
belong to \( L^1((0, R), \mathbb{C}) \). Hence
\[ \lambda^{\sigma_i^+} \left( c_1^i + \int_R \frac{s^{-\sigma_i^+ + 1}}{\sigma_i^+ - \sigma_i^-} \zeta_i(s) \, ds \right) = o(\lambda^{\sigma_i^-}) \quad \text{as} \quad \lambda \to 0^+, \]
and then, since \( \frac{\mu}{|\psi|} \in L^2(B_R, \mathbb{C}) \) and \( \frac{|\xi_i^-|}{|\xi_i^+|} \notin L^2(B_R, \mathbb{C}) \), we conclude that there must be
\[ c_2^i = -\int_0^R \frac{s^{-\sigma_i^- + 1}}{\sigma_i^- - \sigma_i^+} \zeta_i(s) \, ds. \]
Using (83), we then deduce that
\[ \lambda^{\sigma_i^-} \left( c_2^i + \int_R \frac{s^{-\sigma_i^- + 1}}{\sigma_i^- - \sigma_i^+} \zeta_i(s) \, ds \right) = \lambda^{\sigma_i^-} \left( \int_0^\lambda \frac{s^{-\sigma_i^- + 1}}{\sigma_i^- - \sigma_i^+} \zeta_i(s) \, ds \right) = O(\lambda^{\sigma_i^+ + \varepsilon}) \]
as \( \lambda \to 0^+ \). From (79), (82), and (84), we obtain that
\[ c_1^i + \int_0^R \frac{s^{-\sigma_i^+ + 1}}{\sigma_i^+ - \sigma_i^-} \zeta_i(s) \, ds = 0, \]
thus implying, together with (83),
\[ \lambda^{\sigma_i^+} \left( c_1^i + \int_R \frac{s^{-\sigma_i^+ + 1}}{\sigma_i^+ - \sigma_i^-} \zeta_i(s) \, ds \right) = \lambda^{\sigma_i^+} \int_0^\lambda \frac{s^{-\sigma_i^+ + 1}}{\sigma_i^+ - \sigma_i^-} \zeta_i(s) \, ds = O(\lambda^{\sigma_i^+ + \varepsilon}) \]
as \( \lambda \to 0^+ \). Collecting (79), (84), and (85), we conclude that
\[ \varphi_i(\lambda) = O(\lambda^{\sigma_i^+ + \varepsilon}) \quad \text{as} \quad \lambda \to 0^+ \quad \text{for every} \quad i \in \{1, \ldots, m\}, \]

namely, setting \( u^\lambda(\theta) = u(\lambda \theta) \),
\[ (u^\lambda, \psi)^{L^2(S^{N-1}, \mathbb{C})} = O(\lambda^{\gamma + \varepsilon}) \quad \text{as} \quad \lambda \to 0^+ \]
for every \( \psi \in A_0 \), where \( A_0 \) is the eigenspace of the operator \( L_{A,a} \) associated to the eigenvalue \( \mu_{j_0}(A, a) = \mu_{j_0 + 1}(A, a) = \cdots = \mu_{j_0 + m - 1}(A, a) \). Let \( w^\lambda(\theta) = (H(\lambda))^{-1/2} u(\lambda \theta) \). From (59), there exists \( C(\varepsilon) > 0 \) such that \( \sqrt{H(\lambda)} \geq C(\varepsilon) \lambda^{\gamma + \varepsilon} \) for \( \lambda \) small, and therefore
\[ (w^\lambda, \psi)^{L^2(S^{N-1}, \mathbb{C})} = O(\lambda^{\gamma / 2}) = o(1) \quad \text{as} \quad \lambda \to 0^+ \]
for every \( \psi \in A_0 \). From Lemma 6.1, for every sequence \( \lambda_n \to 0^+ \), there exist a subsequence \( \{\lambda_{n_j}\}_{j \in \mathbb{N}} \) and an eigenfunction \( \tilde{\psi} \in A_0 \) such that
\[ \int_{S^{N-1}} |\tilde{\psi}(\theta)|^2 \, dS = 1 \quad \text{and} \quad w^\lambda_{n_j} \to \tilde{\psi} \quad \text{in} \quad L^2(S^{N-1}, \mathbb{C}). \]
From (86) and (87), we infer that
\[ 0 = \lim_{j \to +\infty} (w^\lambda_{n_j}, \tilde{\psi})^{L^2(S^{N-1}, \mathbb{C})} = \|\tilde{\psi}\|^2_{L^2(S^{N-1}, \mathbb{C})} = 1, \]
thus reaching a contradiction.
The analysis carried out in this section leads to a complete description of the behavior of solutions to (2) near the singularity and hence to the proof of Theorem 1.3.

**Proof of Theorem 1.3**  Identity (9) follows from part (i) of Lemma 6.1, thus there exists \( k_0 \in \mathbb{N}, \ k_0 \geq 1 \), such that \( \lim_{r \to 0^+} \mathcal{N}_{a,h}(r) = -\frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_{k_0}(A,a)} \). Let us denote as \( m \) the multiplicity of \( \mu_{k_0}(A,a) \), so that, for some \( j_0 \in \mathbb{N}, \ j_0 \geq 1, \ j_0 \leq k_0 \leq j_0 + m - 1 \), \( \mu_{j_0}(A,a) = \mu_{j_0+1}(A,a) = \cdots = \mu_{j_0+m-1}(A,a) \) and let \( \{\psi_i : j_0 \leq i \leq j_0 + m - 1\} \) be an \( L^2(\mathbb{S}^{N-1},\mathbb{C}) \)-orthonormal basis for the eigenspace of \( L_{A,a} \), associated to \( \mu_{k_0}(A,a) \). Set

\[
\gamma = -\frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_{k_0}(A,a)}
\]

and let \( \{\lambda_n\}_{n \in \mathbb{N}} \subset (0, +\infty) \) such that \( \lim_{n \to +\infty} \lambda_n = 0 \). Then, from part (ii) of Lemma 6.1 and Lemmas 6.4 and 6.5 there exist a subsequence \( \{\lambda_{n_k}\}_{k \in \mathbb{N}} \) and \( m \) real numbers \( \beta_{j_0}, \ldots, \beta_{j_0+m-1} \in \mathbb{R} \) such that \( (\beta_{j_0}, \beta_{j_0+1}, \ldots, \beta_{j_0+m-1}) \neq (0, 0, \ldots, 0) \) and

\[
\lambda_{n_k}^{-\gamma} u(\lambda_{n_k} \theta) \to \sum_{i=j_0}^{j_0+m-1} \beta_i \psi_i(\theta) \quad \text{in } C^{1,\tau}(\mathbb{S}^{N-1},\mathbb{C}) \quad \text{as } k \to +\infty
\]

and

\[
\lambda_{n_k}^{-\gamma} \nabla u(\lambda_{n_k} \theta) \to \sum_{i=j_0}^{j_0+m-1} \beta_i (\gamma \psi_i(\theta) + \nabla_{\mathbb{S}^{N-1}} \psi_i(\theta)) \quad \text{in } C^{0,\tau}(\mathbb{S}^{N-1},\mathbb{C}^N) \quad \text{as } k \to +\infty
\]

for any \( \tau \in (0, 1) \). We now prove that the \( \beta_i \)'s depend neither on the sequence \( \{\lambda_n\}_{n \in \mathbb{N}} \) nor on its subsequence \( \{\lambda_{n_k}\}_{k \in \mathbb{N}} \).

Let us fix \( R > 0 \) such that \( \overline{B}_R \subset \Omega \). Defining \( \varphi_i \) and \( \zeta_i \) as in (78) and expanding \( u \) as in (77), from (88) it follows that, for any \( i = j_0, \ldots, j_0 + m - 1 \),

\[
\lambda_{n_k}^{-\gamma} \varphi_i(\lambda_{n_k}) = \int_{\mathbb{S}^{N-1}} \frac{u(\lambda_{n_k} \theta)}{\lambda_{n_k}^{\gamma}} \psi_i(\theta) \, dS(\theta) \to \sum_{j=j_0}^{j_0+m-1} \beta_j \int_{\mathbb{S}^{N-1}} \psi_j(\theta) \psi_i(\theta) \, dS(\theta) = \beta_i
\]

as \( k \to +\infty \). As deduced in the proof of Lemma 6.5, for any \( i = j_0, \ldots, j_0 + m - 1 \) and \( \lambda \in (0, R] \), there holds

\[
\varphi_i(\lambda) = \lambda^{\sigma_i^+} \left( c_i^1 + \int_{0}^{\lambda} \frac{s^{-\sigma_i^+} \zeta_i(s) \, ds}{\sigma_i^+ - \sigma_i^-} \right) + \lambda^{\sigma_i^-} \left( \int_{0}^{\lambda} \frac{s^{-\sigma_i^-} \zeta_i(s) \, ds}{\sigma_i^- - \sigma_i^+} \right)
\]

\[
= \lambda^{\sigma_i^+} \left( c_i^1 + \int_{0}^{R} \frac{s^{-\sigma_i^+} \zeta_i(s) \, ds}{\sigma_i^+ - \sigma_i^-} \right) + O(\lambda^{\sigma_i^+ + \epsilon}) \quad \text{as } \lambda \to 0^+,
\]

for some \( c_i^1 \in \mathbb{R} \), where

\[
\sigma_i^+ = \gamma = -\frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_{k_0}(A,a)}, \quad \sigma_i^- = -\frac{N-2}{2} - \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_{k_0}(A,a)}.
\]

Choosing \( \lambda = R \) in the first line of (91), we obtain

\[
c_i^1 = R^{-\sigma_i^+} \varphi_i(R) - R^{\sigma_i^- - \sigma_i^+} \int_{0}^{R} \frac{s^{-\sigma_i^+} \zeta_i(s) \, ds}{\sigma_i^+ - \sigma_i^-}.
\]
Thus implying that the convergences in (88) and (89) actually hold as $y \to 0^+$, so it remains to prove that

Statement (i) follows directly from (10). Statement (iii) is an immediate proof of Corollary 1.4. In particular the $\beta$ theorem.

Hence (91) yields

$$
\lambda^{-\gamma} \varphi_1(\lambda) \to \mathcal{R}^{-\sigma^+_1} \varphi_1(R) - \mathcal{R}^{\sigma^-_1 - \sigma^+_1} \int_0^R \frac{s^{-\sigma^+_1 + 1}}{\sigma^+_1 - \sigma^-_i} \zeta_i(s) \, ds + \int_0^R \frac{s^{-\sigma^+_1 + 1}}{\sigma^+_1 - \sigma^-_i} \zeta_i(s) \, ds \quad \text{as } \lambda \to 0^+,
$$

and therefore, from (91) we deduce that

$$
\beta_i = R^{-\gamma} \int_{S^{N-1}} u(R\theta) \overline{\psi_i(\theta)} \, dS(\theta)
- R^{-2\gamma - N + 2} \int_0^R \frac{s^{\gamma + N - 2}}{2\gamma + N - 2} \left( \int_{S^{N-1}} h(s \eta) u(s \eta) \overline{\psi_i(\eta)} \, dS(\eta) \right) \, ds
+ \int_0^R \frac{s^{1-\gamma}}{2\gamma + N - 2} \left( \int_{S^{N-1}} h(s \eta) u(s \eta) \overline{\psi_i(\eta)} \, dS(\eta) \right) \, ds.
$$

In particular the $\beta_i$'s depend neither on the sequence $\{\lambda_n\}_{n \in \mathbb{N}}$ nor on its subsequence $\{\lambda_{n_k}\}_{k \in \mathbb{N}}$, thus implying that the convergences in (88) and (89) actually hold as $\lambda \to 0^+$ and proving the theorem. \hfill \Box

**Proof of Corollary 1.4** Statement (i) follows directly from (10). Statement (iii) is an immediate consequence of (10) and (11). To prove (ii), we notice that classical elliptic regularity theory yields Hölder continuity away from 0, so it remains to prove that $u$ is Hölder continuous in every $\overline{B_r} \subset \Omega$. To this aim, we argue by contradiction and assume that there exist sequences $\{x_n\}_{n \in \mathbb{N}}$, $\{y_n\}_{n \in \mathbb{N}} \subset \overline{B_r}$ such that

$$
\lim_{n \to +\infty} \frac{|u(x_n) - u(y_n)|}{|x_n - y_n|^\gamma} = +\infty.
$$

Hölder continuity away from 0 implies that either $|x_n| \to 0$ or $|y_n| \to 0$ along a subsequence. Hence we can assume without loss of generality that $|y_n| \to 0$ and $|x_n| \geq |y_n|$. Two cases can occur.

**Case 1:** there exists a positive constant $c > 1$ such that $\frac{|x_n|}{|y_n|} \leq c$. In this case, $|x_n| \to 0$ and, letting $\lambda_n = 2c|x_n|$ and observing that $\frac{x_n}{\lambda_n}, \frac{y_n}{\lambda_n} \in \overline{B_1/(2c)} \setminus B_1/(2c^2) \subset B_1 \setminus \{0\}$, from part (ii) of Lemma 6.1 and Lemmas 6.4 and 6.3 it follows

$$
\lim_{n \to +\infty} \frac{\lambda_n^{-\gamma} u(\lambda_n \frac{x_n}{\lambda_n}) - (2c)^{-\gamma} \psi\left(\frac{x_n}{|x_n|}\right) - \lambda_n^{-\gamma} u(\lambda_n \frac{y_n}{\lambda_n}) + \frac{|y_n|^\gamma}{\lambda_n} \psi\left(\frac{y_n}{|y_n|}\right)}{\frac{|x_n - y_n|^\gamma}{\lambda_n}} = 0
$$

for some eigenfunction $\psi$ of the operator $L_{A,a}$. Since the function $|x|^\gamma \psi\left(\frac{x}{|x|}\right)$ is Hölder continuous away from 0, from above we conclude that

$$
\frac{|u(x_n) - u(y_n)|}{|x_n - y_n|^\gamma} = \frac{\lambda_n^{-\gamma} u(\lambda_n \frac{x_n}{\lambda_n}) - \lambda_n^{-\gamma} u(\lambda_n \frac{y_n}{\lambda_n})}{\frac{|x_n - y_n|^\gamma}{\lambda_n}}
$$

is bounded uniformly in $n$, thus giving rise to a contradiction.

**Case 2:** There exists subsequences $\{x_{n_k}\}_{k \in \mathbb{N}}$ and $\{y_{n_k}\}_{k \in \mathbb{N}}$ such that $\frac{|x_{n_k}|}{|y_{n_k}|} \to +\infty$. In particular $|y_{n_k}| = o(|x_{n_k}|)$ as $k \to +\infty$. From (92) we deduce that $|x_{n_k}| \to 0$ as $k \to +\infty$.
and by Corollary 6.3
\[
\frac{|u(x_n) - u(y_n)|}{|x_n - y_n|^{\gamma}} = |x_n|^{-\gamma} \frac{|u(x_n) - u(y_n)|}{|x_n|^{\gamma} - |y_n|^{\gamma}}
\]
\[
\leq \text{const} \frac{|x_n|^{-\gamma} |x_n|^\gamma + |y_n|^\gamma}{|x_n|^{\gamma} - |y_n|^{\gamma}} \leq \text{const}
\]
thus giving rise to a contradiction with (92).

\[\square\]

Invariance by Kelvin’s transform allows rewriting equations in exterior domains as equations in bounded neighborhoods of 0, thus reducing the problem of asymptotics at infinity to the problem of asymptotics at 0. Hence we can deduce Theorem 1.3 from Theorem 1.3.

**Proof of Theorem 1.5.** Let \( u \) be a weak solution of (2) where \( \Omega \) is an external domain as in the statement of the theorem. Let \( v \) be the Kelvin transform of \( u \), i.e.
\[
v(x) = |x|^{2-N} u \left( \frac{x}{|x|^2} \right), \quad x \in \tilde{\Omega} = \{ x \in \mathbb{R}^N : x/|x|^2 \in \Omega \}.
\]
If we put \( y = \frac{x}{|x|^2} \), then we have
\[
\Delta u(x) = |y|^{N+2} \Delta v(y) \quad \text{for all } y \in \tilde{\Omega},
\]
and
\[
\frac{a(x/|x|) - |A(x/|x|)|^2 + i \text{div}_{\mathbb{S}^{N-1}} A(x/|x|)}{|x|^2} u(x)
\]
\[
= |y|^{N+2} \frac{a(y/|y|) - |A(y/|y|)|^2 + i \text{div}_{\mathbb{S}^{N-1}} A(y/|y|)}{|y|^2} v(y) \quad \text{for all } y \in \tilde{\Omega}.
\]
Moreover, by the transversality assumption \( \text{[A.3]} \) we also have
\[
\frac{A(x/|x|)}{|x|} \cdot \nabla u(x) = |y|^{N+2} \frac{A(y/|y|)}{|y|} \cdot \nabla v(y) \quad \text{for all } y \in \tilde{\Omega}.
\]
Therefore, by (93–95) we obtain
\[
\mathcal{L}_{A,a} v(y) = |y|^{-4} \left( \frac{y}{|y|^2} \right) v(y) \quad \text{in } \tilde{\Omega} \setminus \{0\}.
\]
From a direct computation we infer that \( \nabla v \in L^2(\tilde{\Omega}, \mathbb{C}^N) \), \( \frac{y}{|y|^2} \in L^2(\tilde{\Omega}, \mathbb{C}) \), and hence \( v \in H^1(\tilde{\Omega}, \mathbb{C}) \). This is sufficient for proving that \( v \) is a \( H^1 \)-weak solution of equation (97) in \( \tilde{\Omega} \).

On the other hand, by (14)
\[
|y|^{-4} \left( \frac{y}{|y|^2} \right) = O(|y|^{-2+\varepsilon}), \quad \text{as } |y| \to 0^+
\]
and hence \( v \) satisfies all the assumptions of Theorem 1.3. The proof of (16) and the asymptotic estimate for \( u \) then follows by Theorem 1.3 (93), and the fact that
\[
\mathcal{N}_{\nu,|y|^{-\varepsilon}}(y/|y|^2)(r) = \tilde{N}_{u,H}(\frac{r}{y}) - N + 2
\]
with $N_{u,h}$ as in (15). For proving the estimate on the gradient one may proceed as follows. Let $\tilde{\gamma}$ be as in the statement of the theorem and let $\gamma = \lim_{r \to 0^+} N_{r,|y|^{-1}h(y)|y|^2}(r)$. From (98) it follows that $\gamma = \tilde{\gamma} - N + 2$, hence by (98) we have

\begin{equation}
\lambda^{-\tilde{\gamma}} \nabla v(\lambda \theta) = (2 - N) \lambda^{-\tilde{\gamma}} u \left( \frac{\theta}{\lambda} \right) + \lambda^{-\tilde{\gamma} - 1} \nabla u \left( \frac{\theta}{\lambda} \right) - 2 \lambda^{-\tilde{\gamma} - 1} \left( \nabla u \left( \frac{\theta}{\lambda} \right) \cdot \theta \right) \theta
\end{equation}

for any $\lambda$ such that $B_\lambda \subset \Omega$ and for any $\theta \in S^{N-1}$. Applying Theorem 1.3 to the function $v$, from the previous identity we infer

\begin{equation}
(2 - N) \lambda^{-\tilde{\gamma}} u \left( \frac{\theta}{\lambda} \right) - \lambda^{-\tilde{\gamma} - 1} \left( \nabla u \left( \frac{\theta}{\lambda} \right) \cdot \theta \right) \to \gamma \sum_{i=0}^{j_0 + m - 1} \tilde{\beta}_i \psi_i(\theta)
\end{equation}

in $C^{0,\tau}(S^{N-1}, \mathbb{C})$ for any $\tau \in (0,1)$ as $\lambda \to 0^+$. From the first part of the theorem we also have that

\begin{equation}
\lambda^{-\tilde{\gamma}} u \left( \frac{\theta}{\lambda} \right) \to \sum_{i=0}^{j_0 + m - 1} \tilde{\beta}_i \psi_i(\theta)
\end{equation}

from which we obtain

\begin{equation}
\lambda^{-\tilde{\gamma} - 1} \left( \nabla u \left( \frac{\theta}{\lambda} \right) \cdot \theta \right) \to -\gamma \sum_{i=0}^{j_0 + m - 1} \tilde{\beta}_i \psi_i(\theta)
\end{equation}

in $C^{0,\tau}(S^{N-1}, \mathbb{C})$ for any $\tau \in (0,1)$ as $\lambda \to 0^+$. Letting $\lambda \to 0^+$ in (99), applying again Theorem 1.3 to the function $v$ and using (100), (101) we deduce that

\begin{equation}
\lambda^{-\tilde{\gamma} - 1} \nabla u \left( \frac{\theta}{\lambda} \right) \to \sum_{i=0}^{j_0 + m - 1} \tilde{\beta}_i (-\tilde{\gamma} \psi_i(\theta) \theta + \nabla_{S^{N-1}} \psi_i(\theta))
\end{equation}

in $C^{0,\tau}(S^{N-1}, \mathbb{C})$ for any $\tau \in (0,1)$ as $\lambda \to 0^+$. By replacing $\lambda$ with $1/\lambda$ we obtain the desired estimate.

7. An example: Aharonov-Bohm magnetic potentials in dimension 2

In this section we discuss an application of Theorem 1.3 to Schrödinger equations with Aharonov-Bohm vector potentials (1), i.e. we let $N = 2$, $A(\cos t, \sin t) = \alpha(- \sin t, \cos t)$, $a(\cos t, \sin t) = a_0$ for some $a_0 \in \mathbb{R}$, and consider the corresponding equation

\begin{equation}
\left( -i \nabla + A \left( \frac{x_2}{|x|^2}, -\frac{x_1}{|x|^2} \right) \right)^2 u - \frac{a_0}{|x|^2} u = h u,
\end{equation}

with $x = (x_1, x_2)$ in a bounded domain of $\mathbb{R}^2$ containing 0 and $h$ verifying (3). In this case, an explicit calculation yields

\begin{equation}
\{ \mu_k(A, a) : k \in \mathbb{N} \setminus \{0\} \} = \{ (\alpha - j)^2 - a_0 : j \in \mathbb{Z} \}
\end{equation}

hence, in particular,

\begin{equation}
\mu_1(A, a) = (\text{dist}(\alpha, Z))^2 - a_0.
\end{equation}

If $\text{dist}(\alpha, Z) \neq \frac{1}{2}$, then all eigenvalues are simple and the eigenspace associated to the eigenvalue $(\alpha - j)^2 - a_0$ is generated by $\psi(\cos t, \sin t) = e^{-ijt}$. If $\text{dist}(\alpha, Z) = \frac{1}{2}$, then all eigenvalues have multiplicity 2. Theorem 1.3 hence yields:
We notice that, by the Caffarelli-Kohn-Nirenberg inequality (see [5] and [6]),
\[ \lambda^{-\sqrt{(\alpha-j_0)^2-a_0}u(\lambda \cos t, \lambda \sin t) \to \beta e^{-ij_0t} \quad \text{as } \lambda \to 0^+, \]
in \(C^{1,\gamma}(0, 2\pi, \mathbb{C})\) for all \(\gamma \in (0, 1)\);

ii) if \(a_0 < \sqrt{(\alpha, Z)} \neq 1\), then there exists \(j_0 \in \mathbb{Z}\) and \(\beta \in \mathbb{C}\) such that
\[ \lambda^{-\sqrt{(\alpha-j_0)^2-a_0}u(\lambda \cos t, \lambda \sin t) \to \beta_1 e^{-ij_0t} + \beta_2 e^{-i(2\alpha-j_0)t} \quad \text{as } \lambda \to 0^+, \]
in \(C^{1,\gamma}(0, 2\pi, \mathbb{C})\) for all \(\gamma \in (0, 1)\).

The constants \(\beta, \beta_1, \beta_2\) can be computed as in [12]. Furthermore, in view of Corollary 1.4, if \((\text{dist}(\alpha, Z))^2 < 1 + a_0\) then \(u \in C^0_{\text{loc}}(\Omega, \mathbb{C})\) with \(\gamma = \sqrt{(\text{dist}(\alpha, Z))^2 - a_0}\), whereas \(u\) is locally Lipschitz continuous in \(\Omega\) if \((\text{dist}(\alpha, Z))^2 \geq 1 + a_0\).

8. Magnetic Hardy-Sobolev type inequalities

This section is devoted to the proof of a weighted electromagnetic Hardy-Sobolev inequality in dimension \(N \geq 3\). We start by observing that, from Lemma 2.2 and classical Sobolev’s inequality, the following electromagnetic Hardy-Sobolev inequality holds.

**Proposition 8.1.** Let \(N \geq 3\) and \(a, A\) satisfying (A.2), (A.3), and (A.4). Then
\[ S(A, a) := \inf_{u \in D^{1,2}([\mathbb{R}^N, \mathbb{C}] \setminus \{0\})} \frac{Q_{A,a}(u)}{(\int_{\mathbb{R}^N} |u(x)|^2 \, dx)^{2/2'}} > 0. \]

**Proof.** The proof follows from Lemma 2.2, part (i) of Lemma 2.3 and Sobolev’s inequality. □

We assume \(N \geq 3\) and (A.5) so that the number
\[ \sigma = \sigma(a, N) := -\frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_1(0, a)} \]
is well defined. Let \(\phi \in H^1(S^{N-1}, \mathbb{R}), \|\phi\|_{L^2(S^{N-1}, \mathbb{R})} = 1\), be the first positive eigenfunction of the eigenvalue problem
\[-\Delta_{S^{N-1}} \phi(\theta) - a(\theta) \phi(\theta) = \mu_1(0, a) \phi(\theta) \quad \text{in } S^{N-1}.\]
We recall from [12] Lemma 2.1 that \(\mu_1(0, a)\) is simple and \(\min_{S^{N-1}} \phi > 0\). Let
\[ w(x) = |x|^\sigma \phi \left( \frac{x}{|x|} \right) \quad \text{for all } x \in \mathbb{R}^N \setminus \{0\} \]
and introduce the weighted space \(D^{1,2}_w([\mathbb{R}^N, \mathbb{C}]\) as the closure of \(C^\infty_c([\mathbb{R}^N, \mathbb{C}]\) with respect to the norm
\[ \|v\|_{D^{1,2}_w([\mathbb{R}^N, \mathbb{C}] := \left( \int_{\mathbb{R}^N} w^2(x)|\nabla v(x)|^2 \, dx \right)^{1/2}. \]
We notice that, by the Caffarelli-Kohn-Nirenberg inequality (see [3] and [9]), \(v \in D^{1,2}([\mathbb{R}^N, \mathbb{C}]\) if and only if \(wv \in D^{1,2}([\mathbb{R}^N, \mathbb{C}]\) and there exists \(C_w > 0\) such that
\[ C_w \int_{\mathbb{R}^N} w^2(x) \frac{|v(x)|^2}{|x|^2} \, dx \leq \int_{\mathbb{R}^N} w^2(x)|\nabla v(x)|^2 \, dx \]
for every \( v \in \mathcal{D}^{1,2}_w(\mathbb{R}^N, \mathbb{C}) \).

**Proposition 8.2.** Let \( N \geq 3 \) and \( A \) satisfying (A.2), (A.3), (A.5), and let \( w \) be the function defined in (105). Then

\[
\int_{\mathbb{R}^N} w^2(x) |\nabla v(x)|^2 + i \frac{A(x/|x|)}{|x|} v(x) |^2 \, dx \geq S(A,a) \left( \int_{\mathbb{R}^N} w^2(x) |v(x)|^2 \, dx \right)^{\frac{2}{N}}
\]

for all \( v \in \mathcal{D}^{1,2}_w(\mathbb{R}^N, \mathbb{C}) \).

**Proof.** First of all, one can check by explicit computation that the function \( w \) solves the equation

\[
-\Delta w(x) - \frac{a(x/|x|)}{|x|^2} w(x) = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}.
\]

Let \( v \in C^0_c(\mathbb{R}^N \setminus \{0\}, \mathbb{C}) \subset \mathcal{D}^{1,2}_w(\mathbb{R}^N, \mathbb{C}) \) so that \( u(x) := w(x)v(x) \in C^0_c(\mathbb{R}^N \setminus \{0\}, \mathbb{C}) \subset \mathcal{D}^{1,2}(\mathbb{R}^N, \mathbb{C}) \). By (105) and integration by parts we have

\[
\int_{\mathbb{R}^N} \nabla w(x) \nabla (w(x)v(x))^2 \, dx - \int_{\mathbb{R}^N} \frac{a(x/|x|)}{|x|^2} w^2(x)v(x)^2 \, dx = 0.
\]

By a direct computation we infer

\[
\nabla u \nabla (w|v|^2) = |\nabla w|^2 |v|^2 + w \nabla w (\nabla v + v \nabla \overline{v})
\]

and

\[
\left\| \nabla u + i \frac{A(x/|x|)}{|x|} u \right\|^2 = |\nabla w|^2 |v|^2 + w \nabla w (\nabla v + v \nabla \overline{v}) + w^2 |v|^2
\]

\[
- 2 \Re \left( \frac{A}{|x|} w^2 v \nabla \overline{v} \right) + \left| \frac{A}{|x|} \right|^2 |w^2|v|^2.
\]

From (106), (107), and (108), we obtain that

\[
\int_{\mathbb{R}^N} \left| \nabla u(x) + i \frac{A(x/|x|)}{|x|} u(x) \right|^2 \, dx - \int_{\mathbb{R}^N} \frac{a(x/|x|)}{|x|^2} |u(x)|^2 \, dx
\]

\[
= \int_{\mathbb{R}^N} w^2(x) |\nabla v(x)|^2 \, dx - \int_{\mathbb{R}^N} 2 \Re \left( \frac{A(x/|x|)}{|x|} w^2(x)v(x) \nabla \overline{v}(x) \right) \, dx
\]

\[
+ \int_{\mathbb{R}^N} \left| \frac{A(x/|x|)}{|x|} \right|^2 w^2(x)v(x)^2 \, dx
\]

\[
= \int_{\mathbb{R}^N} w^2(x) \left| \nabla v(x) + i \frac{A(x/|x|)}{|x|} v(x) \right|^2 \, dx.
\]

By the above identity and Proposition 8.1, we obtain (104) for any \( v \in C^0_c(\mathbb{R}^N \setminus \{0\}, \mathbb{C}) \). By a density argument (see [6] Lemma 2.1), we deduce that (104) holds for any \( v \in \mathcal{D}^{1,2}_w(\mathbb{R}^N, \mathbb{C}) \). \( \square \)
9. A Brezis-Kato type lemma for $N \geq 3$

This section is devoted to the proof of a Brezis-Kato type result in dimension $N \geq 3$. Let $w$ be the function defined in (103). We define the weighted space $H^1_w(\Omega, \mathbb{C})$ as the closure of $H^1(\Omega, \mathbb{C}) \cap C^\infty(\Omega, \mathbb{C})$ with respect to the norm

$$
\|v\|_{H^1_w(\Omega, \mathbb{C})} := \left( \int_\Omega w^2(x) \left[ |\nabla v(x)|^2 + |v(x)|^2 \right] \, dx \right)^{1/2},
$$

and the space $D^{1,2}_w(\Omega, \mathbb{C})$ as the closure of $C^\infty(\Omega, \mathbb{C})$ with respect to

$$
\|v\|_{D^{1,2}_w(\Omega, \mathbb{C})} := \left( \int_\Omega w^2(x)|\nabla v(x)|^2 \, dx \right)^{1/2}.
$$

It is easy to verify that $v \in H^1_w(\Omega, \mathbb{C})$ if and only if $wv \in H^1(\Omega, \mathbb{C})$. For $N \geq 3$ and any $q \geq 1$, we also denote as $L^q(w^2, \Omega, \mathbb{C})$ the weighted $L^q$-space endowed with the norm

$$
\|v\|_{L^q(w^2, \Omega, \mathbb{C})} := \left( \int_\Omega w^{2q}(x)|v(x)|^q \, dx \right)^{1/q},
$$

where $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent.

**Lemma 9.1.** Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$, be a bounded open set containing 0, (A.2), (A.3) and (A.5) hold, and $v \in H^1_w(\Omega, \mathbb{C}) \cap L^q(w^2, \Omega, \mathbb{C})$, $q > 2$, be a weak solution to

$$
- \text{div}(w^2(x)\nabla v(x)) - \frac{2iA(x/|x|)}{\phi(x/|x|)} \nabla \phi_{N-1}(\frac{x}{|x|}) - \frac{|A(\frac{x}{|x|})|^2 + i \text{div} A_{N-1}(\frac{x}{|x|})}{|x|^2} w^2(x)v(x)
$$

$$
- 2i w^2(x) \frac{A(\frac{x}{|x|})}{|x|} \cdot \nabla v(x) = w^{2^*}(x)V(x)v(x), \quad \text{in } \Omega,
$$

where $(\Re(V))_+ \in L^s(w^2, \Omega, \mathbb{C})$ for some $s > N/2$. Then, for any $\Omega' \Subset \Omega$ such that $0 \in \Omega'$, $v \in L^{2s/N}(w^{2^*}, \Omega', \mathbb{C})$ and

$$
\|v\|_{L^{2s/N}(w^{2^*}, \Omega', \mathbb{C})} \leq S(A, a)^{-\frac{4}{q}} \|v\|_{L^q(w^2, \Omega, \mathbb{C})} \left( \frac{32}{C(q)} \frac{M^{2-2^*}(\tilde{C}(\Omega, \Omega'))^{q(2-2^*)}}{(\text{dist}(\Omega', \partial\Omega))^2} + \frac{2\ell_q}{C(q)} \right)^{\frac{1}{q}},
$$

where $C(q) := \min \left\{ \frac{1}{4}, \frac{1}{q+4} \right\}$, $M = \min_{\mathbb{S}^{N-1}} \phi > 0$, 

$$
\tilde{C}(\Omega, \Omega') = \begin{cases} 
\text{diam } \Omega & \text{if } \mu_1(0, a) \leq 0, \\
\text{dist}(0, \mathbb{R}^N \setminus \Omega') & \text{if } \mu_1(0, a) > 0,
\end{cases}
$$

and

$$
\ell_q = \frac{8}{S(A, a)} \|\Re(V)\|_{L^q(w^2, \Omega, \mathbb{C})} + \frac{q+4}{2S(A, a)} \|\Re(V)\|_{L^{2s/N}(w^{2^*}, \Omega, \mathbb{C})} \left[ \frac{2s}{s+2} \right]^{\frac{2N}{s+2}}.
$$
Proof. Hölder’s inequality and (104) yield for any $u \in D^{1,2}_w(\Omega, \mathbb{C})$

\begin{equation}
\int_\Omega w^2(x)(\text{Re}(V(x)))^+ |u(x)|^2 \, dx
\leq \ell_q \int_\Omega w^2(x)|u(x)|^2 \, dx + \int_{(\text{Re}(V(x)))^+ \geq \ell_q} w^{(2^*-2)}(x)(\text{Re}(V(x)))^+ w^2(x)|u(x)|^2 \, dx
\end{equation}

\begin{equation}
\leq \ell_q \int_\Omega w^2(x)|u(x)|^2 \, dx + \left( \int_\Omega w^2(x)|u(x)|^2 \, dx \right)^{\frac{q}{q-2}} \left( \int_{(\text{Re}(V(x)))^+ \geq \ell_q} w^2(x)(\text{Re}(V(x)))^+ \, dx \right)^{\frac{2-q}{q-2}}
\end{equation}

\begin{equation}
\leq \frac{1}{S(A,a)} \left( \int_\Omega w^2(x)|\nabla u(x) + i \frac{A(\bar{a}|x|)}{|x|} u(x)|^2 \, dx \right) \times \left( \int_{(\text{Re}(V(x)))^+ \geq \ell_q} w^2(x)(\text{Re}(V(x)))^+ \, dx \right)^{\frac{q}{q-2}} + \ell_q \int_\Omega w^2(x)|u(x)|^2 \, dx.
\end{equation}

By Hölder’s inequality and by the choice of $\ell_q$ it follows that

\begin{equation}
\int_{(\text{Re}(V(x)))^+ \geq \ell_q} w^2(x)(\text{Re}(V(x)))^+ \, dx \leq \left( \int_{(\text{Re}(V(x)))^+ \geq \ell_q} w^2(x)(\text{Re}(V(x)))^+ \, dx \right)^{\frac{q}{q-2}} \left( \int_{(\text{Re}(V(x)))^+ \geq \ell_q} w^{2^*-2}(x)(\text{Re}(V(x)))^+ \, dx \right)^{\frac{2-q}{q-2}}
\end{equation}

\begin{equation}
\leq \left( \int_\Omega w^2(x)(\text{Re}(V(x)))^+ \, dx \right)^{\frac{q}{q-2}} \left( \int_{(\text{Re}(V(x)))^+ \geq \ell_q} \left( \text{Re}(V(x)) \right)^+ \, dx \right)^{\frac{2-q}{q-2}}
\end{equation}

\begin{equation}
\leq \|\text{Re}(V)\|_{L^q(\Omega, C)}^{s-\frac{q}{q-2}} \leq \min \left\{ \frac{S(A,a)}{8}, \frac{2 S(A,a)}{q+4} \right\},
\end{equation}

and hence from (112) we obtain that for any $u \in D^{1,2}_w(\Omega, \mathbb{C})$

\begin{equation}
\int_\Omega w^2(x)(\text{Re}(V(x)))^+ |u(x)|^2 \, dx \leq \ell_q \int_\Omega w^2(x)|u(x)|^2 \, dx
\end{equation}

\begin{equation}
+ \min \left\{ \frac{1}{8}, \frac{2}{q+4} \right\} \left( \int_\Omega w^2(x)|\nabla u(x) + i \frac{A(\bar{a}|x|)}{|x|} u(x)|^2 \, dx \right).
\end{equation}

Let $\eta \in C^\infty(\Omega, \mathbb{R})$ be a nonnegative cut-off function such that

\begin{equation}
\text{supp}(\eta) \subseteq \Omega, \quad \eta \equiv 1 \text{ on } \Omega', \text{ and } |\nabla \eta(x)| \leq \frac{2}{\text{dist}(\Omega', \partial \Omega)}.
\end{equation}

Set $v^\eta := \min(n, |v|) \in H^{1}_w(\Omega, \mathbb{C})$. Let us test (110) with $\eta^2(v^\eta)^{q-2} \bar{v} \in D^{1,2}_w(\Omega, \mathbb{C})$ and take the real part. Observing that $\text{Re}(\bar{v} \nabla v) = |v| \nabla |v|$ and using the elementary inequality $2ab \leq 1/2a^2 + 2b^2$
and the diamagnetic inequality (see Lemma A.1), we thus obtain
\[
(q - 2) \int_{\Omega} w^2(x)\eta^2(x)(v^n(x))^{q-2} \chi_{\{y \in \Omega; |v(y)| < n\}}(x) |\nabla|v(x)|^2 \, dx
+ \int_{\Omega} w^2(x)\eta^2(x)(v^n(x))^{q-2} |\nabla v(x)|^2 \, dx + \int_{\Omega} \frac{|A(\frac{x}{|x|})|^2}{|x|^2} w^2(x)\eta^2(x)(v^n(x))^{q-2} |v(x)|^2 \, dx
+ 2 \int_{\Omega} w^2(x)\eta^2(x)(v^n(x))^{q-2} \frac{A(\frac{x}{|x|})}{|x|} \cdot \Im(\bar{v}(x)\nabla v(x)) \, dx
\]
\[
= \int_{\Omega} w^2(x)\Re(V(x))\eta^2(x)|v(x)|^2(v^n(x))^{q-2} \, dx - 2 \int_{\Omega} w^2(x)\eta(x)(v^n(x))^{q-2}|v(x)|\nabla |v(x)| \cdot \nabla \eta(x) \, dx
\]
\[
\leq \int_{\Omega} w^2(x)\Re(V(x))\eta^2(x)|v(x)|^2(v^n(x))^{q-2} \, dx + 2 \int_{\Omega} w^2(x)|\nabla \eta(x)|^2(v^n(x))^{q-2} |v(x)|^2 \, dx
+ \frac{1}{2} \int_{\Omega} w^2(x)\eta^2(x)(v^n(x))^{q-2} \left| \nabla v(x) + i \frac{A(\frac{x}{|x|})}{|x|} v(x) \right|^2 \, dx
\]
and hence
\[
(q - 2) \int_{\Omega} w^2(x)\eta^2(x)(v^n(x))^{q-2} \chi_{\{y \in \Omega; |v(y)| < n\}}(x) |\nabla|v(x)|^2 \, dx
+ \frac{1}{2} \int_{\Omega} w^2(x)\eta^2(x)(v^n(x))^{q-2} \left| \nabla v(x) + i \frac{A(\frac{x}{|x|})}{|x|} v(x) \right|^2 \, dx
\]
\[
\leq \int_{\Omega} w^2(x)\Re(V(x))\eta^2(x)|v(x)|^2(v^n(x))^{q-2} \, dx + 2 \int_{\Omega} w^2(x)|\nabla \eta(x)|^2(v^n(x))^{q-2} |v(x)|^2 \, dx.
\]
Furthermore, by diamagnetic inequality (see Lemma A.1) we have that
\[
(116) \quad \left| \nabla ((v^n)^\frac{q-2}{2} v\eta) + i \frac{A(\frac{x}{|x|})}{|x|} (v^n)^\frac{q-2}{2} v\eta \right|^2
= \left| \nabla ((v^n)^\frac{q-2}{2} v\eta) \right|^2 + 2 \frac{A(\frac{x}{|x|})}{|x|} \eta (v^n)^{q-2} \Im(\bar{v} \nabla v) + \frac{|A(\frac{x}{|x|})|^2}{|x|^2} (v^n)^{q-2} |\eta|^2 |v|^2
\]
\[
\leq \frac{(q + 2)(q - 2)}{4} (v^n)^{q-2} \eta^2 |\nabla v|^2 + 2 (v^n)^{q-2} \eta^2 \left| \nabla v + i \frac{A(\frac{x}{|x|})}{|x|} v \right|^2
+ \frac{q + 2}{2} (v^n)^{q-2} |\eta|^2 |\nabla \eta|^2.
\]
Letting \( C(q) := \min \left\{ \frac{q}{2}, \frac{q}{q+2} \right\} \) from (115) and (116) we obtain
\[
(117) \quad C(q) \int_{\Omega} w^2(x) \left| \nabla ((v^n)^\frac{q-2}{2} v\eta) + i \frac{A(\frac{x}{|x|})}{|x|} (v^n(x))^{\frac{q-2}{2}} v(x) \eta(x) \right|^2 \, dx
\]
\[
\leq \int_{\Omega} w^2(x)\Re(V(x))\eta^2(x)|v(x)|^2(v^n(x))^{q-2} \, dx
+ 2 \int_{\Omega} w^2(x)(v^n(x))^{q-2} |v(x)|^2 |\nabla \eta(x)|^2 \, dx + C(q) \frac{q + 2}{2} \int_{\Omega} w^2(x)(v^n(x))^{q-2} |v(x)|^2 |\nabla \eta(x)|^2 \, dx.
\]
Estimate (114) applied to $\eta(v^n)^{\frac{q}{q-1}}v$ gives
\begin{equation}
\int_{\Omega} w^2(x)(\mathbb{R}(V(x)))_{+} |\eta(x)(v^n(x))^{\frac{q}{q-1}}v(x)|^2 \, dx \leq \ell_q \int_{\Omega} w^2(x)|\eta(x)(v^n(x))^{\frac{q}{q-1}}v(x)|^2 \, dx \\
+ \min \left\{ \frac{1}{8}, \frac{2}{q+4} \right\} \left( \int_{\Omega} w^2(x) |\nabla (\eta(v^n)^{\frac{q}{q-1}}v(x)) + i \frac{A(x/|x|)}{|x|} \eta(x)(v^n(x))^{\frac{q}{q-1}}v(x)|^2 \, dx \right).
\end{equation}

Using (118) to estimate the term with $V$ in (117), (104) yields
\begin{equation}
\left( \int_{\Omega} w^2(x)|v^n(x)|^{\frac{q}{q-1}2}|v(x)|^2 \eta^2(x) \, dx \right)^{\frac{q}{2}} \leq \frac{2\ell_q}{C(q)S(A,a)} \int_{\Omega} w^2(x)\eta^2(x)|v^n(x)|^{q-2}|v(x)|^2 \, dx \\
+ \frac{4 + C(q)(q + 2)}{C(q)S(A,a)} \int_{\Omega} w^2(x)|v^n(x)|^{q-2}|v(x)|^2 |\nabla \eta(x)|^2 \, dx \\
\leq \frac{2\ell_q}{C(q)S(A,a)} \int_{\Omega} w^2(x)\eta^2(x)|v^n(x)|^{q-2}|v(x)|^2 \, dx \\
+ \frac{8}{C(q)S(A,a)} \int_{\Omega} w^2(x)|v^n(x)|^{q-2}|v(x)|^2 |\nabla \eta(x)|^2 \, dx.
\end{equation}

Letting $n \to \infty$ in the above inequality, (111) follows. \qed

**Remark 9.2.** It is possible to extend the result of Lemma 9.1 also to the case
\[(\mathbb{R}(V))_+ \in L^{N/2}(w^2, \Omega, C)\]
and obtain estimate (111). Indeed, by the previous summability assumption on $\mathbb{R}(V)_+$, it is possible to find $\ell_q$ such that
\[\int_{(\mathbb{R}(V(x)))_{+} \geq \ell_q} w^2(x)(\mathbb{R}(V(x)))_{+}^{\frac{q}{2}} \, dx \leq \min \left\{ \frac{S(A,a)}{8}, \frac{2 S(A,a)}{q+4} \right\}.\]

But we have not a control on the constant $\ell_q$ in terms of $q$ as in Lemma 9.1 since it is not possible to apply Hölder’s inequality in (113) when $s = N/2$. The rest of the proof in the case $s = N/2$ coincide with the proof of Lemma 9.1.

The previous lemma allows starting a Brezis-Kato type iteration.

**Theorem 9.3.** Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$, be a bounded open set containing $0$, [A.2], [A.3] and [A.5] hold.

i) If $V$ is such that $(\mathbb{R}(V))_+ \in L^s(w^2, \Omega, C)$ for some $s > N/2$, then, for any $\Omega' \Subset \Omega$, there exists a positive constant
\[C_\infty = C_\infty(N, A, a, \|(\mathbb{R}(V))_+\|_{L^{s}(w^2, \Omega, C)}, \text{dist}(\Omega', \partial \Omega), \tilde{C}(\Omega, \Omega'))\]
depending only on $N, A, a, \|(\mathbb{R}(V))_+\|_{L^{s}(w^2, \Omega, C)}, \text{dist}(\Omega', \partial \Omega)$, and $\tilde{C}(\Omega, \Omega')$, such that for any weak $H^1(\Omega', C)$-solution $u$ to
\begin{equation}
L_{A,a} u(x) = w^{2s-2}(x)V(x)u(x), \quad \text{in } \Omega,
\end{equation}
there holds $|x|^{-\sigma} u \in L^\infty(\Omega', C)$ and
\[\| |x|^{-\sigma} u \|_{L^\infty(\Omega', C)} \leq C_{\infty} \| u \|_{L^{2s}(\Omega, C)}.\]
ii) If $V$ is such that $(\Re(V))_+ \in L^{N/2}(w^2, \Omega, C)$, then, for any $\Omega' \subseteq \Omega$ and for any $s \geq 1$, there exists a positive constant
$$C_s = C_s(N, A, \| (\Re(V))_+ \| L^{N/2}(w^2, \Omega, C), s, \text{dist}(\Omega', \partial \Omega), \tilde{C}(\Omega, \Omega'))$$
depending only on $N$, $A$, $\| (\Re(V))_+ \| L^{N/2}(w^2, \Omega, C)$, $s$, $\text{dist}(\Omega', \partial \Omega)$, and $\tilde{C}(\Omega, \Omega')$, such that for any $H^1(\Omega, C)$-solution $u$ to (114) in $\Omega$ there holds $|x|^{-\sigma} u \in L^s(w^2, \Omega', C)$ and
$$\| |x|^{-\sigma} u \|_{L^s(w^2, \Omega', C)} \leq C_s \| u \|_{L^s(w^2, \Omega, C)}.$$

**Proof.** i) Let $u$ be a weak $H^1(\Omega, C)$-solution to (114). It is easy to verify that $v := w^{-1}u$ belongs to $H^1_w(\Omega, C)$ and is a weak solution to (113). Let $R > 0$ be such that $\Omega' \subseteq \Omega' + B(0, 2R) \subseteq \Omega$.

Using Lemma 9.1 in $\Omega_1 := \Omega' + B(0, R(2 - r_1)) \subseteq \Omega' + B(0, 2R)$, $r_1 = 1$, with $q = q_1 = 2^*$, we infer that $v \in L^{2^{*^2} + 3}(w^2, \Omega_1, C)$ and the following estimate holds
$$\| v \|_{L^{2^{*^2} + 3}(w^2, \Omega_1, C)} \leq S(A, a)^{-\frac{1}{q_1}} \| v \|_{L^{2^*}(w^2, \Omega_1, C)} \left( \frac{32}{C(q_1)} \frac{M^{2^* - 2^*} (\tilde{C}(\Omega, \Omega'))^{\sigma(2 - 2^*)}}{(Rr_1)^2} + \frac{2q_1}{C(q_1)} \right)^{\frac{1}{q_1}}.$$

Using again Lemma 9.1 in $\Omega_2 := \Omega' + B(0, R(2 - r_1 - r_2)) \subseteq \Omega_1$, $r_2 = \frac{1}{2}$, with $q = q_2 = (2^*)^2/2$, we infer that $v \in L^{2^{*^2} + 3}(w^2, \Omega_2, C)$ and
$$\| v \|_{L^{2^{*^2} + 3}(w^2, \Omega_2, C)} \leq S(A, a)^{-\frac{1}{q_2}} \left( \frac{32}{C(q_2)} \frac{M^{2^* - 2^*} (\tilde{C}(\Omega, \Omega'))^{\sigma(2 - 2^*)}}{(Rr_2)^2} + \frac{2q_2}{C(q_2)} \right)^{\frac{1}{q_2}} \| v \|_{L^{2^*}(w^2, \Omega_2, C)}.$$

Setting, for any $n \in N$, $n \geq 1$,
$$q_n = 2 \left( \frac{2^*}{2} \right)^n, \quad \Omega_n := \Omega' + B \left( 0, R \left( 2 - \sum_{k=1}^n r_k \right) \right), \quad \text{and} \quad r_n = \frac{1}{n^z},$$
and using iteratively Lemma 9.1 we obtain that, for any $n \in N$, $n \geq 1$,
$$\| v \|_{L^{2^{*^2} + 3}(w^2, \Omega_n, C)} \leq \| v \|_{L^{2^{*^2} + 3}(w^2, \Omega_{n-1}, C)} \leq \| v \|_{L^{2^*}(w^2, \Omega_{n-1}, C)} \left( S(A, a) \right)^{-\sum_{k=1}^n \frac{1}{q_k}} \times \prod_{k=1}^n \left( \frac{32}{C(q_k)} \frac{M^{2^* - 2^*} (\tilde{C}(\Omega, \Omega'))^{\sigma(2 - 2^*)}}{(Rr_k)^2} + \frac{2q_k}{C(q_k)} \right)^{\frac{1}{q_k}}.$$

We notice that
$$\prod_{k=1}^n \left( \frac{32}{C(q_k)} \frac{M^{2^* - 2^*} (\tilde{C}(\Omega, \Omega'))^{\sigma(2 - 2^*)}}{(Rr_k)^2} + \frac{2q_k}{C(q_k)} \right)^{\frac{1}{q_k}} = \exp \left[ \sum_{k=1}^n b_k \right]$$
where
\[ b_k = \frac{1}{q_k} \log \left( \frac{32}{C(q_k)} \frac{M^{2-2'}(\bar{C}(\Omega, \Omega'))^{\sigma(2-2')}}{(R_k)^2} + 2\ell_{q_k} C(q_k) \right), \]
and, for some constant \( C = C(N, A, a, \| (\mathcal{R}(V))_+ \|_{L^r(w^2, \Omega, \mathbb{C})}, \text{dist}(\Omega', \partial \Omega), C(\Omega, \Omega')) \),
\[ b_k \sim \frac{1}{2} \left( \frac{2}{2'} \right) k \log \left[ C \left( 2 \left( \frac{2'}{2} \right) \right)^{\frac{2}{2'-2}} \right] \quad \text{as } k \to +\infty. \]
Hence \( \sum_{n=1}^{\infty} b_n \) converges to some positive sum depending only on \( N, A, a, \| (\mathcal{R}(V))_+ \|_{L^r(w^2, \Omega, \mathbb{C})}, \text{dist}(\Omega', \partial \Omega), C(\Omega, \Omega') \), hence
\[ \lim_{n \to +\infty} (S(A, a))_{n+1} = \frac{\sum_{k=1}^{n+1}}{\sum_{k=1}^{n}} \prod_{k=1}^{n} \left( \frac{32}{C(q_k)} \frac{M^{2-2'}(\bar{C}(\Omega, \Omega'))^{\sigma(2-2')}}{(R_k)^2} + 2\ell_{q_k} C(q_k) \right)^{\frac{1}{2}} \]
is finite and depends only on \( N, A, a, \| (\mathcal{R}(V))_+ \|_{L^r(w^2, \Omega, \mathbb{C})}, \text{dist}(\Omega', \partial \Omega), C(\Omega, \Omega') \). Hence, from \ref{thm:9.3}, we deduce that there exists a positive constant \( C \) depending only on \( \| (\mathcal{R}(V))_+ \|_{L^r(w^2, \Omega, \mathbb{C})}, N, A, a, \text{dist}(\Omega', \partial \Omega), C(\Omega, \Omega') \), such that
\[ \| v \|_{L^{n+1}(w^2, \Omega, \mathbb{C})} \leq C \| v \|_{L^2(w^2, \Omega, \mathbb{C})} \]
for all \( n \in \mathbb{N} \).

Letting \( n \to +\infty \) we deduce that \( |v| \) is essentially bounded in \( \Omega' \) with respect to the measure \( w^2 \, dx \) and
\[ \| v \|_{L^\infty(w^2, \Omega', \mathbb{C})} \leq C \| v \|_{L^2(w^2, \Omega', \mathbb{C})} = C \| u \|_{L^2(\Omega, \mathbb{C})}, \]
where \( \| v \|_{L^\infty(w^2, \Omega', \mathbb{C})} \) denotes the essential supremum of \( v \) with respect to the measure \( w^2 \, dx \). Since \( w^2 \, dx \) is absolutely continuous with respect to the Lebesgue measure and vice versa, there holds \( \| v \|_{L^\infty(w^2, \Omega', \mathbb{C})} = \| v \|_{L^\infty(\Omega', \mathbb{C})} \), hence \( v \in L^\infty(\Omega', \mathbb{C}) \) and
\[ \| v \|_{L^\infty(\Omega', \mathbb{C})} \leq C \| u \|_{L^2(\Omega, \mathbb{C})}, \]
thus completing the proof of part i). We recall that for any \( x \in \Omega \setminus \{0\} \) we have
\[ |x|^{-\sigma} u(x) = w^{-1}(x) \phi(x/|x|) u(x) = \phi(x/|x|) v(x) \leq (\max_{S_{N-1}} \phi) v(x). \]

ii) Since \( u \in H^1(\Omega, \mathbb{C}) \) is a weak solution to \ref{eq:111} then \( v := w^{-1} u \in H^1_0(\Omega, \mathbb{C}) \) is a weak solution of \ref{eq:110}. Using Remark \ref{rem:A.3} and the iterative scheme used to prove part i), for any \( 1 \leq s < \infty \), after a finite number of iterations we arrive to \( v \in L^s(w^2, \Omega, \mathbb{C}) \) and
\[ \| v \|_{L^s(w^2, \Omega, \mathbb{C})} \leq C_s \| v \|_{L^2(w^2, \Omega, \mathbb{C})}. \]
This completes the proof. \hfill \( \square \)

Applying Theorem \ref{thm:A.3} to the nonlinear equation \ref{eq:3}, we can obtain a pointwise estimate for solutions to \ref{eq:3}.

**Theorem 9.4.** Let \( \Omega \subset \mathbb{R}^N, N \geq 3 \), be a bounded open set containing 0, \ref{eq:A.2}, \ref{eq:A.3} and \ref{eq:A.5} hold. Let \( u \) be a weak \( H^1(\Omega, \mathbb{C}) \)-solution of \ref{eq:3} with \( f(x, u) \) satisfying \ref{eq:7}. Then for any \( \Omega' \subset \Omega \) there exists a positive constant
\[ \bar{C}_\infty = \bar{C}_\infty(N, A, a, C_f, \text{dist}(\Omega', \partial \Omega), \bar{C}(\Omega, \Omega')) \]
depending only on \( N, A, a, C_f, \) \( \text{dist}(\Omega', \partial\Omega), \) and \( \tilde{C}(\Omega, \Omega'), \) such that \( |x|^{-\sigma} u \in L^\infty(\Omega', \mathbb{C}) \) and

\[
| |x|^{-\sigma} u |_{L^\infty(\Omega', \mathbb{C})} \leq \tilde{C}_\infty \| u \|_{L^2(\Omega, \mathbb{C})}.
\]

**Proof.** If we put

\[
V(x) := \begin{cases}
    w^{2-2^s} f(x, u(x)) / u(x), & \text{if } u(x) \neq 0, \\
    0, & \text{if } u(x) = 0,
\end{cases}
\]

then, by (7) and the Sobolev embedding \( H^1(\Omega, \mathbb{C}) \subset L^{2^*}(\Omega, \mathbb{C}) \), we have that \( V \in L^{N/2}(w^{2^s}, \Omega, \mathbb{C}) \) and \( u \) weakly solves

\[
L_{A, a} u(x) = w^{2^* - 2} V(x) u(x) \quad \text{in } \Omega.
\]

From part ii) of Theorem 9.3, it follows that \( |x|^{-\sigma} u \in L^s(w^{2^s}, \Omega', \mathbb{C}) \) for any \( \Omega' \subset \Omega \) and for any \( s \geq 1 \). Fix now \( s_0 = N/2 + \varepsilon_0 \) with \( 0 < \varepsilon_0 < \frac{N(N-2)}{4|\sigma|} \). By (7) we easily deduce that \( V \in L^{s_0}(w^{2^s}, \Omega', \mathbb{C}) \). The proof of the theorem follows now by part i) of Theorem 9.3. \( \square \)

The a-priori estimate of solutions to the nonlinear problem obtained above, allows deducing Theorem 1.6 from Theorem 1.3.

**Proof of Theorem 1.6 for \( N \geq 3 \).** Note that all the assumptions of Theorem 9.4 are verified and hence

\[
|u(x)| = O(|x|^{\sigma}) \quad \text{as } |x| \to 0,
\]

where \( \sigma > -\frac{N-2}{2} \) is defined by (102). Therefore, by (7) and (122),

\[
\left| \frac{f(x,u)}{u} \right| \leq \text{const} \left( 1 + |x|^{-2 + \frac{\varepsilon}{N-2}} \sqrt{(\frac{N-2}{2}) + \mu_1(0,a)} \right)
\]

for some constant \( \text{const} > 0 \). Hence, the function

\[
h(x) := \begin{cases}
    \frac{f(x,u(x))}{u(x)}, & \text{if } u(x) \neq 0 \\
    0, & \text{if } u(x) = 0
\end{cases}
\]

satisfies \( h(x) = O(|x|^{-2 + \varepsilon}) \) as \( |x| \to 0^+ \) for some \( \varepsilon > 0 \). On the other hand, by Remark 1.2 we also have \( u \in L^\infty_{loc}(\Omega \setminus \{0\}) \) and in turn by (7), \( h \in L^\infty_{loc}(\Omega \setminus \{0\}) \). This shows that all the assumptions of Theorem 1.3 are satisfied and the proof of Theorem 1.6 follows in the case \( N \geq 3 \). The proof of Theorem 1.6 in the case \( N = 2 \) is postponed to section 10. \( \square \)

**Proof of Theorem 1.7 for \( N \geq 3 \).** It follows from Theorems 1.3 and 1.6 by the use of the Kelvin transform. \( \square \)

Since the proof of the pointwise a-priori estimate (121) (and then of Theorems 1.6 and 1.7) in dimension \( N = 2 \) originates from a different inequality than (104) and requires a little bit different notation, we devote the next section to a sketched description of the modifications to be made in the above argument to treat the case \( N = 2 \).
10. A Brezis-Kato type lemma in dimension $N = 2$

Similarly to section 9 for $N = 2$ we define the spaces $\mathcal{D}_r^{1,2}(\Omega, \mathbb{C})$ and $\mathcal{D}^{1,2}_{s,w}(\Omega, \mathbb{C})$ as the completion of $C_c^\infty(\Omega \setminus \{0\}, \mathbb{C})$ respectively with the norms

$$
\|u\|_{\mathcal{D}_r^{1,2}(\Omega, \mathbb{C})} := \left( \int_{\Omega} \left( |\nabla u(x)|^2 + \frac{|u(x)|^2}{|x|^2} \right) dx \right)^{1/2}
$$

and

$$
\|v\|_{\mathcal{D}^{1,2}_{s,w}(\Omega, \mathbb{C})} := \left( \int_{\Omega} w^2 \left( |\nabla u(x)|^2 + \frac{|u(x)|^2}{|x|^2} \right) dx \right)^{1/2}
$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain containing the origin and $w$ is defined by (103). We observe that the space $\mathcal{D}_r^{1,2}(\Omega, \mathbb{C})$ is smaller than $H^1_0(\Omega, \mathbb{C})$. Moreover, it easy to verify that $v \in \mathcal{D}_r^{1,2}(\Omega, \mathbb{C})$ if and only if $uv \in \mathcal{D}_r^{1,2}(\Omega, \mathbb{C})$. Similarly, we define the space $H^{1}_{s,w}(\Omega, \mathbb{C})$ as the completion of $\{v \in H^1(\Omega, \mathbb{C}) \cap C^\infty(\Omega, \mathbb{C}) : v \text{ vanishes in a neighborhood of } 0\}$ with respect to the norm

$$
\|v\|_{H^{1}_{s,w}(\Omega, \mathbb{C})} := \left( \int_{\Omega} w^2 \left( |\nabla v(x)|^2 + \frac{|v(x)|^2}{|x|^2} + |v(x)|^2 \right) dx \right)^{1/2}.
$$

The following weighted Poincaré-Sobolev inequality holds.

**Proposition 10.1.** Let $N = 2$ and $a, A$ satisfying [A.2], [A.3] and [A.5]. Then, for any $1 \leq p < \infty$,

$$
S(A, a, p, \Omega) = \inf_{u \in \mathcal{D}_r^{1,2}(\Omega, \mathbb{C}) \setminus \{0\}} \frac{\int_{\Omega} \left[ \left| \left( \nabla + i \frac{A}{|x|} \right) u(x) \right|^2 - \frac{a(x/|x|)}{|x|^2} |u(x)|^2 \right] dx}{\left( \int_{\Omega} |u(x)|^p dx \right)^{2/p}} > 0.
$$

Moreover

$$
\int_{\Omega} w^2 \left| \nabla v(x) + i \frac{A(x/|x|)v(x)}{|x|} \right|^2 dx \geq S(A, a, p, \Omega) \left( \int_{\Omega} w^p |v(x)|^p dx \right)^{2/p}
$$

for all $v \in \mathcal{D}_r^{1,2}(\Omega, \mathbb{C})$.

**Proof.** Inequality (123) follows by Lemma 2.2 and classical Poincaré-Sobolev inequality. To obtain the second part of the statement, by density it is sufficient to prove inequality (124) for functions $v \in C_c^\infty(\Omega \setminus \{0\}, \mathbb{C})$ as one can easily do by following the same procedure developed in the proof of Proposition 8.2.

**Remark 10.2.** We notice that the constant in (124) depends on the domain $\Omega$, unlike the constant appearing in (103) in the case $N = 3$ and $p = 2^*$. Moreover $S(A, a, p, \Omega)$ is decreasing with respect to $\Omega$, i.e. if $\Omega_1 \subset \Omega_2$ then $S(A, a, p, \Omega_1) \geq S(A, a, p, \Omega_2)$.

We are now ready to prove the following 2-dimensional version of Lemma 9.1.
Lemma 10.3. Let $\Omega \subset \mathbb{R}^2$ be a bounded open set containing $0$, (A.2), (A.3), (A.5) hold, and, for some $p > 2$ and $q > 2$, let $v \in H^1_{\text{div}}(\Omega, \mathcal{C}) \cap L^q(w^p, \Omega, \mathcal{C})$ be a weak solution to
\[
-\text{div}(w^2(x)\nabla v(x)) - \frac{2i\text{Re}(\overline{w(x)}z)/|x|^2}{2} \nabla \phi(\frac{x}{|x|}) - |A(\frac{x}{|x|})|^2 + i\text{div}_{\mathbb{S}^N} A(\frac{x}{|x|}) \quad w^2(x)v(x)
\]
\[-2i w^2(x) |x|^2 \cdot \nabla v(x) = w^p(x)V(x)v(x), \quad \text{in} \quad \Omega,
\]
where $(\Re(V))_+ \in L^s(w^p, \Omega, \mathcal{C})$ for some $s > \frac{p}{p-2}$. Then, for any $\Omega' \subseteq \Omega$ such that $0 \in \Omega'$, $v \in L^{\frac{p2}{p-2}}(w^p, \Omega', \mathcal{C})$ and
\[
\|v\|_{L^{\frac{p2}{p-2}}(w^p, \Omega', \mathcal{C})} \leq S(A, a, p, \Omega)^{\frac{4}{q}} \|v\|_{L^q(w^p, \Omega, \mathcal{C})} \times \left( \frac{32}{C(q)} M^{-p}(\tilde{C}(\Omega, \Omega'))^{p(2-p)} + \frac{2\ell_q}{C(q)} \right)^{\frac{4}{q}},
\]
where $C(q) := \min \left\{ \frac{q}{2}\frac{p}{p-2}, \frac{q}{p} \right\}$, $\tilde{C}(\Omega, \Omega') = \text{dist}(0, \mathbb{R}^N \setminus \Omega')$, $M = \min_{\mathbb{S}^N} \phi > 0$ and
\[
\ell_q = \left[ \max \left\{ \frac{8\|v\|_{L^s(w^p, \Omega, \mathcal{C})}}{S(A, a, p, \Omega)} \|v\|_{L^s(w^p, \Omega, \mathcal{C})}, 2 \frac{q + 4}{\ell_q S(A, a, p, \Omega)} \right\} \right]^{\frac{q}{p(2-p)-1}}.
\]

PROOF. The proof may be obtained proceeding as in the proof of Lemma 9.3 and using (124) in place of (108).

The counterpart in dimension $N = 2$ of Theorem 9.3 is the following Brezis-Kato type result.

Theorem 10.4. Let $\Omega \subset \mathbb{R}^2$ be a bounded open set containing $0$, (A.2), (A.3), (A.5) hold, and let $p > 2$.

i) If $V$ is such that $(\Re(V))_+ \in L^s(w^p, \Omega, \mathcal{C})$ for some $s > \frac{p}{p-2}$, then, for any $\Omega' \subseteq \Omega$, there exists a positive constant
\[
C_{s,2} = C_{s,2}(\Omega, p, A, a) \|v\|_{L^s(w^p, \Omega, \mathcal{C})}, \text{dist}(\Omega', \partial\Omega), \tilde{C}(\Omega, \Omega'))
\]
depending only on $\Omega$, $p$, $A$, $a$, $\|v\|_{L^s(w^p, \Omega, \mathcal{C})}$, $\text{dist}(\Omega', \partial\Omega)$, and $\tilde{C}(\Omega, \Omega')$, such that for any weak $H^1(\Omega, \mathcal{C})$-solution $u$ to
\[
\mathcal{L}_{A, a} u(x) = w^{p-2} V(x) u(x), \quad \text{in} \quad \Omega,
\]
there holds $|x|^{-s} u \in L^\infty(\Omega', \mathcal{C})$ and
\[
\| |x|^{-s} u \|_{L^\infty(\Omega', \mathcal{C})} \leq C_{s,2} \|u\|_{L^p(\Omega, \mathcal{C})}.
\]

ii) If $V$ is such that $(\Re(V))_+ \in L^{\frac{p2}{p-2}}(w^p, \Omega, \mathcal{C})$, then, for any $\Omega' \subseteq \Omega$ and for any $1 \leq s < \infty$, there exists a positive constant
\[
C_{s,2} = C_{s,2}(\Omega, p, A, a) \|v\|_{L^s(w^p, \Omega, \mathcal{C})}, \text{dist}(\Omega', \partial\Omega), \tilde{C}(\Omega, \Omega'))
\]
depending only on $\Omega$, $p$, $A$, $a$, $\|v\|_{L^s(w^p, \Omega, \mathcal{C})}$, $\text{dist}(\Omega', \partial\Omega)$, and $\tilde{C}(\Omega, \Omega')$, such that for any weak $H^1(\Omega, \mathcal{C})$-solution $u$ to
\[
| |x|^{-s} u \|_{L^s(w^p, \Omega, \mathcal{C})} \leq C_{s,2} \|u\|_{L^p(\Omega, \mathcal{C})}.
\]
Proof. This theorem can be proved by iterating the estimate proved in Lemma 10.3 and following the same scheme as in the proof of Theorem 9.3. We notice that the constants $S(\mathbf{A}, a, p, \Omega_i)$ appearing at each step (at a negative power) can be uniformly controlled with $S(\mathbf{A}, a, p, \Omega)$ in view of Remark 10.2. □

From the above analysis, the proofs of Theorems 1.6 and 1.7 in dimension $N = 2$ follow.

Proof of Theorem 1.6 for $N = 2$. Arguing as in the proof of Theorem 9.4 from Theorem 10.4 we deduce that $|u(x)| = O(|x|^2)$ as $|x| \to 0$. In particular, from (17), the function $\frac{f(x,u(x))}{u(x)}\chi_{\{x:u(x)\neq 0\}}$ is bounded. The conclusion then follows from Theorem 1.3. □

Proof of Theorem 1.7 for $N = 2$. As in dimension $N \geq 3$, it follows from Theorems 1.5 and 1.6 by the use of the Kelvin transform. □

Appendix

We recall the following well known result proved in [22].

Lemma A.1. (Diamagnetic inequality) Let $N \geq 2$. If $u \in D^{1,2}_*(\mathbb{R}^N, \mathbb{C})$ then

$$|\nabla |u|(x)| \leq \left| \nabla u(x) + i \frac{\mathbf{A}(x/|x|)}{|x|} u(x) \right|$$

for a.e. $x \in \mathbb{R}^N$.

Proof. We only give an idea of the proof. We have

$$|\nabla |u|(x)| = \Re \left( \frac{\nabla u(x)}{|u(x)|} \right)$$

$$\leq \left| \Re \left( \left( \nabla u(x) + i \frac{\mathbf{A}(x/|x|)}{|x|} u(x) \right) \frac{\nabla u(x)}{|u(x)|} \right) \right| = \left| \nabla u(x) + i \frac{\mathbf{A}(x/|x|)}{|x|} u(x) \right|$$

for a.e. $x \in \mathbb{R}^N$. □

An analogous result can be easily shown also for $H^1_*(\Omega, \mathbb{C})$-functions. The following lemma allows comparing assumptions (A.4) and (A.5).

Lemma A.2. Let $N \geq 2$ and assume that (A.2) and (A.3) hold. Then $\mu_1(\mathbf{A}, a) \geq \mu_1(0, a)$ with equality holding if and only if $\text{curl}\mathbf{A} = 0$ in a distributional sense.

Proof. The fact that $\mu_1(\mathbf{A}, a) \geq \mu_1(0, a)$ follows by (21) and the diamagnetic inequality on the sphere

$$|\nabla S^{-1} |\psi|(|\theta|)| \leq |\nabla S^{-1} \psi(|\theta|) + i A(|\theta|) \psi(|\theta|)|$$

for a.e. $\theta \in S^{N-1}$ which holds for any function $\psi \in H^1(S^{N-1})$. Indeed if $\psi_1 \in H^1(S^{N-1})$ is a nontrivial eigenfunction of $\mu_1(\mathbf{A}, a)$ then

$$\mu_1(\mathbf{A}, a) = \frac{\int_{S^{N-1}} |\nabla S^{-1} \psi_1(|\theta|) + i A(|\theta|) \psi_1(|\theta|)|^2 dS - \int_{S^{N-1}} a(|\theta|) |\psi_1(|\theta|)|^2 dS}{\int_{S^{N-1}} |\psi_1(|\theta|)|^2 dS} \geq \frac{\int_{S^{N-1}} |\nabla S^{-1} \psi_1(|\theta|)|^2 dS - \int_{S^{N-1}} a(|\theta|) |\psi_1(|\theta|)|^2 dS}{\int_{S^{N-1}} |\psi_1(|\theta|)|^2 dS} \geq \mu_1(0, a).$$

We start by assuming that $\mu_1(\mathbf{A}, a) = \mu_1(0, a)$. Let $\psi_1$ be as in (128) so that by (127) we infer

$$|\nabla S^{-1} \psi_1(|\theta|) + i A(|\theta|) \psi_1(|\theta|)| = |\nabla S^{-1} \psi_1(|\theta|)|$$

for a.e. $\theta \in S^{N-1}$. 

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Similarly to (129) we have
\[
|\nabla_{S^{N-1}}|\psi_{1}|(\theta)| \leq |\mathcal{R} \left( \frac{\psi_{1}(\theta)}{|\psi_{1}(\theta)|} \right) (\nabla_{S^{N-1}}\psi_{1}(\theta) + i\mathbf{A}(\theta)\psi_{1}(\theta)) | \leq |\nabla_{S^{N-1}}\psi_{1}(\theta) + i\mathbf{A}(\theta)\psi_{1}(\theta)|
\]
which together with (129) gives
\[
\Re(\psi_{1}(\theta)(\nabla_{S^{N-1}}\psi_{1}(\theta) + i\mathbf{A}(\theta)\psi_{1}(\theta))) = 0 \quad \text{for a.e. } \theta \in S^{N-1}
\]
and in turn
\[
\mathbf{A}(\theta) = -\Re \left( \frac{\nabla_{S^{N-1}}\psi_{1}(\theta)}{\psi_{1}(\theta)} \right) \quad \text{for a.e. } \theta \in S^{N-1}.
\]
This implies
\[
\frac{\mathbf{A}(x/|x|)}{|x|} = -\Re \left( \frac{\nabla(\psi_{1}(x/|x|))}{\psi_{1}(x/|x|)} \right) \quad \text{for a.e. } x \in \mathbb{R}^{N}.
\]
By direct computation this gives \( \text{curl} \frac{\mathbf{A}}{|x|} = 0 \) in a distributional sense.

Suppose now that \( \text{curl} \frac{\mathbf{A}}{|x|} = 0 \) in a distributional sense and let us prove that \( \mu_{1}(\mathbf{A}, a) = \mu_{1}(0, a) \).

By (20) we have that there exists \( \phi \in L^{1}_{\text{loc}}(\mathbb{R}^{N}) \) such that \( \nabla \phi = \frac{\mathbf{A}}{|x|} \) in a distributional sense. From (A.3) it follows that \( \phi(x) = \phi(\frac{x}{|x|}) \) and \( \nabla_{S^{N-1}}\phi = \mathbf{A} \). Let \( \Psi \) be a nontrivial eigenfunction of \( \mu_{1}(0, a) \) and define the angular function \( \psi(\theta) \) by
\[
\psi(\theta) = e^{-i\phi(\theta)}\Psi(\theta).
\]
Then
\[
\mu_{1}(\mathbf{A}, a) \leq \int_{S^{N-1}} |\nabla_{S^{N-1}}\psi(\theta) + i\mathbf{A}(\theta)\psi(\theta)|^2 dS - \int_{S^{N-1}} a(\theta)|\psi(\theta)|^2 dS
= \int_{S^{N-1}} |\nabla_{S^{N-1}}\Psi(\theta)|^2 dS - \int_{S^{N-1}} a(\theta)|\Psi(\theta)|^2 dS
= \mu_{1}(0, a).
\]
Since the reverse inequality is always verified the proof is complete. \( \square \)

The following Hardy type inequality with boundary terms is due to Wang and Zhu [27].

**Lemma A.3 (Wang and Zhu).** For every \( r > 0 \) and \( u \in H^{1}(B_{r}, \mathbb{C}) \) there holds
\[
\int_{B_{r}} |\nabla u(x)|^2 dx + \frac{N-2}{2r} \int_{\partial B_{r}} |u(x)|^2 dS \geq \left( \frac{N-2}{2} \right)^2 \int_{B_{r}} \frac{|u(x)|^2}{|x|^2} dx.
\]

**Proof.** See [27] Theorem 1.1. \( \square \)

The following lemma establishes the relation between the classical \( H^{1} \)-space on the sphere and its magnetic counterpart,

**Lemma A.4.** If \( N \geq 2 \) and \( \mathbf{A} \in L^{\infty}(S^{N-1}, \mathbb{R}^{N}) \), then the space \( H^{1}_{\mathbf{A}}(S^{N-1}) \) defined in (14) coincides with the Sobolev space
\[
H^{1}(S^{N-1}, \mathbb{C}) := \{ \psi \in L^{2}(S^{N-1}, \mathbb{C}) : \nabla_{S^{N-1}}\psi \in L^{2}(S^{N-1}, \mathbb{C}^{N}) \}.
\]
Moreover the norms \( \| \cdot \|_{H^{1}(S^{N-1}, \mathbb{C})} \) and
\[
\| \cdot \|_{H^{1}_{\mathbf{A}}(S^{N-1})} := \left( \int_{S^{N-1}} |\nabla_{S^{N-1}}|^{2} + \| \cdot \|_{L^{2}(S^{N-1}, \mathbb{C})}^{2} \right)^{1/2},
\]
are equivalent.

**Proof.** It follows easily from boundedness of the function \( \theta \mapsto |A(\theta)| \).

We finally describe the spectrum of angular operator \( L_{A,a} \).

**Lemma A.5.** Let \( a \in L^\infty(S^{N-1}, \mathbb{R}) \) and \( A \in C^1(S^{N-1}, \mathbb{R}^N) \). Then the spectrum of the operator \( L_{A,a} \) on \( S^{N-1} \) consists in a diverging sequence of real eigenvalues with finite multiplicity \( \mu_1(A,a) \leq \cdots \leq \mu_k(A,a) \leq \cdots \) the first of which admits the variational characterization (27).

**Proof.** For \( \lambda = 1 + \|a\|_{L^\infty(S^{N-1}, \mathbb{R})} \), the operator \( T : L^2(S^{N-1}, \mathbb{R}) \to L^2(S^{N-1}, \mathbb{R}) \) defined as
\[
Tf = u \quad \text{if and only if} \quad (-i \nabla_{S^{N-1}} + A)^2 u - au + \lambda u = f
\]
is well-defined, symmetric, and compact. The lemma follows then from classical spectral theory.

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