Asymptotics of the Mittag-Leffler function $E_a(z)$ on the negative real axis when $a \to 1$

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Abstract

We consider the asymptotic expansion of the single-parameter Mittag-Leffler function $E_a(-x)$ for $x \to +\infty$ as the parameter $a \to 1$. The dominant expansion when $0 < a < 1$ consists of an algebraic expansion of $O(x^{-1})$ (which vanishes when $a = 1$), together with an exponentially small contribution that approaches $e^{-x}$ as $a \to 1$. Here we concentrate on the form of this exponentially small expansion when $a$ approaches the value 1.

Numerical examples are presented to illustrate the accuracy of the expansion so obtained.

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1. Introduction

The single-parameter Mittag-Leffler function $E_a(z)$ is defined by

$$E_a(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(an+1)} \quad (|z| < \infty)$$

(1.1)

where $a > 0$. This function has recently found application in fractional calculus and in the modelling of ‘non-standard’ processes; see, for example, [3, 4, 5, 12]. When $0 < a < 1$, it also arises in the standard model of fractional diffusion [6]. In particular, when $z = -x$ ($x > 0$), the limit $a \to 1$ corresponds to the transition from fractional (slow) diffusion to classical diffusion.

In this paper we shall restrict the parameter $a$ to satisfy $0 < a < 1$ and pay particular attention to the above-mentioned limit $a \to 1$. The standard asymptotic expansion of $E_a(z)$ for $|z| \to \infty$ when $0 < a < 1$ is [2, §18.1], [11, §5.1.4]

$$E_a(z) \sim \begin{cases} \frac{1}{a} \exp\left(\frac{z}{a}\right) + H(z) & (|\arg z| < \pi a), \\ H(z) & (|\arg(-z)| < \pi(1 - \frac{1}{2}a)), \end{cases}$$

(1.2)
where the algebraic expansion \( H(z) \) is given by the formal asymptotic sum
\[
H(z) = -\sum_{k=1}^{\infty} \frac{z^{-k}}{\Gamma(1-ak)} = -\frac{1}{\pi} \sum_{k=1}^{\infty} \Gamma(ak) \sin(\pi ak) z^{-k}.
\]

When \( a = 1 \), \( H(z) \equiv 0 \) and the Mittag-Leffler function reduces to the simple exponential function \( e^z \).

In the first expansion in \((1.2)\) we have extended the domain of validity of the compound expansion up to the Stokes lines \( \arg z = \pm \pi a \). In the sector \( | \arg z | < \frac{1}{2} \pi a \), the exponential term is dominant for large \( |z| \), becoming oscillatory in character on \( \arg z = \pm \frac{1}{2} \pi a \). In the sectors \( \frac{1}{2} \pi a < | \arg z | < \pi a \) the exponential term is subdominant and, although exponentially small, can still make a significant contribution in high-precision asymptotics. On the rays \( \arg z = \pm \pi a \), the exponential term is maximally subdominant relative to the algebraic expansion \( H(z) \).

Across these rays a Stokes phenomenon occurs, where in the sense of increasing \( | \arg z | \) the exponential term "switches off" in a smooth manner described approximately by an error function \( \text{erf} \).

\[\text{erf}(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.\]

The finite sum on the right-hand side of \((2.1)\) corresponds to the first \( M \) terms of the asymptotic expansion \( H(z) \) in \((1.3)\). We put \( z = xe^{i\theta} \), where it is sufficient to consider \( 0 \leq \theta \leq \pi \) since \( E_a(xe^{-i\theta}) \) is given by the conjugate value.

We shall choose \( M \) to be the optimal truncation index of \( H(z) \) (corresponding to truncation at, or near, the least term in modulus) given by \( aM \sim |z|^{1/a} \) as \( |z| \to \infty \). More specifically, we set
\[
aM = X + \nu, \quad X = x^{1/a},
\]

### 2. The expansion of \( E_a(-x) \) when \( 0 < a < 1 \)

The two-parameter Mittag-Leffler function \( E_{a,b}(z) \) satisfies the recursion property
\[
E_{a,b}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(an+b)} = z^{-1} E_{a,b-a}(z) - \frac{z^{-1}}{\Gamma(b-a)}.
\]

Application of this result \( M \) times, where \( M \) is an arbitrary positive integer, yields the result for \( E_{a,1}(z) \equiv E_{a}(z) \)
\[
E_{a}(z) = -\sum_{k=1}^{M} \frac{z^{-k}}{\Gamma(1-ak)} + R_{M}(a;z), \quad R_{M}(a;z) = z^{-M} E_{a,1-aM}(z).
\]

The finite sum on the right-hand side of \((2.1)\) corresponds to the first \( M \) terms of the asymptotic expansion \( H(z) \) in \((1.3)\). We put \( z = xe^{i\theta} \), where it is sufficient to consider \( 0 \leq \theta \leq \pi \) since \( E_a(xe^{-i\theta}) \) is given by the conjugate value.

We shall choose \( M \) to be the optimal truncation index of \( H(z) \) (corresponding to truncation at, or near, the least term in modulus) given by \( aM \sim |z|^{1/a} \) as \( |z| \to \infty \). More specifically, we set
\[
aM = X + \nu, \quad X = x^{1/a},
\]
where \( \nu \) is bounded. From [2 §18.1], [10 (2.4)], we have the integral representation

\[
R_M(a; z) = \frac{z^{-M}}{2\pi i} \int_C \frac{u^{aM-1}e^u}{u-z} \, du = \frac{e^{-iM\theta}}{2\pi i} \int_C \frac{\tau^{aM-1}}{\tau^a-e^{i\theta}} \, e^{X\tau} d\tau,
\]

where \( C \) denotes a loop surrounding the unit disc with endpoints at \(-\infty\) on either side of the branch cut along the negative \( \tau \)-axis (with \( C' \) being the map of this loop in the \( u \)-plane). The integrand has poles at the points \( P_k = \exp [i(\theta+2\pi k)/a] \) \( (k = 0, \pm 1, \pm 2, \ldots) \) and, since \( aM \sim X \), it also has saddle points at \( e^{\pm \pi i} \); see Fig. 1. When \( \theta < \pi a \), the pole \( P_0 \) is situated in \( 0 \leq \arg \tau < \pi \). The contour \( C \) can be deformed over \( P_0 \) and round the branch point at \( \tau = 0 \) (which is integrable) to yield the expansion given in [10 (2.9)]. This expansion, however, breaks down in the vicinity of \( \theta = \pi a \) (a Stokes line) since the pole \( P_0 \) becomes coincident with the saddle point at \( \tau = e^{\pi i} \) in this limit.

In what follows we consider the expansion of the remainder term \( R_M(a; z) \) in a region enclosing the Stokes line \( \theta = \pi a \), which will enable us to deal with the case \( \theta = \pi \). For \( 0 \leq \theta \leq \pi a \), the pole \( P_0 \) lies on the principal Riemann sheet \( (\arg \tau < \pi) \) in the \( \tau \)-plane; when \( \theta = \pi a \) the pole \( P_0 \) lies on \( \arg \tau = \pi \) and when \( \theta = \pi \) the pole \( P_0 \) has passed onto the adjacent sheet that connects with the principal sheet along \( \arg \tau = \pi \). The pole \( P_{-1} = e^{i(\theta-2\pi)/a} \) lies on the adjacent sheet that connects with the principal sheet along \( \arg \tau = -\pi \). We observe that when \( \theta = \pi \), the poles \( P_0 \) and \( P_{-1} \) are situated symmetrically at \( e^{\pm \pi i/a} \) on the (separate) adjacent Riemann sheets.

The loop \( C \) in (2.2) is now deformed round the branch point \( \tau = 0 \), with the path on the upper side of the cut passing above the pole \( P_0 \) and saddle at \( \tau = e^{\pi i} \) and the path on the lower side of the cut passing below the pole \( P_{-1} \) and saddle at \( \tau = e^{-\pi i} \). We concentrate on the contribution from the integral taken along the upper side of the cut since, when \( \theta = \pi \), that from the integral along the lower side of the cut will yield the conjugate value. The details of this calculation are given in the appendix.

3. The expansion when \( \theta = \pi \)

From [A.9], the expansion of the integral along the upper side of the branch cut in the \( \tau \)-plane when \( \theta = \pi \) is

\[
\frac{e^{Xe^{\pi i/a}}}{2a} \text{erfc} [c(\pi)|X/2|] - \frac{ie^{-X-X(\pi)iX}}{a\sqrt{2\pi X}} \sum_{k=0}^{\infty} B_{2k}(\pi)(\frac{1}{2})_{k}(\frac{1}{2}X)^{-k}
\]
as $X \to +\infty$ with $\omega(\pi) = \pi(1-a)/a$. The contribution to the integral (2.3) from the lower side of the branch cut in the $\tau$-plane will yield the conjugate of the above expansion.

Hence, on the negative real $z$-axis we have the following result:

**Theorem 1.** The expansion of the Mittag-Leffler function $E_a(-x)$ for $x \to +\infty$ and $\frac{1}{3} < a < 1$ is

$$E_a(-x) = -\sum_{k=1}^{M} \frac{(-x)^{-k}}{\Gamma(1 - ak)} + R_M(a; -x),$$

where the remainder $R_M(a; -x)$ has the exponentially small expansion

$$R_M(a; -x) \sim \frac{2}{a} \Re \left\{ \frac{\exp[X e^{-\pi i/a}]}{2} \text{erfc}[c(\pi) \sqrt{X/2}] - \frac{i e^{-X - \omega(\pi)X}}{\sqrt{2\pi X}} \sum_{k=0}^{\infty} B_{2k}(\pi) \left( \frac{1}{2}\right)^{k} \frac{X}{\pi} \right\}.$$

Here, $M$ is the optimal truncation index of the algebraic expansion given in (2.2), $X = x^{1/a}$, $\omega(\pi) = \pi(1-a)/a$ and $c(\pi)$ is determined from (A.7) and, in the limit $a \to 1$, by (A.7).

It now remains to discuss the coefficients $B_{2k}(\pi)$ appearing in (3.2), which is carried out in the next sub-section.

### 3.1 The coefficients $B_{2k}(\pi)$

If we use the Series command in *Mathematica*, we can obtain the coefficients in the expansion $f(u) = \sum_{r=0}^{\infty} \alpha_r u^r$, where $f(u)$ is defined in (A.4). Upon inversion of the transformation in (A.2) to obtain

$$t - 1 = u + \frac{1}{3} u^2 + \frac{1}{36} u^3 - \frac{1}{270} u^4 + \frac{1}{4320} u^5 + \ldots,$$

the first three even-order coefficients $\alpha_r$ are found to be

$$\begin{align*}
\alpha_0 &= \frac{1}{1-T}, \\
\alpha_2 &= \frac{1}{12(1-T)^3} \left\{ 1 + 6\nu^2 (1-T)^2 + (6a^2 + 6a - 2)T + (6a^2 - 6a + 1)T^2 \\
&\quad - 6\nu(1-T)(1 + (2a - 1)T) \right\}, \\
\alpha_4 &= \frac{1}{864(1-T)^5} \left\{ 1 + 36\nu^4 (1-T)^4 + 4(-1 + 9a + 30a^2 + 30a^3 + 9a^4)T \\
&+ 6(1 - 18a - 20a^2 + 60a^3 + 66a^4)T^2 + 4(-1 + 27a - 30a^2 - 90a^3 + 99a^4)T^3 \\
&+ (1 - 36a + 120a^2 - 120a^3 + 36a^4)T^4 - 24\nu^3 (1-T)^3 (5 + (-5 + 6a)T) \\
&+ 24\nu^2 (1-T)^2 (5 + (-10 + 15a + 9a^2)T + (5 - 15a + 9a^2)T^2) \\
&+ 12\nu (1-T) (3 + (-9 + 20a + 30a^2 + 12a^3)T + (9 - 40a + 48a^3)T^2 \\
&+ (-3 + 20a - 30a^2 + 12a^3)T^3) \right\},
\end{align*}$$

where we have put $T = e^{i\omega(\pi)}$ for brevity. We recall that the quantity $\nu$ appears in the definition of the optimal truncation index $M$ in (2.2). It is impractical to present higher coefficients as they depend on three quantities ($a$, $\nu$ and $T$) and rapidly become too complicated. However, when dealing with specific cases, where the numerical values of $a$, $\nu$ and $T$ are known, it is feasible to evaluate many more coefficients $\alpha_{2k}$ by this method; see Section 4 for an example.
From (A.7) and (A.8), we have when $\theta = \pi$

$$f(u) = A\left\{\frac{1}{u - u_0} + \sum_{r=0}^{\infty} B_r(\pi)u^r\right\} = A\left\{\frac{1}{u_0} \left(1 + \frac{u}{u_0} + \frac{u^2}{u_0^2} + \ldots\right) + \sum_{r=0}^{\infty} B_r(\pi)u^r\right\} \quad (u < u_0),$$

whence it follows that the coefficients $B_{2k}(\pi)$ are given by

$$B_{2k}(\pi) = ae^{-i\omega(\pi)}\alpha_{2k} + \frac{1}{(ic(\pi))^{2k+1}}. \quad (3.4)$$

The leading coefficient consequently has the value

$$B_0(\pi) = \frac{ae^{-i\omega(\pi)}}{1 - e^{i\omega(\pi)}} + \frac{1}{ic(\pi)}. \quad (3.5)$$

The form (3.4) has the inconvenient feature of a removable singularity since $\omega(\pi)$ and $c(\pi) \to 0$ as $a \to 1$. For $a \approx 1$, we can expand the coefficients $B_{2k}(\pi)$ in ascending powers of $\omega(\pi)$, viz.

$$B_{2k}(\pi) = \sum_{r=0}^{\infty} b_{2k,r} \omega^r, \quad \omega \equiv \omega(\pi). \quad (3.6)$$

Using *Mathematica* to carry out this procedure and the expansion of $c(\pi)$ in powers of $\omega(\pi)$ in (A.7), we obtain the first few values of the coefficients $b_{2k,r}$ in the form:

$$b_{0,0} = \frac{1}{2}a + \nu - \frac{1}{6}, \quad b_{0,1} = -\frac{i}{12}(2a^2 + 6a\nu + 6\nu^2),$$

$$b_{0,2} = -\frac{1}{1080}(1 + 90\nu(a + \nu)(a + 2\nu)), \quad b_{0,3} = \frac{i}{12960}(1 + 18a^4 - 540\nu^2(a + \nu)^2), \quad (3.7)$$

$$b_{0,4} = \frac{1}{181440}(-1 - 252a^4\nu + 2520a^2\nu^3 + 3780a\nu^4 + 1512\nu^5), \ldots,$$

$$b_{2,0} = \frac{1}{1080}(-2 + 45a - 45a^2 + 90\nu - 270a\nu + 90a^2\nu - 270\nu^2 + 270a\nu^2 + 180\nu^3),$$

$$b_{2,1} = \frac{i}{1440}(-1 - 10a^2 + 6a^4 - 60a\nu + 120a^2\nu - 60\nu^2 + 360a^2\nu^2 - 180a\nu^2 + 240\nu^3$$

$$-360a\nu^3 - 180a^4), \quad (3.8)$$

$$b_{2,2} = \frac{1}{60480}(1 - 126a^4 - 420a^2\nu + 504a^4\nu - 1260a^2\nu^2 + 3780a^2\nu^3 - 840\nu^3$$

$$+7560a^4\nu^3 - 5040a^2\nu^3 + 3780a\nu^4 - 7560a\nu^4 - 3024\nu^5), \ldots$$

and

$$b_{4,0} = \frac{1}{181440}(65 + 105a - 630a^2 + 210a^4 + 210\nu - 3780a\nu + 4200a^2\nu - 252a^4\nu - 3780\nu^2$$

$$+12600a\nu^2 - 6300a^2\nu^2 + 8400a^3\nu - 12600a^2\nu^3 + 2520a^2\nu^3 - 6300\nu^4 + 3780a\nu^4 + 1512\nu^5),$$

$$b_{4,1} = \frac{i}{1088640}(2 + 105a^2 - 1260a^4 + 180a^6 + 630a\nu - 7560a^2\nu + 5040a^4\nu + 630\nu^2$$

$$-22680a\nu^2 + 37800a^2\nu^2 - 3780a^2\nu^2 - 15120\nu^3 + 75600a\nu^3 - 50400a\nu^3 + 37800\nu^4$$

$$-7560a\nu^4 + 18900a^2\nu^4 - 30240\nu^5 + 22680a\nu^5 + 75600\nu^5), \ldots.$$
3.2 Approximate form of $R_M(a; -x)$ as $a \to 1$

An estimate of the value of the exponentially small term $R_M(a; -x)$ as $a \to 1$ can be obtained from Theorem 1 and the fact that, from (3.7), $c(\pi) = \omega + O(\omega^2)$. Then, from (3.2), it follows that to leading order

$$R_M(a; -x) \simeq \frac{1}{a} e^{X \cos \pi/a} \text{erfc}\left[\frac{\omega}{a} X/2\right] - \frac{2e^{-X}}{a \sqrt{2\pi X}} \Re\{ie^{-\omega X} B_0(\pi)\}.$$

Since $B_0(\pi) = b_{0,0} + b_{0,1} \omega + O(\omega^2)$, where $b_{0,1}$ is imaginary and $\omega = \pi (1 - a)/a$, we finally obtain

$$R_M(a; -x) \simeq \frac{1}{a} e^{X \cos \pi/a} \text{erfc}\left[\frac{\pi(1 - a)}{a} \sqrt{\frac{X}{2}}\right] - \frac{2e^{-X}}{a \sqrt{2\pi X}} \left\{ b_{0,0} \sin \omega X + |b_{0,1}| \omega \cos \omega X \right\}$$

(3.10)
as $x \to +\infty$, where we recall that $X = x^{1/a}$ and $b_{0,0}, b_{0,1}$ are given in (3.7).

The behaviour of $R_M(a; -x)$ as one approaches the limit $a = 1$ is seen to be controlled by a complementary error function, which increases rapidly as $a \to 1$ to the value $\text{erfc}(0) = 1$ since $\text{erfc}\xi \approx \exp[-\xi^2]/\sqrt{\pi} \xi$ for $\xi \gg 1$. When $a = 1$, the quantity $\omega \equiv \omega(\pi) = 0$ and we recover the limiting value of the exponentially small term $R_M(a; -x) = e^{-x}$. Thus the formula (3.10) correctly describes the appearance of the exponential $e^{-x}$ when $a = 1$.

4. Numerical results and concluding remarks

To verify the accuracy of the expansion in Theorem 1 we subtract the optimally truncated algebraic expansion from $E_a(-x)$ and define

$$E(a; x) := E_a(-x) + \sum_{k=1}^{M} \frac{(-x)^{-k}}{\Gamma(1 - ak)},$$

(4.1)

where the optimal index $M$ is defined in (2.2). This quantity is then compared to the exponentially small contribution $R_M(a; -x)$ for different $a$ and $x$.

Table 1: The coefficients $B_{2k}(\pi)$ when $a = 0.99$ and $x = 40$ ($M = 42$).

| $k$ | $B_{2k}(\pi)$ |
|-----|----------------|
| 0   | $+3.8975364113 \times 10^{-1} - 3.6166205223 \times 10^{-3}i$ |
| 1   | $-6.4791569264 \times 10^{-3} - 2.2873163550 \times 10^{-5}i$ |
| 2   | $+1.193771912 \times 10^{-3} + 2.9428888000 \times 10^{-5}i$ |
| 3   | $+6.7326294689 \times 10^{-5} - 3.3561255923 \times 10^{-7}i$ |
| 4   | $+6.4497172230 \times 10^{-6} - 2.2913466614 \times 10^{-7}i$ |
| 5   | $-4.9612005443 \times 10^{-7} + 4.0896790580 \times 10^{-9}i$ |
| 6   | $-3.8100530725 \times 10^{-8} + 1.6905896799 \times 10^{-9}i$ |

To illustrate we consider the case $x = 40$ and $a = 0.99$. From (2.2) we find $M = 42$ with the parameter $\nu = 0.0614272718 \ldots$. The first three coefficients $B_{2k}(\pi)$ can be computed from
The Mittag-Leffler function $E_a(z)$ as $a \to 1$

The values of $\alpha_0, \alpha_2$ and $\alpha_4$ stated in Section 3.1. However, since we have numerical values the higher coefficients can be obtained by the approach discussed at the end of Section 3.1, whereby we expand $f(u)$ in (3.4) using the Series command in Mathematica together with (3.4). The values of $B_{2k}(\pi)$ for $0 \leq k \leq 6$ so obtained are presented in Table 1. We note that the values of $B_{2k}(\pi)$ depend on $x$ through the quantity $\nu$ defined in (2.2). In Table 2 we show the values of $R_M(a; x)$ for different truncation index $k$ and two values of $a \simeq 1$ compared with the computed values of $\mathcal{E}(a; x)$. It is seen that there is excellent agreement between the computed value of $\mathcal{E}(a; x)$ and the asymptotic estimate for the exponentially small contribution.

Table 2: The values of $R_M(a; -x)$ for different truncation index $k$ when $x = 40$: (i) $a = 0.99, M = 42$ and (ii) $a = 0.995, M = 20$. The final row gives the values of $\mathcal{E}(a; x)$ defined in (3.4) for comparison.

| $k$ | $R_M(a; -x)$, $a = 0.99$ | $R_M(a; -x)$, $a = 0.995$ |
|-----|-------------------------|--------------------------|
| 0   | 1.56895 52145 63456 × 10^{-19} | 1.37899 77500 62528 × 10^{-09} |
| 1   | 1.56913 08832 53406 × 10^{-19} | 1.37891 00449 63445 × 10^{-09} |
| 2   | 1.56913 32394 39717 × 10^{-19} | 1.37890 98868 81488 × 10^{-09} |
| 3   | 1.56913 32235 20415 × 10^{-19} | 1.37890 99084 34786 × 10^{-09} |
| 4   | 1.56913 32232 61265 × 10^{-19} | 1.37890 99085 29609 × 10^{-09} |
| 5   | 1.56913 32232 65555 × 10^{-19} | 1.37890 99085 08090 × 10^{-09} |
| 6   | 1.56913 32232 65644 × 10^{-19} | 1.37890 99085 08192 × 10^{-09} |
| $\mathcal{E}(a; x)$ | 1.56913 32232 65642 × 10^{-19} | 1.37890 99085 08192 × 10^{-09} |

In Table 3 we show values of $\mathcal{E}(a; x)$ and $R_M(a; -x)$ (with truncation index $k = 5$) for a range of $a$-values. At the end of Section 2.1 it was argued that the parameter $a > \frac{1}{3}$ for the sector of validity of the expansion (3.2) to include the negative real axis. It is noteworthy that there continues to be good agreement between $\mathcal{E}(a; x)$ and $R_M(a; -x)$ even when $a \leq \frac{1}{3}$. The value of $x$ chosen in the cases $a = \frac{1}{3}$ and $a = \frac{1}{2}$ in Table 3 is small; larger values would result in very large optimal truncation index $M$ (for example, if $x = 10$ when $a = \frac{1}{2}$, we find $M = 4 \times 10^4$). This would produce extreme accuracy from just the algebraic expansion, with $R_M(a; -x)$ so small as to be negligible in most applications. The validity of this agreement when $a \leq \frac{1}{2}$ would require further investigation, which is not carried out here as our main interest is in the limit $a \to 1$.

Table 3: The values of $\mathcal{E}(a; x)$ and $R_M(a; -x)$ (with truncation index $k = 5$) for different values of $a$ and $x$.

| $a$ | $x$ | $M$ | $\mathcal{E}(a; x)$ | $R_M(a; -x)$ |
|-----|-----|-----|---------------------|---------------|
| 0.95| 20  | 25  | -2.521343 284521 × 10^{-11} | -2.521343 284522 × 10^{-11} |
| 0.90| 20  | 21  | -2.706560 459479 × 10^{-13} | -2.706560 459478 × 10^{-13} |
| 0.80| 53  | 53  | -4.827618 810882 × 10^{-20} | -4.827618 810882 × 10^{-20} |
| 0.70| 68  | -3.052228 407002 × 10^{-23} | -3.052228 407002 × 10^{-23} |
| 0.60| 77  | -6.895973 422484 × 10^{-22} | -6.895973 422484 × 10^{-22} |
| 0.50| 50  | -1.106145 146730 × 10^{-12} | -1.106145 146730 × 10^{-12} |
| 0.33| 81  | +8.345377 837784 × 10^{-14} | +8.345377 837735 × 10^{-14} |
| 0.25| 324 | -1.220075 244872 × 10^{-37} | -1.220075 244872 × 10^{-37} |
Appendix: Estimation of the contribution along the upper side of the cut

The procedure we employ is a slight modification of that described by Olver \[8\] in the treatment of the generalised exponential integral; see also [11, \S 6.2.6]. If we make the change of variable $t = e^{-\pi \tau}$ in (A.1), the integral taken along the upper side of the cut in the $\tau$-plane becomes

$$J = e^{-i M(\theta - \pi \alpha)} \frac{e^{-X}}{2\pi i} \int_0^\infty e^{-X \psi(t)} \frac{t^a + \nu - 1}{t^a - t_0^a} \, dt,$$

where

$$\psi(t) = t - \log t - 1, \quad t_0 = e^{i \omega(\theta)}, \quad \omega(\theta) = (\theta - \pi \alpha)/a.$$

In the $t$-plane, the branch cut is now situated on $[0, \infty)$ and the integration path in (A.1) passes below the image of the pole $P_0$ and the saddle at $t = 1$. Setting

$$\frac{1}{2} u^2 = t - \log t - 1, \quad \frac{dt}{du} = \frac{ut}{t - 1},$$

we can express the integral (A.1) in the form

$$J = e^{-i M(\theta - \pi \alpha)} \frac{e^{-X}}{2\pi i} \int_{-\infty}^\infty e^{-\frac{1}{2} X u^2} f(u) \, du,$$

where

$$f(u) := \frac{t^a + \nu - 1}{t^a - t_0^a} \, \frac{dt}{du} = \frac{ut^a + \nu}{(t - 1)(t^a - t_0^a)}.$$

The function $f(u)$ can be expanded in the form

$$f(u) = A \left\{ \frac{1}{u - u_0} + g(u) \right\},$$

where the pole at $u = u_0 \equiv ic(\theta)$ corresponds to the pole in the $t$-plane at $t_0 = e^{i \omega(\theta)}$ and $g(u)$ is analytic at the point $u = u_0$. We have from (A.2)

$$\frac{1}{2} c^2(\theta) = 1 + i \omega(\theta) - e^{i \omega(\theta)},$$

where the branch of the square root is chosen so that near $\theta = \pi a$ the expansion of $c(\theta)$ has the form

$$c(\theta) = \omega(\theta) + \frac{i}{2} \omega^2(\theta) - \frac{1}{3!} i \omega^3(\theta) + \frac{1}{2!} \omega^4(\theta) + \frac{1}{2!} \omega^5(\theta) + \ldots.$$  (A.7)

The constant $A$ appearing in (A.5) can be determined by a limiting process. If we let $t = t_0 + \epsilon$, $\epsilon \to 0$, so that from (A.2) $u - u_0 = \epsilon(t_0 - 1)/(u_0 t_0) + O(\epsilon^2)$, we find

$$A = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \frac{u - u_0}{f(u)} = \frac{e^{i \omega(\theta)}}{a}.$$

Substitution of the above expansion for $f(u)$ in (A.3) then yields

$$J = \frac{e^{-X - i \omega(\theta) X}}{2\pi i} \left\{ \int_{-\infty}^\infty e^{-\frac{1}{2} X u^2} \frac{dt}{u - u_0} + \int_{-\infty}^\infty e^{-\frac{1}{2} X u^2} g(u) \, du \right\}.$$

The first integral on the right-hand side of the above expression (where the path is indented to pass below the pole $u_0$) can be evaluated in terms of the complementary error function

$$\int_{-\infty}^\infty e^{-\frac{1}{2} X u^2} \frac{dt}{u - u_0} = \pi i e^{\frac{1}{2} X c^2(\theta)} \text{erfc} \left[ c(\theta) \sqrt{X/2} \right].$$
The Mittag-Leffler function $E_a(z)$ as $a \to 1$

In the second integral the path may be taken as the real axis with no indentation, since the integrand has no singularity on the integration path. If we expand $g(u)$ as a Maclaurin series

$$g(u) = \sum_{r=0}^{\infty} B_r(\theta) u^r,$$

we find

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2} X u^2} g(u) \, du \sim \pi^{1/2} \sum_{k=0}^{\infty} B_{2k}(\theta) \left(\frac{1}{2}\right)^k \left(\frac{1}{X}\right)^{-k} \quad (X \to \infty).$$

Collecting together these results and noting that

$$e^{\frac{1}{2} X c(\theta)^2} = \exp \left[ z^{1/a} + X + i\omega(\theta) X \right]$$

by (A.6), we finally obtain from Theorem 1 of [8, p. 1473] the desired expansion

$$J \sim \frac{1}{a} \left\{ \exp \left[ \frac{z^{1/a}}{2} \right] \text{erfc} \left[ c(\theta) \sqrt{X/2} \right] - \frac{i e^{X - i\omega(\theta) X}}{2\pi X} \sum_{k=0}^{\infty} B_{2k}(\theta) \left(\frac{1}{2}\right)^k \left(\frac{1}{X}\right)^{-k} \right\}$$

as $|z| \to \infty$ in the sector $-\pi a < \theta < 3\pi a$, where $c(\theta)$ is defined by (A.6) with the expansion in ascending powers of $\omega(\theta)$ given in (A.7). The coefficients $B_{2k}(\theta) \equiv B_{2k}(\theta, \nu)$ in the case $\theta = \pi$ are discussed in Section 3.2. The above sector clearly includes the negative real axis arg $z = \pi$ when $a > \frac{1}{3}$.

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There is an error in the sign of the second term in this expansion in [10] (2.8).