HYPERBOLICITY OF LINKS COMPLEMENTS IN SEIFERT FIBERED SPACES

TOMMASO CREMASCHI AND JOSE ANDRES RODRIGUEZ-MIGUELES

Abstract: Let $\bar{\gamma}$ be a link in a Seifert fibered space $M$ over a hyperbolic 2-orbifolds $O$ that projects injectively to a filling multicurve of closed geodesics $\gamma$ in $O$. We prove that the complement $M_\bar{\gamma}$ of $\bar{\gamma}$ in $M$ admits a hyperbolic structure of finite volume and give combinatorial bounds of its volume.

1. Introduction

Given a hyperbolic surface of finite type $\Sigma$ in the projective unit tangent bundle $PT^1(\Sigma)$ there is a very special family of links $\hat{\gamma}$ coming from canonical lifts of a geodesic multicurve $\gamma$ in $\Sigma$, namely the image under the map $T^1\Sigma \to PT^1(\Sigma)$ of the corresponding finite set of periodic orbits of the geodesic flow. Foulon and Hasselblatt [FH13] gave a topological criterium, depending only on the immersion of $\gamma$ in $\Sigma$, that guarantees the existence of a complete hyperbolic metric of finite volume in the canonical lift complement of $\gamma$ in $PT^1(\Sigma)$.

Theorem 1.1 (Foulon-Hasselblatt,[FH13]). Let $\gamma$ be a closed geodesic on a hyperbolic surface $\Sigma$. Then, the complement of the canonical lift admits a finite volume complete hyperbolic structure if and only if $\gamma$ is filling.

The previous theorem was stated in a more general setting. The authors of [FH13] considered any embedded lift $\bar{\gamma}$ in the unit tangent bundle of the hyperbolic surface as long as the projection was injective outside the double points of the closed geodesic $\gamma$.

After reading their proof carefully, we noticed that an argument relative to the atoridality of these knot complements was only stated for the particular case of knots coming from periodic orbits of the geodesic flow; on the other hand, the arguments for the other cases worked in greater generality.

One of the steps of the proof of the atoroidalty of $M_\bar{\gamma} = PT^1(\Sigma) \setminus \hat{\gamma}$ is to show that no essential torus $T \subset M_\bar{\gamma}$ is null-homotopic in $PT^1(\Sigma)$. To do so the authors of [FH13] argue that since the geodesic flow is product covered in the universal cover $\widetilde{PT^1(\Sigma)}$ the complement of all the lifts $\tilde{\{\hat{\gamma}\}}$ of $\hat{\gamma}$ is homeomorphic to $(\mathbb{R}^2 \setminus X) \times \mathbb{R}$, for $X$ a discrete set.

Since $\pi_1((\mathbb{R}^2 \setminus X) \times \mathbb{R})$ is free and the essential torus $T$ lifts to $\widetilde{PT^1(\Sigma)} \setminus \tilde{\{\hat{\gamma}\}}$ we reach a contradiction. This is because a free group does not contain any $\mathbb{Z}^2$ subgroup. To avoid using the geodesic flow we will directly show that $\pi_1(\widetilde{PT^1(\Sigma)} \setminus \tilde{\{\hat{\gamma}\}})$ is free for any lift $\tilde{\gamma}$ in $\widetilde{PT^1(\Sigma)}$ of a filling multicurve of closed geodesics on $\Sigma$.

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Moreover, by adding our argument to their proof we can achieve a version of Theorem 1.1 in a more general setting for links complements in Seifert fibered spaces, whose projection to their hyperbolic 2-orbifold base is a filling multicurve of closed geodesics.

Our main result is:

**Theorem 1.2.** Suppose $O$ is a hyperbolic 2-orbifold and $\gamma$ a link in an orientable Seifert fibered space $M$ over the orbifold $O$ that projects injectively to a filling multicurve of closed geodesics $\gamma$ in $O$. Then the complement of $\gamma$ on $M$, denoted by $M_\gamma$, is a hyperbolic manifold of finite volume.

The aim of this paper is to prove the missing argument for the atoroidality of these links complements and extend some results of the second author from the unit tangent bundle to this setting. This question was posed in a beautiful blog-post of Calegari [Cal] where he gives a geometric proof of Theorem 1.1.

The main ingredient of the proof of 1.2 is the following Theorem:

**Theorem 1.3.** Let $M$ be a Seifert fibered space over a hyperbolic surface $\Sigma$, let $\gamma \subset \Sigma$ a filling multicurve of closed geodesics, $q : \tilde{M} \to M$ the universal covering map of $M$ and $\{\tilde{\gamma}\}$ the total preimage of the link $\gamma$ under $q$. Then, the group $\pi_1(\tilde{M} \setminus \{\tilde{\gamma}\})$ is free.

Once the hyperbolicity of $M_\gamma$ is settled then, by the Mostow’s Rigidity Theorem we can pursue the problem of estimating the volume of $M_\gamma$.

By constructing a particular ideal triangulation on $M_\gamma$ one can give a volume upper bound to $M_\gamma$, independent of the lift $\tilde{\gamma}$, which is linear in terms of the self-intersection number of $\gamma$. What might be remarkable is that, as a consequence of results found in [HP18], this upper bound is sharp, up to some constant, for some closed geodesics on any punctured hyperbolic surface and for particular lifts in its unit tangent bundle:

**Corollary 1.4.** Let $\Sigma_{g,n}$ be an $n$-punctured hyperbolic surface, $n \geq 1$, then there exist a sequence of $\{\gamma_n\}_{n \in \mathbb{N}}$ filling closed geodesics with $i(\gamma_n, \gamma_n) \nearrow \infty$, and respective lifts $\{\tilde{\gamma_n}\}_{n \in \mathbb{N}}$ in $PT^1(\Sigma_{g,n})$ such that,

$$\frac{v_3}{2}(i(\gamma_n, \gamma_n) - (2 - 2g)) \leq \text{Vol}(M_{\gamma_n}) \leq 44v_3i(\gamma_n, \gamma_n),$$

where $v_3$ (or $v_8$) is the volume of the regular ideal tetrahedron (octahedron) and $i(\gamma_n, \gamma_n)$ the self-intersection number of $\gamma_n$.

We also give a lower bound by generalising the arguments of the second author’s (Theorem 1.5, [RM17]) to the Seifert fibered setting.

**Theorem 1.5.** Given a pants decomposition $\Pi$ on a hyperbolic 2-orbifold $O$, a Seifert fibered space $M$ over $O$, and a filling geodesics multicurve $\gamma$ on $O$, for any closed continuous lift $\tilde{\gamma}$ we have that :

$$\frac{v_3}{2} \sum_{P \in \Pi} (\sharp\text{ isotopy classes of } \tilde{\gamma}\text{-arcs in } p^{-1}(P) - 3) \leq \text{Vol}(M_\gamma),$$

where $v_3$ is the volume of a regular ideal tetrahedron.
Outline: In section 2 we recall some basic facts about Seifert Fibered spaces and orbifolds. In section 3 we prove Theorem 1.2 and Theorem 1.3. In section 4 by using results of the second author 4 we prove some volume bounds.

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2. Seifert fibered spaces and orbifolds

In this section we recall some known facts about the topology of Seifert fibered spaces and orbifolds. For more details see [Jac80, Hem76].

Definition 2.1. A compact 3-manifold $M$ is a Seifert fibered space if $M$ is the union of a collection $\{C_\alpha\}_{\alpha \in A}$ of pairwise disjoint simple closed curves called fibers such that every fiber $C_\alpha$ has a closed neighbourhood $V_\alpha$ homeomorphic to a solid torus and a covering map $p_\alpha : \mathbb{D}^2 \times S^1 \to V_\alpha$ satisfying:

(i) for all $x \in \mathbb{D}^2$ we have that $p_\alpha(\{x\} \times S^1) = C_\beta$ for some $\beta \in A$ so that $V$ is a union of fibers;

(ii) $p_\alpha^{-1}(C_\alpha)$ is connected;

(iii) the group of covering transformation is generated by $r_{n,m}$ for $n,m$ relatively prime integers such that:

$$r_{n,m}(re^{i\theta}, e^{i\phi}) = (re^{i(\theta + 2\frac{m}{n}\pi)}, e^{i(\phi + \frac{2\pi}{n})})$$

If $|n| = 1$ we have that $p_\alpha$ is a homeomorphism and we say that $C_\alpha$ is a regular fiber, otherwise we say it is a singular fiber.

Note that whenever $|n| > 1$ by (ii) $C_\alpha = p_\alpha(\{0\} \times S^1)$ and for $x \neq 0$ we have that $p(\{x\} \times S^1)$ is mapped to a fiber $C_\beta$ which crosses the meridional disk $p_\alpha(\mathbb{D}^2 \times \{1\})$ $n$ times and wraps $m$ times around $C_\alpha$. Also, since every fiber in a neighbourhood of a singular fiber is regular we get that if $M$ is compact it has finitely many singular fibers.

Definition 2.2. We say that an Hausdorff topological space $O$ is an orbifold if we have a covering $U = \{U_i\}_{i \in \mathbb{N}}$ closed under finite intersections and continuous maps: $\phi_i : V_i \to U_i$ for $V_i$ open subsets of $\mathbb{R}^2$ invariant under a faithful linear action of a finite group $\Gamma_i$ such that $\phi_i : V_i/\Gamma_i \to U_i$ is a homeomorphism. Moreover, we say that the charts $\{U_i\}_{i \in \mathbb{N}}$ form an orbifold atlas if:

- for $U_i \subset U_j$ we have a monomorphism $f_{ij} : \Gamma_i \to \Gamma_j$;
- for $U_i \subset U_j$ we have a $\Gamma_i$-equivariant homeomorphism $\psi_{ij}$, called a gluing map, from $V_i$ to an open subset of $V_j$;
- for all $i, j$ we have $\phi_j \circ \psi_{ij} = \phi_i$;
- the gluing maps are unique up to compositions with group elements.

Remark. Even though general orbifold can have reflections in the rest of this case we will only consider orbifolds with conical points. That is the set of singular points in any orbifold $O$ will always be a discrete set.
If $M$ is a Seifert fibered space we have a natural projection map: $\pi : M \to O$ obtained by mapping every fiber $C_\alpha$ to a point, the space $O$ is called the orbit-manifold. Given a neighbourhood of $C_\alpha$ the map $\pi \circ p_\alpha : \mathbb{D}^2 \times \{1\} \to O$ is an embedding if $C_\alpha$ is a regular fiber and is equivalent to the projection onto the orbit space of $\mathbb{D}^2 \times \{1\}$ under a periodic rotation otherwise. Therefore, the quotient space $O$ is naturally an orbifold.

**Remark.** From the classification theorem of Seifert fibered spaces [Sei33] follows that any Seifert fibered space $M$ is homeomorphic to an $S^1$-bundle over a compact surface $S$ where we glue some singular neighbourhoods along some tori boundary components. Equivalently, we can think of a Seifert fibered space as an orientable $S^1$-bundle on a compact orbifold $O$.

3. Proof of the Theorem 1.3

**Definition 3.1.** Given a Seifert fibered space $M$ with its bundle map $p : M \to O$ we say that a link $\gamma \subset M$ projects injectively to a multicurve $\gamma \subset O$ if distinct components of $\gamma$ map under $p$ to distincts components of $\gamma$ such that the projection $p$ is injective except at self-intersection points of $\gamma$ which have two pre-images.

Let $M$ be a Seifert fibered space over a hyperbolic 2-orbifold $O$, $\gamma$ a geodesic multicurve on $O$ and $\overline{\gamma}$ a link in $M$ which projects injectively to $\gamma$ under $p$. Then we have the following diagram:

\[
\begin{array}{c}
\begin{array}{c}
\gamma \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\overline{\gamma} \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
M \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\uparrow p \\
\downarrow \bigcup_{i=1}^n S^1 \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
O \\
\end{array}
\end{array}
\end{array}
\]

From now on we denote by $M_{\overline{\gamma}}$ the complement of a normal neighborhood of $\overline{\gamma}$ in $M$.

**Definition 3.2.** For a hyperbolic 2-orbifold $O$ with a discrete set of singular points $S$ we say that a multicurve $\gamma$ of closed geodesics is filling if $\gamma$ is disjoint from $S$ and if $O \setminus \gamma$ is a collection of disks, once-puncture disks or disks with one conical point.

![Figure 1. A filling geodesic on a 2-orbifold.](image)

In order to prove Theorem 1.2.

**Theorem 1.2.** Suppose $O$ is a hyperbolic 2-orbifold and $\overline{\gamma}$ a link in an orientable Seifert fibered space $M$ over the orbifold $O$ that projects injectively to a filling multicurve of closed geodesics $\gamma$ in $O$. Then $M_{\overline{\gamma}}$ is a hyperbolic manifold of finite volume.
We first reduce it to the case where the orbifold $O$ is a surface, i.e. to the case where the Seifert fibered space has a hyperbolic surface $\Sigma$ as base.

**Lemma 3.3.** If $\Sigma \to O$ is a finite cover from a surface $\Sigma$ and $\gamma$ is a filling multicurve of closed geodesics in the orbifold $O$ then the union of all lifts $\gamma_0$ is also a filling geodesic multicurve on $\Sigma$.

**Proof.** Let $\Sigma_0 \cong O \setminus S$, then $\gamma$ is filling in $\Sigma_0$. Consider the induced cover $q : q^{-1}(\Sigma_0) \to \Sigma_0$ for $q^{-1}(\Sigma_0)$ a connected subsurface of $\Sigma$. Then $q^{-1}(\gamma)$ is filling in $q^{-1}(\Sigma_0)$. However, $\Sigma = q^{-1}(\Sigma_0) \cup D$ for $D$ a collection of disks each covering a disk with a cone point. Thus $q^{-1}(\gamma)$ is filling in $\Sigma$ and since $q^{-1}(\gamma) = \gamma_0$ we are done. $\square$

**Lemma 3.4.** Given a finite cover $\pi : \hat{M}_{\gamma_0} \to M_{\gamma}$ we have that if $\hat{M}_{\gamma_0}$ is atoroidal so is $M_{\gamma}$.

**Proof.** Given an essential torus $T \subset M_{\gamma}$ the restriction $\pi : \pi^{-1}(T) \to T$ is a finite cover hence, every component of $\pi^{-1}(T)$ is an essential torus $\hat{T} \subset \hat{M}_{\gamma_0}$. Thus, since $\hat{M}_{\gamma_0}$ is atoroidal the essential torus $\hat{T}$ is homotopic into a torus component $\hat{S}$ of $\partial \hat{M}_{\gamma_0}$. The torus component $\hat{S}$ must cover a torus component $S$ of $\partial M_{\gamma}$. By pushing the homotopy via $\pi$ we see that $T$ is also homotopic into a torus component $\pi(\hat{T})$ of $\partial M_{\gamma}$. $\square$

We now reduce the proof of the main theorem to the case in which the orbifold is an actual surface.

**Proposition 3.5.** If Theorem 1.3 holds for hyperbolic surfaces then it holds for orbifolds.

**Proof.** By the geometrization conjecture [Sco83] the Seifert fibered space $M$ over a compact hyperbolic orbifold $O$ has a geometry modelled on either $H^2 \times \mathbb{R}$ or $SL_2$.

Assume that $G \cong \pi_1(M)$ is a discrete group of isometries of $H^2 \times \mathbb{R}$ that acts freely and has quotient an orientable $S^1$-bundle $M$. Notice that the isometry group of $H^2 \times \mathbb{R}$ can be naturally identified with $\text{Isom}(H^2) \times \text{Isom}(\mathbb{R})$ and we regard the factors as subgroups in the usual way. As $G$ is discrete and $M$ is a Seifert bundle, then $G \cap \text{Isom}(\mathbb{R}) = \mathbb{Z}$. Let $\Gamma$ denote the image of the projection $G \twoheadrightarrow \text{Isom}(H^2)$. Then we have the exact sequence:

$$1 \to \mathbb{Z} \to G \to \Gamma \to 1,$$

where $\Gamma$ is a discrete group of isometries of $H^2$.

On the other hand, if $G$ is a discrete group of isometries of $\widetilde{SL_2}$ which acts freely and has quotient an orientable $S^1$-bundle $M$ we have the following exact sequence:

$$0 \to \mathbb{R} \to \text{Isom}(\widetilde{SL_2}) \xrightarrow{p} \text{Isom}(H^2) \to 1.$$

If $\Gamma$ denotes $p(G)$, we have the exact sequence:

$$1 \to \mathbb{Z} \to G \to \Gamma \to 1,$$

for $\Gamma$ a discrete group of isometries of $H^2$.

In either case $\Gamma$ is a finitely generated subgroup of $\text{Isom}(H^2)$, thus by [Bau62] $\Gamma$ is residually finite. Hence, $\Gamma$ has a torsion free subgroup $\hat{\Gamma}$ of finite index. Let $\hat{G}$ be the subgroup of $G$ projecting onto $\hat{\Gamma}$ and let $\hat{M} \cong SL_2 / \hat{G}$ or $(H^2 \times \mathbb{R}) / \hat{G}$. By the first isomorphism Theorem [DF04] we have that: $G / \hat{G} \cong \Gamma / \hat{\Gamma}$ hence $\hat{G}$ is also finite index in $G$. Therefore, we have the following commutative diagram:
\[ \hat{M} \xrightarrow{\pi} M \]
\[ \Sigma \cong \mathbb{H}^2 / \hat{\Gamma} \xrightarrow{\hat{\pi}} \mathbb{H}^2 / \Gamma \cong O \]

where \( \pi : \hat{M} \rightarrow M \) is a finite index cover. Thus by lifting \( \bar{\gamma} \subset M \) to \( \bar{\gamma}_0 = \pi^{-1}(\bar{\gamma}) \subset \hat{M} \), we get a finite cover:

\[ \pi : \hat{M}_{\bar{\gamma}_0} \rightarrow M_{\bar{\gamma}} \]

Moreover, by the commutativity of the previous diagram the link \( \bar{\gamma}_0 \) projects injectively onto the filling multicurve \( \gamma_0 = \hat{\pi}^{-1}(\gamma) \).

By Lemma 3.3 we get that \( \gamma_0 \) satisfy the conditions of 1.2 for \( \Sigma \). Then by Proposition 3.5 if \( \hat{M}_{\bar{\gamma}_0} \) is atoroidal, we get that \( M_{\bar{\gamma}} \) is also atoroidal. \( \square \)

Therefore, to show Theorem 1.2 it suffices to prove:

**Theorem 3.6.** Suppose \( \Sigma \) is a hyperbolic surface and \( \bar{\pi} \) is a link in a orientable Seifert fibered space \( M \) over \( \Sigma \) that projects injectively to a filling multicurve \( \gamma \) of closed geodesics in \( \Sigma \). Then \( M_\gamma \) is a hyperbolic manifold of finite volume.

3.1. Proof of Theorem 3.6

**Definition 3.7.** We say that a triangulation \( \tau = \{T_i\}_{1 \leq i \leq m} \) of a hyperbolic surface \( \Sigma \) is **simple** for a geodesic multicurve \( \gamma \) if the edges of \( \tau \) are geodesic arcs transversal to \( \gamma \) and in every triangle \( T \in \tau \) we have that if \( \gamma \cap T \neq \emptyset \) then it contains either two intersecting subarcs of \( \gamma \) or a single subarc of \( \gamma \), see Figure 2.

![Figure 2. Two possible γ-arcs configuration inside T.](image)

Now we will give a partition of the universal cover \( \hat{M} \) of \( M \) induced by a given simple triangulation \( \tau \) on \( \Sigma \) relative to a geodesic multicurve \( \gamma \). In our setting we have:
\[ \widetilde{M} \xrightarrow{q} M \xrightarrow{p} \Sigma \]

Since \( p \) is a bundle map we have that for all \( T \in \tau \) the preimage: \( p^{-1}(T) \) is homeomorphic to the solid torus \( S^1 \times \mathbb{D}^2 \cong V_T \). Moreover, the solid torus \( V_{T_i} \) inherits from the triangulation \( \{ T_i \}_{1 \leq i \leq m} \) a decomposition of \( \partial V_{T_i} \) into:

1. three edges \( w_1^i, w_2^i, w_3^i \) each one homeomorphic to \( S^1 \) and corresponding to the vertices \( v_1^i, v_2^i, v_3^i \) of \( T_i \);
2. three faces \( F_1^i, F_2^i, F_3^i \) homeomorphic to annuli \( I \times S^1 \) and corresponding to the edges \( e_1^i, e_2^i, e_3^i \) of \( T_i \).

By going to the universal cover \( \widetilde{M} \) of \( M \) one gets that each \( V_{T_i} \) lifts to a collection: \( \coprod_{\alpha \in A_i} T_1^\alpha \times \mathbb{R} \), with \( T_1^\alpha \cong \mathbb{D}^2 \), and the previous decompositions of \( V_{T_i} \) induces a decomposition of each \( T_1^\alpha \times \mathbb{R} \) into:

1. three edges \( \overline{w}_1^i, \overline{w}_2^i, \overline{w}_3^i \) each one homeomorphic to \( \mathbb{R} \) and corresponding to the vertices \( v_1^i, v_2^i, v_3^i \) of \( T_i \);
2. three faces \( \overline{F}_1^i, \overline{F}_2^i, \overline{F}_3^i \) homeomorphic to \( I \times \mathbb{R} \) and corresponding to the edges \( \overline{e}_1^i, \overline{e}_2^i, \overline{e}_3^i \) of \( T_i \).

Remark 3.8. Notice that since for all \( 1 \leq i \leq m \) we have that \( (p \circ q)^{-1}(T_i) = \coprod_{\alpha \in A_i} T_1^\alpha \times \mathbb{R} \) and the universal cover \( \widetilde{M} \) can be rewritten as \( \bigcup_{i=1}^{m} \coprod_{\alpha \in A_i} T_1^\alpha \times \mathbb{R} \).

Lemma 3.9. Let \( \widetilde{M} \xrightarrow{q} M \) be the universal covering map of the Seifert fibered manifold \( M \) and \( M \xrightarrow{p} \Sigma \) be the Seifert map for \( \Sigma \) not a sphere. Given any simple triangulation \( \tau \) on \( \Sigma \) we have that \( \widetilde{M} = \bigcup_{n=1}^{\infty} K_n \) where each \( K_n \) is compact, simply connected and \( K_n = K_{n-1} \cup S_n T_1^{\alpha_n} \times \mathbb{R} \) for \( S_n \) either one or two faces.

Proof. By remark 3.8 we have that \( \widetilde{M} = \bigcup_{i=1}^{m} \coprod_{\alpha \in A_i} T_1^\alpha \times \mathbb{R} \).

Step 1: For \( i \neq j \) the thick cylinders \( T_1^\alpha \times \mathbb{R} \) and \( T_1^\beta \times \mathbb{R} \) are either disjoint, share at most two faces or share only one edge.

Proof. Suppose that they are not disjoint so that \( T_i = p \circ q(T_1^\alpha \times \mathbb{R}) \) and \( T_j = p \circ q(T_1^\beta \times \mathbb{R}) \) intersect in \( \Sigma \). Then, since they are distinct elements of the triangulation \( \tau \) they must intersect in their boundary. Since \( \Sigma \) is not a sphere it follows that \( T_i \) and \( T_j \) either intersect in a vertex or they share at most two edges and the result follows.

Step 2: There are nested simply connected subsets \( \{ K_n \}_{n \in \mathbb{N}} \) of \( \widetilde{M} \) such that \( \widetilde{M} = \bigcup_{n=1}^{\infty} K_n \) and \( K_n = K_{n-1} \cup S_n T_1^{\alpha_n} \times \mathbb{R} \) where \( S_n \) are at most two faces of \( T_1^{\alpha_n} \times \mathbb{R} \) sharing an edge.

Proof. Pick \( T_1 \) and let \( K_1 \) be any component \( T_1^\alpha \times \mathbb{R} \) of \( (p \circ q)^{-1}(T_1) \) in \( \widetilde{M} \). Then, mark the edges \( \overline{w}_1, \overline{w}_2, \overline{w}_3 \) in \( \partial K_1 \) and develop around them. That is let \( \{ T_{1k}^\alpha \times \mathbb{R} \}_{1 \leq k \leq n_1} \) be the finitely many components containing \( \overline{w}_1 \) as an edge. Then at least one of the \( \{ T_{1k}^\alpha \times \mathbb{R} \}_{1 \leq k \leq n_1} \), say \( T_{11}^\alpha \times \mathbb{R} \), shares one or two faces \( S \) with \( K_1 \). We then let \( K_2 = K_1 \cup S T_{11}^\alpha \times \mathbb{R} \). By repeating this for all \( \{ T_{1k}^\alpha \times \mathbb{R} \}_{1 \leq k \leq n_1} \) we have added all solid tori \( T_1^\alpha \times \mathbb{R} \) having \( \overline{w}_1 \) as an edge to \( K_1 \). The sets \( \{ K_n \}_{1 \leq n \leq n_1+1} \) so constructed are simply connected.
since they are all homeomorphic to $\mathbb{D}^2 \times \mathbb{R}$. This is because at every stage we glue a thick cylinder to another thick cylinder along a simply connected subset of their boundary.

By repeating this with $\tilde{w}_2, \tilde{w}_3$ we get new simply connected subsets $\{K_n\}_{n=1}^{m}$, $m \in \mathbb{N}$ (see Figure 3):

![Figure 3. Schematic of the simply connected subset $K_m$.](image)

Moreover, all the $K_n, n \leq m$, so constructed are simply connected and properly embedded in $\tilde{M}$ and $\tilde{w}_1, \tilde{w}_2, \tilde{w}_3$ are contained in the interior of $K_m$. We then mark all edges $\tilde{w}_1^m, \ldots, \tilde{w}_n^m$ of $\partial K_m$ and repeat the previous construction by first adding all the thickened cylinders sharing an edge with $\tilde{w}_1^m$ and so on.

This yields a collection $\{K_n\}_{n \in \mathbb{N}}$ of properly embedded nested simply connected subset of $\tilde{M}$ such that for all $n \in \mathbb{N}$ $K_{n+1} \setminus K_n = T_{n+1}^\alpha \times \mathbb{R}$. Moreover, since the universal cover $\tilde{M} = \bigcup_{i=1}^m \bigcap_{n \in A_i} T_i^n \times \mathbb{R}$ we have that $\tilde{M} \cup_\infty K_n$ since each $T_i^n \times \mathbb{R}$ is in $\cup_\infty K_n$. □

Which concludes the proof. □

We now prove a key Lemma:

**Lemma 3.10.** Let $\tau$ be a simple triangulation for a geodesic multicurve $\gamma$ in $\Sigma$, let $M$ be a Seifert fibered space over $\Sigma$ with projection $p : M \rightarrow \Sigma$ and $\{\tilde{\gamma}\}$ all the lifts on $\tilde{M}$ of the link $\tilde{\gamma}$. Then, for every $T \in \tau$, we have that $\pi_1 \left( p^{-1}(T) \setminus \{\tilde{\gamma}\} \right)$ is a free group. Moreover, the set of $\{\tilde{\gamma}\}$-arcs in $p^{-1}(T)$ forms a free basis.

**Proof of 3.10.** For a simple triangulation $\tau$ of $\Sigma$ transversal to $\gamma$, we claim:

**Claim.** For each $T \in \tau$, there exists a smooth lift $\overline{T}$ of $T$ embedded in $p^{-1}(T) \subset M$ such that $\overline{\gamma} \cap \overline{T} = \emptyset$.

**Proof.** Suppose $\gamma_0$ and $\gamma_1$ are two $\gamma$-arcs contained in $T$ with unique self-intersection point $x = \gamma_1(t) = \gamma_0(s)$. First consider a new lift $\tilde{\gamma}_0$ of the arc $\gamma_0$ on $p^{-1}(T)$ which is at positive constant $S^1$-fiber distance from $\tilde{\gamma}_0$ and does not intersects $\tilde{\gamma}_1$. Then consider the unique lift $\tilde{\gamma}_1$ passing through $y = \tilde{\gamma}_0 \cap p^{-1}(x)$ and at $S^1$-fiber distance $d_{S^1}(y, \tilde{\gamma}_1(s))$ from $\tilde{\gamma}_1$. Finally consider any smooth lift $\overline{T}$ of $T$ which contains $\tilde{\gamma}_1 \cup \tilde{\gamma}_0$. As $\tilde{\gamma}_1 \cup \tilde{\gamma}_0$ does not intersect $\tilde{\gamma}$ and the projection of $\overline{T} \setminus (\tilde{\gamma}_1 \cup \tilde{\gamma}_0)$ under $p$ is disjoint from $\gamma \cap T$, then $\overline{\gamma} \cap \overline{T} = \emptyset$.

The case of only one $\gamma$-arc inside $T$ follows similarly from the previous case. □

By cutting $p^{-1}(T)$ along the lift $\overline{T}$ coming from the previous claim we can associate to $p^{-1}(T) \setminus \overline{T} \cup \overline{\gamma}$ a string diagram on $T$ such that there is at most one self-crossing. Following
the method of Wirtinger to give a presentation to knot complements in $\mathbb{S}^3$ by using a knot diagram associated to it, see \cite[Chap.3 Sec.D]{Rol76}, one can show that:

$$\pi_1(p^{-1}(T) \setminus \overline{T} \cup \overline{\gamma})$$

is free,

and the generators are in bijection with the $\gamma$-arcs inside $p^{-1}(T) \setminus \overline{T}$. Equivalently the bijection is with $\gamma$-arcs inside $T$.

Finally, notice that $p^{-1}(T) \setminus \overline{\gamma}$ is obtained by gluing copies of $p^{-1}(T) \setminus \overline{T} \cup \overline{\gamma}$ along the lifts of $T$ inside $\overline{p^{-1}(T)}$. Since each $T$ is simply connected, by the Van Kampen Theorem, we have that: $\pi_1\left(p^{-1}(T) \setminus \overline{\gamma}'\right)$ is a free product of free groups. Moreover, the generators are in correspondence with the $\gamma$-arcs inside $\overline{p^{-1}(T)}$. \hfill $\Box$

**Theorem 1.3.** Let $M$ be an orientable Seifert fibered space over a hyperbolic surface $\Sigma$, let $\gamma \subset \Sigma$ a filling multicurve of closed geodesics, $q : \widetilde{M} \to M$ the universal covering map of $M$ and let $\overline{\gamma}$ be the total preimage of the link $\overline{\gamma}$ under $q$. Then, the group $\pi_1\left(\widetilde{M} \setminus \overline{\gamma}\right)$ is free and the set $\overline{\gamma}$ forms a free basis.

**Proof.** By using the decomposition of $\widetilde{M}$ as $\bigcup_{n=1}^{\infty} K_n$ coming from Lemma 3.9 relative to the chosen simple triangulation $\tau$ on $\Sigma$, we define:

$$C_{T_1^n} = T_1^n \times \mathbb{R} \setminus \overline{\gamma},$$

and let $C_n = K_n \setminus \overline{\gamma}$.

**Claim:** For every $n \in \mathbb{N}$, we have that $\pi_1(C_n)$ is free. Moreover, the generators are in bijection with the $\overline{\gamma}$-arcs inside $C_n$.

**Proof.** The proof is by induction over $n$, where the base case is Lemma 3.10. Suppose that the claim is true for $C_n$ we will show it is also true for $C_{n+1} = K_n \cup Z_n \cap C_{T_1^n}$. Notice that the intersection of $C_n$ with $C_{T_1^n}$ is either one or two punctured faces $Z_n = S_n \setminus \overline{\gamma}$ such that each puncture comes from a subset of $\overline{\gamma}$-arcs inside $C_n$ and the same holds for $C_{T_1^n}$. Then the natural inclusions:

$$(i_1)_* : \pi_1\left(C_{T_1^n} \cap C_n\right) \to \pi_1\left(C_{T_1^n}\right)$$

and $$(i_2)_* : \pi_1\left(C_{T_1^n} \cap C_n\right) \to \pi_1(C_n)$$

map generators to generators. Thus, by Van Kampen’s Theorem $\pi_1(C_{n+1})$ is also free because the new relations are given by:

$$(i_1)_*(s)(i_2)_*(s))^{-1} = 1$$

where $s$ is a generating element of $\pi_1\left(C_{T_1^n} \cap C_n\right)$. Thus, we are just pairing the generators of the two free groups. Therefore, the relations either rename the generators of $\pi_1\left(C_{T_1^n}\right)$ with generators of $\pi_1(C_n)$ or reduced the number of generators of $\pi_1(C_n)$. \hfill $\Box$

Finally, notice that $\widetilde{M} \setminus \overline{\gamma} = \bigcup_{n=1}^{\infty} C_n$. By the previous claim we have that each $\pi_1(C_n)$ is free and that the inclusion $j_n : C_n \to C_{n+1}$ induce maps $(j_n)_* : \pi_1(C_n) \to \pi_1(C_{n+1})$ that map generators to generators. Therefore we have that each free basis of $C_n$ can be extended to a free basis of $C_{n+1}$. Hence we obtain that:

$$\lim_{\overline{\gamma}} \pi_1(C_n) \cong \pi_1\left(\widetilde{M} \setminus \overline{\gamma}\right)$$
is a free group as well. Moreover, the \( \{ \tilde{\gamma} \} \) form a generating set for \( \pi_1(\tilde{M} \setminus \{ \tilde{\gamma} \}) \).

As a consequence of the proof we get:

**Corollary 3.11.** Given a component \( \alpha \in \{ \tilde{\gamma} \} \) then \( \alpha \) is unknotted in \( \tilde{M} \cong \mathbb{R}^2 \times \mathbb{R} \).

We can now show:

**Theorem 3.6** Suppose \( \Sigma \) is a hyperbolic surface and \( \tilde{\gamma} \) is a link in an orientable Seifert fibered space \( M \) over \( \Sigma \) projecting injectively to a filling multicurve \( \gamma \) of closed geodesics in \( \Sigma \). Then the complement of \( \gamma \) in \( M \), denoted by \( M_\gamma \), is a hyperbolic manifold of finite volume.

**Proof.** By Thurston geometrization Theorem [Thu82] it suffices to show that \( M_\gamma \) is atoroidal, irreducible and with infinite \( \pi_1 \). The last two claims are immediate since \( \gamma \neq 0 \) in \( \pi_1(\Sigma) \) then \( M_\gamma \) is irreducible and as the natural inclusion \( \iota: M_\gamma \hookrightarrow M \) is \( \pi_1 \)-surjective by transversality, then we have that \( \pi_1(M_\gamma) \) is infinite. Thus, we will only need to prove the atoroidality condition. The proof will be by contradiction and it involves three cases. Let \( T \subset M_\gamma \) be an incompressible torus, then \( \pi_1(\iota(T)) \) has either rank zero, one or two in \( \pi_1(M) \).

**Case 1:** the rank of \( \pi_1(\iota(T)) \) is zero.

**Proof.** This means that \( \iota(T) \) is null-homotopic in \( M \). Hence, the map: \( \iota: T \to M \) lifts to an embedded torus \( \tilde{T} \) in \( \tilde{M} \). Moreover, for \( \{ \tilde{\gamma} \} \) the lifts of \( \tilde{\gamma} \subset \tilde{M} \) we get that \( \tilde{T} \subset \tilde{M} \setminus \{ \tilde{\gamma} \} \) and is essential. However, by Theorem 1.3 \( \pi_1(\tilde{M} \setminus \{ \tilde{\gamma} \}) \) is a free group which does not contain any \( \mathbb{Z}^2 \) subgroup. \( \square \)

**Case 2:** The rank of \( \pi_1(\iota(T)) \) is one.

**Proof.** If \( \text{rank}(\pi_1(\iota(T))) = 1 \) it means that \( \iota(T) \) is compressible in \( M \). Therefore we have a compression disk \( D \) such that compressing \( \iota(T) \) along \( D \) gives us a 2-sphere \( S^2 \hookrightarrow M \). Since \( M \) is irreducible it means that \( S^2 \) bounds a 3-ball \( B \subset M \). Thus, we see that \( \iota(T) \) bounds a solid torus \( V \) in \( M \) and by incompressibility of \( T \) in \( M_\gamma \) we must have that \( \tilde{\gamma} \cap V \neq \emptyset \). Since \( T \cap \tilde{\gamma} = \emptyset \) we have that every component \( \tilde{\gamma}_i \in \pi_0(\tilde{\gamma}) \) intersecting \( V \) is contained in \( V \).

**Claim:** There is a unique component \( \tilde{\gamma}_i \in \pi_0(\tilde{\gamma}) \) contained in \( V \) and it is a generator of \( \pi_1(V) \).

Let \( \alpha \) be a generator of \( \pi_1(V) \) in \( \pi_1(M) \). Then every component \( \tilde{\gamma}_i \subset V \) of \( \tilde{\gamma} \) is homotopic, in \( V \), to \( \alpha^{n_i} \) for some \( n_i \in \mathbb{N} \). But every \( \tilde{\gamma}_i \) is the lift of a geodesic in \( \Sigma \) and so it is primitive. Hence, every \( \tilde{\gamma}_i \subset V \) generates \( \pi_1(V) \). Thus, any two \( \tilde{\gamma}_i, \tilde{\gamma}_j \) in \( V \) must be homotopic contradicting the fact that \( \tilde{\gamma} \) projects injectively to a geodesic multicurve on \( \Sigma \). Thus there is a unique component \( \tilde{\eta} \in \pi_0(\tilde{\gamma}) \) that is contained in \( V \) and it generates \( \pi_1(V) \).

**Claim:** The torus \( T \) is boundary parallel in \( M_\gamma \).

Consider a lift \( \tilde{V} \) of \( V \) in \( \tilde{M} \). Then \( \tilde{V} \) is homeomorphic to \( \mathbb{D}^2 \times \mathbb{R} \) and it contains \( \tilde{\eta} \). If \( \tilde{V} \) is not boundary parallel in \( \tilde{M} \setminus \{ \tilde{\gamma} \} \) we have that the lift \( \tilde{\eta} \) is knotted in \( \tilde{V} \) contradicting Corollary 3.11. Therefore, the infinite cylinder \( \partial \tilde{V} \) is isotopic into \( \partial N_\epsilon(\tilde{\eta}) \). Therefore, \( \pi_1(T) \) is conjugated into \( \pi_1(\partial N_\epsilon(\tilde{\eta})) \) contradicting the fact that \( T \) was essential in \( M_\gamma \). \( \square \)

**Case 3:** The rank of \( \pi_1(\iota(T)) \) is two.
Proof. If \( \iota(T) \) is essential in \( M \) then by Proposition [Hat07, 1.11] we see that \( \iota(T) \) is isotopic to either a horizontal surface or a vertical surface in \( M \). If \( \iota(T) \) is horizontal it means that the hyperbolic surface \( \Sigma \) is covered by a torus which is impossible. Therefore, \( \iota(T) \) is isotopic to a vertical torus \( T' \). Then if we consider the projection \( p : M \to \Sigma \) we see that \( p(T') \) is an essential simple closed curve \( \alpha \subset \Sigma \), moreover since \( T' \cap \tilde{\gamma} = \emptyset \) we have that \( \alpha \cap \gamma = \emptyset \). However, this contradicts the fact that \( \gamma \) is a filling multicurve. □

Thus, \( M_\tilde{\gamma} \) is atoroidal and hence admits a complete hyperbolic metric of finite volume. □

4. Volume of \( M_\tilde{\gamma} \)

Once the hyperbolicity of \( M_\tilde{\gamma} \) is settled then by Mostow’s Rigidity we can pursue the problem of estimating geometric invariants in terms of topological relations between the multicurve \( \gamma \) and the hyperbolic orbifold \( O \).

4.1. Volume. The volume invariant has been studied in the particular case of canonical lifts of geodesics in the projective unit tangent bundle \( PT^1(\Sigma) \) of a hyperbolic surface \( \Sigma \) or of the modular orbifold. Upper bounds have been found in terms of the geodesic length in [BPS17] and a combinatorial lower bound by the second author in [RM17].

In ([RM17], Sec. 5) the second author noticed that the behaviour of the volume of \( M_\tilde{\gamma} \) among different lifts of \( \gamma \) does not depend on the diagram given by the couple \((\gamma, O)\). More precisely the second author proved that:

**Proposition 4.1.** For any hyperbolic metric \( X \) on \( \Sigma \), there exist a sequence of \( \{\gamma_n\}_{n \in \mathbb{N}} \) filling closed geodesics and respective lifts \( \{\tilde{\gamma}_n\}_{n \in \mathbb{N}} \) in \( PT^1(\Sigma) \) with \( \ell_X(\gamma_n) \nearrow \infty \), such that,

\[
   k_X \ell_X(\gamma_n) \leq \text{Vol}(M_{\tilde{\gamma}_n}),
\]

where \( k_X \) is a positive constant that depends on the metric \( X \). Moreover, there exists a constant \( V_0 > 0 \) such that \( \text{Vol}(M_{\tilde{\gamma}_n}) < V_0 \) for every \( n \in \mathbb{N} \), where \( \tilde{\gamma}_n \) is the canonical lift of \( \gamma_n \) on \( PT^1(\Sigma) \).

Furthermore, by ([HP18], Thm. 1.1) one can construct a continuous lift inside the projective unit tangent bundle, see Corollary [L.4] of a punctured hyperbolic surface over some closed geodesics such that the knot complement’s hyperbolic volume is, up to a multiplicative factor, the self-intersection number of the geodesic multicurve.

The upper bound of Corollary [L.4] on the volume of \( M_\gamma \) actually holds for any continuous lift \( \tilde{\gamma} \) inside a Seifert fibered spaces \( M \) over a geodesic multicurve \( \gamma \) on its hyperbolic 2-orbifold base, not just for one lift on the the projective unit tangent bundle of a punctured surface. For completeness, we will give the arguments of this claim. Firstly, the idea is to give a link \( L_\gamma \) inside \( M \) which reduces the complexity of \( \tilde{\gamma} \), in the sense that \( M_\tilde{\gamma} \) is obtained performing Dehn filling along some knot components of \( L_\gamma \). Since Dehn filling does not increase the simplicial volume ([Buc13], Theorem 1.3), we have that:

\[
   \text{Vol}(M_\tilde{\gamma}) \leq \|M \setminus L_\gamma\| \leq 2T_{L_\gamma},
\]

where \( T_{L_\gamma} \) is a regular ideal tetrahedra triangulation of \( M \) such that \( L_\gamma \) belongs to the 1-skeleton. After constructing the link \( L_\gamma \), we will argue that there exist \( T_{L_\gamma} \) with the number of tetrahedra comparable to the self-intersection number of \( \gamma \).

In order to find \( L_\gamma \), we start by taking the fibers under \( p \) corresponding to pair of points \( v_{\alpha_+}, v_{\alpha_-} \) next to each edge \( \alpha \) of the graph associated to \( \gamma \) on \( O \). Notice that for each edge
There exist an integer $n_\alpha$ such that by performing a $\frac{1}{n_\alpha}$ Dehn filling along $p^{-1}(v_{\alpha+})$ and $\frac{-1}{n_\alpha}$ along $p^{-1}(v_{\alpha-})$, then we modify the corresponding $\gamma$-arc $\bar{\alpha}$ over the edge $\alpha$ to a straight arc $\alpha_0$ linking the points of $\gamma$ intersecting the fibers corresponding to the endpoints of $\alpha$ (see [Oso06], Theorem 2.1). We will denote the new lift of $\gamma$ obtained in this way as $\gamma_0$ (Figure 4).

**Remark.** The manifolds $M \setminus L_\gamma$ is not a hyperbolic manifold and the link $L_\gamma$ depends on the intersection points of $\gamma$ with the fibers corresponding to self-intersection points of $\gamma$.

To give the tetrahedra decomposition of $M$ with $L_\gamma$ as part of its 1-skeleton, we start by triangulating the fibers corresponding to the self-intersection points of $\gamma$ by adding two vertices on each fiber corresponding to the intersection of $\gamma_0$ with them (Figure 5).

**Figure 5.** Triangulation of the fibers coming from self-intersection points of $\gamma$.

We then extend it to a triangulation of the complex $p^{-1}(\gamma)$ by triangulating each annulus coming from the pre-image, under $p$, of an edge $\alpha$ of the graph associated to $\gamma$. We do this by adding two edges, one corresponding to the embedded $\gamma_0$-arc in the annulus and the second which is an embedded arc connecting the other vertices in each boundary fibers that do not intersect $\gamma_0$ (up to isotopy this arc is unique). This induces a regular quadrilateral decomposition of each annulus by two quadrilateral (Figure 6).

Since $\gamma$ is filling on $\mathcal{O}$, it allow us to consider the disjoint family of solid tori, coming from $p^{-1}(\mathcal{O} \setminus \gamma)$, whose boundary we have just given a quadrilateral decomposition. Consider a connected component $D$ of $\mathcal{O} \setminus \gamma$. We can then extend the quadrilateral decomposition given in $p^{-1}(\partial D)$ to $p^{-1}(D)$. To do so, first consider a minimal triangulation of the polygon $D$ (without considering the punctures inside $D$), see (Figure 7).

Suppose that $T$ is one of the components of the triangulation on $D$ and consider the solid torus $p^{-1}(T)$. Notice that some of its faces, the ones corresponding to edges of $\gamma$, have
already a quadrilateral decomposition. For the new faces we just add a pair of parallel straight arcs connecting the vertices on the boundary (Figure 7).

By using the straight arcs in each face of $p^{-1}(\partial T)$, we can find a simple closed path formed by some of its edges, such that it represents the homotopy class of a meridian relative to $p^{-1}(T)$. Then we can split $p^{-1}(T)$ along a compression disk bounding this meridian and obtain a piece homeomorphic to a 3-ball whose boundary has a quadrilateral decomposition. By extending the quadrilateral decomposition of each sphere into a triangulation of its bounded ball we obtain a triangulation on $M$ which is comparable with the number of edges of the 4-valent graph induced by the image of $\gamma$ in $O$. Finally, this graph has at most a number of edges equal to two times the self-intersection number of $\gamma$.

We can now prove:

**Corollary 1.4.** Let $\Sigma_{g,n}$ be an $n$-punctured hyperbolic surface, $n \geq 1$, then there exist a sequence of $\{\gamma_n\}_{n \in \mathbb{N}}$ filling closed geodesics with $i(\gamma_n, \gamma_n) \nearrow \infty$, and respective lifts $\{\bar{\gamma}_n\}_{n \in \mathbb{N}}$
in $PT^1(\Sigma_{g,n})$ such that,

$$\frac{v_8}{2} (i(\gamma_n, \gamma_n) - (2 - 2g)) \leq \text{Vol}(M_{\gamma_n}) \leq 44v_3 i(\gamma_n, \gamma_n),$$

where $v_3$ ($v_8$) is the volume of the regular ideal tetrahedron (octahedron) and $i(\gamma_n, \gamma_n)$ the self-intersection number of $\gamma_n$.

Proof. For the sake of concreteness, we will prove first the result for the once-punctured torus $\Sigma_{1,1}$. Let $\gamma_n$ be constructed as in Figure 9.

As $PT^1(\Sigma_{1,1}) \cong \Sigma_{1,1} \times S^1$, so consider a global section $S_{1,1}$ embedded in $PT^1(\Sigma_{g,n})$. Let $\overline{S_{1,1}}$ be constructed in $N(S_{1,1})$ a normal neighbourhood of $S_{1,1}$, such that its corresponding diagram on $S_{1,1}$ is the alternating diagram in Figure 10.

**Figure 9.** a) Pants decomposition of $\Sigma_{1,1}$, b) $\gamma_1$, c) $\gamma_2$, and d) $\gamma_3$.

**Figure 10.** An alternating diagram for a) $\gamma_1$, and b) $\gamma_2$. 
Notice that by making trivial Dehn filling around the torus coming from the puncture of $\Sigma_{1,1}$, then $PT^1(\Sigma_{1,1})$ becomes $\Sigma_1 \times S^1$ and the global section $S_1$ and the position of our knots $\gamma_n$ is the same. Then we are in the case of ([HP18], Thm.1.1) where the projection of $\gamma_n$ has a weakly twist-reduced, weakly generalised alternating diagram on a generalised projection surface ([HP18], Sec.2) $S_1$ in $\Sigma_1 \times S^1$. As $Y = (\Sigma_1 \times S^1) \setminus N(S_1)$ is atoroidal, its boundary is incompressible and $\partial$-annular[1]. Also, by the fact that $\gamma_n$ filling in $\Sigma_1$, we have that:

$$\frac{v_8}{2}(tw(\gamma_n) - \chi(\Sigma_1)) \leq \text{Vol}((\Sigma_1 \times S^1)_{\gamma_n}) < \text{Vol}(M_{\gamma_n}),$$

where $tw$ is the number of twisting regions of the link diagram ([HP18], Def.6.4), which in the case of closed geodesics, as it does not have bigons in its diagram then this number is equivalent to the self-intersection number of the corresponding geodesic. The second inequality holds because volume decreases under Dehn filling operation, which we are doing along the unique cusp of $PT^1(\Sigma_{1,1})$. The upper bound comes from our previous discussions and thus we complete the proof.

To generalise this result to any be an $n$-punctured hyperbolic surface $\Sigma_{g,n}$ notice that:

1) The number of connected components of the sequence of $\{\Sigma_{1,1} \setminus \gamma_k\}_{k \in \mathbb{N}}$ tends to infinity. Then the subsequence of closed geodesics constructed previously such that $\Sigma_{1,1} \setminus \gamma_k$ has more than $n$ connected components can be seen in $\Sigma_{1,n}$, just by removing one puncture in $n$ simply connected components of $\Sigma_{1,1} \setminus \gamma_k$.

2) It is a straightforward exercise to show that any projection of a link on the 2-sphere can be made alternating by changing crossings. Then any closed geodesic in $\Sigma_{0,n}$ admits an alternating diagram.

---

**Figure 11.** a) $\alpha_1$ and $\alpha_2$ in a neighbourhood of the glued boundary component, b) The induced projection diagram of $\alpha_1 \hat{*} \alpha_2$ around a neighbourhood of the glued boundary component, c) Changing to the opposite crossing projection on one of the $\alpha_2$ subarcs (green) to obtain an alternating diagram.

---

1This means that $Y$ has no essential annuli whose boundary is not contained in the boundary components isotopic to the removed fiber.
To prove the $\Sigma_{g,1}$ case we can proceed by induction on the genus, using the following claim:

**Claim:** Let $\alpha_1$ and $\alpha_2$ be a filling closed geodesics admitting an alternating diagram on $\Sigma_{g,1}$ and $\Sigma_{1,2}$ respectively. Then the closed geodesic homotopic to $\alpha_1 \ast \alpha_2$ (see [RM17], Subsec. 4.2) is filling and admits an alternating diagram on $\Sigma_{g+1,1}$.

The filling property is proven in ([RM17], Claim. 4.13) and the existence of an alternating diagram follows from fixing one alternating diagram on each $\alpha_1$ and $\alpha_2$, if after connecting both geodesics the corresponding diagram is not alternating (Figure 11) then just change the crossing orientation of all crossings in one of the subarcs $\alpha_i$ making the diagram of the geodesic corresponding to $\alpha_1 \ast \alpha_2$ alternating. \[\square\]

**Remark 4.2.** Not every closed geodesic on a surface of genus greater or equal than 1 admits an alternating diagram (Figure 12). Even though, for each hyperbolic surface one can find an infinite number of distinct types of closed geodesics which admit an alternating diagram.

![Figure 12](image)

**Figure 12.** a) Pants decomposition on $\Sigma_2$, b) Closed geodesic not admitting an alternating diagram, c) Closed geodesic with an alternating diagram.

Similarly to [RM17], given any geodesic multicurve $\gamma$ and any continuous lift $\tilde{\gamma}$, one has a combinatorial lower bound for the volume of $M_\tilde{\gamma}$. Recall that a pants decomposition on an orbifold $O$, is a maximal family of disjoint simple closed geodesics on the underlying topological surface $\Sigma_O$ which do not intersect the singular points of $O$. We will show:

**Theorem 1.5.** Given a pants decomposition $\Pi$ of a hyperbolic orbifold $O$, a Seifert fibered space $M$ with base $O$, and $\gamma$ a filling geodesics multicurve on $O$, we have that for any continuous lift $\tilde{\gamma}$:

$$\frac{\nu_3}{2} \sum_{P \in \Pi} (\# \{\text{isotopy classes of } \tilde{\gamma} \text{-arcs in } p^{-1}(P)\} - 3) \leq \text{Vol}(M_\tilde{\gamma})$$
for $v_3$ the volume of the regular ideal tetrahedron.

Given a pair of pants $P$, we say that two arcs $\tilde{\alpha}, \tilde{\beta} : [0,1] \to p^{-1}(P)$ with $\tilde{\alpha}(\{0,1\}) \cup \tilde{\beta}(\{0,1\}) \subset \partial(p^{-1}(P))$ are in the same isotopy class in $p^{-1}(P)$, if there exist an isotopy $h : [0,1] \times [0,1] \to p^{-1}(P)$ such that:

$$h_0(t_2) = \tilde{\alpha}(t_2), \ h_1(t_2) = \tilde{\beta}(t_2) \text{ and } h([0,1] \times \{0,1\}) \subset \partial(p^{-1}(P)).$$

**Remark 4.3.** The 3 in the lower bound of Theorem 1.5 comes from the fact that the projection of $\gamma$-arcs configurations on $P$ which are equivalent up to isotopy classes to a family of simple arcs without intersections, have at most 3 isotopy classes of $\gamma$-arcs on $p^{-1}(P)$. These configurations do not contribute to the lower bound on Theorem 1.5. In fact, there are only six such configurations:

![Image of six configurations](image.png)

**Figure 13.** The projection on $P$ of the only six $\gamma$-arcs configuration, up to isotopy, whose $\gamma$-arcs project to pairwise disjoint simple arcs in $P$.

Before stating the main result to prove Theorem 1.5 we recall some definitions.

If $N$ is a hyperbolic 3-manifold and $S \subset N$ is an embedded incompressible surface, we will use $N|S$ to denote the manifold obtained from $N$ by cutting along $S$. The manifold $N|S$ is homeomorphic to the complement in $N$ of an open regular neighbourhood of $S$. If one takes two copies of $N|S$, and glues them along their boundary by using the identity diffeomorphism, one obtains the double of $N|S$, which we denote by $D(N|S)$.

**Definition 4.4.** Let $P$ be a pair of pants belonging to a pant decomposition of a orbifold $O$ and let $\gamma$ be a closed geodesic in $O$ that is not contained in $P$. Moreover, assume that $P \cap \gamma$ is a finite set of geodesic arcs $\{\alpha_i\}$ connecting boundary components of $P$. We define $P_\gamma$ to be the set:

$$p^{-1}(P) \setminus \bigcup_i \tilde{\alpha}_i.$$

We also define $D(P_\gamma)$, as the gluing, via the identity homeomorphism, of two copies of $P_\gamma$ along the punctured tori coming from:

$$\partial(p^{-1}(P)) \setminus \bigcup_i \tilde{\alpha}_i.$$

Moreover, $D(P_\gamma)$ is a link complement in the Seifert fibered space $D(p^{-1}(P))$, described as :

$$D(p^{-1}(P)) \setminus \bigcup_i D(\tilde{\alpha}_i),$$
where the projection orbifold of $D(p^{-1}(P))$, whose underlying surface will be denoted by $S^0$, is either a genus two surface (if $\sharp(\partial(\Sigma_O) \cap \partial P) = 0$), a surface of type $(1,2)$ (if $\sharp(\partial(\Sigma_O) \cap \partial P) = 2$) or a surface of type $(0,4)$ (if $\sharp(\partial(\Sigma_O) \cap \partial P) = 1$). Each $D(p^{-1}(P))$ is a knot in $D(p^{-1}(P))$ obtained by gluing $\alpha_i$ along the two points $\partial(p^{-1}(P)) \cap \alpha_i$ via the identity.

**Figure 14.** a) A pair of pants and a set of geodesic arcs connecting the boundary. b) $P_\gamma$ associated to (a), and c) $D(P_\gamma)$ with the induced projection to $S^0$.

**Definition 4.5.** Given a connected, orientable 3-manifold $M$ with boundary we let $S_k(M; \mathbb{R})$ be the singular chain complex of $M$. That is, $S_k(M; \mathbb{R})$ is the set of formal linear combination of $k$-simplices, and we set as usual $S_k(M, \partial M; \mathbb{R}) = S_k(M; \mathbb{R}) / S_k(\partial M; \mathbb{R})$. We denote by $\|c\|$ the $l_1$-norm of the $k$-chain $c$.

If $\alpha$ is a homology class in $H^3_k(M, \partial M; \mathbb{R})$, the Gromov norm of $\alpha$ is defined as

$$\|\alpha\| = \inf_{\|c\| = \alpha} \{ \sum_{\sigma} |r_{\sigma}| \text{ such that } c = \sum_{\sigma} r_{\sigma} \sigma \}.$$ 

The simplicial volume of $M$ is the Gromov norm of the fundamental class of $(M, \partial M)$ in $H^3_k(M, \partial M; \mathbb{R})$ and is denoted by $\|M\|$.

The key ingredient to prove Theorem 1.5 is the following result due to Agol, Storm and Thurston ([AST07], Theorem 9.1):

**Theorem (Agol-Storm-Thurston).** Let $N$ be a compact manifold with interior a hyperbolic 3-manifold of finite volume. Let $S$ be an embedded incompressible surface in $N$, then:

$$\frac{\nu_2}{2} \|D(N|S)\| \leq \text{Vol}(N)$$

We now prove the lower bound for the volume of the canonical lift complement:

**Proof of Theorem 1.5.** Let $\{\eta_i\}_{i=1}^{3g-3}$ be the simple closed geodesics on the boundary of the pants decomposition $\Pi$. Consider the incompressible surface $S = \bigcup_{i=1}^{3g-3} (T_{\eta_i})_{\gamma}$ in $M_{\gamma}$. 

2By a surface of type $(n, m)$ we mean a genus $n$ surface with $m$ punctures.
(RM17, Lemma 2.5). From (AST07, Theorem 9.1) we deduce that:

\[ \frac{v_3}{2} \sum_{i=1}^{\chi(S)} \| D((P_i)_{cusp}) \| = \frac{v_3}{2} \| D(M_{\gamma}) \| \leq \text{Vol}(M_{\gamma}) \]

For each pair of pants \( P \) we have:

\[ v_3 \# \{ \text{cusps of } D(P_{s_{\gamma}})_{hyp} \} \leq \text{Vol}(D(P_{s_{\gamma}})_{hyp}) \leq v_3 \| D(P_{s_{\gamma}})_{hyp} \| = v_3 \| D(P_{\gamma}) \| \]

where \( D(P_{s_{\gamma}})_{hyp} \) is the atoroidal piece of \( D(P_{s_{\gamma}}) \) (Remark 4.7), complement of the characteristic submanifold, with respect to its JSJ-decomposition. The first and third inequality come from [Gro82] and [Ada88] respectively.

In Lemma 4.6 we show that there is a injection between the homotopy classes of \( s_{\gamma} \)-arcs in \( p^{-1}(P) \) and the cusps of \( D(P_{s_{\gamma}})_{hyp} \), which completes the proof.

**Lemma 4.6.** The number of isotopy classes of \( \bar{\gamma} \)-arc in \( p^{-1}(P) \) is less or equal to the number of cusps of \( D(P_{\gamma})_{hyp} \).

**Proof.** We will define an injective function:

\[ \{ \text{\( \bar{\gamma} \)-arcs in } p^{-1}(P) \} \xrightarrow{\varphi} \{ \text{cusps of } D(P_{\gamma})_{hyp} \} \]

where the target can be decomposed as:

\[ \{ \text{cusps of } D(P_{\gamma})_{hyp} \} = \begin{cases} \text{splitting tori of the } \text{JSJ-decomposition of } D(P_{\gamma}) \\ \text{tori contained in } \partial D(P_{\gamma}) \cap D(P_{\gamma})_{hyp} \end{cases} \]

The function \( \varphi \) is defined as follows: if the boundary component on \( D(P_{\gamma}) \) induced by the \( \bar{\gamma} \)-arc in \( p^{-1}(P) \) belongs to the characteristic submanifold of \( D(P_{\gamma}) \), \( \varphi \) maps it to a splitting tori connecting the hyperbolic piece with the component of the characteristic submanifold where it is contained. Otherwise, \( \varphi \) sends it to the torus coming from its double.

Assume that there are more isotopy classes of \( \bar{\gamma} \)-arcs in \( p^{-1}(P) \) than the number of cusps of \( D(P_{\gamma})_{hyp} \). Then, there are two tori, associated to non isotopic \( \bar{\gamma} \)-arcs in \( p^{-1}(P) \), that belong to the same connected component of the characteristic submanifold. Since each

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**Figure 15.** The JSJ-decomposition of \( D(P_{\gamma}) \) of Figure (14c).
component of the characteristic submanifold is a Seifert fibered space over a punctured surface we have that all such arcs correspond to regular fibers. Thus, they are isotopic in the corresponding component hence homotopic in \( p^{-1}(P) \), contradicting the fact that they were not isotopic.

In the following remark will be using Theorem 1.2 for links complements in \( S^1 \times S \), where \( S \) is a hyperbolic surface.

**Remark 4.7.** Let \( \Omega \) be the subset of \( \gamma \)-arcs on \( P \) having one arc for each isotopy class of \( \gamma \)-arcs on \( p^{-1}(P) \) and in minimal position, then:

1. If the \( \Omega \)-arc configuration on \( P \) is in the list of Remark 4.3, then \( D(P) \) does not have a hyperbolic piece hence, it has zero simplicial volume. This is because \( D(P) \) can be seen as a link complement in \( D(p^{-1}(P)) \) (Definition 4.4) whose projection to the base surface \( S^0 \) is a union of pairwise disjoint simple loops that are isotopic to simple closed geodesics.

2. If the \( \Omega \)-arc configuration on \( P \) is not in the list of Remark 4.3, then there is at least one geometric intersection point. This means that \( D(P) \) can be seen as a link complement in \( D(p^{-1}(P)) \) (Definition 4.4) whose projection to \( S^0 \) is a union of closed loops transversally homotopic to a union closed loops in minimal position, with at least one intersection point. Notice that the atoroidal piece of \( D(P) \) corresponds to the subsurface of \( S^0 \) which \( D(\Omega) \) fills. Furthermore,

\[
D(P)^{hyp} \cong D(P)^{hyp}.
\]

This result implies that there exist a filling multicurve of closed geodesics \( \gamma \) on \( O \) with bounded components such that the Vol(\( M_\gamma \)) can be as big as we want. Let us fix a pants decomposition on \( O \), then for any \( N \in \mathbb{N} \) there exist a closed geodesic with at least \( N \) homotopy classes of geodesic arcs in one pair of pants. This is constructed by taking \( N \) non-homotopic geodesic arcs in a pair of pants and linking them to form a filling multicurve of closed geodesics on \( O \).

The lower bound of the volume of \( M_\gamma \) obtained in Theorem 1.5 does not have control on the length of the geodesic multicurve, even if each homotopy class of \( \gamma \)-arcs contributes to the length of \( \gamma \).

**Question 4.8.** Given a hyperbolic orbifold, estimate the volume of \( M_\gamma \) among the filling geodesic multicurve \( \gamma \) whose length is bounded by a fixed constant.

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Department of Mathematics, Boston College.
140 Commonwealth Avenue Chestnut Hill, MA 02467.
Maloney Hall
e-mail: cremasch@bc.edu

Department of Mathematics and Statistics, University of Helsinki.
e-mail: joe_serdn@ciencias.unam.mx