A NEGATIVE MINIMUM MODULUS THEOREM AND 
SURJECTIVITY OF ULTRADIFFERENTIAL OPERATORS

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Dedicated to the memory of Professor Ciprian Foiaș

Abstract. In 1979 I. Ciorănescu and L. Zsidó have proved a minimum modulus theorem for entire functions dominated by the restriction to $(0, +\infty)$ of entire functions of the form $\omega(z) = \prod_{j=1}^{\infty} \left(1 + \frac{iz}{t_j}\right)$, $z \in \mathbb{C}$, with $0 < t_1 \leq t_2 \leq t_3 \leq \ldots \leq +\infty$, $t_1 < +\infty$, $\sum_{j=1}^{\infty} \frac{1}{t_j} < +\infty$, and such that

$$\int_{1}^{+\infty} \frac{\ln |\omega(t)|}{t^2} \ln \frac{t}{\ln |\omega(t)|} \, dt < +\infty.$$  

It implies that for $\omega$ as above, every $\omega$-ultradifferential operator with constant coefficients and of convergence type maps some $\mathcal{D}_\omega' \supset \mathcal{D}_\omega'$ onto itself. Here we show that the above results are sharp, by proving the negative counterpart of the above minimum modulus theorem: if

$$\int_{1}^{+\infty} \frac{\ln |\omega(t)|}{t^2} \ln \frac{t}{\ln |\omega(t)|} \, dt = +\infty,$$

then always there exists an entire function dominated by the restriction to $(0, +\infty)$ of $\omega$, which does not satisfy the minimum modulus conclusion in the 1979 paper. It follows that for such $\omega$ there exists an $\omega$-ultradifferential operator with constant coefficients and of convergence type, which does not map any $\mathcal{D}_\omega' \supset \mathcal{D}_\omega'$ onto itself.

1. Introduction

The main purpose of this paper is to expose (in a slightly completed form) the surjectivity criterion for ultradifferential operators with constant coefficients, given in [10], Proposition 2.7, and to prove that this criterion is sharp.

To avoid ambiguity, we notice that we will use Bourbaki’s terminology: ”positive” and ”strictly positive” instead of ”non-negative” and ”positive”, as well as ”increasing” and ”strictly increasing” instead of ”non-decreasing” and ”increasing”.

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In Section 2 we present, following [9], the current ultradistribution theories on \( \mathbb{R} \). Up to equivalence, there are two of them.

The first one is parametrized by entire functions of the form

\[
\omega(z) = \prod_{j=1}^{\infty} \left( 1 + \frac{iz}{t_j} \right), \quad z \in \mathbb{C},
\]

where \( 0 < t_1 \leq t_2 \leq t_3 \leq ... \leq +\infty \), \( t_1 < +\infty \), \( \sum_{j=1}^{\infty} \frac{1}{t_j} < +\infty \),

whose set is denoted by \( \Omega \). \( \mathcal{D}_{\omega} \) is a strict inductive limit of a sequence of nuclear Fréchet spaces, whose elements are infinitely differentiable functions of compact support. The strong dual \( \mathcal{D}_{\omega}' \) is the space of \( \omega \)-ultradistributions. \( \mathcal{D}_{\omega} \) can be naturally considered a subspace of \( \mathcal{D}_{\omega}' \). If \( \omega, \rho \in \Omega \) are such that

\[
|\omega(t)| \leq c |\rho(t)|
\]

for some constant \( c > 0 \) and all \( t \in \mathbb{R} \), then \( \mathcal{D}_{\rho} \subset \mathcal{D}_{\omega} \) and \( \mathcal{D}_{\omega}' \subset \mathcal{D}_{\rho}' \).

A second ultradistribution theory is obtained by considering the spaces \( \mathcal{D}_{\omega} \) and \( \mathcal{D}_{\omega}' \) only for entire functions \( \omega \) as above with the \( t_j \)'s satisfying additionally

\[
0 < t_1 \leq \frac{t_2}{2} \leq \frac{t_3}{3} \leq ... .
\]

\( \Omega_0 \) will denote the set of these entire functions.

In Section 3 we discuss ultradifferential operators and formulate the main results.

We call a linear map \( T : \mathcal{D}_{\omega} \rightarrow \mathcal{D}_{\omega} \) \( \omega \)-ultradifferential operator whenever the support of \( T \varphi \) is contained in the support of \( \varphi \in \mathcal{D}_{\omega} \). It is of constant coefficients if it commutes with the translation operators.

\( T \) is an \( \omega \)-ultradifferential operator of constant coefficients if and only if there exists an entire function \( f \) of exponential type 0 such that \( |f(it)| \leq c |\omega(t)|^n \), \( t \in \mathbb{R} \), for some \( c > 0 \) and integer \( n \geq 1 \), such that the Fourier transform of \( T \varphi \) is the product of the Fourier transform of \( \varphi \) multiplied by \( \mathbb{R} \ni t \mapsto f(it) \). In order that \( T \) be the convergent Taylor series \( f(D) \) of the derivation operator \( D \), \( f \) must satisfy the stronger majorization property

\[
|f(z)| \leq c |\omega(|z|)|^n, z \in \mathbb{C}, \text{ with } c > 0 \text{ a constant and } n \geq 1 \text{ an integer}.\]

In this case \( T \) is called of convergence type.

Any \( \omega \)-ultradifferential operator \( T \) of constant coefficients can be uniquely extended to a continuous linear operator \( \mathcal{D}_{\omega}' \rightarrow \mathcal{D}_{\omega}' \), still denoted by \( T \). A central issue is the characterization of the situation \( T \mathcal{D}_{\omega}' = \mathcal{D}_{\omega}' \), when the equation \( f(D)X = F \) has a solution \( X \in \mathcal{D}_{\omega}' \) for each \( F \in \mathcal{D}_{\omega}' \), in terms of the entire function \( f \) associated to \( T \). Such a criterion was obtained by I. Ciorânescu in [8], Proposition 2.4 and Theorem 3.4: \( T \mathcal{D}_{\omega}' = \mathcal{D}_{\omega}' \) if and only if \( f \) satisfies a certain minimum modulus condition.

In [10] a minimum modulus theorem was obtained, which implies that if

\[
\int_{1}^{+\infty} \frac{\ln |\omega(t)|}{t^2} \ln \frac{t}{\ln |\omega(t)|} \, dt < +\infty
\]
then, for every $\omega$-ultradifferential operator $T$ of constant coefficients and of convergent type, there exists some $\rho \in \Omega$, $|\omega(t)| \leq c|\rho(t)|$ for some constant $c > 0$ and all $t \in \mathbb{R}$, hence such that $\mathcal{D}_\omega' \subset \mathcal{D}_\rho'$, for which the surjectivity $T\mathcal{D}_\rho' = \mathcal{D}_\rho'$ holds true. We complete this result by proving that if $\omega \in \Omega_0$ (Theorem 3.9). To do this, we completed the minimum modulus theorem from [10] correspondingly (Theorem 3.8).

On the other hand we prove (Theorem 3.11) that if

$$\int_{1}^{+\infty} \frac{\ln |\omega(t)|}{t^2} \ln \frac{t}{\ln |\omega(t)|} \, dt = +\infty$$

then, there exists an $\omega$-ultradifferential operator $T$ of constant coefficients and of convergent type, such that the surjectivity $T\mathcal{D}_\rho' = \mathcal{D}_\rho'$ can not hold for any $\rho \in \Omega$ (Theorem 3.11). This is consequence of the negative minimum modulus theorem (Theorem 3.10), claiming that for $\omega$ as above there exists an entire function $f$ such that $|f(z)| \leq |\omega(|z|)|^2$, $z \in \mathbb{C}$, but for no increasing function

$$\beta : (0,+\infty) \rightarrow (0,+\infty)$$

with

$$\int_{1}^{+\infty} \beta(t) \frac{t}{t^2} \, dt < +\infty$$

the minimum modulus condition

$$\sup_{t \in \mathbb{R}} \ln |f(s)| \geq -\beta(t), \quad t > 0.$$ 

This negative minimum modulus theorem is the hearth of the paper and is proved in the last, 6th section.

In Section 4 we investigate the majorization of positive functions defined on $(0,+\infty)$ with functions $\alpha : (0,+\infty) \rightarrow (0,+\infty)$ belonging to different regularity classes and satisfying the non-quasianalyticity condition

$$\int_{1}^{+\infty} \frac{\alpha(t)}{t^2} \, dt < +\infty.$$ 

(like $(0,+\infty) \ni t \mapsto \ln |\omega(t)|$ for $\omega \in \Omega$). These topics are used in the proof of Theorem 3.8. Lemma 4.2 could be of interest for itself.

Section 5 is devoted to increasing functions $\alpha : (0,+\infty) \rightarrow (0,+\infty)$ satisfying

$$\int_{1}^{+\infty} \frac{\alpha(t)}{t^2} \, dt < +\infty \quad \text{and} \quad \int_{1}^{+\infty} \frac{\alpha(t)}{t^2} \ln \frac{t}{\alpha(t)} \, dt = +\infty.$$ 

Discretization of the above conditions is investigated (Propositions 5.3 and 5.4) and the case $\alpha(t) = \ln |\omega(t)|$, $\omega \in \Omega$, is characterized (Theorem 5.6).
In particular, for $0 < t_1 \leq t_2 \leq t_3 \leq \ldots \leq +\infty, t_1 < +\infty$, 

$$\alpha : (0, +\infty) \ni t \mapsto \ln \left| \prod_{j=1}^{\infty} \left( 1 + \frac{it}{t_j} \right) \right|$$

satisfies the above two conditions if and only if

$$\sum_{j=1}^{\infty} \frac{1}{t_j} < +\infty \quad \text{and} \quad \sum_{j=1}^{\infty} \frac{\ln t_j}{t_j} = +\infty,$$

what happens, for example, if $t_j = j (\ln j)(\ln \ln j)^p, j \geq 3$, with $1 < p \leq 2$. Section 5 actually prepares Section 6.

Finally, in Section 6 the negative minimum modulus theorem Theorem 3.10 is proved. The proof uses the machinery developed in Section 5 and the key ingredient Lemma 6.1. I am indebted to Professor W. K. Hayman for the proof of a statement very close to Lemma 6.1 in the case of $n_j = \alpha(2^j), j \geq 2$, where $\alpha(t) = \frac{t}{(\ln t)(\ln \ln t)^2}, t > e$, sent to me in [15]. The proof of Lemma 6.1 is based on Hayman’s ideas, it is actually an adaptation of Hayman’s draft to the general case.

2. Ultradistribution theories

In order to enlarge the family of L. Schwartz’s distributions, I. M. Gelfand and G. E. Shilov proposed in [12] (see also [13], Chapters II and IV) the following extension of L. Schwartz’s strategy: consider an appropriate locally convex topological vector space $\mathcal{B}$ of infinitely differentiable functions such that

- $\mathcal{B}$ is a Fréchet space or a countable inductive limit of Fréchet spaces,
- the topology of $\mathcal{B}$ is stronger than the topology of pointwise convergence.

The elements of $\mathcal{B}$ are called basic functions, and the elements of the dual $\mathcal{B}'$, generalized functions. If we ”shrink” $\mathcal{B}$, then $\mathcal{B}'$ becomes larger.

The generalized functions $\mathcal{B}'$ are usually called ultradistributions when, roughly speaking, disjoint compact sets can be separated by functions which belong to $\mathcal{B}$. This yields a ”lower bound” for $\mathcal{B}$. Ultradistribution theories are mostly based on non-quasianaliticity.

Let us briefly sketch, following [9], Section 7, what we will here understand by an ultradistribution theory on the real line $\mathbb{R}$ (a slightly different picture is given in [24]).

Let $\mathcal{S}$ be a parameter set and assume that to each $\sigma \in \mathcal{S}$ is associated a locally convex topological vector space $\mathcal{D}_\sigma$ of infinitely differentiable functions $\mathbb{R} \rightarrow \mathbb{C}$ with compact support such that, for every $\sigma \in \mathcal{S}$,

(i) $\mathcal{D}_\sigma$ is an inductive limit of a sequence of Fréchet spaces;
(ii) the topology of $\mathcal{D}_\sigma$ is stronger than the topology of pointwise convergence;
(iii) $\mathcal{D}_\sigma$ is an algebra under pointwise multiplication;
(iv) for $K \subset D \subset \mathbb{R}$, $K$ compact and $D$ open, there exists $\varphi \in \mathcal{D}_\sigma$ such that

$$0 \leq \varphi \leq 1, \quad \varphi(s) = 1 \text{ for } s \in K, \quad \text{supp}(\varphi) \subset D;$$

(v) denoting by $\mathcal{E}_\sigma$ the multiplier algebra of $\mathcal{D}_\sigma$, that is the set of all functions $\psi : \mathbb{R} \rightarrow \mathbb{C}$ satisfying $\varphi \psi \in \mathcal{D}_\sigma, \varphi \in \mathcal{D}_\sigma$, and endowing it with the projective limit topology defined by the linear mappings $\mathcal{E}_\sigma \ni \psi \mapsto \varphi \psi \in \mathcal{D}_\sigma, \varphi \in \mathcal{D}_\sigma$,

the set $\mathcal{A}$ of all real analytic complex functions on $\mathbb{R}$ is a dense subset of $\mathcal{E}_\sigma$.

We will say that $\{\mathcal{D}_\sigma\}_{\sigma \in \mathcal{S}}$ is a theory of ultradistributions and the elements of the dual $\mathcal{D}_\sigma'$ will be called $\sigma$-ultradistributions.

For $\sigma \in \mathcal{S}$ and $F \in \mathcal{D}_\sigma'$, there is a smallest closed set $S \subset \mathbb{R}$ such that

$$\varphi \in \mathcal{D}_\sigma, S \cap \text{supp}(\varphi) = \emptyset \implies F(\varphi) = 0.$$

Then $S$ is called the support of $F$ and is denoted by supp($F$). The dual $\mathcal{E}_\sigma'$ can be identified with the vector space of all $\sigma$-ultradistributions of compact support, since the restriction map $\mathcal{E}_\sigma' \ni G \mapsto G[\mathcal{D}_\sigma]$ is a linear isomorphism of $\mathcal{D}_\sigma'$ onto $\{F \in \mathcal{D}_\sigma'; \text{supp}(F) \text{ compact}\}$.

By a $\sigma$-ultradifferential operator we mean a linear operator $T : \mathcal{D}_\sigma \rightarrow \mathcal{D}_\sigma$ which doesn’t enlarge the support:

$$\text{supp}(T \varphi) \subset \text{supp}(\varphi), \quad \varphi \in \mathcal{D}_\sigma.$$

Let $\{\mathcal{D}_\sigma\}_{\sigma \in \mathcal{S}}$ and $\{\mathcal{D}_\tau\}_{\tau \in \mathcal{T}}$ be two ultradistribution theories. We say that the ultradistribution theory $\{\mathcal{D}_\tau\}_{\tau \in \mathcal{T}}$ is larger than $\{\mathcal{D}_\sigma\}_{\sigma \in \mathcal{S}}$ if for every $\sigma \in \mathcal{S}$ there exists some $\tau \in \mathcal{T}$ such that $\mathcal{D}_\tau \subset \mathcal{D}_\sigma$, or equivalently, $\mathcal{E}_\tau \subset \mathcal{E}_\sigma$. When this happens then the inclusion maps $\mathcal{D}_\tau \hookrightarrow \mathcal{D}_\sigma$ and $\mathcal{E}_\tau \hookrightarrow \mathcal{E}_\sigma$ are continuous and have a dense range.

We notice that if $\{\mathcal{D}_\sigma\}_{\sigma \in \mathcal{S}}$ and $\{\mathcal{D}_\tau\}_{\tau \in \mathcal{T}}$ are ultradistribution theories and $\{\mathcal{D}_\tau\}_{\tau \in \mathcal{T}}$ is larger than $\{\mathcal{D}_\sigma\}_{\sigma \in \mathcal{S}}$, then

$$\mathcal{A} \subset \bigcap_{\tau \in \mathcal{T}} \mathcal{E}_\tau \subset \bigcap_{\sigma \in \mathcal{S}} \mathcal{E}_\sigma.$$

We say that two ultradistribution theories $\{\mathcal{D}_\sigma\}_{\sigma \in \mathcal{S}}$ and $\{\mathcal{D}_\tau\}_{\tau \in \mathcal{T}}$ are equivalent whenever each one of them is larger than the other.

Let us recall the usual ultradistribution theories. They are labeled by one of the following parameter sets $\mathcal{S}$:

• $\mathcal{M}$ is the set of all sequences $(M_p)_{p \geq 0}$ in $(0, +\infty)$, $M_0 = 1$, satisfying

$$M_p^2 \leq M_{p-1}M_{p+1}, \quad p \geq 1 \text{ (logarithmic convexity)},$$

$$\sum_{p \geq 1} \frac{M_{p-1}}{M_p} < +\infty \text{ (non-quasianalyticity)}.$$
\( M_0 \) is the set of all sequences \( (M_p)_p \in \mathcal{M} \) which satisfy the stronger logarithmic convexity condition
\[
\left( \frac{M_p}{p!} \right)^2 \leq \frac{M_{p-1}}{(p-1)!} \cdot \frac{M_{p+1}}{(p+1)!}, \quad p \geq 1.
\]
\( \mathcal{A} \) is the set of all continuous functions \( \alpha : \mathbb{R} \to (0, +\infty) \) satisfying
\[
\alpha(0) = 0, \quad \alpha(t + s) \leq \alpha(t) + \alpha(s) \text{ for } t, s \in \mathbb{R} \text{ (subadditivity)},
\]
there exist \( a \in \mathbb{R} \) and \( b > 0 \) such that \( \alpha(t) \geq a + b \ln(1 + |t|), t \in \mathbb{R} \),
\[
\int_{-\infty}^{+\infty} \frac{\alpha(t)}{1 + t^2} dt < +\infty.
\]
\( \Omega \) is the set of all entire functions \( \omega \) of the form
\[
(2.1) \quad \omega(z) = \prod_{j=1}^{\infty} \left( 1 + \frac{iz}{t_j} \right), \quad z \in \mathbb{C},
\]
where \( 0 < t_1 \leq t_2 \leq t_3 \leq ... \leq +\infty, \quad t_1 < +\infty, \quad \sum_{j=1}^{\infty} \frac{1}{t_j} < +\infty. \)

\( \Omega_0 \) is the set of all entire functions \( \omega \) of the form
\[
\omega(z) = \prod_{j=1}^{\infty} \left( 1 + \frac{iz}{t_j} \right), \quad z \in \mathbb{C},
\]
where \( 0 < t_1 \leq \frac{t_2}{2} \leq \frac{t_3}{3} \leq ... \leq +\infty, \quad t_1 < +\infty, \quad \sum_{j=1}^{\infty} \frac{1}{t_j} < +\infty. \)

Let \( (M_p)_p \in \mathcal{M} \) be fixed. For \( K \subset \mathbb{R} \) compact and \( h > 0 \), let \( \mathcal{D}_{\{M_p\}_p, h}(K) \) denote the vector space of all infinitely differentiable functions \( \varphi : \mathbb{R} \to \mathbb{C} \) with \( \text{supp}(\varphi) \subset K \), satisfying
\[
\|\varphi\|_{\{M_p\}_p, h} := \sup_{s \in K, p \geq 0} \frac{1}{h^p M_p} |\varphi^{(p)}(s)| < +\infty.
\]
Then \( \mathcal{D}_{\{M_p\}_p, h}(K) \), endowed with the norm \( \| \cdot \|_{\{M_p\}_p, h} \), becomes a Banach space.

The Roumieu ultradifferentiable functions of class \( (M_p)_p \in \mathcal{M} \) on \( \mathbb{R} \), having compact support, are
\[
\mathcal{D}_{\{M_p\}_p} := \lim_{K \subset \mathbb{R} \text{ compact}} \lim_{0<h \to \infty} \mathcal{D}_{\{M_p\}_p, h}(K)
\]
(see \cite{22} or \cite{17}), while the Beurling-Komatsu ultradifferentiable functions of class \( (M_p)_p \in \mathcal{M} \) on \( \mathbb{R} \), having compact support, are
\[
\mathcal{D}_{\{M_p\}_p} := \lim_{K \subset \mathbb{R} \text{ compact}} \lim_{0<h \to 0} \mathcal{D}_{\{M_p\}_p, h}(K)
\]
(see \cite{17}). \( \{\mathcal{D}_{\{M_p\}_p}\}_{(M_p)_p \in \mathcal{M}} \) and \( \{\mathcal{D}_{\{M_p\}_p}\}_{(M_p)_p \in \mathcal{M}} \) are the Roumieu resp. Beurling-Komatsu ultradistribution theories.
Let now $\alpha \in \mathcal{A}$ be fixed. For $K \subset \mathbb{R}$ compact we denote by $D_\alpha(K)$ the vector space of all continuous functions $\varphi : \mathbb{R} \rightarrow \mathbb{C}$ with $\text{supp}(\varphi) \subset K$, for which

$$
\|\varphi\|_{\alpha,\lambda} := \int_{-\infty}^{+\infty} |\hat{\varphi}(t)| e^{\lambda \alpha(t)} dt < +\infty, \quad \lambda > 0,
$$

where $\hat{\varphi}$ stands for the Fourier transform of $\varphi$:

$$
\hat{\varphi}(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \varphi(s) e^{-its} ds
$$

Then $D_\alpha(K)$, endowed with the family of norms $\| \cdot \|_{\alpha,\lambda}$, $\lambda > 0$, becomes a Fréchet space.

The Beurling-Björck ultradifferentiable functions of class $\alpha \in \mathcal{A}$ on $\mathbb{R}$, having compact support, are

$$
D_\alpha := \lim_{K \subset \mathbb{R} \text{ compact}} D_\alpha(K)
$$

(see [2] and [11]). $\{D_\alpha\}_{\alpha \in \mathcal{A}}$ is the Beurling-Björck ultradistribution theory.

Finally, for $\omega \in \Omega$ and $K \subset \mathbb{R}$ compact, let $D_\omega(K)$ be the vector space of all continuous functions $\varphi : \mathbb{R} \rightarrow \mathbb{C}$ with $\text{supp}(\varphi) \subset K$, for which

$$
p_{\omega,n}(\varphi) := \sup_{t \in \mathbb{R}} |\hat{\varphi}(t) \omega(t)^n| < +\infty, \quad n \geq 1.
$$

Then $D_\omega(K)$, endowed with the family of norms $p_{\omega,n}$, $n \geq 1$, becomes a Fréchet space.

The $\omega$-ultradifferentiable functions on $\mathbb{R}$, having compact support, are

$$
D_\omega := \lim_{K \subset \mathbb{R} \text{ compact}} D_\omega(K)
$$

(see [9], Section 2). $\{D_\omega\}_{\omega \in \Omega}$ is the $\omega$-ultradistribution theory.

We have to remark that in [9], Definition III, $D_\omega(K)$ is defined by using the norms $p_{\omega,L,n}$, $L > 0$, $n \geq 1$, where

$$
p_{\omega,L,n}(\varphi) := \sup_{t \in \mathbb{R}} |\hat{\varphi}(t) \omega(Lt)^n|.
$$

However, with the notation of (2.1), we have

$$
(2.2) \quad |\omega(Lt)| = \prod_{j=1}^{\infty} \left(1 + \frac{L^2 t^2}{t_k}\right)^{1/2} \leq \prod_{j=1}^{\infty} \left(1 + \frac{L^2}{t_k}\right)^{L^2/2} = |\omega(t)|^{L^2},
$$

so the two definitions are equivalent.

We notice that the Roumieu, the Beurling-Komatsu and the Beurling-Björck ultradistribution theories were considered also on open subsets of $\mathbb{R}^d$ (see [23], [17], [4]), while the $\omega$-ultradistribution theory, originally considered in [9] only on $\mathbb{R}$, was subsequently extended to the multidimensional setting (see [6] and [11]). However, in this paper we will restrict us to the one-dimensional case of $\mathbb{R}$. 

In [9], 7.4 it was shown that the ultradistribution theories
(2.3) \{D_{\{M_p\}}\}_{(M_p) \in \mathbb{M}}, \{D_{(M_p)}\}_{(M_p) \in \mathbb{M}}, \{D_{\omega}\}_{\omega \in \Omega}
are equivalent. Thus they are just different labelings of the same global set of ultradistributions. To work with ultradifferential operators, the setting of the \(\omega\)-ultradistribution theory seems to be the most advantageous. Therefore we will adopt this setting in the sequel.

We notice that, according to [11], Theorem 1, also the ultradistribution theories
(2.4) \{D_{\{M_0\}}\}_{(M_0) \in \mathbb{M}_0}, \{D_{(M_0)}\}_{(M_0) \in \mathbb{M}_0}, \{D_{\alpha}\}_{\alpha \in \mathcal{A}}, \{D_{\omega}\}_{\omega \in \Omega_0}
are equivalent. As was pointed out in [9], Section 7.7, \(\bigcap_{\omega \in \Omega} E_{\omega} \neq \bigcap_{\omega \in \Omega_0} E_{\omega}\), so the ultradistribution theories (2.3) are larger than those in (2.4), but not equivalent to them.

3. ULTRADIFFERENTIAL OPERATORS AND THE MAIN RESULTS

For \(\omega \in \Omega\), let us consider the \(\omega\)-ultradifferentiable function spaces \(D_{\omega}, E_{\omega}\), as defined in Section 2 (\(D_{\omega}\) on page 7, and \(E_{\omega}\) as indicated in (v) on page 5).

\(D_{\omega}\) is strict inductive limit of a sequence of nuclear Fréchet spaces and it is stable under a series of elementary operations like pointwise multiplication, convolution, differentiation, translations etc. Moreover, these operations are continuous.

\(E_{\omega}\) is a nuclear Fréchet space and has similar stability properties as \(D_{\omega}\).

The set \(\mathcal{A}\) of all real analytic complex functions on \(\mathbb{R}\), as well as \(D_{\omega}\), are dense subsets of \(E_{\omega}\).

The space of the \(\omega\)-ultradistributions is the strong dual \(D'_{\omega}\) of \(D_{\omega}\) and, associating to each \(\varphi \in E_{\omega}\) the linear functional
\[D_{\omega} \ni \psi \mapsto \int_{-\infty}^{+\infty} \varphi(s)\psi(s)\,ds,\]
we obtain an inclusion map with dense range \(E_{\omega} \hookrightarrow D'_{\omega}\).

For all the above facts we send to [9], Section 2.

Let now \(T : D_{\omega} \rightarrow D_{\omega}\) be an \(\omega\)-ultradifferential operator, that is a linear operator satisfying the condition
\[\text{supp}(T\varphi) \subset \text{supp}(\varphi), \quad \varphi \in D_{\omega}.\]
Then \(T\) is continuous and can be (uniquely) extended to a continuous linear operator \(E_{\omega} \rightarrow E_{\omega}\), which will be still denoted by \(T\) ([9], Theorem 2.16).

We say that an \(\omega\)-ultradifferential operator is with constant coefficients if it commutes with every translation operator. An immediate consequence of [9], Theorem 2.21 is

**Proposition 3.1.** If \(f\) is an entire function of exponential type 0 such that
(3.1) \[|f(it)| \leq d_0 |\omega(t)^{\alpha_0}|, \quad t \in \mathbb{R}\]
for some integer $n_0 \geq 1$ and real number $d_0 > 0$, then the formula
\[
(f(D)\varphi)(t) = f(it) \hat{\varphi}(t), \quad \varphi \in \mathcal{D}_\omega, \ t \in \mathbb{R}
\]
defines an $\omega$-ultradifferential operator $f(D)$ with constant coefficients. Conversely, any $\omega$-ultradifferential operator $f(D)$ with constant coefficients is of this form.

If $f$ is an entire function of exponential type 0, satisfying (3.1) for some $n_0 \geq 1$ and $d_0 > 0$, then the $\omega$-ultradifferential operator $f(D): \mathcal{E}_\omega \rightarrow \mathcal{E}_\omega$ with constant coefficients can be extended to a continuous linear operator $\mathcal{D}_\omega' \rightarrow \mathcal{D}_\omega'$, which we will still denote by $f(D)$ (see [9], discussion before Theorem 3.5).

Denoting by $\delta_{s_0}$ the Dirac measure concentrated at $s_0 \in \mathbb{R}$, considered an $\omega$-ultradistribution of support $\{s_0\}$, for each $\omega$-ultradistribution $F$ with support $\{s_0\}$ there exists an entire function as above such that $T = f(D)\delta_{s_0}$ (see [9], Theorem 3.5).

If $\omega(z) = \infty \prod_{j=1}^{\infty} \left(1 + \frac{i z}{t_j}\right)$, $z \in \mathbb{C}$, where
\[
0 < t_1 \leq t_2 \leq t_3 \leq \ldots \leq +\infty, \quad t_1 < +\infty, \quad \sum_{j=1}^{\infty} \frac{1}{t_j} < +\infty,
\]
then, for $n \geq 1$ and $k \geq 0$ integers, we denote by $a_{\omega,n}^k$ the square root of the coefficient of $z^k$ in the power series expansion of the entire function
\[
\mathbb{C} \ni z \mapsto (\omega(z)\overline{\omega(z)})^n = \prod_{j=1}^{\infty} \left(1 + \frac{z^2}{t_j^2}\right)^n
\]
($\overline{\omega(z)}$ stands here, as usual, for $\omega(\overline{z})$). We recall (see [9], page 109):
\[
(3.2) \quad \sup_{p \geq 0} a_{\omega,n}^p |t|^p \leq |\omega(t)|^n \leq \sqrt{2} \sup_{p \geq 0} a_{\omega,n}^p \sqrt{2} |t|^p, \quad t \in \mathbb{R}.
\]

We have also, according to [9], Corollary 2.9,
\[
(3.3) \quad (a_{\omega,n}^k)^2 \geq a_{\omega,n}^{k-1} \cdot a_{\omega,n}^{k+1}, \quad n \geq 1, \ k \geq 0.
\]

A useful consequence of (3.3) is
\[
(3.4) \quad \left(\frac{a_{\omega,n}^k}{a_{\omega,n}^{k-1}}\right)^k \sup_{p \geq 0} a_{\omega,p}^n \left(\frac{a_{\omega,n}^{k-1}}{a_{\omega,n}^k}\right)^p = a_{\omega,n}^k, \quad n, k \geq 1 \text{ integers}.
\]

Indeed, since
\[
\left(\frac{a_{\omega,n}^k}{a_{\omega,n}^{k-1}}\right)^k \sup_{p \geq 0} a_{\omega,p}^n \left(\frac{a_{\omega,n}^{k-1}}{a_{\omega,n}^k}\right)^p = \left(\frac{a_{\omega,n}^k}{a_{\omega,n}^{k-1}}\right)^k \max \left(1, \sup_{p \geq 1} \prod_{q=1}^{p} a_{\omega,q}^n \cdot a_{\omega,n}^{k-1} / a_{\omega,n}^k\right)
\]
and, by (3.3),
\[ \frac{a_{q,n}^{\omega,n}}{a_{q-1,n}^{\omega,n}} \quad \begin{cases} \geq 1 \text{ for } q \leq k \\ \leq 1 \text{ for } q \geq k \end{cases}, \]
we deduce:
\[ \left( \frac{a_{k,n}^{\omega,n}}{a_{k-1,n}^{\omega,n}} \right)^k \sup_{p \geq 0} a_{p,n}^{\omega,n} t^p = \prod_{q=1}^k \left( \frac{a_{q,n}^{\omega,n}}{a_{q-1,n}^{\omega,n}} \cdot \frac{a_{k,n}^{\omega,n}}{a_{k-1,n}^{\omega,n}} \right) = a_{k,n}^{\omega,n}. \]

(3.4) implies immediately:
\[ \min_{t > 0} \frac{1}{t^k} \sup_{p \geq 0} a_{p,n}^{\omega,n} t^p = a_{k,n}^{\omega,n}, \quad n \geq 1, k \geq 0. \]

We notice also the inequality
\[ (3.6) \quad |\omega(z)| = \prod_{j=1}^\infty \left| 1 + \frac{iz}{t_j} \right| \leq \prod_{j=1}^\infty \left( 1 + \frac{|z|}{t_j} \right) = \omega(-i|z|), \quad z \in \mathbb{C}. \]

If $P$ is a polynomial with complex coefficients and $P(z) = \sum_{k=0}^n c_k z^k$, then
\[ P(D) = \sum_{k=0}^n c_k D^k \] where $D$ is the derivation operator. The next proposition, a variant of [9], Theorem 2.25, characterizes those $\omega$-ultradifferential operators with constant coefficients, which can be expanded in power series in $D$.

**Proposition 3.2.** Let $f$ be an entire function of exponential type 0 such that (3.7) holds true for some $n_0 \geq 1$ and $d_0 > 0$, and $f(z) = \sum_{k=0}^\infty c_k z^k$ its expansion in a power series. Then the following statements are equivalent:

(i) There exist an integer $n_1 \geq 1$ and a real number $d_1 > 0$ such that
\[ |f(z)| \leq d_1 |\omega(|z|)^{n_1}|, \quad z \in \mathbb{C}. \]

(ii) There exist an integer $n_2 \geq 1$ and real numbers $L_2, d_2 > 0$ such that
\[ |c_k| \leq d_2 L_2^k a_{k,n_2}^{\omega,n_2}, \quad k \geq 0. \]

(iii) We have $f(D) = \sum_{k=0}^\infty c_k D^k$, where the series converges in the vector space of all continuous linear maps $E_\omega \rightarrow E_\omega$, endowed with the topology of the uniform convergence on the bounded subsets of $E_\omega$.

(iv) We have $f(D) = \sum_{k=0}^\infty c_k D^k$, where the series converges in the vector space of all continuous linear maps $E_\omega \rightarrow E_\omega$, endowed with the topology of the pointwise convergence.
Proof. For (i) ⇒ (ii). Using the Cauchy estimate, (i) and \( (3.2) \), we obtain for any integer \( k \geq 0 \) and real \( r > 0 \):

\[
|c_k| \leq \frac{1}{r^k} \sup_{|z| = r} |f(z)| \leq d_1 \frac{1}{r^k} \sup_{|z| = r} |\omega(|z|)|^{n_1} \leq \sqrt{2} d_1 \frac{1}{r^k} \sup_{p \geq 0} a_p^{\omega,n_1} (\sqrt{2} r)^p.
\]

Using now \( (3.5) \), we infer:

\[
|c_k| \leq \sqrt{2} d_1 \inf_{0 < r_0} \frac{1}{r^k} \sup_{p \geq 0} a_p^{\omega,n_1} (\sqrt{2} r)^p = \sqrt{2} d_1 \inf_{t > 0} \left( \frac{\sqrt{2}}{t} \right)^k \sup_{p \geq 0} a_p^{\omega,n_1} t^p
\]

\[
= \sqrt{2} d_1 \left( \frac{\sqrt{2}}{t} \right)^k a_k^{\omega,n_1}, \quad k \geq 0.
\]

Thus (ii) holds with \( n_2 = n_1, L_2 = \sqrt{2}, d_2 = \sqrt{2} d_1 \).

For (ii) ⇒ (i). Using (ii) and the first inequality in \( (3.2) \), we deduce:

\[
|f(z)| \leq \sum_{k=0}^{\infty} |c_k| |z|^k \leq d_2 \sum_{k=0}^{\infty} a_k^{\omega,n_2} (L_2 |z|)^k = d_2 \sum_{k=0}^{\infty} \frac{1}{2k} a_k^{\omega,n_2} (2L_2 |z|)^k
\]

\[
\leq d_2 \left( \sum_{k=0}^{\infty} \frac{1}{2k} \right) \sup_{k \geq 0} a_k^{\omega,n_2} (2L_2 |z|)^k \leq 2d_2 \left( \omega(2L_2 |z|)^n \right).
\]

Choosing some integer \( m \geq 2L_2 \) and using \( (2.2) \), we obtain

\[
|f(z)| \leq 2d_2 \left( \omega(m |z|)^n \right) \leq 2d_2 \left( \omega(|z|)^{m^2n_2} \right),
\]

hence (i) holds with \( n_1 = m^2n_2 \) and \( d_1 = 2d_2 \).

Implication (ii) ⇒ (iii) follows by \( [9] \), Proposition 2.24, and implication (iii) ⇒ (iv) is trivial.

Finally, for the proof of (iv) ⇒ (ii) we adapt the proof of (iv) ⇒ (ii) in \( [9] \), Theorem 2.25 as follows.

(iv) implies that the sequence \((c_k D^k \varphi)_{k \geq 0} = (c_k \varphi^{(k)})_{k \geq 0}\) converges in \( E_\omega \) to 0 for every \( \varphi \in E_\omega \). Therefore the sequence \( E_\omega \ni \varphi \mapsto c_k \varphi^{(k)}(0), k \geq 0, \)

is pointwise convergent to 0 in \( E_\omega \), in particular it is pointwise bounded. Since \( E_\omega \) is a Fréchet space, and hence barrelled, if follows that the above sequence in \( E_\omega \) is equicontinuous (see e.g. \( [5] \), Ch. III, §4, Section 1).

Recalling that the topology of \( E_\omega \) is defined by the semi-norms

\[
r_{\omega,L,n}^K : E_\omega \ni \varphi \mapsto \sup_{p \geq 0} \left( L_p a_p^{\omega,n} \sup_{s \in K} |\varphi^{(p)}(s)| \right),
\]

where \( K \subset \mathbb{R} \) is compact, \( L > 0 \) and \( n \geq 1 \) is an integer (see \( [9] \), Definition V on page 110), we deduce the existence of some \( K, L, n \) and a constant \( d > 0 \) such that

\[
|c_k \varphi^{(k)}(0)| \leq d \cdot r_{\omega,L,n}^K(\varphi), \quad k \geq 0, \varphi \in E_\omega.
\]

Applying this inequality to \( \varphi = e^{i\alpha}, \alpha > 0 \), we obtain

\[
|c_k| \cdot \alpha^k \leq d \cdot \sup_{p \geq 0} \left( L_p a_p^{\omega,n} \alpha^p \right), \quad k \geq 0, \alpha > 0.
\]
Therefore, using (3.5), we infer:

$$|c_k| \leq d \inf_{\alpha > 0} \frac{1}{\alpha^k} \sup_{p \geq 0} a_p^\omega, n (L \alpha)^p = d \inf_{t > 0} \left( \frac{L}{t} \right)^k \sup_{p \geq 0} a_p^\omega, n t^p = d L^k a_k^\omega, n, \ k \geq 0.$$ 

In other words, (ii) holds with $$n_2 = n, L_2 = L, d_2 = d.$$

□

If \( f \) is an entire function satisfying the equivalent conditions in Proposition 3.2, then we will say that the \( \omega \)-ultradifferential operator with constant coefficients \( f(D) \) is of convergence type.

Proposition 3.2 enables to prove a description of those \( \omega \in \Omega \), for which every \( \omega \)-ultradifferential operator with constant coefficients is of convergence type. This description is essentially \[9\], Theorem 2.25.

**Corollary 3.3.** The following statements concerning \( \omega \in \Omega \) are equivalent:

(i) There exist an integer \( n_1 \geq 1 \) and a real number \( d_1 > 0 \) such that

$$|\omega(-it)| \leq d_1 |\omega(t)|^{n_1}, \quad t \in \mathbb{R}.$$ 

(jj) There exist an integer \( n_2 \geq 1 \) and a real number \( d_2 > 0 \) such that

$$|\omega(z)| \leq d_2 |\omega(|z|)|^{n_2}, \quad z \in \mathbb{C}.$$ 

(jjj) The \( \omega \)-ultradifferential operator with constant coefficients \( \omega(-iD) \) is of convergence type.

(jw) Every \( \omega \)-ultradifferential operator with constant coefficients is of convergence type.

**Proof.** (j) \( \Rightarrow \) (jj) follows easily by using (3.6):

$$|\omega(z)| \leq |\omega(-i|z|)| \leq d_1 |\omega(|z|)|^{n_1}, \quad z \in \mathbb{C}.$$ 

On the other hand, implication (jj) \( \Rightarrow \) (j) is trivial:

$$|\omega(-it)| \leq d_2 |\omega(|-it|)|^{n_2} = d_2 |\omega(|t|)|^{n_2} = d_2 |\omega(t)|^{n_2}, \quad t \in \mathbb{R}.$$ 

Thus (j) \( \Leftrightarrow \) (jj).

Next, equivalence (jj) \( \Leftrightarrow \) (jjj) is an immediate consequence of the definition of the convergence type by using condition (i) in Proposition 3.2, while implication (jw) \( \Rightarrow \) (jjj) is trivial. Thus it remains only to prove, for example, (jj) \( \Rightarrow \) (jw).

For let us assume that (jj) is satisfied and \( f \) is an arbitrary entire function of exponential type 0, satisfying (3.1). Denoting, for convenience,

$$\rho(z) := d_0 \cdot \omega(z)^{n_0}, \quad z \in \mathbb{C},$$ 

\( \rho \) is an entire function of exponential type 0, which has no zeros in the open lower half-plane. Therefore, using the terminology of [13], Chapter VII, §4, \( \rho \) is an entire function of class \( P \). Since \( f(i \cdot) \) is an entire function of exponential type 0 and, by (3.1),
\[ |f(it)| \leq |\rho(t)|, \quad t \in \mathbb{R}, \]

applying [18], Chapter IX, §4, Lemma 1, we obtain:
\[ |f(iz)| \leq |\rho(z)| \text{ and } |f(i\bar{z})| \leq |\rho(z)| \text{ for all } z \in \mathbb{C} \text{ with } \text{Im}z \leq 0, \]
that is
\[ |f(iz)| \leq \begin{cases} \frac{d_0}{d_0} |\omega(z)^{n_0}| & \text{for } z \in \mathbb{C}, \text{Im}z \leq 0 \\ \frac{d_0}{d_0} |\omega(\bar{z})^{n_0}| & \text{for } z \in \mathbb{C}, \text{Im}z \geq 0 \end{cases} \tag{3.7} \]

On the other hand, (ji) yields for every \( z \in \mathbb{C} \)
\[ |\omega(z)| \leq d_2 |\omega(|z|)^{n_2}| \text{ and } |\omega(\bar{z})| \leq d_2 |\omega(|\bar{z}|)^{n_2}| = d_2 |\omega(|z|)^{n_2}|. \tag{3.8} \]

Now, by (3.7) and (3.8) we deduce:
\[ |f(iz)| \leq \frac{d_0}{d_0} (d_2)^{n_0} |\omega(|z|)^{n_2-n_0}|, \quad z \in \mathbb{C}. \]

Consequently condition (i) in Proposition 3.2 holds true with \( n_1 = n_2 \cdot n_0 \)
and \( d_1 = d_0 (d_2)^{n_0} \), and we conclude that the \( \omega \)-ultradifferential operator with constant coefficients \( f(D) \) is of convergence type.

\[ \square \]

Following [9], Definition XI, we will say that \( \omega \) satisfies the strong non-quasianalyticity condition whenever it fulfills the equivalent conditions in Corollary 3.3.

**Remark 3.4.** If \( \omega(z) = \prod_{j=1}^{\infty} \left( 1 + \frac{iz}{t_j} \right), z \in \mathbb{C}, \) where
\[ 0 < t_1 \leq t_2 \leq t_3 \leq ... \leq +\infty, \quad t_1 < +\infty, \quad \sum_{j=1}^{\infty} \frac{1}{t_j} < +\infty, \]
then, in order that \( \omega \) satisfy the strong non-quasianalyticity condition, a necessary condition is
\[ \sum_{j=1}^{\infty} \frac{\ln j}{t_j} < +\infty \]
(see [9], Corollary 1.9), while a sufficient condition is the existence of a constant \( c > 0 \) such that
\[ \sum_{j=k}^{\infty} \frac{1}{t_j} \leq c \frac{k}{t_k}, \quad k \geq 1 \]
(see [17], Proposition 4.6 or [9], comments after Proposition 5.15). If we assume also
\[ 0 < t_1 \leq \frac{t_2}{2} \leq \frac{t_3}{3} \leq ... , \]
then \( \omega \) satisfies the strong non-quasianalyticity condition if and only if there exists a constant \( c > 0 \) such that
\[
\sum_{j=k}^{\infty} \frac{1}{t_j} \leq c \frac{k}{t_k} \left( 1 + \ln \frac{t_k}{(t_1 \cdots t_k)^{1/k}} \right), \quad k \geq 1
\]
(see \cite{9}, Proposition 5.15).

Central issue in the theory of \(\omega\)-ultradifferential operators with constant coefficients \(f(D) : \mathcal{D}_\omega^\prime \to \mathcal{D}_\omega^\prime\) is the characterization of its surjectivity, that is of the existence of a solution \(X \in \mathcal{D}_\omega^\prime\) of the equation \(f(D)X = F\) for each \(F \in \mathcal{D}_\omega^\prime\), in terms of \(f\). A surjectivity criterion was proved by I. Ciorănescu in \cite{8}, Proposition 2.4 and Theorem 3.4:

**Proposition 3.5.** For \(f\) an entire function of exponential type \(0\) such that (3.1) holds true for some \(n_0 \geq 1\) and \(d_0 > 0\), the following statements are equivalent:

(i) There exists some \(E \in \mathcal{D}_\omega^\prime\) such that \(f(D)E = \delta_0\).

(ii) \(f(D) : \mathcal{D}_\omega^\prime \to \mathcal{D}_\omega^\prime\) is surjective, that is \(f(D)\mathcal{D}_\omega^\prime = \mathcal{D}_\omega^\prime\).

(iii) there are constants \(c, c' > 0\) such that

\[
\sup_{s \in \mathbb{R}} \ln |f(s)| \geq -c \ln |\omega(t)| - c', \quad t \in \mathbb{R}.
\]

\(\square\)

If \(f\) satisfies the equivalent conditions of Proposition 3.5 then, following \cite{7}, Définition III.1-4, and by abuse of language, we will say that \(f(D)\) is invertible in \(\mathcal{D}_\omega^\prime\).

If \(\rho \in \Omega\) and

\[|\omega(t)| \leq c|\rho(t)|, \quad t \in \mathbb{R}\]

for some \(c > 0\), then \(\mathcal{D}_\rho \subset \mathcal{D}_\omega\), where the inclusion is continuous and with dense range. Consequently also \(\mathcal{D}_\omega^\prime \subset \mathcal{D}_\rho^\prime\), where the inclusion is continuous and with dense range. Any \(\omega\)-ultradifferential operator with constant coefficients \(f(D)\) is clearly also a \(\rho\)-ultradifferential operator with constant coefficients, hence we can consider the problem of the invertibility of \(f(D)\) in \(\mathcal{D}_\rho^\prime\). We notice that if \(f(D)\) is of convergence type as \(\omega\)-ultradifferential operator, then it is of convergence type also as \(\rho\)-ultradifferential operator.

The main goal of this paper is to give an exact answer to the question: for which \(\omega \in \Omega\) is every \(\omega\)-ultradifferential operator with constant coefficients and of convergence type, \(\rho\)-invertible for some \(\rho \in \Omega\) with \(\omega \leq c\rho\), where \(c > 0\) is a constant?

A sufficient condition for this was already found in \cite{10}, Proposition 2.7, namely

\[
(3.9) \quad \int_{1}^{+\infty} \frac{\ln |\omega(t)|}{t^2} \ln \frac{t}{\ln |\omega(t)|} \, dt < +\infty.
\]
Let us call condition (3.9) the mild strong non-quasianalyticity condition. This denomination is justified by the fact that (3.9) is implied by the strong non-quasianalyticity property. More precisely, we have:

**Proposition 3.6.** For \( \omega(z) = \prod_{j=1}^{\infty} \left( 1 + \frac{i z}{t_j} \right), z \in \mathbb{C} \), where

\[
0 < t_1 \leq t_2 \leq t_3 \leq \ldots \leq +\infty, \quad t_1 < +\infty, \quad \sum_{j=1}^{\infty} \frac{1}{t_j} < +\infty,
\]
we have

\( \omega \) satisfies the strong non-quasianalyticity condition

\[
\implies \sum_{j=1}^{\infty} \frac{\ln j}{t_j} < +\infty \iff \sum_{j=1}^{\infty} \frac{\ln t_j}{t_j} < +\infty
\]

\[
\implies \sum_{j=1}^{\infty} \frac{\ln t_j}{j}{t_j} < +\infty \iff \omega \text{ satisfies the mild strong non-quasianalyticity condition.}
\]

**Proof.** The first implication was already pointed out in Remark 3.4. A proof of equivalence \( \sum_{j=1}^{\infty} \frac{\ln j}{t_j} < +\infty \iff \sum_{j=1}^{\infty} \frac{\ln t_j}{t_j} < +\infty \) was given in the comments after [9], Corollary 1.9 (Page 92). The second implication is trivial, while the last equivalence is (i) \( \iff \) (iv) in [10], Lemma 2.1.

We will need the next calculus lemma:

**Lemma 3.7.** Let \( \alpha, \gamma : (0, +\infty) \to (0, +\infty) \) be two functions such that

\[
\frac{\gamma(t)}{\alpha(t)} \geq e, \quad t > 0.
\]

(i) If \( \alpha \) and \( \gamma \) are increasing, then also the function

\[
(0, +\infty) \ni t \mapsto \alpha(t) \ln \frac{\gamma(t)}{\alpha(t)} \in (0, +\infty)
\]

is increasing.

(ii) If \( \alpha \) and \( \gamma \) are twice differentiable and concave, then also the function

\[
(3.10)
\]

is twice differentiable and concave.

**Proof.** For (i). Assume that \( \alpha, \gamma \) are increasing and let \( 0 < t_1 < t_2 \) be arbitrary. Then

\[
\alpha(t_1) \ln \frac{\gamma(t_1)}{\alpha(t_1)} \leq \alpha(t_1) \ln \frac{\gamma(t_2)}{\alpha(t_1)} = \gamma(t_2) \left( \frac{\alpha(t_1)}{\gamma(t_2)} \ln \frac{\gamma(t_2)}{\alpha(t_1)} \right).
\]
Theorem 3.8. For (ii). Assume that \(\alpha, \gamma\) are twice differentiable and concave, hence \(\alpha'', \gamma'' \leq 0\). Function (3.10) is clearly twice differentiable, its first derivative at \(t > 0\) is
\[
\alpha'(t) \ln \frac{\gamma(t)}{\alpha(t)} + \frac{\gamma'(t)\alpha(t) - \alpha'(t)\gamma(t)}{\gamma(t)},
\]
while its second derivative at \(t > 0\) is
\[
\alpha''(t) \left(\ln \frac{\gamma(t)}{\alpha(t)} - 1\right) + \frac{\gamma''(t)\alpha(t) - \left(\gamma'(t)\alpha(t) - \alpha'(t)\gamma(t)\right)^2}{\alpha(t)\gamma(t)^2}.
\]
Since \(\alpha''(t) \gamma''(t) \leq 0\) and \(\ln \frac{\gamma(t)}{\alpha(t)} - 1 \geq \ln e - 1 = 0\), the second derivative is \(-\) at all \(t > 0\).

The next theorem is a slightly extended version of [10], Theorem 2.2. For its proof we adapted the proof of [10], Theorem 2.2.

**Theorem 3.8.** Let us assume that \(\omega \in \Omega\) satisfies the mild strong non-quasianalyticity condition. Then there exists some \(\rho \in \Omega\) satisfying
\[
|\omega(t)| \leq c_0 |\rho(t)|, \quad t \in \mathbb{R}
\]
with \(c_0 > 0\) a constant, such that:

If \(f\) is an entire function and
\[
|f(z)| \leq d_0 |\omega(|z|)^{n_0}|, \quad z \in \mathbb{C}
\]
for some integer \(n_0 \geq 1\) and \(d_0 > 0\), then there are constants \(c, c' > 0\) such that
\[
\sup_{s \in \mathbb{R}, |s-t| \leq c \ln |\rho(t)| + c'} \ln |f(s)| \geq -c \ln |\rho(t)| - c', \quad t \in \mathbb{R}.
\]
Moreover, if \(\omega \in \Omega_0\), then we can choose \(\rho \in \Omega_0\).

**Proof.** In the case of a general \(\omega \in \Omega\), let \(\alpha\) denote the function
\[
(0, +\infty) \ni t \mapsto \ln |\omega(t)| \in (0, +\infty).
\]
In the case of \(\omega \in \Omega_0\) we need for \(\alpha\) an infinitely differentiable, increasing, concave function satisfying
\[
\int_1^{+\infty} \frac{\alpha(t)}{t^2}\,dt < +\infty, \quad \int_1^{+\infty} \frac{\alpha(t)}{t^2} \ln \frac{t}{\alpha(t)}\,dt < +\infty
\]
and \(\ln |\omega(t)| \leq \alpha(t), t > 0\). To obtain it, let
be such that \( \omega(z) = \prod_{j=1}^{\infty} \left( 1 + \frac{iz}{t_j} \right), \ z \in \mathbb{C}, \) and set
\[
\alpha(t) := \ln 3 + 2 \ln \left( 1 + \sum_{k=1}^{\infty} \frac{(4t)^k}{t_1 \ldots t_k} \right), \quad t > 0.
\]
Then \((0, +\infty) \ni t \mapsto -\alpha(t) \in (0, +\infty)\) is infinitely differentiable, increasing and, according to Lemma 4.2, concave. On the other hand, by [9], Lemma 1.7, we have
\[
\ln |\omega(t)| \leq \alpha(t), \quad t > 0.
\]
Finally, since by [9], Lemma 1.7,
\[
\alpha(t) \leq \ln 3 + 2 \left( \ln 4 + 2 \ln |\omega(8t)| \right) = \ln(48) + 4 \ln |\omega(8t)|, \quad t > 0,
\]
and \(\omega\) satisfies the mild strong non-quasianalyticity condition, Lemma 5.1 yields (3.14).

An inspection of the proof of [10], Corollary 1.2 shows that there exists a constant \(\lambda > 0\) such that
\[
1 + t^{\lambda \alpha(2et)} > 8e, \quad t > 0,
\]
and the function
\[
\beta : (0, +\infty) \ni t \mapsto 6 \alpha(2et) \ln \frac{1 + t}{\alpha(2et)} + 8 \sum_{j=1}^{\infty} \alpha(2^j et) 4^j \in (0, +\infty),
\]
which is, according to Lemma 3.7, increasing and, in the case of \(\omega \in \Omega_0\), also concave, satisfies
\[
\int_{1}^{+\infty} \frac{\beta(t)}{t^2} \, dt < +\infty \text{ and has the property:}
\]
If \(f\) is any entire function satisfying
\[
\ln |f(z)| \leq d \alpha(|z|) + d', \quad z \in \mathbb{C}
\]
for some \(d, d' > 0\), then there exist constants \(c_1, c'_1 > 0\) such that
\[
\sup_{t \leq r \leq t + c_1 \alpha(t)} \inf_{|z| = r} \ln |f(z)| \geq -c_1 \beta(t) - c'_1, \quad t > 0.
\]
We notice that \(\beta(t) \geq 6 \alpha(2et) \ln(8e) > \alpha(t)\) for all \(t > 0\).

Now, according to a result of O. I. Inozemcev and V. A. Marcenko ([16], Theorem 1, see also [9], Theorem 1.6), there exist \(\rho \in \Omega\) and a constant \(d_1 > 0\) such that
\[
\beta(t) \leq \ln |\rho(t)| + d_1, \quad t > 0.
\]
Moreover, in the case of \(\omega \in \Omega_0\), when \(\beta\) is increasing and concave, Theorem 4.4 ensures that \(\rho\) can be chosen belonging to \(\Omega_0\).

Since
\[
|\omega(t)| \leq e^{\alpha(t)} < e^{\beta(t)} \leq e^{\ln |\rho(t)| + d_1} = e^{d_1} |\rho(t)|, \quad t < 0,
\]
(3.11) holds true with \(c_0 = e^{d_1}\).
Let $f$ be an entire function satisfying (3.12). Then
\[ \ln |f(z)| \leq n_0 \ln |\omega(|z|)| + \ln d_0 \leq n_0 \alpha(|z|) + \ln d_0, \quad z \in \mathbb{C}. \]
By the choice of $\beta$ there exist then constants $c_1, c'_1 > 0$ such that
\[ \sup_{t \leq r \leq t + c_1 \alpha(t)} \inf_{|z| = r} \ln |f(z)| \geq -c_1 \beta(t) - c'_1, \quad t > 0. \]
It follows for every $t \in \mathbb{R}$
\[ \sup_{s \in \mathbb{R}} \ln |f(s)| \geq \sup_{|s-t| \leq c_1 \ln |\rho(t)| + c'_1 + c_1 d_1} \inf_{|z| = r} \ln |f(z)| \geq -c_1 \beta(|t|) - c'_1 \]
\[ \geq -c_1 |\rho(t)| - c'_1 - c_1 d_1 \]
\[ = -c_1 |\rho(t)| - c'_1 - c_1 d_1. \]
Consequently (3.13) holds true with $c = c_1$ and $c' = c'_1 + c_1 d_1$.

Theorem 3.8 implies that mild strong non-quasianalyticity of $\omega \in \Omega$ is a sufficient condition in order that every $\omega$-ultradifferential operator with constant coefficients and of convergence type be invertible in $D^\rho_\rho$ for some $\rho \in \Omega$ satisfying $|\omega(t)| \leq c_0|\rho(t)|, t \in \mathbb{R}$, with $c_0 > 0$ a constant. This is the statement of [10], Proposition 2.7:

**Theorem 3.9.** If $\omega \in \Omega$ is satisfying the mild strong non-quasianalyticity condition, then there exist $\rho \in \Omega$ and a constant $c_0 > 0$ with
\[ |\omega(t)| \leq c_0 |\rho(t)|, \quad t \in \mathbb{R}, \]
such that every $\omega$-ultradifferential operator with constant coefficients and of convergence type is invertible in $D^\rho_\rho$.

Moreover, if $\omega \in \Omega_0$, then we can choose $\rho \in \Omega_0$.

**Proof.** Choose $\rho$ and $c_0$ as in Theorem 3.8.
According to Propositions 3.1 and 3.2, every $\omega$-ultradifferential operator with constant coefficients and of convergence type is of the form $f(D)$ with $f$ an entire function satisfying condition (i) in Proposition 3.2. By the choice of $\rho$ and $c_0$, there exist constants $c, c' > 0$ such that (3.13) is satisfied.
Applying now Proposition 3.5 we conclude that $f(D)$ is invertible in $D^\rho_\rho$. \( \square \)

The main result of this paper is the following theorem, which shows that Theorem 3.8 is sharp. It will be proved in Section 6.
Theorem 3.10. Let us assume that $\omega \in \Omega$ does not satisfy the mild strong non-quasianalyticity condition, that is such that
\[
\int_{1}^{+\infty} \frac{\ln |\omega(t)|}{t^2} \frac{\ln t}{\ln |\omega(t)|} \, dt = +\infty.
\]
Then there exists an entire function $f$ such that
\[
|f(z)| \leq |\omega(|z|)|^2, \quad z \in \mathbb{C}
\]
but for no increasing $\beta : (0, +\infty) \to (0, +\infty)$ with
\[
\int_{1}^{+\infty} \frac{\beta(t)}{t^2} \, dt < +\infty
\]
can hold the condition
\[
\sup_{s \in \mathbb{R}} \ln |f(s)| \geq -\beta(t), \quad t > 0.
\]

Using Theorem 3.10, we infer that also Theorem 3.9 is sharp:

Theorem 3.11. Let us assume that $\omega \in \Omega$ does not satisfy the mild strong non-quasianalyticity condition. Then there exists some $\omega$-ultradifferential operator with constant coefficients and of convergence type, which is not invertible in $D_{\rho}'$ for any $\rho \in \Omega$ satisfying
\[
|\omega(t)| \leq c_0 |\rho(t)|, \quad t \in \mathbb{R}
\]
for some constant $c_0 > 0$.

Proof. Let $f$ be an entire function $f$ as in Theorem 3.10. Then, according to Propositions 3.1 and 3.2, we can consider the $\omega$-ultradifferential operator with constant coefficients $f(D)$, and it is of convergence type.

If it would exist some $\rho \in \Omega$ satisfying
\[
|\omega(t)| \leq c_0 |\rho(t)|, \quad t \in \mathbb{R}
\]
with $c_0 > 0$ a constant, such that $f(D)$ is invertible in $D_{\rho}'$, then Proposition 3.5 would imply the existence of constants $c, c' > 0$ such that
\[
\sup_{s \in \mathbb{R}} \ln |f(s)| \geq -c \ln |\rho(t)| - c', \quad t \in \mathbb{R}.
\]
But this is not possible because $\beta : (0, +\infty) \ni t \mapsto c \ln |\rho(t)| + c' \in (0, +\infty)$ would be a function with
\[
\int_{1}^{+\infty} \frac{\beta(t)}{t^2} \, dt < +\infty
\]
(see e.g. [16], Theorem 1 or [9], Theorem 1.6) such that
\[
\sup_{s \in \mathbb{R}} \ln |f(s)| \geq -\beta(t), \quad t \in \mathbb{R},
\]
in contradiction with the choice of $f$. 

\[ \square \]

4. On the non-quasianalyticity condition

For sake of convenience, we will say that a Lebesgue measurable function $\alpha : (0, +\infty) \rightarrow (0, +\infty)$ satisfies the non-quasianalyticity condition if

\[ \int_1^{+\infty} \frac{\alpha(t)}{t^2} \, dt < +\infty. \]

This denomination is suggested by the classical Denjoy-Carleman Theorem (see e.g. [19], 4.III) in which non-quasianalyticity is characterized by this condition.

Examples of functions satisfying the non-quasianalyticity condition:

**Remark 4.1.** If $0 < t_1 \leq t_2 \leq t_3 \leq \ldots \leq +\infty$, $t_1 < +\infty$, $\sum_{j=1}^{\infty} \frac{1}{t_j} < +\infty$,

then the increasing functions

1. $(0, +\infty) \ni t \mapsto n(t)$ with $n(t)$ the number of the elements of the set $\{k \geq 1; t_k \leq t\}$ (the distribution function of the sequence $(t_j)_{j \geq 1}$),

2. $(0, +\infty) \ni t \mapsto \alpha(t) := \ln \left(1 + \sum_{k=1}^{\infty} \frac{t^k}{t_1 \ldots t_k}\right) \in (0, +\infty)$,

3. $(0, +\infty) \ni t \mapsto N(t) := \ln \max \left(1, \sup_{k \geq 1} \frac{t^k}{t_1 t_2 \ldots t_k}\right) \in (0, +\infty)$,

4. $(0, +\infty) \ni t \mapsto \ln |\omega(t)| \in (0, +\infty)$, where $\omega \in \Omega$ is defined by

\[ \omega(z) = \prod_{j=1}^{\infty} \left(1 + \frac{iz}{t_j}\right), \quad z \in \mathbb{C}, \]

satisfy the non-quasianalyticity condition.

Since

\[ \int_{t_1}^{t_{k+1}} n(t) \frac{dt}{t^2} = \sum_{j=1}^{k} \int_{t_j}^{t_{j+1}} n(t) \frac{dt}{t^2} = \sum_{j=1}^{k} \left( \frac{1}{t_j} - \frac{1}{t_{j+1}} \right) = \left( \sum_{j=1}^{k} \frac{1}{f_j} \right) - \frac{k}{t_{k+1}}, \]

we have

\[ \int_{t_1}^{+\infty} n(t) \frac{dt}{t^2} \leq \sum_{j=1}^{\infty} \frac{1}{f_j} < +\infty. \]

A proof of the non-quasianalyticity of $\ln |\omega(\cdot)|$ can be found, for example, in the proof of implication (iii) $\Rightarrow$ (i) in [9], Theorem 1.6.

The non-quasianalyticity of $N(\cdot)$ follows from the non-quasianalyticity of $\ln |\omega(\cdot)|$ and the clear inequality $N(t) \leq \ln |\omega(t)|$, $t > 0$.

Finally, the non-quasianalyticity of $\alpha$ is consequence of the inequality

\[ 1 + \sum_{k=1}^{\infty} \frac{t^k}{t_1 \ldots t_k} = 1 + \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{t^k}{(t_1/2) \ldots (t_k/2)}. \]
and of the non-quasianalyticity of \( N(\cdot) \) with \( t_k \) replaced by \( t_k/2 \).

The goal of this section is to show, how we can majorize functions of a certain regularity, satisfying the non-quasianalyticity condition, with more regular or more explicit functions, still satisfying the non-quasianalyticity condition. We consider three function groups:

- **Increasing functions** \( (0, +\infty) \rightarrow (0, +\infty) \).
- "Concave like functions" \( \alpha : (0, +\infty) \rightarrow (0, +\infty) \), which can be
  1. concave: \( \alpha((1-\lambda)t_1 + \lambda t_2) \geq (1-\lambda)\alpha(t_1) + \lambda\alpha(t_2) \) for \( 0 \leq \lambda \leq 1 \) and \( t_1 t_2 > 0 \);
  2. such that \( (0, +\infty) \ni t \mapsto \alpha(t) \) is decreasing;
  3. subadditive: \( \alpha(t_1 + t_2) \leq \alpha(t_1) + \alpha(t_2) \) for \( t_1, t_2 > 0 \).

We notice that 1) \( \Rightarrow \) 2) \( \Rightarrow \) 3). Indeed, if \( \alpha \) is concave and \( 0 < t_1 < t_2 \) are arbitrary, then we have for any \( 0 < \varepsilon < t_1 \)

\[
\alpha(t_1) = \alpha\left(\frac{t_2 - t_1}{t_2 - \varepsilon} t_1 + \frac{t_1 - \varepsilon}{t_2 - \varepsilon} t_2\right) \geq \frac{t_2 - t_1}{t_2 - \varepsilon} \alpha(t_1) + \frac{t_1 - \varepsilon}{t_2 - \varepsilon} \alpha(t_2) \geq \frac{t_1 - \varepsilon}{t_2 - \varepsilon} \alpha(t_2).
\]

Letting \( \varepsilon \rightarrow 0 \) we conclude that \( \alpha(t_1) \geq \frac{t_1}{t_2} \alpha(t_2) \iff \frac{\alpha(t_1)}{t_1} \geq \frac{\alpha(t_2)}{t_2} \).

On the other hand, if \( (0, +\infty) \ni t \mapsto \alpha(t) \) is decreasing, then we have for all \( t_1, t_2 > 0 \):

\[
\alpha(t_1 + t_2) = t_1 \frac{\alpha(t_1 + t_2)}{t_1 + t_2} + t_2 \frac{\alpha(t_1 + t_2)}{t_1 + t_2} \leq t_1 \frac{\alpha(t_1)}{t_1} + t_2 \frac{\alpha(t_2)}{t_2} = \alpha(t_1) + \alpha(t_2).
\]

- \( (0, +\infty) \ni t \mapsto \ln |\omega(t)| \) with \( \omega \) an entire function belonging to \( \Omega \) or \( \Omega_0 \).

The next lemma extends [9], Lemma 1.7:

**Lemma 4.2.** If

\[
0 < t_1 \leq \frac{t_2}{2} \leq \frac{t_3}{3} \leq \ldots \leq +\infty, \quad t_1 < +\infty,
\]

then the function

\[
\alpha : (0, +\infty) \ni t \mapsto \ln \left(1 + \sum_{k=1}^{\infty} \frac{t^k}{t_1 \ldots t_k}\right) \in (0, +\infty)
\]
is strictly increasing and concave. Assuming additionally that \( \frac{t_k}{k} \neq \frac{t_{k+1}}{k+1} \) for at least one \( k \geq 1 \), \( \alpha \) turns out to be even strictly concave.

**Proof.** \( \alpha \) is clearly strictly increasing.

For the proof of the concavity it is convenient to denote \( c_k = \frac{t_k}{k} \), \( k \geq 1 \). Then \( c_1 \geq c_2 \geq c_3 \geq \ldots \geq 0 \), \( c_1 > 0 \) and

\[
\alpha(t) = \ln \left( \sum_{k=0}^{\infty} \left( \prod_{j=1}^{k} c_j \right) \frac{t^k}{k!} \right) \quad t > 0,
\]

where we agree that \( \prod_{j=1}^{k} c_j = 1 \) for \( k = 0 \).

If \( c_k = \frac{t_k}{k} = \frac{t_{k+1}}{k+1} = c_{k+1} \) for all \( k \geq 1 \), then \( \alpha(t) = \ln e^{c_1 t} = c_1 t \), so \( \alpha \) is linear, hence concave. We will show that, assuming \( c_k > c_{k+1} \) for some \( k \geq 1 \), \( \alpha \) is strictly concave, by proving that \( \alpha''(t) < 0 \) for all \( t > 0 \). Let \( k_0 \) denote the least integer \( k \geq 1 \) for which \( c_k > c_{k+1} \).

Denoting \( f(t) = \sum_{k=0}^{\infty} \left( \prod_{j=1}^{k} c_j \right) \frac{t^k}{k!} \), we have

\[
\alpha''(t) = \left( \ln f(t) \right)'' = \left( \frac{f'(t)}{f(t)} \right)' = \frac{f''(t)f(t) - f'(t)^2}{f(t)^2}.
\]

Therefore out task is to prove that \( f(t)^2 - f''(t)f(t) > 0 \) for all \( t > 0 \).

Computation yields \( f'(t) = \sum_{k=0}^{\infty} \left( \prod_{j=1}^{k} c_j \right) \frac{t^k}{k!} \), \( f''(t) = \sum_{k=0}^{\infty} \left( \prod_{j=1}^{k} c_j \right) \frac{t^k}{k!} \) and

\[
f(t)^2 - f''(t)f(t) = \sum_{k=0}^{\infty} \left( \sum_{p,q \geq 0 \atop p+q=k} \frac{1}{plq!} \left( \prod_{j=1}^{p+1} c_j \right) \left( \prod_{j=1}^{q+1} c_j \right) - \sum_{p,q \geq 0 \atop p+q=k} \frac{1}{plq!} \left( \prod_{j=1}^{p+2} c_j \right) \left( \prod_{j=1}^{q} c_j \right) \right) t^k.
\]

Hence the proof will be done once we show that

\[
C_k := \sum_{p,q \geq 0 \atop p+q=k} \frac{1}{plq!} \left( \prod_{j=1}^{p+1} c_j \right) \left( \prod_{j=1}^{q+1} c_j \right) - \sum_{p,q \geq 0 \atop p+q=k} \frac{1}{plq!} \left( \prod_{j=1}^{p+2} c_j \right) \left( \prod_{j=1}^{q} c_j \right) \geq 0
\]

for all \( k \geq 0 \), and \( C_{k_0-1} > 0 \).

Since \( C_0 = c_1 c_1 - c_1 c_2 = c_1 (c_1 - c_2) \geq 0 \), where the inequality is strict if \( k_0 = 1 \), it remains that we prove that \( C_k \geq 0 \) for all \( k \geq 1 \) and \( C_{k_0-1} > 0 \) if \( k_0 \geq 2 \).

For each \( k \geq 1 \), using
and
\[ \sum_{p,q \geq 0 \atop p+q=k} \frac{1}{plq!} \left( \prod_{j=1}^{p+q} c_j \right) = \frac{1}{k!} \prod_{j=1}^{k+1} c_j + \sum_{p \geq 1, q \geq 0 \atop p+q=k} \frac{1}{plq!} \left( \prod_{j=1}^{p} c_j \right) \left( \prod_{j=1}^{q} c_j \right) \]
we obtain
\[ (4.1) \quad C_k = \frac{1}{k!} \left( c_1 - c_{k+2} \right) \prod_{j=1}^{k+1} c_j + S_k, \]
where
\[ (4.2) \quad S_k = \sum_{p \geq 1, q \geq 0 \atop p+q=k} \left( \frac{1}{plq!} - \frac{1}{(p-1)!(q+1)!} \right) \left( \prod_{j=1}^{p+1} c_j \right) \left( \prod_{j=1}^{q+1} c_j \right). \]
Therefore it is enough to show that \( S_k \geq 0 \) for all \( k \geq 1 \). Indeed, then (4.1) yields \( C_k \geq 0 \) for all \( k \geq 1 \). Moreover, if \( k_0 \geq 2 \) and so \( k_0 - 1 \geq 1 \), then (4.1) and \( c_1 - c_{k_0+1} \geq c_{k_0} - c_{k_0+1} > 0 \) yield also
\[ C_{k_0-1} = \frac{1}{(k_0-1)!} \left( c_1 - c_{k_0+1} \right) \prod_{j=1}^{k_0} c_j + S_{k_0-1} > S_{k_0-1} \geq 0. \]
Direct computation shows that \( S_k \geq 0 \) for \( 1 \leq k \leq 5 \):
\[ S_1 = 0, \quad S_2 = \frac{1}{2} c_2^2 c_2 (c_1 - c_3) \geq 0, \quad S_3 = \frac{2}{3} c_2^2 c_3 c_2 (c_2 - c_4) \geq 0, \]
\[ S_4 = \frac{1}{8} c_1^2 c_2 c_3 c_2 (c_2 - c_3) + \frac{1}{12} c_1^2 c_2^2 c_3 (c_3 - c_4) \geq 0, \]
\[ S_5 = \frac{1}{30} c_1^2 c_2 c_3 c_4 (c_2 - c_5) + \frac{1}{24} c_1^2 c_2^2 c_3 c_4 (c_3 - c_5) \geq 0. \]
It remains to show that \( S_k \geq 0 \) for all \( k \geq 6 \).

Let in the sequel the integer \( k \geq 6 \) be arbitrary. For \( p = \frac{k+1}{2} \) (what can happen only for odd \( k \)) we have
\[ \frac{1}{plq!} - \frac{1}{(p-1)!(q+1)!} = 0, \]

hence
\[ S_k = \sum_{1 \leq p \leq k/2 \atop p+q=k} \left( \frac{1}{plq!} - \frac{1}{(p-1)!(q+1)!} \right) \left( \prod_{j=1}^{p+1} c_j \right) \left( \prod_{j=1}^{q+1} c_j \right). \]
Denoting by \( p_0 \) the unique integer for which \( \frac{k-1}{2} \leq p_0 \leq \frac{k}{2} \), it follows:

\[
S_k = \sum_{1 \leq p \leq p_0-1 \atop p+q=k} \left( \frac{1}{p!q!} - \frac{1}{(p-1)!(q+1)!} \right) \left( \prod_{j=1}^{p+1} c_j \right) \left( \prod_{j=1}^{q+1} c_j \right) + \frac{1}{p_0!(k-p_0)!} - \frac{1}{(p_0-1)!(k-p_0+1)!} \left( \prod_{j=1}^{p_0+1} c_j \right) \left( \prod_{j=1}^{k-p_0+1} c_j \right) + \sum_{1 \leq p \leq p_0-1 \atop p+q=k} \left( \frac{1}{p!q!} - \frac{1}{(p+1)!(q-1)!} \right) \left( \prod_{j=1}^{p+1} c_j \right) \left( \prod_{j=1}^{q+1} c_j \right) + \left( \frac{1}{k!} - \frac{1}{(k-1)!} \right) c_1 \left( \prod_{j=1}^{k+1} c_j \right) = \left( \frac{1}{p_0!(k-p_0)!} - \frac{1}{(p_0-1)!(k-p_0+1)!} \right) \left( \prod_{j=1}^{p_0+1} c_j \right) \left( \prod_{j=1}^{k-p_0+1} c_j \right) + \left( \frac{1}{k!} - \frac{1}{(k-1)!} \right) c_1 \left( \prod_{j=1}^{k+1} c_j \right) + \sum_{1 \leq p \leq p_0-1 \atop p+q=k} \left( \frac{2}{p!q!} - \frac{1}{(p-1)!(q+1)!} - \frac{1}{(p+1)!(q-1)!} \right) \left( \prod_{j=1}^{p+1} c_j \right) \left( \prod_{j=1}^{q+1} c_j \right).
\]

Set

\[
d_{k,p_0} := \frac{1}{p_0!(k-p_0)!} - \frac{1}{(p_0-1)!(k-p_0+1)!} = \frac{k-2p_0+1}{p_0!(k-p_0+1)!} > 0
\]
and, for \( 1 \leq p \leq p_0 - 1 \),
\[
d_{k,p} := \frac{2}{p!(k-p)!} - \frac{1}{(p-1)!(k-p+1)!} - \frac{1}{(p+1)!(k-p-1)!}.
\]
Then
\[
S_k = d_{k,p_0} \left( \prod_{j=1}^{p_0+1} c_j \right) \left( \prod_{j=1}^{k-p_0+1} c_j \right) - \left( \frac{1}{(k-1)!} - \frac{1}{k!} \right) c_1 \left( \prod_{j=1}^{k+1} c_j \right)
\]
\[
+ \sum_{1 \leq p \leq p_0 - 1} d_{k,p} \left( \prod_{j=1}^{p+1} c_j \right) \left( \prod_{j=1}^{k-p+1} c_j \right).
\]

(4.3)

It is easy to see that
\[
d_{k,p} \geq 0 \text{ if } p \geq \frac{k - \sqrt{k + 2}}{2}, \quad d_{k,p} < 0 \text{ if } p < \frac{k - \sqrt{k + 2}}{2}.
\]

Let \( p_1 \) denote the unique integer for which
\[
\frac{k - \sqrt{k + 2}}{2} \leq p_1 < \frac{k - \sqrt{k + 2}}{2} + 1.
\]
If \( k = 6 \), then \( p_0 = 3 \) and \( p_1 = 2 \), while if \( k \geq 7 \), then
\[
2 \leq \frac{k - \sqrt{k + 2}}{2} \leq \frac{k - 1}{2} \leq p_1 < \frac{k - \sqrt{k + 2}}{2} \leq \frac{k - 1}{2} + 1 \leq p_0.
\]
Thus we always have \( 2 \leq p_1 \leq p_0 - 1 \).

Since the function
\[
\{0, 1, 2, \ldots, p_0\} \ni p \mapsto \left( \prod_{j=1}^{p+1} c_j \right) \left( \prod_{j=1}^{k-p+1} c_j \right) = \left( \prod_{j=1}^{p+1} c_j \right) \left( \prod_{j=p+2}^{k+1} c_j \right)
\]
is increasing and, according to (4.4),
\[
d_{k,p} \geq 0 \text{ if } p \geq p_1, \quad d_{k,p} < 0 \text{ if } p \leq p_1 - 1,
\]
we deduce
\[
d_{k,p} \left( \prod_{j=1}^{p+1} c_j \right) \left( \prod_{j=1}^{k-p+1} c_j \right) \geq d_{k,p} \left( \prod_{j=1}^{p_1+1} c_j \right) \left( \prod_{j=1}^{k-p_1+1} c_j \right), \quad 1 \leq p \leq p_0 - 1.
\]
We have also
\[
\left( \prod_{j=1}^{p_0+1} c_j \right) \left( \prod_{j=1}^{k-p_0+1} c_j \right) \geq \left( \prod_{j=1}^{p_1+1} c_j \right) \left( \prod_{j=1}^{k-p_1+1} c_j \right),
\]
\[
c_1 \left( \prod_{j=1}^{k+1} c_j \right) \leq \left( \prod_{j=1}^{p_1+1} c_j \right) \left( \prod_{j=1}^{k-p_1+1} c_j \right),
\]
so (4.3) yields
\[
S_k \geq \left( d_{k,p_0} - \frac{1}{(k-1)!} - \frac{1}{k!} \right) + \sum_{1 \leq p \leq p_0 - 1} d_{k,p} \left( \prod_{j=1}^{p_1+1} c_j \right) \left( \prod_{j=1}^{k-p_1+1} c_j \right)
\]
(4.5)
In order to compute the sum
\[ s_k := d_{k,p_0} - \left( \frac{1}{(k-1)!} - \frac{1}{k!} \right) + \sum_{1 \leq p \leq p_0 - 1} d_{k,p}, \]
we notice that, according to (4.3), \( s_k \) is equal to \( S_k \) with \( c_1 = c_2 = \cdots \).
Computing \( S_k \) in this case by using the formula (4.2) instead of (4.3), we obtain
\[ s_k = \sum_{p=1}^{k} \left( \frac{1}{p!(k-p)!} - \frac{1}{(p-1)!(k-p+1)!} \right). \]
But this is a telescoping sum, hence it is equal to \( \frac{1}{k!0!} - \frac{1}{0!0!} k! = 0 \).

Using now (4.5), we deduce the desired result: \( S_k \geq 0 \).

The next majorization theorem is essentially [16], Theorem 1 and [9], Theorem 1.6, claiming that any increasing function, which satisfies the non-quasianalyticity condition, can be majorized by some function \( c + \ln |\omega(\cdot)| \) with \( \omega \in \Omega \) and \( c \geq 0 \) a constant:

**Theorem 4.3.** For \( f : (0, +\infty) \to (0, +\infty) \) the following conditions are equivalent:

(i) \( f(t) \leq \alpha(t), t > 0 \), for \( \alpha : (0, +\infty) \to (0, +\infty) \) some increasing function satisfying the non-quasianalyticity condition.

(ii) There exist
\[ 0 < t_1 \leq t_2 \leq t_3 \leq \ldots \leq +\infty, \quad t_1 < +\infty, \quad \sum_{j=1}^{\infty} \frac{1}{t_j} < +\infty \]
and a constant \( c \geq 0 \), such that
\[ f(t) \leq c + \ln \max \left( 1, \sup_{k \geq 1} \frac{1}{t_1 t_2 \cdots t_k} \right), \quad t > 0. \]

(iii) \( f(t) \leq c + \ln |\omega(t)|, t > 0 \), for some \( \omega \in \Omega \) and constant \( c \geq 0 \).

A necessary condition that \( f \) satisfies the above equivalent conditions is
\[ \lim_{t \to +\infty} \frac{f(t)}{t} = 0. \]

**Proof.** The equivalences (i) ⇔ (ii) ⇔ (iii) are immediate consequences of the corresponding equivalences in [9], Theorem 1.6.

Also the necessary condition (4.6) is well-known. Here is a short proof of it:

Let \( \alpha \) be as in (i). Then
\[ 0 \leq \frac{f(t)}{t} \leq \frac{\alpha(t)}{t} = \int_{t}^{+\infty} \frac{\alpha(t)}{s^2} ds \leq \int_{t}^{+\infty} \frac{\alpha(s)}{s^2} ds \xrightarrow{t \to +\infty} 0. \]

□
The second majorization theorem is an extended version of [16], Theorem 2 and [9], Theorem 1.8. It claims essentially that Lebesgue measurable positive subadditive functions on \((0, +\infty)\), which are bounded on \((0, 1]\) and satisfy the non-quasianalyticity condition, can be majorized by a continuous, increasing, concave function satisfying the non-quasianalyticity condition, or by a function of the form \(c + \ln |\omega(\cdot)|\) with \(\omega \in \Omega_0\) and \(c \geq 0\) a constant:

**Theorem 4.4.** For \(f : (0, +\infty) \to (0, +\infty)\) the following conditions are equivalent:

(i) \(f(t) \leq \alpha(t), t > 0, \) for \(\alpha : (0, +\infty) \to (0, +\infty)\) some Lebesgue measurable, subadditive function, bounded on \((0, 1]\) and satisfying the non-quasianalyticity condition.

(ii) \(f(t) \leq \alpha(t), t > 0, \) for \(\alpha : (0, +\infty) \to (0, +\infty)\) some continuous function, bounded on \((0, 1]\) and satisfying the non-quasianalyticity condition, such that \((0, +\infty) \ni t \mapsto \frac{\alpha(t)}{t}\) is decreasing.

(iii) \(f(t) \leq \alpha(t), t > 0, \) with \(\alpha : (0, +\infty) \to (0, +\infty)\) some increasing, concave function satisfying the non-quasianalyticity condition.

(iv) \(f(t) \leq \alpha(t), t > 0, \) with \(\alpha : (0, +\infty) \to (0, +\infty)\) some infinitely differentiable, increasing, concave function satisfying the non-quasianalyticity condition.

(v) There exist

\[
0 < t_1 \leq \frac{t_2}{2} \leq \frac{t_3}{3} \leq \ldots \leq +\infty, \quad t_1 < +\infty, \quad \sum_{j=1}^{\infty} \frac{1}{t_j} < +\infty
\]

and a constant \(c \geq 0\), such that

\[
f(t) \leq c + \ln \left(1 + \sum_{k=1}^{\infty} \frac{t^k}{t_1 \ldots t_k}\right), \quad t > 0.
\]

(vi) There exist

\[
0 < t_1 \leq \frac{t_2}{2} \leq \frac{t_3}{3} \leq \ldots \leq +\infty, \quad t_1 < +\infty, \quad \sum_{j=1}^{\infty} \frac{1}{t_j} < +\infty
\]

and a constant \(c \geq 0\), such that

\[
f(t) \leq c + \ln \max \left(1, \sup_{k \geq 1} \frac{t^k}{t_1 t_2 \ldots t_k}\right), \quad t > 0.
\]

(vii) \(f(t) \leq c + \ln |\omega(t)|, t > 0, \) for some \(\omega \in \Omega_0\) and constant \(c \geq 0\).

A necessary condition that \(f\) satisfies the above equivalent conditions is

\[
(4.7) \quad \lim_{t \to +\infty} \frac{f(t) \ln t}{t} = 0.
\]

**Proof.** The equivalences (i) \(\Leftrightarrow\) (vi) \(\Leftrightarrow\) (vii) are immediate consequences of the equivalences (iii) \(\Leftrightarrow\) (ii) \(\Leftrightarrow\) (iv) in [9], Theorem 1.8.

Implications (vi) \(\Rightarrow\) (v) and (iv) \(\Rightarrow\) (iii) are trivial, while implication (v) \(\Rightarrow\) (iv) follows by Lemma [4.2] and Remark [4.1]
Finally, the implications (iii) \(\Rightarrow\) (ii) \(\Rightarrow\) (i) where proved in the discussion before Lemma 4.2.

The necessary condition (4.7) is, like (4.6) in Theorem 4.3, well-known. We provide a short proof of it, essentially reproducing the proof of [4], Corollary 1.2.8:

Let \(\alpha\) be as in (ii). We have for every \(t > 1\)

\[
\int_{\sqrt{t}}^{+\infty} \frac{\alpha(s)}{s^2} ds \geq \int_{\sqrt{t}}^{t} \frac{\alpha(s)}{s^2} ds \geq \int_{\sqrt{t}}^{t} \frac{\alpha(t)}{t} \frac{1}{s} ds = \frac{\alpha(t)}{t} \ln \frac{t}{\sqrt{t}} = \frac{1}{2} \frac{\alpha(t)}{t} \ln t,
\]

so

\[
0 \leq \frac{\alpha(t) \ln t}{t} \leq 2 \int_{\sqrt{t}}^{+\infty} \frac{\alpha(s)}{s^2} ds \xrightarrow{t \to +\infty} 0.
\]

\[\square\]

We notice that (i) \(\Leftrightarrow\) (iii) in Theorem 4.4 was originally proved by A. Beurling (see [2], lemma 1, [4], Theorem 1.2.7, [3], Lemma V), [14], Lemma 3.3). A new feature of Theorem 4.4 consists in the exhibition (thanks to Lemma 4.2) of a rather explicite \(\alpha\) in (iii), obtaining thus the equivalent conditions (iv) and (v).

Clearly, every \(f\), which satisfies the equivalent conditions in Theorem 4.4, satisfies also the equivalent conditions in Theorem 4.3. It is an intriguing question: does it exist \(f\) satisfying the conditions in Theorem 4.3 but not those in Theorem 4.4? The answer is yes:

**Corollary 4.5.** There exists an increasing function \((0, +\infty) \rightarrow (0, +\infty)\) satisfying the non-quasianalyticity condition, which can not be majorized by any Lebesgue measurable, subadditive function on \((0, +\infty)\), which is bounded on \((0, 1]\) and satisfies the non-quasianalyticity condition.

Consequently there exists \(\omega \in \Omega\) such that \(|\omega(\cdot)|\) can not be majorized by a scalar multiple of some \(|\rho(\cdot)|\) with \(\rho \in \Omega_0\).

**Proof.** Let \(e = t_1 < t_2 < t_3 < \ldots\) be a sequence such that \(\sum_{k=1}^{\infty} \frac{1}{\ln t_k} < +\infty\) (for example, \(t_k = e^{k^2}\)). Defining the function \(f : (0, +\infty) \rightarrow (0, +\infty)\) by

\[
f(t) := 0 \quad \text{for} \quad 0 < t < t_1,
\]

\[
f(t) := \frac{t_k}{\ln t_k} \quad \text{for} \quad t_k \leq t < t_{k+1}, k \geq 1,
\]

\(f\) will be increasing and satisfying the non-quasianalyticity condition:

\[
\int_{1}^{+\infty} \frac{f(t)}{t^2} dt = \sum_{k=1}^{\infty} \int_{t_k}^{t_{k+1}} \frac{f(t)}{t^2} dt = \sum_{k=1}^{\infty} \frac{t_k}{\ln t_k} \left( \frac{1}{t_k} - \frac{1}{t_{k+1}} \right) \leq \sum_{k=1}^{\infty} \frac{1}{\ln t_k} < +\infty.
\]
The above defined $f$ cannot be majorized by any Lebesgue measurable, subadditive function on $(0, +\infty)$, which is bounded on $(0, 1]$ and satisfies the non-quasianalyticity condition. Indeed, otherwise (4.7) would hold true by Theorem 4.4, contradicting $f(t_k)\ln t_k = 1, k \geq 1$.

5. The mild strong non-quasianalyticity condition

First at all we notice that if $\alpha, \beta : (0, +\infty) \rightarrow (0, +\infty)$ are increasing, $\int_1^{+\infty} \frac{\alpha(t)}{t^2} dt < +\infty$ and $\lim_{t \to +\infty} \frac{\beta(t)}{t} = 0$, then $\int_1^{+\infty} \frac{\alpha(t)}{t^2} \ln \frac{t}{\beta(t)} dt > -\infty$ is a well defined improper integral. Indeed, if $t_0 \geq 1$ is such that $\frac{\beta(t)}{t} < 1$ for $t \geq t_0$, then $[t_0, +\infty) \ni t \mapsto \frac{\alpha(t)}{t^2} \ln \frac{t}{\beta(t)}$ is a positive Lebesgue measurable function.

In particular, if $\alpha : (0, +\infty) \rightarrow (0, +\infty)$ is an increasing function and $\int_1^{+\infty} \frac{\alpha(t)}{t^2} dt < +\infty$, then $\int_1^{+\infty} \frac{\alpha(t)}{t^2} \ln \frac{t}{\alpha(t)} dt > -\infty$ is a well defined improper integral. Indeed, we have $\lim_{t \to +\infty} \frac{\alpha(t)}{t} = 0$ by Theorem 4.3.

Let us say that an increasing function $\alpha : (0, +\infty) \rightarrow (0, +\infty)$ satisfies the mild strong non-quasianalyticity condition if

$$\int_1^{+\infty} \frac{\alpha(t)}{t^2} dt < +\infty \quad \text{and} \quad \int_1^{+\infty} \frac{\alpha(t)}{t^2} \ln \frac{t}{\alpha(t)} dt < +\infty.$$ 

We notice that, for $\omega \in \Omega$, $(0, +\infty) \ni t \mapsto |\omega(t)| \in (0, +\infty)$ satisfies the mild non-quasianalyticity condition exactly when condition (3.9) is satisfied, that is when $\omega$ satisfies the mild non-quasianalyticity condition as defined in Section 2.

**Proposition 5.1.** Let $\alpha : (0, +\infty) \rightarrow (0, +\infty)$ be an increasing function satisfying the mild strong non-quasianalyticity condition. Then

(i) $c \cdot \alpha$ and $\alpha(L \cdot)$ satisfy the mild strong non-quasianalyticity condition for each $c > 0$ and $L > 0$;

(ii) $\alpha + \beta$ satisfies the mild strong non-quasianalyticity condition for each increasing $\beta : (0, +\infty) \rightarrow (0, +\infty)$ satisfying the mild strong non-quasianalyticity condition;

(iii) any increasing $\beta : (0, +\infty) \rightarrow (0, +\infty), \beta \leq \alpha$, satisfies the mild strong non-quasianalyticity condition.

**Proof.** The proof of (i) is immediate. Also (ii) is easily seen by using that
\[
\frac{\alpha(t) + \beta(t)}{t^2} \ln \frac{t}{\alpha(t) + \beta(t)} \, dt \leq \frac{\alpha(t)}{t^2} \ln \frac{t}{\alpha(t)} \, dt + \frac{\beta(t)}{t^2} \ln \frac{t}{\beta(t)} \, dt.
\]

For the proof of (iii) we notice that, according to Theorem 4.3, there exists a \( t_0 \geq 1 \) such that
\[
\frac{\alpha(t)}{t^2} < \frac{1}{e} \iff \alpha(t) < \frac{t}{e} \text{ for all } t \geq t_0.
\]
Then
\[
\beta(t) \ln \frac{t}{\beta(t)} \leq \alpha(t) \ln \frac{t}{\alpha(t)}, \quad t \geq t_0.
\]
Indeed, \( \left(0, \frac{t}{e}\right) \ni x \mapsto -x \ln \frac{t}{x} \) is increasing and \( 0 < \beta(t) \leq \alpha(t) < \frac{t}{e} \).

At first view, the next characterization of mild strong non-quasianalyticity (more precisely, of its negation) can appear surprising:

**Proposition 5.2.** Let \( \alpha : (0, +\infty) \to (0, +\infty) \) be an increasing function such that
\[
\int_1^{+\infty} \frac{\alpha(t)}{t^2} \ln \frac{t}{\alpha(t)} \, dt < +\infty.
\]

Then the following conditions are equivalent:

(i) \( \alpha \) does not satisfy the mild strong non-quasianalyticity condition, that is
\[
\int_1^{+\infty} \frac{\alpha(t)}{t^2} \ln \frac{t}{\alpha(t)} \, dt = +\infty.
\]

(ii) For any increasing function \( \beta : (0, +\infty) \to (0, +\infty) \) such that
\[
\int_1^{+\infty} \frac{\beta(t)}{t^2} \, dt < +\infty,
\]
we have
\[
\int_1^{+\infty} \frac{\alpha(t)}{t^2} \ln \frac{t}{\beta(t)} \, dt = +\infty.
\]

The above two conditions imply the condition
\[
(iii) \int_1^{+\infty} \frac{\alpha(t)}{t^2} \ln \ln(t) \, dt = +\infty
\]

and, if \( \alpha \) is also subadditive or \( \alpha = \ln |\omega(\cdot)| \) with \( \omega \in \Omega_0 \), then all the above three conditions are equivalent.

**Proof.** Implication (ii) \( \Rightarrow \) (i) is trivial. For (i) \( \Rightarrow \) (ii) : since
\[
\frac{\alpha(t)}{t^2} \ln \frac{t}{\beta(t)} \geq \frac{\alpha(t)}{t^2} \ln \frac{t}{\alpha(t) + \beta(t)} = \frac{\alpha(t)}{t^2} \ln \frac{t}{\alpha(t)} - \frac{\alpha(t)}{t^2} \ln \frac{\alpha(t) + \beta(t)}{\alpha(t)} \geq \frac{\alpha(t)}{t^2} \ln \frac{t}{\alpha(t)} - \frac{\alpha(t)}{t^2} \ln \frac{\alpha(t)}{t^2},
\]
we have
\[
\int_1^{+\infty} \frac{\alpha(t)}{t^2} \ln \frac{t}{\beta(t)} \, dt \geq \int_1^{+\infty} \frac{\alpha(t)}{t^2} \ln \frac{t}{\alpha(t)} \, dt - \int_1^{+\infty} \frac{\alpha(t) + \beta(t)}{t^2} \, dt = +\infty.
\]
(ii)⇒(iii) follows by applying (ii) to
\[
\beta(t) = \begin{cases} 
\frac{t}{(\ln t)^2} & \text{for } t \geq e^2, \\
\frac{t}{4} & \text{for } 0 < t < e^2.
\end{cases}
\]

Finally we prove that, if \( \alpha \) is also subadditive or \( \alpha = \ln |\omega(\cdot)| \) with \( \omega \in \Omega_0 \), then (iii)⇒(i). For we recall that, according to Theorem 4.4,
\[
\lim_{t \to +\infty} \frac{\alpha(t) \ln t}{t} = 0.
\]
Consequently there exists some \( t_0 \geq e \) such that
\[
\frac{\alpha(t) \ln t}{t} < 1 \iff \frac{t}{\alpha(t)} > \ln t \text{ for } t \geq t_0.
\]
We deduce:
\[
\int_{t_0}^{+\infty} \frac{\alpha(t) \ln t}{t^2} \, dt \geq \int_{t_0}^{+\infty} \frac{\alpha(t)}{t^2} \ln(t) \, dt = +\infty.
\]
\[\square\]

The non-quasianalyticity and mild strong non-quasianalyticity conditions for increasing functions can be rewritten in discretized form:

**Proposition 5.3.** Let \( \alpha : (0, +\infty) \to (0, +\infty) \) be an increasing function.

(i) \[\int_1^{+\infty} \frac{\alpha(t)}{t^2} \, dt < +\infty \text{ if and only if } \sum_{j=1}^{\infty} \frac{\alpha(2^j)}{2^j} < +\infty.\]

(ii) Assuming that \[\int_1^{+\infty} \frac{\alpha(t)}{t^2} \, dt < +\infty, \] we have \[\int_1^{+\infty} \frac{\alpha(t)}{t^2} \ln \frac{t}{\alpha(t)} \, dt = +\infty\]
if and only if \[\sum_{j=1}^{\infty} \frac{\alpha(2^j)}{2^j} \ln \frac{2^j}{\beta(2^j)} = +\infty \text{ for any increasing function } \beta : (0, +\infty) \to (0, +\infty) \text{ satisfying } \int_1^{+\infty} \frac{\beta(t)}{t^2} \, dt = +\infty.\]

(iii) \[\int_\epsilon^{+\infty} \frac{\alpha(t)}{t^2} \ln(t) \, dt < +\infty \text{ if and only if } \sum_{j=1}^{\infty} \frac{\alpha(2^j)}{2^j} \ln j < +\infty.\]

**Proof.** (i) follows by noticing that
\[
\int_{2^j}^{2^{j+1}} \frac{\alpha(t)}{t^2} \, dt \leq \alpha(2^{j+1}) \int_{2^j}^{2^{j+1}} \frac{1}{t^2} \, dt = \frac{\alpha(2^{j+1})}{2^{j+1}}
\]
and
\[
\int_{2^j}^{2^{j+1}} \frac{\alpha(t)}{t^2} \, dt \geq \alpha(2^j) \int_{2^j}^{2^{j+1}} \frac{1}{t^2} \, dt = \frac{\alpha(2^j)}{2^{j+1}}.
\]
Let us now assume that \( \int_1^{+\infty} \frac{\alpha(t)}{t^2} \, dt < +\infty \) and \( \int_1^{+\infty} \frac{\alpha(t)}{t^2} \ln \frac{t}{\alpha(t)} \, dt = +\infty \).

Let further \( \beta : (0, +\infty) \rightarrow (0, +\infty) \) be an arbitrary increasing function, such that \( \int_1^{+\infty} \frac{\beta(t)}{t^2} \, dt < +\infty \). Then also \( \beta(2 \cdot) \) is increasing and such that
\[
\int_1^{+\infty} \frac{\alpha(t)}{t^2} \ln \frac{t}{\beta(2t)} \, dt = +\infty.
\]

Since \( 2^{j+1} \int_2^{2^j} \frac{\alpha(t)}{t^2} \ln \frac{t}{\alpha(t)} \, dt \leq \frac{\alpha(2^j)}{2^j} \ln \frac{2^j}{\alpha(2^j)} \int_2^{2^j} \frac{1}{t^2} \, dt = \frac{\alpha(2^j)}{2^j+1} \ln \frac{2^j}{\alpha(2^j+1)}, \)
we obtain
\[
\sum_{j=1}^{+\infty} \frac{\alpha(2^j)}{2^j} \ln \frac{2^j}{\beta(2^j)} \geq \int_1^{+\infty} \frac{\alpha(t)}{t^2} \ln \frac{t}{\beta(2t)} \, dt = +\infty.
\]

Assume now that
\[
\int_1^{+\infty} \frac{\alpha(t)}{t^2} \, dt < +\infty \text{ and } \sum_{j=1}^{+\infty} \frac{\alpha(2^j)}{2^j} \ln \frac{2^j}{\beta(2^j)} = +\infty
\]
for any increasing function \( \beta : (0, +\infty) \rightarrow (0, +\infty) \) with \( \int_1^{+\infty} \frac{\beta(t)}{t^2} \, dt < +\infty \).

Since \( \alpha(2 \cdot) \) is such a function, we have \( \sum_{j=1}^{+\infty} \frac{\alpha(2^j)}{2^j} \ln \frac{2^j}{\alpha(2^j)} = +\infty \). Since
\[
\int_2^{2^{j+1}} \frac{\alpha(t)}{t^2} \ln \frac{t}{\alpha(t)} \, dt \geq \frac{\alpha(2^j)}{2^j} \ln \frac{2^j}{\alpha(2^j)} \int_2^{2^{j+1}} \frac{1}{t^2} \, dt = \frac{\alpha(2^j)}{2^j+1} \ln \frac{2^j}{\alpha(2^j+1)},
\]
we deduce
\[
\int_2^{+\infty} \frac{\alpha(t)}{t^2} \ln \frac{t}{\alpha(t)} \, dt \geq \frac{1}{2} \sum_{j=1}^{+\infty} \frac{\alpha(2^j)}{2^j} \ln \frac{2^j}{\alpha(2^j+1)} = +\infty.
\]

Finally, (iii) follows by using the estimations
\[
\int_2^{2^j} \frac{\alpha(t)}{t^2} \ln \ln(t) \, dt \leq \frac{\alpha(2^{j+1})}{2^{j+1}} \ln \ln(2^{j+1}) \int_2^{2^j} \frac{1}{t^2} \, dt \leq \frac{\alpha(2^{j+1})}{2^{j+1}} \ln(j + 1),
\]
\[
\int_2^{2^{j+1}} \frac{\alpha(t)}{t^2} \ln \ln(t) \, dt \geq \frac{\alpha(2^j)}{2^j} \ln \frac{2^j}{2^{j+1}} \int_2^{2^j} \frac{1}{t^2} \, dt \geq \frac{\alpha(2^j)}{2^{j+1}} \ln \frac{j}{2^j} = \frac{1}{4} \frac{\alpha(2^j)}{2^j} \ln j,
\]
the second one, in which $\ln \ln(2^j) \geq \frac{\ln j}{2}$ was used, valid only for $j \geq 3$.

\[\square\]

The condition for the sequence $(\alpha(2^j))_{j \geq 1}$, formulated in Proposition 5.3 to characterize the negation of the mild strong non-quasianalyticity for an increasing $\alpha : (0, +\infty) \to (0, +\infty)$ satisfying the non-quasianalyticity condition, has an important permanence property which will be used in the next section to prove Theorem 3.10.

**Proposition 5.4.** Let $(a_j)_{j \geq 1}$ be a sequence in $[0, +\infty)$ such that

\[\sum_{j=1}^{\infty} \frac{a_j}{2^j} < +\infty\]

and

\[\sum_{j=1}^{\infty} \frac{a_j}{2^j} \ln \frac{2^j}{\beta(2^j)} = +\infty\]

for any increasing function $\beta : (0, +\infty) \to (0, +\infty)$ with $\int_{1}^{+\infty} \frac{\beta(t)}{t^2} \, dt < +\infty$.

Then the sequence $((a_{j+1} - a_j)^+)_{j \geq 1}$, where

\[\lambda^+ = \begin{cases} 
\lambda & \text{for } \lambda \geq 0, \\
0 & \text{for } \lambda < 0,
\end{cases}\]

has the same two properties.

**Proof.** First of all,

\[\sum_{j=1}^{\infty} \frac{(a_{j+1} - a_j)^+}{2^j} \leq \sum_{j=1}^{\infty} \frac{a_{j+1} + a_j}{2^j} \leq 2 \sum_{j=1}^{\infty} \frac{a_j}{2^j} < +\infty.\]

Now let $\beta : (0, +\infty) \to (0, +\infty)$ be any increasing function satisfying $\int_{1}^{+\infty} \frac{\beta(t)}{t^2} \, dt < +\infty$. By Theorem 4.3, $\lim_{t \to +\infty} \frac{\beta(t)}{t} = 0$, so there is an integer $n_0 \geq 1$ such that $\beta(2^n) \leq 2^n$ for $n \geq n_0$.

For each $n > n_0$,

\[\sum_{j=n_0}^{n} \frac{(a_{j+1} - a_j)^+}{2^j} \ln \frac{2^j}{\beta(2^j)} \geq \sum_{j=n_0}^{n} \frac{a_{j+1} - a_j}{2^j} \ln \frac{2^j}{\beta(2^j)} = -\frac{a_{n_0}}{2^{n_0}} \ln \frac{2^{n_0}}{\beta(2^{n_0})} + \sum_{j=n_0+1}^{n} \frac{a_j}{2^j} \left(2 \ln \frac{2^{j-1}}{\beta(2^{j-1})} - \ln \frac{2^j}{\beta(2^j)}\right) + \frac{a_{n+1}}{2^n} \ln \frac{2^n}{\beta(2^n)}
\]

\[\geq -\frac{a_{n_0}}{2^{n_0}} \ln \frac{2^{n_0}}{\beta(2^{n_0})} + \sum_{j=n_0+1}^{n} \frac{a_j}{2^j} \ln \left(\frac{2^{2j-2}}{\beta(2^{j-1})^2} \cdot \frac{\beta(2^j)}{2^j}\right)\]
\[ \geq - \frac{a_{n_0}}{2^{n_0}} \ln \frac{2^{n_0}}{\beta(2^{n_0})} + \sum_{j=n_0+1}^{n} \frac{a_j}{2^j} \ln \frac{2^{j-2}}{\beta(2^j)}, \]

so

\[ \sum_{j=n_0}^{n} \frac{(a_{j+1} - a_j)^+}{2^j} \ln \frac{2^j}{\beta(2^j)} \]

\[ \geq - \frac{a_{n_0}}{2^{n_0}} \ln \frac{2^{n_0}}{\beta(2^{n_0})} - \sum_{j=n_0+1}^{n} \frac{a_j}{2^j} \ln 4 + \sum_{j=n_0+1}^{n} \frac{a_j}{2^j} \ln \frac{2^j}{\beta(2^j)} = +\infty. \]

We denote

\[ \ln^+ t := \begin{cases} 
\ln t & \text{for } t > 0 \\
0 & \text{for } t \leq 0
\end{cases} \]

The next lemma completes Proposition 3.6:

**Lemma 5.5.** For \( 0 < t_1 \leq t_2 \leq t_3 \leq \ldots \leq +\infty, \ t_1 < +\infty \), are equivalent:

(i) \( \sum_{j=2}^{\infty} \frac{\ln \ln j}{t_j} < +\infty \);

(ii) \( \lim_{j \to \infty} t_j = +\infty \) and \( \sum_{j=1}^{\infty} \frac{\ln \ln t_j}{t_j} < +\infty \).

Moreover, (i) and (ii) imply

(iii) \( \lim_{j \to \infty} \frac{t_j}{j} = +\infty \) and \( \sum_{j=1}^{\infty} \frac{\ln^+ \frac{t_j}{j}}{t_j} < +\infty \),

and, if \( 0 < t_1 \leq \frac{t_2}{2} \leq \frac{t_3}{3} \leq \ldots \), then (i), (ii) and (iii) are all equivalent.

**Proof.** First we prove that (i) implies (ii) and (iii).

Clearly, \( t_j \to +\infty \). For each \( j \geq 3 \), if \( t_j \leq j^2 \) then

\[ \frac{\ln^+ \ln t_j}{t_j} \leq \frac{\ln \ln(j^2)}{t_j} = 2 \frac{\ln \ln j}{t_j}, \]

while if \( t_j > j \), then

\[ \frac{\ln^+ \ln t_j}{t_j} = \frac{\ln \ln t_j}{t_j} < \frac{\ln \ln(j^2)}{j^2} = 2 \frac{\ln \ln j}{j^2}. \]

Therefore

\[ \sum_{j=3}^{\infty} \frac{\ln^+ \ln t_j}{t_j} \leq 2 \sum_{j=3}^{\infty} \frac{\ln \ln j}{t_j} + 2 \sum_{j=3}^{\infty} \frac{\ln \ln j}{j^2} < +\infty. \]
On the other hand, since \( \ln \ln j > 1 \) for all \( j \geq 16 \), we have \( \sum_{j=1}^{\infty} \frac{1}{t_j} < +\infty \).

Furthermore, for each \( j \geq 6 \), if \( \frac{t_j}{j} \leq (\ln j)^2 \) then

\[
\frac{\ln^+ \frac{t_j}{j}}{t_j} \leq \frac{\ln(\ln j)^2}{t_j} = 2 \frac{\ln \ln j}{t_j},
\]

while if \( \frac{t_j}{j} > (\ln j)^2 > e \), then

\[
\frac{\ln^+ \frac{t_j}{j}}{t_j} = \frac{\ln \frac{t_j}{j}}{t_j} < \frac{\ln(\ln j)^2}{(\ln j)^2} = 2 \frac{\ln \ln j}{(\ln j)^2}, \quad \text{hence} \quad \frac{\ln^+ \frac{t_j}{j}}{t_j} < 2 \frac{\ln \ln j}{(\ln j)^2}.
\]

We conclude that

\[
\sum_{j=6}^{\infty} \frac{\ln^+ \frac{t_j}{j}}{t_j} \leq 2 \sum_{j=6}^{\infty} \frac{\ln \ln j}{t_j} + 2 \sum_{j=6}^{\infty} \frac{\ln \ln j}{j(\ln j)^2} < +\infty.
\]

Next we prove implication \( \text{(ii)} \Rightarrow (i) \).

Since \( t_j \rightarrow +\infty \), we have eventually \( \ln^+ \ln t_j \geq 1 \), hence \( \sum_{j=1}^{\infty} \frac{1}{t_j} < +\infty \).

Consequently, (see e.g. [9], Lemma 1.5 (ii)), \( \lim_{j \to \infty} \frac{t_j}{j} = +\infty \). In particular, we have eventually \( j \leq t_j \) and the convergence of \( \sum_{j=2}^{\infty} \frac{\ln \ln j}{t_j} \) follows.

Finally we show that if \( 0 < t_1 \leq \frac{t_2}{2} \leq \frac{t_3}{3} \leq \ldots \), then \( (iii) \Rightarrow (i) \).

Since \( \frac{t_j}{j} \rightarrow +\infty \), we have eventually \( \ln^+ \frac{t_j}{j} \geq 1 \), and thus \( \sum_{j=1}^{\infty} \frac{1}{t_j} < +\infty \).

By [9], Lemma 1.5 (iii) it follows that \( \frac{t_j}{j \ln j} \rightarrow +\infty \). In particular, we have eventually \( \ln j \leq \frac{t_j}{j} \) and the convergence of \( \sum_{j=2}^{\infty} \frac{\ln \ln j}{t_j} \) follows.

We end this section with a summary of several characterizations the mild strong non-quasianalyticity condition for functions of the form \( |\omega(\cdot)| \) with \( \omega \in \Omega \).
Theorem 5.6. For \( 0 < t_1 \leq t_2 \leq t_3 \leq \ldots \leq +\infty \), \( t_1 < +\infty \), \( \sum_{j=1}^{\infty} \frac{1}{t_j} < +\infty \), let us denote
\[
n(t) = \# \{ k \geq 1; t_k \leq t \}, \quad t > 0,
\]
\[
N(t) = \ln \max \left( 1, \sup_{k \geq 1} \frac{t^k}{t_1 t_2 \ldots t_k} \right), \quad t > 0,
\]
\[
\omega(z) = \prod_{j=1}^{\infty} \left( 1 + \frac{iz}{t_j} \right), z \in \mathbb{C}.
\]
Then the following conditions are equivalent:

(i) \( \sum_{j=1}^{+\infty} \frac{\ln t_j}{t_j} < +\infty \);

(ii) \( \int_{1}^{+\infty} \frac{n(t)}{t^2} \ln \frac{t}{n(t)} \, dt < +\infty \);

(iii) \( \int_{1}^{+\infty} \frac{N(t)}{t^2} \ln \frac{t}{N(t)} \, dt < +\infty \);

(iv) \( \int_{1}^{+\infty} \frac{\ln |\omega(t)|}{t^2} \ln \frac{t}{\ln |\omega(t)|} \, dt < +\infty \).

(In the above conditions we take \( 0 \ln \frac{1}{0} = 0 \) when it occurs.)

The above conditions are implied by the next equivalent conditions:

(v) \( \sum_{j=2}^{\infty} \frac{\ln \ln t_j}{t_j} < +\infty \);

(vi) \( \sum_{j=2}^{\infty} \frac{\ln^+ \ln t_j}{t_j} < +\infty \).

Finally, if \( 0 < t_1 \leq \frac{t_2}{2} \leq \frac{t_3}{3} \leq \ldots \), then all the above six conditions are equivalent.

Proof. Statement (i) \( \Leftrightarrow \) (ii) \( \Leftrightarrow \) (iii) \( \Leftrightarrow \) (iv) is \[10\], Lemma 2.1, while (v) \( \Leftrightarrow \) (vi) and the relationship between the above two groups of equivalent conditions is Lemma 5.5.

6. PROOF OF THE NEGATIVE MINIMUM MODULUS THEOREM

In this section we provide a proof for Theorem 3.10, a negative minimum modulus theorem. The idea of the proof, located in the proof of the next
Lemma 6.1 is due to W. K. Hayman ([15]), while the technical execution is based upon the topics of Section 5.

**Lemma 6.1.** Let \( n_1, n_2, \ldots \geq 0 \) be integers such that

\[
\sum_{j=1}^{\infty} \frac{n_j}{2^j} < +\infty
\]

and

\[
\sum_{j=1}^{\infty} \frac{n_j}{2^j} \ln \frac{2^j}{\beta(2^j)} = +\infty
\]

for any increasing \( \beta : (0, +\infty) \to (0, +\infty) \) such that \( \int_{1}^{+\infty} \frac{\beta(t)}{t^2} \, dt < +\infty \).

The formulas

\[
\omega_0(z) = \prod_{j=1}^{\infty} \left(1 + \frac{i|z|}{2^j}\right)^{n_j}, \quad f(z) = \prod_{j=1}^{\infty} \left(1 - \left(\frac{z}{2^j}\right)^2\right)^{n_j}, \quad z \in \mathbb{C}
\]

define a function \( \omega_0 \in \Omega \) and an entire function \( f \) with

\[
|f(z)| \leq \left|\omega_0(|z|)\right|^2, \quad z \in \mathbb{C}, \quad |f(z)| \leq \omega_0(2|z|)^2,
\]

such that there exists no increasing function \( \beta : (0, +\infty) \to (0, +\infty) \) with

\[
\int_{1}^{+\infty} \frac{\beta(t)}{t^2} \, dt < +\infty
\]
satisfying

\[
\sup_{|s-t| \leq \beta(t)} \ln |f(s)| \geq -\beta(t), \quad t > 0.
\]

**Proof.** (6.1) yields \( \omega_0 \in \Omega \) and, since

\[
1 - \left(\frac{z}{2^j}\right)^2 \leq 1 + \left(\frac{|z|}{2^j}\right)^2 = 1 + i\frac{|z|}{2^j},
\]

(6.3) holds true.

For the remaining part of the proof, we need a particular upper estimate of \( |f(z)| \) for \( z \) in the disk of radius \( 2^j \), centered at \( 2^j \).

Let \( j \geq 1 \) be arbitrary. Since \( 2^j \) is a zero of multiplicity \( n_j \) of \( f \), we can apply the general Schwarz’ lemma (see e.g., [20], Chapter XII, §3, Section 2, page 359, or [21], Chapter 9, §2, Exercise 1, page 274), obtaining for any \( z \in \mathbb{C}, |z - 2^j| \leq 2^j, \)

\[
|f(z)| \leq \left(\sup_{|z' - 2^j| = 2^j} |f(z')|\right) \left|\frac{z - 2^j}{2^j}\right|^{n_j}
\]

\[
\leq \left(\sup_{|z' - 2^j| = 2^j} |\omega_0(|z'|)|\right)^2 \left|\frac{z - 2^j}{2^j}\right|^{n_j} = |\omega_0(2^{j+1})| \left(\frac{z - 2^j}{2^j}\right)^{n_j}.
\]

Thus, for each \( 0 < \delta \leq 1, \)

\[
\ln |f(z)| \leq 2 \ln |\omega_0(2^{j+1})| + n_j \ln \delta, \quad z \in \mathbb{C}, |z - 2^j| \leq 2^j \delta.
\]
Now we assume that for some increasing \( \beta : (0, +\infty) \to (0, +\infty) \) with \( \int_1^{+\infty} \frac{\beta(t)}{t^2} \, dt < +\infty \) we have (6.4) and show that this assumption leads to a contradiction.

By (6.4) we have

\[
\sup_{s \in \mathbb{R}} \frac{\ln |s|}{|s - 2^j|} \leq \frac{\beta(2^j)}{2^j}, \quad j \geq 1.
\]

On the other hand, using Theorem 4.3, we deduce \( \lim_{t \to \infty} \frac{\beta(t)}{t} = 0 \), so there exists \( j_0 \geq 1 \) such that

\[
\frac{\beta(2^j)}{2^j} \leq 1, \quad j \geq j_0.
\]

Applying (6.5) with \( \delta = \frac{\beta(2^j)}{2^j} \), we obtain

\[
\sup_{z \in \mathbb{C}} \frac{\ln |f(z)|}{|z - 2^j|} \leq 2 \ln \left| \prod_{j=1}^{j_0} \left( 1 + \frac{iz}{2^j} \right) \right| + n_j \ln \frac{\beta(2^j)}{2^j}, \quad j \geq j_0.
\]

(6.6) and (6.7) imply successively for every \( j \geq j_0 \)

\[
- \beta(2^j) \leq 2 \ln \left| \prod_{j=1}^{j_0} \left( 1 + \frac{iz}{2^j} \right) \right| + n_j \ln \frac{\beta(2^j)}{2^j},
\]

\[
n_j \ln \frac{2^j}{\beta(2^j)} \leq 2 \ln \left| \prod_{j=1}^{j_0} \left( 1 + \frac{iz}{2^j} \right) \right| + \beta(2^j).
\]

Consequently

\[
\sum_{j=j_0}^{\infty} n_j \ln \frac{2^j}{\beta(2^j)} \leq 4 \sum_{j=j_0}^{\infty} \ln \left| \prod_{j=1}^{j} \left( 1 + \frac{iz}{2^j} \right) \right| + \sum_{j=j_0}^{\infty} \frac{\beta(2^j)}{2^j}.
\]

But this is not possible, because the left-hand side of the above inequality is \( +\infty \) according to the assumption (6.2), while the right-hand side is finite because of Remark 4.4 (4) and Proposition 5.3 (i).

□

Now we are ready to prove Theorem 3.10, the main goal of this section:

**Proof (of Theorem 3.10).** Let \( 0 < t_1 \leq t_2 \leq t_3 \leq \ldots \leq +\infty, t_1 < +\infty, \sum_{j=1}^{\infty} \frac{1}{t_j} < +\infty \), be such that

\[
\omega(z) = \prod_{j=1}^{\infty} \left( 1 + \frac{iz}{t_j} \right), \quad z \in \mathbb{C}.
\]

Let us denote, for every \( t > 0 \), by \( n(t) \) the number of the elements of
the set \( \{ k \geq 1; t_k \leq t \} \). By Remark 4.1 (1) we have \( \int_1^\infty \frac{n(t)}{t^2} \, dt < +\infty \) and, according to Proposition 5.3 (i), it follows \( \sum_{j=1}^\infty \frac{n(2^j)}{2^j} < +\infty \).

On the other hand, Theorem 5.6 yields
\[
\int_1^\infty \frac{n(t)}{t^2} \ln \frac{t}{n(t)} \, dt = +\infty.
\]
Applying Proposition 5.3 (ii) to \( (0, +\infty) \ni t \mapsto n(t) \), we deduce that
\[
\sum_{j=1}^\infty \frac{n(2^j)}{2^j} \ln \frac{2^j}{\beta(2^j)} = +\infty
\]
for any increasing \( \beta : (0, +\infty) \to (0, +\infty) \) satisfying \( \int_1^\infty \frac{\beta(t)}{t^2} \, dt < +\infty \).

Set
\[
n_1 := n(2), \quad n_j := n(2^j) - n(2^{j-1}) \text{ for } j \geq 2.
\]
By Proposition 5.4 we infer that \( \sum_{j=1}^\infty \frac{n_j}{2^j} < +\infty \) and
\[
\sum_{j=1}^\infty \frac{n_j}{2^j} \ln \frac{2^j}{\beta(2^j)} = +\infty
\]
for any increasing \( \beta : (0, +\infty) \to (0, +\infty) \) satisfying \( \int_1^\infty \frac{\beta(t)}{t^2} \, dt < +\infty \).

In other words, the sequence \( (n_j)_{j \geq 1} \) satisfies conditions (6.1) and (6.2), so Lemma 6.1 implies that the formula
\[
f(z) = \prod_{j=1}^\infty \left( 1 - \left( \frac{|z|}{2^j} \right)^2 \right)^{n_j}, \quad z \in \mathbb{C}
\]
defines an entire function \( f \) such that there exists no increasing function \( \beta : (0, +\infty) \to (0, +\infty) \) with \( \int_1^\infty \frac{\beta(t)}{t^2} \, dt < +\infty \) satisfying (6.4) = (3.16).

It remains only to verify (3.15): we have for every \( z \in \mathbb{C} \)
\[
|f(z)| \leq \prod_{j=1}^\infty \left( 1 + \left( \frac{|z|}{2^j} \right)^2 \right)^{n_j} = \left( 1 + \left( \frac{|z|}{2} \right)^2 \right)^{n(2)} \prod_{j=2}^\infty \left( 1 + \left( \frac{|z|}{2^j} \right)^2 \right)^{n(2^j) - n(2^{j-1})}
\]
\[
\left[ \prod_{k=1}^{n(2)} \left( 1 + \left( \frac{|z|}{t_k} \right)^2 \right) \right] \prod_{j=2}^{\infty} \left[ \prod_{k=n(2j-1)+1}^{n(2j)} \left( 1 + \left( \frac{|z|}{t_k} \right)^2 \right) \right] = \prod_{k=1}^{\infty} \left( 1 + \left( \frac{|z|}{t_k} \right)^2 \right) = |\omega(|z|)|^2.
\]

\[ \square \]

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