Bose-Einstein condensation of diquark molecules in three-flavor quark matter

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We study the phase diagram of strongly interacting matter with three quark flavors at low and intermediate densities and non-zero temperatures in the framework of an NJL-type model with four-point interactions. At large densities, when the interactions are weak due to asymptotic freedom, quarks form loosely bound Cooper pairs. However, when the density decreases, interactions become stronger and quark Cooper pairs transform smoothly into tightly bound diquark molecules. We find that such molecules are stable at low density and temperature and that they dissociate above a temperature $T_{\text{diss}}$ of the order of the chiral phase transition temperature $T_c \sim 170$ MeV. We also explore the conditions under which these molecules undergo Bose-Einstein condensation (BEC). We find that BEC is only possible if we increase the attractive interaction in the diquark channel to (probably unrealistically) large values.

§1. Introduction

At asymptotically large densities, due to asymptotic freedom of QCD\textsuperscript{1}, the interaction between quarks becomes arbitrarily weak. Then, single-gluon exchange constitutes the dominant contribution to the quark-quark interaction. Single-gluon exchange is attractive in the color-antitriplet channel. Thus, we expect the formation of quark Cooper pairs which destabilize the Fermi surface\textsuperscript{2} and lead to color superconductivity,\textsuperscript{3} see\textsuperscript{4} for reviews.

At sufficiently large densities, the coupling constant is sufficiently small so that color superconductivity can be analyzed rigorously in the framework of QCD, using resummation techniques based on perturbative methods and power counting.\textsuperscript{5-8} This treatment is the analogue of weak-coupling BCS theory in condensed matter physics. The zero-temperature gap parameter turns out to be parametrically small in the coupling constant, $\phi_0 \sim \mu \exp(-1/g)$, where $\mu$ is the quark chemical potential. This, in turn, leads to “large” Cooper pairs, i.e., the quark correlation length $\xi \sim \phi_0^{-1} \sim \mu^{-1} \exp(1/g)$ is parametrically larger than the interparticle distance $\sim \mu^{-1}$.\textsuperscript{9}

When the density decreases, however, the strength of the quark-quark interaction and, thus, the gap parameter, increase, and the correlation length decreases. Let us note that a signature of strong correlations in the normal phase is the appearance

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of a so-called pseudogap in the vicinity of the transition to the superconducting phase.\textsuperscript{10,11} When the correlation length $\xi \lesssim \mu^{-1}$, quark Cooper pairs should be regarded as tightly bound diquark molecules. These molecules are stable at all temperatures below their so-called dissociation temperature $T_{\text{diss}}$.\textsuperscript{12} Above $T_{\text{diss}}$, they decay into the two quarks constituting the molecule. At small temperatures, diquark molecules may undergo Bose-Einstein condensation (BEC).\textsuperscript{9,12–20}

In these proceedings, we investigate bound diquark states and the possibility that they undergo BEC in the phase diagram of the quark matter. We use an NJL-type model with four-quark interactions. The strength of the attractive diquark interaction in the color-antitriplet channel is regarded as a free parameter. We show that bound diquark molecules appear at low densities and temperatures for all values of the diquark coupling strength studied here. We also find that BEC of diquarks occurs for (probably unrealistically) large values of the diquark coupling.

The remainder of this work is organized as follows. In Sec. 2 we introduce the model and the formalism in order to study diquark correlations. In Sec. 3 we present our numerical results. We conclude our work in Sec. 4. Our units are $\hbar = k_B = c = 1$, the metric tensor is $g^{\mu\nu} = \text{diag}(+, -, -, -)$.

\section{Model and Formalism}

In this work, we employ an NJL-type model Lagrangian for three quark flavors,

$$
\mathcal{L} = \bar{\psi}(i\slashed{D} - m)\psi + G_S \sum_{a=0}^{8} \left[ (\bar{\psi}\lambda_a\psi)^2 + (\bar{\psi}i\gamma_5\lambda_a\psi)^2 \right]
$$

$$
+ G_D \sum_{\gamma,C} \left[ \bar{\psi}_a^\alpha \gamma_5 \epsilon^{\alpha\beta\gamma} \epsilon_{abc} (\psi_C)^b_{\beta} \right] \left[ (\bar{\psi}_C)^b_{\alpha} \gamma_5 \epsilon^{\rho\sigma\gamma} \epsilon_{rsc} \psi_s^\rho \right], \quad (2.1)
$$

where the quark field $\psi_a^\alpha$ has color, $a = r, g, b$, and flavor, $\alpha = u, d, s$, indices. The current quark mass matrix is given by $m = \text{diag}(m_u, m_d, m_s)$, and $\lambda_a$ are (twice) the generators of $U(3)$. The charge-conjugate spinors are $\psi_C = C\bar{\psi}^T$ and $\bar{\psi}_C = \psi^T C$, where $C = i\gamma_2\gamma^0$ is the charge conjugation matrix. A sum over doubly appearing upper and lower indices in color and flavor space is implied (but not if both are upper or lower indices).

In the Lagrangian (2.1), the terms proportional to $G_S$ are quark-antiquark four-point interactions in the scalar and pseudoscalar channel, respectively. The terms proportional to $G_D$ parametrize the diquark four-point interaction in the color-antitriplet, flavor-antitriplet channel. For one-gluon exchange in QCD, this channel is attractive and thus leads to color superconductivity. Note that the diquark term can also be obtained from the quark-antiquark term by a Fierz transformation; in this case, the diquark coupling strength is fixed, $G_D = 0.75 G_S$.\textsuperscript{21} For the sake of simplicity, we neglect the effect of the $U(1)_A$ anomaly, so there is no ’tHooft-type six-point interaction term in Eq. (2.1).
In mean-field approximation, the thermodynamic potential is

$$\Omega = \sum_{c=1}^{3} \frac{\Delta_c^2}{4G_D} + \sum_{\alpha=1}^{3} \frac{(M_\alpha - m_\alpha)^2}{8G_S} - \frac{T}{2} \sum_n \int \frac{d^3p}{(2\pi)^3} \text{Tr}_{D,f,c} \ln \left[ S^{-1}(i\omega_n, p) \right],$$  

(2.2)

where

$$M_\alpha = m_\alpha - 4G_S \langle \bar{\psi}_\alpha \psi_\alpha \rangle,$$

(2.3)

$$\Delta_c = 2G_D \langle \bar{\psi}_\alpha (P_c)^{\alpha\beta} \psi_\beta \rangle,$$

(2.4)

are the constituent quark masses and the gap parameters for color superconductivity, respectively, with $(P_c)^{\alpha\beta} = i\gamma_5 \epsilon^{\alpha\beta} \epsilon_{abc}$. The $72 \times 72$ Nambu-Gor’kov propagator is defined by

$$S^{-1}(i\omega_n, p) = \left( \phi + \mu \gamma_0 - p \cdot \gamma \right) \sum_c \bar{P}_c \Delta_c - \mu \gamma_0 + M,$$

(2.5)

with $\phi = i\omega_n \gamma_0 - p \cdot \gamma$. Here, $\omega_n = (2n + 1)\pi T$, $n = 0, \pm 1, \pm 2, \ldots$, are the fermionic Matsubara frequencies. In this first exploratory study of diquark molecules and BEC, we assume the quark chemical potential $\mu$ to have a common value for all flavors. In application to compact stellar objects, this changes due to the conditions of overall electric (and color) neutrality, as well as temperature for the transition can be determined from

$$\Delta_c \equiv 0 \quad \text{and} \quad \Delta_{c \neq 0} \equiv 0.$$

(2.6)

In mean-field approximation, the phase transition from the color-superconducting to the normal phase is of second order. This means that the order parameter for condensation, $\Delta_c$, goes to zero smoothly as a function of temperature and the critical temperature for the transition can be determined from

$$\left. \frac{1}{\Delta_c} \frac{\partial \Omega}{\partial \Delta_c} \right|_{\Delta_c = 0} = 0.$$

(2.7)

Since up and down flavors are treated symmetrically in our model, we assume $M_u = M_d$ and $\Delta_1 = \Delta_2$. Because of explicit chiral symmetry breaking by a nonzero current quark mass, $\langle \bar{\psi}_\alpha \psi_\alpha \rangle$ is nonzero for all $T$ and $\mu$. The following, we refer to the phase with $\Delta_3 \neq 0$ and $\Delta_{1,2} = 0$ as the 2SC phase, and $\Delta_3 \neq 0$ and $\Delta_{1,2} \neq 0$ as the CFL phase. The phase with $\Delta_1 = \Delta_2 = \Delta_3 = 0$ corresponds to unpaired quark matter.

At nonzero temperature, the order parameters $\Delta_c$ and $M_\alpha$ fluctuate around their mean-field values. The propagation of these fluctuations in unpaired quark matter is characterized by the retarded propagator

$$D_c^R(x, t; x', t') = -i\theta(t - t') \langle [\bar{\psi}(x, t) P_c \psi_C(x, t), \bar{\psi}_C(x', t') P_c \psi(x', t')] \rangle$$

$$= \int \frac{d\omega d^3k}{(2\pi)^4} D_c^R(\omega, k) e^{-i\omega(t-t') + ik \cdot (x-x')}.$$

(2.8)
where \( c = 1, 2, \) or 3 denotes the down-strange, up-strange, or up-down diquark field, respectively. In the random phase approximation, the diquark propagators are given by

\[
D^R_c(\omega, \mathbf{p}) = \frac{1}{2} \frac{Q^R_c(\omega, \mathbf{p})}{1 + G_D Q^R_c(\omega, \mathbf{p})},
\]

where \( Q^R_c(\omega, \mathbf{p}) \) is the one-loop quark-quark polarization function. For imaginary energies \( \omega = i\nu_n, \) where \( \nu_n = 2n\pi T, \) \( n = 0, \pm 1, \pm 2, \ldots, \) are the bosonic Matsubara frequencies, it is given by

\[
Q_c(i\nu_n, \mathbf{p}) = 2T \sum_m \frac{d^3q}{(2\pi)^3} |\epsilon_{\beta\gamma}| \text{Tr}_{D,c}[\mathcal{G}_\beta(i\omega_m, \mathbf{q})\mathcal{G}_\gamma(i\nu_n + i\omega_m, \mathbf{p} + \mathbf{q})], \tag{2.10}
\]

with the trace taken over Dirac and color indices. Here, \( \mathcal{G}_\alpha(i\omega_n, \mathbf{p}) = [(i\omega_n + \mu)\gamma_0 - \mathbf{p} \cdot \gamma - M_\alpha]^{-1} \) are the Matsubara Green’s functions for quarks of flavor \( \alpha. \) Substituting these Green’s function into Eq. (2.10) and taking the analytic continuation \( Q^R_c(\omega, \mathbf{p}) = Q_c(i\nu_n, \mathbf{p})|_{i\nu_n = -\omega + i\eta}, \) we obtain

\[
Q_c^R(\omega, \mathbf{p}) = -2 \sum_{\beta,\gamma} |\epsilon_{\beta\gamma}| \int \frac{d^3q}{(2\pi)^3} \sum_{s,t=\pm} st \frac{(sE_\beta + tE_\gamma)^2 - |\mathbf{p}|^2 - \delta M_c^2}{E_\beta E_\gamma} \times \frac{f(tE_\gamma - \mu) - f(-sE_\beta + \mu)}{\omega + 2\mu - sE_\beta - tE_\gamma + i\eta}, \tag{2.11}
\]

where \( E_\beta = \sqrt{|\mathbf{q} - \mathbf{p}|^2 + M_\beta^2}, \) \( E_\gamma = \sqrt{|\mathbf{q}|^2 + M_\gamma^2}, \) \( \delta M_c = |M_\beta - M_\gamma| \) and \( f(E) = [\exp(E/T) + 1]^{-1} \) is the Fermi-Dirac distribution function. The imaginary part of \( Q^R_c(\omega, \mathbf{p}) \) denotes the difference of decay and production rates of the diquark field. At \( \mathbf{p} = 0, \) it is given by

\[
\text{Im}Q_c^R(\omega, 0) = 2\pi \sum_{\beta,\gamma} |\epsilon_{\beta\gamma}| \int \frac{d^3q}{(2\pi)^3} \frac{(\omega + 2\mu)^2 - \delta M_c^2}{E_\beta E_\gamma} \times \left\{ - \left[ (1 - f_\beta^+) (1 - f_\gamma^+) - f_\beta^+ f_\gamma^+ \right] e_\beta(\omega + 2\mu - E_\beta - E_\gamma) 
            + \left[ (1 - f_\beta^+) (1 - f_\gamma^-) - f_\beta^- f_\gamma^- \right] e_\beta(\omega + 2\mu + E_\beta + E_\gamma) 
            - \left[ f_\beta^+ (1 - f_\gamma^+) - (1 - f_\beta^-) f_\gamma^+ \right] e_\beta(\omega + 2\mu + E_\beta - E_\gamma) 
            - \left[ f_\beta^+ (1 - f_\gamma^-) - (1 - f_\beta^-) f_\gamma^- \right] e_\beta(\omega + 2\mu - E_\beta + E_\gamma) \right\}, \tag{2.12}
\]

where \( f_\alpha^\pm = \{\exp[(E_\alpha \mp \mu)/T] + 1\}^{-1}. \) The first (second) term in curly brackets corresponds to the decay of a diquark into two quarks (anti-quarks) and assumes nonzero values for \( \omega > 2M_c - 2\mu \) and \( \omega < -2M_c - 2\mu, \) with \( M_c = (M_\beta + M_\gamma)/2. \) The third and fourth terms represent Landau damping of a diquark. They are nonzero for \( -\delta M_c - 2\mu < \omega < \delta M_c - 2\mu. \) The numerical results show that the latter processes do not affect the stability of diquark excitations in our model, since the pole of the diquark field never appears around energies where Landau damping occurs.
The poles of the diquark propagator $D^R_c$ are determined by solving $D^R_c(\omega, \mathbf{p})^{-1} = 0$ or, equivalently,
\[ 1 + G_D Q^R_c(\omega, \mathbf{p}) = 0. \tag{2.13} \]
Setting $\omega = |\mathbf{p}| = 0$ in this equation, one can show that Eq. (2.13) is equivalent to the condition Eq. (2.7). This means that there exists a pole of the diquark propagator for zero energy and momentum at the critical temperature $T_c$ of a second-order phase transition. This fact is known as the Thouless criterion.\(^{22}\) Above $T_c$, the pole moves continuously to the fourth quadrant of the complex $\omega$-plane. The corresponding mode is called the soft mode. If $\bar{M}_c < \mu$ at $T = T_c$, $\omega = 0$ is in the continuum and the soft mode has a decay width above $T_c$.\(^{10}\) On the other hand, if $\bar{M}_c > \mu$, the soft mode does not have a decay width and the pole stays on the real axis. Then, this mode is nothing but a stable diquark molecule.\(^{12}\) As $T$ increases, the pole will move along the real axis until, at the dissociation temperature $T_{\text{diss}}^c$, it eventually arrives at the threshold of the decay process into two quarks $\omega_{\text{thr}}^c = 2(\bar{M}_c - \mu)$. (Note that $\bar{M}_c - \mu$ is the energy required to put one additional quark into the system. If the diquark energy exceeds twice this value, the diquark will decay.) The dissociation temperature $T_{\text{diss}}^c$ is determined by solving
\[ 1 + G_D Q^R_c(2\bar{M}_c - 2\mu, 0) = 0. \tag{2.14} \]
If stable diquarks are formed above $T_c$, it is natural to associate the superfluid phase below $T_c$ with BEC of these diquark molecules. In the following, we regard the region of a color-superconducting phase where $\mu < \bar{M}_c$ is satisfied as a Bose-Einstein condensed phase.\(^{12}\) Note, however, that this is just a rough estimate to separate BEC and BCS regimes; these two limits are connected continuously and there is no sharp phase boundary.\(^{12}\)

§3. Numerical Results

In this section, we present the phase diagram in the $T$-$\mu$ plane for several values of the diquark coupling constant $G_D$. In Fig. 1, we show the phase diagram for $G_D/G_S = 0.75$, which is the canonical value arising from a Fierz transformation of the NJL-type four-point interaction.\(^{21,23}\) One observes two color-superconducting phases, the 2SC and CFL phases, at high $\mu$ and low $T$. At $T = 0$, these phases are separated by a first-order phase transition; the first-order transition terminates at nonzero temperature. Also shown in Fig. 1 are the dissociation temperatures for stable diquark molecules; below the curve labelled $T_{\text{diss}}^{3,1,2}$, bound states of up and down quarks are stable, and below $T_{\text{diss}}^{1,2}$, bound states of up with strange and down with strange quarks are stable.

As discussed in the previous section, BEC of diquark molecules requires that $\mu < \bar{M}_c$ inside a color-superconducting phase. In Fig. 1, we show where $\mu = \bar{M}_c$; the regions to the left of these lines satisfy $\mu < \bar{M}_c$. One sees that these lines terminate at the first-order transition between normal and superconducting phase and that $\mu < \bar{M}_c$ is not satisfied inside a color-superconducting phase. Therefore, there is no BEC for $G_D/G_S = 0.75$. 


In Fig. 2, we show the phase diagram for $G_D/G_S = 1.1$. We see that the regions of the 2SC and CFL phases expand towards lower $\mu$ and higher $T$. Now there appears BEC inside the 2SC phase, namely where $\mu < \bar{M}_3$, shown by the shaded area in Fig. 2. One also sees that the dissociation temperatures $T_{\text{diss}}$ increase with $G_D$. The values of $T_{\text{diss}}$ at $\mu = 0$ are comparable to or even larger than the critical temperature of the QCD phase transition as determined by lattice QCD calculations, $T_{\text{c,LQCD}} \simeq 170$ MeV. Therefore, bound diquarks can survive even in the quark-gluon plasma phase if the diquark coupling is sufficiently large.

The other interesting feature shown in Fig. 2 is the fate of the first-order phase transition. One sees that the corresponding transition line terminates at smaller temperatures for $G_D/G_S = 1.1$ than for $G_D/G_S = 0.75$. This result can be interpreted as a result of the interplay between chiral symmetry breaking and color superconductivity. As far as we have checked, the endpoint of the first-order transition smoothly approaches $T = 0$ as $G_D$ increases. *)

Finally, let us investigate the phase diagram for an unrealistically large diquark coupling of $G_D/G_S = 1.5$, cf. Fig. 3. We observe that the BEC region becomes rather wide. The critical temperatures $T_c$ for the 2SC and CFL phases and $T_{\text{diss}}$ also increase substantially. If the diquark coupling is raised further, even the vacuum at $T = \mu = 0$ becomes a Bose-Einstein condensate of diquark molecules.

*) We do not observe a new endpoint at a lower temperature, found in Ref.25)
In this paper, we studied the phase diagram of three-flavor quark matter with particular emphasis on the formation of bound diquark molecules and the possibility that they undergo Bose-Einstein condensation (BEC). We found that diquark molecules appear at low densities and temperatures smaller than their dissociation...
temperature $T_{\text{diss}}$. This dissociation temperature may exceed the deconfinement temperature and implies that diquarks can be tightly bound even in the quark-gluon-plasma phase. We also found that BEC of diquarks is possible for large diquark interaction strengths $G_D$. This finding is in agreement with the results of Ref.\textsuperscript{16}. However, the required values for $G_D$ are probably unrealistically large, which makes it rather unlikely that a BEC of diquarks will be observed in compact stellar objects or even heavy-ion collisions.

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