Quasisymmetric Maps on the Boundary of a Negatively Curved Solvable Lie Group

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Abstract

We describe all the self quasisymmetric maps on the ideal boundary of a particular negatively curved solvable Lie group. As applications, we prove a Liouville type theorem, and derive some rigidity properties for quasiisometries of the solvable Lie group.

Keywords. quasiisometry, quasisymmetric map, negatively curved solvable Lie groups.

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1 Introduction

In this paper we study quasisymmetric maps on the ideal boundary of a particular negatively curved solvable Lie group.

Let

\[ A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \]

Let \( \mathbb{R} \) act on \( \mathbb{R}^2 \) by \((t, v) \to e^{tA}v (t \in \mathbb{R}, v \in \mathbb{R}^2)\). We denote the corresponding semi-direct product by \( G_A = \mathbb{R}^2 \rtimes_A \mathbb{R} \). That is, \( G_A = \mathbb{R}^2 \times \mathbb{R} \) as a smooth manifold, and the group operation is given by:

\[(v, t) \cdot (w, s) = (v + e^{tA}w, t + s)\]

for all \((v, t), (w, s) \in \mathbb{R}^2 \times \mathbb{R}\). The group \( G_A \) is a simply connected solvable Lie group.

We endow \( G_A \) with the left invariant Riemannian metric determined by taking the standard Euclidean metric at the identity of \( G_A = \mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^3 \). With this metric \( G_A \) has pinched negative sectional curvature (and so is Gromov hyperbolic). Hence \( G_A \) has a well defined ideal boundary \( \partial G_A \). There is a so-called cone topology on \( \overline{G_A} = G_A \cup \partial G_A \), in which \( \partial G_A \) is homeomorphic to the 2-dimensional sphere and \( \overline{G_A} \) is homeomorphic to the closed 3-ball in the Euclidean space. For each \( v \in \mathbb{R}^2 \), the map \( \gamma_v : \mathbb{R} \to G_A, \gamma_v(t) = (v, t) \) is a geodesic. We call such a geodesic a vertical geodesic. It can be checked that all vertical
geodesics are asymptotic as $t \to +\infty$. Hence they define a point $\xi_0$ in the ideal boundary $\partial G_A$.

Each geodesic ray in $G_A$ is asymptotic to either an upward oriented vertical geodesic or a downward oriented vertical geodesic. The upward oriented geodesics are asymptotic to $\xi_0$ and the downward oriented vertical geodesics are in 1-to-1 correspondence with $\mathbb{R}^2$. Hence $\partial G_A \setminus \{\xi_0\}$ can be naturally identified with $\mathbb{R}^2$.

For any proper Gromov hyperbolic geodesic space $X$ and any $\xi \in \partial X$, there are so-called parabolic visual (quasi)metrics on $\partial X \setminus \{\xi\}$. See [SX], Section 5. In our case, a parabolic visual quasimetric $D$ on $\partial G_A \setminus \{\xi_0\}$ is given by:

$$D((x_1, y_1), (x_2, y_2)) = \max \{ |y_2 - y_1|, |(x_2 - x_1) - (y_2 - y_1) \ln |y_2 - y_1| | \}$$

for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2 = \partial G_A \setminus \{\xi_0\}$, where $0 \ln 0$ is understood to be 0.

We remark that $D$ is not a metric on $\mathbb{R}^2$, but merely a quasimetric. Recall that a quasimetric $\rho$ on a set $A$ is a function $\rho : A \times A \to \mathbb{R}$ satisfying the following three conditions:

1. $\rho(x, y) = \rho(y, x)$ for all $x, y \in A$; (2) $\rho(x, y) \geq 0$ for all $x, y \in A$ and $\rho(x, y) = 0$ if and only if $x = y$; (3) there is some $M \geq 1$ such that $\rho(x, z) \leq M(\rho(x, y) + \rho(y, z))$ for all $x, y, z \in A$. For each $M \geq 1$, there is a constant $c_0 > 0$ such that $\rho^2$ is biLipschitz equivalent to a metric for all quasimetric $\rho$ with constant $M$ and all $0 < \epsilon \leq c_0$, see Proposition 14.5. in [Hn].

Let $\eta : [0, \infty) \to [0, \infty)$ be a homeomorphism. A bijection $F : X \to Y$ between two quasimetric spaces is $\eta$-quasisymmetric if for all distinct triples $x, y, z \in X$, we have

$$\frac{d(F(x), F(y))}{d(F(x), F(z))} \leq \eta \left( \frac{d(x, y)}{d(x, z)} \right).$$

A map $F : X \to Y$ is quasisymmetric if it is $\eta$-quasisymmetric for some $\eta$.

The following is the main result of the paper.

**Theorem 1.1.** Every quasisymmetric map $F : (\mathbb{R}^2, D) \to (\mathbb{R}^2, D)$ is a biLipschitz map. Furthermore, a bijection $F : (\mathbb{R}^2, D) \to (\mathbb{R}^2, D)$ is a quasisymmetric map if and only if it has the following form: $F(x, y) = (ax + c(y), ay + b)$ for all $(x, y) \in \mathbb{R}^2$, where $a \neq 0$, $b$ are constants and $c : \mathbb{R} \to \mathbb{R}$ is a Lipschitz map.

One should compare this with quasiconformal maps on the sphere or the Euclidean space, where there are plenty of non-biLipschitz quasiconformal maps. On the other hand, the conclusion of Theorem 1.1 is not as strong as in the cases of quaternionic hyperbolic spaces, Cayley plane $(\mathbb{P}^2)$ and Fuchsian buildings $(\mathbb{B}P, \mathbb{X})$, where every quasiconformal map of the ideal boundary is actually a conformal map. In our case, there are many non-conformal quasisymmetric maps of the ideal boundary of $G_A$.

As applications, we describe all the isometries and all the similarities of $(\mathbb{R}^2, D)$, see Proposition 6.1. We also prove a Liouville type theorem for $(\mathbb{R}^2, D)$.

**Theorem 1.2.** Every conformal map $f : (\mathbb{R}^2, D) \to (\mathbb{R}^2, D)$ is the boundary map of an isometry $G_A \to G_A$.

Theorem 1.2 also has geometric consequences. Let $L \geq 1$ and $C \geq 0$. A (not necessarily continuous) map $f : X \to Y$ between two metric spaces is an $(L, A)$-quasiisometry if:
The parabolic visual quasimetric $D$ on the horosphere $\partial G$ is defined as follows: $D(\gamma, \gamma') = \sup_{t \in \mathbb{R}} |e^{-tA} \gamma - e^{-tA} \gamma'|$, where $A$ is a symmetric matrix given by $Q(t) = e^{-tA}$. Here $A^T$ denotes the transpose of $A$. With this metric $G_A$ has sectional curvature $-\frac{(6 + \sqrt{29})}{4} = -b^2 \leq K \leq -\frac{6 - \sqrt{29}}{4}$. Hence $G_A$ has a well defined ideal boundary $\partial G_A$. Each geodesic ray in $G_A$ is asymptotic to either an upward oriented vertical geodesic or a downward oriented vertical geodesic. The upward oriented geodesics are asymptotic to $\xi_0$ and the downward oriented vertical geodesics are in 1-to-1 correspondence with $\mathbb{R}^2$. Hence $\partial G_A \setminus \{\xi_0\}$ can be naturally identified with $\mathbb{R}^2$.

We next define three parabolic visual quasimetrics on $\partial G_A \setminus \{\xi_0\} = \mathbb{R}^2$. Given $v, w \in \mathbb{R}^2 \simeq \partial G_A \setminus \{\xi_0\}$, the parabolic visual quasimetric $D_e(v, w)$ is defined as follows: $D_e(v, w) = e^t$, where $t$ is the unique real number such that at height $t$ the two vertical geodesics $\gamma_v$ and $\gamma_w$ are at distance one apart in the horosphere; that is, $d_{\mathbb{R}^2 \times \{t\}}(|v, t) - (w, t)| = |e^{-tA}(v - w)| = 1$. Here the subscript $e$ in $D_e$ means it corresponds to the Euclidean norm.

Recall that the super norm on $\mathbb{R}^2$ is given by: $|x, y|_s = \max\{|x|, |y|\}$ for all $(x, y) \in \mathbb{R}^2$. The parabolic visual quasimetric $D_s$ on $\partial G_A \setminus \{\xi_0\}$ is defined as follows: $D_s(v, w) = e^t$, where
\( t \) is the smallest real number such that at height \( t \) the two vertical geodesics \( \gamma_v \) and \( \gamma_w \) are at distance one apart with respect to the norm \( \cdot |_s \); that is, \( |e^{-tA}(v - w)|_s = 1 \). Here the subscript \( s \) in \( D_s \) means it corresponds to the super norm \( \cdot |_s \).

Notice that \( |v|_s \leq |v| \leq \sqrt{2} |v|_s \) for all \( v \in \mathbb{R}^2 \). Using this, one can verify the following lemma, whose proof is left to the reader.

**Lemma 2.1.** For all \( v, w \in \mathbb{R}^2 \) we have \( D_s(v, w) \leq D_e(v, w) \leq 2^{1/2a} D_s(v, w) \), where \( a = \sqrt{6 - \sqrt{29}}/2 \).

The following result provides a parabolic visual quasimetric \( D \) which admits an explicit formula and is also biLipschitz equivalent with \( D_e \) and \( D_s \).

**Proposition 2.2.** For all \( v = (x_1, y_1), w = (x_2, y_2) \in \mathbb{R}^2 \),

\[
D(v, w)/3 \leq D_s(v, w) \leq 3D(v, w),
\]

where \( D(v, w) = \max \{|y_2 - y_1|, |(x_2 - x_1) - (y_2 - y_1)\ln |y_2 - y_1||\} \) and \( 0 \ln 0 \) is understood to be 0.

Let \( g = ((x, y), t) \in \mathbb{R}^2 \times \mathbb{R} \) and denote by \( L_g : G_A \to G_A \) the left translation by \( g \). We calculate

\[
L_g((x', y'), t') = ((x + e^t(x' + ty'), y + e^t y'), t' + t).
\]

We see that \( L_g \) maps vertical geodesics to vertical geodesics. It follows that \( L_g \) induces a map \( T_g : \mathbb{R}^2 \to \mathbb{R}^2 \),

\[
T_g(x', y') = (x + e^t(x' + ty'), y + e^t y').
\]

Since \( L_g \) is an isometry of \( G_A \) and it translates by \( t \) in the vertical direction, the definition of the quasimetric \( D_e \) shows that

\[
D_e(T_g(x_1, y_1), T_g(x_2, y_2)) = e^t D_e((x_1, y_1), (x_2, y_2))
\]

for all \( (x_1, y_1), (x_2, y_2) \in \mathbb{R}^2 \). In other words, \( T_g \) is a similarity of \((\mathbb{R}^2, D_e)\) with similarity constant \( e^t \). When \( t = 0 \), \( T_g \) is simply a Euclidean translation and it is an isometry with respect to \( D_e \). Similar statements also hold for the quasimetric \( D_s \).

Notice that Euclidean translations are also isometries with respect to the function \( D \). This together with the same statement about \( D_s \) implies that we can assume \((x_1, y_1) = (0, 0)\) in order to prove Proposition 2.2.

**Proof of Proposition 2.2.** By the preceding remark, we may assume \((x_1, y_1) = (0, 0)\), and write \((x, y)\) for \((x_2, y_2)\). Recall \( D_s((x, y), (0, 0)) = e^t \) if \( t \) is the smallest real number such that \( |e^{-tA}(x, y)|_s = 1 \). We calculate \( |e^{-tA}(x, y)|_s = \max\{e^{-t|x - ty|}, e^{-t|y|}\} \). We consider several cases:

Case 1: \( y = 0 \). In this case, \( |e^{-tA}(x, 0)|_s = e^{-t|x|} \). Hence \( D_s((x, 0), (0, 0)) = e^t = |x| = D((x, 0), (0, 0)) \).

When \( y \neq 0 \), we let \( t_0 = \ln |y| \) and \( a = x/y - \ln |y| \).
Case 2: \( y \neq 0 \) and \( |x - t_0 y| \leq |y| \). In this case, \( |e^{-t_0 A}(x, y)|_s = \max\{e^{-t_0}|x - t_0 y|, e^{-t_0}|y|\} = e^{-t_0}|y| = 1 \). Notice also \( |e^{-t A}(x, y)|_s \geq e^{-t}|y| > 1 \) if \( t < t_0 \). Hence \( D_s((x, y), (0, 0)) = e^{t_0} = |y| = D((x, y), (0, 0)) \).

When \( y \neq 0 \) and \( |x - t_0 y| > |y| \), we have \( |a| > 1 \).

Case 3: \( y \neq 0 \), \( |x - t_0 y| > |y| \) and \( a > 1 \). In this case, \( D((x, y), (0, 0)) = |x - y \ln |y|| \).

Let \( t_1 > t_0 \) be the smallest real number \( t \) satisfying \( e^{-t}|x - ty| = 1 \). Notice that \( e^{-t_1}|y| < 1 \) and so \( D_s((x, y), (0, 0)) = e^{t_1} \). Set \( u = t_1 - t_0 > 0 \). The equality \( e^{-t_1}|x - t_1 y| = 1 \) implies \( e^u = a - u \). Clearly \( e^u = a - u \leq a \). We claim \( e^u = a - u \geq a/3 \). Otherwise, \( u > 2a/3 \). This contradicts \( a = u + e^u > u + (1 + u) \). Hence \( a/3 \leq e^u \leq a \) and

\[
|x - y \ln |y||/3 = a|y|/3 \leq |y|e^u = e^{t_1} = D_s((x, y), (0, 0)) \leq a|y| = |x - y \ln |y||.
\]

Case 4: \( y \neq 0 \), \( |x - t_0 y| > |y| \) and \( a < -1 \). In this case, \( D((x, y), (0, 0)) = |x - y \ln |y|| \).

Let \( t_1 > t_0 \) be the smallest real number \( t \) satisfying \( e^{-t}|x - ty| = 1 \). Again we have \( D_s((x, y), (0, 0)) = e^{t_1} \). Set \( u = t_1 - t_0 > 0 \). The equality \( e^{-t_1}|x - t_1 y| = 1 \) implies \( e^u = u - a \). Clearly \( e^u = u - a > -a \). We claim \( e^u = u - a \leq -3a \). Otherwise, \( u > -2a \) and hence \( -a = e^u - u > 1 + u^2/2 > u > -2a \), a contradiction. Hence \( |a| = -a \leq e^u \leq -3a = 3|a| \) and

\[
|x - y \ln |y|| = |ay| \leq |y|e^u = e^{t_1} = D_s((x, y), (0, 0)) \leq 3|ay| = 3|x - y \ln |y||.
\]

We describe some isometries and similarities of the space \((\mathbb{R}^2, D)\). The following proposition can be easily proved by using the formula for \( D \).

**Proposition 2.3.** Let \((\mathbb{R}^2, D)\) be as above.

1. Then Euclidean translations of \(\mathbb{R}^2\) are isometries with respect to \(D\);
2. Let \( \pi : \mathbb{R}^2 \to \mathbb{R}^2 \) be defined by \( \pi(x, y) = (-x, -y) \). Then \( \pi \) is an isometry with respect to \(D\);
3. For any real number \( t \), let \( \lambda_t : \mathbb{R}^2 \to \mathbb{R}^2 \) be defined by \( \lambda_t(x, y) = (e^t(x + ty), e^t y) \). Then \( D(\lambda_t(x_1, y_1), \lambda_t(x_2, y_2)) = e^t \cdot D((x_1, y_1), (x_2, y_2)) \) for all \((x_1, y_1), (x_2, y_2) \in \mathbb{R}^2\).

We notice that the three classes of maps in Proposition 2.3 are boundary maps of isometries of \(G_A\). Euclidean translations of \(\mathbb{R}^2 = \partial G_A \setminus \{\xi_0\} \) are boundary maps of left translations \(L_g\) of \(G_A\) for elements \(g\) of type \(g = ((x, y), 0) \in \mathbb{R}^2 \times \mathbb{R} = G_A\). The map \(\lambda_t\) is the boundary map of left translation \(L_g\) for \(g = ((0, 0), t)\). Finally, \(\tau\) is the boundary map of the automorphism \(\tau' : G_A \to G_A, \tau'((x, y), t) = ((-x, -y), t)\). Notice that \(\tau'\) is indeed an automorphism of \(G_A\) and the tangential map of \(\tau'\) at the identity is an isometry. It follows that \(\tau'\) is an isometry of \(G_A\).

### 3 Quasisymmetric maps preserve horizontal foliation

In this section we prove that every self quasisymmetric map of \((\mathbb{R}^2, D)\) maps horizontal lines to horizontal lines. The proof belongs to Bruce Kleiner. Here I am trying to provide
more details and I am responsible for the inaccuracies that might result from this. I would like to express my gratitude towards Bruce for allowing me to include his argument.

**Definition 3.1.** Let \((X, \rho)\) be a quasimetric space and \(L \geq 1\). A subset \(A \subset X\) is called an \(L\)-quasi-ball if there is some \(x \in X\) and some \(r > 0\) such that \(B(x, r) \subset A \subset B(x, Lr)\). Here \(B(x, r) = \{y \in X : \rho(y, x) < r\}\).

The following notion is key to the proof.

**Definition 3.2.** (Kleiner) Pick \(Q \geq 1\). Let \(u : X \to \mathbb{R}\) be a function (not necessarily continuous) defined on a quasimetric space, and let \(P\) be a collection of subsets of \(X\). The \(Q\)-variation of \(u\) over \(P\) – denoted \(V_Q(u, P)\) – is the quantity

\[
\sum_{P \in P} \left[ \text{osc}(u|_P) \right]^Q,
\]

where \(\text{osc}(u|_P)\) denotes the oscillation (sup minus inf) of the restriction of \(u\) to the subset \(P \subset X\). The \(Q\)-variation \(V_Q(u)\) of \(u\) is \(\sup\{V_Q(u, P)\}\) where \(P\) ranges over all disjoint collections of balls in \(X\). The \((Q,K)\)-variation \(V_{Q,K}(u)\) of \(u\) is \(\sup\{V_Q(u, P)\}\) where \(P\) ranges over all disjoint collections of \(K\)-quasi-balls in \(X\).

There are useful variants of this definition, for instance one can look at the infimum over all coverings. Or one can take the infimum over all coverings followed by the sup as the mesh size tends to zero. The definition performs the same function as Pansu’s modulus [P1], but it seems easier to digest.

**Lemma 3.1.** Let \(F : X \to Y\) be an \(\eta\)-quasisymmetric map between two quasimetric spaces. Then for every function \(u : X \to \mathbb{R}\) we have \(V_{Q,K}(u) \leq V_{Q,\eta(K)}(u \circ F^{-1})\).

**Proof.** For any subset \(A \subset X\), the oscillation of \(u\) on \(A\) equals the oscillation of \(u \circ F^{-1}\) on \(F(A)\). Let \(P\) be a disjoint collection of \(K\)-quasi-balls in \(X\). Then \(F(P) = \{F(P) : P \in P\}\) is a disjoint collection of \(\eta(K)\)-quasi-balls in \(Y\), and \(V_Q(u, P) = V_Q(u \circ F^{-1}, F(P))\). Hence

\[
V_{Q,K}(u) = \sup_P V_Q(u, P) = \sup_P V_Q(u \circ F^{-1}, F(P)) \leq V_{Q,\eta(K)}(u \circ F^{-1}).
\]

\(\square\)

By Proposition 2.3 and the discussion preceding the proof of Proposition 2.2, for each \(g \in G_A\), the map \(T_g : \mathbb{R}^2 \to \mathbb{R}^2\) is a similarity with respect to the quasimetrics \(D_e, D_s\) and \(D\). Hence, in particular, the images of the unit square \(S\) under the action of \(G_A\) on \(\mathbb{R}^2\) are \(K\)-quasi-balls in these quasimetrics for some fixed \(K\). In Lemmas 3.2 through 3.4, \(\mathbb{R}^2\) is equipped with one of the three quasimetrics.

**Lemma 3.2.** The coordinate function \(y : \mathbb{R}^2 \to \mathbb{R}\) has locally finite \((2, L)\)-variation for any \(L\).

**Proof.** Let \(U \subset \mathbb{R}^2\) be any bounded open subset. First observe that if two \(L\)-quasi-balls have comparable size, then the oscillation of \(y\) over the two quasi-balls will be comparable. Hence
when we calculate the 2-variation, it suffices to consider only packings of the form $T_g(S)$ where $g \in G_A$. For each such square, we clearly have

$$|\text{osc}(y|_B)|^2 = \text{area}(B)$$

where $\text{area}(B)$ denotes the Euclidean area. It follows that the 2-variation of $y|_U$ is bounded by the area of $U$.

\[\square\]

**Lemma 3.3.** Let $U \subset \mathbb{R}^2$ be an open subset. If $u : U \to \mathbb{R}$ is a continuous function which is not constant along some horizontal line segment in $U$, then $V_{2,K}(u) = \infty$.

**Proof.** Since $u$ is continuous and is not constant along a horizontal line segment in $U$, after composing $u$ with an affine function, we may assume that there is a rectangle $C = [a, b] \times [c, d] \subset U$ such that $u \leq 0$ on $F_0 := \{(x, y) \in C : x = a\}$ and $u \geq 1$ on $F_1 := \{(x, y) \in C : x = b\}$. Let $\mathcal{G}$ be the standard unit coordinate grid. Pick $t \in \mathbb{R}$, $t << 0$. The image of $\mathcal{G}$ under

$$\lambda_t = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}$$

is a “sheared grid”, whose tiles have area $e^{2t}$. Organize these into nearly horizontal chains (which correspond to the image of vertical strips under $\lambda_t$). Notice that these chains have slope $1/t$ and intersect vertical lines in segments with Euclidean length $e^t/|t|$. It follows that there are at least

$$\frac{(d - c) - (b - a)/|t|}{e^t/|t|} = \frac{(d - c)|t| - (b - a)}{e^t}$$

such chains connecting the left edge $F_0$ of $C$ to the right edge $F_1$ of $C$.

Now consider a chain as above that connects $F_0$ and $F_1$. Orient the chain from left to right. Let $T$ be the last tile in the chain that intersects $F_0$ and $T'$ the first tile in the chain that intersects $F_1$. Order the tiles in the chain between $T$ and $T'$ from left to right and denote them by $T_1, \ldots, T_k$. Set $T_0 = T$, $T_{k+1} = T'$. Let $p_i$ ($i = 1, \ldots, k + 1$) be the upper left vertex of $T_i$. Also choose any $p_0 \in T_0 \cap F_0$ and $p_{k+2} \in T_{k+1} \cap F_1$. Notice that the difference between the $x$-coordinates of $p_{i+1}$ and $p_i$ ($i = 1, \ldots, k$) is $|t|e^t$. It follows that $k < \frac{b-a}{|t|e^t}$.

Let $a_i$ be the oscillation of $u$ on $T_i \cap S$. Then $a_i \geq |u(p_{i+1}) - u(p_i)|$. By the triangle inequality, we have

$$\sum_{i=0}^{k+1} a_i \geq \sum_{i=0}^{k+1} |u(p_{i+1}) - u(p_i)| \geq |u(p_{k+2}) - u(p_0)| \geq 1.$$

In the last inequality we used the facts that $u \leq 0$ on $F_0$ and $u \geq 1$ on $F_1$. Hence

$$\sum_{i=0}^{k+1} a_i^2 \geq \frac{1}{k + 2} (\sum_{i=0}^{k+1} a_i)^2 \geq \frac{1}{k + 2} \geq \frac{|t|e^t}{(b - a) + 2|t|e^t}.$$

Since there are at least $\frac{(d - c)|t| - (b - a)}{e^t}$ chains connecting $F_0$ and $F_1$, the $(2, K)$ – variation of $u$ over this particular packing is at least

$$\frac{|t|e^t}{(b - a) + 2|t|e^t} \times \frac{(d - c)|t| - (b - a)}{e^t} \geq \frac{|t| \{ (d-c)|t| - (b-a) \}}{(b - a) + 2|t|e^t}.$$
As $t \to -\infty$, we see that $V_{2,K}(u) = \infty$.

Lemma 3.4. Let $U, V \subset \mathbb{R}^2$ be two open subsets, and $F : U \to V$ be a quasisymmetric map. Then $F$ maps each horizontal line segment in $U$ to a horizontal line segment in $V$.

Proof. Assume $F : U \to V$ is $\eta$-quasisymmetric. Suppose that the claim in the lemma is false. Then there are two points $p, q$ on the same horizontal line segment in $U$ such that $F(p)$ and $F(q)$ are not on the same horizontal line. Then $F(p)$ and $F(q)$ have different $y$ coordinates. Hence $y \circ F$ is not constant along horizontal lines. By Lemma 3.3, $V_{2,K}(y \circ F) = \infty$. On the other hand, applying Lemma 3.1 to the function $y \circ F : U \to \mathbb{R}$ and $F : U \to V$, we obtain $V_{2,\eta(K)}(y) \geq V_{2,K}(y \circ F) = \infty$. This contradicts Lemma 3.2.

Proposition 3.5. Let $F : \partial G_A \to \partial G_A$ be a quasisymmetric homeomorphism, where $\partial G_A$ is equipped with a visual metric. Then $F$ fixes the point $\xi_0$ and maps horizontal lines to horizontal lines.

Proof. Suppose $F(\xi_0) \neq \xi_0$. Then $F$ induces a homeomorphism

$$F_1 : \partial G_A \setminus \{\xi_0, F^{-1}(\xi_0)\} \to \partial G_A \setminus \{F(\xi_0), \xi_0\}$$

between two open subsets of $\mathbb{R}^2$. Since a visual metric (away from $\xi_0$) is locally quasisymmetrically equivalent with a parabolic visual metric (say a metric of the form $D^\epsilon$ with $\epsilon$ sufficiently small) (see [SX] Section 5), $F_1$ is locally quasisymmetric with respect to any one of $D_e$, $D_s$ and $D$. Now Lemma 3.4 implies that $F_1$ maps horizontal line segments to horizontal line segments. Let $L$ be a complete horizontal line in $\mathbb{R}^2$ which does not contain $F^{-1}(\xi_0)$. Then $L \cup \{\xi_0\}$ is a circle in $\partial G_A$ and hence $F(L \cup \{\xi_0\})$ is a circle in $\mathbb{R}^2$. By the above argument, $F(L)$ is horizontal and is dense in the circle $F(L \cup \{\xi_0\}) \subset \mathbb{R}^2$. This is clearly impossible. Hence $F$ fixes $\xi_0$. Now Lemma 3.4 implies $F$ maps horizontal lines to horizontal lines.

We omit the proof of the following consequence of Proposition 3.5 since the proof is more or less routine and is already contained in [SX], Section 6.

Corollary 3.6. The group $G_A$ is not quasiisometric to any finitely generated group.

4 Quasisymmetric maps are $D$-biLipschitz

In this section we show that every quasisymmetric map of $\partial G_A$ is biLipschitz with respect to $D$. One should contrast this with the round sphere or the Euclidean space, where there are plenty of non-biLipschitz quasisymmetric maps. On the other hand, $(\mathbb{R}^2, D)$ is not as rigid as the ideal boundary of a quarternionic hyperbolic space or a Cayley plane ([P2]) or a Fuchsian building ([BP], [X]), where each self quasisymmetry is a conformal map.
Let $K \geq 1$ and $C > 0$. A bijection $F : X_1 \to X_2$ between two quasimetric spaces is called a $K$-quasisimilarity (with constant $C$) if

$$\frac{C}{K} d(x, y) \leq d(F(x), F(y)) \leq C K d(x, y)$$

for all $x, y \in X_1$. When $K = 1$, we say $F$ is a similarity. It is clear that a map is a quasisimilarity if and only if it is a biLipschitz map. The point of using the notion of quasisimilarity is that sometimes there is control on $K$ but not on $C$.

**Theorem 4.1.** Let $F : (\mathbb{R}^2, D) \to (\mathbb{R}^2, D)$ be an $\eta$-quasisymmetry. Then $F$ is a $K$-quasisimilarity, where $K = (\eta(1)/\eta^{-1}(1))^6$.

We first recall some definitions.

Let $g : (X_1, \rho_1) \to (X_2, \rho_2)$ be a bijection between two quasimetric spaces. Suppose $g$ satisfies the following condition: for any fixed $x \in X_1$, $\rho_1(y, x) \to 0$ if and only if $\rho_2(g(y), g(x)) \to 0$. We define for every $x \in X_1$ and $r > 0$,

$$L_g(x, r) = \sup \{\rho_2(g(x), g(x')) : \rho_1(x, x') \leq r\},$$

$$l_g(x, r) = \inf \{\rho_2(g(x), g(x')) : \rho_1(x, x') \geq r\},$$

and set

$$L_g(x) = \limsup_{r \to 0} \frac{L_g(x, r)}{r}, \quad l_g(x) = \liminf_{r \to 0} \frac{l_g(x, r)}{r}.$$ 

Then

$$L_g^{-1}(g(x)) = \frac{1}{l_g(x)} \quad \text{and} \quad l_g^{-1}(g(x)) = \frac{1}{L_g(x)}$$

for any $x \in X_1$. If $g$ is an $\eta$-quasisymmetry, then $L_g(x, r) \leq \eta(1)l_g(x, r)$ for all $x \in X_1$ and $r > 0$. Hence if in addition

$$\lim_{r \to 0} \frac{L_g(x, r)}{r} \quad \text{or} \quad \lim_{r \to 0} \frac{l_g(x, r)}{r}$$

exists, then

$$0 \leq l_g(x) \leq L_g(x) \leq \eta(1)l_g(x) \leq \infty.$$ 

We notice that for every $y_1, y_2 \in \mathbb{R}$, the Hausdorff distance with respect to $D$,

$$HD(\mathbb{R} \times \{y_1\}, \mathbb{R} \times \{y_2\}) = |y_1 - y_2|.$$ 

(4.1)

Also, for any $p = (x_1, y_1) \in \mathbb{R}^2$ and any $y_2 \in \mathbb{R}$,

$$D(p, \mathbb{R} \times \{y_2\}) = |y_1 - y_2|.$$ 

(4.2)

Let $F : (\mathbb{R}^2, D) \to (\mathbb{R}^2, D)$ be an $\eta$-quasisymmetry. By Lemma 3.3 $F$ preserves the horizontal foliation on $\mathbb{R}^2$. Hence it induces a map $G : \mathbb{R} \to \mathbb{R}$ such that for any $y \in \mathbb{R}$, $F(\mathbb{R} \times \{y\}) = \mathbb{R} \times \{G(y)\}$. For each $y \in \mathbb{R}$, let $H(\cdot, y) : \mathbb{R} \to \mathbb{R}$ be the map such that $F(x, y) = (H(x, y), G(y))$ for all $x \in \mathbb{R}$. Notice that the restriction of $D$ to a horizontal line agrees with the Euclidean distance. Because $F : (\mathbb{R}^2, D) \to (\mathbb{R}^2, D)$ is an $\eta$-quasisymmetry, for each fixed $y \in \mathbb{R}$, the map $H(\cdot, y) : (\mathbb{R}, |\cdot|) \to (\mathbb{R}, |\cdot|)$ is also an $\eta$-quasisimmetry. The following lemma together with equations (4.1) and (4.2) imply that $G : \mathbb{R} \to \mathbb{R}$ is also an $\eta$-quasisimilarity with respect to the Euclidean metric on $\mathbb{R}$.
Lemma 4.2. ([1] Lemma 15.9) Let $g : X_1 \to X_2$ be an \( \eta \)-quasisymmetry and $A, B, C \subset X_1$. If $HD(A, B) \leq t HD(A, C)$ for some $t \geq 0$, then there is some $a \in A$ such that

$$HD(g(A), g(B)) \leq \eta(t)d(g(a), g(C)).$$

We recall that if $g : X_1 \to X_2$ is an \( \eta \)-quasisymmetry, then $g^{-1} : X_2 \to X_1$ is an \( \eta_1 \)-quasisymmetry, where $\eta_1(t) = (\eta^{-1}(t^{-1}))^{-1}$. See [1], Theorem 6.3.

Theorem 4.1 is proved in Lemmas 4.3 through 4.7. In these proofs, the quantities $l_G, L_G, l_{G^{-1}}, L_{G^{-1}}$, $l_{H(y,y)}$, $L_{H(y,y)}$, $l_I$ and $L_I$ are all defined with respect to the Euclidean metric on $\mathbb{R}$, where $I := H(\cdot, y)^{-1} : \mathbb{R} \to \mathbb{R}$.

Lemma 4.3. The following hold for all $y \in \mathbb{R}$, $x \in \mathbb{R}$:

1. $L_G(y, r) \leq \eta(1) l_{H(y,y)}(x, r)$ for any $r > 0$;
2. $\eta^{-1}(1) l_{H(y,y)}(x) \leq L_G(y) \leq \eta(1) l_{H(y,y)}(x)$;
3. $\eta^{-1}(1) L_{H(y,y)}(x) \leq L_G(y) \leq \eta(1) L_{H(y,y)}(x)$.

Proof. (1) Let $y \in \mathbb{R}$, $x \in \mathbb{R}$ and $r > 0$. Let $y' \in \mathbb{R}$ with $|y - y'| \leq r$ and $x' \in \mathbb{R}$ with $|x - x'| \geq r$. Denote $x'' = x + (y' - y) \ln|y' - y|$. Then $D((x, y), (x'', y')) \leq r \leq D((x, y), (x', y))$. Since $F$ is $\eta$-quasisymmetric, we have

$$|G(y) - G(y')| \leq D(F(x'', y'), F(x, y)) \leq \eta(1) D(F(x, y), F(x', y)) = \eta(1)|H(x, y) - H(x', y)|.$$

Since $y'$ and $x'$ are arbitrary, (1) follows.

(2) and (3). It follows from $l_G(y, r) \leq L_G(y, r)$, $l_{H(y,y)}(x, r) \leq L_{H(y,y)}(x, r)$ and (1) that $L_G(y, r) \leq \eta(1) L_{H(y,y)}(x, r)$ and $L_G(y, r) \leq \eta(1) l_{H(y,y)}(x, r)$ for any $r > 0$. Hence $L_G(y) \leq \eta(1) L_{H(y,y)}(x)$ and $L_G(y) \leq \eta(1) l_{H(y,y)}(x)$. Notice that the inverse map $F^{-1} : (\mathbb{R}^2, D) \to (\mathbb{R}^2, D)$ is an $\eta_1$-quasisymmetry. Applying the inequality $l_G(y) \leq \eta(1) l_{H(y,y)}(x)$ to $I := H(\cdot, y)^{-1}$ and $G^{-1}$ we obtain:

$$\frac{1}{L_G(y)} = l_{G^{-1}}(G(y)) \leq \eta(1) \cdot l_I(H(x, y)) = \frac{\eta^{-1}(1) \cdot \eta(1)}{L_{H(y,y)}(x)},$$

hence $L_G(y) \geq \eta^{-1}(1) L_{H(y,y)}(x)$. Similarly we prove $l_G(y) \geq \eta^{-1}(1) l_{H(y,y)}(x)$.

Because $G : \mathbb{R} \to \mathbb{R}$ is a quasisymmetry, it is differentiable a.e. (with respect to the Lebesgue measure).

Lemma 4.4. Let $y \in \mathbb{R}$ be such that $G'(y)$ exists. Then $0 < l_G(y) = L_G(y) = G'(y) < \infty$.

Proof. Let $y \in Y$ be such that $G'(y)$ exists. Then $0 \leq l_G(y) = L_G(y) = G'(y) < \infty$. Suppose $G'(y) = 0$. Then Lemma 4.3 (3) implies $L_{H(y,y)}(x) = 0$ for all $x \in \mathbb{R}$. It follows that $H(\cdot, y) : \mathbb{R} \to \mathbb{R}$ is a constant function, contradicting the fact that $H(\cdot, y)$ is a homeomorphism. Hence $G'(y) \neq 0$.

\begin{flushright} \Box \end{flushright}
Lemma 4.5. Let \( y \in \mathbb{R} \) be such that \( G'(y) \) exists. Then the map \( H(\cdot, y) : \mathbb{R} \to \mathbb{R} \) is an \( \eta(1)/\eta^{-1}(1) \)-quasisimilarity with constant \( G'(y) \).

Proof. By Lemma 4.3 (2) we have \( l_{H(\cdot, y)}(x) \geq l_G(y)/\eta(1) \) for all \( x \in \mathbb{R} \). Lemma 4.3 (3) and Lemma 4.6 imply \( L_{H(\cdot, y)}(x) \leq L_G(y)/\eta^{-1}(1) = l_G(y)/\eta^{-1}(1) \) for all \( x \in \mathbb{R} \). Because \( \mathbb{R} \) is a geodesic space, the map \( H(\cdot, y) \) is an \( \eta(1)/\eta^{-1}(1) \)-quasisimilarity with constant \( l_G(y) = G'(y) \).

Lemma 4.6. There exists a constant \( C > 0 \) with the following properties:

(1) For each \( y \in \mathbb{R} \), \( H(\cdot, y) \) is an \( \eta(1)/\eta^{-1}(1) \)^4-quasisimilarity with constant \( C \);

(2) \( G : \mathbb{R} \to \mathbb{R} \) is an \( \eta(1)/\eta^{-1}(1) \)^5-quasisimilarity with constant \( C \).

Proof. (1) Fix any \( y_0 \in \mathbb{R} \) such that \( G'(y_0) \) exists and set \( C = G'(y_0) \). Let \( y \in \mathbb{R} \) be any point such that \( G'(y) \) exists. By Lemma 4.3 the map \( H(\cdot, y) : \mathbb{R} \to \mathbb{R} \) is an \( \eta(1)/\eta^{-1}(1) \)-quasisimilarity with constant \( G'(y) \). Let \( x_0 \in \mathbb{R} \) and choose \( x \in \mathbb{R} \) such that \( |x - x_0| \geq |y - y_0| \). Let \( x' = x + (y_0 - y) \ln |y_0 - y| \). Then

\[
D((x', y_0), (x_0, y)) = D((x, y), (x_0, y)) = |x - x_0|.
\]

By picking \( x \) so that in addition

\[
\kappa := |H(x', y_0) - H(x_0, y) - (G(y_0) - G(y))\ln |G(y_0) - G(y)|| > |G(y_0) - G(y)|,
\]

by the \( \eta \)-quasisymmetry of \( F \) we have

\[
\kappa = D(F(x', y_0), F(x_0, y)) \leq \eta(1)D(F(x, y), F(x_0, y)) = \eta(1)|H(x, y) - H(x_0, y)|.
\]

By Lemma 4.5 and the choice of \( y \), we have

\[
|H(x, y) - H(x_0, y)| \leq (\eta(1)/\eta^{-1}(1))l_G(y)|x - x_0|.
\]

On the other hand, letting \( \tau = (G(y_0) - G(y))\ln |G(y_0) - G(y)| \), we have

\[
\kappa \geq |H(x', y_0) - H(x_0, y_0)| - |H(x_0, y_0) - H(x_0, y)| - |\tau|
\]

\[
\geq -\frac{G'(y_0)}{\eta(1)/\eta^{-1}(1)}|x' - x_0| - |H(x_0, y_0) - H(x_0, y)| - |\tau|.
\]

Combining the above inequalities and letting \( |x - x_0| \to \infty \), we obtain

\[
G'(y) = l_G(y) \geq \frac{1}{(\eta(1))^{3}(\eta^{-1}(1))^{-2}}G'(y_0) = \frac{C}{(\eta(1))^{3}(\eta^{-1}(1))^{-2}}.
\]

Switching the roles of \( y \) and \( y_0 \) we obtain

\[
G'(y_0) \geq \frac{1}{(\eta(1))^{3}(\eta^{-1}(1))^{-2}}G'(y).
\]
Hence \( \frac{C}{(\eta(1)\eta^{-1}(1))^{2}} \leq G'(y) \leq C(\eta(1))^{3}(\eta^{-1}(1))^{-2} \). By Lemma 4.3 for all \( x \in \mathbb{R} \),

\[
L_{H(\cdot,y)}(x) \leq L_{G(y)}/\eta^{-1}(1) \leq C(\eta(1))^{3}(\eta^{-1}(1))^{-3}
\]

and

\[
l_{H(\cdot,y)}(x) \geq \frac{1}{\eta(1)}l_{G}(y) \geq \frac{C}{(\eta(1))^{4}(\eta^{-1}(1))^{-2}}.
\]

Hence for a.e. \( y \in \mathbb{R} \), the map \( H(\cdot,y) \) is an \((\eta(1)/\eta^{-1}(1))^{4}\)-quasisimilarity with constant \( C \). A limiting argument shows that this is true for all \( y \).

(2) Statement (1) implies the following for all \( x, y \in \mathbb{R} \),

\[
\frac{C}{(\eta(1)/\eta^{-1}(1))^{4}} \leq l_{H(\cdot,y)}(x) \leq L_{H(\cdot,y)}(x) \leq C(\eta(1)/\eta^{-1}(1))^{4}.
\]

Now Lemma 4.3 implies

\[
\frac{C}{(\eta(1)/\eta^{-1}(1))^{5}} \leq l_{G}(y) \leq L_{G}(y) \leq C(\eta(1)/\eta^{-1}(1))^{5}
\]

for all \( y \in \mathbb{R} \). Hence (2) holds.

\( \square \)

**Lemma 4.7.** \( F \) is an \((\eta(1)/\eta^{-1}(1))^{6}\)-quasisimilarity with constant \( C \), where \( C \) is the constant in Lemma 4.6.

**Proof.** Set \( K = (\eta(1)/\eta^{-1}(1))^{5} \). Let \((x_1, y_1), (x_2, y_2) \in \mathbb{R}^2\). We shall first establish a lower bound for \( D(F(x_1, y_1), F(x_2, y_2)) \). Set \( \tau = x_1 - x_2 - (y_1 - y_2) \ln |y_1 - y_2| \). If \( |\tau| \leq |y_1 - y_2| \), then \( D((x_1, y_1), (x_2, y_2)) = |y_1 - y_2| \) and by Lemma 4.6 (2),

\[
D(F(x_1, y_1), F(x_2, y_2)) \geq |G(y_1) - G(y_2)| \geq \frac{C}{K}|y_1 - y_2| = \frac{C}{K}D((x_1, y_1), (x_2, y_2)).
\]

If \( |\tau| > |y_1 - y_2| \), then

\[
D((x_1, y_1), (x_2, y_2)) = |\tau| = D((x_1 - (y_1 - y_2) \ln |y_1 - y_2|, y_2), (x_2, y_2)),
\]

and since \( F \) is an \( \eta \)-quasisymmetry, we have

\[
D(F(x_1, y_1), F(x_2, y_2)) \geq \frac{1}{\eta(1)}D(F(x_1 - (y_1 - y_2) \ln |y_1 - y_2|, y_2), F(x_2, y_2))
\]

\[
= \frac{1}{\eta(1)}|H(x_1 - (y_1 - y_2) \ln |y_1 - y_2|, y_2) - H(x_2, y_2)|
\]

\[
\geq \frac{C}{\eta(1)K}|x_1 - x_2 - (y_1 - y_2) \ln |y_1 - y_2||
\]

\[
= \frac{C}{\eta(1)K}D((x_1, y_1), (x_2, y_2)),
\]

with the second inequality following from Lemma 4.6 (1). Hence we have a lower bound for \( D(F(x_1, y_1), F(x_2, y_2)) \).
By Lemma 4.6 (2), $G^{-1} : \mathbb{R} \to \mathbb{R}$ is a $K$-quasisimilarity with constant $C^{-1}$. Similarly, Lemma 4.6 (1) implies that for each $y \in \mathbb{R}$, $(H(\cdot, y))^{-1}$ is a $K$-quasisimilarity with constant $C^{-1}$. Also recall that $F^{-1}$ is an $\eta_1$-quasisymmetry and $F$ is an $\eta$-quasisymmetry. Now the argument in the previous paragraph applied to $F^{-1}$ implies
\[
D(F^{-1}(x_1, y_1), F^{-1}(x_2, y_2)) \geq \frac{1}{CK\eta(1)}D((x_1, y_1), (x_2, y_2)).
\]
It follows that
\[
D(F(x_1, y_1), F(x_2, y_2)) \leq CK\eta(1)D((x_1, y_1), (x_2, y_2)) = CK/\eta^{-1}(1)D((x_1, y_1), (x_2, y_2))
\]
for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$. Hence we also obtain an upper bound for the quantity $D(F(x_1, y_1), F(x_2, y_2))$. \hfill \qed

5 Characterization of quasisymmetric maps

In this section we give a complete description of all self quasisymmetric maps of $\partial G_A$.

**Theorem 5.1.** A map $F : (\mathbb{R}^2, D) \to (\mathbb{R}^2, D)$ is a quasisymmetric map if and only if it has the following form: $F(x, y) = (ax + c(y), ay + b)$ for all $(x, y) \in \mathbb{R}^2$, where $a \neq 0$, $b$ are constants and $c : \mathbb{R} \to \mathbb{R}$ is a Lipschitz map.

Let $F : (\mathbb{R}^2, D) \to (\mathbb{R}^2, D)$ be a quasisymmetric map. From Section 4 we know there is a quasisymmetric map $G : \mathbb{R} \to \mathbb{R}$, and for each $y \in \mathbb{R}$ there is a quasisymmetric map $H(\cdot, y) : \mathbb{R} \to \mathbb{R}$ such that $F(x, y) = (H(x, y), G(y))$ for all $(x, y) \in \mathbb{R}^2$. Then $G'(y)$ exists almost everywhere. Similarly, for each $y \in \mathbb{R}$, the map $H(\cdot, y)$ has derivative $H_x(x, y)$ for a.e. $x \in \mathbb{R}$.

**Lemma 5.2.** Let $F : (\mathbb{R}^2, D) \to (\mathbb{R}^2, D)$ be a quasisymmetric map. Let $y \in \mathbb{R}$ be such that $G'(y)$ exists, and $x \in \mathbb{R}$ such that $H_x(x, y)$ exists at $x$. Then $G'(y) = H_x(x, y)$.

**Proof.** Let $F : (\mathbb{R}^2, D) \to (\mathbb{R}^2, D)$ be an $\eta$-quasisymmetric map. By replacing $F$ with $T_{(-H(x, y), -G(y))} \circ F \circ T_{(x, y)}$, we may assume $(x, y) = (H(x, y), G(y)) = 0$. Here $T_{(x, y)}$ denotes the Euclidean translation by $(x, y)$. Lemma 4.4 implies $G'(0) \neq 0$. By composing $F$ with a dilation $\lambda_t$ for a suitable $t$ we may assume $G'(0) = 1$ or $-1$. If $G'(0) = -1$, we further compose $F$ with the rotation $\pi : \mathbb{R}^2 \to \mathbb{R}^2$, $\pi(x, y) = (-x, -y)$. Hence we may assume $G'(0) = 1$. Denote $\lambda = H_x(x, 0)$. By Lemma 4.6 (2) we have $\lambda \neq 0$. We shall prove that $\lambda = 1$.

Since $\lambda_t$ is a similarity, the family of maps $\{F^t := \lambda_t \circ F \circ \lambda_{-t} | t \in \mathbb{R}\}$ consists of $\eta$-quasisymmetric maps. Write $F^t(x, y) = (H^t(x, y), G^t(y))$. We notice that $H^t(x, 0) = e^tH(e^{-t}x, 0)$ and $G^t(y) = e^tG(e^{-t}y)$. Since the derivative $H_x(0, 0)$ exists, the maps $H^t(\cdot, 0) : \mathbb{R} \to \mathbb{R}$ converge (as $t \to \infty$) in the pointed Gromov-Hausdorff distance towards the map $x \to \lambda x$. Similarly, the maps $G^t : \mathbb{R} \to \mathbb{R}$ converge (as $t \to \infty$) in the pointed Gromov-Hausdorff distance towards the map $y \to y$. The compactness property of quasisymmetric maps implies that there is a sequences $t_i \to \infty$ such that $F^{t_i}$ converges in the pointed
Gromov-Hausdorff distance towards an \( \eta \)-quasisymmetric map \( \tilde{F} : (\mathbb{R}^2, D) \to (\mathbb{R}^2, D) \). If we write \( \tilde{F}(x, y) = (\tilde{H}(x, y), \tilde{G}(y)) \), then \( \tilde{G}(y) = y \) and \( \tilde{H}(x, 0) = \lambda x \).

By Theorem 4.1, the map \( \tilde{F} \) is \( L \)-biLipschitz for some \( L \geq 1 \). Fix some \( x \in \mathbb{R} \) and a positive integer \( n \geq 1 \). For \( i = 0, \ldots, n \), let \( (x_i, y_i) = (x - \frac{i}{n} \ln n, \frac{i}{n}) \). Then \( D((x_i, y_i), (x_{i+1}, y_{i+1})) = 1/n \). Hence

\[
|\tilde{H}(x_i, y_i) - \tilde{H}(x_{i+1}, y_{i+1}) - \frac{1}{n} \ln n| \leq D(\tilde{F}(x_i, y_i), \tilde{F}(x_{i+1}, y_{i+1})) \leq L \cdot \frac{1}{n}.
\]

Adding up all these inequalities for \( i = 0, \ldots, n - 1 \) and using the triangle inequality we obtain

\[
|\tilde{H}(x_0, y_0) - \tilde{H}(x_n, y_n) - \ln n| \leq L.
\]  

(5.1)

On the other hand, \( D((x_n, y_n), (x - \ln n, 0)) = 1 \) and hence

\[
|\tilde{H}(x_n, y_n) - \tilde{H}(x - \ln n, 0)| \leq D(\tilde{F}(x_n, y_n), \tilde{F}(x - \ln n, 0)) \leq L.
\]  

(5.2)

It follows from (5.1) and (5.2) that \( |\tilde{H}(x_0, y_0) - \tilde{H}(x - \ln n, 0) - \ln n| \leq 2L \). Notice that \( \tilde{H}(x_0, y_0) = \tilde{H}(x, 0) = \lambda x \) and \( \tilde{H}(x - \ln n, 0) = \lambda (x - \ln n) \). So we have \(|(\lambda - 1) \ln n| \leq 2L \).

Since this is true for all \( n \geq 1 \), we must have \( \lambda = 1 \).

\[ \square \]

**Lemma 5.3.** There exist constants \( a \neq 0 \) and \( b \) and also a function \( c : \mathbb{R} \to \mathbb{R} \) such that

(1) \( G(y) = ay + b \);

(2) \( H(x, y) = ax + c(y) \) for all \( (x, y) \in \mathbb{R}^2 \).

**Proof.** Let \( y \in \mathbb{R} \) be any point where \( G \) is differentiable. By Lemma 5.2, the quasisymmetric map \( H(\cdot, y) : \mathbb{R} \to \mathbb{R} \) a.e. has derivative \( G'(y) \). It follows that \( H(\cdot, y) \) is an affine map; to be more precise, there is a constant \( c(y) \) depending only on \( y \) such that \( H(x, y) = G'(y)x + c(y) \) for all \( x \in \mathbb{R} \).

We claim that \( G'(y_1) = G'(y_2) \) holds for any two points \( y_1, y_2 \in \mathbb{R} \) at which \( G \) is differentiable. Set \( \tau = (y_2 - y_1) \ln |y_2 - y_1| \). Let \( x > 0 \) and denote \( p = (0, y_1), q = (x, y_1), p' = (\tau, y_2) \) and \( q' = (x + \tau, y_2) \). One checks that \( D(p, q) = D(p', q') = x \) and \( D(p, p') = D(q, q') = |y_2 - y_1| \). By Theorem 4.1, \( F \) is \( L \)-biLipschitz for some \( L \geq 1 \). We have \( D(F(p), F(p')) \leq L|y_2 - y_1| \) and \( D(F(q), F(q')) \leq L|y_2 - y_1| \). On the other hand, by the preceding paragraph, we have \( F(p) = (c(y_1), G(y_1)), F(q) = (G'(y_1)x + c(y_1), G(y_1)) \) and \( F(p') = (G'(y_2)x + c(y_2), G(y_2)) \). \( F(q') = (G'(y_2)(x + \tau) + c(y_2), G(y_2)) \). Set \( \tau' = (G'(y_2) - G'(y_1)) \ln |y_2 - y_1| \).

\[ |G'(y_2)(x + \tau) + c(y_2)| - |G'(y_1)x + c(y_1)| - \tau'| \leq D(F(q), F(q')) \leq L|y_2 - y_1| \]

for all \( x > 0 \), we must have \( G'(y_1) = G'(y_2) \).

Since \( G \) is differentiable a.e., it follows from the above claim that \( G \) a.e. has constant derivative, hence must be an affine map. That is, there are constants \( a \neq 0, b \) such that \( G(y) = ay + b \) for all \( y \in \mathbb{R} \). This proves (1). Now (2) follows from (1) and the first paragraph.

\[ \square \]
Completing the proof of Theorem 5.1. First suppose \( F : (\mathbb{R}^2, D) \to (\mathbb{R}^2, D) \) is a quasisymmetric map. Then by Lemma 5.3 \( F \) has the form \( F(x, y) = (ax + c(y), ay + b) \), where \( a \neq 0 \), \( b \) are constants, and \( c : \mathbb{R} \to \mathbb{R} \) is a function. Now fix \( y_1, y_2 \in \mathbb{R} \). Let \( \tau = (y_2 - y_1) \ln |y_2 - y_1| \) and denote \( p = (0, y_1) \), \( q = (\tau, y_2) \). One checks that \( D(p, q) = |y_2 - y_1| \). By Theorem 4.1 \( F \) is a \( L \)-biLipschitz map for some \( L \geq 1 \). Hence the above calculation shows that \( \lambda_\pi \) is Lipschitz, as follows. Let \( \lambda = (y_2 - y_1) \ln |y_2 - y_1| \) and denote \( p = (0, y_1) \), \( q = (\lambda, y_2) \). One checks that \( D(p, q) = |y_2 - y_1| \). By Theorem 4.1 \( F \) is a \( L \)-biLipschitz map for some \( L \geq 1 \). Hence \( D(F(p), F(q)) \leq LD(p, q) = |y_2 - y_1| \). On the other hand, \( F(p) = \lambda \) and \( F(q) = \lambda \). We have \( D(F(p), F(q)) \geq |(a\tau + c(y_2) - c(y_1)) - a(y_2 - y_1) \ln |a(y_2 - y_1)|| \)

\[
\begin{align*}
D(F(p), F(q)) &\geq |(a\tau + c(y_2) - c(y_1)) - a(y_2 - y_1) \ln |a(y_2 - y_1)|| \\
&= |c(y_2) - c(y_1)| - (a \ln |a|)(y_2 - y_1)|.
\end{align*}
\]

Now the triangle inequality implies \( |c(y_2) - c(y_1)| \leq (L + |a \ln |a||)|y_2 - y_1| \), that is, \( c \) is \( (L + |a \ln |a||) \)-Lipschitz.

Conversely, suppose \( F \) has the form \( F(x, y) = (ax + c(y), ay + b) \), where \( a \neq 0 \), \( b \) are constants, and \( c : \mathbb{R} \to \mathbb{R} \) is \( L \)-Lipschitz. One checks by direct calculation that \( F \) is Lipschitz, as follows. Let \( p = (x, y) \), \( q = (x', y') \in \mathbb{R}^2 \) be two arbitrary points. Then \( F(p) = (ax + c(y), ay + b) \) and \( F(q) = (ax' + c(y'), ay' + b) \). Set \( \tau = (x' - x) - (y' - y) \ln |y' - y| \). We have \( D(p, q) = \max \{|y' - y|, |\tau|\} \) and

\[
D(F(p), F(q)) = \max \{|a(y' - y)|, |a\tau + c(y') - c(y)| - (a \ln |a|)(y' - y)|\}.
\]

Now \( |a(y' - y)| = |a| \cdot |y' - y| \leq |a|D(p, q) \) and

\[
\begin{align*}
|a\tau + c(y') - c(y)| - (a \ln |a|)(y' - y)| &\leq |a\tau| + |c(y') - c(y)| + |(a \ln |a|)(y' - y)| \\
&\leq |a|D(p, q) + L|y' - y| + |a \ln |a|| \cdot |y' - y| \\
&\leq (|a| + L + |a \ln |a||)D(p, q).
\end{align*}
\]

It follows that \( F \) is Lipschitz with Lipschitz constant \( |a| + L + |a \ln |a|| \). On the other hand, \( F^{-1} \) has the form

\[
F^{-1}(x, y) = \left( \frac{1}{a} \cdot x - \frac{1}{a} \cdot c \left( \frac{1}{a} y - \frac{b}{a} \right), \frac{1}{a} \cdot y - \frac{b}{a} \right).
\]

As a composition of Lipschitz maps, the map \( c' : \mathbb{R} \to \mathbb{R} \), \( c'(y) = -\frac{1}{a} c \left( \frac{1}{a} y - \frac{b}{a} \right) \) is also Lipschitz. Hence the above calculation shows that \( F^{-1} \) is also Lipschitz.

\[\square\]

6 A Liouville type theorem for \((\mathbb{R}^2, D)\)

In this section we prove a Liouville type theorem for \((\mathbb{R}^2, D)\), which says that all conformal maps of \((\mathbb{R}^2, D)\) are boundary maps of isometries of \( G_A \). We first identify all the conformal maps of \((\mathbb{R}^2, D)\).

Using Theorem 5.1 we can identify all the isometries and similarities of \((\mathbb{R}^2, D)\). Recall that the map \( \pi \) and similarities \( \lambda_i \) are defined in Proposition 2.3.
Proposition 6.1. (1) The group of all isometries of \((\mathbb{R}^2, D)\) is generated by Euclidean translations and \(\pi\);
(2) The group of all similarities of \((\mathbb{R}^2, D)\) is generated by Euclidean translations, \(\pi\) and the similarities \(\lambda_t\) \((t \in \mathbb{R})\).

Proof. We only prove (2), the proof of (1) being similar. Let \(F : (\mathbb{R}^2, D) \to (\mathbb{R}^2, D)\) be a similarity. By composing \(F\) with a suitable \(\lambda_t\), we may assume \(F\) is an isometry. By Theorem 5.1, \(F\) has the form \(F(x, y) = (ax + c(y), ay + b)\), where \(a \neq 0\), \(b\) are constants and \(c : \mathbb{R} \to \mathbb{R}\) is a Lipschitz map. By composing \(F\) with \(\lambda_t\) and \(c\), following form:
\[F(H(x, y)) = (x + c(y), y)\]
We only prove (2), the proof of (1) being similar. Let \(F : (\mathbb{R}^2, D) \to (\mathbb{R}^2, D)\)
Proof. Since \(F\) is a similarity, it is quasisymmetric in particular. By Theorem 1.1, \(F\) has the form \(F(x, y) = (x + c(y), ay + b)\), where \(a \neq 0\), \(b\) are constants and \(c : \mathbb{R} \to \mathbb{R}\) is a Lipschitz map. By composing \(F\) with a similarity, we may assume \(a = 1\) and \(b = 0\); that is, \(F(x, y) = (x + c(y), y)\). We shall prove that \(c(y)\) is a constant function.

Since \(c : \mathbb{R} \to \mathbb{R}\) is a Lipschitz function, it is differentiable a.e. We shall show that \(c'(y) = 0\) for a.e. \(y \in \mathbb{R}\). By the definition of a conformal map, \(L_F(x, y) = l_F(x, y)\) for a.e. \((x, y) \in \mathbb{R}^2\) with respect to the Lebesgue measure in \(\mathbb{R}^2\). It follows from Fubini’s theorem that for a.e. \(y \in \mathbb{R}\), the derivative \(c'(y)\) exists and \(L_F(x, y) = l_F(x, y)\) for a.e. \(x \in \mathbb{R}\). Let \(y_0\) be an arbitrary such point and \(x_0 \in \mathbb{R}\) be such that \(L_F(x_0, y_0) = l_F(x_0, y_0)\). We will show \(c'(y_0) = 0\).

By pre-composing and post-composing with Euclidean translations if necessary, we may assume that \(L_F(x_0, y_0) = l_F(x_0, y_0)\) = 0. We need to show \(c'(0) = 0\). We will suppose \(c'(0) \neq 0\) and get a contradiction. Notice that \(F(x, 0) = (x, 0)\) for all \(x \in \mathbb{R}\). It follows that \(L_F(0, 0) \geq 1\) and \(l_F(0, 0) \leq 1\). Combining this with the assumption \(L_F(0, 0) = l_F(0, 0)\), we obtain \(L_F(0, 0) = l_F(0, 0) = 1\). First suppose \(c'(0) > 0\). Then \(c(y) > 0\) for sufficiently small \(y > 0\). Let \(p = (0, 0)\) and \(q = (r + r \ln r, r)\) with \(r > 0\). Then \(F(p) = p\) and \(F(q) = (r + r \ln r + c(r), r)\). One calculates \(D(p, q) = r\) and \(D(F(p), F(q)) = r + c(r)\). It follows that \(L_F(p, r) \geq r + c(r)\) and hence \(L_F(p, r) \geq 1 + c'(0) > 1\), contradicting \(L_F(0, 0) = 1\). If \(c'(0) < 0\), then letting \(q = (-r + r \ln r, r)\) one similarly obtains a contradiction.
Theorem 6.3. Let \( F : (\mathbb{R}^2, D) \to (\mathbb{R}^2, D) \) be a conformal map. Then \( F \) is the boundary map of some isometry \( f : G_A \to G_A \).

Proof. By Lemma 6.2, \( F \) is a similarity. By Proposition 6.1 (2), \( F \) is the composition of Euclidean translations, \( \tau \) and similarities \( \lambda_t \). Now the theorem follows from the following facts (see the end of Section 2): (1) Euclidean translations of \( \mathbb{R}^2 \) are boundary maps of the Lie group left translations \( \mathcal{L} \) for elements of the form \( g = ((x, y), 0) \in G_A \); (2) \( \tau \) is the boundary map of the isometry \( \tau : G_A \to G_A \); (3) \( \lambda_t \) is the boundary map of the Lie group left translation \( \mathcal{L} \) for \( g = ((0, 0), t) \).

\[ \qed \]

7 Quasiisometries of \( G_A \)

In this section we calculate the quasiisometry group of \( G_A \) and identify all the quasiisometries of \( G_A \) up to bounded distance. From this it is easy to see that all quasiisometries of \( G_A \) are almost isometries and are height-respecting.

We first discuss the structure of the group \( QS(\mathbb{R}_2^2, D) \) of all quasisymmetric maps of \( (\mathbb{R}_2^2, D) \). We identify three subgroups of \( QS(\mathbb{R}_2^2, D) \). Let \( H_1 = \{ \lambda_t : t \in \mathbb{R} \} \cong \mathbb{R} \). Let \( H_2 = < \tau > \cong \mathbb{Z}_2 = \{ 0, 1 \} \) be the order 2 cyclic group generated by \( \tau \). Let \( H_3 \) be the group of homeomorphisms of \( \mathbb{R}_2^2 \) of the form \( F_{C,b}(x, y) = (x + C(y), y + b) \), where \( b \in \mathbb{R} \) and \( C : \mathbb{R} \to \mathbb{R} \) is a Lipschitz function. Direct calculations show that \( H_1 \) and \( H_2 \) commute, both \( H_1 \) and \( H_2 \) normalize \( H_3 \), and \( H_3 \cap < H_1, H_2 > \) is trivial. On the other hand, Theorem 5.1 implies that \( QS(\mathbb{R}_2^2, D) \) is generated by \( H_1, H_2 \) and \( H_3 \). It follows that we have the following isomorphism:

\[ QS(\mathbb{R}_2^2, D) \cong H_3 \times (H_1 \oplus H_2). \]

Let \( L \) be the additive group consisting of Lipschitz functions \( C : \mathbb{R} \to \mathbb{R} \). Let \( \mathbb{R} \) act on \( L \) by \( b \ast C = C \circ T_b \), for \( b \in \mathbb{R} \) and \( C \in L \), where \( T_b \) is the translation on \( \mathbb{R} \) by \( b \). Then it is easy to check that the map given by \( F_{C,b} \mapsto (C, b) \) defines an isomorphism from the group \( H_3 \) to the opposite group \( L \times \mathbb{R} \) of \( L \times \mathbb{R} \). It now follows that we have the following isomorphism:

\[ QS(\mathbb{R}_2^2, D) \cong (L \times \mathbb{R}) \times (\mathbb{R} \times \mathbb{Z}_2). \]

Here the action of \( \mathbb{R} \times \{ 0 \} \) on \( L \times \mathbb{R} \) is given by \((t, 0) \ast (C, b) = (C', b') \) for \((t, 0) \in \mathbb{R} \times \{ 0 \} \) and \((C, b) \in L \times \mathbb{R} \), where

\[ C'(y) = e^t \cdot C(e^{-t}y) + bte^t \quad \text{and} \quad b' = e^tb; \]

and the action of \( \{ 0 \} \times \mathbb{Z}_2 \) on \( L \times \mathbb{R} \) is given by \((0, 1) \ast (C, b) = (C'', -b) \), where \( C''(y) = -C(-y) \).

Two quasiisometries \( f, g : X \to Y \) between two metric spaces are said to be equivalent if \( \sup \{ d(f(x), g(x)) : x \in X \} < \infty \). For any metric space \( X \), the quasiisometry group \( QI(X) \) consists of equivalence classes of quasiisometries \( X \to X \) and has group operation given by composition.

For each quasiisometry \( f : G_A \to G_A \), let \( \partial f : \partial G_A \to \partial G_A \) be its boundary map.
Theorem 7.1. We have $QI(G_A) \cong QS(\mathbb{R}^2, D) \cong (L \times \mathbb{R}) \times (\mathbb{R} \times \mathbb{Z}_2)$, where $L$ and the various actions are as described above.

Proof. By Proposition 3.5 each quasiisometry $f : G_A \to G_A$ induces a quasisymmetric map $\partial f : (\mathbb{R}^2, D) \to (\mathbb{R}^2, D)$. Notice that an quasiisometry $f$ of $G_A$ is at finite distance from the identity map of $G_A$ if and only if the boundary map of $f$ is the identity map on $(\mathbb{R}^2, D)$. Hence the map $\partial_1 : QI(G_A) \to QS(\mathbb{R}^2, D)$, $\partial_1([f]) = \partial f$ is well-defined and is injective. Here $[f]$ denotes the equivalence of $f$. On the other hand, by [BS] each element in $QS(\mathbb{R}^2, D)$ is the boundary map of an quasiisometry. Hence $\partial_1 : QI(G_A) \to QS(\mathbb{R}^2, d)$ is also surjective.

We now identify a group of quasiisometries of $G_A$ that is isomorphic to $QI(G_A)$. Let $H'_1 = \{L_g : g = ((0, 0), t) \in G_A\}$. Let $H'_2 = \langle \tau' \rangle$, where $\tau'(x, y, t) = ((-x, -y), t)$ is the automorphism of $G_A$ defined in Section 2. The two groups $H_1$ and $H_2$ consist of isometries of $G_A$. Let $H'_3$ be the group of homeomorphisms of $G_A$ of the following form:

$$f_{C,b} : G_A \to G_A, \quad f_{C,b}(x, y, t) = ((x + C(y), y + b), t),$$

where $b \in \mathbb{R}$ and $C \in L$. It is clear that $H'_3$ is isomorphic to $H_3 \cong \mathbb{L} \times \mathbb{R}$. Using Lemma 6.3 in [SX] and the fact that $F_{C,b} : (\mathbb{R}^2, D_e) \to (\mathbb{R}^2, D_e)$ is biLipschitz, it is easy to check that each $f_{C,b}$ ($b \in \mathbb{R}$, $C \in L$) is an almost isometry of $G_A$. In particular, $H_3$ consists of quasiisometries of $G_A$.

Let $QI'(G_A)$ be the group of homeomorphisms of $G_A$ generated by $H'_1$, $H'_2$ and $H'_3$. A similar discussion as above shows that $QI'(G_A) \cong H'_3 \rtimes (H'_1 \times H'_2) \cong L \times \mathbb{R} \times (\mathbb{R} \times \mathbb{Z}_2)$, where the various actions are as described above. Let $\partial : QI'(G_A) \to QS(\mathbb{R}^2, D)$ be the map that assigns to each $f \in QI'(G_A)$ its boundary map. It is easy to see that $\partial$ maps $H'_i$ ($i = 1, 2, 3$) isomorphically onto $H_i$. It follows that $\partial$ is an isomorphism. Let $p : QI'(G_A) \to QI(G_A)$ be the group homomorphism that assigns to each $f \in QI'(G_A)$ its equivalence class. Since $\partial = \partial_1 \circ p$ (where $\partial_1$ is defined in the proof of Theorem 7.1), it follows from Theorem 7.1 that $p$ is an isomorphism. We obtain:

Theorem 7.2. Every quasiisometry $f : G_A \to G_A$ is at a finite distance from exactly one element of $QI'(G_A)$.

Now we can provide a proof of Corollary 1.3

Proof of Corollary 1.3. Since each element in $H'_1$ and $H'_2$ is an isometry of $G_A$, and every element of $H'_3$ is an almost isometry, we see that every element of $QI'(G_A)$ is an almost isometry. Now Corollary 1.3 follows from this and Theorem 7.2.

Under the identification of $G_A$ with $\mathbb{R}^2 \times \mathbb{R}$, we view the map $h : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$, $h((x, y), t) = t$ as the height function. A quasiisometry $f : G_A \to G_A$ is height-respecting if $|h(f((x, y), t)) - t|$ is bounded independent of $((x, y), t) \in G_A$. Since every element of $H'_1$, $H'_2$ and $H'_3$ is height-respecting, we have

Corollary 7.3. All self quasiisometries of $G_A$ are height-respecting.
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