MAYER-VIETORIS SEQUENCE FOR GENERATING FAMILIES IN DIFFEOLOGICAL SPACES

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ABSTRACT. We prove a version of the Mayer-Vietoris sequence for De Rham differential forms in diffeological spaces. It is based on the notion of a generating family instead of that of a covering by open subsets.

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INTRODUCTION

Diffeological spaces are a generalization of differentiable manifolds which provides a unified framework for non-classical objects in differential geometry like quotients of manifolds, spaces of leaves of foliations, spaces of smooth functions or groups of diffeomorphisms. Roughly speaking, a diffeology on an set $X$ specifies which of the maps from a domain in $\mathbb{R}^m$, $m \geq 0$, into $X$ (called plots) are smooth. In this setting one can define many constructions and invariants analogous to the classical ones, but with a much larger scope. In particular, the differential graded algebra $\Omega(X)$ of differential forms and the De Rham cohomology can be generalized to this context. The standard reference for the subject of diffeological spaces is Iglesias-Zemmour’s book [6].

In the classical De Rham theory, the Mayer-Vietoris sequence is considered the main basic tool for computing the cohomology groups of a manifold that can be written as the union of two open subspaces. In diffeological spaces, versions of this Mayer-Vietoris result have been given. For instance, Iwase and Izumida [8] introduced a new version of differential forms (cubical differential forms) in order to obtain partitions of unity and an exact Mayer-Vietoris sequence. Also, Haraguchi [4] considered the Mayer-Vietoris sequence for general differential forms and gave sufficient conditions for the existence of partitions of unity. In [9], Kuribayashi developed cohomological methods in diffeological

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spaces and used a Mayer-Vietoris argument for different cochain complexes.

In all these cases, the classical theory is imitated by considering two open sets of the so-called $D$-topology. Those are the sets $W \subseteq X$ such that $\alpha^{-1}(W)$ is open in $U$ for any plot $\alpha: U \to X$ of the diffeology. However, it is our opinion that introducing the $D$-topology fits poorly with the general philosophy of diffeological spaces, where topology is often irrelevant and differential objects are defined intrinsically: we can give a space a differentiable structure without first giving it a topology. See [12] for a discussion on this issue. In other words, smoothness is a property that is not based on continuity: for instance, the $D$-topology is determined by the smooth curves, while diffeologies are not [3, Theorem 3.7].

There is another notion more intimately linked to the foundations of the theory, namely that of a generating family (Definition 1.5.1). Many diffeologies are constructed by only giving a generating family of plots, often much smaller than the whole diffeology. This mode of construction of diffeologies is very useful because it reduces the study of a diffeological space to a subset of its plots.

Then, instead of a covering \{U, V\} of $X$ by two open sets, which would give the classical sequence

$$0 \to \Omega(X) \to \Omega(U) \oplus \Omega(V) \to \Omega(U \cap V) \to 0,$$

our version of the Mayer-Vietoris result is based on a generating family \{\alpha, \beta\} formed by two plots $\alpha, \beta$. We get an exact sequence

$$0 \to \Omega(X) \to \Omega(\alpha) \oplus \Omega(\beta) \to \Omega(P),$$

where $\Omega(\alpha)$ (resp. $\Omega(\beta)$) is the subcomplex of $\Omega(U)$ of $\alpha$-horizontal (resp. $\beta$-horizontal) forms (see [2,3]) and $P$ is the pullback of two plots (see [1,6]). This idea is natural, because $U \cap V$ can be viewed as the pullback of the inclusions $U, V \subseteq X$.

Our objective is to understand how the De Rham complex of $X$ is determined by $\alpha$ and $\beta$. This is achieved in our main Theorem 3.1.2. Our procedure is close to the diffeological gluing procedure in [11], which can be considered as a form of Mayer-Vietoris construction.

Note that we do not obtain a long exact sequence in cohomology, which would need the existence of partitions of unity, a practically impossible objective in such a general context. This is coherent with the fact that diffeological spaces is a wide category that allows abstract constructions but where the results that depend on the local structure need additional hypothesis [1].
The contents of the paper are as follows. In Section 1 we present the basic definitions that we need along the paper, including that of a generating family and the pullback of two plots. In Section 2 we explain Cartan calculus and De Rham cohomology in diffeological spaces, and we introduce the notion of horizontal forms. In Section 3 we state our main Theorem 3.1.2, which is the Mayer-Vietoris sequence associated to a generating family by two plots. Finally, in Section 4 we compute the cohomology of an illustrative example.

1. DIFFEORELOGICAL SPACES

An open subset $U \subseteq \mathbb{R}^m$ of some Euclidean space, $m \geq 0$, is called an $m$-domain. A differentiable $C^\infty$ map $h: V \subseteq \mathbb{R}^n \to U \subseteq \mathbb{R}^m$ between domains will be called a change of coordinates. Let $X$ be a set. Any set map $\alpha: U \subseteq \mathbb{R}^m \to X$ defined on an $m$-domain is called a parametrization on $X$.

1.1. Diffeology. A diffeology (of class $C^\infty$) on the set $X$ is a family of parametrizations satisfying the following axioms.

**Definition 1.1.1.** Let $X$ be a set. A diffeology on $X$ is any family $D$ of parametrizations on $X$ such that

1. any constant parametrization on $X$ belongs to $D$;  
2. if $\alpha: U \subseteq \mathbb{R}^n \to X$ is a parametrization that locally belongs to $D$ (i.e. for every $x \in U$ there exists a neighbourhood $U_x$ such that $\alpha|_{U_x} \in D$), then $\alpha \in D$;  
3. if $\alpha \in D$ and $h: V \to U$ is a change of coordinates, then $\alpha \circ h \in D$.

A set endowed with a diffeology is called a diffeological space. The parametrizations of the diffeology $D$ of $X$ are called plots. For the sake of clarity, sometimes we shall make explicit the domain of a plot $\alpha: U \to X$ by denoting it as $(U, \alpha)$.

**Example 1.1.2.** If $U \subseteq \mathbb{R}^m$ is an $m$-domain, the $C^\infty$ changes of coordinates $V \to U$ with codomain $U$ form its usual diffeology. More generally, if $M$ is a finite dimensional differentiable manifold, the $C^\infty$ maps $V \subseteq \mathbb{R}^m \to M$, with domain any $m$-domain, form the usual or standard diffeology on $M$. In this case, all the diffeological objects that we shall define correspond to the usual ones.

1.2. Differentiable maps.

**Definition 1.2.1.** A map $(X, D) \xrightarrow{f} (X', D')$ between diffeological spaces is differentiable if $f \circ \alpha \in D'$ for all $\alpha \in D$. 

The composition of differentiable maps is differentiable. The plots \( U \rightarrow X \) of \( \mathcal{D} \) are differentiable for the usual diffeology on \( U \).

1.3. **Subspaces.** A nice characteristic of diffeological spaces is that any subspace inherits a diffeology, making the theory much more clean than the usual theory of differentiable manifolds and submanifolds.

**Definition 1.3.1.** If \( (X, \mathcal{D}_X) \) is a diffeological space and \( Y \subseteq X \) is any subset, we define the *induced diffeology* or *subspace diffeology* \( \mathcal{D}_Y \) on \( Y \) as the family of plots \((U, \alpha)\) in \( \mathcal{D}_X \) whose image is contained in \( Y \).

With this diffeology, not only the inclusion \( \iota_Y : Y \hookrightarrow X \) is differentiable, but a map \( F : Z \rightarrow Y \) is differentiable if and only if the composition \( \iota_Y \circ F : Z \rightarrow X \) is differentiable.

1.4. **Comparing diffeologies.** We shall avoid the words *finer* and *coarser* when comparing diffeologies on the same set \( X \). If \( \mathcal{D} \subseteq \mathcal{D}' \) we say that \( \mathcal{D}' \) is *larger* than \( \mathcal{D} \), equivalently \( \mathcal{D} \) is *smaller* than \( \mathcal{D}' \).

The smallest diffeology on \( X \) is the *discrete* diffeology, formed by all the locally constant parametrizations on \( X \).

The largest diffeology on \( X \) is formed by all possible parametrizations. It is called the *indiscrete* or *coarse* diffeology.

**Proposition 1.4.1.** The intersection of an arbitrary collection of diffeologies on \( X \) is a diffeology on \( X \).

1.5. **Generating families.** The diffeologies can be built starting with a generating family. We refer to [6, Art. 1.66 and fol.] for the proofs of the basic results that we will need here.

**Definition 1.5.1.** Let \( \mathcal{F} = \{ \alpha_j \} \) be an arbitrary family of parametrizations \( \alpha_j : U_j \rightarrow X \) and let \( \mathcal{D} = \langle \mathcal{F} \rangle \) be the intersection of all the diffeologies on \( X \) containing \( \mathcal{F} \). We say that \( \mathcal{F} \) is a generating family for \( \mathcal{D} \).

The diffeology \( \mathcal{D} = \langle \mathcal{F} \rangle \) generated by the family \( \mathcal{F} \) is then the smallest diffeology containing it. The intersection is not void because \( \mathcal{F} \) is always contained in the indiscrete diffeology. It is always possible to add to a generating family any family of parameterizations that generate the discrete diffeology, without altering the generated diffeology; consequently, we shall always assume that the family \( \mathcal{F} \) contains all the locally constant parametrizations. Then \( \langle \mathcal{F} \rangle \) can be characterized by the following property, that we will use heavily.

**Theorem 1.5.2.** The parametrization \((U, \alpha)\) is a plot in \( \langle \mathcal{F} \rangle \) if and only if for all \( x \in U \) there exists a neighborhood \( V = U_x \) of \( x \) in \( U \).
such that there exist a plot \( f: W \to X \) in the family \( F \) and a change of coordinates \( h: V \to W \) such that \( \alpha|_V = f \circ h \).

**Example 1.5.3.** The usual diffeology on \( \mathbb{R}^m \) is generated by the plot \( \text{id}: \mathbb{R}^m \to \mathbb{R}^m \).

### 1.6. Pullbacks

The category of diffeological spaces and differentiable maps allows any kind of categorical limits \[1\]. In particular, we shall need the following construction of the pullback (or **fibered product**) of two plots.

Let \((U, \alpha)\) and \((V, \beta)\) be two plots on the diffeological space \((X, \mathcal{D})\). We consider the subspace \(P\) (endowed with the induced diffeology) of the product \(U \times V\) (endowed with the usual diffeology) given by

\[
P = \{(u, v) \in U \times V : \alpha(u) = \beta(v)\};
\]

also we consider the differentiable maps

\[
p_U: P \to U, \quad p_U(u, v) = u,
\]

and

\[
p_V: P \to V, \quad p_V(u, v) = v,
\]

induced by the projections. They verify

\[
\alpha \circ p_U = \beta \circ p_V.
\]

**Proposition 1.6.1.** \(P\) is a pullback in the category of diffeological spaces. That means that it verifies the following universal property: given any diffeological space \(Y\) and two differentiable maps \(a: Y \to U\) and \(b: Y \to V\) such that \(\alpha \circ a = \beta \circ b\), then there exists a unique differentiable map \(F: Y \to P\) such that \(p_U \circ F = a\) and \(p_V \circ F = b\).

![Diagram of pullback](https://via.placeholder.com/150)

### 2. De Rham cohomology

Exterior differential forms and De Rham cohomology can be generalized to the context of diffeological spaces. We refer to \[2, 10\] and \[6, Art. 6.28 and fol.\] for the basics on Cartan-De Rham Calculus.
2.1. **Differential forms.** A differential form on a diffeological space is defined as the family of its pullbacks by the plots of the diffeology, which are usual differential forms in $m$-domains.

**Definition 2.1.1.** Let $(X, \mathcal{D})$ be a diffeological space. A differential $k$-form on $X$, $k \geq 0$, is any collection $w = \{w_\alpha\}$, where for each plot $\alpha: U \to X$ in $\mathcal{D}$ we have a usual differential form $w_\alpha \in \Omega^k(U)$. Moreover, it must verify the following compatibility condition: for any $C^\infty$ change of coordinates $h: V \to U$ we must have

$$w_\alpha \circ h = h^* w_\alpha,$$

where $h^* w_\alpha \in \Omega^k(V)$ is the usual pullback of a differential form.

We shall denote by $\Omega^k(X, \mathcal{D})$ the vector space of $k$-forms on $(X, \mathcal{D})$.

It is another nice feature of diffeological theory that this set of forms can be endowed with a diffeology in a natural way ([6, Art. 6.29].

**Remark 2.1.2.** The form $\omega_\alpha$ is not an arbitrary form on $U$. Notice that if $h, h': V \to U$ are two changes of coordinates such that $\alpha \circ h = \alpha \circ h'$, then

$$h^* \omega_\alpha = \omega_{\alpha \circ h} = \omega_{\alpha \circ h'} = (h')^* \omega.$$

We shall need this property later (see Section 2.3).

**Example 2.1.3.** If $M$ is a differentiable manifold, the diffeological forms for the usual diffeology are the usual differential forms [6, 5]

**Example 2.1.4.** Let $X = \{\ast\}$ be a one-point set (with the discrete diffeology). Then $\Omega^0(\{\ast\}) = \mathbb{R}$ and $\Omega^k(\{\ast\}) = 0$ for $k \geq 1$.

**Proposition 2.1.5** ([6, Art. 6.31]). *The zero forms on a diffeological space $(X, \mathcal{D})$ are identified with the differentiable maps $X \to \mathbb{R}$, for the usual diffeology on $\mathbb{R}$.***

**Proof.** If $f: X \to \mathbb{R}$ is a differentiable map, we define the form $\omega(f) \in \Omega^0(X)$ as

$$\omega(f)_\alpha = f \circ \alpha \in \Omega^0(U) = C^\infty(U, \mathbb{R})$$

for any plot $(U, \alpha)$ on $X$.

Conversely, if $\omega \in \Omega^0(X)$, we define the function

$$f(\omega): X \to \mathbb{R}, \quad f(\omega)(x) = \omega_{c_x},$$

where for each $x \in X$ we take the constant plot $c_x: \mathbb{R}^0 = \{0\} \to X$ with $c_x(0) = x$.

The function $f = f(\omega)$ is differentiable because for any plot $(U, \alpha)$ on $X$, we have

$$(f \circ \alpha)(u) = \omega_{c_u(u)} = \omega_{\alpha \circ c_u} = c_u^* (\omega_\alpha) = \omega_\alpha(u),$$

where $c_u: \mathbb{R}^0 \to U$ is the constant map $u$, and $\omega_\alpha \in C^\infty(U, \mathbb{R})$. □
Definition 2.1.6. If $F : (X, \mathcal{D}) \to (X', \mathcal{D}')$ is a differentiable map between diffeological spaces and $\omega \in \Omega^k(X', \mathcal{D}')$ is a $k$-form on $X'$ then we define the pull-back $F^* \omega \in \Omega^k(X, \mathcal{D})$ as
\[(F^* \omega)_\alpha := \omega_{F \circ \alpha}\]
for any plot $\alpha$ on $X$.

Notice that if $\omega \in \Omega^k(X, \mathcal{D})$, then $\omega_\alpha = \alpha^* \omega$ for any plot $\alpha \in \mathcal{D}$.

Definition 2.1.7. The exterior derivative $d_X : \Omega^k(X, \mathcal{D}) \to \Omega^{k+1}(X, \mathcal{D})$ is defined, for any plot $(U, \alpha) \in \mathcal{D}$, by
\[(d_X \omega)_\alpha = d_U(\omega_\alpha),\]
where $d_U : \Omega^k(U) \to \Omega^{k+1}(U)$ is the usual exterior derivative.

The most important property of exterior differential is that $d_X \circ d_X = 0$, as follows immediately from the analogous property for $m$-domains.

The pullback behaves well with the usual constructions in Cartan calculus, among them the exterior differential: if $F : X \to X'$ is a differentiable map, then
\[d_X(F^* \omega) = F^*(d_X \omega).\]

2.2. De Rham Cohomology. We already have all the tools necessary to define a De Rham cohomology theory on diffeological spaces.

A $k$ form $\omega \in \Omega^k(X)$ is closed if $d_X \omega = 0$. It is exact if $\omega = d_X \mu$ for some $(k - 1)$-form $\mu$. Every exact form is closed. In other words, we have a De Rham complex
\[0 \to \Omega^0(X, \mathcal{D}) \xrightarrow{d_0} \Omega^1(X, \mathcal{D}) \xrightarrow{d_1} \cdots \xrightarrow{d_k} \Omega^k(X, \mathcal{D}) \xrightarrow{d_{k+1}} \Omega^{k+1}(X, \mathcal{D}) \to \cdots\]
such that $\text{im } d_{k-1} \subseteq \ker d_k$. We define the De Rham’s $k$-cohomology group of $(X, \mathcal{D})$ as the quotient of vector spaces
\[H^k(X, \mathcal{D}) = \ker d_k / \text{im } d_{k-1}.\]
It measures the failure of closed forms to be exact.

Example 2.2.1. It was proved in [3] that the cohomology of the leaf space $M/\mathcal{F}$ of a foliated manifold $(M, \mathcal{F})$, endowed with the quotient diffeology, equals the so-called basic cohomology of the foliation.

Each differentiable map $F : (X, \mathcal{D}) \to (X', \mathcal{D}')$ induces a morphism
\[F^* : H^k(X', \mathcal{D}') \to H(X, \mathcal{D}), \quad F^*([\omega]) = [F^* \omega],\]
and the usual properties hold,
\[(G \circ F)^* = F^* \circ G^*, \quad \text{id}_X^* = \text{id}_{H(X)}^*.\]
2.3. Horizontal forms.

Definition 2.3.1. In view of Remark 2.1.2, if \((U, \alpha)\) is a plot on \(X\) we define the subspace \(\Omega^k(\alpha) \subseteq \Omega^k(U)\) of \(\alpha\)-horizontal forms as those forms \(\mu \in \Omega^k(U)\) verifying: if \(h, h' : V \to U\) are changes of coordinates such that \(\alpha \circ h = \alpha \circ h'\), then \(h^*\mu = (h')^*\mu\).

They form a subcomplex of \(\Omega(U)\) because if \(\omega \in \Omega^k(\alpha)\) and \(\alpha \circ h = \alpha \circ h'\) then
\[
h^*(d_U\omega) = d_V h^*\omega = d_V(h')^*\omega = (h')^*d_U\omega,
\]
hence \(d_U\omega \in \Omega^{k+1}(\alpha)\).

3. The Mayer-Vietoris sequence

We shall consider a generating family of \((X, D)\) formed by two plots \((U, \alpha)\) and \((V, \beta)\). We want to understand to what extent the cohomology \(H(X, D)\) is determined by these two plots.

3.1. Restriction and difference morphisms. Let \((P, p_U, p_V)\) be the pullback defined in Section 1.6 and Proposition 1.6.1:
\[
\begin{align*}
P & \xrightarrow{p_V} V \subseteq \mathbb{R}^n \\
& \downarrow p_U \\
U \subseteq \mathbb{R}^m & \xrightarrow{\alpha} X
\end{align*}
\]
We have the sequence of complexes
\[
0 \to \Omega^k(X) \xrightarrow{\partial} \Omega^k(\alpha) \oplus \Omega^k(\beta) \xrightarrow{\delta} \Omega(P)
\]
where we call \(\partial\) a restriction morphism and \(\delta\) the difference morphism. They are defined by
\[
\partial(\omega) = (\alpha^*\omega, \beta^*\omega)
\]
and
\[
\delta(\mu, \nu) = p_U^*\mu - p_V^*\nu.
\]
The complexes \(\Omega(\alpha)\) and \(\Omega(\beta)\) of horizontal forms were considered in Section 2.3

Lemma 3.1.1. \(\partial\) and \(\delta\) are well defined morphisms of complexes.

Proof. If \(\omega \in \Omega^k(X)\), and \(h, h' : V \to U\) are two changes of coordinates such that \(\alpha \circ h = \alpha \circ h'\), then
\[
h^*(\alpha^*\omega) = (\alpha \circ h)^*\omega = (h')^*(\alpha^*\omega).
\]
Then \(\alpha^*\omega\) is a horizontal form. Analogously for \(\beta\). This proves that \(\partial\) is well defined.
The morphism $r$ commutes with the differentials, $(d_U \oplus d_V) \circ r = r \circ d_X$, because
\[(d_U \oplus d_V)r(\omega) = (d_U \oplus d_V)(\alpha^* \omega, \beta^* \omega) = (d_U \alpha^* \omega, d_V \beta^* \omega) = (\alpha^* d_X \omega, \beta^* d_X \omega) = r(d_X(\omega)).\]

Finally,
\[(d_P \circ \delta)(\mu, \nu) = d_P(p_U^* \mu - p_V^* \nu) = d_P p_U^* \mu - d_P p_V^* \nu = p_U^* d_U \mu - p_V^* d_V \nu = \delta(d_U \mu, d_V \nu).\]

This proves $d_P \circ \delta = \delta \circ (d_U \oplus d_V)$.

**Theorem 3.1.2.** The sequence (3.2) is exact.

**Proof.** We shall make the proof in several steps.

Step 1. The morphism $r$ is injective.

Let $\omega \in \Omega^k(X)$ such that $r(\omega) = 0$. We must show that $w_\gamma = 0$ for any plot $\gamma: W \to X$. Since $\{\alpha, \beta\}$ is a generating family, for each $p \in W$ there exists a neighborhood $W$ of $p$ such that either $\gamma|_W$ is constant, in which case $\gamma$ factors through $c_0: W \to \mathbb{R}^0 = \{\ast\}$ and we have $\omega_\gamma = c_0^* 0 = 0$ on $W$, or there is a change of coordinates $h: W \to U$ with $\gamma|_W = \alpha \circ h$ (analogously $\gamma|_W = \beta \circ h'$ for some $h': W \to V$). Then, on $W$ we have
\[\omega_\gamma = \gamma^* \omega = (\alpha \circ h)^* \omega = h^* \alpha^* \omega = 0.\]

This proves that $\omega = 0$.

Step 2. $\text{im } r \subseteq \ker \delta$.

We show $\delta \circ r = 0$:
\[p_U^* \alpha^* \omega - p_V^* \beta^* \omega = (\alpha \circ p_U)^* \omega - (\beta \circ p_V)^* \omega = 0.\]

because Diagram [3.1] is commutative.

Step 3. $\ker \delta \subseteq \text{im } r$.

Let $(\mu, \nu) \in \Omega^k(\alpha) \oplus \Omega^k(\beta)$ such that $p_U^* \mu = p_V^* \nu$. We need to define a form $\omega \in \Omega^k(X)$ such that $\alpha^* \omega = \mu$ and $\beta^* \omega = \nu$. For that, we need to define $\omega_\gamma \in \Omega^k(W)$ for any plot $(W, \gamma)$ on $X$.

Obviously, for $\gamma = \alpha$ we take $\omega_\alpha = \mu$. If $h: W \to U$ is a change of coordinates, we put $\omega_{\alpha \circ h} = h^* \mu$. This form does not depend on $h$ due to the definition of horizontal form (Section [2.3]), that is, if $\alpha \circ h = \alpha \circ h'$ then $h^* \mu = (h')^* \mu$. Analogously, for $\gamma = \beta$ we take $\omega_\beta = \nu$, and for a change of coordinates $h': W \to V$ we put $\omega_{\beta \circ h'} = (h')^* \nu$. It may happen that $\alpha \circ h = \beta \circ h'$, for some changes of coordinates $h: W \to U$ and $h': W \to V$. In this case, by the pullback property of Proposition
there exists a differentiable map \( F: W \to P \) such that \( p_U \circ F = h \) and \( p_V \circ F = h' \). Hence
\[
(h')^* \nu = (p_V \circ F)^* \nu = F^* p_V^* \nu = F^* p_U^* \mu = (p_U \circ F)^* \mu = h^* \mu.
\]
This proves that the preceding definitions are consistent.

Finally, let \( \gamma: W \to X \) be an arbitrary plot. For each point \( p \in W \) there is a neighbourhood \( W_p \) such that the restriction of \( \gamma \) to \( W_p \) either is constant, in which case we define \( \omega_\gamma = 0 \) on the neighbourhood \( W_p \), or \( \gamma|_{W_p} = \alpha \circ h \) for some \( h: W_p \to U \), in which case we define \( \omega_\gamma = h^* \omega_\alpha \), or \( \gamma|_{W_p} = \beta \circ h' \), in which case we state \( \omega_\gamma \) to be \((h')^* \omega_\beta \) on \( W_p \).

The compatibility condition can be checked easily. This proves that we have a well defined \( k \) form \( \omega \) on \( X \) such that \( r(\omega) = (\mu, \nu) \). □

The latter Theorem shows that the complex \( \Omega^k(X) \) is isomorphic to the subcomplex \( \ker \delta \) of \( \Omega(\alpha) \oplus \Omega(\beta) \).

Note that we do not prove the surjectiveness of \( \delta \). This would require the existence of partitions of unity, a practically impossible objective in such a wide context.

4. An Example

We shall compute the De Rham cohomology of the following Example as an application of our Theorem 3.1.2.

Let \( X \) be the set obtained as the union of the two coordinate axis in \( \mathbb{R}^2 \), that is,
\[
X = \{(x, y) \in \mathbb{R}^2 : xy = 0\}.
\]
We take on \( X \) the diffeology \( \mathcal{D} \) generated by the two axis inclusions, that is, by the plots
\[
\alpha: U = \mathbb{R} \to X, \quad \alpha(s) = (s, 0),
\]
and
\[
\beta: V = \mathbb{R} \to X, \quad \beta(t) = (0, t).
\]
The pullback of these two plots is \( P = \{(0, 0)\} \), because \( \alpha(s) = \beta(t) \) if and only if \( s = t = 0 \). Hence Example 2.1.4 applies.

Remark 4.0.1. Notice that this diffeology \( \mathcal{D} \) is strictly smaller than the subspace diffeology \( \mathcal{D}' \) of \( X \) as a subset of \( \mathbb{R}^2 \). In fact, the 1-plot \( \alpha: \mathbb{R} \to X \subseteq \mathbb{R}^2 \) given by the \( C^\infty \) map
\[
\alpha(t) = \begin{cases} (e^{1/t}, 0) & \text{if } t < 0, \\ (0, 0) & \text{if } t = 0, \\ (0, e^{-1/t}) & \text{if } t > 0, \end{cases}
\]
does not belong to the diffeology, by Proposition [1.5.2]. Hence the computation of the 1-differential forms for $(X, D')$ (the so-called cross) in [7] does not apply here.

We compute the horizontal forms for $D$: since $p_\ast \alpha = \text{id}_\mathbb{R}$, for the first projection $p_1: \mathbb{R}^2 \rightarrow \mathbb{R}$, the condition $\alpha \circ h = \alpha \circ h'$ for $h, h': W \rightarrow U = \mathbb{R}$ means in fact $h = h'$. Hence $\Omega(\alpha) = \Omega(U)$. Analogously $\Omega(\beta) = \Omega(V)$.

Now, we know that $\Omega^k(X) \cong \text{im} r = \ker \delta \subseteq \Omega^k(\alpha) \oplus \Omega^k(\beta)$. For $k \geq 2$ that means $\Omega^k(X) = 0$. For $k = 0$ we have that $(F, G) \in \Omega^0(\mathbb{R}) \oplus \Omega^0(\mathbb{R})$ is a pair of functions $F, G: \mathbb{R} \rightarrow \mathbb{R}$, and being in the kernel of $\delta$ means $F(0) = G(0)$. For $k = 1$, we have a pair of 1-forms $(\mu, \nu) \in \Omega^1(\mathbb{R}) \oplus \Omega^1(\mathbb{R})$, say $\mu = f(t)dt$ and $\nu = g(t)dt$. The pair of functions $(f, g)$ is arbitrary because $p_\ast^\ast \mu = 0 = p_\ast^\ast \nu$ in $\Omega^1(P) = 0$.

We consider the resulting De Rham complex of $X$, say,

$$0 \rightarrow \Omega^0(X) \xrightarrow{d} \Omega^1(X) \rightarrow 0 \rightarrow \cdots$$

which can be written as

$$0 \rightarrow C^\infty(\mathbb{R}) \oplus_0 C^\infty(\mathbb{R}) \xrightarrow{d} C^\infty(\mathbb{R}) \oplus C^\infty(\mathbb{R}) \rightarrow 0 \rightarrow \cdots$$

where

$$d(F, G) = (F', G').$$

The kernel of $d$ is a pair of constant functions, which must be equal by the condition $F(0) = G(0)$, hence

$$H^0(X, D) = \mathbb{R}.$$

We compute the image of $d$. Since any $f \in C^\infty(\mathbb{R})$ is the derivative of some $F$, unique up to a constant, we can always write $f = dF$, $g = dG$ and we can assume that $F(0) = G(0)$, by taking $G - G(0) + F(0)$. Hence $d$ is surjective and

$$H^1(X, D) = 0.$$

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