An Approximate Solution of the Dynamical Casimir Effect in a Cavity with a Two–Level Atom

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Abstract

In this paper we treat the so–called dynamical Casimir effect in a cavity with a two–level atom and give an analytic approximate solution under the general setting.

The aim of the paper is to show another approach based on Mathematical Physics to the paper [arXiv : 1112.0523 (quant-ph)] by A. V. Dodonov and V. V. Dodonov.

We believe that our method is simple and beautiful.

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1 Introduction

In this paper we revisit the so-called dynamical Casimir effect (DCE). This means the photon generation from vacuum due to the motion (change) of neutral boundaries, which corresponds to a kind of quantum fluctuation of the electro-magnetic field.

This phenomenon is a typical example of interactions between the microscopic and the macroscopic levels and is very fascinating from the point of view of not only (pure) Physics but also Mathematical Physics. See for example [1] and its references.

Then, how do we detect the photons generated? This is also an important problem. Recently, Dodonov and Dodonov in [2] treated this problem by use of a two-level system of an atom in a cavity. They called it “the cavity dynamical Casimir effect in the presence of a two-level atom”, and gave an effective Hamiltonian by simplifying the Hamiltonian given by Law [3] and adding a two-level system of an atom to it, and constructed an approximate analytical solution.

In this paper we treat this problem once more and present another approach to construct an analytic approximate solution under the general setting. We believe that our method is clearer than that of [2].

2 Model

First of all let us make a brief review of [2] within our necessity. In the following we set $\hbar = 1$ for simplicity.

(i) Cavity DCE ([3], [11]). For this we take the simplest Hamiltonian which is the special case of Law [3] (namely, $\epsilon(x, t) = \epsilon(t)$)

$$H_{DCE} = \omega(t)a^\dagger a + i\chi(t)\{(a^\dagger)^2 - a^2\}$$ (1)
where $\omega(t)$ is a periodic function depending on the cavity form and $\chi(t)$ is given by

$$
\chi(t) = \frac{1}{4\omega(t)} \frac{d\omega(t)}{dt} = \frac{1}{4} \frac{d}{dt} \log |\omega(t)|
$$

, and $a$ and $a^\dagger$ are the cavity photon annihilation and creation operators respectively. Therefore, the physics that we are treating is two-photon generation processes from the vacuum state.

(ii) Detection ([4], [5]). For this we take a two-level system of an atom inserted in the cavity and the Rabi Hamiltonian as interaction

$$
H_D = \frac{\Omega}{2} \sigma_3 \otimes 1 + g(\sigma_+ + \sigma_-) \otimes (a + a^\dagger)
$$

$$
= \frac{\Omega}{2} \sigma_3 \otimes 1 + g(\sigma_+ \otimes a + \sigma_+ \otimes a^\dagger + \sigma_- \otimes a + \sigma_- \otimes a^\dagger)
$$

(2)

where $1$ is the identity operator on the Fock space generated by $\{a, a^\dagger, N \equiv a^\dagger a\}$ and

$$
\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad 1_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
$$

However, to solve this system is very hard. In fact, our aim is to obtain some analytic approximate solution. Therefore, as a rule we replace the Rabi Hamiltonian with the Jaynes-Cummings one

$$
\tilde{H}_D = \frac{\Omega}{2} \sigma_3 \otimes 1 + g(\sigma_+ \otimes a + \sigma_- \otimes a^\dagger).
$$

(3)

Here, let us note the well-known $su(2)$ relations

$$
[(1/2)\sigma_3, \sigma_+] = \sigma_+, \quad [(1/2)\sigma_3, \sigma_-] = -\sigma_-, \quad [\sigma_+, \sigma_-] = 2 \times (1/2)\sigma_3.
$$

(iii) Total System. By adding (3) to (1) we treat the (effective) Hamiltonian like

$$
H = H(t) = H_{DCE} + \tilde{H}_D
$$

$$
= \omega(t) 1_2 \otimes N + i\chi(t) 1_2 \otimes (a^\dagger)^2 - a^2 \} + \frac{\Omega}{2} \sigma_3 \otimes 1 + g(\sigma_+ \otimes a + \sigma_- \otimes a^\dagger).
$$

(4)

Our aim is to solve the Schrödinger equation

$$
i \frac{d}{dt} |\Psi(t)\rangle = H |\Psi(t)\rangle = H(t) |\Psi(t)\rangle
$$

(5)
under the general setting.

If $H$ is time–independent, then the general (formal) solution is given by

$$|\Psi(t)\rangle = e^{-itH}|\Psi(0)\rangle.$$  

To calculate $e^{-itH}$ exactly is another problem (which is in general very hard). However, in our case $H$ is time–dependent, so solving (5) becomes increasingly difficult. We must give a further approximation to the Hamiltonian.

(iv) Interaction Picture. From here let us change to the interaction picture. Namely, for $V = V(t)$ we set

$$|\Phi(t)\rangle = V^\dagger|\Psi(t)\rangle \iff |\Psi(t)\rangle = V|\Phi(t)\rangle. \quad (6)$$

Then it is easy to see that the equation (5) can be changed to

$$i\frac{d}{dt}|\Phi(t)\rangle = \left(V^\dagger HV - iV^\dagger \frac{d}{dt}V\right)|\Phi(t)\rangle. \quad (7)$$

This is in general called the interaction picture.

In the following we impose some restrictions on the model. Namely, we take $\omega(t)$ as in [2]

$$\omega(t) = \omega_0(1 + \epsilon \sin(\eta t))$$

where $\omega_0$, $\epsilon$ and $\eta$ are real constants. We assume that $\omega_0 > 0$, $0 < \epsilon \ll 1$ and $\eta$ is large enough. Then $\omega(t) \approx \omega_0$ and

$$\chi(t) = \frac{\epsilon \eta \cos(\eta t)}{4(1 + \epsilon \sin(\eta t))} \approx \frac{\epsilon \eta}{4} \cos(\eta t)$$

, and here we take $V$ as

$$V = V(t) = e^{-it\frac{\eta}{2}\sigma_3} \otimes e^{-it\frac{\eta}{2}N} = \begin{pmatrix} e^{-it\frac{\eta}{2} - it\frac{\eta}{2}N} & 0 \\ 0 & e^{it\frac{\eta}{2} - it\frac{\eta}{2}N} \end{pmatrix}. \quad (8)$$

Some calculation by use of (8) gives

$$\hat{H}(t) \equiv V^\dagger HV - iV^\dagger \frac{d}{dt}V = \begin{pmatrix} \omega_0 - \eta \sqrt{2} & \epsilon \eta \sqrt{2} N \\ -\epsilon \eta \sqrt{2} N & \omega_0 + \eta \sqrt{2} \end{pmatrix} + \begin{pmatrix} \Omega \sigma_3 \otimes 1 + g(\sigma_+ \otimes a + \sigma_- \otimes a^\dagger) \\ \Omega \sigma_3 \otimes 1 - g(\sigma_+ \otimes a + \sigma_- \otimes a^\dagger) \end{pmatrix}$$
and we can also use the rotating wave approximation \( e^{2i\eta t} \approx 0 \) when \( \eta \) is large enough. As a result we finally obtain the time-independent Hamiltonian (see [2]) like

\[
\hat{H} = \left( \omega_0 - \frac{\eta}{2} \right) 1_2 \otimes N + \frac{\epsilon \eta}{8} 1_2 \otimes \left\{ (a^\dagger)^2 - a^2 \right\} + \frac{\Omega - \eta/2}{2} \sigma_3 \otimes 1 + g(\sigma_+ \otimes a + \sigma_- \otimes a^\dagger). \tag{9}
\]

(v) Our Target. The aim of this paper is to solve the equation

\[
i \frac{d}{dt} |\Phi(t)\rangle = \hat{H} |\Phi(t)\rangle \tag{10}
\]

under the general setting and the formal solution is given by

\[
|\Phi(t)\rangle = e^{-it\hat{H}} |\Phi(0)\rangle. \tag{11}
\]

Therefore, what we must do in the following is to calculate the term \( e^{-it\hat{H}} \) explicitly.

Before closing this section we must present an important problem.

**Problem** We used approximation (a kind of rotating wave approximation) two times. Make an adaptive range (or region) of the model clear.

### 3 Approximate Solution

In this section we try to calculate the term \( e^{-it\hat{H}} \) in (11). For simplicity we set

\[
-it\hat{H} = X + Y
\]

where

\[
X = -itA, \quad A = \left( \omega_0 - \frac{\eta}{2} \right) 1_2 \otimes N + \frac{\epsilon \eta}{8} 1_2 \otimes \left\{ (a^\dagger)^2 - a^2 \right\} ,
\]

\[
Y = -itB, \quad B = \frac{\Omega - \eta/2}{2} \sigma_3 \otimes 1 + g(\sigma_+ \otimes a + \sigma_- \otimes a^\dagger). \tag{12}
\]

For our purpose the Zassenhaus formula is useful:

**Zassenhaus Formula** We have an expansion

\[
e^{X+Y} = \cdots e^{-\frac{1}{6}[2[X,Y],Y]+[X,Y],X]} e^\frac{1}{2}[X,Y] e^Y e^X.
\]
Note that the formula is a bit different from that of [6]. In this paper we use
\[ e^{X+Y} \approx e^{\frac{1}{2}[X,Y]} e^Y e^X = e^{-\frac{1}{2}e^{A,B}} e^{-itB} e^{-itA}. \] (13)

Therefore, let us calculate each term in the following.

[I] First, we calculate \( e^{-itA} \). If we set \( \alpha = 2(\omega_0 - \eta) \), \( \beta = \frac{1}{4} \epsilon \eta \) \( \alpha \) then \( A \) in (12) becomes
\[
A = \frac{\alpha}{2} 1_2 \otimes N + i \frac{\beta}{2} 1_2 \otimes \{(a^\dagger)^2 - a^2\}
= 1_2 \otimes \left\{ \alpha \frac{1}{2} N + i \beta \left( \frac{1}{2}(a^\dagger)^2 - \frac{1}{2}a^2 \right) \right\}.
\]

Here, we use a well–known Lie algebraic method, see for example [7]. Namely, if we set
\[
K_+ = \frac{1}{2}(a^\dagger)^2, \quad K_- = \frac{1}{2}a^2, \quad K_3 = \frac{1}{2} \left( N + \frac{1}{2} \right)
\]
(15)

it is easy to see both \( K_+^\dagger = K_- \), \( K_3^\dagger = K_3 \) and the \( su(1,1) \) relations
\[
[K_3, K_+] = K_+, \quad [K_3, K_-] = -K_-, \quad [K_+, K_-] = -2K_3
\]
(16)
by use of the relation \([a, a^\dagger] = 1\). Then, \( A \) in (12) can be written as
\[
A = 1_2 \otimes \left\{ -\frac{1}{4} \alpha + \alpha K_3 + i \beta (K_+ - K_-) \right\}
= \begin{pmatrix}
-\frac{1}{4} \alpha + \alpha K_3 + i \beta (K_+ - K_-) \\
-\frac{1}{4} \alpha + \alpha K_3 + i \beta (K_+ - K_-)
\end{pmatrix}
\]
and from this we have
\[
e^{-itA} = \begin{pmatrix}
e^{\frac{it}{4} \alpha} e^{-it\{aK_3 + i\beta(K_+ - K_-)\}} \\
e^{\frac{it}{4} \alpha} e^{-it\{aK_3 + i\beta(K_+ - K_-)\}}
\end{pmatrix}.
\]

Therefore, we have only to calculate the term
\[
U(t) \equiv e^{-it\{aK_3 + i\beta(K_+ - K_-)\}} = e^{-itaK_3 + i\beta(K_+ - K_-)}.
\]

6
For the purpose we want to look for the following disentangling form

\[ U(t) = e^{f(t)K}e^{g(t)K_3}e^{h(t)K^{-}} \quad (17) \]

with functions \( f(t), \ g(t), \ h(t) \) \( (f(0) = g(0) = h(0) = 0) \)

The result is as follows.

\[
\begin{align*}
  f(t) &= -\frac{\beta}{\sqrt{-\frac{\alpha^2}{4} + \beta^2}} \sinh \left( t \sqrt{-\frac{\alpha^2}{4} + \beta^2} \right) \\
  &\quad \cosh \left( t \sqrt{-\frac{\alpha^2}{4} + \beta^2} \right) + \frac{i\frac{\beta}{\sqrt{-\frac{\alpha^2}{4} + \beta^2}}}{\sqrt{-\frac{\alpha^2}{4} + \beta^2}} \sinh \left( t \sqrt{-\frac{\alpha^2}{4} + \beta^2} \right), \\
  g(t) &= -2 \log \left( \cosh \left( t \sqrt{-\frac{\alpha^2}{4} + \beta^2} \right) + \frac{i\frac{\beta}{\sqrt{-\frac{\alpha^2}{4} + \beta^2}}}{\sqrt{-\frac{\alpha^2}{4} + \beta^2}} \sinh \left( t \sqrt{-\frac{\alpha^2}{4} + \beta^2} \right) \right), \quad (18) \\
  h(t) &= -\frac{\beta}{\sqrt{-\frac{\alpha^2}{4} + \beta^2}} \sinh \left( t \sqrt{-\frac{\alpha^2}{4} + \beta^2} \right) \\
  &\quad \cosh \left( t \sqrt{-\frac{\alpha^2}{4} + \beta^2} \right) + \frac{i\frac{\beta}{\sqrt{-\frac{\alpha^2}{4} + \beta^2}}}{\sqrt{-\frac{\alpha^2}{4} + \beta^2}} \sinh \left( t \sqrt{-\frac{\alpha^2}{4} + \beta^2} \right) = -f(t).
\end{align*}
\]

The derivation is analogous to that of [8]. See also the appendix.

[I] Second, we calculate \( e^{-itB} \). The result is well–known and is given by

\[
\begin{align*}
  e^{-itB} &= \exp \left\{ -it \begin{pmatrix} \frac{\Omega-\eta/2}{2} & ga \\ ga^\dagger & -\frac{\Omega-\eta/2}{2} \end{pmatrix} \right\} \\
  &= \begin{pmatrix} \\
  \cos t\sqrt{\varphi + g^2} - \frac{ig}{2} \sin t\sqrt{\varphi + g^2} & -ig \frac{\sin t\sqrt{\varphi + g^2}}{\sqrt{\varphi + g^2}} a \\
  -ig \frac{\sin t\sqrt{\varphi + g^2}}{\sqrt{\varphi + g^2}} a^\dagger & \cos t\sqrt{\varphi + g^2} + \frac{i\delta}{2} \sin t\sqrt{\varphi + g^2} \end{pmatrix}, \quad (19)
\end{align*}
\]

where we have set

\[
\delta \equiv \Omega - \frac{\eta}{2}, \quad \varphi \equiv \frac{\delta^2}{4} + g^2 N
\]

for simplicity.

\[^1\text{this is a standard method}\]
[III] Third, we calculate $e^{-\frac{t^2}{2}[A,B]}$. From (12) simple calculation gives

$$[A,B] = g\frac{\alpha}{2}(-\sigma_+ \otimes a + \sigma_- \otimes a^\dagger) - ig\beta(\sigma_+ \otimes a^\dagger + \sigma_- \otimes a)$$

$$= g \begin{pmatrix} 0 & -\frac{\alpha}{2} a - i\beta a^\dagger \\ \frac{\alpha}{2} a^\dagger - i\beta a & 0 \end{pmatrix}$$

$$\equiv g \begin{pmatrix} 0 & -D \\ D^\dagger & 0 \end{pmatrix}$$

(20)

where we have set

$$D = \frac{\alpha}{2} a + i\beta a^\dagger, \quad D^\dagger = \frac{\alpha}{2} a^\dagger - i\beta a$$

for simplicity. Note that

$$[D, D^\dagger] = \left(\frac{\alpha^2}{4} - \beta^2\right) 1.$$  

The result is

$$e^{-\frac{t^2}{2}[A,B]} = \begin{pmatrix} \cos \left( g \frac{t^2}{2} \sqrt{DD^\dagger} \right) & \frac{1}{\sqrt{DD^\dagger}} \sin \left( g \frac{t^2}{2} \sqrt{DD^\dagger} \right) D \\ -\frac{1}{\sqrt{DD^\dagger}} \sin \left( g \frac{t^2}{2} \sqrt{DD^\dagger} \right) D^\dagger & \cos \left( g \frac{t^2}{2} \sqrt{DD^\dagger} \right) \end{pmatrix}. \quad (21)$$

[IV] As a result, our approximate solution to the equation (5) is given by

$$|\Psi(t)\rangle \approx e^{-\frac{t^2}{2}[A,B]} e^{-itB} e^{-itA} |\Psi(0)\rangle$$

(22)

under any initial value $|\Psi(0)\rangle$. If we write

$$\hat{U}(t) \equiv e^{-\frac{t^2}{2}[A,B]} e^{-itB} e^{-itA} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$$

(23)
then each component is given by

\[ U_{11} = e^{\frac{4}{\alpha}} \left\{ \cos \left( \frac{g^2}{2} \sqrt{D^\dagger D} \right) \left( \cos t \sqrt{\varphi + g^2} - \frac{i \delta \sin t \sqrt{\varphi + g^2}}{2 \sqrt{\varphi + g^2}} \right) - \right. \]

\[ \left. ig \frac{1}{\sqrt{D^\dagger D}} \sin \left( \frac{g^2}{2} \sqrt{D^\dagger D} \right) D \frac{\sin t \sqrt{\varphi}}{\sqrt{\varphi}} a^\dagger \right\} e^{\hat{f}(t)K_+} e^{\hat{g}(t)K_3} e^{\hat{h}(t)K_-}, \]

\[ U_{12} = e^{\frac{4}{\alpha}} \left\{ -ig \cos \left( \frac{g^2}{2} \sqrt{D^\dagger D} \right) \frac{\sin t \sqrt{\varphi + g^2}}{\sqrt{\varphi + g^2}} a + \right. \]

\[ \left. \frac{1}{\sqrt{D^\dagger D}} \sin \left( \frac{g^2}{2} \sqrt{D^\dagger D} \right) D \left( \cos t \sqrt{\varphi} + \frac{i \delta \sin t \sqrt{\varphi}}{2 \sqrt{\varphi}} \right) \right\} e^{\hat{f}(t)K_+} e^{\hat{g}(t)K_3} e^{\hat{h}(t)K_-}, \]

\[ U_{21} = e^{\frac{4}{\alpha}} \left\{ - \frac{1}{\sqrt{D^\dagger D}} \sin \left( \frac{g^2}{2} \sqrt{D^\dagger D} \right) D^\dagger \left( \cos t \sqrt{\varphi + g^2} + \frac{i \delta \sin t \sqrt{\varphi + g^2}}{2 \sqrt{\varphi + g^2}} \right) - \right. \]

\[ \left. ig \cos \left( \frac{g^2}{2} \sqrt{D^\dagger D} \right) \frac{\sin t \sqrt{\varphi}}{\sqrt{\varphi}} a^\dagger \right\} e^{\hat{f}(t)K_+} e^{\hat{g}(t)K_3} e^{\hat{h}(t)K_-}, \]

\[ U_{22} = e^{\frac{4}{\alpha}} \left\{ ig \frac{1}{\sqrt{D^\dagger D}} \sin \left( \frac{g^2}{2} \sqrt{D^\dagger D} \right) D^\dagger \frac{\sin t \sqrt{\varphi + g^2}}{\sqrt{\varphi + g^2}} a + \right. \]

\[ \left. \cos \left( \frac{g^2}{2} \sqrt{D^\dagger D} \right) \left( \cos t \sqrt{\varphi} + \frac{i \delta \sin t \sqrt{\varphi}}{2 \sqrt{\varphi}} \right) \right\} e^{\hat{f}(t)K_+} e^{\hat{g}(t)K_3} e^{\hat{h}(t)K_-}. \]

This is our main result.

4 Closing Remarks

In this paper we treated the model by A. V. Dodonov and V. V. Dodonov and presented another approach based on Mathematical Physics and obtained some analytic approximate solution. Details with applications and developments will be published separately.

We have neglected Dissipation (for example, the Cavity loss) in this paper, which is not realistic. If we take dissipation into consideration the model will become very complicated. For example, see our papers [8], [9], [10] and [11], [12] (the last two are highly recommended).

Then, to obtain an analytic approximate solution will become increasingly difficult. Further study and new ideas are needed.

Appendix
A note is added. In the text we must calculate

\[ U(t) = e^{-it\{\alpha K_3 + i\beta (K_+ - K_-)\}} = e^{-i\alpha K_3 + t\beta (K_+ - K_-)}. \]

From this we have the differential equation

\[ \frac{d}{dt} U(t) = \{-i\alpha K_3 + \beta (K_+ - K_-)\} U(t). \]

On the other hand, under the ansatz \(^{(17)}\) \( U(t) = e^{f(t)K_+} e^{g(t)K_3} e^{h(t)K_-} \) some calculation gives

\[ \frac{d}{dt} U(t) = \left\{ \left( \dot{f} - \dot{g} f + \dot{h} e^{-g} f^2 \right) K_+ + \left( \dot{g} - 2\dot{h} e^{-g} f \right) K_3 + \left( \dot{h} e^{-g} \right) K_- \right\} U(t) \]

where we have used \( \dot{f} = \frac{df}{dt} \), etc for simplicity. Therefore, by comparing two equations we obtain

\[
\begin{align*}
\dot{f} - \dot{g} f + \dot{h} e^{-g} f^2 &= \beta \\
\dot{g} - 2\dot{h} e^{-g} f &= -i\alpha \\
\dot{h} e^{-g} &= -\beta
\end{align*}
\]

\[
\begin{align*}
\dot{f} - \dot{g} f - \beta f^2 &= \beta \\
\dot{g} + 2\beta f &= -i\alpha \\
\dot{h} e^{-g} &= -\beta
\end{align*}
\]

The equation

\[ \dot{f} + i\alpha f + \beta f^2 = \beta \]

is a (famous) Riccati equation of general type. If we can solve the equation we have the solutions like \( f(t) \implies g(t) \implies h(t) \). See \(^{(18)}\) as these solutions.

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