ON AN ACTION OF THE BRAID GROUP $B_{2g+2}$ ON THE FREE GROUP $F_{2g}$

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Abstract. We construct an action of the braid group $B_{2g+2}$ on the free group $F_{2g}$ extending an action of $B_4$ on $F_2$ introduced earlier by Reutenauer and the author. Our action induces a homomorphism from $B_{2g+2}$ into the symplectic modular group $Sp_{2g}(\mathbb{Z})$. In the special case $g=2$ we show that the latter homomorphism is surjective and determine its kernel, thus obtaining a braid-type presentation of $Sp_4(\mathbb{Z})$.

1. Introduction

In [9] Christophe Reutenauer and the present author considered the automorphisms $G$, $D$, $\tilde{G}$, $\tilde{D}$ of the free group $F_2$ on two generators $a$ and $b$ defined by

$$G : (a, b) \mapsto (a, ab), \quad D : (a, b) \mapsto (ba, b),$$

$$\tilde{G} : (a, b) \mapsto (a, ba), \quad \tilde{D} : (a, b) \mapsto (ab, b),$$

(see also [11 Sect. 2.2.2]). Their images under the natural surjection (the abelianization map) $\pi : \text{Aut}(F_2) \to \text{Aut}(\mathbb{Z}^2) = \text{GL}_2(\mathbb{Z})$ are the matrices

$$\pi(G) = \pi(\tilde{G}) = A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \pi(D) = \pi(\tilde{D}) = B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$  

The matrices $A$ and $B$ generate the subgroup $SL_2(\mathbb{Z})$ and satisfy the braid relation

$$AB^{-1}A = B^{-1}AB^{-1}.$$  

In [9 Lemma 2.1] we observed that $G$, $D$, $\tilde{G}$, $\tilde{D}$ satisfy similar braid relations in the automorphism group $\text{Aut}(F_2)$, namely

$$GD^{-1}G = D^{-1}GD^{-1}, \quad \tilde{G}D^{-1}\tilde{G} = D^{-1}\tilde{G}D^{-1},$$

$$\tilde{G}\tilde{D}^{-1}\tilde{G} = \tilde{D}^{-1}\tilde{G}\tilde{D}^{-1}, \quad \tilde{G}D^{-1}G = \tilde{D}^{-1}G\tilde{D}^{-1},$$

together with the commutation relations

$$G\tilde{G} = \tilde{G}G \quad \text{and} \quad D\tilde{D} = \tilde{D}D.$$  

These relations allowed us to define a group homomorphism $f$ from the braid group $B_4$ on four strands to $\text{Aut}(F_2)$ by

$$f(\sigma_1) = G, \quad f(\sigma_2) = D^{-1}, \quad f(\sigma_3) = \tilde{G},$$

where $\sigma_1, \sigma_2, \sigma_3$ are the standard generators of $B_4$. In [9] we proved that the image $f(B_4)$ of $f$ is the index-two subgroup $\pi^{-1}(SL_2(\mathbb{Z}))$ of $\text{Aut}(F_2)$ and that the kernel of $f$ is the center of $B_4$.

After the article [9] was published, Etienne Ghys suggested that the action of $B_4$ on $F_2$ given by (1.3) might be derived from the fact that a punctured torus is a
double covering of a disk branched over four points. We checked that this fact indeed led to (1.3).

The present article is a continuation of [9]; it is based on the fact that a punctured surface of arbitrary genus \( g \) can be realized as a ramified double covering of a disk with \( 2g + 2 \) ramification points. Our main result yields an action of the braid group \( B_{2g+2} \) on \( 2g + 2 \) strands on the free group \( F_{2g} \) on \( 2g \) generators: this action is given by an explicit group homomorphism \( f : B_{2g+2} \to \text{Aut}(F_{2g}) \), extending (1.3).

The formulas for this homomorphism are given in Section 2. We show in Section 3 how to derive them geometrically. In Section 2.3 in an attempt to define higher analogues of the special Sturmian monoid, introduced in [9], we search for automorphisms in \( f(B_{2g+2}) \subset \text{Aut}(F_{2g}) \) that preserve the free monoid on \( 2g \) generators. It turns out that the situation for \( g \geq 2 \) is less satisfactory than for \( g = 1 \).

Finally, in Section 4 we first observe that the image \( f(B_{2g+2}) \) maps under the abelianization map \( \pi : \text{Aut}(F_{2g}) \to \text{GL}_{2g}(\mathbb{Z}) \) into the symplectic modular group \( \text{Sp}_{2g}(\mathbb{Z}) \). Concentrating on the case \( g = 2 \), we show that the map \( \pi \circ f : B_6 \to \text{Sp}_4(\mathbb{Z}) \) is surjective and we determine its kernel; we thus obtain a braid-type presentation of \( \text{Sp}_4(\mathbb{Z}) \) with generators the standard generators of \( B_6 \) and with relations the usual braid relations together with four additional relations.

2. The main result

Let \( g \) be an integer \( \geq 1 \) and \( F_{2g} \) be the free group on \( 2g \) generators \( a_1, \ldots, a_g, b_1, \ldots, b_g \).

2.1. A family of automorphisms of \( F_{2g} \). We consider the \( 2g + 1 \) automorphisms \( u_1, \ldots, u_{2g+1} \) of \( F_{2g} \) defined as follows.

- The automorphism \( u_1 \) fixes all generators, except \( b_1 \) for which we have
  \begin{equation}
  u_1(b_1) = a_1 b_1.
  \end{equation}

- The automorphism \( u_{2g+1} \) fixes all generators, except \( b_g \) for which
  \begin{equation}
  u_{2g+1}(b_g) = b_g a_g.
  \end{equation}

- For \( i = 1, \ldots, g \), the automorphism \( u_{2i} \) fixes all generators, except \( a_i \) for which
  \begin{equation}
  u_{2i}(a_i) = b_i^{-1} a_i.
  \end{equation}

- For \( i = 1, \ldots, g-1 \), the automorphism \( u_{2i+1} \) fixes all generators, except \( b_i \) and \( b_{i+1} \) for which we have
  \begin{equation}
  u_{2i+1}(b_i) = b_i a_i a_{i+1}^{-1} \quad \text{and} \quad u_{2i+1}(b_{i+1}) = a_{i+1} a_i^{-1} b_{i+1}.
  \end{equation}

When \( g = 1 \), the above \( 2g + 1 \) automorphisms reduce to three, namely \( u_1, u_2, u_3 \); these coincide respectively with the automorphisms \( G, D^{-1}, \tilde{G} \) of \( F_2 \) defined by (1.1).

2.2. An action of the braid group \( B_{2g+2} \). Let \( B_{2g+2} \) be the braid group on \( 2g + 2 \) strands with its standard presentation by generators \( \sigma_1, \ldots, \sigma_{2g+1} \) satisfying the relations \( (1 \leq i, j \leq 2g + 1) \)

\begin{equation}
\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if} \quad |i - j| > 1
\end{equation}

and

\begin{equation}
\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \quad \text{if} \quad |i - j| = 1.
\end{equation}

We now state our main result: it yields an action of \( B_{2g+2} \) on the free group \( F_{2g} \) by group automorphisms.

**Theorem 2.1.** There is a group homomorphism \( f : B_{2g+2} \to \text{Aut}(F_{2g}) \) such that \( f(\sigma_i) = u_i \) for all \( i = 1, \ldots, 2g + 1 \).
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The proof is straightforward: it suffices to check that the automorphisms $u_{1}, \ldots, u_{2g+1}$ satisfy Relations (2.5) and (2.6). (Formulas (2.1)–(2.4) and Theorem 2.1 were first publicized in [10, Exercise 1.5.2].)

For $g = 1$, the homomorphism of Theorem 2.1 coincides with the homomorphism $f : B_{4} \to \text{Aut}(F_{2})$ of [3, Lemma 2.5] defined in the introduction. In loc. cit. we showed that its kernel is exactly the center of $B_{4}$. For $g > 1$ we have the following weaker result.

**Proposition 2.2.** The kernel of $f : B_{2g+2} \to \text{Aut}(F_{2g})$ contains the center of $B_{2g+2}$.

**Proof.** Let $\delta = \sigma_{1} \cdots \sigma_{2g+1}$. It is well known (see [3, 10]) that the center of $B_{2g+2}$ is generated by $\delta^{2g+2}$. Let $\delta = f(\delta) = u_{1} \cdots u_{2g+1}$ be the corresponding automorphism of $F_{2g}$. Using (2.1)–(2.4), it is easy to check that for all $i = 1, \ldots, g$ we have

\[ \tilde{\delta}(a_{i}) = (b_{1} \ldots b_{i})^{-1} \quad \text{and} \quad \tilde{\delta}(b_{i}) = \begin{cases} a_{i}a_{i+1}^{-1} & \text{if } i \neq g, \\ a_{g}^{-1} & \text{if } i = g. \end{cases} \]

We deduce that

\[ \tilde{\gamma}^{2}(a_{i}) = \begin{cases} a_{i+1}a_{i}^{-1} & \text{if } i \neq g, \\ a_{1}^{-1} & \text{if } i = g, \end{cases} \quad \text{and} \quad \tilde{\gamma}^{2}(b_{i}) = \begin{cases} b_{i+1} & \text{if } i \neq g, \\ (b_{1} \ldots b_{i})^{-1} & \text{if } i = g. \end{cases} \]

From these formulas it is easy to conclude that $\tilde{\gamma}^{2g+2}$ is the identity. \qed

2.3. **Preserving the free submonoid $M_{2g}$ of $F_{2g}$.** Let $M_{2g}$ be the submonoid of $F_{2g}$ generated by $a_{1}, \ldots, a_{g}, b_{1}, \ldots, b_{g}$; it is a free monoid. By construction the automorphisms $u_{1}$ and $u_{2g+1}$ preserve $M_{2g}$. So do the inverses $u_{2i}^{-1}$ of the automorphisms $u_{2i}$, since

\[ u_{2i}^{-1}(a_{i}) = b_{i}a_{i}, \]

the other generators being fixed.

Unfortunately, as we can see from (2.4), the automorphisms $u_{2i+1}$ ($1 \leq i \leq g-1$), which exist only when $g \geq 2$, do not preserve the monoid $M_{2g}$. Nor do their inverses since

\[ u_{2i+1}(b_{i}) = b_{i}a_{i+1}a_{i}^{-1} \quad \text{and} \quad u_{2i+1}^{-1}(b_{i}) = a_{i}a_{i+1}^{-1}b_{i+1}. \]

Thus, the action of the higher braid groups $B_{2g+2}$ on $F_{2g}$ with $g \geq 2$ is quite different from the action of $B_{4}$ on $F_{2}$ when it comes to preserving the monoid $M_{2g}$.

Let us consider the case $g = 1$. In [3] we observed that the $M_{2}$-preserving automorphisms $u_{1} = G, u_{2}^{-1} = D, u_{3} = \tilde{G}$ of $F_{2}$ are so-called Sturmian morphisms. Together with the Sturmian morphism $\tilde{D}$, the morphisms $G, D, \tilde{G}$ generate the special Sturmian monoid $S_{0}$, for which we gave a presentation by generators and relations, and which we proved to be isomorphic to the submonoid of $B_{4}$ generated by $\sigma_{1}, \sigma_{2}^{-1}, \sigma_{3}$, and $(\sigma_{1}\sigma_{3}^{-1})^{-1}$. See [3, Sect. 3] for details.

In the case $g \geq 2$, consider the submonoid $\Omega_{2g}$ of $\text{Aut}(F_{2g})$ generated by the $g + 2$ elements $u_{1}, u_{2g+1}$, and $u_{2i}^{-1}$ ($1 \leq i \leq g$). It follows from the observation above that all elements of $\Omega_{2g}$ preserve the monoid $M_{2g}$.

We now express $\Omega_{2g}$ in terms of the free monoid $M_{2}$ on two generators and the free monoid $M_{1}$ on one generator, which we identify with the monoid of non-negative integers.

**Proposition 2.3.** Let $g \geq 2$. We have an isomorphism of monoids

\[ \Omega_{2g} \cong M_{2} \times (M_{1})^{g-2} \times M_{2}. \]

Moreover, $\Omega_{2g}$ is isomorphic to the submonoid of $B_{2g+2}$ generated by $\sigma_{1}, \sigma_{2g+1}, \text{and } \sigma_{2i}^{1}$ ($1 \leq i \leq g$).
Proof. In view of the commutation relations \(2.5\) for the automorphisms \(u_i\), any product of \(u_1, u_2^{-1}, u_4^{-1}, \ldots, u_{2g-2}^{-1}, u_{2g-1}^{-1}, u_{2g+1}^{-1}\) can be uniquely written as

\[
w(u_1, u_2^{-1}) u_4^{-n_2} \cdots u_{2g-2}^{-n_{2g-2}} w'(u_{2g}^{-1}, u_{2g+1}^{-1}),
\]

where \(w(u_1, u_2^{-1})\) belongs to the submonoid of \(\text{Aut}(F_{2g})\) generated by \(u_1\) and \(u_2^{-1}\), the exponents \(n_2, \ldots, n_{g-1}\) are non-negative integers, and \(w'(u_{2g}^{-1}, u_{2g+1}^{-1})\) belongs to the submonoid of \(B_{2g+2}\) generated by \(u_{2g}^{-1}\) and \(u_{2g+1}^{-1}\). It remains to show that the submonoid generated by \(u_1, u_2^{-1}\) and the submonoid generated by \(u_{2g}^{-1}, u_{2g+1}^{-1}\) are both isomorphic to the free monoid \(M_2\). We give a proof of this claim for the first submonoid (there is a similar proof for the second one). Since \(u_1\) and \(u_2^{-1}\) move only the generators \(a_1\) and \(b_1\), we may consider them in \(\text{Aut}(F_2)\). Now let \(w(u_1, u_2^{-1})\) be a non-trivial word in \(u_1, u_2^{-1}\) and consider its image in \(\text{GL}_2(\mathbb{Z})\); we have

\[
\pi \left( w(u_1, u_2^{-1}) \right) = w(A, B)
\]

where \(A\) and \(B\) are the matrices defined by \(1.2\). It is well known (and easy to check) that any non-trivial word in \(A, B\) cannot be the identity matrix. Therefore, \(w(u_1, u_2^{-1}) \neq 1\) in \(\text{Aut}(F_2)\). This proves our claim.

Set \(\iota(u_1) = \sigma_1, \iota(u_{2g+1}) = \sigma_{2g+1}, \) and \(\iota(u_{2i}^{-1}) = \sigma_{2i}^{-1}\) for \(i = 1, \ldots, g\). In view of the first assertion and the braid relations \(2.5, 2.6\), these formulas define a monoid homomorphism \(\iota : \Omega_{2g} \to B_{2g+2}\). Since \(f \circ \iota = \text{id}\) on \(\Omega_{2g}\), the homomorphism \(\iota\) is injective, which proves the second assertion. \(\Box\)

3. RAMIFIED DOUBLE COVERINGS OF THE DISK

We now explain how we found Formulas \(2.1, 2.4\) which define the automorphisms \(u_1, \ldots, u_{2g+1}\) of Section 2. The material in this section is standard; we nevertheless give details for the sake of non-topologists.

It is well known that any closed surface \(\Sigma_g\) of genus \(g > 0\) can be realized as a ramified double covering of the sphere \(S^2\) with \(2g + 2\) ramification points. It suffices to embed \(\Sigma_g\) into \(\mathbb{R}^3\) in such a way that it is invariant under the hyperelliptic involution, which is the reflexion in a line \(L\) intersecting \(\Sigma_g\) in \(2g + 2\) points as in Figure 1. The quotient of \(\Sigma_g\) by this involution is a sphere equipped with \(2g + 2\) distinguished points, namely the projections of the points of \(\Sigma_g \cap L\); these are the ramification points of the double covering.

![Figure 1. A surface \(\Sigma_g\) invariant under the hyperelliptic involution.](image)

From the interior of \(\Sigma_g \setminus (\Sigma_g \cap L)\) remove a small open disk \(D\) with center \(P\) (represented by \(\ast\) in the figures), as well as the disk \(D'\) obtained from \(D\) under the reflection in \(L\). Then \(\Sigma_g' = \Sigma_g \setminus (D \cup D')\) is a ramified double covering of the sphere deprived of a disk, in other words, a double covering of a disk \(D_0\) with \(2g + 2\) ramification points.

As is well known (see \[9\] or \[10\] Sect. 1.6]), the braid group \(B_{2g+2}\) is isomorphic to the mapping class group of \(D_0\) consisting of the isotopy classes of orientation-preserving homeomorphisms that fix each point of the boundary of \(D_0\) and permute
the \(2g + 2\) distinguished points. If \(\varphi\) is such an homeomorphism, we pick the lift \(\tilde{\varphi}\) of \(\varphi\) to \(\Sigma^g\) that fixes \(D\) and \(D'\) pointwise. The correspondence \(\varphi \mapsto \tilde{\varphi}\) induces a homomorphism from \(B_{2g+2}\) to the mapping class group of \(\Sigma^g\), hence a homomorphism \(B_{2g+2} \to \text{Aut}(\Pi)\), where \(\Pi\) is the fundamental group of \(\Sigma^g\), which is a free group. We wish to determine this homomorphism.

It is enough to determine the action of the generators of \(B_{2g+2}\) on a smaller free group, namely the fundamental group of \(\Sigma^g - D'\), whose elements we represent as loops in \(\Sigma^g - D'\) based at the point \(P\). This fundamental group is the free group \(F_{2g}\) generated by the loops \(a_1, \ldots, a_g, b_1, \ldots, b_g\) as depicted in Figure 1.

Consider the curves \(C_0, C_1, \ldots, C_g, C'_1, \ldots, C'_g\) of Figure 2. It is easy to check that a lift \(\tilde{\sigma}_1\) of the homeomorphism of \(D_0\) representing the generator \(\sigma_1\) of \(B_{2g+2}\) is the Dehn twist \(T_0\) along the curve \(C'_0\) (for a definition of Dehn twists, see [10, Sect. 3.2.4]). The action of \(T_0\) on the generators of the fundamental group of \(\Sigma^g - D'\) is easy to compute; clearly it leaves the generators \(a_1, \ldots, a_g\) as well as \(b_2, \ldots, b_g\) unchanged. On \(b_1\) it acts as in Figure 3, which leads to Formula (2.1).

Similarly, a lift \(\tilde{\sigma}_{2g+1}\) of the homeomorphism representing the last generator \(\sigma_{2g+1}\) of \(B_{2g+2}\) is the Dehn twist \(T_g\) along the curve \(C_g\). This twist fixes all generators of the fundamental group except \(b_g\). The reader is encouraged to draw the corresponding figure and derive (2.2) from it.

A lift \(\tilde{\sigma}_{2i}\) of the homeomorphism representing the generator \(\sigma_{2i} (1 \leq i \leq g)\) is the Dehn twist \(T'_i\) along the curve \(C'_i\). This twist affects only the generator \(a_i\), on which it acts as in Figure 4; we thus obtain (2.3).

Finally, when \(1 \leq i \leq g-1\), a lift \(\tilde{\sigma}_{2i+1}\) of the homeomorphism representing \(\sigma_{2i+1}\) is the Dehn twist \(T_i\) along the curve \(C_i\). This twist fixes all generators of the fundamental group except \(b_i\) and \(b_{i+1}\); it acts on \(b_i\) as in Figure 5 and on \(b_{i+1}\) as in Figure 6. This yields Formula (2.4).

The above computation appeared in [4] with different notation; the correspondence between that paper and ours is given by \(s_i = b_i^{-1}, t_i = a_i^{-1}, h_{C_i} = T_0^{-1}, h_{U_i} = T_i^{-1}, h_{Z_i} = T_i^{-1}\) (see also [8, Sect. 4] and [3, Sect. 2]). A proof of Proposition 2.2 can also be derived from Relation (10) in [4].

\[\text{Figure 2. The curves } C_i \text{ and } C'_i.\]

\[\text{Figure 3. Action of the Dehn twist } T_0 \text{ on } b_1.\]
4. The symplectic modular group

In this section we first observe that, after abelianizing our action, we obtain a symplectic action of the braid group $B_{2g+2}$ on the free abelian group $\mathbb{Z}^{2g}$. In the second part of the section we elaborate on the case $g = 2$ and obtain a braid-type presentation of $\text{Sp}_4(\mathbb{Z})$.

4.1. Symplectic automorphisms. Let $g \geq 1$ be a positive integer. Pick a basis $(\bar{a}_1, \ldots, \bar{a}_g, \bar{b}_1, \ldots, \bar{b}_g)$ of $\mathbb{Z}^{2g}$. Using this basis, we identify the group $\text{Aut}(\mathbb{Z}^{2g})$ of automorphisms of $\mathbb{Z}^{2g}$ with the general linear group $\text{GL}_{2g}(\mathbb{Z})$.

We equip $\mathbb{Z}^{2g}$ with the standard alternating bilinear form $(\ , \ )$ which vanishes on all pairs of basis elements, except the following ones:

$$\langle \bar{a}_i, \bar{b}_i \rangle = -\langle \bar{b}_i, \bar{a}_i \rangle = 1 \quad (i = 1, \ldots, g).$$

The *symplectic modular group* $\text{Sp}_{2g}(\mathbb{Z})$ (formerly called Siegel’s modular group) is the group of automorphisms of $\mathbb{Z}^{2g}$ preserving this alternating form; it can be
described as the group of matrices $M \in \text{GL}_{2g}(\mathbb{Z})$ such that

\[ (4.1) \quad M^T \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} M = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}, \]

where $M^T$ is the transpose of $M$ and $I_g$ is the identity matrix of size $g$.

Consider the composition

\[ \overline{f} : B_{2g+2} \rightarrow \text{GL}_{2g}(\mathbb{Z}) \]

of the homomorphism $f : B_{2g+2} \rightarrow \text{Aut}(F_{2g})$ of Section 2 with the natural surjection $\pi : \text{Aut}(F_{2g}) \rightarrow \text{GL}_{2g}(\mathbb{Z})$.

**Proposition 4.1.** We have $\overline{f}(B_{2g+2}) \subset \text{Sp}_{2g}(\mathbb{Z})$.

**Proof.** It is enough to check that the image $\pi(u_i)$ in $\text{GL}_{2g}(\mathbb{Z})$ of each automorphism $u_i$ belongs to $\text{Sp}_{2g}(\mathbb{Z})$. Now the automorphisms $u_i$ are induced by elements of a mapping class group which are well known to induce symplectic linear maps (see [12, Sect. 5.8]). Alternatively, one checks that each matrix $\pi(u_i)$ satisfies Relation (4.1).

When $g = 1$, the symplectic group $\text{Sp}_2(\mathbb{Z})$ identifies naturally with the modular group $\text{SL}_2(\mathbb{Z})$. We proved in [9] that the image $\overline{f}(B_1)$ is the entire group $\text{SL}_2(\mathbb{Z})$.

**Remark 4.2.** Taking an adequate specialization of the Burau representation, Magnus and Peloso [13] also constructed a homomorphism $B_{2g+2} \rightarrow \text{Sp}_{2g}(\mathbb{Z})$, for which they showed that it is surjective if and only if $g = 1$ or $g = 2$. We do not know if their symplectic representation is related to ours.

4.2. The case $g = 2$. We now consider the homomorphism $\overline{f} : B_{2g+2} \rightarrow \text{Sp}_{2g}(\mathbb{Z})$ in the special case $g = 2$.

**Theorem 4.3.** The homomorphism $\overline{f} : B_6 \rightarrow \text{Sp}_4(\mathbb{Z})$ is surjective and its kernel is the normal subgroup of $B_6$ generated by $\Delta^2$, $\alpha^2$, $\alpha \beta$, $(\alpha \gamma)^2$, where

\[ \Delta = \sigma_1 \sigma_2 \sigma_3 \sigma_4 \sigma_5 \sigma_6, \quad \alpha = (\sigma_4 \sigma_5)^3, \quad \beta = \sigma_3^{-1} \sigma_1 \sigma_2, \quad \gamma = \sigma_1 \sigma_2 \sigma_3. \]

As a consequence, we obtain the following braid-type presentation of $\text{Sp}_4(\mathbb{Z})$.

**Corollary 4.4.** The symplectic group $\text{Sp}_4(\mathbb{Z})$ has a presentation with generators $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5$ and 14 relations consisting of Relations (2.5) - (2.8) and of the four relations

\[ (\sigma_1 \sigma_2 \sigma_3 \sigma_4 \sigma_5)^2 = 1, \]

\[ (\sigma_4 \sigma_5)^6 = 1, \quad (\sigma_1 \sigma_2)^3 = \sigma_3 (\sigma_4 \sigma_5)^3 \sigma_3^{-1}, \]

\[ (\sigma_1 \sigma_2 \sigma_3)^{-1} = (\sigma_4 \sigma_5)^3 (\sigma_1 \sigma_2 \sigma_3) (\sigma_4 \sigma_5)^{-3}. \]

Previously Behr [11] gave a quite different presentation of this group, with six generators and 18 relations (see below); Bender [2] improved Behr’s presentation by reducing it to one with two generators and 8 relations (see also [5]).

**Proof of Theorem 4.3** The generators in Behr’s presentation [11] of $\text{Sp}_4(\mathbb{Z})$ are the following symplectic matrices:

\[ x_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad x_\alpha + \beta = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad x_{2\alpha + \beta} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \]
The images under $\overline{f}$ of the elements $\sigma_1, \ldots, \sigma_5$, and $\Delta$ of $B_6$ are given by

$$M_1 = \overline{f}(\sigma_1) = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_2 = \overline{f}(\sigma_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$M_3 = \overline{f}(\sigma_3) = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_4 = \overline{f}(\sigma_4) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$M_5 = \overline{f}(\sigma_5) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_\Delta = \overline{f}(\Delta) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$
Using the braid applet of \[\text{[7]},\] we deduce from (4.5) and (4.6) that
\[
(4.7) \quad \gamma_1 = \sigma_3\gamma_2\sigma_3^{-1}.
\]
Behr’s Relation (7) is equivalent to \(r_7 = 1\) with \(r_7 = w\omega w_\beta^2\omega_\alpha w_\beta^{-2}\). We thus have
\[
\gamma_7 = (\sigma_4\sigma_5)^3\Delta(\sigma_4\sigma_5\sigma_4)^{-2}(\sigma_4\sigma_5)^3\Delta(\sigma_4\sigma_5\sigma_4)^2
\]
\[
= (\sigma_4\sigma_5)^3\Delta(\sigma_4\sigma_5\sigma_4)^{-2}(\sigma_4\sigma_5\sigma_4)^2\Delta(\sigma_4\sigma_5)^3
\]
\[
= (\sigma_4\sigma_5)^3\Delta^2(\sigma_4\sigma_5).
\]
Since \(\Delta^2\) is central, as is well known (see \[\text{[6, 10]}\]), we obtain
\[
(4.8) \quad \gamma_7 = (\sigma_4\sigma_5)^6\Delta^2.
\]
Behr’s Relation (10), which is equivalent to \(w_\beta^{-4} = 1\), yields
\[
(4.9) \quad \gamma_{10} = (\sigma_4\sigma_5\sigma_4)^4 = (\sigma_4\sigma_5)^6.
\]
Relation (13) in \[\text{[11]}\] is equivalent to \(r_{13} = 1\), where \(r_{13} = \omega_\alpha w_\alpha w_\alpha^{-1}w_\alpha^{-3}w_\alpha\). Therefore,
\[
(4.10) \quad \gamma_{13} = (\sigma_4\sigma_5)^3(\sigma_1\sigma_3\sigma_1^{-1}\sigma_5)(\sigma_4\sigma_5)^{-3}(\sigma_1\sigma_3\sigma_1^{-1}\sigma_5).
\]
In view of the relations \(\Delta\sigma_i = \sigma_{6-i}\Delta\) \((i = 1, \ldots, 5)\), we obtain
\[
(4.11) \quad \gamma_{13} = (\sigma_4\sigma_5)^3(\sigma_1\sigma_3\sigma_1^{-1}\sigma_5)(\sigma_4\sigma_5)^{-3}(\sigma_1\sigma_3\sigma_1^{-1}\sigma_5).
\]
Using again the braid applet \[\text{[7]},\] we find
\[
(4.12) \quad \gamma_{14} = \gamma_2 = \gamma_{13}^{-1}.
\]
Relation (14) in \[\text{[11]}\] is equivalent to \(r_{14} = 1\), where \(r_{14} = \omega_\beta^{-1}w_\alpha^{-1}w_\beta\). Therefore,
\[
(4.13) \quad \gamma_{14} = (\sigma_5^{-1}\sigma_4^{-1}\sigma_1^{-1}\sigma_3\sigma_5^{-1}\sigma_4\sigma_5)(\sigma_4\sigma_5\sigma_4)(\sigma_1^{-1}\sigma_3\sigma_5^{-1}sigma_1^{-1})(\sigma_4^{-1}\sigma_5^{-1}\sigma_4^{-1}).
\]
We similarly find
\[
(4.14) \quad \gamma_{15} = \gamma_2 = \gamma_{13}^{-1}.
\]
Relation (17) in \[\text{[11]}\] is equivalent to \(r_{17} = 1\) with \(r_{17} = \omega_\alpha w_\alpha w_\alpha^{-1}w_\alpha w_\alpha\). Thus,
\[
(4.15) \quad \gamma_{17} = (\sigma_4\sigma_5)^3\Delta(\sigma_5^{-1}\sigma_3^{-1}\sigma_3\sigma_5^{-1}\sigma_4\sigma_5)\Delta^{-1}(\sigma_4\sigma_5)^{-3}\times
\]
\[
(\sigma_5^{-1}\sigma_4^{-1}\sigma_3\sigma_5^{-1}\sigma_4\sigma_5)(\sigma_4\sigma_5)^3\Delta(\sigma_5^{-1}\sigma_3^{-1}\sigma_3\sigma_5^{-1}\sigma_4\sigma_5).
\]
Letting both \(\Delta\) jump to the right and letting the leftmost one merge with \(\Delta^{-1}\), we obtain
\[
(4.16) \quad \gamma_{17} = (\sigma_4\sigma_5)^3(\sigma_1\sigma_2\sigma_1^{-1}\sigma_3\sigma_5^{-1}\sigma_2^{-1}\sigma_1^{-1})(\sigma_4\sigma_5)^{-3}\times
\]
\[
(\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_3\sigma_5^{-1}\sigma_4\sigma_5)(\sigma_4\sigma_5)^3(\sigma_1\sigma_2\sigma_1^{-1}\sigma_3\sigma_5^{-1}\sigma_2^{-1}\sigma_1^{-1})\Delta.
\]
It follows from (4.7), (4.11), (4.12) that the kernel of \(\mathcal{F}\) is the normal closure of the subgroup generated by \(\gamma_7, \gamma_{10}, \gamma_{13},\) and \(\gamma_{17}\). Now, by (4.8)–(4.10) we have \(\gamma_7 = \alpha^2\Delta^2, \gamma_{10} = \alpha^2,\) and \(\gamma_{13} = \alpha\gamma\gamma^{-1}\gamma = (\alpha\gamma)^2\gamma^{-1}(\alpha^2)^{-1}\gamma,\)
where \(\Delta, \alpha, \gamma\) have been defined in the statement of the theorem. Hence, the kernel of \(\mathcal{F}\) is generated as a normal subgroup by \(\Delta^2, \alpha^2, (\alpha\gamma)^2,\) and \(\gamma_{17}\). To complete the proof of the theorem, we consider the normal subgroup \(N\) of \(B_{2g}\) generated by \(\Delta^2, \alpha^2,\) and \((\alpha\gamma)^2,\) and we show that \(\gamma_{17}\) is conjugate to \(\alpha\beta\) modulo \(N\).

Let us now prove this. From (4.13) we first derive
\[
\gamma_{17} = \sigma_1\sigma_2\alpha\gamma^{-1}\sigma_2^{-1}\sigma_1^{-1}\alpha^{-1}\sigma_3\sigma_5^{-1}\sigma_4^{-1}\gamma^{-1}\sigma_4\sigma_5\alpha\sigma_1\sigma_2\gamma^{-1}\sigma_2^{-1}\sigma_1^{-1}\Delta.
\]
Now, \((\alpha \gamma)^2 \equiv 1 \mod N\). Hence, \(\gamma^{-1} \equiv \alpha \gamma \alpha\). Therefore, in view of this and of \(\alpha^2 \equiv 1\), we obtain

\[
\gamma_{17} \equiv \sigma_1 \sigma_2 \sigma^2 \gamma \sigma_2^{-1} \sigma_1^{-1} \alpha^{-1} \sigma_1^{-1} \sigma_2^{-1} \sigma_4^{-1} \alpha \gamma \sigma_4 \sigma_5 \sigma_1 \sigma_2 \alpha \gamma^{-1} \alpha^2 \sigma_2^{-1} \sigma_1^{-1} \Delta \\
\equiv (\sigma_1 \sigma_2 \sigma_3^{-1} \sigma_2^{-1} \sigma_1^{-1} \sigma_5^{-1} \sigma_4^{-1} \alpha \gamma \sigma_4 \sigma_5 \sigma_1 \sigma_2 \alpha \gamma^{-1} \alpha^2 \sigma_2^{-1} \sigma_1^{-1} \Delta \\
\equiv (\sigma_1 \sigma_2 \sigma_3^{-1} \sigma_2^{-1} \sigma_1^{-1} \sigma_5^{-1} \sigma_4^{-1} \alpha \gamma \sigma_4 \sigma_5 \sigma_1 \sigma_2 \alpha \gamma^{-1} \alpha^2 \sigma_2^{-1} \sigma_1^{-1} \Delta \\
\equiv (\sigma_1 \sigma_2 \sigma_3^{-1} \sigma_2^{-1} \sigma_1^{-1} \sigma_5^{-1} \sigma_4^{-1} \alpha \gamma \sigma_4 \sigma_5 \sigma_1 \sigma_2 \alpha \gamma^{-1} \alpha^2 \sigma_2^{-1} \sigma_1^{-1} \Delta)
\]

modulo \(N\). Using the applet [27], we obtain

\[
\alpha^{-1} \sigma_2^{-1} \sigma_1^{-1} \Delta = \sigma_3 \sigma_4 \sigma_5 \sigma_2 \sigma_1 \sigma_2 \sigma_3 \sigma_1 \sigma_2 \sigma_1 \sigma_3^{-1} \sigma_2^{-1} \sigma_1^{-1} \sigma_4^{-1} \sigma_3^{-1}.\]

Therefore, \(\gamma_{17} \equiv (\sigma_1 \sigma_2 \sigma_3^{-1} \sigma_2^{-1} \sigma_4^{-1}) \gamma' (\sigma_1 \sigma_2 \sigma_3^{-1} \sigma_2^{-1} \sigma_4^{-1})^{-1}\), where

\[
\gamma' = \alpha \sigma_3^{-1} \sigma_1 \sigma_2 \sigma_3 \sigma_4 \sigma_5 \sigma_4 \sigma_5 \sigma_1 \sigma_3 \sigma_2 \sigma_1 \sigma_3 \\
= \alpha \sigma_3^{-1} (\sigma_1 \sigma_2 \sigma_3 \sigma_4 \sigma_5 \sigma_4 \sigma_5 \sigma_1 \sigma_3 \sigma_2 \sigma_1 \sigma_3) \\
= \alpha \sigma_3^{-1} (\sigma_1 \sigma_2 \sigma_3 \sigma_4 \sigma_5 \sigma_4 \sigma_5 \sigma_1 \sigma_3 \sigma_2 \sigma_1 \sigma_3) \\
= \alpha \sigma_3^{-1} (\sigma_1 \sigma_2 \sigma_3 \sigma_4 \sigma_5 \sigma_4 \sigma_5 \sigma_1 \sigma_3 \sigma_2 \sigma_1 \sigma_3) \\
= \alpha \sigma_3^{-1} (\sigma_1 \sigma_2 \sigma_3 \sigma_4 \sigma_5 \sigma_4 \sigma_5 \sigma_1 \sigma_3 \sigma_2 \sigma_1 \sigma_3) \\
= \alpha \sigma_3^{-1} (\sigma_1 \sigma_2 \sigma_3 \sigma_4 \sigma_5 \sigma_4 \sigma_5 \sigma_1 \sigma_3 \sigma_2 \sigma_1 \sigma_3) \\
= \alpha \sigma_3^{-1} (\sigma_1 \sigma_2 \sigma_3 \sigma_4 \sigma_5 \sigma_4 \sigma_5 \sigma_1 \sigma_3 \sigma_2 \sigma_1 \sigma_3)
\]

This completes the proof of Theorem [4,3].

\[\square\]

**Remark 4.5.** We may wonder whether the four elements \(\Delta^2, \alpha^2, \alpha \beta, (\alpha \gamma)^2\) of the kernel of \(\overline{f}\) already belong to the kernel of \(f\). This holds true for the central element \(\Delta^2\); indeed, \(f(\Delta^2) = \text{id}\) by Proposition [2,2]. The other three elements map under \(f\) to non-inner automorphisms of \(F_4\). For instance, the automorphism \(f(\alpha^2)\) is the following: it fixes \(a_1\) and \(b_1\), and on \(a_2\) and \(b_2\) we have

\[
a_2 \mapsto (a_2^{-1} b_2 a_2 b_2^{-1}) a_2 (a_2^{-1} b_2 a_2 b_2^{-1})^{-1} \quad \text{and} \quad b_2 \mapsto (a_2^{-1} b_2 a_2 b_2^{-1}).
\]

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