THE SYMMETRIC KAZDAN–WARNER PROBLEM AND APPLICATIONS

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Abstract. After R. Schoen completed the solution of the Yamabe problem, compact manifolds could be categorized into three classes, depending on whether they admit a metric with positive, non-negative, or only negative scalar curvature. Here we follow Yamabe’s first attempt to solve his problem through variational methods and provide an analogous equivalent classification for manifolds equipped with actions by non-discrete compact Lie groups. Moreover, we apply the method and the results to: classify total spaces of fiber bundles with compact structure groups (concerning scalar curvature), to conclude density results and compare realizable scalar curvature functions between some exotic manifolds and their standard counterpart. We also provide an extended range of prescribed scalar curvature functions of warped products, especially with Calabi–Yau manifolds.

Declaration

Declaration and Statements Our manuscript has no associated data.

1. Introduction

As exemplified by the Bonnet–Meyers Theorem, Differentiable Sphere Theorem ([Bre10]) and Poincaré Conjecture ([Per02a, Per03a, Per03b]), to assume the existence of a special type of geometry is a consistent way to understand a manifold’s topology. The converse question, however,

Given a fixed manifold, which are its admissible geometries?

is widely open. The existence of manifolds such as the very interesting exotic spheres, first introduced by J. Milnor in [Mil56], adds to the relevance of the question.

On the other hand, as a generalization of the classification of closed surfaces, Yamabe [Yam60] proposed the existence of metrics of constant scalar curvature on manifolds. Once there found a mistake in Yamabe’s proof, the problem came to be known as the Yamabe problem. A complete solution came later, through the works of N. Trudinger [Tru68], T. Aubin

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and R. Schoen [Sch84]. R. Schoen and S-T Yau [SY79a, SY79b, SY81]. By combining it with Kazdan–Warner [KW75a, KW75b], one can now state Yamabe’s classification in the following way. Every \( n \)-dimensional closed manifold \( M \) with \( n \geq 3 \) falls in one of the following classes:

- \( \mathcal{P} \): Every function is the scalar curvature of a Riemannian metric on \( M \);
- \( \mathcal{Z} \): A function is the scalar curvature of a Riemannian metric on \( M \) if and only if it is the zero function or it is negative somewhere;
- \( \mathcal{N} \): A function is the scalar curvature of a Riemannian metric on \( M \) if and only if it is negative somewhere.

On the one hand, it is known that every class is nonempty and admits varieties of manifolds. On the other hand, curiously, all obstructions to the realizability of scalar curvature functions vanish in the presence of non-Abelian symmetry (see Lawson–Yau [LY74]). However, in the realm of \( G \)-invariant metrics, one only knows that the scalar curvature must be basic, i.e., invariant by the group action and hence constant along orbits; but it is unknown how large the class of realizable ones is.

Here we give a complete answer to this question by providing a classification analogous to the Yamabe one in the \( G \)-invariant setting. It is also worth pointing out that, since we are dealing essentially with manifolds with isometric actions, we are never interested in homogeneous spaces, that is, \( G \)-manifolds (i.e., manifolds with isometric) with transitive group actions.

**Theorem A.** Let \((M, G)\) be a closed smooth manifold with dimension \( n \geq 3 \) equipped with an effective action by a non-discrete compact Lie group \( G \). If no \( G \)-orbit has zero dimension then \((M, G)\) belongs to one of the following classes:

- \( \mathcal{P}^G \): Every basic function is the scalar curvature of a \( G \)-invariant metric on \( M \);
- \( \mathcal{Z}^G \): A basic function is the scalar curvature of a \( G \)-invariant metric on \( M \) if and only if it is the zero function or it is negative somewhere;
- \( \mathcal{N}^G \): A basic function is the scalar curvature of a \( G \)-invariant metric on \( M \) if and only if it is negative somewhere.

Moreover, if \( G \) has a non-Abelian Lie algebra, \((M, G)\) \( \in \mathcal{P}^G \).

We also observe that the respective classes do depend on both the manifolds and the equipped action, as L. Bergery [Ber83] provides examples of manifolds in \( \mathcal{P} \) that do not admit a \( G \)-invariant metric as in \( \mathcal{P}^G \).

Related to that, we stress that the proof of Theorem A is divided into two main results (Theorems B and C) and an auxiliary result (Theorem E). The main results are \( G \)-invariant versions of the solution of the Yamabe problem and of Kazdan–Warner [KW75a, Theorem A], respectively.

**Theorem B.** Let \((M, G)\) be a closed smooth manifold with dimension \( n \geq 3 \) equipped with an effective action by a non-discrete compact Lie group \( G \). If no orbit has zero dimension then
(1) every \((M,G)\) admits a \(G\)-invariant metric with constant negative scalar curvature;
(2) if \((M,G)\) admits a \(G\)-invariant metric with non-negative scalar curvature, then it admits a metric with zero scalar curvature;
(3) if \((M,G)\) admits a \(G\)-invariant metric with a non-vanishing non-negative scalar curvature, then it admits a metric with positive constant scalar curvature.

The desired metric for Theorem B is obtained by a \(G\)-invariant conformal change of the original metric, relating to Yamabe’s problem. Yamabe’s first attempt to prove the existence of constant scalar curvature in a fixed conformal class dates to 1960. The proof presented an error, as pointed out by Trudinger [Tru68], due to the lack of compactness caused by a critical exponent in the Rellich–Kondrachov Theorem. By combining the works of Aubin and Schoen [Amb76, Sch84], which used Green functions among other techniques, a complete solution was given in 1984.

Surprisingly, the proof of Theorem B takes a complete turnaround and follows like Yamabe’s original attempt via variational methods. The merit and novelty of the present paper is to show how it is possible to variationally approach PDE problems associated with basic quantities by restricting the analysis to slices. This dimension reduction improves the classical Rellich–Kondrachov compactness embedding, naturally handling the critical exponent (see Theorem 2.5). In particular, we provide a simpler proof for Yamabe’s problem in the large class of invariant metrics. Related works that inspired the discussion can also be seen in [Heb90, HebVa93, HebVa97].

The second main result places Kazdan–Warner [KW75a, Theorem A] in the \(G\)-invariant setting. We observe that \(G\) can be discrete in this theorem.

**Theorem C.** Let \((M,G,g)\) be a closed Riemannian manifold where \(G\) is a compact Lie group acting by isometries. Suppose that \(f:M \to \mathbb{R}\) is a basic smooth function and let \(c > 0\) be such that

\[
 c \min_M f < \operatorname{scal}_g(x) < c \max_M f
\]

for every \(x \in M\). Then there is a \(G\)-invariant metric on \(M\) whose scalar curvature is \(f\).

From Kazdan–Warner [KW75a], one knows that the metric \(\tilde{g}\) realizing \(\operatorname{scal}_{\tilde{g}} = f\) is arbitrarily close to the orbit of \(g\) by the diffeomorphism group. Using this fact, reducing the diffeomorphism group to a given closed subgroup of the isometry group, we can show that:

**Corollary D.** Let \((M,G)\) be a closed smooth manifold equipped with an effective isometric action of a compact Lie group \(G\). If \(M \in \mathcal{A} \cap \mathcal{A}^G\), \(\mathcal{A} \in \{\mathcal{P}, \mathcal{Z}\}\), then any smooth function which is the scalar curvature of a metric on \(M\) is realized by a metric arbitrarily \(L^p\)-close to a \(G\)-invariant metric, for \(1 \leq p < \infty\).
In fact, the proof of Corollary D is a direct application of Theorem C once one observes that we can regard $M$ with appropriate $G$-invariant constant scalar curvature metrics, then verifying that any given function can be chosen to satisfy the required pinching.

To refine Theorem A, we study some very classical metric deformations. A deformation commonly used in the construction of metrics with positive/non-negative sectional curvature is the Cheeger deformation (we refer to [Zil87] or [CeSS18] for details). To prove Theorem E below, we base our methods on the improved version of Lawson–Yau Theorem from [CeSS18], which guarantees that every $G$-invariant metric $g$ develops positive scalar curvature after Cheeger deformation, as long as $G$ has a non-Abelian Lie algebra. Theorem E implies the last statement in Theorem A.

**Theorem E.** Suppose that $(M, G) \notin \mathcal{P}^G$ where $M, G$ are compact. Then $G$ has an Abelian Lie algebra and there is no $G$-invariant metric on $M$ such that the induced metric on the manifold part of $M/G$ has positive scalar curvature.

We refer to [Wie16] for results on the existence of positive scalar curvature under Abelian symmetry. A more detailed study of the evolution of the scalar curvature under Cheeger deformation improves Corollary D in interesting cases. Recall that an action is called semi-free if every isotropy group is discrete.

**Theorem F.** Suppose that $(M, g)$ is a compact Riemannian manifold and $G$ is a compact Lie group acting semi-freely by isometries in $M$. Assume that $G$ has a non-Abelian Lie algebra and that $f : M \to \mathbb{R}$ is smooth and basic function. Then there is a metric $\tilde{g}$ satisfying $\text{scal}_{\tilde{g}} = f$ which is arbitrarily $L_p$-close to a Cheeger deformation $g_t$ of $g$, for $1 \leq p < \infty$, up to diffeomorphism.

This is equivalent to saying that, fixing a $G$-invariant metric $g$, the set of scalar curvatures realized by the orbits $\text{Diff}(M) \cdot \{g_t \mid t \in (1, \infty)\}$ is dense in the set of smooth functions. For instance, Grove–Ziller [GZ00] and Goette–Kerin–Shankar [GKS20] construct several interesting 7-manifolds with semi-free $S^3$-actions. Theorem F applies directly to them.

A precursor of the Cheeger deformation is the canonical variation. By combining it with Theorem A and [Ber83], a result in the realm of (locally trivial) fiber bundles arises in a natural way: there exists strong restrictions whenever the total space do not admit a metric with positive scalar curvature.

**Corollary G.** Suppose that $\pi : F \hookrightarrow \bar{M} \to M$ is a compact fiber bundle with compact structure group (acting effectively) $G$. If $\bar{M} \notin \mathcal{P}$ then $M \notin \mathcal{P}$ and $(F, G) \notin \mathcal{P}^G$. In particular, $G$ has an Abelian Lie algebra.

In the context of a fiber bundle $\pi : \bar{M} \to M$, a partial conformal change, called *general vertical warping*, is in place. Given a basic function
\[ \phi : M \to \mathbb{R}, \text{ the general vertical warping defined by } \phi \text{ is defined by multiplying by } e^{2\phi} \text{ the component of the metric tangent to the fibers.} \]

Common examples of fiber bundles are vector bundles and warped products, which are often non-compact.

On the other hand, the problem of prescribing scalar curvature of complete metrics on non-compact manifolds is still open. Here, the variational method used to prove Theorem H is applied to extend Ehrlich–Yoon-Tae–Kim [EYTK96]. In particular, we realize a great range of (basic) scalar curvature functions as warped products, including the case of non-compact fibers, while there the authors focus on prescribing constant scalar curvature.

**Theorem H.** Let \((M, h)\) be a closed \(n\)-dimensional Riemannian manifold and \(F\) be a \(k\)-dimensional manifold with a (complete) metric \(g_F\) with constant scalar curvature \(c\). Given a basic smooth function \(f : M \times F \to \mathbb{R}\), assume one of the following conditions:

1. \(c > 0\) and \(\text{scal}_h > 0\);
2. \(c \leq 0\) and \(f : M \times F \to \mathbb{R}\) satisfies the condition:
   \[ -\left(\frac{k+1}{8k}\right)(f - \text{scal}_h) + \left(\frac{(k+1)^2}{8(k-1)k}\right)c(\text{vol}(M))^{\frac{2}{k-1}} \geq 0. \]

Then,

(a) if \(|c| > 0\) there are \(\mu_1, \mu_2 > 0\) and a (complete) warped product metric \(\bar{g}\) on \((M \times F, \mu_1 h \times \mu_2 g_F)\) such that \(\text{scal}_{\bar{g}}(x,y) = f(x)\). In particular, if \(M\) admits a metric with constant scalar curvature \(c'\), \(M \times F\) admits a warped product metric with scalar curvature \(c'\).

(b) If \(c = 0\) there exists \(A \geq 0\) such that the previous conclusion holds to \(f + A\).

Despite stating in this manner, we prove a more general statement to Theorem H indeed, ensuring that the result holds for Riemannian submersions which are locally products, changing the word “warped product metrics” on the thesis in the former statements, to “general vertical warping metrics”. See Proposition 4.1.

A family of manifolds fundamental to the study of scalar and Ricci curvature consists of the Calabi–Yau manifolds. We call a compact Kähler manifold Calabi–Yau if its first Chern class vanishes, or, equivalently, if it admits a Ricci flat Kähler metric. For simplicity, we call such a metric as Calabi–Yau. To the authors’ knowledge, Theorem H is the first result of prescribing scalar curvature functions on non-compact fiber bundles over Calabi–Yau manifolds. On the other hand, Tosatti–Zhang [TZ14] proved that holomorphic submersions from compact Calabi–Yau manifolds are trivial, up to covering. We combine Theorem H with this result of Tosatti–Zhang to obtain:

**Corollary I.** Suppose \(\pi : X \to Y\) is a holomorphic submersion with connected fiber \(F\) satisfying one of the following:
(1) \(X, Y\) are projective manifolds with \(X\) Calabi–Yau;
(2) \(X, Y\) are compact Kähler manifolds; \(Y, F\) are Calabi–Yau; and either 
\(b_1(F) = 0\) or \(b_1(Y) = 0\) and \(F\) is a torus.

If \(f : Y \to \mathbb{R}\) is a scalar curvature function in \(Y\), then there is a metric \(\tilde{g}\) on \(X\) whose scalar curvature is \(f \circ \pi\). Moreover, if \(g\) is a Calabi–Yau metric on \(Y\), \(\tilde{g}\) can be chosen so that \(\pi\) is a Riemannian submersion over \(g\).

To illustrate the theory, in Section 5 we present families of examples of exotic manifolds closely related to their standard counterpart. Specifically, we recall the manifolds in [GZ00, Dur01, DPR10, Spe16, CS18, GKS20], which are realized as \(\star\)-diagrams. That is, a cross-diagram

\[
\begin{array}{ccc}
G & \text{\bullet} & M' \\
\downarrow & & \downarrow \\
G \times G & \text{\bullet} & M' \\
\downarrow & & \downarrow \\
P & \pi' & M' \\
\end{array}
\]

where \(P\) is a smooth manifold admitting two free commuting \(G\)-actions, \(\bullet\) and \(\star\), whose quotients we denote by \(M\) and \(M'\), respectively. Observe that the \(G \times G\)-action on \(P\) descends to \(G\)-actions on both \(M\) and \(M'\). For brevity, we denote the \(\star\)-diagram by \(M \leftarrow P \rightarrow M'\). Examples of manifolds realized as the \(M'\)-term in \(\star\)-diagrams are Kervaire, and exotic 7-, 8- and 10-spheres, their connected sums with various other manifolds, bundles, and candidates for 'exotic' homogeneous manifolds (see Section 5 for explicit constructions).

We justify our interest in such diagrams in a natural way: the first construction of exotic manifolds, due to Milnor [Mil56], makes use of the classical Reeb’s Theorem to show that certain 7-dimensional total spaces of sphere bundles are homeomorphic to a standard sphere. Moreover, one can recover the smooth structure of a manifold through its space of smooth functions (see, for example, [MS74, Problem 1-C]). However, in the presence of a \(\star\)-diagram, the set of basic functions of \(M\) and \(M'\) are naturally identified, since they are naturally identified with the space of \(G \times G\)-invariant functions on \(P\), proving that the set of basic functions does not recover \((M, G)\).

Seeing \(G \times G\)-invariant function \(f : P \to \mathbb{R}\) as a basic function on both \(M\) and \(M'\), Corollary [C] gives:

**Corollary J.** Consider a \(\star\)-diagram \(M \leftarrow P \rightarrow M'\) with \(G\) connected. Then, a basic function on \(M\) is the scalar curvature of a \(G\)-invariant metric on \(M\) if and only if it is the scalar curvature of a \(G\)-invariant metric on \(M'\) if and only if it lifts to the scalar curvature of a \(G \times G\)-invariant metric on \(P\).
Further examples and applications of the theory can be found in Section 5.1.

The paper is organized as follows. The inaugural Section 2.1 deals with basic facts on manifolds with isometric group actions. It is followed by Section 2.2 where we make some comments on Sobolev spaces of $G$-invariant functions or sections of some fiber bundles. Next, in Section 2.3 we deal with the Kazdan–Warner problem on Riemannian manifolds with isometric actions. Namely, we start with the proof of Theorem C and then approach the Yamabe problem in the $G$-invariant setting (Section 2.4), proving Theorem [A]. Such a combination leads to the proof of Theorem [A].

Section 3 is devoted to a deeper study of the relation between scalar curvature and the group of isometries. We combine classical methods such as Cheeger deformations and Canonical Variations to prove Theorems [E] [F] and Corollaries [D] [C].

In Section 4 we apply the variational method again to prove Theorem [E]. Section 5 is devoted to the construction of several examples to which the developed theory is applicable. In Section 5.1 we prove Corollary [I] and its applications are proved in Section 5.2.

2. Prescribing $G$-invariant scalar curvature

This section encompasses not only the proofs of our results from Theorem [A] to Corollary [D] but the needed elements in order to a better understanding of the full paper. We shall begin with two preliminary sections containing, respectively, the basic aspects of manifolds with isometric actions, followed by the corresponding Sobolev space of $G$-invariant sections, also referred to as basic sections.

2.1. A self-containing minimum on isometric actions. The main reference for this section is [AB15]. We stress out that our main goal in including the content written here is the sake of self-containment plus clarifying some needed aspects for the to-come proofs of several results, with particular interest being for the proof of Lemma 2.3.

We recall that for a Riemannian manifold $(M, g)$ to admit an isometric action by a Lie group $G$ it means there exists a representation of $G$ in the Lie group $\text{Iso}(g)$, of isometries of $g$. It is sometimes useful to say a little bit further, meaning that a left $G$-action can be translated into a smooth map $\mu : G \times M \to M$ defined as $\mu(g, x) := gx$, in a way that the representation of $G$ in $\text{Iso}(g)$ preserves the Lie group homomorphism composition rules, see Definition 3.1 in [AB15] if needed.

It is worth recalling that the Lie group action of $G$ on $(M, g)$ is said to be effective if no element other than the unit of $G$, which we shall very often denote by $e$, fixes every element in $M$, that is, $gx = x$ for every $x \in M$ implies that $g = e$. Every result in this paper, when dealing with isometric actions, also makes the assumption of effectiveness. In addition to that, we
shall assume $G$ to be compact, which ensures that the action representation of $G$ in $\text{Iso}(g)$ is proper, meaning that it defines a smooth embedding as well. Under the former hypotheses, there exists a dimensional constraint relating to the dimensions $n$ of $M$ and $\dim G$. More precisely, in order for $(M, g)$ to admit an effective isometric action is needed that

$$\dim G \leq \frac{1}{2}n(n + 1).$$

The embedding condition guaranteed by the isometric effective action by a compact Lie group guarantees that for each $x \in M$ the orbit $Gx = \{gx : g \in G\}$ is an embedded submanifold of $M$ which happens to be diffeomorphic to $G/G_x$, where $G_x := \{g \in G : gx = x\}$ is the closed Lie subgroup of $G$ named as the isotropy subgroup at $x$. Also, for our considered actions, there exists an open dense and geodesically convex subset $M^{\text{reg}} \subset M$ which enjoys the property that any two points $x, y \in M^{\text{reg}}$ have conjugate isotropies, and so, each two orbits are diffeomorphic two eachother (see Theorem 3.82 in [AB15]). One is naturally led to ask about $M \setminus M^{\text{reg}}$.

In $M \setminus M^{\text{reg}}$ it is very useful to discuss the concept of stratification (see Definition 3.100 in [AB15]). One first observes that if the $G$-action is free, meaning that: for any $x \in M$, the condition $gx = x$ implies that $g = e$; then $M^{\text{reg}} = M$ since each orbit is diffeomorphic to $G$ it self. This way, it also becomes well defined a submersion $\pi : M \to M/G$ via the quotient map. Moreover, $\dim M/G = \dim M - \dim G$ and so $\dim M = \dim G$ if, and only if, $M/G$ is zero-dimensional and so the $G$-action has to be transitive, or in other words, there is only one $G$-orbit on $M$ and it coincides with $G = M$. In this case, we say that $M$ is a homogenous space.

Though in the last paragraph we have considered free actions to talk about homogeneous space, weakening the assumptions on the $G$-action to not require that to be free, whenever we have a transitive isometric action by a closed subgroup of $\text{Iso}(g)$ we say that $M$ is a homogeneous space, but in this case, $M \cong G/G_x$ for any $x \in M$. It is necessary to talk about homogeneous spaces at this point, since in the next section, when dealing with invariant functions, we shall have to observe that any smooth $G$-invariant function on a homogeneous space is necessarily a constant, so this constraint shall be present when discussing some of the results.

We thus finish the discussion here, coming back to the stratification of $\Sigma := M \setminus M^{\text{reg}}$. Thereon, connectdeness is often assumed for each orbit. For any two points $x, y \in \Sigma$, we say that $x, y$ have the same orbit type if there is a $G$-equivariant diffeomorphism between the orbits $Gx, Gy$, say $\varphi : Gx \to Gy$. Or in other words, the isotropy subgroups at $x$ and $y$ are conjugate to each other.

It also can be proved that, see Theorem 3.57 in [AB15], there exists well defined $G$-invariant tubular neighborhoods centered on $Gx$ and $Gy$, so a refinement to orbit type is the equivalence relation defined by local orbit type. Denoting by $M_x^\sim$ the connected components of the partition on $\Sigma$ by orbits with the same type as $Gx$, we say that any $y, z \in M_x^\sim$ have the same
local orbit type if there is a $G$-equivariant map between some $G$-invariant tubular neighborhood $\text{Tub}(Gx)$ and $\text{Tub}(Gy)$.

Orbit type promotes an equivalence relation in $\Sigma$ which makes $\Sigma$ be decomposed as a disjoint union of submanifolds with the same dimension, that is, 

$$\Sigma = \Sigma_0 \cup \ldots \cup \Sigma_d,$$

where each $i \in \{0, \ldots, d\}$ corresponds to the dimension of an orbit type, see Theorem 3.102 in [AB15]. As a last remark, we observe that the restriction $\pi|_{M^{\text{reg}}} : M^{\text{reg}} \to M^{\text{reg}}/G$ defines a Riemannian submersion, the same holding as well to $\pi|_{\Sigma_i} : \Sigma_i \to \Sigma_i/G$, for $i > 0$. For our purposes, considering invariant functions allows us to choose appropriate coordinate charts in a way that any $G$-invariant smooth function $f : M \to \mathbb{R}$ can be locally seen as a smooth function on the quotient, either in $M^{\text{reg}}/G$ or $\Sigma_i/G$. Such a dimension reduction in the domain generally allows better compact embedding results, such as some improvement of the Rellich–Kondrachov Embedding Theorem: see Theorem 2.5.

2.2. General facts about Sobolev Spaces of $G$-invariant functions. Here we proceed by discussing the ring of $G$-invariant (or basic) functions. Some of the already mentioned concepts in the former section shall be expanded.

Let $(M, g)$ be a closed Riemannian manifold with an isometric effective action by a compact Lie group $G$. Let $Q$ denote a biinvariant Riemannian metric on $G$. Recalling that the $G$-action defines a smooth map 

$$\mu : G \times M \to M$$

$$\mu(g, x) := gx,$$

for each point $x \in M$, the map $\mu_x(g) = gx$ induces a diffeomorphism between $G/G_x$ and the orbit of $G$ through $x$, denoted by $Gx$, where $G_x$ is the isotropy subgroup at $x$. Moreover, the biinvariant metric $Q$ induces an orthogonal decomposition $g = g_x \oplus m_x$, where $g_x$ is the Lie algebra of $G_x$. In particular, $m_x$ is isomorphic to $T_xGx$, the isomorphism being induced by computing action fields: given an element $U \in m_x$, if $\exp$ denotes the exponential map between $g$ and $G$ and $e$ denotes the unit element in $G$, then the vectors

$$U_x^* := d(\mu_x)_e(U) = \frac{d}{dt} \bigg|_{t=0} \exp(tU)x.$$

generate $T_xGx$.

Let $W^{k,p}(M), k \geq 0, p \in [1, \infty)$ denote the Sobolev space of $M$ with respect to any invariant metric $g$. We reinforce that since $M$ is assumed to be closed, hence compact, we do not need to fix some specific metric since any metric induces equivalent spaces: we know that $W^{k,p}(M)$ can be recovered by smooth functions on $M$ via taking the closure of $C^\infty(M)$ by
an an appropriate norm:
\[ \|u\|_{k,p}^p := \sum_{|\alpha|=0}^k \int_M |D^{[\alpha]}u|^p, \]
where \( \alpha = (\alpha_1, \ldots, \alpha_{\dim M}) \in \mathbb{N}^{\dim M} \) with \( |\alpha| = \alpha_1 + \ldots + \alpha_{\dim M} \) and we can locally write \( D^{[\alpha]} := \frac{\partial^{[\alpha]}}{\partial x_1^{\alpha_1} \ldots \partial x_{\dim M}^{\alpha_{\dim M}}} \).

On the other hand, once fixed a Haar measure in \( G \), any element \( u \in W^{k,p}(M) \) can be averaged along \( G \) to produce a basic function \( \bar{u} \), i.e., \( \bar{u}(gx) = \bar{u}(x) \) for almost every \( x \in M \) and every \( g \in G \). We denote the set of such elements by \( W^{k,p}_G(M) \). Therefore, by denoting by \( C^\infty_G(M;\mathbb{R}) \) the set of smooth basic functions in \( (M,G) \), it can be shown ([Heb96, Lemma 5.4, p. 91]) that \( C^\infty_G(M;\mathbb{R}) \|_{W^{k,p}(M)} = W^{k,p}_G(M) \). We also adopt the convention \( W^{0,p}_G(M) = L^p_G(M) \).

It is important to remark once more that if the \( G \)-action on \( M \) is transitive, then each \( G \)-invariant function is necessarily a constant. Moreover, throughout this paper, unless explicitly contrarily said, we shall assume that \( G \) is a compact Lie group with positive dimension, i.e., it is not discrete.

Let \( \pi_E : E \to M \) be any smooth Euclidean vector bundle over \( M \). It is possible to consider the Sobolev Space of \( G \)-invariant local sections of \( \pi_E \). More precisely, first recall that due to the local triviality of fiber bundles one has has the identification \( \pi^{-1}_E(U) \cong U \times V \), where \( V \) is the typical fiber of \( E \) and \( U \subset M \). Moreover, let \( \rho \) be a representation of \( G \) in \( \text{GL}(V) \). Then, \( E \big|_U \) is obtained as the quotient of the following induced \( G \)-action on \( U \times G \times V \), locally described as
\[ g \cdot (x, g', v) = (x, gg', \rho(g)v). \]
Hence, if \( s : U \subset M \to E \) is a smooth local section, we say that \( s \) is \( G \)-invariant if
\[ s(gx) = (x, g, \rho(g)1) = g \cdot (x, e, 1) \sim (x, e, 1) = s(x), \quad x \in U, \ v \in V, \ g \in G, \]
where 1 denotes the identity in \( V \) and \( e \) the unit element in \( g \). That is, \( s \) can be uniquely characterized by \( x \), being fiberwise constant: a \( G \)-invariant local section is decomposed by \( G \)-invariant functions on any local frame. Needless to say that \( G \) also defines an action on the frame bundle related to \( E \to M \), via changing basis. We denote the set of these elements by \( C^\infty_G(M;E) \) and define
\[ W^{k,p}_G(M;E) := C^\infty_G(M;E) \|_{k,p}, \]
where
\[ \|s\|_{k,p} := \left( \sum_{j=0}^k |\nabla^j_E s|_{p,\alpha}^p \right)^{1/p}, \]
\( \langle, \rangle \) is a family of \( G \)-invariant Euclidean inner products on the fibers and \( \nabla_E \) a compatible connection on \( E \).
In Section 2.3 we will consider $E = S^2TM \rightarrow M$, i.e., the vector bundle over $M$ whose fiber at $x$ consists of the vector space of symmetric bilinear forms defined in $T_xM$, and the respective Sobolev space $W^{2,p}_G(M; S^2TM)$.

### 2.3. A direct approach to prescribing $G$-invariant scalar curvature.

One of the main results of the problem of prescribing curvature is

**Theorem 2.1** ([KW75a], Theorem A). Let $(M, g)$ be a closed Riemannian manifold. Then any smooth function $f : M \rightarrow \mathbb{R}$ satisfying

$$\min_M cf < \text{scal}_g < \max_M cf,$$

for some $c > 0$, is the scalar curvature of a Riemannian metric on $M$.

In this section we prove Theorem C which is a natural generalization for the Kazdan–Warner’s result in the $G$-invariant setting. For convenience, we restate it below.

**Theorem C.** Let $(M, g)$ be a closed Riemannian manifold with an isometric effective action by a compact Lie group $G$. Then any smooth basic function $f : M \rightarrow \mathbb{R}$ satisfying

$$\min_M cf < \text{scal}_g < \max_M cf,$$

for some $c > 0$, is the scalar curvature of a $G$-invariant Riemannian metric on $M$.

Let $(M^n, g)$ be a closed Riemannian manifold with an isometric effective action by a compact Lie group $G$. Denote by $W^{2,p}_G(M; S^2TM)$ the Sobolev space of symmetric bilinear forms in $M$ that are invariant by the $G$-action (if needed, see the discussion presented on Section 2.2).

Let $K : M \rightarrow \mathbb{R}$ be a smooth $G$-invariant function. Following Kazdan and Warner, finding a $G$-invariant metric $g$ whose scalar curvature is $K$ consists of solving a quasilinear PDE that we simply write as

$$F(g) = K.$$

Here we start with $G$-invariant both metric $g$ and operator $F$. Moreover, the linearization $A$ of $F$ at $g$,

$$Ah := \left. \frac{d}{dt} \right|_{t=0} F(g + th) =: F'(g)(h),$$

is given in coordinates by

$$Ah = -\Delta_g u + h^{ij} - h_{ij}R_{ij},$$

where $R_{ij}$ are the coordinates of the Ricci tensor of $g$. Its formal $L^2_G$-adjoint is given by

$$A^*u = -(\Delta_g u) g + \text{Hess } u - u\text{Ric}(g)$$

We remark that, as in [KW75a], a straightforward computation shows that the symbol of $A^*$ is injective.
The proof of Theorem C follows easily from the following Lemmas (compare with Lemmas 1, 2 and the Approximation Lemma in [KW75a]):

**Lemma 2.2.** Let \((M^n, g)\) be a closed manifold with an isometric effective action by a compact Lie group \(G\) and \(A = F'(g)\). Then \(\ker A^* = \{0\}\) except maybe if \(F(g)\) is either a positive constant or \(F(g) = 0\) and \(\text{Ric}(g) = 0\). Moreover, if \(\ker A^* = \{0\}\) then the map \(A : W^{2,p}_G(M; S^2 TM) \to L^p_G(M; \mathbb{R})\), \(p > 1\), is a surjection.

**Proof.** Our proof follows very closely the proof of [FM74, Theorem 3, p. 480].

It is easy to see that the image of \(A \big|_{W^{2,p}_G(M; S^2 TM)} \subset L^p_G(M; \mathbb{R})\). Assume first that \(F(g) = 0\) but \(\text{Ric}(g) \neq 0\) and let \(u\) be such that \(A^* u = 0\). Taking the trace on equation (4) leads to

\[
(n-1)(-\Delta u) = F(g)u = 0.
\]

Hence, by the Hopf Lemma it follows that \(u = \text{constant}\). However, the equation (4) hence implies that \(u \text{Ric}(g) = 0\), which is a contradiction except if \(u \equiv 0\).

Assume now that \(F(g)\) is not a positive constant and take \(u\) such that \(A^* u = 0\). Taking the divergence in equation (4) gives:

\[
\frac{1}{2} u dF(g) = 0
\]

where we have used that

\[
\delta \text{Ric}(g) = -\frac{1}{2} dF(g),
\]

\[
\delta \text{Hess } u + d\Delta u + g(du, \text{Ric}(g)) = 0,
\]

being \(\delta\) the divergence operator.

Assume by contradiction that \(u \not\equiv 0\). If \(u\) never vanishes, then \(dF(g) = 0\) and hence \(F(g) = \text{constant}\). According to the first equality of (5), it follows that \(u\) is an eigenvector of \(-\Delta\) so \(F(g) \geq 0\). Now our hypotheses guarantee that \(F(g) = 0\), although the equation (4) implies that \(\text{Ric}(g) = 0\), which is again a contradiction. Therefore, \(u\) vanishes at a point \(x\).

If \(du(x) = 0\), by considering a geodesic \(\gamma\) starting at \(x\), the real valued function \(h(s) = u \circ \gamma(s)\) satisfies a linear second order ODE with initial conditions \(u(0) = 0, u'(0) = g(\nabla u, \gamma'(0)) = du(x)(\gamma'(0)) = 0\). Hence, \(h \equiv 0\) and, since \(M\) is assumed to be complete, \(u \equiv 0\). Therefore, \(du(x) \neq 0\) and 0 is a regular value of \(u\). In this case, (6) implies that \(dF(g) = 0\) on an open dense set, what is again a contradiction with the first equality in (5).

For the second claim, note that \(\ker AA^* = \ker A^* = \{0\}\). Therefore, according to [BE69, Lemma 4.4, p. 383], the composition operator \(AA^* : W^{4,p}_G(M; \mathbb{R}) \to L^p_G(M; \mathbb{R})\) is elliptic. Moreover, according to [BE69, Theorem 4.1, p. 383], it is surjective and consequently invertible, so the surjectivity of \(A\) follows. \(\Box\)
Thereon assume that \( p > n = \dim M \). The Sobolev Embedding Theorem implies that any \( g \in W^{2,p}_G(M; S^2 TM) \) is continuously differentiable. Denoting by \( S^2 TM \) the fiber bundle over \( M \) whose fiber at \( x \in M \) consists of the cone of symmetric bilinear positive definite forms in \( T_x M \), since \( F \) is quasi-linear, it follows that \( F : W^{2,p}_G(M; S^2 TM) \to L^p_G(M; \mathbb{R}) \) is continuous and that \( F' \) is continuous in the sense that: given any \( \epsilon > 0 \) there exists \( \delta > 0 \) such that if \( \| g - \tilde{g} \|_{2,p} < \delta \) then
\[
\| F'(g)h - F'(\tilde{g})h \|_p < \epsilon \| h \|_{2,p}.
\]
That is, \( F \) is Fréchet differentiable.

**Lemma 2.3.** Let \( A = F'(g) \) and \( f \in L^p_G(M; \mathbb{R}) \). Assume that \( A^* \) is injective. Then \( F : W^{2,p}_G(M; S^2 TM) \to L^p_G(M; \mathbb{R}) \) is locally surjective.

**Proof.** This is done using the classical Inverse Function Theorem on Banach spaces. Let \( U \subset W^{2,p}_G(M) \) be a small \( G \)-invariant neighborhood of \( 0 \) such that \( V := A^*(U) \) satisfies \( g + V \subset W^{1,2}_G(M; S^2 TM) \). That is, \( g + V \) consists of positive definite symmetric tensors. Define the quasilinear elliptic operator
\[
Q(u) := F(g + A^* u).
\]
Note that \( Q \) satisfies \( Q(0) = AA^* \), which is an elliptic operator with a trivial kernel, being therefore invertible (see [BE69, Theorem 4.1, p. 383]). Moreover, since \( \ker A^* = \{ 0 \} \) then \( A \) is surjective (Lemma 2.2). Therefore, the linear operator \( AA^* \) is bounded with a bounded inverse. The Inverse Function Theorem then implies that \( Q(U) \) contains a \( G \)-invariant neighborhood \( \tilde{U} \) of \( Q(0) = F(g) \) where \( Q|_{\tilde{U}} \) is invertible, concluding the proof. \( \square \)

The next result is crucial to the proof of Theorem C. It consists of an adaptation of the Approximation Lemma by Kazdan and Warner, presented in [KW75b]. We stress that there are some gaps in the original proof of theirs, that we only circumvent by adding the missing ingredients. Also, it is worth remarking that for this result, \( G \) is allowed to be a discrete group.

**Lemma 2.4** (Approximation Lemma). Let \( (M, g) \) be a closed Riemannian manifold with an isometric action by a compact Lie group \( G \). Let \( f, g : M \to \mathbb{R} \) be continuous and \( G \)-invariant functions belonging to \( L^p_G(M) \). Assume that \( \min_M f \leq g \leq \max_M f \). Then for any \( \epsilon > 0 \) and \( 1 \leq p < \infty \), there exists a \( G \)-equivariant diffeomorphism \( \tilde{\Phi} : M \to M \) such that
\[
\| f \circ \tilde{\Phi} - h \|_p < \epsilon.
\]

**Sketch of the proof.** We follow closely Kazdan and Warner’s original proof ([KW75b, Theorem 2.1]) by observing that the original construction can be done in the manifold part of \( M/G \) through flows of vector fields, so they can be lifted to \( M \) via basic horizontal fields: we shall see that every required diffeomorphism can be realized as a flow. We only sketch the proof by adding the missing elements to it.
Observe that we can, without loss of generality, assume that the $G$-action is not transitive, that is, that $M$ itself is not a homogeneous space, otherwise both $f$ and $g$ are constant and the result should hold trivially since then, the condition $\min_M f \leq g \leq \max_M f$ implies that $f = g$.

Fix $\delta > 0$ throughout. Recall that the regular stratum of the $G$-action on $M$ is an open and dense geodesically convex subset of $M$ such that every two orbits are diffeomorphic, being usually denoted by $M^{\text{reg}}$. Since the singular strata, $\Sigma := M \setminus M^{\text{reg}}$ can be described as the union of less-dimensional submanifolds, it has null-measure. Gathering the former in formation, it is straightforward to check that there is an open invariant subset $\tilde{E} \subset M$ such that

$$\int_{M \setminus \tilde{E}} |f(x) - g(x)|^p \, dx \leq \delta.$$ 

Moreover, $\tilde{E}$ can be chosen to contain a tubular neighborhood of the regular stratum at $M$. Therefore, $\tilde{E}$ is contained in the regular stratum and $\tilde{E}, M \setminus \tilde{E}$ are $G$-invariant subsets. This observation allows us to work in the manifold part of the quotient $M/G$ and use the same steps as Kazdan and Warner. We also remark that if $M/G$ is a manifold itself, then on it the argumentation holds similarly with $\tilde{E} = M$. We focus on the explanation, however, in the case that the singular strata is not empty.

Denote the manifold part $M^{\text{reg}}/G$ of $M/G$ by $M^*$. Recall that $M^*$ is a convex open dense subset of $M/G$ and write $E = \tilde{E}/G$, $\bar{f}, \bar{g}$, for the respective functions defined on $M/G$, that is, $\bar{f} \circ \pi = f$, $\bar{g} \circ \pi = g$, where $\pi : M^{\text{reg}} \to M^*$ is a submersion. Observe that the dimension of $M^*$ is the codimension of any principal orbit on $M$. Also, that the good definition of $\bar{f}$ and $\bar{g}$ is guaranteed since $f, g$ are $G$-invariant.

Choose a triangulation $\{\Delta_i\}$ of some subset $M^* \supset M_1 \supset E$ such that $\bar{g}$ is arbitrarily close to a constant function in every triangle $\Delta_i$. That is, we can find a triangulation $\{\Delta_i\}$ of $M_1$ such that

$$\sup_{x,y \in \Delta_i} |\bar{g}(x) - \bar{g}(y)| < \delta.$$ 

Now, for each $i$, fix a point $b_i$ in the interior of $\Delta_i$ and take a collection of different pair wise points $\{x_i\} \subset M^*$ such that $|f(x_i) - \bar{g}(b_i)| < \delta$. Such a collection exists since the required condition on $f$ and $g$ is open and the image of $g$ is contained in the image of $f$. We then decompose $\Phi$ into two diffeomorphisms: $\Phi = \Phi_2 \circ \Phi_1$. We require that $\Phi_2$ fix $M^* \setminus E$ and that $\Phi_1(b_i) = x_i$. It is well-known that such a diffeomorphism can be realized as the time-$1$ flow of a vector field, but we include here a construction for completeness’ sake.

First assume that $\dim E \geq 2$. Given a pair $x_i, b_i$, choose an embedding $\phi : S^1 \to E$ of the unit circle $S^1 \subset \mathbb{C}$ such that $\phi(1) = b_i$ and $\phi(-1) = x_i$. The map $z \mapsto \text{d}\phi_z(iz)$ defines a vector field of $\phi(S^1)$ which can be extended as a vector field that vanishes outside an arbitrarily thin tubular neighborhood of $\phi(S^1)$. Note that the time-$\pi$ flow of this vector field
field interchanges the position of $b_i$ and $x_i$, as wanted. Moreover, we can assume that no other $b_j, x_j$ is contained in the support of the field, allowing us to construct $\Phi_1$ as the composition of a finite number of time-$\pi$ flows. It would only be left to consider the case where $E$ is an open interval, but a *mutatis mutandis* argument would provide the result: let $X_i$ to be the only tangent vector to $E$ emerging from $x_i$ and which its time-1 flow is $b_i$. We can consider appropriate bump functions in a way that such a flow vanishes close to the connected components of the boundary of $E$, so the claim follows similarly also under the assumption that the support of the flow of $X_i$ does not contain $b_i, x_i$. The precompactness of $E$ allows us to consider $\Phi_1$ to be a finite composition of flows, as wanted.

Now, choose an open neighborhood $\Omega$ of the $(n-1)$-skeleton $\cup_i \partial \Delta_i$ such that

$$\int_{\Omega} |f \circ \Phi_1(x) - \bar{g}(x)|^p \, dx < \delta.$$  

Next, for every $b_i$, choose an open neighborhood $V_i$ such that $V_i \subset M_1 \setminus \Omega$ and $|f \circ \Phi_1(b_i) - f \circ \Phi_1(y)| < \delta$. Construct $\Phi_2$ so that it is the identity in some neighborhood $\Omega_1$ satisfying

$$\cup_i \partial \Delta_i \subset \Omega_1 \subset \overline{\Omega}_1 \subset \Omega,$$

and $\Phi_2(\Delta_i \setminus \Omega_1) \subset V_i$. Such a diffeomorphism can be easily realized as a flow by identifying the interior of $\Delta_i$ with an open ball $B$. For instance, one can consider some sort of radial contraction which is the identity near the boundary and choose $\Omega_1$ accordingly.

The arguments above, together with the last computation in the proof of [KW75b, Theorem 2.1] show that, on each $\Delta_i$, we can choose $\Phi$ such that $f \circ \Phi$ is $L^p$-close to the constant function $\bar{g}(b_i)$. The proof is then finished by recalling that one can always lift vector fields from $M^s$ to $M^{reg}$ in a special way: for every point $p \in M^{reg}$ and vector $\bar{v} \in TM^*$, there is a unique $v \in (T_p G_p)^{\perp}$ that projects to $\bar{v}$. A vector field in $M^{reg}$ which is the result of such a lift is called a *basic horizontal field*. Its flow is known to be $G$-equivariant. \hfill $\square$

We now complete the proof of Theorem C and Corollary D. We shall employ the same notations in both proofs.

**Proof of Theorem C.** Assume that $\text{scal}_g$ satisfies inequality (3). If needed to fulfill the hypotheses of Lemma 2.2 make a $G$-invariant perturbation of $g$ in a way that the inequality (3) still holds. According to Lemma 2.2, there is a $G$-equivariant diffeomorphism $\tilde{\Phi}$ such $cf \circ \tilde{\Phi}$ is arbitrarily close to $F(g)$ in the $L^p$-norm. Lemma 2.3 implies that there is a $G$-invariant Riemannian metric $\tilde{g}$ such that $\text{scal}_{\tilde{g}} = cf \circ \tilde{\Phi}$. A direct computation shows that the $G$-invariant metric $(\tilde{\Phi})^{-1}(c\tilde{g})$ has scalar curvature equals to $f$. Regularity theory for elliptic PDE’s implies that since $p > n$ then $(\tilde{\Phi})(c\tilde{g})$ is smooth since $\tilde{g} = g + A^* u$ for some smooth $G$-invariant function $u : M \to \mathbb{R}$. \hfill $\square$
Just for convenience, we restate Corollary D to next prove it.

**Corollary D.** Let \((M, G)\) be a closed smooth manifold equipped with an effective isometric action of a compact Lie group \(G\). If \(M \in \mathcal{A} \cap \mathcal{A}^G\), then any smooth function which is the scalar curvature of a metric on \(M\) is realized by a metric arbitrarily \(L_p\)-close to a \(G\)-invariant metric, for \(1 \leq p < \infty\).

**Proof.** If \((M, G)\) belongs to \(\mathcal{P} \cap \mathcal{P}^G\), then Theorems A and B guarantee that \(M\) admits a \(G\)-invariant Riemannian metric \(g\) with positive constant scalar curvature. Moreover, it is a consequence of Theorem 2.1 that any smooth function \(f : M \rightarrow \mathbb{R}\) can be realized as the scalar curvature of a metric \(\tilde{\phi}^{-1}(c\tilde{g})\) where \(\tilde{g} = g + A^* u\) for some \(c > 0\).

Indeed, if \(g\) has positive constant scalar curvature, for any smooth function \(f\) we can choose \(c\) in a way that the pinching hypothesis in Theorem 2.1 holds. Note, however, that even though \(g\) is \(G\)-invariant, it doesn’t need to be true that \(u\) is as well: it shall be if the data \(f\) is \(G\)-invariant. However, since \(A^* u\) belongs to an arbitrarily small neighborhood of \(g\) (see the proof of Lemma 2.3), we have the claim. Finally, if \(M \in \mathcal{Z} \cap \mathcal{Z}^G\) then we can pick a \(g\)-invariant Riemannian metric with zero scalar curvature, and the former argumentation adapts. \(\Box\)

2.4. A variational approach to the \(G\)-invariant Kazdan-Warner problem: the case of conformally equivalent metrics. Throughout this section, let \((M^n, g)\), \(n \geq 3\), be a closed Riemannian manifold with an isometric effective action by a compact non-discrete Lie group \(G\). We will also always assume that \(M\) is not a homogenous space, since we are interested in invariant functions that are not constant.

2.4.1. The strategy for the to-come proofs and its self-motivation. We begin by describing the ideas behind the to-come proofs.

Pursuing the analogous Yamabe problem, given any \(c \in \mathbb{R}\), to obtain a Riemannian metric \(\tilde{g}\) for which \(\text{scal}_{\tilde{g}} = c\), we could search for smooth positive \(G\)-invariant solutions to the following PDE, obtained by a suitable conformal change of \(g\):

\[
4 b_n \Delta_g u - \text{scal}_g u + cu^{\gamma_n} = 0,
\]

where \(\gamma_n := \frac{n + 2}{n - 2}\) and \(b_n := \frac{n - 1}{n - 2}\).

In a more general manner, we could, for instance, ask whether if we change \(c\) to any \(G\)-invariant smooth function \(f : M \rightarrow \mathbb{R}\) it could exist a solution to the PDE (7). Such a problem was already treated by Kazdan and Warner without symmetric assumptions.

Moreover, in contrast to Kazdan–Warner’s approach ([KW75a]), we shall proceed by variational methods to obtain analogous results to theirs but in
the invariant setting. Namely, we shall consider the following functional

$$J(u) = 2b_n \int_M |\nabla u|^2 + \frac{1}{2} \int_M \text{scal}_g u^2 - \frac{c}{2^*} \int_M u^{2^*}$$

defined in $W^{1,2}(M)$, where $2^* := 2n/(n - 2)$. Note that $J$ is well defined and of class $C^1$ according to the Sobolev Embedding Theorem. Moreover, since $J$ is $G$-invariant, it admits a restriction to $W^{1,2}_G(M)$.

On the one hand, to find a critical point for $J$ restricted to $W^{1,2}_G(M)$ means to exist $u \in W^{1,2}_G(M)$ such that $dJ(u)(v) = 0 \forall v \in W^{1,2}_G(M)$. It is in this very point that a result such as the principle of symmetric criticality is commonly needed. Observe that if one has the former, we say to have a symmetric critical point. To obtain a critical point indeed, one is usually required to verify that for every $v \in W^{1,2}(M)$ it holds that $dJ(u)(v) = 0$. Fortunately, the Principle of symmetric criticality due to Palais [Pal79] is in hand: any symmetric critical point is a critical point when the group action is by isometries.

Moreover, we justify our interest in finding a $G$-invariant solution to ensure that the conformal changed metric is compatible with the Singular Riemannian Foliation geometric structure induced by the partition of $M$ into the orbits of $G$. It turns out, however, that such a restriction plays a huge role in the argumentation of finding a minimum for $J$ via variational methods due to the existence of better compactness results such as Theorem 2.5 below. Note for instance that $\gamma_n$ is a critical exponent for the classical Rellich–Kondrachov Theorem, i.e, it is such that it is not necessarily true that there is a compact embedding of $W^{1,2}(M)$ in $L^{\gamma_n + 1}(M)$. Symmetries, however, improve compact embeddings, so we naturally proceed to find a basic critical point.

To do so, we shall look for a local minimum for $J$ with specific constraints in a way that routine arguments in variational methods, trivial in this scenario given the Rellich–Kondrachov type result, then ensure the existence of such a minimum point $u$. Since the constraint will consist of a submanifold, we observe that the Lagrange Multiplier equation needs to play some role. However, as we shall see, in this problem, we can reduce the solution to the one of $dJ(u)(v) = 0 \forall v \in W^{1,2}_G(M)$, after scaling the original metric, concluding that $u$ is then the desired critical point.

The effective combination of such Yamabe problem solution with Theorem 2.5 will lead to the Kazdan–Warner type results we have obtained.

2.4.2. The proofs. The presence of an isometric action makes, in some cases, it possible to deal with the equation (7) by variational methods: denote by $d := \min_{x \in M} \dim Gx$. According to Theorem [Heb96, Theorem 5.6, p. 92], it follows that

(i) if $n - d \leq p$, then for any real number $q \geq 1$ one has that $W^{1,p}_G(M)$ can be compactly embedded in $L^q(M)$;
(ii) if \( n - d > p \), then for any real number \( 1 \leq q \leq \frac{(n-d)p}{(n-d-p)} \) one has that \( W^{1,q}_G(M) \) embeds in \( L^q_G(M) \) and this embedding is compact if \( q < \frac{(n-d)p}{(n-d-p)} \).

Observe that it is then required that every orbit has positive dimension if there is at least one singular orbit, that is, if \( \Sigma := M \setminus M^{reg} \) is not-empty. This holds since the results formerly stated are based on considering regular sub-foliations on both \( M^{reg} \) and on the induced stratification on \( \Sigma \), recall the introductory aspects presented in the end of Section 2.1.

One then concludes that (i) and (ii) imply that (compare with [Heb96, Corollary 5.7, p. 93], [HebVa93, HebVa97]) if \( d \geq 1 \), then there are better compact embedding results:

**Theorem 2.5** (Larger embeddings). Let \((M, g)\) be a closed Riemannian manifold with isometric effective action by a compact non-discrete Lie group \( G \). If \( d = \min_{x \in M} \dim Gx \geq 1 \), then for any \( p \in [1, n) \) there is \( p_0 > \frac{np}{(n-p)} =: p^* \) such that \( W^{1,p}_G(M) \) embeds compactly in \( L^q_G(M) \) for every \( 1 \leq q \leq p_0 \).

In what follows, we shall make extensive use of Theorem 2.5. We proceed with furnishing the proof of Theorem B after proving several minor results.

We first observe that:

(a) if \( \text{scal}_g \equiv 0 \) then the PDE (7) is reduced to \( 4b_n \Delta_g u = -cu^n \). Therefore, if one takes \( c \neq 0 \), the only possible solution to it is \( u \equiv 0 \). Similarly, if \( c = 0 \) the only solution is \( u \equiv \text{constant} \);
(b) on the other hand, by assuming \( c > 0 \) one can multiply (7) by \( u^{-1} \) and integrate it by parts to get
\[
4b_n \int_M \frac{1}{u^2} |\nabla u|^2_g - \int_M \text{scal}_g + \int_M cu^2 = 0.
\]
Thus concluding that if a positive solution \( u \) exists, then \( \int_M \text{scal}_g \geq 0 \);
(c) finally, if \( c = 0 \) then (7) is reduced to
\[
4b_n \Delta_g u = \text{scal}_g u.
\]
Therefore, also assuming the existence of a positive solution \( u \), by multiplying the former equation by \( u^{-1} \) and integrating it, one concludes that \( \int_M \text{scal}_g \geq 0 \).

Once we have established the previous analyses and considering the obtained possible obstructions, we prove Lemma 2.6 next. We remark that a completely analogous proof holds for every similar statement to be stated, for instance, compare with the to-come proof of Lemma 4.3.
Lemma 2.6. Assume that $\text{scal}_g$ is a continuous nonzero function such that $\min \text{scal}_g \geq 0$ and let $c > 0$. Given $\epsilon > 0$, consider the set

$$M_G := \left\{ u \in W^{1,2}_G(M) : u \geq 0, \frac{c}{2^*} \int_M u^{2^*} = \epsilon \right\}.$$ 

Then for

$$J(u) = 2b_n \int_M |\nabla u|^2_g + \frac{1}{2} \int_M \text{scal}_g u^2 - \frac{c}{2^*} \int_M u^{2^*},$$

it holds that

1. $M_G$ is not empty and weakly closed and $J|_{M_G}$ is weakly lower semicontinuous;
2. $J|_{M_G}$ is coercive.

Proof. (1) It is very straightforward to check, by picking $u$ some appropriate constant, that $M_G$ is not empty, so take a sequence $\{u_m\} \subset M_G$ weakly converging to some $u \in W^{1,2}_G(M)$: shortly, $u_m \rightharpoonup u$. Once $\{u_m\} \subset W^{1,2}_G(M)$ and this is a Banach space (so it is convex and closed, then weakly closed) one has that $u \in W^{1,2}_G(M)$. According to Theorem 2.5 one has the compact embedding of $W^{1,2}_G(M)$ in $L^{2^*}(M)$, from where it follows that $c \int_M u^{2^*} = \lim_{m \to \infty} c \int_M u^{2^*}_m = c2^*$ due to the strong convergence in $L^{2^*}(M)$. Moreover, since we also have $L^1(M)$ convergence, up to passing to a subsequence, we can assume that the sequence $\{u_m\}$ has a pointwise convergent subsequence, so that $u \geq 0$ almost everywhere. Therefore, $u \in M_G$ and hence $M_G$ is weakly closed.

On the other hand, the weak convergence implies that

$$\liminf_{m \to \infty} \|u_m\|_{1,2} \geq \|u\|_{1,2}.$$ 

Moreover, since $\int_M u^2_m \to \int_M u^2$ is a convergent sequence, one has that

$$\liminf_{m \to \infty} \|u_m\|^2_{1,2} = \int_M u^2 + \liminf_{m \to \infty} \int_M |\nabla u_m|^2.$$ 

Hence

$$\liminf_{m \to \infty} \int_M |\nabla u_m|^2 \geq \int_M |\nabla u|^2.$$ 

Finally, due to the continuity of $\text{scal}_g$ one concludes that

$$\liminf_{m \to \infty} J(u_m) \geq 2b_n \int_M |\nabla u|^2 + \frac{1}{2} \int_M \text{scal}_g u^2 - \frac{c}{2^*} \int_M u^{2^*}. \quad \Box$$

(2) Note that

$$J(u) \geq 2b_n \int_M |\nabla u|^2_g + \frac{\min M \text{scal}_g}{2^*} \int_M u^2 - \frac{c}{2^*} \int_M u^{2^*}$$

$$= 2b_n \int_M |\nabla u|^2_g + \frac{\min M \text{scal}_g}{2} \int_M u^2 - \epsilon.$$
Therefore, according to the Poincaré’s Inequality, since $\min_M \text{scal}_g \geq 0$, $J(u) \rightarrow \infty$ if $\|u\|_{W^{1,2}_G(M)} \rightarrow \infty$.

Observe that Lemma 2.6 ensures that $J|_{M_G}$ has a point of minimum $u \in M_G$, which we will show that is corresponds to a symmetric minimum point of $J$, previous to the restriction to $M_G$. Indeed, a routine argument allows us to check that $M_G$ is a codimension-1 submanifold of $W^{1,2}_G(M)$.

So observe that for every $v \in W^{1,2}_G(M)$, the Lagrange Multiplier Theorem states that there is $\lambda \in \mathbb{R}$ such that

$$J'(u)(v) = 4b_n \int_M \langle \nabla u, \nabla v \rangle + \int_M \text{scal}_g uv = (1 + \lambda)c \int_M u^{\gamma_n}v.$$ 

Therefore, $u$ is a weak solution of (7) with $c$ replaced by $c' = (1 + \lambda)c$. On the other hand, by computing $J'(u)(u)$, we conclude that $1 + \lambda > 0$ and so that $c' > 0$.

It is left to check whether the weak solution can be represented as a classical (smooth) solution. To do so, observe that, due to the regularity theory of elliptic PDE’s, one concludes that the solution $u$ is smooth as long as $\text{scal}_g$ is smooth. In fact, note that the non-linearity on the PDE (7) corresponds to the term $u^{\gamma_n}$. Since the function $F : x \rightarrow x^{\gamma_n}$ is of class $C^1$ and the solution $u$ has a finite essential supremum, then $F(u) \in W^{1,2}(M)$. An iterative application of Theorem 3.58 in [Aub98, p. 87] implies the result. In addition to it, the maximum principle [Aub98, Proposition 3.75, p.98] implies that the obtained solution is positive. A simple rescaling of the resulting metric (corresponding to $u \mapsto u + a$, for some $a$) provides the right constant scalar curvature metric.

We thus have proved that an analogous to the Yamabe problem is true:

**Proposition 2.7** (Prescribing positive constant scalar curvature). Let $(M^n, g), n \geq 3$ be a closed Riemannian manifold with an isometric effective action by a compact non-discrete Lie group $G$. Also assume that every orbit has positive dimension.

If $\text{scal}_g \geq 0$ and $\text{scal}_g \not\equiv 0$ then any $c > 0$ is the scalar curvature of a $G$-invariant metric.

As an immediate consequence, one gets:

**Corollary 2.8.** Let $(M^n, g), n \geq 3$ be a closed Riemannian manifold with effective isometric action by a compact non-discrete Lie group $G$. Assume that no orbit has zero dimension and that $M$ does not carry a $G$-invariant metric of positive scalar curvature. If $\text{scal}_g \geq 0$ then $\text{scal}_g \equiv 0$.

We proceed to the proof of item (1) in the Theorem B:

**Proposition 2.9** (Every invariant metric has a metric with negative constant scalar curvature). Let $(M^n, g), n \geq 3$ be a closed Riemannian manifold with an isometric effective action by a compact non-discrete Lie group $G$. Also
assume that no orbit has zero dimension. Then there is \( c \geq 0 \) such that for any \( c' \geq c \) there exists a \( G \)-invariant Riemannian metric \( g \) such that \( \text{scal}_g = -c' \).

**Proof.** Take \( c \geq 0 \) such that

\[
\left( \frac{2^*}{2} \right) \min_M \text{scal}_g \text{vol}(M)^{1-2^*/2} + c \geq 0,
\]

and consider the functional

\[
J(u) = 2b_n \int_M |\nabla u|_g^2 + \frac{1}{2} \int_M \text{scal}_g u^2 + \frac{c}{2^*} \int_M u^{2^*}.
\]

Hölder's Inequality implies that

\[
\left( \int_M u^2 \right) \leq \text{vol}(M)^{1-2^*/2} \left( \int_M u^{2^*} \right)^{2/2^*}.
\]

(10)

We now consider two separate cases. Depending if \( \min \text{scal}_g \leq 0 \) or \( \min \text{scal}_g > 0 \), respectively, we have

\[
J(u) \geq 2b_n \int_M |\nabla u|_g^2 + \frac{1}{2} \min_M \text{scal}_g \text{vol}(M)^{1-2^*/2} \left( \int_M u^{2^*} \right)^{2/2^*} + \frac{c}{2^*} \int_M u^{2^*};
\]

\[
J(u) \geq 2b_n \int_M |\nabla u|_g^2 + \frac{c}{2^*} \int_M u^{2^*}.
\]

To show that \( J \) is coercive, observe that if \( \int_M u^{2^*} \to \infty \) then \( \left( \int_M u^{2^*} \right)^{2/2^*} < \int_M u^{2^*} \). Hence,

\[
J(u) \geq 2b_n \int_M |\nabla u|_g^2 + \left( \frac{1}{2} \min_M \text{scal}_g \text{vol}(M)^{1-2/2^*} + \frac{c}{2^*} \right) \int_M u^{2^*};
\]

\[
J(u) \geq 2b_n \int_M |\nabla u|_g^2 + \frac{c}{2^*} \int_M u^{2^*}.
\]

from where it follows that \( J \) is coercive if \( \frac{2^*}{2} \min_M \text{scal}_g \text{vol}(M)^{1-2/2^*} + c > 0 \) or \( c > 0 \), respectively. Whenever one of these terms vanishes, we can proceed in a similar fashion as in item 1. in Lemma 2.6.

Following further the proof of Lemma 2.6 we can show that \( J \) has a critical point in \( W_{G}^{1,2} \). The Principle of Symmetric Criticality of Palais [Pal79, p. 23] implies that this is a weak solution for the PDE (7). Combining Theorem 3.58 on [Aub98, p. 87] and the maximum principle [Aub98, Proposition 3.75, p.98] we conclude that such a weak solution is smooth and positive. \( \square \)

We thus conclude the proof of Theorem B: items (1), (2) and (3) follows from Proposition 2.9, Corollary 2.8 and Proposition 2.7, respectively. We finish this section by proving Theorem A.

**Proof of Theorem A.** We first observe that Theorem C implies that a manifold admitting a \( G \)-invariant metric with constant scalar curvature \( c \) guarantees that every basic function that is negative somewhere (resp.
positive) is a scalar curvature of \((M, G)\), if \(c < 0\) (resp. \(c > 0\)). Therefore, Proposition 2.9 shows that every \((M, G)\) admits any scalar curvature function that is negative somewhere. Moreover, if a non-vanishing non-negative scalar is admissible, Proposition 2.7 shows that \((M, G) \in \mathscr{P}^G\).

The last assertion in the statement follows from Proposition 2.7 and a slightly improved version of Theorem A in \([LY74]\), due to the maintenance of symmetry group, proved in \([CeSS18]\):

**Theorem 2.10** (Cavenaghi–e Silva–Sperança). Let \((M, g)\) be a compact Riemannian manifold with an effective isometric \(G\)-action, where \(G\) is a compact non-Abelian Lie group. Then \(g\) develops positive scalar curvature after a finite Cheeger deformation.

\[
\square
\]

### 3. Improved Curvature Estimates

The proof of Theorem 2.10 together with Theorem 2.7 provide a better view on the variety of functions that are realizable as scalar curvature functions of \((G\)-invariant\) metrics. Specifically, the proof of Theorem 2.10 gives a pinching estimate for a \(G\)-invariant metric after a long Cheeger deformation, which is described only regarding orbit types and isotropy representations. On the other hand, a similar behavior is found after a canonical deformation. We study both cases in this section.

#### 3.1. Scalar curvature Pinching under Cheeger deformations.

Given a singular point \(x \in M\), its isotropy representation \(\tilde{\rho}_x : G_x \to O(T_xM)\), naturally acts on the space normal to the orbit. Fixed \(X \in \nu_xG_x\), the differential of \(\tilde{\rho}_x\), computed at the identity, defines a map

\[
\rho_X : g_x \to \nu_xG_x
\]

\[
U \mapsto (d\tilde{\rho}_x)_U(U)X.
\]

Following \([CeSS18]\), we call an element in the image of \(\rho_X\) as a *fake horizontal vector with respect to \(X\), and denote \(\rho_X(g_x) = H_X\). We (abuse notation and) denote by \(\rho_X^{-1}\) the inverse of \(\rho_X\) restricted to \(H_X\).

If \(g\) is non-Abelian, we can describe the scalar curvature pinching of a long-time Cheeger deformation by using the following parameters:

\[
\overline{\text{scal}}_{G/G_x} = \frac{1}{4} \sum_{i,j=1}^k \| [v_i, v_j] \|_g^2,
\]

\[
\xi(g_x) = \frac{\sum_{i,j=1}^{n-k} \| e_{i,j}^H \|_g^4}{\sum_{i,j=1}^{n-k} \| \rho_{e_{i,j}}^{-1} e_j^H \|_Q^4}
\]

where \(\{e_i\}\) and \(\{v_i\}\) are \(g\)- and \(Q\)-orthonormal basis for \(\nu_xG_x\) and \(m_x\), respectively.
Proposition 3.1. Suppose $g$ non-Abelian and let $g_t$ be the Cheeger deformation of $g$. Then,
\[
\lim_{t \to \infty} \frac{\max_{x \in M} \text{scal}_{g_t}(x)}{\min_{x \in M} \text{scal}_{g_t}(x)} = \frac{\max_{x \in M} \left\{ \text{scal}_{G/G_x} + 3\xi(\rho_x) \right\}}{\min_{x \in M} \left\{ \text{scal}_{G/G_x} + 3\xi(\rho_x) \right\}}.
\]

Proof. Given a $G$-invariant metric $g$, the scalar curvature of its Cheeger deformation was computed in [CeSS18]:
\[
\text{scal}_{g_t}(p) = \sum_{i,j=1}^{n} K_g(C_t^{1/2} e_i, C_t^{1/2} e_j) + \sum_{i,j=1}^{n} z_t(C_t^{1/2} e_i, C_t^{1/2} e_j) + \sum_{i,j=1}^{k} \lambda_i \lambda_j t^3 \frac{1}{(1 + t\lambda_i)(1 + t\lambda_j)} \frac{1}{Q} \| [v_i, v_j] \|_Q^2.
\]

Some definitions are required:

1. Given $x \in M$, let $P : m_x \to m_x$ is the unique endomorphism that satisfies $Q(PU, V) = g(U^*, V^*)$ for every $U, V \in m_x$. Consider $\{\lambda_i\}$ as the set of eigenvalues of $P$;
2. $C_t : T_x M \to T_x M$ is defined by $C_t|_{\nu_x G_x} = 1$ and $C_t U^* = (1 + tP)^{-1}(U^*)$.

$z_t$, on its turn, is quadratic in each entry and is described in [CeSS18, Lemma 3.1.1]. We gather the relevant information below:

Lemma 3.2. For every $\overline{X} = X + U^*$, $\overline{Y} = Y + V^*$, $X, Y \in \nu_x G_x$, $z_t$ satisfies
\[
z_t(\overline{X}, \overline{Y}) = 3t \max_{Z \in \mathcal{E}} \frac{\{dw_Z(\overline{X}, \overline{Y}) + \frac{1}{2}Q([PU, PV], Z)\}^2}{tg(Z^*, Z^*) + 1},
\]
where $w_Z(\overline{X}) = \frac{1}{2}g(\overline{X}, Z^*)$ is a one-form in $TM$ satisfying
\[
dw_Z(V^*, X) = \frac{1}{2} X g(V^*, Z^*),
\]
\[
dw_Z(Y, X) = g(\nabla_X Y, Z^*),
\]
\[
dw_Z(U^*, V^*) = Q([PU, V] + [U, PV] - P[U, V], Z).
\]

Moreover, if $U \in \mathcal{H}_X$,
\[
dw_Z(U, X) = g(U, \nabla_X Z^*).
\]

One concludes that the second and third summands in (13) are either zero or grow like $t$. It is sufficient to study these terms to prove the theorem.
since every other term is bounded. Moreover,

\[ \lim_{t \to \infty} \frac{1}{t} K_g(C_t^{1/2} e_i, C_t^{1/2} e_j) = 0 \]  

(19)

\[ \lim_{t \to \infty} \frac{1}{t} \sum_{i,j=1}^k \frac{\lambda_i \lambda_j t^3}{(1 + t \lambda_i)(1 + t \lambda_j)} \frac{1}{4} \|[v_i, v_j]\|_Q^2 = \text{scal}_{G/G_x}; \]

(20)

\[ \lim_{t \to \infty} \frac{1}{t} \sum_{i,j=1}^n z_t(C_t^{1/2} e_i, C_t^{1/2} e_j) = \lim_{t \to \infty} \frac{1}{t} \sum_{i,j=1}^{n-k} z_t(e_i, e_j). \]

(21)

It is just left to study the last term. Using Lemma 3.2, one concludes that every term in (21) with either \( i > n - k \) or \( j > n - k \) is bounded, since \( C_1 \frac{1}{2t} |T_{Gx}| = (1 + tP)^{-1/2} \) goes to zero as \( t^{-1/2} \). To analyze the case \( i, j \leq n - k \), we use the compactness of the unit ball \( \|Z\|_Q = 1 \), and take a fixed \( Z \) such that

\[ \lim_{t \to \infty} \frac{1}{t} z_t(e_i, e_j) = \lim_{t \to \infty} \frac{1}{t} \frac{d w_Z(e_i, e_j)^2}{g(Z^*, Z^*) + 1}. \]

One readily sees that this limit is zero unless \( g(Z^*, Z^*) = 0 \). On the other hand, if \( Z \in g_x \), by observing that

\[ \nabla e_j Z^*(x) = (d \rho_x)e(Z)(e_j) = \rho_{e_j}(Z), \]

Lemma 3.2 guarantees that \( d w_Z(e_i, e_j) = d w_Z(e_i^{H_{e_j}}, e_j) \). In this case, \( Z \) can be taken as \( \rho_{e_j}^{-1}(e_i^{H_{e_j}}) / \|\rho_{e_j}^{-1}(e_i^{H_{e_j}})\| \), which necessarily realizes the maximum and shows that

\[ \sum_{i,j=1}^{n-k} \lim_{t \to \infty} \frac{1}{t} z_t(e_i, e_j) = 3 \xi(\rho_x). \]

\[ \square \]

Theorem [E] now follows directly from Proposition 3.1.

Proof of Theorem [E] Since a semi-free action has no singular orbits, the \( \xi \)-term is not present. Moreover, since the regular part of \( M \) is convex (see [AB15, Theorem 3.49, p. 65]), every pair of isotropy Lie subalgebra are conjugate to each other. Thus, we conclude that \( \text{scal}_{G/G_x} \) is constant with respect to \( x \), making the limit in Proposition 3.1 equal to 1. In addition to that, due to the non-Abelian assumption, we have that \( \text{scal}_{g_t} \) is positive as \( t \) grows large, which is crucial to what follows. Indeed, Theorem [E] holds since, given a smooth and basic function \( f \) and \( \epsilon > 0 \) arbitrarily small, there is \( t^* > 0 \) such that for any \( t > t^* \) we have \( \frac{\min_{x \in M} \text{scal}_{g_t}}{\max_{x \in M} \text{scal}_{g_t}} - 1 < \epsilon \). Moreover, it is easy to check that if \( \min_{x \in M} f \leq 0 \) but \( \max_{x \in M} f > 0 \) we can find \( c > 0 \) such that

\[ c \min_{x \in M} f < \text{scal}_{g_t} < c \max_{x \in M} f. \]

so Theorem [E] with its proof conclude the result. On the other hand, if \( f \equiv 0 \) then via a conformal change, we can assume that \( u^{4/n-2} g_t \) has the constant
0 as its scalar curvature for some $t$ large enough, ensuring the result. This is Proposition 2.7.

We finish the argument assuming that $\min_{x \in M} f > 0$. Since $\frac{\min_{x \in M} f}{\max_{x \in M} f} \leq 1 < \min_{x \in M} \text{scal}_{g_t}$ we have

$$\epsilon' + \left( \frac{\min_{x \in M} \text{scal}_{g_t}}{\min_{x \in M} f} \right) > \left( \frac{\max_{x \in M} \text{scal}_{g_t}}{\max_{x \in M} f} \right)$$

for $\epsilon' = \left( \frac{\max_{x \in M} \text{scal}_{g_t}}{\min_{x \in M} f} \right) \epsilon$. It is then straightforward to check that

$$c = \left( \frac{\min_{x \in M} \text{scal}_{g_t}}{\min_{x \in M} f} \right)$$

recovers equation (22) and so Theorem C applies once more. □

We now complete the proof of Theorem E by considering $\mathfrak{g}$ Abelian. Before doing so, we remark that throughout the proof we only focus on the regular stratum and, using the fact that the scalar curvature function is $G$-invariant, it naturally descends to the manifold part of the quotient $M/G$. This is the same kind of argumentation we did in the proof of Lemma 2.4.

**Proof of Theorem [E]**. As in the proof of Lemma 2.4 we work in the regular part of the $M/G$. Denote by $\pi : M \to M/G$ the quotient map and note that the restriction of $\pi$ to the regular stratum defines a smooth principal bundle with image a smooth manifold. We proceed by a proof by contrapositive observing that if the Lie Algebra $\mathfrak{g}$ of $G$ is non-Abelian, then $(M, G) \in \mathcal{P}_G$ due to Theorem [B] so we denote by $\mathring{g}$ the Riemannian metric induced by this restriction of $\pi$ to its image and assume that $\mathring{g}$ has positive scalar curvature. We shall show that $\text{scal}_{\mathring{g}}$ has positive scalar curvature for $t$ arbitrarily large, which another application of Theorem [E] will ensure the result.

Due to continuity, we can assume that $\text{scal}_{\mathring{g}}$ has a positive lower bound in any compact set inside the regular stratum. On the other hand, Lemma 3.2 guarantees that $\lim_{t \to \infty} \text{scal}_{\mathring{g}}(p) = +\infty$ at non-regular points, it is then sufficient, once positivity is then guaranteed, to prove that $\text{scal}_{\mathring{g}}(p)$ converges to $\text{scal}_{\mathring{g}}(\pi(p))$ in the regular stratum.

To this aim, assume that $\mathfrak{g}$ is Abelian and recall that the respective equations for the Ricci curvature of $g_t$ are, as in [CeSS18, Lemma 2.6],

$$\text{Ric}_{g_t}(C_t^{1/2} V^*) = \lim_{t \to \infty} \sum_{i=1}^{n} K_{g}(C_t^{1/2} e_i, C_t^{1/2} V^*) + \sum_{i,j=1}^{n} z_t(C_t^{1/2} e_i, C_t^{1/2} V^*),$$

$$\text{Ric}_{g_t}(X) = \sum_{i=1}^{n} K_{g}(C_t^{1/2} e_i, C_t^{1/2} X) + \sum_{i,j=1}^{n} z_t(C_t^{1/2} e_i, C_t^{1/2} X),$$

where $V \in \mathfrak{g}$ and $X \in (T_pGp)\perp$. 


On the one hand, observe that \( C_t^{1/2} V^* \to 0 \) for every \( V \in \mathfrak{g} \) and \( z_t \) is bounded uniformly on \( t \) at points in the regular stratum (Lemma 3.2). Thus, \( \text{Ric}_{g_t}(V^*) \to 0 \) whenever \( \mathfrak{g} \) is Abelian.

On the other hand, for \( p \) in the regular stratum, [CeSS18, Lemma 4.2] or [SW15, Proposition 4.3] implies that

\[
\lim_{t \to \infty} \text{Ric}_{g_t}(X) = \text{Ric}_{g}(\pi X)
\]

for every \( X \in (T_pG_p)^\perp \). Since \( C_t^{1/2} X = X \), we conclude that \( \text{scal}_{g_t}(p) \) converges to \( \text{scal}_{g}(\pi(p)) \), as wanted. \( \square \)

### 3.2. Canonical variations on Fiber Bundles.

Let \( F \hookrightarrow \overline{M}, g' \xrightarrow{\pi} (M, h) \) be a fiber bundle with compact structure group \( G \). From Theorem B, there is a \( G \)-invariant metric \( g_F \) on \( F \) with constant scalar curvature \( c \).

Therefore, there is a metric \( g \) on \( 
\overline{M} \), where \( \pi \) is a Riemannian submersion over \( (M, h) \) and every fiber is isometric to \( (F, g_F) \) (this is a standard construction using associated bundles. See, for example, Proposition 2.7.1 in [GW09], for details). Moreover, the fibers of \( \pi : (\overline{M}, g) \to (M, h) \) can be taken as totally geodesic.

Corollary \( \mathcal{G} \) is proved by using a widely-known deformation called canonical variation. To simplify the text, we denote

\[
\mathcal{V} = \ker d\pi, \quad \mathcal{H} = (\ker d\pi)^\perp.
\]

Given \( s > 0 \), the canonical variation \( \tilde{g} \) of \( g \) is defined as:

\[
\tilde{g} = g \big|_{\mathcal{H} \times \mathcal{H}} + s \, g \big|_{\mathcal{V} \times \mathcal{V}}.
\]

Since the fibers are contracted, it is expected that the positive curvature of the fibers increases the overall curvature, somehow. It is easy to see that this expected behavior is the right one and this is the first step towards Theorem \( \mathcal{G} \). The remaining is a direct application of a result in [Ber83] and Theorem \( \mathcal{E} \).

**Lemma 3.3.** Suppose \( F \hookrightarrow \overline{M} \to M \) is a fiber bundle with a compact structure group \( G \). If \( (F,G) \in \mathcal{P}^G \), then \( \overline{M} \in \mathcal{P} \).

**Proof.** We first recall the curvature formulae in [Besse87].

**Proposition 3.4.** Let \( \pi : (F,g_F) \hookrightarrow (\overline{M}, g) \to (M, h) \) be a Riemannian submersion with totally geodesic fibers. Denote by \( K_{\tilde{g}}, K_g, K_h, K_{g_F} \) the non-reduced sectional curvatures of \( \tilde{g}, g, h \) and \( g_F \), respectively. If \( X,Y,Z \in \mathcal{H} \) and \( V,W \in \mathcal{V} \), then

\[
\begin{align*}
K_{\tilde{g}}(X,Y) &= K_g(d\pi X, d\pi Y)(1 - s) + s K_g(X,Y), \\
K_{\tilde{g}}(X,V) &= s^2 K_g(X,V), \\
K_{\tilde{g}}(V,W) &= s K_{g_F}(V,W).
\end{align*}
\]
Fix $x \in M$ and let $\{e_i\}_{i=1}^n$ be an orthonormal basis for $\mathcal{H}_x$ and $\{e_j\}_{j=n+1}^{n+k}$ be an orthonormal basis for $\mathcal{V}_x$. Note that $\{e_i\}_{i=1}^n \cup \{s^{-\frac{1}{2}}e_j\}_{j=n+1}^{n+k}$ is a $\tilde{g}$-orthonormal basis for $T_xM$. Using Proposition 3.3, we obtain:

$$\operatorname{scal}_{\tilde{g}}(x) = \operatorname{scal}_{\tilde{g}}(x) + 2s \sum_{j=n+1}^{n+k} \sum_{i=1}^n K_{\tilde{g}}(e_i, e_j) + s^{-1} \operatorname{scal}_{g_F}(x),$$

where

$$\operatorname{scal}_{g_F}(x) = \operatorname{scal}_{h}(x)(1-s) + s \sum_{i,j=1}^n K_{\tilde{g}}(e_i, e_j).$$

Therefore, if $c = \operatorname{scal}_{g_F} > 0$, $\operatorname{scal}_{\tilde{g}} > 0$ for $s$ sufficiently small. □

**Proof of Corollary 3.** Consider a fiber bundle $\pi : \overline{M} \to M$ with compact structure group $G$ and suppose that $\overline{M} \notin \mathcal{P}$. From Lemma 3.3 we conclude that $(F,G) \notin \mathcal{P}^G$, therefore, $G$ is a compact group whose identity component is a torus $T$.

Recall that $\pi$ can be seen as a bundle associated with a principal $G$-bundle $P \to M$. More specifically, $\overline{M}$ is diffeomorphic to the twisted product $P \times_G F$. On the other hand, a result of Bergery [Ber83] shows that a manifold $N$ with a free $T$-action admits an invariant metric with positive scalar curvature if and only if the quotient $N/T$ does. We conclude that $\overline{M} = P \times T F$ admits a metric with positive scalar curvature if and only if $P \times F$ admits a $T$-invariant metric with positive scalar curvature. On the other hand, if $M = P/T$ would have a metric with positive scalar curvature, $P$ would have a $T$-invariant metric with positive scalar curvature. Therefore, $P \times F$ would have a $T \times T$-invariant metric with positive scalar curvature. Since $\overline{M} \notin \mathcal{P}$, we conclude that no metric on $M$ or $T$-invariant metric on $(F,T)$ has positive scalar curvature. □

**4. Prescribing scalar curvature via general vertical warpings**

Our goal in this section is to study how to prescribe scalar curvature in fiber bundles which are, essentially, locally isometric to products. To this aim, we resort to a natural generalization of the canonical variation, called general vertical warping. Given a Riemannain submersion $\pi : (\overline{M}, g) \to (M, h)$ and a smooth function $\phi : M \to \mathbb{R}$, we define the metric $g_\phi$ along the same lines as the canonical variation:

$$g_\phi = g\big|_{\mathcal{H} \times \mathcal{H}} + e^{2\phi} \big|_{\mathcal{V} \times \mathcal{V}}.$$

The submersion $\pi : (\overline{M}, g_\phi) \to (M, h)$ is Riemannian and its fibers are $e^{2\phi}$-homothetic to the original fibers in $(\overline{M}, g)$. The main result in this section is Proposition 4.1 from which Theorem 4 follows.
In order to well introduce its statement, define
\[
b_k := \frac{k+1}{8k}, \quad c_k := \frac{(k+1)^2}{8(k-1)k}, \quad \theta_k := \frac{2k-1}{k+1}.
\]

In what follows, the superscripts always denote the dimensions of the underlined manifolds.

**Proposition 4.1.** Let \( \pi : F^k \rightarrow (\overline{M}^{n+k}, g) \rightarrow (M^n, h) \) be a Riemannian submersion from a closed oriented manifold \( \overline{M} \). Assume that \( (M, g) \) is locally isometric to a product whose fibers have constant scalar curvature \( c \in \mathbb{R} \).

(a) If \( c > 0 \) and \( \min_{\overline{M}} \text{scal}_h > 0 \) then any basic smooth function is the scalar curvature of some general vertical warping metric;

(b) Assume \( c \leq 0 \) and suppose that a basic function \( f : \overline{M} \rightarrow \mathbb{R} \) satisfies the condition:
\[
-b_k (f - \text{scal}_h) + c_k c (\text{vol}(M))^{2/\theta_k - 1} \geq 0.
\]

If \( |c| > 0 \) then \( f \) is the scalar curvature of some general vertical warping metric. In contrast to it, if \( c = 0 \), then there exists some \( A \geq 0 \) such that \( f + A \) is the scalar curvature of a general vertical warping metric.

Before proceeding to the proof, we observe that the scope of application of Proposition 4.1 is broader than local products of the form \( M \times F \). Indeed, every warped product \( (\overline{M}, g) = (M \times F, h \times \phi g_F) \) admits a metric where it is locally a product. Lemma 4.1 also holds for submersions which are locally warped products (see Section 5 for further details).

### 4.1. The case \( c \leq 0 \)

To begin with, assume that \( (M^n, h) \) is a closed Riemannian manifold and \( (F, g_F) \) is a Riemannian manifold with constant scalar curvature \( c \leq 0 \). Let \( f : M \times F \rightarrow \mathbb{R} \) be a smooth basic function. We start by discussing the proof of Proposition 4.1 to the case where \( \overline{M} \) is a Riemannian product. To do so, we apply standard variational methods to guarantee the existence of positive solutions for the following PDE:

\[
\Delta_h u + \frac{k+1}{4k} (f - \text{scal}_h) u - \frac{k+1}{4k} u^{k+3} c = 0.
\]

Equation (25) is obtained from a general vertical warping through Lemma 4.2, a proof of that is obtained as a consequence of the proof of Lemma 4.4 to be stated and proved later in this paper.

**Lemma 4.2.** Let \( (M, h) \) and \( (F, g_F) \) be Riemannian manifolds and \( \phi : M \rightarrow \mathbb{R} \) be a smooth function. Denote by \( g \) the warped metric of \( M \times \phi g_F \). Then the scalar curvature of \( g \) is given by
\[
\text{scal}_g = \text{scal}_h + e^{-2\phi} \text{scal}_{g_F} - k(k-1)|\nabla \phi|_h^2 - 2k|\nabla \phi|_h^2 - 2k \Delta_h \phi.
\]
Indeed, given a basic function \( f \in C^\infty(M \times F; \mathbb{R}) \), consider the PDE

\[
\text{scal}_h + e^{-2\phi}\text{scal}_{g_F} - f = k(k-1)|\nabla \phi|^2_h + 2k \left\{ |\nabla \phi|^2_h + \Delta_h \phi \right\}.
\]

The solution \( \phi \) is such that \( f \) is the scalar curvature of the warped metric \( \tilde{g} = h + e^{2\phi}g_F \). By setting \( \phi = \log u^{2/k+1} \) a direct computation shows that equation (26) is reduced to equation (25). To show that the PDE (25) has a positive solution, assume that \( f, \text{scal}_h \) are continuous functions. Define the following functional

\[
J(u) = \frac{1}{2} \int_M |\nabla u|^2_h - b_k \int_M (f - \text{scal}_h) u^2 + c_k \int_M cu^{\theta_k}.
\]

Note that \( J \) is well defined and, according to the classical Sobolev Embedding Theorem, it is of class \( C^1 \).

4.1.1. The case \( c < 0 \). Assume that \( c < 0 \). We will obtain the desired solution \( u \) to equation (25) as a minimum for \( J \) restricted to the set \( M := \{ u \in W^{1,2}(M) : u \geq 0, \int_M u^{\theta_k} = 1 \} \). This is the content of Lemma 4.3.

**Lemma 4.3.** Define \( M := \{ u \in W^{1,2}(M) : u \geq 0, \int_M u^{\theta_k} = 1 \} \). Assume that \( f, \text{scal}_h \) are continuous functions, that \( c < 0 \) and that the following inequality holds

\[
-b_k (f - \text{scal}_h) + c_k c (\text{vol}(M))^{2/\theta_k - 1} \geq 0.
\]

Then,

1. there is a constant \( c_0 > 0 \) such that \( J|_M > c_0 \),
2. \( J|_M \) is coercive,
3. \( M \) is weakly closed and \( J|_M \) is weakly lower semi-continuous.

Before proving it, let us show how Proposition 4.1 follows from it. Lemma 4.3 provides that \( J|_M \) has a minimum point in \( M \). Therefore, we have a Lagrange Multiplier problem: there exists \( \Lambda \in \mathbb{R} \) satisfying

\[
\int_M (\nabla u, \nabla v) - 2b_k \int_M (f - \text{scal}_h) uv + c_k \theta_k \int_M u^{\theta_k-1} v = \theta_k \Lambda \left( \int_M u^{\theta_k-1} v \right),
\]

\( \forall v \in W^{1,2}(M) \). We claim that \( \Lambda \geq 0 \): by taking \( v = u \), we have

\[
\int_M |\nabla u|^2 - 2b_k \int_M (f - \text{scal}_h) u^2 + c_k \theta_k \int_M u^{\theta_k} = \theta_k \Lambda \int_M u^{\theta_k}
\]

However, since \( \theta_k \leq 2 \) the Hölder’s inequality implies

\[
\int_M u^{\theta_k} = \left( \int_M u^{\theta_k} \right)^{2/\theta_k} \leq (\text{vol}(M))^{2/\theta_k - 1} \int_M u^2.
\]
Therefore,
\[ \theta_k \Lambda \int_M u^{\theta_k} \geq \int_M |\nabla u|^2 + 2 \int_M \left( -b_k (f - \text{scal}_h) + \frac{\theta_k}{2} c c_k (\text{vol}(M))^{2/\theta_k - 1} \right) u^2 \]
\[ \geq 2 \int_M \left( -b_k (f - \text{scal}_h) + c c_k (\text{vol}(M))^{2/\theta_k - 1} \right) u^2 \geq 0, \]
so \( \Lambda \geq 0 \).

We now multiply both sides of (29) by \( \zeta > 0 \) to get
\[ \int_M \langle \nabla(\zeta u), \nabla v \rangle - 2b_k \int_M (f - \text{scal}_h)(\zeta u) v + \zeta^{-\theta_k + 2}(c c_k - \Lambda) \theta_k \int_M (\zeta u)^{\theta_k - 1} v = 0, \]
\[ \forall v \in W^{1,2}(M). \]
But since \( \Lambda \geq 0 \) we can assume that \( \zeta \) is such that
\[ \zeta^{-\theta_k + 2}(c c_k - \Lambda) = c c_k, \]
concluding that \( \tilde{u} = \zeta u \) is a weak solution for equation (25).

To finish the argument, note that if \( f \) and \( \text{scal}_h \) are smooth functions, the classical regularity theory for elliptic PDE’s (see [Aub98, Theorem 3.58, p. 87]) guarantees that the solution \( u \) is smooth. Finally, we observe that the solution \( u \) is positive. Otherwise, a Maximum Principle argument, as in [Aub98, Proposition 3.75, p.98] guarantees that \( u \equiv 0 \). We thus have proved that any basic smooth function \( f : M \times F \to \mathbb{R} \) satisfying (24) can be realized as the scalar curvature of a warped product metric, thus concluding item (2) of Proposition 4.1 in the case of Riemannian products. It is left to prove Lemma 4.3.

Proof of Lemma 4.3. It is easy to see that \( M \neq \emptyset \). Moreover, the Poincaré’s Inequality implies that
\[ J(u) \geq \frac{1}{2} \int_M |\nabla u|^2 + \int_M \left( -b_k (f - \text{scal}_h) + c c_k (\text{vol}(M))^{2/\theta_k - 1} \right) u^2 \]
\[ \geq \int_M \left( \frac{\lambda_1}{2} - b_k (f - \text{scal}_h) + c c_k (\text{vol}(M))^{2/\theta_k - 1} \right) u^2 - \frac{\lambda_1}{2 \text{vol}(M)} (\int_M u)^2, \]
where \( \lambda_1 \) is the first positive eigenvalue of \( -\Delta_h \). On the other hand, the Hölder’s inequality gives:
\[ \int_M u \leq \text{vol}(M) \left( \int_M u^{\theta_k} \right)^{1/\theta_k} = \text{vol}(M), \]
leading to an explicit bound for the last term above. We thus conclude that \( J(u) \) has a lower bound, since \( -b_k (f - \text{scal}_h) + c c_k (\text{vol}(M))^{2/\theta_k - 1} \geq 0 \) by hypothesis. Likewise, coercivity follows directly from (30): if \( ||u||_{L^2} \to \infty \) then equation (30) implies that \( J(u) \to \infty \). If \( ||u||_{L^2} \) is bounded but \( |||\nabla u|||_{L^2} \to \infty \), the same equation ensures the result.

To see that \( M \) is weakly closed, take \( \{ u_m \} \subset M \) weakly converging to \( u \in W^{1,2}(M) \). According to the Rellich–Kondrachov’s Theorem, the Sobolev space \( W^{1,2}(M) \) compactly embeds in \( L^{\theta_k}(M) \). Hence, since \( 1 = \int_M |u_m|^{\theta_k} \to \int_M |u|^{\theta_k} \),
\[ \int_M |u|^\theta_k, \] we have the result: not only \( \int_M |u|^\theta_k = 1 \) but since we have the strong \( L^{\theta_k}(M) \) convergence, hence strong \( L^1(M) \) convergence, it holds that, up to passing to a subsequence, \( \{u_m\} \) pointwisely converges to \( u \), so \( u \geq 0 \) almost everywhere. These all gather imply that \( M \) is weakly closed, as desired.

Finally, to ensure that \( J \) is weakly lower semi-continuous, note that since \( \operatorname{scal}_h, \operatorname{scal}_{g_F} \) and \( f \) are continuous functions and
\[
\int_M |u|^2 \leq \liminf_{m \to \infty} \int_M |u_m|^2,
\]
we conclude that
\[
\liminf_{m \to \infty} J(u_m) = \liminf_{m \to \infty} \left( \int_M |u_m|^2_h - b_k \int_M (f - \operatorname{scal}_h) u_m^2 + c_k \int_M u_m^{\theta_k} \right)
\geq \int_M |u|^2_h - b_k \int_M (f - \operatorname{scal}_h) u^2 + c_k \int_M u^{\theta_k} = J(u). \quad \Box
\]

4.1.2. The case \( c = 0 \). We finally deal with the case \( c = 0 \). We only sketch it, since it follows closely to the proof of Lemma 4.3 as we shall make clear.

In this case, the functional \( J \) is reduced to
\[
J(u) = \frac{1}{2} \int_M |\nabla u|^2_h - b_k \int_M (f - \operatorname{scal}_h) u^2
\]
For this case, we can simply search for a minimum of \( J \) restricted to the set
\[
M = \left\{ u \in W^{1,2}(M) : \int_M u^2 = 1 \right\}.
\]

Since the Hölder’s Inequality implies that there is \( C > 0 \) such that for any \( u \in M \) we have \( (\int_M u^2) \leq C (\int u^2) = C \), it is immediate to see that the Poincaré’s Inequality implies that \( J \) has a lower-bound restricted to \( M \). Moreover, under the assumption that \( f - \operatorname{scal}_h \leq 0 \) the coercivity also follows. The same routine argument used in Lemma 4.3 implies that \( M \) is weakly-closed. Therefore, \( J \) has a critical point in \( M \).

Once more, given any function \( v \in W^{1,2}(M) \) tangent to \( M \), one has a Lagrange Multiplier problem:
\[
\int_M \langle \nabla u, \nabla v \rangle - 2b_k \int_M (f - \operatorname{scal}_h) uv = 2\Lambda \int_M uv,
\]
for every \( v \in W^{1,2}(M) \). Making \( v = u \) we see that since \( -2b_k \max_M (f - \operatorname{scal}_h) \geq 0 \) and \( \int_M u^2 = 1 \) then \( \Lambda \geq 0 \).

Now let us rewrite equation (32) as
\[
\int_M \langle \nabla u, \nabla v \rangle - 2b_k \int_M (f - \operatorname{scal}_h + \Lambda b_k^{-1}) uv = 0.
\]
Note that equation (33) is the weak formulation of the warped product approach to prescribing the function \( f + \Lambda b_k^{-1} \), which includes the result defining \( A := +\Lambda b_k^{-1} \).
We finish this subsection by sketching the proof of the general case of item (2) of Proposition 4.1—it follows via straightforward modifications of the previous arguments. We start by recalling the following result (compare with the proof of Lemma 3.3):

**Lemma 4.4.** Let $F^k \hookrightarrow (M^{n+k}, \bar{g}) \xrightarrow{\bar{\pi}} (M^n, h)$ be a Riemannian submersion. Let $\tilde{g}$ be a general vertical warping metric on $M$ via the function $u^{4k+1}$, where $u : M \to \mathbb{R}$ is a smooth basic function. Then the scalar curvature of $\tilde{g}$ is given by

\begin{equation}
\text{scal}_{\tilde{g}} = \text{scal}_g - \frac{4k}{k+1}u^{-1}\Delta_h u + \left(\frac{2}{k+1}u^{-1}\text{du}(H)\right) + \left(u^{-\frac{4}{k+1}} - 1\right)\text{scal}_{g_F} - \left(1 - u^{\frac{4}{k+1}}\right)\|A\|^2,
\end{equation}

where $\|A\|^2$ is a term that depends on the non-integrability of the horizontal distribution $\mathcal{H} = (\ker d\pi)^\perp$. In particular, $\|A\|^2 = 0$ if $\pi$ is locally a warped product.

**Proof.** We first have to quickly introduce some notation. We recall that $A_XY := \frac{1}{2}[X,Y]_{\ker d\pi}$ stands for the O’Neill integrability tensor and, fixed any $X \in \mathcal{H}$, we denote by $A_X^*$ its $g$-dual. We shall use the formulae described in Chapter 2 in [GW09]. Next, $K$ always describes the non-reduced sectional curvature with respect to the subscripted metric. Moreover, fixed $X \in \mathcal{H}$ we denote by $S_X$ the shape operator of the submersion $\pi$. Namely, if $\sigma : \ker d\pi \times \ker d\pi \to \mathcal{H}$ denotes the second fundamental form associated with $\pi$, then $S_X$ is the $g$-dual

$g(S_X(\cdot), \cdot) = g(\sigma(\cdot, \cdot), X)$.

In what shall follow, observe that the equations here appearing are natural generalizations to the ones first appearing in the proof of Lemma 3.3.

Let $T_i = e^{-\phi}v_i$, $v_i \in \ker d\pi$ and $X \in \mathcal{H}$. Then,

$\tilde{K}(e^{-\phi}v_1, e^{-\phi}v_2) = (e^{-2\phi} - 1)K_{g_F}(v_1, v_2) + K_g(v_1, v_2) - |\nabla\phi|^2 + d\phi(\sigma(v_1, v_1) + \sigma(v_2, v_2))$,

$\tilde{K}(X, e^{-\phi}v) = K_g(X, v) - \left(1 - e^{2\phi}\right)|A_X^* v|^2 - \text{Hess } \phi(X, X) - d\phi(X)^2 + 2d\phi(X)g(S_X v, v)$.

Take a $g$-orthonormal basis $\{e_i\}$ to $\mathcal{H}$. Then,

$$\sum_{i,j} \tilde{K}(e_i, e_j) = (1 - e^{2\phi})\text{scal}_h + e^{2\phi}\text{scal}_\mathcal{H},$$

where $\text{scal}_\mathcal{H}$ means the corresponding sum of sectional curvatures appearing in a scalar curvature expression, but only restricted to horizontal vectors, that is, vectors belonging to $\mathcal{H}$.
We also have
\[
\sum_{r,s} \tilde{K}(e^{-\phi}v_r, e^{-\phi}v_s) = (e^{-2\phi} - 1) \text{scal}_{g_F} + \text{scal}^V - k(k-1) |\nabla \phi|^2 + 2(k-1) \text{d}\phi(H)
\]
\[
2 \sum_{i,r} \tilde{K}(e^{-\phi}v_r, e_i) = 2 \sum_{i,r} K(e_i, v_r) - 2(1 - e^{2\phi}) \sum_{i,r} |A^*_{e_i} v_r|^2
\]
\[
- 2k (\Delta_h \phi + |\nabla \phi|^2) + 4 \sum_{r,i} \text{d}\phi(e_i) g(S_{e_i} v_r, v_r),
\]
where \text{scal}^V means the corresponding sum of sectional curvatures appearing in a scalar curvature expression, but only restricted to \textit{vertical vectors}, that is, vectors belonging to ker \text{d}\pi.

Without loss of generality we assume that \text{d}\phi(e_1) = |\nabla \phi|, i.e, \text{e}_1 = \frac{\nabla \phi}{|\nabla \phi|}, \text{d}\phi(e_i) = 0, \ i \geq 2, \text{from where we obtain}
\[
4 \sum_{r,i} \text{d}\phi(e_i) g(\nabla_{e_i} v_r, v_r) = 4 \text{tr} \ S_{\nabla \phi} = 4 \text{d}\phi(H).
\]

So we conclude that
\[
\text{scal} = \text{scal}_g - 2(1 - e^{2\phi}) \sum_{i,r} |A^*_{e_i} v_r|^2 + (1 - e^{2\phi})(\text{scal}_B + \text{scal}^H) + (e^{-2\phi} - 1) \text{scal}_{g_F}
\]
\[
- k(k-1) |\nabla \phi|^2 - 2k (\Delta_h \phi + |\nabla \phi|^2) + (4 + 2(k-1)) \text{d}\phi(H).
\]

Once equation (35) only differs from equation (26) by the terms
\[
2(1 - e^{2\phi}) \sum_{i,r} |A^*_{e_i} v_r|^2 + (4 + 2(k-1)) \text{d}\phi(H),
\]
by introducing the change of variables \phi = \log \varphi \text{ and } \varphi = \frac{u^2}{k+1} \text{ we conclude that}
\[
\text{scal} = \text{scal}_g - \frac{4k}{k+1} u^{-1} \Delta_h u + (4 + 2(k-1)) \frac{2}{k+1} u^{-1} \text{d}u(H)
\]
\[
+ \left( u^{-\frac{k+1}{k+1}} - 1 \right) \text{scal}_F + \left( 1 - u^{\frac{k+1}{k+1}} \right) \left( \text{scal}_h - \text{scal}^H - 2 \sum_{i,r} |A^*_{e_i} v_r|^2 \right).
\]

We proceed by proving that:

\textbf{Lemma 4.5.} Let \( F^k \hookrightarrow (M^{n+k}, g) \xrightarrow{\pi} (M^n, h) \) be a Riemannian submersion which is locally isometric to a product. If \( \Delta_g \) denotes the Laplace operator on the metric \( g \), then the restriction of \( \Delta_g \) to basic functions defines a strongly elliptic operator.

\[\square\]
Let $u : M \to \mathbb{R}$ be a basic function. Once $M$ is compact, it is possible to choose a collection of open sets $\{W_l\} \subset M$ trivializing the submersion $\pi$ in the following sense (see [Her60])

$$W_l = U_l \times F, \ U_l \subset M.$$ 

Once $u$ is a basic function,

$$u|_{W_l}(x) = u(x_1, x_2) = u(x_1, x'_2), \ \forall x_2, x'_2 \in F, \ \forall x_1 \in U_l.$$ 

If $\{\psi_l\}$ denotes a partition of unity subordinated to $\{U_l\}$, then

$$u = \sum_l \psi_l u$$ 

and there is a well defined injection

$$\zeta : W^{1,2}(\mathcal{M}) \to W^{1,2}(M)$$ 

$$u \mapsto v,$$ 

where $v = \sum_l v_l$, $v_l(x_1) = \psi_l(x_1)u(x_1, x_2)$, $\forall x_1 \in U_l$.

Since the submersion is locally a product, it follows that its fibers are totally geodesic, so $H = 0$. Hence, by identifying $\zeta u = u$ one has that $\Delta_h u = \Delta_g u$ once $\Delta_h u = \Delta_g u - du(H) = \Delta_g u$ (see [GW09, Section 2.1.4, p.53]).

To prove Proposition 4.1 item (2), we observe that since the Riemannian submersion $\pi$ is locally isometric to a product, its fibers are totally geodesic submanifolds and its horizontal distribution is integrable. In particular, $H \equiv 0$ and $\|A\| = 0$. Hence, according to Lemmas 4.4 and 4.5, given a basic function $f : M \to \mathbb{R}$, we need to study the following elliptic problem

$$\left(k + \frac{1}{4k}\right) u(f - \text{scal}_g) = -\Delta_h u + \left(k + \frac{1}{4k}\right) \left(u^{\frac{k+1}{2k}} - u\right) c,$$ 

where $0 \geq c = \text{scal}_F$.

The proof of item (2) on Proposition 4.1 follows by adapting the proof of Lemma 4.3 to solve the PDE (37). This is done by searching for a minimum point in the set

$$\mathcal{M}_F := \{u \in W^{1,2}(\mathcal{M}) : u \geq 0, \ u \text{ is basic and } \int_M u^{\theta_k} = 1\}.$$ 

We use the subscript $\mathcal{F}$ to emphasize that we are considering classes of elements whose function representatives are constant along the foliation induced by the fibers $F$.

### 4.2. The case $c > 0$.

We consider this case separately since we need a different version of Lemma 4.3. Our result is a natural generalization of [EYTK96, Theorem 4.10, p. 253]. We first remark that this case suffers no obstruction as it is substantiated by the fact that a product of a manifold with positive scalar curvature always admits positive scalar curvature (compare this fact with the mentioned Kazdan–Warner tricotomies).
Following the notation established in Section 4.1 and motivated by Lemma 4.5, we consider the following functional of class $C^1$:

$$J(u) = \frac{1}{2} \int_M |\nabla u|^2 + 2 b_k \int_M \left( (\text{scal}_g - c - f) \frac{u^2}{2} + c \theta_k^{-1} u \theta_k \right)$$

defined in

$$M_F := \{ u \in W^{1,2}(M) : u \geq 0, \text{ u is basic and } 2 b_k c \theta_k^{-1} \int_M u \theta_k = 1 \}.$$

By following arguments analogous to the ones in the proof of Lemma 4.3 it can be shown that if

$$\min_M (\text{scal}_g - c) \geq \max_M f,$$

then $J|_{M_F}$ has a minimum point that is a positive smooth solution to equation (37). The item is proved since we can prescribe any smooth function: it is always possible to divide $f$ by an arbitrary positive constant and then scale the resulting metric, so the above constraint is always overcome. \qed

5. Examples

We finish the paper by presenting examples of manifolds with the desired symmetric properties. The intention is to present to the reader how vast the variety of such examples can be. We begin with examples where Theorem \ref{thm:existence} and Proposition 4.1 can be applied. Then we follow with standard and less-standard constructions of $G$-manifolds.

5.1. Polar foliations, toric symmetry and Calabi–Yau bundles. Observe that a manifold is locally a metric product if it admits some cover which splits as a metric product. Specifically, $(\overline{M}, g)$ is locally isometric to a product if there are $(M, h)$ and $(F, g_F)$ such that $\overline{M}$ is isometric to the quotient of $(M \times F, h \times g_F)$ by a subgroup $\Gamma < \pi(M)$ acting as deck transformations. Although the condition of being locally isometric to a product is quite restrictive, we can slightly weaken it by considering manifolds which are locally warped products. A manifold is locally a warped product if there is a function $\psi$ such that the general vertical warping induced by $\psi, \tilde{g}$, makes $(\overline{M}, \tilde{g})$ locally a metric product. To characterize the latter, we recall a result in Gromoll–Walschap \cite{GW09}.

**Proposition 5.1** (Proposition 2.2.1, \cite{GW09}). Let $\pi : \overline{M}^{n+k} \to M^n$ be a Riemannian submersion. Then $\pi$ is locally a warped product if and only if

1. the distribution $\mathcal{H} = (\ker d\pi)^\perp$ is integrable;
2. the fibers are totally umbilic submanifolds of $\overline{M}$;
3. $\pi$ is isoparametric; i.e., the mean curvature form $\kappa$ (dual to the mean curvature vector field) is basic.
5.1.1. Calabi–Yau bundles. We proceed with a brief discussion about prescribing scalar curvature on some Calabi–Yau bundles. We first observe that realizing basic scalar functions on a trivial bundle \( Y \times F \to Y \) follows from previous works: assume that \( F \) has a metric \( g_F \) with constant scalar curvature \( c \). If \( f : Y \to \mathbb{R} \) is a smooth function, consider a metric \( g \) on \( Y \) that realizes \( \text{scal}_g = f - c \). Then, the scalar curvature of the product \( \tilde{g} = g \times g_F \) satisfy \( \text{scal}_{\tilde{g}}(x, y) = f(x) \). The originality of Theorem H and Proposition 4.1 is the possibility to fix and preserve an initial metric \( g \) on the base \( Y \).

An interesting application revolves around Calabi–Yau manifolds. Specifically, we apply Proposition 4.1 to the following result of Tosatti–Zhang:

**Theorem 5.2** (Theorems 1.2 and 1.3 in [TZ14]). Suppose \( \pi : X \to Y \) is a holomorphic submersion with connected fiber \( F \) satisfying one of the following:

1. \( X, Y \) are projective manifolds with \( X \) Calabi–Yau;
2. \( X, Y \) are compact Kähler manifolds; \( Y, F \) are Calabi–Yau; and either \( b_1(F) = 0 \) or \( b_1(Y) = 0 \) and \( F \) is a torus.

Then \( Y \) is Calabi–Yau and there is a finite unramified cover \( p : \tilde{Y} \to Y \) such that the pullback bundle \( \tilde{\pi} : p^*X \to \tilde{Y} \) is holomorphically trivial.

In the present context, a *finite unramified covering map* means a smooth covering map with a finite number of sheets. Theorem 5.2 is proved in the holomorphic context, without considering any compatibility between the metrics and the submersion. We observe here that the holomorphic submersion in the theorem can be taken as a Riemannian one while choosing a Calabi–Yau metric \( g_Y \) on \( Y \).

To this aim, note that, since \( Y \) is Calabi–Yau, its fundamental group is Abelian, therefore every covering is regular. In particular, \( p : \tilde{Y} \to Y \) is a principal bundle with a finite principal group \( \Gamma = \pi_1(Y)/\pi_1(\tilde{Y}) \). Since \( \Gamma \) is finite, there is a \( \Gamma \)-invariant metric \( g_F \) on \( F \), therefore there is a metric \( g \) on \( X \) such that both \( \tilde{\pi} : (p^*X, p^*g_Y \times \pi_2) \to (X, g) \) and \( \pi : (X, g) \to (Y, g_Y) \) are Riemannian submersions. We thus conclude that \( g \) is locally a metric product and Theorem H applies. The proof of Corollary I is concluded by observing that the construction carries over with any arbitrary metric on \( Y \).

5.1.2. Toric symmetry. Theorem A states that, if \( G \) is non-Abelian, every \( G \)-invariant function can be realized as the scalar curvature of a \( G \)-invariant metric on a compact \( (M, G) \). On the other hand, there is a wealth of manifolds with abelian symmetry as it is illustrated by the following Theorem of Corro–Galaz-García ([CGG20 Theorem A]):

Since the metric on the factor \( (M, h) \) is preserved by this construction, Proposition 4.1 can be applied to manifolds satisfying the conditions in Proposition 5.1.
Theorem 5.3 (Corro–Galaz-García). For each integer \( n \geq 1 \), the following holds:

1. There exist infinitely many diffeomorphism types of closed simply-connected smooth \((n+4)\)-manifolds \( M \) with a \( T^n \)-invariant Riemannian metric with positive Ricci curvature;
2. The manifolds \( M \) realize infinitely many spin and non-spin diffeomorphism types;
3. Each manifold \( M \) supports a smooth, effective action of a torus \( T^{n+2} \) extending the isometric \( T^n \)-action in item (i).

In particular,

Corollary 5.4. For each integer \( n \geq 1 \), \( \mathcal{P}^G \) contains infinitely many diffeomorphism types of both spin and non-spin closed simply-connected smooth \((n+4)\)-manifolds \( M \) with Abelian symmetry.

We observe that the quotients \( M/G \) in Theorem 5.3 are connected sums of finitely many copies of products of 2-spheres and complex projective planes. Therefore, these manifolds admit metrics of positive scalar curvature.

On the other hand, a manifold with toric symmetry which does not lie in \( \mathcal{P}^G \) can be constructed as follows: take an enlargeable manifold \( M_1 \). Recall that a Riemannian manifold \((M, g)\) is said to be enlargeable if, for any given \( \epsilon > 0 \), \( M \) has an oriented Riemannian covering that admits an \( \epsilon \)-contracting map with non-trivial degree to the unit sphere \( S^n(1) \). Tori and every other manifold with a metric of non-positive sectional curvature is enlargeable. Furthermore, enlargeability is a homotopy invariant and it is closed by products, connected sums and even by maps of non-trivial degree (if a manifold admits a map of non-trivial degree to an enlargeable manifold, the source manifold is itself enlargeable).

Now consider the product of \( M_1 \) with a torus, \( M = M_1 \times T \). Due to the closedness property of enlargeability under taking products, we have \((M, T) \notin \mathcal{P}^T\). More generally, Bergery [Ber83] shows that a compact manifold with a free \( T \)-action \((M, T)\) do not belong to \( \mathcal{P}^T \) if and only if it is a \( T \)-bundle over a manifold in the complement of \( \mathcal{P} \). More interestingly, [Ber83] presented examples of manifolds \((M, T) \in (\mathcal{P}^T)^c \cap \mathcal{P} \), where \((\mathcal{P}^T)^c \) stands for the complement of \( \mathcal{P}^T \).

Based on the information above, we explicitly leave open the questions on the structure of manifolds in \((\mathcal{P}^T)^c \cap \mathcal{P} \); and if there is a manifold \((M, T) \notin \mathcal{P}^T \) whose quotient \( M/T \) admits positive scalar curvature (in a generalized sense).

5.2. Equivariant constructions and examples. In [Mil56], John Milnor introduced the first examples of exotic manifolds, which were topologically equivalent to spheres, but not diffeomorphic. Since then, lots of new exotic spaces have been produced. For instance, there are uncountable many pairwise non-diffeomorphic structures on \( \mathbb{R}^4 \) (see [Tan87]) as exotic
manifolds not bounding spin manifolds [Hit74], exotic projective spaces and some of their connected sums [CS18].

On the other hand, Carlos Durán [Dur01] explored symmetries to produce an exotic sphere out of its classical counterpart. It was promptly generalized in [DPR10, Spe16, CS18, CS19] in a procedure that enjoys a rough dictionary between invariant objects on them. More concretely, consider a compact connected principal bundle $G \to P \to M$ with a principal action $\bullet$. Assume that there is another free action on $P$ that commutes with $\bullet$, which we denote by $\star$. These make $P$ a $G \times G$-manifold resulting in a $\star$-diagram:

\[
\begin{array}{c}
G \\
\bullet \\
\downarrow \\
G \star P \xrightarrow{\pi} M' \\
\downarrow \pi \\
M
\end{array}
\]

Here $M, M'$ are the quotients of $P$ by the $\bullet$- and $\star$-actions, respectively. Since the two actions commute, they can be put together to define a $G \times G$-action on $P$. Moreover, $\bullet$ descends to an action on $M'$ and $\star$ to an action in $M$. We simply denote the orbit spaces associated with the descending actions by $M'/G$ and $M/G$, respectively. Both orbit spaces are canonically identified with $P/(G \times G)$. In particular, they are naturally identified with themselves.

This section is dedicated to exploring the relation between the manifolds $M$ and $M'$ and to recall some of the examples in literature. We begin by proving Corollary J, which follows from the lemma below and Corollary G.

**Proof of Corollary J.** Let $M \leftarrow P \to M'$ be a $\star$-diagram and $p \in P$. Denote $\pi(p) = x$ and $\pi'(p) = x'$. First, a straightforward calculation shows that the isotropy groups $G_x$ and $G_{x'}$ are isomorphic. Therefore, if $G$ acts effectively on $M$, so it does on $M'$. Therefore, if $G$ has a non-Abelian Lie algebra, then $(M, G), (M', G') \in \mathcal{P}^G$. If $G$ is Abelian, then it is a torus. Therefore, we can apply [Wie16, Theorem 2.2] to conclude that $P$ admits a $G \times G$-invariant metric with positive scalar curvature if and only if both $M$ and $M'$ admit $G$-invariant metrics with positive scalar curvature, as wanted. □

We finish by sketching concrete examples where Corollary J can be applied.

**Example 1** (The Gromoll–Meyer exotic sphere). This construction first appeared in [GM72] and was first put in a $\star$-diagram in [Dur01] (see also [CS19]). Consider the compact Lie group

\[
Sp(2) = \left\{ \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in S^7 \times S^7 \mid ab + cd = 0 \right\},
\]
where \( a, b, c, d \in \mathbb{H} \) are quaternions, equipped with their usual conjugation and multiplication and norm. The projection \( \pi : Sp(2) \to S^7 \) of an element to its first row defines a principal \( S^3 \)-bundle with principal action:

\[
\begin{pmatrix} a & c \\ b & d \end{pmatrix} \bar{q} = \begin{pmatrix} a & \bar{c} q \\ b & \bar{d} q \end{pmatrix}.
\]

Gromoll–Meyer [GM74] introduced the \( \star \)-action

\[
q \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} qa & qc \\ qb & qd \end{pmatrix},
\]

whose quotient is an exotic 7-sphere. It all fits in the following diagram

\[
\begin{array}{ccc}
S^3 & \xrightarrow{\star} & Sp(2) \\
\downarrow & & \downarrow \\
S^3 & \xrightarrow{\pi'} & \Sigma_{GM}^7 \\
\downarrow & & \downarrow \\
S^7 & \xrightarrow{\pi} & \Sigma_{GM}^7
\end{array}
\]

which is a building block for several examples. If fact, if \( \phi : N \to M \) is a \( G \)-equivariant function, the whole diagram can be pulled back, producing a new quotient \( (\phi^*P)/G = N' \). The pull-back construction was applied in [Spe16, CS18] to obtain the following examples:

\( (\Sigma_k^7) \): consider \( \phi : S^7 \to S^7 \) as the octonionic \( k \)th fold power. Then the corresponding \( \star \)-diagram \( S^7 \leftarrow \phi^*Sp(2) \to (S^7)' \) yields \( (S^7)' \) diffeomorphic to the connected sum of \( k \) times \( \Sigma_{GM}^7 \);  

\( (\Sigma^8) \): there is a \( S^3 \)-equivariant suspension \( \eta : S^8 \to S^7 \) of the Hopf map \( \eta : S^3 \to S^2 \) whose quotient \( (S^8)' = \eta^*Sp(2)/S^3 \) is the only exotic 8-sphere;  

\( (\Sigma^{10}) \): there is a \( S^3 \)-equivariant suspension \( \theta : S^{10} \to S^7 \) of a generator of \( \pi_6S^3 \) whose induced \( \star \)-quotient \( (S^{10})' \) is a generator of the index 2 subgroup of homotopy 10-spheres that bound spin manifolds;  

\( (\Sigma^{4n+1}) \): the frame bundle \( pr_n : SO(2n+2) \to S^{2n+1} \) can be also seen as a \( \star \)-diagram: one can endow \( SO(2n+2) \) with both the right and left multiplication by \( SO(2n+1) \). In this case, \( M = M' = S^{2n+1} \). However, there is a pull-back map \( J\tau : S^{4n+1} \to S^{2n+1} \), whose \( \star \)-diagram \( S^{4n+1} \leftarrow (J\tau)^*SO(2n+2) \to (S^{4n+1})' \) has \( (S^{4n+1})' \) diffeomorphic to a Kervaire sphere. Moreover, one can ‘reduce’ \( G = SO(2n+1) \) in to either \( U(n) \) or \( Sp(n) \) (supposing \( n \) odd for the last).

Another class of examples are the manifolds constructed in Grove–Ziller [GZ00]. Given integers \( p_+ , q_+ , p_- , q_- \equiv 1 \mod 4 \), [GZ00] produces a cohomogeneity-one manifold \( (P^{10}_{p_-, q_- , p_+ , q_+ }, (S^3)^3) \). We further assume that \( \gcd(p_-, q_-) = \gcd(p_+, q_+) = 1 \). Then one can observe that the subactions
of $S^3_3 = S^3 \times \{1\} \times \{1\}$ and of $S^3_\star = \{1\} \times \Delta S^3$ are commuting and free, where $\Delta S^3$ is the diagonal in $S^3 \times S^3$. They fit in the diagram

\[
\begin{array}{ccc}
S^3 & \rightarrow & M'_{p-q+p+q+} \\
\downarrow \pi & & \downarrow \pi' \\
S^3 & \rightarrow & M_{p-q+p+q+}
\end{array}
\]

Here, $M'_{p-q+p+q+}$ is the $S^3$-bundle over $S^4$ classified by the transition function $\alpha: S^3 \to SO(4)$ defined by $\alpha(x)v = x^k v x^l$, where $k = (p_-^2 - p_+^2)/8$ and $l = -(q_-^2 - q_+^2)/8$. However, we find ourselves in a curious situation here since the ‘exotic’ manifold $M'$ is topologically better recognized than $M$ itself: the only descriptions the literature presents for $M$ are its cohomogeneity-one diagram and its appearance in the family $P_7$ in [Hoe10]. Maybe the $\star$-diagram together with the techniques in [CS18] could be used to determine something about its structure, compared to the one in $M'_{p-q+p+q+}$.

**Example 2** (Gluing and connected sums). Consider $W_1, W_2$ manifolds with boundaries and equipped with $G$-actions. Assume that $f: \partial W_1 \to \partial W_2$ is an equivariant diffeomorphism. Then one can produce a new manifold $W = W_1 \cup_f W_2$, by gluing $W_1, W_2$ via $f$. $W$ thus inherits a natural smooth $G$-action whose restrictions to $W_1, W_2 \subset W$ coincide with the original actions.

Interesting examples arise in the following way: let $(M_1, G), (M_2, G)$ be closed manifolds with $G$-actions. Suppose that $x_i \in M_i$, $i = 1, 2$ have the same orbit type, that is, $G_{x_1}, G_{x_2}$ are subgroups in the same conjugacy class and their isotropy representations are equivalent. Then one can remove small tubular neighborhoods of the orbits $Gx_1, Gx_2$ and glue the boundaries together. In particular, if $x_1, x_2$ are fixed points with equivalent isotropy representations, one can perform a connected sum.

More generally, one can consider the case where $(M, G)$ admits an equivariant embedding of $(S^k \times D^{l+1}, G)$, where $G$ acts on $S^k \times D^{l+1}$ is equipped with a linear action. In this case, one can perform surgery along $\psi$. That is,

At this point $\star$-diagrams becomes quite useful, since corresponding orbits in $M, M'$ often have the same orbit type, as the next lemma points out.

**Lemma 5.5.** Let $M \leftrightarrow P \rightarrow M'$ be a $\star$-diagram and $p \in P$. Then, there is a group isomorphism $\phi: G_{\pi(p)} \to G_{\pi'(p)}$ and a linear isomorphism $\psi$ such that $\rho_{\pi(p)} = \psi \rho_{\pi'(p)} \phi$.

**Proof.** Denote $\pi(p) = x$ and $\pi'(p) = x'$. For simplicity, we only prove the last assertion, since the existence of $\phi$ follows by direct computation.
Note that $d\pi_p, d\pi'$ define isomorphisms between normal spaces:

$$
\frac{T_x M}{T_x Gx} \xleftarrow{\overline{d\pi}} \frac{T_p P}{T_p (G \times G)p} \xrightarrow{\overline{d\pi'}} \frac{T_{x'} M'}{T_{x'} Gx'}
$$

Moreover, since $\pi, \pi'$ commute with the (respective complementary) actions,

$$
\rho_x(h) \overline{d\pi}(\overline{\pi}) \leftarrow \rho_p(h, \phi(h)) \overline{\pi} \rightarrow \rho_{x'}(\phi(h)) \overline{d\pi'}(\overline{\pi}),
$$

inducing the desired identification. \qed

For instance, the isomorphism $\phi$ in the examples $(\Sigma^7_k$)-(0) are all of the form $\phi(h) = ghg^{-1}$ for $g \in G$ only depending on $p$. In such cases, every pair of points $x, x'$, $\pi^{-1}(x) \cap (\pi')^{-1}(x') \neq \emptyset$, have the same orbit type.

We claim that several surgeries can be done equivariantly on the manifolds resulting in these examples. Moreover, such surgeries can be done by keeping the $\star$-diagram apparatus. We give more details on the $(\Sigma^7_k$)-case to illustrate the assertion.

In this case, $S^3$ acts on $S^7$ as $q(a, b)^T = (gaq, gbq)^T$. This action is inherited from the representation $\tilde{\rho} : S^3 \rightarrow SO(8)$ defined by $2\rho_0 \oplus 2\rho_1$, where $\rho_0$ and $\rho_1$ stands for the trivial representation and the representation defined by the composition of the double-cover $S^3 \rightarrow SO(3)$ and the standard action of $SO(3)$ in $\mathbb{R}^3$. That is, $\tilde{\rho}$ is the double suspension of the bi-axial action of $SO(3)$ in $\mathbb{R}^6$, up to a double-cover.

Note that $(a, b)^T$ is a fixed point of $\tilde{\rho}$ whenever $a, b \in \mathbb{R}$ and consider another manifold $(M^7, S^3)$ with a fixed point $p$ whose isotropy representation is $\rho_0 \oplus 2\rho_1$. One can produce a standard degree-one equivariant map $\phi : M^7 \rightarrow S^7$ by ‘wrapping’ $S^7$ with an open ball centered at the fixed point, and sending the remaining o $M$ to the antipodal of $\phi(p)$. As in [CS18, Theorem 4.1], the induced $\star$-diagram results in $M \leftarrow \phi^* P \rightarrow M \# \Sigma^7_k$.

To proceed with the surgery process, note that $(S^7, S^3)$ admits the equivariant submanifolds below. We omit the explicit embeddings and use the representation instead of $G$ in the notation $(M, G)$, to present more explicit information.

$$(S^1, 2\rho_0) \times (D^6, 2\rho_1) = \{(a, b)^T \in S^7 \mid (\text{Re}(a), \text{Re}(b)) \neq (0, 0)\};
$$

$$(S^2, \rho_1) \times (D^5, 2\rho_0 \oplus \rho_1) = \{(a, b)^T \in S^7 \mid \text{Im}(a) \neq 0\};
$$

$$(S^3, \rho_0 \oplus \rho_1) \times (D^4, \rho_0 \oplus \rho_1) = \{(a, b)^T \in S^7 \mid a \neq 0\};
$$

$$(S^4, 2\rho_0 \oplus \rho_1) \times (D^3, \rho_1) = \{(a, b)^T \in S^7 \mid (a, \text{Re}(b)) \neq (0, 0)\};
$$

$$(S^5, 2\rho_1) \times (D^2, 2\rho_0) = \{(a, b)^T \in S^7 \mid (\text{Im}(a), \text{Im}(b)) \neq (0, 0)\};
$$

$$(S^6, \rho_0 \oplus 2\rho_1) \times (D^1, \rho_0) = \{(a, b)^T \in S^7 \mid (\text{Im}(a), b) \neq (0, 0)\}.
$$

Save $S^1 \times D^6$ and $S^4 \times D^3$, every submanifold above can be chosen to lie in an arbitrarily small region of a fixed $\text{Re}(a)$. In particular, arbitrarily many of these surgeries can be performed.

Moreover, we conclude that such surgeries can be performed by preserving infinitely many fixed points. Therefore, the degree-one map mentioned...
above can be considered, producing a \( * \)-diagram over the new manifold \( M \). Although the connected sum applied to this context seems ad-hoc, the resulting manifold \( (\phi^*P)/G = M\#\Sigma \) is the same manifold one obtains by performing the same surgeries on \( \Sigma \). The construction is part of a more general framework, which we state without proof.

**Proposition 5.6.** Suppose that \( p \) is a fixed point in \((M,G)\) and \( M \leftarrow P \to M' \) is a \( * \)-diagram. Suppose there is an equivariant embedding \( \psi : (S^k \times D^{l+1}, G) \to (M, G) \), where \( G \) acts linearly in both \( S^k \) and \( D^{l+1} \), and whose image lies in a sufficiently small neighborhood of \( p \). Then, there is a \( * \)-diagram over the resulting surgery, \( \tilde{M} : \)

\[
\tilde{M} \leftarrow \tilde{P} \to \tilde{M}'
\]

where \( \tilde{M} = (M \setminus \psi(S^k \times D^{l+1})) \cup D^{l+1} \times S^l \). Moreover, there is an equivariant embedding \( \psi' : (S^k \times D^{l+1}, G) \to (M', G) \), with the same action on \( S^k \times D^{l+1} \) such that

\[
\tilde{M}' = (M' \setminus \psi'(S^k \times D^{l+1})) \cup D^{l+1} \times S^l.
\]

We further recall that \( * \)-diagrams were originally used in [CS18] to produce positive Ricci curvature in \( M' \). Indeed, they provide a natural setting to apply Searle–Wilhelm’s positive Ricci lifting theorem [SW15, Theorem A]:

**Theorem 5.7** (Theorem 6.6, [CS18]). Let \( M \leftarrow P \to M' \) be a \( * \)-diagram with both \( G \) and \( M \) compact. Suppose that \((M,G)\) has a regular orbit with a finite fundamental group and that \( M \) admits a \( G \)-invariant metric whose induced quotient metric satisfies \( \text{Ric}_{M/G} \geq 1 \). Then, both \( M \) and \( M' \) have a \( G \)-invariant metric of positive Ricci curvature.

**Example 3** (More connected sums). A list of manifolds whose fixed points have isotropy representations isomorphic to the ones of the examples \((\Sigma^7_k)-\Sigma^4n+1)\) are found in [CS18]. We compile it here, to reinforce Proposition 5.6.

**Proposition 5.8** (Cavenaghi–Sperançã). The following manifolds have fixed points whose isotropy representations are isomorphic to the ones in \((\Sigma^7_k)-(\Sigma^{4n+1})\):

- (i) \((\Sigma^7_k)\): any 3-sphere bundle over \( S^4 \);
- (ii) \((\Sigma^8)\): every 3-sphere bundle over \( S^5 \) or a 4-sphere bundle over \( S^4 \);
- (iii) \((\Sigma^{10})\):
  - (a) \( M^8 \times S^2 \) with \( M^8 \) as in item (ii);
  - (b) any 3-sphere bundle over \( S^7 \), 5-sphere bundle over \( S^5 \) or 6-sphere bundle over \( S^4 \);
- (iv) \((\Sigma^{4m+1}, U(n))\):
  - (a) a sphere bundle \( S^{2m} \hookrightarrow M^{4m+1} \to S^{2m+1} \) associated to any multiple of \( O(2m+1) \hookrightarrow O(2m+2) \to S^{2m+1} \), the frame bundle of \( S^{2m+1} \).
(b) a $\mathbb{C}P^m$-bundle $\mathbb{C}P^m \hookrightarrow M^{4m+1} \to S^{2m+1}$ associated to any multiple of the bundle of unitary frames $U(m) \hookrightarrow U(m + 1) \to S^{2m+1}$

(c) $M^{4m+1} = \frac{U(m+2)}{SU(2) \times U(m)}$

(v) $(\Sigma^{8r+5}, Sp(r))$: $M \times N^{5-k}$, where $N$ is any manifold and

(a) $S^{4r+k-1} \hookrightarrow M^{8r+k} \to S^{4r+1}$ is the $k$-th suspension of the unitary tangent $S^{4r-1} \hookrightarrow T_1S^{4r+1} \to S^{4r+1}$,

(b) $k = 1$ and $\mathbb{H}P^m \hookrightarrow M^{8m+1} \to S^{4m+1}$ is the $\mathbb{H}P^m$-bundle associated to any multiple of $Sp(m) \hookrightarrow Sp(m + 1) \to S^{4m+1}$

(c) $k = 0$ and $M = \frac{Sp(m+2)}{Sp(2) \times Sp(m)}$

(d) $k = 1$ and $M = M^{8m+1}$ is as in item (iv)

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