A RAÏKOV-TYPE THEOREM FOR RADIAL POISSON DISTRIBUTIONS: A PROOF OF KINGMAN’S CONJECTURE.

THU VAN NGUYEN

Abstract. In the present paper we prove the following conjecture in Kingman, J.F.C., Random walks with spherical symmetry, Acta Math.,109, (1963), P. 11-53, concerning a famous Raikov’s theorem of decomposition of Poisson random variables: “If a radial sum of two independent random variables X and Y is radial Poisson, then each of them must be radial Poisson.”

Keywords and phrases: radial characteristic functions; radial Poisson distribution.

AMS2000 subject classification: 60B99, 60E07, 60E99.

1. Introduction, Notations and Preliminaries

It is well-known that for each \( s \geq -\frac{1}{2} \) the Bessel function \( \Lambda_s(\cdot) \) defined by

\[
\Lambda_s(x) = \Gamma(s + 1)J_s(x)/(1/2x)^s,
\]

where \( J_s(x) \) denotes the Bessel function of the first kind,

\[
J_s(x) := \sum_{j=0}^{\infty} \frac{(-1)^j (x/2)^{\nu+2j}}{j! \Gamma(\nu+j+1)}
\]

can be used as a kernel of radial characteristic functions (rad.ch.f.’s) which possesses many of the properties usually associated with the familiar univariate characteristic functions (ch.f.’s). In particular, in terms of Bessel functions, one can introduce important classes of probability distributions in the context of the Kingman convolutions such as radial Gaussian, radial Poisson distributions, etc.

Let \( \mathcal{P} := \mathcal{P}(\mathbb{R}^+) \) denote the set of all probability measures (p.m.’s) on the positive half-line \( \mathbb{R}^+ \) equipped with the weak convergence. The Kingman convolution (cf. Kingman [3], Urbanik [12]) is defined as follows. For each continuous bounded function \( f \) on \( \mathbb{R}^+ \) we put

\[
\int_0^\infty f(x)\mu*_{1,\delta} \nu(dx) = \frac{\Gamma(s+1)}{\sqrt{\pi}\Gamma(s+\frac{3}{2})} \int_0^\infty \int_0^\infty \int_{-1}^1 f((x^2 + 2uxy + y^2)^{1/2})(1 - u^2)^{s-1/2}\mu(dx)\nu(dy)du,
\]

where \( \mu \) and \( \nu \in \mathcal{P} \) and \( \delta = 2(s+1) \geq 1 \). In the sequel, for the sake of simplicity, we will denote \( *_{1,\delta} = *_s \). The convolution algebra \( (\mathcal{P},*_s) \) is the most important example of Urbanik convolution algebras. In language of the Urbanik convolution
algebras, the characteristic measure, say $\sigma_s$, of the Kingman convolution has the Rayleigh density

$$d\sigma_s(y) = \frac{2(s+1)^{s+1}}{\Gamma(s+1)}y^{2s+1}\exp\left(-(s+1)y^2\right)dy$$

with the characteristic exponent $\kappa = 2$. It is known (cf. Kingman [3], Theorem 1), that the kernel $\Lambda_s$ itself is an ordinary ch.f. of a symmetric p.m., say $F_s$, defined on the interval $[-1,1]$. Thus, if $\theta_s$ denotes a random variable (r.v.) with distribution $F_s$ then for each $t \in \mathbb{R}^+$,

$$\Lambda_s(t) = E\exp(it\theta_s) = \int_{-1}^{1} \cos(tx)dF_s(x).$$

Suppose that $X$ is a nonnegative r.v. with distribution $\mu \in \mathcal{P}$ and $X$ is independent of $\theta_s$. The radial characteristic function (rad.ch.f.) of $\mu$, denoted by $\hat{\mu}(t)$, is defined by

$$\hat{\mu}(t) = E\exp(itX\theta_s) = \int_{0}^{\infty} \Lambda_s(tx)\mu(dx),$$

for every $t \in \mathbb{R}^+$. The characteristic measure of the Kingman convolution $*_s$, denoted by $\sigma_s$, has the Maxwell density function

$$d\sigma_s(x) = \frac{2(s+1)^{s+1}}{\Gamma(s+1)}x^{2s+1}\exp\{-2(s+1)x^2\}, \quad (0 < x < \infty).$$

and the rad.ch.f.

$$\hat{\sigma}_s(t) = \exp\left\{-t^2/4(s+1)\right\}.$$  

2. Transforms $\tau_s, s \geq -\frac{1}{2}$

We begin this Section with the definition of a class of the following transforms $\tau_s, s \geq -\frac{1}{2}$,

$$\tau_s(G)(E) = \int_{0}^{\infty} T_c(F_s)(E)G(dc), \quad G \in \mathcal{P},$$

where for $c \geq 0$ and $E \subset \mathbb{R}$ and a p.m. $\mu \in \mathbb{R}$

$$T_c\mu(E) = \mu(c^{-1}E).$$

Let us denote by $\mathcal{S}$ the class of all symmetric p.m.’s on $\mathbb{R}$. Obviously, this class of p.m.’s is closed with respect to the ordinary convolution $\ast$. Let $\mathcal{S}_s$ be a subclass of $\mathcal{S}$ consisted of p.m.’s of the form (12). Moreover, by (12) and by Fourier transforms and rad.ch.f.’s,

**Theorem 1.** For any $\mu, \nu \in \mathcal{P}$ and $\alpha \geq 0, \beta \geq 0, \alpha + \beta = 1$,

$$\tau_s(\alpha\mu + \beta\nu) = \tau_s(\mu) + \beta\tau_s(\nu)$$

$$\tau_s(\mu \ast_s \nu) = (\tau_s\mu) \ast (\tau_s\nu)$$

$$\tau_s(\sigma_s) = N(0, 2(s+1)).$$

where the $\ast$ denotes the ordinary convolution.

Hence, it follows that each pair $(\mathcal{S}_s, \ast, s \geq -1/2)$, is a topological semigroup of symmetric p.m.’s with the unit element $\delta_0$. Moreover, the map $\tau_s$ stands for a homeomorphism between the two convolution algebras $(\mathcal{P}, \ast)$ and $(\mathcal{S}_s, \ast)$. 
3. Raikov’s type theorem for radial Poisson distributions

Given nonnegative r.v.’s $X$ and $Y$ with the corresponding distributions $\mu$ and $\nu$ such that $X$, $Y$ and $\theta_s$ are independent. Following Kingman ([3], p.19) we put

\[ X \oplus_s Y := \sqrt{X^2 + Y^2 + 2XY \theta_s} \]

and call it (and any one of the equivalent r.v.’s $X \oplus_s Y$) a radial sum of $X$ and $Y$. It should be noted that

\[ \mu * \nu \overset{d}{=} (X \oplus_s Y). \]

In a recent paper [9] a multi-dimensional analogue of the Cramér-Lévy theorem was obtained (see also Urbanik [13] for one-dimensional case). For the sake of simplicity, we state below the univariate version.

**Theorem 2.** Let $X$, $Y$, and $Z$ be nonnegative independent r.v.’s such that

\[ \sigma_s \overset{d}{=} X \overset{d}{=} (Y \oplus_s Z). \]

Then, $Y \overset{d}{=} \alpha Y_1$ and $Z \overset{d}{=} \beta Y_2$ for some nonnegative constants $\alpha$ and $\beta$ with $\alpha^2 + \beta^2 = 1$ and, $\sigma_s \overset{d}{=} Y_1 \overset{d}{=} Y_2$.

It is remarkable that Rayleigh distributions share important properties with ordinary normal distributions. Our further aim is to study a similar situation for radial Poisson distributions which are defined by

**Definition 1.** (cf. Kingman [3], P. 33-34) A distribution $\pi_s$ is said to be radial Poisson, if its r.d.f. is of the form

\[ \hat{\pi}_s(t) = \exp \{\Lambda_s(ct) - 1\} \]

(a, $c > 0$ and $t \geq 0$)

or, equivalently,

\[ \pi_s \overset{d}{=} (X_1 \oplus_s X_2 \oplus_s \ldots \oplus_s X_N) \]

where all the $X_i$ are equal to $c$ with probability one, and $N$ has a Poisson distribution with mean $a$, and r.v.’s $N, X_i, i = 1, 2, \ldots$ are independent.

It should be noted that, in the case $s = -\frac{1}{2}$ and $c = 1$ we have $*_s = *$ and $\pi_{-\frac{1}{2}}$ becomes the ordinary Poisson distribution.

The distribution has atoms at 0 and at $c$, together with an absolutely continuous component in $x > 0$. A question posed by Kingman ([3], P. 34) has been existing for many years: It would be interesting to prove for this distribution analogues of some of the well-known results (such as Raikov’s theorem, [4], P.174) about the Poisson distribution”. Our main aim in this paper is to prove a confirmative answer to Kingman’s question as follows.

**Theorem 3.** Suppose, $X$, $Y$, and $Z$ are nonnegative independent r.v.’s such that $\pi_s \overset{d}{=} X$ and the following equation holds

\[ X \overset{d}{=} (Y \oplus_s Z). \]

Then, each of r.v.’s $X$ and $Y$ has radial Poisson distribution.

**Proof.** First, we consider the case $s = -\frac{1}{2}$.

A symmetric distribution $\nu \in S$ is called symmetric Poisson, if its r.d.f. (Fourier transform) is of the form

\[ F_\nu(t) = \exp \{\cos(ct) - 1\} \]

(a, $c > 0$ and $t \geq 0$).
Consider Urbanik’s symmetric convolution $*_{1,1}$ which is identical with the Kingman convolution $*_{-1}$ (cf. Urbanik [12]). Since, the transform $\tau_{-1/2}$ is the symmetrization of p.m.’s in $\mathcal{P}$, it follows that every symmetric distribution $H \in \mathbb{R}$ is uniquely represented by
\begin{equation}
H = \tau_{-1/2}(G), \quad G \in \mathcal{P}
\end{equation}
which implies that $\mathcal{S} = \{\tau_{-1/2}(G), G \in \mathcal{P}\}$.

Moreover, the rad.ch.f. of a distribution $G \in \mathcal{P}$ is the same as the Fourier transform of its symmetrisation, that is
\begin{equation}
\hat{G}(t) = \mathcal{F}_{\tau_{-1/2}(G)}(t) \quad t \geq 0.
\end{equation}
In particular, if $G = \pi_{-1/2}$ the Equation (19) reads
\begin{equation}
\tau_{-1/2}(\pi_{-1/2}) = \tau_{-1/2}(G_1) \star \tau_{-1/2}(G_2)
\end{equation}
which implies that
\begin{equation}
\pi_{-1/2} = G_1 \star G_2.
\end{equation}

By virtue of the classical Raikov’s decomposition theorem ([4], P.174) we infer that the distributions $G_1$ and $G_2$ are both Poisson distributions and, consequently, the symmetric distributions $\tau_{-1/2}(G_1)$ and $\tau_{-1/2}(G_2)$ are both symmetric Poisson distributions which shows that the Raikov’s decomposition theorem holds true also for symmetric distributions on $\mathbb{R}$. Now, by (15) it follows that the Raikov’s type theorem is true also for the Kingman convolution algebra $(\mathcal{P}, *_{-1/2})$. In general, let $\pi_s, s > -1/2$ be a radial Poisson distribution with the rad.ch.f. given by the equation (15) and the decomposition (16) holds. Let $G_1$ and $G_2$ be distributions of the r.v.’s $Y$ and $Z$, respectively. Then we have
\begin{equation}
\pi_s = G_1 * s G_2.
\end{equation}
Applying the transform (9) to both sides of the above equation one get
\begin{equation}
\tau_s(\pi_s) = \tau_s(G_1) \star \tau_s(G_2)
\end{equation}
which by virtue of the Raikov’s theorem for symmetric Poisson distributions implies that the distributions $\tau_s(G_1)$ and $\tau_s(G_2)$ are symmetric Poisson distributions and consequently, since for each $t \geq -1/2$ and for each $G \in \mathcal{P}$
\begin{equation}
\hat{G}(t) = \mathcal{F}_{\tau_s(G)}(t) = \exp a\{\lambda_s(ct) - 1\} \quad (a, c > 0 \text{ and } t \geq 0)
\end{equation}
the distributions $G_1$ and $G_2$ are radial Poisson distributions. Finally, the equation (16) holds. □
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Department of Mathematics; International University, HCM City; No.6 Linh Trung ward, Thu Duc District, HCM City; Email: nvthu@hcmiu.edu.vn
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\[
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\]

where \( J_s(x) \) denotes the Bessel function of the first kind,

\[
J_s(x) := \sum_{j=0}^{\infty} \frac{(-1)^j (x/2)^{\nu+2j}}{j! \Gamma(\nu+j+1)}
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can be used as a kernel of radial characteristic functions (rad.ch.f.’s) which possesses many of the properties usually associated with the familiar univariate characteristic functions (ch.f.’s). In particular, in terms of Bessel functions, one can introduce important classes of probability distributions in the context of the Kingman convolutions such as radial Gaussian, radial Poisson distributions, etc.

Let \( \mathcal{P} := \mathcal{P}(\mathbb{R}^+) \) denote the set of all probability measures (p.m.’s) on the positive half-line \( \mathbb{R}^+ \) equipped with the weak convergence. The Kingman convolution (cf. Kingman [3], Urbanik [12]) is defined as follows. For each continuous bounded function \( f \) on \( \mathbb{R}^+ \) we put

\[
\int_{0}^{\infty} f(x) \mu *_{1,\delta} \nu(dx) = \frac{\Gamma(s+1)}{\sqrt{\pi} \Gamma(s+\frac{1}{2})} \int_{0}^{\infty} \int_{0}^{\infty} \int_{-1}^{1} f((x^2+2uxy+y^2)^{1/2})(1-u^2)^{s-1/2} \mu(dx) \nu(dy) du,
\]

where \( \mu \) and \( \nu \in \mathcal{P} \) and \( \delta = 2(s+1) \geq 1 \). In the sequel, for the sake of simplicity, we will denote \( *_{1,\delta} = *_{s} \). The convolution algebra \((\mathcal{P}, *_{s})\) is the most important
example of Urbanik convolution algebras. In language of the Urbanik convolution algebras, the characteristic measure, say $\sigma_s$, of the Kingman convolution has the Rayleigh density

\[
d\sigma_s(y) = \frac{2(s+1)^{s+1}}{\Gamma(s+1)} y^{2s+1} \exp\left(-\frac{s+1}{2}y^2\right) \, dy
\]

with the characteristic exponent $\kappa = 2$. It is known (cf. Kingman [3], Theorem 1), that the kernel $\Lambda_s$ itself is an ordinary c.h.f. of a symmetric p.m., say $F_s$, defined on the interval $[-1,1]$. Thus, if $\theta_s$ denotes a random variable (r.v.) with distribution $F_s$ then for each $t \in \mathbb{R}^+$,

\[
\Lambda_s(t) = E \exp(it\theta_s) = \int_{-1}^{1} \cos(tx) dF_s(x).
\]

Suppose that $X$ is a nonnegative r.v. with distribution $\mu \in \mathcal{P}$ and $X$ is independent of $\theta_s$. The radial characteristic function (rad.ch.f.) of $\mu$, denoted by $\hat{\mu}(t)$, is defined by

\[
\hat{\mu}(t) = E \exp(itX\theta_s) = \int_{0}^{\infty} L_s(tx)\mu(dx),
\]

for every $t \in \mathbb{R}^+$. The characteristic measure of the Kingman convolution $\ast_s$, denoted by $\sigma_s$, has the Maxwell density function

\[
d\sigma_s(x) = \frac{2(s+1)^{s+1}}{\Gamma(s+1)} x^{2s+1} \exp\left\{-\frac{(s+1)}{4}x^2\right\}, \quad (0 < x < \infty).
\]

and the rad.ch.f.

\[
\hat{\sigma}_s(t) = \exp\left\{-\frac{t^2}{4(s+1)}\right\}.
\]

2. Transforms $\tau_s$, $s \geq -\frac{1}{2}$

Let us denote by $S$ the class of all symmetric p.m.’s on $\mathbb{R}$. Obviously, this class of p.m.’s is closed with respect to the ordinary convolution $\ast$ and convex combinations of p.m.’s and the weak convergence. Define transforms $\tau_s : \mathcal{P} \to S$, $s \geq -\frac{1}{2}$ as follows

\[
\tau_s(\mu)(E) = \int_{0}^{\infty} T_{c}(F_s)(E)\mu(dc),
\]

where $c \geq 0$ and $E \subset \mathbb{R}$ and for a p.m. $\mu \in \mathcal{P}$

\[
T_{c}\mu(E) = \mu(c^{-1}E).
\]

By Sonine’s first finite integral for Bessel function ([14], p. 373), for $s > v \geq -\frac{1}{2}$ we have

\[
\Lambda_s(t) = \frac{2\Gamma(s+1)}{\Gamma(v+1)\Gamma(s-v)} \int_{0}^{1} x^{2v+1}(1-x^2)^{s-v-1} L_v(tx) \, dx,
\]

which implies that the transform $\tau_s$ is subordinate to $\tau_v$. In particular, each transform $\tau_s, \tau \geq -\frac{1}{2}$ is subordinate to $\tau_{-\frac{1}{2}}$ which is defined via the cosine convolution.
Specifically, for \( s > v \geq -\frac{1}{2} \) and \( \mu \in \mathcal{P} \),
\[
\hat{\mu}(t) = \int_0^\infty \Lambda_s(tx) \mu(dx) = \int_0^\infty \cos(tx)(\tau_s \mu)(dx)
\]
(11)
\[
= \int_0^\infty \frac{2\Gamma(s+1)}{\Gamma(v+1)\Gamma(s-v)} \int_0^1 u^{2v+1}(1-u^2)^{s-v-1} \Lambda_v(tux)du \mu(dx)
\]
\[
= \int_0^\infty \Lambda_v(tx) \frac{2\Gamma(s+1)}{\Gamma(v+1)\Gamma(s-v)} \int_0^1 u^{2v+1}(1-u^2)^{s-v-1} \mu(dx/u)du] = \int_0^\infty \cos(tx)(\tau_s \mu)(dx),
\]
where \( \nu(dx) := \frac{2\Gamma(s+1)}{\Gamma(s+1)\Gamma(s-v)} \int_0^1 u^{2v+1}(1-u^2)^{s-v-1} \mu(dx/u)du \), which implies that for every bounded continuous function on \( \mathbb{R}^+ \)
(12)
\[
= \int_0^\infty \frac{2\Gamma(s+1)}{\Gamma(v+1)\Gamma(s-v)} \int_0^\infty f(x) \int_0^1 (\tau_s \nu)(dx/u)u^{2v+1}(1-u^2)^{s-v-1}du.
\]

Let \( S_s \) be a subclass of \( S \) consisted of p.m.'s of the form \( \{\} \). By \( \{\} \) and by Fourier transforms and rad.ch.f.'s and, by a similar proof of Proposition 1b in Bingham [1], p. 176.

**Theorem 1.** For any \( \mu, \nu \in \mathcal{P} \) and \( \alpha \geq 0, \beta \geq 0, \alpha + \beta = 1 \),
(13) \( \tau_s(\alpha \mu + \beta \nu) = \alpha \tau_s(\mu) + \beta \tau_s(\nu) \)
(14) \( \tau_s(\mu * \nu) = (\tau_s \mu) * (\tau_s \nu) \)
(15) \( \tau_s(\sigma_s) = N(0, 2(s+1)) \).

where the \( * \) denotes the ordinary convolution.

Hence, it follows that each pair \((S_s, *, s \geq -1/2)\), is a topological semigroup of symmetric p.m.'s with the unit element \( \delta_0 \). Moreover, the map \( \tau_s \) stands for a homeomorphism between the two convolution algebras. \((\mathcal{P}, *)\) and \((S_s, *)\). Finally, by virtue of (12) the family \( S_s, s \geq -\frac{1}{2} \) is monotone decreasing in \( s \). The sets \( S_s \) are continuous in \( s \) in that
(16) \( \cap_{0 \leq u < s} S_u = S_s, \cup_{s < u \leq \infty} S_u = S_s, \quad (s \in [0, \infty]) \).

3. Raikov’s type theorem for radial Poisson distributions

Given nonnegative r.v.'s \( X \) and \( Y \) with the corresponding distributions \( \mu \) and \( \nu \) such that \( X, Y, \theta_s \) are independent. Following Kingman \( \{\} \), p.19 we put
(17) \( X \oplus_s Y := \sqrt{X^2 + Y^2 + 2XY\theta_s} \)
and call it (and any one of the equivaleent r.v.'s \( X \oplus_s Y \)) a radial sum of \( X \) and \( Y \). It should be noted that
(18) \( \mu * \nu \overset{d}{=} (X \oplus_s Y) \).

In a recent paper \( \{\} \) a multi-dimensional analogue of the Cramér-Lévy theorem was obtained (see also Urbanik \{\} for one-dimensional case). For the sake of simplicity, we state below the univariate version.
Theorem 2. Let $X$ and $Y$ be nonnegative independent r.v.’s such that
\[(18) \quad \sigma_s^d = (X \oplus_s Y).\]
Then, $X = \alpha X_1$ and $Y = \beta Y_1$ for some nonnegative constants $\alpha$ and $\beta$ with $\alpha^2 + \beta^2 = 1$ and $\sigma_s^d = X_1^d = X_2^d$.

It is remarkable that Rayleigh distributions share important properties with ordinary normal distributions. Our further aim is to study a similar case for radial Poisson distributions which are defined as follows.

Definition 1. (cf. Kingman [3], p. 33-34) A r.v. $X$ or its distribution $\pi$ is said to be radial Poisson, or precisely, $s$-radial Poisson, if its rad.ch.f. is of the form
\[(19) \quad \hat{\pi}(t) = E(\Lambda_s(tX)) = \exp\{a(\Lambda_s(ct) - 1)\} \quad (a, c > 0 \text{ and } t \geq 0)\]
where all the $X_i$ are equal to $c$ with probability one, and $N$ has a Poisson distribution with mean $a$, and r.v.’s $N, X_i, i = 1, 2, \ldots$ are independent.

By virtue of (19, 20) it follows that each radial Poisson distribution defined above is uniquely determined by $a$ and $c, a = EN$ and $c$ is a scale parameter and will be denoted by $\pi(s, a, c)$. If $s = -\frac{1}{2}$ and $c = 1$ we have $\pi_s = \pi_{-\frac{1}{2}}$ becomes an ordinary Poisson distribution. The distribution has atoms at 0 and at $c$, together with an absolutely continuous component in $x > 0$.

Definition 2. A symmetric distribution $\nu \in S$ is called symmetric Poisson, if its ch.f. (Fourier transform) is of the form
\[(21) \quad F_\nu(t) = \exp\{a(\cos(ct) - 1)\} \quad (a, c > 0 \text{ and } t \geq 0).\]

A question posed by Kingman ([3], p. 34) has been existing for many years: “It would be interesting to prove for this distribution analogues of some of the well-known results (such as Raikov’s theorem, [4], p.174) about the Poisson distribution”. Our main aim in this paper is to give a complete answer to Kingman’s question.

Theorem 3. Suppose, $X, Y,$ and $Z$ are nonnegative independent r.v.’s such that $\pi(s, a, c)^d = X$ and the following equation holds
\[(22) \quad X^d = (Y \oplus_s Z).\]
Then, there exist nonnegative numbers $a_1$ and $a_2$ such that $a = a_1 + a_2$ and r.v.’s $X$ and $Y$ have radial Poisson distributions $\pi(s, a_1, c)$ and $\pi(s, a_2, c)$, respectively.

Proof. First, observe that if $X \stackrel{d}{=} \pi(s, a, c)$, then $X/c \stackrel{d}{=} \pi(s, a, 1)$. Therefore, without loss of generality, one may assume that $c = 1$. Next, consider the case $s = -\frac{1}{2}$. It is evident that the Urbanik’s symmetric convolution $*_{1,1}$ (cf. Urbanik[12]) is identical with the Kingman convolution $*_{-\frac{1}{2}}$ or cosine convolution. Since, the transform $\tau_{-\frac{1}{2}}$ is the symmetrization of p.m.’s in $\mathcal{P}$, it follows that every symmetric distribution $H \in \mathbb{R}$ is uniquely represented by
\[(23) \quad H = \tau_{-\frac{1}{2}}(G), \quad G \in \mathcal{P}\]
which implies that $S = \{\tau_{-\frac{1}{2}}(G), G \in \mathcal{P}\}$. 
Moreover, the rad.ch.f. of a distribution $G \in (\mathcal{P}, \ast \frac{1}{2})$ is the same as the Fourier transform of its symmetrisation, that is
\begin{equation}
\hat{G}(t) = \mathcal{F}_{\tau_{-\frac{1}{2}}(G)}(t) \quad t \geq 0. \tag{24}
\end{equation}
In particular, if $G = \pi(-\frac{1}{2}, a, c)$ the Equation (24) reads
\[\hat{\pi}(-\frac{1}{2}, a, c)(t) = \int_{-1}^{1} \cos(ut)\pi(-\frac{1}{2}, a, c)(du) = \exp\{\cos(ct)-1\} \quad (a, c > 0 \text{ and } t \geq 0)\]
which shows that $\tau_{-\frac{1}{2}}(\pi(-\frac{1}{2}, a, c))$ is a symmetric Poisson distribution.

Suppose now that for symmetric distributions $\tau_{-\frac{1}{2}}(G_i), i = 1, 2$ we have
\[\tau_{-\frac{1}{2}}(\pi(-\frac{1}{2}, a, c)) = \tau_{-\frac{1}{2}}(G_1) \ast \tau_{-\frac{1}{2}}(G_2)\]
which implies that
\[\pi(-\frac{1}{2}, a, c) = G_1 \ast \frac{1}{2} G_2.\]

Proceedings successively, by virtue of the classical Raikov’s decomposition theorem ([4], P.174) we infer that the distributions $G_1$ and $G_2$ are both Poisson distributions and, consequently, the symmetric distributions $\tau_{-\frac{1}{2}}(G_1)$ and $\tau_{-\frac{1}{2}}(G_2)$ are both symmetric Poisson distributions which shows that the Raikov’s decomposition theorem holds true also for symmetric distributions on $\mathbb{R}$. Now, by (21, 24) it follows that, Theorem [4] is true for the Kingman convolution algebra $(\mathcal{P}, \ast \frac{1}{2})$. In general, let $\pi(s, a, c)$ be a radial Poisson distribution with the rad.ch.f. given by the equation (19) and the decomposition (22) holds. Let $G_1$ and $G_2$ be distributions of the r.v.’s $Y$ and $Z$, respectively. Then we have
\[\pi(s, a, c) = G_1 \ast \frac{1}{2} G_2.\]
Applying the transform (9) to both sides of the above equation one get
\[\tau_s(\pi(s, a, c)) = \tau_s(G_1) \ast \tau_s(G_2)\]
which by virtue of the Raikov’s theorem for symmetric Poisson distributions implies that the distributions $\tau_s(G_1)$ and $\tau_s(G_2)$ are symmetric Poisson distributions and consequently, since for each $s > -\frac{1}{2}$ and for each $G_i \in \mathcal{P}$
\[\hat{G}_i(t) = \mathcal{F}_{\tau_s(G_i)}(t) = \exp a_i\{\Lambda_s(ct) - 1\} \quad (a_i, c > 0, i = 1, 2 \text{ and } t \geq 0)\]
we have $G_i = \pi(s, a_i, c), a = a_1 + a_2$ and the equation (22) holds. \hfill \Box

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