Spin Networks and Recoupling in Loop Quantum Gravity

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Abstract

I discuss the role played by the spin-network basis and recoupling theory (in its graphical tangle-theoretic formulation) and their use for performing explicit calculations in loop quantum gravity. In particular, I show that recoupling theory allows the derivation of explicit expressions for the eigenvalues of the quantum volume operator. An important side result of these computations is the determination of a scalar product with respect to which area and volume operators are symmetric, and the spin network states are orthonormal.

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1 Introduction

A promising attempt towards a quantum theory of gravitation is provided by the loop quantization [1]. It amounts to the direct canonical quantization of the (Poisson bracket) holonomy algebra generated by:

\[ \mathcal{T}[\alpha] = -\text{Tr}[U_\alpha], \]
\[ \mathcal{T}^a[\alpha](s) = -\text{Tr}[U_\alpha(s, s) \tilde{E}^a(s)], \]

where \( U_\alpha \) is the parallel propagator of the Ashtekar-Sen connection \( A_a \) along a loop \( \alpha \). The crucial point is that to each loop \( \alpha \) is uniquely associated a \( \mathcal{T} \) observable. This suggests to use as carrier space of the representation a suitable subspace \( \mathcal{V} \) of the free loop algebra defined as the finite linear combinations \( \Phi \) of finite products of loops, as in:

\[ \Phi = c_0 + \sum_i c_i[\alpha_i] + \sum_{jk} c_{jk}[\alpha_j][\alpha_k] + \ldots . \]

In this talk I shall summarize the results obtained in [5]; in particular, I shall show that the \( \mathcal{V} \)-space admits a tangle-theoretic description in terms of the Temperley-Lieb recoupling theory [6], and that a convenient basis in this space is given by the spin-network states.

It is important to note that an element of the \( \mathcal{T} \) holonomy algebra

\[ \mathcal{T}[\Phi] = c_0 + \sum_i c_i \mathcal{T}[\alpha_i] + \sum_{ij} c_{ij} \mathcal{T}[\alpha_i] \mathcal{T}[\alpha_j] + \ldots \]

is associated to each element of the free loop algebra, and that the connection representation is defined by introducing an Hilbert space structure on a suitable extensions \( \mathcal{A}/\mathcal{G} \) of this \( \mathcal{T} \) algebra [2]. The relation between the two representations is given by the loop transform [3]. For a more detailed analysis of this point see [4].

2 The Carrier Space of the Loop Representation and the Spin-Networks Basis

The loop representation of quantum gravity is assumed to be the linear representation of the Poisson algebra of the \( \mathcal{T} \) variables over \( \mathcal{V} \) defined by:

\[ \langle \Phi \mid \mathcal{T}[\beta] = \langle \Phi \cdot [\beta] \mid. \]

It may happen that two elements \( \Phi \) and \( \Psi \) of the free loop algebra correspond to the same function on the holonomy algebra \( \mathcal{T}[\Phi] = \mathcal{T}[\Psi] \) (i.e., they give the same value for any values of the connection). As a consequence, the carrier space \( \mathcal{V} \) of the representation must be defined
as the space of the equivalence classes of the free loop algebra under the equivalence defined by all the relations (Mandelstam relations)

\[ \Phi \sim \Psi \text{ if } \mathcal{T}[\Phi] = \mathcal{T}[\Psi], \quad (4) \]

namely by the equality of the corresponding holonomies. The principal consequences of the Mandelstam relations are that the \( \mathcal{V} \)-space does not depend on the orientation and parameterization of the loops. Moreover, the following identities hold true: [Retracing]: if \( \gamma \) is a segment starting in a point of \( \alpha \) then \( [\alpha \circ \gamma \circ \gamma^{-1}] \sim [\alpha] \); [Binor identity]: \( [\alpha] \cdot [\beta] \sim -[\alpha \# \beta] - [\alpha \# \beta^{-1}] \).

According to \( [5] \), a natural way to represent an element \( \Phi \) of \( \mathcal{V} \) is the following: first, one introduces the extended planar graph \( \Gamma_{ex} \) of \( \Phi \) as the two-dimensional surface obtained by “thickening out” a planar representation of the image \( \gamma \) of all the loop in \( \Phi \); then, one draws the loops of \( \Phi \) over \( \Gamma_{ex} \). In this way, to each element of the \( \mathcal{V} \)-space corresponds an element in the algebra of all possible tangles over \( \Gamma_{ex} \). Moreover, in the tangle-theoretic planar representation of the states \( \langle \Phi \rangle \), the retracing and binor identities become the following tangle identities

\[
\begin{align*}
\begin{array}{c}
\scriptstyle{\top} \\
-2
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\scriptstyle{-2} \\
\scriptstyle{-2}
\end{array}
\end{align*}
\]

(5)

These identities are the key axioms of the tangle-theoretic recoupling theory \( [6] \), if the value of the \( A \) parameter set to \(-1\). This gives the possibility of using the whole machinery of the tangle-theoretic recoupling theory \( [6] \) for dealing with the graphical representation of the states. In particular, a basis in this tangle-theoretic algebra is provided by the spin-networks, where a spin network \( S \) is the sum of tangles given as follow: consider a three-valent “virtual” graph \( \Gamma^v \) over \( \Gamma_{ex} \) and a “compatible coloring” \( \{p_e\} \) of the edges of \( \Gamma^v \); replace each edge with the full-antisymmetric sum of \( p_e \) parallel lines; then, in each three-valent vertices connect all the incoming lines in the unique possible planar way.

In this way, any element of the \( \mathcal{V} \)-space can be characterized in terms of the tangle-theoretic spin network quantum state \( \langle S \rangle = \langle \gamma, \Gamma_{ex}, S \rangle \), defined as the element of \( \mathcal{V} \) determined by the graph \( \gamma \), its extended planar graph \( \Gamma_{ex} \), and the spin-network tangle \( S \) over \( \Gamma_{ex} \). These states are essentially characterized by the graph \( \gamma \), the number \( p_e \) of loops in each edge \( e \) of \( \gamma \), and the recoupling (in terms of three-valent vertices) of the edges connected to each vertex of \( \gamma \) (see figure \( [6] \)).

Note that, representing a vertex of a spin network state \( \langle S \rangle \) by means of
Figure 1: An example of spin-network state and of its extended planar representation.

the portion of its spin-network $S$ contained in the vertex

$$
\langle V^{(n)}[P_0, \ldots, P_{n-1}]\rangle_{t_i} = \langle P_1 \bullet \cdots \bullet P_{n-3} \rangle_{t_i} P_{n-2} \bigg| P_{n-1},
$$

and considering two vertices, $V_n[a_1, \ldots, a_n]$ and $V'_n[b_1, \ldots, b_n]$, they can connected only if all the corresponding incoming lines have the same colors. In this case, it is possible to attach the $a_i$ lines and the corresponding $b_i$ lines to each other. Then, it is obtained a tangle over $\Gamma_{ex}$ constituted only of loops contractible to a point. By recoupling, a tangle in which all the loops are contractible to a point reduce to a number. The operation of computing this number is named chromatic evaluation. This evaluation, denoted by $\langle V_n(a_1, \ldots, a_n)|V'_n(b_1, \ldots, b_n)\rangle$, is a scalar product in the space of the possible vertices. In particular, the chromatic evaluation of a 2-vertex (i.e. of a line) and that of a 3-vertex will be denoted as $\Delta$ function and $\theta$ function, respectively:

$$
\Delta_n = \langle V_2[n], V_2[n]\rangle = \frac{a}{b},
$$

$$
\theta(a, b, c) = \langle V_3[a, b, c], V_3[a, b, c]\rangle = \frac{a}{b}.
$$

For the explicit values of this evaluation and all the details see \[5\] and \[6\]. In this tangle-theoretic interpretation of the loop representation, one still has the analogous of the Wigner-Eckart theorem: the recoupling theorem of \[3\] (pg. 60) states, as a tangle relation, that

$$
\sum_i \left\{ \begin{array}{ccc} a & b & i \\ c & d & j \end{array} \right\} = \sum_i \left\{ \begin{array}{ccc} b & i \\ a & d \\ c \end{array} \right\}.
$$

3
where the quantities \( \{a \ b \ i \ c \ d \ j\} \) are \( su(2) \) six-j symbols (normalized as in [6]).

One of the main advantages of the spin-network basis is that the action of the \( T^a[\alpha](s) \) operators is particularly simple. In fact, one has:

\[
\hat{V}[V] = \sum_{i \in \{s \in V\}} \hat{V}_i,
\]

\[
\hat{V}_i = \frac{l_0^3}{\sqrt{\sum_{r \neq s \neq t = 0}^{n_i-1} \frac{i}{16 \ 3!} \hat{W}^{(ni)}_{rts} \hat{W}^{(ni)}_{rts} \hat{W}^{(ni)}_{rts}}}
\]

The first sum is over all the vertices and the second sum is over the triples of edges adjacent to each vertex. In [5], it has also been shown that the vertex operators \( \hat{W}^{(ni)}_{rts} \) are represented by diagonalizable matrices with real eigenvalues. In fact, a normalization of the vertex exists such that, in this basis, the vertex operator \( \hat{W}^{(ni)}_{rts} \) are represented by the real and antisymmetric matrices

\[
\langle V^{(n_i)}(K_{I_i}) \rangle_N \hat{W}^{(ni)}_{rts} = \sum_{K_{I_i}} \hat{W}^{(ni)}_{rts} K_{I_i} \langle V^{(n_i)}(K_{I_i}) \rangle_N
\]
In the case of 4-valent vertices, this normalization is explicitly given by:

\[
\left \langle V^{(4)}(i) \right \rangle_N = \sqrt{\frac{\langle \begin{array}{c} a \\ b \\ c \\ d \end{array} \rangle}{\langle \begin{array}{c} i \\ a \\ b \\ c \end{array} \rangle}},
\]

and the matrix elements of the vertex operator are given by the chromatic evaluation:

\[
\tilde{W}^{(4)}_{[012]}(a, b, c, d)^k_i = a \cdot b \cdot c \cdot \sqrt{\frac{\langle \begin{array}{c} a \\ b \\ c \\ d \end{array} \rangle}{\langle \begin{array}{c} i \\ a \\ b \\ c \end{array} \rangle}}.
\]

The matrices \(\tilde{W}^{(4)}_{[012]}(a, b, c, d)^k_i\) have elements different from zero only if \(|i - k| = 2\). Using the equations

\[
k_i^c \mathcal{C}^d = \frac{k}{\bar{k}} \cdot \mathcal{C}^d = \frac{k}{\bar{k}} \cdot \mathcal{C}^d,
\]

\[
r \cdot \mathcal{R}^g_r = p \cdot \mathcal{R}^g_r + q \cdot \mathcal{R}^g_r,
\]

it is possible to reduce eq. (14) to the following chromatic evaluation:

\[
\tilde{W}^{(4)}_{[012]}(a, b, c, d)^k_i = \frac{k}{\bar{k}} \cdot \sqrt{\frac{\langle \begin{array}{c} a \\ b \\ c \\ d \end{array} \rangle}{\langle \begin{array}{c} i \\ a \\ b \\ c \end{array} \rangle}} \cdot \left[ b \begin{array}{c} 2 \\ 2 \end{array} \right] - k \begin{array}{c} 2 \\ 2 \end{array} \left[ b \begin{array}{c} 2 \\ 2 \end{array} \right] \cdot \left[ b \begin{array}{c} 2 \\ 2 \end{array} \right] \cdot \left[ c \begin{array}{c} 2 \\ 2 \end{array} \right].
\]

Define \(t = (i+k)/2\) and \(\epsilon = (k-i)/2\). The matrix element \(\tilde{W}^{(4)}_{[012]}(a, b, c, d)^k_i(a, b, c, d)\) is different from zero only if \(\epsilon = \pm 1\) and all the 3-vertices in equation (17) are admissible. Indeed, by performing the chromatic evaluations of eq. (17),
one gets the following explicit formula for the matrix elements $(\epsilon = \pm 1)$:

$$\tilde{W}^{(4)}_{[012]}(a, b, c, d)_{t+\epsilon} = -\epsilon(-1)^{\frac{a+b+c+d}{2}} \left[ \frac{1}{4t(t+2)} \frac{a+b+t+3}{2} \frac{c+d+t+3}{2} \right] \frac{1+a+b-t}{2} \frac{1+a+t-b}{2} \frac{1+b+t-a}{2} \frac{1+c+d-t}{2} \frac{1+c+t-d}{2} \frac{1+d+t-c}{2}$$

(18)

This formula can be used to obtain explicit formulas for the eigenvalues of the volume operator. For example, in the case $d = a + b + c - 2$, where the matrices $\tilde{W}_{[rst]}$ are two-dimensional, one obtains the following result for the (degenerate) eigenvalue of the volume associated to the 4-vertex $\langle V^{(4)}[a, b, c, a + b + c - 2] \mid v(a, b, c, d) = \frac{l_0^3}{\sqrt{2}} \left[ a b c (a + b + c) \right] \frac{1}{16} \right]^{\frac{1}{2}}$, (19)

that can be compared with the results of [9].

4 The Normalized State and the Scalar Product

A scalar product between two spin-network states can be defined by assuming that it is different from zero only if the two states have exactly the same graph and the same coloring of the real edges. Then, the normalization ([13]), with respect to which the volume operator is represented by symmetrical matrices, determines this scalar product uniquely. Its value is determined by the chromatic evaluation of the vertices:

$$\langle s, s' \rangle = \delta_{\gamma,\gamma'} \prod_{e \in \mathcal{E}} \frac{\delta_{n_e, n'_e}}{\Delta_{n_e}} \prod_{i \in \mathcal{V}} \langle V_i, V'_i \rangle$$

(20)

where the products are extended only to the real edge and to the real vertices, and $\langle V_i, V'_i \rangle$ is the chromatic evaluation obtained by gluing the vertices $V_i$ and $V'_i$. In [4], it has been shown that the scalar product defined in this way is precisely the loop transform [3] of the Ashtekar-Lewandowski measure [2].

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