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Trigonometric Rosen–Morse Potential as a Quark–Antiquark Interaction Potential for Meson Properties in the Non-relativistic Quark Model Using EAIM

1 Introduction

Hadrons consist of more fundamental particles called quarks and gluons. Quantum theory of these particles is quantum chromodynamics (QCD). This theory has essential properties including asymptotic freedom, spontaneous symmetry breaking, and confinement. For high-energy physics, the perturbative QCD calculations work due to the asymptotic freedom. On the other hand, to investigate non-perturbative QCD effects, the low-energy effective field theory is applied. The Schrödinger equation (SE) plays an important role in describing many phenomena as in high energy physics. Thus, the solutions of the SE are important for calculating mass of quarkonia and thermodynamic properties. To obtain the exact and approximate solutions of SE, the various methods have been used for specific potentials such as Nikiforov–Uvarov method [1–3]. The analytical iteration method (AIM) reproduces exact solutions to many differential equations which are important in the physical applications, such as the equations of Hermite, Laguerre, Legendre, and Bessel. The AIM also gives complete exact solutions of Schrödinger equation for Pösch–Teller potential, the harmonic oscillator potential, the complex cubic, quartic [4], and sextic anharmonic oscillator potentials [5–7].

Recently, the trigonometric Rosen–Morse potential (TRM) plays a role as an effective potential for QCD. Refs. [8, 9], the potential is employed in one dimensions space. In Ref. [3], the author extended the study to D-dimensional space using the Nififorov–Uvorov method. In Ref. [10], the trigonometric quark confinement potential is considered to provide an efficient tool for quark model calculations of spectroscopic characteristics of baryons.

Therefore, this work aims to extend the TRM to calculate heavy-meson properties in the free and hot media that are not considered in the previous works, in particular, spectra of heavy and heavy–light meson masses and thermodynamic properties. For this purpose, the N-radial Schrödinger equation (SE) is analytically solved using the exact analytical iteration method (EAIM) for the present potential.
The paper is organized as follows: In Sect. 2, the shape of the trigonometric Rosen–Morse potential is studied and the energy eigenvalues and the corresponding wave functions are calculated in the \( N \)-dimensional space with TRM potential. In Sect. 3.2, the thermodynamic properties are calculated. In Sect. 3, the results are discussed. In Sect. 4, summary and conclusion are presented.

2 Exact Solution of Schrödinger Equation with Trigonometric Rosen–Morse Potential (TRM)

In this section, we discuss the features of Rosen–Morse potential that takes the following form as in Refs. [10–12].

\[
V(z) = \frac{1}{2\mu d^2} \left( -2b \cot|z| + \frac{a(a+1)}{(\sin|z|)^2} \right),
\]

(1)

where \( a = 1, 2, 3 \), and \( z = r/d \). \( \mu, b, d \) are parameters will be determined later. It is quite instructive to expand the potential in a Taylor series for small \( z \). The potential in Eq. (1), takes the following form

\[
V(r) = -\frac{A}{r} + B r + C r^2 + Dr^2,
\]

(2)

where, \( A = \frac{b}{\mu d}, B = \frac{b}{3\mu d^3}, C = \frac{a(a+1)}{2\mu}, \) and \( D = \frac{a(a+1)}{30\mu d^4} \).

In Fig. 1, the TRM potential is plotted, we note that TRM potential has two features the Coulomb potential and confinement potential, Coulomb potential describes the short distance and confinement part describes the long distances. The approximate potential is concise with exact potential up to 0.8 fm. In Ref. [10], the author discussed and they show that TRM has features of QCD for short and long distances. Thus, the approximate potential is a good potential for the description of quark–antiquark interaction potential.

The Schrödinger equation for two particles interacting via symmetric potential in the \( N \)-dimensional space takes the form as in Ref. [13].

\[
\left[ \frac{d^2}{dr^2} + \frac{N-1}{r} \frac{d}{dr} - \frac{L(L+N-2)}{r^2} + 2\mu(E - V(r)) \right] \Psi(r) = 0,
\]

(3)

where \( L, N, \) and \( \mu \) are the angular momentum quantum number, the dimensionality number, and the reduced mass for the quarkonium particle, respectively. Setting wave function \( \Psi(r) = r^{\frac{1-N}{2}} R(r) \), then Eq. (3) takes the following form

\[
\left[ \frac{d^2}{dr^2} + 2\mu(E_{nl} - V(r)) - \frac{(L + \frac{N-2}{2})^2 - \frac{1}{4}}{2\mu r^2} \right] R(r) = 0,
\]

(4)
By substituting Eq. (2) into Eq. (4), we obtain the following equation.

$$\frac{d^2}{dr^2} + 2\mu \left( E_{nl} + \frac{A}{r} - Br - \frac{C}{r^2} - Dr^2 \right) - \frac{(L + \frac{N-2}{2})^2 - \frac{1}{4}}{2\mu r^2} R(r) = 0,$$

Equation (5) is reduced to the following form

$$\frac{d^2}{dr^2} + \left( \varepsilon_{nl} \frac{C_1}{r} - b_1 r - \frac{d_1}{r^2} + a_1 r^2 \right) R(r) = 0,$$

where,

$$\varepsilon_{nl} = 2\mu E_{nl}, \quad C_1 = 2\mu A, \quad b_1 = 2\mu B,$$
$$d_1 = 2\mu C + \left( L + \frac{N-2}{2} \right)^2 - \frac{1}{4}, \quad a_1 = -2\mu D.$$

The analytical exact iteration method (AEIM) requires making the following ansatz [14] as follows

$$R_{nl}(r) = f_n(r) \exp\left[ g_l(r) \right],$$

where,

$$f_n(r) = \begin{cases} 1, & n = 0, \\ \prod_{i=1}^{n} \left( r - \alpha_i^{(n)} \right), & n = 1, 2, \ldots, \end{cases}$$

and,

$$g_l(r) = -\frac{1}{2} 2\alpha r^2 - \beta r + \delta \ln r, \quad \alpha > 0, \beta > 0.$$

It is clear that $f_n(r)$ are equivalent to the Leguere polynomials. From Eq. (8), we obtain

$$R^n(r) = \left[ g^n_1(r) + g^n_1(r) + \frac{\varepsilon(r) + 2\varepsilon'(r) g'(r)}{f'(r)} \right] R_{nl}(r).$$

By comparing Eqs. (10) and (6), at $n = 0$ we get

$$-a_1 r^2 + b_1 r - \frac{C_1}{r} + \frac{d_1}{r^2} - \varepsilon_{0l} = \alpha^2 r^2 + 2\alpha \beta r - \alpha [1 + 2(\delta + 0)] + \beta^2 - \frac{2\beta(\delta + 0)}{r} + \frac{\delta(\delta - 1)}{r^2}.$$

Comparing the corresponding powers of $r$ on both sides of Eq. (11), we get

$$\alpha = \sqrt{a_1}, \quad \varepsilon_{0l} = \alpha [1 + 2(\delta + 0)] - \beta^2, \quad \beta = \frac{b_1}{2\sqrt{a_1}},$$
$$C_1 = 2\beta(\delta + 0), \quad d_1 = \delta(\delta - 1),$$

then by comparing Eq. (12) with Eq. (7), thus we get

$$\delta = \frac{1}{2} (1 \mp \ell'),$$

where,

$$\ell' = \sqrt{8\mu C + 4 \left( L + \frac{N-2}{2} \right)^2},$$

the positive sign is taken of $\delta$ that solutions of the radial wave function at the boundaries well behaved.
Now, we calculate the energy eigenvalue equation for the ground state. By using Eqs. (7) and (12), we get:

\[ E_{0l} = \frac{1}{2\mu} \left[ \sqrt{\frac{a(a+1)}{15d^4}}(2 + l') - \frac{5b^2}{3a(a+1)d^2} \right] \],

(15)

also we obtain ground state function from the following equation

\[ \psi_{0l}(r) = \frac{1}{r}R_{0l}(r). \]

(16)

By substituting \( \alpha, \beta \) and \( \delta \) from Eqs. (12) and (13) respectively. We finally obtain the following ground state wave function:

\[ \psi_{0l}(r) = N_{0l}r^{l+1/2}e^{-\frac{a(a+1)r^2}{60d^2}}\left( -\sqrt{\frac{a(a+1)}{3a(a+1)d^2}} \right). \]

(17)

Secondly, the first node \((n = 1)\), using \( f_1(r) = (r - \alpha_1^{(1)}) \) and \( g_1(r) \) from Eqs. (9) and (10) is written as follows

\[-a_1r^2 + b_1r - \frac{C_1}{r} + \frac{d_1}{r^2} - \epsilon_{11} = -\alpha[1 + 2(\delta + 1)] + \alpha^2r^2 + 2\beta(\delta - 1) + \frac{\delta(\delta - 1)}{r^2} \]

\[ + 2\alpha\beta r - \frac{2\beta\delta}{r} - \frac{2[\beta + \alpha\alpha_1^{(1)}]}{r - \alpha_1^{(1)}} + \frac{2\delta}{r(r - \alpha_1^{(1)})}, \]

(18)

multipliying both sides of Eq. (18) by \((r - \alpha_1^{(1)})\), the relationship between the potential parameters and coefficient \( \alpha, \beta, \delta, \) and \( \alpha_1^{(1)} \) are

\[-a_1r^2 + b_1r - \frac{C_1}{r} + \frac{d_1}{r^2} - \epsilon_{11} = \alpha^2r^2 + 2\alpha\beta r + \beta^2 - \alpha[1 + 2(\delta + 1)]

\[ - \frac{2\beta(\delta + 1)}{r} + \frac{\alpha\alpha_1^{(1)}}{r} + \frac{\delta(\delta - 1)}{r^2}, \]

(19)

therefore for the relations between the potential parameters and the coefficient \( \alpha, \beta, \delta, \) and \( \alpha_1^{(1)} \) are calculated, as by comparing both sides of Eq. (19), then \( \alpha = \sqrt{a_1}, \beta = \frac{b_1}{2\sqrt{a_1}}, a_1 > 0, \delta = \frac{1}{2}(1 + l'), \)

\[ \epsilon_{11} = \alpha[1 + 2(\delta + 1)] - \beta^2, C_1 - 2\beta(\delta + 1) = 2\alpha\alpha_1^{(1)}, \]

and

\[ d_1 = -\delta(\delta - 1) - C_1. \]

(20)

Hence, the energy eigenvalue is

\[ E_{11} = \frac{1}{2\mu} \left[ \sqrt{\frac{a(a+1)}{15d^4}}(4 + l') - \frac{5b^2}{3a(a+1)d^2} \right]. \]

(21)

The corresponding wave function

\[ \psi_{11}(r) = N_{11}(r - \alpha_1^{(1)})r^{l+1/2}e^{-\frac{1}{2} \frac{a(a+1)r^2}{15d^2}}\left( -\frac{1}{2} \sqrt{\frac{a(a+1)}{3a(a+1)d^2}} - \frac{b}{3a(\alpha+1)r} \right). \]

(22)

Following the analytical procedures node \((n = 2)\) with \( f_2(r) = (r - \alpha_1^{(2)})(r - \alpha_2^{(2)}) \) and \( g_1(r) \) as defined in Eq. (9), at \( n = 2, l = 1, \) thus
\[ -a_1 r^2 + b_1 r - \frac{C_1}{r} + \frac{d_1}{r^2} - \varepsilon_{21} = -\alpha [1 + 2(\delta + 1)] + \alpha^2 r^2 + \beta^2 + \frac{\delta(\delta - 1)}{r^2} + 2\alpha\beta r - \frac{2\beta \delta}{r} - 2 + 2 \left( -\alpha r + \beta + \frac{\delta}{2} \right) \left( 2r - \alpha_1^{(2)} - \alpha_2^{(2)} \right) \left( r - \alpha_1^{(2)} \right) \left( r - \alpha_2^{(2)} \right) , \quad (23) \]

by multiplying both sides of Eq. (23), by \((r - \alpha_1^{(2)}) (r - \alpha_2^{(2)})\),

we get

\[ -a_1 r^2 + b_1 r - \frac{C_1}{r} + \frac{d_1}{r^2} - \varepsilon_{21} = \alpha^2 r^2 + 2\alpha\beta r + \beta^2 - \alpha [1 + 2(\delta + 2)] \]

\[ -2\beta (\delta + 2) + 2\alpha (\alpha_1^{(2)} + \alpha_2^{(2)}) \]

\[ + \frac{\delta(\delta - 1)}{r^2} , \quad (24) \]

then the relationship between the potential parameters coefficients \(\alpha, \beta, \delta, \alpha_1^{(2)},\) and \(\alpha_2^{(2)}\) are \(\alpha = \sqrt{a_1}, \beta = \frac{b_1}{2\sqrt{a_1}}, a_1 > 0,\)

\[ \delta = \frac{1}{2} (1 + l^\prime), \varepsilon_{21} = \alpha [1 + 2(\delta + 2)] - \beta^2, \]

\[ C_1 - 2 \beta (\delta + 2) = 2\alpha \sum_{i=1}^{2} \alpha_i^{(2)} . \quad (25) \]

Hence, the energy eigenvalue

\[ E_{21} = \frac{1}{2\mu} \left[ \sqrt{\frac{a(a+1)}{15} d^3} (6 + l^\prime) - \frac{5b^2}{3\alpha (a+1)} \right] . \quad (26) \]

The corresponding wave function is

\[ \Psi_{21}(r) = N_{21} \prod_{i=1}^{2} \left( r - \alpha_i^{(2)} \right) r^{\left( -1 + l^\prime \right) / 2} e^{-\frac{1}{2} \sqrt{\frac{a(a+1)}{15} d^3} r^2 - \frac{b_1 r}{2\sqrt{a_1}}}. \quad (27) \]

We can repeat this iteration procedure several times to write the exact energy formula for the TRM potential with any arbitrary \(n\) state as

\[ E_{nl} = \frac{1}{2\mu} \left[ \sqrt{\frac{a(a+1)}{15} d^3} (2 + 2n + l^\prime) - \frac{5b^2}{3\alpha (a+1) d^2} \right] . \quad (28) \]

The corresponding wave function for any \(n\) state is

\[ \Psi_{nl}(r) = N_{nl} \prod_{i=1}^{n} \left( r - \alpha_i^{(n)} \right) r^{\left( -1 + l^\prime \right) / 2} e^{-\frac{1}{2} \sqrt{\frac{a(a+1)}{15} d^3} r^2 - \frac{b_1 r}{2\sqrt{a_1}}}. \quad (29) \]

Then, the relationship between the potential parameters coefficients \(\alpha, \beta, \delta, \alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_3^{(n)}\) are \(\alpha = \sqrt{a_1}, \beta = \frac{b_1}{2\sqrt{a_1}}, a_1 > 0,\)

\[ \delta = \frac{1}{2} (1 + l^\prime), \varepsilon_{nl} = \alpha [1 + 2(\delta + 3)] - \beta^2, \]

\[ C_1 - 2 \beta (\delta + n) = 2\alpha \sum_{i=1}^{n} \alpha_i^{(n)}, n = 1, 2, 3, \quad (30) \]

the coefficient \(\alpha_1^{(3)}, \alpha_2^{(3)},\) and \(\alpha_3^{(3)}\) are found from the constraint relation,

\[ \alpha \sum_{i=1}^{3} \alpha_i^{(3)} + 2 + \beta \sum_{i=1}^{3} \alpha_i^{(3)} - 3(\delta + 1) = 0. \quad (31) \]
Table 1 Mass spectra of charmonium (in GeV) \((m_c = 1.275 \text{ GeV} \ [16], \mu = 0.638 \text{ GeV}, d= 1.687 \text{ GeV}^{-1}, b = 3.99 \text{ GeV}^2, a = 2.4)\)

| State | Present paper | [17] | [18] | [19] | [20] | [21] | \(N = 4\) | Exp. [2] |
|-------|---------------|------|------|------|------|------|-----------|---------|
| 1S    | 3.239         | 3.078| 3.096| 3.096| 3.078| 3.096| 3.360     | 3.096   |
| 1P    | 3.372         | 3.415| 3.433| 3.433| 3.415| 3.255| 3.673     | 3.525   |
| 2S    | 3.646         | 4.187| 3.686| 3.686| 3.581| 3.686| 3.698     | 3.649   |
| 1D    | 3.604         | 3.752| 3.767| 3.770| 3.749| 3.704| 3.895     | 3.769   |
| 2P    | 3.779         | 4.143| 3.910| 4.023| 3.917| 3.779| 3.827     | 3.900   |
| 3S    | 4.052         | 5.297| 3.984| 4.040| 4.085| 4.040| 3.966     | 4.040   |
| 4S    | 4.459         | 6.407| 4.150| 4.355| 4.589| 4.269| 3.986     | 4.415   |
| Total error | 0.05788 | 0.9513 | 0.11065 | 0.0501 | 0.11152 | 0.19012 |

3 Results and Discussion

3.1 Quarkonium Masses

In this section, we calculate spectra of the heavy quarkonium system such as charmonium and bottomonium that have the quark and antiquark flavor, the mass of quarkonium is calculated in 3-dimensional space \((N = 3)\). So we apply the following relation as in Refs. [14,15].

\[
M = 2m + E_{nl}, \tag{32}
\]

where, \(m\) is bare quark mass for quarkonium. By using Eq. (28), we write Eq. (32) as follows

\[
M = 2m + \frac{1}{2\mu} \left[ \sqrt{\frac{a(a+1)}{15}d^4} \left(2 + 2n + \frac{i}{l} \right) - \frac{5b^2}{3a(a+1)d^2} \right], \tag{33}
\]

In Table 1, we calculated mass spectra of charmonium for states from 1S to 4S, this by using Eq. (33). The free parameters of the present calculations are \(a\), \(b\) and \(d\) are fitted with experimental data. Also, bare quark masses are obtained from Ref. [16]. We note calculations of masses of charmonium are in a good agreement with experimental data and are improved in comparison with Refs. [17–21] by calculating total error in comparison with experimental data. In Table 1, we note the spectra masses of bottomonium from states 1S to 4S agree with experimental data, and the present calculations are improved in comparison with Refs. [17,20] in which the total error is reduced in comparison with these works. In Table 3, we calculate mass spectra of meson \(b\bar{c}\) mesons from using \(2m = m_b + m_c\) part of Eq. (33) for states from 1S to 3S. We find that the 1S and 2S states close with experimental data, but the values of the experimental data for other states are not available. We note that the present calculations of the \(b\bar{c}\) mass improved in comparison with Refs. [22–24] by calculating total error for these works. In Table 4, we calculate mass spectra of c\(\bar{s}\) mesons from 1S to 1D state, by using \(2m = m_c + m_s\) part of Eq. (33). 1S and 2S close with experimental data. Other states are improved in comparison with power potential, screened potential, and phenomenological potential [24] by calculating total error for each potential. Thus, we deduce that TRM gives good results for charmonium, bottomonium, \(b\bar{c}\), and c\(\bar{s}\) meson in comparison with experimental data and improved in comparison with recent works.

The higher dimensional space plays an important role in particle physics. Based on superstring theory such as Ref. [3] and references therein, the number of dimensions in the universe restricted to ten spatial dimensions and one for the time dimension. If the amount of spatial dimensional increases more than ten, the universe unstable and collapse. In present work, we note that the energy eigenvalues increase with increasing the dimensionality number \(N\) for every state as in Tables 1, 2, 3 and 4. Therefore, the spectra of states of masses increase with increasing dimensionality number. This lead to the limitation of non-relativistic quark models when we applied on quark systems.

3.2 Thermodynamics properties

In Fig. 2, we note that partition function \((Z)\) decreases with increasing \(\beta\). The range of \(\beta = 4.0\) to 5.88 Mev\(^{-1}\) corresponding to \(T = 0.25\) to 0.170 GeV which represents the range of temperature above the critical temperature. In Ref. [24], the authors studied the thermodynamic properties for diatomic molecules in the
Table 2 Mass spectra of bottomonium (in GeV) ($m_b = 4.18$ GeV [16], $\mu = 2.09$ GeV, $d = 0.96$ GeV$^{-1}$, $b = 3.9$ GeV$^2$, $a = 2.3$)

| State | Present paper | [17] | [20] | [18] | [25] | $N = 4$ | Exp. [16] |
|-------|---------------|------|------|------|------|--------|-----------|
| 1S    | 9.495         | 9.510| 9.510| 9.460| 9.460| 9.610  | 9.444     |
| 1P    | 9.657         | 9.862| 9.862| 9.840| 9.811| 10.022 | 9.9       |
| 2S    | 10.023        | 10.627| 10.038| 10.023| 10.023| 10.072 | 10.023    |
| 1D    | 10.161        | 10.214| 10.214| 10.140| 10.161| 10.205 | 10.161    |
| 2P    | 10.26         | 10.944| 10.390| 10.160| 10.374| 10.269 | 10.26     |
| 3S    | 10.355        | 11.726| 10.566| 10.280| 10.355| 10.306 | 10.355    |
| 4S    | 10.579        | 12.834| 11.094| 10.420| 10.655| 10.344 | 10.579    |
| Total error | 0.09612 | 0.488529| 0.09777| 0.08207| 0.02897|

Table 3 Mass spectra of $b\bar{c}$ meson (in GeV) ($m_b = 4.18$ GeV, $m_c = 1.275$ GeV, $\mu = 0.976$ GeV, $d = 0.949$, $a = 1.699$, $b = 2.465$ GeV$^2$)

| State | Present paper | [22] | [23] | [24] | $N = 4$ | Exp. [26] |
|-------|---------------|------|------|------|--------|-----------|
| 1S    | 6.268         | 6.349| 6.264| 6.270| 6.355  | 6.274     |
| 1P    | 6.529         | 6.715| 6.700| 6.699| 6.883  | –         |
| 2S    | 6.895         | 6.821| 6.856| 6.835| 6.878  | 6.871     |
| 2P    | 7.156         | 7.102| 7.108| 7.091| 7.161  | –         |
| 3S    | 7.522         | 7.175| 7.244| 7.193| 8.035  | –         |
| Total error | 0.00444 | 0.01922| 0.00377| 0.0087|

Table 4 Mass spectra of $c\bar{s}$ meson in (GeV) ($m_c = 1.275$, $m_s = 0.419$) GeV, $\mu = 0.315$ GeV, $d = 2.132$ GeV$^{-1}$, $a = 1.5b = 2.23$ GeV$^2$)

| State | Present paper | Power | Screened | Phenomenological | $N = 4$ | Exp. [26] |
|-------|---------------|-------|----------|------------------|--------|-----------|
| 1S    | 1.969         | 1.9724| 1.9685   | 1.9683           | 2.3001 | 1.968     |
| 1P    | 2.126         | 2.540 | 2.7485   | 2.5665           | 2.742  | 2.112     |
| 2S    | 2.318         | 2.6506| 2.8385   | 2.8155           | 2.797  | 2.317     |
| 3S    | 2.667         | 2.9691| 3.2537   | 3.2805           | 2.967  | 2.700     |
| 1D    | 2.374         | –     | –        | –                | 3.934  | 2.318     |
| Total error | 0.04392 | 0.44803| 0.73041 | 0.64471|

Fig. 2 Partition function is plotted as a function of $\beta$

relativistic models using NU method. They found that the partition function decreases with increasing $\beta$. In addition, in Ref. [27], the authors employed oscillator plus inverse square potential in the SE and found the partition function decreases with increasing $\beta$. Therefore, we found the present behavior is a qualitative
agreement with these works. In Fig. 3, we note that the specific heat (C) decreases with increasing of $\beta$. We found a qualitative agreement with these works [18,24,27].

In Fig. 4, we note that the free energy (F) increases with increasing $\beta$. In Ref. [4], the free energy increases with decreasing of $\beta$. In Refs. [16,28–32], the free energy increases with decreasing temperature. Therefore, the present result is an agreement with Refs. [28–31,33,34]. In Fig. 5, we note that the entropy (S) decreases with increasing $\beta$. This finding is an agreement with Refs. [24,27,33–35], in which the entropy increases with increasing temperature for the diatomic molecules.

In Fig. 6, we note that internal mean energy (U) decreases with increasing of $\beta$. In Ref. [4], the authors found that U increases with decreasing of $\beta$. This observation is noted in Ref. [33] for the diatomic molecules HCl and H2. In addition, In Refs. [33,34], U is plotted as a function of the dimensionless temperature and the authors found that U increases with increasing of temperature. Thus, the present work has the same conclusion for the charm medium.

4 Summary and Conclusion

In this work, the trigonometric Rosen–Morse potential is suggested as an effective potential for quark–antiquark interaction, in which the potential satisfied the features of QCD. By using the analytical exact iteration method (AEIM), the eigenvalues of energy and corresponding wave functions are obtained in the N-dimensional radial Schrodinger equation. The present results are applied for calculating the mass of heavy mesons such as
Fig. 5 Entropy $S$ is plotted as a function $\beta$

Fig. 6 Internal energy $U$ is plotted as a function $\beta$

charmion c $\bar{c}$, bottomion $b$ $\bar{b}$, $b$ $\bar{c}$, and $c$ $\bar{s}$ mesons and thermodynamic properties such as the internal energy, the specific heat, the free energy, and the entropy.

I—At $N = 3$, the heavy meson spectra masses are calculated. We obtained the total errors as 0.05788 for charmion, 0.09612 for bottomion, 0.00444 for the $b$ $\bar{c}$ meson, and 0.04392 for the $c$ $\bar{s}$ meson in comparison with experimental data. For $N > 3$, we found that the binding energy increases with increasing dimensional number space which leads to the limitation of non-relativistic quark models.

II—At $N = 3$, thermodynamic properties are calculated such as the internal energy, the free energy, the specific heat, and the entropy. We found thermodynamic properties for the charm quark plasma are a qualitative agreement with Refs. [31,33]. In these works, thermodynamic properties the light quark, the strange quark, and natural particles are studied. We conclude that the trigonometric Rosen–Morse potential gives good results in comparison with other recent works and the present results are good agreement with experimental data. In addition, the TRM potential provides satisfied results for thermodynamic properties for charm medium.

Appendix A

In this appendix, thermodynamics properties of the Trigonometric Rosen–Morse potential are studied, the partition function is given $Z = \sum_{n=0}^{\infty} e^{-\beta E}$, where $\beta = \frac{1}{kT}$, $K$ is the Boltzmann constant as in Ref. [1]
by substituting Eq. (28), we obtain

\[ Z(\beta) = \frac{e^{-\beta(C_1+C_2-C_3)}}{1 - e^{-\beta C_1}} \]  

(A2)

where,

\[ C_1 = \frac{1}{\mu} \sqrt{\frac{a(a+1)}{15d^4}}, \quad C_2 = \frac{l'}{2\mu} \sqrt{\frac{a(a+1)}{15d^4}}, \quad C_3 = \frac{5}{6\mu} \frac{b^2}{a(a+1)d^2}, \quad l' = \sqrt{8\mu C+4 \left( \frac{N-2}{2} \right)^2}. \]  

(A3)

Mean energy \( U \)

\[ U(\beta) = -\frac{d}{d\beta} \ln Z(\beta), \]  

(A4)

\[ U(\beta) = -e^{\beta(C_1+C_2-C_3)} \left( 1 - e^{-\beta C_1} \right) \left( \frac{-e^{-\beta C_1 - \beta(C_1+C_2-C_3)} C_1}{(1 - e^{-\beta C_1})^2} + \frac{e^{-\beta(C_1+C_2-C_3)} (-C_1 - C_2 + C_3)}{1 - e^{-\beta C_1}} \right). \]  

(A5)

Specific heat \( C \)

\[ C(\beta) = \frac{dU}{dT} = -K\beta^2 \frac{dU}{d\beta^2}. \]  

(A6)

\[ C(\beta) = -K\beta^2 \left( -e^{-\beta C_1 + \beta(C_1+C_2-C_3)} C_1 \left( \frac{-e^{-\beta C_1 - \beta(C_1+C_2-C_3)} C_1}{(1 - e^{-\beta C_1})^2} + \frac{e^{-\beta(C_1+C_2-C_3)} (-C_1 - C_2 + C_3)}{1 - e^{-\beta C_1}} \right) 
- e^{\beta(C_1+C_2-C_3)} (1 - e^{-\beta C_1})(C_1 + C_2 - C_3) \left( \frac{-e^{-\beta C_1 - \beta(C_1+C_2-C_3)} C_1}{(1 - e^{-\beta C_1})^2} \right) 
+ \frac{e^{-\beta(C_1+C_2-C_3)} (-C_1 - C_2 + C_3)}{1 - e^{-\beta C_1}} \right) 
- e^{\beta(C_1+C_2-C_3)} (1 - e^{-\beta C_1}) \left( \frac{2e^{-2\beta C_1 - \beta(C_1+C_2-C_3)} C_1^2}{(1 - e^{-\beta C_1})^3} \right) 
- e^{-\beta C_1 - \beta(C_1+C_2-C_3)} C_1 (-2C_1 - C_2 + C_3) \left( \frac{-e^{-\beta C_1 - \beta(C_1+C_2-C_3)} C_1}{(1 - e^{-\beta C_1})^2} \right) 
+ \frac{e^{-\beta(C_1+C_2-C_3)} (-C_1 - C_2 + C_3)}{1 - e^{-\beta C_1}} \right) \right). \]  

(A7)

Free energy

\[ F(\beta) = -KT \ln Z(\beta), \]

\[ F(\beta) = -\frac{\log \left[ \frac{e^{-\beta(C_1+C_2-C_3)}}{1 - e^{-\beta(C_1)}} \right]}{\beta} \]  

(A8)
Entropy

\[ S(\beta) = K \ln Z(\beta) - K \beta \frac{\partial}{\partial \beta} \ln Z(\beta). \]

\[ S(\beta) = K \log \left[ \frac{e^{-\beta(C_1+C_2-C_3)}}{1 - e^{-\beta C_1}} \right] - e^{\beta(C_1+C_2-C_3)}(1 - e^{-\beta C_1})K\beta \]
\[ \left( -e^{-\beta C_1} - (C_1 + C_2 - C_3)C_1 \right) + \frac{e^{\beta(C_1+C_2-C_3)}(-C_1 - C_2 + C_3)}{1 - e^{-\beta C_1}} \right). \quad (A9) \]

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