Fourier-Mukai Transform and Mirror Symmetry for D-Branes on Elliptic Calabi-Yau

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Fibrewise T-duality (Fourier-Mukai transform) for D-branes on an elliptic Calabi-Yau three-fold \(X\) is seen to have an expected adiabatic form for its induced cohomology operation only when an appropriately twisted operation resp. twisted charge is defined. Some differences with the case of \(K3\) as well as connections with the spectral cover construction for bundles on \(X\) are pointed out. In the context of mirror symmetry Kontsevich’s association of line bundle twists (resp. a certain ’diagonal’ operation) with monodromies (esp. the conifold monodromy) is made explicit and checked for two example models. Interpreting this association as a relation between FM transforms and monodromies, we express the fibrewise FM transform through known monodromies. The operation of fibrewise duality as well as the notion of a certain index relevant to the computation of the moduli space of the bundle is transported to the sLag side. Finally the moduli space for D4-branes and its behaviour under the FM transform is considered with an application to the spectral cover.
1. Introduction and summary

This paper treats some connections between four different, although related topics: D-branes, mirror symmetry, elliptic Calabi-Yau and Fourier-Mukai transform.

The last year saw an intense study on BPS $D$-branes in type II string theories on a Calabi-Yau manifold (cf. [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11]), focussing on the behaviour of the D-brane spectrum under variation in the Calabi-Yau moduli space (related to the phenomenon of marginal stability) as well as on the relations at a special point in moduli space (such as the relation with boundary conformal field theory).

This development has some close connections with a reformulation of mirror symmetry given by Kontsevich on the one hand and Strominger/Yau/Zaslow and Vafa on the other which brings supersymmetric D-branes on both sides of the mirror correspondence into the play (cf. [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24]). This relates bundles or better sheaves (think of these here as bundles supported on holomorphic subvarieties) on a Calabi-Yau $X$ in type IIA string theory with special Lagrangian submanifolds (with an $U(1)$ bundle over them) in the mirror Calabi-Yau $Y$ (or more precisely the derived category $D(X)$ of the category of sheaves on $X$ with Fukaya’s $A_\infty$ category of Lagrangian submanifolds of $Y$); cohomological invariants relate then the $H^{\text{even}}(X)$ of the bundle side and $H^3(Y)$ on the sLag side. Furthermore monodromies around divisors $D$ in moduli space where some such even-dim. cycle (say a divisor $D$) vanish correspond conjecturally to twisting with the line bundle $L_D$ associated to the divisor $D$ and a similar more complicated relation, which generalizes the twisting with the help of the concept of a Fourier-Mukai transform, relates the conifold monodromy.

Now for the description of bundles on a Calabi-Yau it was taken a great step forward when Friedman/Morgan/Witten made explicit (via spectral covers) a construction of bundles for the case of an elliptically fibered Calabi-Yau. This was then intended as compactification space for the heterotic string and allowed detailed studies on such issues as the moduli space, brane impurities, relation with F-theory and model building (cf. [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24]).

Note that the T-duality on the $T^3$ fibre in that construction is not easily related to the T-duality on the holomorphic elliptic fibre considered later in the framework of the fibrewise FM transform as the holomorphic $T^2$ is not contained in the $T^3$. 

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This class of elliptic Calabi-Yau's is also especially interesting as it allows for a version of the Fourier-Mukai transform of a bundle $V$ on non-toroidal spaces which in contrast to earlier transforms in such cases (on $K3$ say) keeps completely the idea of using the duality on a torus by building a fibrewise FM transform (cf. [46],[47],[48],[49],[50],[51],[52],[53]).

This is what the physicist would call T-duality on the fibre and operates in an interesting way on the spectrum of D-branes on the one hand and has on the other hand a close connection with the spectral cover construction (a relation of an FM transform to Kontsevich’s version of mirror symmetry was already mentioned).

A number of related points in the aforementioned web of connections will be studied in the paper. In section 2 the cohomological invariants of a bundle and its fibrewise dual are computed and it is shown how by using an appropriately twisted charge (in analogy with the $\sqrt{Td(X)}$ twist of the Chern character to get the Mukai vector in the usual full FM transform) the adiabatic character of the operation is confirmed. I.e. by using a decomposition of the cohomology into base and fibre parts the operation of the fibrewise duality on the cohomology will be seen to take the form one gets from an adiabatic extension of the same operation on the cohomology of a $T^2$ (fulfilling the expectations from the interpretation as T-duality on D-branes); a simpler variant of the twisting idea is seen also to be necessary for the case of $K3$ where, in accordance with earlier treatments of that case in the literature, still an operation relating the untwisted Chern characters themselves can be given by considering a natural twist in the duality functor itself; after these cases of Calabi-Yau’s of complex dimension one and two our presentation follows the line of ascending complexity and demonstrates how for a Calabi-Yau three-fold one has to use the twisted charge definition (which also naturally incorporates via an reinterpretation the findings for $K3$) or an adapted version of Mukai’s $f$-map (which uses the usual Mukai vector as charge but a slightly twisted operation).

In section 3 we treat some well-studied two parameter CY (represented by hypersurfaces of degree 8 and 12 in weighted projective space) and make explicit Kontsevich’s association of monodromies with twists by line bundles resp. a more complicated operation

\footnote{with (appendix) and without (main body of the paper) use of the spectral cover construction}
for the conifold monodromy.

In section 4 a connection between the two main themes of the foregoing chapters is given: the fibrewise FM transform is given its place in Kontsevich’s general association of FM transforms with monodromies. For this note that the relative FM transform on elliptic fibrations is an autoequivalence $S:D(X) \to D(X)$ of the derived category, its inverse functor being (up to a shift) the functor $\hat{S}$ described in Section 2.2. Then $S$ should correspond in the mirror to a monodromy. As part of Kontsevich’s generalization of mirror symmetry can be formulated without the mirror we can test this already on $X$.

In section 5 we point to another connection between our fibrewise FM transform and mirror symmetry. Namely it is pointed out how via the mirror identification one can transport operation of fibrewise FM on cohomology studied in section 2 to the mirror side and get there a corresponding operation on the middle cohomology (this is basically just a careful comparison of base choices as one has to relate natural symplectic bases of periods with the decomposition into base and fibre parts used to make the fibrewise FM transform most easily visible). It is then of course a very interesting question whether that operation on $H^3(Y)$ transported from $H^{even}(X)$ actually can be derived from a certain operation already on the space level as it can be done on the original bundle side. As a second instance of that transport philosophy we point out how the more precise information one has on the moduli space of bundles in the elliptic set-up (as given essentially by a certain index) can be transported to a ”sLag index” on the sLag side which should give more structure to conjectured relations such as the one between $h^1(EndV)$ and $h^1(Q)$ (Q the sLag 3-cycle).

Finally in section 6 again some moduli space questions in connection with $h^1(EndV)$

\[\text{To be precise, one should use here the term quasi-inverse instead of inverse and the equality signs in }\hat{S} \circ S(\cdot) = (\cdot)[-1] \text{ and } S \circ \hat{S}(\cdot) = (\cdot)[-1] \text{ have to be understood as natural functor isomorphisms (see pag. 71 of \cite{54}; there is also given a quite illuminating discussion about how inappropriate the notion of isomorphism of categories is).}\]

\[\text{The twists with the line bundle associated to a divisor are related to the monodromies around the locus in moduli space where the divisor vanishes, but this already in the Kähler moduli space of type IIA, i.e. } H^{even}(X) - \text{ the identification of the period monodromy in the complex structure moduli space of the mirror CY } Y \text{ (related to } H^3(Y')) \text{ are then reached by combining the first identification with the mirror map which identifies the monodromies}\]
on the bundle side are treated; this shows how these quantities are related to corresponding expressions in the spectral cover construction; as that construction relates D6-brane (the bundle $V$) to a D4-brane (its spectral cover divisor with a certain line bundle over it) this is considered from the general perspective of studying the action of the fibrewise FM transform relating D6-branes to D4-branes on $X$.

The appendix recalls the computation of cohomological invariants of a bundle $V$ and its fibrewise dual in the spectral cover construction and shows why the twist of the duality functor alone, which was successful in the $K3$ case to get the ‘adiabatic’ transformation matrix, is insufficient in the three-fold case.

Let us state the technical framework the paper will be moving in. We will assume the elliptically fibered CY $X$ has a smooth Weierstrass model, having singular $I_1$ fibers over a one-dimensional locus in the base, which furthermore has a section $\sigma$. All sheaves are coherent. Quite often we will deal with bundles which are fibrewise of degree zero. They will be in addition semistable on the generic fibre in case we refer to the spectral cover construction. For those sheaves the fibrewise FM transforms preserves fibrewise semistability. The preservation under fibrewise FM of the absolute stability (with respect to some particular kind of polarization in $X$) is only known in the two dimensional case (cf. [48],[53],[55]). The natural normalization of the Poincaré bundle we use will be recalled in section 2. As usual in consideration of the FM transform the ‘dual’ CY $\tilde{X}$, given as a compactification of the Jacobian of the original fibration (Jacobian fibrewise), will be identified with $X$ when appropriate. Concerning the notion of a Dp-brane which has its p-dimensional spatial world-volume wrapped on a holomorphic p-cycle in $X$ the mathematical oriented reader should think of a bundle concentrated on that p-cycle, i.e. a sheaf on $X$ with support on this cycle (cf. subsection 2.5 and the presentations [6],[2],[3]).

2. Fibrewise Fourier-Mukai transform (T-duality) on elliptic Calabi-Yau

We consider an $SU(n)$ bundle $V$ over an elliptically fibered $X$ of $c_1(V) = 0$ or equivalently $n$ D6-branes wrapped over $X$ with induced lower-dimensional D2-and D0-brane charges (D2i-charges meaning here for now just $ch_i(V)$). The case of elliptically fibered
Calabi-Yau three-folds $X$ will have a double advantage: one can describe $V$ explicitly via the associated spectral cover $\mathbb{P}$ and has furthermore the action of T-duality on the elliptic fibre on the bundles. So our procedure in this section will be first to make the bundle description more explicit using the spectral cover method and then to describe the change in $ch(V)$ induced by fibrewise T-duality, an operation which should be mirrored on the sLag side by a corresponding operation.

The T-duality on the $T^2$ fiber maps in general (the subscripts indicate whether fibre $F$ or base $B$ is contained (resp. contains) the wrapped world-volume)

$$
\begin{align*}
D6 & \rightarrow \tilde{D}4_B \\
D4_B & \rightarrow \tilde{D}6 , \quad D4_F & \rightarrow \tilde{D}2_B \\
D2_B & \rightarrow \tilde{D}4_{\tilde{F}} , \quad D2_F & \rightarrow \tilde{D}0 \\
D0 & \rightarrow \tilde{D}2_{\tilde{F}}
\end{align*}
$$

(2.1)

Our goal in this section is to understand the operation of fibrewise T-duality on the cohomological data representing the bundle $V$ and its Fourier-Mukai (FM) dual $\tilde{V}$, i.e. we will mirror the mentioned D-brane relations as a map between $ch(V)$ and $ch(\tilde{V})$ (in a first approximation; the modification needed to make this work will be made precise along the way). For this we will assume the following decomposition of the vertical cohomology (the $C$ in $H^4(X)$ resp. $H^6(X)$ are $H^4(B)$ resp. $\sigma H^4(B)$; the latter $\sigma$ will be often suppressed)

$$
\begin{align*}
H^0(X) & = C \\
\oplus \\
H^2(X) & = C\sigma \oplus H^2(B) \\
\oplus \\
H^4(X) & = H^2(B)\sigma \oplus C \\
\oplus \\
H^6(X) & = C
\end{align*}
$$

(2.2)

Then essentially the six entries of the Chern character vector in our decomposition are
pairwise interchanged and the transformation has the block-diagonal form

\[
Q = \begin{pmatrix}
0 & * & & & & \\
* & 0 & & & & \\
& & 0 & * & & \\
& & * & 0 & & \\
& & & & 0 & * \\
& & & & * & 0
\end{pmatrix} \tilde{Q}
\]

This has the following interpretation: the fibrewise Fourier-Mukai transform (fibrewise T-duality) is given here just by adiabatic extension of the T-duality on the fibre; this adiabatic relation should be reflected by a corresponding adiabatic relation between the matrices representing the operation on the cohomology. Now for the case of an one-dimensional Calabi-Yau consisting just of an elliptic curve (representing the fibre $T^2$) the matrix is [10]

\[
A = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\]

So one would like to see that the actual form of the transformation matrix induced by the mathematical operation of fibrewise Fourier-Mukai transformation on the bundle (which represents the fibrewise T-duality process) is given by (in the order of arrangement given in the decomposition above)

\[
Q = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -1 & 0
\end{pmatrix} \tilde{Q}
\]

thereby then confirming the 'adiabatic' interpretation of the Fourier-Mukai transform as

\[
\begin{pmatrix}
0_3 & 1_3 \\
-1_3 & 0_3
\end{pmatrix}
\]

with entries now consisting of $3 \times 3$ blocks corresponding to

\[
H^*(X) = \sigma H^*(B) \oplus \pi^* H^*(B)
\]

A major part of the discussion will be concerned with the question whether one has to transform just the Chern characters $ch(V)$ and $ch(\tilde{V})$ themselves or to use appropriately twisted charges $Q, \tilde{Q}$ (like in the usual Fourier-Mukai transformation where the charges have twists with $\sqrt{Td(X)}$) and similarly the related question whether the transformation process itself has to be somewhat twisted.
2.1. Description of the bundle by spectral cover data

Let us first recall the spectral cover description. The $SU(n)$ bundle $V$ on $X$ of $c_1(V) = 0$ decomposes on the typical fibre $E$ (where $V$ is assumed to be semistable) as a sum $\oplus_i L_i$ of line bundles of degree zero and each of the $L_i$ corresponds (having chosen the distinguished reference point $p$ as origin) to a welldefined point $Q_i$ on $E$ (these points sum up to zero as $\det(V) = 1$). When the reference point is globalized by the section $\sigma$ the variation of the $n$ points in a fibre lead to a hypersurface $i : C \hookrightarrow X$, a ramified $n$-fold cover (the ‘spectral cover’) of $B$. The equation $s = 0$ of $C$ involving the section $s$ of $O(\sigma)^n$ can in the process of globalization still be twisted by a line bundle $M$ over $B$ of $c_1(M) = \eta$, i.e. $S$ can be actually a section of $O(\sigma)^n \otimes M$ and the cohomology class of $C$ in $X$ is given by

$$C = n\sigma + \eta$$

Now $V$ will be induced as $V = p_* R$ from a line bundle $R$ over the $n$-fold cover $p : X \times_B C \to X$, i.e. generically will the fibre of $V$ over a point $x \in X$ with the $N$ preimages $\tilde{x}_i$ be given by the sum of the fibre of $R$ at the $\tilde{x}_i$. If one takes for $R$ the global version of $P$ on has indeed that fibrewise $V = p_* P$ as $p_*$ sums up the line bundles which makes $P$ out of the points collected in the fibre $C_b$ of $C$ over $b \in B$, which themselves corresponded to the line bundles summands of $V$ on $E_b$. As the twist by a line bundle $L$ over $C$ leaves the fibrewise isomorphism class unchanged ($L$ being locally trivial along $C$) the construction generalizes to

$$V = p_*(p_C^* L \otimes P)$$

where $p$ and $p_C$ are the projections on the first and second factor of $X \times_B C$

$$\begin{array}{ccc}
X \times_B C & \overset{p_C}{\longrightarrow} & C \\
p \downarrow & & \pi_C \downarrow \\
X & \overset{\pi_1}{\longrightarrow} & B
\end{array}$$

5 The double interpretation of $E$ as pointset $E_1$ resp. parameter space $E_2$ for degree zero line bundles on $E_1$ is formalized by introducing the Poincaré bundle $P$ on $e_1 \times E_2$ which restricts on $E_1 \times Q$ to $L_Q$; actually one uses the symmetrized version $P = O(\Delta - p \times E_2 - E_1 \times p)$. 

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The condition $c_1(V) = 0$ translates to a fixing of $c_1(L)$ in $H^{1,1}(C)$ up to a class in $\ker \pi_* : H^{1,1}(C) \to H^{1,1}(B)$; such a class is known to be of the form $\gamma = \lambda(n\sigma - (\eta - nc_1))$ with $\lambda$ half-integral.

2.2. FM transform

For the description of the FM transform we will instead of working on $X \times_B C$ work on $X \times_B \tilde{X}$

$$
\begin{array}{c}
X \times_B \tilde{X} \\
\downarrow p_1 \\
X \\
\end{array} \xrightarrow{p_2} \xrightarrow{\pi_2} \begin{array}{c}
\tilde{X} \\
\downarrow \pi_2 \\
B \\
\end{array}
$$

(2.9)

where $\tilde{X}$ is the compactified relative Jacobian of $X$. $\tilde{X}$ parameterizes torsion-free rank 1 and degree zero sheaves of the fibres of $X \to B$ and it is actually isomorphic with $X$ (see [17] or [15]). We will then identify $\tilde{X}$ and $X$.

The bundle $V$ is then given by $^6$

$$
V = R^0_{p_1*} (p_2^*(i_*L) \otimes \mathcal{P})
$$

(2.10)

where

$$
\mathcal{P} = \mathcal{O}(\Delta - \sigma \times \tilde{X} - X \times \tilde{\sigma} - c_1(B))
$$

(2.11)

is the Poincaré sheaf normalized to make $\mathcal{P}$ trivial along $\sigma \times \tilde{X}$ and $X \times \tilde{\sigma}$.

Now let us determine the FM-transform. For this we make use of the fact that the representation of $V$ by the $(C, L)$ data already looks in itself like a FM transform; so when we want to describe now the FM transform of $V$ this is practically the double transform of $i_*L$; but it is known that the inverse transform of FM is not precisely FM itself again but a slightly twisted version of the first transform as we shall see later in this section; so only this twisted version would bring us back from $V$ to $i_*L$, or said differently, if we now start on $V$ by making the original FM transform we will get $i_*L$ times the inverse twist.

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6 $\pi_*(c_1(L)) = -\pi_*(c_1(C) - c_1)/2$

7 In what sequel we will identify a bundle with the locally free sheaf of its sections, so that the terms bundle and locally free sheaf are used interchangeable.

8 Here we denote by $\mathcal{O}(\Delta)$ the dual of the ideal sheaf of the diagonal. Neither $\mathcal{O}(\Delta)$ nor $\mathcal{P}$ are line bundles due to the presence of singular fibres, but they are torsion-free and rank 1.
So with $V = p_1^*(p_2^*(i_*L) \otimes \mathcal{P})$ the 'fibrewise dual' bundle is given\(^9\) in terms of the spectral cover data by

\[
\tilde{V} = R^1p_2^*(p_1^*(V) \otimes \mathcal{P}^*)
\]

\[= i_*L \otimes \pi_2^*K_B\] (2.12)

Note that one wants now to show that there exists a matrix $M$ in the block diagonal form as above in (2.5) which relates $V$ to $\tilde{V}$ or $V$ to $i_*L$ (this is not a big difference because the latter option is simply the inverse process as $V$ is the dual of $i_*L$).

Before going on we would like to recall some well-known facts about the FM transform for elliptic fibrations (cf. [47], [46] or [48]) that explain the facts mentioned above.

We define the Fourier-Mukai functors $S^i, i = 0, 1$ by associating with every sheaf $V$ on $X$ the sheaf $S^i(F)$ on $X$ (where $X$ and $\tilde{X}$ are identified)

\[
S^i(V) = R^i p_1^*(p_2^*(V) \otimes \mathcal{P})
\] (2.13)

where $\mathcal{P}$ denotes the Poincaré sheaf (2.11) on the fibre product. It can be also described as (cf. [25])

\[
\mathcal{P} = \mathcal{I}^* \otimes p_1^*\mathcal{O}(-\sigma) \otimes p_2^*\mathcal{O}(-\sigma) \otimes q^*K_B
\] (2.14)

with $q = \pi .p_1 = \pi .p_2$ and $\mathcal{I} = \mathcal{O}(\Delta)^*$ the ideal sheaf of the diagonal immersion $\delta : X \to X \times_B X$.

We can also define the inverse Fourier-Mukai functors $\hat{S}^i, i = 0, 1$ by associating with every sheaf $V$ on $X$ the sheaf

\[
\hat{S}^i(V) = R^i p_2^*(p_1^*(V) \otimes \mathcal{P}^* \otimes q^*K_B^{-1})
\] (2.15)

The relationship between these functors is more neatly stated if we consider the associated functors between the derived categories of complexes of coherent sheaves bounded from

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\(^9\) As mentioned above the logic of the procedure is that this follows from the fact that the inverse FM transform is given by $W = R^1p_2^*(p_1^*V \otimes \mathcal{P}^* \otimes p_2^*\pi_2^*K_B^{-1}) = R^1p_2^*(p_1^*V \otimes \mathcal{P}^*) \otimes \pi_2^*K_B^{-1}$ so that we have (with $W = i_*L$) $W = \tilde{V} \otimes \pi_2^*K_B^{-1}$.

\(^{10}\) This is fibrewise just the fact that $\tilde{V} = \oplus_i (\pi_2)_*(\pi_1^*(\mathcal{L}_{Q_i} \otimes \mathcal{P}^*)) = \oplus_i \mathcal{O}_{Q_i} = i_*1_L$ with the trivial line bundle $1_L$ on $C = \cup\{Q_i\}$, for note that $(\pi_1^*(\mathcal{L}_{Q_i} \otimes \mathcal{P}^*))|_{E \times q \neq E} = \mathcal{O}_E$ for $q \neq Q_i$. 
condition (S defined by $x$

In this case, the vanishing of $S$ of degree zero. Conversely, if $V$ at the “degree zero” position. ˆ

points (may-by counted more than once); $i$ $V$ one cohomology sheaf, which is

or a sheaf $i$ cohomology sheaves of the shifted complex $G$

fibre over $S$ fibre, that is $G$ is the rank one torsion free sheaf of degree 0 on $X$

defined by $x$; a sheaf supported fibrewise by points, as $i$ $L$ where $L$ is a line bundle on a spectral cover $C$ flat of degree $n$ over $B$

In this case, the vanishing of $S^1(V)$ or WIT$_0$ condition, is equivalent to WIT$_0$ on every fibre, that is $S^1(V) = 0 \iff H^1(X, V \otimes L_x) = 0$ for every point $x$ where $X$ is the fibre over $t = \pi_2(x)$, $F_t = F|X_t$ and $L_x$ is the rank one torsion free sheaf of degree 0 on $X$

defined by $x$; a sheaf supported fibrewise by points, as $i$ $L$, is always WIT$_0$. The WIT$_1$

condition $(S^0(V) = 0)$ is not a fibrewise condition. If $V$ is WIT$_1$ on every fibre, (that is, $H^0(X, V \otimes L_x) = 0$ for every point $x$) then $V$ is globally WIT$_1$ and the only FM transform $S^1(V)$ is flat over $B$ (2.11). This happens for a vector bundle $V$ fibrewise semistable of degree zero. Conversely, if $V$ is WIT$_1$ the flatness of $S^1(V)$ is necessary to ensure that

\[ S(\hat{S}(G)) = G[-1], \quad \hat{S}(S(G)) = G[-1] \quad (2.17) \]

Remark: Even when $G$ is a single sheaf $V$, that is, a complex with $V$ at degree 0 and no other terms, $S(V)$ is an object of the derived category, or a complex whose cohomology sheaves are the sheaf FM transforms $S^i(V)$. It is then interesting to know when only one of the FM sheaf transforms is different from zero, the so-called WIT$_i$ condition. This condition is better studied when the sheaf $V$ is flat over $B$ (for instance, a vector bundle or a sheaf $i_*L$ where $L$ is a line bundle on a spectral cover $C$ flat of degree $n$ over $B$

11 For any complex $G$ (or any element in the category) with cohomology sheaves $G^i$, the cohomology sheaves of the shifted complex $G[n]$ are $G[n]^i = G^{i+n}$

12 Concerning the meaning of the -1 shift, consider a complex given by a single sheaf $V$ located at the “degree zero” position. $\hat{S}(S(V)) = V[-1]$ means that the complex $\hat{S}(S(V))$ has only one cohomology sheaf, which is $V$, but located at “degree 1”, $[\hat{S}(S(V))]^1 = V$, $[\hat{S}(S(V))]^i = 0, i \neq 1$. When $S^0(V) = 0$ the complex $S(V)$ reduces to a single sheaf, which is the unique FM transform $S^1(V)$, but located at “degree 1”, that is, $S(V) = S^1(V)[-1]$ and the complex $\hat{S}(S(V)) = \hat{S}(S^1(V))[-1]$ has two cohomology sheaves, one at degree 1 which is $\hat{S}(S^1(V))$, and one at degree 2 which is $\hat{S}(S^1(V))$. So one has $\hat{S}(S^1(V)) = V$, $\hat{S}(S^1(V)) = 0$.

13 We mean that $\pi|_C : C \to B$ is a flat morphism of degree $n$, that is, all its fibres consist of $n$ points (may-by counted more than once); $i : C \to X$ is the embedding.
V is fibrewise WIT\textsubscript{1}. A typical example is a rank n vector bundle V fibrewise of degree zero, which is only semistable on the generic fibre. It is still WIT\textsubscript{1} but fails to be so at those fibres where it is unstable; this reflects the fact that the spectral cover C contains those fibres so that neither C nor $S^1(V) = i_*L$ are flat over B.

Later we will assume $c_1(V) = 0$ or at least that V has degree 0 and is semistable when restricted to the fibre; this is the most important case to our purposes and the bundles given by the spectral cover construction are of this kind.

2.3. K3 case

Let us consider first the two-dimensional case of $X = K3$ and assume here the decomposition

\begin{align*}
H^0(X) &= C \\
\oplus \\
H^2(X) &= C\sigma \oplus C \\
\oplus \\
H^4(X) &= C
\end{align*}

The class of the spectral curve C on which $i_*L$ is supported is $C = n\sigma + kF$.

The Chern characters of $i_*L$ can be obtained using Grothendieck-Riemann-Roch for the embedding $i: C \to \tilde{X}$

$$ch(i_*L)Td(\tilde{X}) = i_*(ch(L)Td(C))$$

one gets

$$ch(i_*L) = (0, C, n)$$

further $ch(V)$ ($V$ is the only FM transform of $i_*L$) is given by

$$ch(V) = (n, 0, -k)$$

Now let us introduce the new functor $T(\cdot) = S(\cdot) \otimes \pi^*K_B^{-1/2} = S(\cdot) \otimes \mathcal{O}(F)$ (and similarly
\( T^i(\cdot) = S^i(\cdot) \otimes \pi^* K_B^{-1/2} = S^i(\cdot) \otimes \mathcal{O}(F) \) so that \( T^0(i_*L) = V \otimes \mathcal{O}(F) \). We get\(^\text{14}\)

\[
ch(T^0(i_*L)) = ch(V)(1 + F) \\
= M \cdot ch(i_*L)
\] (2.22)

where we reach the matrix we wanted

\[
M = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{pmatrix}
\]

The functor \( T \) was introduced \cite{47} because its inverse transform in the sense of (2.17) is the “natural” one: \( T \) is the FM transform w.r.t. the sheaf \( \mathcal{P} = \mathcal{P} \otimes q^* \mathcal{O}(F) \) and its inverse functor \( \hat{T} \) is the FM transform w.r.t. the dual sheaf \( \mathcal{P}^* \). \( S \) does not have this property as its inverse transform is not the FM transform with respect to \( \mathcal{P}^* \), but this twisted by \( q^* K_B^{-1} \). So we just divided this up in two parts of \( K_B^{-1/2} \), distributed among the original and the inverse transform.

2.4. Calabi-Yau threefold case

So far we considered FM transformation of \( V \) with vanishing first Chern class in the context of \cite{25}. Let us now describe the topological invariants of the relative FM transform for a coherent sheaf \( \mathcal{F} \) on an elliptic Calabi-Yau threefold. Therefore we start again from (2.19). Applying GRR for \( p_1 \) we get (\( \mathcal{G} \) can be an object of the derived category)

\[
ch(S(\mathcal{G})) = p_{1*}[p_2^*(\mathcal{G}) \cdot ch(\mathcal{P}) \cdot Td(T_{X/B})]
\] (2.23)

where \( Td(T_{X/B}) = 1 - \frac{1}{2} c_1 + \frac{1}{12}(13c_1^2 + 12\sigma c_1) - \frac{1}{2}\sigma c_1^2 \) (with \( c_1 = \pi^* c_1(B) \)) denotes the Todd class of the relative tangent bundle \( T_{X/B} = T_X/\pi^* T_B \). Note that \( S(\mathcal{G}) \) is a complex (or an object of the derived category) and then its Chern character is

\[
ch(S(\mathcal{G})) = \sum_i (-1)^i ch(S^i(\mathcal{G}))
\] (2.24)

\(^\text{14}\) This holds more generally: if only \( T^i(\mathcal{F}) \) is non-zero, one has \((-1)^i ch(T^i(\mathcal{F})) = M \cdot ch(i_*L) \) and the same formula for \( \hat{T} \) (\cite{47}, eqn.(4.1)). As the only inverse transform of \( T^i(\mathcal{F}) \) is \( \hat{T}^{1-i} \), one has \( ch(\hat{T}^{1-i}(T^i(\mathcal{F}))) = -M^2 \cdot ch(\mathcal{F}) \), consistent with \( \hat{T}^{1-i}(T^i(\mathcal{F})) = F \) and \( M^2 = -id_4 \).
To compute (2.23) note first that \( ch(\mathcal{I}) = 1 - ch(\delta_* \mathcal{O}_X) \) with the diagonal immersion \( \delta \).
Riemann–Roch gives
\[
ch(\delta_* \mathcal{O}_X) Td(X \times_B X) = \delta_* (ch(\mathcal{O}_X) Td(X))
\] (2.25)
where one has the expressions for \( Td(X) \) and \( Td(X \times_B X) \) given by
\[
Td(X) = 1 + \frac{1}{12}(c_2 + 11c_1^2 + 12\sigma c_1)
\] (2.26)
\[
Td(X \times_B X) = p_*^2 Td(X) p_*^1 Td(T_{X/B})
\]
The Chern character of the ideal sheaf is then given by (with the diagonal class \( \Delta = \delta_* (1) \))
\[
ch(\mathcal{I}) = 1 - \delta_* (1) - \frac{1}{2} \delta_* (c_1) + \delta_* (\sigma \cdot c_1) + \frac{5}{6} \delta_* (c_1^2) + \frac{1}{2} \delta_* (\sigma c_1^2)
\] (2.27)
\[
= 1 - \Delta - \frac{1}{2} \Delta \cdot p_*^2 c_1 + \Delta \cdot p_*^2 (\sigma \cdot c_1) + \frac{5}{6} \Delta \cdot p_*^2 (c_1^2) + \frac{1}{2} \Delta \cdot p_*^2 (\sigma c_1^2)
\]
Defining the numerical invariants (with \( F \) the elliptic fibre class)
\[
n = rk \mathcal{G}, \quad s = ch_3(\mathcal{G})
\]
\[
d = ch_1(\mathcal{G}) \cdot F, \quad g = ch_1(\mathcal{G}) \cdot \sigma \cdot c_1
\] (2.28)
\[
c = ch_2(\mathcal{G}) \cdot \sigma \quad f = ch_2(\mathcal{G}) \cdot c_1
\]
we get for the Chern characters of \( S(\mathcal{G}) \)
\[
ch_0(S(\mathcal{G})) = d
\]
\[
ch_1(S(\mathcal{G})) = ch_1(\mathcal{G}) - (d + n) \sigma - p_1*(p_*^2 ch_1(\mathcal{G}) \sigma) - p_1*(p_*^2 ch_2(\mathcal{G})) - \frac{3}{2} dc_1
\]
\[
ch_2(S(\mathcal{G})) = ch_2(\mathcal{G}) - 2ch_1(\mathcal{G})(c_1 + \sigma) + \sigma p_1*(p_*^2 (ch_1(\mathcal{G}) \sigma)) - \sigma p_1*(p_*^2 ch_2(\mathcal{G}))
\] (2.29)
\[
+ \frac{25}{12} dc_1^2 + (s + 2g - c - \frac{3}{2} f) F + (\frac{1}{2} n + d) \sigma c_1
\]
\[
ch_3(S(\mathcal{G})) = -\left( \frac{1}{6} n \sigma c_1^2 - \frac{1}{2} \sigma c_1^2 d - \frac{1}{2} g + f + c \right)
\]
Note that if there is only one non-vanishing transform \( S^i \), its Chern character is computed from (2.23) by \( ch(S^i(\mathcal{G})) = (-1)^i ch(S(\mathcal{G})) \) due to (2.24).

Similar calculations can be done for the inverse FM transform.
\[
ch_0(\hat{S}(\mathcal{G})) = d
\]
\[
ch_1(\hat{S}(\mathcal{G})) = -ch_1(\mathcal{G}) + (d - n) \sigma + p_2*(p_*^1 ch_1(\mathcal{G}) \sigma) + p_2*(p_*^1 ch_2(\mathcal{G})) + \frac{3}{2} dc_1
\]
\[
ch_2(\hat{S}(\mathcal{G})) = -ch_2(\mathcal{G}) - 2ch_1(\mathcal{G})(c_1 + \sigma) + \sigma p_2*(p_*^1 (ch_1(\mathcal{G}) \sigma)) + \sigma p_2*(p_*^1 ch_2(\mathcal{G}))
\] (2.30)
\[
+ \frac{25}{12} dc_1^2 + (s + 2g + c + \frac{3}{2} f) F + (d - \frac{1}{2} n) \sigma c_1
\]
\[
ch_3(\hat{S}(\mathcal{G})) = -\left( \frac{1}{6} n \sigma c_1^2 - \frac{1}{2} \sigma c_1^2 d + \frac{1}{2} g + f + c \right)
\]
If we consider now a sheaf $V$ and write its Chern character as

\[
ch_0(V) = n, \quad ch_1(V) = x\sigma + S, \quad ch_2(V) = \sigma\eta + aF, \quad ch_3(V) = s
\]

($\eta, S \in p_2^*H^2(B)$), then by (2.29) and (2.30), the Chern character of the FM of $V$ and of the inverse FM of $V$ are

\[
ch_0(S(V)) = x
\]
\[
ch_1(S(V)) = -n\sigma + \eta - \frac{1}{2}xc_1
\]
\[
ch_2(S(V)) = (\frac{1}{2}nc_1 - S)\sigma + (s - \frac{1}{2}\eta c_1\sigma + \frac{1}{12}xc_1^2\sigma)F
\]
\[
ch_3(S(V)) = -\frac{1}{6}n\sigma c_1^2 - a + \frac{1}{2}\sigma c_1 S
\]

and

\[
ch_0(\hat{S}(V)) = x
\]
\[
ch_1(\hat{S}(V)) = -n\sigma + \eta + \frac{1}{2}xc_1
\]
\[
ch_2(\hat{S}(V)) = (-\frac{1}{2}nc_1 - S)\sigma + (s + \frac{1}{2}\eta c_1\sigma + \frac{1}{12}xc_1^2\sigma)F
\]
\[
ch_3(\hat{S}(V)) = -\frac{1}{6}n\sigma c_1^2 - a - \frac{1}{2}\sigma c_1 S + x\sigma c_1^2
\]

Using the decomposition of the cohomology we find

\[
ch(V) = \begin{pmatrix} n \\ x \\ S \\ \eta \\ a \\ s \end{pmatrix}, \quad ch(\hat{S}(V)) = \begin{pmatrix} 0 \\ -n \\ \eta + \frac{1}{2}xc_1 \\ -\frac{1}{2}nc_1 - S \\ s + \frac{1}{2}\eta c_1\sigma + \frac{1}{12}xc_1^2\sigma \\ -\frac{1}{6}n\sigma c_1^2 - a - \frac{1}{2}\sigma c_1 S + x\sigma c_1^2 \end{pmatrix}
\]

If we multiply $ch(\hat{S}(V))$ by the Todd class $Td(N) = 1 - \frac{1}{2}c_1 + \frac{1}{12}c_1^2 \square$ we get

\[
ch(\hat{S}(V)) \cdot Td(N) = \begin{pmatrix} x \\ -n \\ \eta \\ -S \\ s - \frac{1}{12}x\sigma c_1^2 \\ -a + x\sigma c_1^2 \end{pmatrix}
\]

\[\square^{15}\text{ We always confuse the normal bundle } N \text{ with its pull-back } \pi^* N \text{ to } X \]
When $V$ is fibrewise of degree 0, which is the case we are mainly interested in (bundles constructed from spectral covers are of this kind), we have $x = 0$ and then

\[
\begin{pmatrix}
  x \\
  \eta \\
  \eta \\
  s \\
  -a
\end{pmatrix} =
\begin{pmatrix}
  0 & 1 & 0 & 0 & 0 & 0 \\
  -1 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 1 & 0 & 0 \\
  0 & 0 & -1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 0 & -1 & 0
\end{pmatrix}
\begin{pmatrix}
  n \\
  x \\
  S \\
  \eta \\
  a \\
  s
\end{pmatrix}
\]

that is

\[
\text{Td}(N) \cdot \text{ch}(\tilde{S}(V)) = M \cdot \text{ch}(V). \tag{2.34}
\]

So for fibrewise degree 0 and semistable bundles $V$ we have $\tilde{S}^{0}(V) = 0$ and $\tilde{S}^{1}(V) = i_{*}L$, so that (2.34) is equivalent to

\[
\text{Td}(N) \cdot \text{ch}(i_{*}L) = \text{Td}(N) \cdot \text{ch}(\tilde{S}^{1}(V)) = -M \cdot \text{ch}(V).
\]

There is an analogous equation of (2.35) for the direct FM transform $S(V)$; proceeding as above one shows that if $x = 0$ then

\[
\text{Td}(N^{-1}) \cdot \text{ch}(S(V)) = M \cdot \text{ch}(V). \tag{2.36}
\]

For fibrewise degree 0 and semistable bundles $V$ we have $S^{0}(V) = 0$ and if we write $S^{1}(V) = i_{*}\overline{L}$, we have

\[
\text{Td}(N^{-1}) \cdot \text{ch}(i_{*}\overline{L}) = \text{Td}(N^{-1}) \cdot \text{ch}(S^{1}(V)) = -M \cdot \text{ch}(V).
\]

How have these results to be interpreted? We will show in the appendix why in the three-fold case the sole use of the T-functor known from the $K3$-case to give the map between the Chern classes of the bundle and its dual is insufficient to exhibit as transformation matrix the adiabatic extension (2.5) of the usual T-duality matrix (2.4) on the fibre.

Rather one has to invoke the precise definition of the twisted charge relevant here. Recall from the usual (full, not fibrewise) Fourier-Mukai transform that one has actually to take the Mukai vector which has a twist by $\sqrt{\text{Td}(X)}$ and not just the Chern character.
Now there are two possible alternative routes of procedure: one can either twist the operation a little bit and stick to the usual Mukai-vector (this is described in the next subsection) or keep the operation and make the twist in the usual Mukai-vector more 'relative' as described in what follows.

Similarly to the usual twist in the Mukai-vector here in the fibrewise situation a twist by \( \sqrt{Td(N)} \) with the normal bundle \( N = j^*TX/TB = j^*T_{X/B} \) plays a role (where \( j : B \hookrightarrow X \)). Now suppose you have a complex of sheaves \( \mathcal{G} \) on \( X \) and you twist the standard definition of \( ch \) to define a \textit{charge}

\[
Q(\mathcal{G}) = \sum (-1)^i ch(\mathcal{G}^i)(\sqrt{Td(N)})^{i+1}
\]

(2.37)

where \( \mathcal{G}^i \) are the cohomology sheaves of the complex \( \mathcal{G} \) and \(|i + 1| = (-1)^{i+1} \).

Then

\[
Q(\hat{S}(V)) = -ch(\hat{S}^1(V))(\sqrt{Td(N)}) \quad Q(V) = ch(V)(\sqrt{Td(N)})^{-1}
\]

(2.38)

and thus\(^{16}\)

\[
Q(\hat{S}(V)) = M \cdot Q(V)
\]

(2.39)

Since \( \hat{S}^0(V) = 0 \) and \( \hat{S}^1(V) = i_*L \) we have \( \hat{S}(V) = i_*L[-1] \) as complexes and then (2.38) is equivalent to

\[
Q(i_*L[-1]) = M \cdot Q(V) = M \cdot Q(S(i_*L))
\]

This represents one way to arrange the quantities involved to get the M matrix. In view of the internal twist by \( Td(T_{X/B}) \) in the FM transform (2.23) the charge definition (2.37) may not come as a surprise\(^{17}\). As outlined above still a different route can be taken. This is described in what follows.

\(^{16}\) For a single sheaf \( V \), we are writing \( Q(V) \) in the sense that \( V \) is understood as a complex with \( V \) at degree 0 and no other terms.

\(^{17}\) note the close relation \( j^*T_{X/B} = N \) between \( T_{X/B} = TX/\pi^*TB \) and \( N = (j^*TX)/TB \) where \( j : B \hookrightarrow X \); cf. also the motivation for the T-functor in the K3 case described at the end of that subsection and the appendix for the role of \( Td(N) \)
2.5. f-map

The effect of the FM transform in cohomology is usually described by means of the so-called f-map. For a fibrewise FM transform, we can also introduce a relative version $f_r : H(X) \to H(X)$ of the f-map. It is defined by

$$f_r(x) = p_1^*(p_2^*(x) \cdot Z_r)$$

where $Z_r = \sqrt{p_2^*Td(TX/B)} \cdot ch(P) \cdot \sqrt{p_1^*Td(TX/B)}$. Then

$$\sqrt{Td(TX/B) \cdot ch(S(V))} = f_r(ch(V) \cdot \sqrt{Td(TX/B)})$$

If we consider instead of $ch(V)$ the effective charge given by the Mukai vector

$$Q(V) = ch(V) \cdot \sqrt{Td(X)} \quad (2.40)$$

then the effect of $f_r$ on $Q(V)$ is described by

$$f_r(Q(V)) = Q(S(V)) \quad (2.41)$$

But if we modify the definition of $f_r$ to $f : H(X) \to H(X)$,

$$f(x) = p_1^*(p_2^*(x) \cdot Z) \quad (2.42)$$

with $Z = \sqrt{p_2^*Td(X)} \cdot ch(P) \cdot \sqrt{p_1^*Td(X)}$, then by (2.36), the effective charge of $V$ transforms to

$$f(Q(S(V))) = M \cdot Q(V) \quad (2.43)$$

when $x = 0$. Then if $V$ is moreover fibrewise semistable so that $S^0(V) = 0$ and $S^1(V) = i_*L$, then

$$-f(Q(i_*L)) = -f(Q(S^1(V))) = f(Q(S(V))) = M \cdot Q(V)$$

2.6. Fibrewise T-duality on D-branes at the sheaf level

Our next goal is to describe T-duality on the $T^2$ fiber maps given in (2.1) at the sheaf level.
Let us consider the skyscraper sheaf $\mathcal{C}(x)$ at a point $x$ of $X$. It is a WIT$_0$ sheaf and its FM transform $S^0(\mathcal{C}(x))$ is a torsion-free rank one sheaf $L_x$ on the fibre of $X$ over $\pi(x)$ as we expect from (2.1) and thus we see $D0 \to D2\mathcal{F}$.

For the topological invariants we have indeed $n = x = a = 0, S = \eta = 0, s = 1$ and then

$$
ch_i(S^0(\mathcal{C}(x))) = 0, \quad i = 0, 1, 3, \quad ch_2(S^0(\mathcal{C}(x))) = F \tag{2.44}
$$

If we start with $\mathcal{F} = \mathcal{O}_\sigma$; proceeding as in (3.16) of [48] we have

$$
\begin{align*}
S^0(\mathcal{O}_\sigma) &= \mathcal{O}_X, & S^1(\mathcal{O}_\sigma) &= 0 \\
S^0(\mathcal{O}_X) &= 0, & S^1(\mathcal{O}_X) &= \mathcal{O}_\sigma \otimes \pi^* K_B \tag{2.45}
\end{align*}
$$

Then $\mathcal{O}_\sigma$ transforms to the structure sheaf of $X$ and $\mathcal{O}_X$ transforms to a line bundle on $\sigma$ as we expect from (2.1) since $D4_B \leftrightarrow D6$. We have as before the transformations at the cohomology level;

$$
n = 0, \quad x = 1, \quad S = 0, \quad \eta = \frac{1}{2}c_1, \quad a = 0, \quad s = \frac{1}{6}\sigma^2 c^2_i \tag{2.46}
$$

and then we get

$$
ch_0(S^0(\mathcal{O}_\sigma)) = 1, \quad ch_i(S^0(\mathcal{O}_\sigma)) = 0, \quad i = 1, 2, 3 \tag{2.47}
$$

Finally, let us consider a sheaf $\mathcal{F}$ on $B$; by (2.44) we have

$$
\begin{align*}
S^0(\mathcal{O}_\sigma \otimes \pi^* \mathcal{F}) &= \pi^* \mathcal{F}, & S^1(\mathcal{O}_\sigma \otimes \pi^* \mathcal{F}) &= 0 \\
S^0(\pi^* \mathcal{F}) &= 0, & S^1(\pi^* \mathcal{F}) &= \mathcal{O}_\sigma \otimes \pi^* \mathcal{F} \otimes \pi^* K_B \tag{2.48}
\end{align*}
$$

Then, a sheaf $\mathcal{O}_\sigma \otimes \pi^* \mathcal{F} = j_* \mathcal{F}$ ($j : B \to X$ is the section) supported on a curve $\tilde{C}$ in $B$ embedded in $X$ via $j$ transforms to a sheaf on the elliptic surface supported on the inverse image of $\tilde{C}$ in $X$ and vice versa. This is what we expected form the map $D2_B \leftrightarrow D4\mathcal{F}$ of (2.1).

Then, even at the sheaf level we have the relations (2.1) appropriate for the fibrewise T-duality on D-branes

$$
\begin{align*}
D4_B &\to \tilde{D}6 \\
D2_B &\to \tilde{D}4\mathcal{F} \tag{2.49}
\end{align*}
$$

---

18 With the identification $X \simeq \tilde{X}$ the point $x$ corresponds precisely to $L_x$ (see [17] or [48]).

19 Our formulas differ from those in [48] because we are using a different Poincaré sheaf.
3. Period monodromies and bundle automorphisms

Kontsevich proposed to consider mirror symmetry as an categorical equivalence between the bounded derived category $D(X)$ of $X$ and Fukaya’s $A_\infty$ category of Lagrangian submanifolds for the mirror $\hat{X}$. The object of the derived category $D(X)$ is a complex of coherent sheafs on $X$ resp. the object of Fukaya’s $A_\infty$ category is a Lagrangian submanifold with flat $U(1)$ bundles on it.

It is now expected that under the proposed categorical equivalence between $D(X)$ and $A_\infty$ the monodromy of the SLAG 3-cycle is mapped to certain automorphisms in $D(X)$ (Fourier-Mukai transformations). We are going to make these relations explicit and check them thereby for two models.

3.1. The models

So we consider the models $\mathbb{P}^4_{1,1,2,2,2}[8]$ and $\mathbb{P}^4_{1,1,2,2,6}[12]$ given by degree 8 resp. 12 hypersurfaces in the respective weighted projective spaces. Among these hypersurfaces are

$$z_1^8 + z_2^8 + z_3^4 + z_4^4 + z_5^4 = 0 \quad (3.1)$$

and

$$z_1^{12} + z_2^{12} + z_3^6 + z_4^6 + z_5^2 = 0 \quad (3.2)$$

These models were studied in [15], [17]. At $z_1 = z_2 = 0$ both models have a curve $C$ (of genus three resp. two) of $A_1$ singularities, which when blown up leads to an exceptional divisor $E$ in $X$. $E$ has the structure of a ruled surface ($P^1$ fibration) over $C$; let $l$ denote its fibre. A second divisor class $L$ (besides $E$) comes from the fact that both models are $K3$ fibrations over $P^1$ (the degree one polynomials generate a linear system $|L|$ which projects $X$ to $\mathbb{P}^1$ with fiber $L = K3$). $L$ and $E$ together generate $H_4(X, \mathbb{Z})$. The generators of the complexified Kähler cone in $H^2(X, \mathbb{Z})$ are chosen to be $(E, L)$, thus the generic Kähler class can be written as $t = t_1E + t_2L$ where $(t_1, t_2)$ are coordinates on the Kähler moduli space of $X$.

---

20 which are just $K3$ fibrations, not elliptic ones, so somewhat outside our main applications
A second linear system $|H|$ is generated by degree two polynomials and related to the first by $|H| = |2L + E|$ (for both models). Let $4h$ denote the intersection of two general members of $H$ and $L$ in the degree 8 model, which is a plane quartic (think of a hyperplane section of the $K3 = P^3[4]$) which can be specialized to a sum of four lines. Similarly one can define $2h = H \cdot L$ in the degree 12 model. $h = \frac{1}{4}H \cdot L$ and $l = \frac{1}{4}H \cdot E$ in the degree 8 model resp. $h = \frac{1}{2}H \cdot L$ and $l = \frac{1}{2}H \cdot E$ in the degree 12 model generate $H_2(X, \mathbb{Z})$ in both cases. They are dual to $H$ and $L$ in the sense that the relations $L \cdot l = 1, H \cdot l = 0, H \cdot h = 1$ hold (which mean for $E$ that $E \cdot l = -2, E \cdot h = 1$).

Let us recall now the intersection relations for both models. As they are $K3$ fibrations one has $L^2 = 0$. The remaining intersections are as follows. For $\mathbf{P}^4_{1,1,2,2,2}[8]$ one has

$$E^3 = -16, \quad E^2 \cdot L = 4$$

$$c_2(X) \cdot L = 24, \quad c_2(X) \cdot E = 8$$

and for $\mathbf{P}^4_{1,1,2,2,6}[12]$ one has

$$E^3 = -8, \quad E^2 \cdot L = 2$$

$$c_2(X) \cdot L = 24, \quad c_2(X) \cdot E = 4$$

The mirror family $\hat{X}$ of $X$ can be obtained by applying the Greene-Plesser construction [18], which is given by $p = 0/G$ with

$$p = z_1^8 + z_2^8 + z_3^4 + z_4^4 + z_5^4 - 8\psi z_1 z_2 z_3 z_4 z_5 - 2\phi z_1^4 z_2^4 = 0$$

(3.5)

for $\mathbf{P}^4_{1,1,2,2,2}[8]$ with $G = \mathbb{Z}_4^3$ and for $\mathbf{P}^4_{1,1,2,2,6}[12]$

$$p = z_1^{12} + z_2^{12} + z_3^6 + z_4^6 + z_5^2 - 12\psi z_1 z_2 z_3 z_4 z_5 - 2\phi z_1^6 z_2^6 = 0$$

(3.6)

with $G = \mathbb{Z}_6^2 \times \mathbb{Z}_2$. Note that $\phi$ and $\psi$ parameterize the moduli space of complex structures of the mirror $\hat{X}$ which does not get any $\alpha'$ corrections.

Of course the complex structure moduli space of the mirror is identified with the Kähler moduli space of the original Calabi-Yau and under this identification monodromies

\[\text{21} \text{ one has also the relations } H^3 = 8, \quad H^2 \cdot L = 4 \quad \text{and } H \cdot E \cdot L = 4\]

\[\text{22} \text{ one has also the relations } H^3 = 4, \quad H^2 \cdot L = 2 \quad \text{and } H \cdot E \cdot L = 2\]
on the Kähler side are replaced by there transpose-inverse. The $N=2$ prepotential $F$ determines the vector of periods of the holomorphic three-form $\hat{\Omega}$ on the mirror manifold $\hat{X}$ which is given in the $(E, L)$ basis by (note the identification just mentioned)

$$\Pi = \begin{pmatrix}
2F - t^i F_i \\
F^1 - 2F^2 \\
F^2 \\
t_1 \\
t_2
\end{pmatrix}$$

Note that the order here is $H^6, H^4, H^0, H^2$. So the degree in the $t$'s is $i$ on $H^{2i}$. Here the cubic prepotential is given for $\bf{P}_{1,1,2,2}^4$[8] by[15]

$$F = -\frac{4}{3} t_1^3 - 2t_1^2 t_2 - 2t_1 t_2^2 + \frac{7}{3} t_1 + t_2$$

and for $\bf{P}_{1,1,2,6}^4$[12] by

$$F = -\frac{2}{3} t_1^3 - t_1^2 t_2 + \frac{13}{6} t_1 + t_2$$

with $\partial F^i / \partial t_i, i = 1, 2$. It contains in the purely cubic part the properly normalized (with $-1/3!$) intersection numbers $C_{ijk}$. One then gets in a neighborhood of the large radius limit for $\bf{P}_{1,1,2,2}^4$[8] resp. $\bf{P}_{1,1,2,2,6}^4$[12] (or of the large complex structure limit for the mirrors)

$$\Pi = \begin{pmatrix}
\frac{2}{3} t_1^3 + t_1^2 t_2 + \frac{13}{6} t_1 + t_2 \\
\frac{1}{6} - 2t_1 t_2 \\
1 - t_1^2 \\
1 \\
t_1 \\
t_2
\end{pmatrix}, \quad \Pi = \begin{pmatrix}
\frac{4}{3} t_1^3 + 2t_1 t_2 + \frac{7}{3} t_1 + t_2 \\
-4t_1 t_2 + 4t_1 - 2t_2 + \frac{1}{3} \\
-2t_2^2 - 2t_1 + 1 \\
1 \\
t_1 \\
t_2
\end{pmatrix}$$

3.2. Monodromy

The vectors just mentioned have a monodromy behaviour (represented by a matrix $S_i$) under $t_i \rightarrow t_i + 1$ easy to read off. It belongs to a divisor $D_i$ in the moduli space where the corresponding cycle becomes small. Note that infinity is the fixpoint of the

23 The 2. entry is not $F^1$ as one comes from the $H, L$ basis and then $E = H - 2L$ (cf. also [7] where the monodromy transformations correspond to shifts in the $(H, L)$ basis not the $(E, L)$ basis)
mentioned shift in the respective variable; furthermore in the matrices \( R_i := S_i - 1 \) one finds \( R_i R_j R_k = C_{ijk} Y \) with \( Y \) a matrix independent of \( i, j, k \), i.e. the \( R_i \) fulfill the algebra of the \( D_i \). Note also (cf. [15]) that multiplication with \( D_i \) corresponds to \( N_i := \log(1+R_i) \). This relation means that \( ch([D_i]) = e^{D_i} \) (where \([D_i]\) is the line bundle associated with the divisor) corresponds to the monodromy matrix \( S_i = 1 + R_i \) on the Kaehler period vector. This is the Kontsevich relation between tensoring by the line bundle and monodromy matrix purely on the \( H^{\text{even}} \) side; mirror symmetry identifies this with the monodromy of the type IIB periods.

Let us write down the monodromy matrices \( S_i \) about the two divisors (following the notation of [15]) \( \mathcal{D}_H = D_{(0,-1)} \) and \( \mathcal{D}_L = C_{\infty} = D_{(1,0)} \) for the degree 8 and the degree 12 model

\[
S_L = \begin{pmatrix}
1 & 0 & -1 & 2 & -4 & 0 \\
0 & 1 & 0 & -2 & 4 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}, \quad S_H = \begin{pmatrix}
1 & -1 & -2 & 6 & 4 & 0 \\
0 & 1 & 0 & 4 & 0 & -4 \\
0 & 0 & 1 & -4 & -4 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

\[
S_L = \begin{pmatrix}
1 & 0 & -1 & 2 & 0 & 0 \\
0 & 1 & 0 & 0 & -2 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}, \quad S_H = \begin{pmatrix}
1 & -1 & -2 & 5 & 2 & 1 \\
0 & 1 & 0 & 0 & 0 & -2 \\
0 & 0 & 1 & -1 & -2 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

The monodromy about the conifold locus is given for both models in the \((E, L)\) basis by

\[
T = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

For easier comparison let us note that Fig. 1 of [15] shows among other things the divisors (of compactified moduli space) \( C_{\infty} \) given by \( \phi \to \infty, \psi \to \infty \), the conifold locus \( C_{\text{con}} \) given by \( 864\phi^6 + \phi = \pm 1 \) (or \((1-x)^2 - x^2y = 0\) in the variables \( x := -\phi/(864\psi^6), \ y := 1/\phi^2\)), \( C_1 \) given by \( y = 1 \) and \( C_0 \) given by \( \psi = 0 \) (with its singular point \( \psi = 0 = \phi \)); in a
heterotic language one would describe $C_\infty$ by $S = \infty$, $C_1$ by $S = T$ and the points of intersection of $C_{\text{con}}$ with $C_\infty$ resp. $C_1$ with $C_\infty$ by $T = i$ resp. $S = T = \infty$, leading to the Seiberg-Witten point resp. the large radius point (or the large complex structure point in the mirror interpretation); 'leading to' because actually and more precisely both of these points belong to the points in the moduli space which have to be blown up to get divisors with normal crossings only (cf. Fig. 4 of [15]). In the course of the resolution process at these two points are the divisors $E_2$ resp. $D_{(0,-1)}$ introduced which intersect $C_\infty = D_{(1,0)}$ in the point $u = \infty$ at infinity of the Seiberg-Witten $u$-plane resp. in the large complex structure point. One finds that $S_L = (ATB)^{-1}$, $S_H = (AT)^{-2}$ where $A$, $B$, $T$ are the monodromies around $C_0$, $C_1$, $C_{\text{con}}$. Note also that, for example, the association between the divisor $L$, the $K_3$, and the divisor $D_L$ in the moduli space becomes explicit as the making the basis $P^1$ of the $K3$ fibration large is equivalent to making the $K3$ itself hierarchically small.

Note that our $S_D$ matrices (and the $T$ matrix) are related to those of [13] by

$$m \cdot D_D \cdot m^{-1} = \tilde{D}_D$$
$$K \cdot \tilde{D}_D \cdot K^{-1} = S_D$$

with

$$m = \begin{pmatrix}
-1 & 1 & 0 & 0 & 0 & 0 \\
1 & 3 & 3 & 2 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 1 & 1 & 0 \\
2 & 0 & 0 & -2 & 1 & 1
\end{pmatrix}, \quad K = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -3 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

3.3. D6-branes charges and bundle data

We are now interested in an explicit map between the topological invariants of the characteristic classes of the Chan-Paton sheaf $V$ and the brane charges.\footnote{Cf. the analysis of [7] for another two-parameter Calabi-Yau and [14] for the quintic.} Therefore let us consider the BPS charge lattice which can be identified with the middle cohomology lattice of the mirror manifold $H^3(\hat{X}, \mathbb{Z})$ and consider the central charge associated to the integral vector $n = (n_6, n_4^1, n_4^2, n_0, n_2^1, n_2^2)$ which is

$$Z(n) = n_6 \Pi_1 + n_4^1 \Pi_2 + n_4^2 \Pi_3 + n_0 \Pi_4 + n_2^1 \Pi_5 + n_2^2 \Pi_6.$$  \hfill (3.10)
One the other side, one has in the large volume limit of $X$ the lattice of microscopic D-brane charges (which is identified with the K-theory lattice $K(X)$). Here one considers the effective charge $Q$ of a D-brane state $\eta$ given by the Mukai vector

$$Q = ch(\eta) \sqrt{Td(X)} \in H^0(X) \oplus H^2(X) \oplus H^4(X) \oplus H^6(X)$$  \hspace{1cm} (3.11)

with the associated central charge

$$Z(t) = \int Z(t) = \int Z(t^3 - \frac{1}{2}ch_1(\eta) + (ch_2(V) + \frac{r}{24} c_2(X))t - (ch_3(V) + \frac{1}{24} ch_1(V)c_2(X))$$  \hspace{1cm} (3.12)

where $t = t_1 H + t_2 L$ is the generic Kähler class and expansion of (3.11) leads to

$$Q = (r, ch_1(V), ch_2(V) + \frac{r}{24} c_2(X), ch_3(V) + \frac{1}{24} ch_1(V)c_2(X))$$  \hspace{1cm} (3.13)

so one has

$$Z(Q) = \frac{r}{6} t^3 - \frac{1}{2} ch_1(\eta) t^2 + (ch_2(V) + \frac{r}{24} c_2(X)) t - (ch_3(V) + \frac{1}{24} ch_1(V)c_2(V))$$  \hspace{1cm} (3.14)

The comparison of $Z(n)$ and $Z(t)$ leads then to a map between the low energy charges $n$ and the topological invariants of the K-theory class $\eta$. We find for $\mathbf{P}^4_{1,1,2,2,2}$ \cite{8}

$$r(V) = n_6$$

$$ch_1(V) = n_1^1 E + n_4^2 L$$

$$ch_2(V) = (4n_4^1 - 2n_4^2 + n_2^1)h + (-2n_4^1 + n_2^2)l$$

$$ch_3(V) = -n_0 - \frac{2}{3} n_4^1 - 2n_4^2$$

and for $\mathbf{P}^4_{1,1,2,2,6}$ \cite{12} we get

$$r(V) = n_6$$

$$ch_1(V) = n_1^1 E + n_4^2 L$$

$$ch_2(V) = n_4^1 h + n_2^2 l$$

$$ch_3(V) = -n_0 - \frac{1}{3} n_4^1 - 2n_4^2$$

\footnote{Note that for $N$ coincident D6-branes wrapping $X$, the gauge field has to satisfy the Hermitian Yang Mills equations in order to preserve supersymmetry $F_{ij} = 0$, $\omega^2 \wedge tr F = \omega^2 \wedge c_1(V) = 0$. The first equation tells us that the vacuum gauge field $A$ is holomorphic connection on a holomorphic vector bundle $V \to X$. The second condition is the integrability condition which guarantees a unique solution to the Donaldson-Uhlenbeck-Yau equation $g^{ij} F_{ij} = 0$.}
3.4. Kontsevich’s association

A part of Kontsevich’s association can be described just on the bundle side; it is just the question of transport between the ‘fibration’ basis (2.7) and the Kähler period vector basis (cf. [15], p.12 and [4], p. 7; mirror symmetry is then ”only” a rephrasing of the Kähler period vector in $H^{\text{even}}(X)$ as complex structure period vector in IIB for the $H^3$ of mirror); the relations (3.13), (3.16) and (4.22) express the mentioned basis transformations.

We find that the monodromy transformations $S_L, S_H$ correspond to the following automorphisms $M(D)$

$$[V] \rightarrow [V \otimes \mathcal{O}_X(L)], \quad [V] \rightarrow [V \otimes \mathcal{O}_X(H)]$$

with the topological invariants of $V$ changed according to for a ‘twisted’ sheaf $V' = V \otimes \mathcal{O}_X(D)$ to

$$r(V') = r(V)$$

$$ch_1(V') = ch_1(V) + rD$$

$$ch_2(V') = ch_2(V) + ch_1(V)D + \frac{r}{2}D^2$$

$$ch_3(V') = ch_3(V) + ch_2(V)D + \frac{1}{2}ch_1(V)D^2 + \frac{r}{6}D^3$$

and we see that the linear transformations acting on $\mathbf{n}$ corresponding to $D = H, L$ are

$$M(D) = S_D^{-1}$$

Let us consider now a second type of monodromy transformation proposed by Kontsevich. He proposed that the monodromy $T$ about the conifold locus of the mirror corresponds to the automorphism of the derived category whose effect on cohomology can be denoted by (where $1_X$ is the standard generator of $H^0(X, \mathbb{Q})$)

$$S : \gamma \rightarrow \gamma - \left( \int \gamma \wedge Td(T_X) \right) \cdot 1_X$$

(3.20)

corresponding to a change in the topological invariants of $V$

$$ch(V) \rightarrow ch(V) - \left( \frac{ch_1(V)c_2(X)}{12} + ch_3(V) \right)$$

(3.21)
using the expression of the prepotential in the large radius limit

\[ F = \frac{1}{6} (t \cdot J)^3 - \frac{c_2(X)}{24} (t \cdot J) + \ldots \]  

(3.22)

where \((t \cdot J) = \sum t_a J^a\) (in particular we have \(J_1 = E + 2L\) and \(J_2 = L\)) and using the period vector \(\Pi\) we find the expression valid for both models

\[
\begin{align*}
ch_1(V) &= n_1^1 (J_1 - 2J_2) + n_4^2 J_2 \\
ch_3(V) &= -(n_1^1 (J_1 - 2J_2) + n_4^2 J_2) \frac{c_2(X)}{12} - n_0
\end{align*}
\]

(3.23)

leading to the universal shift

\[ n_6 \rightarrow n_6 + n_0 \]  

(3.24)

comparing this to the monodromy we find that the linear transformation acting on \(n\) corresponds to

\[ S = T^{-1} \]  

(3.25)

4. FM-transform as Monodromy

Perhaps most important in view of the other investigations in the paper is the question whether the Fourier-Mukai transform will be related to a monodromy matrix in the sense of Kontsevich's proposal. As the corresponding matrices are by mirror symmetry identifiable already on the bundle side (as worked out for some two parameter examples in the foregoing section) this comes down, in a similar elliptic example, exxentially to the question whether the matrix \(M\) in (2.5) (or the corresponding matrix in the sLag side), is generated by \(S_H, S_L, T\). This will be made precise below.

Now recall that the transformations in the Kontsevich association considered in the foregoing section can themselves be considered as FM transforms. This time we will not work on the fibre product but rather on the ordinary product.

\[
\begin{array}{ccc}
X \times \tilde{X} & \xrightarrow{q_2} & \tilde{X} \\
q_1 \downarrow & & \pi_2 \downarrow \\
X & \xrightarrow{\pi_1} & B
\end{array}
\]

(4.1)
One defines the Fourier-Mukai functors $S^i$ by associating with every sheaf $V$ on $X$ the complex of sheaves $S^i_\mathcal{E}(V)$ on $X$ (where $X$ and $\tilde{X}$ are identified as before)

$$S^i_\mathcal{E}(V) = R^i p_1_*(p_2^*(V) \otimes \mathcal{E}) \quad (4.2)$$

where the kernel $\mathcal{E} \in D(X \times X)$ is an object in the derived category. We can also define the full FM transformation at the derived category level

$$S^i_\mathcal{E}(\mathcal{G}) = Rp_{1_*(p_2^*(\mathcal{G}) \otimes \mathcal{E})} \quad (4.3)$$

For example the twist transformation $\mathcal{G} \rightarrow \mathcal{G} \otimes \mathcal{L}$ considered in (3.17) comes from $\mathcal{E} = O_\Delta \otimes q_2^*(\mathcal{L})$ where $\Delta$ is the diagonal of $X \times X$. In particular $S_{O_\Delta}$ is the identity. The other operation (corresponding to the 'conifold monodromy') considered before corresponds to an $\mathcal{E}$ whose cohomology is the ideal sheaf $\mathcal{I}_\Delta$ of the diagonal of $X \times X$. The topological invariants of this Kontsevich FM transform can be easily obtained: The exact sequence

$$0 \rightarrow \mathcal{I}_\Delta \rightarrow O_{X \times X} \rightarrow O_\Delta \rightarrow 0$$

leads to a triangle triangle in the derived category $D(X)$ relating the corresponding FM transforms with these kernels:

$$S_{\mathcal{I}_\Delta}(\mathcal{G}) \rightarrow S_{O_{X \times X}}(\mathcal{G}) \rightarrow S_{O_\Delta}(\mathcal{G}) \rightarrow S_{\mathcal{I}_\Delta}(\mathcal{G})[1] \quad (4.4)$$

and then

$$ch(S_{\mathcal{I}_\Delta}(\mathcal{G})) = ch(S_{O_{X \times X}}(\mathcal{G})) - ch(S_{O_\Delta}(\mathcal{G})) = ch(S_{O_{X \times X}}(\mathcal{G})) - ch(\mathcal{G}) \quad (4.5)$$

Riemann-Roch gives now that $ch_i(S_{O_{X \times X}}(\mathcal{G})) = 0$ for $i > 0$ and then

$$ch(S_{\mathcal{I}_\Delta}(\mathcal{G})) = ch_0(S_{O_{X \times X}}(\mathcal{G})) - ch(\mathcal{G}) = \left(\int ch(\mathcal{G}) \wedge Td(T_X)\right) - ch(\mathcal{G}) \quad (4.6)$$

so that we recover the “gamma shift” (3.20) as expected when $\mathcal{G}$ is a WIT sheaf.

Now our fibrewise FM transform is specified by using $\mathcal{E} = j_* \mathcal{P}$ where $\mathcal{P}$ is the Poincaré of (2.14) and $j : X \times B \rightarrow X \times X$ is the natural embedding, so it is build up out of the

\footnote{The relationship between fibrewise FM and full FM is that (cf. also [13] ) FM on $X \times_B X$ with respect to any sheaf $\mathcal{P}$ is the same as FM on $X \times X$ with respect to $j_* \mathcal{P}$ as is clear from the fact that $p_i = q_i \circ j$ implies $Rp_{1_*(p_2^*(V) \otimes \mathcal{P})} = R(q_{1_*(j^*(q_2^*(V)) \otimes \mathcal{P})) = Rq_1_*(j^*(q_2^*(V)) \otimes \mathcal{P})) = Rq_1_*(q_2^*(V) \otimes \mathcal{P})$ by the projection formula.}
structures given by the divisors (resp. their associated line bundles) \( \sigma \) and \( \pi^*K_B \) on the one hand and by \( \mathcal{O}_\Delta \) on the other.

Since we have the \( M \) matrix equation when we transform bundles \( V \) with the inverse FM, we concentrate on it. This FM is \( \hat{S} = S_{j_* \mathcal{Q}} \) with

\[
Q = \mathcal{P}^* \otimes q^*(K_B^{-1}) = \mathcal{I} \otimes p_1^*\mathcal{O}(\sigma) \otimes p_2^*\mathcal{O}(\sigma) \otimes q^*K_B^{-2}
\]

where \( \mathcal{I} \) is the ideal of the diagonal of \( X \times_B X \). Then

\[
\hat{S} = (\otimes q^*K_B^{-2}) \circ (\otimes \mathcal{O}(\sigma)) \circ S_{j_* \mathcal{I}} \circ (\otimes \mathcal{O}(\sigma)) \quad (4.7)
\]

or

\[
\hat{S} = S_{\mathcal{O}_\Delta(2c_1)} \circ S_{\mathcal{O}_\Delta(\sigma)} \circ S_{j_* \mathcal{I}} \circ S_{\mathcal{O}_\Delta(\sigma)}
\]

We have then written our fibrewise FM \( \hat{S} \) as a composition of three Kontsevich full FM transforms and one FM transform \( S_{j_* \mathcal{I}} \).

Technically one can proceed now in two ways. One can read off the relevant matrix from our earlier treatment of the relative FM transform in section 2. Alternatively, and presented first, we are going to describe \( S_{j_* \mathcal{I}} \) in terms of the Kontsevich full FM transform \( S_{\mathcal{I}_\Delta} \), to make the closest contact to that quantity.

To this end, we use the exact sequence

\[
0 \to \mathcal{J} \to \mathcal{I}_\Delta \to j_* \mathcal{I} \to 0
\]

\( \mathcal{J} \) being the ideal of the closed immersion \( j : X \times_B X \to X \times X \). We have as in (4.4) a triangle in the derived category \( D(X) \)

\[
S_{\mathcal{J}}(\mathcal{G}) \to S_{\mathcal{I}_\Delta}(\mathcal{G}) \to S_{j_* \mathcal{I}}(\mathcal{G}) \to S_{\mathcal{J}}(\mathcal{G})[1]
\]

and then

\[
ch(S_{\mathcal{I}_\Delta}(\mathcal{G})) = ch(S_{\mathcal{I}}(\mathcal{G})) + ch(S_{\mathcal{J}}(\mathcal{G})) \quad (4.8)
\]

The term \( ch(S_{\mathcal{I}_\Delta}(\mathcal{G})) \) is given by (4.10) whereas \( ch(S_{\mathcal{J}}(\mathcal{G})) \) can be computed from

\[
ch(\mathcal{J}) = 1 - j_*(1) + \frac{1}{2} j_*(1) \cdot q_1^*(c_1) - \frac{1}{2} j_*(1) \cdot q_1^*(c_1^2) = 1 - j_*(1) + \frac{1}{2} j_*(1) \cdot q_2^*(c_1) - \frac{1}{2} j_*(1) \cdot q_2^*(c_1^2)
\]
(\(j_*(1)\) is the class of \(X \times_B X\) in \(H^4(X \times X)\)).

If we write

\[
\begin{align*}
ch_0(G) &= n_G, \\
ch_1(G) &= x_G \sigma + S_G, \\
ch_2(G) &= \sigma \eta_G + a_G F, \\
ch_3(G) &= s_G
\end{align*}
\]

(4.6) now reads

\[
\begin{align*}
ch_0(S_{I \Delta}(G)) &= s_G - \frac{1}{12} x_G \sigma c_1^2 + \sigma c_1 S_G + \frac{1}{12} x_G \sigma c_2 - n_G \\
&= (ch_3 G + \frac{ch_1 G c_2(X)}{12}) - ch_0 G \\
ch_1(S_{I \Delta}(G)) &= -ch_1 G \\
ch_2(S_{I \Delta}(G)) &= -ch_2 G \\
ch_3(S_{I \Delta}(G)) &= -ch_3 G
\end{align*}
\]

and \(ch(S_{\mathcal{J}}(G))\) is given by

\[
\begin{align*}
ch_0(S_{\mathcal{J}}(G)) &= s_G - \frac{1}{12} x_G \sigma c_1^2 + \sigma c_1 S_G + \frac{1}{12} x_G \sigma c_2 - x_G \\
&= (ch_3 G + \frac{ch_1 G c_2(X)}{12}) - x_G \\
ch_1(S_{\mathcal{J}}(G)) &= -n_G c_1 - \eta_G + \frac{1}{2} x_G c_1 \\
ch_2(S_{\mathcal{J}}(G)) &= \frac{1}{2} n_G - \frac{1}{12} x_G c_1^2 - c_1 S_G + \frac{1}{2} n_G c_1 - s_G F \\
ch_3(S_{\mathcal{J}}(G)) &= 0
\end{align*}
\]

This gives then the needed information about \(ch(S_{I}(G))\) which can alternatively be computed also from (2.27). If one writes

\[
\begin{align*}
ch_0(G) &= n_G, \\
ch_1(G) &= x_G \sigma + S_G, \\
ch_2(G) &= \sigma \eta_G + a_G F, \\
ch_3(G) &= s_G
\end{align*}
\]

then

\[
\begin{align*}
ch_0(S_{I}(G)) &= x_G - n_G \\
ch_1(S_{I}(G)) &= -x_G \sigma - S_G + (n_G - \frac{1}{2} x_G) c_1 + \eta_G \\
ch_2(S_{I}(G)) &= -\sigma \eta_G - a_G F - \left(\frac{1}{2} n_G - \frac{1}{12} x_G\right) c_1^2 + c_1 S_G - \frac{1}{2} \eta_G c_1 + s_G F \\
ch_3(S_{I}(G)) &= -s_G
\end{align*}
\]
Let us now start with a sheaf $V$ whose Chern character is written as in (2.31) in the form $ch_0(V) = n$, $ch_1(V) = x\sigma + S$, $ch_2(V) = \sigma\eta + aF$, $ch_3(V) = s$. By applying the composition (4.7) we obtain

$$ch(\hat{S}(V)) = ch(O(2c_1)) \cdot ch(O(\sigma)) \cdot ch(S_T(G))$$

with $G = V \otimes O(\sigma)$. The Chern character of $G$ is given in terms of $ch(V)$ by (3.18). We then have

$$n_G = n, \quad x_G = x + n, \quad S_G = S, \quad \eta_G = \eta - \frac{1}{2} nc_1 + S - xc_1$$
$$a_G = a, \quad s_G = s - \sigma c_1 \eta + a + \frac{1}{2} x\sigma c_1^2 - \frac{1}{2} \sigma c_1 S + \frac{1}{6} n \sigma c_1^2$$

Now from (4.7) we expect the following relation for the monodromy corresponding to the relative FM transform expressed as a product of known Kontsevich monodromies and the monodromy around the diagonal of the fibre product whose ideal sheaf is $I$

$$S_V = S_\sigma \cdot S^2_{c_1} \cdot S_T \cdot S_{\sigma}$$

Let us make this more explicit by considering the degree 18 model.

**Example**

Consider the elliptic fibration given by $P^4_{1,1,1,6,9}[18]$. This model has been extensively studied in the context of mirror symmetry [16] and in the context of D-branes on elliptic Calabi-Yau [7]. Among the degree 18 hypersurfaces is

$$z_1^{18} + z_2^{18} + z_3^{18} + z_4^3 + z_5^2 = 0$$

At $z_1 = z_2 = z_3 = 0$ the ambient space has a singular line which intersects $X$ in a single point. The blow up of this line gives an exceptional divisor $E = P^2$ in $X$. A second divisor $L$ (defined by the first order polynomials) is given by the elliptic surface over a line in $P^2$ and together with $E$ generates $H_4(X, \mathbb{Z})$. The elliptic fibration structure is induced by the linear system $|L|$ generated by $z_1, z_2, z_3$ mapping $X$ to $P^2$. The section of the fibration is given by $B_2 = E$. The homology class of the elliptic fibre in $H_2(X)$ will be denoted by $h = L^2$. Further intersection relations are given by

$$E \cdot L^2 = 1, \quad E^2 \cdot L = -3, \quad L^3 = 0, \quad E^3 = 9$$
Working in the $E,L$ basis (cf. [7]) the generic Kähler class is given by $J = t_1 E + t_2 L$ with $t_1, t_2$ coordinates on the Kähler moduli space.

In the degree 18 model again one has $\sigma = E = B = P^2$ and $\pi^* K_B = 3L$; note that $H = 3L + E$ with a corresponding multiplicative relation $S_E = S_H \cdot S_L^{-3}$ for the matrices (as the Chern character is multiplicative).

Here the relation becomes

$$S_V = S_E \cdot S_L^6 \cdot S_I \cdot S_E$$

whereas the matrices are given by

$$S_H = \begin{pmatrix} 1 & -1 & -3 & 10 & 9 & 3 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -3 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad S_L = \begin{pmatrix} 1 & 0 & -1 & 3 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

$$S_E = \begin{pmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -9 & 0 & 3 \\ 0 & 0 & 1 & 3 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 1 \end{pmatrix}$$

Note that these matrices commute (cf. also [15], p. 12) and that $S_E = S_H \cdot S_L^{-3}$.

In order to obtain the matrices $S_I$ and $S_V$ we have to use the comparison (which has been performed for this model in [7]) of the central charges $Z(n)$ and $Z(Q)$ which gives the relation between the middle cohomology charges $n$ and the cohomological invariants of the vector bundle $G$

$$\begin{align*}
ch_0(G) &= n \\
ch_1(G) &= \alpha E + \beta L \\
ch_2(G) &= \gamma EL + \delta L^2 \\
ch_3(G) &= \epsilon
\end{align*}$$

(4.18)
where the coefficients are given in terms of the BPS charge vector \( n \)

\[
\begin{align*}
  n &= n_6 \\
  \alpha &= n_4^1, \quad \beta = n_4^2 \\
  \gamma &= \frac{3}{2} n_4^1 + n_2^2, \quad \delta = \frac{3}{2} n_4^2 + n_1^1, \\
  \epsilon &= -n_0 + \frac{1}{2} n_1^1 - 3 n_3^2
\end{align*}
\]

(4.19)

Now the FM transforms \( \hat{S}(.) \) and \( S_{\mathcal{I}}(.) \) induce a linear transformation on the BPS charge lattice (i.e. one compares the coefficients of (4.18) with the new coefficients given by the cohomological invariants of \( \hat{S}(.) \) and \( S_{\mathcal{I}}(.) \)).

\[
S_{\mathcal{I}} = \begin{pmatrix}
  1 & 0 & 3 & -9 & 0 & 0 \\
  1 & 1 & 3 & -9 & -1 & 0 \\
  0 & 0 & 1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 1 & 1 & 0 \\
  0 & 0 & 0 & 0 & 1 & 0 \\
  0 & 0 & 1 & -3 & 0 & 1
\end{pmatrix}, \quad S_{\mathcal{V}} = \begin{pmatrix}
  0 & -1 & 0 & 1 & 0 & 3 \\
  1 & 0 & 0 & 0 & -1 & 0 \\
  0 & 0 & 0 & 0 & -3 & -1 \\
  0 & 0 & 0 & 0 & -1 & 0 \\
  0 & 0 & 0 & 1 & 0 & 0 \\
  0 & 0 & 1 & -3 & -3 & 0
\end{pmatrix}
\]

(4.20)

with \( S_{\mathcal{I}}^{-1} = P(S_{\mathcal{I}}(\cdot)) \), \( S_{\mathcal{V}}^{-1} = P(\hat{S}(\cdot)) \)

denoting the linear transformations on the lattice \( n \).

Finally, using these matrices we can write the \( M \) matrix as (for \( \alpha = n_4^1 = 0 \))

\[
M = l \cdot [S_{\mathcal{V}}^{-1}]^t \cdot l^{-1} \cdot S_{td}
\]

(4.21)

where \( l \) relates (cf. (4.19)) the period basis with the 'fibration' basis (2.7) and \( S_{td} \) represents the \( Td(N) \) twist for this model (cf. appendix)

\[
l = \begin{pmatrix}
  1 & 0 & 0 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 & 0 & 0 \\
  0 & 0 & 1 & 0 & 0 & 0 \\
  0 & 3/2 & 0 & 0 & 0 & 1 \\
  0 & 0 & 3/2 & 0 & 1 & 0 \\
  0 & 1/2 & -3 & -1 & 0 & 0
\end{pmatrix}, \quad S_{td} = \begin{pmatrix}
  1 & 0 & 0 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 & 0 & 0 \\
  -3/2 & 0 & 1 & 0 & 0 & 0 \\
  0 & -3/2 & 0 & 1 & 0 & 0 \\
  3/4 & 0 & -3/2 & 0 & 1 & 0 \\
  0 & 3/4 & 0 & -3/2 & 0 & 1
\end{pmatrix}
\]

(4.22)
5. Some analogues on the mirror side

Let us note that (4.21) also shows that if we perform a linear transformation on $H^\ast_{\text{odd}}(Y)$ (transforming in the “fibre base”) by $l$ the $M$ matrix naturally operates on the charge lattice as

$$
\begin{pmatrix}
\alpha \\
\gamma \\
\delta \\
\epsilon
\end{pmatrix}
= M \cdot
\begin{pmatrix}
-\alpha \\
-\gamma \\
-\epsilon
\end{pmatrix}
$$

and thus (4.21) gives just the right transport $^\ast$ of the $M$ matrix to $H^\ast_{\text{odd}}(Y)$ (where we have a priori only the period basis)

$$
\begin{array}{ccc}
H^{\text{even}}(X) & \xrightarrow{M} & H^{\text{even}}(X) \\
\downarrow & & \downarrow \\
H^3(Y) & \xrightarrow{M} & H^3(Y)
\end{array}
$$

Thus the $l$ transformation shows how to get the ‘fibration’ basis (2.7) from the Kähler period vector basis, resp. (after identification with the mirror side) how the (2.7) basis (when transported via identification to the mirror side) is related to the complex structure period vector basis there in IIB, making explicit the decomposition in $H^{\text{odd}}(X_{\text{mirror}})$ corresponding to (2.7)

$$
H^{\text{odd}}(X_{\text{mirror}}) = H^{\text{non-ell}} \oplus H^{\ell}
$$

Then the interesting question remains whether this decomposition and ‘duality’ transformation on the middle cohomology of the mirror is actually induced by a natural decomposition and map on the mirror space. It would be interesting to unfold that question about mirror transport of a certain involution also in the context of involutions such as the involution on $X$ discussed later in sect. 5 in connection with the $\mathbb{Z}_2$ index theorem or Poincaré duality or (fibrewise) complex structure conjugation.

$^27$ Note as a caveat that the $T^2$ which is given by the holomorphic elliptic curve used in the fibrewise T-duality on the bundle side (an operation inside $H^{\text{even}}(X)$) is not contained in the $T^3$ used in the Strominger/Yau/Zaslow T-duality employed to go from $X$ to $Y$; only one $S^1$ of the elliptic curve re-occurs in the $T^3$. 

33
5.1. D6-brane moduli space and extended mirror conjecture

Note that as the relevant object to consider on the bundle side is a holomorphic cycle (so even-dimensional) with a bundle over it correspondingly the relevant object on the sLag side is a special Lagrangian submanifold $C$ (three-dimensional) with a $U(1)$ bundle over it. According to McLean’s theorem the number of real (extrinsic) motions of $C$ inside $Y$ is the same as the (intrinsic) number $b_1(C)$; these real moduli then pair up with the same number of real moduli of the $U(1)$ bundle, combining to $b_1(C)$ complex moduli.

$$h^1(\text{End}(V), X) = h^1(C)$$

(5.1)

For a complex surface the complex dimension of the moduli space of irreducible bundles (sheafs) is completely determined by the Mukai vector $Q$

$$\dim_{\mathbb{C}} \mathcal{M}(Q) = Q^2 + 2$$

(5.2)

which can be derived using the fact that we have a non-vanishing index

$$\chi(X, \text{End}(V)) = \sum_{i=0}^{2} (-1)^i \dim H^i(X, \text{End}(V))$$

(5.3)

Now this index becomes in the case of a Calabi-Yau three-fold trivial. One has (cf. section 6) by self-duality of $\text{End}(V) = V \otimes V^*$ and Serre duality

$$\sum_{i=0}^{3} (-1)^i \dim H^i(X, \text{End}(V)) = 0$$

(5.4)

$\mathbb{Z}_2$ Index Theorem

Let us consider manifolds $X$ with a group of non-trivial automorphisms which extend to automorphisms of the vector bundle $V$ over it. For those $X$ the Atiyah-Singer index theorem has a natural generalization [56], [57], the character-valued index theorem which describes how the zero modes of the Dirac operator transform under this automorphism group.

Now the class of elliptic fibered $X$ carry such a symmetry due to the involution $\tau$, the “sign-flip” in the elliptic fibers. We will assume that at some point in the moduli space,
the \( \tau \)-invariant point, the symmetry can be lifted to an action on the bundle. Then we can think of \( \tau \) as decomposing \( H^i(End(V)) \) into even and odd subspaces \( H^i_e(End(V)) \) and \( H^i_o(End(V)) \). Now Serre duality involves multiplying by a holomorphic three-form (which is odd) and thus maps \( H^i_e(End(V)) \) to \( H^{3-i}_o(End(V)) \). If one projects on the \( \tau \)-invariant part of the index problem one gets \[25\]

\[-\frac{1}{2} \sum_{i=0}^3 (-1)^i Tr_{H^i(X,End(V))} \tau = - \sum_{i=0}^3 (-1)^i \dim H^i(X,End(V)) \]

(5.5)

which is a “character valued index” and can be effectively computed by a fix point theorem \[25\]. Now, using the fact that the components of the fixed point set are of codimension two and orientable in the case of elliptic fibered Calabi-Yau three-folds one gets for the sum

\[ \sum_{i=0}^3 (-1)^i Tr_{H^i(X,End(V))} \tau = \sum_i \int_{U_i} \frac{\text{ch}(End(V)_{i,e}) - \text{ch}(End(V)_{i,o})}{1 + e^{c_1(N)}} Td(U_i) \]

(5.6)

where \( End(V)_{i,e} \) denotes the restriction of the even resp. odd subspaces of \( End(V) \) to \( U_i \) and \( N_i \) is the normal bundle of \( U_i \) in \( X \). This leads \[25\] to the dimension of the moduli space (of \( \tau \) invariant bundles; we will then assume that \( n_{odd} = h^{1,0}(C) = 0 \), cf. also \[32\])

\[ \dim_{\mathbb{C}} \mathcal{M}(Q) = I + 2h^{1,0}(C) = r(V) - \sum_j \int_{U_j} c_2(V) + 2h^{1,0}(C) \]

(5.7)

For a vector bundle as given in chapter 2 which has (with \( x = 0 \))

\[ c_2(V) = \frac{L^2}{2} - \sigma \eta - aF \]

(5.8)

one gets (using the fact that \( L^2 c_1 = c_1 F = 0 \))

\[ \sum_j \int_{U_j} c_2(V) = 2L^2 \sigma + \sigma \eta c_1 - 4a \]

(5.9)

**Example**

Let us consider \( X \) again being given by \( \mathbb{P}^4_{1,1,1,6,9} [18] \) (cf. \[16\]). The fixed point set can be best described using the Weierstrass model of \( X \) given by \( x_3^2 + x_4^3 + f x_4 + g = 0 \) where
\[ f = f(x_1, x_2, x_3) \] has degree 12, and \[ g = g(x_1, x_2, x_3) \] has degree 18 and \((x_1, x_2, x_3)\) are coordinates on the base \(B = \mathbb{P}^2\). The \(\tau\) symmetry manifests itself in \(x_5 \rightarrow -x_5\) and the two components of the fixed point set are given by \(x_4 = 0, x_5 \neq 0\) which is isomorphic to a copy of the base (the section \(E \) of \(X\)) on the one hand; the other component is a triple cover of the base \(B\) given by \(0 = x_3^4 + fx_4 + g\). For this model the Chern-classes can be expressed as

\[
\begin{align*}
ch_1(V) &= n_4^2 L \\
ch_2(V) &= (\frac{3}{2} n_4^2 + n_2^1) L^2 + (\frac{3}{2} n_4^1 + n_2^2) EL \\
ch_3(V) &= -n_0 + \frac{1}{2} n_4^1 - 3n_2^2
\end{align*}
\]  

(assuming here again \(n_4^1 = 0\).) Since \(c_1(V)\) is non zero we get from the integrability condition \(\omega^2 \wedge c_1(V) = 0\) - guaranteeing a unique solution to the Donaldson-Uhlenbeck-Yau equation - that (using \(\omega = t_1 H + t_2 L\))

\[
\int_X (3t_1^2 + 2t_1 t_2)n_4^2 = 0
\]

For the dimension of the moduli space we get therefore

\[
h^1(\text{End}(V), X) = n_6 - 3n_2^2 + 6n_4^2 + 4n_2^1 - 2(n_4^2)^2
\]

**Connection to FMW bundles**

Let us now see which BPS vectors \(n\) describe the bundles constructed by Friedman, Morgan and Witten \[25\]. The bundles which are invariant under the involution of the elliptic fiber have \(c_1(V) = c_3(V) = 0\) and \(\eta \equiv c_1(B) \mod 2\) and \(n\) is even. One has

\[
c_2(V) = \eta \sigma - \frac{(n^3 - n)}{24} c_1^2 - \frac{n}{8} \eta(n - nc_1)
\]

therefore these bundles are described by BPS vectors

\[
n = (n_6, 0, 0, 0, n_2^1, n_2^2)
\]

In order to get a dictionary between the BPS charges and the bundles data in the FMW set-up we have to express (5.13) in terms of the base \((E, S)\). Therefore setting \(\eta = ac_1(B)\) and \(a\) odd we get

\[
c_2(V) = 3aES - \frac{3(n^3 - n) + 9a(a - n)n}{8} S^2
\]
and comparing with
\[ c_2(V) = -n_2^2ES - n_1^2S^2 \] (5.16)
leads then to the dictionary
\[ n_2^2 = -3a \]
\[ n_1^2 = \frac{3(n^3 - n) + 9a(a - n)n}{8} \] (5.17)
\[ n_6 = n \]

6. Moduli space for D4-Branes and applications to FM-transform and spectral covers

We will be now interested in the dimension of the moduli space of a D4-brane configuration on a divisor $D$ in $X$, so we consider the embedding $i : D \rightarrow X$. Further let consider a vector bundle $E$ over $D$. The conditions for unbroken supersymmetry are now replaced by the generalized Hitchin equations. The associated K-theory class is now given by the torsion sheaf $i_*E$ (being the extension of $E$ by zero to $X$). The Mukai vector is then given by applying GRR for the embedding $i$

\[ i_*(\text{ch}(E)Td(D)) = \text{ch}(i_*E)Td(X) \] (6.1)

We will first compare $\text{Ext}^1_X(i_*E, i_*E)$ and $\text{Ext}^1_D(E, E)$ where the first one can be bigger by deformations (movements) of $D$ in $X$ and then for the case of $E$ a line bundle $L$ on the spectral cover $C$ compare $\text{Ext}^1_X(V, V)$ and $\text{Ext}^1_X(i_*L, i_*L)$ explicitly.

The Moduli-Space

The dimension of the associated moduli space relevant here is given by the dimension of $\text{Ext}^1_D(E, E)$ respectively $\text{Ext}^1_X(i_*E, i_*E)$. One can in general expect that the dimension of the moduli space associated to $i_*E$ living over $X$ is bigger then the dimension of the moduli space of $E$ over $D$. This is because, naively speaking, $D$ can move inside $X$ and therefore leads to additional deformations (the number of global deformations of $D$ in $X$) which are related to the number of sections of the normal bundle, i.e. the dimension of $H^0(N)$ (cf. in the context of $F$-theory [33]). This additional deformations play an important role in
the comparison of D-brane moduli with the number of CFT moduli as pointed out in the
$K3$-fibration case [9]. The naive picture can be made precise by considering the long exact
sequence (first written down and proven in [58])

$$0 \to \text{Ext}^1_D(E, E) \to \text{Ext}^1_X(i_*E, i_*E) \to \text{Ext}^0_D(E, E \otimes N) \to \text{Ext}^2_D(E, E) \to 0$$

(6.2)

The above exact sequence can be derived from Grothendieck duality for the closed immersion $i : D \to X$ (see [59] Section §6). One has an isomorphism in the derived category

$$R\text{Hom}_X(i_*E, i_*E) = R\text{Hom}_D(i_*E, i^!(i_*E))$$

and $i^!(i_*E)$ is determined by the equation $i_*(i^!(i_*E)) = R\text{Hom}_D(i_*O_D, i_*E)$ (where $\text{Hom}$ stands for the Hom-sheaf). Form the exact sequence

$$0 \to O_X(-D) \to O_X \to i_*O_D \to 0$$

we read that $R\text{Hom}_D(i_*O_D, i_*E)$ is represented by the complex

$$i_*E \xrightarrow{d=0} \text{Hom}_O(X(-D), i_*E)$$

that is,

$$i^!(i_*E) = \{ E \xrightarrow{d=0} E \otimes N \}$$

(6.3)

in the derived category. Then, since $E$ is a vector bundle, we have

$$R\text{Hom}_X(i_*E, i_*E) = R\text{Hom}_D(i_*E, \{ E \xrightarrow{d=0} E \otimes N \})$$

$$= R\Gamma(D, \{ \text{End}(E) \xrightarrow{d=0} \text{End}(E) \otimes N \})$$

(6.4)

Taking into account the natural isomorphisms $\text{Ext}^i_D(E, E) = H^i(D, \text{End}(E))$ and $\text{Ext}^i_D(E, E \otimes N) = H^i(D, \text{End}(E) \otimes N)$, the exact sequence (6.2) is identified with the sequence of the low terms

$$0 \to E_2^{1,0} \to H^1(M) \to E_2^{0,1} \xrightarrow{d_2} E_2^{2,0} \to H^2(M)$$

of the spectral sequence approaching $\text{Ext}^i_X(i_*E, i_*E)$ from the double complex associated to an injective resolution of $\{ \text{End}(E) \xrightarrow{d=0} \text{End}(E) \otimes N \}$. We have here a more complete
information: since $d = 0$ the first differential of this double complex is zero and then $d_2 : E^{0,1}_2 \to E^{2,0}_2$ is zero as well. The sequence (6.2) takes now the form

$$0 \to \Ext^1_D(E, E) = H^1(D, End(E)) \to \Ext^1_X(i_*E, i_*E) \to H^0(D, End(E) \otimes N) \to 0$$

(6.5)

When $E = L$ is a line bundle on a spectral cover $D = C$, $End(L) = \mathcal{O}_C$ and we have

$$0 \to H^1(C, \mathcal{O}_C) \to \Ext^1_X(i_*L, i_*L) \to H^0(C, \mathcal{O}_C) \to 0$$

so that

$$\dim \Ext^1_D(L, L) = h^1(C, \mathcal{O}_D) = h^{(0,1)}(C)$$
$$\dim \Ext^1_X(i_*L, i_*L) = h^1(C, \mathcal{O}_D) + h^0(C, N) = h^{(0,1)}(C) + h^{(2,0)}(C)$$

(6.6)

where the very last formula uses that the ambient $X$ is CY.

If we consider the vector bundle $V = S^0(i_*L)$ derived from $L$ by the spectral cover construction (or by the FM transform), then $V$ is WIT$_1$ and its unique inverse FM transform goes back to $\hat{S}^1(V) = i_*L$. Then, by the “Parceval isomorphism” (see [50], [51] or [18]), we have

$$\Ext^1_X(V, V) = \Ext^1_X(\hat{S}^1(V), \hat{S}^1(V)) = \Ext^1_X(i_*L, i_*L)$$

(6.7)

and then

$$\dim \Ext^1_D(L, L) = h^1(C, \mathcal{O}_D) = h^{(0,1)}(C)$$
$$\dim \Ext^1_X(V, V) = h^1(C, \mathcal{O}_D) + h^0(C, N) = h^{(0,1)}(C) + h^{(2,0)}(C)$$

(6.8)

We now want to show explicitly that $\dim \Ext^1_X(V, V) = \dim \Ext^1_X(i_*L, i_*L)$ without using (6.7). Let us recall that from Serre duality one has

$$\sum_{i=0}^{3} (-1)^i \dim \Ext^i(V, V) = 0$$

(6.9)

for $V$ on a Calabi-Yau. However, since we work on elliptic CY we can use the character valued index and compute in the spectral cover representation that (cf. [32])

$$\dim H^1(X, \text{End}(V)) = h^{(2,0)}(C) + h^{(1,0)}(C)$$

(6.10)
showing together with (6.6) the wished for agreement.

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Appendix

In this appendix we illustrate why in the three-fold case the sole use of the $T$-functor known from the $K3$-case to map the Chern classes of the bundle and its dual is insufficient to exhibit as transformation matrix the adiabatic extension (2.5) of the usual T-duality matrix (2.4) on the fibre.

Let us recall the findings in this case (we phrase them here in the language of [25]).

The Chern character of $V$ is given by

$$ r(V) = n $$
$$ ch_1(V) = 0 $$
$$ ch_2(V) = -\eta \sigma - \eta c_1 + \pi_*(\frac{c_1(C)^2 + c_2(C)}{12}) + \frac{c_1(L)c_1(C)}{2} + \frac{c_1(L)^2}{2} - \frac{n(c_1^2 + c_2)}{12} $$

$$ = -\eta \sigma + \pi_*(\frac{C^2}{24} + \frac{1}{2}\gamma^2) - \frac{n c_1^2}{24} $$

$$ ch_3(V) = \lambda \eta (\eta - nc_1) \cdot \sigma = -\gamma C \cdot \sigma $$

Using the decomposition of the cohomology we find

$$ ch(V) = \begin{pmatrix}
  n \\
  0 \\
  0 \\
  \pi_*(\frac{C^2}{24} + \frac{1}{2}\gamma^2) - \frac{n c_1^2}{24} \\
  -\gamma C
\end{pmatrix} $$

---

Note that $ch_2(V) = -c_2(V)$ can be computed [25] from its restriction to $\sigma = B$ via $(\pi|_{C})_* (e^{c_1(L)Td(C)}) = ch(V|_{\sigma})Td(B)$ where then the term $-\eta c_1$ is corrected/lifted to $\eta \sigma$ leading to the correction $-(\eta \sigma + \eta c_1)$ above. For the different evaluations of $ch_3(V) = c_3(V)/2$ cf. [27]; note also that $C \sim n\sigma + \eta$, $c_1(C) = -C|_{C}$, $c_2(C) = c_2(X)|_{C} + C^2$, $c_1(L) = \frac{c_1^2 + c_1}{2} + \gamma$ and $c_2(X) = 12\sigma c_1 + c_2 + 11c_1^2$. 

40
Using GRR we get on the other hand for $ch(i_*L)$

$$r(i_*L) = 0$$
$$ch_1(i_*L) = C$$
$$ch_2(i_*L) = i_*(c_1(L) + \frac{c_1(C)}{2})$$
$$= i_*(\frac{c_1}{2} + \gamma)$$
$$ch_3(i_*L) = i_*(\frac{C^2 + 3c_1^2}{24} + \frac{\gamma(c_1 + \gamma)}{2})$$

The T-functor, analogous to $K3$, in the three-fold case

If one tries for the T-functor again

$$T^3(\cdot) = S(\cdot) \otimes \pi^* K_{B}^{-1/2}$$

one will get not completely the vector needed for the M matrix:

The Chern-characters of $T^3(V) = \tilde{V} \otimes K_{B}^{-1/2} = i_*L \otimes K_{B}^{1/2}$ are by (3.18) given by

$$r(T^3(V)) = 0$$
$$ch_1(T^3(V)) = C = n\sigma + \eta$$
$$ch_2(T^3(V)) = \gamma C$$
$$ch_3(T^3(V)) = i_*(\frac{C^2}{24} + \frac{\gamma^2}{2})$$

and so

$$ch(T^3(V) = \begin{pmatrix}
0 \\
n \\
\eta \\
0 \\
i_*(\frac{C^2}{24} + \frac{\gamma^2}{2})
\end{pmatrix}$$

---

29 Note that the intersection products in $C$ pushed down to $B$ by $\pi_*$ (as used in the computations for $V$) equal the same intersection products pushed forward to $X$ via $i_*$ (as used in the computations for $V$) as only those triple intersections survive which have one $\sigma$ factor and then the remaining intersection product of two classes in $B$ equals the expression obtained on the first route (also note that $i_*1 = C$, $\pi_*\sigma = \eta - nc_1$ and $\pi_*1 = n$).
showing the mismatch of $nc_1^2/24$ between the fifth entry of $ch(V)$ and the sixth entry of $ch(T^2(V))$. Even without explicit computation of both sides one can see directly why this can not work as (the last step uses $V|_B = \pi_{C*} L$ following from $V = \pi_1^*(\pi_2^* L \otimes P)$)

$$
\int_X ch(i_* L)Td(X) = \int_X i_* (ch(L)Td(C)) = \int_C ch(L)Td(C)
= \int_B \pi_{C*} (ch(L)Td(C)) = \int_B ch(\pi_{C*} L)Td(B)
= \int_B ch(V|_B)Td(B)
$$

This shows that

$$
\int_X ch_3 (i_* L) + C \frac{c_2(X)}{12} = \int_B ch_2 (V|_B) + n \frac{c_1^2 + c_2}{12}
$$
or, in other words 30 (where in the last step $-\omega ch_2(V) = ch_2 (V|_B) - \eta c_1$ is used 31) 

$$
\int_X ch_3 (T^2(V)) = \int_B ch_2 (V|_B) + (chK_B^{1/2} - Td(X) + Td(B))|_{(2)} \cdot C
= \int_B ch_2 (V|_B) + (\frac{c_1^2}{8} - \frac{c_2(X)}{12} + \frac{c_1^2 + c_2}{12}) \cdot C
= \int_B ch_2 (V|_B) + n \frac{c_1^2}{24} - \eta c_1
= -\omega ch_2 (V) + n \frac{c_1^2}{24}
$$

The amended T-functor

However, if we introduce the T functor

$$
T(\cdot) = S(\cdot) \times Y
$$

(so that for example $T(V) = \tilde{V} \otimes Y$ and $T(i_* L) = V \otimes Y$) with a sheaf $Y$ of

$$
ch(Y) = (1 + \frac{c_1}{2} + \frac{c_1^2}{6}) = 1/(1 - \frac{c_1}{2} + \frac{c_1^2}{12}) = Td(N)^{-1}
$$

30 here $|_{(2)}$ denotes the terms of complex dimension 2; note that it is $-(Td(X) - Td(B))|_{(2)}$ what is used and not $-\frac{Td(X)}{Td(B)}|_{(2)}$; the latter would have a $\frac{c_1^2}{4}$ correction.

31 with the notation $c_2(V) = \eta \sigma + \omega$ where $\omega \in H^4(B)$ (pullback for $\omega$ understood)
where $N$ denotes the normal bundle of $B$ in $X$, then we find

$$ch(T(i_*L)) = ch(V)ch(Y) = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -1 & 0
\end{pmatrix} ch(i_*L)$$

Equivalently we can write (note that $Td(N) = ch(K_B^{1/2}) \cdot (1 - \frac{c_1^2}{24})$)

$$ch(V) = M \cdot ch(i_*L)Td(N)$$

$$= M \cdot ch(i_*L)(1 - \frac{c_1}{2} + \frac{c_1^2}{12}) = M \cdot ch(i_*L)(1 - \frac{c_1}{2} + \frac{c_1^2}{8} - \frac{c_1^2}{24})$$

But note that the appearance of the $Td(N)^{-1}$ term cannot be understood by taking for example just $Y = j_*(\mathcal{O}_\sigma)$, as GRR for the embedding $j : B \hookrightarrow X$ gives actually

$$ch(j_*(\mathcal{O}_\sigma)) = j_*(ch((\mathcal{O}_\sigma))Td(B))Td(X)^{-1} = j_*(ch((\mathcal{O}_\sigma))Td(B)/j^*Td(X))$$

$$= j_*(ch((\mathcal{O}_\sigma))Td(N)^{-1}) = j_*(Td(N)^{-1}) = \sigma Td(N)^{-1}$$

**$Td(N)$ twist as matrix**

One can present the $Td(N)$ twist as a matrix if one considers its operation on a general cohomology vector

$$v = \begin{pmatrix}
n \\
x \\
S \\
\eta \\
a \\
s
\end{pmatrix} = n + (x\sigma + S) + (\eta\sigma + aF) + s$$

that for a twist by $Td(N) = 1 - \frac{c_1}{2} + \frac{c_1^2}{12}$ (or similar for $Td(N)^{-1} = 1 + \frac{c_1}{2} + \frac{c_1^2}{6}$)

$$Td(N)v = n + (x\sigma + S) + (\eta\sigma + aF) + s = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
-\frac{c_1}{2} & 0 & 1 & 0 & 0 & 0 \\
0 & -\frac{c_1}{2} & 0 & 1 & 0 & 0 \\
\frac{c_1^2}{12} & 0 & -\frac{c_1}{2} & 0 & 1 & 0 \\
0 & \frac{c_1^2}{12} & 0 & -\frac{c_1}{2} & 0 & 1
\end{pmatrix} v$$

which when one also decomposes the base cohomology becomes a matrix of numbers.
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