A Block Decomposition Algorithm for Sparse Optimization

Ganzhao Yuan\textsuperscript{1}, Li Shen\textsuperscript{2}, Wei-Shi Zheng\textsuperscript{3}
\textsuperscript{1} Center for Quantum Computing, Peng Cheng Laboratory, Shenzhen, China
\textsuperscript{2} Tencent AI Lab, Shenzhen, China \hspace{0.5cm} \textsuperscript{3} Sun Yat-sen University, Guangzhou, China
yuanganzhao@foxmail.com, mathshenli@gmail.com, zhwshi@mail.sysu.edu.cn

Abstract

Sparse optimization is a central problem in machine learning and computer vision. However, this problem is inherently NP-hard and thus difficult to solve in general. Combinatorial search methods find the global optimal solution but are confined to small-sized problems, while coordinate descent methods are efficient but often suffer from poor local minima. This paper considers a new block decomposition algorithm that combines the effectiveness of combinatorial search methods and the efficiency of coordinate descent methods. Specifically, we consider a random strategy or a greedy strategy to select a subset of coordinates as the working set, and then perform a global combinatorial search over the working set based on the original objective function. We show that our method finds stronger stationary points than Amir Beck et al.’s coordinate-wise optimization method. In addition, we establish the global convergence and convergence rate of our block decomposition algorithm. Our experiments on solving sparse regularized and sparsity constrained least squares optimization problems demonstrate that our method achieves state-of-the-art performance in terms of accuracy.

1 Introduction

In this paper, we mainly focus on the following sparse optimization problem (‘≜’ means define):

\[ \min_{x \in \mathbb{R}^n} F(x) \triangleq f(x) + h(x) \] (1)

where \( f(x) \) is a smooth convex function with its gradient being \( L \)-Lipschitz continuous, and \( h(x) \) is the following nonconvex sparse regularized/sparsity constrained function:

\[ \{ h_{\text{regu}}(x) \triangleq \lambda \|x\|_0 + I_{\Omega}(x) \} \quad \text{or} \quad \{ h_{\text{cons}}(x) \triangleq I_{\Psi}(x), \Psi \triangleq \{ x \mid \|x\|_0 \leq s \} \}. \] (2)

Here, \( \Omega \triangleq \{ x \mid \|x\|_\infty \leq \rho \} \), \( I_\Psi(\cdot) \) is an indicator function on the set \( \Psi \) with \( I_\Psi(x) = \{ 0, x \in \Psi, \|x\|_\infty \leq \rho \} \), \( \|\cdot\|_0 \) is a function that counts the number of nonzero elements in a vector, \( \lambda \) and \( \rho \) are positive constants, and \( s \in [n] \) is a positive integer. Problem (1) captures a variety of applications of interest in both machine learning and computer vision (e.g., sparse coding \cite{10, 16, 11, 2}, sparse subspace clustering \cite{17}).

This paper proposes a block decomposition algorithm using a proximal strategy and a combinatorial search strategy for solving the sparse optimization problem as in (1). We review existing methods in the literature and summarize the merit of our approach.

• The Relaxed Approximation Method. One popular method to solve (1) is the convex or nonconvex relaxed approximation method. Many approaches such as \( \ell_1 \) norm, top-k norm, Schatten \( \ell_p \) norm, re-weighted \( \ell_1 \) norm, capped \( \ell_1 \) norm, and half quadratic function have been proposed for solving

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sparse optimization problems in the last decade. It is generally believed that nonconvex methods often achieve better accuracy than the convex counterparts. However, minimizing the approximate function does not necessarily lead to the minimization of the original function in Problem (1). Our method directly controls the sparsity of the solution and minimize the original problem.

• **The Greedy Pursuit Method.** This method is often used to solve sparsity constrained optimization problems. It greedily selects at each step one coordinate of the variables which have some desirable benefits [39] [4] [50] [29]. This method has a monotonically decreasing property and achieves optimality guarantees in some situations, but it is limited to solving problems with smooth objective functions (typically the square function). Furthermore, the solutions must be initialized to zero and may cause divergence when being incorporated to solve the bilinear matrix factorization problem [2]. Our method is a greedy coordinate descent algorithm without forcing the initial solution to zero.

• **The Combinatorial Search Method.** This method is typically concerned with NP-hard problems [14]. A naive strategy is an exhaustive search which systematically enumerates all possible candidates for the solution and picks the best candidate corresponding to the lowest objective value. The cutting plane method solves the convex linear programming relaxation and adds linear constraints to drive the solution towards binary variables, while the branch-and-cut method performs branches and applies cuts at the nodes of the tree having a lower bound that is worse than the current solution. Although in some cases these two methods converge without much effort, in the worse case they end up solving all $2^n$ convex subproblems. Our approach leverages the effectiveness of combinatorial search methods.

• **The Proximal Gradient Method.** Based on the current gradient $\nabla f(x^k)$, the proximal gradient method [3] [25] [21] [13] [34] [35] [23] iteratively performs a gradient update followed by a proximal operation: $x^{k+1} = \text{prox}_{\nu,h}(x^k - \nu \nabla f(x^k))$. Here the proximal operator $\text{prox}_{\nu,h}(a) = \underset{x}{\arg\min} \frac{1}{2\nu} \|x - a\|^2 + h(x)$ can be evaluated analytically, and $\nu = 1/L$ is the step size with $L$ being the Lipschitz constant. This method is closely related to (block) coordinate descent [31] [11] [9] [15] [37] [5] [18] [26] [42] in the literature. Due to its simplicity, many strategies (e.g., variance reduction [22] [41] [12], asynchronous parallelism [24] [58], and non-uniform sampling [45]) have been proposed to accelerate proximal gradient method. However, existing works use a scalar step size and solve a first-order majorization/surrogate function via closed-form updates. Since Problem (1) is nonconvex, such a simple majorization function may not necessarily be a good approximation. Our method significantly outperforms proximal gradient method and inherits its computational advantages.

**Contributions:** The contributions of this paper are three-fold. (i) Algorithmically, we introduce a novel block decomposition method for sparse optimization (See Section 2). (ii) Theoretically, we establish the optimality hierarchy of our algorithm and show that it always finds stronger stationary points than existing methods (See Section 3). Furthermore, we prove the global convergence and convergence rate of our algorithm (See Section 4). (iii) Empirically, we have conducted experiments on some sparse optimization tasks to show the superiority of our method (See Section 5).

**Notation:** All vectors are column vectors and superscript $T$ denotes transpose. For any vector $x \in \mathbb{R}^n$ and any $i \in \{1, 2, ..., n\}$, we denote by $x_i$ the $i$-th component of $x$. The Euclidean inner product between $x$ and $y$ is denoted by $\langle x, y \rangle$ or $x^T y$. $e_i$ is a unit vector with a 1 in the $i$th entry and 0 in all other entries. The number of possible combinations choosing $k$ items from $n$ without repetition is denoted by $C_n^k$. For any $B \in \mathbb{N}_k$ containing $k$ unique integers selected from $\{1, 2, ..., n\}$, we define $\bar{B} \triangleq \{1, ..., n\} \setminus B$.

2 **Proposed Block Decomposition Algorithm**

This section presents our block decomposition algorithm for solving (1). Our algorithm is an iterative procedure. In every iteration, the index set of variables is separated into two sets $B$ and $\bar{B}$, where $B$ is the working set. We fix the variables corresponding to $\bar{B}$, while minimizing a sub-problem on variables corresponding to $B$. We use $x_B$ to denote the sub-vector of $x$ indexed by $B$. The proposed method is summarized in Algorithm [1].

At first glance, Algorithm [1] might seem to be merely a block coordinate descent algorithm [40] applied to (1). However, it has some interesting properties that are worth commenting on.

• **Two Novel Strategies.** (i) Instead of using majorization techniques for optimizing over the block of the variables, we consider minimizing the original objective function. Although the subproblem
We uniformly select one combination (which contains $c$ and $\bar{c}$). We sort the vectors $i = 1, \ldots, n$ in increasing order and then pick the top-$k$ coordinates as the working set. One may use a cyclic strategy to alternatingly select all the choices of the working set.

Algorithm 1 The Proposed Block Decomposition Algorithm

1: Input: the size of the working set $k$, an initial feasible solution $x^0$. Set $t = 0$.
2: while not converge do
3: (S1) Employ some strategy to find a working set $B$ of size $k$. We define $\bar{B} = \{1, \ldots, n\} \setminus B$.
4: (S2) Solve the following subproblem globally using combinatorial search methods:
   \[
   x^{t+1} = \arg\min_{z} F(z) + \frac{\rho}{2} \|z - x^t\|^2, \text{ s.t. } z_{\bar{B}} = (x^t)_\bar{B}
   \] (3)
5: (S3) Increment $t$ by 1
6: end while

is NP-hard and admits no closed-form solution, we can use an exhaustive search to solve it exactly. (ii) We consider a proximal point strategy for the subproblem in (3). This is to guarantee sufficient descent condition for the optimization problem and global convergence of Algorithm 1 (refer to Theorem 2).

- **Solving the Subproblem Globally.** The subproblem in (3) essentially contains $k$ unknown decision variables and can be solved exactly within sub-exponential time $O(2^k)$. Using the variational reformulation of $\ell_0$ pseudo-norm [7] Problem (3) can be reformulated as a mixed-integer optimization problem and solved by some global optimization solvers such as ‘CPLEX’ or ‘Gurobi’. For simplicity, we consider a simple exhaustive search (a.k.a. generate and test method) to solve it. Specifically, for every coordinate of the $k$-dimensional subproblem, it has two states, i.e., zero/nonzero. We systematically enumerate the full binary tree to obtain all possible candidate solutions and then pick the best one that leads to the lowest objective value as the optimal solution.

- **Finding the Working Set.** We observe that it contains $C^k_n$ possible combinations of choice for the working set. One may use a cyclic strategy to alternatingly select all the choices of the working set. However, past results show that the coordinate gradient method results in faster convergence when the working set is chosen in an arbitrary order [19] or in a greedy manner [40, 20]. This inspires us to use a random strategy or a greedy strategy for finding the working set. We remark that the combination of the two strategies is preferred in practice.

**Random strategy.** We uniformly select one combination (which contains $k$ coordinates) from the whole working set of size $C^k_n$. One good benefit of this strategy is that our algorithm is ensured to find a block-$k$ stationary point (discussed later) in expectation.

**Greedy strategy.** Generally speaking, we pick the top-$k$ coordinates that lead to the greatest descent when one variable is changed and the rest variables are fixed based on the current solution $x^t$. We denote $Z = \{i: x^t_i = 0\}$ and $\bar{Z} = \{j: x^t_j \neq 0\}$. For $Z$, we solve a one-variable subproblem to compute the possible decrease for all $i \in Z$ of $x^t$ when changing from zero to nonzero:

$$\forall i = 1, \ldots, |Z|, \ c_i = \min_\alpha F(x^t + \alpha e_i) - F(x^t).$$

For $\bar{Z}$, we compute the decrease for each coordinate $j \in \bar{Z}$ of $x^t$ when changing from nonzero to exactly zero:

$$\forall j = 1, \ldots, |\bar{Z}|, \ d_j = F(x^t + \alpha e_j) - F(x^t), \ \alpha = x^t_j.$$ We sort the vectors $c$ and $d$ in increasing order and then pick the top-$k$ coordinates as the working set.

3 Optimality Analysis

This section provides an optimality analysis of our method. In the sequel, we present some necessary optimal conditions for [1]. Since the block-$k$ optimality condition is novel in this paper, it is necessary to clarify its relations with existing optimality conditions formally. We use $\bar{x}$, $\check{x}$, and $\hat{x}$ to denote an arbitrary basic stationary point, an $L$-stationary point, and a block-$k$ stationary point, respectively.

**Definition 1.** (Basic Stationary Point) A solution $\check{x}$ is called a basic stationary point if the following holds. $h \triangleq h_{\text{regu}} : \check{x} = \arg\min_{\rho_1 \leq y \leq \rho_1} f(y), \ s.t. \ y_Z = 0; \ h \triangleq h_{\text{const}} : \bar{x} = \arg\min_{\rho} f(y), \ s.t. \ |Z| \leq k, \ y_Z = 0. \ \text{Here,} \ Z \triangleq \{i|\bar{x}_i = 0\}, \ Z \triangleq \{j|\bar{x}_j \neq 0\}.$

\[1\] For all $x \in \mathbb{R}^n$ with $\|x\|_\infty \leq \rho$, it always holds that $\|x\|_0 = \min_\nu \langle 1, \nu \rangle, \ s.t. \ \nu \in \{0, 1\}^n, \ |x| \leq \rho \nu.$
Remarks: The basic stationary point states that the solution achieves its global optimality when the support set is restricted. One good feature of the basic stationary condition is that the solution set is enumerable and its size is $2^n$. This makes it possible to validate whether a solution is optimal for the original sparse optimization problem.

Definition 2. (L-Stationary Point) A solution $\hat{x}$ is an L-stationary point if it holds that: $\hat{x} = \arg\min_y g(y, \hat{x}) + h(y)$ with $g(y, x) \triangleq f(x) + \langle \nabla f(x), y - x \rangle + \frac{L}{2} ||y - x||^2$.

Remarks: This is the well-known proximal thresholding operator [3]. The term $g(y, x)$ is a majorization function of $f(y)$ and it always holds that $f(y) \leq g(y, x)$ for all $x$ and $z$. Although it has a closed-form solution, this simple surrogate function may not be a good majorization/surrogate function for the non-convex problem.

Definition 3. (Block-k Stationary Point) A solution $\bar{x}$ is a block-k stationary point if it holds that:

$$\bar{x} \in \arg\min_{z \in \mathbb{R}^n} \mathcal{P}(z; \bar{x}, B) \triangleq \{ F(z), \ s.t. \ z_B = \bar{x}_B \}, \forall |B| = k.$$ (4)

Remarks: (i) The concept of block-k stationary point is novel in this paper. (ii) The sub-problem $\min_{x} \mathcal{P}(z; \bar{x}, B)$ is NP-hard, and it takes sub-exponential time $O(2^k)$. However, since $k$ is often very small, it can be tackled by some practical global optimization methods. (iii) Testing whether a solution $\bar{x}$ is a block-k stationary point deterministically requires solving $C_n^k$ subproblems, therefore leading to a total time complexity of $C_n^k \times O(2^k)$. However, using a random strategy for finding the working set $B$ from $C_n^k$ combinations, we can test whether a solution $\bar{x}$ is the block-k stationary point in expectation within a time complexity of $T \times O(2^k)$ with the constant $T$ being the number of iterations which is related to the confidence of the probability.

The following proposition states the relations between the three types of the stationary point.

Proposition 1. Optimality Hierarchy between the Necessary Optimality Conditions. The following optimality hierarchy holds:

- Basic Stat. Point $\iff$ L-Stat. Point $\iff$ Block-k Stat. Point $\iff$ Block-(k + 1) Stat. Point $\iff$ Block-(k + 2) Stat. Point $\iff$ ... $\iff$ Block-n Stat. Point $\iff$ Optimal Point

with $k = 1$ for $h \triangleq h_{\text{regu}}$, and $k = 2$ for $h \triangleq h_{\text{cons}}$.

Proof. We denote $\Pi(\mathbf{a}) \triangleq \min(\rho, \max(-\rho, \mathbf{a}))$. $\Gamma_s(\mathbf{x})$ is the operator that sets all but the largest (in magnitude) $s$ elements of $\mathbf{x}$ to zero.

(1) First, we prove that an L-stationary point $\bar{x}$ is also a basic stationary point $\bar{x}$ when $h \triangleq h_{\text{regu}}$. Using Definition 2 we have the following closed-form solution for $\bar{x}$:

$$\bar{x}_i = \begin{cases} \Pi(\bar{x}_i - \nabla_i f(\bar{x})/L), & (\bar{x}_i - \nabla_i f(\bar{x})/L)^2 > 2\lambda/L; \\ 0, & \text{else}. \end{cases}$$

This implies that there exists a support set $S$ such that $\bar{x}_S = \Pi(\bar{x}_S - (\nabla f(\bar{x}))/L)$, which is the optimal condition for a basic stationary point. Defining $Z \triangleq \{i|\bar{x}_i = 0\}$, $\bar{Z} \triangleq \{j|\bar{x}_j \neq 0\}$, we notice that $|\nabla f(\bar{x})|_i \leq \sqrt{2\lambda L}, \forall i \in Z$, and $|\nabla f(\bar{x})|_j = 0, |x_j| \geq \min(\rho, \sqrt{2\lambda/L}), \forall j \in \bar{Z}$. Second, we prove that an L-stationary point $\bar{x}$ is also a basic stationary point $\bar{x}$ when $h \triangleq h_{\text{cons}}$. For an L-stationary point, we have $\bar{x} = \Gamma_s(\bar{x} - (\nabla f(\bar{x}))/L)$. This implies that $\exists S, \bar{x}_S = \bar{x}_S - (\nabla f(\bar{x}))/L$, which is the optimal condition for a basic stationary point.

(2) First, we prove that a block-1 stationary point is also an L-stationary point for $h \triangleq h_{\text{regu}}$. Assume that the convex objective function $f(\cdot)$ has coordinate-wise Lipschitz continuous gradient with constant $s_i$. For all $x \in \mathbb{R}^n, \epsilon \in \mathbb{R}, i = 1, 2, ..., n$, it holds that [31]: $f(x + \epsilon e_i) \leq Q_i(x, \epsilon) \triangleq f(x) + \langle \nabla f(x), \epsilon e_i \rangle + \frac{s_i}{2} ||\epsilon e_i||^2$. Any block-1 stationary point must satisfy the following relation: $0 \in \arg\min_{x} Q_i(x, \epsilon) + \lambda ||x_i + \epsilon||_0, \forall i$. We have the following optimal condition for $\bar{x}$ with $k = 1$:

$$\bar{x}_i = \begin{cases} \Pi(\bar{x}_i - \nabla_i f(\bar{x})/s_i), & (\bar{x}_i - \nabla_i f(\bar{x})/s_i)^2 > 2s_i; \\ 0, & \text{else}. \end{cases}$$

Since $\forall i, s_i \leq L$, the latter formulation implies the former one.

Second, we prove that a block-2 stationary point is also an L-stationary point for $h \triangleq h_{\text{cons}}$. Given a vector $\mathbf{a} \in \mathbb{R}^n$, we consider the following optimization problem:

$$\{z_B^2 = \arg\min_{z_B} \|z - \mathbf{a}\|^2_2, \ s.t. \ \|z_B\|_0 + \|z_B\|_0 \leq s, \ \forall |B| = 2\},$$ (5)
which in fact contains $C^2_n$ 2-dimensional subproblems. It is not hard to validate that \( \mathbf{5} \) achieves the optimal solution with $z^* = \Gamma_s(a)$. For any block-2 stationary point $\bar{x}$, we have $x_E = \arg \min_{z_E} \|z - (\bar{x} - \nabla f(\bar{x})/L)\|_2^2$, s.t. $\|z_E\|_0 + \|z_E\|_0 \leq s$. Applying this conclusion with $a = \bar{x} - \nabla f(\bar{x})/L$, we have $x = \Gamma_s(x - \nabla f(\bar{x})/L)$.

(3) It is not hard to notice that the subproblem for the block-$k_2$ stationary point is a subset of those of the block-$k_1$ stationary point when $k_1 \geq k_2$. Therefore, the block-$k_1$ stationary point implies the block-$k_2$ stationary point when $k_1 \geq k_2$.

(4) It is clear that any block-$n$ stationary point is also the optimal global solution.

\[\square\]

Remarks: It is worthwhile to point out that the seminal works of \[6\] also present a coordinate-wise optimality condition for sparse optimization. However, our block-$k$ condition is stronger since their optimality condition corresponds to $k = 1$ in our optimality condition framework.

| $k \triangleq h_{\text{regu}}$ | Basic-Stat. | $t$-Stat. | Block-1 Stat. | Block-2 Stat. | Block-3 Stat. | Block-4 Stat. | Block-5 Stat. | Block-6 Stat. |
|-----------------------------|-------------|------|-------------|-------------|-------------|-------------|-------------|-------------|
| 57                          | 57          | 56   | 9           | 5           | 1           | 1           | 1           | 1           |
| 1                           | 1           | 1    | 1           | 1           | 1           | 1           | 1           | 1           |

Table 1: Number of points satisfying optimality conditions.

A Running Example. We consider the quadratic optimization problem $\min_{x \in \mathbb{R}^n} \frac{1}{2} x^T Q x + x^T p + h(x)$ where $n = 6$, $Q = cc^T + I$, $p = 1$, $c = [1 2 3 4 5 6]^T$. The parameters for $h_{\text{regu}}(x)$ and $h_{\text{cons}}(x)$ are set to $(\lambda, \rho, s) = (0.01, \infty, 4)$. The stationary point distribution of this example can be found in Table[1].

Table[1] This problem contains $\sum_{i=0}^{6} C^i_6 = 64$ basic stationary points for $h \triangleq h_{\text{regu}}$, while it has $\sum_{i=0}^{6} C^i_6 = 57$ basic stationary points for $h \triangleq h_{\text{cons}}$. As $k$ becomes large, the newly introduced type of local minimizer (i.e., block-$k$ stationary point) become more restricted in the sense that they have a smaller number of stationary points.

4 Convergence Analysis

This section provides some convergence analysis for Algorithm[1]. We assume that the working set of size $k$ is selected randomly and uniformly (sample with replacement). Due to space limitations, some proofs are placed into the supplementary material.

Proposition 2. Global Convergence. Letting $\{x^t\}_{t=0}^\infty$ be the sequence generated by Algorithm[1] we have the following results. (i) It holds that: $F(x^{t+1}) \leq F(x^t) - \frac{\bar{\theta}}{2} \|x^{t+1} - x^t\|^2$, $\lim_{t \to \infty} \mathbb{E}[\|x^{t+1} - x^\dagger\|^2] = 0$. (ii) As $t \to \infty$, $x^t$ converges to the block-$k$ stationary point $\bar{x}$ of \[1\] in expectation.

Remarks: Coordinate descent may cycle indefinitely if each minimization step contains multiple solutions \[6\]. The introduction of the strongly convex parameter $\theta > 0$ is necessary for our nonconvex problem since it guarantees sufficient decrease condition, which is essential for global convergence. Our algorithm is guaranteed to find the block-$k$ stationary point, but it is in expectation.

The following theorem establishes some convergence properties of our algorithm for sparse regularized optimization with $h \triangleq h_{\text{regu}}$.

Theorem 1. Convergence Properties for Sparse Regularized Optimization. For $h \triangleq h_{\text{regu}}$, we have the following results:

(a) It holds that $\|x^t\| \geq \delta$ for all $i$ with $x^t_i \neq 0$, where $\delta \triangleq \min(\rho, \sqrt{2\lambda/(\theta + L)})$, $\min(\|x^0\|)$. Whenever $x^{t+1} \neq x^t$, we have $\|x^{t+1} - x^t\|^2 \geq \frac{\theta \delta^2}{n}$ and the objective value is decreased at least by $D$. The solution changes at most $D$ times in expectation for finding a block-$k$ stationary point $\bar{x}$. Here $D$ and $J$ are respectively defined as:

$$D \triangleq \frac{k \theta \delta^2}{2\pi n}, \quad J \triangleq \frac{F(x^0) - F(\bar{x})}{\delta^2}. \tag{6}$$

(b) Assume that $f(\cdot)$ is generally convex. If the support set of $x^t$ does not changes for all $t = 0, 1, \ldots, \infty$, Algorithm[1] takes at most $V_1$ iterations in expectation to converge to a stationary point $\bar{x}$
satisfying $F(x^t) - F(x) \leq \epsilon$. Moreover, Algorithm 1 takes at most $V_1 \times \bar{J}$ in expectation to converge to a stationary point $\bar{x}$ satisfying $F(x^t) - F(\bar{x}) \leq \epsilon$. Here, $V_1$ is defined as:

$$V_1 = \max\left(\frac{4\nu^2}{\kappa}, \sqrt{\frac{2\nu^2(F(x^0) - F(\bar{x}))}{\kappa}}\right)/\epsilon,$$  

with $\nu \triangleq \frac{2n\sqrt{\kappa}0}{\epsilon}$.

(7)

(c) Assume that $f(\cdot)$ is $\sigma$-strongly convex. If the support set of $x^t$ does not change for all $t = 0, 1, \ldots, \infty$, Algorithm 1 takes at most $V_2$ iterations in expectation to converge to a stationary point $\bar{x}$ satisfying $F(x^t) - F(\bar{x}) \leq \epsilon$. Moreover, Algorithm 1 takes at most $V_2 \times \bar{J}$ in expectation to converge to a stationary point $\bar{x}$ satisfying $F(x^t) - F(\bar{x}) \leq \epsilon$. Here, $V_2$ is defined as:

$$V_2 = \log_\alpha(\epsilon/(F(x^0) - F(\bar{x}))), \ \text{with} \ \alpha \triangleq \frac{n0}{\epsilon} / \left(1 + \frac{n0}{\epsilon\sigma}\right).$$

(8)

Remarks: (i) When the support set is fixed, the optimization problem reduces to a convex problem. (ii) We derive a upper bound for the number of changes $\bar{J}$ for the support set in (a), and a upper bound on the number of iterations $V_1$ (or $V_2$) performed after the support set is fixed in (b) or (c). Multiplying these two bounds, we can establish the actual number of iterations for Algorithm 1. However, this bound may not be tight enough.

In what follows, we establish a better convergence rate of our algorithm with $h \triangleq h_{\text{regu}}$.

**Theorem 2. Convergence Rate for Sparse Regularized Optimization.** For $h \triangleq h_{\text{regu}}$, we have the following results:

(a) Assume that $f(\cdot)$ is generally convex, Algorithm 1 takes at most $N_1$ iterations in expectation to converge to a block-$k$ stationary point $\bar{x}$ satisfying $F(x^t) - F(\bar{x}) \leq \epsilon$, where $N_1 = (\frac{J}{D} + \frac{1}{k}) \times \max\left(\frac{4\nu^2}{\kappa}, \sqrt{\frac{2\nu^2(F(x^0) - F(\bar{x}) - D)}{\nu}}\right)$.  

(b) Assume that $f(\cdot)$ is $\sigma$-strongly convex, Algorithm 1 takes at most $N_2$ iterations in expectation to converge to a block-$k$ stationary point $\bar{x}$ satisfying $F(x^t) - F(\bar{x}) \leq \epsilon$, where $N_2 = J \log_\alpha\left(\frac{D}{F(x^0) - F(\bar{x})}\right) + \log_\alpha\left(\frac{F(x^0) - D - F(\bar{x})}{\nu}\right)$.  

Remarks: (i) Our proof of Theorem 2 is based on the results in Theorem 1 and a similar bounding technique as in [25]. (ii) If $\bar{J} \geq 2$ and the accuracy $\epsilon$ is sufficiently small such that $\epsilon \leq \frac{D}{2}$, we have $\frac{J}{D} + \frac{1}{k} \leq \frac{2}{\epsilon}$, leading to $(\frac{J}{D} + \frac{1}{k}) \times \max\left(\frac{4\nu^2}{\kappa}, \sqrt{\frac{2\nu^2(F(x^0) - F(\bar{x}) - D)}{\nu}}\right) \leq J \times \max\left(\frac{4\nu^2}{\kappa}, \sqrt{\frac{2\nu^2(F(x^0) - F(\bar{x}))}{\nu}}\right)$. Using the same assumption and strategy, we have $J \log_\alpha\left(\frac{D}{F(x^0) - F(\bar{x})}\right) + \log_\alpha\left(\frac{F(x^0) - D - F(\bar{x})}{\nu}\right) \leq J \times \log_\alpha\left(\frac{D}{F(x^0) - F(\bar{x})}\right)$. In this situation, the bounds in Theorem 2 are tighter than those in Theorem 1.

We prove the convergence rate of our algorithm for sparsity constrained optimization with $h \triangleq h_{\text{cons}}$.

**Theorem 3. Convergence Rate for Sparsity Constrained Optimization.** Let $f(\cdot)$ be a $\sigma$-strongly convex function. We assume that $f(\cdot)$ is Lipschitz continuous such that $\forall t$, $\|\nabla f(x^t)\|_2^2 \leq \kappa$ for some positive constant $\kappa$. Denote $\alpha \triangleq \frac{2\nu}{\kappa} / (1 + \frac{2\nu}{\kappa})$. We have the following results: $E[F(x^t) - F(\hat{x})] \leq (F(x^0) - F(\hat{x}))\alpha^t + \frac{2\nu^2}{\kappa} \alpha^t$, and $E[\|x^{t+1} - \hat{x}\|_2^2] \leq \frac{2\nu^2}{\kappa} (F(x^0) - F(\hat{x}))\alpha^t + \frac{2\nu}{\kappa} \alpha^t$.

Remarks: Our convergence rate results are similar to those of the gradient hard thresholding pursuit as in [44]. The first term and the second term for our convergence rate are called parameter estimation error and statistical error, respectively. While their analysis relies on the conditions of restricted strong convexity or smoothness, our study only relies on the requirements of generally strong convexity or smoothness.

5 Experimental Validation

This section demonstrates the effectiveness of our algorithm on two sparse optimization tasks, namely the sparse regularized least squares problem and the sparsity constrained least squares problem. Given a design matrix $A \in \mathbb{R}^{m \times n}$ and an observation vector $b \in \mathbb{R}^m$, we solve the following optimization problem:

$$\min_x \frac{1}{2}\|Ax - b\|_2^2 + \lambda\|x\|_0 \quad \text{or} \quad \min_x \frac{1}{2}\|Ax - b\|_2^2, \ s.t. \ \|x\|_0 \leq s,$$

(9)
Figure 1 Experimental results on sparse regularized least squares problems on different data sets.

Figure 2 Experimental results on sparsity constrained least squares problems on different data sets.

Figure 3 Convergence curve for solving sparsity constrained least squares problems with $k = 20$. 

(a) random-256-1024, $\lambda = 1$
(b) E2006-5000-1024, $\lambda = 1$
(c) E2006-5000-2048, $\lambda = 1$
(d) random-256-1024-corrupted, $\lambda = 10$
(e) E2006-5000-1024-corrupted, $\lambda = 10$
(f) E2006-5000-2048-corrupted, $\lambda = 10$
where $\lambda$ and $s$ are given parameters.

- **Experimental Settings.** We use DEC (R:G:j) to denote our block decomposition method along with selecting $i$ coordinates using the Random strategy and $j$ coordinates using the Greedy strategy. Since the two strategies may select the same coordinate, the working set at most contains $i + j$ coordinates. We keep a record of the relative changes of the objective function values by $r_t = (f(x^t) - f(x^{t-1}))/f(x^t)$. We let DEC run up to $T$ iterations and stop it at iteration $t < T$ if $\text{mean}(\{r_{t-min(t,g)+1}, r_{t-min(t,g)+2}, \ldots, r_t\}) \leq \epsilon$. We use the default value $(\theta, \epsilon, g, T) = (10^{-3}, 10^{-3}, 50, 1000)$ for DEC. All codes were implemented in Matlab on an Intel 3.20GHz CPU with 8 GB RAM. We measure the quality of the solution by comparing the objective values for different methods. Note that although DEC uses the randomized strategy to find the working set, we can always measure the quality of the solution by computing the deterministic objective value.

- **Data Sets.** Four types of data sets for $\{A, b\}$ are considered in our experiments. (i) ‘random-m-n’: We generate the design matrix as $A = \text{randn}(m, n)$, where randn$(m, n)$ is a function that returns a standard Gaussian random matrix of size $m \times n$. To generate the sparse original signal $\hat{x} \in \mathbb{R}^n$, we select a support set of size 100 uniformly at random and set them to arbitrary number sampled from standard Gaussian distribution, the observation vector is generated via $b = Ax + o$ with $o = 10 \times \text{randn}(m, 1)$. (ii) ‘E2006-m-n’: We use the real-world data set ‘E2006’. We uniformly select $m$ examples and $n$ dimensions from the original data set. (iii) ‘random-m-n-corrupted’: To verify the robustness of DEC, we generate design matrices containing outliers by $P(A)$. Here, $P(X) \in \mathbb{R}^{m \times p}$ is a noisy version of $X \in \mathbb{R}^{m \times p}$ where 2% of the entries of $X$ are corrupted uniformly by scaling the original values by 100 times. We use the same strategy to generate $A$ as in ‘random-m-n’. Note that the Hessian matrix may be ill-conditioned. (iv) ‘E2006-m-n-corrupted’: We use a similar strategy to generate the corrupted real-world data as in ‘random-m-n-corrupted’.

- **Sparse Regularized Least Squares Problem.** We compare DEC with four state-of-the-art methods: (i) Proximal Gradient Method (PGM) [32], (ii) Accelerated Proximal Gradient Method (APGM) [32], (iii) Random Coordinate Descent Method (RCDM) [28], and (iv) Greedy Coordinate Descent Method (GCDM) [20]. All the initial solutions for the comparing methods are set to zero.

Several observations can be drawn from Figure 1. (i) PGM and APGM achieve similar performance and they get stuck into poor local minima. (ii) DEC is more effective than PGM. In addition, we find that as the parameter $k$ becomes larger, more higher accuracy is achieved. (iii) DEC (R(G10)) converges quickly but it generally leads to worse solution quality than DEC (R(10G0)). Based on this observation, we consider a combined random and greedy strategies for finding the working set in our forthcoming experiments.

- **Sparsity Constrained Least Squares Problem.** We compare DEC with seven state-of-the-art sparse optimization algorithms: (i) Regularized Orthogonal Matching Pursuit (ROMP) [30], (ii) Subspace Pursuit (SSP) [14], (iii) Orthogonal Matching Pursuit (OMP) [39], (iv) Gradient Pursuit (GP) [8], (v) Compressive Sampling Matched Pursuit (CoSaMP) [29], (vi) Proximal Gradient Method (PGM) [2], and (vii) Quadratic Penalty Method (QPM) [27]. We remark that ROMP, SSP, OMP, GP and CoSaMP are greedy algorithms and their support sets are selected iteratively. They are non-gradient type algorithms, it is hard to incorporate these methods into other gradient-type based optimization algorithms [2]. We use the Matlab implementation in the ‘sparsify’ toolbox [4]. Both PGM and QPM are based on iterative hard thresholding. Since the optimal solution is expected to be sparse, we initialize the solutions of PPA, QPM and DEC to $10^{-7} \times \text{randn}(n, 1)$ and project them to feasible solutions. The initial solution of greedy pursuit methods is initialized to zero points implicitly. We vary $s = \{3, 8, 13, 18, \ldots, 50\}$ on different data sets and show the average results of using 5 random initial points.

Several conclusions can be drawn from Figure 2. (i) The iterative hard thresholding-based methods PPA and QPM generally lead to the worst performance. (ii) ROMP and COSAMP are not stable and sometimes they present bad performance. (iii) DEC presents comparable performance to the

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1 For the purpose of reproducibility, we provide our code in the supplementary material.

2 We use the Matlab implementation in the ‘sparsify’ toolbox [4].

3 Matlab script: $I = \text{randperm}(m \times p, \text{round}(0.02 \times m \times p)); X(I) = X(I) \times 100$.

4 For the purpose of reproducibility, we provide our code in the supplementary material.

5 http://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/
greedy methods on ‘random-256-1024’, ‘E2006-5000-1024’, and ‘random-256-1024-corrupted’, but it significantly outperforms the greedy techniques on the other data sets.

- **Computational Efficiency.** Figure 1 and Figure 3 show the convergence curve of different methods for sparse regularized optimization and sparsity constrained optimization, respectively. Generally speaking, DEC is effective and practical for large-scale sparse optimization. Although it takes longer time to converge than the comparing methods, the computational time is acceptable and it generally takes less than 30 seconds to converge in all our instances. This computation time pays off as DEC achieves significantly higher accuracy. The main bottleneck of computation is on solving the small-sized subproblems using sub-exponential time $O(2^k)$. The parameter $k$ in Algorithm 1 can be viewed as a tuning parameter to balance the efficacy and efficiency.

6 Conclusions

This paper presents an effective and practical method for solving sparse optimization problems. Our approach takes advantage of the effectiveness of the combinatorial search and the efficiency of coordinate descent. We provided rigorous optimality analysis and convergence analysis for the proposed algorithm. Our experiments show that our method achieves state-of-the-art performance. Our block decomposition algorithm can be extended to solve binary optimization problems. A full version of this paper can be found in [43].

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Appendix

The appendix section is organized as follows. Section 7 presents the global convergence for general sparse optimization of our algorithm. Section 8 and Section 9 present the convergence rate for sparse regularized optimization and sparsity constrained optimization, respectively. Section 10 provides additional discussions for our method.

7 Proof of Global Convergence

Proposition 2. Global Convergence. Letting \( \{x^t\}_{t=0}^{\infty} \) be the sequence generated by Algorithm 1, we have the following results. (i) It holds that: \( F(x^{t+1}) \leq F(x^t) - \frac{\theta}{2} \|x^{t+1} - x^t\|^2 \), \( \lim_{t \to \infty} E[\|x^{t+1} - x^t\|] = 0 \). (ii) As \( t \to \infty \), \( x^t \) converges to the block-k stationary point \( \bar{x} \) of (4) in expectation.

Proof. (i) Due to the optimality of \( x^{t+1} \), we have: \( F(x^{t+1}) + \frac{\theta}{2} \|x^{t+1} - x^t\|^2 \leq F(u) + \frac{\theta}{2} \|u - x^t\|^2 \) for all \( u_B = (x^t)_B \). Letting \( u = x^t \), we obtain the sufficient decrease condition:

\[
F(x^{t+1}) \leq F(x^t) - \frac{\theta}{2} \|x^{t+1} - x^t\|^2 \tag{10}
\]

Taking the expectation of \( B \) for the sufficient descent inequality, we have \( E[F(x^{t+1})] \leq E[F(x^t)] - E[\frac{\theta}{2} \|x^{t+1} - x^t\|^2] \). Summing this inequality over \( i = 0, 1, 2, ..., t - 1 \), we have: \( \frac{\theta}{2} \sum_{i=0}^{t} E[\|x^{i+1} - x^i\|^2] \leq E[F(x^t)] - F(x^t) \).

Using the fact that \( F(\bar{x}) \leq F(x^t) \), we obtain:

\[
\min_{i=1,...,t} E[\frac{\theta}{2} \|x^{i+1} - x^i\|^2] \leq \frac{\theta}{2} \sum_{i=0}^{t} E[\|x^{i+1} - x^i\|^2] \leq \frac{F(x^t) - F(\bar{x})}{t} \tag{11}
\]

Therefore, we have \( \lim_{t \to \infty} E[\|x^{t+1} - x^t\|] = 0 \).

(ii) We assume that the stationary point is not a block-k stationary point. In expectation there exists a block of coordinates \( B \) such that \( x^t \not\in \arg \min_{z} \mathcal{P}(z; x^t, B) \) for some \( B \), where \( \mathcal{P}(\cdot) \) is defined in Definition 3. However, according to the fact that \( x^t = x^{t+1} \) and subproblem (3) in Algorithm 1 we have \( x^{t+1} \in \arg \min_{z} \mathcal{P}(z; x^t, B) \). Hence, we have \( x^t \neq x^{t+1} \). This contradicts with the fact that \( x^t = x^{t+1} \) as \( t \to \infty \). We conclude that \( x^t \) converges to the block-k stationary point.

\[ \square \]

8 Proof of Convergence Rate for Sparse Regularized Optimization

The following lemma is useful in our proof.

Lemma 1. Assume a nonnegative sequence \( \{u^t\}_{t=0}^{\infty} \) satisfies \( (u^{t+1})^2 \leq C(u^t - u^{t+1}) \) for some constant \( C \). We have:

\[
u^t \leq \frac{\max(2C, \sqrt{Cu^0})}{t+1} \tag{12}\]

Proof. We denote \( C_1 \triangleq \max(2C, \sqrt{Cu^0}) \). Solving this quadratic inequality, we have:

\[
u^{t+1} \leq -\frac{C}{2} + \frac{C}{2} \sqrt{1 + \frac{4u^0}{C}} \tag{13}\]

We now show that \( u^{t+1} \leq \frac{C_1}{t+1} \), which can be obtained by mathematical induction. (i) When \( t = 0 \), we have \( u^1 \leq -\frac{C}{2} + \frac{C}{2} \sqrt{1 + \frac{4u^0}{C}} \leq -\frac{C}{2} + \frac{C}{2} (1 + \sqrt{\frac{4u^0}{C}}) = \frac{C}{2} \sqrt{\frac{4u^0}{C}} = \sqrt{C}u^0 \leq \frac{C_1}{1+1} \). (ii) When \( t \geq 1 \), we assume that \( u^t \leq \frac{C_1}{t} \) holds. We derive the following results: \( t \geq 1 \Rightarrow \frac{t}{t+1} \leq 2 \Rightarrow (a) C\frac{t+1}{t} \leq C_1 \Rightarrow (b) C \frac{1}{t+1} \leq \frac{C_1}{t+1} \Rightarrow C_1 \frac{t}{t+1} \leq \frac{C_1}{t+1} \Rightarrow C_1 \frac{1}{t+1} \leq \frac{C_1}{t+1} \Rightarrow C_1 \frac{2}{t+1} \leq \frac{C_1}{t+1} \Rightarrow C_1 \frac{2}{t+1} \leq \frac{C_1}{t+1} \Rightarrow C_1 \frac{2}{t+1} \leq \frac{C_1}{t+1} \Rightarrow \).

Here, step (a) uses \( 2C \leq C_1 \); step (b) uses \( \frac{1}{t+1} \); step (c) uses \( u^t \leq \frac{C_1}{t} \).

\[ \square \]
Theorem 1. Convergence Properties for Sparse Regularized Optimization. For $h \triangleq h_{\alpha, \theta}$, we have the following results:

(a) It holds that $|x_i^t| \geq \delta$ for all $i$ with $x_i^t \neq 0$, where $\delta \triangleq \min(p, \sqrt{2/\gamma/((\theta + L))})$. Whenever $x^{t+1} \neq x^t$, we have $\|x^{t+1} - x\|_2^2 \geq \frac{\theta \delta^2}{n}$ and the objective value is decreased at least by $D$. The solution changes at most $J$ times in expectation for finding a block-k stationary point $\bar{x}$. Here $D$ and $J$ are respectively defined as

$$D \triangleq \frac{\theta \delta^2}{2n}, \quad J \triangleq \frac{F(x^0) - F(\bar{x})}{\nu}.$$  

(b) Assume that $f(\cdot)$ is generally convex. If the support set of $x^t$ does not change for all $t = 0, 1, ..., \infty$, Algorithm 1 takes at most $V_1$ iterations in expectation to converge to a stationary point $\bar{x}$ satisfying $F(x^t) - F(\bar{x}) \leq \epsilon$. Moreover, Algorithm 1 takes at most $V_1 \times J$ in expectation to converge to a stationary point $\bar{x}$ satisfying $F(x^t) - F(\bar{x}) \leq \epsilon$. Here, $V_1$ is defined as:

$$V_1 = \max\left(4n^2, \sqrt{\frac{2n^2(F(x^0) - F(\bar{x}))}{\epsilon}}\right),$$

with $\nu \triangleq \frac{2n\sqrt{\gamma\theta}}{\kappa}$.  

(c) Assume that $f(\cdot)$ is $\sigma$-strongly convex. If the support set of $x^t$ does not change for all $t = 0, 1, ..., \infty$, Algorithm 1 takes at most $V_2$ iterations in expectation to converge to a stationary point $\bar{x}$ satisfying $F(x^t) - F(\bar{x}) \leq \epsilon$. Moreover, Algorithm 1 takes at most $V_2 \times J$ in expectation to converge to a stationary point $\bar{x}$ satisfying $F(x^t) - F(\bar{x}) \leq \epsilon$. Here, $V_2$ is defined as:

$$V_2 = \log\left(\frac{\epsilon}{(F(x^0) - F(\bar{x}))}\right),$$

with $\alpha \triangleq \frac{\theta}{\kappa^2}/(1 + \frac{\sigma^2}{\kappa^2})$.  

Proof. (a) Note that Algorithm 1 solves problem (5) in every iteration. Using Proposition 1 we have that the solution $x^t_{i+1}$ is also a $L$-stationary point. Therefore, we have $|x_i^{t+1}| \geq \min(p, \sqrt{2\gamma/((\theta + L))})$ for all $x_i^{t+1} \neq 0$. Taking the initial point of $x$ for consideration, we have that

$$|x^t_{i+1}| \geq \min(p, |x^t_i|, \sqrt{2\gamma/((\theta + L))}), \forall i = 1, 2, ..., n.$$  

Therefore, we have the following results: $\|x^{t+1} - x\|_2 \geq \delta$. Taking the expectation of $B$, we have the following results: $E[\|x^{t+1} - x\|_2^2] = \frac{k}{n}\|x^{t+1} - x\|_2^2 \geq \frac{k\delta^2}{n}$. Every time the index set of $x$ is changed, the objective value is decreased at least by $E[\frac{\theta}{\kappa}\|x^{t+1} - x\|_2^2] \geq \frac{k\delta^2}{2n} \triangleq D$. Moreover, we obtain: $\frac{2F(x^t) - 2F(\bar{x})}{16} \geq \frac{\delta^2 k}{n}$ from (11). Therefore, the number of iterations is upper bounded by $J$.

(b) We notice that when the support set is fixed, the original problem reduces to the following convex composite optimization problem:

$$F(x) \triangleq f(x) + p(x) + \text{const}, \quad \text{with } p(x) \triangleq I_B(x).$$

We use $\partial F(x)$ to denote the sub-gradient of $F(\cdot)$ in $x$. Since the algorithm solves the following optimization: $x^{t+1} = \arg\min_z f(z) + p(z) + \frac{\theta}{2}\|z - x^t\|^2$, s.t. $z_B = x^t_B$, we have the following optimality condition for $x^{t+1}$:

$$w_B + \theta(x^{t+1} - x^t)_B = 0, \quad (x^{t+1})_B = (x^t)_B, \forall w \in \partial F(x^{t+1})$$

We now consider the case when $f(\cdot)$ is generally convex. We derive the following inequalities:

$$\begin{align*}
E[F(x^{t+1})] - F(\bar{x}) &\leq E[(w, x^{t+1} - \bar{x})], \forall w \in \partial F(x^{t+1}) \\
&\leq E[\frac{\theta}{\kappa} (w_B, (x^{t+1} - \bar{x})_B)], \forall w \in \partial F(x^{t+1}) \\
&\leq E[\frac{\theta}{\kappa} ((x^{t+1} - x^t)_B \cdot \|x^{t+1} - \bar{x}\|_2)] \\
&\leq E[\frac{\theta}{\kappa} \cdot \|x^{t+1} - x^t\|_2 \cdot \|x^{t+1} - \bar{x}\|_2] \\
&\leq \frac{\theta}{\nu} \cdot \frac{2n\sqrt{\gamma\theta}}{\kappa} \|x^{t+1} - x^t\|_2.
\end{align*}$$
where step (a) uses the convexity of $F$; step (b) uses the fact that each block $B$ is picked randomly with probability $k/n$; step (c) uses the optimality condition of $x^{t+1}$ in (18); step (d) uses the Cauchy-Schwarz inequality; step (e) uses $\|x^{t+1} - x^t\|_2 = \|x^{t+1} - x^t\|_2$; step (f) uses $\|x^{t+1} - x\|_B \leq \sqrt{k}\|x^{t+1} - x\|_B \leq \sqrt{k}\|x^{t+1} - x\|_\infty \leq \sqrt{k}(\|x^{t+1}\|_\infty + \|x\|_\infty) \leq 2p\sqrt{k}$.

For any $x^t, x^{t+1}, \bar{x} \in \Omega$, we derive the following results:

$$E[F(x^{t+1}) - F(\bar{x})] \leq E[\nu \|x^{t+1} - x^t\|_2] \leq E[\nu \sqrt{\frac{2}{\rho} (F(x^t) - F(x^{t+1}))}]$$

(20)

where the step (a) uses (19); step (b) uses the sufficient decent condition in (10). Denoting $\Delta^t \triangleq E[F(x^t) - F(\bar{x})]$ and $C \triangleq \frac{2\nu^2}{\rho}$, we have the following inequality:

$$\left(\Delta^{t+1}\right)^2 \leq C(\Delta^t - \Delta^{t+1})$$

Combining with Lemma I, we have:

$$E[F(x^t) - F(\bar{x})] \leq \frac{\max\left(4\nu^2, \sqrt{\frac{2\nu^2}{\rho}} \right)}{1 - \frac{2\nu^2}{\rho}}$$

Therefore, we obtain the upper bound for the number of iterations to converge to a stationary point $\bar{x}$ satisfying $F(x^t) - F(\bar{x}) \leq \epsilon$ with fixing the support set. Combining the upper bound for the number of changes $J$ for the support set in (a), we naturally establish the actual number of iterations for Algorithm I.

(e) We now consider the case when $f(\cdot)$ is strongly convex. For any $x^t, x^{t+1}, \bar{x} \in \Omega$, we derive the following results:

$$E[F(x^{t+1}) - F(\bar{x})] \leq E[\frac{1}{\rho} \|x - x^{t+1}\|_2^2 + \langle x - x^{t+1}, w \rangle], \forall w \in \partial F(x^{t+1})$$

(21)

where step (a) uses the strongly convexity of $f(\cdot)$; step (b) uses the fact that $\frac{1}{\rho} \|x\|_2^2 - \langle x, y \rangle \leq \frac{1}{\rho} \|y\|_2^2$; step (c) uses the fact that $E[\|w_B\|_2^2] = \frac{1}{k} \|w\|_2^2$; step (d) uses the optimality of $x^{t+1}$; step (e) uses the sufficient condition in (10).

Rearranging terms for (21), we have:

$$\frac{E[F(x^{t+1}) - F(\bar{x})]}{E[F(x^t) - F(\bar{x})]} \leq \frac{1}{1 + \frac{\nu^2}{\rho}} = \alpha$$

Solving the recursive formulation, we obtain:

$$E[F(x^t) - F(\bar{x})] \leq E[\alpha^t [F(x^0) - F(\bar{x})]],$$

and it holds that $t \leq \log_{\alpha} \left( \frac{F(x^t) - F(\bar{x})}{F(x^0) - F(\bar{x})} \right)$ in expectation. Using similar techniques as in (b), we establish the actual number of iterations for Algorithm I when $f(\cdot)$ is strongly convex.

Theorem 2. Convergence Rate for Sparse Regularized Optimization. For $h \triangleq h_{\text{reg}}$, we have the following results:

(a) Assume that $f(\cdot)$ is generally convex, Algorithm I takes at most $N_1$ iterations in expectation to converge to a block-$k$ stationary point $\bar{x}$ satisfying $F(x^t) - F(\bar{x}) \leq \epsilon$, where $N_1 = \left(\frac{1}{h} + \frac{1}{2}\right) \times \max\left\{\frac{4\nu^2}{\rho}, \sqrt{\frac{2\nu^2}{\rho} \frac{F(x^0) - F(\bar{x})}{F(x^0) - F(\bar{x})}}\right\}$. 
(b) Assume that \( f(\cdot) \) is \( \sigma \)-strongly convex, Algorithm 1 takes at most \( N_2 \) iterations in expectation to converge to a block-\( k \) stationary point \( \bar{x} \) satisfying \( F(x^j) - F(\bar{x}) \leq \epsilon \), where \( N_2 = J \log_a \left( \frac{D}{F(x^0) - F(\bar{x})} \right) + \log_a \left( \frac{\epsilon}{F(x^0) - D - F(\bar{x})} \right) \).

Proof. (a) We first consider the case when \( f(\cdot) \) is generally convex. We denote \( Z_t = \{ i : x_i^t = 0 \} \). We known that the \( Z_t \) only changes for a finite number of times. We assume that \( Z_t \) only changes at \( t = c_1, c_2, ..., c_j \) and we define \( c_0 = 0 \). Therefore, we have:

\[
Z_0 = Z_1 = ..., Z_{1+c_1} \neq Z_{c_1} = Z_{1+c_1} = Z_{2+c_1} = ..., = Z_{1+c_j} \neq Z_{c_j} = ... \neq Z_{c_j} = ...
\]

with \( j = 1, ..., J \). We denote \( \bar{x}^{c_j} \) as the optimal solution of the following optimization problem:

\[
\min_x f(x) + p(x), \text{ s.t. } x_{Z_{c_j}} = 0
\]

with \( 1 \leq j \leq J \).

The solution \( \bar{x}^{c_j} \) changes \( j \) times, the objective values decrease at least by \( jD \), where \( D \) is defined in (14). Therefore, we have:

\[
F(\bar{x}^{c_j}) \leq F(\bar{x}) - j \times D
\]

Combining with the fact that \( F(\bar{x}) \leq F(\bar{x}^{c_j}) \), we obtain:

\[
0 \leq F(\bar{x}^{c_j}) - F(\bar{x}^{c_{j-1}}) \leq F(\bar{x}) - F(\bar{x}^{c_{j-1}}) \leq D
\]

We now focus on the intermediate solutions \( x_{c_{j-1}}, x_{1+c_{j-1}}, ..., x_{-1+c_{j-1}}, x_{c_j} \). Using part (b) in Theorem 1, we conclude that to obtain an accuracy such that \( F(\bar{x}^{c_j}) - F(\bar{x}^{c_j}) \leq D \), it takes at most

\[
\max \left( \frac{4\nu^2}{\theta}, \sqrt{\frac{2\nu^2(F(\bar{x}^{c_j}) - F(\bar{x}))}{\theta}} \right) \text{ iterations to converge to } \bar{x}^{c_j}, \text{ that is,}
\]

\[
c_j - c_{j-1} \leq \max \left( \frac{4\nu^2}{\theta}, \sqrt{\frac{2\nu^2(F(\bar{x}^{c_j}) - F(\bar{x}))}{\theta}} \right) \text{ iterations to converge to } \bar{x}^{c_j}
\]

Summing up the inequality (24) for \( j = 1, 2, ..., J \) and using the fact that \( j \geq 1 \) and \( c_0 = 0 \), we obtain that:

\[
c_J \leq J \leq \frac{\max \left( \frac{4\nu^2}{\theta}, \sqrt{\frac{2\nu^2(F(\bar{x}^{c_j}) - F(\bar{x}))}{\theta}} \right)}{D}
\]

After \( c_j \) iterations, Algorithm 1 becomes the proximal gradient method applied to the problem as in (22). Therefore, the total number of iterations for finding a block-\( k \) stationary point \( N_1 \) is bounded by:

\[
N_1 \leq c_J \leq \frac{\max \left( \frac{4\nu^2}{\theta}, \sqrt{\frac{2\nu^2(F(\bar{x}^{c_j}) - F(\bar{x}))}{\theta}} \right)}{D}
\]

where step (a) uses the fact that the total number of iterations for finding a stationary point after \( x_{c_j} \) is upper bounded by \( \max \left( \frac{4\nu^2}{\theta}, \sqrt{\frac{2\nu^2(F(\bar{x}^{c_j}) - F(\bar{x}))}{\theta}} \right) / \epsilon \); step (b) uses (23) and \( j \geq 1 \).

(b) We now discuss the case when \( f(\cdot) \) is strongly convex. Using part (c) in Theorem 1, we have:

\[
c_j - c_{j-1} \leq \log_a \frac{D}{F(x^{c_j}) - F(\bar{x})},
\]

Summing up the inequality (26) for \( j = 1, 2, ..., J \), we obtain the following results:

\[
c_J \leq \log_a \frac{D}{(F(x^0) - F(\bar{x}))^2} = J \log_a \frac{D}{(F(x^0) - F(\bar{x}))^2}
\]
Therefore, the total number of iterations $\tilde{N}_2$ is bounded by:

$$
\tilde{N}_2 \leq c_j + \log_\alpha \left( \frac{F(x^*) - F(\bar{x})}{\epsilon} \right)
$$

where step (a) uses the fact that the total number of iterations for finding a stationary point after $x_{c_j}$ is upper bounded by $\log_\alpha \left( \frac{F(x^*) - F(\bar{x})}{\epsilon} \right)$; step (b) uses \ref{eq:lip} that $0 \leq F(x^0) - F(\bar{x}) - t \times D$ and $t \geq 1$.

\section{Proof of Convergence Rate for Sparsity Constrained Optimization}

We now prove the convergence rate of Algorithm \ref{alg:main} for sparse constrained optimization with $h \triangleq h_{\text{cons}}$. Our results are based on the strongly convexity and Lipschitz continuity of the objective function. We naturally derive the following theorem.

\textbf{Theorem 3. Convergence Rate for Sparsity Constrained Optimization.} Let $f(\cdot)$ be a $\sigma$-strongly convex function. We assume that $f(\cdot)$ is Lipschitz continuous such that $\forall t$, $||\nabla f(x)\||_{2} \leq \kappa$ for some positive constant $\kappa$. Denote $\alpha \triangleq \frac{\alpha t}{1 + \frac{\alpha t}{\kappa}}$. We have the following results:

$$
\begin{align*}
\mathbb{E}[F(x^t) - F(\bar{x})] &\leq (F(x^0) - F(\bar{x}))\alpha^t + \frac{\kappa}{2\sigma} \frac{\alpha}{1 - \alpha}, \\
\mathbb{E}\left[\frac{\sigma}{2} ||x^{t+1} - \bar{x}||_{2}^2\right] &\leq \frac{\kappa}{2\sigma} \frac{\alpha}{1 - \alpha}.
\end{align*}
$$

\textbf{Proof.} (a) First of all, we define the zero set and nonzero set as follows:

$$
S \triangleq \{i \mid i \in B, \; x_i^{t+1} \neq 0\}, \; Q \triangleq \{i \mid i \in B, \; x_i^{t+1} = 0\},
$$

Using the optimality of $x_i^{t+1}$ for the subproblem, we obtain

$$
(\nabla f(x_i^{t+1}))_S + \theta(x_i^{t+1} - x_S^i) = 0 \Rightarrow (\nabla f(x_i^{t+1}))_S / \theta + x_S^{t+1} = x_S^i \tag{27}
$$

We derive the following inequalities:

$$
\begin{align*}
\mathbb{E}[f(x^{t+1}) - f(\bar{x})] &\leq \mathbb{E}[(x^{t+1} - \bar{x}, \nabla f(x^{t+1})) - \frac{\sigma}{2}||x^{t+1} - \bar{x}||_{2}^2] \\
&\leq \frac{\kappa}{2} \cdot \mathbb{E}[(x_B^{t+1} - x_B^*, (\nabla f(x^{t+1}))_B) - \frac{\sigma}{2}||x_B^{t+1} - \bar{x}_B||_{2}^2] \\
&\leq \frac{\kappa}{2} \cdot \frac{\sigma}{2} \cdot \mathbb{E}[\|\nabla f(x^{t+1})\|_B/\sigma ||_2^2] \\
&\leq \mathbb{E}\left[\|\theta(x_S^t - x_S^{t+1})/\sigma ||_2^2 + ||(\nabla f(x^{t+1}))_Q/\sigma ||_2^2\right] \\
&\leq \mathbb{E}\left[\|\theta(x_S^t - x_S^{t+1})/\sigma ||_2^2 + ||(\nabla f(x^{t+1}))_Q/\sigma ||_2^2\right] \\
&\leq \mathbb{E}\left[\|f(x^t) - f(x^{t+1})\|_2 + \frac{\kappa}{2\sigma} \cdot ||(\nabla f(x^{t+1}))_Q/\sigma ||_2^2\right] \\
&\leq \mathbb{E}\left[\|f(x^t) - f(x^{t+1})\|_2 + \frac{\kappa}{2\sigma} \cdot \|\nabla f(x^{t+1})\|_Q^2\right] \\
&\leq \left[\frac{\kappa}{2\sigma} \cdot ||f(x^t) - f(x)\|_2 - \frac{\kappa}{2\sigma} \cdot \|f(x^{t+1}) - f(\bar{x})\|_2 + \frac{\kappa}{2\sigma} \cdot \|f(x^{t+1}) - f(\bar{x})\|_2 + \frac{\kappa}{2\sigma} \cdot \|f(x^{t+1}) - f(\bar{x})\|_2\right] \
&\leq \left[\frac{\kappa}{2\sigma} \cdot ||f(x^t) - f(x)\|_2 - \frac{\kappa}{2\sigma} \cdot \|f(x^{t+1}) - f(\bar{x})\|_2 + \frac{\kappa}{2\sigma} \cdot \|f(x^{t+1}) - f(\bar{x})\|_2\right] \
&\leq \left[\frac{\kappa}{2\sigma} \cdot ||f(x^t) - f(x)\|_2 - \frac{\kappa}{2\sigma} \cdot \|f(x^{t+1}) - f(\bar{x})\|_2 + \frac{\kappa}{2\sigma} \cdot \|f(x^{t+1}) - f(\bar{x})\|_2\right]
\end{align*}
$$

where step (a) uses the strongly convexity of $f(\cdot)$; step (b) uses the fact that the working set $B$ is selected with $\frac{\kappa}{2\sigma}$ probability; step (c) uses the inequality that $\langle x, a \rangle - \frac{\sigma}{2} ||x||_2^2 = \frac{\sigma}{2} ||a||_2^2 - \frac{\sigma}{2} ||x - a/\sigma||_2^2 \leq \frac{\sigma}{2} ||a/\sigma||_2^2$ for all $a, x$; step (d) uses the fact that $B = S \cup Q$, step (e) uses \ref{eq:lip}; step (f) uses $||x_S||_{2}^2 \leq ||x||_2^2$ for all $x$; step (g) uses the sufficient decrease condition that $\frac{\sigma}{2} ||x^{t+1} - x^t||_2^2 \leq F(x^t) - F(x^{t+1})$; step (h) uses the Lipschitz continuity of $f(\cdot)$ that $||\nabla f(x^{t+1})||_2^2 \leq \kappa$, $\forall t$. 

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From (28), we have the following inequalities:
\[
\mathbb{E}[(1 + \frac{\sigma}{\kappa^2})(f(x^{t+1}) - f(\bar{x}))] \leq \mathbb{E}[\frac{\sigma}{\kappa^2} \cdot \langle f(x^t) - f(\bar{x}) \rangle + \frac{\kappa}{\sigma^2}\kappa]
\]
\[
\mathbb{E}[f(x^{t+1}) - f(\bar{x})] \leq \mathbb{E}[\alpha(f(x^t) - f(\bar{x})) + \frac{n\alpha}{\sigma^2}\kappa]
\]
\[
\mathbb{E}[f(x^{t+1}) - f(\bar{x})] \leq \mathbb{E}[\alpha(f(x^t) - f(\bar{x})) + \frac{n\alpha^2}{\sigma^2}]
\]
Solving this recursive formulation, we have:
\[
\mathbb{E}[f(x^t) - f(\bar{x})] \leq \mathbb{E}[\alpha^t(f(x^0) - f(\bar{x}))] + \kappa \sum_{i=1}^{t} \alpha^t
\]
\[
= \mathbb{E}[\alpha^t(f(\bar{x}) - f(\bar{x}))] + \frac{n\alpha}{\sigma^2} \cdot \frac{1}{1 - \alpha}
\]
\[
\leq \mathbb{E}[\alpha^t(f(x^0) - f(\bar{x}))] + \frac{n}{\sigma^2} \cdot \frac{\alpha}{1 - \alpha}
\]
Since \(x^t\) is always a feasible solution for all \(t = 1, 2, ..., \infty\), we have \(F(x^t) = f(x^t)\). Therefore, we obtain (27).

(b) We now prove the second part of this theorem. First, we derive the following inequalities:
\[
\mathbb{E}[\|x^{t+1} - x^t\|_2^2] \leq \mathbb{E}[\|\bar{x}^{t+1} - x^t, \nabla f(x^{t+1})\| + f(\bar{x}) - f(x^{t+1})]
\]
\[
\leq \mathbb{E}[\|\bar{x}^{t+1} - x^t, \nabla f(x^{t+1})\|]
\]
\[
\leq \mathbb{E}[\|x^{t+1} - \bar{x}\| \cdot \|\nabla f(x^{t+1})\|]
\]
where step (a) uses the sufficient decrease condition in (10); step (b) uses the fact that \(F(\bar{x}) \leq F(x^{t+1})\); step (c) uses the result in (29).

Second, we have the following results:
\[
\mathbb{E}[\|x^{t+1} - \bar{x}\|_2^2] \leq \mathbb{E}[\|\bar{x}^{t+1} - x^t, \nabla f(x^{t+1})\| + f(\bar{x}) - f(x^{t+1})]
\]
\[
\leq \mathbb{E}[\|\bar{x}^{t+1} - x^t, \nabla f(x^{t+1})\|]
\]
\[
\leq \mathbb{E}[\|x^{t+1} - \bar{x}\| \cdot \|\nabla f(x^{t+1})\|]
\]
where step (a) uses the strongly convexity of \(f(\cdot)\); step (b) uses the fact that \(f(\bar{x}) \leq f(x^{t+1})\); step (c) uses the Cauchy-Schwarz inequality.

From (30), we further have the following results:
\[
\mathbb{E}[\|x^{t+1} - \bar{x}\|_2^2] \leq \mathbb{E}[\|\nabla f(x^{t+1})\|_2^2]
\]
\[
= \frac{\kappa}{\sigma} \mathbb{E}[\|\nabla_B f(x^{t+1})\|_2^2]
\]
\[
= \frac{\kappa}{\sigma} \mathbb{E}[\|\nabla S f(x^{t+1})\|_2^2 + \|\nabla Q f(x^{t+1})\|_2^2]
\]
\[
\leq \frac{\kappa}{\sigma} \mathbb{E}\left[\theta^2 \|x^{t+1} - x^t\|_2^2 + \frac{n}{\sigma^2} \kappa \right]
\]
\[
= \frac{\kappa}{\sigma} \mathbb{E}[2\theta \alpha^t(f(x^0) - f(\bar{x})) + \frac{\kappa}{1 - \alpha}]
\]
where step (a) uses the strongly convexity of \(f(\cdot)\); step (b) uses the fact that \(B = S \cup Q\); step (c) uses the assumption that \(\|\nabla f(x^t)\|_2^2 \leq \kappa\) for all \(x^t\) and the optimality of \(x^{t+1}\) in (27); step (d) uses (29). Therefore, we finish the proof of this theorem.

\[\square\]

10 Additional Discussions

This section provides additional discussions for the proposed method.
10.1 When the objective function is complicated

In step (S2) of the proposed algorithm, a global solution is to be found for the subproblem. When \( f(\cdot) \) is simple (e.g., a quadratic function), we can find efficient and exact solutions to the subproblems. We now consider the situation when \( f \) is complicated (e.g., logistic regression, maximum entropy models). One can still find a quadratic majorizer \( Q(x, z) \) for the convex function \( f(x) \) with

\[
f(x) \leq Q(x, z) \triangleq f(z) + (x - z)^T \nabla f(z) + \frac{1}{2} (x - z)^T M(z)(x - z), \quad \forall \, z, \ x, \ M(z) \succ \nabla^2 f(z).
\]

By minimizing the upper bound of \( f(x) \) (i.e., the quadratic surrogate function) at the current estimate \( x^t \), i.e.,

\[
x^{t+1} = \arg \min_x Q(x, x^t) + h(x),
\]

we can drive the objective downward until a stationary point is reached. We will obtain a stationary point \( \bar{x} \) satisfying:

\[
\bar{x} = \arg \min_z h(z) + f(\bar{x}) + (z - \bar{x})^T \nabla f(\bar{x}) + \frac{1}{2} (z - \bar{x})^T M(\bar{x})(z - \bar{x}), \quad s.t. \ \bar{x}_B = z_B
\]

for all \( B \). However, this is weaker than the block-\( k \) stationary point \( \tilde{x} \): \( \tilde{x} = \arg \min_z h(z) + f(z), \quad s.t. \ (\tilde{x})_B = (z)_B \) for all \( B \).

10.2 Practical computational efficiency

Block coordinate descent is shown to be very efficient for solving convex problems (e.g., support vector machines [11, 19], LASSO problems [40], nonnegative matrix factorization [20]). The main difference of our block coordinate descent from existing ones is that our method needs to solve a small-sized NP-hard subproblem globally which takes subexponential time \( O(2^k) \). As a result, our algorithm finds a block-\( k \) approximation solution for the original NP-hard problem within \( O(2^k) \) time. When \( k \) is large, it is hard to enumerate the full binary tree since the subproblem is equally NP-hard. However, \( k \) is relatively small in practice (e.g., 2 to 20). In addition, real-world applications often have some special (e.g. unbalanced, sparse, local) structure and block-\( k \) stationary point could also be the global stationary point (refer to Table 1 of this paper).