Relative concentration bounds for the spectrum of kernel matrices

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Abstract: In this paper we study some concentration properties of the kernel matrix associated with a kernel function. More specifically, we derive concentration inequalities for the spectrum of a kernel matrix, quantifying its deviation with respect to the spectrum of an associated integral operator. The main difference with most results in the literature is that we do not assume the positive definiteness of the kernel. Instead, we introduce Sobolev-type hypotheses on the regularity of the kernel. We show how these assumptions are well-suited to the study of kernels depending only on the distance between two points in a metric space, in which case the regularity only depends on the decay of the eigenvalues. This is connected with random geometric graphs, which we study further, giving explicit formulas for the spectrum and its fluctuations.

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1. Introduction and motivations

Kernel methods have become nowadays a major tool in machine learning with a wide range of applications such as principal component analysis (PCA), clustering and more recently statistical analysis of networks. Many of these methods rely on the spectral decomposition of a data dependent random matrix. In this context, the study of eigenvalues fluctuation of such a matrix is a central point to get theoretical guarantees of the error on these methods.

Given a kernel $W$, we study the spectrum of two related objects: an integral operator associated with $W$ and the so-called kernel matrix, which is built from the kernel by sampling. The reason is that as the size of the kernel matrix grows, an appropriate normalization of it will converge to the kernel. In particular, the spectrum of the matrix converges to the spectrum of the integral operator, which has been proven in [1]. This is connected with the convergence of a growing sequence of graphs to a kernel (or graphon), which is made precise in theory of limits of dense graphs (see [2]). It seems reasonable to study the finite sample approach, that is to quantify the way this convergence takes place as a function of the sample size. We focus on obtaining concentration inequalities that describe how each eigenvalue of the kernel matrix deviates from the corresponding eigenvalue of its continuous counterpart, giving explicit rates of convergence.

There are many contributions which, in a direct or indirect way, have addressed this issue. For instance, in [1] is obtained an asymptotic result for the convergence of the spectrum of the kernel matrix to the spectrum of the integral operator, under a properly defined metric $\ell_2$ type metric. Is also achieved a central limit type theorem describing the precise limit law for the fluctuation of eigenvalues of the kernel matrices, under mild integrability conditions on the kernel. Other works have followed a finite sample approach. In [3] the authors applied
concentration theorems for random matrices in order to obtain a bound for the $\delta_2$ metric, introduced in [1], between the spectrum of kernel matrix and the operator. In [4] are proposed similar results focusing on the concentration of the tail sum of eigenvalues. Following a different approach, Braun [5] studied the concentration of each eigenvalue of the kernel matrix with respect to the corresponding eigenvalue of the integral operator, when the kernel is assumed to be positive. Our results take place in the continuity of these developed by Braun [5], as we follow its strategy to obtain a bound for each eigenvalue, although we use different concentration inequalities. Indeed, instead of scalar concentration, we use matrix concentration inequalities following the work of [6]. Another major difference relies in the fact that we do not assume the positivity of the kernel, because in the framework of networks (our original motivation), this hypothesis is restrictive. Instead, our main hypothesis relies on the decreasing rate of the eigenvalues and the growth rate of the eigenvectors, which we quantify by introducing two parameters. Their interaction is crucial for describing the eigenvalues concentration. We also show how these decay/growth hypotheses take a simpler form in the specific case of kernels that only depends on the distance. When one takes the sphere $\mathbb{S}^{d-1}$ as our underlying space, the eigenfunctions of the integral operator are fixed (they are the spherical harmonics) and our regularity hypothesis translates as an eigenvalue decay rate, which is related to the classical Sobolev-type regularity conditions. This makes our results well-studied to study the spectrum of random geometric graphs on the sphere. We apply our result to examples in this context, computing explicitly its spectrum. To the best of our knowledge, these results are published for the first time.

Overview

The present paper is structured as follows: in Section 2 we introduce the basic material for integral operators, explaining in Section 2.3 its connection with random graphs and the graphon model. In Section 3 we state the main results and give the main steps of their proof, recalling some concentration inequalities for random matrices. In Section 4 we compare the obtained rates of convergence with results in the literature. In Section 5 we dig further on the main hypothesis of the theorem and explicit its relation with some classical smoothness assumptions. We study the case of kernels that only depend on the distance between two points, and give some examples in the case of geometric random graphs, obtaining explicit formulas for the spectrum.

2. Preliminary results

2.1. Integral operator and empirical matrices

We introduce the main objects used throughout this paper, namely the kernel function and the integral operator associated with it. We also highlight the connection with the graphon model, which allows to apply our main results in the context of dense networks. The particular case when the underlying space is the sphere and the kernel only depends on the geodesic distance between two points corresponds to geometric random graph model, which will be carefully studied in Section 6.
2.2. Integral operator and kernel matrices

Given a measurable space \((\Omega, \mu)\), a kernel is a symmetric measurable function \(W : \Omega^2 \to \mathbb{R}\). For a kernel function \(W\), their eigenvalues and eigenvectors are those of the integral operator \(T_W : L^2(\Omega) \to L^2(\Omega)\) defined by the relation

\[
T_W f(x) = \int_{\Omega} W(x, y) f(y) d\mu(y)
\]

By a classical result of functional analysis, \(T_W\) is a compact operator and by the spectral theorem for compact operators the spectrum of \(T_W\) is a numerable set and the only accumulation point is 0. We index the sequence of eigenvalues \(\{\lambda_k\}_{k \in \mathbb{N}}\) in decreasing order with respect to its absolute value, that is \(|\lambda_1| \geq |\lambda_2| \geq \cdots \geq 0\). For \(i \in \mathbb{N}\), \(\lambda_i(T_W)\) will denote the \(i\)-largest eigenvalue in absolute value. On the other hand, \(\lambda(T_W) \in \mathbb{R}^N\) will denote the infinite ordered spectrum (in decreasing order). A consequence of the spectral theorem is that the eigenvectors sequence \(\{\phi_k\}_{0 \leq k \leq \infty}\) form a Hilbertian basis of \(L^2(\Omega, \mu)\). Furthermore, we can decompose the kernel in the \(L^2\) sense as follows

\[
W = \sum_{k \in \mathbb{N}} \lambda_k \phi_k \otimes \phi_k \tag{1}
\]

where \(\phi_k \otimes \phi_k(x, y) = \phi_k(x)\phi_k(y)\). The latter implies that \(\{\phi_k \otimes \phi_k\}_{0 \leq k \leq \infty}\) is a Hilbertian basis of \(L^2(\Omega^2, \mu \times \mu)\). When we say that a kernel \(W\) has rank \(R\), it means that in \(\{\lambda_k\}_{0 \leq k \leq \infty}\) there are exactly \(R\) non-zero elements. If there are infinitely non-zero eigenvalues, we say that the kernel has infinite rank. Under stronger assumptions the convergence in (1) will hold pointwise or uniformly on \(\Omega\). For instance, the classical Mercer theorem tells that if we assume the kernel to be continuous and positive, then the convergence in (1) is uniform (see [7]).

Given the i.i.d random \(\Omega\)-valued random variables \(\{X_i\}_{1 \leq i \leq n}\) with law \(\mu\), we construct an empirical version of this operator, which is a \(n \times n\) matrix. For any \(n \in \mathbb{N}\) we define

\[
\Theta_{ij}(n) = W(X_i, X_j)
\]

which is a random matrix, called kernel or empirical matrix. It has also been called probability matrix in the graphon framework, as in [8]. We mainly deal with its normalized version \(T_n := \frac{1}{n}\Theta(n)\). We use the notation \(\lambda(T_n)\) and \(\lambda_i(T_n)\) for the spectrum in the same way as in the infinite dimensional case, with the exception that, for comparison purposes with the infinite dimensional case, in the matrix case we complete the eigenvalue sequence with zeroes, so its spectrum is also in \(\mathbb{R}^N\). We focus our work on quantifying the difference between \(\lambda_i(T_W)\) and \(\lambda_i(T_n)\) as is done in [5]. The difference between the whole spectral sequences \(\lambda(T_W)\) and \(\lambda(T_n)\) can be quantified by using the \(\ell_2\) metric, which is defined given two sequences \(x\) and \(y\) on \(\ell_2\) by

\[
\delta_2(x, y) := \inf_{\pi \in P} \sum_{i} (x_i - y_{\pi(i)})^2
\]

where \(P\) is the set of permutations with finite support. In [1] there are asymptotic results which prove the convergence of \(\lambda(T_n)\) to \(\lambda(T_W)\) as \(n \to \infty\) in the \(\delta_2\) metric. They also derived a central limit theorem, giving the law of fluctuations of \(\lambda(T_n)\) around \(\lambda(T_W)\). In the same way, in [3] the authors obtained finite sample concentration results for the ensemble of eigenvalues. More specifically, they obtained a bound for \(\delta_2(\lambda(T_W), \lambda(T_n))\) that holds with high probability. Their results are included in the framework of random geometric graphs, although many of them apply in the wider context of bounded kernels. The aforementioned
results already give some information about the convergence of each eigenvalue of the kernel matrix to the eigenvalue of the same index of the continuous counterpart, by the trivial relation
$$|\lambda_i(T_W) - \lambda_i(T_n)| \leq \delta_2(\lambda(T_W), \lambda(T_n)),$$
but as is highlighted in [5], we have to adjust to the scale of each eigenvalue to avoid overestimating the error.

In other words, we are looking for results of the following type: with high probability, the error satisfies
$$|\lambda_i(T_W) - \lambda_i(T_n)| = O(n^{-h})$$
where $h$ is a positive number (usually $0 < h < 1$). Here more parameters are usually introduced, most commonly the parameter that fixes the number of terms in a finite rank approximation, which introduces a bias term. A priori this impede the convergence to zero of the error as $n$ tends to infinity. We choose to make these finite rank approximation parameter to depend on $n$ and to optimize this dependence. Doing so, allows to get a convergence to zero error, obtaining the explicit rates that appears in Theorem 1.

Although it is always possible to obtain inequalities for individual eigenvalues using Weyl inequalities for smaller eigenvalues, this will give a bound in the scale of the operator norm (or the maximum eigenvalue in absolute value). For eigenvalues different of the largest, this inequality would be pessimistic as quoted in [5] (see also Remark 3 in Appendix A). There the authors obtain concentration inequalities scaled with respect to the eigenvalue in consideration. They called this inequalities \textit{scaling bounds} because the magnitude of each eigenvalue appears in the bound, obtaining a relative bound of the form
$$\frac{|\lambda_i(T_W) - \lambda_i(T_n)|}{|\lambda_i(T_W)|} = O(n^{-h})$$
for the eigenvalues such as $|\lambda_i(T_W)| > 0$ (in the case $\lambda_i(T_W) = 0$, the previous result becomes $|\lambda_i(T_n)| = O(n^{-h'})$). Thus each inequality is scaled in terms of each particular eigenvalue which should be more accurate, specially for smaller eigenvalues. On the other hand, the results in [5] assume positive definiteness of the kernel, which in the context of graphs or graphons might be too restrictive.

Here we follow a similar approach to the one in [5] obtaining scaling bounds specialized to each eigenvalue. The main difference is that we do not assume positive definiteness of the kernel functions, which renders our results more suitable for the study of networks. The counterpart is that we need a stronger hypothesis. However there are many cases where this hypothesis holds, as we exemplify in the context of geometric random graphs in Section 6.

\textbf{2.3. Connections with random graphs}

One general way to construct a random graph models is to define its random adjacency matrix. For a simple graph with $n$ nodes, we consider the $n \times n$ symmetric matrix $A$, where the entries $(A_{ij})_{1 \leq i < j \leq n}$ are independent Bernoulli random variables with corresponding mean $\Theta_{ij}$. The latter matrix is called the probability matrix in [8] and is in fact a bounded kernel matrix. We will mainly focus on the properties of the matrix $\Theta_{ij}$, when is considered a generative random model that depends on a kernel.

In graph limit theory, a symmetric kernel of the form $W : \Omega^2 \rightarrow [0,1]$ is called \textit{graphon}. We could consider, without loss of generality, that any graphon $W$ is defined on $\Omega = [0,1]$, in the sense that there exists an equivalent graphon defined on $[0,1]$. We choose no to do so, because sometimes the representation on the interval $[0,1]$ could be less revealing than using a different space as is exemplified in [9, p. 190]. Later, in the geometric graph setting we will make an explicit use of the sphere geometry throughout the geodesic distance.
3. Main results

We are looking for quantifying the difference between the spectrum of \( T_W \) and \( T_n \) by probabilistic bounds, one for the deviation of each eigenvalue of the ordered spectrum and the other for the ensemble of eigenvalues.

Our main result depends on two hypotheses, one on the eigenvectors and the other on the eigenvalues, which together will allow to have a fine control on the terms in the expansion (1). It is well-known that the eigenvalues of the compact operator \( T_W \) decrease to 0. And the more regular the kernel is, the faster they decrease. The eigenvectors are continuous functions, but they may not be bounded uniformly, even for very regular kernels [11, ex. 1].

In brief, the eigenvalues sequence \( \{\lambda_k\}_{k \in \mathbb{N}} \) will decrease to 0 and the eigenvectors norm sequence \( \{\|\phi_k\|\}_{k \in \mathbb{N}} \) could grow to \( \infty \). We parametrize this by introduction of \( \delta \) and \( s \), that will express respectively the behavior of the eigenvalues and eigenvectors. More precisely, we will assume for \( k \in \mathbb{Z} \) that \( |\lambda_k| = O(f(|k|)) \) where \( f \) is either \( f(k) = k^{-\delta} \) (polynomial decay) or \( f(k) = e^{-\delta k} \) (exponential or superpolynomial decay). These two decreasing rates are common assumptions in theoretical analysis of kernel methods, as in [5] or [12]. In addition, they are satisfied for many widely used kernels.

To quantify the growth in the eigenvectors, we introduce the function \( \eta(R) = \| \sum_{k=1}^{R} \phi_k^2 \|_{\infty} \) and we assume that \( \eta(R) = O(g(R)) \), where \( g \) can be either \( g(R) = R^s \) or \( g(R) = e^{sR} \). We note that this quantity has appeared before in the literature of concentration inequalities for kernels [3, thm. 2]. We could certainly have assumed a bound in the eigenvector of the form \( \|\phi_k\|_{\infty} = O(|k|^s) \) instead, as the latter will imply that \( \eta(k) = O(k^{2s+1}) \). Both type of hypotheses are then equivalent. We introduce the hypotheses we will work with

| Assumption                             | Assumption                             |
|----------------------------------------|----------------------------------------|
| \( H_1: \lambda_i = O(k^{-\delta}); \eta(R) = O(R^s) \) | \( H_2: \lambda_i = O(e^{-\delta k}); \eta(R) = O(R^s) \) |
| \( H_3: \lambda_i = O(k^{-\delta}); \eta(R) = O(e^{sR}) \) | \( H_4: \lambda_i = O(e^{-\delta k}); \eta(R) = O(e^{sR}) \) |

Table 1: Hypotheses for the decrease of eigenvalues and growth of eigenvectors

We delay until Section 5 the discussion of these hypotheses and examples when they hold. The following theorem is our main result describing the fluctuation of single individual eigenvalues for infinite rank kernels

**Theorem 1.** If \( W \) is a \( L^2(\Omega \times \Omega) \) infinite rank kernel satisfying the polynomial regularity condition \( H_1 \), then for \( \alpha \in (0,1) \) there exists a sequence \( w_{\alpha,w}(n) \), such as \( w_{\alpha,w}(n) \to \infty \) if

Given a kernel (or probability) matrix we define the connection probability between two nodes as follows

\[
P(A_{ij} = 1) = W(X_i, X_j) = \Theta_{ij}
\]

Here \( A_{ij} \) is the adjacency matrix of a graph of node size \( n \). In order to sample a graph from a graphon, we first construct the probability matrix and then the adjacency is obtained by considering the variables \((A_{ij})_{i<j}\) to be independent Bernoulli variables with mean \( \Theta_{ij} \). Note that by choosing \( W(x,y) = p \) we recognize the well-known Erdös-Rényi model. On the other hand, if we consider a metric \( \rho \) on \( \Omega \) and the graphon \( W(x,y) = 1_{\rho(x,y)<\tau} \) we recover the classical random geometric graph of threshold parameter \( \tau \). A comprehensive probabilistic study of the latter model can be found in [10].
Then for $1 \leq i \leq w_{\alpha,W}(n)$ we have with probability bigger than $1 - \alpha$

$$|\lambda_i(T_W) - \lambda_i(T_n)| \leq C(\alpha) i^{-2t^2} \left( \frac{n}{\log n} \right)^{\frac{\delta}{2t+\gamma}}$$

where $C(\alpha)$ is decreasing in $\alpha$ and independent of $n$. On the other hand, if $W$ satisfies the exponential regularity $H_2$ we have for $1 \leq i \leq w'_{\alpha,W}(n)$ with probability bigger than $1 - \alpha$

$$|\lambda_i(T_W) - \lambda_i(T_n)| \leq C'(\alpha) i^{\frac{45 - 4t^2 - 12s \lambda}{4t + 3s}} n^{-\frac{28}{4t + 3s}}$$

where $C'(\alpha)$ is decreasing in $\alpha$ and independent of $n$. In the case $W$ satisfies the hypothesis $H_3$, there exists a sequence $1 \leq i \leq w''_{\alpha,W}(n)$ such as with probability bigger than $1 - \alpha$

$$|\lambda_i(T_W) - \lambda_i(T_n)| \leq C''(\alpha) e^{\frac{2t^2}{2t+\gamma}} \left( \frac{n}{\log n} \right)^{\frac{\delta}{2t+\gamma}}$$

**Remark 1** (About the functions $w_{\alpha,W}, w'_{\alpha,W}$ and $w''_{\alpha,W}$ in Theorem 1). Introducing the functions $w_{\alpha,W}, w'_{\alpha,W}$ and $w''_{\alpha,W}$ is mainly technical and related to the deterministic perturbation results used in the proof of Theorem 1. The number of indexes for which the concentration inequalities hold grows with $n$. For instance we have $w_{\alpha,W} = O(n^{1/4})$, see Remark 10 in Appendix C.1.2.

Given the dependence in $i$ on the right hand side, the theorem represents a scaling bound for individual eigenvalues, in the same way that [5, thm. 3 and thm. 4]. This allows for more accurate estimation of the error for smaller eigenvalues, in contrast to non scaling bound that for smaller eigenvalues tends to be too pessimistic (as the dominant eigenvalues "set the scale" in that case). This have potential applications for graphs, as the ratio between the smallest and the largest eigenvalue give information about certain graph parameters, like the chromatic number of a graph (see [13, thm. 6.7]) and the discussion above. Now we give a concentration theorem in the finite rank case, that is when there are exactly $R$ terms involved in the expansion (1), which immediately implies that the expansion holds pointwise.

**Theorem 2.** Let $W_R$ be a rank $R$ kernel and $T_n$ its normalized kernel matrix, then for $t > 0$

$$\mathbb{P}(|\lambda_i(T_{W_R}) - \lambda_i(T_n)| \geq t) \leq d \cdot \exp \left( \frac{-nt^2}{(\eta(R) - 1)(2(\eta(R) + 1)\lambda_i^2 + \frac{2\lambda_i}{\sqrt{d}})} \right)$$

(2)

The latter implies that for $\alpha \in (0, 1)$, it holds with probability bigger than $1 - \alpha$

$$|\lambda_i(T_{W_R}) - \lambda_i(T_n)| \leq \frac{\lambda_i \eta(R)}{n} \log d/\alpha + \frac{2\lambda_i}{\sqrt{n}} \sqrt{2(\eta^2(R) - 1) \log d/\alpha}$$

### 3.1. Techniques used and idea of the proof

The proof of Theorem 1 builds upon a series of intermediate probabilistic bounds, which have interest on their own, and is presented in detail in the Appendix B. Here we present a stylized version of some bounds from which Theorem 1 follows, omitting some details to avoid overly technical arguments. We introduce the finite rank operator

$$W_R(x, y) = \sum_{k=1}^{R} \lambda_k \phi_k(x) \phi_k(y)$$

(3)
where the parameter $R$ represents the number of eigenvalues/eigenvectors used to approximate $W$. Many of our results, as these in [5] and [4], will include the tail sum term $\nu_R := \sum_{k=1}^R |\lambda_k|$. An intermediate step in our proof will be a bound of the following type

**Proposition 3.** Given $R \in \mathbb{N}$ fixed. It holds, with probability bigger than $1 - \alpha$, for $1 \leq i \leq R$,

$$|\lambda_i(T_W) - \lambda_i(T_n)| \leq C\frac{|\lambda_i|}{\sqrt{n}} \vartheta(R, \alpha) + \nu_R + \frac{\varsigma(R, \alpha)}{\sqrt{n}}$$

(4)

and, for $i \geq R$,

$$|\lambda_i(T_W) - \lambda_i(T_n)| \leq C'|\lambda_i| + \nu_R + \frac{\varsigma(R, \alpha)}{\sqrt{n}}$$

where $\vartheta$ and $\varsigma$ are two increasing functions in first argument and decreasing in the second. $C$ and $C'$ are constants which do not depend on $n$ or $R$.

The strategy to prove Proposition 3 is to use the following perturbation theorem for each eigenvalue, obtained applying Weyl and Ostrowsky theorems. This is the perturbation bound which appears as Theorem 1 in [5]

**Proposition 4.** If $R, i \in \{1, \ldots, n\}$ we have

$$|\lambda_i(T_n) - \lambda_i(T_W)| \leq \lambda_i \|E_{R,n}\| + \|T_n - T_{n,R}\| \quad \text{for } i \leq R$$

$$|\lambda_i(T_n) - \lambda_i(T_W)| \leq \lambda_i + \|T_n - T_{n,R}\| \quad \text{for } i > R$$

We remark that Proposition 4 is deterministic, so it will hold pointwise. Also note that hypotheses $H_1$ and $H_2$ are not needed in this step. In order to obtain probabilistic bounds, we will resort on matrix concentration inequalities for the sum of random matrices. Particularly, we make repeated use of the following Matrix Bernstein inequality (see [6, thm. 6.1]).

**Theorem 5** (Matrix Bernstein). Let $\{X_k\}_{1 \leq k \leq n}$ a sequence of independent, self-adjoint, random matrices of dimension $d$. We assume that $\mathbb{E}(X_k) = 0$ and $\|X_k\|_{op} \leq R$ a.s. Defining $\sigma^2 = \|\sum_{1 \leq k \leq n} \mathbb{E}(X_k^2)\|_{op}$, we have

$$\mathbb{P}(\|\sum_{1 \leq k \leq n} X_k\|_{op} \geq t) \leq d \exp \frac{-t^2/2}{\sigma^2 + Rt/3}$$

Furthermore, we have

$$\mathbb{E}(\|\sum_{1 \leq k \leq n} X_k\|_{op}) \leq \sqrt{2\sigma^2 \log d} + \frac{R}{3} \log d$$

Note that the main role of our hypotheses is to ensure that the expansion (1) holds not only in $L^2(\Omega)$, but also pointwise. We prove Theorem 1 under a weaker hypothesis than $H_1$ and $H_2$, which its explicit form is given in (H) in Appendix B. The decomposition of the kernel in the pointwise sense is central for the application of Theorem 5 and will also immediately allows for the application of other potentially useful concentration inequalities for sum of random matrices. See [6] for a comprehensive overview.

The end of the proof of Theorem 1 relies on the optimization of the parameter $R$ in Proposition 3 part (4). We find in that way an $R$ that will depend on $n$ and $i$. Assuming that $R$ depends on $n$ changes the way of the inequality (4) and its potential applications. To see
this, note that in Theorem 1 we have a bound that decreases to 0 as \( n \to \infty \). Contrarily, in Proposition 3 we have that \( R \) is fixed and the error \(|\lambda_i(T_W) - \lambda_i(T_n)|\) does not tend to 0 with \( n \), because of the remaining tail (or bias) term \( \nu_R \). In [5] they consider that \( R \) can depend on \( n \) and \( i \), but not both at the same time. In this sense, our result would have particular interest if we want to obtain precise information about a singular predefined eigenvalue. This is sometimes the case when studying networks, e.g., where particular eigenvalues, the second largest of the adjacency matrix for instance, gives information about its connectivity.

**Remark 2.** As is noted in [6], the direct use of Bernstein inequality is not the only way for obtaining tail bounds for the sum of symmetric random matrices. We can instead use one dimensional concentration inequalities for the suprema of empirical process, as is exemplified in [14, chap. 13]. These inequalities, however do not provide information about the mean of the operator norm. This is why Theorem 5 is useful and both inequalities can be used in a complementary way.

4. Asymptotic rate analysis

We can compare our asymptotic rates with the ones obtained in [5] in the case of bounded eigenfunctions. As they assume uniformly bounded eigenfunctions, this will translate to \( \eta(R) = O(R) \) in our setting, that is \( s = 1 \). Their error rate is \( O(i^{-\delta}n^{-1/2\delta} \sqrt{\log n}) \) and ours is \( O\left(i^{-2\delta^2/2\delta+1}\left(\frac{n}{\log n}\right)^{-\delta/2\delta+1}\right) \). We observe that for \( i \) fixed we obtain a slightly better rate for the error in terms of \( n \). Indeed, as \( \delta > 1 \) we have that \( \frac{1-\delta}{2\delta} > \frac{-\delta}{2\delta+1} \). For instance, if \( \delta = 2 \) we have \( \frac{1-\delta}{2\delta} = \frac{1}{4} \) and \( \frac{-\delta}{2\delta+1} = \frac{-2}{3} \). This difference is explained by the finer control on each eigenfunction norm, given by the hypotheses H1, H2 and H3, which results in a better control on the variance proxy \( \sigma^2 \) in Theorem 5. On the other hand, note that in terms of \( i \) our rate is worse, but if the regularity is big enough they will be similar. This will be important in the case when \( i \) tends to infinity with \( n \). For \( i \) fixed, the analysis in Appendix C reveals that the error rate is maintained. In that case we are choosing a parameter \( R \) that do not consider the particular eigenvalue \( i \). This could serve if we are interested, for instance, in a portion of the spectrum, other that a single eigenvalue. Here is hidden the fact that when we consider the order we are omitting constants and in a finite sample analysis, they could play an important role. In that sense, our analysis permits not only to find bounds to the asymptotic order of the error in terms of \( n \) (as we do in Propositions 12 and 13), but also propose proxies for \( N \) and \( R \).

5. Eigenvalue decay revisited

In Theorem 1 we assumed certain decreasing rate in the eigenvalues \( \{\lambda_k\}_k \) of the integral operator, along with an hypothesis on the growth of the eigenvectors. Given a kernel, the fact that their eigenvalues and eigenfunctions will fall into this regime is not necessarily easy to verify. On the other hand, we know that the decay rate of the eigenvalues is closely related to the notion of smoothness of the kernel, when such a notion can be properly defined. In this section we highlight this connection, which has obvious practical consequences. This type of hypothesis appears naturally as regularity hypothesis of the Sobolev-type. Indeed, given a measurable metric space \((\mathcal{X}, \kappa, \nu)\) where \( \kappa \) is a distance and \( \nu \) a probability measure, we suppose that \( \{\varphi_k\}_{k \in J} \) is an orthonormal basis of \( L^2(\mathcal{X}, \nu) \), where \( J \) is a countable set. We
define the weighted Sobolev space $S_\omega$ with associated positive weights $\omega = \{\omega_j\}_{j \in \mathcal{J}}$ as

$$S_\omega(\mathcal{X}) := \left\{ f \in L^2(\mathcal{X}\times \mathcal{X},\omega) \mid \nabla f \in L^2(\mathcal{X}\times \mathcal{X},\omega) \right\}$$

We can take, as in Section 2.3, the measurable metric space $(\mathcal{X}, \rho, \mu)$ and consider $\mathcal{X} = \Omega^2$ and $\nu = \mu \times \mu$. If $\phi_k$ is a basis of $L^2(\Omega, \mu)$ then a basis for $L^2(\mathcal{X}, \mu)$ is $\varphi_k = \phi_k \otimes \phi_k$. We note that for a kernel $W$ in $S_\omega(\mathcal{X})$ with eigenvalues $\lambda_k$ and eigenvectors $\phi_k$, we have $\hat{f}(k) = \lambda_k$. If we want that the series in definition of the Sobolev space to converge, it must hold $\lambda_k^2 \frac{1}{\omega_k} = O\left(\frac{1}{k^{1+\delta'}}\right)$ where $\delta' > 0$. The latter implies for $\lambda_k$ a decrease of order $O(k^{-\delta})$ with $\delta > 1$ which is related to the behavior of the coefficients $\omega_k$.

When $\Omega$ is an open subset of $\mathbb{R}^d$, the classical definition of weighted Sobolev spaces makes use of the (weak)-derivatives of a function. If $\omega : \Omega \to [0, \infty)$ is a locally integrable function, we define the weighted Sobolev space $W^p_2(\Omega, \omega)$ as the normed space of locally integrable functions $f : \Omega \to \mathbb{R}$ with $p$ weak derivatives such as the following norm is finite

$$\|f\|_{p, \omega} = \left( \int_\Omega |f(x)|^p \omega(x)^{\frac{1}{p}} \right)^{\frac{1}{p}} + \left( \sum_{|\alpha|=p} |D^\alpha f(x)|^2 \omega(x) \right)^{\frac{1}{2}}$$

For a symmetric kernel $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ we could think in defining the Sobolev regularity by the embedding $\mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{2d}$, but it seems more natural (see [12, sect. 2.2]) to say that the kernel satisfies the weighted Sobolev condition if $K(\cdot, x) \in W^p_2(\Omega, \omega)$ for any $x \in \Omega$. However, in the case of the dot product kernels, where there exists a function $f : \mathbb{R} \to [0, 1]$ such as $K(x, y) = f((x, y))$ it is more natural to say that $K$ that satisfies the Sobolev condition with weight $\omega : \mathbb{R} \to \mathbb{R}$ if $f \in W^p_2(\mathbb{R}, \omega)$. That is, given that $f$ is defined on $\mathbb{R}$, we can carry out the analysis in one dimension.

In [15] is proved that in the one dimensional case, both definitions of weighted Sobolev spaces are coincident. Otherwise stated, the following equality between metric spaces $S_\omega([-1, 1]) = W^p_2([-1, 1], \omega')$ holds, where $\omega = \{w_k\}_{k \in \mathbb{N}} = \frac{1}{1+\nu_k^d}$, with $\nu_k = k(k+d-1)$ and $\omega'(x) = (1-x^2)^{\frac{d-2}{2}}$. Here we recognize in $\nu_k$ the sequence of eigenvalues of the Laplacian operator and $\omega'$ is the weight associated with the Gegenbauer polynomials $C_\gamma^d(\cdot)$ with $\gamma = \frac{d-2}{2}$. In the next section, we explore this case in more detail and highlight the connection with random geometric graphs.

6. Random geometric graphs on the sphere

Now we will consider the space $\Omega = S^{d-1}$ equipped with $\rho$ the geodesic distance and the measure $\sigma$, which is the surface (or uniform) measure. We will say that a graphon of the form $W(x, y) = p(\rho(x, y))$, where $p : [-1, 1] \to [0, 1]$, is a random geometric graphon. The name came from the classical random geometric graph model where $p(t) = 1_{(x,y) \ge \tau}$. Here $\tau \in [-1, 1]$ is called the threshold parameter. For a complete study of this model, we refer the interested readers to [10]. We keep the name although the model we consider is more general. Note that the geodesic distance on the sphere is codified by the internal product, that is $\rho(x, y) = \arccos(\langle x, y \rangle)$. Thus we directly assume that $W$ only depends on the internal product

$$W(x, y) = p(\langle x, y \rangle)$$
For this type of kernels, the integral operator $T_W$ is a convolution operator. In this case we have a fixed basis of eigenvectors that only depends on the space and not on the particular graphon $W$. These are the well-known spherical harmonics \cite[chap. 1]{16}. For each $l \in \mathbb{N}$ we have an associated eigenspace $Y_l$, known as the space of spherical harmonics of order $l$. It is a finite dimensional vector space. Let $\{Y_{jl}\}_{j=1}^{d_l}$ be an orthonormal basis of $Y_l$ and define $d_l = \dim(Y_l)$, then by \cite[cor. 1.1.4]{16}

$$d_l = \binom{l + \gamma - 1}{l} - \binom{l + \gamma - 3}{l - 2} = O(l^{d-2})$$

for $l \geq 2$ and $d_0 = 1, d_1 = d$. The second equality is justified in the Appendix D.1. The eigenvalues $\lambda_l$, which we index without repetition, are associated with the corresponding space $Y_l$. Note that the indexation of the eigenvalues here follows the index of the spherical harmonics, and is not necessarily in decreasing order. We need to have this in mind when applying Theorem 1 or Theorem 2. In this setting, the expansion (1) becomes

$$p(\langle x, y \rangle) = \sum_{l \geq 0} \lambda_l \sum_{j=0}^{d_l} Y_{jl}(x)Y_{jl}(y)$$

(6)

On the other hand, the addition theorem for spherical harmonics \cite[eq. 1.2.8]{16} gives

$$Z_l(x, y) = \sum_{j=0}^{d_l} Y_{jl}(x)Y_{jl}(y)$$

and the preceding equality does not depend on the particular choice of basis $\{Y_{jl}\}_{j=1}^{d_l}$. The $Z_l$ are called the zonal harmonics. So based on (6) we have the following

$$p(\langle x, y \rangle) = \sum_{l \geq 0} \lambda_l Z_l(x, y)$$

(7)

A main property is that each zonal harmonic $Z_l(x, y)$ is a multiple of the Gegenbauer (ultraspherical) polynomial of level $l$, hence it only depends on the internal product of $x, y \in S^{d-1}$. More precisely, we have the following

**Proposition 6.** For any $x, y \in S^{d-1}$, $l \in \mathbb{N}$, $d \geq 3$ and $\gamma = \frac{d-2}{2}$

$$Z_l(x, y) = c_l G_l^\gamma(\langle x, y \rangle) = c_l \sqrt{d_l} \tilde{G}_l^\gamma(\langle x, y \rangle)$$

where $c_l := \frac{l+\gamma}{\Gamma(\frac{\gamma}{2})} = \frac{2l+d-2}{d-2}$, $G_l^\gamma$ is the $l$-th Gegenbauer (ultraspherical) polynomial and $\tilde{G}_l^\gamma = G_l^\gamma/\|G_l^\gamma\|_{2,\gamma}$. Furthermore, for any $l \in \mathbb{N}$, $Z_l$ attains its maximum in the diagonal, that is

$$\max_{x,y \in S^{d-1}} |Z_l(x, y)| = |Z_l(x, x)| = d_l$$

This gives a simple formula to compute the eigenvalues. Defining $b_d = \frac{\Gamma(\frac{\gamma}{2})}{\sqrt{\pi} \Gamma(\frac{d-1}{2})}$ we have

$$\lambda_l = \left(\frac{c_l b_d}{d_l}\right) \int_{-1}^{1} p(t) G_l^\gamma(t) w_\gamma(t) dt = \frac{\Gamma(\frac{\gamma}{2})}{\sqrt{\pi} \Gamma(\frac{d-1}{2})} \frac{l!}{(2d-2)^{l}} \int_{-1}^{1} p(t) G_l^\gamma(t) w_\gamma(t) dt$$

(8)
where \((a)^{(i)} = a \cdot (a + 1) \cdots (a + i - 1)\) is the rising factorial or (rising) Pochhammer symbol. What precedes means that in this framework, the growth rate of the eigenvector is known and fixed, and the fulfillment of the hypotheses depends on the eigenvalue decrease rate only, which can be verified using formula (8) above. We exhibit explicit calculations for graph models in Section 7 below.

Given Proposition 6, the hypothesis (H) translates to

\[
\sum_{k \in \mathbb{N}} |\lambda_k| d_k < \infty
\]

Because of the explicit value of \(d_k\), we get that in order to satisfy the hypothesis (H) \(|\lambda_l|\) should be at least \(O(l^{1-d})\). In this framework, the eigenvectors are fixed, by Proposition 6, which implies that \(\eta(R)\) is fixed for all kernels. More explicitly, we have

\[
\eta(R) = O(R^{d-1})
\]

thus the parameter \(s\) in Theorem 1 is fixed and equal to \(d - 1\). The hypothesis \(H_1\) and \(H_2\) became

\[
\lambda_k = O(k^{-\delta}), \eta(R) = O(R^{d-1}), \quad (H'_1)
\]

\[
\lambda_k = O(e^{-\delta k}), \eta(R) = O(R^{d-1}) \quad (H'_2)
\]

If \(W\) is a kernel in \(W_2^p([-1, 1], \omega')\) where \(\omega'(t) = (1 - t^2)^{\frac{d-3}{2}}\), then as is noted in Section 5, its eigenvalues \(\lambda_k\) satisfy

\[
\sum_k \lambda_k d_k (1 + \nu_k^{2p})
\]

where \(\nu_k = k(k + d - 1)\). If \(p > \delta - \frac{\delta}{2}\) then \(W\) satisfies \(H'_1\) and applying Theorem 1 we get

\[
|\lambda_i(T_W) - \lambda_i(T_u)| = O\left(\frac{n^{-\frac{2\delta^2}{2d+2}}}{\log n}\right)
\]

with high probability.

7. Illustrative examples

We consider examples of geometric graphons defined on the sphere of dimension \(d - 1\) for a fixed \(d\), that is we work with the space \(\Omega = S^{d-1}\) equipped with \(\sigma\) the uniform measure on the sphere and a graphon \(K : S^{d-1} \times S^{d-1} \rightarrow [0, 1]\). In addition, we will consider that \(K(x, y) = f((x, y))\) where \(f : [-1, 1] \rightarrow [0, 1]\). A common fact for those models is that the (normalized) degree function, as defined in [9, p. 116], is constant. Indeed

\[
d_K(x) = \int_{S^{d-1}} f(\langle x, y \rangle)d\sigma(y) = \int_{S^{d-1}} f(\langle x', y \rangle)d\sigma(y) = d_K(x')
\]

for all \(x, y \in S^{d-1}\). We call this constant \(d_K\). Taking the function \(g : S^{d-1} \rightarrow S^{d-1}\) with \(g(x) = 1\) for all \(x \in S^{d-1}\), we see that \(d_K\) is an eigenvalue associated with the function \(g\).
7.1. Constant and linear graphons

Since smooth graphons on the sphere can be conveniently approximated by series of Gegenbauer (ultraspherical) polynomials, we describe here what its spectrum looks like in the finite rank case.

As in Section 6, we consider \( \gamma = \frac{d-2}{2} \). We start by considering the first polynomial in the Gegenbauer basis which is \( C_0^0(t) = 1 \). A graphon \( K(1)(x, y) = p_0C_0^0((x, y)) = p_0 \), where \( p_0 \in [0, 1] \), that is a rank 1 graphon. This is the well-known Erdős-Rényi graphon. If we generate a graph with this model, following Section 2.3, with \( \{X_i\}_{1 \leq i \leq n} \) a uniform sample on the sphere, then the probability that \( X_i \) and \( X_j \) are connected for any \( i, j \in \{1, \cdots, n\} \) is \( p_0 \). That is, for any two nodes the probability that they are connected is the same, regardless of its position on the sphere. For this reason, this model is considered as structureless (see [17]). Its eigenvalues are

\[
\begin{align*}
\lambda_0^{(1)} &= p_0 \\
\lambda_l^{(1)} &= 0, \text{ for all } l > 1
\end{align*}
\]

In this case, the eigenvalue \( \lambda_0^{(1)} \) which has multiplicity one has a clear interpretation in the context of graphon theory. Indeed, if we consider the normalized degree we have

\[
d_K(x) = \int_\mathbb{S}^d p_0 d\sigma(y) = p_0
\]

thus the non-zero eigenvalue \( \lambda_0^{(1)} \) is just the mean degree of the graphon, which is constant on the sphere.

We now consider the rank \( 1 + d_1 = 1 + d \) graphon \( K(2), (x, y) = p_0C_0^0((x, y)) + p_1C_1^0((x, y)) = p_0 + p_12\gamma t \), where the \( d_1 \) is the first spherical harmonic space dimension given in (5). This graphon is based on the first two Gegenbauer (ultraspherical) polynomials \( C_0^0(t) = 1 \) and \( C_0^0(t) = 2\gamma t \) and we call it linear graphon. The eigenvalues for this model are given by

\[
\begin{align*}
\lambda_0^{(2)} &= p_0 \\
\lambda_1^{(2)} &= p_1 \frac{d-2}{d} \\
\lambda_l^{(2)} &= 0, \text{ for all } l \geq 2
\end{align*}
\]

The eigenvalue \( \lambda_0^{(2)} \) has multiplicity one and the eigenvalue \( \lambda_1^{(2)} \) has multiplicity \( d \). At first glance \( \lambda_1^{(2)} \) is \( O(d) \), but since by definition a graphon takes values \( 0 \leq K(t) \leq 1 \), the values \( p_0 \) and \( p_1 \) must satisfy certain constraints. In this particular case, we see that \( p_0 \in [0, 1] \) and \( p_0 \pm p_1 2\gamma \geq 0 \) so \( |p_1| \leq \frac{p_0}{2\gamma} \). That implies that \( \lambda_1^{(2)} \) is decreasing on \( d \). More specifically, since \( \gamma = \frac{d-2}{2} \) we have \( \lambda_1^{(2)} = O\left(\frac{1}{d}\right) \). An application of Theorem 2 gives us, for \( \alpha \in (0, 1) \)

\[
\begin{align*}
|\lambda_0(T_n^{(2)}) - \lambda_0^{(2)}| &\leq \frac{d^2p_0}{n} \log d/\alpha + \frac{2p_0}{\sqrt{n}} \sqrt{2(d^2 - 1)} \log d/\alpha \\
|\lambda_1(T_n^{(2)}) - \lambda_1^{(2)}| &\leq \frac{d^2\lambda_1^{(2)}}{n} \log d/\alpha + \frac{2\lambda_1^{(2)}}{\sqrt{n}} \sqrt{2(d^2 - 1)} \log d/\alpha
\end{align*}
\]

with probability bigger than \( 1 - \alpha \).

7.2. Geometric threshold graphon

We consider the kernel on the sphere \( K^{(\gamma)}(x, y) = 1_{(x,y) \geq 0} \), which generates the geometric random graph model with threshold parameter \( \tau = 0 \). If we generate a random graph by
the model described in Section 2.3, with \( \{X_i\}_{i \in \mathbb{N}} \) a uniform sample on the sphere, then the corresponding nodes \( X_i \) and \( X_j \) will be connected if and only if they belong to the same semi-sphere. Applying (8) we get
\[
\lambda_l^{(g)} = a_{l,d} \int_0^1 G_t^m(t)^2 w(t) dt = a_t b_{l,d} \int_0^1 \frac{d^l}{dt^l} w_{\gamma_i}(t) dt
\]
where \( a_{l,d} = c_{l,d}^\alpha \) and \( b_{l,d} = \frac{(-1)^l (2d-2)^{(l)}}{2^l \Gamma\left(\frac{d}{2}+1\right)} \). In the second equality we have used the Rodrigues formula for Gegenbauer polynomials. The eigenvalues for this model are given by
\[
\begin{cases}
\lambda_0^{(g)} = \frac{1}{2} \\
\lambda_l^{(g)} = 0, \text{ for } l > 0 \text{ even} \\
\lambda_l^{(g)} = 2(-1)^{\frac{d+1}{2}} \frac{\pi}{\beta(d, \frac{d}{2}+1)}, \text{ for } l \text{ odd}
\end{cases}
\]
where \( \beta(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)} \) is the Beta function. More details can be found in Appendix E. Since this function is neither regular nor finite rank we cannot apply directly Theorems 1 and 2. Indeed, from the expression for eigenvalues \( \lambda_l^{(g)} \) we deduce that for \( d \) fixed, asymptotically as \( l \) tends to infinity
\[
|\lambda_l^{(g)}| \sim \Gamma\left(\frac{d}{2}\right) \cdot \left(\frac{l}{2}\right)^{-\frac{d}{2}}
\]
using the Stirling asymptotic approximation of the Beta function.

Clearly the eigenvalues do not fulfill hypothesis (H). Nevertheless, we can apply the results to the \( m \)-fold composition of the operator \( T_m^{\alpha,K^{(g)}} \) with \( m \in \mathbb{N} \), which is an integral operator with kernel:
\[
K_{om}^{(g)}(x, y) = \int_{(S^{d-1})^{m-1}} K^{(g)}(x, z_1) \cdot K^{(g)}(z_1, z_2) \cdots K^{(g)}(z_{m-1}, y) \, d\sigma(z_1) \cdots d\sigma(z_{m-1})
\]
where \((S^{d-1})^{m-1}\) is the \( m - 1 \) product space of the \( d \)-dimensional unit sphere.

The following \( L^2(S^{d-1} \times S^{d-1}) \) holds
\[
K_{om}^{(g)}(x, y) = \sum_{k=0}^{\infty} \lambda_k^m \phi_k(x) \phi_k(y)
\]
Taking \( m \geq 2 \) and using the previous estimation, we have that
\[
|\lambda_l^{(g)}|^m \sim \left(\frac{l}{2}\right)^{-\frac{dm}{2}}
\]
Thus, we can use Theorem 1 for this kernel. In the case \( m = 2 \), the kernel matrix is
\[
(T_n^{o,2})_{ij} := \frac{1}{n} \int_{S^{d-1}} K^{(g)}(X_i, z) K^{(g)}(z, X_j) \, d\sigma(z)
\]
and we have \( \lambda_1^{(g)} = O(l^{-d}) \) and \( \eta(R) = O(R^{-d+1}) \). Then we use Theorem 1 with \( \delta = d \) and \( s = d - 1 \) which proves that there exists \( w_{\alpha,K^{(g)}}(n) \) increasing in terms of \( n \) such that for \( 1 \leq i \leq w_{\alpha,K^{(g)}}(n) \) and \( \alpha \in (0, 1) \)
\[
|\lambda_i^{(g)}|^2 - \lambda_i(T_n^{o,2})^2 | \leq C(\alpha)/i^{-\frac{2d}{5d-4}} \left(\frac{n}{\log n}\right)^{-\frac{d}{5d-4}}
\]
with probability higher than \( 1 - \alpha \).
7.3. Logistic graphon

We consider the graphon $K^{(g)}(x, y) = f(\langle x, y \rangle)$ on the sphere, where $f(t) := \frac{e^{rt}}{1 + e^{rt}} = \frac{1}{1 + e^{-rt}}$. This function is symmetric with respect to $\frac{1}{2}$, which means that $f(t) - \frac{1}{2}$ is even. By (8) its eigenvalues are

$$\lambda_l^{(g)} = a_{l,d} \int_{-1}^{1} f(t) G_l^\gamma(t) w_\gamma(t) dt$$

where $a_{l,d} = \frac{\Gamma\left(\frac{d}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{d-1}{2}\right)} \frac{n}{(2d-2)!}$. We have

$$\lambda_l^{(g)} = a_{l,d} \int_{-1}^{1} (f(t) - \frac{1}{2}) G_l^\gamma(t) w_\gamma(t) dt + \frac{a_{l,d}}{2} \int_{-1}^{1} G_l^\gamma(t) w_\gamma(t) dt \tag{9}$$

Gegenbauer polynomials $G_l^\gamma(t)$ are even for $l$ even and odd for $l$ odd. For $l$ even the first term on the right hand side of (9) vanishes, because we have the symmetric integral of a product of even functions which is itself even. Note that by definition the weight function $w_\gamma(t)$ is also even. Then, for $l$ even we have

$$\lambda_l^{(g)} = \frac{a_{l,d}}{2} \int_{-1}^{1} G_l^\gamma(t) w_\gamma(t) dt \begin{cases} \frac{1}{2} & \text{if } l = 0 \\ 0 & \text{if } l = 2k, \text{ for } k \geq 1, k \in \mathbb{N} \end{cases}$$

where we make use of the orthogonality properties of Gegenbauer polynomials. For $l$ odd, the second term of the right hand side of (9) also vanishes, again by orthogonality of Gegenbauer polynomials, but we can not use the same argument for the first term given the parity of the function within the integral. For $l$ odd we have

$$\lambda_l^{(g)} = a_{l,d} b_{l,d} \int_{0}^{1} \frac{1 - e^{-rt}}{1 + e^{-rt}} \frac{d^l}{dt^l} w_{r+l}(t) dt \tag{10}$$

where $b_{l,d} = \frac{(-1)^l (2d-2)!}{2^d d! (d-1)^{tl}}$. From the expression (10) we see the eigenvalue relation

$$\lambda_l^{(1)} \leq \lambda_l^{(g)} \leq \lambda_l^{(g)}$$

for any value of the parameter $r$. In addition, if $r = 0$ we have $\lambda_l^{(1)} = \lambda_l^{(g)}$ and if $r \to \infty$ we have $\lambda_l^{(g)} = \lambda_l^{(g)}$.

In fact, we have the following formula, for $\gamma \in \mathbb{N}$

$$\lambda_l^{(g)} = 2 \sum_{k=0}^{\infty} \sum_{i=0}^{\gamma+[l/2]} (-1)^k a_{l,d} b_{l,d} g_{l,t} \gamma(2i + 1, kr) + \lambda_l^{(g)}$$

where $g_{l,t} := \binom{\gamma+1}{l+1} (2i+l)$. Here $(a)_l = a \cdot (a - 1) \cdots (a - l + 1)$ is the falling factorial symbol. A similar expression can be obtained in the case $\gamma \notin \mathbb{N}$. The details are in the Appendix E.
7.4. Gaussian kernel

One of the most studied kernels is the Gaussian kernel. Here we consider the space \( \Omega = \mathbb{R} \) with the measure \( \mu \) with density with respect to the Lebesgue measure \( d\mu(x) = \frac{1}{\sqrt{\pi}} e^{-x^2} dx \) and \( K(x, y) = e^{-\frac{1}{2}(x-y)^2} \). Its eigenvalues and normalized eigenfunctions are given in [18, sec. 6.2], which in this example gives for \( k = 0, 1, \ldots \)

\[
\lambda_k = \frac{2^{-2k}}{(\frac{1}{2}(1 + \sqrt{2}) + \frac{1}{4})^{k + \frac{1}{2}}} \leq \frac{2}{5^{k + \frac{1}{2}}}
\]

\[
\phi_k(x) = \frac{\sqrt{2}}{\sqrt{2k}!} \exp \left( -\left( \sqrt{2} - 1 \right) \frac{x^2}{2} \right) H_k(\sqrt{2}x)
\]

where \( H_k(\cdot) \) is the \( k \)-th order Hermite polynomial (see [19, Ch. 5]). We note that the eigenvalues have an exponential decreasing rate. On the other hand, using [20, eq.22.14.17] we have for all \( x \)

\[
\exp \left( -\frac{x^2}{\sqrt{2}} \right) H_k(\sqrt{2}x) \leq \kappa \sqrt{2k}!
\]

where \( \kappa = 1.086435 \). Combining this with the bound \( x^* \leq \sqrt{2k} \) for \( x^* = \arg \max_{x \in \mathbb{R}} e^{-\frac{x^2}{2}} H_k(x) \), given in [21], we obtain

\[
\phi_k(x) \leq \phi_k(x^*) \leq \frac{\sqrt{2} \kappa}{\sqrt{2k}!} \kappa
\]

This model falls into the \( H_3 \) hypothesis and then Theorem 1 can be applied with \( \delta = \log 5 \) and \( s = 1 \). If \( T_n \) is the normalized kernel matrix, we have that there exists \( w''_{\alpha,K}(n) \) which grows with \( n \), such that for \( 1 \leq i \leq w''_{\alpha,K}(n) \) and with \( \alpha \in (0, 1) \)

\[
|\lambda_i(T_K) - \lambda_i(T_n)| \leq C''(\alpha) e^{-\frac{2(\log 5)^2}{2\log 5 + 1}} \left( \frac{n}{\log n} \right)^{-\frac{\log 5}{2\log 5 + 1}}
\]

with probability larger than \( 1 - \alpha \).

7.5. Smale-Zhou counterexample

In [11] the authors, based on an original idea of Smale, exhibited a continuous positive definite kernel in \( \Omega = [0, 1] \) equipped with the Lebesgue measure, with \( C^\infty \) eigenfunctions which are not uniformly bounded. This construction has been done using a wavelet-type basis for \( L^2[0, 1] \), which \( \| \cdot \|_\infty \) norm is unbounded. It has been built in such a way that the norm \( \| \phi_k \|_\infty \) grows exponentially and that the eigenvalues decrease also exponentially. So this example falls in the \( H_3 \) hypothesis. Nevertheless the kernel has been constructed by hand resulting into a convergent series.

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Appendix A: Proof of Theorem 2

Given a $L^2(\Omega, \mu)$ Hilbertian basis $\{\phi_k\}_{k=0}^{\infty}$ we will consider the rank $R$ graphon

$$W_R(x, y) = \sum_{k=0}^{R-1} \lambda_k \phi_k(x) \phi_k(y)$$

Based on the sample $\{X_i\}_{i=1}^n$ we define the kernel matrix

$$(T_n)_{ij} = \frac{1}{n} W_R(X_i, X_j)$$

If we define $\phi_k \in \mathbb{R}^n$ by $(\phi_k)_i = \frac{1}{\sqrt{n}} \phi_k(X_i)$, we get

$$T_n = \sum_{k=0}^{R-1} \lambda_k \phi_k \phi_k^T$$

Define $\Phi_{n,R}$ the $n \times R$ matrix where the $k$-th column is $\phi_k$, then we get

$$T_n = \Phi_{n,R} \Lambda_R \Phi_{n,R}^T$$

where $\Lambda_R$ is the $R \times R$ diagonal matrix

$$\Lambda_R = \begin{bmatrix} 
\lambda_0 & 0 \\
0 & \ddots \\
0 & 0 & \lambda_R 
\end{bmatrix}$$

Note that $T_n$ is a multiplicative perturbation of $\Lambda_R$. By Ostrowsky theorem [22, thm. 4.5.9],

$$|\lambda_i(T_n) - \lambda_i| \leq |\lambda_i| \|\Phi_{n,R}^T \Phi_{n,R} - I_R\|_{op}$$

In order to obtain the tail concentration bounds for $|\lambda_i(T_n) - \lambda_i|$ it is enough to do it for $\|\Phi_{n,R}^T \Phi_{n,R} - I_R\|_{op}$. Indeed,

$$\mathbb{P}(|\lambda_i(T_n) - \lambda_i| \geq t) \leq \mathbb{P}(|\lambda_i| \|\Phi_{n,R}^T \Phi_{n,R} - I_R\|_{op} \geq t)$$

We note that

$$\Phi_{n,R}^T \Phi_{n,R} = \sum_{j=1}^{n} Z_j Z_j^T$$

where $Z_j \in \mathbb{R}^R$ is given by

$$(Z_j)_k = \frac{1}{\sqrt{n}} \phi_k(X_j)$$

To obtain a tail bound we use matrix concentration results for sum of symmetric random matrices. More specifically, we will use a matrix Bernstein theorem, for the sum

$$\Phi_{n,R}^T \Phi_{n,R} - I_R = \sum_{j=1}^{n} (Z_j Z_j^T - \frac{1}{n} I_R)$$
It is a direct consequence of the definition that $\mathbb{E}[Z_j Z_j^T] = \frac{1}{n} I_R$. We note that

$$
\|Z_j Z_j^T - I_R\|_{op} \leq |\lambda_{\text{max}}(Z_j Z_j^T) - \frac{1}{n}| \vee |\lambda_{\text{min}}(Z_j Z_j^T) - \frac{1}{n}|
$$

As $Z_j Z_j^T$ is positive definite, we have $\lambda_{\text{max}}(Z_j Z_j^T) \geq 0$ and $\lambda_{\text{min}}(Z_j Z_j^T) \geq 0$. Also

$$
\lambda_{\text{min}}(Z_j Z_j^T) \leq \lambda_{\text{max}}(Z_j Z_j^T) = \|Z_j\|_2 = \sum_{k=0}^{R-1} \frac{1}{n} \phi_k^2(X_j)
$$

Given that $\sum_{k=0}^{R-1} \phi_k^2(X_j) \leq K$, where $K$ only depends on the eigenbasis and the degree $R$, we have

$$
\|Z_j Z_j^T - I_R\|_{op} \leq \frac{K - 1}{n}
$$

On the other hand

$$
\mathbb{E}[(Z_j Z_j^T - \frac{1}{n} I_R)^2] = \mathbb{E}[\|Z_j\| Z_j Z_j^T - \frac{1}{n^2} I_R]
$$

$$
\leq \frac{K}{n} \mathbb{E}[Z_j Z_j^T] - \frac{1}{n} I_R
$$

$$
\leq \frac{K^2 - 1}{n^2}
$$

then

$$
\sum_{k=1}^{n} \mathbb{E}[(Z_j Z_j^T - \frac{1}{n} I_R)^2] \leq \frac{K^2 - 1}{n}
$$

Using the matrix Bernstein theorem with $S_j = Z_j Z_j^T - \frac{1}{n} I_R$, $L = \frac{K - 1}{n}$ and $\sigma^2 = \frac{K^2 - 1}{n}$, we get

$$
P(\|\Phi_{n,R}^T \Phi_{n,R} - I_R\|_{op} \geq t) \leq d \exp \frac{-t^2}{2(\sigma^2 + \frac{L t}{3})}
$$

$$
\leq d \exp \frac{-nt^2}{(K - 1)(2(K + 1) + \frac{2t}{3})}
$$

From (12) we deduce

$$
P(|\lambda_i(T_n) - \lambda_i| \geq t) \leq d \exp \frac{-nt^2}{(K - 1)(2(K + 1)\lambda_i^2 + \frac{2\lambda_i t}{3})} \quad (13)
$$

Remark 3 (Comparison with non-scaling bound). It is possible to use Weyl inequalities

$$
|\lambda_i(A + B) - \lambda_i(A)| \leq \|B\|_{op}
$$

instead of Ostrowsky theorem in (11), although with suboptimal results. Indeed, using this we will get

$$
P(|\lambda_i(T_n) - \lambda_i| \geq t) \leq d \exp \frac{-nt^2}{(K - 1)(2(K + 1)\lambda_i^2 + \frac{2\lambda_i t}{3})} \quad (14)$$
Appendix B: Proof of Theorem 1

Here we gather some technical results that lead to the proof of Theorem 1. In Appendix B.1 we prove Lemma 7 which is a more explicit version of Proposition 3. The concentration inequalities proven in that section do not make use of hypothesis $H_1, H_2$ or $H_3$ and are therefore more general. Then in Appendix B.2 we use those concentration inequalities to prove Theorem 1. In Appendix C we detail the optimization of parameters. We recall that for $R \geq 1$, the finite rank version of $W$ called $W_R$ which is such as

$$W_R(x, y) = \sum_{k=1}^{R} \lambda_k \phi_k(x) \phi_k(y)$$

We will introduce a condition that force the eigenvalues to decrease faster than the possible growth of the eigenvectors (in the norm $\| \cdot \|_{\infty}$). The latter will imply in particular the uniform convergence in the expansion (1). This restricted hypothesis is the price to pay to extend the approach to non-positive, and possibly non-continuous, kernels. We will instead assume that

$$\sum_{k \geq 0} \| \lambda_k \phi_k^2 \|_{\infty} \leq K < \infty$$

This implies the convergence in (1) and the following decomposition of the kernel $W$

$$\sum_{k \in I^+} \lambda_k \phi_k \otimes \phi_k - \sum_{k \in I^-} |\lambda_k| \phi_k \otimes \phi_k$$

where $I^+$ (resp. $I^-$) is the set of indices such as $\lambda_k$ is positive (resp. negative). We will denote $W_+ = \sum_{k \in I^+} \lambda_k \phi_k \otimes \phi_k$ and $W_- = \sum_{k \in I^-} |\lambda_k| \phi_k \otimes \phi_k$. The imposed condition (H) gives the bound $\max \{\|W_+\|_{\infty}, \|W_-\|_{\infty}\} \leq K$.

**Remark 4.** The hypothesis (H) is stronger than the bounded kernel assumption, considered in [5] for positive and continuous kernels. The bounded kernel hypothesis implies that $\| \sum_k \lambda_k \phi_k^2 \|_{\infty} < \infty$ and in the case on non-positive kernels we can not infer (H) from the previous.

We will note $\nu_R = \sum_{k \geq R} |\lambda_k|$ and $\nu_{R,N} \sum_{k=R}^{N} |\lambda_k|$, and we define the functions

$$\rho_R := \min \{K, \sup_{R \leq k \leq N} \| \phi_k \|_{\infty} \nu_{R,N} \}$$

and

$$\sigma_R^2 := \lambda_R \rho_R$$

that will help us to simplify notation in the theorems.

Let $\Phi_{n,R}$ is the $n \times R$ matrix where its $k-th$ column is

$$(\phi_k(X_1), \phi_k(X_2), \ldots, \phi_k(X_n))^T$$

and

$$T_{n,R} := \Phi_{n,R} \text{Diag}((\lambda_1, \lambda_2, \ldots, \lambda_R)) \Phi_{n,R}^T$$

Here $\text{Diag}((\lambda_1, \lambda_2, \ldots, \lambda_R))$ is the diagonal matrix with $\lambda_1, \ldots, \lambda_R$ in the diagonal. We also define $E_{n,R} := \Phi_{n,R} \Phi_{n,R}^T - I_n$. 




We recall that in Proposition 4 we had the bound

\[ |\lambda_i(T_n) - \lambda_i(T_W)| \leq \lambda_i\|E_{R,n}\| + \|T_n - T_{n,R}\| \quad \text{for } i \leq R \]

We need now to study \(\|E_{n,R}\|\) and \(\|T_{n,R} - T_n\|\) to obtain explicit bounds for each eigenvalue. Here we follow a different approach to one in [5]. We bound \(\|E_{n,R}\|\) and \(\|T_{n,R} - T_n\|\) with the help of the concentration results for matrices, developed in [6]. First we introduce the number \(N > R\) (which should be later estimated) such that

\[ T_n - T_{n,R} = \frac{1}{n} \left( \sum_{k=R+1}^{N} \lambda_k \phi_k \phi_k^T + \sum_{k=N+1}^{\infty} \lambda_k \phi_k \phi_k^T \right) \]

### B.1. Proof of concentration inequalities

Applying the aforementioned concentration results bounds we obtain the following

**Lemma 7.** Let \(W\) be an infinite rank symmetric kernel in \(L^2(\Omega \times \Omega, \mu \times \mu)\) with eigenfunction basis \(\{\phi_k\}_{k \geq 0}\) and eigenvalues \(\{\lambda_k\}_{k \geq 0}\). We assume that \(W\) satisfies (II) and that \(\eta(R) \log R/\alpha \leq n\). Then with probability bigger than \(1 - \alpha\) that

\[
|\lambda_i(T_W) - \lambda_i(T_n)| \leq \min_{i \leq R < N} \left\{ \lambda_i \sqrt{\frac{3\eta(R) \log(2R/\alpha)}{n}} + 2\lambda_R + \nu_N + \frac{4}{3n} \rho_R \log(2N/\alpha) + 2\sqrt{\frac{2\sigma_R^2 \log(2N/\alpha)}{n}} \right\}
\]

and

\[
|\lambda_i(T_W) - \lambda_i(T_n)| \leq \min_{i > R} \left\{ 3\lambda_R + \nu_N + \frac{4}{3n} \rho_R \log(2N/\alpha) + 2\sqrt{\frac{2\sigma_R^2 \log(2N/\alpha)}{n}} \right\}
\]

**Remark 5.** If we assume that, in addition,

\[ \frac{\rho_R}{\lambda_R} \log \frac{2N}{\alpha} < n \]

the result of Lemma 7 simplifies to

\[
|\lambda_i(T_W) - \lambda_i(T_n)| \leq \min_{i \leq R < N} \left\{ \lambda_i \sqrt{\frac{3\eta(R) \log(2R/\alpha)}{n}} + 2\lambda_R + \nu_N + 3\sqrt{\frac{2\sigma_R^2 \log(2N/\alpha)}{n}} \right\}
\]

Here we observe the typical interaction between a bias term that decrease with \(R\) and the variance term that increase with \(R\). Lemma 7 will follow from Proposition 4 and Lemmas 8 and 9 bellow. To bound the quantity \(\|E_{n,R}\|\) we will resort on the following lemma (see [3, lemma.12]) that gives the required control

**Lemma 8.** With the notation in Lemma 7 we have, for \(\alpha \in (0, 1)\) and \(n \geq \eta(R) \log(2R/\alpha)\)

\[ \|E_{n,R}\| \leq \sqrt{\frac{3\eta(R) \log(2R/\alpha)}{n}} \]

with probability bigger than \(1 - \alpha\)
To bound $\|T_n - T_{n,R}\|_{op}$, we can use the concentration of measure theorems for sums of rank 1 matrices developed in [6].

**Lemma 9.** Let $W$ be an infinite rank symmetric kernel in $L^2(\Omega \times \Omega, \mu \times \mu)$ with eigenfunction basis $\{\phi_k\}_{k \geq 0}$ and eigenvalues $\{\lambda_k\}_{k \geq 0}$. Assuming (H) and given $N > R$, we have with probability bigger than $1 - \alpha$

$$\|T_n - T_{n,R}\|_{op} \leq \nu_N + 2\lambda_R + \frac{4}{3n}\rho_R \log \left(\frac{(N - R + 1)}{\alpha}\right) + 2\sqrt{\frac{2\sigma_R^2 \log ((N - R + 1)/\alpha)}{n}} \tag{16}$$

**Proof.** We know that

$$T_n - T_{n,R} = \frac{1}{n} \sum_{k \geq R} \lambda_k \phi_k \phi_k^T$$

where $\phi_k$ is a vector of dimension $n$ such as $(\phi_k)_i = \phi_k(X_i)$. Given a number $N$ we have the following decomposition

$$T_n - T_{n,R} = \frac{1}{n} \left( \sum_{k = R+1}^N \lambda_k \phi_k \phi_k^T + \sum_{k = N+1}^\infty \lambda_k \phi_k \phi_k^T \right)$$

$$= \frac{1}{n} \left( \sum_{k \in I_{N,R}^+} \lambda_k \phi_k \phi_k^T - \sum_{k \in I_{N,R}^-} |\lambda_k| \phi_k \phi_k^T + \sum_{k = N+1}^\infty \lambda_k \phi_k \phi_k^T \right)$$

where $k \in I_{N,R}^+ \ (I_{N,R}^- \text{ respectively})$ if and only if $\lambda_k \geq 0 \ (\lambda_k < 0 \text{ respectively})$ and $R+1 \leq k \leq N$. The idea is to obtain a bound with high probability for each of these three terms, and then by the triangular inequality the bound for $\|T_n - T_{n,R}\|_{op}$ will follow.

Let us focus on $\frac{1}{n} \sum_{k \in I_{N,R}^+} \lambda_k \phi_k \phi_k^T$ first, as the term involving negative eigenvalues is similar.

We can define the $n \times I_{N,R}^+$ matrix $A$ such as the its columns are the vectors $\sqrt{\frac{n}{\lambda_k}} \phi_k$ with $k \in I_{N,R}^+$.

So we have $\frac{1}{n} \sum_{k \in I_{N,R}^+} \lambda_k \phi_k \phi_k^T = AA^T$. We will use the fact that $\|AA^T\|_{op} = \|A^TA\|_{op}$ and the following decomposition

$$A^TA = \frac{1}{n} \sum_{i=1}^n Z_i$$

where $Z_i$ is a $I_{N,R}^+ \times I_{N,R}^+$ positive semidefinite matrix such that $(Z_i)_{kl} = \sqrt{\lambda_k \lambda_l} \phi_k(X_i) \phi_l(X_i)$.

Note that the $Z_i$ are of the form $Z_i = u_i u_i^T$ with

$$u_i = (\sqrt{\lambda_{k_1}} \phi_{k_1}(X_i), \sqrt{\lambda_{k_2}} \phi_{k_2}(X_i), \ldots, \sqrt{\lambda_{k_{|I_{N,R}^+|}}} \phi_{k_{|I_{N,R}^+|}}(X_i))^T$$

hence $Z_i$ is positive semidefinite. Here $k_1, \ldots, k_{|I_{N,R}^+|}$ are the indices of the eigenvalues in $I_{N,R}^+$.

We compute the mean of this matrix

$$\mathbb{E}((Z_i)_{kl}) = \mathbb{E}(\sqrt{\lambda_k \lambda_l} \phi_k(X_i) \phi_l(X_i)) = \delta_{kl} \lambda_k$$
where the second equality follows from the orthonormality of the eigenfunctions $\phi_k$. So the mean of $Z_i$ is a diagonal matrix

$$
\mathbb{E}Z_i = \begin{bmatrix}
\lambda_{R+1} & 0 \\
0 & \ddots \\
0 & 0 & \lambda_N
\end{bmatrix}
$$

We define $\tilde{Z}_i := Z_i - \mathbb{E}Z_i$, which satisfy

$$
\|\tilde{Z}_i\|_{op} \leq \max \{ |\lambda_{\max}(Z_i) - \lambda_{\min}(\mathbb{E}(Z_i))|, |\lambda_{\min}(Z_i) - \lambda_{\max}(\mathbb{E}(Z_i))| \}
$$

where what precedes is given by the dual Weyl inequalities. As $Z_i$ is positive definite, we have $\|Z_i\|_{op} = \lambda_{\max}(Z_i)$ and, by inspection of the explicit form of $\mathbb{E}(Z_i)$, we have $\lambda_{\min}(\mathbb{E}(Z_i)) = \lambda_N$ and $\lambda_{\max}(\mathbb{E}(Z_i)) = \lambda_{R+1}$. So we have

$$
\|\tilde{Z}_i\|_{op} \leq \max \{ \|Z_i\|_{op} - \lambda_N, |\lambda_{\min}(Z_i) - \lambda_{R+1}| \}
$$

in the last equality we use the positive semidefiniteness of $Z_i$. On the other hand, we note that

$$
\|Z_i\|_{op} = \|u_i u_i^T\|_{op} = \|u_i\|_2^2
$$

and

$$
\|u_i\|_2^2 = \sum_{k \in I_{N,R}^+} \lambda_k \phi_k(X_i)^2 \leq \sup_{X_i, R \leq k \leq N} \phi_k^2(X_i) \sum_{k \in I_{N,R}^+} \lambda_k
$$

Note also that, by hypothesis, $\sum_{k \in I_{N,R}^+} \lambda_k \phi_k(X_i)^2 \leq K$. Using these bounds in (17) we obtain

$$
\|\tilde{Z}_i\|_{op} \leq \sup_{X_i, R \leq k \leq N} \phi_k^2(X_i) \sum_{k \in I_{N,R}^+} \lambda_k \lor \lambda_{R+1}
$$

and

$$
\|\tilde{Z}_i\|_{op} \leq K \lor \lambda_{R+1}
$$

On the right hand side of the first inequality, the first term dominates. Indeed, we note that, for all $j$

$$
1 = \int_{\Omega} \phi_j^2(x) d\mu(x) \leq \|\phi_j\|_\infty \mu(\Omega) = \|\phi_j\|_\infty
$$

We used that $\mu(\Omega) = 1$, as $\mu$ is a probability measure. Then we have

$$
\sup_{X_i, R \leq k \leq N} \phi_k^2(X_i) \sum_{k \in I_{N,R}^+} \lambda_k \geq \lambda_{R+1}
$$

So, defining $\rho^+_R = \min \{ K \lor \lambda_{R+1}, \sup_{X_i, R \leq k \leq N} \phi_k^2(X_i) \sum_{k \in I_{N,R}^+} \lambda_k \}$ we have $\|\tilde{Z}_i\|_{op} \leq \rho^+_R$. 

For the variance, we have
\[
\|\text{Var}(\tilde{Z}_i)\|_\infty = \|\mathbb{E}((Z_i - \mathbb{E}Z_i)^2)\|_\infty \\
= \|\mathbb{E}(Z_i^2 - 2X_i\mathbb{E}Z_i + (\mathbb{E}Z_i)^2)\|_\infty \\
= \|\mathbb{E}(Z_i^2) - (\mathbb{E}Z_i)^2\|_\infty \\
= \|\mathbb{E}(\|u_i\|_2^2 Z_i) - (\mathbb{E}Z_i)^2\|_\infty \\
\leq \|\mathbb{E}(\|u_i\|_2^2 Z_i)\|_\infty - \lambda_N^2
\]

Here we apply the dual Weyl inequalities and the fact that \(\mathbb{E}(Z_i^2)\) and \(\mathbb{E}(Z_i)^2\) are positive semidefinite, and the fact that \(\mathbb{E}(Z_i^2) \neq \mathbb{E}(Z_i)^2\). Using the bound for \(\|u_i\|_2^2\) above, we deduce
\[
\|\text{Var}(\tilde{Z}_i)\|_\infty \leq \lambda + \sup_{X_i,R \leq k \leq N} \sigma_k^2(X_i) \sum_{k \in \mathcal{I}_{N,R}} \lambda_k
\]

Defining \(\sigma_R^2 := \lambda + \sup_{X_i,R \leq k \leq N} \sigma_k^2(X_i) \sum_{k \in \mathcal{I}_{N,R}} \lambda_k\), we have for the operator norm \(\|\text{Var}(\tilde{Z}_i)\|_\infty \leq \sigma_R^2\), thus \(\|\text{Var}(\sum_{i=1}^n \tilde{Z}_i)\|_\infty \leq n \sigma_R^2\). Applying the matrix Bernstein theorem (see [6, thm. 6.1]) we obtain the tail bound
\[
P(\|\sum_{i=1}^n \tilde{Z}_i\|_\infty \geq t') \leq |\mathcal{I}_{N,R}^+| \exp \left( \frac{-t'^2/2}{n \sigma_R^2 + \rho_R^+ t'/3} \right)
\]
and the bound for the expectation
\[
\mathbb{E}(\|\sum_{i=1}^n \tilde{Z}_i\|_\infty) \leq \sqrt{2\sigma_R^2 \log 2 |\mathcal{I}_{N,R}^+|} + \frac{\rho_R^+}{3} \log 2 |\mathcal{I}_{N,R}^+|
\]
or
\[
\mathbb{E}(\|\sum_{i=1}^n Z_i\|_\infty - n \lambda_{R+1}) \leq \sqrt{2\sigma_R^2 \log 2 |\mathcal{I}_{N,R}^+|} + \frac{\rho_R^+}{3} \log 2 |\mathcal{I}_{N,R}^+| \tag{18}
\]
Using \(t' = tn\) we get
\[
P(\|\sum_{i=1}^n \tilde{Z}_i\|_\infty \geq t) \leq |\mathcal{I}_{N,R}^+| \exp \left( \frac{-t^2n}{2(\sigma_R^2 + \rho_R^+ t/3)} \right)
\]
For \(\alpha \in (0, 1)\) we have
\[
\left|\|\frac{1}{n} \sum_{i=1}^n X_i\|_\infty - \lambda_R\right| \leq \frac{2}{3n} \rho_R \log \left( I_{R,N}^+ / \alpha \right) + \sqrt{\frac{2\sigma_R^2 \log (I_{N,R}^+ / \alpha)}{n}}
\]
with probability higher than \(1 - \alpha\). We can carry out the same analysis for the matrix \(\|\sum_{k \in \mathcal{I}_{N,R}} |\lambda_k| \phi_k^T \phi_k^T\) obtaining
\[
\|T_n - T_{n,R}\|_\infty \leq \sum_{k > N} |\lambda_k| + 2 \left( \lambda_R + \frac{2}{3n} \rho_R \log (I_{R,N}^+ / \alpha) + \sqrt{\frac{2\sigma_R^2 \log (I_{N,R}^+ / \alpha)}{n}} \right)
\]
where \( \sigma^2_R := \max \{ \sigma^+_R, \sigma^-_R \} \) and \( \hat{I}_{N,R} := \max \{ I^+_R, I^-_R \} \leq N - R \)

\[
\| T_n - T_{n,R} \|_{\text{op}} \leq \sum_{k > N} |\lambda_k| + 2 \left( \lambda_R + \frac{2}{3n} \rho_R \log \left( \frac{(N - R)}{\alpha} \right) \right) + \sqrt{\frac{2\sigma^2_R \log \left( \frac{(N - R)}{\alpha} \right)}{n}} \tag{19}
\]

**Remark 6.** Using the matrix Chernoff bound [6, thm. 5.1.1], instead of matrix Bernstein as we already did before, we obtain the following bound for the expectation

\[
\mathbb{E}(\| \sum_{i=1}^{n} Z_i \|_{\text{op}}) \leq \frac{e^{\theta} - 1}{\theta} \mu_{\max} + \frac{1}{\theta} \rho_R^+ \log |I^+_{N,R}|
\]

(20)

here \( \mu_{\max} = \| \sum_{i=1}^{n} \mathbb{E}(Z_i) \| \), which in this case is equal to \( \sum_{i=1}^{n} \| \mathbb{E}(Z_i) \|_{\text{op}} = n\lambda_{R+1} \), because as stated in the proof \( \mathbb{E}(X_k) \) is diagonal and positive. This bound should be compared with (18).

For instance, if \( \theta = 1 \) we have

\[
\mathbb{E}(\| \sum_{i=1}^{n} Z_i \|) \leq 1.72 \times n\lambda_{R+1} + \rho_R^+ \log |I^+_{N,R}|
\]

**B.2. Inserting hypotheses \( H_1, H_2 \) and \( H_3 \) and concluding the proof**

We prove Theorem 1 assuming \( H_1 \) and \( H_3 \), defined in Table 1. The proof for \( H_2 \) follows from the other two easily. We first prove a lemma that leads to more precise estimations of the approximation error in Lemma 7. More precisely, we have the following

**Lemma 10.** Given an infinite rank \( L^2(\Omega^2) \) kernel \( W \), we assume \( H_1 \) holds and, in addition, that \( \frac{\rho_R}{\lambda_R} \log \frac{2N}{\alpha} < n \). Then, given \( N > R \) we have with probability bigger than \( 1 - \alpha \), for \( i < R \)

\[
|\lambda_i(T_W) - \lambda_i(T_n)| \leq C \min_{R,N} \left\{ i^{-\delta} \sqrt{\frac{R^s \log (R/\alpha)}{n}} + R^{-\delta} + N^{1-\delta} + R^{-\delta} \sqrt{\frac{K \log N/\alpha}{n}} \right\} \tag{21}
\]

implying that the error \( |\lambda_i(T_W) - \lambda_i(T_n)| \) is order \( O \left( i^{\frac{-2\delta^2 - \delta}{2(\delta + s)}} \left( \frac{n}{\log n} \right)^{\frac{\delta}{2(\delta + s)}} \right) \).

On the other hand, if we assume \( H_3 \), given \( N > R \), we have with probability bigger than \( 1 - \alpha \), for \( i < R \)

\[
|\lambda_i(T_W) - \lambda_i(T_n)| \leq C \min_{R,N} \left\{ e^{-\delta i} \sqrt{\frac{e^{nR} \log (R/\alpha)}{n}} + 2e^{-\delta R} + e^{-\delta N} + e^{-\delta R} \sqrt{\frac{\log N/\alpha}{n}} \right\}
\]

with error rate \( O \left( e^{-\delta i} \frac{\delta}{i^{2+s}} \frac{1}{n^{s+2\delta}} \sqrt{\log n} \right) \)

It is clear that Theorem 1 will follow directly from Lemma 10 above. We note that the concentration inequalities follow directly by plugging the decrease assumptions \( H_1 \) and \( H_2 \) into Lemma 7. The end of the proof is related to the optimization of the parameters \( N \) and \( R \), finding their optimal dependence on \( n \).
B PROOF OF THEOREM 1

We start by considering that hypothesis $H_1$ holds. We follow the approach of two step optimization for $R$ and $N$. First we consider a fixed $R$ and optimize only on $N$. This problem can be written

$$m_R = \min_N N^{(1-\delta)} + \frac{R^{-\delta}}{\sqrt{n}} \sqrt{\log(N/\alpha)}$$

which corresponds to the last two terms of (21). Appendix C.1 explains how the relaxation of this problem can be solved using $V_{-1}$, the $-1$ branch of the W-Lambert function, see [23] (we use $V_{-1}$ instead of the classical notation $W_{-1}$ in order to avoid any confusion with the kernel function). Using (26) in Appendix C.1.1, with $\gamma = \delta - 1$, $K_1 = (\delta - 1)\alpha^{\delta-1}$ and $K_2 = \frac{2R^{-\delta}}{\sqrt{n}}$, we obtain

$$N^* = \alpha \exp \frac{-1}{2(\delta - 1)} V_{-1}\left(\frac{-2R^{-\delta}}{n(\delta - 1)^{\delta}2^{(1-\delta)}}\right)$$

At first glance, we do not know whether $N^*$ is bigger than $R$ or not. So we can take

$$N^* = \min \{\alpha N^*, R + 1\}$$

It is easy to see, based on the computation of the derivatives of the objective function (see Appendix C.1.1), that the condition for the minimum to be $N = R + 1$ is given by

$$\gamma K_1 \leq \frac{(R + 1)\gamma}{\log R + 1}$$

Remark 7. If $N = R + 1$ and parameter $s = 1$, the result is similar to [5, thm. 4].

We substitute $N^*$ in the expression for the minimum

$$m_R = (\delta - 1)\alpha^{1-\delta} \exp \frac{1}{2} V_{-1}\left(\frac{-2R^{-\delta}}{n(\delta - 1)^{\delta}2^{(1-\delta)}}\right) + 2R^{-\delta} \sqrt{-\frac{V_{-1}\left(\frac{-2R^{-\delta}}{n(\delta - 1)^{\delta}2^{(1-\delta)}}\right)}{2(\delta - 1)}}$$

Although with this method we are able to obtain the exact minimum $m_R$, we are more interested in the order of $m_R$ in terms of $n$ and $R$. In this sense, Proposition 11 gives us $m_R = O(R^{-\delta/2}n^{-1/2})$.

In the sequel we will work with the order we have found for $N$ in terms of $R$ introducing generic constants that will depend on the parameters of the problem, as we will later specify. We next want to obtain the order of $R$ (an upper bound) in function of $n$, that is

$$m^*_n \leq \min_R C_1 R^{\frac{\delta}{2}} \sqrt{\log R} + C_2 R^{-\delta} + C_3 R^{-\frac{\delta}{2}}$$

We can optimize on $R$ using the expression obtained for $N$, but in this case it is harder to solve analytically. We point out the fact that $C_1, C_2$ and $C_3$ depend on the rest of parameters, that is on $i, n, s$ and $\delta$. We obtain an upper bound for $R$, given by $\overline{R}$, see (31) in Appendix C.1.2.

$$\overline{R} = \exp \frac{1}{2(\frac{s}{2} + \frac{\delta}{2})} \left\{ V_0\left(\frac{8(\delta C_2 + \frac{s}{2}C_3)^2}{s^2C_1^2} (\frac{\delta}{2} + \frac{s}{2})\right)\right\}$$

Since we have worked with the orders of the different quantities involved, the constants have been just implicitly defined. In Appendix C.1.2 we discuss the order of $R$ in terms of $n$, finding that $R = O(i \frac{n^{\frac{\delta}{2}}}{\log n})^\frac{\delta}{2(\alpha + s)}$ (see Proposition 12).
We can carry out a similar analysis under the hypothesis $H_2$. The details can be found in Appendix C.2. In particular Proposition 13 implies that the error is $O(e^{-\delta i} \frac{\delta}{\delta n \sqrt{\log n}})$. Summarizing, we have

| Assumption | $R(n)$ | Error bound |
|------------|--------|-------------|
| $\lambda_i = O(i^{-\delta})$; $\eta(R) = O(R^\gamma)$ | $O\left(\frac{2^{i+\frac{1}{2}}}{\log n} \left(\frac{1}{\log n} \right)^{\frac{1}{2^{i+\frac{1}{2}}}}\right)$ | $O\left(\frac{2^{i+\frac{1}{2}}}{\log n} \left(\frac{1}{\log n} \right)^{\frac{1}{2^{i+\frac{1}{2}}}}\right)$ |
| $\lambda_i = O(e^{-\delta i})$; $\eta(R) = O(e^{\delta R})$ | $O\left(\frac{\log i}{\log n} \left(\frac{1}{\log n} \right)^{\frac{1}{\log i}}\right)$ | $O\left(\frac{\log i}{\log n} \left(\frac{1}{\log n} \right)^{\frac{1}{\log i}}\right)$ |
| $\lambda_i = O(e^{-\delta i})$; $\eta(R) = O(R^\gamma)$ | $O\left(e^{\frac{2^{i+\frac{1}{2}}}{\log n} \left(\frac{1}{\log n} \right)^{\frac{1}{2^{i+\frac{1}{2}}}}}\right)$ | $O\left(e^{\frac{2^{i+\frac{1}{2}}}{\log n} \left(\frac{1}{\log n} \right)^{\frac{1}{2^{i+\frac{1}{2}}}}}\right)$ |

**Appendix C: Optimization of parameters in Theorem 1**

Here we gather some results related to the optimization of parameters $N$ and $R$ in Lemma 10. Results rely mostly on the use of the W-Lambert function and its approximations. For $N$ we find an analytic solution for the optimization in terms of $R$ and the rest of parameters, notably $i$ the index of the eigenvalue considered. The problem for $R$ is more complicate to solve analytically, but we find an approximation for its order in terms of the rest of parameters, being its order in terms of $n$ the most interesting for our purposes.

**C.1. Optimization of $R$ and $N$ under polynomial eigenvalue decay**

We follow a two step optimization strategy which leads to the approximate optimal parameters $N$ and $R$. We begin with parameter $N$.

**C.1.1. Optimization of $N$**

in order to optimize $N$ we need to solve a problem of the form

$$m_N := \min_{N \geq R} f(z) = \min_{N \geq R} \left\{ \frac{C_1}{N^\gamma} + \frac{C_2}{\sqrt{R^8 n \sqrt{\log(N)}}} \right\}$$

(24)

where $C_1$ and $C_2$ are constants that do not depend on $n$ or $R$. Note that for this optimization, $n$ and $R$ can be considered as constants. We start by analyzing the problem

$$m_N := \min_{z \in \mathbb{N}} f(z) = \min_{z \in \mathbb{N}} \left\{ \frac{K_1}{z^\gamma} + K_2 \sqrt{\log(z)} \right\}$$

(25)

where $K_1$, $K_2$ and $\gamma$ are constants, at least for the optimization problem (they will depend on $R$, $n$ and the other parameters). First we find $z \in \mathbb{R}$ that solves the relaxed version of the optimization problem. We derive and make the derivative equal to 0 obtaining

$$-\gamma \frac{K_1}{z^{\gamma+1}} + \frac{K_2}{2z \sqrt{\log(z)}} = 0$$

Solving this equation for $z$ we find

$$z' = \exp\left\{ \frac{-1}{2\gamma} V_{-1} \left( -\frac{K_2}{2\gamma K_1^2} \right) \right\}$$

(26)
where $V_{-1}$ is the $-1$ branch of the $W$-Lambert function (see [23]). The approximate minimizer in $\mathbb{N}$, called $z^*$ is

$$z^* = \begin{cases} \lfloor z' \rfloor & \text{if } f(\lfloor z' \rfloor) \leq f(\lceil z' \rceil) \\ \lceil z' \rceil & \text{if } f(\lfloor z' \rfloor) > f(\lceil z' \rceil) \end{cases}$$

Replacing $z'$ on the expression for $m_N$ we obtain

$$m_N = K_1 \exp \left( \frac{1}{2} V_{-1} \left( \frac{-K_2}{2 \gamma K_1^2} \right) \right) + K_2 \sqrt{\frac{1}{2 \gamma} V_{-1} \left( \frac{-K_2}{2 \gamma K_1^2} \right)}$$

We have the following inequalities for the function $V_{-1}$ [24, thm. 1], valid for $u > 0$

$$-1 - \sqrt{2u} - u < V_{-1}(-e^{-u-1}) < -1 - \sqrt{2u} - \frac{2}{3}u \quad (27)$$

In our problem, we will use

$$u := \log \frac{2 \gamma K_1^2}{K_2} - 1$$

On the other hand, defining $\zeta = \frac{K_2^2}{2 \gamma K_1^2}$, we can rewrite the minimum as

$$m_N = K_1 \exp \left( \frac{1}{2} V_{-1}(-\zeta) \right) + K_1 \sqrt{-\zeta V_{-1}(-\zeta)}$$

The inequality in terms of $\zeta$ is

$$-\sqrt{2 \log \left( \frac{1}{\xi} \right) - 2 - \log \left( \frac{1}{\xi} \right)} < V_{-1}(-\zeta) < -\frac{1}{3} - \sqrt{2 \log \left( \frac{1}{\xi} \right) - 2 - \frac{2}{3} \log \left( \frac{1}{\xi} \right)}$$

So we obtain the bound

$$\exp \left( \frac{1}{2} V_{-1}(-\zeta) \right) < e^{-\frac{1}{3} \left( 2 \log \left( \frac{1}{\xi} \right) - 2 \right)^{-\theta}}$$

where $\theta > \frac{c}{2}$. The latter comes from the inequality $\log n \leq \frac{2}{e} \sqrt{n}$ for any $n > 1$. On the other hand,

$$-\zeta V_{-1}(-\zeta) < \zeta \left( \sqrt{2 \log \left( \frac{1}{\xi} \right) - 1} + \log \left( \frac{1}{\xi} \right) \right)$$

So we obtain for the minimum $m$

$$m_N \leq K_1 \left[ \xi^{\frac{1}{3}} \left( 2 \log \left( \frac{1}{\xi} \right) - 2 \right)^{-\theta} + \zeta \left( \sqrt{2 \log \left( \frac{1}{\xi} \right) - 1} + \log \left( \frac{1}{\xi} \right) \right) \right] \quad (28)$$

and also

$$m_N \geq K_1 \left[ \xi^{\frac{1}{3}} + \zeta \left( \sqrt{2 \log \left( \frac{1}{\xi} \right) - 1} + \log \left( \frac{1}{\xi} \right) \right) \right] \quad (29)$$

If $K_1 = (\delta - 1)\alpha^{\delta - 1}$ and $K_2 = \frac{2R^2}{\sqrt{n}}$, then $\zeta = \frac{2R^{1-\delta}}{\gamma(\delta - 1)\alpha^{\delta - 1}}$ the inequalities (28) and (29) together imply the following

**Proposition 11.** There exists two constants $\kappa_1, \kappa_2$ such as

$$\frac{\kappa_1}{\sqrt{R^3 n}} \leq m_N \leq \frac{\kappa_2}{\sqrt{R^3 n}}$$
Proof. From the analysis below (inequality (29)) we found that exists a constant $\kappa_1$ such as 
$\kappa_1 R^{-\delta/2} n^{-1/2} \leq m_N$. On the other hand, we note that putting $N = R^{\delta/2} n^{1/2}$ in (24) we obtain that $m_N \leq \kappa_2 R^{-\delta/2} n^{-1/2}$ for some constant $\kappa_2$, thus proving the upper bound.

Remark 8. Note that this analysis is independent of the value of $\lambda_i(T_W)$, so it will also apply to the case $\lambda_i(T_W) = 0$.

C.1.2. Optimization of $R$

The idea is, given the order for $N$, to obtain an upper bound for $R$ that will hint the right order in terms of $n$. Here we try to solve the optimization in $R$

$$m_R := \min_R g(R) := \min_R \frac{i-\delta}{\sqrt{n}} R^{\frac{\delta}{2}} \sqrt{\log R/\alpha} + R^{-\delta} + \frac{C'}{\sqrt{n}} R^{\frac{-\delta}{2}}$$  \hspace{1cm} (30)

To solve the relaxed problem of the right hand side we derivate to obtain the following equation

$$\frac{i-\delta}{2\sqrt{n}} R^{\frac{\delta}{2}+\delta} \sqrt{\log R/\alpha} = \delta + R^{\frac{\delta}{2}} \frac{\delta C'}{2\sqrt{n}} \left( -\frac{R^{\frac{\delta}{2}+\delta}}{\sqrt{\log R/\alpha}} \right)$$

This problem is difficult to solve exactly, but we can simplify it by introducing some assumptions. First, we assume that $\delta C' - R^{\frac{\delta}{2}+\delta} < 0$, which will be true for $R$ large enough. That allows to find an upper bound for the maximizer by noting that the left hand side is increasing and the right hand side is decreasing, in terms of $R$, and solving

$$\frac{i-\delta}{2\sqrt{n}} R^{\frac{\delta}{2}+\delta} \sqrt{\log R} = \delta$$

By what precedes discussion, we obtain:

$$R = \exp \frac{1}{s + 2\delta} \left( V_0 \frac{4i^2 n \delta^2}{s^2} (s + 2\delta) \right)$$ \hspace{1cm} (31)

where $V_0$ is the principal branch of the W-Lambert function. We note that in the expression above the term inside $V_0(\cdot)$ is order $O(i^2 n)$.

For the principal branch of the W-Lambert function, we have that for $x \geq e$, (see [25, thm. 2.7])

$$\log x - \log \log x + \frac{1}{2} \log \log x \leq V_0(x) \leq \log x - \log \log x + \frac{e}{e - 1} \frac{\log \log x}{\log x}$$ \hspace{1cm} (32)

For $x \geq e$ we have $\frac{\log \log x}{\log x} \leq 1$. In addition, $\lim_{x \to \infty} \frac{\log \log x}{\log x} = 0$. So for $x \geq e$ we have that the difference between the right extreme and the left extreme in the inequality (32) is

$$\left( \frac{e}{e - 1} - \frac{1}{2} \right) \frac{\log \log x}{\log x} \leq \frac{1}{e}$$

So for large values of $x$ we can consider that $V_0(x) = \log x - \log \log x$ with small error. For a constant $x$ we have $V_0(x t^{2\delta} n) = \log x t^{2\delta} n - \log \log x t^{2\delta} n$ and

$$\exp \frac{1}{s + 2\delta} V_0(x t^{2\delta} n) = (x t^{2\delta} n)^{\frac{1}{s + 2\delta}} \exp \left\{ -\frac{1}{s + 2\delta} \log \log x t^{2\delta} n \right\}$$

$$= \left( \frac{x t^{2\delta} n}{\log x t^{2\delta} n} \right)^{\frac{1}{s + 2\delta}}$$
So $R = \left(\frac{2^\delta}{\log n}\right)^{\frac{1}{\eta + 2s}}$ with a small error. Replacing this in (30), we obtain

\[ m_R = O\left(\frac{n}{\log n}\right)^{\frac{2^\delta}{2s + \eta}} + O\left(\frac{n^{2^\delta}}{\log n}\right)^{\frac{2^\delta}{2s + \eta}} + O\left(\frac{n}{\log n}\right)^{\frac{2^\delta}{2s + \eta}} \]

It is easy to see that $\frac{2^\delta}{2s + \eta} > \frac{3^\delta - s}{4}$. Also note that for $s \geq 2$ we have that $\frac{2^\delta}{2s + \eta} > \frac{2^\delta}{2s + \delta}$, and for $s < 2$, the inverse holds. This implies that

\[ g(R) = \begin{cases} \frac{2^\delta}{2s + \eta} \left(\frac{n}{\log n}\right)^{\frac{2^\delta}{2s + \eta}} & \text{if } s \geq 2 \\ \frac{2^\delta}{2s + \eta} \left(\frac{n}{\log n}\right)^{\frac{2^\delta}{2s + \eta}} & \text{if } s < 2 \end{cases} \]

As $m_R \leq g(R)$, we deduce that there exists a constant $\kappa_1$ such as

\[ m_R \leq \begin{cases} \kappa_1 \left(\frac{n}{\log n}\right)^{\frac{2^\delta}{2s + \eta}} & \text{if } s \geq 2 \\ \kappa_1 \left(\frac{n}{\log n}\right)^{\frac{2^\delta}{2s + \eta}} & \text{if } s < 2 \end{cases} \]

**Remark 9.** Here the explicit assumption on the eigenvalue decay does not affect the rate order in terms of $n$, so the same dependence on $n$ in the decreasing rate will be valid for the hypothesis $H_2$. It is easy to see, replacing $i^{2\delta}$ by $e^{2\delta}$, in what precedes, that the rate will be $O(e^{2\delta})$ under $H_2$.

**Remark 10** (Order of the function $w_{\alpha,W}$ in Theorem 1). The perturbation result Proposition 4 gives a bound for $1 \leq i \leq R$. From the previous discussion we have that $R = \left(\frac{2^\delta}{i^{2s + \eta}} n^{\frac{1}{2s + \eta}}\right)$, then the condition $i \leq R$ implies that

\[ i \leq Cn^{1/s} \]

where $C$ is a constant. Then $w_{\alpha,W}(n) = O(n^{1/s})$.

**C.1.3. Lower bound**

We define

\[ m'_R = \min_R h(r) := \min_R \frac{i^{-\delta}}{\sqrt{n}} R^2 \sqrt{\log R/\alpha} + R^{-\delta} \]

and we note that $m'_R \leq m_R$. Also we note that

\[ m'_R \geq \min_R \frac{i^{-\delta}}{\sqrt{n}} R^2 + R^{-\delta} \]

On the right hand side, we have that the minimizer $R_\alpha = \left(\frac{2^\delta \sqrt{n}}{s}\right)^{\frac{2}{2s + \eta}}$ and then

\[ \min_R \frac{i^{-\delta}}{\sqrt{n}} R^2 + R^{-\delta} = \left(\frac{2^\delta}{s}\right)^{\frac{2}{2s + \eta}} \left(\frac{n^{2\delta}}{i^{2s + \eta}} - \delta \right) n^{\frac{-\delta}{2s + \eta}} + \left(\frac{2^\delta}{s}\right)^{\frac{2}{2s + \eta}} \left(\frac{n^{2\delta}}{i^{2s + \eta}} - \delta \right) n^{\frac{-\delta}{2s + \eta}} \]
So there exists a constant \( \kappa_2 \) such as

\[
m'_R \geq \begin{cases} 
\kappa_2^2 n^{\frac{\delta}{2 \delta + \gamma}} & \text{if } s < 2 \\
\kappa_2^2 n^{\frac{-\delta}{2 \delta + \gamma}} & \text{if } s \geq 2 
\end{cases}
\]

what precedes discussion serve as proof of the following

**Proposition 12.** There exist two constants \( \kappa_1 \) and \( \kappa_2 \) such as

\[
\kappa_2^2 n^{\frac{-\delta}{2 \delta + \gamma}} \leq m_R \leq \kappa_1^2 n^{\frac{-\delta}{2 \delta + \gamma}} \left( \frac{n}{\log n} \right)^{\frac{-\delta}{2 \delta + \gamma}}
\]

if \( s < 2 \) and

\[
\kappa_2^2 n^{\frac{-\delta}{2 \delta + \gamma}} \leq m_R \leq \kappa_1^2 n^{\frac{-\delta}{2 \delta + \gamma}} \left( \frac{n}{\log n} \right)^{\frac{-\delta}{2 \delta + \gamma}}
\]

if \( s \geq 2 \).

### C.2. Optimization of \( N \) and \( R \) under exponential eigenvalue decay

Given an exponential decrease on \( \lambda_k \), that is \( \lambda_k = O(e^{-\tau k}) \), and an exponential growth in the eigenfunctions, more precisely \( \eta(R) = O(e^{\sigma R}) \) we need to optimize for \( N \)

\[
m_{exp}^R := \min_N C_1 e^{-\tau N} + C_2 \frac{\sqrt{\log N}}{\log n}
\]

So the equation we get is

\[
2\tau C_1 e^{-\tau N} N \sqrt{\log N} = C_2 e^{-\tau R}
\]

Instead we will solve for \( N \) the equation

\[
2\tau C_1 e^{-\tau N} N = C_2 e^{-\tau R}
\]

which gives an upper bound on \( N \). Defining \( \nu = -\tau N \) we get

\[
\nu e^\nu = -\frac{C_2}{2 C_1} e^{-\tau R}
\]

So \( \nu = V_{-1}(\frac{C_2}{2 C_1} e^{-\tau R}) \). Using (27) we obtain

\[
N < R + \frac{1}{\tau} \sqrt{2(\log \left( \frac{2C_1}{C_2} \right) + \tau R - 1) + \frac{1}{\tau} \log \left( \frac{2C_1}{C_2} \right)}
\]

Also we observe that in the particular case of the problem of optimization with exponential decrease, we have that \( C_2 \) depends on \( n \) (is order \( O(\frac{1}{\sqrt{n}}) \)), but \( C_1 \) does not. Since \( N \) appears with negative sign in (33), we use the lower bound \( N > R \) to get the problem

\[
\min_R K_1 e^{\frac{sR}{2}} \sqrt{\log R} + K_2 e^{-\delta R} + K_3 e^{-\delta R} \sqrt{\log (R + \log n)}
\]

We derivate and equalize to 0 to get

\[
K_1 e^{\frac{sR}{2}} \left( \frac{s}{2} \sqrt{\log R} + \frac{1}{2R \sqrt{\log R}} \right) = \delta K_2 e^{-\delta R} + K_3 e^{-\delta R} \left( \delta \sqrt{\log R} - \frac{1}{2R \sqrt{\log R}} \right)
\]
In order to obtain a bound for the order of $R$ in terms of $n$ we will apply the following reasoning. First we will reduce the equation, in a similar way as we did in the polynomial decrease case, to the simpler problem of finding $R$ that satisfies

$$e^{R(\frac{4}{3}+\delta)}\sqrt{\log R} = \frac{2\delta}{s K_1} (K_2 + K_3)$$

In what precedes we use the assumption that $K_1 \frac{s R}{2R\sqrt{\log R}} > K_3 e^{-\delta R}(\delta \sqrt{\log R} - \frac{1}{2R\sqrt{\log R}})$, which is reasonable since $K_1$ and $K_3$ are $O(\frac{1}{\sqrt{n}})$. From what precedes, we know that the $R$ that solves the equation satisfies $R \leq \frac{2}{s+2\kappa} \log \frac{2\delta}{s K_1} K_2$. As $K_1$ is $O(\frac{1}{\sqrt{n}})$, and $K_2$ does not depend on $n$. The order of the term inside log is $\sqrt{n}$. Thus $R$ is at most $O(\log (\frac{2s}{s+2\kappa} n^{\frac{1}{2}}))$. If we introduce this in the expression for the minimum we obtain an order of $O(e^{-\delta \frac{s}{s+2\kappa} n^{\frac{1}{2}} \log n})$. In the exponential case under hypothesis H2 we have the following

**Proposition 13.** There exists a constant $K$ such as

$$m_{\exp}^R \leq K e^{-\delta \frac{s}{s+2\kappa} n^{\frac{1}{2}} \log n}$$

### Appendix D: Gegenbauer polynomials and spherical harmonic dimension

The Gegenbauer (ultraspherical) polynomials $G^\gamma_l$ are themselves multiples of the Jacobi polynomials $P^{(\gamma-\frac{1}{2},\gamma-\frac{1}{2})}_l$, satisfying

$$G^\gamma_l(t) = \frac{(2\gamma)^{(l)}}{(\gamma+\frac{1}{2})^{(l)}} P^{(\gamma-\frac{1}{2},\gamma-\frac{1}{2})}_l(t)$$

where $(\cdot)^{(l)}$ is the rising Pochhammer symbol. The Jacobi polynomials are a well-studied family of orthogonal polynomials (see [19, chap. 4]). A convenient way to define them is as the solutions of the following differential equation

$$L_{\gamma-\frac{1}{2}} x^{(\gamma-\frac{1}{2},\gamma-\frac{1}{2})}_l(t) = \beta_l x^{(\gamma-\frac{1}{2},\gamma-\frac{1}{2})}_l(t)$$

where

$$L_{\gamma-\frac{1}{2}} u = -(1-t^2)^{\frac{1}{2}} \gamma \frac{d}{dt} (1-t^2)^{\frac{1}{2} \gamma} \frac{du}{dt}$$

and $\beta_l = l(l+2\gamma)$. The Gegenbauer polynomials satisfy the following orthogonality relations with respect to the weight function $w_{\gamma}(t) = (1-t^2)^{\gamma-\frac{1}{2}}$

$$c_\gamma \int_{-1}^{1} G^\gamma_k(t) G^\gamma_l(t) w_{\gamma}(t) dt = \frac{\gamma}{n+\gamma} G^\gamma_l(1) \delta_{kl}$$

where $c_\gamma = 2^{-2\gamma} \frac{\Gamma(2\gamma+1)}{\Gamma(\gamma+\frac{1}{2})^2}$ and $G^\gamma_l(1) = \frac{(2\gamma)^{(l)}}{l!}$. From (6), we have

$$p(t) = \sum_{l \geq 0} \lambda_l c_l G^\gamma_l(t) = \sum_{l \geq 0} \lambda_l \sqrt{\delta_l} G^\gamma_l(t)$$
where $\tilde{G}_t^\gamma(t) = \frac{G_t^\gamma(t)}{\|G_t^\gamma\|_2^2}$, \{\lambda_l\}_{l \in \mathbb{N}} are the eigenvalues of the operator $T_K$ and \{\lambda_l \sqrt{\gamma_l}\}_{l \in \mathbb{N}} are the Fourier-Gegenbauer coefficients. We recall that in the case $\gamma = \frac{d-2}{2}$ we have, by (8)
\[
\lambda_l = \frac{\Gamma\left(\frac{d}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{d-1}{2}\right)} \frac{l!}{(2d-2)!} \int_{-1}^{1} p(t)G_t^\gamma(t)w_\gamma(t)dt
\]
A useful tool to compute the previous integral is the Rodrigues formula (see [19, eq. 4.3.1])
\[
G_t^\gamma(t)w_\gamma(t) = b_{l,d} \frac{d^l}{dt^l} w_{\gamma+l}(t) = b_{l,d} \frac{d^l}{dt^l} (1 - t^2)^{\frac{d-3}{2} - l}
\]
where $b_{l,d} = \frac{(-1)^l (2d-2)!}{2^d l! \left( \frac{d-1}{2} \right)^{l+1}}$.

D.1. Estimation of $d_t$ coefficients

The dimension of the spherical harmonic spaces is well-known (see [16, cor. 1.1.4]),
\[
d_t = \frac{1}{l + d - 1} - \frac{1}{l + d - 3} - \frac{1}{l - 2}.
\]
This is a polynomial in $l$. Routinary computations lead to
\[
d_t = 2 \frac{(l + d - 3)!}{dl(l-1)!} + \frac{(l + d - 3)}{d - 3}
\]
which is necessary to determine the asymptotic order with respect to $l$, because the term $2 \frac{(l + d - 3)!}{dl(l-1)!}$ has the leading term in $l$, which determines that $d_t = O(l^{d-2})$. We can also determine the order of $\kappa(R) = \sum_{l=0}^{R} d_t$. As we know that $d_0 = 1$ and $d_1 = 1$ we can take them out of the sum
\[
\kappa(R) = 1 + d + \sum_{l=2}^{R} d_t
\]
and $\sum_{l=2}^{R} d_t$ is of the form $\sum_{l=2}^{R} F(l) - F(l - 2) = \sum_{l=2}^{R} F(l) - F(l - 1) + (F(l - 1) - F(l - 2))$, with $F(l) = \frac{l + d - 1}{l}$. Then
\[
\kappa(R) = 1 + d + F(R) + F(R - 1) - F(0) - F(1)
\]
The leading term is contained in $F(R)$ and is a constant times $R^{d-1}$. Thus $\kappa(R) = O(R^{d-1})$. Note that in the case of kernels that only depends on the distance, as described in Section 6 we have that the eigenfunction growth parameter $\eta$ satisfies
\[
\eta(R) = \kappa(R) = O(R^{d-1})
\]

Appendix E: Eigenvalues computations

E.1. Logistic function

We will compute the eigenvalues for the logistic graphon $f(t) = \frac{e^{rt}}{1 + e^{rt}}$.
\[
\lambda_l = a_{l,d} \int_{-1}^{1} f(t)G_t^\gamma(t)w_\gamma(t)dt = a_{l,d} \int_{-1}^{1} \frac{1}{1 + e^{-rt}}G_t^\gamma(t)w_\gamma(t)dt
\]
Using the Rodrigues formula for Gegenbauer polynomials eq. (36) we get

\[ \lambda_l = a_{l,d}b_{l,d} \int_{-1}^{1} \frac{1}{1 + e^{-rt}} \frac{d^l}{dt^l} t^{d-3/2+l} dt \]

We have the following power series expansion \( \frac{1}{1 + e^{-rt}} = \sum_{k=0}^{\infty} (-1)^k e^{-krt} \), which is valid for any \( t > 0 \). Then we get

\[
\begin{align*}
\lambda_l &= a_{l,d}b_{l,d} \left( \int_{0}^{1} \frac{1}{1 + e^{-rt}} \frac{d^l}{dt^l} w_{\gamma+l}(t) dt + \int_{-1}^{0} \frac{1}{1 + e^{-rt}} \frac{d^l}{dt^l} w_{\gamma+l}(t) dt \right) \\
&= a_{l,d}b_{l,d} \left( \int_{0}^{1} \frac{1}{1 + e^{-rt}} \frac{d^l}{dt^l} w_{\gamma+l}(t) dt + \int_{0}^{1} \frac{1}{1 + e^{rt}} \frac{d^l}{dt^l} w_{\gamma+l}(-t) dt \right) \\
&= a_{l,d}b_{l,d} \left( \int_{0}^{1} \frac{1}{1 + e^{-rt}} \frac{d^l}{dt^l} w_{\gamma+l}(-t) dt + \int_{0}^{1} \frac{1}{1 + e^{-rt}} \frac{d^l}{dt^l} (w_{\gamma+l}(t) - w_{\gamma+l}(-t)) dt \right)
\end{align*}
\]

The parity of the Gegenbauer polynomial \( G^\gamma_l(t) \) is the same that \( l \), then the previous relation reduces to

\[
\lambda_l = a_{l,d}b_{l,d} \left( -\int_{0}^{1} \frac{d^l}{dt^l} w_{\gamma+l}(t) dt + \int_{0}^{1} \frac{2}{1 + e^{-rt}} \frac{d^l}{dt^l} w_{\gamma+l}(t) dt \right) = a_{l,d}b_{l,d} \int_{0}^{1} \frac{1 - e^{-rt}}{1 + e^{-rt}} \frac{d^l}{dt^l} w_{\gamma+l}(t) dt
\]

Already from this expression we see that in the case \( r \to 0 \), being the rest of the parameters fixed, the eigenvalues will be

\[ \lambda_l = \begin{cases} 
\frac{1}{2} & \text{if } l = 0 \\
0 & \text{if } l > 0
\end{cases} \]

In the case \( r \to \infty \) gives

\[ \lambda_l = \begin{cases} 
\frac{1}{2} & \text{if } l = 0 \\
a_{l,d}b_{l,d} \int_{0}^{1} \frac{d^l}{dt^l} w_{\gamma+l}(t) dt & \text{if } l > 0
\end{cases} \]

Indeed, for \( t \) fixed \( g_r(t) = \frac{1 - e^{rt}}{1 + e^{rt}} \) satisfies \( g_r(t) \to 0 \) for \( r = 0 \) and \( g_r(t) = 1 \) for \( r \to \infty \). Also, note that \( g_r(t) \) is also continuous on \( r \) and increasing for \( r \geq 0 \). For \( 0 < r < \infty \), we have to compute the quantity

\[ A_{r,l} := \int_{0}^{1} \frac{1}{1 + e^{-rt}} \frac{d^l}{dt^l} w_{\gamma+l}(t) \]

We can use the power series expansion of the logistic function

\[
A_{r,l} = \int_{0}^{1} \sum_{k=0}^{\infty} (-1)^k e^{-krt} \frac{d^l}{dt^l} w_{\gamma+l}(t) dt = \sum_{k=0}^{\infty} (-1)^k \int_{0}^{1} e^{-krt} \frac{d^l}{dt^l} w_{\gamma+l}(t) dt
\]

Let us first assume that \( \gamma \in \mathbb{N} \), then we expand

\[
w_{\gamma+l}(t) = \sum_{i=0}^{\gamma - \frac{1}{2} + l} \binom{\gamma - \frac{1}{2} + l}{i} (-1)^i t^{2i}
\]
Thus

\[
\begin{align*}
\frac{d}{dt} w_{\gamma-\frac{1}{2}+i}(t) &= \sum_{i=\lfloor l/2 \rfloor}^{\gamma-\frac{1}{2}+l} \left( \gamma - \frac{1}{2} + l \right) (-1)^i (2i) t^{2i-l} \\
&= \sum_{i=0}^{\gamma-\frac{1}{2}+l/2} g_{i,l} t^{2i}
\end{align*}
\]

(40)

where \( g_{i,l} := (\gamma - \frac{1}{2} + l) (-1)^{i+l/2} (2i+l) t \). Plugging into the expression (38) we get

\[
\begin{align*}
A_{r,l} &= \sum_{k=0}^{\infty} \gamma + \lfloor l/2 \rfloor \sum_{i=0}^{\gamma + \lfloor l/2 \rfloor} (-1)^k g_{i,l} \int_0^1 e^{-krt} t^{2i} dt \\
&= \sum_{k=0}^{\infty} \gamma + \lfloor l/2 \rfloor \sum_{i=0}^{\gamma + \lfloor l/2 \rfloor} \frac{(-1)^k}{(kr)^{2i+1}} g_{i,l} \int_0^{kr} e^{-t} t^{2i} dt \\
&= \sum_{k=0}^{\infty} \gamma + \lfloor l/2 \rfloor \sum_{i=0}^{\gamma + \lfloor l/2 \rfloor} \frac{(-1)^k}{(kr)^{2i+1}} g_{i,l} (2i+1, kr)
\end{align*}
\]

(41)

On the other hand, we have

\[
\int_0^1 \frac{d}{dt} w_{\gamma+i}(t) = \left. \frac{d^{l-1}}{dt^{l-1}} w_{\gamma+i}(t) \right|_{t=1} - \left. \frac{d^{l-1}}{dt^{l-1}} w_{\gamma+i}(t) \right|_{t=0}
\]

It is easy to see that \( \left. \frac{d^{l-1}}{dt^{l-1}} w_{\gamma+i}(t) \right|_{t=1} = 0 \), because this derivative is a multiple of \((1-t^2)^\gamma\).

Using (39) we obtain

\[
\left. \frac{d^{l-1}}{dt^{l-1}} w_{\gamma+i}(t) \right|_{t=0} = \left( \gamma - \frac{1}{2} + l \right) (-1)^{\lfloor l/2 \rfloor} (l)_{l-1}
\]

By definition, we get

\[
\left. \frac{d^{l-1}}{dt^{l-1}} w_{\gamma+i}(t) \right|_{t=0} = \left( \gamma - \frac{1}{2} + l \right) (-1)^{\lfloor l/2 \rfloor} (l)_{l-1}
\]

\[
= \left( \frac{d-1}{2} + l \right) (-1)^{l+1} l!
\]

\[
= (-1)^{l+1} \frac{\Gamma\left(\frac{d-1}{2} + l\right) \Gamma(l+1)}{\Gamma\left(\frac{l+3}{2}\right) \Gamma\left(\frac{2d-1}{2} + 1\right)}
\]

On the other hand,

\[
a_{t,d} b_{t,d} = \frac{\Gamma\left(\frac{d}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{d-1}{2}\right) (2d-2)_{l/2} 2^l l!} \frac{(-1)^l (2d-2)_{l}}{(d-1)_{l/2} l}
\]

\[
= \frac{(-1)^l \Gamma\left(\frac{d}{2}\right)}{2^l \sqrt{\pi} \Gamma\left(\frac{d-1}{2}\right) (2d-2)_{l}} l
\]

\[
= \frac{(-1)^l \Gamma\left(\frac{d}{2}\right)}{2^l \sqrt{\pi} \Gamma\left(\frac{d-1}{2} + l\right)}
\]
Then we have

\[ a_{l,d} b_{l,d} \int_0^1 \frac{d^l}{dt^l} w_{\gamma+1}(t) \, dt = \frac{(-1)^{\frac{d+1}{2}}}{2^l \sqrt{\pi}} \frac{\Gamma(l+1) \Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{l+3}{2}\right) \Gamma\left(\frac{d+1}{2}\right)} = \frac{2(-1)^{\frac{d+1}{2}} \Gamma\left(\frac{l+1}{2}\right) \Gamma\left(\frac{d}{2}\right)}{\pi(l+1) \Gamma\left(\frac{d+1}{2}\right)} = \frac{2(-1)^{\frac{d+1}{2}}}{\pi(l+1)} \text{Beta}\left(\frac{d}{2}, \frac{l}{2} + 1\right) \]

The eigenvalues are for \( l \) even:

\[ \lambda_l = 2 \sum_{k=0}^{\infty} \frac{\gamma+\lfloor l/2 \rfloor}{(kr)^{2i+1}} g_{i,l} \gamma(2i+1, kr) + \frac{2(-1)^{\frac{d+1}{2}}}{\pi(l+1)} \text{Beta}\left(\frac{d}{2}, \frac{l}{2} + 1\right) \tag{42} \]

In the case \( \gamma \notin \mathbb{N} \) we can use the generalized binomial expansion

\[ w_{\gamma+1}(t) = (1 - t^2)^{\gamma - \frac{1}{2}} = \sum_{k=0}^{\infty} \binom{\gamma - \frac{1}{2}}{k} (-1)^k t^{2k} \]

which is absolutely convergent for \(-1 \leq t \leq 1\). The generalized binomial coefficient is defined for \( \alpha, \beta \in \mathbb{R} \) by \( \binom{\alpha}{\beta} := \frac{\Gamma(\alpha+1)}{\Gamma(\beta+1) \Gamma(\alpha-\beta+1)} \). With that we obtain

\[ \frac{d^l}{dt^l} w_{\gamma+1}(t) = \sum_{i=0}^{\infty} \tilde{g}_{i,l} t^{2i} \]

where \( \tilde{g}_{i,l} = \binom{\gamma+i}{i+\lfloor l/2 \rfloor} (-1)^i \gamma + \lfloor l/2 \rfloor (2i + l) \). Note that definition is the same that in the particular case \( \gamma \in \mathbb{N} \), but the binomial in \( \tilde{g}_{i,l} \) is the generalized one. Of course, one might take the more general definition in the case of \( g_{i,l} \) too.

**E.2. Threshold function**

If \( \gamma \in \mathbb{N} \) and \( l \) is odd, the computations in Appendix E.1 can be used for the eigenvalues of the threshold function \( f(t) = 1_{t \leq 0} \). Indeed,

\[ \lambda_l = a_{l,d} b_{l,d} \int_0^1 \frac{d^l}{dt^l} w_{\gamma+1}(t) \, dt \]

\[ = \frac{2(-1)^{\frac{d+1}{2}}}{\pi(l+1)} \text{Beta}\left(\frac{d}{2}, \frac{l}{2} + 1\right) \]

If \( \gamma \in \mathbb{N} \), \( l \) is even and \( l \neq 0 \) we have \( \lambda_l = 0 \), because

\[ \frac{d^l}{dt^l} w_{\gamma+1}(t) \bigg|_{t=0} = 0, \]

Indeed, we have \( \frac{d^{l-1}}{dt^{l-1}} w_{\gamma+1}(t) \bigg|_{t=1} = 0 \), because that derivative is a multiple of \((1 - t^2)^\gamma\). On the other hand \( \frac{d^{l-1}}{dt^{l-1}} w_{\gamma+1}(t) \bigg|_{t=0} = 0 \), because when we derivate an odd number of times the
function $w_{\gamma+1}(t)$ there will be no constant term. This can be readily seen from the expansion 39. If $l = 0$ we finally have

$$\lambda_0 = a_{l,d} \int_0^1 w_\gamma(t) dt = \frac{1}{2}$$