THE MAXIMUM SIZE OF $L$-FUNCTIONS

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Dedicated to Hugh Montgomery on his 60th birthday.

Abstract. We conjecture the true rate of growth of the maximum size of the Riemann zeta-function and other $L$-functions. We support our conjecture using arguments from random matrix theory, conjectures for moments of $L$-functions, and also by assuming a random model for the primes.

1. Introduction and statement of results

A fundamental problem in analytic number theory is to calculate the maximum size of $L$-functions in the critical strip. For example, the importance of the Lindelöf Hypothesis, which is a consequence of the Riemann Hypothesis, is that it provides at least a crude estimate for the maximum in the case of the Riemann zeta-function. In this paper we use a variety of methods to conjecture the true rate of growth.

Consider first the Riemann zeta-function, which is a prototypical $L$-function. The Lindelöf Hypothesis asserts that for every $\varepsilon > 0$, $\zeta(\frac{1}{2} + it) = O(t^\varepsilon)$ (here we assume $t$ is positive). Under the Riemann Hypothesis, one can show that

$$\zeta(\frac{1}{2} + it) = O\left(\exp\left(C\frac{\log t}{\log \log t}\right)\right)$$

(1.1)

for some constant $C$ (see Theorem 14.14A of [19], for example). Several results of the form

$$|\zeta(\frac{1}{2} + it)| = \Omega\left(\exp\left(C'\sqrt{\frac{\log t}{\log \log t}}\right)\right)$$

(1.2)

have also been established. Assuming the Riemann Hypothesis (RH), Montgomery [16] showed $C' \geq \frac{1}{20}$. Balasubramanian and Ramachandra [2] improved the constant $C'$ and removed the assumption of RH. Soundararajan [18] further improved the estimate to $C' \geq 1$, and he has also obtained similar results for the central values of a family of $L$-functions. In fact, Soundararajan’s calculations show that the proportion of $t$ for which $\zeta(\frac{1}{2} + it)$ is this big is quite large, suggesting that it may get bigger still. Numerical calculations of Kotnik [15] indicate that $C' > 2$ and perhaps $C'$ can be much larger.

We are interested in finding out which of equations (1.1) or (1.2) is closer to the truth. This is part of a class of problems that has recently come to be known as the “1 or 2?” question, where one has an $O$-result and an $\Omega$-result which, suitably interpreted, differ by a factor of 2. In the case here, the unknown factor is the power of $\log t$ in the exponential. The calculations in this paper support the view that “1” is the correct answer in this case, and we make the following conjecture:

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Conjecture A.

\[ \max_{t \in [0, T]} |\zeta(\frac{1}{2} + it)| = \exp \left( (1 + o(1)) \sqrt{\frac{1}{2} \log T \log \log T} \right). \quad (1.3) \]

Similar arguments to those presented for \( \zeta(\frac{1}{2} + it) \) work for \( S(t) \), the error term in the number of zeros of the zeta-function up to height \( t \), and lead to

Conjecture B.

\[ \limsup_{t \to \infty} \frac{S(t)}{\sqrt{\log t \log \log t}} = \frac{1}{\pi \sqrt{2}}. \quad (1.4) \]

Much recent progress in understanding analytic properties of \( L \)-functions has come from the idea of a “family” of \( L \)-functions with an associated symmetry type. The idea is that to a collection of \( L \)-functions, with appropriate natural conditions, one can associate a classical compact group: unitary, symplectic, or orthogonal. One expects the analytic properties of the \( L \)-functions to be largely governed only by the symmetry type. Here we apply this philosophy to conjecture the maximal size of the critical values of \( L \)-functions.

A family \( \mathcal{F} \) of \( L \)-functions is partially ordered by the “conductor” \( c(F) \) for \( F \in \mathcal{F} \). Our calculations assume that \( \# \{ F \in \mathcal{F} : c(F) < D \} \approx D \). Straightforward modifications can handle the case in which the family grows like \( D^A \) for any \( A > 0 \).

For a more detailed discussion of families of \( L \)-functions, see [4, 5]. (However, note that [5] introduces a refined notion of “conductor”, which, asymptotically, is the logarithm of the “usual” conductor used here.)

Conjecture C. Suppose \( \mathcal{F} \) is a family of \( L \)-functions and, for \( F \in \mathcal{F} \), let \( c(F) \) denote the conductor of \( F \). With \( B = 1/2 \) for unitary families and \( B = 1 \) for symplectic and orthogonal families, we have

\[ \max_{F \in \mathcal{F}, c(F) \leq D} |F(\frac{1}{2})| = \exp \left( (1 + o(1)) \sqrt{B \log D \log \log D} \right). \quad (1.5) \]

The implied constant depends only on \( \mathcal{F} \).

For example, for the symplectic family of real primitive Dirichlet \( L \)-functions, \( L(s, \chi_d) \), where \( \chi_d = (d)^{\frac{s-1}{2}} \), we conjecture that

\[ \max_{d \leq D} |L(\frac{1}{2}, \chi_d)| = \exp \left( (1 + o(1)) \log D \log \log D \right). \quad (1.6) \]

Similarly, for the orthogonal family of Dirichlet series associated with holomorphic cusp forms, \( L(s, f) \), where \( f \in S_k(\Gamma_0(N)) \), the conductor is \( kN \), so we conjecture that

\[ \max_{kN \leq D} |L(\frac{k}{2}, f)| = \exp \left( (1 + o(1)) \log D \log \log D \right). \quad (1.7) \]

Note that Conjecture C contains Conjecture A because any primitive \( L \)-function, \( L(s) \), has associated with it the unitary family

\[ \mathcal{F}_L := \{ F_y(s) := L(s + iy) \mid y \in \mathbb{R} \} \quad (1.8) \]

with conductor \( c(F_y) \sim |y| \).

Our conjectures suggest that on the critical line the answer to the “1 or 2” question is “1”. Work of Montgomery and Vaughan [17] and Granville and Soundararajan [9, 10] has suggested that the answer also is “1” on the 1-line. Thus in both cases, the maximum value the \( L \)-function attains appears to be closer to the \( \Omega \)-result than to the \( O \)-result.

In Section 2 we use a rigorous approximation to the zeta-function due to Gonek, Hughes, and Keating [8] to justify Conjecture A. This approximation represents \( \zeta(s) \) as a product...
over primes times a product over zeros. We use characteristic polynomials of random unitary matrices to model the product over zeros, and a separate probabilistic model due to Granville and Soundararajan for the product over primes. The approximation to the zeta-function has a parameter which controls the relative contribution of the primes and the zeros. We show that the predicted maximal order of the zeta-function is the same independent of the choice of parameter. That is, whether we use only the primes, or only the zeros, or some combination of the two, we obtain Conjecture A.

In Section 3 we use recent conjectures for moments of \( L \)-functions to give an alternative justification for Conjecture A. Our approach also provides new limits on the range of validity of those conjectured moments.

In Section 4 we modify the treatment in Section 2 to obtain Conjecture B.

In Section 5 we describe how to extend our approach to obtain Conjecture C. We also describe some other approaches to obtaining these conjectures and then indicate possible arguments against the conjectures.

Finally, in Appendix A we prove a theorem about random matrix polynomials that is used in Section 2.

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2. A probabilistic model for the zeta-function

Gonek, Hughes, and Keating \[8\] have proved that if \( s = \sigma + it \), with \( 0 \leq \sigma \leq 1 \) and \( |t| \geq 2 \), then for \( X > 2 \) and \( K \) any positive integer,

\[
\zeta(s) = P_X(s)Z_X(s) \left( 1 + O \left( \frac{X^{2-\sigma+K}}{|t| \log X} \right) + O(X^{-\sigma} \log X) \right),
\]

where

\[
P_X(s) := \exp \left( \sum_{n \leq X} \frac{\Lambda(n)}{n^s \log n} \right), \tag{2.2}
\]

\( \Lambda(n) \) is the von-Mangoldt function, and

\[
Z_X(s) := \exp \left( -\sum_{\rho} U((s - \rho) \log X) \right). \tag{2.3}
\]

Here the \( \rho \) are non-trivial zeros of \( \zeta(s) \) and \( U(z) = \int_0^\infty u(x) E_1(z \log x) \, dx \), where \( E_1(z) = \int_z^\infty \frac{e^{-w}}{w} \, dw \) is the exponential integral and \( u \) is any smooth function supported in \([e^{1-1/X}, e] \).

The parameter \( X \) controls the relative influence of the primes and the zeros. If \( X \) is large, there are many primes in \( P_X(s) \), and only the zeros very close to \( s \) affect the product in \( Z_X(s) \), while if \( X \) is small, the zeros further away from \( s \) make a contribution to \( Z_X(s) \), but the number of primes in \( P_X(s) \) is diminished. When \( X \) is not too large, we expect \( Z_X \) and \( P_X \) to behave somewhat independently, and Gonek, Hughes, and Keating \[8\] give evidence of this. In the remainder of this section we describe probabilistic models for \( P_X \) and \( Z_X \) which, assuming independence, will give Conjecture A. In Section 2.1 we describe our model for the large values of \(|Z_X|\), establish some new results on the size of characteristic polynomials of random unitary matrices, and justify Conjecture A by choosing \( X \) small. In Section 2.2 we describe Granville and Soundararajan’s model for \( P_X \) and justify Conjecture A by choosing \( X \) large. Then, in Section 2.3 we combine \( Z_X \) and \( P_X \), showing that intermediate values of \( X \) also lead to Conjecture A.
2.1. A random matrix model for large values of $Z_X(\frac{1}{2} + it)$. Here we study the characteristic polynomial
\[ \Lambda_U(\theta) = \det \left( I - U e^{-i\theta} \right) = \prod_{n=1}^{N} \left( 1 - e^{i(\theta_n - \theta)} \right) \] (2.4)
of a random unitary matrix $U \in U(N)$ chosen uniformly with respect to Haar measure. The characteristic polynomial $\Lambda_U(\theta)$ was first developed as a model for the Riemann zeta-function by Keating and Snaith \[13\]. In \[8\], building on \[13\], it is argued that for $t \approx T$, $Z_X(\frac{1}{2} + it)$, given by (2.3), can be modeled by $\Lambda_U(\theta)$, where $U \in U(N)$ with
\[ N = \left[ \frac{\log T}{\epsilon^\gamma \log X} \right]. \] (2.5)
We will prove a result about the value distribution and maximal size of $|\Lambda_U(\theta)|$ in Appendix A and use it to conjecture the distribution of large values of $|Z_X|$. 

The largest value of $|\Lambda_U(\theta)|$ is $2^N$, and values near this occur when $U$ is in a small neighborhood of scalar multiples of the identity matrix. If $X = e^{o(\log \log T)}$, this violates the known bound on $|\zeta(\frac{1}{2} + it)|$, so our model for the large values of $|Z_X(\frac{1}{2} + it)|$ must do something more subtle than just take the maximum of $|\Lambda_U|$ over all $U \in U(N)$. 

If $T$ and $X$ are thought of as fixed, then matrices of size $N = \frac{\log T}{e^\gamma \log X}$ should model the zeta-function as long as $T/X^{\epsilon^\gamma} < t < T$. If $X > 2$, say, then up to constants there are $T \log T$ zeros in this interval. Therefore, in order to have the same number of eigenvalues, one needs
\[ M = \frac{T \log T}{N} \approx \exp(e^\gamma N \log X) \log X \] (2.6)
matrices. Thus, one plausible guess for the maximum value of $|Z_X(\frac{1}{2} + it)|$ for $0 < t < T$ is $K = K(M, N)$, where $N$ and $M$ are given in (2.5) and (2.6), respectively, and $K$ is the smallest possible function of $M$ and $N$ such that
\[ \mathbb{P} \left\{ \max_{1 \leq j \leq M} \max_{\theta} |\Lambda_{U_j}(\theta)| \leq K \right\} \to 1 \] (2.7)
as $N \to \infty$. Such a $K$ is found in Theorem 2.1.

We have glossed over some issues here, but we argue that they are not essential. First, we have claimed that matrices of size $N$ model $Z_X(\frac{1}{2} + it)$ for $T/X^{\epsilon^\gamma} < t < T$, whereas we want $0 < t < T$. However, if $X \to \infty$, then $[T/X^{\epsilon^\gamma}, T]$ will cover almost all of $[0, T]$, and so should capture the maximum. Secondly, we have been slightly cavalier in dropping the condition that $N$ should be an integer, which will have an effect on the number of matrices, $M$, we maximize over. However, as we will see below, the answer depends only on the logarithmic size of $M$, so this is not a serious problem. Finally, we remark that the placement of $e^\gamma$ in our definitions of $N$ and $M$ is actually irrelevant: our heuristics are sufficiently robust that increasing $N$ by any fixed constant and decreasing $M$ correspondingly leads to the same conjectured maximum. We include the $e^\gamma$ factor to be consistent with \[8\], where the precise choice of $N$ does matter.

We now find an explicit $K$ satisfying (2.7).

**Theorem 2.1.** Fix $\delta > 0$. Let $M = \exp(N^{\beta})$, with $\delta < \beta < 2 - \delta$, and set
\[ K_\varepsilon(N) = \exp \left( \left( \sqrt{1 - \frac{1}{2} \beta + \varepsilon} \right) \sqrt{\log M \log N} \right). \] (2.8)
If $U_1, \ldots, U_M$ are chosen independently from $U(N)$, then as $N \to \infty$,
\[ \mathbb{P} \left\{ \max_{1 \leq j \leq M} \max_{\theta} |\Lambda_{U_j}(\theta)| \leq K_\varepsilon(N) \right\} \to 1 \] (2.9)
for all $\varepsilon > 0$ and for no $\varepsilon < 0$. 

Proof. Note that by the independence of the $U_j$,
\[
P \left\{ \max_{1 \leq j \leq M} \max_{\theta} |\Lambda_U(\theta)| \leq K_\varepsilon(N) \right\} = P \left\{ \max_{\theta} |\Lambda_U(\theta)| \leq K_\varepsilon(N) \right\}^M
\]
and, for this to tend to 1 as $N \to \infty$, we must have
\[
M \log \left( 1 - P \left\{ \max_{\theta} |\Lambda_U(\theta)| > K_\varepsilon(N) \right\} \right) \to 0.
\]
Thus, the proof of the theorem (and all similar ones in this paper) requires knowledge of the tails of the distribution, and this is given by Lemma A.1. If $\varepsilon > 0$, we have $N \to \infty$ for all $\varepsilon > 0$, but for no $\varepsilon < 0$. \qed

To summarize, we use the characteristic polynomials $\Lambda_U(\theta)$ of random unitary matrices $U \in U(N)$ to model $Z_X(\frac{1}{2} + it)$. To model the large values of $|Z_X(\frac{1}{2} + it)|$ for $t \in [0, T]$ we choose $N$ as in (2.5) with $\log N < (\log N)^A$ for some $A$, and we take about $M = N^c \exp(\varepsilon N \log X)$ different matrices. Here $c \geq 0$ is fixed, and we include it to allay concerns that choosing too few matrices may miss some large values. With these values for $M$ and $N$, it follows that $\beta$ in Theorem 2.1 is $\sim 1$, and this leads to the following conjecture:

**Conjecture D.** If $2 < X < \log^A T$, then
\[
\max_{t \in [0,T]} |Z_X(\frac{1}{2} + it)| = \exp \left( (1 + o(1)) \sqrt{\frac{1}{2} \log T \log \log T} \right)
\]
as $T \to \infty$.

We can now complete our argument for Conjecture A.

**Justification of Conjecture A.** By the prime number theorem and (2.2), we see that
\[
|P_X(\frac{1}{2} + it)| \leq \exp \left( \sum_{n \leq X} \frac{\Lambda(n)}{\sqrt{n \log n}} \right) = O \left( \exp \left( 3 \sqrt{\frac{X}{\log X}} \right) \right).
\]
Thus, if $X = O(\log T)$ and $T/X^{\varepsilon^T} < t < T$, then
\[
P_X(\frac{1}{2} + it) = O \left( \exp \left( C \frac{\log t}{\log \log t} \right) \right).
\]
Combining this, (2.1), and Conjecture A we obtain Conjecture A. \qed

The argument above essentially splits the critical line into blocks of size 1, maximizes over each block, and then finds the maximum of the maxima. However, one might instead wish to sample the critical line at many evenly spaced points. If they are not too sparse, then a value close to the global maximum in $[0, T]$ will be found. The following lemma explains why this is the case.

**Lemma 2.2.** Suppose $|\zeta(\frac{1}{2} + it_0)| = m_T := \max_{t \in [0,T]} |\zeta(\frac{1}{2} + it)|$. There is an absolute constant $A > 0$ such that if $|t - t_0| < A/\log T$, then $|\zeta(\frac{1}{2} + it)| > \frac{1}{2} m_T$. 


Proof. We can estimate the size of the derivative of the zeta-function near \( \frac{1}{2} + it \) using Cauchy’s theorem. If we integrate around a circle of size \( \frac{1}{\log T} \) and use the functional equation, we find that there exists an absolute constant \( c_1 \) such that if \( |s - (\frac{1}{2} + it)| < c_1 / \log T \), then \( \zeta'(s) \ll m_T \log T \). This gives the lemma. \( \square \)

As a random matrix model for \(|Z_X(\frac{1}{2} + it)|\) when it is sampled at evenly spaced points, one might consider the largest value of \( K = K(N, X) \) such that

\[
\mathbb{P} \left\{ \max_{1 \leq j \leq N^c \exp(N \log X)} |\Lambda U_j(0)| \leq K \right\} \to 1. 
\] (2.17)

The following theorem determines \( K \) explicitly as a function of \( N \) and \( X \) and shows that such sampling is sufficient to capture the large values.

**Theorem 2.3.** Let \( M = N^c e^{N \log X} \), where \( 2 < X < N \) and \( c > 0 \) is fixed. If

\[
K_\varepsilon(N) = \exp \left( \frac{1}{\sqrt{2}} + \varepsilon \right) \sqrt{N \log N \log X}, \tag{2.18}
\]

and \( U_1, \ldots, U_M \) are chosen independently from \( U(N) \), then as \( N \to \infty \),

\[
\mathbb{P} \left\{ \max_{1 \leq j \leq M} |\Lambda U_j(0)| \leq K_\varepsilon(N) \right\} \to 1 
\] (2.19)

for all \( \varepsilon > 0 \) and for no \( \varepsilon < 0 \).

Proof. As in the proof of Theorem 2.1, since the \( U_j \) are independent we have

\[
\log \mathbb{P} \left\{ \max_{1 \leq j \leq M} |\Lambda U_j(0)| \leq K_\varepsilon(N) \right\} = M \log \left( 1 - \mathbb{P} \{|\Lambda U(0)| > K_\varepsilon(N)\} \right). \tag{2.20}
\]

Theorem 3.5 of [12] asserts that if \( \delta > 0 \) is fixed and \( \exp(N\delta) \leq K \leq \exp(N^{1-\delta}) \), then

\[
\mathbb{P} \{|\Lambda U(0)| > K\} = \exp \left( -\frac{\log^2 K}{\log N - \log \log K(1 + o(1))} \right) \tag{2.21}
\]

as \( N \to \infty \). Hence, if \( M = N^c \exp(N \log X) \) and \( K_\varepsilon(N) \) is given by (2.18), then the left-hand side of (2.20) tends to zero for all \( \varepsilon > 0 \), but for no \( \varepsilon < 0 \). \( \square \)

Note that the statements of Theorems 2.1 and 2.3 are almost identical and, in particular, one can capture the largest values of \( |\Lambda U| \), hence \( |Z_X| \), just by sampling at individual points; it is not necessary to find the maxima of the individual polynomials. This is significant for our modeling of the prime contribution \( P_X \), for in that case we are only able to sample at individual points, and there is nothing comparable to a sequence of polynomials over which we can maximize individually.

### 2.2. Probabilistic model for large values of \( P_X \)

The material in this section was provided to us by Granville and Soundararajan.

First note that

\[
\log P_X(\frac{1}{2} + it) = \sum_{p \leq X} \frac{1}{p^{\frac{1}{2} + it}} + O \left( \sum_{p \leq \sqrt{X}} \frac{1}{p} \right) \\
= \sum_{p \leq X} \frac{1}{p^{\frac{1}{2} + it}} + O(\log \log X). \tag{2.22}
\]

Hence

\[
P_X(\frac{1}{2} + it) = \exp \left( \sum_{p \leq X} \frac{1}{p^{\frac{1}{2} + it}} \right) \times \exp(O(\log \log X))
\]


\[
\exp \left( P_X \left( \frac{1}{2} + it \right) \right) \times \exp(O(\log \log X)),
\]
(2.23)
say. We will see that this approximation is adequate as long as \( X = \exp(o(\sqrt{\log T \log \log T}) \).

The method is based on treating the \( p^{-it} \) as independent random variables. The large values of \( P_X \left( \frac{1}{2} + it \right) \) can then be obtained from the following lemma, the proof of which involves calculating the moments of the distribution.

**Lemma 2.4.** Let \( \{z_j\} \) be a sequence of independent random variables distributed uniformly on the unit circle and let \( \{a_j\} \) be a sequence of bounded real numbers such that for all \( n \geq 3 \),
\[
\frac{1}{V_J} \sum_{1 \leq j \leq J} a_j^n \to 0
\]
as \( J \to \infty \), where
\[
V_J := \sum_{1 \leq j \leq J} a_j^2.
\]
Then, as \( J \to \infty \), the distribution of
\[
Y_J := \text{Re} \sum_{1 \leq j \leq J} a_j z_j
\]
tends to a Gaussian with mean 0 and variance \( \frac{1}{2} V_J \).

Applying the lemma to \( P_X \left( \frac{1}{2} + it \right) \) with \( z_j = p_j^{-it} \), where \( p_j \) is the \( j \)th prime, and \( a_j = 1/\sqrt{p_j} \), we see that \( V_J \sim \log \log X \). For \( X = \exp(\sqrt{\log T}) \) we model the maximum of \( |P_X \left( \frac{1}{2} + it \right)| \) by independently choosing \( T \log^c T \) values of \( t \). By (2.23) this yields
\[
\max_{t \in [0,T]} |P_X \left( \frac{1}{2} + it \right)| = \exp \left( (1 + o(1)) \sqrt{\frac{1}{2} \log T \log \log T} \right).
\]
(2.27)
If \( X = \exp(\sqrt{\log T}) \), the method of Section 2.1 predicts that
\[
\max_{t \in [0,T]} |Z_X \left( \frac{1}{2} + it \right)| = O \left( \exp \left( \sqrt{\log T} \right) \right).
\]
(2.28)
These estimates together with (2.11) give an alternative justification of Conjecture A.

### 2.3. Combining \( Z_X \) and \( P_X \).

If \( X = \exp(\log^\alpha T) \) with \( 0 < \alpha < \frac{1}{2} \), then the largest values of \( |Z_X| \) and \( |P_X| \) are approximately the same size and both will contribute to the largest values of \( |\zeta(\frac{1}{2} + it)| \). Specifically, applying Theorem 2.1 with \( N = \log T/\log X \) (so that \( Z_X \) is modeled by \( \Lambda_U \)) and \( M = T \log X \) (so that we sample enough characteristic polynomials to cover the critical line between \( t = 0 \) and \( t = T \)), the previous analysis using characteristic polynomials predicts that \( |Z_X \left( \frac{1}{2} + it \right)| \) gets as large as
\[
\exp \left( \frac{1}{\sqrt{2}} \sqrt{(1 - 2\alpha) \log T \log \log T} \right),
\]
(2.29)
and \( |P_X \left( \frac{1}{2} + it \right)| \) gets as large as
\[
\exp \left( \sqrt{\alpha \log T \log \log T} \right).
\]
(2.30)
The product of these two quantities is larger than our conjectured maximum of \( |\zeta(\frac{1}{2} + it)| \), as it should be, because we do not expect \( |Z_X| \) and \( |P_X| \) to attain their maximum values simultaneously. Instead of multiplying the maxima, we must find the distribution of the large values of the product \( |Z_X P_X| \) in order to check that our method is consistent throughout the range \( 0 < \alpha < \frac{1}{2} \). This is a calculation involving the tails of the distributions of \( Z_X \) and \( P_X \).
Thus, Conjecture E.

and this leads to the following conjecture:

\[ x \]

where \( x \) yields the solution \( K \). If we write \( K = \exp \left( d \sqrt{\log T \log \log T} \right) \), then solving

\[ 0 = f_K'(x_0) = \frac{2 \left( x_0 - d \sqrt{\log T \log \log T} \right)}{\alpha \log \log T} + \frac{2x_0}{(1 - \alpha) \log \log T - \log x_0} + \frac{x_0}{(1 - \alpha) \log \log T - \log x_0^2} \]  

yields the solution \( x_0 \sim d(1 - 2\alpha) \sqrt{\log T \log \log T} \) (which is justified so long as \( 0 < \alpha < 1/2 \)). Thus,

\[ f_K(x_0) = (2 + o(1)) d^2 \log T, \]

and this leads to the following conjecture:

**Conjecture E.** For \( d > 0 \) fixed and \( T \to \infty \), we have

\[ \frac{1}{T} \mathrm{meas} \left\{ 0 < t < T : |\zeta\left(\frac{1}{2} + it\right)| > \exp(d \sqrt{\log T \log \log T}) \right\} = \exp \left( -2d^2 \log T(1 + o(1)) \right). \]

By Lemma 2.2, \( |\zeta\left(\frac{1}{2} + it\right)| \) is close to its maximum value over a window of size \( C/\log T \), so we wish to find the smallest \( d \) such that

\[ \mathrm{meas} \left\{ 0 < t < T : |\zeta\left(\frac{1}{2} + it\right)| > \exp(d \sqrt{\log T \log \log T}) \right\} \ll \frac{1}{\log T}, \]

that is, such that \( T \log T \exp(-2d^2(1 + o(1)) \log T) \ll 1 \). This happens if \( d = \sqrt{\frac{1}{2} + \varepsilon} \) for any \( \varepsilon > 0 \), but for no \( \varepsilon < 0 \). Once more this gives Conjecture A and this time in a way that is independent of the choice of \( X = \exp(\log^\alpha T) \) for \( 0 < \alpha < 1/2 \).

**3. Bounds based on conjectures for moments**

In this section we obtain Conjecture A by using conjectures for moments of the zeta-function. Our method also leads to limits on the possible uniformity of the conjectured moments.

Our approach here is based on the work of Conrey and Gonek [4]. Let

\[ m_T := \max_{t \in [0, T]} |\zeta\left(\frac{1}{2} + it\right)|, \]
and note that we have the trivial inequality
\[ m_T^{2k} \geq \left( \frac{1}{T} \int_0^T |\zeta(\frac{1}{2} + it)|^{2k} \, dt \right). \] (3.2)

It follows that estimates for the right-hand side of the inequality imply lower bounds for the maximum size of the zeta-function.

Keating and Snaith [13] used random matrix theory to conjecture that if \( k > -1/2 \) is fixed, then as \( T \to \infty \),
\[ \frac{1}{T} \int_0^T |\zeta(\frac{1}{2} + it)|^{2k} \, dt \sim \frac{G^2(k + 1)}{G(2k + 1)} a(k) \log^{k^2} T, \] (3.3)
where \( G \) is the Barnes \( G \)-function, and
\[ a(k) = \prod_{p \text{ prime}} \left( 1 - \frac{1}{p} \right)^{k^2} \sum_{m=0}^{\infty} \left( \frac{\Gamma(m + k)}{m! \Gamma(k)} \right)^2 p^{-m}. \] (3.4)

This conjecture is for \( k \) fixed, but we would like to let \( k \to \infty \), because the \( 2k \)th root of the right-hand side of (3.2) then actually tends to \( m_T \). Thus, we would like to know how large \( k \) can be as a function of \( T \).

Conrey and Gonek showed that if formula (3.3) holds for \( k = \sqrt{\log T / \log \log T} \) then
\[ m_T \geq \exp \left( C_1 \sqrt{\frac{\log T}{\log \log T}} \right), \] (3.5)
and if it holds for \( k \) as large as \( \log T / \log \log T \) then
\[ m_T \geq \exp \left( C_2 \frac{\log T}{\log \log T} \right), \] (3.6)
where \( C_1 \) and \( C_2 \) are given explicitly. Hughes [11] gave a convexity argument to show that formula (3.3) must fail before \( k = \log \delta T / \log \log T \). However, using the last point at which convexity holds for (3.3), one still obtains (3.6), but with a smaller constant \( C_2 \).

In all of these cases one only requires a lower bound for the right-hand side of (3.3). Our approach here is to use the mean value formula (3.3) to obtain upper bounds instead of lower bounds for \( m_T \). As a consequence, we also obtain restrictions on the possible range of validity of (3.3) for \( k \) growing with \( T \). Specifically we prove the following:

**Theorem 3.1.** Formula (3.3) does not hold for
\[ k \geq (2 \sqrt{2} + \varepsilon) \sqrt{\frac{\log T}{\log \log T}} \] (3.7)
for any fixed \( \varepsilon > 0 \).

Our method allows us to get upper bounds for \( m_T \) and, in particular, we obtain

**Theorem 3.2.** If formula (3.3) holds for \( k = \log^\delta T \) for some \( \delta < \frac{1}{2} \), then
\[ m_T \ll \exp \left( \log^{1-\delta} T \right). \] (3.8)
Moreover, if formula (3.3) holds for \( k = \sqrt{2 \log T / \log \log T} \), then
\[ m_T \ll \exp \left( \sqrt{\frac{1}{2} \log T \log \log T + O \left( \sqrt{\frac{\log T \log \log \log T}{\log \log T}} \right)} \right), \] (3.9)
and if formula (3.3) holds for \( k = \sqrt{8 \log T / \log \log T} \), then
\[
m_T \gg \exp \left( \sqrt{\frac{1}{2} \log T \log \log T} + O \left( \frac{\log \log \log T}{\log \log T} \right) \right),
\] (3.10)

Theorem 3.2 says that if formula (3.3) holds true until \( k = \sqrt{8 \log T / \log \log T} \) (after which we know it must fail), then we have
\[
m_T = \exp \left( \sqrt{\frac{1}{2} \log T \log \log T} \left( 1 + O \left( \frac{\log \log \log T}{\log \log T} \right) \right) \right)
\] (3.11)

Note that this implies the conjecture found in the previous section. This is not surprising because, as we will show, the arithmetic factor in the conjectured moments is smaller than the other factors. Thus, this bound is coming just from the random matrix model.

If the true order of the zeta-function is larger than the bound in (3.11), then one would like to know where our calculation fails. Since formula (3.3) is only the leading order term in the asymptotic expansion for the 2th moment of the zeta-function, it is possible that the lower order terms dominate when \( k = \log^\delta T \). However, this is unlikely. In the random matrix case, the 2th moment is given by
\[
\mathbb{E} \left\{ |A_U(\theta)|^{2k} \right\} = \frac{G^2(k+1) G(N+1)G(N+2k+1)}{G^2(N+k+1)}
\] (3.12)
\[
= \frac{G^2(k+1)}{G(2k+1)} \exp \left( k^2 \log N + \frac{k^3}{N} - \frac{7k^4}{12N^2} + \frac{k^5}{2N^3} + \ldots \right),
\] (3.13)
and one sees that the first term dominates even for \( k \) as large as \( N^\delta T \) provided that \( \delta < 1 \).

In the zeta-function case, the complete main term of the 2th moment has been conjectured (see [5], Conjecture 1.5.1). One can check that the contribution from the primes is bounded by \( \exp(ck^2) \), which is insufficient to affect the estimate for \( m_T \). Thus, if our conjecture for the growth of \( |\zeta(1/2 + it)| \) is incorrect, then the main term in the mean value must take a new form for \( k = \log^\delta T \) for some \( \delta < \frac{1}{2} \). If \( (1.1) \) is the true maximal size, then by equation (3.20), the conjectured mean value can only hold for \( k \ll \log \log T \).

Our main tool for finding upper bounds is the following lemma.

**Lemma 3.3.** For all positive real \( k \) we have
\[
m_T^{2k} \ll 2^{2k} \log T \int_0^T |\zeta(1/2 + it)|^{2k} dt,
\] (3.14)
where the implied constant is absolute.

**Proof.** Suppose that \( |\zeta(1/2 + it_0)| = m_T \), where \( 0 < t_0 < T \). By Lemma 2.2, there is an absolute constant \( A > 0 \) such that if \( t - t_0 \leq A/\log T \) then \( |\zeta(1/2 + it)| \geq \frac{1}{2} m_T \). This gives
\[
\int_0^T |\zeta(1/2 + it)|^{2k} dt \geq \int_{t_0 - A/\log T}^{t_0 + A/\log T} |\zeta(1/2 + it)|^{2k} dt
\begin{align*}
&\geq \frac{A}{\log T} \left( \frac{m_T}{2} \right)^{2k} ,
\end{align*}
(3.15)
as claimed. \( \square \)

**Proof of Theorems 3.1 and 3.2**. By Lemma 3.3 and (3.2), there exists an absolute constant \( C \) such that
\[
\left( \frac{1}{T} \int_0^T |\zeta(1/2 + it)|^{2k} dt \right)^{1/2k} \leq m_T \leq 2(C T \log T)^{1/2t} \left( \frac{1}{T} \int_0^T |\zeta(1/2 + it)|^{2t} dt \right)^{1/2t}.
\] (3.16)
We use these inequalities to prove Theorem 3.2 first.
The asymptotic expansion for the Barnes G-function (see Barnes [3]) gives
\[
\frac{G(1+k)^2}{G(1+2k)} = \exp\left(k^2 \left(-\log k + \frac{3}{2} - 2 \log 2\right) - \frac{1}{12} \log k + \frac{1}{12} \log 2 + \zeta'(-1) + O\left(\frac{1}{k}\right)\right)
\] (3.17)
for \( k \geq 1 \). Furthermore, Conrey and Gonek [6] have shown that
\[
\log a(k) \sim -k^2 \log(2e^\gamma \log k) + o(k^2) \quad \text{for} \quad k \to \infty.
\] (3.18)
Thus, if (3.3) holds, then
\[
\log \left(\frac{1}{T} \int_0^T |\zeta(\frac{1}{2} + it)|^{2\ell} \, dt\right)^{1/2\ell} = \frac{1}{2} \ell \log \log T - \frac{1}{2} \ell \log \ell + O(\ell \log \log \ell).
\] (3.19)
It therefore follows from (3.16) that
\[
\log m_T \ll \frac{\log T + \log \log T}{2\ell} + \frac{1}{2} \ell \log \log T - \frac{1}{2} \ell \log \ell + O(\ell \log \log \ell).
\] (3.20)
Setting \( \ell = \log^\delta T \), we find that
\[
m_T \ll \exp\left(\frac{1}{2} \log^{1-\delta} T + O(\log^\delta T \log \log T)\right),
\] (3.21)
which gives the first inequality in Theorem 3.2. Setting \( \ell = c \sqrt{\frac{\log T}{\log \log T}} \), we find that
\[
m_T \ll \exp\left(\left(\frac{1}{2c} + \frac{c}{4}\right) \sqrt{\log T \log \log T} + O\left(\frac{\sqrt{\log T \log \log T}}{\sqrt{\log \log T}}\right)\right),
\] (3.22)
which is minimized by taking \( c = \sqrt{2} \). This gives the second estimate in Theorem 3.2.

By (3.16) and (3.19) we also have
\[
m_T \gg \exp\left(\frac{1}{2} k \log \log T - \frac{1}{2} k \log k + O(k \log \log k)\right).
\] (3.23)
If we set \( k = c \sqrt{\frac{\log T}{\log \log T}} \), we find that
\[
m_T \gg \exp\left(\frac{c}{4} \sqrt{\log T \log \log T} + O\left(\frac{\sqrt{\log T \log \log T}}{\sqrt{\log \log T}}\right)\right).
\] (3.24)
Choosing \( c = 2 \sqrt{2} \), we obtain the third inequality in Theorem 3.2. If \( c > 2 \sqrt{2} \), this contradicts (3.22) and thereby establishes Theorem 3.1. □

4. Bounds for \( S(t) \)

Recall that \( S(t) \) is the error term in the counting function for the number of non-trivial zeros of the zeta-function with imaginary part less than \( t \). It may also be expressed as \( S(t) = \frac{1}{\pi} \Im \log \zeta\left(\frac{1}{2} + it\right) \) (see Titchmarsh [19]). Since \( \zeta\left(\frac{1}{2} + it\right) \) is essentially \( P_X\left(\frac{1}{2} + it\right)Z_X\left(\frac{1}{2} + it\right) \) and \( Z_X \) is modeled by \( \Lambda_U(\theta) \), one would expect that if \( X \) is sufficiently small, so that the contribution from \( P_X \) is negligible, then \( \frac{1}{\pi} \Im \log \zeta\left(\frac{1}{2} + it\right) \) too can be modeled by random matrix theory, in particular, by
\[
\frac{1}{\pi} \Im \log \Lambda_U(0).
\] (4.1)
Evidence for this is presented in [13]. This is the basis for our Conjecture [13].
Theorem 4.1. Set
\[ K_\varepsilon(N) = \left( \frac{1}{\sqrt{2}} + \varepsilon \right) \sqrt{N \log N} \]  
and \( M = N^c e^N \), where \( c > 0 \) is fixed. If \( U_1, \ldots, U_M \) are chosen independently from \( U(N) \), then as \( N \to \infty \),
\[ P \left\{ \max_{1 \leq j \leq M} \Im \log \Lambda_{U_j}(0) \leq K_\varepsilon(N) \right\} \to 1 \]  
for all \( \varepsilon > 0 \) and for no \( \varepsilon < 0 \).

Proof. This follows along the same lines as the proof of Theorem 2.3. Since we are making independent choices,
\[ P \left\{ \max_{1 \leq j \leq M} \Im \log \Lambda_{U_j}(0) \leq K \right\} = P \{ \Im \log \Lambda_U(0) \leq K \}^M. \]  
For this to tend to 1, we need
\[ M \log P \{ \Im \log \Lambda_U(0) \leq K \} \to 0. \]  
By Theorem 3.6 of Hughes, Keating and O’Connell \[12\], if \( K = N^\lambda \), where \( \delta < \lambda < 1 - \delta \) and \( \delta > 0 \) is fixed, then
\[ P \{ \Im \log \Lambda_U(0) \leq K \} = 1 - \exp \left( -\frac{K^2}{\log N - \log K} (1 + o_\delta(1)) \right). \]  
One can easily check that if \( K_\varepsilon(N) \) is given by (4.2), then (4.5) holds for all \( \varepsilon > 0 \), but for no \( \varepsilon < 0 \). \( \square \)

Conjecture B now follows in the same manner as the justification of Conjecture A; that is, by controlling the prime contribution from \( \Im \log P_X(\frac{1}{2} + it) \).

5. Other families and other arguments

5.1. Other families: symplectic and orthogonal. The analogue of Gonek, Hughes, and Keating’s approximation to the zeta-function has not yet been extended to the case of other \( L \)-functions near the critical point. However, it is believed that the characteristic polynomials of symplectic (or orthogonal) matrices model the central value of \( L \)-functions taken from a symplectic (or orthogonal) family of \( L \)-functions \[14\]. Thus, the methods developed in section 2.1 can be applied. Moreover, we can still estimate the maximal size of critical values by using a partial Euler product and modifying the method of Granville and Soundararajan. Finally, we can also apply the method involving mean values.

We give as an example finding the large values of the characteristic polynomials of the symplectic group at the critical point. The orthogonal family is treated in an almost identical way. The characteristic polynomial of an \( N \times N \) symplectic matrix (\( N \) must be even) with eigenvalues \( e^{\pm i \theta_n} \) is
\[ Z(U,0) = \prod_{j=1}^{N/2} (1 - e^{i \theta_n})(1 - e^{-i \theta_n}). \]  
Keating and Snaith \[14\] calculated the moment generating function and found that
\[ E_{Sp(N)} \{ Z(U,0)^s \} = 2^{Ns} \prod_{j=1}^{N/2} \frac{\Gamma(N/2 + j + 1) \Gamma(s + j + 1/2)}{\Gamma(j + 1/2) \Gamma(s + N/2 + j + 1)}. \]
A long but straightforward calculation using Stirling’s asymptotic series for the gamma function shows that if \( \delta > 0 \) is fixed and \( \delta < \lambda < 1 - \delta \), then for \( A(N) = N^\lambda \), \( B(N) = \frac{N^{2\lambda}}{1 - \lambda} \log N \), and fixed \( s \geq 0 \), we have
\[
\lim_{N \to \infty} \frac{1}{B} \log \mathbb{E}_{Sp(N)} \left\{ Z(U, 0)^{sB/A} \right\} = \frac{1}{2} s^2. 
\] (5.3)

From this, large deviation theory (for example, see [7]) allows us to deduce that if \( \exp(N^\delta) \leq K \leq \exp(N^{1-\delta}) \), then
\[
\mathbb{P}_{Sp(N)} \left\{ Z(U, 0) > K \right\} = \exp \left( -\frac{\log^2 K}{2 \log N - 2 \log \log K} (1 + o_3(1)) \right) 
\] as \( N \to \infty \). Comparing this with (5.21), the analogous statement for the unitary group, we note the extra factor of 2 in the denominator. This difference explains why the constant \( B \) in Conjecture 5.1 equals 1 rather than 1/2.

We now see, by methods identical to those of the previous section, that if \( M = N^c e^N \) for any fixed \( c \geq 0 \), and if \( K_\varepsilon = \exp \left( (1 + \varepsilon) \sqrt{N \log N} \right) \), then if \( U_1, \ldots, U_M \) are chosen independently from \( Sp(N) \),
\[
\mathbb{P}_{Sp(N)} \left\{ \max_{1 \leq j \leq M} Z(U_j, 0) \leq K_\varepsilon(N) \right\} \to 1 
\] (5.5)
as \( N \to \infty \) for all \( \varepsilon > 0 \) and for no \( \varepsilon < 0 \).

Consider for instance the family of all quadratic Dirichlet \( L \)-functions \( L(s, \chi_d) \). For characters with modulus around \( D \), random matrix theory suggests (see Keating and Snaith [14]) that \( N = \log D \) is the correct identification between the size of the matrix and the conductor (though note that in [14] \( N \) is half the size of the symplectic matrix). Furthermore, it is well known that there are asymptotically \( 6D/\pi^2 \) primitive discriminants less than \( D \). Thus, we conjecture that
\[
\max_{|d| \leq D} |L(\frac{1}{2}, \chi_d)| = \exp \left( (1 + o(1)) \sqrt{\log D \log \log D} \right). 
\] (5.6)

Similarly, the moment generating function has been calculated for the orthogonal case (see [14] and, if \( N \) is even, we have
\[
\mathbb{E}_{SO(N)} \left\{ Z(U, 0)^s \right\} = 2^{Ns} \prod_{j=1}^{N/2} \frac{\Gamma(N/2 + j - 1) \Gamma(s + j - 1/2)}{\Gamma(j - 1/2) \Gamma(s + N/2 + j - 1)}. 
\] (5.7)

Equations (5.3) and (5.4) apply to the orthogonal case without change, so by the same reasoning as previously, if \( M = N^c e^N \) for any fixed \( c \geq 0 \), and \( K_\varepsilon = \exp \left( (1 + \varepsilon) \sqrt{N \log N} \right) \), then
\[
\mathbb{P}_{SO(N)} \left\{ \max_{1 \leq j \leq M} Z(U_j, 0) \leq K_\varepsilon(N) \right\} \to 1 
\] as \( N \to \infty \) for all \( \varepsilon > 0 \), but for no \( \varepsilon < 0 \).

Next we consider how to adapt the Granville-Soundararajan argument involving the product over primes to the symplectic case. We require the following lemma.

**Lemma 5.1.** Let \( \{x_j\} \) be a sequence of independent real random variables with mean 0 and variance 1, and let \( \{a_j\} \) be a bounded sequence of real numbers such that for all \( n \geq 3 \),
\[
\frac{1}{V_J^2} \sum_{1 \leq j \leq J} a_j^n \to 0 
\] as \( J \to \infty \), where
\[
V_J := \sum_{1 \leq j \leq J} a_j^2. \] (5.10)
Then as $J \to \infty$, the distribution of

$$Y_J := \sum_{1 \leq j \leq J} a_j x_j$$

(5.11)
tends to a Gaussian with mean 0 and variance $V_J$.

Just as in the treatment involving characteristic polynomials, the fact that the variance is $V_J$ for these families instead of $V_J/2$ leads to the constant $B = 1$ in Conjecture C, instead of $B = 1/2$ for the unitary family dealt with previously.

5.2. Other arguments, for. Although the conjectures in this paper are based on very recent work, Hugh Montgomery has pointed out to us that a similar conjecture can be obtained by viewing $\log |\zeta(\frac{1}{2} + i t)|$ as a Gaussian distributed random variable with variance $C \log \log T$, where one estimates $m_T$ by sampling $T^A$ times.

Soundararajan suggests a different way to use the moments of the zeta-function to conjecture an upper bound. The proof of Lemma 3.3 showed that if

$$\frac{1}{T} \text{meas}\{t \in [0, T] : |\zeta(\frac{1}{2} + i t)| \geq \tau\} \leq \frac{1}{T},$$

(5.12)
then $m_T \leq 2\tau$. For when $|\zeta(\frac{1}{2} + i t)|$ is very large, it must remain large over an interval of size $c/\log T$. Now

$$\tau^{2k} \text{meas}\{t \in [0, T] : |\zeta(\frac{1}{2} + i t)| \geq \tau\} \leq \int_0^T |\zeta(\frac{1}{2} + i t)|^{2k} \, dt,$$

(5.13)
so if equation (3.3) holds, then we have

$$\frac{1}{T} \text{meas}\{t \in [0, T] : |\zeta(\frac{1}{2} + i t)| \geq \tau\} \leq \tau^{-2k} \frac{G^2(k+1)}{G(2k+1)} (\log T)^{k^2}.$$

(5.14)
The right-hand side is less than $c/(T \log T)$ (which means there is only one place where the maximum occurs) when

$$\tau \geq \exp\left(\frac{\log T}{2k} + \frac{k}{2} \log \log T - \frac{k}{2} \log k\right).$$

(5.15)
The minimum of this is $\exp\left(\sqrt{\frac{1}{2} \log T \log \log T}\right)$ and it occurs when $k = \sqrt{2 \log T / \log \log T}$.

5.3. Other arguments, against. We now discuss potential arguments against the conjectures in this paper. One possibility, so fundamental that it cannot be addressed, is that the large values of an $L$-function may be so rare that these statistical models cannot detect them. Indeed, since these are problems in number theory, there may be number-theoretic constructions of large values which contradict our conjectures. The two examples below, due to Brian Conrey, suggest the kinds of things we have in mind.

The first argument invokes an analogy with the divisor function $d(n) = \sum_{d|n} 1$ and the related function $\omega(n) = \sum_{p|n} 1$, where here the sum is over prime divisors of $n$. Since $\omega(n)$ is $\log \log n$ on average and has a Gaussian distribution, the relation $d(n) = 2^{\omega(n)}$ for square-free $n$ might lead one to conjecture that for such $n$, $d(n)$ is bounded by

$$\exp(c \sqrt{\log n / \log \log n}).$$

(5.16)
However, we know how to construct large values of $d(n)$, and it is easy to see that for $n$ squarefree, $d(n)$ can get as large as

$$\exp(c \log n / \log \log n).$$

(5.17)
The second argument concerns the Fourier coefficients $a_n$ of cusp forms. For integer weight cusp forms, rescaled so that for $p$ prime we have $|a_p| \leq 2$, the coefficients $a_n$ can get as large as

$$\exp(c \log n / \log \log n).$$

(5.18)
In other words, they can get about as large as d(n). The question is: can the coefficients of half-integral weight forms also get this large? If they can, then our conjecture on the maximal size of the critical values of a symplectic family of L-functions is incorrect. For if \( f \in S_k(\Gamma_0(N)) \), then \( L_f(\frac{1}{2}, \chi_d) = c_d^2/\sqrt{d} \), where \( c_d \) is a Fourier coefficient of the half-integral weight form associated with \( f \) by the Shimura correspondence.

Our methods cannot be directly applied to produce a version of Conjecture [14] for symplectic and orthogonal families. But if one were to believe that for a family \( F \) of L-functions with \( c(F) \) denoting the conductor of \( F \in \mathcal{F} \),

\[
\limsup_{c(F) \to \infty} \frac{3m \log F(\frac{1}{2})}{\log \log c(F)} = \sqrt{B}
\]  
(5.19)

where \( B = 1/2 \) for unitary families and \( B = 1 \) for symplectic and orthogonal families, then since the rank of an elliptic curve is related to the order of vanishing of its associated L-function, this could lead to new information about large ranks. That is, if (5.19) is true, then it suggests that for rational elliptic curves we have

\[
\limsup_{c_E \to \infty} \frac{\text{rank}(E)}{\sqrt{\log c_E \log \log c_E}} = 1,
\]  
(5.20)

where \( c_E \) is the conductor of \( E \). Note that this is smaller than the ranks of elliptic curves found by Ulmer [20] in the function field case.

**Appendix A. The Tail of the Distribution of \( \max_\theta |\Lambda_U(\theta)| \)**

Here we prove the random matrix polynomial result used in Section 2.

**Lemma A.1.** If \( \delta > 0 \) is fixed and \( \delta \leq \lambda \leq 1 - \delta \), then

\[
\mathbb{P} \left\{ \max_\theta |\Lambda_U(\theta)| \geq \exp(N^\lambda) \right\} = \exp \left( - \frac{N^{2\lambda}}{(1 - \lambda) \log N} (1 + o(1)) \right). \tag{A.1}
\]

**Proof of Lemma A.1.** Bernstein’s inequality for polynomials implies that for any matrix \( U \),

\[
\max_\theta |\Lambda_U(\theta)| \leq N \max_\theta |\Lambda_U(\theta)|. \tag{A.2}
\]

Thus, if \( \phi \) is a point at which the maximum of \( |\Lambda_U(\theta)| \) occurs, and if we indicate the maximum by \( m_U \), then for \( |\theta - \phi| \leq 1/N \),

\[
|\Lambda_U(\theta)| \geq m_U - |\theta - \phi|N m_U. \tag{A.3}
\]

It follows that

\[
\int_0^{2\pi} |\Lambda_U(\theta)|^{2k} \, d\theta \geq m_U^{2k} \int_{-1/N}^{1/N} (1 - |x|)^{2k} \, dx
\]  
(\ref{eq:4})

\[
= m_U^{2k} \frac{2}{N^{2k+1}}. \tag{A.4}
\]

Combining this with the trivial lower bound for \( m_U \), we find that

\[
\frac{1}{2\pi} \int_0^{2\pi} |\Lambda_U(\theta)|^{2k} \, d\theta \leq m_U^{2k} \leq \frac{2k + 1}{2} N \int_0^{2\pi} |\Lambda_U(\theta)|^{2k} \, d\theta. \tag{A.5}
\]

This bound holds for any matrix. We now average over all \( N \times N \) unitary matrices with respect to Haar measure. That is, we calculate the expectation \( E_N \) of \( |\Lambda_U(\theta)|^{2k} \). Set

\[
E_N \left\{ |\Lambda_U(\theta)|^{2k} \right\} = M_N(2k). \tag{A.6}
\]

Keating and Snaith [13] have shown that

\[
M_N(2k) = \frac{G^2(k + 1) G(1 + N)G(1 + N + 2k)}{G(2k + 1) G^2(1 + N + k)}, \tag{A.7}
\]
where \( G \) is the Barnes \( G \)-function. Note that this is independent of \( \theta \). Therefore, by (A.6)

\[
M_N(2k) \leq \mathbb{E}\{m_B^2\} \leq \pi(2k + 1)NM_N(2k).
\] (A.9)

Hughes, Keating and O’Connell [12] have shown that if \( A(N) = N^\lambda \) with \( \delta < \lambda < 1 - \delta \) and \( \delta > 0 \) fixed, and if

\[
B(N) = \frac{N^{2\lambda}}{(1 - \lambda) \log N},
\] (A.10)

then for \( s \geq 0 \),

\[
\lim_{N \to \infty} \frac{1}{B(N)} \log M_N^{\frac{sB(N)}{A(N)}} = \frac{1}{4}s^2.
\] (A.11)

Since

\[
\frac{1}{B(N)} \log M_N^{\frac{sB(N)}{A(N)}} \leq \frac{1}{B(N)} \log \mathbb{E}\{m_U^{sB(N)/A(N)}\}
\]

\leq \frac{1}{B(N)} \log M_N^{sB(N)/A(N)} + O\left(\frac{(\log N)^2}{N^{2\lambda}}\right),
\] (A.12)

we conclude that for \( s \geq 0 \),

\[
\lim_{N \to \infty} \frac{1}{B(N)} \log \mathbb{E}\left\{\exp\left(\frac{sB(N)\log\max_\theta |A_U(\theta)|}{A(N)}\right)\right\} = \frac{1}{4}s^2.
\] (A.13)

From this, large deviation theory (see, for example, [2]) allows us to deduce that

\[
\lim_{N \to \infty} \frac{1}{B(N)} \log \mathbb{P}\left\{\log \max_\theta |A_U(\theta)| \geq A(N)\right\} = -1.
\] (A.14)

Inserting \( A(N) = N^\lambda \) and \( B(N) \) from (A.10), we obtain the statement in the lemma. \( \square \)

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