Maximal Function Characterizations of Hardy Spaces on $\mathbb{R}^n$ with Pointwise Variable Anisotropy

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Abstract In 2011, Dekel et al. developed highly geometric Hardy spaces $H^p(\Theta)$, for the full range $0 < p \leq 1$, which are constructed by continuous multi-level ellipsoid covers $\Theta$ of $\mathbb{R}^n$ with high anisotropy in the sense that the ellipsoids can change shape rapidly from point to point and from level to level. In this article, if the cover $\Theta$ is pointwise continuous, then the authors further obtain some real-variable characterizations of $H^p(\Theta)$ in terms of the radial, the non-tangential and the tangential maximal functions, which generalize the known results on the anisotropic Hardy spaces of Bownik.

1 Introduction

As a generalization of the classical isotropic Hardy spaces $H^p(\mathbb{R}^n)$ [10], anisotropic Hardy spaces $H^p_A(\mathbb{R}^n)$ were introduced and investigated by Bownik [3] in 2003. These spaces were defined on $\mathbb{R}^n$ associated with a fixed expansive matrix which act on ellipsoid instead of the Euclidean balls. In [1, 2, 5, 11, 14, 15], many authors also studied Bownik’s anisotropic Hardy spaces. In 2011, Dekel et al. further [9] generalized Bownik’s spaces by constructing Hardy spaces with pointwise variable anisotropy $H^p(\Theta)$, $0 < p \leq 1$, associated with an ellipsoid cover $\Theta$. The anisotropy in Bownik’s Hardy spaces is the same one in each point in $\mathbb{R}^n$, while the anisotropy in $H^p(\Theta)$ can change rapidly from point to point and from level to level. Moreover, the ellipsoid cover $\Theta$ is a very general setting which includes the classical isotropic setting, non-isotropic setting of Calderón and Torchinsky [6] and the anisotropic setting of Bownik [3] as special cases, see more details in [3, pp.2-3] and [8, p. 157].

On the other hand, maximal function characterizations are very fundamental characterizations of Hardy spaces and they are crucial to conveniently apply the real-variable theory of Hardy spaces $H^p(\mathbb{R}^n)$ with $p \in (0,1]$. Maximal functions characterizations was first shown for the classical isotropic Hardy spaces $H^p(\mathbb{R}^n)$ by Fefferman and Stein in their fundamental work [10], [12, Chapter III]. Analogous results were shown by Calderón and Torchinsky [6, 7] for parabolic $H^p$ spaces and Uchiyama [13] for $H^p$ on the space of homogeneous type. In 2003, Bownik [3, p.42] obtained the maximal function characterizations of the anisotropic Hardy space $H^p_A(\mathbb{R}^n)$. Motivated by the above mentioned facts,
A natural question arises: Do anisotropic Hardy spaces $H^p(\Theta)$ have maximal function characterizations? In this article, we shall answer the problem affirmatively.

This article is organized as follows.

In Section 2, we recall some notation and definitions concerning anisotropic continuous ellipsoid cover $\Theta$, several maximal functions, anisotropic Hardy spaces $H^p(\Theta)$ defined via the grand radial maximal function. We also give some propositions about $H^p(\Theta)$, several classes of variable anisotropic maximal functions and Schwartz functions since they provide tools for further work. In Section 3, we first state main result: if ellipsoid cover $\Theta$ is pointwise continuous (see Definition 2.1), we may obtain some real-variable characterizations of $H^p(\Theta)$ in terms of the radial, the non-tangential and the tangential maximal functions (see Theorem 3.1). Then we present several lemmas which are isotropic extensions in the setting of variable anisotropy and finally we show the proof of main result.

It is worth pointing out that the pointwise continuity for cover $\Theta$ is added, and this makes the ellipsoids impossible to change rapidly from point to point. However, in the process of proving main result, this assumption is necessary for us to obtain the following result:

$$\{ x \in \mathbb{R}^n : F^* t_0(x) > \lambda \} \text{ is open for any } \lambda > 0 \text{ (see Proposition 2.12).}$$

Moreover, to obtain the atomic decompositions of $H^p(\Theta)$, Dekel, Petrushev and Weissblat also use the following result without proof (see [9, p. 1080]):

$$\{ x \in \mathbb{R}^n : M^0 f(x) > \lambda \} \text{ is open for any } \lambda > 0, \quad (1.1)$$

where $M^0 f$ is the grand radial maximal operator of distribution $f$ in $H^p(\Theta) \cap L^1(\mathbb{R}^n)$. We can only cover the gap under the pointwise continuity for cover $\Theta$ (see Proposition 2.6) as well. Furthermore, the pointwise continuity of $\Theta$ is also a mute assumption which holds automatically in the classical isotropic setting, non-isotropic setting of Calderón and Torchinsky [6] and the anisotropic setting of Bownik [3]; see more details in [8, p. 157, Examples].

Finally, we make some conventions on notation. Let $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$. For any $\alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, $|\alpha| := \alpha_1 + \cdots + \alpha_n$ and $\partial^\alpha := \left( \frac{\partial}{\partial x_1} \right)^{\alpha_1} \cdots \left( \frac{\partial}{\partial x_n} \right)^{\alpha_n}$. Throughout the whole paper, we denote by $C$ a positive constant which is independent of the main parameters, but it may vary from line to line. For any sets $E, F \subset \mathbb{R}^n$, we use $E^c$ to denote the set $\mathbb{R}^n \setminus E$. If there are no special instructions, any space $X(\mathbb{R}^n)$ is denoted simply by $X$. Denote by $S$ the space of all Schwartz functions, $S'$ the space of all tempered distributions.

2 Preliminary and Some Basic Propositions

In this section, we first recall the notion of continuous ellipsoid covers $\Theta$ and we introduce the pointwise continuity for $\Theta$. An ellipsoid $\xi$ in $\mathbb{R}^n$ is an image of the Euclidean unit ball $B^n := \{ x \in \mathbb{R}^n : |x| < 1 \}$ under an affine transform, i.e.

$$\xi := M_\xi(B^n) + c_\xi,$$
where $M_{\xi}$ is a nonsingular matrix and $c_{\xi} \in \mathbb{R}^n$ is the center.

Let us begin with the definition of continuous ellipsoid covers, which was introduced in [8, Definition 2.4].

**Definition 2.1.** We say that

$$\Theta := \{\theta(x, t) : x \in \mathbb{R}^n, t \in \mathbb{R}\}$$

is a **continuous ellipsoid cover** of $\mathbb{R}^n$, or shortly an **ellipsoid cover**, if there exist positive constants $p(\Theta) := \{a_1, \ldots, a_6\}$, such that:

(i) For every $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$, there exists an ellipsoid $\theta(x, t) := M_{x,t}(B_n) + x$ satisfying

$$a_1 2^{-t} \leq |\theta(x, t)| \leq a_2 2^{-t}. \quad (2.1)$$

(ii) Intersecting ellipsoids from $\Theta$ satisfy a “shape condition”, i.e., for any $x, y \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $s \geq 0$, if $\theta(x, t) \cap \theta(y, t + s) \neq \emptyset$, then

$$a_3 2^{-a_4 s} \leq \frac{1}{\| (M_{y,t+s}^{-1} M_{x,t}^{-1}) \|} \leq \| (M_{x,t})^{-1} M_{y,t+s} \| \leq a_5 2^{-a_6 s}. \quad (2.2)$$

Here, $\| \cdot \|$ is the matrix norm given by $\| M \| := \max_{|x| = 1} |Mx|$ for an $n \times n$ real matrix $M$.

Moreover, for ellipsoid cover $\Theta$, we say $\Theta$ is **pointwise continuous**, if for any $x, x' \in \mathbb{R}^n$ and $t \in \mathbb{R}$,

$$\| M_{x',t} - M_{x,t} \| \to 0 \quad \text{as} \quad x' \to x. \quad (2.3)$$

**Remark 2.2.** Here we remark that the pointwise continuity of $\Theta$ guarantees that, for any $x, x' \in \mathbb{R}^n$ and $t \in \mathbb{R}$,

$$\theta(x', t) \to \theta(x, t) \quad \text{as} \quad x' \to x,$$

i.e., for any $y \in \mathbb{B}^n$, $x' + M_{x',t} y \to x + M_{x,t} y$ as $x' \to x$.

Taking $M_{y,t+s} = M_{x,t}$ in (2.2), we have

$$a_3 \leq 1 \quad \text{and} \quad a_5 \geq 1. \quad (2.4)$$

More properties about ellipsoid covers, see [8, 9].

For any $N, \bar{N} \in \mathbb{N}_0$ with $N \leq \bar{N}$, let

$$S_{N, \bar{N}} := \left\{ \psi \in S : \| \psi \|_{S_{N, \bar{N}}} := \max_{\alpha \in \mathbb{N}_0^n, |\alpha| \leq N \bar{N}} \sup_{y \in \mathbb{R}^n} (1 + |y|)^{\bar{N}} |\partial^\alpha \psi(y)| \leq 1 \right\}$$

Let $\Theta$ be an ellipsoid cover. For any $\varphi \in S$, $x \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $\theta(x, t) = M_{x,t}(B_n) + x$, denote

$$\varphi_{x,t}(y) := |\det(M_{x,t}^{-1})| \varphi(M_{x,t}^{-1} y), \quad y \in \mathbb{R}^n.$$
Definition 2.3. Let \( f \in S', \varphi \in S \) and \( N, \tilde{N} \in \mathbb{N}_0 \) with \( N \leq \tilde{N} \). We define the non-tangential, the grand non-tangential, the radial, the grand radial and the tangential maximal functions, respectively as

\[
M_{\varphi}f(x) := \sup_{t \in \mathbb{R}} \sup_{y \in \theta(x,t)} |f \ast \varphi_{x,t}(y)|, \quad x \in \mathbb{R}^n,
\]

\[
M_{N,\tilde{N}}f(x) := \sup_{\varphi \in S_{N,\tilde{N}}} M_{\varphi}f(x), \quad x \in \mathbb{R}^n.
\]

\[
M_0^0 f(x) := \sup_{t \in \mathbb{R}} |f \ast \varphi_{x,t}(x)|, \quad x \in \mathbb{R}^n,
\]

\[
M_0^0_{N,\tilde{N}} f(x) := \sup_{\varphi \in S_{N,\tilde{N}}} M_0^0 \varphi f(x), \quad x \in \mathbb{R}^n.
\]

\[
T_{\varphi}^N f(x) := \sup_{t \in \mathbb{R}} \sup_{y \in \mathbb{R}^n} |f \ast \varphi_{x,t}(y)| (1 + |M_{x,t}^{-1}(x-y)|)^{-N}, \quad x \in \mathbb{R}^n.
\]

Remark 2.4. It is immediate that we have the following pointwise estimate among the radial, the non-tangential and the tangential maximal functions:

\[
M_0^0 \varphi f(x) \leq M_{\varphi} f(x) \leq 2^N T_{\varphi}^N f(x), \quad x \in \mathbb{R}^n.
\]

Next, we recall the definition of Hardy spaces with pointwise variable anisotropy [9, Definition 3.6] via the grand radial maximal function.

Let \( \Theta \) be an ellipsoid cover of \( \mathbb{R}^n \) with parameters \( p(\Theta) = \{a_1, \cdots, a_6\} \) and \( 0 < p \leq 1 \). We define \( N_p(\Theta) \) as the minimal integer satisfying

\[
N_p := N_p(\Theta) > \frac{\max(1, a_4) n + 1}{a_6 p}, \tag{2.5}
\]

and then \( \tilde{N}_p(\Theta) \) as the minimal integer satisfying

\[
\tilde{N}_p := \tilde{N}_p(\Theta) > \frac{a_4 N_p(\Theta) + 1}{a_6}. \tag{2.6}
\]

Definition 2.5. Let \( \Theta \) be an ellipsoid cover and \( 0 < p \leq 1 \). Define \( M^0 := M^0_{N_p,\tilde{N}_p} \) and the anisotropic Hardy space is defined as

\[
H^0_{N_p,\tilde{N}_p}(\Theta) := \{ f \in S' : M^0 f \in L^p \}
\]

with the (quasi-)norm \( \| f \|_{H^p(\Theta)} := \| M^0 f \|_{L^p} \).

The gap (1.1) appeared in the proof of the atomic decomposition of \( H^p(\Theta) \) from Dekel, Petrushev and Weissblat can be covered by the following Proposition 2.6.
Proposition 2.6. Let $\Theta$ be an ellipsoid cover which is pointwise continuous. For any $q \in [1, \infty)$, $f \in L^q \cap H^p(\Theta)$ and $\lambda > 0$, the set 
$$\Omega := \{x \in \mathbb{R}^n : M^0 f(x) > \lambda\}$$
is open.

Proof. For any $x, x', y \in \mathbb{R}^n$, $t \in \mathbb{R}$, by (2.3), we have
$$|x - M_{x,t} y - (x' - M_{x',t} y)| \leq |x - x'| + \|M_{x,t} - M_{x',t}\| |y| \to 0 \text{ as } x' \to x.$$From this, $\varphi \in S(\mathbb{R}^n)$, Hölder’s inequality and the continuity of the $L^q$-integral of $f$, $q \in [1, \infty)$, it follows that
$$|f * \varphi_{x,t}(x) - f * \varphi_{x',t}(x')| \leq \int_{\mathbb{R}^n} |f(x - M_{x,t} y) - f(x' - M_{x',t} y)| |\varphi(y)| dy \leq \|f(x - M_{x,t}) - f(x' - M_{x',t})\|_{L^q} \|\varphi\|_{L^{q'}} \to 0 \text{ as } x' \to x,$$
where $q'$ is the conjugate index of $q$, namely, $1/q + 1/q' = 1$.

For any $x \in \Omega$, there exist $t_0 \in \mathbb{R}$ and $\varphi \in \mathcal{S}_{N,\tilde{N}}$ such that
$$|f * \varphi_{x,t_0}(x)| > \lambda.$$By this and (2.7), we know that there exists $\epsilon > 0$ such that for any $x' \in B(x, \epsilon) := \{x \in \mathbb{R}^n : |x| < \epsilon\}$,
$$|f * \varphi_{x',t_0}(x')| > \lambda,$$which implies that $M^0 f(x') > \lambda$ and hence $\Omega$ is open. \hfill $\square$

Proposition 2.7. Let $\Theta$ be an ellipsoid cover which is pointwise continuous, $0 < p \leq 1 \leq q \leq \infty$, $p < q$ and $l \geq N_p$ with $N_p$ as in (2.5). If $N \geq N_p$ and $\tilde{N} \geq (a_4 N + 1)/a_6$, then
$$H^p_{N_p,\tilde{N}_p} (\Theta) = H^p_{q,l}(\Theta) = H^p_{N,\tilde{N}}(\Theta)$$with equivalent (quasi-)norms, where $H^p_{q,l}(\Theta)$ denotes the atomic Hardy space with pointwise variable anisotropy; see [9, Definition 4.2].

Proof. This proposition is a corollary of [9, Theorems 4.4 and 4.19]. Indeed, by Definition 2.5, we obtain that, for any $N \geq N_p$ and $\tilde{N} \geq (a_4 N + 1)/a_6$,
$$H^p_{N_p,\tilde{N}_p} (\Theta) \subseteq H^p_{N,\tilde{N}}(\Theta).$$Combining this and $H^p_{q,l}(\Theta) \subseteq H^p_{N_p,\tilde{N}_p} (\Theta)$ (see [9, Theorem 4.4]), we obtain
$$H^p_{q,l}(\Theta) \subseteq H^p_{N,\tilde{N}}(\Theta).$$ (2.8)
By checking the definition of anisotropic \((p, q, l)\)-atom (see [9, Definition 4.1]), we know that every \((p, \infty, l)\)-atom is also a \((p, q, l)\)-atom and hence

\[ H^p_{\infty, l}(\Theta) \subseteq H^p_{q, l}(\Theta). \]

Let \(l' \geq \max(l, N)\). By a similar argument to the proof of [9, Theorem 4.19] with the fact that \(\Omega\) is open for any \(f \in H^p(\Theta) \cap L^1\) (see Proposition 2.6), we obtain

\[ H^p_{N, \tilde{N}}(\Theta) \subseteq H^p_{\infty, l'}(\Theta), \]

where \(N \geq N_p\) and \(\tilde{N} \geq (a_4 N + 1)/a_6\). Thus,

\[ H^p_{N, \tilde{N}}(\Theta) \subseteq H^p_{\infty, l'}(\Theta) \subseteq H^p_{\infty, l}(\Theta) \subseteq H^p_{q, l}(\Theta). \]  \(\text{(2.9)}\)

Combining (2.8) and (2.9), we conclude that

\[ H^p_{N, \tilde{N}}(\Theta) = H^p_{q, l}(\Theta) = H^p_{N, \tilde{N}}(\Theta) \]

with equivalent (quasi-)norms.

**Remark 2.8.** From Proposition 2.7, we deduce that, for any integers \(N \geq N_p\) and \(\tilde{N} \geq (a_4 N + 1)/a_6\), the definition of \(H^p_{N, \tilde{N}}(\Theta)\) is independent of \(N\) and \(\tilde{N}\). Therefore, from now on, we denote \(H^p_{N, \tilde{N}}(\Theta)\) with \(N \geq N_p\) and \(\tilde{N} \geq (a_4 N + 1)/a_6\) simply by \(H^p(\Theta)\).

**Proposition 2.9.** [9, Lemma 2.3] Let \(\Theta\) be an ellipsoid cover. Then there exists a constant \(J := J(p(\Theta)) \geq 1\) such that for any \(x \in \mathbb{R}^n\) and \(t \in \mathbb{R}\),

\[ 2 M_{x, t}(B) + x \subset \theta(x, t - J). \]

Here and hereafter, let \(J\) always be as in Proposition 2.9.

**Definition 2.10.** [9, Definition 3.1] Let \(\Theta\) be an ellipsoid cover. For any locally integrable function \(f\), maximal function of Hardy-Littlewood type of \(f\) is defined by

\[ M_{\Theta} f(x) := \sup_{t \in \mathbb{R}} \frac{1}{|\theta(x, t)|} \int_{\theta(x, t)} |f(y)| \, dy, \quad x \in \mathbb{R}^n. \]

**Proposition 2.11.** [9, Theorem 3.3] Let \(\Theta\) be an ellipsoid cover. Then

\begin{enumerate}
  \item[(i)] there exists a constant \(C\) depending only on \(p(\Theta)\) and \(n\) such that for all \(f \in L^1\) and \(\alpha > 0\),

  \[ |\{x : M_{\Theta} f(x) > \alpha\}| \leq C \alpha^{-1} \|f\|_{L^1}; \]  \(\text{(2.10)}\)

  \item[(ii)] for \(1 < p < \infty\) there exists a constant \(C_p\) depending only on \(C\) and \(p\) such that for all \(f \in L^p\),

  \[ \|M_{\Theta} f\|_{L^p} \leq C_p \|f\|_{L^p}. \]  \(\text{(2.11)}\)
\end{enumerate}
We give some useful results about variable anisotropic maximal functions with different apertures. They also play important roles to obtain the maximal function characterizations of $H^p(\Theta)$. Suppose that $F : \mathbb{R}^n \times \mathbb{R} \to [0, \infty)$ is an arbitrary function. In our case $F$ is going to be (at least) Lebesgue measurable. For fixed $l \in \mathbb{Z}$ and $t_0 < 0$ define the maximal function of $F$ with aperture $l$ as

$$F_{l}^{\star t_0}(x) := \sup_{t \geq t_0} \sup_{y \in \theta(x, t-lJ)} F(y, t). \quad (2.12)$$

**Proposition 2.12.** For any $l \in \mathbb{Z}$ and $t_0 < 0$, let $F_{l}^{\star t_0}$ be as in (2.12). If the ellipsoid cover $\Theta$ is pointwise continuous, then $F_{l}^{\star t_0} : \mathbb{R}^n \to [0, \infty]$ is lower semicontinuous, i.e.,

$$\{ x \in \mathbb{R}^n : F_{l}^{\star t_0}(x) > \lambda \} \text{ is open for any } \lambda > 0.$$

**Proof.** If $F_{l}^{\star t_0}(x) > \lambda$ for some $x \in \mathbb{R}^n$, i.e., there exist $t \geq t_0$ and $y \in \theta(x, t-lJ)$ so that $F(y, t) > \lambda$. Since $\theta(x, t)$ is continuous for variable $x$ (see Remark 2.2), there is a sufficiently small neighborhood of $x$ so that $y \in \theta(x', t-lJ)$ for any $x'$ in the sufficiently small neighborhood of $x$, which implies $F_{l}^{\star t_0}(x') > \lambda$. Thus, $\{ x \in \mathbb{R}^n : F_{l}^{\star t_0}(x) > \lambda \}$ is open.

By Proposition 2.12, we obtain that $\{ x \in \mathbb{R}^n : F_{l}^{\star t_0}(x) > \lambda \}$ is Lebesgue measurable. Based on this and inspired by [3, Lemma 7.2], the following Proposition 2.13 shows some estimates for maximal function $F_{l}^{\star t_0}$.

**Proposition 2.13.** Let $E$ be a Lebesgue measurable set in $\mathbb{R}^n$ and $\Theta$ an ellipsoid cover which is pointwise continuous. Then there exists a constant $C > 0$ which depends on parameters $p(\Theta)$ such that for any function $F : \mathbb{R}^n \times \mathbb{R} \to [0, \infty)$, $\lambda > 0$, integers $l, l'$ with $l \leq l'$ and $t_0 < 0$, we have

$$| \{ x \in E : F_{l}^{\star t_0}(x) > \lambda \} | \leq C 2^{l(l-l')J} | \{ x \in E : F_{l'}^{\star t_0}(x) > \lambda \} | \quad (2.13)$$

and

$$\int_{E} F_{l}^{\star t_0}(x) \, dx \leq C 2^{l(l-l')J} \int_{E} F_{l'}^{\star t_0}(x) \, dx. \quad (2.14)$$

**Proof.** Let $\Omega := \{ x \in E : F_{l}^{\star t_0}(x) > \lambda \}$. We claim that

$$\{ x \in E : F_{l}^{\star t_0}(x) > \lambda \} \subset \{ x \in \mathbb{R}^n : M_{\Theta}(\chi_{\Omega})(x) \geq C_1 2^{(l'-l)J} \}, \quad (2.15)$$

where $C_1$ is a positive constant to be fixed later. Assuming the claim holds for the moment, from this and weak type (1,1) of $M_{\Theta}$ (see (2.10)), we deduce

$$| \{ x \in E : F_{l}^{\star t_0}(x) > \lambda \} | \leq \left| \{ x \in \mathbb{R}^n : M_{\Theta}(\chi_{\Omega})(x) \geq C_1 2^{(l'-l)J} \} \right| \leq C_1^{-1} 2^{(l-l')J} \| \chi_{\Omega} \|_{L^1} \leq C 2^{l(l-l')J} | \Omega |$$

and hence (2.13) holds true, where $C := 1/C_1$. Furthermore, integrating (2.13) on $(0, \infty)$ with respect to $\lambda$ yields (2.14). Therefore, it remains to show (2.15).
Suppose $F^*t_0(x) > \lambda$ for some $x \in E$. Then there exist $t$ with $t \geq t_0$ and $y \in \theta(x, t-lJ)$ such that $F(y, t) > \lambda$. For any $l, l' \in \mathbb{Z}$ and $l \geq l'$, we first prove that the following holds true:

$$a_5^{-1} \theta(y, t - l'J) \subseteq \theta(x, t - (l + 1)J) \cap \Omega. \quad (2.16)$$

For any $z \in a_5^{-1} \theta(y, t - l'J)$, by (2.4), we have $z \in \theta(y, t - l'J)$ and hence

$$\theta(z, t - l'J) \cap \theta(y, t - l'J) \neq \emptyset.$$  

Thus, by (2.2), we have

$$\left\| M_{z, t-l'J}^{-1} M_{y, t-l'J} \right\| \leq a_5.$$

From this, it follows that

$$a_5^{-1} M_{z, t-l'J}^{-1} M_{y, t-l'J} \subseteq \mathbb{B}^n$$

and hence

$$a_5^{-1} M_{y, t-l'J} \subseteq M_{z, t-l'J} \subseteq \mathbb{B}^n.$$

By this and $y \in a_5^{-1} M_{y, t-l'J} + z$, we obtain $y \in \theta(z, t - l'J)$. From this and $F(y, t) > \lambda$ with $t \geq t_0$, we deduce that $F^*t_0(z) > \lambda$ and hence $z \in \Omega$, which implies

$$a_5^{-1} \theta(y, t - l'J) \subseteq \Omega. \quad (2.17)$$

Besides, by $y \in \theta(x, t - lJ)$, (2.2) and $l \geq l'$, we have

$$\left\| M_{x, t-lJ}^{-1} M_{y, t-l'J} \right\| \leq a_5 2^{-a_6(l-l')J} \leq a_5.$$

From this, it follows that

$$a_5^{-1} M_{x, t-lJ}^{-1} M_{y, t-l'J} \subseteq \mathbb{B}^n$$

and hence

$$a_5^{-1} M_{y, t-l'J} \subseteq M_{x, t-lJ} \subseteq \mathbb{B}^n.$$

By this, (2.4), $y \in \theta(x, t - lJ)$ and Proposition 2.9, we obtain

$$a_5^{-1} M_{y, t-l'J} + y \subseteq 2M_{x, t-lJ} + x \subseteq \theta(x, t - (l + 1)J).$$

From this and (2.17), we deduce that (2.16) holds true.

Next, let’s prove (2.15). By (2.16) and (2.1), we obtain

$$|\theta(x, t - (l + 1)J) \cap \Omega| \geq (a_5)^{-n} |\theta(y, t - l'J)| \geq \frac{a_1}{(a_5)^n} 2^{l'J-t}.$$  

(2.18)

Taking $b_0 := t - (l + 1)J$, by (2.1) and (2.18), we have

$$\frac{1}{|\theta(x, b_0)|} \int_{\theta(x, b_0)} |\chi_\Omega(y)| dy \geq a_2^{-1} 2^{b_0} |\theta(x, b_0) \cap \Omega| \geq \frac{a_1}{(a_5)^n a_2} 2^{(l' - l)J},$$

which implies $M_\Omega(\chi_\Omega)(x) \geq C_2 2^{(l' - l)J}$ and hence (2.15) holds true, where $C_1 := 2^{-J} a_1/[(a_5)^n a_2]$.  

\[\square\]
The following result enables us to pass from one function in $\mathcal{S}$ to the sum of dilates of another function in $\mathcal{S}$ with nonzero mean, which is a variable anisotropic extension of [12, p. 93, Lemma 2] of Stein and [3, Lemma 7.3] of Bownik.

**Proposition 2.14.** Suppose $\Theta$ is an ellipsoid cover, $\varphi \in \mathcal{S}$ and $\int_{\mathbb{R}^n} \varphi(x) dx \neq 0$. For every $\psi \in \mathcal{S}$, $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$ there exists a sequence of test functions $\{\eta^k\}_{k=0}^{\infty}$, $\eta^k \in \mathcal{S}$ such that

$$\psi(\xi) = \sum_{k=0}^{\infty} \eta^k * \varphi^k(\xi) \text{ converges in } \mathcal{S},$$

(2.19)

where $\varphi^k(\xi) := \varphi_{k,t}^\xi(\xi) := 2^{kJ} \varphi(M_k^{-1}M_{x,t}\xi)$. Here and hereafter, for any $\theta(x,t+kJ) = x + M_{x,t+kJ}(\mathbb{R}^n) \in \Theta$, we denote $M_k := M_{x,t+kJ}$.

Furthermore, for any positive integers $L$, $N$, $\tilde{N}$ and $\tilde{N} \geq N$, there exist integers $M \geq N + 2(n+1)$, $\tilde{M} \geq \max\{\tilde{N} + 2(n+1) + L((1/(a_0J)] + 1), M\}$ and constant $C$ (depending on $L$, $N$, $\tilde{N}$ and parameters $p(\Theta)$, but independent of the choice of $\psi$) such that

$$\|\eta^k\|_{S_{N,\tilde{N}}} \leq C2^{-kL}\|\psi\|_{S_{\tilde{M},\tilde{N}}}.$$

(2.20)

**Proof.** The proof is divided into 3 steps since it is a little complicated.

**Step 1.** Show (2.19) holds pointwise everywhere. By scaling of $\varphi$ and $\mathbb{B}^n$ we can assume that $\int_{\mathbb{R}^n} \varphi(x) dx = 1$ and $|\hat{\varphi}(\xi)| \geq 1/2$ for $\xi \in 2a_5 \mathbb{B}^n$. Consider an infinitely differentiable function $\zeta$ such that $\zeta \equiv 1$ on $\mathbb{B}^n$ and supp $\zeta \subset 2\mathbb{B}^n$. Fix $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Define a sequence of functions $\{\zeta_k\}_{k=0}^{\infty}$, where $\zeta_0 := \zeta$,

$$\zeta_k(\xi) := \zeta(M_k^T(M_{x,t}^T)^{-1}\xi) - \zeta(M_{k-1}^T(M_{x,t}^T)^{-1}\xi) \quad \text{for } k \geq 1, \quad \xi \in \mathbb{R}^n,$$

where $M^T$ denotes the transposed matrix of $M$. By $\|M\| = \|M^T\|$ and (2.2), for any given $\xi \in \mathbb{R}^n$, we obtain

$$|M_k^T(M_{x,t}^T)^{-1}\xi| \leq \|M_k^T(M_{x,t}^T)^{-1}\| \|\xi\| \leq a_5 2^{-a_0kJ||\xi||} \to 0 \text{ as } k \to \infty.$$

From this, we deduce that, for any $\xi \in \mathbb{R}^n$, there exists $k$ large enough such that $M_k^T(M_{x,t}^T)^{-1}\xi \in \mathbb{B}^n$. By this, we have

$$\sum_{k=0}^{\infty} \zeta_k(\xi) = 1, \quad \xi \in \mathbb{R}^n.$$

Thus,

$$\hat{\psi}(\xi) = \sum_{k=0}^{\infty} \frac{\zeta_k(\xi)}{\hat{\varphi}(M_k^T(M_{x,t}^T)^{-1}\xi)} \hat{\varphi}(M_k^T(M_{x,t}^T)^{-1}\xi).$$

For $k \in N_0$ define $\eta^k$ by

$$\eta^k(\xi) := \frac{\zeta_k(\xi)}{\hat{\varphi}(M_k^T(M_{x,t}^T)^{-1}\xi)} \hat{\varphi}(\xi).$$
By \( \hat{\varphi}(M^T_k (M^T_{k,t})^{-1} \xi) = \hat{\varphi}^k(\xi) \) and the choice of \( \{\eta^k\}_{k=0}^\infty \), we know that (2.19) holds true pointwise everywhere.

**Step 2.** Prove (2.19) holds in \( \mathcal{S} \). We first claim that for any positive integers \( N, \tilde{N} \in \mathbb{N}_0 \) and \( \tilde{N} \geq N \), there exists a constant \( C > 0 \) so that

\[
\sup_{x \in \mathbb{R}^n} \sup_{t \in \mathbb{R}} \|\eta^k \ast \hat{\varphi}^k\|_{\mathcal{S}_{N,\tilde{N}}} \leq C\|\varphi\|_{\mathcal{S}_{N+n+1,\tilde{N}+n+1}} \|\eta^k\|_{\mathcal{S}_{N,\tilde{N}}}.
\]

(2.21)

Indeed, for any multi-index \( \alpha, |\alpha| \leq N, k \in \mathbb{N}_0 \), and \( x, \xi \in \mathbb{R}^n \), by (2.2), we have

\[
(1 + |\xi|)^{\tilde{N}} |\partial^\alpha (\eta^k \ast \hat{\varphi}^k)(\xi)| = (1 + |\xi|)^{\tilde{N}} \left| \left( \partial^\alpha \eta^k \ast \hat{\varphi}^k \right)(\xi) \right|
\leq \int_{\mathbb{R}^n} (1 + |\xi - y|)^{\tilde{N}} |\partial^\alpha \eta^k(\xi - y)| (1 + |y|)^{\tilde{N}} |\hat{\varphi}^k(y)| \, dy
\leq 2^{k \beta} \left\| \eta^k \right\|_{\mathcal{S}_{N,\tilde{N}}} \left\| \varphi \right\|_{\mathcal{S}_{N+n+1,\tilde{N}+n+1}}
\times \int_{\mathbb{R}^n} (1 + \left| M_{x,t}^{-1} M_k \right| \left| M_k^{-1} M_{x,t} y \right|)^{\tilde{N}} (1 + \left| M_k^{-1} M_{x,t} y \right|)^{-(\tilde{N}+n+1)} \, dy
\leq (a_5)^{\tilde{N}} 2^{k \beta} \left\| \eta^k \right\|_{\mathcal{S}_{N,\tilde{N}}} \left\| \varphi \right\|_{\mathcal{S}_{N+n+1,\tilde{N}+n+1}} \int_{\mathbb{R}^n} (1 + \left| M_k^{-1} M_{x,t} y \right|)^{-(n+1)} \, dy
\leq C\left\| \varphi \right\|_{\mathcal{S}_{N+n+1,\tilde{N}+n+1}} \left\| \eta^k \right\|_{\mathcal{S}_{N,\tilde{N}}},
\]

which implies (2.21) holds true.

Now, by Step 1, (2.21) and (2.20), we may obtain that the convergence of the series (2.19) is in \( \mathcal{S} \). Therefore, it remains to show (2.20).

**Step 3.** Show (2.20). Firstly, we claim that, for any \( \eta \in \mathcal{S} \), positive integers \( N, N' \) and \( N' \geq N \), there exists a constant \( C > 0 \) such that

\[
\left\| \hat{\eta} \right\|_{\mathcal{S}_{N,N'}} \leq C\left\| \eta \right\|_{\mathcal{S}_{N+n+1,N'+n+1}}.
\]

(2.22)

Indeed, for any multi-indices \( |\alpha|, |\beta| \leq N \), we have

\[
(2\pi i)^{|eta|} \xi^\beta \partial^\alpha \hat{\eta}(\xi) = (-2\pi i)^{|\alpha|} \int_{\mathbb{R}^n} \partial^\beta [x^\alpha \eta(x)] e^{-2\pi i (x,\xi)} \, dx.
\]

Hence, by multiplying and dividing the right hand side by \( (1 + |x|)^{n+1} \), we have

\[
\left| \xi^\beta \partial^\alpha \hat{\eta}(\xi) \right| \leq (2\pi)^{|\alpha|-|\beta|} \sup_{x \in \mathbb{R}^n} (1 + |x|)^{n+1} \left| \partial^\beta [x^\alpha \eta(x)] \right| \int_{\mathbb{R}^n} (1 + |x|)^{-n-1} \, dx,
\]

which implies (2.22).

Thus, to show (2.20), by (2.22), it suffices to show that there exists a constant \( C \) (independent of \( \psi \in \mathcal{S} \)) such that

\[
\left\| \hat{\eta}^k \right\|_{\mathcal{S}_{N+n+1,\tilde{N}+n+1}} \leq C 2^{-kL} \|\psi\|_{\mathcal{S}_{M_x,\tilde{M}_x}}, \quad k \in \mathbb{N}_0.
\]

(2.23)
We first claim
\[
\sup_{\xi \in \mathbb{R}^n} \sup_{|\alpha| \leq N+n+1} \left| \partial^\alpha \left( \hat{\zeta}(\cdot) / \hat{\varphi} \left( M_k^T (M_{x,t}^T)^{-1} \right) \right) (\xi) \right| \leq C_1, \tag{2.24}
\]
where \( C_1 := C(\zeta, N, n, p(\Theta)) \) is a positive constant. Indeed, let
\[
\tilde{\zeta}(\xi) := [\zeta(\xi) - \zeta(M_{(k-1)}^T (M_{x,k}^T)^{-1} \xi)] / \hat{\varphi}(\xi).
\]
Obviously,
\[
\text{supp} \tilde{\zeta} \subseteq \text{supp} \zeta \cup \text{supp} \zeta(M_{(k-1)}^T (M_{x,k}^T)^{-1} \xi).
\]
If \( M_{(k-1)}^T (M_{x,k}^T)^{-1} \xi \in 2\mathbb{B}^n \), then we have
\[
\xi \in 2M_{k}^T (M_{k-1}^T)^{-1} (\mathbb{B}^n). \tag{2.26}
\]
Furthermore, by \( ||M^T|| = ||M|| \) and (2.2), we obtain
\[
||M_{k}^T (M_{k-1}^T)^{-1}|| \leq a_5 2^{-a_0 \cdot J} \leq a_5,
\]
which implies \( M_{k}^T (M_{k-1}^T)^{-1} (\mathbb{B}^n) \subseteq a_5 \mathbb{B}^n \). By this, (2.26), (2.25), \( \text{supp} \zeta \subset 2\mathbb{B}^n \) and \( a_5 \geq 1 \) (see (2.4)), we have
\[
\text{supp} \tilde{\zeta} \subset 2\mathbb{B}^n \cup 2a_5 \mathbb{B}^n = 2a_5 \mathbb{B}^n.
\]
Using this and \( |\hat{\varphi}(\xi)| \geq 1/2 \) for any \( \xi \in 2a_5 \mathbb{B}^n \), we obtain
\[
\sup_{\xi \in \mathbb{R}^n} \sup_{|\alpha| \leq N+n+1} |\partial^\alpha \tilde{\zeta}(\xi)| \leq C, \tag{2.27}
\]
where \( C := C(\zeta, N, n, p(\Theta)) \) is a positive constant.

From \( \tilde{\zeta}(M_{x,t}^{-1})(M_{x,t}^{-1}) = \tilde{\zeta}(\xi) / \hat{\varphi}(M_{x,t}^{-1}) (M_{x,t}^{-1}) \), the chain rule, \( ||M^T|| = ||M|| \), (2.2) and (2.27), it follows that
\[
\sup_{\xi \in \mathbb{R}^n} \sup_{|\alpha| \leq N+n+1} \left| \partial^\alpha \left( \tilde{\zeta}(\cdot) / \hat{\varphi} \left( M_k^T (M_{x,t}^T)^{-1} \right) \right) (\xi) \right| \\
= \sup_{\xi \in \mathbb{R}^n} \sup_{|\alpha| \leq N+n+1} \left| \partial^\alpha \left( \tilde{\zeta} \left( M_k^T (M_{x,t}^T)^{-1} \right) \right) (\xi) \right| \\
\leq C \sup_{\xi \in \mathbb{R}^n} \sup_{|\alpha| \leq N+n+1} \left| M_k^T (M_{x,t}^T)^{-1} \right|^{[\alpha]} \left| \left( \partial^\alpha \tilde{\zeta} \right) \left( M_k^T (M_{x,t}^T)^{-1} \xi \right) \right| \\
\leq C \sup_{\xi \in \mathbb{R}^n} \sup_{|\alpha| \leq N+n+1} \left( a_5 2^{-a_0 \cdot J} \right)^{[\alpha]} \left| \left( \partial^\alpha \tilde{\zeta} \right) \left( M_k^T (M_{x,t}^T)^{-1} \xi \right) \right| \\
\leq C a_5^{N+n+1} \sup_{\xi \in \mathbb{R}^n} \sup_{|\alpha| \leq N+n+1} \left| \partial^\alpha \tilde{\zeta}(\xi) \right| \leq C_1,
\]
which means (2.24) holds true.
Notice that for any $\xi \in a_5^{-1} M_{x,t}^T(M_{k-1}^T)^{-1}(\mathbb{B}^n)$, by (2.2) and $\zeta \equiv 1$ on $\mathbb{B}^n$, we have $\zeta_k(\xi) = 0$. By this, the product rule and (2.24), we obtain

$$\left\| \hat{\eta}^k \right\|_{S_{N+n+1, N+n+1}} = \sup_{\xi \notin a_5^{-1} M_{x,t}^T(M_{k-1}^T)^{-1}(\mathbb{B}^n)} \sup_{|a| \leq N+n+1} \left\{ (1 + |\xi|)^{N+n+1} \right\} \times \left| \partial^a \left[ \zeta_k(\cdot) \hat{\psi}(\cdot)/\widehat{\varphi} \left( M_k^T(M_{k-1}^T)^{-1} \cdot \right) \right] (\xi) \right\}$$

$$\leq C_{N,n} C_1 \sup_{\xi \notin a_5^{-1} M_{x,t}^T(M_{k-1}^T)^{-1}(\mathbb{B}^n)} \sup_{|a| \leq N+n+1} (1 + |\xi|)^{N+n+1} \left| \partial^a \hat{\psi}(\xi) \right|$$

$$\leq C_{N,n} C_1 \sup_{\xi \notin a_5^{-1} M_{x,t}^T(M_{k-1}^T)^{-1}(\mathbb{B}^n)} (1 + |\xi|)^{-L(\lfloor \frac{1}{a_0 J} \rfloor + 1)} \left\| \hat{\psi} \right\|_{S_{M-n-1, M-n-1}},$$

where $M \geq N + 2(n + 1)$ and $\tilde{M} \geq \max \{ \tilde{N} + 2(n + 1) + L(1/(a_0 J) + 1), M \}$. From $\|M^T\| = \|M\|$ and (2.2), we deduce that

$$\|M_{k-1}^T(M_{x,t}^T)^{-1}\| \leq a_5^2 - a_6(k-1)J$$

and hence

$$M_{k-1}^T(M_{x,t}^T)^{-1}(\mathbb{B}^n) \subseteq a_5^2 - a_6(k-1)J \mathbb{B}^n,$$

which implies

$$\frac{2a_6(k-1)J}{a_5} \mathbb{B}^n \subseteq M_{x,t}^T(M_{k-1}^T)^{-1}(\mathbb{B}^n).$$

By this and $\xi \notin a_5^{-1} M_{x,t}^T(M_{k-1}^T)^{-1}(\mathbb{B}^n)$, we have

$$|\xi| \geq 2a_6(k-1)J/(a_5)^2.$$

Consequently, inserting (2.29) into (2.28) and by (2.22), we have

$$\left\| \hat{\eta}^k \right\|_{S_{N+n+1, N+n+1}} \leq C2^{-kL} \left\| \hat{\psi} \right\|_{S_{M-n-1, M-n-1}} \leq C2^{-kL} \left\| \psi \right\|_{S_{M, \tilde{M}}}$$

and hence (2.23) holds true. This finishes the proof of Proposition 2.14. \(\square\)

3 Maximal Function Characterizations of $H^p(\Theta)$

In this section, if ellipsoid cover $\Theta$ is pointwise continuous, then we may obtain the maximal function characterizations of $H^p(\Theta)$ using the radial, the non-tangential and the tangential maximal function of a single test function $\varphi \in S$.

**Theorem 3.1.** Let $\Theta$ be an ellipsoid cover which is pointwise continuous, $0 < p < \infty$ and $\varphi \in S$ satisfy $\int_{\mathbb{B}^n} \varphi(x) \, dx \neq 0$. Then for any $f \in S'$, the following are mutually equivalent:

$$f \in H^p(\Theta);$$

$$M\varphi f \in L^p;$$

$$M\varphi f \in L^p;$$

where $M\varphi f \in L^p$.

Proof. The proof of this theorem follows from the properties of maximal functions and the characterizations of $H^p(\Theta)$ established in the previous sections. The details of the proof are omitted due to space constraints.
The Maximal Function Characterizations

\[ M^0_\varphi f \in L^p; \]  
(3.3)

\[ T^N_\varphi f \in L^p, \quad N > \frac{1}{a_6 p}. \]  
(3.4)

In this case,

\[ \|f\|_{H^p(\Theta)} = \|M^0 f\|_{L^p} \leq C_1 \|T^N_\varphi f\|_{L^p} \leq C_2 \|M_\varphi f\|_{L^p} \leq C_3 \|M^0_\varphi f\|_{L^p} \leq C_4 \|f\|_{H^p(\Theta)}, \]

where the positive constants \( C_1, C_2, C_3 \) and \( C_4 \) are independent of \( f \).

The framework to prove Theorem 3.1 is motivated by Fefferman and Stein [10], [12, Chapter III], and Bownik [3, p.42, Theorem 7.1].

Inspired by Fefferman and Stein [12, p.97] and Bownik [3, p.47], we now start with maximal functions obtained from truncation with an additional extra decay term. Namely, for \( t_0 < 0 \) representing the truncation level and real number \( L \geq 0 \) representing the decay level, we define the radial, the non-tangential, the tangential, the grand radial, and the grand non-tangential maximal functions, respectively as

\[ M^0_{\varphi}(t_0, L) f(x) := \sup_{t \geq t_0} |(f * \varphi_{x, t})(x)| \left( 1 + |M^{-1}_{x, t_0} x| \right)^{-L} (1 + 2^{t+t_0})^{-L}, \]

\[ M^1_{\varphi}(t_0, L) f(x) := \sup_{t \geq t_0, y \in \Theta(x, t)} |(f * \varphi_{x, t})(y)| \left( 1 + |M^{-1}_{x, t_0} y| \right)^{-L} (1 + 2^{t+t_0})^{-L}, \]

\[ T^N_{\varphi}(t_0, L) f(x) := \sup_{t \geq t_0, y \in \mathbb{R}^n} \frac{|(f * \varphi_{x, t})(y)|}{(1 + |M^{-1}_{x, t} (x-y)|)^N} \left( 1 + 2^{t+t_0} \right)^L \left( 1 + |M^{-1}_{x, t_0} y| \right)^L, \]

\[ M^0_{N, \tilde{N}}(t_0, L) f(x) := \sup_{\varphi \in \mathcal{S}_{N, \tilde{N}}} M^0_{\varphi}(t_0, L) f(x) \]

and

\[ M^{(t_0, L)}_{N, \tilde{N}} f(x) := \sup_{\varphi \in \mathcal{S}_{N, \tilde{N}}} M^{(t_0, L)}_{\varphi} f(x). \]

The following Lemma 3.2 guarantees the control of the tangential by the non-tangential maximal function in \( L^p(\mathbb{R}^n) \) independent of \( t_0 \) and \( L \).

**Lemma 3.2.** Let \( E \) be Lebesgue measurable set in \( \mathbb{R}^n \) and \( \Theta \) an ellipsoid cover which is pointwise continuous. Suppose \( p > 0, N > 1/(a_6 p) \) and \( \varphi \in \mathcal{S} \). Then there exists a positive constant \( C \) such that for any \( t_0 < 0, L \geq 0 \) and \( f \in \mathcal{S}' \),

\[ \left\| T^N_{\varphi}(t_0, L) f \right\|_{L^p(E)} \leq C \left\| M^{(t_0, L)}_{\varphi} f \right\|_{L^p(E)}. \]

**Proof.** Consider function \( F : \mathbb{R}^n \times \mathbb{R} \rightarrow [0, \infty) \) given by

\[ F(y, t) := \|(f * \varphi_{x, t})(y)|^p \left( 1 + |M^{-1}_{x, t_0} y| \right)^{-pL} (1 + 2^{t+t_0})^{-pL}. \]
Fix \( x \in E \) and let \( F^*_{x,t_0} \) be as in (2.12) with \( l = 0 \). When \( y \in \theta(x, t) \), we have \( M^{-1}_{x,t} (x - y) \in \mathbb{B}^n \) and hence \( |M^{-1}_{x,t} (x - y)| < 1 \). If \( t \geq t_0 \), then

\[
F(y, t) \left[ 1 + \left| M^{-1}_{x,t} (x - y) \right| \right]^{-p_N} \leq F^*_{x,t_0}(x).
\]

When \( y \in \theta(x, t - kJ) \setminus \theta(x, t - (k - 1)J) \) for some \( k \geq 1 \), we have

\[
M^{-1}_{x,t} (x - y) \notin M^{-1}_{x,t} M_{x, t-(k-1)J}(\mathbb{B}^n).
\]  

By (2.2), we obtain

\[
\left\| M^{-1}_{x,t-(k-1)J} M_{x,t} \right\| \leq a_5 2^{-a_6(k-1)J}
\]

and hence

\[
M^{-1}_{x,t-(k-1)J} M_{x,t}(\mathbb{B}^n) \subseteq a_5 2^{-a_6(k-1)J} \mathbb{B}^n,
\]

which implies

\[
(2^{a_6(k-1)J}/a_5) \mathbb{B}^n \subseteq M^{-1}_{x,t-(k-1)J}(\mathbb{B}^n).
\]

From this and (3.5), it follows that \( |M^{-1}_{x,t} (x - y)| \geq 2^{a_6(k-1)J}/a_5 \). Thus, for any \( t \geq t_0 \), we have

\[
F(y, t) \left[ 1 + \left| M^{-1}_{x,t} (x - y) \right| \right]^{-p_N} \leq a_5^{-p_N 2^{-pN_a(k-1)J}} F^*_{x,t_0}(x).
\]

By taking supremum over all \( y \in \mathbb{R}^n \) and \( t \geq t_0 \), we know that

\[
\left[ T_{\phi}^{N(t_0,L)} f(x) \right]^p \leq a_5^{-p_N} \sum_{k=0}^{\infty} 2^{-pN_a(k-1)J} F^*_{k,t_0}(x).
\]

Therefore, using this and Proposition 2.13, we obtain

\[
\left\| T_{\phi}^{N(t_0,L)} f \right\|_{L^p(E)}^p \leq a_5^{-p_N} \sum_{k=0}^{\infty} 2^{-pN_a(k-1)J} \int_E F^*_{k,t_0}(x) dx
\]

\[
= C' \left\| f \right\|_{L^p(E)}^p,
\]

where \( C' := a_5^{p_N} 2^{pN_a J} \sum_{k=0}^{\infty} 2^{(1-pN_a)k} J = a_5^{p_N} 2^J / (1 - 2(1-pN_a)J) \).

The following Lemma 3.3 gives the pointwise majorization of the grand radial maximal function by the tangential one, which is a variable anisotropic extension of [3, Lemma 7.5].

**Lemma 3.3.** Suppose \( \varphi \in S \) and \( \int_{\mathbb{R}^n} \varphi(x) dx \neq 0 \). For given positive integers \( N \) and \( L \), there exist integers \( M > 0, \tilde{M} \geq M \) and constant \( C > 0 \) such that, for any \( f \in S' \) and \( t_0 < 0 \), it holds true that

\[
M^0_{M, \tilde{M}} f(x) \leq C T_{\varphi}^{N(t_0,L)} f(x), \quad x \in \mathbb{R}^n.
\]
Proof. For any $\psi \in \mathcal{S}$, by Proposition 2.14, there exists a sequence of test functions $\{\eta^k\}_{k=0}^{\infty}$ such that (2.19) and (2.20) hold true. Thus, for fixed $x \in \mathbb{R}^n$ and $t \geq t_0$, we have

$$|(f \ast \psi_{x,t})(x)| = \left| \left[ f \ast \sum_{k=0}^{\infty} \left( \eta^k \ast \varphi^k \right)_{x,t} \right](x) \right|$$

\[ \leq C \left| \left[ f \ast \sum_{k=0}^{\infty} 2^{t+kJ} \int_{\mathbb{R}^n} \eta^k(y) \varphi \left( M_{k}^{-1} \cdot - M_{k}^{-1} M_{x,t} y \right) dy \right](x) \right| \]

\[ = C \left| \left[ f \ast \sum_{k=0}^{\infty} 2^{2t+kJ} \int_{\mathbb{R}^n} \eta^k \left( M_{x,t} y \right) \varphi \left( M_{k}^{-1} \cdot - y \right) dy \right](x) \right| \]

\[ \leq C \sum_{k=0}^{\infty} \left| \left[ f \ast \left( \eta^k \right)_{x,t} \ast \varphi_{x,t+kJ} \right](x) \right| \]

\[ \leq C \sum_{k=0}^{\infty} \int_{\mathbb{R}^n} \left| f \ast \varphi_{x,t+kJ}(x-y) \right| \left| \left( \eta^k \right)_{x,t}(y) \right| dy \]

\[ \leq C T_{\varphi}^{N}(t_0, L) f(x) \sum_{k=0}^{\infty} \int_{\mathbb{R}^n} \left( 1 + |M_{k}^{-1} y| \right)^{N} \]

\[ \times \left( 1 + |M_{x,t_0}^{-1} (x-y)| \right)^{L} \left( 1 + 2^{t_0+kJ} \right)^{L} \left| \eta^k \right|_{x,t}(y) dy. \]

Therefore,

$$M_{\psi}^{0}(t_0, L) f(x) \leq T_{\varphi}^{N}(t_0, L) f(x) \sup_{t \geq t_0} \sum_{k=0}^{\infty} \int_{\mathbb{R}^n} \left( 1 + |M_{k}^{-1} y| \right)^{N} \]

\[ \times \left( 1 + |M_{x,t_0}^{-1} (x-y)| \right)^{L} \left( 1 + 2^{t_0+kJ} \right)^{L} \left| \eta^k \right|_{x,t}(y) dy \]

\[ =: T_{\varphi}^{N}(t_0, L) f(x) I_{t_0}(x). \]

To estimate $I_{t_0}(x)$, by

$$\frac{1 + 2^{t_0+kJ}}{1 + 2^{t_0}} = \frac{2^{kJ}(2^{kJ}+2^{t_0})}{1 + 2^{t_0}} \leq C 2^{kJ}$$

and

$$1 + |x+y| = 1 + |x| + |y| \leq (1 + |x|)(1 + |y|), \quad x, y \in \mathbb{R}^n, \quad (3.7)$$

we obtain

$$I_{t_0}(x) \leq C \sum_{k=0}^{\infty} 2^{t_0+kJ} \int_{\mathbb{R}^n} \left( 1 + |M_{x,t_0}^{-1} y| \right)^{N} \left( 1 + |M_{x,t_0}^{-1} y| \right)^{L} \left| \eta^k \left( M_{x,t} y \right) \right| dy$$

\[ \leq C \sum_{k=0}^{\infty} 2^{t_0+kJ} \int_{\mathbb{R}^n} \left( 1 + |M_{x,t_0}^{-1} y| \right)^{N} \left( 1 + |M_{x,t_0}^{-1} y| \right)^{L} \left| \eta^k \left( M_{x,t} y \right) \right| dy. \]
\[ \leq C \sum_{k=0}^{\infty} 2^{kJL} \int_{\mathbb{R}^n} (1 + \|M_k^{-1}M_{x,t}\| |y|)^N (1 + \|M_{x,t_0}^{-1}M_{x,t}\| |y|)^L \eta^k(y) \, dy, \]

which, together with

\[ \|M_k^{-1}M_{x,t}\| \leq a_3 2^{4k} \quad \text{and} \quad \|M_{x,t_0}^{-1}M_{x,t}\| \leq a_5 2^{-(t-t_0)} \leq a_5 \quad \text{(by \( t \geq t_0 \) and (2.2))}, \]

further implies that

\[ I_{t_0}(x) \leq C \sum_{k=0}^{\infty} 2^{kJ(L+a_4N)} \int_{\mathbb{R}^n} (1 + |y|)^{N+L} \eta^k(y) \, dy \tag{3.8} \]

\[ \leq C \sum_{k=0}^{\infty} 2^{kJ(L+a_4N)} \left\| \eta^k \right\|_{S_{N+n+L, \tilde{N}+n+L}}. \]

By Proposition 2.14, there exist positive integers \( M \) and \( \tilde{M} \) such that

\[ \left\| \eta^k \right\|_{S_{N+n+L, \tilde{N}+n+L}} \leq C 2^{-kJ[L+(a_{4}+1)N]} \left\| \eta \right\|_{S_{M, \tilde{M}}}. \tag{3.9} \]

Thus, combining with (3.6), (3.8) and (3.9), we finally obtain

\[ M_{M, \tilde{M}}^{0(t_0, L)} f(x) = \sup_{\psi \in S_{M, \tilde{M}}} M_{\psi}^{0(t_0, L)} f(x) \]

\[ \leq C \sum_{k=0}^{\infty} 2^{kJN[a_{4}-(a_{4}+1)]} T_{\psi}^{N}(t_0, L) f(x) = C T_{\psi}^{N}(t_0, L) f(x). \]

This finishes the proof of Lemma 3.3. \( \square \)

The following Lemma 3.4 shows that the radial and the grand non-tangential maximal functions are pointwise equivalent, which is a variable anisotropic extension of [3, Proposition 3.10].

**Lemma 3.4.** [4, Theorem 3.4] For any \( N, \tilde{N} \in \mathbb{N} \) with \( N \leq \tilde{N} \), there exists a positive constant \( C := C(\tilde{N}) \) such that for any \( f \in S' \),

\[ M_{N, \tilde{N}}^{0} f(x) \leq M_{N, \tilde{N}}^{0} f(x) \leq C M_{N, \tilde{N}}^{0} f(x), \quad x \in \mathbb{R}^n. \]

The following Lemma 3.5 is a variable anisotropic extension of [3, p. 46, Lemma 7.6].

**Lemma 3.5.** Let \( \varphi \in S \), \( f \in S' \) and \( K \in (0, \infty) \). Then for every \( M > 0 \) and \( t_0 < 0 \) there exist \( L > 0 \) and \( N' > 0 \) large enough such that

\[ M_{\varphi}^{(t_0, L)} f(x) \leq C 2^{-t_0(2a_{4}N'+2L+a_{4}L)(1 + |x|)^{-M}}, \quad x \in \mathbb{B}_K := \{ y \in \mathbb{R}^n : |y| < K \}, \tag{3.10} \]

where \( C \) is a positive constant dependent on \( p(\Theta) \), \( N' \), \( f \), \( \varphi \) and \( K \).
Proof. For any \( \varphi \in \mathcal{S} \), there exist an integer \( N > 0 \) and positive constant \( C := C(\varphi) \) such that, for any \( N' \geq N \) and \( y \in \mathbb{R}^n \),
\[
|f * \varphi(y)| \leq C\|\varphi\|_{S_{N',N'}}(1 + |y|)^{N'}.
\]
(3.11)

Therefore, for any \( t_0 < 0 \), \( t \geq t_0 \) and \( x \in B_K \), by (3.11), we have
\[
|(f \ast \varphi_{x,t})(y)| (1 + |M_{x,t_0}^{-1}y|)^{-L} (1 + 2^{t+t_0})^{-L}
\]
\[
\leq C2^{-L(t+t_0)}\|\varphi_{x,t}\|_{S_{N',N'}}(1 + |y|)^{N'} (1 + |M_{x,t_0}^{-1}y|)^{-L}.
\]
(3.12)

Let us first estimate \( \|\varphi_{x,t}\|_{S_{N',N'}} \). By the chain rule and (2.1), we have
\[
\|\varphi_{x,t}\|_{S_{N',N'}} = |\det M_{x,t}^{-1}| \sup_{z \in \mathbb{R}^n} (1 + |z|)^{N'} |\partial^\alpha \varphi(M_{x,t}^{-1}z)|
\]
\[
\leq C2^{t} \sup_{z \in \mathbb{R}^n} (1 + |z|)^{N'} \|M_{x,t}^{-1}\|^{\alpha} |\partial^\alpha \varphi(M_{x,t}^{-1}z)|
\]
\[
\leq C2^{t} \sup_{z \in \mathbb{R}^n} (1 + |M_{x,t}z|)^{N'} \|M_{x,t}^{-1}\|^{\alpha} |\partial^\alpha \varphi(z)| .
\]
(3.13)

Notice that for any given \( K \in (0, \infty) \), there exists \( t_K \in \mathbb{R} \) such that \( B_K \subset \theta(0,t_K) \). Here we might assume \( t_K < 0 \) as well. Then, for any \( x \in B_K \), we get \( \theta(x,0) \cap \theta(0,t_K) \neq \emptyset \). Thus, by (2.2), we have
\[
\|M_{x,0}\| = \left\| M_{0,t_K}M_{0,t_K}^{-1}M_{x,0} \right\| \leq \|M_{0,t_K}\| \left\| M_{0,t_K}^{-1}M_{x,0} \right\|
\]
\[
\leq a_5 2^{\alpha_6 t_K} \|M_{0,t_K}\| := C_1.
\]
(3.14)

Similarly, we also have
\[
\|M_{x,0}^{-1}\| \leq C_2.
\]
(3.15)

Here, \( C_1 \) and \( C_2 \) are positive constants depending on \( K \) and \( p(\Theta) \).

Now, let's further estimate (3.13) in the following two cases.

**Case 1: \( t \geq 0 \).** By (2.2), (3.14) and (3.15), we have
\[
\|M_{x,t}^{-1}\| = \left\| M_{x,t}^{-1}M_{x,0}^{-1} \right\| \leq \|M_{x,t}^{-1}M_{x,0}\| \|M_{x,0}^{-1}\| \leq \|M_{x,0}^{-1}\| \|2^{-\alpha t} \| = C2^{as t}
\]
and
\[
|M_{x,t}| = \left\| M_{x,0}M_{x,0}^{-1}M_{x,t} \right\| \leq \|M_{x,0}\| \left\| M_{x,t}^{-1}M_{x,0} \right\| \|M_{x,0}^{-1}\| \|M_{x,0}M_{x,0}^{-1}M_{x,t} \| \|z| \leq \|M_{x,0}\| \|a_5 2^{-\alpha t} \| |z| \leq C|z|.
\]

Inserting the above two estimates into (3.13) with \( t \geq 0 \), we know that
\[
\|\varphi_{x,t}\|_{S_{N',N'}} \leq C2^{t} \sup_{z \in \mathbb{R}^n} (1 + |M_{x,t}z|)^{N'} \|M_{x,t}^{-1}\|^{\alpha} |\partial^\alpha \varphi(z)|
\]
(3.16)
\[ \leq C2t^{2a_4t}N \| \varphi \|_{S_{N,N'}}. \]

**Case 2:** \( t_0 \leq t < 0 \). By (2.2), (3.14) and (3.15), we have
\[ \| M^{-1}_{x,t} \| = \| M^{-1}_{x,t} M_{x,0} M^{-1}_{x,0} \| \leq \| M^{-1}_{x,t} M_{x,0} \| \leq \| M^{-1}_{x,0} \| a_5 2^{a_6t} \leq C \]
and
\[ |M_{x,t}| = |M_{x,0} M^{-1}_{x,0} M_{x,t} | \leq \| M_{x,0} \| \| M^{-1}_{x,0} M_{x,t} | \leq \| M_{x,0} \| \| M^{-1}_{x,0} M_{x,t} | \leq \| M_{x,0} \| a_3^{-1} 2^{-a_4t} |z| = C 2^{-a_4t_0} |z|. \]

Inserting the above two estimates into (3.13) with \( t_0 \leq t < 0 \), we know that
\[ \| \varphi_{x,t} \|_{S_{N,N'}} \leq C 2^t \sup_{z \in \mathbb{R}^n} \sup_{|a| \leq N'} (1 + |M_{x,t}z|)^N |M^{-1}_{x,t}| |\mathcal{P} a \varphi(z)| (3.17) \]
\[ \leq C 2^{-a_4t_0} N' \| \mathcal{P} \|_{S_{N,N'}}. \]

For any \( M > 0 \), let \( L := M + N' \). For any \( t_0 < 0 \), \( t \geq t_0 \) and taking some integer \( N' > 0 \) large enough, by (3.16) and (3.17), we obtain
\[ 2^{-L(t+t_0)} \| \varphi_{x,t} \|_{S_{N,N'}} \leq C 2^{-t_0(a_4 N'+2L)} \| \varphi \|_{S_{N,N'}}. \]

Inserting (3.18) into (3.12), we further obtain
\[ \| (f * \varphi_{x,t})(y) \| (1 + |M_{x,t_0}y|)^{-L} (1 + 2^{t+t_0})^{-L} \]
\[ \leq C 2^{-t_0(a_4 N'+2L)} \| \varphi \|_{S_{N,N'}} (1 + |y|)^{N'} (1 + |M_{x,t_0}y|)^{-L}. \]

For any \( y \in \theta(x, t) \), there exists \( z \in \mathbb{B}^n \) such that \( y = x + M_{x,t}z \). By (3.7), we have
\[ 1 + |y| = 1 + |x + M_{x,t}z| \leq (1 + |x|)(1 + |M_{x,t}z|). \]

If \( t \geq 0 \), by (2.2) and (3.14), then
\[ |M_{x,t}| = |M_{x,0} M^{-1}_{x,0} M_{x,t} | \leq \| M_{x,0} \| \| M^{-1}_{x,0} M_{x,t} | \leq \| M_{x,0} \| \| M^{-1}_{x,0} M_{x,t} | \leq \| M_{x,0} \| a_5 2^{a_6t} |z| \leq C. \]

If \( t_0 \leq t < 0 \), by (2.2) and (3.14), then
\[ |M_{x,t}| = |M_{x,0} M^{-1}_{x,0} M_{x,t} | \leq \| M_{x,0} \| \| M^{-1}_{x,0} M_{x,t} | \leq \| M_{x,0} \| \| M^{-1}_{x,0} M_{x,t} | \leq \| M_{x,0} \| a_3^{-1} 2^{-a_4t} |z| = C 2^{-a_4t_0}. \]

Therefore, for any \( t \geq t_0 \), by using the above two estimates, we have
\[ |M_{x,t}z| \leq C 2^{-a_4t_0}. \]
From this and (3.20), it follows that
\[(1 + |y|) \leq C 2^{-a t_0} (1 + |x|). \tag{3.21}\]

Besides, for any \(t_0 < 0\), by (2.2) and (3.14), we have
\[
1 + |x| \leq 1 + \|M_{x,0}\| \left| M_{x,0}^{-1} M_{x,t_0} \right| \left| M_{x,t_0}^{-1} x \right| \leq C 2^{-a t_0} (1 + |M_{x,t_0}^{-1} x|).
\]

Furthermore, for any \(y \in \theta(x, t)\), we have \(x \in M_{x,t} (B^n) + y\). Thus, there exists \(z \in B^n\) such that \(x = M_{x,t} z + y\). Hence, for any \(t \geq t_0\), by (3.7) and (2.2), we obtain
\[
(1 + |M_{x,t}^{-1} x|) = (1 + |M_{x,t_0}^{-1}(y + M_{x,t} z)|) \leq (1 + |M_{x,t_0}^{-1} y|) (1 + \|M_{x,t_0}^{-1} M_{x,t} \| |z|) \leq (1 + |M_{x,t_0}^{-1} y|) (1 + a 2^{-a_0(t-t_0)} |z|) \leq C (1 + |M_{x,t_0}^{-1} y|).
\]

Combining with the above two inequalities, we have
\[(1 + |M_{x,t_0}^{-1} y|) \geq C 2^{a t_0} (1 + |x|). \tag{3.22}\]

Thus, for any \(t \geq t_0\) and \(y \in \theta(x, t)\), inserting (3.21) and (3.22) into (3.19) with \(L = M + N\), we obtain
\[
|f \ast \varphi_x(t)(y)| (1 + |M_{x,t_0}^{-1} y|)^{-L} (1 + 2^{t+t_0})^{-L} \leq C 2^{-t_0(2a_4 N' + 2L + a_4L)} (1 + |x|)^{-M},
\]
which implies (3.10) holds true and hence completes the proof of Lemma 3.5. \(\square\)

Note that the above argument gives the same estimate for the truncated grand maximal function \(M_{\varphi}^{0(t_0, L)} f(x)\). As a consequence of Lemma 3.5, we can get that for any choice of \(t_0 < 0\) and any \(f \in S'\), we can find an appropriate \(L > 0\) so that the maximal function, say \(M_{\varphi}^{0(t_0, L)} f\), is bounded and belongs to \(L^p(B_K)\). This becomes crucial in the proof of Theorem 3.1, where we work with truncated maximal functions. The complexity of the preceding argument stems from the fact that a priori we do not know whether \(M_{\varphi}^{0} f \in L^p\) implying \(M_{\varphi} f \in L^p\), instead we must work with variants of maximal functions for which this is satisfied.

**Proof of Theorem 3.1.** Let \(\varphi \in S\) satisfy \(\int_{\mathbb{R}^n} \varphi(x) \, dx \neq 0\). From Remark 2.4 and the definition of the grand radial maximal function, it follows that
\[
(3.4) \Rightarrow (3.2) \Rightarrow (3.3)
\]
and
\[
(3.1) \Rightarrow (3.3).
\]
By Lemma 3.2 applied for \(L = 0\), we have
\[
\left\| T_{\varphi}^{N(t_0, 0)} f \right\|_{L^p} \leq C \left\| M_{\varphi}^{0(t_0, 0)} f \right\|_{L^p} \quad \text{for any } f \in S' \text{ and } t_0 < 0.
\]
As \( t_0 \to -\infty \), by the monotone convergence theorem, we obtain
\[
\| T^N_\varphi f \|_{L^p} \leq C \| M_\varphi f \|_{L^p},
\]
which shows \((3.2) \Rightarrow (3.4)\).

Combining Lemma 3.3 applied for \( N > 1/(a_0 p) \) and \( L = 0 \) and Lemma 3.2 applied for 
\( L = 0 \), we conclude that there exist integers \( M > 0, \tilde{M} \geq M \) large enough and positive constant \( C \) such that
\[
\left\| M^{0(t_0, 0)}_{M, \tilde{M}} f \right\|_{L^p} \leq C \left\| M^{(t_0, 0)}_\varphi f \right\|_{L^p} \text{ for any } f \in \mathcal{S}' \text{ and } t_0 < 0.
\]

As \( t_0 \to -\infty \), by the monotone convergence theorem, we obtain
\[
\left\| M^0_{M, \tilde{M}} \tilde{\varphi} f \right\|_{L^p} \leq C \| M_\varphi f \|_{L^p}.
\]

From this and Proposition 2.7, we deduce that
\[
\| f \|_{H^p(\Theta)} = \left\| M^0_{N_p, \tilde{N}_p} f \right\|_{L^p} \leq C \left\| M^0_{M, \tilde{M}} \tilde{\varphi} f \right\|_{L^p} \leq C \| M_\varphi f \|_{L^p}
\]
and hence \((3.2) \Rightarrow (3.1)\). It remains to show \((3.3) \Rightarrow (3.2)\).

Suppose now \( M_\varphi^0 f \in L^p \). For any given \( t_0 < 0 \) and \( K \in (0, \infty) \), let
\[
\Omega^K_{t_0} := \left\{ x \in \mathbb{B}_K : M^{0(t_0, L)}_{M, \tilde{M}} f(x) \leq C_2 M^{(t_0, L)}_\varphi f(x) \right\},
\]
(3.23)
where \( C_2 := 2^{1/p} C_1 \) with \( C_1 \) to be specified later and \( \mathbb{B}_K \) is a ball as in Lemma 3.5. Combining Lemmas 3.2 and 3.3, we know that there exist integer \( M > 0 \) large enough and integer \( \tilde{M} \geq M \) such that
\[
\left\| M^{0(t_0, L)}_{M, \tilde{M}} f \right\|_{L^p(\mathbb{B}_K / \Omega^K_{t_0})} \leq C_1 \left\| M^{(t_0, L)}_\varphi f \right\|_{L^p(\mathbb{B}_K / \Omega^K_{t_0})},
\]
(3.24)
where constant \( C_1 \) is independent of \( t_0 < 0 \).

Next, we claim that
\[
\int_{\mathbb{B}_K} \left[ M^{(t_0, L)}_\varphi f(x) \right]^p dx \leq 2 \int_{\Omega^K_{t_0}} \left[ M^{(t_0, L)}_\varphi f(x) \right]^p dx.
\]
(3.25)
Indeed, this follows from \((3.24)\), \( M^{(t_0, L)}_\varphi f \in L^p(\mathbb{B}_K) \) and
\[
\int_{\mathbb{B}_K / \Omega^K_{t_0}} \left[ M^{(t_0, L)}_\varphi f(x) \right]^p dx \leq C_2 \int_{\mathbb{B}_K / \Omega^K_{t_0}} \left[ M^{0(t_0, L)}_{M, \tilde{M}} f(x) \right]^p dx
\]
\[
\leq (C_1/C_2)^p \int_{\mathbb{B}_K} \left[ M^{(t_0, L)}_\varphi f(x) \right]^p dx,
\]
where \((C_1/C_2)^p = 1/2\).
For given $t_0 < 0$, let
\[
\Omega_{t_0} := \left\{ x \in \mathbb{R}^n : M_0^{(t_0, L)} f(x) \leq C_2 M^{(t_0, L)} f(x) \right\}.
\] (3.26)

Observe that the set $\Omega_{t_0}^K$ is monotonically increasing with respect to $K$ and
\[
\lim_{K \to \infty} \Omega_{t_0}^K = \Omega_{t_0}.
\] (3.27)

By (3.25) and (3.27), and letting $K \to \infty$, we obtain
\[
\int_{\mathbb{R}^n} M_0^{(t_0, L)} f(x)^p \, dx \leq 2 \int_{\Omega_{t_0}} M_0^{(t_0, L)} f(x)^p \, dx.
\] (3.28)

We also claim that for $0 < q < p$ there exists a constant $C_3 > 0$ such that for any $t_0 < 0$,
\[
M_0^{(t_0, L)} f(x) \leq C_3 \left[ M_\Theta \left( M_0^{(t_0, L)} f \right)^q (x) \right]^{1/q},
\] (3.29)

where $M_\Theta$ is as in Definition 2.10. Indeed, let $t \geq t_0$, $y \in \theta(x, t)$ and
\[
F(y, t) := |(f \ast \varphi_x, t)(y)| (1 + |M_x^{-1} f|^L (1 + 2^t))^{-L}.
\]

Suppose $x \in \Omega_{t_0}$ and let $F_0^{t_0}(x)$ be as in (2.12) with $l = 0$. Then there exist $t' \geq t_0$ and $y' \in \theta(x, t')$ such that
\[
F(y', t') \geq F_0^{t_0}(x)/2 = M_0^{(t_0, L)} f(x)/2.
\] (3.30)

Consider $x' \in y' + M_{x, t'+l}(\mathbb{R}^n)$ for some integer $l \geq 1$ to be specified later. Let $\Phi(z) := \varphi \left( z + M_{x, t'}^{-1} (x' - y') \right) - \varphi(z)$. Obviously, we have
\[
f \ast \varphi_{x, t'}(x') - f \ast \varphi_{x, t'}(y') = f \ast \Phi_{x, t'}(y').
\] (3.31)

Let us first estimate $\| \Phi \|_{\mathcal{S}_{M, \bar{M}}}$ From $x' \in y' + M_{x, t'+l}(\mathbb{R}^n)$, we deduce that
\[
M_{x, t'}^{-1} (x' - y') \in M_{x, t'}^{-1} M_{x, t'+l}(\mathbb{R}^n).
\]

By this and the mean value theorem, we obtain
\[
\| \Phi \|_{\mathcal{S}_{M, \bar{M}}} \leq \sup_{h \in M_{x, t'}^{-1} M_{x, t'+l}(\mathbb{R}^n)} \| \varphi(\cdot + h) - \varphi(\cdot) \|_{\mathcal{S}_{M, \bar{M}}}
\] (3.32)
\[
= \sup_{h \in M_{x, t'}^{-1} M_{x, t'+l}(\mathbb{R}^n)} \sup_{z \in \mathbb{R}^n} \sup_{|z| \leq \bar{M}} (1 + |z|)^{\bar{M}} |(\partial^\alpha \varphi)(z + h) - \partial^\alpha \varphi(z)|
\]
\[
\leq C \sup_{h \in M_{x, t'}^{-1} M_{x, t'+l}(\mathbb{R}^n)} \sup_{z \in \mathbb{R}^n} \sup_{|z| \leq M+1} (1 + |z|)^{\bar{M}} |(\partial^\alpha \varphi)(z + h)|
\]
We choose integer \( l \geq 1 \) large enough such that \( C_4 C_2 C x_{\theta} \leq 1/4 \). Therefore, for any \( x \in \Omega_{t_0} \) and \( x' \in M_{x, t' + L} + y' \), we further have

\[
2^L a_5 \Phi(x', t') \geq M_{x, t' + L} f(x) / 2 - C_4 C_2 C x_{\theta} M_{x, t'} f(x) \geq M_{x, t'} f(x) / 4.
\]  

Besides, by \( y' \in \theta(x, t') \) and Proposition 2.9, we have

\[
M_{x, t' + L} + y' \leq M_{x, t' + L} + M_{x, t} + x
\]
\[ \leq 2M_{x,t'}(\mathbb{R}^n) + x \leq \theta(x, t' - J). \]

Thus, for any \( x \in \Omega_{t_0} \) and \( t \geq t_0 \), by (3.35) and (3.36), we obtain

\[
\left[ M_{\varphi}^{(t_0, L)} f(x) \right]^q \leq \frac{4^{2Ld_x a_0^L}}{|M_{x,t'+1J}(\mathbb{R}^n)|} \int_{y'+M_{x,t'+1J}(\mathbb{R}^n)} |F(z, t')|^q dz \\
\leq C4^{2Ld_x a_0^L} 2(\theta(x, t' - J)) \int_{\theta(x,t' - J)} \left[ M_{\varphi}^{(t_0, L)} f(z) \right]^q dz \\
\leq C_3 M_\Theta \left( \left( M_{\varphi}^{(t_0, L)} f \right)^q \right)(x),
\]

which shows the above claim (3.29).

Consequently, by (3.28), (3.29) and Proposition 2.11 with \( p/q > 1 \), we have

\[
\int_{\mathbb{R}^n} \left[ M_{\varphi}^{(t_0, L)} f(x) \right]^p dx \leq 2 \int_{\Omega_{t_0}} \left[ M_{\varphi}^{(t_0, L)} f(x) \right]^p dx \\
\leq 2C_3^p \int_{\Omega_{t_0}} M_\Theta \left( \left( M_{\varphi}^{(t_0, L)} f \right)^q \right)(x)^{p/q} dx \\
\leq C_5 \int_{\mathbb{R}^n} \left[ M_{\varphi}^{(t_0, L)} f(x) \right]^p dx,
\]

where the constant \( C_5 \) depends on \( p/q > 1 \), \( L \geq 0 \) and \( p(\Theta) \), but is independent of \( t_0 < 0 \). This inequality is crucial as it gives a bound of the non-tangential by the radial maximal function in \( L^p \). The rest of the proof is immediate.

For any \( x \in \mathbb{R}^n, y \in \mathbb{R}^n \) and \( t < 0 \), by (2.2), we obtain

\[
|M_{x,t}^{-1}y| = |M_{x,t}^{-1}M_{x,0}M_{x,0}^{-1}y| \leq \|M_{x,t}^{-1}M_{x,0}\| \|M_{x,0}^{-1}\| |y| \\
\leq a_x 2^{a_x t} \|M_{x,0}^{-1}\| |y| \to 0 \text{ as } t \to -\infty.
\]

Hence, we obtain \( M_{\varphi}^{(t_0, L)} f(x) \) converges pointwise and monotonically to \( M_{\varphi} f(x) \) for all \( x \in \mathbb{R}^n \) as \( t_0 \to -\infty \), which, together with (3.37) and the monotone convergence theorem, further implies that \( M_{\varphi} f \in L^p \). Therefore, we can now choose \( L = 0 \) and again by (3.37) and the monotone convergence theorem, we have \( \|M_{\varphi} f\|_p \leq C_5 \|M_{\varphi} f\|_p^p \), where \( C_5 \) corresponds to \( L = 0 \) and is independent of \( f \in S' \). This finishes the proof of Theorem 3.1. \( \square \)

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