THE 6-TH NORM OF A STEINHAUS CHAOS

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abstract. We prove that for the Steinhaus Random Variable $z(n)$

$$
E \left( \left| \sum_{n \in E_{N,m}} z(n) \right|^{6} \right) \asymp |E_{N,m}|^{3} \text{ for } m \leq \left( \log \log N \right)^{\frac{1}{3}},
$$

where

$$
E_{N,m} := \{1 \leq n : \Omega(n) = m\}
$$

and $\Omega(n)$ denotes the number of prime factors of $N$.

1. Introduction

Let $z(p_{\text{prime}})$ be the Steinhaus random variable, equidistributed on the unit circle $\mathbb{T} := \{s \in \mathbb{C} : |s| = 1\}$. This function can be extended to all natural numbers by defining it completely multiplicatively. We define

$$
S_{N}(z) := \sum_{1 \leq n \leq N} z(n) \text{ and } S_{N,m}(z) := \sum_{n \in E_{N,m}} z(n), \text{ where } E_{N,m} := \{1 \leq n : \Omega(n) = m\}.
$$

Expectations of such Steinhaus chaoses, $S_{N}$ and $S_{N,m}$, received attention from several mathematicians in recent years due to its connections to number theory and harmonic analysis [6]. In [4], Helson observed that if

$$
E(|S_{N}|) = o(\sqrt{N})
$$

then Nehari’s theorem on boundedness of Hankel forms does not extend to $\mathbb{T}^{\infty}$ (the infinite dimensional torus). While Nehari’s theorem has shown to fail on $\mathbb{T}^{\infty}$ by means of another Dirichlet polynomial [6], the question of whether (1) holds remained open and was proved only recently by Harper [3]. In an interesting approach to obtain a lower bound for $E(|S_{N}|)$, Bondarenko and Seip [2] showed that

$$
||S_{N,m}\|_{2} \asymp ||S_{N,m}\|_{4} \text{ for } m < \frac{1}{2} \log \log N,
$$

where $||S_{N,m}\|_{q} := E(|S_{N}|^{q})^{1/q}$. This implies

$$
E(|S_{N}|) \gg \frac{\sqrt{N}}{(\log \log N)^{0.05616}} \text{ and } |S_{N,m}|_{q} \gg q \left( \frac{\sqrt{N}}{(\log \log N)^{0.07672}} \right) \text{ for } q > 0.
$$

In this article, we will investigate the following question:

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Is there a constant $c(k)$, for each $k$, such that $\|S_{N,m}\|_{2k} \asymp \|S_{N,m}\|_2$ when $m < c(k) \log \log N$?

We conjecture that such a constant exists for each $k$, but proving this statement seems difficult. Instead we will show the following:

**Theorem 1.** For $m \leq (\log \log N)^{\frac{1}{2}}$,

$$\|S_{N,m}\|_6 \asymp \|S_{N,m}\|_4$$

as $N \to \infty$.

Certain computations in the proof of the above theorem indicate us to conjecture that $c(3) = \frac{1}{6}$.

We may compare the above result to a result of Hough [5] on Rademacher random variable. Let $f$ denotes the Rademacher random variable defined on primes, and takes the values $\pm 1$ with probability $\frac{1}{2}$ each. Further, extend $f$ to all natural numbers by defining it as a completely multiplicative function. Let

$$S_{N,m,f} := \sum_{n \in E_{N,m}} f(n).$$

Then (see Proposition 10 [5])

$$E(|S_{N,m,f}|^{2k}) \asymp |E_{N,m}|^k$$

for $m = o(\log \log \log N)$.

Our theorem gives a better range for $m$ when $k = 3$ in case of the Steinhaus random variable.

We will simplify $E(|S_{N,m}|^6)$ in Section 2 and prove some preparatory lemmas in Section 3 and 4. In Section 5 we will give a proof of Theorem 1.

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## 2. An Identity For 6th Norm

To integrate $|S_{N,m}|^6$, we will obtain workable expressions for $|S_{N,m}|^4$ and $|S_{N,m}|^2$.

$$|S_{N,m}|^4 = \sum_{a_i, b_i \in E_{N,m}} z(a_1)z(a_2)z(b_1)z(b_2)$$

$$= \|S_{N,m}\|^4 + \sum_{a_i, a_i \neq b_i, b_i \in E_{N,m}} z(a_1)z(a_2)z(b_1)z(b_2)$$

(2) $$= \|S_{N,m}\|^4 + \sum_{k=1}^{2m} \sum_{(a,b) \in E_{N,2k}} \sum_{a_i, a_i', a_i'' \leq N} \sum_{b_i, b_i' \leq N} \sum_{\substack{\Omega(a_i') \leq m \quad \Omega(b_i') \leq m \quad \Omega(a_i'') = m - \Omega(a_i') \quad \Omega(b_i'') = m - \Omega(b_i') \quad a_i' \leq \frac{a_i}{2} \quad b_i'' \leq \frac{b_i}{2}}} z(a)z(b).$$

In the above sum, we factored $a_i$, $b_i$ as $a_i = a_i'a_i''$ and $b_i = b_i'b_i''$ such that $(a_1a_2, b_1b_2) = a_1'a_1''b_1'b_2''$; and denoted $a = a_1'a_2', b = b_1'b_2'$. Later, we will use the notation $\Omega(a_i) = k_i$ and $\Omega(b_i) = k_i'$ for $i = 1, 2$. 
Also (see [2])

\[ |S_{N,m}|^2 = |E_{N,m}| + \sum_{k=1}^{m} \sum_{a,b \in E_{N,k}} |E_{\max(a,b),m-k}| z(a) \overline{z(b)}. \]  

From (2) and (3), we write the 6\(^{th}\) norm of \(S_{N,m}\) as follows:

\[ \|S_{N,m}\|^6 = \|S_{N,m}\|^4 |E_{N,m}| + \sum_{k=1}^{m} \sum_{a,b \in E_{N,k}} |E_{\max(a,b),m-k}| \sum_{a=a'_1 a'_2, b=b'_1 b'_2} S \left( \frac{N}{a_i'}, \frac{N}{b_i'}, m - \Omega(a'_i), m - \Omega(b'_i) : i = 1, 2 \right), \]

where

\[ S(N_i, N'_i, m_i, m'_i : i = 1, 2) = \sum_{a_1 a_2 = b_1 b_2} 1, \quad \Omega(a_i) = m_i, \Omega(b_i) = m'_i, \quad a_i \leq N_i, \ b_i \leq N'_i. \]

3. Some Useful Estimates

Now we will give upper bounds for some expression involving \(|E_{N,m}|\), which will be used later in the proof of our theorem.

The function \(|E_{N,m}|\) is of interest in number theory, and studied extensively in the literature starting from the prime number theorem. Following estimate for \(|E_{N,m}|\) is due to Sathe[7].

Lemma 1. For \(N \to \infty\) and \(1 \leq m \leq (2 - \epsilon) \log \log N\) for \(0 < \epsilon < 1\), we have

\[ |E_{N,m}| \asymp \frac{N (\log \log N)^{m-1}}{(m-1)! \log N}. \]

Later Balazard, Delange and Nicolas[1] generalized this result to an uniform range of \(m\):

Lemma 2. For \(m \geq 1\) and \(\frac{m}{2m} \to \infty\), we have

\[ |E_{N,m}| \asymp \frac{N}{2^m (\log \frac{N}{2m})} \sum_{j=0}^{m-1} \frac{(2 \log \log \frac{N}{2m})^j}{j!}. \]

We will use Lemma 1 and Lemma 2 to prove the following results. Lemma 3 and Lemma 4 below have also appeared in [2].

Lemma 3. For \(k \leq \log \log N\) and \(N, k \to \infty\), we have

\[ \sum_{b \in E_{N,k}} \frac{|E_{b,k}|}{b^2} \ll 2^{2k}. \]
Proof.

\[
\sum_{b \in E_{N,k}} \frac{|E_{b,k}|}{b^2} \ll \int_{2k}^{N} \frac{1}{2^k x \log \frac{x}{2^k}} \sum_{0 \leq j_1 < k} \frac{(2 \log \log \frac{x}{2^k})^{j_1}}{j_1!} dE_{x,k}
\]

\[
\ll \int_{2k}^{N} \frac{1}{2^k x (\log \frac{x}{2^k})^2} \sum_{0 \leq j_1, j_2 < k} \frac{(2 \log \log \frac{x}{2^k})^{j_1+j_2}}{j_1! j_2!} dx
\]

\[
\ll \frac{1}{2^{2k}} \sum_{0 \leq j_1, j_2 < k} \frac{2^{j_1+j_2}}{j_1! j_2!} \int_{2k}^{N} \frac{(\log \log \frac{x}{2^k})^{j_1+j_2}}{x (\log \frac{x}{2^k})^2} dx
\]

\[
\ll \frac{1}{2^{2k}} \sum_{0 \leq l < 2k-1} 2^l \ll 2^{2k}.
\]

\[\square\]

**Lemma 4.** Let \(k, k' \leq \log \log N\) and \(N, k, k' \to \infty\). Then

\[
\sum_{b \in E_{N,k}} \frac{|E_{b,k}|}{b^{N/k'}} \ll 2^{2k'} |E_{N,k}|^2.
\]

Proof.

\[
\sum_{b \in E_{N,k}} \frac{|E_{b,k}|}{b^{N/k'}} \ll |E_{N,k}| \int_{\sqrt{N}}^{N/2^k} \frac{x}{\sqrt{N}} \left| E_{x,k} \right|^2 dE_{x,k}
\]

\[
\ll \frac{|E_{N,k}|^2}{2^{2k'}} \sum_{0 \leq l < 2k-1, j_1 + j_2 = l} \frac{2^{j_1+j_2}}{j_1! j_2!} \int_{\sqrt{N}}^{N/2^k} \frac{(\log \log \frac{x}{2^k})^{j_1+j_2}}{x (\log \frac{x}{2^k})^2} dx
\]

\[
\ll \frac{|E_{N,k}|^2}{2^{2k'}} \sum_{0 \leq l < 2k-1} 2^l \ll 2^{2k'} |E_{N,k}|^2.
\]

\[\square\]

**Lemma 5.** Let \(k, k' \leq \log \log N\) and \(N \to \infty\). Then

\[
\sum_{b \in E_{N,k}} \frac{|E_{b,k'}|}{b^{N/k'}} \ll |E_{N,k}| \left( \log \log N \right)^{k'}/k'.
\]
Proof. Using Lemma 2, we simplify the above sum as follows:

\[
\sum_{b \in E_{N,k}, b > \sqrt{N}} \left| E_{N,k}^b \right| \ll \int_{\sqrt{N}}^{N/2} \frac{N}{x^{2k'}} \sum_{j=0}^{k'-1} \left( \frac{2 \log \log N}{x^{2k'}} \right)^j \frac{1}{j!} d|E_{x,k}|
\]

\[
\ll \left| E_{N,k} \right| \sum_{j=0}^{k'-1} \frac{1}{j^{2k'-j}} \int_{\sqrt{N}}^{N/2} \frac{1}{x^{2k'}} \left( \frac{\log \log N}{x^{2k'}} \right)^j dx
\]

\[
\ll \left| E_{N,k} \right| \sum_{j=0}^{k'-1} \frac{1}{j^{2k'-j}} \int_{\log(\frac{k'}{k-k'}) \log 2}^{\log N+(k-k') \log 2} y^j dy
\]

\[
\ll \left| E_{N,k} \right| \sum_{j=0}^{k'-1} \left( \log \log N \right)^{j+1} \frac{1}{(j+1)!^{2k'-j}} \ll \left| E_{N,k} \right| \frac{\left( \log \log N \right)^{k'}}{k!}.
\]

\[\square\]

**Lemma 6.** Let \( m \leq \frac{1}{4} \log \log N, k \leq m \) and \( N \to \infty \). Then

\[
\sum_{b \in E_{N,k}, b > \sqrt{N}} \frac{|E_{b,k}|}{b} \left| E_{N,k}^b \right| \ll |E_{N,k}| (\log N)^{-1/5}.
\]

**Proof.** From the proof of Lemma 5, we observe that

\[
\frac{1}{N} \sum_{b \in E_{N,k}, b > \sqrt{N}} \left| E_{N,k}^b \right| \ll \frac{(\log \log N)^{m-1}}{(m-k)(k-1)! \log N} \ll (\log N)^{-1} \left( \frac{2 \log \log N}{m} \right)^m.
\]

We may also verify that \( \left( \frac{2 \log \log N}{m} \right)^m \) is an increasing function of \( m \) for \( 1 \leq m \leq \frac{1}{4} \log \log N \). So

\[
\sum_{b \in E_{N,k}, b > \sqrt{N}} \frac{|E_{b,k}|}{b} \left| E_{N,k}^b \right| \ll \left( \frac{2 \log \log N}{m} \right)^m \ll \frac{|E_{N,k}|}{\log N} \left( 8e \right)^{\frac{1}{4} \log \log N} \ll \frac{|E_{N,k}|}{(\log N)^{0.2}}.
\]

\[\square\]

### 4. Upper Bound For \( S \)

In this section we will obtain some estimates for upper bound of \( S \).

**Lemma 7.** Let

\[
N = N_1'N_2' = \min(N_1N_2, N_1'N_2'),
\]

\[
m_1 + m_2 = m_1' + m_2'.
\]

Then

\[
S(N_i, N_i', m_i, m_i' : i = 1, 2) \ll \left\{ \left| E_{N_i,m_i} \right| \left| E_{N_i',m_i'} \right| \left( \left| E_{N_1,m_1} \right| \left| E_{N_2,m_2} \right| + \left| E_{N_1',m_1'} \right| \left| E_{N_2,m_2'} \right| \right) \right\}^{\frac{1}{2}}.
\]
Proof. Note that by Cauchy-Schwarz inequality

\[ S(N_i, N_i', m_i, m_i' : i = 1, 2) = \int_{\mathbb{T}^2(N)} \left( \sum_{n \leq N_{i,n}} z(n) \right) \left( \sum_{n' \leq N_{i',n'}} z(n') \right) d\tilde{z} \]

\[ \leq \left( \int_{\mathbb{T}^2(N)} \left| \sum_{n \leq N_{i,n}} z(n) \right|^2 d\tilde{z} \right)^{\frac{1}{2}} \left( \int_{\mathbb{T}^2(N)} \left| \sum_{n' \leq N_{i',n'}} z(n') \right|^2 d\tilde{z} \right)^{\frac{1}{2}} \]

\[ \leq \left\{ E_{N_i',m_i'} \mid E_{N_2,m_2} \right\} \left( \left| E_{N_1,m_1} \right| E_{N_2,m_2} + \left| E_{N_1,m_1} \right| E_{N_2,m_2} \right) \frac{1}{2} . \]

\[ \square \]

Lemma 8. Let \( N_i, N_i', m_i, m_i' \) for \( i = 1, 2 \) and \( N, m \) be as in the previous theorem. Then we have the following trivial bound for \( S \):

\[ S(N_i, N_i', m_i, m_i' : i = 1, 2) \leq \left( \frac{m}{m_1} \right) |E_{N_i',m_i'}||E_{N_2,m_2}|. \]

5. Proof of Theorem

To prove Theorem\( \Box \) we will divide the sum in (4) in the following 3-parts and estimate each of them separately:

\[ \sum_{k=1}^{m} \sum_{a \in E_{N_k}} \sum_{b \in E_{N_k}} \left| E_{\frac{N_{a,b},m_k}} \right| \sum_{a=a'_1a'_2} S(\ldots) \]

\[ = \sum_{k=1}^{m} \sum_{b \in E_{N_k}} \left| E_{\frac{N_{a,b},m_k}} \right| \sum_{a=a'_1a'_2} \sum_{b=b'_1b'_2} S(\ldots) \]

\[ = \sum_{k=1}^{m} \left( A_1(k) + A_2(k) + A_3(k) + A_4(k) \right) , \]
where

\[ A_1(k) = \sum_{\substack{b \in E_{N,k} \atop b \leq \sqrt{N}}} \cdots, \]

\[ A_2(k) = \sum_{\substack{b \in E_{N,k} \atop b > \sqrt{N}}} \cdots \sum_{a \in E_{b,k}} \cdots \sum_{a = a_1', a_2', b = b_1', b_2'} \sum_{b > b_2'} \sum_{b < b_2'} \cdots, \]

\[ A_3(k) = \sum_{\substack{b \in E_{N,k} \atop b > \sqrt{N}}} \cdots \sum_{a \in E_{b,k}} \cdots \sum_{a = a_1', a_2', b = b_1', b_2'} \sum_{b > b_2'} \sum_{b < b_2'} \cdots, \]

Recall that we have already assumed \( a < b, a_1' \leq a_2', b_1' \leq b_2' \).

5.1. \( A_1 \). Note that

\[
\sum_{k=1}^{m} A_1(k) \left| E_{N,m} \right|^3 \ll \sum_{k=1}^{2^{2k}} \left| E_{N,m-k} \right| \sum_{k=k_1+k_2 \atop k_1, k_2 = 1} \frac{\left| E_{N,m-k_1} \right| \left| E_{N,m-k_2} \right| \left| |E_{N,m-k_1}| \right| \left| |E_{N,m-k_2}| \right|}{|E_{N,m}|^3} \sum_{b \in E_{N,k}} \left| E_{b,k} \right| \left| E_{N,m-k} \right| \max_{k_1, k_2 = k} \left| E_{N,m-k_1} \right| \left| E_{N,m-k_2} \right|
\]

\[
\ll \sum_{k=1}^{m} \left( \frac{4m}{\log \log N} \right)^{2k}.
\]

The above sum is convergent when \( m < c \log \log N \) for any \( c < 1/4 \). This is also the reason for our conjecture that the critical homogeneity is \( \frac{1}{4} \log \log N \).

5.2. \( A_2 \).

\[
\sum_{k=1}^{m} A_2(k) \left| E_{N,m} \right|^3 \ll \frac{2^{2m}}{|E_{N,m}|^3} \sum_{k=1}^{2^{2k}} \sum_{b \in E_{N,k} \atop b > \sqrt{N}} \left| E_{b,k} \right| \left| E_{N,m-k} \right| \max_{k_1, k_2 = k} \left| E_{N,m-k_1} \right| \left| E_{N,m-k_2} \right|
\]

\[
\ll \sum_{k=1}^{m} \frac{2^{2m+k} \left| E_{N,k} \right|}{(\log N)^{1/5}} \frac{\left| E_{N,m} \right|}{|E_{N,m}|} \max_{k_1, k_2 = k} \left( \frac{m}{\log \log N} \right)^{k_1+k_2}
\]

\[
\ll \sum_{k=1}^{m} (\log N)^{-1/5} \left( \frac{4m}{\log \log N} \right)^{m-k} \left( \frac{8m}{\log \log N} \right)^{k}.
\]

The above sum is bounded for \( m \leq \frac{1}{8} \log \log N \).
5.3. $A_3$

$A_3(k)$

$$A_3(k) = \sum_{b \in E_{N,k}} \left| E_{\frac{N}{a_1},m-k} \right| \sum_{a \in E_{b,k}} \sum_{a=a', a_2, b=b', b_2} S \left( \frac{N}{a_1}, \frac{N}{b_1}, m - \Omega(a_1'), m - \Omega(b_1') : i = 1, 2 \right)$$

$$= \sum_{b \in E_{N,k}} \left| E_{\frac{N}{a_1},m-k} \right| \sum_{a \in E_{b,k}} \sum_{a=a', a_2, b=b', b_2} \max \left( \left| E_{\frac{N}{a_1},m-k_1} \right|, \left| E_{\frac{N}{a_1},m-k_2} \right|, \left| E_{\frac{N}{a_1},m-k_1}' \right|, \left| E_{\frac{N}{a_1},m-k_2}' \right| \right)^{\frac{1}{2}}$$

It is sufficient to consider only the summand involving $a_1$:

$$\leq \sum_{b \in E_{N,k}} \left| E_{\frac{N}{a_1},m-k} \right| \sum_{a \in E_{b,k}} \sum_{a=a', a_2, b=b', b_2} \max \left( \left| E_{\frac{N}{a_1},m-k_1} \right|, \left| E_{\frac{N}{a_1},m-k_2} \right|, \left| E_{\frac{N}{a_1},m-k_1}' \right|, \left| E_{\frac{N}{a_1},m-k_2}' \right| \right)^{\frac{1}{2}}$$

$$\leq \left( \sum_{k'=0}^{k} \sum_{k=0}^{k} \left( \frac{k}{k_1} \right) \left( \frac{k}{k_2} \right) \left| E_{N,m-k_1} \right| \left| E_{N,m-k_2} \right| \right)^{\frac{1}{2}}$$

$$= \left( \sum_{b \in E_{N,k}} \left| E_{\frac{N}{a_1},m-k} \right| \sum_{a \in E_{b,k}} \max \left( \left| E_{\frac{N}{a_1},m-k_1} \right|, \left| E_{\frac{N}{a_1},m-k_2} \right|, \left| E_{\frac{N}{a_1},m-k_1}' \right|, \left| E_{\frac{N}{a_1},m-k_2}' \right| \right)^{\frac{1}{2}} \right)$$

$$\leq \left( \sum_{b \in E_{N,k}} \left| E_{\frac{N}{a_1},m-k} \right| \sum_{a \in E_{b,k}} \max \left( \left| E_{\frac{N}{a_1},m-k_1} \right|, \left| E_{\frac{N}{a_1},m-k_2} \right|, \left| E_{\frac{N}{a_1},m-k_1}' \right|, \left| E_{\frac{N}{a_1},m-k_2}' \right| \right)^{\frac{1}{2}} \right)$$

By Lemma 5.

So

$$\sum_{k=1}^{m} A_3(k) \lesssim \sum_{k=1}^{m} \sum_{k'=0}^{k} \sum_{k=0}^{k} \frac{1}{k! k_1!} \left( \frac{k^2 m}{\log \log N} \right)^{k_1/k_1} \left( \frac{m^2}{\log \log N} \right)^{m-k}. $$

The above sum is bounded when $m \leq (\log \log N)^{\frac{1}{2}}$.

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