Quantum chaos in the framework of complex probability processes

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Abstract

The problem of random motion of 1D quantum reactive harmonic oscillator (QRHO) is formulated in terms of a wave functional regarded as a complex probability process \( \Psi_{stc}(x, t; \{\xi\}) \) in an extended space \( \Xi = \mathbb{R}^1 \otimes \mathbb{R}^{\{\xi\}} \). In the complex stochastic differential equation (SDE) for \( \Psi_{stc}(x, t; \{\xi\}) \) the variables are separated with the help of the Langevin-type model SDE introduced in the functional space \( R^{\{\xi\}} \). The complete positive Fokker-Plank measure of the space \( R^{\{\xi\}} \) is obtained. The average wave function of roaming QRHO is obtained by means of functional integration over the process \( \Psi_{stc}(x, t; \{\xi\}) \) with the complete Fokker-Plank measure in the space \( R^{\{\xi\}} \). The local and averaged transition matrices of roaming QRHO were constructed. The thermodynamics of nonrelativistic vacuum is investigated in detail and expressions for the internal energy, Helmholtzian energy and entropy are obtained. The oscillator’s ground state energy, its shift and broadening are calculated.

1 Introduction

From the point of view of classical dynamics all phenomena, that are described in the framework of conventional theory of quantum mechanics are stochastic processes. The natural symmetry between the Schrödinger and Fokker-Plank equations was used both for the formulation of quantum mechanics as a classical stochastic theory and quantization of classical theory by introduction of the concept of random process.

Note, that in all representations the main object of quantum mechanics, the wave function, is deterministic. It is worthwhile to stress that the deterministic nature of a physical theory is a consequence of the symmetry of basic equations with respect to time inversion.

It is known that the observability is an essential property of objects that is realized when the contact with an external world is established. Therefore, all objects under study are open physical systems. This conclusion implies, in particular, that it is impossible to conform in any reasonable way the probabilistic, irreversible nature of measurements to the reversibility of quantum theory concepts.

Moreover, the stochastic nature is the basic property of the physical world. This assertion becomes especially conclusive if one allows for the real existence of physical vacuum in the nature that is the cause of casual quantum jumps in the isolated systems. In other words, all objects that are investigated taking into account the interaction with fluctuations of the physical vacuum are open physical systems. Whereas the isolated systems are studied by means of methods of conventional quantum mechanics, the behaviour of open systems, in particular of systems that are in equilibrium with the environment, is usually described by the laws of thermodynamics.

Since due to permanent fluctuations the open system has no any definite quantum state, i.e., the state vector of the system randomly changes with time, the construction of a theory alternative to the reversible quantum mechanics becomes urgent. One can construct such a theory if, in particular, the random processes taking place in the system are directly included in the foundations of the theory and are not regarded as external perturbations.

In the present work the problem of stochastic quantum mechanics is investigated using the model of random motion of 1D quantum reactive oscillator (QRHO). Mathematically the problem of an evolution of quantum states is formulated in the framework of probabilistic complex process in an extended space with stochastic space-time continuum. For roaming classical oscillator the conditions of reducibility of a model stochastic
differential equation (SDE) to nonlinear Langevin SDE are investigated. With the help of model SDE the variables in the SDE for a wave functional - the complex probability process, have been separated. Based on the Langevin-type SDE the Fokker-Plank equation is derived and the measure of functional space is obtained. For roaming QRHO the average wave function is constructed in the form of functional integral with positive measure. The local stochastic and averaged transition matrix of the roaming QRHO is calculated analytically. The probability of "vacuum-vacuum" transition is calculated numerically and its behavior is shown versus the fluctuation parameter. The thermodynamics of nonrelativistic vacuum is builded and the expressions for the main thermodynamic potentials are obtained.

2 Formulation of the problem

Let us consider a closed system "quantum object+thermostat" and assume that the quantum object moves in the Euclidean space $R^1$ and the state of the thermostat is characterized by an unlimited set of modes in the functional space $R(\xi)$, where $\{\xi\} \equiv \xi(t, \{W\})$ is a complex functional of some random process $\{W\} \equiv W(t)$. One can mathematically describe the wave state of the system with the help of a functional - a complex asymptotic wave function $(7)$, that admits exact solution (see [6]). In the expression (3) $\Omega$ the wave functional can be written in the most general case as:

\[ \partial^2 = \partial x^2. \]

Let the frequency be given as:

\[ \Omega(t; \{W\}) = \Omega_0(t) + \Omega_1(t; \{W\}) \]

where $\Omega_0(t) > 0$ satisfies the following boundary conditions:

\[ \lim_{t \to \pm \infty} \Omega_0(t) = \Omega_{in(out)} > 0. \]

Note, that when $\Omega_1(t; \{W\}) \equiv 0$, the equations (1)-(3) describes the case of regularly moving QRHO with asymptotic wave function $(3)$, that admits exact solution (see [3]). In the expression $\{3\} \Omega_1(t; \{W\})$ respectively denotes the random part of frequency and satisfies the following conditions:

\[ \lim_{t \to -\infty} \Omega_1(t; \{W\}) = 0, \quad \lim_{t \to +\infty} \Omega_1(t; \{W\}) \neq 0. \]

Next, assume that the total wave functional meets the natural boundary conditions:

\[ \lim_{|x| \to \infty} \Psi_{stc}(x, t; \{\xi\}) = \lim_{|x| \to \infty} \partial_x \Psi_{stc}(x, t; \{\xi\}) = 0. \]

We denote as $\Psi_{stc}(n|x, t; \{\xi\})$ the total wave functional that is evolved in the Euclidean subspace $R^1_{in}$ from the purely $n$-th vibrational state of the quantum harmonic oscillator:

\[ \Psi_{stc}(n|x, t; \{\xi\}) \to_{t \to -\infty} \Psi_{in}(n|x, t) = \]

\[ = \left[ \frac{(\Omega_{in}/\pi)^{1/2}}{2^n n!} \right]^{1/2} \exp \left\{ -i \left( n + \frac{1}{2} \right) \Omega_{in} t - \frac{1}{2} \Omega_{in} x^2 \right\} H_n \left( \sqrt{\Omega_{in}} x \right), \]

\[ n = 0, 1, 2... \]
The objectives of the present investigation are:

a) to establish the criteria for separation of variables in SDE (1)–(2) and determine the random wave functional \( \Psi_{s}^{(+)}(n|x, t; \{ \xi \}) \) in an explicit form;

b) to calculate the wave function of roaming QRHO

\[
\Psi_{br}^{(+)}(n|x, t) = \langle \Psi_{s}^{(+)}(n|x, t; \{ \xi \}) \rangle_{\{ \xi \}},
\]

where the brackets \( \langle \ldots \rangle_{\{ \xi \}} \) stand for the functional integration including the integration over the distribution of coordinate \( \xi \) in the functional space \( R(\xi) \) at instant \( t \);

c) to calculate the transition matrix \( S_{br}^{nm} \) of the roaming QRHO.

3 Derivation of SDE for roaming classical oscillator

We begin our consideration with a second-order equation

\[
\ddot{\xi} + \Omega^{2}(t; \{W\}) \xi = 0,
\]

where

\[
\dot{\xi} = d_{t}\xi(t; \{W\}), \quad d\xi(t; \{W\}) = \left( \partial_{t}\xi + \frac{1}{2}\delta_{W}\xi \right) dt + (\delta_{W}\xi) dW(t).
\]

Remember that the second formula in (10) denotes the Ito differential of the random functional \( \xi(t; \{W\}) \) (see [7]), where the functional derivative is determined in the usual way:

\[
\delta_{W}\xi = \left\{ \frac{\delta\xi(t; W(t'))}{\delta W(t')} \right\}_{t=t'}.
\]

Taking into account the conditions (3)–(4) we find the asymptotical solution of (9) in the \((in)\) channel to be

\[
\xi(t; \{W\}) \underset{t \to -\infty}{\sim} \exp(i\Omega_{in}t).
\]

One is to stress here, that the equation (9) makes sense and describes the behaviour of roaming classical oscillator only when it is reducible to the Langevin-type SDE.

**Theorem:** There exists a set of nonsingular functionals \( \xi(t; \{W\}) \) that reduce the equation (9) to a nonlinear complex Langevin-type SDE, or, that is the same, to a system of two real nonlinear SDE.

**Proof:** Assume that the solution of model equation (9) has the following form:

\[
\xi(t; \{W\}) = \xi_{0}(t) \exp \left( \int_{-\infty}^{t} \phi(t'; \{W'\}) dt' \right),
\]

where \( \xi_{0}(t) \) is the solution of (9) with regular frequency \( \Omega_{0}(t) \) (see [6]). Consider the first total derivative of the random functional (13) with respect to time. Taking into account (10) one has

\[
d_{t}\xi(t; \{W\}) = \left[ \xi_{0t}(t)/\xi_{0}(t) + \phi(t; \{W\}) + \frac{1}{2} \int_{-\infty}^{t} \delta_{W}\phi(t'; \{W\}) dt' + (d_{t}W(t)) \int_{-\infty}^{t} \delta_{W'}\phi(t'; \{W'\}) dt' \right] \xi(t; \{W\}),
\]

\[
\xi_{0t}(t) = d_{t}\xi_{0}(t).
\]

Since for all \( t' < t \)

\[
\delta_{W'}\phi(t'; \{W'\})|_{t'<t} = 0,
\]

\[
\delta_{W'}\phi(t'; \{W'\})|_{t'<t} = 0,
\]

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we can rewrite the expression (14) as
\[ d_t \xi (t; \{ W \}) = [\xi_0 (t) / \xi_0 (t) + \phi (t; \{ W \})] \xi (t; \{ W \}) = \partial_t \xi (t; \{ W \}). \] (16)
Substituting the solution (13) to (9) and taking into account (16) one can find:
\[ \dot{\phi} + 2[\xi_0 (t) / \xi_0 (t)] \phi + \phi^2 + \Omega_0^2 (t) + U (t; \{ W \}) = 0, \] (17)
where the notation was made
\[ U (t; \{ W \}) = \Omega_0^2 (t; \{ W \}) + 2\Omega_1 (t; \{ W \}) \Omega_0 (t). \] (18)
In particular, from (15) and equation (17) the following relations results
\[ \delta_W \xi (t; \{ W \}) = 0, \quad \delta_W \{ \partial_t \xi (t; \{ W \}) \} \neq 0 \] (19)
from which it is seen that the operators \( \delta_W \) and \( \partial_t \) do not commute.
Making the substitution
\[ \phi (t; \{ W \}) = \Phi (t; \{ W \}) - \xi_0 (t) / \xi_0 (t), \] (20)
onelast obtain finally from (17)
\[ \dot{\Phi} + \Phi^2 + \Omega_0^2 (t) + F_0 + F (t; \{ W \}) = 0. \] (21)
In SDE (21) the following notations were made
\[ U (t; \{ W \}) = F_0 + F (t; \{ W \}), \quad F_0 = \langle U (t; \{ W \}) \rangle, \] (22)
and in (22) the averaging of expression (18) is made over the ensemble.
The solution (21) of SDE, taking into account (12), satisfies the initial condition of the type
\[ \lim_{t \to -\infty} \Phi (t; \{ W \}) = i\Omega_{in}, \] (23)
that, in its turn, suggests the complex solution
\[ \Phi (t; \{ W \}) = \theta (t; \{ W \}) + i\phi (t; \{ W \}). \] (24)
After substitution of (24) to (21) and separation of real and imaginary parts we have the following system of SDE:
\[ \dot{\theta} + \theta^2 - \varphi^2 + U_0 (t) + F (t; \{ W \}) = 0, \quad U_0 (t) = \Omega_0^2 (t) + F_0 > 0, \] (25)
\[ \dot{\varphi} + 2\varphi \theta = 0. \] (26)
The system of SDE (25)-(26) can be rewritten in a vector equation:
\[ \dot{\Phi} + \mathbf{K} (t; \{ \Phi \}) + \mathbf{F} (t; \{ W \}) = 0, \] (27)
where the vectors have the following projections
\[ \Phi (t; \{ W \}) = \{ \theta, \varphi \}, \quad \mathbf{K} (t; \{ \Phi \}) = \left\{ [\theta^2 - \varphi^2 + U_0 (t) + F_0] ; 2\theta \right\}, \]
\[ \mathbf{F} (t; \{ W \}) = \left\{ F (t; \{ W \}) ; 0 \right\}. \] (28)
The theorem is proved.
For subsequent investigations of SDE (1)-(2) it is convenient to write SDE (1) in the form
\[ \xi (t; \{ W \}) = \sigma (t; \{ W \}) \exp [ir (t; \{ W \})], \] (29)
where $\sigma(t; \{W\})$ and $r(t; \{W\})$ are yet unknown real functionals:

$$\text{Im} \sigma(t; \{W\}) = \text{Im} r(t; \{W\}) = 0. \quad (30)$$

Taking into account (12) one can establish the asymptotic behaviour of these functionals:

$$\lim_{t \to -\infty} \sigma(t; \{W\}) = 1, \quad r(t; \{W\}) \to t \to -\infty \Omega_in.$$  

(31)

After substitution of (29) to (19) and taking into account (24), (26) and (30) we find that

$$\delta W \sigma(t; \{W\}) = 0, \quad \delta W \partial_t \sigma(t; \{W\}) = 0, \quad (32)$$

as well as that

$$\delta W r(t; \{W\}) = 0, \quad \delta W \partial_t r(t; \{W\}) = 0. \quad (33)$$

It must be noted that the conditions (32)-(33) are highly important for our further discussions.

4 Solution of SDE for complex probabilistic process - the wave functional

Now pass to the solution of SDE (1)-(2) using the Langevin-type model nonlinear SDE for separation of variables.

**Theorem:** If the relations (32)-(33) are fulfilled, then the complex SDE (1)-(2) has an exact solution in the form of orthonormal in $L^2(R^1 \otimes R_{\{\xi\}})$ random complex functionals

$$\Psi_{stc}^{(+)}(n|x, t; \{\xi\}) = \left[ \frac{(\Omega_{in} / \pi)^{1/2}}{2\pi n! \sigma} \right]^{1/2} \exp \left\{ -i \left( n + \frac{1}{2} \right) \Omega_{in} \tau + \frac{\sigma_t x^2}{2\sigma} - \frac{1}{2} r_t x^2 \right\} H_n \left( \sqrt{2\Omega_{in} \frac{x}{\sigma}} \right), \quad (34)$$

where $\tau(t; \{W\}) = r(t; \{W\})/\Omega_{in}$, $\sigma_t = \partial_t \sigma(t; \{W\})$ and $r_t = \partial_t r(t; \{W\})$. Moreover, $L_2(R^1 \otimes R_{\{\xi\}})$ denotes the space of square integrable functions on the real axis $R^1$ with values in some functional space $R_{\{\xi\}}$.

**Proof:** Changing the variables in (1)-(2)

$$x \to y = \frac{x}{\sigma(t; \{W\})}, \quad (35)$$

and taking into account (32)-(33) the equation for complex stochastic process will be as follows:

$$\begin{align*}
\partial_t \Psi_{stc}(y, t; \{\xi\}) &= \left\{ \frac{\sigma_t}{\sigma} y \delta_y - \frac{1}{2\sigma^2} \delta_y^2 + \frac{\sigma^2}{2} \Omega^2(t; \{W\}) y^2 \right\} \tilde{\Psi}_{stc}(y, t; \{\xi\}), \\
\tilde{\Psi}_{stc}(y, t; \{\xi\}) &= \Psi_{stc}(x, t; \{\xi\}).
\end{align*} \quad (36)$$

Now, let us rewrite the solution of equation (36) as

$$\tilde{\Psi}_{stc}(y, t; \{\xi\}) = \left[ \frac{\exp \left[ i2\Lambda(t; \{W\}) y^2 \right]}{\sigma(t; \{W\})} \right]^{1/2} \chi(y, \tau(t; \{W\})). \quad (37)$$

Making the substitution of (37) into (36) and the transformation

$$t \to \tau = \frac{r(t; \{W\})}{\Omega_{in}}, \quad (38)$$

with regard to (32)-(33) one finds
\[i \left( \Lambda - \frac{1}{2} \sigma_t \sigma \right) (\chi + 2y \delta_y \chi) + i \frac{r_t \sigma_t^2}{\Omega_{in}} \delta_r \chi = -\frac{1}{2} \left( \sigma_y^2 - \sigma^2 \left[ 2\Lambda - 4\sigma_t \sigma^{-1} \Lambda + 4\sigma^{-2} \Lambda^2 + \sigma^2 \Omega^2 (t; \{W\}) \right] y^2 \right) \chi, \quad (39)\]

where the functionals \(\sigma (t; \{W\}), r (t; \{W\})\) and \(\Lambda (t; \{W\})\) are still unknown. For their determination we shall suppose that the following relations were observed:

\[r_t (t; \{W\}) = \Omega_{in} \sigma_t (t; \{W\}), \quad (40)\]
\[\Lambda (t; \{W\}) = \frac{1}{2} \sigma_t (t; \{W\}) \sigma (t; \{W\}), \quad (41)\]
\[2 \dot{\Lambda} - 4 \sigma_t \sigma^{-1} \Lambda + 4 \sigma^{-2} \Lambda^2 + \sigma^2 \Omega^2 (t; \{W\}) = \frac{1}{2} \Omega_{in}^2. \quad (42)\]

It is easy to see that the system of equations (40)-(42), taking into account (28) and relations (32)-(33), is equivalent to the model equation (1) and, thereby to Langevin-type nonlinear SDE (21) with the complex initial condition (23). Using the expressions (40)-(42) one can finally obtain from (39):

\[i \delta_r \chi (y, \tau) = \frac{1}{2} \left( \sigma_y^2 + \Omega_{in}^2 y^2 \right) \chi (y, \tau). \quad (43)\]

The equation (43) describes an autonomous system of the quantum harmonic oscillator on the stochastic space-time continuum. The solution of equation (43) is obtained by conventional methods and has the form

\[\chi (y, \tau) = \left[ \frac{(\Omega_{in}/ \pi)^{1/2}}{2^n n!} \right]^{1/2} \exp \left\{ -i \left( n + \frac{1}{2} \right) \Omega_{in} \tau - \frac{1}{2} \Omega_{in} y^2 \right\} H_n \left( \sqrt{\Omega_{in}} y \right). \quad (44)\]

Combining (37) and (44) for the wave functional, evolving from the initial pure state (4), it is easy to find the expression (34). Taking into account (22)-(24) one can find the relation between the stochastic time \(\tau (t; \{W\})\) and the apparent time, the natural parameter \(t\),

\[\tau = \int_{-\infty}^{t} \frac{dt'}{\sigma^2 (t'; \{W\})}. \quad (45)\]

Now take note of the very important property of the wave functional

\[\langle \Psi_{stc}^{(+)} (m|x, t; \{\xi\}) \Psi_{stc}^{(+)} (n|x, t; \{\xi\}) \rangle_x = \delta_{mn}, \quad \langle \ldots \rangle = \int_{-\infty}^{+\infty} dx, \quad (46)\]

pointing to the fact that the basis formed in the space \(L_2 (R^1 \otimes R_{\{\xi\}})\) is orthonormal.

*The theorem is proved.*

5 Derivation of Fokker-Plank equation for conditional probability \(P (\Phi, t|\Phi', t')\)

Let us consider the functional of the form:

\[P (\Phi, t|\Phi', t') = \langle \delta [\Phi (t) - \Phi (t')] \rangle, \quad (47)\]

where \(\Phi (t) \equiv \Phi (t; \{W\})\) is the solution of SDE (27). After the differentiation of functional (47) over the time and use of (27) one can obtain:
\[ \partial_t P(\Phi, t|\Phi', t') = -\partial_{\Phi} \left\langle \Phi \, \delta [\Phi(t) - \Phi(t')] \right\rangle = \]

\[ = \partial_{\Phi} \{ K(t; \{ \Phi \}) P(\Phi, t|\Phi', t') + \langle F(t; \{ W \}) \rangle \} \delta [\Phi(t) - \Phi(t')] \}. \quad (48) \]

To obtain the equation for conditional probability (47) in an explicit form one has to specify the stochastic frequency \( F(t; \{ W \}) \). Consider the most common case when the stochastic component \( \epsilon \) is the Gaussian random function \( F(t; \{ W \}) = W = F(t) \). This assumption implies that one can completely define \( F(t) \) with the help of correlation function that is a white noise type correlator in case when \( F(t) \) changes faster than the solution \( \theta(t) \)

\[ \langle F(t) F(t') \rangle = 2\varepsilon \delta(t - t'), \quad \langle F(t) \rangle = 0, \quad \varepsilon > 0. \quad (49) \]

Now, using the Vick's theorem (see [7])

\[ \langle F(t) N(t; \{ F \}) \rangle = 2 \left\langle \frac{\delta N}{\delta F} \right\rangle_{\{ F \}}, \quad (50) \]

where \( N(t; \{ F \}) \) is an arbitrary functional of \( F(t) \), one can write the following expression

\[ \langle F(t; \{ W \}) \rangle \delta [\Phi(t) - \Phi(t')] = 2\varepsilon \left\langle \frac{\delta \theta(t)}{\delta F(t)} \cdot \delta [\Phi(t) - \Phi(t')] \right\rangle. \quad (51) \]

Due to the stochasticity of \( \theta(t) \) the variational derivative of \( F(t) \) is \( \varepsilon \cdot \text{sgn}(t - t') + O(t - t') \). After common regularization procedure (in the sense of Fourier decomposition) one can find the value at \( t = t' \):

\[ \varepsilon \cdot \text{sgn}(0) = \frac{1}{\varepsilon} \varepsilon. \]

Taking into account the aforesaid we finally obtain the following Fokker-Plank equation for conditional probability:

\[ \partial_t P(\Phi, t|\Phi', t') = \sum_{i,j=1}^2 \partial_{\Phi_i} \left[ K_i(t; \{ \Phi \}) + \varepsilon_{ij} \partial_{\Phi_j} \right] P(\Phi, t|\Phi', t'), \quad (52) \]

where

\[ \varepsilon_{11} = \varepsilon, \quad \varepsilon_{12} = \varepsilon_{21} = \varepsilon_{22} = 0, \quad \Phi_1 = \theta, \quad \Phi_2 = \varphi \]

\[ K_1(t; \{ \Phi \}) = [\theta^2 - \varphi^2 + U_0(t)], \quad K_2(t; \{ \Phi \}) = 2\theta \varphi. \quad (53) \]

Note, that the equation (52) determines a diffusion process for which \( \Phi(t) \) is continuous.

Let the conditional probability meet the boundary condition

\[ P(\Phi, t|\Phi', t') = \delta(\Phi - \Phi'), \quad (54) \]

then for small time intervals it is easy to determine the solution of equation (52):

\[ P(\Phi, t|\Phi', t') = \frac{1}{2\pi \sqrt{\varepsilon \Delta t}} \times \]

\[ \exp \left\{ -\frac{1}{2\Delta t} \left[ \Phi - \Phi' - K(t; \{ \Phi \}) \Delta t \right] \right\} \exp \left\{ -\frac{1}{2\varepsilon \Delta t} \left[ \theta - \theta' - (\theta^2 - \varphi^2 + U_0(t)) \Delta t \right] \right\}, \quad t = t' + \Delta t. \quad (55) \]

Thus, one can state that the evolution of the system in the functional space \( R(\Phi) \) is characterized by a regular shift with the velocity \( K(t; \{ \Phi \}) \), against the background of which the Gaussian fluctuations with diffusion matrix \( \varepsilon_{ij} \) take place. As to the trajectory \( \Phi(t) \) in the space \( R(\Phi) \), it is determined by the formula (see [7])
\[ \Phi (t + \Delta t) = \Phi (t) + K(t; \Phi) \Delta t + F(t) \Delta t^{1/2}. \] (56)

The trajectory is seen from (56) to be continuous everywhere, i.e. \( \Phi (t + \Delta t) \rightarrow \Phi (t) \), but is undifferentiable everywhere owing to the presence of term \( \sim \Delta t^{1/2} \). If we write the time interval in the form \( \Delta t = t/N \), where \( N \rightarrow \infty \), then one can interpret the expression (55) as the probability of transition from \( \Phi_k = \Phi (t') \) to \( \Phi_{k+1} = \Phi (t) \) during the time \( \Delta t \) in the model of Brownian motion.

### 6 Solution of Fokker-Plank equation for the distribution of \( \theta \) coordinate in the limit \( t \rightarrow +\infty \)

First, let us consider the equation (26). Taking into account (24) as well as the initial condition (23) one can have the solution of equation (26) in the explicit form:

\[ \varphi = \Omega_{in} \exp \left( -2 \int_{-\infty}^{t} \theta (t') dt' \right) \] (57)

Using (57) one can reduce the system of SDE (25)-(26) to one nonlinear and nonlocal SDE of the form:

\[ \dot{\theta} + \theta^2 - \Omega_{in}^2 \exp \left( -4 \int_{-\infty}^{t} \theta (t') dt' \right) + U_0 (t) + F(t) = 0. \] (58)

Using SDE (58) one can obtain the equation for conditional probability \( Q (\theta, t) = P (\theta, t|0, 0) \) in the sense of distribution of coordinate \( \theta \) at the time \( t \):

\[ \partial_t Q (\theta, t) + \partial_{\theta} J \left( \theta, \int_{-\infty}^{t} \theta (t') dt' ; t \right) = 0 \] (59)

with the flow of probability

\[ J \left( \theta, \int_{-\infty}^{t} \theta (t') dt' ; t \right) = - \left[ \theta^2 - \Omega_{in}^2 \exp \left( -4 \int_{-\infty}^{t} \theta (t') dt' \right) + U_0 (t) \right] Q - \varepsilon \partial_{\theta} Q \] (60)

In view of the fact that in the equation (59) the factor of the shift \( \left[ \theta^2 - \Omega_{in}^2 \exp \left( -4 \int_{-\infty}^{t} \theta (t') dt' \right) + U_0 (t) \right] \)
tends to infinity at the limits \( \theta \rightarrow \mp \infty \), a symmetric non-vanishing flows of probability \( J (-\infty, +\infty ; t) = J (+\infty, -\infty ; t) \neq 0 \) there arise on the borders. For solution of equation (59)-(60) one has to require the natural initial and boundary conditions

\[ \lim_{t \to t_c} Q (\theta; t) = \delta (\theta), \] (61)

\[ \lim_{|\theta| \to \infty} Q (\theta; t) = 0. \] (62)

The investigation of equation (59)-(60) with initial and boundary conditions for arbitrary \( t \) is a very difficult problem. However, below we shall see that for the construction of averaged matrix in the model of roaming QRHO it is important to know the distribution of \( \theta \) coordinate when \( t \rightarrow +\infty \).

Turning again to the solution of (57) note, that in the limit of large time intervals the self-averaging of trajectory \( \theta (t) \) takes place by virtue of the ergotic hypothesis, so that

\[ \lim_{t \to +\infty} \varphi (t) = \alpha \Omega_{in}, \quad \alpha = \exp \left( -2 \int_{-\infty}^{+\infty} \theta (t') dt' \right). \] (63)
We should like to remind, that by virtue of the same ergodic hypothesis the constant \( \alpha \) can be calculated by averaging over the ensemble.

Now taking into account that for \( t \to +\infty \) the density of flow reaches its limiting value

\[
J_{0 f} = -\lim_{t \to +\infty} \int \left( \theta, \int \theta \left( t' \right) dt' ; t \right),
\]

we obtain the equation for the probability of distribution of a stationary process,

\[
J_{0 f} = [\theta^2 - \alpha^2 \Omega_{in}^2 + U_+] Q_s - \varepsilon d\theta Q_s, \quad d\theta = d/d\theta,
\]

\[
U_+ = \lim_{t \to +\infty} U_0 \left( t \right).
\]

In Eq. (65) the probability distribution for a stationary process is denoted by \( Q_s \) that is obtained from Eq. (65) to be [9]

\[
Q_s \left( \varepsilon, \lambda; \gamma; \theta \right) = \varepsilon^{-1/3} \tilde{Q}_s \left( \lambda; \gamma; \tilde{\theta} \right) = \varepsilon^{-2/3} J_{0 f} \exp \left( -\frac{\tilde{\theta}^3}{3} - \lambda \gamma \tilde{\theta} \right) \int_{-\infty}^{\tilde{\theta}} d\tilde{z} \exp \left( \left( \frac{\tilde{z}^3}{3} + \lambda \gamma \tilde{z} \right) \right),
\]

where \( \lambda = \left( \Omega_{in}/\varepsilon^{1/3} \right)^2, \gamma = \left( \Omega_{out}/\Omega_{in} \right)^2 - \alpha^2 + F_0/\Omega_{in}^2 \) and \( \tilde{\theta} = \theta/\varepsilon^{1/3} \). The constant \( J_{0 f} \) is calculated from the condition of normalization of stationary distribution \( Q_s \) to unity and has the form

\[
J_{0 f}^{-1} = \pi^{1/2} \varepsilon^{-1/3} \int_0^{\infty} dz \ v^{-1/2} \exp \left( -\frac{\tilde{z}^3}{12} - \lambda \gamma \tilde{z} \right).
\]

One can obtain representation of \( J_{0 f} \) in terms of special functions. Passing to Fourier components in equation (63) we find [10]:

\[
\mathcal{F}^{-1}_{0 f} = \left[ A i^2 \left( -\lambda \gamma \right) + B i^2 \left( -\lambda \gamma \right) \right],
\]

where \( A i \left( x \right) \) and \( B i \left( x \right) \) are linear independent solutions of Airy equation [11]

\[
y'' - xy = 0.
\]

Note, that the normalization constant \( J_{0 f} \) has one important feature, namely, it defines the number of states with energy values \( E \) from \( -\infty \) up to \( \Omega_{out}^2 \) per unit of distance. In the case when \( \left( E/\varepsilon^{2/3} \right) \gg 1 \) and \( E > 0 \) the expression (67) is calculated asymptotically:

\[
N_{\Sigma} = J_{0 f} \approx \pi^{-1} E^{1/2} \left[ 1 + \frac{5}{32} \frac{\varepsilon^2}{E^3} + O \left( \frac{\varepsilon^4}{E^6} \right) \right].
\]

One can use expression (63) for determining of states distribution for concrete energy value. Taking into account the fact that bound states are characterized by negative energy values and making in equation (67) the formal substitution \( \Omega_{out}^2 + F_0 \to -E \) one can obtain expression for the number of states with concrete localization energy:

\[
N_E = \pi^{-1} \varepsilon^{1/2} \int_0^{\infty} dz z^{-1/2} \exp \left( -\frac{\varepsilon^3}{12} + \frac{E}{\varepsilon^{2/3}} z \right).
\]

In the case when \( \left( E/\varepsilon^{2/3} \right) \gg 1 \) and \( E > 0 \) the expression (71) is calculated asymptotically and have the following form:
\[ N_E \approx \pi^{-1} E^{1/2} \exp \left( -\frac{4}{3} \frac{E^{3/2}}{\varepsilon} \right) \left[ 1 + O \left( \frac{\varepsilon}{E^{3/2}} \right) \right]. \] (72)

It is easy to calculate now the states distribution for concrete value of bound state energy:

\[ P_E = \frac{N_E}{N_\Sigma} \approx \exp \left( -\frac{4}{3} \frac{E^{3/2}}{\varepsilon} \right) \left[ 1 + O \left( \frac{\varepsilon}{E^{3/2}} \right) \right]. \] (73)

It is worthwhile at the end of this section to pay attention to one important property of the stationary distribution \( Q_s \), that follows, in particular, from numerical analyses. The fact is that for all \( \gamma \in (-\infty, +\infty) \) the probability of positive values of the \( \theta \) coordinate is higher than that for negative values (see FIG.1). It follows, hence, that \( \alpha = 0 \) and \( \varphi_s = 0 \), where \( \varphi_s \) is the imaginary part of complex coordinate \( \Phi \) in the stationary process limit.

### 7 Calculation of the wave function of roaming QRHO

Having at hand the expression for conditional probability \( \tilde{\mu} \) one can now pass to the averaging of the wave functional in the \( R(\xi) \) space, or, in other words, to the calculation of wave function of roaming QRHO. Writing the wave functional \( \tilde{\Psi}_{stc} \) in the moving reference system (see (24)), and taking into account the formula (7) for the average value of wave functional, we obtain

\[ \Psi_{br}^{(+)}(x,t) = \left\langle \tilde{\Psi}_{stc}^{(+)}(x,t;\{\xi\}) \right\rangle_{\{\xi\}} = \left\langle \tilde{\Psi}_{stc}^{(+)}(x,t;\{\Phi\}) \right\rangle_{\{\Phi\}} = \frac{1}{\alpha} \int D\mu \{ \Phi \} \tilde{\Psi}_{stc}^{(+)}(x,t;\{\Phi\}), \] (74)

In this formula \( D\mu \{ \Phi \} \) determines the total Fokker-Plank measure of the functional space \( R(\Phi) \):

\[ D\mu \{ \Phi \} = d\mu \{ \Phi_0 \} \cdot d\mu \{ \Phi_t \} \lim_{N \to \infty} \left( \frac{1}{\sqrt{\pi \varepsilon t}} \right)^N \times \prod_{k=0}^{N} \exp \left\{ -\frac{N}{\varepsilon t} \left[ \theta - \theta' - (\theta^2 + \varphi^2 + U_0(t)) \right] \right\} d\theta_{k+1} d\varphi_{k+1}, \] (75)

where \( \alpha, d\mu \{ \Phi_0 \} \) and \( d\mu \{ \Phi_t \} \) are determined with the help of formulae

\[ \alpha (\varepsilon, t) = \int D\mu \{ \Phi \}, \]

\[ d\mu \{ \Phi_0 \} = \delta (\theta_0) \delta (\Omega_{in} - \varphi_0) d\theta d\varphi, \]

\[ d\mu \{ \Phi_t \} = P (\Phi, t | 0, 0) d\theta d\varphi. \] (76)

In equations (73)–(76) \( \alpha(\varepsilon, t) \) denotes the normalization factor for functional integral (74) with total Fokker-Plank measure and in the limit \( t \to +\infty \) it acquires a constant value that, in general, is different from the unity. The integration over the measure \( d\mu \{ \Phi_0 \} \) means the averaging in the initial distribution of complex coordinate \( \Phi \) at the moment of time \( t = -\infty \). As for the integration over the measure \( d\mu \{ \Phi_t \} \), it provides the averaging in the distribution of complex coordinate \( \Phi \) at an arbitrary instant \( t \). In particular, one can show that in the limit \( t \to +\infty \) it assumes the following form:

\[ d\mu \{ \Phi_\infty \} = d\mu \{ \Phi_s \} = Q_s(\varepsilon, \lambda, \gamma; \theta) d\theta d\varphi. \] (77)

Now we shall try to study the (out) asymptotical state of the "oscillator+thermostat" system.

Taking into account the relation (3) one can separate the fluctuation process at the frequency (3) in the span of time \( (t_c, +\infty) \), where \( t_c \) is the zero time reference point in the (out) channel, by giving it with the help of white noise correlator with the diffusion constant \( \varepsilon_+ \). It is evident, hence, that the (out) asymptotic state of the "oscillator+thermostat" system will be also characterized by a complex probabilistic process \( \Psi_{out} (m|x,t;\{\eta_+\}) \).
in the extended space $\Xi_{\text{out}} = R_{\text{out}}^1 \otimes R_{\{\eta\}}$, where $\eta_+ = \exp \left(i \Omega_{\text{out}} t + \int_{t_0}^t \Psi_+(t') dt'\right)$ is the corresponding solution of SDE (3). Proceeding with the same reasoning one can construct the wave function of the (out) asymptotic state $\Psi_{br} (m|x, t)$.

At the end of this section we should like to note that in the limit $\varepsilon \to 0$ the measure of the functional space $R_{\{\xi\}}$ turns out to be of the Wiener Type, and the wave function of roaming QRHO, $\Psi_{\text{out}}^{(+)} (n|x, t)$ steadily transforms to the exact solution $\Psi^{(+)} (n|x, t)$ that describes the motion of regularly moving QRHO.

8 The local stochastic matrix of transitions of roaming QRHO

As was mentioned above, the roaming QRHO has two asymptotic states: $\Psi_{\text{in}} (n|x, t) \in L_2 \left(R_{\text{in}}^1\right)$ when $t \to -\infty$ (see [11]) and respectively $\Psi_{\text{out}} (m|x, t; \{\eta\}) \in L_2 \left(R_{\text{out}}^1 \otimes R_{\{\eta\}}\right)$ when $t \to +\infty$. The goal of the theory of scattering is to construct a unitary operator that will change one of these states to the other [10].

Prior to that let us consider the wave functional $\Psi_{\text{stc}}^{(-)} (m|x, t; \{\eta\})$ that is the solution of SDE (1)-(2) and passes to the asymptotic wave functional $\Psi_{\text{out}} (m|x, t; \{\eta\})$ when $t = t_+ > t_c$, which in its turn passes to the wave function $\Psi_{\text{out}} (m|x, t)$ when $t = t_c$. Since the sets of functionals $\Psi_{\text{stc}}^{(+)} (m|x, t; \{\xi\})$ and $\Psi_{\text{stc}}^{(-)} (m|x, t; \{\eta\})$ form completely orthonormalized bases in $L_2$, one can write the following decompositions:

$$\Psi_{\text{stc}}^{(+)} (n|x, t; \{\xi\}) = \sum_k S_{kn} (t; \{\xi\} | t'; \{\eta\}) \Psi_{\text{stc}}^{(-)} (k|x, t'; \{\eta\}),$$

$$\Psi_{\text{stc}}^{(-)} (m|x, t; \{\eta\}) = \sum_k S^*_{km} (t'; \{\xi\} | t; \{\eta\}) \Psi_{\text{stc}}^{(+)} (k|x, t'; \{\xi\}),$$

where the coefficients $S_{kn}$ and $S^*_{km}$ are some random local evolutionary operators. Taking into account the orthogonality relations [11], one can obtain from (78)

$$S_{nm} (t; \{\xi\} | t'; \{\eta\}) = \langle \Psi_{\text{stc}}^{(+)} (n|x, t; \{\xi\}) \Psi_{\text{stc}}^{(-)} (m|x, t', \{\eta\}) \rangle_{x}.$$  (80)

One can easily find from expressions (78) and (79), with due regard for (16) and (80),

$$\sum_k S_{kn} (t; \{\xi\} | t'; \{\eta\}) S^*_{km} (t; \{\xi\} | t'; \{\eta\}) = \delta_{nm},$$  (81)

that is basically a generalization of the unitarity condition for the stochastic matrix $S_{\text{stc}}$, the matrix elements of which are determined by means of the formula (80).

Note that the unitarity of the stochastic matrix $S_{\text{stc}}^* S_{\text{stc}} = I$ is a consequence of conservation of laws in the closed system "oscillator+thermostat", that is defined in the extended space $\Xi$. It is convenient to carry out explicit calculations of the elements of stochastic matrix (80) by means of the method of generating functionals. Let us consider the sum

$$\Psi_{\text{stc}}^{(+)} (z|x, t; \{\xi\}) = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} \Psi_{\text{stc}}^{(+)} (n|x, t; \{\xi\}),$$  (82)

where $z$ is an auxiliary function. If we substitute the expression for wave functional (14) in (82) and make an appropriate summation (see [11]), then we shall have

$$\Psi_{\text{stc}}^{(+)} (z|x, t; \{\xi\}) = \left(\frac{\Omega_{\text{in}}}{\pi}\right)^{\frac{1}{4}} \exp \left\{ -\frac{1}{2} \left(ax^2 - 2bx + c\right) \right\},$$  (83)

where the coefficients $a$, $b$ and $c$ are
\[ a(t; \{\xi\}) = -i \frac{\dot{\xi}(x; \{W\})}{\xi(x; \{W\})}, \]
\[ b(t; \{\xi\}) = \sqrt{2\Omega_{in}} \frac{\xi(t; \{W\})}{\xi(t; \{W\})}, \]
\[ c(t; \{\xi\}) = z^2 \exp [-i2r(t; \{W\})]. \]  

As it is seen from (83)-(84), the generating functional in \( x \) coordinate is a stochastic Gaussian packet that in the limit \( t \to -\infty \) goes into an ordinary Gaussian packet:

\[ \Psi_{stc}^{(+)}(z|x, t; \{\xi\}) \to_{t\to-\infty} \Psi_{in}(z|x, t) = \left( \frac{\Omega_{in}}{\pi} \right)^{\frac{1}{4}} \times \]
\[ \exp \left\{ -\frac{1}{2} \left( \Omega_{in}x^2 - 2\sqrt{\Omega_{in}}zx \exp (-i\Omega_{in}t) + z^2 \exp (-i2\Omega_{in}t) + i\Omega_{in}t \right) \right\}, \]  

(85)

As for the generating functional of the (out) state, it is easily obtained from (83)-(84) by making simple formal substitutions \( z \to z_+ \), \( \Omega_{in} \to \Omega_{out} \), \( \xi \to \eta_+ \) and \( t \to t_+ \):

\[ \Psi_{out}(z_+|x, t_+; \{\eta_+\}) = \left( \frac{\Omega_{in}}{\pi} \right)^{\frac{1}{4}} \eta_+^{-1/2} \exp \left\{ -\frac{1}{2} \left( a_+x^2 - 2b_+x + c_+ \right) \right\}, \]  

(86)

where the following notations are made:

\[ a_+(t_+; \{\eta_+\}) = -\frac{\eta_{(t_+; \{W_+\})}}{\eta_+(t_+; \{W_+\})}, \]
\[ b_+(t_+, z_+; \{W_+\}) = \frac{\sqrt{2\Omega_{out}}z_+}{\eta_+(t_+; \{W_+\})}, \]
\[ c_+(t_+; \{\eta_+\}) = z_+^2 \exp [-i2r_+(t_+; \{W_+\})], \]
\[ r_+(t_+; \{W_+\}) = \Omega_{out} \int_{t_0}^{t_+} \frac{dt'}{\eta_+(t'; \{\eta_+^{*}\})}. \]  

Now consider the following integral:

\[ I(z, t; \{\xi\} | z_+, t_+; \{\eta_+^{*}\}) = \left\langle \Psi_{stc}^{(+)}(z|x, t; \{\xi\}) \Psi_{out}(z_+|x, t_+; \{\eta_+\}) \right\rangle_x. \]  

(88)

Making appropriate substitutions for the generating functionals from (83)-(84) and (85)-(86) and integrating over the coordinate \( x \) we obtain

\[ I(z, t; \{\xi\} | z_+, t_+; \{\eta_+^{*}\}) = (\Omega_{in}\Omega_{out})^{1/4} \left[ \frac{2}{A\eta_+^{*}} \right]^{1/2} \exp \left\{ -\frac{1}{2} \left( C - \frac{B^2}{A} \right) \right\}, \]  

(89)

with the following notations:

\[ A(t; \{\xi\} | t_+; \{\eta_+^{*}\}) = -\frac{\dot{\xi}}{\xi} + i \frac{\dot{\eta}_+^{*}}{\eta_+^{*}}, \]
\[ B(t, z; \{\xi\} | t_+, z_+; \{\eta_+^{*}\}) = \frac{\sqrt{2\Omega_{out}}z_+}{\xi} + \frac{\sqrt{2\Omega_{out}}z_+^{*}}{\eta_+^{*}}, \]
\[ C(t; \{\xi\} | t_+; \{\eta_+^{*}\}) = \exp (-i2r) z^2 + \exp (i2r_+) \left( z_+^{*} \right)^2. \]  

(90)

It is easy to show that \( I(z, t; \{\xi\} | z_+, t_+; \{\eta_+^{*}\}) \) is a generating functional for stochastic matrix elements:

\[ I(z, t; \{\xi\} | z_+, t_+; \{\eta_+^{*}\}) = \sum_{n, m=0}^{\infty} \frac{z^n z'^m}{\sqrt{n!m!}} S_{nm}(t; \{\xi\} | t_+; \{\eta_+^{*}\}). \]  

(91)
Making expansion in the left hand side of expression (91) in the Taylor series in \( z \) and \( z'_+ \), we find

\[
S_{nm} (t; \{\xi\} | t_+; \{\eta'_+\}) = \frac{1}{\sqrt{n!}} \left[ \partial^n_{z_+} \partial^n_{z'_+} I (z, t; \{\xi\} | z'_+, t_+; \{\eta'_+\}) \right]_{z = z'_+ = 0}.
\]  

(92)

To calculate several first elements of the local stochastic transition matrix one can substitute (89) to (92):

\[
S_{00} (t; \{\xi\} | t_+; \{\eta'_+\}) = (4\Omega_{in}\Omega_{out})^{1/4} \exp \left( \frac{\xi^2}{2} \right) \left( \xi \xi^{-1} - \eta'_+ (\eta'_+)^{-1} \right)^{-1/2},
\]

\[
S_{11} (t; \{\xi\} | t_+; \{\eta'_+\}) = [S_{00} (t; \{\xi\} | t_+; \{\eta'_+\})]^3,
\]

\[
S_{20} (t; \{\xi\} | t_+; \{\eta'_+\}) = S_{00} \left[ - \exp (-i2r) + \left( \frac{\Omega_{in}}{\Omega_{out}} \right)^{1/2} \xi^{-1} \eta'_+ S_{00}^2 \right],
\]

\[
S_{02} (t; \{\xi\} | t_+; \{\eta'_+\}) = S_{00} \left[ - \exp (i2r) + \left( \frac{\Omega_{out}}{\Omega_{in}} \right)^{1/2} \xi (\eta'_+)^{-1} S_{00}^2 \right].
\]  

(93)

One can see from expressions (93) that in analogy to the case of regular QRHO the transitions will occur only between asymptotic states of similar parity irrespective of the value of diffusion constant \( \varepsilon \). But in contrast to the regular case, it is characteristic that the symmetry in local matrix elements (93) with respect to the transposition of quantum numbers of initial \( n \) and final \( m \) channels is violated.

9 The averaged transition matrix of roaming QRHO. The probability of ”vacuum-vacuum” transition

The ultimate objective of our study is the calculation of transition probabilities of roaming QRHO. This aim in view it is necessary first to make the averaging of local stochastic elements (93).

**Definition 1.** The expression

\[
S_{nm}^{br} = \lim_{t \to +\infty} \lim_{t_+ \to +\infty} \left\langle \left\langle \left\langle \hat{S}_{nm} (t; \{\Phi + f\} | t_+; \{\Phi_+ + f_+\}) \right| \mathcal{F} \right| \{\Phi_+\} \right\rangle \right\rangle
\]

(94)

is termed the averaged (generalized) transition matrix of roaming QRHO, where the local \( S_{nm} (t; \{\Phi + f\} | t_+; \{\Phi_+ + f_+\}) \) matrix element in (94) is obtained from (93) by writing the latter in an arbitrary movable reference frame with the help of transformation (20). Remember that the first bracket in (94) means a functional integration over the measure

\[
\mathcal{D} f (t) = \lim_{N \to \infty} \prod_{k=0}^{N} \delta (f (t_k) - f_0 (t_k)) df (t_k), \quad t_k = \frac{k}{N},
\]

(95)

\[
f_0 (t) = \left\{ \text{Re} \left[ \frac{\xi_{in} (t)}{\xi_{out} (t)} \right]; \text{Im} \left[ \frac{\xi_{in} (t)}{\xi_{out} (t)} \right] \right\}.
\]

The second bracket respectively means the integration over the measure

\[
\mathcal{D} f_+ (t) = \mathcal{D} f_0 (t) = \lim_{N \to \infty} \prod_{k=0}^{N} \delta (f_+ (t_k) - f_{0+} (t_k)) df_+ (t_k), \quad t_k = \frac{k}{N},
\]

(96)

\[
f_{0+} = f_{0+} = \{0; \Omega_{out}\}.
\]

In (94) the third and forth brackets respectively stand for integration in \( R_{\{\Phi_+\}} \) and \( R_{\{\Phi_+\}} \) spaces.

Note that the integration over variables \( f (t) \) and \( f_+ (t) \) gives the average value of transition matrix in the steady (laboratory) frame of reference. It is also noteworthy that, as it follows from expressions (80)-(81), the generalized transition matrix \( S^{br} \) is usually not a unitary matrix, i.e., \( S^{br} (S^{br})^+ \neq 1 \). Nevertheless, there are
some limiting cases when one can simplify both the measures of functional spaces $R_{\{\Phi\}}$ and $R_{\{\Phi_+\}}$ (by making these of the Wiener type), and the corresponding wave functionals. As a result of these simplifications, the contribution to the expression \(B\) is made only by the integrals over the final distributions of coordinates $\theta$ and $\theta_+$ and that, in its turn, makes unitary the generalized transition matrix $S_W^{br}(S_W^{br})^+ = I$, where the index $w$ means simplification of the matrix in the above sense.

As an illustration of the proposed approach, consider now the probability of "vacuum-vacuum" transition in the case when the spaces $R_{\{\Phi\}}$ and $R_{\{\Phi_+\}}$ have the Wiener-type measures. After the substitution of the expression for $S^{00}(t; \{\Phi\} | t_+; \{\Phi_+\})$ to (94) and simple integration we find

$$S^{br}_{00}(\lambda, \lambda_+; \rho) = (1 - \rho)^{1/4} \{I_1(\lambda, \lambda_+; \rho) - iI_2(\lambda, \lambda_+; \rho)\},$$

where the following notations were made:

$$I_{1,2}(\lambda, \lambda_+; \rho) = \int_{-\infty}^{+\infty} d\bar{\theta}Q_s(\lambda, \gamma; \bar{\theta}) \int_{-\infty}^{+\infty} d\bar{\theta}_+Q_s(\lambda_+, \gamma; \bar{\theta}_+) \left( a + \frac{1}{2a^2} \right)^{1/2},$$

$$a(\lambda; \bar{\theta} | \lambda_+; \bar{\theta}_+) = \left[ 1 + \frac{1}{\lambda \gamma} \left( \bar{\theta} - \sqrt{\frac{\lambda}{\lambda_+}} \bar{\theta}_+ \right) \right]^{1/2}.$$

In above formulae $\rho$ stands for the reflection coefficient from a barrier in the one-dimensional problem of quantum mechanics with momentum $K(x) = \Omega_0(x)$, where $t$ was replaced by $x$ (see [12]). As for the function $\gamma(\rho)$, for the frequency model in the form of step barrier

$$\Omega_0(t) = \Omega_{in} + (\Omega_{out} - \Omega_{in}) \text{sign}(t)$$

it has the form

$$\gamma(\rho) = \left( \frac{\Omega_{out}}{\Omega_{in}} \right)^2 = \left( 1 + \rho^{1/2} \right)^2.$$ (101)

Now one can write the final expression for the amplitude of "vacuum-vacuum" transition probability

$$\Delta^{br}_{0 \to 0}(\lambda, \lambda_+; \rho) = \sqrt{1 - \rho} \{I_1^2(\lambda, \lambda_+; \rho) + I_2^2(\lambda, \lambda_+; \rho)\},$$

with due regard for the modification of the nature of fluctuation process in the "oscillator-thermostat" system.

In case when $\varepsilon_+ \to 0$, i.e., the final state of the system is described by the wave function, the expression (101) is strongly simplified

$$\Delta^{br}_{0 \to 0}(\lambda; \rho) = \sqrt{1 - \rho} \{I_1^2(\lambda; \rho) + I_2^2(\lambda; \rho)\},$$

$$I_{1,2}(\lambda; \rho) = \int_{-\infty}^{+\infty} d\bar{\theta}Q_s(\lambda, \gamma; \bar{\theta}) \left( \frac{\bar{a} + 1}{2\bar{a}^2} \right)^{1/2},$$

$$\bar{a}(\lambda, \gamma; \bar{\theta}) = \left( 1 + \frac{\bar{\theta}^2}{\lambda \gamma} \right)^{1/2}.$$ (105)

The numerical calculations of "vacuum-vacuum" transition probability by the formulae (103)-(105) show its nonmonotonic behaviour (see FIG.2).
10 Calculation of average values of dynamical variables. Thermodynamics of the nonrelativistic vacuum in asymptotic subspace $R^1_{as}$

It is well known that the main object of quantum statistical mechanics is the density matrix that can be written in the nonstationary representation as (see [19])

$$\rho(x, x', t) = \sum_m w_m \varphi_m(x, t) \varphi_m(x', t),$$

(106)

where $w_m$ is the probability that at the moment of time $t = 0$ the system is in the state $\varphi_m(x, t)$. The function $\varphi_n(x)$ is the solution of Schrödinger equation that satisfies the initial condition

$$\varphi_m(x) = \varphi_m(x, t)|_{t=0},$$

(107)

At $t = 0$ the following density matrix is defined with the help of a set of wave functions $\varphi_m(x)$:

$$\rho(x, x') = \sum_m w_m \varphi_m(x) \varphi_m(x').$$

(108)

Note that the density matrix in the form of (108) was first defined by Dirac and von Neuman (see [20]), the following microcanonical distribution being introduced for the coefficient $w_m$:

$$w_m = \exp \left( -\frac{E_m}{kT} \right),$$

(109)

where $E_m$ is the energy of quantum level $m$, $k$ is the Boltzmann constant and $T$ is the temperature of the thermostat.

In particular, it follows from the definition (106)-(108) that if at an initial moment of time $t = 0$ the system is in the state $\varphi_m(x, 0)$ with probability $w_m$, then the probability for the system to be in the state $\varphi_m(x, t)$ at the moment of time $t$ will be the same. One should note that the representation (108) is valid only if the quantum system weakly interacts with the thermostat.

Below we shall study some relaxation processes that occur with the energy spectrum of vacuum-immersed quantum harmonic oscillator (QHO), as well as calculate the thermodynamic potentials of vacuum state when no any limitations are imposed on the amplitude of interaction between the quantum oscillator and vacuum.

Because each quantum state $m$ in the problem under consideration is described by a complex random process, it makes sense to apply here the thermodynamic description.

**Definition 2.** The partial density matrix is defined to be

$$\rho^{(m)}_{stc} = \{x, t; \{\xi\}|x', t'; \{\xi'\}} = \{\Psi_{stc}(m|x, t; \{\xi\}) \Psi_{stc}^*(m'|x', t'; \{\xi'\}) \},$$

(110)

Let us remember, that the wave functional $\Psi_{stc}(m|x, t; \{\xi\})$ in (110) describes QHO state with the frequency

$$\Omega(t) = \Omega_{as} + \Omega_+ (t; \{W_+\}), \quad \Omega_{as} = \text{const},$$

(111)

where the stochastic term of frequency squared have the following form

$$U_+(t) = 2\Omega_{as}\Omega_+ (t; \{W_+\}) + \Omega_+^2 (t; \{W_+\}) = F_{0+} + F_+ (t), \quad \langle U_+(t) \rangle = F_{0+} = \text{const}. $$

(12)

We shall suppose that

$$\langle F_+(t) \rangle = 0, \quad \langle F_+(t) F_+(t') \rangle = 2\varepsilon_+ \delta (t - t').$$

(133)

**Definition 3.** The mathematical expectation of the stochastic operator $\hat{A}(x, t; \{\theta\})$ in the quantum state $m$ is defined as

$$A = \lim_{t \to +\infty} \left\{ \text{Sp}_x \left[ \text{Sp}_\xi \hat{A}_{stc}^{(m)} \right] / \text{Sp}_x \left[ \text{Sp}_\xi \rho^{(m)}_{stc} \right] \right\}. $$

(144)
where $Sp_{\{x\}}$ means the integration along the diagonal line $x = x'$, and $Sp_{\{\theta\}}$ denotes respectively the functional integration along the diagonal $\{\theta\} = \{\theta'\}$ with measure $\Omega_{as}$-take into account that in the asymptotic subspace $R_{as}$ the following notations were made $\Omega_0(t) \rightarrow \Omega_{as}$, $F_0 \rightarrow F_{0+}$, $\gamma \rightarrow \gamma_+ = 1 + F_{0+}/\Omega_{as}^2$, and $\varepsilon \rightarrow \varepsilon_+$.  

**Definition 4.** We shall call the following distribution function “the nonequilibrium partial” one

$$
\vartheta (\varepsilon_+, E, t) = \frac{\partial}{\partial \varepsilon_+} \left\{ \frac{N_E}{N_\Sigma} \right\} = \frac{\partial}{\partial \varepsilon_+} \left\{ \frac{\Delta E}{\Delta \varepsilon_+} \right\},
$$

(115)

where $E$ is the average value of energy in the quantum state characterized by the index $m$, $Sp_{\{\theta\}}$ standing for the functional integration along the diagonal line $\{\theta\} = \{\theta'\}$ but with the measure

$$
D\mu \{\theta\} = \alpha^{-1} (\varepsilon_+, \Omega_{as}; t) \frac{\Delta E}{\Delta \varepsilon_+} D\mu \{\theta\},
$$

(116)

where $\Delta E(\varepsilon_+, \theta; t)$ is the solution of the Fokker-Plank equation (59)-(60) after replacement of $U_0(t)$ by $-E$, $E > 0$, and $\Delta \Sigma$, respectively, after replacement of $U_0(t)$ by $E$. It is easy to see that the ratio $\Delta E/\Delta \Sigma$ gives the distribution of quantum states in the vicinity of binding energy in the functional space $R_{\{\theta_+\}}$. It is necessary to stress that the average energy $E$ here is determined with the help of formula (114) and, hence, depends on the quantum number $m$. After simple integration in (115) with due regard for (116) and passing to a limit $t \rightarrow +\infty$ we obtain for the equilibrium distribution function of quantum states

$$
\vartheta (m) (\varepsilon_+, E) = \frac{N_E}{N_\Sigma}, \quad E > 0, \quad N_\Sigma = N_{-E},
$$

(117)

where

$$
N_E^{-1} = \pi^{-1/2} \varepsilon_+^{1/2} \int_0^\infty dzz^{-1/2} \exp \left( -\frac{z^3}{12} + \frac{E z^{3/2}}{\varepsilon_+^{3/2}} \right).
$$

(118)

Having in view the asymptotic estimates (70)-(73) one can conclude that when the condition $E/\varepsilon_+^{2/3} \ll 1$ is met, the partial distribution function $\vartheta (m) (\varepsilon_+, E)$ passes into the microcanonical distribution when the constant $\varepsilon_+ = \frac{4}{3} E^* kT$. Owing to that, the integration of stochastic density matrix (110) over the functional space $R_{\{\theta_+\}}$ with the measure (116) and summation over the index $m$ gives the density matrix in the representation of Dirac and von Neuman (106)-(108).

Now, having the partial distribution function $\vartheta (m) (\varepsilon_+, E)$ one can determine the thermodynamic potentials of the specific level $m$:

- a) the average internal energy

$$
U (m) (\varepsilon_+, E) = -\partial_{\varepsilon_+} \left\{ \ln \vartheta (m) (\varepsilon_+, E) \right\},
$$

(119)

- b) the free Helmholtzian energy

$$
F (m) (\varepsilon_+, E) = -\varepsilon_+^{-1} \ln \vartheta (m) (\varepsilon_+, E),
$$

(120)

- c) the entropy

$$
S (m) (\varepsilon_+, E) = \varepsilon_+ k \left\{ U (m) (\varepsilon_+, E) + F (m) (\varepsilon_+, E) \right\}.
$$

(121)

Having in view the illustration of results we shall carry out the calculations for vacuum state, i.e., $m = 0$. An appropriate partial stochastic matrix in the inhomogeneous reference will have the form (see Section 4)

$$
\rho_{stc}^{(0)} (x, t; \{\theta\} | x', t'; \{\theta'\}) = \left( \frac{\Omega_{as}}{\Omega_{0+}} \right)^{1/2} \exp \left\{ -\frac{\Omega_{as}}{\Omega_{0+}} (x^2 + x'^2) -
$$

$$
- \frac{1}{2} \int_{-\infty}^t \theta (\tau) d\tau - \frac{1}{2} \int_{-\infty}^{t'} \theta (\tau) d\tau - i [\theta (t) x^2 - \theta (t') x'^2] \right\},
$$

(122)

where $\Omega_{as}$ is the permanent frequency of QHO in the asymptotic space $R_{as}^1$. 

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Substituting the expression for stochastic potential (2) and stochastic density matrix (122) to (114) and making simple calculations we obtain the following expression for the average value of energy of "vacuum-oscillator" system in the ground state:

\[ E^{(0)}(\lambda_+; \Omega_{as}) = \frac{1}{2} \Omega_{as} J_+ \int_0^\infty dz z^{-3/2} \exp \left( -\frac{3}{12} \lambda_+ z \right) + \]

\[ + \frac{1}{2} \Omega_{as} \left\{ 1 - \frac{J_+}{\lambda_+} \int_0^\infty dz z^{3/2} \exp \left( -\frac{3}{12} \lambda_+ z \right) \right\} + \]

\[ + \frac{\Omega_{as}}{2 \sqrt{\lambda_+}} J_+ \int_0^\infty dz z^{1/2} \exp \left( -\frac{z^3}{12} \lambda_+ z \right), \]  

where the notation was made

\[ J_+^{-1} = \pi/2 \varepsilon_+^{-1/3} \int_0^\infty dz z^{-1/2} \exp \left( -\frac{z^3}{12} \lambda_+ z \right). \]

Note that the expression (123) was obtained taking into account that \( \langle F \rangle_{\{\psi_+\}} = 0 \). One can see that the first term in (123) is diverging that corresponds to the infinite energy of physical vacuum. The second term corresponds to the energy of oscillator in the ground state that is shifted as a result of interaction with the nonrelativistic vacuum (see FIG.3)

\[ E^{(0)}_{osc}(\lambda_+; \Omega_{as}) = \frac{1}{2} \Omega_{as} \left\{ 1 - \frac{1}{\lambda_+} \left[ (\partial_\alpha \ln A(-\lambda_+ + \alpha)^2 + \partial_\alpha^2 \ln A(-\lambda_+ + \alpha) \right]|_{\alpha=0} \right\}, \]  

\[ A(-\lambda_+ + \alpha) = A^2(-\lambda_+ + \alpha) + B^2(-\lambda_+ + \alpha). \]

It is noteworthy that the second term in (124) is an analogue of the Lamb shift of the energy level, that is well known from the QED [21]. The third term in (123) corresponds to a broadening of the ground level and, hence, is inverse proportional to its decay time

\[ \Delta t^{(0)} \sim 2 \sqrt{\frac{\lambda_+}{\Omega_{as}}} \left\{ \partial_\alpha \ln A(-\lambda + \alpha) \right\}^{-1}|_{\alpha=0}. \]  

(125)

Now, based on the formulae (117)-(121), one can write an expression for the entropy of QHO in the m-th quantum state immersed in the physical vacuum

\[ S^{(m)}(\varepsilon_+, E_{osc}) = -k \varepsilon_+ \left\{ \partial_\alpha \ln \left[ A\left( -\frac{E_{osc}}{\varepsilon_+^2} + \alpha \right) A\left( \frac{E_{osc}}{\varepsilon_+^2} + \alpha \right) \right] + \right\} \]

\[ + \varepsilon_+^{-1} \ln \left[ A\left( \frac{E_{osc}}{\varepsilon_+^2} \right) / A\left( -\frac{E_{osc}}{\varepsilon_+^2} \right) \right]|_{\alpha=0}. \]  

(126)

To obtain an expression for the entropy of oscillator in the ground state one has to make a substitution \( E_{osc} \rightarrow E^{(0)}_{osc} \) in the formula (126)

\[ S^{(0)}(\beta^{(0)}_+) = k \left\{ \beta^{(0)}_+ \left[ \frac{A_{1/2}(\beta^{(0)}_+)}{A_{-1/2}(\beta^{(0)}_+)} + \frac{A_{1/2}(\beta^{(0)}_+)}{A_{-1/2}(\beta^{(0)}_+)} \right] + \ln \frac{A_{-1/2}(\beta^{(0)}_+)}{A_{-1/2}(\beta^{(0)}_+)} \right\}, \]  

(127)

where the notations were made

\[ A_p \left( q \beta^{(0)}_+ \right) = \int_0^\infty dz z^p \exp \left( -\frac{z^3}{12} + q \beta^{(0)}_+ z \right), \quad \beta^{(0)}_+ = \frac{E^{(0)}_{osc}}{\varepsilon_+^{1/3}}. \]  

(128)
11 Conclusions

At present the three following major schemes for the quantum chaos initiation are discussed:

a) the dynamical one, when the classical analogue of the quantum object under study is a nonintegrable system \([\mathbb{R}]\). An evident example of such a case is the three-body problem \([\mathbb{R}]\).

b) the measurement problem comprising the issues of mesophysics, of the irreversibility problem and quantum jumps \([\mathbb{R}]\).

c) randomness, and, therefore, the irreversibility as the basic properties of the physical world, owing to which the really existing object - the physical vacuum (see \([\mathbb{R}]\)), is included in theoretical consideration.

The objective of the present paper was to investigate the third case. The main point here is the idea that the quantum object and the physical vacuum are considered to make a joint system described in the framework of single equation. Since such a system has infinite number of the degrees of freedom, it is convenient to mathematically formulate the problem in the framework of complex SDE in the extended space \(\Xi = \mathbb{R}^1 \otimes R(\xi)\). In this case the system "quantum object + thermostat" will be described by the wave functional, - a complex probabilistic process, and not by the wave function. As a specific example, the problem of one-dimensional QRHO motion in the Euclidean space \(\mathbb{R}^1\) has been considered in the work. It was shown that in the complex SDE \((1)-(2)\) the variables were separated by means of Langevin-type standard real nonlinear SDE and the solutions for the wave state were obtained in the form of complex orthonormalized functionals \(\Psi^{(+)}_{br}(n \mid x, t; \{\xi\})\) in the space \(L_2(\mathbb{R}^1 \otimes R(\xi))\). For nonlinear Langevin SDE the corresponding Fokker-Plank equation has been obtained and with its help a complete positive measure for the functional space was constructed. This last fact permits the construction of uniformly converging representation for the averaged wave function \(\Psi^{(+)}_{br}(n \mid x, t)\) of moving QRHO on the conventional space-time continuum \((\mathbb{R}^1, t)\) by means of functional integration of the wave process \(\Psi^{(+)}_{br}(n \mid x, t; \{\xi\})\) in the space \(R(\xi)\). The obtained mixed continual-undulatory representation unites two concepts that at the first sight seem contradictory: the quantum analogue of the Arnold transformations \([\mathbb{R}]\) that excludes the rise of chaos inside the given trajectory beam, i.e., forbids any changes of beam topology, and the functional integration that allows for the contribution of various topological pipes and thereby assists to the generation of the chaos. With the help of wave functionals \(\Psi^{(+)}_{br}(n \mid x, t; \{\xi\})\) an expression for the elements of transition matrices (see \([\mathbb{R}]\)) was constructed. It was shown that the stochastic matrix formed in such a way is unitary \(S(\xi)^+ = L\). The generalized transition matrix \(S(\xi)^+\) for QRHO was obtained by averaging of the stochastic matrix \(S^{stc}\) in the \(R(\xi)\) space and it was shown to be generally nonunitary. In other words, in the theoretical scheme in question the quantum oscillator is an open system in the space-time continuum \((\mathbb{R}^1, t)\) and here the conservation laws are violated. Nevertheless, when the oscillator weakly interacts with the thermostat the transition matrix is simplified and turns to be unitary, \(S(\xi)^+ = L\), i.e., the conservation laws are effective again.

In the present paper a detailed analysis of the probability of "vacuum-vacuum" transition \(\Delta^{br}_{\text{vac-vac}}(\lambda, \rho)\) is given in terms of \(\lambda\) that characterizes random fluctuations of the thermostat before and after the crossover, as well as in terms of barrier reflection coefficient \(\rho\) in the relevant problem of the one-dimensional quantum mechanics. It was shown that for relatively strong interactions of the oscillator with the thermostat, i.e., for relatively small parameter \(\lambda\), the \(\rho\)-dependence of the probability is nonmonotonic (see FIG.2).

Based on the example of vacuum-immersed QHO, a novel formalism of quantum statistical mechanics has been developed in the framework of density matrix approach \([\mathbb{R}]\). It was shown that in the limit of weak oscillator-vacuum interaction, i.e., at \(\lambda_+ \to \infty\) it can be reduced to the Dirac and von Neuman density matrix. In the framework of novel approach the relaxation effects related to the energy spectrum of the oscillator were studied in detail. In particular, analytical expressions for the energy of ground state with broadening and shift (analogous to the well known Lamb shift from the Quantum Electrodynamics) as a function of constant \(\lambda_+\), and for the entropy of the ground (vacuum) state of the oscillator were obtained.

We should like to draw attention to one very important property of the quantum representation under consideration, viz., that it permits to relate the domain of quantum chaos with that of classical chaos \([\mathbb{R}]\), the transition to the classical chaotic dynamics taking place in the limit \(\hbar \to 0\) when \(\varepsilon \neq 0\). In case when \(\varepsilon \to 0\) and \(\hbar \neq 0\) SDE \((1)\) goes into the usual Schrödinger equation and describes a reversible quantum process. The classical theory of reversible processes is obtained when one tends to zero \(\hbar \to 0\) and \(\varepsilon \to 0\).

An explicit relation between the quantum mechanics and quantum statistical thermodynamics was also established to specify, in particular, the range of applicability of the statistical matrix of Dirac and von Neumann.

In conclusion we should like to note that one can successfully develop the quantum mechanics in the frame-
work of complex SDE (1) both for the model of multidimensional oscillator and also for other exactly solved nonstationary problems of quantum mechanics. Its covariant generalization is also not difficult, and we postpone the detailed discussions of these problems to our later publications.

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Figure 1: The dependence of the distribution of stationary process \( \tilde{Q}_s (\lambda; \bar{\theta}) \) of over \( \bar{\theta} \) for different values of parameter \( 1/\lambda \sim \varepsilon \).
Figure 2: "Vacuum-vacuum" transition probability in dependence of \( \lambda \) and \( \rho \).
Figure 3: Dependence of oscillator ground state energy, its shift and entropy of vacuum over the parameter $\lambda_+$. 
\bar{Q}_s(\theta;\lambda,\rho)
$\Delta_{0 \rightarrow 0}(\lambda, \rho)$
Entropy, Energy, and Shift of Energy as functions of $\lambda$. The graph shows the behavior of these variables over a range of $\lambda$ values.