Fixing the rotating-wave approximation for strongly-detuned quantum oscillators

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Periodically-driven oscillators are commonly described in a frame co-rotating with the drive and using the rotating-wave approximation (RWA). This description, however, is known to induce errors for off-resonant driving. Here we show that the standard quantum description, using creation and annihilation of particles with the oscillator’s natural frequency, necessarily leads to incorrect results when combined with the RWA. We demonstrate this on the simple harmonic oscillator and present an alternative operator basis which reconciles the RWA with off-resonant driving. The approach is also applicable to more complex models, where it accounts for known discrepancies. As an example, we demonstrate the advantage of our scheme on the driven quantum Duffing oscillator.

Nature displays many degrees of freedom that are coupled to one another. For simplicity, we describe them as linear resonators that drive each other as they oscillate with a bare harmonic around their respective equilibrium points. Hence, finding the response of a system to an applied drive is an essential task in all areas of science, ranging from fundamental effects in classical and quantum physics to applications in spectroscopy and diagnostics of our daily lives.

The bare harmonic translates to a drive that is periodic in time with a given frequency. To describe the system’s response to the drive, a common approach involves moving to a rotating frame, where the system’s behavior appears stationary on short timescales. In the rotating frame, the so-called rotating-wave approximation (RWA) is used, whereby any remaining rapidly oscillating terms are dropped. This procedure becomes less precise when the system and drive frequencies are far detuned from one another, i.e., when the neglected terms are less rapid. Nevertheless, the RWA is a standard approach to find the response of driven systems [1, 2], such as nanomechanical resonators [3–5], optical cavities [6–12], phononic and magnonic modes [13–19] and superconducting junctions [20–22]. Similarly, it is a common starting point for Floquet engineering and perturbative expansions of nonlinear driven systems [23–27].

Despite its ubiquity, the RWA is known to induce errors at large detuning [28]. There the neglected oscillating terms become significant and must be included as perturbations that affect the system’s response [29–33]. Yet, such a perturbative treatment becomes very tasking in many cases that rely on detuned driving, such as optomechanical cooling [24, 34] and the formation of frequency combs [36–40]. These cases are therefore commonly treated using the RWA. The issue is particularly salient in the case of the simple quantum harmonic oscillator, where the solution obtained with the RWA disagrees with the exactly soluble classical limit.

In this work, we pinpoint the source for the quantum-to-classical discrepancy in the driven harmonic oscillator and correct for it. Specifically, we show the description of the oscillator in terms of its bare excitations is ill-suited for the RWA. Instead, we propose a non-standard operator basis (frame) in which the RWA yields the correct solution for the driven case. We furthermore demonstrate that this basis choice is also beneficial for describing driven nonlinear systems, as it significantly improves the fidelity of the RWA in the driven Duffing oscillator at large detuning. We expect that our approach will lead to a significant improvement in the description of numerous scenarios where the response of an oscillator to a coherent drive is calculated.

The Hamiltonian of a periodically driven harmonic oscillator reads

\[ H = p^2/2m + \omega_0^2 x^2/2 - F_0 \cos(\omega t) x, \]

with position \( x \), its conjugate momentum \( p \), mass \( m \), and resonance frequency \( \omega_0 \). The driving force has amplitude \( F_0 \) and oscillates at frequency \( \omega \). The time evolution of the system is given by Hamilton’s equations of motion (EOMs),

\[ \dot{p} = -m \omega_0^2 x + F_0 \cos(\omega t), \quad \dot{x} = p/m. \]

These EOMs can be solved exactly by transforming to the Fourier domain, where the driving term appears as a Dirac delta function. As a result, the stationary response of the oscillator to the drive reads \[41\]

\[ x = X \cos(\omega t), \quad p = -m \omega X \sin(\omega t), \]

with \( X = F_0/\left[m(\omega_0^2 - \omega^2)\right] \). We can parameterize the oscillator’s phase space by \( x \) and \( p/m\omega_0 \). The trajectory described by Eq. \[3\] is then an ellipse with vertices at \( X \) and \( x \omega/\omega_0 \), see Fig. \[1\](a).

Within the framework of quantum mechanics, the Hamiltonian \[1\] is an operator, written in terms of a pair of non-commuting operators \( \hat{x} \) and \( \hat{p} \). To find the oscillator’s response in the quantum realm, one commonly introduces raising and lowering operators \( a \) and \( a^\dagger \) that diagonalize the time-independent part of \( H \),

\[ \hat{x} = \sqrt{\frac{\hbar}{2m\omega_0}} (a^\dagger + a), \quad \hat{p} = i\sqrt{\frac{\hbar\omega_0}{2}} (a^\dagger - a), \]

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resulting in

\[ H = \hbar \left[ \omega_0 (a^\dagger a + 1/2) - F_0 (e^{i\omega t} + e^{-i\omega t})(a^\dagger + a) \right], \quad (5) \]

with \( F_0 = F_0/(2\sqrt{2m\omega_0\hbar}) \), see Fig. 1(b). The driving renders the Hamiltonian \( \tilde{H} \) time-dependent and several methods exist to find the resulting response. For example, one approach involves time-dependent perturbation theory \([42]\), where we move to the interaction picture using the transformation \( V(t) = e^{-i\omega_0^t a^\dagger a} \), and diagrammatically expand the remaining interaction Hamiltonian \( H_{\text{int}} \). While this approach is possible for the harmonic oscillator, it rapidly becomes infeasible when additional terms are included in \( H \). Hence, in many cases only linear response is considered, e.g., using the input-output method \([13]\).

Another common approach involves the rotating-wave approximation (RWA). Here, we aim at removing the oscillatory time-dependence altogether. To do so, we first use the unitary transformation \( U_a(t) = e^{-i\omega_0^t a^\dagger a} \). In the resulting rotating frame, we obtain the (still time-dependent) effective Hamiltonian

\[ \tilde{H} = U_a^\dagger H U_a - i h U_a^\dagger \dot{U}_a \]

\[ = \hbar \left\{ -\Delta \dot{a} - F_0 \left[ \dot{a} \left( 1 + e^{-2i\omega t} \right) + a^\dagger \left( 1 + e^{2i\omega t} \right) \right] \right\}, \quad (6) \]

where \( \dot{a} \equiv U_a(t)^\dagger a U_a(t) \) denotes the operator \( a \) in the rotating frame, and \( \Delta = \omega - \omega_0 \) is the so-called detuning away from resonance.

In the rotating frame, the remaining time-dependent terms are far-detuned since \( 2\omega \gg |\Delta| \) and therefore elicit negligible response. Hence, we can apply the RWA and remove all of these from the Hamiltonian \( \tilde{H} \) to obtain \( \tilde{H}_{\text{RWA}} \). We may then find the response by solving Heisenberg’s EOM, \( i\hbar \partial_t \langle \hat{a}, \tilde{H}_{\text{RWA}} \rangle / \hbar = -\Delta \langle \hat{a} \rangle - F_0 \), and obtain the stationary solution \( \langle \hat{a} \rangle = -F_0/\Delta \). Then, by inverting the transformation \( U_a(t) \) we have \( \langle \hat{a} \rangle = \langle \hat{a} \rangle e^{-i\omega t} \). An analogous EOM is similarly solved for \( \langle \hat{a}^\dagger \rangle \). Combining the two solutions [cf. Eq. (4)], we obtain in the classical limit

\[ x_{\text{RWA}}(t) = -F_0/m \frac{2\omega_0^\Delta}{2\omega_0^2} \cos(\omega t). \quad (7) \]

Crucially, the response amplitude found with the RWA does not match the analytical solution in Eq. (3), see Fig. 1(a). Indeed, the ratio

\[ x(t)/x_{\text{RWA}}(t) \rightarrow 2\omega_0/\omega^2 - \omega_0^2/\omega_0 + \omega^2, \quad (8) \]

only approaches unity for \( \Delta \to 0 \), which demonstrates the limitation of the RWA to near-resonant drives. A comparison between the exact and RWA results in phase space is also illustrative. To this end, we take a stationary amplitude \( \langle \hat{a} \rangle \equiv |\langle \hat{a} \rangle| e^{i\theta} \), and using Eq. (4), we find the equivalent trajectory in phase space, \( x(t) = 2|\langle \hat{a} \rangle| \sqrt{\frac{\hbar}{2m\omega_0}\sin(\omega_0 t - \theta)} \) and \( p = -2|\langle \hat{a} \rangle| \sqrt{\frac{\hbar}{2m\omega_0}} \sin(\omega_0 t - \theta) \). The resulting path is always circular, unlike the elliptical path of the exact solution, cf. Eq. (3) and Fig. 1(a). Indeed, the dropped time-dependent oscillating terms, dubbed micromotion, add on to the circular path to produce the full elliptical motion, cf. small circle in Fig. 1(a). This implies that to describe the elliptical path, \( \langle \hat{a} \rangle \) must be time-dependent, which cannot be obtained only by time-independent corrections from a high-frequency expansion \([23–27]\). In other words, the assertion of a stationary \( \langle \hat{a} \rangle \) violates Hamilton’s EOMs, specifically the relation \( p = m\dot{x} \).

As the driven quantum harmonic oscillator is exactly solvable, the RWA was not necessary in the aforementioned analysis. Without the approximation, we would still have obtained the exact result. However, when the system becomes nonlinear, its EOMs are usually not exactly solvable. The RWA, being effectively the lowest order of a perturbative expansion, is then a common method to find the stationary response. As we have seen, however, it neglects crucial corrections due to the detuned drive frequency. We identify that this systemic mismatch stems from the usual starting point \([14]\), which relies on operators that diagonalize the bare oscillator Hamiltonian and describes excitations with energies \( h\omega_0 \),

FIG. 1. (a) Response amplitude [cf. Eq. (3)] of the driven harmonic oscillator as a function of the driving frequency \( \omega \) for the exact solution (solid, blue) and the standard RWA solution (dashed, red) with \( F_0 = m = \omega_0 = 1 \). The inset shows the corresponding phase space plots for large detuning, \( \omega = 0.5 \). Spheres indicate solutions in the rotating frame. The ansatz of stationary \( \langle \hat{a} \rangle \) (large red dashed circle) does not match the exact result. For this, an additional time-dependent term in the solution is needed (small red circle). This term oscillates at \( 2\omega \), which results in an elliptical path. (b) Energy quantization of the harmonic oscillator Hamiltonian \( \tilde{H} \) with energy spacing \( h\omega_0 \) using the operators \( a, a^\dagger \) defined in Eq. (4). (c) With the operators \( b, b^\dagger \), the harmonic oscillator is not diagonalized, showing a level spacing of \( h(\omega_0^2 + \omega^2)/2\omega \) and processes which create or annihilate two particles with amplitude \( \hbar(\omega_0^2 - \omega^2)/4\omega \).
To demonstrate this, we compare the results obtained
which are not exactly solvable. Our result (12) implies that
the correct result is obtained re-
sponding Hamiltonian (11) now reads

\[ H = \hbar \left( \frac{\omega_a^2 + \omega^2}{2\omega} (b^\dagger b + \frac{1}{2}) + \frac{\omega_a^2 - \omega^2}{4\omega} (b^2 + (b^\dagger)^2) \right) - F_b (e^{i\omega t} + e^{-i\omega t})(b^\dagger + b), \]

with \( F_b = F_0/(2\sqrt{2m\omega_0\hbar}) \), see Fig. 1(c). Note that we effectively rotated from the \( b \) to the \( b^\dagger \) operators using a unitary (Bogoliubov) transformation [44]. With the new operators, Eq. (10) contains squeezing terms that do not conserve the particle number, since \( b \) and \( b^\dagger \) do not diagonalize the bare harmonic oscillator unless \( \omega = \omega_0 \).

We repeat the same procedure in the new basis. We first apply the unitary transformation \( U_b(t) = e^{-i\omega t b^\dagger b} \) to move to a rotating frame, and then obtain the corre-
\[ i \frac{d}{dt} \langle \hat{b} \rangle = \frac{\omega_0^2 - \omega^2}{2\omega} \left( \langle \hat{b} \rangle + \langle \hat{b}^\dagger \rangle e^{2i\omega t} \right) - F_b \left( 1 + e^{2i\omega t} \right). \]
\[ \langle \hat{b} \rangle = \frac{2F_b \omega}{\omega_0^2 - \omega^2}, \]

which upon inverting \( U_b(t) \) matches the exact result in
Eq. (3). Importantly, the correct result is obtained re-
gardless of whether we take the RWA or not. In other
words, when the operators \( b, b^\dagger \) are used, the sta-
tary solution in the rotating frame is exact and satisfies
Eq. (11). This is the main result of this work.

**Example: Duffing oscillator.**—The RWA is a common
starting point for dealing with time-dependent systems
that are not exactly solvable. Our result (12) implies that
a suitable choice of operators made prior to applying the
RWA can significantly reduce errors at large detuning.
To demonstrate this, we compare the results obtained
using the RWA with the two operator definitions, \( a, a^\dagger \)
and \( b, b^\dagger \). We consider a Duffing oscillator, described by the Hamiltonian

\[ H_D = p^2/(2m) + m\omega_0^2 x^2/2 + \alpha x^4/4 - F_0 \cos(\omega t)x, \]

where \( \alpha \) is the Duffing nonlinearity. We plug both opera-
tor definitions [Eqs. (4) and (9)] into Eq. (13), move to a
rotating frame [cf. Eq. (9)], and apply the RWA in both
procedures. Note that the RWA drops multiple oscillat-
ting terms that describe frequency conversion processes
due to the Duffing nonlinearity. We furthermore apply a semi-
classical mean-field ansatz [45] to obtain Heisen-
berg’s EOMs

\[ i \frac{d}{dt} \langle \hat{a} \rangle = -\Delta \langle \hat{a} \rangle - \frac{3\alpha \hbar}{4m^2\omega_0^2} \left( \langle \hat{a} \rangle + \langle \hat{a}^\dagger \rangle \right), \]
\[ i \frac{d}{dt} \langle \hat{b} \rangle = \frac{\omega_0^2 - \omega^2}{2\omega_0^2} \langle \hat{b} \rangle - \frac{F_0}{\hbar} + \frac{3\alpha \hbar}{4m^2\omega^2} \left( \langle \hat{b} \rangle + \langle \hat{b}^\dagger \rangle \right). \]

We can now search for stationary solutions in either of the rotating frames [\( \frac{d}{dt} \langle \hat{a} \rangle = 0 \) or \( \frac{d}{dt} \langle \hat{b} \rangle = 0 \)]. Both Eqs. (14) and (15) generate a cubic polynomial condition which has up to three solutions [11, 46, 47, see Fig. 2(a)]. Both approaches produce the expected tail-shaped re-
sponse, i.e., a single solution in the \( \omega < \omega_0 \) regime that tends up towards a high-amplitude solution and a co-
existence region in the \( \omega > \omega_0 \) regime, where both low-
and high-amplitude solutions appear. The coexistence
region manifests mathematically as a bifurcation point for the roots of the polynomial condition. Despite qual-
itative agreement, the solutions in the two frames differ quantitatively, i.e., in their amplitudes as a function of
detuning and in the positions of their bifurcation points.

To find out which of the two approximate procedures
describes the driven Duffing oscillator [13] more ac-
curately, we compare their predictions with a “numerical experiment” in the classical picture. Specifically, we nu-
merically evolve Hamilton’s equations for the Hamilto-
nian (13) until we reach stationary motion for different
detunings and initial conditions and then adiabatically sweep the detuning to explore the solution landscape.
Note that we add an infinitesimal dissipation term to
force convergence in the simulation, which negligibly
shifts the stationary outcome. Additionally, the nu-
merical time-trace describes the oscillatory response at the
drive frequency \( \omega \) including high harmonic generation.
Hence, the relevant benchmark for the RWA solutions
is the Fourier component \( x_\omega \) at frequency \( \omega \) of the
\text{time} trace. In Figs. 2(b) and (c), we plot the relative discrep-
ancy \( \delta_x = |(x_\omega - x_\omega^{\text{RWA}})|/x_\omega \) between the numerically
obtained \( x_\omega \) and the two RWA results in the high-
and low-amplitude solution branches, respectively. We ob-
serve that our new RWA approach agrees much better
for all values of \( \omega \). In the low-amplitude regimes, the
system responds quasi-linearly and our scheme performs
better as discussed above in the harmonic oscillator case.
Slight deviations are seen in the high-amplitude regime,
where nonlinear effects dominate the physics.

In driven-dissipative systems, phase diagrams depict
the number or type of stationary solutions as a function of
system parameters [48, 49]. Here, we focus on the
FIG. 2. A comparison of the Duffing oscillator response obtained using the $a, a^\dagger$ and $b, b^\dagger$ operator bases [cf. Eqs. (1), (3)] with a time-dependent simulation. The parameters used are $m = \omega_0 = \alpha = 1$. (a) The response amplitude $|x_\omega|$ plotted against the drive frequency $\omega$ for $F_0 = 0.2$ using $a, a^\dagger$ (dashed, red) and $b, b^\dagger$ (solid, blue). The time-dependent simulation result (dot-dashed grey, behind blue) consists of adiabatic up and downsweeps of $\omega$. (b), (c) The relative discrepancy in log scale of the $a, a^\dagger$ (dashed, red) and the $b, b^\dagger$ (solid, blue) results from the time-dependent simulation. The circles and crosses in (a) and (c) represent the jump frequency in the downsweep for operators $a$ and $b$. (d) A phase diagram as a function of $\omega$ and $F_0$. Blue (red) fill denotes regions with one (two) stable solution(s). (e) Relative discrepancy $\delta_\omega = |(\omega - \omega_{\text{RWA}})|/\omega$ in log scale of the phase boundaries obtained using $a, a^\dagger$ (dashed, red) and $b, b^\dagger$ (solid, blue) from the time-dependent result. The numerical approach is exact up to $10^{-5}$ which becomes visible at low $F_0$, where the discrepancy of our scheme is of that order. The green dotted lines in (d) and (e) indicate where the simulation of (a) takes place.

In summary, we have presented an operator basis for periodically-driven systems which anticipates a response at the driving frequency and is thus better suited for using the RWA. For both the harmonic and the Duffing oscillators, this basis choice significantly improves the fidelity of the RWA while retaining its simplicity. Our approach is applicable to systems in many areas of physics beyond mechanical oscillators described by $x$ and $p$. In quantum optics [40] and optomechanics [50], a driven cavity mode is described by Eq. (1) with the magnetic and electric fields playing the roles of $x$ and $p$. For models with inherent detuning such as optomechanical cooling and frequency-comb generation, our approach may result in significant numerical corrections. A specific case of interest are models coupling a cavity to one or more two-level systems [51, 52]; here the so-called general RWA is used, where a basis diagonalizing the spin-cavity interaction reduces the RWA error [53]. As general RWA uses the standard operators $a$ and $a^\dagger$, we expect that it can be improved further by using our scheme. Detuned driven systems also appear in circuit QED where the magnetic flux and charge appear in place of $x$ and $p$ [20]. Overall, a sizable body of work exists elaborating on the errors due to the RWA at large detuning [54–57]. Our work suggests that in many cases, these errors can be dramatically reduced by choosing an appropriate operator basis.

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