Approach to Universality in Axisymmetric Bubble Pinch-Off

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The pinch-off of an axisymmetric air bubble surrounded by an inviscid fluid is compared in four
cases to different physical realizations: (i) cavity collapse in the wake of an impacting disc, (ii) gas bubbles injected
through a small orifice, (iii) bubble rupture in a straining flow, and (iv) a bubble with an initially
necked shape. Our boundary-integral simulations suggest that all systems eventually follow the
universal behavior characterized by slowly varying exponents predicted in [Eggers et al., PRL 98,
094502 (2007)]. However, the time scale for the onset of this final regime is found to vary by orders
of magnitude depending on the system in question: while for the impacting disc it is well in the
millisecond range, for the gas injection needle universal behavior sets in only a few microseconds
before pinch-off. These findings reconcile the different views expressed in recent literature about the
universal nature of bubble pinch-off.

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The precise nature of axisymmetric bubble collapse in
a low-viscosity fluid has been a subject of controversy
over the last years. Such a collapse may be initiated
by a variety of different forces (e.g. surface tension, hy-
drostatic pressure, and external flows). In a later stage,
however, it is only the requirement of mass conservation
that forces the liquid to accelerate more and more as
the shrinking bubble neck closes in on the axis of sym-
metry. This purely inertial nature of the final collapse
motivated the first hypotheses about universality of the
final collapse regime [1, 2]. A simple power-law was pre-
dicted with the neck radius scaling as the square root of
the time remaining until the pinch-off singularity. Nei-
ther numerically nor experimentally could this behavior
be confirmed. Instead, for different systems and initial
conditions a variety of scaling exponents all slightly above
1/2 have been obtained [3, 4, 5, 6, 7, 8, 9, 10, 11] leading
to doubts about the universal nature of bubble collapse.

Recently, the idea of universality has been revived by
[12, 13] who suggested an intricate coupling between the
radial and axial length scale. The authors of [13] ex-

cplicitly predict the existence of a final universal regime
which however is no longer a simple power-law, but char-

acterized by a local exponent that slowly varies in time.
The value of 1/2 is recovered in the asymptotic limit in-
finitesimally close to pinch-off. According to this theory
the variety of observed exponents corresponds to different
time averages of this local exponent.

In the present work we aim to reconcile the different
views about universality in axisymmetric bubble pinch-
off expressed over the last years. The key aspect is that
we examine in detail how and when different physical
realizations of bubble pinch-off reach the universal regime.
We present detailed numerical simulations which are able
to follow the neck evolution over more than 12 decades
in time even for complex realistic systems. We first show
that all systems that have recently been studied in the
context of bubble pinch-off eventually follow the same
universal behavior predicted by [13]. The time scale on
which universality is reached, however, varies enormou-

sly from one setup to another. For an impacting disc [3, 13]
universality can be observed during several milliseconds
prior to pinch-off and thus on a time scale which is experi-
mentally accessible. However, for gas bubbles injected
through a small needle [1, 2, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14]
universality sets in only in a few microseconds (or even less,
depending on the precise initial conditions) before pinch-
off. This may well be the reason why universality has
thus far never been observed even in very precise gas in-
jection experiments and why non-inertial effects like sur-
face tension have been claimed to play a dominant role
in this geometry [10, 11].

Four different physical systems have been reported in
the literature on bubble pinch-off, numerically and ex-
perimentally, and will be compared in this Letter:

(i) *Impacting disc*. The bubble is created by the impact
of a circular disc on a liquid surface [3, 14] as shown in
Fig. 1(i). Upon impact an axisymmetric air cavity forms
and eventually pinches off halfway down the cavity under
the influence of hydrostatic pressure. Immediately after
pinch-off the ejection of a violent jet can be observed
whose formation however is not caused by the singularity
alone [15] as one might naively expect. Since surface
tension is negligible [3, 14, 17] the only relevant control
parameter is the Froude number $Fr = V_0^2/gR_0$ with the
impact velocity $V_0$, gravity $g$ and the disc radius $R_0$. In
the data reported here the disc radius varies between 1
and 3cm and the impact velocity ranges from 1 to 20m/s.

(ii) *Gas injection through an orifice*. A small needle
sticks through the bottom of a quiescent liquid pool
[1, 2, 4, 6, 7, 8, 9, 10, 11, 12, 14, 16] as illustrated in
Fig. 1(ii). A pressure reservoir connected to the needle
slowly pushes a gas bubble out of the needle’s orifice. The
bubble then rises under the influence of buoyancy. When
the air thread between the orifice and the main bubble
becomes long enough, surface tension causes the thin-
ning of the neck which eventually leads to the pinch-off
of the bubble. We present data for three sample config-
Here, $F$ represents the force driving the collapse, and dots denote the derivative with respect to time $t$. $R_{\infty}$ is a cut-off radius required to saturate the pressure at large distances. Assuming $R_{\infty}$ to be a constant leads to the neck radius $r_0$ shrinking as a power-law (possibly with logarithmic corrections \cite{3, 5, 12}) with exponent $1/2$ as a function of the time to pinch-off $\tau = t_c - t$, where $t_c$ is the closure time. At first sight, this expectation seems to be very well confirmed for all four systems by the lines in Fig. 4(b) which to the naked eye appear perfectly straight over more than 12 decades. The slope which corresponds to the scaling exponent is slightly larger than $1/2$, in agreement with previous experiments and simulations \cite{3, 4, 5, 10, 11, 12}.

A more detailed look at the local exponent, defined as the slope in Fig. 2(b), $\alpha(\tau) = \partial \ln r_0 / \partial \ln \tau$ (Fig. 2(c) – (e)) however, reveals that the behavior of the neck radius cannot be described by a simple power-law. The local exponent $\alpha$ varies during the approach to pinch-off \cite{13}. In fact, the relevant equation for the time-dependence of $\alpha$ in \cite{13} can be derived directly from Eq. (1) by letting $R_{\infty} = 2\sqrt{r_0r_c}$. Here $r_c$ is the local axial radius of curvature (see Fig. 2(a)). The combination $\sqrt{r_0r_c}$ is the scale by which the axial coordinate has to be rescaled in order to collapse neck profiles at different times when rescaling radial coordinates by $r_0$ \cite{3, 5}. This leads to the aspect ratio of the cavity naturally being defined as $\gamma = r_0 / \sqrt{r_0r_c}$. With the above substitutions and working in the limit of vanishing $F$ we obtain from Eq. (1):

$$
\frac{d}{d \ln \tau} \left( -\alpha + 2\alpha^2 \right) \ln \left( \frac{4}{\gamma^2} \right) = -\alpha^2
$$

which is exactly identical to Eq. (4) in \cite{13} (being $\Gamma_1 = 8$ \cite{23} and $a_0'' = 2\gamma^2$ in the original notation) . Equation (2) with the $\alpha d/\ln \tau$ term neglected due to the slow variation of $\alpha$ \cite{26} is shown as the dashed red line in Figs. 2(c)-(e). It represents the universal regime where the only driving is provided by inertia and all external forces have become negligible. We will now proceed to compare the approach of the different systems (i)-(iv) to this universal curve. For this we use the aspect ratio $\gamma$ as a universal “clock” replacing the time to pinch-off $\tau$ \cite{23}. Note that $\gamma \rightarrow 0$ as $\tau \rightarrow 0$ meaning that the cavity becomes more and more slender \cite{7, 12}.

One of the key points to address is if and how this behavior can be observed experimentally. Besides the obvious difficulty of obtaining a sufficient number of decades to observe the slow variation of the local exponent, the crucial question is: at what time (before pinch-off) does the system exhibit universal behavior? This is crucial because, firstly, the duration of the universal regime needs to be within the time resolution of the experimental equipment. And secondly, the onset of universality needs to happen before other effects such as air flow, viscosity, non-axisymmetric instabilities, etc. unavoidably destroy the purely inertial regime. We will now provide those time scales for the various systems based on numerical boundary-integral simulations which do not have these

\[\begin{align*}
\frac{d (r_0^2)}{dt} + \frac{r_0}{R_{\infty}} \ln \frac{r_0}{R_{\infty}} + \frac{1}{2} \frac{\dot{r}_0^2}{r_0} = F.
\end{align*}\]
FIG. 2: (a) Illustration of the cavity surface (blue line), the minimal neck radius \( r_0 \) and the local radius of curvature \( r_c \). (b) “Classical” plot of the neck radius versus the time to pinch-off for system (i) with \( Fr=5.1 \) (blue), system (ii) in setup B (red), system (iii) (dark gray) and system (iv) (magenta). The curves are offset by one decade from each other to improve readability. (c) The local exponent \( \alpha \) as a function of the aspect ratio \( \gamma \) for system (i) with \( Fr=3.4 \) (cyan), \( Fr=5.1 \) (blue), \( Fr=46 \) (brown), \( Fr=500 \) (green), and \( Fr=4000 \) (black). After an initial transient all curves follow the same universal regime. The dashed line is Eq. (2). The local maxima correspond roughly to the start of the universal regime. (d) The local exponent for system (ii) in Fr=500 (green), and Fr=4000 (black). After an initial transient all curves follow the same universal regime. The dashed line is Eq. (2). Figure 2 (c) further gives us a good measure at what aspect ratio the universal regime is attained: approximately after passing their respective local maxima all curves follow the same behavior. The aspect ratio of this maximum can then easily be related to the physical time before pinch-off \( \tau_u \). We find \( \tau_u \approx 6\, \text{ms} \) and \( \tau_u \approx 1\, \text{ms} \) for \( Fr=3.4 \) and \( Fr=4000 \), respectively. That the high Froude case reaches universality later can be intuitively understood: at high Froude the cavity closes deeper and therefore the hydrostatic driving pressure is larger and its effects on the neck dynamics can be felt longer. It is remarkable nevertheless that the duration of the universal regime changes only by a factor of less than 10 while the corresponding control parameter varies over three orders of magnitude. At the same time both values are easily within experimentally accessible time scales.

We now compare this to system (ii), the bubble injection through a small needle (1, 2, 4, 6, 7, 8, 10, 11, 15, 16) in Fig. 2 (d). The approach to the universal regime is much less abrupt than in system (i). Due to this more gradual approach, it is difficult to specify precisely the time when universality is reached for the gas injection needle. We thus choose to keep our previous definition of \( \tau_u \) being the time corresponding to the local maximum in Fig. 2 (d). This gives a good upper bound for the time when universality sets in. Surprisingly, we find even these times to be of the order of 5µs in case A, 60ns in case C and as low as 10ns in case B, respectively [27]. Thus, the duration of the universal regime in the needle setup is dramatically (by at least three orders of magnitude) shorter than for the impacting disc. This may well explain why, among other experimental limitations, the universal regime has never been observed experimentally for this widely used system.

Figure 2 (e) confirms that also the systems (iii) and (iv) follow the universal regime. System (iii) does so even over the entire plotted range. Both are somewhat idealized systems and we are not aware of any experimental investigations regarding the approach to pinch-off in these systems. Without a relevant length and time scale it is therefore impossible to specify the physical time to universality in these cases.

The different behaviors of the individual systems can intuitively be understood as follows. System (iii) contains no external driving force other than liquid inertia which makes it the ideal system to compare with Eq. (2). Indeed, this entirely inertial system follows the universal regime over the widest range in aspect ratios of all systems studied. Similarly, the very violent disc impact should introduce a large amount of inertia in system (i) which consequently follows the universal regime also for a rather long time. On the other hand, the two systems governed by surface tension (ii) and (iv) contain little
inertia and thus approach the universal regime only relatively late and in a similar fashion.

To make the above arguments more quantitative we realize that the universal regime sets in when the inertial driving of the collapse becomes dominant over the external driving force. This can be expressed by a local balance between inertia and the respective driving force. For system (i) the driving force is hydrostatic pressure and the relevant parameter thus the local Froude number $F_r^{\text{local}} = \frac{\rho \dot{r}^2}{2 g z_c}$ with gravity $g$ and $z_c$ the depth below the surface where the cavity eventually closes. For system (ii) the local Weber number $W_e^{\text{local}} = \frac{\rho r^2 \dot{r}}{\sigma}$ (with density $\rho$ and surface tension $\sigma$ of water) gives the balance between inertia and surface tension as the relevant driving force. The duration of the universal regime can then be estimated as the time before pinch-off when these local quantities become of order unity. Figure 3 shows the local Froude and Weber numbers as a function of time to pinch-off $\tau$ for a number of representative cases of system (i) and (ii), respectively. One can clearly appreciate that $W_e^{\text{local}}$ for the needle system becomes unity later than $F_r^{\text{local}}$ for the impacting disc. This explains the large discrepancy in $\tau_u$ for the two systems.

At the same time the distance between the two disc impacts with $F_r=3.4$ and $F_r=4000$ is much smaller than the distance between the two needle setups A and B. Accordingly, the duration of the universal regime varies only between $\sim1\text{ms}$ and $\sim6\text{ms}$ for the disc while its duration in the needle setup depends much stronger on initial conditions varying from micro- down to several nanoseconds as seen above.

Our results allow us to explain the Froude-dependence of the experimentally and numerically observed exponent in the impacting disc. Based on Eq. 2 (c) these exponents can be viewed as a time-average of the local exponent. Due to the limited resolution and the onset of other effects (e. g. air flow) only the right part of these plots is accessible in experiments and the time-average will be heavily weighted towards the beginning of the universal regime. We can thus assume the experimentally observed effective exponent to be roughly equal to the maximum value of the local exponent. Using Eq. 2 we can predict this value once the characteristic initial aspect ratio $\gamma_i$ for each cavity is known as a function of the impact Froude number.

This characteristic aspect ratio can be estimated to be proportional to the ratio between the (non-dimensional) maximum cavity expansion $R_{\text{max}}$ and the closure depth $z_c$. The first quantity scales with the Froude number approximately as $R_{\text{max}} \sim F_r^{-1/4}$ while the latter behaves as $z_c \sim F_r^{1/2}$. The characteristic aspect ratio of the cavity thus becomes $\gamma_i = R_{\text{max}}/z_c = C F_r^{-1/4}$ with $C$ a constant of order unity. Inserting $\gamma_i$ into Eq. 2 and solving for $\alpha(F_r)$, again neglecting $d\alpha/d\ln \tau$, we can thus predict the experimentally observable averaged exponent which is found in excellent agreement with [5] as demonstrated by Fig. 7 (b).

In conclusion, we have demonstrated that the universal theory of \cite{13, 23} faithfully predicts the approach of the neck radius for inviscid, axisymmetric bubble pinch-off in four different systems widely studied in the literature over the past years. Remarkably, however, the duration of the final regime is shown to be strongly dependent on the type of system and the various control parameters employed. While it lies easily within experimentally accessible time scales ($\sim\text{ms}$) for an impacting circular disc, it can be as low as a few nanoseconds for gas bubbles injected through a small orifice into a quiescent liquid pool. We were able to trace this difference back to the relative importance of the respective driving forces. Our findings reconcile the prediction of universality in bubble pinch-off \cite{12, 13} with an apparent dependence on initial conditions \cite{5}, an apparently constant scaling exponent \cite{6, 5, 9}, and with the observation that non-inertial forces can be dominant in many experimental settings \cite{10, 11, 10}.

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[25] See EPAPS Document No. XXX for supplementary material. For more information on EPAPS, see http://www.aip.org/pubservs/epaps.html
[26] In the numerical data one can verify that indeed $|d\alpha/d\ln \tau| \ll \alpha - 2\alpha^2$.
[27] The long duration for the quasi-static case A can be understood by considering that it has the smallest (non-dimensional) bubble size and thus the neck possesses the most symmetrical shape.