$M|G|\infty$ Queue Parameters Values
Approximation Through a Markov Renewal Process

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Abstract: Some $M|G|\infty$ queue systems parameters values approximations, obtained through the consideration of an adequate Markov renewal process, are presented, and studied.

Keywords: $M|G|\infty$; Markov renewal process; queue.

1. Introduction

In the $M|G|\infty$ queue system the customers arrive according to a Poisson process at rate $\lambda$, receive a service which time is a positive random variable with distribution function $G()$ and mean $\alpha$ and, when they arrive, find immediately an available server. Each customer service is independent from the other customers’ services and from the arrivals process. The traffic intensity is $\rho = \lambda \alpha$.

A suggestion to obtain approximate results for these systems, when the exact ones are still not known, is to use a Markov renewal process, see (1-2).

Along this work, some of the approximations so obtained are evaluated.

2. Sojourn Time Mean Value in State $k$

For the process referred above, the sojourn time mean value in state $^1k, k = 0,1,...$ is given by

$$m_k = \int_0^\infty e^{-\lambda t} \left[ \int_t^\infty \frac{(1 - G(x))dx}{\alpha} \right]^k dt, k = 0,1,... \quad (2.1).$$

Proposition 2.1

$$m_0 = \frac{1}{\lambda} \quad (2.2).$$

Obs.: The sojourn time mean value in state 0, does not depend neither on $G(.)$ nor on the arrivals process. It depends only on the arrivals process.

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$^1$The state of the $M|G|\infty$ queue in instant $t$ is the number of customers being served in the system at instant $t$. 

Proposition 2.2

\[ m_k \leq \frac{1}{\lambda} , \ k = 0,1, \ldots \] (2.3).

**Dem:** It is enough to note that \( \alpha^{-1} \int_{t}^{\infty} [1 - G(x)] dx \leq 1. \ □ \)

**Obs:** So, the sojourn time mean value in any state does not exceed the one of the state 0. Put

\[ E_0 = \frac{1}{\lambda} (2.4). \]

Proposition 2.3

\[ m_k \leq \alpha \frac{\gamma_s^2 + 1}{\sqrt{2 \rho (2k + 1)}} , \ k = 1,2, \ldots \] (2.5)

being \( \gamma_s \) the service coefficient of variation.

**Dem:** Using the Schwartz's inequality, \( m_k^2 \leq \int_{0}^{\infty} e^{-2\lambda t} dt \int_{0}^{\infty} \left[ \frac{\int_{t}^{\infty} [1 - G(x)] dx}{\alpha} \right]^{2k} dt = \frac{1}{2\lambda \alpha^{2k}} \int_{t}^{\infty} [1 - G(x)] dx \right]^{2k} dt = \frac{1}{2\lambda \alpha^{2k}} \frac{2ka^2}{2} (\gamma_s^2 + 1) \frac{\alpha^{2k-1}b_{2k-1}}{2(2k+1)} \leq \alpha \frac{\gamma_s^2 + 1}{\lambda (2k+1)} \]

since, see (3),

\[ \int_{0}^{\infty} \left[ \int_{t}^{\infty} [1 - G(x)] dx \right]^{n} dt = \frac{n\alpha^2}{2} (\gamma_s^2 + 1) \frac{\alpha^{n-1}b_{n-1}}{n(n+1)} \] with \( b_n \leq 2, \ n = 0,1, \ldots \) (2.6). □

**Obs:** Define

\[ E_1 = \alpha \frac{\gamma_s^2 + 1}{\sqrt{2 \rho (2k + 1)}} (2.7). \]

Proposition 2.4

If \( k \geq \frac{1}{4} \rho (\gamma_s^2 + 1) - \frac{1}{2} \), \( E_1 \leq E_0 \).

**Dem:**

\[ \alpha \frac{\gamma_s^2 + 1}{\sqrt{2 \rho (2k + 1)}} \leq \frac{1}{\lambda} \Leftrightarrow \frac{\gamma_s^2 + 1}{2 \rho (2k + 1)} \leq \frac{1}{\rho^2} \Leftrightarrow \rho (\gamma_s^2 + 1) \leq 4k + 2 \Leftrightarrow k \]

\[ \geq \frac{1}{4} \rho (\gamma_s^2 + 1) - \frac{1}{2}. \ □ \]

**Proposition 2.5**

\[ m_k \leq \alpha \frac{\gamma_s^2 + 1}{k + 1} , \ k = 1,2, \ldots \] (2.8).

**Dem:**
\[ m_k \leq \int_0^\infty \left[ \int_t^\infty (1 - G(x)) dx \right]^k \frac{dt}{\alpha} \leq \frac{1}{\alpha^k} \frac{k \alpha^2}{2} (y_s^2 + 1) \frac{2\alpha^{k-1}}{k(k+1)} = \frac{\alpha y_s^2 + 1}{k+1} \]

after (2.6). □

**Obs:** Define

\[ E_2 = \frac{\alpha y_s^2 + 1}{k+1} \] (2.9).

**Proposition 2.6**

If \( k \geq \rho(y_s^2 + 1) - 1 \), \( E_2 \leq E_0 \).

**Dem:**

\[ \frac{\alpha y_s^2 + 1}{k+1} \leq \frac{1}{\lambda} \Leftrightarrow \rho(y_s^2 + 1) \leq k + 1 \Leftrightarrow k \geq \rho(y_s^2 + 1) - 1. \]

**Proposition 2.7**

If \( k \leq 2\rho(y_s^2 + 1) - 1 \), \( E_1 \leq E_2 \).

**Dem:**

\[ \frac{E_1}{E_2} = \frac{\alpha \sqrt{\frac{y_s^2 + 1}{2\rho(2k+1)}}}{\alpha \frac{y_s^2 + 1}{k+1}} = \frac{\sqrt{y_s^2 + 1} + 1}{(y_s^2 + 1) \sqrt{2\rho(2k+1)}} = \frac{k + 1}{\sqrt{y_s^2 + 1} \sqrt{2\rho(2k+1)}} \]

\[ \leq \frac{k + 1}{\sqrt{2\rho(y_s^2 + 1)}} \frac{k + 1}{2\rho(y_s^2 + 1)} \leq 1 \Leftrightarrow k + 1 \leq 2\rho(y_s^2 + 1) \Leftrightarrow k \leq 2\rho(y_s^2 + 1) - 1. \]

**Proposition 2.8**

If \( k \geq 4\rho(y_s^2 + 1) - 1 \), \( E_2 \leq E_1 \).

**Dem:**

\[ \frac{E_1}{E_2} = \sqrt{\frac{k + 1}{\sqrt{2\rho(y_s^2 + 1)} \sqrt{2k + 1}}} \geq \sqrt{\frac{k + 1}{4\rho(y_s^2 + 1)^{\frac{1}{2}} k + 1}} = \frac{k + 1}{\sqrt{4\rho(y_s^2 + 1)^{\frac{1}{2}}}} \sqrt{\frac{k + 1}{4\rho(y_s^2 + 1)}} \]

\[ \geq 1 \Leftrightarrow k \geq 4\rho(y_s^2 + 1) - 1. \]

The Propositions 2.4, 2.6, 2.7 and 2.8 lead to the following upper bounds choice for \( m_k, k = 1, 2, \ldots : \)
A) $\rho(y^2_s + 1) > \frac{2}{3}$

\[
\begin{align*}
  k &< \frac{1}{4} \rho(y^2_s + 1) - \frac{1}{2} & m_k &\leq \frac{1}{\lambda} \\
  \frac{1}{4} \rho(y^2_s + 1) - \frac{1}{2} \leq k &\leq 2\rho(y^2_s + 1) - 1 & m_k &\leq \alpha \frac{y^2_s + 1}{2\rho(2k + 1)} \\
  2\rho(y^2_s + 1) - 1 &< k < 4\rho(y^2_s + 1) - 1 & m_k &\leq \min \left\{ \alpha \frac{y^2_s + 1}{2\rho(2k + 1)}, \alpha \frac{y^2_s + 1}{k + 1} \right\} \\
  k &\geq 4\rho(y^2_s + 1) - 1 & m_k &\leq \alpha \frac{y^2_s + 1}{k + 1}
\end{align*}
\]

B) $\frac{1}{2} < \rho(y^2_s + 1) \leq \frac{2}{3}$

\[
\begin{align*}
  k = 1 &\quad m_1 \leq \min \left\{ \alpha \frac{y^2_s + 1}{6\rho}, \alpha \frac{y^2_s + 1}{2} \right\} \\
  k = 2, 3, ... &\quad m_k \leq \alpha \frac{y^2_s + 1}{k + 1}
\end{align*}
\]

C) $\rho(y^2_s + 1) \leq \frac{1}{2}$

\[
\begin{align*}
  m_k &\leq \alpha \frac{y^2_s + 1}{k + 1}, k = 1, 2, ...
\end{align*}
\]

Proposition 2.9

If the service time distribution is NBUE (New Better than Used in Expectation)

\[
m_k \leq \frac{\alpha}{k + \rho}, k = 1, 2, \ldots \quad (2.10)
\]

Dem: It is enough to note that if the service time is NBUE with mean $\alpha$, \[
\int_b^\infty [1 - G(x)]dx \leq \int_b^\infty e^{-\frac{x}{\alpha}}dx, \text{ for any } b \geq 0. \Box
\]

Obs: If the service time is NWUE (New Worse than Used in Expectation) with mean $\alpha$, \[
\int_b^\infty [1 - G(x)]dx \geq \int_b^\infty e^{-\frac{x}{\alpha}}dx, \text{ for any } b \geq 0 \text{ and}
\]

\[
m_k \geq \frac{\alpha}{k + \rho}, k = 1, 2, \ldots \quad (2.11).\]
Proposition 2.10

If the service time distribution is IMRL

\[ m_k \geq e^{k\left(1 - \frac{2a}{3\mu_2^2}\right)\frac{\mu_2}{\mu_2\lambda + 2k\alpha}}, k = 1, 2, \ldots \]  

(2.12)

being \( \mu_2 \) and \( \mu_3 \) the 2\(^{\text{nd}}\) and the 3\(^{\text{rd}}\) \( \mathcal{G}(\cdot) \) moments around the origin

Dem: If the service time\(^2\) is IMRL (Increasing Mean Residual Life)

\[ 1 - G^*(x) = 1 - \frac{1}{\alpha} \int_0^x \left[ 1 - G(y) \right] dy = \frac{\int_0^x [1 - G(y)] dy}{\alpha} \geq e^{\frac{-2a}{\mu_2}x - \frac{2a}{3\mu_2^2}\mu_3 + 1}. \]

So,

\[ m_k \geq \int_0^\infty e^{-\lambda t} \left( e^{\frac{-2a}{\mu_2}t - \frac{2a}{3\mu_2^2}\mu_3 + 1} \right)^k dt = \]

\[ = e^{\frac{k\left(1 - \frac{2a}{3\mu_2^2}\mu_3\right)}{\mu_2\lambda + 2k\alpha}} \cdot \frac{\mu_2}{\mu_2\lambda + 2k\alpha}, \square \]

Proposition 2.11

If the service time distribution is DFR\(^3\) (Decreasing Failure Rate)

\[ m_k \geq e^{k\left(1 - \frac{\gamma^2}{2}\right)\frac{\alpha}{k + \rho}}, k = 1, 2, \ldots \]  

(2.13).

Dem:

If the service is DFR \( 1 - G(x) \geq e^{-\frac{x}{\alpha} - \frac{\gamma^2}{2} + \frac{1}{2}}. \)

So,

\[ m_k \geq \frac{1}{\alpha k} \int_0^\infty e^{-\lambda t} \left[ \int_t^\infty e^{-\frac{x}{\alpha} - \frac{\gamma^2}{2} + \frac{1}{2}} dx \right]^k dt = \frac{e^{k\left(1 - \frac{\gamma^2}{2}\right)\frac{\alpha}{k + \rho}}}{\alpha^k} \int_0^\infty e^{-\lambda t} \left[ \int_t^\infty e^{-\frac{x}{\alpha} dx} \right]^k dt = \]

\[ = e^{k\left(1 - \frac{\gamma^2}{2}\right)\frac{\alpha}{k + \rho}}. \square \]

\(^2\) \( G^*(x) = \frac{1}{\alpha} \int_0^x [1 - G(y)] dy \) is the service time equilibrium distribution.

\(^3\) For more details about NBUE (New Better than Used in Expectation), NWUE (New Worse than Used in Expectation), IMRL (Increasing Mean Residual Life) and DFR (Decreasing Failure Rate) distributions, important in reliability theory, see (4).
Note: It is not known an expression to the sojourn time value in state $k$ for the $M|G|\infty$ queue systems, except for

a) $k = 0$, for every $G(.)$, being

$$m_0 = \frac{1}{\lambda} \quad (2.14)$$

b) Every $k$, for exponential service time, where

$$m_k = \frac{\alpha}{k+\rho}, k = 0,1,\ldots \quad (2.15).$$

In the same circumstances, the Markov renewal process supplies the same results: in fact $(2.14)$ is equal to $(2.2)$ and if $G(x) = 1 - e^{-x}, x \geq 0$ in $(2.1)$ it is obtained $(2.15)$.

- The bounds given by $(2.10), (2.11)$, match the exact value given by $(2.15)$. The expression $(2.13)$ is coincident with $(2.15)$ for $\gamma_s = 1$.

3. State 0 Recurrence Mean Time

For the Markov renewal process, the state 0 mean recurrence time is given by

$$\mu_0 = \frac{1}{\lambda}\left[1 + \sum_{j=1}^{\infty} \prod_{k=1}^{j} \frac{\lambda m_k}{1-\lambda m_k}\right] \quad (3.1).$$

Proposition 3.1

If $\rho \leq \frac{1}{\gamma_s^2 + 1}$, $\mu_0 \leq \frac{e^{\rho(\gamma_s^2 + 1)}}{\lambda} \quad (3.2)$

Dem: To use an upper bound of $m_k$ in $(3.1)$ it is necessary to certify that it is lesser than $\frac{1}{\lambda}$. The condition $\rho(\gamma_s^2 + 1) \leq 1$, due to Proposition 2.6, guaranties that $E_2$ fulfills that request for $k \geq 1$.

So

$$\mu_0 \leq \frac{1}{\lambda}\left[1 + \sum_{j=1}^{\infty} \prod_{k=1}^{j} \frac{\rho(\gamma_s^2 + 1)}{k+1} \right] = \frac{1}{\lambda}\left[1 + \sum_{j=1}^{\infty} \prod_{k=1}^{j} \frac{\rho(\gamma_s^2 + 1)}{k+1} \right] \leq \frac{1}{\lambda}\left[1 + \sum_{j=1}^{\infty} \frac{\rho(\gamma_s^2 + 1)}{j!} \right] = \frac{e^{\rho(\gamma_s^2 + 1)}}{\lambda}. \quad \square$$

Obs: For the $M|G|\infty$ queue systems

$$\mu_0 = \frac{e^{\rho}}{\lambda} \quad (3.3).$$

So, in these conditions, the relative error arising from considering $(3.2)$ instead of $(3.1)$ is

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$^4$It is in fact the $M|G|\infty$ queue busy cycle mean time, see (5).
\[
\frac{e^\rho(y_s^2 + 1)}{\lambda} \cdot \frac{e^\rho}{\lambda} = e^\rho y_s^2 - 1 \leq \frac{y_s^2}{1 - e^{-\rho}} - 1 < e - 1.
\]

But \( e^{y_s^2 + 1} - 1 \leq r \iff \frac{y_s^2}{y_s^2 + 1} \leq \log(r + 1) \iff y_s^2 \leq \frac{\log(r + 1)}{1 - \log(r + 1)} \).

That is: if \( \rho(y_s^2 + 1) \leq 1 \), the relative error arising from taking the bound given by (3.2) instead of the true value given by (3.3) for \( \mu_0 \) is such that:

a) \( \varepsilon \leq e^{y_s^2 + 1} - 1 \),

b) \( \varepsilon = 0 \) if \( y_s^2 = 0 \),

c) \( \varepsilon < e - 1 \),

d) \( \varepsilon \leq r \) \( (r < e - 1) \) since \( y_s^2 \leq \frac{\log(r + 1)}{1 - \log(r + 1)} \).

So, requesting that \( \varepsilon \) is lesser than a given \( r \), it results a criterion to measure the goodness of the \( m_k \) approximation by \( E_2 \) for a certain \( y_s^2 \), since \( \rho(y_s^2 + 1) \leq 1 \).

Being \( B \) the \( M[G] \infty \) queue busy period length, see (5-6),

\[ E[B] = \frac{e^\rho - 1}{\lambda} \quad (3.4). \]

For the Markov renewal process, since \( \rho(y_s^2 + 1) \leq 1 \),

\[ E[B] = \frac{e^\rho(y_s^2 + 1) - 1}{\lambda} \quad (3.5). \]

Now, the relative error own to take (3.5) instead (3.4), is

\[ \frac{e^\rho(y_s^2 + 1) - 1}{\lambda} \cdot \frac{e^\rho - 1}{\lambda} = \frac{e^\rho(y_s^2 + 1) - e^\rho}{e^\rho - 1} \leq \frac{e^y_k^2 - 1}{1 - e^{-\rho}} \leq \frac{y_s^2}{1 - e^{-\rho}} < \frac{e - 1}{1 - e^{-\rho}}. \] So

a) \( \delta \leq \frac{y_s^2}{1 - e^{-\rho}} \),

b) \( \varepsilon = 0 \) if \( y_s^2 = 0 \),

c) \( \delta < \frac{e - 1}{1 - e^{-\rho}} \),

d) \( \delta \leq r \left( r < \frac{e - 1}{1 - e^{-\rho}} \right) \) since \( y_s^2 \leq \frac{\log(r(1 - e^{-\rho}) + 1)}{1 - \log(r(1 - e^{-\rho}) + 1)} \).
So, the bounds for $\delta$ are greater than the obtained to $\epsilon$. Then it is preferable to use a criterion based on $\epsilon$ than on $\delta$, to measure the goodness of the approximation of $m_k$ by $E_2$.

## 4. Mean Number of Entries in State $k$ between Two Entries in State 0

For the Markov renewal process, the mean number of entries in state $k$ between two entries in state 0 is

$$v_k = \lambda^{k-1} \frac{m_1 \cdots m_k}{(1-m_1) \cdots (1-m_k)}, k = 1, 2, \ldots \quad (4.1)$$

**Proposition 4.1**

If $\rho \leq \frac{1}{\gamma_s^2 + 1}$

$$v_k \leq (k + 1) \frac{\rho^{k-1} (\gamma_s^2 + 1)^{k-1}}{k!}, k = 1, 2, \ldots \quad (4.2)$$

**Obs.:** Values for $v_k$, $k = 1, 2, \ldots$ for the $M|G|\infty$ queue system is not known.

From (2.2), (2.9) and (4.2) it follows that

$$m_kv_k \leq \frac{\alpha \rho^{k-1} (\gamma_s^2 + 1)^{k-1}}{k!}, k = 0, 1, \ldots \quad (4.3)$$

Since $\rho (\gamma_s^2 + 1) \leq 1$.

For the $M|G|\infty$ queue system

$$m_kv_k = \frac{\alpha \rho^{k-1}}{k!}, k = 0, 1, \ldots \quad (4.4)$$

But

$$\frac{\alpha \rho^{k-1} (\gamma_s^2 + 1)^{k-1}}{k!} = (\gamma_s^2 + 1)^k - 1,$$

that is null for $\gamma_s = 0$ or $k = 0$ and increases with $k$ if $\gamma_s^2 > 0$.

**Note:** For $\rho \leq 1$ and $\gamma_s^2 = 0$ the Markov renewal process supplies the following results:

a) $m_0 = \frac{1}{\lambda}$,

b) $m_k \leq \frac{\alpha}{k+1}, k = 1, 2, \ldots$

c) $\mu_0 \leq \frac{e\rho}{\lambda}$.
d) \( E[B] \leq \frac{e^{\rho - 1}}{\lambda} \),

e) \( v_k \leq (k + 1) \frac{\rho^{k-1}}{k!}, k = 1, 2, ... \)

f) \( m_k v_k = \frac{\alpha \rho^{k-1}}{k!}, k = 0, 1, ... \)

So, the value obtained for \( m_0 \) coincides with the \( M|G|\infty \) one. And the bounds obtained for \( \mu_0, E[B] \) and \( m_k v_k \) coincide with the true values for the same \( M|G|\infty \) quantities. But the bounds obtained for \( m_k \) and \( v_k \) coincide with the true value obtained when the service time distribution is exponential, and the traffic intensity is 1. In opposition, the bound got for \( m_k \) cannot coincide with the one given by (2.15) for \( \rho < 1 \). So, it is excluded the hypothesis of having an expression for \( m_k \) independent from the service time distribution and equal to the one given by (2.15). Then, only rarely the Markov renewal process gives values for \( m_0, E[B] \) and \( m_k v_k \) identical to the \( M|G|\infty \) ones.

If \( \rho (y_s^2 + 1) \leq 1 \) it is possible, after the Markov renewal process, to get upper bounds for the \( M|G|\infty \) system quantities \( \mu_0, E[B] \) and \( m_k v_k \). So, it is admissible to consider that at least \( E_2 \), beyond being a \( m_k \) upper bound for the Markov renewal process, also plays the same role for the \( M|G|\infty \) queue.

Note still that if \( y_s^2 = 0 \), for the Markov renewal process

\[
m_k = \int_0^\alpha e^{-\lambda t} \left[ 1 - \frac{t}{\alpha} \right]^k dt, \quad k = 0, 1 \ldots \quad (4.5).
\]

So, for \( k \geq 1 \), \( m_k \leq \int_0^\alpha \left( 1 - \frac{t}{\alpha} \right)^k dt = \left[ \frac{-\alpha}{k+1} (1 - \frac{t}{\alpha})^{k+1} \right]_0^\alpha \),

That is \( m_k \leq \frac{\alpha}{k+1}, k = 1, 2, \ldots \).

But, requesting that \( \frac{\alpha}{k+1} \leq \frac{1}{\lambda} \Leftrightarrow k \geq \rho - 1 \) that leads to \( \rho - 1 \leq 0 \Leftrightarrow \rho \leq 1 \).

\[
- m_1 = \int_0^\alpha e^{-\lambda t} \left( 1 - \frac{t}{\alpha} \right) dt = \left[ \frac{-e^{-\lambda t}}{\lambda} \right]_0^\alpha - \int_0^\alpha \frac{e^{-\lambda t}}{\lambda} \left( -\frac{1}{\alpha} \right) dt = \frac{1}{\lambda} - \frac{1}{\rho} \int_0^\alpha e^{-\lambda t} dt = \frac{1}{\lambda} - \frac{1}{\rho} \left[ \frac{e^{-\lambda t}}{-\lambda} \right]_0^\alpha = \frac{1}{\lambda} - \frac{1}{\rho} \left( -\frac{e^{-\rho}}{\lambda} + \frac{1}{\lambda} \right). \quad \text{So,}
\]

\[
m_1 = \frac{\alpha^{\rho+e^{-\rho}-1}}{\rho^2} \quad (4.6).
\]

And, integrating by parts,

\[
m_{k+1} = \frac{1}{\lambda} - \frac{k+1}{\rho} m_k, k = 1, 2, \ldots \quad (4.7).
\]
With (4.6) and (4.7) it is possible to obtain \( m_k, \; k = 1,2, \ldots \) for \( \gamma_s^2 = 0 \) and it is possible to conclude that, in this case, (2.15) does not hold.

**Proposition 4.2**

If the service time distribution is **NBUE**

a) \( \mu_0 \leq \frac{e^\rho}{\lambda} \),

b) \( E[B] \leq \frac{e^{\rho-1}}{\lambda} \),

c) \( u_k \leq (k + 1) \frac{\rho^{k-1}}{k!}, \; k = 1,2, \ldots \)

d) \( m_k u_k \leq \frac{\alpha \rho^{k-1}}{k!}, \; k = 0,1, \ldots \)

**Obs:** The bounds obtained for \( \mu_0, E[B] \) and \( m_k u_k \) coincide with the true value of these quantities for the \( M|G|\infty \) queue.

If the service time distribution is **NWUE**

a) \( \mu_0 \geq \frac{e^\rho}{\lambda} \),

b) \( E[B] \geq \frac{e^{\rho-1}}{\lambda} \),

c) \( u_k \geq (k + 1) \frac{\rho^{k-1}}{k!}, \; k = 1,2, \ldots \)

d) \( m_k u_k \geq \frac{\alpha \rho^{k-1}}{k!}, \; k = 0,1, \ldots \)

with a comment identical to the one in the case **NBUE**.

So, it is admissible that \( \frac{\alpha}{k+\rho}, \; k = 0,1, \ldots \) is an upper bound (lower bound) for the true value of \( m_k \) in the \( M|G|\infty \) queue systems in the case of **NBUE (NWUE)** service time distributions.

After (2.1) and integrating by parts

\[
m_{k+1} = \int_0^\infty e^{-\lambda t} \left[ \frac{\int_0^t [1 - G(x)] \, dx}{\alpha} \right]^{k+1} \, dt = \left[ -\frac{e^{-\lambda t}}{\lambda} \left( \frac{\int_0^t [1 - G(x)] \, dx}{\alpha} \right)^{k+1} \right]_0^\infty - \int_0^\infty -\frac{e^{-\lambda t}}{\lambda} (k + 1) \frac{\int_0^t [1 - G(x)] \, dx}{\alpha} \frac{G(t) - 1}{\alpha} \, dt = \frac{1}{\lambda} - \frac{k+1}{\rho} \int_0^\infty e^{-\lambda t} \left[ \frac{\int_0^t [1 - G(x)] \, dx}{\alpha} \right]^k (1 - G(t)) \, dt \\
\geq \frac{1}{\lambda} - \frac{k+1}{\rho} m_k \]

So

\[
m_{k+1} \geq \frac{1}{\lambda} - \frac{k+1}{\rho} m_k, \; k = 1,2, \ldots (4.8).
\]
Note: According to (4.7), when the service is constant, the equality holds in (4.8).

5. Sojourn Time in State k Distribution

The sojourn time in state k distribution function for the Markov renewal process is

\[ C_k(t) = 1 - e^{-\lambda t} \left( \int_{0}^{\infty} [1 - G(x)] dx \right)^k, \quad t \geq 0, \ k = 0, 1, \ldots \quad (5.1). \]

Evidently,

**Proposition 5.1**

\[ C_k(t) \geq 1 - e^{-\lambda t}, \ t \geq 0, \ k = 0, 1, \ldots \quad (5.2). \]

**Proposition 5.2**

If the service time distribution is exponential

\[ C_k(t) = 1 - e^{-\frac{1}{\alpha} (k+\rho)t}, \ t \geq 0, \ k = 0, 1, \ldots \quad (5.3). \]

**Obs.** This result is coincident with the one known for the M|G|\(\infty\) queue.

**Proposition 5.3**

\[ C_0(t) = 1 - e^{-\lambda t}, \ t \geq 0, \quad (5.4). \]

**Obs.** Result obvious for any M|G|\(\infty\) queue and for any queue with Poisson arrivals process.

**Proposition 5.4**

If the service time distribution is NBUE

\[ C_k(t) \geq 1 - e^{-\frac{1}{\alpha} (k+\rho)t}, \ t \geq 0, \ k = 0, 1, \ldots \quad (5.5). \]

**Obs.** As emphasized before, (5.5), beyond supplying a lower bound for \(C_k(t)\) in the Markov renewal process, also gives a lower bound for that quantity in the M|G|\(\infty\) system for the case of NBUE service time.

If the service time distribution is NWUE

\[ C_k(t) \leq 1 - e^{-\frac{1}{\alpha} (k+\rho)t}, \ t \geq 0, \ k = 0, 1, \ldots \quad (5.6) \]

And it is pertinent a comment analogous to the former one with the change of lower bound by upper bound.

**Proposition 5.5**
If the service time distribution is IMRL

\[ C_k(t) \leq 1 - e^{-\lambda t + k \left( -\frac{2\alpha}{\mu_2} t - \frac{2\alpha}{3\mu_2^2} \mu_3 + 1 \right)}, t \geq 0, k = 0, 1, \ldots \]  \hspace{1cm} (5.6)

**Proposition 5.6**

If the service time distribution is DFR

\[ C_k(t) \leq 1 - e^{-\frac{1}{\alpha}(k+\rho)t + k\frac{1 - \gamma^2}{2}}, t \geq 0, k = 0, 1, \ldots \]  \hspace{1cm} (5.7)

**Conclusions**

When analytical exact results are not available, numerical methods are used to try to find approximations for the interesting quantities under study. It is what is done in this work for the $M|G|\infty$ queue, trying to approximate it for a Markov renewal process. An alternative is using simulation methods. For this approach see, for instance (6, 7).

Still another is to determine service time distributions for which it is possible to determine most of the interesting quantities for the $M|G|\infty$ queue. This is made solving differential equations induced for the study of the transient behavior, see (8-10).

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