SESHADRI CONSTANTS FOR CURVE CLASSES

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Abstract. We develop a local positivity theory for movable curves on projective varieties similar to the classical Seshadri constants of nef divisors. We give analogues of the Seshadri ampleness criterion, of a characterization of the augmented base locus of a big and nef divisor, and of the interpretation of Seshadri constants as an asymptotic measure of jet separation.

1. Introduction

Let $X$ be a projective variety of dimension $n$ over an algebraically closed field. Denote by $N^1(X)$ the space of Cartier divisors modulo numerical equivalence with $\mathbb{R}$ coefficients. Fix $L$ a nef line bundle (or class in the nef cone of divisors $\text{Nef}^1(X) \subset N^1(X)$), and a closed point $x \in X$. The Seshadri constant $\varepsilon(L;x)$ is a measure of local positivity for $L$ at $x$. It can be defined as

\begin{equation}
\varepsilon(L;x) := \max \{ t \geq 0 \mid \pi^*L - tE \text{ is a nef divisor} \},
\end{equation}

where $\pi : \text{Bl}_x X \to X$ denotes the blow-up of $x$, and $-E$ is the divisor associated to the Serre relative $\mathcal{O}(1)$ sheaf. An equivalent interpretation is

\begin{equation}
\varepsilon(L;x) = \inf \{ L \cdot C \mid C \text{ reduced irreducible curve on } X \text{ through } x \}.
\end{equation}

The Seshadri constant $\varepsilon(L;x)$ is a homogeneous numerical invariant of $L$. Its properties (carefully detailed in [Dem92, BDRH+09] and [Laz04, Chapter 5]) motivate our results.

In this paper we study a similar local positivity measure for curve classes.

The role of $\text{Nef}^1(X) \subset N^1(X)$ is taken by the movable cone in the dual space $N^1(X)$ of curve classes modulo numerical equivalence. Recall that $\text{Mov}^1(X) \subset N^1(X)$ is the closure of the convex cone generated by irreducible curves whose deformations sweep out a dense subset of $X$. If $C \in \text{Mov}^1(X)$, then $C \cdot L \geq 0$ for all effective Cartier divisors $L$. Inspired by (1.0.2) we consider the following

Definition 1.1. Let $C \in \overline{\text{Mov}^1}(X)$ and $x \in X$. Set

$$
\varepsilon(C;x) := \inf \left\{ \frac{C \cdot L}{\text{mult}_x L} \mid L \text{ effective Cartier divisor on } X \text{ through } x \right\}.
$$

When $X$ is singular, the regularity condition that $L$ be Cartier (and not just Weil) is necessary for defining $C \cdot L$. The positivity condition on $C$ (its movability) ensures that the infimum is nonnegative (and not $-\infty$) for all $x \in X$. Later on we also consider Seshadri constants for “nef Cartier curves”. The distinction is only relevant when $X$ is singular.

We have an analogous interpretation to (1.0.1).

Proposition 1.2. Let $x \in X$ be a smooth point on a projective variety. Let $\ell$ be a line in the exceptional divisor $E \simeq \mathbb{P}^{n-1}$ of the blow-up $\pi : \text{Bl}_x X \to X$. Let $C \in \overline{\text{Mov}^1}(X)$. Then

$$
\varepsilon(C;x) = \max \{ t \mid \pi^*C - t\ell \in \overline{\text{Mov}^1}(\text{Bl}_x X) \}.
$$

The condition that $x$ be a smooth point of $X$ is necessary for constructing $\pi^*C$ as in Definition 3.7.
1.1. Examples.

**Example 1.3** (Surfaces are a familiar picture). Let \( X \) be a smooth projective surface. There is a canonical isomorphism \( N^1(X) \cong N_1(X) \) coming from the intersection pairing. Via this identification, being nef is equivalent to being movable. In this case the theory of Seshadri constants is the same for curves as it is for divisors.

When \( n = \dim X \) is arbitrary and \( L \) is a nef divisor while \( C \) is a movable curve, the classical \( \varepsilon(L;x) \) is a codimension 1 generalization of the case \( n = 2 \), while our \( \varepsilon(C;x) \) is a dimension 1 generalization of the same case.

**Example 1.4** (Projective space, cf. 3.2). Let \( X = \mathbb{P}^n \), let \( \Lambda \) be the class of a line, and \( H \) the class of a linear hyperplane. Then \( \varepsilon(\Lambda;x) = \varepsilon(H;x) = 1 \) for all \( x \in X \).

**Example 1.5** (Smooth toric varieties, cf. 3.13). Let \( X = X(\Delta) \) be a smooth projective toric variety, and let \( x_\sigma \in X \) be a torus-invariant point corresponding to an \( n \)-dimensional regular cone \( \sigma \in \Delta \). If \( C \in \text{Mov}_1(X) \), then

\[
\varepsilon(C;x_\sigma) = \min \{ C \cdot D_\tau \mid \tau \text{ ray of } \sigma \}.
\]

Here \( D_\tau \) is the torus-invariant divisor corresponding to the ray \( \tau \). A similar formula holds classically for \( \varepsilon(L;x_\sigma) \) if \( L \) is a nef divisor.

**Example 1.6** (Picard rank 1, cf. 7.7). Let \( X \) be a smooth projective variety such that rank \( N^1(X) = 1 \). Let \( H \) be an ample generator of \( N^1(X) \). The curve intersection class \( H^{n-1} \) generates \( N_1(X) \). Seshadri constants (for curves and for divisors) determine the nef and the pseudo-effective cones of divisors for the blow-up \( \text{Bl}_x X \):

\[
\text{Nef}(\text{Bl}_x X) = \langle \pi^* H, \pi^* H - \varepsilon(H;x) E \rangle \quad \text{and} \quad \text{Eff}(\text{Bl}_x X) = \langle E, \pi^* H - \frac{(H^n)}{\varepsilon(H^{n-1};x) E} \rangle.
\]

Recall that the pseudo-effective cone of divisors \( \text{Eff}(X) \) is the closure in \( N^1(X) \) of the convex cone generated by classes of effective Cartier divisors.

A particular case of this brings to light a first difference between curves and divisors.

**Example 1.7** (Grassmann varieties, cf. 6.2). Let \( X = G(k;n) \) be the Grassmann variety of \( k \)-dimensional subspaces of \( \mathbb{C}^n \). Let \( \Lambda \subset X \) be a line generating \( N_1(X) \). Let \( H \) be a hyperplane section of \( X \) in its Plücker embedding. Then for all \( x \in X \) we have \( \varepsilon(H;x) = 1 \) ([Laz04 Example 5.1.7]), but

\[
\varepsilon(\Lambda;x) = \frac{1}{\min\{k, n - k\}}.
\]

**Remark 1.8.** There are a couple of reasons why this may be surprising.

- If \( L \) is a very ample or just ample and globally generated divisor on a projective variety, then \( \varepsilon(L;x) \geq 1 \) for all \( x \in X \) ([Laz04 Example 5.1.8]). While \( \Lambda \in G(k;n) \) satisfies any reasonable analogue of ampleness and most immediate analogues of global generation, the corresponding result does not hold.

- As \( k \) grows, we see that \( \sup_{x \in X} \varepsilon(C;x) \) can be arbitrarily small. For big and nef divisors on projective varieties over \( \mathbb{C} \) it is conjectured that \( \varepsilon(L;x) \geq 1 \) for very general \( x \in X \).

**Example 1.9** (Genus 3 curve in its Jacobian, cf. 7.16). Let \( C \) be a curve of genus 3. Let \( X = J(C) \) be its Jacobian, and assume that rank \( N^1(X) = 1 \). This holds for very general (non-hyperelliptic) curves, or for curves that are very general among the hyperelliptic ones. Let \( \theta \) be the principal polarization. Let \( C \subset X \) be an Abel–Jacobi inclusion. For all \( x \in X \), by a computation of ([Kon03],

\[
\varepsilon(\theta;x) = \begin{cases} 
\frac{12}{5} & \text{in the non-hyperelliptic case} \\
\frac{3}{2} & \text{in the hyperelliptic case}
\end{cases}
\]
We prove that in both cases
\[ \varepsilon(C; x) = \frac{3}{2}. \]
In particular the Seshadri constant of the theta divisor is a finer invariant in this case.

**Remark 1.10.** In genus 2, under the same assumption that the Jacobian has Picard rank 1, [Ste98] proves that \( \varepsilon(C; x) = \varepsilon(\theta; x) = \frac{1}{3} \).

1.2. **The main results.** Note that the function \( \varepsilon(\cdot; x) : \overline{\text{Mov}}_1(X) \to \mathbb{R} \) is 1-homogeneous, nonnegative, and concave. It is positive and locally uniformly continuous on the strict interior of the cone. Concerning positivity on the strict interior, a stronger result is true:

**Theorem A** ("Ampleness" criterion). Let \( X \) be a projective variety over an algebraically closed field, and let \( C \in \overline{\text{Mov}}_1(X) \). Then \( C \) is in the strict interior of the movable cone if and only if \( \inf_{x \in X} \varepsilon(C; x) > 0 \).

The motivation comes from a result of Seshadri ([Har70] Chapter 10) or [Laz04 Theorem 1.4.13]). It states that a nef divisor \( L \) is ample (that is in the strict interior of the nef cone) if and only if \( \inf_{x \in X} \varepsilon(L, x) > 0 \). [BDPP13] (and [FL17b, Theorem 2.22]) in positive characteristic prove that \( \overline{\text{Mov}}_1(X) \) is the dual of the pseudo-effective cone of divisors \( \overline{\text{Eff}}^1(X) \subset N^1(X) \). Thus we may see the movable cone of curves as the "nef" cone of curves. Theorem A gives a characterization of its interior, which is why we may consider it an ampleness criterion.

The proof of the Seshadri criterion for divisors is inductive, relying on the Nakai–Moishezon criterion of ampleness, so it does not readily extend to curves. Instead, the proof of Theorem A relies on an analysis of Zariski decompositions for divisors, and on multiplicity estimates for divisors in large linear series.

**Theorem B** (A characterization of the Null locus). Let \( X \) be smooth projective over an algebraically closed field. Let \( C \in \overline{\text{Mov}}_1(X) \) and assume there exists \( x_0 \in X \) such that \( \varepsilon(C; x_0) > 0 \). Then

i) If \( [L] \in \overline{\text{Eff}}^1(X) \) satisfies \( C \cdot L = 0 \), then \( L \equiv N_\sigma(L) \), where \( N_\sigma(L) \) denotes the negative part in the divisorial Zariski decomposition of [Nak04]. Furthermore the cone
\[ C^\perp \cap \overline{\text{Eff}}^1(X) := \{ [L] \in \overline{\text{Eff}}^1(X) \mid C \cdot L = 0 \} \]
is simplicial, generated by the classes of finitely many effective irreducible divisors \( L_1, \ldots, L_r \) that do not pass through \( x_0 \). As a cycle, any effective \( \mathbb{R} \)-Cartier \( \mathbb{R} \)-divisor \( L \) whose class is in this cone is necessarily a nonnegative linear combination of the \( L_i \).

ii) The class \( C \) is big, that is in the strict interior of the Mori cone of curves \( \overline{\text{Eff}}^1(X) \subset N^1(X) \).

In fact it is in the strict interior of the possibly smaller dual cone \( \overline{\text{Mov}}^{-1}(X)^\vee \). Recall that the movable cone of divisors \( \overline{\text{Mov}}^{-1}(X) \subset N^1(X) \) is the closure of the convex cone generated by classes of Cartier divisors in linear series without fixed divisorial components.

iii) The “null locus” \( \text{Null}(C) := L_1 \cup \ldots \cup L_r \) coincides with the set \( \{ x \in X \mid \varepsilon(C; x) = 0 \} \). In particular the latter is Zariski closed in \( X \).

Manifestly \( \varepsilon(C; x) = 0 \) whenever \( x \) belongs to the support of some effective divisor \( L \) with \( C \cdot L = 0 \). The content of the theorem is the reverse inclusion \( \{ x \in X \mid \varepsilon(C; x) = 0 \} \subseteq \text{Null}(C) \). The motivation comes from [ELM+09 Corollary 5.6 and Remark 6.5]. There the authors prove that if \( L \) is a big and nef divisor, then the locus \( \{ x \in X \mid \varepsilon(L; x) = 0 \} \) coincides with the non-ample locus \( B_+(L) \), also known as the augmented base locus, and with \( \bigcup_{V \subset X \text{ closed subvariety, } \dim V > 0} V \not\text{big} \Leftrightarrow L_{|V} \text{ not big} \). This is a good analogue of Theorem B, since for curves the only nontrivial restrictions are to divisors.

**Theorem C** ("Jet separation"). Let \( X \) be a projective variety over an algebraically closed field, and let \( x \) be a smooth point on it. Let \( [C] \) be an \( \mathbb{R} \)-class in the strict interior of \( \overline{\text{Mov}}_1(X) \). Then \( \varepsilon(C; x) \)
is the supremum of all $s \geq 0$ such that for all effective $\mathbb{R}$-Weil $\mathbb{R}$-divisors $L$ with $x \in \text{Supp} \ L$ there exists an effective $\mathbb{R}$-1-cycle $C' \equiv C$ such that $\text{Supp} \ C'$ and Supp $L$ meet properly and $\text{mult}_x C' \geq s$. The same statement holds with $\mathbb{Q}$ replacing $\mathbb{R}$ throughout.

It is classical that with notation as in the theorem we have $C' \cdot L \geq \text{mult}_x C' \cdot L \geq s \cdot \text{mult}_x L$, leading to $\varepsilon(C; x) \geq s$. The deeper content is that the inequality becomes an equality when considering the supremum of such $s$. The motivation came from the alternate interpretation of Seshadri constants of divisors as an asymptotic measure of jet separation. We say that a Cartier divisor $L$ separates $s$-jets at a smooth point $x \in X$ if the natural map

$$H^0(X, \mathcal{O}_X(L)) \to H^0(X, \mathcal{O}_X(L) \otimes \mathcal{O}_X/m_x^{s+1})$$

is surjective. We denote by $m_x$ the ideal sheaf of $x \in X$. If $L$ separates $s$-jets at $x$, then the linear series $|L|$ verifies the following incidence condition: for every irreducible curve $C$ through $x$ there exists a member $L' \in |L|$ such that $\text{mult}_x L' \geq s$ and $L'$ meets $C$ properly. If $L$ is nef, this implies that $\varepsilon(L; x) \geq s$. When $L$ is ample, [Dem92, Theorem 6.4] and [Laz04, Theorem 5.1.17] prove that this inequality becomes an equality asymptotically. Denote by $s(L; x)$ the largest $s$ such that $L$ separates $s$-jets at $x$. Then $\varepsilon(L; x) = \lim_{k \to \infty} \frac{s(kL; x)}{k}$ if $L$ is ample.

In Theorem [6.7 we also prove an asymptotic version of Theorem C giving control on the coefficients of $C'$.

### 1.3. Bounds on Seshadri constants

Finding global lower bounds on Seshadri constants for ample divisors has proved useful in approaching versions of the famous Fujita conjecture. It predicts that if $A$ is an ample Cartier divisor on the complex projective manifold $X$ of dimension $n$, then $|K_X + (n+1)A|$ is basepoint free, and $|K_X + (n+2)A|$ is very ample. [EKL93, Corollary 3] prove that $|K_X + 2n^2A|$ determine a map which is birational onto its image. We refer to [Dem92, Proposition 6.8] or [Laz04, Proposition 5.1.19] for another result in this direction.

In [EL93 (see also Laz04, Proposition 5.2.3)] it is proved that if $X$ is a smooth projective surface, and $L$ an ample divisor on $X$, then $\varepsilon(L; x) \geq 1$ except for possibly countably many points $x \in X$. A generalization for divisors appears in [EKL93, Theorem 1]. There it is shown that if $X$ is a smooth projective variety of dimension $n$ and $L$ is big and nef, then $\varepsilon(L; x) \geq \frac{1}{n}$ for very general $x \in X$. It is conjectured that $\varepsilon(L; x) \geq 1$ for very general $x$.

Using the techniques of [EL93], we give lower bounds for complete intersection curves.

**Proposition 1.11.** Let $X$ be a projective variety of dimension $n$ over $\mathbb{C}$. Let $H$ be an ample Cartier $\mathbb{Z}$-divisor on $X$ such that $(H^n) \geq n^{n-2}$, or $\varepsilon(H; x_0) \geq 1$ for some $x_0 \in X$. Then $\varepsilon(H^{n-1}; x) \geq 1$ for very general $x \in X$.

The surface case of [EL93] is a particular case of this. See Proposition [7.3] for a sharper version.

**Remark 1.12.** The example of the Grassmanian proves that there cannot exist a lower bound independent of dimension that holds for all classes with $\mathbb{Z}$-coefficients in the strict interior of $\text{Mov}_1(X)$.

Upper bounds are easier to find.

**Definition 1.13.** Let $X$ be a projective variety and fix $x \in X$. For any $L \in \text{Eff}^1(X)$, consider the Fujita–Nakayama-type invariant

$$\mu(L; x) := \sup \{ t \geq 0 \mid \pi^*L - tE \in \text{Eff}^1(\text{Bl}_x X) \},$$

where $\pi : \text{Bl}_x X \to X$ is the blow-up with exceptional divisor $E$.

In a sense this measures the maximal multiplicity at $x$ of effective $\mathbb{R}$-divisors $\mathbb{R}$-linearly equivalent to $L$. An important bound on $\mu(L; x)$ is obtained by counting conditions for a function to vanish at a smooth point with prescribed multiplicity.
Proposition 1.14. If $x$ is a smooth point of $X$ and $L \in \overline{\text{Eff}}^1(X)$, then
\[ \mu(L; x) \geq \text{vol}^{1/n}(L). \]

From the easy observation that
\[ (1.14.1) \quad \varepsilon(C; x) \mu(L; x) \leq C \cdot L \]
for all $C \in \overline{\text{Mov}}_1(X)$ and $L \in \text{Eff}^1(X)$ we find that

Proposition 1.15. If $x$ is a smooth point of $X$, then
\[ \varepsilon(C; x) \leq \inf \left\{ \frac{C \cdot L}{\text{vol}^{1/n}(L)} \mid L \text{ is a big } \mathbb{R}\text{-Cartier } \mathbb{R}\text{-divisor} \right\}. \]

In particular $\varepsilon(H^{n-1}) \leq (H^n)^{\frac{1}{n}}$ for every ample divisor class $H$.

The right hand side is $\mathfrak{m}^{n-1}_C$ as defined by [Xia15] and further studied in [LX16]. The function $\mathfrak{m}$ is a volume-type function on $\overline{\text{Mov}}_1(X)$ obtained by polar Legendre–Fenchel transform from the volume function on $\text{Eff}^1(X)$.

1.4. Relations with the independent work of [MX17]. Seshadri constants for curves have been studied independently at the same time by [MX17]. They see $\varepsilon(\cdot; x) : \overline{\text{Mov}}_1(X) \to \mathbb{R}$ as the polar transform of $\mu(\cdot; x) : \overline{\text{Eff}}^1(X) \to \mathbb{R}$ in the sense that
\[ \varepsilon(C; x) = \inf \left\{ \frac{C \cdot L}{\mu(L; x)} \mid L \text{ is a big } \mathbb{R}\text{-Cartier } \mathbb{R}\text{-divisor} \right\}. \]

There is significant overlap between our work and theirs. They also prove Theorem A, Theorem B, and Propositions 1.14 and 1.15. Their version of Theorem B is sharper in a sense. It also identifies a movable big $\mathbb{R}$-divisor class $L_C$ (an "$(n-1)$ divisorial root of $C$" coming from [LX16 Theorem 1.8]) such that $\text{Null}(C)$ agrees with the union of the divisorial components of $B_+(L_C)$. The class $C$ can be reconstructed from $L_C$.

They also consider a polar transform of $\varepsilon(\cdot; x) : \text{Nef}^1(X) \to \mathbb{R}$ giving rise to a dual function $\mu(\cdot; x) : \overline{\text{Eff}}_1(X) \to \mathbb{R}$. With notation as in Proposition 1.2 assuming that $x$ is a smooth point, the latter also has a geometric interpretation:
\[ \mu(C; x) = \sup\{ t \geq 0 \mid \pi^*C - t\ell \in \overline{\text{Eff}}_1(\text{Bl}_x X) \}. \]

Implicit is the statement that $\pi^*$ preserves the pseudo-effectivity of curves when $\pi$ is the blow-up of a smooth point. This is not true for arbitrary blow-ups (even with smooth centers) because it is dual to saying that the cycle pushforward $\pi_*$ preserves the nefness of divisors.

1.5. Seshadri constants for nef dual classes. Pseudo-effective cones $\overline{\text{Eff}}^k(X) \subset N^k(X)$ exist for all cycle dimensions $0 \leq k \leq n$. Dually, we have nef cones $\text{Nef}^k(X) := \overline{\text{Eff}}^k(X)^\vee$ in all codimensions inside the dual numerical groups $N^k(X) := N_k(X)^\vee$. For example $\text{Nef}^1(X)$ is the usual cone of nef Cartier divisor classes by [Kle66]. When $X$ is smooth, $\text{Nef}^{n-1}(X) = \overline{\text{Mov}}_1(X)$ by [BDPP13], but not for general singular $X$. Working with nef classes allows us to remove the regularity conditions from $L$ or $x$ in our definitions for $\varepsilon(C; x)$.

Definition 1.16. Let $X$ be a projective variety and fix a possibly singular point $x \in X$. For any $\alpha \in \text{Nef}^k(X)$ set
\[ \varepsilon(\alpha; x) = \inf \left\{ \frac{\alpha \cdot V}{\text{mult}_x V} \mid V \text{ effective } k\text{-cycle through } x \right\}. \]
It is enough to consider the irreducible subvarieties $V$ through $x$, with no regularity conditions. As for divisors,

$$
\varepsilon (\alpha; x) = \max \{ t \geq 0 \mid \pi^* \alpha + t(-E)^k \in \text{Nef}^k(\text{Bl}_x X) \}.
$$

Nefness is preserved by proper pullback and the exceptional divisor $E$ of $\pi : \text{Bl}_x X \to X$ is Cartier, so the smoothness assumption on $x$ is not needed here. Seshadri constants for nef classes share many of the formal properties we saw for nef divisors or movable curves. The function $\varepsilon (\cdot; x) : \text{Nef}^k(X) \to \mathbb{R}$ is 1-homogeneous, nonnegative, and concave. It is positive and locally uniformly continuous on the strict interior of the cone.

For nef dual curve classes, meaning $k = n - 1$, we find an analogue of Theorem A (cf. Proposition 8.13) and a weaker version of Theorem B (see Proposition 8.15). Note that Theorem C does not hold for arbitrary classes in the strict interior of $\text{Nef}^k(X)$. By [DLY11], there exist abelian varieties that carry nef classes that are not pseudo-effective. Example 8.5 suggests that the Seshadri constants of nef dual classes are finer than those of movable curve classes.

### 1.6. Organization

In section 2 we review some background on numerical groups and their duals and on the various positivity notions that we need. In section 3 we develop the basic properties of Seshadri constants for curves. The next three sections correspond to the proofs of Theorems A, B, and C respectively. Section 7 deals with bounds on Seshadri constants. Lastly, we develop a theory of Seshadri constants for nef dual classes in arbitrary codimension.

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## 2. Notation and Background

Let $X$ be a projective variety of dimension $n$ over an algebraically closed field. The real space of Cartier divisors on $X$ modulo numerical equivalence is denoted $N^1(X)$. Its dual space $N_1(X)$ is the real space of 1-dimensional cycles on $X$ modulo numerical equivalence. Generalizations for arbitrary dimension are provided by [Ful84, Chapter 12]. A $k$-cycle $Z$ is numerically trivial if $\int_X P = 0$ for all polynomials $P$ of weight $k$ in Chern classes of possibly several vector bundles on $X$. The real space of cycles modulo numerically trivial cycles is denoted $N_k(X)$. Its elements are called numerical cycle classes. Its abstract dual $N^k(X) := N_k(X)^\vee$ is also the space of Chern polynomials of weight $k$ modulo the dual equivalence relation: $P$ is numerically trivial if $\int_X P = 0$ for all $k$-cycles $Z$. The elements of $N^k(X)$ are called dual cycle classes. We routinely denote numerical classes of cycles $V$ or Chern polynomials $P$ by $[V]$ and $[P]$ respectively. When $X$ is also smooth, the intersection pairing induces an isomorphism $N^k(X) \simeq N_{n-k}(X)$. In general we have a natural linear “cyclification” morphism $N^k(X) \cap [X] \to N_{n-k}(X) : P \mapsto P \cap [X]$. It is injective when $k = 1$, and surjective when $k = n - 1$.

These vector spaces contain important closed convex cones. The closure of the cone generated by effective Cartier divisors on $X$ is the pseudo-effective cone $\text{Eff}^1(X) \subset N^1(X)$. It contains the nef cone $\text{Nef}^1(X)$, the closure of the convex span of ample divisors. In the dual space $N_1(X)$ we find the duals of these two cones. The Mori cone $\text{Eff}^1(X)$ is the closure of the cone of effective cycle classes, dual to $\text{Nef}^1(X)$ in view of [Kle66]. The movable cone $\text{Mov}^1(X)$ is the closure of the convex span of irreducible curves that deform in families that dominate $X$. This cone is also dual to $\text{Eff}^1(X)$ by [BDPP13] (see also [Laz04, Theorem 11.4.19], and [FL17b, Theorem 2.22] for the case of positive characteristic).

In $N_k(X)$ we can analogously define the pseudo-effective cone $\text{Eff}^k(X)$ and the movable cone $\text{Mov}^k(X)$. The nef cone $\text{Nef}^k(X) \subset N^k(X)$ is the dual of $\text{Eff}^k(X) \subset N^k(X)$.
$N_k(X)$. We often say that a (dual) cycle is nef/movable when its numerical class has the same property.

Let $x \in X$ be a closed point and let $\pi : \text{Bl}_x X \to X$ be the blow-up of $X$ at $x$ with exceptional divisor $E$, such that $-E$ is the divisor associated to the relative Serre $\mathcal{O}(1)$ sheaf. By [Ful84, p.79], for any irreducible and reduced closed subset $V \subseteq X$ (except $V = \{x\}$), one has

$$\text{mult}_x V = -\nabla \cdot (-E)^\dim V,$$

where $\nabla$ denotes the strict transform of $V$. Since the strict transform of a union of subvarieties (different from $\{x\}$) is the union of the strict transforms, one can extend $V \mapsto \nabla$ linearly to $k$-cycles with arbitrary coefficients. Then the same holds for $V \mapsto \text{mult}_x V$.

3. **Seshadri constants for curves**

Recall that if $C$ is a curve with movable numerical class and $x \in X$, then

$$\varepsilon(C; x) := \inf \left\{ \frac{C \cdot L}{\text{mult}_x L} \mid L \text{ effective Cartier divisor on } X \text{ through } x \right\}.$$

**Remark 3.1.** From the definition we see that $\varepsilon(C; x)$ depends only on the numerical class of $C$. Furthermore $[C] \mapsto \varepsilon(C; x)$ is 1-homogeneous, nonnegative, and concave on $\overline{\text{Mov}_1(X)}$.

**Example 3.2** (Projective space). Let $\ell$ be a line in $\mathbb{P}^n$. By Bézout: $\deg \frac{L}{\text{mult}_x L} \geq 1$ for every effective divisor $L$ through a point $x \in \mathbb{P}^n$. Equality is achieved when $L$ is a linear hyperplane through $x$. Thus $\varepsilon(\ell; x) = 1$ for every $x \in \mathbb{P}^n$.

**Remark 3.3.** If $X$ is smooth, then the infimum can be computed over irreducible divisors $L$. This is because of the inequality $\frac{a \cdot c}{a \cdot d} \geq \min \left\{ \frac{a}{b}, \frac{c}{d} \right\}$ for $a, b \geq 0$ and $c, d > 0$. When $X$ is singular and $Z$ is a component of $L$, the intersection number $C \cdot Z$ may be undefined.

**Remark 3.4** (Real coefficients). By the proof of [Fuj99, Lemma 0.14], we may work with $\mathbb{R}$-Cartier $\mathbb{R}$-divisors $L$ in the definition of $\varepsilon(C; x)$. The claim is that being effective for an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor is equivalent to being a nonnegative combination of (possibly nonreduced and reducible) effective Cartier divisors.

**Lemma 3.5.** If $[C]$ is in the strict interior of $\overline{\text{Mov}_1(X)}$, then there exists a constant $\varepsilon > 0$ such that $\varepsilon(C; x) \geq \varepsilon$ for all $x \in X$.

**Proof.** Denote $n := \dim X$. There exists an ample $\mathbb{Q}$-divisor $H$ on $X$ and $\alpha \in \overline{\text{Mov}_1(X)}$ such that $[C] = [H^{n-1}] + \alpha$. Let $\pi : \text{Bl}_x X \to X$ be the blow-up of $x$ with exceptional divisor $E$. For all effective Cartier divisors $L$ through fixed $x$,

$$\frac{C \cdot L}{\text{mult}_x L} \geq \frac{H^{n-1} \cdot L}{\text{mult}_x L} = \frac{\pi^* H^{n-1} \cdot \nabla}{(-E)^{n-1} \cdot (-\nabla)},$$

where $\nabla$ denotes the strict transform of $L$ (it is still Cartier, since $E$ is Cartier). Since $H$ is ample, by the Seshadri criterion of ampleness ([Har70, Chapter 10] or [Laz04, Theorem 1.4.13]) there exists $m > 0$ such that $\varepsilon(H; x) \geq \frac{1}{m}$ for all $x \in X$. Then $\pi^* H - \frac{1}{m} E$ is nef for all $x \in X$ and

$$0 \leq \left( \pi^* H - \frac{1}{m} E \right)^{n-1} \cdot \nabla = \pi^* H^{n-1} \cdot \nabla + \frac{1}{m^{n-1}} (-E)^{n-1} \cdot \nabla.$$

Note that the intermediary intersections $(\pi^* H)^i \cdot E^{n-1-i} \cdot \nabla$ with $0 < i < n - 1$ vanish. In fact $\pi^* H \cdot E = 0$ since up to taking multiples $H$ can be moved away from $x$. This proves $\varepsilon(C; x) \geq \frac{1}{m^n}$ for all $x \in X$. $\square$

**Corollary 3.6.** The function $\varepsilon(\cdot; x)$ is locally uniformly continuous on the strict interior of $\overline{\text{Mov}_1(X)}$. 
Proof. The lemma proves that the function is positive on the interior of the movable cone. Then apply [Leh16, Lemma 2.7].

For an interpretation equivalent to (1.0.1), we need to be able to pullback. This requires some regularity condition on $\pi$.

Definition 3.7. Let $x \in X$ be a smooth point. In this case $E \simeq \mathbb{P}^{n-1}$. Fix $\ell$ a line in $E$, so that $E \cdot \ell = -1$. For any curve $C \subseteq X$, set

$$\pi^*C := \overline{C} + (\text{mult}_x C) \cdot \ell.$$

Proposition 3.8. The pullback $\pi^*$ defined above is linear, respects numerical equivalence, and satisfies the projection formula

$$D \cdot \pi^*C = (\pi_*D) \cdot C$$

for any Cartier divisor $D$ on $\text{Bl}_x X$.

Proof. Linearity holds because $V \mapsto \overline{V}$ and $V \mapsto \text{mult}_x V$ are linear operations on cycles of the same dimension. If $D$ is Cartier on $\text{Bl}_x X$, then $\pi_*D$ is Cartier on $X$ (because $\pi$ is an isomorphism away from the smooth point $x$) and

$$(3.8.1) \quad D = \pi^*\pi_*D - (D \cdot \ell)E.$$

If $C$ is a numerically trivial 1-cycle, then $\pi^*C \cdot E = 0$. This is straightforward from the relation $\overline{C} \cdot E = \text{mult}_x C$. (see (2.0.11)) and proves that $\pi^*$ respects numerical equivalence.

From the definition, $\pi_*\pi^*C = C$. Then the projection formula is clear for divisors $D = \pi^*L$. Using (3.8.1), it remains to treat the case $D = E$, which follows from $\pi^*C \cdot E = 0$ above.

Lemma 3.9. If $x \in X$ is a smooth point and $[C] \in \overline{\text{Mov}}_1(X)$, then $[\pi^*C] \in \overline{\text{Mov}}_1(\text{Bl}_x X)$.

Proof. Use the projection formula and the duality between $\overline{\text{Mov}}_1(\text{Bl}_x X)$ and $\overline{\text{Eff}}^1(\text{Bl}_x X)$ (cf. [BDPP13]).

The next result interprets the Seshadri constant of a curve as the distance from $[\pi^*C]$ to the boundary of $\overline{\text{Mov}}_1(\text{Bl}_x X)$ in the $[-\ell]$ direction:

Proposition 3.10. Let $x \in X$ be a smooth point on a projective variety. Let $C \subseteq X$ be a curve with $[C] \in \overline{\text{Mov}}_1(X)$. Then

$$\varepsilon(C; x) = \max\{t \mid [\pi^*C - t\ell] \in \overline{\text{Mov}}_1(\text{Bl}_x X)\}.$$

Note that the maximum is nonnegative by the previous lemma, and well-defined (finite) because for fixed ample $H$ on $\text{Bl}_x X$, we have $(\pi^*C - t\ell) \cdot H < 0$ for $t \gg 0$.

Proof. Let $t \geq 0$ with $\pi^*C - t\ell$ movable. Note that

$$(3.10.1) \quad \overline{L} = \pi^*L - (\text{mult}_x L) \cdot E$$

is Cartier on $\text{Bl}_x X$ for any Cartier divisor $L$ on $X$. Then $(\pi^*C - t\ell) \cdot \overline{L} \geq 0$ for all effective Cartier divisors $L$ on $X$. By (2.0.11) and the projection formula this is equivalent to $C \cdot L \geq t \cdot \text{mult}_x L$. If $L$ ranges through effective Cartier divisors that pass through $x$, then we obtain $t \leq \varepsilon(C; x)$. It follows that $\max\{t \geq 0 \mid [\pi^*C - t\ell] \in \overline{\text{Mov}}_1(\text{Bl}_x X)\} \leq \varepsilon(C; x)$.

Conversely, if $0 \leq t \leq \frac{C \cdot L}{\text{mult}_x L}$ for all effective Cartier $L$ through $x$, then $C \cdot L \geq t \cdot \text{mult}_x L$. This is also clearly true for $L$ effective Cartier not passing through $x$. As above, $(\pi^*C - t\ell) \cdot \overline{L} \geq 0$ for all effective Cartier $L$ on $X$. We also observe $(\pi^*C - t\ell) \cdot E = t \geq 0$. Since $x$ is a smooth point of $X$, any effective Cartier divisor on $\text{Bl}_x X$ is the sum of an effective Cartier divisor of form $\overline{L}$ and a nonnegative multiple of $E$. Using [BDPP13] we conclude that $\pi^*C - t\ell$ is movable. Consequently $\pi^*C - \varepsilon(C; x)\ell$ is movable.
Seshadri constants for curves also verify a semi-continuity type statement analogue to the case of divisors ([Laz04, Example 5.1.11]).

**Proposition 3.11.** Let $T$ be a smooth variety over an uncountable algebraically closed field, and let $p : \mathcal{X} \to T$ be a smooth projective morphism with connected fibers. Assume that $p$ admits a section $x : T \to \mathcal{X}$. Let $\mathcal{C} \subseteq \mathcal{X}$ be a cycle of dimension $\text{dim} T + 1$. Denote by $X_t$ the scheme theoretic fiber of $p$ over $t \in T$ and by $[C_t]$ the class of the restriction $[C]|_{X_t}$ (in the sense of [Ful84, Chapter 8]). Assume that $[C_t] \in \text{Mov}_1(X_t)$ for all $t \in T$. Then $\varepsilon([C_t]; x_t)$ is constant for very general $t \in T$. For special $t$ it may only decrease.

There are no effectiveness assumptions on $\mathcal{C}$. Recall that a property is said to hold for very general $t \in T$ if there exists an at most countable collection of proper closed subsets $V_i \subseteq T$ such that the property holds for all $t \in T \setminus \bigcup_i V_i$.

**Proof.** Let $\pi : \widetilde{\mathcal{X}} \to \mathcal{X}$ be the blow-up of $\mathcal{X}$ along the image of $x$ with induced smooth morphism $q : \widetilde{\mathcal{X}} \to T$ and exceptional divisor $\mathcal{E}$. Let $\Lambda$ be a cycle on $\widetilde{\mathcal{X}}$ such that the class of its restriction to each $E_t$ is the same as the class of a line in the exceptional divisor of $\text{Bl}_{x_t}X_t$. Such cycles exist, even effective ones. For fixed $t_0$, apply the lemma below for $\pi^*\mathcal{C} - \varepsilon([C_{t_0}]; x_{t_0})\Lambda$ (here $\pi^*\mathcal{C}$ denotes a choice of a cycle representing the Chow class $\pi^*[\mathcal{C}]$ in the sense of [Ful84, Chapter 8]) to show that $\varepsilon([C_t]; x_t) \geq \varepsilon([C_{t_0}]; x_{t_0})$ for very general $t \in T$. Then apply it for $t_0$ very general to find that $\varepsilon([C_t]; x_t)$ is constant for $t$ very general.

**Lemma 3.12.** Let $T$ be a smooth variety over an uncountable algebraically closed field, and let $p : \mathcal{X} \to T$ be a smooth projective morphism with connected fibers. Let $\mathcal{C} \subseteq \mathcal{X}$ a cycle of dimension $\text{dim} T + 1$. Assume there exists $t_0 \in T$ such that $[C_{t_0}] \in \text{Mov}_1(X_{t_0})$. Then $[C_t]$ is movable for very general $t \in T$.

**Proof.** Let $(L_t, t)$ be the set of pairs with $t \in T$ and $L_t$ an irreducible divisor in $X_t$ with $[L_t] \cdot [C_t] < 0$. By usual Hilbert scheme arguments, these are parameterized by countably many schemes $H_i$. If the conclusion fails, then by the main result of [BDPP13], one of the $H_i$ dominates $T$. Up to replacing $H_i$ by a closed subset, we may assume that the map $\epsilon : H_i \to T$ is generically finite and dominant. Let $\mathcal{D} \subseteq \mathcal{X}$ be the closure of the union of the divisors $L_t$ parameterized by $H_i$. It is an effective divisor that may contain some of the fibers of $p$ in its support. For general $t \in T$ though, the fiber $D_t$ is the sum $\sum_{\epsilon(h_i) = t} L_{h_i}$. In particular $[D_t] \cdot [C_t] < 0$.

Since $D$ is effective and $p$ is smooth, $[D_{t_0}] \in N^1(X_{t_0})$ is a pseudo-effective divisor class. See [FL16, Lemma 4.10]. For general $t \in T$ we find the contradiction $0 \leq [D_{t_0}] \cdot [C_{t_0}] = [D_t] \cdot [C_t] < 0$. The intersection numbers are equal to $\mathcal{D} \cdot \mathcal{E} \cdot X_t$, which makes sense after considering a smooth projective completion of $\mathcal{X}$.

**Example 3.13** (Toric varieties). Let $X = X(\Delta)$ be a smooth complete toric variety. Let $[C] \in \text{Mov}_1(X)$. Let $x = x_{\sigma}$ be a torus invariant point. Then the Seshadri constant is computed by one of the irreducible invariant divisors through $x_{\sigma}$. Specifically,

$$\varepsilon(C; x) = \min\{C \cdot D_{\tau} \mid \tau \in \sigma(1)\}.$$  

(A deformation argument shows that $\varepsilon(C; x) = \min\{\frac{C \cdot D_{\tau}}{\mult_{x_{\sigma}} D_{\tau}} \mid x_{\sigma} \in D_{\tau}\}$. All invariant divisors on a smooth toric variety are smooth.) When $x$ is not a torus invariant point, the Seshadri constant is potentially bigger by the results above.

4. A Seshadri-type criterion

**Proof of Theorem A.** One implication is provided by Lemma 3.5. For the converse, assume first that $X$ is also smooth and $\varepsilon(C; x) \geq \varepsilon > 0$ for some fixed $\varepsilon$ independent of $x \in X$. If $[C]$ is not an interior class, then by the duality result of [BDPP13] (see [FL17b, Theorem 2.22] for the case of positive characteristic) there exists a pseudo-effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor $L$ with $[L] \neq 0$ such that
$C \cdot L = 0$. Let $L = P_\sigma(L) + N_\sigma(L)$ be the divisorial Zariski decomposition in the sense of \cite{Nak04} (or \cite{Mum13}, \cite{CHMS14}, or \cite{FKL16} Section 4) for the case of positive characteristic. We can assume that $[L]$ is extremal in $\operatorname{Eff}^1(X)$, and then either $L \equiv N_\sigma(L)$, or $L = P_\sigma(L)$. If $L \equiv N_\sigma(L)$, then $L$ is effective (up to numerical equivalence). Choose $x$ in the support of some effective representative $L'$ of $[L]$. We obtain the contradiction $0 = C \cdot L \geq \varepsilon \cdot \operatorname{mult}_x L' > 0$.

We now treat the case when $L = P_\sigma(L)$ with $[L] \neq 0$. By \cite{Nak04} V.1.11. Theorem (see \cite{CHMS14} for characteristic $p$), there exists $\beta > 0$ and $A$ ample such that

$$\dim C H^0(X, \mathcal{O}_X([mL] + A)) \geq \beta \cdot m$$

for all $m$ sufficiently large. For any $D_m \in [[mL] + A]$ we have $C \cdot (|mL| + A) \geq \varepsilon \cdot \operatorname{mult}_x D_m$ for any $x \in X$. Since $C \cdot L = 0$, the left hand side is bounded independently of $m$. It follows that there exists some integer $B > 0$ such that $B > \operatorname{mult}_x D_m$ for all $x \in X$, all sufficiently large $m$, and all $D_m \in [[mL] + A]$. However, passing through a fixed smooth $x$ with multiplicity at least $B + 1$ is a constant number of conditions on any linear series on $X$. Since $[[mL] + A]$ has arbitrarily large dimension, for large $m$ we may find $D_m \in [[mL] + A]$ with $\operatorname{mult}_x D_m > B$. This is a contradiction.

Consider now the case when $X$ is an arbitrary projective variety over $\mathbb{C}$. Let $\pi : \widetilde{X} \to X$ be a resolution of singularities of $X$. Let $H$ be a large ample on $X$. Since $\pi$ is birational, the divisor $\pi^* H$ is big, so it can be written as $\pi^* H = A + E$ with $A$ ample on $\widetilde{X}$ and $E$ effective. As before, there exists a pseudo-effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor $L$ on $X$ such that $C \cdot L = 0$ and $[L] \neq 0$ in $N^1(X)$.

Assume that $P_\sigma(\pi^* L)$ is not numerically trivial. Up to replacing $H$ (so $A$ and $E$) by high multiples (see the proof of \cite{Nak04} V.1.11. Theorem), we may assume that $A$ and $P_\sigma(\pi^* L)$ are as in the smooth and movable case. Fix $x$ in the smooth locus of $X$ such that $\pi$ is an isomorphism in a neighborhood of $x$, and denote $\tilde{x} = \pi^{-1}\{x\}$. Choose $D_m \in [[mP_\sigma(\pi^* L)] + A]$ such that $\lim_{m \to \infty} \operatorname{mult}_{\tilde{x}} D_m = \infty$. Note that

$$C \cdot H = C \cdot (mL + H) = C \cdot \pi_*(D_m + \langle mP_\sigma(\pi^* L) \rangle + mN_\sigma(\pi^* L) + E) \geq \varepsilon \cdot \operatorname{mult}_x \pi_* D_m = \varepsilon \cdot \operatorname{mult}_{\tilde{x}} D_m,$$

where by $\langle \cdot \rangle$ we denote the fractional part of a divisor, as in \cite{Nak04} Section II.2.d. The second equality is true because the two divisors that we intersect with are linearly equivalent. The inequality holds because $D_m + \langle mP_\sigma(\pi^* L) \rangle + mN_\sigma(\pi^* L) + E$ is a sum of effective $\mathbb{R}$-Weil divisors, and it stays so after pushforward. Furthermore the pushforward is an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor linearly equivalent to $mL + H$. The last equality holds because $\pi$ is an isomorphism above $x$. As $m$ grows we get a contradiction.

It remains to consider the case where $P_\sigma(\pi^* L)$ is numerically trivial. For all $m > 0$ we have that $\frac{1}{m} A + P_\sigma(\pi^* L)$ is an ample $\mathbb{R}$-divisor on $\widetilde{X}$, in particular $\mathbb{R}$-linearly equivalent to an effective $\mathbb{R}$-divisor $F_m$. Since $[L] \neq 0$, it follows that $\pi_* N_\sigma(\pi^* L)$ is a nonzero effective $\mathbb{R}$-divisor on $X$. This is easily seen by intersecting $L = \pi_* P_\sigma(\pi^* L) + \pi_* N_\sigma(\pi^* L)$ with $H^{n-1}$, where $n = \dim X$. Note that $L \cdot H^{n-1} > 0$ since $[L] \neq 0$ is pseudo-effective, and $H^{n-1}$ is in the strict interior of $\overline{\operatorname{Mov}}_1(X)$ (\cite{FL17b} Lemma 3.9)). Then

$$\frac{1}{m} C \cdot H = C \cdot \left( \frac{1}{m} H + L \right) = C \cdot \pi_* (F_m + N_\sigma(\pi^* L) + \frac{1}{m} E) \geq \varepsilon \cdot \operatorname{mult}_x \pi_* N_\sigma(\pi^* L)$$

for every $x$ in the support of $\pi_* N_\sigma(\pi^* L)$. As $m$ grows, we get a contradiction.

Finally, we consider the case of arbitrary characteristic. Instead of a resolution, consider $\pi : \widetilde{X} \to X$ a nonsingular alteration (\cite{Li19}). For $H$ ample, $\pi^* H$ is big, so we can construct $A$ and $E$ as before. The proof goes through as above with minimal changes. In the projection formula there is a correction by $\deg \pi$, e.g., $\deg \pi \cdot L = \pi \cdot \pi^* L$. The point $x$ is chosen in the regular locus of $X$ and in the finite locus of $\pi$, and $\tilde{x}$ is any point in $\pi^{-1}\{x\}$. A bounding relation between $\operatorname{mult}_{\tilde{x}} D_m$ and $\operatorname{mult}_x \pi_* D_m$ is provided by the lemma below. \[\square\]
Lemma 4.1. Let $\pi : X \to Y$ be a proper generically finite morphism of varieties. Let $x \in X$ be a closed point such that $\pi$ is finite in a neighborhood of $x$. Let $y := \pi(x)$. Then $\mult_y \pi_* X \geq \mult_x X$.

Proof. Let $z := \pi^{-1} y$ denote the scheme theoretic preimage of $y$. Assume first that $z$ is a finite length subscheme. Let $\tilde{\pi} : \tilde{X} \to \tilde{Y}$ be the induced morphism between $\text{Bl}_y Y$ and $\text{Bl}_z X$. Let $E$ denote the exceptional divisor of $\tilde{Y}$ and let $F$ denote the exceptional divisor of $\tilde{X}$. We have $\tilde{\pi}^* E = F$. By $F_x$ we denote the (connected) component of $F$ over $x$. It is also a Cartier divisor. Let $x' \subseteq E$ denote the primary component of $z$ centered at $x$. For $n := \dim X$, we have $\mult_x X \leq \mult_{x'} X = -(F_x)^n \leq -(z)^n = -(\deg \pi) \cdot -(E)^n = \deg \pi \cdot \mult_y Y = \mult_y \pi_* X$.

When $z$ has components of positive dimension, automatically not through $x$, the inequality $-(F_x)^n \leq -(z)^n$ is unclear. In this case, let $X \xrightarrow{f} Z \xrightarrow{g} Y$ be the Stein factorization of $\pi$. Apply the previous case twice. $\square$

The following lemma provides some control for $\inf_{x \in X} \varepsilon (C; x)$ under blow-ups of smooth varieties along smooth centers. We hope that it will inspire arguments following the steps of the MMP.

Lemma 4.2. Let $X$ be a smooth projective variety over $\mathbb{C}$, and let $C$ be a movable curve such that $\varepsilon (C; x) \geq \varepsilon > 0$ for all $x \in X$. Let $Z \subseteq X$ be a smooth closed subvariety with $\text{codim}_X (Z) \geq 2$, and let $Y := \text{Bl}_Z X$ with blow-down morphism $\pi$. Let $E$ be the exceptional divisor of $\pi$, and let $\ell$ be a line in any fiber of $E \to Z$. Set $C' := \pi^* C - \varepsilon \ell$. Then $C'$ is movable, $\pi_* C' = C$, and $\varepsilon (C'; y) \geq \frac{\varepsilon}{2}$ for all $y \in Y$.

Proof. It is clear that $\pi_* C' = C$. We want to prove the Seshadri inequality $C' \cdot L \geq \frac{\varepsilon}{2} \mult_y L$ for all irreducible divisors $L$ on $Y$ and for all $y \in Y$. If this holds, then in particular $C' \cdot L \geq 0$ for all effective divisors on $Y$, hence $C'$ is movable. Consider first the case where $Z = \{z\}$ is a point. Let $y \in Y$ with $\pi(y) = x \neq z$. Let $D$ be an irreducible divisor on $Y$ containing $y$. It follows that $\overline{D} \neq E$, hence it is the strict transform of a divisor $D$ on $X$. We have

$$C' \cdot \overline{D} = C' \cdot D - \frac{\varepsilon}{2} \mult_z D \geq C \cdot D - \frac{1}{2} C \cdot D \geq \frac{\varepsilon}{2} \mult_x D = \frac{\varepsilon}{2} \mult_y \overline{D},$$

since $\pi$ is an isomorphism around $x \neq z$. Let now $y \in E$. If $\overline{D}$ is the strict transform of an effective divisor $D$ on $X$, then

$$C' \cdot \overline{D} = C' \cdot D - \frac{\varepsilon}{2} \mult_z D \geq (\varepsilon (C; x) - \frac{\varepsilon}{2}) \cdot \mult_z D \geq \frac{\varepsilon}{2} \mult_z D.$$ 

We claim that $\mult_z D \geq \mult_y \overline{D}$. This is because $\overline{D}$ and some line in $E$ through $y$ meet properly, so $\mult_z D = \overline{D} \cdot \ell \geq \mult_y \overline{D}$. Furthermore

$$C' \cdot E = \frac{\varepsilon}{2} = \frac{\varepsilon}{2} \mult_y E$$

for all $y \in E$, since $E$ is smooth.

In the general case, let $y \not\in E$ and let $\overline{D}$ be an irreducible divisor containing $y$. Set $x := \pi(y)$. As above, $\overline{D}$ is the strict transform of an irreducible divisor $D$ on $X$. Furthermore

$$C' \cdot \overline{D} = C' \cdot D - \frac{\varepsilon}{2} \mult Z D = C' \cdot D - \frac{\varepsilon}{2} \mult_z D \geq C \cdot D - \frac{1}{2} C \cdot D = \frac{1}{2} C \cdot D \geq \frac{\varepsilon}{2} \mult_x D = \frac{\varepsilon}{2} \mult_y \overline{D},$$

since $\pi$ is an isomorphism around $x$. Here $z$ is a general point of $Z$ so that $\mult Z D = \mult_z D$. The basic properties of $\mult Z D$ are listed in the remark below. Let now $y \in E$ so $x := \pi(y) \in Z$. If $\overline{D}$ is the strict transform of an effective divisor $D$ on $X$, then

$$C' \cdot \overline{D} = C' \cdot D - \frac{\varepsilon}{2} \mult Z D \geq \frac{\varepsilon}{2} \cdot (2 \mult Z D - \mult Z D).$$

We show that $2 \mult_z D - \mult Z D \geq \mult_y \overline{D}$. This is a local computation. In a small analytic neighborhood of $y$ we have that $\pi$ is given by

$$(y_1, y_2, \ldots, y_n) \mapsto (y_1 y_2, \ldots, y_1 y_d, y_{d+1}, \ldots, y_n).$$
where $d$ is the codimension of $Z$, and $Z$ is given by $x_1 = \ldots = x_d = 0$. A local equation for $D$ around $x$ is

$$f(x_1, \ldots, x_n) = f_m(x_1, \ldots, x_n) + f_{m+1}(x_1, \ldots, x_n) + \cdots,$$

where $f_i$ is homogeneous of degree $i$ and $m := \text{mult}_x D$. Then a local equation for $\overline{D}$ around $y$ is

$$\frac{f_m(y_1, y_1 y_2, \ldots, y_1 y_d, y_{d+1}, \ldots, y_n)}{y_1^{\text{mult}_Z D}} + \frac{f_{m+1}(y_1, y_1 y_2, \ldots, y_1 y_d, y_{d+1}, \ldots, y_n)}{y_1^{\text{mult}_Z D}} + \cdots$$

If a monomial $x_1^{a_1} \cdots x_n^{a_n}$ appears in the equation of $D$, then $y_1^{a_1 + \ldots + a_d - \text{mult}_Z D} \cdot y_2^{a_2} \cdots y_n^{a_n}$ appears in the equation of $\overline{D}$. The correspondence is reversible, hence one-to-one. The degree is $a_1 + 2(a_2 + \ldots + a_d) + a_{d+1} + \ldots + a_n - \text{mult}_Z D$. Since $f_m \neq 0$, at least one of these has degree at most $2m - \text{mult}_Z D$ as desired.

It remains to verify the Seshadri inequality for the divisor $E$ and a point $y \in E$. Using that $E$ is smooth,

$$C' \cdot E = \frac{\varepsilon}{2} = \frac{\varepsilon}{2} \text{mult}_y E.$$

\[\square\]

**Remark 4.3** (Multiplicity of a divisor along a subvariety). Let $X$ be a variety, $Z \subset X$ a subvariety of codimension $c \geq 2$, not contained in $\text{Sing}(X)$, and $D \subset X$ a Weil divisor. Let $\pi : \text{Bl}_Z X \to X$ be the blow-up with exceptional divisor $E$. The general fiber of $\pi|_E$ is isomorphic to $\mathbb{P}^{c-1}$. Let $\ell$ be a line in one such fiber. Denote by $\overline{D}$ the strict transform of $D$. Put

$$\text{mult}_Z D := \overline{D} \cdot \ell.$$

The intersection is well defined, and agrees with the definition from [Ful84, Section 4.3]: $\pi_* (\overline{D} \cdot (-E)^{c-1}) = (\text{mult}_Z D) \cdot [Z]$. (The intersection can be computed by restricting $\overline{D}$ to the Cartier $E$, then further restricting to the fiber over general $z \in Z$, which is regularly embedded in $E$. Finally, the fiber is smooth, and $\ell$ is a complete curve, hence Cartier divisors on it have well defined degree. The result clearly does not depend on the choice of the general $z \in Z$. By the projection formula, the coefficient of $[Z]$ in the pushforward is $\overline{D} \cdot (-E)^{c-1} \cdot \pi^*[z] = \overline{D} \cdot \ell$, where $z$ is a general point on $Z$ and $\ell$ a line in $\pi^{-1}z \simeq \mathbb{P}^{c-1}$.)

If $D$ is Cartier, then $\pi^* D = \overline{D} + (\text{mult}_Z D) \cdot E$. (Intersect with $\ell$.)

Since $x \mapsto \text{mult}_x D$ is upper-semicontinuous for the Zariski topology ($x$ is not necessarily closed), we also have $\text{mult}_Z D = \text{mult}_z D$ for general $z \in Z$. This is [Laz04, Definition 5.2.10].

5. **Null locus**

*Proof of Theorem B. i)*. Note that $\varepsilon(C; x_0) > 0$ implies $C \neq 0$. If $[L] \in \mathbb{E}^1(X)$ verifies $C \cdot L = 0$, then $C \cdot P_r(L) = 0$. If $P_r(L) \neq 0$, then the multiplicity arguments in the proof of Theorem A applied to the point $x_0$ contradict $\varepsilon(C; x_0) > 0$. We conclude that $L \equiv N_0(L)$. In particular $L$ is numerically equivalent to an effective divisor. For any effective $L$ with $[L]$ in $C^1 \cap \mathbb{E}^1(X)$, the irreducible components of Supp $L$ are also in the cone, and [Nak04, III.1.10, Proposition] proves that the numerical classes of these components are linearly independent in $N^1(X)$. Furthermore, no component of Supp $L$ may pass through $x_0$ because $\varepsilon(C; x_0) > 0$. Choose finitely many effective divisors $G_j$ whose classes span the subspace $V$ generated by $C^1 \cap \mathbb{E}^1(X)$ in $N^1(X)$. Let $L_i$ be the finite set of irreducible divisors that appear as components of any of the $G_j$. Clearly $[L_i]$ also generate $V$. By looking at $\sum_j G_j$, we see that the irreducible divisors above have classes that are linearly independent in $N^1(X)$. In particular these classes form a basis of $V$. If $E$ is any effective divisor with $[E] \in C^1 \cap \mathbb{E}^1(X)$, let $E_0$ denote any of the irreducible components of its support. By looking at $\sum_i L_i + E_0$, we deduce that $E_0$ is one of the $L_i$, or else $L_1, \ldots, L_r, E_0$ also have linearly independent classes.
ii). If $C$ is not big or in the strict interior of $\Mov^1(X)^\vee$, then there exists some nef/movable divisor $L$ with $C \cdot L = 0$ and $[L] \neq 0$. Since nef divisors are movable, in both cases we have $L = P_\sigma(L)$. Using $i)$, we find the contradiction $[L] = 0$.

iii). If $x \in L_i$ for some $i$, then $\varepsilon(C; x) = 0$ since $C \cdot L_i = 0$ and $\mult_x L_i > 0$. Conversely, assume $\varepsilon(C; x) = 0$. Then there exists a sequence $D_j$ of effective divisors with irreducible supports such that $\lim_{j \to \infty} C \cdot D_j = 0$ and $x \in \Supp D_j$ for all $j$. Up to rescaling, we may assume that $H^{n-1} \cdot D_j = 1$ for some very ample divisor $H$ on $X$, where $n := \dim X$. By Bézout, $\mult_x D_j \leq H^{n-1} \cdot D_j = 1$. By [FL17a Theorem 1.4.(3)], the sequence $[D_j] \in N^1(X)$ is bounded, so up to passing to a subsequence, we may assume that $\lim_{j \to \infty} [D_j] = [D]$ for some $D \in \Eff^1(X)$. Furthermore $H^{n-1} \cdot D = 1$, so $[D] \neq 0$. From the bound on the multiplicity of $D_j$, we also deduce $C \cdot D = 0$.

By part i), we see that $D \neq P_\sigma(D)$. Thus there exists some irreducible divisor $E$ (in fact one of the $L_i$) on $X$ with associated valuation $\sigma_E$ such that $\sigma_E(D) > 0$. By the lower semi-continuity of $\sigma_E$ (cf. [Nak04 III 1.7.(1) Lemma]), we have $\sigma_E(D_j) > 0$ for large $j$ (after maybe passing to a subsequence). Since $\Supp D_j$ is irreducible, it follows that $\Supp D_j = E$. Using $D_j \cdot H^{n-1} = 1$ and the irreducibility of $E$, we find that $D_j$ is an eventually constant sequence, so $[D] = [D_j]$ for large $j$. By part i) we deduce that $D_j$ is a nonnegative linear combination of $L_i$ and in particular that $x \in \Supp D_j$ is contained in $\Null(C)$.

Remark 5.1. One can also use the work of [LX16] as explained in Remark 7.13 to prove that if $\varepsilon(C; x) > 0$, and $C \cdot L = 0$ for $[L] \in \Eff^1(X)$, then $L \equiv N_\sigma(L)$ and $[C]$ is in the strict interior of $\Mov^1(X)^\vee$, in particular it is big.

We see this again for a special class of curves.

Definition 5.2 ([Voi10]). Let $X$ be a smooth projective variety, and let $V \subseteq X$ be a subvariety of dimension $k$. Say that $V$ is very moving if for a very general $x \in X$ we have that for a very general $k$-dimensional subspace $W \subseteq T_x X$ there exists a deformation $V'$ of $V$ passing through $x$ with $V'$ smooth and $T_x V' = W$.

Remark 5.3. Note that if $C$ is a very moving curve, then $\varepsilon(C; x) \geq 1$ for very general $x \in X$. The theorem implies then that $[C]$ is in the strict interior of $\Mov^1(X)^\vee$. [Voi10 Proposition 2.7] proves that very moving curves are in the strict interior of $\Eff^1(X)$. Voisin also conjectures that very moving subvarieties of arbitrary dimension $k$ have classes in the strict interior of $\Eff^k(X)$ and shows that this implies the generalized Grothendieck–Hodge conjecture for complete intersections of coniveau 2 in projective spaces.

6. Jet separation

Remark 6.1. Let $x \in X$ be a smooth point. If $C$ is a curve such that for all irreducible divisors $D$ through $x$ we have that some deformation of $C$ meets $D$ properly and passes through $x$, then $\varepsilon(C; x) \geq 1$. (For any effective Cartier divisor $L$ through $x$, we can find a deformation $C'$ of $C$ that meets $L$ properly and also passes through $x$. Then $C \cdot L \geq \mult_x C' \cdot \mult_x L \geq \mult_x L$ by [Ful84 Theorem 12.4].)

In particular, when $X$ is smooth and the above condition holds for all $x \in X$, such curves $C$ have classes in the strict interior of $\Mov^1(X)$ by Theorem A.

We see the above as a counterpart to the statement that $\varepsilon(L; x) \geq 1$ if $L$ is ample and globally generated ([Laz04 Example 5.1.18]).

Example 6.2 (Grassmannian varieties). Let $\ell$ be a line in the Grassmannian variety $X = G(k, n)$ of $k$-dimensional subspaces of $\mathbb{C}^n$. Then

$$\varepsilon(\ell; x) = \frac{1}{\min\{k, n-k\}}$$
for all \( x \in X \). (Using \( G(k, n) \simeq G(n - k, n) \), we may assume that \( 2k \leq n \). Since \( X \) is homogeneous, \( \varepsilon (\ell; x) \) is independent of \( x \). Denote it \( \varepsilon (\ell) \).) Let \( L \) be the very ample divisor induced by the Plücker embedding. By \cite[Proposition 14.6.3]{Ful84} we have \( \ell \cdot L = 1 \). Let \( d := (L^{k(n-k)}) \) be the degree of the Grassmann variety.\footnote{Though we don’t use the formula here, it can be computed (for example by \cite[Proposition 1.10]{Muk93}) as \( d = (k(n-k))! \prod_{1 \leq i < j \leq n} (j-i)^{-1} \).} We have \( \ell = \frac{1}{d} L^{k(n-k)-1} \). From Example \cite[7.7]{Laz04} we deduce \( \varepsilon (\ell) = \frac{L^{k(n-k)-1}}{d \mu(L)} = \frac{1}{n (L)} \). The effective cone of the blow-up of the Grassmann variety at one point is computed in \cite[Corollary 3.2]{Kop16} and \cite[Lemma 7.2.2]{Ris17}. It gives (also by \cite[Example 2]{RZ01}) that \( \mu(L) = k \). The conclusion follows.

For \( X = G(2, 4) \) we find \( \varepsilon (\ell) = \frac{1}{2} \). Geometrically we can see this as follows: Let \( x \in X \) and let \( D_x := X \cap T_x X \), the intersection taking place in \( \mathbb{P}^5 \). Then \( \text{mult}_x D_x = 2 \) and \( \frac{\ell \cdot D_x}{\text{mult}_x D_x} = \frac{1}{2} \). This gives the bound \( \varepsilon (\ell) \leq \frac{1}{2} \). Since \( \varepsilon (\ell) < 1 \), the line \( \ell \) cannot satisfy the conditions of Remark \ref{rem:jet}. This can also be checked directly. Any line on \( X \) through \( x \) is contained in \( T_x X \), hence also in \( D_x \) and cannot be moved to meet \( D_x \) properly through \( x \). However, \( 2\ell \) is the class of the complete intersection of 3 general members of \( |L| \), and so it does satisfy the conditions of Remark \ref{rem:jet}. We deduce the reverse inequality \( \varepsilon (2\ell) \geq 1 \).

**Definition 6.3** ("Jet separation" for curves). Let \( X \) be a projective variety, and let \( x \in X \). Let \( [C] \in \overline{\text{Mov}}_1(X) \) be a \( \mathbb{Z} \)-class. Denote by \( s(\mathcal{C}; x) \) the largest nonnegative integer \( s \) for which there exists an integer \( N \geq 1 \) such that for every effective divisor \( L \) through \( x \) there exists an effective \( \mathbb{Z} \)-cycle \( C' \equiv N \cdot C \) with \( \text{mult}_x C' \geq NS \), and \( C' \) meets \( L \) properly. When no such \( s \) exists, set \( s(\mathcal{C}; x) = -1 \).

**Remark 6.4.** By asking that \( C' \) is merely a \( \mathbb{Q} \)-cycle, one can leave out the integer \( N \). It is a restriction on the denominators of \( C' \), slightly weaker than the condition \( L \in |L| \) in the case of divisors.

**Remark 6.5.** If \( C \) is movable and \( L \) is a Cartier divisor through a smooth point \( x \), then \( C \cdot L \geq \frac{1}{N} \text{mult}_x C' \cdot \text{mult}_x L \) for any effective \( \mathbb{R} \)-cycle \( C' \) with \( C' \equiv N \cdot C \). It follows that \( \varepsilon (C; x) \geq s(\mathcal{C}; x) \).

By passing to multiples, \( \varepsilon (C; x) \geq \sup_k \frac{s(kC; x)}{k} \).

**Remark 6.6.** If \( C_1 \) and \( C_2 \) are movable \( \mathbb{Z} \)-classes with \( s(C_1; x) \geq 0 \) and \( s(C_2; x) \geq 0 \), then \( s(C_1 + C_2; x) \geq s(C_1; x) + s(C_2; x) \). (If integers \( N_1, N_2 \) are as in the definition for \( s(C_1; x) \) and \( s(C_2; x) \), respectively, then \( \text{lcm}(N_1, N_2) \) satisfies the condition in the definition for \( s = s(C_1; x) + s(C_2; x) \).) In particular \( \sup_k \frac{s(kC; x)}{k} = \limsup_{k \to \infty} \frac{s(kC; x)}{k} \).

**Theorem 6.7.** Let \( X \) be a projective variety over an algebraically closed field. Let \( x \in X \) be a smooth point, and let \( [C] \) be an integral class in the strict interior of \( \overline{\text{Mov}}_1(X) \). Then

\[
\varepsilon (C; x) = \lim_{k \to \infty} \frac{s(kC; x)}{k}.
\]

**Proof.** It is enough to prove that \( \liminf_{k \to \infty} \frac{s(kC; x)}{k} \geq \varepsilon (C; x) \). Since \( [C] \) is in the strict interior of \( \overline{\text{Mov}}_1(X) \), we have that \( \varepsilon (C; x) > 0 \). Let \( 0 < \frac{p}{q} < \varepsilon (C; x) \) be a rational approximation of \( \varepsilon (C; x) \). Lemma \ref{lem:rank} below gives that \( [q \pi^* C - p\ell] \) is in the strict interior of \( \overline{\text{Mov}}_1(Bl_x X) \). The version of the main result of \cite{BDP13} in \cite[Theorem 11.4.19]{Laz04} implies that there exist finitely many birational morphisms \( Y_i \to X \) and \( Z_j \to Bl_x X \) such that \( [C] \) and respectively \( [q \pi^* C - p\ell] \) are \( \mathbb{Q} \)-linear combinations of complete intersection curves (of very ample divisors) from \( Y_i \) and \( Z_j \), respectively. We use here that \( [C] \) and \( [q \pi^* C - p\ell] \) are in the strict interior of \( \overline{\text{Mov}}_1(X) \) and \( \overline{\text{Mov}}_1(Bl_x X) \) respectively.
Choose \( N \) large enough to clear all denominators in the \( \mathbb{Q}_+ \)-linear combinations above. For any effective divisor \( L \) through \( x \) choose effective divisors \( L_i \) on each of \( Y_i \) and \( L_j \) on each \( Z_j \) that surject onto it. Find general complete intersections on \( Y_i \) that meet \( L_i \) properly and similarly for \( Z_j \). Pushing from \( Y_i \) to \( X \) and adding up we find an effective curve of class \( N \cdot [C] \) that meets \( L \) properly. Pushing from \( Z_j \) to \( \text{Bl}_x X \) and adding up, we find an effective curve of class \( N \cdot [q\pi^*C - p\ell] \). Its pushforward to \( X \) is an effective curve of class \( Nq \cdot [C] \) with multiplicity at least \( Np \) at \( x \). It also meets \( L \) properly.

For any positive integer \( k \), write \( k = mq + q_1 \) with \( 0 \leq q_1 < q \). From the discussion above we construct an effective \( \mathbb{Z} \)-1-cycle of class \( Nk \cdot [C] \) with multiplicity at least \( Nmp \) at \( x \) that meets \( L \) properly. Then \( \frac{s(t(x,C))}{k} \geq \frac{mp}{mq+q_1} \). As \( k \) grows, the latter approximates \( \frac{p}{q} \). We conclude by letting \( \frac{q}{p} \) tend to \( \varepsilon (C;x) \).

**Lemma 6.8.** Let \( x \in X \) be a smooth point and let \([C]\) be a class in the strict interior of \( \overline{\text{Mov}}(X) \). With the usual notations for the blow-up of \( x \), we have that \( \pi^*C - t\ell \) is in the strict interior of \( \overline{\text{Mov}}(\text{Bl}_x X) \) for all \( 0 < t < \varepsilon (C;x) \).

**Proof.** Note that \( \varepsilon (C;x) > 0 \) since \([C]\) is in the strict interior of \( \overline{\text{Mov}}(X) \). If for some \( t \in (0, \varepsilon (C;x)) \) the class \([\pi^*C - t\ell]\) is not in the strict interior of \( \overline{\text{Mov}}(\text{Bl}_x X) \), there exists a pseudo-effective \( \mathbb{R} \)-Cartier \( \mathbb{R} \)-divisor \( D \) with \( [D] \neq 0 \) on \( \text{Bl}_x X \) and with \( (\pi^*C - t\ell) \cdot D = 0 \). Since \( (\pi^*C - \varepsilon (C;x)\ell) \cdot D > 0 \) and \( \pi^*C \) is movable so \( \pi^*C \cdot D > 0 \), using \( t < \varepsilon (C;x) \) we find \( \pi^*C \cdot D = \ell \cdot D = 0 \). Set \( D := \pi_*D \). It is still an \( \mathbb{R} \)-Cartier \( \mathbb{R} \)-divisor, since \( x \) is smooth. It is pseudo-effective, and \( C \cdot D = 0 \) by the projection formula. Since \([C]\) is in the strict interior of \( \overline{\text{Mov}}(X) \), necessarily \([D] = 0 \). For example by \([\text{FlL16}] \) Theorem 4.13, it follows that \( [D] \) is effective. The only effective divisors on \( \text{Bl}_x X \) with numerically trivial pushforward via \( \pi \) are the multiples of the exceptional \( E \). From \( E \cdot l = -1 \) we deduce \( [D] = 0 \), which contradicts our choice. \( \square \)

An easier asymptotics-free result holds for \( \mathbb{R} \)-classes.

**Corollary 6.9 (Theorem C).** Let \( x \in X \) be a smooth point and let \([C]\) be an \( \mathbb{R} \)-class in the strict interior of \( \overline{\text{Mov}}(X) \). Then \( \varepsilon (C;x) \) is the supremum of all \( s \geq 0 \) such that for all effective \( \mathbb{R} \)-Weil \( \mathbb{R} \)-divisors \( L \) with \( x \in \text{Supp} \ L \) there exists an effective \( \mathbb{R} \)-1-cycle \( C' \equiv C \) such that \( \text{Supp} \ C' \) and \( \text{Supp} \ L \) meet properly and \( \text{mult}_x C' \geq s \). The same statement holds with \( \mathbb{Q} \) replacing \( \mathbb{R} \) throughout.

**Proof.** The Bézout arguments of Remark 6.1 prove that \( \varepsilon (C;x) \geq s \) for all such \( s \). For the reverse inequality, let \( 0 < t < \varepsilon (C;x) \). By Lemma 6.8 the class \([\pi^*C - t\ell]\) is in the strict interior of \( \overline{\text{Mov}}(\text{Bl}_x X) \), by \([\text{BDPP13}] \) (the version in \([\text{Laz04}] \) Theorem 11.4.19) it is represented by some \( \mathbb{R}_+ \)-combination of complete intersection curves. Its pushforward represents \([C]\), has multiplicity \( t \) at \( x \), and by genericity can be chosen to meet any given divisor on \( X \) properly. This proves that \( \varepsilon (C;x) \) is bounded above by the supremum of all \( s \) as in the statement. The same arguments hold for rational coefficients. \( \square \)

**Corollary 6.10.** Let \( \pi : X \to Y \) be a dominant morphism of projective varieties. Let \( C \in \overline{\text{Mov}}(X) \). Then \( \varepsilon (C;x) \leq \varepsilon (\pi_*C;\pi(x)) \) for all \( x \in X \) such that \( x \) and \( \pi(x) \) are both smooth.

**Proof.** Immediate from the previous corollary and from Lemma 4.11 One can also give a direct proof using the projection formula by observing \( \text{mult}_{\pi(x)} D \leq \text{mult}_x \pi^*D \) for any Cartier divisor \( D \) through \( \pi(x) \).

\( \square \)

7. Bounds on Seshadri constants

7.1. Lower bounds.

**Lemma 7.1.** Let \( X \) be a projective variety of dimension \( n \geq 2 \) over \( \mathbb{C} \). Let \( H \) be an ample \( \mathbb{R} \)-divisor. Then \( \varepsilon (H^{n-1};x) \geq \varepsilon (H;x)^{n-1} \) for all \( x \in X \).
Proof. Replace $\frac{1}{m}$ in the proof of Lemma 7.5a by $\varepsilon(H;x)$. 

**Corollary 7.2.** Let $H$ be an ample $\mathbb{Z}$-divisor on a smooth projective variety $X$ of dimension $n$ over $\mathbb{C}$. Then $\varepsilon(H^{n-1}; x) \geq \frac{1}{n}$ for very general $x \in X$.

**Proof.** Use the bound $\varepsilon(H;x) \geq \frac{1}{n}$ of \cite[Theorem 1]{EL95}.

We can generalize the original \cite{EL93} in the following way:

**Proposition 7.3.** Let $X$ be a projective variety of dimension $n$ over $\mathbb{C}$. Let $H$ be an ample Cartier $\mathbb{Z}$-divisor on $X$ such that $(H^n) \geq n^{n-2}$, or $\varepsilon(H; x_0) \geq 1$ for some $x_0 \in X$, or more generally

$$
\varepsilon(H; x_0)^{n-2} \cdot (H^n) \geq 1.
$$

Then $\varepsilon(H^{n-1}; x) \geq 1$ for very general $x \in X$. In fact we can choose $x$ outside the union of the singular locus of $X$ with countably many closed subsets of codimension two or more.

The artificial condition (7.3.1) is automatically satisfied when $X$ is a surface.

**Proof.** The proof mimics the surface case from \cite{EL93}. If the result fails, then by usual Hilbert scheme arguments one can find a variety $T$ and a family of irreducible divisors on $X$ denoted $L \subseteq T \times X$, flat over $T$, with a section $T \to L : t \mapsto (t, x_t)$ such that $H^{n-1} \cdot L_t < \operatorname{mult}_{x_t} L_t$ for all $t \in T$ and $\bigcup_{t \in T} x_t \subseteq X$ is a constructible set whose closure has dimension at least $n - 1$ and meets the smooth locus of $X$. Since $H^{n-1} \cdot L_t < \operatorname{mult}_{x_t} L_t$, we deduce $\operatorname{mult}_{x_t} L_t \geq 2$, so that $x_t$ is a singular point of $L_t$. Each $L_t$ is reduced, hence generically smooth. We conclude that $L_t$ is not a constant divisor, so in particular the family $L$ covers $X$. If we let $m := \operatorname{mult}_{x_t} L_t$ for general $t \in T$, then we can guarantee that an infinitesimal deformation $L'_t$ of $L_t$ has multiplicity at least $m - 1$ at $x_t$ and meets $L_t$ properly.

For any $k > 0$ denote by $s_k$ the largest integer $s$ such that the linear series $|kH|$ separates $s$-jets at $x_t$. By choosing $n - 2$ elements $D_1, \ldots, D_{n-2}$ in $|kH|$ general among those with multiplicity at least $s_k$ at $x_t$, we then have

$$
\begin{align*}
&
\varepsilon(H; x_0)^{n-2} \cdot (H^n) \\
&
= (L^2 \cdot H^{n-2}) \cdot (H^n) \\
&
\leq (L_t \cdot H^{n-1})^2 \\
&
\leq (m - 1)^2.
\end{align*}
$$

The first inequality is because $L^2 \cdot (kH)^{n-2} = L_t \cdot L'_t \cdot D_1 \cdot \ldots \cdot D_{n-2}$ and we apply \cite[Example 12.4.9]{Ful84} to the infinitesimal situation of $n$ divisors meeting properly with multiplicities at least $m, m-1, s_k, \ldots, s_k$ respectively at $x_t$. The second is a Hodge inequality on a surface obtained as complete intersection of members of $|kH|$. The last inequality is because $L_t \cdot H^{n-1} < m$ and $L_t \cdot H^{n-1}$ is an integer.

By \cite[Theorem 5.1.17]{Laz04}, we have $\lim_{k \to \infty} \frac{s_k}{k} = \varepsilon(H;x)$. After taking limits, the assumption (7.3.1) leads to the contradiction $2 \leq m(m-1) \leq (m-1)^2$. $
$

**Remark 7.4.** Example 6.2 shows that if $C$ is a $\mathbb{Z}$ 1-cycle with class in the strict interior of $\operatorname{Mov}_1(X)$, then the inequality $\varepsilon(C;x) \geq 1$ may fail for all $x \in X$. Also we do not have an universal lower bound on $\varepsilon(C;x)$ for very general points $x \in X$ independent of dim $X$, as is expected to hold for ample Cartier divisors.

7.2. Upper bounds.

**Definition 7.5.** Let $X$ be a projective variety and let $x \in X$. For any pseudo-effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor $L$ on $X$, define the Fujita–Nakayama-type invariant

$$
\mu(L;x) := \sup \{ t \geq 0 \mid \pi^* L - tE \text{ pseudo-effective} \},
$$

where $\pi : \operatorname{Bl}_x X \to X$ is the blow-up with exceptional divisor $E$. 


Remark 7.6. Let $x \in X$ be a smooth point. For any movable curve $C$ and pseudo-effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor $L$ we have

$$\varepsilon(C; x) \cdot \mu(L; x) \leq C \cdot L.$$  

(This is because $(\pi^*C - \varepsilon(C; x)\ell) \cdot (\pi^*L - \mu(L; x)E) \geq 0$.) Note that $\mu(L; x) \geq \varepsilon(L; x)$ when $L$ is nef.

Equality may be achieved in $\varepsilon(C; x) \cdot \mu(L; x) \leq C \cdot L$.

Example 7.7 (Picard rank 1). Let $X$ be a smooth projective variety of dimension $n$ with $\text{rk} \, N^1(X) = 1$. Let $H$ be an ample generator of $N^1(X)$. Let $x \in X$ and let $\tilde{X} := \text{Bl}_x X$ with blow-up morphism $\pi : \tilde{X} \to X$. As usual denote the exceptional divisor by $E$ and the class of a line in $E = \mathbb{P}^{n-1}$ by $\ell$.

Then

$$\mathbb{E}ff^1(\tilde{X}) = \langle \pi^*H - \mu(H; x)E, E \rangle \quad \text{and} \quad \text{Nef}^1(\tilde{X}) = \langle \pi^*H - \varepsilon(H; x)E, \pi^*H \rangle.$$  

For curves we have

$$\mathbb{E}ff^1(X) = \langle \pi^*H^{n-1} - \mu(H^{n-1}; x)\ell, \ell \rangle \quad \text{and} \quad \text{Mov}^1(X) = \langle \pi^*H^{n-1} - \varepsilon(H^{n-1}; x)\ell, \pi^*H^{n-1} \rangle.$$  

We define $\mu(H^{n-1}; x)$ as for divisors. The known dualities between these cones give

$$\mu(H^{n-1}; x) \cdot \varepsilon(H; x) = (H^n) \quad \text{and} \quad \varepsilon(H^{n-1}; x) \cdot \mu(H; x) = (H^n).$$  

The inequality $\mu(L; x) \geq \varepsilon(L; x)$ may be strict.

Example 7.8. When $X$ is an irreducible principally polarized abelian surface with Picard number 1 and $H$ is a theta divisor,

$$\varepsilon(H; x) = \frac{4}{3} < \sqrt{2} = \sqrt{(H^2)} < \frac{3}{2} = \mu(H; x)$$  

for all $x \in X$. See [Ste98, Proposition 2].

Example 7.9. Let $X$ be a smooth projective variety. Let $[C] \in \text{Mov}_1(X)$. For every $x \in X$ there exists $L_x$ a nonzero pseudo-effective divisor on $X$ such that $\varepsilon(C; x) \cdot \mu(L_x; x) = C \cdot L_x$. (If $\varepsilon(C; x) = 0$, use one of the divisors $L_i$ in Theorem B. Assume henceforth that $\varepsilon(C; x) > 0$.) Since $\pi^*[C] - \varepsilon(C; x)\ell$ is not in the interior of the movable cone, there exists a pseudo-effective divisor $L_x$ on $\text{Bl}_x X$ such that $(\pi^*[C] - \varepsilon(C; x)\ell) \cdot L_x = 0$. Manifestly $[L_x]$ and $[E]$ are not proportional. In particular $L_x := \pi_* L_x$ is nonzero. Using [BDPP13] we find that $L_x$ is not big. We deduce that

$$\mu(L; x) \geq \varepsilon(L; x)$$  

Example 7.10. When the Picard rank is bigger than 1, it is not always the case that if $H$ is ample and $C = H^{n-1}$, then $\varepsilon(C; x) \cdot \mu(H; x) = (H^n)$ for all $x \in X$. What could motivate the question is that $H$ minimizes the expression $\frac{C \cdot L}{\text{vol}^{1/n}(L)}$ from the definition of $\text{M}(C)$ below.

Take $X = \text{Bl}_p \mathbb{P}^2$ and consider the ample divisor $3H - E$, where $H$ is the pullback of the line class from $\mathbb{P}^2$, and $E$ is the exceptional divisor. We have $(3H - E)^2 = 8$. If $x$ is one of the torus invariant points on $E$, then $\varepsilon(3H - E; x) = 1$ by [BDPP13] and $\mu(3H - E) = 5$. The latter is because the effective cone of $\text{Bl}_p X$ is $\langle H - E - F, E - F, F \rangle$, where $F$ is the new exceptional line, and by $H$ and $E$ we denote the pullbacks of the respective classes from $X$.

Lemma 7.11. Let $L$ be a big (for $\mathbb{R}$-Weil divisors, this is taken in the sense of [PKL16]) $\mathbb{R}$-divisor on the projective variety $X$ of dimension $n$, and let $x \in X$ be a smooth point. Then

$$\mu(L; x) \geq \text{vol}^{1/n}(L).$$

Proof. There are \((n+e-1) \leq \frac{(e+n-1)^n}{n!}\) conditions for a member of a linear series on \(X\) to vanish at \(x\) with multiplicity at least \(e\). Since \(\text{vol}(L) = \lim_{m \to \infty} \frac{\dim H^0(X, mL)}{m^n/n!}\), a limiting argument yields the result. \(\square\)

**Corollary 7.12.** Let \(X\) be a projective variety of dimension \(n\), and let \(x \in X\) be a smooth point. Then \(\varepsilon(C; x) \leq \frac{C \cdot L}{\text{vol}^{1/n}(L)}\) for all big \(\mathbb{R}\)-Cartier \(\mathbb{R}\)-divisors \(L\).

**Remark 7.13.** Let \(X\) be a smooth projective variety of dimension \(n\), and let \(\alpha \in \overline{\text{Mov}}_1(X)\). \([\text{Xia15}], [\text{LX16}]\) consider

\[\mathcal{M}(\alpha) := \inf \left\{ \left( \frac{\alpha \cdot L}{\text{vol}^{1/n}(L)} \right)^{\frac{n-1}{n}} \mid L \text{ is a big } \mathbb{R}\text{-Cartier } \mathbb{R}\text{-divisor} \right\}.\]

They prove that it can naturally be extended to a continuous function on \(N_1(X)\) that is positive precisely in the interior of the dual cone \(\overline{\text{Mov}}_1(X)^\circ\), which contains \(\overline{\text{Mov}}_1(X)\). Note that

\[\varepsilon(\alpha; x) \leq \mathcal{M}^{\frac{n-1}{n}}(\alpha)\]

for all \(x \in X\).

**Corollary 7.14.** Let \(X\) be a projective variety of dimension \(n\), and let \(x \in X\) be a smooth point. Let \(H\) be a big and nef \(\mathbb{R}\)-divisor class on \(X\). Then \(\varepsilon([H^{n-1}]; x) \leq (H^n)^{\frac{n-1}{n}}\).

**Example 7.15** (Curve in its Jacobian). Let \(C \subseteq J(C)\) be a smooth projective curve of genus \(g \geq 1\) embedded in its Jacobian with theta divisor \(\theta\). Since \(J(C)\) acts transitively on itself by translations, \(\varepsilon(C; x)\) is independent of \(x\). We denote it \(\varepsilon(C)\). By the previous corollary we find

\[\varepsilon(C) \leq \frac{\sqrt{g}}{\sqrt{g!}} < e,\]

where \(e\) is Euler’s constant. (Recall that \([C] = \frac{\theta^{g-1}}{(g-1)!}\) and \((\theta^g) = g!\).) By studying the singularities of \(\theta\), we find \(\varepsilon(C) \leq \frac{2}{\nu(\theta)} \leq \frac{2}{\sqrt{g}}\), where \(\nu(\theta)\) is the maximal multiplicity of \(\theta\) at a point. While the result is weaker than the asymptotic result, it is stronger than what the method above could do for the 1-dimension linear series \(|\theta|\). (By the Riemann singularity theorem this is the maximal dimension of \(H^0(C; L)\) as \(L\) ranges over all effective divisors of degree \(g-1\) in \(C\). From the existence theorem in Brill–Noether theory \([\text{ACGHH85}], \text{VII.}(2.3)\) Theorem] or \([\text{Laz04}], \text{Theorem 7.2.12}\), we know that \(\nu(\theta) \geq \lceil \sqrt{g} \rceil\), while the Clifford index theorem implies \(\nu(\theta) \leq \frac{2 + 1}{2}\). We have \(\varepsilon(C) \leq \frac{C \cdot \theta}{\nu(\theta)} = \frac{g}{\nu(\theta)}\).

When \(C\) is hyperelliptic of odd genus, then by the Clifford index theorem the bound \(\nu(\theta) = \frac{g + 1}{2}\) is achievable by points corresponding to multiples of the unique \(g_1\). We get the better bound

\[\varepsilon(C) \leq \frac{2g}{g + 1} < 2 < e.\]

When \(C\) is not hyperelliptic of genus \(g \geq 3\),

\[\varepsilon(C) \leq \frac{g}{2}.\]

This result is only sharper than the first for \(g = 3\). (A result of J. Fay \((\text{PP01}, (1.2))\) states that the elements of \(2|\theta|\) with multiplicity at least 4 at the neutral element \(o \in J(C)\) are precisely those that contain the difference variety \(C - C\). The dimension of this subspace is \(2^g - 1 - \binom{g}{2}\) (see \([\text{PP01}], (1.3)\)), so it is nonempty. Then \(\varepsilon(C) \leq \frac{C \cdot 2g}{4} = \frac{2g}{4} = \frac{g}{2}\). This result can be improved for some low genera by \([\text{PP01}, \text{Chapter 6}]\).

By \([\text{Kon03}]\), at least when \(J(C)\) has Picard rank 1, the Seshadri constant \(\varepsilon(\theta)\) differentiates between hyperelliptic and non-hyperelliptic curves. This is no longer the case for \(\varepsilon(C)\).
**Example 7.16** (Curves of genus 3 whose Jacobian has Picard rank 1). In this case we prove that

\[ \varepsilon(C) = \frac{3}{2}. \]

(Assume first that \( C \) is not hyperelliptic. The difference divisor \( C - C \), which has multiplicity \( 2g - 2 = 4 \) at the origin and class \( 2\theta \), gives the upper bound \( \varepsilon(C) \leq \frac{3}{2} \). For the lower bound, assume there exists an irreducible divisor \( D \) of class \( ab \) (necessarily proportional to \( \theta \) since \( J(C) \) has Picard rank 1 by assumption) with multiplicity \( b \) at the origin \( o \in J(C) \), such that \( \frac{C \cdot D}{\text{mult}_o D} < \frac{3}{2} \), equivalently \( b > 2a \). On the blow-up \( \pi : \text{Bl}_o J(C) \to J(C) \) with exceptional divisor \( E \) we consider the product

\[ (\pi^*\theta - \frac{12}{7}E) \cdot (\pi^*a\theta - bE) \cdot (\pi^*2\theta - 4E) = 12a - \frac{48}{7}b < 12a - \frac{96}{7}a = -\frac{12}{7}a < 0. \]

The first class is nef since \( \varepsilon(\theta) = \frac{12}{7} \) by [Kon03 Theorem 1.1.(2)]. The next two classes are represented by the strict transforms of the distinct irreducible divisors \( D \) and \( C - C \), hence intersect properly. Then the product of the 3 classes is nonnegative, which is a contradiction.

When \( C \) is hyperelliptic, then \( C - C \) is a theta divisor (now the difference map \( C \times C \to J(C) \) is generically 2-to-1 over its image). By the Clifford index theorem and the Riemann singularity theorem, the singularities of any theta divisor have multiplicity 2. Then \( C - C \) gives the bound \( \varepsilon(C) \leq \frac{3}{2} \). Consider the product on the blow-up of \( J(C) \) at a singular point of \( C - C \)

\[ (\pi^*\theta - \frac{3}{2}E) \cdot (\pi^*a\theta - bE) \cdot (\pi^*\theta - 2E) = 6a - 3b < 0, \]

where \( a, b \) are as in the non-hyperelliptic case. The class \( \pi^*\theta - \frac{3}{2}E \) is nef because \( \varepsilon(\theta) = \frac{3}{2} \) by [Kon03 Theorem 1.1.(1)]. We get a contradiction as in the previous case.)

**Remark 7.17.** For very general (non-hyperelliptic) curves \( C \) we have that \( \text{rk} N^1(J(C)) = 1 \) (for example by [ACCHS5 Lemma on page 359]). This is also true for very general hyperelliptic curves of positive genus (The group \( N^1(J(C))_0 \) injects into \( \text{End}(J(C)) \) by [Mum08 Corollary 19.1] and Corollary 19.2]. Then [Zar00] and [Cla11] prove that \( \text{End}(J(C)) = \mathbb{Z} \) for very general hyperelliptic curves.)

**Corollary 7.18.** If \( C \) is a curve of genus 3 with \( \text{rk} N^1(J(C)) = 1 \), let \( \pi : X \to J(C) \) be the blow-up of the origin with exceptional divisor \( E \). Then \( \text{Eff}^1(X) = \langle E, \pi^*\theta - 2E \rangle \), while \( \text{Nef}^1(X) = \langle \pi^*\theta, \pi^*\theta - aE \rangle \), where \( a = \frac{12}{7} \) if \( C \) is not hyperelliptic, and \( a = \frac{3}{2} \) if \( C \) is hyperelliptic.

**Proof.** The boundary of the nef cone is determined by \( \varepsilon(\theta) \), which is computed in [Kon03]. The boundary of the effective cone is determined by \( \varepsilon(C) \) as in Example 7.16. Use \( [C] = \frac{\theta^2}{2} \) and \( \theta^3 = 6 \). \( \square \)

8. **Seshadri constants for nef dual classes**

When \( X \) is smooth projective of dimension \( n \), the space \( N^1(X) \) is naturally identified with \( N_1(X) \). The identification sends \( \text{Nef}^n(X) \) to \( \text{Mov}_1(X) \) by [Laz04 Theorem 11.4.19]. In general we only have a linear surjection \( N^{n-1}(X) \to N_1(X) \).

**Example 8.1.** Let \( X \subseteq \mathbb{P}^N \) be a projective cone over a smooth projective variety \( Y \subseteq \mathbb{P}^{N-1} \) of dimension \( n-1 \). By [FL17a Example 2.8], we have that \( N_k(X) \simeq N_{k-1}(Y) \) for all \( 1 \leq k \leq n-1 \). In particular \( N_1(X) \simeq \mathbb{R} \), while \( N_{n-1}(X) \simeq N^1(Y) \) can be arbitrarily large. The cone \( \text{Nef}^{n-1}(X) \) is full-dimensional ([FL17a Lemma 3.7]) and salient ([FL17a Remark 2.14]) in \( N^{n-1}(X) \). In this case it surjects onto \( \text{Mov}_1(X) \), which is the non-negative half line in \( N_1(X) \simeq \mathbb{R} \).
The definition of Seshadri constants for movable curves carries the inconvenience of asking that \( L \) is Cartier. In particular, in the singular case, we can not restrict to irreducible divisors as in Remark 5.3. This difficulty is no longer present when considering dual classes, even in arbitrary codimension.

**Definition 8.2.** Let \( X \) be a projective variety, and let \( \alpha \in \text{Nef}^k(X) \). For any \( x \in X \) define

\[
\varepsilon (\alpha; x) := \inf \left\{ \frac{\alpha \cdot Z}{\text{mult}_x Z} \mid Z \text{ effective } k\text{-cycle on } X \text{ containing } x \right\}.
\]

We may restrict to the case where \( Z \) is an irreducible subvariety of dimension \( k \).

Another advantage of dual classes over numerical classes is that they pullback. Moreover, the projection formula shows that pulling back preserves nefness.

**Example 8.3.** When \( k = 1 \) we recover the case of nef \( \mathbb{R} \)-Cartier \( \mathbb{R} \)-divisor. When \( k = n - 1 \) and \( X \) is smooth we recover the case of movable curves.

**Example 8.4.** Let \( X \) be a possibly singular projective variety of dimension \( n \) and choose \( x \in X \). Let \( \alpha \in \text{Nef}^{n-1}(X) \). Then \( \alpha \cap [X] \) is a movable curve class. We have

\[
\varepsilon (\alpha; x) \leq \varepsilon (\alpha \cap [X]; x).
\]

(The infimum on the right runs over Cartier divisors, while one the one the left runs over the larger set of Weil divisors.)

**Example 8.5** (The inequality may be strict). Let \( X \subseteq \mathbb{P}^N \) be a smooth projective surface, and let \( C \) be a projective cone over \( X \) with vertex \( o \). We have a (noncommutative) diagram

\[
\begin{array}{ccc}
Z & \xrightarrow{\pi} & X \\
\sigma \downarrow & & \downarrow \iota \\
\mathbb{P}^N & \xrightarrow{\iota} & \mathbb{P}^N
\end{array}
\]

where \( Z = \mathbb{P}_X(\mathcal{O}_X(1) \oplus \mathcal{O}_X) \) with projective bundle map \( \pi : Z \to X \). The morphism \( \sigma \) is the blow-down of the section \( X_0 \) of \( \pi \) corresponding to the projection on the second component \( \mathcal{O}_X(1) \oplus \mathcal{O}_X \to \mathcal{O}_X \). The embedding \( \iota \) is the image via \( \sigma \) of the section \( X_1 \) of \( \pi \) corresponding to the projection onto the first component \( \mathcal{O}_X(1) \oplus \mathcal{O}_X \to \mathcal{O}_X(1) \). It is also the intersection of \( C \subseteq \mathbb{P}^{N+1} \) with the \( \mathbb{P}^N \) supporting \( X \).

Let \( \alpha \in \text{Nef}^1(X) \). Since \( \iota \) is the embedding of a very ample divisor, it is not hard to see that the class \( \eta \alpha \in N^2(C) \) defined by \( \eta \alpha \cap [D] := \alpha \cap \iota^*[D] \) for every \( [D] \in N_2(C) \) is also nef. We compute

\[
\varepsilon (\eta \alpha; o) = \min \left\{ t \geq 0 \mid \alpha - th \in \text{Nef}^1(X) \right\},
\]

where \( h := c_1(\mathcal{O}_X(1)) \). (Let \( D \) be an effective divisor on \( C \) all of whose components pass through \( o \). Considering such divisors is sufficient for computing the Seshadri constant. Since \( D \) and \( \iota(X) \) meet properly, the class \( \iota^*[D] \) is represented by a well defined effective 1-cycle \( L \). Then \( \eta \alpha \cap [D] = \alpha \cap \iota^*[D] = \alpha \cap [L] \). We have \([D] = \sigma_* \pi^*[L] \). For the cycle \( D' := \sigma_* \pi^{-1}[L] \), denote by \( \overline{D'} \) the strict transform of \( D' \) on \( Z \). Clearly \( \overline{D'} = \pi^{-1}[L] \). Then \( \text{mult}_o D' = -\overline{D'} \cdot [X_0]^2 = -\pi^*[L] \cdot [X_0]^2 = h \cap [L] \).

We prove that \( \text{mult}_o D' \geq \text{mult}_o D \). Assuming this, and using that for every effective \( L \) on \( X \) the divisor \( \sigma_* \pi^* L \) is effective and all of its components pass through \( o \), we deduce

\[
\varepsilon (\eta \alpha; o) = \inf \left\{ \frac{\alpha \cap [L]}{h \cap [L]} \mid L \text{ effective on } X \right\}.
\]

The claimed Seshadri constant computation is then a consequence of the duality between \( \text{Nef}^1(X) \) and \( \text{Eff}_1(X) \). For the claim on multiplicities, let \( \overline{D} \) be the strict transform of \( D \). Since \( \sigma_* \overline{D} = [D] \),
necessarily \( |D| = \pi^*[L] + a[X_0] \) for some \( a \in \mathbb{R} \). Since \( \pi_*|D| = a[X] \) is effective on \( X \), necessarily \( a \geq 0 \). Then \( \text{mult}_o D = -[D] \cdot [X_0]^2 = (\pi^*[L] + a[X_0]) \cdot \pi^*h \cdot [X_0] = ([L] - ah) \cdot h \leq [L] \cdot h \).

Next we compute
\[
\varepsilon (\eta \alpha \cap [C]; o) = \frac{\alpha \cdot h}{h^2}.
\]

(Every nonzero effective Cartier divisor \( D \) on \( C \) is equivalent to \( a\xi(X) \) for some \( a \geq 0 \). Furthermore \( \pi^*[D] = ah \). Then \( (\eta \alpha \cap [C]) \cap [D] = \alpha \cap (\pi^*[D] \cap [X]) = \alpha \cdot h \). For an arbitrary effective divisor \( L \) on \( C \), \( \pi^*[L] \simeq O_X(a) \), the divisor \( D' = \sigma \pi^*L \) is Cartier on \( C \) passing through \( o \). The multiplicity at \( o \) is \( -\pi^*L \cdot [X_0]^2 = ah^2 \). As before, we prove \( \text{mult}_o D' \geq \text{mult}_o D \). Let \( \overline{\mathcal{D}} \) be the strict transform of \( D \) on \( Z \). The only classes that push to \( [D] \) have form \( \pi^* \varepsilon [D] + b[X_0] \) for some \( b \). Set then \( \overline{\mathcal{D}} = a \pi^*h + b[X_0] \). By pushing to \( X \), we find \( b \geq 0 \). Using that \( \overline{\mathcal{D}} \) and \( X_0 \) meet properly, we deduce that \( (a - b)h \) is effective, so \( a \geq b \). It follows that \( \text{mult}_o \underbrace{D - [D] \cdot [X_0]^2} = (a \pi^*h + b[X_0]) \cdot \pi^*h \cdot [X_0] \leq (a - b)h^2 \leq ah^2 = \text{mult}_o D' \). The formula for the Seshadri constant follows easily.)

If \( X \) is the blow-up of \( \mathbb{P}^2 \) at one point and \( \alpha \) is the pullback of a line class from \( \mathbb{P}^2 \), then \( \alpha \cdot L = 0 \) for \( L \) the exceptional \( \mathbb{P}^1 \) and we find \( \varepsilon (\eta \alpha \cap [C]; o) = 0 < \varepsilon (\eta \alpha \cap [C]; o) \) for any ample \( h \) on \( X \). \( \square \)

**Remark 8.6.** In the previous example, we see that \( \varepsilon (\eta \alpha \cap [C]; o) \) is a more subtle invariant than \( \varepsilon (\eta \alpha \cap [C]; o) \). This suggests that on singular varieties it may be more fruitful to study nef “dual” classes than movable classes.

**Example 8.7** (Irrational Seshadri constant for nef classes). Let \( X \) be an abelian surface with a round nef cone as in [Laz04] Example 2.3.8. Then for general choices of integral classes \( h \) and \( \alpha \), the number \( \min \{ t \mid \alpha - th \in \text{Nef}^1(X) \} = \varepsilon (\eta \alpha; o) \) is quadratic irrational.

**N.B.** It is not obvious what one should mean by an “integral” dual class in \( N^k(X) \) (coming from a weight \( k \) Chern polynomial with integer coefficients vs. taking integer values on \( N_k(X) \)). The class \( \eta \alpha \) takes integer values on \( N_2(C)_\mathbb{Z} \), so with either definition it is at least rational. \( \square \)

**Remark 8.8.** For fixed \( x \in X \), the function \( \alpha \mapsto \varepsilon (\alpha; x) \) is 1-homogeneous, nonnegative, and concave on \( \text{Nef}^k(X) \).

**Remark 8.9.** Semi-continuity-type statements analogous to Proposition 3.11 and Lemma 3.12 hold for nef classes as well.

**Example 8.10** (Toric varieties). Let \( X = X(\Delta) \) be a projective toric variety, possibly singular and let \( x_\sigma \) be a torus-invariant point. Let \( \alpha \in \text{Nef}^k(X) \). Then
\[
\varepsilon (\alpha; x_\sigma) = \min \left\{ \frac{\alpha \cdot V_\theta}{\text{mult}_{x_\sigma} V_\theta} \mid \theta \text{ is an } (n - k)\text{-dimensional subcone of } \sigma \right\}.
\]

Here \( V_\theta \) denotes the \( k \)-dimensional torus-invariant subvariety of \( X \) corresponding to \( \theta \). (Multiplicity is upper semi-continuous in families, so the maximal multiplicity at \( x_\sigma \) of effective cycles in any given class is achieved by torus-invariant ones.)

**Lemma 8.11.** If \( \alpha \) is in the strict interior of \( \text{Nef}^k(X) \), then there exists a constant \( \varepsilon > 0 \) such that \( \varepsilon (\alpha; x) \geq \varepsilon \) for all \( x \in X \).

**Proof.** By [FL17, Corollary 3.15], complete intersections are in the strict interior of \( \text{Nef}^k(X) \). Thus by concavity and nonnegativity it is enough to consider the case where \( \alpha = [H^k] \) for some ample divisor \( H \) on \( X \). This case is treated analogously to Lemma 3.3 \( \square \)

When \( x \) is a smooth point, we could also deduce the previous lemma from the following

**Lemma 8.12.** Let \( x \in X \) be a smooth point. Let \( \alpha \in \text{Nef}^k(X) \), and let \( H \) be an ample \( \mathbb{R} \)-divisor class. Then \( \varepsilon (\alpha \cdot H; x) \geq \varepsilon (\alpha; x) \varepsilon (H; x) \).
Proof. Jet separation for divisors implies that for every \( k + 1 \)-dimensional subvariety \( Z \) through \( x \) we can find an \( \mathbb{R} \)-divisor of class \( H \) through \( x \) meeting \( Z \) properly and with multiplicity arbitrarily close to \( \varepsilon (H; x) \) at \( x \). The intersection \( H \cdot Z \) is then represented by the limit of (the classes of the elements of) a sequence of effective \( k \)-cycles \( Z'_m \) with \( \text{mult}_x Z'_m \geq (\varepsilon (H; x) - \frac{1}{m}) \cdot \text{mult}_x Z \). The last part uses that \( x \in X \) is smooth. From \( \frac{\alpha (H) \cdot Z}{\text{mult}_x Z} \geq (\varepsilon (H; x) - \frac{1}{m}) \cdot \varepsilon (\alpha; x) (\varepsilon (H; x) - \frac{1}{m}) \) we conclude by taking \( m \to \infty \) and then the infimum over all \( Z \).

As we did for movable curves we deduce

**Corollary 8.13.** The function \( \varepsilon (\cdot; x) \) is locally uniformly continuous on the strict interior of \( \text{Nef}^k (X) \).

**Conjecture 8.14** (Seshadri criterion). Let \( X \) be a projective variety. Let \( \alpha \in \text{Nef}^k (X) \) such that there exists \( \varepsilon > 0 \) with \( \varepsilon (\alpha; x) \geq \varepsilon \) for all \( x \in X \). Then \( \alpha \) is in the strict interior of \( \text{Nef}^k (X) \).

We verify this for curves.

**Proposition 8.15.** Let \( X \) be a projective variety of dimension \( n \) over an algebraically closed field. Let \( \alpha \in \text{Nef}^{n-1} (X) \) such that there exists \( \varepsilon > 0 \) such that \( \varepsilon (\alpha; x) \geq \varepsilon \) for all \( x \in X \). Then \( \alpha \) is in the strict interior of \( \text{Nef}^{n-1} (X) \).

Note that the converse is provided by Lemma 8.11. The case when \( X \) is smooth is covered by Theorem A, since in this case \( N^{n-1} (X) = N_1 (X) \) and \( \text{Nef}^{n-1} (X) = \overline{\text{Mov}}_1 (X) \). The proof is similar to Theorem A.

**Proof.** If \( \alpha \) is not an interior class, then there exists \( 0 \neq [L] \in \overline{\text{Eff}}_{n-1} (X) \) such that \( \alpha \cdot L = 0 \). Let \( \pi : \tilde{X} \to X \) be a nonsingular alteration. By [Fujita, 1987, Corollary 3.22] there exists \([\tilde{L}] \in \overline{\text{Eff}}_{n-1} (\tilde{X})\) with \( \pi_* [\tilde{L}] = [L] \). Consider the divisorial Zariski decomposition \( \tilde{L} = P_\sigma (\tilde{L}) + N_\sigma (\tilde{L}) \). From the projection formula and the nefness of \( \alpha \), we deduce \( \pi^* \alpha \cdot P_\sigma (\tilde{L}) = 0 \) and \( \alpha \cdot \pi_* N_\sigma (\tilde{L}) = 0 \). Note that \( N_\sigma (\tilde{L}) \) is an effective \( \mathbb{R} \)-divisor. If \( \pi_* N_\sigma (\tilde{L}) \neq 0 \), for any point \( x \) in its support we find a contradiction \( 0 = \alpha \cdot \pi_* N_\sigma (\tilde{L}) \geq \varepsilon \cdot \text{mult}_x \pi_* N_\sigma (\tilde{L}) > 0 \). We deduce that \( \pi_* N_\sigma (\tilde{L}) = 0 \). Since \( L \) is not numerically trivial, neither is \( P_\sigma (\tilde{L}) \). It is movable, and hence there exists \( A \) ample on \( \tilde{X} \) such that \( \text{dim} H^0 (\tilde{X}, \mathcal{O}_{\tilde{X}} ([mP_\sigma (\tilde{L})] + A)) \) grows at least linearly with \( m \). Let \( x \in X \) be a point in the finite locus of \( \pi \), and let \( \tilde{x} \) be a closed point in \( \pi^{-1} \{ x \} \). Let \( D_m \in ([mP_\sigma (\tilde{L})] + A) \) with \( \lim_{m \to \infty} \text{mult}_{\tilde{x}} D_m = \infty \). Then \( \lim_{m \to \infty} \text{mult}_x \pi_* D_m = \infty \). In fact \( \text{mult}_x \pi_* D_m \geq \text{mult}_{\tilde{x}} D_m \) by Lemma 4.1. As in Theorem A we find that \( \alpha \cdot (mL + \pi_* A) = \pi^* \alpha \cdot (mP_\sigma (\tilde{L}) + A) = \pi^* \alpha \cdot D_m = \alpha \cdot \pi_* D_m \geq \varepsilon \cdot \text{mult}_x \pi_* D_m \) grows to infinity because \( \varepsilon (x; \alpha) \geq \varepsilon \), and on the other hand is constant because \( \alpha \cdot L = 0 \).

**Proposition 8.16.** Let \( X \) be a projective variety, and let \( x \in X \) be a possibly singular point. Let \( \alpha \in \text{Nef}^k (X) \). Let \( E \) be the exceptional divisor of the blow-up \( \pi : \text{Bl}_x X \to X \). Then

\[
\varepsilon (\alpha; x) = \max \{ t \mid \pi^* \alpha + t(-E)^k \in \text{Nef}^k (\text{Bl}_x X) \}.
\]

**Proof.** The proof is analogous to Proposition 3.10. The only part that may require additional explanation is that \( (\pi^* \alpha + t(-E)^k) \cdot Z \geq 0 \) when \( Z \) is an effective cycle mapped to \( x \) by \( \pi \) and \( t \geq 0 \). This is because \( -E |_Z \) is ample, and because \( \pi^* \alpha \cdot Z = 0 \) by the projection formula.

**Lemma 8.17.** Let \( \pi : X \to Y \) be a generically finite dominant morphism of projective varieties. Let \( \alpha \in \text{Nef}^k (Y) \). Then \( \varepsilon (\pi^* \alpha; x) \geq \varepsilon (\alpha; \pi (x)) \) for all \( x \) such that \( \pi \) is finite around \( x \).

**Proof.** Let \( Z \) be a \( k \)-dimensional subvariety through \( x \). By Lemma 4.1 we have \( \text{mult}_x Z \geq \text{mult}_x \pi_* Z \), hence \( \frac{\alpha (\pi_* Z)}{\text{mult}_x \pi_* Z} \geq \frac{\alpha (\pi_* Z)}{\text{mult}_x \pi_* Z} \geq \alpha (\pi (x)) \).

**Proposition 8.18** (Null locus). Let \( X \) be a projective variety of dimension \( n \), and let \( \alpha \in \text{Nef}^{n-1} (X) \) contained in the strict interior of \( \overline{\text{Mov}}_{n-1} (X)^\circ \). Then
i) If $[L] \in \overline{\text{Eff}_{n-1}(X)}$ satisfies $\alpha \cdot [L] = 0$, then $L$ is numerically equivalent to an effective divisor. Furthermore there are only finitely many irreducible effective divisors $L_1, \ldots, L_r$ such that $\alpha \cdot [L_i] = 0$.

ii) Set $\text{Null} (\alpha) := L_1 \cup \ldots \cup L_r$, and $\text{SV} (\alpha) := \{ x \in X \mid \varepsilon (\alpha; x) = 0 \}$. Then $\text{Null} (\alpha) \subseteq \text{SV} (\alpha) \subseteq \text{Null} (\alpha) \cup \text{Sing} X$.

Proof. i). Let $\pi : \tilde{X} \to X$ be a nonsingular alteration. We claim that $\pi^* \alpha$ is also in the strict interior of $\overline{\text{Mov}_{n-1}(\tilde{X})}^\vee$. If not, since $\pi^* \alpha$ is nef, there exists $\bar{P} \neq 0$ movable on $\tilde{X}$ with $\pi^* \alpha \cdot [\bar{P}] = 0$. Since $\pi_* \bar{P} \in \overline{\text{Mov}_{n-1}(X)}$, and $\alpha$ is in the strict interior of the dual cone, by the projection formula we find that necessarily $\pi_* [\bar{P}] = 0$. Let $H$ be an ample divisor on $X$. By the projection formula $[\bar{P}] \cdot \pi^* H^{n-1} = 0$. A repeated application of [FL17a Corollary 3.11, Lemma 3.12] proves $[\bar{P}] = 0$.

Assume now $\alpha : [L] = 0$. By [FL17a Corollary 3.22] there exists $[\bar{L}] \in \overline{\text{Eff}_{n-1}(\tilde{X})}$ with $\pi_* [\bar{L}] = [L]$. By the projection formula $\pi^* \alpha \cdot [\bar{L}] = 0$. Since $\pi^* \alpha$ is in the interior of $\overline{\text{Mov}_{n-1}(\tilde{X})}^\vee$, we deduce that $\bar{L} = N_\sigma (L)$. Pushing to $X$, it follows that $L = \pi_* N_\sigma (\bar{L})$. The latter is an effective divisor as desired. The finiteness statement follows from Theorem B.

ii). Clearly $\varepsilon (\alpha; x) = 0$ for $x \in L_i$. Assume now that a smooth point $x \in X$ satisfies $\varepsilon (\alpha; x) = 0$. Let $D_i$ be a sequence of irreducible divisors through $x$ such that $\lim_{i \to \infty} \alpha \cdot [D_i] = 0$. Let $H$ be a very ample divisor on $X$. As in Theorem B we may assume $[D_i] = 1$ (that is $H^{n-1} \cdot [D_i] = 1$), that $\lim_{i \to \infty} [D_i] = [D]$ for some (nonzero) $[D] \in \overline{\text{Eff}_{n-1}(X)}$ with $\alpha \cdot [D] = 0$. Let $\pi : \tilde{X} \to X$ be a nonsingular alteration. Let $\bar{D}_i$ be a divisor with irreducible support on $\tilde{X}$ with $\pi_* \bar{D}_i = D_i$. We claim that $[\bar{D}_i]$ form a bounded sequence in $N^1(\tilde{X})$. Assuming this, by passing to a subsequence, we may assume that $\lim_{i \to \infty} [\bar{D}_i] = [\bar{D}]$ for some $\bar{D}$ with $\pi_* [\bar{D}] = [D]$. By the semicontinuity arguments in Theorem B, we may assume that $\bar{D}_i$ and so also $D_i$ is a constant sequence. It follows that $[D]$ is represented by a divisor with irreducible support containing $x$. Since $\alpha \cdot [D] = 0$, the support is one of the $L_i$.

For the claim, since $\pi^* H$ is big, there exist divisors $A$ and $E$ on $\tilde{X}$ with $A$ ample and $E$ effective such that $\pi^* H = A + E$. If $\bar{D}_i$ meets $E$ properly, then by [FL16 Lemma 4.12], we have $A^{n-1} \cdot [\bar{D}_i] \leq \pi^* H^{n-1} \cdot [\bar{D}_i] = H^{n-1} \cdot D_i = 1$. Since multiplying by $A^{n-1}$ is a norm ([FL17a Theorem 1.4.3)], the conclusion follows. If $\bar{D}_i$ does not meet $E$ properly, then its (irreducible) support is contained in $\text{Supp} E$, so it is one of the irreducible components of $\text{Supp} E$. From $[D_i] = 1$, we again deduce that up to passing to a subsequence $\bar{D}_i$ is constant, hence so is $D_i$. \hfill $\square$

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