On Sinai billiards on flat surfaces with non-flat horns

Henk Bruin *

August 3, 2020

Abstract

We show that certain billiard flows on planar billiard tables with horns can be modeled as suspension flows over Young towers with exponential tails. Because the height function of the suspension flow itself is polynomial when the horns are Torricelli-like trumpets, one can derive Limit Laws, including Stable Limits if the parameter of the Torricelli trumpet is chosen in $(1, 2)$, for the corresponding flow.

1 Introduction

Recent results on the statistical properties of non-uniformly hyperbolic flows in dynamical systems include polynomial mixing rate, when the flow can be modeled as suspension flow over a Gibbs-Markov map or over a Young tower, and the roof function $h$ of the suspension flow has polynomial tails:

\[
\mu(\{x \in M : h(x) > t\}) = \ell(t)t^{-\beta},
\]

where $\ell(t)$ is a slowly varying function. There are geodesics flows on non-compact surfaces of curvature $-1$ where the above tail condition with $\beta = 1$ or $2$ applies (although properties of Kleinian groups rather than Young towers are used in the modeling). For Lorentz gas with infinite horizon, we get tail behavior with $\beta = 1$ or $\beta = 2$, depending on the dimension of the lattice of the scatterers. So, although the theory puts no restriction on the parameter $\beta$ in (1), classical examples provide us only with very specific values of $\beta$. In dimension 1, the Pomeau-Manneville maps allow for inducing schemes where the induce time satisfies (1) for $\beta = 1/\alpha$ and $\alpha$ is the order of contact between the graph and the tangent at the neutral fixed point. Thus $\beta$ is variable, but despite some higher-dimensional variants, Pomeau-Manneville maps remain too specific to play a substantial role in the modeling of billiards or other mechanical models. In [8, 10, 4] it was shown that almost Anosov diffeomorphisms (and flows [5]) also allow inducing schemes with tails satisfying (1). These are non-uniformly hyperbolic invertible systems, and, in contrast to Pomeau-Manneville maps, can be chosen to be $C^\infty$ or real analytic, even if $\beta$ is non-integer.

The purpose of this paper is to provide a class of examples that fit directly in the context of billiard maps, and which can be modeled by suspension flows over Young towers with tails as in (1). The basic ingredient is the geodesic flow over a surface of revolution, which we call horns, of which the pseudo-sphere and the Torricelli trumpet...
are examples. The pseudo-sphere has constant curvature $-1$; in fact it is the largest manifold with this property that can be embedded in $\mathbb{R}^3$, [7]. Pseudo-spheres are part of non-compact hyperbolic surfaces with cusps’, such as the “three-horned sphere”. As such, they are covered by the classical theory of geodesic flows on surfaces of curvature $-1$, and they lead to exponential tails. “New” (or at least we are not aware of explicit calculations in the literature) is a one-parameter family of Torricelli trumpets, which provide tails as in (1) where the exponent $\beta$ is equal to the parameter of the family.

Let the billiard table $O$ be a flat compact manifolds, such as a torus or a rectangle with reflecting boundaries. We assume that there are multiple convex scatterers $O_i$ with $C^3$ boundaries and curvature bounded away from zero. Disks will do. These are “hard balls”, i.e., the collision rule of the particle with such scatterers is the rule of fully elastic reflection. Also, the scatterers have disjoint closures and there are enough of them that the horizon is finite. Additionally we replace finitely many disk-shape neighborhoods in $O \setminus \bigcup_i O_i$ horns $H_j$, but always such that here is no straight line between horns without a scatterer in between. Thus $\partial H_j$ are circles in $O$, say with radius $r_0$, and a particle that exits a horn first meets a regular scatterer before entering another horn. Thus the “horizon”, i.e., the maximal flight time $\tau_{\text{max}}$ between collisions is finite, and since the $O_i$’s and $H_j$ have disjoint closures, also the minimal flight time between collisions $\tau_{\text{min}} > 0$.

A unit mass, unit speed particle moves on this surface with scatterers. It reflects fully elastically at the scatterers, but when it meets a horn $H_j$, it moves up, keeping its speed but observing the law of preservation of angular momentum and the holonomic constraint keeping it in $H_j$, until it exits $H_j$ again and resumes its trajectory on $O$. The excursion time is $2t_{\text{max}} = 2t_{\text{max}}(\varphi_0)$ where $\varphi_0 \in \left[\frac{\pi}{2}, \frac{\pi}{2}\right]$ is the angle of incidence that the particle trajectory makes with the normal vector to $\partial H_j$ and $t_{\text{max}}$ is the time for an excursion to reach the highest point in the horn. Due to the radial symmetry of $H_j$, the angle “of reflection” $\varphi_1 = -\varphi_0$. We denote the flow on $Q \cup \bigcup_j H_j$ by $\phi^\prime$.

One can consider the horn as a “soft-ball” scatterer in the sense that it does reflect the particle, but not according to the law of elastic collision: although the angle of incidence equals the angle of reflection (up to a minus sign: $\varphi_1 = -\varphi_0$), the point of entrance on $\partial H$ is not the point of exit. (However, the excursions of the particle in the horn can take an unbounded amount of time, and the effects of this resemble the effects of infinite horizon Lorentz gas.) Let us parametrize the circle $\partial H_j$ by the angle $\theta$. The exit angle $\theta_1$ is a function of the entrance angle $\theta_0$ and the angle of incidence $\varphi_0$. The angle displacement

$$\theta_1 - \theta_0 =: \Delta \theta = \Delta \theta(\varphi_0)$$

depends on $\varphi_0$ but (due to radial symmetry) not on $\theta_0$. Since $\varphi_1 = -\varphi_0$, the resulting billiard map

$$T : M \to M, \quad M := \left( \bigcup_i \partial O_i \cup \bigcup_j \partial H_j \right) \times \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right],$$

This map decomposes into a flight map $F : M \to M, (\theta_0, 0, \varphi_1) \mapsto (\theta_0, \theta_0 + \varphi_1 - \theta_1)$ taking the outgoing coordinates $(\theta_1, \varphi_1)$ from one horn (or scatterer) to the incoming coordinates $(\theta_0, \varphi_0)$ of the next, and reflection map $R : M \to M, (\theta_0, \varphi_0) = (\theta_0 + \Delta \theta(\varphi_0), -\varphi_0)$ taking the incoming coordinates into outgoing coordinates (and $\Delta \theta(\varphi_0) = 0$ at scatterers.) As usual in billiards, $T = R \circ F$ preserves a measure $\mu$ that is absolutely continuous to Lebesgue measure, with density $d\mu = \cos \varphi_1 \, d\theta_1 \, d\varphi_1$.

**Theorem 1.1** The above billiard flow can be modeled as a suspension flow over a Young tower with exponential tails, see Section 2.4. The height function $h$ of the suspension
1. has polynomial tails $\mu(\{x \in M : h(x) > t\}) \sim Ct^{-\beta}$ for some constant $C > 0$ if the horns\(^1\) are Torricelli trumpets (Section 3.2) with $\beta > 0$ equal to the parameter of the trumpet.

2. has polynomial tails if the horns are pseudo-spheres (Section 3.3).

3. is bounded above if the horns are sections of spheres (Section 3.4).

**Notation:** We will write $(\varphi, \theta)$ for the position and angle at incoming collisions, and use $(\theta_0, \varphi_0)$ only if we want to emphasize that it is about the incoming collision (and then $(\theta_1, \varphi_1) = R(\theta_0, \varphi_0)$ is used for the outcoming collision). We write $a_n \sim b_n$ if $\lim_n a_n/b_n = 1$ and $a_n \approx b_n$ if $a_n/b_n$ have a bounded and positive lim sup and lim inf.

**Acknowledgments:** HB gratefully acknowledges the support of FWF grant P31950-N45. He also wants to thank Péter Bálint for his explanations of his and Chernov’s papers [2, 3, 6], and Homero Canales for wholesome doubts about some of the lengthier computations in this paper.

## 2 Billiards with horns

### 2.1 Wavefronts

The horns can be interpreted as “soft balls”, and this allows us to adapt the approach of Bálint & Tóth [2, 3], which in turn adapts the approach of Chernov [6], obtaining conditions on the type and distance of “soft ball” scatterers in order to prove that $T$ is uniformly hyperbolic (albeit with singularities) and to build a Young tower with exponential tails for it (leading to exponential mixing of Hölder observables).

The hyperbolicity proof consists foremost of an analysis of wave fronts, specifically their focusing and defocusing nature, and how these are altered by a collision. It is the defocusing fronts that indicate the unstable directions of the billiard map $T$; the stable directions are then given by the defocusing fronts of the inverse $T^{-1}$.

The property of a soft ball scatterer is that the entrance and exit positions of a particle are not the same; for round scatterers the difference is expressed as a difference in angles $\Delta \theta$, which is a function of the angle of incidence $\varphi$. The key quantity is the derivative

$$\kappa(\varphi) = \frac{d}{d\varphi} \Delta \theta(\varphi).$$

If $\kappa(\varphi) \notin [-2, 0]$, defocused wavefronts remain defocused in the collision. If $\kappa(\varphi) \in [-2, 0]$ then the wave front can start to focus during the free flight. However, $\kappa(\varphi)$ is not too close to $-2$ and the distance between the scatterers sufficiently large, then wave fronts can focus, go “through the focus” and then defocus again before reaching the next scatterer, see Figure 1. In [2] the conditions for this mechanism is captured in Definition 2, which gives conditions on the distortion of $\kappa(\varphi)$, and its values in connection to the distance between scatterers.

In detail, Definition 2 in [2] requires that

- $\inf_{\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]} |2 + \kappa(\varphi)| > 0$, and
- $\tau_{min} > \max_{\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]} \{-2r_0\kappa(\varphi)\frac{\cos \varphi}{2+\kappa(\varphi)}\}$ where $r_0$ stands for the radius of the horn.

\(^1\)The statistical behavior is governed by the trumpet with the smallest parameter.
For the elastic scatterers $O_i$, the displacement in collision angle $\Delta \theta(\varphi) \equiv 0$, so $\kappa(\varphi) \equiv 0$, and the above conditions follow automatically as long as $\tau_{\min} > 0$.

In the Section 3, we compute sojourn times as well as $\Delta \theta(\varphi)$ and $\kappa(\varphi)$ for horns of various types. An important conclusion is that the range of $\kappa$ includes the interval $[-2, 0]$ entirely, as $\varphi$ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, so the mechanism of collision-focusing-defocusing-collision cannot hold everywhere. But since there will be a hard scatterer between every two excursions in a horn, wave fronts that are so close to parallel fronts when they leave a horn that they cannot defocus before the next scatterer, will collide with a hard scatterer and then become defocused. In Proposition 2.1 we quantify this mechanism, and determine the minimal distances between scatterers to guarantee uniform hyperbolicity of the billiard map.

**Proposition 2.1** Assume that the distance between the horn $H$ and any scatterer $O$ is at least $2r_Hr_O$ (where $r_H$ and $r_O$ are the radii of $H$ and $O$ respectively), then the billiard map is hyperbolic.

**Proof.** We follow the argument of [2, Section 2] on the behavior of wavefronts at scatterers. Let $q_-$ parametrize the wavefront $W_-$ coming from the horn $H$ and approaching the scatterer $O$, so $q_-$ and $q_- + dq_-$ are to infinitesimally close points on $W_-$ with unit velocity vectors $v_-$ and $v_- + dv_-$. Note that $W_-$ is parallel if $\frac{dv_-}{dq_-} = 0$ and focusing if $\frac{dv_-}{dq_-} < 0$. After collision, we obtain point $q_+$ and $q_+ + dq_+$ on the outgoing wavefront $W_+$, with velocities $v_+ \text{ ad } v_+ + dv_+$, see Figure 2 left.

Since for a scatterer $\Delta \theta(\varphi) = 0$ and hence $\kappa(\varphi_0) = 0$, [2, Formula (3.2)] gives

$$dq_+ = dq_-, \quad dv_+ = dv_- + \frac{2}{r_H \cos \varphi} dq_-.$$  

Note here that in the formula for $dv_+$ we get a plus-sign (as opposed to [2, Formula (3.2)]) because we parametrize $W_+$ in Figure 2 in the opposite way of [2, Figure 1]. If $\frac{dv_-}{dq_-} \geq -\frac{1}{r_O}$, then $\frac{dv_+}{dq_+} > \frac{1}{r_O}$, so the outgoing front is dispersive. If $\frac{dv_-}{dq_-} < -\frac{1}{r_O}$, then extrapolating backwards and using that the distance between $H$ and $O$ is $> 2r_Hr_O$, congruence of triangles (see Figure 2 (right)) shows that the wavefront leaving $H$ is actually wider than $H$ itself. This means that the actual wavefront $W_-$ focuses before it reaches the scatterer $O$. $\quad \Box$
2.2 Conditions to build a Young tower

The construction of the Young tower that we need relies on the approach of Chernov [6]. Just as is in [2], we will state and verify abstract properties in the billiard map that allow one to put Chernov’s construction of the Young tower in place. In concreto, the properties in Definition 3 of [2], and the comments relevant to our situation, are the following:

1. Hölder continuity of the angle displacement $\Delta \theta(\varphi)$. This holds in our case for $\varphi \sim \pm \frac{\pi}{2}$, but fails for $\varphi \sim 0$, i.e., the straight approach to the horn. In fact, if $\varphi = 0$, the particle will never leave the horn again. As we will see, $\Delta \theta(\varphi)$ blows up near the discontinuity $\varphi = 0$, and this requires the largest adjustment to the argument in [2].

A second issue is the unboundedness of $\Delta \theta$ (and hence of $\kappa$) near head-on collisions with horns (i.e., angle of incidence $\varphi \approx 0$). This situation is not covered in [3]; it requires extra measures (in the shape of adding more “homogeneity strips”) to control the distortion. However, we expect the extremal expansion in this neighborhood to help in overcoming the effects of the extra chopping that this entails. Indeed, for fixed angle of entry $\theta_0$, we can partition a punctured neighborhood of $\varphi_0 = 0$ into intervals $I_j$ such that the exit angle $\theta_1$ ranges from 0 to $2\pi$ on each $I_j$. Unstable leaves become automatically long in this way, and there is no need for growing lemmas. However, we need to compute distortion of $\Delta \theta(\varphi)$ on these $I_j$, see Proposition 2.3.

2. The angle displacement $\Delta \theta(\varphi)$ is $C^2$ on all the intervals of continuity in $[\frac{\pi}{2}, \frac{\pi}{2}]$. Because $\Delta \theta(\varphi)$ blows up near $\{\varphi = 0\}$, we have to resort to $C^2$ smoothness on the (artificial) homogeneity strips near $\{\varphi = 0\}$, but this is unproblematic.

3. There is a constant $C \in \mathbb{R}$ such that the derivative $|\kappa'(\varphi)| \leq C |2\kappa(\varphi)|^3$ for all $\varphi$ where this derivative is defined.

4. The expression $\omega(\varphi) = \frac{2 + \kappa(\varphi)}{\cos \varphi}$ is monotone on every sufficiently small one-sided neighborhood of a discontinuity point of $\kappa(\varphi)$.

2.3 Distortion control near $\{\varphi = 0\}$.

For each horn $H_j$ select a scatterer $O_i$ opposite to it. The line connecting the centers of $H_j$ and $O_j$ contains a trajectory $\phi'(x)$ such that $x \in \partial O_i$ and $\lim_{t \to \pm \infty} \phi'(x) = C_j$, the center of $H_j$. This trajectory meets no other scatterers or horns. There is a maximal arc $A_j$ in $\partial H_j$ such that the billiard map $T^2(A_j \times \{0\}) \subset \partial H_j \times [-\frac{\pi}{2}, \frac{\pi}{2}]$, i.e., a particle starting in $A_j \subset \partial H_j$ with outgoing angle $\varphi = 0$ hits $\partial H_j$ again at the second iterate of the billiard map, bouncing once against $O_j$ in between, see Figure 3.
Let \( A'_j = F \circ T(A_j \times \{0\}) \); it is a smooth curve in \( M_j := \partial H_j \times [-\frac{\pi}{2}, \frac{\pi}{2}] \) that stretches across \( M_j \) in the vertical direction and is transversal to \( \{\varphi = 0\} \). Let \( a_j^- = (\theta_j^-, +\frac{\pi}{2}) \) and \( a_j^+ = (\theta_j^+, -\frac{\pi}{2}) \) be the endpoints of \( A'_j \). The reflection map

\[
R : M_j \setminus \{\varphi = 0\} \to M_j \setminus \{\varphi = 0\}, \quad (\theta, \varphi) \mapsto (\theta + \Delta \theta(\varphi), -\varphi),
\]

is a bijection, sending \( a_j^\pm \) to \( (\theta_j^\pm, \pm\frac{\pi}{2}) \) and the closer to the equator \( \partial H_j \times \{0\} \), the stronger the shear. That is, an arc \( \{\theta\} \times (0, \frac{\pi}{2}] \) is mapped by \( F \) to a spiral curve wrapping infinitely often around the annulus \( M_j \), compactifying onto \( \partial H_j \times \{0\} \) from below, while \( \{\theta\} \times [\frac{\pi}{2}, 0) \) is mapped by \( R \) to a spiral curve wrapping infinitely often around the annulus \( M_j \), compactifying onto \( \partial H_j \times \{0\} \) from above. Conversely, \( \Psi_j := R^{-1}(A'_j \setminus \{\varphi = 0\}) \) consists of two spirals wrapping infinitely often around \( M_j \) and compactifying on the equator, one from each direction. For every point \( (\theta, \varphi) \in \Psi_j \), \( F \circ R \circ R(\theta, \varphi) \) represents a particle outgoing from \( O_j \) and frontally approaching \( H_j \), so that \( T \circ F \circ R(\theta, \varphi) \) is not defined: the particle never leaves \( H_j \) again.

Now \( M_j \setminus \overline{\Psi_j} \) consists of two strips that wrap around \( M_j \) infinitely often and approaching \( \{\varphi = 0\} \) in a spiral fashion from above and below. Take \( \psi_j^- = F^{-1}(a_j^-) \) and \( \psi_j^+ = R^{-1}(a_j^+) \), and let \( E_j \) be the straight line connecting \( \psi_j^- \) and \( \psi_j^+ \) in \( M_j \). Then \( E_j \) cuts \( M_j \setminus \overline{\Psi_j} \) into infinitely many strips \( I_{\pm k}, k \in \mathbb{N} \), whose closures are curvilinear rectangles except that they coincide at two opposite corners; note that they wrap around \( M_j \) once.

\[
\begin{array}{c}
\varphi = 0 \\
\xi_j^- \\
E_j \\
\xi_j^+ \\
\end{array} \quad \overset{R}{\longrightarrow} \quad 
\begin{array}{c}
\varphi = 0 \\
a_j^- \\
A'_j' \\
a_j^+ \\
\end{array}
\]

Figure 4: The curves \( \Psi_j, E_j \) and \( R(\Psi_j), A'_j = R(\Psi_j) \) in \( M_j \).

The set \( I_{\pm k} \) play the role of homogeneity strips, within which unstable derivatives are uniformly bounded.
Lemma 2.1 Let \( \tilde{I}_{\pm k} \) be the projections of \( I_{\pm k} \) into \( \{0\} \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \). Then \( |\tilde{I}_{\pm k}| \approx k^{-(1+\frac{\beta}{2})} = o(d(\tilde{I}_k, 0)) \) as \( k \to \infty \) for some \( \beta > 0 \), where \( d \) is the Euclidean distance on \( \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \).

Proof. The precise computation depends on the shape of the horn \( H_j \), but there is always a leading term of the map \( \Delta \theta : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R} \) of the form \( g : \varphi \mapsto C\varphi^{-\beta} \) for some \( 0 \neq C \in \mathbb{R} \) and \( \beta > 0 \). Now

\[
\tilde{I}_k \sim g^{-1}(2\pi k, 2\pi(k + 1)) = \left(\frac{C}{2\pi k}\right)^{1/\beta}, \quad \left(\frac{C}{2\pi(k + 1)}\right)^{1/\beta},
\]

so \( |\tilde{I}_k| \sim \left(\frac{C}{2\pi k}\right)^{1/\beta} \). The same argument works for \(-k\). \qed

Proposition 2.2 (Un)stable leaves are uniformly transversal to the coordinate axes: there is \( C_{\text{tran}} > 0 \) such that

\[
\frac{1}{C_{\text{tran}}} \leq \frac{d\omega_s(\varphi)}{d\varphi} \leq C_{\text{tran}},
\]

for all (un)stable leaves \( \omega_s(\varphi) \) parametrized as function of the outgoing angle. Also the angles between stable and unstable leaves are uniformly bounded away from zero.

Proof. Let us use coordinates \((d\theta, d\varphi)\) for the tangent space \( T_{(\theta, \varphi)} \) (at outgoing angles).

In the theory of elastic (hard) convex scatterers, the (un)stable cone-fields

\[
C^u_{(\theta, \varphi)} := \{d\theta \cdot d\varphi \geq 0\} \quad \text{and} \quad C^s_{(\theta, \varphi)} := \{d\theta \cdot d\varphi \leq 0\}
\]

under \( DT \) and \( DT^{-1} \) respectively, see [6, Section 6]. Basically, the reflection changes the sign of the angle, and fixes the position, so that \( d\varphi^+ = -d\varphi^- \) and \( d\theta^+ = d\theta^- \). Then the next collision incoming at \((\theta_1^-, \varphi_1^-)\) reverses orientation again, because \( d\varphi_1^- = \psi - d\varphi^+ \) (where \( \psi \) is the angle between the normal vectors at \( \theta \) and \( \theta_1 \)), and \( d\theta_1^- = (1 + \kappa(\theta_1^-))(1 + \kappa(\theta^+))\). Here \( \kappa \) denotes the curvature at the boundary of the scatterers, and the flight time associated with the billiard map \( T \).

For horns we have \( d\varphi^+ = -d\varphi^- \) and \( d\theta^+ = d\theta^- \) because \( \theta^+ = \theta^- + \Delta \theta \). Also \( DT \) maps \( C^u_{(\theta, \varphi)} \) strictly into \( C^u_{T(\theta, \varphi)} \) and \( DT^{-1} \) maps \( C^s_{(\theta, \varphi)} \) strictly into \( C^s_{T^{-1}(\theta, \varphi)} \). Therefore (un)stable leaves \( W_s \) are transversal to the coordinate axes, i.e., the level sets of \( \varphi \) and \( \theta \).

The discussion so far dealt with the strict invariance of the unstable cone-field \( \{d\theta \cdot d\varphi \geq 0\} \). It remains to establish the strictly positive constant \( C_{\text{tran}} \). But here (6.1)-(6.4) in [6] for both scatterers and horns apply with no other change that the variable \( r \) is called \( \theta \) in our setting.

The same argument applies to stable leaves as well, and since stable and unstable curves belong to disjoint cones (parallel to coordinate axes), uniform transversality between stable and unstable leaves follows. \qed

Also the backward singularity sets \( S^{-m} := T^{-m}(\bigcup_i \partial O_i \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]) \bigcup \bigcup_j \partial H_j \times \{0\}, \) \( m \geq 1 \), belong to the stable cone field, and the forward singularity sets \( S^m := T^m((\bigcup_i \partial O_i \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]) \bigcup \bigcup_j \partial H_j \times \{0\}), m \geq 1 \), belong to the unstable cone field, see [6, Section 6].

Proposition 2.3 Let \( W \) be an unstable leaf contained in \( I_{\pm k} \) and bounded away from \( \{\varphi = \pm \frac{\pi}{2}\} \). Then there is \( C_{\text{dist}} \in \mathbb{R} \) such that the distortion in the unstable direction

\[
\log \left| \frac{J^u_x R(y)\dd y}{J^u_x R(x)} \right| \leq C_{\text{dist}} d(W(x), R(y)),
\]

(3)
where $d_{R(W)}$ indicates arc-length in $R(W)$. In fact, $\log \frac{|J^u_R(y)|}{|J^u_R(x)|} = o(d_{R(W)}(R(x), R(y))$ as $k \to \infty$, that is as $\varphi \to 0$.

**Proof.** First we assume that $W$ is transversal to horizontal lines in $M$, i.e., transversal to lines of constant $\varphi$. Thus $W$ can be written as the graph of a $C^1$-function $w : I_{x,k} \to \partial H_j$. The length-element of arc-length along $W$ is $ds = ds(\varphi) = \sqrt{1 + (w'(\varphi))^2}$. Recall that $R(\theta, \varphi) = (\theta + \Delta(\varphi), -\varphi)$ is the reflection map, and $\Delta(\varphi) = \kappa(\varphi)$. Then the image of $ds$ under $R$ is

$$dR(s) = \sqrt{d\varphi^2 + (w'(\varphi) + \kappa(\varphi))^2} \, d\varphi = \sqrt{1 + (w'(\varphi) + \kappa(\varphi))^2} \, d\varphi.$$ 

Transversality of $W$ means that $|w'|$ is bounded, so

$$\frac{dR(s)}{ds} = \sqrt{1 + (w'(\varphi) + \kappa(\varphi))^2} \approx \kappa(\varphi) \approx C\beta \varphi^{-(\beta + 1)}.$$ 

where we used that leading term of $\kappa$ is $C\beta \varphi^{-(\beta + 1)}$ as in Lemma 2.1. Hence, taking $x = (w(\varphi), \varphi)$ and $y = (w(\varphi + \varepsilon), \varphi + \varepsilon)$ the distortion

$$\log \frac{|J^u_R(y)|}{|J^u_R(x)|} \approx - (\beta + 1) \log(1 + \frac{\varepsilon}{\varphi}) \approx -\varepsilon(1 + \beta) \varphi.$$ 

Now to find $d_{R(W)}(R(y), R(x))$ we integrate $dR(s)$ over $[\varphi, \varphi + \varepsilon]$. This gives

$$\int_{\varphi}^{\varphi + \varepsilon} \sqrt{1 + (w'(v) + \kappa(v))^2} \, dv \approx \int_{\varphi}^{\varphi + \varepsilon} \kappa(v) \, dv \approx \Delta\theta(\varphi + \varepsilon) - \Delta\theta(\varphi) \approx C \left( ((\varphi + \varepsilon)^{-\beta} - \varphi^{-\beta}) \right) \approx C\varphi^{-\beta} \left( (1 + \frac{\varepsilon}{\varphi})^{-\beta} - 1 \right) \sim -\varepsilon C\varphi^{-1+\beta},$$

where in the last approximation we used that $|\varepsilon|$ is small compared to $|\varphi|$ as shown in Lemma 2.1. Formula (3) follows.

For $\varphi \to 0$, we get $1 + \sup |w'|^2 = o(\kappa(\varphi))$, so that the $\approx$ becomes $\sim$ in this case, and the extra factor $\varphi^{-\beta}$ accounts for the little $o$. □

**Corollary 2.1** The unstable derivative on unstable manifolds in $I_{x,k}$ satisfies:

$$\frac{1}{C_{exp}} k^{1+\beta} \leq J^u_W(x) \leq C_{exp} k^{1+\beta},$$

for some expansion constant $C_{exp} > 0$.

### 2.4 Building a Young tower with exponential tails

A Young tower [13] is a schematic dynamical system, in fact an extension over a dynamical system $(X, T)$, of the form $(\Delta, T_{\Delta}, \mu_{\Delta})$, where the space

$$\Delta = \bigcup_i \Delta_i = \bigcup_i \bigcup_{\ell=0}^{\sigma_i-1} \Delta_i, \ell,$$

where $\Delta_0$ is a subset of $X$, partitioned into subsets $\Delta_{i,0}$ such that $T^{\sigma_i} : \cup_i \Delta_{i,0} \to \Delta_0$ is a Gibbs-Markov map (so uniformly hyperbolic), preserving an SRB-measure $\mu_0$. Here
$r : \Delta_0 \to \mathbb{N}$ with $r|_{\Delta_{i,0}} =: \sigma_i$ constant for all $i$ is called the roof function. The sets $\Delta_{i,\ell}$ are copies of the $\Delta_{i,0}$ and the tower map $T_\Delta$ acts as

$$T_\Delta : x \in \Delta_{i,\ell} \mapsto \begin{cases} x \in \Delta_{i,\ell+1} & \text{if } 0 \leq \ell < \sigma(x) - 1; \\ T^{\sigma(x)}(x) \in \Delta_0 & \text{if } \ell = \sigma(x) - 1, \end{cases}$$

and $(\Delta, T_\Delta)$ factors over $(X, T)$ via $\pi : \Delta \to X$, $\pi(x \in \Delta_{i,\ell}) = T^\ell(x)$. We speak of exponential tails if there is $\lambda \in (0, 1)$ such that $\mu_0(\{x : \sigma(x) > n\}) = O(\lambda^n)$. We can extend $\mu_0$ to a $T\Delta$-invariant measure by setting $\mu_\Delta|_{\Delta_{i,\ell}} = \hat{\sigma}^{-1}\mu_0|_{\Delta_{i,0}}$ for normalizing constant $\hat{\sigma} = \sum_{n \geq 1} n\mu_0(\{\sigma(x) = n\})$. This measure pushes down to a $T$-invariant SRB-measure on $(X, T)$ via $\mu = \mu_\Delta \circ \pi^{-1}$. The existence of a Young tower with exponential tails implies that the underlying system $(X, T, \mu)$ is exponentially mixing (provided $\gcd(\sigma_i : i \in \mathbb{N}) = 1$) and satisfies the Central Limit Theorem for Hölder observables, see [14].

Chernov [6, Theorem 2.1] proved a general theorem on the existence of a Young tower with exponential tails for non-uniformly hyperbolic invertible maps, based on a set of conditions concerning expansion and distortion control along unstable leaves and a specific “growth of unstable manifolds” condition (2.6)-(2.8) in [6]. He continues to verify these conditions for various billiard systems, of which the standard Sinai billiard maps (disjoint strictly convex fully elastic scatterers with $C^3$ boundaries on a compact flat table, first fully treated in [13]) is the most relevant to us, see [6, Section 6 & 7]. Bálint & Tóth verify the conditions for soft scatterers, leading them to condition Definition 3 in [3]. In the previous sections we verified most of the Chernov resp. Bálint & Tóth conditions, and here we combine these steps to the final verification. That is, we indicate which adaptations in the arguments of [6, Section 7] are still required.

Chernov [6, Section 7] uses two metrics to obtain hyperbolic expansion:

- The $p$-(pseudo-)metric which has the best expansion properties, but only that after a close-to-grazing collision with corresponding cut into homogeneity strips, the expansion has a one iterate delay. (Note that for horns, grazing collisions induce no discontinuities, so for the such homogeneity strips can be skipped.)

- The Euclidean metric. Now the expansion factor in unstable directions occurs instantaneously at collisions, but it is not always $\geq 1$. Therefore a particular iterate $T^m$ of the billiard map $T$ is chosen, which multiplies the number of discontinuity curves $\bigcup_{j=0}^{m-1} S^{-j}$ and $\bigcup_{n=0}^{m-1} T^{-n}(\bigcup_{j} \partial H_j \times \{0\})$, and corresponding boundaries of homogeneity strips $\bigcup_{k \geq k_0} \bigcup_{n=0}^{m-1} T^{-n}(\partial I_{\pm k})$.

However, combining the two metrics, one can prove uniform expansion (contraction) of unstable (stable) leaves, see [6, Lemma 7.1].

Let $W$ be any unstable leave of length $\leq \delta_0$. It may be cut into at most $K_m + 1$ pieces by $S^{-m}$, where $K_m$ depends only on $m$ and the number of scatterers and horns. In the next $m$ iterate, it may be cut again, even into countably many pieces, by curves in $\bigcup_{n=0}^{m-1} T^{-n}(\{\varphi = 0 \text{ at horns}\} \cup \{\varphi = \pm \frac{\pi}{2}\} \cup_{k \geq k_0} \partial I_{\pm k})$ We label these pieces as $W_{k_1,\ldots,k_m,j}$, where $1 \leq j \leq K_m + 1$, $k_i \in \mathbb{Z}$ and $T^{-m-n}(W_{k_1,\ldots,k_m,j}) \subset I_{k_n}$. Bear in mind that some of these labels can refer to the empty set. Since horns and scatterers have their own homogeneity strips $I_{\pm k}$ where the expansion of the billiard map is $\approx k^{1+\beta}$ and $\approx k^2$ respectively, we will use $\nu := \min\{2, 1 + \beta\}$ for the worst case of the two. The unstable expansion for $T_1 = T^m$ on a piece $W_{k_1,\ldots,k_m}$ of unstable manifold thus becomes

$$J^u_1(x) \geq L_{k_1,\ldots,k_m} := \max\{\Lambda_1, \frac{1}{C_{exp}} \prod_{k_i \neq 0} k_i^\nu\}.$$
This product \( \prod_{k_i \neq 0} \) then reappears in the definition\(^2\) of \( \Theta := 2 \sum_{k \geq k_0} k^{-\nu} \leq \frac{2^{1/2}}{\nu} \). We need to choose \( k_0 \) so large that, as in [6, Formula (7.5)] with corresponding constant \( B_0 \),

\[
(K_m + 1)(\Lambda_1^{-1} + 2B_0\Theta) < 1. \tag{5}
\]

Also [6, Lemma 7.2] needs to be adjusted to:

**Lemma 2.2** For all \( \delta > 0 \), there is \( B = B(m) \) such that

\[
\sum_{k_1, \ldots, k_m \geq 2} \min\{\delta, (k_1 \cdots k_m)^{-\nu}\} < B(m)\delta^{\frac{1}{2m}}.
\]

But the proof goes as in [6, Appendix], with some minor and obvious adaptations.

Thus we can apply Chernov’s main theorem for the billiard map, which we restate here:

**Theorem 2.1** For any type of horn discussed in this paper, there are \( \alpha, \lambda \in (0,1) \) such that the billiard map \( (M, T) \) has exponential decay of correlations:

\[
\left| \int_M v \cdot w \circ T^n d\mu - \int_M v d\mu \int_M w d\mu \right| = O(\lambda^n)
\]

for the SRB-measure \( \mu \) and \( \alpha \)-Hölder functions \( v, w : M \to \mathbb{R} \) and also the Central Limit Theorem holds for observables not cohomologous to a constant.

### 2.5 The billiard flow

The billiard flow can now be modeled as a suspension flow over this Young tower, i.e., the space is now \( \Delta^h := \sqcup_{i,t} \Delta_{i,t} \times [0, h(x)]/\sim \) where \( (x, h(x)) \sim (T_\Delta(x), 0) \), and the flow \( \phi^t_\Delta(x, u) = (x, u + t) \in \Delta^h \). The height function \( h \) is either equal to the (bounded) flight time \( \tau(x) \) between two scatterers, or equal to the flight time \( \tau(x) \) between a scatterer and a horn plus the sojourn time \( 2t_{\text{max}}(\varphi_0) \), where \( \varphi \) is the incoming angle at the collision with the horn. The \( \phi^t_\Delta \)-invariant measure \( \mu^h_\Delta = \frac{1}{h} \mu_\Delta \otimes \text{Leb} \) for the normalizing constant \( h = \int h(x) d\mu_\Delta \) or \( h = 1 \) if this integral is infinite, because in this infinite measure case, there is no point in normalizing. The corresponding flow-invariant measure \( \mu^h \) is the push-down \( \mu^h_\Delta \circ \pi^{-1}_h \) where \( \pi^h(x, u) = \phi^u \circ \pi(x) \).

The computations in Section 3.2 show that the tails of \( h \) have the asymptotics

\[
\mu(\{x \in M : h(x) > t\}) = \frac{1}{2\#(H_j \cup O_i)} \int_{\{x \in M : h(x) > 2t_{\text{max}}(\varphi_0) > t\}} \cos \varphi_0 d\mu \sim Ct^{-\beta}
\]

for some constant depending only on the shape of the horns and \( \#(H_j \cup O_i) \) stand for the number of horns and scatterers. In fact, the exponent \( \beta \) is equal to the parameter \( \beta \) of the Torricelli trumpet, and therefore \( \mu^h \) is finite if and only if \( \beta > 1 \).

**Theorem 2.2** Here I want to say something about limit theorems.

For the pseudo-sphere in Section 3.3, we get exponential tails for \( h \), and for the sphere section in Section 3.4, \( h \) is bounded above. In these two cases, the billiard flow has exponential decay of correlations and again satisfies the Central Limit Theorem.

\(^2\)This \( \Theta \) is called \( \theta_0 \) in [6].
3 Dynamics of the flow on horns

Let \( H \) be a surface of revolution in \( \mathbb{R}^3 \) obtained by revolving the curve \( x = x(z) \) around the \( z \)-axis. We will use the radius \( r = r(z) = \sqrt{x^2 + y^2} \) as radius of \( H \) and \( z = z(r) \) is the inverse function. Thus \( H \) has the parametrization

\[
\sigma(z, \theta) = (r(z) \cos \theta, r(z) \sin \theta, z), \quad z \geq z_0, \theta \in [0, 2\pi).
\]

Abbreviate \( r_0 := r(z_0) \).

**Example 3.1** The area and volume of \( H \) are

\[
A := 2\pi \int_{z_0}^{\infty} (1 + r^2) r \, dz \quad \text{and} \quad V := \pi^2 \int_{z_0}^{\infty} r^2 \, dz
\]

respectively. For \( r(z) = z^{-\beta} \), we get \( A = \infty, V = \frac{\pi}{2\beta - 1} < \infty \) (painter’s paradox) if \( \beta \in (\frac{1}{2}, 1] \). This holds specifically for the case \( \beta = 1 \), i.e., \( r = 1/z \), when \( H \) is called the trumpet of Torricelli (or Gabriel’s horn, see Figure 5, right). For \( \beta > 1 \) we have \( A, V < \infty \) and for \( \beta \leq \frac{1}{2} \), both \( A, V = \infty \). Also the Gaussian curvature of such a surface is

\[
\kappa_G = -\frac{r''}{r(1 + r^2)^2} = -\frac{\beta(\beta + 1)z^{-(2+\beta)}}{z^{-\beta}(1 + \beta^2 z^{-2(1+\beta)})^2} = -\frac{(\beta^2 + \beta)}{z^2(1 + \beta^2 z^{-2(1+\beta)})^2}.
\]

**Figure 5:** Geodesics on the pseudo-sphere and Torricelli’s trumpet.

For the next exposition, see [1, Section 4C]. A geodesic \( \Gamma \) is the path on \( H \) traced out by a unit mass particle moving along \( H \) at unit speed with no external forces other than the holonomic constraints keeping it on \( H \). Assume the geodesic starts at \((z_0, \theta_0) \in \partial H\), making an incoming angle \( \varphi_0 \in [-\frac{\pi}{2}, \frac{\pi}{2}] \) with the vertical meridian, see Figure 5, left. The kinetic energy

\[
E_{\text{kin}} = \frac{1}{2} |v|^2 = \frac{1}{2} ((1 + r'^2) \dot{z}^2 + r^2 \dot{\theta}^2) = \frac{1}{2}
\]

is one constant of motion. Due to the rotational symmetry (using Noether’s Theorem), the \( z \)-component of the angular momentum

\[
r^2 \dot{\theta} = r_0 |v| \sin \varphi_0 = r_0 \sin \varphi_0
\]

(8)
is the second constant of motion. Inserting $r\dot{\theta} = |v| \sin \alpha$ we get a derived constant of motion (Clairaut’s Theorem)

$$r(z) \sin \alpha = r_0 \sin \varphi_0. \quad (9)$$

The value $\sin \alpha$ takes its largest value at the highest point of the geodesic (where $\sin \varphi_{\text{max}} = 1$, $r_{\text{min}} = r_0 \sin \varphi_0$, $\theta = \theta_{\text{max}}$ and $z = z_{\text{max}} = z(r_{\text{min}})$), then the geodesic spirals down again (symmetrically to the upwards spiral), until it hits $\partial H$ with an angle $\varphi_1 = -\varphi_0$.

The question we pose ourselves is

What is the time $t_{\text{max}}$ needed of the geodesic particle to reach the top at $z_{\text{max}}$?

This has a direct consequence for the tails of geodesic flow if these sojourns inside the horns are modeled by suspension flow with height function $2t_{\text{max}}$ and base map

$$R : M_j := \partial H \times [-\pi/2, \pi/2] \rightarrow M_j, \quad (\theta, \varphi) \mapsto (\theta + \Delta \theta \Theta, -\varphi).$$

### 3.1 Computation of $t_{\text{max}}$ and $\theta_{\text{max}}$

From (7) combined with (8) we find

$$\dot{z} = \frac{dz}{dt} = \sqrt{\frac{1 - r(z)^2 \theta^2}{1 + r'(z)^2}} = \sqrt{\frac{1 - r_0^2 r(z)^{-2} \sin^2 \varphi_0}{1 + r'(z)^2}}.$$ 

Therefore

$$dt = \sqrt{\frac{1 + r'(z)^2}{1 - r_0^2 r(z)^{-2} \sin^2 \varphi_0}} \, dz$$

and

$$t_{\text{max}} = \int_0^{t_{\text{max}}} dt = \int_{z_0}^{z_{\text{max}}} \sqrt{\frac{1 + r'(z)^2}{1 - r_0^2 r(z)^{-2} \sin^2 \varphi_0}} \, dz.$$ 

Using the change of coordinates $u = \frac{z_0}{r(z)} \sin \varphi_0$, so $z = z(r_0 \frac{\sin \varphi_0}{u})$, $z = z_0 \Leftrightarrow u = |\sin \varphi_0|$, $z = z_{\text{max}} \Leftrightarrow u = 1$ and $dz = -\frac{r_0}{u^2} \frac{\sin \varphi_0}{r'(z(r_0 \frac{\sin \varphi_0}{u}))} du$, we find

$$t_{\text{max}} = r_0 |\sin \varphi_0| \int_1^1 \frac{1}{u^2} \sqrt{\frac{1 + r'(z(r_0 \frac{\sin \varphi_0}{u})))^{-2}}{1 - r_0^2 r(z)^{-2} \sin^2 \varphi_0}} \, du. \quad (10)$$

Now for the displacement in angle $\theta$ between the initial angle $\theta_0$ and the angle $\theta_{\text{max}}$ reached at the top of the geodesic, we find, using (8) and the previous computation for $\dot{z}$:

$$\theta_{\text{max}} - \theta_0 = \int_{\theta_0}^{\theta_{\text{max}}} d\theta = \int_0^{t_{\text{max}}} \dot{\theta} \, dt = \int_0^{t_{\text{max}}} \frac{r_0 \sin \varphi_0}{r^2} \, dt$$

$$= \int_{z_0}^{z_{\text{max}}} \frac{r_0 \sin \varphi_0}{r^2} \frac{1}{\dot{z}} \, dz$$

$$= r_0 \sin \varphi_0 \int_{z_0}^{z_{\text{max}}} \frac{1}{r^2} \sqrt{\frac{1 + r'(z)^2}{1 - r_0^2 r(z)^{-2} \sin^2 \varphi_0}} \, dz.$$ 

---

3Because $r' < 0$, we obtain an extra minus sign when moving $r'$ into the square-root.
This should be compared to Formula (5.2) in [2] expressing $\Delta \theta$ in terms of the potential of a soft scatterer. Applying the transformation $u = \frac{r_0}{\sin \varphi |} \sin \varphi_0$ as before, we get

$$\theta_{\text{max}} - \theta_0 = \int_{|\sin \varphi_0|}^{1} \sqrt{1 + r'(z(r_0|\sin \varphi_0|)u)} \frac{2}{1 - u^2} du. \quad (11)$$

Throughout (and following the notation of [2, 3]) we let

$$\Delta \theta = 2(\theta_{\text{max}} - \theta_0) = 2 \int_{|\sin \varphi_0|}^{1} \sqrt{1 + r'(z(r_0|\sin \varphi_0|)u)} \frac{2}{1 - u^2} du. \quad (12)$$

be the difference in incoming and outgoing angle of the obstacle as function of angle of incidence $\varphi_0$. Its derivative w.r.t. $\varphi_0$ is denoted as

$$\kappa(\varphi_0) = \frac{\partial \Delta \theta(\varphi_0)}{\partial \varphi_0} = -2 \operatorname{sgn}(\varphi_0) \sqrt{1 + (r'(z))^{-2}} - 2r_0 \operatorname{sgn}(\varphi_0) \cos \varphi_0 \times$$

$$\int_{|\sin \varphi_0|}^{1} \frac{z'(r_0|\sin \varphi_0|)}{1 + r'(z(r_0|\sin \varphi_0|))} \frac{r''(z(r_0|\sin \varphi_0|))}{\left(r'(z(r_0|\sin \varphi_0|))\right)^2} \frac{1}{u \sqrt{1 - u^2}} du. \quad (13)$$

For convex obstacles, i.e., with $z' < 0$ and $r'' > 0$, the two terms in this expression have opposite signs. The first term $< -2$, whereas the second varies between $0$ (as $\varphi_0 \to \pm \pi/2$) and potentially $\infty$ (as $\varphi \to 0$). Therefore we cannot expect that $\kappa(\varphi_0)$ is bounded way from $[-2, 0]$ as required in [3] to obtain uniform hyperbolicity.

### 3.2 Torricelli’s trumpets

To simplify formulas in these subsections, we write $\varphi$ again for $\varphi_0$. Assume that the horn is the surface of revolution of the curve $r(z) = z^{-\beta}$, with $r'(z) = -\beta z^{-(1+\beta)}$, $r''(z) = \beta(1 + \beta) z^{-(2+\beta)}$ and $z(r) = r^{-1/\beta}$. Inserting the equations for $r(z)$ into (10) gives $r'(\frac{r_0|\sin \varphi|}{u}) = \beta^{-2} z(\frac{r_0|\sin \varphi|}{u})^{2(1+\beta/\beta)} = \beta^{-2} z(\frac{r_0|\sin \varphi|}{u})^{2(1+1/\beta)}$ and

$$t_{\text{max}} = |\sin \varphi|^{-\frac{1}{\beta}} \beta^{-1} r_0^{-\frac{3}{\beta}} \int_{|\sin \varphi|}^{1} \frac{1}{u^2} \sqrt{\frac{\beta^2(r_0|\sin \varphi|)^{2(1+\beta)}}{1 - u^2} + u^2} du. \quad (14)$$

The integral $I(\varphi)$ tends to a positive constant $\mathcal{I}_0 = \beta^{-1} r_0^{-\frac{3}{\beta}} \int_{0}^{\frac{\pi}{2}} \sin^{\beta-1} \alpha \, d\alpha$ as $\varphi \to 0$, so the leading asymptotics of $t_{\text{max}}$ is $|\sin \varphi|^{-1/\beta} \mathcal{I}_0$. This gives tails on the height function over base $M_j := -\partial H_j \times [-\frac{\pi}{2}, \frac{\pi}{2})$,

$$\mu((\theta, \varphi) \in M_j : 2t_{\text{max}} > t) = 2\pi \mu(|\sin \varphi| < \left(\frac{t}{2r_0 I(\varphi)}\right)^{\frac{1}{\beta}}) \sim 4\pi(\frac{t}{2r_0 I_0})^{-\beta}.$$

Applying the same formulas to (11), we get

$$\Delta \theta(\varphi) = \frac{2 \operatorname{sgn}(\varphi)}{(\sin \varphi)^{\frac{1}{\beta}}} \int_{|\sin \varphi|}^{1} \sqrt{\frac{\beta^2(r_0|\sin \varphi|)^{2(1+\beta)}}{1 - u^2} + u^2} \frac{1}{\beta r_0} \int_{|\sin \varphi|}^{1} \sqrt{\frac{\beta^2(r_0|\sin \varphi|)^{2(1+\beta)}}{1 - u^2} + u^2} du, \quad (15)$$
and \( J(\varphi) \to \beta^{-1} r_0 \frac{1 + \beta \varphi^2}{\beta} \int_0^{\varphi} \sin(\alpha) \frac{1 + \beta}{\beta} d\alpha \) as \( \varphi \to 0 \).

Therefore the base map \( R : M \to M \) for base \( M_j := \partial H_j \times [-\frac{T}{2}, \frac{T}{2}] \) becomes

\[
R : (\theta, \varphi) \mapsto (\theta + 2 \text{sgn}(\varphi) |\sin \varphi|^{-\frac{1+\beta}{\beta}} J(\varphi), -\varphi).
\]

Since \( J(\varphi) \to 0 \) as \( \varphi \to \pm \frac{T}{2} \) we get \( F(\theta, \alpha) \to (\theta, -\alpha) \) as \( \alpha \to \pm \frac{T}{2} \).

Inserting the above into (13), we find

\[
\kappa(\varphi) = -2 \text{sgn}(\varphi) \sqrt{1 + \beta^2 r_0^{-2(1+\beta)/\beta}} + \frac{2(1 + \beta)}{\beta^2} r_0^{-\frac{(1+\beta)}{\beta}} |\sin \varphi|^{\frac{(1+\beta)}{\beta}} |\tan \varphi| \times \\
\int |\sin \varphi| \sqrt{u^{(1+\beta)/\beta} + \beta^2 (r_0 \sin \varphi)^{(1+\beta)/\beta}}\sqrt{1 - u^2} du.
\]

As \( \varphi \) increases from 0 to \( \pi/2 \), \( \kappa(\varphi) \) decreases from \( \infty \) to \( -2\sqrt{1 + \beta^2 r_0^{-2(1+\beta)/\beta}} \), so it crosses the interval \([-2, 0]\) where hyperbolicity is compromised.

The leading term of \( \kappa(\varphi) \), however, is \( C \varphi^{-\frac{1+2\beta}{\beta}} \) for some \( C > 0 \), so, since \( \kappa \) is a smooth function of \( \varphi \) for \( \varphi \neq 0 \), the leading term of \( \kappa'(\varphi) \) in absolute value is

\[
\frac{1 + 2\beta}{\beta} C \varphi^{-\frac{1+3\beta}{\beta}} \leq C \varphi^{-\frac{3+2\beta}{\beta}} = O(|2 + \kappa(\varphi)|) \quad \text{as} \quad \varphi \to 0,
\]

whenever \( \beta > -2/3 \). Hence, for every \( \beta > 0 \), Condition 3 in Section 2.2 holds. By the same token, recalling that \( \omega(\varphi) = \frac{2 + \kappa(\varphi)}{\cos \varphi} \),

\[
\omega'(\varphi) := \frac{\kappa'(\varphi) + \kappa(\varphi) \tan \varphi}{\cos \varphi}
\]

is bounded away from 0 for \( \varphi \) close to 0. Therefore \( \omega(\varphi) \) is monotone in one-sided neighborhood of \( \{ \varphi = 0 \} \), and Condition 4 in Section 2.2 holds.

### 3.3 The pseudo-sphere

To treat the pseudo-sphere (which is the surface of revolution of the tractrix), we insert the tractrix into (10): \( z(r) = \int_r^1 \frac{s - x^2}{s} ds \), with \( z'(r) = -\frac{\sqrt{1 - x^2}}{s} \). Therefore \( r'(z) = -\frac{v}{\sqrt{1 - v^2}} \).

According to Hilbert’s Theorem, see [7] or e.g. [12, Section 11.1] says that the pseudo-sphere is the largest surface of constant curvature \( \kappa_G = -1 \) that can be embedded in \( \mathbb{R}^3 \).

Inserting these formulas into (10) gives

\[
t_{\text{max}} = r_0 |\sin \varphi| \int_{|\sin \varphi|}^{1} \frac{1}{u} \sqrt{1 - u^2} \sqrt{1 + \left( \frac{r_0 |\sin \varphi|}{u} \right)^2 - \frac{1}{u} \sqrt{1 - r_0^2 \sin^2 \varphi}} du
\]

\[
= \int_{|\sin \varphi|}^{1} \frac{1}{u} \sqrt{1 - u^2} du = \int_{|\sin \varphi|}^{1} \frac{1}{|\sin \varphi|} d\alpha \quad (u = \sin \alpha)
\]

\[
= \int_{0}^{\cos \varphi} \frac{dv}{1 - v^2} = \frac{1}{2} \int_{0}^{\cos \varphi} \frac{1}{1 - v} + \frac{1}{1 + v} dv \quad (v = \cos \alpha)
\]

\[
= \log \left( \frac{1 + \cos \varphi}{1 - \cos \varphi} \right) = \log \left( \frac{1 + \cos \varphi}{\sin \varphi} \right).
\]
Alternatively, see also [12, Example 9.3.3], in the hyperbolic half-plane \( \mathbb{H} \), we can obtain a modular surface by taking the quotient under a Kleinian group \( \langle M_t, M_i \rangle \) for the translation \( M_t : z \mapsto z + 2\pi \) and involution \( M_i : z \mapsto (2\pi)^2/z \). The region \( S = [0, 2\pi] + [1, \infty)i \) is then comparable to the pseudo-sphere with boundary \( \partial H = [0, 1\pi] \times \{i\} \). A geodesic \( \Gamma \) starting at \( \partial H \) making an angle \( \phi \) with the vertical meridian is then a circular arc of radius \( R = 1/\sin \phi \), connecting \( a := (1 - \cos \phi)R \equiv (\text{mod} 2\pi) + iR \sin \phi = (1 - \cos \phi)R \equiv (\text{mod} 2\pi) + i \) to \( b := (1 + \cos \phi)R \equiv (\text{mod} 2\pi) + iR \sin \phi = (1 + \cos \phi)R \equiv (\text{mod} 2\pi) + i \), see Figure 6. This is part of the semi-circle connection 0 to 2\( R \), and the the maximal point of this geodesic is \( c := (1 + i)R \). To compute the length of this geodesic, we transform it to a vertical arc in \( \mathbb{H} \) using the isometry (Mobius transformation)

\[
f(z) = \frac{-z}{z - 2R},
\]

so that \( f(0) = 0 \), \( f(c) = 1 \) and \( f(2R) = \infty \).

Then \( f(a) = \frac{\sin \phi}{1 + \cos \phi}i \) and \( f(a) = \frac{\sin \phi}{1 - \cos \phi}i \). Therefore the hyperbolic length of \( \Gamma \) is

\[
\ell(\Gamma) = \int_{f(a)}^{f(b)} \frac{1}{y} \, dy = \log \frac{f(b)}{f(a)} = \log \frac{1 + \cos \phi}{1 - \cos \phi} = 2 \log \frac{1 + \cos \phi}{\sin \phi}.
\]

As the unit mass particle goes through this curve at unit speed (in hyperbolic metric), the time it takes to go from \( a \) to \( b \) is \( 2t_{\text{max}} = 2 \log \frac{1 + \cos \phi}{\sin \phi} \).

This gives tails on the height function over base \( M_j \):

\[
\mu((\theta, \varphi) \in M_j : 2t_{\text{max}} > t) = 2\pi \mu \left( \left| \frac{\sin \varphi}{1 + \cos \varphi} \right| < e^{-t/2} \right) \sim 8\pi e^{-t/2}.
\]

The corresponding reflection map \( R : M_j \to M_j \) is given by

\[
R : (\theta, \varphi) \mapsto (\theta + 2\cos \varphi \sin \varphi, -\varphi).
\]

Again, \( R(\theta, \varphi) \to (\theta, -\varphi) \) as \( \varphi \to \pm \frac{\pi}{2} \). In this case \( \Delta \theta = 2\frac{\cos \varphi}{\sin \varphi} \). Hence \( \kappa(\varphi) = 2\sin^{-2} \varphi \) and \( \omega(\varphi) = 2 + 2\sin^{-2} \varphi \) are easily seen to satisfy the conditions in Section 2.2, except that \( \kappa(\varphi) \) does not avoid the interval \([-2, 0]\). So the use of scatterer to regain hyperbolicity of wavefronts is necessary.
3.4 The sphere

Assuming again that \( r_0 = 1 \), the radius of the stunted sphere \( H \) centered at the origin containing the circle \( \{ r = r_0 = 1, z = z_0 \} \) as boundary is \( R = r_0 \sqrt{1 + r_0^2} \) and \( z_0 = -r_0 r'_0 \).

This gives a parametrization \( r(z) = \sqrt{R^2 - z^2} \), so \( r'(z) = \frac{z}{\sqrt{R^2 - z^2}} \), and \( z(r) = \sqrt{R^2 - r^2} \).

Inserting this in (10) we obtain

\[
\begin{align*}
\tau_{\text{max}} &= r_0 |\sin \varphi| \int_{|\sin \varphi|}^{1} \frac{1}{u} \frac{1}{\sqrt{1 - u^2}} \frac{1}{\sqrt{u^2 - r_0^2 R^2 - 2 \sin^2 \varphi}} \, du \\
&= r_0 |\sin \varphi| \int_{|\varphi|}^{\frac{\pi}{2}} \frac{1}{\sin \alpha} \frac{1}{\sqrt{\sin^2 \alpha - r_0 R^2 - 2 \sin^2 \varphi}} \, d\alpha.
\end{align*}
\]

By inserting this curve in (11) gives us the change in \( \theta \):

\[
\Delta \theta(\varphi) = 2 \text{sgn}(\varphi) \int_{|\sin \varphi|}^{1} \frac{1}{\sqrt{1 - u^2}} \frac{1}{u} \frac{1}{\sqrt{u^2 - r_0^2 R^2 - 2 \sin^2 \varphi}} \, du
\]

\[
= 2 \text{sgn}(\varphi) \int_{|\varphi|}^{\frac{\pi}{2}} \frac{\sin \alpha}{\sqrt{\sin^2 \alpha - r_0^2 R^2 - 2 \sin^2 \varphi}} \, d\alpha,
\]

with corresponding reflection map. As \( \varphi \to 0 \), \( \Delta \theta(\varphi) \to \pi \) (the geodesic goes over the “North” pole), and as \( \varphi \to \pm \frac{\pi}{2} \), \( \Delta \theta(\varphi) \to 0 \) (the geodesic is tangent to \( \partial H \)). Since \( 0 \leq \sin^2 \varphi \leq \sin^2 \alpha \) in this integrand, we get

\[
\frac{\pi - 2|\varphi|}{\sqrt{1 - r_0^2/R^2}} \geq |\Delta \theta| \geq \pi - 2|\varphi| \quad \text{and} \quad \lim_{\varphi \to \pm 0} \Delta \theta(\varphi) = \pm \pi.
\]

The derivative is

\[
\kappa(\varphi) = -2 \text{sgn}(\varphi) \cos \varphi \left( \frac{1}{z_0} + r_0^2 \sin \varphi \int_{|\varphi|}^{\frac{\pi}{2}} \frac{\sin \alpha}{(R^2 \sin^2 \alpha - r_0^2 \sin \varphi)^{3/2}} \, d\alpha \right)
\]

and this is bounded on \([-\frac{\pi}{2}, \frac{\pi}{2}]\). It tends to \( \mp 2 \left( \frac{R}{z_0} + \frac{r'_0}{R^2} \right) \) as \( \varphi \to \pm 0 \), and to \( 0 \) as \( \varphi \to \pm \frac{\pi}{2} \).

References

[1] V. I. Arnol’d, *Mathematical methods of celestial mechanics*, Springer Verlag, New York Heidelberg Berlin 4th ed. (1984).

[2] P. Bálint, P. I. Tóth, *Correlation decay in certain soft billiards*, Communications in Mathematical Physics, 243 (2003) 55-91.

[3] P. Bálint, P. I. Tóth, *Mixing and its rate in soft and hard billiards motivated by the Lorentz process*, Physica D, 187 (2004) 128-135.

[4] H. Bruin, D. Terhesiu, *Regular variation and rates of mixing for infinite measure preserving almost Anosov diffeomorphisms*, Ergod. Th. and Dyn. Sys. 40 (2020), 663–698.

[5] H. Bruin, D. Terhesiu, M. Todd, *Pressure function and limit theorems for almost Anosov flows*, Preprint 2018, arXiv:1811.08293, to appear in Commun. Math. Phys.
[6] N. Chernov, *Decay of correlations and dispersive billiards*, J. Statist. Phys. 94 (1999) 513–556.

[7] D. Hilbert, *Über Flächen von konstanter Krümmung*, Trans. Amer. Math. Soc. 2 (1901), 87–99.

[8] H. Hu, *Conditions for the existence of SBR measures of “almost Anosov” diffeomorphisms*, Trans. Amer. Math. Soc. 352 (2000) 2331–2367.

[9] H. Hu, L-S. Young, *Nonexistence of SBR measures for some diffeomorphisms that are "almost Anosov"*, Ergod. Th. Dynam. Sys. 15 (1995), 67–76.

[10] H. Hu, X. Zhang, *Polynomial decay of correlations for almost Anosov diffeomorphisms*, Ergod. Th. Dynam. Sys. 39 (2019), 832–864.

[11] A. Katok, *Bernoulli diffeomorphisms on surfaces*, Ann. of Math. 110 (1979), 529–547.

[12] A. Pressley, *Elementary differential geometry*, 2nd ed. Springer-Verlag, London (2010).

[13] L.-S. Young, *Statistical properties of dynamical systems with some hyperbolicity*, Ann. of Math. 147 (1998) 585–650.

[14] L.-S. Young, *Recurrence times and rates of mixing*, Israel J. Math. 110 (1999) 153–188.