Gauge invariant spectral Cauchy characteristic extraction

Casey J Handmer¹, Béla Szilágyi¹ and Jeffrey Winicour²

¹Theoretical Astrophysics 350-17, California Institute of Technology, Pasadena, CA 91125, USA
²Department of Physics and Astronomy, University of Pittsburgh, Pittsburgh, PA 15260, USA

E-mail: chandmer@caltech.edu

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Abstract
We present gauge invariant spectral Cauchy characteristic extraction. We compare gravitational waveforms extracted from a head-on black hole merger simulated in two different gauges by two different codes. We show rapid convergence, demonstrating both gauge invariance of the extraction algorithm and consistency between the legacy Pitt null code and the much faster spectral Einstein code (SpEC).

Keywords: spectral, gravitational radiation, general relativity, numerical methods, gauge invariance, characteristic extraction, waveform

(Some figures may appear in colour only in the online journal)

1. What is CCE? What is gravitational waveform gauge invariance?

The strong gravitational radiation produced in the inspiral and merger of binary black holes has been a dominant motivation for the construction of gravitational wave observatories. The details of the gravitational waveform supplied by numerical simulation is a key theoretical tool to fully complement the sensitivity of the LIGO, Virgo, GEO, and KAGRA observatories, by enhancing detection and providing useful scientific interpretation of the gravitational signal [1–4]. Characteristic evolution coupled to Cauchy characteristic extraction (CCE) provides the most accurate numerical computation of the Bondi news function, which determines both the waveform and the radiated energy and momentum at null infinity.

In CCE, the Cauchy evolution is used to supply boundary data on a timelike inner worldtube necessary to carry out a characteristic evolution extending to future null infinity \( J^+ \), where the radiation is computed using the geometric methods developed by Bondi et al.
as depicted in figure 1. More intuitive methods, including intrinsically inertial compactified hyperboloidal formulations [8–10] have not yet found adoption in the evolution of binary black holes. CCE is an initial-boundary value problem based upon a timelike worldtube [11]. It has been implemented as a characteristic evolution code, the Pitt null code [12, 13], which incorporates a Penrose compactification of the spacetime, and which has subsequently been extended to higher order methods by Reisswig et al [14]. It has more recently been implemented as a spectral code within the spectral Einstein code (SpEC) by Handmer and Szilágyi [15], upon which the present work is based.

One technical complication introduced by CCE is that the coordinates induced on $I^+$ by the computational Cauchy coordinates on the inner worldtube do not correspond to inertial observers, i.e., to the coordinates intrinsic to a distant freely falling and non-rotating observatory. The gravitational waveform first obtained in the ‘computational coordinates’ of CCE is in a scrambled form. This gauge ambiguity in the waveform is removed by constructing the transformation between computational coordinates and inertial coordinates at $I^+$. There still remains the freedom in the choice of inertial observers. In special relativistic theories, this freedom is reduced to the translations and Lorentz transformations of the Poincaré group. As explained in section 3, in an asymptotically flat space time the corresponding asymptotic symmetry group consists of supertranslations and Lorentz transformations. This freedom governs the redshift and initial phase of the waveform.

A physically relevant calculation of the radiation flux must also be referred to such inertial coordinates at $I^+$. In this paper, the calculation of the energy–momentum flux via the Bondi news function is first carried out in the induced worldtube coordinates and then transformed to the inertial coordinates.
2. Characteristic formalism

The characteristic formalism is based upon a family of outgoing null cones emanating from an inner worldtube and extending to infinity where they foliate \( \mathcal{I}^+ \) into spherical slices. We let \( u \) label these hypersurfaces, \( y^A = (u, r, y^A) \) be angular coordinates that label the null rays, and \( r \) be a surface area coordinate along the outgoing null cones.

Employing the conventions used in [15], in the resulting coordinates, the metric takes the Bondi–Sachs form

\[
\begin{align*}
\text{ds}^2 &= - \left( e^{2\beta}(rW + 1) - r^2 h_{AB} U^A U^B \right) du^2 \\
&\quad - 2e^{2\beta} dudr - 2r^2 h_{AB} U^B dudy^A + r^2 h_{AB} dy^A dy^B,
\end{align*}
\]

where \( h^{AB} h_{BC} = \delta^A_C \) and \( \det(h_{AB}) = \det(q_{AB}) \), with \( q_{AB} \) a unit sphere metric. In analyzing the Einstein equations, we also use the intermediate variable

\[
Q_A = r^2 e^{-2\beta} h_{AB} U^B.
\]

The metric coefficients \( W, h_{AB}, U^A, Q_A, \beta \) represent respectively the mass aspect, the spherical two-metric, the shift and its radial derivative, and the lapse. The vector and tensor fields \( h_{AB}, U^A, Q_A \) are expressed as spin-weighted fields by contracting them with the complex dyad \( q_A \) for the unit sphere metric satisfying

\[
\begin{align*}
q_A q_A &= 1, \\
q_A q_B &= \frac{1}{2} \delta^{AB} + \frac{1}{2} \delta^{AB} q^C \Omega_{CB},
\end{align*}
\]

with \( \Omega_{AB} = \delta_{AB} - q_A q_B / 2 \). Under this convention, the spin-weighted functions \( U = U^A q_A \) and \( Q = Q_A q_A \), while \( J = h_{AB} q^A q^B / 2 \) uniquely determines the spherical two-metric component of the general four-metric [13]. We chose a dyad consistent with the computational formulation of the spin-weight raising \( \delta \) operator [16], given by \( q_A = (-1, -i \sin \theta) \) in standard spherical coordinates \((\theta, \phi)\). This is regular everywhere except the poles, which we can avoid through careful choice of grid points. It is worth noting that any choice of angular coordinates are possible. Other conventions use multiple patches to avoid singularities at the poles.

A key feature of the Bondi–Sachs formulation is that the Einstein equations can be integrated along the outgoing characteristics in a sequential order. We use a form which first appeared in [17] and was implemented as the Pitt code in [13, 18]:

\[
\begin{align*}
\beta_{,\tau} &= N_{,\tau}, \\
(r^2 Q)_{,\tau} &= - r^2 \left( \delta_{,\tau} J + \delta_{,\tau} K \right)_{,\tau} + 2 r^4 \delta \left( r^2 \beta_{,\tau} \right)_{,\tau} + N_{,\tau}, \\
U_{,\tau} &= r^2 e^{2\beta} Q + N_{,\tau}, \\
(r^2 W)_{,\tau} &= \frac{1}{2} e^{2\beta} R - 1 - e^{3\beta} \delta \delta e^{3\beta} + \frac{1}{4} r^{-2} \left( r^4 \left( \delta^2 U + \delta U \right) \right)_{,\tau} + N_{,\tau},
\end{align*}
\]

and the evolution equation

\[
2(rJ)_{,\tau} = \left( (1 + rW) (rJ)_{,\tau} - r^{-1} \left( r^2 \delta U \right)_{,\tau} + 2 r^{-1} e^{3\beta} \delta^2 e^{\beta} - (rW)_{,\tau} \right) J + N_{,\tau},
\]

where

\[
R = 2 K - \delta \delta K + \frac{1}{2} \left( \delta^2 J + \delta^2 J \right) + \frac{1}{4K} \left( \delta J \delta J - \delta J \delta J \right),
\]

is the curvature scalar associated with \( h_{AB} \). \( K^2 = 1 + JJ \) and \( N_{,\tau}, N_{,\tau}, N_{,\tau}, N_{,\tau} \) are nonlinear terms given in [13].

On each constant \( u \) hypersurface of the spacetime foliation, these equations are integrated in turn. Given \( J, \beta \) is solved, then \( U, Q, \) and \( W \) in turn, enabling the computation of \( J_{,\tau}, J_{,\tau} \).
permits a step forward in time and $J$ is thus defined on the next hypersurface. The radial compactification of infinity is given by

$$r = r_m \rho / (1 - \rho), \quad \frac{1}{2} \leq \rho \leq 1,$$

(2.9)

where the compactification parameter $r_m (u, x^A)$ is the (not necessarily constant) areal radius coordinate on the worldtube.

Angular derivatives are implemented using the action of the $\delta$ operator on spin-weighted spherical harmonics, e.g., $\partial U = \delta^A_B U_A$, where a colon denotes the covariant derivative with respect to $g_{AB}$ [16]. In spherical coordinates, this takes the explicit form for a spin-weight-s field

$$\partial_{\eta} = -\left( \sin^s \theta \right) \left( \frac{\partial}{\partial \theta} + \frac{i}{\sin \theta} \frac{\partial}{\partial \phi} \right) \left( \sin^{-s} \theta \eta \right),$$

(2.10)

$$\tilde{\partial}_{\eta} = -\left( \sin^{-s} \theta \right) \left( \frac{\partial}{\partial \theta} - \frac{i}{\sin \theta} \frac{\partial}{\partial \phi} \right) \left( \sin^s \theta \eta \right).$$

(2.11)

$\tilde{\partial}$ is the associated spin-weight lowering operator.

The spectral algorithm used to solve these equations and the treatment of the nonlinear terms $N_3, N_4, N_5, N_6, N_7$ are detailed in Handmer and Szilágyi [15]. Here, we extend the characteristic spectral algorithm to calculating the gauge invariant Bondi news at $I^+$.  

3. Waveforms at $I^+$

For technical simplicity, the theoretical derivation of the waveform at infinity is best presented in terms of an inverse surface-area coordinate $\ell = 1/r$, where $\ell = 0$ at $I^+$. In the resulting $x^\mu = (u, \ell, x^A)$ conformal Bondi coordinates, the physical spacetime metric $g_{\mu
u}$ has the conformal compactification $\hat{g}_{\mu
u} = \ell^4 g_{\mu
u}$, where $\hat{g}_{\mu
u}$ is smooth at $I^+$ and, referring to the metric (2.1), takes the form [11]

$$\hat{g}_{\mu\nu} dx^\mu dx^\nu = - \left( e^{2/4(\ell^2 + \ell W)} - h_{AB} U^A U^B \right) du^2 + 2 e^{2/4} du d\ell - 2 h_{AB} U^B du dx^A + h_{AB} dx^A dx^B.$$  

(3.1)

As described in [19, 20], both the Bondi news function $N(u, x^A)$ and the Newman–Penrose Weyl tensor component [21]

$$\Psi^0 \left( u, x^A \right) = \lim_{r \to \infty} r \sqrt{3},$$

(3.2)

which describe the waveform, are determined by the asymptotic limit at $I^+$ of the tensor field

$$\hat{\Sigma}_{\mu\nu} = \ell \left( \hat{\nabla}_\mu \hat{\nabla}_\nu - \frac{1}{4} \hat{g}_{\mu\nu} \hat{\nabla}_\lambda \hat{\nabla}^\lambda \ell \right).$$

(3.3)

This limit is constructed from the leading coefficients in an expansion of the metric about $I^+$ in powers of $\ell$. We thus write

$$h_{AB} = H_{AB} + \ell e_{AB} + O \left( \ell^2 \right).$$

(3.4)

Conditions on the asymptotic expansion of the remaining components of the metric follow from the Einstein equations:
\[ \beta = H + O(\ell^2), \quad (3.5) \]
\[ U^A = L^A + 2\ell e^{2H}H^{AB}D_B H + O(\ell^2), \quad (3.6) \]
and
\[ W = D_A L^A + \ell \left( e^{2H} R/2 + D_A D^A e^{2H} - 1 \right) + O(\ell^2), \quad (3.7) \]
where \( H \) and \( L \) are the asymptotic limits of \( \beta \) and \( U \) and where \( R \) and \( D_A \) are the two-dimensional curvature scalar and covariant derivative associated with \( H_{AB} \).

The expansion coefficients \( H, H_{AB}, c_{AB}, \) and \( L^A \) (all functions of \( u \) and \( x^A \)) completely determine the radiation field. One can further specialize the Bondi coordinates to be inertial at \( \mathscr{I}^+ \), i.e., have Minkowski form, in which case \( H = L = 0, H_{AB} = g_{AB} \) (the unit sphere metric) so that the radiation field is completely determined by \( c_{AB} \). However, the characteristic extraction of the waveform is carried out in computational coordinates (determined by the Cauchy data on the extraction worldtube) so this inertial simplification cannot be assumed.

In order to first compute the Bondi news function in the \( \tilde{g}_{\mu\nu} \) computational frame, it is necessary to determine the conformal factor \( \omega \) relating \( H_{AB} \) to a unit sphere metric \( Q_{AB} \), i.e., to an inertial conformal Bondi frame \([11]\) satisfying
\[ Q_{AB} = \omega^2 H_{AB}. \quad (3.8) \]
(See \([22]\) for a discussion of how the news in an arbitrary conformal frame is related to its expression in this inertial Bondi frame.) We can determine \( \omega \) by solving the elliptic equation governing the conformal transformation of the curvature scalar (2.8) to a unit sphere geometry:
\[ R = 2\left( \omega^2 + H^{AB}D_A D_B \log \omega \right). \quad (3.9) \]
The elliptic equation (3.9) need only be solved at the initial time where, with initial data \( J \mid_{\mathscr{I}^+} = 0, H^{AB}D_A D_B \) simplifies to the two-Laplacian on the unit sphere. Then, as described in the next section, application of the Einstein equations on \( \mathscr{I}^+ \) determines the time dependence of \( \omega \) according to
\[ 2\tilde{\eta}^\mu \partial_\mu \log \omega = -e^{-2H} D_A L^A. \quad (3.10) \]
where \( \tilde{\eta}^\mu = \tilde{g}^{\mu\nu} \nabla_\nu \ell \) is the null vector tangent to the generators of \( \mathscr{I}^+ \). We use (3.10) to evolve \( \omega \) along the generators of \( \mathscr{I}^+ \) given a solution of (3.9) as initial condition.

First recall some basic elements of Penrose compactification. In a general conformal frame with metric \( \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \), where \( \Omega = 0 \) on \( \mathscr{I}^+ \), the vacuum Einstein equations \( G_{\mu\nu} = 0 \) take the form
\[ \Omega^2 G_{\mu\nu} + 2\Omega \nabla_\mu \nabla_\nu - \tilde{g}_{\mu\nu} \left( 2\Omega \nabla_\rho \nabla_\sigma - 3 \left( \nabla_\rho \Omega \nabla_\sigma \Omega \right) \right) = 0. \quad (3.11) \]
It immediately follows that
\[ \left. \left( \nabla_\mu \Omega \right) \nabla_\nu \Omega \right|_{\mathscr{I}^+} = 0, \quad (3.12) \]
so that \( \mathscr{I}^+ \) is a null hypersurface and that
\[ \left. \left[ \nabla_\mu \nabla_\nu \Omega - \frac{1}{4} \tilde{g}_{\mu\nu} \nabla_\rho \nabla_\sigma \Omega \right] \right|_{\mathscr{I}^+} = 0. \quad (3.13) \]

With respect to this frame, the construction of an inertial conformal frame proceeds as follows. We introduce a new conformal factor \( \tilde{\Omega} = \omega \Omega \), with \( \tilde{g}_{\mu\nu} = \omega^2 \tilde{g}_{\mu\nu} \) by requiring, in accord with (3.10),
As a result, it follows from a straightforward calculation that
\[ \nabla^\rho \nabla_\rho \hat{\Omega} \big|_{\mathcal{J}^+} = 0, \]
which means in the \( g_{\mu\nu} \) conformal frame \( \mathcal{J}^+ \) is null, shear- and divergence-free.

It also follows that
\[ \hat{\rho}_\mu \partial_\mu \hat{x}^A \big|_{\mathcal{J}^+} = 0, \]
where \( \hat{\rho}_\mu = \hat{g}^{\mu\nu} \hat{\nabla}_\nu \hat{\Omega} \), i.e., in the \( g_{\mu\nu} \) frame, \( \hat{\rho}_\mu \) is an affinely parametrized null generator of \( \mathcal{J}^+ \).

To construct inertial coordinates \( (\bar{u}, x^A) \) on \( \mathcal{J}^+ \), we first assign angular coordinates \( x^A \) to each point of the initial spacelike spherical slice \( u = u_0 \) of \( \mathcal{J}^+ \). We then propagate these coordinates along the generators of \( \mathcal{J}^+ \) according to
\[ \hat{\rho}_\mu \partial_\mu \hat{\rho}_\mu \big|_{\mathcal{J}^+} = 0, \]
so that \( \hat{\rho} \) is an affine parameter along the generators in the \( g_{\mu\nu} \) conformal frame.

4. News

The Bondi news function \( N \) is computed in the computational coordinates with the appropriate conformal transformation. It is then interpolated onto the inertial coordinates. The formalism follows that of [13], appendix B (with a sign error in \( s_3 \) corrected):
\[ N = \frac{1}{4\omega A} \left( s_1 + s_2 + \frac{1}{4} (\partial U + \partial U) s_3 - 4\omega^{-2} s_4 + 2\omega^{-4} s_5 \right), \]
where \( A = \omega^{2/3} \) and the \( s_i \) terms are
\[ s_1 = \left( J^2 J_{tu} + J J_{tu} - 2 J K K_{tu} - 2 J K_{tu} + 2 J_{tu} K + 2 J_{tu} \right)/(K + 1), \]
\[ s_2 = \left( \partial J J \hat{U} + 2 \partial J K \hat{U} + 2 \partial J \hat{U} + \partial J J \hat{J} \hat{U} \right) / (K + 1), \]
\[ s_3 = - \left( J^2 J_{t\ell} + J J_{t\ell} - 2 J K K_{t\ell} - 2 J K_{t\ell} + 2 J_{t\ell} K + 2 J_{t\ell} \right)/(K + 1), \]
\[ s_4 = \left( \partial \partial \omega J J + 2 \partial \partial \omega K + 2 \partial \partial \omega - \partial \partial \omega J K \right) / (K + 1). \]
\[
\begin{align*}
\Delta s_5 &= \left( 2\Delta^2 A J J + 4\Delta^2 A K + 4\Delta^2 A L + 2\Delta^2 A J^2 - 4\Delta A J K \\
&- 4\Delta A J + \partial A \partial J J^2 + 2\partial A \partial J K J^2 + 2\partial A \partial J J^2 - 4\partial A \partial K J K^2 J \\
&+ 2\partial A \partial J K J K + 2\partial A \partial J J J - 2\partial A \partial K J J K - 4\partial A \partial K J J - 4\partial A \partial K K \\
&- 4\partial A K - \partial A \partial J J J K + 2\partial A \partial J K + 2\partial A \partial J - \partial A \partial J J^2 K \\
&+ 2\partial A \partial K J^2 J - \partial J \partial A J J K - 2\partial J \partial A J J - 2\partial J \partial A K \\
&- 2\partial J \partial A - \partial J \partial A J J^2 K - 2\partial J \partial A J J + 2\partial K \partial A J^2 J \\
&+ 4\partial K \partial A J K + 4\partial K \partial A J + \partial A \partial J J^2 J \\
&+ \partial A \partial J J^3 - 2\partial A \partial K J^2 K)/(4(K + 1)).
\end{align*}
\]

(4.2)

In our implementation, \(s_5\) derivatives are derived from spectrally calculated \(s_p\) derivatives using the appropriate Jacobian.

5. Results

In our comparison tests of CCE, the worldtube boundary data were extracted from a simulation of an equal mass non-spinning head-on black hole collision, with initial separation of 30\(M\). The control run (Isotropic) utilized the standard harmonic gauge damping identical to that in [23] throughout the head-on merger and ring-down. Harmonic gauge damping adds a dissipative forcing term to the wave equations satisfied by the harmonic Cartesian spatial coordinates \((x, y, z)\). In order to diminish the effects of a custom designed gauge, we also compare with results of another run (HyZero) which turns off gauge damping in the harmonic \(y\)-direction, transverse to the \(x\)-direction motion of the black holes. These two high-resolution runs were used as boundary data for all the subsequent CCE runs. These runs include three different resolutions, two different codes, two different gauges, and three different extraction radii, for a total of 36 runs.

As described in [15], the SpEC characteristic evolution algorithm exploits spectral methods and innovative integral methods that greatly improve upon the speed and accuracy of the Pitt null code. This is seen as essential for for taking advantage of the efficiency of SpEC Cauchy evolution. The necessary improvement in efficiency has been preserved in the SpEC extraction module, as displayed in table 1. The comparison runs were performed using the current version of the Pitt code [20], which forms part of the Einstein toolkit.

The initial conditions and extraction parameters were deliberately chosen as a stressful test of the algorithms. In particular, at the beginning of the run the black hole excision boundary extends out to Cartesian radius \(R = 16M\), which is very close to our smallest choice of extraction radius at \(R = 30M\). At this radius, gauge effects are highly significant and would make perturbative extraction schemes meaningless, in accordance with our intentions. One consequence of such an extreme choice is that differences between the Pitt and SpEC inertial frame and worldtube initialization procedures lead to noticeably different waveforms. Worldtube initialization involves supplying the ‘integration constants’ from the Cauchy code, which allows radial integration of the characteristic hypersurface and evolution equations (2.3)–(2.7) from the worldtube to \(r^+\). In both Pitt null code and SpEC, the initial condition on \(J\) is determined by the inner boundary value, supplied by the Cauchy evolution, with a smooth roll off to zero at \(r^+\).

The extraction worldtube \(\Gamma\) is determined by a surface of constant Cartesian radius \(R\). In the Pitt CCE code, the areal radius \(r_{wt}\) of \(\Gamma\) lies between two surfaces of constant Cartesian
radii \( R_1 \leq r_{\text{ext}} \leq R_2 \) and this carries over to the compactified radial coordinate. As a result, interpolation is necessary to supply the integration constants, which introduces numerical error. In the SpEC CCE code, this interpolation error is avoided by introducing the compactified radial coordinate \((2.9)\), with range \( 1/2 \leq \rho \leq 1 \) between \( \Gamma \) and \( \mathcal{I} \).

Worldtube data from each run were extracted using both Pitt and SpEC CCE, at three different Cartesian radii: \( R = 30M \), \( R = 100M \) and \( R = 250M \), as illustrated by the news function waveforms in figures 2, 4, and 6, respectively. In these figures, the HytZero and Isotropic waveforms are so close that they appear on top of one another. The major discrepancy between the Pitt and SpEC waveforms is due to the worldtube interpolation error in the Pitt code. This is especially evident at small extraction radii, where there is strong ‘junk’ radiation near the worldtube, which is inherent in the initial Cauchy data and its mismatch with the initial characteristic data.

This interpolation error in the Pitt code converges away at larger radii, where the field gradients between \( R_1 \) and \( R_2 \) become smaller. This is seen in figures 3, 5, and 7, where the

**Figure 2.** Waveforms of the real part of the \((2, 2)\) spherical harmonic mode of the news function, as computed by the Pitt null code and SpEC with extraction worldtube at \( R = 30M \). Different initialization procedures at the worldtube give rise to a difference between the Pitt and SpEC waveforms, which is most pronounced at this small extraction radius. The different gauge choices, Isotropic and HytZero, do not have noticeable effect on this scale, indicating successful gauge effect removal in both codes.

**Table 1.** Resolution parameters used for code convergence comparisons, with time steps \( \Delta t \), \( N_r \) represents the radial grid sizes. The Pitt null code uses two stereographic patches with \( 2N^2 \) total number of angular grid points. The SpEC code has \( 2L^2 \) total angular grid points. \( T \) is the CPU time taken for \( R = 30M \), \( t_{\text{final}} = 450M \) runs in the Isotropic gauge, and is representative for the other runs. All resolutions and codes were run from the same initial data.

| Run       | Pitt1 | Pitt2 | Pitt3 | SpEC1 | SpEC2 | SpEC3 |
|-----------|-------|-------|-------|-------|-------|-------|
| \( N_r \) | 100   | 150   | 200   | 10    | 12    | 14    |
| \( N \) or \( L \) | 40    | 60    | 80    | 12    | 14    | 17    |
| \( \Delta t/M \) | 0.1   | 0.0666... | 0.05 | 1.0   | 0.666... | 0.5   |
| \( T \) (CPU hours) | 173   | 274   | 374   | 0.7   | 1.9   | 3.1   |
Each run was computed at three different resolutions to monitor convergence, as indicated in table 1. In the following subsections, we first show convergence and the removal of gauge effects, separately for the Pitt and SpEC codes. Next, we compare $\Psi^2$ waveforms and establish further agreement between the two codes. Finally, we examine the evolution of the relative difference between the Pitt and SpEC news function waveforms is compared with the relative numerical error implied by convergence tests.

Figure 3. Graphs showing the relative difference between the real part of the $(2, 2)$ mode Pitt and SpEC news function waveforms for extraction radius $R = 30M$, in comparison to the relative numerical error implied by convergence tests, corresponding to the waveforms in figure 2. While both SpEC and Pitt have comparable and consistent levels of error, the codes do not agree within that level of error at this extraction radius.

Figure 4. Waveforms of the real part of the $(2, 2)$ mode of the news function, as computed by the Pitt and SpEC codes with extraction worldtube at $R = 100M$. Compared to figure 2, at this larger extraction radius the worldtube initialization differences lead to a much smaller difference between waveforms, which appear nearly identical here. The main discrepancy arises from the treatment of the junk radiation at early times. Here, too, gauge differences between HytZero and Isotropic are not visible at this scale.
inertial coordinates at $\mathcal{I}^+$ relative to the worldtube coordinates induced by the Cauchy evolution.

Comparison of the relative error $E_{\text{rel}}$ between dataset $A$ and dataset $B$ is computed according to

$$E_{\text{rel}} = \log_{10} \left( \frac{|A - B|}{|B|} \right), \quad (5.1)$$

where in convergence tests $B$ is the highest resolution dataset, and the real parts of the $(\ell, m) = (2, 2)$ spherical harmonic modes are compared.

Figure 5. Graph showing the relative difference between the real part of the $(2, 2)$ mode SpEC and Pitt news waveforms for extraction radius $R = 100M$, and the relative numerical error, corresponding to the waveforms in figure 4. In comparison with figure 3, by $R = 100M$ the difference between the Pitt and SpEC algorithms has dropped to the level of numerical error in each algorithm.

Figure 6. Waveforms of the real part of the $(2, 2)$ mode of the news function, as computed by the Pitt and SpEC codes with extraction worldtube at $R = 250M$. At this large extraction radius there is a barely noticeable difference between all the waveforms, limited to the junk radiation at early times.
5.1. Pitt code convergence and removal of gauge effects

Here, in order to establish a baseline, we examine the self convergence of the Pitt code for each of the extraction radii, using the three resolutions (Pitt1, Pitt2, Pitt3) indicated in table 1. In figures 8–10, we see in both the Isotropic and HytZero gauges that the news function converges over the entire run. Indeed, Isotropic (solid lines) and HytZero (dashed lines) overlap completely. The figures also plot the relative error in the news computed in both gauges, which is consistently below the numerical error implied by convergence tests for extraction worldtubes at $R = 30M$ and $R = 100M$. This verifies that the Pitt code successfully removes gauge effects. Furthermore, the figures plot the relative error between the news computed in the worldtube coordinates and the inertial coordinates at $I^+$. In the $R = 30M$ case shown in figure 8, the initial discrepancy is high due to the strong gauge effects of junk radiation. It does not fall below the relative error between the Isotropic and HytZero gauges until well after the signal has passed. This confirms that the transformation to inertial coordinates is essential for correctly removing gauge effects from the waveform. For extraction at $R = 100M$ shown in figure 9, the relative error between worldtube and inertial coordinates has dropped below the Isotropic-HytZero gauge effect. At $R = 250M$ shown in figure 10, the predominant error is the Isotropic-HytZero gauge effect.

These results show that the selected runs do produce a substantial gauge error between the worldtube and inertial coordinates and that the Pitt code effectively removes it, while remaining convergent for the duration of the run.

5.2. SpEC code convergence and removal of gauge effects

Here we examine the SpEC code’s self convergence for each extraction radii, in the same way that the Pitt code was examined in section 5.1. In figures 11–13, we see that convergence, measured with the resolutions indicated in table 1, is comparable to the Pitt code’s convergence, while the potential gauge contamination is consistently removed at all worldtube
Figure 8. Graphs of the relative error $\log_{10} |\Delta N_{22}/N_{22}|$ in the $(2, 2)$ mode of the news function for the $R = 30M$ Pitt extraction run in both gauges. The relative errors for the Pitt1 (low) and Pitt2 (high) resolutions (compared to Pitt2 and Pitt3 respectively) are rescaled to demonstrate convergence. The dashed blue line indicates the relative error (Isotropic versus HytZero) between the news computed in both gauges. The dotted-dashed purple line (Inertial versus worldtube) indicates the relative error between the news computed in the worldtube coordinates and the inertial coordinates. At this small extraction radius, this discrepancy is high due to the strong gauge effect of junk radiation.

Figure 9. Graphs of the relative error $\log_{10} |\Delta N_{22}/N_{22}|$ in the news function for Pitt extraction at $R = 100M$. Again, the errors for the Pitt1 and Pitt2 resolutions demonstrate convergence. Compared to figure 8, the relative error (Inertial versus worldtube) between inertial and worldtube coordinates has now dropped below the Isotropic versus HytZero gauge effect.
radii. As in figures 8–10, the solid lines (Isotropic) and dashed lines (HytZero) overlap due to consistency in gauge removal. The SpEC extraction code effectively removes gauge error at all radii while remaining convergent throughout the runs.

5.3. Comparison of \( \Psi^4_0 \) between the Pitt and SpEC codes

In sections 5.1 and 5.2, we have shown that both codes are convergent and remove potential gauge effects. We have also demonstrated that the difference in the news computed by the
two codes disappears as the extraction worldtube radius increases. Here we provide further
evidence that even at a small worldtube radius the waveform computed by the SpEC code is
valid.

After the gauge freedom is removed by extraction, there is still supertranslation and
Lorentz freedom in the choice of inertial coordinates, which affect the phase and velocity of
the inertial observers. This effect is highly sensitive to initial conditions and also to the
 evolution of the inertial conformation transformation factor $\omega$, especially in the extreme gauge
conditions of extraction at $R = 30M$. It feeds into the worldtube interpolation error in the Pitt
code. In order to verify that the discrepancy illustrated in figure 2 between the news computed
by the Pitt null code and SpEC is partially due to this inertial coordinate freedom, we compute
the time derivative of the news, which is related to the Weyl curvature in inertial coordinates

\[
\frac{\Delta N_{22}/N_{22}}{2^n} \log_{10} 2
\]

Figure 12. Graphs of the relative error $\log_{10} [\Delta N_{22}/N_{22}]$ in the news function for the
$R = 100M$ SpEC extraction run in both gauges. The graphs show convergence in both
gauges. The Isotropic versus HytZero gauge error is relatively small.

Figure 13. Graphs of the relative error $\log_{10} [\Delta N_{22}/N_{22}]$ in the news function for the
$R = 250M$ SpEC extraction run in both gauges, showing convergence in both gauges as
well as small gauge error throughout the run.
This suppresses phase differences between the two waveforms. In making the comparisons, \( \Psi^4 \) is computed semi-independently using the Weyl tensor waveform module in the current version of the Pitt code [20]. In these runs, \( \Psi^4 \) was found to be convergent with truncation error comparable to the consistency between \( \Psi^4 \) and \( \partial_t N \) in the Pitt code.

In figure 14, we see that the time derivative of the news and \( \Psi^4 \) have much less discrepancy than figure 2 would suggest. In figures 15, 17, and 19, we compare relative errors according to \( \partial_t N = \Psi^4 \). This suppresses phase differences between the two waveforms. In making the comparisons, \( \Psi^4 \) is computed semi-independently using the Weyl tensor waveform module in the current version of the Pitt code [20]. In these runs, \( \Psi^4 \) was found to be convergent with truncation error comparable to the consistency between \( \Psi^4 \) and \( \partial_t N \) in the Pitt code.
between $\Psi_4^0$ and $\partial_t N$ computed by the Pitt and SpEC codes. Not only is there agreement between the codes at $R = 30M$, this agreement persists for larger extraction radii, as shown in figures 16 and 18. Both codes show comparable agreement throughout the run.

5.4. Relative motion between inertial and worldtube coordinates

In section 3, we discussed the construction of an inertial coordinate system and its evolution with respect to the worldtube coordinates constructed from the Cartesian coordinates of the Cauchy code. Here, we describe the motion of the inertial $(\theta, \phi)$ angular coordinates relative
to the worldtube angular coordinates, constructed in the standard way from the worldtube Cartesian coordinates. Figure 20 illustrates the global pattern of this relative motion for the Isotropic gauge SpEC run with the highest resolution extraction at \( R = 30 \, M \). Generally speaking, the coordinates wiggle back and forth in the direction corresponding to the motion of the black holes. The complete movement amounts to at most a few percent of their initial values, but even this is sufficient to introduce considerable gauge error in the waveform, as already seen in figure 8.

In figure 21, the relative \( \phi \) motion of the point circled in figure 20 is plotted as a function of inertial time. Initial junk radiation causes considerable wobble, followed by a smooth return almost to its starting point. The maximum excursion of the \( \phi \)-coordinate shift from its initial value is about 3.5%.

**Figure 18.** Waveforms of \( \Psi^0_4 \) and \( \partial_t N \) as computed for the SpEC and Pitt codes using the Isotropic gauge with extraction radius \( R = 250 \, M \).

**Figure 19.** The relative error \( \log_{10} \left[ \Delta \Psi^0_4 / \Psi^0_4 \right] \) between \( \Psi^0_4 \) and \( \partial_t N \) computed for the SpEC and Pitt runs using the Isotropic gauge with extraction radius \( R = 250 \, M \), corresponding to the waveforms in figure 18. Both codes show good agreement throughout the run.
In addition to the head-on collision tests which we have described, we have also investigated stability and convergence of the Pitt and SpEC CCE modules, together with the inertial-worldtube coordinate transformation, using the generic test run of precessing, spinning binary black holes in [15], as taken from Taylor et al [24]. Its parameters are mass ratio $q = 3$, black hole spins $c_1 = (0.7, 0, 0.7)/\sqrt{2}$ and $c_2 = (-0.3, 0, 0.3)/\sqrt{2}$, number of orbits 26, total time $T = 7509M$, initial eccentricity $10^{-3}$, initial frequency $\omega_{ini} = 0.032/M$, and extraction radius $R = 100M$. The Pitt and SpEC waveforms displayed in figure 22 are fairly typical waveforms, spanning the initial junk radiation through inspiral, merger, and ringdown.

Figures 23 and 24 show a log scale comparison of the waveforms with absolute error. The codes agree strongly throughout the run.
The relative inertial-worldtube coordinate motion of a representative point in the extended generic run is illustrated in figure 25. At early times, the helix has two loops per cycle corresponding to each of the black holes. At later times, precession dominates the evolution of this particular coordinate. Throughout the run, the deviation is around 0.5%.

6. Conclusion

The SpEC characteristic evolution algorithm has now been furnished with a convergent, efficient news extraction module. SpEC is now capable of rapidly producing accurate, gauge free waveforms as required.
Figure 24. Same graphs as in figure 23 but focusing on the merger part of the waveform. Absolute error shows consistency between the Pitt and SpEC news waveforms through merger and ringdown.

Figure 25. Relative motion of a representative angular coordinate taken from SpEC with extraction radius $R = 100M$ showing a long term helical pattern concordant with the black hole inspiral and merger. Globally, initial oscillation due to junk radiation is aligned primarily along the Cartesian $x$ direction, corresponding to the initial black hole orientation. Coordinate motion is an epicyclic helix whose amplitude is modulated by precession of the orbital plane.
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