On the total $H$-irregularity strength of graphs: A new notion

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Abstract. A total edge irregularity strength of $G$ has been already widely studied in many papers. The total $\alpha$-labeling is said to be a total edge irregular $\alpha$-labeling of the graph $G$ if for every two different edges $e_1$ and $e_2$, it holds $w(e_1) \neq w(e_2)$, where $w(uv) = f(u) + f(uv) + f(v)$, for $e = uv$. The minimum $\alpha$ for which the graph $G$ has a total edge irregular $\alpha$-labeling is called the total edge irregularity strength of $G$, denoted by $t\alpha(G)$. A natural extension of this concept is by considering the evaluation of the weight is not only for each edge but we consider the weight on each subgraph $H \subseteq G$. We extend the notion of the total $\alpha$-labeling into a total $H$-irregular $\alpha$-labeling. The total $\alpha$-labeling is said to be a total $H$-irregular $\alpha$-labeling of the graph $G$ if for $H \subseteq G$, the total $H$-weights $W(H) = \sum_{v \in V(H)} f(v) + \sum_{e \in E(H)} f(e)$ are distinct. The minimum $\alpha$ for which the graph $G$ has a total $H$-irregular $\alpha$-labeling is called the total $H$-irregularity strength of $G$, denoted by $t\alpha s(G)$. In this paper we initiate to study the $t\alpha s(G)$ of shackle and amalgamation of any graphs and their bound.

Keywords: Total $\alpha$-labeling, Total $H$-irregularity strength, shackle of any graph, amalgamation of any graph.

1. Introduction

All graphs in this paper are simple, nontrivial and undirected graphs. A total labeling $f : V(G) \cup E(G) \rightarrow \{1, 2, 3, \ldots, \alpha\}$ is called a total $\alpha$-labeling of a graph $G$. The weight of an edge $uv$ of $G$, denoted by $w(uv)$, is the sum of the labels of end vertices $u$ and $v$ and also edge $uv$, i.e. $w(uv) = f(u) + f(uv) + f(v)$. The total $\alpha$-labeling is said to be a total edge irregular $\alpha$-labeling of the graph $G$ if for every two different edges $e_1$ and $e_2$, it holds $w(e_1) \neq w(e_2)$. The minimum $\alpha$ for which the graph $G$ has a total edge irregular $\alpha$-labeling is called the total edge irregularity strength of $G$, denoted by $t\alpha s(G)$. A natural extension of this concept is by considering the evaluation of the weight is not only for each edge but we consider the weight on each subgraph $H \subseteq G$. Thus, we extend the notion of the total $\alpha$-labeling into a total $H$-irregular $\alpha$-labeling. The total $\alpha$-labeling is said to be a total $H$-irregular $\alpha$-labeling of the graph $G$ if for $H \subseteq G$, the total $H$-weights $W(H) = \sum_{v \in V(H)} f(v) + \sum_{e \in E(H)} f(e)$ are distinct. The minimum $\alpha$ for which the graph $G$ has a total $H$-irregular $\alpha$-labeling is called the total $H$-irregularity strength of $G$, denoted by $t\alpha s(G)$. The minimum $\alpha$ for which the graph $G$ has a subgraph irregular total $\alpha$-labeling is called the total $H$-irregularity strength of $G$, $t\alpha s(G)$.
The beginning of the study of the irregularity strength is introduced by Togni et al. [10] and Frieze et al. [4]. By then, there are some result related to the total $H$-irregularity strength study. Jendrol et al. [6] determined the total edge irregularity strength of complete and bipartite complete graph, Jeyanthi et al. [7] studied about total edge irregularity strength of disjoin union wheel graph, and Baca et al. [2], [3] studied about total edge irregularity strength of generalized of prism graph and any graphs. Furthermore Ahmad et al. [1] found total edge irregularity strength of wheel graph, and Baca et al. [2], [3] studied about total edge irregularity strength of generelized series parallel graph. In this paper, we study the existence of the total $H$-irregularity labeling of some graph operations, namely shackle and amalgamation of graph $G$. A shackle of $G_1, G_2, \ldots, G_k$, denoted by $Shack(G_1, G_2, \ldots, G_k)$, is any graph constructed from non-trivial connected and ordered graphs $G_1, G_2, \ldots, G_k$ such that for every $1 \leq i, j \leq k$ with $|i-j| \geq 2$, $G_i$ and $G_j$ have no common vertex and for every $1 \leq i \leq k-1$, $G_i$ and $G_{i+1}$ share exactly one common vertex, called a linkage vertex, where the $k-1$ linkage vertices are all distinct. Meanwhile, let $\{G_1, G_2, \ldots, G_n\}$ be a finite collection of graphs and each $G_i$ has a fixed vertex $v_0$, or edge $e_0$, called a terminal vertex or edge, respectively [5]. The vertex-amalgamation of $G_1, G_2, \ldots, G_n$ denoted by $Amal\{G_i, v_0\}$, is formed by taking all the $G_i$‘s and identifying their terminal vertices. Similarly, the edge-amalgamation of $G_1, G_2, \ldots, G_n$, denoted by $Amal\{G_i, e_0\}$, is formed by taking all the $G_i$‘s and identifying their terminal edges. Furthermore, if $G_i$’ are isomorphic graphs then we denote such graphs as $Shack\{G, v, n\}$ and $Amal\{G, v, n\}$ for vertex, or $Shack\{G, e, n\}$ and $Amal\{G, e, n\}$ for edge. In this paper we will study the $tHs$ of shackle and amalgamation of any graphs and as well as determine their bound.

2. The Results

Prior to show the values of $tHs$ of those graphs, we will show the lower bound of $tHs$ in general graph by the following lemma.

**Lemma 2.1** Given a graph $H \subseteq G$. Let $p_H, q_H$ be respectively be number of vertices and edges of $H$ and $|H|$ be the number of subgraphs. The total $H$-irregularity strength satisfies

$$tHs(G) \geq \left\lceil \frac{p_H + q_H + |H| - 1}{p_H + q_H} \right\rceil$$

**Proof.** A total $\alpha$-labeling is a labeling $f : V(G) \cup E(G) \rightarrow \{1, 2, 3, \ldots, \alpha\}$. The $H$-irregularity total $\alpha$-labeling of graph $G$ is a total $\alpha$-labeling such that for each subgraph $H \subseteq G$, the weight $W(H) = \sum_{v \in V(K)} f(v) + \sum_{e \in E(K)} f(e)$ are all distinct. Furthermore, since we require the minimum $\alpha$ for which the graph $G$ has a total $H$-irregular $\alpha$-labeling, the set of the total $H$-weight should be consecutive, otherwise it will not give a minimum $tHs$. Thus, the set of total $H$ weight is $W(H) = \{p_H + q_H, p_H + q_H + 1, p_H + q_H + 2, \ldots, p_H + q_H + (|H| - 1)\}$. On the other hand the maximum possible $H$ weight of graph $G$ is at most $tHs(G)(p_H + q_H)$. It implies

$$tHs(G)(p_H + q_H) \geq p_H + q_H + |H| - 1$$

$$tHs(G) \geq \left\lceil \frac{p_H + q_H + |H| - 1}{p_H + q_H} \right\rceil$$

Since $tHs(G)$ should be integer, and we need a sharpest lower bound, it implies

$$tHs(G) \geq \left\lceil \frac{p_H + q_H + |H| - 1}{p_H + q_H} \right\rceil.$$

It completes the proof. \qed

Now, we are ready to show our main results.
Theorem 2.1 Let $G = \text{Shack}(H, v, n)$ be a shackle of any graph $H$. Then the total $H$-irregularity strength satisfies

$$tHs(\text{Shack}(H, v, n)) = \left\lfloor \frac{m + n + 1}{m + 2} \right\rfloor$$

where $p_H$ and $q_H$ are respectively the number of vertices and edges in subgraph $H \subseteq G$ and $m = p_H + q_H - 2$ and $n = |H|$.

Proof. The vertex set and edge set of the graph $\text{Shack}(H, v, n)$ can be split into two following sets: $V(\text{Shack}(H, v, n)) = \{v_{ij}; 1 \leq i \leq p_H - 2, 1 \leq j \leq n\} \cup \{x_k; 1 \leq k \leq n + 1\}$ and $E(\text{Shack}(H, v, n)) = \{e_{ij}; 1 \leq l \leq q_H, 1 \leq j \leq n\}$. Thus, the graph $\text{Shack}(H, v, n)$ has $|V(\text{Shack}(H, v, n))| = (n - 1)p_H + 1$, $|E(\text{Shack}(H, v, n))| = nq_H$. Since $m = p_H + q_H - 2$, then by Lemma 2.1, we have $tHs(\text{Shack}(H, v, n)) \geq \left\lfloor \frac{p_H + q_H + |H| - 1}{m + 2} \right\rfloor = \left\lfloor \frac{m + n + 1}{m + 2} \right\rfloor$. Thus, $tHs(\text{Shack}(H, v, n)) \geq \left\lfloor \frac{m + n + 1}{m + 2} \right\rfloor$.

Now we will show that $tHs(\text{Shack}(H, v, n)) \leq \left\lfloor \frac{m + n + 1}{m + 2} \right\rfloor$. Define $f$ as a vertex and edge labeling of graph $G$, $f : V(G) \cup E(G) \to \{1, 2, \ldots, \alpha\}$ by the following function.

$$f(x_k) = \left\lfloor \frac{k}{m + 2} \right\rfloor$$

$$f(v_{ij}) \cup f(e_{ij}) = \left\{ \left. \frac{j}{m + 2} \right\rfloor; 1 \leq j \leq m - t + 1, 1 \leq t \leq m \right\} \cup \{j - (m - 1) + 1; m - t + 2 \leq j \leq n, 1 \leq t \leq m\}$$

Under the labeling $f$, the total $H$-weight $W(H) = \sum_{v \in V(H)} f(v) + \sum_{e \in E(H)} f(e)$ is $W(H) = \{m + 2, m + 3, \ldots, m + n + 1\}$ forms a consecutive sequence. It implies the set of $H$-weights are distinct. By considering the above label $f$, the minimum $tHs(\text{Shack}(H, v, n))$ can be achieved by the following:

$$tHs(\text{Shack}(H, v, n)) \leq \left\lfloor \frac{j + t - (m + 1)}{m + 2} \right\rfloor + 1, \text{ for } j = n, t = m$$

$$= \left\lfloor \frac{n + m - m - 1}{m + 2} \right\rfloor + \left\lfloor \frac{m + 1}{m + 2} \right\rfloor$$

$$= \left\lfloor \frac{n + m - 1}{m + 2} \right\rfloor + \left\lfloor \frac{m + n + 1}{m + 2} \right\rfloor$$

Thus, $tHs(\text{Shack}(H, v, n)) \leq \left\lfloor \frac{m + n + 1}{m + 2} \right\rfloor$. It concludes that $tHs(\text{Shack}(H, v, n)) = \left\lfloor \frac{m + n + 1}{m + 2} \right\rfloor$.

Theorem 2.2 Let $G = c\text{Shack}(H, v, n)$ be disjoint union of multiple copies $c$ of shackle of graph $H$. Then

$$tHs(c\text{Shack}(H, v, n)) = \left\lfloor \frac{m + cn + 1}{m + 2} \right\rfloor$$

where $m = p_H + q_H - 2$, $p_H$ and $q_H$ are the number of vertices and edges in $H$ respectively, $n = |H|$ and $c$ is number of copies of $G$.

Proof. The graph $G = c\text{Shack}(H, v, n)$ is a diconnected graph with vertex set $V(c\text{Shack}(H, v, n)) = \{v_{ij}; 1 \leq i \leq p_H - 2, 1 \leq j \leq n, 1 \leq u \leq c\} \cup \{x_k; 1 \leq k \leq n + 1, 1 \leq u \leq c\}$ and edge set $E(c\text{Shack}(H, v, n)) = \{e_{ij}; 1 \leq l \leq q_H, 1 \leq j \leq n, 1 \leq u \leq c\}$. Thus, the graph $c\text{Shack}(H, v, n)$ has $|V(c\text{Shack}(H, v, n))| = c(n - 1)p_H + 1$ and $|E(c\text{Shack}(H, v, n))| = c(q_H)$. Since $m = p_H + q_H - 2$, then by Lemma 2.1

$$tHs(c\text{Shack}(H, v, n)) \geq \left\lfloor \frac{2p_H + q_H + |H| - 1}{m + 2} \right\rfloor$$

$$= \left\lfloor \frac{m + 2c(n - 1) + 1}{m + 2} \right\rfloor$$

$$= \left\lfloor \frac{m + cn + 1}{m + 2} \right\rfloor$$
Now we will show that \( tHS(cShack(H, v, n)) \leq \left\lceil \frac{m+cn+1}{m+2} \right\rceil \). The vertex and edge labeling \( f \) is a bijective function \( f : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, \alpha\} \). Let \( w = jw; 1 \leq j \leq n, 1 \leq u \leq c \) such that \( 1 \leq w \leq cn \).

\[
f(x_i^u) = \left\lceil \frac{u}{m+2} \right\rceil, 1 \leq u \leq c
\]

\[
f(v_i^u) \cup f(e_{ij}) = \left\{ \begin{array}{ll}
\left\lceil \frac{w}{m-(t+1)} \right\rceil; 1 \leq w \leq m-t+1, 1 \leq t \leq m \\
\left\lceil \frac{w+(t+1)-m}{m+1} \right\rceil + 1; m-t+2 \leq w \leq cn, 1 \leq t \leq m
\end{array} \right.
\]

Under the labeling \( f \), the total \( H \)-weight \( W(H) = \sum_{v \in V(H)} f(v) + \sum_{e \in E(H)} f(e) \) is \( W(H) = \{m+2, m+3, \ldots, m+cn+1\} \) which form a consecutive sequence. It implies the set of \( H \)-weights are distinct. Now considering the lower label of \( f \), the minimum \( tHS(cShack(H, v, n)) \) can be achieved by the following:

\[
tHS(cShack(H, v, n)) \leq \left\lceil \frac{cn+m-(m+1)}{m+2} \right\rceil + 1
\]

\[
= \left\lceil \frac{cn+1}{m+2} \right\rceil
\]

\[
= \left\lceil \frac{cn+1}{m+2} \right\rceil + \left\lceil \frac{m+1}{m+2} \right\rceil
\]

\[
= \left\lceil \frac{cn+1}{m+2} \right\rceil + 1
\]

Thus, \( tHS(cShack(H, v, n)) \leq \left\lceil \frac{m+cn+1}{m+2} \right\rceil \). It implies that \( tHS(cShack(H, v, n)) = \left\lceil \frac{m+cn+1}{m+2} \right\rceil \). \( \square \)

**Theorem 2.3** Let \( G \) be an amalgamation of any connected graph \( H \), denoted by \( G = Amal(H, v, n) \). Then the following holds

\[
tHS(Amal(H, v, n)) = \left\lceil \frac{r+n-1}{r} \right\rceil
\]

where \( r = p_H + q_H - 1 \), \( p_H \) and \( q_H \) is the number of vertices and edges in \( H \) respectively and \( n = |H| \).

**Proof.** The vertex set and edge set of the graph \( Amal(H, v, n) \) can be split into following sets: \( V(Amal(H, v, n)) = \{A\} \cup \{x_{ij}; 1 \leq i \leq p_H - 1, 1 \leq j \leq n\} \) and \( E(Amal(H, v, n)) = \{e_{ij}; 1 \leq l \leq q_H, 1 \leq j \leq n\} \). Thus, the graph \( Amal(H, v, n) \) has \( |V(Amal(H, v, n))| = p_G \), and \( |E(Amal(H, v, n))| = q_G \). Let \( n, m \) be positive integers with \( n \geq 2 \) and \( m \geq 3 \). Thus \( |V(Amal(H, v, n))| = p_G = n(p_H - 1) + 1 \) and \( |E(Amal(H, v, n))| = q_G = nq_H \). Then by lemma 2.1,

\[
tHS(Amal(H, v, n)) \geq \left\lceil \frac{p_H + q_H + |H|-1}{r} \right\rceil
\]

\[
= \left\lceil \frac{r+n-1}{r} \right\rceil
\]

\[
= \left\lceil \frac{r+n-1}{r} \right\rceil
\]

Thus, the lower bound \( tHS(Amal(H, v, n)) \geq \left\lceil \frac{r+n-1}{r} \right\rceil \). Now we will prove that \( tHS(Amal(H, v, n)) \leq \left\lceil \frac{r+n-1}{r} \right\rceil \). The vertex and edge labeling \( f \) is a bijective function \( f : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, \alpha\} \).

\[
f(A) = 1
\]

\[
f(x_{ij}) \cup f(e_{ij}) = \left\{ \begin{array}{ll}
\left\lceil \frac{j-t}{r-(t+1)} \right\rceil; 1 \leq j \leq r-t+1, 1 \leq t \leq r \\
\left\lceil \frac{t-1-(r+1)}{r} \right\rceil + 1; r-i+2 \leq j \leq n, 1 \leq t \leq r.
\end{array} \right.
\]

Under the labeling \( f \), the total \( H \)-weight \( W(H) = \sum_{v \in V(H)} f(v) + \sum_{e \in E(H)} f(e) \) is \( W(H) = \{r+1, r+2, \ldots, r+n\} \) form a consecutive sequence. It implies the set of \( H \)-weights are distinct.
Now by considering the above label $f$, the minimum $tHs(\text{Amal}(H, v, n))$ can be achieved by the following:

$$tHs(\text{Amal}(H, v, n)) \leq \left\lceil \frac{n + r - (r + 1)}{r} \right\rceil + 1 = \left\lceil \frac{n - 1 + r}{r} \right\rceil = \left\lceil \frac{r + n - 1}{r} \right\rceil$$

It is clear to conclude that $tHs(\text{Amal}(H, v, n)) = \left\lceil \frac{r + n - 1}{r} \right\rceil$.

\[\text{Theorem 2.4} \]

Let $G$ be a disjoint union of multiple copies $c$ of amalgamation of graph $H$, denoted by $G = c\text{Amal}(H, v, n)$. Then

$$tHs(c\text{Amal}(H, v, n)) = \left\lceil \frac{r + cn - 1}{r} \right\rceil$$

where $r = p_H + q_H - 1$, $p_H$ and $q_H$ is the number of vertices and edges in $H$ respectively, $n = |H|$ and $c$ is number of copies of $G$.

\[\text{Proof.}\]

The vertex set and edge set of the graph $G = c\text{Amal}(H, v, n)$ can be split into following sets: $V(G) = \{A^k; 1 \leq k \leq c\} \cup \{x^k_{ij}; 1 \leq i \leq p_H - 1, 1 \leq j \leq n, 1 \leq k \leq c\}$ and $E(G) = \{e^k_{ij}; 1 \leq j \leq n, 1 \leq l \leq q_H, 1 \leq k \leq c\}$. Thus the graph $c\text{Amal}(H, v, n)$ has with $|V(c\text{Amal}(H, v, n))| = p_G$, and $|E(c\text{Amal}(H, v, n))| = p_G$. Let $n$, $r$, and odd $c$ be positive integers with $n \geq 2$ and $r$, $c \geq 3$. Thus $|V(G)| = p_G = c(n(p_H - 1) + 1)$ and $|E(G)| = q_G = cnq_H$.

Then by lemma 2.1,

$$tHs(c\text{Amal}(H, v, n)) \geq \left\lceil \frac{p_H + q_H + |H| - 1}{r} \right\rceil = \left\lceil \frac{r + cn - 1}{r} \right\rceil$$

Thus, the lower bound $tHs(c\text{Amal}(H, v, n)) \geq \left\lceil \frac{r + cn - 1}{r} \right\rceil$. Now we will show that $tHs(c\text{Amal}(H, v, n)) \leq \left\lceil \frac{r + cn - 1}{r} \right\rceil$. For any $V$ and $E$, the labeling as follows. Let $w = jk; 1 \leq j \leq n, 1 \leq k \leq c$ such that $1 \leq w \leq cn$.

$$f(A^k) = 1, 1 \leq k \leq c$$

$$f(x^k_{ij}) \cup f(e^k_{ij}) = \begin{cases} \left\lceil \frac{w}{r - (t - 1)} \right\rceil; 1 \leq w \leq r - t + 1, 1 \leq t \leq r, 1 \leq k \leq c \\ \left\lceil \frac{w + t - (r + 1)}{r} \right\rceil + 1; r - t + 2 \leq w \leq cn, 1 \leq t \leq r, 1 \leq k \leq c. \end{cases}$$

Under the labeling $f$, the total $H$-weight $W(H) = \sum_{v \in V(H)} f(v) + \sum_{e \in E(H)} f(e)$ is $W(H) = \{r + 1, r + 2, \ldots, r + cn\}$ form a consecutive sequence. It implies the set of $H$-weights are distinct. Now considering the above label of $f$, the minimum $tHs(c\text{Amal}(H, v, n))$ can be achieved by the following:

$$tHs(c\text{Amal}(H, v, n)) \leq \left\lceil \frac{w + t - (r + 1)}{r} \right\rceil + 1 = \left\lceil \frac{cn + r - 1}{r} \right\rceil$$

It concludes the proof. \[\square\]

\[\text{Theorem 2.5} \]

Let $G$ be a shackle of connected graph $C_m$ graph, denoted by $G = \text{Shack}(C_m, v, n)$. Then

$$tHs(\text{Shack}(C_m, v, n)) = \left\lceil \frac{2m + n - 1}{2m} \right\rceil$$

where $m$ is an order of the cycle graph and $n$ number of $C_m$.
The graph $\text{Shack}(C_m, v, n)$ is a connected graph with vertex set $V(\text{Shack}(C_m, v, n)) = \{v_{ij}; 1 \leq i \leq p_{C_m}, 2 \leq j \leq n\} \cup \{x_k; 1 \leq k \leq n+1\}$ and edge set $E(\text{Shack}(C_m, v, n)) = \{e_{ij}; 1 \leq i \leq l \leq q_{C_m}, 1 \leq j \leq n\}$. The cardinalities of the graph $\text{Shack}(C_m, v, n)$ are $|V(\text{Shack}(C_m, v, n))| = (n-1)p_{C_m} + 1$, and $|E(\text{Shack}(C_m, v, n))| = nq_{C_m}$, where $p_{C_m} = |V(C_m)|$, and $q_{C_m} = |E(C_m)|$. Then by Lemma 2.1,
\[
\text{tHS}(\text{Shack}(C_m, v, n)) \geq \left\lceil \frac{p_{C_m} + q_{C_m} + |C_m| - 1}{2m} \right\rceil \geq \left\lceil \frac{m + n - 1}{2m} \right\rceil
\]

Now we will show that $\text{tHS}(\text{Shack}(C_m, v, n)) \leq \left\lceil \frac{2m + n - 1}{2m} \right\rceil$. Define the vertex and edge labelings $f : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, \alpha\}$ as follows
\[
f(x_k) = \left\lceil \frac{k}{2m} \right\rceil
\]
\[
f(v_{ij}) \cup f(e_{ij}) = \left\lceil \frac{2m - 2}{2m} \right\rceil; 1 \leq j \leq 2m - t - 1, 1 \leq t \leq 2m - 2
\]
Under the labeling $f$, the total $H$-weight $W(H) = \sum_{v \in V(H)} f(v) + \sum_{e \in E(H)} f(e)$ is $W(H) = \{2m, 2m + 1, \ldots, 2m + n - 1\}$ form a consecutive sequence. It implies the set of $H$-weights are distinct. Now considering the above label of $f$, the minimum $\text{tHS}(\text{Shack}(C_m, v, n))$ can be achieved by the following:
\[
\text{tHS}(\text{Shack}(C_m, v, n)) \leq \left\lceil \frac{2m + n - 1}{2m} \right\rceil
\]
Thus $\text{tHS}(\text{Shack}(C_m, v, n)) \leq \left\lceil \frac{2m + n - 1}{2m} \right\rceil$, it implies that $\text{tHS}(\text{Shack}(C_m, v, n)) = \left\lceil \frac{2m + n - 1}{2m} \right\rceil$.

**Theorem 2.6** Let $G$ be a disjoint union of multiple copies $c$ of shackles of graph $C_m$, denoted by $G = c\text{Shack}(C_m, v, n)$. Then
\[
\text{tHS}(c\text{Shack}(C_m, v, n)) = \left\lceil \frac{2m + cn - 1}{2m} \right\rceil
\]
where $m$ is an order of the cycle graph, $n$ is a number of $C_m$, and $c$ is number of multiple copies of $G$.

**Proof.** Suppose we denote the vertex and edge sets of the graph $G = c\text{Shack}(C_m, v, n)$ as follows:
$V(c\text{Shack}(C_m, v, n)) = \{v_{ij}; 1 \leq i \leq p_{cC_m} - 2, 1 \leq j \leq n, 1 \leq u \leq c\} \cup \{x_k; 1 \leq k \leq n + 1, 1 \leq u \leq c\}$ and $E(c\text{Shack}(C_m, v, n)) = \{e_{ij}; 1 \leq l \leq q_{cC_m}, 1 \leq j \leq n, 1 \leq u \leq c\}$. Thus, the graph $c\text{Shack}(C_m, v, n)$ has $|V(c\text{Shack}(C_m, v, n))| = c(n - 1)p_{C_m} + 1$, and $|E(c\text{Shack}(C_m, v, n))| = cnq_{C_m}$, where $p_{C_m} = |V(C_m)|$ and $q_{C_m} = |E(C_m)|$. Then by Lemma 2.1
\[
\text{tHS}(c\text{Shack}(C_m, v, n)) \geq \left\lceil \frac{p_{C_m} + q_{cC_m} + |C_m| - 1}{2m} \right\rceil \geq \left\lceil \frac{2m + cn - 1}{2m} \right\rceil
\]
Next we will show that $\text{tHS}(c\text{Shack}(C_m, v, n)) \leq \left\lceil \frac{2m + cn - 1}{2m} \right\rceil$ by defining the vertex and edge labeling $f : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, \alpha\}$ by the following. Let $w = ju; 1 \leq j \leq n, 1 \leq u \leq c$ such that $1 \leq w \leq cn$.
\[
f(x_k) = \left\lceil \frac{w}{2m} \right\rceil, 1 \leq u \leq c
\]
\[
f(x_k) \cup f(e_{ij}) = \left\lceil \frac{2m - 2}{2m} \right\rceil; 1 \leq w \leq 2m - t - 1, 1 \leq t \leq 2m - 2
\]
Under the labeling \( f \), the total \( H \)-weight \( W(H) = \sum_{v \in V(H)} f(v) + \sum_{e \in E(H)} f(e) \) is \( W(H) = \{2m, 2m + 1, \ldots, 2m + cn - 1\} \) form a consecutive sequence. It implies the set of \( H \)-weights are distinct. Now considering the above label of \( f \), the minimum \( tHs(cShack(C_m, v, n)) \) can be achieved by the following.

\[
tHs(Shack(C_m, v, n)) \leq \left\lfloor \frac{w+t-(2m-2+1)}{2m} \right\rfloor + 1
= \left\lfloor \frac{cn+(2m-2)-(2m-2+1)}{2m} + \frac{2m}{2m} \right\rfloor
= \left\lfloor \frac{cn-1 + \frac{2m}{2m}}{2m} \right\rfloor
= \left\lfloor \frac{2m+cn-1}{2m} \right\rfloor.
\]

It is clear to conclude that \( tHs(Shack(C_m, v, n)) = \left\lfloor \frac{2m+cn-1}{2m} \right\rfloor \).

\[\Box\]

**Theorem 2.7** Let \( G \) be an amalgamation of connected graph \( C_3 \), denoted by \( G = Amal(C_3, v, n) \). Then the following holds

\[
tHs(\text{Amal}(C_3, v, n)) = \left\lfloor \frac{n+4}{5} \right\rfloor
\]

where \( n \) is a number of \( C_3 \).

**Proof.** Let the graph \( \text{Amal}(C_3, v, n) \) has with \( |V(G)| = p_G, |E(G)| = q_G, |V(H)| = |V(C_3)| = p_H = p_{C_3} \), and \( |E(H)| = |E(C_3)| = q_H = q_{C_3} \). Suppose we denote the vertex and edge sets of the graph \( G = Amal(C_3, v, n) \) as follows: \( V(G) = \{A\} \cup \{x_{ij}: 1 \leq i \leq 2, 1 \leq j \leq n\} \) and \( E(G) = \{Ax_{ij}: 1 \leq i \leq 2, 1 \leq j \leq n\} \cup \{x_{ij}x_{2j}: 1 \leq j \leq n\} \). Thus, the graph \( \text{Amal}(C_3, v, n) \) has \( |V(\text{Amal}(C_3, v, n))| = 2n + 1, \) and \( |E(\text{Amal}(C_3, v, n))| = 3n, \) where \( p_{C_3} = |V(C_3)| \) and \( q_{C_3} = |E(C_3)| \).

Then by Lemma 2.1, we have the following

\[
tHs(\text{Amal}(C_3, v, n)) \geq \left\lfloor \frac{p_H + q_H + |H|-1}{6n+1} \right\rfloor
= \left\lfloor \frac{n+4}{5} \right\rfloor
= \left\lfloor \frac{n+4}{5} \right\rfloor.
\]

Thus, the lower bound \( tHs(\text{Amal}(C_3, v, n)) \geq \left\lfloor \frac{n+4}{5} \right\rfloor \). Now we will show that \( tHs(\text{Amal}(C_3, v, n)) \leq \left\lfloor \frac{n+4}{5} \right\rfloor \). The vertex and edge labelings \( f \) is a bijective function \( f : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, \alpha\} \).

\[
f(A) = 1
f(x_{i,j}) \cup f(Ax_{i,j}) \cup f(x_{ij}x_{2j}) = \left\{
\begin{aligned}
&\left\lfloor \frac{j}{5} \right\rfloor: 1 \leq j \leq 6 - i, 1 \leq i \leq 5 \\
&\left\lfloor \frac{j+i-6}{5} \right\rfloor + 1: 7 - i \leq j \leq n, 1 \leq i \leq 5.
\end{aligned}
\right.
\]

Under the labeling \( f \), the total \( H \)-weight \( W(H) = \sum_{v \in V(H)} f(v) + \sum_{e \in E(H)} f(e) \) is \( W(H) = \{6, 7, \ldots, 6 + (n - 1)\} \) which form a consecutive sequence. It implies the set of \( H \)-weights are distinct. Now considering the above label of \( f \), the minimum \( tHs(\text{Amal}(C_3, v, n)) \) can be achieved by the following.

\[
tHs(\text{Amal}(C_3, v, n)) \leq \left\lfloor \frac{j+i-6}{5} \right\rfloor + 1
= \left\lfloor \frac{n+4}{5} \right\rfloor
\]

It concludes the proof. \( \Box \)
Acknowledgement

Theorem 2.8 Let G be a disjoint union of amalgamation of $C_3$ graph, denoted by $cAmal(C_3, v, n)$. Then

$$tHs(cAmal(C_3, v, n)) = \left\lceil \frac{cn + 4}{5} \right\rceil$$

Proof. Let the graph $cAmal(C_3, v, n)$ has with $|V(G)| = p_G$, $|E(G)| = q_G$, $|V(H)| = |V(C_3)| = p_H = p_{C_3}$, and $|E(H)| = |E(C_3)| = q_H = q_{C_3}$. The vertex set and edge set of the graph $G = cAmal(C_3, v, n)$ can be split into following sets: $V(G) = \{A^k; 1 \leq k \leq c\} \cup \{x_{ij}; 1 \leq i \leq 2, 1 \leq j \leq n, 1 \leq k \leq c\}$ and $E(G) = \{A^k x_i^k; 1 \leq i \leq 2, 1 \leq j \leq n, 1 \leq k \leq c\} \cup \{x_{ij}^k x_{ij}; 1 \leq j \leq n, 1 \leq k \leq c\}$. Let $n, m, l$ and odd $s$ be positive integers with $n \geq 2$ and $r, c \geq 3$. Thus, $|V(G)| = p_G = c(2n + 1)$ and $|E(G)| = q_G = 3cn$. Then by lemma 2.1,

$$tHs(cAmal(C_3, v, n)) \geq \left\lceil \frac{p_G + q_G + |H| - 1}{r + c} \right\rceil = \left\lceil \frac{5 + cn}{6} \right\rceil = \left\lceil \frac{cn + 4}{5} \right\rceil$$

Thus, the lower bound $tHs(cAmal(C_3, v, n)) \geq \left\lceil \frac{cn + 4}{5} \right\rceil$. Now we will prove that $tHs(cAmal(H, v, n)) \leq \left\lceil \frac{cn + 4}{5} \right\rceil$. Let $l = jk; 1 \leq j \leq n, 1 \leq k \leq c$ such that $1 \leq l \leq cn$. For any $V$ and $E$, the labeling as follows.

$$f(A^k) = 1, 1 \leq k \leq c$$

$$f(x_{ij}^k) \cup f(A^k x_i^k) \cup f(x_{ij}^k x_{ij}) = \left\{ \begin{array}{ll} [\frac{l - i}{c - 1}]; 1 \leq l \leq 6 - i, 1 \leq i \leq 5, 1 \leq k \leq c \\
[\frac{l + i - (6)}{c - 1}] + 1; 7 - i \leq l \leq cn, 1 \leq i \leq 5, 1 \leq k \leq c. \end{array} \right.$$  

Under the labeling $f$, the total $H$-weight $W(H) = \sum_{v \in V(H)} f(v) + \sum_{e \in E(H)} f(e)$ is $W(H) = \{6, 7, \ldots, 6 + (cn - 1)\}$ which form a consecutive sequence. It implies the set of $H$-weights are distinct. Now considering the above label of $f$, the minimum $tHs(cAmal(C_3, v, n))$ can be achieved by the following.

$$tHs(cAmal(C_3, v, n)) \geq \left\lceil \frac{\frac{l + i - (6)}{c - 1} + 1}{\frac{5 + cn - 6}{5}} \right\rceil \geq \left\lceil \frac{cn + 4}{5} \right\rceil$$

Thus $tHs(cAmal(C_3, v, n)) \leq \left\lceil \frac{cn + 4}{5} \right\rceil$, it implies that $tHs(cAmal(C_3, v, n)) = \left\lceil \frac{cn + 4}{5} \right\rceil$. □

Concluding Remarks

We have found the total $H$-irregularity strength of shackel and amalgamation of G, namely $tHs(Shack(H, v, n))$, $tHs(c(Shack(H, v, n)))$, $tHs(Shack(H, v, n))$ and $tHs(c(Shack(H, v, n)))$. Apart from those graphs, the study of the values of $tHs$ are considered to be interesting research topic as it is a new extension of total edge irregularity strength of $G$. Therefore, we propose the following open problem.

Open Problem 2.1 Let $G$ be any connected and disconnected graph, apart from the above graphs determine the value of $tHs(G)$.

Open Problem 2.2 Let $tes(G)$ and $tHs(G)$ be total edge irregularity strength and total $H$-irregularity strength of graph $G$. Characterize the connection between $tes(G)$ and $tHs(G)$.

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