The Zakharov-Shabat spectral problem on the semi-line: Hilbert formulation and applications

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Abstract
The inverse spectral transform for the Zakharov-Shabat equation on the semi-line \( x > 0 \) is reconsidered as a Hilbert problem. The boundary data induce an essential singularity as \( k \to \infty \) to one of the basic solutions. Then solving the inverse problem means solving a Hilbert problem with particular prescribed behavior. It is demonstrated that the direct and inverse problems are solved in a consistent way as soon as the spectral transform vanishes as \( O(1/k) \) at infinity in the whole upper half plane (where it may possess single poles) and is continuous and bounded on the real \( k \)-axis. The method is applied to stimulated Raman scattering and sine-Gordon (light cone) for which it is demonstrated that time evolution conserves the properties of the spectral transform.

1 Introduction
Since the discovery of the inverse spectral transform (IST) in 1967 \[1\] to solve nonlinear evolutions (Korteweg-de Vries and later nonlinear Schrödinger \[2\], then sine-Gordon, etc... \[3\]) with Cauchy conditions on the infinite space (and vanishing asymptotic boundary values), the application of the method to boundary value problems on semi-infinite (or finite) intervals has been the object of intense research with few general results, which is reminiscent from the fact that even for linear partial differential equations, boundary value problem much differ from one another. It would be useless to quote all papers that has dealt with boundary value problems for nonlinear evolutions, only the most recent ones will be cited here, see \[4\]-[25] where more complete reference will be found.

Taking for instance the nonlinear Schrödinger equation (NLS) \[2\] for a field \( q(x,t) \), saying that it is solvable by IST on the infinite line actually means that, given Dirichlet condition \( q(x,0) \) on the line \( x \in (-\infty, +\infty) \) and Cauchy conditions \( q(\pm\infty, t) = 0 \) and \( q_x(\pm\infty, t) = 0 \) for any \( t > 0 \), the method IST allows to construct the solution \( q(x,t) \) at any \( t \) for all \( x \). Although Dirichlet conditions at \( x = \pm\infty \), namely \( q(\pm\infty, t) = 0 \), constitute with \( q(x,0) \) a well-posed problem, the solution of NLS with IST requires the adjunction of the Neuman condition \( q_x(\pm\infty, t) = 0 \). This simply results from the fact that the Lax pair contains
explicitely the derivative \( q_x \). The point is that this supplementary constraint is compatible with the rest of the boundary values and remains compatible at any time. This \textit{stability} is a property of the spectral transform itself that conserves the class of function (potentials) chosen at \( t = 0 \). For instance if \( q(x, 0) \) is in the Schwartz space (the function and all its derivatives are continuous and vanish faster than any power of \( x \) as \( x \to \infty \)), then the solution \( q(x, t) \) obtained by IST is also in the Schwartz space which makes the method work.

Such is not anymore the case on finite (or semi-infinite) interval. Indeed, while the constraint \( q_x(\pm\infty, t) = 0 \) is not restrictive when we already have \( q(\pm\infty, t) = 0 \), it is not so on a finite domain where imposing the data of both the function and its derivative produce an over-constrained problem. Then, although NLS keeps many of its \textit{integrability properties} on the finite interval (with mixed Dirichlet-Neumann conditions on the semi-axis) \cite{4, 5}, and Bäcklund transformation can be used to determine explicit solvable subclasses \cite{6}, the general problem is still unsolved.

A detailed study of the Dirichlet boundary problem (data of \( q(x, 0) \), \( q(0, t) \) and \( q(\infty, t) = 0 \)) has been performed in \cite{10} where it has been shown that the problem of the determination of \( q_x(0, t) \) from \( q(0, t) \) is transformed into the problem of determining some \textit{missing spectral datum}. This approach allows also to construct classes of solutions but not to solve the problem in general.

Another point of view is presented in \cite{11} where the method is based on the scattering problem on a line in the \((x, t)\)-plane in the Gelfand-Levitan-Marchenko formulation. This is applied to the Korteveg-de Vries equation and also provides much insight in the boundary-value problem but shows that indeed the problem is still open.

An interesting approach has been proposed in \cite{13} where both operators of the Lax pair are treated as joint spectral problems. There the boundary-value problem becomes a Dirichlet condition on a closed domain, namely data of \( q(x, 0) \), \( q(0, t) \), \( q(\infty, t) = 0 \) and \( q(t, \infty) = 0 \). Applied to NLS (defocusing case), this method provides the solution as a system of coupled integral equations which is then difficult to use. But this approach allows for particular solutions, see also \cite{12}, and for large time asymptotic behaviors. Moreover it has been successfully applied to sine-Gordon in the light cone (SG) \cite{13} for which the Lax pair is compatible with the boundary values, and then extended to boundary values on arbitrary convex domain \cite{14}. The method has also proved being very useful to treat linear partial differential equations with moving boundaries \cite{15}. The method furnishes the solution under the form of a set of Cauchy-Green integral equations.

There is another case where the general solution has been obtained by IST: the stimulated Raman scattering equations (SRS) on the semi-axis \cite{16} (the \textit{sharp-line} version on the finite interval has been treated in \cite{17} by the method of \cite{13}). The boundary-value problem there is an open-end problem and the stability of direct/inverse spectral transformation allows indeed to get the general solution on the semi-axis (as well as the finite interval case) which has been successfully checked on numerical simulations \cite{18}. The method works also for the discrete integrable version of SRS \cite{19} and it is based on the approach of
the dispersion relation \[2\] where the principal Lax operator is taken as the spectral operator and the auxiliary Lax operator is the tool to define altogether the dispersion relation and the time-evolution of the spectral transform. The advantage of such a method is that, in the physically relevant case of a medium initially at rest, it allows to get all the information on the nonlinear Fourier spectrum explicitly from the input radiation fields. An interesting consequence is the possibility to build literally the spectral transform out of measurements and compare it to the theory, which has been done successfully on numerical simulations \[18\] of the finite interval, as well as experimental data \[21\] with infinite line damped model. The method has also been extended to treat perturbation of SRS \[22\], confirming the validity of approximations used for comparison with experiments in \[21\].

As a general result, it has been shown that the dispersion relation will depend explicitly not only on the boundary values (as for infinite line \[23\]) but also on the spectral transform itself hence producing nonlinear time evolution of spectral data (of Riccati type) \[16\]. Interesting enough, such an evolution induces the motion of the poles (discrete eigenvalues of the spectral problem, associated to solitons) and thus results in creation (and anihilation) of solitons due to input boundaries. The same property has been found for the karpman-Kaup system \[24\] with a remarkable adequation to numerical simulations \[25\].

The spectral problem underlying stimulated Raman scattering and sine-Gordon is the Zakharov-Shabat system on the semi-axis with prescribed potentials values in \(x = 0\). We show here that the solution can be written out of a system integral equations of Cauchy type where the contour of integration differs from the one working in the infinite line case. This results from the presence of an essential singularity in the asymptotic behaviors of one of the Jost solutions, a problem that has been underestimated in the literature so far.

Then we shall begin with solving the direct and inverse spectral problems in sections 3 and 4. In doing this we shall discover that the reconstructed solutions (Jost solutions) must obey a sequence of properties, named P1 to P9, that are central to the method. This is then applied to SRS in section 3 and to sine-Gordon in section 4. In both cases it is demonstrated that the time evolution of the spectral transform is such that it conserves the constraints necessary to ensure the 9 fundamental properties of the Jost solutions. In order to illustrate the method for the sine-Gordon, we consider the explicitly solvable case of a piecewise constant boundary datum which generically stands for the discretization of an arbitrary boundary value.

2 The direct problem
2.1 Basic solutions

The time \( t \) being an external parameter it is omitted in the whole section. We consider the Zakharov-Shabat spectral problem \(^2\) on the semi-line

\[
x > 0 : \quad \phi_x = ik[\phi, \sigma_3] + Q(x)\phi, \quad Q = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}
\]  

(2.1)

with given boundary values

\[
q(x)|_{x=0^+} = q_0, \quad r(x)|_{x=0^+} = r_0.
\]  

(2.2)

Jost solutions \(^2\) can be defined as the solution of the integral equations

\[
\begin{pmatrix} \psi_{11}(k, x) \\ \psi_{21}(k, x) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \left( \int_0^x dy \ q(y)\psi_{21}^+(k, y) \right),
\]  

(2.3)

\[
\begin{pmatrix} \phi_{11}(k, x) \\ \phi_{21}(k, x) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \left( -\int_0^x dy \ r(y)\phi_{11}^+(k, y)e^{2ik(x-y)} \right),
\]  

(2.4)

\[
\begin{pmatrix} \phi_{11}(k, x) \\ \phi_{21}(k, x) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \left( -\int_x^\infty dy \ q(y)\phi_{21}^+(k, y) \right),
\]  

(2.5)

\[
\begin{pmatrix} \phi_{12}(k, x) \\ \phi_{22}(k, x) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \left( \int_0^x dy \ r(y)\phi_{12}^+(k, y)e^{2ik(x-y)} \right).
\]  

(2.6)

2.2 Boundary behaviors

The so-called physical solution

\[
\Phi(k, x) = (\phi_1^-, \phi_2^+) ,
\]  

(2.7)

obeys the bounds

\[
\Phi(k, x)|_{x=0^+} = \begin{pmatrix} 1 & \rho^+ \\ \rho^- & 1 \end{pmatrix},
\]  

(2.8)

\[
\Phi(k, x)|_{x=\infty} = \begin{pmatrix} \tau^- & 0 \\ 0 & \tau^+ \end{pmatrix},
\]  

(2.9)

where the reflection coefficients \( \rho^\pm(k) \) are defined by reading the above integral equations at \( x = 0^+ \), namely

\[
\rho^+(k) = -\int_0^\infty dy \ q(y)\phi_{22}^+(k, y)e^{2iky}
\]  

\[
\rho^-(k) = -\int_0^\infty dy \ r(y)\phi_{11}^-(k, y)e^{-2iky} ,
\]  

(2.10)
and the transmission coefficients $\tau^{\pm}(k)$ by making the limit as $x \to \infty$ i.e.

$$
\begin{align*}
\tau^+(k) &= 1 + \int_0^\infty dy \, r(y) \phi_{12}^+(k, y), \\
\tau^-(k) &= 1 + \int_0^\infty dy \, q(y) \phi_{21}^-(k, y).
\end{align*}
$$

(2.11)

By direct computation with (2.1) we demonstrate that the determinant of any solution is independent on $x$ and hence can be calculated at $x = 0$ or at $x \to \infty$.

Applied to $\Phi$ it gives

$$
1 - \rho^+ \rho^- = \tau^+ \tau^-,
$$

(2.12)

which is called the unitarity relation.

The rest of the solution, namely

$$
\Psi(k, x) = (\psi_1^+, \psi_2^-),
$$

(2.13)

is used to solve the inverse problem and we call it the intermediate solution. It obeys the following bounds for $k \in \mathbb{R}$

$$
\Psi(k, x)|_{x=0^+} = \begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix},
$$

(2.14)

$$
\Psi(k, x)|_{x \to \infty} \sim \begin{pmatrix}1/\tau^+ & -e^{-2ikx}\rho^-/\tau^- \\ -e^{2ikx}\rho^-/\tau^- & 1/\tau^-\end{pmatrix},
$$

(2.15)

(the boundary behavior as $x \to \infty$ is actually obtained by using the relations that follow). The intermediate solution is related to the physical solution by the following relations for $k \in \mathbb{R}$

$$
\begin{align*}
\psi_1^+ - \phi_1^- &= -e^{2ikx}\rho^- \psi_2^- , \\
\phi_2^+ - \psi_1^- &= e^{-2ikx}\rho^+ \psi_1^+ ,
\end{align*}
$$

(2.16)

which can be proved in different ways, for instance by the computation of the differences out of (2.4)-(2.6) and the comparison the resulting integral equation with the one for the r.h.s. of (2.16). A simpler but equivalent method is to express the vector solution $\phi_1^-$ on the basis $\psi_1^+$ and $e^{2ikx}\psi_2^-$. The coefficients of this expansion are then calculated by comparing the boundary values in $x = 0^+$.

Another proof goes with expressing the solution $\Psi$ in terms of $\Phi$ as

$$
\Phi(k, x) = \Psi(k, x) \, e^{-ik\sigma_3 x} \begin{pmatrix}1 & \rho^+(k) \\ \rho^-(k) & 1\end{pmatrix} \, e^{ik\sigma_3 x},
$$

(2.17)

(computed by matching the values in $x = 0^+$) which in particular readily provides the behavior (2.13) from (2.9).
2.3 Analytical properties

Using standard method \cite{26} one obtains that, for potentials vanishing fast enough as \(x \to \infty\), the vector \(\psi^+\) is analytic in the upper half \(k\)-plane and \(\psi^-\) in the lower half. Next, the vector \(\phi^+_2\) is meromorphic function of \(k\) in the upper half-plane with \(N^+\) simple poles \(k^+_n\) which are the zeroes of the analytic function \(1/\tau^+(k)\). Indeed from (2.15) we have

\[
\frac{1}{\tau^+} = 1 + \int_0^\infty q\psi^+_{21},
\]

with \(\psi^+_{21}\) analytic. Moreover the vector

\[
\tilde{\phi}^+_2 = \frac{1}{\tau^+}\phi^+_2
\]

can be written from (2.4) as the solution of

\[
\left( \begin{array}{c} \tilde{\phi}^+_1(k,x) \\ \tilde{\phi}^+_2(k,x) \end{array} \right) = \left( \begin{array}{c} 0 \\ 1 \end{array} \right) + \left( \begin{array}{c} \int_0^x dy q(y)\tilde{\phi}^+_2(k,y)e^{-2ik(x-y)} \\ \int_0^x dy r(y)\tilde{\phi}^+_1(k,y) \end{array} \right),
\]

which shows that \(\tilde{\phi}^+_2\) is analytic in \(k\) and thus that \(\phi^+_2\) has poles where the analytic function \(1/\tau^+(k)\) has zeroes.

Similarly, \(\phi^-_1\) is meromorphic in the lower half-plane with \(N^-\) poles \(k^-_n\), zeroes of \(1/\tau^-(k)\). Consequently, from their very definitions \(2.10\), \(2.11\), \(\tau^\pm\) and \(\rho^\pm\) have meromorphic extensions in their respective half-planes \(\pm\text{Im}(k) > 0\) where they possess the \(N^\pm\) simple poles \(k^\pm_n\) (the bound states locations). In particular we have the relation, obtained simply by taking the residues on \(2.17\)

\[
\text{Res}\{\phi^+_2\} = \rho^+_n\psi^+_1(k^+_n)\exp[-2ik^+_nx], \quad \rho^+_n = \text{Res}\{\rho^+_n\},
\]

\[
\text{Res}\{\phi^-_1\} = \rho^-_n\psi^-_2(k^-_n)\exp[2ik^-_nx], \quad \rho^-_n = \text{Res}\{\rho^-_n\}.
\]

The spectral data is then constituted by the set

\[
\mathcal{S} = \{\text{Im}(k) > 0, \rho^+(k) \ ; \ \text{Im}(k) < 0, \rho^-(k)\},
\]

which is complete if we prove that it allows to reconstruct the potentials \(r(x)\) and \(q(x)\) with \(q(0^+) = q_0\) and \(r(0^+) = r_0\).

2.4 Large \(k\) behaviors

In order to solve the Hilbert problem \(2.16\) with the singularities \(k^\pm_n\) in the complex plane, one must calculate first the behaviour of the solutions \(\Phi(k,x)\) and \(\Psi(k,x)\) as \(|k| \to \infty\). Note that we follow Mushkelishvili \cite{27} to refer to \(2.16\) as a Hilbert problem.
By using repeatedly the integration by part we obtain from the integral equations (2.4) and (2.6) the following regular asymptotic behavior of the physical solution

\[(\phi^-_1, \phi^+_2) = 1 + \frac{1}{2ik} \left( -\int_0^x rq - \frac{q}{\int_0^x rq} \right) + \frac{1}{(2ik)^2} \left( -\int_0^x [r_0 q - \int_0^x rq] - q_x + q \int_0^x rq \right) + O\left(\frac{1}{k^3}\right), \quad (2.24)\]

which has to be understood in \(\text{Im}(k) \leq 0\) for the first vector \(\phi^-_1\) and in \(\text{Im}(k) \geq 0\) for \(\phi^+_2\).

The same procedure applied to the intermediate solution shows that it possesses an essential singularity at \(k = \infty\) on the real axis as indeed we get

\[(\psi^+_1, \psi^-_2) = 1 + \frac{1}{2ik} \left( -\int_0^x rq - \frac{q}{\int_0^x rq} \right) + \frac{1}{2ik} \left( 0 - q_0 e^{-2ikx} \right) + \frac{1}{(2ik)^2} \left( -\int_0^x [r_0 q - \int_0^x rq] - q_x + q \int_0^x rq \right) + \frac{1}{(2ik)^2} \left( q e^{2ikx} - q_0 \right) r_0 + \left( q_0 \int_0^x rq + q_0 e^{-2ikx} \right) \left( r e^{-2ikx} - r_0 q_0 \right) + \cdots. \quad (2.25)\]

Consequently, when \(q_0\) and \(r_0\) do not vanish, the intermediate solution has not the same behavior as the physical solution, in contrast with the infinite line case. In particular the formulation of the solution of the Hilbert problem will differ from the one of the infinite line.

Note for future use that from the definitions (2.10) of \(\rho^\pm\) follow the large \(k\) expansions

\[\rho^+(k) = \frac{1}{2ik} q_0 - \frac{1}{2ik} q_0' r_0 + \frac{1}{(2ik)^3} (q_0' - q_0^2 r_0) + O\left(\frac{1}{k^4}\right), \quad (2.26)\]

\[\rho^-(k) = -\frac{1}{2ik} r_0 - \frac{1}{2ik} r_0' q_0 + \frac{1}{(2ik)^3} (r_0' - r_0^2 q_0) + O\left(\frac{1}{k^4}\right), \quad (2.26)\]

where we have defined

\[q_0' = \frac{\partial q(x)}{\partial x} \bigg|_{x=0^+}, \quad q_0'' = \frac{\partial^2 q(x)}{\partial x^2} \bigg|_{x=0^+}, \quad (2.27)\]

so as for \(r(x)\).

### 3 Solution of the Hilbert problem

#### 3.1 Statement of the problem

Let be given the set of spectral data

\[S = \{\text{Im}(k) \geq 0, \ \rho^+(k) : \text{Im}(k) \leq 0, \ \rho^-(k)\}, \quad (3.1)\]
where \( \rho^+(k) \) (resp. \( \rho^-(k) \)) is continuous and bounded on the real axis and meromorphic in the upper (resp. lower) half plane with a finite number of single poles \( k_n^+ \) (resp. \( k_n^- \)) and related residues \( \rho_n^+ \) (resp. \( \rho_n^- \)). Moreover they obey in their respective half-planes

\[
\rho^+(k) = \frac{1}{2i k} q_0 + O\left(\frac{1}{k^2}\right), \quad \rho^-(k) = -\frac{1}{2i k} r_0 + O\left(\frac{1}{k^2}\right)
\]

(3.2)

where \( q_0 \) and \( r_0 \) are given numbers. The problem is to construct the two matrices \( \Phi(k,x) \) and \( \Psi(k,x) \) solution of the Hilbert problem (2.16) obeying the following set of properties.

**P.1** : Alternative expression of the Hilbert problem

\[
\Phi = \Psi M, \quad M = e^{-ik\sigma_3 x} \begin{pmatrix} 1 & \rho^+(k) \\ \rho^- & 1 \end{pmatrix} e^{ik\sigma_3 x}, \quad x \geq 0.
\]

**P.2** : The vector \( \psi^+_1 \) (resp. \( \psi^-_2 \)) is analytic in \( \text{Im}(k) > 0 \) (resp. \( \text{Im}(k) < 0 \)).

The vector \( \phi^+_2 \) (resp. \( \phi^-_1 \)) is meromorphic in \( \text{Im}(k) > 0 \) (resp. \( \text{Im}(k) < 0 \)) where it possess \( N^+ \) (resp. \( N^- \)) simple poles \( k_n^+ \) (resp. \( k_n^- \)) with residues

\[
\text{Res}_{k_n^+} \{ \phi^+_2 \} = \rho_n^+ \psi^+_1(k_n^+) \exp\{-2ik_n^+ x\}, \quad \text{Res}_{k_n^-} \{ \phi^-_1 \} = \rho_n^- \psi^-_2(k_n^-) \exp[2ik_n^- x].
\]

**P.3** : The functions \( \Psi(k,x) \) and \( \Phi(k,x) \) obey the bounds

\[
\Psi(k,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Phi(k,0) = \begin{pmatrix} 1 & \rho^+(k) \\ \rho^- & 1 \end{pmatrix}.
\]

**P.4** : the function \( \Psi(k,x) - 1 \) vanishes for \( x < 0 \).

**P.5** : the function \( \Phi(k,x) \) has a Laurent expansion with a pole of zero order at \( k = \infty \):

\[
\Phi(k,x) = 1 + \frac{1}{k} \Phi^{(1)}(x) + O(1/k^2)
\]

(in the lower half-plane for the first vector, in the upper half-plane for the second).

**P.6** : the function \( \Psi(k,x) \) has a Laurent expansion with an essential singularity at \( k = \infty \) on the real axis

\[
\Psi = 1 + \frac{1}{k} \Psi^{(1)}(x) + \frac{1}{k} \Psi^{(1)}_0 e^{2ik\sigma_3 x} + \cdots
\]

**P.7** : There exists an off-diagonal matrix \( Q(x) \) such that for \( x > 0 \)

\[
\Psi_x = i k [\Psi, \sigma_3] + Q(x) \Psi,
\]

which is called the Zakharov-Shabat spectral equation.

**P.8** : The function \( Q \) obeys

\[
Q(x)|_{x=0^+} = \begin{pmatrix} 0 & q_0 \\ r_0 & 0 \end{pmatrix}, \quad Q(x)|_{x=0^-} = 0, \quad Q(x) \xrightarrow{x \to \infty} 0.
\]

**P.9** : The above defined function \( Q(x) \) is also given by

\[
Q(x) = i[\sigma_3, \Phi^{(1)}(x)].
\]
3.2 The solution

We prove now that the solution of the Hilbert problem (2.16) obeying the properties P.1-P.9 above is given by the following equations. First \( \Phi(k, x) \) is given by the explicit expressions

\[
\begin{align*}
\phi_1^-(k, x) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{1}{2i\pi} \int_{-\infty}^{+\infty} \frac{d\lambda}{\lambda - k} \rho^-(\lambda)e^{2i\lambda x}\psi^-_2(\lambda, x) \\
&\quad - \sum_{n=1}^{N^-} \frac{\rho_n^-}{k_n - k} \psi_2^-(k_n^-, x)e^{2ik_n^-x}, \quad \text{Im}(k) \leq 0 \quad (3.3)
\end{align*}
\]

\[
\begin{align*}
\phi_2^+(k, x) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{2i\pi} \int_{-\infty}^{+\infty} \frac{d\lambda}{\lambda - k} \rho^+(\lambda)e^{-2i\lambda x}\psi^+_1(\lambda, x) \\
&\quad - \sum_{n=1}^{N^+} \frac{\rho_n^+}{k_n^+ - k} \psi_1^+(k_n^+, x)e^{-2ik_n^+x}, \quad \text{Im}(k) \geq 0 \quad (3.4)
\end{align*}
\]

where the function \( \Psi(k, x) \) solves the following Cauchy-Green integral system of equations

\[
\begin{align*}
\psi_1^+(k, x) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{1}{2i\pi} \int_{-\infty}^{+\infty} \frac{d\lambda}{\lambda - k} \rho^-(\lambda)e^{2i\lambda x}\psi_2^-(\lambda, x) \\
&\quad - \sum_{n=1}^{N^-} \frac{\rho_n^-}{k_n - k} \psi^-_2(k_n^-, x)e^{2ik_n^-x}, \quad \text{Im}(k) \geq 0 \quad (3.5)
\end{align*}
\]

\[
\begin{align*}
\psi_2^-(k, x) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{2i\pi} \int_{-\infty}^{+\infty} \frac{d\lambda}{\lambda - k} \rho^+(\lambda)e^{-2i\lambda x}\psi^+_1(\lambda, x) \\
&\quad - \sum_{n=1}^{N^+} \frac{\rho_n^+}{k_n^+ - k} \psi_1^+(k_n^+, x)e^{-2ik_n^+x}, \quad \text{Im}(k) \leq 0 \quad (3.6)
\end{align*}
\]

In contrast with what occurs in the one dimensional case \( x \in (-\infty, +\infty) \), here the integrals do not run on the real axis but on contours along the real axis in the upper or lower half-plane as indicated. We shall see that this apparently slight difference has a lot of consequences and allows in particular to reconstruct the very solutions \( \Phi \) and \( \Psi \) defined in the direct problem.

It is worth stressing here that we do not address the problem of the complete characterization of the spectral data \( \rho^\pm(k) \), namely to find the necessary and sufficient conditions which would ensure a unique solution to the above Cauchy-Green integral equations (such a problem has been worked out in [12]). However we shall work with a set of sufficient conditions on \( \rho^\pm(k) \) ensuring that the reconstructed potential does coincide with the one we started with. In particular for what concerns solution of nonlinear evolutions, we will demonstrate that a set of actual spectral data (constructed from a given potential at time 0) evolve in time such as to conserve this set of sufficient conditions.

We proceed now step by step to demonstrate that the expressions (3.3)-(3.6) do produce functions \( \Phi \) and \( \Psi \) that verify properties P.1 to P.9.
Property P.1: For real values of $k$ the integral equations (3.5) (3.6) can be written by virtue of Cauchy theorem (by assumption $\rho^\pm(k)$ have no poles on the real axis)

$$
\begin{align*}
\psi_1^+(k,x) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{1}{2i\pi} \int_{-\infty}^{+\infty+i0} \frac{d\lambda}{\lambda - k} \rho^-(\lambda)e^{2i\lambda x} \psi_2^-(\lambda,x) \\
&\quad - \rho^-(k)e^{2ikx}\psi_2^-(k,x) - \sum_{n=1}^{N^-} \frac{\rho_n^-}{k_n^- - k}\psi_2^-(k_n^-,x)e^{2ik_n^- x}, \quad k \in \mathbb{R},
\end{align*}
$$

(3.7)

$$
\begin{align*}
\psi_2^-(k,x) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{2i\pi} \int_{-\infty}^{+\infty-i0} \frac{d\lambda}{\lambda - k} \rho^+(\lambda)e^{-2i\lambda x}\psi_1^+(\lambda,x) \\
&\quad - \rho^+(k)e^{-2ikx}\psi_1^+(k,x) - \sum_{n=1}^{N^+} \frac{\rho_n^+}{k_n^+ - k}\psi_1^+(k_n^+,x)e^{-2ik_n^+ x}, \quad k \in \mathbb{R}.
\end{align*}
$$

(3.8)

Hence follow the Hilbert relations

$$
\begin{align*}
\psi_1^- - \phi_1^- &= -e^{2ikx}\rho^-\psi_2^- , \quad \phi_2^+ - \psi_2^- = e^{-2ikx}\rho^+\psi_1^+ ,
\end{align*}
$$

(3.9)

and by rearrangement of the vectors into matrices, P.1 is proved.

Property P.2: The analytical properties of the various vectors follow from their very expressions. Indeed, as the set of single poles of $\Phi$ is explicitly displayed, one has only to prove that the Cauchy-like integrals produce holomorphic functions [27]. Such is the case if the integrant is a continuous bounded function of $\lambda \in \mathbb{R}$ vanishing as $\lambda \to \infty$. This actually results from the assumed spectral data $\rho^\pm(\lambda)$ and from the large $k \in \mathbb{R}$ properties of $\Psi(k,x)$ deduced from the integral equations (3.3) and (3.6) as the property P.6 that follows.

Property P.3: The equations (3.5) (3.6) evaluated at $x = 0$ ($x$ is a parameter) lead to the system

$$
\begin{align*}
\psi_1^+(k,0) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{1}{2i\pi} \int_{-\infty}^{+\infty+i0} \frac{d\lambda}{\lambda - k} \rho^-(\lambda)e^{2i\lambda (0)} \\
&\quad - \sum_{n=1}^{N^-} \frac{\rho_n^-}{k_n^- - k}\psi_2^-(k_n^-,0),
\end{align*}
$$

(3.10)

$$
\begin{align*}
\psi_2^-(k,0) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{2i\pi} \int_{-\infty}^{+\infty+i0} \frac{d\lambda}{\lambda - k} \rho^+(\lambda)e^{-2i\lambda 0} \\
&\quad - \sum_{n=1}^{N^+} \frac{\rho_n^+}{k_n^+ - k}\psi_1^+(k_n^+,0).
\end{align*}
$$

(3.11)
Now, as we assume a unique solution to the integral equations, it is sufficient to verify that \( \Psi(k,0) = 1 \) is indeed solution of the above system. For \( \psi_1^+(k,0) \) we have then the equation

\[
\psi_1^+(k,0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{1}{2i\pi} \int_{-\infty}^{+\infty} \frac{d\lambda}{\lambda - k} \rho^-(\lambda) \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \sum_{n=1}^{N^-} \rho_n^- \begin{pmatrix} 0 \\ k_n^- - k \end{pmatrix},
\]

which by contour integration in the lower half plane, and thanks to the properties of \( \rho^- \), easily gives \( \psi_1^+(k,0) = (1,0)^T \) (remember that \( k \) lives in the upper half-plane). The same procedure applies to the second vector and gives \( \psi_2^+(k,0) = (0,1)^T \).

Next, by using exactly the same method with \( \Phi(k,0) \), and taking care of the pole \( \lambda = k \), we do obtain the property P.3 (note that one could also use P.1 to derive the boundary value of one of the functions from the other one).

**Property P.4:** For \( x < 0 \), thanks to the factor \( e^{2i\lambda x} \) appearing in the expression (3.5), we can perform the integration in the lower half-plane for \( \psi_1^+ \). The contribution of the poles exactly cancels out and \( \psi_1^+(k,x) = (1,0)^T \) for \( x < 0 \). The same procedure holds for \( \psi_2^- \) which proves P.4.

**Property P.5:** The expansion of \( \Phi(k,x) \) at large \( k \) readily results from the expressions (3.3) and (3.4) where the Cauchy integral behave well. Indeed, taking e.g. \( \phi_1^- \), we integrate above the real axis where, for \( x > 0 \), the exponential is bounded.

We obtain in particular for the matrix \( \Phi^{(1)}(x) = (\phi_1^{(1)}, \phi_2^{(1)}) \)

\[
\phi_1^{(1)} = \frac{1}{2i\pi} \int_{-\infty}^{+\infty} \rho^-(\lambda)e^{2i\lambda x} \psi_2^-(\lambda,x) + \sum_{n=1}^{N^-} \rho_n^- \psi_2^-(k_n^-,-x)e^{2ik_n^-x}, \tag{3.12}
\]

\[
\phi_2^{(1)} = -\frac{1}{2i\pi} \int_{-\infty}^{+\infty} \rho^+(\lambda)e^{-2i\lambda x} \psi_1^+(\lambda,x) + \sum_{n=1}^{N^+} \rho_n^+ \psi_1^+(k_n^+,x)e^{-2ik_n^+x}. \tag{3.13}
\]

**Property P.6:** The expansion of \( \Psi(k,x) \) at large \( k \in \mathbb{R} \) cannot be obtained directly from the integral equations (3.3) and (3.6) (one sees on (3.3) that \( \exp[2i\lambda x] \) is integrated for \( x > 0 \) on a contour lying in the lower half \( \lambda \)-plane), but rather form the versions (3.7) and (3.8) where the Cauchy integrals behave well. Then the same procedure as for P.5 applies to get the expansion given in P.6 (it is actually easier to use P.5 and P.1 together with the expansion of the spectral data \( \rho^\pm(k) \) to demonstrate P.6). We obtain in particular, according to the notations in P.6,

\[
\Psi^{(1)}(x) = \Phi^{(1)}(x), \quad \Psi_0^{(1)} = \begin{pmatrix} 0 & -\phi_0/2i \\ \phi_0/2 & 0 \end{pmatrix}, \tag{3.14}
\]

which matches the expressions (2.25) (and can be pursued to next orders).
Property P.7: This is a fundamental property in the process of solving the inverse problem as indeed it gives the expression of the potential from the solution of the Cauchy-Green integral equations (3.5), in other words in terms of the spectral data \( \rho^\pm(k) \).

To perform the proof of P.7 it is first more convenient to rewrite the integral equations as follows

\[
\begin{align*}
\psi_1^+ &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{1}{2i\pi} \int_{C_-} \frac{d\lambda}{\lambda - k} \rho^- e^{2i\lambda x} \psi_2^- , \\
\psi_2^- &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{2i\pi} \int_{C_+} \frac{d\lambda}{\lambda - k} \rho^+ e^{-2i\lambda x} \psi_1^+ ,
\end{align*}
\]

where \( C_+ \) (resp. \( C_- \)) is a smooth contour in the upper (resp. lower) half-plane extending from \(-\infty\) to \(\infty\) and passing over all poles \( k_n^+ \) (resp. under all poles \( k_n^- \)). Note that \( \psi_1^+ \) is written for \( \text{Im}(k) > 0 \) and hence there is no contribution of the pole \( \lambda = k \) in the lower half-plane. The same holds reversely for \( \psi_2^- \).

In short these integral equations read

\[
\begin{align*}
\Psi(k, x) &= 1 + \frac{1}{2i\pi} \int_C \frac{d\lambda}{\lambda - k} \Psi(\lambda, x) R(\lambda, x) ,
\end{align*}
\]

where

\[
R(k, x) = e^{-ik\sigma_3 x} \begin{pmatrix} 0 & \rho^+(k) \\ -\rho^-(k) & 0 \end{pmatrix} e^{ik\sigma_3 x} ,
\]

and where the contour of integration has to be understood as being \( C_- \) for the first vector of the product \( \Psi R \), and \( C_+ \) for the second.

Thanks to the particular dependence in \( x \) of \( R(k, x) \) it is easy to compute \( \Psi_x + ik[\sigma_3, \Psi] \) and by algebraic manipulations to get

\[
\begin{align*}
\Psi_x + ik[\sigma_3, \Psi] &= -\frac{1}{2\pi} \int_C d\lambda [\sigma_3, \Psi R] \\
&\quad + \frac{1}{2i\pi} \int_C \frac{d\lambda}{\lambda - k} (\Psi_x + i\lambda[\sigma_3, \Psi]) R .
\end{align*}
\]

Then by defining the off-diagonal matrix

\[
Q(x) = -\frac{1}{2\pi} \int_C d\lambda [\sigma_3, \Psi(\lambda, x) R(\lambda, x)] ,
\]

we can compare the integral equation (3.19) to the integral equation for the matrix \( Q(x)\Psi(\lambda, x) \). As the solution is assumed to be unique we arrive at the desired result

\[
\Psi_x + ik[\sigma_3, \Psi] = Q\Psi .
\]
Property P.8: We derive now from the above definition of $Q(x)$ the properties P.8 and to that end rewrite (3.20) as

$$q(x) = -\frac{1}{\pi} \int_{C_+} \rho^+(\lambda) \psi_1^+(\lambda, x)e^{-2i\lambda x} ,$$

$$r(x) = -\frac{1}{\pi} \int_{C_-} \rho^-(\lambda) \psi_2^-(\lambda, x)e^{2i\lambda x} .$$

(3.22)

First, for $x < 0$, we can close the contours with the circle of infinite radius in the related half-planes and readily obtain that $q$ and $r$ vanish (remember that the contours pass beyond the poles of $\rho$).

To evaluate the boundary values in $x = 0$ we use

$$\psi_1^+(k, 0) = \psi_2^-(k, 0) = 1$$

(the function $\Psi$ is continuous in $x = 0$), but keep the exponentials as they produce discontinuous functions in $x = 0$. Then with help of the large $k$ behaviors (3.2) and the formulae

$$x > 0 \Rightarrow -\frac{1}{2i\pi} \int_{C_+} e^{-2i\lambda x} \lambda = \frac{1}{2i\pi} \int_{C_-} e^{2i\lambda x} \lambda = 1 ,$$

(3.23)

we do obtain $q(0^+) = q_0$ and $r(0^+) = r_0$, where $q_0$ and $r_0$ are given from the spectral data $\rho^\pm$ in (3.2).

Note that the procedure can be applied to compute the $x$-derivative of the potential by first regularizing it in $x = 0$ and by using the next order expansion of $\rho(k)$ given in (2.26). Indeed we have

$$\frac{\partial}{\partial x} \left\{ q(x) - q_0 \theta(x) \right\} = -\frac{1}{\pi} \frac{\partial}{\partial x} \int_{C_+} \left[ \rho^+ \psi_1^+ - \frac{q_0}{2i\lambda} \right] e^{-2i\lambda x} ,$$

(3.24)

and applying the same procedure we obtain eventually

$$\lim_{x \to 0^+} \left\{ \frac{\partial}{\partial x} q(x) \right\} = \frac{\partial}{\partial x} \left\{ q(x) - q_0 \theta(x) \right\} \bigg|_{x=0} = q'_0 .$$

(3.25)

The large $x$ vanishing behaviors of $q$ and $r$ follow from their expressions (3.22) rewritten as (we consider only $q(x)$, the same computation works as well for $r(x)$)

$$q(x) = -\frac{1}{\pi} \int_{-\infty+i0}^{+\infty+i0} \rho^+(\lambda) \psi_1^+(\lambda, x)e^{-2i\lambda x}$$

$$+ 2i \sum_{n=1}^{N^+} \rho_n^+ \psi_1^+(k_n^+, x)e^{-2ik_n^+ x} .$$

(3.26)

The above expression vanishes at large $x$ because the path integral vanishes and because

$$\psi_1^+(k, x) \to \frac{1}{\tau^+(k)} , \quad \frac{1}{\tau^+(k)} \bigg|_{k=k_n^+} = 0 ,$$

(3.27)

which results from (2.15) and from the very definition of the poles $k_n^+$. 

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Property P.9 results from the fact that, due to P.1 and P.7, the function $\Phi$ also solves Zakharov-Shabat, i.e.

$$\Phi_x = ik[\Phi, \sigma_3] + Q(x)\Phi. \quad (3.28)$$

Then by inserting in it the large $k$ behavior given by P.5 we must arrive at p.9. To check consistency we have then to demonstrate that the following two expressions of $q(x)$

$$-\frac{1}{\pi} \int_{-\infty+i0}^{+\infty+i0} \rho^+(\lambda)\psi_{11}^+(\lambda, x)e^{-2i\lambda x} + 2i \sum_{n=1}^{N^+} \rho_n^+ \psi_{11}^+(k_n^+, x)e^{-2ik_n^+x}, \quad (3.29)$$

$$-\frac{1}{\pi} \int_{-\infty-i0}^{+\infty-i0} \rho^+(\lambda)\psi_{11}^+(\lambda, x)e^{-2i\lambda x} + 2i \sum_{n=1}^{N^+} \rho_n^+ \psi_{11}^+(k_n^+, x)e^{-2ik_n^+x}, \quad (3.30)$$
do coincide. This simply results from the starting hypothesis of a spectral transform $\rho(k)$ which is continuous and bounded on the real axis (no poles). Then both path integral hereabove are equal.

Note that the relation P.9 giving $Q(x)$ can be evaluated in $x = 0$, which does lead to the expected result thanks to the boundary behavior (2.8) of $\Phi$ in $x = 0$ and the large $k$ behaviors of $\rho^+$ and $\rho^-$. Note also that, in order to compute $q(0^+)$ one should first deform the contour to pass above all poles $k_n^+$ and then make use of (3.23).

3.3 Reductions

The 3 particular evolutions that we shall study then result from a reduction from the two potentials $r$ and $q$ to only one. The first reduction concerns SRS and reads

$\overline{Q} = \sigma_2 Q \sigma_2 \iff r = -\bar{q}$. \quad (3.31)

Then by direct computation one easily proves that $\sigma_2\overline{\Psi(k,x)}\sigma_2$ solves the same spectral equation (2.1) as $\Psi(k,x)$ and possess the same boundary behavior (2.14) in $x = 0$. Those two functions then coincide,

$$\Psi(k,x) = \sigma_2\overline{\Psi(k,x)}\sigma_2, \quad (3.32)$$

and examination of their boundary behavior as $x \to \infty$ gives the reduction constraint on the spectral transform

$$\overline{\tau^-(k)} = \tau^+(k), \quad \overline{\rho^-(k)} = -\rho^+(k). \quad (3.33)$$

Considering the poles and residues, the above expressions imply

$$N^+ = N^-, \quad \rho_n^- = -\rho_n^+, \quad k_n^- = k_n^+. \quad (3.34)$$
The second case is related to SG in the light cone, it reads

$$Q = \sigma_2 Q \sigma_2 \iff r = -q .$$

(3.35)

In that case it is easy to prove that

$$\Psi(k, x) = \sigma_2 \Psi(-k, x) \sigma_2 ,$$

(3.36)

and examination of the boundary behavior as \( x \to \infty \) and \( x \to 0 \) gives the reduction constraint on the spectral transform

$$\rho^+(k, t) = -\rho^-(k, t) , \quad \tau^-(k) = \tau^+(k) ,$$

$$N^+ = N^- , \quad \rho_n^- = -\rho_n^+ .$$

(3.37)

In the case of sine-Gordon, we also need to implement the reality constraint \( q \in \mathbb{R} \) for which both relations (3.31) and (3.35) have to hold together. As a consequence, by direct manipulations of relations (3.33) and (3.37), the resulting constraint on the spectral transform \( \rho = \rho^+ \) which comes in addition to (3.37)

$$\overline{\rho}(-k) = \rho(k) .$$

(3.38)

As a consequence the poles in the complex plane come by pair, namely if \( k_n \) is a pole of \( \rho(k) \) with residue \( \rho_n \), then \( -k_n \) is also a pole with residue \( -\overline{\rho}_n \). In short

$$\text{Res}_{k_n} \{ \rho(k) \} = \rho_n \Rightarrow \text{Res}_{-k_n} \{ \rho(k) \} = -\overline{\rho}_n .$$

(3.39)

4 The Dirichlet problem for SRS

4.1 Boundary values and Lax pair

The formalism of the direct and inverse problem for the Zakharov-Shabat system on the semi-line \( x > 0 \) is now applied to the Stimulated Raman Scattering (SRS) equations for the pulse envelopes of the pump \( a(k, x, t) \), the Stokes \( b(k, x, t) \) and the medium excitation \( q(x, t) \)

$$\partial_x a = qbe^{2ikx} , \quad \partial_x b = -qae^{-2ikx}$$

$$\partial_t q = -\int g(k) dk \ a\overline{b}e^{-2ikx} ,$$

(4.1)

where \( g(k) \) measures the coupling (related to Raman gain) and the parameter \( k \) is a mismatch wave number resulting from group velocity dispersion [16].

The Lax pair for SRS has been written in the case \( g(k) = \delta(k - k_0) \) in [28] and regularized by means of the parameter \( k \) in [20] which found its physical meaning in [16].

Dirichlet conditions are prescribed on the line \( x = 0 , \quad t > 0 , \)

$$a(k, x, t)|_{x=0} = A(t) , \quad b(k, x, t)|_{x=0} = B(t) ,$$

(4.2)
and on the line $x > 0$, $t = 0$,

$$g(x, t)|_{t=0} = c(x).$$  \tag{4.3}

Note that we could assume boundary values $A(k, t)$ and $B(k, t)$ instead of $A(t)$ and $B(t)$, but here we have chosen to leave the $k$-dependence in the arbitrary distribution $g(k)$. This is equivalent to assuming separable dependence of $A$ and $B$ in $k$ and $t$. The given boundary value $c(x)$ is assumed to be of Schwartz type (it decreases at large $x$ with all its derivatives faster that any polynomial) but the physically relevant case is a medium initially at rest, namely $c(x) = 0$. Note that the compatibility of this set of Dirichlet boundary data is inferred from the system (4.1) which gives

$$\partial_x a(k, x, 0)|_{x=0} = c(0)B(0), \quad \partial_x b(k, x, 0)|_{x=0} = -c(0)A(0).$$  \tag{4.4}

The above Dirichlet problem has been solved in [16] and, for sake of completeness, we give hereafter the main lines of the derivation of the solution. The vector $\begin{pmatrix} a \\ be^{2ikx} \end{pmatrix}^T$ is expanded on the basis $\psi_1^+$ and $φ_2^+ e^{2ikx}$, the coefficients being obtained by examination of the values in $x = 0$. The result reads

$$\begin{pmatrix} a \\ be^{2ikx} \end{pmatrix} = (A - \rho B)\psi_1^+ + B\phi_2^+.$$  \tag{4.5}

As a consequence we have

$$\begin{pmatrix} a \\ b \end{pmatrix} \to \begin{pmatrix} (A - \rho B)/\tau \\ |A - \rho B|/\tau + B\tau \end{pmatrix}.$$  \tag{4.6}

The time dependence is plug into the spectral data by requiring that the solution $\mu_+ = (\psi_1^+, \phi_2^+)$ of the Zakharov-Shabat equation be also solution for $k \in \mathbb{R}$ of

$$\mu_+ = V_+ \mu_+ + \mu_+ e^{-ikx} \Omega_+ e^{ikx},$$

$$V_+ = \frac{1}{4i} \int \frac{g(\lambda) d\lambda}{\lambda - (k + i0)} \begin{pmatrix} |a|^2 - |b|^2 & 2a\mathfrak{T}e^{-2ix} \\ 2b\mathfrak{T}e^{2ix} & |b|^2 - |a|^2 \end{pmatrix}.$$  \tag{4.7}

Note that, as the matrix $V(k, x, t)$ is discontinuous when $k$ crosses the real axis, we have to chose a given half-plane. A similar equation could be written for $\mu^- = (\psi_1^-, \phi_2^-)$ with $V^-$ and $\Omega^-$ but, due to the reduction symmetry, it is redundant. The compatibility between the two operators (4.29)-(4.7) with the constraint $\Omega_x = 0$, results in the equation (4.1) within the reduction (3.31).

### 4.2 Evolution of the spectral transform

The free entry $\Omega_+ = \Omega_+^+(k, t)$ is the dispersion relation and, according to the general theory [20], it is strictly dependent of the choice of the solution. It is called dispersion relation because, in the infinite line case, it coincides with the dispersion law of the linearized evolution. The differential equation (4.7) is
evaluated at \(x = 0\) and \(x \to \infty\). This provides 8 equations for the 4 components of \(\Omega^+(k,t)\) and the 4 evolutions equations of \(\rho(k,t), \tau(k,t)\) and their conjugates (these evolutions are compatible which actually results from the reduction). We obtain

\[
\Omega^+(k,t) = -\frac{i}{4} \mathcal{I}(k) \left[ 2\rho(k,t)\overline{AB}\sigma_3 + \begin{pmatrix} |A|^2 - |B|^2 & 0 \\ 2\overline{AB} & -|A|^2 + |B|^2 \end{pmatrix} \right],
\]

(4.8)

(note that the dispersion relation depends on the spectral transform itself), and the Riccati evolution

\[
\rho_t = \frac{i}{2} \mathcal{I} \left[ \overline{AB} \rho^2 - (|A|^2 - |B|^2) \rho - \overline{AB} \right],
\]

(4.9)

where we have defined

\[
\mathcal{I}(k) = \int g(\lambda) d\lambda / (\lambda - k), \quad \text{Im}(k) > 0.
\]

(4.10)

The main fact is that the above evolution conserves the Laurent expansion of \(\rho(k,t)\). Indeed, seeking a solution of (4.9) as the power series

\[
\rho(k,t) = \sum_{j=1}^{p} \rho_j(t) k^j + \sum_{n=0}^{\infty} \frac{1}{k^n} \rho^{(n)}(t),
\]

(4.11)

allows to prove by induction that \(\rho_j(t) = 0\) for \(j \geq 2\). For \(j = 1\) we have the following evolution equation

\[
\frac{\rho_{1t}}{\rho_1^2} = \frac{i}{2} \overline{AB} \int d\lambda \, g(\lambda) = \alpha(t),
\]

(4.12)

which is solved to get

\[
\rho_1(t) = \frac{\rho_1(0)}{1 - \rho_1(0) \int_0^t \alpha(t)}.
\]

(4.13)

Then if \(\rho_1\) vanishes at time \(t = 0\), it vanishes for all \(t\). Such is also the case for \(\rho^{(0)}\) whose evolution (with \(\rho_1(t) = 0\)) simply reads

\[
\rho_{1}^{(0)} = 0.
\]

(4.14)

Consequently, as \(\rho_1\) and \(\rho^{(0)}\) vanish at initial time (because \(\rho(k,0)\) is the spectral transform of the Dirichlet datum \(q(x,0)\)), they vanish for all time and the solution of the Riccati equation (4.9) possess the right behavior for large \(k\), compatible with (2.26).

The coefficient of \(k^{-1}\) gives rise to

\[
\partial_t \rho^{(1)}(t) = \frac{i}{2} \overline{AB} \int d\lambda \, g(\lambda).
\]

(4.15)
The evolution (4.1) written in $x = 0$ gives

$$\partial_t q_0(t) = -A^B \int d\lambda \, g(\lambda) ,$$

(4.16)

and moreover we have, by construction,

$$\rho^{(1)}(0) = \frac{q_0(0)}{2i} .$$

(4.17)

Consequently the unique solution to (4.15) is

$$\rho^{(1)}(t) = \frac{q_0(t)}{2i}$$

(4.18)

as required by (2.20).

The time evolution (4.9) of the spectral transform has coefficients depending only on the input boundary values $A(t)$ and $B(t)$ and contains all the relevant information on the nonlinear scattering of the three waves $q$, $a$ and $b$. In some cases this Riccati equation is explicitely solvable, for instance when $A(t)$ and $B(t)$ are proportional (which is physically meaningful) as illustrated in [16]. It can then be checked on the explicit expression of $\rho(k,t)$ of [16] that it does obey the good large $k$ asymptotics. The fact of being able to provide an explicit expression of the spectral transform in a class of physically interesting cases makes the very interest of the present approach of boundary value problems.

4.3 Why it works

The Dirichlet open-end problem is well posed if the solution $q(x,t)$ constructed hereabove is proved to obey for all $t$ the boundary value

$$q(x,t)|_{x=0^+} = q_0(t) ,$$

(4.19)

and to belong to the class of functions such that

$$\lim_{x \to \infty} q(x,t) = 0 .$$

(4.20)

This proof goes along the following steps.

**Step 1:** the solution $\rho(k,t)$ of the Riccati equation (4.9) possess the large $k$ expansion

$$\rho(k,t) = \frac{q_0(t)}{2ik} + O(1/k^2)$$

(4.21)

resulting from (4.18). This result is central to the theory as we see here that the time evolution of the spectral transform conserves the properties it has at $t = 0$, see (2.20). Note that the expansion can be pursued at any order.
Step 2: with such a behavior of $\rho(k,t)$ the property P.3 still holds, namely

$$\Psi(k,0,t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$  \tag{4.22}

The proof of the above boundary value is obtained by following exactly the method of section 3 where $q_0$ has simply to be understood as $q_0(t)$.

Step 3: compute the limit of $q(x,t)$ as $x \to 0^+$ out of the expression (3.26) by following the method of section 3 used to prove property P.8.

Step 4: compute the limit of $q(x,t)$ as $x \to \infty$ out of the expression (3.26) again by following the proof of property P.8. Note that the essential property to be used is the fact that $\rho(k,t)$ is continuous and bounded for $k \in \mathbb{R}$ at any given $t$. Particular attention is required when, as $t$ evolves, a pole of $\rho$ is “seen” to cross the real axis. It has been demonstrated in [18] that the process is continuous, or put in other words, a resonance (pole in the lower half-plane) dissapears and a pole appears while never belonging to the real axis. The same situation arises for the Schrödinger eigenvalue problem [25].

4.4 Summary of the method of solution of SRS

The solution of SRS with Dirichlet condition is achieved along the following steps:

1 - The spectral transform at time zero $\rho(k,0)$ is obtained form the Dirichlet condition $\tilde{q}(x)$ by finding the vector solution $\phi_2(k,x,0)$ of the integral equations (2.4) in the upper half $k$-plane, namely

$$\phi_{12}^+(k,x,0) = -\int_x^\infty dy \, \tilde{q}(y) \phi_{22}^+(k,y,0) e^{-2ik(x-y)},$$

$$\phi_{22}^+(k,x,0) = 1 - \int_0^x dy \, \tilde{q}(y) \phi_{12}^+(k,y,0), \tag{4.23}$$

and then $\rho(k,0)$ results from

$$\rho(k,0) = \phi_{12}^+(k,x,0)\big|_{x=0^+}. \tag{4.24}$$

2 - $\rho(k,t)$ is calculated in the upper half $k$-plane from $\rho(k,0)$ by solving the Riccati time-evolution

$$\rho_t = \frac{i}{2} \mathcal{I} \left[ \begin{array}{c} \mathcal{A}B \rho^2 - (|A|^2 - |B|^2) \rho - A\overline{B} \end{array} \right], \tag{4.25}$$

with entries $A(t)$ and $B(t)$. The $N$ poles $k_n$ in the upper half-plane and related residues $\rho_n$ of $\rho(k,t)$ must be identified for each value of the parameter $t$. 

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3 - The integral system (3.5) (3.6) which, within the reduction (3.33), simplifies to

\[
\psi_{11}^+(k, x, t) = 1 + \frac{1}{2i\pi} \int_{-\infty-i0}^{+\infty-i0} \frac{d\lambda}{\lambda - k} \rho(\lambda, t) e^{2i\lambda x} \psi_{12}^-(\lambda, x, t)
\]

\[
+ \sum_{n=1}^{N(t)} \frac{\rho_n(t)}{k_n(t) - k} \psi_{12}^-(k_n(t), x, t) e^{2ik_n(t)x} ,
\]

(4.26)

\[
\psi_{12}^-(k, x, t) = \frac{1}{2i\pi} \int_{-\infty+i0}^{+\infty+i0} \frac{d\lambda}{\lambda - k} \rho(\lambda, t) e^{-2i\lambda x} \psi_{11}^+(\lambda, x, t)
\]

\[
- \sum_{n=1}^{N(t)} \frac{\rho_n(t)}{k_n(t) - k} \psi_{11}^+(k_n(t), x, t) e^{-2ik_n(t)x} ,
\]

(4.27)

is solved to get \(\psi_{11}^+(k, x, t)\). This system of equations holds for any fixed value of the “parameters” \(x\) and \(t\).

4 - Finally the solution \(q(x, t)\) is obtained from expression (3.26), i.e.

\[
q(x, t) = -\frac{1}{\pi} \int_{-\infty+i0}^{+\infty+i0} d\lambda \rho(\lambda, t) \psi_{11}^+(\lambda, x, t) e^{-2i\lambda x}
\]

\[
+ 2i \sum_{n=1}^{N(t)} \rho_n(t) \psi_{11}^+(k_n(t), x, t) e^{-2ik_n(t)x} .
\]

(4.28)

5 The Dirichlet problem for sine-Gordon

5.1 Boundary values and Lax pair

Another case where the method applies successfully is the sine-Gordon equation in the light cone

\[
\theta_{xt} + \sin \theta = 0 ,
\]

(5.1)

which is solved here for Dirichlet conditions \(\theta_0(t)\) and \(\tilde{\theta}(x)\), namely

\[
t \in [0,T] : \theta(0,t) = \theta_0(t) , \quad x \in [0,\infty) : \theta(x,0) = \tilde{\theta}(x) ,
\]

(5.2)

in the class of functions vanishing (modulo \(2\pi\)) with all derivatives as \(x \to \infty\) for every value of time \(t\) (the value \(T\) is arbitrary).

This equation is fundamental in many areas of physics and has been consequently widely studied. The solution of the boundary value problem on the quadrant also has been the subject of intense studies, see e.g. [8, 10, 13, 14] and the references therein.
The Lax pair, found in [29], is given by the Zakharov-Shabat equation for a particular choice of potentials \( r \) and \( q \), namely

\[
\Phi_x = ik[\Phi, \sigma_3] + Q\Phi, \quad Q = -\frac{i}{2}\sigma_2 \theta_x ,
\]  

(5.3)
together with the following evolution

\[
\Phi_t = -\frac{i}{4k} (\sigma_3 \cos \theta + \sigma_1 \sin\theta) \Phi + \Phi e^{-ik\sigma_3 x} \Omega e^{ik\sigma_3 x} .
\]  

(5.4)

We have chosen to write this Lax pair for the particular solution \( \Phi \) defined in section 2. Note that the above particular structure of the potential \( Q \) obeys the reduction constraint (3.35).

5.2 Evolution of the spectral transform

By using the behaviors (2.8) and (2.9) in (5.4) we calculate \( \Omega \) and get the time evolution of the spectral transform. Within the reduction relations (3.37) we eventually obtain

\[
\Omega = -\frac{i}{4k} \begin{pmatrix} \cos \theta_0 - \rho(-k, t) \sin \theta_0 & 0 \\ 0 & -\cos \theta_0 + \rho(k, t) \sin \theta_0 \end{pmatrix} ,
\]  

(5.5)

\[
\rho_t = \frac{i}{4k} [\rho^2 \sin \theta_0 - 2\rho \cos \theta_0 - \sin \theta_0] .
\]

(5.6)

Note that the reduction constraint \( \rho(-k) = \rho(k) \) which ensures real value potential \( \theta(x, t) \) is conserved by the above time evolution (as soon as \( \theta_0 \in \mathbb{R} \) of course).

The evolution of \( \rho(k, t) \) is of Riccati type and depends exclusively on the boundary value \( \theta_0(t) \), while the other Dirichlet condition \( \tilde{\theta}(x) \) fixes the initial value \( \rho(k, 0) \). Following exactly the same method as for SRS we show that, as \( \rho_1(0) \) and \( \rho^{(0)}(0) \) vanish (by construction), the large \( k \) asymptotic behavior of the solution \( \rho(k, t) \) reads

\[
\rho(k, t) = \frac{1}{k} \rho^{(1)} + \mathcal{O}(1/k^2) , \quad \partial_t \rho^{(1)} = -\frac{i}{4} \sin \theta_0 .
\]  

(5.7)

Thanks to the sine-Gordon equation evaluated in \( x = 0 \) we can integrate the above time evolution and obtain

\[
\rho^{(1)}(t) = -\frac{1}{4i} \frac{\theta_x(0, t)}{2i} = \frac{q_0(t)}{2i} ,
\]  

(5.8)
as indeed, by construction of \( \rho(k, 0) \) from \( \tilde{\theta}(x) \), we have

\[
\rho^{(1)}(0) = -\frac{1}{4i} \left. \tilde{\theta}_x \right|_{x=0} = -\frac{1}{4i} \theta_x(0, 0) .
\]  

(5.9)
The last point to consider is the apparent singularity of $\rho(k,t)$ when $k \to 0$ by real values. Actually for $k \to 0$ the spectral transform possess the following regular behavior

$$\rho(k,t) = r_0(t) + kr_1(t) + \mathcal{O}(k^2),$$

(5.10)

that can be deduced from the very definition (2.10) of $\rho(k,t)$ (or equivalently by (5.18) below), by expanding these expressions in power of $k$. Inserting the above expansion in (5.6) gives the following unique solution vanishing for $\theta_0 = 0$,

$$\rho(0,t) = r_0(t) = \frac{\cos \theta_0 - 1}{\sin \theta_0}.$$

(5.11)

The other terms $r_n(t)$ in the expansion are then recursively given in terms of $r_j$, ($j < n$), without constraint.

5.3 Why it works

The Dirichlet open-end problem is well posed if the solution $\theta(x,t)$ constructed hereabove is proved to obey for all $t$ the boundary value

$$\theta(x,t)|_{x=0^+} = \theta_0(t),$$

(5.12)

and to belong to the class of functions such that

$$\lim_{x \to \infty} \theta_x(x,t) = 0.$$

(5.13)

This proof goes along the following steps.

Step 1: the solution $\rho(k,t)$ of the Riccati equation (5.3) possess the large $k$ expansion

$$\rho(k,t) = \frac{q_0(t)}{2ik} + \mathcal{O}(1/k^2)$$

(5.14)

where we have

$$q_0(t) = -\frac{1}{2} \theta_x(0) + \frac{1}{2} \int_0^t \sin \theta_0$$

(5.15)

resulting from (5.7).

Step 2: then property P.3 holds for all $t$, namely

$$\Psi(k,0,t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$ 

(5.16)

The proof is obtained by following the method of section 3 where $q_0$ has simply to be understood as $q_0(t)$, given from Dirichlet conditions by (5.15).
Step 3: the limit of $q(x,t)$ as $x \to 0^+$ is computed out of the expression (5.26) by following the method of section 3 used to prove property P.8.

Step 4: the limit of $q = -\theta_x/2$ as $x \to \infty$ is computed out of the expression (5.26) again by following the proof of property P.8. Note that the essential property to be used is the fact that $\rho(k,t)$ is continuous and bounded for $k \in \mathbb{R}$ at any given $t$.

5.4 Summary of the method of solution of sine-Gordon

The solution of sine-Gordon with Dirichlet conditions is achieved along the following steps:

1 - The spectral transform at time zero, $\rho(k,0)$, is obtained form the Dirichlet condition $\tilde{\theta}(x)$ by finding the vector solution $\phi_2(k,x,0)$ of the integral equations (5.17) in the upper half $k$-plane, namely by solving
\begin{align}
\phi_{12}^+(k,x,0) &= - \int_x^\infty dy \ \tilde{q}(y)\phi_{22}^+(k,y,0)e^{-2ik(x-y)} \\
\phi_{22}^+(k,x,0) &= 1 + \int_0^x dy \ \tilde{q}(y)\phi_{12}^+(k,y,0),
\end{align}

where $\tilde{q} = -\tilde{\theta}_x/2$. Then $\rho(k,0)$ results from
\[ \rho(k,0) = \phi_{12}^+(k,x,0)|_{x=0^+}. \]

2 - The spectral transform at time $t$, $\rho(k,t)$, is calculated in the upper half $k$-plane from $\rho(k,0)$ and Dirichlet condition $\theta_0(t)$ by solving for $k \neq 0$ the Riccati time-evolution
\[ \rho_t = \frac{i}{4k} \left[ \rho^2 \sin \theta_0 - 2\rho \cos \theta_0 - \sin \theta_0 \right], \]
and for $k = 0$
\[ \rho(0,t) = \frac{\cos \theta_0 - 1}{\sin \theta_0}. \]

The $N$ poles $k_n$ in the upper half-plane and related residues $\rho_n$ of $\rho(k,t)$ must be identified for each value of the parameter $t$.

3 - The integral system (3.5)-(3.6) which, within the reduction (3.37), simplifies to
\begin{align}
\psi_{11}^+(k,x,t) &= 1 - \frac{1}{2i\pi} \int_{-\infty}^{+\infty-0} \frac{d\lambda}{\lambda - k} \rho(-\lambda,t)e^{2i\lambda x}\psi_{12}^-(\lambda,x,t) \\
&- \sum_{n=1}^{N(t)} \frac{\rho_n(t)}{k_n(t)+k} \psi_{12}^-(-k_n(t),x,t)e^{-2ik_n(t)x},
\end{align}

23
\[ \psi_{12}^{-}(k, x, t) = \frac{1}{2i\pi} \int_{-\infty+i0}^{+\infty+i0} \frac{d\lambda}{\lambda - k} \rho(\lambda, t)e^{-2i\lambda x} \psi_{11}^{+}(\lambda, x, t) \]
\[ - \sum_{n=1}^{N(t)} \frac{\rho_n(t)}{k_n(t) - k} \psi_{11}^{+}(k_n, x, t)e^{-2ik_n(t)x} , \]  

(5.22)

is solved to get \( \psi_{11}^{+}(k, x, t) \). This system of equation holds for any fixed value of the “parameters” \( x \) and \( t \).

4 - Finally the solution \( \theta(x, t) \) is obtained under the form \( \theta_x(x, t) = -2q(x, t) \) where \( q(x, t) \) results from \( \psi_{11}^{+} \) and \( \rho \) as

\[ q(x, t) = \frac{1}{2i\pi} \int_{-\infty+i0}^{+\infty+i0} d\lambda \rho(\lambda, t)\psi_{11}^{+}(\lambda, x, t)e^{-2i\lambda x} \]
\[ + 2t \sum_{n=1}^{N(t)} \rho_n(t)\psi_{11}^{+}(k_n(t), x, t)e^{-2ik_n(t)x} . \]  

(5.23)

5.5 An explicit example

To illustrate the above method we consider the explicitly solvable case of a piecewise constant boundary datum \( \theta_0(t) \), namely

\[ t \in [0, t_1] : \theta_0(t) = 0 , \]
\[ t \in [t_j, t_{j+1}] : \theta_0(t) = \varphi_j , \]
\[ t \in [t_n, \infty] : \theta_0(t) = 0 , \]  

(5.24)

compatible with a vanishing initial condition

\[ \theta(x, 0) = \tilde{\theta}(x) = 0 . \]  

(5.25)

Note that such a boundary datum represents an arbitrary localized function \( \theta_0(t) \) discretized with \( n \)-step time grid.

We adopt the following notations

\[ \rho_j(t) = \rho(k, t) \text{ for } t \in [t_j, t_{j+1}] , \]
\[ \rho_j = \rho_j(t_{j+1}) \equiv \rho(k, t_{j+1}) . \]  

(5.26)

Then the solution of the evolution \([5,6]\) can be written for \( t \in [t_j, t_{j+1}] \)

\[ \rho_j(t) = \frac{b_j(\rho_{j-1} + a_j)e^{i(t-t_j)/2k} - a_j(\rho_{j-1} + b_j)}{(\rho_{j-1} + a_j)e^{i(t-t_j)/2k} - (\rho_{j-1} + b_j)} , \]  

(5.27)

with the following definitions

\[ a_j = \frac{1 + \cos \varphi_j}{\sin \varphi_j} , \quad b_j = \frac{1 - \cos \varphi_j}{\sin \varphi_j} . \]  

(5.28)
Note that the limit $\varphi_j \to 0$ out of (5.27) does produces the solution to evolution (5.6) where $\theta_0 = \varphi_j$ is set to zero.

The large-k behaviour of solution (5.27) is obtain by induction and reads

$$\rho_j(t) = -\frac{i}{4k} \left[ (t - t_j) \sin \varphi_j + \sum_{\ell=1}^{j} (t_{\ell} - t_{\ell-1}) \sin \varphi_{\ell-1} \right] + O\left(\frac{1}{k^2}\right). \quad (5.29)$$

Hence we have

$$\partial_t \rho_j(t) = -\frac{i}{4k} \sin \varphi_j + O\left(\frac{1}{k^2}\right) \quad (5.30)$$

compatible with (5.7) and solution (5.27) has the expected large-k behaviour.

At the last step, that is for $\varphi_n = 0$, (5.27) can be writen

$$\rho_n(t) = \rho_{n-1} e^{i(t - t_n)/2k} + O\left(\frac{1}{k^2}\right), \quad (5.31)$$

which eventually simplifies to

$$\rho_n(t) = e^{i(t - t_n)/2k} \quad (5.32)$$

As a consequence, the spectral transform evaluated after the total effect of the boundary datum $\theta_0(t)$, i.e. for $t > t_n$, is completely determined by $\rho_{n-1} = \rho(k, t_n)$. This function of $k$ is itself obtained as the following mapping

$$\rho_0 = 0, \quad \rho_j = -\frac{b_j(\rho_{j-1} + a_j) e^{i(t_j + t_j)/2k} - a_j(\rho_{j-1} + b_j)}{(\rho_{j-1} + a_j) e^{i(t_j + t_j)/2k} - (\rho_{j-1} + b_j)}, \quad j = 1 \cdots n - 1. \quad (5.33)$$

This is an explicit expression that allows to get insight in the effect of a generic input datum $\theta_0(t)$ on a vanishing background under suitable (arbitrary) discretization. We report the study of the above mapping (in the complex k-plane) to forthcoming work and simply mention here that the main point concerns the study of the creation (anihilation) of poles of $\rho(k, t)$ in the upper half k-plane.

Remembering the reality constraint (3.39), poles of $\rho(k)$ come by pair, and a pole $k = k_n(t)$ of $\rho_j(t)$ is given by

$$e^{i\frac{t_{j+1}}{2k_n}} = \frac{\rho_{j-1} + b_j}{\rho_{j-1} + a_j}, \quad (5.34)$$

where $\rho_{j-1}$ is itself a function of $k_n$.

This equation for the pole $k_n$ is in general highly complicated but if $\rho_{j-1} = 0$ (as for the initial stage $t = t_1$), the above relation becomes

$$e^{i\frac{t_{j+1}}{2k_n}} = \frac{1 - \cos \varphi_1}{1 + \cos \varphi_1}. \quad (5.35)$$
Defining
\[ \gamma = \ln \frac{1 - \cos \varphi_1}{1 + \cos \varphi_1}, \quad (5.36) \]
the solution of (5.35) can be written
\[ k_n(t) = \frac{1}{2} \left[ (2n + 1)\pi + i\gamma \right] \frac{(t - t_1)}{(2n + 1)^2\pi^2 + \gamma^2}, \quad (5.37) \]
where \( n \) assumes values in \( \mathbb{Z} \). The above expression is identical to the one obtained in [13] (see formula 5.52) for \( t - t_1 = T \). A property of our approach is to explicitly display the \emph{time dynamics} of the spectral transform (such as poles generation and motion).

Figure 1: Positions of the poles of \( \rho(k, t) \) for \( n \in [-20, 19] \), \( t - t_1 = 10 \) and \( \varphi_1 = 0.65\pi \).

Among the above set of poles, only those with positive imaginary part contribute to the soliton spectrum. It is then remarkable that
\[ \varphi_1 \in \left[ 0, \frac{\pi}{2} \right] \Rightarrow \gamma < 0 \Rightarrow \text{Im}(k_n) < 0, \quad (5.38) \]
\[ \varphi_1 \in \left[ \frac{\pi}{2}, \pi \right] \Rightarrow \gamma > 0 \Rightarrow \text{Im}(k_n) > 0, \quad (5.39) \]
which shows that no poles are generated by a constant boundary datum \( \theta_0(t) = \varphi_1 \) smaller than \( \pi/2 \), while an infinity of them are generated for \( \varphi_1 > \pi/2 \). An example of poles locations is shown on the figure where we see that the point
$k = 0$ is an accumulation point for the infinity of poles. Indeed when $n \to \infty$ we have from (5.37) $k_n \to 0$. The case $\varphi_1 = \pi/2$ is singular as $\rho(k,t)$ would then possess poles on the real axis, for which the theory fails.

Note that the discretization of a smooth localized boundary datum $\theta_0(t)$ produces small $\varphi_j$’s. Hence the soliton generation would then result from the contributions in equation (5.34) of the $\rho_j-\ell$. In the case of a single constant boundary datum of amplitude greater than $\pi/2$, the assignation of solitons to the presence of an infinity of poles is not clear (usually an infinity of poles is related to a self-similar structure).

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