Abstract. Fractal random media can exhibit a dramatic topological phase transition, changing from a dust-like set of isolated points into a connected cluster that spans the entire system. The precise transition points are typically unknown and difficult to estimate. In many classical percolation models the percolation thresholds have been approximated well using additive geometric functionals, known as Minkowski functionals or intrinsic volumes. Motivated by the question whether a similar approach is possible for fractal models, we introduce and study corresponding geometric functionals for Mandelbrot’s fractal percolation process $F$. More precisely, our functionals arise as rescaled limits of expected intrinsic volumes of (I) the construction steps of $F$ or (II) their closed complements. These new functionals are closely related to (expected) fractal curvatures, but in contrast to them they can be computed explicitly and are easily determined from simulations with high precision, for which we provide a freely available code. They may serve as geometric descriptors of the fractal percolation process and can be generalized to other random self-similar sets. While it turns out that these functionals cannot be used directly to improve the known bounds on percolation thresholds, they provide further geometrical insights.

1. Introduction

Fractal percolation in $\mathbb{R}^d$ is a family of random subsets of the unit cube $J := [0,1]^d \subset \mathbb{R}^d$ depending on two parameters $M \in \mathbb{N}_{\geq 2}$ and $p \in [0,1]$, which is informally defined as follows: In the first step divide $J$ into $M^d$ closed subcubes of side length $1/M$. Each of these subcubes is kept with probability $p$ and discarded with probability $1-p$ independently of all other subcubes. Then this construction is iterated for each subcube. Let $F_n$, $n \in \mathbb{N}$, denote the union of the subcubes kept in the $n$-th step. Assuming that $F_{n-1}$ is already constructed, in the $n$-th step each cube in $F_{n-1}$ (of side length $1/M^{n-1}$) is divided into $M^d$ subcubes (of side length $1/M^n$) and each of these subcubes is kept (and included in $F_n$) with probability $p$ independently of all other subcubes and of the previous steps. This way one obtains a decreasing sequence $F_0 := J \supset F_1 \supset F_2 \supset \ldots$ of (possibly empty) random compact sets. The limit set

$$F := \bigcap_{n \in \mathbb{N}_0} F_n$$

is known as fractal percolation or Mandelbrot percolation, see e.g. [1, 3]. It is well known that $F$ is almost surely empty if $p \leq 1/M^d$, i.e. if on average not more than one of the $M^d$ subcubes of any cube in the construction survives. For $p > 1/M^d$, however, there is a positive probability (depending on $p, M$ and $d$) that $F \neq \emptyset$. 

[1] and [3] provide further references and details.

2000 Mathematics Subject Classification. 28A80, 60K35, 82B43.

Key words and phrases. fractal percolation, Mandelbrot percolation, Minkowski functionals, intrinsic volumes, curvature measures, fractal curvatures, random self-similar set, percolation threshold.
and conditioned on $F$ being nonempty, the Hausdorff dimension and equally the Minkowski dimension of $F$ are almost surely given by the number

$$\dim_H F = D := \frac{\log(M^d p)}{\log(M)} = d - \frac{\log(1/p)}{\log(M)},$$

see e.g. [4]. The sets $F$ are among the simplest examples of self-similar random sets as introduced in [6, 8, 13]. Beside many other properties, in particular their connectivity has been studied. Fractal percolation exhibits a dramatic topological phase transition – for all dimensions $d \geq 2$ and all $M \in \mathbb{N}_{\geq 2}$ – when the parameter $p$ increases from 0 to 1: there is a critical probability $p_c = p_c(M,d) \in (0,1)$ such that, for $p < p_c$, the set $F$ is almost surely totally disconnected (‘dustlike’), and, for $p \geq p_c$, there is a positive probability that $F$ percolates, meaning here that $F$ has a connected component which intersects the left and the right boundary of $J$, that is $\{0\} \times [0,1]^{d-1}$ and $\{1\} \times [0,1]^{d-1}$. Chayes, Chayes and Durrett [4] were the first to prove the existence of this transition rigorously in dimension $d = 2$ and their arguments show it in fact for any dimension $d \geq 2$. Note that the phase transition is discontinuous: at $p_c$ there is a positive probability that $F$ percolates.

Like in many other percolation models, the exact values of $p_c(M,d)$ are not known. In fact, for this model the situation is even worse than usual. Classical techniques using finite size scaling apparently fail in this fractal model, since a proper scaling regime is inaccessible with modern hardware. Some rigorous lower and upper bounds on $p_c(M,d)$ have been obtained in particular for $d = 2$, see Sec. 2 but they are not tight.

Morphometric methods to estimate thresholds in percolation models have been proposed in [15] and intensively studied in the physics literature [14, 16, 9]. They are based on additive functionals from integral geometry, in particular the Euler characteristic, and rely on the observation that in many percolation models the expected Euler characteristic per site (as a function of the model parameter $p$)–which can easily be computed analytically in many models–has a zero close to the percolation threshold of the model. Based on empirical evidence and heuristic arguments,
these zeros provide for many classes of percolation models reasonable approximations and putative bounds on the thresholds that capture their dependence on system parameters like the degree of anisotropy \( R \).

In analogy to these findings for discrete and continuum percolation models, we introduce and study here some corresponding geometric functionals for fractal percolation and ask the question whether one can use them to predict or at least approximate percolation thresholds. It is natural to expect that the dramatic phase transition (from dust to strong connectivity) in these fractal models should leave at least some trace in geometric functionals such as the Euler characteristic. Due to the self-similarity of the model, there is even some hope that – although percolation is a global property – the thresholds can be predicted by local information alone.

Note that \( F \) as well as the construction steps \( F_n \) are random compact subsets of the unit cube \([0,1]^d\). Moreover, due to their construction, the sets \( F_n \) are finite (random) unions of cubes (of side length \( 1/M^n \)). Therefore, each \( F_n \) is almost surely polyconvex, i.e., a finite union of convex sets, and so intrinsic volumes \( V_0(F_n), V_1(F_n), \ldots, V_d(F_n) \) (also known as Minkowski functionals) and even curvature measures are well defined for \( F_n \) almost surely.

Recall that intrinsic volumes are defined for any compact, convex set \( K \subset \mathbb{R}^d \) as the unique coefficients (up to normalization) in the Steiner formula, expressing the Lebesgue measure of \( K_{\varepsilon} := \{ x \in \mathbb{R}^d : \inf_{y \in K} ||x - y|| \leq \varepsilon \} \) as a polynomial in \( \varepsilon \). They are additive functionals, i.e., for any \( k = 0, \ldots, d \) and any compact, convex sets \( K, L \subset \mathbb{R}^d \) the relation

\[
V_k(K) + V_k(L) = V_k(K \cup L) + V_k(K \cap L)
\]

holds, provided \( K \cup L \) is again convex. They can be extended additively to the convex ring \( \mathcal{R}^d \), the family of all compact polyconvex sets. Since the set family \( \mathcal{R}^d \) is closed under unions and intersections, equation (1.3) holds for any \( K, L \in \mathcal{R}^d \), see e.g. [17, Ch. 4] or [18, Ch. 14.2] for more details on intrinsic volumes.

While for the sets \( F_n \) intrinsic volumes are well defined, the limit set \( F \) is a fractal and so these functionals are not directly defined. Since the set family \( \mathcal{R}^d \) is closed under unions and intersections, equation (1.3) holds for any \( K, L \in \mathcal{R}^d \), see e.g. [17, Ch. 4] or [18, Ch. 14.2] for more details on intrinsic volumes.

Our first main result is a general formula which expresses these limits in terms of lower dimensional mutual intersections of certain parts of the construction steps \( F_n \). Let \( J_1, \ldots, J_{M^d} \) be the \( M^d \) closed subcubes into which \([0,1]^d\) is divided in the first step of the construction of \( F \). Denote by \( F_{n,j}^d \), \( j = 1, \ldots, M^d \), the union of all subcubes kept in the \( n \)-th step, that are contained in \( J_j \) (see (3.1) for a formal definition).

**Theorem 1.1.** Let \( F \) be a fractal percolation on \([0,1]^d\) with parameters \( M \in \mathbb{N}_{\geq 2} \) and \( p \in (M^{-\min(3,d)},1] \). Let \( D \) be the Hausdorff dimension of \( F \) (given by (1.2)) and let \( r := 1/M \). Then, for each \( k \in \{0, \ldots, d\} \), the limit

\[
\bar{V}_k(F) := \lim_{n \to \infty} r^{n(D-k)}E V_k(F_n)
\]

exists and is given by the expression

\[
q_{d,k} + \sum_{T \subset \{1, \ldots, M^d\}, |T| \geq 2} (-1)^{|T|-1} \sum_{n=1}^{\infty} r^{n(D-k)}E V_k(\bigcap_{j \in T} F_{n,j}^d),
\]

where \( q_{d,k} := V_k([0,1]^d) \) is the \( k \)-th intrinsic volume of the unit cube in \( \mathbb{R}^d \).

We point out that all the intersections occurring in (1.4) consist of at least two of the cubes \( F_{n,j}^d \) and are thus contained in some hyperplane. Hence, on the right hand
side of the formula only sets appear which can be studied in a lower dimensional ambient space allowing to use fractal percolations in lower dimensional cubes for the computations. This makes the formula practically useful for explicit calculations as carried out in $\mathbb{R}^1$ and $\mathbb{R}^2$ below. Note also that many of the intersections are actually empty and that there are a lot of symmetries between the remaining ones.

The formula holds for all parameters $p$ such that $M^{\min\{3,d\}} > 1$, which in dimensions $d \leq 3$ includes all parameters, for which $F$ is nonempty with positive probability. For dimensions $d \geq 4$, however, some interval $M^{-d} < p \leq M^{-3}$ remains, for which the formula is probably still true but for which we do not provide a proof here (see also Remark 6.4).

In $\mathbb{R}^2$ (and similarly for $\mathbb{R}$, see Corollary 4.3) we use the formula in Theorem 1.1 to derive more explicit expressions for the limits $\mathcal{V}_k(F)$.

**Theorem 1.2.** Let $F$ be a fractal percolation in $[0,1]^2$ with parameters $M \in \mathbb{N}_{\geq 2}$ and $p \in (1/M^2,1]$. Then,

$$
\mathcal{V}_2(F) = 1, \quad \mathcal{V}_1(F) = \frac{2M(1-p)}{M-p} \quad \text{and}
$$

$$
\mathcal{V}_0(F) = 1 - \frac{2p(M-1)^2}{M-p} \left( \frac{3}{M-1} - \frac{4p}{M-p} + \frac{p^2}{M-p^2} \right) + \frac{2p(M^2-1)}{M^2-p} - \frac{4p^2(M-1)^2}{(M-p)^2} + \frac{p^3(M-1)^2(M+p^2)}{(M-p^2)(M^2-p^3)}.
$$

While $\mathcal{V}_2(F)$ (the rescaled limit of the expected area) is constant and thus independent of $M$ and $p$, the functional $\mathcal{V}_1(F)$ (the rescaled limit of the expected boundary length) is monotone decreasing in $p$ (for each fixed $M$). Most interesting is the limit $\mathcal{V}_0(F)$ of the expected Euler characteristics of $F_n$.

Figure 2 (left) shows $\mathcal{V}_0(F(p))$ as a function of the survival probability $p$ for different $M$ (black curves). The dotted vertical line indicates the threshold below which $F$ is almost surely empty. The coloured curves depict the analytic expressions for finite approximations of the limit $\mathcal{V}_0(F(p))$ by the rescaled functionals $p \mapsto r^{nD(p)}E\mathcal{V}_0(F_n(p))$ for different $n$ (obtained in the proof of Theorem 1.2). Already for $n = 12$ the curves are almost indistinguishable from $n = 10$, indicating a fast convergence, which is rigorously confirmed below, see Remark 4.7. The formulae for finite approximations are compared to simulations, see Remark 4.8. The marks depict the arithmetic mean over 2500 to 75000 samples (depending on $n$). The error bars depict the standard error of the mean. The simulation results are in excellent agreement with the analytic curves.

The functionals $\mathcal{V}_k(F)$ which are based on the approximation of $F$ by the sequence $F_n$, provide a natural and intuitive first approach to quantify the geometry of fractal percolation $F$. One should however keep in mind that these limits most likely also depend on the approximation sequence. There are other natural sequences of sets which approximate $F$ well and which may even be better suited to capture certain aspects of the geometry of $F$. In particular, the parallel sets $F_{(r^\varepsilon)}$, $\varepsilon > 0$ of $F$ are considered a good means of approximation, preserving many properties, and have been studied extensively also for (deterministic and random) self-similar sets, cf. e.g. [23, 24, 25]. Although the existence of the resulting limits (known as fractal curvatures) has been established for random self-similar sets in [25], parallel set approximation seems technically too difficult in order to derive explicit expressions for these limits even for the simplest examples.

Note that in the current approach using the sets $F_n$ we consider closed cubes, meaning that two surviving subcubes in any finite approximation $F_n$ are connected.
even if they touch each other at a single corner. In the limit set $F$ such connections cannot survive (because it would require an infinite number of consecutive successes in a Bernoulli experiment with success probability $p$: the survival at each level $n$ of the two level-$n$ squares touching the corner). Therefore, it might be advisable to seek for an approximation which avoids diagonal connections from the beginning. Such an approximation is provided by the closed complements of the $F_n$. By connecting the subcubes in the complement, such non-surviving connections get disconnected already in the finite approximations $F_n$. More precisely, we study in Section 5 expectations $\mathbb{E}V_k(C_n)$, where $C_n := [0,1]^d \setminus F_n$ are the closed complements of the $F_n$ in the unit cube, and the limits

$$\mathbb{E}^F_k(n) := \lim_{m \to \infty} r^{n(D-k)} \mathbb{E}V_k(C_n),$$

with $D$ as in (1.2). We obtain for these limits a general formula (see Theorem 5.1), which is very similar to the one obtained in Theorem 1.1 for $\mathbb{E}V_k(F)$. Again, for the case $d = 2$, we have computed explicit expressions. Here we state only the formula for the Euler characteristic (i.e., the case $k = 0$), the most interesting functional in connection with the percolative behaviour to be discussed in the next section (for the case $k = 1$ see Proposition 5.13).

**Theorem 1.3.** Let $F$ be a fractal percolation in $[0,1]^2$ with parameters $M \in \mathbb{N}_{\geq 2}$ and $p \in (1/M^2, 1]$. Then,

$$V_0^F(F) = M^2(1-p)^2 + (M-1)p^2 + (M-1)p - M \over (M^2 - p^3)(M - p).$$

Note that in $\mathbb{R}^2$, $-V_0(C_n)$ is essentially the Euler characteristic of the set $F_n$ with all diagonal connections between cubes removed (up to some boundary effects along the boundary of $[0,1]^2$). Therefore, $-V_0(F)$ will be the functional of interest in the sequel in connection with the percolative properties of $F$.

More precisely, $1 - V_0(C_n) + V_0(C_n \cap \partial [0,1]^2)$ corresponds to the Euler characteristic of the cell complex with vertex set given by the squares of $F_n$, edges between any two squares if they intersect in a common side and faces given by four edges forming a square. It can be shown that the effect of the last summand $V_0(C_n \cap \partial [0,1]^2)$ is asymptotically negligible. The approach corresponds to considering nearest neighbors in $\mathbb{Z}^2$—as opposed to taking also next-to-nearest neighbors into account, as done before.

2. Relation with percolation thresholds

We start by recalling some known results concerning the percolation thresholds $p_c = p_c(M)$ of fractal percolation in the plane. Already Chayes, Chayes and Durrett [4] established that, for any $M \in \mathbb{N}_{\geq 2}$,

$$\sqrt{1/M} \leq p_c(M) \leq 0.9999.$$  

In [3] it is shown that the percolation threshold $p_{c,NN}$ of site percolation on the nearest neighbor (NN) graph on $\mathbb{Z}^2$ is a lower bound, i.e. $p_{c,NN} \leq p_c(M)$ for any $M \in \mathbb{N}_{\geq 2}$. Since $0.556 \leq p_{c,NN}$, cf. [21], this improves the above lower bound for any $M \geq 4$. Moreover, $\lim_{M \to \infty} p_c(M) = p_{c,NN}$, see [3]. It is believed that $p_c(M') \leq p_c(M)$ for $M' \geq M$ but this monotonicity is only established in special cases, e.g. if $M' = M^2$. These bounds have been improved in [22, 5] for some small $M$. The best known bounds for $M = 2$ and 3 are

$$0.881 \leq p_c(2) \leq 0.993 \quad \text{and} \quad 0.784 \leq p_c(3) \leq 0.940,$$

respectively, and $p_c(4) \leq 0.972$, cf. [5].
Figure 2. Rescaled expected Euler characteristic of finite approximations $F_n$ (left) and their closed complements $C_n$ (right) as functions of the survival probability $p$ for $M = 2$ (top), $M = 3$ (center) and $M = 4$ (bottom). Each plot compares finite approximations with increasing $n$ to the limit curve ($n = \infty$), that is, to $p \mapsto V_0(F(p))$ given in Theorem 1.2 (left) and $p \mapsto -V_0(F(p))$ given in Theorem 1.3. The coloured areas indicate the rigorous known bounds on the percolation threshold, see (2.2).

In view of the aforementioned observations in [15, 14, 16, 9], that the zero of the expected Euler characteristic per site is close to the percolation thresholds in many percolation models, let us now discuss the connections between the limit functionals for $F$ introduced above and the connectivity properties in fractal percolation.

$p_0$ is a lower bound for $p_c$. Our first observation is that, for any $M \in \mathbb{N}_{\geq 2}$, the function $p \mapsto V_0(F(p))$ has a unique zero $p_0 = p_0(M)$ in the open interval $(1/M^2, 1)$, as suggested by Figure 2 (left). Moreover, $V_0(F(p)) > 0$ for $p < p_0$ and $V_0(F(p)) < 0$ for $p > p_0$. By comparing $p_0$ with the known lower bounds for $p_c$, it is not hard to see that $p_0(M)$ is a lower bound for $p_c(M)$, i.e.

$$p_0(M) \leq p_c(M), \quad \text{for all } M \in \mathbb{N}_{\geq 2}.$$
Indeed, for $M = 2$ and 3, $p_0(M)$ is below the lower bounds for $p_c(M)$ of Don, cf. (2.2), while for $M \geq 4$, $p_0(M) \leq 0.556$, which is the lower bound for the site percolation threshold $p_{c,NN}$ due to van den Berg and Ermankov [21]. Although $p_0$ is a lower bound for $p_c$, unfortunately it is not very tight. In particular, it does not improve the known bounds.

The large-$M$ limit of $p_0(M)$ is not $p_{0,NN}$ but $p_{0,NNN}$.

In analogy with $p_c$, for which $p_c(M) \rightarrow p_{c,NN}$ as $M \rightarrow \infty$, we observe that also the zeros $p_0(M)$ converge to a limit, as $M \rightarrow \infty$. The (pointwise) limit of the functions $p \mapsto \overline{v}_0(F(p))$, as $M \rightarrow \infty$, is the function $v$ given by

$$v(p) := 1 - 4p + 4p^2 - p^3, \quad p \in (0, 1),$$

which is the red curve depicted in Figure 3 (left). It turns out that $v(p)$ coincides (up to a factor $p$) with the mean Euler characteristic per site $\overline{v}_0(Z^2, NNN; p)$ of site percolation on the next-to-nearest neighbor (NNN) graph on $Z^2$, cf. [10]. In particular, this implies for the zeros that

$$\lim_{M \rightarrow \infty} p_0(M) = p_{0,NNN},$$

where $p_{0,NNN} = (3 - \sqrt{5})/2$ is the unique zero of $v$ in $(0, 1)$, i.e. of $p \mapsto \overline{v}_0(Z^2, NNN; p)$.

At first glance it might be surprising that a different site percolation model appears in the limit (NNN instead of NN, which showed up for the percolation thresholds). But this is consistent with the discussion before Theorem 1.3 – there is too much connectivity in the approximation sets $F_n$. We will get back to this in a moment.

$p_{\min} - a bound for p_c$? For any $M \in \mathbb{N}_{\geq 2}$, the function $p \mapsto \overline{v}_0(F(p))$ has a unique minimum $p_{\min} = p_{\min}(M)$ in the open interval $(1/M^2, 1)$, which lies always to the right of $p_0$ (i.e., potentially closer to $p_c$). This is another natural candidate to bound the percolation threshold. For $M = 2$, $p_{\min}$ is clearly a lower bound for $p_c(2)$, but as $M \rightarrow \infty$, $p_{\min}(M) \rightarrow 2/3$, which is above $p_{c,NN}$. So, for large $M$, $p_{\min}(M)$ is clearly not a lower bound for $p_c(M)$. This implies that $p_{\min}(M)$ can neither be a general lower nor a general upper bound for the percolation thresholds. Interesting open questions are at which $M$, $p_c(M)$ and $p_{\min}(M)$ change their order and whether $p_{\min}(M)$ (which can be interpreted as the parameter for which the difference between number of holes and the number of connected components is maximal) is related in some way to the percolation transition.

\[\text{Figure 3. For increasing values of the number of subdivisions } M \text{ (color coded), the rescaled expected Euler characteristic of fractal percolation (left) and its complement (right) are plotted as functions of } p. \text{ The limiting curve (red) for } M \rightarrow \infty \text{ corresponds to the mean Euler characteristic per site (rescaled by the intensity) of site percolation on } Z^2 \text{ with eight or four neighbors, respectively.}\]
As the discussion before Theorem 1.3 suggests, there might be approximation sequences for $F$ which better capture the percolative behaviour of $F$ and one candidate sequence are the modified sets $F_n$ with all diagonal connections between cubes removed, which we studied by looking at the closed complements $C_n := \mathcal{F} \setminus F_n$. Let us now discuss possible connections with percolation thresholds of the corresponding limit functionals $V_0(F)$.

$p_1$ – a lower bound for $p_c$? Figure 2 (right) shows plots of the functions $p \mapsto \nabla_0(F(p))$ for different $M$ (the black curves labelled ‘$n = \infty$’), again accompanied by some finite approximations for different $n$. In Figure 3 (right) there are plots of the functions $p \mapsto -\nabla_0(F(p))$ for all $M$ together with the limit curve as $M \to \infty$. Each of these curves possesses again a unique zero $p_1 = p_1(M)$ in $(1/M^2, 1)$. It is apparent from the plots in Figure 2 that $p_1(M)$ is larger than $p_0(M)$ and thus potentially closer to the percolation threshold $p_c(M)$. At least for $M = 2, 3$, $p_1(M)$ is a better lower bound for $p_c(M)$. But is this true in general? Unfortunately not, as will become clear from looking at large $M$.

Large-$M$ limit of $p_1(M)$. The (pointwise) limit of the functions $p \mapsto -\nabla_0(F(p))$, as $M \to \infty$, is

$$v^c(p) := -(1-p)(p^2 + p - 1) = p^3 - 2p + 1, \quad p \in (0, 1]$$

which is the red curve depicted in Figure 3 (right). It turns out to coincide (up to a factor $p$) with the mean Euler characteristic per site $V_0(Z^{2,NN}, p)$ of site percolation on the nearest neighbor graph on $\mathbb{Z}^2$ as a function of $p \in [0, 1]$, see e.g. [18] eq. (5), p. 4. In particular, one gets for the zeros that

$$\lim_{M \to \infty} p_1(M) = p_{0,NN},$$

where $p_{0,NN} = (\sqrt{5} - 1)/2 \approx 0.618$ is the unique zero of $v^c$ in $(0, 1)$. Note that $p_{0,NN}$ is strictly larger than $p_{c,NN} \approx 0.59$. Thus for large $M$, $p_c(M) < p_1(M)$, while for $M = 2, 3$, one has $p_c(M) > p_1(M)$. So $p_1$ can neither be a general lower bound nor a general upper bound for $p_c$. This observation also rules out the minimum of $p \mapsto \nabla_0(F(p))$ to be a good general bound in any way.

These findings show that there is not such a close connection between the Euler characteristics and percolation thresholds in this fractal model as there are in other percolation models. An explanation, why the phase transition leaves no signature in the studied functionals might be that percolation happens in fact on lower dimensional subsets. Recently it has been shown, see [2], that for $p \geq p_c$ (and conditioned on $F$ being nonempty), the union $Z$ of all connected components of $F$ larger than one point forms almost surely a set of strictly smaller Hausdorff dimension than the remaining set $F \setminus Z$ (the dust), which has dimension $\dim_H F \setminus Z = \dim_H F = D$ almost surely. The rescaling with $r^{Dn}$ of the geometric functionals essentially means that they do not see the lower dimensional set $Z$ on which percolation occurs. So from the point of view of the Hausdorff dimension, our result is consistent with the findings in [2]. But in [2], it is also shown that in contrast the Minkowski dimensions of $Z$ and $F \setminus Z$ coincide almost surely for $p \geq p_c$. Since our approximation of $F$ by unions of boxes $F_n$ is rather related to the Minkowski (or box) dimension than to the Hausdorff dimension, our results support the hypothesis that, also in the Minkowski setting, the effect of the dust dominates that of the larger components, though not on the level of dimension but on the refined level of associated measures or contents as provided by our functionals. Long before percolation occurs (i.e. for $p < p_c$), the expected Euler characteristic $E V_0(F_n)$ becomes negative, i.e. it detects more holes than components in the approximations $F_n$, which indicates that the $n$-th approximation of the dust must have a lot of structure which only disappears in the limit. More refined methods are necessary to separate the dust from the
larger clusters. It might for instance be worth to look at the Euler characteristic of the percolation cluster in finite approximations.

We emphasize that, although our work is motivated by questions regarding the percolation properties, our focus here is on establishing the existence of the geometric limit functionals \( \mathcal{V}_k(F) \) and \( \mathcal{V}_k^c(F) \), and on computing them explicitly. The methods developed and the results obtained can be transferred to many other random (self-similar) models. The functionals may have other applications. Just as fractal curvatures, they clearly carry geometric information beyond the fractal dimension. But unlike fractal curvatures, they can be computed explicitly for random sets (at least in some cases). Even more importantly, they can be estimated well from the finite approximations, see Remarks 4.7 and 5.14 for a discussion of the speed of convergence of \( V^n_k E_k(F_n) \) and \( V^n_k R_k(C_n) \) as \( n \to \infty \), and see Remark 4.8 for a practical demonstration. Hence the functionals may serve as robust and efficient geometric descriptors in applications and may e.g. help to distinguish different geometric structures of the same fractal dimension. It is an aim of future research to develop this “box counting” approach further to work for general (random) fractals.

The remainder of this article is organized as follows. In Section 3 we describe fractal percolation as a random self-similar set and introduce some notation and basic concepts. In Section 4 we study in detail the approximation of \( F \) by the sets \( F_n \), and in Section 5 the approximation by the sets \( C_n \). In both cases we prove first a general formula for arbitrary dimensions (Theorems 1.1 and 5.1) which we then use to compute the limit functionals in \( \mathbb{R} \) and \( \mathbb{R}^2 \). A careful analysis of the model in \( \mathbb{R} \) is essential for the computations in \( \mathbb{R}^2 \). Additionally, it is necessary to understand the intersection of two independent copies of \( F \) in \( \mathbb{R} \), the analysis of which also provides a new point of view on the lower bound for \( p_c \) in (2.1) obtained in [3], see Remark 4.9. In the course of the proofs not only explicit expressions for the limit functionals are derived but also exact formulas for the \( n \)-th approximations, see in particular Remarks 4.7 and 5.14. In the last section some estimates are proved which ensure the convergence of the series occurring in the main formulas in Theorems 1.1 and 5.1. They are not needed for the further results in \( \mathbb{R} \) and \( \mathbb{R}^2 \), as the convergence can be checked directly in these cases but ensure their validity in higher dimensions.

3. Fractal percolation as a random self-similar set

Fractal percolation \( F \) in \( \mathbb{R}^d \) with parameters \( p \in [0, 1] \) and \( M \in \mathbb{N}_{\geq 2} \) is a random self-similar set generated by the following random iterated function system (RIFS) \( S \) constructed on the basic set \( J = [0, 1]^d \). Denote the \( M \) subsets of side length \( r = 1/M \) into which \( J \) is divided in the first step of the construction of \( F \) described above by \( J_1, \ldots, J_{M^d} \). \( S \) is a random subset of the set \( \Phi := \{ \phi_1, \ldots, \phi_{M^d} \} \), where \( \phi_j, j = 1, \ldots, M^d \), is the similarity which maps \( J \) to \( J_j \) (rotation and reflection free, for simplicity and uniqueness). Each map \( \phi_j \) is included in \( S \) with probability \( p \) independent of all the other maps. It is obvious that \( S \) satisfies the open set condition (OSC) with respect to the interior \( \text{int}(J) \) of \( J \), since \( S \) is a random subset of \( \Phi \) and even the full set \( \Phi \) satisfies OSC with respect to \( \text{int}(J) \).

For obtaining \( F \) as an invariant set of the RIFS \( S \), we employ a Galton-Watson tree on the set of all finite words \( \Sigma_n := \bigcup_{n=0}^\infty \Sigma_n \), where \( \Sigma_n := \{1, \ldots, M^d\}^n \), \( n \in \mathbb{N}_0 \). In particular, \( \Sigma_0 = \{\varepsilon\} \) where \( \varepsilon \) is the empty word of length \( |\varepsilon| = 0 \). For each \( \sigma \in \Sigma_n \), let \( \mathcal{S}_n \) be an independent copy of the RIFS \( S \). \( \mathcal{S}_n \) contains a random number \( \nu_\sigma \) of maps (with \( \nu_\sigma \) being binomially distributed with \( p \) and \( M^d \)). Let \( I_\sigma \subseteq \{1, \ldots, M^d\} \) be the set of indices of the maps in \( \mathcal{S}_n \). It is convenient to denote these maps by \( \phi_\sigma_i, i \in I_\sigma \). Note that \( |I_\sigma| = \nu_\sigma \). In particular, \( I_\sigma \) may be empty.
We build a random tree $T$ in $\Sigma_*$ as follows: set $T_0 := \{\varepsilon\}$ and define, for $n \in \mathbb{N}_0$, $T_{n+1} := \emptyset$, if $T_n = \emptyset$, and 

$$
T_{n+1} := \{\sigma i : \sigma \in T_n, i \in I_\sigma\},
$$

if $T_n \neq \emptyset$. Finally, we set 

$$
T := \bigcup_{n=0}^{\infty} T_n.
$$

$T$ can be interpreted as the population tree of a Galton-Watson process in which $T_n$ represents the $n$-th generation and $\sigma i \in T_{n+1}$, $i \in I_\sigma$ are the descendants of an individual $\sigma \in T_n$. The self-similar random set associated with the RIFS $S$ is the set 

$$
F := \bigcap_{n=1}^{\infty} \bigcup_{\sigma \in T_n} J_\sigma,
$$

where, for any $\sigma \in \Sigma_*$ of length $|\sigma| = n \in \mathbb{N}$ and any set $K \subset \mathbb{R}^d$, 

$$
K_\sigma := \phi_{\sigma|1} \circ \phi_{\sigma|2} \circ \ldots \circ \phi_{\sigma|n}(K).
$$

Here $\sigma|k$, $k \in \{1, \ldots, n\}$ denotes the word formed by the first $k$ letters of $\sigma$. $F$ is called self-similar because of the following stochastic self-similarity property (which characterizes $F$ uniquely): if $F(i)$, $i \in \{0, 1, \ldots, M^d\}$ are i.i.d. copies of $F$ and $S$ is the corresponding RIFS as above, independent of the $F(i)$, then 

$$
F^{(0)} = \bigcup_{\phi_i \in S} \phi_i(F(i)).
$$

In the language of the tree and the associated sets considered above, the construction steps $F_n$, $n \in \mathbb{N}$ of the fractal percolation process are given by 

$$
F_n = \bigcup_{\sigma \in T_n} J_\sigma.
$$

Here the sets $J_\sigma$, with $|\sigma| = n$ encode the subcubes of level $n$ of the construction, and the above union extends over those subcubes $J_\sigma$, which survived all the previous steps, i.e. over all $\sigma$ for which all the cubes $J_{\sigma|i}$, $i \in \{1, \ldots, n\}$, have been kept in the $i$-th step of the construction. We also introduce, for each $j \in \{1, \ldots, M^d\}$ and each $n \in \mathbb{N}$, the set 

$$
F_n^j := \bigcup_{\sigma \in T_n, \sigma|1=j} J_\sigma,
$$

being the union of those cubes of level $n$ which are subcubes of $J_j = \phi_j(J)$. We will not make much use of the limit objects and their self-similarity in the sequel, we will mainly use the following basic properties of the construction steps $F_n$ and their parts $F_n^j$: For any $j \in \{1, \ldots, M^d\}$ and any $n \in \mathbb{N}$, we have 

$$
F_n^j = \phi_j(\tilde{F}_{n-1}^j)
$$

in distribution, where $\tilde{F}_{n-1}$ is the random set which equals $F_{n-1}$ with probability $p$ and is empty otherwise (i.e., $\tilde{F}_{n-1}^j = F_{n-1}^j \cap J_j$, where $J_j$ is a random set independent of $F_{n-1}^j$, which equals $J_j$ with probability $p$ and is empty otherwise). The homogeneity and motion invariance of the intrinsic volumes implies now in particular that 

$$
\mathbb{E}V_k(F_n^j) = p\mathbb{E}V_k(\phi_j(F_{n-1})) = pr^k\mathbb{E}V_k(F_{n-1}),
$$

for any $k \in \{0, \ldots, d\}$ and any $j \in \{1, \ldots, M^d\}$, where $r = 1/M$ is the scaling ratio of $\phi_j$. 


4. Approximation of $F$ by the sequence $(F_n)_n$.

Our first aim in this section is to prove Theorem 1.1. For the moment let $M \in \mathbb{N}_{\geq 2}$ and $p \in (0,1]$ be arbitrary. We will only later need to restrict the range of $p$ as assumed in Theorem 1.1. Let $D$ be as defined in (1.2). ($D$ is the Minkowski dimension of $F$ in case $M^d p \geq 1$ and negative otherwise.)

Set

$$
\tau_k(n) := r^{n(D-k)} \mathbb{E} V_k(F_n), \quad n \in \mathbb{N}_0,
$$

where $F_0 := J = [0,1]^d$. Since the latter is a deterministic set, we have $\tau_k(0) = V_k(F_0) = q_{d,k}$. We are going to show that the limit $\overline{\tau}_k(F) = \lim_{n \to \infty} \tau_k(n)$ exists for any $k$ and, moreover, that it coincides with the expression stated in (1.4). The first step is to derive a kind of renewal equation for the $\overline{\tau}_k$. (The approach is similar to the methods in [23, 25] which are based on renewal theory. However, here we do not need the Renewal theorem as it is possible to argue directly.)

Setting

$$
w_k(n) := \overline{\tau}_k(n) - \overline{\tau}_k(n-1), \quad n \in \mathbb{N},
$$

it is easy to see that

$$
\lim_{n \to \infty} \tau_k(n) = \overline{\tau}_k(0) + \sum_{j=1}^\infty w_k(j),
$$

i.e., the limit on the left exists if and only if the sum on the right converges. (Indeed, by definition of $w_k$, we have $\overline{\tau}_k(n) = \overline{\tau}_k(n-1) + w_k(n) = \ldots = \overline{\tau}_k(0) + \sum_{j=1}^n w_k(j)$ for any $n \in \mathbb{N}$, and so in (4.2) the limit on the left exists if and only if the partial sums on the right converge.) Therefore, it is enough to compute the functions $w_k$, which turns out to be easier than computing the $\overline{\tau}_k$ directly. The relation

$$
\overline{\tau}_k(n) = \overline{\tau}_k(n-1) + w_k(n), \quad n \in \mathbb{N}
$$

can be viewed as a (discrete) renewal equation with $w_k$ being the error term. By definition of $w_k$, we have

$$
w_k(n) = \overline{\tau}_k(n) - \overline{\tau}_k(n-1) = r^{n(D-k)} \mathbb{E} V_k(F_n) - r^{(n-1)(D-k)} \mathbb{E} V_k(F_{n-1})
$$

$$
\quad = r^{n(D-k)} \left( \mathbb{E} V_k(F_n) - r^{-D} \mathbb{E} V_k(F_{n-1}) \right)
$$

$$
\quad = r^{n(D-k)} \left( \mathbb{E} V_k(F_n) - M^d p r^k \mathbb{E} V_k(F_{n-1}) \right),
$$

where we employed the relation $M^d p = r^{-D}$ in the last step. Now the similarity relation (3.3) implies $\sum_{j=1}^{M^d} \mathbb{E} V_k(F_j) = M^d p r^k \mathbb{E} V_k(F_{n-1})$, which we can insert in the above expression to obtain

$$
w_k(n) = r^{n(D-k)} \left( \mathbb{E} V_k(F_n) - \sum_{j=1}^{M^d} \mathbb{E} V_k(F_j) \right).
$$

Using the inclusion-exclusion principle, this can be expressed in a more convenient form. Since $F_n = \bigcup_{j=1}^N F_n^j$, we get

$$
V_k(F_n) - \sum_{j=1}^N V_k(F_n^j) = \sum_{T \subseteq \{1, \ldots, M^d\} : \# T \geq 2} (-1)^{|T|-1} V_k(\bigcap_{j \in T} F_n^j).
$$

Taking expectations and plugging the resulting equation into (4.3), we obtain for each $n \in \mathbb{N}$ and each $k \in \{0, \ldots, d\}$ the representation

$$
w_k(n) = \sum_{T \subseteq \{1, \ldots, M^d\} : \# T \geq 2} (-1)^{|T|-1} r^{n(D-k)} \mathbb{E} V_k(\bigcap_{j \in T} F_n^j).
$$
Note that this is a finite sum with a fixed number of terms (independent of \(n\)). Combined with (4.2), it yields

\[
(4.5) \quad \overline{V}_k(F) = qd,k + \sum_{n=1}^{\infty} \sum_{T \subseteq \{1, \ldots, M^d\}, |T| \geq 2} (-1)^{|T|-1} r^{n(D-k)} \overline{V}_k(\bigcap_{j \in T} F_n^j).
\]

This is almost the formula stated in Theorem 1.1 except for the different order of summation. The summations can be interchanged (and thus the formula (1.4) is verified) provided that the summations over \(n\) in (1.4) converge for each set \(T\). This convergence is ensured by Proposition 4.1 below in case the parameter \(p\) satisfies \(p > M^{-\min(3,d)}\). Recall that the \(k\)-th intrinsic volume of a polyconvex set \(K\) can be localized to a signed measure on \(K\), the \(k\)-th curvature measure \(C_k(K, \cdot)\). Denote by \(C_k^{\infty}(K)\) the total mass of the total variation measure of \(C_k(K, \cdot)\).

**Proposition 4.1.** Let \(F\) be a fractal percolation in \([0,1]^d\) with parameters \(M \geq 2\) and \(p \in (M^{-\min(3,d)}, 1]\). For each \(k \in \{0, \ldots, d\}\) and each \(T \subseteq \{1, \ldots, M^d\}\) with \(|T| \geq 2\),

\[
\sum_{n=1}^{\infty} r^{n(D-k)} \overline{V}_k(\bigcap_{j \in T} F_n^j) < \infty.
\]

In particular, the sums

\[
\sum_{n=1}^{\infty} r^{n(D-k)} \overline{V}_k(\bigcap_{j \in T} F_n^j)
\]

converge absolutely.

We postpone the proof of Proposition 4.1 to the last section where we will discuss it together with the proof of a similar assertion needed in Section 5. With this statement at hand we can now complete the proof of the main theorem.

**Proof of Theorem 1.1.** To obtain the formula (1.4), all we have to do is to interchange the order of the summations in the formula (4.5). This is justified, since, by Proposition 4.1, for \(p > M^{-\min(3,d)}\) all the series occurring in (1.4) converge. \(\Box\)

**The case \(d = 1\).** Fractal percolation in one dimension is not very interesting as a percolation model. However, the limiting behaviour of the studied geometric functionals is of independent interest. Moreover, the one dimensional case (for which we use throughout the letter \(K\) instead of \(F\)) is essential for the computations in the two dimensional case. First, we derive explicit expressions for the expected intrinsic volumes of each approximation step \(K_n, n \in \mathbb{N}_0\) of a fractal percolation \(K\) in \([0,1]\), from which it is easy to determine the rescaled limits \(\overline{V}_k(K)\). Then we study the intersection of two such random sets, which is what we actually need for dimension 2.

**Proposition 4.2.** Let \(K\) be a fractal percolation on the interval \([0,1]\) with parameters \(M \in \mathbb{N}_{\geq 2}\) and \(p \in (0,1]\). Denote by \(K_n^j\) the \(n\)-th step of the construction of \(K\). Then, for any \(n \in \mathbb{N}_0\),

\[
\overline{V}_1(K_n^j) = p^n \quad \text{and} \quad \overline{V}_0(K_n^j) = (Mp)^n \left(1 - \frac{(M-1)p}{M-p} \left[1 - \left(\frac{p}{M}\right)^n\right]\right).
\]

**Proof.** For \(j = 1, \ldots, M\), let \(K_n^j\) be the union of the surviving intervals of level \(n\) contained in \(J_j = \phi_j([0,1])\), cf. (1.1). Then \(K_n = \bigcup_{j=1}^M K_n^j\) and since in this union
only sets $K^n_j$ with consecutive indices can have a nonempty intersection, by the inclusion-exclusion formula, we get

\begin{equation}
\mathbb{E}V_k(K_n) = \sum_{j=1}^{M} \mathbb{E}V_k(K^n_j) - \sum_{j=1}^{M-1} \mathbb{E}V_k(K^n_j \cap K^{n+1}_j).
\end{equation}

For $k = 1$, the second sum vanishes, since these intersections consist of at most one point. Moreover, by (5.3), the terms in the first sum satisfy

\begin{equation}
\mathbb{E}V_k(K^n_j) = p^k \mathbb{E}V_k(K_{n-1}^j), \quad n \in \mathbb{N}.
\end{equation}

Since $V_1(K_0) = V_1([0, 1]) = 1$, this yields

$$\mathbb{E}V_1(K_n) = \sum_{j=1}^{M} \frac{p}{M} \mathbb{E}V_1(K_{n-1}^j) = p^{\mathbb{E}V_1(K_{n-1})} = \ldots = p^n$$

as claimed. For $k = 0$, the terms in second sum in (4.6) contribute. The Euler characteristic $V_0(K^n_j \cap K^{n+1}_j)$ equals 1 with probability $p^{2n}$ (and is 0 otherwise), since for a nonempty intersection at each level from 1 to $n$ the two intervals containing the possible intersection point need to survive (which has probability $p$ for each of these intervals). Using this and (4.7), we conclude from (4.6) that

$$\mathbb{E}V_0(K_n) = \sum_{j=1}^{M} p^{\mathbb{E}V_0(K_{n-1}^j)} - \sum_{j=1}^{M-1} p^{2n} = M p^{\mathbb{E}V_0(K_{n-1})} - (M - 1) p^{2n}.$$ 

This is a recursive relation for the sequence $(\mathbb{E}V_0(K_n))_{n \in \mathbb{N}_0}$ where $\mathbb{E}V_0(K_0) = 1$. By an induction argument, it is easy to obtain the explicit representation

$$\mathbb{E}V_0(K_n) = (M p)^n - (M - 1) \sum_{i=1}^{n} (M p)^{n-i} p^{2i} = (M p)^n \left(1 - (M - 1) \sum_{i=1}^{n} \left(\frac{p}{M}\right)^i\right),$$

which yields the asserted formula.

\textbf{Corollary 4.3.} Let $K$ be a fractal percolation on the interval $[0, 1]$ with parameters $M \in \mathbb{N}_{\geq 2}$ and $p \in (0, 1]$. Then

$$\nabla V_1(K) = 1 \quad \text{and} \quad \nabla V_0(K) = \frac{M(1 - p)}{M - p}.$$ 

\textbf{Proof.} Since $D = \frac{\log(M p)}{\log M}$, we have $M p = M^D = r^{-D}$ and so, by Proposition 4.2

$$\nabla V_0(K) = \lim_{n \to \infty} r^{Dn} \mathbb{E}V_0(K_n) = \lim_{n \to \infty} 1 - \frac{(M - 1)p}{M - p} \left[1 - \left(\frac{p}{M}\right)^n\right] = 1 - \frac{(M - 1)p}{M - p}$$

and

$$\nabla V_1(K) = \lim_{n \to \infty} r^{(D-1)n} \mathbb{E}V_1(K_n) = \lim_{n \to \infty} r^{-n} p^n = 1,$$

as claimed.

Figure 3 (left) shows plots of $\nabla V_0(K)$ as a function of $p$ for different parameters $M$. It is apparent that these are positive and monotone decreasing functions in $p$ for any $M$ and that the limit as $M \to \infty$ is given by $f(p) = 1 - p$.

\textbf{Proposition 4.4.} Let $K^{(1)}, K^{(2)}$ be independent fractal percolations on the interval $[0, 1]$ with the same parameters $M \in \mathbb{N}_{\geq 2}$ and $p \in [0, 1]$. Then, for any $n \in \mathbb{N}_0$,

$$\mathbb{E}V_1(K^{(1)}_n \cap K^{(2)}_n) = p^{2n} \quad \text{and} \quad \mathbb{E}V_0(K^{(1)}_n \cap K^{(2)}_n) = (M p^2)^n \times\left(3 - 2M^{-n} - 4p \frac{M - 1}{M - p} \left[1 - \left(\frac{p}{M}\right)^n\right] + \frac{(M - 1)p^2}{M - p^2} \left[1 - \left(\frac{p^2}{M}\right)^n\right]\right).$$
Proof. For $i \in \{1, 2\}$ and $j \in \{1, \ldots, M\}$, let $K^{(i), j}_n$ be the union of those level-$n$ intervals in the union $K^{(i)}_n$ which are contained in $J_j$ (similarly as in (3.1)). Then $K^{(i)}_n = \bigcup_{j=1}^M K^{(i), j}_n$. Since $K^+_j \subset J_j$ and $J_j \cap J_l \neq \emptyset$ if and only if $|j-l| \leq 1$, we can write the intersection $K^{(1)}_n \cap K^{(2)}_n$ as

$$K^{(1)}_n \cap K^{(2)}_n = \bigcup_{j=1}^M K^{(1), j}_n \cap \bigcup_{l=1}^M K^{(2), l}_n = \bigcup_{j=1}^M \left( K^{(1), j}_n \cap \bigcup_{l=j-1}^{j+1} K^{(2), l}_n \right) =: \bigcup_{j=1}^M L_j,$$

where we have set $K^{(2), 0}_n = K^{(2), M+1}_n := \emptyset$ for convenience. The random sets $L_j$ (whose dependence on $n$ we suppress in the notation) satisfy $L_j \subset J_j$ a.s. and therefore in the union $\bigcup_j L_j$ only sets with consecutive indices can have a nonempty intersection. Thus, by the inclusion-exclusion principle, we conclude for the expected intrinsic volumes

$$(4.8) \quad \mathbb{E} V_k(K^{(1)}_n \cap K^{(2)}_n) = \sum_{j=1}^M \mathbb{E} V_k(L_j) - \sum_{j=1}^{M-1} \mathbb{E} V_k(L_j \cap L_{j+1}).$$

Now observe that, for $j = 1, \ldots, M-1$,

$$L_j \cap L_{j+1} = K^{(1), j}_n \cap K^{(1), j+1}_n \cap \left( K^{(2), j}_n \cup K^{(2), j+1}_n \right),$$

and this random set is either empty or consists of exactly one point $z_j$ (namely, the unique point in the intersection $J_j \cap J_{j+1}$). The latter event occurs if and only if for each of the two sets $K^{(1), j}_n, K^{(1), j+1}_n$ at each level $k = 1, \ldots, n$ the subinterval of level $k$ that contains $z_j$ survives (which has each probability $p^k$) and if a similar survival of all subintervals containing $z_j$ also occurs for at least one of the sets $K^{(2), j}_n, K^{(2), j+1}_n$. The probability for this latter event is $2p^n - p^{2n}$. Hence $\mathbb{E} V_k(L_j \cap L_{j+1}) = (2p^n - p^{2n}) V_k(\{z_j\})$ and therefore

$$(4.9) \quad \mathbb{E} V_0(L_j \cap L_{j+1}) = 2p^{3n} - p^{4n} \quad \text{and} \quad \mathbb{E} V_k(L_j \cap L_{j+1}) = 0, \quad k \geq 1.$$ 

It remains to determine $\mathbb{E} V_k(L_j)$. By definition of $L_j$, we have $L_j = \bigcup_{l=j-1}^{j+1} K^{(1), j}_n \cap K^{(2), l}_n$ and therefore the inclusion-exclusion formula gives

$$\mathbb{E} V_k(L_j) = \sum_{l=j-1}^{j+1} \mathbb{E} V_k(K^{(1), j}_n \cap K^{(2), l}_n) - \sum_{l=j-1}^{j+1} \mathbb{E} V_k(K^{(1), j}_n \cap K^{(2), l}_n \cap K^{(2), l+1}_n).$$

Now again $K^{(1), j}_n \cap K^{(2), l}_n$ is a singleton with probability $p^{2n}$ and empty otherwise, provided $l = j-1$ or $l = j+1$ (and $l \not\in \{0, M+1\}$). Similarly, $K^{(1), j}_n \cap K^{(2), l}_n \cap K^{(2), l+1}_n$ is a singleton with probability $p^{3n}$ and empty otherwise, provided $l \not\in \{0, M\}$. (For the exceptional $l$, these intersections are empty a.s.) This implies

$$\mathbb{E} V_0(L_j) = \mathbb{E} V_0(K^{(1), j}_n \cap K^{(2), j}_n) + \begin{cases} 2(p^{2n} - p^{3n}), & j \in \{2, \ldots, M-1\}, \\ 2p^n, & j \in \{1, M\}, \end{cases}$$

and $\mathbb{E} V_k(L_j) = \mathbb{E} V_k(K^{(1), j}_n \cap K^{(2), j}_n)$ for any $k \geq 1$. Plugging this and (4.9) into equation (4.8), we conclude that, for $k = 0, 1$ and any $n \in \mathbb{N}$,

$$(4.10) \quad \mathbb{E} V_k(K^{(1)}_n \cap K^{(2)}_n) = p_k(n) + \sum_{j=1}^M \mathbb{E} V_k(K^{(1), j}_n \cap K^{(2), j}_n),$$
where \( p_0(n) := (M - 1)(2p^{2n} - 4p^{3n} + p^{4n}) \) and \( p_1(n) := 0, n \in \mathbb{N} \). Now observe that, by (3.2), we have \( K_n^{(1), j} \cap K_n^{(2), j} = \phi_j(K_{n-1}^{(1)} \cap K_{n-1}^{(2)}) \) in distribution, where \( K_{n-1}^{(i)} \) is similarly as in (3.2) the random set which equals \( K_n^{(i)} \) with probability \( p \) and is empty otherwise. This implies

\[
\mathbb{E}V_k(K_n^{(1), j} \cap K_n^{(2), j}) = p^2 r^k \mathbb{E}V_k(K_{n-1}^{(1)} \cap K_{n-1}^{(2)}),
\]

for any \( j = 1, \ldots, M \) and any \( n \in \mathbb{N} \), where \( K_0^{(i)} = [0, 1] \) and thus \( \mathbb{E}V_k(K_0^{(1)} \cap K_0^{(2)}) = V_k([0, 1]) = 1 \) for \( k = 0, 1 \). Setting \( \alpha_n := \mathbb{E}V_1(K_n^{(1)} \cap K_n^{(2)}), n \in \mathbb{N}_0 \), we have \( \alpha_0 = 1 \) and we infer from (4.10) that

\[
\alpha_n = \sum_{j=1}^M \mathbb{E}V_1(K_n^{(1), j} \cap K_n^{(2), j}) = M p^2 r \alpha_{n-1} = p^2 \alpha_{n-1}, \quad n \in \mathbb{N}.
\]

It is easy to see now that \( \alpha_n = p^{2n} \), proving the first formula in Proposition 4.4.

Setting \( \beta_n := \mathbb{E}V_0(K_n^{(1)} \cap K_n^{(2)}), n \in \mathbb{N}_0 \), we infer in a similar way that \( \beta_0 = 1 \) and

\[
\beta_n = \sum_{j=1}^M \mathbb{E}V_0(K_n^{(1), j} \cap K_n^{(2), j}) + p_0(n) = M p^2 \beta_{n-1} + p_0(n), \quad n \in \mathbb{N},
\]

which provides a recursive relation for the sequence \( (\beta_n)_n \). By an induction argument, we obtain

\[
\beta_n = (M p^2)^n + \sum_{j=1}^n (M p^2)^{n-j} p_0(j), \quad n \in \mathbb{N}_0.
\]

Plugging in the \( p_0(j) \) and computing the sum, we conclude that, for any \( n \in \mathbb{N}_0 \),

\[
\beta_n = (M p^2)^n \left( 3 - \frac{2}{M^n} - 4p \frac{M - 1}{M - p} \left( 1 - \frac{p}{M} \right)^n + \frac{(M - 1)p^2}{M^2} \left( 1 - \frac{p^2}{M} \right)^n \right),
\]

which shows the second formula in Proposition 4.4 and completes the proof. \( \square \)

Remark 4.5. It is easy to see from Proposition 4.4 that for \( D' := \log(M p^2)/\log M \) the rescaled expressions \( r^{n(D'-k)} \mathbb{E}V_k(K_n^{(1)} \cap K_n^{(2)}) \) converge as \( n \to \infty \). Indeed, since

\[
r^{D'-1} = p^{-2} \quad \text{and} \quad r^{D'} = (1/M)^{D'} = (M p^2)^{-1},
\]

we obtain

\[
\overline{V}_1(K^{(1)} \cap K^{(2)}) := \lim_{n \to \infty} r^{n(D'-1)} \mathbb{E}V_1(K_n^{(1)} \cap K_n^{(2)}) = 1 \quad \text{and}
\]

\[
\overline{V}_0(K^{(1)} \cap K^{(2)}) := \lim_{n \to \infty} r^{nD'} \mathbb{E}V_0(K_n^{(1)} \cap K_n^{(2)}) = 3 - 4p \frac{M - 1}{M - p} + p^2 \frac{M - 1}{M - p^2}.
\]

Again the rescaled constant \( \overline{V}_1(K^{(1)} \cap K^{(2)}) \) is constant, while the rescaled Euler characteristic of \( K^{(1)} \cap K^{(2)} \) depends on \( p \) and \( M \). Figure 4 (right) shows plots of \( \overline{V}_0(K^{(1)} \cap K^{(2)}) \) as a function of \( p \) for different parameters \( M \). It is apparent that these are positive and monotone decreasing functions in \( p \) for any \( M \) and the limit as \( M \to \infty \) is given by \( f(p) = 3 - 4p + p^2 \). From the existence of the limits \( \overline{V}_k(K^{(1)} \cap K^{(2)}) \) it is clear that \( D' \) as chosen above is the correct scaling exponent. The notation for the limit is justified by the fact that \( D' \) is almost surely the Hausdorff dimension of \( K^{(1)} \cap K^{(2)} \), as the following statement clarifies.

Proposition 4.6. Let \( K^{(1)}, K^{(2)} \) be independent fractal percolations on \([0, 1] \) with the same parameters \( M \in \mathbb{N}_{\geq 2} \) and \( p \in [0, 1] \). If \( p \leq 1/\sqrt{M} \), then the set \( K^{(1)} \cap K^{(2)} \) is almost surely empty. If \( p > 1/\sqrt{M} \), there is a positive probability that \( K^{(1)} \cap K^{(2)} \neq \emptyset \) and, conditioned on \( K^{(1)} \cap K^{(2)} \neq \emptyset \), we have \( \dim_H(K^{(1)} \cap K^{(2)}) = D' \) almost surely.
Fractal Percolation in 1D \( \mathcal{V}_0(K) \)  
Intersecting Fractal Percolations in 1D \( \mathcal{V}_0(K_1 \cap K_2) \)

\[ \text{Prob of survival } p \]

**Figure 4.** The rescaled limits \( \mathcal{V}_0(K) \) (left) and \( \mathcal{V}_0(K_1 \cap K_2) \) (right) as functions of \( p \in [0,1] \) for different values of \( M \) (color coded) as given by Corollary 4.3 and Remark 4.5 respectively. The limit curves as \( M \to \infty \) are shown in red.

**Proof.** For any \( p \in [0,1] \), the set \( K^{(1)} \cap K^{(2)} \) can be coupled with a fractal percolation \( F \) on \([0,1]\) with parameter \( p^2 \) (and the same \( M \)) by retaining an interval \( I_\sigma \) of level \( n \) if and only if it is contained in both \( K^{(1)}_n \) and \( K^{(2)}_n \). Then \( K^{(1)} \cap K^{(2)} \) dominates \( F \). Hence, almost surely, \( \dim_H (K^{(1)} \cap K^{(2)}) \geq \dim_H F \). Now observe that conditioning on the event \( \{K^{(1)} \cap K^{(2)} \neq \emptyset\} \) is the same as conditioning on \( \{F \neq \emptyset\} \). Indeed, on the one hand the first event is obviously satisfied whenever the latter is. On the other hand, if \( \{F = \emptyset\} \) holds, then there is some \( n \in \mathbb{N} \) such that \( F_n = \emptyset \). This implies that, for any \( m \geq n \), \( K^{(1)}_m \cap K^{(2)}_m \) consists of finitely many isolated points contained in the set \( \{\frac{k}{\sqrt{N}} : k \in \{1, \ldots, M^n - 1\}\} \) (cf. the proof of Proposition 4.4). In particular, there are no new points generated after the \( n \)-th step. Each point at level \( m \geq n \) is independently retained in the next step with probability \( p^2 \). This means in particular that \( K^{(1)} \cap K^{(2)} \) is empty almost surely under the condition \( F = \emptyset \). We conclude that, for \( p \leq \sqrt{M} \), the set \( K^{(1)} \cap K^{(2)} \) is empty almost surely, since \( F \) has this property. Moreover, since conditioned on \( F \neq \emptyset \) we have \( \dim_H F = D' \) almost surely for any \( p \geq \sqrt{M} \), we infer from the above inequality that conditioned \( K^{(1)} \cap K^{(2)} \neq \emptyset \), \( D' \) is almost surely a lower bound for \( \dim_H (K^{(1)} \cap K^{(2)}) \). (The same is true for the Minkowski dimension.)

We show that \( D' \) is also an upper bound for \( \dim_H (K^{(1)} \cap K^{(2)}) \). For any realization of \( K^{(1)} \cap K^{(2)} \) and any \( \delta > 0 \), a \( \delta \)-cover of \( K^{(1)} \cap K^{(2)} \) is obtained by taking the cubes of level \( n \) (for some \( n \) large enough that \( M^{-n} < \delta \)) contained in \( F_n \) (which cover \( F \)) and adding the finitely many singletons \( \{\frac{k}{\sqrt{N}} : k \in \{1, \ldots, M^n - 1\}\} \) which clearly cover the additional isolated points in \( K^{(1)} \cap K^{(2)} \) not already covered by the chosen intervals. Using these covers and noting that the singletons have diameter zero and the intervals diameter \( M^{-n} \), we get for any \( s > 0 \), that \( \mathcal{H}_s^D(K^{(1)} \cap K^{(2)}) \leq Z_n M^{-ns} \), where \( Z_n \) is the number of cubes in \( F_n \). Since \( Z_n \) is of the \( n \)-th generation of a Galton-Watson process in which the expected number of offspring of an individuum is \( M^{D'} = M^2 p \), it is well known that \( Z_n M^{-n D'} \to 1 \) almost surely as \( n \to \infty \). This shows \( \mathcal{H}_s^D(K^{(1)} \cap K^{(2)}) \ll M^{-n D'} \) almost surely and thus \( \dim_H (K^{(1)} \cap K^{(2)}) \leq D' \).

**The case** \( d = 2 \). Now we provide proofs of the formulas for the three limit functionals \( \mathcal{V}_k(F) \), \( k = 0, 1, 2 \) for fractal percolation \( F \) in \( \mathbb{R}^2 \) stated in Theorem 1.2. The starting point is again the general formula in Theorem 1.1 which can be simplified...
Figure 5. Possible mutual positions of the basic cubes $J_j$ which produce nonempty intersections.

Further by using on the one hand the various symmetries in the fractal percolation model and on the other hand the properties of the functionals.

Proof of Theorem 1.2 Let $M \in \mathbb{N}_{\geq 2}$ and $p \in (1/M^2, 1]$ and $k \in \{0, 1, 2\}$. By (1.4) in Theorem 1.1 we have

$$V_k(F) = q_{2,k} + \sum_{T \subset \{1, \ldots, M^2\}, |T| \geq 2} (-1)^{|T|-1} \sum_{n=1}^{\infty} r^n(D-k) \mathbb{E} V_k(\bigcap_{j \in T} F^n_j).$$

Observe that among the intersections $\bigcap_{j \in T} F^n_j$ occurring in (4.11) only those need to be considered for which the corresponding intersection $\bigcap_{j \in T} J_j$ of the subcubes $J_j = \phi_j(J)$ is nonempty. All other intersections are empty almost surely and hence their expected intrinsic volumes are zero. The nonempty intersections of subcubes can be reduced to four basic cases, see Figure 5. There are only two ways in which two subcubes can have a nonempty intersection, namely they can intersect in a common face (like $J_1$ and $J_4$ in Fig. 5) or in a common corner (like $J_1$ and $J_2$). Three subcubes can only have a nonempty intersection at a common corner (like $J_1$, $J_2$ and $J_3$) and similarly four subcubes can only intersect in a common corner (like $J_1$, $J_2$, $J_3$ and $J_4$). Only the number of intersections of each of these four types changes with $M$. These numbers are given by $2M(M-1)$, $2(M-1)^2$, $4(M-1)^2$ and $(M-1)^2$, respectively, independent of $p$ and $n$. Hence formula (4.11) reduces to

$$V_k(F) = q_{2,k} - 2M(M-1) \sum_{n=1}^{\infty} r^n(D-k) \mathbb{E} V_k(F^n_1 \cap F^n_4)$$
$$- 2(M-1)^2 \sum_{n=1}^{\infty} r^n(D-k) \mathbb{E} V_k(F^n_1 \cap F^n_2)$$
$$+ 4(M-1)^2 \sum_{n=1}^{\infty} r^n(D-k) \mathbb{E} V_k(F^n_1 \cap F^n_2 \cap F^n_3)$$
$$- (M-1)^2 \sum_{n=1}^{\infty} r^n(D-k) \mathbb{E} V_k \left( \bigcap_{j=1}^{4} F^n_j \right).$$

For $k = 2$, i.e. for the area $V_2$ in $\mathbb{R}^2$, it is enough to observe that the area of all the intersections of the level sets $F^n_j$ in this formula are almost surely zero (they are all contained in a line segment), implying that $\mathbb{E} V_2(\bigcap_{j \in T} F^n_j) = 0$ for all $n \in \mathbb{N}$ and all index sets $T$ with $|T| \geq 2$. Therefore,

$$\overline{V}_2(F) = q_{2,2} = V_2([0,1]^2) = 1,$$

independent of $M$ and $p$ as asserted in Theorem 1.2.
For $k = 1$, i.e. for the “boundary length” $V_1$, only the intersections of the first type $F^1_n \cap F^4_n$ need to be considered, while for the other three types the intersection is at most one point, implying that the expected boundary length vanishes independent of $n$. This yields

$$\mathbf{V}_1(F) = q_{2,1} - 2M(M-1) \sum_{n=1}^{\infty} r^{n(D-1)} \mathbb{E}V_1(F^1_n \cap F^4_n).$$

We claim that, for each $n \in \mathbb{N}$,

$$\mathbb{E}V_1(F^1_n \cap F^4_n) = p^{2n}/M.$$  

We will show below that this follows from Proposition 4.4. Plugging (4.14) into equation (4.13) and computing the remaining series yields the missing terms of

$$\mathbf{V}_1(F) = 2 - 2(M-1) \sum_{n=1}^{\infty} (M p)^{-n} p^{2n} = 2 - 2(M-1) \sum_{n=1}^{\infty} (p/M)^n$$

$$= 2 - 2(M-1) \frac{p}{M-p} = \frac{2M(1-p)}{M-p}.$$ 

For $k = 0$, i.e. for the Euler characteristic $V_0$, all terms in the above formula (4.12) are relevant and contribute to the limit. It is rather easy to see that $V_0(F^1_n \cap F^2_n \cap F^3_n \cap F^4_n) = 1$ with probability $p^{4n}$, since at all levels $n = 1, \ldots, n$, in each of the four cubes $J_i, i = 1, \ldots, 4$ the subcube of level $n$ which intersects the common corner needs to survive (which happens with probability $p$, independently of all the other subcubes of any level). Otherwise the intersection of the four sets $F^0_n$ is empty. Hence, for each $n \in \mathbb{N}$ (and each $M \geq 2$),

$$\mathbb{E}V_0(F^1_n \cap F^2_n \cap F^3_n \cap F^4_n) = p^{4n}.$$ 

Therefore, the sum in the last line of formula (4.12) is given by

$$\sum_{n=1}^{\infty} r^{nD} \mathbb{E}V_0 \left( \bigcap_{j=1}^{4} F^j_n \right) = \sum_{n=1}^{\infty} (r^D p^4)^n = \frac{r^D p^4}{1 - r^D p^4} = \frac{p^3}{M^2 - p^2},$$

where the last equality is due to the relation $p r^D = r^2 = M^{-2}$. (Note that the geometric series above converges, since $p^3 r^D = p^3 M^{-2} < 1$ for any $p \in [0, 1]$ and any integer $M \geq 2$.)

Similarly, one observes that $V_0(F^1_n \cap F^2_n \cap F^3_n) = 1$ with probability $p^{3n}$ and $V_0(F^1_n \cap F^2_n) = 1$ with probability $p^{2n}$ for $n \in \mathbb{N}$, which yields for the sums in the third and the second line in formula (4.12) the expressions

$$\sum_{n=1}^{\infty} r^{nD} \mathbb{E}V_0(F^1_n \cap F^2_n \cap F^3_n) = \sum_{n=1}^{\infty} (r^D p^3)^n = \frac{p^2}{M^2 - p^2}$$

and

$$\sum_{n=1}^{\infty} r^{nD} \mathbb{E}V_0(F^1_n \cap F^2_n) = \sum_{n=1}^{\infty} (r^D p^2)^n = \frac{p}{M^2 - p^2}.$$ 

It remains to compute the expected Euler characteristic for the type $F^1_n \cap F^4_n$. We claim that, for any $n \in \mathbb{N}$,

$$\mathbb{E}V_0(F^1_n \cap F^4_n) = (M p^2)^n \times \left( \frac{3}{M} - 2M^{-n} - 4 \frac{M-1}{M-p} \frac{p}{M} - \left( \frac{p}{M} \right)^n \right) + \frac{M-1}{M-p} \left[ \frac{p^2}{M} - \left( \frac{p^2}{M} \right)^n \right].$$

We will demonstrate below that this follows from Proposition 4.4. Plugging (4.16) - (4.19) into (4.12) and computing the remaining series yields the missing terms of
\( V_0(F) \). More precisely, we get for the last sum on the first line of equation (4.12) the expression
\[
E_1 := \frac{2(M-1)^2p}{M-p} \left( \frac{3}{M-1} - \frac{4p}{M} + \frac{p^2}{M-p^2} \right) - 2M(M-1)^2p \times \\
\times \left( \frac{2}{(M-1)(M^2-p)} - \frac{4p}{(M-p)(M^2-p^2)} + \frac{p^2}{(M-p^2)(M^2-p^3)} \right)
\]
and therefore
\[
\nabla_0(F) = 1 - E_1 + (M-1)^2 \left( \frac{-2p}{M^2-p} + \frac{4p^2}{M^2-p^2} - \frac{p^3}{M^2-p^3} \right) .
\]
Combining some of the terms gives the formula stated in Theorem 1.2 for \( \nabla_0(F) \).

To complete the proof, it remains to verify equations (4.14) and (4.19). To understand the structure of \( F_1 \cap F_n \), it is enough to study the intersection of two independent 1-dimensional fractal percolations \( K^{(1)} \) and \( K^{(2)} \) defined on a common interval \([0,1]\) (with the same parameters \( M \) and \( p \) as \( F \)). For \( n \in \mathbb{N} \) and \( i = 1, 2 \), let \( K_n^{(i)} \) denote the \( n \)-th steps of their construction. Similarly as in (3.2), let \( \tilde{K}_n^{(i)} \), \( i = 1, 2 \) be the random set, which equals \( K_n^{(i)} \) with probability \( p \) and is empty otherwise, i.e. we add an additional \( 0 \)-th step to decide whether the set \( K_n^{(i)} \), \( n \in \mathbb{N} \), is kept or discarded. This is to account for the first step of the construction of \( F \) (in which the cubes \( J_i \) are discarded with probability \( 1-p \)). Then, for each \( n \), we have the following equality in distribution
\[
(4.20) \quad F_n^1 \cap F_n^4 = \psi(\tilde{K}_{n-1}^{(1)} \cap \tilde{K}_{n-1}^{(2)}),
\]
where \( \psi : \mathbb{R} \to \mathbb{R}^2, x \mapsto (t/M)a + ((1-t)/M)b \) is the similarity, which maps \([0,1]\) to the segment \( J_1 \cap J_4 \) with endpoints \( a \) and \( b \). Since intrinsic volumes are independent of the ambient space dimension, motion invariant and homogeneous, this implies in particular
\[
(4.21) \quad \mathbb{E} V_k(F_n^1 \cap F_n^4) = \mathbb{E} V_k(\psi(\tilde{K}_{n-1}^{(1)} \cap \tilde{K}_{n-1}^{(2)})) = r^k p^2 \mathbb{E} V_k(K_{n-1}^{(1)} \cap K_{n-1}^{(2)}).
\]

Now the claims (4.14) and (4.19) follow by combining (4.21) with Proposition 4.3.

**Remark 4.7.** From the proof of Theorem 1.3, we also get explicit expressions for the expected intrinsic volumes of the approximation sets \( F_n \) for each \( n \in \mathbb{N} \). To determine \( \nabla_0(m) := r^{m(D-k)} \mathbb{E} V_k(F_m) \), it is enough to truncate all the sums in formula (4.12) after the \( m \)-th step and compute the resulting finite geometric sums. This yields for \( k = 0 \) and \( n \in \mathbb{N} \),
\[
\tau_0(n) = 1 - \frac{2p(M-1)^2}{M-p} \left( \frac{3}{M-1} - \frac{4p}{M} + \frac{p^2}{M-p^2} \right) \left[ 1 - \left( \frac{p}{M} \right)^n \right] \\
+ \frac{2p(M^2-1)}{M^2-p} \left[ 1 - \left( \frac{p}{M^2} \right)^n \right] - \frac{4p^2(M-1)^2}{(M-p)^2} \left[ 1 - \left( \frac{p^2}{M^2} \right)^n \right] \\
+ \frac{p^3(M-1)^2(M+p^2)}{(M-p^2)(M^2-p^3)} \left[ 1 - \left( \frac{p^3}{M^2} \right)^n \right] .
\]
It is easy to see that this sequence converges exponentially fast to \( \nabla_0(F) \) as \( n \to \infty \).

More precisely, we have
\[
\tau_0(n) - \nabla_0(F) \sim c (p/M)^n, \quad \text{as } n \to \infty,
\]
(i.e. the quotient of the left and the right hand side converges to 1) with the constant
\[
c := \frac{2p(M-1)^2}{M-p} \left( \frac{3}{M-1} - \frac{4p}{M-p} + \frac{p^2}{M-p^2} \right)
\]
being positive for each \( p \in (0,1) \) and \( M \in \).
\( \mathbb{N}_{\geq 2} \). Moreover, the sequence \( \tau_0(n) \) is eventually strictly decreasing, i.e. strictly decreasing from some index \( n_0 \in \mathbb{N} \). This exemplifies that the convergence \( \tau_k(n) \to \tau_k(F) \) is extremely fast and that the functionals \( \tau_k(F) \) can be approximated well by the \( \tau_k(n) \). This was also observed in simulations, where already for small \( n \) (like \( n = 8 \), even for \( M = 2 \), see Fig. 2) \( \tau_k(n) \) is virtually indistinguishable from the limit \( \tau_k(F) \), see also Remark 4.8 below. Fast convergence can also be expected for the limits \( \tau_k(K) \) of other random self-similar sets \( K \), for which no exact formula may be available. It is another intriguing question whether a similar speed of convergence can be expected for the percolation probabilities of \( F_n \).

**Remark 4.8.** (On the simulation study) Due to the fast convergence of the studied geometric functionals, their numerical estimation is efficient and accurate. A simulation study demonstrates their potential as robust shape descriptors for applications, see Fig. 3. To generate the approximations of fractal percolation, we create black-and-white pixel images by hierarchically simulating the survival or death of squares (given by patches of pixels). We use the MT19937 generator [12] (known as “Mersenne Twister”) to generate the required Bernoulli variables. Taking advantage of the additivity of the Minkowski functionals, we compute the Euler characteristic using an efficient algorithm, where the computation time grows linearly with the system size. We simply iterate over all \( 2 \times 2 \) neighborhoods of pixels and add the corresponding values from a look-up table as described in [1]. In two separate simulations using analogous parameters, we have computed the Euler characteristic \( F_n \) and \( C_n \) (see Section 4).

We simulate realizations of finite approximations for \( M = 2, 3, \) and \( 4 \). For each value of \( M \), we choose three levels \( n \) of the approximation \( n = 32/(2^M), 32/(2^M) + 2, \) or \( 32/(2^M) + 4 \). Since the rate of convergence increases with \( M \), for larger \( M \) smaller values of \( n \) are sufficient. For each chosen value of the probability of survival \( p = 0.11, 0.13, \ldots, 0.99 \), we simulate 75000, 5000, or 2500 samples for \( M = 2, 3, \) or \( 4 \), respectively. Only in the case of \( M = 2, p \leq 0.31 \) for \( F_n \), the number of samples is increased by a factor 10 for improved statistics.

The mean values are unbiasedly estimated by the arithmetic mean of the Euler characteristic of the samples. The error bars in the plots represent the sample standard deviations. The simulation results, shown in Figure 4, are in excellent agreement with the analytic curves, see Remarks 4.7 and 5.14. The code is freely available via GitHub [10].

**Remark 4.9.** An essential observation used in the proof of Theorem 1.2 (see [1,20]) is that any intersection \( F^{(1)} \cap F^{(2)} \) of two fractal percolations constructed in neighboring squares sharing a common side, can be modelled by the intersection \( K^{(1)} \cap K^{(2)} \) of two fractal percolations on that side (with the same parameters \( M \) and \( p \) as the \( F^{(1)} \)). Not only the intrinsic volumes of the intersections of the corresponding construction steps coincide. Also the Hausdorff dimension of \( F^{(1)} \cap F^{(2)} \) coincides with that of \( K^{(1)} \cap K^{(2)} \), which has been determined in Proposition 4.6. Moreover, the intersection \( F^{(1)} \cap F^{(2)} \) is almost surely empty for \( p \leq 1/\sqrt{M} \).

This provides an alternative proof of the lower bound \( 1/\sqrt{M} \) of Chayes, Chayes and Durrett [4] for the percolation threshold of fractal percolation \( F \) in \([0,1]^2\) (see [2,1]). The intersection of \( F \) with any vertical line \( y = k/M^n \), where \( n \in \mathbb{N} \) and \( k \in \{1, \ldots, M^n - 1\} \), can be modeled as a union of \( M^n \) small copies of \( K^{(1)} \cap K^{(2)}. \) Any path in \( F \) from left to right need to pass this line, which is impossible if these intersections are empty a.s., i.e. for any \( p \leq 1/\sqrt{M} \).

**Remark 4.10.** It is easy to see from Theorem 1.1 that also for fractal percolation in \( \mathbb{R}^d \), the rescaled limit \( \overline{\tau}_d(F) \) of the volume equals \( 1 \) for any \( p \) and \( M \). Indeed,
none of the intersections occurring in formula \([1.4]\) will contribute to the limit as they are contained in lower dimensional subsets of \(\mathbb{R}^d\).

5. Approximation of \(F\) by the closed complements of \((F_n)_n\).

Now we consider the closed complements \(C_n := J \setminus F_n, n \in \mathbb{N}_0\), of the construction steps \(F_n\) of the fractal percolation process inside the unit cube \(J = [0,1]^d\).

Note that \(C_0 = \emptyset\), since \(F_0 = J\). The random sets \(C_n\) are also given by

\[
C_n = \bigcup_{\sigma \in \Sigma_n \setminus \mathcal{T}_n} J_{\sigma},
\]

cf. Section 3 implying in particular that each realization of \(C_n\) consists of a finite number of closed cubes and is thus polyconvex. Hence intrinsic volumes are well defined. The set \(C_n\) consists of those subcubes \(J_{\sigma}\) of level \(n\) for which at least one of the cubes \(J_{\sigma,i}, i \in \{1, \ldots, n\}\), was discarded. We also introduce, for each \(j \in \{1, \ldots, M^d\}\) (and each \(n \in \mathbb{N}_0\)), the set

\[
C_n^j := \bigcup_{\sigma \in \Sigma_n \setminus \mathcal{T}_n, |\sigma| = j} J_{\sigma},
\]
as the union of those cubes of level \(n\) which are contained in \(J_j \cap C_n\).

We are interested in the expected intrinsic volumes \(\mathbb{E} V_k(C_n)\), \(k = 0, \ldots, d\) and in particular in the limiting behaviour as \(n \to \infty\), for which we have the following general formula analogous to \([1.4]\) in Theorem 1.1.

**Theorem 5.1.** Let \(F\) be a fractal percolation in \(\mathbb{R}^d\) with parameters \(M \in \mathbb{N}_{\geq 2}\) and \(p \in (1/M,1]\) and let \(D\) be the Minkowski dimension of \(F\) (see \([1.2]\)). Then, for each \(k \in \{0, \ldots, d\}\) such that \(k < D\), the limit

\[
\overline{V}_k(F) := \lim_{n \to \infty} n^{D-k} \mathbb{E} V_k(C_n)
\]
exists and is given by the expression

\[
(5.1) \quad q_{d,k} \frac{M^{d-k}(1-p)}{M^{d-k}p-1} + \sum_{T \subseteq \{1, \ldots, M^d\}, |T| \geq 2} (-1)^{|T|} \sum_{n=1}^{\infty} n^{D-k} \mathbb{E} V_k\left(\bigcap_{j \in T} C_n^j\right),
\]

where as before \(q_{d,k} = V_k([0,1]^d)\).

**Remark 5.2.** The condition \(k < D\) is natural. If the dimension \(D\) of \(F\) is smaller than the homogeneity index \(k\) of the functional, then the edge effects caused by the common boundary of \(C_n\) with \(J = [0,1]^d\) will dominate the limiting behaviour and therefore a different rescaling will be necessary. More precisely, since for the cube \(J\) no rescaling is necessary for the intrinsic volumes, one would expect the limit \(\lim_{n \to \infty} n^{k-k} \mathbb{E} V_k(C_n)\) to converge instead which is too rough to see the lower-dimensional set \(F\). For \(D < d - 1\), for instance, it is easy to see that the surface area \(C_{d-1}(C_n, \partial J) \to V_{d-1}(J)\) as \(n \to \infty\), while \(C_{d-1}(C_n, \partial F_n) \approx r^{(d-1-D)n} \to 0\).

The restriction \(p > 1/M = r\) on the other hand is due to the method of proof. It can probably be improved to \(p > r^{d-k}\), which would be equivalent to \(D > k\). This is indeed true for \(d = 1\) and \(d = 2\), as the explicit computations below of the limit functionals in these cases show.

**Proof.** We follow the lines of the proof of Theorem 1.1. Let

\[
\tau_k(n) := n^{D-k} \mathbb{E} V_k(C_n), \quad n \in \mathbb{N}_0.
\]

Since \(C_0 = \emptyset\), we have \(\tau_k(0) = \mathbb{E} V_k(\emptyset) = 0\). Setting

\[
w_k(n) := \tau_k(n) - \tau_k(n-1), \quad n \in \mathbb{N},
\]

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we observe that, similarly as in (4.2) above,
\begin{equation}
\nabla_k^c(F) = \lim_{n \to \infty} \tau_k^c(n) = \sum_{n=1}^{\infty} w_k(n).
\end{equation}

By definition of \(w_k\), we have
\begin{equation}
w_k(n) = \tau_k^c(n) - \tau_k^c(n-1) = r^{n(D-k)} \left( E_k(C_n) - M^d pr^k E_k(C_{n-1}) \right).
\end{equation}

Now recall from (3.2) that, for each \(j \in \{1, \ldots, M^d\} \), \(F_0\) survives the first construction step with probability \(p\) (in which case it is distributed like \(\phi_j(F_{n-1})\)) and it is empty otherwise. Thus we get for the closed complements \(C_n^j\)
\begin{align*}
E_k(C_n^j) &= p E_k(\phi_j(C_{n-1})) + (1-p) E_k(J_j) \\
&= pr^k E_k(C_{n-1}) + (1-p)r^k q d k,
\end{align*}

and therefore
\begin{equation}
\sum_{j=1}^{M^d} E_k(C_n^j) = M^d pr^k E_k(C_{n-1}) + M^d (1-p)r^k q d k.
\end{equation}

Plugging this into (5.3) and recalling that \(r^D = M^d p\) yields
\begin{equation}
w_k(n) = r^{n(D-k)} \left( E_k(C_n) - \sum_{j=1}^{M^d} E_k(C_n^j) \right) + r^{(n-1)(D-k)} \frac{1-p}{p} q d k.
\end{equation}

Since \(C_n = \bigcup_{j=1}^{M^d} C_n^j\), by the inclusion-exclusion principle, this can be expressed in a more convenient form: for each \(n \in \mathbb{N}\) and each \(k \in \{0, \ldots, d\}\),
\begin{equation}
w_k(n) = r^{(n-1)(D-k)} \frac{1-p}{p} q d k + \sum_{T \subseteq \{1, \ldots, M^d\}, \mid T \mid \geq 2} (-1)^{|T|-1} r^{(D-k)} E_k(\bigcap_{j \in T} C_n^j).
\end{equation}

Note that this is again a finite sum with a fixed number of terms (independent of \(n\)) and that all the intersections appearing in this formula are at most \(d-1\)-dimensional. The summation over \(n\) can be shown to converge for each summand separately (see Proposition 5.3 below; this is where the hypothesis \(p > r\) is used).

Inserting the representation (5.4) for \(w_k\) into equation (5.2) yields
\begin{equation}
\nabla_k^c(F) = \sum_{n=1}^{\infty} \left( r^{(n-1)(D-k)} \frac{1-p}{p} q d k + \sum_{T \subseteq \{1, \ldots, M^d\}, \mid T \mid \geq 2} (-1)^{|T|-1} r^{(D-k)} E_k(\bigcap_{j \in T} C_n^j) \right)
\end{equation}
\begin{align*}
&= q d k \frac{1-p}{p} \sum_{n=0}^{\infty} r^{(D-k)} + \sum_{T \subseteq \{1, \ldots, M^d\}, \mid T \mid \geq 2} (-1)^{|T|-1} \sum_{n=1}^{\infty} r^{(D-k)} E_k(\bigcap_{j \in T} C_n^j),
\end{align*}

where the convergence of the geometric series in the first term is due to the assumption \(D > k\), and the convergence of the series in the last expression (for each index set \(T\)) follows from Proposition 5.3 just below, justifying in particular the interchange of the summations and showing the existence of the limit of the \(\nabla_k^c(n)\) as \(n \to \infty\). Now formula (5.1) follows easily by computing the series in the first term and recalling that \(r^{-D} = M^d p\). □
Proposition 5.3. Let $F$ be a fractal percolation in $[0,1]^d$ with parameters $M \in \mathbb{N}_{\geq 2}$ and $p \in (1/M,1]$. For each $k \in \{0,\ldots,d\}$ and each $T \subset \{1,\ldots,M^d\}$ with $|T| \geq 2$,
\[ \sum_{n=1}^{\infty} r_n^{(D-k)} \mathcal{E}C_{k+}(\bigcap_{j \in T} C_n^j) < \infty, \]
where as before $C_{k+}(K)$ denotes the total mass of the total variation measure of the $k$-th curvature measure of a polyconvex set $K$. In particular, the sums
\[ \sum_{n=1}^{\infty} r_n^{(D-k)} \mathcal{E}V_k(\bigcap_{j \in T} C_n^j) \]
converge absolutely.

We postpone the proof of Proposition 5.3 to the last section.

Remark 5.4. Note that the assertion of Proposition 5.3 is only needed in the last lines of the proof of Theorem 5.1 to ensure that the series appearing in (5.1) converge. If one can show this convergence directly, as we do e.g. in the direct computations for $\mathbb{R}$ and $\mathbb{R}^2$ below, then Proposition 5.3 is not needed and the assumption $p > 1/M$ in Theorem 5.1 can be weakened.

The case $d = 1$. In order to derive explicit formulas for the limits $\overline{V}_k(F)$ in $\mathbb{R}^2$ it is again necessary to discuss these functional in $\mathbb{R}$ first. We start with a general formula to determine the intrinsic volumes of a polyconvex set $C \subset \mathbb{R}$ from the intrinsic volumes of its closed complement. This will be used to derive expressions for $\mathcal{E}V_k(D_n^{(1)})$ and $\mathcal{E}V_k(D_n^{(1)} \cap D_n^{(2)})$ from the ones already obtained in Section 4 for $\mathcal{E}V_k(K_n^{(1)})$ and $\mathcal{E}V_k(K_n^{(1)} \cap K_n^{(2)})$, where $D_n^{(i)} := I \setminus K_n^{(i)}$.

Lemma 5.5. Let $I := [0,1] \subset \mathbb{R}$ be the unit interval and let $K \subset I$ be polyconvex (i.e. a finite union of intervals). Then the closed complement $C := \overline{I \setminus K}$ of $K$ within $I$ is polyconvex, $V_1(C) = 1 - V_1(K)$ and
\[ V_0(C) = 1 + V_0(K) - 1_K(0) - 1_K(1) - N(K), \]
where $1_A$ denotes the indicator function of a set $A$ and $N(A)$ is the number isolated points in $A$. Moreover, if $K' \subset I$ is a second polyconvex set and $C' := \overline{I \setminus K'}$, then
\[ V_1(C \cap C') = 1 - V_1(K) - V_1(K') + V_1(K \cap K') \]
and
\[ V_0(C \cap C') = 1 + V_0(K) + V_0(K') - V_0(K \cap K') - 1_{K \cup K'}(0) \]
\[ - 1_{K \cup K'}(1) - N(K) - N(K') + N(K \cap K'). \]

Proof. The first formula is an easy consequence of the additivity of $V_1$ noting that $I = K \cup C$, $V_1(I) = 1$ and $V_1(C \cap K) = 0$. The second formula for $V_1$ follows from the first one and additivity by noting that
\[ C \cup C' = \overline{I \setminus (K \cap K')} \]

In $\mathbb{R}$ the Euler characteristic $V_0$ equals the number of connected components of a polyconvex set. If $K \subset I$ has $k$ connected components, $k \in \mathbb{N}_0$, then $K^c = \mathbb{R} \setminus K$ has
Corollary 5.6. Let 

$$V_0(C \cap C') = V_0(C) + V_0(C') - V_0(C \cup C')$$

$$= 1 + V_0(K) - 1_K(0) - 1_K(1) - N(K)$$

$$+ 1 + V_0(K') - 1_{K'}(0) - 1_{K'}(1) - N(K')$$

$$- 1 - V_0(K \cap K') + 1_{K \cap K'}(0) + 1_{K \cap K'}(1) + N(K \cap K')$$

$$= 1 + V_0(K) + V_0(K') - V_0(K \cap K') - 1_{K \cup K'}(0) - 1_{K \cup K'}(1)$$

$$- N(K) - N(K') + N(K \cap K'),$$

where we have used the additivity of the indicator function, implying $$1_K + 1_{K'} = 1_{K \cup K'} + 1_{K \cup K'}$$. This completes the proof of the last formula.

It is clear that corresponding formulas hold for the expected intrinsic volumes of random polyconvex subsets $$K$$ and $$K'$$ of $$[0, 1]$$ and their closed complements. Note that the functional $$N$$ counting the number of isolated points is not additive. Below we always have the situation that $$K$$ and $$K'$$ have no isolated points, while isolated points may appear in the intersection $$K \cap K'$$. 

Corollary 5.6. Let $$K^{(1)}$$, $$K^{(2)}$$ be two independent fractal percolations on the interval $$I = [0, 1]$$ both with the same parameters $$M \in \mathbb{N}_{\geq 2}$$ and $$p \in (0, 1]$$. For $$n \in \mathbb{N}_0$$, let $$K_n^{(i)}$$ denote the $$n$$-th step of the construction of $$K_i$$, $$i = 1, 2$$ and let $$D_n^{(i)} := I \setminus K_n^{(i)}$$. Then, for any $$n \in \mathbb{N}_0$$,

$$\mathbb{E}V_1(D_n^{(1)} \cap D_n^{(2)}) = 1 - 2\mathbb{E}V_1(K_n^{(1)}) + \mathbb{E}V_1(K_n^{(1)} \cap K_n^{(2)})$$

and

$$\mathbb{E}V_0(D_n^{(1)} \cap D_n^{(2)}) = 2\mathbb{E}V_0(K_n^{(1)}) - \mathbb{E}V_0(K_n^{(1)} \cap K_n^{(2)}) + \mathbb{E}N(K_n^{(1)} \cap K_n^{(2)})$$

$$+ 1 - 4p^n + 2p^{2n}.$$ 

Moreover, we have

$$\mathbb{E}V_1(D_n^{(1)}) = 1 - \mathbb{E}V_1(K_n^{(1)})$$

and

$$\mathbb{E}V_0(D_n^{(1)}) = \mathbb{E}V_0(K_n^{(1)}) + 1 - 2p^n.$$ 

Proof. Applying Lemma 5.5 to realizations $$C, C'$$ of the random sets $$D_n^{(1)}$$, $$D_n^{(2)}$$, respectively, and taking expectations, we obtain, for $$k = 1$$, directly the formula stated above and, for $$k = 0$$,

$$\mathbb{E}V_0(D_n^{(1)} \cap D_n^{(2)}) = 1 + \mathbb{E}V_0(K_n^{(1)}) + \mathbb{E}V_0(K_n^{(2)}) - \mathbb{E}V_0(K_n^{(1)} \cap K_n^{(2)})$$

$$- P(0 \in K_n^{(1)} \cup K_n^{(2)}) - P(1 \in K_n^{(1)} \cup K_n^{(2)})$$

$$- \mathbb{E}N(K_n^{(1)}) - \mathbb{E}N(K_n^{(2)}) + \mathbb{E}N(K_n^{(1)} \cap K_n^{(2)}).$$

Now observe that almost surely $$K_n^{(1)}$$ contains no isolated points, implying that

$$\mathbb{E}N(K_n^{(1)}) = 0.$$ 

Moreover,

$$P(0 \in K_n^{(1)} \cup K_n^{(2)}) = P(0 \in K_n^{(1)}) + P(0 \in K_n^{(2)}) - P(0 \in K_n^{(1)} \cap K_n^{(2)})$$

$$= 2p^n - p^{2n},$$
and similarly for the point 1 instead of 0. This shows the second formula. The third formula is a direct application of the first formula in Lemma 5.5 to the realizations $C$ of the random set $D_n^{(1)}$. Similarly, the last formula follows by applying the second formula in Lemma 5.5 to the realizations $C$ of $D_n^{(1)}$ and taking expectations:

$$
\mathbb{E}V_0(D_n^{(1)}) = 1 + \mathbb{E}V_0(K_n^{(1)}) - \mathbb{P}(0 \in K_n^{(1)}) - \mathbb{P}(1 \in K_n^{(1)}) - \mathbb{E}N(K_n^{(1)})
$$

$$
= \mathbb{E}V_0(K_n^{(1)}) + 1 - 2p^n,
$$

since $\mathbb{P}(0 \in K_n^{(1)}) = \mathbb{P}(1 \in K_n^{(1)}) = p^n$ and $\mathbb{E}N(K_n^{(1)}) = 0$. □

To get more explicit expressions for $\mathbb{E}V_k(D_n^{(1)})$ and $\mathbb{E}V_k(D_n^{(1)} \cap D_n^{(2)})$ from Corollary 5.6, we can employ Propositions 4.2 and 4.4 where formulas for $\mathbb{E}V_k(K_n^{(1)})$ and $\mathbb{E}V_k(K_n^{(1)} \cap K_n^{(2)})$ have been derived. The missing piece is an explicit expression for the expected number $\mathbb{E}N(K_n^{(1)} \cap K_n^{(2)})$ of isolated points.

**Proposition 5.7.** Let $K^{(1)}$, $K^{(2)}$ be independent fractal percolations on the interval $[0, 1]$ both with the same parameters $M$ and $p$. Then, for any $n \in \mathbb{N}_0$,

$$
\mathbb{E}N(K_n^{(1)} \cap K_n^{(2)}) = (Mp^2)^n \left(2 - 2M^{-n} - 4pM - p \left[1 - \left(\frac{p}{M}\right)^n\right] + 2p^2 M - p^2 \left[1 - \left(\frac{p^2}{M}\right)^n\right]\right).
$$

**Proof.** First observe that for $n = 0$ both sides of the formula equal zero and thus the formula holds in this case. Let $N(n) := N(K_n^{(1)} \cap K_n^{(2)})$ and let $N_j(n) := N(K_n^{(1)} \cap K_n^{(2)} \cap \cap O_j)$, $j = 1, \ldots, M$, be the number of those isolated points contained in the open subinterval $O_j := \text{int}(J_j) = ((j - 1)/M, j/M)$. Then obviously

$$
(5.7) \quad N(n) = \sum_{j=1}^M N_j(n) + \sum_{j=1}^{M-1} 1\{j/M \text{ isolated in } K_n^{(1)} \cap K_n^{(2)}\}.
$$

Due to the self-similarity, $N_j(n)$ has the same distribution as $\tilde{N}(n - 1)$, where $\tilde{N}(n - 1)$ is the random variable which equals $N(n - 1)$ with probability $p^2$ and is zero otherwise (which accounts for the effect that $J_j$ may be discarded in the first construction step of $K^{(1)}$ or $K^{(2)}$, in which case there are no isolated points generated). Moreover, by symmetry, the indicator variables in the second sum all have the same distribution given by

$$
\mathbb{P}(\{1/M \text{ isolated in } K_n^{(1)} \cap K_n^{(2)}\}) = 2p^{2n}(1 - p^n)^2 =: q_n(p).
$$

Indeed, in order for $1/M$ to be isolated, either both $K_n^{(1)}$ and $K_n^{(2)}$ (with $K_n^{(i), j}$ as defined in (3.1)) need to have a nonempty intersection with $1/M$ (which happens with probability $p^{2n}$) while at the same time both $K_n^{(1), 2}$ and $K_n^{(2), 1}$ do not intersect $1/M$ (probability $(1 - p^n)^2$) or we have the same situation exactly reversed, i.e. $K_n^{(1), 2}$ and $K_n^{(2), 1}$ intersect $1/M$ while $K_n^{(1), 1}$ and $K_n^{(2), 2}$ do not. Taking expectations in (5.7), we get

$$
\mathbb{E}N(n) = \sum_{j=1}^M p^2 \mathbb{E}N(n - 1) + (M - 1)\mathbb{P}(\{1/M \text{ isolated in } K_n^{(1)} \cap K_n^{(2)}\})
$$

$$
= Mp^2 \mathbb{E}N(n - 1) + (M - 1)q_n(p),
$$

which is a recursion relation for the sequence $(\gamma_n)_{n \in \mathbb{N}}$ with $\gamma_n := \mathbb{E}N(n)$. By induction, we infer that

$$
\gamma_n = (M - 1)q_n(p) +Mp^2\gamma_{n-1} = \ldots = (M - 1)\sum_{s=1}^{n} (Mp^2)^{n-s} q_s(p).
$$
Since \(q_n(p) = 2p^{2s}(1 - 2p^s + p^2s) = 2(p^{2s} - 2p^{3s} + p^{4s})\), we conclude that

\[
\gamma_n = 2(M - 1)(Mp^2)^n \sum_{s=1}^{n} \frac{(1/M)^s - 2(p/M)^s + (p^2/M)^s}{n} = 2(M - 1)(Mp^2)^n \sum_{s=1}^{n} \frac{1 - \left( \frac{1}{M} \right)^s}{n}.
\]

Then, for any \(n \in \mathbb{N}_0\),

\[
\text{from which the expression stated in Proposition 5.7 easily follows. \(\square\)}
\]

Now we are ready to derive explicit expressions for \(E_{V_0}(D_n^{(1)})\) and \(E_{V_0}(D_n^{(1)} \cap D_n^{(2)})\) from Corollary 5.6.

**Theorem 5.8.** Let \(K^{(1)}, K^{(2)}\) be independent fractal percolations on the interval \(I = [0, 1]\) both with the same parameters \(M \in \mathbb{N}_{\geq 2}\) and \(p \in (0, 1)\). For \(n \in \mathbb{N}_0\), let \(K_n^{(i)}\) be the \(n\)-th step of the construction of \(K_i\), \(i = 1, 2\) and let \(D_n^{(i)} := I \setminus K_n^{(i)}\).

Then, for any \(n \in \mathbb{N}_0\), \(E_{V_0}(D_n^{(1)} \cap D_n^{(2)}) = 1 - 2p^n + p^{2n}\) and

\[
E_{V_0}(D_n^{(1)}) = (Mp)^n \left[ 1 - \left( \frac{M - 1}{M - p} \right)^n \right] + 1 - 4p^n + 2p^{2n}.
\]

Moreover, we have \(E_{V_0}(D_n^{(1)} \cap D_n^{(2)}) = 1 - p^n\) and

\[
E_{V_0}(D_n^{(1)}) = (Mp)^n \left[ 1 - \left( \frac{M - 1}{M - p} \right)^n \right] + 1 - 2p^n.
\]

**Proof.** Combine Corollary 5.6 with Propositions 1.2, 4.4 and 5.7. The two formulas for \(V_1\) and also the one for \(E_{V_0}(D_n^{(1)})\) follow at once. In case of \(E_{V_0}(D_n^{(1)} \cap D_n^{(2)})\) observe that, for any \(n \in \mathbb{N}_0\),

\[
\begin{align*}
E_{V_0}(D_n^{(1)} \cap D_n^{(2)}) & = (Mp)^n \left[ 2 - 2M^n - 4pM \frac{M - 1}{M - p} \left[ 1 - \left( \frac{p}{M} \right)^n \right] + 2p^2 \frac{M - 1}{M - p^2} \left[ 1 - \left( \frac{p^2}{M} \right)^n \right] \right] \\
& - (Mp)^n \left[ 3 - 2M^n - 4pM - 1 \frac{M - 1}{M - p} \left[ 1 - \left( \frac{p}{M} \right)^n \right] + \frac{p^2}{M - p^2} \left[ 1 - \left( \frac{p^2}{M} \right)^n \right] \right] \\
& = (Mp)^n \left[ -1 + p^2 \frac{M - 1}{M - p^2} \left[ 1 - \left( \frac{p^2}{M} \right)^n \right] \right] \\
& = (Mp)^n \left[ M(p^2 - 1) \frac{M - 1}{M - p^2} - (M - 1)p^2 \left( \frac{p^2}{M} \right)^n \right]
\end{align*}
\]

and thus \(E_{V_0}(D_n^{(1)} \cap D_n^{(2)})\) equals

\[
2E_{V_0}(K_n^{(1)}) + 1 - 4p^n + 2p^{2n} + (Mp)^n \left[ -1 + p^2 \frac{M - 1}{M - p^2} \left[ 1 - \left( \frac{p^2}{M} \right)^n \right] \right]. \quad \square
\]

It is now easy to derive explicit expressions for the limit functionals \(V_k(K) = \lim_{n \to \infty} r^{(D-k)n} E_{V_k}(D_n)\) of fractal percolation \(K\) in \(\mathbb{R}\).
Corollary 5.9. Let $K$ be a fractal percolation on the interval $[0,1]$ with parameters $M \in \mathbb{N}_{\geq 2}$ and $p \in (1/M,1]$. Then, with $D_n := I \setminus K_n$,

$$\lim_{n \to \infty} \mathbb{E} V_1(D_n) = 1 \quad \text{ (while } \mathbb{V}_1^n(K) = \lim_{n \to \infty} r^{(D-1)n} \mathbb{E} V_1(D_n) = 0).$$

Moreover,

$$\mathbb{V}_0^n(K) = \frac{M(1-p)}{M-p} \quad (= \mathbb{V}_0^n(K), \text{ cf. Corollary 4.3}).$$

Proof. Recall from (1.2), that the Hausdorff dimension of $K$ (provided $K \neq \emptyset$) is given by $D = \frac{\log M}{\log p}$, implying in particular that $Mp = r^{-D}$. Therefore, Theorem 5.8 yields

$$r^{Dn} \mathbb{E} V_0^n(D_n) = 1 - p \frac{M-1}{M-p} \left( 1 - \left( \frac{p}{M} \right)^n \right) + (Mp)^{-n}(1-2p^n).$$

Letting now $n \to \infty$, the stated limit for $\mathbb{V}_0^n(K)$ follows. □

The case $d = 2$. In $\mathbb{R}^2$, formula (5.1) in Theorem 5.1 reduces to

$$(5.8) \quad \mathbb{V}_k^n(F) = q_{2,k} \frac{M^{2-k}(1-p)}{M^{2-k}p - 1} - E_1 - E_2 + E_3 - E_4,$$

where

$$E_1 := 2M(M-1) \sum_{n=1}^{\infty} r^{n(D-k)} \mathbb{E} V_k(C_1^n \cap C_1^n),$$

$$E_2 := 2(M-1)^2 \sum_{n=1}^{\infty} r^{n(D-k)} \mathbb{E} V_k(C_1^n \cap C_2^n),$$

$$E_3 := 4(M-1)^2 \sum_{n=1}^{\infty} r^{n(D-k)} \mathbb{E} V_k(C_1^n \cap C_2^n \cap C_3^n),$$

$$E_4 := (M-1)^2 \sum_{n=1}^{\infty} r^{n(D-k)} \mathbb{E} V_k \left( \bigcap_{j=1}^{4} C_1^n \right).$$

Here the sets $C_1^n, \ldots, C_4^n$ are four of the $M^2$ sets $C_j^n = J_j \setminus F_k^n$ chosen such that the corresponding sets $J_j, j = 1, \ldots, 4$ intersect in a point $x$ and are numbered as indicated in Figure 5. The factor in front of the summation in each $E_i$ indicates how many times this particular intersection configuration occurs in the union $C_n = \bigcup_{j=1}^{M^2} C_j^n$, taking into account all symmetries.

In view of Remark 5.4, formula (5.8) is valid for $k = 0$ for all $p$ such that $D > 0$, i.e. for $p \in (1/M^2,1]$, and for $k = 1$ for all $p$ such that $D > 1$, i.e. $p \in (1/M,1]$, provided that all the sums in the terms $E_1, \ldots, E_4$ are finite (which we show below). While for $E_1$ we will again employ the one-dimensional case, the last three summands $E_2, E_3$ and $E_4$ vanish for $k = 1$, since the involved intersections contain at most one point. For $k = 0$ these terms can be obtained by direct inspection of the intersections of the $C_j^n$. 
Lemma 5.10. Suppose $p > 1/M^2$. Then, for $k = 0$, the terms $E_2, E_3$ and $E_4$ in (5.8) are given by

$$E_2 = 2(M - 1)^2\left(\frac{1}{M^2p - 1} - \frac{2}{M^2 - 1} + \frac{p}{M^2 - p}\right),$$

$$E_3 = 4(M - 1)^2\left(\frac{1}{M^2p - 1} - \frac{3}{M^2 - 1} + \frac{3p}{M^2 - p} - \frac{p^2}{M^2 - p^2}\right),$$

and

$$E_4 = (M - 1)^2\left(\frac{1}{M^2p - 1} - \frac{4}{M^2 - 1} + \frac{6p}{M^2 - p} - \frac{4p^2}{M^2 - p^2} + \frac{p^3}{M^2 - p^3}\right).$$

Proof. In all three intersection configurations of the sets $C_n^i$ considered here, the intersection contains at most one point, $x$, cf. Figure 3. For each of the sets $C_n^i$ to contain $x$ it is necessary, that at least in one of the sets $F_k^n$, $k = 1, \ldots, n$ the $k$-th level subsquare intersecting $x$ is discarded (which happens with probability $1 - p^n$). Thus, by independence, we obtain for the intersections of $\ell = 2, 3$ or 4 of these sets

$$\mathbb{E}V_0\left(\bigcap_{j=1}^{\ell} C_n^j\right) = \mathbb{P}\left(\bigcap_{j=1}^{\ell} C_n^j = \{x\}\right) = (1 - p^n)^\ell,$$

and therefore

$$(5.9) \qquad \sum_{n=1}^{\infty} r^n D \mathbb{E}V_0\left(\bigcap_{j=1}^{\ell} C_n^j\right) = \sum_{k=0}^{\ell} \binom{\ell}{k} (-1)^k \frac{p^{k-1}}{M^2 - p^k - p^{k-1}}.$$

Indeed, employing the binomial theorem and the relation $r^{-D} = M^2 p$, we get

$$\sum_{n=1}^{\infty} r^n D \mathbb{E}V_0\left(\bigcap_{j=1}^{\ell} C_n^j\right) = \sum_{n=1}^{\infty} (M^2 p)^{-n} (1 - p^n)^\ell = \sum_{n=1}^{\infty} \sum_{k=0}^{\ell} \binom{\ell}{k} (-1)^k \frac{p^{k-1}}{M^2} \left(\sum_{n=1}^{\infty} \frac{p^{k-1}}{M^2}\right)^n,$$

where in the last expression all the geometric series converge due to the assumption $p > 1/M^2$. Computing these series yields the expression stated in (5.9). Finally, the assertion of the lemma follows by plugging (5.9) into the expressions for $E_2, E_3$ and $E_4$ given by (5.8). \qed

The expressions derived in Theorem 5.8 for intersections of fractal percolations in one dimension will now be used to compute the expected intrinsic volumes of the (1-dimensional) intersections $C_n^1 \cap C_n^4$ appearing in the term $E_1$ in formula (5.8) for fractal percolation in $\mathbb{R}^2$.

Proposition 5.11. Let $F$ be a fractal percolation in $\mathbb{R}^2$ with parameters $M \in \mathbb{N}_{\geq 2}$ and $p \in [0,1]$. Then, for any $n \in \mathbb{N},$

$$\mathbb{E}V_1(C_n^1 \cap C_n^4) = \frac{1}{M} \left(1 - 2p^n + p^{2n}\right)$$

and

$$\mathbb{E}V_0(C_n^1 \cap C_n^4) = 2(Mp)^n \left(\frac{1 - p}{M - p} + \frac{M - 1}{M - p} \left(\frac{p}{M}\right)^n\right) + 1 - 4p^n + 2p^{2n}$$

$$- (Mp^n)^n \left(\frac{1 - p^2}{M - p^2} + \frac{M - 1}{M - p^2} \left(\frac{p^2}{M}\right)^n\right).$$

Proof. Let $K^{(1)}, K^{(2)}$ be two independent fractal percolations on the interval $I = [0,1]$ with the same parameters $M$ and $p$ as $F$ and independent of $F$. For $n \in \mathbb{N}_0$, let $K^{(i)}_n$ be the $n$-th step of the construction of $K^{(i)}$, $i = 1, 2$ and let $F^{(i)}_n := I \setminus K^{(i)}_n$ (just as in Corollary 5.6). Denote by $\tilde{K}^{(i)}_n$, $i = 1, 2$, the random set which equals $K^{(i)}_n$ with probability $p$ and is empty otherwise. Recalling from (4.20) that in distribution

$$F^{(1)}_n \cap F^{(2)}_n = \psi(\tilde{K}^{(1)}_{n-1} \cap \tilde{K}^{(2)}_{n-1}),$$
we infer that also the following equation holds in distribution, since $C_n^i$ is determined by $F_n^i$ and similarly $D_n^{(i)}$ is determined by $K_n^{(i)}$. For each $n \in \mathbb{N}$, we have

\begin{equation}
C_n^1 \cap C_n^4 = \psi(\hat{D}_{n-1}^{(1)} \cap \hat{D}_{n-1}^{(2)}),
\end{equation}

where $\hat{D}_n^{(i)}$ is the random set which equals $D_n^{(i)}$ with probability $p$ and $I$ with probability $1 - p$. This implies for each $n \in \mathbb{N}_0$,

\[
\mathbb{E}V_0(C_{n+1} \cap C_{n+1}^4) = \mathbb{E}V_0(\hat{D}_{n}^{(1)} \cap \hat{D}_{n}^{(2)})
\]

\[
= p^2 \mathbb{E}V_0(D_n^{(1)} \cap D_n^{(2)}) + p(1 - p)\mathbb{E}V_0(D_n^{(1)} \cap I)
\]

\[
+ p(1 - p)\mathbb{E}V_0(I \cap D_n^{(2)}) + (1 - p)^2\mathbb{E}V_0(I \cap I)
\]

\[
= p^2 \mathbb{E}V_0(D_n^{(1)} \cap D_n^{(2)}) + 2p(1 - p)\mathbb{E}V_0(D_n^{(1)}) + (1 - p)^2.
\]

Employing now the formulas derived in Theorem 5.8 for $\mathbb{E}V_0(D_n^{(1)} \cap D_n^{(2)})$ and $\mathbb{E}V_0(D_n^{(1)})$, we obtain for each $n \in \mathbb{N}_0$

\[
\mathbb{E}V_0 \left( C_{n+1}^4 \cap C_{n+1}^4 \right)
\]

\[
= p^2 \left( 2(Mp)^n \left( 1 - p \frac{M - 1}{M - p} \left[ 1 - \left( \frac{p}{M} \right)^n \right] \right) + 1 - 4p^n + 2p^{2n} + (Mp^2)^n \left( -1 + p^2 \frac{M - 1}{M - p^2} \left[ 1 - \left( \frac{p^2}{M} \right)^{n-1} \right] \right) 
\]

\[
+ 2p(1 - p) \left( (Mp)^n \left( 1 - p \frac{M - 1}{M - p} \left[ 1 - \left( \frac{p}{M} \right)^n \right] \right) + 1 - 2p^n \right) + (1 - p)^2
\]

\[
= 2p(Mp)^n \left( 1 - p \frac{M - 1}{M - p} \left[ 1 - \left( \frac{p}{M} \right)^n \right] \right) + 1 - 4p^{n+1} + 2p^{2(n+1)}
\]

\[
+ p^2(Mp^2)^n \left( -1 + p^2 \frac{M - 1}{M - p^2} \left[ 1 - \left( \frac{p^2}{M} \right)^n \right] \right),
\]

where we combined some of the terms to get to the last expression. Replacing now $n + 1$ by $n$, this simplifies to

\[
\mathbb{E}V_0 \left( C_n^1 \cap C_n^4 \right) = \frac{2}{M}(Mp)^n \left( 1 - p \frac{M - 1}{M - p} \left[ 1 - \left( \frac{p}{M} \right)^{n-1} \right] \right) + 1 - 4p^n + 2p^{2n} + \frac{1}{M}(Mp^2)^n \left( -1 + p^2 \frac{M - 1}{M - p^2} \left[ 1 - \left( \frac{p^2}{M} \right)^{n-1} \right] \right)
\]

\[
= 2(Mp)^n \left( 1 - p \frac{M - 1}{M - p} \left[ 1 - \left( \frac{p}{M} \right)^n \right] \right) + 1 - 4p^n + 2p^{2n} - (Mp^2)^n \left( -1 + p^2 \frac{M - 1}{M - p^2} \left[ 1 - \left( \frac{p^2}{M} \right)^n \right] \right),
\]

for any $n \in \mathbb{N}$, completing the proof of the formula for $V_0$ in Proposition 5.11. The formula for $V_1$ follows similarly from (5.10) using the corresponding formulas from Theorem 5.8.

\[
\mathbb{E}V_1(C_{n+1}^1 \cap C_{n+1}^4) = \mathbb{E}V_1(\psi(\hat{D}_n^{(1)} \cap \hat{D}_n^{(2)})) = \frac{1}{M}\mathbb{E}V_1(\hat{D}_n^{(1)} \cap \hat{D}_n^{(2)})
\]

\[
= \frac{1}{M} \left( p^2 \mathbb{E}V_1(D_n^{(1)} \cap D_n^{(2)}) + 2p(1 - p)\mathbb{E}V_1(D_n^{(1)}) + (1 - p)^2 \right)
\]

\[
= \frac{1}{M} \left( p^2 \left[ 1 - 2p^n + p^{2n} \right] + 2p(1 - p) [1 - p^n] + (1 - p)^2 \right)
\]

\[
= \frac{1}{M} \left( 1 - 2p^{n+1} + p^{2(n+1)} \right).
\]
Corollary 5.12. If $p > 1/M^2$, then for $k = 0$ the term $E_1$ in (5.8) is given by
\[
E_1 = \frac{2M(M - 1)}{M - p} \left( \frac{2(1 - p)}{M - 1} + \frac{2(M - 1)p - p(1 - p^2)}{M - p^2} \right) - \frac{2M(M - 1)^2p^3}{(M - p^2)(M^2 - p^3)} + \frac{2M(M - 1)}{M^2p - 1} - \frac{8M}{M + 1} + \frac{4M(M - 1)p}{M^2 - p}.
\]
Similarly, if $p > 1/M$, then for $k = 1$, $E_1 = 2(M - 1) \left( \frac{1}{M - 1} - \frac{2}{M^2 - 1} + \frac{p^3}{M^2 - p^3} \right)$.

Proof. To determine $E_1$ for $k = 0$, we multiply the expression derived in Proposition 5.11 for $\mathbb{E}_0(C_0^a \cap C_0^b)$ by $r^Dn = (M^2p)^{-n}$ and sum over $n$ to obtain
\[
E_1 = 2M(M - 1) \sum_{n=1}^{\infty} (M^2p)^{-n} \left[ (2M)\left( \frac{1 - p}{p - M} + \frac{M - 1}{p - M} \left( \frac{p}{M} \right)^n \right) - (M^2p)^n \left( \frac{1 - p}{p - M} + \frac{M - 1}{p - M} \left( \frac{p^2}{M} \right)^n \right) + 1 - 4p^n + 2p^{2n} \right].
\]
and the expression stated above is then derived by computing the various geometric series (which do all converge due to the assumption $p > 1/M^2$, justifying thus in particular the above interchange of summations) and combining some of the terms. For $k = 1$, the stated expression for $E_1$ follows similarly by multiplying the expression for $\mathbb{E}_1(C_1^a \cap C_1^b)$ from Proposition 5.11 by $r^{(D-1)n} = (M^2)^{-n}$ and summing over $n$. The involved series converge due to the assumption $p > 1/M$. □

Now we are ready to prove Theorem 1.3. For convenience, we repeat the statement here, and we add a corresponding formula for the limit $\overline{V}_1(F)$ of the rescaled boundary lengths:

Proposition 5.13. Let $F$ be a fractal percolation in $\mathbb{R}^2$ with parameters $M \in \mathbb{N}_{\geq 2}$ and $p \in [0, 1]$. Then, for any $p > 1/M^2$,
\[
\overline{V}_0(F) = M^2(1 - p)M + (M - 1)p^2 + (M - 1)p - M
\]
Moreover, for any $p > 1/M$,
\[
\overline{V}_1(F) = 2M \frac{1 - p}{M - p} \quad (= \overline{V}_1(F), \quad \text{cf. (1.2)}).
\]

Proof of Proposition 5.13 and thus in particular of Theorem 1.3. All one has to do is to insert the expressions for $E_1, \ldots, E_4$ obtained in Lemma 5.10 and Corollary 5.12 into formula (5.8) for $\overline{V}_k(F)$. For $k = 0$, we obtain (recalling that $q_{2,0} = V_0(J) = 1$)
\[
(5.12) \quad \overline{V}_0(F) = \frac{M^2(1 - p)}{M^2p - 1} - E_1 - E_2 + E_3 - E_4
\]
where
\[
-E_2 + E_3 - E_4 = (M - 1)^2 \left( \frac{1}{M^2p - 1} - \frac{4}{M^2 - 1} + \frac{4p}{M^2 - p} - \frac{p^3}{M^2 - p^3} \right).
\]
Fortunately, this can be simplified to the expression stated above. Similarly, we get for \( k = 1 \) (taking into account that \( E_2 = E_3 = E_4 = 0 \) in this case)

\[
\nabla_1^c(F) = 2M \frac{1-p}{Mp-1} - 2(M-1) \left( \frac{1}{Mp-1} - \frac{2}{M-1} + \frac{p}{M-p} \right),
\]

which simplifies to the expression stated above for \( \nabla_1(F) \). This completes the proof.

\[ \square \]

**Remark 5.14.** From the proof of Proposition 5.13, we also get explicit expressions for the expected intrinsic volumes of the approximation sets \( C_m \) for each \( m \in \mathbb{N} \). To determine \( \tau_k(m) := r^{m(D-k)} \mathbb{E} V_k(C_m), m \in \mathbb{N} \), it is enough to truncate all the sums in formula (5.5) after the \( m \)-th term (including the very first one which appears already in summed form in (5.8), cf. (5.5)). For \( k = 0 \), we obtain for any \( m \in \mathbb{N} \),

\[
\tau_0(m) = \frac{1-p}{p} \sum_{n=0}^{m} r^nD - E_1(m) - 2E_2(m) + 4E_3(m) - E_4(m),
\]

where \( E_1(m) \), the truncated term corresponding to \( E_1 \), can be read off from equation (5.11) in the proof of Corollary 5.12:

\[
E_1(m) := 2M(M-1) \sum_{n=1}^{m} \left\{ \frac{1}{M} - \frac{1}{M-p} \left( \frac{1}{M} \right)^n + \frac{M-1}{M-p} \left( \frac{p}{M} \right)^n - \frac{1-p^2}{M-p^2} \left( \frac{p}{M} \right)^n \right\} - \frac{M-1}{M-p^2} \left( \frac{p^3}{M^2} \right)^n + \left( \frac{1}{M^2} \right)^n - 4 \left( \frac{1}{M^2} \right)^n + 2 \left( \frac{p}{M} \right)^n. \]

Similarly, \( E_\ell(m), \ell = 2, 3, 4 \), are derived by truncating the corresponding sums \( E_\ell \) and computing the resulting finite geometric sums, cf. (5.9):

\[
E_\ell(m) := (M-1)^\ell \sum_{n=1}^{m} (M^2p)^{-n}(1-p^n)^\ell
\]

\[
= (M-1)^\ell \sum_{k=0}^{\ell} \binom{\ell}{k} (-1)^k \frac{p^{k-1}}{M^2 - p^{-k-1}} \left[ 1 - \frac{p^{k-1}}{M^2} \right]^m. \]

Computing all the finite geometric sums, we get for each of the terms in equation (5.12), a corresponding one for \( \tau_0(m) \) with a factor of the form \((1 - q^m)\) for a suitable \( q \) (just as in the last line of the formula above). The constant terms add up to \( \nabla_0(F) \), such that we end up with the following exact expansion in \( m \):

\[
\tau_0(m) = \nabla_0(F) + \frac{4M(1-p)}{M-p} M^{-m} - \frac{2M(M-1)p(1-p^2)}{(M-p)(M-p^2)} M^{(D-3)m} + M^{-Dm}
\]

\[
- 4M^{-2m} + \frac{4p(M-1)}{M-p} M^{(D-4)m} + \tilde{c} M^{(3D-8)m},
\]

where \( \tilde{c} := \frac{(M-1)p^3}{M-p^2} (M-1)(M-p^2) - 2M(M-p) \). It is easy to see that this sequence converges again exponentially fast to \( \nabla_0(F) \) as \( m \to \infty \). Moreover, multiplying by \( M^{Dm} \), we obtain an exact expansion for \( \mathbb{E} V_0(C_m) \):

\[
\mathbb{E} V_0(C_m) = \nabla_0(F) M^{Dm} + \frac{4M(1-p)}{M-p} M^{(D-1)m} - \frac{2M(M-1)p(1-p^2)}{(M-p)(M-p^2)} M^{(2D-3)m}
\]

\[
+ 1 - 4M^{(D-2)m} + \frac{4p(M-1)}{M-p} M^{(2D-4)m} + \tilde{c} M^{(4D-8)m}. \]
Since $D < 2$ for $p < 1$, the last three terms vanish as $m \to \infty$. The remaining terms determine the subdimensions of $F$ in the sense of [20]. We obtain,

$$
EV_0(C_m) = \nabla^r_0(F)M^{D_m} + c_2M^{D_2m} + c_3M^{D_3m} + 1 + o(1),
$$
as $m \to \infty$, where $c_2 = \frac{4M(1-p)}{M-p}$ is the amplitude of the first subdimension $D_2 := D - 1$ and $c_3 = \frac{-2(M(1-p) - p^2)}{(M-p)(M-p^2)}$ is the amplitude of second subdimension $D_3 := 2D - 3 = \frac{\log M^2}{\log M}$. Recall from Proposition 4.6 and Remark 4.9, that $D_3$ (which is positive for $p > 1/\sqrt{M}$) is the dimension of the intersection of two (small) copies of $F$ constructed in neighboring squares sharing a common side. Similarly $D_2$ (which is positive for $p > 1/M$) is the dimension of a fractal percolation on an interval with the same parameters as $F$ or equally the dimension of $F \cap \partial[0,1]^2$. Hence two subdimensions appear for these random fractals as suggested by [20] and they carry geometric meaning as in the deterministic setting studied there.

6. Proof of Propositions 4.4 and 5.3

Let $n \in \mathbb{N}$ and $W_1, W_2, \ldots$ be unions of subcubes of $J = [0,1]^d$ of level $n$. More precisely, if $\Omega^1, \Omega^2, \ldots$ are arbitrary subsets of $\{1, \ldots, M^d\}^n$, then we let

$$W_i := \bigcup_{\sigma \in \Omega^i} J_{\sigma}, \quad i \in \mathbb{N}.$$  

Our first aim is to establish a general bound on the curvature of the intersection $W_1 \cap W_2 \cap \ldots \cap W_\ell$ for an arbitrary number $\ell \in \mathbb{N}_{\geq 2}$ of these sets. For this we will employ an estimate from [23]. Recall from [23] that for any finite family $X = \{X_1, \ldots, X_m\}$ of sets, the intersection number $\Gamma = \Gamma(X)$ is defined by

$$\Gamma := \max_{i \in \{1, \ldots, m\}} |\{j : X_j \cap X_i \neq \emptyset\}|.$$  

If $\Gamma$ is small compared to $m$, then the following estimate is particularly useful, which is a special case of [23, Corollary 3.0.5] for a family of convex sets.

**Lemma 6.1.** Let $\{X_1, \ldots, X_m\}$ be a family of compact, convex subsets of $\mathbb{R}^d$ and let $\Gamma$ be its intersection number. Then, for any $k \in \{0, \ldots, d\}$,

$$C^\varphi_k(\bigcup_{j=1}^m X_j) \leq m 2^\ell b_k,$$

where $b_k := \max\{C_k(X_j) : j = 1, \ldots, m\}$.

**Proof.** Since the $X_i$ are convex, any intersection $X_I := \bigcap_{i \in I} X_i, I \subseteq \{1, \ldots, m\}$ is also convex and contained in any of the sets $X_i, i \in I$. Therefore, the monotonicity and positivity of the intrinsic volumes implies that $C^\varphi_k(X_I) = C_k(X_I) \leq \max_{i \in I} C_k(X_i) \leq b_k$. Thus (for $B := \mathbb{R}^d$) the assumptions of [23, Corollary 3.0.5] are satisfied with $b := b_k$ and the assertion follows. $\square$

Recall that $r = 1/M$.

**Lemma 6.2.** There is a constant $c_{d,k}$ such that for any $n \in \mathbb{N}$ and $\ell \in \mathbb{N}_{\geq 2}$ and any collection $W_1, \ldots, W_\ell$ of unions of cubes of level $n$ as defined in (6.1) the following estimate holds

$$C^\varphi_k(W_1 \cap \ldots \cap W_\ell) \leq c_{d,k} |\Omega^1|^{\ell - kn},$$

where $|\Omega^1|$ is the number of cubes in $W_1$.

Note that $|\Omega^1| \leq M^{dn} = e^{-dn}$, which implies that the right hand side of (6.2) is always bounded from above by $c_{d,k}^{(k-d)n}$. 

Proof. First let \( \ell = 2 \). We write the intersection \( W_1 \cap W_2 \) as a union of convex sets. It is clear that each set \( J_{\sigma} \) in the union \( W_1 \) intersects at most \( 3^d \) cubes (the neighboring ones) from the union \( W_2 \). Let \( \Omega^C \subset \Omega \) be the set of indices of the cubes from \( W_2 \) intersecting \( J_{\sigma} \). Then \( |\Omega^C| \leq 3^d \) and we have

\[
W_1 \cap W_2 = \bigcup_{\sigma \in \Omega} \left( J_{\sigma} \cap \bigcup_{\omega \in \Omega^C} J_{\omega} \right) = \bigcup_{\sigma \in \Omega} \bigcup_{\omega \in \Omega^C} \left( J_{\sigma} \cap J_{\omega} \right).
\]

This way we have represented \( W_1 \cap W_2 \) as a union of at most \( |\Omega| \cdot 3^d \leq (M^d)^n 3^d \) convex sets \( R_{\sigma,\omega} := J_{\sigma} \cap J_{\omega} \). Note that each of the sets \( R_{\sigma,\omega} \) is the intersection of two cubes and thus a \( k \)-face of some cube of level \( n \) (of some dimension \( k \in \{0, \ldots, d\} \)). We may reduce the number of sets in this representation by deleting the double occurrences of any face without changing the union set. Then the reduced family \( \mathcal{F} \subset \{ R_{\sigma,\omega} \} \) has an intersection number \( \Gamma = \Gamma(\mathcal{F}) \) bounded from above by \( 3^d \) times the number of faces of a cube in \( \mathbb{R}^d \) (which also equals \( 3^d \)). Indeed, each set \( R \in \mathcal{F} \) is contained in a cube of dimension \( d \) and any other set \( R' \in \mathcal{F} \) intersecting \( R \) must be a face of the same cube or of one of the neighboring cubes. Note also that any of the sets \( R \in \mathcal{F} \) is convex and contained in a cube of sidelength \( r \). Therefore,

\[
C_{\mathcal{F}}(R) = C_k(R) \leq C_k(r^n J) = r^{kn} C_k(J) = r^{kn} q_{d,k},
\]

where he have used the monotonicity, motion invariance and homogeneity of the intrinsic volumes. Now we can apply Lemma \([6.1]\) to the family \( \mathcal{F} \) consisting of \( m \leq (M^d)^n 3^d \) sets and satisfying \( b_k := \max \{ C_k(R) : R \in \mathcal{F} \} \leq r^{kn} q_{d,k} \). We obtain

\[
C_{\mathcal{F}}(W_1 \cap W_2) \leq |\Omega| \cdot 3^d 2^{3d} q_{d,k} = c_{d,k} |\Omega|^1 \cdot |\Omega|^k,
\]

where the constant \( c_{d,k} := 3^d 2^{3d} q_{d,k} \) is independent of \( n \). This proves the case \( \ell = 2 \). For the general case, fix some \( \ell > 2 \) and note that \( W_1 \cap \ldots \cap W_\ell \) can be represented by

\[
W_1 \cap \ldots \cap W_\ell = \bigcup_{\sigma \in \Omega} \bigcup_{\omega_1 \in \Omega^C_1} \ldots \bigcup_{\omega_\ell \in \Omega^C_\ell} J_{\sigma} \cap J_{\omega_1} \cap \ldots \cap J_{\omega_\ell},
\]

where similarly as before \( \Omega^C_\ell \) is the family of those words \( \omega \in \Omega^C \) for which \( J_{\sigma} \cap J_{\omega} \neq \emptyset \), \( j = 2, \ldots, \ell \). Now observe that \( J_{\sigma} \cap J_{\omega_1} \cap \ldots \cap J_{\omega_\ell} \) is a finite intersection of cubes of the grid and thus a \( k \)-face of \( J_{\sigma} \) (of some dimension \( k \in \{0, \ldots, d\} \)) if not empty. Hence, for fixed \( \sigma \), there are at most \( 3^d \) distinct sets in the union corresponding to the faces of \( J_{\sigma} \). Deleting all multiplicities such that no set appears more than once in the union on the right of \( (6.3) \), we again end up with a representation of \( W_1 \cap \ldots \cap W_\ell \) by at most \( |\Omega| \cdot 3^d \) convex sets and, as before, the intersection number of the reduced family will not exceed \( 3^d \). Hence \( (6.2) \) follows again from Lemma \([6.1]\) with the same constant \( c_{d,k} \) as before. This completes the proof for arbitrary integers \( \ell \geq 2 \).

\( \square \)

**Proposition 6.3.** For \( \ell \in \mathbb{N}, \ell \geq 2 \), let \( F^{(1)}, \ldots, F^{(\ell)} \) be independent fractal percolations in \( \mathbb{R}^d \) with parameters \( M \geq 2 \) and \( p \in (0, 1] \) such that \( p > \max \{ r^3, r^d \} \). Then there exist some constants \( c > 0 \) and \( 0 < \gamma < 1 \) such that, for each \( n \in \mathbb{N} \),

\[
\mathbb{E} C_k^{\mathcal{F}}(F_n^{(1)} \cap \ldots \cap F_n^{(\ell)}) \leq cr^{(k-\ell+1)n} p^{2n \gamma}.
\]

**Proof.** Let \( N_n \) be the number of cubes in \( F_n^{(1)} \). The sequence \( (N_n)_n \) is a Galton-Watson process with a binomial offspring distribution with parameters \( M^d \) and \( p \). Hence \( N_1 \) has mean \( \mu := \mathbb{E} N_1 = M^d p \) (with \( \mu > 1 \) due to the assumption \( p > r^3 \)) and variance \( \nu^2 := \mathbb{V} (N_1) = M^d p (1 - p) \). Moreover, mean and variance of \( N_n \) are given by \( \mathbb{E} N_n = \mu^n \) and \( \mathbb{V} (N_n) = \frac{\nu^2 \mu^{n-1}}{\mu^n} \). Choose \( t > 1 \) such that \( r/p < t^2 < r^{-2} \). (The assumption \( r^3 < p \) ensures this is
possible. Set \(b_n := \mathbb{E}N_n(1 + t^n) = \mu^n(1 + t^n)\), \(n \in \mathbb{N}\). For \(S(n) := F_n^{(1)} \cap \ldots \cap F_n^{(t)}\), we have
\[
\mathbb{E}C_k^{\varphi}(S(n)) = \mathbb{E}(C_k^{\varphi}(S(n)) | N_n \leq b_n)\mathbb{P}(N_n \leq b_n)
+ \mathbb{E}(C_k^{\varphi}(S(n)) | N_n > b_n)\mathbb{P}(N_n > b_n).
\]

To prove the upper bound asserted in (6.4), it suffices to show that each of the two summands above satisfies such a bound. For the first summand, we bound the probability by 1 and apply Lemma 6.2 to see that
\[
\mathbb{E}(C_k^{\varphi}(S(n)) | N_n \leq b_n) \leq c_{d,k}b_n \mu^{kn} = c_{d,k}\mu^{r(k-d)n}p^n(1 + t^n) \leq c_{d,k}\mu^{r(k-d-1)n}p^n2\alpha^n,
\]
where \(\alpha := rt\). For the last inequality, note that \(t > 1\) implies \((1 + t^n) < 2t^n\). Due to the choice of \(t\), we have \(0 < \alpha < 1\) and thus an estimate of the type (6.4) for the first summand.

For the second term, we apply Chebychev’s inequality to see that
\[
\mathbb{P}(N_n \geq b_n) = \mathbb{P}(N_n \geq \mu^n(1 + t^n)) \leq \mathbb{P}(|N_n - \mu^n| \geq t^n\mu^n) \leq \frac{\mathbb{V}(N_n)}{t^n\mu^{2n}} \leq \bar{c}t^{-2n},
\]
where \(\bar{c} := \frac{\mu^2}{\mu^{\mu-1}}\) is independent of \(n\). Setting now \(\beta := r/(pt^2)\), we have \(0 < \beta < 1\) by the choice of \(t\), and applying again Lemma 6.2 we conclude
\[
\mathbb{E}(C_k^{\varphi}(S(n)) | N_n > b_n)\mathbb{P}(N_n > b_n) \leq c_{d,k}\mu^{r(k-d)n} \beta(p/r)^n\beta^n \leq \bar{c}c_{d,k}\mu^{r(k-d-1)n}p^n\beta^n.
\]
Combining the two summands, it follows that the claimed estimate (6.4) holds with \(\gamma := \max\{\alpha, \beta\}\) and \(c := (2 + \bar{c})c_{d,k}\).

With Lemma 6.2 and the refined estimate Proposition 6.3 at hand, we will now prove Propositions 4.1 and 5.3.

**Proof of Propositions 4.1 and 5.3.** Given \(T \subseteq \{1, \ldots, M^d\}, |T| \geq 2\), let \(U := \bigcap_{j \in T} J_j\) (where \(J_j\) is the cube of sidelength \(r\) containing \(F^j\)). \(U\) is a cube of some dimension \(u \in \{0, \ldots, d - 1\}\) and the intersection \(\bigcap_{j \in T} F_j\) is contained in \(U\) (and similarly \(\bigcap_{j \in T} C_{i_n} \subseteq U\)). Let \(H\) be the affine hull of \(U\), which is a \(u\)-dimensional affine space. Since intrinsic volumes are independent of the dimension of the ambient space, it is enough to study the intersection of the sets \(F_j \cap H\), \(j \in T\) (or \(C_{i_n} \cap H\), respectively). It is easy to see that, given \(F_j\) is nonzero, and up to scaling by a factor \(r\), the sets \(F_j \cap H\) coincide in distribution with the \(n\)-th approximation \(K_n^{(j)}\) of a fractal percolation \(K^{(j)}\) in \([0, 1]^u\) with the same parameters \(M\) and \(p\) as the given \(F\). More precisely, if \(\tilde{K}^{(j)}\) denotes the random set which equals \(K^{(j)}\) with probability \(p\) and is empty otherwise, we have \(F_j \cap H = \psi(\tilde{K}_n^{(j)}), j \in T\), in distribution, where \(\psi : H \to \mathbb{R}^u\) is one of the similarities (with factor \(1/r\)) mapping \(U\) to \([0, 1]^u\). (Analogously, we have \(C_{i_n} \cap H = \psi(\tilde{D}_n^{(j)}), j \in T\), in distribution, where \(\tilde{D}_n^{(j)} := [0, 1]^u \setminus \tilde{K}_n^{(j)}\), is the random set which equals \(D_n^{(j)} := [0, 1]^u \setminus K_n^{(j)}\) with probability \(p\) and \([0, 1]^u\) with probability \(1 - p\).)

Now we can apply Proposition 6.3 to the sets \(K_n^{(j)}, j \in T\), which are contained in \(\mathbb{R}^u\), to infer that there exist positive constants \(c\) and \(\gamma < 1\) such that
\[
\mathbb{E}C_k^{\varphi}(\bigcap_{j \in T} F_j) = \mathbb{E}C_k^{\varphi}(\bigcap_{j \in T} \psi(\tilde{K}_n^{(j)})) \leq r^{-k}\mathbb{E}C_k^{\varphi}(\bigcap_{j \in T} K_n^{(j)})
\leq r^{-k}\mathbb{E}C_k^{(k-u-1)n}p^n\gamma^n.
\]
Since \(u \leq d - 1\) and \(r^D = r^{d}p^{-1}\), by definition of \(D\), we infer that the last expression is bounded by \(c'r^{(k-d)n}p^n\gamma^n = c'r^{(k-D)n}n\gamma^n\) for some positive constants \(c’\) and...
Remark 6.4. \( \gamma < 1 \), from which it is easy to see that the series \( \sum_{n=0}^{\infty} r^{n(D-k)} E C^\var(r^{nTj_n}) \) converges. This shows the first assertion in Proposition 4.1. The second assertion is now obvious from the fact that, for any polyconvex set \( K \), \( |C_k(K)| \leq C^\var(K) \), completing the proof.

For the intersection of the sets \( D^{(j)}_n \) we do not have the refined estimate of Proposition 6.3 available (and it is clear that a similar argument will not work for the \( D^{(j)}_n \)), but we can still employ Lemma 6.2 to get analogous bounds at least for the most interesting interval \( p \in (r, 1] \). Recall that each realization of the set \( D^{(j)}_n \) is a union of (at most \( M'^{n} \)) \( u \)-dimensional cubes of sidelength \( r^n \). We obtain

\[
E C^\var(\bigcap_{j \in T} C^{(j)}_n) = E C^\var(\bigcap_{j \in T} \psi(D^{(j)}_n)) \leq r^{-k} E C^\var(\bigcap_{j \in T} D^{(j)}_n) \leq r^{-k} c_{d,k} r^{(k-u)n} \leq cr^{(k-d)n} p^n \gamma^n,
\]

where \( c := r^{-k} c_{d,k} \) and \( \gamma := r/p \). In the last inequality we used that \( u \leq d - 1 \) implying \( r^{-u} \leq r^{-d} r \). Observe that for \( p > r \) we have \( \gamma < 1 \) and that, by definition of \( D \), \( r^{(k-d)n} p^n = r^{(k-d)n} \). Now the assertions of Proposition 5.3 are obvious, completing the proof.

\[\Box\]

Remark 6.4. We conjecture that for any \( m \in \mathbb{N} \) there is some constant \( c \) such that \( E|N_n - E N_n|m \leq c(M'u)^{mn} \) for all \( n \in \mathbb{N} \). In the above proof we have used that this is true for \( m \leq 2 \). If this was true for some larger \( m \), then one could derive from the general Markov inequality that the estimate in Proposition 6.3 (and thus the assertions of Proposition 5.3) hold for any \( p \in (r^{m+1}, 1] \).

Acknowledgements. We thank Klaus Mecke and Philipp Schönöfer, for inspiring discussions and their preliminary work \[20\] \[19\] which motivated the authors to look at this model more closely. During the work on this project both authors have been members of the DFG research unit Geometry and Physics of Spatial Random Systems at Karlsruhe. We gratefully acknowledge support from grants number HU1874/3-2, and LA965/6-2. Part of this research was carried out while the second author was staying at the Institut Mittag-Leffler participating in the 2017 research programme Fractal Geometry and Dynamics. He would like the staff as well as the participants and organizers for the stimulating atmosphere and support. \[10\]

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1Department of Physics, Princeton University, Princeton, New Jersey 08544, USA mklatt@princeton.edu

2Karlsruhe Institute of Technology, Department of Mathematics, 76128 Karlsruhe, Germany, steffen.winter@kit.edu