EXACT, $E = 0$, SOLUTIONS FOR GENERAL POWER-LAW POTENTIALS.
II. QUANTUM WAVE FUNCTIONS

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ABSTRACT

For zero energy, $E = 0$, we derive exact, quantum solutions for all power-law potentials, $V(r) = -\gamma/r^\nu$, with $\gamma > 0$ and $-\infty < \nu < \infty$. The solutions are, in general, Bessel functions of powers of $r$. For $\nu > 2$ and $l \geq 1$ the solutions are normalizable; they correspond to states which are bound by the angular-momentum barrier. Surprisingly, the solutions for $\nu < -2$ are also normalizable, they are discrete states but do not correspond to bound states. For $2 > \nu \geq -2$ the states are unnormalizable continuum states. The $\nu = 2$ solutions are also unnormalizable, but are exceptional solutions. Finally, we find that by increasing the dimension of the Schrödinger equation beyond 4 an effective centrifugal barrier is created, due solely to the extra dimensions, which is enough to cause binding. Thus, if $D > 4$, there are $E = 0$ bound states for $\nu > 2$ even for $l = 0$. We discuss the physics of the above solutions are compare them to the classical solutions of the preceding paper.

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1 Introduction

In paper I [1], we gave the general \( E = 0 \), classical-orbit solutions for all classical potentials of the form

\[
V(r) = -\frac{\gamma}{r^\nu} = -\frac{\gamma}{r^{2\mu+2}}, \quad \gamma > 0, \quad -\infty < \nu, \mu < \infty,
\]

where, as in the classical case, we will use for convenience two parameters \( \nu \) and \( \mu \), which are related by

\[
\mu \equiv \frac{\nu - 2}{2}, \quad \nu = 2(\mu + 1).
\]

In this paper we will obtain the \( E = 0 \) quantum solutions for the power-law potentials (1). We will find some remarkable similarities between the classical and quantum solutions, and demonstrate unusual quantum effects.

In Section 2 we derive the general solutions of the Schrödinger equation for the potentials (1), for all \( \nu \) (except \( \nu = 2 \) or \( \mu = 0 \), which is discussed separately in Section 5.2). In Section 3 we find out which of these solutions are normalizable. Then, in Section 4, we show that these normalizable states belong to two classes, which have a different physical nature: For \( \nu > 2 \) or \( \mu > 0 \), the normalizable solutions are bound states if the angular-momenta satisfy \( l > 0 \), just as the classical orbits were bound if the angular momentum was not zero. In contrast, the normalizable states for \( \nu < -2 \) or \( \mu < -2 \), cannot be interpreted as bound states, and has an unusual interpretation.

For \( -2 \leq \nu \leq 2 \) or \( -2 \leq \mu \leq 0 \), the solutions are not normalizable and hence correspond to free (unbounded) wave functions. These are discussed in Section 5. We discuss the solutions for special cases of \( \nu \) in Section 6. As a final result, in Section 7 we demonstrate a quantum mechanical effect which is due to the dimension of the Schrödinger equation.

In Section 8 we discuss the correspondence between the classical solutions of paper I with the quantum solutions. This returns us to the starting point of paper I, the “Folk Theorem” about the similar solvability or lack thereof of equivalent classical and quantum systems. We close, in Section 9, with a short summary. (Two appendices deal with the choice of physical solutions and with the \( \gamma < 0 \) problem.)

2 General Quantum-Mechanical Solutions for All
\( \nu \neq 2 \) or \( \mu \neq 0 \)

The radial Schrödinger equation with angular-momentum quantum-number, \( l \), is

\[
ER_t = \left[-\frac{\hbar^2}{2m} \left( \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{l(l+1)}{r^2} \right) + V(r)\right] R_t.
\]
It is useful to express the Schrödinger equation in terms of a dimensionless radius

\[ \rho \equiv r/a , \] (4)

where \( a \) is a convenient quantity with the dimension of length. Throughout this paper we also use a quantity with the dimension of energy, given by

\[ \mathcal{E}_0 \equiv \frac{\hbar^2}{2ma^2} , \] (5)

Dividing Eq. (3) by \( \mathcal{E}_0 \) and using \( \rho \) yields

\[ \frac{E}{\mathcal{E}_0} R_l(\rho) = \left[ -\left( \frac{d^2}{d\rho^2} + \frac{2}{\rho} \frac{d}{d\rho} - \frac{l(l+1)}{\rho^2} \right) + \frac{V(r)}{\mathcal{E}_0} \right] R_l(\rho) . \] (6)

If we express the potentials (1) in terms of \( \rho \), we have

\[ V(r) = -\frac{\gamma}{r^\nu} \equiv -\mathcal{E}_0 \frac{g^2}{\rho^\nu} , \] (7)

where \( g^2 \) manifestly is a dimensionless coupling constant.

Finally, multiplying Eq. (3) by \(-\rho^2\), setting \( E = 0 \), and substituting the power-law potentials of Eq. (7) into Eq. (3), we obtain

\[ 0 = \left[ \rho^2 \frac{d^2}{d\rho^2} + 2\rho \frac{d}{d\rho} - l(l+1) + \frac{g^2}{\rho^{2\nu}} \right] R_l . \] (8)

Eq. (8) is a form of the differential equation for a product of a power times a Bessel function of another power. It can be written in the form [2]

\[ 0 = \left[ z^2 \frac{d^2}{dz^2} + (1 - 2\alpha)z \frac{d}{dz} + (\beta \eta z^\eta)^2 + \left( \alpha^2 - \sigma^2 \eta^2 \right)^2 \right] w(z) , \] (9)

with the general solution being [3]:

\[ w = A \ z^\alpha J_\sigma(\beta z^\eta) + B \ z^\alpha Y_\sigma(\beta z^\eta) , \] (10)

where \( A \) and \( B \) are arbitrary constants. By considering the asymptotic behavior of the two independent solutions, \( J_\sigma(z) \) and \( Y_\sigma(z) \), we are led to choose the \( J_\sigma(z) \) as the physical solutions. This choice is explained in Appendix A. Therefore, we have

\[ R_l(\rho) = \frac{1}{\rho^{1/2}} \int \left( \frac{g}{|\mu| \rho^\mu} \right) = \frac{1}{\rho^{1/2}} \int \left( \frac{2g}{|\nu - 2| \mu^2} \right) , \ \mu \neq 0 . \] (11)
Note, in particular, that the \( \mu \) that labels the power of \( \rho \) in the argument of \( J \) does not have an absolute value.

It is interesting to observe that for special indices, \( \mu \), the radial wave functions \( \Pi \) become proportional to spherical Bessel functions

\[
j_n(z) = \sqrt{\frac{\pi}{2z}} J_{n+1/2}(z) = -z^n \left( -\frac{1}{z} \frac{d}{dz} \right)^n \frac{\sin z}{z}.
\]

This happens for a specific \( l \) if \( |\mu| \) can be written as a ratio of the form

\[
|\mu| = \frac{l + 1/2}{n + 1/2} = \frac{2l + 1}{2n + 1}.
\]

In particular, for the specific index, \( \mu = 1 \), the condition (13) is always true for all \( l \). (The example discussed in Section 4.2 corresponds to \( \mu = -3, l = 1 \) and \( n = 0 \).)

### 3 Normalizable States

The normalization constants for these wave functions, if they exist, would be equal to

\[
N^{-2}_l = \int_0^\infty \frac{r^2 dr}{\rho} \frac{\rho^2}{|\mu| |\rho|^\mu} \left( \frac{g}{|\mu| \rho^\mu} \right), \quad \mu \neq 0.
\]

Changing variables from \( r \) to \( \rho \) and then from \( \rho \) to \( t = \rho(|\mu|/g)^{1/\mu} \), one obtains

\[
N^{-2}_l = a^3 \left( \frac{g}{\mu} \right)^{2/\mu} \left[ -\frac{1}{\mu} \int_0^\infty t \, dt \, J^2_{(l+1/2)\left( |\mu| / \rho^\mu \right)} \left( \frac{1}{t^\mu} \right) \right].
\]

We now want to change variables to \( z = t^{-\mu} \). In doing so, we must be careful when converting the limits of integration. For \( \mu > 0 \), the change inverts the limits of integration, introducing another minus sign. So, the integral’s prefactor \(-1/\mu \rightarrow +1/\mu = 1/|\mu|\). Contrariwise, for \( \mu < 0 \) the limits stay the same. But in this case, we have, \(-1/\mu = 1/|\mu|\). Therefore, in all cases we have

\[
N^{-2}_l = a^3 \left( \frac{g}{|\mu|} \right)^{2/\mu} \frac{I_l}{|\mu|},
\]
where

\[ I_l = \int_0^{\infty} \frac{dz}{z^{(1+2/\mu)}} \left( \frac{J_2^2}{l+1/2} \right)(z) . \]  

(17)

The integral (17) is a very special case of the difficult Weber-Schafheitlin discontinuous integral, whose integrand involves a product of two Bessel functions of different orders and different arguments. The general integral is not continuous as a function of the arguments of the Bessel functions as they approach each other. Details may be found in Watson’s book [4]. The end result is of the form [5, 6, 7, 8]

\[ I_l(\sigma, \lambda) = \int_0^{\infty} \frac{dz}{z^{\lambda}} J_\sigma^2(z) = \frac{\Gamma(\lambda)}{2\lambda \Gamma^2 \left( \frac{1+\lambda}{2} \right)} \frac{\Gamma \left( \frac{1-\lambda+2\sigma}{2} \right)}{\Gamma \left( \frac{1+\lambda+2\sigma}{2} \right)} , \]

(18)

valid provided that the following two conditions are satisfied:

\[ 1 + 2 \text{Re}(\sigma) > \text{Re}(\lambda) > 0 . \]

(19)

[The last expression in Eq. (18) is obtained [9] by using the duplication formula

\[ \Gamma(2x) = \Gamma(x) \Gamma(x + 1/2) 2^{2x-1} \pi^{-1/2} \]

(20)

to factorize \( \Gamma(\lambda)\).]

Applying Eq. (18) to the integral (17) we obtain

\[ I_l = \frac{\Gamma \left( \frac{1}{2} + \frac{1}{\mu} \right) \Gamma \left( \frac{l+1/2}{|\mu|} - \frac{1}{\mu} \right)}{2\pi^{1/2} \Gamma \left( 1 + \frac{1}{\mu} \right) \Gamma \left( 1 + \frac{l+1/2}{|\mu|} + \frac{1}{\mu} \right)} , \]

(21)

which is finite, if

\[ 1 + \frac{2l + 1}{|\mu|} > 1 + \frac{2}{\mu} > 0 . \]

(22)

4 Classes of Normalizable States

4.1 Bound states at the scattering threshold: \( \nu > 2 \) or \( \mu > 0 \) with \( l \geq 1 \)

The conditions (22) are satisfied for \( \mu > 0 \) or \( \nu > 2 \) and \( l > 1/2 \). These are normal bound states, and exactly correspond to the conditions for classically bound orbits.
described in paper I. In the classical problem, there were $E = 0$ bound states for any non-zero angular momentum with $\nu > 2$. In quantum mechanics, the smallest non-zero angular momentum allowed is $l = 1 > 1/2$.

This physics is not what one normally expects in quantum mechanics. Usually one thinks that the $E = 0$ state at the edge of the continuum is free. That intuition comes from thinking of the Coulomb system, where the potential asymptotes to zero from below at $r \to \infty$. (See Fig. 10 of Paper I.)

Here, the potential asymptotes to zero from above, as in Figure 1 of Paper I. Thus, for $E = 0$ the wave function which is concentrated at the origin must tunnel to infinity to reach another free region. That takes forever, and so the state is bound.

### 4.2 Normalizable yet “unbound” states: $\nu < -2$ or $\mu < 0$

We now come to an unusual set of normalizable states. The conditions (22) are satisfied for all $\nu < -2$ or $\mu < -2$, for any $l$. (Technically, for all $l > -1/2$.) Since in this case the potentials are strongly repulsive, this result at first seems counter-intuitive.

Note that for $\nu < 2$ the power potentials (1) are more repulsive than the inverse harmonic oscillator. The unusual spectral properties of such potentials are known to the mathematical physics community. What we have found here is an explicit solution for $E = 0$ and all potentials (1) with $\nu < -2$.

It can be shown, using asymptotic arguments, that normalizable solutions also exact for the continuum spectrum $E \neq 0$. However, by imposing proper boundary conditions, a discrete subset can be chosen, which makes the Hamiltonian self adjoint.[11, 12, 14, 15].

Physically, one can see a classical analogue for this unusual behavior. Recall the result given at the end of Sec. 4.4 of paper I. We noted that for $\nu < -2$ or $\mu < -2$ it took a finite time for the particle to go from its turning point, $a$, to infinity. Thus, although these states are “unbound,” the proportion of time they spend outside a sphere of finite radius $R$ can be made as small as desired.

Classically, when proper boundary conditions are imposed, the solutions are periodic in time [10] in the sense that the particle goes back and forth between infinity and the turning point, $a$, with definite time intervals. Such classical orbits correspond to the normalizable, discrete quantum solutions.

It can be useful to the reader to look at a simple, special case. If one considers $\mu = -3$ and $l = 1$, the solution is a spherical Bessel function with one term, and hence easily normalized.

### 5 Unnormalizable Solutions

#### 5.1 Normal bound states: $\nu > 2$ or $\mu > 0$ with $l \geq 1$

The unnormalizable solutions for the potentials (1) fall into three categories. The first two are normal, unbound states.

i) The S-wave ($l = 0$) solutions of the attractive potentials, with $\nu > 2$ or $\mu > 0$.  

6
Here, there is no forbidden tunneling region to prevent the wave function from traveling to infinity.

ii) The solutions of the potentials, with $-2 \leq \nu < 2$ or $-2 \leq \mu < 0$ and all $l$.

In this case, the corresponding effective potentials, $U(\rho) = E_0(l(l+1))/\rho^2 - g^2/\rho^2$, further split into two types. The potentials are Kepler like, for $0 < \nu < 2$, in that they have a minimum for $l \neq 0$ at a finite $\rho$. Otherwise, for $-2 < \nu \leq 0$, the potentials are “weakly” repulsive, meaning the classical particle takes an infinite time to reach infinity.

5.2 The exceptional “free” solution: $\nu = 2$ or $\mu = 0$

iii) The exceptional power $\nu = 2$.

Just as in the classical theory, the $\nu = 2$ or $\mu = 0$ case is the one exceptional case which does not fit into the general solution, which is given here in Eq. (11). This is the quantum-mechanical analogue of having an inverse-square potential $V(r) = -\gamma/r^2$, that cannot be differentiated from the angular-momentum barrier.

The value of $g^2$ relative to $l(l+1)$ leads to three kinds of effective potentials, $U(\rho)$: purely positive, identically zero, and purely negative. In quantum mechanics there are zero-energy solutions for each of the above 3 cases. (Contrariwise, since the kinetic energy is classically nonnegative, there are nontrivial $E = 0$ solutions only for negative effective potentials, $U < 0$.)

The Schrödinger equation,

$$0 = \left[ \frac{d^2}{d\rho^2} + \frac{2}{\rho} \frac{d}{d\rho} + \frac{g^2 - l(l+1)}{\rho^2} \right] R_l,$$

is homogeneous in $d/d\rho$ and $1/\rho$. Therefore, its solutions are expected to be powers of $\rho$:

$$\left[ \rho^2 \frac{d^2}{d\rho^2} + 2\rho \frac{d}{d\rho} + g^2 - l(l+1) \right] \rho^\alpha = [\alpha^2 + \alpha + g^2 - l(l+1)]\rho^\alpha = 0.$$

The two roots of the above quadratic equation are

$$\alpha_{\pm} = -\frac{1}{2} \pm \sqrt{(l+1/2)^2 - g^2} = -\frac{1}{2} \pm \sqrt{\Delta},$$

where

$$\Delta = (l+1/2)^2 - g^2.$$
Depending on the value of $\Delta$, we obtain three types of solutions of Eq. (23):

$$R_l = \begin{cases} 
\rho^{-1/2}[A \rho^{\sqrt{\Delta}} + B \rho^{-\sqrt{\Delta}}] & l + 1/2 > g \\
\rho^{-1/2}[A + B \ln \rho] & l + 1/2 = g \\
\rho^{-1/2}[A \cos(|\Delta|^{1/2} \ln \rho) + B \sin(|\Delta|^{1/2} \ln \rho)] & l + 1/2 < g
\end{cases} \quad (27)$$

where $A$ and $B$ are arbitrary constants.

The solutions in the first line of Eq. (27) hold when $\Delta > 0$. They correspond to powers, whose orders are the two real square roots of $\Delta$, which have opposite signs. These square roots approach each other as $\Delta \to 0$. Then, as $\sqrt{\Delta} \to +0$ and $\sqrt{\Delta} \to -0$, the solutions become the constant and logarithmic terms, respectively, given on the second line of Eq. (27).

Finally, for $\Delta < 0$, the square roots become pure imaginary, and thus leads to trigonometric behavior.

Note that Eq. (23) for $g = 0$ corresponds to a free particle with zero energy. For this case, the solutions with $l + 1/2 > g = 0$ in (27) are relevant. They then yield

$$R_l = A r^l + B r^{-(l+1)} \quad (28)$$

as expected.

## 6 Special Quantum-Mechanical Cases

### 6.1 $\nu = 4$ or $\mu = 1$

The potential $V(r) = -\gamma/r^4$ classically leads to the bound orbit which is a circle. The quantum solution Eq. (11) simplifies to

$$R_l = \rho r^l J_l(g \rho) \quad (29)$$

where $j_n$ is the spherical Bessel function defined in Eq. (12). It corresponds to a bound states, for $l \geq 1$.

### 6.2 Coulomb potential: $\nu = 1$ or $\mu = -1/2$

This is the zero-energy solution for the Coulomb problem. From Eq. (11) above, we have

$$R_l = \rho^{-1/2} J_{2l+1}(2g \rho^{1/2}) \quad (30)$$

These are continuum unnormalizable states.
6.3 Constant potential: \( \nu = 0 \) or \( \mu = -1 \)

This corresponds to a constant negative potential of depth \( g^2 \) below the angular-momentum barrier. So, its solution is similar to that of a free particle with angular-momentum \( l \) and positive total energy \( g^2 E_0 = (\hbar k)^2 / 2m \), where \( k = p / \hbar = g / a \) is the effective wave number. The solution is a spherical Bessel function,

\[
R_l = j_l(g \rho) = j_l(kr) , \quad \text{where} \quad k = g / a .
\] (31)

6.4 Inverted harmonic-oscillator potential: \( \nu = \mu = -2 \)

Our final special case is that of a repulsive (negative) harmonic-oscillator potential, \( V = -\gamma r^2 \). The solution is

\[
R_l = \rho^{-1/2} J_{(l/2+1/4)}(g \rho^2 / 2) .
\] (32)

7 Bound States in Arbitrary Dimensions

One can easily generalize the problem of the last section to arbitrary \( D \) space dimensions. Doing so yields another surprising physical result.

To obtain the \( D \)-dimensional analogue of Eq.(8), one simply has to replace \( 2 \rho \) by \( (D - 1) \rho \) and \( l(l + 1) \) by \( l(l + D - 2) \) \[20\]:

\[
0 = \left[ \rho^2 \frac{d^2}{d \rho^2} + (D - 1) \rho \frac{d}{d \rho} - l(l + D - 2) + \frac{g^2}{\rho^2 \mu} \right] R_{l,D} .
\] (33)

This equation is again a generalized Bessel equation. Comparing (33) with (9), yields the following physical solutions:

\[
R_{l,D} = \frac{1}{\rho^{D/2-1}} J_{s(D/2-1)} \left( \frac{g}{|\mu| \rho^2} \right)
\] (34)

\[
= \frac{1}{\rho^{D/2-1}} J_{2s(D/2-1)} \left( \frac{2g}{|\nu - 2| \rho^{(\nu - 2)/2}} \right) .
\]

To find out which states are normalizable one first has to change the integration measure from \( r^2 dr \) to \( r^{D-1} dr \) and again proceed as before. The end result is that if the wave functions are normalizable, the normalization constant is given by

\[
N_{l,D}^{-2} = \frac{\alpha^D}{|\mu|} \left( \frac{g}{|\mu|} \right)^{2/\mu} I_{l,D} ,
\] (35)
where

\[ I_{l,D} = \int_{0}^{\infty} \frac{dz}{z^{(1+2/\mu)}} J^2_{(l+D/2-1)\text{Re}(z)}(z) . \]  \hspace{1cm} (36)

We see that the above integral is exactly equal to that in Eq. (17), except that \( l \) is replaced by the effective quantum number

\[ l_{\text{eff}} = l + \frac{D-3}{2} . \]  \hspace{1cm} (37)

Therefore,

\[ I_{l,D} = \frac{1}{2\pi^{1/2}} \frac{\Gamma\left(\frac{1}{2} + \frac{1}{\mu}\right)}{\Gamma\left(1 + \frac{1}{\mu}\right)} \frac{\Gamma\left(\frac{l+D/2-1}{|\mu|} - \frac{1}{\mu}\right)}{\Gamma\left(\frac{l+D/2-1}{|\mu|} + \frac{1}{\mu}\right)} , \]  \hspace{1cm} (38)

which is defined and convergent for

\[ \frac{2l + D - 2}{|\mu|} + 1 > \frac{2}{\mu} + 1 > 0 . \]  \hspace{1cm} (39)

This yields the surprising result that there are bound states for all \( \nu > 2 \) or \( \mu > 0 \) when \( l > 2 - D/2 \). Explicitly this means that the minimum allowed \( l \) for there to be zero-energy bound states are:

\[
D = 2 , \quad l_{\text{min}} = 2 , \]
\[
D = 3 , \quad l_{\text{min}} = 1 , \]
\[
D = 4 , \quad l_{\text{min}} = 1 , \]
\[
D > 4 , \quad l_{\text{min}} = 0 . \]  \hspace{1cm} (40)

This effect of dimensions is purely quantum mechanical and can be understood as follows: Classically, the number of dimensions involved in a central potential problem has no intrinsic effect on the dynamics. The orbit remains in two dimensions, and the problem is decided by the form of the effective potential, \( U \), which contains only the angular momentum barrier and the dynamical potential.

In quantum mechanics there are actually two places where an effect of dimension appears. The first is in the factor \( l(l+D-2) \) of the angular-momentum barrier. The second is more fundamental. It is due to the operator

\[
U_{\text{qm}} = -\frac{(D-1)}{\rho} \frac{d}{d\rho} . \]  \hspace{1cm} (41)
The contribution of $U_{qm}$ to the “effective potential” can be calculated by using the ansatz

$$R_{l,D}(\rho) \equiv \frac{1}{\rho^{(D-1)/2}} \chi_{l,D}(\rho) .$$

This transforms the $D-$dimensional radial Schrödinger equation into a 1$-$dimensional Schrödinger equation in the $\rho$ variable:

$$0 = \left[-\frac{d^2}{d\rho^2} + U_{l,D}(\rho)\right] \chi_{l,D} .$$

In Eq. (43), the effective potential $U_{l,D}(\rho)$ is given by

$$U_{l,D}(\rho) = \frac{(D-1)(D-3)}{4\rho^2} + \frac{l(l+D-2)}{\rho^2} + V(\rho)$$

$$= \frac{l_{\text{eff}}(l_{\text{eff}} + 1)}{\rho^2} + V(\rho) ,$$

with $l_{\text{eff}}$ given in Eq. (37). Since the Schrödinger equation (43) depends only on the combination $l_{\text{eff}}$, the solution $\chi_{l,D}(\rho)$ does not depend on $l$ and $D$ separately. This explains, in particular, the values of $l_{\text{min}}$ given in Eq. (40).

Although the above ansatz is well known, the dimensional effect has apparently not been adequately appreciated. One reason may be attributed to the fact that in going from $D = 3$ to $D = 1$, $l_{\text{eff}}$ remains equal to $l$.

However, in our problem this effect leads to a physical result, which is so counter-intuitive, that it cannot be overlooked.

The dimensional effect essentially produces an additional centrifugal barrier which can bind the wave function at the threshold, even though the expectation value of the angular momentum vanishes. Note that this is in distinction to the classical problem, where there would be no “effective” centrifugal barrier to prevent the particle from approaching $r \to \infty$.

### 8  Classical vs. Quantum: The “Folk Theorem”

We found here, exactly as in classical physics [1], that except for the $\nu = 2$ case, the $E = 0$ radial solutions of the Schrödinger equation (8) are given in terms of a single function, presented in Eq. (11). Although the functional form of the solutions is the same for all $\nu \neq 2$, the properties of these solutions change drastically as one passes over three regions of the index, $\nu$.

(1) For $\nu > 2$, the radial wave functions are normalizable if $l > 1/2 > 0$. These are bound states, analogous to the classical bound orbits. The intuitive explanation
for the existence of these bound states is that the tunneling distance is infinite. The forbidden region itself is due to the repulsive angular-momentum barrier. (For \( l = 0 \) there is no such infinite barrier, and so the solutions are free, as are the \(-2 \leq \nu < 2\) cases described below.)

Observe that here, as in the classical case, the virial theorem is violated. Neither \( \langle V \rangle \) nor \( \langle -\nabla^2 \rangle \) is finite.

(2) For \(-2 \leq \nu \leq 2\) and all \( l > 0 \) the effective potential, \( U(r) \), is repulsive near the origin, because in this region the centripetal potential dominates. This classically insures that the particle does not approach the origin beyond a certain minimal distance, \( r = a \). Further, \( U(\rho) \) for \( \rho \to \infty \) approaches zero from below. This means that the wave functions are unnormalizable and are part of the continuum.

(3) For \( \nu < -2 \), the unusual quantum solutions mimic the unusual classical solutions. Classically, the travel time to infinity is finite! This raises an interesting question: What happens classically to the particle after it reaches infinity?

The classical answer is that the solutions are periodic in time. Quantum mechanically this corresponds to imposing boundary conditions on the wave functions. These boundary conditions imposed on the normalizable solutions yield a discrete spectrum instead of a continuous one. In this way one obtains a proper self adjoint extension of the Hamiltonian, and the unitarity of the transition operator \( U(t) = \exp[-itH] \) is thus assured [11, 12, 13, 14]. The special potential with \( \nu = 2 \) has exceptional solutions, both in classical physics and also in quantum physics.

Therefore, we see that the mathematical similarities between all the classical and quantum-mechanical solutions are intriguing. So, too, are the physical meanings of the solutions. How true, one must ask, is the “Folk Theorem?”

9 Summary

The reason the \( E = 0 \) solutions could be solved analytically was because there is one less \( \rho \)-power-term in the differential equations of Newton and Schrödinger which must be dealt with. This enabled us to obtain exact results in both cases. We could then study the properties of the solutions explicitly and make concrete comparisons between the classical and the quantum cases. Sometimes the classical solutions were intuitively helpful in understanding the quantum solutions, and sometimes the opposite was true. In any event, the interest in the “Folk Theorem” was amply rewarding.

Along the way there were unusual mathematical problems, due to the singular behavior of the potentials [1], which stood in the way of physical understanding. For example, instead of the standard requirement that \( R(0) \) be finite at the origin, we only demanded the physically reasonable condition, that the probability of finding the particle near the origin should be finite. In this way we could accept, as physical solutions, functions which are not analytic at \( \rho = 0 \). (See appendix A.3)

The study of the quantum solutions led us to demonstrate three interesting effects.

(1) There exist permanent bound states at the threshold, for all \( l > 0 \) and all \( \gamma > 0 \). That is, there are \( E = 0 \) states which persist if we change the coupling constant \( \gamma \) by a positive factor.

In contrast, \( E = 0 \) bound states which exist in the literature [16, 17, 18, 19] occur only “accidentally,” i.e., for special values of the coupling constants. These
states occur when the bound-state pole in the partial-wave amplitude is just crossing the scattering threshold to reach the second sheet. Thus, they become a decaying resonance. Such an accidental crossing of the threshold does not occur for all \( l > 0 \) simultaneously. A review of these accidental bound states is given in [10].

2) There exist normalizable solutions for highly repulsive potentials (\( \nu < -2 \)).

3) For higher-space dimensions, each additional dimension adds a half unit to the effective angular-momentum quantum number, \( l_{\text{eff}} \), of Eq. (37). An effective centripetal barrier, solely due to this dimensional effect, i.e., even for \( L^2 = 0 \), is capable of producing a bound state when \( D > 4 \). This result is a remarkable manifestation of quantum mechanics [21] and has no classical counterpart.
Appendix A: Asymptotics and Choice of Physical Solutions

A.1: Asymptotics of Bessel functions

The physical solutions of the radial Schrödinger equation must be linear combinations of the solutions of the generalized Bessel equation \((9)\). To determine these combinations, we first consider the behavior of the wave functions at the origin and for large radii, \(\rho\).

Recall the asymptotic behavior of the Bessel functions:

\[
J_\sigma(z) \sim z^\sigma, \quad Y_\sigma(z) \sim z^{-\sigma}, \quad z \to 0 ,
\]

and

\[
J_\sigma(z) \sim \frac{1}{\sqrt{z}} \cos(z - c_\sigma), \quad Y_\sigma(z) \sim \frac{1}{\sqrt{z}} \cos(z - \tilde{c}_\sigma), \quad z \to \infty ,
\]

where the \(c_\sigma\) and \(\tilde{c}_\sigma\) are constant phases, which depend on the index \(\sigma\). The argument of the Bessel functions in the solutions \((11)\) depend on \(\rho\) as

\[
z = \frac{g}{|\mu|\rho^\mu} , \quad \mu \neq 0 .
\]

Therefore, the \(z \to 0\) limit of Eq. \((45)\) gives

\[
R_l(\rho) = \frac{1}{\sqrt{\rho}} \int \frac{(g/|\mu|\rho^\mu)}{||\mu|^{|\mu|}} \sim \frac{1}{\sqrt{\rho}} (\rho^{-n})^{|\mu|/|\mu|} = \frac{1}{\sqrt{\rho}} \rho^{-(l+1/2)\text{sgn} \mu}
\]

\[
= \left\{ \begin{array}{ll}
\rho^{-l(l+1)} & \to 0 \quad \text{for} \quad \rho \to \infty \quad \text{and} \quad \mu > 0 , \\
\rho^l & \to 0 \quad \text{for} \quad \rho \to 0 \quad \text{and} \quad \mu < 0 ,
\end{array} \right.
\]

Similarly, the \(z \to \infty\) limit of Eq. \((46)\) yields the following upper bounds:

\[
|R_l(\rho)| = \left| \frac{1}{\sqrt{\rho}} \int \frac{(g/|\mu|\rho^\mu)}{|\mu|^{|\mu|}} \right| \sim \left| \rho^{\mu-1} \cos \left( \frac{g}{|\mu|\rho^\mu} - \text{const.} \right) \right| \leq \rho^{\mu-1}
\]

\[
= \left\{ \begin{array}{ll}
\rho^{\mu(|\mu|)/2} & \to 0 \quad \text{for} \quad \rho \to 0 \quad \text{and} \quad \mu > 1 , \\
\rho^{-\mu(|\mu|)/2} & \to \infty \quad \text{for} \quad \rho \to 0 \quad \text{and} \quad 1 > \mu > 0 , \\
\rho^{-\mu(|\mu|+1)/2} & \to 0 \quad \text{for} \quad \rho \to \infty \quad \text{and} \quad \mu < 0 .
\end{array} \right.
\]
A.2: The physical solutions for $\mu < 0$

From Eq. (49) it follows that the solutions (11) for $\mu < 0$ behave as $R_l(r) \sim \rho^l$, for $\rho \to 0$. Therefore, the $R_l(0)$ are finite for all $l \geq 0$, as one usually requires for physical solutions. In contrast, the $Y_\sigma$ solutions would lead to $R_l(r) \sim \rho^{-l-1}$. This shows that the choice $R_l(\rho) \sim J_\sigma(z)$ gives the correct physical solutions.

For $\rho \to \infty$ the above solutions behave as

$$R_l(\rho) \sim \rho^{\frac{\mu-1}{2}} \cos\left(\frac{1}{\rho^\mu} - \text{const.}\right).$$

Thus, the probability integral in the asymptotic region can be estimated as follows:

$$\int_0^\infty |R_l(\rho)|^2 \rho^2 d\rho \simeq \int_0^\infty \rho^{\mu+1} \cos^2\left(\frac{g}{|\mu| \rho^\mu} - \text{const.}\right) d\rho \leq \int_0^\infty \rho^{\mu+1} d\rho$$

$$= -\frac{1}{\mu + 2} R^{\mu+2}, \quad \mu < -2.$$  

This result can also be understood as follows: In order for the probability integral to converge at infinity, the integrand must fall off stronger than $1/\rho$, which is the limiting behavior that gives a logarithmic divergence. Hence, for $\mu + 1 < -1$ or $\mu < -2$, we obtain convergence at infinity.

Since, our solutions are finite at the origin, we conclude that our solutions (11) are normalizable for $\mu < -2$ for all $l \geq 0$. These are the surprising discrete, yet unbound, states.

A.3: Physical solutions for $\mu > 0$

The choice of physical solutions is much more complicated for $\mu > 0$. In this case, to obtain the $\rho \to 0$ limit we must use the $z \to \infty$ limit in Eq. (50). The solution (11) behaves as

$$R_l(\rho) \sim \rho^{(\mu-1)/2} \cos\left(g/(|\mu| \rho^\mu) - \text{const.}\right), \quad \rho \to 0.$$  

$R_l(\rho)$ is a rapidly oscillating function of $\rho$, which is bounded by $\rho^{(\mu-1)/2}$. This bound stays finite for $\mu \geq 1$, but goes to infinity as $\rho \to 0$ for $1 > \mu > 0$. In both cases, $R_l(\rho)$ is not an analytic function of $\rho$ at the origin. This is not a customary behavior at the origin. However, it cannot be avoided since both solutions, $\psi \propto J_\sigma(z)$ and $\psi \propto Y_\sigma(z)$, $\sigma \equiv (l + 1/2)/|\mu|$, have the same type of asymptotic behavior at infinite argument $z \equiv g/(|\mu| \rho^\mu)$.

However, recalling that our potentials for $\mu > 0$ are unusually singular, we can relax the condition that $R_l(0)$ be finite. Instead, we require that the probability of finding the particle in any finite neighborhood of the origin should be finite:

$$\int_0^R |R_l(\rho)|^2 \rho^2 d\rho = \int_0^R \rho^{\mu+1} \cos^2\left(\frac{g}{|\mu| \rho^\mu} - \text{const.}\right) d\rho \leq \int_0^R \rho^{\mu+1} d\rho$$
This shows that the probability is finite at the origin for all positive $\mu$, including $1 > \mu > 0$.

But the $\rho \to 0$ behavior for $\mu > 0$ still does not enable us to choose between the $J_\sigma$ and the $Y_\sigma$ solutions. Both choices, or any linear combinations of them, yield physically acceptable solutions at the origin. However, the $\rho \to \infty$ limit can settle the matter. In this limit, we have

$$R_l \sim J_\sigma(z) / \sqrt{\rho} \sim \rho^{-(l+1)} , \quad \rho \to \infty ,$$

whereas

$$R_l \sim Y_\sigma(z) / \sqrt{\rho} \sim \rho^l , \quad \rho \to \infty .$$

The asymptotic behavior (56) shows that our choice (11) leads to decaying solutions for $\rho \to \infty$. In fact, this choice for $\mu > 0$ insures that the whole probability integrated up to $\rho \to \infty$ remains finite, and thus these solutions correspond to bound states.

### Appendix B: A Note on the Power-Law Potentials with $\gamma < 0$

Throughout the present paper, and in paper I [1], we have discussed the power-law potentials only for $\gamma > 0$. For completeness, we add a few comments on the $\gamma < 0$ case. There the power-law potentials are everywhere positive, $V(r) > 0$. Therefore, they do not have any classical solutions with $E = 0$, since then the kinetic energy would have to be negative.

However, quantum mechanically the $E = 0$ situation is a little more interesting. From Eq. (11), our solutions would now be of the form

$$\hat{R}_l(\rho) = \frac{1}{\rho^{l/2}} J^{(i l + 1/2)}(|\mu| \rho^{1/2}) \sim \frac{1}{\rho^{l/2}} I^{(i l + 1/2)}(|\mu| \rho^{1/2}) \left( \frac{g}{|\mu| \rho^l} \right) , \quad \mu \neq 0 ,$$

where $g \equiv |\sqrt{\gamma}|$ and the $I$’s are the modified Bessel functions [22]. The general solutions would also involve the modified Bessel functions of the second kind, the $K$’s, which can be written as linear combinations of the $I$’s.

We now can distinguish between two index regions:

1. For $\nu > 0$ or $\mu > -1$, the negative-$\gamma$ potentials are positive definite and repulsive from the origin. Hence, we expect scattering solutions.

2. For $\nu < 0$ or $\mu < -1$, the negative-$\gamma$ potentials are confining potentials, which can only have discrete bound states. The corresponding energies must all be positive, similar to the energy levels of the spherical harmonic oscillator. Therefore, the solution of Eq. (58) is not physical.

In this connection, and although not directly related to the nonexisting $E = 0$ physical states, we find it interesting to note that Grosche and Steiner [23] were able to calculate the $E = 0$ propagator for these $\nu < 0$, negative-$\gamma$ potentials, by using path integrals. They used this propagator to obtain interesting sum rules.
References

[1] J. Daboul and M. M. Nieto, preceding paper.

[2] W. Magnus, F. Oberhettinger, and R. P. Soni, Formulas and Theorems for the Special Functions of Mathematical Physics, 3rd Edition (Springer-Verlag, New York, 1966).

[3] P. 77 of [2].

[4] G. N. Watson, A Treatise on the Theory of Bessel Functions, 2nd Ed. (Cambridge University Press, Cambridge, 1966), Secs. 13.41-13.43.

[5] P. 407, Eq. (1) of Ref. [1].

[6] C. J. Tranter, Bessel Functions With Some Physical Applications (Hart Publishing Co., New York, 1968), § 5.6.

[7] The third integral on page 99 of Ref. [2].

[8] I. S. Gradshteyn and I. M Ryzhik, Table of Integrals, Series, and Products (Academic Press, New York, 1965), p. 693, integral 6.576.2.

[9] P. 3 of Ref. [2].

[10] J. Daboul and M. M. Nieto, Phys. Lett. A 190, 357-362 (1994).

[11] C. Zhu and J. R. Klauder, Am. J. Phys. 61, 605-611 (1993); Found. Phys. 23, 617-631 (1993).

[12] F. A. Berezin and M. A. Shubin, The Schrödinger Equation (Kluwer, Dordrecht, 1991).

[13] M. J. Bertrand, Comptes Rendus Acad. Sci. (Paris) 77, 849-853 (1875).

[14] A. S. Wightman, in: 1964 Cargèse Lectures in Theoretical Physics, Vol. 2, ed. by M. Lévy (Gordon and Breach, New York, 1967), pp. 171-291. See Sec. VIII, p. 262.

[15] K. M. Case, Phys. Rev. 80, 797-806 (1950).

[16] B. W. Downs, Am. J. Phys. 30, 248-255 (1962).

[17] See the discussions around pages 127 and 279 in: L. I. Schiff, Quantum Mechanics, 3rd Edition (McGraw-Hill, New York, 1968)

[18] Y. N. Demkov and V. N. Ostrovskii, Soviet Phys. JETP 35 66-69 (1972). [Zh. Eksp. Teor. Fiz. 62, 125-132 (1972).]

[19] Y. Kitagawa and A. O. Barut, J. Phys. B 16, 3305-3327 (1983); ibid. 17, 4251-4259 (1984).
[20] M. M. Nieto, Am. J. Phys. 47, 1067-1072 (1979).

[21] As a final note we observe that the exceptional $\nu = 2$ solutions of Section 5.1 will also change their natures as a function of $D$.

[22] P. 66 of [2]

[23] F. Steiner, in: *Path Integrals from meV to MeV*, ed. by M. C. Gutzwiller, A. Inomata, J. R. Klauder, and L. Streit (World Scientific, Singapore, 1986), pp. 335-359.