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The annihilating-submodule graph of modules over commutative rings II

Abstract Let $M$ be a module over a commutative ring $R$. The annihilating-submodule graph of $M$, denoted by $\text{AG}(M)$, is a simple graph in which a non-zero submodule $N$ of $M$ is a vertex if and only if there exists a non-zero proper submodule $K$ of $M$ such that $NK = (0)$, where $NK$, the product of $N$ and $K$, is denoted by $(N : M)(K : M)M$ and two distinct vertices $N$ and $K$ are adjacent if and only if $NK = (0)$. This graph is a submodule version of the annihilating-ideal graph. We prove that if $\text{AG}(M)$ is a tree, then either $\text{AG}(M)$ is a star graph or a path of order 4 and in the latter case $M \cong F \times S$, where $F$ is a simple module and $S$ is a module with a unique non-trivial submodule. Moreover, we prove that if $M$ is a cyclic module with at least three minimal prime submodules, then $gr(\text{AG}(M)) = 3$ and for every cyclic module $M$, $cl(\text{AG}(M)) \geq |\text{Min}(M)|$.

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1 Introduction

Throughout this paper, $R$ is a commutative ring with a non-zero identity and $M$ is a unital $R$-module. By $N \leq M$ (resp., $N < M$) we mean that $N$ is a submodule (resp., proper submodule) of $M$.

Define $(N : R) M$ or simply $(N : M) = \{r \in R | rM \subseteq N\}$ for any $N \leq M$. We denote $((0) : M)$ by $\text{Ann}_R(M)$ or simply $\text{Ann}(M)$. $M$ is said to be faithful if $\text{Ann}(M) = (0)$.

Let $N, K \leq M$. Then, the product of $N$ and $K$, denoted by $NK$, is defined by $(N : M)(K : M)M$ (see [6]).

There are many papers on assigning graphs to rings or modules (see, for example, [4,7,10,11]). The annihilating-ideal graph $\text{AG}(R)$ was introduced and studied in [11]. $\text{AG}(R)$ is a graph whose vertices are

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ideals of \( R \) with non-zero annihilators and in which two vertices \( I \) and \( J \) are adjacent if and only if \( IJ = (0) \). Later, it was modified and further studied by many authors (see [1–3]).

In [7,8], we generalized the above idea to submodules of \( M \) and defined the (undirected) graph \( AG(M) \), called the annihilating-submodule graph, with vertices \( V(AG(M)) = \{ N \leq M \mid \text{there exists } (0) \neq K < M \text{ with } NK = (0) \} \). In this graph, distinct vertices \( N, L \in V(AG(M)) \) are adjacent if and only if \( NL = (0) \). Let \( AG(M)^* \) be the subgraph of \( AG(M) \) with vertices \( V(AG(M)^*) = \{ N < M \} \). Let \( AG(M)^+ \) be the submodule of \( AG(M) \) with vertices \( N < M \text{ with } (N : M) \neq \text{Ann}(M) \). Let \( \phi \) be a submodule \( K < M \) with \( (K : M) \neq \text{Ann}(M) \) and \( NK = (0) \). Note that \( M \) is a vertex of \( AG(M) \) if and only if there exists a non-zero proper submodule \( N \) of \( M \) with \( (N : M) = \text{Ann}(M) \) if and only if every non-zero submodule of \( M \) is a vertex of \( AG(M) \).

In this work, we continue our study in [7,8] and we generalize some results related to annihilating-ideal graph obtained in [1–3] for annihilating-submodule graph.

A prime submodule of \( M \) is a submodule \( P \neq M \), such that whenever \( re \in P \) for some \( r \in R \) and \( e \in M \), we have \( r \in (P : M) \) or \( e \in P \) [14].

The prime radical \( \text{rad}_M(N) \) or simply \( \text{rad}(N) \) is defined to be the intersection of all prime submodules of \( M \) containing \( N \), and in case \( N \) is not contained in any prime submodule, \( \text{rad}_M(N) \) is defined to be \( M \) [14].

The notations \( Z(R) \), \( \text{Nil}(R) \), and \( \text{Min}(M) \) will denote the set of all zero-divisors, the set of all nilpotent elements of \( R \), and the set of all minimal prime submodules of \( M \), respectively. In addition, \( Z_R(M) \) or simply \( Z(M) \), the set of zero divisors on \( M \), is defined to be the intersection of all prime submodules of \( M \) containing \( N \), and in case \( N \) is not contained in any prime submodule, \( Z_R(M) \) is defined to be \( M \) [14].

A clique of a graph is a complete subgraph and the supremum of the sizes of cliques in \( G \), denoted by \( cl(G) \), is called the clique number of \( G \). Let \( \chi(G) \) denote the chromatic number of the graph \( G \), that is, the minimal number of colors needed to color the vertices of \( G \), so that no two adjacent vertices have the same color. Obviously \( \chi(G) \geq cl(G) \).

In Sect. 2, we prove that if \( AG(M) \) is a tree, then either \( AG(M) \) is a star graph or is the path \( P_4 \) and in this case, \( M \cong F \times S \), where \( F \) is a simple module and \( S \) is a module with a unique non-trivial submodule (see Theorem 2.7). Next, we study the bipartite annihilating-submodule graphs of modules over Artinian rings (see Theorem 2.8). In Sect. 3, we study coloring of the annihilating-submodule graph and investigate the interplay between \( \chi(AG(M)) \), \( cl(AG(M)) \), and \( \text{Min}(M) \) (see Theorems 3.5 and 3.8). In Corollary 3.7, we prove that if \( M \) is a cyclic module with at least three minimal prime submodules, then \( gr(AG(M)) = 3 \) and for every cyclic module \( M \), \( cl(AG(M)) \geq |\text{Min}(M)| \).

Let us introduce some graphical notions and denotations that are used in what follows: a graph \( G \) is an ordered triple \((V(G), E(G), \psi_G)\) consisting of a non-empty set of vertices, \( V(G) \), a set \( E(G) \) of edges, and an incident function \( \psi_G \) that associates an unordered pair of distinct vertices with each edge. The edge \( e \) joins \( x \) and \( y \) if \( \psi_G(e) = \{x, y\} \), and we say \( x \) and \( y \) are adjacent. A path in graph \( G \) is a finite sequence of vertices \( \{x_0, x_1, \ldots, x_n\} \), where \( x_{i-1} \) and \( x_i \) are adjacent for each \( 1 \leq i \leq n \) and we denote \( x_{i-1} \) for existing an edge between \( x_{i-1} \) and \( x_i \).

A graph \( H \) is a subgraph of \( G \), if \( V(H) \subseteq V(G) \), \( E(H) \subseteq E(G) \), and \( \psi_H \) is the restriction of \( \psi_G \) to \( E(H) \). A bipartite graph is a graph whose vertices can be divided into two disjoint sets \( U \) and \( V \), such that every edge connects a vertex in \( U \) to one in \( V \); that is, \( U \) and \( V \) are each independent sets and complete bipartite graph on \( n \) and \( m \) vertices, denoted by \( K_{n,m} \), where \( V \) and \( U \) are of size \( n \) and \( m \), respectively, and \( E(G) \) connects every vertex in \( V \) with all vertices in \( U \). Note that a graph \( K_{1,m} \) is called a star graph and the vertex in the singleton partition is called the center of the graph. For some \( U \subseteq V(G) \), we denote by \( V(U) \), the set of all vertices of \( G \setminus U \) adjacent to at least one vertex of \( U \). For every vertex \( v \in V(G) \), the size of \( V(v) \) is denoted by \( d(v) \). If all the vertices of \( G \) have the same degree \( k \), then \( G \) is called \( k \)-regular, or simply regular. An independent set is a subset of the vertices of a graph, such that no vertices are adjacent. We denote by \( P_n \) and \( C_n \), a path and a cycle of order \( n \), respectively. Let \( G \) and \( G' \) be two graphs. A graph homomorphism from \( G \) to \( G' \) is a mapping \( \phi : V(G) \rightarrow V(G') \), such that for every edge \( \{u, v\} \) of \( G \), \( \phi(u), \phi(v) \) is an edge of \( G' \). A retract of \( G \) is a subgraph \( H \) of \( G \), such that there exists a homomorphism \( \phi : G \rightarrow H \) such that \( \phi(x) = x \), for every vertex \( x \) of \( H \). The homomorphism \( \phi \) is called the retract (graph) homomorphism (see [12]).

2 Cycles in the annihilating-submodule graphs

An ideal \( I \leq R \) is said to be nil if \( I \) consist of nilpotent elements.

Proposition 2.1 Suppose that \( e \) is an idempotent element of \( R \). We have the following statements.

(a) \( R = R_1 \times R_2 \), where \( R_1 = eR \) and \( R_2 = (1 - e)R \).
(b) $M = M_1 \times M_2$, where $M_1 = eM$ and $M_2 = (1 - e)M$.
(c) For every submodule $N$ of $M$, $N = N_1 \times N_2$ such that $N_1$ is an $R_1$-submodule $M_1$, $N_2$ is an $R_2$-submodule $M_2$, and $(N : R) = (N_1 : R_1 \times (N_2 : R_2)$.
(d) For submodules $N$ and $K$ of $M$, $NK = N_1K_1 \times N_2K_2$ such that $N = N_1 \times N_2$ and $K = K_1 \times K_2$.
(e) Prime submodules of $M$ are $P \times M_2$ and $M_1 \times Q$, where $P$ and $Q$ are prime submodules of $M_1$ and $M_2$, respectively.

Proof This is clear. \[ \square \]

We need the following lemmas.

**Lemma 2.2** [5, Proposition 7.6] Let $R_1, R_2, \ldots, R_n$ be non-zero ideals of $R$. Then, the following statements are equivalent:

(a) $R = R_1 \times \cdots \times R_n$;
(b) As an abelian group, $R$ is the direct sum of $R_1, \ldots, R_n$;
(c) There exist pairwise orthogonal idempotents $e_1, \ldots, e_n$ with $1 = e_1 + \cdots + e_n$, and $R_i = Re_i$, $i = 1, \ldots, n$.

**Lemma 2.3** [13, Theorem 21.28] Let $I$ be a nil ideal in $R$ and $u \in R$ be such that $u + I$ is an idempotent in $R/I$. Then, there exists an idempotent $e$ in $uR$ such that $e - u \in I$.

**Lemma 2.4** [8, Lemma 2.4] Let $N$ be a minimal submodule of $M$ and let $\text{Ann}(M)$ be a nil ideal. Then, we have $N^2 = (0)$ or $N = eM$ for some idempotent $e \in R$.

**Proposition 2.5** Let $M$ be a finitely generated $R$-module such that $R/\text{Ann}(M)$ is Artinian. Then, every non-zero proper submodule $N$ of $M$ is a vertex in $\text{AG}(M)$.

Proof Let $N$ be a non-zero submodule of $M$. Therefore, there exists a maximal submodule $K$ of $M$, such that $N \subseteq K$. Hence, we have $0 : M (K : M) \subseteq 0 : M (N : M)$. Since $R/\text{Ann}(M)$ is an Artinian ring, $(K : M)$ is a minimal prime ideal containing $\text{Ann}(M)$. Thus, $(K : M) \in \text{Ass}(M)$. It follows that $(K : M) = (0 : m)$ for some $0 \neq m \in M$. Therefore, $N(Rm) = (0)$, as desired. \[ \square \]

**Lemma 2.6** Let $M = M_1 \times M_2$, where $M_1 = eM$, $M_2 = (1 - e)M$, and $e (e \neq 0, 1)$ is an idempotent element of $R$. If $\text{AG}(M)$ is a triangle-free graph, then one of the following statements holds.

(a) Both $M_1$ and $M_2$ are prime $R$-modules.
(b) One $M_i$ is a prime module for $i = 1, 2$ and the other one is a module with a unique non-trivial submodule.

Moreover, $\text{AG}(M)$ has no cycle if and only if either $M = F \times S$ or $M = F \times D$, where $F$ is a simple module, $S$ is a module with a unique non-trivial submodule, and $D$ is a prime module.

Proof If none of $M_1$ and $M_2$ is a prime module, then there exist $r \in R_1$ ($R_1 = Re$ and $R_2 = R(1 - e)$), $0 \neq m_i \in M_i$ with $r_i m_i = 0$, and $r_i \notin \text{Ann}_{R_i}(M_i)$ for $i = 1, 2$. Therefore, $r_1 M_1 \times (0), (0) \times r_2 M_2$, and $r_1 m_1 \times r_2 m_2$ form a triangle in $\text{AG}(M)$, a contradiction. Thus, without loss of generality, one can assume that $M_1$ is a prime module. We prove that $\text{AG}(M_2)$ has at most one vertex. On the contrary suppose that $\{N, K\}$ is an edge of $\text{AG}(M_2)$. Therefore, $M_1 \times (0), (0) \times N$, and $(0) \times K$ form a triangle, a contradiction. If $\text{AG}(M_2)$ has no vertex, then $M_2$ is a prime module and so part (a) occurs. If $\text{AG}(M_2)$ has exactly one vertex, then by [7, Theorem 3.6] and Proposition 2.5, we obtain part (b). Now, suppose that $\text{AG}(M)$ has no cycle. If none of $M_1$ and $M_2$ is a simple module, then choose non-trivial submodules $N_i$ in $M_i$ for some $i = 1, 2$. Therefore, $N_1 \times (0), (0) \times N_2$, $M_1 \times (0)$, and $(0) \times M_2$ form a cycle, a contradiction. The converse is trivial. \[ \square \]

**Theorem 2.7** If $\text{AG}(M)$ is a tree, then either $\text{AG}(M)$ is a star graph or $\text{AG}(M) \cong P_4$. Moreover, $\text{AG}(M) \cong P_4$ if and only if $M = F \times S$, where $F$ is a simple module and $S$ is a module with a unique non-trivial submodule.

Proof If $M$ is a vertex of $\text{AG}(M)$, then there exists only one vertex $N$ such that $\text{Ann}(M) = (N : M)$ and since $\text{AG}(M)^+$ is an empty subgraph, $\text{AG}(M)$ is a star graph. Therefore, we may assume that $M$ is not a vertex of $\text{AG}(M)$. Suppose that $\text{AG}(M)$ is not a star graph. Then, $\text{AG}(M)$ has at least four vertices. Obviously, there are two adjacent vertices $N$ and $K$ of $\text{AG}(M)$, such that $|V(N) \setminus \{K\}| \geq 1$ and $|V(K) \setminus \{N\}| \geq 1$. Let $V(N) \setminus \{K\} = \{N_i\}_{i \in A}$ and $V(K) \setminus \{N\} = \{K_j\}_{j \in \Gamma}$. Since $\text{AG}(M)$ is a tree, we have $V(N) \cap V(K) = \emptyset$. By [7, Theorem 3.4], $\text{diam}(\text{AG}(M)) \leq 3$. So every edge of $\text{AG}(M)$ is of the form $\{N, K\}, \{N, N_i\}$ or $\{K, K_j\}$, for some $i \in A$ and $j \in \Gamma$. Now, consider the following claims:

Claim 1 Either $N^2 = (0) = K^2$. Pick $p \in A$ and $q \in \Gamma$. Since $\text{AG}(M)$ is a tree, $N_p K_q$ is a vertex of $\text{AG}(M)$. If $N_p K_q = N_a$, for some $u \in \Lambda$, then $KN_u = (0)$, a contradiction. If $N_p K_q = K_v$, for some $v \in \Gamma$, ...
then \( NK_e = (0) \), a contradiction. If \( N_pK_q = N \) or \( N_pK_q = K \), then \( N^2 = (0) \) or \( K^2 = (0) \), respectively, and the claim is proved.

Here, without loss of generality, we suppose that \( N^2 = (0) \). Clearly, \((N : M)M \not\subset K\) and \((K : M)M \not\subset N\).

Claim 2 Our claim is to show that \( N \) is a minimal submodule of \( M \) and \( K^2 \neq (0) \). To see that, first, we show that for every \( 0 \neq m \in N \), \( Rm = N \). Assume that \( 0 \neq m \in N \) and \( Rm \neq N \). If \( Rm = K \), then \( K \subset N \), a contradiction. Thus \( Rm \neq K \), and the induced subgraph of \( AG(M) \) on \( N, K \), and \( Rm \) is \( K_3 \), a contradiction. Therefore, \( Rm = N \). This implies that \( N \) is a minimal submodule of \( M \). Now, if \( K^2 = (0) \), then we obtain the induced subgraph on \( N, K \), and \((N : M)M + (K : M)M \) is \( K_3 \), a contradiction. Thus, \( K^2 \neq (0) \), as desired.

Claim 3 For every \( i \in \Lambda \) and every \( j \in \Gamma \), \( N_i \cap K_j = N \). Let \( i \in \Lambda \) and \( j \in \Gamma \). Since \( N_i \cap K_j \) is a vertex and \( N(N_i \cap K_j) = K(N_i \cap K_j) = (0) \), either \( N_i \cap K_j = N \) or \( N_i \cap K_j = K \). If \( N_i \cap K_j = K \), then \( K^2 = (0) \), a contradiction. Hence, \( N_i \cap K_j = N \) and the claim is proved.

Claim 4 We complete the claim by showing that \( M \) has exactly two minimal submodules \( N \) and \( K \). Let \( L \) be a non-zero submodule properly contained in \( K \). Since \( NL \neq NK = (0) \), either \( L = N \) or \( L = N_i \) for some \( i \in \Lambda \). Thus, by Claim 3, \( N \subseteq L \subset K \), a contradiction. Hence, \( K \) is a minimal submodule of \( M \). Suppose that \( L' \) is another minimal submodule of \( M \). Since \( N \) and \( K \) both are minimal submodules, we deduce that \( N \cap K = (0) \), a contradiction. Therefore, the claim is proved.

Now by Claims 2 and 4, \( K^2 \neq (0) \) and \( K \) is a minimal submodule of \( M \). Then, by Lemma 2.4, \( K = eM \) for some idempotent \( e \in R \). Now, we have \( M \cong eM \times (1 - e)M \). By Lemma 2.6, we deduce that either \( M = F \times S \) and \( AG(M) \cong P_4 \) or \( R = F \times D \) and \( AG(M) \) is a star graph. Conversely, we assume that \( M = F \times S \). Then, \( AG(M) \) has exactly four vertices \((0) \times S, F \times (0), (0) \times N, \) and \( F \times N \). Thus, \( AG(M) \cong P_4 \) with the vertices \((0) \times S, F \times (0), (0) \times N, \) and \( F \times N \).

Theorem 2.8 Let \( R \) be an Artinian ring and \( AG(M) \) is a bipartite graph. Then, either \( AG(M) \) is a star graph or \( AG(M) \cong P_4 \). Moreover, \( AG(M) \cong P_4 \) if and only if \( M = F \times S \), where \( F \) is a simple module and \( S \) is a module with a unique non-trivial submodule.

Proof First, suppose that \( R \) is not a local ring. Hence, by [9, Theorem 8.9], \( R = R_1 \times \cdots \times R_n \), where \( R_i \) is an Artinian local ring for \( i = 1, \ldots, n \). By Lemma 2.2 and Proposition 2.1, since \( AG(M) \) is a bipartite graph, we have \( n = 2 \) and \( M \cong M_1 \times M_2 \). If \( M_1 \) is a prime module, then it is easy to see that \( M_1 \) is a vector space over \( R/\text{Ann}(M_1) \) and so is a semisimple \( R \)-module. Hence, by Lemma 2.6 and Theorem 2.7, we deduce that either \( AG(M) \) is isomorphic to \( P_2 \) or \( P_4 \). Now, we assume that \( R \) is an Artinian local ring. Let \( m \) be the unique maximal ideal of \( R \) and \( k \) be a natural number such that \( m^kM = (0) \) and \( m^{k-1}M \neq (0) \). Clearly, \( m^{k-1}M \) is adjacent to every other vertex of \( AG(M) \) and, therefore, \( AG(M) \) is a star graph.

Proposition 2.9 Assume that \( \text{Ann}(M) \) is a nil ideal of \( R \).

(a) If \( AG(M) \) is a finite bipartite graph, then either \( AG(M) \) is a star graph or \( AG(M) \cong P_4 \).
(b) If \( AG(M) \) is a regular graph of finite degree, then \( AG(M) \) is a complete graph.

Proof (a) If \( M \) is a vertex of \( AG(M) \), then \( AG(M) \) has only one vertex \( N \), such that \( \text{Ann}(M) = (N : M) \) and since \( AG(M)^+ \) is an empty submodule, \( AG(M) \) is a star graph. Thus, we may assume that \( M \) is not a vertex of \( AG(M) \), and hence, by [7, Theorem 3.3], \( M \) is not a prime module. Therefore, \[7, Theorem 3.6\] follows that \( R/\text{Ann}(M) \) is an Artinian ring. If \((R/\text{Ann}(M), m/\text{Ann}(M))\) is a local ring, then there exists a natural number \( k \), such that \( m^kM = (0) \) and \( m^{k-1}M \neq (0) \). Clearly, \( m^{k-1}M \) is adjacent to every other vertex of \( AG(M) \) and, therefore, \( AG(M) \) is a star graph. Otherwise, by [9, Theorem 8.9] and Lemma 2.2, there exist pairwise orthogonal idempotents modulo \( \text{Ann}(M) \). By Lemma 2.3, it is easy to see that \( M \cong eM \times (1 - e)M \), where \( e \) is an idempotent element of \( R \) and Lemma 2.6 implies that \( AG(M) \) is a star graph or \( AG(M) \cong P_4 \).

(b) If \( M \) is a vertex of \( AG(M) \), since \( AG(M) \) is a regular graph, \( AG(M) \) is a complete graph. Hence, we may assume that \( M \) is not a vertex of \( AG(M) \). Thus, \( M \) is not a prime module, and hence, \( rm = 0 \), such that \( 0 \neq m \in M, r \not\in \text{Ann}(M) \). It is easy to see that \((rM) (0 :_MM r) = (0) \). If the set of \( R \)-submodules of \( rM \) (resp., \((0 :_MM r)) \) is infinite, then \((0 :_MM r) \) (resp., \( rM \)) has infinite degree, a contradiction. Thus, \( rM \) and \((0 :_MM r) \) have finite length. Since \( rM \cong M/(0 :_MM r) \), \( M \) has finite length, so that \( R/\text{Ann}(M) \) is an Artinian ring. As in the proof of part (a), \( M \cong M_1 \times M_2 \). If \( M_1 \) has one non-trivial submodule \( N \), then \( \deg((0) \times M_2) > \deg(N \times M_2) \) and this contradicts the regularity of \( AG(M) \). Hence, \( M_1 \) is a simple module. Similarly, \( M_2 \) is a simple module. Therefore, \( AG(M) \cong K_2 \). Now, suppose that
Let $S$ be a multiplicatively closed subset of $R$. A non-empty subset $S^*$ of $M$ is said to be $S$-closed if $s \in S^*$ for every $s \in S$ and $e \in S^*$. An $S$-closed subset $S^*$ is said to be saturated if the following condition is satisfied: whenever $ae \in S^*$ for $a \in R$ and $e \in M$, then $a \in S$ and $e \in S^*$.

We need the following result due to Chin-Pi Lu.

**Theorem 2.10** [16, Theorem 4.7] Let $M = Rm$ be a cyclic module. Let $S^*$ be an $S$-closed subset of $M$ relative to a multiplicatively closed subset $S$ of $R$, and $N$ a submodule of $M$ maximal in $M \setminus S^*$. If $S^*$ is saturated, the ideal $(N : M)$ is maximal in $R \setminus S$, so that $N$ is prime in $M$.

**Theorem 2.11** If $M$ is a cyclic module, $\text{Ann}(M)$ is a nil ideal, and $|\text{Min}(M)| \geq 3$, then $AG(M)$ contains a cycle.

**Proof** If $AG(M)$ is a tree, then by Theorem 2.7, either $AG(M)$ is a star graph or $M \cong F \times S$, where $F$ is a simple module and $S$ has a unique non-trivial submodule. The latter case is impossible, because $|\text{Min}(F \times S)| = 2$. Suppose that $AG(M)$ is a star graph and $N$ is the center of star. Clearly, one can assume that $N$ is a minimal submodule of $M$. If $N^2 \neq (0)$, then by Lemma 2.4, there exists an idempotent $e \in R$ such that $N = eM$, so that $M \cong eM \times (1-e)M$. Now, by Proposition 2.1 and Lemma 2.6, we conclude that $|\text{Min}(M)| = 2$, a contradiction. Hence, $N^2 = 0$. Thus, one may assume that $N = Rm$ and $(Rm)^2 = (0)$. Suppose that $P_1$ and $P_2$ are two distinct minimal prime submodules of $M$. Since $(Rm)^2 = (0)$, we have $(Rm : M)^2 \subseteq \text{Ann}(M) \subseteq (P_i : M)$, $i = 1, 2$. So $(Rm : M)^2 \subseteq (P_1 : M)$, $i = 1, 2$. Hence, $m \in P_1$, $i = 1, 2$. Choose $z \in (P_1 : M) \setminus (P_2 : M)$ and set $S_1 = \{1, z, z^2, \ldots\}$, $S_2 = M \setminus P_1$, and $S^* = S_1 \cup S_2$. If $0 \notin S^*$, then $\Sigma = \{N < M | N \cap S^* = \emptyset\}$ is not empty. Then, $\Sigma$ has a maximal element, say $N$. Hence, by Theorem 2.10, $N$ is a prime submodule of $M$. Since $N \subseteq P_1$, we have $N = P_1$, a contradiction because $z \notin (N : M)$. So $0 \in S^*$. Therefore, there exists positive integer $k$ and $m' \in S_2$ such that $z^k m' = 0$. Now, consider the submodules $(m)$, $(m')$, and $z^k M$. It is clear that $(m) \neq (m')$ and $(m) \neq z^k M$. If $(m) = z^k M$, then $z \in (P_2 : M)$, a contradiction. Thus $(m)$, $(m')$, and $z^k M$ form a triangle in $AG(M)$, a contradiction. Hence, $AG(M)$ contains a cycle. □

**Theorem 2.12** Suppose that $M$ is a cyclic module, $\text{rad}_M(0) \neq (0)$, and $\text{Ann}(M)$ is a nil ideal. If $|\text{Min}(M)| = 2$, then either $AG(M)$ contains a cycle or $AG(M) \cong P_4$.

**Proof** A similar argument to the proof of Theorem 2.11 shows that either $AG(M)$ contains a cycle or $M \cong F \times S$, where $F$ is a simple module and $S$ is a module with a unique non-trivial submodule. The latter case implies that $AG(M) \cong P_4$ (note that $\text{rad}_{F \times D}(0) = (0)$, where $F$ is a simple module and $D$ is a prime module). □

The radical of $I$, defined as the intersection of all prime ideals containing $I$, is denoted by $\sqrt{I}$. Before stating the next theorem, we recall that if $M$ is a finitely generated module, then $\sqrt{(Q : M)} = (\text{rad}(Q) : M)$, where $Q < M$ (see [18, Theorem 4.4]). In addition, we know that if $M$ is a finitely generated module, then for every prime ideal $p$ of $R$ with $p \supseteq \text{Ann}(M)$, there exists a prime submodule $P$ of $M$, such that $(P : M) = p$ (see [15, Theorem 2]).

**Theorem 2.13** Assume that $M$ is a finitely generated module, $\text{Ann}(M)$ is a nil ideal, and $|\text{Min}(M)| = 1$. If $AG(M)$ is a triangle-free graph, then $AG(M)$ is a star graph.

**Proof** Suppose first that $P$ is the unique minimal prime submodule of $M$. Since $M$ is not a vertex of $AG(M)$, $Z(M) \neq (0)$. Therefore, there exist non-zero elements $r \in R$ and $m \in M$, such that $rm = 0$. It is easy to see that $rM$ and $Rm$ are vertices of $AG(M)$, because $(rM)(Rm) = (0)$. Since $AG(M)$ is triangle-free, $Rm$ or $rM$ is a minimal submodule of $M$. Without loss of generality, we can assume that $Rm$ is a minimal submodule of $M$, so that $(Rm)^2 = (0)$ (if $rM$ is a minimal submodule of $M$, then there exists $0 \neq m' \in M$ such that $rM = rM$).

We claim that $Rm$ is the unique minimal submodule of $M$. On the contrary, suppose that $K$ is another minimal submodule of $M$. So either $K^2 = K$ or $K^2 = (0)$. If $K^2 = K$, then by Lemma 2.4, $K = eM$ for some idempotent element $e \in R$ and hence, $M \cong eM \times (1-e)M$. This implies that $|\text{Min}(M)| > 1$, a contradiction. If $K^2 = (0)$, then we have $C_3 = (K : M) + (Rm : M)$ and $Rm = Rm - K$, a contradiction. Therefore, $Rm$ is the unique minimal submodule of $M$. Let $V_1 = V(Rm)$, $V_2 = V(AG(M)) \setminus V_1$, $A = \{K \in V_1 \mid Rm \subseteq K\}$, $B = V_1 \setminus A$, and $C = V_2 \setminus \{Rm\}$. We prove that $AG(M)$ is a bipartite graph with parts $V_1$ and $V_2$. We may assume
that $V_1$ is an independent set because $AG(M)$ is triangle-free. We claim that one end of every edge of $AG(M)$ is adjacent to $Rm$ and another end contains $Rm$. To prove this, suppose that $(N, K)$ is an edge of $AG(M)$ and $Rm \neq N, Rm \neq K$. Since $N(Rm) \subseteq Rm$, by the minimality of $Rm$, either $N(Rm) = (0)$ or $Rm \subseteq N$. The latter case follows that $K(Rm) = (0)$. If $N(Rm) = (0)$, then $K(Rm) \neq (0)$ and hence $Rm \subseteq K$. So, our plain is proved. This gives that $V_2$ is an independent set and $V(C) \subseteq V_1$. Since every vertex of $A$ contains $Rm$ and $AG(M)$ is triangle-free, all vertices in $A$ are just adjacent to $Rm$ and so by [7, Theorem 3.4], $V(C) \subseteq B$.

Since one end of every edge is adjacent to $Rm$ and another end contains $Rm$, we also deduce that every vertex of $C$ contains $Rm$ and so every vertex of $A \cup V_2$ contains $Rm$. Note that if $Rm = P$, then one end of each edge of $AG(M)$ is contained in $Rm$, and since $Rm$ is a minimal submodule of $M$, $AG(M)$ is a star graph with center $Rm = P$. Now, suppose that $P \neq Rm$. We claim that $P \in A$. Since $Rm \subseteq P$, it suffices to show that $(Rm)P = (0)$. To see this, let $r \in (P : M)$. We prove that $rm = 0$. Clearly, $(Rm) \subseteq Rm$. If $rm = 0$, then we are done. Thus $Rm = Rm$ and so $m = rms$ for some $s \in R$. We have $m(1-rs) = 0$. By [15, Theorem 2], we have $\text{Nil}(R) = (P : M)$ (note that $\sqrt{\text{Ann}(M)} = (\text{rad}(0) : M) = (P : M)$). Therefore, $1-rs$ is unit, a contradiction, as required. Since $N(C) \subseteq B$, if $B = \emptyset$, then $C = \emptyset$ and, therefore, $AG(M)$ is a star graph with center $Rm$. It remains to show that $B = \emptyset$. Suppose that $K \in B$ and consider the vertex $K \cap P$ of $AG(M)$. Since every vertex of $A \cup V_2$ contains $Rm$, yields $K \cap P \in B$. Pick $0 \neq m' \in K \cap P$. Since $AG(M)$ is triangle-free, one can find an element $m'' \in Rm'$ such that $Rm''$ is a minimal submodule of $M$ and $(Rm'')^2 = (0)$. Since $Rm$ is the unique minimal submodule of $M$, we have $Rm = Rm'' \subseteq Rm'$. Thus $Rm \subseteq K \cap P$, a contradiction. So $B = \emptyset$ and we are done. Hence, $AG(M)$ is a star graph whose center is $Rm$, as desired.

\begin{corollary}
Assume that $M$ is a finitely generated module, $\text{Ann}(M)$ is a nil ideal, and $|\text{Min}(M)| = 1$. If $AG(M)$ is a bipartite graph, then $AG(M)$ is a star graph.
\end{corollary}

\section{On the coloring of the annihilating-submodule graphs}

We recall that $N < M$ is said to be a semiprime submodule of $M$ if for every ideal $I$ of $R$ and every submodule $K$ of $M$, $I^2K \subseteq N$ implies that $IK \subseteq N$. Furthermore, $M$ is called a semiprime module if $(0) \subseteq M$ is a semiprime submodule. Every intersection of prime submodules is a semiprime submodule (see [20]).

\begin{theorem}
Let $S$ be a multiplicatively closed subset of $R$ containing no zero-divisors on finitely generated module $M$. Then, $cl(AG(M_S)) \subseteq cl(AG(M))$. Moreover, $AG(M_S)$ is a retract of $AG(M)$ if $M$ is a semiprime module. In particular, $cl(AG(M_S)) = cl(AG(M))$, whenever $M$ is a semiprime module.
\end{theorem}

\begin{proof}
Consider a vertex map $\phi : V(AG(M)) \longrightarrow V(AG(M_S)), N \longrightarrow N_S$. Clearly, $N_S \neq K_S$ implies $N \neq K$ and $NK = (0)$ if and only if $N_SK_S = (0)$. Thus, $\phi$ is surjective, and hence, $cl(AG(M_S)) \subseteq cl(AG(M))$. In what follows, we assume that $M$ is a semiprime module. If $N \neq K$ and $NK = (0)$, then we show that $N_S \neq K_S$. Without loss of generality, we can assume that $M$ is not a vertex of $AG(M)$, and On the contrary, suppose that $N_S = K_S$. Then, $N_S^2 = N_SK_S = (NK)_S = (0)$ and so $N^2 = (0)$, a contradiction. This shows that the map $\phi$ is a graph homomorphism. Now, for any vertex $N_S$ of $AG(M_S)$, we can choose the fixed vertex $N$ of $AG(M)$. Then, $\phi$ is a retract (graph) homomorphism which clearly implies that $cl(AG(M_S)) = cl(AG(M))$ under the assumption.\end{proof}

\begin{corollary}
If $M$ is a finitely generated semiprime module, then $cl(AG(T(M))) = cl(AG(M))$, where $T = R \setminus Z(M)$.
\end{corollary}

Since the chromatic number $\chi(G)$ of a graph $G$ is the least positive integer $r$, such that there exists a retract homomorphism $\psi : G \longrightarrow K_r$, the following corollaries follow directly from the proof of Theorem 3.1.

\begin{corollary}
Let $S$ be a multiplicatively closed subset of $R$ containing no zero-divisors on finitely generated module $M$. Then, $\chi(AG(M_S)) \leq \chi(AG(M))$. Moreover, if $M$ is a semiprime module, then $\chi(AG(M_S)) = \chi(AG(M))$.
\end{corollary}

\begin{corollary}
If $M$ is a finitely generated semiprime module, then $\chi(AG(T(M))) = \chi(AG(M))$, where $T = R \setminus Z(M)$.
\end{corollary}

Eben Matlis in [17, Proposition 1.5] proved that if $\{p_1, \ldots, p_n\}$ is a finite set of distinct minimal prime ideals of $R$ and $S = R \setminus \bigcup_{i=1}^n p_i$, then $R_{p_1} \times \cdots \times R_{p_n} \cong R_S$. In [19], this result was generalized to finitely generated multiplication modules. In Theorem 3.6, we use this generalization for a cyclic module.
Theorem 3.5  [19, Theorem 3.11] Let \( \{P_1, \ldots, P_n\} \) be a finite set of distinct minimal prime submodules of finitely generated multiplication module \( M \) and \( S = R \setminus \bigcup_{i=1}^n (P_i : M) \). Then, \( M_{P_1} \times \cdots \times M_{P_n} \cong MS, \) where \( p_i = (P_i : M) \) for \( 1 \leq i \leq n. \)

Theorem 3.6  Let \( M \) be a cyclic module and \( \{P_1, \ldots, P_n\} \) be a finite set of distinct minimal prime submodules of \( M \). Then, there exists a clique of size \( n. \)

Proof  Let \( M \) be a cyclic module and \( S = R \setminus \bigcup_{i=1}^n p_i \), where \( p_i = (P_i : M) \) for \( 1 \leq i \leq n. \) Then, since \( M \) is a multiplication module, by Theorem 3.5, there exists an isomorphism \( \phi : M_{P_1} \times \cdots \times M_{P_n} \longrightarrow MS. \)

Let \( M = Rm, e_i = (0, \ldots, 0, m/1, \ldots, 0, 0) \) and \( \phi(e_i) = n_i / l_i \), where \( m \in M, \) \( 1 \leq i \leq n, \) and \( m/1 \) is in the \( i \)th position of \( e_i. \) Consider the principal submodules \( N_i = (n_i/l_i) = (n_i/1) \) in the module \( MS. \) By Lemma 2.2 and Proposition 2.1, the product of submodules \( (0) \times \cdots \times (0) \times (m/1)R_{P_j} \times (0) \times \cdots \times (0) \) and \( (0) \times \cdots \times (0) \times (m/1)R_{P_j} \times (0) \times \cdots \times (0) \) are zero, \( i \neq j. \) Since \( \phi \) is an isomorphism, there exists \( t_{ij} \in S, \) such that \( t_{ij}r_in_j = 0, \) for every \( i, j, 1 \leq i < j \leq n, \) where \( n_i = r_im \) for some \( r_i \in R. \) Let \( t = \Pi_{1 \leq i < j \leq n} t_{ij}. \) We show that \( (tn_1, \ldots, tn_n) \) is a clique of size \( n \) in \( AG(M). \) For every \( i, j, 1 \leq i < j \leq n, \) \( (Rtn_i)(Rtn_j) = (Rtn_j : M)Rtn_j = (Rtn_j : M)Rtn_i, \) for some \( r_i \in R. \) Since \((tn_1)S = (n_1/1) \) \( N_1, \) we deduce that \( (tn) \) are distinct non-trivial submodules of \( M. \)

Corollary 3.7  For every cyclic module \( M, cl(AG(M)) \geq |Min(M)| \) and if \( |Min(M)| \geq 3, \) then \( gr(AG(M)) = 3. \)

Theorem 3.8  Let \( M \) be a cyclic module and \( rad_M(0) = (0). \) Then, \( \chi(AG(M)) = cl(AG(M)) = |Min(M)|. \)

Proof  If \( |Min(M)| = \infty, \) then by Corollary 3.7, there is nothing to prove. Thus, suppose that \( |Min(M)| = \{P_1, \ldots, P_n\} \) for some positive integer \( n. \) Let \( p_j = (P_j : M) \) and \( S = R \setminus \bigcup_{i=1}^n p_i. \) By Theorem 3.5, we have \( M_{P_1} \times \cdots \times M_{P_n} \cong MS. \) Clearly, \( cl(AG(M)) \geq n. \) Now, we show that \( \chi(AG(M)) \leq n. \) By [15, Corollary 3], \( P_iR_{P_i} \) is the only prime submodule of \( M \) and since \( rad_M(0) = (0), \) every \( M_{P_i} \) is a simple \( R_{n_i} \) module. Define the map \( C : V(AG(M)) \longrightarrow \{1, 2, \ldots, n\} \) by \( C(N_1 \times \cdots \times N_n) = \min\{i | N_i \neq 0\}. \) Since each \( M_{P_i} \) is a simple \( R_{n_i} \) module, \( C \) is a proper vertex coloring of \( AG(M). \) Thus \( \chi(AG(M)) \leq n \) and so \( \chi(AG(M)) = cl(AG(M)) = n. \) Since \( rad_M(0) = (0), \) it is easy to see that \( S \cap Z(M) = \emptyset. \) Now, by Theorem 3.1 and Corollary 3.3, we obtain the desired.

Theorem 3.9  For every module \( M, cl(AG(M)) = 2 \) if and only if \( \chi(AG(M)) = 2. \) In particular, \( AG(M) \) is bipartite if and only if \( AG(M) \) is triangle-free.

Proof  For the first assertion, we use the same technique in [3, Theorem 13]. Let \( cl(AG(M)) = 2. \) On the contrary assume that \( AG(M) \) is not bipartite. Therefore, \( AG(M) \) contains an odd cycle. Suppose that \( C := N_1 - N_2 - \cdots - N_{2k+1} - N_1 \) be a shortest odd cycle in \( AG(M) \) for some natural number \( k. \) Clearly, \( k \geq 2. \) Since \( C \) is a shortest odd cycle in \( AG(M), \) \( N_3N_{2k+1} \) is a vertex. Now, consider the vertices \( N_1, N_2, \) and \( N_3N_{2k+1}. \) If \( N_1 = N_3N_{2k+1}, \) then \( N_4N_1 = (0). \) This implies that \( N_1 - N_4 - \cdots - N_{2k+1} - N_1 \) is an odd cycle, a contradiction. Thus, \( N_1 \neq N_3N_{2k+1}. \) If \( N_2 = N_3N_{2k+1}, \) then we have \( C_3 = N_2 - N_3 - N_4 - N_2, \) again a contradiction. Hence, \( N_2 \neq N_3N_{2k+1}. \) It is easy to check \( N_1, N_2, \) and \( N_3N_{2k+1} \) form a triangle in \( AG(M), \) a contradiction. The converse is clear. In particular, we note that empty graphs and the isolated vertex graphs are bipartite graphs.

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