ON THE NUMBER OF CONGRUENCE CLASSES OF PATHS

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Abstract. Let \( P_n \) denote the undirected path of length \( n - 1 \). The cardinality of the set of congruence classes induced by the graph homomorphisms from \( P_n \) onto \( P_k \) is determined. This settles an open problem of Michels and Knauer (Disc. Math., 309 (2009) 5352-5359). Our result is based on a new proven formula of the number of homomorphisms between paths.

Keywords: Graph, graph endomorphisms, graph homomorphisms, paths, lattice paths

1. Introduction

We use standard notations and terminology of graph theory in \([3]\) or \([6, \text{Appendix}]\). The graphs considered here are finite and undirected without multiple edges and loops. Given a graph \( G \), we write \( V(G) \) for the vertex set and \( E(G) \) for the edge set. A homomorphism from a graph \( G \) to a graph \( H \) is a mapping \( f : V(G) \to V(H) \) such that the images of adjacent vertices are adjacent. An endomorphism of a graph is a homomorphism from the graph to itself. Denote by \( \text{Hom}(G, H) \) the set of homomorphisms from \( G \) to \( H \) and by \( \text{End}(G) \) the set of endomorphisms of a graph \( G \). For any finite set \( X \) we denote by \( |X| \) the cardinality of \( X \). A path with \( n \) vertices is a graph whose vertices can be labeled \( v_1, ..., v_n \) so that \( v_i \) and \( v_j \) are adjacent if and only if \( |i - j| = 1 \); let \( P_n \) denote such a graph with \( v_i = i \) for \( 1 \leq i \leq n \). Every endomorphism \( f \) on \( G \) induces a partition \( \rho \) of \( V(G) \), also called the congruence classes induced by \( f \), with vertices in the same block if they have the same image.

Let \( \mathcal{C}(P_n) \) denote the set of endomorphism-induced partitions of \( V(P_n) \), and let \( |\rho| \) denote the number of blocks in a partition \( \rho \). For example, if \( f \in \text{End}(P_4) \) is defined by \( f(1) = 3, f(2) = 2, f(3) = 1, f(4) = 2 \), then the induced partition \( \rho \) is \( \{\{1\}, \{2, 4\}, \{3\}\} \) and \( |\rho| = 3 \).

The problem of counting the homomorphisms from \( G \) to \( H \) is difficult in general. However, some algorithms and formulas for computing the number of homomorphisms of paths have been published recently (see \([1, 2, 5]\)). In particular, Michels and Knauer \([5]\) give an algorithm based on the epispectrum \( \text{Epi}(P_n) \) of a path \( P_n \). They define \( \text{Epi}(P_n) = (l_1(n), ..., l_{n-1}(n)) \), where

\[
l_k(n) = |\{\rho \in \mathcal{C}(P_n) : |\rho| = n - k + 1\}|.
\]

Here a misprint in the definition of \( l_k(n) \) in \([5]\) is corrected.

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In [5], based on the first values of $l_k(n)$, Michels and Knauer speculated the following conjecture.

**Conjecture 1.** There exists a polynomial $f_k \in \mathbb{Q}[x]$ with $\deg(f_k) = \lceil (k - 2)/2 \rceil$ such that for a fixed $n_k$ (most probably $n_k = 2k$) the equality $l_k(n) = f_k(n)$ holds for $n \geq n_k$.

The aim of this paper is to confirm this conjecture by giving an explicit formula for the polynomial $f_k$. For this purpose, we shall prove a new formula for the number of homomorphisms from $P_n$ to $P_k$, which is the content of the following theorem.

**Theorem 2.** For any positive integers $n$ and $k$,

$$|\text{Hom}(P_n, P_k)| = k \times 2^{n-1} - \sum_{i=0}^{n-2} \sum_{j \in \mathbb{Z}} \left( \left( \left\lfloor \frac{i}{2} \right\rfloor - j(k+1) \right) - \left( \left\lfloor \frac{i+k+1}{2} \right\rfloor - j(k+1) \right) \right).$$

(1.2)

From the above theorem we are able to derive the following main result.

**Theorem 3.** If $n \geq 2k$, then

$$l_k(n) = \begin{cases} n-1 & \text{if } k = 2 \left\lfloor \frac{n}{2} \right\rfloor - 1 \\ n-1 & \text{if } k = 2 \left\lfloor \frac{n}{2} \right\rfloor - 1 \end{cases}.$$  

(1.3)

Equivalently, the above formula can be rephrased as follows

$$l_{2k}(n) = 2 \binom{n-1}{k-1}, \quad l_{2k+1}(n) = \binom{n}{k}.$$  

(1.4)

When $n \geq 2k$, Theorem 3 shows immediately that $l_k(n)$ is a polynomial in $n$ of degree $\lceil (k - 2)/2 \rceil$. This proves Conjecture 1. In particular, we have $l_1(n) = 1, l_2(n) = 2, l_3(n) = n, l_4(n) = 2(n-1), l_5(n) = \frac{1}{2}n(n-1)$ and $l_6(n) = (n-1)(n-2)$, which coincide with the conjectured values in [5] after shifting the index by 1.

In the next section, we first recall some basic counting results about the lattice paths and then prove Theorem 2. In Section 3 we give the proof of Theorem 3.

2. **The number of homomorphisms between paths**

One can enumerate homomorphisms from $P_n$ to $P_k$ by picking a fixed point as image of 1 and moving to vertices which are adjacent to this vertex, as

$$f \in \text{Hom}(P_n, P_k) \iff \forall x \in \{1, \ldots, n-1\} : \{f(x), f(x+1)\} \in E(P_k).$$

Hence, one can describe all possible moves through the edge structure of the two paths.

For $1 \leq j \leq k$, let

$$\text{Hom}^j(P_n, P_k) = \{ f \in \text{Hom}(P_n, P_k) : f(1) = j \}.$$  

(2.1)

Obviously, we have

$$|\text{Hom}^j(P_n, P_k)| = |\{ f \in \text{Hom}(P_n, P_k) : f(n) = j \}|.$$  

(2.2)
**Definition 1.** A lattice path of length $n$ is a sequence $(\gamma_0, \ldots, \gamma_n)$ of points $\gamma_i$ in the plan $\mathbb{Z} \times \mathbb{Z}$ for all $0 \leq i \leq n$ and such that $\gamma_{i+1} - \gamma_i = (1, 0)$ (east-step) or $(0, 1)$ (north-step) for $1 \leq i \leq n-1$.

As shown by Arworn [1], we can encode each homomorphism $f \in \text{Hom}^1(P_n, P_k)$ by a lattice path $\gamma = (\gamma_0, \ldots, \gamma_{n-1})$ in $\mathbb{N} \times \mathbb{N}$ between the lines $y = x$ and $y = x - k + 1$ as follows:

- $\gamma_0 = (0, 0)$, and for $j = 1, \ldots, n-1$,
- $\gamma_{j+1} = \gamma_j + (1, 0)$ if $f(j) > f(j-1)$,
- $\gamma_{j+1} = \gamma_j + (0, 1)$ if $f(j) < f(j-1)$.

For example, if the images of successive vertices of $f \in \text{Hom}(P_{15}, P_{11})$ are $1, 2, 3, 2, 3, 4, 5, 4, 3, 4, 5, 6, 5, 6, 5$; then the corresponding lattice path is given by Figure 1.

**Definition 2.** For nonnegative integers $n, m, t, s$, Let $\mathcal{L}(n, m)$ be the set of all the lattice paths from the origin to $(n, m)$ and $\mathcal{L}(n, m; t, s)$ the set of lattice paths in $\mathcal{L}(n, m)$ that stay between the lines $y = x + t$ and $y = x - s$ (being allowed to touch them), where $n + t \geq m \geq n - s$.

**Lemma 4.** Let $K = \min(\lceil \frac{n+k}{2} \rceil, n)$, then

$$|\text{Hom}^1(P_n, P_k)| = \sum_{l=\lceil \frac{n-1}{2} \rceil}^{K-1} |\mathcal{L}(l, n - l - 1; 0, k - 1)|.$$  \hspace{1cm} (2.3)

**Proof.** It follows from the above correspondence that each homomorphism from $P_n$ to $P_k$ is encoded by a lattice path in some $\mathcal{L}(\#E, \#N; 0, k - 1)$, where $\#E$ is the number of
east-steps and \( \#N \) the number of north-steps. The path structures require that
\[
\#E + \#N = n - 1, \quad \#E - \#N \leq k - 1, \quad \#E - \#N \geq 0.
\]
Therefore, we must have \( \#E \geq (n - 1)/2, \#E \leq n - 1 \) and \( \#E \leq (k + n - 2)/2 \).

To evaluate the sum in (2.3), we need a formula for the cardinality of \( \mathcal{L}(n, m; t, s) \).
First of all, each lattice path in \( \mathcal{L}(n, m) \) can be encoded by a word of length \( n + m \) on the alphabet \{A, B\} with \( n \) letters A and \( m \) letters B. So, the cardinality of \( \mathcal{L}(n, m) \) is given by the binomial coefficient \( \binom{n+m}{n} \). Next, each lattice path in \( \mathcal{L}(n, m) \) which passes above the line \( y = x + t \) (or reaching the line \( y = x + t + 1 \)) can be mapped to a lattice path from \((-t - 1, t + 1)\) to \((n, m)\) by the reflection with respect to the line \( y = x + t + 1 \) (see Figure 2). Hence, there are \( \binom{n+m}{n+t+1} \) such lattice paths. Therefore, the number of lattice paths in \( \mathcal{L}(n, m) \) which do not pass above the line \( y = x + t \) (or not reaching the line \( y = x + t + 1 \)), where \( m \leq n + t \), is given by
\[
\binom{n+m}{n} - \binom{n+m}{n+t+1}.
\]

By a similar reasoning, we can prove the following known result (see [4, Lemma 4A], for example). For the reader’s convenience, we provide a sketch of the proof.

**Lemma 5.** The cardinality of \( \mathcal{L}(n, m; t, s) \) is given by
\[
|\mathcal{L}(n, m; t, s)| = \sum_{k \in \mathbb{Z}} \left( \begin{pmatrix} n + m \\ n - k(t + s + 2) \end{pmatrix} - \begin{pmatrix} n + m \\ n - k(t + s + 2) + t + 1 \end{pmatrix} \right), \quad (2.4)
\]
where \( \begin{pmatrix} n \\ k \end{pmatrix} = 0 \) if \( k > n \) or \( k < 0 \).
**Sketch of proof.** Let $T$ and $S$ be the lines $y = x + t + 1$ and $y = x - s - 1$, respectively. Let $A_1$ denote the set of lattice paths in $\mathcal{L}(n, m)$ reaching $T$ at least once, regardless of what happens at any other step, and let $A_2$ denote the set of lattice paths in $\mathcal{L}(n, m)$ reaching $T, S$ at least once in the order specified. Generally, let $A_i$ denote the set of lattice paths in $\mathcal{L}(n, m)$ reaching $T, S, \ldots$, alternatively ($i$ times) at least once in the specified order. Let $B_i$ be the set defined in the same way as $A_i$ with $S, T$ interchanged. A standard Inclusive-Exclusive principle argument yields:

$$|\mathcal{L}(n, m; t, s)| = \binom{n+m}{n} + \sum_{i \geq 1} (-1)^i(|A_i| + |B_i|). \quad (2.5)$$

As the symmetric point of $(a, b)$ with respect to the line $y = x + c$ is $(b - c, a + c)$, by repeatedly applying the reflection principle argument, we obtain

$$|A_{2j}| = \binom{n+m}{n + j(t + s + 2)}, \quad |A_{2j+1}| = \binom{n+m}{n - j(t + s + 2) - (t + 1)},$$

and

$$|B_{2j}| = \binom{n+m}{n - j(t + s + 2)}, \quad |B_{2j+1}| = \binom{n+m}{n + j(t + s + 2) - (s + 1)}.$$

Substituting this into (2.5) leads to (2.4).

**Lemma 6.** For each positive integers $n$ and $k$,

$$|\text{Hom}^1(P_n, P_k)| = \sum_{j \in \mathbb{Z}} (\left\lfloor \frac{n-1}{2} \right\rfloor - j(k + 1)) = \left( \left\lfloor \frac{n+k}{2} \right\rfloor - j(k + 1) \right). \quad (2.6)$$

**Proof.** Substituting (2.4) into (2.3) and exchanging the order of the summations,

$$|\text{Hom}^1(P_n, P_k)| = \sum_{j \in \mathbb{Z}} \sum_{l = \left\lfloor \frac{n-1}{2} \right\rfloor}^{K-1} \left( \left\lfloor \frac{n-1}{2} \right\rfloor - j(k + 1) \right) - \left( \left\lfloor \frac{n+k}{2} \right\rfloor - j(k + 1) \right)) = \sum_{j \in \mathbb{Z}} \left( \left\lfloor \frac{n-1}{2} \right\rfloor - j(k + 1) \right) - \left( \left\lfloor \frac{n+k}{2} \right\rfloor - j(k + 1) \right)). \quad (2.7)$$

Now, if $n \geq k$, then $K = \left\lfloor \frac{n+k}{2} \right\rfloor$,

$$\left( \left\lfloor \frac{n-1}{2} \right\rfloor - j(k + 1) \right) = \left( \left\lfloor \frac{n+k}{2} \right\rfloor - j(k + 1) \right),$$

if $k > n$, then $K = n$, since

$$\left( \left\lfloor \frac{n-1}{2} \right\rfloor - j(k + 1) \right) = \left( \left\lfloor \frac{n+k}{2} \right\rfloor - j(k + 1) \right) = 0,$$

the equation (2.8) is also valid. Hence (2.7) and (2.6) are equal. \qed
Proof of Theorem 1. For \( f \in \text{Hom}(P_{i+1}, P_k) \) with \( i = 1, \ldots, n-1 \), consider the following three cases:

(i) if \( f(i) = 1 \), then \( f(i+1) = 2 \) and there are \( |\text{Hom}^1(P_i, P_k)| \) such homomorphisms.

(ii) if \( f(i) = k \), then \( f(i+1) = k-1 \) and there are \( |\text{Hom}^k(P_i, P_k)| \) such homomorphisms.

(iii) if \( f(i) = j \) with \( j \in \{2, 3, \ldots, k-1\} \), then \( f(i+1) = j-1 \) or \( j+1 \) and there are \( 2|\text{Hom}^j(P_i, P_k)| \) such homomorphisms.

Summarizing, we get

\[
|\text{Hom}(P_{i+1}, P_k)| = |\text{Hom}^1(P_i, P_k)| + 2 \sum_{j=2}^{k-1} |\text{Hom}^j(P_i, P_k)| + |\text{Hom}^k(P_i, P_k)|.
\]

Since \( |\text{Hom}(P_i, P_k)| = \sum_{j=1}^k |\text{Hom}^j(P_i, P_k)| \) and \( |\text{Hom}^1(P_i, P_k)| = |\text{Hom}^k(P_i, P_k)| \), it follows that

\[
|\text{Hom}(P_{i+1}, P_k)| = 2|\text{Hom}(P_i, P_k)| - 2|\text{Hom}^1(P_i, P_k)|.
\]

By iteration, we derive

\[
|\text{Hom}(P_n, P_k)| = 2^{n-1}|\text{Hom}(P_1, P_k)| - \sum_{i=1}^{n-1} 2^{n-i}|\text{Hom}^1(P_i, P_k)|
\]

\[
= k \times 2^{n-1} - \sum_{i=1}^{n-1} 2^{n-i}|\text{Hom}^1(P_i, P_k)|. \tag{2.9}
\]

Plugging (2.6) into (2.9), we obtain (1.2).

□

Remark. The key point in the above proof is to reduce the counting problem of \( |\text{Hom}(P_n, P_k)| \) to \( |\text{Hom}^1(P_i, P_k)| \) for \( i = 1, \ldots, n-1 \). Arworn and Wojtylak [2] give a formula for \( |\text{Hom}(P_n, P_k)| = \sum_{j=1}^k |\text{Hom}^j(P_n, P_k)| \) without using this reduction. Moreover, their expression for \( |\text{Hom}^j(P_n, P_k)| \) depends on the parity of \( n-j \):

\[
|\text{Hom}^j(P_n, P_k)| = \begin{cases} 
\sum_{t=-n+1}^{n-1} (-1)^t \sum_{u=0}^{\frac{k-1}{2}} \binom{n-1-t}{u} \binom{n-1-u}{\frac{k-1}{2}} 
& \text{if } n-j \text{ is odd,} \\
\sum_{t=-n+1}^{n-1} (-1)^t \sum_{u=0}^{\frac{k-1}{2}} \binom{n-1-t}{u} \binom{n-1-u}{\frac{k-1}{2}} 
& \text{if } n-j \text{ is even.} 
\end{cases} \tag{2.10}
\]

Note that Lemma 2.6 unifies the two cases in (2.10) when \( j = 1 \).

When \( k = n \), we can deduce a simple formula for the number of endomorphisms of \( P_n \) (see \texttt{http://oeis.org/A102699}) by applying two binomial coefficient identities.
Lemma 7. For \( m \geq 1 \), the following identities hold
\[
\sum_{k=0}^{m-1} \binom{2k}{k} 2^{2m-1-2k} = m \binom{2m}{m}, \quad (2.11)
\]
\[
\sum_{k=0}^{m-1} \binom{2k+1}{k} 2^{2m-1-2k} = (m+1) \binom{2m+1}{m} - 2^{2m}. \quad (2.12)
\]

Proof. We prove (2.11) by induction on \( m \). Clearly (2.11) is true for \( m = 1 \). If it is true for \( m \geq 1 \), then for \( m+1 \), the left-hand side after cutting out the last term, can be written as
\[
2^2 \sum_{k=0}^{m-1} \binom{2k}{k} 2^{2m-1-2k} + 2 \binom{2m}{m} = 4m \binom{2m}{m} + 2 \binom{2m}{m} = (m+1) \binom{2m+2}{m+1}.
\]
Thus (2.11) is proved. Similarly we can prove (2.12). \( \square \)

Proposition 8. For \( n \geq 1 \),
\[
|\text{End}(P_n)| = \begin{cases} 
(n+1)2^{n-1} - (2n-1)\binom{n-1}{(n-1)/2} & \text{if } n \text{ is odd}, \\
(n+1)2^{n-1} - n\binom{n}{n/2} & \text{if } n \text{ is even}.
\end{cases} \quad (2.13)
\]

Proof. When \( k = n \), Theorem 2 becomes
\[
|\text{End}(P_n)| = n \times 2^{n-1} - \sum_{i=0}^{n-2} 2^{n-1-i} \times \binom{i}{\lfloor i/2 \rfloor}. \quad (2.14)
\]
By Lemma 7 if \( n \) is even, say \( n = 2m \), then
\[
\sum_{i=0}^{n-2} 2^{n-1-i} \times \binom{i}{\lfloor i/2 \rfloor} = \sum_{k=0}^{m-2} \binom{2k+1}{k} 2^{2m-2-2k} + \sum_{k=0}^{m-1} \binom{2k}{k} 2^{2m-1-2k}
\]
\[
= 2m \binom{2m-1}{m-1} - 2^{2m-1} + m \binom{2m}{m};
\]
if \( n \) is odd, say \( n = 2m+1 \), then
\[
\sum_{i=0}^{n-2} 2^{n-1-i} \times \binom{i}{\lfloor i/2 \rfloor} = \sum_{k=0}^{m-1} \binom{2k+1}{k} 2^{2m-1-2k} + \sum_{k=0}^{m-1} \binom{2k}{k} 2^{2m-2k}
\]
\[
= (m+1)\binom{2m+1}{m} - 2^{2m} + 2m \binom{2m}{m}.
\]
Substituting these into (2.14) we obtain the desired result. \( \square \)
3. Proof of theorem 3

We first establish three lemmas. For any \( n \geq 1 \), let \( [n] = \{1, \ldots, n\} \), which is \( V(P_n) \). Denote by \( \mathcal{G}_r \) the set of permutations of \([n] \). For \( 1 \leq k \leq n \), denote by \( \text{Epi}(P_n, P_k) \) the set of epimorphisms from \( P_n \) to \( P_k \), namely,
\[
\text{Epi}(P_n, P_k) = \{ f \in \text{Hom}(P_n, P_k) : f([n]) = [k] \}.
\]

**Lemma 9.** For \( 1 \leq k \leq n-1 \),
\[
l_k(n) = |\text{Epi}(P_n, P_{n-k+1})|/2.
\]

*Proof.* Let \( r = n - k + 1 \). Denote by \( \text{End}_r(P_n) \) the subset of endomorphisms in \( \text{End}(P_n) \) such that \( |f([n])| = r \) and \( L_k(n) \) the set of partitions induced by endomorphisms in \( \text{End}_r(P_n) \). By definition (see (3.1)), the integer \( l_k(n) \) is the cardinality of \( L_k(n) \).

For each \( f \in \text{End}_r(P_n) \), if \( f([n]) = \{a, a+1, \ldots, a+r-1\} \) for some integer \( a \in [n-r+1] \), we define \( \hat{f} \in \text{Epi}(P_n, P_r) \) by \( \hat{f}(x) = f(x)-a+1. \) Then \( f \) and \( \hat{f} \) induce the same partition in \( L_k(n) \). Hence, we can consider \( L_k(n) \) as the set of partitions induced by epimorphisms in \( \text{Epi}(P_n, P_r) \).

If \( \{A_1, \ldots, A_r\} \) is a partition of \([n]\) induced by an \( f \in \text{Epi}(P_n, P_r) \), then, we can assume that \( \min(A_1) \leq \min(A_2) \leq \ldots \leq \min(A_r) \). Hence, we can identify \( f \) with a permutation \( \sigma \in \mathcal{G}_r \) by \( f(A_{\sigma(i)}) = i \) for \( i \in [r] \). Moreover, two blocks \( A_i \) and \( A_j \) are adjacent in the arrangement \( A_{\sigma(1)} \ldots A_{\sigma(r)} \) if and only if there are two consecutive integers \( \alpha \) and \( \beta \) such that \( \alpha \in A_i \) and \( \beta \in A_j \). We show that there are exactly two such permutations for a given induced partition.

Starting from a partition \( \{A_1, \ldots, A_r\} \) of \([n]\) induced by an \( f \in \text{Epi}(P_n, P_r) \), we arrange step by step the blocks \( A_1, \ldots, A_i \) for \( 2 \leq i \leq r \) such that \( A_i \) is adjacent to the block \( A_j \) containing \( \min(A_i) - 1 \) and \( j < i \). Since \( \min(A_1) = 1 \) and \( \min(A_2) = 2 \), there are two ways to arrange \( A_1 \) and \( A_2 \): \( A_1A_2 \) or \( A_2A_1 \). Suppose that the first \( i \geq 2 \) blocks have been arranged as \( W_i := A_{\sigma_1(1)} \cdots A_{\sigma(i)} \) with \( \sigma_i \in \mathcal{G}_i \), then \( \min(A_{i+1}) - 1 \) must belong to \( A_{\sigma_i(1)} \) or \( A_{\sigma_i(i)} \) because any two adjacent blocks in \( W_i \) should stay adjacent in all the \( W_j \) for \( i \leq j \leq r \). Hence there is only one way to insert \( A_{i+1} \) in \( W_i \): at the left of \( W_i \) (resp. right of \( W_i \)) if \( \min(A_{i+1}) - 1 \in A_{\sigma_i(1)} \) (resp. \( A_{\sigma_i(i)} \)) for \( i \geq 2 \). As there are two possibilities for \( i = 2 \) we have thus proved that there are exactly two corresponding epimorphisms in \( \text{Epi}(P_n, P_r) \) for a given induced partition with \( r \) blocks. For example, starting from the induced partition \( \{1,3,5,9\}, \{2,4,10\}, \{6,8\}, \{7\}, \{11\} \) of \( V(P_{11}) \), we obtain the two corresponding arrangements:

\[
\{7\}\{6,8\}\{1,3,5,9\}\{2,4,10\}\{11\} \quad \text{and} \quad \{11\}\{2,4,10\}\{1,3,5,9\}\{6,8\}\{7\}.
\]

This is the desired result. \( \square \)

**Lemma 10.** For \( 1 \leq k \leq n \),
\[
l_k(n) = \frac{1}{2} |\text{Hom}(P_n, P_{n-k+1})| - |\text{Hom}(P_n, P_{n-k})| + \frac{1}{2} |\text{Hom}(P_n, P_{n-k-1})|.
\]
Proof. By definition we have $\text{Hom}(P_n, P_k \setminus \text{Epi}(P_n, P_k)) = A \cup B$, where
\[
A = \{ f \in \text{Hom}(P_n, P_k) : f([n]) \subseteq [k-1]\},
\]
\[
B = \{ f \in \text{Hom}(P_n, P_k) : f([n]) \subseteq [k] \setminus [1]\}.
\]
Hence
\[
|\text{Hom}(P_n, P_k)| - |\text{Epi}(P_n, P_k)| = |A| + |B| - |A \cap B|.
\] (3.3)
Since $|A| = |B| = |\text{Hom}(P_n, P_k_1)|$, and
\[
|A \cap B| = |\{ f \in \text{Hom}(P_n, P_k) : f([n]) \subseteq [k-1] \setminus [1]\}| = |\text{Hom}(P_n, P_k_2)|,
\]
we derive from (3.3) that
\[
|\text{Epi}(P_n, P_k)| = |\text{Hom}(P_n, P_k)| - 2|\text{Hom}(P_n, P_k_1)| + |\text{Hom}(P_n, P_k_2)|.
\]
The result follows then by applying Lemma 9.

It follows from Lemma 10 and Theorem 2 that
\[
l_k(n) = \sum_{i=0}^{n-2} 2^{n-i-2} \sum_{j \in \mathbb{Z}} (-A_{i,j} + 2B_{i,j} - C_{i,j}),
\] (3.4)
where
\[
A_{i,j} = A_{i,j}^+ - A_{i,j}^-, \quad B_{i,j} = B_{i,j}^+ - B_{i,j}^-, \quad C_{i,j} = C_{i,j}^+ - C_{i,j}^-,
\]
with
\[
A_{i,j}^+ = \left(\left\lceil \frac{i}{2}\right\rceil - j(n-k+2)\right), \quad A_{i,j}^- = \left(\left\lceil \frac{i+n-k}{2}\right\rceil + 1 - j(n-k+2)\right),
\]
\[
B_{i,j}^+ = \left(\left\lceil \frac{i}{2}\right\rceil - j(n-k+1)\right), \quad B_{i,j}^- = \left(\left\lceil \frac{i+n-k-1}{2}\right\rceil + 1 - j(n-k+1)\right),
\]
\[
C_{i,j}^+ = \left(\left\lceil \frac{i}{2}\right\rceil - j(n-k)\right), \quad C_{i,j}^- = \left(\left\lceil \frac{i+n-k-2}{2}\right\rceil + 1 - j(n-k)\right).
\]

Lemma 11. For $n \geq 2k$,
\[
\sum_{j \in \mathbb{Z}} (-A_{i,j} + 2B_{i,j} - C_{i,j}) = \left\{\left(\left\lceil \frac{i+1}{2}\right\rceil + 1\right) + \left(\left\lceil \frac{i+n-k}{2}\right\rceil + 1 \right)\right\}
\]
\[
- 2 \left\{\left(\left\lceil \frac{i+n-k-1}{2}\right\rceil + 1\right) + \left(\left\lceil \frac{i+n-k-2}{2}\right\rceil + 1 \right)\right\}.
\] (3.5)

Proof. Since $0 \leq k \leq \frac{n}{2}$, we have $\frac{n}{2} \leq n - k \leq n - 1$. Therefore,

(1) if $j < 0$, then $\left\lceil \frac{i}{2}\right\rceil - j(n-k) \geq \left\lceil \frac{i}{2}\right\rceil + n-k \geq \left\lceil \frac{i}{2}\right\rceil + \frac{n}{2} \geq \left\lceil \frac{i}{2}\right\rceil + \frac{1}{2} + 1 \geq i + 1$ because $i \leq n - 2$. Similarly we have $\left\lceil \frac{i+n-k-2}{2}\right\rceil + 1 - j(n-k) \geq i + 1$. Hence, all the summands $A_{i,j}$, $B_{i,j}$ and $C_{i,j}$ vanish.
(2) if \( j > 0 \), then \( \left\lceil \frac{i}{2} \right\rceil - j(n-k) \leq \left\lceil \frac{i}{2} \right\rceil - (n-k) \leq \left\lceil \frac{n}{2} \right\rceil - \left\lfloor \frac{n}{2} \right\rfloor \leq -1 \) because \( i \leq n-2 \). Hence, all \( A_{i,j}^+, B_{i,j}^+ \) and \( C_{i,j}^+ \) vanish;

(3) if \( j \geq 2 \), then \( \left\lfloor \frac{i+n-k}{2} \right\rfloor + 1 - j(n-k+2) \leq \left\lfloor \frac{n-2+n-k}{2} \right\rfloor + 1 - 2(n-k+2) \leq \frac{3}{2} k - n - 5 \leq -1 \). Similarly we have \( \left\lfloor \frac{i+n-k-1}{2} \right\rfloor + 1 - j(n-k+1) \leq -1 \) and \( \left\lfloor \frac{i+n-k-2}{2} \right\rfloor + 1 - j(n-k) \leq -1 \), so all \( A_{i,j}^-, B_{i,j}^- \) and \( C_{i,j}^- \) vanish.

It follows that the summation over \( j \in \mathbb{Z} \) in (3.5) reduces to

\[-A_{i,0}^- + 2B_{i,0}^- - C_{i,0}^- - A_{i,1}^- + 2B_{i,1}^- - C_{i,1}^-.

Using \( \binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k} \) to combine \( A_{i,0}^- \) with \( C_{i,0}^- \) and \( A_{i,1}^- \) with \( C_{i,1}^- \), respectively, we derive the desired formula. \( \square \)

Now, we are in position to prove Theorem 1. When \( n \geq 2k \), by Lemma 11 the summands in (3.4) can be written as

\[ 2^{n-i-2} \sum_{j \in \mathbb{Z}} (-A_{i,j} + 2B_{i,j} - C_{i,j}) = D_{i+1} - D_i, \]

where

\[ D_i = 2^{n-i-1} \left\{ \binom{i}{n-k-1} + 1 + \binom{i+n-k-1}{n} \right\}. \]

Substituting this into (3.4) we obtain

\[ l_k(n) = \sum_{i=0}^{n-2} (D_{i+1} - D_i) = D_{n-1}, \]

which is clearly equivalent to (1.3).

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