A linear time algorithm to compute
the impact of all the articulation points

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Abstract. The articulation points of an undirected connected graphs
are those vertices whose removal increases the number of connected
components of the graph, i.e. the vertices whose removal disconnects the
graph. However, not all the articulation points are equal: the removal
of some of them might end in a single vertex disconnected from the
graph, whilst in other cases the graph can be split in several small
pieces. In order to measure the effect of the removal of an articulation
point, in [1] has been proposed the impact, defined as the number of
vertices that get disconnected from the main (largest) surviving connected
component (CC). In this paper we present the first linear time algorithm
(\(O(m + n)\) for a graph with \(n\) vertices and \(m\) edges) to compute the
impact of all the articulation points of the graph, thus improving from
the \(O(a(m + n)) \approx O(nm + n^2)\) of the naïve algorithm, with \(a\) being the
number of articulation points of the graph.

1 Introduction

The connectivity is one of the most basic structural properties of a graph. If
an undirected graph is connected, a natural related question is which are the
vertices whose removal disconnects the graph; this question has been answered by
Hopcroft and Tarjan in [3], that proposed the now classical linear time algorithm,
described in many textbooks (see, e.g., [2]), based on the Depth First Search
visit of a graph [5].

From the topological point of view all the articulation points are equal, in the
sense that they split the graph in more than one connected components; however,
it is natural, especially when studying the resiliency of a graph, to ask which
are the vertices (i.e., articulation points) whose removal is more disrupting. In
order to measure the effect of the removal of an articulation point, in [1] has been
proposed the impact, defined as the number of vertices that get disconnected from the main (largest) surviving connected component (CC). Consider, as an
example, the graph shown in Figure 1: in this graph it is possible to see that
the vertex 4 is the one with the largest impact, and its removal disconnects six
vertices from largest connected component (i.e, the one that includes vertices
\{5, 6, 7, 8, 9, 10\}).

From its definition, the impact can be naïvely computed in the following
way: we first compute, using Hopcroft and Tarjan’s algorithm, the articulation

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Fig. 1. A connected undirected graph. The vertex 4 is the one with largest impact: the removal of vertex 4 leaves a largest connected components made by vertices \{5, 6, 7, 8, 9, 10\} and, thus, six vertices (in two connected components) are detached from the largest connected component.

Point of the graph, then, we remove each of them, one at a time, and perform any linear visit of the graph, i.e. a DFS or a BFS, to compute the number of reachable vertices. If the graph has \(a\) articulation points, this naïve algorithm costs \(O(a(m + n)) \approx O(nm + n^2)\), since \(a\) can be of the same order of \(n\), and indeed it holds \(0 \leq a \leq n - 2\).

In this paper we present the first linear time algorithm to compute the impact for all the articulation points of an undirected graph in linear time, i.e. \(O(m + n)\) for a graph with \(n\) vertices and \(m\) edges. The algorithm is built over three main ingredients: the block forest data structure, proposed by Westbrook and Tarjan in [6] to maintain online graph biconnected components, a novel algorithm to recursively compute this data structure offline, and a depth first based traversal of this structure, to compute the impact of all the vertices.

This paper is organized as follows: the few necessary preliminary notions are discussed below. The algorithm is described in Section 2, whilst concluding remarks are addressed in Section 3. Due to space constraint, we provide only some intuition of the algorithm properties, and do not prove them formally. In the Appendix we present the algorithm’s pseudocode and some other remarks.

**Preliminaries.** Given an undirected graph \(G = (V, E)\), we define connected component: a maximal set of vertices \(V' \subseteq V\) such that, given \(u, v \in V'\), there is at least one path between \(u\) and \(v\) in \(G\); articulation point: a vertex \(v \in V\) such that its removal from the graph \(G\) increases the number of connected components in \(G\); bridge: an edge \(e \in E\) such that its removal from the graph \(G\) increases the number of connected components in \(G\); biconnected component: a maximal set of nodes \(V'' \subseteq V\) such that, given \(u, v \in V''\), there are at least two distinct paths between \(u\) and \(v\) in \(G\).

In Figure 2 we can see the graph from Figure 1 rearranged in order to emphasize the biconnected components. It is important to note that all the vertices adjacent to bridges are articulation points unless the bridge is their only incident edge, as is the case for node 1 in Figure 2: the removal of the bridge leaves vertex 1 isolated, but the removal of vertex 1 (together with all its adjacent edges, i.e. only the bridge) does not increase the number of connected components of \(G\).
Fig. 2. The graph of Figure 1, rearranged to emphasize the biconnected components, the articulation points (bold vertices), and the bridges (bold edges) [left]; the effect of the removal of vertex 4 [right].

2 The algorithm

In this section we describe our linear time algorithm to compute the impact of all the vertices (articulation points). As we mentioned in the introduction, the algorithm uses the block forest (BF) data structure, proposed by Westbrook and Tarjan in [6] to maintain online the biconnected components. In order to provide some intuition for the algorithm we propose, we briefly recall some properties of the block forest data structure. In Figure 3 it is possible to see the graph from Figure 1, rearranged to emphasize the biconnected components, and the corresponding block forest data structure that, since the graph is connected, is a single tree. This tree has two distinct type of vertices: the square vertices, that correspond to vertices of the original graph, and are depicted as square vertices in Figure 3, and vertices that correspond to the biconnected components of the graph, and are depicted as round vertices in the figure. Each vertex of the graph (square vertex) is connected, in the BF, to all the biconnected components it belongs to (round vertices). The articulation points of the graph, shown in figure as bold square vertices, are the square vertices that are not leaves in the BF.

Now that we described the properties of the BF data structure, the algorithm can be roughly divided into two main steps:

- it computes the BF data structure offline, using a novel recursive offline algorithm (see the pseudocode in the appendix);
- it computes the impact of each (square non-leaf) vertex in the BF, using a novel DFS based algorithm (see the pseudocode in the appendix).

We provide some intuition for both the steps of the algorithm: the first one builds the BF data structure one tree at a time, starting from an unvisited vertex of the graph at each step; the construction of a single tree can be seen as a modified version of Hopcraft and Tarjan’s algorithm[3]. The second step is slightly more involved, from a high level point of view, thus let us clarify it with an example. In the BF tree shown in Figure 3, in order to compute the impact of vertex 4, one could imagine that each of the round vertices connected to the square vertex 4 store the number of square vertices that belong to their subtree. These are the values shown in the picture, in the small gray badges. It
can be proven that the impact, for each square vertex, can be easily calculated by looking at the values written in its round neighbours. We compute these values for each square vertex using a DFS based algorithm.

3 Conclusions

In this paper we presented a linear time algorithm to compute, in linear time, the impact of all the articulation points of an undirected graph. The approach described in this paper could be extended to directed graphs as well but, even if recently has been proposed a linear time algorithm to compute all the strong articulation points, it is still open the problem of how to compute the 2-vertex connected components [4].

References

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A Pseudocode

In this section we can see the pseudocode of the algorithm to compute the impact values of all the articulation points of an undirected input graph \( G = (V, E) \); it can be divided in two main parts, as described in Section 2:

- computation of the block forest (BF) data structure, depicted in Algorithm 1, that uses Algorithm 2 as a subroutine;
- the actual computation of the impact values, depicted in Algorithm 3

### Algorithm 1 Computation of the block forest (BF) data structure

1: procedure make-block-forest
2: \( \triangledown G = (V, E) \) is the input graph
3: \( \text{global} \) \( \text{timer} \leftarrow 0 \)
4: \( \text{global} \) \( \text{edges} \leftarrow \) empty stack
5: create a square node for each vertex in \( V \)
6: for \( v \in V \) do
7: if \( v \) not yet numbered then
8: recursive-make-bf\((v)\)
9: end if
10: end for
11: end procedure
Algorithm 2 BF data structure recursive subroutine

1: procedure RECURSIVE-MAKE-BF(v) \(\triangleright\) It uses the globals of MAKE-BLOCK-FOREST
2: \hspace{1cm} NUMBER(v) ← TIMER
3: \hspace{1cm} LOWPT(v) ← TIMER
4: \hspace{1cm} TIMER ← TIMER + 1
5: \hspace{1cm} for u adjacent to v and u is not father of v in the dfs tree do
6: \hspace{1cm} if u not yet numbered then
7: \hspace{2cm} EDGES_PUSH((v, u)) \(\triangleright\) Proceed in the dfs.
8: \hspace{1cm}\hspace{1cm} RECURSIVE-MAKE-BF(u)
9: \hspace{1cm}\hspace{1cm} LOWPT(v) ← \(\text{min}\{\text{LOWPT}(v), \text{LOWPT}(u)\}\)
10: \hspace{1cm}\hspace{1cm} if LOWPT(u) ≥ NUMBER(v) then
11: \hspace{2cm}\hspace{1cm} BCC ← new round node
12: \hspace{2cm}\hspace{1cm} while EDGES_TOP = (a, b) and NUMBER(a) ≥ NUMBER(u) do
13: \hspace{3cm} add an edge to the block forest between a (square node) and BCC
14: \hspace{3cm} add an edge to the block forest between b (square node) and BCC
15: \hspace{2cm}\hspace{1cm} EDGES_POP()
16: \hspace{2cm}\hspace{1cm} end while
17: \hspace{2cm} add edges (u, BCC) and (v, BCC) to the block forest
18: \hspace{1cm} EDGES_POP() \(\triangleright\) Pop edge (u, v) from the stack
19: \hspace{1cm} end if
20: \hspace{1cm} else if NUMBER(u) < NUMBER(v) then
21: \hspace{2cm} EDGES_PUSH((v, u))
22: \hspace{2cm} LOWPT(v) ← \(\text{min}\{\text{LOWPT}(v), \text{NUMBER}(u)\}\)
23: \hspace{1cm} end if
24: \hspace{1cm} end for
25: end procedure
Algorithm 3: Computation of \textit{impact} values, given a BF data structure

\begin{enumerate}
\item \textbf{procedure} \textsc{recursive-compute-sq-size}(v) \hspace{1em} $\triangleright$ v is the vertex of the BF, whose \textsc{type} is either \texttt{round} or \texttt{square}
\item \textsc{sq-size}(v) $\leftarrow$ 0
\item for $u$ child of $v$ in the BF do
\item \quad \textsc{recursive-compute-sq-size}(u)
\item \quad \textsc{sq-size}(v) $\leftarrow$ \textsc{sq-size}(v) + \textsc{sq-size}(u)
\item \end for
\item if $v$.\textsc{type} = \texttt{square} then
\item \quad \textsc{sq-size}(v) $\leftarrow$ \textsc{sq-size}(v) + 1
\item \end if
\item \end procedure

\item \textbf{procedure} \textsc{compute-impact}(v) \hspace{1em} $\triangleright$ v is a square vertex of the BF
\item \textsc{max-cc} $\leftarrow$ (size of the graph CC containing $v$) - \textsc{sq-size}(v)
\item for $u$ (round) child of $v$ in the BF do
\item \quad \textsc{max-cc} $\leftarrow$ max\{\textsc{max-cc}, \textsc{sq-size}(v)\}
\item \end for
\item \textsc{impact}(v) $\leftarrow$ (size of the graph CC containing $v$) - \textsc{max-cc} - 1
\item \end procedure

\item \textbf{procedure} \textsc{compute-all-impacts}
\item for each connected component (tree) $C$ in the BF do
\item \quad call \textsc{recursive-compute-sq-size} on the root of $C$.
\item \end for
\item for each square vertex $v$ in the BF do
\item \quad \textsc{compute-impact}(v).
\item \end for
\item \end procedure
\end{enumerate}