A TRACE FORMULA APPROACH TO CONTROL THEOREMS FOR
OVERCONVERGENT AUTOMORPHIC FORMS

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Abstract. We present an approach to proving control theorems for overconvergent automorphic forms on Harris-Taylor unitary Shimura varieties based on a comparison between the rigid cohomology of the multiplicative ordinary locus and the rigid cohomology of the overlying Igusa tower, the latter which may be computed using the Harris-Taylor version of the Langlands-Kottwitz method.

1. Introduction

In [Col], Coleman proved that if $f$ is an overconvergent modular form of weight $k$ and tame level $\Gamma_1(N)$ which is an eigenform for $U_p$ with slope (i.e. the $p$-adic valuation of the eigenvalue) less than $k-1$, then $f$ is in fact a (classical) modular form of weight $k$ for the congruence subgroup $\Gamma_1(N) \cap \Gamma_0(p)$ using an analysis of the cohomology of the ordinary locus of the modular curve. This theorem was later reproved by Kassaei ([Kas]) using a very different geometric method inspired by work of Buzzard and Taylor on the Artin conjecture ([BT]). Since then this method has been generalised to more general Shimura varieties (for some of the most recent progress, see [Bij, PS1, PS2, Tia]), but no progress had been made on the cohomological method. In [Joh] we presented a cohomological argument essentially in the context of Hilbert modular varieties, combining the main conceptual ideas of Coleman with some slope calculations in rigid cohomology. Independently, Tian and Xiao ([TX]) gave a similar cohomological argument for Hilbert modular varieties, utilising geometric results on the Ekedahl-Oort stratification instead of slopes in rigid cohomology. In this paper we wish to present an alternative to these two arguments using the cohomology of Igusa varieties, illustrated in the context of some unitary Shimura varieties studied by Harris and Taylor in their proof of the Local Langlands correspondence.

Let us briefly sketch the argument of [Col], in a language closer to that of [Joh], [TX] and this paper. Let $Y$ be the (non-compact) modular curve of level $\Gamma_1(N)$ over $\mathbb{Z}_p$ $(p \nmid N)$ and let $\mathcal{E}_k$ be the local system $\text{Sym}^{k-2}(H^1(E/Y))$ in any appropriate cohomology theory where $E/Y$ is the universal elliptic curve. Coleman proved that the rigid cohomology group $H^1_{\text{rig}}(Y_{\text{ord}}^p, \mathcal{E}_k)$ is isomorphic to a quotient $M^1_{k}/\theta^{k-1}M^1_{2-k}$ where $M^1_{k}$ is the sheaf of weight $k$ overconvergent modular forms, $Y_{\text{ord}}^p$ is the ordinary locus, and further showed that if $f \in M^1_{k}$ is a $U_p$-eigenvector of slope less than $k-1$, then $f \notin \theta^{k-1}M^1_{2-k}$.

By comparing $H^1_{\text{rig}}(Y_{\text{ord}}^p, \mathcal{E}_k)$ to the parabolic cohomology $H^1_{\text{par}}(Y_{\text{ord}}^p, \mathcal{E}_k)$ and some dimension counting Coleman computed $H^1_{\text{rig}}(Y_{\text{ord}}^p, \mathcal{E}_k)$ in terms of classical modular forms and proved that such $f$ as in the previous sentence are classical. A key point here is that the small slope forms whose cohomology class comes from $H^1_{\text{par}}(Y_{\text{ord}}^p, \mathcal{E}_k)$ makes up "most" of $H^1_{\text{rig}}(Y_{\text{ord}}^p, \mathcal{E}_k)$.
and the contribution from the classical forms not of this form may be quantified and shown to exhaust the rest of $H^1_{\text{rig}}(Y^\ord_{F_p}, \mathcal{E}_k)$.

The isomorphism $H^1_{\text{rig}}(Y^\ord_{F_p}, \mathcal{E}_k) \cong M^1_k/\theta^{k-1}M^1_{2-k}$ may be generalised to very general PEL Shimura varieties (and probably Hodge/abelian type as well, if one has a suitable definition of the ordinary locus) using Faltings’s dual BGG complex; the argument presented in [Joh] readily generalises as long as the basic ingredients are present. It is also insensitive to whether one works with the Shimura variety with full level or with Iwahori level at $p$ (the former is the case covered in [Joh], the latter is covered in this paper). Similarly the observation that if $f$ has small slope then $f \notin \theta^{k-1}M^1_{2-k}$ also generalises to the same generality, using the observation in [Joh] that this is a consequence of the fact that the central character remains constant throughout the dual BGG complex. These results, though proven in slightly different ways in the cases at hand, are common to [Joh], [TX] and this paper and form the first half of the argument. It remains to carry out the analogue of computing $H^1_{\text{rig}}(Y^\ord_{F_p}, \mathcal{E}_k)$. In [Joh], we only computed a part of this group, which sufficed to deduce a control theorem; in [TX] the whole group was computed using the geometric results mentioned above.

In this paper we propose that the Euler characteristic

$$\sum (-1)^{i+1} H^i_{\text{rig}}(Y^\ord_{F_p}, \mathcal{E}_k)$$

in terms of classical automorphic representations using the Igusa tower above $Y^\ord_{F_p}$ and a comparison of the Lefschetz trace formula in rigid cohomology and the Arthur-Selberg trace formula. We will be working in the context of the book [HT] in the special case when the totally real field is $\mathbb{Q}$; let us now briefly recall this setting (for precise definitions see §2). We let $G$ be a unitary group over $\mathbb{Q}$ coming from a division algebra $B$ over an imaginary quadratic field $F$ satisfying a list of conditions, the most important being that $G(\mathbb{R}) \cong GU(n-1,1)$ and that for our fixed prime $p$, $G(\mathbb{Q}_p) \cong GL_n(\mathbb{Q}_p) \times \mathbb{Q}_p$. Associated with these groups are Shimura varieties $X_U$ ($U$ compact open subgroup of $G(\mathbb{A}^{\infty})$) with reflex field $F$; when $U$ is of a certain type at $p$ the $X_U$ have proper integral models that were extensively studied by [HT] and [TX]. We will mainly be concerned with the case when the level at $p$ is Iwahori; let us call this open compact $I_w = U^p I_{w,p}$. The integral models of $X_{I_w}$ have strictly semistable reduction; the special fibre $Y_{I_w}$ has an ordinary-multiplicative locus $Y^0_{I_w,1}$ we will be interested in together with its tubular neighbourhood $X^\ord_{I_w}$ inside the analytification of $X_{I_w}$. Our Shimura varieties have overconvergent $F$-isocrystals $V^! (\xi)$ associated with algebraic representations $\xi$ of $G$ as well as coherent sheaves $W(\mu)$ associated with algebraic representations $\mu$ of the parabolic subgroup $Q$ of $G_C$ attached to the Shimura datum; overconvergent automorphic forms are defined as overconvergent sections of the $W(\mu)$ on $X^\ord_{I_w}$.

After carrying out the program sketched above, one arrives with an overconvergent automorphic form of small slope appearing in $H^d_{\text{rig}}(Y^0_{I_w,1}, V^! (\xi))$, where $d = \dim X_{I_w}$. In fact, we make a slight refinement of the arguments to make sure that it appears $\sum (-1)^{d+i} H^i_{\text{rig}}(Y^0_{I_w,1}, V^! (\xi))$. On top of $V^0_{I_w,1}$ (as well the other strata of $Y_{I_w}$) Harris-Taylor and Taylor-Yoshida constructed the tower of Igusa varieties of the first kind; a tower of finite Galois covers $(Ig_{U^p,m})_{m \geq 1} \to Ig_{I_w} \cong Y^0_{I_w,1}$, and computed the corresponding Euler characteristics $\sum (-1)^{d+i} H^i_{\text{et}}(Ig_{I_w}, V(\xi))$ in terms of $\sum (-1)^{d+i} H^i_{\text{dR}}(X_{I_w}, V(\xi))$; this is one of the main technical results of [HT] (Theorem V.5.4), known as the “second basic identity”. We use a comparison of Lefschetz trace formulas in étale and rigid cohomology to transfer this result to rigid cohomology. The comparison between $\sum (-1)^{d+i} H^i_{\text{et}}(Ig_{I_w}, V(\xi))$ and $\sum (-1)^{d+i} H^i_{\text{dR}}(X_{I_w}, V(\xi))$ is straightforward for Hecke operators at primes away from $p$; the remaining work is to work out what happens at $p$. In the end, one arrives at the following theorem, which is the main result of this paper (part 2) of Theorem 17):
Theorem 1. Let $f \in H^0(X^{rig}_{Iw}, W^i(k_1, ..., k_n, w))$ be a simultaneous eigenvector for $A[U_p]$ of $U_p$-slope less than

$$-\frac{w + k_1 + ... + k_{n-2} + k_n - k_{n-1} + n}{2} - (n - 1).$$

Then $f$ is classical, i.e. $f$ occurs inside the virtual $A[U_p]$-module $(\pi^\infty)^{Iw}$ for some automorphic representation $\pi$ of $G(k)$. Here $U_p$ is a Hecke operator at $p$ generalising the classical $U_p$ for $GL_2$ (see §3.3 for the definition) and $A$ is an algebra of Hecke operators commuting with $U_p$.

Here we use tuples of integers $(k_1 \geq ... \geq k_{n-1}, k_n, w)$ as dominant weights to classify the algebraic representations $\mu$ of the Levi of $P$. The main advantages of the strategy employed in this paper compared that of [Joh] is that one gets stronger results, one can work at the Iwahori level as opposed to full level at $p$, and although one needs an integral model, one needs to know very little about it apart from the properness of the whole model and smoothness and moduli interpretation of the ordinary locus (note that we do not use the moduli interpretation of the rest of the special fibre). The second point is important, because the Iwahori level is the right one for constructing the eigenvariety ($[AIP]$). In [Joh] this was not a problem, as in that setting the canonical subgroup allows one to pass from full level to Iwahori level, but in general (e.g. in our setting here for $n \geq 3$) the canonical subgroup only takes one to some parahoric (non-Iwahori) level. Thus any control theorem proved on the full level Shimura variety needs to be complemented with descent results before it can be applied to the eigenvariety. The main disadvantage is of course that it relies heavily on the technical results on the cohomology of the Igusa tower proved in [HT]. The construction of the Igusa tower has been generalised to the unramified PEL (type $A$ and $C$) setting by Mantovan ([Man]) and their points have been counted and the resulting formula stabilised by Shin ([Shi1, Shi2]). It remains however to compare the resulting formula to the Arthur-Selberg trace formula in full generality. We should remark though that many compact cases with “no endoscopy” were done in [Shi4] and some compact cases with endoscopy were done in [Shi3]; we would expect that our methods generalise directly to these cases.

Let us now outline the contents of the papers. Section 2 is devoted to setting up the definitions of the groups, Shimura varieties and the Igusa varieties that we will be working with. In §3 we discuss the relation between our Shimura varieties and Igusa varieties as towers with the action of adelic groups and define the automorphic sheaves and the Hecke algebras that we will be working with, and record a few relations. Finally, the computation of the cohomology is carried out in §4, first from the point of view of overconvergent automorphic forms and then in terms of classical automorphic representations. The comparison between the two viewpoints gives the main theorem.

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2. Groups and Shimura Varieties

2.1. The groups. The main reference for this section is [HT] §I.7. We will, to the best of our ability, follow the notation and terminology set up there. Let \( u \geq 2 \) be a fixed integer, \( p \) a fixed rational prime and \( F \) an imaginary quadratic field in which \( p \) splits as \( p = u \omega_c \), where \(-c\) denotes the nontrivial automorphism of \( F \) (complex conjugation). We will let \( B \) be a division algebra with centre \( F \) such that

- \( \dim_F B = n^2 \)
- \( B^{\text{op}} \cong B \otimes F, c \)
- \( B \) is split at \( u \).
- at any non-split place \( x \) of \( F \) which is not split, \( B_x \) is split
- at any split place \( x \) of \( F \) either \( B_x \) is split or \( B_x \) is a division algebra
- if \( n \) is even, then \( 1 + n/2 \) is congruent modulo 2 to the number of places of \( Q \) above which \( B \) is ramified

We will write \( \det_{B/F} \) resp. \( \text{tr}_{B/F} \) for the reduced norm resp. trace of \( B/F \), and define \( \det_{B/Q} = \det_{F/Q} \circ \det_{B/F} \) and \( \text{tr}_{B/Q} = \text{tr}_{F/Q} \circ \text{tr}_{B/F} \). Pick an involution \( * \) of the second kind on \( B \), which we assume is positive (i.e. \( \text{tr}_{B/Q}(xx^*) > 0 \) for all nonzero \( x \in B \)). Pick \( \beta \in B^{*,-1} \); the pairing

\[
(x, y) = \text{tr}_{B/Q}(x \beta y^*)
\]

is an alternating \(*\)-Hermitian pairing \( B \times B \to Q \), and we define a new involution of the second kind by \( x^# = \beta x^* \beta^{-1} \). We have that

\[
(bxc, y) = \text{tr}_{B/Q}(bxc \beta y^*) = \text{tr}_{B/Q}(xc \beta y^* b) = \text{tr}_{B/Q}(x \beta (\beta^{-1} c \beta) y^* b) = (x, ((\beta^{-1} c \beta) y^* b)^*) = (x, b^* yc^#)
\]

as \( (\beta^{-1})^* = (\beta^*)^{-1} = -\beta^{-1} \). We let \( G/Q \) be algebraic group whose \( R \)-points (\( R \) any \( Q \)-algebra) are

\[
G(R) = \{(g,\lambda) \in (B^{\text{op}} \otimes Q R)^* \times R^* \mid g g^# = \lambda \}
\]

When \( x \) is place of \( Q \) that splits as \( y f^c \) in \( F \), \( y \) induces an isomorphism

\[
G(Q_x) \cong (B^{\text{op}} \otimes F_y)^* \times Q_x^\
\]

In particular, we get an isomorphism

\[
G(Q_p) = (B^{\text{op}}_u)^* \times Q_p^\
\]

where we have written \( B^{\text{op}}_u = B^{\text{op}} \otimes F u \).

We assume that (see Lemma I.7.1 of [HT])

- if \( x \) is a rational prime which does not split in \( F \), then \( G \) is quasi-split at \( x \)
- the pairing \((-,-)\) on \( B \otimes Q \mathbb{R} \) has invariants \((1, n - 1)\).

We will fix a maximal order \( \Lambda_u = \mathcal{O}_{B_u} \) in \( B_u \). The pairing \((-,-)\) gives a perfect duality between \( B_u \) and \( B_{u^*} \) and we define \( \Lambda_u^* \subseteq B_{u^*^*} \) to be the dual of \( \Lambda_u \subseteq B_u \). Then

\[
\Lambda = \Lambda_u \oplus \Lambda_u^* \subseteq B \otimes Q_p
\]

is a \( \mathbb{Z}_p \)-lattice in \( B \otimes Q \mathbb{Q}_p \) and \((-,-)\) restricts to a perfect pairing \( \Lambda \times \Lambda \to \mathbb{Z}_p \). There is a unique \( \mathbb{Z}_p \)-order \( \mathcal{O}_B \) of \( B \) such that \( \mathcal{O}_B^* = \mathcal{O}_B \) and \( \mathcal{O}_{B,u} = \mathcal{O}_{B,u} \) (where \( \mathcal{O}_{B,u} = \mathcal{O}_B \otimes_{\mathcal{O}_{F,(u)}} \mathcal{O}_{F_u}, \mathcal{O}_{F,(u)} \) being the algebraic localization of \( \mathcal{O}_F \) at \( u \)). The stabilizer of \( \Lambda \) in \( G(Q_p) \) is \( (\mathcal{O}_{B,u}^{\text{op}})^* \times \mathbb{Z}_p^* \). Fix an isomorphism \( \mathcal{O}_{B,u} \cong M_n(\mathcal{O}_{F_u}) = M_n(\mathbb{Z}_p) \). By composing it with the transpose map \(-^t \) we
get an isomorphism $\mathcal{O}^{op}_{B,u} \cong M_n(\mathbb{Z}_p)$. If we let $\epsilon \in M_n(\mathbb{Z})$ denote the idempotent whose $(i,j)$th entry is 1 if $i = j = 1$ and 0 otherwise, as well as the corresponding idempotent in $\mathcal{O}^{op}_{B,u}$ (using our isomorphism), we get an isomorphism

$$\epsilon A_u \cong (\mathcal{O}^{op}_{F_u})^\vee$$

We give $\epsilon A_u$ the basis $e_1,...,e_n$ corresponding to the standard dual basis of $(\mathcal{O}^{op}_{F_u})^\vee$. The induced isomorphism $B^{op}_u \cong M_n(F_u) = M_n(\mathbb{Q}_p)$ gives us an isomorphism

$$G(\mathbb{Q}_p) \cong \text{GL}_n(F_u) \times \mathbb{Q}_p^\times = \text{GL}_n(\mathbb{Q}_p) \times \mathbb{Q}_p^\times$$

### 2.2. Shimura varieties and integral models.

The main reference for this section is §III of [HT]. Define

$$C = (B^{op} \otimes \mathbb{Q}) \times \mathbb{R}$$

Then there is a unique conjugacy class $\mathcal{S}$ of $\mathbb{R}$-algebra homomorphisms

$$h : C \rightarrow C$$

such that $h(\bar{z}) = h(z)^*$ and the pairing $u,v \mapsto (u,v,h(i))$ is positive or negative definite and symmetric. The pair $(G, \mathcal{S})$ is Shimura datum whose reflex field is $F$ with respect to the unique infinite place of $F$. For any compact open subgroup $U \subseteq G(\mathbb{A}^\infty)$ let us write $Sh_U$ for the canonical model of the Shimura variety

$$G(\mathbb{Q}) \backslash G(\mathbb{A}^\infty) \times \mathcal{S} / U$$

Following [HT] §III.1, we define a contravariant functor $X_U$ from locally Noetherian schemes over $F$ to sets by letting $X_U(S)$, for any connected locally Noetherian $F$-scheme $S$, be the set of equivalence classes of quadruples $(A, \lambda, i, \bar{\eta})$ where

- $A$ is an abelian scheme over $S$ of dimension $n^2$
- $\lambda : A \rightarrow A^\vee$ is a polarisation
- $i : B \hookrightarrow \text{End}_S(A) \otimes \mathbb{Z}_p \mathbb{Q}$ is an algebra homomorphism such that $(A, i)$ is compatible ([HT] p. 90) and $\lambda \circ i(b) = i(b^\ast)^\vee \circ \lambda$ for all $b \in D$
- $\bar{\eta}$ is a $U$-level structure on $A$

Two quadruples are equivalent if there is a quasi-isogeny between the corresponding abelian schemes preserving the structure. See [HT] p. 91 for more details. When $U$ is neat, $X_U$ is representable by a smooth projective variety over $F$, which we will also denote by $X_U$, and above it we have an abelian scheme $A_U$ fitting in a universal quadruple $(A_U, \lambda_U, i_U, \bar{\eta}_U)$. We have

$$(2.1) \quad X_U = Sh_U$$

because the relevant $\ker(\mathbb{Q}, G)$ that complicates the situation in general vanishes since $F^+ = \mathbb{Q}$. By varying $U$, we get a projective system $(X_U)_U$ of varieties where the morphisms between are finite, surjective and etale and similarly for the $A_U$.

Next we recall the integral models defined in §III.4 of [HT]. Put, for $m \in \mathbb{Z}_{\geq 0}$,

$$U_p(m) = K(m) \times \mathbb{Z}_p^\times \subseteq \text{GL}_n(\mathbb{Q}_p) \times \mathbb{Q}_p^\times = G(\mathbb{Q}_p)$$

where $K(m) \subseteq \text{GL}_n(\mathbb{Z}_p)$ is the subgroup of matrices reducing to the identity modulo $p^m$. For $U^p \subseteq G(\mathbb{A}^{p,\infty})$, we denote the product $U^p \times U_p(m) \subseteq G(\mathbb{A}^{\infty})$ by $U(m)$. We will denote $U(0)$ by $U$. Define a contravariant functor $X_{U(m)}$ from locally Noetherian $\mathbb{Z}_p$-schemes to sets by letting $X_{U(m)}(S)$, for any connected locally Noetherian $\mathbb{Z}_p$-scheme $S$, be the set of equivalence classes of quintuples $(A, \lambda, i, \bar{\eta}, \alpha)$ where
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- $A/S$ is an abelian scheme of dimension $n^2$;
- $\lambda : A \rightarrow A^\vee$ is a prime-to-$p$ polarisation;
- $i : \mathcal{O}_B \rightarrow \text{End}_S(A) \otimes \mathbb{Z}_p(\varphi)$ is an algebra homomorphism such that $(A, i)$ is compatible and $\lambda \circ i(b) = i(b^*)^\vee \circ \lambda$ for all $b \in \mathcal{O}_B$;
- $\tilde{\eta}^p$ is a $U^p$-level structure on $A$;
- $\alpha : p^{-m}\epsilon A_u/\epsilon A_u \rightarrow \mathcal{G}_A[p^m](S) = \epsilon A[u^m](S)$ is a Drinfeld $p^m$-level structure.

Equivalence here is as before, but using prime-to-$p$ quasi-isogenies. See [HT] p. 109 for more details and the proof of the representability of $\mathcal{X}_{U(m)}$ for $U(m)$ neat. We will also denote the representing scheme over $\mathbb{Z}_p = \mathcal{O}_{F_u}$ by $\mathcal{X}_{U(m)}$. Its generic fibre is $X_{U(m)}/\mathbb{Q}_p$ and we will denote its special fibre by $\mathcal{X}_{U(m)}$. Similarly we will let $\mathcal{A}_{U(m)}$ denote the universal abelian variety over $\mathcal{X}_{U(m)}$; its generic fibre is $A_{U(m)}/\mathbb{Q}_p$ and its special fibre will be denoted by $\mathcal{X}_{U(m)}$. We will let $\mathcal{G}$ denote the Barsotti-Tate group $\mathcal{G}_{A_{U(m)}[\mu^\infty]}$; over any base in which $p$ is nilpotent it is one-dimensional and compatible of height $n$ (see [HT] §II for terminology). $\mathcal{G}$ defines a stratification $(\mathcal{X}_{U(m)}, \{\mathcal{G}(h)\})_{0 \leq h \leq n-1}$ of $\mathcal{X}_{U(m)}$ where $\mathcal{X}_{U(m),[h]}$ is the closed reduced subscheme of $\mathcal{X}_{U(m)}$ whose geometric points $s$ are those for which the maximal etale quotient of $\mathcal{G}_s$ has height $\leq h$. In the case $m = 0$, each $\mathcal{X}_{U,[h]}$ is smooth (see remark near the end of p. 113 of [HT]; we will not need this). We let $\mathcal{X}_{U(m),[h]} - \mathcal{X}_{U(m),[h-1]}$ for $h \geq 1$ and $\mathcal{X}_{U(m),[0]} - \mathcal{X}_{U(m)[0]}$; these are smooth of pure dimension $h$ for arbitrary $m$ (HT Corollary III.4.4). The stratification $\mathcal{X}_{U(m)} = \bigsqcup_{h=0}^{n-1} \mathcal{X}_{U(m),[h]}$ is the Newton stratification of $\mathcal{X}_{U(m)}$. Denote by $\mathcal{G}(h)$ the restriction of $\mathcal{G}$ to $\mathcal{X}_{U(m),[h]}$. All these varieties etc. fit into projective systems as we vary $U^p$ and $m$, with transition maps being finite, flat and surjective. If we keep $m$ fixed and vary $U^p$ the maps are etale as well.

2.3. Semistable integral models of Iwahori level. The reference for this section is [TY] §3. Let $B_n$ denote the standard upper triangular Borel subgroup of $\text{GL}_n$ and let $I_n$ denote the corresponding Iwahori subgroup of $\text{GL}_n(\mathbb{Z}_p)$ defined as the pullback of $B_n(k_u) = B_n(\mathbb{F}_p)$ under the reduction map $\text{GL}_n(\mathbb{Z}_p) \rightarrow \text{GL}_n(\mathbb{F}_p)$ (here $k_u$ denotes the residue field of $F_u$). We set, for any compact open subgroup $U^p \subseteq G(A^{p,\infty})$,

$$I_{w_p} = I_n \times \mathbb{Z}_p^X \subseteq \text{GL}_n(\mathbb{Q}_p) \times \mathbb{Q}_p^X = G(\mathbb{Q}_p)$$

$$I_w = U^p \times I_{w_p} \subseteq G(A^\infty)$$

Define a contravariant functor $X_{I_w}$ from locally Noetherian $\mathbb{Z}_p$-schemes to sets by letting $X_{I_w}(S)$, for any connected locally Noetherian $\mathbb{Z}_p$-scheme $S$, be the set of equivalence classes of quintuples $(A, \lambda, i, \tilde{\eta}^p, C)$ where $(A, \lambda, i, \tilde{\eta}^p)$ is as for $A_I$ and $C$ is a chain of isogenies $(\mathcal{G}_A = \epsilon A[\mu^\infty])$:

$$\mathcal{G}_A = \mathcal{G}_0 \rightarrow \mathcal{G}_1 \rightarrow ... \rightarrow \mathcal{G}_n = \mathcal{G}_A/\mathcal{G}_A[p]$$

of compatible Barsotti-Tate groups, each of degree $p = \#k_u$, with composite equal to the canonical map $\mathcal{G}_A \rightarrow \mathcal{G}_A/\mathcal{G}_A[p]$. Equivalently, we may view it as a flag

$$0 = C_0 \subseteq C_1 \subseteq ... \subseteq C_n = \mathcal{G}_A[p]$$

of finite flat subgroup schemes, with each successive quotient of degree $p$. $X_{I_w}$ is representable by a scheme over $\mathbb{Z}_p$ which we will also denote by $X_{I_w}$; its generic fibre is $X_{I_w}/\mathbb{Q}_p$ and we will denote its special fibre by $Y_{I_w}$. We let $A_{I_w}$ denote the universal abelian variety over $X_{I_w}$; its generic fibre is $A_{I_w}/\mathbb{Q}_p$ and we denote by $\mathcal{A}_{I_w}$ its special fibre. Furthermore, we let

$$\mathcal{G} = \mathcal{G}_0 \rightarrow \mathcal{G}_1 \rightarrow ... \rightarrow \mathcal{G}_n$$
denote the universal chain of isogenies on $X_{Iw}$, and let $Y_{Iw,i}$ denote the closed subscheme of $Y_{Iw}$ over which $G_{i-1} \rightarrow G_i$ has connected kernel. Here $G = cA_{Iw}[u^\infty]$ by abuse of notation; since this $G$ is the pullback of the previous $G$ via the natural map $X_{Iw} \rightarrow X_U$, we hope that this should not cause any confusion. We have the following:

**Proposition 2.** ([TY] Proposition 3.4)

1) $X_{Iw}$ is regular, has pure dimension $n$ and the natural map $X_{Iw} \rightarrow X_U$ is finite and flat.

2) Each $Y_{Iw,i}$ is smooth over $\mathbb{F}_p$ of pure dimension $n - 1$, $Y_{Iw} = \bigcup_i Y_{Iw,i}$ and for $i \neq j$ $Y_{Iw,i}$ and $Y_{Iw,j}$ have no common connected component. In particular, $X_{Iw}$ has strictly semistable reduction and for each $S \subseteq \{1, ..., h\}$, $Y_{Iw,S} = \bigcap_{i \in S} Y_{Iw,i}$ is smooth of pure dimension $n - \# S$.

We write

$$Y_{0, Iw,S} = Y_{Iw,S} - \bigcup_{T \supseteq S} Y_{Iw,T}$$

and we let $\overline{\mathcal{T}}_{Iw,S}$ and $\mathcal{G}_S$ (resp. $\overline{\mathcal{T}}_{Iw,S,0}$ and $\mathcal{G}^0_S$) denote the restriction of $\overline{\mathcal{T}}_{Iw}$ resp. $\mathcal{G}$ to $Y_{Iw,S}$ (resp. $Y^0_{Iw,S}$). When $S$ is singleton $\{i\}$ we will write the subscript as $i$ instead of $\{i\}$. All varieties defined here fit into projective systems as we vary $U^p$, and the transition maps are finite, etale and surjective. Finally, note that under the natural map $Y_{Iw} \rightarrow \overline{X}_U$, $\bigcup_{S = n - k} Y^0_{Iw,S}$ is the inverse image of $\overline{X}_{U(h)}$, in particular we have maps $Y^0_{Iw,S} \rightarrow \overline{X}_{U(n - \# S)}$ for all $S$. We also have a forgetful map $X_{U(1)} \rightarrow X_{Iw}$. It is given by sending a quintuple $(A, \lambda, i, \tilde{\eta}, \alpha)$ to the quintuple $(A, \lambda, i, \tilde{\eta}, C)$, where $C$ is the flag of subgroup schemes given by letting $C_i$ be the unique finite flat subgroup scheme of $G_u[u]$ that has $a(M_i)$ is a full set of sections of $C$, where $M_i$ is the subspace of $p^{-1}cA_u/cA_u$ generated by $p^{-1}c_1, ..., p^{-1}c_i$ (see [HT] Lemma II.2.4).

### 2.4. Igusa varieties of the first kind.

The main reference for this section is [HT] Chapter 4. Let $s = \text{Spec } \mathbb{F}_p$ be a geometric point of $\overline{X}_{U(n-1)}$. As a Barsotti-Tate $\mathcal{O}_{B,p} = \mathcal{O}_B \otimes_{\mathbb{Z}(p)} \mathbb{Z}_p$-module, $\overline{\mathcal{A}}_{U(n-1)|p^\infty}_s$ decomposes as

$$\overline{\mathcal{A}}_{U(n-1)|p^\infty}_s \times \overline{\mathcal{A}}_{U(n-1)}[u^\infty]_s \times \overline{\mathcal{A}}_{U(n-1)}[(u^c)^\infty]_s$$

with $\overline{\mathcal{A}}_{U(n-1)|p^\infty}_s = (\overline{\mathcal{A}}_{U(n-1)}[u^\infty]_s)^\vee$ (Cartier dual), $\overline{\mathcal{A}}_{U(n-1)|p^\infty}_s = (G^{(n-1)})^n$ as Barsotti-Tate $\mathbb{Z}_p$-modules and $G^{(n-1)}_u \cong (\mathbb{Q}/\mathbb{Z})_p[n-1]$ as Barsotti-Tate groups. The Igusa variety of the first kind $Ig_{U^p,m}/\overline{X}_{U(n-1)}$ is defined to be the moduli space for isomorphisms

$$\alpha^{et} : (p^{-m}\mathbb{Z}_p/\mathbb{Z}_p)^{n-1} \cong \mathcal{G}^{(n-1)}[u^m]$$

See [HT] p.121. The forgetful morphism $Ig_{U^p,m}/\overline{X}_{U(n-1)/k}$ is a Galois cover with Galois group $\text{GL}_{n-1}(\mathcal{O}_{F_c}/u^m) = \text{GL}_{n-1}(\mathbb{Z}/p^m)$. We think of $\text{GL}_{n-1}$ as a factor of the Levi subgroup $GL_1 \times GL_{n-1}$ of $GL_n$ of block diagonal matrices with square blocks of side lengths 1 and $n - 1$. The $Ig_{U^p,m}$ fit together in a projective system when varying $U^p$ and $m$, with the transition maps being finite and etale.

We also need to recall (a special case of) the Iwahori-Igusa variety of the first kind $Ig_{Iw}$ defined in the beginning of §4 of [TY] (there denoted by $I_{U(n-1)}^{(n-1)}$). It is defined as the moduli space of chains of isogenies

$$\mathcal{G}^{(n-1),et} = G_1 \rightarrow G_2 \rightarrow ... \rightarrow G_n = \mathcal{G}^{(n-1),et}/\mathcal{G}^{(n-1),et}[u]$$
over $\mathcal{X}_{U,(n-1)}$. The natural map $Ig_{Iw} \to \mathcal{X}_{U,(n-1)}$ is finite and etale, and the natural map $Ig_{U^r,1} \to Ig_{Iw}$ is Galois with Galois group $B_{n-1}(\mathbb{F}_p)$. Moreover, there is a natural map $Y_{Iw,1}^0 \to Ig_{Iw}$ defined by taking etale quotients in the chain of isogenies on $Y_{Iw,1}^0$; by Lemma 4.1 of [TY] it is a finite map that is bijective on geometric points. In fact it is an isomorphism, the map in the other direction coming from augmenting the chain of isogenies
\[
\mathcal{G}(n-1), et = \mathcal{G}_1 \to \mathcal{G}_2 \to \ldots \to \mathcal{G}_n = \mathcal{G}(n-1), et / \mathcal{G}(n-1), et [w]
\]
with the $p$-th power Frobenius morphism $\mathcal{G}(n-1) = \mathcal{G}_0 \xrightarrow{F_p^{\alpha}} \mathcal{G}_1 = \mathcal{G}(n-1), (q) = \mathcal{G}(n-1), et$ (note that this is special to $Y_{Iw,1}^0$, it does not hold for $Y_{Iw,1}^i$ for $i \neq 0$). To summarize, we have that the Igusa varieties $Ig_{U^r,m}$ form a projective system of Galois covers of $Y_{Iw,1}^0$, and the Galois group of $Ig_{U^r,1} \to Y_{Iw,1}^0$ is $B_{n-1}(\mathbb{F}_p)$.

3. HECKE ACTIONS AND AUTOMORPHIC FORMS

In this section we will first relate our Igusa varieties to the Shimura varieties and then define automorphic vector bundles and overconvergent $F$-isocrystals on them. Finally, the last subsection is devoted to a discussion of the various Hecke algebras acting on each finite level and the commutative subalgebra (the Atkin-Lehner ring) that acts on automorphic forms and that we will use in our control theorems.

3.1. HECKE ACTIONS ON THE SHIMURA VARIETIES AND IGUSA VARIETIES. The action of $G(\mathbb{A}^{\infty})$ on the system $(\mathcal{X}_{U(m)})_{U^r,m}$ is described on p.109 of [HT]. On p.116 a decomposition
\[
\mathcal{X}_{U(m),h} = \prod_M \mathcal{X}_{U(m),M}
\]
coming from the Drinfeld level structure (see [HT] Lemma II.2.1(5) ) is defined. Here the $M$ range over free rank $n - h$ direct summands of $p^{-n} \epsilon \Delta_u / \epsilon \Gamma_u$. Now let $M$ be a free rank $n - h$ direct summand of $\epsilon \Delta_u$ and let $P_M \subseteq \text{Aut}(\epsilon \Delta_u)$ denote the parabolic subgroup associated with $M$. If we set $\mathcal{X}_{U(m),M} = \mathcal{X}_{U(m),p^{-n} \times M/M}$, then the $\mathcal{X}_{U(m),M}$ form a projective system with an action of the group
\[
G_M(\mathbb{A}^{\infty}) = G(\mathbb{A}^{p,\infty}) \times P_M(\mathbb{Q}_p) \times \mathbb{Q}_p^{\times}
\]
From now on, we fix $M = \langle \epsilon_1 \rangle \subseteq \epsilon \Delta_u$. If we look at the forgetful map $\mathcal{X}_{U(1)} \to Y_{Iw}$, we see that the preimage of $Y_{Iw,1}^0$ consists of those $(A, \lambda, \epsilon, \eta^p, \alpha)$ for which $\alpha(\epsilon^{-1} \epsilon_1) = 0$, i.e. the preimage is $\mathcal{X}_{U(1),M}$. Thus we see that $\mathcal{X}_{U(1),M} \to Y_{Iw,1}^0$ is Galois with Galois group $B_{n-1}(\mathbb{F}_p)$ and that the $(\mathcal{X}_{U(m),M})_{M \geq 1}$ form a tower above $Y_{Iw,1}^0$ (as well as above $\mathcal{X}_{U,(n-1)} = \mathcal{X}_{U,M}$).

Next let us consider Igusa varieties. We follow [HT] p. 122 ff (but note our choices of $h = n - 1$ and $M = \langle \epsilon_1 \rangle$). The projective system $Ig_{U^r,m}$ carries an action of
\[
G(\mathbb{A}^{p,\infty}) \times (\mathbb{Z} \times \text{GL}_{n-1}(\mathbb{Q}_p))^+ \times \mathbb{Q}_p^{\times}
\]
where $(\mathbb{Z} \times \text{GL}_{n-1}(\mathbb{Q}_p))^+ = \{ (c, g) \in \mathbb{Z} \times \text{GL}_{n-1}(\mathbb{Q}_p) \mid p^{-n} g \in M_{n-1}(\mathbb{Z}_p) \}$. We denote by $j$ the surjection $\epsilon \Delta_u \to \mathbb{Z}_p^{n-1}$ with kernel $M$ given by identifying $\mathbb{Z}_p^{n-1}$ with the direct summand $\langle \epsilon_2, \ldots, \epsilon_n \rangle$. Then define a homomorphism $j_\ast : P_M(\mathbb{Q}_p) \to \mathbb{Z} \times \text{GL}_{n-1}(\mathbb{Q}_p)$ by $g \mapsto (\text{ord}_p(g|M), j \circ g \circ j^{-1})$ which induces a homomorphism
\[
G_M(\mathbb{A}^{\infty}) \to G(\mathbb{A}^{p,\infty}) \times \mathbb{Z} \times \text{GL}_{n-1}(\mathbb{Q}_p) \times \mathbb{Q}_p^{\times}
\]
that we will also denote by $j_\ast$. There is a morphism

$$j^\ast : Ig_{U, m} \to X_{U(m), M}$$

defined on p.124 of [HT], given by sending $(A, \lambda, i, \bar{n}, \alpha, \tau)$ to $(A^{(p^m)}, \lambda^{(p^m)}, i, \bar{n}^{p}, \alpha)$, where $-(p^m)$ denote twisting by the $p^m$-power Frobenius $F^{p^m}$ and $\alpha = F^{p} \circ \alpha^{et} \circ j$. In fact, $j^\ast$ is an isomorphism.

The map $j^\ast F_{p^m}$ is therefore a finite purely inseparable morphism and for $g \in G_M(\mathbb{A}^\infty)$ such that $j_\ast(g) \in G(\mathbb{A}^{p, \infty}) \times (\mathbb{Z} \times \text{GL}_{n-1}(\mathbb{Q}_p))^+ \times \mathbb{Q}_p^\infty$, the diagram

$$(3.1)$$

\[
\begin{array}{ccc}
Ig_{U, m} & \xrightarrow{j_\ast(g)} & Ig_{U, m'} \\
\bigg| & \downarrow & \bigg| \\
X_{U(m), M} & \xrightarrow{g} & X_{U(m'), M}
\end{array}
\]

commutes (where $m$ and $m'$ are chosen so that one may define $g$ and $j_\ast(g)$). It follows that this action extends to an action of $G(\mathbb{A}^{p, \infty}) \times \mathbb{Z} \times \text{GL}_{n-1}(\mathbb{Q}_p) \times \mathbb{Q}_p^\infty$ on cohomology.

3.2. Representations, motives and sheaves. We wish to define various sheaves on our varieties that we want to work with. Before we can do so we need to recall some basic facts about the finite dimensional algebraic representations of our group $G$. We have $G(\mathbb{R}) = GU(n - 1, 1)$. As is well known, the finite dimensional representations of $\text{GL}_n(\mathbb{C})$ are parametrized by integers $k_1 \geq \ldots \geq k_n$, which are dominant weights for the diagonal torus $T_n$ with respect to the upper triangular Borel $B_n$. We parametrize complex representations of $GU(n - 1, 1)$ by $n + 1$-tuples of integers $(k_1, \ldots, k_n, w)$ where $k_1 \geq \ldots \geq k_n$ parametrises an irreducible representation of $U(n - 1, 1)$ and $w \equiv \sum k_i \mod 2$ parametrizes a character of the split part of the center of $GU(1, n - 1)$.

Algebraic representations $\xi$ of $GU(n - 1, 1)$ define sheaves $V(\xi)$ in various cohomology theories (etale, de Rham or singular on the generic fibre, and rigid cohomology on the special fibre); see §III.2 of [HT]. For us it will be convenient to define these as realizations of motives, defined using self-products of the universal abelian variety together with an idempotent. Let us recall the construction from §3.2 of [HT] and §2 of [TY]. Given $\xi$, there are integers $t_\xi$ and $m_\xi$ and an idempotent $a_\xi$ such that

- $a_\xi R^{m_\xi} \pi_{m_\xi} \mathbb{Q}_p(t_\xi) \cong V(\xi)$;
- $a_\xi H^j(A, \mathbb{Q}_p(t_\xi)) = H^{j-m_\xi}(X, V(\xi))$ for $j \geq m_\xi$ and 0 otherwise.

for any of the above mentioned cohomology theories, where we have used $A$ for the universal abelian variety, $X$ for some Shimura variety, $\mathbb{Q}_p$ the constant sheaf in the relevant cohomology theory and $\pi_{m_\xi} : A^{m_\xi} \to X$ denotes the projection. $(A^{m_\xi}, a_\xi, t_\xi)$ defines a classical motive (see e.g. [Sch]). Moreover $a_\xi$ commutes with the action of $G(\mathbb{A}^\infty)$.

We will be interested in the overconvergent $F$-isocrystals $V^+ = a_\xi R^{m_\xi} \pi_{m_\xi}(\hat{A}_{Iw, 1, 0}/Y_{Iw, 1}^0)$ on $Y_{Iw, 1}^0$. Using the frame ([LeSt] Definition 3.1.5)

$$Y_{Iw, 1}^0 \subseteq Y_{Iw} \subseteq \hat{X}_{Iw}$$

(the hat for the completion along the special fibre) we may construct the underlying overconvergent isocrystal by considering the de Rham realisation of $V(\xi)$ on $X_{Iw}/\mathbb{Q}_p$; analytifying it to the Raynaud generic fibre $X_{Iw}^{rig}$ of $\hat{X}_{Iw}$ (which agrees with the Tate analytification of $X_{Iw}/\mathbb{Q}_p$ by properness) and
finally applying Berthelot’s $\mathfrak{J}_{I_w}^\dagger$-functor ([LeSt] §5.1; it is probably easiest to use Proposition 5.1.12 as the definition).

Next, consider the parabolic $Q = Q_{n-1,1} \times \mathbb{C}^\times$ of $G(\mathbb{C}) \cong \text{GL}_n(\mathbb{C}) \times \mathbb{C}^\times$ where $Q_{n-1,1}$ is the standard $(n - 1, 1)$ parabolic of $\text{GL}_n(\mathbb{C})$ with last row of the form $(0 \ldots 0 \ast)$. Weights $(k_1, \ldots, k_n, w)$ for $G$ are dominant for the Levi $L_Q$ of $Q$ with respect to $B_n \cap L_Q$ if and only if $k_1 \geq \ldots \geq k_{n-1}$. The automorphic vector bundle construction of Mihăilescu and Harris ([Mil], see also [HT] p.101 for the construction over the complexes) produces out of finite-dimensional algebraic representations $\mu$ of $Q$ vector bundles $W(\mu)$ on $X_{I_w}(\mathbb{C})$ that descend to a number field; in fact we may base change them to $\mathbb{Q}_p$ (this uses that $G$ is split over $\mathbb{Q}_p$ and that $p$ is totally split in $F$). We define $W^+(\mu) = \mathfrak{J}_{I_w}^\dagger W(\mu)$ using the frame above. If $sp : X_{I_w}^{rig} \to Y_{I_w}$ denotes the specialization map, let us put $X_{I_w}^{rig,d} = sp^{-1}(Y_{I_w,0})$. We remark that this is not the full ordinary locus in $X_{I_w}^{rig}$, but rather what could be called the ordinary-multiplicative locus.

Definition 3. 1) An automorphic form of weight $\mu$, tame level $U^p$ and Iwahori level at $p$ is an element of $H^0(X_{I_w}^{rig}, W(\mu))$.

2) An overconvergent automorphic form of weight $\mu$ and tame level $U^p$ is an element of $H^0(X_{I_w}^{rig}, W^+(\mu))$.

3) A $p$-adic automorphic form of weight $\mu$ and tame level $U^p$ is an element of $H^0(X_{I_w}^{ord}, W(\mu))$.

Remark 4. We will usually suppress any mention of the tame level and the level at $p$. The tame level is assumed to be fixed throughout this paper; it plays no explicit role.

We have natural inclusions $H^0(X_{I_w}^{rig}, W(\mu)) \subseteq H^0(X_{I_w}^{rig}, W^+(\mu)) \subseteq H^0(X_{I_w}^{ord}, W(\mu))$.

3.3. Hecke algebras. In this section we will define the Hecke algebras that we will be working with in the next section and recall some results from the theory of smooth admissible representations of $p$-adic groups. For any compact open subgroup $V = U^pV_p \subseteq G(\mathbb{A}^\infty)$, we will let $\mathcal{H}_V$ denote the full Hecke algebra of smooth, $\mathbb{Q}$-valued compactly supported bi-$V$-invariant function on $G(\mathbb{A}^\infty)$ (generated over $\mathbb{Q}$ by the double cosets $VgV$, $g \in G(\mathbb{A}^\infty)$ via their characteristic functions), and we will similarly let $\mathcal{H}_{U^p}$ denote the full Hecke algebra away from $p$ (which is independent of $V$). Throughout the rest of the paper $\mathcal{H}^p \subseteq \mathcal{H}_{U^p}$ will denote a fixed commutative subalgebra, assumed to the full spherical Hecke algebra at all places for which $U^p$ is a hyperspecial maximal compact subgroup.

Now consider $V = I_w$, corresponding to the compact open $I_wU_p = I_n \times \mathbb{Z}_p^\times \subseteq \text{GL}_n(\mathbb{Q}_p) \times \mathbb{Q}_p^\times$. Let $\mathcal{H}_{I_w,p}$ denote the full Hecke algebra over $\mathbb{Q}_p$ generated by the double cosets $I_wU_pG I_wU_p$, $g \in \text{GL}_n(\mathbb{Q}_p) \times \mathbb{Q}_p^\times$. We let $\mathcal{H}_p$ denote the subalgebra generated by double cosets of the forms

$$U(a_1, \ldots, a_n, b) = I_w_p(\text{diag}(p^{a_1}, \ldots, p^{a_n}), p^b)I_w_p$$

for integers $a_1 \geq \ldots \geq a_n$ and $b$. If we forget the $\mathbb{Q}_p^\times / \mathbb{Z}_p^\times$-part this algebra is sometimes called the Atkin-Lehner ring or the “dilating” Hecke algebra, see e.g. [BC] §6.4.1. $\mathcal{H}_p$ is commutative and we have that

$$U(a_1, \ldots, a_n, b)U(a'_1, \ldots, a'_n, b') = U(a_1 + a'_1, \ldots, a_n + a'_n, b + b'),$$

so the generators $U(a_1, \ldots, a_n)$ form a monoid that we will denote by $\mathcal{U}^\dagger$. We will be interested in one particular element of $\mathcal{H}_p$, namely $U(0, -1, \ldots, -1, -1)$, which we will denote by $U^-$. We have a
coset decomposition
\[ I_w P (\text{diag}(1, p^{-1}, ..., p^{-1}), p^{-1}) I_w P = \prod_{E \in \mathbb{F}_{p^{-1}}} \left( \begin{array}{cc} 1 & p^{-1} E \\ 0 & p^{-1} \text{Id}_{n-1} \end{array} \right) I_w P \]

where we abuse notation and let \( E \) also denote an arbitrary lift of \( E \) to \( \mathbb{Z}_p^{-1} \). We let \( \mathcal{H} = \mathcal{H}^P \otimes \mathbb{Q} \mathcal{H}_P \); this is the commutative subalgebra of \( \mathcal{H}_{Iw} \) that we will use when talking about forms. \( \mathcal{H}_{Iw} \) acts on our Shimura varieties and universal abelian varieties (including integral models) of level \( Iw \) and their cohomology via the action of \( G(\mathbb{A}^\infty) \) on the towers; this corresponds to choosing a Haar measures on \( G(\mathbb{A}^{p, \infty}) \) and \( G(\mathbb{Q}) \) so that \( U^P \) and \( I_w P \) have measure 1.

We also need to consider Hecke algebras at \( p \) acting on our Igusa varieties. Recall from §3.1 that the group \( (\mathbb{Z} \times \text{GL}_{n-1}(\mathbb{Q})^+) \times \mathbb{Q}_p^x \) acts on \( \text{Ig}_{U^P, m} \), and that this action extends to an action of \( \mathbb{Z} \times \text{GL}_{n-1}(\mathbb{Q}) \times \mathbb{Q}_p^x \) on cohomology. This gives us an action of the Hecke algebra \( \mathcal{H}_{Ig, P} \) generated by the double cosets in
\[ I_{wIg}((\mathbb{Z} \times \text{GL}_{n-1}(\mathbb{Q}) \times \mathbb{Q}_p^x)/I_{wIg} \]
where \( I_{wIg} = 0 \times I_{n-1} \times \mathbb{Z}_p^x \). Put \( \mathcal{H}_{Ig} = \mathcal{H}^P \otimes \mathbb{Q} \mathcal{H}_{Ig, P} \). The compatibility in equation 3.1 gives us the following:

**Lemma 5.** \( \lim_{\rightarrow} \mathcal{H}^t_{rig,c}(\text{Ig}_{U^P, m}, V(\xi)) \cong \lim_{\rightarrow} \mathcal{H}^t_{rig,c}((\mathbb{Z} \times \text{GL}_{n-1}(\mathbb{Q})^+) \times \mathbb{Q}_p^x) \) as \( G_M(\mathbb{A}^\infty) \)-modules, where the group \( \lim_{\rightarrow} \mathcal{H}^t_{rig,c}(\text{Ig}_{U^P, m}, V(\xi)) \) has the action of \( G_M(\mathbb{A}^\infty) \) given by pulling back its natural action of \( G(\mathbb{A}^{p, \infty}) \times \mathbb{Z} \times \text{GL}_{n-1}(\mathbb{Q}) \times \mathbb{Q}_p^x \) via \( j_p \).

We embed the monoid \( U^- \) into \( \mathcal{H}_{Ig, P} \) by sending \( U(a_1, ..., a_n, b) \) to \( I_{wIg}(a_1, \text{diag}(a_2, ..., a_n), b) I_{wIg} \). Note that \( \mathcal{H}_{Ig, P} \) is isomorphic to the Hecke algebra generated by the double cosets in
\[ ((\mathbb{Z}_p^x \times I_{n-1} \times \mathbb{Z}_p^x) \times ((\mathbb{Q}_p^x \times \text{GL}_{n-1}(\mathbb{Q}) \times \mathbb{Q}_p^x)/((\mathbb{Z}_p^x \times I_{n-1} \times \mathbb{Z}_p^x)) \].

We will need the following lemma, which is a rewrite of Proposition 6.4.3 of [BC] in §4.2:

**Lemma 6.** Let \( \pi \) be an irreducible admissible representation of \( \text{GL}_n(\mathbb{Q}_p) \times \mathbb{Q}_p^x \). Consider the parabolic \( P = P_M \times \mathbb{Q}_p^x \) and its Levi subgroup \( L_P = \mathbb{Q}_p^x \times \text{GL}_{n-1}(\mathbb{Q}) \times \mathbb{Q}_p^x \) embedded in \( P \) in the obvious way. Let \( N \) be the unipotent radical of \( P \). Then the natural map
\[ \pi^{I_{wP}} \rightarrow (\pi_N)^{I_{wIg}} \otimes \delta_p^{-1} \]
is an isomorphism of \( U^- \)-modules. Here \( \pi_N \) denote the \( N \)-coinvariants (the unnormalized Jacquet module), and \( \delta_p \) is the modular character with respect to the parabolic \( P \).

**Proof.** Let \( B = TU \) denote the Borel inside \( P \) (notation for this proof only). Then by Proposition 6.4.3 of [BC] the natural maps
\[ \pi^{I_{wP}} \rightarrow (\pi_U)^{T_0} \otimes \delta_B^{-1}, \]
\[ (\pi_N)^{I_{wIg}} \rightarrow ((\pi_N)_{U \cap L_P})^{T_0} \otimes \delta_B^{L_P} \]
are isomorphisms; here \( T_0 = T_n((\mathbb{Z}_p) \times \mathbb{Z}_p^x) \) inside \( T \). The Lemma now follows since \( (\pi_N)_{U \cap L_P} \cong \pi_U \) via the natural map, \( \delta_p^{-1} = \delta_B^{-1} \delta_B \), and the natural map \( \pi^{I_{wP}} \rightarrow (\pi_U)^{T_0} \) is the composition of the natural maps \( \pi^{I_{wP}} \rightarrow (\pi_N)^{I_{wIg}} \) and \( (\pi_N)^{I_{wIg}} \rightarrow ((\pi_N)_{U \cap L_P})^{T_0} \). \( \square \)
Let us consider the operator $U_p \in U^-$ from above. The corresponding element inside $\mathcal{H}_{Iw,p}$ is the double coset
\[ Iw_{Ig}(0, \text{diag}(p^{-1}, \ldots, p^{-1}))Iw_{Ig}. \]
$(0, \text{diag}(p^{-1}, \ldots, p^{-1}))$ is central, so the double coset acts by $(0, \text{diag}(p^{-1}, \ldots, p^{-1}))$. Lemma 6 tells us that $U_p$ acts on $\pi_{Iw}$ as $p^{n-1}Iw_{Ig}(0, \text{diag}(p^{-1}, \ldots, p^{-1}))Iw_{Ig}$ does on $(\pi_{N})_{Iw}$ (with $\pi$ as in the statement of Lemma). On the other hand, geometrically, we see from Lemma 5 that the double coset $U_p$, decomposed as
\[ Iw_{p}(\text{diag}(1, p^{-1}, \ldots, p^{-1}), p^{-1})Iw_{p} = \prod_{E \in \mathbb{F}_p^{-1}} x_{E}Iw_{p}, \]
x_E = \left( \begin{array}{cc} 1 & p^{-1}E \\ 0 & p^{-1}I_{d_{n-1}} \end{array} \right), \]
acts the same way as $\prod_{E} j_{s}(x_{E})Iw_{Ig} = p^{n-1}(0, \text{diag}(p^{-1}, \ldots, p^{-1}))Iw_{Ig}$. These observations will be used in §4.2. Note that similar observations apply to all elements of $U^-$. 

4. Computation of Cohomology and Classicality

In this section we will compute the Euler characteristic $\sum_{i}(-1)^{i}H_{rig}^{i}(V_{Iw,1}^{g}, V^{\dagger}(\xi)^{\ast})$ in two ways in the Grothendieck group of Hecke modules and deduce a control theorem for systems of Hecke eigenvalues of overconvergent automorphic forms of small slope. We will make use of rigid cohomology; for a short recollection of the terminology we need see [Joh] §4 and for a good reference see [LeSt]. In this section, all integral models will be over $\mathbb{Z}_p$, all characteristic 0 schemes and rigid analytic spaces will be over $\mathbb{Q}_p$ and all characteristic $p$ schemes will be over $\mathbb{F}_p$ (so e.g. we will just write $X_{Iw}$ for what was previously called $X_{Iw/q_p}$).

4.1. Computation in terms of overconvergent automorphic forms. Let us begin by recalling the generalized BGG resolution for the pair $(g = \text{Lie}(G), p = \text{Lie}(P))$. See [Hum] for a good reference on BGG resolutions for semisimple Lie algebras; the extension to reductive Lie algebras is trivial and just amounts to inserting a central character (that remains constant throughout the resolution).

**Theorem 7.** (“generalized” BGG resolution) If $\xi$ is an irreducible representation of the reductive Lie algebra $g$ of dominant weight $\lambda = (k_1, \ldots, k_n, w)$, then we have a resolution
\[ 0 \to C_{n-1}^{\xi} \to \ldots \to C_{0}^{\xi} \to \xi \to 0 \]
with $C_{s}^{\xi} = U(g) \otimes_{U(p)} M(w_{k}(\lambda + \rho) - \rho)$, where $w_s$ for $0 \leq s \leq n - 1$ is the element of the Weyl group $S_n$ of $G$ sending $(k_1, \ldots, k_n, w)$ to $(k_1, \ldots, k_{n-s-1}, k_{n-s+1}, \ldots, k_n, k_{n-s}, w)$, $\rho = \frac{1}{2}(n - 1, n - 3, \ldots, 1 - n, 0)$ is half the sum of the positive roots for $G$, and $M(k_{1}', \ldots, k_{n}', w')$ for $k_{1}' \geq \ldots \geq k_{n-1}'$, $w' \equiv \sum_{i=1}^{n} k_i' \mod 2$ denotes the irreducible algebraic representation of $L$ with dominant weight $(k_1', \ldots, k_n', w')$. The chain complex $C_{\ast}^{\xi}$ is a quasi-isomorphic direct summand of the bar resolution $D_{\ast}^{\xi}$ of $\xi$ (where $D_{\ast}^{\xi} = U(g) \otimes_{U(p)} \wedge^{\ast}(g/p) \otimes_{C} \xi$). Moreover if $\xi$ is an irreducible representation of $G$, then the above sequence is a resolution of $(U(g), P)$-modules (and similarly for the quasi-isomorphism).

Applying the automorphic vector bundle construction one obtains Faltings’s dual BGG complex for the vector bundle $V(\xi)$ with connection on $X_{Iw}$.
Theorem 8. ([Fal] Theorem 3, [ChFa]) If $\xi$ is an irreducible representation of $G$ with dominant weight $\lambda = (k_1, \ldots, k_n, w)$ we have a complex
\[ 0 \to K^0_\lambda \to \cdots \to K^{n-1}_\lambda \to 0 \]
called the dual BGG complex, with $K^0_\lambda = W(w_\lambda(\lambda + \rho) - \rho)^\vee = W(k_1, \ldots, k_{n-s}, k_{n-s+1} - 1, \ldots, k_n - 1, k_n - s, w)^\vee$ on $X_{Iw/C}$, where the maps are Hecke-equivariant differential operators, which is a quasi-isomorphic direct summand of the de Rham complex $V(\xi)^\vee \otimes_{\mathcal{O}_{X_{Iw/C}}} \Omega^1_{X_{Iw/C}}$. Moreover these constructions descend to a number field and may be base changed to $\mathbb{Q}_p$ (because $G$ is split over $\mathbb{Q}_p$ and $p$ is totally split in $F$).

Next we apply $j_{Y_{Iw,1}}^\dagger$ to the complex $K^\bullet_\lambda$ to obtain a complex $K^\dagger_\lambda = W^\dagger(w_\lambda(\lambda + \rho) - \rho)^\vee$ which, by the exactness of $j_{Y_{Iw,1}}^\dagger$, is a quasi-isomorphic of the overconvergent de Rham complex of $V^\dagger(\xi)^\vee$ and hence may be used to compute the $H^{i}_{\text{rig}}(Y_{Iw,1}^0, V^\dagger(\xi)^\vee)$.

Proposition 9. $H^{i}_{\text{rig}}(Y_{Iw,1}^0, V^\dagger(\xi)^\vee)$ is equal to $h^i(H^0(X^\text{rig}_{Iw}, W^\dagger(w_\lambda(\lambda + \rho) - \rho)^\vee)$, where $h^i$ stands for “$i$-th cohomology of the complex”.

Proof. We have
\[ H^{i}_{\text{rig}}(Y_{Iw,1}^0, V^\dagger(\xi)^\vee) = H^i_{\text{dR}}(X^\text{rig}_{Iw}, V^\dagger(\xi)^\vee) = H^i(X^\text{rig}_{Iw}, K^\dagger_\lambda) \]
by the discussion above. Next we claim that the coherent cohomology groups $H^j(X^\text{rig}_{Iw}, W^\dagger(w_\lambda(\lambda + \rho) - \rho)^\vee)$ vanish for $j \geq 1$. This follows from the fact that $Y_{Iw,1}^0$ is affine as in Theorem 20 of [Joh]. The affineness of $Y_{Iw,1}^0$ follows from that of $\mathfrak{X}_{U,n-1}$ by finiteness of the morphism $Y_{Iw,1}^0 \to \mathfrak{X}_{U,n-1}$, and the affineness of $\mathfrak{X}_{U,n-1}$ follows from the fact that $\mathfrak{X}_{U,n-1}$ is the non-vanishing locus of a non-zero section (the Hasse invariant) of an ample line bundle on the projective variety $\mathfrak{X}_U$ (see e.g. Proposition 7.8 of [LS]).

The following theorem is a consequence of Hida’s calculations of $p$-integrality for Hecke operators (see [Hid]):

Theorem 10. The slopes of the operator $U_p$ on $H^0(X^\text{rig}_{Iw}, W^\dagger(k_1, \ldots, k_n, w))$ are greater than or equal to
\[ -\frac{w + k_1 + \cdots + k_n - 1}{2} - (n - 1). \]

From now on let $\xi$ be the algebraic representation of $G$ with dominant weight $\lambda = (1 - n - k_n, 1 - k_{n-1}, \ldots, 1 - k_1 - w)$ for $k_1, \ldots, k_n$ such that $k_1 \geq k_2 \geq \ldots \geq k_n + n$. By above we have $H^i_{\text{rig}}(Y_{Iw,1}^0, V^\dagger(\xi)^\vee) = h^i(H^0(X^\text{rig}_{Iw}, W^\dagger(w_\lambda(\lambda + \rho) - \rho)^\vee)$. Here $\xi^\vee$ has dominant weight $(k_1 - 1, \ldots, k_n - 1, k_n + n - 1, w)$ and the dual BGG complex has, for $0 \leq s \leq n - 2$,
\[ W^\dagger(w_\lambda(\lambda + \rho) - \rho)^\vee = W^\dagger(k_1, \ldots, k_s, k_{s+2} - 1, \ldots, k_{n-1} - 1, k_n + n - 1, k_{n+1} - 1 - s, w) \]
and $W^\dagger(w_{n-1}(\lambda + \rho) - \rho)^\vee = W^\dagger(k_1, \ldots, k_n, w)$. We deduce the following consequence of Theorem 10:
Corollary 11. With notation as above, if \( f \in H^0(X_{Iw}^{rig}, W^1(k_1, \ldots, k_n, w)) \) is an eigenvector for \( U_p \) of slope (strictly) less than
\[
\alpha_{n-2} := -w + k_1 + \ldots + k_n - k_{n-2} + k_n - (n-1),
\]
then the system of eigenvalues of \( f \) occurs in \( \sum_i (-1)^{d+i} H_{rig}^i(Y_{Iw,1}^0, V^1(\xi)^\vee) \) (viewed as a virtual \( A[U_p]\)-module, for any algebra \( A \) of linear operators on the \( H_{rig}^i(Y_{Iw,1}^0, V^1(\xi)^\vee) \) commuting with \( U_p \) for which \( f \) is a simultaneous eigenvector).

Proof. As noted above, we have \( H_{rig}^i(Y_{Iw,1}^0, V^1(\xi)^\vee) = h^i(H^0(X_{Iw}^{rig}, W^1(w_s(\lambda + \rho) - \rho)^\vee) \) and
\[
W^1(w_s(\lambda + \rho) - \rho)^\vee = W^1(k_1, ..., k_s, k_{s+2} - 1, ..., k_{n-1} - 1, k_n + n - 1, k_{s+1} - 1 - s, w),
\]
for \( 0 \leq s \leq n - 2 \) and \( W^1(w_{n-1}(\lambda + \rho) - \rho)^\vee = W^1(k_1, ..., k_n, w) \), moreover the differentials in the dual BGG complex are Hecke-equivariant. By Theorem 10, the slopes of \( U_p \) are greater than or equal to
\[
\alpha_s := \frac{w + \left( \sum_{i \neq s+1} k_i \right) - k_{s+1} + s + 2}{2} - (n-1)
\]
on \( H^0(X_{Iw}^{rig}, W^1(w_s(\lambda + \rho) - \rho)^\vee) \) for \( 0 \leq s \leq n - 2 \) and greater than or equal to \( \alpha_{n-1} := -w + k_1 + \ldots + k_{n-1} - k_n) / 2 - (n-1) \) on \( H^0(X_{Iw}^{rig}, W^1(w_{n-1}(\lambda + \rho) - \rho)^\vee) \). We have that, for \( 0 \leq s \leq n - 2 \),
\[
\alpha_s - \alpha_{n-1} = k_{s+1} - (k_n + 1 + \frac{s}{2})
\]
and hence \( \alpha_1 \geq \ldots \geq \alpha_{n-2} \geq \alpha_{n-1} \). Thus if the slope of \( f \) is less than \( \alpha_{n-2} \), then the system of eigenvalues of \( f \) can only occur in \( H^0(X_{Iw}^{rig}, W^1(w_s(\lambda + \rho) - \rho)^\vee) \), hence occurs in \( H_{rig}^{n-1}(Y_{Iw,1}^0, V^1(\xi)^\vee) \) but not in \( H_{rig}^i(Y_{Iw,1}^0, V^1(\xi)^\vee) \) for \( i \neq n-1 \), and hence must occur in \( \sum_i (-1)^{d+i} H_{rig}^i(Y_{Iw,1}^0, V^1(\xi)^\vee) \) as desired.

Remark 12. If we choose \( w \) to make \( \alpha_{n-1} = 0 \), then \( \alpha_{n-2} = k_{n-1} - k_n - (n+1)/2 \).

4.2. Computation in terms of classical automorphic forms. We continue to assume that \( \xi \) is the irreducible algebraic representation of \( G \) with dominant weight \( \lambda = (1 - n, k_1, k_2 - 1, ..., 1 - k_1, w) \).

Proposition 13. \( \sum_i (-1)^{d+i} H_{rig}^i(Y_{Iw,1}^0, V^1(\xi)^\vee) = \sum_i (-1)^{d+i} H_{rig,c}^i(Y_{Iw,1}^0, V^1(\xi)^\vee) = \sum_i (-1)^{d+i} H_{rig,c}^i(Y_{Iw,1}^0, V^1(\xi)^\vee) \) as virtual \( H^p[U^-]\)-modules.

Proof. By Poincare duality \( H_{rig}^i(Y_{Iw,1}^0, V^1(\xi)^\vee) = H^{2(n-1) - i}_{rig,c}(Y_{Iw,1}^0, V^1(\xi)^\vee) \), and Poincare duality is Hecke equivariant follows since Hecke operators act by correspondences. The proposition follows by taking the sum. \( \square \)

Since
\[
\sum_i (-1)^{d+i} H_{rig,c}^i(Y_{Iw,1}^0, V^1(\xi)^\vee) = \left( \sum_i (-1)^{d+i} H_{rig,c}^i(Y_{Iw,1}^0, V^1(\xi)^\vee) \right)^\vee
\]
we may calculate \( \sum_i (-1)^{d+i} H_{rig,c}^i(Y_{Iw,1}^0, V^1(\xi)^\vee) \) via \( \sum_i (-1)^{d+i} H_{rig,c}^i(Y_{Iw,1}^0, V^1(\xi)^\vee) \). The calculation of the latter is essentially done by Theorem V.5.4 ("the second basic identity") of [HT]. To state it we will need some notation. We will also explain how to pass from the \( \ell \)-adic setting of that theorem to the \( p \)-adic rigid setting that we are in. We have \( H_{rig,c}(Y_{Iw,1}^0, V^1(\xi)) = a_\xi H_{rig,c}(Y_{Iw,1}^0 \otimes Q_p). \)
By Deligne’s conjecture/Fujiwara’s trace formula (proved by Fujiwara ([Fuj]) in the \( \ell \)-adic setting and by Mieda ([Mie]) in the rigid setting) we may deduce by a comparison of geometric sides of this trace formula (in the \( \ell \)-adic resp. rigid setting) that \( H^i_{rig,c}(Y^0_{Iw,1}, V^\dagger(\xi)) \) is isomorphic, as a Hecke module, to the \( \ell \)-adic cohomology group \( H^i_{et,c}(Y^0_{Iw,1}, V^\dagger(\xi)) \) after applying an isomorphism \( \mathbb{Q}_p \cong \mathbb{Q}_\ell \) (note that the Hecke algebra is generated by elements whose traces, by Deligne’s conjecture, are rational numbers, so this makes sense). In the cases we are interested in Deligne’s conjecture is a simple consequence of the usual Lefschetz trace formula (as in [Kle]). Namely, \( \mathcal{A}_{Iw,1,0}^m \subseteq \mathcal{A}_{Iw,1}^m \) and the latter is projective and smooth, and moreover the complement is a normal crossings divisor (since \( Y_{Iw,1} \setminus Y^0_{Iw,1} \) is a normal crossings divisor), so the Lefschetz trace formula together with the excision sequence and the purely geometric fact that after twisting with a high enough power of Frobenius any correspondence intersects the diagonal transversally implies Deligne’s conjecture for \( \mathcal{A}_{Iw,1,0}^m \) for any integer \( m \geq 1 \) for correspondences on \( \mathcal{A}_{Iw,1}^m \) respecting the stratification. One may check that the Hecke operators accept this stratification, as do the various \( a_\zeta \).

We may now state Theorem V.5.4 of [HT] in \( p \)-adic form:

**Theorem 14.** \( n \sum_i (-1)^{d+i} H^i_{rig,c}(Y^0_{Iw,1}, V^\dagger(\xi)) = \left( \lim_{\to} \left( \sum_i (-1)^{d+i} H^i_{dR}(X_{U(m)}, V(\xi)) \right) \right) \big|_{N_M^p}^{U^p \times Iw_{Ig}} \)

as virtual \( \mathcal{H}_{Ig} \)-modules, where \( N_M \) denotes the unipotent radical of \( P_M \) and \( N_M^p \) denotes the unipotent radical of the opposite parabolic of \( P_M \).

Note that we have used here the fact that the Iwahori-Igusa variety \( I_{Iw} \) is naturally isomorphic to \( Y^0_{Iw,1} \) (respecting the relevant Hecke actions; see §2.4). Thus, in light of this theorem, what remains is to understand the local factor at \( p \) of the right hand side in terms of that of \( \sum_i (-1)^i H^i_{dR}(X_{Iw}, V(\xi)) \) for the monoid \( U^- \), now viewed as a multiplicative submonoid of \( \mathcal{H}_{Iw_p} \).

To do this, we start by recalling the following result (see [Cas], Corollary 4.2.5):

**Lemma 15.** Let \( G \) be a connected reductive group over \( \mathbb{Q}_p \), \( P = LN \) a parabolic and \( \pi \) an irreducible admissible representation of \( G(\mathbb{Q}_p) \). Let \( P^{op} = LN^{op} \) be the opposite parabolic of \( P \). Then \( (\pi_{N^{op}})^\vee \cong (\pi^\vee)_N \) as representations of \( L \).

We can now put the various technical results together to obtain:

**Proposition 16.** \( n \sum_i (-1)^{d+i} H^i_{rig}(Y^0_{Iw,1}, V^\dagger(\xi)^\vee) \otimes \delta_{P^-}^{-1} = \sum_i (-1)^{d+i} H^i_{dR}(X_{Iw}, V(\xi)^\vee) \) as virtual \( \mathcal{H}^p[U^-] \)-modules.

**Proof.** By Proposition 13, Lemma 15 and Poincare duality we have

\[
n \sum_i (-1)^{d+i} H^i_{rig}(Y^0_{Iw,1}, V^\dagger(\xi)^\vee) = \left( \lim_{\to} \left( \sum_i (-1)^{d+i} H^i_{dR}(X_{U(m)}, V(\xi)^\vee) \right) \right) \big|_{N_M^p}^{U^p \times Iw_{Ig}} \cdot \delta_{P^-}^{-1}
\]

Tensoring by \( \delta_{P^-}^{-1} \) and applying Lemma 6 to the right hand side we obtain the result. \( \Box \)

In what follows we put \( g_\infty = Lie(G(\mathbb{R})) \) and let \( K_\infty \) be a maximal compact modulo center subgroup of \( G(\mathbb{R}) \). For any \( (g_\infty, K_\infty) \)-module \( \tau \), we let \( H^i(\mathfrak{g}_\infty, K_\infty, \tau) \) denote the \( i \)-th \( (g_\infty, K_\infty) \)-cohomology of \( \tau \) (see [BW]). Let \( \iota: \mathbb{C} \to \mathbb{Q}_p \). We may now state and prove the main result of this paper:
Theorem 17. 1) We have
\[ n \sum \frac{(-1)^{d+i}H_{\text{rig}}^i(Y^0_{Iw,1}, V^\dagger(\xi)^\vee) \otimes \delta_p^{-1}}{d + i} \otimes_{Q_p} \mathfrak{C}_p = \]
\[ = \left( \bigoplus \pi \right) m(\pi) (\pi^\infty)^{Iw} \otimes \left( \sum \frac{(-1)^{d+i}H^i(\mathfrak{g}_\infty, K_\infty, \pi^\infty \otimes \xi^\vee)}{d + i} \right) \]
as virtual $\mathcal{H}^p[U^-]$-modules, where the sum is taken over all automorphic representations of $G(\mathbb{A})$ (necessarily cuspidal) and $m(\pi)$ denotes the multiplicity of $\pi$.

2) Let $f \in H^0(X_{Iw}^{\text{rig}}, W^1(k_1, ..., k_n, w))$ be a simultaneous eigenvector for $\mathcal{H}^p[U^-]$ of $U_p$-slope less than
\[ -\frac{w + k_1 + ... + k_{n-2} + k_n - k_{n-1} + n}{2} - (n - 1). \]
Then $f$ is classical, i.e. $f$ occurs inside the virtual $\mathcal{H}^p[U^-]$-module $(\pi^\infty)^{Iw}$ for some automorphic representation $\pi$ of $G(\mathbb{A})$.

Proof. 1) is direct consequence of applying Matsushima’s formula ([BW] VII.5.2)
\[ H_{2R}^p(X_{Iw}(\mathbb{C}), V(\xi)^\vee) = \bigoplus \pi m(\pi)(\pi^\infty)^{Iw} \otimes H^i(\mathfrak{g}_\infty, K_\infty, \pi^\infty \otimes \xi^\vee) \]
to the right hand side of Proposition 16. If $f$ is as in the statement of 2), then by Corollary 11 the $\mathcal{H}^p[U^-]$-eigensystem of $f$ occurs in $\sum \frac{(-1)^{d+i}H_{\text{rig}}^i(Y^0_{Iw,1}, V^\dagger(\xi)^\vee)}{d + i}$. The result now follows from 1) and the remarks at the end of section 3.3. □

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