THE MÖBIUS TRANSFORMATION OF CONTINUED FRACTIONS WITH
BOUNDED UPPER AND LOWER PARTIAL QUOTIENTS

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Abstract. Let \( h: x \mapsto \frac{ax+b}{cx+d} \) be the nondegenerate Möbius transformation with integer
entries. We get a bound of the continued fraction of \( h(x) \) by the upper and lower bound of
continued fraction of \( x \), which extends a result of Stambul [7].

1. Introduction

A continued fraction representation of a number \( x \in \mathbb{R} \) is an expansion of the form

\[
    x = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots}}}
\]

where \( a_0 \in \mathbb{Z} \) and \( a_i \in \mathbb{N}^+ \), \( i = 1, 2, \ldots \). A continued fraction may be finite or infinite. If
\( x \) is a finite continued fraction, we denote it by \([a_0; a_1, a_2, \ldots, a_n]\); if \( x \) is infinite, then we
 denote it by \([a_0; a_1, a_2, \ldots]\). We call \( a_j \) the \( j \)th partial quotient. It is a well known fact that
the continued fraction of \( x \) is infinite iff \( x \) is irrational.

Given a nondegenerate \( 2 \times 2 \) matrix \( M \) with integer entries, that is \( M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \),
where \( a, b, c, d \in \mathbb{Z} \) and the determinant \( ad - bc \neq 0 \), we can define the associated Möbius
transformation \( h: x \mapsto \frac{ax+b}{cx+d} \). We also denote by

\[
    h(x) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x = \frac{ax+b}{cx+d}
\]

In this paper, we study the bound of partial quotients under the Möbius transformation.
We will use \( \lfloor x \rfloor = \max\{ j \in \mathbb{Z} : j \leq x \} \). Our main result is

Theorem 1.1. Let \( M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) be a nondegenerate matrix with entries in \( \mathbb{Z} \) and \( h \)
be the associated Möbius transformation. Let \( x = [a_0; a_1, a_2, \ldots] \) be a real number such that
\( B_1 \leq a_i \leq B_2 \) for \( j \) large enough. Let \( h(x) = [a_0^*; a_1^*, a_2^*, \ldots] \). Then \( a_j^* \leq \left\lfloor \frac{D-1}{B_1} \right\rfloor +
\left\lfloor \frac{D B_1 + \sqrt{B_1^2 B_2^2 + 4 B_1 B_2}}{2B_1} \right\rfloor \) for large \( j \), where \( D = |\det(M)| \).

Now we always assume \( x = [a_0; a_1, a_2, \ldots] \) and \( \frac{ax+b}{cx+d} = [a_0^*; a_1^*, a_2^*, \ldots] \) with \( D = |ad - bc| \geq 1 \).

1. Set \( M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) and \( h(x) = \frac{ax+b}{cx+d} \).

It is an old result that a real number \( \frac{ax+b}{cx+d} \) has bounded partial quotients if \( x \) does \( \mathbb{Z} \mathbb{Z} \mathbb{Z} \mathbb{Z} \mathbb{Z} \),
so the quantitative bound becomes an interesting question. Lagarias-Shallit \( \mathbb{Z} \mathbb{Z} \mathbb{Z} \mathbb{Z} \mathbb{Z} \) and Cusick-
France \( \mathbb{Z} \mathbb{Z} \mathbb{Z} \mathbb{Z} \mathbb{Z} \) obtained a quantitative bound, which stated that if \( x \) has bounded partial quotients
with \( a_j \leq K \) eventually, then the associated partial quotients \( a_j^* \) of \( \frac{ax+b}{cx+d} \) satisfy \( a_j^* \leq D(K+2) \)
eventually.
Using an algorithm developed by Liardet-Stambul [4] to calculate the partial quotients of $h(x)$, Stambul gave an upper bound $a_j^* = D - 1 + \frac{K + \sqrt{K^2 - 16}}{2}$, which is the $B_1 = 1$ case of Theorem [1, 7]. In this paper, we concern the partial quotients with lower and upper bound at the same time. Our methods are based on the refining of analysis in papers [4, 7].

2. Algorithm for partial quotients

In this section, we will introduce some notations and the algorithm developed by Liardet-Stambul [4, 7] to calculate the partial quotients of $h$. Let $M_{2,N}$ be the set of all matrices $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $(a, b, c, d \in \mathbb{N})$ such that $ad - bc \neq 0$. $M$ is said to be in $\mathcal{D}_2$ when $a \geq c$ and $b \geq d$, in $\mathcal{D}'_2$ when $a \leq c$ and $b \leq d$, and in $\mathcal{E}_2$ when $(a - c)(b - d) < 0$. \{$\mathcal{D}_2, \mathcal{D}'_2, \mathcal{E}_2$\} is a partition of $M_{2,N}$.

It is easy to see that $M \in \mathcal{E}_2$ satisfies

\[
\max\{|a| + |b|, |c| + |d|\} \leq |\det M| = D.
\]

For all matrices $M \in \mathcal{D}_2 \cup \mathcal{D}'_2$, there exists a unique factorization

\[
M = \begin{pmatrix} c_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} c_n & 1 \\ 1 & 0 \end{pmatrix} M'
\]

such that $c_0 \in \mathbb{N}$, $c_1, \ldots, c_n \in \mathbb{N}^+$ and $M' \in \mathcal{E}_2$. This factorization will be denoted by $M = \Pi_{0 \leq j \leq n} c_j$. Moreover, $\{c_0; c_1, c_2, \ldots, c_{n-1}\}$ is the common sequence of partial quotients of $\frac{a}{c}$ and $\frac{b}{d}$ if $n \neq 1$. $c_n$ can be determined by the following several cases [4].

Case 1: If $\frac{a}{c} = [c_0; c_1, c_2, \ldots, c_{n-1}]$, then $c_n$ is the $n$th partial quotient of $\frac{a}{c}$.

Case 2: If $\frac{b}{d} = [c_0; c_1, c_2, \ldots, c_{n-1}]$, then $c_n$ is the $n$th partial quotient of $\frac{b}{d}$.

Case 3: Otherwise, $c_n$ is the smaller one of $n$th partial quotients of $\frac{a}{c}$ and $\frac{b}{d}$.

Assume $M \in \mathcal{E}_2$ and $h$ is the associated M"obius transformation. Let $x = [a_0; a_1, a_2, \ldots] > 1$. Recall the algorithm in [4, 7] to compute the partial quotients of $h(x)$.

Step 0: $M_0 = M \in \mathcal{E}_2, j = 0, n = 0$.

Let $j_1$ be the smallest positive integer (see [3] for the existence) such that $M_0 \Pi_{j_0=1} \cdots a_j = 1 \\ \cdots \cdots a_j \in \mathcal{E}_2 \cdots \cdots \mathcal{E}_2$. Factorizing $M_0 \Pi_{j_0=1} \cdots a_j$ as [3], we get

(Output-0) $M_0 \Pi_{j_0=1} \cdots a_j = \begin{pmatrix} c_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} c_n & 1 \\ 1 & 0 \end{pmatrix} M_1$

with $M_1 \in \mathcal{E}_2$.

Step 1: $M_1 \in \mathcal{E}_2, j = j_1 + 1, n = n_1 + 1$.

Let $j_2 \geq j_1 + 1$ be the smallest positive integer such that $M_1 \Pi_{j_0=1} \cdots a_j = 1 \\ \cdots \cdots a_j \in \mathcal{E}_2 \cdots \cdots \mathcal{E}_2$. Factorizing $M_1 \Pi_{j_0=1} \cdots a_j$ as [3], we get

(Output-1) $M_1 \Pi_{j_0=1} \cdots a_j = \begin{pmatrix} c_{n_1+1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_{n_1+2} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} c_{n_2} & 1 \\ 1 & 0 \end{pmatrix} M_2$

with $M_2 \in \mathcal{E}_2$.

Step 2: $M_2 \in \mathcal{E}_2, j = j_2 + 1, n = n_2 + 1$.

Let $j_3 \geq j_2 + 1$ be the smallest positive integer such that $M_2 \Pi_{j_0=1} \cdots a_j = 1 \\ \cdots \cdots a_j \in \mathcal{E}_2 \cdots \cdots \mathcal{E}_2$. Factorizing $M_2 \Pi_{j_0=1} \cdots a_j$ as [3], we get

(Output-2) $M_2 \Pi_{j_0=1} \cdots a_j = \begin{pmatrix} c_{n_2+1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_{n_2+2} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} c_{n_3} & 1 \\ 1 & 0 \end{pmatrix} M_3$

with $M_3 \in \mathcal{E}_2$. 


Step k: $M_k \in \varepsilon_2, j = j_k + 1, n = n_k + 1$.

Let $j_k+1 \geq j_k + 1$ be the smallest positive integer such that $M_k \Pi_{a_{j_k+1} a_{j_k+2} \cdots a_{j_k+1-1}} \in \varepsilon_2$ and $M_k \Pi_{a_{j_k+1} a_{j_k+2} \cdots a_{j_k+1-1}} \in D_2 \cup D_2'$. Factorizing $M_k \Pi_{a_{j_k+1} a_{j_k+2} \cdots a_{j_k+1-1}}$ as (3), we get

\[(\text{Output-k}) \quad M_k \Pi_{a_{j_k+1} a_{j_k+2} \cdots a_{j_k+1-1}} = \left( \begin{array}{ccc} c_{n_k+1} & 1 & 0 \\ 1 & 0 & 0 \\ \end{array} \right) \left( \begin{array}{ccc} c_{n_k+2} & 1 & 0 \\ 1 & 0 & 0 \\ \end{array} \right) \cdots \left( \begin{array}{ccc} c_{n_k+1} & 1 & 0 \\ 1 & 0 & 0 \\ \end{array} \right) M_{k+1} \]

Putting all the Output (Output-k) together, we get a sequence

\[(\text{Alloutput-k}) \quad c_0 c_1 c_2 c_3 \cdots c_{n_k} \]

Unfortunately, many $c_i$ maybe zero, thus we must introduce the contraction map $\mu$. For any word $c_0 c_1 c_2 c_3 \cdots c_n \in \mathbb{N}^n$, let $\mu$ be the contraction map which transforms a word into a word where all letters are positive integers (except perhaps the first one), replacing from left to right factors $a+b$ by the letter $a+b$.

By the fact

\[\Pi \mu(c_0 c_1 c_2 c_3 \cdots c_n) = \Pi c_0 c_1 c_2 c_3 \cdots c_n \]

we have

\[\text{(4)} \quad \Pi \mu(c_0 c_1 c_2 c_3 \cdots c_n) = \Pi c_0 c_1 c_2 c_3 \cdots c_n \]

Let $\mu$ act on (Alloutput-k), then we get

\[(\text{Partialquotients}) \quad c_0^* c_1^* c_2^* c_3^* \cdots c_{n_k}^* = \mu(c_0 c_1 c_2 c_3 \cdots c_{n_k}) \]

By the arguments in [4], $n_k'$ goes to infinity as $k$ does, moreover,

\[\text{(5)} \quad \frac{ax + b}{cx + d} = [c_0^*; c_1^*, \cdots, c_{n_k-1}^*, \cdots] \]

and the $n_k'$ th partial quotient following $c_{n_k-1}^*$ is no less than $c_{n_k}^*$.

Now, we give a quantitative estimate about $c_k$ in (Alloutput-k).

Lemma 2.1. Assume $M \in \varepsilon_2$ and $x = [a_0; a_1, a_2, \cdots] > 1$. Let $h$ be the associated Möbius transformation and $D = |\det M| \geq 1$. Let $h$ be the associated Möbius transformation and $D = |\det M| \geq 1$. Suppose $a_j \leq K$ for some $K \in \mathbb{N}^+$. We do the algorithm as above, then the following three claims hold,

(i): For any $n_k < j \leq n_k + 1 - 1$, $c_j \leq D - 1$

(ii): For any $k$, $c_{n_k+1} \leq DK$

(iii): If for some $k$, $c_{n_k+1} \geq D$, then the right upper entry of $M_{k+1}$ must be zero, that is $M_{k+1}$ has the form

\[\text{(6)} \quad M_{k+1} = \left( \begin{array}{ccc} \star & 0 & 0 \\ \star & \star & \star \\ \end{array} \right) \]

Proof. The three claims are from [7]. We rewrite the proof here to make the paper more readable. By the algorithm, we already have $M_k \Pi_{a_{j_k+1} a_{j_k+2} \cdots a_{j_k+1-1}} \in \varepsilon_2$ and $M_k \Pi_{a_{j_k+1} a_{j_k+2} \cdots a_{j_k+1-1}} \in D_2 \cup D_2'$. For simplicity, let $M' = \left( \begin{array}{ccc} \alpha & \beta \\ \gamma & \delta \end{array} \right) = M_k \Pi_{a_{j_k+1} a_{j_k+2} \cdots a_{j_k+1-1}} \in \varepsilon_2$ and $f = a_{j_k+1} \leq K$. Then $M' \Pi f \in D_2 \cup D_2'$. 


If $\gamma = 0$, then
\[
M'\Pi f = \begin{pmatrix} \alpha f + \beta & \alpha \\ \delta & 0 \end{pmatrix} \in D_2 \cup D'_2
\]
and we must have $\alpha f + \beta \geq \delta$. Thus
\[
M'\Pi f = \begin{pmatrix} \alpha f + \beta & \alpha \\ \delta & 0 \end{pmatrix} = \begin{pmatrix} \lfloor \frac{\alpha f + \beta}{\delta} \rfloor & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \delta \\ 0 \end{pmatrix} = \begin{pmatrix} \lfloor \frac{\alpha f + \beta}{\delta} \rfloor \\ 0 \end{pmatrix} \mod \begin{pmatrix} \alpha & \delta \\ 0 & 1 \end{pmatrix}.
\]
In this case, in order to prove the Lemma, it suffices to show that
\[
\left\lfloor \frac{\alpha f + \beta}{\delta} \right\rfloor \leq DK.
\]
Otherwise, one has
\[
DK + 1 \leq \left\lfloor \frac{\alpha f + \beta}{\delta} \right\rfloor = \left\lfloor \frac{\alpha f}{\delta} + \frac{\beta}{\delta} \right\rfloor \leq \left\lfloor \frac{\alpha K}{\delta} + \frac{\beta}{\delta} \right\rfloor,
\]
since $f \leq K$.

By the fact $M' = \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \in \mathcal{D}_2$, we have $\beta < \delta$, $|\alpha| + |\beta| \leq D$. This is contradicted to (8).

If $\alpha = 0$, then
\[
M'\Pi f = \begin{pmatrix} \beta & 0 \\ \gamma f + \delta & \gamma \end{pmatrix} \in D_2 \cup D'_2
\]
and we must have $\gamma f + \delta \geq \beta$. Thus
\[
M'\Pi f = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \lfloor \frac{\gamma f + \delta}{\gamma} \rfloor & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} \gamma f \mod \beta \\ \beta \gamma \end{pmatrix}.
\]
In this case, we can still prove the Lemma like the case $\gamma = 0$.

If $\alpha, \gamma \geq 1$, then
\[
M'\Pi f = \begin{pmatrix} \alpha f + \beta & \alpha \\ \gamma f + \delta & \gamma \end{pmatrix} \in D_2 \cup D'_2.
\]

By the algorithm, $n_k \leq j \leq n_{k+1} - 1$, $c_j$ is the common partial quotient of $\frac{a}{\gamma}$ and $\frac{a f + \beta}{\gamma f + \delta}$.

We first show claim 1 holds. Indeed, $\alpha \leq D$ and $\gamma \geq 1$. If $\alpha = D$ and $\gamma = 1$, we must have $\beta = 0$ and $\delta = 1$. This implies claim 1 when we consider the partial quotient of $\frac{a f + \beta}{\gamma f + \delta}$.

Otherwise ($\alpha = D$ and $\gamma = 1$ do not hold) claim 1 holds if we consider the partial quotient of $\frac{a}{\gamma}$.

Suppose the last letter, i.e. $c_{n_{k+1}} \geq D$, then we must have $\frac{a}{\gamma} = [c_{j_{k+1}}; c_{j_{k+2}}, c_{j_{k+3}}, \ldots, c_{j_{k+1} - 1}]$ by the (Case1-Caese3) and $c_{n_{k+1}} \geq D$ is the $n_{k+1} - n_k + 1$th partial quotient of $\frac{a f + \beta}{\gamma f + \delta}$. This implies claims 2 and 3 if we can show
\[
\frac{1}{DK} \leq \frac{\alpha f + \beta}{\gamma f + \delta} \leq DK.
\]
We only prove the fact $\frac{1}{DK} \leq \frac{\alpha f + \beta}{\gamma f + \delta}$ is the same.

If $\gamma f + \delta \geq 2$, then $\frac{1}{\gamma f + \delta} \leq \frac{DK + 1}{2} \leq DK$. If $\gamma f + \delta \leq 1$, then we have $\delta = 0$ and $\gamma = K = 1$. This implies $\beta = D$ and $\alpha = 0$. We still have $\frac{1}{\gamma f + \delta} \leq DK$. \(\square\)
3. Some Lemmas

We say a Möbius transformation \( h(\cdot) = M \cdot \) can not change the continued fraction eventually, if for any \( x \), there exists some \( N \in \mathbb{N} \) such that the \( n \)th partial quotients of \( h(x) \) and \( x \) are the same for any \( n \geq N \).

**Lemma 3.1.** The following forms of Möbius transformations can not change the continued fraction eventually.

\[
S = \{ \begin{pmatrix} 1 & k_1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} k_2 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ k_3 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \},
\]

where \( k_1, k_2, k_3 \in \mathbb{Z} \).

**Proof.** The proof is based on direct computation. \( \square \)

**Remark:** The determinant of each matrix in \( S \) is \( \pm 1 \).

**Lemma 3.2.** Assume \( a, b, c, d \in \mathbb{Z} \) and \( ad - bc \neq 0 \), then \( M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) can be rewritten in the following form

\[
M = S_1 S_2 \cdots S_n M'
\]

with \( M' \in \varepsilon_2 \). Moreover if \( D = \det M = 1 \), then \( M' \) can be \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \).

**Proof.** Using Möbius transformation \( \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in S \) and \( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in S \), we can assume \( a, c \geq 0 \).

Using Möbius transformation \( \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix} \in S \) and \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in S \), \( M \) can be changed to \( M_1 = \begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix} \) with \( a_1 \geq 1 \).

Using Möbius transformation \( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in S \) and \( \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \in S \), \( M_1 \) can be changed to \( M' = \begin{pmatrix} a_1 & b_1 \mod |d_1| \\ 0 & \frac{d_1}{|d_1|} \end{pmatrix} \in \varepsilon_2 \).

Moreover, if \( D = 1 \), we must have \( a_1 = 1, |b_1| = 1 \) and \( b_1 \mod |d_1| = 0 \). \( \square \)

**Remark:** If \( |\det M| = 1 \), then the associated Möbius transformations can not change the continued fraction eventually.

**Lemma 3.3.** Let \( M \in \varepsilon_2 \) and \( D = |\det M| \geq 2 \). Let \( x = [a_0; a_1, a_2, \cdots] \) such that \( B_1 \leq a_j \leq B_2 \) for all \( j \geq 0 \). Using the Algorithm in section 2, we get a sequence \( c_0^* c_1^* c_2^* \cdots \) by (Partialquotients). If \( c_0^* = 0 \), then

\[
c_i^* \leq |Dy_0|
\]

where \( y_0 = [B_2; B_1, B_2, B_1, \cdots] \triangleq [B_2, B_1] = \frac{B_1 B_2 + \sqrt{B_1^2 B_2^2 + 4 B_1 B_2}}{2 B_1} \). Moreover, the equality in (11) holds iff \( a = 0, b = 1, c = D \) and \( d = 0 \).

In addition, assume \( M \neq \begin{pmatrix} 0 & 1 \\ D & 0 \end{pmatrix} \), then

\[
c_i^* \leq \max \{ \lfloor \frac{D}{4} y_0 + 1 \rfloor, D - 1 \}
\]
if \( c_0^* = 0 \).

**Proof.** Let

\[
\frac{p_n}{q_n} = [a_0; a_1, a_2, \ldots, c_n],
\]

then

\[
\Pi_{a_0, a_1, \ldots, a_n} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}.
\]

Thus we have the following simple facts

(13) \[ M \Pi_{a_0, a_1, \ldots, a_n} = \begin{pmatrix} ap_n + bq_n & ap_{n-1} + bq_{n-1} \\ cp_n + dq_n & cp_{n-1} + dq_{n-1} \end{pmatrix}, \]

and

\[
\lim_{n \to \infty} \frac{ap_n + bq_n}{cp_n + dq_n} = \frac{ax + b}{cx + d}
\]

If \( c_0^* = 0 \), then \( c_1^* \) is the second common partial quotient of \( \frac{ap_n + bq_n}{cp_n + dq_n} \) and \( \frac{ap_{n-1} + bq_{n-1}}{cp_{n-1} + dq_{n-1}} \) for any large \( n \). Combining with (13), we must have

(14) \[ c_1^* = \left\lfloor \frac{cx + d}{ax + b} \right\rfloor. \]

Now we are in a position to prove the Lemma, based on (14).

**Case 1:** \( a \geq 1 \)

Using \( x > 1 \), one has

\[
\frac{cx + d}{ax + b} \leq \frac{cx + d}{ax} < \frac{c + d}{a} \leq D
\]

where the third inequality holds by (2). This implies \( c_1^* \leq D - 1 \).

**Case 2:** \( a = 0 \)

In this case, we have \( b > d \), \( bc = D \) and \( c + d \leq D \) by \( M \in \varepsilon_2 \), and

(15) \[ c_1^* = \left\lfloor \frac{D}{bx} + \frac{d}{b} \right\rfloor. \]

If \( b \geq 2 \), by (15), one has

\[ c_1^* \leq \left\lfloor \frac{D}{4} x + 1 \right\rfloor. \]

Notice that if a real number with bounded partial quotients in \([B_1, B_2] \cap \mathbb{Z}\) is such that \( x \leq y_0 \), then

\[ c_1^* \leq \left\lfloor \frac{D}{4} y_0 + 1 \right\rfloor \leq \lfloor Dy_0 \rfloor - 1, \]

since \( y_0 \geq \frac{\sqrt{5} + 1}{2} \) and \( D \geq 2 \).

If \( b = 1 \), we must have \( c = D \) and \( d = 0 \).

Putting all the cases together, we complete the proof. \( \square \)

**Lemma 3.4.** Let \( M \in \varepsilon_2 \) with the form \( \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \) and \( D = |\det M| \geq 1 \). Let \( x = [a_0; a_1, a_2, \ldots] \) such that \( B_1 \leq a_j \leq B_2 \) for all \( j \geq 0 \). Applying the Algorithm in section 2 to \( M \cdot x \), we get a sequence \( c_0^* c_1^* c_2^* c_3^* \ldots \) by [Partialquotients]. If \( c_0^* = 0 \), we must have

\[ c_1^* \leq \left\lfloor \frac{D}{x_0} \right\rfloor, \]
where \( x_0 = [B_1; B_2, B_1, B_2, \cdots] \triangleq [B_1, B_2] = \frac{B_2 B_1 + \sqrt{B_1^2 B_2^2 + 4B_1 B_2}}{2B_2} \).

**Proof.** Let \( b = 0 \) in \((14)\), then we get

\[
(16) \quad c_1^* = \lfloor \frac{cx + d}{ax} \rfloor.
\]

Notice that if a real number with bounded partial quotients in \([B_1, B_2] \cap \mathbb{Z}\) is such that \( x \geq x_0 \), then

\[
(17) \quad c_1^* \leq \lfloor \frac{cx_0 + d}{ax_0} \rfloor.
\]

Thus in order to prove this Lemma, it suffices to show

\[
(18) \quad \frac{cx_0 + d}{ax_0} \leq \frac{D}{x_0}.
\]

If \( a = 1 \), we must have \( c = 0 \) and \( d = D \), this implies \((18)\).

If \( a \geq 2 \), we already have \( ad = D \) and \( c \leq a - 1 \).

**Case 1:** \( D \geq 2x_0 > 2 \)

One has

\[
\begin{align*}
\frac{cx_0 + d}{ax_0} &\leq (a - 1)x_0 + \frac{D}{2} \\
&\leq \frac{D(a - 1)}{2} + \frac{D}{2} \\
&\leq Da
\end{align*}
\]

This implies \((18)\).

**Case 2:** \( x_0 \leq D < 2x_0 \)

It suffices to show

\[
(19) \quad \frac{cx_0 + d}{ax_0} < 2.
\]

This is obvious by the following computation,

\[
\begin{align*}
\frac{cx_0 + d}{ax_0} &\leq (a - 1)x_0 + D \\
&< ax_0 + 2x_0 \\
&\leq 2ax_0
\end{align*}
\]

This implies \((23)\).

**Case 3:** \( D < x_0 \)

By direct computation,

\[
\begin{align*}
\frac{cx_0 + d}{ax_0} &= \frac{c}{a} + \frac{D}{a^2 x_0} \\
&< \frac{a - 1}{a} + \frac{1}{a^2} \\
&< 1.
\end{align*}
\]

This also implies \((18)\). \(\square\)
4. Proof of Theorem 1.1

Proof of Theorem 1.1

Proof. Suppose \( x = [a_0; a_1, a_2, \cdots] \) is such that \( B_1 \leq a_j \leq B_2 \) for \( j \geq j_0 \), and \( M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) is such that \( D = |\text{det} M| \geq 1 \). By Lemmas 3.1 and 3.2 we may assume \( M \in \varepsilon_2 \). By the fact

\[
(20) \quad h(x) = M \cdot x = M \Pi_{a_{a_1a_2\cdots a_{j_0}}} [a_{j_0+1}; a_{j_0+2}, \cdots]
\]

combining with (3), in order to prove Theorem 1.1, we only need to prove the case when all the partial quotients of \( x \) satisfy \( B_1 \leq a_i \leq B_2 \).

By the Algorithm, it suffices to show that for any word \( k_1 k_2 \cdots k_p \) in \( \text{Alloutput-k} \) with \( k_i \in \mathbb{N}^+, i = 1, 2, \cdots, p \), we have

\[
(21) \quad k_1 + k_2 + \cdots + k_p \leq \left\lfloor \frac{D - 1}{B_1} \right\rfloor + \left\lfloor \frac{DB_1B_2 + \sqrt{B_1^2B_3^2 + 4B_1B_2}}{2B_1} \right\rfloor.
\]

Assume \( k_1 \) is the last letter of \( k \)th step \( \text{Alloutput-k} \). Then the output of \( k + 1 \)th step is \( 0k_2, k + 2 \)th step is \( 0k_3, \cdots \).

Case 1: \( k_1 \geq D \)

By (iii) of Lemma 2.1, \( M_{k+1} \) has the form

\[
M_{k+1} = \begin{pmatrix} a_k & 0 \\ c_k & d_k \end{pmatrix} \in \varepsilon_2.
\]

By Lemma 3.4 we have

\[
\sum_{j=2}^p k_j \leq \left\lfloor \frac{D}{x_0} \right\rfloor.
\]

By (ii) of Lemma 2.1, \( k_1 \leq DB_2 \), then

\[
\sum_{j=1}^p k_j \leq \left\lfloor \frac{D}{x_0} \right\rfloor + DB_2
\]

\[
\leq \left\lfloor \frac{DB_2B_1 + \sqrt{B_1^2B_3^2 + 4B_1B_2}}{2B_1} \right\rfloor
\]

\[
\leq \left\lfloor \frac{D - 1}{B_1} \right\rfloor + \left\lfloor \frac{DB_1B_2 + \sqrt{B_1^2B_3^2 + 4B_1B_2}}{2B_1} \right\rfloor.
\]

This implies the Theorem in this case.

By the Remark following Lemma 3.2 we can assume \( D \geq 2 \).

Case 2: \( k_1 \leq D - 1 \)

If \( M_{k+1} \neq \begin{pmatrix} 0 & 1 \\ D & 0 \end{pmatrix} \), by (13) one has

\[
\sum_{j=2}^p k_j \leq \max\{\left\lfloor \frac{D}{4} y_0 + 1 \right\rfloor, D - 1\}.
\]

Direct computation (spliting the computation into \( B_1 = 1 \) or \( B_1 \geq 2 \)),

\[
\sum_{j=1}^p k_j \leq D - 1 + \max\{\left\lfloor \frac{D}{4} y_0 + 1 \right\rfloor, D - 1\}
\]

\[
\leq \left\lfloor \frac{D - 1}{B_1} \right\rfloor + \left\lfloor \frac{DB_1B_2 + \sqrt{B_1^2B_3^2 + 4B_1B_2}}{2B_1} \right\rfloor.
\]
This implies the Theorem in this case.

If $M_{k+1} = \begin{pmatrix} 0 & 1 \\ D & 0 \end{pmatrix}$, by (21) one has
\[ c_1' \leq \lfloor Dy_0 \rfloor. \]

Thus in order to prove the Theorem in this case, it suffices to show
\[ k_1 \leq \frac{D - 1}{B_1}. \]

By the Algorithm of $k$th step, we have
\[ M_{k+1} = \Pi_{c, \ldots, c_{N-1}} \left( \begin{array}{cc} k_1 & 1 \\ 1 & 0 \end{array} \right) \left( \begin{array}{cc} 0 & 1 \\ D & 0 \end{array} \right) \in D_2 \cup D_2', \]
and $M_{k+1} \Pi_{a_1, a_2, \ldots, a_N-1} \in \varepsilon_2$.

This implies
\[ M_{k} \Pi_{a_1, a_2, \ldots, a_N-1} = \Pi_{c_{i_1}, c_{i_2}, \ldots, c_{i_{N-1}}} \left( \begin{array}{cc} k_1 & 1 \\ 1 & 0 \end{array} \right) \left( \begin{array}{cc} 0 & 1 \\ D & 0 \end{array} \right) \left( a_N & 1 \\ 1 & 0 \right) \right)^{-1}. \]

By direct computation, one has
\[ M_{k} \Pi_{a_1, a_2, \ldots, a_N-1} = \Pi_{c_{i_1}, c_{i_2}, \ldots, c_{i_{N-1}}} \left( k_1 \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \left( a_N & 1 \\ 1 & 0 \end{array} \right) \right)^{-1}. \]

Since all entries of $M_{k} \Pi_{a_1, a_2, \ldots, a_N-1}$ are non-negative, we must have
\[ -k_1 a_N + D \geq 1. \]

This implies
\[ k_1 \leq \left\lfloor \frac{D - 1}{B_1} \right\rfloor, \]
since $a_N \geq B_1$. We complete the proof. \(\square\)

**ACKNOWLEDGMENTS**

I would like to thank Svetlana Jitomirskaya for comments on earlier versions of the manuscript. This research was partially supported by the AMS-Simons Travel Grant (2016-2018) and NSF DMS-1401204.

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