The Laplacian Eigenvalues and Invariants of Graphs

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Abstract

In this paper, we investigate some relations between the invariants (including vertex and edge connectivity and forwarding indices) of a graph and its Laplacian eigenvalues. In addition, we present a sufficient condition for the existence of Hamiltonicity in a graph involving its Laplacian eigenvalues.

Key words: Laplacian eigenvalue, Connectivity, Hamiltonicity, Forwarding index.

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1 introduction

Let $G = (V, E)$ be a simple graph with vertex set $V(G) = \{v_1, \cdots, v_n\}$ and edge set $E(G) = \{e_1, \cdots, e_m\}$. Denote by $d(v_i)$ the degree of vertex $v_i$. If $D(G) = diag(d_u, u \in V)$ is the diagonal matrix of vertex degrees of $G$ and $A(G)$ is the $0 - 1$ adjacency
matrix of $G$, the matrix $L(G) = D(G) - A(G)$ is called the Laplacian matrix of a graph $G$. Moreover, the eigenvalues of $L(G)$ are called Laplacian eigenvalues of $G$. Furthermore, the Laplacian eigenvalues of $G$ are denoted by

$$0 = \sigma_0 \leq \sigma_1 \leq \cdots \leq \sigma_{n-1},$$

since $L(G)$ is positive semi-definite. In recent years, the relations between invariants of a graph and its Laplacian eigenvalues have been investigated extensively. For example, Alon in [1] established that there are relations between an expander of a graph and its second smallest eigenvalue; Mohar in [13] presented a necessary condition for the existence of Hamiltonicity in a graph in terms of its Laplacian eigenvalues. The reader is referred to [3], [9] and [11] etc.

The purpose of this paper is to present some relations between some invariants of a graph and its Laplacian eigenvalues. In Section 2, the relations between the vertex and edge connectivities of a graph and its Laplacian eigenvalues are investigated. In Section 3, we present a sufficient condition for the existence of Hamiltonicity in a graph involving its Laplacian eigenvalues. In last Section, the lower bounds for forwarding indices of networks are obtained. Before finishing this section, we present a general discrepancy inequality from Chung[4], which is very useful for later.

For a subset $X$ of vertices in $G$, the volume $\text{vol}(X)$ is defined by $\text{vol}(X) = \sum_{v \in X} d_v$, where $d_v$ is the degree of $v$. For any two subsets $X$ and $Y$ of vertices in $G$, denote

$$e(X,Y) = \{(x,y) : x \in X, y \in Y, \{x,y\} \in E(G)\}.$$

**Theorem 1.1** Let $G$ be a simple graph with $n$ vertices and average degree $d = \frac{1}{n} \text{vol}(G)$. If the Laplacian eigenvalues $\sigma_i$ of $G$ satisfy $|d - \sigma_i| \leq \theta$ for $i = 1, 2, \cdots, n-1$, then for any two subsets $X$ and $Y$ of vertices in $G$, we have

$$|e(X,Y) - \frac{d}{n} |X||Y| + d|X \cap Y| - \text{vol}(X \cap Y)| \leq \frac{\theta}{n} \sqrt{\frac{1}{n} |X| (n-|X|)|Y| (n-|Y|)}.$$

### 2 Connectivity

The vertex connectivity of a graph $G$ is the minimum number of vertices that we need to delete to make $G$ is disconnected and denoted by $\kappa(G)$. Fiedler in [6] proved that if $G$ is not the complete graph, then $\kappa(G)$ is at least the value of the second smallest Laplacian eigenvalue. In here, we present another bound for the vertex connectivity of a graph.
**Theorem 2.1** Let $G$ be a simple graph of order $n$ with the smallest degree $\delta \leq \frac{n}{2}$ and average degree $d$. If the Laplacian eigenvalues $\sigma_i$ satisfies $|d - \sigma_i| \leq \theta$ for $i \neq 0$, then

$$\kappa(G) \geq \delta - (2 + 2\sqrt{3})^2 \theta^2 \frac{\delta}{n}.$$ 

**Proof.** Let $c = 2 + 2\sqrt{3}$. If $\theta \geq \frac{\delta}{c}$, there is nothing to show. We assume that $\theta < \frac{\delta}{c}$.

Suppose that there exists a subset $S \subset V(G)$ with $|S| < \delta - (c\theta)^2 \frac{\delta}{n}$ such that the induced graph $G[V \setminus S]$ is disconnected. Denote by $U$ the set of vertices of the smallest connected component of $G[V \setminus S]$ and $W = V \setminus (S \cup U)$. Since the smallest degree of $G$ is $\delta$, $|S| + |U| > \delta$, which implies $|U| \geq \frac{\delta}{c\theta}$. Moreover, $|W| = n - (|U| + |S|) \leq \frac{n - \delta}{2} \leq \frac{n}{4}$. Because $U$ and $W$ are disjoint for two subsets of $G$, by 1.1, we have

$$\frac{d}{n} |U||W| \leq \theta \frac{n}{d} \sqrt{|U||W|(n - |U|)(n - |W|)} \leq \sqrt{|U||W|}.$$ 

Hence

$$|U| \leq \frac{\theta^2 n^2}{d^2 |W|} \leq \frac{\theta}{d} \frac{n}{d} \frac{\theta n}{d} \frac{4 \theta n}{c} \frac{n}{d},$$

since $\frac{\theta}{d} < \frac{\theta}{c} < \frac{1}{c}$. By using Corollary 4 in [4], we have

$$|2e(U)| - \frac{d|U|(|U| - 1)}{n} \leq \frac{2\theta}{n} |U|(n - \frac{|U|}{2}).$$

Then

$$2|e(U)| \leq 2\theta |U| + \frac{d}{n} |U|^2$$

$$\leq (2\theta + \frac{4 \theta n}{nc})|U|$$

$$= (2 + \frac{4}{c})\theta |U|.$$ 

Hence, by $\theta < \frac{\delta}{c}$ and $c = 2 + 2\sqrt{3}$,

$$|e(U, S)| \geq \delta |U| - 2|e(U)|$$

$$\geq (\delta - (2 + \frac{4}{c})\theta)|U|$$

$$> (1 - (2 + \frac{4}{c}))\delta |U|$$

$$> (\frac{1}{2} + \frac{1}{c})\delta |U|.$$
On the other hand, by 1.1 and $|S| \leq \delta$, $|U| \geq \frac{2\theta^2}{\delta}$ and $\frac{d}{n} \leq \frac{1}{2}$, we have

$$|e(U, S)| \leq \frac{d}{n}|U||S| + \theta \sqrt{|U||S|}$$

$$\leq \left( \frac{d\delta}{n} + \theta \frac{\delta \sqrt{\delta}}{c\theta} \right) |U|$$

$$= \left( \frac{1}{2} + \frac{1}{c} \right) \delta |U|.$$ 

It is a contradiction. Therefore the result holds. □

**Corollary 2.2 ([10])** Let $G$ be a $d$–regular graph of order $n$ with $d \leq \frac{n}{2}$. Denote by $\lambda$ the second largest absolute eigenvalue of $A(G)$. Then

$$\kappa(G) \geq d - \frac{36\lambda^2}{d}.$$ 

**Proof.** Since $G$ is a $d$–regular graph, the eigenvalues of $A(G)$ are $d - \sigma_0, d - \sigma_1, \ldots, d - \sigma_{n-1}$. Hence $\lambda$ satisfies $|d - \sigma_i| \leq \lambda$ for $i \neq 0$. It follows from Theorem 2.1 that $\kappa(G) \geq d - \frac{(2+2\sqrt{3})d^2}{d} \geq d - \frac{36\lambda^2}{d}$. □

From [10], for a $d$–regular graph, the lower bound for $\kappa(G)$ in Corollary 2.2 is tight up to a constant factor, which implies Theorem 2.1 is tight up to a constant factor.

It is known that the edge connectivity $\kappa'(G)$ of a graph $G$ is the minimum number of edges that need to delete to make disconnected. In [7], Goldsmith and Entringer gave a sufficient condition for edge connectivity equal to the smallest degree. In here, we present also a sufficient condition for edge connectivity equal to the smallest degree in terms of its Laplacian eigenvalues.

**Theorem 2.3** Let $G$ be a graph of order $n$ with average degree $d$ and the smallest degree $\delta$. If the Laplacian eigenvalues satisfy $2 \leq \sigma_1 \leq \sigma_{n-1} \leq 2d - 2$, then $\kappa'(G) = \delta$.

**Proof.** Let $U$ be a subset of vertices of $G$ with $|U| \leq \frac{n}{2}$.

If $1 \leq |U| \leq \delta$, then for every vertex $u \in U$, $u$ is adjacent to at least $\delta - |U| + 1$ vertices in $G \setminus U$. Therefore,

$$|e(U, G \setminus U)| \geq |U|(|\delta - |U| + 1) \geq \delta.$$ 

If $\delta < |U| \leq \frac{n}{2}$, let $\theta = d - 2$. Since $2 \leq \sigma_1 \leq \sigma_{n-1} \leq 2d - 2$, $|d - \sigma_i| \leq \theta$ for $i \neq 0$. By Theorem 1.1,

$$||e(U, V \setminus U)| - \frac{d}{n}|U||V \setminus U|| \leq \frac{\theta}{n}|U|(n - |U|).$$
Thus,
\[ |e(U, V \setminus U)| \geq \frac{d - \theta}{n} |U| (n - |U|) \geq \frac{d - \theta}{n} \delta (n - \delta) \geq \frac{2 \delta (n - \delta)}{n} \geq \delta. \]

Hence there are always at least \( \delta \) edges between \( U \) and \( V \setminus U \). Therefore \( \kappa'(G) = \delta \). 

\[ \blacksquare \]

## 3 Hamiltonicity and the chromatic number

In this section, we first give an upper bound for the independence number \( \alpha(G) \), which is used to present a sufficient condition for a graph to have a Hamilton cycle. Moreover, a lower bound for the chromatic number of a graph is obtained. The independence number is the maximum cardinality of a set of vertices of \( G \) no two of which are adjacent.

**Lemma 3.1** Let \( G \) be a graph of order \( n \) with average \( d \). If the Laplacian eigenvalues satisfies \( |d - \sigma_i| \leq \theta \) for \( i \neq 0 \), then

\[ \alpha(G) \leq \frac{2n \theta + d}{d + \theta}. \]

**Proof.** Let \( U \) be an independent set with the size \( \alpha(G) \). By Corollary 4 in [4], we have

\[ |2|e(U)| - d|U||(|U| - 1)| \leq \frac{2 \theta}{n} |U| (n - |U|/2). \]

Hence \( |U| \leq \frac{2 \theta + d}{d + \theta} \). \( \blacksquare \)

**Lemma 3.2** [5] Let \( G \) be a graph. If the vertex connectivity of \( G \) is at least as large as its independence number, then \( G \) is Hamiltonian.

**Theorem 3.3** Let \( G \) be a graph of order \( n \) with average \( d \) and the smallest degree \( \delta \). If the Laplacian eigenvalues satisfies \( |d - \sigma_i| \leq \theta \) for \( i \neq 0 \) and \( \delta - (2 + 2 \sqrt{3}) \frac{2 \theta^2}{\delta} \geq \frac{2n \theta + d}{d + \theta} \), then \( G \) is Hamiltonian.

**Proof.** By Theorem 2.1, \( G \) has at least \( \delta - (2 + 2 \sqrt{3}) \frac{2 \theta^2}{\delta} \) vertex connected. On the other hand, by Lemma 3.1, the independence number of \( G \) is at most \( \frac{2n \theta + d}{d + \theta} \). It follows from Lemma 3.2 that \( G \) is Hamiltonian. \( \blacksquare \)
Theorem 3.4 Let $G$ be a connected graph of order $n$ with the smallest degree $\delta$. If $\sigma_1 \geq \frac{\sigma_{n-1} - \delta}{\sigma_{n-1}} n$, then $G$ is Hamiltonian.

Proof. By a theorem in [6], $\kappa(G) \geq \sigma_1$. On the other hand, by Corollary 3.3 in [15], the independence number $\alpha(G) \leq \frac{\sigma_{n-1} - \delta}{\sigma_{n-1}} n$. It follows from Lemma 3.2 that $G$ is Hamiltonian.

The proper coloring of the vertices of $G$ is an assignment of colors to the vertices in such a way that adjacent vertices have distinct colors. The chromatic number, denoted by $\chi(G)$, is the minimal number of colors in a vertex coloring of $G$.

Theorem 3.5 Let $G$ be a graph of order $n$ with the smallest degree $\delta \geq 1$. Then

$$\chi(G) \geq \frac{\sigma_{n-1}}{\sigma_{n-1} - \delta}.$$ 

Moreover, if $G$ is a $d-$ regular bipartite graph, or a complete $r-$partite graph $K_{s,s,\ldots,s}$, then equality holds.

Proof. Let $V_1, V_2, \ldots, V_\chi$ denote the color class of $G$. Denote by $e$ the vector with all component equal to 1. Let $s_i$ be the restriction vector of $\frac{1}{|V_i|} e$ to $V_i$; that is, $(s_i)_j = \frac{1}{|V_i|}$, if $j \in V_i$; $(s_i)_j = 0$, otherwise. Thus $S = (s_1, \ldots, s_\chi)$ is an $n \times \chi$ matrix and $S^T S = I_n$. Let $B = S^T L(G) S = (b_{ij})$ and its eigenvalues $\mu_0 \leq \mu_1 \leq \cdots \leq \mu_{\chi-1}$.

By eigenvalue interlacing, it is easy to see that $\mu_0 = 0$ and $\mu_{\chi-1} \leq \sigma_{n-1}$. Moreover, $b_{ii} = \frac{1}{|V_i|} \sum_{v \in V_i} d_v \geq \delta$. Hence

$$\delta \chi \leq \text{tr}B = \mu_0 + \cdots \mu_{\chi-1} \leq (\chi - 1)\sigma_{n-1},$$

which yields the desired inequality. If $G$ is a $d-$ regular graph, then $\chi = 2$, $\delta = d$ and $\sigma_{n-1} = 2d$. So equality holds. If $G$ is a complete $r-$partite graph, then $\chi = r$, $\delta = (r - 1)s$ and $\sigma_{n-1} = \frac{r}{r-1}s$. Hence equality holds.

4 Forwarding indices of graphs

In this section, we discuss some relations between the Laplacian eigenvalues of a graph and its forwarding indices.

A routing $R$ of a graph $G$ of order $n$ is a set of $n(n-1)$ paths specified for all ordered pairs $u$ and $v$ of vertices of $G$. Denote $\xi(G, R, v)$ by the number of paths of
Let $X$ be a proper subset of $V$. The vertex cut induced by $X$ is

$$N(X) = \{ y \in V \mid \{x, y\} \in E(G) \}.$$ 

Moreover, denote $X^+$ by the complement of $X \cup N(X)$ in $V$.

The vertex expanding factor is defined by

$$\gamma(G) = \min \{ \frac{|N(X)|}{|X||X^+|} \mid X \subseteq V, 1 \leq |X| \leq n - 1, |X^+| \geq 1 \},$$

where the min on a void set of $X$ is taken to be infinite.

**Theorem 4.1** Let $G$ be a graph of order $n$ with average degree $d$. If the Laplacian eigenvalues satisfies $|d - \sigma_i| \leq \theta$ for $i \neq 0$, then

$$\gamma(G) \geq \frac{d^2 - \theta^2}{n\theta^2}.$$ 

**Proof.** Let $U$ be a subset of $G$ such that

$$\gamma(G) = \frac{|N(U)|}{|U||U^+|}, \quad 1 \leq |U| \leq n - 1, \quad |U^+| \geq 1.$$ 

Set $W = V \setminus (U \cup N(U))$. By Theorem 1.1, we have

$$||e(U, W)| - \frac{d}{n}|U||W|| \leq \frac{\theta}{n}\sqrt{|U|(n - |U|)|W|(n - |W|)}.$$ 

Hence

$$d^2|U||W| \leq \theta^2(|U| + |N(U)|)(|W| + |N(W)|).$$ 

Then

$$\frac{|N(U)|}{|U||U^+|} = \frac{|N(U)|}{|U|(n - |W|)} \geq \frac{d^2 - \theta^2}{n\theta^2}.$$ 

We complete the proof. □

**Theorem 4.2** Let $G$ be a graph of order $n$. If $\sigma_1 \leq \frac{1}{2}$, then $\xi(G) \geq \sqrt{\frac{1 - 2\sigma_1}{\sigma_1}}$. 

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Proof. By Lemma 2.4 in [1], we have
\[ \sigma_1 \geq \frac{c^2}{4 + 2c^2}, \]
where \( c \) satisfies \( \frac{|N(X)|}{|X|} \geq c \) for every \( |X| \leq \frac{n}{2} \) and \( X \subset U \). Hence
\[ \gamma(G) \leq c \leq \sqrt{\frac{4\sigma_1}{1 - 2\sigma_1}}. \]
On the other hand, there exists a subset \( U \) such that \( \gamma(G) = \frac{|N(U)|}{|U||U^+|} \). It follows from the definition of \( \xi(G) \) that \( 2|U||U^+| \geq \xi(G)|N(U)| \), since there does not exist edges between \( U \) and \( U^+ \). Hence
\[ \xi(G) \geq \frac{2|U||U^+|}{|N(U)|} = \frac{2}{\gamma(G)} \geq \frac{1 - 2\sigma_1}{\sigma_1}. \]
We finish the proof. 

Lemma 4.3 Let \( G \) be a graph of order \( n \) with average degree \( d \) and let \( \beta(G) = \min \{ \frac{|e(U,V \setminus U)|}{|U|(n-|U|)} \mid 1 \leq |U| \leq n-1 \} \). If the Laplacian eigenvalues satisfy \( |d - \sigma_i| \leq \theta \) for \( i \neq 0 \), then
\[ \beta(G) \leq \frac{d + \theta}{n}. \]

Proof. By the definition of \( \beta(G) \), there exists a subset \( U \) such that \( \beta(G) = \frac{|e(U,V \setminus U)|}{|U|(n-|U|)} \). On the other hand, by Theorem 1.1, we have
\[ ||e(U,V \setminus U)| - \frac{d}{n}|U|(n-|U|)||e \theta |n|U|(n-|U|). \]
Hence \( \beta(G) \leq \frac{d + \theta}{n} \).

Theorem 4.4 Let \( G \) be a graph of order \( n \) with average degree \( d \). If the Laplacian eigenvalues satisfy \( |d - \sigma_i| \leq \theta \) for \( i \neq 0 \), then
\[ \pi(G) \geq \frac{2n}{d + \theta}. \]

Proof. It follows from Theorem 1 \( \pi(G) \beta(G) \geq 2 \) in [14] and Lemma 4.3 that the result holds.

Remark The lower bounds for \( \xi(G) \) and \( \pi(G) \) are tight up to a constant factor. For example, Let \( P_n \) be a path of order \( n \). It is easy to see that \( \xi(P_n) = 2(\lfloor \frac{n}{2} \rfloor (\lceil \frac{n}{2} \rceil - 1) \), \( \pi(G) = 2(\lceil \frac{n}{2} \rceil (\lfloor \frac{n}{2} \rfloor ) \); while \( \sigma_1 = 4 \sin^2 \frac{\pi}{2n} \).
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