CONGRUENCES OF GALOIS REPRESENTATIONS ATTACHED TO EFFECTIVE $A$-MOTIVES OVER GLOBAL FUNCTION FIELDS

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Abstract. This article investigates congruences of $p$-adic representations arising from effective $A$-motives defined over a global function field $K$. We give a criterion for two congruent $p$-adic representations coming from strongly semi-stable effective $A$-motives to be isomorphic up to semi-simplification when restricted to decomposition groups of suitable places of $K$. This is a function field analog of Ozeki-Taguchi’s criterion for $\ell$-adic representations of number fields. Motivated by a non-existence conjecture on abelian varieties over number fields stated by Rasmussen and Tamagawa, we also show that there exist no strongly semi-stable effective $A$-motives with some constrained.

Contents

1. Introduction 1
2. Preliminary 5
3. Effective $A$-motives 6
4. Galois representations arising from local shtukas 11
5. Rigid analytic aspects of effective $A$-motives 18
6. Congruences of Galois representations 22
Appendix A. Proof of Thoreom 2.2 27
References 29

1. INTRODUCTION

1.1. Motivation and results. Let $Q$ be a global function field whose constant field is a finite field $\mathbb{F}_q$ with $q$-elements of characteristic $p$. We fix a place $\infty$ of $Q$ and suppose that its residue field $\mathbb{F}_\infty$ is equal to $\mathbb{F}_q$. Define $A \subset Q$ to be the ring of rational functions regular outside $\infty$. Let $K$ be a global function field equipped with an injective $\mathbb{F}_q$-algebra homomorphism $\gamma: A \to K$. Then $\gamma$ canonically extends to $\gamma: Q \to K$, and $K$ is a finite extension of $\gamma(Q)$. Denote by $G_K := \text{Gal}(K^{\text{sep}}/K)$ the absolute Galois group of $K$. For a place $v$ of $K$, we denote by $G_v$ the decomposition group of $G_K$ at $v$.

Let $p \subset A$ be a maximal ideal and set $\mathbb{F}_p := A/p$. Let $Q_p$ be the $p$-adic completion of $Q$ and let $A_p$ be its valuation ring. In this paper, we mean by $p$-adic representations of $G_K$ that finite-dimensional $Q_p$-vector spaces $V$ on which $G_K$ acts continuously. We write $V_v = V|_{G_v}$ for the restriction to $G_v$. Now let $V$ and $V'$ be two $p$-adic representations of $G_K$ with the same dimension and let $v$ be a finite place (i.e., a place not lying above $\infty$) of $K$. We write

$$V_v \simeq_{\text{ss}} V'_v$$

if their semi-simplifications are isomorphic as $p$-adic representations of $G_v$. Since $V_v$ and $V'_v$ have many common properties when $V_v \simeq_{\text{ss}} V'_v$, it is worth considering when two $p$-adic representations
are isomorphic up to semi-simplification. To do this, we focus on the case where $V_v$ and $V_v'$ are congruent modulo $p$ in the following sense: For a choice of $G_v$-stable $A_p$-lattices $T \subset V_v$ and $T' \subset V_v'$, we write

$$V_v \equiv_{ss} V_v' \pmod{p}$$

if $T/pT$ and $T'/pT'$ have isomorphic semi-simplifications $(T/pT)^{ss}$ and $(T'/pT')^{ss}$ as $\mathbb{F}_p$-representations of $G_v$. As we will see in Remark 2.5, two $p$-adic representations isomorphic up to semi-simplification are congruent modulo $p$, and so we are interested in the converse:

**Question 1.1.** When does the implication

$$V_v \equiv_{ss} V_v' \pmod{p} \implies V_v \simeq_{ss} V_v'$$

hold?

In number field case, Ozeki and Taguchi [OT14] prove a criterion for two $\ell$-adic Galois representations of the absolute Galois group $G_F$ of a number field $F$ as the following type: Let $v$ be a finite place of $F$. If the prime $\ell$ is sufficiently large (with respect to $F$, $v$, and the type of the $\ell$-adic representations under considerations), then for any place $u$ lying above $\ell$ and any $\ell$-adic representations $V$ and $V'$ of $G_F$ satisfying a certain set of conditions (which in particular contains semi-stability at $v$ and $u$), the two congruence relations $V_v \equiv_{ss} V_v'$ (mod $\ell$) and $V_u \equiv_{ss} V_u'$ (mod $\ell$) imply $V_v \simeq_{ss} V_v'$. The method of the proof is, as in [Oze11] Lemma 3.9 and [RT17] §3.3, to recover the Frobenius characteristic polynomials of $V_v$ and $V_v'$ at $v$ from their reduction modulo $\ell$ by computing a bound of the coefficients of them, which follows from the facts in $p$-adic Hodge theory due to Caruso [Car06] and Caruso-Savitt [CS09] that describe relations between Hodge-Tate weights and tame inertia weights of torsion semi-stable $p$-adic representations. Here tame inertia weights are numerical invariants determined by inertia action on the semi-simplifications of residual representations.

Question 1.1 is meaningful from the point of view of Serre’s uniformity problem. It is proved by Serre [Ser72] that for any elliptic curve $E$ over $\mathbb{Q}$ without CM, there exists a constant $C(E) > 0$ such that for any prime number $p > C(E)$, the residue representation

$$\rho_{E,p} : G_{\mathbb{Q}} \to \text{GL}_2(\mathbb{F}_p) \simeq \text{GL}_2(\mathbb{F}_p)$$

is surjective. In contrast, Serre asked whether $C(E)$ can be made independent of $E$: Does there exist $C > 0$ such that for any non-CM elliptic curve $E$ over $\mathbb{Q}$ and any prime number $p > C$, the residue representation $\rho_{E,p}$ is surjective? Replacing the surjectivity condition with another, one can get a variant of Serre’s uniformity problem. For example, as we will see below, a conjecture of Rasmussen and Tamagawa [RT08] is a kind of this problem. In a function field setting, Pink and Rötschke [PR09] show a Drinfeld module analog of Serre’s result. Thus one can consider the uniformity problem for Drinfeld modules. To solve these problems, we can expect that it is useful to consider the congruences of Galois representations. For instance, under the assumption of the Generalized Riemann Hypothesis, the Rasmussen-Tamagawa conjecture is proven in [RT17] by showing the implication as in Question 1.1 for $\ell$-adic representations coming from abelian varieties. In this context, we can interpret the work of Ozeki and Taguchi [OT14] as giving a general framework for studying Serre’s uniformity problem in number field case, and so it is natural to be interested in function field case.

The purpose of this article is to study the congruence relation between two $p$-adic Galois representations arising from effective $A$-motives over $K$. To be more precise, set $A_K := A \otimes_{\mathbb{F}_q} K$ and consider the endomorphism $\sigma$ of $A_K$ given by $\sigma(a \otimes \lambda) = a \otimes \lambda^q$ for $a \in A$ and $\lambda \in K$. Let $J = J_K \subset A_K$ be the ideal generated by $\{a \otimes 1 - 1 \otimes \gamma(a) \mid a \in A\}$, so that it is the kernel of the map $A_K \to K$ with $a \otimes \lambda \mapsto \gamma(a)\lambda$. An effective $A$-motive over $K$ is a pair $M = (M, \tau_M)$ consisting of a locally free $A_K$-module $M$ of finite rank and an injective $A_K$-homomorphism $\tau_M : \sigma^* M := M \otimes_{A_K} \sigma A_K \to M$ whose cokernel is a finite-dimensional $K$-vector space annihilated by a power of $J$. The rank and dimension of $M$ are defined by $\text{rk} M := \text{rk}_{A_K} M$ and $\dim M := \dim_K \text{Coker}(\tau_M)$, and $M$ is said to
be of height $\leq h$ if $J^h \cdot \text{Coker}(\tau_M) = 0$. The notion of effective $A$-motives is a variant of $t$-motives in the sense of Anderson [And86] and it is also a generalization of Drinfeld $A$-modules (cf. [Dri74], [DH87], and [Hay92]) and abelian Anderson $A$-modules (cf. [And86], [Cos96], and [Har19]), which are a function field analog of abelian varieties over number fields. In this article, we consider two kinds of “cohomology realizations” of $M$; the first is the $p$-adic realizations; the second is the de Rham realizations. The $p$-adic realization $H^1_p(M, Q_p)$ of $M$ is a $p$-adic representation of $G_K$ with dimension equal to $\text{rk} \ M$. Taking the $Q_p$-linear dual of $H^1_p(M, Q_p)$, we get the $p$-adic rational Tate module $V_pM$, which is the central consideration of our study. The de Rham realization $H^1_{dR}(M, K)$ of $M$ is a finite-dimensional $K$-vector space equipped with a descending filtration by $K$-subspaces so-called the Hodge-Pink filtration. As jumps of this filtration, we define the multi-set $\mathcal{HP}(M)$ of Hodge-Pink weights of $M$, which play the role of Hodge-Tate weights. We should note that we adopt the negatives of usual Hodge-Pink weights as the definition of Hodge-Pink weights; see Remark 3.20.

In order to investigate the congruence of $V_pM$, we impose the strong semi-stability on effective $A$-motives, which is originally defined for analytic $\tau$-sheaves by Gardeyn; see [Gar02] and [Gar02]. The Tate module $V_pM$ associated with strongly semi-stable $M$ behaves as “semi-stable representations” in the following sense. If $M$ has strongly semi-stable reduction at a finite place $u$ of $K$ lying above $p$, the restriction $V_pM|_{G_u}$ is isomorphic (up to semi-simplification) to a direct sum of $p$-adic representations arising from effective local shtukas. Local shtukas are linear algebraic objects over the valuation ring of $K_u$ describing lattices of equal-characteristic crystalline representations in Hodge-Pink theory (cf. [Har09], [Kim09], [GL11], [Har11], and [HK20]). Although there are still no definitions of equal-characteristic analog of Hodge-Tate and semi-stable $p$-adic representations, we can see that $V_pM|_{G_u}$ has a behavior expected of equal-characteristic semi-stable representations. As we will see in Theorem 5.15, the Hodge-Pink weights (and the dimension) of $M$ relate with the tame inertia weights of $V_pM|_{G_u}$ under some condition on ramification. This is an analogue of results on semi-stable $p$-adic representations in [Car00] and [CS09]. On the other hand, the strong semi-stability of $M$ at a finite place $v \mid p$ implies that $V_pM|_{G_v}$ is “semi-stable” in the sense that the inertia subgroup $I_v$ of $G_v$ acts on $V_pM$ unipotently. This allows us to consider the Frobenius characteristic polynomial of $V_pM$ at $v$ and define the multi-set $W_v(M)$ of Weil weights at $v$ via $\infty$-adic absolute values of roots of the characteristic polynomial. Then we will see in Proposition 6.5 that the Weil weights are related to the Hodge-Pink weights of $M$. In consequence, if $M$ has strongly semi-stable reduction at both $u \mid p$ and $v \mid p$, then the coefficients of the Frobenius characteristic polynomial at $v$ can be described by the reduction modulo $p$ of $V_pM|_{G_u}$ from the perspective of tame inertia weights.

To state our first result, let $\text{Mot}_{K,r,v}(u, h)$ be the set of effective $A$-motives of rank $r$ and of height $\leq h$ which are strongly semi-stable at $v$ and $u$, and satisfy a set of conditions; see Definition 6.6. Note that we have $v \nmid p$ and $u \nmid p$ under the assumptions of our results (Theorems 1.2 and 1.3). We denote by $d_p = [\mathbb{F}_p : \mathbb{F}_q]$ the degree of $p$. Let $[K : Q]_i$ be the inseparable degree of $K/\gamma(Q)$, that is, $[K : Q]_i = [K : K_i]$ for the maximal separable extension $K_i$ of $\gamma(Q)$ in $K$. We define

$$D_K = \begin{cases} \max\{d_q \mid q \subset A \text{ is a maximal ideal dividing } \mathfrak{d}\} & \text{if } \mathfrak{d} \neq A, \\ 1 & \text{if } \mathfrak{d} = A, \end{cases}$$

where $\mathfrak{d}$ is the relative discriminant of $K_\mathfrak{s}/\gamma(Q)$. Using the strategy as in [OT14], we obtain the following result:

**Theorem 1.2** (= Theorem 6.7). Let $r$ and $h$ be as above and fix a finite place $v$ of $K$ with degree $d_v = [\mathbb{F}_v : \mathbb{F}_q]$. For any maximal ideal $p$ of $A$ with $d_p > \max\{d_q r^2 h, [K : Q]_i h, D_K\}$ and any finite place $u$ of $K$ lying above $p$, the following holds: For any $M \in \text{Mot}_{K,r,v}(u, h)$ and $M' \in \text{Mot}_{K,r,v}(u, h)$, if $M \equiv M' \pmod{p}$, then $M \equiv M' \pmod{p}$.
\[ \text{Mot}_{K,r,v}(u, (q_p - 2)[K : Q]_i^{-1}), \text{ if both} \]

\[ \begin{cases} V_p M |_{G_v} \equiv_{ss} V_p M' |_{G_v} \pmod{p}, \\ V_p M |_{G_u} \equiv_{ss} V_p M' |_{G_u} \pmod{p} \end{cases} \]

hold, then one has \( V_p M |_{G_v} \simeq_{ss} V_p M' |_{G_v}, \dim M = \dim M', \text{ and } \mathcal{W}_v(M) = \mathcal{W}_v(M') \).

Our second result (Theorem 6.9) is motivated by a non-existence conjecture on constrained abelian varieties over number fields due to Rasmussen and Tamagawa in [RT08, Conjecture 1] as follows. For a fixed number field \( F / \mathbb{Q} \) and a fixed integer \( g > 0 \), the conjecture says that for a prime number \( \ell \) large enough (with respect to \( F \) and \( g \)), there exist no \( g \)-dimensional abelian varieties \( A \) over \( F \) with good reduction outside \( \ell \) such that the \( G_F \)-action on the residue Galois representations \( A[\ell] \) are given by upper-triangular matrices whose diagonal components are powers of the modulo \( \ell \) cyclotomic character. The condition on \( A[\ell] \), in other words, is that the congruence relation \( V_r A \equiv_{ss} \bigoplus_{i=1}^g \mathbb{Q}_\ell(\chi_i^\ell) \pmod{\ell} \) holds for some integers \( n_i \), where \( \mathbb{Q}_\ell(\chi_i^\ell) \) is the representation space of the \( n_i \)-power of the \( \ell \)-adic cyclotomic character \( \chi_\ell : G_F \to \mathbb{Z}_\ell^\times \). Although the conjecture of Rasmussen and Tamagawa is an open problem in general, various partial results are known; see [Oze11, Oze13, AM14, Bou15, RT17], and [Lom18]. For example, in the case where considered abelian varieties \( A \) have everywhere semi-stable reduction, the non-existence of such abelian varieties is proved in [RT17, Theorem 3.6].

Motivated by the above conjecture, we would like to consider the non-existence of effective \( A \)-motives \( M \) whose \( p \)-adic Tate module \( V_p M \) are “residually Borel”. In the case where \( Q \) is the rational function field \( F_q(t) \) and \( A = F_q[t] \), the author’s previous work [Oku19] proves that, under the assumption \( r \geq 2 \) and \( r \nmid [K : Q]_i \), if \( d_p \) is sufficiently large, then there exist no rank-\( r \) Drinfeld \( F_q[t]-\)modules \( \mathcal{E} \) over \( K \) with good reduction outside \( p \) such that the \( G_K \)-action on \( T_p \mathcal{E} \otimes \mathbb{F}_p \) is given by upper-triangular matrices whose diagonal components are \( \mathbb{F}_p \)-valued characters arising from tensor powers of the Carlitz motive \( \mathcal{C} \) (Example 3.5). Here the Carlitz motive \( \mathcal{C} \) is a one-dimensional rank-one effective \( F[t]-\)motive with everywhere good reduction, which gives a function field analog of the theory on cyclotomic extensions of number fields. For general \( A \), replacing \( \mathcal{C} \) by one-dimensional rank-one effective \( A \)-motives with good reduction, we show the following non-existence result:

**Theorem 1.3** (\( = \) Theorem 6.9). Let \( r \geq 2 \) be an integer and \( h \geq 0 \). Let \( v \) be a finite place of \( K \). For any maximal ideal \( p \) of \( A \) with \( d_p > \max\{d_v r^2 h[K : Q]_i, D_K\} \) and any finite place \( u \) of \( K \) lying above \( p \), there exist no effective \( A \)-motives \( M \in \text{Mot}_{K,r,v}(u,h) \) satisfying the following conditions:

- There exist at least one Weil weight \( w \in \mathcal{W}_v(M) \) such that \( [K : Q]_i w \) is a non-integer;
- There exist one-dimensional rank-one effective \( A \)-motives \( M_1, \ldots, M_r \) over \( K \) having good reduction at both \( v \) and \( u \) such that

\[ \begin{cases} V_p M |_{G_v} \equiv_{ss} \bigoplus_{i=1}^r V_p (M_i^\otimes m_i) |_{G_v} \pmod{p}, \\ V_p M |_{G_u} \equiv_{ss} \bigoplus_{i=1}^r V_p (M_i^\otimes m_i) |_{G_u} \pmod{p} \end{cases} \]

hold for some non-negative integers \( m_1, \ldots, m_r \).

Remark that we cannot expect that the bounds of \( d_p \) in Theorems 1.2 and 1.3 are optimal because they are merely a combination of conditions necessary for proof. However, in terms of the uniformity problem, it is worth the fact that these bounds are independent of effective \( A \)-motives themselves.
1.2. **Organization of this article.** The purpose of §2 is to review the definitions and basic facts on Galois representations used in this article. In particular, we state that the characteristic polynomials completely determine the semi-simplifications of Galois representations. (This is well-known, but few references describe it, so we give the proof in Appendix.) In §3, we introduce basics about effective $A$-motives, often without proof. First, we recall the fundamental concepts on effective $A$-motives over general $A$-fields, and then we study $p$-adic and de Rham cohomology realizations of effective $A$-motives over $K$. In §4, following [Kim09] and [HK20], over a complete discrete valuation field $L$ with a perfect residue field, we study the equal-characteristic Fontaine theory described by local shtukas and torsion local shtukas. As an analog of Raynaud’s classification of finite flat group schemes of $p$-power rank, we classify torsion local shtukas with “coefficients” of rank one and investigate tame inertia weights of residue representations arising from local shtukas; this will be extended to strongly semi-stable effective $A$-motives in the next section. We introduce in §5 the notion of analytic $A(1)$-motives, a rigid analytic version of effective $A$-motives. Considering rigid analytification, we define and investigate the notion of strongly semi-stable reduction for effective $A$-motives. In §6, after preparing properties on Weil weights, we prove our results.

1.3. **Notation and convention.** Let $\mathbb{F}_q, Q, \infty, A$, and $\gamma: A \to K$ be as in the beginning of §§1.1. We use the following notation in this article:

- We mean by a **multi-set** a collection of unordered numbers allowing for multiplicity. For a multi-set $\mathcal{X}$, we define
  $$\Sigma \mathcal{X} := \sum_{x \in \mathcal{X}} x$$
  if the right hand side is well-defined. For example, if $\mathcal{X} = \{x, x, x\}$, then $\Sigma \mathcal{X} = 3x$.
- For a ring homomorphism $\sigma: R \to R$ and an $R$-module $M$, we set $\sigma^* M := M \otimes_{R, \sigma} R$ and $\sigma^* f := f \otimes \text{id}: \sigma^* M \to \sigma^* N$ for an $R$-homomorphism $f: M \to N$. For any $m \in M$, we write $\sigma^* m := m \otimes 1 \in \sigma^* M$.
- Let $v$ be a place of $K$. Then the symbols $K_v, O_v, \text{ and } \mathbb{F}_v$ mean the completion of $K$ at $v$, the valuation ring of $K_v$, and the residue field of $K_v$, respectively. Denote by $d_v := [\mathbb{F}_v : \mathbb{F}_q]$ the degree of $v$ over $\mathbb{F}_q$. For each $v$, we fix an embedding $\bar{K} \to \bar{K}_v$ between fixed algebraic closures of $K$ and $K_v$, and identify the decomposition group $G_v \subset G_K$ as $G_{K_v} = \text{Gal}(K_v^{\text{sep}}/K_v)$.
- For the fixed place $\infty$ of $Q$ whose residue field is $\mathbb{F}_q$, let $| \cdot |_\infty: Q \to \mathbb{R}$ be the corresponding absolute value normalized such as $|a|_\infty = \# A/aA$ for any $a \in A$. We fix an algebraic closure $\bar{Q}$ of $Q$ and extend the absolute value to $\bar{Q}$, which is again denoted by $| \cdot |_\infty$.

For the absolute value $| \cdot |_\infty$, we have the following elementally lemma used to compare Frobenius characteristic polynomials to prove the main results:

**Lemma 1.4.** Let $p$ be a maximal ideal of $A$. If $a \in A$ satisfies $a \equiv 0 \pmod{p}$ and $|a|_\infty < q^{d_p}$, then $a = 0$.

**Proof.** Assume that $a \neq 0$. Then by $a \equiv 0 \pmod{p}$, the principal ideal $aA$ decomposes as $aA = p^e a$, where $e > 0$ and $p \nmid a$. Thus it follows that
$$|a|_\infty = \#A/aA = (\#A/p^e) \cdot (\#A/a) \geq q^{d_p}.$$ This is a contradiction. \qed

2. **Preliminary**

This section recalls some basic facts on Galois representations used in this article. Let $G$ be a profinite group and $F$ a topological field. (We consider the discrete topology on $F$ if it is finite.) On finite-dimensional $F$-vector spaces, we consider the product topology induced by $F$. By an $F$-**representation of $G$,** we mean a finite-dimensional $F$-vector space with a continuous $G$-action. For
an \( F \)-representation \( V \) of \( G \), the Jordan-Hölder theorem gives a descending filtration by \( G \)-stable \( F \)-subspaces

\[
\{0\} = V_0 \subset V_1 \subset \cdots \subset V_n = V
\]
such that each \( V_i/V_{i-1} \) is irreducible. Then the semi-simplification of \( V \) is defined by

\[
V^{ss} := \bigoplus_{i=1}^n V_i/V_{i-1},
\]
which is completely determined by \( V \) up to isomorphisms.

**Definition 2.1.** For two \( F \)-representations \( V \) and \( V' \) of \( G \), we write \( V \simeq_{ss} V' \) if their semi-simplifications \( V^{ss} \) and \( V'^{ss} \) are isomorphic as \( F \)-representations of \( G \).

For any element \( s \in G \), we denote the characteristic polynomial of \( s \) by \( P_{V,s}(X) := \det(X - s \mid V) \). Since the characteristic polynomials are multiplicative in short exact sequences, two \( F \)-representations \( V \) and \( V' \) with \( V \simeq_{ss} V' \) satisfy \( P_{V,s}(X) = P_{V',s}(X) \) for any \( s \in G \). It is well-known that the converse also holds. This fact is crucial for our study.

**Theorem 2.2** (Brauer-Nesbitt). Two \( F \)-representations \( V \) and \( V' \) of \( G \) satisfy \( V \simeq_{ss} V' \) if and only if \( P_{V,s}(X) = P_{V',s}(X) \) for any \( s \in G \).

**Proof.** The proof is given in Appendix. Note that it is often called the Brauer-Nesbitt theorem in the case where \( G \) is a finite group; see [CR62, Theorem 30.16]. \( \square \)

Now for a maximal ideal \( \mathfrak{p} \) of \( A \) and a place \( v \) of \( K \), let us consider the case where \( F = Q_\mathfrak{p} \) and \( G = G_v = \text{Gal}(K_v^{\text{sep}}/K_v) \). We denote by \( I_v \) the inertia subgroup of \( G_v \). For a \( p \)-adic representation \( V \) of \( G_v \), we say that \( V \) is unramified (at \( v \)) if \( I_v \) trivially acts on \( V \). Let \( \varphi_v \) be the arithmetic Frobenius of \( G_{F_v} = \text{Gal}(F_v^{\text{sep}}/F_v) \), that is, \( \varphi_v(x) = x^{q_v} \) for \( x \in F_v^{\text{sep}} \). Take a lift \( \text{Frob}_v \in G_v \) of \( \varphi_v \) via \( G_v \to G_{F_v} \) and set \( P_{V,v}(X) := P_{V,\text{Frob}_v}(X) \). Then the Brauer-Nesbitt theorem particularly implies the following:

**Proposition 2.3.** Let \( V \) and \( V' \) be \( p \)-adic representations of \( G_v \) with the same dimension. Suppose that \( I_v \) acts unipotently on both \( V \) and \( V' \). Then \( V \simeq_{ss} V' \) if and only if \( P_{V,v}(X) = P_{V',v}(X) \).

**Proof.** The “only if” part is trivial, so we prove the converse. Suppose that \( P_{V,v}(X) = P_{V',v}(X) \). Considering semi-simplification, we may assume that \( V \) and \( V' \) are semi-simple so that they are unramified at \( v \) by assumption. Thus \( V \) and \( V' \) are \( F \)-representations of \( G_{F_v}(\cong G_v/I_v) \), and so we have \( P_{V,\varphi_v}(X) = P_{V',\varphi_v}(X) \). Since the pro-cyclic group \( G_{F_v} \) is topologically generated by \( \varphi_v \), it follows by continuity that \( P_{V,s}(X) = P_{V',s}(X) \) for any \( s \in G_{F_v} \). Thus we obtain the conclusion by the Brauer-Nesbitt theorem. \( \square \)

**Definition 2.4.** Let \( V \) and \( V' \) be \( p \)-adic representations of \( G_v \) with the same dimension. We say that \( V \) and \( V' \) are congruent modulo \( p \) and write \( V \equiv_{ss} V' \mod p \) if for a choice of \( G_v \)-stable \( A_p \)-lattices \( T \subset V \) and \( T' \subset V' \), we have \( T/pT \simeq_{ss} T'/pT' \) as \( F_p \)-representations of \( G_v \).

**Remark 2.5.** Since the \( G_v \)-action is continuous, there exists a \( G_v \)-stable \( A_p \)-lattice \( T \) of \( V \) and hence \( P_{V,s}(X) = \det(X - s \mid T) \in A_p[X] \) for any \( s \in G_v \). For any choice of \( T \subset V \), the characteristic polynomial of \( s \in G_v \) on \( T/pT \) equals to \( P_{V,s}(X) \mod p \in F_p[X] \). By the Brauer-Nesbitt theorem, the semi-simplification \( (T/pT)^{ss} \) is independent of the choice of lattices and hence that \( V \simeq_{ss} V' \) implies \( V \equiv_{ss} V' \mod p \).

3. Effective \( A \)-motives

3.1. **Definitions.** In this first subsection, we recall the notion of effective \( A \)-motives over general fields. Let \((F, \gamma)\) be an \( A \)-field, that is, an extension field of \( F_q \) equipped with a (not necessary injective) \( F_q \)-algebra homomorphism \( \gamma : A \to F \). We set \( A_F := A \otimes_{F_q} F \) and consider the ideal
Let $J := J_F := \text{Ker}(A_F \to F; a \otimes \lambda \mapsto \gamma(a) \lambda) \subset A_F$, which is generated by $\{a \otimes 1 - 1 \otimes \gamma(a) \mid a \in A\}$. Note that $A_F$ is a Dedekind domain. For the endomorphism $\sigma = \sigma_F := \text{id}_A \otimes (\cdot)\gamma$ of $A_F$ and an $A_F$-module $M$, we set $\sigma^* M := M \otimes_{A,F} A_F$ and write $\sigma^* m = m \otimes 1 \in \sigma^* M$ for $m \in M$.

**Definition 3.1.** An effective $A$-motive over an $A$-field $F$ of rank $r$ is a pair $M = (M, \tau_M)$ consisting of a locally free $A_F$-module of rank $r$, and an injective $A_F$-homomorphism $\tau_M: \sigma^* M \to M$ such that $\text{Coker}(\tau_M)$ is a finite-dimensional $F$-vector space annihilated by some power of $J$. We set $\text{rk} M := r$ and define the dimension of $M$ by $\dim M := \dim_F \text{Coker}(\tau_M)$. For $h \geq 0$, we say that $M$ is of height $\leq h$ if $J^h \cdot \text{Coker}(\tau_M) = 0$.

A morphism $f: M \to N$ between effective $A$-motives over $F$ is an $A_F$-homomorphism $f: M \to N$ such that $f \circ \tau_M = \tau_N \circ \sigma^* f$. A morphism $f: M \to N$ is called an isogeny if $f$ is injective with torsion cokernel.

**Remark 3.2.** For a locally free $A_F$-module $M$ and an $A_F$-homomorphism $\tau_M: \sigma^* M \to M$, if $\text{Coker}(\tau_M)$ is annihilated by a power of $J$, then $\tau_M$ is injective and $\text{Coker}(\tau_M)$ is finite-dimensional over $F$ by [Har19 Proposition 2.3].

**Example 3.3 (Unit objects).** Let $M = A_F$ and define $\tau_M: \sigma^* M \to M$ by $\tau_M := \sigma \otimes \text{id}$. Then under the natural isomorphism $\sigma^* M = A_F \otimes_{A,F,F} A_F \cong A_F; a \otimes b \mapsto \sigma(a)b$, the map $\tau_M$ is identified as $\text{id}: A_F \to A_F$. We denote by $A_F = (A_F, \text{id})$, which is an effective $A$-motive over $F$ with $\text{rk} A_F = 1$ and $\dim A_F = 0$, so that it is of height $\leq 0$.

**Example 3.4 (Abelian Anderson $A$-modules).** Let $r$ and $d$ be positive integers. An abelian Anderson $A$-module over $F$ of rank $r$ and dimension $d$ is a pair $E = (E, \varphi)$ consisting of an affine group scheme $E$ satisfying $E \cong \mathbb{G}_a^d$ as $F_q$-module schemes over $F$ and a ring homomorphism $\varphi: A \to \text{End}_F(E); a \mapsto \varphi_a$ such that

1. $(\text{Lie} \varphi_a - a)^d = 0$ on $\text{Lie} E$ for any $a \in A$,
2. the set $M := M(E) := \text{Hom}_{\text{G}_a,F,G_{a,F}}(E, \mathbb{G}_a,F)$ of $F_q$-linear homomorphisms of $F$-group schemes is a locally free $A_F$-module of rank $r$ under the action given on $m \in M$ by $(a \otimes \lambda)m := \lambda \circ m \circ \varphi_a$ for $a \in A$ and $\lambda \in F$.

If $d = 1$, then $E$ is called a Drinfeld $A$-module. Note that in the case where $A = F_q[t]$ and $F$ is perfect, an abelian Anderson $A$-module coincides with an abelian $t$-module in the sense of Anderson [And86]. Let $\tau$ be the relative $q$-Frobenius endomorphism of $\mathbb{G}_a, F = \text{Spec} F[x]$ given by $x \mapsto x^q$. For an abelian Anderson $A$-module $E$ of rank $r$ and dimension $d$ over $F$ and $M = M(E)$, define $\tau_M: \sigma^* M \to M$ by $\tau_M(\sigma^* m) = \tau \circ m$ for $m \in M$. Then it is known that the pair $M(E) = (M(E), \tau_M)$ becomes an effective $A$-motive over $F$ of rank $r$ and dimension $d$, and that the correspondence $E \mapsto M(E)$ determines a contravariant fully faithful functor; see [And86 Theorem 1] and [Har19, Theorem 3.5].

**Example 3.5 (the Carlitz motive).** Let $A = F_q[t]$ and $F = F_q(\vartheta)$ the rational function field in variable $\vartheta$. Suppose that $\gamma: A \to F$ is given by $\gamma(t) = \vartheta$, so that $A_F = F_q(\vartheta)[t]$ and $J = (t - \vartheta)$. Then the Carlitz motive is given by $C = (F_q(\vartheta)[t], \tau_C = t - \vartheta)$. It is known that $C$ is an effective $F_q[t]$-motive associated with the Carlitz module, which is a Drinfeld $F_q[t]$-module of rank one.

**Definition 3.6.** Let $M$ and $M'$ be effective $A$-motives over $F$. Then the direct sum of $M$ and $M'$ is defined by $M \otimes M' := (M \oplus M', \tau_M \oplus \tau_M')$, and the tensor product of $M$ and $M'$ is defined by $M \otimes M' := (M \otimes_{A,F} M', \tau_M \otimes \tau_M')$. Note that both $M \oplus M'$ and $M \otimes M'$ are effective $A$-motives over $F$. We also define the tensor powers of $M$ by $M \otimes^n := A_F$ and by $M \otimes^n := M \otimes^{(n-1)} M$ for $n > 0$. We define the determinant of $M$ by $\det M := \lambda^{\text{rk} M} M$.

The next formulas follow easily from the elementary divisor theorem (cf. [HJ20 Page 51]):

**Proposition 3.7.** Let $M$ and $M'$ be effective $A$-motives over $F$.

1. $\text{rk} M \oplus M' = (\text{rk} M) + (\text{rk} M')$ and $\dim M \otimes M' = (\dim M) + (\dim M')$. 
(2) $\text{rk} M \otimes M' = (\text{rk} M) \cdot (\text{rk} M')$ and $\dim M \otimes M' = (\text{rk} M') \cdot (\dim M) + (\text{rk} M) \cdot (\dim M')$.

(3) $\text{rk} \det M = 1$ and $\dim \det M = \dim M$.

Let $F \to F'$ be a field embedding over $\mathbb{F}_q$ and regard $F'$ as an $A$-field via $A \overset{\gamma}{\to} F \to F'$. Then the induced ring homomorphism $A_F \to A_{F'}$ commutes the diagram

\[
\begin{array}{ccc}
A_F & \longrightarrow & A_{F'} \\
\sigma_F \downarrow & & \downarrow \sigma_{F'} \\
A_F & \longrightarrow & A_{F'}.
\end{array}
\]

Thus for an effective $A$-motive $M = (M, \tau_M)$ over $F$, we have

$$\sigma^*(M \otimes_{A_p} A_{F'}) \cong \sigma^* M \otimes_{A_p} A_{F'} \tau_M \otimes \text{id} M \otimes_{A_p} A_{F'},$$

which provides the base change $M \otimes A_{F'}$ of $M$ as follows:

**Proposition 3.8.** If $F$ and $F'$ be as above, then $M = (M, \tau_M) \mapsto M \otimes A_{F'} := (M \otimes_{A_p} A_{F'}, \tau_M \otimes \text{id})$ defines a functor from the category of effective $A$-motives over $F$ of height $\leq h$ to that of effective $A$-motives over $F'$ of height $\leq h$ such that $\text{rk} M = \text{rk} M \otimes A_{F'}$ and $\dim M = \dim M \otimes A_{F'}$.

3.2. $p$-adic realizations. In what follows, let us consider effective $A$-motives defined over the $A$-field $(K, \gamma: A \to K)$ as in §1 (in particular, $\gamma$ is injective). Let $p$ be a maximal ideal of $A$. For an effective $A$-motive $M$ over $K$, we obtain $p$-adic representations of $G_K$ as follows. Let $A_{p,K^{\text{sep}}} := A_p \otimes_{\mathbb{F}_q} K^{\text{sep}} = \lim \frac{A_{K^{\text{sep}}}}{p^n A_{K^{\text{sep}}}}$ be the $p$-adic completion of $A_{K^{\text{sep}}}$. Then the endomorphism $\sigma$ of $A_{K^{\text{sep}}}$ canonically extends to $\sigma: A_{p,K^{\text{sep}}} \to A_{p,K^{\text{sep}}}$ and so $\tau_M$ induces $\tau_M^*: \sigma^* M \otimes_{A_K} A_{p,K^{\text{sep}}} \to M \otimes_{A_K} A_{p,K^{\text{sep}}}$. Note that the $\sigma$-invariant subring of $A_{p,K^{\text{sep}}}$ is $(A_{p,K^{\text{sep}}})^\sigma = A_p$. Then the $p$-adic realization of $M$ is defined to be the $A_p$-module

$$H_{p,1}^1(M, A_p) := (M \otimes_{A_K} A_{p,K^{\text{sep}}})^\sigma := \{ m \in M \otimes_{A_K} A_{p,K^{\text{sep}}}, \tau_M(\sigma^* m) = m \},$$

where $\sigma^* m := m \otimes 1$ is the image by the natural map $M \otimes_{A_K} A_{p,K^{\text{sep}}} \to \sigma^* M \otimes_{A_K} A_{p,K^{\text{sep}}}$. It is known that $H_{p,1}^1(M, A_p)$ is a free $A_p$-module of rank equal to $\text{rk} M$ with a continuous action of $G_K$; see [TW96]. Define the rational $p$-adic realization of $M$ by

$$H_{p,1}^1(M, Q_p) := H_{p,1}^1(M, A_p) \otimes_{A_p} Q_p.$$

Taking dual of them, we define the $p$-adic Tate module and the rational $p$-adic Tate module of $M$ by

$$T_p M := \text{Hom}_{A_p}(H_{p,1}^1(M, A_p), A_p) \text{ and } V_p M := T_p M \otimes_{A_p} Q_p.$$

It follows that $M \mapsto V_p M$ defines a contravariant exact faithful tensor functor from the category of effective $A$-motives over $K$ to that of $p$-adic representations of $G_K$.

**Remark 3.9.** Set $M_{p,K^{\text{sep}}} = M \otimes_{A_K} A_{p,K^{\text{sep}}}$. Then $T_p M$ can be defined directly by

$$T_p M = \{ f \in \text{Hom}_{A_{p,K^{\text{sep}}}}(M_{p,K^{\text{sep}}}, A_{p,K^{\text{sep}}}) | f(\tau_M(\sigma^* m)) = \sigma(f(m)) \text{ for } m \in M_{p,K^{\text{sep}}} \}.$$

If $M$ comes from an abelian Anderson $A$-module $E$ over $K$, then $T_p M$ is canonically isomorphic to the $p$-adic Tate module $T_p E = \lim_{\leftarrow} E[p^n]$ of $E$.

**Remark 3.10.** For a finite place $v$ of $K$, the (rational) $p$-adic Tate modules of effective $A$-motives over $K_v$ are defined in the same way. By construction, for an effective $A$-motive $M$ over $K$, we have $T_p M|_{G_v} \cong T_p(M \otimes A_{K_v})$ and $V_p M|_{G_v} \cong V_p(M \otimes A_{K_v})$.

To review the notion of good reduction for effective $A$-motives, let $v$ be a finite place of $K$ and regard $K_v$ as an $A$-field via $\gamma: A \to K \subset K_v$. Since this map factors through the valuation ring $\mathcal{O}_v$, the residue field $\mathbb{F}_v$ also becomes an $A$-field via $A \overset{\gamma}{\to} \mathcal{O}_v \to \mathbb{F}_v$. We set $A_{\mathcal{O}_v} = A \otimes_{\mathbb{F}_q} \mathcal{O}_v$.

**Definition 3.11.** Let $M$ be an effective $A$-motive over $K_v$. 

(1) A model of $M$ is a pair $(M, \tau_M)$ consisting of a finite locally free $A_{O_v}$-module and an injective $A_{O_v}$-homomorphism $\tau_M: \sigma^*M \to M$ such that there is an isomorphism $\iota: M \cong \mathcal{M} \otimes_{A_{O_v}} A_{K_v}$ with $\tau_M \circ \sigma^* \iota = \iota \circ \tau_M$.

(2) A model $\mathcal{M}$ of $M$ is called a good model if the induced $A_{F_v}$-homomorphism

$$
\tau_M \otimes \text{id}: \sigma^*\mathcal{M} \otimes_{A_{O_v}} A_{F_v} \to \mathcal{M} \otimes_{A_{O_v}} A_{F_v}
$$

is injective. In this case, we say that $\mathcal{M}$ has good reduction.

We say that an effective $A$-motive $\underline{M}$ over $K$ has good reduction at $v$ if so does $\mathcal{M} \otimes K_v$.

**Remark 3.12.** For a model $(M, \tau_M)$ of an effective $A$-motive $\underline{M}$ over $K_v$, \cite[Theorem 4.7]{HH16} implies that it is a good model if and only if $\text{Coker}(\tau_M)$ is a finite free $O_{v}$-module annihilated by a power of $J_{O_v}$, where $J_{O_v}$ is the ideal of $A_{O_v}$ generated by $\{a \otimes 1 - 1 \otimes \gamma(a) \mid a \in A\}$. Therefore, in this case, the pair $(M \otimes_{A_{O_v}} A_{F_v}, \tau_M \otimes \text{id})$ becomes an effective $A$-motive over $F_v$ which has the same rank and dimension of $\mathcal{M}$. Note that by \cite[Lemma 2.10]{Gar02} an effective $A$-motive $\underline{M}$ over $K$ has good reduction at almost all finite places $v$.

As an analog of the Néron-Ogg-Shafarevich criterion for abelian varieties, Gardeyn \cite{Gar02} proves the following theorem:

**Theorem 3.13 (Gardeyn).** Let $v$ be a finite place of $K$. For an effective $A$-motive $\underline{M}$ over $K$, the following statements are equivalent.

1. $\underline{M}$ has good reduction at $v$.
2. For all maximal ideal $p$ of $A$ with $v \nmid p$, the $p$-adic representation $V_p\underline{M}$ is unramified at $v$.
3. There is a maximal ideal $p$ with $v \nmid p$ such that $V_p\underline{M}$ is unramified at $v$.

For a finite place $v$ of $K$ with $v \nmid p$ and an effective $A$-motive $\underline{M}$ with good reduction at $v$, the characteristic polynomial

$$
P_{\underline{M},v}(X) = \det(X - \text{Frob}_v \mid V_p\underline{M})
$$

of $\text{Frob}_v$ is well-defined.

**Proposition 3.14.** If $\underline{M}$ has good reduction at $v \nmid p$, then $P_{\underline{M},v}(X)$ has coefficients in $A$ which are independent of $p$.

**Proof.** This is a special case of \cite[Theorem 7.3]{Gar02}. \qed

We will recall the notion of purity for $A$-motives. Let $\Kbar$ be a place of $K$ lying above $\infty$, and denote by $C := \Kbar_{\infty \Kbar}$ a completion of an algebraic closure of a completion of $K$ at $\Kbar$. We fix an embedding $K \to C$ and regard $C$ as an $A$-field via $\gamma: A \to C \to C$. Choose a uniformizer $z_\infty \in \mathcal{O} \setminus \{0\}$ at $\infty$. Since we now assume $F_q = F_q$, we have $Q_\infty = \mathbb{F}_q((z_\infty))$ and $A \subset Q_\infty$, so that we have $Q_\infty \otimes_{\mathbb{Z}_q} C = \mathbb{C}((z_\infty))$ and an injection $A_C \to \mathbb{C}((z_\infty))$. The endomorphism $\sigma$ of $A_C$ extends to $\sigma: \mathbb{C}((z_\infty)) \to \mathbb{C}((z_\infty))$ by $\sigma(z_\infty) = z_\infty$ and $\sigma(\lambda) = \lambda^q$ for $\lambda \in C$.

**Definition 3.15.** An effective $A$-motive $\underline{M}$ over $\mathcal{O}$ is said to be pure if there is a $C[[z_\infty]]$-lattice $M_\infty$ of $M \otimes_{A_C} C((z_\infty))$ such that for some integers $d, r$ with $r > 0$, the map $\tau^r_M := \tau_M \circ \sigma^r \tau_M \circ \cdots \circ \sigma^{r(\ell - 1)} \tau_M: \sigma^{r*}M \to M$ induces an isomorphism

$$
z_\infty^d \tau^r_M: \sigma^{r*}M_\infty \cong M_\infty.
$$

In this case, the weight of $\underline{M}$ is defined by $\text{wt} \underline{M} := \frac{d}{r}$. An effective $A$-motive $\underline{M}$ over $K$ is said to be pure if so is $\underline{M} \otimes A_C$.

**Example 3.16.** Let $\mathcal{E}$ be a Drinfeld $A$-module over $K$ of rank $r$. Then the associated effective $A$-motive $\underline{M}(E)$ is pure of weight $\frac{1}{r}$.

**Proposition 3.17.** Let $\underline{M}$ and $\underline{M}'$ be effective pure $A$-motives over $K$.
(1) The weight of $M$ is $\text{wt } M = (\dim M)/(\text{rk } M)$.
(2) The tensor product $M \otimes M'$ is pure of weight $(\text{wt } M) + (\text{wt } M')$.
(3) If $M$ has good reduction at a finite place $v$ of $K$, then any root $\alpha \in Q$ of $P_{M,v}(X)$ satisfies $|\alpha|_\infty = q_v^{\text{wt } M} M$.

Proof. (1) and (2) are well-known; see [HJ20, Proposition 2.3.11] for example. For a maximal ideal $p$ of $A$ with $v \not\mid p$, each eigenvalue of $\text{Frob}_v$ on $H^1_F(M, A_p)$ has the absolute value $q_v^{\text{wt } M}$ by [HJ20 Proposition 2.3.36], which implies (3).

Lemma 3.18. Under the assumption $\mathbb{F}_\infty = \mathbb{F}_q$, any rank-one effective $A$-motive over $K$ is pure. In particular, the determinant $\det M$ of an effective $A$-motive $M$ is pure of weight equal to $\dim M$.

Proof. Suppose that $M$ is of rank one and set $d = \dim M$. Then by [HJ20 Examples 2.3.6 and 2.3.9], there is an isogeny $f: M \otimes A \to N_{\otimes d}$, where $N = (N, \tau_N)$ is an effective pure $A$-motive over $\mathbb{C}$ of rank one satisfying $\tau_N(\sigma^*N) = J \cdot \bar{N}$, so that $\dim N = 1$. (Such $N$ is so-called a Carlitz-Hayes $A$-motives.) Since $N_{\otimes d}$ is pure, it follows by [HJ20 Proposition 2.3.11. (d)] that $M$ is also pure.

3.3. de Rham realizations and Hodge-Pink weights. Let $z \in Q$ be a uniformizer at some place of $Q$ and set $\zeta := \gamma(z) \in K$. Then for the ideal $J = \text{Ker}(A_K \to K)$, [HJ20 Lemma 2.1.3] arranges us to identify $\lim_{\longrightarrow} A_K/J^n = K[z - \zeta]$, where $K[z - \zeta]$ is the power series ring over $K$ in “variable” $z - \zeta$. Thus we obtain an injective flat homomorphism $A_K \to K[z - \zeta]$ and $J \cdot K[z - \zeta] = (z - \zeta)$.

Let $M = (M, \tau_M)$ be an effective $A$-motive over $K$. The de Rham realizations of $M$ are defined as

\[
\begin{align*}
H^1_{\text{dR}}(M, K[z - \zeta]) &:= \sigma^* M \otimes_{A_K} K[z - \zeta], \\
H^1_{\text{dR}}(M, K((z - \zeta))) &:= H^1_{\text{dR}}(M, K[z - \zeta]) \otimes_{K[z - \zeta]} K((z - \zeta)), \\
H^1_{\text{dR}}(M, K) &:= \sigma^* M \otimes_{A_K} A_K/J = H^1_{\text{dR}}(M, K[z - \zeta]) \otimes_{K[z - \zeta]} K[z - \zeta]/(z - \zeta).
\end{align*}
\]

Since $\text{Coker}(\tau_M)$ is annihilated by a power of $J$, the map $\tau_M : \sigma^* M \to M$ extends to an isomorphism $\tau_M : H^1_{\text{dR}}(M, K((z - \zeta))) \xrightarrow{\sim} M \otimes_{A_K} K((z - \zeta))$. The Hodge-Pink lattice of $M$ is defined as the $K[z - \zeta]$-submodule $Q_M := \tau_M^{-1}(M \otimes_{A_K} K[z - \zeta])$ of $H^1_{\text{dR}}(M, K((z - \zeta)))$. We set $P_M = H^1_{\text{dR}}(M, K[z - \zeta])$, so that $P_M \subset Q_M$. Then the Hodge-Pink filtration $F^i H^1_{\text{dR}}(M, K) = \{F^i H^1_{\text{dR}}(M, K)\}_{i \in \mathbb{Z}}$ of $M$ is the descending filtration by $K$-subspaces of $H^1_{\text{dR}}(M, K)$ defined by

\[
F^i H^1_{\text{dR}}(M, K) := (P_M \cap (z - \zeta)^i Q_M) / ((z - \zeta)P_M \cap (z - \zeta)^i Q_M).
\]

In other words, the filtration is defined by setting $F^i H^1_{\text{dR}}(M, K)$ to be the image of $(P_M \cap (z - \zeta)^i Q_M)$ via the surjection $P_M \to P_M/(z - \zeta)P_M = H^1_{\text{dR}}(M, K)$. Thus we obtain the following diagram

\[
\begin{array}{ccc}
H^1_{\text{dR}}(M, K((z - \zeta))) & \xrightarrow{\sim} & M \otimes_{A_K} K((z - \zeta)) \\
\cup & \downarrow & \cup \\
Q_M & \xrightarrow{\sim} & M \otimes_{A_K} K[z - \zeta] \\
\cup & \downarrow & \cup \\
P_M = H^1_{\text{dR}}(M, K[z - \zeta]) & \xrightarrow{\tau_M} & M \otimes_{A_K} K[z - \zeta].
\end{array}
\]

Since $K[z - \zeta]$ is a principal ideal domain, the elementary divisor theorem implies that there are non-negative integers $h_1 \leq \cdots \leq h_{kr_M}$ and $\lambda_1, \ldots, \lambda_{kr_M} \in K[z - \zeta]^\times$ such that $\tau_M : P_M \to
$M \otimes_{A_k} K[z - \zeta]$ is of the form

$$\tau_M = \begin{pmatrix}
\lambda_1(z - \zeta)^{h_1} & \lambda_2(z - \zeta)^{h_2} & \cdots \\
\lambda_1(z - \zeta)^{h_1} & \lambda_2(z - \zeta)^{h_2} & \cdots \\
& & \ddots \\
\lambda_{r_k M}(z - \zeta)^{h_{r_k M}} & & & & \ddots
\end{pmatrix}$$

(3.1)

with respect to suitable $K[z - \zeta]$-bases of $P_M$ and $M \otimes_{A_k} K[z - \zeta]$.

**Definition 3.19.** We call the integers $h_1, \ldots, h_{r_k M}$ the Hodge-Pink weights of $M$, and denote by $\mathcal{H}P(M) := \{h_1, \ldots, h_{r_k M}\}$ the multi-set consisting of them.

**Remark 3.20.** The above discussion provides a $K[z - \zeta]$-basis $e_i \in P_M$ such that $P_M = \bigoplus_{i=1}^{r_k M} K[z - \zeta] \cdot e_i$ and $Q_M = \bigoplus_{i=1}^{r_k M} K[z - \zeta] \cdot (z - \zeta)^{-h_i} \cdot e_i$. Although the negatives $-h_1 \geq \cdots \geq -h_{r_k M}$ are often called the Hodge-Pink weights of $M$, we adopt Definition 3.19 to simplify the description of Proposition 3.22 below.

**Remark 3.21.** By the matrix representation (3.1), we see that the Hodge-Pink weights are characteristic of the jumps of the Hodge-Pink filtration, that is, an integer $h$ is a Hodge-Pink weight of $M$ if and only if

$$F^h H^1_{\mathrm{dR}}(M, K) \supseteq F^{h+1} H^1_{\mathrm{dR}}(M, K),$$

and in addition the multiplicity of $h \in \mathcal{H}P(M)$ equals to $\dim_K (F^h H^1_{\mathrm{dR}}(M, K)/F^{h+1} H^1_{\mathrm{dR}}(M, K))$.

For an effective $A$-motive $M = (M, \tau_M)$ over $K$, since $\text{Coker}(\tau_M) \cong \bigoplus_{i=1}^{r_k M} K[z - \zeta]/(z - \zeta)^{h_i}$ as $K$-vector spaces if $\mathcal{H}P(M) = \{h_1, \ldots, h_{r_k M}\}$, we immediately obtain the next assertion:

**Proposition 3.22.** Let $M$ be an effective $A$-motive over $K$.

1. $\dim M = \sum \mathcal{H}P(M)$.
2. For $h \geq 0$, $M$ is of height $\leq h$ if and only if $\mathcal{H}P(M) \subset [0, h]$.

4. Galois representations arising from local shtukas

Throughout this section, let $L$ be a complete discrete valuation field containing $K$ and suppose that the residue field $k$ of $L$ is perfect, and that the composite $A \to K \to L$ factors through the valuation ring $O$ of $L$ and satisfies $\text{Ker}(A \to O \to k) = p$ for some maximal ideal $p$ of $A$. We set $q := q^h = \#F_p$. Let $\pi \in L$ be a uniformizer and denote by $\text{ord}_L(\cdot): L \to \mathbb{Z} \cup \{\infty\}$ the discrete valuation of $L$ normalized as $\text{ord}_L(\pi) = 1$. Choose and fix a uniformizer $z \in O$ at $p$. This allows us to identify $A_p = F_p[z]$ and $Q_p = F_p((z))$. By continuity, the map $\gamma: A \to O$ extends to $\gamma: A_p \to O$, and we set $\zeta := \gamma(z) \in O$. Let $O[z]$ be the power series ring over $O$ in “variable” $z$. Let us consider the endomorphism $\hat{\sigma}$ of $O[z]$ determined by $\hat{\sigma}(z) = z$ and $\hat{\sigma}(\lambda) = \lambda^h$ for $\lambda \in O$. For an $O[z]$-module $M$, we set $\hat{\sigma}^* M := M \otimes_{O[z], \hat{\sigma}} O[z]$.

4.1. Local shtukas and torsion local shtukas.

**Definition 4.1.** A local shtuka over $O$ of rank $r$ is a pair $\hat{M} = (\hat{M}, \tau_{\hat{M}})$ consisting of a free $O[z]$-module $\hat{M}$ of rank $r$ and an isomorphism $\tau_{\hat{M}}: \hat{\sigma}^* \hat{M} \left[\frac{1}{z-\zeta}\right] \xrightarrow{\sim} \hat{M} \left[\frac{1}{z-\zeta}\right]$. If $\tau_{\hat{M}}(\hat{\sigma}^* M) \subset M$, then $\hat{M}$ is said to be effective, and if $(z - \zeta)^{h}\hat{M} \subset \tau_{\hat{M}}(\hat{\sigma}^* \hat{M}) \subset \hat{M}$ for $h \geq 0$, then $\hat{M}$ is said to be effective of height $\leq h$.

A morphism of local shtukas $f: (\hat{M}, \tau_{\hat{M}}) \to (\hat{M}', \tau_{\hat{M}'})$ over $O$ is a morphism of $O[z]$-modules $f: \hat{M} \to \hat{M}'$ such that the induced morphism $f: \hat{M} \left[\frac{1}{z-\zeta}\right] \to \hat{M}' \left[\frac{1}{z-\zeta}\right]$ satisfies $\tau_{\hat{M}} \circ \hat{\sigma}^* f = f \circ \tau_{\hat{M}}$. It is called an isogeny if $f$ induces an isomorphism $f: \hat{M} \left[\frac{1}{z}\right] \to \hat{M}' \left[\frac{1}{z}\right]$ of $O((z))$-modules. Note that if $f: \hat{M} \to \hat{M}'$ is an isogeny, then $f$ is injective and $\text{rk} \hat{M} = \text{rk} \hat{M}'$. 

Lemma 4.2. Let $\hat{M} = (\hat{M}, \tau_M)$ be a local shtuka over $\mathcal{O}$.

1. There is an integer $d \in \mathbb{Z}$ such that $\det \tau_{\hat{M}} \in (z - \zeta)^d \cdot \mathcal{O}[z]^\times$.
2. If $\hat{M}$ is effective, then the integer $d$ in (1) satisfies $d \geq 0$ and $\hat{M}/\tau_{\hat{M}}(\hat{\sigma}^* \hat{M})$ is a free $\mathcal{O}$-module of rank $d$ which is annihilated by $(z - \zeta)^d$.

Proof. This is [HK20, Lemma 3.2.3]. \hfill \Box

Definition 4.3. Define the dimension of $\hat{M}$ by the integer $d$ as in Lemma 4.2, and set $\dim \hat{M} := d$. Clearly $\hat{M}$ is of height $\leq d$ if it is effective.

Let $\hat{M} = (\hat{M}, \tau_M)$ be a local shtuka over $\mathcal{O}$. Then $\tau_M$ induces an isomorphism

$$\tau_M : \hat{\sigma}^* \hat{M} \otimes \mathcal{O}[z] L^{\text{sep}}[z] \xrightarrow{\sim} \hat{M} \otimes \mathcal{O}[z] L^{\text{sep}}[z]$$

since $z - \zeta$ is invertible in $L^{\text{sep}}[z]$. We define the action of $G_L = \text{Gal}(L^{\text{sep}}/L)$ on $\hat{M} \otimes \mathcal{O}[z] L^{\text{sep}}[z]$ via the trivial action on $\hat{M}$ and the natural action on $L^{\text{sep}}[z]$. Now the $\hat{\sigma}$-invariant subring of $L^{\text{sep}}[z]$ is $\mathbb{F}_p[z] = A_p$. Then we define the dual Tate module of $\hat{M}$ by

$$T_p \hat{M} := (\hat{M} \otimes \mathcal{O}[z] L^{\text{sep}}[z])^\tau := \{ m \in \hat{M} \otimes \mathcal{O}[z] L^{\text{sep}}[z] \mid \tau_{\hat{M}}(\hat{\sigma}^* m) = m \},$$

which is a free $A_p$-module of rank equal to $\text{rk} \hat{M}$ with a continuous $G_L$-action. We also define the rational dual Tate module of $\hat{M}$ by the $\mathbb{Q}_p$-vector space with a continuous $G_L$-action $V_q \hat{M} := T_p \hat{M} \otimes \mathbb{Q}_p$. As dual of them, the Tate module and the rational Tate module of $\hat{M}$ are defined by

$$T_p \hat{M} = \text{Hom}_{A_p}(T_p \hat{M}, A_p) \text{ and } V_q \hat{M} = T_p \hat{M} \otimes \mathbb{Q}_p.$$

Example 4.4 (cf. [HK20, Example 3.2.2]). It is known that local shtukas relate with effective $A$-motives with good reduction. To describe this, let $\hat{M}$ be an effective $A$-motive over $L$ with good reduction. Let $M = (M, \tau_M)$ be a good model of $\hat{M}$. Now we consider the $\mathfrak{p}$-adic completion $A_{p, \mathcal{O}} := A_p \otimes_{\mathbb{F}_q} \mathcal{O}$ of $A_{\mathcal{O}}$ and set $\hat{M} \otimes A_{p, \mathcal{O}} := (M \otimes_{A_{\mathcal{O}}} A_{p, \mathcal{O}}, \tau_M \otimes \text{id})$. Then we get the local shtuka associated with $\hat{M}$ denoted by $\hat{M}_{\mathfrak{p}}(\hat{M})$ as follows.

- If $\mathbb{F}_p = \mathbb{F}_q$, and so $\hat{\sigma} = \sigma$ and $\hat{\sigma} = \sigma$, then we have $A_{p, \mathcal{O}} = \mathcal{O}[z]$ and $J \cdot A_{p, \mathcal{O}} = (z - \zeta)$. Hence $M \otimes A_{p, \mathcal{O}}$ itself becomes an effective local shtuka over $\mathcal{O}$. We set $\hat{M}_{\mathfrak{p}}(M) := M \otimes A_{p, \mathcal{O}}$.
- On the other hand, let us assume $d_p = [\mathbb{F}_p : \mathbb{F}_q] > 1$. For each $0 \leq i \leq d_p - 1$, we consider the ideals $A_{i, \mathcal{O}}$ generated by $b \otimes 1 - 1 \otimes \gamma(b)^q = b \in \mathbb{F}_p$. Then these ideals satisfy $\prod_{b=0}^{d_p-1} A_i = (0)$ because for any $b \in \mathbb{F}_p$, the polynomial $\prod_{b=0}^{d_p-1} (X - b^q)$ is a multiple of the minimal polynomial of $b$ over $\mathbb{F}_q$ and even equal to it when $\mathbb{F}_p = \mathbb{F}_q(b)$. The Chinese remainder theorem yields the decomposition

$$A_{p, \mathcal{O}} = \prod_{i=0}^{d_p-1} A_{p, \mathcal{O}}/A_i$$

whose factors have canonical isomorphisms $A_{p, \mathcal{O}}/A_i \cong \mathcal{O}[z]$. In addition, the factors are cyclically permuted by $\sigma$ since $\sigma(A_i) = A_{i+1}$, and hence $\hat{\sigma} = \sigma$ stabilizes each factor. Here it follows that the ideal $J$ decomposes as $J \cdot A_{p, \mathcal{O}}/A_0 = (z - \zeta)$ and $J \cdot A_{p, \mathcal{O}}/A_i = (1)$ for $i \neq 0$. Considering the $d_p$-th iteration $\tau_{\hat{M}} := \tau_M \circ \sigma^* \tau_M \circ \cdots \circ \sigma^* (\sigma^{(d_p-1)}) \tau_M$, we get the effective local shtuka $\hat{M}_{\mathfrak{p}}(\hat{M}) := (M \otimes A_{p, \mathcal{O}}(A_{p, \mathcal{O}}/A_0); (\tau_M \otimes \text{id})^{d_p})$. This definition coincides with the before one when $d_p = 1$. By construction, we see that $\text{Coker}(\tau_M)$ is canonically isomorphic to $\hat{M}/\tau_M(\hat{\sigma}^* \hat{M})$ as free $\mathcal{O}$-modules. Hence we have $\dim \hat{M} = \dim \hat{M}_{\mathfrak{p}}(\hat{M})$. 

Y. Okumura
Remark 4.5. The local shtuka \( \hat{M}_p(M) \) allows to recover \( M \otimes A_p, O \) via the isomorphism
\[
\bigoplus_{i=0}^{d_p-1} (\tau_M \otimes \text{id})^i \otimes a_i : \left( \bigoplus_{i=0}^{d_p-1} \sigma^i(M \otimes A_p, O/a_0), (\tau_M \otimes \text{id}) \oplus \bigoplus_{i \neq 0} \text{id} \right) \sim M \otimes A_p, O.
\]
Sometimes \( M \otimes A_p, O \) is called an effective local shtuka. In fact, the category of locally free \( A_p, O \)-modules \( N \) equipped with injective \( A_p, O \)-homomorphisms \( \tau_N : \sigma^* N \rightarrow N \) satisfying \( J^n \cdot \text{Coker}(\tau_N) = 0 \) for some \( n > 0 \) is equivalent to the category of effective local shtukas over \( O \) via the functor \( (N, \tau_N) \mapsto (N/a_0, (\tau_N \otimes \text{id})^d_p) \); see \[BH11\], Propositions 8.8 and 8.5.

The local shtuka \( \hat{M}_p(M) \) associated with \( M \) satisfies \( T_p M \cong T_p \left( \hat{M}_p(M) \right) \) by the following:

Proposition 4.6. Let \( M \) be an effective A-motive over \( L \) with good reduction and set \( \hat{M} = \hat{M}_p(M) \). Then there is a canonical functorial isomorphism \( H_p^1(M, A_p) \sim T_p \hat{M} \) as representations of \( G_L \).

Proof. See \[HK20\], Proposition 3.4.6. \( \square \)

We next introduce a torsion version of equi-characteristic Fontaine’s theory given by torsion local shtukas, which are a function field analog of finite flat group schemes of \( p \)-power order.

Definition 4.7. A torsion local shtuka over \( O \) is a pair \( \mathfrak{M} = (M, \tau_M) \) consisting of a finitely presented \( O[[z]] \)-module \( M \) which is \( z \)-power torsion and finite free over \( O \), and an isomorphism \( \tau_M : \hat{\sigma}^* M[z^{-1} - 1] \sim M[z^{-1} - 1] \). If \( \tau_M(\hat{\sigma}^* M) \subset M \), then \( M \) is said to be effective, and if \( (z - \zeta)^h M \subset \tau_M(\hat{\sigma}^* M) \subset M \) for \( h \geq 0 \), then \( M \) is said to be of effective height \( \leq h \). Define the rank of \( M \) by \( \text{rk} \mathfrak{M} := \text{rk}_O M \).

A morphism of torsion local shtukas \( f : \mathfrak{M} \rightarrow \mathfrak{M}' \) over \( O \) is a morphism of \( O[[z]] \)-modules \( f : M \rightarrow M' \) such that \( \tau_M' \circ \hat{\sigma}^* f = f \circ \tau_M \).

Remark 4.8. Note that \( M \) injects into \( M[z^{-1} - 1] \) because \( \zeta \) is \( M \)-regular and \( M \) has \( z \)-power torsion.

One can associate torsion local shtukas with torsion Galois representations in the same way for local shtukas as follows. Let \( \mathfrak{M} = (M, \tau_M) \) be a torsion local shtuka over \( O \). Then \( \tau_M \) induces an isomorphism \( \tau_M : \hat{\sigma}^* M \otimes O[[z]] \sim M \otimes O[[z]] L^{\text{sep}}[z] \). Define
\[
\hat{T}_p \mathfrak{M} := (M \otimes O[[z]] L^{\text{sep}}[z])^\tau := \{ m \in M \otimes O[[z]] L^{\text{sep}}[z] | \tau_M(\hat{\sigma}^* m) = m \},
\]
which has a discrete \( G_L \)-action induced by that on \( L^{\text{sep}}[z] \). In fact, it is known by \[HK20\], Proposition 3.7.8] that \( \hat{T}_p \mathfrak{M} \) is a torsion \( A_p \)-module of length equal to \( \text{rk} \mathfrak{M} = \text{rk}_O M \) with discrete \( G_L \)-action, and that the inclusion \( \hat{T}_p \mathfrak{M} \hookrightarrow M \otimes O[[z]] L^{\text{sep}}[z] \) induces a canonical \( G_L \)-equivariant isomorphism of \( L^{\text{sep}}[z] \)-modules
\[
\hat{T}_p \mathfrak{M} \otimes A_p, L^{\text{sep}}[z] \sim M \otimes O[[z]] L^{\text{sep}}[z]
\]
such that it commutes with the diagram
\[
\begin{array}{ccc}
\hat{T}_p \mathfrak{M} \otimes A_p & \stackrel{\sim}{\longrightarrow} & M \otimes O[[z]] L^{\text{sep}}[z] \\
\text{id} \otimes \hat{\sigma} & \downarrow & \\
\hat{T}_p \mathfrak{M} \otimes A_p & \stackrel{\sim}{\longrightarrow} & M \otimes O[[z]] L^{\text{sep}}[z],
\end{array}
\]
where \( M \otimes O[[z]] L^{\text{sep}}[z] \rightarrow M \otimes O[[z]] L^{\text{sep}}[z] \) is given by \( m \otimes \lambda \mapsto \tau_M(\hat{\sigma}^* m) \otimes \hat{\sigma}(\lambda) \) for \( m \in M \) and \( \lambda \in L[[z]] \). Also we define
\[
\hat{T}_p \mathfrak{M} := \text{Hom}_{A_p}(\hat{T}_p \mathfrak{M}, Q_p/A_p).
\]
Then \( \mathfrak{M} \mapsto \hat{T}_p \mathfrak{M} \) yields a contravariant exact functor from the category of torsion local shtukas to that of torsion \( A_p \)-modules with discrete \( G_L \)-action.
For an isogeny of local shtukas \( f: (\hat{M}, \tau_{\hat{M}}) \to (\hat{M}', \tau_{\hat{M}'}) \) over \( \mathcal{O} \), set \( \mathcal{M}_f := \text{Coker}(f) \). We see that \( \tau_{\hat{M}'} \) induces an isomorphism \( \tau_f: \hat{\sigma}^*\mathcal{M}_f[z_{\hat{M}'}] \to \mathcal{M}_f[z_{\hat{M}}] \). Then we have the following (cf. \cite[Example 3.7.3]{HK20}): \[
\]

**Lemma 4.9.** Let \( f: \hat{M} \to \hat{M}' \) be an isogeny of local shtukas over \( \mathcal{O} \). Then \( \mathcal{M}_f = (\mathcal{M}_f, \tau_f) \) is a torsion local shtuka over \( \mathcal{O} \), which is effective of height \( \leq h \) if \( \hat{M}' \) is effective of height \( \leq h \).

**Proof.** Recall that \( f \) is injective and \( \text{rk } \hat{M} = \text{rk } \hat{M}' \). By the definition of isogenies, there is a positive integer \( n \) such that \( z^n\hat{M}' \subset f(\hat{M}) \), and so \( \mathcal{M}_f \) is annihilated by \( z^n \). Tensoring with the residue field \( k \) over \( \mathcal{O} \), we have an exact sequence \[
0 \to \text{Tor}_1^\mathcal{O}(k, \mathcal{M}_f) \to \hat{M}' \otimes_{\mathcal{O}[z]} k[[z]] \to \hat{M} \otimes_{\mathcal{O}[z]} k[[z]] \to \mathcal{M}_f \otimes_{\mathcal{O}} k \to 0.
\]

Since \( \hat{M} \otimes_{\mathcal{O}[z]} k[[z]] \) and \( \hat{M}' \otimes_{\mathcal{O}[z]} k[[z]] \) are free \( k[[z]] \)-modules of rank \( \text{rk } \hat{M} = \text{rk } \hat{M}' \) and \( \mathcal{M}_f \otimes_{\mathcal{O}} k \) is annihilated by \( z^n \), the elementary divisor theorem implies that \( f \otimes \text{id}_k \) is injective. Hence \( \text{Tor}_1^\mathcal{O}(k, \mathcal{M}_f) = 0 \). Applying the snake lemma to the diagram

\[
\begin{array}{ccc}
0 & \to & \hat{M} \oplus \mathcal{M}_f \\
\downarrow f & & \downarrow \text{id} \\
\hat{M}' & \to & \mathcal{M}_f \oplus \hat{M}' \\
\downarrow z^n & & \downarrow z^n \\
0 & \to & \hat{M} \to \mathcal{M}_f \to 0,
\end{array}
\]

we see that \( \mathcal{M}_f = \text{Coker}(f: \hat{M}/z^n\hat{M} \to \hat{M}'/z^n\hat{M}') \) and hence \( \mathcal{M}_f \) is finitely presented over \( \mathcal{O} \). Thus Nakayama’s lemma shows that \( \mathcal{M}_f \) is finite free over \( \mathcal{O} \). If \( \hat{M}' \) is effective of height \( \leq h \), then \( \mathcal{M}_f \) is effective. The surjection \( \hat{M}'/\tau_{\hat{M}'}(\hat{\sigma}^*\hat{M}') \to \mathcal{M}_f/\tau_f(\hat{\sigma}^*\mathcal{M}_f) \) shows that \( \mathcal{M}_f \) is of height \( \leq h \).

**Remark 4.10.** Conversely, it is known that any torsion local shtuka \( \mathcal{M} \) over \( \mathcal{O} \) is of the form \( \mathcal{M} = \mathcal{M}_f \) for some isogeny \( f: \hat{M} \to \hat{M}' \) of local shtukas over \( \mathcal{O} \); see \cite[Lemma 3.7.5]{HK20}.

**Example 4.11.** For a local shtuka \( \hat{M} = (\hat{M}, \tau_{\hat{M}}) \) over \( \mathcal{O} \) and any positive integer \( n \), the map \( f_n: \hat{M} \to \hat{M}; m \mapsto z^n m \) gives an isogeny \( f_n: \hat{M} \to \hat{M} \). Thus we get the torsion local shtuka \( \hat{M}/z^n\hat{M} := (\hat{M}/z^n\hat{M}, \tau_{\hat{M}/z^n\hat{M}} \mod z^n) \).

**Lemma 4.12.** Let \( f: \hat{M} \to \hat{M}' \) be an isogeny of local shtukas over \( \mathcal{O} \). Then for the torsion local shtuka \( \mathcal{M}_f \), induced by \( f \), we have \( T_p\mathcal{M}_f \cong \text{Coker}(T_p f) \) as torsion \( A_p \)-modules with discrete \( G_L \)-action, where \( T_p f: T_p \hat{M}' \to T_p \hat{M} \) is the map induced by \( f \). In particular, for a local shtuka \( \hat{M} \), we have a natural isomorphism \( T_p \hat{M} \otimes A_p (A_p/p^n A_p) \cong T_p \left( \hat{M}/z^n \hat{M} \right) \).

**Proof.** Applying \cite[Lemma 5.1.9]{Kim09} to the map \( f: \hat{M} \otimes_{\mathcal{O}[z]} L[z] \to \hat{M}' \otimes_{\mathcal{O}[z]} L[z] \) induced by \( f \), we obtain the conclusion.

**Proposition 4.13.** Let \( \mathcal{M} \) be a torsion local shtuka over \( \mathcal{O} \). Suppose that there is an exact sequence
\[
0 \to T' \to T_p \mathcal{M} \to T'' \to 0
\]
of torsion \( A_p \)-modules of finite length with discrete \( G_L \)-action. Then the sequence is induced by an exact sequence
\[
0 \to \mathcal{M}'' \to \mathcal{M} \to \mathcal{M}' \to 0
\]
of torsion local shtukas over \( \mathcal{O} \) via the functor \( \mathcal{M} \mapsto T_p \mathcal{M} \). Moreover, if \( \mathcal{M} \) is effective of height \( \leq h \), then so are \( \mathcal{M}' \) and \( \mathcal{M}'' \).
Proof. This is \cite[Proposition 3.7.15]{HK20} (or \cite[Proposition 9.2.2]{Kim09}), and so we just show how to give $\mathfrak{M}'$ and $\mathfrak{M}''$. We set $T = T_p\mathfrak{M}$, $T' = \text{Hom}_{\mathbb{A}_p}(T', Q_p/A_p)$, and $T'' = \text{Hom}_{\mathbb{A}_p}(T'', Q_p/A_p)$. Taking $G_L$-fixed parts of the inverse of (4.1), we have an isomorphism

$$\mathfrak{M} \otimes \mathcal{O}_K [z] \overset{\sim}{\longrightarrow} (\bar{T} \otimes \mathbb{A}_p L^\text{sep}[z])^G_L.$$ 

As the composition with $(\bar{T} \otimes \mathbb{A}_p L^\text{sep}[z])^G_L \to (T' \otimes \mathbb{A}_p L^\text{sep}[z])^G_L$, we obtain a surjection

$$f: \mathfrak{M} \otimes \mathcal{O}_K [z] \longrightarrow (T' \otimes \mathbb{A}_p L^\text{sep}[z])^G_L.$$ 

Set $\mathfrak{M}' = f(\mathfrak{M})$. Then $\tau_{\mathfrak{M}}$ induces an isomorphism $\tau_{\mathfrak{M}'}: \hat{\vartheta}^*\mathfrak{M}'[\frac{1}{z-\zeta}] \cong \mathfrak{M}'[\frac{1}{z-\zeta}]$ and the pair $\mathfrak{M}' = (\mathfrak{M}', \tau_{\mathfrak{M}'})$ is a torsion local shtuka over $\mathcal{O}$ with $T_p\mathfrak{M}' \cong T'$. Next let $\mathfrak{M}''$ be the kernel of $f_{|\mathfrak{M}}: \mathfrak{M} \to \mathfrak{M}'$ and $\tau_{\mathfrak{M}''} = \tau_{\mathfrak{M}'}|_{\mathfrak{M}''}$. Then $\mathfrak{M}'' = (\mathfrak{M}'', \tau_{\mathfrak{M}''})$ is a torsion local shtuka over $\mathcal{O}$ with $T_p\mathfrak{M}'' \cong T''$. \hfill $\Box$

4.2. Tame inertia weights. Let $I_L \subset G_L$ be the inertia subgroup, so that $I_L \cong \text{Gal}(L^\text{sep}/L^\text{ur})$. Let $I^\text{wild}_L \subset I_L$ be the wild inertia subgroup, that is, the maximal pro-$p$ subgroup of $I_L$. Define the tame inertia group by $I^\text{tame}_L := I_L/I^\text{wild}_L \cong \text{Gal}(L^\text{tame}/L^\text{ur})$, where $L^\text{tame}$ is the maximal tamely ramified extension of $L^\text{ur}$. We note that $I^\text{tame}_L$ is pro-cyclic.

Let $\mathbb{F}$ be an intermediate finite subfield of $k/F_p$ and set $r = [\mathbb{F} : F_p]$. Recall that we now set $\hat{q} = #\mathbb{F}$, so that $#F = \hat{q}^r$. Denote by $\mu_{q^r-1}(L)$ the group of $(\hat{q}^r - 1)$-st roots of unity in $L$ and choose an isomorphism $\mu_{q^r-1}(L) \cong \mathbb{F}^\times$. Let $\pi \in L$ be a uniformizer. Denote by $L\langle \pi \rangle$ the finite Galois extension of $L$ generated by the roots of $X^{q^r-1} - \pi$. Then $L\langle \pi \rangle / L$ is a totally tamely ramified extension whose ramification index is $\hat{q}^r - 1$. For a root $\pi_r \in L\langle \pi \rangle$ of $X^{q^r-1} - \pi$, we obtain the surjective character

$$\theta_r = \theta_{r/L}: I_L \to \mu_{q^r-1}(L) \cong \mathbb{F}^\times; \quad s \mapsto \frac{s(\pi_r)}{\pi_r},$$

which is called a fundamental character of level $r$ (cf. \cite{Ser72}). It factors through $I^\text{tame}_L$ and does not depend on the choice of $\pi$. For a field extension $L'/L$ with finite ramification index $e_{L'/L}$, we have $\theta_{r/L'}|_{I_{L'}} = (\theta_{r/L})^{e_{L'/L}}$ by construction.

Let $\psi: I_L \to \mathbb{F}^\times$ be an $\mathbb{F}^\times$-valued character. Then it factors through $I^\text{tame}_L$ and there is an integer $0 \leq n \leq \hat{q}^r - 1$ such that $\psi = \theta^n$. If the integer $n$ decomposes $n = n_0\hat{q}^{r-1} + n_1\hat{q}^{r-2} + \cdots + n_{r-1}$ with integers $0 \leq n_i \leq \hat{q} - 1$, then $n_0, \ldots, n_{r-1}$ are independent of the choice of the $\mu_{q^r-1}(L) \cong \mathbb{F}^\times$.

Definition 4.14. We call these integers $n_0, \ldots, n_{r-1}$ the tame inertia weights of $\psi: I_L \to \mathbb{F}^\times$ and denote by $\mathcal{T}\mathcal{I}(\psi) = \{n_0, \ldots, n_{r-1}\}$ the multi-set consisting of them.

To define the tame inertia weights of $p$-adic representations of $G_L$, we need the following:

Lemma 4.15. An irreducible $\mathbb{F}_p$-representation $\bar{V}$ of $I_L$ becomes a one-dimensional representation of $I^\text{tame}_L$ over a finite extension $\mathbb{F}$ of $F_p$ with $[\mathbb{F} : F_p] = \dim_{F_p} \bar{V}$.

Proof. By irreducibility, the $I^\text{wild}_L$-action on $\bar{V}$ is trivial and so the action of $I_L$ factors through $I^\text{tame}_L$. Thus we obtain a group homomorphism $\rho: I^\text{tame}_L \to \text{GL}_{\mathbb{F}_p}(\bar{V})$. Let us consider the commutant $\mathbb{F} := \text{End}_{\mathbb{F}_p[I^\text{tame}_L]}(\bar{V}) \subset \text{GL}_{\mathbb{F}_p}(\bar{V})$. Then Schur’s lemma implies that $\mathbb{F}$ is a finite division algebra, so it is a finite field. Since $I^\text{tame}_L$ is abelian, the image $\rho(I^\text{tame}_L)$ is contained in the commutant $\mathbb{F}$. Thus we see by irreducibility that $\bar{V}$ is one-dimensional over $\mathbb{F}$ and so $[\mathbb{F} : F_p] = \dim_{\mathbb{F}_p} \bar{V}$. \hfill $\square$

Let $V$ be a $p$-adic representation of $G_L$. Take a $G_L$-stable $\mathbb{A}_p$-lattice $T \subset V$ and consider the $F_p$-representation $\bar{V} := T/pT$ of $G_L$. Then by the Brauer-Nesbitt theorem, the isomorphism class of the semi-simplification $\bar{V}^{ss} = \bigoplus_i \bar{V}_i$ is independent of the choice of $T$. We see that each simple factor $\bar{V}_i$ is irreducible as a representation of $I_L$ because the $I_L$-fixed part $\bar{V}_i I_L$ is a subrepresentation of $\bar{V}_i$. Hence Lemma 4.15 implies that $I^\text{tame}_L$-action on $\bar{V}^{ss}$ is described by a sum of characters $\psi_i: I^\text{tame}_L \to \mathbb{F}_i^\times$. 


where $F_i$ are some finite extensions of $F_p$. Replacing $L$ by a suitable finite unramified extension, we may assume that each $F_i$ can be embedded into $k$ and so each $\psi_i$ is a power of the fundamental character $\theta_i : L_{\text{tame}}^\times \to F_i^\times$ of level $r_i = [F_i : F_p]$.

**Definition 4.16.** We define the multi-set of tame inertia weight $T \mathcal{I}(V)$ of $V$ by the union of the $T \mathcal{I}(\psi_i)$ for all $i$.

### 4.3. Torsion local shtukas with coefficients

To study tame inertia weights of Galois representations coming from local shtukas, we shall give an overview of the theory on torsion local shtukas with “coefficients”. Note that we only consider the finite field coefficients case; see [Kim09] and [HK20] in general.

As before, let $F$ be an intermediate finite field of $k/F_p$ with degree $r = [F : F_p]$. We regard $F$ as an $A_p$-algebra via $A_p = F_p[[z]] \to F_p \subset F; z \mapsto 0$. Let denote $O[z]_F := O[z] \otimes_{A_p} F$ and define $\hat{\sigma} : O[z]_F \to O[z]_F$ by $F$-linearly extending $\hat{\sigma} : O[z] \to O[z]$.

**Definition 4.17.** A torsion local $F$-shtuka over $O$ is a pair $\mathfrak{M}_F = (\mathcal{M}_F, \tau_{\mathcal{M}_F})$ consisting of a free $O[z]_F$-module of finite rank and an $O[z]_F$-isomorphism $\tau_{\mathcal{M}_F} : \hat{\sigma}^* \mathcal{M}_F[z^{-1}] \cong \mathcal{M}_F[z^{-1}]$. By $\mathcal{M}_F$-rank of $\mathfrak{M}_F$, we mean the rank of the $O[z]_F$-module $\mathcal{M}_F$. We consider the obvious notion of morphisms.

We may regard $\mathfrak{M}_F$ as a torsion local shtuka over $O$ by forgetting the $F$-action. We say that $\mathfrak{M}_F$ is effective (resp. effective of height $\leq h$ for some $h \geq 0$) if it is so as a torsion local shtuka.

**Remark 4.18.** By [HK20] Lemma 3.7.24], any torsion local shtuka $\mathfrak{M} = (\mathfrak{M}, \tau_{\mathfrak{M}})$ equipped with a $\tau_{\mathfrak{M}}$-compatible $F$-action is a torsion local $F$-shtuka, that is, the underlying module $\mathfrak{M}$ is free over $O[z]_F$.

**Lemma 4.19.** For a torsion local $F$-shtuka $\mathfrak{M}_F$ over $O$, we see that $T_p \mathfrak{M}_F$ becomes an $F$-vector space of dimension equal to the $O[z]_F$-rank of $\mathfrak{M}_F$ such that the $F$-action on $T_p \mathfrak{M}_F$ is induced by that on $\mathfrak{M}_F$.

**Proof.** This is [HK20] Lemma 3.7.23 (a). \qed

For a uniformizer $\pi \in L$, we identify $L = k((\pi))$ and $O = k[[\pi]]$, and fix an embedding $F \hookrightarrow k \subset O$. Set $e := \text{ord}_Q(\gamma)$. Canonically extending $\gamma : A \subset Q \to K$ to the embedding $\gamma : Q_p \to L$, we see that the integer $e$ is the ramification index of $L/\gamma(Q_p)$. Let $h \geq 0$. For any $\lambda \in k^\times$ and any $r$-tuple $n = (n_0, n_1, \ldots, n_{r-1})$ of integers with $0 \leq n_i \leq eh$, we obtain an important effective torsion local $F$-shtuka $\mathfrak{M}_{(\lambda, \underline{n})} = (\mathfrak{M}_{(\lambda, \underline{n})}, \tau)$ of $O[z]_F$-rank one as follows. We view $O$ as an $O[z]$-algebra over $z \mapsto 0$. We set
\[
\mathfrak{M}_{(\lambda, \underline{n})} := \bigoplus_{i=0}^{r-1} O \cdot e_i
\]
and define $\tau : \hat{\sigma}^* \mathfrak{M}_{(\lambda, \underline{n})} \to \mathfrak{M}_{(\lambda, \underline{n})}$ by
\[
\tau(\hat{\sigma}^* e_i) := \pi^n_i e_{i+1} \quad \text{if } i \neq r - 1,
\]
\[
\tau(\hat{\sigma}^* e_{r-1}) := \lambda \pi^{n_{r-1}} e_0.
\]
Since $\text{Coker}(\tau) \cong \bigoplus_{i=0}^{r-1} O/(\pi^{n_i})$ and $(z - \xi)k \equiv e \xi^{eh}$ (mod $z$) for some $\xi \in O\times$, we see that $\mathfrak{M}_{(\lambda, \underline{n})}$ is an effective torsion local shtuka over $O$ of height $\leq h$. Let us consider the $F$-action on $\mathfrak{M}_{(\lambda, \underline{n})}$ given by $b \cdot e_i := b^\ell e_i$ for $b \in F$. Then it is compatible with $\tau$ and hence $\mathfrak{M}_{(\lambda, \underline{n})}$ is a torsion local $F$-shtuka whose $O[z]_F$-rank is one. Thus the $G_L$-action on $T_p \mathfrak{M}_{(\lambda, \underline{n})}$ is given by an $F^\times$-valued character $\psi : G_L \to F^\times$.

**Lemma 4.20.** Let $\lambda \in k^\times$ and $\underline{n} = (n_0, \ldots, n_{r-1})$ be as above. Then the character $\psi : G_L \to F^\times$ determined by $T_p \mathfrak{M}_{(\lambda, \underline{n})}$ satisfies $\psi|_{G_L} = \theta_r^\psi$, where $n = n_0q^{-1} + n_1q^{-2} + \cdots + n_{r-1}$. Therefore, if $eh \leq q - 1$, then we have $T \mathcal{I}(T_p \mathfrak{M}_{(\lambda, \underline{n})}) = \{n_0, \ldots, n_{r-1}\}$ as multi-sets.
Proof. It suffices to check that the $I_L$-action on the dual $\bar{T}_p\mathfrak{M}(\lambda,\omega)$ of $T_p\mathfrak{M}(\lambda,\omega)$ is given by $\theta_r^n$ for $n = \sum_{i=0}^{r-1} n_i q^{r-1-i}$. Since $\mathfrak{M}(\lambda,\omega) \otimes \mathcal{O}[z] L^{\text{sep}}[z] = \bigoplus_{i=0}^{r-1} L^{\text{sep}} \cdot e_i$, any element $m \in \bar{T}_p\mathfrak{M}(\lambda,\omega)$ is of the form $m = \sum_{i=0}^{r-1} x_i e_i$ such that $x_i \in L^{\text{sep}}$ satisfy $x_i^\lambda = (\pi^n)^{-1} x_{i+1}$ for $i \neq r-1$ and $x_{r-1}^\lambda = (\pi^n)^{-1} x_0$. Since giving such $m$ is equivalent to giving $x_0 \in L^{\text{sep}}$ with $x_0^\lambda = (\pi^n)^{-1} x_0$, we can regard $\bar{T}_p\mathfrak{M}(\lambda,\omega)$ as an $\mathbb{F}$-submodule of $L^{\text{sep}}$ via the map $m = \sum_{i=0}^{r-1} x_i e_i \mapsto x_0$. Then the $G_L$-action on $\bar{T}_p\mathfrak{M}(\lambda,\omega)$ is given by $s(x_0) = \varepsilon(s) \theta_r^n x_0$ for $s \in G_L$, where $\varepsilon : G_L \to \mathbb{F}^\times$ is the unramified character determined by $s(\lambda') = \varepsilon(s) \lambda'$ for a $(\tilde{q}^{-1} - 1)$-st root $\lambda'$ of $\lambda^{-1}$. This provides the conclusion. \hfill $\square$

**Example 4.21.** Let $\hat{M}$ be an effective local shtuka over $\mathcal{O}$ with $\text{rk} \hat{M} = 1$. Then it is of the form $\hat{M} = (\mathcal{O}[z], \tau_M(\hat{\sigma}^* e) = \xi(z-\zeta)^d e)$, where $d = \text{dim} M$ and $\xi \in \mathcal{O}[z]^\times$. Since $\mathcal{O}[z]^\times = \mathcal{O}^\times + z\mathcal{O}[z]$, the effective torsion local shtuka $\hat{M} = \hat{M}/z\hat{M}$ is given by $\mathfrak{M} = \mathcal{O} \cdot e$ and $\tau_{\mathfrak{M}}(\hat{\sigma}^* e) = \lambda_\mathfrak{M} d^{-1}$ for some $\lambda \in \mathcal{O}^\times$. Hence, if $ed \leq \hat{q} - 1$, then $\mathfrak{M} = \mathfrak{M}(\lambda,\omega)$ for $n = (ed)$ and so the $I_L$-action on $T_p\mathfrak{M}$ is given by $\theta_r^n$.

**Proposition 4.22.** Let $\mathbb{F} \subset k$ be as above. Then any effective torsion local $\mathbb{F}$-shtuka $\mathfrak{M}_\mathbb{F}$ over $\mathcal{O}$ of height $\leq h$ with $\mathbb{F}[z]_\mathbb{F}$-rank one is isomorphic to $\mathfrak{M}_\mathbb{F}(\lambda,\omega)$ for a unique $\lambda \in k^\times$ up to $(k^\times)^{\hat{q}^{-1}-1}$-multiple, and a unique $n = (n_0, n_1, \ldots, n_{r-1})$ with $0 \leq n_i \leq eh$.

Proof. This is [Kim09] Corollary 9.1.4]. \hfill $\square$

Based on the above preparation, we can get an important formula for tame inertia weights of $p$-adic representations arising from effective local shtukas under the condition $eh < \hat{q} - 1$.

**Proposition 4.23.** Let $\hat{M}$ be an effective local shtuka over $\mathcal{O}$ of height $\leq h$. Suppose that the ramification index $e = \text{ord}_L(\zeta)$ of $L/\gamma(Q_p)$ satisfies $eh < \hat{q} - 1$. Then $TI(V_p\hat{M}) \subset [0,eh]$ and the equation

$$e \cdot \text{dim} \hat{M} = \Sigma TI(V_p\hat{M})$$

holds.

Proof. We may assume that $k$ is algebraically closed and $G_L = I_L$ for the following reason. Let $\hat{L}^\text{ur}$ be the completion of the maximal unramified extension $L^\text{ur}$ of $L$ and denote by $\hat{\mathcal{O}}^\text{ur}$ the valuation ring of $\hat{L}^\text{ur}$. Then $\hat{L}^\text{ur}$ has residue field $k^\text{sep}$ and $G_L^\text{ur} \cong I_L$. In addition, it follows that $\hat{M} \otimes \hat{\mathcal{O}}^\text{ur}[z] := (\hat{M} \otimes \mathcal{O}[z]) \otimes \mathcal{O}^\text{ur}[z] ; \tau_{\hat{M}} \otimes \text{id}$ becomes an effective local shtuka over $\hat{\mathcal{O}}^\text{ur}$ which has the same rank and dimension of $\hat{M}$, and that $T_p\hat{M}|_{I_L}$ is canonically isomorphic to $T_p(\hat{M} \otimes \hat{\mathcal{O}}^\text{ur}[z])$; see [Kim09] Proposition 8.1.13 for more details.

Now we set $r = \text{rk} \hat{M}$ and $d = \text{dim} \hat{M}$, so that $\text{det} \tau_{\hat{M}} \in (z-\zeta)^d : \mathcal{O}[z]^\times$ by Lemma 4.22. Consider the effective torsion local shtuka $\mathfrak{M}_p := (\mathfrak{M}, \tau_{\mathfrak{M}}) := \hat{M}/z\hat{M}$. Then $\mathfrak{M}_p$ is of height $\leq h$. Since $T_p\hat{M} \otimes \mathcal{O}_p \cong T_p\mathfrak{M}_p$ as $\mathcal{O}_p$-representations of $G_L$, we have $TI(V_p\hat{M}) = TI(T_p\mathfrak{M}_p)$. Since $\text{det} \tau_{\mathfrak{M}_p} \equiv \text{det} \tau_{\hat{M}} \mod z$ and $\mathcal{O}[z]^\times = \mathcal{O}^\times + z\mathcal{O}[z]$, we have $\text{ord}_L(\text{det} \tau_{\mathfrak{M}_p}) = ed$. Thus it suffices to check the equation $\text{ord}_L(\text{det} \tau_{\mathfrak{M}_p}) = \Sigma TI(T_p\mathfrak{M}_p)$.

To do this, we first consider the case where $T_p\mathfrak{M}$ is irreducible. Then Lemma 4.13 implies that $T_p\mathfrak{M}$ is one-dimensional representation of $I_L$ over the finite field $\mathbb{F} := \text{End}_{\mathcal{O}_p}[T^\text{ur}\mathfrak{M}]$ of degree $[\mathbb{F} : \mathbb{F}_p] = r$. Taking $G_L$-invariant parts of both sides of (4.1), we get an isomorphism

$$(T_p\mathfrak{M} \otimes \mathcal{O}_p L^{\text{sep}}[z])^{G_L} \cong \mathfrak{M} \otimes \mathcal{O}[z] L[z].$$

Via this isomorphism, the $F$-action of $T_p\mathfrak{M}$ induces an $F$-action on $\mathfrak{M} \otimes \mathcal{O}[z] L[z]$, which commutes with $\tau_{\mathfrak{M}} \otimes \text{id}$. By the assumption $eh < \hat{q} - 1$, it follows by [Kim09] Proposition 9.3.3. (2) that this $F$-action preserves $\mathfrak{M}$ and so $\mathfrak{M}$ becomes an effective torsion local $\mathbb{F}$-shtuka. Since $\mathfrak{M}$ is of $\mathcal{O}[z]$-$\mathbb{F}$-rank one, it follows by Proposition 4.22 that $\mathfrak{M}$ is isomorphic to $\mathfrak{M}(\lambda,\omega)$ for some $\lambda \in k^\times$ and
$\sigma = (n_0, \ldots, n_{r-1})$ with $0 \leq n_i \leq eh$. Hence Lemma 4.20 shows that $\mathcal{T}(T_p \hat{\mathcal{M}}) = \{n_0, \ldots, n_{r-1}\}$. Since $\text{Coker}(\tau_{\mathfrak{m}}) \cong \bigoplus_{i=0}^{r-1} \mathcal{O}/(\pi^{n_i})$, we obtain $\text{ord}_L(\det \tau_{\mathfrak{m}}) = n_0 + \cdots + n_{r-1} = \sum \mathcal{T}(T_p \hat{\mathcal{M}})$.

Next, we suppose that $T_p \hat{\mathcal{M}}$ is not irreducible. Then there is a descending filtration by $G_L$-stable $\mathbb{F}_p$-subspaces of $T_p \hat{\mathcal{M}}$

$$0 = T_0 \subset T_1 \subset \cdots \subset T_n = T_p \hat{\mathcal{M}}$$

by the Jordan-Hölder theorem, and consider the semi-simplification $(T_p \hat{\mathcal{M}})^{ss} = \bigoplus_{j=1}^{n} T_j/T_j-1$. Then $\sum \mathcal{T}(T_p \hat{\mathcal{M}})$ is the sum of all $\sum \mathcal{T}(T_j/T_j-1)$. Proposition 4.13 shows that all sequences $0 \to T_j-1 \to T_j \to T_j/T_j-1 \to 0$ come from exact sequences of effective torsion local shtukas of height $\leq h$

$$0 \to \hat{\mathcal{M}}_j \to \hat{\mathcal{M}}_j \to \hat{\mathcal{M}}_{j-1} \to 0$$

with $T_p \hat{\mathcal{M}}_j = T_j/T_j-1$ and $T_p \hat{\mathcal{M}}_j = T_j$. Since $\det \tau_{\mathfrak{m}} = (\det \tau_{\mathfrak{m}_1}) \cdot (\det \tau_{\mathfrak{m}_{n-1}})$ and $\det \tau_{\mathfrak{m}_0} = \det \tau_{\mathfrak{m}_n}$, we have $\det \tau_{\mathfrak{m}} = \prod_{j=1}^{n} \det \tau_{\mathfrak{m}_j}$. Therefore $\text{ord}_L(\det \tau_{\mathfrak{m}}) = \sum_{j=1}^{n} \text{ord}_L(\det \tau_{\mathfrak{m}_j})$. Since each $T_p \hat{\mathcal{M}}_j$ is irreducible, we get $\text{ord}_L(\det \tau_{\mathfrak{m}}) = \sum \mathcal{T}(T_j/T_j-1)$ for any $1 \leq j \leq n$. Hence $\text{ord}_L(\det \tau_{\mathfrak{m}}) = \sum \mathcal{T}(T_p \hat{\mathcal{M}})$ holds.

\[\Box\]

Remark 4.24. \textit{The relation $\mathcal{T}(V_p \check{\mathcal{M}}) < [0, eh]$ is of course true even if $eh = \check{q} - 1$, but the equation $e \cdot \dim \check{M} = \sum \mathcal{T}(V_p \check{\mathcal{M}})$ does not hold in general. For example, if $e = \check{q} - 1$ and $\dim \check{M} = \dim \check{M} = 1$ (hence, $\check{M}$ is of height $\leq 1$), then $I_L$ acts on $T_p \check{M} \otimes_{A_p} \mathbb{F}_p$ via $\theta_1 = \theta_1^{\check{q} - 1} = \theta_1$. Thus $\sum \mathcal{T}(V_p \check{\mathcal{M}}) = 0$.}

5. Rigid analytic aspects of effective $A$-motives

In this section, we introduce the notion of \textit{strongly semi-stable reduction} for effective $A$-motives and prove a formula between tame inertia weights and dimensions of such effective $A$-motives. We fix a maximal ideal $\mathfrak{p}$ of $A$. Let $L$ be a completion of $K$ at a finite place lying above $\mathfrak{p}$. Let us denote its valuation ring by $\mathcal{O}$ and its residue field by $k$. Thus the $A$-field structure $\gamma: A \to K \subset L$ factors through $\mathcal{O}$ and we have $\mathfrak{p} = \text{Ker}(A \xrightarrow{\gamma} \mathcal{O} \to k)$. We take a uniformizer $\pi \in L$.

5.1. Analytic $A(1)$-motives. To introduce “analytification” of effective $A$-motives, let us first prepare some rigid analytic rings (cf. [HH16] and [Hus13]). Recall that $A_L$ and $A_k$ are Dedekind domains. By the isomorphism $A_\mathcal{O}/\pi A_\mathcal{O} \cong A_k$, the uniformizer $\pi$ is a prime element of $A_\mathcal{O}$.

\textbf{Definition 5.1.} \textit{Denote by $\tilde{A}_\mathcal{O} := A_\mathcal{O}[\pi]$ the $\pi$-adic completion of $A_\mathcal{O}$, and set $\tilde{A}_L := A_\mathcal{O}[\frac{1}{\pi}]$.}

It follows that the topological $\mathcal{O}$-algebra $\tilde{A}_\mathcal{O}$ is \textit{admissible} in the sense of Raynaud, that is, it is of topologically finite presentation and has no $\pi$-torsion; see [Bos14, §7.2, Definition 3] for instance. (It is also known from [Hus13, Proposition 1.7] that $A_\mathcal{O}$ is a regular integral domain.) Hence $\tilde{A}_L$ is a reduced affinoid $L$-algebra. Let $(p, \pi)$ denote the ideal of $A_\mathcal{O}$ generated by $p$ and $\pi$. By [HH16, Lemma 2.3] (or [Hus13, Lemma 1.16]), the $(p, \pi)$-adic completion of $A_\mathcal{O}$ is canonically isomorphic to $A_{p, \mathcal{O}}(= A_p \hat{\otimes} \mathcal{O})$ as $A_\mathcal{O}$-algebras. Since the canonical map $A_\mathcal{O} \to A_{p, \mathcal{O}}$ uniquely factors through $\tilde{A}_\mathcal{O}$, the induced map $\tilde{A}_\mathcal{O} \to A_{p, \mathcal{O}}$ identifies $A_{p, \mathcal{O}}$ with the $(p, \pi)$-adic completion of $\tilde{A}_\mathcal{O}$, and hence it is flat. Thus we obtain the commutative diagram

$$
\begin{array}{ccc}
A_\mathcal{O} & \longrightarrow & \tilde{A}_\mathcal{O} & \longrightarrow & \tilde{A}_L \\
\downarrow & & \downarrow & & \downarrow \\
A_\mathcal{O} & \longrightarrow & A_{p, \mathcal{O}} & \longrightarrow & A_{p, L},
\end{array}
$$

where all arrows are injective and flat.

Let us next consider the lifting of the Frobenius map $\sigma = \text{id}_A \otimes (\cdot)^q$ of $A_\mathcal{O}$ to the above rings. We see that the map $\sigma$ is $\pi$-adically and $p$-adically continuous, and therefore it canonically extends to
$\sigma: \tilde{A}_O \to \tilde{A}_O$ and $\sigma: A_{p,O} \to A_{p,O}$. Since $\sigma: A_O \to A_O$ is finite flat by [HH16, Remark 3.2], we get the commutative diagram

$$
\begin{array}{ccc}
A_O & \longrightarrow & \tilde{A}_O \\
\sigma \downarrow & & \sigma \downarrow \\
A_O & \longrightarrow & \tilde{A}_O
\end{array}
$$

satisfying that both squares are co-Cartesian and all vertical arrows are finite flat; see [HH16, Lemma 3.1].

Now let $\tilde{J}_O$ be the ideal of $\tilde{A}_O$ generated by $\{a \otimes 1 - 1 \otimes \gamma(a) \mid a \in A\} \subset A_{O}$, so that $\tilde{J}_O = J_O \tilde{A}_O$. Since $A_{O}$ is Noetherian and $\mathcal{O}$ is $\pi$-adically complete, the map $A_{O} \to \mathcal{O}; a \otimes \lambda \mapsto \gamma(a)\lambda$ induces a surjection $\tilde{A}_{O} \to \mathcal{O}$ and hence $\tilde{J}_O$ is the kernel of this map. Consequently, the ideal $\tilde{J} = J \tilde{A}_L$ of $\tilde{A}_L$ is a maximal ideal with residue field $\mathcal{L}$.

**Definition 5.2.** An analytic $A(1)$-motive over $L$ of rank $r$ and dimension $d$ is a pair $\tilde{M} = (\tilde{M}, \tau_{\tilde{M}})$ consisting of a locally free $\tilde{A}_L$-module of rank $r$ and an injective $\tilde{A}_L$-homomorphism $\tau_{\tilde{M}}: \sigma^* \tilde{M} \to \tilde{M}$ such that $\text{Coker}(\tau_{\tilde{M}})$ is a $d$-dimensional $L$-vector space annihilated by a power of $\tilde{J}$. Set $\text{rk} \tilde{M} = r$ and $\dim \tilde{M} = d$. For $h \geq 0$, we say that $\tilde{M}$ is of height $\leq h$ if $\tilde{J}^h \cdot \text{Coker}(\tau_{\tilde{M}}) = 0$.

A morphism $f: \tilde{M} \to \tilde{N}$ between analytic $A(1)$-motives over $L$ is an $\tilde{A}_L$-homomorphism $f: \tilde{M} \to \tilde{N}$ such that $f \circ \tau_{\tilde{M}} = \tau_{\tilde{N}} \circ \sigma^* f$.

**Remark 5.3.** The prefix “$A(1)$-”, which is used in [HH16] to reflect the notation in [BH07], indicates that analytic $A(1)$-motives are variants of effective $A$-motives over the rigid analytic “unit disc” in $\text{Spec} A_L$. To elaborate, let us recall a geometric interpretation of the rings $\tilde{A}_O$ and $\tilde{A}_L$.

In contrast, another (and well-known) analytic variant of effective $A$-motives is the notion of analytic $\tau$-sheaves, which are locally free $\mathcal{O}_{\text{Spec} A_L}^{\text{an}}$-modules $\mathcal{F}$ of finite rank endowed with suitable injective Frobenius structures $\tau_{\mathcal{F}}: \sigma^* \mathcal{F} \to \mathcal{F}$; see [Gar02] and [Gar02] for details. If the cokernel of $\tau_{\mathcal{F}}$ is supported on the point corresponding to $\tilde{J}$, the restriction of analytic $\tau$-sheaf $(\mathcal{F}, \tau_{\mathcal{F}})$ to the unit disc $\text{Sp} \tilde{A}_L$ gives rise to an analytic $A(1)$-motive over $L$. Consequently, various properties of analytic $\tau$-sheaves are also valid for analytic $A(1)$-motives.

**Definition 5.4.** Let $\tilde{M}$ be an analytic $A(1)$-motive over $L$. An analytic $A(1)$-motive $\tilde{N}$ is called an $A(1)$-submotive of $\tilde{M}$ if $\tilde{N}$ is an $\tilde{A}_L$-submodule of $\tilde{M}$ and $\tau_{\tilde{N}} = \tau_{\tilde{M}}|_{\sigma^* \tilde{N}}$. We say that an $A(1)$-submotive $\tilde{N}$ of $\tilde{M}$ is saturated if the map $\tau_{\tilde{M}} (\mod \tilde{N})): \sigma^* \tilde{M}/\tilde{N} \to \tilde{M}/\tilde{N}$ induced by $\tau_{\tilde{M}}$ is injective, which means that the pair $\tilde{M}/\tilde{N} = (\tilde{M}/\tilde{N}, \tau_{\tilde{M}} (\mod \tilde{N}))$ is an analytic $A(1)$-motive over $L$.

**Lemma 5.5.** For an exact sequence $0 \to \tilde{M}' \to \tilde{M} \to \tilde{M}'' \to 0$ of analytic $A(1)$-motives, we have $\dim \tilde{M} = \dim \tilde{M}' + \dim \tilde{M}''$. If $\tilde{M}$ is of height $\leq h$, then $\tilde{M}'$ and $\tilde{M}''$ are so.

**Proof.** The snake lemma yields an exact sequence of $L$-vector spaces

$$0 \longrightarrow \text{Coker}(\tau_{\tilde{M}'}) \longrightarrow \text{Coker}(\tau_{\tilde{M}}) \longrightarrow \text{Coker}(\tau_{\tilde{M}'}) \longrightarrow 0,$$

which proves the claim. \qed
For an effective $A$-motive $M = (M, \tau_M)$ over $L$, we define the analytification of $M$ by
\[ \tilde{M} \otimes \tilde{A}_L := (M \otimes_{A_L} \tilde{A}_L, \tau_M \otimes id). \]
We immediately obtain the following:

**Proposition 5.6.** The correspondence $M \mapsto \tilde{M} \otimes \tilde{A}_L$ yields a functor from the category of effective $A$-motives over $L$ to that of analytic $A(1)$-motives over $L$. If $M$ is of rank $r$, dimension $d$, and height $\leq h$, then so is $\tilde{M} \otimes \tilde{A}_L$.

Let $\tilde{M} = (\tilde{M}, \tau_{\tilde{M}})$ be an analytic $(1)$-motive over $L$ of rank $r$. With $\tilde{M}$, one can associate Galois representations by the same way as effective $A$-motives. For a maximal ideal $q$ of $A$, consider the $q$-adic completion $A_{q, \text{sep}} = A_q \otimes_{\mathbb{Q}_q} \mathbb{L}_{\text{sep}}$ of $A_{L, \text{sep}}$ and define the $q$-adic realization of $\tilde{M}$ by
\[ H^1_q(\tilde{M}, A_q) := \{ m \in \tilde{M} \otimes_{\tilde{A}_L} A_{q, \text{sep}} | \tau_{\tilde{M}}(\sigma^* m) = m \}, \]
which is a free $A_q$-module of rank $r$ equipped with a continuous action $G_L$. We denote the (rational) $q$-adic Tate modules of $\tilde{M}$ by
\[ T_q \tilde{M} := \text{Hom}_{A_q}(H^1_q(\tilde{M}, A_q), A_q) \quad \text{and} \quad V_q \tilde{M} := T_q \tilde{M} \otimes_{A_q} Q_q. \]

**Remark 5.7.** If $\tilde{M}$ is the analytification of an effective $A$-motive $M$ over $L$, it follows by construction that $T_q \tilde{M} \cong T_q M$ as $A_q[G_L]$-modules.

### 5.2. Reduction theory.

**Definition 5.8.** Let $\tilde{M}$ be an analytic $(1)$-motive over $L$.

1. A formal model of $\tilde{M}$ is a pair $\tilde{M} = (\tilde{M}, \tau_{\tilde{M}})$ consisting of a finite locally free $\tilde{A}_O$-module and an injective $\tilde{A}_O$-homomorphism $\tau_{\tilde{M}}: \sigma^* \tilde{M} \to \tilde{M}$ such that there is an isomorphism $\tau: \tilde{M} \cong \tilde{M} \otimes_{\tilde{A}_O} \tilde{A}_L$ with $\tau_{\tilde{M}} \circ \sigma^* \tau \mathrel{\sim} \tau \circ \tau_{\tilde{M}}$.
2. A formal model $\tilde{M}$ of $M$ is called a good model if the induced $A_k$-homomorphism
\[ \tau_{\tilde{M}} \otimes id: \sigma^* \tilde{M} \otimes_{\tilde{A}_O} A_k \to \tilde{M} \otimes_{\tilde{A}_O} A_k \]
is injective. In this case, we say that $\tilde{M}$ has good reduction.
3. We say that $\tilde{M}$ has strongly semi-stable reduction if there is a filtration (called a semi-stable filtration) of $\tilde{M}$ by saturated $(1)$-submodules
\[ 0 = \tilde{M}_0 \subset \tilde{M}_1 \subset \cdots \subset \tilde{M}_n = \tilde{M} \]
such that all the quotients $\tilde{M}_i/\tilde{M}_{i-1}$ have good reduction.

We say that an effective $A$-motive $M$ over $K$ has strongly semi-stable reduction at a finite place $v$ of $K$ if its analytification $\tilde{M} \otimes_{A_k} \tilde{A}_K$, does so.

**Remark 5.9.** By [H16] Theorem 4.7], it follows that $\tilde{M}$ is a good model of $M$ if and only if $\text{Coker}(\tau_{\tilde{M}})$ is a finite free $O$-module annihilated by some power of $\tilde{J}_O$. This means that the reduction $(\tilde{M} \otimes_{\tilde{A}_O} A_k, \tau_{\tilde{M}} \otimes id)$ becomes an $A$-motive over the residue field $k$ having the same rank and dimension of $M$. In addition, if $\tilde{M}$ is the analytification of an effective $A$-motive $M$ over $L$, then we can see that $\tilde{M}$ has good reduction if and if $\tilde{M}$ has good reduction, and that the reduction $(\tilde{M} \otimes_{\tilde{A}_O} A_k, \tau_{\tilde{M}} \otimes id)$ is isomorphic to $(M \otimes_{A_0} A_k, \tau_M \otimes id)$ for a good model $(M, \tau_M)$ of $\tilde{M}$.

**Remark 5.10.** Gardeyn [Gar02] Definition 4.6] provides a more general notion of semi-stability as follows. An analytic $\tau$-sheaf $F$ on the rigid analytic space $(\text{Spec } A_L)^{\text{an}}$ is said to be semi-stable if there is a non-empty open subsheaf $\mathcal{X}$ of $\text{Spec } A_L$ such that the restriction $F|_{\mathcal{X}_{\text{an}}}$ to the associated rigid analytic space $\mathcal{X}^{\text{an}}$ of $\mathcal{X}$ admits a suitable “semi-stable filtration”
\[ 0 = F_0 \subset F_1 \subset \cdots \subset F_n = F|_{\mathcal{X}_{\text{an}}}. \]
If \( X \) can be taken as \( X = \text{Spec} A_L \), then \( F \) is said to be strongly semi-stable. Although any analytic \( \tau \)-sheaf always becomes semi-stable after extending \( L \) to a finite extension, there is an example of an analytic \( \tau \)-sheaf which is never strongly semi-stable even if one extends \( L \) to any extensions; see [Gar02, §§5.3].

By the result [Gar02, Theorem 5.7] on analytic \( \tau \)-sheaves due to Gardeyn, we obtain an analytic \( A(1) \)-motive analog of the Néron-Ogg-Shafarevich criterion. Recall that now \( L \) is a completion of \( K \) at a place lying above \( p \).

**Theorem 5.11** (Gardeyn). For an analytic \( A(1) \)-motive \( \tilde{M} \) over \( L \), the following statements are equivalent.

1. \( \tilde{M} \) has good reduction.
2. For any maximal ideal \( q \) of \( A \) with \( q \neq p \), the \( q \)-adic representation \( V_q \tilde{M} \) is unramified.
3. There is a maximal ideal \( q \neq p \) such that \( V_q \tilde{M} \) is unramified.

**Corollary 5.12.** If an analytic \( A(1) \)-motive \( \tilde{M} \) over \( L \) has strongly semi-stable reduction, then for any maximal ideal \( q \) of \( A \) with \( q \neq p \), the \( I_L \)-action on \( V_q \tilde{M} \) is unipotent.

**Proof.** By the semi-stable filtration of \( \tilde{M} \), we see that \( V_q \tilde{M} \) is isomorphic up to semi-simplification to a direct sum of \( p \)-adic representations coming from analytic \( A(1) \)-motives with good reduction. Thus the semi-simplification \( (V_q \tilde{M})^{ss} \) is unramified. \( \square \)

For abelian varieties over a field \( F \), there is a known criterion for good reduction other than that of Néron-Ogg-Shafarevich; Namely, a good reduction criterion from the viewpoint of Barsotti-Tate groups is known; see [SGA7, Proposition IX. 5.13] (for ch(\( F \)) = 0) and [H198, 2.5] (for ch(\( F \)) > 0). In an equal-characteristic setting, effective local shtukas play the role of \( F \)-crystals of Barsotti-Tate groups. As this analogy, Hartl and Hüsken [HH16] provide the following criterion:

**Theorem 5.13** (Hartl and Hüsken). Let \( \tilde{M} \) be an analytic \( A(1) \)-motive over \( L \) of height \( \leq h \). Then the following assertions are equivalent.

1. \( \tilde{M} \) has good reduction.
2. There is a \( \tau \)-equivariant isomorphism \( f : \tilde{M} \otimes \tilde{A}_L A_{p,L} \xrightarrow{\sim} (N, \tau_N) \otimes_{A_{p,L}} A_{p,L} \), where \( N \) is a finite locally free \( A_{p,L} \)-module equipped with an injective \( A_{p,L} \)-homomorphism \( \tau_N : \sigma^* N \rightarrow N \) whose cokernel is annihilated by \( \tilde{j}^h \).

Especially, for a good model \( \tilde{M} \) of \( \tilde{M} \), the above pair \((N, \tau_N)\) can be taken as \((N, \tau_N) = (\tilde{M} \otimes \tilde{A}_O A_{p,O}, \tau_M \otimes \text{id})\). As we have observed in Remark 4.3, such \((N, \tau_N)\) corresponds to an effective local shtuka \( \tilde{M} \) over \( O \) of height \( \leq h \). In particular, we obtain the following:

**Proposition 5.14.** Let \( \tilde{M} \) be an analytic \( A(1) \)-motive over \( L \) of height \( \leq h \). If \( \tilde{M} \) has good reduction, then there is an effective local shtuka \( \tilde{M} \) of height \( \leq h \) over \( O \) which has the same rank and dimension of \( \tilde{M} \), and satisfies \( T_p \tilde{M} \cong T_p \tilde{M} \) as \( A_p[G_L] \)-modules.

**Proof.** Let \( \tilde{M} = (\tilde{M}, \tau_M) \) be a good model of \( \tilde{M} \). As with Example 4.3, for the ideals \( a_i \) of \( A_{p,O} \) generated by \( \{b \otimes 1 - 1 \otimes \gamma(b)q^i \mid b \in F_p\} \), we consider the decomposition \( A_{p,O} = \prod_{i=0}^{d_p-1} A_{p,O}/a_i \) and define \( \tilde{M} := (\tilde{M}, \tau_M) := (\tilde{M} \otimes \tilde{A}_O A_{p,O}/a_0, \tau_M \otimes \text{id}) \), which is an effective local shtuka over \( O \) of height \( \leq h \). It has the same rank of \( \tilde{M} \) by construction and admits an isomorphism

\[
\begin{pmatrix}
\bigoplus_{i=0}^{d_p-1} (\tau_M \otimes \text{id})^i \mod a_i : (\bigoplus_{i=0}^{d_p-1} \sigma^* \tilde{M}, \tau_M \otimes \bigoplus_{i \neq 0} \text{id}) \xrightarrow{\sim} \tilde{M} \otimes A_{O,A} A_{p,O}.
\end{pmatrix}
\]
From this, it follows that $\text{Coker}(\tau_M) \cong \text{Coker}(\tau_M^\lambda)$ and hence $\dim \tilde{M} = \dim \tilde{M}$. Now, using the decomposition $A_{p,O} = \prod_{i=0}^{d_p-1} A_{p,O}/a_i$, we have

$$\tilde{M} \otimes_{\tilde{A}_L} A_{p,L^\text{sep}} \cong (\tilde{M} \otimes_{\tilde{A}_O} A_{p,O}) \otimes_{A_{p,O}} A_{p,L^\text{sep}} \cong \prod_{i=0}^{d_p-1} \tilde{M} \otimes_{\tilde{A}_O} A_{p,L^\text{sep}}/a_i.$$

Thus the element $m = (m_i) \in \tilde{M} \otimes_{\tilde{A}_L} A_{p,L^\text{sep}} = \prod_{i=0}^{d_p-1} \tilde{M} \otimes_{\tilde{A}_O} A_{p,L^\text{sep}}/a_i$ satisfies $\tau_M^\lambda(\sigma^*m) = m$ if and only if $m_{i+1} = \tau_M^\lambda(\sigma^*m_i)$ for $i \neq d_p - 1$ and $m_0 = \tau_M^\lambda(\sigma^*m_{d_p-1})$, so that $m_0 = \tau_M^\lambda(\sigma^*m_0)$ and $m = (m_i)$ can be recovered by $m_0$. Hence $(m_i) \mapsto m_0$ gives rise to the isomorphism $H_1^p(\tilde{M}, A_p) \cong T_p\tilde{M}$ and so we get $T_p\tilde{M} \cong T_p\tilde{M}$ by taking dual of both sides.

**Theorem 5.15.** Let $M$ be an effective $A$-motive over $L$ of height $\leq h$ and suppose it has strongly semi-stable reduction. If the ramification index $e$ of the extension $L/\gamma(Q_p)$ satisfies $eh < q_p - 1$, then $\mathcal{I}(V_p\tilde{M}) \subset [0,eh]$ and the equation

$$e \cdot \dim \tilde{M} = \sigma \mathcal{I}(V_p\tilde{M})$$

holds.

**Proof.** By assumption, the analytification $\tilde{M} := \tilde{M} \otimes \tilde{A}_L$ of $\tilde{M}$ admits a semi-stable filtration

$$0 = \tilde{M}_0 \subset \tilde{M}_1 \subset \cdots \subset \tilde{M}_n = \tilde{M}$$

and so we have $V_p\tilde{M} \cong \bigoplus_{i=1}^n V_p(\text{Gr}_i \tilde{M})$, where $\text{Gr}_i \tilde{M} := \tilde{M}_i/\tilde{M}_{i-1}$ for each $i$. This implies that $\mathcal{I}(V_p\tilde{M}) = \prod_{i=1}^n \mathcal{I}

(V_p(\text{Gr}_i \tilde{M}))$. By Lemma 5.5, each $\text{Gr}_i \tilde{M}$ is of height $\leq h$ and $\dim \tilde{M} = \sum_{i=1}^n \dim \text{Gr}_i \tilde{M}$. Since all $\text{Gr}_i \tilde{M}$ have good reduction, Proposition 5.14 implies that there are effective local shtukas $\tilde{M}_i$ of height $\leq h$ such that $\dim \text{Gr}_i \tilde{M} = \dim \tilde{M}_i$ and $V_p(\text{Gr}_i \tilde{M}) \cong V_p \tilde{M}_i$. Applying Proposition 4.23 to each $\tilde{M}_i$, we get the conclusion. \hfill \Box

6. **Congruences of Galois Representations**

In what follows, we consider effective $A$-motives over the $A$-field $(K, \gamma; A \to K)$ as in §3 and associated $\mathfrak{p}$-adic representations of $G_K$. The goal of this section is to prove our results on two $\mathfrak{p}$-adic representations of $G_K$ arising from effective $A$-motives with strongly semi-stable reduction at finite places $v, u$ of $K$, where often $v$ is not lying above $\mathfrak{p}$ and $u \mid \mathfrak{p}$. The key ingredient is the *Weil weights* at $v$ of strongly semi-stable effective $A$-motives, which are determined by eigenvalues of a lift $\text{Frob}_v \in G_v$ of the arithmetic Frobenius $\varphi_v \in G_{\mathbb{F}_v}$. Recall that we denote by $d_v = [\mathbb{F}_v : \mathbb{F}_\mathfrak{p}]$ and $q_v = \# \mathbb{F}_v = q^d_v$, and that we write $V_v = V|_{G_v}$ for a representation $V$ of $G_K$. As in §3.2.3, we fix an algebraic closure $\bar{Q}$ of $Q$ and an extension $| \cdot |_\infty$ of the normalized absolute value at the place $\infty$ of $Q$.

6.1. **Weil weights.** Let $v$ be a finite place of $K$.

**Lemma 6.1.** Let $\tilde{M} = (\tilde{M}; \tau_M)$ be an effective $A$-motive over $\mathbb{F}_v$ and $\mathfrak{p} \subset A$ a maximal ideal with $v \not| \mathfrak{p}$. Then the characteristic polynomial $\det(X - \varphi_v \mid T_p\tilde{M})$ has coefficients in $A$ which are independent of $\mathfrak{p}$.

**Proof.** It is known that there is a group homomorphism $\tau: \tilde{M} \to \tilde{M}$ satisfying $\tau(\lambda m) = \sigma(\lambda)\tau(m)$ for $\lambda \in A_{\mathbb{F}_v}$ such that $\tau_M^\lambda(\sigma^*m) = \tau(m)$ for any $m \in \tilde{M}$. Since $A_{\mathbb{F}_v}$ is finite flat over $A$, it follows that $\tilde{M}$ is finite projective over $A$. Thus there is an $A$-module $P$ such that $\tilde{M} \oplus P$ is finite free over $A$. Let $x$ be an indeterminate and consider the zero map $0: P \to P$. Then the characteristic polynomial of $\tau \oplus 0$ on the free $A$-module $\tilde{M} \oplus P$

$$\det(id - x\tau \mid \tilde{M}) := \det(id - x(\tau \oplus 0) \mid \tilde{M} \oplus P) \in A[x]$$
is independent of the choice of $P$; see [BP09, Lemma-Definition 8.1.1]. Now the $d_v$-th iteration $\sigma^{d_v} = \sigma \circ \cdots \circ \sigma$ is identity on $F_v$, and hence $\tau^{d_v}: M \to A_{F_v}$ is linear. By [BP09 Lemma 8.1.4], we see that $\det(id - x\tau_M | M) = \det(id - x^{d_v} \tau_M | M)$, where $M$ is regarded as an $A_{F}$-module in the right hand side. In particular, we have $\det(id - x\tau_v | T_pM) = \det(id - x^{d_v} | M)$. Here by [TW96 §6], we have $\det(id - x^{d_v} \varphi_v | T_pM) = \det(id - x^{d_v} \tau_M | M)$. Setting $X = x^{d_v}$ and $r = \text{rk } M$, we get $\det(X - \varphi_v | T_pM) = X^r \cdot \det(id - x\tau_v | M) \in A[X]$.

\[ \Box \]

**Proposition 6.2.** Let $M$ be an effective $A$-motive over $K$ with strongly semi-stable reduction at $v$ and $p$ is a maximal ideal of $A$ with $v \nmid p$. Then for $V = V_pM$, the characteristic polynomial

\[ P_{M,v}(X) := \det(X - \text{Frob}_v | V_v^{ss}) \]

has coefficients in $A$ which are independent of $p$.

**Proof.** Let $\tilde{M} := M \otimes \tilde{A}_K$ be the analytification of $M$ and consider a semi-stable filtration

\[ 0 = \tilde{M}_0 \subset \tilde{M}_1 \subset \cdots \subset \tilde{M}_n = \tilde{M} \]

of $\tilde{M}$. Set $Gr_i \tilde{M} := \tilde{M}_i/\tilde{M}_{i-1}$ for each $1 \leq i \leq n$. Since $V_v \simeq_{\text{ss}} \bigoplus_{i=1}^n V_p(\text{Gr}_i \tilde{M})$ and each $V_p(\text{Gr}_i \tilde{M})$ is unramified at $v$, we obtain

\[ P_{M,v}(X) = \prod_{i=1}^n \det(X - \text{Frob}_v | V_p(\text{Gr}_i \tilde{M})) = \prod_{i=1}^n \det(X - \text{Frob}_v | T_p(\text{Gr}_i \tilde{M})). \]

Now let $\tilde{M}_i = (\tilde{M}_i, \tau_i)$ be a good model of $\text{Gr}_i \tilde{M}$ for each $i$. Then the reductions $\tilde{M}_i \otimes A_{F_v} = (\tilde{M}_i \otimes A_{F_v}, \tau_i \otimes id)$ are effective $A$-motives over $F_v$. Since $T_p(\text{Gr}_i \tilde{M})$ is unramified at $v$, it is isomorphic to $T_p(\tilde{M}_i \otimes A_{F_v})$ as representations of $G_v/I_v \cong G_{F_v}$. Hence $\det(X - \text{Frob}_v | T_p(\text{Gr}_i \tilde{M})) = \det(X - \varphi_v | T_p(\tilde{M}_i \otimes A_{F_v}))$ holds for any $i$ and so the claim is proved by Lemma [6.1]. \[ \Box \]

**Definition 6.3.** Suppose that $M$ has strongly semi-stable reduction at $v$. Let $\alpha_1, \ldots, \alpha_{rkM} \in \tilde{Q}$ be the roots of $P_{M,v}(X)$ and define the rational numbers $w_i \in \tilde{Q}$ by $|\alpha_i|_\infty = q_i^{w_i}$. We call $w_1, \ldots, w_{rkM}$ the Weil weights of $M$ at $v$, and denote by $W_v(M) := \{w_1, \ldots, w_{rkM}\}$ the multi-set consisting of them. Note that $W_v(M)$ is independent of the choice of extensions of $|\cdot|_\infty$ to $\tilde{Q}$.

**Example 6.4.** Let $M$ be an effective pure $A$-motive over $K$ of rank $r$ and dimension $d$. If $M$ has good reduction at $v \nmid p$, then $P_{M,v}(X) = \det(X - \text{Frob}_v | T_pM)$ and so Proposition [3.17] implies

\[ W_v(M) = \left\{ \frac{d}{\lfloor \frac{d}{r} \rfloor}, \ldots, \frac{d}{r} \right\}, \]

where the multiplicity of $\frac{d}{r}$ is $r$. Then we particularly obtain $\Sigma W_v(M) = d$.

We can see that the equation $\Sigma W_v(M) = d$ also holds in the strongly semi-stable case:

**Proposition 6.5.** Let $M$ be an effective $A$-motive over $K$ of rank $r$. Suppose that $M$ has strongly semi-stable reduction at a finite place $v$ of $K$. We set $P_{M,v}(X) = \sum_{i=0}^r a_i X^{r-i} \in A[X]$. Then the following assertions hold.

1. \[ \Sigma W_v(M) = \dim M. \]
2. Suppose that all Weil weights of $M$ at $v$ are non-negative. Then $|a_i|_\infty \leq q_i^{\dim M}$ for any $0 \leq i \leq r$.

**Proof.** (1) Let $p$ be a maximal ideal of $A$ with $v \nmid p$. As we see in Proposition [3.17], the determinant $\det M$ of $M$ is an effective $A$-motive over $K$ with $\text{rk} \det M = 1$ and $\dim \det M = \dim M$. Then $\det(V_p(M) \cong V_p(\det M)$ as $p$-adic representations of $G_K$. Since the $I_v$-action on $V_p(M)$ is unipotent by Corollary [7.12], $V_p(\det M)$ is unramified at $v$ and hence $\det M$ has good reduction at $v$. From this, we have $W_v(\det M) = \{\Sigma W_v(M)\}$ because the $\text{Frob}_v$-eigenvalue of $V_p(\det M)$ is the product...
of all roots of $P_{M,v}(X)$. Since $\det M$ is pure of weight equal to $\dim M$ by Lemma 3.18, we get $\dim M = \Sigma W_v(\overline{M})$.

(2) Let $\alpha_1, \ldots, \alpha_r$ be the roots of $P_{M,v}(X) = \sum_{i=0}^{r} a_i X^{r-i}$. Then obviously $a_0 = 1$. For any $1 \leq i \leq r$, we have

$$a_i = \sum_{1 \leq s_1 < \cdots < s_i \leq r} (-\alpha_{s_1})(-\alpha_{s_2}) \cdots (-\alpha_{s_i}).$$

Since all $\alpha_i$ satisfy $|\alpha_i|_\infty \geq 1$ by assumption, we have $|a_i|_\infty \leq |a_r|_\infty = q_v^{\dim M}$. Therefore we have $|a_i|_\infty \leq \max_{1 \leq s_1 < \cdots < s_i \leq r} \{ |\alpha_{s_1} \cdots \alpha_{s_i}|_\infty \} \leq q_v^{i \dim M}$.

\[ \square \]

6.2. Results. To state our results, we recall the notation introduced in §1. Let $r > 0$ be a positive integer as usual. The injective $A$-field structure $\gamma: A \to K$ extends to the field embedding $\gamma: Q \to K$ and then $K/\gamma(Q)$ is a finite extension. Let $K_s$ denote the maximal separable extension of $\gamma(Q)$ in $K$, so that $K/K_s$ is purely inseparable if $K \neq K_s$. Then we write $[K : Q]_i := [K : K_s]$ for the inseparable degree of $K/\gamma(Q)$. Denote by $\mathfrak{d} = \mathfrak{d}_{K_s/\gamma(Q)}$ the relative discriminant of $K_s/\gamma(Q)$. Then we define

$$D_K = \begin{cases} \max\{ d_q \mid q \subset A \text{ is a maximal ideal dividing } \mathfrak{d} \} & \text{if } \mathfrak{d} \neq A, \\ 1 & \text{if } \mathfrak{d} = A. \end{cases}$$

Now it is known that purely inseparable extensions of global function fields have to be totally ramified at all places by [Ros02, Proposition 7.5]. Hence, for any finite place $u$ of $K$ lying above a maximal ideal $\mathfrak{p}$ of $A$ with $d_\mathfrak{p} > D_K$, the ramification index $e_{u|\mathfrak{p}}$ of $u \mid \mathfrak{p}$ is equal to $[K : Q]_i$.

**Definition 6.6.** For a non-negative $h \geq 0$ and two finite places $v, u$ of $K$, denote by

$$\text{Mot}_{K,r,v}(u,h)$$

the set of effective $A$-motives $M$ over $K$ of rank $r$ which satisfy the following conditions:

- $M$ has strongly semi-stable reduction at both $v$ and $u$,
- All Weil weights of $M$ at $v$ are non-negative,
- $M$ is of height $\leq h$.

Our first result is the following criterion for congruent $p$-adic representations arising from strongly semi-stable effective $A$-motives, which is a function field analog of the results on $\ell$-adic representations due to Ozeki and Taguchi [OT14]:

**Theorem 6.7.** Let $r$ and $h$ be as above and fix a finite place $v$ of $K$. For any maximal ideal $\mathfrak{p}$ of $A$ with $d_\mathfrak{p} > \max\{ d_v r^2 h, [K : Q]_i h, D_K \}$ and any finite place $u$ of $K$ lying above $\mathfrak{p}$, the following holds: For any $M \in \text{Mot}_{K,r,v}(u,h)$ and $M' \in \text{Mot}_{K,r,v}(u, (q_p - 2)[K : Q]_i^{-1})$, if both

\[
\begin{aligned}
V_p M|_{G_v} &\equiv_{ss} V_p M'|_{G_v} \pmod{p}, \\
V_p M|_{G_u} &\equiv_{ss} V_p M'|_{G_u} \pmod{p}
\end{aligned}
\]

hold, then one has $V_p M|_{G_v} \equiv_{ss} V_p M'|_{G_v}$, $\dim M = \dim M'$, and $W_v(M) = W_v(M')$.

**Proof.** Let us first see what we immediately get from the assumption $d_\mathfrak{p} > \max\{ d_v r^2 h, [K : Q]_i h, D_K \}$. Since particularly $d_v < d_\mathfrak{p}$, we see that $v$ is not lying above $\mathfrak{p}$. For the maximum of Hodge-Pink weights $h' = \max H_P(M')$ of $M'$, it follows that $M'$ is of height $\leq h'$ and $h' \leq (q_p - 2)[K : Q]_i^{-1}$. As mentioned previously, we have $e_{u|\mathfrak{p}} = [K : Q]_i$ by $d_\mathfrak{p} > D_K$. Therefore we have $e_{u|\mathfrak{p}} h' < q_p - 1$. On the other hand, we have $e_{u|\mathfrak{p}} h < q_p - 1$ as

$$e_{u|\mathfrak{p}} h = [K : Q]_i h < d_\mathfrak{p} < q_p^d - 1 = q_p - 1.$$ 

Note that the inequality $d_\mathfrak{p} < q_p^d - 1$ follows as $d_\mathfrak{p} > D_K \geq 1$. 


Now write \( V := V_p^M \) and \( V' := V_p^{M'} \) for the associated \( p \)-adic representations of \( G_K \). Then \( V_s \cong V_v' \) holds if and only if \( P_{M,v}(X) = P_{M',v}(X) \) by Proposition 2.3 and \( W_v(M) = W_v(M') \) in this case. Let \( P_{M,v}(X) = \sum_{i=0}^{r} a_i X^{r-i} \) and \( P_{M',v}(X) = \sum_{i=0}^{r} a'_i X^{r-i} \), and set \( c_i := a_i - a'_i \in A \). To see \( P_{M,v}(X) = P_{M',v}(X) \), it suffices to check that \( c_i = 0 \) for any \( i \). Since \( V_s \equiv V_v' \mod p \), we obtain \( P_{M,v}(X) \equiv P_{M',v}(X) \mod p \). Hence \( c_i \equiv 0 \mod p \) for any \( i \). Applying Proposition 6.5 (2) to \( P_{M,v}(X) \) and \( P_{M',v}(X) \), we get the estimate
\[
|c_i|_\infty \leq \max\{|a_i|_\infty, |a'_i|_\infty\} \leq q^{d_r} \max\{\dim M, \dim M'\}
\]
for any \( 0 \leq i \leq r \). On the other hand, the assumption \( V_s \equiv V_v' \mod p \) implies \( T \mathcal{I}(V_s) = T \mathcal{I}(V_v') \) as multi-sets. Since we have \( e_u|p h < q_p - 1 \) and \( e_u|p h' < q_p - 1 \), Theorem 5.15 provides the equation
\[
e_{u|p} \cdot \dim M = \sum T \mathcal{I}(V_s) = \sum T \mathcal{I}(V_v') = e_{u|p} \cdot \dim M'.
\]
Thus \( M \) and \( M' \) have the same dimension. Since \( M \) is of height \( \leq h \) and the multi-set \( \mathcal{H}P(M) \) consists of \( r \) integers, it follows by Proposition 3.22 that \( \dim M = \sum \mathcal{H}P(M) \leq rh \). Consequently, we obtain
\[
|c_i|_\infty \leq q^{d_r} h^2 < q^d_p
\]
and so Lemma 5.3 implies \( c_i = 0 \) for any \( i \). Thus we get \( V_s \cong V_v' \) and \( W_v(M) = W_v(M') \). \( \square \)

For a maximal ideal \( p \) of \( A \), we consider the union \( \text{Mot}_{K,r,v}(p,h) := \bigsqcup_{u|h} \text{Mot}_{K,r,v}(u,h) \), where \( u \) runs through all places of \( K \) lying above \( p \). As a corollary of Theorem 6.7 we have:

**Corollary 6.8.** Let \( r, h, \text{ and } v \) be as above. Fix an effective \( A \)-motive \( M' \) over \( K \) of rank \( r \) such that \( M' \) has strongly stable reduction at \( v \) and all Weil weights at \( v \) are non-negative. Then there is a constant \( C = C(K, r, v, M') > 0 \) such that if a maximal ideal \( p \) of \( A \) satisfies \( d_p > C \), then there exist no effective \( A \)-motives \( M \in \text{Mot}_{K,r,v}(p,h) \) satisfying \( V(M) \neq V(M') \) and \( V_p^M \equiv V_p^{M'} \mod p \) as representations of \( G_K \).

**Proof.** Since \( M' \) has good reduction at all but finitely many places of \( K \), we may choose a constant \( C > \max\{d_r h^2 |K : Q|, D_K\} \) so large with respect to \( M' \) such that if \( d_p > C \), then \( M' \) is of height \( \leq (q_p - 2) \cdot [K : Q]^{-1} \) and has good reduction at all places lying above \( p \). Hence if \( M \in \text{Mot}_{K,r,v}(p,h) \) satisfies \( V_p^M \equiv V_p^{M'} \mod p \), then \( W_v(M) = W_v(M') \) must hold by Theorem 6.7. \( \square \)

Our second result is motivated by the non-existence conjecture on abelian varieties with constrained torsion due to Rasmussen and Tamagawa [RT08]:

**Theorem 6.9.** Let \( r \geq 2 \) be an integer and \( h \geq 0 \). Let \( v \) be a finite place of \( K \). For any maximal ideal \( p \) of \( A \) with \( d_p > \max\{d_r h^2 |K : Q|, D_K\} \) and any finite place \( u \) of \( K \) lying above \( p \), there exist no effective \( A \)-motives \( M \in \text{Mot}_{K,r,v}(u,h) \) satisfying the following conditions:

- There exist at least one Weil weight \( w \in W_v(M) \) such that \( |K : Q| w \) is a non-integer,
- There exist one-dimensional rank-one effective \( A \)-motives \( M_1, \ldots, M_r \) over \( K \) having good reduction at both \( v \) and \( u \) such that

\[
\begin{align*}
V_p^M &\equiv \bigoplus_{i=1}^{r} V_p(M_i^{m_i}) |_{G_v} \mod p, \text{ and} \\
V_p^M &\equiv \bigoplus_{i=1}^{r} V_p(M_i^{m_i}) |_{G_u} \mod p 
\end{align*}
\]

hold for some non-negative integers \( m_1, \ldots, m_r \).

**Remark 6.10.** If \( r = 1 \), then \( W_v(M) = \{\dim M\} \) and so the first condition of the theorem never holds. Hence it suffices to consider only the case \( r \geq 2 \).
Proof. Set \( V := V_p M \) and \( V' := \bigoplus_{i=1}^r V_p (M_i^{\otimes m_i}) \), so that \( V' \) is semi-simple. Let \( \chi_i : G_K \to \mathbb{F}_p^\times \) be the character determined by \( T_p M_i \otimes_{A_p} \mathbb{F}_p \) for each \( i \). Since \( \chi_i|_{G_A} \) is coming from a rank-one effective local shtuka of dimension one, it follows by Example 1.21 that \( \chi_i|_{I_u} = \theta_i^{u_p} \), where \( \theta_i : I_u \to \mathbb{F}_p^\times \) is the level-one fundamental character. Now we have \( e_{u_p} = [K : Q] \) by \( d_p > D_K \) and therefore the \( I_u \)-action on \( T_p M_i^{\otimes m_i} \otimes_{A_p} \mathbb{F}_p \) is given by \( \theta_i^{e_{u_p}} \). For each \( i \), let \( n_i \) be the integer satisfying \( \chi_i^{m_i}|_{I_u} = \theta_i^{n_i} \) and \( 0 \leq n_i \leq q_p - 2 \). Then we have \( T_I(V_u') = \{ n_1, \ldots, n_r \} \) as multi-sets and \( [K : Q] m_i \equiv n_i \pmod{q_p - 1} \) for any \( i \). Since \( V_u \equiv_{ss} V_u' \pmod{p} \) by assumption, we have \( T_I(V_u) = T_I(V_u') \). Applying Theorem 5.13 to \( M \), we have \( T_I(V_u) \subseteq [0, e_{u_p}] \) and so all \( n_i \) satisfy \( 0 \leq n_i \leq h[K : Q] \).

Now let us consider the \( [K : Q] \)-power \( \text{Frob}_i^{[K : Q]} \) of \( \text{Frob}_v \). Then the characteristic polynomials of it on \( V_i^{\text{ss}} \) and \( V'_i \) have coefficients in \( A \). We set \( P_{M, v}^{[K : Q]}(X) := \det(X - \text{Frob}_i^{[K : Q]} | V_i^{\text{ss}}) \). Define \( M' := \bigoplus_{i=1}^r M_i^{\otimes m_i} \) (Recall that \( M_i^{\otimes m_i} = A_K \) if \( n_i = 0 \)). Then by \( V_u \equiv_{ss} V'_u \pmod{p} \) and \( [K : Q] m_i \equiv n_i \pmod{q_p - 1} \), we obtain

\[
P_{M, v}^{[K : Q]}(X) = \det(X - \text{Frob}_i^{[K : Q]} | V_i^{\text{ss}}) \equiv \det(X - \text{Frob}_i^{[K : Q]} | V'_i) = \prod_{i=1}^r (X - \chi_i^{m_i} \text{Frob}_v^{[K : Q]}) \equiv \prod_{i=1}^r (X - \chi_i^{n_i} \text{Frob}_v) \equiv P_{M', v}(X) \pmod{p}.
\]

Since each \( M_i \) is pure of weight one by Lemma 3.18, the multi-set of Weil weights at \( v \) of \( M' \) is \( \mathcal{W}_v(M') = \{ n_1, \ldots, n_r \} = T_I(V_u'). \) By \( 0 \leq n_i \leq h[K : Q] \), we have \( \dim M' = \sum \mathcal{W}_v(M') \leq r h[K : Q] \). If we write \( P_{M, v}(X) = \sum_{i=1}^r a_i X^{r-i} \in A[X] \), then it follows by Proposition 6.5 that

\[
|a'_i|_\infty \leq q_e^{r \dim M'} \leq q^{d_e r^2 h[K : Q]}
\]

for each \( 0 \leq i \leq r \). On the other hand, since any root \( \beta_i \in \bar{Q} \) of \( P_{M, v}^{[K : Q]}(X) \) is a \( [K : Q] \)-power of a root of \( P_{M, v}(X) \), we have \( |\beta_i|_\infty = q_{w_i[K : Q]} \) for some \( w_i \in \mathcal{W}_v(M) \). Thus if we write \( P_{M, v}^{[K : Q]}(X) = \sum_{i=0}^r b_i X^{r-i} \in A[X] \), then

\[
|b_i|_\infty \leq |b_r|_\infty = |\beta_1 \cdots \beta_r|_\infty = q^{r \dim M[K : Q]} \leq q^{d_e r^2 h[K : Q]}
\]

for each \( 0 \leq i \leq r \). Hence by \( d_p > d_{v q} r^2 h[K : Q] \), Lemma 1.4 implies that \( P_{M, v}^{[K : Q]}(X) = P_{M', v}(X) \). From this, we see that all \( w \in \mathcal{W}_v(M') \) satisfy \( [K : Q] w \in \mathcal{W}_v(M') \). However, the multi-set \( \mathcal{W}_v(M') = \{ n_1, \ldots, n_r \} \) contains only integers, which contradicts the assumption. \( \square \)

Specializing Theorem 6.9 to effective pure \( A \)-motives, we immediately obtain the next corollary:

**Corollary 6.11.** Let \( r \geq 2 \) be an integer and \( q \) a maximal ideal of \( A \). Let \( d > 0 \) be a positive integer such that \( d[K : Q] \) is not divisible by \( r \). If a maximal ideal \( p \) of \( A \) satisfies \( d_p > \max\{ d_{q} r^2 d[K : Q] : [K : Q], D_K \} \), then there exist no \( d \)-dimensional effective pure \( A \)-motives \( M \) over \( K \) of rank \( r \) such that

- \( M \) has good reduction at a place of \( K \) lying above \( q \),
- \( M \) has strongly semi-stable reduction at a place of \( K \) lying above \( p \),
- There exist one-dimensional rank-one effective \( A \)-motives \( M_1, \ldots, M_r \) over \( K \) with good reduction outside \( \infty \), and non-negative integers \( m_1, \ldots, m_r \) such that

\[
V_p M \equiv_{ss} \bigoplus_{i=1}^r V_p (M_i^{\otimes m_i}) \pmod{p}
\]
holds as $p$-adic representations of $G_K$.

Proof. Let $p$ be as $d_p > \max\{dq^2d[K : Q] \cdot [K : Q], D_K\}$ and assume that there is an $M$ satisfying the conditions as above. Let $v \mid p$ and $u \mid p$ be finite places of $K$ such that $\overline{M}$ has good (resp. strongly semi-stable) reduction at $v$ (resp. at $u$). Then we have $\overline{M} \in \text{Mot}_{K,v,u}(u, d)$. By $d_v = d_q[F_v : F_\mathbb{q}] < d_q[K : Q]$, we have $d_v^2d[K : Q] < d_p$. Since $\overline{M}$ is pure and has good reduction at $v$, the Weil weights at $v$ are $W_v(M) = \{d_1, \ldots, d_v\}$ and $d_v[K : Q]_i$ is a non-integer by assumption. This contradicts the non-existence in Theorem 6.9. \hfill $\square$

APPENDIX A. PROOF OF THEOREM 2.2

This Appendix based on [Yam09] aims to give a proof of Theorem 2.2. In fact, Theorem 2.2 follows from a general statement (Proposition 4.4 below) under the following setting. From now on, let $A$ be a (not necessarily commutative) ring with unity and consider a (left) $A$-module $M$. (What we have in mind as $A$ and $M$ are $A = F[G]$ and $M = V^{ss}$.) Denote by $\text{End}(M)$ the ring of endomorphisms of $M$ as an abelian group, and by $\text{End}_A(M)$ its subring consisting of $A$-endomorphisms. Then the bicommutant $B_M$ of $M$ is defined as

$$
B_M = \{f \in \text{End}(M) \mid f \circ g = g \circ f \text{ for any } g \in \text{End}_A(M)\},
$$

which is a subring of $\text{End}(M)$. For any $a \in A$, we write $[a]_M$ for the homothety $[a]_M : M \rightarrow M; x \mapsto ax$ and denote by $A_M = \{[a]_M \mid a \in A\}$ the ring of homotheties. Then $A_M \subset B_M$.

Theorem A.1 (Density theorem). Suppose that $M$ is semi-simple. Then for any $b \in B_M$ and any finite sequence $x_1, \ldots, x_n \in M$, there exists an element $a \in A$ such that $[a]_M(x_i) = b(x_i)$ for each $1 \leq i \leq n$.

Proof. See [Bou81, Chapitre 8, §5, Théorème 3] and its proof. \hfill $\square$

Corollary A.2. Let $M$ be a semi-simple $A$-module. If $M$ is finitely generated as a left $\text{End}_A(M)$-module, then $A_M = B_M$.

Proof. We have already seen $A_M \subset B_M$. To prove the converse, we take an element $b \in B_M$ and a generating set $\{x_1, \ldots, x_n\}$ of $M$ as a left $\text{End}_A(M)$-module. By Theorem A.1, there exists $[a]_M \in A_M$ such that $b(x_i) = [a]_M(x_i)$ for each $i$. Since any $x \in M$ is written by $x = \sum_{i=1}^n g_i(x_i)$ for $g_i \in \text{End}_A(M)$, it follows that

$$
[a]_M(x) = a \sum_{i=1}^n g_i(x_i) = \sum_{i=1}^n g_i(ax_i) = \sum_{i=1}^n g_i(b(x_i)) = \sum_{i=1}^n b(g_i(x_i)) = b(x).
$$

Thus $B_M \subset A_M$. \hfill $\square$

Corollary A.3. Let $M_1, \ldots, M_n$ be simple $A$-modules which are not isomorphic to each other. Suppose that each $M_i$ is finitely generated as a left $\text{End}_A(M_i)$-module. For each $1 \leq i \leq n$, take $a_i \in A_M$. Then there is an $a \in A$ such that $[a]_M = a_i$ for each $i$.

Proof. Set $M := \bigoplus_{i=1}^n M_i$. Then $M$ is finitely generated as a left $\text{End}_A(M_i)$-module because so is each $M_i$ as a left $\text{End}_A(M_i)$-module. Thus $A_M = B_M$ by Corollary A.2. Here since $M_i$ and $M_j$ are not isomorphic if $i \neq j$, we see that $\text{End}_A(M) = \prod_{i=1}^n \text{End}_A(M_i)$. Therefore the endomorphism $(a_i)_i : M \rightarrow M$ satisfies $(a_i)_i \in B_M = A_M$. \hfill $\square$

Now let $F$ be a field and $A$ an $F$-algebra. For an $A$-module $M$ which is finite-dimensional over $F$, denote by $P_{M,a}(X) = \det(X - a | M)$ the characteristic polynomial of $a \in A$. Let $\dim_F M = r$ and set $P_{M,a}(X) = \sum_{i=0}^r c_i X^{r-i}$. Then the coefficients of $P_{M,a}(X)$ are characterized by $c_i = (-1)^i \text{tr}_A(M)(a) \in F$. Then we have:

Proposition A.4. Let $M$ and $M'$ be semi-simple $A$-modules and suppose they have the same finite dimension over $F$. Set $r := \dim_F M = \dim_F M'$. Then the following assertions hold:
(1) Suppose that $F$ has characteristic $p > 0$. Then $M$ is isomorphic to $M'$ if and only if $P_{M,a}(X) = P_{M',a}(X)$ for all $a \in A$.
(2) Suppose either $F$ has characteristic $0$ or has characteristic $p > 0$ satisfying $r < p$. Then $M$ is isomorphic to $M'$ if and only if $\text{tr}_M(a) = \text{tr}_{M'}(a)$ for all $a \in A$.

**Proof.** The “only if” parts are obvious, so we prove the “if” parts. Let $\mathcal{S}$ denote the category of isomorphism classes of simple $A$-modules. For an object $\mu$ of $\mathcal{S}$, if $M$ has a simple factor belonging to $\mu$, we denote it by $M_\mu$ and write $m_\mu$ for its multiplicity. If $M$ has no simple factors belonging to $\mu$, then we define $M_\mu = \{0\}$ and $m_\mu = 0$. For $M'$, we also define $M'_\mu$ and $m'_\mu$ in the same way.

Then we can take a finite set $H$ of objects of $\mathcal{S}$ such that

$$M = \bigoplus_{\mu \in H} (M_\mu)_{\oplus m_\mu}, \quad M' = \bigoplus_{\mu \in H} (M'_\mu)_{\oplus m'_\mu},$$

and either $m_\mu \neq 0$ or $m'_\mu \neq 0$ holds for any $\mu \in H$. Thus, to prove $M \cong M'$, it is enough to show that $m_\mu = m'_\mu$ for all $\mu \in H$.

(1) Suppose that $P_{M,a}(X) = P_{M',a}(X)$ for all $a \in A$. For each $\mu \in H$, we fix a simple $A$-module $N_\mu$ which represents $\mu$. (Notice that if $m_\mu \neq 0$ and $m'_\mu \neq 0$, then $M_\mu \cong N_\mu \cong M'_\mu$ and $P_{M,a}(X) = P_{N,a}(X) = P_{M',a}(X)$ for all $a \in A$.) We first claim that $m_\mu \equiv m'_\mu \pmod{p}$ holds for any $\mu \in H$. To see this, we fix an arbitrary $\eta \in H$. Applying Corollary 3.3 to $\bigoplus_{\mu \in H} N_\mu$ and the sequence $(a_\mu)_{\mu \in H}$ in $A$ with $a_\eta = 1$ and $a_\mu = 0$ for $\mu \neq \eta$, we can take an element $c \in A$ such that $[c]_{N_\eta} = 1$ and $[c]_{N_\mu} = 0$ for $\mu \neq \eta$. Thus for any integer $0 \leq k \leq \nu$, we have $\text{tr}_{A,M}(c) = \text{tr}_{A,M}(c) = \text{tr}_{M,M}(c) = \text{tr}_{M,M}(c)$.

Hence if $\dim_F N_\eta$ is not divisible by $p$, then $m_\nu \equiv m'_\nu \pmod{p}$. On the other hand, suppose that $\dim_F N_\eta$ decomposes as $\dim_F N_\eta = p^\nu d$, where $0 < \nu$ and $d$ is not divisible by $p$. Then of course $p^\nu d \leq r$ and the binomial coefficient $\binom{p^\nu d}{p^\nu}$ is coprime to $p$. For any integer $m \geq 0$, we see that

$$\dim_F \left( \bigwedge_{s_i}^\nu N_\eta^{\oplus m} \right) = \dim_F \left( \bigoplus_{s_1 + \cdots + s_m = p^\nu} \left( \bigotimes_{i=1}^m \bigwedge_{s_i} N_\eta \right) \right) = \sum_{s_1 + \cdots + s_m = p^\nu} \prod_{s_i = 1}^m \left( \binom{p^\nu}{s_i} \right),$$

and that $\binom{p^\nu d}{p^\nu}$ is not divisible by $p$ if and only if $s_i = p^\nu$. Hence we obtain

$$\text{tr}_{A^{p^\nu}} (N_\eta^{\oplus m}) = \dim_F \left( \bigwedge_{s_i}^{p^\nu} N_\eta^{\oplus m} \right) = m \left( \binom{p^\nu}{p^\nu} \right) \in F,$$

because $F$ has characteristic $p$. Hence $\text{tr}_{A^{p^\nu}}(M_\eta^{\oplus m}(1)) = \text{tr}_{A^{p^\nu}}(M'_\eta^{\oplus m}(1))$ implies $(m_\nu - m'_\nu) \binom{p^\nu}{p^\nu} = 0 \in F$. Since $\binom{p^\nu}{p^\nu} \in F^\times$, we get $m_\nu \equiv m'_\nu \pmod{p}$.

Consequently, there exist semi-simple $A$-modules $L_1, L'_1, M_1$, and $M'_1$ such that

$$M = L_1 \oplus (M_1)_{\oplus p}, \quad M' = L'_1 \oplus (M'_1)_{\oplus p}, \quad \text{and} \quad L_1 \cong L'_1.$$

In particular, $\dim_F M_1 = \dim_F M'_1$. Now we have $P_{(M_1)_{\oplus p}, a}(X) = (P_{M_1,a}(X))^p$ and $P_{(M'_1)_{\oplus p}, a}(X) = (P_{M'_1,a}(X))^p$ for any $a \in A$. We see that the assumption $P_{M,a}(X) = P_{M',a}(X)$ implies $(P_{M_1,a}(X))^p = (P_{M'_1,a}(X))^p$. Hence $P_{M,a}(X) = P_{M',a}(X)$. Repeating the same argument, we can take semi-simple $A$-modules satisfying

$$M = L_h \oplus (M_h)_{\oplus p^h}, \quad M' = L'_h \oplus (M'_h)_{\oplus p^h}, \quad \text{and} \quad L_h \cong L'_h.$$
for any $h$, but $M$ and $M'$ are finite-dimensional and so $M_h = \{0\}$ and $M'_h = \{0\}$ for some $h$. Hence we have $M \cong M'$.

(2) Suppose that $\text{tr}_M(a) = \text{tr}_{M'}(a)$ for all $a \in A$. As with the argument in the proof of (1), we have $(m_\eta - m'_\eta) \text{tr}_{N_\eta}(1) = 0$ in $F$ for any $\eta \in H$. If $F$ has characteristic 0, then $m_\eta = m'_\eta$ and so $M \cong M'$. If $F$ has characteristic $p$ with $r < p$, then we have already seen that $m_\eta \equiv m'_\eta \pmod{p}$ for any $\eta \in H$. On the other hand, we have $|m_\eta - m'_\eta| \leq r < p$. Hence $m_\eta = m'_\eta$. 

Finally, applying Proposition [A.4] to $A = F[G]$, $M = V^{ss}$, and $M' = (V')^{ss}$, we get Theorem 2.2.

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