OPERATOR $\theta$-HÖLDER FUNCTIONS WITH RESPECT TO $\|\cdot\|_p$, $0 < p \leq \infty$

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Abstract. Let $\theta \in (0, 1)$ and $(\mathcal{M}, \tau)$ be a semifinite von Neumann algebra. We consider the function spaces introduced by Sobolev \cite{Sobolev78, Sobolev79} (denoted by $S_{d,\theta}$), showing that there exists a constant $d > 0$ depending on $\theta$, $0 < p \leq \infty$, only such that every function $f : \mathbb{R} \to \mathbb{C} \in S_{d,\theta}$ is operator $\theta$-Hölder with respect to $\|\|_p$, that is, there exists a constant $C_{p, f}$ depending on $p$ and $f$ only such that the estimate

$$\|f(A) - f(B)\|_p \leq C_{p, f} \|A - B|^\theta\|_p$$

holds for arbitrary self-adjoint $\tau$-measurable operators $A$ and $B$. In particular, we obtain a sharp condition such that a function $f$ is operator $\theta$-Hölder with respect to all quasi-norms $\|\|_p$, $0 < p \leq \infty$, which complements the results on the case for $\frac{1}{\theta} < p < \infty$ by Aleksandrov and Peller \cite{AleksandrovPeller2}, and the case when $p = \infty$ treated by Aleksandrov and Peller \cite{AleksandrovPeller3}, and by Nikol’skaya and Farforovskaya \cite{Nikol’skayaFarforovskaya62}. As an application, we show that this class of functions is operator $\theta$-Hölder with respect to a wide class of symmetrically quasi-normed operator spaces affiliated with $\mathcal{M}$, which unifies the results on specific functions due to Birman, Koplienko and Solomjak \cite{BirmanKoplienkoSolomjak14, BirmanKoplienkoSolomjak16}, Bhatia \cite{Bhatia12}, Ando \cite{Ando8}, and Ricard \cite{Ricard74, Ricard75} with significant extension. In addition, when $\theta > 1$, we obtain a reverse of the Birman-Koplienko-Solomjak inequality, which extends a couple of existing results on fractional powers $t \mapsto t^\theta$ by Ando et al.

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1. Introduction

Let $0 < \theta < 1$. A function $f : \mathbb{R} \to \mathbb{C}$ is called a $\theta$-Hölder function if it satisfies the inequality

$$(1) \quad |f(x) - f(y)| \leq \text{const}|x - y|^\theta, \quad x, y \in \mathbb{R}.$$ 

Let $\mathcal{M}$ be a semifinite von Neumann algebra equipped with a semifinite faithful normal trace $\tau$. We denote by $S(\mathcal{M}, \tau)$ the $*$-algebra of all $\tau$-measurable operators affiliated with $\mathcal{M}$. The main object of the present paper is the so-called operator $\theta$-Hölder (or operator-Hölder of order $\theta$ \cite{Sobolev77}) functions $f$ with respect to $\|\|_p$, where $\|\|_p$ stands for the quasi-norm of the non-commutative
$L_p$-space $L_p(\mathcal{M}, \tau)$, $0 < p \leq \infty$ [67]. That is, for a given $p > 0$, there exists a constant $C(p, f)$ depending on $p$ and $f$ (and, obviously, on $\theta$) only such that

\begin{equation}
\|f(X) - f(Y)\|_p \leq C(p, f) \|X - Y\|_p
\end{equation}

for an arbitrary semifinite von Neumann algebra $\mathcal{M}$ and any self-adjoint operators $X, Y \in S(\mathcal{M}, \tau)$.

The principal results of this paper were motivated by the following question:

**Question 1.1.** Let $\theta \in (0, 1)$. What is the class of functions $f : \mathbb{R} \to \mathbb{C}$ such that for an arbitrary $p \in (0, \infty]$, there exists a constant $C(p, f)$ depending on $p$ and $f$ only (independent of $\mathcal{M}$) such that

\[\|f(X) - f(Y)\|_p \leq C(p, f) \|X - Y\|_p, \quad \forall X = X^*, Y = Y^* \in S(\mathcal{M}, \tau)\]

This question has a deep history in operator theory. The starting point is the so-called Powers-Størmer inequality [73] (see also [12, 52, 85]):

\[\|X^{\frac{1}{2}} - Y^{\frac{1}{2}}\|_2 \leq \|X - Y\|_2^{\frac{1}{2}}, \quad X, Y \in B(\mathcal{H})^+\]

where $B(\mathcal{H})^+$ stands for the positive part of the $*$-algebra $B(\mathcal{H})$ of all bounded linear operators on a Hilbert space $\mathcal{H}$. A remarkable extension of the above inequality is due to Birman, Kopleienko and Solomjak by using double operator integrals techniques [14] (an alternative proof was given in [70]): if $0 < \theta < 1$, $t(t) := |t|^\theta$, $t \in \mathbb{R}$ and $\|\cdot\|$ is an arbitrary fully symmetric norm on $B(\mathcal{H})$, then

\begin{equation}
\|X^\theta - Y^\theta\| \leq \|X - Y\|^\theta, \quad X, Y \in B(\mathcal{H})^+.
\end{equation}

After that, many mathematicians enlarged the classes of functions $f$ or the (quasi)-norms for which (3) holds. There is a vast literature concerned with problems of this type, with a large number of deep results (see e.g. [1–5, 7, 8, 10, 21, 25, 51, 71, 75]). In particular, Ando [8] replaced the function $t \to t^\theta$ with any non-negative operator monotone function $f : \mathbb{R}^+ \to \mathbb{R}^+$ (classic examples are $t \to t^\theta$, $0 < \theta < 1$, $t \to \log(t+1)$ and $t \to \frac{1}{r+t}$, $r > 0$), showing that

\[\|f(X) - f(Y)\| \leq \|f(|X - Y|)|, \quad X, Y \in B(\mathcal{H})^+\]

for any fully symmetric norm $\|\cdot\|$ on $B(\mathcal{H})$. This result was later extended by P. Dodds and T. Dodds [25] to the case of fully symmetric spaces affiliated with a semi-finite von Neumann algebra (see also [9, 10, 85] for related topics on operator monotone functions). Kosaki [44] proved (3) for the Haagerup $L_p$-spaces, $p \geq 1$ (see also [54, Proposition 7]).

If $\theta = 1$, then operator-$H$ölder functions are the so-called operator-Lipschitz functions. It is well-known [41, 50] that a Lipschitz function on the real line is not necessarily operator-Lipschitz with respect to $\|\cdot\|_\infty$, i.e., the condition

\[|f(x) - f(y)| \leq \text{const}|x - y|, \quad x, y \in \mathbb{R},\]

does not imply the that estimate

\[\|f(A) - f(B)\|_\infty \leq \text{const}'\|A - B\|_\infty\]

holds for any self-adjoint operators $A, B \in B(\mathcal{H})$. Davies [23] showed that the absolute value function $t \to |t|$ is not operator-Lipschitz for $\|\cdot\|_1$. However, it has been shown recently by Potapov and Sukochev [71] that every Lipschitz function is necessarily operator-Lipschitz in the Schatten–von Neumann ideal $S_p$ (or noncommutative $L_p$-space) for any $1 < p < \infty$ (see [19] for the best constant for operator Lipschitz functions). A class of operator-Lipschitz functions for Schatten–von Neumann ideal $S_p$ for $p < 1$ has been identified in [60].

An interesting extension of (3) was obtained recently by Aleksandrov and Peller [2, 3] (the case for $\|\cdot\|_\infty$ was obtained also by Nikol’skaya and Farforovskaya [62]). They showed that the situation changes dramatically if we consider $\theta$-Hölder functions instead of Lipschitz functions, i.e., for any $0 < \theta < 1$, condition (1) implies that for every $1 < p \leq \infty$ and any self-adjoint $X, Y \in B(\mathcal{H})$, one has

\[\|f(X) - f(Y)\|_{p/\theta} \leq C_{p,f}\|X - Y\|_p^\theta,\]
with $C_{p,f}$ depending on $p$ and $f$ only (see [7] for the case of normal operators in $(B(\mathcal{H}), \|\cdot\|_\infty)$).

Equivalently, every $\theta$-Hölder function $f$ on $\mathbb{R}$ is necessarily operator $\theta$-Hölder with respect to $\|\cdot\|_p$ provided that $p > \frac{1}{\theta}$, that is,

$$\|f(X) - f(Y)\|_p \leq C_{p,f} \|X - Y\|^\theta_\theta,$$

However, a $\theta$-Hölder function $f$ is not necessarily operator $\theta$-Hölder with respect to $\|\cdot\|_p$, when $p = \frac{1}{\theta}$ (see [2, Section 9]). Very little is known for the case of $\theta$ when $p \leq \frac{1}{\theta}$, and for other symmetric quasi-norms. Birman and Solomjak [16, Proposition 4.9] obtained a result for weak $L_p$-spaces, showing that for every $0 < p < \infty$, $0 < \theta < 1$ and any positive $X, Y \in B(\mathcal{H})$, one has

$$\|X^\theta - Y^\theta\|_{p,\infty} \leq C_{p,\theta} \|X - Y\|_{p,\infty},$$

with $C_{p,\theta}$ depending on $p$ and $\theta$ only.

For the general case of quasi-norms $\|\cdot\|_p$, $p > 0$, a weak-type result (in terms of quasi-commutator estimates) was recently obtained by Sobolev [78]. He introduced a class

$$S_{d,\theta} := \left\{ f \in C^d(\mathbb{R}\setminus\{0\}) \cap C(\mathbb{R}) : \|f\|_{S_{d,\theta}} := \sup_{0 \leq k \leq d} \sup_{x \neq 0} |f^{(k)}(x)||x|^{-\theta+k} < \infty \right\},$$

which also plays a crucial role in the study of Wiener–Hopf operators [57, 79]. Note that all functions in $S_{d,\theta}$ are $\theta$-Hölder when $d \geq 1$ (see [79, (2.3)] or Lemma 2.3). Sobolev considered $f \in S_{d,\theta}$, supported on a compact interval $[-r, r]$ and proved the inequality

$$(4) \quad \|f(A) - f(B)\|_{\|\cdot\|_{\|\cdot\|}} \leq C_{d,f,\sigma,\theta,r} \|A - B\|_{\theta}^{\sigma}, \quad A = A^*, B = B^* \in B(\mathcal{H})$$

for $\sigma < \theta$ such that $(d - \sigma)^{-1} < p \leq 1$ and for every symmetrically $p$-normed ideal $(\mathcal{S}, \|\cdot\|_\mathcal{S})$ of $B(\mathcal{H})$. However, since $\sigma$ is strictly smaller than $\theta$, $C_{d,f,\sigma,\theta,r} \to \infty$ as $\sigma \to \theta$, and since the function $f : \mathbb{R} \to \mathbb{R}$, $t \mapsto |t|^\theta$ has unbounded support, until recently it remained unknown whether the classic $\theta$-Hölder function $t \mapsto |t|^\theta$ is operator $\theta$-Hölder (i.e. (2)) even for Schatten $p$-class, $0 < p < 1$.

A recent breakthrough is due to Ricard [75], who established (3) for noncommutative $L_p$-spaces affiliated with $\mathcal{M}$, $0 < p \leq \infty$. Precisely, Ricard [75] proved that there exists a constant $C_{p,\theta}$ depending on $p$ and $\theta$ only such that

$$(5) \quad \|\|X\|^\theta - |Y|^\theta\|_p \leq C_{p,\theta} \|X - Y\|_{p,\theta}, \quad X = X^*, Y = Y^* \in S(\mathcal{M}, \tau)$$

(see also [8, 25] for the case when $p \geq 1$). This result demonstrates that $f : t \mapsto |t|^\theta$ provides a non-trivial positive (partial) answer to Question 1.1. However, the argument in [75] heavily rests on the homogeneity of fractional power functions, which does not seem to extend to more general functions.

For the sake of convenience, we denote

$$S_{\infty,\theta} := \cap_{0 \leq d < \infty} S_{d,\theta}.$$

In particular, all functions in $S_{\infty,\theta}$ are $\theta$-Hölder (note that $\sup_{d \geq 0} \|f\|_{S_{d,\theta}}$ is not necessarily finite).

It is immediate that $t \mapsto |t|^\theta \in S_{\infty,\theta}$. We show that the class $S_{\infty,\theta}$ generated by the function spaces $S_{d,\theta}$ introduced in [78] provides a condition such that Question 1.1 has an affirmative answer. The following is the main result of the present paper, which complements the results in [2, 3] (see also [62] and [7]; note that these results do not require differentiability imposed on the functions) for Schatten $p$-classes when $p > \frac{1}{\theta}$, and extends those in [14, 16, 74, 75].

**Theorem 1.2.** Let $\theta \in (0, 1)$. Then, for every $p > 0$, there exists a constant $d = d(p)$ such that every function $f \in S_{d,\theta}$ is operator $\theta$-Hölder with respect to $\|\cdot\|_p$. In particular, every function $f \in S_{\infty,\theta}$ is operator $\theta$-Hölder with respect to $\|\cdot\|_p$ for arbitrary $p > 0$, that is, for $p > 0$ and $0 < \theta < 1$, there exists a constant $C_{p,\theta}$ such that

$$(6) \quad \|f(X) - f(Y)\|_p \leq C_{p,\theta} \|f\|_{S_{d,\theta}} \|X - Y\|^\theta_\theta, \quad X = X^*, Y = Y^* \in S(\mathcal{M}, \tau).$$
As an application of (6), we obtain a submajorization inequality for general self-adjoint $\tau$-measurable operators $X$ and $Y$, i.e., there exists a constant $C_{p,\theta}$ depending only on $p$ and $\theta$ such that for any $X = X^*, Y = Y^* \in S(\mathcal{M}, \tau)$, we have (see Theorems 6.1 and 6.3)

$$
(\mu(f(X) - f(Y)))^p \prec \prec C_{p,\theta} \|f\|_{S_{d,\theta}}^p \mu(\|X - Y\|^\theta)^p.
$$

(7)

Here, $\prec \prec$ denotes submajorisation in the sense of Hardy–Littlewood–Polya and $\mu(\cdot)$ is the generalised singular value function [34, 58]. The estimate (7) extends results in [8, 14, 25].

The majority of Banach symmetric spaces used in analysis are fully symmetric rather than just symmetric. Moreover, a wide class of symmetric $p$-normed spaces can be constructed from the $1/p$-th power of symmetrically normed spaces [32, 49] (e.g., $\|\cdot\|_{p,\infty}$ is equivalent to the $\frac{1}{p+\epsilon}$-power of a fully symmetric norm for any $\epsilon > 0$ [11, Chapter 4, Lemma 4.5]). Denote by $E(\mathcal{M}, \tau)^{(p)}$ the $1/p$-power of a fully symmetric space $E(\mathcal{M}, \tau)$ affiliated with $\mathcal{M}$ [32]. That is, $E(\mathcal{M}, \tau)^{(p)}$ is the class of operators $X \in S(\mathcal{M}, \tau)$ with $\|X\|^p \in E(\mathcal{M}, \tau)$ and $\|X\|_{E(\mathcal{p})} = \|\|X\|^p\|_E^{1/p}$. We obtain the following corollary.

**Corollary 1.3.** Let $\theta \in (0,1)$ and $p \in (0,\infty)$. Then, there exists a constant $d = d(p)$ such that every function $f \in S_{d,\theta}$ is operator $\theta$-Hölder with respect to $\|\cdot\|_{E(\mathcal{p})}$, i.e., there exists a constant $C_{p,\theta}$ such that for any self-adjoint $X, Y \in S(\mathcal{M}, \tau)$ with $X - Y \in E(\mathcal{p})(\mathcal{M}, \tau)$, we have

$$
\|f(X) - f(Y)\|_{E(\mathcal{p})} \leq C_{p,\theta} \|f\|_{S_{d,\theta}} \|\|X - Y\|^\theta\|_{E(\mathcal{p})}.
$$

(8)

This corollary extends results in [14], [16] [74] and [75] in two directions. Firstly, instead of considering only noncommutative $L_p$-spaces, we prove that (5) holds true for a very wide class of quasi-Banach symmetric spaces. Secondly, we extend significantly the class of functions $f$ for which the result is applicable. In particular, letting $f(t) = |t|^\theta$ and $p = 1$ in (8), we obtain (3); letting $E(\mathcal{M}, \tau) = L_1(\mathcal{M}, \tau)$, we obtain (5). Indeed, the space $S_{\infty,\theta}$ embraces a wide class of functions, e.g., Schwartz class functions and classic operator monotone functions such as $t \mapsto \frac{t}{|t|^\theta}$, $t > 0$. As an application of (7), we obtain the inequality for fractional powers when $\theta > 1$ (see Corollaries 6.5 and 6.6), extending existing results for fully symmetric ideals on $B(\mathcal{H})$ in [8, 12, 85].

Finally, we obtain the estimates for absolute value map, commutators and quasi-commutators, extending results in [13], [75] and [53, 54]. In contrast with the recent result by Sobolev [78, Theorem 2.4], the compactness of the supports of the functions imposed in (4) is no longer required and we can also treat the case when $\theta = \alpha$. Comparing with the constant $d$ obtained in [78], the constant $d(p)$ obtained in our paper is sharp (see Remark 7.4), which nicely complements [78, Theorem 2.4]. The main result presented in this paper could be used to provide an alternative way to prove main results in [79] and [57] even for unbounded domains $\Lambda$ in $\mathbb{R}^d$. To keep this paper to a reasonable length, this application will be published separately.

2. Preliminaries

In this section, we recall main notions from the theory of noncommutative integration and recall some properties of the generalised singular value function. In what follows, $\mathcal{H}$ is a Hilbert space and $B(\mathcal{H})$ is the $*$-algebra of all bounded linear operators on $\mathcal{H}$, and $1$ is the identity operator on $\mathcal{H}$. Let $\mathcal{M}$ be a von Neumann algebra represented on $\mathcal{H}$. For details on von Neumann algebra theory, the reader is referred to e.g. [24]. General facts concerning measurable operators may be found in [61], [76] (see also the forthcoming book [34]).

2.1. $\tau$-measurable operators and generalised singular values. A linear operator $X : \mathcal{D}(X) \to \mathcal{H}$, where the domain $\mathcal{D}(X)$ of $X$ is a linear subspace of $\mathcal{H}$, is said to be affiliated with $\mathcal{M}$ (denoted by $X \in \mathcal{M}$) if $XY \subseteq XY$ for all $Y \in \mathcal{M}$, where $\mathcal{M}$ is the commutant of $\mathcal{M}$. A linear operator $X : \mathcal{D}(X) \to \mathcal{H}$ is termed measurable with respect to $\mathcal{M}$ if $X$ is closed, densely defined, affiliated with $\mathcal{M}$ and there exists a sequence $(P_n)_{n=1}^{\infty}$ in the lattice of all projections of $\mathcal{M}$, $\mathcal{P}(\mathcal{M})$, such that $P_n \uparrow 1$, $P_n(\mathcal{H}) \subseteq \mathcal{D}(X)$ and $1 - P_n$ is a finite projection (with respect to $\mathcal{M}$) for all $n$. It should be noted that the condition $P_n(\mathcal{H}) \subseteq \mathcal{D}(X)$ implies that $XP_n \in \mathcal{M}$. The collection of all
measurable operators with respect to \( \mathcal{M} \) is denoted by \( S(\mathcal{M}) \), which is a unital *-algebra with respect to strong sums and products (denoted simply by \( X + Y \) and \( XY \) for all \( X, Y \in S(\mathcal{M}) \)).

Let \( X \) be a self-adjoint operator affiliated with \( \mathcal{M} \). We denote its spectral measure by \( \{E^X\} \).

It is well known that if \( X \) is a closed operator affiliated with \( \mathcal{M} \) with the polar decomposition \( X = U|X| \), then \( U \in \mathcal{M} \) and \( E \in \mathcal{M} \) for all projections \( E \in \{E^{|X|}\} \). Moreover, \( X \in S(\mathcal{M}) \) if and only if \( X \) is closed, densely defined, affiliated with \( \mathcal{M} \) and \( E^{|X|}(\lambda, \infty) \) is a finite projection for some \( \lambda > 0 \). It follows immediately that in the case when \( \mathcal{M} \) is a von Neumann algebra of type \( II \) or a type \( I \) factor, we have \( S(\mathcal{M}) = \mathcal{M} \). For type \( II \) von Neumann algebras, this is no longer true. From now on, let \( \mathcal{M} \) be a semifinite von Neumann algebra equipped with a faithful normal semifinite trace \( \tau \).

For any closed and densely defined linear operator \( X : \mathcal{D}(X) \to \mathcal{H} \), the *null projection* \( n(X) = n(|X|) \) is the projection onto its kernel \( \text{Ker}(X) \), the *range projection* \( r(X) \) is the projection onto the closure of its range \( \text{Ran}(X) \) and the *support projection* \( s(X) \) of \( X \) is defined by \( s(X) = 1 - n(X) \).

An operator \( X \in S(\mathcal{M}) \) is called \( \tau \)-measurable if there exists a sequence \( \{P_n\}_{n=1}^\infty \) in \( P(\mathcal{M}) \) such that \( P_n \uparrow 1 \), \( P_n(\mathcal{H}) \subseteq \mathcal{D}(X) \) and \( \tau(1 - P_n) < \infty \) for all \( n \). The collection \( S(\mathcal{M}, \tau) \) of all \( \tau \)-measurable operators is a unital *-subalgebra of \( S(\mathcal{M}) \) denoted by \( S(\mathcal{M}, \tau) \). We denote by \( S(\mathcal{M}, \tau)_h \) the real subspace of self-adjoint elements of \( S(\mathcal{M}, \tau) \). It is well known that a linear operator \( X \) belongs to \( S(\mathcal{M}, \tau) \) if and only if \( X \in S(\mathcal{M}) \) and there exists \( \lambda > 0 \) such that \( \tau(E^{|X|}(\lambda, \infty)) < \infty \). Alternatively, an unbounded operator \( X \) affiliated with \( \mathcal{M} \) is \( \tau \)-measurable [40] if and only if \( \tau(E^{|X|}(n, \infty)) \to 0 \) as \( n \to \infty \).

**Definition 2.1.** Let a semifinite von Neumann algebra \( \mathcal{M} \) be equipped with a faithful normal semifinite trace \( \tau \) and let \( X \in S(\mathcal{M}, \tau) \). The generalised singular value function \( \mu(X) : t \to \mu(t; X) \) of the operator \( X \) is defined by setting

\[
\mu(s; X) = \inf\{\|XP\|_\infty : P = P^* \in \mathcal{M} \text{ is a projection, } \tau(1 - P) \leq s\}.
\]

An equivalent definition in terms of the distribution function of the operator \( X \) is the following. For every \( X \in S(\mathcal{M}, \tau)_h \), we define a right-continuous function \( d_X(\cdot) \) by \( d_X(t) = \tau(E^X(t, \infty)) \), \( t > 0 \) (see e.g. [40]). We have (see e.g. [40] and [58])

\[
\mu(t; X) = \inf\{s \geq 0 : d_X(s) \leq t\}.
\]

For every \( \varepsilon, \delta > 0 \), we define the set

\[
V(\varepsilon, \delta) = \{x \in S(\mathcal{M}, \tau) : \exists P \in \mathcal{P}(\mathcal{M}) \text{ such that } \|X(1 - P)\|_\infty \leq \varepsilon, \tau(P) \leq \delta\}.
\]

The topology generated by the sets \( V(\varepsilon, \delta) \), \( \varepsilon, \delta > 0 \), is called the measure topology \( \tau_\tau \) on \( S(\mathcal{M}, \tau) \) [34, 40, 61]. A further important vector space topology on \( S(\mathcal{M}, \tau) \) is the *local measure topology* [31, 34]. A neighbourhood base for this topology is given by the sets \( V(\varepsilon, \delta; P) \), \( \varepsilon, \delta > 0 \), \( P \in \mathcal{P}(\mathcal{M}) \) with \( \tau(P) \leq \varepsilon \), where

\[
V(\varepsilon, \delta; P) = \{X \in S(\mathcal{M}, \tau) : PX \in V(\varepsilon, \delta)\}.
\]

If \( \{X_\alpha\} \subset S(\mathcal{M}, \tau) \) is a net and if \( X_\alpha \rightharpoonup X \in S(\mathcal{M}, \tau) \) in local measure topology, then \( X_\alpha Y \to XY \) and \( YX_\alpha \to YX \) in the local measure topology for all \( Y \in S(\mathcal{M}, \tau) \) [31, 34].

### 2.2. Quasi-Banach symmetric spaces.

It is convenient to recall the notion of a quasi-normed space [47, 48]. Let \( X \) be a linear space. A strictly positive, absolutely homogeneous functional \( \|\cdot\| : X \to [0, \infty) \) is called a quasi-norm if there exists a constant \( K > 0 \) such that

\[
\|x_1 + x_2\| \leq K (\|x_1\| + \|x_2\|), \quad x_1, x_2 \in X.
\]

The optimal choice of \( K \) will be called the modulus of concavity of the quasi-norm.

**Definition 2.2.** Let a semifinite von Neumann algebra \( \mathcal{M} \) be equipped with a faithful normal semifinite trace \( \tau \). Let \( E(\mathcal{M}, \tau) \) be a linear subset in \( S(\mathcal{M}, \tau) \) equipped with a quasi-norm \( \|\cdot\|_E \).

We say that \( E(\mathcal{M}, \tau) \) is a symmetrically quasi-normed space if for \( X \in E(\mathcal{M}, \tau) \), \( Y \in S(\mathcal{M}, \tau) \) and \( \mu(Y) \leq \mu(X) \) imply that \( Y \in E \) and \( \|Y\|_E \leq \|X\|_E \). In particular, if \( \|\cdot\|_E \) is a norm, then \( E(\mathcal{M}, \tau) \) is called a symmetrically normed space.
A symmetrically (quasi-)normed space is called a (quasi-Banach) symmetric space if it is complete. It is well-known that any quasi-symmetrically normed space $E$ is a quasi-normed $\mathcal{M}$-bimodule, that is $AXB \in E$ for any $X \in E$, $A, B \in \mathcal{M}$ and $\|AXB\|_E \leq \|A\|_\infty \|B\|_\infty \|X\|_E [31, 34, 81]$.

If $X, Y \in S(\mathcal{M}, \tau)$, then $X$ is said to be submajorized by $Y$, denoted by $X \prec Y$ (Hardy–Littlewood–Polya submajorization, see $[31, 58]$), if

$$
\int_0^t \mu(s; X)ds \leq \int_0^t \mu(s; Y)ds, \forall t > 0.
$$

A (quasi-)Banach symmetrically normed space is called a (quasi-)Banach symmetric space. A symmetric space $E(\mathcal{M}, \tau) \subset S(\mathcal{M}, \tau)$ is called strongly symmetric if its norm $\|\cdot\|_E$ has the additional property that $\|X\|_E \leq \|Y\|_E$ whenever $X, Y \in E(\mathcal{M}, \tau)$ satisfy $X \prec Y$. In addition, if $X \in S(\mathcal{M}, \tau)$, $Y \in E(\mathcal{M}, \tau)$ and $X \prec Y$ imply that $X \in E(\mathcal{M}, \tau)$ and $\|X\|_E \leq \|Y\|_E$, then $E(\mathcal{M}, \tau)$ is called fully symmetric space (of $\tau$-measurable operators). We denote by $E(\mathcal{M}, \tau)_h$ the real subspace of a symmetrically quasi-normed space $E(\mathcal{M}, \tau)$.

A wide class of quasi-Banach symmetric operator spaces associated with the von Neumann algebra $\mathcal{M}$ can be constructed from concrete quasi-Banach symmetric function spaces studied extensively in e.g. $[56]$. Let $(E(0, \infty), \|\cdot\|_{E(0, \infty)})$ be a (quasi-Banach) symmetric function space on the semi-axis $(0, \infty)$. That is, a quasi-Banach symmetric space for the von Neumann algebra $L_\infty(0, \infty)$ with trace given by Lebesgue integration. The pair

$$
E(\mathcal{M}, \tau) = \{X \in S(\mathcal{M}, \tau) : \mu(X) \in E(0, \infty)\}, \quad \|X\|_{E(\mathcal{M}, \tau)} := \|\mu(X)\|_{E(0, \infty)}
$$

is a (quasi-Banach) symmetric operator space affiliated with $\mathcal{M}$ (see e.g. $[45, 49, 58, 81]$). For convenience, we denote $\|\cdot\|_{E(\mathcal{M}, \tau)}$ by $\|\cdot\|_E$. Many properties of quasi-Banach symmetric spaces, such as reflexivity, Fatou property, order continuity of the norm as well as Köthe duality carry over from commutative symmetric function space $E(0, \infty)$ to its noncommutative counterpart $E(\mathcal{M}, \tau)$ (see e.g. $[28, 30, 32, 34, 46]$). In particular, $E(\mathcal{M}, \tau)$ is fully symmetric whenever $E(0, \infty)$ is a fully symmetric function space on $(0, \infty)$ $[31, 34]$.

### 2.3. $\theta$-Hölder functions.

Throughout this section, we always assume that $0 < \theta < 1$. A function $f : \mathbb{R} \to \mathbb{C}$ is called a $\theta$-Hölder function if it satisfies the inequality

$$
|f(x) - f(y)| \leq \text{const}|x - y|^{\theta}, \quad x, y \in \mathbb{R}.
$$

Let $\Lambda_\theta$ denote the space of all $\theta$-Hölder functions. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and $p > 0$. If $f(A) - f(B) \in S_p$ for any bounded self-adjoint operators $A, B \in B(H)$ with $\text{rank}(A - B) < \infty$, then $g := f \circ h \in B_{1/p}^{1/p}$ for any rational function $h$ that is real on $\mathbb{R}$ and has no pole at $\infty$ $[2, \text{Theorem 9.2}]$, where $B_{s,q}^p \supset 0 < s, p, q < \infty$ denotes the Besov space on $\mathbb{R}$ defined by using Littlewood-Paley decomposition (see $[84, \text{Section 2.6.1}]$). In particular, if $f$ is operator $\theta$-Hölder with respect to $\|\cdot\|_p$, then $f \in B_{1/p}^{1/p}$ locally, i.e., the restriction of $f$ to an arbitrary finite interval can be extended to a function of class $B_{1/p}^{1/p}$ (see $[2, \text{Theorem 9.2}]$ and the comments below). However, this necessary condition is very loose. The main goal of the present paper to obtain a simple criterion for a function to be operator $\theta$-Hölder.

Recently, Sobolev $[78, 79]$ introduced the space $S_{d,\theta}$ consists of functions $f \in C^d(\mathbb{R}\setminus 0) \cap C(\mathbb{R})$ such that

$$
\|f\|_{S_{d,\theta}} := \max_{0 \leq k \leq d} \|f\|_{-\theta+k,k} := \max_{0 \leq k \leq d} \sup_{x \neq 0} |x|^{-\theta+k} |f^{(k)}(x)| < \infty.
$$

The following fact is well-known (see e.g. $[79, (2.3)]$). For the sake of completeness, we provide a short proof.

**Lemma 2.3.** For $d \geq 1$, we have $S_{d,\theta} \subset \Lambda_\theta$.

**Proof.** When $xy \leq 0$, it is clear that

$$
|f(x) - f(y)| \leq |f(x)| + |f(y)| \leq \|f\|_{S_{0,\theta}} |x|^\theta + \|f\|_{S_{0,\theta}} |y|^\theta \leq 2 \|f\|_{S_{0,\theta}} |x - y|^\theta.
$$
When $xy \geq 0$, we have

$$|f(x) - f(y)| = \left| \int_x^y f'(t) dt \right| \leq \|f\|_{S_{1,\theta}} \left| \int_x^y |t|^{\theta - 1} dt \right| = \frac{1}{\theta} \|f\|_{S_{1,\theta}} |x|^{\theta} - |y|^{\theta}. $$

Note that

$$|x|^{\theta} - |y|^{\theta} \leq \Theta \left| \frac{x}{y} \right|^{\theta} \leq |x - y|^{\theta}, \quad x, y \in \mathbb{R}. $$

Hence, for $xy \geq 0$, we have

$$|f(x) - f(y)| \leq \frac{1}{\theta} \|f\|_{S_{1,\theta}} |x - y|^{\theta}. $$

In either case, we have

$$|f(x) - f(y)| \leq \frac{2}{\theta} \|f\|_{S_{1,\theta}} |x - y|^{\theta}. $$

\[\square\]

**Remark 2.4.** The definition of $S_{d,\theta}$ can be modified to the case of $f \in C_d^0(\mathbb{R} \setminus x_0) \cap C(\mathbb{R})$, $x_0 \in \mathbb{R}$, and the results of the present paper remain true (see [78, Remark 2.6]).

We note that $\|\cdot\|_{S_{-\theta+k,k}}$ is an analogue of the Schwartz semi-norm with multi-indices $(\theta - \theta, k)$, $0 \leq k \leq d$. The space $S_{d,\theta}$ embraces a wide class of functions, e.g., the Schwartz space is a subspace of $S_{\infty,\theta}$. It is easy to see that classic operator-monotone functions such as $t \mapsto |t|^\theta$, $t \mapsto \log(|t| + 1)$ and $t \mapsto \frac{|t|}{|t| + 1}$, $t \in \mathbb{R}$ are in $S_{\infty,\theta}$.

In this paper, we mainly consider $\theta$-Hölder functions of the class $S_{d,\theta}$. The following proposition demonstrates that $f \in S_{d,\theta}$ is a very mild condition for a $\theta$-Hölder function $f$.

**Proposition 2.5.** Let $f : \mathbb{R} \to \mathbb{R} \in C^0(\mathbb{R} \setminus \{0\}) \cap C(\mathbb{R})$, $n \geq 0$, with $f(0) = 0$ and $\theta \in (0, 1)$. Assume that there exist positive numbers $r, C, \varepsilon > 0$ such that

$$\frac{|f^{(n)}(x)|}{|x|^{\theta - r - n}} \geq C > 0 $$

for $x \in (0, \varepsilon)$. Then, $f$ is not $\theta$-Hölder.

**Proof.** Without loss of generality, we assume that $r \in (0, \theta)$. We assert that if $k \geq 2$ and a constant $C$, there exists a constant $C_{k,\theta, r} > 0$ such that

$$\left| f^{(k)}(x) - C \right| \geq C_{k,\theta, r} x^{\theta - r - k}, \quad x \in (0, \varepsilon), $$

then there exist constants $C_{k-1,\theta, r} > 0$ and $C'$ such that

$$\left| f^{(k-1)}(x) - C' \right| \geq C_{k-1,\theta, r} x^{\theta - r - k + 1}, \quad x \in (0, \varepsilon). $$

Without loss of generality, we assume that

$$f^{(k)}(x) - C \geq C_{k,\theta, r} x^{\theta - r - k}, \quad x \in (0, \varepsilon). $$

Integrating from $x$ to $\varepsilon$, we obtain that

$$f^{(k-1)}(\varepsilon) - f^{(k-1)}(x) - C(\varepsilon - x) \geq \frac{C_{k,\theta, r}}{\theta - r - k + 1} \left( \varepsilon^{\theta - r - k + 1} - x^{\theta - r - k + 1} \right), \quad x \in (0, \varepsilon).$$

Since any polynomial is bounded on $(0, \varepsilon)$, it follows that there exists a constant $C'$ such that

$$f^{(k-1)}(x) - C' \leq \frac{C_{k,\theta, r}}{\theta - r - k + 1} x^{\theta - r - k + 1}, \quad x \in (0, \varepsilon).$$

Noting that $\theta - r - k + 1 < 0$, we obtain the validity of (11).

Assume that there exist constants $C_{1,\theta, r} > 0$ and $C'$ such that

$$\left| f'(x) - C \right| \geq C_{1,\theta, r} x^{\theta - r - 1}, \quad x \in (0, \varepsilon).$$

Without loss of generality, we assume that $f'(x) - C \geq C_{1,\theta, r} x^{\theta - r - 1}, \quad x \in (0, \varepsilon)$. Hence,

$$f(x) - C x = f(x) - f(0) - C x \geq \frac{C_{1,\theta, r}}{\theta - r} x^{\theta - r}, \quad x \in (0, \varepsilon).$$
Therefore,
\[ \sup_{x \neq 0} \frac{|f(x)|}{x^{\beta}} = \sup_{x \neq 0} \frac{|f(x)| - Cx^{1-\theta}|}{x^{\beta}} \geq \sup_{x \in (0, \varepsilon)} \frac{|C_{1,0,\varepsilon} - x^{-\frac{1}{\beta}}|}{x^{\beta}} = |C|^{1-\theta} \]

This together with (11) implies that \( f \) is not \( \theta \)-Hölder. \( \square \)

3. Double operator integrals

In this section, we review some aspects of the beautiful theory of double operator integrals. Double operator integrals appeared in the paper by Daletskii and Krein [22]. For details of the theory of double operator integrals, the reader is referred to [6, 17, 70, 77]. For basic properties of noncommutative \( L_p \)-spaces \( L_p(\mathcal{M}, \tau) \), we refer to [67].

Symbolically, a double operator integral is defined by the formula
\[ T_a^{A,B}(V) = \int_{\mathbb{R}^2} a(\lambda, \mu)dE_A(\lambda)VdE_B(\mu), \quad V \in L_2(\mathcal{M}, \tau) \]
for a bounded Borel function \( a \) on \( \mathbb{R}^2 \) and for self-adjoint operators \( A \) and \( B \) affiliated with \( \mathcal{M} \) (see e.g. [38], see also [17] [77, Section 3.5]). Here, \( (\mathcal{A}, E_A) \) and \( (\mathcal{B}, E_B) \) are spectral measures of \( A \) and \( B \), respectively.

We write \( a \in \mathcal{M}_p \) if
\[ \|a\|_{\mathcal{M}_p} := \sup_{\mathcal{M} = A = A^*, B = B^*} \sup_{\|\cdot\|_{L_p(\mathcal{M}, \tau)}} \|T_a^{A,B}\|_{L_p(\mathcal{M}, \tau)} < \infty, \]
where \( \|T_a^{A,B}\|_{L_p(\mathcal{M}, \tau)} \) is the operator quasi-norm of \( T_a^{A,B} \) from \( (L_p \cap L_2)(\mathcal{M}, \tau) \) into itself. It is clear that for every \( p > 0 \), we have
\[ \|ab\|_{\mathcal{M}_p} \leq \|a\|_{\mathcal{M}_p}\|b\|_{\mathcal{M}_p} \]
for every \( a, b \in \mathcal{M}_p \). It follows from the \( p \)-triangle inequality for \( L_p(\mathcal{M}, \tau) \), \( 0 < p \leq 1 \), and every \( a, b \in \mathcal{M}_p \), we have (see e.g. [1, (1.2)])
\[ \|a + b\|_{\mathcal{M}_p}^p \leq \|a\|_{\mathcal{M}_p}^p + \|b\|_{\mathcal{M}_p}^p. \]
Moreover, if \( a_n \in \mathcal{M}_p \), \( n \geq 0 \), such that \( \sum_{n=0}^{\infty} a_n \) is bounded Borel function, then
\[ \left\| \sum_{n=0}^{\infty} a_n \right\|_{\mathcal{M}_p}^p \leq \sum_{n=0}^{\infty} \|a_n\|_{\mathcal{M}_p}^p. \]

Let \( f \) be a function admitting a representation
\[ f(x, y) = \sum_{n \geq 0} \varphi_n(x) \psi_n(y), \]
where \( \varphi_n \in L^\infty(E_A) \), \( \psi_n \in L^\infty(E_B) \) are Borel functions, and
\[ \sum_{n \geq 0} \|\varphi_n\|_{\infty} \|\psi_n\|_{\infty} < \infty. \]
Then, for arbitrary \( V \in B(\mathcal{H}) \),
\[ T_f^{A,B}(V) := \sum_{n \geq 0} \left( \int_{\mathcal{A}} \varphi_n(\lambda)dE_A(\lambda) \right) V \left( \int_{\mathcal{B}} \psi_n(\mu)dE_B(\mu) \right). \]
Here, the convergence is understood in the uniform norm topology, and therefore, in (local) measure topology. In particular, \( T_f^{A,B} \) is a bounded operator from \( \mathcal{M} \) into \( \mathcal{M} \) [64] (see also [69, Theorem 4]).

The following lemma provides a criterion for verifying whether a function is in \( \mathcal{M}_p \) or not (see also [1, p. 279] and [75, Lemma 2.3]).
Lemma 3.1. Let $p \in (0,1]$ and let $a$ be a Borel function admitting a representation
\begin{equation}
\alpha(\lambda, \mu) = \sum_{n \geq 0} \varphi_n(\lambda)\psi_n(\mu), \ \lambda, \mu \in \mathbb{R},
\end{equation}
where $(\|\varphi_n\|_\infty)_n \in \ell_\infty$ and $(\|\psi_n\|_\infty)_n \in \ell_p$. We have
\begin{equation}
\|a\|_{\mathfrak{M}_p} \leq \left\| \left\{ \|\varphi_n\|_\infty \right\}_{n \geq 0} \right\|_\infty \left\| \left\{ \|\psi_n\|_\infty \right\}_{n \geq 0} \right\|_p.
\end{equation}

Proof. Let $a$ be a Borel function admitting representation (15) and such that the right hand side of (16) is finite. Let $A$ and $B$ two self-adjoint operators affiliated with $\mathcal{M}$. Since $\ell_p \subset \ell_1$, it follows that
\[ \sum_{n \geq 0} \|\varphi_n\|_\infty \|\psi_n\|_\infty < \infty. \]
We have
\[ T_a^{A,B}(V) = \sum_{n \geq 0} \varphi_n(A)V\psi_n(B), \quad V \in \mathcal{M}, \]
where convergence is in the norm topology of $\mathcal{M}$. If $V \in L_p(\mathcal{M}, \tau) \cap \mathcal{M}$, then
\[ \|T_a^{A,B}(V)\|^p \leq \sum_{n \geq 0} \|\varphi_n(A)V\psi_n(B)\|^p \leq \left( \sum_{n \geq 0} \|\varphi_n(A)\|_{\infty} \|\psi_n(B)\|_{\infty}^p \right) \cdot \|V\|^p \]
\[ \leq \left( \sum_{n \geq 0} \|\varphi_n\|_{\infty} \|\psi_n\|_{\infty}^p \right) \cdot \|V\|^p. \]
Therefore,
\[ \|T_a^{A,B}\|_{L_p \rightarrow L_p} \leq \left( \sum_{n \geq 0} \|\varphi_n\|_{\infty} \|\psi_n\|_{\infty}^p \right)^{\frac{1}{p}} \leq \left\{ \left\{ \|\varphi_n\|_\infty \right\}_{n \geq 0} \right\|_\infty \left\{ \|\psi_n\|_\infty \right\}_{n \geq 0} \right\|_p. \]
Taking the supremum over $A = A^*$, $B = B^*\eta\mathcal{M}$ and then over $\mathcal{M}$, we complete the proof. \qed

The following is an easy corollary of Lemma 3.1. Similar results are obtained in [75, Corollary 2.8].

Corollary 3.2. Let $0 < p \leq 1$. Let
\[ \alpha(s,t) := \frac{1}{s-t} \chi_{(\frac{1}{2},1)}(|s|)\chi_{(0,\frac{1}{2})}(|t|), \quad s, t \in \mathbb{R} \]
and
\[ \beta(s,t) := \frac{t}{t-s} \chi_{(\frac{1}{2},1)}(|s|)\chi_{(2,\infty)}(|t|), \quad s, t \in \mathbb{R}. \]
Then, we have
\[ \|\alpha\|_{\mathfrak{M}_p}, \|\beta\|_{\mathfrak{M}_p} < \infty. \]

Proof. Note that
\[ \frac{1}{1-x} = \sum_{n \geq 0} x^n, \ |x| < 1. \]
We write
\[ \alpha(s,t) = \frac{1}{s-t} \chi_{(\frac{1}{2},1)}(|s|)\chi_{(0,\frac{1}{2})}(|t|) = \frac{1}{s} \sum_{n \geq 0} (2s)^{-n} \chi_{(\frac{1}{2},1)}(|s|)(2t)^n \chi_{(0,\frac{1}{2})}(|t|) \]
for any $s, t \in \mathbb{R}$. Since $\sup_s (2s)^{-n} \chi_{(\frac{1}{2},1)}(|s|) < 1$, it follows from Lemma 3.1 that
\[ \|\alpha\|_{\mathfrak{M}_p} \leq 2 \left\{ \left\{ (2s)^{-n} \chi_{(\frac{1}{2},1)}(|s|) \right\}_{n \geq 0} \right\|_\infty \left\{ (1/2)^n \right\}_{n \geq 0} \right\|_p < \infty. \]
This completes the proof that $\|\alpha\|_{\mathfrak{M}_p} < \infty$.

For $\beta$, we write
\[ \beta(s,t) = \frac{1}{t} \chi_{(\frac{1}{2},1)}(|s|)\chi_{(2,\infty)}(|t|) = \sum_{n \geq 0} s^n \chi_{(\frac{1}{2},1)}(|s|) \cdot t^{-n} \chi_{(2,\infty)}(|t|) \]
for any \( s, t \in \mathbb{R} \). By Lemma 3.1, we have
\[
\|\beta\|_{\mathfrak{M}_p} \leq \|\{2^{-n}\}_{n \geq 0}\|_{\infty} \|\{2^{-n}\}_{n \geq 0}\|_p < \infty.
\]
\[\square\]

**Remark 3.3.** For a bounded Borel function \( g : \mathbb{R} \to \mathbb{C} \), define a bounded Borel function \( f : \mathbb{R}^2 \to \mathbb{C} \) by setting \( f(x, y) = g(x) \). We have
\[
\|f\|_{\mathfrak{M}_p} = \|g\|_{\infty}.
\]

If \( f \) is a Lipschitz function on \( \mathbb{R} \), then we have
\[
\|f\|_{\mathfrak{M}_p} = \|g\|_{\infty}.
\]

\[\square\]

**Lemma 3.4.** Let \( b \in \mathbb{Z} \) be such that \( b > 1/p, p \in (0, 1] \). Let \( a : \mathbb{R}^2 \to \mathbb{C} \) be a bounded Borel \( 2\pi \)-periodic function in the both arguments. If the mixed partial derivative \( \frac{\partial^{m+n} a}{\partial x^m \partial y^n} \) exists (in the usual sense) for every multi-index \( m, n \), \( 0 \leq m + n \leq b + 1 \) with \( |\alpha| \leq k \), the mixed partial derivative exists in the weak sense and is in \( L_p(\Omega) \) \[84\]. Here, \( \mathbb{T}^2 \) stands for the 2-dimensional torus, i.e., \( \mathbb{T} = \mathbb{R}/2\pi \mathbb{Z} \) (equipped with the Lebesgue measure). The following lemma is a modification of [75, Lemma 2.7]. Note that the function considered below is not required to be infinitely differentiable for the second argument as in [75, Lemma 2.7].

**Lemma 3.4.** Let \( b \in \mathbb{Z} \) be such that \( b > 1/p, p \in (0, 1] \). Let \( a : \mathbb{R}^2 \to \mathbb{C} \) be a bounded Borel \( 2\pi \)-periodic function in the both arguments. If the mixed partial derivative \( \frac{\partial^{m+n} a}{\partial x^m \partial y^n} \) exists (in the usual sense) for every multi-index \( m, n \), \( 0 \leq m + n \leq b + 1 \) with \( |\alpha| \leq k \), then we have
\[
\|a\|_{\mathfrak{M}_p} \leq c_{p,b} \|a\|_{W^{b+1,2}(\mathbb{T}^2)} < \infty.
\]

**Proof.** We use Fourier representation in the second argument
\[
a(x, y) = \sum_{n \in \mathbb{Z}} a_n(x) e_n(y) = a_0(x) + \sum_{n \neq 0} |n|^b a_n(x) \cdot |n|^{-b} e_n(y),
\]
where \( e_n(y) = e^{iny}, y \in \mathbb{R} \) and \( a_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} a(x, y) e^{-iny} dy, x \in \mathbb{R}, n \in \mathbb{Z} \). By Lemma 3.1, we have
\[
\|a\|_{\mathfrak{M}_p} = \|a_0\|_{\infty} + \left\{ \left\| \sum_{n \neq 0} |n|^b a_n \right\|_{\infty} \right\}_{n \neq 0, 0} \| n^{-b} e_n \|_{\infty}
\]
\[= \|a_0\|_{\infty} + c_{p,b} \left\| \sum_{n \neq 0} |n|^b a_n \right\|_{\infty} \| n^{-b} e_n \|_{\infty},
\]
where \( c_{p,b} := (\sum_{n \neq 0} |n|^{-b})^{1/p} < \infty \). For every continuously differentiable function \( h \) on \( \mathbb{T} \), integrating by part, we get (see e.g. [80, Theorem 5.1])
\[
\hat{h}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} h(t) e^{-int} dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} 1 \cdot \frac{1}{ni} \int_{-\pi}^{\pi} \hat{h}'(t) e^{-int} dt = \frac{1}{ni} \hat{h}'(n),
\]
where \( \hat{h}(n) \) stands for the \( n \)-th Fourier coefficient of \( h \). By Cauchy’s inequality, we have

\[
\|h\|_\infty \leq \sum_n |\hat{h}(n)| \leq \|h(0)| + \left( \sum_{n \neq 0} |\hat{h}(n)|^2 \cdot n^2 \right)^{1/2} \left( \sum_{n \neq 0} n^{-2} \right)^{1/2} \\
\leq \|h\|_2 + \left( \sum_{n \neq 0} |\hat{h}'(n)|^2 \right)^{1/2} \left( \frac{\pi^2}{3} \right)^{1/2} \\
(21)
\leq \|h\|_2 + \left( \frac{\pi^2}{3} \right)^{1/2} \|h'\|_2,
\]

Applying (21) to every \( a_n \) in (19), we have

\[
\|a\|_{\mathfrak{M}_p} \overset{(19)}{\leq} \|a_0\|_\infty + c_{p,b} \left\{ \|n^b a_n\|_\infty \right\}_{n \neq 0} \overset{(21)}{\leq} \|a_0\|_2 + \left( \frac{\pi^2}{3} \right)^{1/2} \|a'_0\|_2 \\
+ c_{p,b} \left\{ \|n^b a_n\|_2 \right\}_{n \neq 0} \left( \frac{\pi^2}{3} \right)^{1/2} \|a'_0\|_\infty \leq \|a_0\|_2 + \left( \frac{\pi^2}{3} \right)^{1/2} \|a'_0\|_2 \\
+ c_{p,b} \left\{ \|n^b a_n\|_2 \right\}_{n \neq 0} \left( \frac{\pi^2}{3} \right)^{1/2} \|a'_0\|_2.
\]

By Pythagorean identity and for every \( x \in \mathbb{R} \), letting \( a_x(y) = a(x, y) \), \( y \in \mathbb{R} \), we obtain that

\[
\left\{ |n|^b a_n e_n \right\}_{n \neq 0} \overset{(20)}{=} \sum_{n \neq 0} |n|^b a_n e_n \leq \left\| \frac{\partial^b}{\partial^2_{x,0} a} \right\|_2,
\]

and, for every \( x \in \mathbb{R} \), letting \( c_x(y) = \frac{\partial}{\partial x} a(x, y) \), \( y \in \mathbb{R} \), we have

\[
\left\{ |n|^b a_n' e_n \right\}_{n \neq 0} \overset{(20)}{=} \sum_{n \neq 0} |n|^b a_n' e_n \leq \left\| \frac{\partial^{b+1}}{\partial^2_{x,0} a} \right\|_2.
\]

This completes the proof. \( \square \)

For \( r > 0 \), the dilation operator \( \sigma_r \) are acting on any bounded Borel function \( a : \mathbb{R} \to \mathbb{C} \) such that

\[
(\sigma_r a)(s) := a \left( \frac{s}{r} \right), \quad s \in \mathbb{R}.
\]

In particular, if \( f \in S_{d,0} \), then

\[
\|\sigma_r f\|_{S_{d,0}} = \max_{0 \leq k \leq d} \left| (\sigma_r f)^{(k)}(x) \right| = \max_{0 \leq k \leq d} \sup_{x \neq 0} \left| \frac{f^{(k)}(x)}{|x|^{d-k}} \right| \\
\overset{(22)}{=} \max_{0 \leq k \leq d} \sup_{x \neq 0} \left| \frac{f^{(k)}(x)}{|x|^{d-k}} \right| = 1 \frac{1}{r^d} \|f\|_{S_{d,0}}.
\]
Similarly, for \( r > 0 \), the dilation operator \( \sigma_r \) are acting on any bounded Borel function \( a : \mathbb{R}^2 \to \mathbb{C} \) such that

\[
(\sigma_r a)(s, t) := a \left( \frac{s}{r}, \frac{t}{r} \right), \quad s, t \in \mathbb{R}.
\]

**Remark 3.5.** For any bounded Borel function \( a : \mathbb{R}^2 \to \mathbb{C} \) and self-adjoint operators \( A, B \in \mathcal{M} \), and for any \( V \in L_2(\mathcal{M}, \tau) \), we have

\[
T_a^{A,B}(V) = \int_{\mathbb{R}^2} a(\lambda, \mu) dE_A(\lambda)V E_B(\mu) = \int_{\mathbb{R}^2} a \left( \frac{\lambda}{r}, \frac{\mu}{r} \right) dE_{rA}(\lambda)V E_{rB}(\mu) = T_{\sigma_r a}^{A,rB}(V).
\]

In particular, we have

\[
\|\sigma_r a\|_{2^p_a} = \|a\|_{2^p_a}.
\]

The following result is an easy consequence of [68, Theorem 3.1] (by taking \( x = pq \)). Similar results have been proven in [39, Proposition 6.11] and [38, Lemma 7.3].

**Lemma 3.6.** Assume that \( f : \mathbb{R} \to \mathbb{C} \) is a Lipschitz function. Let \( A, B \in S(\mathcal{M}, \tau) \) be self-adjoint operators and let \( p, q \) be spectral projections of \( A \) and \( B \), respectively. If \( p(A - B)q \in L_2(\mathcal{M}, \tau) \), then

\[
T_{\sigma_f}^{A,B}(p(A - B)q) = pf(A) - f(B))q.
\]

4. **Double operator integrals of divided differences on \( L_p(\mathcal{M}, \tau) \), \( 0 < p \leq 1 \)**

Let \( 0 < \theta < 1 \). Assume that \( \mathcal{H} \) is a separable Hilbert space. In this section, we deal with the subspace \( S_{d, \theta} \) of the \( \theta \)-Hölder class \( \Lambda_\theta \).

We define \( d := d(p) \), \( p > 0 \), by setting that \( d \) is the minimal integer such that

\[
d > \frac{1}{p} + 2, \quad p \leq 1
\]

and

\[
d(p) = d(1) = 4, \quad p > 1.
\]

As mentioned before, when \( p = \infty \), the operator \( \theta \)-Hölder functions has been described in [3, 7, 62]. Hence, we only consider the case for \( 0 < p < \infty \). Without loss of generality, we only prove the case for \( 0 < p \leq 1 \). Indeed, the case when \( p > 1 \) follows immediately from the case for \( p = 1 \), the Hardy–Littlewood–Polya inequality [46, Corollary 2.5] (see also [37, Theorem 11] and [59, Chapter I, Theorem D.2] and Theorem 6.1).

The argument in [75] heavily rests on the homogeneity of the function \( t \mapsto |t|^\theta \), \( t \in \mathbb{R} \), which is one of the difficulties we encountered. A different method from that in [75] is required in order to consider a much more general class of operator \( \theta \)-Hölder functions.

Throughout this section, we always assume that \( p \in (0, 1] \) and \( f \in S_{d, \theta} \).

We note that a result similar to the following lemma was claimed in [15, Theorem 9.6] under slightly weaker conditions. We would like to thank Edward McDonald for providing us a proof of [15, Theorem 9.6].

**Lemma 4.1.** Let \( f \in S_{d, \theta} \) and let \( \phi \) be a smooth function. If \( \phi \) is supported on \( (\varepsilon, \pi) \), then

\[
\| (\phi \otimes \phi) \cdot \mathcal{D}f \|_{2^p_a} \leq c_{p,\theta}\|f\|_{S_{d, \theta}}.
\]

Here, \( (\phi \otimes \phi) \cdot \mathcal{D}f \) is defined by setting \( (\phi \otimes \phi) \cdot \mathcal{D}f(x, y) = \phi(x)\phi(y)\mathcal{D}f(x, y), x, y \in (\varepsilon, \pi) \).

**Proof.** Let \( a : \mathbb{R}^2 \to \mathbb{C} \) be a Borel function \( 2\pi \)-periodic in both arguments such that

\[
a = (\phi \otimes \phi) \cdot \mathcal{D}f \text{ on } [\varepsilon, \pi]^2.
\]

By Lemma 3.4, for any \( b > 1/p \), we have

\[
\| (\phi \otimes \phi) \cdot \mathcal{D}f \|_{2^p_a} \leq c_{p, b}\|a\|_{W^{b+1,2}(T^2)}.
\]
Since $\phi = 0$ on $(-\varepsilon, \varepsilon)$, it follows that there exists a constant $c_0$ such that
\[
\|a\|_{W^{1,2}([T])} = \|(\phi \otimes \phi) \cdot D f\|_{W^{1,2}([-\varepsilon, \varepsilon]^2)}
\]
(23)
\[
\leq c_0 \left( \max_{0 \leq k \leq b+1} \left\| \phi^{(k)} \right\|_\infty \right)^2 \|D f\|_{W^{1,2}([-\varepsilon, \varepsilon]^2)}
\]
where
\[
\|D f\|_{W^{1,2}([-\varepsilon, \varepsilon]^2)} = \sum_{0 \leq m+n \leq b+1} \left\| \frac{\partial^{m+n}}{\partial y^m \partial x^n} (D f) \chi_{[-\varepsilon, \varepsilon]^2} \right\|_{\infty}.
\]
Take $b = d - 2$. For $x, y \in [-\varepsilon, \varepsilon]$, we have $tx + (1 - t)y \geq \varepsilon$. By the Leibniz integral rule, we obtain that for any $m, n \geq 0$ with $m + n \leq d - 1$,
\[
\left| \frac{\partial^{m+n}}{\partial y^m \partial x^n} (D f) (x, y) \right| \leq \left| \int_0^1 \frac{\partial^{m+n}}{\partial y^m \partial x^n} f'(tx + (1 - t)y) dt \right|
\]
(18)
\[
\leq \int_0^1 t^m (1 - t)^n \left| f'(tx + (1 - t)y) \right| dt
\]
\[
\leq \int_0^1 \left| f'(tx + (1 - t)y) \right| dt
\]
(9)
\[
\leq \left| tx + (1 - t)y \right|^{-m-n-1+\theta} \|f\|_{S_{d,\theta}} dt
\]
\[
\leq \varepsilon^{-m-n-1} \|f\|_{S_{d,\theta}},
\]
which together with (23) completes the proof. 

The following lemma contain crucial estimates for our main result in this section, Theorem 4.4.

**Lemma 4.2.** There exists a constant $c_p > 0$ depending on $p$ only such that

(i)
\[
\left\| (D f) \cdot \chi_{[-\varepsilon, \varepsilon]} \right\|_{H^p} \leq c_p \|f\|_{S_{d,\theta}}.
\]

(ii)
\[
\left\| (D f) \cdot \chi_{[-\varepsilon, \varepsilon]} \right\|_{(2, \infty)} \leq c_p \|f\|_{S_{d,\theta}}.
\]

(iii)
\[
\left\| (D f) \cdot \chi_{[-\varepsilon, \varepsilon]} \right\|_{[1, 0]} \leq c_p \|f\|_{S_{d,\theta}}.
\]

**Proof.** (i). Let $\alpha$ be defined as in Corollary 3.2 and let $a_0$ and $a_1$ be such that
\[
a_0(s, t) := f(s) \chi_{[0, 1]}(s), \quad a_1(s, t) := f(t) \chi_{[0, 1]}(t), \quad s, t \in \mathbb{R}.
\]
It is obvious that
\[
(D f) \cdot \chi_{[-\varepsilon, \varepsilon]} = a_0 \alpha - a_1 \alpha.
\]

By (12) and (13), we have
\[
\left\| (D f) \cdot \chi_{[-\varepsilon, \varepsilon]} \right\|_{H^p} \leq \left\| a_0 \right\|_{H^p} \|\alpha\|_{H^p} + \|a_1\|_{H^p} \|\alpha\|_{H^p}.
\]

By Remark 3.3, we have
\[
\left\| a_0 \right\|_{H^p} \leq \left\| f \chi_{(0, 1)} \right\|_{H^p} \leq \|f\|_{S_{d,\theta}}, \quad \|a_1\|_{H^p} \leq \left\| f \chi_{(0, 1)} \right\|_{H^p} \leq \|f\|_{S_{d,\theta}}.
\]
The assertion follows now from Corollary 3.2.

(ii). Let $\beta$ be defined as in Corollary 3.2 and let $b_0$ and $b_1$ be such that
\[
b_0(t, s) = \frac{f(t)}{t} \chi_{[1, \infty)}(t), \quad b_1(t, s) = \frac{1}{t} \chi_{[1, \infty)}(t), \quad s, t \in \mathbb{R}.
\]
It is obvious that
\[
(D f) \cdot \chi_{[-\varepsilon, \varepsilon]} = b_0 \beta - a_0 b_1 \beta.
\]

By (12) and (13), we have
\[
\left\| (D f) \cdot \chi_{[-\varepsilon, \varepsilon]} \right\|_{H^p} \leq \left\| b_0 \right\|_{H^p} \|\beta\|_{H^p} + \left\| a_0 \right\|_{H^p} \|b_1\|_{H^p} \|\beta\|_{H^p}.
\]
By Remark 3.3, we have
\[ \|b_0\|_{\mathfrak{M}_p} \leq \sup_{t > 1} \left| \frac{f(t)}{t} \right| \leq \|f\|_{S_{\theta,\vartheta}}, \]
\[ \|a_0\|_{\mathfrak{M}_p} \leq \|f \chi_{(0,1)}\|_{\infty} \leq \|f\|_{S_{\theta,\vartheta}}, \quad \|b_1\|_{\mathfrak{M}_p} \leq \sup_{t > 1} \frac{1}{t} = 1. \]

The assertion follows now from Corollary 3.2.

(iii) Let \( \phi : [0,2\pi] \to [0,1] \) be a smooth function supported in \([\frac{1}{4}, \pi]\) which is identically 1 on \([\frac{1}{4}, 2]\). We have
\[ (Df) \cdot \chi_{\left(\frac{1}{4}, 1\right) \times \left(\frac{1}{4}, 2\right)} = \left((\phi \otimes \phi)Df\right) \cdot \chi_{\left(\frac{1}{4}, 1\right) \times \left(\frac{1}{4}, 2\right)}, \]
Therefore,
\[ \left\| (Df) \cdot \chi_{\left(\frac{1}{4}, 1\right) \times \left(\frac{1}{4}, 2\right)} \right\|_{\mathfrak{M}_p} \leq \left\| (\phi \otimes \phi)Df \right\|_{\mathfrak{M}_p} \left\| \chi_{\left(\frac{1}{4}, 1\right) \times \left(\frac{1}{4}, 2\right)} \right\|_{\mathfrak{M}_p}. \]
Note that \( d \) depends on \( p \), while \( \phi \) is independent of \( f, \theta \) and \( p \). The assertion follows now from Lemma 4.1.

**Proposition 4.3.** There exists a constant \( c_p \) depending on \( p \) only such that
\[ \left\| (Df) \cdot \chi_{\left(\frac{1}{4}, 1\right) \times (0,\infty)} \right\|_{\mathfrak{M}_p} \leq c_p \|f\|_{S_{\theta,\vartheta}}, \]
\[ \left\| (Df) \cdot \chi_{\left(-1, -\frac{1}{4}\right) \times (-\infty, 0)} \right\|_{\mathfrak{M}_p} \leq c_p \|f\|_{S_{\theta,\vartheta}}. \]

**Proof.** The first inequality immediately follows from Lemma 4.2 (i), (ii) and (iii) and (13). The same argument in Lemma 4.2 yields the validity of the second inequality.

The following is a consequence of Remark 3.5, which is the first main result of this section.

**Theorem 4.4.** There exists a constant \( C_p \) such that
\[ \left\| (Df) \cdot \chi_{\left(\frac{1}{4}, 1\right) \times (0,\infty)} \right\|_{\mathfrak{M}_p} \leq C_p \|f\|_{S_{\theta,\vartheta}} 2^{-k(\theta-1)}, \quad k \in \mathbb{Z}. \]

**Proof.** We define the following functions:
\[ g_k(s, t) := (Df) \cdot (s, t) \chi_{\left(\frac{1}{4}, 1\right) \times [0,\infty)}(s)\chi_{(0,\infty)}(t), \quad k \in \mathbb{Z}, \ s, t \in \mathbb{R}. \]

By Proposition 4.3, we have
\[ \left\| g_0 \right\|_{\mathfrak{M}_p} \leq C_p \|f\|_{S_{\theta,\vartheta}}. \]
Applying the latter result to function \( \sigma_{2^{k}}f \) and noting that
\[ g_k = 2^k \sigma_{2^{-k}} \left((D\sigma_{2^{k}}f) \cdot \chi_{\left(\frac{1}{4}, 1\right) \times [0,\infty)}\right), \]
by Remark 3.5, we have
\[ \left\| g_k \right\|_{\mathfrak{M}_p} = 2^k \left\| (D\sigma_{2^{k}}f) \chi_{\left(\frac{1}{4}, 1\right) \times [0,\infty)} \right\|_{\mathfrak{M}_p} \leq C_p 2^k \|\sigma_{2^{k}}f\|_{S_{\theta,\vartheta}} \leq C_p 2^k \|f\|_{S_{\theta,\vartheta}}. \]

The indicator function featuring in Proposition 4.3 and Theorem 4.4 are related to measurable sets positioned in the first and third quadrants, which are needed in the next section. The following lemmas allow us to consider the case of the indicator functions of sets in the 2nd or 4th quadrant.

**Lemma 4.5.** Let \( a > 0 \). Let
\[ b_0(s, t) := \frac{|s|^\theta}{s-t} \chi_{(-\infty, -a) \times (0,\infty)}, \ s, t \in \mathbb{R}, \]
and
\[ b_1(s, t) := \frac{|t|^\theta}{s-t} \chi_{(-\infty, -a) \times (0,\infty)}, \ s, t \in \mathbb{R}. \]
There exists a constant \( C_{p, \theta} \) such that
\[ \left\| b_0 \right\|_{\mathfrak{M}_p}, \left\| b_1 \right\|_{\mathfrak{M}_p} \leq C_{p, \theta} a^{\theta-1}. \]
Proof. We only prove the assertion for $b_0$, since the case for $b_1$ is very similar. To show that $\|b_0\|_{\mathcal{M}_p} < \infty$, it is equivalent to show that the function $b$ defined by

$$b(s, t) = \frac{|s|^\theta}{s + t} \chi_{(a, \infty) \times (0, \infty)}, \ s, t \in \mathbb{R},$$

satisfies that $\|b\|_{\mathcal{M}_p} < \infty$.

Define

$$I_k := [2^k a, 2^{k+1} a], \ k \geq 0, \text{ and } I_{-1} = [0, a].$$

Assume that $k \geq l \geq -1$. By Remark 3.5, for any $k \geq 0$ and $l \neq -1$, we have

$$\left\| \left( \frac{1}{s + t} \right)_{s \in I_k, t \in I_l} \right\|_{\mathcal{M}_p} \leq \frac{1}{2^k a} \left\| \left( \frac{1}{s + t} \right)_{s \in [1, 2], t \in [2^{l-k-1}, 2^{l-k}]} \right\|_{\mathcal{M}_p}.$$ 

when $l = -1$, we have

$$\left\| \left( \frac{1}{s + t} \right)_{s \in I_k, t \in I_l} \right\|_{\mathcal{M}_p} \leq \frac{1}{2^k a} \left\| \left( \frac{1}{s + t} \right)_{s \in [1, 2], t \in [0, 2]} \right\|_{\mathcal{M}_p}.$$ 

Consider smooth functions $\phi_1$ supported in $[\frac{3}{4}, \frac{9}{4}]$ with $\phi_1(t) = 1$ on $[1, 2]$ and $\phi_2$ supported in $[-\frac{1}{4}, \frac{1}{4}]$ with $\phi_2(t) = 1$ on $[0, 2]$. We define a function $c$ by $c(s, t) := \phi_1(s) \phi_2(t) \left( \frac{1}{s + t} \right)$, which is a smooth function supported in $[\frac{1}{2}, \frac{3}{2}] \times [-\frac{1}{4}, \frac{1}{4}]$. By Lemma 3.4, we have

$$\|c\|_{\mathcal{M}_p} \leq c_p,$$

where $c_p$ depends on $p$, $\phi_1$ and $\phi_2$ only. Hence,

$$\left\| \left( \frac{1}{s + t} \right)_{s \in I_k, t \in I_l} \right\|_{\mathcal{M}_p} \leq \frac{1}{2^k a} c_p.$$ 

The case when $l \geq k \geq 0$ can be obtained by a similar argument. Therefore,

$$\left\| \left( \frac{1}{s + t} \right)_{s \in I_k, t \in I_l} \right\|_{\mathcal{M}_p} \leq \frac{1}{2 \max(k, l) a} c_p, \ k \geq 0, \ l \geq -1.$$

On the other hand, setting $c_1(s) = |s|^\theta$, by Remark 3.3, we have

$$\|c_1|_{s \in I_k, t \in I_l}\|_{\mathcal{M}_p} \leq \sup \{|s|^\theta : s \in I_k\} = 2^{(k+1)\theta} a^\theta \leq 2 \cdot 2^{\max(k, l)\theta} a^\theta.$$ 

Thus, we obtain that there exists a constant $C_p$ such that

$$\|b\|_{\mathcal{M}_p}^p \leq a^{p(\theta - 1)} \sum_{k \geq 0, l \geq -1} C_p 2^{p \theta \max(k, l)} 2^{-p \max(k, l)}$$

$$\leq a^{p(\theta - 1)} \sum_{k \geq 0, l \geq -1} C_p 2^{-p(1-\theta) \max(k, l)}$$

$$\leq a^{p(\theta - 1)} \sum_{k \geq 0} (k + 2) C_p 2^{-p(1-\theta)k} < \infty,$$

which completes the proof. \hfill \Box

Lemma 4.6. Given $a > 0$, we have

$$\|\langle \mathcal{O} \rangle \chi_{(-\infty, -a] \times [0, \infty)}\|_{\mathcal{M}_p} \leq C_p, \|f\|_{S_{a, \theta}^p} a^{\theta - 1}.$$
Proof. Let \( b_0 \) and \( b_1 \) be defined as in Lemma 4.5. Let
\[
a_0(s, t) := \frac{f(s)}{|s|^p}, \quad s \neq 0, \quad \text{and} \quad a_1(s, t) := \frac{f(t)}{|t|^p}, \quad t \neq 0.
\]
Then, we have
\[
(\mathcal{D}f) \cdot \chi_{(-\infty,-a] \times [0,\infty)} = a_0 b_0 - a_1 b_1.
\]
By Remark 3.3,
\[
\|a_0\|_{\mathfrak{m}_p}, \|a_1\|_{\mathfrak{m}_p} \leq \|f\|_{S_{0,0}}.
\]
The assertion follows from the \( p \)-th triangular inequality (13) and Lemma 4.5. \( \Box \)

5. Operator \( \theta \)-Hölder functions with respect to \( \|\cdot\|_p, p > 0 \)

In this section, without loss of generality (see Remark 5.10), we may assume that \( \mathcal{M} \) is a semifinite von Neumann algebra acting on a separable Hilbert space \( \mathcal{H} \). We study the operator \( \theta \)-Hölder functions with respect to \( \|\cdot\|_p, p > 0 \). Again, since the case when \( p = \infty \) has been thoroughly treated in [3, 7, 62], we only consider the case when \( 0 < p < \infty \). Moreover, the assertion for \( p > 1 \) is a consequence of that for \( p = 1 \) together with the Hardy–Littlewood–Polya inequality [46, Corollary 2.5] and Theorem 6.1. Therefore, unless otherwise stated, we always assume that
\[
0 < p \leq 1.
\]
For the sake of convenience, we denote \( s(X)_+ := E^X(0, \infty) \) and \( s(X)_- := E^X(-\infty, 0) \), and \( n(X) = 1 - s(X) \) for \( X \in S_h(\mathcal{M}, \tau) \) (see Section 2). Assume that \( f \in S_{d(\rho),\theta} \). Note that
\[
\begin{align*}
f(A) - f(B) &= (f(A) - f(B)) \cdot s(B)_+ + s(A)_- \cdot (f(A) - f(B)) \cdot s(B)_- \\
&\quad + n(A) \cdot (f(A) - f(B)) \cdot s(B) + s(A) \cdot (f(A) - f(B)) \cdot n(B).
\end{align*}
\]
(25)
The main result of this section is Theorem 5.9, which provides the estimates of the quasi-norm \( \|f(A) - f(B)\|_p \) in terms of \( \|A - B\|_p^\theta \). We shall provide a detailed proof of the estimates of the quasi-norms of \( s(A)_+ \cdot (f(A) - f(B)) \cdot s(B)_+ \) and \( s(A) \cdot (f(A) - f(B)) \cdot n(B) \). The proof for the other cases is exactly the same, where the proof in the case of \( s(A)_+ \cdot (f(A) - f(B)) \cdot s(B)_- \) and \( s(A) \cdot (f(A) - f(B)) \cdot s(B)_+ \) require Lemma 4.6 instead of Theorem 4.4.

For the sake of convenience, we use the following notations.

Notation 5.1. Set \( I_k := [2^{-k-1}, 2^{-k}) \) and \( J_k := (0, 2^{-k}) \), \( k \in \mathbb{Z} \). Assume that \( A \) and \( B \) are in \( S(\mathcal{M}, \tau) \). We set
\[
\begin{align*}
g_k(s, t) &:= (\mathcal{D}f)(s, t) \chi_{[2^{-k-1}, 2^{-k})}(s) \chi_{(0,\infty)}(t), \\
h_k(s, t) &:= (\mathcal{D}f)(s, t) \chi_{[2^{-k-1}, 2^{-k})}(s) \chi_{(0,\infty)}(t), \\
p_k &:= \chi_{[2^{-k-1}, 2^{-k})}(A), \\
q_k &:= \chi_{[2^{-k-1}, 2^{-k})}(B), \\
P_k &:= \chi_{(0,2^{-k})}(A), \\
Q_k &:= \chi_{(0,2^{-k})}(B), \\
V_k &:= p_k(A - B)Q_k, \\
W_k &:= P_{k+1}(A - B)q_k.
\end{align*}
\]

Lemma 5.2. Let \( A, B \in S(\mathcal{M}, \tau) \) be such that \( A - B \in L_p(\mathcal{M}, \tau) \). We have
\[
p_k(f(A) - f(B))Q_k = T^{A,B}_{g_k}(V_k), \\
P_{k+1}(f(A) - f(B))q_k = T^{A,B}_{h_k}(W_k).
\]
Proof. Note that \( V_k \in \mathcal{M} \) and \( V_k \in L_p(\mathcal{M}, \tau) \). Hence, \( V_k \in L_2(\mathcal{M}, \tau) \). Let
\[
g_{k,l} = (\mathcal{D}f) \cdot \chi_{[2^{-k-1}, 2^{-k}) \times [2^{-l-1}, 2^{-l})}.
\]
Let \( \phi_{k,l} : \mathbb{R} \to \mathbb{R} \) be a compactly supported smooth function which vanishes near 0 and such that \( \phi_{k,l} = 1 \) on \( [2^{-k-1}, 2^{-k}) \) and on \( [2^{-l-1}, 2^{-l}) \). Let \( h := f_{\phi_{k,l}} \). We have
\[
g_{k,l} = (\mathcal{D}h) \cdot \chi_{[2^{-k-1}, 2^{-k}) \times [2^{-l-1}, 2^{-l})}.
\]
Therefore, by Lemma 3.6, we have
\[ T_{g_k}^{A,B}(V_k q_l) = T_{g_k}^{A,B}(p_k(A - B)q_l) = (T_{g_k}^{A,B})(\chi_{[2^{-k-1},2^{-k-2}]}(p_k(A - B))q_l) = T_{\mathcal{H}}^{A,B}(p_k(A - B))q_l = p_k(h(A) - h(B))q_l = p_k(f(A) - f(B))q_l. \]

Since \( V_k \in L_2(M, \tau) \), it follows that
\[ V_k = \sum_{l \geq k} V_k q_l, \]
where the series converges in \( L_2 \)-topology (see e.g. [33, Theorem 3.1]). Since \( T_{g_k}^{A,B} \) is bounded on \( L_2(M, \tau) \), it follows that
\[ T_{g_k}^{A,B}(V_k) = \sum_{l \geq k} T_{g_k}^{A,B}(V_k q_l) = \sum_{l \geq k} T_{g_k}^{A,B} T_{A,B}^{A,B}(V_k q_l) = \sum_{l \geq k} T_{g_k}^{A,B}(V_k q_l), \]
where the series converges in \( L_2 \)-topology. By the preceding paragraph,
\[ T_{g_k}^{A,B}(V_k) = \sum_{l \geq k} p_k(f(A) - f(B))q_l, \]
where the series again converges in \( L_2 \)-topology and therefore in local measure topology. Moreover,
\[ \sum_{l \geq k} p_k(f(A) - f(B))q_l = p_k(f(A) - f(B))Q_k, \]
where the series converges locally in measure. The assertion follows by comparison of these 2 equalities.

The following lemma allows us to decompose the difference \( f(A) - f(B) \). For brevity, we say that a series \( \sum_{k \in \mathbb{Z}} X_k, X_k \in S(M, \tau) \), converges in local measure topology in the sense of principal value if \( \sum_{k = -n}^n X_k \) converges in local measure topology. We denote such convergence by \( (p.v.) = \sum_{k \in \mathbb{Z}} X_k \).

**Lemma 5.3.** We have
\[ s(A)_+ \cdot (f(A) - f(B)) \cdot s(B)_+ = (p.v.) - \sum_{k \in \mathbb{Z}} \left( T_{g_k}^{A,B}(V_k) + T_{h_k}^{A,B}(W_k) \right). \]

**Proof.** Observe that
\[ \sum_{k = -n}^n (p_k Q_k + P_{k+1} q_k) = \sum_{k = -n}^n p_k(Q_n - q_l) + \sum_{l = -n}^{k-1} q_l + \sum_{k = -n}^n (P_{n+1} + \sum_{l = k+1}^n p_l)q_k \]
\[ = \sum_{k = -n}^n p_k Q_n - \sum_{-n \leq l < k \leq n} p_k q_l + \sum_{k = -n}^n P_{n+1} q_k + \sum_{-n \leq k < l \leq n} p_l q_k \]
\[ = \sum_{k = -n}^n p_k Q_n + P_{n+1} \sum_{k = -n}^n q_k \]
\[ = \chi_{[2^{-n-1},2^n)}(A)\chi(0,2^n)(B) + \chi(0,2^{-n-1})(A)\chi_{[2^{-n-1},2^n)}(B). \]

Noting that \( \chi_{[2^{-n-1},2^n)}(A) \to_n s(A)_+, \chi(0,2^n)(B) \to_n s(B)_+ \), \( \chi(0,2^{-n-1}) (A) \to_n 0 \) and \( \chi_{[2^{-n-1},2^n)}(B) \to_n 0 \) in local measure topology (see e.g. [31] and [34, Chapter II, Section 7]), we obtain that \( \sum_{k = -n}^n (p_k Q_k + P_{k+1} q_k) \to_n s(A)_+ \cdot s(B)_+ \) in the local measure topology. We write
\[ s(A)_+ s(B)_+ = (p.v.) - \sum_{k \in \mathbb{Z}} (p_k Q_k + P_{k+1} q_k). \]
Therefore,
\[
\begin{align*}
    s(A) f(A) s(B) &= f(A) \cdot s(A) s(B) = f(A) \cdot (p.v.) - \sum_{k \in \mathbb{Z}} \left( p_k Q_k + P_{k+1} q_k \right) \\
    &= (p.v.) - \sum_{k \in \mathbb{Z}} \left( p_k f(A) Q_k + P_{k+1} f(A) q_k \right)
\end{align*}
\]
and
\[
\begin{align*}
    s(A) f(B) s(B) &= s(A) s(B) \cdot f(B) = \left( (p.v.) - \sum_{k \in \mathbb{Z}} p_k Q_k + P_{k+1} q_k \right) \cdot f(B) \\
    &= (p.v.) - \sum_{k \in \mathbb{Z}} \left( p_k f(B) Q_k + P_{k+1} f(B) q_k \right)
\end{align*}
\]
It follows that
\[
\begin{align*}
    s(A) (f(A) - f(B)) s(B) &= (p.v.) - \sum_{k \in \mathbb{Z}} \left( p_k (f(A) - f(B)) Q_k + P_{k+1} (f(A) - f(B)) q_k \right).
\end{align*}
\]
It remains to note that
\[
p_k (f(A) - f(B)) Q_k \overset{\text{(26)}}{=} T_{A,B}^{A,B}(V_k), \quad P_{k+1} (f(A) - f(B)) q_k \overset{\text{(26)}}{=} T_{-A,B}^{A,B}(W_k).
\]
An alternative proof for the following special case in the setting of (discrete) finite von Neumann algebras was sketched in [75, case 1 of Theorem 3.2] by the complex interpolation theorem for $L_p$, $0 < p < 1$ (see e.g. [86] and [66, Lemma 2.5], see also [72, Section 3.2]). We present a straightforward and complete proof for $S(M, \tau)$ below.

**Lemma 5.4.** We have
\[
\| s(A) (f(A) - f(B)) n(B) \|_p \leq \| f \|_{S_{d,t}} \| A - B \|^\theta \|_p.
\]
and
\[
\| n(A) (f(A) - f(B)) s(B) \|_p \leq \| f \|_{S_{d,t}} \| A - B \|^\theta \|_p.
\]

**Proof.** We only prove the first inequality. The proof for the second inequality is exactly the same. Note that
\[
s(A) (f(A) - f(B)) n(B) = s(A) f(A) n(B) - s(A) f(B) n(B) = f(A) n(B).
\]
By the Spectral Theorem, we have
\[
|f(A) n(B)|^2 = n(B) |f(A)|^2 n(B) \leq n(B) \left( \sup_{t \neq 0} \frac{|f(t)|}{|t|^{\theta}} \right)^2 |A|^{2\theta} n(B)
\]
\[
= \sup_{t \neq 0} \frac{|f(t)|}{|t|^{\theta}} \| A \|^{\theta} n(B)^2.
\]
By the operator-monotonicity (see e.g. [25, Proposition 1.2]) of function $t \mapsto \sqrt{t}$, $t \in \mathbb{R}^+$, that
\[
\| f(A) n(B) \|_p \leq \sup_{t \neq 0} \frac{|f(t)|}{|t|^{\theta}} \| A \|^{\theta} n(B)^2.
\]
The Araki-Lieb-Thirring inequality (see [42, Lemma 2.5], see also [55]) states that
\[
h(X^\theta Z^\theta X^\theta) \ll h((X Z^2 X)^\theta), \quad 0 \leq X, Z \in S(M, \tau),
\]
where $h : [0, \infty) \to [0, \infty)$ is a continuous increasing function such that $t \mapsto h(e^t)$ is convex and $h(0) = 0$. Setting $h(t) := t^{p/2}$, $t \geq 0$, we infer
\[
|Z^\theta X^\theta|^p \ll |X^\theta Z^2 X^\theta|^p \ll |X Z^2 X^\theta|^p = |Z X|^p.
\]
Setting $Z := |A|$ and $X := n(B)$, we obtain that, we obtain that
\[
(\mu(|A|^\theta n(B)))^p \ll (\mu(A n(B)))^\theta.
\]
Hence, \( \| A^n(B) \|_p \leq \| A(B)^n \|_p \), which together with (27) and the fact that \( Bn(B) = 0 \) implies that
\[
\| f(A)n(B) \|_p \leq \sup_{t \neq 0} \frac{|f(t)|}{|t|^\theta} \cdot \| An(B)^\theta \|_p = \sup_{t \neq 0} \frac{|f(t)|}{|t|^\theta} \cdot \| (A - B)n(B)^\theta \|_p \leq \sup_{t \neq 0} \frac{|f(t)|}{|t|^\theta} \cdot \| A - B \|_p.
\]

Before proceeding to the main result, we prove the following scalar inequality.

**Proposition 5.5.** For every \( 0 < \theta < 1 \) and \( 0 < q < \infty \), there exists a \( c_{\theta,q} > 0 \) such that
\[
\sum_{l \in \mathbb{Z}} 2^{q(l-1)\theta} \cdot \min\{\alpha, 2^{1-l}\theta\} \leq c_{\theta,q} \alpha^q, \quad \alpha \in \mathbb{R}^+.
\]

**Proof.** Let \( k \) be an integer be such that \( \alpha \in (2^{-k}, 2^{1-k}] \). If \( \alpha \leq 2^{1-k} \), then \( 2^{k(1-\theta)} \leq (2\alpha^{-1})^{1-\theta} \).

Noting that
\[
\sum_{l \leq k} 2^{q(l-1)\theta} = 2^{k(l-1)\theta} \frac{1}{1 - 2^{(\theta-1)q}},
\]
we have
\[
\sum_{l \leq k} 2^{q(l-1)\theta} \min\{\alpha, 2^{1-l}\theta\} \leq \sum_{l \leq k} 2^{q(l-1)\theta} \alpha^q = 2^{k(l-1)\theta} \frac{\alpha^q}{1 - 2^{(\theta-1)q}} \leq (2\alpha^{-1})^{1-\theta} \frac{\alpha^q}{1 - 2^{(\theta-1)q}} = 2^{q(1-\theta)} \frac{\alpha^q}{1 - 2^{(\theta-1)q}} \alpha^q.
\]

On the other hand, since \( \alpha > 2^{-k} \), it follows that
\[
\sum_{l > k} 2^{q(l-1)\theta} \min\{\alpha, 2^{1-l}\theta\} \leq \sum_{l > k} 2^{q(l-1)\theta} 2^{q(1-l)} = \sum_{l > k} 2^{q(l-1)\theta}
\]
\[
= 2^q \cdot 2^{-qk\theta} \sum_{l > 0} 2^{-ql\theta} = 2^{q(1-\theta)} \frac{1}{1 - 2^{q\theta}} 2^{-qk\theta} \leq 2^{q(1-\theta)} \frac{1}{1 - 2^{q\theta}} \alpha^q,
\]
which together with (29) implies (28). \( \square \)

The proof of the next lemma requires a combination of Proposition 5.5, Lemma 5.3 and Theorem 4.4. This is the final intermediate step before our main result.

**Lemma 5.6.** Let \( A, B \in S(M, \tau) \) be such that \( A - B = x \varepsilon \) for some \( \tau \)-finite projection \( \varepsilon \) and a real number \( x \). There exists a constant \( c_{p,q} \) such that
\[
\| s(A)_+ \cdot (f(A) - f(B)) \cdot s(B)_+ \|_p \leq c_{p,q} \| f \|_{S_{d,\theta}} \| A - B \|_p,
\]
\[
\| s(A)_- \cdot (f(A) - f(B)) \cdot s(B)_+ \|_p \leq c_{p,q} \| f \|_{S_{d,\theta}} \| A - B \|_p,
\]
\[
\| s(A)_+ \cdot (f(A) - f(B)) \cdot s(B)_- \|_p \leq c_{p,q} \| f \|_{S_{d,\theta}} \| A - B \|_p,
\]
\[
\| s(A)_- \cdot (f(A) - f(B)) \cdot s(B)_- \|_p \leq c_{p,q} \| f \|_{S_{d,\theta}} \| A - B \|_p.
\]

**Proof.** We only prove the first inequality. The proof of the rest is exactly the same.

It follows from Lemma 5.3 that (see Notation 5.1)
\[
(30) \quad s(A)_+ \cdot (f(A) - f(B)) \cdot s(B)_+ = (p.v.) - \sum_{k \in \mathbb{Z}} \left( T_{gk}^{A,B}(V_k) + T_{hk}^{A,B}(W_k) \right).
\]

Theorem 4.4 implies that there exists a constant \( c_{p,q} \) such that for every \( k \in \mathbb{Z}, \)
\[
\| T_{gk}^{A,B}(V_k) \|_p \leq c_{p,q} \cdot 2^{k(1-\theta)} \| f \|_{S_{d,\theta}} \| V_k \|_p.
\]

Note that \( \| V_k \|_\infty \leq \| p_k A \|_\infty + \| BQ_k \|_\infty \leq 2^{1-k} \). Hence,
\[
\| V_k \|_p \leq \| A - B \|_p = \| A - B \|_\infty \tau(\varepsilon)^{\frac{1}{\theta}}
\]
and
\[ \|V_k\|_p \leq \|V_k\|_\infty \tau(\text{supp}(V_k))^{\frac{1}{p}} \leq \|V_k\|_\infty \tau(e)^{\frac{1}{p}} = 2^{1-k} \tau(e)^{\frac{1}{p}}. \]

There exists a constant \( C_{p,\theta} \) such that
\[
(p.v.) - \sum_{k \in \mathbb{Z}} \|T_{\theta k}^{A,B}(V_k)\|_p^p \\
\leq (p.v.) - \sum_{k \in \mathbb{Z}} \delta_{p,\theta}^p \cdot 2^{pk(1-\theta)} \|f\|_{S_{d,\theta}}^p \min\{\|A - B\|_\infty, 2^{1-k}\} \tau(e) \\
\leq C_{p,\theta} \|f\|_{S_{d,\theta}}^p \|A - B\|_{S_{d,\theta}}^{\theta} \tau(e) = C_{p,\theta} \|f\|_{S_{d,\theta}}^p \|A - B\|_{p}^{\theta}.
\]

The \( p \)-triangle inequality implies that \( (p.v.) - \sum_{k \in \mathbb{Z}} T_{\theta k}^{A,B}(V_k) \) converges in quasi-norm topology (similar for \( (p.v.) - \sum_{h \in \mathbb{Z}} T_{\theta h}^{A,B}(W_k) \)). A fortiori these series converges in the local measure topology [81]. Hence, the series in (30) converges to \( s(A) \cdot (f(A) - f(B)) \cdot s(B) \) in the \( L_p \) topology. Therefore, there exists a constant \( C_{p,\theta} \) such that
\[
\|s(A) \cdot (f(A) - f(B)) \cdot s(B)\|_p \leq C_{p,\theta} \|f\|_{S_{d,\theta}} \|A - B\|_{p}^{\theta},
\]
which completes the proof. \( \square \)

**Corollary 5.7.** Let \( A, B \in S(M, \tau) \) be bounded and such that \( A - B = xe \) for some \( \tau \)-finite projection \( e \) and a real number \( x \). There exists a constant \( C_{p,\theta} \) such that
\[
\|f(A) - f(B)\|_p \leq C_{p,\theta} \|f\|_{S_{d,\theta}} \|A - B\|_{p}^{\theta}.
\]

**Proof.** Recall that \( f(A) - f(B) \) can be rewritten as in the form of (25). The assertion follows from Lemmas 5.6 and 5.4. \( \square \)

**Lemma 5.8.** Let \( A, B \in S_h(M, \tau) \), with \( A - B = \sum_{k=1}^{n} x_k e_k \), where \( 1 \leq n < \infty \), \( x_k \) are real numbers and \( e_k \) are mutually orthogonal projections. Then, there exists a constant \( C_{p,\theta} \) such that
\[
\|f(A) - f(B)\|_p \leq C_{p,\theta} \|f\|_{S_{d,\theta}} \|A - B\|_{p}^{\theta}.
\]

**Proof.** Set \( A_0 := B \) and
\[ A_m := B + \sum_{k=1}^{m} x_k e_k, \quad 1 \leq m \leq n. \]

We have
\[
f(A_n) - f(B) = \sum_{m=0}^{n-1} f(A_{m+1}) - f(A_m).
\]

Hence, we have
\[
\|f(A_n) - f(B)\|_p^p \leq \sum_{m=0}^{n-1} \|f(A_{m+1}) - f(A_m)\|_p^p \leq C_{p,\theta} \|f\|_{S_{d,\theta}}^p \sum_{m=0}^{n-1} \|A_{m+1} - A_m\|_{p}^\theta \\
= C_{p,\theta} \|f\|_{S_{d,\theta}}^p \|A - B\|_{p}^\theta,
\]
which completes the proof. \( \square \)

The latter lemma allows us to extend the result to the case when the difference \( A - B \) (possibly unbounded) belongs to the noncommutative \( L_p \)-space, \( p > 0 \). A similar proof of the special case of \( t \mapsto t^\theta \) was given in [75, Theorem 3.4]. We present a straightforward proof below.

**Theorem 5.9.** Let \( 0 < p \leq \infty \) and \( 0 < \theta < 1 \). There exists \( C_{p,\theta} \) such that for any semi-finite von Neumann algebra \( (M, \tau) \), and \( A, B \in S(M, \tau)_h \) such that \( A - B \in L_{\theta p}(M, \tau) \) and any \( f \in S_{d,\theta} \), then
\[
\|f(A) - f(B)\|_p \leq C_{p,\theta} \|f\|_{S_{d,\theta}} \|A - B\|_{p}^{\theta}.
\]

In particular, \( f \in S_{\infty,\theta} \) is operator \( \theta \)-Hölder function with respect to all \( \|\cdot\|_p \), \( p > 0 \).
Proof. We first assume that $0 < p \leq 1$. Let $B - A = U|B - A|$ be the polar decomposition. We define a sequence $\{K_n := U \sum_{k=1}^{n^2} \frac{k-1}{n} E^{[B-A]}(\frac{k-1}{n}, \frac{k}{n})\}$, which converges to $B - A$ in measure topology. In particular, $A + K_n \overset{L^2}{\to} A + B = A$ as $n \to \infty$. By the continuity of functional calculus in $S(M, \tau)_h$, we have

$$f(A + K_n) \overset{L^2}{\to} f(B).$$

Applying Lemma 5.8 to $A$ and $A + K_n$ and observing that $|K_n| \leq |B - A|$, we obtain that

$$\|f(A) - f(A + K_n)\|_p \leq C_{p, \theta} \|f\|_{S_{d, \theta}} \|K_n\|_p \leq C_{p, \theta} \|f\|_{S_{d, \theta}} \|A - B\|_p.$$  

Fatou’s lemma (see e.g. [28, Proposition 3.3] or [40, Lemma 3.4]) implies that

$$\liminf \|f(A) - f(B)\|_p \leq \liminf \|f(A) - f(A + K_n)\|_p \leq C_{p, \theta} \|f\|_{S_{d, \theta}} \|A - B\|_p.$$  

The case when $p > 1$ is a consequence of the case for $p = 1$ together with Theorem 6.1 and the Hardy–Littlewood–Polya inequality [46, Corollary 2.5].

Remark 5.10. Although the results in the present section are proved under the assumption that the von Neumann algebra $M$ acts on a separable Hilbert space, the above result indeed also holds for non-separable Hilbert spaces.

Sketch of proof. Assume that $M$ is a general semifinite von Neumann algebra. We first consider self-adjoint operators $X, Y \in L_{\theta p}(M, \tau)$. There exist sequences $\{X_n := UX \sum_{k=1}^{n^2} \frac{k-1}{n} E^{[X]}(\frac{k-1}{n}, \frac{k}{n})\}$ converging to $X$ and $\{Y_n := UY \sum_{k=1}^{n^2} \frac{k-1}{n} E^{[Y]}(\frac{k-1}{n}, \frac{k}{n})\}$ converging to $Y$, where $UX$ and $UY$ are partial isometries such that $X = UX|X|$ and $Y = UY|Y|$. Consider the standard representation of $M$ on $L_2(M, \tau)$. For every $n$,

$$\left\{ E^{[X]}(\frac{1}{n}, \frac{2}{n}), \ldots, E^{[X]}(\frac{n^2-1}{n}, \frac{n^2}{n}), E^{[Y]}(\frac{1}{n}, \frac{2}{n}), \ldots, E^{[Y]}(\frac{n^2-1}{n}, \frac{n^2}{n}) \right\}$$

is a finite subset of $L_2(M, \tau)$, which generates a separable Hilbert subspace of $L_2(M, \tau)$. Hence,

$$\|f(X_n) - f(Y_n)\|_p \leq C_{p, \theta} \|f\|_{S_{d, \theta}} \|X_n - Y_n\|_p.$$  

By the continuity of functional calculus and the Fatou Lemma, we obtain that

$$\|f(X) - f(Y)\|_p \leq C_{p, \theta} \|f\|_{S_{d, \theta}} \|X - Y\|_p.$$  

The proof of [75, Theorem 3.4] indeed allows us to extend the result to the general case when $X, Y \in S(M, \tau)$ with $X - Y \in L_{\theta p}(M, \tau)$ and $M$ acts on a non-separable Hilbert space. □

6. OPERATOR $\theta$-HÖLDER FUNCTIONS FOR $p$-TH POWER OF A SYMMETRIC SPACE

Let $(M, \tau)$ be a semifinite von Neumann algebra represented on a Hilbert space. In the following theorem, we obtain a submajorization inequality related to $\theta$-Hölder functions, which is the key tool in proving Corollary 1.3. Indeed, the following theorem holds under very general assumptions.

Theorem 6.1. Fix $p \in (0, \infty)$ and let $g : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous increasing function. Suppose that $f : \mathbb{R} \to \mathbb{C}$ is a continuous function such that

(32)  $$\|f(X) - f(Y)\|_p \leq C_{p, f, g} \|g(|X - Y|)\|_p, \forall X, Y \in S_h(M, \tau), \ g(|X - Y|) \in L_p(M, \tau)$$

and such that

(33)  $$\|f(X) - f(Y)\|_\infty \leq C_{\infty, f, g} \|g(|X - Y|)\|_\infty, \forall X, Y \in S_h(M, \tau), \ g(|X - Y|) \in M$$

for some constants $C_{p, f, g}$ and $C_{\infty, f, g}$ depending on $g$, $p$ and $f$ only. Then, there exists a constant $C'_{p, f, g}$ depending on $f, g$ and $p$, such that for any $X, Y \in S_h(M, \tau)$, we have

$$\mu(f(X) - f(Y))^p \leq C'_{p, f, g} g(|X - Y|)^p.$$  

1See [83] or the comments below [31, Proposition 2]. See also the forthcoming book [34, Chapter II, Theorem 8.7].
Proof. Without loss of generality, we may assume that \( \mathcal{M} \) is atomless (see e.g. [58, Lemma 2.3.18]). We first consider the case when \( X - Y \) is \( \tau \)-compact.

For every \( t > 0 \), we can find a projection \( e_t \in \mathcal{M} \) such that \( \mu(s; X - Y) = \mu(s; |X - Y| e_t) \) for all \( 0 \leq s < t \) with \( \tau(e_t) = t \) and \( \|(X - Y)e_t^\perp\|_\infty \leq \mu(t; X - Y) \) (see [34, Chapter III, Lemma 7.7] or [27, p.953]). Moreover, \( e_t \) can be chosen to commute with \( X - Y \). By Definition 2.1, we have
\[
\|(X - Y)e_t^\perp\|_\infty \geq \mu(t; X - Y).
\]
That is, \( \|(X - Y)e_t^\perp\|_\infty = \mu(t; X - Y) \). Let
\[
X_t := Y + (X - Y)e_t.
\]

Letting \( c_p := \max\{1, 2^{p-1}\} \), by [58, Corollary 2.3.16] and the fact that \( (a + b)^p \leq c_p(a^p + b^p) \) for any \( a, b \geq 0 \) (see [48, (2.2) and (2.3)]), we have
\[
\mu(f(X) - f(Y)) \leq \mu(f(X_t) - f(Y)) + \|f(X) - f(X_t)\|_\infty, \forall t > 0,
\]
and therefore,
\[
\int_0^t \mu(s; f(X) - f(Y))^p \, ds \leq \int_0^t \left( \mu(s; f(X_t) - f(Y)) + \|f(X) - f(X_t)\|_\infty \right)^p \, ds
\]
\[
\leq c_p \int_0^t \mu(s; f(X_t) - f(Y))^p \, ds + c_p \int_0^t \|f(X) - f(X_t)\|_\infty^p \, ds
\]
\[
= c_p \int_0^t \mu(s; f(X_t) - f(Y))^p \, ds + tc_p \|f(X) - f(X_t)\|_\infty^p
\]
\[
\leq c_p \|f(X_t) - f(Y)\|_p^p + tc_p \|f(X) - f(X_t)\|_\infty^p.
\]

(34)

Note that since \( g \) is assumed to be monotone, we have \( \mu(\lfloor (X - Y)e_t \rfloor) = g(\mu((X - Y)e_t)) = g(\mu(|X - Y|) \chi_{(0,1)}) \) and \( \|g(\lfloor (X - Y)e_t \rfloor)^\#:\|_\infty = g(\mu(t; X - Y)) \) [58, Corollary 2.3.17]. Now, appealing to the hypothesis, we obtain that
\[
\int_0^t \mu(s; f(X) - f(Y))^p \, ds
\]
\[
\leq \int_0^t \mu(s; f(X) - f(Y))^p \, ds
\]
\[
\leq c_p C_{p,f,g}^p \|g(\lfloor X_t - Y \rfloor)\|_p^p + c_p C_{\infty,f,g}^p \|g(\lfloor X_t - X \rfloor)\|_\infty^p
\]
\[
= c_p C_{p,f,g}^p \|g(\lfloor X - Y \rfloor)\|_p^p + c_p C_{\infty,f,g}^p \|g(\lfloor X - Y \rfloor)\|_\infty^p
\]
\[
\leq c_p (C_{p,f,g}^p + C_{\infty,f,g}^p) \int_0^t \mu(s; g(|X - Y|))^p \, ds,
\]

which completes the proof for the case when \( X - Y \) is a \( \tau \)-compact operator. For simplicity, we denote \( C_1 := c_p(C_{p,f}^p + C_{\infty,f}^p) \).

Now, assume that \( X - Y \) is not necessarily \( \tau \)-compact. Let
\[
Z := (X - Y - \mu(\infty; X - Y))^+ - (X - Y + \mu(\infty; X - Y))^-.
\]

It is easy to see that \( Z \) is \( \tau \)-compact (see e.g. [58, Corollary 2.3.17 (d)]). By the Spectral Theorem, we have
\[
\|X - Y - Z\|_\infty \leq \mu(\infty; X - Y),
\]
and
\[
\mu(f(X) - f(Y)) \leq \mu(f(Y + Z) - f(Y)) + \|f(X) - f(Y + Z)\|_\infty.
\]
Hence, by the result for \( \tau \)-compact operators, there exists a constant \( C'_{p,f,g} \) such that
\[
\int_0^t \mu(s; f(X) - f(Y))^p ds \\
\leq c_p \int_0^t \mu(s; f(Y + Z) - f(Y))^p ds + c_p \int_0^t \|f(X) - f(Y + Z)\|_{\infty}^p ds \\
\leq c_p C_1 \int_0^t \mu(s; g(|Z|))^p ds + C_{\infty,f,g} \cdot c_p \cdot t \|g(|X - Y - Z|)\|_{\infty}^p \\
\leq c_p C_1 \int_0^t \mu(s; g(|X - Y|))^p ds + C_{\infty,f,g} \cdot c_p \cdot t \mu(\infty; g(|X - Y|))^p \\
\leq C'_{p,f,g} \int_0^t \mu(s; g(|X - Y|))^p ds,
\]
which completes the proof.

\[\square\]

**Remark 6.2.** By the above theorem, numerous results concerning operator inequalities in the setting of noncommutative \( L_p \)-spaces, \( p \geq 1 \) (see e.g. [2, 21]), can be extended to the case of fully symmetric spaces.

The main object of this section is the so-called \( p \)-th power of symmetric spaces, which play an important role in analysis (see e.g. [18, 63, 82]). Following the notation introduced in [87] (see also [27, 63]), for \( 0 < p < \infty \) and a quasi-Banach symmetric space \( E(M, \tau) \), the \( \frac{1}{p} \)-th power of \( E(M, \tau) \) is defined by
\[
E(M, \tau)^{(p)} = \{ X \in S(M, \tau) : |X|^p \in E(M, \tau), \quad \|X\|_{E(p)} = \|X^p\|_{E}^{1/p} \}.
\]
It is known (see e.g. [32, Proposition 3.1]) that \( E^{(p)}(M, \tau) = E(M, \tau)^{(p)} \), where \( E^{(p)}(M, \tau) \) is the quasi-Banach symmetric space corresponding to the \( \frac{1}{p} \)-th power \( E(0, \infty)^{(p)} \) of the quasi-Banach symmetric function space \( E(0, \infty) \). If \( E(0, \infty) \) is a symmetric space, then \( E^{(p)}(M, \tau) \) is \( p \)-convex (see [32, Proposition 3.1]), that is, there exists a constant \( M \) such that for any finite sequence \( \{X_k\}_{k=1}^n \subset E^{(p)}(M, \tau) \), we have
\[
\left( \sum_{k=1}^n |X_k|^p \right)^{1/p} \leq M \left( \sum_{k=1}^n \|X_k\|_{E(p)}^p \right)^{1/p}.
\]
If \( E(0, \infty) \) is a fully symmetric space and \( 0 < p < \infty \), then it is clear that for every \( X \in E^{(p)}(M, \tau) \) and \( Y \in S(M, \tau) \) with \( \mu(Y)^p \preccurlyeq \mu(X)^p \), we have \( Y \in E^{(p)}(M, \tau) \) with \( \|Y\|_{E(p)} \leq \|X\|_{E(p)} \). We note that \( \|\cdot\|_{E(p)} \) is a \( p \)-norm when \( 0 < p \leq 1 \) (see e.g. [35, Corollary 5.4] or [49, Theorem 8.10]). Most quasi-Banach symmetric spaces which occur in the literature (such as \( L_p \)-spaces, \( L_{p,q} \)-spaces, etc.) can be constructed as the \( 1/p \)-th power of a fully symmetric space.

Taking \( g(s) = s^q \), \( s \in \mathbb{R}^+, \theta < 1 \) and \( f \in S_{d, \theta(p)}, \ 0 < p < \infty \). Conditions (32) and (33) in Theorem 6.1 are satisfied (see Theorem 5.9 and [3, Theorem 4.1], respectively). By Theorem 5.9, we obtain (7), i.e., for any \( X = X^*, Y = Y^* \in S(M, \tau) \), we have
\[
(\mu(f(X) - f(Y))^p \preccurlyeq C_{p,\theta} \|f\|_{S_{d, \theta}} \cdot \mu(|X - Y|^\theta)^p).
\]
The following theorem is an immediate consequence of this inequality.

**Theorem 6.3.** For any fully symmetrically normed space \( E(0, \infty), \ 0 < p < \infty, \ 0 < \theta < 1 \) and \( f \in S_{d, \theta}, \) where \( d = d(p), \) there exists a constant \( C_{p,\theta} \) such that for any \( X, Y \in S(M, \tau)_h \) with \( X - Y \in E^{(p)}(M, \tau) \), we have
\[
\|f(X) - f(Y)\|_{E(p)} \leq C_{p,\theta} \|f\|_{S_{d, \theta}} \|X - Y\|_h^p \|X - Y\|^p_{E(p)}.
\]
Now, we consider two classic operator-monotone functions \( t \mapsto \log(|t| + 1) \in S_{\infty, \theta} \) and \( t \mapsto \frac{|t|}{r+|t|} \in S_{\infty, \theta}, \) where \( r > 0, \) obtaining analogue of [8, (ii) of Corollary 2] (see also [53, Theorem 3.4] for similar estimates).
Corollary 6.4. Assume that $E(0, \infty)$ is an arbitrary fully symmetrically normed space and let 
$f(t) := \log(|t| + 1)$ (or $f(t) = \text{sgn}(t) \log(|t| + 1)$), \( \frac{t}{r + |t|}, \frac{r}{r + |t|}, r > 0 \), $t \in \mathbb{R}$. Then, for any 
$0 < p < \infty$ and $0 < \theta < 1$, there exists a constant $C_{p, \theta}$ such that 
$$
\| f(X) - f(Y) \|_{E(p)} \leq C_{p, \theta} \| f \|_{S_{d, \theta}} \| |X - Y|^\theta \|_{E(p)}, \ X, Y \in S(M, \tau)_h.
$$

For invertible functions, we have the following result (similar results have been obtained in [8] and [10]).

Corollary 6.5. Let $\theta \in (1, \infty)$ and $p \in (0, \infty)$. If $f : \mathbb{R} \to \mathbb{R} \in S_{d, 1/\theta}$ is invertible, then there 
exists a constant $C_{p, \theta}$ such that 
$$
C_{p, \theta} \| f \|_{S_{d, 1/\theta}} \| f^{-1}(X) - f^{-1}(Y) \|_{p/\theta} \gg \| X - Y \|, \ X, Y \in S(M, \tau)_h.
$$

In particular, assuming $E(0, \infty)$ is a fully symmetrically normed space, we have
$$
\| f^{-1}(X) - f^{-1}(Y) \|_{E(p)} \geq \| (X - Y)^\theta \|_{E(p)}.
$$

Proof. By (7), we have 
$$
(\mu(f(X) - f(Y)))^p \ll C_{p, \theta} \| f \|_{S_{d, 1/\theta}} (\mu(X - Y))^{p/\theta}.
$$
Substituting $X$ and $Y$ with $f^{-1}(X)$ and $f^{-1}(Y)$, we obtain that 
$$
C_{p, \theta} \| f \|_{S_{d, 1/\theta}} \mu(f^{-1}(X) - f^{-1}(Y))^{p/\theta} \gg \| X - Y \|^p.
$$

Note that $\theta > 1$. Inequality (38) follows from the Hardy–Littlewood–Pólya inequality [46, Corollary 2.5].

In particular, we obtain the following reverse inequality of [75], which extends [8, Corollary 4] and [12, Corollary 3 and Corollary 4]. Note that the following corollary holds even for operators in $S(M, \tau)_h$ rather than positive operators as in [8] (see similar results in [10, Corollaries 1 and 2]). We note that the “sgn” below can not be omitted (consider the case when $X = -Y$).

Corollary 6.6. For any $\theta \in (1, \infty)$, $p \in (0, \infty)$ and fully symmetrically normed space $E(0, \infty)$, there 
exists a constant $C_{p, \theta}$ depending on $p$ and $\theta$ only such that 
$$
\| \text{sgn}(X) |X|^\theta - \text{sgn}(Y) |Y|^\theta \|_{E(p)} \geq C_{p, \theta} \| X - Y \|_{E(p)}, \ X, Y \in S_h(M, \tau),
$$
and 
$$
\| \text{sgn}(X) \left( e^{X} - 1 \right) - \text{sgn}(Y) \left( e^{Y} - 1 \right) \|_{E(p)} \geq C_{p, \theta} \| X - Y \|^\theta_{E(p)}, \ X, Y \in S_h(M, \tau).
$$

Proof. Appealing to Corollary 6.5 with $g(t) = \text{sgn}(t)|t|^\theta$ and $g(t) = \text{sgn}(t) \left( e^{t} - 1 \right)$.

7. Applications

7.1. Commutator and quasi-commutator estimates. We consider commutator and quasi-commutator estimates for 
operator $\theta$-Hölder functions, which complement [2, Theorem 11.7] and [3, Theorem 10.5]. The proof of the following corollary is essentially the same as the implication in [74, Lemma 2.4] via Cayley transform (for similar results for Lipschitz estimates, see [29, Theorem 2.2], [20, Theorem 6.1] and [3, Theorem 10.1]).

Corollary 7.1. Let $0 < p < \infty$ and $0 < \theta < 1$. Let $f \in S_{d, \theta}$. Let $E(0, \infty)$ be a fully symmetrically 
normed space. Then, there exists a constant $C_{p, \theta}$ such that for $X \in S(M, \tau)_h$ and $B \in M$, we have 
$$
\| [f(X), B] \|_{E(p)} \leq C_{p, \theta} \| f \|_{S_{d, \theta}} \| |X - B|^\theta \|_{E(p)} \| B \|^1_{\infty} - \theta.
$$

Proof. By homogeneity, it suffices to prove the case when $\| B \|_{\infty} = 1$. Let $K_q := \max \{ 2^{\frac{1}{q}} - 1, 1 \}$, 
which is the modulus of concavity of the quasi-norm $\| \|_{E(q)}$, $q > 0$ [47, page 8]. We denote by 
$\text{re}(B)$ and $\text{im}(B)$ the real part and the imaginary part of $B$, respectively. Note 
$$
\| [f(X), B] \|_{E(p)} \leq K_p (\| [f(X), \text{re}(B)] \|_{E(p)} + \| [f(X), \text{im}(B)] \|_{E(p)})
$$
and
\[
\left\| [X, \text{re}(B)] \right\|_{E(p)}^{\theta} \leq \left( \frac{1}{2} K_{\theta p} \| [X, B^*] \|_{E(\theta p)} + \frac{1}{2} K_{\theta p} \| [X, B] \|_{E(\theta p)} \right)^{\theta} \\
= (K_{\theta p} \| [X, B] \|_{E(\theta p)})^{\theta} \\
= K_{\theta p}^{\theta} \left\| [X, B] \right\|_{E(\theta p)}^{\theta},
\]

where we use the fact that
\[
\mu([X, B]) = \mu([X, B^*]) = \mu([B^*, X]) = \mu([X, B^*])
\]
for the first equality. We may assume that \( B = B^* \).

Next, we use the Cayley transform defined by
\[
U = (B - i)(B + i)^{-1}, \quad B = 2i(1 - U)^{-1} - i.
\]
Clearly, \( U \) is unitary. The functional calculus together with the assumption that \( \|B\|_{\infty} = 1 \) yields that
\[
\| (1 - U)^{-1} \|_{\infty} \leq \frac{1}{\sqrt{2}} \quad \text{and} \quad \| (B + i)^{-1} \|_{\infty} \leq 1.
\]

Hence, we have
\[
\left\| f(X), B \right\|_{E(p)} = \left\| f(X)B - Bf(X) \right\|_{E(p)}
\]
\[
\leq 2 \left\| f(X)(1 - U)^{-1} - (1 - U)^{-1}f(X) \right\|_{E(p)}
\]
\[
\leq 2 \left\| f(X)(1 - U)^{-1} \right\|_{E(\infty)} \left\| (1 - U) \right\|_{E(p)} - \left\| (1 - U) - (1 - U)f(X) \right\|_{E(p)}
\]
\[
\leq \left\| f(X)U - Uf(X) \right\|_{E(p)}
\]
\[
= \left\| U^*f(X) - f(X) \right\|_{E(p)}
\]
\[
= \left\| f(U^*XU - X) \right\|_{E(p)}
\]
\[
\leq C_{p, \theta} \left\| f \right\|_{\text{S}_d, \theta} \left\| U^*XU - X \right\|_{E(p)}^{\theta}
\]
\[
\leq C_{p, \theta} \left\| f \right\|_{\text{S}_d, \theta} \left\| XU - UX \right\|_{E(\theta p)}^{\theta}.
\]

By (39), we have
\[
[X, U] = [X, (B - i)(B + i)^{-1}] = X(B - i)(B + i)^{-1} - (B - i)(B + i)^{-1}X
\]
\[
= (B + i)^{-1} \left( (B + i)X(B - i) - (B - i)X(B + i) \right)(B + i)^{-1}.
\]

Hence,
\[
\left\| f(X), B \right\|_{E(p)}^{\theta}
\]
\[
\leq C_{p, \theta} \left\| f \right\|_{\text{S}_d, \theta} \left\| (B + i)^{-1} \right\|_{E(\infty)}^{2\theta} \left\| (B + i)X(B - i) - (B - i)X(B + i) \right\|_{E(\theta p)}^{\theta}
\]
\[
\leq C_{p, \theta} \left\| f \right\|_{\text{S}_d, \theta} \left\| [X, B] \right\|_{E(p)}^{\theta}.
\]

A further consequence may be obtained for quasi-commutator estimates. The following corollary extends [75, Proposition 5.1] and [53, Corollary 3.2],

**Corollary 7.2.** Let \( 0 < p < \infty \) and \( 0 < \theta < 1 \). Let \( f \in \text{S}_{d, \theta} \). Let \( E(0, \infty) \) be a fully symmetrically normed space. Then, there exists a constant \( C_{p, \theta} \) such that for \( A, B \in S(M, \tau) \) and \( R \in M \), we have
\[
\left\| f(A)R - Rf(B) \right\|_{E(p)} \leq C_{p, \theta} \left\| f \right\|_{\text{S}_{d, \theta}} \left\| [AR - RB]^\theta \right\|_{E(p)} \left\| R \right\|_{\infty}^{1-\theta}.
\]
Proof. By homogeneity, it suffices to consider the special case when \( \|R\|_{\infty} = 1 \). Let us first assume that the special case when \( A, B \) are unitarily equivalent in \( \mathcal{M} \), i.e., \( A = U^*BU \) for a unitary operator \( U \in \mathcal{M} \) and we shall prove that

\[
\|f(A)R - Rf(B)\|_{E(p)} = \|U^*f(B)UR - Rf(B)\|_{E(p)} \\
\leq C_{p,\theta} \|f\|_{S_{d,\theta}} \|U^*BUR - RB\|^{\theta}_{E(\theta p)}.
\]

(41)

By Corollary 7.1, we have

\[
\|U^*f(B)UR - Rf(B)\|_{E(p)} = \|f(B)UR - URF(B)\|_{E(p)} \\
\leq C_{p,\theta} \|f\|_{S_{d,\theta}} \|BUR - URB\|^{\theta}_{E(\theta p)}.
\]

Hence, we obtain the validity of (41).

Now, we consider the case of arbitrary self-adjoint operators \( A, B \in S(\mathcal{M}, \tau) \). We consider the algebra \( \bar{\mathcal{M}} = M_2 \otimes \mathcal{M} \) equipped with the trace \( \bar{\tau} = Tr \otimes \tau \), where \( Tr \) is the standard trace on \( M_2 \). We consider \( E(\mathcal{M}, \bar{\tau})^{(p)} \) instead of \( E(\mathcal{M}, \tau)^{(p)} \). For any \( X, Y \in E(\mathcal{M}, \tau)_h \), we have

\[
\tilde{Z} := \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \in E(\bar{\mathcal{M}}, \bar{\tau})
\]

with

\[
\|X\|_{E(p)} \leq \|\tilde{Z}\|_{E(p)} = \|\mu(X \oplus Y)\|_{E(p)} \leq K_p \left( \|X\|_{E(p)} + \|Y\|_{E(p)} \right),
\]

where \( K_p \) stands for the modulus of concavity of \( E^{(p)}(\mathcal{M}, \tau) \). Put

\[
\tilde{A} = \begin{pmatrix} A & 0 & 0 \\ 0 & B \end{pmatrix}, \tilde{B} = \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix}, \quad \text{and} \quad \tilde{R} = \begin{pmatrix} R & 0 \\ 0 & R^* \end{pmatrix}.
\]

Then, \( \tilde{A} \) and \( \tilde{B} \) are unitarily equivalent in \( \bar{\mathcal{M}} \). We have

\[
f(\tilde{A})\tilde{R} = \begin{pmatrix} f(A)R & 0 \\ 0 & f(B)R^* \end{pmatrix} \quad \text{and} \quad \tilde{R}f(\tilde{B}) = \begin{pmatrix} Rf(B) & 0 \\ 0 & R^*f(A) \end{pmatrix}.
\]

Hence, by (42), we have

\[
\|f(A)R - Rf(B)\|_{E(p)} \leq \left\| f(\tilde{A})\tilde{R} - \tilde{R}f(\tilde{B}) \right\|_{E(p)}
\]

and (noting that \( A \) and \( B \) are self-adjoint and therefore \( (AR - RB)^* = R^*A - BR^* \))

\[
\left\| \tilde{A}R - \tilde{R}B \right\|_{E(\theta p)} \leq K_{\theta p} \left( \|AR - RB\|_{E(\theta p)} + \|BR^* - R^*A\|_{E(\theta p)} \right) \\
= 2K_{\theta p} \|AR - RB\|_{E(\theta p)},
\]

which together with (41) implies that

\[
\|f(A)R - Rf(B)\|_{E(p)} \leq \left\| f(\tilde{A})\tilde{R} - \tilde{R}f(\tilde{B}) \right\|_{E(p)}^{(41)} \leq C_{p,\theta} \|f\|_{S_{d,\theta}} \left\| \tilde{A}R - \tilde{R}B \right\|^{\theta}_{E(\theta p)} \\
\leq (2K_{\theta p})^\theta C_{p,\theta} \|f\|_{S_{d,\theta}} \|AR - RB\|^{\theta}_{E(\theta p)}.
\]

This completes the proof. \( \square \)

Remark 7.3. The proof in Corollary 7.1 and Corollary 7.2 indeed show that, in the setting of a symmetrically quasi-normed space, the commutator estimates, quasi-commutator estimates and the difference estimates for operator \( \theta \)-Hölder functions are equivalent.

Remark 7.4. In Corollary 7.2, no restrictions on the supports of \( f \) are needed. That is, we can consider functions having possibly unbounded supports rather than functions having compact supports as in [78, Theorem 2.4].

Moreover, in [78, Theorem 2.4], the estimate of the quasi-norm of \( [f(X), B] \) is obtained in terms of \( \|X, B\|^{\sigma} \) only when \( \sigma \) is strictly less than \( \theta \), \( \theta \in (0,1) \). Moreover, the constant obtained in [78, page 168] goes to infinity when we take \( \sigma \uparrow \theta \). However, in Corollary 7.2, we obtain the estimate in terms of \( \|X, B\|^{\theta} \)
Note that the integer \(d\) obtained in [78, Theorem 2.4] satisfies that \(d > \frac{1}{p} + \sigma \in (\frac{1}{p}, \frac{1}{p} + 1), p \in (0, 1]\) and \(\sigma \in (0, \theta)\) (one can take \(d = \lfloor \frac{1}{p} + \sigma \rfloor + 1\), while the integer \(d(p)\) in our paper is the minimal integer such that \(d(p) > \frac{1}{p} + 2\) (see Section 5). That is, \(d(p) = \lfloor \frac{1}{p} + 2 \rfloor + 1\). There exist \(0 < p \leq 1, \theta \) and \(\sigma\) such that \(d(p) - 1 = \lfloor \frac{1}{p} + 2 \rfloor = \lfloor \frac{1}{p} + \sigma \rfloor + 1 = \lfloor \frac{1}{p} + \theta \rfloor + 1\). In other words, \(d(p)\) is the minimal integer such that \(d(p) > d = \lfloor \frac{1}{p} + \sigma \rfloor + 1\) but \(d(p)\) does not depend on \(\sigma\) and \(\theta\).

7.2. Estimates for absolute value map. The estimates of the distance between the absolute values of two operators \(A\) and \(B\) have been obtained by several mathematicians (see e.g. [13, 53, 54]). In particular, Kosaki [54] proved that if \(A, B \in (\mathcal{N}, \|\cdot\|_{1})\) (\(\mathcal{N}\) stands for the predual of a general von Neumann algebra \(\mathcal{N}\)), then

\[
\|A - |B|\|_{1} \leq 2^{1/2} (\|A + B\|_{1} \|A - B\|_{1}).
\]

This result was later extended by Kittaneh and Kosaki [53] to the von Neumann-Schatten \(p\)-class, \(p \geq 2\), in \(B(\mathcal{H})\):

\[
\|A - |B|\|_{p} \leq \left(\|A + B\|_{p} \|A - B\|_{p}\right)^{1/2}, A, B \in B(\mathcal{H}).
\]

The case when \(1 \leq p \leq 2\) was proved by Bhatia [13]:

\[
\|A - |B|\|_{p} \leq 2^{1/p - 1/2} \left(\|A + B\|_{p} \|A - B\|_{p}\right)^{1/2}, A, B \in B(\mathcal{H}).
\]

Bhatia [13] also proved the estimates for an arbitrary fully symmetric norm \(\|\cdot\|\) on \(B(\mathcal{H})\):

\[
\|A - |B|\| \leq 2^{1/2} (\|A + B\| \|A - B\|)^{1/2}, A, B \in B(\mathcal{H}).
\]

In this subsection, we consider the quasi-norm estimates of the absolute value map in \(S(M, \tau)\), which extends the results in [13, 53, 54].

**Corollary 7.5.** Let \(E(0, \infty)\) be a fully symmetrically normed space and let \(p \in (0, \infty)\). There exists a constant \(C_{p}\) such that

\[
\|A - |B|\|_{E(p)} \leq C_{p} (\|A + B\|_{E(p)} \|A - B\|_{E(p)})^{1/2}, A, B \in S(M, \tau).
\]

**Proof.** Without loss of generality, we may assume that \(0 < p \leq 1\). Applying Theorem 6.3 to \(f(t) = |t|^{\frac{p}{2}}, t \in \mathbb{R}, \|A\|^{2}\) and \(\|B\|^{2}\), we obtain that there exists a constant \(C_{p}\) such that

\[
\|A - |B|\|_{E(p)} \leq C_{p} \left\|\|A\|^{2} - |B|^{2}\right\|_{E(p)}^{1/2}, A, B \in S(M, \tau).
\]

By [58, Lemma 2.3.15], there exists partial isometries \(U\) and \(V\) such that

\[
\|A^{2} - |B|^{2}\| = \left|\frac{2A^{*}A - 2B^{*}B}{2}\right| = \left|\frac{(A + B)^{*}(A - B) + (A - B)^{*}(A + B)}{2}\right|
\]

\[
\leq U \left|\frac{(A + B)^{*}(A - B)}{2}\right| U^{*} + V \left|\frac{(A - B)^{*}(A + B)}{2}\right| V^{*}.
\]

By the monotonicity of function \(t \mapsto t^{1/2}, t \in \mathbb{R}^{+}\) (see e.g. [25]), we have

\[
\|A^{2} - |B|^{2}\|^{1/2} \leq \left(U \left|\frac{(A + B)^{*}(A - B)}{2}\right| U^{*} + V \left|\frac{(A - B)^{*}(A + B)}{2}\right| V^{*}\right)^{1/2}.
\]
Hence,
\[
\left\| \left( A^2 - |B|^2 \right)^{\frac{1}{2}} \right\|_{E(p)}^2 \\
\leq \left\| \left( \frac{(A + B)^*(A - B)}{2} \right)^{U} + V \left( \frac{(A - B)^*(A + B)}{2} \right)^{V*} \right\|_{E(p)}^{\frac{1}{2}}
\]
\[
= \left\| \left( \frac{(A + B)^*(A - B)}{2} \right)^{U} + V \left( \frac{(A - B)^*(A + B)}{2} \right)^{V*} \right\|_{E(p)}^{\frac{1}{2}}
\]
(44)
\[
\leq \left( \left\| \left( A + B \right)^*(A - B) \right\|_{E(p/2)}^{p/2} + \left\| (A - B)^*(A + B) \right\|_{E(p/2)}^{p/2} \right)^{2/p},
\]

Recall that [26, Theorem 2.2] (see also [43, Corollary 1.13])
\[
\mu(XY)^{p/2} \prec \mu(X)^{p/2} \mu(Y)^{p/2}, \quad X, Y \in S(M, \tau).
\]

By [82, Theorem 1] (see also [36]). Note that the notation of \( E(p) \) in [82] is different from that in our paper, we have
\[
\left\| (A + B)^*(A - B) \right\|_{E(p/2)}^{p/2} = \left\| (A + B)^*(A - B) \right\|_{E}^{p/2},
\]
\[
\leq \left\| \mu(A + B)^{p/2} \mu(A - B)^{p/2} \right\|_{E}^{p/2},
\]
\[
\leq \left\| \mu(A + B)^{p/2} \right\|_{E(2)}^{p/2} \left\| \mu(A - B)^{p/2} \right\|_{E(2)}^{p/2},
\]
\[
= \|A + B\|_{E(p)}^{p/2} \|A - B\|_{E(p)}^{p/2}.
\]

which together with (43) and (44) completes the proof. \( \square \)

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