Scalings for diffraction-decorated caustics in gravitational lensing

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Abstract
In the astronomically natural transitional approximations for waves near caustics in gravitational lensing, the familiar wavelength scalings associated with short-wave asymptotics are accompanied by a variety of dependences on disparate astronomical lengths, such as the Schwarzschild radius, separation of binary stars, and distance to the lens. These dependences are calculated analytically for spacings of interference fringes and the corresponding intensity amplifications, for two much-studied models: lensing of a distant source by an isolated star and near the fold and cusp caustics from a binary star lens. If the aperture of a telescope looking in the direction of the source is modelled by Gaussian apodization, the image is a complexification of the wave in the observation plane.

Keywords: asymptotics, diffraction catastrophes, scaling, focusing, apodization, cusps

1. Introduction
Gravitational lensing is now firmly established as a valuable tool in astronomy [1–5]. Although observations have largely been guided and interpreted in terms of geometrical optics, i.e. ray deflection, there have been many explorations of anticipated wave effects (chapter 7 of [5]), which could add interferometric precision to inferences about distant objects. Among the areas that have been studied from a wave-optical perspective are the coherence of lensed images from the same source [6–8] (even when these images cannot be resolved [9]), and time-dependent diffraction from sudden events [10–13]. Most studies have concerned electromagnetic waves, but wave effects in the lensing of gravitational waves have also been explored [14, 15]; this is a promising area because gravitational waves associated with astrophysical events are often more coherent than electromagnetic waves from extended sources.

It has long been recognised that lensing is dominated by events and places where focusing occurs, that is, by caustics [1, 5] and that the decoration of these singularities by ‘diffraction catastrophes’ [16, 17] will play a major role in the envisaged wave effects [18]. Especially significant near caustics are the effects of decoherence caused by the finite size of sources [19–21]. Most works have considered decoherence near fold caustics, but the stronger cusp singularity has also been studied [22]. On astronomically relevant scales, the associated diffraction integrals are highly oscillatory, raising challenges for their numerical evaluation; an important recent advance, especially for multidimensional integrals, has been the development and application of sophisticated contour deformation techniques [23, 24].

My purpose here is different from previous studies, so I need to explain why. Earlier works have naturally been directed towards possible observational consequences and opportunities, for example the detection of lenses consisting of dark matter, compact sources such as quasars, and particular
wavelengths, e.g. radio. Effects of finite source size have been emphasised, because these cause decoherence, usually obscuring interference and diffraction or at least reducing their contrast. But from a fundamental wave physics perspective it is important to consider in as much detail as possible the pure wave effects, and this will be the focus here.

In particular, I will concentrate on scalings. For gravitational lensing, geometrical optics almost suffices, so the relevant regime is extreme short-wave asymptotics: small wavelengths $\lambda$, i.e. large wavenumbers $k = 2\pi/\lambda$. It is well understood that scaling with $k$ is fundamental in short-wave optics [25, 26]: away from caustics, the familiar interference-fringe spacings are $O(\lambda)$, but near caustics the spacings are larger; for example near a fold caustic the spacing is $O(\lambda^{2/3})$, and the associated intensity amplification is $O(\lambda^{-1/3})$. (This explains, for example, why the rainbow—which is a fold caustic—is bright, and why our naked eyes can resolve supernumerary rainbows decorating the main bow [27], even though the wavelengths in sunlight are much smaller than raindrops.)

But in addition to the wavelength dependence, the fringe spacings and amplifications are strongly influenced by enormously varied relevant astronomical lengths: Schwarzschild radii, binary star separations, lens-observer distances. It is these additional dependences that will be calculated here, for the fringe spacings and intensity amplifications associated with several kinds of focusing. These results exhibit the wide variety of combinations of astronomical lengths, even for different parts of the same caustic, that complement $\lambda$ in the dimensionless quantities naturally describing diffraction.

Section 2 revisits the basic wave theory. I will also briefly mention a curious aspect of the images in a telescope looking in the direction of the lensing objects: if the aperture is modelled by Gaussian apodization, the images are complexifications of those in the observation plane. The main results of the paper are in sections 3 and 4: explicit calculations for simple lens models. Section 3 concerns the focal spot from a single star (section 3), and slightly extends previous studies [7, 8, 18, 28]. Section 4 concerns diffraction scalings across the fold and cusp caustics from a binary star. The concluding section 5 summarises the main results, in terms of physical variables rather than the scaled variables convenient for calculation.

In short-wave asymptotics there are four different regimes, and it is worth pinpointing the one most relevant to the anticipated wave phenomena in gravitational lensing that will be studied here. Regime 1 is the coarsest scale, in which the wave intensity is simply the sum of intensities of contributing rays: inverse Jacobians between source and observer. This is the regime currently employed in conventional gravitational lensing: geometrical optics, with singularities at the caustics [1]. Next is regime 2, in which the intensity is the square of the sum of interfering wave amplitudes, each with a phase (optical path length) decorating the square root of the inverse Jacobian [5]. This is the ‘geometrical wave’; it gives an accurate description of the wave everywhere except very near the caustics, where the light is brightest and wave effects strongest—the very features of interest here. We need to describe the waves very close to the geometrical caustics. This is regime 3, where the lensed wave is described by canonical special functions—diffraction catastrophes [29, 30]—each representing a different type of caustic. In this ‘transitional asymptotics’, the most familiar case, widely studied in gravitational lensing [5, 19, 20], is the Airy function [29, 31] decorating fold caustics. Transitional approximations do not match smoothly onto the geometrical waves of regime 2. The two are reconciled in the ‘uniform asymptotics’ of regime 4, in which the variables in the diffraction catastrophes are stretched to represent the exact optical path lengths of the contributing rays so as to be valid far from caustics as well as close to them [16, 17, 29, 32–34]. In gravitational lensing, it is only very close to caustics that wave effects are likely to be observable, so the additional sophistication of regime 4 is unnecessary: regime 3 suffices, and has the advantage of leading to simple explicit formulas.

Some numerical calculations will be presented. These are intended to illustrate the theoretical formulas. They are not meant to simulate specific examples of wave effects in astronomical lensing, or assess the feasibility of detecting them; that would require a different perspective and different expertise, namely astronomers’ familiarity with the variety of sources and possible lensing objects, and detection techniques for different wavelength regimes.

2. Reprise of basic wave theory

For present purposes, the simplest formulation suffices: paraxial diffraction of monochromatic scalar waves from a distant point source, in the thin screen approximation [5, 18, 35] in which the lenses are projected along the propagation direction, leading to a phase distribution $W(r)$ in the lens plane $r = (x, y)$ (figure 1). Mass endows space with a refractive index, for which an adequate approximation is

$$n(r, z) = 1 - \frac{2\phi(r, z)}{c^2},$$

where $\phi(r, z)$ is the Newtonian gravitational potential. The lens plane phase is then

$$W(r) = \left( \int_{-\infty}^{\infty} dz (n(r, z) - 1) \right)_{\text{regularised}},$$

in which ‘regularised’ refers to ignoring irrelevant infinite constants.

We choose to study the wave in the observation plane $R = (X, Y)$, distant $Z$ from the phase screen, normalised to unit strength for the wave arriving at the screen. Elementary Fresnel-Kirchhoff diffraction gives

$$\psi(R, Z) = \frac{-ik}{2\pi Z} \iiint_{\text{plane}} dr \exp \left( ik \left( W(r) + \frac{(r - R)^2}{2Z} \right) \right).$$

The intensity across the observation plane is $|\psi(R, Z)|^2$. Essentially the same integral arises in different formulations. For example, if the source is at a finite distance $Z_s$ from the screen,
the only effect on the intensity is that $R$ and $Z$ are scaled by $1 + Z/Z_s$: the wave beyond the screen is stretched and further away. And if the observation point is regarded as fixed (e.g. $R = 0$), and the source position $R_s$ in the plane $Z_s$ is regarded as variable (as in [5]), $R_s$ rather than $R$ appears in equation (3).

Like all wave patterns, the interference detail will depend on $k$; this colour dependence will be a characteristic signal of wave lensing, distinguishing it from geometric gravitational lensing, the latter being achromatic because the index (equation (1)) is wavelength-independent. This means that for polychromatic sources there will be spectral distortion near caustics, i.e. coloured ‘white light fringes’ and coloured diffraction catastrophes ([36], also [37]). These phenomena will not be discussed here; for the Airy pattern near a gravitationally lensed fold caustic, they have been considered elsewhere [19]. Simply put: caustics arising from polychromatic sources will be reddened on the dark side. It is not my purpose here to give a detailed discussion of the wave features of images seen when looking in directions $\Omega = (\theta_\kappa, \phi_\kappa)$ from the point $R_s$, $Z$ with a telescope of finite aperture. Elsewhere [37] I have elaborated some of the peculiarities associated with looking at images when there are caustics in the telescope aperture, including the connection with Husimi functions. But it is worth drawing attention to a curious aspect.

The wave in direction $\Omega$ is the windowed Fourier transform of the wave at $R_s$, $Z$ in the observation plane. If the telescope aperture (‘window’) with radius $L$ is modelled by Gaussian apodization rather than the more realistic sharp edges, this is

$$\tilde{\psi}(\Omega, R, Z) = \frac{1}{2\pi L^2} \int_{\Omega_{\text{plane}}} dR' \exp \left(-\frac{(R' - R)^2}{2L^2}\right) \times \exp \left(ik\Omega \cdot R'\right) \psi(R', Z).$$

(4)

A straightforward calculation using equation (3) leads to

$$\tilde{\psi}(\Omega, R, Z) = \exp \left(-\frac{1}{2}k^2L^2\Omega^2\right) \exp (ik\Omega \cdot R) \times [\psi(R, Z)]_{R \rightarrow R - ikL, \Omega \rightarrow \Omega + ikL}.$$

(5)

Thus the wave corresponding to looking in direction $\Omega$ is a complexification of the wave in the observation plane. The corresponding complexifications of the Airy and Pearcey functions were discussed in [37].

An easy connection with the aperture-blurred geometrical image is obtained by approximating equation (5) by including only the terms that do not vanish for large $k$:

$$\tilde{\psi}(\Omega, R, Z) = \frac{\exp (-ikZ_0^2)}{2\pi L^2} \times \int_{\Omega_{\text{plane}}} dR \exp (ik(W(r) + \Omega \cdot r)) \times \exp \left(-\frac{(R - R - \Omega Z_0)^2}{2L^2}\right).$$

(6)

The saddle points $r$ of the fast-varying first exponential factor in the integral are the lens plane points from which rays emerge in the viewing direction $\Omega$, and with these $r$ the second exponential factor gives the geometric aperture blurring (the diffraction blurring $O(1/(kL))$) requires considering both factors.

3. Transitional asymptotics of focal diffraction from a single star

For a single star with mass $M$, the phase $W$ in equation (3) depends on its Schwarzschild radius $R_s$:

$$W(r) = -2R_s \log r, \quad R_s = \frac{2GM}{c^2}.$$  

(7)

This case has been studied extensively, and the diffraction integral evaluated exactly [8, 18, 28, 38]. For our present purpose, namely extracting the relevant transitional asymptotics, it is convenient to scale observation plane distances $R$, $Z$ and wavenumbers $k$ with $R_s$, and lens plane distances $r$ differently:

$$k \equiv \frac{\kappa}{R_s}, \quad R \equiv (R_s \rho, \theta_R), \quad Z \equiv R_s \zeta, \quad r \equiv \left(\frac{Z}{k}\right)^{\frac{1}{2}}, \phi_r.$$  

(8)

In astronomical lensing, the parameters $\kappa$ and $\zeta$ greatly exceed unity. With these scalings, equation (3) becomes

\begin{align*}
\tilde{\psi}(\Omega, R, Z) & = \frac{1}{2\pi L^2} \int_{\Omega_{\text{plane}}} dR' \exp \left(-\frac{(R' - R)^2}{2L^2}\right) \\
& \times \exp \left(ik\Omega \cdot R'\right) \psi(R', Z). \\
& \quad \times \exp \left(-\frac{1}{2}k^2L^2\Omega^2\right). \\
& \quad \times \int_{\Omega_{\text{plane}}} dR \exp (ik(W(r) + \Omega \cdot r)) \times \exp \left(-\frac{(R - R - \Omega Z_0)^2}{2L^2}\right). \\
& \quad \times \exp (-ikZ_0^2). \\
\end{align*}
\[
\psi(\rho, \zeta, \kappa) = 2 \int_0^\infty dt \exp \left(i \left(-2\kappa \log t + t^2\right)\right) J_0 \left(tp\sqrt{2\kappa / \zeta}\right),
\]
(9)
in which here and hereafter we neglect all external phase factors irrelevant to the wave intensity.

The integral can be expressed in terms of confluent hypergeometric functions [8, 18, 28]:
\[
\psi(\rho, \zeta, \kappa) = \sqrt{\frac{2\pi\kappa}{1 - \exp(-2\pi\kappa)}} \, F_1\left(\frac{ik}{2}, 1; \frac{i\rho^2\kappa}{2\zeta}\right),
\]
(10)
The intensity falls from the initial value, which for \( \kappa \gg 1 \) is
\[
|\psi(0, \zeta, \kappa)|^2 \approx |\psi_0(\zeta, \kappa)| = 2\pi\kappa,
\]
(11)
to the asymptotic value unity for large \( \rho \).

Of the variety of asymptotic approximations for confluent hypergeometric functions [29] (or equivalently Whittaker functions [39]), the transitional asymptotics (regime 3) relevant to astronomy is obtained by applying large \( \kappa \) stationary-phase approximation to the exponential factor in equation (9) and using this saddle value \( t = \sqrt{\kappa} \) in the Bessel factor. Thus
\[
|\psi(\rho, \zeta; \kappa)|^2 \approx 2\pi\kappa \, J_0^2 \left(\rho\kappa \sqrt{2 \zeta / \kappa}\right).
\]
(12)
As the dashed curves in figure 2 show, this transitional approximation accurately describes more interference oscillations as \( \kappa \) increases, that is, as \( \lambda \) decreases for fixed \( Z \).

The canonical star-lensing function \( J_0^2(c) \) is shown in figure 3(a) (figures 3(b) and (c) show the fold and cusp canonical functions to be calculated in section 4).

A measure of the fringe spacing is the distance \( X_{\text{star}} \) between the focus and the nearest maximum. Thus, from the argument of \( J_0 \), the spacing in scaled units is
\[
\Delta \rho_{\text{star}} = \frac{X_{\text{star}}}{\kappa} \sqrt{\frac{\zeta}{2}} \quad X_{\text{star}} = 3.832.
\]
(13)
The intensity amplification factor is \( 2\pi\kappa \).
The transitional approximation (equation (12)) is different from the geometrical wave approximation; this is obtained from equation (9) by using the asymptotic approximation for $J_0$ in the integrand, in the form of a cosine, separated into two exponentials, and then evaluating the integral by stationary phase [28]. The resulting intensity is shown as the dotted curves in figure 2. Of course the approximation fails near the focus at $\rho = 0$, but it agrees with the exact intensity very well everywhere else. In particular, it reaches the necessary large $\rho$ limit unity (as an elementary exercise confirms)—unlike the transitional approximation (equation (12)), which by design works from $\rho = 0$ out to a finite value $\rho(\kappa)$ that increases with $\kappa$.

4. Transitional approximations for caustic diffraction from a binary star

4.1. Diffraction integral, caustic, geometrical optics

For an equal-mass binary with separation $2a$, the phase in equation (3) is

$$W(r) = -R_S \log \left[ \left( \frac{(x-a)^2 + y^2}{(x+a)^2 + y^2} \right) \right].$$

It is convenient to scale distances in the observation and lens planes by $a$, distances $Z$ by $a^2/R_S$, and wavenumber $k$ by $R_S$:

$$k \equiv \frac{\kappa}{R_S}, \quad (x,y) \equiv (au,av), \quad (X,Y) \equiv (a\xi,a\eta), \quad Z \equiv \frac{a^2}{R_S} \zeta.$$  \hspace{1cm} (15)

Astronomically, $\kappa \gg 1$ always. We will often specialise formulas for general $\zeta$ to the asymptotic regime $\zeta \gg 1$, because this gives relatively simple formulas and corresponds to many cases of interest (for a binary consisting of two solar-mass stars ($R_S \sim 3$ km) separated by 1au (i.e. $a = 0.5$ au), $\zeta > 1$ corresponds to $Z > 65$ pc). With the scaling equation (15), the binary star lensed wave becomes

$$\psi(\xi, \eta, \zeta, \kappa) = \frac{-i\kappa}{2\pi \zeta} \int \frac{dudv exp(i\kappa S)}{2(iuv)^2},$$

where

$$S(u,v,\xi,\eta,\zeta) = -\log \left( \frac{(u^2 + v^2 + 1)^2 - 4u^2}{(u^2 + v^2 + 1)^2 - 4v^2} \right) + \frac{(\xi - u)^2 + (\eta - v)^2}{2\zeta}. \hspace{1cm} (16)$$

For large $\kappa$, the wave is dominated by the caustics, where real saddles of the phase $S$ coincide. These are well understood [1, 5, 40], but for later reference we need to repeat known material. For constant $\zeta$ sections the caustic consists of fold curves and cusp points forming closed astroids: two for $\zeta < 1/4$, one for $1/4 < \zeta < 2$, and three for $\zeta > 2$. Figure 4 illustrates the caustics for $\zeta > 2$.

The saddle-points of the integrand in equation (16) determine the ray $(\xi, \eta)$ in the observation plane that originates from the point $(u,v)$ in the lens plane, and, by inversion, the lens plane points from which rays reach given observation points:

$$\partial_v S = 0 \Rightarrow \xi = u \left( 1 - \frac{4\zeta(w^2 - 1)}{(1 + w^2)^2 - 4w^2} \right),$$

$$\partial_u S = 0 \Rightarrow \eta = v \left( 1 - \frac{4\zeta(w^2 + 1)}{(1 - w^2)^2 + 4w^2} \right), \quad w = \sqrt{u^2 + v^2}. \hspace{1cm} (17)$$

There are five saddles, all real within the astroids and three real and two complex outside. The caustic is determined by their coalescence, where the determinant of the Hessian matrix

$$M = \begin{pmatrix} \partial_{uu}S & \partial_{uv}S \\ \partial_{uv}S & \partial_{vv}S \end{pmatrix}.$$  \hspace{1cm} (18)
vanishes:
\[
\det M = 0 \Rightarrow \frac{(u^2 - 1)^2 + 4u^2}{(u^2 + 1)^2 - 4u^2} = \frac{(u^2 + 1)^2 - 4\eta^2}{(u^2 - 1)^2 + 4\eta^2} = \frac{1}{16\eta^2}.
\]
Explicitly, the caustic is given parametrically in terms of \( w \) by the real values of
\[
\begin{align*}
\eta_{\pm}(w, \zeta) &= \pm \sqrt{\frac{32(1 + \eta^2 + 2\zeta^2) + (1 + \eta^2)^2 + 8\zeta^2}{32(1 + \eta^2 + 2\zeta^2) - (1 - \eta^2)^2 - 8\zeta^2}}, \\
\nu_{\pm}(w, \zeta) &= \pm \sqrt{\frac{32(1 + \eta^2 + 2\zeta^2) - (1 - \eta^2)^2 - 8\zeta^2}{32(1 + \eta^2 + 2\zeta^2) + (1 + \eta^2)^2 + 8\zeta^2}}.
\end{align*}
\]

The transitional approximations (regime 3) that will be calculated in the following sections are those associated with the singularities illustrated in figure 4 by red dots: the fold and cusp1 and cusp2 along the \( \eta \) axis, and cusp3 along the \( \xi \) axis. Explicitly, these are located at
\[
\begin{align*}
\eta_{\text{fold}}(\zeta) &= \sqrt{2\sqrt{\zeta(\zeta + 2)} - 1 - 2\zeta} \left( \sqrt{\zeta(\zeta + 2)} - 1 - \zeta \right) \\
&= \frac{16}{3\sqrt{3}} \left( \zeta - \frac{1}{4} \right)^{3/2} + O \left( \left( \zeta - \frac{1}{4} \right)^{3/2} \right) \\
&= 2\zeta - 1 + \frac{1}{2} \zeta^{-1} + O \left( \zeta^{-2} \right), \\
\eta_{\text{cusp1}}(\zeta) &= \sqrt{-2\sqrt{\zeta(\zeta - 2)} - 1 + 2\zeta} \left( \sqrt{\zeta(\zeta - 2)} - 1 - \zeta \right) \\
&= \sqrt{3 + 2\sqrt{3}} \sqrt{\zeta - 2} + O \left( \zeta - 2 \right) \\
&= 2\zeta - 1 - \frac{3}{4} \zeta^{-1} + O \left( \zeta^{-2} \right), \\
\eta_{\text{cusp2}}(\zeta) &= \sqrt{2\sqrt{\zeta(\zeta - 2)} - 1 + 2\zeta} \left( -\sqrt{\zeta(\zeta - 2)} - 1 + \zeta \right) \\
&= \sqrt{3 - 2\sqrt{3}} \sqrt{\zeta - 2} + O \left( \zeta - 2 \right) \\
&= 1/\sqrt{3} + O \left( \zeta^{-3/2} \right), \\
\eta_{\text{cusp3}}(\zeta) &= \sqrt{2\sqrt{\zeta(\zeta + 2)} + 1 + 2\zeta} \left( -\sqrt{\zeta(\zeta + 2)} + 1 + \zeta \right) \\
&= 1 - \zeta + O \left( \zeta^{-3/2} \right) = 1/\sqrt{3} + O \left( \zeta^{-3/2} \right),
\end{align*}
\]
where in each case we also give the limiting forms: near where the caustic is born (\( \zeta = 1/4 \) for the fold, \( \zeta = 2 \) for cusp1 and cusp2, and \( \zeta = 0 \) for cusp3), and far away, i.e. \( \zeta \gg 1 \). We will also need the points \( u, v \) in the lens plane from which these caustics originate:
\[
\begin{align*}
u_{\text{fold}}(\zeta) &= -\sqrt{2\sqrt{\zeta(\zeta + 2)} - 1 - 2\zeta}, \\
u_{\text{cusp1}}(\zeta) &= -\sqrt{2\sqrt{\zeta(\zeta - 2)} - 1 + 2\zeta}, \\
u_{\text{cusp2}}(\zeta) &= -\sqrt{2\sqrt{\zeta(\zeta - 2)} - 1 + 2\zeta}, \\
u_{\text{cusp3}}(\zeta) &= \sqrt{2\sqrt{\zeta(\zeta + 2)} + 1 + 2\zeta}.
\end{align*}
\]

Figure 5. Geometrical wave (regime 2) (equation (23)) for \( \kappa = 200, \zeta = 5 \), showing the central astroid.

In all cases, the procedure will be the same: locally expanding the phase \( S \) in equation (16) about the caustic points \( \xi, \eta, u, v \).

Although we are aiming for the transitional approximations, in subsequent sections it will be convenient also to use the geometrical wave [5] (regime 2), as an useful check on the calculations. This is
\[
\psi_{\text{geom}}(\xi, \eta, \zeta, \kappa) = \sum_{j=1, \text{real rays}}^{5} \exp \left( \frac{1}{4} i \pi \mu_j \right) \frac{\exp \left( i S_j \right)}{\zeta \sqrt{|\det M|}},
\]
in which
\[
S_j = S (\eta_j, \zeta, \xi), \quad \mu_j = \sum_{\text{rays}} \text{sgn}(\text{eigenvalues}(M_j)),
\]
Even though the geometric wave fails on the caustics, it gives an easy global picture of the interference structure, as figure 5 illustrates.

In the geometric optics limit, i.e. regime 1, there is a slight subtlety, arising from the fact that along symmetry axes two or more rays may contribute with the same phase and so their contributions must be added coherently, with each intensity multiplied by a degeneracy factor \( d_j \) (1 or 2 in our examples). The geometric intensity is thus
\[
I_{\text{geom}}(\xi, \eta, \zeta, \kappa) = \sum_{j=1, \text{real rays}}^{5} \frac{d_j}{\zeta^2 |\det M|}.
\]
Figure 6 illustrates the accuracy of the geometric wave approximation (regime 2), and how geometric optics (regime 1) describes the mean intensity, everywhere except near the caustics, to the precise description of which we now turn.
4.2. Fold diffraction: transitional approximation

The calculation will correspond to the red dot with the largest value of \( \eta \) on the fold caustic in figure 4(a). As \( \eta \) increases, this caustic originates at \( \eta = 1/4 \) and then recedes (figure 4(b)). To get the transitional approximation we expand the phase in equation (16) about the caustic points in the lens and observation planes in equations (21) and (22):

\[
S(u, v_{\text{fold}}, \eta, \zeta) = K - \frac{\delta \nu \delta \eta}{\zeta} - \frac{1}{4} \frac{\nu^2}{\zeta} + \cdots
\]

in which \( K \) is an irrelevant constant and

\[
A(\zeta) = \frac{1}{2} \frac{\sqrt{2} \sqrt{2 + \zeta^2}}{2 - 1 - 2 \zeta} \times \left( \sqrt{\zeta(2 + \zeta)} \left(1 + \zeta + \zeta^2 + \zeta\right) \right)
\]

The \( u \) integral is Gaussian, and the contribution to the diffraction integral close to the fold at \( \eta_{\text{fold}} \), from the \( dv \) integral in the neighbourhood of \( \eta_{\text{fold}} = 0, v = v_{\text{fold}} \), is the anticipated Airy function [29, 41]

\[
\psi_{\text{fold}}(\delta \eta, \zeta, \kappa) = \kappa^{1/6} E_{\text{fold}}(\zeta) A_i \left( \delta \eta \kappa^{2/3} F_{\text{fold}}(\zeta) \right)
\]

where

\[
F_{\text{fold}}(\zeta) = \frac{1}{A(\zeta)} = \frac{2}{3^{1/6} \left( \zeta - \frac{1}{4} \right)^{1/6}} \left( 1 + O \left( \zeta - \frac{1}{4} \right) \right)
\]

and

\[
E_{\text{fold}}(\zeta) = \sqrt{\frac{\pi}{\zeta}} \frac{1}{A(\zeta)^{1/3}} = \sqrt{\frac{\pi}{\zeta}} \left( 1 + O \left( \zeta - \frac{1}{4} \right) \right)
\]

with \( \kappa^{1/3} E_{\text{fold}}(\zeta)^2 \) giving the intensity amplification at the fold caustic.

The functions \( F_{\text{fold}}(\zeta) \) and \( E_{\text{fold}}(\zeta) \) embody the scalings that are our central interest here. The separation of the first two intensity maxima (cf figure 3(b)) is thus

\[
\Delta \eta_{\text{fold}} = \frac{X_{\text{fold}} \kappa^{-2/3}}{D(\zeta)} \approx \frac{X_{\text{fold}} \kappa^{-2/3}}{\zeta} \quad \Delta \eta_{\text{fold}} = 2.2294.
\]

This shows that in the short-wave limit the fold intensity grows as \( \kappa^{1/3} \), and the fringes get smaller as \( \kappa \to 2^3 \), as anticipated from catastrophe optics [16]; the fringes also expand linearly as the lens distance \( \zeta \) increases. For \( \zeta > 2 \) this fold lies on an astroid (figure 4) that gets smaller as \( \zeta \) increases: its size (distance from the fold down to cusp 1) is close to \( 1/\zeta \) (cf equation (21)). The local approximation (28) and (29) is valid only while the fold and cusp are well separated, i.e. the distance between these singularities exceeds the fringe separation. This requires

\[
\zeta < \frac{\kappa^{1/3}}{\sqrt{X_{\text{fold}}}}
\]

These are the main results of this section. The Airy function emerges clearly only for very large \( \kappa \). This is because there is an additional and non-negligible contamination from the three geometrical rays that are separate from the two that coalesce on the fold. This contamination gets slowly weaker as \( \kappa \) increases, but is still visible as very fast oscillations even for \( \kappa = 10^8 \) (figure 7(a)). In any practical situation these are likely to be averaged away by decoherence, leaving the Airy approximation emerging clearly: numerical calculations (not shown) confirm this. This phenomenon, of diffraction catastrophes being contaminated by fast oscillations that are often only imperfectly resolved in practice, is familiar in several contexts, including the quantum scattering of atoms and molecules [42, 43] and optical rainbows [44, 45].

4.3. Cusp diffraction: transitional approximations

The procedure for all three cusps is the same; we give the details for cusp 3, i.e. the cusp on the \( \xi \) axis (figure 4(a)), and then state the results for cusp 1 and cusp 2. To get the transitional approximation for cusp 3, we expand the phase in equation (16) about its points \( \xi = \xi_{\text{cusp}}, \eta = 0 \), in the observation plane (equation (21)), and \( u = u_{\text{cusp}}, v = 0 \) in the lens plane (equation (22)).
\[ S(u_{\text{cusp3}}(\zeta) + \delta u, v, \xi_{\text{cusp3}}(\zeta) + \delta \xi, 0, \zeta) = K + A_3 v^4 + \delta u \left( -\frac{\delta \xi}{\zeta} + C_3 v^2 \right) + \frac{\delta u^2}{\zeta} + \cdots \]
\[ = K + \left( A_3 - \frac{1}{4} C_3 \zeta \right) v^4 + \frac{1}{2} \delta \xi C_3 v^2 + \cdots. \quad (33) \]

in which the second equality follows after integrating over \( \delta u \), and

\[ A_3 = \frac{1}{8 \zeta^2} \left( 1 - \zeta - \zeta^2 + \zeta \sqrt{\zeta(2 + \zeta)} \right), \]
\[ C_3 = \frac{1}{2 \zeta^2} \sqrt{1 + 2 \zeta + 2 \sqrt{\zeta(2 + \zeta)}} \times \left( \sqrt{\zeta(2 + \zeta)}(\zeta + 1) - \zeta(\zeta + 2) \right). \quad (34) \]

Thus the contribution to the diffraction integral close to \( \xi_{\text{cusp3}} \), from the neighbourhood of \( v = 0, u = u_{\text{cusp}} \) is

\[ \psi_{\text{cusp3}}(\delta \xi, \zeta, \kappa) = E_3(\zeta) \kappa^{1/4} P(\delta \xi \kappa^{1/2} F_3(\zeta)). \quad (35) \]

\( P \) is the quartic-phase diffraction catastrophe integral (Pearcey function [29, 46]) describing waves along the symmetry axis of the cusp and illustrated in figure 3(c):

\[ P(c) = \sum_{\infty}^{\infty} d \exp \left( i (r^4 + c r^2) \right) = \pi \sqrt{\frac{|r|}{8}} \exp \left( -\frac{1}{8} i c^2 \right) \left( \exp \left( \frac{1}{8} i \pi \right) J_{\frac{1}{2}} \left( \frac{1}{8} c^2 \right) - \text{sgn}(c) \exp \left( -\frac{1}{8} i \pi \right) J_{\frac{1}{2}} \left( \frac{1}{8} c^2 \right) \right). \quad (36) \]

Here the \( \zeta \)-dependent functions in the argument of \( P \), and the prefactor, are

\[ F_3(\zeta) = \frac{C_3}{2 \sqrt{A_3 - \frac{1}{4} C_3 \zeta}} = \frac{1}{\zeta^{3/2}} \left( -\zeta(\zeta + 2) + (\zeta + 1) \sqrt{\zeta(\zeta + 2)} \right) \times \sqrt{1 + 14 \zeta + 22 \zeta^2 + 8 \zeta^3 + 2(2 + 7 \zeta + 4 \zeta^2) \sqrt{\zeta(\zeta + 2)}} \]
\[ = \frac{\sqrt{2}}{\zeta} \left( 1 + O(\sqrt{\zeta}) \right) = 2 \left( 1 + O(\zeta^{-1}) \right) \quad (37) \]

and

\[ E_3(\zeta) = \frac{1}{2 \sqrt{\pi} \zeta (A_3 - \frac{1}{4} C_3 \zeta)^{1/4}} \]
\[ = \frac{1}{\sqrt{\pi} \zeta^{1/4}} \left( 1 + O(\sqrt{\zeta}) \right) = \sqrt{\frac{2}{\pi}} \zeta^{1/4} (1 + O(\zeta^{-1})). \quad (38) \]
The functions $F_3(\zeta)$ and $E_3(\zeta)$ embody the scalings we are concerned with. The intensity amplification at cusp3 is $\kappa^{1/2}E_3(\zeta)^2$. The separation of the first two intensity maxima (cf figure 3(c)) is

$$\Delta \xi_{\text{cusp}3} = \frac{X_{\text{cusp}}}{F_3(\zeta)} \kappa^{1/2} \zeta > 1 \approx \frac{1}{2} X_{\text{cusp}} \kappa^{-1/2}, \quad X_{\text{cusp}} = 3.3770.$$ (39)

The $\kappa^{-1/2}$ dependence was anticipated from catastrophe optics, but the fact the for large $\zeta$ the fringe separation is independent of $\zeta$ was unexpected. An analogous phenomenon occurs in optical ‘diffractionless beams’ [47–50].

For $\zeta > 2$, cusp3 lies on an astroid (figure 4) that gets smaller as $\zeta$ increases: its size (distance across to the corresponding cusp on the other side of the astroid, i.e. for $\zeta < 0$), is close to $2/\sqrt{\zeta}$. The local approximation of (35) is valid only while cusp3 and its opposite counterpart are well separated, i.e. the distance between these singularities exceeds the fringe separation. This requires

$$\zeta < \frac{16\kappa}{X_{\text{cusp}}^2}. \quad (40)$$

Figure 7(d) illustrates the accuracy of the transitional approximation in describing these cusp3 oscillations, and figure 8 illustrates how the agreement emerges with increasing accuracy as $\kappa$ increases.

For cusp1 and cusp2, the procedure is the same, with $\eta$ and $\xi$ replacing $\zeta$ and $\eta$, and $v$ and $u$ replacing $u$ and $v$. For the inward-pointing cusp1, the counterparts of equations (36)–(42) are

$$\psi_{\text{cusp}1}(\delta \eta, \zeta, \kappa) = E_1(\zeta) \kappa^{1/4} P(-\delta \eta \kappa^{1/2} F_1(\zeta)), \quad (41)$$

where

$$F_1(\zeta) = \frac{1}{\sqrt{\pi}} \left( \zeta (\zeta - 2) + (\zeta - 1) \sqrt{\zeta (\zeta - 2)} \right) \times \sqrt{-1 + 14\zeta - 22\zeta^2 + 8\zeta^3 - 2 (2 - 7\zeta + 4\zeta^2) \sqrt{\zeta (\zeta - 2)}}$$

$$= \frac{1}{2} \sqrt{3(\zeta - 2)} \left( 1 + O\left(\sqrt{\zeta - 2}\right) \right) = \frac{1}{\sqrt{\zeta}} (1 + O(\zeta^{-1})), \quad (42)$$

and

$$E_1(\zeta) = \frac{1}{\sqrt{\pi}} \left( 1 - 4\zeta + 2\zeta^2 + 2 (\zeta - 1) \sqrt{\zeta (\zeta - 2)} \right) \frac{1}{1/4}$$

$$= \frac{1}{2^{1/4} \sqrt{\pi}} \left( 1 + O\left(\sqrt{\zeta - 2}\right) \right)$$

$$= \frac{1}{2\pi} \zeta^{-3/4} (1 + O(\zeta^{-1})). \quad (43)$$

The intensity amplification at cusp1 is $\kappa^{1/2}E_1(\zeta)^2$. The separation of the first two intensity maxima (cf figure 3(c)) is

$$\Delta \eta_{\text{cusp}1} = \frac{X_{\text{cusp}}}{F_1(\zeta) \kappa^{1/2} \zeta > 1} \approx \zeta^{1/2} X_{\text{cusp}}^{-1/2}, \quad X_{\text{cusp}} = 3.3770.$$ (44)

This increases with $\zeta$, and gets much bigger than the separation (equation (39)) for cusp3. The validity condition, analogous to equation (32) for the fold and equation (40) for cusp3, is that cusp1 is well separated from the fold on the same astroid:

$$\zeta < \frac{\kappa^{1/3}}{X_{\text{cusp}}^{1/3}}. \quad (45)$$
Table 1. Fringe separations, intensity amplifications and validity conditions, expressed in physical variables, for \( Z \gg R_S \) for star and \( Z R_S a^2 \gg 1 \) for fold and cusps; \( X_{fold} = 2.2294, X_{cusp} = 3.3770, \Delta a = 0.2869, |P_{max}| = 6.9438. \) Also listed are the numbers of the equations giving the dimensionless fringe sizes and amplifications.

| Fringe size | Amplification | Validity |
|-------------|---------------|----------|
| Star \( X_{Star} \frac{\lambda}{2^{3/2}\pi} \sqrt{\frac{Z}{R_S}} \) (13) | \( 4\pi^2 \frac{R_S}{\lambda} \) (11) | – |
| Fold \( X_{Fold}\frac{\lambda^{2/3} R_S^{1/3} Z}{(2\pi)^{2/3} a} \) (31) | \( \frac{\pi a^2}{2\sqrt{2}} \left( \frac{2\pi}{\lambda R_S^2} \right)^{1/3} \) (28), (30) | \( Z < \frac{2\pi a^2}{\sqrt{\lambda R_S} \left( \frac{2\pi}{\lambda \sqrt{R_S}} \right)^{1/3}} \) |
| Cusp 1 \( X_{Cusp} \sqrt{\frac{Z \lambda}{2\pi}} \) (44) | \( \frac{|P_{max}|^2 a^3}{R_S \sqrt{2\pi \lambda Z}} \) (41), (43) | \( Z < \frac{a^2}{\sqrt{\frac{2\pi}{\lambda R_S} \left( \frac{2\pi}{\lambda \sqrt{R_S}} \right)^{1/3}}} \) |
| Cusps 2,3 \( X_{Cusp} \frac{a}{2} \left( \frac{\lambda}{\pi R_S} \right) \) (49), (39) | \( \frac{2|P_{max}|^2 R_S}{a} \sqrt{\frac{2\pi Z}{\lambda}} \) (46), (48), (35), (38) | \( Z < \frac{32\pi a^2}{\lambda} \) |

For the outward-pointing cusp2,

\[
\psi_{cusp2}(\delta \eta, \zeta, \kappa) = E_2 \kappa^{1/4} P\left( \delta \eta \kappa^{1/2} F_2(\zeta) \right),
\]

where

\[
F_2(\zeta) = \frac{1}{\zeta^{1/2}} \left( -\zeta (\zeta - 2) + (\zeta - 1) \sqrt{\zeta (\zeta - 2)} \right)
\times \sqrt{-1 + 14\zeta - 22\zeta^2 + 8\zeta^3 + 2(2 - 7\zeta + 4\zeta^2) \sqrt{\zeta (\zeta - 2)}}

= \left( \frac{1}{2} \right) \sqrt{3(\zeta - 2)} \left( 1 + O\left( \sqrt{\zeta - 2} \right) \right) = 2 \left( 1 + O\left( \zeta^{-1} \right) \right),
\]

and

\[
E_2(\zeta) = \frac{1}{\sqrt{\pi}} \left( 1 - 4\zeta^2 - 2(\zeta - 1) \sqrt{\zeta (\zeta - 2)} \right) ^{1/4}

= \frac{1}{2^{1/4} \sqrt{\pi}} \left( 1 + O\left( \sqrt{\zeta - 2} \right) \right)

= \sqrt{\frac{1}{2} \pi} \zeta^{1/4} \left( 1 + O\left( \zeta^{-1} \right) \right).
\]

The intensity amplification at cusp2 is \( \kappa^{1/2} E_2(\zeta)^2 \). The separation of the first two intensity maxima (cf figure 3(c)) is

\[
\Delta \eta_{cusp2} = \frac{X_{cusp}}{F_2(\zeta) \kappa^{1/2}} \approx \frac{1}{2} \frac{X_{cusp} \kappa^{-1/2}}{X_{cusp}} = 3.3770.
\]

This is asymptotically independent of \( \zeta \), like the separation (equation (39)) for cusp3. The validity condition for these oscillations is the same as for cusp3, on whose astroid it lies, namely

\[
\zeta < \frac{16\kappa}{X_{cusp}^2}.
\]

Figures 7(b) and (c) illustrate the accuracy of the transitional approximation near cusp1 and cusp2. For all three cusps, the fast-oscillating contamination by interfering waves not associated with the caustic is much weaker than for the fold (figure 7(a)). This because there are only two such contaminating rays, rather than three.

5. Concluding remarks

The calculations reported here are intended to indicate scalings of the fringe separations and intensity amplifications that could occur in gravitational lensing, in the transitional approximation local to cusp and fold caustics. For convenience, table 1 presents the results in terms of physical variables. The wavelength dependences are those anticipated from singularity theory, but the dependences on astronomical lengths could not have been anticipated without detailed calculation. In particular the dependences of fringe spacing on Schwarzschild radius, binary separation, and lens distance, are very different for the inward-pointing cusp1 and the outward-pointing cusp2 and cusp3. Further similar calculations can be envisaged, for example of the fringe spacings across a cusp, rather than along its symmetry axis; these spacings are much smaller: \( O(\lambda^{3/4}) \) rather than \( O(\lambda^{1/2}) \) [16] (an example of this diffraction anisotropy is the longitudinal stretching of fringes in cusps seen by people wearing eyeglasses while looking at street lights on rainy nights; the images are distorted by raindrop ‘lenses’ on the glass lenses [26, 30, 51]; the apparent oxymoron that these fringes appear black-and-white, i.e. not coloured, is a psychovisual effect [52]).

Two main effects threaten the detection of wave effects such as those anticipated here. The fringes in the observation plane will be blurred by the finite size of the distant source; its angular size, seen from the lens plane, should be smaller than the angle subtended at the lens plane by these fringes, i.e. (spacings in table 1)/\( Z \). Such decoherence is one reason why interference in lensing has not yet been observed; it has been extensively studied, for example in the papers already cited [6, 7, 9, 11, 19–22]. And of course, the size of the aperture of the telescope directed at the source is also a limiting factor; I have not discussed this here, apart from briefly at the end of section 2 (see also [37]).
Data availability statement

No new data were created or analysed in this study.

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Note added in Proof

Diffraction near caustics from weakly nonspherical lenses (e.g. the sun) has been studied recently (see [53] and references therein).

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