FINITE HORIZON PORTFOLIO SELECTION PROBLEMS WITH
STOCHASTIC BORROWING CONSTRAINTS

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Abstract. In this paper we investigate the optimal consumption and investment problem with stochastic borrowing constraints for a finitely lived agent. To be specific, she faces a credit limit which is a constant fraction of the present value of her stochastic labor income at each time. By using the martingale approach and transformation into an infinite series of optimal stopping problems which has the same characteristic as finding the optimal exercise time of an American option. We recover the value function by establishing a duality relationship and obtain the integral equation representation solution for the optimal consumption and portfolio strategies. Moreover, we provide some numerical illustrations for optimal consumption and investment policies.

1. Introduction. In reality, there is a moral hazard or information asymmetry, so the economic agent can not capitalize the full amount of her labor future income to invest or consume as collateral. Thus, the agents has limited opportunities to borrow against future her labor income.

In this paper we study the portfolio selection problem of a finitely-lived agent who faces stochastic borrowing constraint. That is, she has a credit limit which is a constant proportion of the present value of her stochastic labor income at each time. We use the martingale approach developed by Karatzas et al. [17] and Cox and Huang [7]. Similar to He and pages [15], we transform the stochastic borrowing constraints into a static constraint by using a non-decreasing shadow price process. Then the dual problem transforms the primal problem with stochastic borrowing constraints into an unconstrained problem. Similar to El Karoui and Jeanblanc-Picqué [10] and Dybvig and Rogers [9] we transform the dual problem into a series of optimal stopping problems. The optimal stopping problem has the same characteristics as that of finding the optimal exercise time of an American option. We derive the variational inequality from the optimal stopping problem. The partial differential equation(PDE) method can be applied to identify the free boundary for the optimal stopping problem. We establish the duality theorem and

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recover the value function from the dual value function. Moreover, we obtain an integral equation representation for the optimal consumption and portfolio strategy.

There are many researches on the portfolio selection problem with borrowing constraints. He and Pagés [15] and El Karoui and Jeanblanc-Picqué [10] study the portfolio selection problem of an economic agent with non-negative borrowing constraints. Under a model with uninsurable income risk, Koo [19] investigate the effects of non-negative wealth constraints. Farhi and Panageas [11], Choi et al. [5] and Dybvig and Liu [8] consider the optimal consumption, investment and retirement choice problem with non negative borrowing constraints. Park and Jang [24] study an optimal consumption and investment policies with a negative wealth constraints. Recently, Ahn et al. [2] derived the closed-form solution of the portfolio selection problem with time-varying liquidity constraints which is exogenously specified. Choi et al. [4] study a continuous-time model of consumption and portfolio selection with endogenously determined credit limits.

Our contributions are as follows. First, we extend the result of Ahn et al. [2] to finite horizon. To the best of our knowledge, our paper is the first one to consider stochastic borrowing constraint under the finite horizon. Second, we fully characterize the variational inequality arising in our optimization problem. He and Pagés [15] considered a non-negative borrowing constraint in the finite horizon but induced only the PDE. Furthermore, we provide the integral equation representation solution of the free boundary, optimal consumption, wealth and portfolio strategy.

The rest of the paper is organized as follows. Section 2 present our model. In section 3 we provide the link between the dual and primal problems and prove the duality-relationship theorems. Section 4 presents the optimal consumption and investment policies by numerical solutions. Section 5 concludes.

2. The economy. We consider an economic agent who receives stochastic labor income and the continuous-time financial market with her/his lifetime, \([0, T]\). With the stochastic borrowing constraint, the agent’s objective is to maximize the following utility function:

\[ U \equiv E \left[ \int_0^T e^{-\beta(s-t)} u(c_s) ds \right], \]

where \(\beta > 0\) is the subjective discount rate, and \(u(\cdot)\) is a continuously differentiable, strictly concave, and strictly increasing function.

We assumed that the financial market consists of two assets: a riskless asset and one risky asset. The interest rate, \(r > 0\), on the riskless asset is assumed to be constant. The price process \(S_t\) of the risky asset at time \(t\) follows a geometric Brownian motion (GBM)

\[ \frac{dS_t}{S_t} = \mu dt + \sigma dB_t, \]

with a constant drift \(\mu > r\) and a constant volatility \(\sigma > 0\). Here \((B_t)_{0 \leq t \leq T}\) is a standard Brownian motion on the underlying probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and \((\mathcal{F}_s)_{0 \leq s \leq T}\) is the augmentation under \(\mathbb{P}\) of the natural filtration generated by the standard Brownian motion \((B_t)_{0 \leq t \leq T}\).

The agent receives a stream of stochastic labor income \((I_t)_{t=0}^T\) at time \(t\), which is an \(\mathcal{F}\)-progressively measurable process, and is governed by the following GBM

\[ \frac{dI_t}{I_t} = \mu_I dt + \sigma_I dB_t, \]
where \( \mu_I > 0 \) and \( \sigma_I > 0 \) are positive constants. Here we assume that the income process is perfectly correlated with the risk source of the risky financial asset.\(^1\)

Let \( X_t \) be an agent’s wealth process at time \( t \), \( c_t \) be the consumption at time \( t \), and \( \pi_t \) be dollar-amount of investment in the risky asset at time \( t \). The consumption \( c_t \) is assumed to be non-negative and progressively measurable with respect to \( F_t \) for all \( t \in [0, T] \) and the portfolio process \( \pi_t \) is also assumed to be \( F_t \)-measurable for all \( t \in [0, T] \). They satisfy the following mathematical conditions:

\[
\int_0^T c_t dt < \infty, \quad \text{and} \quad \int_0^T \pi_t^2 dt < \infty, \quad \text{a.s.}
\]

Thus the agent’s wealth dynamics \( X_t \) at time \( t \) is given by

\[
dX_t = \left[ rX_t + \pi_t (\mu - r) - c_t + I_t \right] dt + \sigma \pi_t dB_t, \quad X_0 = \bar{x}, \quad I_0 = \bar{I}, \quad (\bar{x}, \bar{I} > 0)
\]

for \( 0 \leq t \leq T \).

For fixed time \( t > 0 \), let us define the market price of risk, the exponential martingale and the pricing kernel, respectively, as

\[
\theta := \frac{\mu - r}{\sigma}, \quad Z_s^t := e^{-\theta (B_s - B_t) - \frac{1}{2} \theta^2 (s-t)}, \quad H_s^t := e^{-r(s-t)} Z_s^t, \quad s \geq t.
\]

Then, for given time \( T > 0 \), we define the equivalent martingale measure

\[
\tilde{P}(A) := E[Z_T^T A], \quad \text{for} \quad A \in F_T.
\]

By Girsanov Theorem, we obtain the process \( \tilde{B}_t = B_t + \theta t \) which is another standard Brownian motion under the risk-neutral measure \( \tilde{P} \). The wealth process in (3) is also reduced to

\[
dX_t = \left[ rX_t - c_t + I_t \right] dt + \sigma \pi_t d\tilde{B}_t,
\]

for \( t \in [0, T] \).

Then, by Fatou’s Lemma and Bayes’ Rule after multiplying \( e^{-r(s-t)} \) to both sides of wealth process (4), we can derive the following static budget constraint:

\[
E_t \left[ \int_t^T H_s^t c_s ds \right] \leq X_t + M_t \quad (5)
\]

where

\[
M_t = E \left[ \int_t^T H_s^t I_s ds \right] = \frac{(1 - e^{-r_I(T-t)})}{r_I} I_t,
\]

\( r_I = r - \mu_I + \theta \sigma_I \) and \( E_t[\cdot] = E[\cdot | F_t] \) is the conditional expectation at time \( t \) on the filtration \( F_t \). Here \( M_t \) is the present value of the stream of the agent’s labor income, and static budget constraint (5) means that the sum of the present value of consumption and that of retirement wealth cannot exceed the sum of the agent’s financial wealth and the present value of future labor income.

**Assumption 1.** The effective discount rate \( r_I \) for labor income is positive

\[
r_I > 0.
\]

\(^1\) The assumption is made to simplify the analysis, avoiding issues related to incompleteness of the financial/insurance market. Ahn et al. [2] have made a similar assumption.
Similar to Ahn et al. [2] we assume that the agent faces the following borrowing constraints: for $\nu$, $0 \leq \nu < 1$

$$X_t \geq -\nu M_t,$$  

(6)

That is, she faces a credit limit which is a constant fraction of the present value of her labor income at each time $t$.

**Definition 2.1.** We call a consumption-portfolio choice $(c, \pi)$, $(c_t)_{t=0}^T$, $(\pi_t)_{t=0}^T$ admissible if

1. **(admissible consumption)** For all $t \in [0, T]$, $c_t$ is a non-negative measurable process with respect to $F_t$ satisfying

$$\int_0^T c_t dt < +\infty, \text{ a.e.}$$

2. **(admissible portfolio)** For all $t \in [0, T]$, $\pi_t$ is measurable process with respect to $F_t$ satisfying

$$\int_0^T \pi_t^2 dt < +\infty, \text{ a.e.}$$

3. **(stochastic borrowing constraint)** For $t \in [0, T]$,

$$X_t \geq -\nu M_t.$$

We denote by $A(x)$ the class of all admissible pairs.

By the similar argument Proposition 2.6 in Karatzas et al. [17], for any admissible consumption $c$, there exist a portfolio process $\pi$ with corresponding wealth process $X$, such that $(c, \pi) \in A(x)$ and

$$X_s = \mathbb{E}_s \left[ \int_s^T H_u^s \left( c_u - I_u \right) du \right].$$

(7)

From the liquidity constraint (6), any admissible consumption $c$ satisfies

$$0 \leq \mathbb{E}_s \left[ \int_s^T H_u^s \left( c_u - (1 - \nu) I_u \right) du \right], \quad t \leq s \leq T.$$  

(8)

3. **Optimization problem.** In this section we derive the value function and the optimal policies with the constant relative risk aversion (CRRA) utility function. The utility function is given by

$$u(c) = \frac{e^{\gamma - 1} - \gamma}{1 - \gamma}, \quad \gamma > 0, \gamma \neq 1,$$

(9)

where $\gamma$ is the agent’s coefficient of relative risk aversion.

3.1. **Primal problem.** We now state our optimization problem of the agents at time $t$.

**Problem 1.** (Primal problem) Given $x, I > 0$, we consider the following maximization problem:

$$V(t, x, I) = \sup_{(c, \pi) \in A(x)} \mathbb{E}_t \left[ \int_t^T e^{\beta(s-t)} u(c_s) ds \mid X_t = x, I_t = I \right]$$

where the maximum is taken over all admissible consumption/portfolio choices $(c, \pi)$ subject to (6).
As in Myneni [22] we will make the following assumption to guarantee the existence of a solution to Problem 1.

**Assumption 2.**

\[ K \equiv r + \frac{\beta - r}{\gamma} + \frac{(\gamma - 1)}{2\gamma^2} \theta^2 > 0. \]

To obtain the solution of Problem 1, we utilize a method proposed by He and Pagés [15] with a modification to fit our setting. First, we heuristically derive the dual problem by ignoring some technical conditions. We write down the Lagrangian as follows:

\[
L = E_t \left[ \int_t^T e^{-\beta(s-t)} u(c_s) ds \right] + \lambda \left( x - E_t \left[ \int_t^T H^t_s(c_s - I_s) ds \right] \right) + \lambda E_t \left[ \int_t^T \eta_s H^t_s \left( \int_s^T H^t_u(c_u - (1 - \nu) I_u) du \right) ds \right],
\]

where \( \eta_s \geq 0 \) is the Lagrange multiplier associated with the liquidity constraint (8) at each time \( s \in [t, T] \) and \( \lambda > 0 \) is the Lagrange multiplier associated static budget constraint (5).

By integration by parts, the third term of right-hand side in (10) can be changed by

\[
E_t \left[ \int_t^T \eta_s \left( \int_s^T H^t_u(c_u - (1 - \nu) I_u) du \right) ds \right] = E_t \left[ \int_t^T d \left( \int_t^s \eta_u du \right) \left( \int_s^T H^t_u(c_u - (1 - \nu) I_u) du \right) \right] \]

\[
= E_t \left[ \int_t^T \left( \int_t^s \eta_u du \right) H^t_s(c_s - (1 - \nu) I_s) ds \right].
\]

Plugging this equation (11) into the Lagrangian (10), we deduce

\[
L = E_t \left[ \int_t^T e^{-\beta(s-t)} u(c_s) ds \right] + \nu \lambda E_t \left[ \int_t^T H^t_s I_s ds \right] - \lambda E_t \left[ \int_t^T \left( 1 - \int_t^s \eta_u du \right) H^t_s(c_s - (1 - \nu) I_s) ds \right] + \lambda x.
\]

We define a non-increasing process \((D_s)_{s=t}^T\) with \( D_t = 1 \) as the cumulative amounts of the Lagrange multipliers,

\[
D_s \equiv 1 - \int_t^s \eta_u du, \quad s \geq t.
\]

We call the process \((D_s)_{s=t}^T\) the *shadow prices*. Then, the Lagrangian \( L \) becomes

\[
L = E_t \left[ \int_t^T e^{-\beta(s-t)} u(c_s) ds \right] + \lambda \left( x + \nu M_t - E_t \left[ \int_t^T D_s H^t_s(c_s - (1 - \nu) I_s) ds \right] \right).
\]
3.2. Dual problem. To derive the dual problem, we first choose consumption to obtain the maximum of the Lagrangian for Problem 1, which takes the following form:

\[ L(D) \equiv \sup_{c_t} \left\{ E_t \left[ \int_t^T e^{-\beta(s-t)} u(c_s) ds \right] + \lambda \left( x + \nu M_t - E_t \left[ \int_t^T D_s H_1^s (c_s - (1 - \nu) I_s) ds \right] \right) \right\} \]  \hspace{1cm} (15)

where \( \lambda > 0 \) is the Lagrange multiplier.

The first-order condition for the optimization problem in (15) is given as follows

\[ c^*_s = (u')^{-1} \left( \lambda e^{\beta(s-t)} H_1^s D_s \right) = \left( \lambda e^{\beta(s-t)} H_1^s D_s \right)^{-\frac{1}{\beta}}. \hspace{1cm} (16) \]

As in He and Pagès [15], we choose the process \( D \) to minimize \( L(D) \).

\[ \inf_{D \in N \mathcal{I}} L(D), \]

where \( N \mathcal{I} \) denotes the set of all positive non-increasing, adapted with respect to \( \mathcal{F} \) and right continuous processes \( D \) with left limits and \( D_t = 1 \). Define the dual value function \( J(t, \lambda, I) \) as

\[ J(t, \lambda, I) = \inf_{D \in N \mathcal{I}} \left\{ E_t \left[ \int_t^T e^{-\beta(s-t)} u(c^*_s) ds \right] + \lambda \left( \nu M_t - E_t \left[ \int_t^T D_s H_1^s (c^*_s - (1 - \nu) I_s) ds \right] \right) \right\} \]

\[ = \inf_{(D_s) \in N \mathcal{I}} \left\{ E_t \left[ \int_t^T \left( e^{-\beta(s-t)} \tilde{u} \left( \lambda e^{\beta(s-t)} D_s H_1^s \right) + \lambda (1 - \nu) D_s H_1^s I_s \right) ds \right] + \nu \lambda M_t \right\}. \hspace{1cm} (17) \]

where \( \tilde{u}(\cdot) \), the convex dual function of \( u(\cdot) \), is defined by

\[ \tilde{u}(y) \equiv \sup_{x > 0} \left( u(x) - xy \right), \quad u(x) = \inf_{y > 0} \left( \tilde{u}(y) + xy \right). \]

To obtain the optimal shadow price process by solving the minimization problem (17), we transform the problem into an infinite series of optimal stopping problems. The transformation hinges on the existence of a one-to-one correspondence between the set of shadow price processes whose paths are right-continuous,

\[ \{(D_s)_{s=t}^T \mid D_s \text{ is non-increasing and right-continuous}\}, \]

and the set of optimal stopping times,

\[ \{(\tau_w)_{0<w<1} \mid \tau_w \text{ is non-increasing and right-continuous of } w\}. \]

The correspondence is given by

\[ \tau_w \land T = \inf \{s > t \mid D_s \leq w \}. \]

Thus, the choice of shadow price process \((D_s)_{0=t}^T\) is equivalent to that of stopping times. The following lemma provides the transformation.
Lemma 3.1. The dual value function can be written as follows:

\[
J(t, \lambda, I) = -I^{1-\gamma} \int_0^1 \sup_{\tau_w \in [t,T]} \mathbb{E}_t^Q \left[ e^{-\hat{\beta}(s-t)} h(\tau_w, w\gamma \tau_w) \right] \frac{dw}{w} + \hat{J}(t, \lambda, I)
\]

where

\[
\hat{J}(t, \lambda, I) = \gamma \left( 1 - e^{-K(T-t)} \right) \lambda^{\frac{1-\gamma}{\gamma}} + \frac{(1 - e^{-r_I(T-t)})}{r_I} \lambda I
\]

\[
h(t, z) = (1 - \nu) z \frac{1 - e^{-r_I(T-t)}}{r_I} - z^{\frac{1-\gamma}{\gamma}} \frac{1 - e^{-K(T-t)}}{K}
\]

\[
\hat{\beta} = \beta - \mu_I (1 - \gamma) + \frac{1}{2} \gamma (1 - \gamma) \sigma_I^2
\]

\[
y_s = \lambda e^\beta \left( s - t \right) H_t y_s, \text{ for } s \in [t,T].
\]

and under the measure \(Q\) defined in the proof, \(B^Q_s = B_s - (1 - \gamma) \sigma_I s\) is a standard Brownian motion and \(\mathbb{E}_t^Q[\cdot]\) is the expectation with respect to measure \(Q\).

Proof. The proof is given in Appendix A. \(\square\)

3.3. Optimal stopping problem and variational inequality. From Lemma 3.1, the dual problem is equivalent to an infinite series of optimal stopping problems. Similar to Dybvig and Rogers [9] and El Karoui and Jeanblanc-Picqué [10], the problems, however, can be reduced to essentially a single problem as shown below.

Problem 3. We consider the following optimal stopping problem:

\[
g(t, z) = \sup_{\tau \in \mathcal{S}(t,T)} \mathbb{E}_t^Q \left[ e^{-\hat{\beta}(\tau-t)} h(\tau, wy) \mid wy = z \right]
\]

where \(\mathcal{S}(t,T)\) denotes the set of all stopping times of the filtration \(\mathcal{F}\) taking values in \([t,T]\).

Notice that Problem 3 is equivalent to that of finding the optimal exercise time of an American option of the put type written on the marginal wealth process \((y_s)\) with payoff equal to \(h(\tau, y_\tau)\) at the time of exercise \(\tau\).

From Problem 3,

\[
g(t, z) = \sup_{\tau \in \mathcal{S}(t,T)} \mathbb{E}_t^Q \left[ e^{-\hat{\beta}(\tau-t)} h(\tau, y_\tau) \mid y_\tau = z \right]. \tag{18}
\]

where

\[
h(t, z) = \left( (1 - \nu) \frac{1 - e^{-r_I(T-t)}}{r_I} - z - \frac{1 - e^{-K(T-t)}}{K} \right)
\]

and \((y_s)_{s \geq t}\) satisfies the following stochastic differential equation (SDE):

\[
dy_s = (\hat{\beta} - r_I) y_s ds + \sigma_s y_s dB^Q_s \quad y_t = z,
\]

with \(\sigma_s \equiv \gamma \sigma_I - \theta\).

The problem is similar to the standard American put option problem treated in the literature (see e.g., Myneni [23] and Peskir [25]), but different from the latter in the sense that the payoff is time dependent. In this section we provide a complete self-contained derivation of the solution to the problem, following the ideas and proofs in Yang and Koo [26].
Using the standard techniques for an optimal stopping problem, \( g(t,z) \) is a solution of the following variational inequality (VI) (see e.g., Ch. 2 of Karatzas and Shreve [18] or Yang and Koo [26]):

\[
\begin{align*}
-\partial_t g - \mathcal{L}g &= 0, \quad \text{if } g(t,z) > h(t,z) \text{ and } (t,z) \in \mathcal{M}_T, \\
-\partial_t g - \mathcal{L}g &\geq 0, \quad \text{if } g(t,z) = h(t,z) \text{ and } (t,z) \in \mathcal{M}_T, \\
g(T,z) &= h(T,z) = 0 
\end{align*}
\]

where \( \mathcal{M}_T = [0,T] \times (0, +\infty) \) and the operator \( \mathcal{L} \) is given by

\[
\mathcal{L} \equiv \frac{\sigma^2}{2} z^2 \partial_{zz} + (\gamma - r_t) z \partial_z - \hat{\gamma}.
\]

By analyzing the variational inequality (26), we derive the following integral representation of the value function \( g(t,z) \) of Problem 3 (In Appendix B we provide all properties for variational inequality (26) arising from Problem 3)

\[
g(t,z) = (1-\nu) z \int_t^T e^{-r_t(\eta-t)} N\left(d^+ (\eta - t, \frac{z}{z^*(\eta)}) \right) d\eta \\
- z^{-\frac{1-\nu}{2}} \int_t^T e^{-K(\eta-t)} N\left(d^+ (\eta - t, \frac{z}{z^*(\eta)}) \right) d\eta,
\]

for \( z \leq z^*(t) \), where \( z^* \) is the free boundary of Problem 3, \( N(\cdot) \) is the cumulative distribution function of a standard normal random variable, and

\[
d^+ (t,z) = \log z + (\hat{\beta} - r_t + \frac{1}{2} \sigma^2 t) t \\
d^+ (t,z) = \log z + (\hat{\beta} - r_t - \frac{1}{2} \sigma^2 t - \frac{1-\nu}{\gamma} \sigma^2) t.
\]

Define the time-reversed free boundary \( \bar{z}^*(t) = z^*(T-t) \), we can obtain that \( \bar{z}^*(t) \) satisfies the following integral equation:

\[
0 = (1-\nu) \int_0^t e^{-r_t \xi} N\left(-d^+ (\xi, \frac{\bar{z}^*(t)}{\bar{z}^*(t-\xi)}) \right) d\xi \\
- (\bar{z}^*(t))^{-\frac{1}{2}} \int_0^t e^{-K \xi} N\left(-d^+ (\xi, \frac{\bar{z}^*(t)}{\bar{z}^*(t-\xi)}) \right) d\xi.
\]

Therefore, we have the following lemma.

**Lemma 3.2.** For \( y_s = z \), the optimal stopping time \( \tau^* \) for Problem 3 is expressed by

\[
\tau^* \equiv \inf \{ s \geq t \mid y_s \geq z^*(s) \}
\]

By Lemma 3.1 and the integral equation representation of \( g(t,z) \) in (20), we can obtain the following proposition.

**Proposition 1.** The dual value function \( J(t,\lambda, I) \) is given by

\[
J(t,\lambda, I) = -I^{1-\gamma} \int_0^1 \left[ (1-\nu) \int_t^T e^{-r_t(\eta-t)} (w\lambda I^\gamma) \cdot N\left(d^+ (\eta - t, \frac{w\lambda I^\gamma}{z^*(\eta)}) \right) d\eta \\
- \int_t^T e^{-K(\eta-t)} (w\lambda I^\gamma)^{-\frac{1-\nu}{\gamma}} \cdot N\left(d^+ (\eta - t, \frac{w\lambda I^\gamma}{z^*(\eta)}) \right) d\eta \right] \frac{dw}{w} \\
+ \frac{1-e^{-K(T-t)}}{K} \frac{\gamma}{1-\gamma} \lambda^{-\frac{1-\nu}{\gamma}} \frac{1-e^{-r_t(T-t)}}{r_t} \lambda I.
\]
3.4. **Main theorem and verification.** Once we find the dual value function \( J(t, \lambda, I) \), the value function \( V(t, x, I) \) of Problem 1 is determined by

\[
V(t, x, I) = \inf_{\lambda > 0} (J(t, \lambda, I) + \lambda x).
\]

Applying a method of convex analysis, we can prove the above duality relationship valid.

**Theorem 3.3.** (a) The value function of Problem 1 and the dual value function derived in Proposition 1 satisfy the following duality relationship :

\[
V(t, x, I) = \inf_{\lambda > 0} (J(t, \lambda, I) + \lambda x).
\]

Moreover, the shadow price \( D^*_s \) is determined by

\[
D^*_s = \min \left( 1, \min_{t \leq \xi \leq s} \frac{z^*(\xi)}{y^*_s} \right), \quad s \in [t, T].
\]

where \( y^*_s = e^{\beta(s-t)} H^t_s I^s \) and \( \lambda^* \) is the unique solution to the minimization problem (22).

(b) The optimal stopping problem in Lemma 3.1 has the following solution

\[
\tau_w = \inf \{ s \geq t \mid w \cdot y^*_s \geq z^*(s) \}.
\]

Moreover, the shadow price \( D^*_s \) is determined by

\[
D^*_s = \min \left( 1, \min_{t \leq \xi \leq s} \frac{z^*(\xi)}{y^*_s} \right), \quad s \in [t, T].
\]

(c) For \( s \geq t \), optimal wealth process \( X^*_s \) satisfies

\[
\frac{X^*_s}{I_s} = \frac{1}{y^*_s D^*_s} g(s, y^*_s D^*_s) + \frac{1 - e^{-K(T-s)}}{K} \left( y^*_s D^*_s \right)^{-\frac{1}{\gamma}} - \frac{1 - e^{-r_I(T-s)}}{r_I}
\]

\[
= (1 - \nu) \int_s^T e^{-r_I(\eta-s)} N \left( d^+ \left( \eta - s, \frac{y^*_s D^*_s}{z^*(\eta)} \right) \right) d\eta
\]

\[
- \left( y^*_s D^*_s \right)^{-\frac{1}{\gamma}} \int_s^T e^{-K(\eta-s)} N \left( d^+ \left( \eta - s, \frac{y^*_s D^*_s}{z^*(\eta)} \right) \right) d\eta
\]

\[
+ \frac{1 - e^{-K(T-s)}}{K} \left( y^*_s D^*_s \right)^{-\frac{1}{\gamma}} - \frac{1 - e^{-r_I(T-s)}}{r_I} \geq -\nu M_t.
\]

(d) For \( s \geq t \), optimal consumption \( c^*_s \) and optimal portfolio \( \pi^*_s \) are given by

\[
\frac{c^*_s}{I_s} = \left( y^*_s D^*_s \right)^{-\frac{1}{\gamma}}
\]

\[
= \frac{K}{1 - e^{-K(T-s)}} \left( \frac{X^*_s}{I_s} + \frac{1 - e^{-r_I(T-s)}}{r_I} \right)
\]

\[
+ \frac{K}{1 - e^{-K(T-s)}} \left\{ (1 - \nu) \int_s^T e^{-r_I(\eta-s)} N \left( d^+ \left( \eta - s, \frac{y^*_s D^*_s}{z^*(\eta)} \right) \right) d\eta
\]

\[- \left( y^*_s D^*_s \right)^{-\frac{1}{\gamma}} \int_s^T e^{-K(\eta-s)} N \left( d^+ \left( \eta - s, \frac{y^*_s D^*_s}{z^*(\eta)} \right) \right) d\eta \right\}
\]

and

\[
\frac{\pi^*_s}{I_s} = \frac{\theta}{\gamma \sigma} \left( \frac{X^*_s}{I_s} + \frac{1 - e^{-r_I(T-s)}}{r_I} \right) - \frac{\sigma_I}{\sigma} \frac{1 - e^{-r_I(T-s)}}{r_I}
\]
The proof is given in Appendix D.

Remark 1. Theorem 3.4 recovers the result of Ahn et al. [2].

Corollary 1. When \( X_s^* = -\nu M_s \), the optimal portfolio \( \pi^*_s \) is given by

\[
\pi^*_s = -\nu \frac{\sigma}{\sigma} M_s.
\]

Proof. Since \( X_s^* = -\nu M_t \),

\[
y^*_s D^*_s = z^*(s).
\]

By Lemma B.3 and \( g(t, z) = h(t, z) + zQ(t, z) \),

\[
g(s, z^*(s)) = (1 - \nu) \cdot \left( \frac{1 - e^{-r_1(T-t)}}{r_I} \right) z^*(s) - \frac{(1 - e^{-K(T-t)})}{K} (z^*(s))^{-\frac{1 - \gamma}{\gamma}},
\]

and

\[
\partial_z g(s, z^*(s)) = (1 - \nu) \cdot \left( \frac{1 - e^{-r_1(T-t)}}{r_I} \right) + \frac{(1 - \gamma)}{\gamma} \frac{(1 - e^{-K(T-t)})}{K} (z^*(s))^{-\frac{1}{\gamma}}.
\]

From (58), we can deduce that

\[
\pi^*_s = -\nu \frac{\sigma}{\sigma} M_s.
\]

We will now prove the convergence of optimal policies for a finite horizon problem to those for the infinite horizon problem.

Theorem 3.4. The following statements are true :

1. As time to maturity goes to infinity, i.e., \( T-t \to \infty \), the free boundary \( z^*(t) \) converges to \( z^\infty \) defined in (33).

2. For \( s \in [t, T] \), as time to maturity goes to infinity, the agent’s optimal wealth \( X^*_{s,\infty} \), and portfolio \( \pi^*_s \) is given by

\[
\frac{X^*_{s,\infty}}{I_s} = \frac{(1 - \nu)}{(1 + \gamma \alpha)} \left( \frac{y^*_s D^*_s}{z^\infty} \right)^{\alpha+} + \frac{1}{K} \left( y^*_s D^*_s \right)^{\frac{1}{\gamma}} - \frac{1}{r_I},
\]

\[
\frac{\pi^*_s}{I_s} = \frac{\theta}{\gamma \sigma} \left( \frac{X^*_{s,\infty}}{I_s} + \frac{1}{r_I} \right) - \frac{\sigma I_s}{\gamma \sigma} + \left( \frac{y^*_s D^*_s}{z^\infty} \right)^{\alpha+},
\]

where

\[
D^*_s = \min \left( 1, \min_{t \leq \tau \leq s} \frac{z^\infty}{y^*_s} \right).
\]

Proof. The proof is given in Appendix D.

Remark 1. Theorem 3.4 recovers the result of Ahn et al. [2].
4. Numerical illustrations. In this section we give some numerical results for optimal consumption and investment strategies. The parameter values are used as follows:

\[ \mu = 0.05, \sigma = 0.2, r = 0.01, \beta = 0.05, \gamma = 2, \mu_I = 0.012, \sigma_I = 0.1, \nu = 0.3. \]

First, Figure 1 plots the free boundary \( z^*(t) \) defined in (30). By using the recursive integration method proposed by Huang et al. [16], we numerically solved the integral equation of time-reversed free boundary \( \tilde{z}^*(t) \) in (21).

The free boundary \( z^*(t) \) partitions the \((t, y)\)-region into the jump region and no-jump region. The optimal shadow price process \( D^* \) ensures that \((s, y^*_s D^*_s)\) will never leave the no-jump region.

\[ \text{Figure 1. Free boundary } z^*(t). \text{ Parameter values are given by } \mu = 0.05, \sigma = 0.2, r = 0.01, \beta = 0.05, \gamma = 2, \mu_I = 0.012, \sigma_I = 0.1, \nu = 0.3 \text{ and } T = 10. \]

Figure 2 shows the simulated paths of the optimal wealth to income rate \( X^*/I \), optimal portfolio to income rate \( \pi^*/I \), optimal consumption to income rate \( c^*/I \), the process \( y^* D^* \) and the optimal shadow price process \( D^* \). As shown in figure 2, the process \( D^* \) regulates \( y^* D^* \) so that \((s, y^*_s D^*_s)\) stays inside the no-jump region. Whenever \( y^* D^* \) high enough to hit the free boundary \( z^* \), the non-increasing process \( D^* \) decreases. That is, the optimal process \( D^* \) must have the property that it decrease only when \( y^* D^* \) hits the free boundary, at which time the liquidity constraints (8) bind. Moreover, when the agent’s wealth to income ratio \( X/I \) approach the boundary \( -\nu M \), she consumes less than the income so that his/her wealth does not hit the boundary.

5. Concluding remarks. In this paper, we have proposed a martingale approach to solving portfolio selection problems with stochastic borrowing constraints. That is, if the agent’s income process follows a geometric Brownian motion, then she has a credit limit which is a constant proportion of the present value of her labor income at each time. We have proved the duality relationships and transforms the dual problem into an infinite series of optimal stopping problems similar to Dybvig and Rogers [9] and El Karoui and Jeanblanc-Picqué [10]. Using the PDE...
In this paper we have assumed that the agent does not face a bequest motive, consideration of the case that the agent has the bequest utility function is left as a topic for future research. This paper assumes that the markets are complete. However, when markets are incomplete, the individuals have difficulties in managing various risks appropriately due to lack of financial hedging tools and insurance contracts. Cocco et al. [6] and Bensoussan et al. [3] investigates the portfolio selection problems with the market incompleteness. The extension of our work to incomplete market setup is left as a topic for future research.
Acknowledgments. We would like to thank you for following the instructions above very closely in advance. It will definitely save us lot of time and expedite the process of your paper’s publication.

Appendix A. Proof of Lemma 3.1. Let us define a function by

\[ f(w) = e^{-\beta(s-t)} \tilde{u} \left( w \lambda e^{\beta(s-t)} H_s^t \right) + \lambda w (1 - \nu) I_s H_s^t. \]

Then,

\[
J(t, \lambda, I) = \inf_{\{D_s\} \in \mathcal{N}^T} \mathbb{E}_t \left[ \int_t^T f(D_s) ds \right] + \lambda \nu M_t
\]

\[
= \inf_{\{D_s\} \in \mathcal{N}^T} \mathbb{E}_t \left[ \int_t^T \left( f(1) - \int_{D_s} f' \right) dw \right] ds + \lambda \nu M_t
\]

\[
= - \sup_{\{D, \gamma\} \in \mathcal{N}^T} \mathbb{E}_t \left[ \int_t^T \int_0^1 f'(w) 1_{\{w > D_s\}} dw ds \right] + \mathbb{E}_t \left[ \int_t^T f(1) ds \right] + \lambda \nu M_t
\]

\[
= - \sup_{\tau \in [t,T]} \mathbb{E}_t \left[ \int_0^1 \int_t^T f'(w) 1_{\{s > \tau\}} ds dw \right] + \mathbb{E}_t \left[ \int_t^T f(1) ds \right] + \lambda \nu M_t
\]

\[
= - \int_0^1 \sup_{\tau \in [t,T]} \mathbb{E}_t \left[ \int_{\tau}^T f'(w) dw \right] + \mathbb{E}_t \left[ \int_t^T \left( e^{-\beta(s-t)} \tilde{u} \left( \lambda e^{\beta(s-t)} H_s^t \right) + \lambda I_s H_s^t \right) ds \right],
\]

where a stopping time \( \tau \) is defined by

\[ \tau = \inf \{ s \geq t \mid D_s \leq w \} \]

for all \( 0 < w < 1 \).

For \( s \in [t, T] \), let us define a process \( K_s^t \) and an equivalent probability measure \( \mathbb{Q} \) defined by

\[ K_s^t = e^{-\frac{1}{2} (1 - \gamma)^2 \sigma_s^2 (s-t) + (1 - \gamma) \sigma_s (B_s - B_t)}. \]

and

\[ \frac{d\mathbb{Q}}{d\mathbb{P}} = K_s^t, \]

respectively.

Then, by the Girsanov theorem, for \( t \leq s \leq T \),

\[ B_s^\mathbb{Q} = B_s - (1 - \gamma) \sigma_t t \cdot s \]

is a standard Brownian motion under \( \mathbb{Q} \) measure.

Since

\[ f'(w) = \lambda H_s^t \tilde{u} \left( w \lambda e^{\beta(s-t)} H_s^t \right) + (1 - \nu) w \lambda I_s H_s^t, \]
Appendix

\[ \sup_{\tau_w \in [t, T]} \mathbb{E}_t \left[ \int_{\tau_w}^T f'(w) ds \right] \]
\[ = \sup_{\tau_w \in [t, T]} \mathbb{E}_t \left[ \int_{\tau_w}^T \frac{1}{w} e^{-\beta(s-t)} \left( (1 - \nu)w \lambda e^{\beta(s-t)} I_s H_s^t - \frac{1 - \gamma}{\lambda} \right) ds \right] \]
\[ = \frac{1 - \gamma}{w} \sup_{\tau_w \in [t, T]} \mathbb{E}_t \left[ e^{-\beta(t-\tau_w)} K_{\tau_w} \mathbb{E}_{\tau_w} \left[ \int_{\tau_w}^T e^{-\beta(s-\tau_w)} \left( (1 - \nu)w \lambda e^{\beta(s-t)} H_s^t I_s \right) \right. \right. \right. \]
\[ \left. \left. - \left( w \lambda e^{\beta(s-t)} H_s^t I_s \right) \right] ds \right] \]
\[ = \frac{1 - \gamma}{w} \sup_{\tau_w \in [t, T]} \mathbb{E}_t \left[ e^{-\beta(t-\tau_w)} K_{\tau_w} \mathbb{E}_{\tau_w} \left[ \int_{\tau_w}^T e^{-\beta(s-\tau_w)} \left( (1 - \nu)w \lambda e^{\beta(s-t)} H_s^t I_s \right) \right. \right. \right. \]
\[ \left. \left. - \left( w \lambda e^{\beta(s-t)} H_s^t I_s \right) \right] ds \right] \] \hspace{1cm} \text{(24)}

By direct calculation,
\[ \mathbb{E}_t \left[ \int_{t}^{T} \left( e^{-\beta(s-t)} g \left( \lambda e^{\beta(s-t)} H_s^t \right) + \lambda I_s H_s^t \right) ds \right] \]
\[ = \frac{\gamma}{1 - \gamma} \left( 1 - e^{-K(T-t)} \right) \frac{1 - \gamma}{\lambda} + \frac{(1 - e^{-r_1(T-t)}) \lambda}{r_1} \lambda I. \] \hspace{1cm} \text{(25)}

By (23), (24) and (25), we have proved the desired result.

Appendix B. The solution of optimal stopping Problem 3. In this section we provide a complete self-contained derivation of the solution to the problem following the ideas and proofs in citeYK.

Since the lower obstacle \( h(t, z) \) in VI (19) is not monotonic function of \( z \), it is difficult to analyze VI (19). Therefore, we consider the following substitutions.
\[ \tilde{g}(t, z) = \frac{g(t, z)}{z}, \quad \tilde{h}(t, z) = \frac{h(t, z)}{z}, \quad Q(t, z) = \tilde{g}(t, z) - \tilde{h}(t, z). \]

Therefore, the VI (19) can be converted to
\[
\begin{align*}
-\partial_t Q - \mathcal{L}^* Q &= -((1 - \nu) - z^{-\frac{1}{2}}), & \text{if } Q(t, z) > 0 \text{ and } (t, z) \in \mathcal{M}_T, \\
-\partial_t Q - \mathcal{L}^* Q &\geq -((1 - \nu) - z^{-\frac{1}{2}}), & \text{if } Q(t, z) = 0 \text{ and } (t, z) \in \mathcal{M}_T, \\
Q(T, z) &= 0
\end{align*}
\] \hspace{1cm} \text{(26)}
where $\mathcal{L}^*$ is given by

$$
\mathcal{L}^* = \frac{\sigma^2}{2} \partial_{zz} + (\hat{\beta} - r_I + \sigma^2) \partial_z - r_I.
$$

We will prove the existence and uniqueness of $W^{1,2}_{p,\text{loc}}$ solution to VI (26) and describe properties of the solution in the lemmas below.

**Lemma B.1.** VI (26) has a unique strong solution $Q$ satisfying the following properties:

1. $Q \in W^{1,2}_{p,\text{loc}}(\mathcal{M}_T) \cap C(\bar{\mathcal{M}}_T)$ for any $p \geq 1$ and $\partial_z Q \in C(\mathcal{M}_T)$, where $\bar{\mathcal{M}}_T = [0, T] \times (0, +\infty)$.

2. $\partial_t Q \leq 0$ in $\mathcal{M}_T$ and $\partial_t Q \leq 0$, a.e. in $\bar{\mathcal{M}}_T$.

**Proof.**

1. First, we transform the degenerate parabolic problem (26) into the following non-degenerate parabolic problem. Setting

$$
\tilde{Q}(t, w) = Q(t, z),
$$

we have

$$
\begin{align*}
-\partial_t \tilde{Q} - \mathcal{L} \tilde{Q} &= -((1 - \nu) - e^{-\frac{\gamma}{2} w}), & &\text{if } \tilde{Q}(t, x) > 0 \text{ and } (t, x) \in [0, T) \times \mathbb{R}, \\
-\partial_t \tilde{Q} - \mathcal{L} \tilde{Q} &\geq -((1 - \nu) - e^{-\frac{\gamma}{2} w}), & &\text{if } \tilde{Q}(t, x) = 0 \text{ and } (t, x) \in [0, T) \times \mathbb{R}, \\
\tilde{Q}(T, x) &= 0
\end{align*}
$$

where

$$
\mathcal{L} = \frac{\sigma^2}{2} \partial_{ww} + (\hat{\beta} - r_I + \frac{\sigma^2}{2}) \partial_w - \hat{\beta}.
$$

Since the non-homogeneous term, $((1 - \nu) - e^{-\frac{\gamma}{2} w})$, the lower obstacle, 0, and the terminal condition, 0, are smooth function, it is not hard to show that (27) has a unique solution $\tilde{Q} \in W^{1,2}_{p,\text{loc}}([0, T] \times \mathbb{R}) \cap C([0, T] \times \mathbb{R})$ for any $p \geq 1$ and $\partial_w \tilde{Q} \in C([0, T] \times \mathbb{R})$. (See citeF1).

2. Let us temporarily denote $\tilde{Q}(t, z) = Q(t, \lambda z)$ for any $\lambda > 1$ and $\tilde{Q}(t, z) = Q(t - \delta, z)$ with $\delta > 0$ being sufficiently small. Then $\tilde{Q}$ satisfies

$$
\begin{align*}
-\partial_t \tilde{Q} - \mathcal{L}^* \tilde{Q} &= -((1 - \nu) - (\lambda z)^{-\frac{1}{2}}), & &\text{if } \tilde{Q}(t, z) > 0 \text{ and } (t, z) \in \mathcal{M}_T, \\
-\partial_t \tilde{Q} - \mathcal{L}^* \tilde{Q} &\geq -((1 - \nu) - (\lambda z)^{-\frac{1}{2}}), & &\text{if } \tilde{Q}(t, z) = 0 \text{ and } (t, z) \in \mathcal{M}_T, \\
\tilde{Q}(T, z) &= 0.
\end{align*}
$$

Since $-((1 - \nu) - (\lambda z)^{-\frac{1}{2}}) < -((1 - \nu) - z^{-\frac{1}{2}})$ for any $\lambda > 1$ and the non-homogeneous term and the terminal values in VIs of $Q$ and $\tilde{Q}$ are the same, the comparison theory for VI (See citeF2) implies that $\bar{Q}(t, z) = Q(t, \lambda z) \leq Q(t, z)$ for any $\lambda > 1$ and $(t, z) \in \mathcal{M}_T$. Hence, we obtain $\partial_z Q \leq 0$ in $\mathcal{M}_T$.

Also, it is easy to deduce that $\tilde{Q}$ satisfies

$$
\begin{align*}
-\partial_t \tilde{Q} - \mathcal{L}^* \tilde{Q} &= -((1 - \nu) - z^{-\frac{1}{2}}), & &\text{if } \tilde{Q}(t, z) > 0 \\
&\quad \text{and } (t, z) \in \mathcal{M}_{T, \delta} \triangleq [\delta, T) \times (0, +\infty), \\
-\partial_t \tilde{Q} - \mathcal{L}^* \tilde{Q} &\geq -((1 - \nu) - z^{-\frac{1}{2}}), & &\text{if } \tilde{Q}(t, z) = 0 \text{ and } (t, z) \in \mathcal{M}_{T, \delta}, \\
\tilde{Q}(T, z) &= Q(T - \delta, z) \geq 0.
\end{align*}
$$

Since $\tilde{Q}(T, z) \geq Q(T, z)$ for any $\delta > 0$, $z > 0$, applying the comparison principle for VI implies $\tilde{Q}(t, z) = Q(t - \delta, z) \leq Q(t, z)$ for any $\delta > 0$, $(t, z) \in \mathcal{M}_{T, \delta}$. Hence, we deduce that $\partial_t Q \leq 0$, a.e.
Denote
\[
\begin{align*}
JR &= \{(t, z) \mid Q(t, z) = 0\} \text{ (jump region),} \\
NR &= \{(t, z) \mid Q(t, z) > 0\} \text{ (no-jump region),}
\end{align*}
\]

Applying Lemma B.1, \(Q\) is monotone decreasing with respect to \(z\), so we can define the free boundary
\[
z^*(t) = \sup\{z \geq 0 \mid Q(t, z) > 0\}, \text{ any } t \in [0, T]. \tag{30}
\]

Moreover, we can rewrite the jump region \(JR\) and the no-jump region \(NR\) as follows
\[
JR = \{(t, z) \mid z \geq z^*(t), \ t \in [0, T]\}, \quad NR = \{(t, z) \mid 0 < z < z^*(t), \ t \in [0, T]\}.
\]

**Remark 2.** If initially \((t, y_t) \in JR\), then \(D\) should jump up immediately, such that \(y_t\) reaches the boundary. On the other hand, if \((t, y_t) \in NR\), \(D\) must stay constant. Therefore, we call \(JR\) and \(NR\) the jump region and the no-jump region, respectively.

**Lemma B.2.** In no-jump region \(NR\), the function \(Q(t, z)\) is a strictly decreasing function with respect to \(z\).

**Proof.** It is enough to show that for \(\lambda > 1\),
\[
Q(t, \lambda z) < Q(t, z), \quad \text{for } (t, z) \in NR.
\]

Since \(\lambda > 1\) and \(\gamma > 0\), we can find \(\mu\) such that
\[
1 < \mu < \lambda^{\frac{1}{\gamma}}.
\]

Let us temporarily denote \(\tilde{Q}(t, z) = \mu \cdot Q(t, \lambda z)\).
\[
\begin{align*}
&\begin{cases}
-\partial_t \tilde{Q} - \mathcal{L}^* \tilde{Q} = -\mu((1 - \nu) - (\lambda z)^{-\frac{1}{\gamma}}), \quad \text{if } \tilde{Q}(t, z) > 0 \text{ and } (t, z) \in M_T, \\
-\partial_t \tilde{Q} - \mathcal{L}^* \tilde{Q} \geq -\mu((1 - \nu) - (\lambda z)^{-\frac{1}{\gamma}}), \quad \text{if } \tilde{Q}(t, z) = 0 \text{ and } (t, z) \in M_T, \\
\tilde{Q}(T, z) = 0.
\end{cases}
\end{align*}
\]

Since \(-\mu((1 - \nu) - (\lambda z)^{-\frac{1}{\gamma}}) < -((1 - \nu) - z^{-\frac{1}{\gamma}})\) for any \(\lambda > 1\) and \(1 < \mu < \lambda^{\frac{1}{\gamma}}\) and the non-homogeneous term and the terminal values in VIs of \(Q\) and \(\tilde{Q}\) are the same, the comparison principle yields that
\[
\tilde{Q}(t, z) = \mu \cdot Q(t, \lambda z) \leq Q(t, z), \quad \text{for } \lambda > 1.
\]

Since \(Q(t, z) > 0\) in the no-jump region \(NR\), we can obtain that
\[
Q(t, \lambda z) < Q(t, z), \quad \text{for } (t, z) \in NR.
\]

**Lemma B.3.** The free boundary \(z^*\) is smooth, i.e., \(z^*(t) \in C[0, T] \cap C^\infty([0, T])\). Moreover, the solution \(Q \equiv 0\) in \(JR\), and \(Q \in C^\infty([\{(t, z) \mid z \leq z^*(t), \ t \in [0, T]\}])\), and \(\partial_t Q \in C(\mathcal{M}_T)\).

**Proof.** By Lemma B.1, we obtain \(\partial_t Q \leq 0\) a.e. in \(\mathcal{M}_T\). Moreover, the coefficient functions in the operator \(\mathcal{L}\), the lower obstacle function, the terminal function, and the non-homogeneous term \(-(1 - \nu) - z^{-\frac{1}{\gamma}}\) are all smooth. Therefore, the regularity results in the lemma follow from Theorem 3.1 in citeF2.
Consider the following function $Q^\infty$ as follows:

$$Q^\infty(t, z) = \begin{cases} \frac{(1 - \nu)}{r_I (1 + \gamma \alpha_+)} \left( \frac{z}{z^\infty} \right)^{\alpha_+} + \frac{1}{K} z^{-\frac{1}{\gamma}} - \frac{(1 - \nu)}{r_I}, & (t, z) \in [0, T] \times (0, z^\infty], \\ 0, & (t, z) \in [0, T] \times (z^\infty, +\infty), \end{cases}$$

(32)

where

$$z^\infty = \left( \frac{(1 + \gamma \alpha_+) r_I}{\alpha_+ K (1 - \nu)} \right)^{\gamma},$$

(33)

and $\alpha_+$ and $\alpha_-$ are the positive and negative roots of the following quadratic equation, respectively,

$$\frac{\hat{\sigma}^2}{2} \alpha^2 + \left( \hat{\beta} - r_I + \frac{1}{2} \hat{\sigma}^2 \right) \alpha - r_I = 0.$$

(34)

Then, the following lemma is established.

Lemma B.4. $Q \equiv 0$ in the domain $[0, T] \times [z^\infty, +\infty)$, where $z^\infty$ is defined in (33). And $Q(t, z) > 0$ in the domain $[0, T] \times (0, z^T)$ with $z^T = (1 - \nu)^{-\gamma}$.

Proof. Firstly, we will show that

$$z^\infty > z^T.$$

(35)

(1) $r_I > K$

In this case, it is obvious that above inequality (35) holds.

(2) $r_I \leq K$

In this case,

$$z^\infty > z^T \iff \alpha_+ < \frac{r_I}{\gamma (K - r_I)}.$$

Since $\alpha_+ > 0$ and $\alpha_- < 0$, it is sufficient to show that

$$\frac{\hat{\sigma}^2}{2} \left( \frac{r_I}{\gamma (K - r_I)} \right)^2 + \left( \hat{\beta} - r_I + \frac{1}{2} \hat{\sigma}^2 \right) \left( \frac{r_I}{\gamma (K - r_I)} \right) - r_I > 0.$$

Since

$$K = r + \frac{\beta - r}{\gamma} + \frac{\gamma - 1}{2 \gamma^2} \theta^2 = r_I + \frac{\hat{\beta} - r_I}{\gamma} + \frac{\gamma - 1}{2 \gamma^2} \hat{\sigma}^2,$$

$$\hat{\sigma}^2 r_I + 2(\hat{\beta} - r_I + \frac{1}{2} \hat{\sigma}^2) \gamma (K - r_I) - 2 \gamma^2 (K - r_I)^2$$

$$= \hat{\sigma}^2 r_I + (2 \hat{\beta} - 2 r_I + \frac{1}{2} \hat{\sigma}^2) \gamma \left( \hat{\beta} - r_I + \frac{\gamma - 1}{2 \gamma} \hat{\sigma}^2 \right) - 2 \gamma^2 \left( \hat{\beta} - r_I + \frac{\gamma - 1}{2 \gamma} \hat{\sigma}^2 \right)^2$$

$$= \hat{\sigma}^2 \left( \hat{\beta} - \frac{\gamma - 1}{\gamma} (\hat{\beta} - r_I) + \frac{\gamma - 1}{2 \gamma^2} \hat{\sigma}^2 \right) = \hat{\sigma}^2 \cdot K > 0.$$

Therefore, we conclude $z^\infty > z^T$. 

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Secondly, in \([0, T] \times (0, z^\infty)\),
\[
\frac{\partial Q^\infty}{\partial z} = \frac{1}{\gamma K} z^{-1-\frac{1}{\gamma}} \left( \left( \frac{z}{z^\infty} \right)^{\alpha + \frac{1}{\gamma}} - 1 \right) < 0
\]
Also, it is not hard to check that
\[Q^\infty \in \mathcal{C}(\mathcal{M}_T) \cap W^{1,2}_{p, \text{loc}}(\mathcal{M}_T),\]
and
\[-\partial_t Q^\infty - \mathcal{L}^* Q^\infty = \begin{cases} 0, & \text{in } [0, T) \times (z^\infty, +\infty) \\ (1 - \nu) - z^{-\frac{1}{\gamma}}, & \text{in } [0, T) \times (0, z^\infty) \end{cases}\]
Since \(z^\infty > z^T = (1 - \nu)^{-\gamma}\),
\[-\partial_t Q^\infty - \mathcal{L}^* Q^\infty \geq -((1 - \nu) - z^{-\frac{1}{\gamma}}), \quad \text{for } (t, z) \in \mathcal{M}_T.\]
Note that \(\partial_z Q^\infty(t, z^\infty) = Q^\infty(t, z^\infty) = 0\), for \(t \in [0, T] \) and \(\partial_z Q^\infty < 0\), in \([0, T) \times (0, z^\infty)\).
Therefore, we obtain that \(Q^\infty(t, z) > 0\), in \([0, T) \times (0, z^\infty)\) and \(Q^\infty\) satisfies the following VI :
\[
\begin{align*}
-\partial_t Q^\infty - \mathcal{L}^* Q^\infty & = -((1 - \nu) - z^{-\frac{1}{\gamma}}), \quad \text{if } Q^\infty > 0 \text{ and } (t, z) \in \mathcal{M}_T, \\
-\partial_t Q^\infty - \mathcal{L}^* Q^\infty & \geq -((1 - \nu) - z^{-\frac{1}{\gamma}}), \quad \text{if } Q^\infty = 0 \text{ and } (t, z) \in \mathcal{M}_T, \\
Q^\infty(t, z) & \geq 0, \quad \forall z > 0.
\end{align*}
\]
By the comparison principle for VI, we have
\[Q(t, z) \leq Q^\infty(t, z), \quad \text{in } [0, T) \times (0, +\infty).\]
Especially,
\[0 \leq Q \leq Q^\infty \leq 0, \quad \text{in } [0, T) \times (z^\infty, +\infty).\]
By the definition of the free boundary \(z^*\), we obtain that \(z^*(t) \leq z^\infty \) in \([0, T]\).
VI (27) implies that in domain coincidence region JR,
\[Q = 0, \quad 0 = -\partial_t Q - \mathcal{L}^* Q \geq -((1 - \nu) - z^{-\frac{1}{\gamma}}).\]
This leads to \(\text{JR} \subset [0, T) \times [z^T, +\infty)\). Hence, it is obvious that \(\{(t, z) \mid Q(t, z) > 0\} = \text{NR} \supset [0, T) \times (0, z^T)\).
By the definition of the free boundary \(z^*\),
\[z^*(t) \geq z^T, \quad \text{in } [0, T].\]
\]
\[\square\]
**Lemma B.5.** The free boundary \(z^*(t), \ t \in [0, T] \) is strictly decreasing with the terminal point \(z^*(T) = \lim_{t \to T^-} z^*(t) = z^T\). And \(z^T < z^*(t) < z^\infty, \forall t \in [0, T]\).

**Proof.** By Lemma B.1,
\[\partial_z Q \leq 0, \quad \partial_t Q \leq 0 \text{ a.e. in } \mathcal{M}_T, \quad Q \in C(\mathcal{M}_T).\]
Therefore, \(Q\) is decreasing with respect to \(t\) and decreasing with respect to \(z\).
For any fixed \(t_0 \in (0, T) \) and any \(z \in [z^*(t_0), +\infty)\), \(t \in (t_0, T)\), we have
\[0 \leq Q(t, z) \leq Q(t, z^*(t_0)) \leq Q(t_0, z^*(t_0)) = 0.
\]
By the definition of the free boundary, \(z^*(t_0) \geq z^*(t)\) for any \(0 \leq t_0 \leq t \leq T\).
Hence, \(z^*(t)\) is decreasing function in \([0, T]\).
Let us denote \( \nu \). By definition of \( \nu \), we know that \( z^* \geq z^T \) in \([0, T]\). Therefore, it is enough to prove that \( z^*(t) \leq z^T \). Otherwise, \( z^*(T) \geq z^T \), and \([0, T] \times (z^T, z^*(T)) \subset \text{NR} \). It is clear that \( Q \) satisfies

\[
\begin{cases}
-\partial_t Q - \mathcal{L}^* Q = -(1 - \nu) - \nu \frac{1}{z^*} & \text{in } \text{NR}, \\
Q(T, z) = 0, \quad \forall \ z \leq z^*(T), & Q(t, z^*(t)) = 0, \quad \forall \ t \in [0, T],
\end{cases}
\]

Hence,

\[
\partial_t Q(T, z) = -\mathcal{L}^* Q(T, z) + ((1 - \nu) - \nu \frac{1}{z^*}) = (1 - \nu) - \nu \frac{1}{z^*} > 0, \quad \forall \ z \in (z^T, z^*(T)).
\]

Since, by Lemma B.1, \( \partial_t Q \leq 0 \), a.e. in \( \mathcal{M}_T \) and by Lemma B.3, \( Q \in C^\infty \left( \{(t, z) \mid z \leq z^*(t), \ t \in [0, T] \} \right) \), we have \( \partial_t Q(T, z) \leq 0 \), \( \forall \ z \in (z^T, z^*(T)) \). It is a contradiction. Therefore, we have proved \( z^*(T) \leq z^T \), and \( z^*(T) = z^T \).

Finally, we show that \( z^*(t) \) is strictly decreasing function of \( t \) in \([0, T]\). Otherwise, there exist constants \( z_1, t_1, t_2 \) such that \( z_1 \in [z^T, z^\infty] \), \( 0 \leq t_2 < t_1 \leq T \) and \( z^*(t) = z_1 \) for any \( t \in [t_2, t_1] \). Then, it is clear that \( Q(t, z) = 0 \), \( \forall \ (t, z) \in [t_2, t_1] \times (z_1, +\infty) \) for any \( z \in (z^T, z^*(T)) \). Since \( \partial_z Q \) continuously crosses the free boundary, \( \partial_z Q(t, z_1) = 0 \), \( \forall \ t \in [t_2, t_1] \). Therefore, we obtain \( \partial_t Q(t, z_1) = 0 \), \( \partial_t (\partial_z Q)(t, z_1) = 0 \), \( \forall \ t \in [t_2, t_1] \). In domain \([t_2, t_1] \times (0, z_1)\), \( \partial_t Q \) satisfies

\[
\begin{cases}
-\partial_t \partial_t Q - \mathcal{L}^* \partial_t Q = 0, \quad \partial_t Q \leq 0, & \text{in } [t_2, t_1] \times (0, z_1), \\
\partial_t Q(t, z_1) = 0, & \forall \ t \in (t_2, t_1). 
\end{cases}
\]

According to Hopf’s boundary point lemma (See Liu [21]), we obtain that \( \partial_z (\partial_t Q) > 0 \), which contradicts the \( \partial_z (\partial_t Q)(t, z_1) = 0 \), \( \forall \ t \in [t_2, t_1] \). Therefore, the free boundary \( z^* \) is strictly decreasing. Also, we conclude that \( z^T < z^*(t) < z^\infty \) for any \( t \in [0, T] \).

\[
\begin{align*}
0 \leq g(t, z) \leq (1 - \nu) \frac{1 - e^{-rt(T-t)}}{r_t},
\end{align*}
\]

\[
\begin{align*}
Q(T, x) = 0, \quad \forall \ x \in \partial z^*(T), & \quad Q(t, z^*(t)) = 0, \quad \forall \ t \in [0, T],
\end{align*}
\]

Let us denote

\[
\begin{align*}
Q(t, z) = Q(t, z) - \frac{1 - e^{-K(T-t)}}{K} z^* - \frac{1}{z^*}. 
\end{align*}
\]

Then, \( Q(t, z) \) satisfies

\[
\begin{cases}
-\partial_t Q - \mathcal{L}^* Q = -(1 - \nu) \leq 0 & \text{in } \text{NR}, \\
Q(T, x) = 0, \quad \forall \ x \in z^*(T), & Q(t, z^*(t)) = -\frac{1 - e^{-K(T-t)}}{K} (z^*(t))^{-\frac{1}{2}} \leq 0, \quad \forall \ t \in [0, T].
\end{cases}
\]
Applying the comparison principle for PDEs, we deduce that \( Q \leq 0 \). Also, it is clear that \( Q \leq 0 \) in \( JR \) and \( Q(T, z) = 0 \). Hence, we can conclude that

\[
Q(t, z) \leq 0, \, \forall (t, z) \in \mathcal{M}_T.
\]

Since \( Q(t, z) = \frac{g(t, z)}{z} - (1 - \nu) \frac{1 - e^{-r_I(T-t)}}{r_I} \), we have

\[
0 \leq \frac{g(t, z)}{z} \leq (1 - \nu) \frac{1 - e^{-r_I(T-t)}}{r_I}.
\]

\[
\square
\]

Since \( g(t, z) = h(t, z) + z \cdot Q(t, z) \),
\[
JR = \{(t, z) \mid z \geq z^*(t), \, t \in [0, T]\} = \{(t, z) \mid g(t, z) = h(t, z)\},
\]

\[
NR = \{(t, z) \mid 0 < z < z^*(t), \, t \in [0, T]\} = \{(t, z) \mid g(t, z) > h(t, z)\}.
\]

**Theorem B.7.** The solution \( g(t, z) \) of optimal stopping problem in (18) satisfies the following integral equation representation:

\[
g(t, z) = (1 - \nu)z \int_t^T e^{-r_I(s-t)} \mathcal{N} \left( \frac{\log \frac{z^*(t)}{z^*(s)} + (\hat{\beta} - r_I + \frac{1}{2} \hat{\sigma}_z^2)(s-t)}{\hat{\sigma}_z \sqrt{s-t}} \right) ds
- z^{-\frac{1}{\gamma}} \int_t^T e^{-K(s-t)} \mathcal{N} \left( \frac{\log \frac{z^*(t)}{z^*(s)} + (\hat{\beta} - r_I + \frac{1}{2} \hat{\sigma}_z^2)(s-t)}{\hat{\sigma}_z \sqrt{s-t}} \right) ds
\]

where \( \mathcal{N}(\cdot) \) is the cumulative distribution function of a standard normal random variable and the free boundary \( z^*(t) \) satisfies the following integral equation:

\[
(1 - \nu)z^*(t) \cdot \frac{1 - e^{-r_I(T-t)}}{r_I} - (z^*(t))^{-\frac{1}{\gamma}} \cdot \frac{1 - e^{-K(T-t)}}{K}
\]

\[
= (1 - \nu)z^*(t) \int_t^T e^{-r_I(s-t)} \mathcal{N} \left( \frac{\log \frac{z^*(t)}{z^*(s)} + (\hat{\beta} - r_I + \frac{1}{2} \hat{\sigma}_z^2)(s-t)}{\hat{\sigma}_z \sqrt{s-t}} \right) ds
- (z^*(t))^{-\frac{1}{\gamma}} \int_t^T e^{-K(s-t)} \mathcal{N} \left( \frac{\log \frac{z^*(t)}{z^*(s)} + (\hat{\beta} - r_I + \frac{1}{2} \hat{\sigma}_z^2)(s-t)}{\hat{\sigma}_z \sqrt{s-t}} \right) ds
\]

**Proof.** Since \( g \in W^{1,2}_{p,loc}((0, T) \times (0, +\infty)) \), by applying the Itô’s formula (see citeKrylov) to \( e^{-\hat{\beta}s}g(s, y_s) \),

\[
\int_t^T d \left( e^{-\hat{\beta}s}g(t, y_s) \right) = \int_t^T e^{-\hat{\beta}s} \left( \frac{\partial g}{\partial s} + \mathcal{L}g - \hat{\beta}g \right) ds - \int_t^T \hat{\sigma}_x e^{-\hat{\beta}s} y_s \frac{\partial g}{\partial z} dB^{Q}_s. \tag{43}
\]

By taking expectation on both sides of (43) and applying the optional sampling theorem, the integral representation of \( g(t, z) \) is given by

\[
g(t, z) = e^{-\hat{\beta}(T-t)} \mathbb{E}^Q \left[ g(T, y_T) \mid y_0 = z \right] - \int_t^T e^{-\hat{\beta}(s-t)} \mathbb{E}^Q \left[ G(s, y_s) 1_{(y_s \geq z^*(s))} \mid y_t = z \right] ds,
\]

for any \((t, z) \in [0, T] \times (0, +\infty)\), where \( G(t, y_s) = \partial_s g + \mathcal{L}g - \hat{\beta}g \) so that \( G(t, z) = G(z) = -((1 - \nu)z - z^{-\frac{1}{\gamma}}) \) for \( z \geq z^*(t) \).
Since \( g(T, z) = 0 \),

\[
g(t, z) = - \int_t^T e^{-\hat{\beta}(s-t)} \mathbb{E}^Q \left[ G(s, y_s) \mathbb{1}_{\{y_s \geq z^*(s)\}} \mid y_t = z \right] ds
\]

\[
= (1 - \nu) \int_t^T e^{-\hat{\beta}(s-t)} \mathbb{E}^Q \left[ y_s \mathbb{1}_{\{y_s \geq z^*(s)\}} \mid y_t = z \right] ds
\]

\[
- \int_t^T e^{-\hat{\beta}(s-t)} \mathbb{E}^Q \left[ \frac{1}{z^*(s)} \mathbb{1}_{\{y_s \geq z^*(s)\}} \mid y_t = z \right] ds.
\]  

(45)

Let us define an equivalent martingale measure \( Q^1 \) by setting

\[
\frac{dQ^1}{dQ} = \exp \left\{ -\frac{\hat{\sigma}^2}{2} (s-t) + \hat{\sigma} \left( B^Q_s - B^Q_t \right) \right\}
\]  

(46)

to ensure the process \( B^Q_s = B^Q_t + \hat{\sigma} \cdot s \) is a standard Brownian motion under \( Q^1 \).

Then \( y_s \) satisfies the following SDE:

\[
dy_s = (\hat{\beta} - r_I - \frac{1}{2}\hat{\sigma}^2) y_s ds + (\gamma \sigma_I - \theta) y_s dB^Q_s,
\]

under \( Q^1 \) measure.

Hence,

\[
e^{-\hat{\beta}(s-t)} \mathbb{E}^Q \left[ y_s \mathbb{1}_{\{y_s \geq z^*(s)\}} \mid y_t = z \right]
\]

\[
= e^{-r_I(s-t)} y_t \mathbb{E}^{Q^1} \left[ \mathbb{1}_{\{y_s \geq z^*(s)\}} \mid y_t = z \right]
\]

\[
= e^{-r_I(s-t)} z \cdot Q^1 (y_s \geq z^*(s))
\]

\[
= e^{-r_I(s-t)} z \cdot \mathcal{N} \left( \log \frac{\hat{\sigma} z}{\mathbb{E}^{Q^1} \mathbb{1}_{\{y_s \geq z^*(s)\}}} + \frac{\hat{\beta} - r_I + \frac{1}{2}\hat{\sigma}^2(s-t)}{\hat{\sigma} \sqrt{s-t}} \right)
\]  

(47)

Let us again define an equivalent probability measure \( Q^2 \) by setting

\[
\frac{dQ^2}{dQ} = \exp \left\{ -\frac{\hat{\sigma}^2}{2} \left( 1 - \frac{1}{\gamma} \right)^2 (s-t) + \hat{\sigma} \left( 1 - \frac{1}{\gamma} \right) (B^Q_s - B^Q_t) \right\}
\]  

(48)

so that the process \( B^Q_s = B^Q_t + \left( 1 - \frac{1}{\gamma} \right) \hat{\sigma} \cdot s \) is a standard Brownian motion under \( Q^2 \) measure. Then the dynamics of \( y_s \) is given by

\[
dy_s = \left( \hat{\beta} - r_I - \left( 1 - \frac{1}{\gamma} \right) \hat{\sigma}^2 \right) y_s ds + \hat{\sigma} y_s dB^Q_s
\]

under the \( Q^2 \) measure.

Similarly as in (47), we obtain that

\[
e^{-\hat{\beta}(s-t)} \mathbb{E}^Q \left[ y_s \mathbb{1}_{\{y_s \geq z^*(s)\}} \right] = e^{-K(s-t)} z \cdot \mathcal{N} \left( \log \frac{\hat{\sigma} z}{\mathbb{E}^{Q^2} \mathbb{1}_{\{y_s \geq z^*(s)\}}} + \frac{\hat{\beta} - r_I - \frac{1}{2}\hat{\sigma}^2(s-t)}{\hat{\sigma} \sqrt{s-t}} \right)
\]  

(49)
Therefore, by (45)–(49), we obtain that

\[
g(t, z) = (1 - \nu) z \int_t^T e^{-r_1(s-t)} \mathcal{N} \left( \frac{\log \frac{\hat{\sigma}_z}{z(s)} + (\hat{\beta} - r_I + \frac{1}{2} \hat{\sigma}_z^2)(s-t)}{\hat{\sigma}_z \sqrt{s-t}} \right) d\eta
\]

Since \( g(t, z^*(t)) = h(t, z^*(t)) \), we just have proved the desired result.

**Appendix C. Proof of Theorem 3.3.** Let us assume that \( (D^*_s)^T_{s=t} \) and \( \lambda^* > 0 \) minimize Lagrangian (15).

Under this assumption, we first prove that \( (e^*_s)^T_{s=t} \), given by (16), is optimal where \( \lambda \) and \( D \) are replaced by \( \lambda^* \) and \( D^* \), respectively. Then we will show the existence and uniqueness of \( \lambda^* \) satisfying (22) and derive \( (D^*_s)^T_{s=t} \).

(1) \( e^* \) is admissible.

**Proof.** For every \( t \in [0, T] \),

\[
\int_t^T (u')^{-1} \left( \lambda e^{\beta(s-t) H^I_s D_s} \right) ds \leq \int_t^T (u')^{-1} \left( \lambda e^{\beta(s-t) H^I_s} \right) ds
\]

\[= \int_t^T \left( \lambda e^{\beta(s-t) H^I_s} \right)^{-\frac{1}{\beta}} ds < +\infty.\]

We rewrite the Lagrangian in (15) as

\[
\mathbf{L}(t, X_t, I_t; D, \lambda) = \mathbb{E}_t \left[ \int_t^T e^{-\beta(s-t)} \tilde{u} \left( \lambda e^{\beta(s-t) H^I_s D_s} \right) ds \right]
\]

\[+ \lambda \left( X_t + \nu M_t + \mathbb{E}_t \left[ \int_t^T (1 - \nu) H^I_s I_s ds \right] \right)\]

For simplicity, let \( \mathbf{L}(t, X_t, I_t; D, \lambda) = \mathbf{L}(D, \lambda) \). For \( h > 0 \) and any stopping time \( \tau \in \mathcal{S}(t, T) \), we consider

\[
\lambda^h = \lambda^* + h, \quad D^h_s = \frac{1}{\lambda^h} \left( D^*_s \lambda^* + h \cdot 1_{[t, \tau]} \right).
\]

Clearly, \( (D^h_s)^T_{s=t} \) is a non-increasing process with \( D^h_t = 1 \). Then, since \( \lambda^* \) and \( (D^*_s)^T_{s=t} \) minimize the Lagrangian,

\[
\mathbf{L}(D^h, \lambda^h) \geq \mathbf{L}(D^*, \lambda^*).
\]

Since \( 0 \leq \lambda^h D^h_s - \lambda^* D^*_s \leq 1_{[0, \tau]} \),

\[
\limsup_{h \downarrow 0} \frac{\mathbf{L}(\lambda^h, D^h) - \mathbf{L}(\lambda^*, D^*)}{h} \geq 0, \quad \text{and} \quad \overline{u}(\lambda^*_e^{\beta(s-t)} H^I_s D^*_s) \geq \overline{u}(\lambda^h e^{\beta(s-t)} H^I_s D^h_s).
\]

By Fatou’s lemma with \( 0 \leq \lambda^h D^h_s - \lambda^* D^*_s \leq 1_{[t, \tau]} \),

\[
0 \leq \limsup_{h \downarrow 0} \frac{\mathbf{L}(\lambda^h, D^h) - \mathbf{L}(\lambda^*, D^*)}{h}
\]

\[\leq \mathbb{E}_t \left[ \limsup_{h \downarrow 0} \left( \int_t^T e^{-\beta(s-t)} \overline{u} \left( \lambda^h e^{\beta(s-t) H^I_s D^h_s} - \lambda^* e^{\beta(s-t)} \lambda^* H^I_s D^*_s \right) ds \right) \right]
\]
Proof. Define \( \lambda^\pm h = \lambda^\pm h \) with \( h > 0 \). Since \( (D^*_s)_{s=t}^T \) and \( \lambda^* \) minimize the Lagrangian,

\[
L(\lambda^\pm h, D^*) \geq L(\lambda^*, D^*).
\]

It leads to

\[
\lim_{h \downarrow 0} \frac{L(\lambda^\pm h, D^*) - L(\lambda^*, D^*)}{h} \geq 0, \quad \lim_{h \uparrow 0} \frac{L(\lambda^\mp h, D^*) - L(\lambda^*, D^*)}{h} \leq 0.
\]

Therefore,

\[
\pm \left\{ (x + \nu M_t) - \left( E_t \left[ \int_t^T H^*_s (c^*_s - (1 - \nu)I_s) \, ds \right] \right) \right\} \geq 0.
\]
This implies,
\[ E_t \left[ \int_t^T H_s^I \mathcal{D}_s^* (c_s^* - (1 - \nu)I_s) ds \right] = x + \nu M_t. \]

That is, \( c^* \) satisfies the liquidity budget constraint with equality. \( \square \)

(4) \( c^* \) is optimal.

**Proof.** For any admissible consumption process \((c_s)_{s=t}^T\), we have
\[
E_t \left[ \int_t^T e^{-\beta(s-t)} u(c_s) ds \right] 
\leq E_t \left[ \int_t^T e^{-\beta(s-t)} u(c_s^*) ds \right] + \lambda^* \left( x + \nu M_t - E_t \left[ \int_t^T H_s^I \mathcal{D}_s^* (c_s^* - (1 - \nu)I_s) ds \right] \right)
\leq E_t \left[ \int_t^T e^{-\beta(s-t)} u(c_s^*) ds \right] + \lambda^* \left( x + \nu M_t - E_t \left[ \int_t^T H_s^I \mathcal{D}_s^* (c_s^* - (1 - \nu)I_s) ds \right] \right)
= E_t \left[ \int_t^T e^{-\beta(s-t)} u(c_s^*) ds \right].
\]

From the fact that by (3), any admissible consumption \( c \) satisfies the liquidity budget constraint, the first inequality holds. Since the fact that \( c^* \) is an optimal choice for \( \lambda^* \), the second equality also is established. The last equality follows from the fact that \( c^* \) satisfies the liquidity budget constraint with equality by (3). Thus, \((c_s^*)_{s=t}^T\) is optimal. \( \square \)

(5) Proof of duality in (22).

**Proof.** By the previous proof we know that for \( \lambda > 0 \)
\[
V(t, x, I)
= E_t \left[ \int_t^T e^{-\beta(s-t)} u(c_s^*) ds \right]
= E_t \left[ \int_t^T e^{-\beta(s-t)} u(c_s^*) ds \right] + \lambda \left( x + \nu M_t - E_t \left[ \int_t^T H_s^I \mathcal{D}_s^* (c_s^* - (1 - \nu)I_s) ds \right] \right)
\leq \sup \left\{ E_t \left[ \int_t^T e^{-\beta(s-t)} u(c_s) ds \right] + \lambda \left( x + \nu M_t - E_t \left[ \int_t^T H_s^I \mathcal{D}_s^* (c_s^* - (1 - \nu)I_s) ds \right] \right) \right\}
= J(t, \lambda, I) + \lambda x.
\]

Thus,
\[ V(t, x, I) \leq \inf_{\lambda > 0} (J(t, \lambda, I) + \lambda x). \]

However, we know that
\[
V(t, x, I)
= E_t \left[ \int_t^T e^{-\beta(s-t)} u(c_s^*) ds \right]
= E_t \left[ \int_t^T e^{-\beta(s-t)} \left( u(c_s^*) - e^{\beta(s-t)} \lambda^* H_s^I \mathcal{D}_s^* \right) ds \right] + \lambda^* E_t \left[ \int_t^T H_s^I \mathcal{D}_s^* c_s^* ds \right]
= J(t, \lambda^*, I) + \lambda^* x.
\]
Therefore,

\[ V(t, x, I) = \inf_{\lambda > 0} (J(t, \lambda, I) + \lambda x). \]

(6) Proof of the rest of the theorem.

**Proof.** By Lemma 3.2, the optimal stopping time \( \tau_w \) in Lemma 3.1 is given by

\[ \tau_w = \inf \{ s \geq t \mid w \cdot y_s^\ast \geq z_s^\ast \}. \tag{52} \]

By (52), the optimal shadow process \( D_s^\ast \) satisfies

\[ \{ D_s^\ast < w \} = \{ s > \tau_w \} = \left\{ \min_{t \leq \xi \leq s} \frac{z^\ast (\xi)}{y_\xi^\ast} < w \right\}. \]

Thus, the optimal shadow process \( \{ D_s \}_{s=t}^T \) can be represented by

\[ D_s^\ast = \min \left( 1, \min_{t \leq \xi \leq s} \frac{z^\ast (\xi)}{y_\xi^\ast} \right). \]

This proves that \( D_s^\ast \) is given as in part (b) of the theorem.

By defining \( \lambda_s = \lambda e^{\beta (s-t)} H_t^s D_s \), since Problem 2 is time consistent, we can show that dual value function at time \( s \geq t \) is given by

\[ J(s, \lambda_s, I_s) = I_s^{1-\gamma} \left[ (1 - \nu) \int_s^T e^{-r_s (\eta-s)} (w \lambda_s I_\eta^\gamma) \cdot \mathcal{N} \left( d^\gamma (\eta-s, \frac{w \lambda_s I_\eta^\gamma}{z^\ast (\eta)}) \right) d\eta \right. \]

\[ + \int_s^T e^{-K (\eta-s)} (w \lambda_s I_\eta^\gamma)^{-\frac{1-\gamma}{\gamma}} \cdot \mathcal{N} \left( d^{1-\gamma} (\eta-s, \frac{w \lambda_s I_\eta^\gamma}{z^\ast (\eta)}) \right) d\eta \right] \]

\[ + \frac{1 - e^{-K (T-s)}}{K} \left[ \gamma \lambda_s^{-\frac{1-\gamma}{\gamma}} - \frac{1 - e^{-r_s (T-s)}}{r_I} I \right]. \]

We know that the value function satisfies

\[ V(s, X_s, I_s) = \inf_{\lambda_s > 0} (J(s, \lambda_s, I_s) + \lambda_s X_s). \tag{53} \]

We can obtain \( \lambda^* \) and the optimal wealth process from the first-order condition of the dual relationship in (53) for \( s = t \).

The first-order condition is given as follows:

\[ x = -\frac{\partial J}{\partial \lambda} (t, \lambda^*, I) \]

\[ = I \left[ g(t, w \lambda^* I^\gamma) dw \right. \frac{1 - e^{-K (T-t)}}{K} (\lambda^*)^{-\frac{1}{\gamma}} - \frac{1 - e^{-r_I (T-t)}}{r_I} I \]

\[ = I \left. \frac{1}{\lambda^* I \gamma} g(t, \lambda^* I^\gamma w) \right|_{w=0}^{1} \frac{1 - e^{-K (T-t)}}{K} (\lambda^*)^{-\frac{1}{\gamma}} - \frac{1 - e^{-r_I (T-t)}}{r_I} I \]

By Lemma B.6, we can deduce

\[ \lim_{w \to 0} \left[ w \cdot \frac{g(t, \lambda^* I^\gamma w)}{\lambda^* I^\gamma w} \right] = 0. \]
Thus,
\[
\frac{x}{t} = \frac{1}{\lambda^* T^e} \left[ g(t, \lambda^* I^e) + \frac{1 - e^{-K(T-t)}}{K} (\lambda^* I^e)^{1/2} - \frac{1 - e^{-r_l(T-t)}}{r_l} \right] \equiv v(\lambda^* I^e). 
\] (54)

We will prove that there exist a unique $\lambda^*$ satisfying (54) such that $0 < \lambda^* < z^*(t)$ where $z^*(t)$ is the free boundary of Problem 3.

It is easily obtain that
\[
\lim_{y \to 0} v(y) = +\infty, \quad \lim_{y \to z^*(t)} v(y) = -\nu \cdot \frac{1 - e^{-r_l(T-t)}}{r_l}. 
\]

Since $g(t, z) = z \cdot Q(t, z) + h(t, z),$
\[
v(y) = Q(t, y) - \nu \frac{1 - e^{-r_l(T-t)}}{r_l}.
\]

By Lemma B.2, $v(y)$ is a strictly decreasing function of $y$ in $y \in (0, z^*(t))$. Since $-\nu \frac{1 - e^{-r_l(T-t)}}{r_l} \leq \frac{x}{t} < +\infty$, there exist a unique $\lambda^* \in (0, z^*(t))$ such that
\[
v(\lambda^* I) = \frac{x}{t}.
\]

Therefore, by first-order condition of dual-relationship in (54) at time $s$, we have
\[
\frac{X^*_s}{I_s} = \frac{1}{y_s^* D^*_s} \left[ g(s, y_s^* D^*_s) + \frac{1 - e^{-K(T-s)}}{K} (y_s^* D^*_s)^{1/2} - \frac{1 - e^{-r_l(T-s)}}{r_l} \right]
= (1 - \nu) \int_s^T e^{-r_l(\eta-s)} \mathcal{N} \left( d^+ (\eta - s, \frac{y_s^* D^*_s}{z^*(\eta)}) \right) d\eta 
- (y_s^* D^*_s)^{1/2} \int_s^T e^{-K(\eta-s)} \mathcal{N} \left( d^+ (\eta - s, \frac{y_s^* D^*_s}{z^*(\eta)}) \right) d\eta 
+ \frac{1}{K} (y_s^* D^*_s)^{-1/2} - \frac{1 - e^{-r_l(T-s)}}{r_l} \geq -\nu \frac{1 - e^{-r_l(T-s)}}{r_l}. 
\] (55)

where $y_s^* = \lambda^* e^{\beta(s-t)} H^*_s I^*_s$.

Next, we can obtain optimal consumption and portfolio. From (16),
\[
e_s^* = \left( \lambda^* e^{\beta(s-t)} H^*_s I^*_s \right)^{-\frac{1}{2}} 
= (y_s^* D^*_s)^{-\frac{1}{2}} \cdot I_s. 
\] (56)

By (55) and (56)
\[
\frac{c_s^*}{I_s} = \frac{K}{1 - e^{-K(T-s)}} \left( \frac{X^*_s}{I_s} + \frac{1 - e^{-r_l(T-s)}}{r_l} \right) 
+ \frac{K}{1 - e^{-K(T-s)}} \left\{ (1 - \nu) \int_s^T e^{-r_l(\eta-s)} \mathcal{N} \left( d^+ (\eta - s, \frac{y_s^* D^*_s}{z^*(\eta)}) \right) d\eta 
- (y_s^* D^*_s)^{-1/2} \int_s^T e^{-K(\eta-s)} \mathcal{N} \left( d^+ (\eta - s, \frac{y_s^* D^*_s}{z^*(\eta)}) \right) d\eta \right\}. 
\]

The optimal wealth process $X^*_s$ can be expressed by
\[
X^*_s = I_s v(s, y_s^* D^*_s).
\]
where
\[
v(t, z) = \frac{1}{\gamma} g(t, z) + \frac{1 - e^{-K(T-t)}}{K} z - \frac{1 - e^{-r I(T-t)}}{r I}.
\]

By generalized Itô’s lemma (see citeHarrison), we obtain
\[
\begin{align*}
    dX_s^* &= I_s \partial_s v(s, y_s^* D_s^*) ds + v(s, y_s^* D_s^*) dI_s + I_s D_s^* \partial_z v(s, y_s^* D_s^*) dy_s^* \\
    &\quad + \partial_z v(s, y_s^* D_s^*) D_s^* ds + D_s^* \partial_z v(s, y_s^* D_s^*) (dy_s^*, dI_s).
\end{align*}
\]

If \( \frac{z^*(s)}{y_s^*} \neq \min_{t \leq \xi \leq s} \frac{z^*(\xi)}{y_t^*} \), then by part (b) of the theorem we know that \( dD_s^* = 0 \) and we can obtain the following equation for the optimal portfolio policy by comparing the martingale term in (3) and (57):
\[
\begin{align*}
    \pi_s^* &= \frac{\sigma_I}{\sigma} I_s \left( \frac{1}{y_s^* D_s^*} g(s, y_s^* D_s^*) + \frac{1 - e^{-K(T-s)}}{K} (y_s^* D_s^*) - \frac{1 - e^{-r I(T-s)}}{r I} \right) \\
    &\quad - \frac{\sigma_d}{\sigma} I_s \left( \frac{1}{y_s^* D_s^*} g(s, y_s^* D_s^*) \partial_z g(s, y_s^* D_s^*) + \frac{1 - e^{-K(T-s)}}{K} (y_s^* D_s^*) - \frac{1 - e^{-r I(T-s)}}{r I} \right).
\end{align*}
\]

Since \( \frac{z^*(s)}{y_s^*} = \min_{t \leq \xi \leq s} \frac{z^*(\xi)}{y_t^*} \) is valid only for \( s \) belonging to a set of Lebesgue measure zero almost surely, the optimal portfolio strategy at a time \( s \) when \( \frac{z^*(s)}{y_s^*} = \min_{t \leq \xi \leq s} \frac{z^*(\xi)}{y_t^*} \) doesn’t have an effect on the agent’s utility. Thus, we can assume the above portfolio strategy also holds at time \( s \) when \( \frac{z^*(s)}{y_s^*} = \min_{t \leq \xi \leq s} \frac{z^*(\xi)}{y_t^*} \). Thus, we can get the optimal portfolio process in Theorem 3.3. \(\)

By (1)–(6), we have proved the desired result.

**Appendix D. Proof of Theorem 3.4.** (1) Let \( \lim_{t \to \infty} \tilde{z}^*(t) = \lim_{T-t \to \infty} z^*(T-t) = z_{\infty}^* \).

From the integral equation (21), we have
\[
0 = (1 - \nu) \int_0^\infty e^{-r I \xi} N \left( -d^+ \xi, \frac{z^*}{\bar{z}_\infty} \right) d\xi \]
\[
\quad - \left( z_{\infty}^* \right)^{-\frac{1}{2}} \int_0^\infty e^{-K \xi} N \left( -d^+ \xi, \frac{z_{\infty}^*}{\bar{z}_\infty} \right) d\xi \]
\[
= (1 - \nu) \int_0^\infty e^{-r I \xi} N \left( -\frac{\hat{\beta} - r I + \frac{\sigma_d^2}{2}}{\sigma_z} \xi^{\frac{1}{2}} \right) d\xi \]
\[
\quad - \left( z_{\infty}^* \right)^{-\frac{1}{2}} \int_0^\infty e^{-K \xi} N \left( -\frac{\hat{\beta} - r I - \frac{\sigma_d^2}{2} - \frac{1-\gamma}{\gamma} \sigma_z^2}{\sigma_z} \xi^{\frac{1}{2}} \right) d\xi. \tag{59}
\]

**Lemma D.1.** For arbitrary \( c > 0 \) and \( d \in \mathbb{R} \),
\[
\int_0^\infty e^{-c \xi} N \left( d \sqrt{\xi} \right) d\xi = \frac{1}{2c} \left( 1 + \frac{d}{\sqrt{d^2 + 2c}} \right).
\]
Proof. By integration by parts,
\[
\int_0^\infty e^{-c\xi} \mathcal{N}(d\sqrt{\xi}) \, d\xi = \left[-\frac{1}{c} e^{-c\xi} \mathcal{N}(d\sqrt{\xi})\right]_{\xi=0}^\infty + \frac{d}{2c\sqrt{2\pi}} \int_0^\infty e^{-c\xi} \frac{a^2}{\sqrt{\xi}} \, d\xi.
\]

By citeAS (p.304, equation (7.4.33)), for any \(a, b \in \mathbb{R}\),
\[
\int_0^\infty \exp\left\{-a^2 x^2 - \frac{b^2}{x^2}\right\} \, dx = \frac{\sqrt{\pi}}{2|a|} e^{-2|a||b|}.
\]

Therefore,
\[
\int_0^\infty e^{-c\xi} \mathcal{N}(d\sqrt{\xi}) \, d\xi = \frac{\sqrt{\pi}}{\sqrt{c + \frac{d^2}{2}}}.
\]

and then, (60) can be expressed by
\[
\int_0^\infty e^{-c\xi} \mathcal{N}(d\sqrt{\xi}) \, d\xi = \frac{1}{2c} \left(1 + \frac{2d}{\sqrt{d^2 + 2c}}\right).
\]

(2) Let
\[
g_{\infty}(z) \equiv \lim_{T-t \to \infty} g(t, z).
\]
In the integral equation representation of \( g(t, z) \) in (42),

\[
g_{\infty}(z) = \lim_{T-t \to \infty} g(t, z)
\]

\[
= \lim_{T-t \to \infty} \left[ (1 - \nu) z \int_{0}^{\tau} e^{-r_1 \xi N} \left( \log \frac{z}{2^{(\tau - \xi)}} + \frac{(\beta - r_l + \frac{1}{2} \hat{\sigma}_z^2) \xi}{\hat{\sigma}_z \sqrt{\xi}} \right) d\xi \right.
\]

\[
- z^{1-\frac{1}{\nu}} \left. \int_{0}^{\tau} e^{-\kappa \xi N} \left( \log \frac{z}{2^{(\tau - \xi)}} + \frac{(\beta - r_l + \frac{1}{2} \hat{\sigma}_z^2) \xi}{\hat{\sigma}_z \sqrt{\xi}} \right) d\xi \right]
\]

\[
(61)
\]

where \( \tau = T - t \) and \( \xi = \eta - t \).

For \( m \in \mathbb{R} \), let us define function \( \phi_m(z) \) and \( p(m) \) as follow:

\[
\phi_m(z) \equiv \int_{0}^{\infty} e^{-p(m)\xi} N \left( \log \frac{z}{\hat{\sigma}_z} \xi^{-\frac{1}{2}} + \left( \frac{\beta - r_l}{\hat{\sigma}_z} - \left( \frac{1}{2} + m \right) \hat{\sigma}_z \right) \xi^{\frac{1}{2}} \right) d\xi,
\]

and

\[
p(m) \equiv \hat{\beta} + (\beta - r_l - \frac{\hat{\sigma}_z^2}{2})m - \frac{\hat{\sigma}_z^2}{2} m^2.
\]

It is suffice to consider the case \( z < z_{\infty} \). Then, the following lemma is true.

**Lemma D.2.** For \( p(m) > 0 \) and \( z < z_{\infty} \),

\[
\phi_m(z) = -\frac{1}{p(m)} \left( \frac{\alpha_+ + m + 1}{\alpha_+ - \alpha_-} \right) \left( \frac{z}{z_{\infty}} \right)^{\alpha_+ + m + 1}.
\]

**Proof.** Let

\[
c = \log \frac{z}{\hat{\sigma}_z}, \quad d = \frac{\beta - r_l}{\hat{\sigma}_z} - \left( \frac{1}{2} + m \right) \hat{\sigma}_z.
\]

Since \( c < 0 \), by integral by parts,

\[
\phi_m(z) = \left[ -\frac{e^{-p(m)\xi}}{p(m)} N \left( c \xi^{-\frac{1}{2}} + d \xi^{\frac{1}{2}} \right) \right]_{\xi=0}^{\infty}
\]

\[
+ \frac{1}{\sqrt{2\pi p(m)}} \int_{0}^{\infty} e^{-p(m)\xi} \xi e^{-\frac{1}{2} \left( c \xi^{-\frac{1}{2}} + d \xi^{\frac{1}{2}} \right)^2} \left( -\frac{c}{2} \xi^{-\frac{3}{2}} + \frac{d}{2} \xi^{-\frac{1}{2}} \right) d\xi
\]

\[
= e^{-cd} \sqrt{\frac{2\pi p(m)}{2\pi}} \int_{0}^{\infty} e^{-\left( \frac{c}{2} \xi^{\frac{1}{2}} + d \xi^{\frac{1}{2}} \right)^2} \left( -\frac{c}{2} \xi^{-\frac{1}{2}} + \frac{d}{2} \xi^{-\frac{1}{2}} \right) d\xi.
\]

Since citeAS (p.304, equation (7.4.33)),

\[
\int_{0}^{\infty} \exp \left\{ -a^2 x^2 - \frac{b^2}{x^2} \right\} dx = \frac{\sqrt{\pi}}{2|a|} e^{-2|a||b|}
\]

\[
\int_{0}^{\infty} \frac{1}{x^2} \exp \left\{ -a^2 x^2 - \frac{b^2}{x^2} \right\} dx = \frac{\sqrt{\pi}}{2|b|} e^{-2|a||b|},
\]

\[
\phi_m(z) = \frac{1}{2p(m)} \left( 1 + \frac{d}{\sqrt{d^2 + 2p(m)}} \right) e^{c(\sqrt{d^2 + 2p(m)} - d)}.
\]
By direct computation,
\[ \sqrt{d^2 + 2p(m)} = \hat{\sigma}_z \left( \frac{\alpha_+ - \alpha_-}{2} \right), \]
\[ \sqrt{d^2 + 2p(m)} - d = \hat{\sigma}_z (\alpha_+ + m + 1), \]
\[ \sqrt{d^2 + 2p(m)} + d = -\hat{\sigma}_z (\alpha_- + m + 1). \]

Since \( p(-1) = -r_I \) and \( p(1) = -K \), it is easy to check that
\[ g_\infty = (1 - \nu) z \phi_{-1}(z) - z^{-\frac{1}{\gamma_+}} \phi_{1-\gamma}(z). \]

By Lemma D.2,
\[ g_\infty = (1 - \nu) z \left( \frac{1}{r_I} \frac{\alpha_-}{\alpha_+ - \alpha_-} \right) \left( \frac{z}{z_\infty} \right)^{\alpha_+} \]
\[ - z^{-\frac{1}{\gamma_+}} \left( \frac{1}{K} \frac{\alpha_- + \frac{1}{2}}{\alpha_+ - \alpha_-} \right) \left( \frac{z}{z_\infty} \right)^{\alpha_+ + \frac{1}{2}} \]
\[ = z \cdot \frac{(1 - \nu)}{(1 + \gamma_+)} \left( \frac{z}{z_\infty} \right)^{\alpha_+}. \]

Therefore, we have
\[ \lim_{T - t \to \infty} J(t, \lambda, I) = -1^{1-\gamma} \int_0^1 g_\infty(w\lambda J)dw + \frac{1}{K} \frac{\gamma}{1 - \gamma} \lambda^{-\frac{1}{\gamma_+}} - \frac{1}{r_I} \lambda I \]
\[ = -1^{1-\gamma} \frac{(1 - \nu)}{(1 + \gamma_+)(1 + \alpha_+)} \left( \frac{1}{z_\infty} \right)^{\alpha_+} (\lambda J)^{\alpha_+ + 1} \]
\[ + \frac{1}{K} \frac{\gamma}{1 - \gamma} \lambda^{-\frac{1}{\gamma_+}} - \frac{1}{r_I} \lambda I. \]

Similarly to the proof of Theorem 3.3, we can prove that \( J_\infty(y_s, I_s) \) is the dual value function for \( T - t = \infty \) and the optimal wealth process for the case \( T - t = \infty \) satisfies
\[ \frac{X_{s,\infty}}{I_s} = -\frac{\partial J}{\partial \lambda}(y_s, I_s) \frac{1}{I_s} \]
\[ = \frac{1}{y_s D_s^*} g_\infty(y_s D_s^*) + \frac{1}{K} (y_s D_s^*)^{-\frac{1}{2}} - \frac{1}{r_I} \]
\[ = \frac{(1 - \nu)}{(1 + \gamma_+)} \left( \frac{y_s D_s^*}{z_\infty} \right)^{\alpha_+} + \frac{1}{K} (y_s D_s^*)^{-\frac{1}{2}} - \frac{1}{r_I}. \]

(62)

where \( D_s^* \) is given by
\[ D_s^* = \min \left( 1, \min_{\frac{z}{y_s}} \frac{z_\infty}{y_s} \right). \]

Similarly, the optimal portfolio for the case \( T - t = \infty \) is given by
\[ \frac{\pi_{s,\infty}}{I_s} = \theta \gamma \sigma \left( \frac{X_{s,\infty}}{I_s} + \frac{1}{r_I} \right) - \frac{\sigma I}{\sigma r_I} + \frac{\hat{\sigma}_z (1 - \nu)}{\gamma \sigma} \left( \frac{y_s D_s^*}{z_\infty} \right)^{\alpha_+}. \]

This completes the proof.
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