The symmetric branes of the $H_4$ WZW model

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Abstract: We review the solution of the boundary CFTs that describe the symmetric branes in the Nappi-Witten gravitational wave, namely D2 and S1 branes. The D2 branes are the twisted branes of the model while the S1 branes are the Cardy branes. We present in both cases the bulk-boundary couplings and the boundary three-point couplings and discuss the relation with branes in $AdS_3$ and in $S^3$. We also discuss the analogy between the open string couplings in the $H_4$ model and the couplings for magnetized and intersecting branes.
1. Introduction

In the last few years important progress was made in the analysis of some irrational CFTs. These theories are interesting on their own and also relevant in order to describe the string dynamics in curved space-times. Among other developments, the Liouville model \[1\] and the \( H^4 \) model \[2\] were solved and the structure of the string spectrum on \( AdS_3 \) clarified \[3\]. The study of string and brane dynamics in non-compact curved backgrounds remains a rich and difficult subject and all the models that can be solved exactly are a source of useful information.

In \[4\] we started the analysis of a class of WZW models based on the Heisenberg groups \( H_{2n+2} \) \[5, 6, 7\] which from the space-time point of view describe the propagation of a string in gravitational wave backgrounds. The gravitational waves provide a rich family of conformally invariant \( \sigma \)-models \[8\] and a very convenient starting point for analyzing the properties of string theory in curved, non-compact and possibly singular backgrounds. They also play an important role in the gauge/string duality. In fact the notion of Penrose limits \[9\] suggested that such backgrounds are dual to modified large \( N \) limits of gauge theories \[10\].

The models we are interested in describe a class of gravitational waves that are supported by a NS-NS flux and have an exact CFT description as WZW models. For more general gravitational wave backgrounds one can investigate the spectrum of the model in the light-cone gauge (for a recent discussion see \[11\]) but a more detailed analysis of the string dynamics and in particular the computation of the three and four-point amplitudes usually remain beyond reach. On the other hand the presence of the affine symmetry algebra allowed us to obtain a clear and detailed understanding of the string dynamics in the gravitational waves that correspond to the \( H_{2n+2} \) WZW models. In \[4\], we solved the \( H_4 \) model without boundary and computed all the three and four-point correlation functions of primary vertex operators. In \[12\] we performed a similar analysis for the \( H_{2n+2} \) models and in particular for the \( H_6 \) model, the Penrose limit of \( AdS_3 \times S^3 \).

Here we will review the D-branes of the Nappi-Witten gravitational wave and we will discuss the dynamics of their open string excitations \[13\]. The dynamics of open strings in curved space-time is even less understood than its closed counterpart. Only the boundary states have been studied for many of the interesting models that exist in the literature. The other structure constants of the boundary CFT, namely the bulk-boundary and the three-point boundary couplings, proved very difficult to compute and only partial results are available at the moment \[14\]. In \[15\] we found the complete solution of the BCFTs pertaining to the two classes of symmetric branes of the \( H_4 \) model \[15, 16\], \( D2 \) and \( S1 \) branes. We solved in both cases the consistency BCFT conditions \[17\] and obtained the complete BCFT data. To our knowledge, with the notable exception of the Liouville model \[1\], this is the first complete tree-level solution of D-brane dynamics in a curved non-compact background. Our solution of the \( H_4 \) model with boundary should be useful not only to improve our understanding of the closed and open string dynamics in curved space-times but also to clarify some properties of both compact and non-compact WZW models. In the following we will review, among other results obtained in \[13\], the first
examples of structure constants for twisted symmetric branes in a WZW model (the D2 branes) and of open four-point functions in a curved background. D-branes in pp-wave backgrounds have been the object of several other studies and a review can be found in [18].

Before we start the analysis of the symmetric branes in the Nappi-Witten gravitational wave, let us briefly describe two additional reasons for paying attention to the \( H_{2n+2} \) models, besides the fact that they provide a rich and interesting class of non-compact WZW models. The first reason is that, as explained in [19], the non-semi-simple WZW models can be considered as contractions of semi-simple WZW models. If we assume this point of view, we can implement at the worldsheet level the idea of Penrose of realizing the gravitational waves as limits of other space-times. Following [4, 12] we can study in detail how the spectrum of the model and the dynamical quantities such as the three-point functions change in the limit. As far as the branes are concerned, the fact that the \( H_4 \) model is the Penrose limit of \( \mathbb{R} \times S^3 \) and \( AdS_3 \times S^1 \) leads to interesting relations between the symmetric branes in \( H_4 \) and the symmetric branes in \( S^3 \) and in \( AdS_3 \). The second reason is that, according to the free-field representation introduced in [20], the \( H_4 \) primary vertex operators can be expressed using orbifold twist fields. The orbifold in question is the quotient of the plane by a rotation and a dictionary can be established connecting the amplitudes computed in the \( H_4 \) model and the amplitudes computed in the orbifold CFT [4]. Given this free-field representation, it is not surprising that in the following we will find several analogies between the D2 branes of the \( H_4 \) model and configurations of intersecting branes, as well as between the \( S1 \) branes of the \( H_4 \) model and branes with a magnetic field on their world-volume.

2. Branes in \( H_4 \)

The \( H_4 \) WZW model was first analyzed as a string theory background by Nappi and Witten [5]. They showed that the target space of the corresponding \( \sigma \)-model is a gravitational wave supported by a NS flux

\[
\begin{align*}
\text{ds}^2 &= -2du dv - \frac{\mu r^2}{4} du^2 + dr^2 + r^2 d\phi^2, \\
H &= \mu r dr \wedge d\phi \wedge du.
\end{align*}
\]  

(2.1)

The commutation relations of the Heisenberg group are

\[
[P^+, P^-] = -2i\mu K, \quad [J, P^\pm] = \mp i\mu P^\pm,
\]  

(2.2)

where the generators \( J \) and \( K \) are anti-hermitian and \( (P^+)\dagger = P^- \). The only subtlety that arises in the construction of a WZW model based on a non-semi-simple group is that in order to express the stress-energy tensor as a bilinear in the currents one can not use the Killing form which is degenerate but has to identify another non degenerate invariant symmetric form. A detailed discussion of this model can be found in [4] while a brief description of the same results can be found in [21].

Since this gravitational wave is a WZW model, we can study in considerable detail the symmetric branes, that is the branes that preserve a linear combination of the left
and right affine algebras. The symmetric branes fall in families which are in one-to-one correspondence with the automorphisms of the current algebra and without loss of generality one can restrict his attention to families of branes associated with distinct outer automorphisms $\Omega$. The brane world-volumes coincide with the twisted conjugacy classes of the group \cite{22, 23}.

Since the $H_4$ algebra admits a non-trivial outer automorphism $\Omega$ which acts on the currents as charge conjugation

\[ \Omega(P^\pm) = P^\mp, \quad \Omega(J) = -J, \quad \Omega(K) = -K, \] (2.3)

we have two families of symmetric branes, wrapped on the conjugacy classes and on the twisted conjugacy classes respectively \cite{13, 14}.

The $H_4$ conjugacy classes form a two parameters family $C(u, \eta)$, characterized by the constant value of the coordinate $u$ and the constant value of the invariant

\[ \eta = v - \frac{\mu u^2}{4} \cot \frac{\mu u}{2}. \] (2.4)

For convenience we will often denote the two parameters that identify an $S1$ branes with a single letter $a \equiv (u_a, \eta_a)$. Their geometric description is particularly simple in Rosen coordinates where the background takes the following form

\[ ds^2 = -2dx^+dx^- + \sin^2 \frac{\mu x^+}{2}(dy_1^2 + dy_2^2), \quad H = \mu \sin^2 \frac{\mu x^+}{2}dy_1 \wedge dy_2 \wedge dx^+. \] (2.5)

Since in Rosen coordinates $x^- = \eta$, these branes are Euclidean two-planes with an $x^+$-dependent scale factor and a two-form field-strength

\[ F_{12} \equiv B_{12} + 2\pi \alpha' F_{12} = -\frac{\sin \mu x^+}{2}, \] (2.6)

where as usual $F$ is the gauge invariant combination that appears in the Dirac-Born-Infeld action. These branes have a non-trivial boundary condition on the real time coordinate and can be called $S1$-branes \cite{24}. In Brinkman coordinates, the metric induced on the two-dimensional world-volume is trivial and the flux is

\[ F_{r\varphi} = B_{r\varphi} + 2\pi \alpha' F_{r\varphi} = -r \cot \frac{\mu u}{2}. \] (2.7)

When $\mu u = 2\pi n$, the geometry of the conjugacy classes changes. We have either points with $r = 0$ and a fixed value of $v$ or cylindrical branes extended along the null direction $v$ and with a fixed radius $r \neq 0$ in the transverse plane. The relation between the $H_4$ vertex operators and the orbifold twist fields will lead to an analogy between the $S1$ branes in the Nappi-Witten wave and branes with a magnetic field on their world-volume \cite{25}.

The twisted conjugacy classes $C^\Omega(\chi)$ are parameterized by a single invariant

\[ \chi = r \cos \varphi, \] (2.8)

where we set $re^{i\varphi} = \chi + i\xi$. In Brinkman coordinates they have a simple description as $D2$ branes whose world-volume covers the two light-cone directions and one direction in the
transverse plane. The induced metric is that of a pp-wave in one dimension less and they also carry a null world-volume flux \( F_{\alpha \xi} = \frac{k_1}{2} \). The \( D2 \) branes of the \( H_4 \) model thus provide an interesting example of curved branes in a curved space-time. The relation between the \( H_4 \) vertex operators and the orbifold twist fields will lead to an analogy between the \( D2 \) branes in the Nappi-Witten wave and configurations of intersecting branes in flat space \([26]\).

It is not difficult to identify how the symmetric branes in \( H_4 \) arise in the Penrose limit from the symmetric branes in \( \mathbb{R} \times S^3 \) and \( AdS_3 \times S^1 \). In the first case, the \( S1 \) branes are the limit of \( S^2 \) branes in \( S^3 \) with a Dirichlet boundary condition in the time direction and the \( D2 \) branes are the limit of rotated \( S^2 \) branes in \( S^3 \) with a Neumann boundary condition in the time direction. In the second case, the \( S1 \) branes are the limit of the \( H_2 \) branes in \( AdS_3 \) with a Dirichlet boundary condition in \( S^1 \) and the \( D2 \) branes are the limit of the \( AdS_2 \) branes in \( AdS_3 \) with a Neumann boundary condition in \( S^1 \). A more detailed description of the Penrose limit and of the contraction of the boundary \( \mathbb{R} \times SU(2)_k \) WZW model can be found in \([13]\).

3. The sewing constraints

We now turn to the solution of the boundary CFTs that describe the two classes of symmetric branes reviewed in the previous section. For a CFT defined on the upper-half plane, there are two sets of fields. The first set contains the bulk fields \( \varphi_{i,j}(z, \bar{z}) \), inserted in the interior of the upper-half plane and characterized by the quantum numbers \((i, \bar{i})\). These quantum numbers specify the representations of the left and right chiral algebras. The second set contains the boundary fields \( \psi_{i}^{ab}(t) \), inserted on the boundary. They are characterized by two boundary conditions \( a \) and \( b \). They are also characterized by the quantum number \( i \), which labels the representations of the linear combination of the left and right affine algebras left unbroken by the boundary conditions. The boundary conformal field theory is completely specified by three sets of structure constants: the couplings between three bulk or three boundary fields and the couplings between one bulk and one boundary field. These structure constants appear in three corresponding sets of OPEs

\[
\varphi_{i,j}(z_1, \bar{z}_1)\varphi_{j,\bar{j}}(z_2, \bar{z}_2) \sim \sum_k (z_1 - z_2)^{h_k - h_i - h_j}(\bar{z}_1 - \bar{z}_2)^{\bar{h}_k - \bar{h}_i - \bar{h}_j}C_{(i,\bar{i}),(j,\bar{j})}^{(k,\bar{k})}\varphi_{k,\bar{k}}(z_2, \bar{z}_2),
\]

\[
\varphi_{i,j}(t + iy) \sim \sum_j (2y)^{h_j - h_i} a B_{i,j}^{(a)} \psi_{j}^{aa}(t),
\]

\[
\psi_{i}^{ab}(t_1)\psi_{j}^{bc}(t_2) \sim \sum_k (t_1 - t_2)^{h_k - h_i} C_{ij}^{abc,k} \psi_{k}^{ac}(t_2),
\]

where \( t_1 < t_2 \). The structure constants satisfy a set of factorization constraints first derived by Cardy and Lewellen \([17]\). These constraints involve, besides the structure constants, the modular \( S \) matrix and the fusing matrices \( F_{pq}^{ij} \), that implement, by definition, the duality transformations of the conformal blocks pertaining to the four-point amplitudes.

In \([13]\) the bulk-boundary couplings \( a B_{i}^{j} \) and the boundary three-point couplings \( C_{ijk}^{abc} \) for the two classes of symmetric branes of the \( H_4 \) WZW model were derived by solving the
sewing constraints. As an input we used the bulk three-point couplings \( C_{(i,\bar{j}),(j,\bar{k})}^{(k,\bar{k})} \) and the fusing matrices computed in \([4]\).

Before discussing the boundary CFT, we briefly recall the bulk spectrum of the \( H_4 \) WZW model, whose structure is very similar to the one established for \( AdS_3 \) \([3]\). Together with the standard highest-weight representations of the affine algebra, restricted by a unitarity constraint, there are other representations that satisfy a modified highest-weight condition. These new representations are related to the standard ones by the operation of spectral flow, which is an automorphism of the current algebra. In our case the spectrum can be organized in highest-weight and spectral-flowed representations of the affine \( \hat{H}_4 \) algebra and there are two distinct classes of states. For generic values of the light-cone momentum \( p \), the states belong to the discrete representations of the \( \hat{H}_4 \) algebra. They correspond to short strings that are confined by the background fields in closed orbits in the plane transverse to the two light-cone coordinates. Whenever \( \mu \alpha' p \in \mathbb{Z} \) the states belong to the continuous representations of the \( \hat{H}_4 \) algebra and correspond to long strings that move freely in the transverse plane. A more detailed discussion of these representations can be found in \([4]\). To each affine representation we associate a primary chiral vertex operator

\[
\Phi_{\pm x,\bar{j}}^\pm(z, x) , \quad 0 < \mu p < 1 , \quad j \in \mathbb{R} ,
\]

\[
\Phi_{s,\bar{j}}^0(z, x) , \quad s > 0 , \quad j \in [-\mu/2, \mu/2) .
\] (3.4)

For the \( \Phi_{\pm x,\bar{j}}^\pm \) vertex operators, \( p \) is the eigenvalue of \( K \) and \( j \) the highest (lowest) eigenvalue of \( J \). For the \( \Phi_{s,\bar{j}}^0 \) vertex operators, \( s \) is related to the Casimir of the representation and \( j \) is the fractional part of the eigenvalues of \( J \). Here \( z \) is a coordinate on the world-sheet and \( x \) a charge variable we introduced to keep track of the infinite number of components of the \( H_4 \) representations \([4]\). States with \( \mu p = \mu \hat{p} + w \) with \( 0 < \mu \hat{p} < 1 \) and \( w \in \mathbb{N} \) fall into spectral-flowed discrete representations \( \Sigma_{\pm w}(\Phi_{\pm x,\bar{j}}^\pm) \) while states with \( \mu p = w \) with \( w \in \mathbb{Z} \) fall into spectral-flowed continuous representations \( \Sigma_w(\Phi_{s,\bar{j}}^0) \).

4. The D2 branes

The D2 branes of the \( H_4 \) model provide examples of curved branes in a curved space-time and it is therefore very interesting to study in detail the dynamics of their open string excitations. This is also important from a formal point of view. In fact the couplings we discuss in this section are the first example of structure constants for the twisted symmetric branes of a WZW model and as such they could suggest how to extend to all possible symmetric branes the general solution that is only available at the moment for the untwisted branes.

The first thing we want to know is the spectrum of the open strings ending on the D2 branes. The brane spectrum, exactly as the bulk spectrum, can be organized in terms of highest weight and spectral flowed \( \hat{H}_4 \) representations. Not surprisingly the open string spectrum of the D2 branes in \( H_4 \) has the same structure as the open string spectrum of the \( AdS_2 \) branes in \( AdS_3 \) \([27]\). Also the spectral flow acts in the same way, in particular
spectral flow by an odd integer maps strings whose ends are at positions $\chi_1$ and $\chi_2$ to strings whose ends are at $\chi_1$ and $-\chi_2$. As a consequence there is an asymmetry between the even and the odd spectral-flowed continuous representations which is also manifest in the annulus amplitude and in the analysis of the contraction of the boundary $\mathbb{R} \times SU(2)_k$ WZW model [13]. The spectrum of a brane localized at $\chi \neq 0$ can then be summarized as follows

$$
\Sigma_{2w} \left[ \psi_{\chi, \pm p, j} \right], \quad 0 < \mu p < 1, \quad j \in \mathbb{R}, \quad w \in \mathbb{Z}, \\
\Sigma_{2w} \left[ \psi_{\chi, s, j} \right], \quad s \in \mathbb{R}, \quad j \in [-\mu/2, \mu/2), \quad w \in \mathbb{Z}, \\
\Sigma_{2w+1} \left[ \psi_{\chi, s, j} \right], \quad s \in \mathbb{R}, \quad j \in [-\mu/2, \mu/2), \quad w \in \mathbb{Z}. \quad (4.1)
$$

The spectrum of open strings ending on two different branes localized at $\chi_1$ and $\chi_2$ is essentially the same. The only difference is that there is a minimal value for the quantum number $s$ that reflects the tension of the string stretched between the two branes.

We can now derive the structure constants by solving the sewing constraints. The best way to proceed is to first derive the bulk-boundary couplings and then the boundary three-point couplings for strings ending on the D2 brane at $\chi = 0$. Once these couplings are obtained, the general solutions can be easily identified.

The constraints for the bulk-boundary couplings follow from two non-trivial bulk two-point functions, namely $\langle \Phi^0_{s, j_1} \Phi^0_{t, j_2} \rangle$ and $\langle \Phi^+_{p, j_1} \Phi^-_{j_2} \rangle$. The first one, which is very similar to the corresponding amplitude in flat space fixes $\chi B_{s, j_1}^{(t, j_2)}$. The second constraint then gives $\chi B_{s, j_1}^{(2p, 2j_2 \pm n)}$. The bulk-boundary couplings with the identity are particular cases of the previous couplings and they can be used for the construction of the boundary states for the D2 branes [13, 23]. Since the D2 branes are invariant under translations along the two light-cone directions, only operators with $p = 0$ and $j = 0, 1/2$ couple to their world-volume.

We now proceed to the computation of the boundary three-point couplings. For the twisted branes of a WZW model there is no general solution available and we have to solve directly the constraints [13]. The couplings between open strings living on the brane at $\chi = 0$ are very similar to the square root of the bulk couplings. For the complete solution we refer the reader to [13]. Here we display only a particular coupling that nicely illustrate the relation between the D2 branes and configurations of branes at angles

$$
C^{\chi_1 \chi_2 \chi_3, (p+q, j_1, j_2+n)}_{(p, j_1), (q, j_2)} = \frac{2^{2\pi \frac{i}{4} n}}{2^{2\pi n!}} \left[ \frac{\gamma(p+q)}{\gamma(p) \gamma(q)} \right]^{\frac{2\pi}{4} + \frac{i}{4}} e^{-\frac{Q^2}{2}} H_n(-Q), \\
Q = \left( \frac{\chi_1 \sin \pi q - \chi_2 \sin \pi (p + q) + \chi_3 \sin \pi p}{\sqrt{2\pi \sin \pi p \sin \pi (p + q) \sin \pi q}} \right). \quad (4.2)
$$

Let us compare this coupling with the couplings for intersecting branes discussed in [30, 31]. The quantum part of the boundary three-point couplings, which can be computed using orbifold twist fields, coincides with (4.2) if we set $n = Q = 0$. The $Q$ dependent exponential in (4.2) can be interpreted as the contribution of a disc world-sheet instanton

$$
C_{ijk} \sim e^{-\frac{\Lambda_{ijk}}{2n}}, \quad (4.3)
$$
where \( A_{ijk} \) is the area of the triangle formed by the three intersecting branes. In order to reproduce (4.2) one has to identify the angles between the branes with the light-cone momenta of the three open strings in the gravitational wave and the distance of the branes from the origin with the label \( \chi \) of the D2 branes.

5. The \( S_1 \) branes

The analysis of the \( S_1 \) branes is greatly simplified by the fact that they are the Cardy branes of the \( H_4 \) model. As a consequence there is a relation between the parameters that label the conjugacy classes and the quantum numbers of the \( \hat{H}_4 \) representations

\[
\mu u = \pm 2\pi (\mu p + w), \quad 2\eta = \pi (2j \pm 2p \mp 1),
\]

where as usual \( 0 < \mu p < 1 \) and \( w \in \mathbb{N} \). This relation can be established either by studying the annulus amplitudes of the \( H_4 \) model or by taking the Penrose limit of the boundary \( \mathbb{R} \times SU(2)_k \) WZW model [13]. We can now associate to a brane with labels \( (u_a, \eta_a) \) the parameters \( p_a, \hat{\chi}_a \) and \( w_a \). This is very useful because as it is the case for the Cardy branes in a RCFT, the spectrum of the open strings \( \psi^{ab} \) ending on the brane \( b \) and the brane \( a \) can be read from the fusion product \( \Sigma_{\pm w_b} [\Phi^\pm p_b, \hat{\chi}_b] \otimes \Sigma_{\mp w_a} [\Phi^\mp p_a, -\hat{\chi}_a] \) of the two corresponding chiral vertex operators.

Using this information it is not difficult to identify the spectrum of the open strings stretched between two \( S_1 \) branes. The open strings that end on the same brane belong to the continuous representations of the \( \hat{H}_4 \) algebra. The open strings that end on two different \( S_1 \) branes with labels \( a \) and \( b \) can belong to any of the highest-weight representations of the \( \hat{H}_4 \) algebra as well as to their images under spectral flow. The precise representation depends on the distance between the two branes along the \( u \) direction, according to (5.1).

For the \( S_1 \) branes, the spectral-flowed representations appear whenever the distance along the \( u \) direction between two branes exceeds \( \frac{2\pi}{\mu} \) and it maps a string stretched between two branes localized at \( u_a \) and \( u_b \) to a string stretched from \( u_a \) to \( u_b + 2\pi w \).

The bulk-boundary structure constants can be fixed by studying the factorization of three kinds of bulk two-point functions, namely \( \langle \Phi^+_p \Phi^+_q \rangle, \langle \Phi^+_p \Phi^-_q \rangle \) and \( \langle \Phi^+_p \Phi^0_q \rangle \). The only non-vanishing couplings are \( ^aB_{\pm p, j}^s \) and \( ^aB_{s_1, j}^{s_2} \) and can be found in [13]. The bulk-boundary couplings with the identity provide all the information necessary for the construction of the boundary states for the \( S_1 \) branes [13, 28]. Since the \( S_1 \) branes brake the translational invariance in the light-cone directions, all the discrete representations and the continuous representations with \( s = 0 \) have a non-vanishing one-point function.

The explicit form of the three-point couplings for the \( S_1 \) branes is slightly more involved than the form of the couplings for the \( D2 \) branes discussed in the previous section. Their derivation is however simpler. In fact the three-point boundary couplings for Cardy boundary conditions in a RCFT, due to the one-to-one correspondence between the boundary labels and the representations of the chiral algebra, can be expressed using the fusing matrices, as it was first realized for the Virasoro minimal models [2]

\[
C^{abc,k}_{ij} \sim F_{bk} \left[ \begin{array}{c} i \\ j \\ a \\ c \end{array} \right].
\]
In [13] we verified that this relation remains valid also for the non-compact $H_4$ WZW model, although it is not a RCFT. For the general solution we refer the reader to [13]. Here we only discuss a very simple coupling in order to illustrate the relation between the $S1$ branes and branes with magnetic fields. Consider the following coupling between three open strings that live on the same brane

$$C_{ts}^{aaa,r} = \frac{1}{\pi st \sin \theta} e^{\frac{i\pi st \sin \theta}{\pi st \sin \theta}} ,$$

(5.3)

with $r^2 = s^2 + t^2 - 2ts \cos \theta$. The phase in (5.3) can be easily understood if we compare this coupling with the coupling between open tachyon vertex operators on a two-torus with a magnetic field $B$ [25]. We introduce two free bosonic fields $X_1$ and $X_2$ subject to the boundary conditions

$$\partial_\sigma X_1 + F \partial_\tau X_2 = 0 , \quad \partial_\sigma X_2 - F \partial_\tau X_1 = 0 ,$$

(5.4)

where $F = 2\pi \alpha' B$. From the OPE between open tachyon vertex operators

$$e^{i\vec{p}\vec{X}(t_1)} e^{i\vec{q}\vec{X}(t_2)} \sim (t_1 - t_2)^{2\alpha'} e^{i\theta_{ij} \vec{p}_i \vec{q}_j} e^{i(\vec{p} + \vec{q}) \vec{X}(t_2)} , \quad \theta_{ij} = \frac{2\pi \alpha' F}{1 + F^2} e^{ij} ,$$

(5.5)

we can derive $F(u) = -\cot(\mu u/2)$ which is the expected result according to (2.7). Note that the magnetic field vanishes for $u = \pi + 2\pi n$. This corresponds to Neumann boundary conditions on $X_1$ and $X_2$. Changing the value of $u$ we get the mixed Neumann-Dirichlet boundary conditions in (5.4) until we reach $u = 2\pi n$, where the field-strength diverges and therefore the boundary conditions become pure Dirichlet. In fact, precisely for these values of the coordinate $u$, the two-dimensional conjugacy classes degenerate to points. According to the analogy with open strings in a magnetic field, the strings that live on the brane world-volume belong to the continuous representations since their ends are both subject to the same magnetic field and therefore they behave as free strings. On the other hand, the strings stretched between two different branes feel generically different magnetic fields and the corresponding vertex operators are twist fields or, in $H_4$ terminology, they belong to the discrete representations.

6. Four-point amplitudes and generalizations

With all the structure constants now at our disposal, we can in principle compute arbitrary correlation functions by sewing together the basic one, two and three-point amplitudes. Actually in [13] we discussed only those disc amplitudes that can be expressed in terms of the four-point conformal blocks, namely amplitudes containing either two bulk fields $\langle \Phi \Phi \rangle$ or one bulk and two boundary fields $\langle \Phi \psi \psi \rangle$ or four boundary fields $\langle \psi \psi \psi \psi \rangle$. Since explicit expressions for the $H_4$ conformal blocks are available [4], these amplitudes can be studied in great detail. We would like to mention that using the relation between the $H_4$ primary vertex operators and the orbifold twist fields [20], the open string amplitudes in the Nappi-Witten gravitational wave are generating functions for the correlators of arbitrarily excited boundary twist fields in models with branes at angles [29, 30, 31] or with branes with a magnetic field on the world-volume [33].
There are two aspects of our work we think deserve further study. The first is to perform a more detailed analysis of the four-point amplitudes and the second to clarify the relation between the open and closed string channel of the annulus amplitudes. There are also several simple generalizations of our work. In particular one could study the symmetric branes of the $H_{2+2n}$ WZW models, which have larger outer automorphism groups, or the branes of the supersymmetric WZW models.

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