MULTIPLE POSITIVE SOLUTIONS FOR A NONLINEAR THREE-POINT INTEGRAL BOUNDARY-VALUE PROBLEM

FAOUZI HADDOUCHI, SLIMANE BENAICHA

Abstract. We investigate the existence of positive solutions to the nonlinear second-order three-point integral boundary value problem

\[ u''(t) + f(t, u(t)) = 0, \quad 0 < t < T, \]
\[ u(0) = \beta u(\eta), \quad u(T) = \alpha \int_0^\eta u(s) \, ds, \]

where \( 0 < \eta < T, \quad 0 < \alpha < \frac{2T}{\eta^2}, \quad 0 < \beta < \frac{2T - \alpha \eta^2}{\alpha \eta^2 - 2\eta T} \) are given constants.

We establish the existence of at least three positive solutions by using the Leggett-Williams fixed-point theorem.

1. Introduction

The study of the existence of solutions of multipoint boundary value problems for linear second-order ordinary differential equations was initiated by Il’in and Moiseev [17]. Then Gupta [6] studied three-point boundary value problems for nonlinear second-order ordinary differential equations. Since then, nonlinear second-order three-point boundary value problems have also been studied by several authors. We refer the reader to [5, 7, 8, 9, 10, 11, 12, 13, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 31, 32, 33, 34, 35, 36, 37, 38, 40, 41, 42] and the references therein.

This paper is a continuation of our study in [15, 16] and is concerned with the existence and multiplicity of positive solutions of the problem

\[ u''(t) + f(t, u(t)) = 0, \quad t \in (0, T), \]

with the three-point integral boundary condition

\[ u(0) = \beta u(\eta), \quad u(T) = \alpha \int_0^\eta u(s) \, ds, \]

Throughout this paper, we assume the following hypotheses:

(H1) \( f \in C([0, T] \times [0, \infty), [0, \infty)) \) and \( f(t, \cdot) \) does not vanish identically on any subset of \( [0, T] \) with positive measure.

(H2) \( \eta \in (0, T), \quad 0 < \alpha < \frac{2T}{\eta^2} \) and \( 0 < \beta < \frac{2T - \alpha \eta^2}{\alpha \eta^2 - 2\eta T} \).

In this paper, by using the Leggett-Williams fixed-point theorem [18], we will show the existence of at least three positive solutions for a three-point integral boundary value problem. Some papers in this area include [25, 30, 39, 12, 8, 14, 1].

2000 Mathematics Subject Classification. 34B15, 34C25, 34B18.

Key words and phrases. Positive solutions; Three-point boundary value problems; multiple solutions; Fixed points; Cone.
2. Background and definitions

The proof of our main result is based on the Leggett-Williams fixed point theorem, which deals with fixed points of a cone-preserving operator defined on an ordered Banach space. For the convenience of the reader, we present here the necessary definitions from cone theory in Banach spaces.

**Definition 2.1.** Let $E$ be a real Banach space. A nonempty closed convex set $P \subset E$ is called a cone if it satisfies the following two conditions:

(i) $x \in P$, $\lambda \geq 0$ implies $\lambda x \in P$;
(ii) $x \in P$, $-x \in P$ implies $x = 0$.

Every cone $P \subset E$ induces an ordering in $E$ given by $x \leq y$ if and only if $y - x \in E$.

**Definition 2.2.** An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

**Definition 2.3.** A map $\psi$ is said to be a nonnegative continuous concave functional on a cone $P$ of a real Banach space $E$ if $\psi : P \to [0, \infty)$ is continuous and

$$\psi(tx + (1-t)y) \geq t\psi(x) + (1-t)\psi(y)$$

for all $x, y \in P$ and $t \in [0, 1]$. Similarly we say the map $\varphi$ is a nonnegative continuous convex functional on a cone $P$ of a real Banach space $E$ if $\varphi : P \to [0, \infty)$ is continuous and

$$\varphi(tx + (1-t)y) \leq t\varphi(x) + (1-t)\varphi(y)$$

for all $x, y \in P$ and $t \in [0, 1]$.

**Definition 2.4.** Let $\psi$ be a nonnegative continuous concave functional on the cone $P$. Define the convex sets $P_c$ and $P(\psi, a, b)$ by

$$P_c = \{x \in P : \|x\| < c\}, \text{ for } c > 0$$

$$P(\psi, a, b) = \{x \in P : a \leq \psi(x), \|x\| \leq b\}, \text{ for } 0 < a < b.$$
Lemma 3.2 (See [15]). Then the unique solution $u(0) = \beta u(\eta)$, $u(T) = \alpha \int_0^T u(s) \, ds$ (3.2)

For problem (3.1)-(3.2), we have the following conclusions which are derived from [15].

Lemma 3.3 (See [15]). Let $\beta \neq \frac{2T - \alpha \eta^2}{\alpha \eta^2 - 2\eta + 2T}$. Then for $y \in C([0, T], \mathbb{R})$, the problem (3.1)-(3.2) has the unique solution $u(t) = \frac{\beta(2T - \alpha \eta^2) - 2\beta(1 - \alpha \eta)t}{(\alpha \eta^2 - 2T) - \beta(2\eta - \alpha \eta^2 - 2T)} \int_0^T (\eta - s)y(s) \, ds$ \(\frac{\alpha \beta \eta - \alpha(\beta - 1)t}{(\alpha \eta^2 - 2T) - \beta(2\eta - \alpha \eta^2 - 2T)} \int_0^T (\eta - s)y(s) \, ds\) \(\frac{2(\beta - 1)t - 2\beta \eta}{(\alpha \eta^2 - 2T) - \beta(2\eta - \alpha \eta^2 - 2T)} \int_0^T (T - s)y(s) \, ds - \int_0^t (t - s)y(s) \, ds\).

Lemma 3.2 (See [15]). Let $0 < \alpha < \frac{2T}{\eta^2}$, $0 \leq \beta < \frac{2T - \alpha \eta^2}{\alpha \eta^2 - 2\eta + 2T}$. If $y \in C([0, T], \mathbb{R})$, then the unique solution $u$ of problem (3.1)-(3.2) satisfies $u(t) \geq 0$ for $t \in [0, T]$.

Lemma 3.3 (See [15]). Let $0 < \alpha < \frac{2T}{\eta^2}$, $0 \leq \beta < \frac{2T - \alpha \eta^2}{\alpha \eta^2 - 2\eta + 2T}$. If $y \in C([0, T], \mathbb{R})$, then the unique solution $u$ of problem (3.1)-(3.2) satisfies $u(t) \geq 0$ for $t \in [0, T]$.

\[\min_{t \in [0, T]} u(t) \geq \gamma \|u\|, \|u\| = \max_{t \in [0, T]} |u(t)|, \]

where

\[\gamma := \min \left\{ \frac{\eta}{T}, \frac{\alpha(\beta + 1)^2}{2T - \alpha(\beta + 1)\eta^2} \right\} \in (0, 1). \]

4. Existence of triple solutions

In this section, we discuss the multiplicity of positive solutions for the general boundary-value problem (1.1)-(1.2).

In the following, we denote

\[\Lambda := (2T - \alpha \eta^2) - \beta(\alpha \eta^2 - 2\eta + 2T), \]

\[m := \left( \frac{T^2(2T(\beta + 1) + \beta \eta(2\alpha + 2) + \alpha \beta T^2)}{2\Lambda} \right)^{-1}, \]

\[\delta := \min \left\{ \frac{\beta \eta(T - \eta)^2}{\Lambda}, \frac{\alpha \eta^2(1 + \beta)(T - \eta)^2}{2\Lambda} \right\}. \]

Using Theorem 2.1, we established the following existence theorem for the boundary-value problem (1.1)-(1.2).

**Theorem 4.1.** Assume (H1) and (H2) hold. Suppose there exists constants $0 < a < b < b/\gamma \leq c$ such that

(D1) $f(t, u) < ma$ for $t \in [0, T], u \in [0, a]$;

(D2) $f(t, u) \geq \frac{\delta}{k} \eta$ for $t \in [\eta, T], u \in [b, \frac{a}{k}]$;

(D3) $f(t, u) \leq mc$ for $t \in [0, T], u \in [0, c]$,

where $\gamma, m, \delta$ are as defined in (3.4), (4.2), and (4.3), respectively. Then the boundary-value problem (1.1)-(1.2) has at least three positive solutions $u_1, u_2$ and $u_3$ satisfying

\[\|u_1\| < a, \quad \min_{t \in [0, T]} u_2(t) > b, \quad a < \|u_3\| \quad \text{with} \quad \min_{t \in [0, T]} u_3(t) < b.\]
Proof. Let \( E = C([0, T], \mathbb{R}) \) be endowed with the maximum norm, \( \|u\| = \max_{t \in [0, T]} u(t) \), define the cone \( P \subseteq C([0, T], \mathbb{R}) \) by

\[
P = \{ u \in C([0, T], \mathbb{R}) : u \text{ concave down and } u(t) \geq 0 \text{ on } [0, T] \}.
\]

Let \( \psi : P \to [0, \infty) \) be defined by

\[
\psi(u) = \min_{t \in [0, T]} u(t), \quad u \in P.
\]

then \( \psi \) is a nonnegative continuous concave functional and \( \psi(u) \leq \|u\|, u \in P \).

Define the operator \( A : P \to C([0, T], \mathbb{R}) \) by

\[
Au(t) = -\frac{\beta(2T-\alpha t^2) - 2\beta(1-\alpha t) t}{\Lambda} \int_0^\eta (\eta - s) f(s, u(s)) ds
- \frac{\alpha \eta - \alpha(\beta - 1) t}{\Lambda} \int_0^\eta (\eta - s)^2 f(s, u(s)) ds
- \frac{2(\beta - 1) t - 2\beta \eta}{\Lambda} \int_0^T (T-s) f(s, u(s)) ds
- \int_0^t (t-s) f(s, u(s)) ds.
\]

Then the fixed points of \( A \) are just the solutions of the boundary-value problem (1.1)-(1.2) from Lemma 3.1. Since \( \psi \) with (H1) and Lemma 3.2, we see that

\[
\psi(Au) \leq \|Au\|, \quad u \in P.
\]

Moreover, \( A \) is completely continuous.

We now show that all the conditions of Theorem 2.5 are satisfied. From (4.4), we know that \( \psi(u) \leq \|u\|, \) for all \( u \in P \).

Now if \( u \in P_c \), then \( 0 \leq u \leq c \), together with (D3), we find \( \forall \ t \in [0, T] \),

\[
Au(t) \leq \frac{2\beta(1-\alpha t) t - \beta(2T-\alpha t^2)}{\Lambda} \int_0^\eta (\eta - s) f(s, u(s)) ds
+ \frac{\alpha(\beta - 1) t - \alpha \eta}{\Lambda} \int_0^\eta (\eta - s)^2 f(s, u(s)) ds
+ \frac{2\beta \eta - 2(\beta - 1) t}{\Lambda} \int_0^T (T-s) f(s, u(s)) ds
\]

\[
\leq \frac{2\beta T + \alpha \beta \eta^2}{\Lambda} \int_0^\eta (\eta - s) f(s, u(s)) ds + \frac{\alpha \beta T}{\Lambda} \int_0^\eta (\eta - s)^2 f(s, u(s)) ds
+ \frac{2\beta \eta + 2T}{\Lambda} \int_0^T (T-s) f(s, u(s)) ds
\]

\[
\leq \frac{2\beta T + \beta \eta(\alpha \eta + 2T)}{\Lambda} \int_0^T (T-s) f(s, u(s)) ds
+ \frac{\alpha \beta T}{\Lambda} \int_0^\eta (\eta - s)^2 f(s, u(s)) ds
\]

\[
\leq \frac{2\beta T + \beta \eta(\alpha \eta + 2T)}{\Lambda} \int_0^T (T-s) f(s, u(s)) ds
+ \frac{\alpha \beta T}{\Lambda} \int_0^T T(T-s) f(s, u(s)) ds.
\]
\[
\begin{align*}
&= \frac{2(\beta + 1) + T^{-1}\beta \eta (\alpha \eta + 2) + \alpha \beta T}{\Lambda} \int_0^T T(T - s) f(s, u(s)) ds \\
&\leq mc \frac{2(\beta + 1) + T^{-1}\beta \eta (\alpha \eta + 2) + \alpha \beta T}{\Lambda} \int_0^T T(T - s) ds \\
&= mc T^2 (2T(\beta + 1) + \beta \eta (\alpha \eta + 2) + \alpha \beta T^2) \\
&= c
\end{align*}
\]

Thus, \( A : \overline{P}_c \to \overline{P}_c \).

By (D1) and the argument above, we can get that \( A : \overline{P}_a \to P_a \). So, \( \|Au\| < a \) for \( \|u\| \leq a \), the condition (C2) of Theorem 2.3 holds.

Consider the condition (C1) of Theorem 2.3 now. Since \( \psi(b/\gamma) = b/\gamma > b \), let \( d = b/\gamma \), then \( \{u \in P(\psi, b, d) : \psi(u) > b\} \neq \emptyset \). For \( u \in P(\psi, b, d) \), we have \( b \leq u(t) \leq b/\gamma, t \in [0, T] \). Combining with (D2), we get

\[
f(t, u) \geq \frac{b}{\delta}, \quad t \in [\eta, T].
\]

Since \( u \in P(\psi, b, d) \), then there are two cases, (i) \( \psi(Au)(t) = Au(0) \) and (ii) \( \psi(Au)(t) = Au(T) \). In case (i), we have

\[
\psi(Au)(t) = Au(0) = \frac{\beta(2T - \alpha \eta^2)}{\Lambda} \int_0^\eta (\eta - s) f(s, u(s)) ds \\
= \frac{-\alpha \beta \eta}{\Lambda} \int_0^\eta (\eta - s)^2 f(s, u(s)) ds + \frac{2 \beta \eta}{\Lambda} \int_0^T (T - s) f(s, u(s)) ds \\
= \frac{2 \beta \eta}{\Lambda} \int_0^T (T - s) f(s, u(s)) ds - \frac{\beta \eta (2T - \alpha \eta^2)}{\Lambda} \int_0^\eta f(s, u(s)) ds \\
+ \frac{\beta(2T - \alpha \eta^2)}{\Lambda} \int_0^\eta sf(s, u(s)) ds \\
- \frac{\alpha \beta \eta}{\Lambda} \int_0^\eta (\eta^2 - 2\eta s + s^2) f(s, u(s)) ds \\
= \frac{2 \beta \eta}{\Lambda} \int_0^T (T - s) f(s, u(s)) ds - \frac{\alpha \beta \eta}{\Lambda} \int_0^\eta s^2 f(s, u(s)) ds \\
- \frac{2T \beta \eta}{\Lambda} \int_0^T f(s, u(s)) ds + \frac{2T \beta + \alpha \eta^2 \beta}{\Lambda} \int_0^\eta sf(s, u(s)) ds \\
= \frac{2T \beta \eta}{\Lambda} \int_0^T f(s, u(s)) ds - \frac{2 \beta \eta}{\Lambda} \int_0^T sf(s, u(s)) ds \\
- \frac{\alpha \beta \eta}{\Lambda} \int_0^\eta s^2 f(s, u(s)) ds + \frac{2T \beta + \alpha \eta^2 \beta}{\Lambda} \int_0^\eta sf(s, u(s)) ds \\
= \frac{2 \beta \eta}{\Lambda} \int_0^T (T - s) f(s, u(s)) ds + \frac{2 \beta (T - \eta) + \alpha \eta^2 \beta}{\Lambda} \int_0^\eta sf(s, u(s)) ds \\
- \frac{\alpha \beta \eta}{\Lambda} \int_0^\eta s^2 f(s, u(s)) ds
\]
\[
> \frac{2\beta\eta}{\Lambda} \int_{\eta}^{T} (T-s)f(s,u(s))ds + \frac{\alpha\eta^2\beta}{\Lambda} \int_{0}^{\eta} s f(s,u(s))ds \\
- \frac{\alpha\beta\eta}{\Lambda} \int_{0}^{\eta} s^2 f(s,u(s))ds \\
= \frac{2\beta\eta}{\Lambda} \int_{\eta}^{T} (T-s)f(s,u(s))ds + \frac{\alpha\beta\eta}{\Lambda} \int_{0}^{\eta} s(\eta-s)f(s,u(s))ds \\
> \frac{2\beta\eta}{\Lambda} \int_{\eta}^{T} (T-s)f(s,u(s))ds \\
\geq \frac{b}{\delta} \frac{2\beta\eta}{\Lambda} \int_{\eta}^{T} (T-s)ds \\
= \frac{b\beta\eta(\eta - \eta)^2}{\Lambda} \\
\geq b.
\]

In case (ii), we have
\[
\psi(Au)(t) = Au(T) \\
= -\frac{\beta(2T-\alpha\eta^2)}{\Lambda} \int_{0}^{\eta} (\eta-s)f(s,u(s))ds \\
- \frac{\alpha\beta\eta - \alpha(\beta-1)T}{\Lambda} \int_{0}^{\eta} (\eta-s)^2 f(s,u(s))ds \\
- \frac{2(\beta-1)T - 2\beta\eta}{\Lambda} \int_{0}^{T} (T-s)f(s,u(s))ds \\
- \int_{0}^{T} (T-s)f(s,u(s))ds \\
\geq \frac{\alpha\beta\eta(\eta - 2T)}{\Lambda} \int_{0}^{\eta} (\eta-s)f(s,u(s))ds \\
+ \frac{\alpha(\beta-1)T - \alpha\beta\eta}{\Lambda} \int_{0}^{\eta} (\eta-s)^2 f(s,u(s))ds \\
+ \frac{\alpha\eta^2(\beta+1)}{\Lambda} \int_{0}^{T} (T-s)f(s,u(s))ds \\
= \frac{\alpha\eta^2(\beta+1)}{\Lambda} \int_{0}^{T} (T-s)f(s,u(s))ds - \frac{\alpha\eta^2T(\beta+1)}{\Lambda} \int_{0}^{\eta} f(s,u(s))ds \\
+ \frac{\alpha\eta(\beta\eta + 2T)}{\Lambda} \int_{0}^{\eta} s f(s,u(s))ds \\
+ \frac{\alpha(\beta-1)T - \alpha\beta\eta}{\Lambda} \int_{0}^{\eta} s^2 f(s,u(s))ds \\
= \frac{\alpha\eta^2T(\beta+1)}{\Lambda} \int_{0}^{T} f(s,u(s))ds - \frac{\alpha\eta^2(\beta+1)}{\Lambda} \int_{0}^{\eta} s f(s,u(s))ds \\
- \frac{\alpha\eta^2(\beta+1)}{\Lambda} \int_{0}^{T} s f(s,u(s))ds + \frac{\alpha\eta(\beta\eta + 2T)}{\Lambda} \int_{0}^{\eta} s f(s,u(s))ds \\
+ \frac{\alpha(\beta-1)T - \alpha\beta\eta}{\Lambda} \int_{0}^{\eta} s^2 f(s,u(s))ds
\]
\[ = \frac{\alpha \eta^2 (\beta + 1)}{\Lambda} \int_0^T (T - s) f(s, u(s))ds + \frac{\alpha \eta (2T - \eta)}{\Lambda} \int_0^\eta s f(s, u(s))ds \\
+ \frac{\alpha \beta (T - \eta) - \alpha T}{\Lambda} \int_0^\eta s^2 f(s, u(s))ds \\
> \frac{\alpha \eta^2 (\beta + 1)}{\Lambda} \int_\eta^T (T - s) f(s, u(s))ds + \frac{\alpha \eta T}{\Lambda} \int_0^\eta s f(s, u(s))ds \\
- \frac{\alpha T}{\Lambda} \int_0^\eta s^2 f(s, u(s))ds \\
= \frac{\alpha \eta^2 (\beta + 1)}{\Lambda} \int_\eta^T (T - s) f(s, u(s))ds \\
+ \frac{\alpha \eta^2 (\beta + 1)}{\Lambda} \int_\eta^T (T - s)f(s, u(s))ds \\
\geq \frac{b \alpha \eta^2 (\beta + 1)}{\delta} \int_\eta^T (T - s)ds \\
= \frac{b \alpha \eta^2 (\beta + 1)(T - \eta)^2}{\delta} \\
\geq b. \]

So, \( \psi(Au) > b \); \( \forall u \in P(\psi, b, b/\gamma) \).

For the condition (C3) of the Theorem 2.5, we can verify it easily under our assumptions using Lemma 3.3. Here

\[ \psi(Au) = \min_{t \in [0,T]} Au(t) \geq \gamma \|Au\| > \frac{b}{\gamma} = b \]

as long as \( u \in P(\psi, b, c) \) with \( \|Au\| > b/\gamma \).

Since all conditions of Theorem 2.5 are satisfied. Then problem (1.1)-(1.2) has at least three positive solutions \( u_1, u_2, u_3 \) with

\[ \|u_1\| < a, \quad \psi(u_2) > b, \quad a < \|u_3\| \quad \text{with} \quad \psi(u_3) < b. \]

\[ \Box \]

5. Some examples

In this section, in order to illustrate our result, we consider some examples.

**Example 5.1.** Consider the boundary value problem

\[ u''(t) + \frac{40u^2}{u^2 + 1} = 0, \quad 0 < t < 1, \quad (5.1) \]

\[ u(0) = \frac{1}{2} u\left(\frac{1}{3}\right), \quad u(1) = 3 \int_0^1 u(s)ds. \quad (5.2) \]

Set \( \beta = 1/2, \alpha = 3, \eta = 1/3, T = 1, \) and

\[ f(t, u) = f(u) = \frac{40u^2}{u^2 + 1}, \quad u \geq 0. \]

It is clear that \( f(\cdot) \) is continuous and increasing on \([0, \infty)\). We can also seen that...
Now we check that (D1), (D2) and (D3) of Theorem 4.1 are satisfied. By (3.4), (4.2), (4.3), we get \( \gamma = \frac{1}{4}, m = \frac{1}{3}, \delta = \frac{4}{45} \). Let \( c = 124 \), we have
\[
 f(u) \leq 40 < mc = \frac{124}{3} \approx 41,33, \quad u \in [0, c],
\]
from \( \lim_{u \to \infty} f(u) = 40 \), so that (D3) is met. Note that \( f(2) = 32 \), when we set \( b = 2 \),
\[
f(u) \geq b \delta = 22.5, \quad u \in [b, 4b],
\]
holds. It means that (D2) is satisfied. To verify (D1), as \( f\left(\frac{1}{120}\right) = \frac{40}{14401} \), we take \( a = \frac{1}{120} \), then
\[
f(u) < ma = \frac{1}{360}, \quad u \in [0, a],
\]
and (D1) holds. Summing up, there exists constants \( a = 1/120, b = 2, c = 124 \) satisfying
\[
0 < a < b < b \gamma \leq c,
\]
such that (D1), (D2) and (D3) of Theorem 4.1 hold. So the boundary-value problem (5.1)–(5.2) has at least three positive solutions \( u_1, u_2 \) and \( u_3 \) satisfying
\[
\|u_1\| < \frac{1}{120}, \quad \min_{t \in [0, T]} u_2(t) > 2, \quad \frac{1}{120} < \|u_3\| \quad \text{with} \quad \min_{t \in [0, T]} u_3(t) < 2.
\]

Example 5.2. Consider the boundary value problem
\[
 u''(t) + f(t, u) = 0, \quad 0 < t < 1, \quad (5.3)
\]
\[
 u(0) = u(\frac{1}{2}), \quad u(1) = \int_0^1 u(s)ds. \quad (5.4)
\]
Set \( \beta = 1, \alpha = 1, \eta = 1/2, \quad T = 1, \quad f(t, u) = e^{-t}h(u) \) where
\[
h(u) = \begin{cases} \frac{2}{25}u & 0 \leq u \leq 1 \\ \frac{2173}{45}u - \frac{2167}{45} & 1 \leq u \leq 4 \\ \frac{87}{39}u & 4 \leq u \leq 544 \\ \frac{544u}{u+270} & 544 \leq u \leq 546 \\ u \geq 546. \end{cases} \quad (5.5)
\]
By (3.4), (4.2), (4.3) and after a simple calculation, we get \( \gamma = \frac{1}{4}, m = \frac{4}{25}, \delta = \frac{1}{8} \).
We choose \( a = 1/4, b = 4, \) and \( c = 544 \); consequently,
\[
f(t, u) = e^{-t} \frac{2}{25}u \leq \frac{2}{25}u < \frac{4}{25} \times \frac{1}{4} = ma, \quad 0 \leq t \leq 1, \quad 0 \leq u \leq 1/4,
\]
\[ f(t, u) = e^{-t}87 \geq \frac{87}{e} > 32 = \frac{b}{\delta}, \quad 1/2 \leq t \leq 1, \quad 4 \leq u \leq 16, \]
\[ f(t, u) = e^{-t}h(u) \leq 87 < \frac{4}{25} \times 544 = mc, \quad 0 \leq t \leq 1, \quad 0 \leq u \leq 544. \]

That is to say, all the conditions of Theorem 4.1 are satisfied. Then problem (5.3), (5.4) has at least three positive solutions \( u_1, u_2, \) and \( u_3 \) satisfying
\[ \|u_1\| < \frac{1}{4}, \quad \psi(u_2) > 4, \quad \|u_3\| > \frac{1}{4} \text{ with } \psi(u_3) < 4. \]

**References**

[1] D. R. Anderson, Multiple Positive Solutions for a Three-Point Boundary Value Problem, *Math. Comput. Modelling*, Vol. 27, No. 6 (1998), 49-57.

[2] D. R. Anderson, R. Avery and A. Peterson; Three positive solutions to a discrete focal boundary value problem, In Positive Solutions of Nonlinear Problems, (Edited by R. Agarwal), *J. Comput. Appl. Math.*, (1998).

[3] R. I. Avery, D. R. Anderson; Existence of three positive solutions to a second-order boundary value problem on a measure chain, *J. Comput. Appl. Math.*, 141 (2002), 65-73.

[4] R. P. Agarwal, D. O’Regan; Triple solutions to boundary value problems on time scales, *Appl. Math. Lett.*, 13 (2000), 7-11.

[5] Z. Chengbo; Positive solutions for semi-positone three-point boundary value problems, *J. Comput. Appl. Math.*, 228 (2009), 279-286.

[6] C. P. Gupta; Solvability of a three-point nonlinear boundary value problem for a second order ordinary differential equations, *J. Math. Anal. Appl.*, 168 (1992), 540-551.

[7] Y. Guo, W. Ge; Positive solutions for three-point boundary value problems with dependence on the first order derivative, *J. Math. Anal. Appl.*, 290 (2004), 291-301.

[8] C. S. Goodrich; Positive solutions to boundary value problems with nonlinear boundary conditions, *Nonlinear Anal.*, 75 (2012), 417-432.

[9] C. S. Goodrich; On nonlocal BVPs with boundary conditions with asymptotically sublinear or superlinear growth, *Math. Nachr.*, 285 (2012), 1404-1421.

[10] C. S. Goodrich; Nonlocal systems of BVPs with asymptotically sublinear boundary conditions, *Appl. Anal. Discrete Math.*, 6 (2012), 174-193.

[11] C. S. Goodrich; Nonlocal systems of BVPs with asymptotically superlinear boundary conditions, Comment. Math. Univ. Carolin., 53 (2012), 79-97.

[12] C. S. Goodrich; On nonlinear boundary conditions satisfying certain asymptotic behavior, *Nonlinear Anal.*, 76 (2013), 58-67.

[13] X. Han; Positive solutions for a three-point boundary value problem, *Nonlinear Anal.*, 66 (2007), 679-688.

[14] X. He and W. Ge; Triple solutions for second-order three-point boundary value problems, *J. Math. Anal. Appl.*, 268 (2002), 256-265.

[15] F. Haddouchi, S. Benaicha; Positive solutions of nonlinear three-point integral boundary value problems for second-order differential equations, [http://arxiv.org/abs/1205.1844] May, 2012.

[16] F. Haddouchi, S. Benaicha; Existence of positive solutions for a three-point integral boundary-value problem, [http://arxiv.org/abs/1304.5664], April, 2013.

[17] V. A. Il’in, E. I. Moiseev; Nonlocal boundary-value problem of the first kind for a Sturm-Liouville operator in its differential and finite difference aspects, *Differ. Equ.*, 23 (1987), 803-810.

[18] R. W. Leggett and L. R. Williams; Multiple positive fixed points of nonlinear operators on ordered Banach spaces, *Indiana Univ. Math. J.*, 28 (1979), 673-688.

[19] S. Liang, L. Mu; Multiplicity of positive solutions for singular three-point boundary value problems at resonance, *Nonlinear Anal.*, 71 (2009), 2497-2505.

[20] R. Liang, J. Peng, J. Shen; Positive solutions to a generalized second order three-point boundary value problem, *Appl. Math. Comput.*, 196 (2008), 931-940.

[21] J. Li, J. Shen; Multiple positive solutions for a second-order three-point boundary value problem, *Appl. Math. Comput.*, 182 (2006), 258-268.
[22] B. Liu; Positive solutions of a nonlinear three-point boundary value problem, *Appl. Math. Comput.*, 132 (2002), 11-28.
[23] B. Liu; Positive solutions of a nonlinear three-point boundary value problem, *Comput. Math. Appl.*, 44 (2002), 201-211.
[24] B. Liu, L. Liu, Y. Wu; Positive solutions for singular second order three-point boundary value problems, *Nonlinear Anal.*, 66 (2007), 2756-2766.
[25] H. Luo, Q. Ma; Positive solutions to a generalized second-order three-point boundary-value problem on time scales, *Electron. J. Differential Equations*, 2005 (2005), No. 17, 1-14.
[26] R. Ma; Multiplicity of positive solutions for second-order three-point boundary value problems, *Comput. Math. Appl.*, 40 (2000), 193-204.
[27] R. Ma; Positive solutions of a nonlinear three-point boundary-value problem, *Electron. J. Differential Equations*, 1999 (1999), No. 34, 1-8.
[28] R. Ma; Positive solutions for second-order three-point boundary value problems, *Appl. Math. Lett.*, 14 (2001), 1-5.
[29] H. Pang, M. Feng, W. Ge; Existence and monotone iteration of positive solutions for a three-point boundary value problem, *Appl. Math. Lett.*, 21 (2008), 656-661.
[30] J. P. Sun, W. T. Li, and Y. H. Zhao; Three positive solutions of a nonlinear three-point boundary value problem, *J. Math. Anal. Appl.*, 288 (2003), 708-716.
[31] Y. Sun, L. Liu, J. Zhang, R. P. Agarwal; Positive solutions of singular three-point boundary value problems for second-order differential equations, *J. Comput. Appl. Math.*, 230 (2009), 738-750.
[32] H. Sun, W. Li; Positive solutions for nonlinear three-point boundary value problems on time scales, *J. Math. Anal. Appl.*, 299 (2004), 508-524.
[33] J. Tariboon, T. Sitthiwirattham; Positive solutions of a nonlinear three-point integral boundary value problem, *Bound. Value Probl.*, 2010 (2010), ID 519210, 11 pages, doi:10.1155/2010/519210.
[34] J. R. L. Webb; A unified approach to nonlocal boundary value problems, *Dynam. Systems Appl.*, 5 (2008), 510-515.
[35] J. R. L. Webb; Solutions of nonlinear equations in cones and positive linear operators, *J. Lond. Math. Soc.*, (2) 82 (2010), 420-436.
[36] J. R. L. Webb, G. Infante; Positive solutions of nonlocal boundary value problems involving integral conditions, *NoDEA Nonlinear Differential Equations Appl.*, 15 (2008), 45-67.
[37] J. R. L. Webb, G. Infante; Positive solutions of nonlocal boundary value problems: a unified approach, *J. Lond. Math. Soc.*, (2) 74 (2006), 673-693.
[38] X. Xu; Multiplicity results for positive solutions of some semi-positone three-point boundary value problems, *J. Math. Anal. Appl.*, 291 (2004), 673-689.
[39] X. Xian; Three solutions for three-point boundary value problems, *Nonlinear Anal.*, 62 (2005), 1053-1066.
[40] Z. Yang; Existence and nonexistence results for positive solutions of an integral boundary value problem, *Nonlinear Anal.*, (8) 65 (2006), 1489-1511.
[41] Z. Yang; Existence of nontrivial solutions for a nonlinear Sturm-Liouville problem with integral boundary conditions, *Nonlinear Anal.*, (1) 68 (2008), 216-225.
[42] G. Zhang, J. Sun; Positive solutions of m-point boundary value problems, *J. Math. Anal. Appl.*, (2) 291 (2004), 406-418.

**Fauzi Haddouchi, Department of Physics, University of Sciences and Technology of Oran, El Mnaouar, BP 1505, 31000 Oran, Algeria**

*E-mail address: fhaddouchi@gmail.com*

**Slimane Benaicha, Department of Mathematics, University of Oran, Es-senia, 31000 Oran, Algeria**

*E-mail address: slimanebenaicha@yahoo.fr*