Fair Division with Minimal Sharing

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Abstract

A set of objects, some goods and some bads, is to be divided fairly among agents with different tastes, modeled by additive utility-functions. If the objects cannot be shared, so that each of them must be entirely allocated to a single agent, then fair division may not exist. What is the smallest number of objects that must be shared between two or more agents in order to attain a fair division?

We focus on Pareto-optimal, envy-free and/or proportional allocations. We show that, for a generic instance of the problem — all instances except of a zero-measure set of degenerate problems — a fair and Pareto-optimal division with the smallest possible number of shared objects can be found in polynomial time, assuming that the number of agents is fixed. The problem becomes computationally hard for degenerate instances, where the agents’ valuations are aligned for many objects.

1 Introduction

What is a fair way to allocate objects without monetary transfers?

When the objects are indivisible, it may be impossible to allocate them fairly — consider a single object and two people. A common approach to this problem is to look for an approximately-fair allocation. There are several definitions of approximate fairness, the most common of which are envy-freeness except one object (EF1) and maximin share (MMS). An alternative solution is to “make objects divisible” by allowing randomization and ensure that the division is fair ex-ante.

While approximate or ex-ante fairness are reasonable when allocating low-value objects, such as seats in a course or in a school, they are not suitable for high-value objects, e.g., houses or precious jewels. Think of a divorcing couple deciding how to split children, or two siblings who have to
divide three houses among them; it is unlikely that one of them will agree to receive a bundle that is envy-free except one child/house, or a lottery that gives either one or two with equal probability.

In practical cases, when monetary transfers are undesired, the common solution is to find a way to share some of the objects. For example, a shop can be jointly owned by several partners, sharing the costs and revenues of operation. A house can be jointly owned by several people, who live in the house alternately in different times. While such sharing may be inevitable if exact fairness is desired, it may be quite inconvenient due to the overhead in managing the shared property. Therefore, it is desirable to minimize the number of objects that have to be shared.

**Our contribution.** The paper advocates a new approach to the problem of fair division:

*minimize the number of sharings under the constraints of fairness and economic efficiency.*

This approach is a compelling alternative to approximate fairness when the objects to be divided are highly valuable and sharing is technically possible (as in all examples above and many other real-life situations) but unwanted.

We consider problems where the objects to be divided may contain both goods and bads, as in the practice of partnership dissolution when valuable assets are often divided together with liabilities. We assume that agents have additive utilities, so the problem can be represented by a valuation matrix, recording the value of each object to each agent.\(^1\)

We focus on the classic fairness notions of proportionality (each agent gets a bundle worth at least \(1/n\) of the total value, where \(n\) is the number of agents) or envy-freeness (each agent weakly prefers his/her bundle to the bundle of any other agent). Economic efficiency is captured by fractional Pareto optimality: no other allocation, even without any restriction on sharing, improves the well-being of some agent without harming some others. We focus on the case in which \(n\) is a fixed small number, as is common in inheritance cases, so the problem size is determined by the number of objects, denoted by \(m\).

The first ingredient we need for the solution is an upper bound on the number of sharings. In Bogomolnaia, Moulin, Sandomirskiy, and Yanovskaya (2016)[Lemma 1] it was shown that there is always a fair fractionally Pareto-optimal allocation with at most \(n-1\) objects shared if the objects are either goods or bads. The bound is tight: when there are \(n-1\) identical goods, all of them must be shared. In Subsection 2.3 we show that this upper bound extends to mixture of goods and bads, and that such an allocation can be constructed in polynomial time.

Then (Section 3) we consider the algorithmic problem of finding a fair and fractionally Pareto-optimal allocation that minimizes the number of sharings. We find that the computational hardness of the problem depends on the degree of degeneracy of the valuation matrix. Informally, the degree of degeneracy measures how close are the agents’ valuations to identical. A valuation matrix is called non-degenerate if for every pair of agents, the ratios of their values are different across all objects. The degree of degeneracy ranges between 0 (for non-degenerate valuations) and \(m-1\) (for identical).

We demonstrate the following dichotomy:

- Minimizing the number of sharings is algorithmically tractable if the degree of degeneracy of the valuation matrix is at most logarithmic in \(m\), in particular, for non-degenerate valuations.

\(^1\)While additive utilities do not allow to express complementarities between objects (e.g., a garage becomes more valuable together with a car), this class proved to be convenient in practice because of simplicity of formulating and reporting such preferences Goldman and Procaccia (2015).
Table 1: Run-time complexity of dividing $m$ objects (goods and bads) among $n$ agents (where $n$ is fixed), with a bound on the number of sharings. Fairness notion is either EF (envy-freeness, which implies proportionality) or Prop (only proportionality). Under both, a fair and fractionally Pareto-optimal (fPO) allocation with at most $n - 1$ sharings always exist and can be computed in polynomial time. In contrast, computing an allocation with a minimal number of sharings is hard in general, but fPO and non-degeneracy of valuations help to overcome this hardness.

We present an algorithm with a run-time polynomial in $m$ when $n$ is fixed. Since the set of valuations with positive degree of degeneracy has zero measure, the algorithm runs in polynomial time for almost all instances.

- Minimizing the number of sharings is NP-hard for any fixed $n \geq 2$ if the degree of degeneracy is at least of the order of $m^\alpha$ for some $\alpha > 0$, in particular, for identical valuations.

Our main results are summarized in Table 1.

**Surprising source of computational hardness and the role of Pareto optimality**  Our results confirm the common sense that computationally-hard instances of resource-allocation problems are rare and, in practice, such problems can often be solved efficiently.

However, the fact that computationally-hard instances are those in which agents have identical valuations is quite surprising. In many previous papers on fair division (e.g. Oh, Procaccia, and Suksompong (2018), Plaut and Roughgarden (2018)), computational hardness results are presented with a qualifier saying that the problem is hard “even when the valuations are identical”; our results show that the “even” is unwarranted.

Another observation that may seem surprising is that finding a fair and fractionally Pareto optimal allocation with minimal sharing is computationally easier than just fair with minimal sharing (without Pareto-optimality). The underlying reason is that, for non-degenerate problems, fractional-Pareto-optimality is a strong condition that shrinks the search space to a polynomial number of structures (see Proposition 3.8 for a formal statement).

The polynomial “size” of the Pareto frontier for non-degenerate problems is the key observation, which allows us to conduct the exhaustive search over fPO allocations. A somewhat similar observation is known within the framework of smoothed analysis of NP-hard problems: the Pareto frontier for a knapsack problem and its extensions becomes polynomially-sized if the instance is randomly perturbed (see Moitra and ODonnell (2011) for a survey).

Note the important contrast between fractional and discrete Pareto-optimality (an allocation is discrete Pareto-optimal if it is not dominated by any allocation with zero sharings). For discrete
Pareto-optimality, even basic questions are computationally hard: deciding whether a given allocation is indivisible-PO is co-NP hard, and deciding whether there exists an envy-free and indivisible-PO allocation is $\Sigma^P_2$-complete (de Keijzer, Bouveret, Klos, and Zhang, 2009). In contrast, deciding whether a given allocation is fractional-PO is polynomial in $m$ and $n$ (see Lemma 2.2), and deciding whether there exists an envy-free and fractionally PO allocation with no sharings is, for almost all instances, polynomial in $m$ for fixed $n$ (Theorem 3.6). These observations suggest that fractional-Pareto-optimality is a compelling concept of economic efficiency even for indivisible objects; recent results by Barman and Krishnamurthy (2018), Barman, Krishnamurthy, and Vaish (2018) confirm this observation.

Structure of the paper. In Section 2 we introduce the notation and describe useful tools such as characterizations of fractional Pareto-optimality and worst-case bounds on the number of sharings. While most of these results are known for goods, extension to mixed problems turns out to be non-trivial. Our main results are proved in Section 3. Related work is surveyed in Section 4. In Section 5 we consider some possible extensions of the model, such as truthful mechanisms and non-linear utilities. Some open problems are mentioned along the way.

2 Preliminaries

2.1 Agents, objects, and allocations

There is a set $[n] = \{1, \ldots, n\}$ of $n$ agents and a set $[m] = \{1, \ldots, m\}$ of $m$ divisible objects. A bundle $x$ of objects is a vector $(x_o)_{o \in [m]} \in [0,1]^m$, where the component $x_o$ represents the portion of $o$ in the bundle (the total amount of each object is normalized to 1).

Each agent $i \in [n]$ has an additive utility function over bundles: $u_i(x) = \sum_{o \in [m]} v_{i,o} \cdot x_o$. Here $v_{i,o} \in \mathbb{R}$ is agent $i$’s value of receiving the whole object $o \in [m]$; the matrix $v = (v_{i,o})_{i \in [n], o \in [m]}$ is called the valuation matrix; it encodes the information about agents’ preferences and is used below as the input of fair division algorithms.

We make no assumptions on valuation matrix $v$ and allow values of mixed signs: for example, the same object $o$ can bring positive value to some agents and negative to others. We say that an object $o$ is:

- a bad if $v_{i,o} < 0$ for all $i \in [n]$.
- neutral if $v_{i,o} = 0$ for at least one $i \in [n]$ and $v_{j,o} \leq 0$ for all $j \in [n]$;
- a good if $v_{i,o} > 0$ for at least one $i \in [n]$;
  - a pure good if $v_{i,o} > 0$ for all $i \in [n]$;

The motivation for the asymmetry in the definitions will become clear soon.

An allocation $z$ is a collection of bundles $(z_i)_{i \in [n]}$, one for each agent, with the condition that all the objects are fully allocated. An allocation can be identified with the matrix $z := (z_{i,o})_{i \in [n], o \in [m]}$ such that all $z_{i,o} \geq 0$ and $\sum_{i \in [n]} z_{i,o} = 1$ for each $o \in [m]$.

The utility profile of an allocation $z$ is the vector $u(z) := (u_i(z_i))_{i \in [n]}$. A vector $U := (U_1, \ldots, U_n)$ is called a feasible utility profile if there exists an allocation $z$ with $U = u(z)$. 

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Fairness and efficiency concepts. The two fundamental notions of fairness, taking preferences of agents into account, are envy-freeness and a weaker concept of proportionality (also known as equal split lower bound or fair share guarantee).

An allocation \( z = (z_i)_{i \in [n]} \) is called envy-free (EF) if every agent prefers her bundle to the bundles of others. Formally, for all \( i, j \in [n] \): \( u_i(z_i) \geq u_i(z_j) \).

An allocation \( z \) is proportional (Prop) if each agent prefers her bundle to the equal division: \( \forall i \in [n] \) \( u_i(z_i) \geq \frac{1}{n} \sum_{o \in [m]} v_{i,o} \). Every envy-free allocation is also proportional; with \( n = 2 \) agents, envy-freeness and proportionality are equivalent.

The idea that the objects must be allocated in the efficient, non-improvable way is captured by Pareto-optimality. An allocation \( z \) is Pareto-dominated by an allocation \( y \) if \( y \) gives at least the same utility to all agents and strictly more to at least one of them.

An allocation \( z \) is fractionally Pareto-optimal (fPO) if no feasible \( y \) dominates it.\(^2\)

We will also need the following extremely weak but easy-checkable efficiency notion: an allocation \( z \) is non-malicious if each good \( o \) is consumed by agents \( i \) with \( v_{i,o} > 0 \), and each neutral object \( o \) by agents \( i \) with \( v_{i,o} = 0 \). Every fPO allocation is clearly non-malicious.

2.2 Agent-object graphs and a characterization of fPO

Our algorithms will use several kinds of agent-object graphs bipartite graphs in which the nodes on one side are the agents and the nodes on the other side are the objects:

- In the (undirected) consumption-graph \( CG_z \) of a given allocation \( z \), there is an edge between agent \( i \in [n] \) and object \( o \in [m] \) iff \( z_{i,o} > 0 \).

- The weighted directed consumption-graph \( \overrightarrow{CG}_z \) of \( z \) is constructed in the following way. If agent \( i \) consumes some object \( o \) (i.e., \( z_{i,o} > 0 \)), then
  - for \( v_{i,o} \geq 0 \), there is an edge \( (i \rightarrow o) \) with weight \( w_{i \rightarrow o} = v_{i,o} \);
  - for \( v_{i,o} < 0 \), there is an edge \( (o \rightarrow i) \) with weight \( w_{o \rightarrow i} = \frac{1}{|v_{i,o}|} \).

Symmetrically, if \( z_{i,o} < 1 \),
  - for \( v_{i,o} > 0 \), there is an edge \( (o \rightarrow i) \) with weight \( w_{o \rightarrow i} = \frac{1}{v_{i,o}} \);
  - for \( v_{i,o} < 0 \), there is an edge \( (i \rightarrow o) \) with weight \( w_{i \rightarrow o} = |v_{i,o}| \).

Intuitively, \( \overrightarrow{CG}_z \) captures the structure of possible exchanges in which an agent may engage. Outgoing edges represent those objects that an agent \( i \) can use as a “currency” to pay others: either goods \( i \) owns or bads owned by somebody else (in this case \( i \) pays to \( j \) who owns a bad \( b \) by taking some portion of \( b \)). Incoming edges are those objects that \( i \) is ready to accept as a currency: either to receive a valuable good, or to diminish \( i \)’s own bad. An example is shown in Figure 1.

Given \( v \), one can reconstruct \( \overrightarrow{CG}_z \) from \( CG_z \) and vice versa. Indeed, the condition \( z_{i,o} < 1 \) from the definition of \( \overrightarrow{CG}_z \) holds if and only if there is an agent \( j \neq i \) with \( z_{j,o} > 0 \), i.e., if \( o \) is connected to some \( j \neq i \) in \( CG_z \).

\(^2\)The literature on indivisible objects considers a weaker notion of economic efficiency: \( z \) is discrete Pareto-optimal if it is not dominated by any feasible \( y \) with \( y_{i,o} \in \{0,1\} \). While fractional-Pareto-optimality has good algorithmic properties, its discrete version does not, see the discussion in Section 1.
Figure 1: Some examples of weighted directed consumption graphs in an instance with pure goods. Alice gets the farm, Bob gets the car, and they share the house. **Left:** The farm, house and car are valued by Alice at 4, 2.5 and 1, and by Bob at 1.25, 2 and 5. The allocation is fPO. **Right:** The house is valued by Alice at 25 (all other valuations are the same). The allocation is not fPO; the bold arrows denote a cycle \( C \) with \( \pi(C) < 1 \).

The product of a directed path \( P \) in \( \mathcal{G}_{\text{CG}} \), denoted \( \pi(P) \), is the product of weights of edges in \( P \). In particular, the product of a cycle \( C = (i_1 \rightarrow o_1 \rightarrow \ldots \rightarrow o_L \rightarrow i_{L+1} = i_1) \) is:

\[
\pi(C) = \prod_{k=1}^{L} (w_{i_k \rightarrow o_k} \cdot w_{o_k \rightarrow i_{k+1}}).
\]

The importance of this product is justified by the following lemma (proved in App. A):

**Lemma 2.1.** An allocation \( z \) is fractionally Pareto-optimal if-and-only-if it is non-malicious and its directed consumption graph \( \mathcal{G}_{\text{CG}} \) has no cycle \( C \) with \( \pi(C) < 1 \).

We see that the information about Pareto-optimality of an allocation is “encoded” in its consumption graph. An analog of Lemma 2.1 is known for pure goods in a more general cake-cutting context Barbanel (2005) and was recently extended to problems with bads only by Branzei and Sandomirskiy (2019). Lemma 2.1 has a useful computational implication, which is also proved in Appendix A.

**Lemma 2.2.** It is possible to decide in time \( O(nm(n + m)) \) whether a given allocation \( z \) is fractionally Pareto-optimal.

Classic results in economic theory (Negishi, 1960, Varian, 1976) represent the Pareto frontier of economies with convex sets of feasible utilities as the set of allocations \( z \) that maximize the weighted utilitarian welfare \( \sum_{i \in [n]} \lambda_i u_i(z_i) \) for some positive weights \( \lambda_i \). This leads to another characterization of fPO allocations (proof can be found in Branzei and Sandomirskiy (2019) for the case of bads and in Appendix A for the mixture of goods and bads).

**Lemma 2.3.** An allocation \( z \) is fractionally Pareto-optimal if and only if there is a vector of weights \( \lambda = (\lambda_i)_{i \in [n]} \) with \( \lambda_i > 0 \) such that for all agents \( i \in [n] \) and objects \( o \in [m] \)

\[
z_{i,o} > 0 \implies \lambda_i v_{i,o} \geq \lambda_j v_{j,o} \quad \text{for any agent } j \in [n].
\]

If the “certificate” \( \lambda \) is unknown, this lemma does not provide an algorithm for checking Pareto-optimality. However, it provides useful insights into the “threshold” structure of fPO allocations captured by the following necessary condition for fPO.
Corollary 2.4. For a fractionally Pareto-optimal allocation $z$ and any pair of agents $i \neq j$, there is a threshold $t_{i,j} > 0$ (i.e., both agents agree whether $o$ is a good or a bad) such that for any object $o$

- if $v_{i,o} \cdot v_{j,o} > 0$ (i.e., both agents agree whether $o$ is a good or a bad), then
  - for $\frac{|v_{i,o}|}{|v_{j,o}|} > t_{i,j}$, we have $z_{j,o} = 0$ in case of a good and $z_{i,o} = 0$ in case of a bad
  - for $\frac{|v_{i,o}|}{|v_{j,o}|} < t_{i,j}$, we have $z_{i,o} = 0$ in case of a good and $z_{j,o} = 0$ in case of a bad
- if $v_{i,o} \cdot v_{j,o} < 0$, then an agent with negative value cannot consume $o$.

In particular, $i$ and $j$ can share only objects $o$ with $\frac{|v_{i,o}|}{|v_{j,o}|} = t_{i,j}$ or such that $v_{i,o} = v_{j,o} = 0$.

A similar result underlies the Adjusted Winner procedure defined by Brams and Taylor (1996) for goods and which is extended to mixed problems in Aziz, Caragiannis, and Igarashi (2018). The condition is necessary and sufficient for $n = 2$ (see Bogomolnaia et al. (2016) for either goods or bads). For $n \geq 3$, the condition is not sufficient for Pareto-optimality; as one can see, it is equivalent to having no cycles of length 4 with $\pi(C) < 1$ in $\mathbb{S}z$ (for $n = 2$ any simple cycle has length at most 4). However, it does not exclude longer cycles.

2.3 Measures of sharing and worst-case bounds

If for some $i \in [n]$, $z_{i,o} = 1$, then the object $o$ is not shared — it is fully allocated to agent $i$. Otherwise, object $o$ is shared between two or more agents. Throughout the paper, we consider two measures quantifying the amount of sharing in a given allocation $z$.

The simplest one is the number of shared objects $\left| \{ o \in [m] : z_{i,o} \in (0,1) \text{ for some } i \in [n] \} \right|$. Alternatively, one can take into account the number of times each object is shared. This is captured by the number of sharings

$$\#\text{shar}_{z} = \sum_{o \in [m]} \left( \left| \{ i \in [n] : z_{i,o} > 0 \} \right| - 1 \right).$$

For “indivisible” allocations both measures are zero, but they differ, for example, if only one object $o$ is shared but each agent consumes a bit of $o$: the number of shared objects in this case is 1 while the number of sharings is $n - 1$. Clearly, the number of shared objects is always at most the number of sharings.

When there are $n$ agents and $n - 1$ identical pure goods, a fair allocation must give each agent a fraction $(n - 1)/n$ of a good, for any reasonable definition of fairness. This requires sharing all $n - 1$ goods, so $n - 1$ is a lower bound on the number of shared objects and thus also for the number of sharings. This lower bound can always be achieved.

In case of pure goods or bads but not a mixture, Bogomolnaia et al. (2016) showed that for any fractionally Pareto-optimal allocation $z$, there exists an equivalent one $z^*$ (in both allocations, all agents receive the same utilities) with $\#\text{shar}_{z^*} \leq n - 1$. A similar result can be found in Barman and Krishnamurthy (2018) (see Claim 2.2) for the so-called competitive equilibrium allocations of pure goods. The following lemma shows that $z^*$ can be constructed efficiently. It also covers arbitrary allocations $z$ rather than just fractionally Pareto-optimal, and a mixture of goods and bads. It is proved in Appendix B.
Lemma 2.5. For any allocation $z$, there is a fractionally PO allocation $z^*$ such that:

- (a) $z^*$ weakly Pareto dominates $z$, i.e., for any agent $i$, $u_i(z_i^*) \geq u_i(z_i)$.
- (b) the non-directed consumption graph $G_{z^*}$ is acyclic.
- (c) $z^*$ has at most $n-1$ sharings (hence at most $n-1$ shared objects).

The allocation $z^*$ can be constructed in strongly-polynomial time using $O(n^2m^2(n+m))$ operations.

This lemma, combined with known algorithms for computing fair and fractionally PO allocations, yields the following corollary.

Corollary 2.6. In any instance with $n$ agents, there exists a fractionally Pareto-optimal, envy-free (and thus proportional) division with at most $n-1$ sharings. In some special cases, such an allocation can be found in strongly-polynomial time:

- If all objects are pure goods — using $O((n+m)^4 \log(n+m))$ operations;
- If all objects are bads and $n$ is fixed — using $O\left(m \frac{n(n-1)}{2} + 3\right)$ operations;
- If the fairness notion is proportionality only — using $O(n^2m^2(n+m))$ operations.

Proof of Corollary 2.6. Existence of envy-free fPO allocations for mixed problems was proved in Bogomolnaia, Moulin, Sandomirskiy, and Yanovskaya (2017). Such allocations can be obtained as equilibrium allocations in the so-called Fisher market (in economic literature known as competitive exchange economy) associated with the division problem. Equilibria satisfy the property of Pareto-indifference: if $z$ is an equilibrium and $z^*$ gives the same utilities to all agents, then $z^*$ is an equilibrium as well. This allows us to apply Lemma 2.5 and get an envy-free fPO allocation with $\#\text{shar}_z \leq n - 1$.

Algorithms for computing equilibrium are known for pure goods (e.g., Orlin (2010)) or bads only, see Branzei and Sandomirskiy (2019). Together with Lemma 2.5, this yields the first two claims.

For the third claim, consider the equal-split allocation $z$ ($z_{i,o} = 1/n$ for all $i, o$) and construct a fractionally Pareto-optimal dominating allocation $z^*$ by Lemma 2.5. Pareto-improvements preserve proportionality, and thus $z^*$ is proportional, fPO, and has at most $n - 1$ sharings.

Open problem 1. (a) Nothing is known about polynomial computability of fPO+EF allocations for a mixture of goods and bads, as well as about a polynomial algorithm for bads when neither $n$ nor $m$ is fixed. (b) The requirement of the Fisher market equilibrium is stronger than the requirements of envy-freeness and fractional-Pareto-optimality. Can we find an EF+fPO allocation using simpler and/or faster algorithms?

\footnote{In case of goods and without Pareto-optimality, an analogue of Corollary 2.6 can be deduced from results on fair cake-cutting. Place goods arbitrarily on a line (goods are represented by consecutive intervals with piece-wise constant utility functions, see (Robertson and Webb, 1998) for the details of reduction) and find a fair partition with connected pieces. A connected partition makes $n-1$ cuts and hence at most $n-1$ goods are shared. For proportionality, such a partition can be found constructively using $O(mn \log(n))$ operations by the Even-Paz algorithm, Even and Paz (1984); for envy-freeness, existence follows from Stromquist (1980) and Su (1999). However, for general cakes and $n \geq 3$, envy-free partition cannot be found using a finite number of queries, see Stromquist (2008).}
3 Pareto Optimal Fair Division: Minimizing the Sharing

As we saw in Subsection 2.3, in the worst case, it might be required to have \(n-1\) sharings. However, in some cases it may be possible to find a fair allocation with less sharing. This raises the following computational problem:

> Given a specific instance of a fair division problem, find a solution that minimizes the number of sharings.

We will contrast between the two extreme cases: agents with identical valuations and agents with non-degenerate valuations.

**Definition 3.1.** A valuation matrix \(v\) is called degenerate if there exist two agents \(i, j\) and two objects \(o, p\) such that \(v_{i,o} \cdot v_{j,p} = v_{i,p} \cdot v_{j,o}\) (or \(\frac{v_{i,o}}{v_{j,o}} = \frac{v_{i,p}}{v_{j,p}}\) if denominators are non-zero). Otherwise, it is called non-degenerate.\(^4\)

Note that if \(v\) is selected according to any continuous probability distribution, it is non-degenerate with probability 1.

### 3.1 Warm-up: two agents, pure goods

For \(n = 2\), the upper bound on the number of sharings of Section 2.3 is 1, so sharing-minimization boils down to finding a fair allocation with no sharings at all (if such an allocation exists). The following “negative” result is well-known (e.g. Lipton, Markakis, Mossel, and Saberi (2004)); we present it to contrast with the “positive” theorem after it.

**Theorem 3.2.** When there are \(n = 2\) agents with identical valuations over \(m\) pure goods, it is NP-hard to decide whether there exists an allocation with no sharings that is proportional (=EF) / proportional and fractionally-PO.

**Proof.** For two agents with identical valuations, all allocations are fractionally-PO, and envy-freeness is equivalent to proportionality. Thus, an allocation satisfying any of the two fairness requirements with or without PO exists if-and-only-if the set of goods can be partitioned into two subsets with the same sum of values. Hence, the problem is equivalent to the NP-complete problem **Partition**.

The following theorem shows that, under the requirement of fractional Pareto-optimality, the computational problem becomes easier when the valuations are different.

**Theorem 3.3.** For two agents with non-degenerate valuations over \(m\) pure goods, it is possible to find in time \(O(m \cdot \log(m))\) a division that is proportional (=EF) and fractionally-PO, and subject to these requirements, minimizes the number of sharings.

If the goods are pre-ordered by the ratio \(\frac{v_{1,o}}{v_{2,o}}\), the computation takes linear time \(O(m)\).

\(^4\)Bogomolnaia et al. (2016) and Branzei and Sandomirskiy (2019) use a stronger definition of degeneracy: the complete agent-object graph has no cycles \(C\) with \(\pi(C) = 1\). Their condition implies that \(CG\) is acyclic for any fPO allocation \(z\) and that there is a bijection between Pareto-optimal utility profiles and fPO allocations. Our definition addresses only cycles of length 4 and thus can be easily checked in \(O(n^2 \cdot m \log m)\) operations (see Subsection 3.2). For 2 agents the definitions coincide.
Proof. Order the goods in descending order of the ratio \( \frac{v_{1,o}}{v_{2,o}} \), for \( o \in \{1, \ldots, m\} \) (this takes \( O(m \log(m)) \) operations). By the assumption of non-degeneracy, no two ratios coincide.

By Corollary 2.4, any fractionally PO allocation \( z \) takes one of two forms:

- “0 sharings”: there is a good \( o \) such that \( z \) gives all the prefix goods \( 1, \ldots, o \) to agent 1, and all postfix goods \( o + 1, \ldots, m \) to agent 2.

- “1 sharing”: there is a good \( o \) which is split between the two agents, while all goods \( 1, \ldots, o - 1 \) are consumed by agent 1 and all remaining goods \( o + 1, \ldots, m \) by agent 2.

Therefore, we have \( m + 1 \) allocation with 0 sharings and each of them can be tested for fairness. If there are no fair allocations among them, then we look for a fair allocation among those with one sharing. For any fixed \( o \), this leads to solving a system of two linear inequalities with just one variable (the amount of \( o \) consumed by agent 1).

However, without fractional Pareto optimality, we cannot escape hardness of Theorem 3.2: sharing minimization remains hard even for non-degenerate instances.

**Theorem 3.4.** Under the assumptions of Theorem 3.3, it is NP-hard to decide whether there exists an allocation with no sharings that is proportional (=EF).

**Proof.** We reduce from PARTITION. Consider a PARTITION instance with \( p \) positive integers, \( a_1, \ldots, a_p \). We create a fair division problem with \( m = p \) pure goods and two agents with the following valuations:

- Alice values each good \( o \in [p] \) as \( a_o \);

- Bob values each good \( o \) as \( a_o + b_o \), where \( b_o \) are small (\( \sum_{o \in [m]} b_o < 1/2 \)) strictly-positive perturbations such that the valuations are non-degenerate. For example, \( b_o = \frac{o}{3m(o+1)} \).

If there exists an equal-sum partition of the numbers \( a_1, \ldots, a_p \) into two subsets \( (X_1, X_2) \), then there exists a fair allocation of the goods: since \( u_A(X_1) = u_A(X_2) \), we can give Bob either \( X_1 \) or \( X_2 \) for which his utility is higher, and give Alice the other subset.

Conversely, if there exists a fair allocation \( (X_A, X_B) \), then Bob is not envious: \( u_B(X_B) \geq u_B(X_A) \). Since \( |u_B(X) - u_A(X)| < \frac{1}{2} \) for all subsets \( X \) and \( u_A(X) \) is an integer, we get \( u_A(X_B) \geq u_A(X_A) \). But Alice does not envy either: \( u_A(X_A) \geq u_A(X_B) \). Thus, \( u_A(X_A) = u_A(X_B) \), i.e., \( (X_A, X_B) \) is an equal-sum partition of \( a_1, \ldots, a_p \).

### 3.2 Main results: \( n \) agents, mixed valuations, varying degeneracy

Now we come back to the full generality of mixed problems with an arbitrary number of agents.

In order to capture instances “in between” the two extremes of non-degenerate problems and identical valuations, we define the *degree of degeneracy* of a valuation \( v \) as

\[
D_v = \max_{i,j \in [n], i \neq j} \max_{r > 0} \left| \left\{ o \in [m] : v_{i,o} = r \cdot v_{j,o} \right\} \right| - 1.
\]

Informally, \( D_v + 1 \) is the maximal number of objects \( o \) such that some agents \( i \neq j \) have the same ratio \( \frac{v_{i,o}}{v_{j,o}} \) for all of them. Degree of degeneracy can be easily computed in time \( O(n^2 \cdot m \log(m)) \):
for each pair of agents rearrange the ratios in a weakly-decreasing order and then find the longest interval of constancy.

A valuation matrix $v$ is non-degenerate if and only if $D_v = 0$. In particular, $D_v$ equals zero with probability one for any continuous\(^{5}\) probability measure on $\mathbb{R}^{n \times m}$. In case of identical valuation, $D_v$ attains its maximal value, which is $m - 1$.

**Remark 3.5.** By Corollary 2.4, in any fPO allocation, for each pair of agents, at most $D_v + 1$ objects are shared. Hence, in any fPO allocation, the number of sharings is at most $(D_v + 1)\frac{n(n-1)}{2}$.

Contrast this with Lemma 2.5: it says that, for any fPO utility profile, there exists an fPO allocation with these utilities, in which the number of sharings is at most $n - 1$.

The next theorem is our main result: it shows how increasing $D_v$ moves us gradually from the easiness illustrated (for two agents) by Theorem 3.3 to the hardness of Theorem 3.2.

**Theorem 3.6.** Fix the number of agents $n \geq 2$.

(a) Given an $n$-agent instance $v$ with a mixture of goods and bads, an allocation $z$ that minimizes the number of sharings $\#\text{shar}_Z$ subject to fractional-Pareto-optimality and proportionality (or envy-freeness) can be computed using

$$O\left(3^{\frac{n(n-1)}{2}}D_v \cdot m^{\frac{n(n-1)}{2}+2}\right)$$

operations. In particular, for any fixed constant $C > 0$, sharing-minimization can be performed in strongly-polynomial time for any instance $v$ with $D_v \leq C \cdot \log(m)$.

(b) Fix constants $C > 0$ and $\alpha > 0$. Checking the existence of a fractional-Pareto-optimal proportional (or envy-free) allocation $z$ with $\#\text{shar}_Z = 0$ is NP-hard for valuations $v$ such that $D_v \geq C \cdot m^\alpha$.

**Proof.**

(a) The algorithm has two phases. The first phase is to enumerate the set $G_v$ of all fPO graphs — undirected consumption graphs of fractionally PO allocations. This phase is the subject of Proposition 3.8 below.

The second phase is testing each $G \in G_v$:

1. Count the number of sharings in $G$. If it exceeds $n - 1$, skip $G$.

2. For each shared object $o$, and each agent $i$ connected to $o$, create a variable $z_{i,o}$ representing the fraction of $o$ allocated to $i$. The total number of variables is at most $2(n-1)$ — for each shared object, we have one variable for each agent who receives a positive share of it.

3. Represent the required fairness condition (EF / proportionality) as a set of linear inequalities in these variables. Solve the resulting LP.

4. Among those graphs $G$ where the LP has a solution, select the one with the smallest number of sharings and return the corresponding allocation.

---

5 An interesting problem is to estimate the expected magnitude of $D_v$ when the valuations $v_{i,o}$ are selected according to a discrete distribution, for example, from a finite set of integers. For the uniform distribution on $[R]$, fixed $n$, and $m = O(R)$ one can easily show that the expected value of $D_v$ is $O(1)$. We leave finer estimates, as well as other asymptotic regimes and non-uniform distributions for future research.
Step 1 is justified by Lemma 2.5: it ensures that we can restrict our attention to fPO allocations with at most \( n - 1 \) sharings. Since all graphs of such allocations are checked, a fair fPO allocation with the minimal \#shar_{z} will be found.

For fixed \( n \), the number of operations per fPO graph \( G \) is \( O(m) \), the time needed to “read” it. Solving the LP takes constant time since its size does not depend on \( m \) — it depends on \( n \) only and \( n \) is fixed: we have at most \( 2(n - 1) \) variables, at most \( 3(n - 1) \) feasibility constraints (at most \( 2(n - 1) \) of non-negativity and at most \( n - 1 \) of full allocation), \( n \) fairness constraints for proportionality and \( n(n - 1) \) for EF. Thus, the run time of the second phase is \( O(m \cdot |G_{v}|) \) and the first phase determines the overall complexity.

(b) We outline a reduction from PARTITION, similar to the one used for Theorem 3.4. We present the construction for \( n = 2 \); the case \( n > 2 \) can be covered by adding dummy agents. Given an instance \( a_1, a_2, \ldots, a_p \) of PARTITION, pick a minimal \( m = m(p) \) such that \( C \cdot m^{\alpha} \geq p - 1 \). Define a fair division instance with \( m \) pure goods of two types:

- \( p \) “big”: for each \( o \in [p] \), the good \( o \) is equally valued by both agents: \( v_{1,o} = v_{2,o} = a_o \).
- \( m - p \) “small” goods: there are \( Q = \frac{m - p}{2} \) pairs (w.l.o.g., \( m - p \) is even) of goods \((q_k, \bar{q}_k)_{k \in [Q]}\) such that \( v_{1,q_k} = v_{2,q_k} = \frac{k+1}{4m^2} \), \( v_{1,\bar{q}_k} = v_{2,\bar{q}_k} = \frac{1}{4m^2} \).

The sum of values of all the small goods is less than 1/2 for both agents, while the value of each big good is a positive integer. Therefore, in any fair fractionally PO allocation, both agents consume some of the big goods. Thus, by Corollary 2.4, agent 1 consumes all the \( q_k \) goods (since their value-ratio is more than 1) and agent 2 all the \( \bar{q}_k \) goods (since their value-ratio is less than 1).

Thus, a fair fPO allocation with 0 sharings exists if and only if \( a_1, \ldots, a_p \) can be partitioned into two subsets of equal sum.

We have reduced PARTITION to an allocation problem with \( D_{v} = p - 1 \leq C \cdot m^{\alpha} \). Since \( m = m(p) \) is bounded by a polynomial in \( p \), the length of binary representation of \( v \) is bounded by a polynomial of the size of PARTITION instance. \( \square \)

Remark 3.7 (alternative objectives in Theorem 3.6). Instead of the number of sharings, one can optimize other objectives, for example, the number of shared objects, the number of sharings in a given subset of objects \( S \subset [m] \) (where \( S \) is a subset of objects that are particularly “hard to share”), the total value of shared objects \( \sum_{i \in [n]} \sum_{o \text{ shared by } i} v_{i,o} \), or the maximum number of agents who share a single object.

For the latter objectives, we will skip step 1 (i.e., we will not discard fPO graphs with more than \( n - 1 \) sharings). Still, by Remark 3.5, the number of sharings in any fPO allocation is at most \( \frac{n(n-1)}{2}(D_{v} + 1) \). Hence, the total number of variables and constraints in the LP is independent of \( m \), the LP is solvable in constant time (when \( n \) is fixed), and the complexity is dictated by the time needed to enumerate all fPO graphs.

Similarly, instead of envy-freeness we can use other fairness notions, such as weighted envy-freeness (see Branzei and Sandomirskiy (2019)) or weighted proportionality capturing unequal ownership rights, or any other fairness notion that can be represented by a constant (in \( m \)) number of linear inequalities on the allocation matrix. If there exists an fPO allocation satisfying the

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6 What is the run-time complexity of the minimization problem in the intermediate range, in which \( D_{v} \) is super-logarithmic but sub-polynomial in \( m \), for example when \( D_{v} = \log^c m \) for some constant \( c > 1 \)? An answer by Young (2019) to a similar question indicates that this question is related to the well-known Exponential-Time Hypothesis.
chosen fairness notion, our algorithm will find it. Otherwise, the algorithm will indicate that such an allocation does not exist.

The following proposition closes the remaining gap in the proof of Theorem 3.6: it shows that the set of consumption graphs of all fPO allocations can be efficiently enumerated.

**Proposition 3.8.** For every fixed number of agents \( n \geq 2 \), the set of all fPO graphs \( \mathcal{G}_v := \{\mathcal{G}_z : z \text{ is fPO for } v\} \) can be enumerated using \( O(3^{(1+D_v)} m^{(n-1)/2} \cdot m^{(n-1)/2+2}) \) operations. In particular, for \( v \) with logarithmic degeneracy \( D_v \leq C \cdot \log(m) \) as in Theorem 3.6), the algorithm runs in strongly-polynomial time.

The total number of graphs in \( \mathcal{G}_v \) satisfies the upper bound\(^8\)

\[
|\mathcal{G}_v| \leq 3^{(1+D_v)} m^{(n-1)/2} \cdot m^{(n-1)/2}.
\]

Proposition 3.8 is proved by the following two lemmas. We enumerate the fPO graphs by iteratively adding agents. We start by enumerating the fPO graphs for the first two agents (Lemma 3.9 and Figure 2). Then we show that, given all fPO graphs for agents 1, \ldots, \( k \), we can efficiently enumerate all fPO graphs for agents 1, \ldots, \( k+1 \) (Lemma 3.10 and Figure 3).

---

\(^7\)There are several alternative approaches. Branzei and Sandomirskiy (2019) recover a subset of fPO graphs by their 2-agent projections in order to compute the so-called competitive allocations of bads. Devanur, Papadimitriou, Saberi, and Vazirani (2008) use a complicated technique of cell-enumeration from computational algebraic geometry for a similar problem with goods. An alternative dynamic-programming approach for goods was outlined by D.W. (2019). That algorithm sequentially adds new goods and presumably runs in time \( O(2^n m^2 |\mathcal{G}_v|) \), however, the construction does not provide an a-priori polynomial upper bound on \( |\mathcal{G}_v| \).

\(^8\)Note the surprising similarity of this bound and the one derived within smoothed analysis framework in Moitra and ODonnell (2011) (main theorem, page 4) for a problem of multi-objective binary optimization using quite involved probabilistic arguments.
Lemma 3.9. For an instance $\mathbf{v}$ with 2 agents and $m$ objects: (a) the total number of graphs in $\mathcal{G}_\mathbf{v}$ is at most $3m \cdot 3^{D_\mathbf{v}}$. (b) If all the objects $o$ with $v_{1,o} \cdot v_{2,o} > 0$, are ordered by the ratio $\frac{|v_{1,o}|}{|v_{2,o}|}$, then $\mathcal{G}_\mathbf{v}$ can be enumerated using $O(m \cdot 3^{D_\mathbf{v}})$ operations.

Proof. We generalize the construction used in Theorem 3.3, from pure goods and non-degenerate valuations, to arbitrary objects and an arbitrary degeneracy level.

We use the following notation:

- $A_{>0} = \{o \in [m] : v_{1,o} > 0, v_{2,o} > 0\}$ for the set of pure goods;
- $A_{<0} = \{o \in [m] : v_{1,o} < 0, v_{2,o} < 0\}$ for the set of bads,
- $A_0 = \{o \in [m] : v_{1,o} = v_{2,o} = 0\}$ for zero objects,
- $A_\pm$ for all the remaining impure goods and neutral objects;
- $A_\pm(t) := \{o \in A_{>0} \cup A_{<0} : \frac{|v_{1,o}|}{|v_{2,o}|} = t\}$.

(a) By non-maliciousness, there is no flexibility in allocating objects from $A_{\pm}$: they are consumed by the agent with larger $v_{1,o}$ at any fPO allocation. In contrast, objects from $A_0$ can be allocated arbitrarily. Such zero objects contribute $|A_0|$ to $D_\mathbf{v}$ and lead to $3^{|A_0|}$ allocation possibilities (each object is consumed either by agent 1, or by agent 2, or by both).\(^9\)

The allocation of objects from $A_{>0} \cup A_{<0}$ is determined by the value-ratio threshold $t_{1,2}$ of Corollary 2.4. Consider the set $T = \left\{ \frac{|v_{1,o}|}{|v_{2,o}|}, o \in A_{>0} \cup A_{<0} \right\}$. To cover all the fPO allocations, it is enough to consider $|T|$ situations, when $t_{1,2}$ equals one of the elements of $T$. Then objects $o \in A_{>0}$ with $\frac{|v_{1,o}|}{|v_{2,o}|} > t_{1,2}$ are allocated to agent 1 and with $\frac{|v_{1,o}|}{|v_{2,o}|} < t_{1,2}$ to agent 2; symmetrically, for $o \in A_{<0}$, bads with $\frac{|v_{1,o}|}{|v_{2,o}|} > t_{1,2}$ go to agent 2, while those with $\frac{|v_{1,o}|}{|v_{2,o}|} < t_{1,2}$ to agent 1. The remaining objects $A_\pm(t_{1,2})$ are allocated arbitrarily between agents, resulting in $3^{|A_\pm(t_{1,2})|}$ possibilities. All in all:

$$|\mathcal{G}_\mathbf{v}| \leq 3^{|A_0|} \cdot |T| \cdot 3^{\max_{t \in T} |A_\pm(t)|}.$$ 

Since $|T| \leq m$ and $|A_0| + |A_\pm(t)| \leq 1 + D_\mathbf{v}$ for any $t > 0$, we get the claimed upper bound. Note that the bound is not tight — for 2 agents and non-degenerate $\mathbf{v}$, we can get $2m + 1$ instead of $3m$ (see the proof of Theorem 3.3).

(b) If $t$ and $t'$ are two consecutive elements of $T$, then passing from $t$ to $t'$ involves reallocation of objects from $A_\pm(t)$ and $A_\pm(t')$ only. This leads to an overall running time proportional to the total number of graphs $|\mathcal{G}_\mathbf{v}|$.

Given the valuation matrix $\mathbf{v} = (v_{i,o})_{i \in [n], o \in [m]}$ and $k \in [n]$, denote by $\mathbf{v}_k$ the valuation of the first $k$ agents: $\mathbf{v}_k = (v_{i,o})_{i \in [k], o \in [m]}$. The previous lemma tells that $|\mathcal{G}_{\mathbf{v}_2}| \leq 3^{D_\mathbf{v}+1} \cdot m$. The next lemma relates $\mathcal{G}_{\mathbf{v}_{k+1}}$ to $\mathcal{G}_{\mathbf{v}_k}$.

Lemma 3.10. For a valuation $\mathbf{v}$ with $n \geq 3$ agents and $k$ such that $2 \leq k \leq n - 1$:

(a) The number of graphs in $\mathcal{G}_{\mathbf{v}_{k+1}}$ satisfies the upper bound

$$|\mathcal{G}_{\mathbf{v}_{k+1}}| \leq |\mathcal{G}_{\mathbf{v}_k}| \cdot 3^{(D_\mathbf{v}+1)^k} \cdot m^k$$

\(^9\) If we are only interested in final allocations, then zero objects can of course be given to one of the agents arbitrarily. However, in Lemma 3.9 we count all possible consumption graphs; this will be important in later steps of the algorithm. Therefore, we must consider all options for zero objects too.
Figure 3: Enumerating the fPO consumption graphs of allocations among three agents (Lemma 3.10) in a non-degenerate instance with $m = 3$ pure goods. It shows some of the consumption graphs in $\mathcal{G}_{v,3}$ derived from the top-right consumption graph in Figure 2. The graphs in the top row are derived by sharing Alice’s goods (the farm and the house) with Carl; the graphs in the second row are derived by sharing Bob’s goods (the car and the house) with Carl.

(b) All the graphs in $\mathcal{G}_{v,k+1}$ can be enumerated using

$$O \left( |\mathcal{G}_{v,k}| \cdot 3^{D_{v,k}} \cdot m^{2+k} \right)$$

operations if $\mathcal{G}_{v,k}$ is given as the input.

Since $v^n = v$, starting from $v^2$ (already covered by Lemma 3.9) and repeatedly applying Lemma 3.10 we get both the algorithmic part of Proposition 3.8 as well as the upper bound (3.1).

Proof of Lemma 3.10. The idea is that any graph $G' \in \mathcal{G}_{v,k+1}$ can be obtained from some $G \in \mathcal{G}_v$ by erasing some of the edges between objects and “old” agents $i \in [k]$ and tracing new edges to a “newcomer” $k + 1$ in such a way that for each old agent $i$, the allocation of objects between $i$ and the newcomer is fPO in the 2-agent subproblem.

First, we check that any fPO allocation $z'$ among $k + 1$ agents can be obtained from an fPO allocation $z$ among $k$ by reallocating some objects to the newcomer. Indeed, by Lemma 2.3, there exists a vector of weights $\lambda = (\lambda_i)_{i \in [k+1]}$ with strictly positive components such that in allocation $z'$ each object $o$ is consumed by agents $i$ with a highest $\lambda_i v_{i,o}$. Giving the share $z'_{k+1,o}$ of each object $o$ consumed by $k + 1$ to an agent $i \in [k]$ with a highest $\lambda_i v_{i,o}$ defines the desired allocation $z$; fPO follows from the same Lemma 2.3 with vector $(\lambda_i)_{i \in [k]}$.

Second, we describe how the reallocation looks in terms of graphs. For $G \in \mathcal{G}_{v,k}$, consider the set of objects consumed by agent $i$: $A_i(G) = \{ o \in [m] : \text{ there is an edge between } i \text{ and } o \}$. For each $i$ consider a 2-agent problem $v^{i,k+1}(A_i)$, where $i$ and $k + 1$ divide the set $A_i(G)$ of objects between themselves. Pick a graph $G^{i,k+1} \in \mathcal{G}_{v^{i,k+1}(A_i)}$ for each $i \in [k]$, which describes how objects in $A_i(G)$ are reallocated between $i$ and $k + 1$. Define the graph $G'$ as follows: agent $i \in [k]$ and object $o$ are connected by an edge if they are connected in $G^{i,k+1}$; an edge between $k + 1$ and $o$ is traced if this edge exists in at least one $G^{i,k+1}, i \in [k]$.

Denote by $\mathcal{G}_{v,k+1}$ the set of graphs $G'$ that we get, when $G$ ranges over $\mathcal{G}_{v,k}$ and $G^{i,k+1}$ over $\mathcal{G}_{v^{i,k+1}(A_i)}$ for all $i \in [k]$. By the construction, $\mathcal{G}_{v,k+1}$ contains all the consumption graphs of fPO
allocations for $v^{k+1}$, but may contain some non-fPO graphs, since the reallocation preserves the fPO condition only for pairs of agents $i, j \in [k+1]$. In order to get $G_{v,k+1}'$, each graph $G \in G_{v,k+1}'$ must be tested for fPO using Lemma 2.2 and those graphs that do not pass the test must be eliminated.

(a) Let us estimate the total number of graphs in $G_{v,k+1}$. For each $G \in G_{v,k}$, the set $G_{v+1}(A_i)$ contains at most $|A_i(G)| \cdot 3^{Dv+1}$ graphs (see Lemma 3.9). Therefore, the total number of graphs $G'$ obtained from $G$ is bounded by $(3^{(Dv+1)})^k \cdot \prod_{i=1}^k |A_i(G)| \leq (3^{(Dv+1)} \cdot m)^k$ and we get (3.2).

(b) The bound on time-complexity follows from Lemma 3.9 as well. For each $G$ and $i \in [k]$, computing $G_{v+1}(A_i)$ takes $O(m \cdot 3^{Dv})$ operations if prior to that for each pair of agents, $i \in [k]$ and $k+1$, objects with non-zero values are ordered by $v_{i,o}^{v,k+1}$. Thus, all $G'$ for a given $G$ are enumerated in time $O\left((m \cdot 3^{Dv})^k\right)$ and the time needed for reordering the objects is absorbed by this expression. Checking fPO takes additional $O(m^2)$ for each $G'$ by Lemma 2.2. Thus, the overall time complexity is $O\left(|G_{v,k}| \cdot m^{2+k} \cdot 3^{Dv-k}\right)$. \qed

4 Related Work

4.1 Known worst-case and average-case bounds on sharing.

The idea of finding fair allocations with a bounded number of shared goods originated from Brams and Taylor (Brams and Taylor, 1996, 2000). They suggested the Adjusted Winner (AW) procedure, which finds fair and fractionally Pareto-optimal allocation of goods between two agents with additive utilities and at most 1 sharing, i.e. with the worst-case optimal number. The AW procedure was applied (at least theoretically) to division problems in divorce cases and international disputes (Brams and Togman, 1996, Massoud, 2000) and was studied empirically (Schneider and Krümer, 2004, Daniel and Parco, 2005). The AW procedure heavily relies on the simple structure of fPO allocations for two agents (see the proof of Theorem 3.3 and Moulin (2004), Example 7.11a). Brams and Taylor do not extend their AW procedure to three or more agents.

For $n \geq 3$ agents, the number of sharings was studied in an unpublished manuscript of Wilson (1998). He proved worst-case bounds on sharing for fairness criteria that may be incompatible with fractional Pareto-efficiency. For example, he proved the existence of an egalitarian\(^{10}\) allocation of goods with $n-1$ sharings. For such criteria, the approach based on our Lemma 2.5 becomes inapplicable and Wilson uses a different technique based on a linear program.

There is a significant gap between worst-case and average-case numbers of sharings; this further stresses the importance of sharing minimization. Dickerson, Goldman, Karp, Procaccia, and Sandholm (2014) considered random instances and demonstrated that the minimal number of sharings for an envy-free fPO allocation is zero with high probability, if the number of goods is large and values are independent and identically distributed. Manurangsi and Suksompong (2017) extended the result to allocations among agent groups.

\(^{10}\)Egalitarian (also known as max-equitable) allocations are those where agents have maximal possible equal utilities (see Pazner and Schmeidler (1978)). Egalitarian allocations of goods are proportional but not necessary envy-free, and may violate efficiency if the valuation matrix has zeros.
4.2 Fairness with indivisible objects

With indivisible objects, envy-free and even proportional allocations may not exist. The most commonly studied relaxations of these two concepts are Envy-freeness except one good (EF1) and Maximin share guarantee (MMS). EF1 was introduced by Lipton et al. (2004) and studied by Budish (2011), Aleksandrov, Aziz, Gaspers, and Walsh (2015), Oh et al. (2018) and others. Existence of EF1 indivisible Pareto Optimal allocations was proved by Caragiannis, Kurokawa, Moulin, Procaccia, Shah, and Wang (2016) and Barman et al. (2018) strengthened the result to fractional PO. Budish (2011) defined MMS, demonstrated existence under large-market assumption and applied the concept in practice for course allocation in (Budish, Cachon, Kessler, and Othman, 2016). For “small markets”, Procaccia and Wang (2014) showed that MMS allocations may not exist for some knife-edge instances and hence all the results about MMS consider a certain approximation to MMS itself, e.g., Ghodsi, HajiAghayi, Seddighin, Seddighin, and Yami (2018), Babaioff, Nisan, and Talgam-Cohen (2017), Aziz, Rauchecker, Schryen, and Walsh (2016), Segal-Halevi (2018a).

Classic microeconomics mostly works with divisible resources and handles indivisible objects by making them “divisible” via a lottery. This approach results in weaker fairness: an allocation is fair ex-ante, i.e., in expectation before the lottery is implemented. Ex-ante fairness was analyzed in many different contexts. Just to name a few: Hylland and Zeckhauser (1979), Abdulkadiroglu and Sonmez (1998), Bogomolnaia and Moulin (2001) considered the problem of fair assignment and Budish, Che, Kojima, and Milgrom (2013) considered its multi-unit constrained modifications; Kesten and Ünver (2015) evaluated fairness of tie-breaking in matching markets; Bogomolnaia, Moulin, and Sandomirskiy (2019) studied randomized rules for online fair division.

Brams, Kilgour, and Klamler (2013) observed that exact fairness with indivisible goods can be achieved by leaving some of them unallocated while keeping some efficiency guarantees: their AL procedure constructs an allocation that is not Pareto-dominated by another envy-free allocation, see also Aziz (2015a). Recently, Caragiannis, Gravin, and Huang (2019) used a similar idea to construct an allocation that is “Envy-free except any good” (also known as EFx, an approximate fairness notion which strengthens EF1) and has high welfare.

Halpern and Shah (2019) suggested a novel approach to achieve exact fairness with indivisible goods: introduce a monetary subsidy by a third party, while minimizing the transfers. Minimization makes this approach ideologically similar to ours and distinguishes it from other results on fair allocation of indivisible goods with monetary transfers, e.g. the rent-division problem of Gal, Mash, Procaccia, and Zick (2016).

4.3 Checking existence of fair allocations of indivisible objects

Fair allocation of indivisible goods might not exist in all cases, but may exist in some. A natural question is how to decide whether it exists in a given instance. It was studied by Lipton et al. (2004), de Keijzer et al. (2009), Bouveret and Lemaître (2016) for various fairness and efficiency notions, showing that it is computationally hard in general (with some exceptions). The undercut procedure of Brams, Kilgour, and Klamler (2012) finds an envy-free allocation of indivisible goods among two agents with monotone (not necessarily additive) valuations, if-and-only-if it exists (see also Aziz (2015b)).
4.4 Cake-cutting with few cuts

The goal of minimizing the number of “cuts” has also been studied in the context of fair cake-cutting — dividing a heterogeneous continuous resource, see Webb (1997), Shishido and Zeng (1999), Barbanel and Brams (2004, 2014), Alijani, Farhadi, Ghodsi, Seddighin, and Tajik (2017), Seddighin, Farhadi, Ghodsi, Alijani, and Tajik (2018). Since the resource is continuous, the techniques and results are quite different.

4.5 Fair division with mixed valuations

Most of the literature on fair division deals with either goods or bads. Recently, some papers have studied mixed valuations, where objects can be good for some agents and bad for others. This setting was first studied by Bogomolnaia et al. (2017) for divisible objects and quite general class of utilities. It was later studied by Segal-Halevi (2018b), Meunier and Zerbib (2018), Avvakumov and Karasev (2019) for a heterogeneous divisible “cake”, and by Aziz et al. (2018) for indivisible objects and approximate fairness.

5 Extensions and Future Work

In this section we present some extensions of the basic model, for which we only have preliminary partial results.

5.1 Costly sharing and non-linear utilities

Our sharing-minimization approach was motivated by the fact that, in practice, sharing is usually unwanted and may also be costly due to the overhead in managing shared property.

An alternative, approach to sharing minimization is to take these costs into account explicitly, when we compute the utility level of an agent. While being more natural from an economic point of view, this approach proves to be difficult from the very beginning.

Assume that all objects are pure goods and the utilities are additively-separable: consuming a fraction $z_{i,o}$ of a good $o$ contributes $f(z_{i,o}) z_{i,o} v_{i,o}$ to the utility of an agent $i$. Here, $f: [0, 1] \rightarrow [0, 1]$ is an increasing function with $f(1) = 1$; we call $f$ an overhead function. Our model so far corresponded to $f \equiv 1$. As another example, if $f(x) = x$, then having $1/2$ of good $o$ gives an agent only $1/4$ of the utility of having the entire good.

The theory of fair division, as well as the economic theory in general, is developed under the assumption that agents’ utilities are concave. The only choice of the overhead function compatible with this assumption is $f \equiv 1$ representing zero sharing costs.

For non-constant overhead functions, the existence of fPO+EF allocations is not guaranteed even in very simple cases.

**Example 5.1.** There are two goods and two agents with identical valuations: they value one good at 1 and the other good at $V > 1$.

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11 In practice, one can expect that the costs of an agent $i$ may differ depending on those with whom he is sharing, in particular, on their number. Also the benefit of getting a small portion of a good can be overweighted by the sharing costs and, hence, costly sharing can turn goods into bads. We leave these interesting complications to future research.
Envy-freeness dictates that both agents have equal utilities. This implies that the $V$ good must be shared. Now, fractional-Pareto-optimality dictates that the utility of both agents must be strictly larger than 1 — otherwise it is dominated by the allocation with zero sharings. There are at least two classes of overhead functions with which this is impossible.

(a) Discontinuous overhead functions. Often, the very fact that a good is shared, even if only a small fraction of it is shared, already decreases its value by some fraction $r \in (0, 1)$. The overhead function then may have the following form:

$$f_r(z_{i,o}) = \begin{cases} 1, & z_{i,o} = 1 \\ r, & z_{i,o} < 1 . \end{cases}$$

In any allocation in which the $V$ object is shared, the sum of utilities is at most $rV + 1$. If $V \geq 1/r$, then this expression is at most 2, so the utility of both agents is at most 1; hence an fPO+EF allocation does not exist.

(b) Total-loss overhead functions. Often, it is useless to have a too-small fraction of a good, so a fraction of at most $s$ of the good (for some $s \in (0, 1)$) is worth 0. The overhead function then may be continuous and have the following form:

$$f_s(z_{i,o}) = \begin{cases} (z_{i,o} - s)/(1 - s), & z_{i,o} \geq s \\ 0, & z_{i,o} \leq s \end{cases}$$

Obviously, giving any agent a fraction of at most $s$ of any object is not fPO. In particular, if $s \geq 1/2$ then no allocation with one or more sharing is fPO. Suppose $s < 1/2$, and consider an allocation in which one agent gets object 1 and a fraction $x$ of object $V$, while the other agent gets a fraction $1 - x$ of object $V$. By straightforward calculations, the only $x$ for which the agents’ utilities may be equal is $x_{EF} = (1 - 1/V)/2$. However, if $V < 1/(1 - 2s)$, then this $x_{EF}$ is smaller than $s$, so the allocation is not fPO.

Open problem 2. (a) Based on the above impossibility results, it may be interesting to find classes of overhead functions with which fair fPO allocations are guaranteed to exist.

(b) Alternatively, one can pick an approximate notion of efficiency (in contrast to fairness, approximate efficiency does not create disputes) and look for a mechanism that provides the best approximation given fairness as a constraint.

5.2 Truthful division and non-Pareto-optimal goals

A division algorithm is truthful if for every agent $i$, it is a weakly-dominant strategy to report the true values $(v_{i,o})_{o \in [m]}$. It is known that, in general, truthfulness, fairness, and Pareto-optimality are incompatible, see Zhou (1990) and Cole, Gkatzelis, and Goel (2013a). Our approach relies on Pareto-optimality both for the upper bound (Lemma 2.5) and for the minimization (Theorem 3.6); hence, truthfulness requires different techniques.

As an example, consider the simple truthful mechanism of Mossel and Tamuz (2010) and Chen, Lai, Parkes, and Procaccia (2013), which uses a consensus allocation. An allocation $z$ is called a consensus allocation if for every two agents $i, j \in [n]$: $u_i(z_j) = V_i/n$, where $V_i = \sum_{o \in [m]} v_{i,o}$. Given a consensus allocation $z$, a permutation $\pi$ over $[n]$ is selected uniformly at random, and the bundle $z_{\pi(i)}$ is allocated to agent $i$. Thus, the expected utility of any agent $i$, whether truthful or not, is $V_i/n$, so the agent cannot gain by false reporting. Moreover, a truthful agent gets a utility of
exactly $V_i/n$ with certainty, while a non-truthful agent might get more or less than $V_i/n$. Therefore, for a risk-averse agent, truthfulness becomes a strictly dominant strategy.

Although consensus allocations are usually not Pareto-optimal, we can still prove an upper bound on the number of sharings.

**Theorem 5.2.** (a) In any instance with $n$ agents, there exists a consensus allocation with at most $n(n-1)$ sharings (hence at most $n(n-1)$ shared objects), and it can be found by a weakly-polynomial algorithm.

(b) The bound is tight both for sharings and for shared objects.

**Proof.** (a) The proof is based on the linear programming approach, first suggested by Wilson (1998). The set of consensus allocations is defined by a combination of:

- $n \cdot m$ non-negativity constraints: $z_{i,o} \geq 0$ for all $i \in [n], o \in [m]$,
- $m$ equality constraints: $\sum_{i \in [n]} z_{i,o} = 1$ for all $o \in [m]$ (objects are fully allocated);
- $n(n - 1)$ equality constraints: $\sum_{o \in [m]} v_{i,o}z_{j,o} = V_i/n$ for $i \in [n]$ and $j \in [n - 1]$ (each agent $i$ believes that the share of $j \in [n - 1]$ is $V_i/n$; for $j = n$ the equality automatically holds).

The equal division ($z_{i,o} = 1/n$ for every $i, o$) satisfies the LP; thus, the LP is feasible and has a basic feasible solution — a solution with at most $n(n - 1) + m$ nonzero elements, see Matousek and Gärtner (2007). Such a solution corresponds to an allocation with at most $n(n - 1)$ sharings.

(b) For tightness, consider an instance with $n$ agents and $n(n - 1)$ pure goods. For each agent, $(n - 1)$ of the goods are “big” and the other $(n - 1)^2$ goods are “small”. Each agent values each of his big goods at $n - 0.5$ and each of his small goods at $0.5/(n - 1)$.

All $n$ sets of big goods are pairwise-disjoint, so that each good is “big” for exactly one agent and “small” for all other agents. For each agent, the sum of all values is $n(n - 1)$, so in a consensus allocation, the value of each bundle should be exactly $n - 1$. However, the value of each big good to its agent is larger than $n - 1$, so all goods must be shared.

**Open problem 3.** Can we find in strongly-polynomial time, a consensus allocation with at most $n(n - 1)$ sharings?

Finding a consensus allocation with minimal sharing is NP-hard when the valuations may be identical; the proof repeats the one of Theorem 3.2. Moreover, it is NP-hard even with non-degenerate valuations.

**Theorem 5.3.** It is NP-hard to decide whether there exists a consensus allocation with no sharings, even with non-degenerate valuations.

**Proof.** We reduce from **DistinctBalancedPartition** — a variant of **Partition** in which the input integers must be distinct and the output parts must have the same cardinality. See Appendix C for a proof that it is NP-hard. Given a **DistinctBalancedPartition** instance $a_1, \ldots, a_{2p}$ with $\sum_{i=1}^{2p} a_i = 2S$. Pick any number $b \in (0, \frac{1}{4p})$. Create a fair division problem with $m = 2p$ goods and two agents with the following valuations:

- Alice values each good $o \in \{1, \ldots, 2p\}$ at $a_o$.
- Bob values each good $o \in \{1, \ldots, 2p\}$ at $a_o + b$. 

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For each object $o$, the value-ratio is $1 + b/a_o$. Since all the numbers $a_o$ are distinct, the resulting valuations are non-degenerate.

Let $(X_1, X_2)$ be a partition of the numbers $a_1, \ldots, a_{2p}$ into two subsets with equal sum and equal cardinality, $|X_1| = |X_2| = p$. In the induced partition of the objects, we have $u_A(X_1) = u_A(X_2) = S$ and $u_B(X_1) = u_B(X_2) = S + p \cdot b$, so it is a consensus allocation. Conversely, let $(X_A, X_B)$ be a consensus allocation of the objects. Since $u_A(X_A) = u_A(X_B) = S$, the induced partition of the integers has an equal sum. Since $u_B(X_A) = u_B(X_B) = S + p \cdot b$, and $b$ is sufficiently small, we must have $|X_A| = |X_B| = p$, so the induced partition of the integers has equal cardinality.

Open problem 4. The hardness proof of Theorem 5.3 is weaker than that of Theorem 3.4, since it uses only a one-dimensional perturbation. It does not rule out the possibility that all instances except a subset of measure zero are easy.

Combining the algorithm of Mossel and Tamuz (2010) and Chen et al. (2013) with Theorem 5.2 gives:

**Corollary 5.4.** There exists a randomized truthful algorithm that always returns an envy-free allocation with at most $n(n-1)$ sharings.

The consensus allocation is very inefficient; there are better mechanisms for truthful fair division: Cole, Gkatzelis, and Goel (2013b), Cole et al. (2013a), Abebe, Cole, Gkatzelis, and Hartline (2019). While these mechanisms do not guarantee Pareto-optimality, they provide a constant-factor approximation to certain measures of social welfare. Is it possible to keep fairness, truthfulness, and approximate-welfare guarantees of these mechanisms, while upper-bounding or minimizing the number of shared objects? This remains a tempting open question.
In this section we prove Lemma 2.1 and Lemma 2.3 together:

- An allocation \( z \) is fractionally Pareto-optimal if-and-only-if it is non-malicious and its directed consumption graph \( \overline{C}_G(z) \) has no cycle \( C \) with \( \pi(C) < 1 \).

- An allocation \( z \) is fractionally Pareto-optimal if and only if there is a vector of weights \( \lambda = (\lambda_i)_{i \in [n]} \) with \( \lambda_i > 0 \) such that for all agents \( i \in [n] \) and objects \( o \in [m] \):
  \[
z_{i,o} > 0 \text{ implies } \lambda_i v_{i,o} \geq \lambda_j v_{j,o} \text{ for any agent } j \in [n]. \tag{A.1}
\]

**Proof that fPO \( \implies \) no C and no maliciousness.** If an allocation is malicious, then reallocating objects in a non-malicious way strictly improves the utilities of some agents without harming the others. Thus, an fPO allocation \( z \) must be non-malicious.

We now show that there are no directed cycles \( C = (i_1 \rightarrow o_1 \rightarrow i_2 \rightarrow o_2 \rightarrow \ldots \rightarrow i_L \rightarrow o_L \rightarrow i_{L+1} = i_1) \) in \( \overline{C}_G(z) \) with \( \pi(C) < 1 \). Assume, by contradiction, that \( C \) is such a cycle. We show how to construct an exchange of objects among the agents in \( C \) such that their utility strictly increases without affecting the other agents. This will contradict the Pareto-optimality of \( z \).

Define \( R := \pi(C)^{1/L} \); by assumption, \( R < 1 \).

For each \( k \in [L] \), there is an edge from agent \( i_k \) to object \( o_k \). Hence, by the definition of \( \overline{C}_G(z) \):

- either \( i_k \) consumes a positive amount of \( o_k \) and both \( i_k \) and \( i_{k+1} \) agree that \( o_k \) is a good \( (v_{i_k,o_k} > 0 \text{ and } v_{i_{k+1},o_k} > 0) \),

- or \( i_{k+1} \) has a positive amount of \( o_k \) and both \( i_k \) and \( i_{k+1} \) agree that \( o_k \) is a bad \( (v_{i_k,o_k} < 0 \text{ and } v_{i_{k+1},o_k} < 0) \).

Suppose each \( i_k \) gives a small positive amount \( \varepsilon_k \) of \( o_k \) to \( i_{k+1} \) in case of a good or \( i_{k+1} \) gives \( \varepsilon_k \) fraction of \( o_k \) to \( i_k \) in case of a bad (\( \varepsilon_k \in (0, h_k] \) where \( h_k = z_{i_k,o_k} \) for a good and \( h_k = z_{i_{k+1},o_k} \) for a bad). Then, agent \( i_k \) loses a utility of \( \varepsilon_k \cdot |v_{i_k,o_k}| \), but gains \( \varepsilon_{k-1} \cdot |v_{i_{k-1},o_{k-1}}| \) from the previous agent, so the net change in the utility of \( i_k \) is \( \varepsilon_{k-1} |v_{i_{k-1},o_{k-1}}| - \varepsilon_k |v_{i_k,o_k}| \) (where the arithmetic on the indices \( k \) is done modulo \( L \) in a way that the index is always in \( \{1, \ldots, L\} \)). To guarantee that all agents in \( C \) strictly gain from the exchange, it is sufficient to choose \( \varepsilon_1, \ldots, \varepsilon_k \) such that the following inequalities hold for all \( k \in [L] \):
  \[
  \varepsilon_{k-1} |v_{i_{k+1},o_{k-1}}| - \varepsilon_k |v_{i_k,o_k}| > 0 \iff \frac{\varepsilon_k}{\varepsilon_{k-1}} < \frac{|v_{i_{k+1},o_{k-1}}|}{|v_{i_k,o_k}|}. \tag{A.2}
  \]

For any \( \varepsilon > 0 \), define \( \varepsilon_k = \varepsilon_{k-1} \cdot R \cdot \frac{|v_{i_k,o_k}|}{|v_{i_{k-1},o_{k-1}}|} \) for \( k \in \{2, \ldots, L\} \). Since \( R < 1 \), the inequality (A.2) is satisfied for each \( k \in \{2, \ldots, L\} \). It remains to show that it is satisfied for \( k = 1 \), too (note that in this case \( k - 1 = L \)). Indeed:

\[
\varepsilon_L = \varepsilon_1 \cdot R^{L-1} \prod_{k=2}^{L} \frac{|v_{i_k,o_k}|}{|v_{i_{k-1},o_{k-1}}|} = \varepsilon_1 \cdot R^{L-1} \cdot \frac{|v_{i_1,o_1}|}{|v_{i_1,o_L}|} \prod_{k=1}^{L} \frac{|v_{i_k,o_k}|}{|v_{i_{k-1},o_{k-1}}|} = \varepsilon_1 \frac{R^{L-1}}{\pi(C)} \cdot \frac{|v_{i_1,o_1}|}{|v_{i_1,o_L}|} = \varepsilon_1 R^{L-1} \frac{|v_{i_1,o_1}|}{|v_{i_1,o_L}|}.
\]
Thus
\[
\frac{\varepsilon_1}{\varepsilon_L} = R \frac{|v_{1,oL}|}{|v_{1,oi}|} < \frac{|v_{1,oL}|}{|v_{1,oi}|}.
\]
By choosing \( \varepsilon_1 \) sufficiently small, we guarantee \( \varepsilon_k \leq h_k \) for all \( k \in [L] \), so this trade is possible. \( \square \)

**Proof that no \( C \) and no maliciousness \( \implies \) existence of \( \lambda \).** We assume that \( \overline{C_{Gz}} \) contains no directed cycles \( C \) with \( \pi(C) < 1 \) and \( z \) is non-malicious. We prove the existence of weights \( \lambda_i > 0 \) from Lemma 3.9.

Add directed edges \( i \to j \) between each pair of distinct agents \( i, j \in [n] \). All the new edges have the same large positive weight in order to ensure that the new graph \( \bar{G} \) has no cycles \( C = (v_1 \to v_2 \to \cdots \to v_{L+1} = v_1) \) with multiplicative weight \( \pi(C) < 1 \). It is enough to pick
\[
w_{i \to j} = \left( \max \left\{ 1, \frac{1}{|v_{k,o}|} : k \in [m], \ o \in [m], \ v_{k,o} \neq 0 \right\} \right)^{2(n-1)}.
\]
Indeed, any simple cycle \( C \) containing a new edge has at most \( 2(n-1) \) old edges; if none of the old edges has weight zero, then \( \pi(C) \geq 1 \) by the definition of \( w_{i \to j} \). Edges \( k \to o \) with \( w_{k \to o} = 0 \) cannot be a part of any cycle: by the definition of \( \overline{C_{Gz}} \) such edges are possible only if \( k \) consumes \( o \) and \( v_{k,o} = 0 \). Since \( z \) is non-malicious, such \( o \) has no outgoing edges.

Fix an arbitrary agent, say, agent 1. For every other agent \( j \in [n] \), let \( P_{1,j} \) be a directed path from 1 to \( j \) in \( \bar{G} \), for which the product \( \pi(P_{1,j}) \) is minimal. The minimum is well-defined and is attained on an acyclic path, since by the construction there are no cycles with a product smaller than 1, so adding cycles to a path cannot make its product smaller.

Set the **weight** of each agent \( j \) as \( \lambda_j := \pi(P_{1,j}) \) (in particular \( \lambda_1 = 1 \)). We now show that these weights satisfy the conditions (A.1), namely: \( z_{i,o} > 0 \) implies \( \lambda_i v_{i,o} \geq \lambda_j v_{j,o} \) for all \( j \in [n] \). W.l.o.g., we can assume that \( i \neq j \) and both agents agree whether \( o \) is a good or a bad, i.e., \( v_{i,o} \cdot v_{j,o} > 0 \). Indeed, if agents disagree, then by the non-maliciousness, (A.1) is satisfied with any \( \lambda_i, \lambda_j > 0 \).

In case of a good (\( v_{i,o} > 0 \) and \( v_{j,o} > 0 \)), there is an edge \( i \to o \) (since \( i \) consumes \( o \)) and \( o \to j \) (since \( v_{j,o} > 0 \) and \( z_{j,o} \neq 1 \)). Consider the optimal path \( P_{1,i} \) and the concatenated path \( Q_{1,j} = P_{1,i} \to o \to j \). The path \( P_{1,j} \) has the minimal product among all paths from 1 to \( j \). Therefore,
\[
\pi(Q_{1,j}) \geq \pi(P_{1,j}) \iff \pi(P_{1,i}) \frac{v_{i,o}}{v_{j,o}} \geq \pi(P_{1,j}) \iff \lambda_i v_{i,o} \geq \lambda_j v_{j,o}.
\]
The mirror argument for a bad (both \( v_{i,o} \) and \( v_{j,o} \) are negative) is as follows. There is an edge \( j \to o \) (because \( v_{j,o} < 0 \) and \( z_{j,o} \neq 1 \)) and \( o \to i \) (since \( i \) consumes \( o \) and \( o \) is a bad). We define \( Q_{1,i} \) as \( P_{1,j} \to o \to i \) and get
\[
\pi(Q_{1,i}) \geq \pi(P_{1,i}) \iff \pi(P_{1,i}) \frac{|v_{j,o}|}{|v_{i,o}|} \geq \pi(P_{1,i}) \iff \lambda_j |v_{j,o}| \geq \lambda_i |v_{i,o}| \iff \lambda_i v_{i,o} \geq \lambda_j v_{j,o}.
\]
\( \square \)

**Proof that existence of \( \lambda \) \( \implies \) fPO.** If in an allocation \( z \) each object \( o \) is consumed by agents \( i \) with highest \( \lambda_i v_{i,o} \), then \( z \) itself maximizes the weighted sum of utilities \( \sum_{i \in [n]} \lambda_i u_i(z_i) \) over all
allocations. Since all $\lambda_i$ are positive, $z$ is fPO because any Pareto-improvement must increase the weighted sum of utilities as well.

Lemma 2.1 has a useful computational implication, Lemma 2.2: It is possible to decide in time $O(nm(n + m))$ whether a given allocation $z$ is fractionally Pareto-optimal.

Proof. The idea is the following: construct the graph $\overrightarrow{CG}_z$, replace each weight with its logarithm, and look for a negative cycle using one of many existing algorithms (Cherkassky and Goldberg, 1999) (e.g. Bellman-Ford). If there is a cycle $C$ in which the sum of log-weights is negative, then $\pi(C) < 1$, so by Lemma 2.1, $z$ is not fractionally PO. Otherwise, $z$ is fractionally PO. A negative cycle can be found in time $O(|V| \cdot |E|)$. Here $|V| = m + n$ and $|E| \leq mn$.

Because of irrationality, logarithms can be computed only approximately and thus, to ensure the correctness of the algorithm, one has to adjust the quality of approximation depending on the input. However, these difficulties are easy to avoid by using a multiplicative version of any of the algorithms in Cherkassky and Goldberg (1999): multiplication replaces addition, division is used instead of subtraction, and one instead of zero.

This allows one to avoid logarithms and keep the same bound of $O(nm(n + m))$ on runtime.

B Worst-case bound on sharing

In this section we prove Lemma 2.5: For any allocation $z$, there exists a fractionally Pareto-optimal allocation $z^*$ such that:

- (a) $z^*$ weakly Pareto dominates $z$, i.e., for any agent $i$, $u_i(z_i^*) \geq u_i(z_i)$.
- (b) the non-directed consumption graph $CG_{z^*}$ is acyclic.
- (c) $z^*$ has at most $n - 1$ sharings (hence at most $n - 1$ shared objects).

Such allocation $z^*$ can be constructed in time $O(n^2m^2(n + m))$.

Proof. If $z$ is malicious, reallocate the objects:

- for each $o \in [m]$ with $\max_{i \in [n]} v_{i,o} > 0$, reallocate the shares of agents $j$ with $v_{j,o} \leq 0$ to an agent $i$ with $v_{i,o} > 0$;
- for each $o \in [m]$ with $\max_{i \in [n]} v_{i,o} = 0$, reallocate the shares of agents $j$ with $v_{j,o} < 0$ to an agent $i$ with $v_{i,o} = 0$.

Denote the resulting non-malicious allocation by $z'$.

Let’s call a cycle $C = (i_1 \rightarrow o_1 \rightarrow i_2 \rightarrow o_2 \rightarrow \ldots \rightarrow i_L \rightarrow o_L \rightarrow i_{L+1} = i_1)$ in the directed graph $\overrightarrow{CG}_{z'}$ simple if each node is visited at most once and for any $i \in [n]$ and $o \in [m]$ only one of the edges $i \rightarrow o$ or $o \rightarrow i$ is contained in the cycle.

If there is a simple cycle $C$ in $\overrightarrow{CG}_{z'}$ with $\pi(C) \leq 1$, then $C$ can be eliminated by the cyclic trade making all the agents weakly better off (similarly to the proof of Lemma 2.1 in Appendix A). Since both edges $i_k \rightarrow o_k$ and $o_k \rightarrow i_{k+1}$ exist in $\overrightarrow{CG}_{z'}$, the values $v_{i_k,o_k}$ and $v_{i_{k+1},o_k}$ are both non-zero and have the same sign. We conduct the following transfers:
• if \( v_{i_k,o_k} > 0 \) and \( v_{i_{k+1},o_k} > 0 \) (i.e., \( o_k \) is a good for \( i_k \) and \( i_{k+1} \)), then take \( \varepsilon_k \) amount of \( o_k \) from \( i_k \) and give it to \( i_{k+1} \) (\( 0 < \varepsilon_k \leq h_k \), where \( h_k = z_{i_k,o_k} \));

• if \( v_{i_k,o_k} < 0 \) and \( v_{i_{k+1},o_k} < 0 \) (i.e., \( o_k \) is a bad), then transfer \( \varepsilon_k \) of \( o_k \) from \( i_{k+1} \) to \( i_k \) (\( 0 < \varepsilon_k \leq h_k = z_{i_{k+1},o_k} \)).

The amounts \( \varepsilon_k \) are selected in such a way that \( \varepsilon_k |v_{i_k,o_k}| = \varepsilon_{k+1} |v_{i_k,o_{k+1}}| \) for \( k \in [L-1] \). Hence, each agent \( i_k \), \( k = 2, \ldots, L \), remains indifferent between the old and the new allocations while agent \( i_1 \) is weakly better off because of the condition \( \pi(C) \leq 1 \). We select epsilons as big as possible:

\[
\varepsilon_k = \frac{\prod_{q=1}^{k-1} \left| \frac{v_{i_q,a_q}}{v_{i_q,a_{q+1}}} \right|}{\min_{l \in [L]} \left( \frac{1}{m_l} \prod_{q=1}^{n_l-1} \left| \frac{v_{i_q,a_q}}{v_{i_q,a_{q+1}}} \right| \right)}, \quad k \in [L],
\]

thus eliminating one of the edges \( i_k \rightarrow o_k \) in \( \overline{G_S^z} \):

Repeat this procedure again and again until there are no simple cycles with \( \pi(C) \leq 1 \). Note that we need at most \((n-1)m\) repetitions since each time at least one edge is deleted in the undirected graph \( G_S^z \) and the total number of edges is at most \( n \cdot m \). Denote the resulting allocation by \( z^* \).

(a) By construction, \( z^* \) weakly improves the utility of each agent, is non-malicious, and has no cycles with \( \pi(C) < 1 \). Thus, \( z^* \) is fractionally Pareto-optimal by Lemma 2.1.

(b) The undirected consumption graph of \( z^* \) is acyclic. Assume by contradiction that there is a cycle \( C \) in \( G_S^z \). Then in the directed graph \( G_S^z \), there are two cycles: \( C \) passed in one direction and in the opposite. Denote them by \( \overrightarrow{C} \) and \( \overleftarrow{C} \). Since \( \pi(\overrightarrow{C}) = \frac{1}{\pi(\overleftarrow{C})} \), by fractional-Pareto-optimality we get \( \pi(\overrightarrow{C}) = \pi(\overleftarrow{C}) = 1 \); however all such cycles were eliminated in the previous stages of the algorithm.

(c) Since any acyclic graph on \( m + n \) nodes has at most \( m + n - 1 \) edges, and the number of sharings equals the number of edges in \( G_S^z \) minus \( m \), the number of sharings at \( z^* \) is at most \( n - 1 \).

It remains to estimate the complexity of the algorithm. Constructing the non-malicious allocation \( z' \) takes \( O(n \cdot m) \) and the overall complexity is determined by the time needed to perform cyclic trades. Cycles with \( \pi(C) < 1 \) can be found using the multiplicative modification of the Bellman-Ford algorithm, as in Lemma 2.1 which results in \( O(nm(n+m)) \) operations per cycle. For a given cycle \( C \) of length \( L \) transfers are conducted in \( O(L) = O(\min\{n,m\}) \) since no simple cycle is longer than \( 2 \min\{m,n\} \).

When all cycles \( C \) with \( \pi(C) < 1 \) have been eliminated, it remains to delete all the cycles in the undirected consumption graph if any (note that all such cycles have \( \pi(C) = 1 \)). Such cycles can be found using a depth-first search which needs \( O(|V| + |E|) = O(n \cdot m) \) operations per cycle.

The total number of cycles to be eliminated is at most \( (n-1)m \) and we get the upper bound \( O(n^2m^2(n+m)) \) for the overall time-complexity. \( \square \)

**Remark B.1.** An alternative “dual” approach to the existence of \( z^* \) with \( n - 1 \) sharings is based on linear programming (LP), see Wilson (1998): \( z^* \) can be constructed as a basic feasible solution (a vertex of the set of solutions) to a linear program where the objective is to maximize the utilitarian social welfare \( \sum_{i \in [n]} u_i(z^*_i) \) given the feasibility and domination constraints. Feasibility and domination can be expressed by \( m + n - 1 \) constraints. Hence the LP has a “basic feasible solution” — a solution with at most \( m + n - 1 \) non-zeros Matousek and Gärtner (2007). It corresponds to an allocation with at most \( n - 1 \) sharings. We illustrate this approach in Subsection 5.2, where
truthful fair division rules are discussed. Such an LP-based approach leads to a weakly-polynomial algorithm.

C NP-hardness of Distinct-Balanced-Partition

In the DistinctBalancedPartition problem, the input is $2m$ distinct positive integers with sum $2T$, and the goal is to decide whether there is a subset of $m$ integers that sum up to $T$.

Theorem C.1. DistinctBalancedPartition is NP-hard.

Proof. There is a known reduction from Partition to DistinctPartition. We now show a reduction from DistinctPartition to DistinctBalancedPartition. Let the positive integers $a_1, \ldots, a_n$ be an instance of DistinctPartition, and assume that their sum is $2S$. We create an instance of DistinctBalancedPartition with the following numbers:

- $n$ big numbers: $3^{n+1} \cdot a_i$ for each $i \in [n]$.
- $3n$ small numbers: $n$ triplets of numbers, where triplet $i$ contains $3^i$ and two integers whose sum is $3^i$, for example: $(1, 2, 3), (4, 5, 9), (13, 14, 27), (40, 41, 81), \ldots$.

In the new instance, there are $4n$ numbers; the sum of the big numbers is $2 \cdot 3^{n+1} \cdot S$ and the sum of the small numbers is $3^{n+1} - 3$, so the goal is to find a subset of $m = 2n$ numbers with a sum of $T = 3^{n+1} \cdot S + (3^{n+1} - 3)/2$.

Note that all big numbers are distinct (since all $a_i$ are), all small numbers are distinct as well by the construction, and the largest small number ($3^n$) is smaller than the smallest large number ($3^{n+1}$), so no two numbers coincide.

Every large number is a multiple of $3^{n+1}$, which is larger than the sum of all small numbers. Hence, in every solution to the DistinctBalancedPartition instance, the sum of all big numbers must be $3^{n+1} \cdot S$, so it constitutes a solution to the DistinctPartition.

Conversely, suppose we are given a solution to the DistinctPartition instance — a subset $X$ with $|X| = k$ (for some $k < n$) and $\sum_{a \in X} a = S$. Without loss of generality, we assume that $k \leq n/2$, since otherwise we can just replace $X$ with its complement, whose sum is $S$ too.

We construct a solution to DistinctBalancedPartition by taking the $k$ big numbers corresponding to $X$, and adding $2n - k$ small numbers as follows:

- From the first $k$ triplets of small numbers, we take the $3^i$;
- From the last $n - k$ triplets of small numbers, we take the two smaller numbers whose sum is $3^i$.

The number of elements in the generated set is $k + k + (2n - 2k) = 2n = m$. The sum of the generated set is $3^{n+1} \cdot S + (3^{n+1} - 3)/2 = T$. Hence it is indeed a solution to DistinctBalancedPartition. \[\square\]

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12 See Yuval Filmus’ answer here: https://cs.stackexchange.com/a/13032/1342
References

Atila Abdulkadiroglu and Tayfun Sonmez. 1998. Random serial dictatorship and the core from random endowments in house allocation problems. *Econometrica* 66, 3 (1998), 689.

Rediet Abebe, Richard Cole, Vasilis Gkatzelis, and Jason D Hartline. 2019. A Truthful Cardinal Mechanism for One-Sided Matching. arXiv preprint 1903.07797.

Martin Aleksandrov, Haris Aziz, Serge Gaspers, and Toby Walsh. 2015. Online Fair Division: analysing a Food Bank problem. In *IJCAI'15*. 2540–2546. arXiv:1502.07571 http://arxiv.org/abs/1502.07571

Reza Alijani, Majid Farhadi, Mohammad Ghodsi, Masoud Seddighin, and Ahmad S Tajik. 2017. Envy-Free Mechanisms with Minimum Number of Cuts.. In *AAAI*. 312–318.

Sergey Avvakumov and Roman Karasev. 2019. Envy-free division using mapping degree. arXiv preprint 1907.11183.

Haris Aziz. 2015a. A generalization of the AL method for fair allocation of indivisible objects. *Economic Theory Bulletin* (2015), 1–18. https://doi.org/10.1007/s40505-015-0089-1

Haris Aziz. 2015b. A note on the undercut procedure. *Social Choice and Welfare* 45, 4 (2015), 723–728. https://doi.org/10.1007/s00355-015-0877-4

Haris Aziz, Ioannis Caragiannis, and Ayumi Igarashi. 2018. Fair allocation of combinations of indivisible goods and chores. arXiv preprint 1807.10684.

Haris Aziz, Gerhard Rauchecker, Guido Schryen, and Toby Walsh. 2016. Approximation algorithms for max-min share allocations of indivisible chores and goods. arXiv preprint 1604.01435.

Moshe Babaioff, Noam Nisan, and Inbal Talgam-Cohen. 2017. Competitive Equilibria with Indivisible Goods and Generic Budgets. arXiv:1703.08150 http://arxiv.org/abs/1703.08150 arXiv preprint 1703.08150.

Julius B Barbanel. 2005. *The geometry of efficient fair division*. Cambridge University Press.

Julius B. Barbanel and Steven J. Brams. 2004. Cake division with minimal cuts: envy-free procedures for three persons, four persons, and beyond. *Mathematical Social Sciences* 48, 3 (Nov. 2004), 251–269. https://doi.org/10.1016/j.mathsocsci.2004.03.006

Julius B. Barbanel and Steven J. Brams. 2014. Two-Person Cake Cutting: The Optimal Number of Cuts. *The Mathematical Intelligencer* 36, 3 (2014), 23–35. https://doi.org/10.1007/s00283-013-9442-0

Siddharth Barman and Sanath Kumar Krishnamurthy. 2018. On the Proximity of Markets with Integral Equilibria. arXiv:cs.GT/1811.08673

Siddharth Barman, Sanath Kumar Krishnamurthy, and Rohit Vaish. 2018. Finding fair and efficient allocations. In *Proceedings of the 2018 ACM Conference on Economics and Computation*. ACM, 557–574.
Anna Bogomolnaia and Herve Moulin. 2001. A New Solution to the Random Assignment Problem. *Journal of Economic Theory* 100, 2 (Oct. 2001), 295–328. https://doi.org/10.1006/jeth.2000.2710

Anna Bogomolnaia, Herve Moulin, and Fedor Sandomirskiy. 2019. A simple Online Fair Division problem. arXiv:cs.GT/1903.10361

Anna Bogomolnaia, Herve Moulin, Fedor Sandomirskiy, and Elena Yanovskaya. 2016. Dividing goods or bads under additive utilities. arXiv:cs.GT/1608.01540

Anna Bogomolnaia, Herve Moulin, Fedor Sandomirskiy, and Elena Yanovskaya. 2017. Competitive division of a mixed manna. *Econometrica* 85, 6 (2 Feb. 2017), 1847–1871. arXiv:1702.00616 http://arxiv.org/abs/1702.00616

Sylvain Bouveret and Michel Lemaître. 2016. Characterizing conflicts in fair division of indivisible goods using a scale of criteria. *Autonomous Agents and Multi-Agent Systems* 30, 2 (2016), 1–32. https://doi.org/10.1007/s10458-015-9287-3

Steven J. Brams, Kilgour, and Christian Klamler. 2012. The undercut procedure: an algorithm for the envy-free division of indivisible items. *Social Choice and Welfare* 39, 2-3 (15 Oct. 2012), 615–631. https://doi.org/10.1007/s00355-011-0599-1

Steven J. Brams, D. Marc Kilgour, and Christian Klamler. 2013. Two-Person Fair Division of Indivisible Items: An Efficient, Envy-Free Algorithm. *Social Science Research Network Working Paper Series* (5 June 2013), 130–141. https://doi.org/10.1090/noti1075

Steven J. Brams and Alan D. Taylor. 1996. *Fair Division: From Cake Cutting to Dispute Resolution*. Cambridge University Press, Cambridge UK.

Steven J. Brams and Alan D. Taylor. 2000. The Win-Win Solution: Guaranteeing Fair Shares to Everybody (Norton Paperback) (reprint ed.). W. W. Norton & Company. http://www.worldcat.org/isbn/0393320812

Steven J. Brams and Jeffrey M. Togman. 1996. Camp David: Was The Agreement Fair? *Conflict Management and Peace Science* 15, 1 (01 Feb. 1996), 99–112. https://doi.org/10.1177/07389429601500105

Simina Branzei and Fedor Sandomirskiy. 2019. Algorithms for Competitive Division of Chores. arXiv:cs.GT/to appear

Eric Budish. 2011. The Combinatorial Assignment Problem: Approximate Competitive Equilibrium from Equal Incomes. *Journal of Political Economy* 119, 6 (Dec. 2011), 1061–1103. https://doi.org/10.1086/664613

Eric Budish, Gérard P Cachon, Judd B Kessler, and Abraham Othman. 2016. Course match: A large-scale implementation of approximate competitive equilibrium from equal incomes for combinatorial allocation. *Operations Research* 65, 2 (2016), 314–336.

Eric Budish, Yeon-Koo Che, Fuhioto Kojima, and Paul Milgrom. 2013. Designing random allocation mechanisms: Theory and applications. *American Economic Review* 103, 2 (2013), 585–623.
Ioannis Caragiannis, Nick Gravin, and Xin Huang. 2019. Envy-freeness up to any item with high Nash welfare: The virtue of donating items. arXiv preprint 1902.04319.

Ioannis Caragiannis, David Kurokawa, Hervé Moulin, Ariel D Procaccia, Nisarg Shah, and Junxing Wang. 2016. The unreasonable fairness of maximum Nash welfare. In Proceedings of the 2016 ACM Conference on Economics and Computation. ACM, 305–322.

Yiling Chen, John K. Lai, David C. Parkes, and Ariel D. Procaccia. 2013. Truth, justice, and cake cutting. Games and Economic Behavior 77, 1 (Jan. 2013), 284–297. https://doi.org/10.1016/j.geb.2012.10.009

Boris V Cherkassky and Andrew V Goldberg. 1999. Negative-cycle detection algorithms. Mathematical Programming 85, 2 (1999), 277–311.

Richard Cole, Vasilis Gkatzelis, and Gagan Goel. 2013a. Mechanism design for fair division: allocating divisible items without payments. In Proceedings of the fourteenth ACM conference on Electronic commerce. ACM, 251–268.

Richard Cole, Vasilis Gkatzelis, and Gagan Goel. 2013b. Positive results for mechanism design without money. In Proceedings of the 2013 international conference on Autonomous agents and multi-agent systems. International Foundation for Autonomous Agents and Multiagent Systems, 1165–1166.

Terry Daniel and James Parco. 2005. Fair, Efficient and Envy-Free Bargaining: An Experimental Test of the Brams-Taylor Adjusted Winner Mechanism. Group Decision and Negotiation 14, 3 (May 2005), 241–264. https://doi.org/10.1007/s10726-005-1245-z

Bart de Keijzer, Sylvain Bouveret, Tomas Klos, and Yingqian Zhang. 2009. On the Complexity of Efficiency and Envy-Freeness in Fair Division of Indivisible Goods with Additive Preferences. In Algorithmic Decision Theory, Francesca Rossi and Alexis Tsoukias (Eds.). Lecture Notes in Computer Science, Vol. 5783. Springer Berlin Heidelberg, 98–110. https://doi.org/10.1007/978-3-642-04428-1_9

Nikhil R. Devanur, Christos H. Papadimitriou, Amin Saberi, and Vijay V. Vazirani. 2008. Market Equilibrium via a Primal–dual Algorithm for a Convex Program. J. ACM 55, 5 (Nov. 2008), 22:1–22:18. https://doi.org/10.1145/1411509.1411512

John P. Dickerson, Jonathan Goldman, Jeremy Karp, Ariel D. Procaccia, and Tuomas Sandholm. 2014. The Computational Rise and Fall of Fairness. In Proceedings of the Twenty-Eighth AAAI Conference on Artificial Intelligence (AAAI’14). AAAI Press, 1405–1411. http://portal.acm.org/citation.cfm?id=2894091

D.W. 2019. Enumerate all allocations of points in a simplex. Theoretical Computer Science Stack Exchange. arXiv:https://cstheory.stackexchange.com/q/42496 https://cstheory.stackexchange.com/q/42496 URL:https://cstheory.stackexchange.com/q/42496 (version: 2019-03-09).

Shimon Even and Azaria Paz. 1984. A Note on Cake Cutting. Discrete Applied Mathematics 7, 3 (March 1984), 285–296. https://doi.org/10.1016/0166-218x(84)90005-2
Ya’akov Kobi Gal, Moshe Mash, Ariel D Procaccia, and Yair Zick. 2016. Which is the fairest (rent division) of them all?. In Proceedings of the 2016 ACM Conference on Economics and Computation. ACM, 67–84.

Mohammad Ghodsi, MohammadTaghi HajiAghayi, Masoud Seddighin, Saeed Seddighin, and Hadi Yami. 2018. Fair Allocation of Indivisible Goods: Improvements and Generalizations. In Proceedings of the 2018 ACM Conference on Economics and Computation. ACM, 539–556. arXiv preprint 1704.00222.

Jonathan Goldman and Ariel D. Procaccia. 2015. Spliddit: Unleashing Fair Division Algorithms. SIGecom Exch. 13, 2 (Jan. 2015), 41–46. https://doi.org/10.1145/2728732.2728738

Daniel Halpern and Nisarg Shah. 2019. Fair Division with Subsidy. http://www.cs.toronto.edu/~nisarg/papers/subsidy.pdf

Aanund Hylland and Richard Zeckhauser. 1979. The efficient allocation of individuals to positions. Journal of Political economy 87, 2 (1979), 293–314.

Onur Kesten and M Utku Ünver. 2015. A theory of school-choice lotteries. Theoretical Economics 10, 2 (2015), 543–595.

R. J. Lipton, E. Markakis, E. Mossel, and A. Saberi. 2004. On approximately fair allocations of indivisible goods. In Proceedings of the 5th ACM conference on Electronic commerce (EC ’04). ACM, New York, NY, USA, 125–131. https://doi.org/10.1145/988772.988792

Pasin Manurangsi and Warut Suksompong. 2017. Asymptotic existence of fair divisions for groups. Mathematical Social Sciences 89 (2017), 100–108.

Tansa G. Massoud. 2000. Fair Division, Adjusted Winner Procedure (AW), and the Israeli-Palestinian Conflict. Journal of Conflict Resolution 44, 3 (01 June 2000), 333–358. https://doi.org/10.1177/0022002700044003003

Jiri Matousek and Bernd Gartner. 2007. Understanding and using linear programming. Springer Science & Business Media.

Frédéric Meunier and Shira Zerbib. 2018. Envy-free cake division without assuming the players prefer nonempty pieces. arXiv preprint arXiv:1804.00449 (2018).

Ankur Moitra and Ryan ODonnell. 2011. Pareto optimal solutions for smoothed analysts. Proceedings of the 43rd annual ACM symposium on Theory of computing - STOC 11 (2011). https://doi.org/10.1145/1993636.1993667

Elchanan Mossel and Omer Tamuz. 2010. Truthful Fair Division. In SAGT, Spyros Kontogiannis, Elias Koutsoupias, and Paul G. Spirakis (Eds.), Vol. 6386. Springer Berlin Heidelberg, Berlin, Heidelberg, 288–299. https://doi.org/10.1007/978-3-642-16170-4_25 arXiv:1003.5480.pdf

Hervé Moulin. 2004. Fair division and collective welfare. MIT press.

Takashi Negishi. 1960. Welfare economics and existence of an equilibrium for a competitive economy. Metroeconomica 12, 2–3 (1960), 92–97.
Hoon Oh, Ariel D Procaccia, and Warut Suksompong. 2018. Fairly allocating many goods with few queries. arXiv preprint 1807.11367.

James B Orlin. 2010. Improved algorithms for computing fisher’s market clearing prices: computing Fisher’s market clearing prices. In Proceedings of the forty-second ACM symposium on Theory of computing. ACM, 291–300.

Elisha A Pazner and David Schmeidler. 1978. Egalitarian equivalent allocations: A new concept of economic equity. The Quarterly Journal of Economics 92, 4 (1978), 671–687.

Benjamin Plaut and Tim Roughgarden. 2018. Almost envy-freeness with general valuations. In Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms. Society for Industrial and Applied Mathematics, 2584–2603.

Ariel D. Procaccia and Junxing Wang. 2014. Fair Enough: Guaranteeing Approximate Maximin Shares. In Proc. EC-14. ACM, New York, NY, USA, 675–692. https://doi.org/10.1145/2600057.2602835

Jack M. Robertson and William A. Webb. 1998. Cake-Cutting Algorithms: Be Fair if You Can (first ed.). A K Peters/CRC Press.

Gerald Schneider and Ulrike S. Krämer. 2004. The Limitations of Fair Division. Journal of Conflict Resolution 48, 4 (01 Aug. 2004), 506–524. https://doi.org/10.1111/0022002704266148

Masoud Seddighin, Majid Farhadi, Mohammad Ghodsi, Reza Alijani, and Ahmad S. Tajik. 2018. Expand the Shares Together: Envy-Free Mechanisms with a Small Number of Cuts. Algorithmica (14 Aug 2018), 1–28. https://doi.org/10.1007/s00453-018-0500-z

Erel Segal-Halevi. 2018a. Competitive equilibrium for almost all incomes. In Proceedings of the 17th International Conference on Autonomous Agents and MultiAgent Systems. International Foundation for Autonomous Agents and Multiagent Systems, 1267–1275.

Erel Segal-Halevi. 2018b. Fairly dividing a cake after some parts were burnt in the oven. In Proceedings of the 17th International Conference on Autonomous Agents and MultiAgent Systems. International Foundation for Autonomous Agents and Multiagent Systems, 1276–1284.

Harunor Shishido and Dao-Zhi Zeng. 1999. Mark-Choose-Cut Algorithms For Fair And Strongly Fair Division. Group Decision and Negotiation 8, 2 (1999), 125–137. http://dx.doi.org/10.1023/a:1008620404353

Walter Stromquist. 1980. How to Cut a Cake Fairly. The American Mathematical Monthly 87, 8 (Oct. 1980), 640–644. https://doi.org/10.2307/2320951

Walter Stromquist. 2008. Envy-free cake divisions cannot be found by finite protocols. Electronic Journal of Combinatorics 15, 1 (Jan. 2008), #R11. http://www.emis.ams.org/journals/EJC/Volume_15/PDF/v15i1r11.pdf Research paper 11, 10 pp., 91B32.

Francis E. Su. 1999. Rental Harmony: Sperner’s Lemma in Fair Division. The American Mathematical Monthly 106, 10 (Dec. 1999), 930–942. https://doi.org/10.2307/2589747
Hal R Varian. 1976. Two problems in the theory of fairness. *Journal of Public Economics* 5, 3-4 (1976), 249–260.

William A. Webb. 1997. How to cut a cake fairly using a minimal number of cuts. *Discrete Applied Mathematics* 74, 2 (April 1997), 183–190. [https://doi.org/10.1016/s0166-218x(96)00032-7](https://doi.org/10.1016/s0166-218x(96)00032-7)

Stephen J Wilson. 1998. Fair division using linear programming. *preprint, Departement of Mathematics, Iowa State University* (1998).

Neal Young. 2019. A partition problem with order constraints. Theoretical Computer Science Stack Exchange. arXiv:https://cstheory.stackexchange.com/q/44316 [https://cstheory.stackexchange.com/q/44316](https://cstheory.stackexchange.com/q/44316) URL:https://cstheory.stackexchange.com/q/44316 (version: 2019-07-24).

Lin Zhou. 1990. On a conjecture by Gale about one-sided matching problems. *Journal of Economic Theory* 52, 1 (1990), 123–135.