Proofs of two Theorems concerning Sparse Spacetime Constraints

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Abstract

In the SIGGRAPH 2014 paper [SvTSH14] an approach for animating deformable objects using sparse spacetime constraints is introduced. This report contains the proofs of two theorems presented in the paper.

1 Introduction

In this report, we provide proofs of two theorems stated in [SvTSH14]. In Sections 2 and 4, we briefly review the background of the theorems and introduce some notation. Sections 3 and 5 contain the proofs. For more background on animating deformable objects using spacetime optimization, we refer to [WK88, KA08, BdSP09, HTZ+11, HSvTP12, BSG12].

2 Sparse Constraints and Linear Dynamics

We consider a linearized finite elements discretization of an elastic solid. The dynamics of the solid are described by a coupled system of linear ordinary second-order differential equations of the form

\[ M \ddot{u}(t) + (\alpha M + \beta K)\dot{u}(t) + K u(t) + g = 0, \]  

(1)

where \( u \in \mathbb{R}^n \) is the displacement vector, \( M \) is the mass matrix, \( K \) is the stiffness matrix, \( \alpha M + \beta K \) a Rayleigh damping term, and \( g \) a constant vector. We use spacetime constraints to force the object to interpolate a set of keyframes. We will first look at the following simple set of keyframes. For a set of \( m + 1 \) nodes \( \{t_0, t_1, \ldots, t_m\} \), we specify interpolation constraints

\[ u(t_i) = u_i \]  

(2)
and additionally the constraints
\[ \dot{u}(t_0) = v_0 \quad \text{and} \quad \dot{u}(t_m) = v_m \] (3)
on the velocity at the boundary of the time interval. To satisfy these constraints, we need to inject an additional force to the system. This force is determined in an optimization problem, where the objective functional is
\[ E(u) = \frac{1}{2} \int_{t_0}^{t_m} \| M \ddot{u} + (\alpha M + \beta K) \dot{u} + K u + g \|^2_{M^{-1}} \, dt. \] (4)

For some arbitrary \( u \) the \( E(u) \) measures the squared \( L^2 \)-norm of the additional force.

The eigenvalues and eigenmodes of (1) are solutions to the equation
\[ K \phi_i = \lambda_i M \phi_i. \]

We consider a basis \( \{ \phi_1, \phi_2, ..., \phi_n \} \) of \( \mathbb{R}^n \) consisting of eigenmodes. In [HSvTP12] it was shown that the minimizer \( u \) of \( E \) over all \( \tilde{u} \in H^2([t_0, t_m], \mathbb{R}^n) \) subject to the constraints (2) and (3) are of the form
\[ u(t) = \sum_i \omega_i(t) \phi_i, \] (5)
where the \( \omega_i(t) \) are so-called wiggly splines [KA08]. The wiggly splines are solutions to the one-dimensional form of the spacetime optimization problem described above.

### 2.1 Sparse spacetime constraints

Instead of the interpolation constraints (2) and (3), in [SvTSH14] linear constraints of the form
\[ A_k u(t_k) = a_k \quad \text{and} \quad B_k \dot{u}(t_k) - b_k \]
are considered. Here \( A_k, B_k \) are rectangular matrices and \( a_k, b_k \) are vectors. The constraints are sparse in the sense that the number of constraints at each node \( t_k \) is less than \( n \). For example, only the positions of a part of the object are prescribed.

Since the computation is, for efficiency, performed in a low-dimensional subspace of \( \mathbb{R}^n \), the constraints are formulated as least squares constraints
\[ E_C(u) = \frac{1}{2} \sum_{k=0}^{m} \left( c_A \| A_k u(t_k) - a_k \|^2 + c_B \| B_k \dot{u}(t_k) - b_k \|^2 \right), \] (6)
where \( c_A \) and \( c_B \) are constants.
3 The First Theorem

The first theorem, [SvTSH14, Theorem 1], shows that the minimizers of the spacetime optimization problem with sparse (least squares) constraints can be described using the eigenmodes and the wiggly splines.

Theorem 1 The minimizers of the energy $E(u) + E_C(u)$ among all functions in the Sobolev space $H^2((t_0, t_m), \mathbb{R}^n)$ are of the form (5) and are twice differentiable at any node $t_k$ where no velocity is prescribed and once differentiable at all other nodes.

Proof. Assume that $u$ is a minimizer of $E = E + E_C$ and that $v \in H^2((t_0, t_m), \mathbb{R}^n)$. The variation $\delta_v E(u)$ of $E$ at $u$ in the direction of $v$ satisfies

$$\delta_v E(u) = \delta_v E(u) + \delta_v E_C(u).$$

The variation $\delta_v E_C(u)$ is

$$\delta_v E_C(u) = \lim_{h \to 0} \frac{1}{h} (E_C(u + hv) - E_C(u)) = \lim_{h \to 0} \frac{1}{2h} \left( \sum_{k=0}^{m} \left( c_A \|A_k(u(t_k) + hv(t_k)) - a_k\|^2 + c_B \|B_k (\dot{u}(t_k) + hv(t_k)) - b_k\|^2 \right) - E_C(u) \right) = \sum_{k=0}^{m} \left( (A_k v(t_k))^T (A_k u(t_k) - a_k) + (B_k \dot{v}(t_k))^T (B_k \dot{u}(t_k) - b_k) \right).$$

Next, we consider the energy $E$ and abbreviate $D = \alpha M + \beta K$. The variation $\delta_v E(u)$ is given by

$$\delta_v E(u) = \lim_{h \to 0} \frac{1}{h} (E(u + hv) - E(u))$$

$$= \lim_{h \to 0} \frac{1}{2h} \left( \int_{t_0}^{t_m} \|M (\dot{u} + hv) + D (\ddot{u} + hv) + K (u + hv) + g\|^2_{M^{-1}} \, dt - E(u) \right) = \int_{t_0}^{t_m} (\ddot{u}^T + \dot{v}^T DM^{-1} + v^T KM^{-1}) (M \ddot{u} + D \dot{u} + Ku + g) \, dt$$

$$= \int_{t_0}^{t_m} v^T (M \dddot{u} + (2K - DM^{-1}D) \ddot{u} + KM^{-1} (Ku + g)) \, dt$$

$$- \sum_{k=1}^{m} (\ddot{v}^T + v^T DM^{-1}) (M \dddot{u} + D \ddot{u} + Ku + g) \bigg|_{t_{k-1}}^{t_k} + \sum_{k=1}^{m} v^T (M \dddot{u} + D \ddot{u} + Ku + g) \bigg|_{t_{k-1}}^{t_k}.$$

In the last step, we decomposed the integral over $[t_0, t_m]$ into a sum of integrals over the intervals $[t_k, t_{k+1}]$ and used integration by parts twice for each of the
summands. We write \( u, v, \) and \( g \) in the eigenbasis \( \{ \phi_1, \phi_2, \ldots, \phi_n \} \)

\[
    u(t) = \sum_i \omega_i(t) \phi_i, \quad v(t) = \sum_i v_i(t) \phi_i, \quad g = \sum_i g_i \phi_i
\]

to obtain

\[
    \delta_v E(u) = \int_{t_0}^{t_m} \sum_i v_i (\dddot{\omega}_i + 2 (\lambda_i - 2 \delta_i^2) \ddot{\omega}_i + \lambda_i (\lambda_i \dot{\omega}_i + g_i)) \, dt
    
    + \sum_{k=1}^m \sum_i (v_i (\dddot{\omega}_i + 2 \delta_i \dot{\omega}_i + \lambda_i \ddot{\omega}_i) - (\dot{v}_i + 2 \delta_i v_i) (\dddot{\omega}_i + 2 \delta_i \dot{\omega}_i + \lambda_i \ddot{\omega}_i + g_i)) |_{t_{k-1}}^{t_k}.
\]

(7)

The variation \( \delta_v E(u) \) vanishes for any \( v \) because \( u \) is a minimizer of \( \mathcal{E} \). From the calculation of \( \delta_v E_C(u) \) we see that \( \delta_v E_C(u) \) depends only on the values of \( u, \dot{u}, v, \) and \( \dot{v} \) at the nodes \( t_k \) (and is independent of the values \( u, \dot{u}, v, \) and \( \dot{v} \) take at any \( t \) in one of the open intervals \( (t_k, t_{k+1}) \)). Then the integrals

\[
    \int_{t_0}^{t_m} v_i (\dddot{\omega}_i + 2 (\lambda_i - 2 \delta_i^2) \ddot{\omega}_i + \lambda_i (\lambda_i \dot{\omega}_i + g_i)) \, dt
\]

must vanish for all \( v_i \in H^2((t_0, t_m), \mathbb{R}) \). This implies

\[
    \dddot{\omega}_i + 2 (\lambda_i - 2 \delta_i^2) \ddot{\omega}_i + \lambda_i (\lambda_i \dot{\omega}_i + g_i) = 0.
\]

The last equation is exactly the characterization of the wiggly splines, see \cite[SvTSH14, Equation (4)]{SvTSH14}. This shows that \( u \) is of the form (5).

The function \( u \) is once differentiable at the nodes \( t_k \) because any function in \( H^2((t_0, t_m), \mathbb{R}) \) is (by the Sobolev’s embedding theorem) once continuously differentiable. Now what remains is to show that \( u \) is twice differentiable at nodes where no velocity is specified. For this, we reorder the terms of (7):

\[
    \delta_v E(u) = \sum_{k=1}^{m-1} \sum_i v_i(t_k) \left( \dddot{\omega}_i(t_k) - \dddot{\omega}_i(t_k) - 2 \delta_i \left( \dddot{\omega}_i(t_k) - \dddot{\omega}_i(t_k) \right) \right) - v_i(t_k) \left( \dddot{\omega}_i(t_k) - \dddot{\omega}_i(t_k) \right)
    
    + \sum_{i} (v_i (\dddot{\omega}_i + 2 \delta_i \dot{\omega}_i + \lambda_i \ddot{\omega}_i) - (\dot{v}_i + v_i 2 \delta_i) (\dddot{\omega}_i + 2 \delta_i \dot{\omega}_i + \lambda_i \ddot{\omega}_i + g_i)) |_{t_{k-1}}^{t_k}.
\]

Here \( \dddot{\omega}_i(t_k) \) denotes the second derivative at \( t_k \) of the restriction of \( \dddot{\omega}_i \) to the interval \( [t_{k-1}, t_k] \), and \( \dddot{\omega}_i(t_k) \) denotes the second derivative at \( t_k \) of the restriction of \( \dddot{\omega}_i \) to the interval \( [t_k, t_{k+1}] \). If no velocity is prescribed at the node \( t_k \), then \( \dot{v}_i(t_k) (\dddot{\omega}_i(t_k) - \dddot{\omega}_i(t_k)) \) has to vanish for all \( \dot{v}_i \). This implies \( \dddot{\omega}_i(t_k) = \dddot{\omega}_i(t_k) \) for all \( i \). Hence, \( u \) is twice differentiable at \( t_k \).
4 Sparse Constraints and Warping

Rotation strain warping was introduced in [HTZ+11]. The goal there was to remove linearization artifacts from the deformation described by the displacement $u$. The warp map $W$ is a nonlinear map on the space of all possible displacements $u$.

To integrate the warping into the spacetime optimization framework described above the least squares energy (6) is replaced by the nonlinear least squares energy

$$E_{WC}(u) = \frac{1}{2} \sum_{k=0}^{m} \left( c_A \| A_k W(u(t_k)) - a_k \|^2 + c_B \| B_k DW \dot{u}(t_k) - b_k \|^2 \right).$$

Then, the objective functional

$$E(u) + E_{WC}(u)$$

is minimized over the space of displacements. The resulting motion is then warped minimizer $W(u(t))$.

5 The Second Theorem

The second theorem in [SvTSH14] shows that the minimizers of the nonlinear optimization problem can still be described using the eigenmodes and the wiggly splines.

**Theorem 2** The minimizers of the energy $E(u) + E_{WC}(u)$ among all functions in the Sobolev space $H^2((t_0, t_m), \mathbb{R}^n)$ are of the form (5) and are twice differentiable at any node $t_k$ where no velocity is prescribed and once differentiable at all other nodes.

**Proof (Sketch).** The proof is similar to that of Theorem 1. So we only sketch the proof here. We first calculate the variation $\delta_v E_{WC}(u)$

$$\delta_v E_{WC}(u) = \lim_{h \rightarrow 0} \frac{1}{h} (E_{WC}(u + hv) - E_{WC}(u))$$

$$= \lim_{h \rightarrow 0} \frac{1}{2h} \sum_{k=0}^{m} \left( c_A \| A_k W(u(t_k)) + hv(t_k) - a_k \|^2 
+ c_B \| B_k DW (\dot{u}(t_k) + hv(t_k)) - b_k \|^2 \right) - E_{WC}(u)$$

$$= \sum_{k=0}^{m} ((A_k DW(v(t_k)))^T (A_k W(u(t_k)) - a_k)$$

$$+ (B_k D^2 W(\dot{v}(t_k)))^T (B_k DW(\dot{u}(t_k)) - b_k)).$$

The last step used the Taylor expansion

$$W(u(t_k) + hv(t_k)) = W(u(t_k)) + hDW(v(t_k)) + \mathcal{R}(h)$$
and
\[ DW \left( \dot{u}(t_k) + h \dot{v}(t_k) \right) = DW \left( \dot{u}(t_k) \right) + h D^2W \left( \dot{v}(t_k) \right) + R(h), \]
where \( R(h) \) is a remainder term for which \( \lim_{h \to 0} \frac{1}{h} R(h) = 0. \)

The rest is as in the proof of Theorem 1. We calculate the variation \( \delta_v E(u) \) of \( E \) and represent \( u \) and \( v \) in the modal basis. This yields (7). From (9) we see that the variation \( \delta_v E_{WC}(u) \) depends only on the values of \( u, \dot{u}, v, \) and \( \dot{v} \) at the nodes \( t_k \) (and is independent of the values \( u, \dot{u}, v, \) and \( \dot{v} \) take at any \( t \) in one of the open intervals \( (t_k, t_{k+1}) \)). As described in the proof of Theorem 1, the minimizers are of the form (5) and they are twice differentiable at any node \( t_k \), where no velocity is prescribed, and once differentiable at all other nodes. ■

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