Finite Temperature Properties of
the Gauge Theory of Nonrelativistic Fermions

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Abstract

We study the finite temperature properties of the gauge theory of nonrelativistic fermions introduced by Halperin, Lee, and Read. This gauge theory is relevant to two interesting systems: high-$T_c$ superconductors in the anomalous metallic phase and a two-dimensional electron system in a strong magnetic field at the Landau filling factor $\nu = 1/2$. We calculate the self-energies of both gauge bosons and fermions by the random-phase approximation, showing that the dominant term at low energies is generated by the gauge-fermion interaction. The current-current correlation function is also calculated by the ladder approximation. We confirm that the electric conductivity satisfies the Drude formula and obtain its temperature dependence, which is of a non-Fermi-liquid.

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1 Introduction

In the last several years, it has been well recognized that gauge field theories play important roles in some interesting topics in condensed matter physics, like the fractional quantum Hall effect (FQHE) and the high-$T_c$ superconductivity. One of the canonical low-energy models describing such electron systems is believed to be a U(1) gauge theory of nonrelativistic fermions, although its concrete form has not been identified yet.

In the strongly-correlated electron systems like the high-$T_c$ superconductors, it is expected that the phenomenon of charge-spin separation (CSS) takes place at low temperatures ($T$) [1], that is, the charge and spin degrees of freedom of electrons behave independently. Various experiments are explained consistently by assuming the CSS. In the previous papers [2], two of the present authors showed that the CSS can be explained very naturally by a confinement-deconfinement phase transition of strong-coupling gauge theory. As demonstrated there, the CSS occurs at low $T$ and the quasi-excitations there are holons, spinons, and gauge bosons.

For the FQHE, a Ginzburg-Landau (GL) theory has been proposed [3], which explains various experimental results. This GL theory is a Chern-Simons (CS) gauge theory coupled with a complex boson field; so-called bosonized electrons, and the FQH state is characterized as a condensation of the bosonized electrons. Motivated by the success of the GL theory as well as Jain’s idea of composite fermions for the FQHE [4], Halperin, Lee, and Read [5] studied the system of electrons at the Landau filling factor $\nu = 1/2$ by introducing and analyzing a U(1) gauge theory of nonrelativistic fermions. This theory contains a parameter $b$, which controls the strength of gauge-field fluctuations [see (2.1),(2.3) in Sect.2].

Because of gauge invariance, the transverse component of gauge boson may survive as a massless mode, i.e., not shielded by vacuum polarization due to fermions, and so fermions interacting massless gauge bosons may have non-Fermi-liquid behavior at low energies. With this expectation, the effect of gauge field on the low-energy fermionic
excitations has been studied by the random-phase approximation (RPA) \cite{5,6} and by the renormalization-group (RG) equation \cite{7}. The dynamics of transverse gauge field is controlled by the Landau dissipative (damping) term in its propagator. At $T = 0$, the fermions exhibit marginal-Fermi-liquid like behavior \cite{8} due to the coupling to this gauge field. This phenomenon bears a close resemblance to the coherent soft-photon dressing of electrons in quantum electrodynamics (QED). The latter is known to be crucial for resolving the problem of infrared singularities in QED.

Therefore, ample systematic studies have been carried out so far for this gauge theory in case of $T = 0$. In this paper, we shall study its finite-temperature properties. There have appeared some studies on similar topics: Lee and Nagaosa \cite{9} calculated the conductivity in the uniform RVB mean-field theory plus gauge field fluctuations of the t-J model of high-$T_c$ superconductivity. This case corresponds to the special value $b = 0$. Kim et al. \cite{10} calculated the current-current correlation functions at $T = 0$ at the two-loop level, and get the conductivity at finite $T$ by assuming the Drude formula and certain scaling arguments. We shall compare our methods and results with theirs in some details. They are summarized in Sect.4.3 and in Sect. 5.

This paper is organized as follows. In Sect.2, we introduce the gauge model, which is, as announced, relevant to the metallic phase of high-$T_c$ superconductors and the electron system at $\nu = 1/2$. The RG studies of the model at $T = 0$ \cite{7} shows that there is a nontrivial infrared (IR) fixed point, whose location depends on the parameter $b$. This fixed point describes a non-Fermi-liquid. In Sect.3, we calculate the self-energies of both fermion and gauge-field propagators by RPA, to show that the relevant term at low energies appears through the loop corrections. In Sect.4, the current-current correlation function is calculated by the ladder approximation (LA). It is shown that the Drude formula of the conductivity is derived. By using the Kubo formula, we obtain the $T$-dependence of dc conductivity in the leading order of low $T$. It exhibits non-Fermi-liquid behavior for $0 \leq b \leq 1$. In short, the resistivity for $b = 1$ behaves as $\rho(T) \propto T^2|\ln T|$. For $b \neq 1$, we employ the $\varepsilon$-expansion w.r.t. $\varepsilon \equiv 1 - b$. For
0 \leq b < 1$, it behaves as $\rho(T) \propto T^{4/(3-b)}$. For $1 < b < 2$, the Fermi-liquid behavior is obtained, i.e., $\rho(T) \propto T^2$. Special attentions are paid to the gauge invariance of the results. Section 5 is devoted for conclusions. In Appendix, detailed calculations of the current-current correlation function and the conductivity are presented.

## 2 Model

We shall consider a two-dimensional system of nonrelativistic spinless fermion $\psi(x, \tau)$ interacting with a dynamical gauge field $A_i(x, \tau)$ $(i = 1, 2)$. In the imaginary-time formalism, the action of the model at finite $T$ is given by

$$S = \int_0^\beta d\tau \int d^2x \left\{ \bar{\psi} (\partial_\tau - \mu) \psi + \frac{1}{2m} (D_i \psi)(D_i \psi) \right\}$$

$$+ \frac{1}{2} \int_0^\beta d\tau \int d^2x \int d^2y \, B(x, \tau)V_B(x - y)B(y, \tau),$$

(2.1)

where $\beta \equiv (k_B T)^{-1}$. The covariant derivative $D_i$ and the magnetic field $B$ are given by

$$D_i = \partial_i + igA_i, \quad B = \epsilon_{ij} \partial_i A_j.$$  

(2.2)

The fluctuations of $B$ are controlled by the "potential" function,

$$V_B(x - y) = v_B \int dq \frac{e^{iq(x - y)}}{q^b},$$

(2.3)

$$\int dq = \frac{d^2q}{(2\pi)^2},$$

where the parameter $b$ is assumed to be in the region $0 \leq b < 2$. For the high-$T_c$ superconductivity, $b$ is chosen as $b = 0$ [9], and for the electron system of the half-filled Landau level, $b$ is related to the Coulombic-type repulsion between electrons [5, 7], but not yet fixed uniquely. We have also introduced the parameter $v_B$ for dimensional reason.

We shall take the Coulomb gauge $\partial_i A_i = 0$. The vector potential is then expressed in terms of $B$ as

$$A_i(x, \tau) = \epsilon_{ij} \int d^2y \, \partial_j G(x - y)B(y, \tau),$$

(2.4)
\[ G(x) = \int \frac{dq \exp(ig \cdot x)}{q^2}. \]  

(2.5)

We treat \( B(x, \tau) \) as a fundamental dynamical field, instead of \( A_i(x, \tau) \) itself. By substituting (2.4) into (2.1),

\[
S = \int_0^\beta d\tau \int d^2x \left\{ \bar{\psi} (\partial_\tau - \mu) \psi + \frac{1}{2m} \partial_i \bar{\psi} \partial^i \psi \right\} \\
\quad + \frac{1}{2} \int_0^\beta d\tau \int d^2x \; d^2y \; B(x, \tau) V_B(x-y) B(y, \tau) \\
\quad - \int_0^\beta d\tau \int d^2x \; d^2y \; \epsilon_{ij} \partial_j G(x-y) B(y, \tau) \frac{iq}{2m} \bar{\psi} \partial_i \psi(x, \tau) \\
\quad + \int_0^\beta d\tau \int d^2x \; d^2y \; d^2z \; \epsilon_{ij} \partial_j G(x-y) \epsilon_{ik} \partial_k G(x-z) \\
\quad \times B(y, \tau) B(z, \tau) \frac{g^2}{2m} \bar{\psi} \psi(x, \tau).
\]  

(2.6)

We perform Fourier transformation for \( \psi(x, \tau) \) and \( B(x, \tau) \),

\[
\psi(x, \tau) = \frac{1}{\beta} \sum_n \int dk \; \exp(ik \cdot x - i\omega_n \tau) \psi(k, \omega_n) \\
B(x, \tau) = \frac{1}{\beta} \sum_n \int dq \; \exp(iq \cdot x - i\epsilon_n \tau) q A(q, \epsilon_n),
\]  

(2.7) \quad (2.8)

where

\[
\omega_n \equiv \frac{(2n + 1)\pi}{\beta}, \quad \epsilon_n \equiv \frac{2n\pi}{\beta}, \quad n \in \mathbb{Z}.
\]  

(2.9)

Then the action (2.6) becomes

\[
S = \frac{1}{\beta} \sum_n \int dk \; \bar{\psi}(k, \omega_n) \left\{ -i\omega_n + \frac{k^2}{2m} - \mu \right\} \psi(k, \omega_n) \\
\quad + \frac{1}{2} \sum_n \int dq \; A(-q, -\epsilon_n) v_B q^{2-b} A(q, \epsilon_n) \\
\quad + \frac{1}{\beta^2} \sum_{n,l} \int dk \; dq \; \frac{ig}{m} \frac{k \times q}{q} A(q, \epsilon_l) \bar{\psi}(k + q, \omega_n + \epsilon_m) \psi(k, \omega_n) \\
\quad + \frac{1}{\beta^3} \sum_{n,l,l'} \int dk \; dq \; dp \; \frac{g^2}{2m} \frac{(-q \cdot q')}{qq'} A(q, \epsilon_l) A(q', \epsilon_{l'}) \bar{\psi}(k + q + q', \omega_n + \epsilon_l + \epsilon_{l'}) \psi(k, \omega_n),
\]  

(2.10)

where

\[
k \times q \equiv \epsilon_{ij} k_i q_j = kq \sin \phi, \\
k \cdot q \equiv k_i q_i = kq \cos \phi.
\]  

(2.11)
In Sect. 3, we shall study how the fermion and the gauge field propagators are renormalized by the gauge-fermion interactions, $A\bar{\psi}\psi$ and $AA\bar{\psi}\psi$.

3 Self-energies of fermions and gauge bosons

In this section we calculate the self-energies of the fermion and gauge-field propagators at finite $T$ by employing the RPA. At $T = 0$, it has been shown that the loop corrections generate the relevant terms at low energies [3, 4].

From (2.10), the gauge field propagator at the tree level is given by

$$D_0^{-1}(q, \epsilon_i) = v_B q^{2-b}. \quad (3.1)$$

In the RPA, the diagrams in Fig.1 are summed up as a geometric series in order to obtain the corrected propagator $D(q, \epsilon_i)$ of the gauge field:

$$D^{-1}(q, \epsilon_i) = D_0^{-1}(q, \epsilon_i) + \Pi(q, \epsilon_i) = v_B q^{2-b} + \Pi(q, \epsilon_i). \quad (3.2)$$

By the straightforward calculation, we obtain

$$\Pi(q, \epsilon_i) = \Pi_M + \tilde{\Pi}(q, \epsilon_i), \quad (3.3)$$

$$\Pi_M = \frac{g^2}{m} \frac{1}{\beta} \sum_n \int dk \ G_0(k, \omega_n)$$

$$= \frac{g^2}{m} \int dk \ f_\beta(E(k)), \quad (3.4)$$

$$\tilde{\Pi}(q, \epsilon_i) = \left( \frac{g}{m} \right)^2 \frac{1}{\beta} \sum_n \int dk \left( \frac{k \times q}{q} \right)^2 G_0(k, \omega_n) G_0(k + q, \omega_n + \epsilon_i)$$

$$= \left( \frac{g}{2\pi m} \right)^2 \int_0^{\infty} dk \ k^3 \int_0^\pi d\phi \ \sin^2 \phi \ \frac{f_\beta(E(k)) - f_\beta(E(k + q))}{i\epsilon_i - \Delta E(q, k)}, \quad (3.5)$$

where

$$G_0^{-1}(k, \omega_n) = i\omega_n - E(k), \quad (3.6)$$

$$f_\beta(E) = \frac{1}{e^{\beta E} + 1}, \quad (3.7)$$

$$E(k) = \frac{k^2}{2m} - \mu, \quad (3.8)$$
\[
\Delta E(q, k) = E(k + q) - E(k)
= \frac{kq}{m} \cos \phi + \frac{q^2}{2m}. \tag{3.9}
\]

In the later discussion, we shall assume the conditions \(\beta^{-1} \ll \mu\) and \(q \ll k_F\). These imply that we consider low-energy excitations near the Fermi surface. In this case we have

\[
\Pi_M = \frac{g^2}{2\pi \beta} \ln(1 + e^{\beta \mu}) \simeq \frac{g^2 \mu}{2\pi}, \tag{3.10}
\]

\[
\bar{\Pi}(q, \epsilon_l) = - \left( \frac{g}{2\pi m} \right)^2 \int_0^\infty dk \int_{-\pi}^\pi d\phi \frac{k^3 \sin^2 \phi}{i\epsilon_l - \Delta E(q, k)}
\times \left[ \Delta E(q, k) \frac{\partial f_\beta}{\partial E}(E(k)) + \frac{\Delta E(q, k)^2}{2} \frac{\partial^2 f_\beta}{\partial E^2}(E(k)) + \cdots \right]
\simeq -\Pi_M + \frac{g^2 \mu}{\pi} \left\{ \frac{1}{4\mu} \left( i\epsilon_l + \frac{q^2}{2m} \right) + \left( \frac{\epsilon_l}{2\pi i} - \frac{\epsilon_l^2}{4\pi \mu} \right) I \left( \frac{q^2}{2m}, \frac{k_F q}{m}, \epsilon_l \right) \right\}, \tag{3.11}
\]

where

\[
I(u, v, w) = \int_{-\pi}^\pi d\phi \frac{\sin^2 \phi}{u + v \cos \phi - iw}
= \frac{i}{2v} \oint_{C_1} \frac{dz}{z^2 + 2 \left( \frac{u-iw}{v} \right) z + 1}. \tag{3.12}
\]

For \(0 < u \ll v\), \(I(u, v, w)\) is evaluated as follows:

\[
I(u, v, w) \simeq \frac{2\pi i}{v^2} \left\{ \text{sgn}(w)(v^2 + w^2)^{1/2} - iu \right\} \left\{ 1 - \frac{|w|}{(v^2 + w^2)^{1/2}} \right\}. \tag{3.13}
\]

From (3.3), (3.11) and (3.13), we obtain

\[
\Pi(q, \epsilon_l) \simeq \frac{g^2 \mu}{\pi} \left\{ \frac{1}{4\mu} \left( i\epsilon_l + \frac{q^2}{2m} \right) + \left( \frac{\epsilon_l}{2\pi i} - \frac{\epsilon_l^2}{4\pi \mu} \right) I \left( \frac{q^2}{2m}, \frac{k_F q}{m}, \epsilon_l \right) \right\}
\simeq \frac{g^2 \mu}{\pi} \frac{|\epsilon_l|}{v_F q} \left[ 1 + \left( \frac{|\epsilon_l|}{v_F q} \right)^2 \right]^{1/2} - \frac{|\epsilon_l|}{v_F q}, \tag{3.14}
\]
where \( v_F = \frac{k_F}{m} \). Therefore \( \Pi(q, \epsilon_l) \) behaves as

(1) \( \epsilon_l \ll v_F q \)
\[
\Pi(q, \epsilon_l) \simeq \frac{g^2 \mu}{\pi} \cdot \frac{|\epsilon_l|}{v_F q},
\] (3.15)

(2) \( \epsilon_l \gg v_F q \)
\[
\Pi(q, \epsilon_l) \simeq \frac{g^2 \mu}{2\pi}.
\] (3.16)

Eq. (3.15) is nothing but the Landau damping factor, which plays an important role at \( T = 0 \). The above result shows that, at low \( T \), the term (3.15) is dominant, because \( \epsilon_l \propto T \) and the summation over \( l \) goes up to \( l \sim \beta v_F q \). On the other hand, at high \( T \), the effect of the dissipative term is less efficient. For the high-\( T_c \) superconductivity, the above remark is important for the discussion on the confinement-deconfinement phase transition (CDPT). Actually, by using the hopping expansion [4], it is shown that the CDPT occurs in the t-J model at a finite critical temperature, \( T_{CD} > 0 \). This result is strongly related with the above remark on the dissipative term. The CDPT in the present model is under study, and the results will be reported in future publications.

By using the gauge field propagator (3.14) obtained by the RPA, we shall calculate the corrected fermion propagator \( G(k, \omega_n) \). The corresponding diagram is given in Fig. 2, which gives rise to

\[
G^{-1}(k, \omega_n) = G_0^{-1}(k, \omega_n) - \Sigma(k, \omega_n)
\]
\[
= i\omega_n - E(k) - \Sigma(k, \omega_n),
\] (3.17)

\[
\Sigma(k, \omega_n) = \left( \frac{g}{m} \right)^2 \frac{1}{\beta} \sum_l \int dq \left( \frac{k \times q}{q} \right)^2 G_0(k + q, \omega_n + \epsilon_l) D(q, \epsilon_l)
\]
\[
= - \left( \frac{g k}{2\pi m} \right)^2 \frac{1}{\beta} \sum_l \int_{k_F} dq q I \left( \frac{k^2}{2m} + \frac{q^2}{2m} - \mu, \frac{kq}{m}, \omega_n + \epsilon_l \right)
\frac{v_B q^{2-b} + \Pi(q, \epsilon_l)}{v_B q^{2-b} + \Pi(q, \epsilon_l)},
\] (3.18)
\[ \Sigma(k_F, \omega_n) \simeq -i \cdot \frac{g^2 v_F}{2\pi} \cdot \frac{1}{\beta} \sum_l \int_0^{k_F} dq \frac{\text{sgn}(\omega_n + \epsilon_l) \left[ \left\{ 1 + \left( \frac{\omega_n + \epsilon_l}{v_{Fq}} \right)^2 \right\}^{\frac{1}{2}} - \frac{\omega_n + \epsilon_l}{v_{Fq}} \right]}{v_Bq^{2-b} + g^2 \mu \cdot \frac{\epsilon_l}{v_{Fq}} \left[ \left\{ 1 + \left( \frac{|\epsilon_l|}{v_{Fq}} \right)^2 \right\}^{\frac{1}{2}} - \frac{|\epsilon_l|}{v_{Fq}} \right]}, \]

(3.19)

We evaluate the \( q \)-integral in (3.19) as follows. Let us assume that the dominant region satisfies \( v_{Fq} \ll 2\pi \beta^{-1} \). This gives rise to the peak of the integrand to be \( q \sim k_F \), which contradicts the assumption for low \( T \). On the other hand, if the dominant region is assumed to satisfy the opposite inequality, \( v_{Fq} \gg 2\pi \beta^{-1} \), the peak of the integrand, \( q^3 - b \propto |\epsilon_l| \) (for \( b < 2 \)) brings no incompatibility. With this assumption, the integral is simplified as follows:

\[ \Sigma(k_F, \omega_n) \simeq -i \cdot \frac{\mu}{\pi} \cdot \frac{\pi^2 \alpha}{\beta \mu} \sum_l \int_0^1 dq \frac{\text{sgn}(\Omega_n + \mathcal{E}_l) \left[ \left\{ 1 + \left( \frac{\Omega_n + \mathcal{E}_l}{q^{2-b} + \frac{q}{q}} \right)^2 \right\}^{\frac{1}{2}} - \frac{\Omega_n + \mathcal{E}_l}{q^{2-b} + \frac{q}{q}} \right]}{q^{3-b} + |\mathcal{E}|}, \]

(3.20)

where we have introduced the following dimensionless variables for later convenience:

\[ \alpha \equiv \frac{g^2 v_F}{2\pi^2 v_B k_F^{1-b}}, \quad \tilde{q} \equiv \frac{q}{k_F}, \]

\[ \Omega_n \equiv \frac{\pi \alpha}{2} \cdot \frac{\omega_n}{\mu}, \quad \mathcal{E}_l \equiv \frac{\pi \alpha}{2} \cdot \frac{\epsilon_l}{\mu}. \]

(3.21)

\( H_b(c) \) in the final line of (3.20) is given by

\[ H_b(c) \equiv \frac{1}{3 - b} \int_c^{\infty} dy y^{\frac{1-b}{3-b}} \ln(1 + y) \]

\[ \simeq \frac{1}{1 - b} \left( 1 - c^{\frac{1-b}{3-b}} \right) \quad \text{for} \quad c \ll 1 \text{ and } b \sim 1. \]

(3.22)

We get the second line of (3.20) by replacing the \( l \)-sum with an integral and using the formula:

\[ \int_{-\infty}^{\infty} dx \, \text{sgn}(a + x) f(|x|) \]
\[
\begin{align*}
&\quad = \theta(a) \left\{ \int_{-|a|}^{\infty} dx \, f(|x|) - \int_{-\infty}^{-|a|} dx \, f(|x|) \right\} \\
&\quad + \theta(-a) \left\{ \int_{|a|}^{\infty} dx \, f(|x|) - \int_{-\infty}^{|a|} dx \, f(|x|) \right\} \\
&\quad = \text{sgn}(a) \int_{-|a|}^{|a|} dx \, f(|x|). 
\end{align*}
\]

(3.23)

Especially for $b = 1$, we obtain

\[
\Sigma(k_F, \omega_n) \simeq -i \cdot \frac{\mu}{\pi} \cdot \Omega_n \left\{ \ln \left( 1 + \frac{1}{|\Omega_n|} \right) + \frac{1}{|\Omega_n|} \ln \left( 1 + |\Omega_n| \right) \right\}.
\]

(3.24)

By using the above results, we shall calculate the current-current correlation functions by the LA in the following section.

4 Current-current correlation function and the conductivity

In the previous section, we obtained the corrected gauge and the fermion propagators at finite $T$. The loop corrections generate nontrivial relevant terms at low energies. The low-energy behavior of the fermion propagator has a branch cut rather than a pole in the frequency, and this behavior has a close resemblance to the 1D Luttinger liquid and the over-screened Kondo effect. Therefore, one can expect that the gauge-fermion interaction generates non-Fermi-liquid behavior also in gauge-invariant correlation functions. In these non-Fermi liquid systems, the $T$-dependence of the resistivity $\rho$ behaves as $\rho(T) \propto T^\Delta$, $\Delta < 2$, which is different from that of the usual Fermi liquid theory $\rho(T) \propto T^2$. We expect similar properties for the present gauge system.

In this section, we shall calculate the current-current correlation function (CCCF) at finite $T$ by the LA. At $T = 0$, this correlation was calculated by Kim et al. [10] at the two-loop order. They observed important cancellation of the leading singularities between the fermion self-energy and the vertex correction, due to the gauge invariance. In the LA below, we shall also evaluate the diagrams corresponding to their calculations, i.e., the fermion self-energy and the vertex correction.
4.1 Schwinger-Dyson equation

The gauge-invariant electromagnetic current $J_i(x, \tau)$ is given by

$$J_i(x, \tau) = j_i(x, \tau) - \frac{g}{m} \bar{\psi} \gamma_i A_i(x, \tau)$$

$$j_i(x, \tau) \equiv \frac{i}{2m} \left\{ \bar{\psi} \partial^\mu \psi(x, \tau) - \partial^\mu \bar{\psi} \psi(x, \tau) \right\}.$$  \hspace{1cm} (4.1)

The effect of the second (contact) term in $J_i(x, \tau)$ on the conductivity is less dominant at low $T$. This can be seen in a straightforward manner from the calculations by Kim et al. \[10\]. Therefore we consider the CCCF for $j_i(q, \epsilon_i)$ that is given by

$$\langle j_i(q, \epsilon_i) j_j(q', \epsilon_{i'}) \rangle = \frac{1}{\beta^2} \sum_{n,n'} \int \frac{dk}{2m} \frac{dk'}{2m} \frac{2k_i + q_i}{2m} \frac{2k_j' + q_j'}{2m}$$

$$\times \langle \bar{\psi}(k, \omega_n) \psi(k + q, \omega_n + \epsilon_i) \bar{\psi}(k', \omega_{n'}) \psi(k' + q', \omega_{n'} + \epsilon_{i'}) \rangle$$

$$= -\delta(q + q') \beta \delta_{i,-i'} \Pi_{ij}(q, \epsilon_i).$$ \hspace{1cm} (4.2)

It satisfies the “Schwinger-Dyson (SD)” equation which is graphically depicted in Fig.3. To calculate the conductivity, we shall use the Kubo formula. In that calculation, only the limit $q, q' \to 0$ of the above CCCF is needed. Therefore, we focus on the CCCF at zero-momenta below.

In order to solve the SD equation in the LA, it is useful to start with the following expression for the polarization tensor $\Pi_{ij}(0, \epsilon_i)$:

$$\Pi_{ij}(0, \epsilon_i) = \frac{1}{\beta^2} \sum_{n,n'} \int \frac{dk}{m} \frac{dk'}{m} \frac{k_i}{m} \frac{k'_j}{m} Y(k, \omega_n; k', \omega_{n'}).$$ \hspace{1cm} (4.3)

In term of the above function $Y(k, \omega_n; k', \omega_{n'}; \epsilon_m)$, the SD equation is rewritten as

$$Y(k, \omega_n; k', \omega_{n'}; \epsilon_i)$$

$$= R(k, \omega_n; \epsilon_i) \delta(k - k') \beta \delta_{n,n'}$$

$$+ R(k, \omega_n; \epsilon_i) \frac{1}{\beta} \sum_{n'} \int \frac{dk''}{m} \left( \frac{g}{m} \right)^2 \left( \frac{k \times k''}{|k - k''|} \right)^2 D(k'' - k, \omega_{n''} - \omega_n)$$

$$+ \delta(k - k'') \beta \delta_{n,n''} \left\{ \Sigma(k, \omega_n) G_0^{-1}(k'', \omega_{n''} + \epsilon_i) + \Sigma(k'', \omega_{n''} + \epsilon_i) G_0^{-1}(k'', \omega_{n''}) \right\}$$

$$\times Y(k'', \omega_{n''}; k', \omega_{n'}; \epsilon_i),$$ \hspace{1cm} (4.4)
where

\[ R(k, \omega_n; \epsilon_i) \equiv G_0(k, \omega_n) G_0(k, \omega_n + \epsilon_i), \quad (4.5) \]

and \( G_0(k, \omega_n) \) is the fermion propagator at the tree level defined in (3.6). To reduce the SD equation to more tractable form, we consider the following integral of the function \( \mathcal{Y} \):

\[
C_j(k, \omega_n; \epsilon_l) = \frac{1}{\beta} \sum_{n'} \int \! dk' \; k'_j \mathcal{Y}(k, \omega_n; k', \omega_{n'}; \epsilon_l)
\]

(4.6)

In terms of this \( C_j(k, \omega_n; \epsilon_l) \), the SD equation (4.4) becomes

\[
C_j(k, \omega_n; \epsilon_l) = k_j R(k, \omega_n; \epsilon_l) + \frac{\Delta \Sigma(k, \omega_n; \epsilon_l)}{i \epsilon_l} C_j(k, \omega_n; \epsilon_l)
\]

(4.7)

where we write

\[
\Delta \Sigma(k, \omega_n; \epsilon_l) \equiv R^{-1}(k, \omega_n; \epsilon_l) \left\{ \Sigma(k, \omega_n) G_0^2(k, \omega_n) - \Sigma(k, \omega_n + \epsilon_l) G_0^2(k, \omega_n + \epsilon_l) \right\}
\]

(4.8)

In the above we used the relation:

\[
R(k, \omega_n; \epsilon_l) \frac{1}{\beta} \sum_{n''} \int \! dk'' \; \delta(k'' - k) \beta \delta_{n,n''} C_j(k'', \omega_{n''}; \epsilon_l)
\]

\[
\times \left\{ \Sigma(k, \omega_n) G_0^{-1}(k'', \omega_{n''} + \epsilon_l) + \Sigma(k'', \omega_{n''} + \epsilon_l) G_0^{-1}(k'', \omega_{n''}) \right\}
\]

\[
= \frac{R^{-1}(k, \omega_n; \epsilon_l)}{i \epsilon_l} \left\{ \Sigma(k, \omega_n) G_0^2(k, \omega_n) - \Sigma(k, \omega_n + \epsilon_l) G_0^2(k, \omega_n + \epsilon_l) \right\} C_j(k, \omega_n; \epsilon_l)
\]

\[
- \frac{1}{\beta} \sum_{n''} \int \! dk'' \; \left( \frac{g}{m} \right)^2 \frac{1}{|k - k''|} \left( \frac{k \times k''}{|k - k''|} \right)^2 D(k'' - k, \omega_{n''} - \omega_n) R(k'', \omega_{n''}; \epsilon_l) C_j(k, \omega_n; \epsilon_l).
\]

Now, let us make the following ansatz for \( C_j(k, \omega_n; \epsilon_l) \) to solve the SD equation (4.7),

\[
C_j(k, \omega_n; \epsilon_l) = k_j R(k, \omega_n; \epsilon_l) \Psi(k, \omega_n; \epsilon_l),
\]

(4.9)
where \( \Psi(k, \omega_n; \epsilon_l) \) is the unknown function to be determined. This form is natural owing to the rotational symmetry. Then, \( \Psi(k, \omega_n; \epsilon_l) \) must satisfy

\[
\Psi(k, \omega_n; \epsilon_l) = 1 + \frac{\Delta \Sigma(k, \omega_n; \epsilon_l)}{i \epsilon_l} \Psi(k, \omega_n; \epsilon_l) \\
+ \left( \frac{g}{m} \right)^2 \frac{1}{\beta} \sum_{n''} \int \frac{dk''}{|k - k''|} \left( k \times k'' \right)^2 D(k'' - k, \omega_{n''} - \omega_n) \times R(k'', \omega_{n''}; \epsilon_l) \left\{ k' \cdot k'' \Psi(k'', \omega_{n''}; \epsilon_l) - \Psi(k, \omega_n; \epsilon_l) \right\},
\]

(4.10)

where we have used the fact that the following integral \( \Gamma_j(k, \omega_n; \epsilon_l) \) is proportional to \( k_j \),

\[
\Gamma_j(k, \omega_n; \epsilon_l) = \left( \frac{g}{m} \right)^2 \frac{1}{\beta} \sum_{n''} \int \frac{dk''}{|k - k''|} \left( k \times k'' \right)^2 D(k'' - k, \omega_{n''} - \omega_n) \times R(k'', \omega_{n''}; \epsilon_l) k'' \Gamma_j(k'', \omega_{n''}; \epsilon_l) \Psi(k'', \omega_{n''}; \epsilon_l) \\
= k_j \frac{k \cdot \Gamma_j(k, \omega_n; \epsilon_l)}{k^2}.
\]

[In the last line, the definition of \( \Gamma_j(k, \omega_n; \epsilon_l) \) is used.] At this stage, the CCCF is expressed as

\[
\Pi_{ij}(0, \epsilon_l) = \frac{1}{\beta} \sum_n \int \frac{dk}{m} \frac{k_i}{m} \frac{k_j}{m} R(k, \omega_n; \epsilon_l) \Psi(k, \omega_n; \epsilon_l).
\]

(4.11)

Hence, by solving (4.10) for \( \Psi(k, \omega_n; \epsilon_l) \), we get a solution for CCCF.

To solve (4.10) we first note that the \( k'' \) integral in (4.10) is dominated by the region \( k'' \sim k_F \) due to the appearance of \( R(k'', \omega_n; \epsilon_l) \) as long as \( \Psi(k, \omega_n; \epsilon_l) \) is a smooth function of \( k \). Furthermore, we make an assumption that the \( n \)-dependence of \( \Psi(k_F, \omega_n; \epsilon_l) \) is weak. One shall see that this crucial assumption is satisfied in the final solution. So this is a self-consistent solution. With these simplifications, the solution of (4.10) is easily obtained as

\[
\Psi(k_F, \omega_n; \epsilon_l) = \frac{i \epsilon_l}{i \epsilon_l - i \epsilon_l \Gamma_{GI}(k_F, \omega_n; \epsilon_l) - \Delta \Sigma(k_F, \omega_n; \epsilon_l)},
\]

(4.12)
where
\[
\Gamma_{GI}(k, \omega_n; \epsilon_l) \equiv \left( \frac{g}{m} \right)^2 \frac{1}{\beta} \sum_{n''} \int \frac{dk''}{2\pi} \frac{\left( k \times (k'' - k) \right)}{|k'' - k|} \frac{k \cdot (k'' - k)}{k^2} \times D(k'' - k, \omega_n'' - \omega_n) R(k'', \omega_n''; \epsilon_l).
\] (4.13)

4.2 The case of \( b = 1 \)

Below we consider the region of low \( T \) to get the concrete results. We shall discuss the case \( b = 1 \) first, because this case allows us to extract the leading nontrivial term of \( \Psi(k_F, \omega_n; \epsilon_l) \) at low \( T \). Let us evaluate \( \Gamma_{GI}(k_F, \omega_n; \epsilon_l) \) as follows: we present basic steps of calculations and explain the approximations involved. The reader who are interested in more details can find them in Appendix. First, by using the polar coordinate, and setting \( |k|, |k''| = k_F \) in \( D(k'' - k, \omega_n'' - \omega_n) \), we get
\[
\Gamma_{GI}(k_F, \omega_n; \epsilon_l) \approx -\frac{g^2 \mu}{(2\pi)^2} \cdot \frac{1}{|\epsilon_l|} \sum'_{\omega_n''} \int_{-\pi}^{\pi} d\phi \sin^2 \phi \frac{k_F \sqrt{2(1 - \cos \phi), \omega_n'' - \omega_n}}{i\omega_n'' - \frac{1}{i\omega_n'' + i\epsilon_l - E}} \times D(k_F \sqrt{2(1 - \cos \phi), \omega_n'' - \omega_n}) \times D(k_F \sqrt{2(1 - \cos \phi), \omega_n'' - \omega_n}),
\] (4.14)

where the contour integral over \( E \) restricts the \( n'' \)-sum; \( \sum' \) denotes the summation over \( n'' \) satisfying \( \text{sgn}(\omega_n'') = -\text{sgn}(\epsilon_l), |\omega_n''| < |\epsilon_l| \).

By repeating the similar argument as in the \( k'' \) integral in (3.19) above, we find that one should use the damping term (3.15) for \( D(q, \epsilon_l) \) rather than the mass term (3.16). Then we get
\[
\Gamma_{GI}(k_F, \omega_n; \epsilon_l) \approx -\frac{g^2 \mu}{2\pi} \cdot \frac{1}{|\epsilon_l|} \sum'_{\omega_n''} \int_{-\pi}^{\pi} d\phi \sin^2 \phi \frac{k_F \sqrt{2(1 - \cos \phi)}}{2v_B k_F^2 (1 - \cos \phi) + \frac{g^2 \mu}{\pi v_B} |\omega_n'' - \omega_n|} \times D(k_F \sqrt{2(1 - \cos \phi), \omega_n'' - \omega_n}) \times D(k_F \sqrt{2(1 - \cos \phi), \omega_n'' - \omega_n}) \times D(k_F \sqrt{2(1 - \cos \phi), \omega_n'' - \omega_n}),
\] (4.15)
where
\[
Q_1(c) \equiv \int_{-\pi}^{\pi} d\phi \frac{\sin^2 \phi \sqrt{2(1 - \cos \phi)}}{2(1 - \cos \phi) + c} = \int_0^2 dx \frac{x^3 \sqrt{4 - x^2}}{x^2 + c}.
\] (4.16)

To perform the \(\phi\)-integral or \(x\)-integral, \(x \equiv \{2(1 - \cos \phi)\}^{1/2}\), we note that \(Q_1(0)\) can be exactly evaluated. For small \(c\), a scaling argument gives rise to the following leading behavior:
\[
Q_1(c) \simeq \frac{8}{3} + Bc \ln c,
\] (4.17)
where \(B\) is some constant. Then we have
\[
\Gamma_{GI}(k_F, \omega_n; \epsilon_i) \simeq -\frac{\alpha}{4} \cdot \frac{1}{|\mathcal{E}_i|} \cdot \frac{\pi^2 \alpha}{\beta \mu} \sum_{n''} \left\{ \frac{8}{3} - B|\Omega_{n''} - \Omega_n| \ln |\Omega_{n''} - \Omega_n|^{-1} \right\}.
\] (4.18)

Since the \(n''\)-sum is restricted and \(T\) is small, we ignore the weak \(n''\) and \(n\) dependence in the log factor above by replacing it by \(\ln \{\pi^2 \alpha/(\beta \mu)\}^{-1}\). Then the \(n''\)-sum is carried out explicitly to reach the final result,
\[
\Gamma_{GI}(k_F, \omega_n; \epsilon_i) \simeq -\frac{\alpha}{4} \left[ \frac{8}{3} - B \text{sgn}(\mathcal{E}_i) \left( \frac{\pi^2 \alpha}{\beta \mu} \right)^2 \ln \left( \frac{\pi^2 \alpha}{\beta \mu} \right)^{-1} \left\{ \frac{l(l+1)}{2} + (l+1)n + n^2 \right\} \right].
\] (4.19)

From (4.12) and (4.19), we get
\[
\Psi(k_F, \omega_n; \epsilon_i) \simeq \frac{i\mathcal{E}_i}{i\tilde{C}_1(\beta)\mathcal{E}_i + i\gamma_b(k_F, \omega_n; \epsilon_i)},
\] (4.20)
\[
\gamma_b(k_F, \omega_n; \epsilon_i) \simeq -\frac{\alpha}{4} B \text{sgn}(\mathcal{E}_i) \left( \frac{\pi^2 \alpha}{\beta \mu} \right)^2 \ln \left( \frac{\pi^2 \alpha}{\beta \mu} \right)^{-1} \left\{ \frac{l(l+1)}{2} + (l+1)n + n^2 \right\},
\]
\[
\tilde{C}_1(\beta) \simeq 1 + \frac{\alpha}{2} \left\{ \frac{4}{3} + \ln \left( \frac{\pi^2 \alpha}{\beta \mu} \right)^{-1} \right\}.
\] (4.21)

The second term of the coefficient \(\tilde{C}_1(\beta)\) comes from \(\Delta \Sigma(k_F, \omega_n; \epsilon_i)\) of (4.8), which reflect the behavior of the fermion self-energy (3.24),
\[
\Sigma(k, \omega_n) \sim \omega_n \ln |\omega_n|^{-1}.
\] (4.22)
It is straightforward to see that \( \Delta \Sigma(k_F, \omega_n; \epsilon_l) \) is proportional to \( \epsilon_l \) for \( k = k_F \).

The solution (4.20) has certainly a weak \( n \)-dependence in the relevant region of \(|\Omega_n| < |\mathcal{E}_l|\), \( \text{sgn}(\Omega_n) = -\text{sgn}(\mathcal{E}_l) \) for \( \Pi_{ij}(0, \epsilon_l) \) at low \( T \) in a self-consistent manner as we assumed. In Fig.4 we plotted \( \Omega_n \)-dependence of \( \Gamma_{GI}(k_F, \omega_n; \epsilon_l) \), which supports this assumption. By inserting (4.20) into (4.11), we obtain the following result for the CCCF:

\[
\Pi_{ij}(0, \epsilon_l) \sim \delta_{ij} \frac{\mu}{2\pi} \cdot \frac{i\epsilon_l}{iC(\beta)\epsilon_l + i\tau^{-1}(\beta)} \quad \text{for } \epsilon_l > 0, \tag{4.23}
\]

where

\[
\tau^{-1}(\beta) \approx \frac{\mu}{6\pi B} \left( \frac{\pi^2 \alpha}{\beta \mu} \right)^2 \ln \left( \frac{\pi^2 \alpha}{\beta \mu} \right)^{-1} \propto T^2 |\ln T|. \tag{4.24}
\]

The electric conductivity \( \sigma_{ij} \) is obtained by the analytic continuation,

\[
\epsilon_l \to -i\epsilon + \delta, \tag{4.25}
\]

where \( \delta \) is an infinitesimal positive constant. Then we observe that Eq.(4.23) is nothing but the Drude formula of the electric conductivity. Explicitly, the dc conductivity is given as

\[
\text{Re} \sigma_{ij} = \lim_{\epsilon \to 0} \frac{\epsilon^2}{-i\epsilon} \Pi_{ij}(0, -i\epsilon + \delta) \approx \delta_{ij} \frac{e^2}{m} \rho \tau(\beta). \tag{4.26}
\]

In the conventional Fermi-liquid theory, the resistivity behaves as \( \rho(T) \propto T^2 \). In contrast, in the present case, \( \rho(T) \propto T^2 |\ln T| \). This difference supports that the present system with \( b = 1 \) is a "marginal" Fermi liquid.

4.3 The case of \( 0 \leq b < 1 \)

Next, let us consider the case \( b < 1 \) at low \( T \). To obtain a concrete result, we employ the idea of \( \varepsilon \) expansion in the critical phenomena. Here we use \( \varepsilon \equiv 1 - b \) as a small expansion parameter to expand various quantities around the known ones at \( b = 1 \).
The detailed calculations are collected in Appendix. The expression of $\Gamma_{GI}(k_F, \omega_n; \epsilon_l)$ is obtained as follows:

$$
\Gamma_{GI}(k_F, \omega_n; \epsilon_l) \simeq -\frac{\alpha}{4} \frac{1}{|\mathcal{E}_l|} \frac{\pi^2 \alpha}{\beta \mu} \sum_{n''} \left\{ A_b - B_b (|\Omega_n'' - \Omega_n|) |\Omega_n'' - \Omega_n| \right\}, \quad (4.27)
$$

where

$$
B_b(c) \simeq \frac{B}{1 - b} \left\{ c^{-\frac{2(1-b)}{3-b}} - 1 \right\} \quad \text{for } c \ll 1 \text{ and } b \sim 1, \quad (4.28)
$$

and $A_b$ is a constant which depends on $b$. We plot $\Gamma_{GI}(k_F, \omega_n; \epsilon_l)$ as a function of $\omega_n$ in Fig.4. This exhibits that it has a weak $n$-dependence, as we assumed to obtain the solution. It should be also remarked that $\Delta \Sigma(k_F, \omega_n; \epsilon_l)$ of (4.8) getting very large for $\Omega_n \sim 0$ or $-\mathcal{E}_l$, and the region in which $\Delta \Sigma(k_F, \omega_n; \epsilon_l)$ is small scales linearly with respect to $|\mathcal{E}_l|$. This behavior is confirmed numerically as Fig.5 shows. As in the case of $b = 1$, the momentum integral in (4.11) can be carried out and obtain the restriction on the summation over $n$ in the interval $|\Omega_n| < |\mathcal{E}_l|$, $\text{sgn}(\Omega_n) = -\text{sgn}(\mathcal{E}_l)$. With these remarks, we again obtain the Drude formula for conductivity (4.23) and its explicit $T$-dependence. The final results of conductivity at low $T$ in the $(1 - b)$-expansion is given by

$$
\text{Re } \sigma_{ij} \simeq \delta_{ij} \frac{e^2 \rho}{m} \tau(\beta),
$$

$$
\tau^{-1}(\beta) \equiv \frac{\mu}{6\pi} \left( \frac{\pi^2 \alpha}{\beta \mu} \right)^2 B_b \left( \frac{\pi^2 \alpha}{\beta \mu} \right) \propto T^{4/3 + b}. \quad (4.29)
$$

From this result, we conclude that the conductivity behaves as that of a non-Fermi-liquid for general values of $0 \leq b < 1$. As the model of high-$T_c$ superconductivity, the parameter $b = 0$ is chosen, and so we get $\rho(T) \propto T^{4/3}$ in the present calculation. This result coincides with the calculation of the relaxation time obtained by Lee and Nagaosa [9]. In their calculation, however, it is not clear whether the gauge invariance is respected or not. On the other hand, in our calculation, the final expressions of the conductivity, (4.24) and (4.29), come from the quantity $\Gamma_{GI}(k_F, \omega_n; \epsilon_l)$ in $\Psi(k_F, \omega_n; \epsilon_l)$. This $\Gamma_{GI}(k_F, \omega_n; \epsilon_l)$ is a gauge-invariant combination of the fermion-self-energy and the vertex correction. Strictly speaking, the quantity $\Psi(k_F, \omega_n; \epsilon_l)$ itself should be
gauge invariant because the current itself is gauge invariant (up to the irrelevant contact terms). However, due to the LA, the other term $\Delta \Sigma(k_F, \omega_n; \epsilon_l)$ appears in the denominator of $\Psi(k_F, \omega_n; \epsilon_l)$, which is not gauge invariant. However, after taking the limit $\epsilon \to 0$, it does not contribute to the conductivity, a physical quantity, as we have shown. As explained, the gauge-invariant CCCF’s were also examined up to the two-loop order at $T = 0$ by Kim et al. [10]. By assuming the Drude formula and some scaling rules, they obtained the $T$-dependence of conductivity for general values of $0 \leq b < 1$. Their result agrees with ours. The lesson one can learn from the present calculation is that the LA with the one-loop kernel in SD equation naturally leads to the Drude formula at low $T$ without any further assumptions. We believe the $(1 - b)$-expansion employed here works well down to $b = 0$.

4.4 The case of $1 < b < 2$

To close this section, we cite the result also for $1 < b < 2$ which is calculated in Appendix. The resistivity behaves as $\rho(T) \propto T^2$, i.e., just as the behavior of a Fermi-liquid. This sharp asymmetry under $\epsilon \leftrightarrow -\epsilon$ comes from the behavior of the function $B_b$ of (4.28).

5 Conclusion

In this paper, we studied the finite-temperature properties of the gauge theory of nonrelativistic fermions. The self-energies of the gauge field and the fermion were calculated by the RPA, and it was found that relevant terms at low energies appear from the loop corrections. We calculated also the CCCF by the LA. We verified how the Drude formula is satisfied, and obtained the $T$-dependence of the dc conductivity. It coincides with that obtained by Lee and Nagaosa [9] for the special case of the high-$T_c$ superconductivity, i.e. $b = 0$, and also the two-loop calculations by Kim et al. [10] for the CCCF which assumed the Drude formula and some scalings. We stress that,
as in the $T = 0$ case of Kim et al., the cancellation takes place in the CCCF between the singular part in the self-energy of fermion and that of vertex correction of fermion-gauge coupling. This cancellation is reflected in the result $\rho(T) \propto T^{3+\epsilon} (0 \leq b < 1)$ at low $T$. This is less dominant than the naive result $\rho(T) \propto T^{3+\epsilon}$ [11] that one may obtain by considering only the fermion propagator and ignoring the vertex corrections. It is interesting to apply the present method of the LA together with the $(1 - b)$-expansion to a system of bosons interacting with the gauge field. Such a system is relevant also for the high-$T_c$ superconductivity.

Very recently, Nayak and Wilczek [12] made a RG study to discuss the finite-temperature properties of metals like one-dimensional Luttinger liquid, but not of gauge theories. We are studying the RG equations at finite $T$ of the present gauge theory as an extension of the previous analysis at $T = 0$ [7]. The results will be reported in a future publication [13].

Finally we comment on the recent experiment [14] of the mobility at $\nu = 1/2$. They fit their data in the form of $C_1 + C_2 T^2 + C_3 |\ln T|$, where the last term takes care of the effect of gauge-field scattering in a dirty metal with impurities calculated by Halperin, Lee and Read [5]. The fitting looks for us not so definitive to reject out all the other possibilities. Our calculations in the present paper is for a clean metal with no impurities. The similar calculations for a disordered system with impurities are under study. It is an interesting subject to compare such experiments with our result in a systematic way. Such comparison will certainly shed some light on the low-energy effective gauge theory of electrons at $\nu = 1/2$, e.g., by selecting out the best value of $b$. 
Appendix

In this Appendix, we shall present detailed calculations of $\GammaGI(k_F, \omega_n; \epsilon_l)$ and $\Pi_{ij}(0, \epsilon_l)$ for general values of $b$ assuming that $b \sim 1$ and $T$ is small. First, let us start with $\GammaGI(k, \omega_n; \epsilon_l)$ of (4.13),

$$\GammaGI(k, \omega_n; \epsilon_l) \equiv \left(\frac{g_m^2}{m}\right)^2 \frac{1}{\beta} \sum_{n''} \int \frac{dk''}{2k''} \frac{\left\{ k \times (k'' - k) \right\}}{|k'' - k|}^2 \frac{k \cdot (k'' - k)}{k^2} \times D(k'' - k, \omega_{n''} - \omega_n) R(k'', \omega_{n''}; \epsilon_l),$$  \hspace{1cm} (A.1)

where $D(q, \epsilon_l)$ is given by (3.2) and (3.14). By using the polar coordinate for the momentum integral,

$$\GammaGI(k_F, \omega_n; \epsilon_l) \simeq -\frac{g^2 \mu}{(2\pi)^2} \cdot \frac{1}{\beta} \sum_{n''} \int_{-\pi}^{\pi} d\phi \sin^2 \phi \ D(k_F\sqrt{2(1 - \cos \phi)}, \omega_{n''} - \omega_n)$$

$$\times \int_{-\infty}^{\infty} dE \frac{1}{i\omega_{n''} - E} : \frac{1}{i\omega_{n''} + i\epsilon_l - E}$$

$$= -\frac{g^2 \mu}{2\pi} \cdot \frac{1}{|\epsilon_l|} \cdot \frac{1}{\beta} \sum_{n''} \int_{-\pi}^{\pi} d\phi \sin^2 \phi$$

$$\times D(k_F\sqrt{2(1 - \cos \phi)}, \omega_{n''} - \omega_n).$$  \hspace{1cm} (A.2)

In the dominant region of the above integral, $v_F k_F \sqrt{2(1 - \cos \phi)} \gg 2\pi/\beta$, and so

$$\GammaGI(k_F, \omega_n; \epsilon_l) \simeq -\frac{g^2 \mu}{2\pi} \cdot \frac{1}{|\epsilon_l|} \cdot \frac{1}{\beta} \sum_{n''} \int_{-\pi}^{\pi} d\phi \sin^2 \phi$$

$$\times \frac{k_F\sqrt{2(1 - \cos \phi)}}{v_B \left\{ k_F\sqrt{2(1 - \cos \phi)} \right\}^{3-b} + \frac{g^2 \mu}{2\pi \mu} |\omega_{n''} - \omega_n|}$$

$$= -\frac{\alpha}{4} \cdot \frac{1}{|\epsilon_l|} \cdot \frac{2\pi}{\beta} \sum_{n''} \int_{-\pi}^{\pi} d\phi \sin^2 \phi$$

$$\times \frac{\sqrt{2(1 - \cos \phi)}}{\left\{ \sqrt{2(1 - \cos \phi)} \right\}^{3-b} + \frac{\pi \alpha}{2} \cdot \frac{|\omega_{n''} - \omega_n|}{\mu}}$$

$$= -\frac{\alpha}{4} \cdot \frac{1}{|\epsilon_l|} \cdot \frac{\pi^2 \alpha}{\beta \mu} \sum_{n''} Q_b(|\Omega_{n''} - \Omega_n|),$$  \hspace{1cm} (A.3)

where

$$\alpha \equiv \frac{g^2 v_F}{2\pi^2 v_B k_F^{1-b}}, \quad \Omega_n \equiv \frac{\pi \alpha}{2} \cdot \frac{\omega_n}{\mu}, \quad \epsilon_l \equiv \frac{\pi \alpha}{2} \cdot \frac{\epsilon_l}{\mu},$$  \hspace{1cm} (A.4)
and

\[ Q_b(c) \equiv \int_{-\pi}^{\pi} d\phi \frac{\sin^2 \phi \sqrt{2(1 - \cos \phi)}}{\sqrt{2(1 - \cos \phi)}^{3-b} + c} \]

\[ = \int_0^2 dx \frac{x^3 \sqrt{4 - x^2}}{x^{3-b} + c}. \quad (A.5) \]

When \( c \ll 1 \) and \( b \sim 1 \), we can calculate \( Q_b \) approximately. First, we shall consider the following integral.

\[ B_b(c) \equiv \frac{Q_b(0) - Q_b(c)}{c} = \int_0^2 dx \frac{x^b \sqrt{4 - x^2}}{x^{3-b} + c}, \quad (A.6) \]

The maximum of the above integrand is at \( x \sim c^{1\over 4} \), and the value of the integrand at \( x = 1 \) is very small in comparison with the value at the maximum. So the naive evaluation will gives \( B_b(c) \propto c^{-2(1-b)} \). On the other hand, in the case of \( b = 1 \), the singularity of \( c \to 0 \) appears as \( \ln c^{-1} \). Therefore, we can evaluate the above integral more precisely as

\[ B_b(c) \propto (1 - b)^{-1} \left\{ c^{-2(1-b)} - 1 \right\} \]

In this way, we get

\[ Q_b(c) = A_b - B_b(c) c, \quad (A.7) \]

where

\[ A_b \equiv \int_0^2 dx x^b \sqrt{4 - x^2}, \quad B_b(c) \simeq \frac{B}{1 - b} \left\{ c^{-2(1-b)} - 1 \right\}, \quad (A.8) \]

and \( B \) is some numerical constant.

Hereafter we assume \( \epsilon_l > 0 \) without loss of generality. To obtain an explicit expression for the conductivity, we have to assume that the parameter \( b \) is close to 1, i.e., \( b = 1 - \epsilon \) and \( |\epsilon| \) is infinitesimally small. After getting the result by assuming that \( \epsilon \) is small, we put, for example, \( \epsilon = 1 \) for \( b = 0 \) in a similar spirit to the usual \( \epsilon \)-expansion. When \( b = 1 - \epsilon \), we can get

\[ \Gamma_{GI}(k_F, \omega_n; \epsilon_l) \simeq \frac{\alpha^2}{4} \cdot \frac{1}{|\mathcal{E}_l|} \cdot \frac{\pi^2 \alpha}{\beta \mu} \sum_{n'} \left\{ A_b - B_b \left( |\Omega_{n''} - \Omega_n| \right) |\Omega_{n''} - \Omega_n| \right\} \]

\[ \simeq \frac{\alpha}{4} \left[ -A_b + \frac{1}{\mathcal{E}_l} B_b \left( \frac{\pi^2 \alpha}{\beta \mu} \right) \frac{\pi^2 \alpha}{\beta \mu} \left\{ \sum_{l'=1}^{n+1} \mathcal{E}_{l'} + \sum_{l'=1}^{n} \mathcal{E}_{l'} \right\} \right] \]

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\[
\begin{align*}
\alpha 
&= 4 
\left[-A_b + \frac{1}{|E_l|} B_b \left( \frac{\pi^2 \alpha}{\beta \mu} \right) \left( \frac{\pi^2 \alpha}{\beta \mu} \right)^2 \left\{ \sum_{l' = 1}^{n+1} l' + \sum_{l' = 1}^{l+n} l' \right\} \right] \\
&= \frac{\alpha}{4} \left[-A_b + \frac{1}{|E_l|} B_b \left( \frac{\pi^2 \alpha}{\beta \mu} \right) \left( \frac{\pi^2 \alpha}{\beta \mu} \right)^2 \left\{ \frac{l(l+1)}{2} + (l+1)n + n^2 \right\} \right] \\
&= \frac{\alpha}{4} \left[-A_b + \frac{1}{2} \cdot \frac{1}{|E_l|} B_b \left( \frac{\pi^2 \alpha}{\beta \mu} \right) \left\{ |\Omega_n|^2 + |E_l + \Omega_n|^2 - 2(\Omega_0)^2 \right\} \right].
\end{align*}
\]

(A.9)

When we translated the first line of the above equation to the second line, \( n^\epsilon \sim 1 \) was used.

If the \( n \)-dependence of \( \Psi(k_F, \omega_n; \epsilon_l) \) is weak, we have

\[
\Psi(k_F, \omega_n; \epsilon_l) \simeq \frac{i E_l}{i C_b E_l + i \xi_b(k_F, \omega_n; \epsilon_l) + i \gamma_b(k_F, \omega_n; \epsilon_l)},
\]

where

\[
C_b \equiv 1 + \frac{\alpha}{4} A_b \quad (A_0 = \pi, A_1 = \frac{8}{3}),
\]

and

\[
\xi_b(k_F, \omega_n; \epsilon_l) \equiv i \frac{\pi \alpha}{2 \mu} \Delta \Sigma(k_F, \omega_n; \epsilon_l)
\]

\[
\simeq \alpha \left( \frac{\pi^2 \alpha}{\beta \mu} \right)^{\frac{1}{\lambda_b}} H_b \left( \frac{\pi^2 \alpha}{\beta \mu} \right) \Omega_n (E_l + \Omega_n) \left\{ \Omega_n^{-1} - (E_l + \Omega_n)^{-1} \right\}
\]

\[
= \alpha \left( \frac{\pi^2 \alpha}{\beta \mu} \right)^{\frac{1}{\lambda_b}} H_b \left( \frac{\pi^2 \alpha}{\beta \mu} \right) E_l.
\]

(A.12)

Here \( H_b(c) \) is defined in (3.22) and estimated as

\[
H_b(c) \simeq \frac{1}{1 - b} \left( 1 - c^{-\frac{1}{\lambda_b}} \right) \quad \text{for} \quad c \ll 1 \quad \text{and} \quad b \sim 1,
\]

(A.13)

and

\[
\gamma_b(k_F, \omega_n; \epsilon_l) \equiv -E_l \left\{ \Gamma_{GI}(k_F, \omega_n; \epsilon_l) + \frac{\alpha}{4} A_b \right\}
\]

\[
\simeq -\frac{\alpha}{8} B_b \left( \frac{\pi^2 \alpha}{\beta \mu} \right) \text{sgn}(E_l) \left\{ |\Omega_n|^2 + |E_l + \Omega_n|^2 - 2\Omega_0^2 \right\}
\]

\[
= -\frac{\alpha}{4} B_b \left( \frac{\pi^2 \alpha}{\beta \mu} \right) \text{sgn}(E_l) \left\{ \frac{l(l+1)}{2} + (l+1)n + n^2 \right\}.
\]

(A.14)
Using the above results, $\Pi_{ij}(0, \epsilon_l)$ is given as follows:

\[
\Pi_{ij}(0, \epsilon_l) \simeq \delta_{ij} \frac{\mu}{2\pi} \frac{\pi^2 \alpha}{\beta \mu} \sum_n \Psi(k_F, \omega_n; \epsilon_l)
\]

\[
= \delta_{ij} \frac{\mu}{2\pi} \frac{\pi^2 \alpha}{\beta \mu} \sum_n \frac{i \text{ sgn}(\epsilon_l)}{C_b(\beta)|\epsilon_l|} \left\{ 1 - \frac{1}{C_b(\beta)|\epsilon_l|} \gamma_6(k_F, \omega_n; \epsilon_l) + \ldots \right\}
\]

\[
\simeq \delta_{ij} \frac{\mu}{2\pi} \frac{\pi^2 \alpha}{\beta \mu} \frac{1}{C_b(\beta)|\epsilon_l|} \left\{ |\epsilon_l| + \frac{1}{C_b(\beta)} \frac{\alpha}{12} B_b \left( \frac{\pi^2 \alpha}{\beta \mu} \right) (\epsilon_l + 2\Omega_0) (\epsilon_l - 2\Omega_0) + \ldots \right\}
\]

\[
\simeq \delta_{ij} \frac{\mu}{2\pi} \frac{\pi^2 \alpha}{\beta \mu} \frac{|\epsilon_l|}{C_b(\beta)|\epsilon_l| - \frac{\alpha}{12} B_b \left( \frac{\pi^2 \alpha}{\beta \mu} \right) (\epsilon_l + 2\Omega_0) (\epsilon_l - 2\Omega_0)} i\epsilon_l
\]

\[
\simeq \delta_{ij} \frac{\mu}{2\pi} \frac{\pi^2 \alpha}{\beta \mu} \frac{i\epsilon_l}{iC_b(\beta) \epsilon_l + i\frac{\alpha}{12} B_b \left( \frac{\pi^2 \alpha}{\beta \mu} \right) \text{ sgn}(\epsilon_l) (i\epsilon_l)^2 + i\frac{\pi^2 \alpha}{2\mu} \text{ sgn}(\epsilon_l) \tau^{-1}(\beta)}.
\] (A.15)

where

\[
C_b(\beta) \equiv C_b + \alpha \left( \frac{\pi^2 \alpha}{\beta \mu} \right)^{-\frac{1}{1-b}} H_b \left( \frac{\pi^2 \alpha}{\beta \mu} \right),
\] (A.16)

\[
\tau^{-1}(\beta) \equiv \frac{\mu}{6\pi} \left( \frac{\pi^2 \alpha}{\beta \mu} \right)^2 B_b \left( \frac{\pi^2 \alpha}{\beta \mu} \right),
\] (A.17)

and we have used the formula

\[
\sum_n \left\{ l(l+1) \frac{1}{2} + (l+1)n + n^2 \right\} = \frac{\text{ sgn}(l)}{3} l(l+1)(l-1).
\] (A.18)

Eq. (A.17) is the result given in the text. Using (A.8), we can get concrete expressions for general $b \sim 1$ as follows:

(i) $0 \leq b < 1$

\[
\tau^{-1}(\beta) \simeq \frac{\mu}{6\pi} B \left( \frac{\pi^2 \alpha}{\beta \mu} \right)^2 \left\{ \left( \frac{\pi^2 \alpha}{\beta \mu} \right)^{\frac{2(1-b)}{1-b}} - 1 \right\},
\] (A.19)

(ii) $b = 1$

\[
\tau^{-1}(\beta) \simeq \frac{\mu}{6\pi} B \left( \frac{\pi^2 \alpha}{\beta \mu} \right)^2 \ln \left( \frac{\pi^2 \alpha}{\beta \mu} \right)^{-1},
\] (A.20)
(iii) $1 < b < 2$

$$\tau^{-1}(\beta) \simeq \frac{\mu}{6\pi} \cdot \frac{B}{b-1} \left( \frac{\pi^2 \alpha}{\beta \mu} \right)^2 \left\{ 1 - \left( \frac{\pi^2 \alpha}{\beta \mu} \right)^{\frac{2(b-1)}{b-2}} \right\}. \quad (A.21)$$
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Figure Captions

Figure 1: Graphs which contribute to the dressed gauge propagator in the RPA.

Figure 2: A graph which contributes to the dressed fermion propagator.

The wavy line describes the dressed gauge boson propagator.

Figure 3: Graphical representation of the Schwinger-Dyson equation for the CCCFs.

Figure 4: Plots of $\Gamma_{GI}$ as functions of $\Omega_n$. They have weak $\Omega_n$ dependence.

Figure 5: Plots of $\Delta \Sigma$ as functions of $\Omega_n$. They have peaks around $\Omega_n \simeq 0, -E_l$. 