Interplay of type I and type II seesaw contributions to neutrino mass

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ABSTRACT: Type I and type II seesaw contributions to the mass matrix of light neutrinos are inherently related if left-right symmetry is realized at high energy scales. We investigate implications of such a relation for the interpretation of neutrino data. We proved recently that the left-right symmetric seesaw equation has eight solutions, related by a duality property, for the mass matrix of right-handed neutrinos $M_R$. In this paper the eight allowed structures of $M_R$ are reconstructed analytically and analyzed numerically in a bottom-up approach. We study the dependence of right-handed neutrino masses on the mass spectrum of light neutrinos, mixing angle $\theta_{13}$, leptonic CP violation, scale of left-right symmetry breaking and on the hierarchy in neutrino Yukawa couplings. The structure of the seesaw formula in several specific $SO(10)$ models is explored in the light of the duality. The outcome of leptogenesis may depend crucially on the choice among the allowed structures of $M_R$ and on the level crossing between right-handed neutrino masses.

KEYWORDS: neutrino mass and mixing; seesaw mechanism; beyond the standard model.
1. Introduction

Experiments with solar, atmospheric, accelerator and reactor neutrinos have given an unambiguous evidence for neutrino oscillations and therefore for neutrino mass. This implies a physics beyond the standard model (SM) of particle physics, since neutrinos are strictly massless in that model. At the same time, neutrino mass is different from zero in almost every extension of the SM. Moreover, many models contain more than one source of neutrino mass. A crucial issue is therefore the ability to discriminate, within a given theoretical framework, among these sources using neutrino data as well as other neutrino-related experimental input.

If lepton number is not conserved, neutrinos are Majorana particles and their mass originates from the dimension five operator $LL\phi\phi$ via the electroweak symmetry breaking ($L \equiv (\nu_L l_L)^T$ and $\phi \equiv (\phi^+ \phi^0)^T$ are the SM lepton and Higgs isodoublets, respectively). New physics can contribute to this operator either at tree or at loop level. We consider here the first possibility; the super-heavy particles exchanged at tree level can then be either isosinglet fermions $N_R$, that is, right-handed (RH) neutrinos [1], or isotriplet scalars $\Delta_L$ [2].

The two cases are known under the names of type I and type II seesaw mechanism, respectively. Besides providing a very natural explanation of the smallness of the neutrino mass, the seesaw mechanism has a simple and elegant built-in mechanism for generating the observed baryon asymmetry of the universe – baryogenesis via leptogenesis [4]. If the masses of $N_R$ and $\Delta_L$ are much larger than the electroweak scale, then neutrino experiments can only probe the couplings of the effective operator. Therefore in general the data cannot tell type I from type II seesaw contribution to the masses of light neutrinos. Nonetheless, in several (possibly unified) models of fermion masses and mixing, both type I and type II terms are naturally present, and the knowledge of their relative size is crucial for achieving predictivity.

In this paper we study in detail the interplay of type I and type II seesaw contributions to neutrino mass. We consider a theoretically well motivated connection between these two contributions, which stems from the proportionality of the Majorana mass terms of left-handed and RH neutrinos. Let the Lagrangian of the theory contain the following term:

$$\mathcal{L}_M = -\frac{f}{2}(\nu_L^T C \nu_L \Delta_L^0 + N_L^c C N_L^c \Delta_R^0) + \text{h.c.}, \quad (1.1)$$

where $C = i\gamma_2\gamma_0$ is the charge conjugation matrix, $N_L^c \equiv (N_R)^c$ and $\Delta_L^0, \Delta_R^0$ are neutral scalar fields. In the case of more than one generation, $\nu_L$ and $N_L^c$ carry a flavor index and

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1We will not consider the unique remaining possibility, that is, isotriplet fermions [3], which are more difficult to embed in minimal extensions of the SM.

2Even in very constrained scenarios, it is far from trivial to control the relative size of the contributions of the two seesaw types. This problem in the minimal SUSY $SO(10)$ model has been extensively discussed in [5, 6] (for a different model see, e.g., [7]).
$f$ is a symmetric matrix of couplings. When $\Delta^0_{L,R}$ develop non-zero vacuum expectation values (VEVs) $v_{L,R}$, two Majorana mass terms, proportional to each other, are generated, one for left-handed and one for RH neutrinos. In general, a Dirac-type Yukawa coupling term is also present in the theory:

$$\mathcal{L}_D = -y^T_{\nu} C N_L^c \phi^0 + \text{h.c.},$$

(1.2)

where the neutral scalar $\phi^0$ has a VEV $v$. When $v_R \gg v$, the effective Majorana mass matrix for light left-handed neutrinos takes the form

$$m_\nu \simeq v_L f - \frac{v^2}{v_R} y f^{-1} y^T.$$  

(1.3)

The first (second) term on the r.h.s. of this equation is known as type II (type I) contribution to the light neutrino mass.\(^3\) The same coupling $f$ enters in both contributions because of the assumption made in eq. (1.1), which follows from a discrete left-right symmetry of the underlying theory.

In fact, the seesaw formula takes the form given in eq. (1.3) in models with left-right (LR) symmetry above some energy scale larger than the electroweak scale [8, 9]. By this we mean that the gauge group contains (or coincides with) $SU(3)_c \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ and a discrete symmetry guarantees that the $SU(2)_L$ and $SU(2)_R$ gauge couplings are asymptotically equal.\(^4\) Such models incorporate naturally RH neutrinos (as well as isotriplet scalars) and explain the maximal parity violation of low-energy weak interactions as a spontaneous symmetry breaking phenomenon. Note also that LR models can be easily made supersymmetric, and the exact R-parity at low energies can be obtained through the spontaneous breaking of $U(1)_{B-L}$ [11].

In this paper we perform a thorough phenomenological analysis of the seesaw formula (1.3). A duality property of this formula, which will be at the basis of our analysis, was identified in a recent paper by the authors [12]. The implications of such duality in a specific $SO(10)$ model are studied in [13].

The paper is organized as follows. In section 2 we discuss the duality property of the left-right symmetric seesaw. We also give the motivation for the reconstruction of the matrix $f$ (and therefore of the mass matrix of heavy RH neutrinos) from the seesaw formula (1.3) and study the multiplicity of the solutions. In section 3 we develop a method of exact analytic reconstruction of $f$ in the cases of one, two and three lepton generations. In section 4 a number of numerical examples, illustrating our analytic results, are given.

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\(^3\) Sometimes in the literature the mechanism leading to eq. (1.3) is called type II seesaw. We prefer the terminology where it is called type I+II seesaw, whereas the term “type II seesaw” is reserved for the situation when the Higgs triplet $\Delta_L$ is the sole source of neutrino mass.

\(^4\) Classic extensions of this minimal LR symmetric setting are provided by the Pati-Salam model, based on $SU(4)_c \times SU(2)_L \times SU(2)_R \equiv SU_{422}$ [8], and by the unified models based on $SO(10)$ [10].
and the conditions for the existence of light RH neutrinos are discussed. In section 5 we discuss the seesaw duality in specific models with LR symmetry, including several SO(10) models. In section 6 we briefly discuss some further issues pertaining to our analysis – stability of the results with respect to the renormalization group evolution effects and baryogenesis through leptogenesis. Discussion and summary are given in section 7. The appendices contain further analytic treatment of relevant topics: the reconstruction of $f$ in the case of antisymmetric $y$ (Appendix A), the estimate of the relative size of type I and II seesaw contributions to $m_\nu$ (Appendix B), and the generalization of the duality and the reconstruction of $f$ in the case of a different realization of the discrete LR symmetry (Appendix C).

2. Bottom-up approach to left-right symmetric seesaw

The LR symmetric seesaw equation (1.3) relates three vacuum expectation values ($v$, $v_L$ and $v_R$) and three $n \times n$ matrices ($m_\nu$, $y$ and $f$), where $n$ is the number of lepton generations. The matrix $f$ is not directly constrained by any experiment. Moreover, being a Majorana-type Yukawa coupling matrix, $f$ has no analogy with the only Yukawa coupling matrices that can be presently accessed experimentally ($y_u$, $y_d$, $y_e$), which are all of Dirac type.

It is therefore natural to employ the seesaw formula for reconstructing the matrix $f$, taking the quantities $m_\nu$, $y$, $v$, $v_{L,R}$ as input parameters. The purpose is to provide an insight into the underlying theory at the seesaw scale, which is deeply characterized by the structure of $f$ and which is not accessible to direct experimental studies. Notice that eq. (1.3) is a non-linear $n \times n$ matrix equation for $f$ and solving it in the case $n > 1$ is highly non-trivial.

In this section we first review the experimental and theoretical constraints on the input parameters. Then we discuss the duality property of the LR symmetric seesaw formula (1.3) and the multiplicity of its solutions for the matrix $f$.

2.1 Parameter space

Let us examine the possible values of the various quantities involved in eq. (1.3).

The neutrino mass matrix $m_\nu$ may be completely determined, at least in principle, from low energy experiments. In terms of the mass eigenvalues $m_i$ ($i = 1, 2, 3$) and leptonic mixing matrix $U$, one has

$$m_\nu = U^* \text{diag}(m_1, m_2, m_3) U^\dagger,$$

where $U$ depends on three mixing angles $\theta_{12}$, $\theta_{23}$ and $\theta_{13}$, one Dirac-type CP-violating phase $\delta$ and two Majorana-type CP-violating phases $\rho$ and $\sigma$. From global fits of low
energy experiments \cite{14, 15} one finds

\begin{equation}
\Delta m_{sol}^2 \equiv m_2^2 - m_1^2 = (7.9^{+1.0}_{-0.8}) \times 10^{-5} \text{ eV}^2, \quad \Delta m_{atm}^2 \equiv |m_3^2 - m_1^2| = (2.6 \pm 0.6) \times 10^{-3} \text{ eV}^2,
\end{equation}

where the best fit values and $3\sigma$ intervals are taken from the last update of \cite{14} (v5). Cosmology yields the most stringent upper limit on the sum of the neutrino masses, $m_1 + m_2 + m_3 \lesssim 0.4(0.7) \text{ eV}$ at 99.9\% C.L. with(without) including Lyman-$\alpha$ data in the fit \cite{16}. The values of the CP-violating phases are completely unknown at present.

As far as the vacuum expectation values are concerned, we choose the convention of real and positive $v, v_L$ and $v_R$ (this can always be achieved by redefining the phases of $\nu_L, N_L^c$ and $y$ in eqs. \eqref{eq:1.1} and \eqref{eq:1.2}). The electroweak symmetry breaking parameter $v \approx 174 \text{ GeV}$ is accurately known, while only weak constraints are available for the triplet VEVs $v_{L,R}$. No lower bound on $v_L$ exists, and actually conditions are known under which $\Delta L$ does not acquire any induced VEV. In contrast to this, $v_L$ is bounded from above by its contribution to the $\rho$-parameter ($\Delta \rho \approx -2v_L^2/v^2$), so that precision electroweak measurements imply $v_L \lesssim \text{ GeV}$ \cite{17}. Since we do not know the scale for the onset of LR symmetry, there is essentially no upper bound on $v_R$; we will be assuming in the following $v_R < M_{Pl}$. The value of $v_R$ is bounded from below by the non-observation of RH weak currents ($v_R \gtrsim \text{ TeV}$) \cite{17}. It is useful to translate these bounds into bounds on the parameter $x \equiv v_Lv_R/v^2$. If all dimensionless couplings in the scalar potential are of order one, its natural value is $x \sim 1$. However, the scalar sector may well be very complicated and the potential may depend on various mass scales which are not constrained \textit{a priori}. Therefore, $x$ should be considered a free parameter in a model independent analysis. Using the above experimental constraints, one finds $0 \leq x \lesssim 10^{14}$.

The Dirac-type neutrino Yukawa coupling matrix $y$ is not directly measurable, at least at present. Nonetheless, in LR-symmetric models and in their embedding in models with partial or grand unification, $y$ is usually related to known quark and/or charged lepton Yukawa couplings. A detailed discussion of the structure of $y$ in some of these models will be given in section \ref{sec:5}. Here we constrain ourselves to a few examples. In the minimal LR symmetric model, in the supersymmetric case one has $y = \tan \beta y_e$, where $y_e$ is the Yukawa coupling matrix of charged leptons and $\tan \beta$ is the usual ratio of VEVs. In the minimal Pati-Salam model, one obtains instead $y = y_u$, where $y_u$ is the Yukawa coupling matrix of the up-type quarks. This relation also holds in the $SO(10)$ model where the only Higgs multiplet contributing to the fermion masses is in the fundamental representation $10_H$. More complicated (and realistic) examples will be discussed in section \ref{sec:5}.

One should bear in mind, however, that the nature of the high energy theory is essentially unknown and therefore the Yukawa coupling matrix of neutrinos may in fact be
very different from those of charged fermions. In particular, the neutrino flavor sector may have radically different symmetries. As an interesting example, one may consider $y = \text{diag}(a, b, b)$ [18], which is motivated by the maximal or nearly maximal 2-3 mixing observed in the atmospheric neutrino oscillations. It should be also noted that, in the presence of low energy supersymmetry, lepton flavor violating processes can be directly sensitive to the neutrino Yukawa couplings. For example, in constrained mSUGRA seesaw models one may need $y_{ij} \ll 1$ for some $i, j$, in order to suppress processes like $\tau \rightarrow \mu \gamma$ [19].

This brief survey illustrates that although certain choices of $y$ may be well motivated, this matrix is in fact model dependent. In the following we will keep the form of $y$ as general as possible, so as to maintain the generality of our results within the chosen LR symmetric framework.

### 2.2 Seesaw duality

Before discussing the general case, it will be useful to find the matrix $f$ approximately in the limits when either of the two contributions to $m_\nu$ in eq. (1.3) dominates (these limits are well defined as long as all the matrix elements in $m_\nu$ are dominated by the same type of seesaw). In the case of dominant type I seesaw, one obtains

$$f_I = -\frac{1}{x} y^T m^{-1} y + \frac{1}{x^2} (ym^{-1}y)^T m^{-1}(ym^{-1}y)$$

$$- \frac{1}{x^3} (m^{-1}ym^{-1}y)^T (y^T m^{-1} y + ym^{-1}y^T)(m^{-1}ym^{-1}y) + \ldots ,$$

where we used the notation

$$m = \frac{m_\nu}{v_L}, \quad x = \frac{v_L v_R}{v^2} .$$

In the case of dominant type II seesaw, one finds

$$f_{II} = m + \frac{1}{x} ym^{-1} y^T - \frac{1}{x^2} (ym^{-1}y)m^{-1}(ym^{-1}y)^T + \ldots .$$

In this latter case $m_\nu \approx v_L f_{II}$, so that low energy neutrino data allow one to reconstruct directly the mass matrix of RH neutrinos $M_R \approx v_R f_{II}$. In particular, the spectrum of the heavy neutrinos coincides (up to an overall factor) with that of light neutrinos [20].

In general, there is no a priori reason to expect that one type of seesaw dominates. If both contributions to $m_\nu$ are comparable, the solution of eq. (1.3) cannot be obtained expanding around the purely type I or II solution and a different approach is needed.

The LR symmetric type I+II seesaw formula in eq. (1.3) has the following intriguing duality property. Suppose that a matrix $f$ solves this equation. Then one can verify that

$$\hat{f} = m - f = -\frac{1}{x} yf^{-1}y^T$$

(2.7)
is also a solution, provided that the matrix $y$ is invertible and symmetric (or antisymmetric). These conditions on $y$ turn out to be both necessary and sufficient for $\hat{f}$ to solve eq. (1.3). In particular, they guarantee that the matrix $\hat{f}$ is invertible, as it must be in order to satisfy eq. (1.3). It is easy to see that the duality operation is closed, i.e. the dual of $\hat{f}$ coincides with $f$.

An important example of duality is provided by the solutions for $f$ in the case of one seesaw type dominance, given in eqs. (2.4) and (2.6): if $y = \pm y^T$, one immediately sees that $f_I$ and $f_{II}$ are dual to each other. This allows us to establish an important result: if there is a solution of the seesaw equation with dominant type I seesaw, then there is also an alternative one, with dominant type II seesaw, and vice versa. More generally, we will show that the LR symmetric seesaw equation always has a pair of dual solutions which reduce to $f_I$ and $f_{II}$ when the input parameters satisfy certain conditions.

In the following we will focus on the case where $y$ is symmetric. This is true in models with the discrete LR symmetry $\nu_L \leftrightarrow N^c_L$ (see eq. (1.2)), including the minimal LR symmetric model and $SO(10)$ models where only the couplings to $10_H$ and $\overline{126}_H$ Higgs multiplets contribute to $y$ (for further details see section 3). The case of antisymmetric $y$ will be discussed in Appendix A. When $y$ is symmetric, one can write $y = U y_d U^T$, where $y_d$ is diagonal, real and positive and $U$ is a unitary matrix. (We consider $m_\nu$ in the basis where the mass matrix of charged leptons is diagonal, so that $U$ describes the mismatch between the left-handed rotations that diagonalize the Yukawa couplings of charged leptons and neutrinos.) As a consequence, eq. (1.3) can be rewritten as

$$U^\dagger m_\nu U^* = v_L (U^\dagger f U^*) - \frac{v^2}{v_R} y_d (U^\dagger f U^*)^{-1} y_d. \quad \text{(2.8)}$$

Therefore, one can always work (and we will) in the basis where $y$ is diagonal, by redefining $(U^\dagger m_\nu U^*) \rightarrow m_\nu$ and $(U^\dagger f U^*) \rightarrow f$. If $U \approx 1$, the input from the low-energy neutrino data will still (approximately) determine the left-hand side of eq. (2.8). Note that $U$ is the leptonic analogue of the CKM mixing matrix in the quark sector, where mixing is known to be small. Therefore the condition $U \approx 1$ may be motivated by a quark-lepton symmetry.

### 2.3 Multiplicity of solutions

We have found that, if a matrix $f$ solves the seesaw equation (1.3), so does $\hat{f} \equiv m - f$, i.e. this equation does not have a unique solution. This is hardly surprising, as eq. (1.3) is a non-linear matrix equation for $f$, or, equivalently, a system of non-linear coupled equations for its elements $f_{ij}$. From the duality property of eq. (1.3) it immediately follows that the number of its solutions must be even. We shall now show that for $n$ lepton generations the multiplicity of solutions is $2^n$. \footnote{This result was first obtained in [12], though in a less straightforward way. An alternative derivation can be found in [13].}
Let us introduce the matrices $\tilde{m}$ and $\tilde{f}$ through the relations
\[ m = \frac{1}{\sqrt{x}} y^{1/2} \tilde{m} y^{1/2}, \quad f = \frac{1}{\sqrt{x}} y^{1/2} \tilde{f} y^{1/2}, \quad (2.9) \]
where $y^{1/2}$ satisfies $(y^{1/2})^2 = y$. From eq. (1.3) with symmetric and invertible $y$ one then obtains
\[ \tilde{m} = \tilde{f} - \tilde{f}^{-1}. \quad (2.10) \]
The duality we discussed above is especially clearly seen in this equation, where it corresponds to the invariance with respect to $\tilde{f} \leftrightarrow -\tilde{f}^{-1} (= \tilde{m} - \tilde{f})$. Multiplying eq. (2.10) by $\tilde{f}$ on the left or on the right, we find that $\tilde{f}$ satisfies
\[ \tilde{f}^2 - \tilde{m} \tilde{f} - 1 = 0 \quad (2.11) \]
and that $[\tilde{m}, \tilde{f}] = 0$. The commutativity of $\tilde{m}$ and $\tilde{f}$ allows one to find a formal solution of eq. (2.11). Indeed, let us write $\tilde{f}$ in the form $\tilde{f} = \tilde{m}/2 + R$, where the matrix $R$ is yet to be determined. Substituting this into eq. (2.11) in which the term $\tilde{m} \tilde{f}$ is rewritten as $(1/2)\{\tilde{m}, \tilde{f}\}$, one finds that the matrix $R$ satisfies $R^2 = \tilde{m}^2/4 + 1$, so that finally one obtains
\[ \tilde{f} = \frac{\tilde{m}}{2} + \sqrt{\frac{\tilde{m}^2}{4} + 1}. \quad (2.12) \]
Since any non-singular $n \times n$ matrix with non-degenerate eigenvalues has $2^n$ square roots \[21\], eq. (2.11) (and so eq. (1.3)) has $2^n$ solutions. Obviously, if $R_0$ is a square root of $R^2$, so is $-R_0$; therefore the solutions in eq. (2.12) form $2^{n-1}$ dual pairs with $\tilde{f} + \tilde{f} = \tilde{m}$.

3. Analytic reconstruction of the coupling matrix $f$

In this section we will solve analytically the seesaw equation for $f$ in the cases of one, two and three lepton generations. For one generation, the analysis is straightforward; for more than one lepton flavor, the effects of mixing complicate the reconstruction of $f$ considerably and, for this purpose, we shall need to develop new algebraic techniques. We will provide general solutions for the case of symmetric $y$, where the seesaw duality occurs (the generalization to antisymmetric $y$ is given in Appendix A).

We will also identify the criteria to quantify the dominance of one or the other type of seesaw. In the presence of mixing, this identification will turn out to be a subtle issue, whose details are given in Appendix B.

A short account of the main results of this section was published in [12]. An alternative approach for the analytic reconstruction of $f$ was developed in [13], for the case of symmetric $y$. 
3.1 The case of one lepton generation

In this case $m_\nu$, $y$ and $f$ are merely complex numbers. Eq. (1.3), being quadratic in $f$, has two solutions, which we denote $f_{\pm}$:

$$v_L f_{\pm} = \frac{1}{2} \left[ m_\nu \pm \left( m_\nu^2 + \frac{4v^2y^2v_L}{v_R} \right)^{1/2} \right] = \frac{m_\nu}{2} \left[ 1 \pm (1 + d)^{1/2} \right], \quad (3.1)$$

where

$$d \equiv \frac{v_L 4v^2y^2}{v_R m_\nu^2}. \quad (3.2)$$

Obviously, the solutions $f_{\pm}$ are dual to each other: $m = f_+ + f_-$. 

At this point it is useful to state our convention for the assignment of the complex phases. The freedom to rephase the fields in eqs. (1.1) and (1.2) allows us to take $v$, $v_L$, $v_R$ as well as $m_\nu$ real and positive. Next, we define $y^2 \equiv |y|^2 e^{i\chi}$, so that $\text{arg } d = \chi$ and the phase of $f$ is then determined by eq. (3.1).

Type I and II contributions to $m_\nu$ are defined as

$$m_\nu^I \equiv -\frac{v^2y^2}{v_R f}, \quad m_\nu^{II} \equiv v_L f = m_\nu - m_\nu^I. \quad (3.3)$$

Their relative size is determined by the value of $d$:

$$r_{II/I} \equiv \frac{m_{\nu \pm}}{m_{\nu \mp}} = \frac{|1 \pm (1 + d)^{1/2}|^2}{d}. \quad (3.4)$$

We identify three possible physical regimes:

(i) $|d| \ll 1$ \quad $|m_{\nu +}^I| \ll |m_{\nu +}^{II}|$ and $|m_{\nu -}^{II}| \ll |m_{\nu -}^I|$ \quad single type dominance

(ii) $|d| \sim 1$ \quad $|m_{\nu +}^I| \sim |m_{\nu -}^{II}|$ \quad hybrid seesaw

(iii) $|d| \gg 1$ \quad $m_{\nu +}^I \approx -m_{\nu -}^{II}$ \quad cancellation regime

Let us discuss these cases in turn.

(i) The dominant seesaw type is I (II) in the case of the $f_-$ ($f_+$) solution. For $|d| \ll 1$, eq. (3.1) becomes

$$f_- \approx -\frac{v^2y^2}{v_R m_\nu}, \quad f_+ \approx \frac{m_\nu}{v_L} + \frac{v^2y^2}{v_R m_\nu}. \quad (3.5)$$

Therefore a value $|d| \ll 1$ implies that one type of seesaw is dominant, but it does not determine which one. Thus, both type I and type II dominance limits correspond to the same condition $|d| \ll 1$. This may look counter-intuitive, as these two limits correspond to apparently opposite conditions $|m_\nu^I| \gg |m_\nu^{II}|$ and $|m_\nu^I| \ll |m_\nu^{II}|$. However, it is easy to see that, when expressed in terms of the “input” parameters only, i.e. when the corresponding solutions for $f$ are substituted, both conditions reduce to $|d| \ll 1$. 


Figure 1: Moduli of the solutions $v_L f_\pm$ of the seesaw formula, as functions of $|d|$. We chose $m_\nu = \sqrt{\Delta m_{atm}^2} = 0.05$ eV. The solid, dashed and dotted curves correspond to $\arg d = 0$, $\pi/2$ and $\pi$, respectively. Recall that $m^{III}_\nu = v_L f_\pm$ and $m^{II}_\nu = m_\nu - m^{III}_\nu = v_L f_\mp$.

A solution $f$ is meaningful (perturbative) only if $|f| \lesssim 1$: for $v_L \ll m_\nu$, the solution $f_+$ violates perturbative unitarity and should be discarded, so that only $f_-$ is viable (dominant type I). A necessary condition for both solutions to be perturbative is $v_R \gtrsim v^2 |y|^2 / m_\nu$, so that either type I or type II contribution to $m_\nu$ can dominate. Notice that for $|d| \ll 1$ the expansions (2.4) and (2.3) apply for $f_-$ and $f_+$, respectively; eq. (3.5) just gives a simplified version of such expansions.

(ii) For $|d| \sim 1$ both seesaw types give sizable contributions to $m_\nu$. Notice that when $d = -1$ (i.e., $4v_L v^2 |y|^2 = v_R m_\nu^2$ and $\chi \equiv \arg y^2 = \pi$), the equality $m^{I}_\nu = m^{III}_\nu = m_\nu/2$ is realized. This degeneracy point corresponds to the absence of CP violation. 

(iii) The case $|d| \gg 1$ corresponds to a cancellation between the two seesaw contributions to $m_\nu$. To bar a too strong (“unnatural”) cancellation, one may demand $|m^{III}_\nu| \lesssim 1$ eV. Eq. (3.3) then implies a lower bound on the scale $v_R$ of $SU(2)_R$ symmetry breaking: $v_R \gtrsim v^2 |y|^2 / (|f| \cdot 1 \text{ eV}) \approx |y^2 / f| 3 \cdot 10^{13}$ GeV. If one makes the additional (“natural”) assumption $x \equiv v_R v_L / v^2 \sim 1$, eq. (3.3) will also imply $v_R \gtrsim v^2 |f| / eV \approx |f| 3 \cdot 10^{13}$ GeV.

For illustration, let us assume that the light neutrino mass is given by the atmospheric scale, $m_\nu = 0.05$ eV. The moduli $v_L |f_\pm|$ of the solutions of eq. (3.1) as functions of $|d|$ are then shown in fig. 1. The three regimes (i), (ii) and (iii) correspond to the left, central and right regions of the plot, respectively. The value of $|f_+| (|f_-|)$ is a slightly decreasing (increasing) function of $\arg d = \chi$ between 0 and $\pi$, as shown in the figure. For $\chi = \pi$ and $|d| \geq 1$ the moduli of the two solutions coincide.

\footnote{Note that in general CP violation is present in the one-generation case due to the presence of the Higgs triplets. Although it does not manifest itself at low energies, it can lead to a successful leptogenesis (more on this in section 5.2).}
The mass of the RH neutrino is given by

$$M_{R\pm} = v_R |f_\pm| = \frac{v^2}{m_\nu} \left| \frac{2y^2(1 \pm (1 + d)^{1/2})}{d} \right|.$$ (3.6)

In fig. 2 we plot $M_R$ as a function of $|d|$ for $m_\nu = 0.05$ eV and $|y| = 1$. The mass of the scalar triplet $M_\Delta$ is not directly related to the parameters in the seesaw formula. One expects $M_\Delta \sim v^2/v_L \sim v_R$, but very different values are also naturally possible, depending on the details of the mechanism which induces $v_L$.

Consider now the case of $n$ lepton generations with no mixing (which means that $f$ and $y$ are diagonal in the same basis). This case can be described as $n$ replicas of the one-generation case, each with two solutions $f_{\pm}^{(i)}$ ($i = 1, \ldots, n$). For each generation, the relative size of type I and II contribution to the light neutrino mass $m_{\nu_i}$ depends on the dominance parameter

$$d_i \equiv \frac{v_L}{v_R} \frac{4v^2y_i^2}{m_{\nu_i}^2}, \quad i = 1, \ldots, n.$$ (3.7)

For a given value of $v_L/v_R$, one can have $|d_i| \ll 1$ (single type dominance) for an $i$-th generation, while at the same time the hybrid seesaw or cancellation regimes may be realized for $j \neq i$. We will see that in the presence of flavor mixing it is much less trivial to assess the relative size of the two seesaw types.

### 3.2 The case of two lepton generations

In this case in the flavor basis, where the mass matrix of charged leptons is diagonal (say,
m_i = \text{diag}(m_\mu, m_\tau), \) the seesaw formula (3.3) can be explicitly written as

\[
\begin{pmatrix}
  m_{\mu\mu} & m_{\mu\tau} \\
  m_{\mu\tau} & m_{\tau\tau}
\end{pmatrix}
= \begin{pmatrix}
  f_{22} & f_{23} \\
  f_{23} & f_{33}
\end{pmatrix} - \frac{1}{xF} \begin{pmatrix}
  y_{\mu2} & y_{\mu3} \\
  y_{\tau2} & y_{\tau3}
\end{pmatrix} \begin{pmatrix}
  f_{33} - f_{23} \\
  -f_{23} & f_{22}
\end{pmatrix} \begin{pmatrix}
  y_{\mu2} & y_{\tau2} \\
  y_{\mu3} & y_{\tau3}
\end{pmatrix},
\]

(3.8)

where \( F \equiv \text{det} f = f_{22}f_{33} - f_{23}^2. \) The matrix equation (3.8) is equivalent to the system of three coupled non-linear equations for \( f_{22}, f_{33}, \) and \( f_{23}: \)

\[
\begin{align*}
XF(f_{22} - m_{\mu\mu}) &= f_{33}y_{\mu2}^2 - 2f_{23}y_{\mu2}y_{\mu3} + f_{22}y_{\mu3}^2, \\
XF(f_{23} - m_{\mu\tau}) &= f_{33}y_{\mu2}y_{\tau2} - f_{23}(y_{\mu3}y_{\tau2} + y_{\mu2}y_{\tau3}) + f_{22}y_{\mu3}y_{\tau3}, \\
XF(f_{33} - m_{\tau\tau}) &= f_{33}y_{\tau2}^2 - 2f_{23}y_{\tau2}y_{\tau3} + f_{22}y_{\tau3}^2.
\end{align*}
\]

(3.9)

To solve the system (3.9), we use the following procedure. Let us define \( f' = f/\sqrt{\lambda}, \) \( m' = m/\sqrt{\lambda} \) and \( y' = y/\sqrt{\lambda}, \) where \( \lambda \) is an as yet arbitrary complex number. The scaling law was chosen in such a way that in terms of the primed variables the system of equations for \( f'_{ij} \) has the same form as eq. (3.9). Next, we fix the value of \( \lambda \) by requiring \( F' \equiv \text{det} f' = 1. \) The system of equations for \( f'_{ij} \) then becomes linear and can be readily solved. Expressing the primed variables back through the unprimed ones and substituting them into the condition \( F'(\lambda) = 1, \) one obtains a 4th order polynomial equation for \( \lambda. \) In general, it has four complex solutions \( \lambda_i \) \( (i = 1, \ldots, 4), \) leading to four allowed matrix structures \( f_i. \)

Notice that \( F_i \equiv \text{det} f_i = \lambda_i. \) This procedure proves that eq. (3.8) has four solutions for a generic structure of \( y \) and, therefore, it generalizes (for the case \( n = 2 \)) the proof given in section 2.3 for symmetric \( y. \)

We now present the general analytic solution in the case where the matrix \( y \) is invertible and symmetric, so that the duality holds. As follows from eq. (3.8), when \( y = y^T \) we can choose the basis where \( y \) is diagonal: \( y_{\mu3} = y_{\tau2} = 0, \) \( y_{\mu2} \equiv y_2, \) \( y_{\tau3} \equiv y_3. \) With a little abuse of notation, we will still denote the matrix elements of \( m \) and \( f \) in the new basis as \( m_{\alpha\beta} \) and \( f_{ij}. \) The system (3.9) can be linearized as described above and is easily solved:

\[
f = \frac{x\lambda}{(x\lambda)^2 - y_2^2y_3^2} \begin{pmatrix}
  x\lambda m_{\mu\mu} + y_2^2m_{\mu\tau} & m_{\mu\tau}(x\lambda - y_2y_3) \\
  \cdots & x\lambda m_{\tau\tau} + y_2^2m_{\mu\mu}
\end{pmatrix},
\]

(3.10)

where \( \lambda \) is a solution the following quartic equation:

\[
[(x\lambda)^2 - y_2^2y_3^2]^2 - x[\text{det} m(x\lambda - y_2y_3)^2x\lambda + (m_{\mu\mu}y_3 + m_{\tau\tau}y_2)^2(x\lambda)^2] = 0.
\]

(3.11)

Taking the determinant of the equality \( \hat{f} \equiv m - f = -yf^{-1}y/x, \) one obtains

\[
x^2\lambda\hat{\lambda} \equiv x^2F \cdot \hat{F} = y_2^2y_3^2,
\]

(3.12)

where \( \hat{F} \equiv \text{det} \hat{f}. \) With the help of this relation, it is straightforward to check that the four solutions \( f_i \) defined by eqs. (3.10) and (3.11) form two dual pairs. Actually, the duality makes it easy to express the four solutions of eq. (3.11) in the closed form as follows:

\[
x\lambda_i = \frac{1}{4} \left[ x\text{det} m + r_\pm \pm \sqrt{2(x\text{det} m)^2 + 4kx + 2r_\pm x\text{det} m} \right],
\]

(3.13)
where

\[ k = m_{\mu\mu}^2 y_3^2 + 2m_{\mu\tau}^2 y_2 y_3 + m_{\tau\tau}^2 y_2^2, \quad r_\pm = \pm \sqrt{(x \det m)^2 + 4kx + 16y_2^2 y_3^2}. \quad (3.14) \]

One pair of dual solutions corresponds to \( r_+ \) and the other one to \( r_- \); within each pair, a solution is distinguished from its dual by the sign in front of the radical in eq. (3.13). Summarizing, eqs. (3.10) and (3.13) define explicitly the four solutions \( f_i \) as functions of the input parameters \( m = m_\nu/v_L, x = v_{LR}/v^2 \) and \( y \).

We can now study the structure of \( f_i \) in any regions of parameters of physical interest. To start with, let us consider the limit \( y_2 \to 0 \). This can be justified if neutrino Yukawa couplings are related to those of charged fermions, so that \( y_2 \ll y_3 \lesssim 1 \). Then eq. (3.13) can be expanded as

\[ x\lambda_{1,3} = \frac{x \det m + r_0^\pm}{2} + \frac{2xy_3 m_{\mu\tau}^2}{r_0^\pm} y_2 + O(y_2^2), \quad x\lambda_{2,4} \equiv x\hat{\lambda}_{1,3} = \frac{y_2^2 y_3^2}{x\lambda_{1,3}}. \quad (3.15) \]

where \( r_0^\pm = \pm \sqrt{(x \det m)^2 + 4xy_2^2 m_{\mu\mu}^2} \). The solutions for \( f \) take the form

\[ f_{1,3} = m - f_{2,4}, \quad f_{2,4} = \begin{pmatrix} 0 & \frac{y_2 y_3 m_{\mu\tau}}{x\lambda_{1,3}} \\ \frac{y_3^2 m_{\mu\mu}}{x\lambda_{1,3}} & \cdots & \frac{y_3^2 m_{\mu\mu}}{x\lambda_{1,3}} \end{pmatrix} + O(y_2^2). \quad (3.16) \]

Since \( \lambda_{2,4} = \det f_{2,4} \) are proportional to \( y_3^2 \), in the case of the solutions \( f_{2,4} \) one RH neutrino mass becomes much smaller than \( v_R \) for very small \( y_2 \). Notice that, even though \( \lambda_{2,4} \) go to zero for \( y_2 \to 0 \), the matrices \( f_{2,4} \) are finite and invertible (and therefore acceptable solutions) for any \( y_2 \neq 0 \).

Let us define, in analogy with eq. (3.2), a dominance parameter that controls the relative size of type I and type II seesaw contributions:

\[ d \equiv \frac{v_L}{v_R} \frac{4y_3^2}{(m_\nu)_{\tau\tau}^2} = \frac{4y_3^2}{x m_{\tau\tau}^2}. \quad (3.17) \]

Obviously, other analogous parameters can be defined by replacing \( m_{\tau\tau} \) and/or \( y_3 \) with the other entries of \( m \) and \( y \). A detailed discussion of this issue is postponed to Appendix B; here we just study the main features of the dependence of the solutions on \( d \). Fig. 3 shows the dependence of the values of \( |x\lambda_i| \) on \( |d| \) for a specific realistic set of the input parameters.

In the region \( |d| \ll 1 \), there is a solution \( |x\lambda_1| \gg |y_i y_j| \), which leads to \( f_1 \simeq m \) (dominant type II seesaw), while for the dual solution, one has \( |x\lambda_2| \ll |y_i y_j| \) (dominant type I seesaw). In general, the other pair of dual solutions corresponds to hybrid seesaw. Only if the value of \( \det m \) is strongly suppressed, one finds \( f_3 \simeq f_1 \) (dominant type II seesaw) and \( f_4 \simeq f_2 \) (dominant type I seesaw). In the region \( |d| \sim 1 \), all four solutions are of hybrid type. Finally, for \( |d| \gg 1 \) a cancellation between type I and II seesaw contributions to \( m_\nu \) occurs.
Figure 3: The four solutions $x\lambda_i$ of eq. (3.13) as functions of $|d| \equiv 4y_3^2/(xm_{\tau\tau}^2)$. We chose $m_{\mu\mu} = m_{\tau\tau} = (0.02 \text{ eV})/v_L$ and $m_{\mu\tau} = (0.03 \text{ eV})/v_L$, so that $\theta_{23} = \pi/4$ and $\Delta m_{23}^2 = 2.4 \cdot 10^{-3}$ eV. We also took $y_3 = 1$ and $y_2 = 10^{-2}$.

Figure 4: The masses $M_{2,3}$ of the RH neutrinos as functions of $|d|$ for the same set of input parameters as in fig. 3. Thick solid (dashed) curves in the left panel correspond to the solution $f_1$ ($f_2$), thin solid (dashed) curves in the right panel, to the solution $f_3$ ($f_4$).

In fig. 3 we plot the values $M_{2,3}$ of the two RH neutrino masses (that is, the eigenvalues of $M_R = v_R f$) versus $|d|$ for the same set of the input parameters as in fig. 3. In order to assess the size of $M_{2,3}$, it is useful to define a “seesaw scale” $M_s \equiv v^2/m_3$, which for $m_3 = 0.05 \text{ eV}$ is equal to $6 \cdot 10^{14} \text{ GeV}$. Since for $|d| \ll 1$ the solution $f_1$ ($f_2$) corresponds to type II (I) seesaw dominance, one has $M_{2,3} \gg M_s$ for $f_1$ and $M_{2,3} \sim y_{2,3}^2 M_s$ for $f_2$, as shown in the left panel of fig. 4. For the solutions $f_3, f_4$, the situation is intermediate and one finds $M_3 \gg M_s$, $M_2 \sim y_{2,3}^2 M_s$, as shown in the right panel of fig. 4. For $|d| \gg 1$, the asymptotic values of $M_2$ are the same for all four solutions; the same is also true for $M_3$, with $M_2/M_3 \sim y_2/y_3$. For the solution $f_1$, one observes a level-crossing of the two RH neutrino masses for $|d| \sim 20$. One could further characterize the sector of RH neutrinos by
studying the dependence of their mixing on \( d \).

### 3.3 The case of three lepton generations

In this case, the symmetric matrix \( f \) contains 6 independent elements \( f_{ij} \). The LR symmetric seesaw relation (3.3) can be written as a system of 6 nonlinear coupled equations for these elements, analogous to that in eq. (3.9). Here we solve this system analytically for the case of a symmetric neutrino Yukawa coupling matrix \( y \), for which the duality holds. Using the freedom to work in the basis where \( y \) is diagonal, \( y = \text{diag}(y_1, y_2, y_3) \), one can write this system as

\[
x F'(f_{ij} - m_{ij}) = y_i y_j F_{ij},
\]

where \( F \equiv \text{det} f \) and

\[
F_{ij} = \frac{1}{2} \epsilon_{ikl} \epsilon_{jmn} f_{km} f_{ln}.
\]

The six corresponding equations for the dual matrix \( \hat{f} \equiv m - f \) are

\[
x \hat{F}(\hat{f}_{ij} - m_{ij}) = -x \hat{F} f_{ij} = y_i y_j \hat{F}_{ij},
\]

where \( \hat{F} \equiv \text{det} \hat{f} \) and

\[
\hat{F}_{ij} = \frac{1}{2} \epsilon_{ikl} \epsilon_{jmn} \hat{f}_{km} \hat{f}_{ln} = F_{ij} - T_{ij} + M_{ij},
\]

\[
T_{ij} = \epsilon_{ikl} \epsilon_{jmn} f_{km} m_{ln}, \quad M_{ij} = \frac{1}{2} \epsilon_{ikl} \epsilon_{jmn} m_{km} m_{ln}.
\]

Taking the determinant of the equality \( \hat{f} = -y f^{-1} y / x \), one proves immediately that

\[
x^3 F \hat{F} = -y^2 y_1^2 y_3^2,
\]

which is an analogue of eq. (3.12).

To solve the system (3.18) for \( f \), we make use of a procedure similar to that employed in the 2-generation case. Let us define \( f' = f / \lambda^{1/3} \), \( m' = m / \lambda^{1/3} \) and \( y' = y / \lambda^{1/3} \) where \( \lambda \) is determined from the condition \( F'(\lambda) \equiv \text{det} f'(\lambda) = 1 \). At this point the left-hand (right-hand) side of equations (3.18), written in terms of the primed variables, becomes linear (quadratic) in \( f'_{ij} \). Using the duality (eqs. (3.20)-(3.22)), the quadratic terms, \( F'_{ij} \), can be rewritten as

\[
F'_{ij} = \frac{(y_1 y_2 y_3)^2}{x^2 y_i y_j} f''_{ij} - M'_{ij} + T'_{ij}.
\]

This allows one to linearize the system for \( f''_{ij} \):

\[
[x^3 - (y_1 y_2 y_3)^2 f'_{ij} - x^3 m'_{ij} = x^2 y_i y_j (T'_{ij} - M'_{ij}).
\]

After some algebra, the solution for \( f = \lambda^{1/3} f' \) can be written in a rather compact form (to be compared with eq. (3.10)):

\[
f_{ij} = \lambda^2 \left[ (\lambda^2 - Y^2)^2 - Y^2 \lambda^2 \text{det} m + Y^4 S \right] m_{ij} + \lambda \left( \lambda^4 - Y^4 \right) A_{ij} - Y^2 \lambda^2 (\lambda^2 + Y^2) S_{ij} \]

\[
(\lambda^2 - Y^2)^3 - Y^2 \lambda^2 (\lambda^2 - Y^2) S - 2 Y^2 \lambda^3 \text{det} m,
\]

\[
(3.25)
\]
where
\[
Y^2 \equiv \frac{(y_1 y_2 y_3)^2}{x^3}, \quad S \equiv \sum_{k,l=1}^{3} \left( \frac{m_{kl} x}{y_k y_l} \right), \quad A_{ij} \equiv \frac{y_i y_j M_{ij}}{x}, \quad S_{ij} \equiv \sum_{k,l=1}^{3} \left( m_{ik} m_{jl} \frac{m_{kl} x}{y_k y_l} \right).
\]

(3.26)

The value of \( \lambda \) is determined from the equation
\[
F(\lambda) \equiv \det f(\lambda) = \lambda,
\]
where \( f(\lambda) \) was defined in eq. (3.25). Eq. (3.27) turns out to be an 8th order polynomial equation for \( \lambda \),\(^7\) which has in general eight complex solutions. Defining
\[
A \equiv \sum_{k,l=1}^{3} \left( \frac{y_k y_l M_{kl}^2}{x} \right),
\]
(3.28)

this equation reads
\[
\left[ (\lambda^2 - Y^2)^2 - Y^2 \lambda^2 S \right]^2 - \lambda^2 (\lambda^2 + Y^2)^2 A - Y^2 \lambda^4 (\det m)^2
- \lambda \left[ \lambda^6 + Y^2 \lambda^2 (\lambda^2 - Y^2) (5 + S) - Y^6 \right] \det m = 0,
\]
(3.29)
to be compared with eq. (3.11). The eight solutions of eq. (3.29) form four dual pairs \( \lambda_i, \hat{\lambda}_i \) \((i = 1, \ldots, 4)\). The duality is described by the relation \( \lambda_i \hat{\lambda}_i = -Y^2 \), which follows from eq. (3.22). As a consequence, eq. (3.29) can be rewritten as
\[
0 = \prod_{i=1}^{4} (\lambda - \lambda_i) \left( \lambda + \frac{Y^2}{\lambda_i} \right) = \prod_{i=1}^{4} (\lambda^2 - z_i \lambda - Y^2), \quad z_i \equiv \lambda_i - \frac{Y^2}{\lambda_i}.
\]
(3.30)

By comparing eqs. (3.29) and (3.30), it is easy to verify that \( z_i \) are the roots of the following quartic equation:
\[
z^4 - \det m z^3 - (2Y^2 S + A)z^2 - Y^2 (8 + S) \det m z + Y^2 [Y^2 S^2 - 4A - (\det m)^2] = 0.
\]
(3.31)

Thus, although the general order 8 algebraic equation does not admit analytic solutions, the equation for \( \lambda \) can be solved in radicals since, due to duality, it reduces to a quartic equation in \( z \). For the latter a general solution in radicals is known, though complicated. The 8 solutions for \( f \) are obtained by plugging into eq. (3.25) the values of \( \lambda \) given by
\[
\lambda_i(\hat{\lambda}_i) = \frac{z_i \pm \sqrt{z_i^2 + 4Y^2}}{2}, \quad (i = 1, \ldots, 4),
\]
(3.32)
where the sign in front of the radical distinguishes between \( \lambda_i \) and \( \hat{\lambda}_i \).

\(^7\)To recognize that, one has to use duality relations to identify and cancel a common polynomial factor in the numerator and denominator of \( F(\lambda) \).
Given the complicated algebraic form of the solutions for $f$, it may seem a hopeless task to quantify the dominance of one or the other seesaw type in the eight cases. However, it turns out that the techniques developed for analyzing this issue in the one- and two-generation cases can be generalized here. The full details of the analysis will be given in Appendix [3]. Here we only describe the generic case, in which the relative size of type I and type II contributions does not change much from one element of the matrix $m$ to another and, in addition, no special cancellations among the entries of $m$, such as leading to $|\det m| \ll |m_{\alpha \beta}|^3$, occur. Then, in terms of the dominance parameter $|d|$ defined in eq. (3.17), the classification goes as follows: for $|d| \ll 1$ there is one pair of dual solution with one seesaw type dominance, while the other three pairs correspond to hybrid seesaw. All three RH neutrino masses are generically larger than $v^2/m_i$ in the solution with type II dominance, only two of them are larger than $v^2/m_i$ in three of the hybrid solutions, only one mass satisfies this condition in the corresponding three duals, and finally all RH neutrino masses are of the order of $v^2/m_i$ in the case of dominant type I seesaw. When $|d| \gtrsim 1$, all 8 solutions for $f$ lead to hybrid seesaw.

In the next two subsections we will specialize our general analytic solution to some physically interesting limits. The reader more interested in numerical examples may proceed directly to section 4.

3.3.1 Limit of hierarchical neutrino Yukawa couplings

A considerable simplification of the general solution for $f$ occurs when type I contributions to the elements of $m_\nu$, proportional to $y_1$, are negligible, i.e. in the limit $y_1 \ll 1$. In fact, this limit is physically well motivated, in view of the tininess of the Yukawa couplings of the charged fermions of first generation. Strictly speaking, for $y_1 = 0$ the matrix $y$ is not invertible, so that the duality of solutions for $f$ does not hold. Nonetheless, eq. (3.23) is valid for any small but non-zero value of $y_1$ and therefore can be expanded in powers of $y_1$. Assuming that $\lambda$ is finite in the limit $y_1 \to 0$, one finds

$$f = \begin{pmatrix}
    m_{ee} & m_{e\mu} & m_{e\tau} \\
    m_{\mu\nu} + \frac{y_2^2}{x\lambda} (M_{22} - \frac{y_3^2 m_{ee} m_{e\mu}}{x\lambda}) & m_{\mu\tau} + \frac{y_2 y_3 m_{e\mu} m_{e\tau}}{x\lambda} & \frac{M_{33} - \frac{y_2 y_3 m_{ee} m_{e\tau}}{x\lambda}}{1 - \frac{y_2^2 y_3^2 m_{ee}}{(x\lambda)^2}} \\
    \ldots & \ldots & \ldots \\
    \end{pmatrix} + O(y_1).$$

The $e$-row of the mass matrix of light neutrinos $m$ directly determines the first row of $f$ (pure type II seesaw), whereas the $2-3$ sector of $f$ depends on the interplay of the elements

$$f = \begin{pmatrix}
    m_{ee} & m_{e\mu} & m_{e\tau} \\
    m_{\mu\mu} + \frac{y_2^2}{x\lambda} (M_{22} - \frac{y_3^2 m_{ee} m_{e\mu}}{x\lambda}) & m_{\mu\tau} + \frac{y_2 y_3 m_{e\mu} m_{e\tau}}{x\lambda} & \frac{M_{33} - \frac{y_2 y_3 m_{ee} m_{e\tau}}{x\lambda}}{1 - \frac{y_2^2 y_3^2 m_{ee}}{(x\lambda)^2}} \\
    \ldots & \ldots & \ldots \\
    \end{pmatrix} + O(y_1).$$

(3.33)
of the $e$-row and $\mu\tau$-block of $m$, as well as on the values of $y_2$, $y_3$ and $x\lambda$.

Consider now the behavior of $\lambda$ in the limit $y_1 \to 0$. For $y_1 = 0$, the 8th order polynomial equation (3.29) reduces to

$$\lambda^4 \left\{ (x\lambda)^2 - m_{ee}^2 y_2^2 y_3^2 x^2 - x \left[ \det (x\lambda - m_{ee} y_2 y_3)^2 x\lambda + (M_{22} y_2 + M_{33} y_3)^2 (x\lambda)^2 \right] \right\} = 0. \quad (3.34)$$

The zeros of the term in curly brackets in eq. (3.34) are (to leading order) the four solutions $\lambda$ which are finite for $y_1 \to 0$. The four duals $\hat{\lambda} = -y_2 y_3^2 m_{ee}/(x^2\lambda)$ vanish in this limit, so that for them eq. (3.33) does not hold. However, the corresponding solutions can be obtained by duality: $\hat{\lambda} = m - \tilde{f}$ with $\tilde{f}$ given in eq. (3.33). By expanding directly eq. (3.25), it is easy to check that $\hat{f}_{11} \sim y_2^2$, $\hat{f}_{12,13} \sim y_1$, while the other entries are finite.

As a consequence, one RH neutrino mass is much smaller than $v_R$ since it is proportional to $y_2^2$.

When $y_1 = 0$, one has only four (instead of eight) solutions, since their duals become singular. From the comparison of the term in curly brackets in eq. (3.34) with eq. (3.11), a strong analogy with the pure two-generation case becomes evident. In fact, a different duality among the four remaining solutions is present: if $\lambda \neq 0$ satisfies eq. (3.34), also $\tilde{\lambda} \equiv y_2^2 y_3^2 m_{ee}/(x^2\lambda)$ does, and it corresponds to $\tilde{\lambda} = \bar{m} - f$, where $\bar{m}_{\alpha\beta} = m_{\alpha\beta} + m_{e\alpha} m_{e\beta}/m_{ee}$. There are two pairs of such solutions.

A very simple yet non-trivial scenario corresponds to the case when type I contributions to $m_\nu$ proportional to $y_2$ are also negligible ($y_2 \to 0$). In this limit the relation $f_{\alpha\beta} = m_{\alpha\beta}$ (pure type II seesaw) holds for all the entries of $f$ but $f_{33}$ [20, 22]. There are two solutions for $f_{33}$, defined by

$$\left( f_{33} - m_{\tau\tau} \right)^2 + \frac{\det m}{M_{33}} (f_{33} - m_{\tau\tau}) - \frac{y_3^2}{x} = 0. \quad (3.35)$$

In particular, this equation admits $|f_{33}| \gg |m_{\tau\tau}| \approx |m_{\mu\mu}|$, that is, $f$ with hierarchical structure can lead to $m_\nu$ with large $2-3$ mixing.

### 3.3.2 Limit of hierarchical light neutrino masses

When the mass spectrum of light neutrinos has normal (inverted) ordering, the lightest neutrino mass $m_1$ ($m_3$) can be negligibly small. In this case the determinant of $m \equiv m_\nu/v_L$ vanishes. For $\det m = 0$ (more generically, for $|\det m| \ll |Y|$), eq. (3.29) reduces to a pair of quartic equations:

$$(\lambda^2 - Y^2)^2 - [\pm \sqrt{A} \lambda (\lambda^2 + Y^2) + \lambda^2 S \lambda^2] = 0. \quad (3.36)$$

If $\lambda_i$ is a solution of the “sign +” equation, then also $Y^2/\lambda_i$ is, while $-\lambda_i$ and $-Y^2/\lambda_i$ are solutions of the “sign −” equation. Therefore, two pairs of dual solutions for $\lambda$ are equal in absolute value and opposite in sign to the other two pairs. Moreover, eq. (3.36) has the
same form as eq. (3.11), if one makes the identifications
\[ Y^2 \leftrightarrow \frac{y_2^2 y_3^2}{x^2}, \quad \pm \sqrt{A} \leftrightarrow \det m, \quad S \leftrightarrow \frac{xk}{y_2 y_3}, \] (3.37)
where the quantities on the right hand sides refer to the two-generation case. Therefore, the quartic equations (3.36) can be explicitly solved and analyzed as in section 3.2.

Since the solar mass squared difference is much smaller than the atmospheric one, in the case of the normal hierarchy one can neglect, in first approximation, both \( m_1 \) and \( m_2 \). In this limit \( m_\nu \) becomes a rank-1 matrix, and one can write
\[ m = \begin{pmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{pmatrix}. \] (3.38)
Almost maximal atmospheric mixing implies \( b \sim c \) as well as \( \theta_{13} \sim a/b \). The solar mass scale is zero in the limit of eq. (3.38), so that the 1-2 mixing is undefined. Notice that plugging eq. (3.38) in eq. (3.26), one finds \( A_{ij} = 0 \) and \( S_{ij} = m_{ij}S \). As a consequence, the expression for \( f \) given in eq. (3.25) becomes singular in this limit, since \( f \propto m \) and thus is not invertible. One therefore has to resort to a different approach in order to find \( f \). It turns out that in this limit the seesaw equation (3.18) has an infinite number of solutions. To illustrate this phenomenon, consider for simplicity the case \( y_1 = 0 \). Eqs. (3.33) and (3.34) cannot be used since they would yield \( f = m \), an unacceptable result when the matrix \( m \) is not invertible. A direct solution of eq. (3.18) gives in this case
\[ f = \begin{pmatrix} a^2 & ab & ac \\ ab & b^2 + y_2 \cos \alpha & bc + \sqrt{y_2 y_3} \sin \alpha \\ ac & bc + \sqrt{y_2 y_3} \sin \alpha & c^2 - y_3 \cos \alpha \end{pmatrix}, \] (3.39)
where \( \alpha \) is an arbitrary complex number different from zero. (For \( \alpha = 0 \) a different ambiguity appears:
\[ f = \begin{pmatrix} a^2 & ab & ac \\ ab & b^2 \pm y_2 & bc \\ ac & bc & c^2 \pm y_3 \end{pmatrix}, \] (3.40)
where the two “\( \pm \)” signs are uncorrelated so that, for this value of \( \alpha \), there are four solutions.) Summarizing, when the matrix \( m \) is of rank 1 there is an infinite number of solutions for \( f \).

4. Numerical examples

We have shown that, for arbitrary values of \( y_{1,2,3}, v_{L,R} \) and of the elements of \( m_\nu \), one generically finds eight solutions for the matrix \( f \), defined by eqs. (3.25), (3.32) and (3.31).
The rounding off in the numerical values of\( f \) matrix structure of the solutions. We have checked that for the high precision solutions of (4.1) no CP violation and \( \Delta m_{sol}^2/\Delta m_{atm}^2 \approx 0.031 \), in agreement with all the current data. The eigenvalues of \( m \) are \((-0.1, 0.2, 1)\), so that the spectrum has the normal hierarchy and the lightest neutrino has CP parity opposite to that of the other two. The above choice of \( m \) fixes also the value \( v_L \approx \sqrt{\Delta m_{atm}^2} \approx 0.05 \text{ eV} \) (see eq. (2.2)). We also take \( x \equiv v_L v_R/v^2 = 1 \), which is a natural value if the dimensionless parameters in the scalar potential are all of order one. This determines \( v_R \approx 6 \cdot 10^{14} \text{ GeV} \). Finally, we take the hierarchy among the eigenvalues of \( y \) to be slightly weaker than that for Yukawa couplings of the charged fermions: \( y_1 = 10^{-2} \), \( y_2 = 10^{-1} \), \( y_3 = 1 \). We neglect possible (CKM-like) rotations between the flavor basis and the basis where \( y \) is diagonal.

With these choices, the solutions of the LR seesaw formula are

\[
\begin{align*}
f_1 & \approx \begin{pmatrix}
-0.001 & 0.10 & -0.14 \\
\ldots & 0.56 & 0.49 \\
\ldots & \ldots & 0.88
\end{pmatrix}, & \quad \hat{f}_1 & \approx \begin{pmatrix}
0.001 & -0.005 & 0.04 \\
\ldots & -0.006 & -0.04 \\
\ldots & \ldots & -0.33
\end{pmatrix}, \\
f_2 & \approx \begin{pmatrix}
-0.01 & 0.11 & -0.04 \\
\ldots & 0.55 & 0.44 \\
\ldots & \ldots & -0.88
\end{pmatrix}, & \quad \hat{f}_2 & \approx \begin{pmatrix}
0.006 & -0.008 & -0.06 \\
\ldots & -0.004 & 0.01 \\
\ldots & \ldots & 1.44
\end{pmatrix}, \\
f_3 & \approx \begin{pmatrix}
0.02 & 0.07 & -0.02 \\
\ldots & 0.61 & 0.30 \\
\ldots & \ldots & 1.58
\end{pmatrix}, & \quad \hat{f}_3 & \approx \begin{pmatrix}
-0.02 & 0.03 & -0.08 \\
\ldots & -0.06 & 0.15 \\
\ldots & \ldots & -1.03
\end{pmatrix}, \\
f_4 & \approx \begin{pmatrix}
0.01 & 0.08 & 0.08 \\
\ldots & 0.60 & 0.25 \\
\ldots & \ldots & -0.19
\end{pmatrix}, & \quad \hat{f}_4 & \approx \begin{pmatrix}
-0.01 & 0.02 & -0.18 \\
\ldots & -0.05 & 0.20 \\
\ldots & \ldots & 0.74
\end{pmatrix}.
\end{align*}
\]  

The rounding off in the numerical values of \( f_{ij} \) is chosen so as to clearly illustrate the matrix structure of the solutions. We have checked that for the high precision solutions the duality relation \( f_i + \hat{f}_i = m \) is satisfied very accurately for each dual pair. The structure of \( m \) is recovered by plugging \( f_i \) or \( \hat{f}_i \) back into eq. (4.3), but very accurate values of
their entries (not shown in eq. 4.2) should be used. This requirement of high accuracy is a consequence of the strong hierarchy among $y_i$ and exactly vanishing $ee$-entry of $m$.

For the solutions $f_i$ (the first column in eq. (4.2)) the dominant $\mu\tau$-block of $m$ is reflected in a dominant 23-block of $f$. The solutions $\hat{f}_i$ (the second column in eq. (4.2)) exhibit a hierarchical structure, with a dominant 33-entry; the maximal 2-3 mixing angle in $m$ is generated from a small mixing angle in $\hat{f}_i$. All the solutions $f_i$ and $\hat{f}_i$ have one order one eigenvalue, so that for the heaviest RH neutrino mass one finds $M_3 \sim v_R$. The other two eigenvalues can be as small as $\sim 10^{-3}$. Type II (I) seesaw contributions dominate the entries $m_{e\mu}$, $m_{\mu\mu}$ and $m_{\mu\tau}$ for the solutions $f_{1,2}$ ($\hat{f}_{1,2}$). The other entries receive significant contributions from both seesaw types (hybrid seesaw). This is also true for all the entries of the dual pairs of solutions $f_3$, $\hat{f}_3$ and $f_4$, $\hat{f}_4$.

2. The largest uncertainty in the structure of $m_\nu$ is due to the unknown absolute mass scale of the light neutrinos. Therefore, it is interesting to study the dependence of the structure of $f$ on this scale. Let us consider the same set of input parameters as above, but change the eigenvalues of $m_\nu$ as follows: $v_L(0.1,0.2,1) \rightarrow (-m_1, \sqrt{m_1^2 + \Delta m^2_{sol}}, \sqrt{m_1^2 + \Delta m^2_{atm}})$. The solutions for $f$ will now depend on $m_1$, which may vary between zero and $\sim 0.23$ eV (due to the cosmological upper bound $\sum_i m_i \lesssim 0.7$ eV). The structures of $f$ given in eq. (4.2) correspond to $m_1 = 0.1 v_L \approx 0.005$ eV. In fig. 5 we plot the masses $M_{1,2,3}$ of the RH neutrinos, given by the eigenvalues of $M_R \equiv v_R f$, as functions of $m_1$. We chose for illustration the first and the third pairs of dual solutions.8

For the solution $f_1$, the mass spectrum of RH neutrinos spans at most one order of magnitude, all three masses being rather close to $v_R$. The two lightest RH neutrinos undergo a level crossing at $m_1 \approx 0.035$ eV. Such crossing points were already identified in [23] for the case of pure type I seesaw. In these points the lepton asymmetry generated in the decays of the RH neutrinos can be resonantly enhanced, which may be crucial for reproducing the observed value of the baryon asymmetry of the universe [23]. For $m_1 \gtrsim 0.1$ eV (quasi-degenerate light neutrinos), the three RH neutrino masses also become quasi-degenerate, since $f_1$ becomes a solution with dominant type II seesaw, so that $f_1 \approx m$.

The dual solution $\hat{f}_1$ has a substantially different spectrum. The RH neutrino masses span about three orders of magnitude, and two of them can be much smaller than $v_R$. Level crossings occur between the two heaviest and between the two lightest RH neutrinos. For $m_1 \gtrsim 0.1$ eV, by duality $\hat{f}_1$ corresponds to the dominant type I seesaw. The masses $M_i$ of RH neutrinos then scale approximately quadratically with the Dirac-type Yukawa couplings $y_i$.

For the solution $f_3$, two RH neutrino masses are close to $v_R$ for all values of $m_1$, while the lightest RH neutrino mass $M_1$ decreases by almost three orders of magnitude when $m_1$
Figure 5: The masses $M_{1,2,3}$ of the three RH neutrinos versus the light neutrino mass scale $m_1$. In the upper (lower) panel solid curves correspond to the solution $f_1$ ($\hat{f}_3$) and dashed curves to the dual solution $\hat{f}_1$ ($\hat{f}_3$). We chose the tri-bi-maximal mixing, the light neutrino mass spectrum $(-m_1, \sqrt{m_1^2 + \Delta m^2_{sol}}, \sqrt{m_1^2 + \Delta m^2_{atm}})$, $v_L \approx 0.051$ eV, $v_R \approx 5.9 \cdot 10^{14}$ GeV and $y_{1,2,3} = 10^{-2,-1,0}$. The irregularities around $m_1 = 0.011$ eV in the lower panel are a numerical artifact.

increases in its allowed range. For the dual solution, the RH neutrino masses lie between $10^{15}$ and $10^{12}$ GeV. This dual pair corresponds to hybrid seesaw for all values of $m_1$.

3. The ordering (normal or inverted) is another important information about the light neutrino mass spectrum that is presently missing. Let us consider the input parameters chosen in example 1, but replace the normally ordered spectrum of $m$ with an inverted one: $(-0.1, 0.2, 1) \rightarrow (-1 + \epsilon, 1 + \epsilon, \epsilon)$, where $\epsilon \equiv (1/4)\Delta m^2_{sol}/\Delta m^2_{atm} \lesssim 0.01$. This amounts to replace eq. (4.1) by

$$m = \frac{1}{6} \begin{pmatrix} -2 & 4 & -4 \\ 4 & 1 & -1 \\ -4 & -1 & 1 \end{pmatrix} + \epsilon \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (4.3)$$
The masses $M_{1,2,3}$ of the three RH neutrinos versus the light neutrino mass scale $m_3$. We chose the same input parameters as in fig. 5, but the inverted light neutrino mass spectrum ($-\sqrt{m_3^2 + \Delta m_{atm}^2 - \Delta m_{sol}^2/2}$, $\sqrt{m_3^2 + \Delta m_{atm}^2 + \Delta m_{sol}^2/2}$, $m_3$). In the left (right) panel, the solid curves correspond to the solution $f_1$ ($\hat{f}_1$) and the dashed curves to its dual solution $\hat{f}_1$ ($f_3$).

Then, one pair of dual solutions is given by

$$f_1 \approx \begin{pmatrix} -0.33 & 0.67 & -0.67 \\ ... & 0.19 & -0.08 \\ ... & ... & 1.08 \end{pmatrix}, \quad \hat{f}_1 \approx \begin{pmatrix} 4 \cdot 10^{-5} & -0.001 & 0.001 \\ ... & -0.01 & -0.08 \\ ... & ... & -0.91 \end{pmatrix}, \quad (4.4)$$

and the pair $f_2$, $\hat{f}_2$ exhibits a qualitatively similar structure. Type II (I) seesaw dominates the entries of the first row of $f_1$ ($\hat{f}_1$), while hybrid seesaw determines the entries of the 23-block. A third pair of dual solutions is

$$f_3 \approx \begin{pmatrix} -0.35 & 0.51 & -0.52 \\ ... & -0.77 & 0.68 \\ ... & ... & -1.67 \end{pmatrix}, \quad \hat{f}_3 \approx \begin{pmatrix} 0.02 & 0.15 & -0.15 \\ ... & 0.95 & -0.84 \\ ... & ... & 1.85 \end{pmatrix}, \quad (4.5)$$

and the pair $f_4$, $\hat{f}_4$ exhibits a qualitatively similar structure. All entries of these matrices receive significant contribution from both type I and type II seesaw.

In fig. 6 the RH neutrino spectrum is shown as a function of the absolute mass scale $m_3$ of light neutrinos. The eigenvalues of $m_\nu$ are chosen as ($-\sqrt{m_3^2 + \Delta m_{atm}^2 - \Delta m_{sol}^2/2}$, $\sqrt{m_3^2 + \Delta m_{atm}^2 + \Delta m_{sol}^2/2}$, $m_3$), which generalizes the choice $v_L(-1 + \epsilon, 1 + \epsilon, \epsilon)$ that leads to eqs. (4.3)-(4.4). For the solution $f_1$, the masses of the three RH neutrinos are rather close to each other (within a factor 5) and to $v_R$. For the solution $\hat{f}_1$, instead, they are spread over 3–4 orders of magnitude, with the lightest one being around $10^{11}$ GeV. These features of the mass spectrum of RH neutrinos are rather insensitive to variations of $m_3$ between zero and 0.23 eV. For both $f_3$ and $\hat{f}_3$ and in the whole allowed range of $m_3$, the masses of two RH neutrinos are close to $v_R$, while the third mass is significantly smaller.
Figure 7: The masses $M_{1,2,3}$ of the three RH neutrinos as functions of $\sin \theta_{13}$. We chose the same input parameters as in fig. 5, but fixed $m_1 = 0.005$ eV and allowed non-zero $\theta_{13}$. In the left (right) panel, the solid curves correspond to the solution $f_1$ ($\hat{f}_1$) and the dashed curves to its dual solution $\hat{f}_1$ ($\hat{\hat{f}}_1$).

Actually, despite the fact that all the entries of $f_3$ and $\hat{f}_3$ in eq. (4.5) are of order one, the determinants $\lambda_3$ and $\hat{\lambda}_3$ turn out to be much smaller than one, corresponding to one RH neutrino mass being much smaller than $v_R$. Notice also that the value of $M_3$ is almost the same for these two dual solutions. Contrary to the case of the normal mass ordering, the present set of input parameters does not lead to the level crossing phenomenon.

4. Up to now we were assuming in this section the exact tri-bi-maximal mixing. However, the present uncertainties of the values of the leptonic mixing angles are quite sizable. In particular, only an upper bound on the 1-3 mixing exists, $\sin \theta_{13} \lesssim 0.2$. Measuring this parameter is one of the main goals of the near-future experimental neutrino program. Let us discuss the modifications to the structures of $f$ when a non-vanishing $\theta_{13}$ is allowed.

Consider the same input parameters as in example 1, but with $\sin \theta_{13} = 0.1$. Then eq. (4.1) is replaced by

$$m \approx \begin{pmatrix} 0.01 & 0.17 & -0.03 \\ 0.17 & 0.53 & 0.44 \\ -0.03 & 0.44 & 0.56 \end{pmatrix} \quad (4.6)$$

The solutions for $f$ are modified slightly with respect to those in eq. (4.2), but their qualitative features remain the same. However, the relative size of type I and II seesaw contributions to a given element of $f$ may change significantly with $\theta_{13}$. The mass spectrum of RH neutrinos in general depends weakly on $\theta_{13}$, as shown in fig. 5 for the pairs of dual solutions $f_{1,3}$ and $\hat{f}_{1,3}$: the masses $M_i$ change at most by about a factor of 10 for $\sin \theta_{13}$ varying between 0 and 0.2. This indicates that similar underlying theories at the seesaw scale may result in very different values of $\theta_{13}$.

5. An important experimental and theoretical issue is whether CP is violated in the
Figure 8: The masses $M_{1,2,3}$ of the three RH neutrinos versus the Majorana-type CP violating phase $\rho$. In the left (right) panel, the solid curves correspond to the solution $f_1$ ($\tilde{f}_2$) and the dashed curves to its dual solution $\tilde{f}_1$ ($f_2$). We chose the same input parameters as in fig. [3] but assumed the quasi-degenerate mass spectrum of light neutrinos $v_L[e^{-2i\rho}, 1 + \Delta m_{\text{sol}}^2/(2v_L^2), 1 + \Delta m_{\text{atm}}^2/(2v_L^2)]$ with $v_L = 0.2$ eV.

leptonic sector. The Dirac-type CP-violating phase is associated to the parameter $\theta_{13}$ and therefore its effect on the structure of $m_\nu$ is small. On the contrary, the two Majorana-type CP-violating phases, that is, the relative complex phases of the eigenvalues $m_{1,2,3}$ of $m_\nu$, can have large effects on the structure of the mass matrix. In particular, the relative phase $\rho$ between $m_1$ and $m_2$ affects substantially the parameter $m_{ee} \equiv |(m_\nu)_{ee}|$, which determines the decay rate of nuclei undergoing neutrinoless $2\beta$-decay. Since next generation $2\beta0\nu$ decay experiments will be mostly sensitive to the quasi-degenerate neutrino mass spectrum, we consider here the dependence on $\rho$ assuming three light neutrinos with mass $\sim 0.2$ eV.

For definiteness, we once again use the parameters chosen in example 1, except that the mass spectrum of light neutrinos is now taken to be $v_L(e^{-2i\rho}, 1 + \epsilon, 1 + \eta)$, with $v_L = 0.2$ eV, $\epsilon \approx \Delta m_{\text{sol}}^2/(2v_L^2)$ and $\eta \approx \Delta m_{\text{atm}}^2/(2v_L^2)$. The mass matrix of light neutrinos takes the form

$$m = \lambda_3 + \frac{1 - e^{-2i\rho}}{6} \begin{pmatrix} -4 & 2 & -2 \\ 2 & -1 & 1 \\ -2 & 1 & -1 \end{pmatrix} + \frac{\epsilon}{3} \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} + \frac{\eta}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}. (4.7)$$

In particular, one finds $m_{ee} \approx v_L \sqrt{1 - (8/9) \sin^2 \rho}$. For $\rho = 0$ ($\rho = \pi/2$) the CP parity of $\nu_1$ is equal (opposite) to that of the other two light neutrinos.

The eight solutions for $f$ depend on the value of the phase $\rho$ in the interval $[0, \pi]$. The dependence of the masses of RH neutrinos on $\rho$ for two pairs of dual solutions is

---

9In the basis where $y$ is real and diagonal, the matrix $m$ in the seesaw formula depends on 6 CP-violating phases, the three low energy ones plus three phases that are not observable at low energies, but may be relevant for the reconstruction of $f$. They can be taken into account by allowing $y_i$ to be complex. In the examples considered here we set them equal to zero.
Figure 9: The masses $M_{1,2,3}$ of the three RH neutrinos as functions of $y_2$. In the left (right) panel the solid curves correspond to the solution $f_1$ ($f_3$) and the dashed curves to its dual solution $\hat{f}_1$ ($\hat{f}_3$). We chose the same input parameters as in fig. 5, but fixed $m_1 = 0.005$ eV and varied $y_2$ between $y_1 = 0.01$ and $y_3 = 1$.

presented in fig. 5. The figure is invariant under the transformation $\rho \leftrightarrow \pi - \rho$ because, in the absence of the other complex phases, this transformation is equivalent to taking the complex conjugate of $m$ and therefore of $f$, so that the masses of RH neutrinos remain unchanged. The dependence of $M_i$ on $\rho$ is generically weak (they change by less than one order of magnitude). The three RH neutrino masses corresponding to the solution $f_1$ are quasi-degenerate for $\rho = 0$ (type II seesaw dominates: $f_1 \approx m \approx 1_3$), whereas for $\rho = \pi/2$ they are split by an order of magnitude. For the solution $\hat{f}_1$ (dominant type I seesaw) the mass spectrum of RH neutrinos is strongly hierarchical independently of the value of $\rho$. For the solution $f_2$, level crossing between $M_2$ and $M_3$ occurs at $\rho = \pi/2$.

6. Next, we discuss the dependence of $f$ on the values of the Dirac-type neutrino Yukawa couplings $y_i$. In the previous examples we assumed the hierarchical values $y_3 = 1$, $y_2 = 0.1$, $y_1 = 0.01$. However, as pointed out in section 2.1, the flavor structure of the Dirac-type Yukawa couplings of neutrinos may be qualitatively different from that for charged fermions.

Let us keep fixed $y_3 \gg y_1$ and vary $y_2$ in between. In fig. 5 we plot $M_i$ as functions of $y_2$ for the dual pairs $f_{1,3}$ and $\hat{f}_{1,3}$, taking for the other input parameters the same values as in example 1. The left side of the figure corresponds to the limit $y_2 = y_1 = 0.01$, the right side to the limit $y_2 = y_3 = 1$. The central value $y_2 = 0.1$ corresponds to example 1 (more precisely, to the dual pairs $f_{1,3}$ and $\hat{f}_{1,3}$ in eq. (4.2), that is, to the value $m_1 = 0.005$ eV in fig. 5). One can see that for the solutions $f_{1,3}$ the variations of $M_i$ with $y_2$ are small; this is because type I contribution proportional to $y_2$ is subdominant. In contrast to this, for the solutions $\hat{f}_{1,3}$ the two lightest RH neutrino masses decrease significantly with decreasing $y_2$. For the solution $\hat{f}_1$ a level crossing occurs at $y_2 \sim 0.3$. 
Let us focus on the special case $y_2 = y_3$, which may be motivated by the experimental observation $\nu_3 \approx (\nu_\mu + \nu_\tau)/\sqrt{2}$, that is, $\theta_{13} \approx 0$ and $\theta_{23} \approx \pi/4$. For illustration, we show the structures of $f_1$ and $\hat{f}_1$:

$$
f_1 \approx \begin{pmatrix} -0.0003 & 0.14 & -0.14 \\ \ldots & 0.86 & 0.76 \\ \ldots & \ldots & 0.86 \end{pmatrix}, \quad \hat{f}_1 \approx \begin{pmatrix} 0.0003 & -0.04 & 0.04 \\ \ldots & -0.31 & -0.31 \\ \ldots & \ldots & -0.31 \end{pmatrix}.
$$

In this as well as in the other three dual pairs, one finds $f_{12} = -f_{13}$ and $f_{22} = f_{33}$. This is a consequence of the choice $m_{e\mu} = -m_{e\tau}$, $m_{\mu\mu} = m_{\tau\tau}$ and $y_2 = y_3$, as can be directly verified using eq. (3.25). Moreover, this “2-3 symmetry” also connects the dual pairs of solutions to each other. This can be best seen by considering the seesaw formula after the maximal 2-3 rotation:

$$
\begin{pmatrix} m_{ee} & \sqrt{2}m_{e\mu} & 0 \\ \ldots & m_{\mu\mu} - m_{\mu\tau} & 0 \\ \ldots & \ldots & m_{\mu\mu} + m_{\mu\tau} \end{pmatrix} = \begin{pmatrix} f_{11} & \sqrt{2}f_{12} & 0 \\ \ldots & f_{22} - f_{23} & 0 \\ \ldots & \ldots & f_{22} + f_{23} \end{pmatrix}
$$

$$
-\frac{1}{x} \begin{pmatrix} y_1 & 0 & 0 \\ \ldots & y_2 & 0 \\ \ldots & \ldots & y_2 \end{pmatrix} \begin{pmatrix} f_{11} & \sqrt{2}f_{12} & 0 \\ \ldots & f_{22} - f_{23} & 0 \\ \ldots & \ldots & f_{22} + f_{23} \end{pmatrix}^{-1} \begin{pmatrix} y_1 & 0 & 0 \\ \ldots & y_2 & 0 \\ \ldots & \ldots & y_2 \end{pmatrix}.
$$

The equation for the 33-entry decouples from the rest of the system and is quadratic in $f_{22} + f_{23}$. One its root is the third eigenvalue common to four solutions for $f$. The other root is the third eigenvalue common to the four duals. The equation for the 1-2 block has the structure of the LR symmetric seesaw in the case of two generations. As a consequence, there are four solutions for $f_{11}$, $f_{12}$ and $f_{22} - f_{23}$. Each of them can be combined with one or the other solution for $f_{22} + f_{33}$, giving two solutions which have the same first and second eigenvalues. Note that, despite the presence of this “2-3 symmetry” in the neutrino sector in the considered example, charged leptons break this symmetry badly because $m_\mu \neq m_\tau$.

7. Finally, let us examine the dependence of the masses of the RH neutrinos on the dominance parameter $d$ defined in eq. (3.17). For the one- and two-generation cases we illustrated this dependence in figs. 2 and 3, respectively. We now fix all the input parameters as in example 1, but allow $x$ to be different from 1. This gives $d \approx 13/x \approx 8 \times 10^{15}$ GeV/$v_R$. Notice that eqs. (3.25) and (3.29) imply that the dependence of $f$ on $x$ amounts to a common rescaling of the three Dirac-type Yukawa couplings: $f(x, y_i) = f(1, y_i/\sqrt{x})$.

In fig. 10 we plot the masses $M_i$ of the three RH neutrinos as functions of $d$ for the dual pairs of solutions $f_{1,3}$ and $\hat{f}_{1,3}$. Since $M_i$ are given by the eigenvalues of $v_R f$, they generally decrease as $v_R \propto 1/d$ with increasing $d$. However, this is not the case when type I seesaw dominates, since in this case $f \propto 1/v_R$ and $M_i$ tend to constant values, as it was
Figure 10: The masses $M_{1,2,3}$ of the three RH neutrinos as functions of the dominance parameter $|d| \equiv |4v^2 y_3^2/(m_{\nu})_L^2| v_L/v_R$. We chose the same input parameters as in fig. 9, but fixed $m_1 = 0.005$ eV and allowed $v_R$ to vary, so that $d = 7.8 \cdot 10^{15}$ GeV/$v_R$. In the left (right) panel, the solid curves correspond to the solution $f_1 (f_3)$ and the dashed curves to its dual $\hat{f}_1 (\hat{f}_3)$.

already shown in [13]. Such type I seesaw dominance is realized for $d \ll 1$ for the solution $\hat{f}_1$ (left panel in fig. 10). Correspondingly, type II seesaw dominates in $f_1$, so that $M_i$ grow linearly with $v_R$ and the mass spectrum of the RH neutrinos is proportional to that of light neutrinos. As discussed at the end of section 3.3 (see also Appendix E), the other three dual pairs exhibit intermediate features for $d \ll 1$, with only $M_1$ ($M_1$ and $M_2$) tending to a constant for the solutions $f_{2,3,4}$ ($\hat{f}_{2,3,4}$), as shown in the right panel of fig. 10. Large $d$ corresponds, instead, to a cancellation between type I and II seesaw contributions to $m_\nu$, that is, $f_i \approx -\hat{f}_i$, so that the mass spectra of the RH neutrinos for each pair of dual solutions tend to coincide, as can be seen on the right sides of the plots in fig. 10. Further discussion of the dependence of the mass spectrum of RH neutrinos on $v_R$ can be found in [13].

4.1 Right-handed neutrinos at TeV scale

The seesaw mechanism is not accessible to direct experimental tests unless the mass scale of the seesaw particles is as low as $\sim$ TeV. Therefore, from the phenomenological point of view, it is interesting to investigate what regions in the space of the input parameters could lead to new particles with masses at this scale. In this section we present a number of examples where such a scenario is realized.

(1) One vanishing or very small Dirac-type neutrino Yukawa coupling ($y_1 \rightarrow 0$). This limit was studied analytically in section 3.3.1, where we showed that in this case four of the solutions for $f$ have one eigenvalue vanishing as $y_1^2$. We illustrate this phenomenon numerically in fig. 11, where $y_3$ and $y_2$ have fixed values (1 and 0.01, respectively), whereas $y_1$ is varied. As expected, in each pair of dual solutions there is one with $M_1$ decreasing as $y_1^2$. The other RH neutrino masses are practically independent of $y_1$ when it becomes
Figure 11: The masses $M_{1,2,3}$ of the three RH neutrinos as functions of $y_1$ for two dual pairs of solutions. We chose $y_2 = 0.01, m_1 = 0.005$ eV, $\sin \theta_{13} = 0.1$, while the other input parameters are the same as in fig. 4. In both panels dashed curves correspond to the solution with $M_1$ vanishing in the limit $y_1 \to 0$ and solid curves to the dual solution.

smaller than $\sim 10^{-4}$. Notice that for $y_1 \sim 10^{-6}$ the mass of the lightest RH neutrino $N_1$ is already as small as $\sim 10$ TeV. However, its Yukawa couplings to the SM leptons are also suppressed as $y_1$ and therefore there is no hope to detect $N_1$ through these couplings. For yet smaller values of $y_1$, $N_1$ could be in principle so light as to play a role in cosmology or in active-sterile neutrino oscillations.

(2) Lowering of the LR symmetry breaking scale $v_R$ down to $\sim$ TeV. This is allowed by direct searches of RH currents, as discussed in section 2.1. Such a scenario is not viable for generic ("natural") values of the Yukawa couplings, since it would result in the masses of light neutrinos which are much larger than the observed ones. Nonetheless, in view of the unique phenomenological possibility to directly test the breaking of the LR symmetry and the seesaw mechanism, it is worth accepting the necessary tuning of Yukawa couplings and investigate its consequences.

The simplest phenomenologically viable scenario with $v_R \sim$ TeV corresponds to choosing tiny values for all the Dirac-type neutrino Yukawa couplings $y_i$. This allows one to avoid conflict with the low-energy neutrino data while maintaining the perturbative unitarity of the solutions for $f$, $f_{ij} \lesssim 1$. Notice that the seesaw formula is invariant with respect to a simultaneous decrease of $y_i$ and $x$ provided that $y_i^2/x$ remains unchanged. Consider the same choice of the input parameters as in example 1 of section 4, except that the values $x = 1, y_3 = 1$ are replaced with $x = 10^{-10}, y_3 = 10^{-5}$, keeping $v_L$ and the hierarchy between $y_i$ unchanged. Then the solutions for $f$ are unchanged, but since $v_R \approx 6 \cdot 10^4$ GeV in this case, the masses of RH neutrinos are smaller than those in example 1 of section 4 by a factor $10^{10}$. As one can see in fig. 4, this means that the lightest RH neutrino can have a mass $\sim 100$ GeV. Such light RH neutrinos would not be observable through their Dirac-type Yukawa couplings, which are tiny; however, because of the LR
mixing, they may interact strongly with the ordinary matter through the Yukawa couplings \( f \) with scalar triplets as well as through their coupling to RH gauge bosons. More generally, the RH gauge bosons as well as the non-standard Higgs particles related to the LR symmetry breaking would provide clear phenomenological signatures if light enough. If level crossing occurs between two TeV-scale RH neutrinos, they could still lead to a successful leptogenesis via the resonant enhancement of the produced lepton asymmetry (see section 6.2).

(3) It should be noted that very small values of \( x \), as in scenario 2 above, would indicate an approximate symmetry of the scalar potential, since for a generic choice of order one couplings one expects \( v_L v_R \sim v^2 \), that is \( x \sim 1 \). Very interestingly, it is possible to realize a TeV scale LR symmetry breaking even preserving this “naturalness” relation. Indeed, consider a modification of scenario 2 in which \( v_L \) is increased by a large factor \( W \) and, at the same time, all \( y_i^2 \) are reduced by \( W \). Then the solutions \( f \) of the seesaw formula maintain the same matrix structure, but are also reduced by an overall factor \( W \). This means, in particular, that the RH neutrino masses are much smaller than \( v_R \). Using the values of the parameters above and taking \( W = 10^8 \), one finds \( M_1 \sim \text{keV} \). Summarizing, if LR symmetry breaking occurs at scales much lower than the Grand Unification scale and the natural relation \( v_L v_R \sim v^2 \) is preserved, this would indicate that the Dirac-type and triplet Yukawa couplings \( y \) and \( f \) are all much smaller than one, with the related consequences for phenomenology.

(4) Finally, let us investigate if TeV scale RH neutrinos may be compatible with at least some order one Dirac-type Yukawa couplings and \( v_R \sim \text{TeV} \). This would require a strong fine-tuning in the structure of the Yukawa coupling matrices, in order to cancel large contributions to the masses of light neutrinos. To provide a simple example, it is convenient to start in the basis where the light neutrino mass matrix is diagonal, \( m_{\text{diag}} = (m_\nu)_{\text{diag}}/v_L = \text{diag}(m_1, m_2, m_3) \). Let us define \( r^2 \equiv m_1/m_2 \) and choose
\[
y = \begin{pmatrix}
r^2 s & irs & 0 \\
irs & -s & 0 \\
0 & 0 & y_3
\end{pmatrix}.
\]
In this case the seesaw equation for \( f \) has only two solutions, the reason being as follows. The matrix \( y \) in eq. (4.10) has one zero eigenvalue, which reduces the number of solutions for \( f \) from eight to four (section 3.3.1). Furthermore, in the basis where \( y \) is diagonal, one has \( m_{ee} = 0 \), which makes two of the four remaining solutions singular and thus unphysical (see eq. (3.34)). The leftover two solutions are
\[
f_{1,2} = \text{diag} \left( m_1, m_2, \frac{m_3}{2} \left[ 1 \pm \sqrt{1 + \frac{4y_3^2}{xm_3^2}} \right] \right).
\]
Notice that the entries of $y$ proportional to $s$ cancel exactly in the seesaw formula and therefore do not contribute to the light neutrino masses. As a consequence, the coupling $s$ can be as large as one. In contrast to this, $y_3^2/x$ has to be smaller than one to guarantee $f_{33} \lesssim 1$. To ensure perturbative unitarity of $(f_{1,2})_{ii}$, $v_L$ should be $\lesssim$ eV; therefore a low LR scale $v_R \sim$ TeV requires $y_3 \lesssim 10^{-5}$.

To consider the structure of the seesaw mechanism in this scenario in the flavor basis, it is sufficient to rotate on the left and on the right by the leptonic mixing matrix $U$:

$$m \rightarrow U^* m_{\text{diag}} U^\dagger, \quad y \rightarrow U^* y U^\dagger, \quad f_{1,2} \rightarrow U^* f_{1,2} U^\dagger.$$ (4.12)

In this flavor basis, type I contributions to the elements of the light neutrino mass matrix $m_{\alpha\beta}$ are proportional to $U^\dagger \alpha y_3 U^\dagger_3 y_2^2/x$.

The masses of RH neutrinos are given by the eigenvalues of $f_{1,2}$ (i.e. the diagonal entries in eq. (4.13)) multiplied by $v_R \sim$ TeV. The two possible solutions for $f$ are distinguished by the mass of the third RH neutrino, while $M_{1,2}$ are uniquely determined, with $M_2^2 - M_1^2 = \Delta m^2_{\text{sol}} v_R/v_L$. Two of the RH neutrinos ($N_{1,2}$) have order one Yukawa couplings to the light neutrinos and charged leptons.

5. Structure of seesaw in specific left-right symmetric models

In this section, we discuss the basic features of models which incorporate type I+II seesaw mechanism. In particular, we analyze their Yukawa sector and derive the seesaw formula for the light neutrino mass in the form given in eq. (4.3). A crucial issue is the structure of the matrix of Dirac-type Yukawa couplings of neutrinos $y$, which is an input in the bottom-up approach we adopted to reconstruct $f$.

5.1 Minimal LR symmetric model

We begin by reviewing the structure of the minimal LR model based on the gauge group $SU(3)_c \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$. It contains the following (color singlet) Higgs multiplets: $\Phi(2,2,0)$, $\Delta_L(3,1,-2)$ and $\Delta_R(1,3,2)$, where in the brackets we indicate the $SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ quantum numbers. The leptons are assigned to $L(2,1,-1) = (\nu_L, l^c_L)^T$ and $L^c(1,2,1) = (N^c_L, l^c_L)^T$, and their Yukawa couplings can be written as

$$-\mathcal{L}_Y = \frac{f_L}{2} L^{cT} i\sigma_2 C \Delta_L L + \frac{f_R}{2} L^{cT} i\sigma_2 C \Delta_R L^c + g L^{cT} C \sigma_2 \Phi \sigma_2 L^c + h L^{cT} C \Phi^* L^c + h.c.,$$ (5.1)

where $\sigma_2$ is the isospin Pauli matrix and the following conventions are assumed:

$$\Delta_L = \begin{pmatrix} \Delta_L^+ / \sqrt{2} & \Delta_L^+ \\ \Delta_L^0 & -\Delta_L^- / \sqrt{2} \end{pmatrix}, \quad \Delta_R = \begin{pmatrix} \Delta_R^+ / \sqrt{2} & \Delta_R^- \\ \Delta_R^0 & -\Delta_R^- / \sqrt{2} \end{pmatrix}, \quad \Phi = \begin{pmatrix} \phi_1^0 & \phi_1^+ \\ \phi_2^- & \phi_2^0 \end{pmatrix}. \quad (5.2)$$
Denoting the VEVs of the neutral components of $\Phi$ as $v_{1,2}$, for the Dirac-type mass matrices of the charged leptons and neutrinos one finds

$$m_l \equiv vy_e = v_1 g + v_2^* h, \quad m_D \equiv vy = v_1^* h + v_2 g.$$  \hfill (5.3)

At low energies the Dirac-type neutrino Yukawa coupling can be rewritten in the form of eq. (1.2) by defining $\phi^0 \equiv (v_1 \phi_1^0 + v_2 \phi_2^0)/v$, where $v \equiv \sqrt{|v_1|^2 + |v_2|^2} \approx 174$ GeV is the electroweak symmetry breaking parameter.\footnote{The field $\phi^0$ is thus identified with the neutral component of the SM Higgs boson. The orthogonal combination of the neutral scalar fields, $\eta^0 \equiv (-v_2 \phi_1^0 + v_1 \phi_2^0)/v$, has zero VEV and so does not contribute to the fermion masses. Since it mediates flavour changing neutral currents, it must be heavy and so decouples from the low-energy dynamics.} Notice that if the theory is supersymmetrized, the coupling $h$ should be removed from eq. (5.1) due to the requirement of the analyticity of the superpotential. As a consequence, $y = ye(v_2/v_1) \equiv ye \tan \beta$ and the two Higgs doublets, $\phi_u \equiv (\phi_1^+ \phi_1^0)^T$ and $\phi_d \equiv (\phi_0^+ \phi_2^0)^T$, do not mix.\footnote{Similarly to eq. (5.3), for quarks one has $m_d \equiv vy_d = v_1 g_q + v_2^* h_q$ and $m_u \equiv vy_u = v_1^* h_q + v_2 g_q$, where the matrices $g_q$ and $h_q$ are in general independent of $g$ and $h$, and $h_q$ is absent in the SUSY case.}

The Majorana mass matrices of neutrinos are generated by the VEVs $v_{L,R}$ of the neutral components of $\Delta_{L,R}$:

$$m_L = v_L f_L, \quad M_R = v_R f_R.$$ \hfill (5.4)

In the limit where the eigenvalues of of $M_R$ are much larger than those of $m_D$, the mass matrix of light neutrinos takes the form

$$m_\nu \simeq m_L - m_D M_R^{-1} m_T = v_L f_L - \frac{v^2}{v_R} y f_R^{-1} y^T,$$ \hfill (5.5)

which is known as type I+II seesaw formula.

At this point, a discrete LR symmetry should be introduced if one wants to guarantee the equality of $SU(2)_L$ and $SU(2)_R$ gauge couplings, that is, the asymptotic LR symmetry of the model. There are essentially two possibilities: the discrete transformation can act as a charge conjugation or as a parity transformation. In the previous sections we were implicitly assuming the former possibility (we postpone the discussion of the latter one to Appendix C). Then, the discrete LR symmetry acts on matter and Higgs fields as follows:

$$L \leftrightarrow L^c, \quad \Phi \leftrightarrow \Phi^T, \quad \Delta_L \leftrightarrow \Delta_R.$$ \hfill (5.6)

This yields

$$f \equiv f_L = f_R, \quad g = g^T, \quad h = h^T.$$ \hfill (5.7)

In this case the mass matrix of light neutrinos has the form given in eq. (1.3). Moreover, eq. (5.7) implies that $y$ is symmetric, so that the seesaw duality is realized. (If, instead,
the model contained a Higgs bidoublet $\Phi'$ with the transformation law $\Phi' \leftrightarrow -\Phi'^T$ under the discrete symmetry, the corresponding Dirac-type Yukawa coupling $y'$ would be antisymmetric, leading to the seesaw duality as well).

We have seen that in the minimal SUSY LR model the matrix $y$ is proportional to the Yukawa coupling matrix of charged leptons $y_e$. Therefore, in the basis where $y_e$ is diagonal, also $y$ is, and the ratios of its eigenvalues are $m_e/m_\mu$ and $m_\mu/m_\tau$. This scenario is ideal for the reconstruction of $f$, since the input related to the structure of $y$ is completely determined. However, one should keep in mind that this minimal model does not provide a realistic description of the quark sector, as it leads to $m_u \propto m_d$. As follows from eq. (5.3), in the non-SUSY case, the structure of $y$ can be different from $y_e$. For a recent realistic model, see [24].

Let us briefly consider an upgrade of the minimal LR model to the Pati-Salam gauge group $SU(2)_L \times SU(2)_R \times SU(4)_c$, with a Higgs field in the $(2, 2, 1)$ representation. In this case quark-lepton unification occurs, which for the Yukawa sector amounts to identifying the quark and lepton coupling matrices: $g_q = g$ and $h_q = h$. This implies, in particular, $y = y_u$ and $y_e = y_d$. In this case the eigenvalues of $y$ are determined by the up-type quark masses, while the mixing is given by the CKM matrix. In non-minimal Pati-Salam models, also Higgs fields transforming as $(2, 2, 15)$ may couple to fermions. Their contribution to the lepton masses differs from those for the quarks by the factor $-3$ because of the different respective values of $(B - L)$. If both bidoublet types exist and contribute significantly to the fermion masses, the direct connection between $y$ and $y_u$ is lost.

5.2 $SO(10)$ models

A natural grand unified embedding of LR symmetric models is provided by $SO(10)$ models. As is well-known, $SO(10)$ accommodates all the standard model fermions as well as RH neutrinos in a single multiplet $16_F$. Fermion bilinears transform under $SO(10)$ as

$$16_F \times 16_F = 10_s + 120_a + 126_s,$$

where the subscript $s(a)$ indicates symmetry (antisymmetry) in the flavor indexes. As a consequence, there are three possible types of Higgs multiplets which may contribute to fermion masses at the renormalizable level: $10_H$ and $\overline{126}_H$ give symmetric contributions, while $120_H$ gives an antisymmetric one.

The Higgs triplets $\Delta_{L,R}$ are both contained in $\overline{126}_H$, so that the Majorana mass matrices of left-handed and RH neutrinos originate from the coupling

$$f \ 16_F 16_F \overline{126}_H \ni f (L L_L + L' L'_R \Delta_R).$$

One generator of $SO(10)$, known as D-parity, acts as in eq. (5.6). In other words, the discrete LR symmetry discussed above is promoted to an automatic gauge symmetry and
this guarantees the proportionality of the two terms on the r.h.s. of eq. (5.9). The Dirac-type neutrino Yukawa coupling $y$ may receive contributions from the Higgs multiplets of all three types, provided that they acquire VEVs in the appropriate direction. In fact, both $10_H$ and $120_H$ contain one bidoublet $\Phi \sim (2, 2, 1)$ under the Pati-Salam group, and both $120_H$ and $\overline{126}_H$ contain one $(2, 2, 15)$ bidoublet. If only $10_H$ and $\overline{126}_H$ contribute to $y$, it is symmetric; if only $120_H$ does, it is antisymmetric.

Consider first the minimal possibility where only one $\overline{126}_H$ multiplet develops a VEV in the $\nu_L N^c_L$ direction, that is, $y \propto f$. In this case both type I and type II seesaw contributions to $m_\nu$ are proportional to $f$. The latter is uniquely determined by $m_\nu$, up to an overall factor. There is no seesaw duality in that case, since $y$ is not independent of $f$ and so it is not invariant under $f \rightarrow m - f$.

If two (or more) $\overline{126}_{Hi}$ multiplets are introduced, the proportionality between $m_L \equiv v_{L1} f_1 + v_{L2} f_2$ and $M_R \equiv v_{R1} f_1 + v_{R2} f_2$ is in general lost. The matrix $y$ depends on the same Yukawa couplings $f_{1,2}$: $y = -3(v_{126,1}^u f_1 + v_{126,2}^u f_2)$, where $v_{126,i}^u$ is the VEV of the “up-type” Higgs doublet (i.e. of the doublet with non-zero VEV in the up-type isospin direction) contained in the $(2, 2, 15)$ component of $\overline{126}_{Hi}$. Nonetheless, if $v_{R2} = 0$, an effective seesaw duality is realized, with $f_1$ and $f_2$ playing the role of $f$ and $y$, respectively:

$$\tilde{m} = f_1 - \frac{1}{\tilde{x}} \tilde{y} f_1^{-1} \tilde{y},$$

(5.10)

where

$$\tilde{m} \equiv \frac{m_\nu - (v_{L2} - 18 v_{126,2}^u v_{126,1}^u/v_{R1}) f_2}{v_{L1} - (3 v_{126,1}^u)^2/v_{R1}}, \quad \tilde{x} \equiv \frac{v_{L1} v_{R1} - (3 v_{126,1}^u)^2}{v^2}, \quad \tilde{y} \equiv -3 v_{126,2}^u f_2.$$

(5.11)

If all the parameters in eq. (5.10) except $f_1$ were known, one would be able to solve (5.10) for $f_1$ using the techniques developed in the previous sections for solving the usual seesaw formula (5.3) for $f$. For three lepton generation this would yield four dual pairs of solutions.

Consider now the scenario with one $10_H$ and one $\overline{126}_H$ multiplets. Since $10$ is a real representation, $10_H^\ast$ transform as $10_H$ under $SO(10)$, so that in general two new Yukawa couplings should be added to eq. (5.9):

$$g \ 16_F 16_F 10_H + h \ 16_F 16_F 10_H^\ast.$$

(5.12)

One then finds

$$m_D \equiv v y = v_{10}^u g + v_{10}^d h - 3 v_{126}^u f,$$

(5.13)

where $v_{10}^{u,d}$ are the VEVs of the up- and down-type Higgs doublets contained in the $(2, 2, 1)$ component of $10_H$. Since $y$ depends on $f$, one cannot regard it as an input for solving the seesaw formula for $f$. However, one can consider as an input the matrix $v \tilde{y} \equiv v_{10}^u g + v_{10}^d h$; note that in this class of models the couplings to $10_H$ usually give the dominant contribution to the mass matrices of the charged fermions.
Then, the seesaw formula \([13]\) can be written as
\[
\tilde{m} = f - \frac{1}{\tilde{x}} \tilde{y} f^{-1} \tilde{y}, \quad \tilde{m} \equiv \frac{m_\nu - (6 v_{126}^u/v_R) \tilde{y}}{v_L - (3 v_{126}^u)^2/v_R}, \quad \tilde{x} \equiv \frac{v_L v_R - (3 v_{126}^u)^2}{v^2}.
\] (5.14)
This equation can be solved for \(f\) as usual, taking \(\tilde{m}, \tilde{y}\) and \(\tilde{x}\) as input parameters. In some models of this class, the GUT symmetry breaking is such that only the Higgs doublets in \(10_H\) develop VEVs (a mixing between these doublets and those in \(126_H\) is induced only in the presence of \(210_H\) multiplets). In this case \(v_{126}^u = 0\), and one recovers \(\tilde{m} = m, \tilde{y} = y\) and \(\tilde{x} = x\). Moreover, one has \(y = y_u\), that is, the matrix \(y\) is fully determined if we know the Yukawa couplings of up-type quarks. If also \(v_{126}^d = 0\), one finds \(y_e = y_d\), which fails to reproduce correctly the masses of the down-type fermions of the second and first generations. The corrections coming from the \(\overline{126}_H\) (and therefore proportional to \(v_{126}^d f\)) may cure this problem. In this case, however, the form of \(f\) is constrained not only by the seesaw formula, but also by the values of the masses of charged fermions.

A comment on the number of the Yukawa couplings to \(10_H\) multiplets is in order. If one wants to study minimal models with only one such Yukawa coupling matrix, there are two options: (i) if only one \(10_H\) is introduced and the model is supersymmetric, the coupling \(h\) in eq. \([5.12]\) is automatically forbidden by the analyticity of the superpotential; (ii) one can forbid \(h\) also in the non-supersymmetric case by assuming that the multiplet \(10_H\) is real. Then only the coupling \(g\) contributes to the fermion masses, leading to \(y = y_u \propto y_e = y_d\), which needs substantial corrections in order to match the differences among the mass spectra of charged leptons, up- and down-type quarks. This turns out to be extremely constraining, if one tries to fit all the data by adding just one \(\overline{126}_H\) multiplet (see [6] and references therein). If one is willing to consider models with one more Yukawa coupling matrix, in the SUSY case one needs to introduce a second \(10_H\) multiplet (see e.g. [13]), whereas in the non-SUSY case one complex \(10_H\) is sufficient. In this framework the up sector is decoupled from the down sector, but the relation \(y_e = y_d\) persists.

Alternatively, instead of adding \(10_H\) to the \(\overline{126}_H\) multiplet, one can explore the possibility where \(120_H\) is added. The couplings in \([5.12]\) are then replaced by
\[
g \ 16_F 16_F 120_H + h \ 16_F 16_F 120_H^\dagger,
\] (5.15)
where the matrices \(g\) and \(h\) are antisymmetric. Denoting \(v_{120,1}^u\) the VEVs of the up (down)-type Higgs doublets contained in the \((2, 2, 1)\) component of \(120_H\), and similarly \(v_{120,15}^d\) for the \((2, 2, 15)\) component of \(120_H\), one finds
\[
m_D \equiv v y = (v_{120,1}^u - 3 v_{120,15}^u) g + (v_{120,1}^d - 3 v_{120,15}^d) h - 3 v_{120}^u f \equiv v \tilde{y} - 3 v_{126}^u f, \quad \] (5.16)
where \(\tilde{y}\) is antisymmetric. Analogously to eq. \([5.14]\), the seesaw relation takes the form
\[
\tilde{m} = f + \frac{1}{\tilde{x}} \tilde{y} f^{-1} \tilde{y}, \quad \tilde{m} = \frac{m_\nu}{v_L - (3 v_{126}^u)^2/v_R}, \quad \tilde{x} = \frac{v_L v_R - (3 v_{126}^u)^2}{v^2}.
\] (5.17)
This equation can be solved for $f$, taking $\tilde{m}$, $\tilde{y}$ and $\tilde{x}$ as input parameters. Using the results of Appendix A, which apply to the case of antisymmetric $\tilde{y}$, one finds two solutions for $f$.

Finally, let us discuss models with both $\mathbf{10}_H$ and $\mathbf{120}_H$ multiplets added to $\mathbf{126}_H$ (for an early analysis of fermion masses in this case, see [7]). In the search for the minimal realistic SUSY $SO(10)$ model, it has been recently shown [25] that the mass matrices of the charged fermions can be reproduced through $\mathbf{10}_H$ and $\mathbf{120}_H$ Yukawa couplings only, at least in the first approximation (a three generation fit presents some problems [26]). In the case when the coupling $f$ to $\mathbf{126}_H$ is also introduced, satisfactory fits of charged fermion and neutrino parameters have been recently obtained [27]. In this scenario there is no seesaw duality (at least in the form considered here), since $y$ contains both symmetric and antisymmetric contributions. Nonetheless, the non-linearity of the seesaw formula still leads to multiple solutions for $f$. To take this into account, one should (i) fix the $\mathbf{10}_H$ and $\mathbf{120}_H$ Yukawa couplings by fitting (approximately) the masses of charged fermions; (ii) use the seesaw formula to derive the different structures of $f$ that reproduce a given set of neutrino data; (iii) test which solutions for $f$ provide the small corrections needed to achieve a satisfactory fit of the masses of charged fermions.

6. Further considerations

In this section we briefly discuss the following issues pertaining to our analysis: stability of our results with respect to the renormalization group evolution effects and baryogenesis via leptogenesis.

6.1 Stability of the seesaw formula

Up to now we were assuming that the seesaw formula (1.3) describes accurately the mass of light neutrinos at low energy scales. However, due to the breaking of the discrete LR symmetry at a scale $v_{LR}$, the renormalization group (RG) evolution effects below this scale can result in a violation of the conditions in eq. (5.7), which in turn would modify the seesaw formula (1.3). A full study of the RG effects in the type I+II LR symmetric seesaw is beyond the scope of this paper; here we constrain ourselves to a classification of possible effects and estimate of their size.

Above the scale $v_{LR}$ the discrete LR symmetry ensures $f_L = f_R$ and $y = y^T$. Below this scale, the fields related by the LR symmetry acquire in general different masses, so that $f_L$ and $f_R$ may evolve differently and $y$ may get asymmetric corrections. For example, if $v_{LR}$ is larger than $v_R$ (the scale of $SU(2)_R$ breaking), logarithmic in $v_R/v_{LR}$ radiative corrections arise. The size of this effect depends on the details of the mass spectra of RH

\[12\] In $SO(10)$ models, the discrete LR symmetry can be actually broken already at Grand Unification scale, even if $SU(2)_R$ is not [28].
gauge bosons and Higgs particles responsible for the breaking of $SU(2)_R$, which are highly model-dependent. We will therefore not discuss this case any further and will just assume that the conditions $f_L = f_R$ and $y^T = y$ still hold at the scale $v_R = v_{LR}$.

At this point the fate of the seesaw formula resides with the particle spectrum below $v_R$. If the masses of all three RH neutrinos as well as of $\Delta_L$ are close to $v_R$, they can be integrated out all together, leading to the low-energy effective neutrino mass matrix $m_\nu$ of eq. (1.3). If, on the contrary, one or more RH neutrinos and/or $\Delta_L$ have masses that are much smaller than $v_R$, they will contribute to the RG evolution of $f_L$, $f_R$ and $y$ down to their mass scale. Generically, these corrections induce a splitting between $f_L$ and $f_R$ and an asymmetry in $y$. One could envisage two possible approaches to determine the effective mass matrix $m_\nu$ at the lightest seesaw scale $M_s$ (the smallest among $M_1$ and $M_{\Delta_L}$). One option is to evolve $f_L, f_R$ and $y$ from $v_R$ down to $M_s$ and then integrate out the RH neutrinos and $\Delta_L$ all together. This has the advantage of determining $f_L, f_R$ and $y$ as they enter in the seesaw formula (5.5). Another option is to integrate out $N_{Ri}$ and $\Delta_L$ each at its own mass scale, redefining iteratively the effective $m_\nu$. Given the $\beta$-functions to some finite order in perturbation theory, this second approach should provide a more accurate result.

For type I seesaw models, a detailed analysis of RG effects, including the running between the different mass scales of RH neutrinos, has been performed in [29]. For type I+II seesaw, such an analysis is not yet available. Here we just estimate the order of magnitude of possible effects. One-loop corrections to the matrix elements of $f$ and $y$ can be schematically written as

$$\langle \delta f, \delta y \rangle \sim \frac{(f^2, y^2)}{16\pi^2} \log \frac{(M_i, M_{\Delta_L})}{v_R},$$

which, for judicious choices of parameters, should be at or below the percent level. For comparison, the present precision of most of the input parameters in $m_\nu$ is at about 10%. Moreover, the loops induced by $N_{Ri}$ exchanges are typically proportional to $y^2$. The lighter $N_{Ri}$, the smaller the corresponding Yukawa coupling, since $m_\nu \sim y^2/M_{Ri}$. Therefore, for lighter RH neutrinos, a RG evolution over a wider range may be partly compensated by smaller couplings. Similarly, loops induced by $\Delta_L$ exchanges are proportional to $f^2$ and, since $m_\nu \sim v_L f \sim v^2 f/M_{\Delta_L}$, the lighter the $\Delta_L$, the smaller the expected coupling $f$ and the corresponding radiative correction. As a simplified numerical test, we chose certain structures of $f$ and $y$ at $v_R$ and modified their elements in the seesaw formula by order percent corrections. Next, we implemented the usual bottom-up procedure to reconstruct $f$ and found that, among the 8 dual solutions, one reproduces the original structure of $f$ within percent errors. However, this is not the case when strong hierarchies among the entries of $f$ and $y$ are present and small matrix elements receive corrections proportional to the large ones. We expect that only in these special regions of the parameter space...
instabilities may occur and large departures of the reconstructed structures of $f$ from the true ones may arise.

To complete the discussion of RG effects, one should consider the evolution of $m_\nu$ from $M_s$ down to the to electroweak scale, where it can be compared with experimental data. If only the standard model (or MSSM) particles contribute to the RG evolution, the running of the mass squared differences and mixing angles is rather small and negligible for the purposes of this paper, except perhaps in the case of quasi-degenerate light neutrinos of same CP parity. This could affect the reconstruction of $f$ in example 5 of section [3].

6.2 Leptogenesis

As we demonstrated above, in the realistic case of three lepton generations there are eight different matrices $f_i$ which, for a given $y$, result in exactly the same mass matrix of light neutrinos $m_\nu$. A natural question is then how one can discriminate between these eight possible solutions. The seesaw formula (1.3) cannot tell us more than it already did, and a new independent source of information is necessary. Such a source could be provided by the ability of $f_i$ to reproduce the observed baryon asymmetry of the universe.

It is well known that the seesaw mechanism not only explains nicely the smallness of neutrino mass, but also has a built-in mechanism for generating the baryon asymmetry of the universe through leptogenesis [4]. In this mechanism, first a lepton asymmetry is produced in out-of-equilibrium CP-violating decays of heavy RH neutrinos and/or Higgs triplets, which is then reprocessed into baryon asymmetry by electroweak sphalerons. The amount of the produced lepton asymmetry depends on (i) mass matrix of RH neutrinos $M_R$ (masses, mixing angles and CP phases); (ii) Majorana-type Yukawa coupling of leptons to Higgs triplets; (iii) Dirac-type Yukawa coupling of leptons to Higgs doublets. In this paper, we consider theories with $f_L = f_R \equiv f$, and so the parameters involved in (i) and (ii) essentially coincide. This renders the computation of the lepton asymmetry more predictive than in generic type I+II seesaw scenarios.

The impact of multiple solutions for $f$ on leptogenesis has been already analyzed in a class of $SO(10)$ models in [13] and very non-trivial results were found. Here we will not undertake a quantitative analysis of leptogenesis, but rather will make some general remarks, which may be of guidance for the development of specific models:

- It is recognized by now [20, 30, 31] that the presence of both RH neutrinos and Higgs triplets may lead to leptogenesis scenarios that are qualitatively different from those in pure type I seesaw models. We would like to stress, as a minimal possibility, that leptogenesis can work in models with $\Delta L$ and just one RH neutrino species $N_R$ (this effective pattern occurs, e.g., when the other two RH neutrinos are super-heavy and/or very weakly coupled). Since $N_R$ couples to a unique linear combination $L$
of the flavor eigenstates of lepton doublets, this scenario may be called “flavorless leptogenesis”. This is a viable possibility since the relevant interactions of $N_R$ and $\Delta_L$ contain an unremovable CP-violating phase (which manifests itself in the one-generation seesaw formula, see section 3.1). A recent study of this scenario can be found in [32].

- In many realistic cases an extra freedom gained by the interplay of type I and type II seesaw terms may turn out to be insufficient to cure the shortcomings of thermal leptogenesis in the pure type I scenario. In particular, it remains true [30, 33] that the decaying particle should generically be very heavy (above $\gtrsim 10^8$ GeV) to produce a sufficient asymmetry, thus requiring large reheating temperature with the associated problems. Also, the hierarchical structure of Yukawa couplings, which is natural in unified scenarios, may strongly suppress the asymmetry. The existence of various solutions for $f_i$ in particular of those with a non-hierarchical structure, may alleviate this problem. Further details can be found in [13].

- Given the rich structure of the Higgs sector at the seesaw scale in LR symmetric and $SO(10)$ models, sources of lepton asymmetry other than the decays of RH neutrinos and/or $\Delta_L$ into left-handed leptons should not be overlooked. As an example [34], a successful leptogenesis may be due to decays of RH neutrinos into a RH charged lepton and an $SU(2)_L$ singlet charged scalar (contained, e.g., in the $120_H$ multiplet of $SO(10)$). The coupling $f_R$ between the RH leptons and $\Delta_R$ (see eq. (5.1)) may also affect the evolution of the asymmetry.

- In minimal models with “natural” values of the seesaw scale and of Yukawa couplings, in most regions of parameter space the produced lepton asymmetry tends to be too small. A way out may be provided by the resonant enhancement of the lepton asymmetry which occurs when two RH neutrinos are quasi-degenerate in mass [35]. As we demonstrated in several examples presented in section 4, the crossing of the RH neutrino mass levels indeed occurs in the LR symmetric seesaw for certain choices of input parameters and for some of the solutions $f_i$. The requirement of level crossing may be a powerful constraint for model building.

7. Discussion and summary

In models with more than one source of neutrino mass, to disentangle different contributions using only low-energy data is not in general possible. The situation is different when these contributions have a common origin, as it is the case in left-right symmetric seesaw models. In a wide class of such models and their partially unified or Grand Unified extensions, the mass matrix of light neutrinos $m_\nu$ contains type I and type II seesaw contributions which
depend on the same Majorana-like triplet Yukawa coupling matrix \( f \). In this paper we undertook for the first time a thorough phenomenological bottom-up analysis of such a scenario. We have shown that the interplay of type I and type II seesaw terms in \( m_\nu \) may change our interpretation of neutrino data and provide an insight into the underlying theory at the seesaw scale.

We have adopted an approach in which the mass matrix of light neutrinos \( m_\nu \) and the matrix of Dirac-type Yukawa couplings \( y \) are considered known, and the seesaw relation is solved for the matrix of the Majorana-like Yukawa couplings \( f \), which coincides (up to a constant factor) with the mass matrix of RH neutrinos and deeply characterizes the structure of the underlying theory. To this end, we have developed a linearization procedure which allowed us, for symmetric or antisymmetric \( y \), to solve the seesaw non-linear matrix equation and obtain exact analytic expressions for the matrix \( f \) in a compact form.

For symmetric \( y \), the overall number of solutions was shown to be \( 2^n \) for \( n \) lepton generations. Thus, in the realistic case of three generations, there are eight different matrices \( f \) which, for a given \( y \), result in exactly the same mass matrix of light neutrinos \( m_\nu \).

We have studied implications of an intriguing duality property [12] of the LR symmetric seesaw mechanism with symmetric or antisymmetric \( y \), which relates pairwise different solutions for the matrix \( f \). The eight solutions of the seesaw equation for \( f \) form four dual pairs. We have demonstrated that, if one of the solutions \( f \) of the seesaw equation corresponds to type I dominance, then the dual solution corresponds to type II seesaw dominance and vice versa. The other solutions then in general correspond to a hybrid seesaw with type I and type II contributions to \( m_\nu \) being of the same order. An important consequence of this result is that, knowing only the values of the input parameters, one can determine if there are solutions with one seesaw type dominance, but cannot decide which particular seesaw type dominates.

We have explored the behaviour of the analytic solutions for \( f \) in a number of interesting limiting cases: (a) one seesaw type dominates; (b) the two seesaw contributions almost cancel each other; (c) the eigenvalues of \( y \) are strongly hierarchical; (d) one light neutrino mass vanishes. By analyzing several numerical examples, we found that the masses of RH neutrinos exhibit the following generic features: (i) differ strongly for different dual solutions for \( f \); (ii) depend crucially on the light neutrino mass spectrum; (iii) depend weakly on variations of \( \theta_{13} \) within its allowed range and on variations of the low-energy leptonic CP-violating phases; (iv) depend strongly on the hierarchy among the eigenvalues of \( y \) only in the cases with predominant type I seesaw contribution to \( m_\nu \); (v) may become quasi-degenerate in several regions of the parameter space which are close to the points where the level crossings occur; (vi) can be at TeV scale or lighter even when \( v_R \) is much larger and/or some entries of the neutrino Yukawa coupling matrix \( y \) are of order one.
Table 1: Multiplicity of solutions for the Majorana-type Yukawa coupling matrix $f$. The first column indicates the seesaw equation under consideration, the second column specifies the assumption made on the structure of the Dirac-type Yukawa coupling matrix $y$, the third column indicates the section where the multiplicity was derived. Fourth, fifth and sixth columns give the number of solutions in the case of one, two and three lepton generations, respectively. The dash indicates the cases for which the number of solutions was not derived.

We have also studied (in Appendix A) the case of antisymmetric Dirac-type Yukawa coupling matrices $y$ and showed that our linearization procedure works in that case as well, leading to simple analytic expressions for $f$. The multiplicity of solutions in that case was found to be 1, 2 and 2 for one, two and three lepton generations.

In addition, we considered (in Appendix C) an alternative realization of the discrete LR symmetry, in which it acts as parity transformation rather than charge conjugation. This leads to type II and type I contributions to $m_{\nu}$ depending on $f$ and $f^*$ rather than on the same matrix $f$. In that case exactly the same duality of solutions holds, provided that the matrix $y$ is Hermitian or anti-Hermitian and invertible. In the present paper we gave explicit formulas for the solutions for one and two lepton generations, and briefly outlined the approach to the three-generation case. The multiplicity of solutions for $n$ lepton generations was shown to be $2^n$ in the case of Hermitian $y$, whereas for anti-Hermitian $y$ there are either $2^n$ or no solutions. The latter possibility is a consequence of the non-analytic dependence of the seesaw relation on $f$ in this realization of the discrete LR symmetry. Our results on the numbers of solutions for $f$ in various cases are summarized in Table 1.

Since one of our main goals was to provide a guidance for building specific models incorporating the LR-symmetric seesaw, we have discussed the structure of the seesaw formula in the minimal LR model, Pati-Salam model and in several $SO(10)$ models with renormalizable Yukawa couplings. Our analysis shows that in most cases the phenomenon of seesaw duality is realized: one can identify a matrix of couplings $f$ which enters in the seesaw formula both directly and through its inverse. Such matrix can be reconstructed by making use of the methods developed in this paper.

We have discussed briefly the stability of our results with respect to the renormaliza-
tion group running effects, which below the LR symmetry breaking scale may result in modifications of the relations $f_L = f_R$ and $y^T = \pm y$ that were at the basis of our analysis. Conditions were found under which these renormalization effects will not spoil our reconstruction of the matrix $f$ from the input data. A comprehensive study of this issue, however, will require a dedicated effort.

Finally, we pointed out that the usual leptogenesis mechanism of the generation of baryon asymmetry of the universe produces an asymmetry that strongly depends on the adopted solution for $f$ and therefore may in principle help discriminate between the eight allowed solutions. Moreover, the level crossing between the masses of the two lightest RH neutrinos, which occurs for a number of solutions, may provide a resonant enhancement of the produced asymmetry.

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A. Reconstruction of $f$ in the case of antisymmetric $y$

Consider the case of antisymmetric Dirac-type neutrino Yukawa coupling, $y = -y^T$. A crucial difference with respect to the case of symmetric $y$ is the fact that $(2n+1)$-dimensional antisymmetric matrices are not invertible, so that the usual duality among the solutions for $f$ does not hold for an odd number of lepton generations. In the case of one generation, one trivially has $y = 0$, and the seesaw is purely of type II.

The case of two generations is more interesting. An antisymmetric $2 \times 2$ matrix is defined by a single parameter. Consider the system (3.9) for $y_{\mu 2} = y_{\tau 3} = 0$ and $y_{\mu 3} = -y_{\tau 2} \equiv \bar{y}$. Using the same linearization procedure as in section 3.2, one easily finds the general solution:

$$f = \frac{x\lambda}{x\lambda - \bar{y}^2} m \quad \text{with} \quad (x\lambda)^2 - (2\bar{y}^2 + x \det m) x\lambda + \bar{y}^4 = 0. \quad (A.1)$$

The solutions $\lambda_{\pm}$ of the quadratic equation correspond to a pair of dual solutions $f_{\pm}$. The duality relation (3.12) is replaced by $x^2\lambda_+\lambda_- = \bar{y}^4$. Notice that both solutions are proportional to the matrix $m$.

While in general the system (3.9) has four solutions, two of them are singular when $y$ is exactly antisymmetric, as assumed above. This can be understood as follows: consider
the case where the diagonal entries of $y$ are equal to zero and the off-diagonal ones satisfy $y_{\mu 3} \approx \bar{y}$ and $y_{\tau 2} \approx -\bar{y}$, so that $y$ is nearly antisymmetric, Then $f$ is still (approximately) given by eq. (A.1), while the quartic equation for $\lambda$ reads

$$[(x \lambda)^2 - (2\bar{y}^2 + x \det m) x \lambda + \bar{y}^4] (x \lambda - \bar{y}^2)^2 \approx 0,$$  \hspace{1cm} (A.2)

to be compared with eq. (A.1). The pair of dual solutions corresponding to $x \lambda \to \bar{y}^2$ is not physical in the limit, since all the matrix elements of $f$ diverge.

Consider now the relative size of type I and II seesaw contributions to $m_\nu$. The two solutions of the characteristic equation for $\lambda$ in eq. (A.1) can be written as

$$x \lambda = \pm \frac{x \det m}{2} \left[ 1 + \frac{d}{2} \pm (1 + d)^{1/2} \right], \hspace{1cm} \lambda \equiv \frac{4\bar{y}^2}{x \det m}.$$  \hspace{1cm} (A.3)

A straightforward calculation shows that both $m_{\nu I}^I$ and $m_{\nu I}^I$ are proportional to $m_\nu$ and their ratio is

$$r_{+/-}^{II/I} \equiv \frac{(m_{\nu I}^I)_{ij}}{(m_{\nu I}^I)_{ij}} \left|_{\pm} \right. = \frac{2}{d} \left[ 1 + \frac{d}{2} \pm (1 + d)^{1/2} \right].$$  \hspace{1cm} (A.4)

For $|d| \ll 1$, $r_{+/-}^{II/I} \approx -4d/|d|^2$, so that one seesaw type dominates. For $|d| \sim 1$, $r_{+/-}^{II/I} \sim 1$, corresponding to hybrid seesaw. For $|d| \gg 1$, $r_{+/-}^{II/I} \approx -1$, which implies a considerable cancellation of the two seesaw contributions. These three cases resemble closely the corresponding limits in the one lepton generation scenario (section 3.1).

Let us now turn to the realistic case of three lepton generations. As was pointed out above, the usual duality does not apply since $y$ has one vanishing eigenvalue. However, a simple analytic reconstruction of the matrix $f$ is still possible, as we show below.

Any antisymmetric $3 \times 3$ matrix can be written as

$$y = \begin{pmatrix} 0 & y_3 & y_2 \\ -y_3 & 0 & y_1 \\ -y_2 & -y_1 & 0 \end{pmatrix} = U \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -y_{123} \\ 0 & y_{123} & 0 \end{pmatrix} U^T \equiv U y' U^T,$$  \hspace{1cm} (A.5)

where $y_{123} \equiv \sqrt{|y_1|^2 + |y_2|^2 + |y_3|^2}$ and the unitary matrix $U$ is given by

$$U = \frac{1}{y_{123} y_{23}} \begin{pmatrix} y_1^* y_2 & -y_2^*/y_{23} & 0 \\ y_1^* y_3 & -y_3^*/y_{23} & y_1 y_{123} \\ y_2^* y_3 & y_3^*/y_{123} & y_2 y_{123} \end{pmatrix},$$  \hspace{1cm} (A.6)

with $y_{23} \equiv \sqrt{|y_2|^2 + |y_3|^2}$. Defining $m' \equiv U^\dagger m U^*$ and $f' \equiv U^\dagger f U^*$, one can write the seesaw equation as

$$m' = f' + \frac{1}{x} y' f'^{-1} y'.$$  \hspace{1cm} (A.7)
The entries of the first row of $f'$ are uniquely determined as $f'_{1i} = m'_{1i}$, whereas to find the 2-3 block one needs to implement the usual linearization procedure. This yields

$$
\begin{align*}
  f'_{22} &= \frac{x\lambda m'_{22} - y^2_{123}m^2_{12}}{x\lambda - y^2_{123}m^2_{11}}, \\
  f'_{23} &= \frac{x\lambda m'_{23} - y^2_{123}m^2_{12}m'_{13}}{x\lambda - y^2_{123}m^2_{11}}, \\
  f'_{33} &= \frac{x\lambda m'_{33} - y^2_{123}m^2_{13}}{x\lambda - y^2_{123}m^2_{11}},
\end{align*}
$$

(A.8)

where $\lambda$ is the solution of the quadratic equation

$$
(x\lambda)^2 - (2y^2_{123}m^2_{11} + x \det m') x\lambda + y^4_{123}m^2_{11} = 0.
$$

(A.9)

The two solutions $\lambda_{\pm}$ have the same form as in eq. (A.3), but with $d \equiv 4y^2_{123}m^2_{11}/(x \det m')$.

Even though the usual duality property does not hold in this case, a different duality between the two solutions is present: one finds $x\lambda_{\pm} \equiv m'_{1i} + m'_{1j}/m'_{11}$. The two solutions for $f$ in the original basis are obtained as $f_{\pm} = Uf'_{\pm} U^T$, where $U$ is given in eq. (A.6).

Thus, for an antisymmetric $y$ the realistic case of three lepton generations can be reduced to that of an effective two-generation system, so that there are only two solutions of the LR seesaw formula for $f$, which are related by a modified duality.

Note that the method of counting the solutions developed in section 2.3 does not apply to the case of antisymmetric $y$: for one and three generations, because $y$ is not invertible, and for two generations because this is a degenerate case. Indeed, for $n = 2$ one finds $\bar{f}^{-1} = [- \det y/(x \det f)] \cdot \bar{f}$, so that the equation $\bar{m} = \bar{f} + \bar{f}^{-1}$, which replaces eq. (2.10) in the case $y^T = -y$, is linear rather than quadratic in $\bar{f}$. This equation results in a quadratic equation for $\det f$, leading to two solutions for $f$.

**B. Relative size of type I and II seesaw in the presence of flavor mixing**

In the simple case of one or more unmixed generations the relative size of type I and type II seesaw contribution to $m_\nu$ was discussed in section 3.1. Here we analyze this issue in the case of 2 and 3 generations with flavor mixing.

Due to the mixing, each entry of $m_\nu$ receives contributions from type I and II seesaw in different proportions. For simplicity, we will present only the conditions for the dominance of one seesaw type in all the matrix elements of $m$ (extensions to more general cases can be easily obtained). In what follows, the relations involving $m_{\alpha\beta}$ and $y_i$ will therefore be assumed to hold for each $\alpha, \beta$ and $i$, if not otherwise stated.

Consider first the two-generation case. From eq. (3.10) one can see that, when a solution $\lambda_1$ satisfies $|x\lambda_1| \gg |y_i y_j|$ ($i, j = 2, 3$), one obtains $f_1 \simeq m$ (assuming that $m_{\mu\mu}$ and $m_{\tau\tau}$ are of the same order). This corresponds to the dominant type II seesaw. Then the dual solution $x\lambda_2 = x\lambda_1 = y^2_{23}/(x\lambda_1)$ has modulus $\ll |y_i y_j|$ and the corresponding
matrix \( f_2 = \hat{f}_1 \) takes the form obtained in type I seesaw case. In analogy with eq. (3.2), let us define the set of parameters
\[
d_{ij}^{\alpha\beta\gamma\delta} \equiv \frac{v_L}{v_R} \frac{4v^2y_iy_j}{(m_\nu)_\alpha\beta(m_\nu)_\gamma\delta} = \frac{4y_iy_j}{xm_{\alpha\beta}m_{\gamma\delta}}, \quad i, j = 2, 3, \quad \alpha, \beta, \gamma, \delta = \mu, \tau.
\] (B.1)

Notice that eqs. (3.13) and (3.14) imply that a necessary condition to have \(|x\lambda_1| \gg |y_iy_j|\) is \(|m_{\alpha\beta}m_{\gamma\delta}| \gg 4|y_iy_j/x|\), that is,
\[
|d_{ij}^{\alpha\beta\gamma\delta}| \ll 1.
\] (B.2)

Therefore, this limit ensures the existence of a pair of dual solutions \( f_{1,2} \) with the dominance of one seesaw type, in analogy with the limit \( |d| \ll 1 \) in the one-generation case. The solutions \( \lambda_{1,2} \) are defined by choosing \( r_+ \) in eq. (3.13) and can be expanded in terms of the small parameters as
\[
x\lambda_1 = x \det m \left[ 1 + \frac{k}{x(\det m)^2} + \frac{3(\det m)^2y_2^3y_3^2 - k^2}{x^2(\det m)^4} + \ldots \right],
\]
\[
x\lambda_2 \equiv x\hat{\lambda}_1 = \frac{y_2^3y_3^2}{x \det m} \left[ 1 - \frac{k}{x(\det m)^2} + \ldots \right].
\] (B.3)

The solutions for \( f \) are related by \( f_1 = m - f_2 \) with
\[
f_2 \approx \frac{1}{x \det m} \left( -y_2^2 \left[ m_{\tau\tau} + y_3^2m_{\mu\mu} \det m - m_{\tau\tau}k \right] \frac{y_2y_3m_{\mu\tau}}{x(\det m)^2} \right) \left[ 1 - \frac{y_2y_3 \det m + k}{x(\det m)^2} \right] \ldots
\]
\[
- y_3^2 \left[ m_{\mu\mu} + \frac{y_2^2m_{\tau\tau} \det m - m_{\mu\mu}k}{x(\det m)^2} \right].
\] (B.4)

The second terms in the square brackets represent the leading order correction to the pure type I seesaw.

In general, condition (B.2) does not guarantee that the other dual pair of solutions \( f_{3,4} \) corresponds to one seesaw type dominance. If no special cancellations occur, eq. (B.2) implies, in particular, \(|\det m| \gg 4|y_iy_j/x|\). If this condition is satisfied, then \( f_3 \) and \( f_4 \) are of hybrid type. This novel feature with respect to the one-generation case is a genuine effect of flavor mixing. In fact, the solutions corresponding to the choice of \( r_- \) in eq. (3.13) are given, up to higher orders in small parameters, by
\[
x\lambda_{3,4} = - \frac{k}{2 \det m} \left[ 1 + \frac{s}{x(\det m)^2} + \ldots \right], \quad s \equiv \sqrt{k^2 - (2y_2y_3 \det m)^2}.
\] (B.5)

Notice that \( \lambda_{3,4} \) are dual to each other. The corresponding solutions for \( f \) (to leading order) are
\[
f_{3,4} \approx \frac{1}{2s} \left( m_{\mu\mu}(s \pm k) + \frac{2m_{\tau\tau}y_2^2 \det m}{x} \right) \left[ m_{\mu\tau}(s \pm k \pm 2y_3y_2 \det m) \right] \ldots
\]
\[
\left[ m_{\tau\tau}(s \pm k) + \frac{2m_{\mu\mu}y_3^2 \det m}{x} \right].
\] (B.6)
Roughly, \((f_{3,4})_{ij} \sim m_{\alpha\beta}\) and \(F_{3,4} = \lambda_{3,4} \sim y_i y_j / x\). Therefore, type II and type I contributions to \(m_\nu\) are of the same order: \(m_\nu^{II} \sim v_L m_{\alpha\beta}\) and \(m_\nu^{I} \sim v^2 y_i y_j f_{kl} / (v_R F) \sim v_L m_{\alpha\beta}\); this proves that the solutions \(f_{3,4}\) are of hybrid type.

If a special cancellation in \(\det m\) occurs, also the solutions \(f_{3,4}\) are dominated by one type of seesaw. Indeed, in the limit \(\det m \rightarrow 0\), eq. \((3.13)\) becomes

\[
x \lambda_1,3 \approx \pm \frac{1}{2} \left( \sqrt{k x + 4 y_i^2 y_j^2} + \sqrt{k x} \right), \quad x \lambda_{2,4} \equiv x \hat{\lambda}_{1,3} \approx \pm \frac{1}{2} \left( \sqrt{k x + 4 y_i^2 y_j^2} - \sqrt{k x} \right),
\]

where \(k \approx (m_{\mu y_3} + m_{\tau y_2})^2\). When eq. \((B.2)\) holds, one has \(|k x| \gg 4 y_i^2 y_j^2\). As a consequence, \(|x \lambda_1| = |x \lambda_3| \gg |y_i y_j|\), so that \(f_{1,3}\) are solutions with dominant type II seesaw. By duality, \(f_{2,4}\) correspond to the dominant type I seesaw.

Note that a suppression in \(\det m_\nu\) is phenomenologically motivated if the neutrino spectrum has normal mass hierarchy. In this case the full \(3 \times 3\) mass matrix \(m_\nu\) is dominated by large entries in the \(2 \times 2\) \(\mu\tau\)-block. This block has to incorporate the maximal mixing observed in the atmospheric neutrino oscillations as well as the hierarchy \(\Delta m_{sol}^2 \ll \Delta m_{atm}^2\). These two requirements imply that the determinant of the \(\mu\tau\)-block is suppressed.

When one abandons the condition \((B.2)\), that is, when at least one \(|d_{\alpha\beta\gamma\delta}^{ij}| \gtrsim 1\), all four solutions are of hybrid type. In other words, \textit{some sets of input parameters necessarily imply a hybrid seesaw scenario}.

In the limit when all \(|d_{\alpha\beta\gamma\delta}^{ij}| \gg 1\), one has \(x \lambda_i \approx r_\pm / 4 \approx \pm y_2 y_3\), and eq. \((3.10)\) implies that (at least the diagonal) matrix elements of \(f\) grow as \(1 / (|x \lambda| - |y_2 y_3|)\), thus ending up to be much larger than \(m_{\alpha\beta}\). This means that one enters the cancellation region, where type I and II contributions are almost equal in absolute value and opposite in sign. In this limit, the matrix elements of \(f\) will violate perturbative unitarity. This means that for some values of the input parameters, there are no physically acceptable solutions for \(f\). In other words, \textit{the LR symmetric seesaw mechanism cannot reproduce certain sets of the input parameters and thus, in principle, can be ruled out}.

Consider now the three-generation case. In analogy with eq. \((B.3)\), one finds that a necessary condition for the dominance of one seesaw type is

\[
|d_{\alpha\beta\gamma\delta}^{ij}| \equiv v_L \left| \frac{4 v^2 y_i y_j}{(m_\nu)_{\alpha\beta}(m_\nu)_{\gamma\delta}} \right| \ll 1, \quad i, j = 1, 2, 3, \quad \alpha, \beta, \gamma, \delta = e, \mu, \tau.
\]

(B.8)

In this case, the solutions \(\lambda_i\) of eq. \((B.29)\) can be expanded in the small parameters \(d_{\alpha\beta\gamma\delta}^{ij}\), which we denote collectively by \(d\). The largest solution turns out to be

\[
\lambda_1 = \det m \left[ 1 + \frac{A}{(\det m)^2} + \frac{3 Y^2 S(\det m)^2 - A^2}{(\det m)^4} + O(d^3) \right].
\]

(B.9)

Noting that \(Y^2 / \lambda_1^2 \sim d^2\), it is straightforward to verify with eq. \((B.25)\) that \(f_1\) corresponds to the dominant type II seesaw. As a consequence, the three RH neutrino masses satisfy
$M_{1,2,3} \gg y_i y_j M_s$, where $M_s$ is the seesaw scale defined as $M_s \equiv v^2/m_3$ (normal mass ordering) of $v^2/m_2$ (inverted mass ordering).

Three more solutions are given by $\lambda_{2,3,4} = \tilde{\lambda}_{2,3,4}[1 + \mathcal{O}(d)]$, where $\tilde{\lambda}_{2,3,4}$ are the roots of the cubic equation

$$\tilde{\lambda}^3 + \frac{A}{\det m} \tilde{\lambda}^2 + Y^2 S\tilde{\lambda} + Y^2 \det m = 0. \quad (B.10)$$

It is easy to see that $\lambda_{2,3,4}/\lambda_1 \sim d$. From eq. (B.25) one finds

$$(f_k)_{ij} \approx \frac{(\tilde{\lambda}_k^3 - Y^2 \det m) m_{ij} + \tilde{\lambda}_k^2 A_{ij} - Y^2 S_{ij}}{\lambda_k^3 - Y^2 \lambda_k S - 2 Y^2 \det m}, \quad k = 2, 3, 4. \quad (B.11)$$

One can verify that for these three solutions type I and II contributions to $m_\nu$ are of the same order (hybrid seesaw). Since $\lambda_k = \det f_k$ are suppressed with respect to $\lambda_1$, only two RH neutrinos are have masses that are larger than the seesaw scale, $M_{2,3} \gg y_i y_j M_s$, while $M_1 \sim y_i y_j M_s$.

The dual solutions, $\hat{\lambda}_{2,3,4} = -Y^2/\lambda_{2,3,4}$, are suppressed by a factor $\sim d^2$ with respect to $\lambda_1$. The corresponding structures for $f$ are

$$(\hat{f}_k)_{ij} \approx \frac{-Y^2(\hat{\lambda}_k S + \det m) m_{ij} - \hat{\lambda}_k^2 A_{ij} + Y^2 S_{ij}}{\hat{\lambda}_k^3 + Y^2 \lambda_k S - 2 Y^2 \det m}, \quad k = 2, 3, 4. \quad (B.12)$$

Note that $f_k + \hat{f}_k = m$, as required by duality. The solutions $\hat{f}_k$ also correspond to type I and II contributions to $m_\nu$ being of the same order (hybrid seesaw). The masses of the RH neutrinos satisfy $M_3 \gg y_i y_j M_s$, $M_{1,2} \sim y_i y_j M_s$.

Finally, the solution dual to $\lambda_1$ is given by

$$\hat{\lambda}_1 = -\frac{Y^2}{\det m} \left[1 - \frac{A}{(\det m)^2} + \mathcal{O}(d^2)\right]. \quad (B.13)$$

This is the smallest solution, since $\hat{\lambda}_1/\lambda_1 \sim d^3$. By duality, it corresponds to the dominant type I seesaw, as one can check explicitly expanding eq. (B.25). All three RH neutrino masses are approximately at the seesaw scale, $M_{1,2,3} \sim y_i y_j M_s$.

As already mentioned, the parameters $d^{ij}_{\alpha\beta\gamma\delta}$ may not all be small at the same time. In that case, for any of the eight solutions $f$, only some of the elements of the matrix $m_\nu$ may be dominated by one seesaw type, while one or more other elements receive significant contributions from both seesaw types.

A remark is in order on the special case when $\det m$ is suppressed. For example, one may have $|\det m| \ll |Y|$ even if eq. (B.8) holds. Clearly, in the limit $\det m \to 0$ the expansions (B.9)-(B.13) do not apply. We showed in section 3.3.2 that, when $\det m = 0$, two pairs of dual solutions for $\lambda$ are equal in absolute value and opposite in sign to the other two dual pairs. Moreover, the equation for $\lambda$ has the same form as in the two-generation case (with the redefinitions (3.37)). In particular, the relative size of type I and II seesaw
Contributions to $m_\nu$ may be evaluated as we did above for the two-generation case: when eq. (B.8) holds, there are two opposite solutions $\pm \lambda_1$ which lead to the dominant type II seesaw, while their duals correspond to the dominant type I seesaw. The other two opposite pairs of dual solutions correspond in general to hybrid seesaw, unless a special suppression in $\sqrt{A}$ occurs.

C. Alternative realization of the discrete left-right symmetry

Consider a different realization of the discrete LR symmetry in the minimal LR model, in which it acts as parity transformation. Eq. (5.6) is then replaced by

$$L \leftrightarrow (L^c)^c \equiv \begin{pmatrix} N_R \\ l_R \end{pmatrix}, \quad \Phi \leftrightarrow \Phi^\dagger, \quad \Delta_L \leftrightarrow \Delta_R^*. \quad \text{(C.1)}$$

This possibility is almost as popular in the literature (see, e.g., [24, 36]) as the one we adopted before, even though it does not allow an $SO(10)$ embedding of the minimal LR model. Eqs. (C.1) and (5.1) imply

$$f \equiv f_L = f_R^*, \quad g = g^\dagger, \quad h = h^\dagger. \quad \text{(C.2)}$$

In this case the seesaw relation is

$$m_\nu \simeq v_L f - \frac{v_R^2}{v_R} y (f^*)^{-1} y^T, \quad \text{(C.3)}$$

where $y$ is defined in eq. (5.3). The difference in complex phases between the seesaw formulas (1.3) and (C.3) can, in principle, lead to a different phenomenology if CP is violated in the leptonic sector.

It can be easily checked by direct substitution into eq. (C.3) that, if the matrix $y$ is Hermitian or anti-Hermitian and invertible, this equation possesses the same duality property as eq. (1.3): namely if a matrix $f$ solves eq. (C.3), so does its dual $\hat{f} \equiv \frac{m_\nu}{v_L} - f$. The above conditions on $y$ are both necessary and sufficient for the duality to hold.

Eqs. (C.2) and (5.3) imply that $y$ is actually Hermitian if $h = 0$ (which holds, e.g., in the SUSY case) or if the two VEVs of the bidoublet $v_1$ and $v_2$ are both real (this is not the case in general, since both VEVs can be non-zero, and their relative phase cannot be rotated away [36]). If $y$ is Hermitian, one can diagonalize it according to $y = U y_d U^\dagger$ with $U$ unitary, and go to the basis where $y$ is diagonal and real by absorbing $U$ in the definition of $m_\nu$ and $f$ (analogously to eq. (2.8)).

Consider first the one-generation case with an arbitrary complex $y$. Defining $f \equiv |f| e^{i\phi}$, one can obtain from eq. (C.3)

$$e^{i\phi} = \frac{m|f|}{|f|^2 - y^2 / x}. \quad \text{(C.4)}$$
The value of $|f|$ is determined by the requirement that that numerator and denominator on the right hand side have the same moduli (recall that in our conventions all the VEVs are real and positive, and so is $x$):

$$|f|^4 - \left[|m|^2 + \frac{2 \cos \chi |y|^2}{x}\right]|f|^2 + \frac{|y|^4}{x^2} = 0. \quad (C.5)$$

Here we used $y^2 \equiv |y|^2 e^{i\chi}$. Eq. (C.5) has (two) real and positive solutions for $|f|^2$ if and only if

$$|m|^2 > \frac{2|y|^2}{x}(1 - \cos \chi). \quad (C.6)$$

This means that, if the input parameters do not obey this inequality, no value of $f$ satisfies the seesaw formula (C.3). In other words, such a set of input parameters cannot be explained by the LR symmetric seesaw mechanism. This problem never occurs in the case of eq. (1.3), which is analytic in the elements of $f$ and thus always has solutions.

The requirement that $y$ be Hermitian or anti-Hermitian reduces in the one generation case to the condition that $y$ be real ($\chi = 0$) or purely imaginary ($\chi = \pi$). Let us focus on the $\chi = 0$ case. Eq. (C.6) is then always satisfied and the two solutions of eq. (C.5) are

$$|f| = \frac{|m|}{2} \left[(1 + |d|)^{1/2} \pm 1\right], \quad (C.7)$$

where $d$ was defined in eq. (3.2). The discussion of the dependence of $|f|$ on $|d|$ in different limiting cases is analogous to that in section 3.1. Finally, the phases $\phi \equiv \arg f$ are determined uniquely by plugging eq. (C.7) into eq. (C.4). Thus, if $y$ is real, eq. (C.3) has two solutions.

Consider now two lepton generations, limiting ourselves to the case $y = y^\dagger$ and working in the basis where $y$ is diagonal and real. The linearization procedure is the same as in section 3.2. However, to close the system of equations one has to consider both eq. (C.3) and its complex conjugate, so that the linearized system contains 6 equations. Defining $\lambda = |\lambda| e^{i\rho}$, one arrives at the solution which is analogous to that in eq. (3.10):

$$f = \frac{x|\lambda|}{(x|\lambda|)^2 - y_{23}^2 y_3^2} \left\langle x|\lambda|m_{\mu\mu} + y_{23}^2 m_{\tau\tau}^* e^{i\rho} x|\lambda|m_{\mu\tau} - y_{23}^2 m_{\mu\mu}^* e^{i\rho}\right. \right. \left. \ldots \right. \left. \ldots \right. \left. x|\lambda|m_{\tau\tau} + y_{23}^2 m_{\mu\mu}^* e^{i\rho}\right. \right. \left. \right\rangle. \quad (C.8)$$

The parameter $\lambda$ is determined, as usual, from the equation $\lambda = \det f(\lambda)$, which reads:

$$\left(x^2|\lambda|^2 - y_{23}^2 y_3^2\right)^2 - x^3|\lambda|^2(|m_{\mu\mu}|^2 y_3^2 + 2|m_{\mu\tau}|^2 y_{23} y_3 + |m_{\tau\tau}|^2 y_3^2) - x^2|\lambda|^2 \frac{\det me^{-i\rho x^2}|\lambda|^2 + (\det m)^* e^{i\rho y_{23}^2 y_3^2}}{\det me^{-i\rho x^2}} = 0. \quad (C.9)$$

The imaginary part of this equation implies $\rho \equiv \arg \lambda = \arg \det m + l\pi$ with $l = 0, 1$ (barring the special cases $\det m = 0$ and $x|\lambda| = y_{23} y_3$). Then $|\lambda|$ satisfies a quartic equation, which is completely analogous to that in eq. (3.11). Therefore, its four solutions are easily found...
analytically and one can verify that for each $l = 0$ and $l = 1$ there is only one pair of dual solutions which yield real and positive $|\lambda|$ and therefore are acceptable. As a consequence, also in this scenario there are four solutions for the matrix $f$.

The three-generation case can be considered quite analogously. The linearization procedure yields a closed set of 12 linear equations for the elements of the matrices $f$ and $f^*$, and solving the characteristic equation one arrives at eight possible solutions for $f$.

It is actually not difficult to show that the number of solutions for $n$ lepton generations is always $2^n$, provided that the matrix $y$ is Hermitian and invertible. Indeed, going into the basis where $y$ is diagonal and real and performing the same transformations as in section 2.3, one arrives at the equation
\begin{equation}
\bar{m} = \bar{f} - (\bar{f}^*)^{-1},
\end{equation}
where $\bar{m}$ and $\bar{f}$ are symmetric. The matrix $\bar{f}$ can be diagonalized as $\bar{f} = V\bar{f}_d V^T$, where $V$ is unitary and $\bar{f}_d$ diagonal and real; it is easy to see that then $(\bar{f}^*)^{-1}$ is diagonalized by the same transformation. Therefore, multiplying eq. (C.10) by $V^\dagger$ on the left and by $V^*$ on the right, one diagonalizes its right hand side, and so also the left hand side. Thus, $V$ is a unitary matrix that diagonalizes the matrix $\bar{m}$ and therefore is determined by the known quantities $m$ and $y$. In the diagonal basis eq. (C.10) yields
\begin{equation}
\bar{f}_{di}^2 - \bar{m}_{di} \bar{f}_{di} - 1 = 0, \quad i = 1, \ldots, n.
\end{equation}
This gives two values of $\bar{f}_{di}$ for each $i$:
\begin{equation}
\bar{f}_{di} = \frac{\bar{m}_{di}}{2} \pm \sqrt{\frac{\bar{m}_{di}^2}{4} + 1}.
\end{equation}
A full $n$-generation solution is obtained by picking one of these two values for each $i$, which yields $2^n$ solutions for the matrix $\bar{f}$ and therefore for $f$.

The same method can also be applied to the case of anti-Hermitian $y$ (making use of the fact that $iy$ is Hermitian). Eq. (C.12) is then replaced by
\begin{equation}
\bar{f}_{di} = \frac{\bar{m}_{di}}{2} \pm \sqrt{\frac{\bar{m}_{di}^2}{4} - 1},
\end{equation}
This leads to $2^n$ solutions for $f$, provided that all $\bar{m}_{di}$ satisfy $\bar{m}_{di}^2 > 4$; otherwise the seesaw equation (C.3) has no solutions.

A procedure similar to that described below eq. (C.10) (but making use of a complex orthogonal transformation rather than of a unitary one) was first used in [13] in order to solve eq. (1.3) with symmetric $y$. In both cases, this approach allows not only to count the solutions of the seesaw equation, but also to obtain them explicitly.
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