QUANTITATIVE BOUNDS ON THE DISCRETE SPECTRUM OF NON SELF-ADJOINT QUANTUM MAGNETIC HAMILTONIANS

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Abstract. We establish Lieb-Thirring type inequalities for non self-adjoint relatively compact perturbations of certain operators of mathematical physics. We apply our results to quantum Hamiltonians of Schrödinger and Pauli with constant magnetic field of strength $b > 0$. In particular, we use these bounds to obtain some information on the distribution of the eigenvalues of the perturbed operators in the neighborhood of their essential spectrum.

1. Introduction and an abstract result

Recently, a number of results on the spectral properties of the non self-adjoint perturbations of operators of mathematical physics were obtained. We quote the articles by Frank-Laptev-Lieb-Seiringer [5], Borichev-Golinskii-Kupin [1], Demuth-Katriel-Hansmann [3], Hansmann [9], Golinskii-Kupin [8], Pushnitskii-Raikov-Villegas-Blas [17] turned to the study of the discrete spectrum of these perturbations. The purpose of this short note is to announce and to give a brief overview of new results in this direction. The main point is that the current construction applies to a large class of operators containing magnetic Schrödinger and Pauli operators with constant magnetic field, hence generalizing the methods of a recent paper by the author [24].

Let $\mathcal{H}_0$ be an unbounded self-adjoint operator defined on a dense subset of $L^2(\mathbb{R}^m)$, $m \geq 1$. Suppose that the spectrum $\sigma(\mathcal{H}_0)$ of the operator is given by an infinite discrete sequence of (real) eigenvalues of infinite multiplicity, i.e.

$$\sigma(\mathcal{H}_0) = \sigma_{\text{ess}}(\mathcal{H}_0) = \bigcup_{j=1}^{\infty} \{\Lambda_j\},$$

where

$$\Lambda_0 \geq 0, \quad \Lambda_{j+1} > \Lambda_j, \quad |\Lambda_{j+1} - \Lambda_j| \leq \delta$$

with some fixed $\delta > 0$. Concrete examples of operators satisfying these assumptions are Schrödinger operators acting on $L^2(\mathbb{R}^{2d}, \mathbb{C})$, $d \geq 1$, and Pauli operators on $L^2(\mathbb{R}^{2d}, \mathbb{C}^2)$ with constant magnetic field of strength $b > 0$, see sections [3] and [4] respectively.

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On the domain of $\mathcal{H}_0$, we consider a (non self-adjoint) diagonal relatively compact perturbation $V$ of $\mathcal{H}_0$, and the perturbed operator

$$\mathcal{H} = \mathcal{H}_0 + V.$$  

(1.2)

This means that $\text{dom}(\mathcal{H}_0) \subset \text{dom}(V)$, and $V(\mathcal{H}_0 - \lambda)^{-1}$ is compact for $\lambda \in \rho(\mathcal{H}_0)$, the resolvent set of the operator $\mathcal{H}_0$. It is well known (see e.g. [12, Chapter VI]) that under this condition on $V$, there is a $\mu < 0$ such that

$$\sigma(\mathcal{H}) \subset \{ \lambda \in \mathbb{C} : \text{Re} \lambda \geq \mu \}.$$  

(1.3)

Furthermore, we impose an additional restriction on $V$ allowing us to control the numerical range $N(\mathcal{H})$ of the operator $\mathcal{H}$, i.e.

$$N(\mathcal{H}) := \{ (\mathcal{H} f, f) : f \in \text{dom}(\mathcal{H}), ||f||_{L^2(\mathbb{R}^n)} = 1 \}.$$  

(1.4)

Namely, we have

$$\sigma(\mathcal{H}) \subset \overline{N(\mathcal{H})} \subset \{ \lambda \in \mathbb{C} : \text{Re} \lambda \geq \mu_1 \}$$  

for some $\mu_1 < 0$. For convenience, we put $\mu_0 = \mu_1 - 1$.

Recall that a compact operator $L$ defined on a separable Hilbert space belongs to the Schatten-von Neumann class $S_p, p \geq 1$, if $||L||_{S_p} = (\text{Tr} |L|^p)^{1/p}$ is finite. We also require that

$$V(\mathcal{H}_0 - \lambda)^{-1} \in S_p,$$

for some $p \geq 1$, which is a stronger condition than just saying that the operator $V(\mathcal{H}_0 - \lambda)^{-1}$ is compact.

Since $V$ is a relatively compact perturbation with respect to the self-adjoint operator $\mathcal{H}_0$, then the Weyl’s criterion on the invariance of the essential spectrum implies that $\sigma_{\text{ess}}(\mathcal{H}) = \sigma_{\text{ess}}(\mathcal{H}_0) = \bigcup_{j=1}^{\infty} \Lambda_j$. Still, the operator $\mathcal{H}$ can have a (complex) discrete spectrum $\sigma_{\text{disc}}(\mathcal{H})$ accumulating to $\bigcup_{j=1}^{\infty} \Lambda_j$, see Gohberg-Goldberg-Kaashoek [6, Theorem 2.1, p. 373], and the coming theorem gives a necessary condition on its distribution. The conclusion of the theorem is written in the form of a relation which is often called a Lieb-Thirring type inequality, see Lieb-Thirring [14] for original work.

**Theorem 1.1.** Let $\mathcal{H}_0$ be a self-adjoint operator and $\sigma(\mathcal{H}_0) = \bigcup_{j=1}^{\infty} \Lambda_j$ as above. Let $\mathcal{H} = \mathcal{H}_0 + V$, and, for some $p > 1$, the perturbation $V$ satisfies

$$||V(\mathcal{H}_0 - \mu_0)^{-1}||_{S_p} \leq K_0,$$

with a constant $K_0$. Then

$$\sum_{\lambda \in \sigma_{\text{disc}}(\mathcal{H})} \frac{\text{dist}(\lambda, \bigcup_{j=1}^{\infty} \Lambda_j)^p}{(1 + |\lambda|)^{2p}} \leq C_0 K_0,$$

(1.7)

where $C_0 = C(p, \mu_0, \Lambda_0)$ is a constant depending on $p, \mu_0$, and $\Lambda_0$.

The proof of this theorem (see Section 2 for more details) is essentially based on a recent theorem of Hansmann [9], and a technical distortion lemma for a conformal mapping coming from complex analysis, see Lemma 2.1.

Applications of this result to magnetic Schrödinger operators on $L^2(\mathbb{R}^{2d}, \mathbb{C})$ and magnetic Pauli operators on $L^2(\mathbb{R}^{2d}, \mathbb{C}^2)$ are given in Theorems 3.1 and
4.1. respectively. In Golinskii-Kupin [8], similar results are obtained for complex perturbations of finite band Schrödinger operators.

Bound (1.7) can be rewritten in a simpler manner for various subsets of $\sigma_{\text{disc}}(\mathcal{H})$. For instance, let $\tau > 0$ be fixed. Then, for $\lambda$ satisfying $|\lambda| \geq \tau$, one has that

$$\frac{1}{1 + |\lambda|} = \frac{1}{|\lambda|} \frac{1}{1 + \frac{1}{|\lambda|^{-1}}} \geq \frac{1}{|\lambda|} \frac{1}{1 + \tau^{-1}},$$

and

$$(1.8) \sum_{\lambda \in \sigma_{\text{disc}}(\mathcal{H}) \atop |\lambda| \geq \tau} \frac{\text{dist}(\lambda, \cup_{j=1}^{\infty} \{\Lambda_j\})^p}{|\lambda|^{2p}} \leq C_1 \left(1 + \frac{1}{\tau}\right)^{2p} K_0.$$ 

Furthermore, if $$(\lambda_k) \subset \sigma_{\text{disc}}(\mathcal{H})$$ converges to a point of $\sigma_{\text{ess}}(\mathcal{H}) = \cup_{j=1}^{\infty} \{\Lambda_j\}$, one has that

$$(1.9) \sum_{k} \text{dist}(\lambda_k, \cup_{j=1}^{\infty} \{\Lambda_j\})^p < \infty.$$ 

This means that, a priori, the eigenvalues from $\sigma_{\text{disc}}(\mathcal{H})$ are getting less densely distributed in a neighborhood of the $\Lambda_j$ with growing $p$.

Similarly, we can also obtain information on diverging sequences of eigenvalues $$(\lambda_k) \subset \sigma_{\text{disc}}(\mathcal{H})$$. For example, if for some $\tau > 0$ the sequence $$(\lambda_k)$$ is such that

$$\text{dist}(\lambda_k, \cup_{j=1}^{\infty} \{\Lambda_j\}) \geq \tau,$$

then one has that

$$(1.10) \sum_{k=1}^{\infty} \frac{1}{|\lambda_k|^{2p}} < \infty.$$ 

We shall progress as follows. We give the sketch of the proof of our main abstract result (Theorem 1.1) in Section 2. We apply it to magnetic 2d-Schrödinger and 2d-Pauli operators in Sections 3 and 4 respectively. In Section 5 we treat the case of magnetic $(2d+1)$-Pauli operators with constant magnetic field. Here, the essential spectrum of the operator under consideration equals $\mathbb{R}_+$, which is rather different from the case of the essential spectrum coinciding with the (discrete) set of Landau levels $\cup_{j=1}^{\infty} \{\Lambda_j\}$ (1.1). This requires the use of methods close to those from Sambou [24].

We adopt mathematical physics and spectral analysis notation and terminology from Reed-Simon [23]. As for the classes of compact operators (i.e. Schatten-von Neumann ideals), we refer the reader to Simon [25] and Gohberg-Goldberg-Krupnik [7]. Constants are generic, i.e. changing from one relation to another. For a real $x$, $\lfloor x \rfloor$ denotes its integer part.

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2. THE ABSTRACT RESULT: SKETCH OF THE PROOF

The following result of Hansmann (see [9, Theorem 1]) is the first crucial point of the proof. Let $B_0 = B_0^*$ be a bounded self-adjoint operator acting on
a separable Hilbert space, and $B$ a bounded operator satisfying $B - B_0 \in S_p$, $p > 1$. Then
\begin{equation}
\sum_{\lambda \in \sigma_{\text{disc}}(B)} \text{dist}(\lambda, \sigma(B_0))^p \leq C\|B - B_0\|_{S_p}^p,
\end{equation}
where the constant $C$ is explicit and depends only on $p$. Note that we cannot apply \((2.1)\) to unbounded operators $H_0$ and $H$. To fix this, let us consider bounded resolvents
\begin{equation}
B_0(\mu_0) := (H_0 - \mu_0)^{-1} \quad \text{and} \quad B(\mu_0) := (H - \mu_0)^{-1},
\end{equation}
where $\mu_0$ is the constant defined next to \((1.5)\). Furthermore,
\begin{equation}
(H - \mu_0)^{-1} - (H_0 - \mu_0)^{-1} = -(H - \mu_0)^{-1}V(H_0 - \mu)^{-1},
\end{equation}
and we obtain
\begin{equation}
\|B(\mu_0) - B_0(\mu_0)\|_{S_p}^p \leq \|(H - \mu_0)^{-1}\|_p^p \|V(H_0 - \mu)^{-1}\|_{S_p}^p,
\end{equation}
where $\|\cdot\|$ stays for the usual operator norm. By \((1.5)\), we have
\begin{equation}
\sigma(H) \subset \overline{N(H)} \subset \{\lambda \in \mathbb{C} : \Re \lambda \geq \mu_1\}.
\end{equation}
This implies that dist$(\mu_0, \overline{N(H)}) \geq 1$, and using \([2, \text{Lemma 9.3.14}]\) we get
\begin{equation}
\|(H - \mu_0)^{-1}\| \leq \frac{1}{\text{dist}(\mu_0, \overline{N(H)})} \leq 1.
\end{equation}
Consequently, by \((1.6), (2.3)\) and \((2.4)\), we obtain
\begin{equation}
\|B(\mu_0) - B_0(\mu_0)\|_{S_p}^p \leq K_0,
\end{equation}
where $K_0$ is the constant defined in \((1.6)\). Hence, we see
\begin{equation}
\sum_{z \in \sigma_{\text{disc}}(B(\mu_0))} \text{dist}(z, \sigma(B_0(\mu_0)))^p \leq CK_0
\end{equation}
by applying Hansmann’s theorem \((2.1)\) to the resolvents $B(\mu_0)$ and $B_0(\mu_0)$. Putting $z = \varphi_{\mu_0}(\lambda) = (\lambda - \mu_0)^{-1}$, we have by the Spectral Mapping theorem \((2.7)\)
\begin{equation}
z \in \sigma_{\text{disc}}(B(\mu_0)) \quad (z \in \sigma(B_0(\mu_0))) \iff \lambda \in \sigma(H) \quad (\lambda \in \sigma(H_0)).
\end{equation}
So, we come to a distortion lemma for the conformal map $z = \varphi_{\mu_0}(\lambda) = (\lambda - \mu_0)^{-1}$, which is the second important ingredient of the proof of the theorem.

**Lemma 2.1.** Let $\mu_0$ be the constant defined next to \((1.5)\) and $\Lambda_j$ the Landau levels defined by \((1.1)\). Then the following bound holds
\begin{equation}
\text{dist}(\varphi_{\mu_0}(\lambda), \varphi_{\mu_0}(\cup_{j=1}^{\infty}\{\Lambda_j\})) \geq \frac{C \text{dist}(\lambda, \cup_{j=1}^{\infty}\{\Lambda_j\})}{(1 + |\lambda|)^2}, \quad \lambda \in \mathbb{C},
\end{equation}
where $C = C(\mu_0, \Lambda_0)$ is a constant depending on $\mu_0$ and $\Lambda_0$.

The proof of the lemma goes as \([24, \text{Lemma 6.2}]\) and is omitted. Now, combining the above lemma, estimates \((2.6)\) and \((2.7)\), we get
\begin{equation}
\sum_{\lambda \in \sigma_{\text{disc}}(H)} \text{dist}(\lambda, \cup_{j=1}^{\infty}\{\Lambda_j\})^p \leq C_0 K_0,
\end{equation}
where $C_0$ is a constant depending on $\mu_0$.
where $C_0 = C(p, \mu_0, \Lambda_0)$ is a constant depending on $p$, $\mu_0$ and $\Lambda_0$. This concludes the proof of Theorem 1.1. \qed

3. Examples: perturbations of the magnetic 2d-Schrödinger operators, $d \geq 1$

Let $X_\perp := (x_1, y_1, \ldots, x_d, y_d) \in \mathbb{R}^{2d}$, $d \geq 1$ and $b > 0$ a constant. We consider

$$H_0 := \sum_{j=1}^{d} \left\{ \left( D_{x_j} + \frac{1}{2} by_j \right)^2 + \left( D_{y_j} - \frac{1}{2} bx_j \right)^2 \right\},$$

(3.1)

the Schrödinger operator acting on $L^2(\mathbb{R}^{2d}) := L^2(\mathbb{R}^{2d}, \mathbb{C})$ with constant magnetic field of strength $b > 0$. The self-adjoint operator $H_0$ is originally defined on $C^\infty_0(\mathbb{R}^{2d})$, and then closed in $L^2(\mathbb{R}^{2d})$. It is well known (see e.g. [4]) that its spectrum consists of the increasing sequence of Landau levels

$$\Lambda_j = b(d+2j), \quad j \in \mathbb{N},$$

(3.2)

and the multiplicity of each eigenvalue $\Lambda_j$ is infinite. As in (1.2), define the perturbed operator

$$H = H_0 + V,$$

(3.3)

where we identify the non self-adjoint perturbation $V$ with the multiplication by the function $V: \mathbb{R}^{2d} \to \mathbb{C}$. Most of known results on the discrete spectrum of Schrödinger operators deal with self-adjoint perturbations $V$ and study its asymptotic behaviour at the edges of its essential spectrum. For $V$ admitting power-like or slower decay at infinity, see [10, Chap. 11-12], [17], [19], [20], [26], [27], and for potentials $V$ decaying at infinity exponentially fast or having a compact support see [22]. For Landau Hamiltonians in exterior domains see [11], [15] and [18].

Consider first the class of non self-adjoint electric potentials $V$ satisfying

$$\text{Re}(V) f, f \geq \mu_1 \|f\|^2$$

(3.4)

for some $\mu_1 < 0$, and the following estimate

$$|V(X_\perp)| \leq CF(X_\perp), \quad F \in L^p(\mathbb{R}^{2d}), \quad p \geq 2,$$

(3.5)

where $C > 0$ is a constant and $F$ is a positive function. The definition of the numerical range (1.4) implies that

$$\sigma(\mathcal{H}) \subset \overline{\mathcal{N}(\mathcal{H})} \subset \{ \lambda \in \mathbb{C} : \text{Re} \lambda \geq \mu_1 \}.$$

Theorem 3.1 is an immediate consequence of the following lemma and Theorem 1.1 with $\mathcal{H}_0 = H_0$, $\mathcal{H} = H$ and $m = 2d$.

**Lemma 3.1.** [24, Lemma 6.1] Let $\Lambda_j$ be the Landau levels defined by (3.2) and $\lambda \in \mathbb{C} \setminus \cup_{j=1}^{\infty} \{ \Lambda_j \}$. Assume that $F \in L^p(\mathbb{R}^{2d})$, $d \geq 1$ with $p \geq 2\left\lfloor \frac{d}{2} \right\rfloor + 2$. Then there exists a constant $C = C(p, b, d)$ such that

$$\|F(H_0 - \lambda)^{-1}\|_{S_p}^p \leq \frac{C(1 + |\lambda|)^d \|F\|_{L^p}^p}{\text{dist}(\lambda, \cup_{j=1}^{\infty} \{ \Lambda_j \})^p}.$$  

(3.6)
Theorem 3.1. Let $H = H_0 + V$ be the perturbed Schrödinger operator defined by (3.3) with $V$ satisfying conditions (3.4) and (3.5). Assume that $F \in L^p(\mathbb{R}^{2d}), \ d \geq 1$ with $p \geq 2\left[\frac{d}{2}\right] + 2$. Then we have

\begin{align}
\sum_{\lambda \in \sigma_{\text{disc}}(H)} \frac{\text{dist}(\lambda, \bigcup_{j=1}^{\infty} \{\Lambda_j\})^p}{(1 + |\lambda|)^{2p}} \leq C_1 \|F\|_{L^p}^p,
\end{align}

where the constant $C_1 = C(p, \mu_1, b, d)$ depends on $p$, $\mu_0 := \mu_1 - 1$, $b$ and $d$.

Notice that if the electric potential $V$ is bounded, then $\mu_0$ can be eliminated in the constant $C_1 = C(p, \mu_0, b, d)$. The price we pay is the additional factor $(1 + \|V\|_{\infty})^{2p}$ in the RHS of (3.7), see [24, Theorem 2.2].

It goes without saying that we can derive relations similar to (1.8)-(1.10) in the present situation.

It seems appropriate to mention that the assumptions of Theorem 3.1 are typically satisfied by the potentials $V : \mathbb{R}^{2d} \to \mathbb{C}$ such that

\begin{align}
|V(X_\perp)| \leq C \langle X_\perp \rangle^{-m}, \quad m > 0, \quad pm > 2d, \quad p \geq 2,
\end{align}

where $C > 0$ is a constant and $\langle y \rangle := (1 + |y|^2)^{1/2}, \ y \in \mathbb{R}^n, \ n \geq 1$. Certain examples showing the sharpness of our results in the two-dimensional case are discussed in [24, subsection 2.2].

4. Examples: perturbations of the magnetic 2d-Pauli operators, $d \geq 1$

To simplify, we consider the two-dimensional Pauli operator acting in $L^2(\mathbb{R}^2) := L^2(\mathbb{R}^2, \mathbb{C}^2)$ and describing a quantum non-relativistic $\frac{1}{2}$-spin-particle subject to a magnetic field of strength $b$ and electric potential $V$. The general case can be treated in a same manner (see the discussion after Theorem 4.1). The self-adjoint unperturbed Pauli operator $H_0$ given by

\begin{align}
H_0 := \begin{pmatrix}
-i \nabla - A & 0 \\
0 & -i \nabla - A + b
\end{pmatrix} =: \begin{pmatrix}
H_1 & 0 \\
0 & H_2
\end{pmatrix},
\end{align}

is defined originally on $C_0^\infty(\mathbb{R}^2)$ and then closed in $L^2(\mathbb{R}^2)$. Here $A := (A_1, A_2) : \mathbb{R}^2 \to \mathbb{R}^2$ is the magnetic potential, and

\begin{align}
b(X_\perp) := \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}, \quad X_\perp = (x, y) \in \mathbb{R}^2
\end{align}

is the magnetic field. We focus on the case where $b > 0$ is a constant. In this situation, the spectrum $\sigma(H_0)$ of the Pauli operator $H_0$ (see e.g [4]) is given by

\begin{align}
\sigma(H_0) = \bigcup_{j=1}^{\infty} \{\Lambda_j\}, \quad \Lambda_j = 2bj.
\end{align}

Now consider the matrix-valued electric potential

\begin{align}
V(X_\perp) := \{v_{\ell k}(X_\perp)\}_{1 \leq \ell, k \leq 2}, \quad X_\perp = (x, y) \in \mathbb{R}^2,
\end{align}

and introduce the perturbed operator

\begin{align}
H = H_0 + V,
\end{align}
where we identify the potential $V$ with the multiplication operator by the matrix-valued function $V$. As in the case of magnetic Schrödinger operators, most of known results on the discrete spectrum of Pauli operators deal with self-adjoint perturbations $V$ (see e.g. [21]). Let us consider the class of non self-adjoint electric potentials $V$ satisfying

\begin{equation}
(\text{Re} \,(V)f,f) \geq \mu_1 \|f\|^2,
\end{equation}

and the following estimate

\begin{equation}
|v_{\ell k}(x)| \leq CF(x), \quad 1 \leq \ell, k \leq 2, \quad F \in L^p(\mathbb{R}^{2d}), \quad p \geq 2,
\end{equation}

where $C > 0$ is a constant and $F$ a positive function. Note that if the potential $V$ is diagonal, i.e. $v_{12} = v_{21} = 0$, then assumption (4.6) is satisfied trivially if $\text{Re}(v_{11}) \geq \mu_1$ and $\text{Re}(v_{22}) \geq \mu_1$. In the case where $V$ is non-diagonal with $\text{Re}(v_{\ell k}) \geq \omega_0$ for some $\omega_0 < 0$, it can be verified that assumption (4.6) holds with $\mu_1 = -2|\omega_0|$. Furthermore, we have the following lemma giving a quantitative bound on the norm $\|V(H_0 - \lambda)^{-1}\|_{\mathcal{S}_p}$ in terms of the $L^p$-norm of $F$. Its proof goes along the same lines as the proof of [24] Lemma 6.1.

**Lemma 4.1.** Let $d = 1$ and $\Lambda_j$ be the Landau levels defined by (4.3) and $\lambda \in \mathbb{C} \setminus \cup_{j=1}^\infty \{\Lambda_j\}$. Assume that $F \in L^p(\mathbb{R}^2)$ with $p \geq 2$. Then there exists a constant $C = C(p, b, d)$ such that

\begin{equation}
\|F(H_0 - \lambda)^{-1}\|_{\mathcal{S}_p}^p \leq \frac{C(1 + |\lambda|)^d \|F\|_{L^p}^p}{\text{dist}(\lambda, \cup_{j=1}^\infty \{\Lambda_j\})^{p}}.
\end{equation}

Setting $\mathcal{H}_0 = H_0$, $\mathcal{H} = H$, $m = 2d = 2$ and recalling Theorem 1.1 readily yields the following result.

**Theorem 4.1.** Let $H = H_0 + V$ be the perturbed Pauli operator defined by (4.5) with $V$ satisfying (4.6) and (4.7). Assume that $F \in L^p(\mathbb{R}^2)$ with $p \geq 2$. Then the following bound holds true

\begin{equation}
\sum_{\lambda \in \sigma_{\text{disc}}(H)} \frac{\text{dist}(\lambda, \cup_{j=1}^\infty \{\Lambda_j\})^p}{(1 + |\lambda|)^{2p}} \leq C_3 \|F\|_{L^p}^p,
\end{equation}

where the constant $C_3 = C(p, \mu_0, b, d)$ depends on $p$, $\mu_0 := \mu_1 - 1$, $b$ and $d$.

As above, if the electric potential $V$ is bounded, then $\mu_0$ can be eliminated in $C_3 = C(p, \mu_0, b, d)$ with the additional factor $(1 + \|V\|_{\infty})^{2p}$ to pay in counterpart in the RHS of (4.9).

Notice that Theorem 4.1 remains valid if we replace the two-dimensional Pauli operator $H_0$ by the general $2d$-Pauli operators acting on $L^2(\mathbb{R}^{2d}, \mathbb{C}^2)$, $d \geq 1$, defined by

\begin{equation}
H_0 := \begin{pmatrix}
\mathbb{H}^{-}_{0,\perp} - bd & 0 \\
0 & \mathbb{H}^{+}_{0,\perp} + bd
\end{pmatrix} =: \begin{pmatrix}
\mathbb{H}^{-}_{0,\perp} & 0 \\
0 & \mathbb{H}^{+}_{0,\perp}
\end{pmatrix}.
\end{equation}

Here

\begin{equation}
\mathbb{H}^{\pm}_{0,\perp} := \sum_{j=1}^d \left\{ \left( D_{x_j} + \frac{1}{2}by_j \right)^2 + \left( D_{y_j} - \frac{1}{2}bx_j \right)^2 \right\}
\end{equation}
is the $2d$-Schrödinger operator defined by (3.1). In this case, the set of Landau levels is given by $\bigcup_{j=1}^{\infty} \{ \Lambda_j \}$ with $\Lambda_j = 2bdj$, $j \in \mathbb{N}$, and we require that $F \in L^p(\mathbb{R}^{2d})$ with $p \geq 2\left\lceil \frac{d}{2} \right\rceil + 2$.

Of course, the counterparts of relations (1.8)-(1.10) apply as well to magnetic Pauli operators under consideration.

5. On Lieb-Thirring type inequalities for magnetic $(2d + 1)$-Pauli operators, $d \geq 1$

In this section, we focus on $(2d + 1)$-dimensional self-adjoint Pauli operators with constant magnetic field, acting on $L^2(\mathbb{R}^{2d+1}) := L^2(\mathbb{R}^{2d+1}, \mathbb{C}^2)$, $d \geq 1$, defined by

$$\mathbb{P}_0 := \begin{pmatrix} \mathbb{H}_0 - bd & 0 \\ 0 & \mathbb{H}_0 + bd \end{pmatrix} =: \begin{pmatrix} \mathbb{P}_1 & 0 \\ 0 & \mathbb{P}_2 \end{pmatrix}.$$  

Here the constant $b > 0$ is the strength of the magnetic field and for the cartesian coordinates $x := (x_1, y_1, \ldots, x_d, y_d, x) \in \mathbb{R}^{2d+1}$,

$$\mathbb{H}_0 := \sum_{j=1}^{d} \left\{ \left( D_{x_j} + \frac{1}{2} by_j \right)^2 + \left( D_{y_j} - \frac{1}{2} bx_j \right)^2 \right\} + D_x^2, \quad D_\nu := -i \frac{\partial}{\partial \nu}$$

is the $(2d+1)$-self-adjoint Schrödinger operator with constant magnetic field originally defined on $C_0^\infty(\mathbb{R}^{2d+1}, \mathbb{C})$. It is well known (see e.g. [4]) that the spectrum of the operator $\mathbb{P}_0$ is absolutely continuous, coincides with $[0, +\infty)$ and has an infinite set of Landau levels

$$\Lambda_j = 2bdj, \quad j \in \mathbb{N}.$$  

We introduce the perturbed operator on the domain of the operator $\mathbb{P}_0$

$$\mathbb{P} = \mathbb{P}_0 + V,$$

where we identify the perturbation $V$ with the multiplication by the matrix-valued function

$$V(x) := \{ v_{\ell k}(x) \}_{1 \leq \ell, k \leq 2}.$$  

Here we assume that $V$ is a bounded non self-adjoint perturbation such that for any $x \in \mathbb{R}^{2d+1}$ and $1 \leq \ell, k \leq 2$,

$$|v_{\ell k}(x)| \leq CF(x)G(x),$$

where $C > 0$ is a constant, $F$ and $G$ are two positive functions satisfying $F \in L^p(\mathbb{R}^{2d+1})$ for $p \geq 2$, and $G \in (L^2 \cap L^\infty)(\mathbb{R})$. Under this assumption on $V$, we obtain (see Lemma 5.1) for any $\lambda \in \rho(\mathbb{P}_0)$ the resolvent set of the operator $\mathbb{P}_0$, that

$$\| F(H_0 - \lambda)^{-1} G \|_{s_p} < \infty.$$  

Once again, this implies that $V$ is a relatively compact perturbation.

The first ingredient of the proof is the following lemma obtained by methods similar to [24, Lemma 3.1].
Lemma 5.1. Let $d \geq 1$ and $\lambda \in \mathbb{C} \setminus [0, +\infty)$. Assume that $F \in L^p(\mathbb{R}^{2d+1})$ with $p \geq 2\left[\frac{d}{2}\right] + 2$ and $G \in \left(L^2 \cap L^\infty\right)(\mathbb{R})$. Then there exists a constant $C = C(p, b, d)$ such that

\begin{equation}
\left\| F(\mathbb{P}_0 - \lambda)^{-1}G \right\|_{L^p} \leq \frac{C(1 + |\lambda|)^{d + \frac{1}{2} - K_1}}{\text{dist}(\lambda, [0, +\infty))^{\frac{d}{2}} \text{dist}(\lambda, \cup_{j \in \mathbb{N}} \{\Lambda_j\})^\gamma},
\end{equation}

where $\Lambda_j$ are the Landau levels defined by (5.9) and

\begin{equation}
K_1 := \|F\|_{L^p}^p (\|G\|_{L^2} + \|G\|_{L^\infty})^p (1 + \|V\|_\infty)^{\frac{d}{2} + \frac{3}{2} + \varepsilon}.
\end{equation}

Note that since the potential $V$ is bounded, then the numerical range of the operator $\mathbb{P}$ satisfies

\begin{equation}
\sigma(\mathbb{P}) \subset N(\mathbb{P}) \subset \{\lambda \in \mathbb{C} : \text{Re}\lambda \geq -2\|V\|_\infty \text{ and } |\text{Im}\lambda| \leq 2\|V\|_\infty\}.
\end{equation}

The Lieb-Thirring type bound for the eigenvalues of the $(2d + 1)$-Pauli operator $\mathbb{P}$ is as follows.

Theorem 5.1. Let $\mathbb{P} = \mathbb{P}_0 + V$ with $V$ satisfying (5.4) and (5.5). Assume that $F \in L^p(\mathbb{R}^{2d+1})$ with $p \geq 2\left[\frac{d}{2}\right] + 2$, $d \geq 1$ and $G \in \left(L^2 \cap L^\infty\right)(\mathbb{R})$. Define

\begin{equation}
K := \|F\|_{L^p}^p (\|G\|_{L^2} + \|G\|_{L^\infty})^p (1 + \|V\|_\infty)^{d + \frac{3}{2} + \varepsilon}.
\end{equation}

for $0 < \varepsilon < 1$. Then we have

\begin{equation}
\sum_{\lambda \in \sigma_{\text{disc}}(H)} \text{dist}(\lambda, [0, +\infty))^{\frac{d}{2} + 1 + \varepsilon} \text{dist}(\lambda, \cup_{j \in \mathbb{N}} \{\Lambda_j\})^{(\frac{d}{2} - 1 + \varepsilon)} \leq C_5 K,
\end{equation}

where $\Lambda_j$ are the Landau levels defined by (5.9), $\gamma > d + \frac{3}{2}$ and $C_5 = C(p, b, d, \varepsilon)$ is a constant depending on $p$, $b$, $d$ and $\varepsilon$.

As usual, $[x]$ denotes the integer part of $x \in \mathbb{R}$, and $x_+ := \max(x, 0)$.

Sketch of the proof of the theorem. The proof goes along the same lines as the proof of [24] Theorem 2.1 with the help of Lemma 5.1. Since $\sigma_{\text{ess}}(\mathbb{P}) = [0, +\infty)$ with an infinite set of thresholds $\Lambda_j$, we obtain two types of estimates.

First, we bound the sums depending on parts of $\sigma_{\text{disc}}(\mathbb{P})$ concentrated around a Landau level $\Lambda_j$ using the Schwarz-Christoffel formula (see e.g. [13] Theorem 1, p. 176)). Namely, if we consider a rectangle

\[ \Pi_j := \{\lambda \in \mathbb{C} : |\lambda - \text{Re}\lambda| \leq b \text{ and } |\text{Im}\lambda| \leq \text{Const.}\} \]

around a Landau level $\Lambda_j$, then we have

\[ \sum_{\lambda \in \sigma_{\text{disc}}(H) \cap \Pi_j} \text{dist}(\lambda, [\lambda_0, +\infty))^{\frac{d}{2} + 1 + \varepsilon} \text{dist}(\lambda, \cup_{j \in \mathbb{N}} \{\Lambda_j\})^{(\frac{d}{2} - 1 + \varepsilon)} \leq C(p, b, j, \varepsilon) K \]

with the asymptotic property $C(p, b, j, \varepsilon) \sim j^{d + \frac{1}{2}}$.
Second, we get the global bound summing up the previous bounds with appropriate weights as follows,

\[
\sum_j \frac{1}{(1+j)\gamma} \sum_{\lambda \in \sigma_{\text{disc}}(H) \cap \Pi_j} \text{dist}(\lambda, [\Lambda_0, +\infty])^{\frac{p}{4}+1+\varepsilon} \text{dist}(\lambda, \cup_{j=1}^{\infty} \{\Lambda_j\})^{\frac{p}{4}-1+\varepsilon} +
\]

\[
\leq \sum_j \frac{C(p, b, d, j, \varepsilon)}{(1+j)^\gamma} K.
\]

Now, choosing \( \gamma > d + \frac{3}{2} \) and using the fact that for any \( \lambda \in \Pi_j \) we have \( 1+j \simeq 1+|\lambda| \), we get the global bound (5.11). \( \square \)

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