Threshold Tests as Quality Signals: Optimal Strategies, Equilibria, and Price of Anarchy

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October 22, 2021

Abstract

We study a signaling game between two firms competing to have their product chosen by a principal. The products have (real-valued) qualities, which are drawn i.i.d. from a common prior. The principal aims to choose the better of the two products, but the quality of a product can only be estimated via a coarse-grained threshold test: given a threshold θ, the principal learns whether a product’s quality exceeds θ or fails to do so.

We study this selection problem under two types of interactions. In the first, the principal does the testing herself, and can choose tests optimally from a class of allowable tests. We show that the optimum strategy for the principal is to administer different tests to the two products: one which is passed with probability $\frac{1}{3}$ and the other with probability $\frac{2}{3}$. If, however, the principal is required to choose the tests in a symmetric manner (i.e., via an i.i.d. distribution), then the optimal strategy is to choose tests whose probability of passing is drawn uniformly from $\left[\frac{1}{4}, \frac{3}{4}\right]$.

In our second interaction model, test difficulties are selected endogenously by the two firms. This corresponds to a setting in which the firms must commit to their testing (quality control) procedures before knowing the quality of their products. This interaction model naturally gives rise to a signaling game with two senders and one receiver. We characterize the unique Bayes-Nash Equilibrium of this game, which happens to be symmetric. We then calculate its Price of Anarchy in terms of the principal’s probability of choosing the worse product. Finally, we show that by restricting both firms’ set of available thresholds to choose from, the principal can lower the Price of Anarchy of the resulting equilibrium; however, there is a limit, in that for every (common) restricted set of tests, the equilibrium failure probability is strictly larger than under the optimal i.i.d. distribution.

1 Introduction

A principal wants to choose between two firms producing interchangeable products, whose qualities are drawn i.i.d. from a known prior. The principal wants to pick the product of higher quality — however, she cannot directly observe the products’ qualities. In order to learn more about the products’ qualities, the principal can simultaneously subject the products to tests. Specifically, we consider the simplest and most coarse-grained tests: binary (i.e., pass/fail) threshold tests that reveal whether the product’s quality lies above or below a chosen θ. How should the principal choose the tests to administer to the two products, so as to help her maximize the probability of picking the better of the two? We refer to this as the optimal selection problem.

Now consider an alternative setting in which firms conduct their own quality control in-house, according to a fully disclosed and verifiable procedure. This may be necessary if the principal does not possess the expertise to conduct quality control herself. In this setting, while the principal may not be able to conduct a test, we assume that she can verify that a firm correctly followed its disclosed testing protocol; in other

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words, we assume that firms inherently have the power to commit to a test. At the time a firm commits to a testing protocol, it will not know the exact quality of each individual product — for example, due to variations across batches and over time, or because the firm acts as an intermediary (e.g. head hunters who vet candidates for a hiring firm). Indeed, such variation is the reason testing is needed in the first place. As before, we assume that firms have independent common priors for their product qualities. How will firms choose tests in such an endogenous selection setting, if each firm wants to maximize the probability of its own product being selected? Will competition push the firms to subject themselves to very difficult tests, or will they coordinate on easy tests at equilibrium? How much worse off is the principal due to having to outsource quality control tests, rather than conducting them herself? Can she improve her probability of choosing the better product by restricting the set of tests from which the firms can choose, e.g., by prescribing standards that such tests must adhere to?

Endogenous test selection by two firms can be naturally viewed as a form of signaling; committing to a testing procedure takes the role of committing to a signaling scheme. Thus, our work can be construed as a natural game played between two agents whose strategies are signaling schemes from a restricted class of available schemes. This parallels several recent works on Bayesian persuasion games between multiple firms vying for customers (Au and Kawai 2019, 2020, Boleslavsky et al. 2016, Boleslavsky and Cotton 2018, Hwang et al. 2019); we discuss these in detail in Section 2. Our high-level question is what the equilibria of such signaling games look like, and how much efficiency is lost (if any) by letting the agents/firms choose their own signaling scheme rather than the principal being able to control how she receives information about the state of the world.

We investigate such questions using the following simple model (see Section 3). The two firms have products with real-valued qualities $X, Y$ drawn randomly from a common prior with continuous cdf $Ψ$. The principal has at her disposal a collection of tests parametrized by a threshold $θ ∈ \mathbb{R}$ which encodes the difficulty level of each test. When a firm’s product with quality $X$ is subjected to a test with threshold $θ$, the outcome reliably reveals whether $X ≥ θ$ (the product passes the test) or $X < θ$ (the product fails the test). In the language of signaling, this means that we restrict to signaling schemes with binary outcomes, in which the sets mapped to each outcome are intervals.

Based on the chosen test difficulties (which are observable in both optimal and endogenous selection regimes) and their outcomes, the principal selects one of the products. Her objective is to minimize the probability of choosing the worse product, while each firm’s objective is to maximize the probability of having its product chosen. We consider the following models, which endow the principal with varying degrees of control:

1. The principal must give both firms the same test.
2. The principal has full control over the difficulties $θ_X, θ_Y$ of the tests given to the two firms.
3. The principal specifies a distribution from which both firms draw tests in an i.i.d. manner. The restriction to identical distributions may be required to achieve ex-post fairness, compared to, for instance, randomizing which of the two firms gets which of two non-identical tests.
4. The firms may endogenously choose their own tests via equilibrium strategies.
5. The principal can restrict available tests to a set $S$ (common to both firms), and firms endogenously choose their tests from $S$. Such a restriction could arise if the principal is a government agency or sufficiently powerful firm providing binding quality control guidelines.

It is clear — simply from suitable subset relationships on sets of available actions — that in terms of the principal’s error probability, \{1, 4\} $≥$ \{5\} $≥$ \{3\} $≥$ \{2\}. Our goal is to explicitly characterize the optimal or equilibrium outcomes under these five models, thereby inferring which of the preceding comparisons are strict, as well as to quantify the increase in error probability for the principal resulting from a move to a weaker model. When comparing a model in which the principal has control with one in which the agents are allowed to choose tests according to an equilibrium strategy, this ratio exactly corresponds to the Price of Anarchy.

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1This is the more common view of signaling in the economics community: a signaling scheme is interpreted as a device (physical or otherwise) that maps relevant states of the world to observable signals. Fixing a device constitutes committing to a signaling scheme. In contrast, recent works in computer science apply signaling/persuasion to scenarios such as communications where it is less clear whether the sender has the ability to commit to a mapping.
1.1 Other Applications and Model Discussion

While we phrase our work in terms of two firms offering products, our model applies more broadly. In particular, it can be viewed as a generalization of the classic “forum shopping” model of Lerner and Tirole (2006) to multiple firms (property owners, in their language). In this model, firms can choose an external certification agency to issue a recommendation on whether or not their product is “acceptable.” There is a continuum of agencies, ranging from fully aligned with the firm’s interests to fully independent. Under suitable parameters, this model precisely corresponds to being able to choose any quantile threshold for a test. While the model does not place the tests “in house,” in terms of the firms’ choices, it is equivalent to our model. The focus of Lerner and Tirole (2006) is on the interplay of the independence/difficulty of the agency and the owner’s “concessions” — direct transfers to any user of the property, such as price reductions or additional features. As they argue, such a setup not only captures agencies certifying products, but also journals/conferences reviewing papers and similar endeavors. In addition to these applications, some of the literature on multi-sender cheap talk/Bayesian persuasion is motivated in terms of competing proposals, either to a funding agency or internal within an organization; see, e.g., (Boleslavsky and Cotton 2018, Boleslavsky et al. 2016).

Another application, aligned with the classic work of Spence (1973) and Ostrovsky and Schwarz (2010), is in the assessment of students. Here, the test is a pass/fail exam (or class) via which a student is assessed. The optimization problem may guide a teacher aiming to correctly rank the students in a class, while the endogenous test selection model roughly corresponds to students choosing the difficulty of projects to undertake or of classes to enroll in.

In the context of applications, three key assumptions in our model are worth discussing. The first is that firms are unaware of their quality when choosing tests. This power of commitment before the state of the world is revealed is the defining distinction between Bayesian Persuasion and Cheap Talk models, and is covered in depth in Section 2. As we discuss, most works on inter-firm signaling make this assumption. For example, Lerner and Tirole (2006) assume that property owners do not know users’ utilities for their product. Similarly, Ostrovsky and Schwarz (2010) consider early contracting between students and employers, in which students at the time of negotiation only have priors on their future performance. Naturally, as with all models, this assumption is a simplification, with reality lying between full and no commitment power.

The second assumption is that tests have binary and monotone (i.e., pass/fail) outcomes; in particular, we assume that no test can be passed with quality \( x \), but failed with quality \( x' > x \). Restricting to monotone information structures is quite common in the literature: for recent examples, see (Dworczak and Martini 2019), (Onuchic and Ray 2019), and (Candogan 2020) and discussions therein. Other kinds of restricted signal spaces also have significant precedent in the literature. Dughmi et al. (2016) analyze Bayesian Persuasion in which the sender is restricted in terms of the number of signals. Boleslavsky et al. (2016) assume that the state of the world is binary (the product is good or bad) and allow each sender to only send one of two signals; nevertheless, competition between senders results in complex signal distributions at equilibrium. Similarly, the certification models of (Gill and Sgroi 2012, Lerner and Tirole 2006) mostly consider binary outcomes (recommend/don’t recommend). As argued in (Gill and Sgroi 2012) (see, e.g., Footnote 3 in (Gill and Sgroi 2012) and the literature cited there), the main purpose of a test or evaluation is to provide a concise summary of the product. When the outcome of the evaluation must be concise, the number of possible signals that can be sent is necessarily bounded, and a binary signal is a clean and idealized way to capture such a desideratum. Monotonicity is natural to assume when signals should be interpretable by a decision maker. This justification is also borne out by the coarse-grained grading systems (pass/fail, grades A–F) typically used in education contexts. It also closely aligns with the argument made in Sobel (2013) that there is a tradeoff between accuracy and complexity of advice (i.e., signals).

The third assumption is that there are exactly two firms (for most of our results), and that their qualities are drawn i.i.d. This assumption is very standard in the study of related questions in competitive signaling; see, e.g., the in-depth discussion of (Li et al. 2016, Boleslavsky and Cotton 2018, Boleslavsky et al. 2016).

\(^2\)Lerner and Tirole (2006) do briefly discuss a multi-firm setting, but only consider one extremely limited example.

\(^3\)However, we note that in addressing the same real-world scenario, Gill and Sgroi (2012) instead consider a model where the owner knows the state before choosing the certifier; see Section 2 for details.
Hwang et al. 2019) and additional related work in Section 2. We discuss the difficulties with extending the result to $n > 2$ firms or non-identical priors in Section 8.

1.2 Our Results

As we elaborate in Section 3, it is equivalent — and much more convenient — to characterize tests not in terms of their thresholds, but in terms of the probability that a product will fail the test. Thus, we can view each possible test as a real number in $[0, 1]$; in this case, the products’ qualities can be assumed w.l.o.g. to be drawn uniformly from $[0, 1]$.

When both firms’ products have to be subjected to the same test, it is easy to see that the optimum test is the median test, passed with probability exactly $\frac{1}{2}$, which chooses the wrong product with probability $\frac{1}{4}$ (see Section 3). When the principal can give the firms different tests, our main result is summarized by the following theorem. (See Sections 4 and 5 for formal statements.)

**Theorem 1 (Optimal Selection of Tests by Principal: Informal)**

1. If the principal can assign arbitrary tests to the two firms, then it is optimal to give one firm a test of $\frac{1}{3}$ and the other a test of $\frac{2}{3}$. This results in a probability of $\frac{1}{6}$ of incorrect selection.
2. If the principal must draw i.i.d. tests for the firms, then the optimal rule draws test thresholds uniformly from the interval $\left[\frac{1}{4}, \frac{3}{4}\right]$. This results in a probability of $\frac{5}{24}$ of incorrect selection.

The preceding theorem is rather surprising! Even though the firms’ products have i.i.d. qualities, the principal can decrease her failure probability significantly (by 33%) by giving the firms very different tests. Analogously, a teacher trying to optimally rank students by ability should give the students different tests, even if their abilities share a common prior distribution.

For the case of endogenous test selection, the equilibrium and its probability of a mistake are characterized by the following result, stated formally and proved in Section 6:

**Theorem 2 (Equilibrium Distribution)** When firms’ qualities are drawn i.i.d. uniformly from $[0, 1]$, and firms choose their test difficulties endogenously, there is a unique Bayes-Nash Equilibrium, which is symmetric, and consists of each firm choosing difficulty $\theta \in [0, 1]$ from the probability density function (pdf) $f(\theta) = \frac{1}{2(\theta^2 + (1-\theta)^2)^{3/2}}$.

The principal’s resulting probability of incorrect selection is approximately 0.23056, causing a Price of Anarchy of approximately 1.38336 compared to the optimum correlated tests and approximately 1.10653 compared to the optimum i.i.d. test distribution.

Finally, in Section 7, we allow the principal to set “guidelines” for the firms’ quality control tests, by prescribing a set $S \subseteq [0, 1]$ from which the thresholds must be drawn.

**Theorem 3 (Restricted Equilibrium Distribution)** When the firms’ qualities are drawn i.i.d. uniformly from $[0, 1]$, and the firms choose their test difficulties endogenously from an interval $S = [a, b] \subseteq [0, 1]$, there is a unique Bayes-Nash Equilibrium. This unique Bayes-Nash equilibrium is symmetric and can be explicitly characterized in closed form.

Moreover, there exist values $a, b$ for which the resulting probability of a mistake by the principal is strictly smaller than for the interval $[0, 1]$; for example, for the interval $[0, 0.79]$, the probability of a mistake is approximately 0.22975.

However, even compared to a principal restricted to i.i.d. test choices, under symmetric Bayes-Nash Equilibria, the Price of Anarchy is lower-bounded by a constant strictly larger than one: for every set $S \subseteq [0, 1]$ (not just intervals), the probability of a mistake is at least $\frac{5}{24} + \frac{1}{256}$. One interesting interpretation of the preceding theorem is that a somewhat bigger part of the problem with endogenous test selection is that firms skew too much towards harder tests. Making extremely difficult tests (the top 20%) unavailable results in a (slightly) better equilibrium probability for the principal. However, as we will see in the analysis, when restricting the interval of available tests, the equilibrium distribution


has non-trivial point mass at the upper end of the interval; in other words, at equilibrium, firms will still compete by choosing difficult tests.

A visual representation of our results is given in Fig. 1. Taken together, our theorems imply a strict separation of all five models of test selection, and notably show that the principal has a higher probability of incorrect selection when choosing the same test for both firms compared to when they choose tests endogenously.

![Figure 1: The principal’s failure probabilities under different models of threshold choices.](image)

Our work raises a wealth of directions for future inquiry, discussed in detail in Section 8. Most immediate would be extensions to more than two firms and to richer signaling schemes. For an extension to multiple firms, an important point is to decide what the principal’s and the firms’ objectives are. One natural generalization is to have the principal still choose one (or \( k \)) of the firms’ products; this appears difficult. A “friendlier” generalization involves a principal who wants to fully rank the firms by quality (e.g., a teacher in a classroom setting), and aims to minimize the number of inversions compared to the true order. In this setting, a firm/student may try to minimize the expected number of other firms ranked ahead of it. Because the objective functions naturally decompose into pairwise objectives by linearity of expectation, our results carry over to this setting completely. The only necessary generalization is for the case of correlated tests. In fact, in Section 5, we characterize the optimal choice of tests for the principal in the presence of any number of firms.

2 Related Work in Depth

2.1 Multi-Sender Signaling

Our equilibrium analysis can be considered as a case of multi-sender signaling to a single receiver, in which the senders are constrained to using threshold strategies to report on their (real-valued) type. There is a significant body of work on signaling models with multiple senders. Two primary dimensions along which to categorize this body of work is the extent to which the senders have commitment power, and how much of the state of the world each sender can observe. When the senders have commitment power, and do not know the state at the time they commits to a strategy, we obtain the Bayesian persuasion model of Kamenica and Gentzkow (2011). (See also algorithmic results by Dughmi (2017), Dughmi and Xu (2016), Hssaine and Banerjee (2018).) On the other hand, when the senders have no commitment power, the interaction is more accurately modeled as “Cheap Talk” in the sense of Crawford and Sobel (1982). The second dimension has two natural extremes: senders observe the entire state of the world, or only information relevant to their own “product.” The former assumption is more relevant in the context of policy advice, while the latter applies to competition between firms. See the survey by Sobel (2013), though most of the survey is focused on a single sender, and the survey predates much of the recent literature on Bayesian Persuasion.

2.1.1 Cheap Talk Models

Much of the early multi-sender literature focused on cheap talk involving multiple senders with access to the full state of the world. The primary motivation was the study of lobbying, advocacy, and outsourcing of the acquisition of expert advice. Krishna and Morgan (2001) and Battaglini (2002) showed that contrary to the classical Cheap Talk model of Crawford and Sobel (1982), when there are multiple competing senders, full revelation of the state of the world can be achieved even when the interests of the receiver and the senders
are not aligned. Matthews and Postlewaite (1985) and Milgrom and Roberts (1986) studied a model in which each sender can introduce uncertainty only by specifying a set containing the actual state of the world. A skeptical receiver can always force full revelation, and Milgrom and Roberts (1986) show that in some cases, competition between senders also leads to full revelation, even when the receiver is not skeptical.

Incentives for information acquisition by multiple senders are studied by Austen-Smith and Wright (1992) and Dewatripont and Tirole (1999). Their primary focus is on when/whether it is beneficial for a principal to outsource the acquisition of information to senders who may either be exogenously biased towards one outcome, or who can be induced by the principal, via a suitable rewards structure, to advocate in favor of one outcome over the other. A different angle of incentives for information acquisition is studied by Brocas et al. (2012) and Gul and Pesendorfer (2012). In their model, each of two senders wants to convince the receiver that the state of the world matches a particular value. Each sender can pay to obtain another i.i.d. signal biased towards the true state of the world. The focus is on the implications of the tradeoff between costs for signal acquisition and welfare achieved under suitable strategies.

To the best of our knowledge, the only work on competition in a cheap talk setting is by Li et al. (2016). In their model, the two senders each have i.i.d. uniform quality in $[0, 1]$, and can send messages to the receiver, who chooses exactly one of them. Each sender’s objective is to maximize the quality of the selected sender, plus an (additive or multiplicative) bonus for having oneself selected. The main result is that, similar to the equilibrium for the single-sender Cheap Talk model (Crawford and Sobel 1982), each sender still partitions $[0, 1]$ into intervals and simply reveals the interval containing the quality. The number of intervals of the senders differs by at most 1, and each interval intersects at most two of the other sender’s intervals. Li et al. (2016) use this characterization to analyze which of the multiple equilibria are best for the receiver.

2.1.2 Bayesian Persuasion Models

Within the framework of Bayesian Persuasion, Gentzkow and Kamenica (2017a,b) study a model in which the information set is **Blackwell-connected**, meaning that each sender can unilaterally deviate to every information state in which the receiver has more information. For practical purposes, this means that each sender has access to the full state of the world. The focus in (Gentzkow and Kamenica 2017a,b) is on showing that competition instead of collusion between the senders leads to more information being revealed to the receiver, and on analyzing the effects of additional senders and more competition. Li and Norman (2019) study similar questions in a setting where the senders commit to their strategies sequentially rather than simultaneously.

Most directly related to our work is a body of literature studying direct competition between multiple senders in a Bayesian Persuasion framework. The typical setup is that each of $n$ (in most work, $n = 2$) senders has a product or proposal whose quality is drawn from a commonly known distribution over a set (often the set $\{\text{good}, \text{bad}\}$); the realization of the quality draw is private to the sender. The principal (receiver) can choose (at most or exactly) one of the products. Each sender can choose an arbitrary information disclosure policy (i.e., Bayes-plausible signaling scheme) about his quality, and the goal is to analyze the equilibrium outcomes of this game.

Boleslavsky and Cotton (2018) study a model in which each project is either good or bad, independently and with known prior probabilities. The receiver has a “reserve expected quality” and will not accept projects whose expected quality lies below this reserve. Even though the state space is binary and the receiver has at most 3 actions (which would imply a small finite number of signals in a single-sender setting), the competitive nature results in senders choosing from among a larger number of signals. Boleslavsky and Cotton (2018) analyze the equilibria, and show that competition leads to more information being revealed to the receiver, and on analyzing the effects of additional senders and more competition. Li and Norman (2019) study similar questions in a setting where the senders commit to their strategies sequentially rather than simultaneously.

Au and Kawai (2019) extend the model of Boleslavsky and Cotton (2018) to allow for positive correlations between the qualities of the projects. They identify two ways in which positive correlation can affect the senders’ utilities (and hence strategies), and show that when the two senders’ prior quality estimates are very different, in the limit, large correlation leads to more information disclosure. Au and Kawai (2020) extends the analysis of Boleslavsky and Cotton (2018) to $n > 2$ senders, and also allows for larger (though still finite) sets of qualities. Au and Kawai (2020) characterize the equilibrium
distribution in terms of payoff distributions. One of the key results is that as the number of senders grows, the equilibrium in the limit is full disclosure by all senders.

Boleslavsky et al. (2016) study a model in which the agents/senders can choose investments in project success probability. The cost to the agent increases quadratically in the desired project success probability. The principal can observe the agents’ investments, but not whether the projects are actually good. For the latter, the agents can commit to signaling schemes. As in our work, Boleslavsky et al. (2016) restrict the signaling schemes to map to binary outcomes. In other words, agents can commit to distributions with which they will exaggerate their projects’ successes, but cannot send more differentiated fractional messages. Among the key observations in (Boleslavsky et al. 2016) is the fact that the ability to exaggerate (as opposed to being forced to reveal whether projects were successful) typically leads to higher investments by the agents. Notice that similar to our work, the model of Boleslavsky et al. (2016) also involves mapping a continuous variable to a binary signal. Different from our model, however, the agents explicitly control the investment. Furthermore, the quantity that matters to the principal is the coin flip itself (i.e., whether the project was successful), whereas in our case, the continuous quality itself is what matters.

Hwang et al. (2019) study a model in which the receiver is a customer who will buy the product of exactly one of the firms/senders. The customer’s values for the products are drawn i.i.d. from a commonly known distribution. In addition to the information disclosure policy (i.e., signaling scheme), firms also control their prices. The equilibrium characterization shows a strong connection to the convexity/concavity of the value distribution. It alternates intervals of full disclosure with those of uniform randomization, depending on whether the value function is concave or convex in that interval. As part of their analysis, Hwang et al. (2019) also show that for a fixed price with convex value distribution, full disclosure results.

2.2 Other Signaling Work

Two related papers on Bayesian persuasion with a single sender are (Dughmi et al. 2016, Treust and Tomala 2019); both are among the few papers which explicitly impose communication constraints on the sender. Dughmi et al. (2016) restrict the number of signals that the sender may use, and study both welfare and algorithmic implications of such restrictions. Much of the focus is on a model of price discrimination by a seller who is informed by the sender about the buyer’s type. Dughmi et al. (2016) also show that in general, the best signaling strategy with limited signals is NP-hard to approximate to within any constant. Treust and Tomala (2019) study a Bayesian persuasion game that is repeated many times, where communication takes place over a limited and noisy information channel. The focus of the analysis is on the loss of sender utility arising because of the limited communication.

The notion of receivers taking a test to determine their unknown quality is also considered in a very different context by (Hssaine and Banerjee 2018). They study the selective disclosure of such scores in a multi-receiver Bayesian signaling setting, where the principal aims to influence the formation of teams among the receivers. The motivation in that work arises from semi-collaborative competitions such as formation of homework groups or teams for online coding challenges and crowdsourcing competitions. The main assumption is that when participating in any such competition, an agent may not fully know his skill level; however, the principal can determine the skill level via an entrance test whose scores are visible only to her. The principal can then exploit this information asymmetry to manipulate the agents’ posteriors in order to make them form more diverse teams.

Several papers discuss somewhat less standard models of signaling product quality. Hoffmann et al. (2020) consider a model in which the utility of each sender’s product to the receiver is the sum of two i.i.d. terms, and each sender must disclose exactly one of these terms. The sender can choose whether to disclose a random term or the higher of the two, and the analysis distinguishes whether the receiver is aware of the senders’ strategies. Hoffmann et al. (2020) consider this as a simple model of information collection for targeted advertising, and study whether the required data collection and targeted advertising (with or without the consumer’s knowledge) is in the consumer’s interest. When senders cannot commit to their strategy, the unique equilibrium (which also maximizes the consumer’s utility) involves each sender revealing the higher of the two terms. When the senders can commit to revealing a random term, then revelation of the higher
term by all senders becomes the unique equilibrium as the number of senders grows large, but may not be an equilibrium for few senders.

Libgober (2018, Chapter 3) studies a model in which each of \( n \) agents has a set of candidate projects, whose utilities in case of success or failure are commonly known. Agents privately observe the success probabilities of their projects, and communicate information to the principal, who chooses an agent and a project to pursue. The focus in (Libgober 2018) is on simpler strategy spaces, in which each agent selects only one of the projects to propose to the principal, rather than revealing information about all probabilities.

A different model of delegated project selection was analyzed by Kleinberg and Kleinberg (2018), who considered a single agent sampling \( n \) candidate projects and choosing to propose one to a principal, who may or may not choose to accept the proposal. The agent’s and principal’s utilities for a project may differ, and the focus in (Kleinberg and Kleinberg 2018) is on designing mechanisms whose equilibrium outcome for the principal is approximately as good as if the agent’s and principal’s utilities were identical. Unlike in the model we study here, the model in (Kleinberg and Kleinberg 2018) assumes that both the agent and the principal can directly assess the utility of a proposed project; the information asymmetry comes from the fact that the agent evaluates \( n \) candidate projects, but the principal only evaluates the project the agent proposes.

In its focus on signals with binary outcomes, our work also closely relates to the literature on external certification, in particular the work of Lerner and Tirole (2006). Their model, while phrased differently, is mathematically equivalent to the following: a property owner has a property\(^4\) of unknown value drawn from a commonly known distribution. The owner can choose a certifier\(^5\) who will verify whether the value lies above or below a threshold — the assumption is that for each \( \alpha \), there is a certifier with threshold exactly \( \alpha \). Conditioned on the outcome, users will buy the property if the conditional expected value lies above a known reserve. An important additional feature in (Lerner and Tirole 2006) is the ability of the owner to directly transfer utility to users buying the property in the form of “concessions”\(^6\), and the primary focus of Lerner and Tirole (2006) is a study of the interplay between the selection of certifier and the resulting concessions. Without concessions, the model of Lerner and Tirole (2006) directly corresponds to our model of endogenous test selection for a single firm. Lerner and Tirole (2006) also briefly discuss an extension to multiple property owners, but only consider one very specific example. Our work can be considered a more general treatment of the 2-owner setting without a reserve utility, in which users simply want to select the better of the two properties.

Gill and Sgroi (2008, 2012) also study a model of certification and its impact on product acceptance. In their model, the state of the world (product quality) is binary (high or low). Furthermore, the owner is aware of the state of the world before choosing a certifier, and needs the external certifier because he cannot credibly transmit the state himself. Certifiers have different (and known) accuracies and difficulties, and the focus is on how various model parameters affect the choice of certifier and related actions. In (Gill and Sgroi 2008), the owner aims to maximize the probability of a herding cascade on his product (when agents observe each others’ decisions). (Gill and Sgroi 2012) instead focuses on the interplay between the choice of a certifier and adjustments to the price after the certifier’s assessment is publicly revealed. However, neither of (Gill and Sgroi 2008, 2012) study a multi-owner scenario and the resulting competition.

Finally, Ostrovsky and Schwarz (2010) study a multi-sender multi-receiver signaling game between schools and employers. Schools have students whose abilities are drawn independently from known distributions, and can signal to employers (via the students’ transcripts) how good a student is. Each school’s objective is to maximize the expected desirability of the students’ employment, while each employer aims to maximize the ability of the student hired. Ostrovsky and Schwarz (2010) show that typically, at equilibrium, schools will withhold some information. Much of the subsequent focus is on the investigation of early contracting, involving students signing employment contracts before their full transcript is known; Ostrovsky and Schwarz (2010) show that early contracting will not occur if schools disclose the equilibrium amount of information about their students, but can occur otherwise.

\(^4\)such as a product, research proposal, or scientific paper  
\(^5\)such as a standards agency or journal  
\(^6\)such as price reductions, additional features, or added figures or results
2.3 Algorithmic Considerations for Ranking from Limited Data

Apart from the related literature in economics and game theory, our study of optimal selection of tests also relates to a large algorithmic literature on ranking and selection based on limited information. This is a core topic in data mining and Bayesian optimization, with a vast body of work; we briefly outline some ideas in this space which are closely related to our algorithmic approaches.

The most directly related topic in the data mining literature is that of learning-to-rank, which refers to a general framework of constructing probabilistic ranking models for a set of objects, training these based on data, and then using them to sort new objects according to their degrees of relevance, preference, or importance. The classical model here is the ELO ranking; a more popular modern system is the TrueSkill system of Herbrich et al. (2006), which is used by Microsoft and others for ranking online gamers. Liu et al. (2009) provide an overview of this and related approaches to learning-to-rank, and their applications in information retrieval.

Learning-to-rank systems can further be subdivided into pointwise, pairwise and global ranking systems; in this context, our approach bears similarities with information-theoretic variants of ranking based on pairwise comparison tests (Jamieson and Nowak 2011, Negahban et al. 2017). The main idea in these works is to consider obtaining noisy signals of pairwise comparisons between sets of items with a true underlying ordering, where the noise in each signal depends on the distance between the two underlying items. Another related field which looks at similar models and questions regarding ranking under probabilistic models is that of social choice. Here again, our work has close connections to studies of statistical (Shah and Wainwright 2017, Conitzer et al. 2006) and computational (Betzler et al. 2009, Kenyon-Mathieu and Schudy 2007) properties of different ranking algorithms used in social choice theory. The main difference in our treatment, however, is that we are able to design the tests we use for ranking, and also consider strategic aspects in agents choosing which tests to use (which then motivates considering rank aggregation through the lens of a signaling game).

With regards to the idea of endogenous selection games in ranking, our work shares commonalities with work of Altman and Tennenholtz (2007, 2010) on incentives in ranking systems. The main focus of these works is to obtain an axiomatic characterization of ranking systems under which agents are incentivized to reveal their true skill levels. The crucial difference here is that agents are aware of their own skills, in contrast to our setting, where agents must take a test to discover their true skill.

Finally, the rise of collaborative platforms and MOOCs has led to a recent upsurge of interest in the use of testing for selecting teams. In this context, Kleinberg and Raghu (2018) look at the question of how test scores of multiple agents can be used to form teams whose output depends on a complex function of agents’ joint utility profiles. On the other hand, Johari et al. (2018) consider in some sense a dual question wherein a principal observes the scores of different teams, with each score being a complex function of the utility profile of the agents in a team, and must use this to try and rank the agents. The focus in these works has primarily been on the computational challenges of learning rankings and/or forming teams based on such scores, in contrast to our focus on the strategic aspects in such settings.

3 Model and Preliminaries

3.1 Qualities, Tests and Selection

We consider a setting in which a principal wants to pick the better of the products provided by two firms $X$ and $Y$. We will equivalently refer to this process as selecting or choosing a firm or ranking the firms. The two firms’ products have i.i.d. qualities $X, Y$ drawn from a common prior distribution with continuous cdf $\Psi$ on $\mathbb{R}$. Abusing notation, we use $X, Y$ to refer both to the firms themselves and their products’ (random) qualities.

\footnote{We adopt the convention that the cumulative distribution function (cdf) of a probability measure on $\mathbb{R}$ is defined by setting $F(x)$ to be the measure of the set $(-\infty, x]$ under the distribution.}
Information about the products’ qualities is revealed by means of binary threshold tests (henceforth simply tests) administered to the products. More specifically, a test is completely characterized by a threshold $\theta \in \mathbb{R}$. A product of quality $X$ subjected to a test with threshold $\theta$ passes if and only if $X \geq \theta$; otherwise, we say that the product fails the test $\theta$. To avoid unnecessary clutter in writing, we also refer to the firm $X$ or $Y$ as passing or failing the test (instead of its product). The larger $\theta$, the less likely a product is to pass the test, so we can naturally think of $\theta$ as the difficulty of the test. When a product is subjected to a test, the outcome (pass or fail) is revealed to everyone, but no additional information can be inferred about the product. This model is mathematically equivalent to the certification model of Lerner and Tirole (2006).

The principal’s goal is to minimize the probability of selecting the product of lower quality. We refer to this as an incorrect selection, or as an error by the principal, or — by analogy with ranking — as an inversion. Formally, consider a rule $\mathcal{T}$ for assigning tests to firms and selecting a firm based on the tests’ outcome. We define $I(\mathcal{T}) := 1[\mathcal{T}$ chooses the wrong firm] as the indicator of $\mathcal{T}$ inverting the ranking. Note that $I(\mathcal{T})$ is a random variable, with randomness arising from: (1) $\mathcal{T}$’s selection of test thresholds, (2) the firms’ products’ random qualities, and (3) possibly randomized aggregation of test outcomes. The principal’s goal is to choose $\mathcal{T}$ so as to minimize $\mathbb{E}[I(\mathcal{T})]$.

Given a firm’s test result, the principal can form a posterior belief of its product’s quality. The posterior expected quality of a product passing threshold test $\theta$ is $\mathbb{E}_{X \sim \Psi} [X \mid X \geq \theta]$, while the posterior expected quality of a product failing it is $\mathbb{E}_{X \sim \Psi} [X \mid X < \theta]$. Observe that for any product quality cdf $\Psi$, we have that $\mathbb{E}_{X \sim \Psi} [X \mid X < \theta]$ and $\mathbb{E}_{X \sim \Psi} [X \mid X \geq \theta]$ are monotone non-decreasing in $\theta$, and strictly increasing for $\theta$ in the support of $\Psi$. Furthermore

$$\mathbb{E}_{X \sim \Psi} [X \mid X < \theta] \leq \mathbb{E}_{X \sim \Psi} [X] \leq \mathbb{E}_{X \sim \Psi} [X \mid X \geq \theta],$$

and both inequalities are strict if $\theta$ is in the support of $\Psi$. Because both products’ qualities are drawn from the same distribution, these observations imply the following proposition.

**Proposition 1** Let $\theta_X > \theta_Y$ be the thresholds of the tests to be applied to the products of firms $X,Y$. Assume that both $\theta_X, \theta_Y$ lie in the support of $\Psi$.

1. If both firms’ products pass their tests, or both fail their tests, then the principal minimizes the probability of an inversion by selecting $X$.
2. If exactly one of the products of $X,Y$ passes its test, then the principal minimizes the probability of an inversion by selecting the firm that passed.

Proposition 1 characterizes a rational principal’s choice (once test outcomes have been revealed) almost completely. To complete the description, we assume that when there is a tie, the principal picks one of the firms uniformly at random. We will refer to this case as a coin flip, and say that $X$ (or $Y$) wins/loses the coin flip. As an illustration, consider the following example:

**Example 1 (The Median Test)** Suppose that both firms’ products have i.i.d. quality levels $X,Y \sim \text{Uniform}[0,1]$ (i.e., drawn uniformly over $[0,1]$). A natural test is the median test $\mathcal{T}_{\text{median}}$, under which both products are subjected to a test with $\theta = \frac{1}{2}$. A product’s posterior expected quality upon passing is $\mathbb{E}[X \mid X \geq 1/2] = 3/4$, and upon failing $\mathbb{E}[X \mid X \leq 1/2] = 1/4$. Now w.l.o.g. suppose that the two firms’ products have qualities $X < Y$. If $X \leq \frac{1}{2} < Y$, then $Y$ passes and $X$ fails, and the principal ranks them correctly. However, if $Y \leq \frac{1}{2}$, then both fail, and if $X > \frac{1}{2}$, then both pass. In either case, a coin flip is required, and the principal chooses correctly only with probability $\frac{1}{2}$. Thus, the median test achieves $\mathbb{E}[I(\mathcal{T}_{\text{median}})] = \frac{1}{2}$.

More generally, if the principal gives the same test $\theta$ to both agents, then an inversion happens if: (1) either both $X,Y \geq \theta$ or both $X,Y < \theta$, and (2) the coin flip determines the wrong winner. Thus, the probability of an inversion is $\frac{1}{2}(\theta^2 + (1-\theta)^2)$. This is minimized at $\theta = \frac{1}{2}$, showing that the median test is optimal for the principal if she must give the same test to both agents.

Given complete control over the choice of testing rule $\mathcal{T}$, the principal’s goal is to choose the rule that minimizes $\mathbb{E}[I(\mathcal{T})]$. This could be a single threshold for both firms (as with the median test); a distribution
uniformly in \([0, 1]\) corresponds to a threshold \(\sigma \in \mathbb{R}\) for its test; we can equivalently view this as the firm picking a threshold quantile \(\theta = \Psi(\sigma) \in [0, 1]\). Note that a product with quality \(X \sim \Psi\) passes a test with threshold quantile \(\theta\) with probability \(1 - \theta\); moreover, a threshold quantile \(\theta \in [0, 1]\) corresponds to a threshold \(\sigma = \Psi^{-1}(\theta)\) in the quality space, where \(\Psi^{-1}(x) \triangleq \inf\{y \in \mathbb{R} \mid \Psi(y) \geq x\}\) is the generalized inverse function associated with the cdf \(\Psi\). Thus, w.l.o.g., we henceforth focus on product qualities drawn from \(\Psi \sim \text{Uniform}[0, 1]\), and understand “threshold” to refer to the threshold quantile \(\theta \in [0, 1]\).

### 3.2 Endogenous Test Selection and Quantile Thresholds

In many settings, firms may be better equipped than the principal to perform quality control tests in house. In these cases, the firms will typically commit to a verifiable quality control procedure for their products. The principal gets to observe (only) the threshold \(\theta\) and the outcome of the test. In other words, both firms commit to a signaling scheme about their products’ qualities, where the space of signaling schemes is restricted to a binary signal space and threshold functions.

Each firm’s goal is to maximize its probability of being selected, or — equivalently — of being ranked ahead of the other firm. Due to the competitive nature of the game, the appropriate solution concept (which we will study) is a Bayes-Nash Equilibrium. We refer to this setting as endogenous test selection. Because the firms are a priori symmetric, in any equilibrium, each firm’s product must be selected with probability \(\frac{1}{2}\).

In a further generalization, note that the principal may be able to rule out some types of tests. In other words, in a more general model, the principal may specify a closed set \(S\) and restrict the firms to selecting test thresholds \(\theta \in S\) only. We will be primarily interested in the case when \(S\) is an interval, but also consider more general closed sets \(S\).

Before continuing, we note that since the utilities of both the principal and the firms depend only on rankings and not actual qualities, it is convenient to work in the quantile space \([0, 1]\) rather than the quality space \(\mathbb{R}\). To do this, note that for any quality \(X \sim \Psi\), its corresponding (random) quantile \(\Psi(X)\) is distributed uniformly in \([0, 1]\). Now, suppose that firm \(X\) chooses (or is assigned) a threshold \(\sigma \in \mathbb{R}\) for its test; we can equivalently view this as the firm picking a threshold quantile \(\theta = \Psi(\sigma) \in [0, 1]\). Note that a product with quality \(X \sim \Psi\) passes a test with threshold quantile \(\theta\) with probability \(1 - \theta\); moreover, a threshold quantile \(\theta \in [0, 1]\) corresponds to a threshold \(\sigma = \Psi^{-1}(\theta)\) in the quality space, where \(\Psi^{-1}(x) \triangleq \inf\{y \in \mathbb{R} \mid \Psi(y) \geq x\}\) is the generalized inverse function associated with the cdf \(\Psi\). Thus, w.l.o.g., we henceforth focus on product qualities drawn from \(\Psi \sim \text{Uniform}[0, 1]\), and understand “threshold” to refer to the threshold quantile \(\theta \in [0, 1]\).

### 3.3 Extension to More Firms

While we have focused thus far on the paradigmatic case of two firms, the model can be naturally extended to \(n \geq 2\) firms. Several natural generalizations suggest themselves, both in terms of the principal’s objective and the firms’ objective. With \(n\) firms, the principal may try to maximize the probability of choosing the best product, or try to produce a complete ranking of all firms’ products, minimizing the total number of inversions. For a firm, the goal might be to maximize the probability of being selected, or to be ranked as highly as possible in expectation. Our results extend naturally to the latter objectives, namely,

- The utility of a firm is proportional to the number of firms ranked behind it.
- The disutility of the principal is proportional to the (normalized) Kendall tau distance\(^{10}\) between the true and inferred rankings, i.e., the fraction of pairwise inversions between the two lists.

Extending our notation from the case of two firms, for a given rule \(T\) for choosing tests for firms, we denote the (random) Kendall tau distance between the resulting ranking and the correct ranking by \(I(T)\). Again, the principal’s goal is to minimize \(\mathbb{E}[I(T)]\). Using linearity of expectations for both the firms and

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\(^{8}\)Alternatively, the setting may be such that the agents naturally have the choice of test difficulty, such as in external certification of product quality (Lerner and Tirole 2006, Gill and Sgroi 2008, 2012) or students’ selection of which classes to attempt (Spence 1973). In these settings, it is still frequently assumed that agents are not aware of their private quality value when they make their choice of difficulty, see for example (Lerner and Tirole 2006) for a model of certification and (Ostrovsky and Schwarz 2010) for a model of contracting between students and employers.

\(^{9}\)There are naturally other objectives in between these two extremes.

\(^{10}\)Recall that the Kendall \(\tau\) distance between two rankings is the number of inversions between the two rankings, i.e., the number of pairs of elements that are in different order.
the principal, all of our results for two firms carry over immediately to the case of \( n \) firms, with exactly the same guarantees regarding the fraction of misranked pairs. The only exception is that for correlated tests (in Section 5), the optimal choice for the principal will depend on the number \( n \) of firms. These results do not extend to other objectives, and both optimal and equilibrium strategies will typically look different for \( n \geq 3 \) firms. See Section 8 for a discussion.

4 Optimal I.I.D. Tests

In this section, we explicitly characterize the optimal distribution from which the principal should draw thresholds when drawing them i.i.d. for both firms.

4.1 Characterizing the Expected-Inversions Functional

Let \( T_G \) denote the test selection rule under which each firm is given a test with threshold drawn i.i.d. from \( G \). We begin by characterizing the expected number of inversions as a functional of the cdf \( G \) from which the thresholds are drawn. In the next section, we will show how to choose \( G \) to minimize this functional. For notational convenience, we henceforth denote \( I(G) = \mathbb{E}[I(T_G)] \).

**Lemma 1** Assume that the quality distribution \( \Psi \) is uniform\(^{11} \) on \([0,1]\). Suppose that thresholds for both firms are drawn i.i.d. from the distribution \( G \) on \([0,1]\) (not necessarily continuous). The probability of selecting the worse product is given by the functional

\[
I(G) = \int_0^1 \int_0^x (1 - G(x) + G(y))^2 \, dy \, dx. \tag{1}
\]

**Proof.** Assume that the two firms’ products have qualities \( x > y \). An inversion occurs when \( Y \) is selected. We can think of the process as first picking two thresholds \( \theta_0, \theta_1 \) i.i.d. from the distribution with cdf \( G \), letting \( \theta = \min(\theta_0, \theta_1) \) and \( \theta' = \max(\theta_0, \theta_1) \), and then uniformly randomizing which of \( X \) and \( Y \) gets which of \( \theta, \theta' \). Because \( G \) may have point masses, it is possible that \( \theta = \theta' \). We consider the following cases, based on which of the two or three intervals \( I_1 = [0, \theta), I_2 = [\theta, \theta'), I_3 = [\theta', 1] \) the products’ qualities \( x \) and \( y \) fall into (see Fig. 2).

1. If both \( x,y \in I_1 \), then both firms fail, and if \( x,y \in I_3 \), then both firms pass. In either case, an inversion happens with probability \( \frac{1}{2} \); if \( \theta = \theta' \), this is due to the randomized tie-breaking rule, while for \( \theta \neq \theta' \), it is because the assignment of \( \theta, \theta' \) to the two firms is uniformly random, and the mechanism ranks as higher the firm which is assigned \( \theta' \).

2. If both \( x,y \in I_2 \), then the firm with threshold \( \theta \) passes, while the one with \( \theta' \) fails. Because the assignment is uniformly random, again, an inversion is created with probability \( \frac{1}{2} \).

\(^{11}\)Recall from Section 3.2 that this assumption is without loss of generality.
3. If \( x \in \mathcal{I}_3 \) and \( y \in \mathcal{I}_2 \), or \( x \in \mathcal{I}_2 \) and \( y \in \mathcal{I}_1 \), then no inversion can be created, because \( X \) will always be ranked ahead of \( Y \). This is because either \( X \) passes and \( Y \) fails, or otherwise, both firms obtain the same result and \( X \) was assigned the higher threshold \( \theta' \).

4. Finally, if \( x \in \mathcal{I}_3 \) and \( y \in \mathcal{I}_1 \), then \( X \) passes and \( Y \) fails, so no inversion is created.

So an inversion is created with probability \( \frac{1}{2} \) if both \( x \) and \( y \) are in the same interval, and with probability 0 otherwise. The probability that both are in \( \mathcal{I}_1 \) is \((1 - G(x))^2\) (because this case is equivalent to \( \theta_0, \theta_1 > x \)); the probability that both are in \( \mathcal{I}_2 \) is \(2G(y) \cdot (1 - G(x))\) (because this case is equivalent to \( \theta_0 \leq y < x < \theta_1 \) or \( \theta_1 \leq y < x < \theta_0 \)); and the probability that both are in \( \mathcal{I}_3 \) is \(G(y)^2\) (because this case is equivalent to \( \theta_0, \theta_1 \leq y \)). Thus, we have

\[
\mathbb{E}[I(\mathcal{G}) \mid (X, Y) = (x, y), x > y] = \frac{(1 - G(x))^2 + G(y)^2 + 2G(y)(1 - G(x))}{2} = \frac{(1 - G(x) + G(y))^2}{2}.
\]

Therefore, the expected probability of an inversion overall is

\[
I(G) = \int_{0}^{1} \int_{0}^{1} \frac{1}{2}(1 - G(\max(x, y)) + G(\min(x, y)))^2 \, dy \, dx = \int_{0}^{1} \int_{0}^{1} (1 - G(x) + G(y))^2 \, dy \, dx.
\]

\[\Box\]

### 4.2 Optimizing the Objective Function

We now provide a characterization of the i.i.d. distribution \( H^* \) that minimizes \( I(G) \).

**Theorem 4** Assume that the quality distribution \( \Psi \) is uniform\(^{12} \) on \([0, 1]\). Let \( H^* \) be the cdf corresponding to the uniform distribution over the interval \([\frac{1}{4}, \frac{3}{4}]\). For every distribution \( G \) over \([0, 1]\), we have \( I(G) \geq I(H^*) \).

In other words, the optimal way to pick i.i.d. tests is to sample them uniformly from \([\frac{1}{4}, \frac{3}{4}]\). This may seem somewhat surprising. Some intuition for this can be derived from looking at correlated test selection rules in the limit of infinitely many firms (see the discussion after Theorem 5 in Section 5). The following proof provides a way to not only prove optimality of \( H^* \), but to obtain lower bounds on \( I(G) \) for every \( G \) that differs substantially from \( H^* \). We use this in Section 7 to obtain lower bounds on the inversion probability of equilibria with restricted sets of available tests.

**Proof of Theorem 4.** We will show that the uniform distribution on \([\frac{1}{4}, \frac{3}{4}]\) is the unique distribution on \([0, 1]\) that optimizes the functional \( I(G) \) defined by (1). Let

\[
G_0(x) = \begin{cases} 
0 & \text{if } x < \frac{1}{4} \\
2x - \frac{1}{2} & \text{if } x \in [\frac{1}{4}, \frac{3}{4}] \\
1 & \text{if } x > \frac{3}{4}
\end{cases}
\]

be the cdf of the uniform distribution on \([\frac{1}{4}, \frac{3}{4}]\), and let \( G \) be (the cdf of) any other distribution on \([0, 1]\). For \( t \in [0, 1] \) we can consider the hybrid distribution \( G_t \) which draws a sample from \( G_0 \) with probability \( 1 - t \) and from \( G \) with probability \( t \). This hybrid distribution has cdf

\[
G_t(x) = tG_0(x) + (1 - t)G_0(x) = G_0(x) + t(G(x) - G_0(x)).
\]

We now prove that for every \( G \neq G_0 \), the function \( I(G_t) \) is strictly increasing in \( t \). This immediately implies that \( I(G) > I(G_0) \), confirming that \( G_0 = H^* \) is uniquely optimal, as claimed.

Substituting the right side of Equation (3) into the definition of \( I(G_t) \), we have

\[
I(G_t) = \int_{0}^{1} \int_{0}^{x} \left(1 - G_0(x) + G_0(y) + t(G_0(x) - G(x) + G(y) - G_0(y))\right)^2 \, dy \, dx
\]

\[
= I(G_0) + 2t \int_{0}^{1} \int_{0}^{x} (1 - G_0(x) + G_0(y)) \cdot (G_0(x) - G(x) + G(y) - G_0(y)) \, dy \, dx
\]

\[
+ t^2 \int_{0}^{1} \int_{0}^{x} (G_0(x) - G(x) + G(y) - G_0(y))^2 \, dy \, dx.
\]

\(^{12}\)Recall from Section 3.2 that this assumption is without loss of generality.
The right side is a quadratic function of $t$, i.e., we can write $I(G_t) = I(G_0) + A(G) \cdot t + B(G) \cdot t^2$, where the coefficients of $t$ and $t^2$ are given by

$$A(G) = 2 \int_0^1 \int_0^x (1 - G_0(x) + G_0(y)) \cdot (G_0(x) - G(x) + G(y) - G_0(y)) \, dy \, dx,$$
$$B(G) = \int_0^1 \int_0^x (G_0(x) - G(x) + G(y) - G_0(y))^2 \, dy \, dx.$$

If $G$ and $G_0$ are not equal, then — since they both are right-continuous — they must differ on a set of positive measure. Consequently, $B(G)$ is strictly positive. To prove that $I(G_t)$ is strictly increasing in $t$, we need only show that $A(G) \geq 0$. We define

$$C(G) = 2 \int_0^1 \int_0^x (1 - G_0(x) + G_0(y)) \cdot (G(x) - G(y)) \, dy \, dx,$$

and note that $A(G) = C(G_0) - C(G)$. Since $G$ is the cdf of a random variable, it can be expressed as a convex combination of step functions. Specifically, for $z \in [0, 1]$, define the step function $T_z(x) = 1[x \geq z]$. Now if $Z$ is a random sample from $G$, then for all $x \in [0, 1]$, we can write $G(x) = \mathbb{E}_{Z \sim G}[T_z(x)]$. Then observing that $C(G)$ is linear in $G$, we have via linearity of expectation that $C(G) = \mathbb{E}_{Z \sim G}[C(T_Z)]$. Moreover, for any fixed $z \in [0, 1]$, we have

$$C(T_z) = 2 \int_0^1 \int_0^x (1 - G_0(x) + G_0(y))(T_z(x) - T_z(y)) \, dy \, dx = 2 \int_0^1 \int_0^z (1 - G_0(x) + G_0(y)) \, dy \, dx,$$

because for $y < x$, we have $T_z(x) - T_z(y) = 1$ when $y < z \leq x$, and 0 otherwise. Also observe that $1 - G_0(x) = G_0(1 - x)$ (since $G_0$ is a distribution on $[0, 1]$ and is symmetric about $\frac{1}{2}$). Thus

$$2 \int_0^1 \int_0^z 1 - G_0(x) + G_0(y) \, dy \, dx = 2 \int_0^1 \int_0^z G_0(1 - x) + G_0(y) \, dy \, dx$$
$$= 2 \int_0^{1 - z} \int_0^z G_0(x) + G_0(y) \, dy \, dx = 2z \int_0^{1 - z} G_0(x) \, dx + 2(1 - z) \int_0^z G_0(y) \, dy$$
$$= 2z \Upsilon(1 - z) + 2(1 - z) \Upsilon(z),$$

where the function $\Upsilon(\cdot)$ is defined by

$$\Upsilon(z) = \int_0^z G_0(x) \, dx = \begin{cases} 0 & \text{if } z < \frac{1}{4} \\ (z - \frac{1}{2})^2 & \text{if } z \in \left[\frac{1}{4}, \frac{3}{4}\right] \\ z - \frac{1}{2} & \text{if } z > \frac{3}{4}. \end{cases}$$

For $z \in \left[\frac{1}{4}, \frac{3}{4}\right]$ we have

$$z \Upsilon(1 - z) + (1 - z) \Upsilon(z) = z \left(\frac{3}{4} - z\right)^2 + (1 - z) \left(z - \frac{1}{2}\right)^2 = z \left(\frac{9}{16} - \frac{3}{2}z + z^2\right) + (1 - z) \left(\frac{1}{16} - \frac{1}{2}z + z^2\right)$$
$$= \frac{1}{16} + \left(\frac{9}{16} - \frac{1}{16} - \frac{1}{2}\right) z + \left(-\frac{3}{2} + \frac{1}{2} + 1\right) z^2 + (1 - 1) z^3 = \frac{1}{16}.$$

For $z < \frac{1}{4}$ we have $z \Upsilon(1 - z) + (1 - z) \Upsilon(z) = z \left(\frac{1}{4} - z\right) = \frac{1}{16} - \left(\frac{1}{4} - z\right)^2$, while for $z > \frac{3}{4}$ we have $z \Upsilon(1 - z) + (1 - z) \Upsilon(z) = (1 - z) \left(z - \frac{1}{2}\right) = \frac{1}{16} - \left(z - \frac{3}{4}\right)^2$. Summarizing this case analysis, $C(T_z) = 2(z \Upsilon(1 - z) + (1 - z) \Upsilon(z)) \leq \frac{1}{16}$, and the inequality is strict if and only if $z \notin \left[\frac{1}{4}, \frac{3}{4}\right]$. Now, using the characterization that $C(G) = \mathbb{E}_{Z \sim G}[C(T_Z)]$ for every $G$, we have $C(G) \leq \frac{1}{16}$, and the inequality is strict if and only if the support of $G$ is not contained in $\left[\frac{1}{4}, \frac{3}{4}\right]$. On the other hand, since $G_0$ is the cdf of a distribution supported on $\left[\frac{1}{4}, \frac{3}{4}\right]$, we have $C(G_0) = \frac{1}{8}$. Combining these two inequalities, we obtain $A(G) = C(G_0) - C(G) \geq 0$, which demonstrates that $I(G_t)$ is strictly increasing in $t$, and thus $I(G) = I(G_1) > I(G_0)$. □
We can now derive the optimal probability of inversion under i.i.d. tests.

**Corollary 1** The optimum i.i.d. test distribution creates an inversion with probability \( \frac{5}{24} \).

**Proof.** Substituting \( H^* \) into Eq. (1), we get that the probability of an inversion under \( H^* \) is

\[
I(H^*) = \int_0^1 \int_0^x (1 - H^*(x) + H^*(y))^2 \, dy \, dx
\]

\[
= \int_0^{\frac{1}{4}} x \, dx + \int_{\frac{1}{4}}^{\frac{3}{4}} \frac{1}{4} \left( 1 - 2 \left( x - \frac{1}{4} \right) \right)^2 \, dx + \int_{\frac{3}{4}}^x (1 - 2(x - y))^2 \, dy \, dx
\]

\[
+ \int_{\frac{3}{4}}^{1} \frac{1}{4} \left( 2 \left( y - \frac{1}{4} \right) \right)^2 \, dy + \int_1^{\frac{1}{4}} \left( x - \frac{3}{4} \right) \, dx
\]

\[
= \frac{1}{32} + \frac{1}{24} + \int_{\frac{3}{4}}^{\frac{5}{4}} \frac{1}{4} (1 - 2y)^2 \, dy \, dx + \frac{1}{24} + \frac{1}{32}
\]

Simplifying the expression further (with standard integration), we get that \( I(H^*) = \frac{5}{24}. \) \( \square \)

## 5 Optimal Correlated Tests

In Section 4, we derived the optimal distribution to sample tests from if each firm must be assigned a test independently from the same distribution. Here, we consider the problem when the firms’ tests can be chosen in a correlated way.

As we mention in Section 3.3, although most of our analysis looks at two firms, it extends naturally to multiple firms when the goal is to minimize the expected number of inversions. When the test assignments can be correlated, the actual number of firms affects the optimal solution. Hence, in this section, we explicitly characterize the optimal choices when there are \( n \) firms. Surprisingly, this takes the following simple form:

**Theorem 5** Assume that the quality distribution \( \Psi \) is uniform\(^{13} \) on \([0, 1]\). Recall that \( I(T) \) denotes the (random) Kendall tau distance between the true and inferred rankings. For \( n \) firms, the expected fraction of inversions \( \mathbb{E}[I(T)] \) is minimized over all correlated test selection rules \( T \) by one which assigns the test with threshold \( \theta_i = \frac{n + 2(i - 1)}{4n - 2} \) to firm \( i \). The resulting expected fraction of inverted pairs of firms is \( \frac{5n - 4}{12(2n - 1)} \).

To get intuition for this result, it is instructive to consider it for \( n = 2 \). In this case, the optimal \( T \) allocates two tests at thresholds \( \frac{1}{4} \) and \( \frac{3}{4} \), respectively, and this improves the fraction of misclassified pairs from \( \frac{5}{24} \) to \( \frac{1}{6} \). The main reason behind this improvement is that the ability to give different tests to the two firms allows the principal to choose tests to maximally split up the space \([0, 1]\), such that the only way the principal makes a mistake is if the products’ qualities \( X, Y \) are in the same interval (refer again to Fig. 2, cases (1) and (2)).

Theorem 5 is also instructive in the limit as \( n \to \infty \). Here, one sees that the optimal test distribution converges to uniformly spaced tests over the interval \( \left[ \frac{1}{4}, \frac{3}{4} \right] \) (and leads to a \( \frac{1}{12} \) fraction of pairs being inverted). This suggests that a uniform distribution of tests over \( \left[ \frac{1}{4}, \frac{3}{4} \right] \) should be the optimal distribution for i.i.d. tests for any number of firms, since drawing \( n \) tests from a continuous distribution results in all \( n \) tests being unique almost surely, and close to the optimal correlated tests. This intuition is indeed confirmed by the earlier Theorem 4.

**Proof of Theorem 5.** For any fixed \( n \)-tuple of thresholds \( (\theta_1, \ldots, \theta_n) \in [0, 1]^n \), consider the expected (over the draw of the products’ qualities) number of inversions. This quantity must have an actual minimizer, which is the principal’s optimal choice; in other words, the thresholds need not be drawn from a distribution.

\(^{13}\)Recall from Section 3.2 that this assumption is without loss of generality.
Because the products’ qualities are drawn i.i.d., without loss of generality, we assume \( \theta_1 \leq \theta_2 \leq \cdots \leq \theta_n \). Consider two firms \( i < j \), and the partition of \([0, 1]\) into the three intervals \([0, \theta_i), [\theta_i, \theta_j), [\theta_j, 1]\). If the qualities \( x_i, x_j \) fall into distinct intervals, then \( i \) and \( j \) will always be ranked in correct order, as can be seen by the following cases (similar to cases (3) and (4) in Fig. 2):

1. If \( x_i < \theta_i \), then firm \( i \) fails its test. Because firm \( j \)'s threshold is higher, whether it passes or fails, it will be ranked ahead of \( i \), which is correct since \( x_j \geq \theta_i > x_i \).
2. If \( \theta_i < x_i < \theta_j \), then firm \( i \) passes its test. If \( x_j < \theta_i \), then \( j \) fails its test and will be correctly ranked behind \( i \). If \( x_j \geq \theta_i \), it passes its test and will correctly be ranked ahead of \( i \), because its test is more difficult.
3. If \( x_i \geq \theta_j \), then firm \( i \) passes its test. Because \( x_j \) is in a different interval, \( x_j < \theta_j \), so \( j \) fails its test, and will be correctly ranked behind \( i \).

When both \( x_i \) and \( x_j \) are in the same interval, the outcomes (fail/fail in the bottom interval, pass/pass in the middle, pass/fail in the top) determine some ranking. The actual ordering between \( x_i \) and \( x_j \), conditioned on being in the same interval, is uniformly random, so the ranking is correct with probability \( \frac{1}{2} \). Thus, we have derived that the inversion probability for the pair \( i, j \) is \( \frac{1}{2} (\theta_i^2 + (\theta_j - \theta_i)^2 + (1 - \theta_j)^2). \)

Write \((\theta_1, \ldots, \theta_n)\) for the rule that assigns each agent \( i \) the threshold \( \theta_i \). Summing over all pairs \( i < j \), the expected number of inverted pairs for \( \theta_1 \leq \theta_2 \leq \cdots \leq \theta_n \) is

\[
\mathbb{E} [I((\theta_1, \ldots, \theta_n))] = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=i+1}^{n} \theta_i^2 + (\theta_j - \theta_i)^2 + (1 - \theta_j)^2 \tag{5}
\]

\[
= (n - 1) \sum_{i=1}^{n} \theta_i^2 + \frac{1}{2} \left( \frac{n}{2} \right)^2 - \sum_{i=1}^{n} (i - 1) \theta_i - \sum_{i=1}^{n} \sum_{j=i+1}^{n} \theta_i \theta_j. \tag{6}
\]

The right side of (5) is a strongly convex quadratic function of \((\theta_1, \ldots, \theta_n)\), and hence its global minimum over \(\mathbb{R}^n\) is attained at the unique point where its gradient vanishes. Using formula (6) and setting the derivative with respect to all \( \theta_i \) to be zero, we have that \( 0 = 2(n-1)\theta_i - (i-1) - \sum_{j \neq i} \theta_j = (2n-1)\theta_i - (i-1) - \sum_{j=1}^{n} \theta_j \)

for all \( i \in [n] \). Writing \( c = \sum_{j=1}^{n} \theta_j \), we get that \( \theta_i = \frac{i + c - 1}{2n - 1} \). Therefore, \( (2n-1)\cdot c = \sum_{j=1}^{n} (c + j - 1) \), which means that \( c = \frac{1}{n-1} \cdot \sum_{j=0}^{n-1} j = \frac{n}{2} \), and thus \( \theta_i = \frac{n+2(i-1)}{4n-2} \). Thus, the vector \((\theta_1, \ldots, \theta_n)\) that minimizes the right side of (5) over all of \(\mathbb{R}^n\) belongs to the set \(\{ (\theta_1, \ldots, \theta_n) \mid 0 \leq \theta_1 \leq \theta_2 \leq \cdots \leq \theta_n \leq 1 \}\), and therefore minimizes \(\mathbb{E} [I((\theta_1, \ldots, \theta_n))]\) over that set. Denote this test selection rule as \(\mathcal{T}^*\). Substituting the above choice of \( \theta_i \) into \(\mathbb{E} [I((\theta_1, \ldots, \theta_n))]\), and omitting some simplifications, we get that the expected number of inversions is

\[
\mathbb{E} [I(\mathcal{T}^*)] = \frac{1}{4(2n-1)^2} \cdot \left( (n-1) \sum_{i=1}^{n} (n + 2(i-1))^2 + 4(2n-1)^2 \cdot \frac{n(n-1)}{4} \right. \\
- 2(2n-1) \sum_{i=1}^{n} (i-1)(n + 2(i-1)) - \sum_{i=1}^{n} \sum_{j=i+1}^{n} (n + 2(i-1))(n + 2(j-1)) \\
= \frac{n(n-1)(5n-4)}{24(2n-1)}. 
\]

Dividing by \( \binom{n}{2} \), we obtain that the fraction of inverted pairs is \( \frac{5n-4}{12(2n-1)} \). \(\square\)

6 Endogenous Test Selection and Price of Anarchy

In this and the next section, we turn to the question of endogenous test selection. Here, we consider the setting where the principal makes all threshold tests in \([0, 1]\) available to the firms for selection; in the next section, we consider the benefits of being able to restrict the set of offered tests. For the entire section, recall
that we assume without loss of generality (see Section 3.2) that the quality distribution \( \Psi \) is uniform on \([0, 1]\).

The particular equilibrium concept we study for endogenous test selection is that of a Bayes-Nash Equilibrium. We say that a pair of distributions \((F_X, F_Y)\) supported on \([0, 1]\) constitutes a Bayes-Nash Equilibrium of the \textit{endogenous test selection game} if, for any distribution \(\Psi\) which is absolutely continuous, and similarly with the roles of \(X\) and \(Y\) reversed. The case when \(F_X\) and \(F_Y\) are equal is referred to as a symmetric Bayes-Nash Equilibrium. In this case, we will write \(F = F_X = F_Y\) and refer to \(F\) as an equilibrium distribution, or simply an equilibrium. We remind the reader that, as discussed in Section 3.2, even though we focus on quality distributions being Uniform\([0, 1]\], the results extend naturally to any distribution \(\Psi\) which is absolutely continuous.

The following proposition simply formalizes an observation from Section 3: that at equilibrium, each of \(X\) and \(Y\) must be ranked first with probability \(\frac{1}{2}\).

**Proposition 2** Let \((F_X, F_Y)\) be a Bayes-Nash Equilibrium of the endogenous test selection game, where agents may be restricted to an arbitrary set. For every threshold \(\theta\) in the support of \(F_X\), \(X\) must be selected with probability exactly \(\frac{1}{2}\) when choosing \(\theta\).

**Proof.** Since every \(\theta\) in the support of \(F_X\) is a best response to \(F_Y\), the probability of \(X\) being selected when choosing \(\theta\) does not depend on \(\theta\). Denote this probability by \(p_X\). Similarly, the probability of \(Y\) being selected when choosing a threshold in the support of \(F_Y\) is equal to a constant \(p_Y\) independent of the threshold chosen. Since \(X\) can guarantee that it is selected with probability \(\frac{1}{2}\) by "strategy stealing" (i.e., sampling its threshold at random from \(Y\)'s equilibrium distribution \(F_Y\)), the best-response condition implies \(p_X \geq \frac{1}{2}\), and similarly \(p_Y \geq \frac{1}{2}\). The equation \(p_X + p_Y = 1\), reflecting the fact that exactly one firm is always selected, now implies \(p_X = p_Y = \frac{1}{2}\). \(\square\)

We now define some key quantities for reasoning about the structure of the endogenous test-selection equilibria. Consider a firm \(X\) facing firm \(Y\) whose test threshold is drawn from the distribution \(F\) (which may not be continuous). Let \(w_{F, \theta}^{+}\) be the probability that \(X\) is selected conditioned on choosing a threshold of \(\theta\) and \textit{passing} its test, and \(w_{F, \theta}^{-}\) the probability of being selected conditioned on choosing a threshold \(\theta\) and \textit{failing} its test. We define the following notation:

\[
\mathcal{F}(\theta) = \int_0^\theta F(t) \, dt \quad \quad \phi_F = \mathbb{E}_{\Theta \sim F}[\Theta] = 1 - \mathcal{F}(1).
\]

\(\phi_F\) is the failure probability under \(F\); i.e., the probability that a firm using the strategy \(F\) fails its test drawn from \(F\). We will also write \(\phi_X = \phi_{F_X}\) and \(\phi_Y = \phi_{F_Y}\) for brevity. The following lemma characterizes the probabilities of being selected.

**Lemma 2** Let \(\delta_\theta = \mathcal{F}(\theta) - \lim_{t \uparrow \theta} F(t)\) be the discrete probability mass of \(F\) at \(\theta\); if \(F\) is continuous at \(\theta\), then \(\delta_\theta = 0\). We have that

\[
w_{F, \theta}^{+} = \phi_F + (1 - \theta) \cdot \left( \mathcal{F}(\theta) - \frac{\delta_\theta}{2} \right) + \mathcal{F}(\theta) \quad \quad \quad w_{F, \theta}^{-} = \theta \cdot \left( \mathcal{F}(\theta) - \frac{\delta_\theta}{2} \right) - \mathcal{F}(\theta).
\]

**Proof.** Assume that \(Y\)'s test is drawn from \(F\), and \(X\) chooses a test of \(\theta\). If \(X\) passes its test, then it is selected in the following (disjoint) scenarios: (1) \(Y\) fails its test (which has probability \(\phi_F\)); (2) \(Y\) passes a test of exactly \(\theta\), but \(X\) wins the coin flip (which has probability \((1 - \theta)\delta_\theta / 2\)); (3) \(Y\) passes a strictly easier test than \(\theta\), which is an event with probability \((\mathcal{F}(\theta) - \delta_\theta) \cdot \mathbb{E}_{\Theta_Y \sim F}[1 - \Theta_Y < \theta] = (1 - \theta) \cdot (\mathcal{F}(\theta) - \delta_\theta) + \mathcal{F}(\theta)\).

Similarly, when \(X\) fails its test, it is selected whenever (1) \(Y\) fails a test of exactly \(\theta\) and \(X\) wins the coin flip (which has probability \(\delta_\theta / 2\)), or (2) \(Y\) fails a strictly easier test than \(\theta\), which is an event of probability \((\mathcal{F}(\theta) - \delta_\theta) \cdot \mathbb{E}_{\Theta_Y \sim F}[\Theta_Y < \theta] = \theta (\mathcal{F}(\theta) - \delta_\theta) - \mathcal{F}(\theta)\). \(\square\)

By combining the two cases of the preceding lemma, we obtain the following corollary:
Corollary 2 Assume that $Y$’s threshold is drawn from the (possibly discontinuous) cdf $F$. Let $\theta$ be the threshold chosen by $X$, and $\delta_\theta = F(\theta) - \lim_{t\to\theta} F(t)$. Then, $X$’s probability of being selected is

$$(1 - \theta) \cdot \frac{d}{d\theta} F(\theta) + \theta \cdot \frac{d}{d\theta} F(\theta) = (1 - \theta) \cdot \frac{d}{d\theta} F(\theta) + ((1 - \theta)^2 + \theta^2) \cdot \left( F(\theta) - \frac{\delta_\theta}{2} \right) + (1 - 2\theta) \cdot F(\theta).$$

Much of the technical work of solving for equilibria of endogenous test selection games goes into showing that at equilibrium, $F_X$ and $F_Y$ are continuous and have full support. The following lemma is proved in Section 7.2 as a special case of the more general treatment for agents restricted to arbitrary intervals.

Lemma 3 Let $(F_X, F_Y)$ be a Bayes-Nash Equilibrium for unconstrained firms, i.e., allowed to choose tests from $[0, 1]$. Then both $F_X$ and $F_Y$ have full support and are continuous over $[0, 1]$.

Using this lemma, we now return to the characterization of endogenous test selection equilibria. The following theorem characterizes the unique Bayes-Nash Equilibrium on $[0, 1]$.

Theorem 6 There is a unique Bayes-Nash Equilibrium of the endogenous test selection game when firms have access to all tests in $[0, 1]$. The unique Bayes-Nash Equilibrium is symmetric, and its equilibrium distribution $F_X = F_Y = F_{eq}$ has the following cdf $F_{eq}$ and pdf $f_{eq}$.

$$F_{eq}(\theta) = \frac{1}{2} \cdot \left( 1 - \frac{1 - 2\theta}{\sqrt{\theta^2 + (1 - \theta)^2}} \right)$$

$$f_{eq}(\theta) = \frac{1}{2} \cdot \frac{1}{(\theta^2 + (1 - \theta)^2)^{3/2}}.$$
as claimed. Since we only used that both firms’ distributions have full support, the same argument can be applied to \( F_X \), to prove that \( F_X = F_{eq} \). Finally, to verify that \( F_X = F_Y = F_{eq} \) is in fact an equilibrium, we can substitute \( F = F_{eq} \) into Corollary 2 and verify that the selection probability of the firm \( X \), when faced with \( F_Y \), is indeed exactly \( \frac{1}{2} \) for all \( \theta \in [0,1] \).

Finally, to verify that \( F_{eq} \) is in fact an equilibrium, we can substitute \( F_{eq} = F_{eq} \) into Corollary 2 and verify that the selection probability of the firm \( X \), when faced with \( F_Y \), is indeed exactly \( \frac{1}{2} \) for all \( \theta \in [0,1] \).

\[ F_{eq} \]

\[ F_{eq} \]

\[ 1 - \frac{1 - 2\Psi(\sigma)}{\sqrt{\psi(\sigma)^2 + (1 - \Psi(\sigma))^2}} \]

\[ \square \]

Figure 3a shows the cdf and pdf of the equilibrium distribution of Theorem 6. Observe that the cdf satisfies the claims established in Lemma 3, namely, that it is continuous and has support \([0,1]\). Moreover, note that the pdf \( f_{eq} \) is also symmetric about \( \frac{1}{2} \). (This is not a priori obvious, and indeed, will not be the case when we consider restricted test sets in the next section). Finally, as discussed before, observe that if quality levels \( X,Y \) are drawn from any absolutely continuous distribution \( \Psi \), then the unique equilibrium distribution for thresholds \( \sigma \in \mathbb{R} \) is given by \( F_{eq,\Psi}(\sigma) = F_{eq}(\Psi(\sigma)) = \frac{1}{2} \left( 1 - \frac{1 - 2\Psi(\sigma)}{\sqrt{\psi(\sigma)^2 + (1 - \Psi(\sigma))^2}} \right) \).

![Equilibrium cdf and pdf for unrestricted firms.](image)

![Equilibrium cdf for firms restricted to \([0,0.79]\).](image)

Figure 3: Examples of equilibrium cdfs for unrestricted and restricted sets of tests.

### 6.1 Price of Anarchy of (Unrestricted) Endogenous Test Selection

We are now in a position to combine Corollary 1 and Theorem 6 to determine the Price of Anarchy (in terms of the principal’s probability of selecting the wrong firm) of allowing firms to choose their own tests. Substituting the characterizations into the functional (Eq. (1)), the resulting expression unfortunately does not lend itself to closed-form evaluation. However, a numerical calculation establishes the following.

**Corollary 3** The equilibrium cdf \( F_{eq} \) satisfies that \( I(F_{eq}) \approx 0.23056 \). Consequently, compared to the optimal i.i.d. test selection rule, endogenous test selection over unrestricted tests has a Price of Anarchy of roughly \( 1.10653 \) for any number of firms. Compared to the optimal correlated test selection rule, it has a Price of Anarchy of approximately \( 1.38336 \) for two firms, decreasing to \( 1.10653 \) as the number of firms \( n \to \infty \).

### 7 Endogenous Test Selection with Restricted Tests

We now consider a more general treatment: the principal restricts the firms to choose tests from a non-empty closed set \( S \subseteq [0,1] \), and the firms will play according to equilibrium distributions \( F_X, F_Y \) supported on subsets of \( S \). Note that although the firms’ tests are restricted to the set \( S \), their products’ qualities are still drawn uniformly from the entire interval \([0,1]\); this is reflected in the probabilities of passing/failing tests.

The existence of a (mixed, symmetric) Bayes-Nash Equilibrium follows from Lemma 7 of Dasgupta and Maskin (1986). However, we note that in general, the Nash equilibrium may not be unique; for example, when \( S = \{ 1 - \sqrt{2}, \sqrt{2} \} \), every pair of probability distributions on \( S \) constitutes an equilibrium. To see that this is the case, observe that conditional on the firms choosing any ordered pair of tests in the product set \( S \times S \), each firm’s probability of being selected is \( \frac{1}{2} \).
The following pair of theorems shows that by restricting the set \( S \) available to the firms, even to an interval, the principal can achieve a strictly smaller inversion probability than under the equilibrium for \( S = [0,1] \); however, for every non-empty set \( S \), the inversion probability under every symmetric Bayes-Nash Equilibrium is larger by some absolute constant than the one under the optimum i.i.d. distribution.

**Theorem 7** Let \( F_{[0,0.79]} \) be the unique\(^{14} \) symmetric Bayes-Nash equilibrium distribution when firms choose from the interval \([0,0.79]\), and \( F_{[0,1]} \) the unique and symmetric Bayes-Nash equilibrium distribution for unrestricted firms. Then\(^{15} \), \( I(F_{[0,0.79]}) < 0.22975 < 0.23052 < I(F_{[0,1]}) \).

**Theorem 8** Let \( S \subseteq [0,1] \) be an arbitrary non-empty set, and \( F \) any symmetric Bayes-Nash equilibrium distribution of firms restricted to choosing tests from \( S \). The expected probability of choosing the wrong firm under \( F \) is \( I(F) \geq \frac{5}{24} + \frac{1}{82944} \).

We emphasize that Theorem 8 establishes a lower bound only for symmetric equilibria. For general \( S \), there may be asymmetric equilibria, and they may achieve error probabilities strictly smaller than \( \frac{5}{24} \). For example, as observed above, when tests are restricted to the set \( S = \{1-\frac{\sqrt{7}}{2}, \frac{\sqrt{7}}{2}\} \), there is an asymmetric equilibrium in which firm \( X \) always chooses \( \theta_X = 1 - \frac{\sqrt{7}}{2} \), \( Y \) always chooses \( \theta_Y = \frac{\sqrt{7}}{2} \), and the inversion probability is \( \frac{1}{2}((\theta_X + (\theta_Y - \theta_X)^2 + (1 - \theta_Y)^2) = 3 - 2\sqrt{2} \approx 0.17157 \), whereas \( \frac{5}{24} \approx 0.2083 \).

The key to proving Theorem 7 is the following complete characterization of the unique Bayes-Nash equilibrium when \( S \) is restricted to intervals, proved in Section 7.2.

**Theorem 9** Let \( S = [a,b] \) be a non-empty interval, and consider the game when both firms are restricted to choosing tests from \( S \). There is a unique Bayes-Nash equilibrium, which is symmetric. Its cdf \( F_{eq} \) is given by the following:

1. If \((1-a) \cdot b \leq \frac{1}{2}\), then \( F_{eq} \) is a step function at \( b \), i.e., both firms deterministically choose \( b \).

2. Otherwise, let

\[
\delta_b = \frac{1 - a(1-b) - b(1-a)}{(1-a)((1-b)^2 + b^2)}, \quad \gamma = \frac{1 - a - 2b + 4ab - 2ab^2}{1 - 4(1-a)b + 2(1-2a)b^2}.
\]

The equilibrium cdf \( F_{eq} \) is given by:

\[
F_{eq}(\theta) = \begin{cases} 
\frac{1}{2(1-a)} \cdot \left((1 - 2a) + \sqrt{a^2 + (1-a)^2} \cdot \frac{2\theta - 1}{\sqrt{\theta^2 + (1-\theta)^2}} \right) & \text{for } a \leq \theta < \gamma \\
1 - \delta_b & \text{for } \gamma \leq \theta < b \\
1 & \text{for } \theta = b.
\end{cases}
\]  

An example of the equilibrium cdf (for firms restricted to interval \([0,0.79]\)) is shown in Fig. 3b. Using Theorem 9, we can now complete the proof of Theorem 7.

**Proof of Theorem 7.** Even for \( a = 0, b = 1 \), it appears that there is no closed-form solution for the value of \( I(F_{[0,1]}) \) for the equilibrium distribution. The closed-form characterization of \( F_{[a,b]} \) allows a numerical evaluation for all values of \( 0 \leq a < \frac{1}{2} < b \leq 1 \). Numerically, the optimum is achieved at \( a = 0, b \approx 0.79 \), where \( I(F_{[0,0.79]}) \leq 0.22975 \). \( \square \)

\(^{14}\) as will be established in Theorem 9

\(^{15}\) Recall that we write \( I(F) = \mathbb{E}[I(T_F)] \).
7.1 Suboptimality of All Symmetric Equilibria

We next give the proof of Theorem 8. Recall that we use $H^*$ to denote the cdf of the optimal distribution, i.e., the uniform distribution on $[\frac{1}{2}, \frac{3}{2}]$. We begin with an easy proposition, capturing that a sufficient condition for $H^*$ and an arbitrary cdf $G$ to differ by at least $\varepsilon$ at $z$ is for $G$ to be “sufficiently discontinuous” at some point $\theta$.

**Proposition 3** If $G(\theta) \geq \varepsilon + \lim_{t \to 0} G(t)$, then there exists a $z$ with $|H^*(z) - G(z)| \geq \frac{\varepsilon}{2}$.

**Proof.** If $|H^*(\theta) - G(\theta)| \geq \frac{\varepsilon}{2}$, then $z = \theta$ works, so assume that $|H^*(\theta) - G(\theta)| < \frac{\varepsilon}{2}$. Then,

$$\frac{\varepsilon}{2} < |H^*(\theta) - \lim_{\rho \to 0} G(\theta - \rho)| = \lim_{\rho \to 0} |H^*(\theta) - G(\theta - \rho)| \leq \lim_{\rho \to 0} |H^*(\theta - \rho) - G(\theta - \rho)| + 2\rho.$$

For sufficiently small $\rho$, we therefore get that $|H^*(\theta - \rho) - G(\theta - \rho)| > \frac{\varepsilon}{2}$, so choosing $z = \theta - \rho$ for such a small $\rho$ completes the proof. \qed

Proposition 3 is the key ingredient to proving Lemma 4, which shows that symmetric equilibrium distributions deviate far from the optimal distribution,

**Lemma 4** Let $F$ be the cdf of an equilibrium distribution for some non-empty closed set $S$. There exists a $z \in (0, 1)$ with $|F(z) - H^*(z)| \geq \frac{1}{21}$.

**Proof.** Let $p = \frac{1}{22}$, and $\theta = \min \{t \mid F(t) \geq 1 - p\}$. Let $\delta_0 = 1 - \lim_{t \to 0} F(t)$ be the point mass at $\theta$ (if any), and $\delta_a = F(a)$ the point mass at the lower end $a$ of the support. Let $q$ be the probability that firm $Y$ chooses a test $y > \theta$ and fails. Then, $w^+ = (F(\theta) - \delta_0) + \delta_0 \cdot (1 - \frac{1}{2} \theta) + q$ and $w^- = \phi - \delta_0 \cdot \frac{\theta}{2} - q$. Thus, the probability for $X$ to be chosen is

$$\frac{1}{2} \geq (1 - \theta) \cdot (F(\theta) - \delta_0 + \delta_0 \cdot (1 - \frac{1}{2} \theta) + q) + \theta \cdot (\phi - \delta_0 \cdot \frac{\theta}{2} - q)$$

$$= (1 - 2\theta) \cdot q + (1 - \theta) \cdot F(\theta) + \theta \phi - \delta_0 \cdot \frac{\theta}{2} \cdot (1 - \theta)^2 + \theta^2).$$

If $\theta \leq \frac{1}{2}$, then we get that $F(\frac{1}{2}) \geq 1 - p$, so $|F(\frac{1}{2}) - H^*(\frac{1}{2})| \geq \frac{1}{2} - p \geq \frac{1}{23}$. Otherwise, $1 - 2\theta < 0$, so we can lower-bound

$$(1 - 2\theta) \cdot q \geq (1 - 2\theta) \cdot (1 - F(\theta)) \geq (1 - 2\theta) \cdot p.$$

Furthermore, we lower-bound

$$\phi = \frac{1 - \delta_a (a^2 + (1 - a)^2)}{2(1 - a)} \geq \frac{1 - \delta_a}{2}.$$

Substituting these bounds, as well as $(1 - \theta)^2 + \theta^2 \leq 1$, we can lower-bound

$$\frac{1}{2} \geq (1 - 2\theta) \cdot p + (1 - \theta) \cdot (1 - p) + \theta \phi - \frac{\delta_0}{2}$$

$$= 1 - \theta \cdot (1 + p) + \theta \cdot \frac{1 - \delta_a}{2} - \frac{\delta_0}{2}$$

$$\geq 1 - \theta \cdot (\frac{1}{2} + p) - \frac{\delta_a + \delta_0}{2},$$

so $\theta \geq \frac{1 - (\delta_a + \delta_0)}{1 + 2p}$. If $\delta_a \geq \frac{1}{12}$ or $\delta_0 \geq \frac{1}{12}$, then the lemma follows by applying Proposition 3 with $\varepsilon = \frac{1}{12}$ and $z = a$ or $z = \theta$. Otherwise, $\theta > \frac{3}{4}$, so $H^*(\theta) = 1$, while $\lim_{t \to 0} F(t) \leq 1 - p$. Therefore, there must exist a $z$ with $|H^*(z) - H(z)| \geq p > \frac{1}{21}$, completing the proof. \qed

The second key lemma shows that a large deviation at even one point implies a significantly larger error probability.
Lemma 5 Let $G$ be any distribution such that $|G(z) - H^*(z)| \geq \varepsilon$ for some $z \in (0, 1)$ and $\varepsilon > 0$. Then, $I(G) \geq I(H^*) + \frac{1}{6}\varepsilon^3$.

Proof. Write $G_0 = H^*$, and define the one-parameter family of cumulative distribution functions $G_t = tG + (1-t)G_0$ as in the proof of Theorem 4. Recall that $I(G_t) = I(G_0) + A(G)\cdot t + B(G)\cdot t^2$, where $A(G), B(G)$ are non-negative coefficients defined in Eq. (4). Suppose that $G_0(z) - G(z) \geq \varepsilon$; the case $G(z) - G_0(z) \geq \varepsilon$ is handled symmetrically. Since $G_0'(x) \leq 2$ for all $x \in [0, 1]$, we have $G_0(x) - G(x) \geq \varepsilon - 2(x - z)$ for all $x \geq z$. Now recall the definition of $B(G)$ in Eq. (4), and that the integrand in the definition of $B(G)$ is symmetric in $x$ and $y$. Hence,

$$B(G) = \frac{1}{2} \int_0^1 \int_0^1 (G_0(x) - G(x) + G(y) - G_0(y))^2 \, dy \, dx.$$

Letting $H(x) := G_0(x) - G(x)$, we have

$$B(G) = \frac{1}{2} \int_0^1 \int_0^1 (H(x) - H(y))^2 \, dy \, dx$$

$$= \frac{1}{2} \int_0^1 H(x)^2 \, dx \, dx + \frac{1}{2} \int_0^1 H(x)H(y) \, dy \, dx + \frac{1}{2} \int_0^1 H(y)^2 \, dy \, dx$$

$$= \int_0^1 H(x)^2 \, dx + \left( \int_0^1 H(x) \, dx \right)^2$$

$$\geq \int_0^1 H(x)^2 \, dx \geq \int_z^{z+\varepsilon/2} H(x)^2 \, dx \geq \int_z^{z+\varepsilon/2} (\varepsilon - 2(x - z))^2 \, dx = \frac{1}{2} \int_0^\varepsilon u^2 \, du = \frac{1}{6}\varepsilon^3.$$

Using the inequalities $A(G) \geq 0$ and $B(G) \geq \frac{1}{6}\varepsilon^3$ in the expression $I(G_t) = I(G_0) + A(G)\cdot t + B(G)\cdot t^2$, and setting $t = 1$ so that $G_t = G$, we find that $I(G) \geq I(G_0) + \frac{1}{6}\varepsilon^3$, as claimed. \qed

Combining Lemma 4 and Lemma 5, with $\varepsilon = \frac{1}{24}$, immediately implies Theorem 8.

7.2 Proofs of Theorem 9 and Lemma 3

We now characterize the Bayes-Nash equilibria of the game, in particular proving Theorem 9 and Lemma 3. As with other BNE characterizations, much of the technical work focuses on characterizing the supports of the two firms’ distributions and ruling out discontinuities, except possibly at the upper end of the allowed range.

7.2.1 Laying the Groundwork

We begin with a lemma relating the winning probabilities conditioned on passing/failing tests and for tests of different thresholds.

Lemma 6 For any distribution $F$, for any $\theta \in [a, b]$,

$$w^+_{F,\theta} - w^-_{F,\theta} > 0.$$  

(9)

Also, if $a \leq \theta_0 < \theta_1 \leq b$ then

$$w^+_{F,\theta_1} - w^+_{F,\theta_0} = \frac{1}{2} \delta_{\theta_0} - \frac{1}{2} \delta_{\theta_1} + \frac{1}{2} \int_{\theta_0}^{\theta_1} (F(\theta) - F(\theta_0)) \, d\theta + (1 - \theta_1)[F(\theta_1) - F(\theta_0)]$$

$$\leq \frac{1}{2} \delta_{\theta_0} + F(\theta_1) - F(\theta_0)$$

$$w^-_{F,\theta_1} - w^-_{F,\theta_0} = \frac{1}{2} \delta_{\theta_0} - \frac{1}{2} \delta_{\theta_1} + \frac{1}{2} \int_{\theta_0}^{\theta_1} (F(\theta) - F(\theta_0)) \, d\theta + \theta_0[F(\theta_1) - F(\theta_0)]$$

$$\leq \frac{1}{2} \delta_{\theta_0} + F(\theta_1) - F(\theta_0).$$

(10)

(11)
Let \( F \) be a probability distribution on \([a,b]\), and suppose that \( \alpha, \beta \) satisfy \( a \leq \alpha < \beta \leq b \) and \( F(\alpha) = F(\beta) \). Then, there exists an \( \varepsilon > 0 \) such that no test in the open interval \((\alpha, \beta + \varepsilon)\) is a best response to \( F \).

**Proof.** First consider \( \theta \in (\alpha, \beta) \). Let \( \theta' \in (\alpha, \theta) \). The assumption that \( F(\alpha) = F(\beta) \) implies that \( F(\theta) = F(\theta') \). Applying Equations (10) and (11), we find that \( w_{\theta'}^+ = w_{\theta'}^- = w_{\theta'}^+ \) and \( w_{\theta}^- = w_{\theta}^- = w_{\theta}^- \), i.e., the probability of winning conditioned on passing is the same at all three thresholds, and similarly for failing. The difference in winning probability between \( \theta' \) and \( \theta \) is

\[
(1 - \theta')w_{\theta'}^+ + \theta'w_{\theta'}^- - (1 - \theta)w_{\theta}^+ - \theta w_{\theta}^- = (\theta - \theta') \cdot (w_{\theta}^- - w_{\theta}^-) > 0
\]

by Inequality (9). In particular, \( \theta' \) is a strictly better response to \( F \) than \( \theta \), so \( \theta \) cannot be a best response to \( F \).

Next, consider \( \theta = \beta + \varepsilon \) (for sufficiently small \( \varepsilon \)), and let \( \theta' = \alpha + \varepsilon \). The benefit of deviating from \( \theta \) to \( \theta' \) is

\[
(1 - \theta')w_{\theta'}^+ + \theta'w_{\theta'}^- - (1 - \theta)w_{\theta}^+ - \theta w_{\theta}^- = (\beta - \alpha - \varepsilon) \cdot (w_{\theta}^+ - w_{\theta}^-) - (1 - \beta)(w_{\theta}^+ - w_{\theta}^-) - \beta(w_{\theta}^- - w_{\theta}^-) + \varepsilon(w_{\theta}^+ - w_{\theta}^-)
\]

\[
\geq (\beta - \alpha - \varepsilon) \cdot (w_{\theta}^+ - w_{\theta}^-) - (1 - \beta)^2 + \beta^2 \cdot \frac{\delta_{\beta}}{2} - (F(\theta) - F(\beta))
\]

where the penultimate line uses Lemma 6 and the fact that \( w_{\theta}^+ - w_{\theta}^- > 0 \), and the last line uses the observation that \( \delta_{\beta} = 0 \), which follows from the assumption that \( F(\alpha) = F(\beta) \). The first term on the last line is strictly positive, whereas all the other terms on the last line converge to zero as \( \varepsilon \to 0 \). (Recall that cumulative distribution functions such as \( F \) are right-continuous.) Therefore, as \( \varepsilon \to 0 \), the quantity on the last line is positive, so there exists some \( \varepsilon \) such that for all \( \theta \leq \beta + \varepsilon \), playing \( \theta \) is not a best response. \( \square \)

The next lemma shows that Bayes-Nash Equilibrium distributions can have point masses (i.e., discontinuities) at most at the upper and lower end of the allowed ranges. We will later also rule out point masses at the lower end.

**Lemma 8** Let \((F_X, F_Y)\) be a Bayes-Nash Equilibrium for firms constrained to tests from \([a,b]\). The distributions \( F_X \) and \( F_Y \) have no point masses other than possibly at \( a \) or \( b \), and at most one of them has a point mass at \( a \).
Prove. We first show that for each $\theta < b$, at most one firm has point mass at $\theta$. Denote $F_Y$ by $F$, and let $\theta < b$ be a point where firm $Y$ has point mass $\delta_\theta > 0$. We now compare the winning probability for a firm $X$ with threshold $\theta + \varepsilon$ to the winning probabilities for $X$ with thresholds $\theta$ or $\theta - \varepsilon$. First, using Lemma 6 and straightforward calculations, for every $\varepsilon > 0$, we can bound

\[
  w_{\theta + \varepsilon}^+ \geq w_{\theta}^+ + \frac{\delta_\theta}{2} \cdot (1 - \theta) \\
  w_{\theta - \varepsilon}^+ \leq w_{\theta}^+ - \frac{\delta_\theta}{2} \cdot (1 - \theta)
\]

Therefore, the winning probability of firm $X$ with threshold $\theta + \varepsilon$ is

\[
  (1 - \theta - \varepsilon)w_{\theta + \varepsilon}^+ + (\theta + \varepsilon)w_{\theta - \varepsilon}^- \geq (1 - \theta - \varepsilon) \left( w_{\theta}^+ + \frac{\delta_\theta}{2} \cdot (1 - \theta) \right) + (\theta + \varepsilon) \left( w_{\theta}^- - \frac{\delta_\theta}{2} \cdot \theta \right)
\]

\[
  \geq (1 - \theta)w_{\theta}^+ + \theta w_{\theta}^- + \frac{\delta_\theta}{2} \cdot ((1 - \theta)^2 + \theta^2) - \varepsilon
\]

\[
  \geq (1 - \theta + \varepsilon) \left( w_{\theta}^+ - \frac{\delta_\theta}{2} \cdot (1 - \theta) \right) + (\theta - \varepsilon) \left( w_{\theta}^- - \frac{\delta_\theta}{2} \cdot \theta \right)
\]

\[
  + \delta_\theta \cdot ((1 - \theta)^2 + \theta^2) - 2\varepsilon
\]

\[
  \geq (1 - \theta + \varepsilon)w_{\theta - \varepsilon}^+ + (\theta - \varepsilon)w_{\theta - \varepsilon}^- + \delta_\theta \cdot ((1 - \theta)^2 + \theta^2) - 2\varepsilon.
\]

For sufficiently small positive $\varepsilon$, the quantity $\frac{\delta_\theta}{2} \cdot ((1 - \theta)^2 + \theta^2) - \varepsilon$ is strictly positive, so neither $\theta$ nor $\theta - \varepsilon$ can be a best response for firm $X$. Therefore, in a Bayes-Nash Equilibrium, the probability of $X$ choosing a test in the set $[\theta - \varepsilon, \theta]$ is zero.

We have shown that if one firm has a point mass at $\theta$, then the other does not. When $\theta = a$, this is all the lemma requires us to prove. When $\theta \in (a, b)$, we need to show that neither of the distributions $F_X$, $F_Y$ can have a point mass at $\theta$. For the sake of contradiction, assume that $F_Y$ has a point mass at $\theta \in (a, b)$. Let $\varepsilon \in (0, \theta - a)$ be small enough that the probability of $X$ choosing a test in $[\theta - \varepsilon, \theta]$ is zero; such an $\varepsilon$ exists by the preceding argument. It follows that $F_X(\theta - \varepsilon) = F_X(\theta)$. Now, using Lemma 7, we may conclude that for some $\varepsilon' > 0$, no test in the interval $(\theta - \varepsilon, \theta + \varepsilon')$ is a best response to $F_X$. Hence $F_Y$, which has a point-mass at $\theta$, cannot be a best response to $F_X$, in contradiction to our assumption that $F_X$ and $F_Y$ constitute a Bayes-Nash Equilibrium. $\square$

The next lemma pins down the support of equilibrium distributions, showing that both firms’ distributions have the same support, and showing that it must be of a very specific form.

**Lemma 9** Let $(F_X, F_Y)$ be a Bayes-Nash Equilibrium for firms constrained to tests from $[a, b]$. The probability distributions $F_X$ and $F_Y$ have the same support. This support set is one of the following three alternatives.

- An interval $[a, \gamma]$ where $a < \gamma \leq b$.
- A set of the form $[a, \gamma] \cup \{b\}$ where $a \leq \gamma < b$.
- The set $\{b\}$.

**Proof.** Denote the complements of the support sets of $F_X, F_Y$ by $U_X, U_Y$, respectively. Both of these sets are open, since the support of a distribution is, by definition, closed. If $U_X$ and $U_Y$ are both empty, the lemma’s conclusion is satisfied. Otherwise, assume without loss of generality that $U_X$ is non-empty. Consider an arbitrary $\theta \in U_X$ and let $J = (\alpha, \beta)$ be the maximal open subinterval of $U_X$ containing $\theta$. (Here, we also consider a half-open interval of the form $[a, \beta)$ or $(\alpha, b]$ to be an open subinterval of $U_X$.) For any $\beta' \in J$ we have $F_X(\alpha) = F_X(\beta')$ which implies, by Lemma 7, that none of the points in $(\alpha, \beta')$ is a best response to $F_X$. Therefore, none of these points is in the support of $F_Y$, i.e., the interval $(\alpha, \beta')$ is contained in $U_Y$. Taking the union over all $\beta' \in J$, we find that $J = \bigcup_{\beta' \in J} (\alpha, \beta') \subseteq U_Y$; in particular, this means that $\theta \in U_Y$. As $\theta \in U_X$ was arbitrary, we have shown that $U_X \subseteq U_Y$. A symmetric argument establishes that
neither of them has a point mass at $a$, that at most one of them has an isolated support point at $a$, since both distributions have the same support, this means that $F_U = F_Y$ and that $J$ is a maximal open subinterval of $U_Y$.

We have shown that every maximal open subinterval of $U$ has right endpoint $b$. If the left endpoint is $\gamma > a$ then $U_X$ has the form $(\gamma, b)$ or $(\gamma, b]$, which exactly corresponds to the first two types in the statement of the lemma. If the left endpoint of $U$ is $a$, then $U$ is one of the sets $(a, b)$, $(a, b]$, or $[a, b]$. The first two alternatives can be eliminated because they both imply that $F_X$ has an isolated support point at $a$. Since both distributions have the same support, this means that $F_Y$ also has an isolated support point at $a$. However, an isolated point in the support of a distribution must be a point mass, and Lemma 8 guarantees that at most one of $F_X$, $F_Y$ has a point mass at $a$.

Lemma 10 If $(F_X, F_Y)$ is a Bayes-Nash Equilibrium, then both distributions have $b$ in their support, and neither of them has a point mass at $a$. If $(1 - a) \cdot b \leq \frac{1}{2}$ then the only equilibrium is a step function at $b$, i.e., both firms deterministically choose $b$. Otherwise, we have the following characterizations of the failure probability, the point mass at $b$, and the (common) largest support point other than $b$, for both firms:

$$
\phi_X = \phi_Y = \phi = \frac{1}{2(1 - a)} \quad (14)
$$

$$
\delta_{b,X} = \delta_{b,Y} = \delta_b = \frac{1 - 2b + 2b\phi}{(1 - b)^2 + b^2} \quad (15)
$$

$$
\gamma = \frac{1 - a - 2b + 4ab - 2ab^2}{1 - 4(1 - a)b + 2(1 - 2a)b^2}. \quad (16)
$$

Proof. If the equilibrium is not a step function at $b$, then Lemma 9 implies that the (common) support of $F_X$ and $F_Y$ contains an interval $[a, \gamma]$, with $\gamma > a$. Consider firm $Y$ playing $a + \varepsilon$, where $\varepsilon < \gamma - a$. Corollary 2 with $\theta = a + \varepsilon$, where $\delta_\theta = 0$, characterizes the probability of $Y$ being chosen as

$$(1 - a - \varepsilon) \cdot \phi_X + ((1 - a - \varepsilon)^2 + (a + \varepsilon)^2) \cdot F_X(a + \varepsilon) + (1 - 2a - 2\varepsilon) \cdot F_X(a + \varepsilon).$$

This must equal $\frac{1}{2}$ by Proposition 2. Taking the limit as $\varepsilon \to 0$, and observing that $\lim_{\varepsilon \to 0} F_X(a + \varepsilon) = F_X(a)$ and $\lim_{\varepsilon \to 0} F_X(a + \varepsilon) = 0$, we obtain that

$$
\phi_X = \frac{1 - F_X(a) \cdot ((1 - a)^2 + a^2)}{2(1 - a)}. \quad (17)
$$

A symmetric derivation, with the roles of $X$ and $Y$ reversed, establishes that $\phi_Y = \frac{1 - F_Y(a) \cdot ((1 - a)^2 + a^2)}{2(1 - a)}$. Notice that these expressions are equal to the expression for $\phi$ in Equation (15), provided that $F_X(a) = F_Y(a) = 0$, i.e. $F_X$ and $F_Y$ have no point mass at $a$. We will prove this fact below.

By Lemma 8, at most one firm has a point mass at $a$; assume without loss of generality that firm $X$ has no point mass at $a$, so $F_X(a) = 0$, whence the failure probability of firm $X$ is $\phi_X = \frac{1}{2(1 - a)}$. If $(1 - a)b \leq \frac{1}{2}$ this means that $\phi_X \geq b$. However, the only way that the failure probability of a firm sampling a test from $[a, b]$ could be as large as $b$ is if $X$ deterministically chooses a test of difficulty $b$. Since $F_X$ and $F_Y$ have the same support, this means that $Y$ also deterministically chooses $b$, i.e., we have confirmed that when $(1 - a)b \leq \frac{1}{2}$, the only equilibrium is that both firms deterministically choose $b$. 

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Assume henceforth that \((1-a)b > \frac{1}{2}\) and let \(\theta\) be the supremum of the supports of \(F_X\) and \(F_Y\). We shall prove that \(\theta = b\), as claimed by the lemma, while also establishing Equation (15). Consider firm \(X\) or \(Y\) playing \(\theta\), and apply Corollary 2 and Proposition 2 to obtain

\[
\frac{1}{2} = (1-\theta) \cdot \phi_Y + ((1-\theta)^2 + \theta^2) \cdot (F_Y(\theta) - \frac{1}{2}\delta_{\theta,Y}) + (1-2\theta) \cdot F_Y(\theta) \\
\frac{1}{2} = (1-\theta) \cdot \phi_X + ((1-\theta)^2 + \theta^2) \cdot (F_X(\theta) - \frac{1}{2}\delta_{\theta,X}) + (1-2\theta) \cdot F_X(\theta).
\]

(18)

By our choice of \(\theta\), we have that \(F_Y(\theta) = F_X(\theta) = 1\), as well as \(F_Y(1) - (1-\theta) = \theta - \phi_Y\) and \(F_X(1) - (1-\theta) = \theta - \phi_X\). Substituting these into the right-hand sides of Equation (18), we obtain that

\[
\frac{1}{2} = (1-\theta + \theta\phi_Y) - \delta_{\theta,Y} \cdot \frac{(1-\theta)^2 + \theta^2}{2} \\
\frac{1}{2} = (1-\theta + \theta\phi_X) - \delta_{\theta,X} \cdot \frac{(1-\theta)^2 + \theta^2}{2}.
\]

(19)  (20)

Recall that we are assuming without loss of generality that \(F_X(a) = 0\) and \(\phi_X = \frac{1}{2(1-a)}\). If \(\delta_{\theta,X} = 0\), we can rearrange Equation (20) to obtain \(\theta(1 - \phi_Y) = \frac{1}{2}\). Since \(\theta \leq 1\) and \(1 - \phi_Y = 1 - \frac{1}{2(1-a)} \leq \frac{1}{2}\), the only way this equation could hold is if \(a = 0\) and \(\theta = 1\). Hence, either \(F_X\) has a point mass at \(\theta\), or \(\theta = 1\). In the former case, Lemma 8 implies that \(\theta = b\). In the latter case, \(\theta = 1 = b\). Thus, in either case, we have proved that \(\theta = b\).

Substituting \(\theta = b\) into Equations (19) and (20) and rearranging, we obtain (essentially) the expressions for \(\delta_{b,X}\) and \(\delta_{b,Y}\) in Equation (15); more specifically, it only remains to show that \(\phi_X = \phi_Y\), which will follow once we have shown that \(F_Y(a) = 0\) below.

Now, suppose that \(F_X\) and \(F_Y\) are supported on the set \([a, \gamma] \cup \{b\}\), for some \(\gamma \in (a, b]\). We turn to calculating \(\gamma\). Let \(\phi_X, \delta_{b,X}\) denote the failure probability and point mass at \(b\) for firm \(X\). Consider firm \(Y\) playing \(\gamma\). Because \(w_1 = 1 - (1-b)\delta_{b,X}\) (if \(Y\) passes, it will be chosen unless \(X\) plays \(b\) and passes), and \(w^-_1 = \phi_X - b\delta_{b,X}\) (if \(Y\) fails, it will be chosen if \(X\) fails, but did not play \(b\)), \(Y\)'s probability of being selected is

\[
\frac{1}{2} = (1-\gamma) \cdot (1 - (1-b)\delta_{b,X}) + \gamma \cdot (\phi_X - b\delta_{b,X}) = 1 - (1-b)\delta_{b,X} - \gamma \cdot (1 + (2b-1)\delta_{b,X} - \phi_X).
\]

(21)

The same reasoning with the roles of \(X\) and \(Y\) reversed implies

\[
\frac{1}{2} = (1-\gamma) \cdot (1 - (1-b)\delta_{b,Y}) + \gamma \cdot (\phi_Y - b\delta_{b,Y}) = 1 - (1-b)\delta_{b,Y} - \gamma \cdot (1 + (2b-1)\delta_{b,Y} - \phi_Y).
\]

(22)

Solving Equation (22) for \(\gamma\) and substituting the characterizations \(\phi_X = \frac{1}{2(1-a)}\) and \(\delta_{b,X} = \frac{1-2b+2\phi_y}{(1-b)^2+b^2}\) derived above gives us that

\[
\gamma = \frac{\frac{1}{2} - (1-b)\delta_{b,X}}{1 + (2b-1)\delta_{b,X} - \phi_X} = \frac{1 - a - 2b + 4ab - 2ab^2}{1 - 4(1-a)b + 2(1-2a)b^2},
\]

as asserted in the statement of the lemma.

Finally, we must prove that firm \(Y\) has no point mass at \(a\), which will imply that the derivations of \(\phi_X\) and \(\delta_{b,X}\) apply equally to \(\phi_Y\) and \(\delta_{b,Y}\). We may rearrange Equations (21)–(22) to derive

\[
((1-\gamma)(1-b) + \gamma b) \cdot \delta_{b,X} = \frac{1}{2} - \gamma + \gamma\phi_X \\
((1-\gamma)(1-b) + \gamma b) \cdot \delta_{b,Y} = \frac{1}{2} - \gamma + \gamma\phi_Y.
\]
By subtracting the second equation from the first, we obtain that

\[
((1 - \gamma)(1 - b) + \gamma b) \cdot (\delta_{b,X} - \delta_{b,Y}) = \gamma \cdot (\phi_X - \phi_Y).
\]

Substituting the expressions for \(\delta_{b,X}, \delta_{b,Y}\) derived above, we obtain the equation

\[
\frac{((1 - \gamma)(1 - b) + \gamma b) \cdot 2b}{(1 - b)^2 + b^2} \cdot (\phi_X - \phi_Y) = \gamma(\phi_X - \phi_Y).
\]

Consequently, either \(\phi_X = \phi_Y\) or \(\frac{((1 - \gamma)(1 - b) + \gamma b) \cdot 2b}{(1 - b)^2 + b^2} = \gamma\). The latter equation can be rearranged to

\[
2b - 2b^2 = (1 - 2b^2)\gamma.
\]

Recall that we are assuming here that \((1 - a)b > \frac{1}{2}\), which in particular implies that \(2b > 1\), so the equation \(2b - 2b^2 = (1 - 2b^2)\gamma\) implies that \(\gamma > 1\), contradicting the fact that \(\gamma \leq b\). Consequently, the equation \(\frac{((1 - \gamma)(1 - b) + \gamma b) \cdot 2b}{(1 - b)^2 + b^2} = \gamma\) cannot be satisfied, meaning that Equation (23) implies \(\phi_X = \phi_Y\).

Recalling the formula for failure probability in Equation (14), we see that the equation \(\phi_X = \phi_Y\) implies that \(F_Y(a) = F_X(a) = 0\), i.e., neither distribution has a point mass at \(a\), as claimed. \(\square \)

### 7.2.2 Proofs of Lemma 3 and Theorem 9

Lemma 3 follows easily as a corollary of Lemma 8 and Lemma 10:

**Proof of Lemma 3.** Lemma 10, applied with \(a = 0\) and \(b = 1\), implies that \(F_X\) and \(F_Y\) have no point mass at \(a = 0\) or \(b = 1\) and that \(\gamma = 1\). Therefore, the distributions have full support. By Lemma 8, the distributions have no discontinuities on \((a, b)\), either, completing the proof. \(\square \)

Finally, we use the characterization of the equilibrium supports to prove Theorem 9.

**Proof of Theorem 9.** The first case (when \((1 - a) \cdot b \leq \frac{1}{2}\) is explicitly covered by Lemma 10. Therefore, assume from now on that \((1 - a) \cdot b \geq \frac{1}{2}\).

By Lemma 9 and Lemma 8, we know that \(F_X\) and \(F_Y\) are both continuous over \([a, b]\) and that there is some \(\gamma \in (a, b)\) such that both functions are strictly monotone over \([a, \gamma]\), constant over \([\gamma, b]\), and then possibly discontinuous at \(b\). Note that the characterization of \(\delta_b\) and \(\gamma\) from Lemma 10 exactly correspond to the definitions of \(\delta_b\) and \(\gamma\) in Theorem 9.

We can now generalize our approach from Section 6 for computing the equilibrium distribution on the interval \([a, \gamma]\). Let \(F\) refer to either of the equilibrium distributions \(F_X, F_Y\). Consider any threshold \(\theta \in (a, \gamma)\). By Proposition 2 and Corollary 2,

\[
\frac{1}{2} = (1 - \theta) \cdot \phi + (1 - 2\theta) F(\theta) + ((1 - \theta)^2 + \theta^2) \cdot F(\theta);
\]

compared to the derivation for \([0, 1]\), we now do not have that \(\phi = \frac{1}{2}\). Rearranging and dividing the equation by \((\theta^2 + (1 - \theta)^2)^{3/2}\), we obtain the differential equation

\[
(\theta^2 + (1 - \theta)^2)^{-1/2} \cdot F(\theta) + \frac{1 - 2\theta}{(\theta^2 + (1 - \theta)^2)^{3/2}} \cdot \frac{F(\theta)}{ \sqrt{\theta^2 + (1 - \theta)^2} } = \frac{1}{2} - (1 - \theta) \cdot \phi \frac{1}{(\theta^2 + (1 - \theta)^2)^{3/2}}.
\]

As before, we can integrate by parts to obtain that for all \(\theta \in (a, \gamma),

\[
\frac{d}{d\theta} \frac{F(\theta)}{\sqrt{\theta^2 + (1 - \theta)^2}} = \frac{1}{2} - (1 - \theta) \cdot \phi \frac{1}{(\theta^2 + (1 - \theta)^2)^{3/2}}.
\]
so, using the boundary condition $F(a) = 0$, we find that for all $\theta \in [a, \gamma],$

\[
\frac{F(\theta)}{\sqrt{\theta^2 + (1-\theta)^2}} = \int_a^\theta \frac{1}{2} - (1-t) \cdot \phi \left( \frac{t}{(t^2 + (1-t)^2)^{3/2}} \right) dt \\
= \phi \cdot \int_a^\theta \frac{t - a}{(t^2 + (1-t)^2)^{3/2}} dt \\
= \phi \cdot \left[ (1-a) + \frac{t + a - 1 - 2at}{\sqrt{t^2 + (1-t)^2}} \right]_a^\theta \\
= \phi \cdot \left( \frac{\theta + a - 1 - 2a\theta}{\sqrt{\theta^2 + (1-\theta)^2}} - \frac{2a - 1 - 2a^2}{\sqrt{a^2 + (1-a)^2}} \right) \\
= \phi \cdot \left( \frac{\sqrt{a^2 + (1-a)^2} - \frac{1 + 2a\theta - a - \theta}{\sqrt{\theta^2 + (1-\theta)^2}}}{\sqrt{\theta^2 + (1-\theta)^2}} \right),
\]

or

\[
F(\theta) = \phi \cdot \left( \sqrt{a^2 + (1-a)^2} \cdot \frac{2\theta - 1}{\sqrt{\theta^2 + (1-\theta)^2}} + (1-2a) \right).
\]

As $F$ in this proof was interpreted to be either of the equilibrium distributions $F_X$, $F_Y$, we find that the only Bayes-Nash Equilibrium is the symmetric equilibrium in which both firms’ distributions obey the formula (24) for all $\theta \in [\alpha, \gamma)$, and both distribution have all of their remaining probability mass located at $b$. 

\[\Box\]

8 Conclusions

We introduced and studied a problem of optimal and endogenous test selection in a setting where a principal wants to select the product of higher quality from one of two firms, but the products’ qualities can only be measured through threshold tests which reveal whether a product’s quality lies above or below a threshold $\theta$. We explicitly characterized the optimal correlated and i.i.d. distributions for the principal, as well as the equilibrium distribution when the firms can choose their own thresholds from an interval $[a,b]$ (in particular including the case of the interval $[0,1]$). Using these characterizations, we showed that the principal can do strictly better by giving the firms different tests than drawing their tests i.i.d. The best i.i.d. distribution is better than any symmetric equilibrium for any set $S$ offered to the firms (including sets $S$ that are not intervals), and the equilibrium under the best interval gives the principal strictly higher probability of selecting the best product than the equilibrium for the interval $[0,1]$. Our work raises a wealth of questions for future work. An immediate question implicitly raised in Section 7 is which set of tests a principal should offer to achieve the smallest probability of selecting the wrong product at equilibrium.\(^{16}\) There are two variants to this question: when the principal is interested only in symmetric equilibria, or also in asymmetric (non-unique) ones. For the former version, a natural conjecture would be that the optimal set for the principal is an interval, in which case our numerical calculations from

\(^{16}\)Of course, if the principal can choose different sets for different firms, then she can choose $S_X = \{\frac{1}{2}\}$ and $S_Y = \{\frac{2}{3}\}$, which would implement the optimal strategy for her. The more interesting question is to find one set $S$ to restrict all firms to, which naturally corresponds to prescribing standards for quality control.
Section 7 would imply that the optimum set would be the interval \([0, 0.79\ldots]\). While we cannot prove or disprove this conjecture at this point, a similar-looking stronger conjecture is false: there are discrete sets \(S\) the principal can offer under which the unique equilibrium is strictly better than if the principal instead offered the smallest interval containing all of \(S\). For the latter case, we conjecture optimality of the set \(\{1 - \sqrt{2}/2, \sqrt{2}/2\}\), discussed in Section 7.

The endogenous test selection game between the firms can be viewed as a natural instance of a signaling game, in which each firm’s strategy is a signaling scheme. Our problem setup severely restricted the signaling schemes the firms could choose from, to binary threshold tests. Naturally, it would be desirable to extend the results to broader classes of signaling schemes. At the full extreme, when firms may choose any signaling scheme, the unique equilibrium of our game is full disclosure. This follows from Corollary 1 of (Hwang et al. 2019). However, an analysis of the intermediate regime, in which the number of signals is still constrained (as in (Dughmi et al. 2016)), would still be of interest.

Perhaps the most immediate next step along these lines would be signaling schemes in which firms can choose an arbitrary mapping from qualities to \{pass, fail\}. It is not hard to show that w.l.o.g., it suffices to consider signaling schemes with two thresholds \(0 \leq \theta_1 \leq \theta_2 \leq 1\) in which the firm passes the test iff its quality lies in \([\theta_1, \theta_2]\). A natural conjecture would be that even if the firms were allowed to choose such tests, at equilibrium, they would always choose threshold tests only, i.e., set \(\theta_1 = 0\). This conjecture is false! If such a symmetric equilibrium existed, it would have to be the equilibrium we derived in Section 6 — however, against this strategy, there are responses yielding firm \(X\) a selection probability strictly larger than \(\frac{1}{2}\). Explicitly characterizing the equilibrium distribution appears difficult.

Another natural version is to require threshold tests, but allow multiple thresholds \(\theta_1 \leq \theta_2 \leq \cdots \leq \theta_k\). This naturally corresponds to the type of tests encountered in classes, where cutoffs are defined between multiple grades. Even for two thresholds, characterizing the equilibrium outcomes appears difficult — a firm with a difficult-to-attain ‘B’ grade may have to be ranked ahead of a firm with an easy-to-attain ‘A’ grade (similarly between easy ‘B’ and difficult ‘C’ . . . ). This is different from the pass-fail model, where every firm that passes a test is ranked ahead of every firm that fails a test, regardless of the tests’ difficulties.

A very interesting direction for future work is considering firms whose product qualities are drawn independently from different distributions. If one distribution stochastically dominates the other, it would be interesting to see if the weaker firm may at equilibrium follow “moon shot” strategies of taking very hard tests and hoping that this will allow it to win some of the time. Characterizing the equilibrium again appears to be quite challenging, because when both firms pass tests of the same difficulty, their posterior quality distributions will be different — as a result, the principal will not simply rank passing firms by their thresholds, and this results in a possibly infinite-dimensional system of differential equations characterizing the equilibrium distribution.

There are several open directions in terms of alternate objectives when extending the model to \(n > 2\) firms. When the principal’s goal is to obtain a complete ranking minimizing the Kendall tau distance, and the firms’ goal is to be ranked as highly as possible in expectation, we argued that our results carry over immediately; and for correlated tests, we explicitly characterized the optimum distribution. However, when the objectives are changed, this ceases to be true. A natural objective is for the principal to maximize the probability of selecting the best product, and for each firm to maximize the probability of being selected. Even for \(n = 3\) firms, it appears difficult to characterize the equilibria of the endogenous test selection game, or the principal’s optimal test distribution.

Instead of having the principal try to maximize the probability of selecting the better firm, an alternative objective would be for the principal to maximize the expected quality of the selected firm. While this is a natural objective, it requires the model to ascribe meaning to the concrete quality values, rather than using them only for comparison, in contrast to a viewpoint where utilities predominantly encode preferences. Nonetheless, the optimization and equilibrium questions would likely yield a rich set of questions.

Finally, we note a possibly interesting connection to a very different setting.\(^{17}\) One can interpret our setting as a principal trying to allocate an item to one of two agents \(X, Y\) via a price-discriminating posted-price mechanism. Different from standard such setups, the natural correspondence has a welfare-maximizing

\(^{17}\) We thank Nicole Immorlica for suggesting this interpretation.
(rather than \textit{revenue-maximizing}) principal. The mechanism corresponding to our testing setting then has the principal offer the two agents possibly different posted prices. If exactly one agent is interested in buying the item at his posted price, that agent is given the item at the posted price. If both agents are interested in buying at their respective prices, the agent with higher price obtains the item at his posted price. If neither agent is interested in buying, then again, the agent with higher price obtains the item, and pays 0. This model raises the issue of strategic manipulation: an agent might decline the item at his posted price, hoping that his price is higher and he will get the item for free. A natural question is whether the principal can price-discriminate in a way that will provide higher social welfare than offering both agents the same price (and choosing randomly which agent obtains the item if both accept/decline).

\textbf{Acknowledgement}

We would like to thank Odilon Camara, Peter Frazier, Moshe Hoffman, Nicole Immorlica, Jonathan Libgober, Erez Yoeli, and Christina Lee Yu for useful discussions and pointers.

SB gratefully acknowledges support from the NSF under grants CNS-1955997, DMS-1839346 and ECCS-1847393.

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