The pilgrim’s process

Walter Dempsey and Peter McCullagh*

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Abstract
Pilgrim’s monopoly is a probabilistic process giving rise to a non-negative sequence $T_1, T_2, \ldots$ that is infinitely exchangeable. The one-dimensional marginal distributions are exponential. The rules are simple, the process is easy to generate sequentially, and a simple expression is available for both the joint density and the multivariate survivor function. There is close connection with survival models, and in particular with the Kaplan-Meier estimate of the survival distribution. Embedded within the process is an ordered partition or partial ranking of the set $[n]$, which is exchangeable and consistent for different $n$. Some aspects of the process, such as the distribution of the number of blocks, can be investigated analytically and confirmed by simulation.

1 Pilgrim’s monopoly

Although not as simple as the Chinese restaurant process (Pitman, pp. 57–62, (2006)) or the Indian buffet process (Broderick, Jordan, Pitman, (2013)), the rules of pilgrim’s monopoly are straightforward and reminiscent of the board game even though the available real estate is an infinite straight line rather than a square loop. It is a process involving a sequence of pilgrims or travellers who pay toll fees and hotel taxes as they go, proceeding until their funds are depleted. The initial funds $X_1, X_2, \ldots$ for each pilgrim are distributed independently according to the exponential distribution with unit mean. Toll fees are paid continuously at the posted per-mile rate $\tau(s)$, which is reduced after each passing traveller, and taxes are levied by each hotel encountered on the route. If his funds are exhausted, the traveller establishes a new hotel at the point of exhaustion, and collects taxes from subsequent passers-by. If he arrives at a hotel where his remaining funds are insufficient to pay the tax, the funds are forfeit, and he remains at the hotel as a resident.

Initially, there are no hotels, the toll rate is uniform $1/\rho$ dollars per mile, so the first traveller setting out from the origin proceeds to the point $T_1 = \rho X_1$ where he establishes the first hotel. At that stage, the toll rate at point $s < T_1$
is reduced to $1/(1 + \rho)$, the rate at $s > T_1$ is unchanged, and the hotel tax is
the log ratio $\log((1 + \rho)/\rho)$ of the upstream toll rate to the downstream rate.
Upstream is the direction of travel.

After $n$ pilgrims have set out from the origin and reached their destinations,
$T_1, \ldots, T_n$, hotels have been established at points $0 < t_1 < t_2 < \cdots < t_k$. Hotel $r$
contains $d_r \geq 1$ pilgrims, so that $\sum d_r = n$. Let $R(s)$ be the number of travellers
who have proceeded beyond point $s > 0$, i.e., $R(s) = \#\{i \leq n: T_i > s\}$, so that
$n - R(s)$ is the number of travellers whose destination lies in $(0, s]$. The toll rate
$\tau(s) = 1/(\rho + R(s))$ is right-continuous and piecewise constant, increasing
from $1/(\rho + R(t_{r-1}))$ immediately before hotel $r$ to $1/(\rho + R(t_r))$ immediately
after. As always, the hotel tax is the log ratio $\log(\tau(s)/\tau(s^-))$ of upstream to
downstream toll rates. Thus, the sequence of tolls and taxes faced by pilgrim
$n + 1$ on the first three legs of his journey are as follows:

| Interval | Total toll | Hotel tax |
|----------|------------|-----------|
| $(0, t_1)$ | $\frac{t_1}{n + \rho}$ | $\log\left(\frac{n + \rho}{n + \rho - d_1}\right)$ |
| $(t_1, t_2)$ | $\frac{t_2 - t_1}{n + \rho - d_1}$ | $\log\left(\frac{n + \rho - d_1}{n + \rho - d_1 - d_2}\right)$ |
| $(t_2, t_3)$ | $\frac{t_3 - t_2}{R(t_2) + \rho}$ | $\log\left(\frac{R(t_2) + \rho}{R(t_3) + \rho}\right)$ |

where $R(t_2) = n - d_1 - d_2$. The destination $T_{n+1} > 0$ is the point of maximum
progress permitted by the funds available, i.e., the total tolls and taxes are such
that $\text{Tx}(0, T_{n+1}) \leq X_{n+1} \leq \text{Tx}(0, T_{n+1})$. If $X_{n+1} < t_1/(n + \rho)$, the pilgrim
establishes a new hotel at $T_{n+1} = (n + \rho)X_{n+1}$; otherwise he pays the full toll
for the segment $(0, t_1)$. If his remaining funds are sufficient to cover the hotel
tax at $t_1$, he does so, and his pilgrimage continues in the same way until his
funds are exhausted. Otherwise, his destination is $T_2 = t_1$.

2 Illustration by simulation

The pilgrim’s process takes an input sequence $X_1, X_2, \ldots$, and produces an
output sequence $T_1, T_2, \ldots$ such that $T[n]$ is a deterministic function of $X[n]$,
though not an invertible one. The following is an example of a 10-component
pair $X, T$ generated with $\rho = 1$:

$X = (0.36, 0.25, 0.36, 2.24, 0.40, 0.03, 1.17, 1.68, 3.31, 1.24, 0.35, 0.50, \ldots)$
$T_\rho(X) = (0.36, 0.36, 0.36, 1.12, 0.36, 0.18, 0.36, 0.85, 1.89, 0.85, 0.36, 0.36, \ldots)$,

where $X$ is exact and $T$ is given to two decimal places. In other words, the
transformation $X \mapsto T$ is non-stochastic, the only random element being the
stochastic nature of the inputs.

The process is straightforward to simulate, but a simple algorithm with minimal
book-keeping is likely to be considerably less efficient than an algorithm
that keeps track of hotel positions and occupancy numbers, updating them as needed. For greater generality, one may include a second parameter $\nu > 0$ such that the toll rate is $\tau(s) = \nu / (\rho + R(s))$, which does not affect the taxes. Although not entirely obvious, the transformation $X \mapsto T_{\nu, \rho}(X)$ is such that $\nu T_{\nu, \rho}(X) = T_{1, \rho}(X)$, so the effect is merely multiplicative. A reasonable argument may be made for taking $\nu = \rho$ as the default. However, unless otherwise stated, we take $\nu = 1$.

The left panels of Figure 1 show four realizations of the process with the same input sequence, $n = 250$, and parameters $\rho = 1, 4, 12, 24$ respectively. The right panels show the conditional survival distribution (solid line) for which

$$-\log \operatorname{pr}(T_{m+1} > t \mid T[m]) = T_{\text{tx}}((0, t])$$

is the sum of the posted tolls and taxes to be levied on one pilgrim, $m = 51$, travelling from the origin to $t + \epsilon$. 

Figure 1: Simulated pilgrim process for $\rho = 1, 4, 12, 24$. The left panels show the sequence $\rho T_i$ for 250 pilgrims; the right panels show the conditional survival distribution $\operatorname{pr}(T_{51} > t \mid T[50])$ (solid line) together with the Kaplan-Meier curve (dotted line), both based on the first 50 values.
Considering only the taxes, we find that

\[ S_n(t, \rho) = e^{-Tx((0,t])} = \prod_{i.t_i \leq t} \frac{\rho + R(t_i)}{\rho + R(t_i) + d_i} \]

\[ S_n(t, 0) = \prod_{i.t_i \leq t} \frac{R(t_i)}{R(t_i) + d_i} \]

so that the limit as \( \rho \to 0 \) is the Kaplan-Meier survival curve. In other words, \(-\log S_n(t, 0)\) is the limit as \( \rho \to 0 \) of the sum of the hotel taxes posted for pilgrim \( n+1 \) in \( (0, t] \). This is also shown in Fig. 1 for comparison, using only the first 50 values. The sample size \( m = 50 \) was kept sufficiently small to make the differences visible. The Kaplan-Meier distribution has mass only at the hotels, and the limiting tax at the final hotel is infinite. (In the presence of censoring, the total mass or tax may be finite, i.e., the Kaplan-Meier distribution may have positive mass at infinity, and the Kaplan-Meier curve then differs from the limiting conditional survival distribution only in the final interval following the last hotel.)

Exchangeability is apparent in the sense that any fixed permutation of the sequence would look much the same. It is also apparent that the process for small \( \rho \) is much more grainy than the process for large \( \rho \). The number of distinct values, i.e., the number of hotels, in the four simulations is 15, 32, 74, and 100 respectively.

Simulation allows us to keep track not only of hotel development and hotel occupancy, but also the collection of tolls and taxes. The \( r \)th pilgrim travelling through the sector \((t, t+dt)\) contributes \( dt/(r-1+\rho) \) in tolls, so the total toll collected in this sector is

\[ Z(dt) = \zeta(R(t)) dt \]

where

\[ \zeta(n) = \sum_{j=0}^{n-1} \frac{1}{\rho + j} \]

and \( \zeta(0) = 0 \). The total collected in tolls from all pilgrims passing through the inter-hotel zone \((t_{i-1}, t_i)\) is \( \zeta(R(t_{i-1}))(t_i - t_{i-1}) \), and the sum of these tolls is

\[ Z = \int_0^\infty \zeta(R(t)) dt \]

The remainder \( X - Z \) is distributed in taxes and forfeits to the various hotels. In other words, the process generates not only a partition of \([n]\) into \( k \) blocks which are ordered in space, but also a partition of \( X \) by tolls and taxes into \( 2k \) parts that are also linearly ordered along the route.

It is of some interest to examine the relationship between the number of hotels, \( k \), and the total paid in tolls. Both are increasing in \( n \). Figure 2 suggests that they are essentially proportional for large \( n \), with limiting ratio one independent of \( \rho \). Specifically, the limiting ratio is equal to the parameter \( \nu \).

Figure 3 is a plot of \( k^{1/2} \) against \( \log(n) \), i.e., the square root of the number of hotels against the log of the number of pilgrims, one panel for each parameter value \( \rho = 1, 4, 12, 24 \). Also plotted is the square root of the accumulated tolls against \( \log(n) \). The simulation is in agreement with the claim that \( k \propto \log^2(n) \), at least for sufficiently large \( n \).

Simulations also allow us to monitor the accumulated wealth of hotel proprietors, either in temporal order of establishment or in spatial order relative to the origin. The wealth of a hotel is the sum of the forfeits of its occupants plus the taxes paid by passers-by. Figure 4 shows the total tax collected by each
Figure 2: Cumulative tolls plotted against the number of hotels established. Two replicates are shown per panel. The parameter values are $\rho = 1, 4, 12, 24$ for the four pairs.

Proprietor in temporal order of establishment (left) and spatial order (right). In each case, there is an upper boundary linearly decreasing in the index. The distribution of total tax collected is approximately uniform under temporal ordering from zero to the upper boundary. Figure 5 shows the total tax when wealth is distributed among occupants. In this case, the maximal wealth per occupant appears to grow initially and then decrease, reaching its peak at the median index independent of $\rho$.

Finally, we investigate the size of hotels in both spatial and temporal order. Figure 6 is a plot of the total number of residents in each hotel on a log-scale for various choices of $\rho$ under both orderings. As $\rho$ tends to infinity, the number of residents in each hotel tends to one. At the other extreme, as $\rho$ tends to zero, all individuals occupy the same hotel. For intermediate values, there tends to be few large hotels with the remainder of moderate size. The number of hotels of size 1 is approximately Poisson with rate proportional to $\log(n)$. Under the spatial order relative to the origin, the larger hotels tend to exist closer to the origin. Under temporal ordering, the larger hotels occur earlier; however, the bunching around early indices is less pronounced under this ordering. The number of hotels is independent of the parameter $\nu$. 

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3 Technical exercises

The pilgrim’s process is a mathematical exercise that can be studied at various levels and in various ways. The questions of interest are mostly related to the behaviour of the sequence $T_1, \ldots, T_n$ both for finite $n$ and the limit as $n \to \infty$. Some very natural questions are related to the number and the sizes of the blocks, i.e., the hotel occupancy numbers $(d_1, \ldots, d_k)$, possibly, but not necessarily, ignoring order. Other questions are concerned with expenditures, particularly the expenditure on taxes and forfeits versus the expenditure on tolls. The total expenditure on tolls, $Z = \int_0^\infty \zeta(R(s)) \, ds$, is a function of $T_1, \ldots, T_n$, but the remainder is not. As a function of $n$, it appears that $Z$ is closely related to the number of blocks.

The goal of this paper is not to state theorems or to provide proofs, but to ask questions. Some have straightforward technical answers, while others may be investigated numerically, either by simulation or by developing non-stochastic recursive relationships. Others can be answered by exact or asymptotic analysis. No indication is provided on the type of each question, but the first few are straightforward.

Q. 1 Show that if $T_1 = t_1$, the taxes to be paid on the segment $(0, t_2]$ are $t_2/(1 + \rho)$ if $t_2 < t_1$ and

$$\frac{t_1}{1 + \rho} + \log\left(\frac{1 + \rho}{\rho}\right) + \frac{t_2 - t_1}{\rho}$$
Figure 4: Total taxes per hotel proprietor ordered temporally (left) and spatially (right). The parameter values are $\rho = 1, 4, 12, 24$ for the four panels with $n = 1000$.

otherwise. Hence show that the joint density function of $(T_1, T_2)$ is

$$
pr(T_2 \in dt_1 \times dt_2) = \begin{cases}
\frac{dt_1 dt_2 e^{-t_1/\rho-t_2/(1+\rho)}}{\rho(1+\rho)} & t_2 < t_1 \\
\frac{dt_1 e^{-t_1/\rho-t_2/(1+\rho)}}{\rho(1+\rho)} & t_2 = t_1 \\
\frac{dt_1 dt_2 e^{-t_2/\rho-t_1/(1+\rho)}}{\rho(1+\rho)} & t_2 > t_1
\end{cases}
$$

Q. 2 Compute the probability $pr(T_1 = T_2)$ as a function of $\rho$. What about $pr(T_2 = T_1)$?

Q. 3 Suppose $n = 3$. Given that the number of distinct values is $k = 2$, find the probability that $d_1 = 1$. In other words, given that there is one duplicate value in $T[3]$, what is the probability that the smaller value is the singleton?

Q. 4 The density function in question 1 may be written in the form

$$
\rho^n f_n(dt) = \exp\left(-\int_0^\infty \zeta(R_t(s)) ds\right) \prod_{r=1}^k \Gamma(d_r)
$$

where $R_t(s) = \#\{i \leq n; t_i > s\}$ for $t \in \mathbb{R}^n$, and the last factor is automatically one for $n = 2$. Explain what it means for the density $f_2$ to integrate to one.
What about $f_3$? Use the density function to compute the probability $\Pr(T_1 = T_2 = T_3)$.

Q. 5 Show that the conditional distribution of $T_{n+1}$ given $T[n] = (T_1, \ldots, T_n)$ has discrete atoms at each of the hotels $t_1, \ldots, t_k$, plus a remainder that is continuous on $(0, \infty)$. Show that the conditional distribution is such that

$$-\log \Pr(T_{n+1} > t \mid T[n]) = \text{tolls+taxes payable in } (0, t].$$

Q. 6 Deduce by induction that the joint density of $T[n]$ is $f_n(t)$.

Q. 7 Show that the conditional distribution given $T[n]$ is such that $T_{n+1} \sim \min(X, X')$ where $X, X'$ are conditionally independent, $X > 0$ is distributed continuously as a function of the tolls, and $X'$ is distributed on $\{t_1, \ldots, t_k, \infty\}$ as a function of taxes.

Q. 8 Show that as hotels merge, the total taxes collected is the same as the sum of the taxes collected at each hotel individually. Deduce that the conditional distribution is a continuous function of the initial configuration. That is, if $A$ is an open interval,

$$\lim_{\epsilon \to 0} \Pr(T_{n+1} \in A \mid T[n] = t + \epsilon) = \Pr(T_{n+1} \in A \mid T[n] = t)$$

where $\epsilon$ is a perturbation of the initial configuration $t = (t_1, \ldots, t_n)$.

Q. 9 Show that the $n$-dimensional survivor function is

$$S(t) = \Pr(T_1 > t_1, \ldots, T_n > t_n) = \exp\left(-\int_0^\infty \zeta(R_t(s)) \, ds\right)$$

where $\zeta(n) = \sum_{j=0}^{n-1} 1/(\rho + j)$ is the harmonic number function.

Q. 10 Deduce that the process is infinitely exchangeable, i.e., for each $n$ and each permutation $\sigma: [n] \to [n]$ that $(T_1, \ldots, T_n)$ has the same distribution as the permuted sequence $(T_{\sigma(1)}, \ldots, T_{\sigma(n)})$.

Q. 11 The number of blocks, i.e., the number of distinct components in $T_1, \ldots, T_n$, is a random non-decreasing function of $n$. Show that the mean number of blocks satisfies the recursion

$$\mu_n = 1 + \frac{1}{\zeta(n)} \sum_{d=1}^n \frac{\mu_{n-d}}{d + \rho - 1}. \tag{1}$$

Compute the first few hundred values for $\rho = 1$ and plot against $n$.

Q. 12 Deduce that $\mu_n \propto \log^2(n)$.

Q. 13 Show that the total expenditure on tolls by the first $n$ pilgrims is $Z = \int_0^\infty \zeta(R(s)) \, ds$, so that $X - Z$ is the tax levied by hotels. Examine by simulation how the relation between tolls and taxes varies with the number of pilgrims.

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Q. 14 Investigate by simulation the relation between the number of blocks, \( k(T[n]) \), and the total expenditure on tolls. Show that the mean value of the total expenditure on tolls, \( E_n[Z] \), also satisfies equation (1). Deduce that \( Z \to \mu_n \) as \( n \) tends to infinity, and therefore the ratio \( Z/k \) tends to one for each realization of the process.

Q. 15 Consider the two-parameter model indexed by \( (\nu, \rho) \) as described in section 2. For fixed \( \rho \), show that the one-parameter model is of exponential-family form with a two-dimensional sufficient statistic. Find the maximum-likelihood estimate of \( \nu \), and show that it is consistent as \( n \to \infty \).

4 Connections with the literature

The output of the pilgrim’s process is an infinitely exchangeable sequence, which automatically has a de Finetti representation as a conditionally independent and identically distributed sequence. It is in fact one of a large class of similar survival processes investigated in detail by Hjort (1990), in which \( \Lambda \) is a stationary, completely independent, measure on \( \mathbb{R} \), and \( T_1, \ldots \) are conditionally independent given \( \Lambda \), with distribution

\[
\Pr(T_i > t \mid \Lambda) = e^{-\Lambda((0,t])}.
\]

This means that \( \Lambda \) is a Lévy process with infinitely divisible increments, which are necessarily stationary if the marginal distributions are to be exponential. Processes of this type have been investigated by Doksum (1974), Ferguson and Phadia (1979), Kalbfleisch (1978), Clayton (1991) and James (2006). They are sometimes called neutral to the right. In general, the increments need not be stationary, there may be individual-specific weights \( w_i \) such that \( \Pr(T_i > t \mid \Lambda) = e^{-w_i\Lambda((0,t])} \), and the observations may be subject to censoring. It is important that these matters be accommodated in applied work, and the processes in this class have the virtue that inhomogeneities and right-censoring are easily accommodated. We have chosen not to emphasize this aspect of things because it tends to obscure the underlying process.

Each process in the class is associated with an infinitely divisible distribution \( X = \Lambda((0,1)) \) which has a characteristic exponent \( \zeta(t) = -\log(E(e^{-tX})) \) for \( t \geq 0 \), and an associated Lévy measure. This means that the multivariate survival function is

\[
\Pr(T_1 > t_1, \ldots, T_n > t_n) = E(e^{-\Lambda(0,t_1] - \cdots - \Lambda(0,t_n]}) = \exp\left(-\int_0^\infty \zeta(R_t(s)) \, ds\right).
\]

For example, the characteristic exponent and the Lévy measure for the gamma process considered by Kalbfleisch (1978) and Clayton (1991) are

\[
\zeta^*(t) = \log(1 + t/\rho) \quad \text{and} \quad w^*(dz) = z^{-1}e^{-\rho z} \, dz.
\]

The characteristic exponent and the Lévy measure for the pilgrim process are

\[
\zeta(t) = \psi(\rho + t) - \psi(\rho) \quad \text{and} \quad w(dz) = e^{-\rho z} \, dz/(1 - e^{-z}),
\]
where $\psi$ is the derivative of the log gamma function. These two pairs $\zeta, \zeta^*$ and $w, w^*$ are very alike and they produce very similar processes, but the limits as $\rho \to 0$ are not exactly the same. The obvious attraction of the pilgrim process is that it has joint and conditional distributions that are simple to evaluate, which does not, in itself, make it more suited for applied work than any of the other processes in the class. Other processes in the same class include $\zeta(t) = t^\alpha$ for $0 < \alpha \leq 1$, which is quite different because it is not analytic at the origin.

In general, the joint density of $T[n]$ at $t \in \mathbb{R}^n$ may be written in terms of the characteristic exponent

$$
\exp\left(-\int_0^\infty \zeta(R_t(s)) \, ds\right) \times \prod_{r=1}^k (-1)^{d_r-1}(\Delta^d r \zeta)(R_t(t_r)),
$$

(2)

where $0 < t_1 < \cdots < t_k$ are the distinct components of $t$, $d_1, \ldots, d_k$ are their multiplicities, and $\Delta^d r \zeta$ is the $d$th order forward difference

$$(\Delta^d r \zeta)(x) = \sum_{j=0}^d (-1)^{d-j} \binom{d}{j} \zeta(x+j).$$

The conditional density of $T_{n+1}$ given $T[n] = t$ is most easily described in terms of the conditional hazard measure, which has an atom or tax

$$
- \log \left( \frac{(\Delta^d r \zeta)(R_t(t_r) + 1)}{(\Delta^d r \zeta)(R_t(t_r))} \right)
$$

at each distinct component of $t$, plus a continuous component with density $(\Delta \zeta)(R_t(s))$ at $s > 0$. A detailed description is provided by Dempsey and McCullagh (2014). In general, the hotel tax is not equal to the log ratio of upstream to downstream toll rates.

Expression (2) is also correct under right censoring provided that the product is restricted to failures. It is also correct under the inhomogeneous model with weights $w_i$, provided that $\zeta(R(t))$ is replaced with $\zeta(w(R(t)))$, and the forward differences at $t_r$ are re-defined in the obvious way as an alternating sum over subsets of the $d_r$ residents at $t_r$. It is also correct under monotone temporal transformation provided that the integral with respect to Lebesgue measure is transformed to an integral with respect to another measure $\nu$, and the Jacobian is included. These modifications are sufficient to cover the great majority of the non-exchangeable or non-homogeneous processes in this class.

5 Conclusion

In this paper, we presented the pilgrim’s process, a probabilistic process in which initial funds $X_1, X_2, \ldots$ for each pilgrim gives rise to a non-negative sequence $T_1, T_2, \ldots$ that is infinitely exchangeable. Simple expressions for a sequential description of the process, the joint density, and the multivariate survivor function
are provided as well as a connection to the Kaplan-Meier estimate. Simulation allows investigation of the process’s properties; for example, we find that the number of hotels grows in tandem with the total expenditure on tolls, both growing at a rate proportional to \( \log^2(n) \). We end by connecting the pilgrim’s process with the class of survival processes investigated by Hjort (1990), seeing that the pilgrim process is closely approximated by the gamma process originally considered by Kalbfleisch (1978) and Clayton (1991).

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Figure 5: Wealth distributed among occupants versus temporal (left) and spatial (right) ordering. The parameter values are $\rho = 1, 4, 12, 24$ for the four panels for $n = 1000$.

Figure 6: Log number of occupants per hotel versus temporal (left) and spatial (right) ordering. The parameter values are $\rho = 1, 4, 12, 24$ for the four panels for $n = 1000$. 