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PETRAS’s THEORY OF A SPIN-1/2 PARTICLE IN ELECTROMAGNETIC AND GRAVITATIONAL FIELDS

20-component Petras’ theory of 1/2-spin particle with anomalous magnetic momentum in presence of external electromagnetic and gravitational fields is investigated. The gravitation field is described as space-time curvature. Correctness of the constructed equations in the sense of general relativity and gauge local Lorentz group symmetry is proved in detail. Tetrad $P$-symmetry of the equations is demonstrated. A generally covariant representation of the invariant bilinear form matrix is established and the conserved current of the 20-component field is constructed. It is shown that after exclusion of the additional vector-bispinor $\Psi_\beta(x)$ the wave equation for the principal $\Psi$-bispinor looks as generally covariant Dirac’s equation with electromagnetic minimal and Pauli interactions and with an additional gravitational interaction through scalar curvature $R(x)$-term. The massless case is analyzed in detail. The conformal non-invariance of the massless equation is demonstrated and new conformally invariant equations for 20-component field are proposed.

33 pages, references – 80

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Introduction

In order to have a comprehensive theory of higher-spin fields, it is necessary to be able to describe interactions. The most important and the best understood is, of course, the electromagnetic. But the gravity is very important as well, at least from theoretical viewpoint if not from practical. Generally, a standpoint may be brought forward that we should give much attention to those facts of physics in the Minkowski space-time which allow for extension to a generally covariant physics, because only doing so we will be able to reconcile the principles of particle physics with those of general relativity, and thus will be able to construct a deeper theory.

Else one point should be taken as of principal importance. Since wave equations with subsidiary conditions usually lead to consistency difficulties when minimally coupled to an electromagnetic field, it seems best to avoid subsidiary conditions from the start. Furthermore, it seems the best to start with a first-order system, which is linked up to a certain Lagrange function, and in which some physically required restrictive conditions are built in from the very beginning in accordance with the Pauli-Fierz \[1,2\] approach. The more so is in the context of possible background of a non-Euclidean geometry, in view of arising additional subtleties in a consistency problem.

It is well known that if one looks for a first-order differential equation describing a mass \(m\) spin \(s\) field, that is form-invariant under Lorentz transformations, derivable from a Lagrangian, then one does not uniquely obtain the common Dirac or Duffin-Kemmer or some other equations. We have taken the existence of such generalized particles for granted, relying for justification on our experience with Dirac and electromagnetic fields. It does not seem to us to appeal always to experiment; after all, purpose in theoretical physics is not to describe the world as we find it, but to explain – in terms of a few fundamental principles – why the world is just the way it is.

A general theory of such first-order and Lagrangian-based equations has been treated at great length by Fierz and Pauli \[1,2\], Dirac \[3\], Bhabha \[4\], Harish-Chandra \[5\] and many others \[6-12\]. In fact, it was shown that almost an infinite many of such equations is possible. However, quite recently there had been a notable dearth of examples of such theories that had been developed at a large extent comparable to the Dirac or Duffin-Kemmer examples\(^2\). But such particle models, being elaborated in full detail, might shed new light on the theory of general arbitrary-spin wave equations.

Much work in this direction has been done \[13-39\]. For instance, generalized wave equations for particles of spin 1/2 (Petras, Halhil, Capri, Fedorov, Pletjuchov, Bogush, Kisel \[15,17,21,22,24,28,30,34,38,39\] with arbitrary anomalous magnetic momentum have been worked out. Else one extension, less known, has been done for a boson particle: a next simplest theory beyond the Duffin-Kemmer case (of spin 0 and 1) was created by Fedorov and Pletjuchov \[25,26\] and an explicit representation of the basic first-order equation was given.

In this paper we examine the properties of the 20-component spin 1/2 theory in the case of no interaction as well as in presence of external electromagnetic and gravitational

\(^2\)In recent years some interest in the Duffin-Kemmer-Petiau formalism again can be noted – see \[40-45\].
fields. Why another paper on this matter. The reader can choose from a number of them. Another work will be worth while only if it offers something new in content and perspective. It is known that in Minkowski space-time this theory is very simply related to the Dirac equation. In the free-field case, the two theory are equivalent, and the more complicated 20-component model can be reduced to the ordinary 4-component one. In the presence of external electromagnetic fields, one obtains a Dirac-Pauli theory with an additional term corresponding to an anomalous magnetic momentum\(^3\). So, starting from the principles about which we are most certain, relativity, quantum mechanics, and the first order equation theory, an additional interaction term, introduced in quite phenomenological way, emerges anew from prime principles as a natural consequence. It turns out that consistent treatment of the 20-component fermion theory lead us to a Dirac-Pauli fermion which interacts with gravitational field in a manner different to an ordinary Dirac’s particle: through an additional Ricci scalar term. So, the rationale for additional Pauli term might be, reasoning in somewhat speculate manner, found in consistent description of additional gravitational interaction of the fermion through a Ricci scalar.

The present article aims to provide a self-contained, comprehensive, an up-to-date exposition of the matter. So, the development is detailed and logical throughout, with each step carefully motivated by what has gone before, and emphasizing the reason why such a step should be done. As to content, although a paper contains a good amount of new material, the taking into account of the possible presence of gravitational background is its most distinctive thing. The more is so, because the formalism developed may potentially find applications far removed from its original scene of the use. This in part explains that frequently we immersed ourselves in technicalities of specific problems; and we took a firm hold on this line. So we have tried to err on the side of inclusion rather than exclusion\(^4\). We hope that we have improved on the original literature in several places, as for instance, in Lorentz group-based techniques used in general covariant description of a fermion field; analysis of the energy-momentum tensor of the generalized fermion in a curved space-time; in deriving a generally covariant Bargmann-Mishel-Telegdi equation for a fermion with anomalous magnetic moment (this part of the work will be done later and separately); in separating the variables in generally covariant wave equations, and so on. We have tried give citations on topics that are mentioned here. But we did not always know who was responsible for material presented here, and the mere absence of a citation should not be taken as a claim that material presented here is original. But hoping some of it is.

\(^3\)This else one time illustrates explicitly a point emphasized by Wightman that although two field theories may be equivalent in absence of interaction, they may be completely inequivalent in presence of interaction.

\(^4\)We hope this work will suit those who are ready to look with interest at old and conventional methods of work with minimum of modern and rigorous mathematics, in our opinion much of those should be revived.
1. Petras equation in flat space-time

Original 20-component equation for a spin 1/2 particle is taken in the form

\[
\gamma^a \partial_a \Psi + \mu (Kg^{ab} + A\sigma^{ab}) \partial_b \Psi = M \Psi , 
\]

(1.1a)

\[
\mu (N\delta^c_b + B\sigma^c_b) \partial_c \Psi = M \Psi_b . 
\]

(1.1b)

Here, a full wave function includes bispinor \( \Psi(x) \) and vector-bispinor \( \Psi_a(x) \); \( \gamma^a \) denotes Dirac’s four-by-four matrices; \( \sigma^{ab} = \frac{1}{4}(\gamma^a\gamma^b - \gamma^b\gamma^a) \); \( mc/\hbar = iM \); \( \mu \) stands for an additional dimensionless characteristic of the fermion under consideration; \( K, N, A, B \) are certain real numbers.

Rewriting eq. (1.1b) as

\[
\Psi_b(x) = \frac{\mu}{M} (N\delta^c_b + B\sigma^c_b) \partial_c \Psi ,
\]

(1.2)

one can exclude the vector-bispinor from eq. (1.1a):

\[
[ \gamma^a \partial_a + \mu^2 M^{-1} (Kg^{ab} + A\sigma^{ab}) (N\delta^c_b + B\sigma^c_b) \partial_a \partial_c - M ] \Psi = 0 .
\]

(1.3)

Introducing the notation

\[
Z^{ac} = (Kg^{ab} + A\sigma^{ab}) (N\delta^c_b + B\sigma^c_b) ,
\]

eq. 1.3) reads as

\[
[ \gamma^a \partial_a + \frac{\mu^2}{M} Z^{ac} \partial_a \partial_c - M ] \Psi(x) = 0 .
\]

(1.4)

Let us consider more closely the expression for \( Z^{ac} \)

\[
Z^{ac} = KN g^{ac} + (KB + NA)\sigma^{ac} + AB \sigma^{ab}\sigma^c_b ,
\]

on allowing for equalities

\[
\sigma^{ab}\sigma^c_b = \left( \frac{1}{2}\gamma^a\gamma^c + \frac{1}{4}g^{ac} \right), \quad g^{ac} = \frac{1}{2}(\gamma^a\gamma^b + \gamma^b\gamma^c) ,
\]

the \( Z^{ac} \) takes on the form

\[
Z^{ac} = (KB + NA)\sigma^{ac} + \gamma^a\gamma^c \left( \frac{1}{2}KN + \frac{1}{2}AB + \frac{1}{8}AB \right) + \gamma^c\gamma^a \left( \frac{1}{2}KN + \frac{1}{8}AB \right) .
\]

Numerical parameters are to be chosen that the \( Z^{ac} \) coincides, apart from a numerical factor, with the skew combination \( \sigma^{ac} \). This will be so if the following requirements hold

\[
\left( \frac{1}{2}KN + \frac{5}{2}AB \right) = +C , \quad \left( \frac{1}{2}KN + \frac{1}{8}AB \right) = -C ,
\]

(1.5a)

5 Working in the Minkowski space-time, the metric tensor \( g^{ab} = \text{diag}(+1, -1, -1, -1) \) is used throughout. Latin letters take the values 0, 1, 2, 3, Greek letters are reserved for generally covariant indices.
then

\[ Z^{ac} = \left( KB + NA \right) + 4C \] \[ \sigma^{ac} . \quad (1.5b) \]

From eq. (1.5a) it follows

\[ 4KN + 3AB = 0 , \quad C = -\frac{KN}{3} . \quad (1.5c) \]

Petras [15] choose a solution in the form

\[ K = \sqrt{3} , \quad N = \sqrt{3} , \quad A = +2 , \quad B = -2 , \quad Z^{ac} = -4\sigma^{ac} . \quad (1.6) \]

Correspondingly, eqs. (1.1) will read as

\[ \gamma^a \partial_a \Psi + \mu \left( \sqrt{3} g^{ab} + 2\sigma^{ab} \right) \partial_b \Psi_b = M \Psi , \quad (1.7a) \]

\[ \mu \left( \sqrt{3} \delta^c_b - 2\sigma^c_b \right) \partial_c \Psi = M \Psi_b . \quad (1.7b) \]

These are equivalent to the following

\[ \Psi_b = \frac{\mu}{M} \left( \sqrt{3} \delta^c_b - 2\sigma^c_b \right) \partial_c \Psi , \quad (1.8a) \]

\[ \left[ \gamma^a \partial_a - \frac{4\mu^2}{M} \sigma^{ac} \partial_a \partial_c - M \right] \Psi = 0 . \quad (1.8b) \]

In presence of an external electromagnetic field one is to lengthen derivatives (the combination \( \frac{e}{hc} \) will be denoted as \( g \))

\[ \partial_a \rightarrow D_a = \partial_a - i\frac{e}{hc} A_a . \]

at this eqs. (1.7) transform into

\[ \gamma^a D_a \Psi + \mu \left( \sqrt{3} g^{ab} + 2\sigma^{ab} \right) D_a \Psi_b = M \Psi , \quad (1.9a) \]

\[ \mu \left( \sqrt{3} \delta^c_b - 2\sigma^c_b \right) D_c \Psi = M \Psi_b . \quad (1.9b) \]

Obviously, eqs. (1.9) are equivalent to

\[ \Psi_b = \frac{\mu}{M} \left( \sqrt{3} \delta^c_b - 2\sigma^c_b \right) D_c \Psi , \quad (1.10a) \]

\[ \left[ \gamma^a D_a - \frac{4\mu^2}{M} \sigma^{ac} D_a D_c - M \right] \Psi = 0 . \quad (1.10b) \]

Since the lengthened derivatives do not commute mutually, a relation \[ 4\sigma^{ac} D_a D_c = -2ig\sigma^{ac} F_{ac} , \]
holds, where \( F_{ac} = \partial_a A_b - \partial_b A_a \) stands for the electromagnetic tensor. Correspondingly, eq. (1.10b) will take on the Dirac-Pauli form

\[ \left( \gamma^a D_a + i\frac{2g\mu^2}{M} \sigma^{ac} F_{ac} - M \right) \Psi(x) = 0 , \quad (1.11) \]
with an additional interaction term generated by an anomalous magnetic momentum. In dimension unites eq. (1.11) reads

\[
(i\gamma^a D_a - \frac{2e\mu^2}{mc^2} \sigma^{ac} F_{ac} - \frac{mc}{\hbar}) \Psi(x) = 0 .
\]  

(1.12)

Let us look more closely at the matters of dimensions in eq. 91.12). As (notation [...]) stands for 'dimension of (...)')

\[
\left[\frac{e}{\hbar c} A\right] = \frac{1}{l} \implies \left[\frac{e}{\hbar c} F\right] = \frac{1}{l^2} \implies \left[\frac{eF}{mc^2}\right] = \frac{1}{l},
\]

the quantity \(\mu\) does carry no physical dimension.

2. Extension to general relativity

Now we are to proceed to the study the main question: given a first order equation that is form invariant under special relativity, what will it look like on the background of general relativity.

A first step, quite evident one in view of the known absolute necessity to use the vierbein formalism of Tetrode-Weyl-Fock-Ivanenko [46-80] at describing spinor fields on a curved space-time background, is to introduce a new set of field functions:

\[
\Psi(x), \Psi_b(x) \implies \Psi(x), \Psi_\beta(x).
\]

At this, \(\Psi(s)\) transforms as scalar under general coordinate changes, and as bispinor under local Lorentz changes of local tetrads\(^6\); in turn, \(\Psi_\beta(x)\) behaves as a general covariant vector and as a local tetrad bispinor at the same time.

A second step is the postulating of a general covariant equation:

\[
\gamma^\alpha D_\alpha \Psi + \mu \left( \sqrt{3} g^{\alpha\beta} + 2 \sigma^{\alpha\beta} \right) D_\alpha \Psi_\beta = M \Psi ,
\]

(2.1)

\[
\mu \left( \sqrt{3} \delta^\rho_\beta - 2 \sigma^\rho_\beta \right) D_\rho \Psi = M \Psi_\beta .
\]

(2.2)

Here, \(g^{\alpha\beta}(x)\) designates the metric tensor of a curved space-time background; its vierbein make-up is as usual

\[
g^{\alpha\beta}(x) = e^a_{(a)}(x)e^\beta_{(b)}(x) g^{ab},
\]

\(e^a_{(a)}(x)\) stands for a tetrad. Generalized Dirac matrices are defined the relations

\[
\gamma^\alpha(x) = \gamma^a e^a_{(a)},
\]

\[
\sigma^{\alpha\beta}(x) = \sigma^{ab} e^a_{(a)} e^\beta_{(b)} = \frac{1}{4} \left( \gamma^\alpha \gamma^\beta - \gamma^\beta \gamma^\alpha \right).
\]

\(^6\)Terms *tetrad* and *vierbein* will be used interchangeably.
The symbol $D_\alpha$ stands for an extended covariant derivative (acting differently on bispinor and vector-bispinor fields)

$$D_\alpha = \nabla_\alpha + B_\alpha - igA_\alpha(x).$$

As usual, $\nabla_\alpha$ denotes the general covariant derivative; $B_\alpha$ is called a bispinor connection

$$B_\alpha = \frac{1}{2} \sigma^{ab} e_{(a)}^\beta \nabla_\alpha e_{(b)\alpha} = \gamma^\alpha \nabla_\beta \gamma_\alpha;$$

$A_\alpha$ represents an electromagnetic 4-vector.

To investigate symmetry properties of the equations (2.1) and (2.2) it will be convenient to exploit the known spinor (Weyl’s) basis in the bispinor space defined by

$$\gamma^\alpha = \left( \begin{array}{cc} 0 & \bar{\sigma}^\alpha \\ \sigma^\alpha & 0 \end{array} \right), \quad \sigma^a = (I, +\sigma^k), \quad \bar{\sigma}^a = (I, -\sigma^k),$$

$$\Psi = \left( \begin{array}{c} \xi \\ \eta \end{array} \right), \quad \Psi_\beta = \left( \begin{array}{c} \xi_\beta \\ \eta_\beta \end{array} \right),$$

$$\xi = \left( \begin{array}{c} \xi_1 \\ \xi_2 \end{array} \right), \quad \eta = \left( \begin{array}{c} \eta_1 \\ \eta_2 \end{array} \right), \quad \xi_\beta = \left( \begin{array}{c} \xi_1^\beta \\ \xi_2^\beta \end{array} \right), \quad \eta_\beta = \left( \begin{array}{c} \eta_1^\beta \\ \eta_2^\beta \end{array} \right), \quad (2.3a)$$

where ($\sigma^k$ stand for Pauli two-by-two Pauli matrices, $k = 1, 2, 3$). Introducing the notation

$$\sigma^\alpha = \sigma^a e_{(a)}^\alpha, \quad \bar{\sigma}^\alpha = \bar{\sigma}^a e_{(a)}^\alpha,$$

$$B_\alpha = \left( \begin{array}{cc} \Sigma_\alpha & 0 \\ 0 & \bar{\Sigma}_\alpha \end{array} \right)$$

$$\Sigma_\alpha = \frac{1}{2} \Sigma^{ab} e_{(a)}^\beta \nabla_\alpha (e_{(b)\beta}), \quad \bar{\Sigma}_\alpha = \frac{1}{2} \Sigma^{ab} e_{(a)}^\beta \nabla_\alpha (e_{(b)\beta}),$$

$$\Sigma^{ab} = \frac{1}{4} (\sigma^a \sigma^b - \bar{\sigma}^a \bar{\sigma}^b), \quad \bar{\Sigma}^{ab} = \frac{1}{4} (\sigma^a \bar{\sigma}^b - \sigma^b \bar{\sigma}^a), \quad (2.3b)$$

instead of eqs. (2.1) one will obtains four equations\(^8\):

$$\bar{\sigma}^\alpha (\nabla_\alpha + \bar{\Sigma}_\alpha - igA_\alpha)\eta +$$

$$+ \mu(\sqrt{3}g^{\alpha\beta} + 2\Sigma^{\alpha\beta})(\nabla_\alpha + \Sigma_\alpha - igA_\alpha)\xi_\beta = M\xi, \quad (2.4a)$$

$$\sigma^\alpha (\nabla_\alpha + \Sigma_\alpha - igA_\alpha)\bar{\xi} +$$

$$+ \mu(\sqrt{3}g^{\alpha\beta} + 2\Sigma^{\alpha\beta})(\nabla_\alpha + \Sigma_\alpha - igA_\alpha)\eta_\beta = M\eta, \quad (2.4b)$$

$$\mu(\sqrt{3}\delta^\rho_{\beta\rho} - 2\Sigma^\rho_{\beta}) (\nabla_\rho + \Sigma_\rho - igA_\rho)\xi = M\xi_\beta, \quad (2.5a)$$

$$\mu(\sqrt{3}\delta^\rho_{\beta\rho} - 2\Sigma^\rho_{\beta}) (\nabla_\rho + \Sigma_\rho - igA_\rho)\eta = M\eta_\beta. \quad (2.5b)$$

\(^7\text{Spinor indices 1, 2, 1, 2 will be suppressed in the following.}\)

\(^8\text{Those may seem somewhat more complex that the above (2.1) and (2.2), but as will be seen later on it is not the case.}\)
As long as the above equations (2.2) and (2.5) involve a tetrad manifestly, while the tetrad itself at a given metric tensor \( g_{\alpha\beta}(x) \) is not uniquely defined:

\[
g^{kl} e_{(k)\alpha} e_{(l)\beta} = g_{\alpha\beta} \implies e'_{(a)\beta} = L_a^k(x) e_{(k)\beta} ,
\]

any two equations, associated respectively with tetrads \( e'_{(a)\beta} \) and \( e_{(k)\beta} \), must be related to each other by means of a local gauge transformation. We make the mechanism of this gauge symmetry more explicit by seeing how these equations transform at a tetrad change. To this end, one is to subject the constituents the wave function to local Lorentz transformations as prescribed by their spinor nature:

\[
\xi' = B \xi , \quad \eta' = [B^+]^{-1} \eta ,
\]

\[
\xi'_\beta = B \xi_\beta , \quad \eta'_\beta = (B^+)\,^{-1} \eta_\beta .
\]

(2.6)

Here a two-by-two matrix \( B \) belongs to a special linear group \( SL(2,\mathbb{C}) \), universal covering of the restricted Lorentz group \( L^{+}_\perp \); symbol ‘+’ stands for the hermitian conjugate. To demonstrate the required symmetry manifestly, we are to study the following expressions:

\[
B\sigma^a B^+ , \quad (B^+)\,^{-1} \sigma^a B^{-1} ,
\]

(2.8a)

\[
B\Sigma^{\alpha\beta} B^{-1} , \quad (B^+)\,^{-1} \Sigma^{\alpha\beta} B^{-1} ,
\]

(2.8b)

\[
B(\nabla_\alpha + \Sigma_\alpha) B^{-1} , \quad (B^+)\,^{-1} (\nabla_\alpha + \Sigma_\alpha) B^+ .
\]

(2.8c)

First consider (2.8a):

\[
B\sigma^a B^+ = e_{(a)}^\alpha B\sigma^a B^+ , \quad (B^+)\,^{-1} \sigma^a B^{-1} = e_{(a)}^\alpha (B^+)\,^{-1} \sigma^a B^{-1} .
\]

From this, taking in mind the equalities\(^10\) (their positive proof will be done below in Sec.3)

\[
B\sigma^a B^+ = \sigma^b L_b^a , \quad (B^+)\,^{-1} \sigma^a B^{-1} = \sigma^b L_b^a .
\]

(2.9)

where \( L_b^a \) designates four-by-four Lorentz matrix, related to the spinor matrix \( B \) in (2.7), we come to

\[
B\sigma^a B^+ = \sigma'\alpha , \quad (B^+)\,^{-1} \sigma^a B^{-1} = \sigma'\alpha .
\]

(2.10)

Here primed matrices are built on the base of the primed tetrad. Relationships (2.10) are just those we need.

For (2.8b), with the use of eqs. (2.10), one will easily find

\[
B\Sigma^{\alpha\beta} B^{-1} = \Sigma'^{\alpha\beta} , \quad (B^+)\,^{-1} \Sigma^{\alpha\beta} B^{-1} = \Sigma'^{\alpha\beta} .
\]

(2.11)

Let us turn to (2.8c). We are to prove two relationships:

\[
B(\nabla_\alpha + \Sigma_\alpha) B^{-1} = (\nabla_\alpha + \Sigma'_\alpha) ,
\]

\(^9\)For a while, for the sake of simplicity in the formulas we will omit the symbol of an external electromagnetic field \( A_\alpha \).

\(^{10}\)They are sometimes referred to as the Dirac equation’s relativistic invariance conditions.
which are equivalent to the expected gauge transformation laws for 2-spinor connections:

\[
B\Sigma_\alpha B^{-1} + B\partial_\alpha B^{-1} = \Sigma'_\alpha ,
\]

\[
(B^+)^{-1}\Sigma_\alpha B^+ + (B^+)^{-1}\partial_\alpha B^+ = \bar{\Sigma}'_\alpha . \tag{2.12b}
\]

Here again primed connections \(\Sigma'_\alpha(x)\) and \(\bar{\Sigma}'_\alpha(x)\) are constructed with the use of a primed tetrad. The validity of the laws (2.12b) will be demonstrated in the next Sec. 3.

Thus, the gauge invariance of the generally covariant Petras equation with respect to tetrad local changes (2.7) belonging to the \(SL(2,C)\) group, has been established. The set of primed field functions, representing a fermion as observed in the \(e'^\beta\)-tetrad, obeys formally the same equation (see (2.4) and (2.5) ) as the initial ones. In other words, the basic equation is form invariant under local gauge (tetrad) transformations. Of this symmetry fact seems to be quite satisfactory, because the inspiration itself for any particle field theory has most often been found in underlying invariance principles. Thus, we in the first place are interested in theories with definite behavior under special relativity as well as under general relativity.

In closing this Section else one addition may be done. In operating with the covariant derivative \(D_\alpha\) we are very frequently to employ one auxiliary relation, this is a useful time to prove it. With this end in view, turn to the formula \(S\gamma^a S^{-1} = \gamma^b L_b^a\)\(^{12}\) and consider it for a infinitesimal Lorentz transformation; then from the above it follows

\[
\sigma^{kl}\gamma^a - \gamma^a\sigma^{kl} = \gamma^b(V^{kl})^a_b , \quad (V^{kl})^a_b = -g^{ka}\delta^l_b - g^{lb}\delta^k_a . \tag{2.13}
\]

Next, allowing for the explicit form of the vector field generators \((V^{kl})^a_b\), we arrive at the commutation relations

\[
\sigma^{kl}\gamma^a - \gamma^a\sigma^{kl} = \gamma^k g^{lk} - \gamma^l g^{ka} . \tag{2.14}
\]

Now, multiplying eq. (2.13) by a tetrad-based expression \(e^\rho_{(a)} \frac{1}{2} e^\sigma_{(k)} \nabla_\beta e^{(l)\sigma}\), we have got to

\[
\sigma^\rho \nabla_\beta \gamma^\rho = \nabla_\beta \gamma^\rho , \tag{2.14a}
\]

which is just that we need. In turn, eq. (2.14) lead to the following commutation rule:

\[
D_\beta \gamma^\rho(x) = \gamma^\rho(x)D_\beta \tag{2.15a}
\]

In the spinor representation, eqs. (2.14) and (2.15) will be split into

\[
\tilde{\sigma}^\rho \nabla_\beta \gamma^\rho = \nabla_\beta \sigma^\rho , \tag{2.16a}
\]

\[
\sigma^\rho \nabla_\beta \gamma^\rho = \nabla_\beta \sigma^\rho , \tag{2.16b}
\]

\[
(\nabla_\beta + \Sigma_\beta)\tilde{\sigma}^\rho = \sigma^\rho(\nabla_\beta + \Sigma_\beta) , \tag{2.17a}
\]

\[
(\nabla_\beta + \Sigma_\beta)\sigma^\rho = \sigma^\rho(\nabla_\beta + \Sigma_\beta) , \tag{2.17b}
\]

take notice on the placement of the bar in these formulas.

---

\(^{11}\)First, it is technically rather laborious task; second, it will be useful time to introduce a number of facts and formulas on the theory of the Lorentz group and its universal covering just in a manner convenient for exploiting in the present work.

\(^{12}\)In the spinor representation it decomposed into eqs. (2.9).
3. **$SL(2,\mathbb{C})$ group and gauge properties of spinor connections**

In this Section we will be interested in some technicalities of the Lorentz group theory, and its universal covering group $SL(2,\mathbb{C})$. In the first place, we will need the following 'multiplication rules' for two sets of Pauli matrices $\sigma^a, \bar{\sigma}^a$:

\[
\bar{\sigma}^a \sigma^b \bar{\sigma}^c = \sigma^a g^{bc} - \bar{\sigma}^b \sigma^c + \sigma^c g^{ab} - i\epsilon^{abcd} \bar{\sigma}^d, \\
\sigma^a \bar{\sigma}^b \sigma^c = \sigma^a g^{bc} - \sigma^b \sigma^c + \sigma^c g^{ab} + i\epsilon^{abcd} \sigma_d, \\
\epsilon^{0123} = +1 .
\] (3.1)

They can be merged into one formula that is known in the literature as a multiplication rule for four-by-four Dirac matrices [12] [13]:

\[
\gamma^a \gamma^b \gamma^c = \gamma^a g^{bc} - \gamma^b g^{ac} + \gamma^c g^{ab} + i\gamma^5 \epsilon^{abcd} \gamma_d, \\
\gamma^5 = \begin{pmatrix} -I & 0 \\ 0 & +I \end{pmatrix} .
\] (3.2)

Besides, we will need some formulas for the traces:

\[
sp(\bar{\sigma}^a \sigma^b) = 2g^{ab}, \\
sp(\sigma^a \bar{\sigma}^b) = 2g^{ab}, \\
sp(\bar{\sigma}^a \sigma^b \sigma^c \sigma^k) = 2(g^{ab}g^{ck} - g^{ac}g^{bk} + g^{ak}g^{bc} - i\epsilon^{abck}), \\
sp(\sigma^a \bar{\sigma}^b \sigma^c \bar{\sigma}^k) = 2(g^{ab}g^{ck} - g^{ac}g^{bk} + g^{ak}g^{bc} + i\epsilon^{abck}).
\] (3.3)

Now, let us turn again to eqs. (2.9):

\[
B \bar{\sigma}^a B^+ = \sigma^b L^a_b, \\
(B^+)^{-1} \sigma^a B^{-1} = \sigma^b L^a_b .
\] (3.4)

Now, we will show that these are indeed valid, at the same time will establish an explicit representation for a Lorentz four-by-four matrix as a function of parameters $k_a$ and $k_a^*$.

With the use of trace formulas (3.2), eqs. (3.4) can be easily solved under the matrix $L^a_b$:

\[
L^a_b = \frac{1}{2} sp [\sigma_b B \bar{\sigma}^a B^+], \\
L^a_b = \frac{1}{2} sp [\bar{\sigma}_b (B^+)^{-1} \sigma^a B^{-1}] .
\] (3.5)

In the following it will be convenient to utilize a natural parameterization of spinor two-by-two matrices through 4-dimensional complex vector[14]

\[
B(k) = \sigma^a k_a, \quad det B = k^2_0 - k^2_j = +1 , \\
B^+(k) = B(k^*), \quad B^{-1}(k) = B(\bar{k}), \quad \bar{k} = (k_0, -k_j) .
\] (3.6)

Taking this notation in mind, from eqs. (3.5) it follows

\[
L^a_b = \frac{1}{2} sp [\sigma_b B(k) \bar{\sigma}^a B(k^*)] = \frac{1}{2} sp [\sigma_b \sigma^m k_n \bar{\sigma}^a \sigma^n k^*_m] ,
\] (3.7a)

---

13However, the couple (3.1) should be understood as a more fundamental then eq. (3.2).

14Of course, we could have chosen other (more physical one: for example, considering a pure rotation followed by a pure boost, but it will be useful to have an explicit form at hand.
Introducing special Kronecker’s symbol

\[ \delta^c_b = \begin{cases} 
0, c \neq b , \\
+1, c = b = 0 , \\
-1, c = b = 1 = 2 = 3 , 
\end{cases} \]

eqs. (3.7) read as

\[ L_b^a = \delta^c_b \frac{1}{2} sp \left[ \sigma_c \sigma^m \sigma^a \sigma^n \right] k_m k^*_n , \]  
(3.8a)

\[ L_b^a = \delta^c_b \frac{1}{2} sp \left[ \sigma_c \sigma^m \sigma^a \sigma^n \right] k_m k^*_n . \]  
(3.8b)

Next, with the use of the trace formulas (3.3), one can easily find

\[ L_b^a = \delta^c_b \frac{1}{2} sp \left[ \sigma_c \sigma^m \sigma^a \sigma^n \right] k_m k^*_n = \delta^c_b \frac{1}{2} sp \left[ \sigma_c \sigma^m \sigma^a \sigma^n \right] k_m k^*_n = \]

\[ = \delta^c_b \left[ -\delta^b_a k^*_n k^*_n + k_c k^*_n + k^*_c k_n + i \epsilon^{acmn} k_m k^*_n \right] . \]  
(3.9)

By this relation we are supplied with a parametrization of Lorentz four-by-four matrix \( L_b^c \) in terms of two complex vectors \( k_n \) and \( k^*_n \).

It can be straightforwardly shown that the matrix defined by (3.9) possesses the orthogonality property, what is expected for any Lorentz transformation:

\[ L_a^b(k, k^*) = g^{bc} L_c^d(\bar{k}, \bar{k}^*) g_{da} . \]  
(3.10)

Indeed, eq. (3.10) can be rewritten as

\[ \delta^c_a \left[ -\delta^b_a k^*_n k_n + k_c k^*_n + k^*_c k_n + i \epsilon^{acmn} k_m k^*_n \right] = \]

\[ = g^{bc} \left[ -\delta^d_c \bar{k}_n \bar{k}^*_n + \bar{k}_c \bar{k}^*_n + \bar{k}^*_c \bar{k}_n + i \epsilon^{dcmn} \bar{k}_m \bar{k}^*_n \right] g_{da} \]

where the notation \( g^{bc} = g^{ba} \delta^c_a \) is used. From the above it follows

\[ ( \delta^c_a \epsilon^{bmn} ) k_n k^*_m = ( g^{bc} \epsilon^{dmn} g_{da} ) \bar{k}_n \bar{k}^*_m . \]

This identity is valid by a direct verifying that it holds for every choice of indices.

Next, we are to check that it is orthochronic Lorentz transformation, that is \( L_0^0(k, k^*) \geq +1 \). Indeed, an explicit form for \( L_0^0 \) is \( L_0^0 = ( k_0 k^*_0 + k_j k^*_j ) \); from this, with the use of inequality

\[ ( | Z_0 | + | Z_1 | + | Z_2 | + | Z_3 | ) \geq | Z_0 + Z_1 + Z_2 + Z_3 | , \]

at \( Z_0 = k_0 k_0, Z_1 = -k_1 k_1, Z_2 = -k_2 k_2, Z_3 = -k_3 k_3 \), and taking into account \( det B(k) = 1 \), we obtain

\[ L_0^0 \geq | k_0 k_0 - k_j k^*_j | = +1 . \]
Else one property of the Lorentz matrix according to (3.9), det $L(k, k^*) = +1$, should be proved too; this is simple (though laborious) task and its solution is omitted.

Now we are to verify the formulas (3.12b). Obviously, it suffices to consider in detail only one of them; for definiteness let it be the first one

$$B\Sigma_\alpha B^{-1} + B\partial_\alpha B^{-1} = \Sigma'_\alpha.$$  \hspace{1cm} (3.11a)

For the term

$$B \Sigma_\alpha B^{-1} = \frac{1}{2} B \Sigma^{ab} B^{-1} \epsilon^{\beta}_{(a)} (\nabla_a e_{(b)\beta}),$$

expressing the tetrad $e_{(b)\beta}$ in terms of primed $e'_{(b)\beta}$, we have

$$B \Sigma_\alpha B^{-1} = \frac{1}{2} (B \Sigma^{ab} B^{-1}) (L^{-1})_a^k e'_{(k)\beta} \left[ \nabla_a (L^{-1})_b^l e_{(l)\beta} \right],$$

and further

$$B \Sigma_\alpha B^{-1} = \frac{1}{2} (B \Sigma^{ab} B^{-1}) \left[ (L^{-1})_a^k \epsilon'_{(k)\beta} \left[ (\partial_\alpha (L^{-1})_b^l) e'_{(l)\beta} + (L^{-1})_a^k \nabla_a e'_{(l)\beta} \right] \right] =$$

$$= \frac{1}{2} (B \Sigma^{ab} B^{-1}) \left[ (L^{-1})_a^k (\partial_\alpha (L^{-1})_b^l) e'_{(k)\beta} + (L^{-1})_a^k \nabla_a e'_{(l)\beta} \right].$$

Thus, we have arrived to

$$B \Sigma_\alpha B^{-1} = \frac{1}{2} (B \Sigma^{ab} B^{-1}) \times$$

$$\times \left[ (L^{-1})_a^k (\partial_\alpha (L^{-1})_b^l) g_{kl} + (L^{-1})_a^k \nabla_a e'_{(l)\beta} \right].$$  \hspace{1cm} (3.11b)

Next, taking into account the identity, simple consequence of (3.9),

$$B \Sigma^{ab} B^{-1} = \Sigma^{mn} L^a_m L^n_b,$$

from (3.11b) it follows

$$B \Sigma_\alpha B^{-1} = \frac{1}{2} \Sigma^{mn} L^b_m \partial_\alpha (L^{-1})_{bn} + \Sigma'_\alpha.$$ \hspace{1cm} (3.11c)

And finally, allowing for eq. (3.11c), from (3.11a) we get to

$$B\partial_\alpha B^{-1} - \frac{1}{2} \Sigma^{mn} L^b_m \partial_\alpha (L^{-1})_{bn} = 0.$$ \hspace{1cm} (3.12a)

Much in the same manner, from second relation in (3.9b) it can be found

$$(B^+)^{-1} \partial_\alpha B^+ - \frac{1}{2} \Sigma^{mn} L^b_m \partial_\alpha (L^{-1})_{bn} = 0.$$ \hspace{1cm} (3.12b)

It is convenient to alter the equalities (3.12a,b) into

$$B\partial_\alpha B^{-1} - \frac{1}{4} (\tilde{\sigma}^m L^b_m) \partial_\alpha (\sigma^n L^{-1})_{bn} = 0,$$ \hspace{1cm} (3.13a)
\[(B^+)^{-1} \partial_\alpha B^+ - \frac{1}{4} (\sigma^m L^b_m) \partial_\alpha (\bar{\sigma}^n L^{-1}_b) = 0 \, . \quad (3.13b)\]

From where, with the use of eq. (3.9), we get to

\[B \partial_\alpha B^{-1} - \frac{1}{4} B \sigma^b B^+ \partial_\alpha [(B^+)^{-1} \sigma_b B^{-1}] = 0 ,\]

\[(B^+)^{-1} \partial_\alpha B^+ - \frac{1}{4} (B^+)^{-1} \sigma^b B^{-1} \partial_\alpha [(B \sigma_b B^+)] = 0 ,\]

which are equivalent to

\[-\frac{1}{4} B \{ \sigma^a [ B^+ \partial_\alpha (B^+)^{-1}] \sigma_b \} B^{-1} = 0 , \quad (3.14a)\]

\[-\frac{1}{4} (B^+)^{-1} \{ \sigma^b [ B^{-1} \partial_\alpha B ] \sigma_b \} B^+ = 0 . \quad (3.14b)\]

Eqs. (3.14a,b) involve two identities:

\[\sigma^a [ B^+ \partial_\alpha (B^+)^{-1}] \sigma_b \equiv 0 \, , \quad \sigma^b [ B^{-1} \partial_\alpha B ] \sigma_b \equiv 0 \, , \quad (3.15)\]

which, in their turn, follow from the known formulas (indices \( p, s \) take on the values 1, 2, 3)

\[B^+ \partial_\alpha (B^+)^{-1} = \sigma_j (k_0 \partial_\alpha k_j - k_j \partial_\alpha k_0 + i \epsilon_{jps} k_p \partial_\alpha k_s) ,\]

\[B^{-1} \partial_\alpha B = -\sigma_j (k_0 \partial_\alpha k_j - k_j \partial_\alpha k_0 + i \epsilon_{jps} k_p \partial_\alpha k_s) , \quad (3.16)\]

and the evident identities

\[\bar{\sigma}^b \sigma_j \sigma_b = 0 \, , \quad \sigma^b \sigma \sigma_b = 0 \, .\]

So, the gauge transformation laws for Infeld-van der Vaerden connections

\[B \Sigma_\alpha B^{-1} + B \partial_\alpha B^{-1} = \Sigma'_\alpha ,\]

\[(B^+)^{-1} \Sigma_\alpha B^+ + (B^+)^{-1} \partial_\alpha B^+ = \Sigma'_\alpha \quad (3.17)\]

are proven\(^{15}\).

### 4. Invariant form matrix and conserved current

Section 4 will provide a framework for addressing the fundamental dynamical questions: what will the Lagrange function of the generalized fermion look like and in what manner the various bilinear combinations, physically observable quantities are to be constructed.

With this end in view, we are to consider the question about a matrix of invariant bilinear form\(^{16}\) and a generalized conserved current. It is convenient to start with the original equations in the form (see (3.1) and (3.2))

\[\gamma^\alpha D_\alpha \Psi + \mu (\sqrt{\text{det} g^{\alpha\beta} + 2 \sigma^{\alpha\beta}}) D_\alpha \Psi - M \Psi = 0 \, , \quad (4.1)\]

\(^{15}\)It should be mentioned that these gauge properties are commonly known, all justification for else one treatment of those is that their proof is given with the use of finite transformations.

\(^{16}\)Sometimes it is referred to as the hermitian matrix.
\mu \left( \sqrt{3} \delta_\beta^\alpha - 2\sigma_\beta^\alpha \right) D_\alpha \Psi - M \Psi_\beta = 0 , \quad (4.2)

here \( D_\alpha = \nabla_\alpha + B_\alpha - igA_\alpha \).

First, let us find an equation conjugate to eq. (4.1). Acting on eq. (4.1) by the hermitian conjugate operation, we get to

\[ \Psi^+ \hat{D}_\alpha^+ \gamma^\alpha + \mu \Psi^+_\beta \hat{D}_\alpha^+ \left( \sqrt{3} g^{\alpha\beta} + 2\sigma^{\alpha\beta} \right) + M \Psi^+ = 0 . \quad (4.3a) \]

Here \( \hat{D}_\alpha \) stands for an derivative operation on the left

\[ \hat{D}_\alpha^+ = \nabla_\alpha + B_\alpha^+ + igA_\alpha . \quad (4.3b) \]

For definiteness it is convenient to assume the use of the Weyl’s spinor representation of the Dirac matrices\(^\text{17} \), in which the two following properties are readily verified

\[ \gamma^0 \gamma^{\alpha+}(x) \gamma^0 = +\gamma^\alpha(x) , \quad \gamma^0 \sigma^{\alpha\beta+}(x) \gamma^0 = -\sigma^{\alpha\beta}(x) , \quad (4.4) \]

With (4.4), from (4.3a) it follows

\[ \Psi^+ \gamma^0 \hat{D}_\alpha \gamma^\alpha + \mu \Psi^+_\beta \gamma^0 \hat{D}_\alpha \left( \sqrt{3} g^{\alpha\beta} - 2\sigma^{\alpha\beta} \right) + M \Psi^+ \gamma^0 = 0 ; \quad (4.5a) \]

where \( \hat{D}_\alpha \) stands for a new derivative operation on the left:

\[ \hat{D}_\alpha = \nabla_\alpha - B_\alpha + igA_\alpha . \quad (4.5b) \]

It may be noted that the \( \hat{\nabla}_\alpha \) obeys the commutation relation similar to (3.14b)

\[ \hat{D}_\beta \gamma^\rho(x) = \gamma^\rho(x) \hat{D}_\beta . \]

As usual, for a combination \( \Psi^+ \gamma^0 \) the notation \( \bar{\Psi} \) will be used, and the \( \bar{\Psi} \)-function will be interpreted as conjugate to \( \Psi \). However, a similar combination as applied to the vector-bispinor constituent: \( \Psi^+_\beta \gamma^0 \), cannot be understood in a similar way as a conjugate to the \( \Psi_\beta \). But the right answer to this problem follows from requirement that the relation

\[ ( \Psi^+_\beta \gamma^0 \eta^\beta_\nu ) ( \eta^{-1} )^\nu_\rho \hat{D}_\alpha \left( \sqrt{3} g^{\alpha\rho} - 2\sigma^{\alpha\rho} \right) = \bar{\Psi}_\nu \hat{D}_\alpha \left( \sqrt{3} g^{\alpha\nu} + 2\sigma^{\alpha\nu} \right) . \]

be hold. The \( \eta \)-matrix might be identified by the formula

\[ ( \eta^{-1} )^\nu_\rho (x) = F \delta^\nu_\rho + G \sigma^\nu_\rho (x) , \]

where \( F \) and \( G \) are yet not-fixed numbers. From the relation

\[ ( F \delta^\nu_\rho + G \sigma^\nu_\rho ) \left( \sqrt{3} g^{\rho\alpha} + 2\sigma^{\rho\alpha} \right) = \left( \sqrt{3} g^{\nu\alpha} - 2\sigma^{\nu\alpha} \right) \]

\(^17\)In Supplement \ldots , the employing of an arbitrary Dirac’s matrix basis in the context of the given problem will be described in some detail.
we obtain equations for \(F\) and \(G\)

\[
\frac{\sqrt{3}}{2} F - \frac{2F + \sqrt{3}G}{4} + \frac{G}{4} = \frac{\sqrt{3}}{2} + \frac{1}{2} ;
\]

\[
\frac{\sqrt{3}}{2} F + \frac{2F + \sqrt{3}G}{4} + \frac{G}{4} = \frac{\sqrt{3}}{2} - \frac{1}{2} .
\]

Their explicit solution is \(F = 1 + \sqrt{3}, \ G = -2\). Therefore, the \(\eta^{-1}\)-matrix is given by

\[
(\eta^{-1})^\nu_\rho(x) = (1 + \sqrt{3}) \delta^\nu_\rho - 2 \sigma^\nu_\rho(x) .
\] (4.6a)

Remember that this matrix is identified by

\[
(\eta^{-1})^\nu_\rho \left( \sqrt{3} \, g^{\rho\alpha} + 2 \, \sigma^{\rho\alpha} \right) = \left( \sqrt{3} \, g^{\alpha\alpha} - 2 \, \sigma^{\alpha\alpha} \right) .
\] (4.6b)

It is readily verified the expression for \(\eta\)

\[
\eta^\beta_\nu(x) = (1 - \sqrt{3}) \delta^\beta_\nu - 2 \sigma^\beta_\nu(x) .
\] (4.7)

So, a set of functions conjugate to the \(\Psi(x), \Psi_\beta(x)\) is defined as below

\[
\bar{\Psi} = \Psi^+ \, \gamma^0 , \quad \bar{\Psi}_\beta = \Psi^+_\beta \, \eta^\beta_\nu(x) .
\] (4.8)

An equation conjugate to eq. (4.1) has the form (see (4.1))

\[
\bar{\Psi} \overset{\leftarrow}{D}_\alpha^\gamma \alpha + \mu \bar{\Psi}_\beta \overset{\leftarrow}{D}_\alpha \left( \sqrt{3} \, g^{\alpha\beta} + 2 \, \sigma^{\alpha\beta} \right) + M \bar{\Psi} = 0 .
\] (4.9)

Now, we are to obtain a second conjugate equation (see (4.2). To this end, acting on eq. (4.2) the hermitian conjugate operation, we get to

\[
\mu \, \Psi^+ \gamma^0 \overset{\leftarrow}{D}_\rho^\gamma \rho \left( \sqrt{3} \delta_\beta^{\rho} + 2 \sigma_\beta^{\rho} \right) + M \, \Psi^+_\beta \gamma^0 = 0 .
\]

Next, multiplying that by \(\eta^\beta_\alpha(x)\) from the left, and taking into account the relation (this one can be derived by a direct calculation)

\[
(\sqrt{3} \, \delta_\beta^{\rho} - 2 \, \sigma_\beta^{\rho}) \eta^\beta_\alpha = \sqrt{3} \, \delta_\alpha^{\rho} + 2 \, \sigma_\alpha^{\rho} ,
\] (4.10a)

we get the result we need

\[
\mu \, \bar{\Psi} \overset{\leftarrow}{D}_\rho \left( \sqrt{3} \delta_\alpha^{\rho} - 2 \, \sigma_\alpha^{\rho} \right) + M \, \bar{\Psi}_\alpha = 0 .
\] (4.10b)

So, the full system of conjugate equations is as follows:

\[
\bar{\Psi} \overset{\leftarrow}{D}_\alpha^\gamma \alpha + \mu \bar{\Psi}_\beta \overset{\leftarrow}{D}_\alpha \left( \sqrt{3} g^{\alpha\beta} + 2 \sigma^{\alpha\beta} \right) + M \bar{\Psi} = 0 ,
\] (4.11a)

\[
\mu \, \bar{\Psi} \overset{\leftarrow}{D}_\alpha \left( \sqrt{3} \delta_\alpha^{\beta} - 2 \, \sigma_\alpha^{\beta} \right) + M \, \bar{\Psi}_\beta = 0 .
\] (4.11b)
Now, we are ready to derive a conserved current law. To this end, multiplying both eq. (4.1) by $\Psi^\beta$ from the left, and similarly multiplying eq. (4.11a) by $\Psi$ from the right as well as eq. (4.11b) by $\Psi^\beta$ from the right, and summing the results, we get to

$$\left[ \bar{\Psi}_\gamma \rightarrow D_\alpha \Psi + \bar{\Psi} \leftarrow D_\alpha \gamma^\alpha \Psi \right] +$$

$$+ \mu \left[ \bar{\Psi} \left( \sqrt{3}g^{\alpha\beta} + 2\sigma^{\alpha\beta} \right) \rightarrow D_\alpha \Psi + \bar{\Psi} \leftarrow D_\alpha \left( \sqrt{3}g^{\alpha\beta} + 2\sigma^{\alpha\beta} \right) \Psi_\beta \right] +$$

$$+ \mu \left[ \bar{\Psi}_\beta \left( \sqrt{3}g^{\beta\alpha} - 2\sigma^{\beta\alpha} \right) \rightarrow D_\alpha \Psi + \bar{\Psi}_\beta \leftarrow D_\alpha \left( \sqrt{3}g^{\beta\alpha} - 2\sigma^{\beta\alpha} \right) \Psi_\beta \right] = 0 . \quad (4.12)$$

The first term in square brackets in (4.12), with the use of the above commutation relation (3.14a), will reads as

$$\bar{\Psi}_\gamma \rightarrow D_\alpha \Psi + \bar{\Psi} \leftarrow D_\alpha \gamma^\alpha \Psi =$$

$$\bar{\Psi} \left( \gamma^\alpha B_\alpha - B_\alpha \gamma^\alpha \right) \Psi + \bar{\Psi} \gamma^\alpha \rightarrow D_\alpha \Psi = \nabla_\alpha (\bar{\Psi} \gamma^\alpha \Psi) . \quad (4.13)$$

This formula exhibits a notable feature: it provides a rule for acting the conventional covariant derivative $\nabla_\alpha$ on a generally covariant vector constructed as a bilinear combination from the field $\Psi$ and a conjugate field $\bar{\Psi}$. According to (4.13), one may replace the action of the $\nabla_\alpha$ by the action by $\rightarrow D_\alpha$ from the right and by the action by $\leftarrow D_\alpha$ from the left on $\bar{\Psi}$ and $\Psi$ respectively, and ignoring completely the general covariant nature of the matrix $\gamma^\alpha(x)$ in the middle. This fact is representative one, being an example of a general rule.

Actually, let us consider, for instance, the following expression $\nabla^\alpha [ \bar{\Psi} \gamma^\rho \gamma^\sigma \Xi ]$ \(^{18}\). Rewriting it in the form

$$\nabla^\alpha [ \bar{\Psi} \gamma^\rho \gamma^\sigma \Xi ] = [(\partial_\alpha \bar{\Psi}) \gamma^\rho \gamma^\sigma \Xi + \bar{\Psi} (\nabla_\alpha \gamma^\rho) \gamma^\sigma \Xi +$$

$$+ \bar{\Psi} \gamma^\rho (\nabla_\alpha \gamma^\sigma) \Xi + \bar{\Psi} \gamma^\rho \gamma^\sigma (\partial_\alpha \Xi)] ,$$

and replacing the covariant derivatives over $\gamma$-matrices as prescribed by (3.14a), we get

$$\nabla^\alpha [ \bar{\Psi} \gamma^\rho \gamma^\sigma \Xi ] =$$

$$= (\partial_\alpha \bar{\Psi}) \gamma^\rho \gamma^\sigma \Xi + \bar{\Psi} (\gamma^\rho B_\alpha - B_\alpha \gamma^\rho) \gamma^\sigma \Xi +$$

$$+ \bar{\Psi} \gamma^\rho (\gamma^\sigma B_\alpha - B_\alpha \gamma^\sigma) \Xi + \bar{\Psi} \gamma^\rho \gamma^\sigma (\partial_\alpha \Xi) .$$

Here two terms involving the product $\gamma^\rho B_\alpha \gamma^\sigma$ cancel each other, and remaining expression can be read as

$$\nabla^\alpha (\bar{\Psi} \gamma^\rho \gamma^\sigma \Xi ) = \bar{\Psi} \rightarrow D_\alpha \gamma^\rho \gamma^\sigma \Xi + \bar{\Psi} \gamma^\rho \gamma^\sigma \rightarrow D_\alpha \Xi . \quad (4.14a)$$

Relation (4.14a) is equivalent to

$$\nabla_\alpha (\gamma^\rho \gamma^\sigma) = -B_\alpha \gamma^\rho \gamma^\sigma + \gamma^\rho \gamma^\sigma B_\alpha . \quad (4.14b)$$

\(^{18}\)Here $\Xi$ is an arbitrary function with transformation properties that coincides with those for the $\Psi$.  

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Extension to a case of bilinear combination of arbitrary tensor structure is straightforward by induction. Indeed, let \( \gamma^{(n)} \) be

\[
\gamma^{(n)} = \gamma^{\rho_1}\gamma^{\rho_2}...\gamma^{\rho_n}, \quad \nabla_\alpha \gamma^{(n)} = \gamma^{(n)} B_\alpha - B_\alpha \gamma^{(n)}; 
\]  

(4.15a)

then \( \gamma^{(n+1)} = \gamma^{(n)} \gamma^\rho \) obeys

\[
\nabla_\alpha \gamma^{(n+1)} = -B_\alpha \gamma^{(n+1)} + \gamma^{(n+1)} B_\alpha.
\]

(4.15)

From eqs. (4.15) one can get the commutation relations we need

\[
\gamma^{(n)} \rightarrow \nabla_\alpha \gamma^{(n)} = \gamma^{(n)} \nabla_\alpha \rightarrow \gamma^{(n)}.
\]

(4.16)

or in an equivalent form

\[
\nabla_\alpha [\bar{\Psi}^{(n)}(x) \Psi] = \bar{\Psi}^{\nabla_\alpha \gamma^{(n)}(x) \Psi} + \bar{\Psi}^{\gamma^{(n)}(x) \nabla_\alpha \Psi}.
\]

(4.17)

With the use of the established rules (4.17), eq. (4.12) reads as a conserved current law

\[
\nabla_\alpha J^\alpha(x) = 0,
\]

\[
J^\alpha = \bar{\Psi} \gamma^\alpha \Psi + \mu [\bar{\Psi}(\sqrt{3}g^{\alpha\beta} + 2\sigma^{\alpha\beta}) \Psi_{\beta} + \bar{\Psi}(\sqrt{3}g^{\beta\alpha} - 2\sigma^{\beta\alpha}) \Psi_{\beta}].
\]

(4.18)

In connection with eq. (4.18), a couple of facts should be discussed. Really, note that the first term in (4.18) is real-valued. So must be the remaining part of the given current. We are do demonstrate that second and third terms in (4.180 are complex conjugate to each other. To this end, let us look at an expression conjugate to the second term (remember eqs. (4.4)):

\[
[\bar{\Psi} (\sqrt{3}g^{\alpha\beta} + 2\sigma^{\alpha\beta}) \Psi_{\beta}]^+ = \bar{\Psi}_{\beta} \gamma^0 (\sqrt{3}g^{\alpha\beta} - 2\sigma^{\alpha\beta}) \Psi.
\]

Next, inserting after \( \Psi^+ \gamma^0 \) the product \( \eta(x)\eta(x)^{-1} \), get to the relation we need

\[
[\bar{\Psi} (\sqrt{3}g^{\alpha\beta} + 2\sigma^{\alpha\beta}) \Psi_{\beta}]^+ = \bar{\Psi}_{\beta} (\sqrt{3}g^{\beta\alpha} - 2\sigma^{\beta\alpha}) \Psi.
\]

(4.19)

Thus, the current defined by (4.18) is a real-valued generally covariant vector.

Else one question should be clarified. What will the current look like in absence of any external electromagnetic field. Obviously, an expected result is that the generalized current must coincide with the conventional Dirac current. To this end in view, let us compare expressions for generalized and conventional Dirac currents. Having used eqs. (3.1),(3.2) and conjugate eqs. (3.11a,b), for the current according to (3.18) one can easily find the following expression

\[
J^\alpha(x) = \bar{\Psi} \gamma^\alpha(x) \Psi - \frac{4\mu}{M} [\bar{\Psi} \sigma^{\alpha\beta}(x)(\partial_\beta + B_\beta) \Psi + \bar{\Psi}(\partial_\alpha - B_\alpha) \sigma^{\alpha\beta}(x) \Psi],
\]

(4.20a)
that can be brought to the form

\[ J^\alpha(x) = \bar{\Psi} \gamma^\alpha(x) \Psi - \frac{4\mu}{M} \nabla_\beta [ \bar{\Psi} \sigma^{\alpha\beta}(x) \Psi ] . \quad (4.20b) \]

Therefore, in absence of external electromagnetic fields, the above generalized conserved current differs from the conventional current of the Dirac particle only in the term proportional to \( \nabla_\alpha \Omega^{[\alpha\beta]}(x) \). But such a freedom in the choice of any current is inherent in the theory. In view of symmetry of the Ricci tensor, presented in (3.20b) additional \( \mu \)-term will vanish identically in the conservation law; indeed,

\[ \nabla_\alpha \nabla_\beta [ \bar{\Psi} \sigma^{\alpha\beta}(x) \Psi ] = R_{\alpha\beta} [ \bar{\Psi} \sigma^{\alpha\beta}(x) \Psi ] \equiv 0 . \]

5. Equation for a main bispinor \( \Psi \)

In Section 5 we are going to study description of the generalized fermion in the representation when an auxiliary vector-bispinor component is excluded from the equations. To this end, let us turn again to the equations (4.1) and (4.2):

\[ \gamma^\alpha D_\alpha \Psi + \mu \left( \sqrt{3} g^{\alpha\beta} + 2 \sigma^{\alpha\beta} \right) D_\alpha \Psi - M \Psi = 0 , \quad (5.1) \]

\[ \mu \left( \sqrt{3} \delta_\beta^\rho - 2 \sigma_\beta^\rho \right) D_\rho \Psi - M \Psi = 0 . \quad (5.2) \]

Substituting \( \Psi_\beta \) from (5.2) in (5.1), we get to

\[ \gamma^\alpha D_\alpha \Psi + \frac{\mu^2}{M} \left( \sqrt{3} g^{\alpha\beta} + 2 \sigma^{\alpha\beta} \right) \left( \sqrt{3} \delta_\beta^\rho - 2 \sigma_\beta^\rho \right) D_\rho \Psi - M \Psi = 0 , \]

and further (see (1.6))

\[ \gamma^\alpha D_\alpha \Psi - \frac{4\mu^2}{M} \sigma^{\alpha\beta} D_\alpha D_\beta \Psi - M \Psi = 0 . \quad (5.3) \]

The commutator \([D_\alpha, D_\beta] \) is to be considered more closely:

\[ [D_\alpha, D_\beta] \Psi = (D_\alpha D_\beta - D_\beta D_\alpha) \Psi = \]

\[ = [ -igF_{\alpha\beta} + (\partial_\alpha B_\beta - \partial_\beta B_\alpha) + (B_\alpha B_\beta - B_\beta B_\alpha) ] \Psi , \quad (5.4) \]

where \( D_\alpha = \nabla_\alpha + B_\alpha - igA_\alpha \). Take notice on the fact that on the right in (5.4) are present only algebraic expressions (with no derivative over \( \Psi \)). The second term on the right reads in more detailed form as

\[ \partial_\alpha B_\beta - \partial_\beta B_\alpha = \nabla_\alpha B_\beta - \nabla_\beta B_\alpha = \]

\[ = \frac{1}{2} \sigma^{ab} \nabla_\alpha \left( e_{(a)}^{\nu} e_{(b)\nu;\beta} \right) - \frac{1}{2} \sigma^{ab} \nabla_\beta \left( e_{(a)}^{\nu} e_{(b)\nu;\alpha} \right) = \]

\[ = \frac{1}{2} \sigma^{ab} e_{(a)}^{\nu} \left[ e_{(b)\nu;\beta;\alpha} - e_{(b)\nu;\alpha;\beta} \right] + \frac{1}{2} \sigma^{ab} \left[ e_{(a)\nu;\alpha} e_{(b)\nu;\beta} - e_{(a)\nu;\beta} e_{(b)\nu;\alpha} \right] . \quad (5.5) \]
Similarly, the third is

\[ B_\alpha B_\beta - B_\beta B_\alpha = \]

\[ = \left( \frac{1}{2} \sigma^{ab} e_\nu^{(a)} e_{(b)\nu;\alpha} \right) \left( \frac{1}{2} \sigma^{kl} e_\mu^{(k)} e_{(l)\mu;\beta} \right) - \left( \frac{1}{2} \sigma^{kl} e_\mu^{(k)} e_{(l)\mu;\beta} \right) \left( \frac{1}{2} \sigma^{ab} e_\nu^{(a)} e_{(b)\nu;\alpha} \right) = \]

\[ = \frac{1}{4} \left( \sigma^{ab} \sigma^{kl} - \sigma^{kl} \sigma^{ab} \right) \left[ \left( e_\nu^{(a)} e_{(b)\nu;\alpha} \right) e_\mu^{(k)} e_{(l)\mu;\beta} \right], \]

that with the use of the known commutation relations

\[ [\sigma^{ab}, \sigma^{kl}] = -\sigma^{kb} g^{la} + \sigma^{lb} g^{ka} - (\sigma^{ka} g^{lb} + \sigma^{la} g^{kb}) , \]

leads to

\[ (B_\alpha B_\beta - B_\beta B_\alpha) = -\frac{1}{2} \sigma^{ab} \left[ e_{(a)\nu;\alpha} e_{(b)\nu;\beta} - e_{(a)\nu;\beta} e_{(b)\nu;\alpha} \right] \]  

(5.6)

Summarizing (5.5) and (5.6), we arrive at

\[ [D_\alpha D_\beta] = -igF_{\alpha\beta} + \frac{1}{2} \sigma^{ab} e_\nu^{(a)} \left[ e_{(b)\nu;\beta;\alpha} - e_{(b)\nu;\alpha;\beta} \right] = \]

\[ = -igF_{\alpha\beta} + \frac{1}{2} \sigma^{ab} e_\nu^{(a)} \left[ e^{\rho}_{(b)\mu;\alpha} R_{\rho\nu\beta\alpha} \right] = -igF_{\alpha\beta} + \frac{1}{2} \sigma^{\nu\rho} R_{\nu\rho\alpha\beta} , \]  

(5.7)

here \( R_{\nu\rho\alpha\beta}(x) \) designates the Riemann curvature tensor. Allowing for (5.7), the above equation (5.3) reads as

\[ [\gamma^\alpha D_\alpha - \frac{2\mu^2}{M} \left( -ig \sigma^{\alpha\beta} F_{\alpha\beta} + \frac{1}{2} \sigma^{\alpha\beta} \sigma^{\nu\rho} R_{\nu\rho\alpha\beta} \right) - M ] \Psi = 0 , \]  

(5.8)

No we are to look more closely to the term \( \sigma^{\alpha\beta} \sigma^{\nu\rho} R_{\nu\rho\alpha\beta} \). In the first place, taking in mind the symmetry properties of the curvature tensor, it can be rewritten in the form

\[ \sigma^{\alpha\beta} \sigma^{\nu\rho} R_{\nu\rho\alpha\beta} = \frac{1}{4} \gamma^\alpha \gamma^\beta \gamma^\rho \gamma^\nu R_{\nu\rho\alpha\beta} . \]

Next, with the use of the extended multiplication law for Dirac matrices, which can be produced from the known formula (3.2) in the case of Minkowski space by multiplying it with three tetrads

\[ \gamma^\alpha \gamma^\beta \gamma^\nu = \gamma^\alpha g^{\beta\nu} - \gamma^\beta g^{\alpha\nu} + \gamma^\nu g^{\alpha\beta} + i \gamma^5 \epsilon_{\alpha\beta\nu\delta} \gamma^\delta , \]  

(5.9)

and taking into account that in view of the known symmetry property of the curvature tensor with respect to cyclic permutation over any three indices its convolution with Levi-Civita tensor over three indices \( \epsilon^{\alpha\beta\rho\sigma} R_{\alpha\beta\rho\sigma} \) vanishes identically, we arrive at

\[ \sigma^{\alpha\beta} \sigma^{\nu\rho} R_{\nu\rho\alpha\beta} = \frac{1}{4} \left( \gamma^\alpha g^{\beta\nu} - \gamma^\beta g^{\alpha\nu} \right) \gamma^\rho R_{\nu\rho\alpha\beta} , \]

or

\[ \sigma^{\alpha\beta} \sigma^{\nu\rho} R_{\nu\rho\alpha\beta} = -\frac{1}{2} \gamma^\alpha \gamma^\rho R_{\alpha\rho} = -\frac{1}{2} R , \]  

(5.10)
where $R(x)$ stands for the Ricci scalar curvature. Substituting (5.10) in (5.8), the latter takes on the form

$$\{ \gamma^\alpha D_\alpha - \frac{2\mu^2}{M} \left[ -ig\sigma^{\alpha\beta} F_{\alpha\beta} - \frac{1}{4} R(x) \right] - M \} \Psi = 0 ,$$

(5.11)

In absence of electromagnetic fields, eq. (5.11) read as a modified Dirac equation

$$\left[ \gamma^\alpha (\partial_\alpha + B_\alpha) - M + \frac{\mu^2}{2M} R(x) \right] \Psi = 0 ,$$

(5.12)

or in ordinary dimensional units

$$\left[ i \gamma^\alpha (\partial_\alpha + B_\alpha) - \left( \frac{mc}{\hbar} \right) + \frac{\mu^2}{2(\frac{mc}{\hbar})} \frac{R(x)}{2} \right] \Psi = 0 .$$

(5.13)

Take notice that eq. (5.13) is not a common Dirac equation but involving one additional $R$-term. So, 20-component theory of a fermion particle implies both an anomalous magnetic momentum and additional gravitational interaction though scalar Ricci $R$. And what is the most interesting, the same single parameter $\mu$ influences both interaction terms as an intensity multiplier.

6. Masless limit and conformal invariance

In massless case, instead of eqs. (5.1) and (5.2), one is to employ the following equations

$$\gamma^\alpha D_\alpha \Psi + \mu \left( \sqrt{3} g^{\alpha\beta} + 2\sigma^{\alpha\beta} \right) D_\alpha \Psi_\beta = 0 ,$$

(6.1)

$$\mu \left( \sqrt{3} \delta^\rho_\beta - 2\sigma^\rho_\beta \right) D_\rho \Psi = \Psi_\beta .$$

(6.2)

These equations, after exclusion of the auxiliary vector-bispinor component, result in

$$\left[ \gamma^\alpha D_\alpha - 2\mu^2 \left( -ig\sigma^{\alpha\beta} F_{\alpha\beta} - \frac{1}{4} R(x) \right) \right] \Psi = 0 ,$$

(6.3)

If the electromagnetic field vanishes, eq. (6.3) reads as

$$\left[ \gamma^\alpha (\partial_\alpha + B_\alpha) + \frac{1}{2} \mu^2 R(x) \right] \Psi = 0 .$$

(6.4)

In contrast to the massive case, here the $\mu$ represents yet another (in particular, dimensional one) characteristic, which is absolutely different from the old one in the massive case.

\[ \text{To avoid misunderstanding a few comments should be done. From heuristic considerations, in massless case one has at least a theoretical possibility (bearing in mind the gauge invariance principle) to investigate a massless complex-valued field in external vector ("electromagnetic") field. At this, the term "electromagnetic" must be understood with caution, in fact as a matter of convention.} \]
Now we are to deviate from the main line our consideration above and are going to study in some detail one theoretical criterion for correctness of any massless wave equation as applied to the 20-component fermion. As previous experience has shown we may expect that any wave equations for massless particles be invariant under conformal transformations; so is electromagnetic field, so is a massless Dirac equation, and so is a conform-invariant equation for a scalar particle.

With this end in view, let us introduce some conventions and formulas. Ordinary Dirac massless equation in a space-time with metric $g_{\alpha\beta}$ has the form

$$ [i \gamma^\alpha(x) (\partial_\alpha + B_\alpha) \Phi(x) = 0 . \tag{6.5} $$

Let metric tensors of the two space-time models differ in an arbitrary factor-function as shown below

$$ dS^2 = g_{\alpha\beta} dx^\alpha dx^\beta , \quad d\tilde{S}^2 = \tilde{g}_{\alpha\beta} dx^\alpha dx^\beta , \quad \tilde{g}_{\alpha\beta}(x) = \varphi^2(x) g_{\alpha\beta}(x) . \tag{6.6} $$

Correspondingly, it is the most simple to choose tetrads proportional to each other

$$ \tilde{e}_\alpha^a = \frac{1}{\varphi} e_\alpha^a , \quad \tilde{e}^a_{(a)} = \varphi e_{(a)a} . \tag{6.7} $$

Besides, we need to compare two sets of Christoffel symbols. Starting from the relation below

$$ \tilde{\Gamma}_{\alpha\beta,\rho} = \frac{1}{2} \left[ \partial_\alpha \tilde{g}_{\beta\rho} + \partial_\beta \tilde{g}_{\alpha\rho} - \partial_\rho \tilde{g}_{\alpha\beta} \right] = \frac{1}{2} \left[ \partial_\alpha \varphi^2 g_{\beta\rho} + \partial_\beta \varphi^2 g_{\alpha\rho} - \partial_\rho \varphi^2 g_{\alpha\beta} \right] = \varphi^2 \Gamma_{\alpha\beta,\rho} + \varphi \left[ (\partial_\alpha \varphi) g_{\beta\rho} + (\partial_\beta \varphi) g_{\alpha\rho} - (\partial_\rho \varphi) g_{\alpha\beta} \right] , $$

or

$$ \tilde{\Gamma}_{\alpha\beta} = \tilde{g}^\rho_{\sigma} \Gamma_{\alpha\beta,\rho} = \frac{1}{\varphi^2} \varphi^2 \Gamma_{\alpha\beta,\rho} + \varphi \left[ (\partial_\alpha \varphi) g_{\beta\rho} + (\partial_\beta \varphi) g_{\alpha\rho} - (\partial_\rho \varphi) g_{\alpha\beta} \right] . $$

we obtain

$$ \tilde{\Gamma}_{\alpha\beta} = \Gamma_{\alpha\beta} + \frac{1}{\varphi} \left[ (\partial_\alpha \varphi) \delta^\rho_\beta + (\partial_\beta \varphi) \delta^\rho_\alpha - g^{\rho\sigma}(x)(\partial_\rho \varphi) g_{\alpha\beta} \right] . \tag{6.8} $$

Similarly, for two sets of Ricci rotation coefficients we have

$$ \tilde{\gamma}_{abc} = - \left[ \frac{\partial}{\partial x^\alpha} \tilde{e}^a_{(a)} - \tilde{\Gamma}_{\beta\alpha} \tilde{e}^\beta_{(b)} \right] \tilde{e}^\alpha_{(c)} = $$

$$ = - \left[ \frac{\partial}{\partial x^\alpha} (\varphi e_{(a)a}) - \Gamma_{\beta\alpha} \varphi e_{(a)a} - \frac{1}{\varphi} \left( (\partial_\beta \varphi) \delta^\rho_\alpha + (\partial_\alpha \varphi) \delta^\rho_\beta - g^{\rho\sigma}(x)(\partial_\sigma \varphi) g_{\beta\alpha}(x) \right) \varphi e_{(a)a} \right] \frac{1}{\varphi^2} \tilde{e}^\alpha_{(b)} \tilde{e}^\beta_{(c)} . $$

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a further

\[ \tilde{\gamma}_{abc}(x) = - \left[ \varphi_e^{(a)} \alpha \beta + e^{\sigma}_{(a)} \frac{\partial \varphi}{\partial x^\sigma} g_{\alpha \beta}(x) \right] \frac{1}{\varphi^2} e^{\beta}_{(b)} e^{\alpha}_{(c)} . \]

The final formula is

\[ \tilde{\gamma}_{abc}(x) = \frac{1}{\varphi} \tilde{\gamma}_{abc}(x) + \frac{1}{\varphi^2} \left[ e^\sigma_{(b)} g_{ac} \partial_\sigma \varphi - e^\sigma_{(a)} g_{bc} \partial_\sigma \varphi \right] . \] (6.9)

Now, we are ready to relate two Dirac equations associated with different space-time models. Let us start with that in \( \tilde{g}_{\alpha \beta}(x) \)-space:

\[ i \tilde{\gamma}^\alpha(x) \left( \partial_\alpha + \tilde{B}_\alpha(x) \right) \tilde{\Phi}(x) = 0 , \] (6.10)

or in altered form

\[ i \gamma^c \left( \tilde{e}^\alpha_{(c)} \partial_\alpha + \frac{1}{2} \sigma^{ab} \tilde{\gamma}_{abc} \right) \tilde{\Phi} = 0 . \] (6.11)

Substituting here the above expressions of the 'tilded' tetrad and Ricci rotation coefficients in terms of 'untilded' ones we arrive at

\[ \{ \left[ i \gamma^c \left( e^\sigma_{(c)} \partial_\sigma + \frac{1}{2} \sigma^{ab} \gamma_{abc} \right) + \right. \]

\[ \left. + \frac{i}{2} \gamma^c \sigma^{ab} \left[ (e^\sigma_{(b)} g_{ac} - e^\sigma_{(a)} g_{bc}) \frac{1}{\varphi} \partial_\sigma \varphi \right] \} \tilde{\Phi} = 0 , \] (6.12)

which is equivalent to

\[ \left[ i \gamma^\sigma \left( \partial_\sigma + B_\sigma \right) + i \gamma_\alpha \sigma^{ab} e^\sigma_{(b)} \frac{1}{\varphi} \partial_\sigma \varphi \right] \tilde{\Phi} = 0 . \] (6.13)

Because

\[ \gamma_\alpha \sigma^{ab} = \frac{3}{2} \gamma^b , \] (6.14)

eq (6.13) will read as

\[ i \gamma^\sigma(x) \left[ \partial_\sigma + B_\sigma(x) + (3/2) \frac{1}{\varphi} \partial_\sigma \varphi \right] \tilde{\Phi} = 0 . \] (6.15)

And the final step is to have used the following substitution

\[ \tilde{\Phi}(x) = (\varphi^{-1})^{-3/2} \Phi(x) \] (6.16)

at this eq. (6.15) describing a massless fermion field \( \tilde{\Phi}(x) \) in the space-time with \( \tilde{g}_{\alpha \beta}(x) \)-metric will take on the form of Dirac massless wave equation in the 'tilded' space-time:

\[ i \gamma^\sigma(x) \left[ \partial_\sigma + B_\sigma(x) \right] \Phi = 0 . \] (6.17)

This fact of such a simple relationship between two Dirac equations in \( g- \) and \( \tilde{g} \)-model is referred to as the conformal invariance property of the massless Dirac equation [62].
Now, in a similar manner, let us consider the case of a particle with anomalous magnetic momentum. We will need one auxiliary formula. The Ricci scalar behaves under conformal transformations according to the law (Gürsey F. Ann. Phys. 1963. Vol. 24, P. 211-244; also see Appendix A)

\[ \tilde{R} = \frac{1}{\varphi^2} \left( R - \frac{6}{\varphi} \nabla^\beta \nabla_\beta \varphi \right). \] (6.18)

On taking into account this law we immediately have to conclude that eq. (6.4) is not conformally invariant.

Obviously, we might start with a bit different equations (compare with (6.1) and (6.2)):

\[ \gamma^\alpha D_\alpha \Psi + \mu \left[ \sqrt{3} g^{\alpha \beta} + 2 \sigma^{\alpha \beta} \right] D_\alpha \Psi_\beta = \frac{1}{2} \mu^2 \tilde{R}(x) \Psi, \] (6.19)

\[ \mu \left( \sqrt{3} \delta_\beta^\rho - 2 \sigma_\beta^\rho \right) D_\rho \Psi = \Psi_\beta. \] (6.20)

Consequently, after exclusion the vector-bispinor constituent from eqs. (6.19) it follows

\[ \left[ \gamma^\alpha(x) \left( \partial_\alpha + B_\alpha - i \frac{e}{\hbar c} A_\alpha \right) + 2i \mu^2 \frac{e}{\hbar c} \sigma^{\alpha \beta}(x) F_{\alpha \beta}(x) \right] \Psi = 0, \] (6.21)

that is a massless conformally invariant Dirac equation. In absence of an external 'electromagnetic' field it yields

\[ \gamma^\alpha(x)(\partial_\alpha + B_\alpha) \Psi = 0. \] (6.22)

7. On gauge $P$-symmetry of the theory

In this Section we will consider some features concerned with a tetrad $P$-symmetry of the 20-component fermion model. In the first place, several initial fact and notations are to be recalled.

As known, the ordinary general covariant Dirac equation

\[ \left[ i \gamma^\alpha(x)(\partial_\alpha + B_\alpha) - m \right] \Psi(x) = 0 \] (7.1)

possesses a symmetry under the following discrete transformation (we assume the use of the Weyl spinor basis)

\[ \Psi'(x) = \gamma^0 \Psi(x). \] (7.2)

Let us demonstrate this. First, we need two relationships $\gamma^0 \gamma^0 = +\gamma^0, \quad \gamma^0 \gamma^i \gamma^0 = -\gamma^i$, hold, which can be summarized in the formula

\[ \gamma^0 \gamma^\alpha \gamma^0 = \delta^a_b \gamma^b = \gamma^b L^{(p)}_b a \] (7.3)

\[ ^{20}\text{The same } R\text{-term might be added in the massive case too.} \]
with the use of a special designation for Lorentz $P$-inversion:

\[
L_{a}^{(p)b} = \tilde{\delta}_{a}^{b} = \begin{pmatrix}
+1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}.
\]

Further, taking into account eq. (7.3), we produce the transformation laws for the generalized Dirac matrices and bispinor connection:

\[
\gamma^{0}\gamma^{a}(x)\gamma^{0} = \gamma^{a}(x) , \quad \gamma^{0}B_{a}(x)\gamma^{0} = B'_{a}(x) ,
\]

(7.4a)

where primed quantities are built with the use of the (primed) $P$-inverted tetrad

\[
e'_{(b)a}(x) = L_{b}^{(p)a}e_{(a)a}(x) .
\]

(7.4b)

With eqs. (7.4), from (7.2) it follows an equation for $\Psi'(x)$:

\[
[ i\gamma^{a}(x) (\partial_{a} + B'_{a}) - m ] \Psi'(x) = 0 ,
\]

(7.5)

which formally coincides with eq. (7.1). This fact proves the tetrad $P$-symmetry we need.

Now let us turn to 20-component model. By simplicity reasons, it is convenient firstly to study the case of a flat space-time (that is, $D^a = \partial_a - igA_a(x)$)

\[
\begin{align*}
\gamma^{a}D_{a}\Psi + \mu Z^{ab}D_{a}\Psi_{b} &= M\Psi , \\
\mu Y_{b}^{a}D_{a}\Psi &= M\Psi_{b} ,
\end{align*}
\]

(7.6a)  (7.6b)

where

\[
Z^{ab} = (\sqrt{3} g^{ab} + 2\sigma^{ab}) , \quad Y_{a}^{b} = (\sqrt{3} \delta_{a}^{b} - 2\sigma_{a}^{b}) .
\]

As concerns to these notations, two useful equalities should be noted:

\[
\begin{align*}
Z^{ab} &= Y^{ba} , & \gamma^{0} (Z^{ab})^{\pm} \gamma^{0} &= Y^{ab} = Z^{ba} .
\end{align*}
\]

The $P$-inversion here as applied to 20-component field is defined by

\[
x_{a}' = L_{b}^{(p)a}x_{a} , \quad \Psi'(x') = \gamma^{0}\Psi(x) , \quad \Psi_{b}'(x') = \gamma^{0}L_{b}^{(p)a}\Psi_{b}(x) .
\]

(7.7)

With the use of eq. (7.3) and of these

\[
\begin{align*}
\gamma^{0}Z^{ab}\gamma^{0} &= \tilde{\delta}_{a}^{k}\tilde{\delta}_{b}^{l}Z^{kl} , & \gamma^{0}Y_{b}^{a}\gamma^{0} &= \tilde{\delta}_{b}^{k}\tilde{\delta}_{a}^{l}Y_{k}^{l} ,
\end{align*}
\]

(7.8)

from (7.6) one readily gets the primed equations

\[
\begin{align*}
\gamma^{a}D_{a}'\Psi' + \mu Z^{ab}D_{a}'\Psi_{b}' &= M\Psi'(x) , \\
\mu Y_{b}^{a}D_{a}'\Psi' &= M\Psi_{b}' ,
\end{align*}
\]

(7.9a)  (7.9b)
where $D'_a = L^{(p)}_a D_b$. So, 20-component fermion model in Minkowski space-time is $P$-invariant.

In will be useful to have in hand a formulation of this symmetry in two-component spinor basis, which can be derived quite straightforwardly. Here eqs. (7.1) will take the form (see (2.4) (2.5))

\[\bar{\sigma}^a D_b \eta + \mu \bar{z}^{ab} D_a \xi_b = M \xi, \quad (7.10a)\]
\[\sigma^a D_a \xi + \mu \bar{z}^{ab} D_a \eta_b = M \eta, \quad (7.10b)\]
\[\mu y_b^c D_c \xi = M \xi_b, \quad (7.11a)\]
\[\mu \bar{y}_b^c D_c \eta = M \eta_b. \quad (7.11b)\]

The notation is as follows

\[z^{ab} = (\sqrt{3} g^{ab} + 2\Sigma^{ab}), \quad \bar{z}^{ab} = (\sqrt{3} g^{ab} + 2\Sigma^{ab}), \quad (7.12)\]

and relations $z^{ab} = y^{ba}$, $\bar{z}^{ab} = \bar{y}^{ba}$ hold. $P$-transformation in spinor form looks as (see (7.7))

\[\xi'(x') = + \eta(x), \quad \eta'(x') = + \xi(x)\]
\[\xi'_b(x') = + \delta^a_b \eta_a(x), \quad \eta'_b(x') = + \bar{\delta}^a_b \xi_a(x). \quad (7.13)\]

Substituting in (7.10) and (7.11) the components $\xi, \eta, \xi'_b, \eta'_b$ in terms of the $\xi', \eta', \xi'_b, \eta'_b$ according to (7.12), and bearing in mind the formulas

\[z^{ab} = \bar{z}^{ab} = \delta^a_b \sigma^b, \quad z^{ab} = \bar{z}^{ab} = \bar{\delta}^a_b \bar{\sigma}^b, \quad (7.14)\]

from (7.10) and (7.11) $P$-inverted equations follow

\[\bar{\sigma}^a D_b' \eta' + \mu \bar{z}^{ab} D'_a \xi_b' = M \xi', \quad (7.15a)\]
\[\sigma^a D'_a \xi' + \mu \bar{z}^{ab} D'_a \eta'_b = M \eta', \quad (7.15b)\]
\[\mu y_b^c D_c' \xi' = M \xi'_b, \quad (7.16a)\]
\[\mu \bar{y}_b^c D_c' \eta' = M \eta'_b. \quad (7.16b)\]

which formally coincide with those original (7.10), (7.11).

At last, we are to look at the generally covariant 20-component case. The situation in the chosen basis $\Psi(x), \Psi_{\beta}(x)$ turns out to be much simpler than in Minkowski space-time because the equations under consideration

\[\gamma^\alpha D_\alpha \Psi + \mu Z^{\alpha\beta} D_\alpha \Psi_{\beta} = M \Psi, \quad (7.17a)\]
\[\mu Y_{\beta}^\alpha D_\alpha \Psi = M \Psi_{\beta}; \quad (7.17b)\]
are obviously form-invariant under the tetrad $P$-inversion established here in accordance with the formulas (compare with (7.2))

$$
\Psi'(x) = \gamma^0 \Psi(x) , \quad \Psi'_\beta(x) = \gamma^0 \Psi_\beta(x) .
$$

(7.18)

Take notice that here the $P$-inversion operation over the wave function components does not affect its vector index, because it is hidden in generally covariant vector notation: $\Psi_\beta = e^{(l)}_\beta(x) \Psi_l(x)$.\(^{21}\)

8. On matrix formulation of the 20-component theory

Now, let us obtain a matrix form of the generalized fermion model\(^{22}\). To this end, we turn again to eq. (2.1):

$$
\gamma^\alpha D_\alpha \Psi + \mu Z^{\alpha\beta} D_\alpha \Psi_\beta = M \Psi ,
$$

(8.1a)

$$
\mu Y_{\beta\rho} D_\rho \Psi = M \Psi_\beta .
$$

(8.1b)

and produce their form when the generally covariant vector index of the wave function is translated into a tetrad one. We need a few auxiliary relations. From

$$
\nabla_\alpha \Psi_\beta = \nabla_\alpha (\Psi_l e^{(l)}_\beta) = \Psi_l e^{(l)}_\beta + e^{(l)}_\beta \partial_\alpha \Psi_l ,
$$

it follows

$$
D_\alpha \Psi_\beta = e^{(l)}_\beta \partial_\alpha \Psi_l + e^{(l)}_\beta \partial_\alpha \Psi_l + B_\alpha e^{(l)}_\beta \Psi_l - i g A_\alpha e^{(l)}_\beta \Psi_l .
$$

(8.2)

In turn, the vector connection according to Tetrode-Weyl-Fock-Ivanenko method

$$
L_\alpha = \frac{1}{2} (V^{cd})_k^l e^{(l)}_\beta \nabla_\alpha e^{(d)\beta} .
$$

(8.3a)

by taking into account the explicit form of the generator for the vector representation of the Lorentz group $(V^{ab})_k^l = (g^{ab} \delta_k^l + g^{bl} \delta_k^a)$, will read as

$$
(L_\alpha)_k^l = e^{(l)}_\beta (\partial_\alpha + g A_\alpha) \Psi_l = M \Psi .
$$

(8.3b)

With the use of eqs. (8.2) and (8.3), the equations (8.1) yield

$$
e^{(a)}_\alpha \gamma^\alpha (\partial_\alpha + B_\alpha - i g A_\alpha) \Psi + + \mu Z^{ab} e^{(a)}_\alpha \times
$$

$$
\times [ \delta_b^l \partial_\alpha + (L_\alpha)_b^l + \delta_b^l B_\alpha - i g \delta_b^l A_\alpha ] \Psi_l = M \Psi ,
$$

(8.4a)

$$
\mu Y_{\eta d} e^{(a)}_\alpha (\partial_\alpha + B_\alpha - i g A_\alpha) \Psi = M \Psi_l .
$$

(8.4b)

Further, the notation will be used:

$$
\Phi = \left( \begin{array}{c} \Psi \\ \Psi_l \end{array} \right) , \quad \Gamma^a = \left( \begin{array}{ccc} \gamma^\alpha & \mu (Z^a)_b^l \\ \mu (Y^a)_b^l & 0 \end{array} \right) = \left( \begin{array}{ccc} \gamma^\alpha & \mu Z^a_b \\ \mu Z^a_b & 0 \end{array} \right) ,
$$

\(^{21}\)Translating to the explicit use of a tetrad vector index will be given in Sec. 8.

\(^{22}\)As will be seen below, sometimes this formalism has advantages over spin-vector approach.
\[ J^{cd} = \sigma^{cd} \otimes I_5 + I \otimes J^{cd}, \quad j^{cd} = \begin{pmatrix} 0 & 0 \\ 0 & (V^{cd})^t_b \end{pmatrix}, \quad (8.5a) \]

\[ \Gamma^\alpha(x) = \Gamma^\alpha e_{(a)}^\alpha, \quad G_\alpha = \frac{1}{2} J^{cd} e_{(c)}^\beta \nabla^\alpha e_{(d)\beta}, \]
\[ G_\alpha(x) = \begin{pmatrix} B_\alpha & 0 \\ 0 & B_\alpha \otimes I + I \otimes L_\alpha \end{pmatrix}. \quad (8.5b) \]

It is easily verified that eqs. (8.4) can be rewritten as the following generally covariant matrix equation:
\[ [ \Gamma^\alpha(x) (\partial_\alpha + G_\alpha - igA_\alpha) - M ] \Phi = 0. \quad (8.6) \]

Below, we will also exploit the notation
\[ D_{s.}^\alpha = \partial_\alpha + B_\alpha - igA_\alpha, \quad D_{v.}^\alpha = \partial_\alpha + B_\alpha \otimes I_4 + I \otimes L_\alpha - igA_\alpha, \]
\[ D_\alpha = \begin{pmatrix} D_{s.}^\alpha & 0 \\ 0 & D_{v.}^\alpha \end{pmatrix}, \quad (8.7) \]

with which the equation (8.6) can be presented as
\[ [ \Gamma^\alpha(x) D_\alpha - M ] \Phi = 0. \quad (8.8) \]

Now, we consider again the question about the Lorentz invariant form matrix in that approach. Obvious, it suffices to restrict ourselves to the flat space-time case:
\[ \begin{pmatrix} \gamma^a \\ \mu Z^a_l \\ \mu Y^a_l \\ 0 \end{pmatrix} \partial_\alpha \begin{pmatrix} \Psi \\ \Psi_l \end{pmatrix} - M \begin{pmatrix} \Psi \\ \Psi_l \end{pmatrix} = 0, \quad (\Gamma^\alpha \partial_\alpha - M)\Phi = 0. \]

Following the usual procedure
\[ \bar{\Phi}(\Gamma^\alpha \partial_\alpha + M) = 0, \quad \bar{\Phi} = \Phi^+ H, \]
\[ \Phi^+ = (\Psi^+, \Psi^+_l), \quad H = \begin{pmatrix} \gamma^0 & 0 \\ 0 & \gamma^0 h^{kl} \end{pmatrix}, \quad (8.9) \]
we get to a defining relation for \( H \)-matrix:
\[ H^{-1}(\Gamma^\alpha)^+ H = \Gamma^\alpha. \quad (8.10a) \]

This equation, in turn, after taking into account the block-structure of the all matrices involved leads to
\[ \begin{pmatrix} \gamma^0 \gamma^a + \gamma^0 & \mu \gamma^0 Y^a + \gamma^0 h \\ \mu h^{-1} \gamma^0 Z^a + \gamma^0 & 0 \end{pmatrix} = \begin{pmatrix} \gamma^a & \mu Z^a \\ \mu Y^a & 0 \end{pmatrix}. \]

So,
\[ \gamma^0 \gamma^a + \gamma^0 = \gamma^a, \quad \gamma^0 Y^a + \gamma^0 h = Z^a, \quad h^{-1} \gamma^0 Z^a + \gamma^0 = Y^a. \quad (8.10b) \]
The truth of the first of these has been already noted (see (4.4)). Two remaining ones give respectively

\[ \gamma^0 (Z^a)_l + \gamma^0 h^b = Z^{ab} , \quad (h^{-1})_{cl} \gamma^0 (Z^{ab}) + \gamma^0 = Y^a_c . \]

and further

\[ \gamma^0 (Z^a)_l + \gamma^0 h^b = Z^{ab} , \quad (h^{-1})_{cl} Z^{la} = Y^a_c . \] (8.11)

Here, the first one represents already known relation (4.4a) being transformed to a tetrad basis, whereas the second is a tetrad version of the above equation (4.6b); at this an expression for \( h \)-matrix follows immediately in the form

\[ h^b = g^{lk} h_k^b , \quad h_k^b = \eta_k^b . \] (8.12)

Thus, the conserved current expression (4.18) can be rewritten in a matrix formalism as follows

\[ J^\alpha = \bar{\Phi} \Gamma^\alpha \Phi = \bar{\Psi} \gamma^\alpha \Psi + \mu \left( \bar{\Psi} Z^{\alpha\beta} \Psi_{\beta} + \bar{\Psi}_{\beta} Y^{\beta\alpha} \Psi \right) . \] (8.13)

Now let us turn again to a generally covariant equation (8.8) and derive a conjugate one. From (8.8) it follows

\[ \bar{\Phi} \left[ (\partial^\alpha + H^{-1} G^+_{\alpha} H + igA_{\alpha}) H^{-1} \Gamma^\alpha + H + M \right] = 0 . \] (8.14)

Allowing for the equalities

\[ H^{-1} \gamma^\alpha + H = + \gamma^\alpha , \quad H^{-1} G^+_{\alpha} H = - G_{\alpha} , \] (8.15)

from (8.14) we get

\[ \bar{\Phi} \left[ (\overleftarrow{\partial}_{\alpha} - G_{\alpha} + igA_{\alpha}) \Gamma^\alpha + H + M \right] = 0 . \] (8.16)

Evidently, the first relation in (8.15) is a simple consequence from (8.10a). The second in (8.15) is easily verified on accounting for that the \( H \)-matrix is to allow a Lorentz invariant to be of the form

\[ \bar{\Phi}' H \Phi' = \bar{\Phi} H \Phi , \quad \Phi' = S \Phi , \]

where \( S \) represents a Lorentz transformation in 20-component \( \Phi \)-space. Therefore, the following relationship

\[ H^{-1} S^+ H = S^{-1} \] (8.17)

is to hold. From this, taking an infinitesimal transformation

\[ S = I + \delta_{ab} J^{ab} , \quad S^{-1} = I - \delta_{ab} J^{ab} , \quad S^+ = I + \delta_{ab} (J^{ab})^+ , \]

we come to

\[ H^{-1} (J^{ab})^+ H = - J^{ab} \] (8.18)

The second relation in (8.15) is readily proved simply by allowing for the equation (8.18) and the definition for connection \( G_{\alpha} \) in terms of generators \( J^{ab} \).
9. Bilinear combinations in a Riemannian space-time

This Section deals with methods for constructing some bilinear invariant combinations in terms of \( \Phi \)- and \( \bar{\Phi} \)-functions. We start with two equations

\[
\begin{align*}
\left[ \Gamma^{\alpha}(x) \left( \tilde{\partial}_{\alpha} + G_{\alpha} - igA_{\alpha} \right) - M \right] \Phi &= 0 , \\
\bar{\Phi} \left[ \left( \partial_{\alpha} - G_{\alpha} + igA_{\alpha} \right) \Gamma^{\alpha} + M \right] &= 0 .
\end{align*}
\]

Multiplying eq. (9.1) from the left by \( \bar{\Phi} \), and eq. (9.2) — from the right by \( \Phi \), and adding results together, we get to

\[
\bar{\Phi} \tilde{\partial}_{\alpha} \Gamma^{\alpha} \Phi + \bar{\Phi} \Gamma^{\alpha} \tilde{\partial}_{\alpha} \Phi + \bar{\Phi} \left( \Gamma^{\alpha}G_{\alpha} - G_{\alpha}\Gamma^{\alpha} \right) \Phi = 0 .
\]

To proceed with an analysis of eq. (9.3), we need one auxiliary relation. To this end, let us turn again to a relativistic invariance condition for the Petras e quation in matrix form

\[
S \Gamma^{a} S^{-1} = \Gamma^{b} L_{b}^{a} ,
\]

and set a Lorentz transformation to be infinitesimal, from where it follows

\[
J^{kl} \Gamma^{a} - \Gamma^{a} J^{kl} = \Gamma^{k} g^{lk} - \Gamma^{l} g^{ka} .
\]

Now, multiplying eq. (9.5) by an expression \( \epsilon^{\beta}_{\sigma(k)} \frac{1}{2} \epsilon^{\beta}_{\sigma(l)} \nabla_{\sigma} \epsilon_{\beta(l)} \), we obtain a needed formula

\[
\Gamma^{\rho} G_{\sigma} - G_{\sigma} \Gamma^{\rho} = \nabla_{\sigma} \Gamma^{\rho} .
\]

By accounting (9.6) in (9.3), we come to

\[
\bar{\Phi} \tilde{\partial}_{\alpha} \Gamma^{\alpha} \Phi + \bar{\Phi} \left( \nabla_{\alpha} \Gamma^{\alpha} \right) \Phi + \bar{\Phi} \Gamma^{\alpha} \tilde{\partial}_{\alpha} \Phi = 0 ,
\]

which can be rewritten as a generally covariant conserved current law:

\[
\nabla_{\alpha} J^{\alpha} = 0 , \quad J^{\alpha} = \bar{\Phi} \Gamma^{\alpha} \Phi .
\]

It is easily verified that this conserved current \( J_{\alpha} \) coincides with the above form (4.18).

It is particularly remarkable that the formula (9.7a) provides us with a hint about a general rule for covariant \( \nabla_{\alpha} \)-differentiating any bilinear combination constructed on the base of \( \Phi \) and \( \bar{\Phi} \) functions.

Indeed, the rule

\[
\nabla_{\alpha} \left( \bar{\Phi} \Gamma^{\alpha} \Phi \right) = \bar{\Phi} \left( \tilde{\partial}_{\alpha} - G_{\alpha} \right) \Gamma^{\alpha} \Phi + \bar{\Phi} \Gamma^{\alpha} \left( \tilde{\partial}_{\alpha} + G_{\alpha} \right) \Psi
\]

\[\text{(9.8a)}\]
is in force. Further it will be convenient to use the notation
\[(\vec{\nabla}_\alpha + G_\alpha)\Psi = \vec{D}_\alpha \Phi, \quad \Phi(\vec{\nabla}_\alpha - G_\alpha) = \Phi \vec{D}_\alpha. \tag{9.8b}\]

As usual, covariant derivatives \(\vec{\nabla}_\alpha\) and \(\vec{\nabla}_\alpha\) in acting on scalar functions become ordinary derivatives: \(\vec{\partial}_\alpha\) and \(\vec{\partial}_\alpha\) respectively. With the notation (9.8b), the above eq. (9.8a) looks as
\[\nabla_\alpha (\Phi \Gamma^\alpha \Phi) = \vec{\Phi} \vec{D}_\alpha \Gamma^\alpha \Phi + \vec{\Psi} \Gamma^\alpha \vec{D}_\alpha \Phi. \tag{9.8c}\]

As prescribed by the formula (9.8c), one may replace the action of \(\nabla_\alpha\) by action of \(\vec{D}_\alpha\) from the right and action of \(\vec{D}_\alpha\) from the left respectively on \(\Phi\) and \(\Phi\), completely ignoring a generally covariant nature of the matrix \(\Gamma^\alpha\) in the middle.

What is more, such a rule will work always at any complicated bilinear function. For instance, let us consider a combination \(\nabla^\alpha [\vec{\Phi} \Gamma^\rho(x) \Gamma^\sigma(x) \Xi]\). In accordance with defining properties of the covariant derivative we can proceed
\[\nabla^\alpha (\vec{\Phi} \Gamma^\rho \Gamma^\sigma \Xi) = (\partial_\alpha \vec{\Phi}) \Gamma^\rho \Gamma^\sigma \Xi + \vec{\Phi} (\nabla_\alpha \Gamma^\rho) \Gamma^\sigma \Xi + \vec{\Phi} \Gamma^\rho (\nabla_\alpha \Gamma^\sigma) \Xi + \vec{\Phi} \Gamma^\rho \Gamma^\sigma (\partial_\alpha \Xi). \tag{9.9a}\]

Replacing the covariant derivatives of \(\Gamma\)-matrices with commutators as prescribed (9.6), one gets
\[\nabla^\alpha (\vec{\Phi} \Gamma^\rho \Gamma^\sigma \Xi) = \vec{\Phi} \vec{D}_\alpha \Gamma^\rho \Gamma^\sigma \Xi + \vec{\Phi} \Gamma^\rho \Gamma^\sigma \vec{D}_\alpha \Xi. \tag{9.9b}\]

In turn, relation (9.9a) is equivalent to
\[\nabla_\alpha (\Gamma^\rho \Gamma^\sigma) = - \Gamma^\rho \Gamma_\alpha \Gamma^\sigma + \Gamma^\rho \Gamma^\sigma \Gamma_\alpha. \tag{9.9b}\]

An analogous formula for a bilinear combination with any tensor structure follows immediately by induction. Indeed, if
\[\Gamma^{(n)} = \Gamma^{(n-1)} \Gamma^{(n-2)} \ldots \Gamma^{(1)} \Gamma^{(0)}, \quad \text{and} \quad \nabla_\alpha \Gamma^{(n)} = \Gamma^{(n)} \Gamma_\alpha - \Gamma_\alpha \Gamma^{(n)}; \tag{9.10a}\]
then for \(\Gamma^{(n+1)} = \Gamma^{(n)} \Gamma^{(0)}\) one derives
\[\nabla_\alpha \Gamma^{(n+1)} = - \Gamma_\alpha \Gamma^{(n+1)} + \Gamma^{(n+1)} \Gamma_\alpha. \tag{9.10b}\]

From (9.10) one can produce the following commutation relations
\[\Gamma^{(n)} \vec{D}_\sigma = \vec{D}_\sigma \Gamma^{(n)}, \quad \Gamma^{(n)} \vec{D}_\sigma = \vec{D}_\sigma \Gamma^{(n)}. \tag{9.11}\]

To close this Section, let us write down a Lagrange density for the system under consideration. In matrix approach that will be of the form
\[L = \frac{1}{2} (\vec{\Phi} \gamma^\alpha \vec{D}_\alpha \Phi - \vec{\Phi} \Gamma^\alpha \vec{D}_\alpha \Phi) - m \vec{\Phi} \Phi. \tag{9.12}\]
10. On canonical energy-momentum tensor

In Sec. 10 we are going to study a question about energy-momentum tensor for a generalized fermion. At this it is convenient to exploit a matrix formalism, when two basic equations are given by (for much generality, an external electromagnetic field will be taken into account)

\[
(\Gamma^\alpha \vec{D}_\alpha - M) \Phi = 0, \quad \Phi (\Gamma^\alpha \vec{D}_\alpha + M) = 0, \tag{10.1a}
\]

where

\[
\vec{D}_{\alpha} = \nabla_{\alpha} + B_{\alpha} - ig A_{\alpha}, \quad \vec{D}_{\alpha} = \nabla_{\alpha} - B_{\alpha} + ig A_{\alpha}, \quad g \equiv e/\hbar c. \tag{10.1b}
\]

Let us introduce a tensor quantity

\[
W^\alpha_\beta = \Phi (\Gamma^\alpha \vec{D}_\beta \Phi = \Phi (\Gamma^\alpha (\vec{D}_\beta + B_\beta) \Phi - ig A_\beta (\Psi \Gamma^\alpha \Psi) \tag{10.2}
\]

and calculate a divergence over it. To this end, acting on the first equation in (10.1a) from the left by \(\Psi \vec{D}_\beta\), and simultaneously multiplying the second equation in (10.1a) from the right by \(\vec{D}_\beta \Phi\), and summing the results, we get

\[
\Phi \vec{D}_\beta (\Gamma^\alpha \vec{D}_{\alpha} \Phi + \Phi \vec{D}_{\alpha} \Gamma^\alpha \vec{D}_\beta \Phi = 0, \tag{10.3a}
\]

and further

\[
\Phi \Gamma^\alpha [ (\vec{D}_{\beta} \vec{D}_{\alpha} - \vec{D}_{\alpha} \vec{D}_{\beta}) + \vec{D}_{\alpha} \vec{D}_\beta ] \Phi + \Phi \vec{D}_{\alpha} \Gamma^\alpha \vec{D}_\beta \Phi = 0. \tag{10.3b}
\]

Now, placing a term with commutator \([\vec{D}_{\beta}, \vec{D}_{\alpha}]_\_\) on the right, we get to

\[
\Phi (\Gamma^\alpha \vec{D}_\alpha + \vec{D}_\alpha \Gamma^\alpha ) \vec{D}_\beta \Phi = \Phi (\vec{D}_{\alpha} \vec{D}_{\alpha} \Gamma^\alpha \vec{D}_\beta \Phi =
\]

\[
= \Phi[\Gamma^\alpha (\vec{D}_\alpha + B_\alpha - ig A_\alpha) + (\vec{D}_\alpha - B_\alpha + ig A_\alpha)] \vec{D}_\beta \Phi =
\]

\[
= \Phi [ \Gamma^\alpha (\vec{D}_\alpha + B_\alpha - B_\alpha \Gamma^\alpha) + (\vec{D}_\alpha \Gamma^\alpha)] \vec{D}_\beta \Phi. \tag{10.4a}
\]

Taking into consideration eq. (7.7), relation (10.4a) can be transformed into

\[
\Phi (\Gamma^\alpha \vec{D}_\alpha + \vec{D}_\alpha \Gamma^\alpha ) \vec{D}_\beta \Phi = \Phi (\vec{D}_\alpha \Gamma^\alpha + \Gamma^\alpha \vec{D}_\alpha + \Gamma^\alpha \vec{D}_\alpha + \Gamma^\alpha \nabla_\alpha ) \vec{D}_\beta \Phi =
\]

\[
= \nabla_\alpha (\Phi \Gamma^\alpha \vec{D}_\beta \Phi). \tag{10.4b}
\]

Hence, eq. (10.3b) is equivalent to

\[
\nabla_\alpha (W^\alpha_\beta) = \Phi \Gamma^\alpha [\vec{D}_{\alpha}, \vec{D}_\beta]_\_ \Phi. \tag{10.5}
\]
Let us consider in more detail the commutator

$$[\vec{D}_\alpha, \vec{D}_\beta]_- = -igF_{\alpha\beta} + D_{\alpha\beta} ,$$  \hspace{1cm} (10.6a)

where

$$F_{\alpha\beta} = \frac{\partial A_\beta}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\beta} , \quad D_{\alpha\beta} = \frac{\partial B_\beta}{\partial x^\alpha} - \frac{\partial B_\alpha}{\partial x^\beta} + (B_\alpha B_\beta - B_\beta B_\alpha) .$$  \hspace{1cm} (10.6b)

For the first term in the expression for $D_{\alpha\beta}$, it is easily to obtain a representation

$$\partial_\alpha B_\beta - \partial_\beta B_\alpha = \nabla_\alpha B_\beta - \nabla_\beta B_\alpha =
\frac{1}{2} J^{ab} \nabla_\alpha ( e^{\nu}_{(a)} e^{(b)\nu;\beta} ) - \frac{1}{2} J^{ab} \nabla_\beta ( e^{\nu}_{(a)} e^{(b)\nu;\alpha} ) =
\frac{1}{2} J^{ab} e^{\nu}_{(a)} [ e^{(b)\nu;\alpha} - e^{(b)\nu;\beta} ] + \frac{1}{2} J^{ab} [ e^{(a)\nu;\alpha} e^{\nu}_{(b);\beta} - e^{(a)\nu;\beta} e^{\nu}_{(b);\alpha} ] .$$  \hspace{1cm} (10.7)

For second one it follows

$$(B_\alpha B_\beta - B_\beta B_\alpha) =
(\frac{1}{2} J^{ab} e^{\nu}_{(a)} e^{(b)\nu;\alpha} ) \left( \frac{1}{2} J^{kl} e^{\mu}_{(k)} e^{(l)\mu;\beta} \right) - \left( \frac{1}{2} J^{kl} e^{\mu}_{(k)} e^{(l)\mu;\alpha} \right) \left( \frac{1}{2} J^{ab} e^{\nu}_{(a)} e^{(b)\nu;\beta} \right) =
\frac{1}{4} (J^{ab} J^{kl} - J^{kl} J^{ab}) \left[ (e^{\nu}_{(a)} e^{(b)\nu;\alpha}) (e^{\mu}_{(k)} e^{(l)\mu;\beta}) \right] .$$

Now, with the use of the commutation relation (see (6.11))

$$[J^{ab}, J^{kl}]_- = (-J^{kb} g^{la} + J^{lb} g^{ka}) - (-J^{ka} g^{lb} + J^{la} g^{kb}) ,$$

we get

$$(B_\alpha B_\beta - B_\beta B_\alpha) = -\frac{1}{2} J^{ab} [ e^{(a)\nu;\alpha} e^{\nu}_{(b);\beta} - e^{(a)\nu;\beta} e^{\nu}_{(b);\alpha} ] .$$  \hspace{1cm} (10.8)

By accounting eqs. (10.7) and (10.8), the above expression for $D_{\alpha\beta}$ is led to

$$D_{\alpha\beta} = \frac{1}{2} J^{ab} e^{\nu}_{(a)} [ e^{(b)\nu;\beta;\alpha} - e^{(b)\nu;\alpha;\beta} ] =
\frac{1}{2} J^{ab} e^{\nu}_{(a)} R^\alpha_{\nu \beta \alpha}(x) \right] = \frac{1}{2} J^{\nu \rho}(x) R^\alpha_{\nu \rho \alpha}(x) ,$$  \hspace{1cm} (10.9)

where $R^\alpha_{\nu \rho \beta}(x)$ is the Riemann curvature tensor. So, eq. (10.5) reads as

$$\nabla_\alpha W_\beta \alpha = -ig J^\alpha F_{\alpha\beta} + \frac{1}{2} R^\alpha_{\nu \rho \beta} \bar{\Phi} \Gamma^{\alpha} J^{\nu \rho} \Phi .$$  \hspace{1cm} (10.10)

In is useful to perform some transformation over second term on the right:

$$\frac{1}{2} R^\alpha_{\nu \rho \beta} \bar{\Phi} \Gamma^{\alpha} J^{\nu \rho} \Phi =$$

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Now, with the use of eq. (10.13b), we readily produce
\[ \frac{1}{2} R_{\nu\rho\alpha\beta} \Phi \left( \frac{1}{2} [ \Gamma^{\alpha} J^\rho - J^\rho \Gamma^{\alpha} ] + \frac{1}{2} [ \Gamma^{\alpha} J^\rho + J^\rho \Gamma^{\alpha} ] \right) \Phi . \]

Further, with the use of commutation relation (see (7.6))
\[ \Gamma^\alpha J^\rho - J^\rho \Gamma^\alpha = g^{\alpha \nu}(x) \Gamma^\rho - g^{\alpha \nu}(x) \Gamma^\nu , \]
one can produce
\[ \frac{1}{2} R_{\nu\rho\alpha\beta} \Phi \Gamma^\rho \Phi = \]
\[ = \frac{1}{2} R_{\alpha\beta} J^\alpha + \frac{1}{4} R_{\nu\rho\alpha\beta} \Phi \left( \Gamma^\alpha J^\rho + J^\rho \Gamma^{\alpha} \right) \Phi . \] (10.11)

Correspondingly, eq. (10.10) takes on the form
\[ \nabla_{\alpha} \left[ W_\beta^{\alpha} \right] = J^\alpha \left[ -ig F_{\alpha\beta} + \frac{1}{2} R_{\alpha\beta} \right] + \frac{1}{4} R_{\nu\rho\alpha\beta} \Phi \left( \Gamma^\alpha J^\rho + J^\rho \Gamma^{\alpha} \right) \Phi , \] (10.12)
which generalizes a formula established by V.A. Fock ([49], eq. (56)) at studying a spin 1/2 particle on the background of a curved space-time model. A single formal difference consists in occurrence of one additional term proportional to the curvature tensor.

It may be checked that, in the established formula (10.12) being applied to an ordinary spin 1/2 particle, this additional \( R \)-dependent term will vanish identically. To this end, let us consider more closely a combination of the Dirac matrices
\[ (\gamma^\alpha \sigma^\nu + \sigma^\nu \gamma^\alpha) = \frac{1}{4} \left[ \gamma^\alpha (\gamma^\nu \gamma^\rho - \gamma^\rho \gamma^\nu) + (\gamma^\nu \gamma^\rho - \gamma^\rho \gamma^\nu) \gamma^\alpha \right] . \]

We need one auxiliary relation. To produce it, one should multiply the known formula for the product of three Dirac matrices [12]
\[ \gamma^a \gamma^b \gamma^c = \gamma^a g^{bc} - \gamma^b g^{ac} + \gamma^c g^{ab} + i \gamma^5 \epsilon^{abcd} \gamma_d \] (10.13a)
by a tetrad-based expression \( e^\alpha_{(a)} e^\beta_{(b)} e^\rho_{(c)} \), so that we get to
\[ \gamma^\alpha(x) \gamma^\beta(x) \gamma^\rho(x) = \left[ \gamma^\alpha(x) g^{\beta\rho}(x) - \gamma^\beta(x) g^{\alpha\rho}(x) + \right. \]
\[ + \gamma^\rho(x) g^{\alpha\beta}(x) + i \gamma^5 \epsilon^{\alpha\beta\rho\sigma}(x) \gamma_\sigma(x) \] . (10.13b)

Here, a generally covariant Levi-Civita symbol is defined by
\[ \epsilon^{\alpha\beta\rho\sigma}(x) = e^\alpha_{(a)} e^\beta_{(b)} e^\rho_{(c)} e^\sigma_{(d)} \epsilon^{abcd} . \] (10.13c)

Now, with the use of eq. (10.13b), we readily produce
\[ \gamma^\alpha(x) \sigma^\nu(x) + \sigma^\nu(x) \gamma^\alpha(x) = i \gamma^5 \epsilon^{\alpha\nu\rho\sigma}(x) \gamma_\sigma(x) . \] (10.14a)

Thus, the following relation
\[ \frac{1}{4} R_{\nu\rho\alpha\beta}(x) \Phi (\gamma^\alpha \sigma^\nu + \sigma^\nu \gamma^\alpha) \Phi = \frac{1}{4} R_{\nu\rho\alpha\beta}(x) \epsilon^{\alpha\nu\rho\sigma}(x) \Phi \gamma^5 \gamma_\sigma(x) \Phi \equiv 0 \] (10.14b)
holds; at deriving eq. (10.14b) we have taken into account the known symmetry property of the curvature tensor under cyclic permutation over any three indices so that a convolution $R^{...}$ with $\epsilon^{...}$ over three indices equals to zero.

Returning again to eq. (10.12a), and bearing in mind the properties of the $H$-matrix

$$H^{-1}[\Gamma^\alpha(x)]^+H = -\Gamma^\alpha(x), \quad H^{-1}[J^{\rho\nu}(x)]^+H = -J^{\rho\nu}(x), \quad (10.15)$$

one straightforwardly finds that on the right in (10.12a) the first and third terms are real-valued, whereas the second one is imaginary:

$$\nabla_\alpha W^\beta_\alpha = Re(x) + i Im(x), \quad Im(x) = -\frac{i}{2}J^\alpha R_{\alpha\beta},$$

$$Re(x) = -ig J^\alpha F_{\alpha\beta} + \frac{1}{4} R_{\nu\rho\alpha\beta} \Phi (\Gamma^\alpha J^{\rho\nu} + J^{\rho\nu} \Gamma^\alpha) \Phi . \quad (10.16)$$

Now, we need to isolate real and imaginary parts on the left in (10.16) too. With this aim in mind, one is to find a complex conjugate tensor $(W^\alpha_\beta)^*$:

$$(W^\alpha_\beta)^* = [\Phi^+ \eta \Gamma^\alpha (\nabla_\beta + G_\beta - ig A_\beta) \Phi ]^+ =$$

$$= -\Phi \Gamma^\alpha (\nabla_\beta - G_\beta + ig A_\beta) \Phi = -\Phi D_\beta \Gamma^\alpha \Phi . \quad (10.17)$$

With the notation

$$Re (W^\alpha_\beta) = \frac{1}{2} [W^\alpha_\beta + (W^\alpha_\beta)^+] = T^\alpha_\beta ,$$

$$Im (W^\alpha_\beta) = \frac{1}{2i} [W^\alpha_\beta - (W^\alpha_\beta)^+] = U^\alpha_\beta . \quad (10.18)$$

eq. (10.16) will split into two real-valued ones:

$$\nabla_\alpha (T^\alpha_\beta) = -ig J^\alpha (x) F_{\alpha\beta} + \frac{1}{4} R_{\nu\rho\alpha\beta} \Phi (\Gamma^\alpha J^{\rho\nu} + J^{\rho\nu} \Gamma^\alpha) \Phi , \quad (10.19)$$

$$\nabla_\alpha (U^\alpha_\beta) = -\frac{i}{2} J^\alpha R_{\alpha\beta} . \quad (10.20)$$

As readily checked, eq. (10.20) represents in essence a direct consequence of the conserved current law. Indeed, in accordance with definition for $U^\alpha_\beta$, we have

$$U^\alpha_\beta = \frac{1}{2i} \left[ \Phi \Gamma^\alpha (\partial_\beta + G_\beta - ig A_\beta) \Phi + \Phi (\bar{\partial}_\beta - G_\beta + ig A_\beta) \Gamma^\alpha \Phi \right] =$$

$$= \frac{1}{2i} \nabla_\beta (\bar{\Phi} \Gamma^\alpha \Phi) = \frac{1}{2i} \nabla_\beta J^\alpha . \quad (10.21)$$

Therefore, eq. (10.19) can be led to the form

$$\nabla_\alpha \nabla_\beta J^\alpha = R_{\alpha\beta} J^\alpha , \quad (10.22a)$$
and further
\[(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha)J^\alpha + \nabla_\beta J^\alpha = J^\alpha R_{\alpha\beta} .\]

From this, with bearing in mind the conservation law for \(J^\alpha\), it follows immediately an identity
\[J^\rho(x) R_{\rho}^{\ \alpha\beta}(x) \equiv J^\alpha(x) R_{\alpha\beta}(x) . \quad (10.22)\]

Thus, eq. (10.20) does not include anything new in addition to current conservation law. As for eq. (10.19), we have
\[T_\alpha^\beta = \frac{1}{2} [\Phi \Gamma^\alpha \Phi - \Phi \Gamma^\alpha \Phi] = \frac{1}{2} [\Phi \Gamma^\alpha (\nabla_\beta + G_\beta) \Phi - \Phi \Gamma^\alpha (\nabla_\beta - G_\beta) \Phi] - igJ^\alpha A_\beta , \quad (10.23a)\]

and a conservation law reads as follows
\[\nabla_\alpha(T_\beta^\alpha) = -ig J^\alpha F_{\alpha\beta} + \frac{1}{4} R_{\nu\rho\sigma\beta} \Phi (\Gamma^\sigma J^{\nu\rho} + J^{\nu\rho} \Gamma^\sigma) \Phi . \quad (10.23b)\]

To proceed with eq. (10.23b), it is a time to have remembered one property of the energy-momentum tensor in Minkowski space-time, which concerns its ambiguity in determination. In the Minkowski space-time, such a freedom is described as follows: if \(T_\beta^\alpha(x)\) obeys the conservation law
\[\partial_a T_\alpha^\beta = 0 ,\]
then another one
\[T_\beta^\alpha(x) = T_\beta^\alpha(x) + \partial_c [ \Omega_b^{[ac]}(x) ] , \quad \Omega_b^{[ac]}(x) = - \Omega_b^{[ca]}(x) \quad (10.24a)\]
satisfies the same conservation law as well
\[\partial_a T_\alpha^\beta = 0 .\]

Obviously, the simultaneous existence of the two tensors is insured by an elementary formula
\[\partial_a \partial_c \Omega_b^{[ac]}(x) \equiv 0 . \quad (10.24b)\]

In the case of a curved space-time such an equivalence between tensors \(T_\beta^\alpha(x)\) and \(\bar{T}_\beta^\alpha(x)\) holds as well, but in a more complicated manner. Indeed, let two tensors be related to each other by
\[\bar{T}_\beta^\alpha(x) = T_\beta^\alpha(x) + \nabla_\rho [ \Omega_\beta^{[\alpha\rho]}(x) ] . \quad (10.25)\]

Acting on both sides by operation of covariant derivative \(\nabla_\alpha\), one produces
\[\nabla_\alpha \bar{T}_\beta^\alpha(x) = \nabla_\alpha T_\beta^\alpha(x) + \nabla_\alpha [ \nabla_\rho \Omega_\beta^{[\alpha\rho]}(x) ] . \quad (10.26)\]

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Now, bearing in mind symmetry properties of the curvature tensor, one obtains
\[ \nabla_\alpha \left[ \nabla_\rho \Omega_\beta^{[\alpha\rho]}(x) \right] = \frac{1}{2} \left[ R_{\alpha\rho\beta}^\sigma \Omega_\sigma^{[\alpha\rho]} + R_{\alpha\rho}^\sigma \Omega_\beta^{[\sigma\rho]} + R_{\alpha\rho}^\rho \Omega_\sigma^{[\alpha\sigma]} \right] = \]
\[ = \frac{1}{2} \left[ R_{\beta\sigma}^\alpha \Omega_\sigma^{[\alpha\rho]} - R_{\rho\sigma}^\sigma \Omega_\beta^{[\sigma\rho]} - R_{\alpha\sigma}^\rho \Omega_\beta^{[\alpha\sigma]} \right], \]
therefore
\[ \nabla_\alpha \left[ \nabla_\rho \Omega_\beta^{[\alpha\rho]}(x) \right] = \frac{1}{2} R_{\beta\sigma\nu\rho} \Omega_\sigma^{[\nu\rho]} . \quad (10.27) \]
Thus, eq. (10.26) reads as
\[ \nabla_\alpha T_\beta^{\alpha}(x) = \nabla_\alpha T_\beta^{\alpha}(x) + \frac{1}{2} R_{\beta\sigma\nu\rho}(x) \Omega_\sigma^{[\nu\rho]}(x) , \quad (10.28) \]
which on accounting eq. (10.23b) takes the form
\[ \nabla_\alpha T_\beta^{\alpha}(x) = -igJ^\alpha F_\alpha^\beta + \frac{1}{4} R_{\nu\rho\sigma\beta}(x) \Phi (\Gamma^\sigma J^\nu^\rho + J^\nu^\rho\Gamma^\sigma) \Phi + \]
\[ + \frac{1}{2} R_{\beta\sigma\nu\rho}(x) \Omega_\sigma^{[\nu\rho]}(x) . \quad (10.29a) \]
If the quantity \( \Omega_\sigma^{[\nu\rho]}(x) \) is hosen as
\[ \Omega_\sigma^{[\nu\rho]}(x) = +\frac{1}{2} \Phi (\Gamma^\sigma J^\nu^\rho + J^\nu^\rho\Gamma^\sigma) \Phi , \]
then second and third terms on the right in (10.29a) cancel each other, and we reach a conservation law in the form
\[ \nabla_\alpha \bar{T}_\beta^{\alpha}(x) = -igJ^\alpha(x) F_\alpha^\beta(x) . \quad (10.29b) \]

Supplement A. Ricci tensor and conformal transformation

The Ricci tensor is expressed in terms of Kristofell symbols
\[ R_{\alpha\beta} = \partial_\rho \Gamma^\rho_{\alpha\beta} - \partial_\beta \Gamma^\rho_{\alpha\rho} + \Gamma^\rho_{\alpha\beta} \Gamma^\sigma_{\rho\sigma} - \Gamma^\sigma_{\alpha\rho} \Gamma^\rho_{\sigma\beta} . \quad (A.1) \]
Let metric tensors of two space-time models differ by a factor
\[ dS^2 = g_{\alpha\beta}dx^\alpha dx^\beta , \quad d\tilde{S}^2 = \tilde{g}_{\alpha\beta}dx^\alpha dx^\beta , \]
\[ \tilde{g}_{\alpha\beta}(x) = \varphi^2(x) g_{\alpha\beta}(x) . \quad (A.2) \]
Comparing two set Kristoffel symbols
\[ \tilde{\Gamma}_{\alpha\beta\rho}(x) = \frac{1}{2} \left[ \partial_\alpha \tilde{g}_{\beta\rho} + \partial_\beta \tilde{g}_{\alpha\rho} - \partial_\rho \tilde{g}_{\alpha\beta} \right] = \frac{1}{2} \left[ \partial_\alpha \varphi^2 g_{\beta\rho} + \partial_\beta \varphi^2 g_{\alpha\rho} - \partial_\rho \varphi^2 g_{\alpha\beta} \right] = \]
\[ \varphi^2 \Gamma_{\alpha \beta, \rho} + \varphi \left[ (\partial_\alpha \varphi) g_{\beta \rho} + (\partial_\beta \varphi) g_{\alpha \rho} - (\partial_\rho \varphi) g_{\alpha \beta} \right], \]

or

\[ \tilde{\Gamma}_{\alpha \beta}^\rho(x) = g^{\sigma \rho} \Gamma_{\alpha \beta, \rho} \varphi^2 \left\{ \varphi^2 \Gamma_{\alpha \beta, \rho} + \varphi \left[ (\partial_\alpha \varphi) g_{\beta \rho} + (\partial_\beta \varphi) g_{\alpha \rho} - (\partial_\rho \varphi) g_{\alpha \beta} \right] \right\}. \]

Thus we arrive at

\[ \tilde{\Gamma}_{\alpha \beta}^\rho = \Gamma_{\alpha \beta}^\rho + \frac{1}{\varphi^2} \left[ \varphi^2 \Gamma_{\alpha \beta, \rho} + \varphi \left[ (\partial_\alpha \varphi) g_{\beta \rho} + (\partial_\beta \varphi) g_{\alpha \rho} - (\partial_\rho \varphi) g_{\alpha \beta} \right] \right]. \] (A.3)

Now let us compare the Ricci tensors

\[ \partial_\rho \tilde{\Gamma}_{\alpha \beta}^\rho = \partial_\rho \Gamma_{\alpha \beta}^\rho - \frac{1}{\varphi^2} (\partial_\rho \varphi) \left[ \delta_\alpha^\rho \partial_\beta \varphi + \delta_\beta^\rho \partial_\alpha \varphi - g_{\alpha \beta} g^{\rho \sigma} \partial_\sigma \varphi \right] + \frac{1}{\varphi} \left[ 2 \partial_\alpha \partial_\beta \varphi - g_{\alpha \beta} g^{\rho \sigma} \partial_\rho \partial_\sigma \varphi - (\partial_\rho g_{\alpha \beta}) g^{\rho \sigma} \partial_\sigma \varphi - g_{\alpha \beta} \left( \partial_\rho g^{\rho \sigma} \right) \partial_\sigma \varphi \right] \]

and further

\[ \partial_\rho \tilde{\Gamma}_{\alpha \beta}^\rho = \partial_\rho \Gamma_{\alpha \beta}^\rho - \frac{1}{\varphi^2} (\partial_\rho \varphi) \left[ 2 \partial_\alpha \partial_\beta \varphi - g_{\alpha \beta} g^{\rho \sigma} \partial_\rho \partial_\sigma \varphi \right] + \frac{1}{\varphi} \left[ 2 \partial_\alpha \partial_\beta \varphi - g_{\alpha \beta} g^{\rho \sigma} \partial_\rho \partial_\sigma \varphi - (\partial_\rho g_{\alpha \beta}) g^{\rho \sigma} \partial_\sigma \varphi - g_{\alpha \beta} \left( \partial_\rho g^{\rho \sigma} \right) \partial_\sigma \varphi \right] - \frac{1}{\varphi} \left[ (\partial_\rho g_{\alpha \beta}) g^{\rho \sigma} + g_{\alpha \beta} (\partial_\rho g^{\rho \sigma}) \partial_\sigma \varphi \right]. \] (A.4)

Taking into account

\[ -\partial_\beta \tilde{\Gamma}_{\alpha \rho}^\sigma = -\partial_\beta (\Gamma_{\alpha \rho}^\sigma + \frac{4}{\varphi} \partial_\alpha \varphi) = -\partial_\beta \Gamma_{\alpha \rho}^\sigma + \frac{4}{\varphi^2} (\partial_\alpha \varphi)(\partial_\beta \varphi) - \frac{4}{\varphi} \partial_\alpha \partial_\beta \varphi. \] (A.5)

we get

\[ \tilde{\Gamma}_{\alpha \beta}^\rho \tilde{\Gamma}_{\rho \sigma}^\alpha = \left[ \Gamma_{\alpha \beta}^\rho + \frac{1}{\varphi} \left( \delta_\alpha^\rho \partial_\beta \varphi + \delta_\beta^\rho \partial_\alpha \varphi - g_{\alpha \beta} g^{\rho \sigma} \partial_\sigma \varphi \right) \right] \left( \Gamma_{\rho \sigma}^\alpha + \frac{4}{\varphi^2} \partial_\rho \varphi \right) - \frac{4}{\varphi^2} (\partial_\alpha \varphi)(\partial_\beta \varphi) + g_{\alpha \beta} g^{\rho \sigma} \left( \partial_\rho \varphi \right) \left( \partial_\sigma \varphi \right). \] (A.6)

Now

\[ -\tilde{\Gamma}_{\alpha \rho}^\sigma \tilde{\Gamma}_{\sigma \beta}^\rho = -\Gamma_{\alpha \rho}^\sigma \Gamma_{\sigma \beta}^\rho - \frac{4}{\varphi^2} (\partial_\alpha \varphi)(\partial_\beta \varphi) + g_{\alpha \beta} g^{\rho \sigma} \left( \partial_\rho \varphi \right) \left( \partial_\sigma \varphi \right). \]
\[-\frac{1}{\varphi^2} \left[ 6(\partial_\alpha \varphi)(\partial_\beta \varphi) - 2g_{\alpha\beta} g^{\rho\sigma} (\partial_\rho \varphi)(\partial_\sigma \varphi) - \frac{1}{\varphi} \left( (\partial_\beta \varphi) \Gamma^\sigma_{\alpha\sigma} + (\partial_\alpha \varphi) \Gamma^\sigma_{\beta\sigma} \right) \right] \]

\[+ 2(\partial_\rho \varphi) \Gamma^{\rho}_{\alpha\beta} - (\partial_\gamma \varphi) g^{\rho\sigma} \Gamma^\rho_{\rho\sigma} g_{\sigma\beta} - (\partial_\gamma \varphi) g^{\gamma\rho} \Gamma^\rho_{\rho\beta} g_{\sigma\alpha} \]  
(A.7)

Allowing for eqs. (A.4)-(A.7), we get to

\[\tilde{R}_{\alpha\beta} = R_{\alpha\beta} \left[ 6(\partial_\alpha \varphi)(\partial_\beta \varphi) - 2g_{\alpha\beta} g^{\rho\sigma} (\partial_\rho \varphi)(\partial_\sigma \varphi) - \frac{1}{\varphi} \left( (\partial_\beta \varphi) \Gamma^\sigma_{\alpha\sigma} + (\partial_\alpha \varphi) \Gamma^\sigma_{\beta\sigma} \right) \right] + \frac{1}{\varphi^2} \left[ 4(\partial_\alpha \varphi)(\partial_\beta \varphi) - g_{\alpha\beta} g^{\rho\sigma} (\partial_\rho \varphi)(\partial_\sigma \varphi) \right] + \frac{1}{\varphi} \left[ (\partial_\sigma \varphi) g^{\rho\sigma} \partial_\rho \varphi - g_{\alpha\beta} (\partial_\rho \varphi) g^{\rho\sigma} \partial_\sigma \varphi + 2(\partial_\rho \varphi) \Gamma^{\rho}_{\alpha\beta} \right. \]

\[\left. - g_{\alpha\beta} \Gamma^\sigma_{\rho\sigma} g^{\rho\sigma} \partial_\gamma \varphi + (\partial_\gamma \varphi) g^{\gamma\rho} \Gamma^\rho_{\rho\sigma} g_{\sigma\beta} + (\partial_\gamma \varphi) g^{\gamma\rho} \Gamma^\rho_{\rho\beta} g_{\sigma\alpha} \right] \]

(A.8)

Transforming here first two terms

\[-(\partial_\sigma \varphi) g^{\rho\sigma} (\partial_\rho g_{\alpha\beta}) = (\partial_\sigma \varphi) g^{\rho\sigma} \left[ \Gamma^\sigma_{\alpha\beta} + \Gamma^\sigma_{\beta\alpha} \right] = \]

\[= (\partial_\sigma \varphi) g^{\rho\sigma} \left[ g_{\alpha\gamma} \Gamma^\gamma_{\beta\rho} + g_{\beta\gamma} \Gamma^\gamma_{\alpha\rho} \right] = \]

\[= -(\partial_\sigma \varphi) g^{\rho\sigma} \Gamma^\gamma_{\beta\rho} g_{\gamma\alpha} - (\partial_\sigma \varphi) g^{\rho\sigma} \Gamma^\rho_{\rho\gamma} g_{\gamma\beta} , \]

\[-g_{\alpha\beta} (\partial_\sigma \varphi) (\partial_\rho g^{\rho\sigma}) = -g_{\alpha\beta} (\partial_\sigma \varphi) \left[ -\Gamma^\rho_{\rho\gamma} g^{\gamma\sigma} - \Gamma^\rho_{\rho\gamma} g^{\rho\gamma} \right] = \]

\[= +g_{\alpha\beta} (\partial_\sigma \varphi) \Gamma^\rho_{\rho\gamma} g^{\rho\gamma} + g_{\alpha\beta} (\partial_\sigma \varphi) \Gamma^\rho_{\rho\gamma} g^{\rho\gamma} . \]

whith which eq. (A.8) gives

\[\tilde{R}_{\alpha\beta} = R_{\alpha\beta} + \frac{1}{\varphi} \left( -2\partial_\alpha \partial_\beta \varphi - g_{\alpha\beta} g^{\rho\sigma} \partial_\rho \partial_\sigma \varphi \right) + \]

\[+ \frac{1}{\varphi^2} \left[ 4(\partial_\alpha \varphi)(\partial_\beta \varphi) - g_{\alpha\beta} g^{\rho\sigma} (\partial_\rho \varphi)(\partial_\sigma \varphi) \right] + \frac{1}{\varphi} \left( \partial_\sigma \varphi \right) \left( 2\Gamma^\sigma_{\alpha\beta} + g_{\alpha\beta} \Gamma^\sigma_{\rho\beta} g^{\rho\gamma} \right) . \]

(A.9)

From eq. (A.9) it follows the transformation law:

\[\tilde{R} = \frac{1}{\varphi^2} \left[ R - \frac{6}{\varphi} g^{\alpha\beta} \left[ \partial_\alpha (\partial_\beta \varphi) - \Gamma^\sigma_{\alpha\beta} (\partial_\sigma \varphi) \right] \right] = \frac{1}{\varphi^2} \left[ R - \frac{6}{\varphi} \nabla^\beta \nabla_\beta \varphi \right] \]

(A.10)
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