Quantum metrology in the presence of spatially correlated noise:
Restoring Heisenberg scaling

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Environmental noise can hinder the metrological capabilities of entangled states. While the use of entanglement allows for Heisenberg-limited resolution, the largest permitted by quantum mechanics, deviations from strictly unitary dynamics quickly restore the standard scaling dictated by the central limit theorem. Product and maximally entangled states become asymptotically equivalent when the noisy evolution is both local and strictly Markovian. However, temporal correlations in the noise have been shown to lift this equivalence while fully (spatially) correlated noise allows for the identification of decoherence free subspaces. Here we analyze precision limits in the presence of noise with finite correlation length and show that there exist robust entangled state preparations which display persistent Heisenberg scaling despite the environmental decoherence, even for small correlation length. Our results emphasize the relevance of noise correlations in the study of quantum advantage and could be relevant beyond metrological applications.

How small can experimental error bars become? The use of entangled states in quantum metrology shows that the answer of this fundamental question can rely on playing with \textit{quantum advantage}, that is, exploiting a quantitative (quantum) resource to outperform classical strategies. In a typical metrological set up, an atomic transition frequency is estimated from the phase relation accumulated between the two components in a superposition state. In purely unitary evolution, the relative phase of a Greenberger-Horne-Zeilinger (GHZ) state of the form $(|0\rangle^\otimes n + |1\rangle^\otimes n)/\sqrt{2}$ advances $n$ times faster than that of a single qubit and parity measurements allow one to saturate the Heisenberg limit [1, 2]. That is, their associated measurement uncertainty decreases as $1/n$, which provides a $1/\sqrt{n}$ improvement over the standard quantum limit (SQL) obtained by performing $n$ independent queries on uncorrelated particles. Signal-to-noise ratios (SNR) overcoming the spectroscopic resolution achievable in ideal experiments using single qubits have been demonstrated using three entangled ions [3]. When the dimension of the probe state grows, decoherence effects will no longer be negligible and the question arises of whether Heisenberg scaling can still be attainable under a non unitary (noisy) evolution. In the case of the dynamics being strictly Markovian, and provided the noise is local, both pure dephasing and dissipative losses restore the standard scaling dictated by the central limit theorem even in the limit of arbitrarily small noise levels [4]. Rigorous bounds showing standard scaling under this type of noise have been recently put forward [5, 6].

These noise models though make two important assumptions, namely, that the noise stems from a Markovian bath and that this noise acts locally on each subsystem. Relaxing the assumption of Markovianity has been shown to result in a new fundamental limit [7] which lifts the previous metrological equivalence of maximally entangled and product states under time-correlated noise and predicts a novel scaling of the form $\sim 1/n^{3/4}$. In this paper we relax the assumption of noise locality and consider a general model for bath correlation length [8] to show the persistence of Heisenberg scaling under correlated noise of finite length.

In the presence of Markovian dephasing noise, frequency measurements on entangled states generally yield probabilities for population measurements of the form $p = \frac{1}{2} \left[ 1 + \cos(n\omega_0 t) \exp(\Gamma(n, \xi) t) \right]/2$, where $\omega_0$ is the atomic transition frequency and $\Gamma(n, \xi)$ the dephasing rate of the state, dependent on the number $n$ of considered qubits and the spatial correlation length $\xi$ of the environmental noise. Considering as fixed resources the total number of particles $n$ and the total duration of the experiment $T$, the duration of the optimal interrogation time $t$, and with it the number of repetitions $T/t$ for a given noise source has to be determined [4]. This procedure yields the time-optimized frequency uncertainty of the measured frequency

$$\Delta \omega_0 = \sqrt{\frac{2 e[\Gamma(n, \xi)]}{n^2 T}}. \quad (1)$$

For spatially uncorrelated Markovian decoherence, the dephasing rate of GHZ states scales as $\Gamma(n, \xi) \rightarrow \Gamma_{uc} = n\gamma_{uc}$, where $\gamma_{uc}$ is the dephasing rate of a single qubit superposition state. This yields a resolution $\sim 1/\sqrt{n}$. The persistence of the standard scaling under Markovian decoherence is actually valid for optimized (entangled) initial states and generalized measurements, with the optimal achievable resolution $\Delta \omega_0^{opt} = \sqrt{2\gamma_{uc}/(nT)}$ [4, 5].

While the derivation above relies on an independent noise model [9, 10], recent experiments with trapped ions have proven to be dominated by spatially correlated dephasing [11, 12]. Particularly, measurements of the dephasing rate of GHZ states have shown a clear $n^2$ de-
pendence [12] which can only be explained by strongly correlated noise. When the noise acts globally on all qubits, it is possible to identify suitable decoherence free subspaces (DFS). This allows for the accurate determination of frequency shifts, as illustrated in [13] by using an entangled state of two ions for the determination of the quadrupole moment of $^{40}\text{Ca}^+$, a quantity of relevance for the calibration of optical frequency standards [14]. It remains unclear, however, what the situation would be when considering larger qubit arrays so that the noise exhibits a correlation length that is smaller than the total length of the system. Using a formalism to consider realistic partially correlated noise, where the correlations decay over a certain correlation length $\xi$ [8], we will show that, even for small $\xi$, Heisenberg scaling prevails when using certain types of entangled states for the estimation of small frequency shifts, as those involved in the precise estimation of an atomic quadrupole moment.

We consider a system of $n$ hydrogen-like ions with a Zeeman splitting term of the sublevels $J_0$ of the total angular momentum and a small correction term due to the interaction of the atomic electric quadrupole moment with the external electric field gradient. These correction terms are quadratic in $J_z$ [13, 15].

$$H_\varphi = \beta \sum_{j=1}^{n} J_z^{(j)} + \alpha \sum_{j=1}^{n} (J_z^{(j)})^2$$

Laser frequency noise and magnetic field noise make dephasing by far the strongest decoherence source, effectively coupling each ion to a fluctuation via $J_z$, $H_{int} = \nu \sum J_z^{(j)} B_j$. The coupling strength $\nu$ defines the total decoherence strength by the coefficient $\gamma_0 = \nu^2$ in all dephasing rates, so for simplicity we set $\nu = 1$. The bath operators’ $B_j$ spatial and temporal correlations are determined by the function $C(\omega, x_j - x_k) = \int_{-\infty}^{\infty} e^{i\omega t} \langle B_j(t, x_j) B_k(0, x_k) \rangle$, where we assume the ions are spatially arranged in a linear array (figure 1). We employ Bloch-Redfield equations with a Markovian approximation and assume homogeneous, decaying spatial correlations $C(0, x_d) = \exp(-|xd|/\xi)$ with the correlation length $\xi$, the distance $d$ between ions in the one-dimensional array and $x \in \mathbb{N}$. We arrive at the master equation for the system density matrix $\rho$ as [8]:

$$\dot{\rho} = i\hbar [\rho, H_\varphi] + \frac{1}{\hbar^2} 2 \sum_{j,k} \exp \left( -\frac{|x_j - x_k|d}{\xi} \right) \left( 2J_z^{(j)} \rho J_z^{(k)} - J_z^{(j)} J_z^{(k)} \rho - \rho J_z^{(j)} J_z^{(k)} \right)$$

As initial states we consider maximally entangled states of the form $(|m_1, m_2, \ldots, m_n\rangle + |m_{n+1}, \ldots, m_{2n}\rangle)/\sqrt{2}$ where the magnetic quantum numbers $m_j$ of the operators $J_z^{(j)}$ satisfy $\sum_{j=1}^{n} m_j = 0$. The two parts of the superposition are Zeeman-shifted by the same amount but their quadrupole moment can be different. We restrict ourselves to three sublevels in this paper: $m \in \{+1, 0, -1\}$ where $+1 = \epsilon_0 + \epsilon_\Delta$ and $-1 = \epsilon_0 - \epsilon_\Delta$. For example in ref. [13] the levels $m \in \{3/2, -1/2, -5/2\}$ in $^{40}\text{Ca}^+$ ions were used. We choose for one part of the initial entangled state all ions to be in $\epsilon_0$ and in the other part half of the ions in $\epsilon_+$ and half in $\epsilon_-$, i.e. $\langle \epsilon_+, \epsilon_+, \epsilon_-, \epsilon_-, \ldots \rangle + |\epsilon_0, \epsilon_0, \epsilon_0, \ldots \rangle).$ This simplifies $J_z = \text{diag}(\epsilon_+, \epsilon_0, \epsilon_-)$ and the coherent evolution is given by $|\Psi(t)\rangle = (|\epsilon_+, \epsilon_+, \epsilon_-, \epsilon_-, \ldots \rangle + \exp(i\omega_0 t) |\epsilon_0, \epsilon_0, \epsilon_0, \ldots \rangle))\sqrt{2}$ where the relative frequency is given by the quadrupole splitting $\omega_0 = \alpha \xi^2$. This frequency is measured with a parity measurement and we will regard the uncertainty scaling with $n$ of this transition frequency.

Uncorrelated Markovian decoherence ($\xi \to 0$) always restores the standard quantum limit [4], whereas for the chosen states correlated decoherence leads to Heisenberg scaling: In the limit of infinite correlation length $\xi \to \infty$ one can define a collective operator $S = \sum\langle J_z^{(j)} \rangle$ in Eq. 3 which restores coherent evolution for the density matrix element of interest $|\epsilon_+\epsilon_+\epsilon_+\epsilon_+\epsilon_-\epsilon_-\ldots\rangle\langle\epsilon_0\epsilon_0\epsilon_0\epsilon_0\ldots\rangle$, because $S$ acting from the left or from the right on it is equivalent. This decoherence-free subspace of all states with equal excitation number [8] guarantees the coherent Heisenberg scaling.

We will now discuss the persistence of Heisenberg scaling under partially correlated noise and assume a finite but non-vanishing correlation length $\xi > d$. The time-evolution of the coherence of interest $\rho_{\pm, 0}$ is given by:

$$\rho_{\pm, 0}(t) = \rho_{\pm, 0}(0) \exp(-i\omega_0 t + \Gamma(n, \xi)t)$$

The dephasing rate $\Gamma(n, \xi)$ depends on the number of ions $n$ involved in the superposition and the correlation length $\xi$. In stark contrast to uncorrelated decoherence, it also depends on the order of the ions in the initial state. In Eq. 3 all pairs of ions contribute terms to $\Gamma(n, \xi)$. Both autocorrelations and cross-correlations of those pairs which are in the same state (i.e. both $\epsilon_+$ or both $\epsilon_-$) contribute negatively to $\Gamma(n, \xi)$ and increase the dephasing rate. Those pairs which are in opposite states compensate with a positive contribution. Since the correlations decrease with distance, a state is more robust the closer mixed pairs are in space. We
The distance $d$ between ions is fixed and the array becomes longer with increasing $n$. We therefore reorder the ions to give a new initial state $(|\epsilon_+\rangle + |\epsilon_-\rangle + |\epsilon_0\rangle)/\sqrt{2}$. The dephasing rate corresponding to this state is given by (see appendix):

$$
\Gamma(n, \xi) = -\frac{n \Delta^2}{2} + \sum_{x=1}^{n/2} (n - 2x + 1) C(0, (2x - 1)d) \epsilon^2 \Delta + \sum_{x=1}^{n/2} (2x - n) C(0, 2xd) \epsilon^2 \Delta.
$$

With the assumed correlation function $C(0, xd) = \exp(-|x|d/\xi)$ one finds:

$$
\Gamma(n, \xi) = -\frac{2e^{\xi} + 2e^{\frac{1-n}{\xi}} + n - e^{2\xi} n^2}{2 \left(1 + e^{\xi}\right)^2} \epsilon^2 \Delta.
$$

To judge whether entangled states give an advantage over the standard quantum limit ($\propto 1/\sqrt{n}$) we need to determine whether the dephasing rate $\Gamma(n, \xi)$ scales faster or slower than $n$. For comparison, the dephasing rate for uncorrelated decoherence is $\Gamma_{uc} = -n \epsilon^2 \Delta^2/2$ (see appendix or [4]).

The frequency $\omega_0 = \alpha \epsilon^2 \Delta^2$ cannot be measured in a single ion; one needs at least an entangled state of two ions to realize it. To obtain a meaningful comparison for the scaling of the frequency uncertainty this must be taken into account. We therefore define two entangled ions as the minimum entanglement resource for measuring a quadrupole moment. We then compare the scaling of the $n$-entangled state with a product state of $n/2$ entangled pairs which contribute $n/2$ more measurements to the statistics. For this minimal entangled array of ion pairs, we find an uncertainty that scales with the SQL as we increase the number $n/2$ of pairs, $\Delta \omega_{0,p} = \sqrt{\epsilon^2 \Delta^2/(4nT)}$.

In contrast to uncorrelated decoherence, there is no unique way of increasing the number of entangled ions for a given noise correlation length. When considering the scaling of the dephasing rate with increasing numbers of ions, one can either keep the length $L$ of the ion array fixed, or the density of ions fixed. We now analyze the achievable spectroscopic resolution in both cases (figure 1).

First we set the correlation length to a fixed number of ions $\xi = cd$ (figure 1A) which means that the array gets longer relative to the correlation length as we increase $n$ (fixed ion density). This will ultimately restore the SQL when $nd \gg \xi$ (figure 2). The gradient of the dephasing rate in this case can be approximated as $\epsilon^2 \Delta^2 d/(4\xi)$ for $\xi > d$. This gradient is closer to zero than for uncorrelated decoherence $\Gamma_{uc}$ because the finite correlation length reduces the dephasing-rate contribution from each ion slightly. So even though the scaling follows the standard quantum limit, one finds a better coefficient than for uncorrelated decoherence.

Alternatively, we can scale the correlation length as a fraction $c$ of the whole array $\xi = cL$ (figure 1B), fixing the correlations between the first ion and the last ion in the array to a value $C(0, L) = \exp(-1/c)$. Figure 2 shows that in this case the dephasing rate quickly approaches the constant $\Gamma(n, \xi) = [-L/\xi] + \exp(-L/\xi) - 1] \epsilon^2 \Delta^4/4$, which can be approximated as $\epsilon^2 \Delta^2 L/(2\xi)$ for long correlation lengths $\xi \gg L$. With this constant rate the corresponding uncertainty (Eq. 1) displays Heisenberg scaling (figure 3):

$$
\Delta \omega_0 \to \sqrt{\frac{2e^{-L/\xi} + \exp(-L/\xi) - 1] \epsilon^2 \Delta^2}{4T}} \frac{1}{n}.
$$

We now introduce the relative frequency resolution:

$$
r = \frac{\Delta \omega_{0,p}}{\Delta \omega_0},
$$

where the full expression for $r$ is given by equations 1, 6 and 8. We find that with increasing correlation length $\xi$ the uncertainty approaches the noiseless Heisenberg scaling (figure 3). Even for partial correlations, which decay on the length scale of the array, the Heisenberg scaling of the uncertainty is robust.

The previous observations have more general applications beyond quadrupole measurements. Heisenberg scaling in spatially correlated environments can generally be
achieved for entangled states which are superpositions of states with the same number of excitations. A counterexample however are GHZ states. In a noiseless environment a frequency measurement will show Heisenberg resolution becomes insensitive to the environmental spatial correlation length $L$ matches the oscillation length of the environmental spatial correlations $L = 2\pi/k$ (see appendix). Note that uniformly negative correlation functions are impossible due to the requirements of positive autocorrelations and multipartite correlation rules.

Previous experimental evidence suggests that the environmental noise in ion traps is spatially correlated with purely positive correlations. We showed that non-zero spatial correlation length fundamentally changes the decoherence of entangled states. In such environments, a topology dependence emerges so that the order in which the ions are placed in the array changes their decoherence properties. After optimisation in this regard, the entangled states designed to measure the electric quadrupole moment have an approximately constant dephasing rate with increasing number of ions $n$. Precision frequency measurements with these initial states therefore show Heisenberg scaling of the uncertainty $\Delta \omega_0 \propto 1/n$ with the numbers of ions $n$. Besides providing a prescription to achieve Heisenberg-scaled resolution in linear ion traps subject to partially correlated noise, our results illustrate the fundamental role of noise correlations in precision spectroscopy. While local Markovian noise eliminates quantum advantage, this is restored when the noise displays a spatial structure. Heisenberg resolution becomes then attainable by means of suitable state preparation whose decoherence rate decreases inversely with $\xi$ so that the evolution is decoherence-free in the limit of infinite correlation length (global noise).

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Appendix A Perfect correlations yield a decoherence-free subspace

For perfect correlations, i.e. infinite correlation length $\xi \to \infty$ the spatial correlation function becomes $\exp(x/\xi) \to 1$. The master equation becomes:

$$\dot{\rho} = \frac{i}{\hbar} [\rho, H_s] + \frac{1}{\hbar^2} \frac{1}{2} \sum_{j,k} \left( 2J^{(j)}_z \rho J^{(k)}_z - J^{(j)}_z J^{(k)}_z \rho - \rho J^{(j)}_z J^{(k)}_z \right)$$ (9)

Taking the sums into each term we define a new Hermitian operator $S = \sum_j J^{(j)}_z$ with $S^\dagger = S$. We therefore find the equations to be of the simple Lindblad form:

$$\dot{\rho} = \frac{i}{\hbar} [\rho, H_s] + \frac{1}{\hbar^2} \left( S \rho S - \frac{1}{2} \{S^2, \rho\} \right)$$ (10)

The Hermitian operator $S$ yields the same value for all states with the same number of excitations. Particularly for the two constituents of our initial state we find:

$$S |\epsilon_+, \epsilon_+, \ldots, \epsilon_-, \epsilon_- \rangle = S |\epsilon_0, \epsilon_0, \ldots \rangle$$ (11)

In other words for their subspace $S \propto \mathbb{1}$ and the master equation becomes:

$$\dot{\rho} = \frac{i}{\hbar} [\rho, H_s] + \frac{1}{\hbar^2} \left( \mathbb{1} \rho \mathbb{1} - \frac{1}{2} \{\mathbb{1}^2, \rho\} \right)$$ (12)

$$= \frac{i}{\hbar} [\rho, H_s] + 0$$ (13)

This means our entangled initial state is in a decoherence-free subspace (for a perfectly correlated bath). Its time evolution is given by:

$$|\Psi(t)\rangle = |\epsilon_+, \epsilon_+, \ldots, \epsilon_-, \epsilon_- \rangle + e^{i\omega_0 t} |\epsilon_0, \epsilon_0, \ldots \rangle$$ (14)

Since eq. 11 holds for all pairs of states with equal excitation number we find that the order of ions is irrelevant for perfect correlations.

Appendix B Partial correlations and its dephasing rate for $n$ ions

For finite but non-vanishing correlation length $\xi$ we solve the master equation:

$$\dot{\rho} = \frac{i}{\hbar} [\rho, H_s] + \frac{1}{\hbar^2} \frac{1}{2} \sum_{j,k} \exp \left( -\frac{|x_j - x_k|}{\xi} \right) \left( 2J^{(j)}_z \rho J^{(k)}_z - J^{(j)}_z J^{(k)}_z \rho - \rho J^{(j)}_z J^{(k)}_z \right)$$ (15)
For the last equation, note that and the left one once. We distinguish three cases: seen by moving a fixed distance along figure 4. For regard the coefficients of 

The coherent part is easily calculated:

\[ \rho_{\pm,0} = |\epsilon_-, \epsilon_+, \epsilon_+, ...\rangle \langle \epsilon_0, \epsilon_0, ..., \epsilon_0| \]  

(16)

The coherent part is easily calculated:

\[ i(\rho_{\pm,0}H_z - H_z\rho_{\pm,0}) = i(\beta n \epsilon_0 + \alpha n \epsilon_0^2 - \beta n \epsilon_0 - \alpha n (\epsilon_0^2 + \epsilon_\Delta^2))\rho_{\pm,0} = -i n \alpha \epsilon_\Delta^2 \rho_{\pm,0} = -i n \omega_0 \rho_{\pm,0} \]  

(17)

To calculate the decoherent part of eq. 15 for the element \( \rho_{\pm,0} \) we regard how \( J_z^{(j)} \) acts from the left and from the right onto element \( \rho_{\pm,0} \) (figure 4):

\[ J_z^{(j)} \rho_{\pm,0} = \begin{cases} \epsilon_- \rho_{\pm,0} = (\epsilon_0 - \epsilon_\Delta)\rho_{\pm,0} & \text{if } j \text{ is odd} \\
\epsilon_+ \rho_{\pm,0} = (\epsilon_0 + \epsilon_\Delta)\rho_{\pm,0} & \text{if } j \text{ is even} \end{cases} \]  

(18)

\[ \rho_{\pm,0} J_z^{(j)} = \epsilon_0 \rho_{\pm,0} \]  

(19)

Next we calculate the sum over each of the three terms in eq. 15 for a fixed distance of ions \(|j - k| = x\) and then regard the coefficients of \( \exp (-x/\xi) \). There are \((n - x)\) pairs of ions with a distance \( x \) between them, which can be seen by moving a fixed distance along figure 4. For \( x > 0 \) each pair is counted twice because \( j \) will be the right one and the left one once. We distinguish three cases:

**case 1) \( x \) is odd**

In each pair there is one spin in state \( \epsilon_+ \) and one in state \( \epsilon_- \).

\[ \sum_{|j - k| = x} J_z^{(j)} \rho_{\pm,0} J_z^{(k)} = 2(n - x)\epsilon_0^2 \rho_{\pm,0} \]  

(20)

\[ \sum_{|j - k| = x} J_z^{(j)} J_z^{(k)} \rho_{\pm,0} = 2(n - x)(\epsilon_0^2 - \epsilon_\Delta^2)\rho_{\pm,0} \]  

(21)

\[ \sum_{|j - k| = x} \rho_{\pm,0} J_z^{(j)} J_z^{(k)} = 2(n - x)\epsilon_0^2 \rho_{\pm,0} \]  

(22)

For the last equation, note that \( j \) is each ion in the pair once and therefore give once \( \epsilon_+ \) and once \( \epsilon_- \). Furthermore \( \epsilon_+ + \epsilon_- = 2\epsilon_0 \) for each pair.

\[ \frac{1}{2} \sum_{|j - k| = x} \exp \left(-\frac{|j - k|}{\xi}\right) \left(2J_z^{(j)} \rho_{\pm,0} J_z^{(k)} - J_z^{(j)} J_z^{(k)} \rho_{\pm,0} - \rho_{\pm,0} J_z^{(j)} J_z^{(k)}\right) = +(n - x)\epsilon_\Delta^2 \epsilon^x/\xi \rho_{\pm,0} \]  

(23)
In each pair the ions are in the same state. There is an equal number of \((\epsilon_+, \epsilon_+)\) pairs and \((\epsilon_-, \epsilon_-)\) pairs because (moving along the chain in figure 4) the pairs start with one type and finish with the other \((n\text{ is even})\).

\[
\sum_{|j-k|=x} J^{(j)}_{z} \rho_{\pm,0} J^{(k)}_{z} = 2(n-x)\epsilon^2_0 \rho_{\pm,0} \tag{24}
\]

\[
\sum_{|j-k|=x} J^{(j)}_{z} J^{(k)}_{z} \rho_{\pm,0} = 2(n-x)\epsilon_0^2 \rho_{\pm,0} \tag{25}
\]

\[
\sum_{|j-k|=x} \rho_{\pm,0} J^{(j)}_{z} J^{(k)}_{z} = 2(n-x)\epsilon_0^2 \rho_{\pm,0} \tag{26}
\]

\[
\frac{1}{2} \sum_{|j-k|=x} \exp\left(-\frac{|j-k|}{\xi}\right) \left(2J^{(j)}_{z} \rho_{\pm,0} J^{(k)}_{z} - J^{(j)}_{z} J^{(k)}_{z} \rho_{\pm,0} - \rho_{\pm,0} J^{(j)}_{z} J^{(k)}_{z}\right) = -(n-x)\epsilon_0^2 \epsilon^x/\xi \rho_{\pm,0} \tag{27}
\]

**Case 3** \(x = 0\)

For \(x = 0 \Leftrightarrow j = k\) there are \(n\) summands (which should not have the factor of 2 from the other cases).

\[
\sum_{|j-k|=x} J^{(j)}_{z} \rho_{\pm,0} J^{(k)}_{z} = n\epsilon_0^2 \rho_{\pm,0} \tag{28}
\]

\[
\sum_{|j-k|=x} J^{(j)}_{z} J^{(k)}_{z} \rho_{\pm,0} = n(\epsilon_0^2 + \epsilon_0^2) \rho_{\pm,0} \tag{29}
\]

\[
\sum_{|j-k|=x} \rho_{\pm,0} J^{(j)}_{z} J^{(k)}_{z} = n\epsilon_0^2 \rho_{\pm,0} \tag{30}
\]

\[
\frac{1}{2} \sum_{|j-k|=x} \exp\left(-\frac{|j-k|}{\xi}\right) \left(2J^{(j)}_{z} \rho_{\pm,0} J^{(k)}_{z} - J^{(j)}_{z} J^{(k)}_{z} \rho_{\pm,0} - \rho_{\pm,0} J^{(j)}_{z} J^{(k)}_{z}\right) = -\frac{n}{2} \epsilon_0^2 \rho_{\pm,0} \tag{31}
\]

We have now calculated all coefficients for the exponentials \(\exp(x/\xi)\) in the decoherent part of the master equation and can now write it down in the form:

\[
\rho_{\pm,0} = -\text{i} n \alpha \epsilon_{\Delta}^2 \rho_{\pm,0} + \Gamma(n, \xi) \rho_{\pm,0} \tag{32}
\]

\[
\rho_{\pm,0}(t) = \rho_{\pm,0}(0) \exp\left(-\text{i} n \alpha \epsilon_{\Delta}^2 t + \Gamma(n, \xi) t\right) \tag{33}
\]

The dephasing rate \(\Gamma(n, \xi)\) is given by the contributions for all \(x\), which we divide into one summand for \(x = 0\), the sum over odd \(x = 2x_c - 1\) and the sum over even \(x = 2x_c:\)

\[
\Gamma(n, \xi) = \left(-\frac{n}{2} + \sum_{x_c=1}^{n/2} (n - 2x_c + 1) e^{(2x_c - 1)/\xi} + \sum_{x_c=1}^{n/2} (2x_c - n) e^{2x_c/\xi}\right) \epsilon_{\Delta}^2 \tag{34}
\]

\[
\Gamma(n, \xi) = \frac{-2 e^{1/\xi} + 2 e^{-1/\xi} + n - e^{2/\xi} n}{2 \left(1 + e^{1/\xi}\right)^2} \epsilon_{\Delta}^2 \tag{35}
\]

This is the dephasing rate for exponentially decaying partial correlations. For uncorrelated decoherence \(\xi \to 0\) all collective terms \(j \neq k\) vanish and only case 3 gives a non-vanishing term. The uncorrelated dephasing rate is therefore

\[
\Gamma_{uc} = -n \epsilon_{\Delta}^2 / 2 \tag{36}
\]
Appendix C Oscillating spatial correlations facilitate a decoherence-free GHZ state

Up to here we have considered perfect spatial correlations $C(0, x) = 1$ and decaying spatial correlations $C(0, x) = \exp(-x/\xi)$. Both are positive functions for all $x$ and the $n^2$ scaling of the dephasing rates found experimentally\cite{12} for GHZ states indicates that in ion traps these two functional forms are good approximations for the noise correlations. However, it is also physically possible for the spatial correlations to take the homogeneous form $C(0, x) = \cos(k_x x)$, where points of certain distances have noise with negative correlations. For example in reference \[8\] an environmental model is discussed that yields cosine shaped spatial correlations. In such an environment GHZ states can be engineered to be within a decoherence-free subspace in two ways assuming the spatial oscillation length is known. One way is to place the ions at half the oscillation length of the environmental spatial correlations (figure 5A); the other way is to match the array length $L$ with the oscillation length $L = 2\pi/k_x$ (figure 5B).

![Figure 5. Two arrangements of ions in oscillating spatial correlations. Both arrangements achieve Heisenberg scaling of the GHZ state: A) For small oscillation length the ions are arranged to meet half the oscillation length. B) For long oscillation length the whole array is arranged to match up one oscillation length.](image)

For the GHZ state’s relevant coherence $r_3 = |111\ldots \rangle \langle 000\ldots |$ all operator pairings $j, k$ in the master equation have the same effect. The only difference comes from the cosine shaped correlation function:

$$
\dot{r}_3 = i[r_3, H_s] + \sum_{jk} \cos(k_x d | j - k |) \left( -\sigma_z^{(j)} \sigma_z^{(k)} r_3 - r_3 \sigma_z^{(j)} \sigma_z^{(k)} + \sigma_z^{(j)} r_3 \sigma_z^{(k)} + \sigma_z^{(k)} r_3 \sigma_z^{(j)} \right) \tag{37}
$$

where $d$ is the distance between ions.

In the arrangement of figure 5A the correlation function becomes effectively $\cos(k_x d | j - k |) = (-1)^{|j-k|}$, i.e. alternates the sign with increasing distance. This recreates the effect of the coherence

$$
\rho_{\pm,0} = |m_-, m_+, m_-, m_+ \ldots \rangle \langle m_1, m_1, m_1, \ldots, m_1 |
$$

of the previous section, where the alternating sign comes from the arrangement of the ions and the correlation function is always the same sign (positive). Here the alternating sign comes from the correlation function and the operators produce the same term for all pairs of $j$ and $k$. From this equivalence one finds that the arrangement of figure 5A leads to Heisenberg scaling for the GHZ state, even if the cosine correlation function has an additional exponentially decaying envelope.

In the arrangement of figure 5B the length of the array matches the spatial oscillation length of the environmental correlations. Regarding eq. 37 we group the pairs of the first ion $j = 1$ with all the other ions (including the autocorrelation $j = k = 1$). We find that for each positive contribution $C(0, (1-k)d) > 0$ there is an equal negative contribution from the $k$-value $n/2$ further down the chain. The sum of all contributions of $j = 1$ therefore cancels (assuming an even number $n$ of ions). The same argument applies for the sum of all contributions for $j = 2$ or any other value of $j$. For large numbers $n$ of ions these summations for one value of $j$ approach an integration over one oscillation length of a cosine, which illustrates the summation to zero even better.

In both arrangements of figure 5 the GHZ state $(|111\ldots \rangle + |000\ldots \rangle)/\sqrt{2}$ turns out to be a decoherence-free state or dark state for all $n$ and frequency measurements with it will therefore show Heisenberg scaling. The difficulty of an experimental implementation of this is to find and map out a noise environment with cosine spatial correlations.

After discussing purely positive and oscillating spatial correlations we would like to point out that purely negative correlation functions, i.e. perfect anti-correlations are impossible due to the necessity of positive autocorrelations $j = k$ and multipartite correlation rules. Particularly negative noise correlations between position $a$ and $b$ combined with negative correlations between position $b$ and $c$ require positive correlations between position $a$ and $c$. 

![Appendix C Oscillating spatial correlations facilitate a decoherence-free GHZ state](image)