Generalized Second Law in String Cosmology

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Abstract

A generalized second law in string cosmology accounts for geometric and quantum entropy in addition to ordinary sources of entropy. The proposed generalized second law forbids singular string cosmologies, under certain conditions, and forces a graceful exit transition from dilaton-driven inflation by bounding curvature and dilaton kinetic energy.
String theory is a consistent theory of quantum gravity, with the power to describe high curvature regions of space-time [1], and as such we could expect it to teach us about the fate of cosmological singularities, with the expectation that singularities are smoothed and turned into brief epochs of high curvature. However, many attempts to seduce an answer out of string theory regarding cosmological singularities have failed so far, even after the wave of recent new developments and results [2]. The reason is probably that most recent technical advancements in string theory rely heavily on supersymmetry, but generic time dependent solutions break all supersymmetries, and therefore known methods are less powerful when applied to cosmology. We propose to turn to general thermodynamical considerations in the quest to understand cosmological singularities, as first suggested by Bekenstein [3] in the context of Einstein’s general relativity. We propose that entropy considerations, in particular accounting for geometric and quantum entropy, accompanied by a generalized second law (GSL) demanding that entropy never decreases, be added to supplement string theory, and show that under certain conditions GSL forbids cosmological singularities. The proposed GSL is different from GSL for black holes [4], but the idea that in addition to normal entropy other sources of entropy have to be included has some similarities.

The idea that entropy and thermodynamics can play a role in cosmology has been stressed recently by Fischler and Susskind, who suggested that the holographic principle [5] may be useful in constraining cosmological solutions [6], followed by several other investigations in which their suggestion was critically examined or improved [7]. Our discussion is closer to that of [8], and in particular to that of [9] where a new Hubble entropy bound (HEB) was formulated, and its relevance to the analysis of cosmological singularities in string cosmology pointed out. The idea that geometric and quantum entropy should be added, and be accompanied by GSL was introduced in [10].

We will focus on two sources of entropy. The first source is geometric entropy $S_g$, whose origin is the existence of a cosmological horizon [11,12,10]. Geometric entropy has been calculated for special systems, but we assume that it is a general property of a system with a cosmological horizon, resulting from the existence of causal boundaries in space-
time. The concept of geometric entropy is closely related to the holographic principle, and it has appeared in this connection recently in discussion of cosmological entropy bounds. For a system with a cosmological horizon, geometric entropy within a Hubble volume is given roughly, ignoring numerical factors, by the area of the horizon. The second source is quantum entropy $S_q$, associated with quantum fluctuations. Changes in $S_q$ take into account “quantum leakage” of entropy, resulting from the well known phenomenon of freezing and defreezing of quantum fluctuations. For example, quantum modes whose wavelength is stretched by an accelerated cosmic expansion to the point that it is larger than the horizon, become frozen (“exit the horizon”), and are lost as dynamical modes, and conversely quantum modes whose wavelength becomes smaller than the Hubble length during a period of decelerated expansion, thaw (“reenter the horizon”) and become dynamical again. This form of entropy was discussed in [13,14]. Obviously, there is a relation, which we will expose more fully, between quantum and geometric entropy, since the rate of expansion of the universe determines the cosmological horizon and the rate of change of quantum entropy.

We adopt the definition of the total entropy of a domain containing more than one cosmological horizon [9], for a given scale factor $a(t)$, and a Hubble parameter $H(t) = \dot{a}/a$, the number of cosmological horizons within a given comoving volume $V = a(t)^3$ is given by the total volume divided by the volume of a single horizon, $n_H = a(t)^3/|H(t)|^{-3}$. If the entropy within a given horizon is $S^H$, then the total entropy is given by $S = n_H S^H$. We will ignore numerical factors, use units in which $c = 1$, $\hbar = 1$, $G_N = e^\phi/16\pi$, $\phi$ being the dilaton, and discuss only flat, homogeneous, and isotropic string cosmologies in the so-called string frame, in which the lowest order effective action is $S_{lo} = \int d^4x \sqrt{-g} e^{-\phi} \left[ R + (\partial \phi)^2 \right]$.

In ordinary cosmology, geometric entropy within a Hubble volume is given by its area $S_g^H = H^{-2} G_N^{-1}$, and therefore specific geometric entropy is given by $s_g = |H| G_N^{-1}$ [10]. A possible expression for specific geometric entropy in string cosmology is obtained by substituting $G_N = e^\phi$, leading to

$$s_g = |H| e^{-\phi}. \quad (1)$$
Reassurance that $s_g$ is indeed given by (1) is provided by the following observation. The action $S_{lo}$ can be expressed in a $(3 + 1)$ covariant form, using the 3-metric $g_{ij}$, the extrinsic curvature $K_{ij}$, considering only vanishing 3–Ricci scalar and homogeneous dilaton, $S_{lo} = \int d^3x dt \sqrt{g_{ij}} e^{-\phi} \left[ -3K_{ij}K^{ij} - 2g^{ij} \partial_t K_{ij} + K^2 - (\partial_t \phi)^2 \right]$. Now, $S_{lo}$ is invariant under the symmetry transformation $g_{ij} \rightarrow e^{2\lambda}g_{ij}$, $\phi \rightarrow \phi + 3\lambda$, for an arbitrary time dependent $\lambda$.

From the variation of the action $\delta S = \int d^3x dt \sqrt{g_{ij}} e^{-\phi} \dot{\lambda} K_{ij}$, we may read off the current and conserved charge $Q = 4a^3 e^{-\phi} K$. The symmetry is exact in the flat homogeneous case, and it seems plausible that it is a good symmetry even when $\alpha'$ corrections are present [15].

With definition (1), the total geometric entropy $S_g = a^3 |H| e^{-\phi}$, is proportional to the corresponding conserved charge. Adiabatic evolution, determined by $\partial_t S_g = 0$, leads to a familiar equation,

$$\dot{H} H - \dot{\phi} + 3H = 0,$$

satisfied by the $(\pm)$ vacuum branches of string cosmology.

Quantum entropy for a single field in string cosmology is, as in [14,10], given by

$$s_q = \int_{k_{min}}^{k_{max}} d^3k f(k), \quad (2)$$

where for large occupation numbers $f(k) \simeq \ln n_k$. The ultraviolet cutoff $k_{max}$ is assumed to remain constant at the string scale. The infrared cutoff $k_{min}$ is determined by the perturbation equation $\psi''_{k_c} + \left( k_c^2 - \frac{\sqrt{s(\eta)''}}{\sqrt{s(\eta)}} \right) \psi_{k_c} = 0$, where $\eta$ is conformal time $\dot{\eta} = \partial \eta$, and $k_c$ is the comoving momentum related to physical momentum $k(\eta)$ as $k_c = a(\eta) k(\eta)$. Modes for which $k_c^2 \lesssim \frac{k''}{\sqrt{s'}}$ are “frozen”, and are lost as dynamical modes. The “pump field” $s(\eta) = a^{2m} e^{\ell \phi}$, depends on the background evolution and on the spin and dilaton coupling of various fields [16]. We are interested in solutions for which $a'/a \sim \phi' \sim 1/\eta$, and therefore, for all particles $\frac{\sqrt{k''}}{\sqrt{s'}} \sim 1/\eta^2$. It follows that $k_{min} \sim H$. In other phases of cosmological evolution our assumption does not necessarily hold, but in standard radiation domination (RD) with frozen dilaton all modes reenter the horizon. Using the reasonable approximation $f(k) \sim$ constant, we obtain, as in [10],

$$\Delta S_q \simeq -\mu \Delta n_H. \quad (3)$$

Parameter $\mu$ is positive, and in many cases proportional to the number of species of particles, taking into account all degrees of freedom of the system, perturbative and non-perturbative.
The main contribution to $\mu$ comes from light degrees of freedom and therefore if some non-perturbative objects, such as D branes become light they will make a substantial contribution to $\mu$.

We now turn to the generalized second law of thermodynamics, taking into account geometric and quantum entropy. Enforcing $dS \geq 0$, and in particular, $\partial_t S = \partial_t S_g + \partial_t S_q \geq 0$, leads to an important inequality,

$$\left(H^{-2}e^{-\phi} - \mu\right) \partial_t n_H + n_H \partial_t \left(H^{-2}e^{-\phi}\right) \geq 0.$$  \hspace{1cm} (4)

When quantum entropy is negligible compared to geometric entropy, GSL (4) leads to

$$\dot{\phi} \leq \frac{\dot{H}}{H} + 3H,$$  \hspace{1cm} (5)

yielding a bound on $\dot{\phi}$, and therefore on dilaton kinetic energy, for a given $H$, $\dot{H}$. Bound (5) was first obtained in [9], and interpreted as following from a saturated HEB.

When quantum entropy becomes relevant we obtain another bound. We are interested in a situation in which the universe expands, $H > 0$, and $\phi$ and $H$ are non-decreasing, and therefore $\partial_t \left(H^{-2}e^{-\phi}\right) \leq 0$ and $\partial_t n_H > 0$. A necessary condition for GSL to hold is that

$$H^2 \leq \frac{e^{-\phi}}{\mu},$$  \hspace{1cm} (6)

bounding total geometric entropy $He^{-\phi} \leq \frac{e^{-\frac{4\phi}{\sqrt{\mu}}}}{\sqrt{\mu}}$. A bound similar to (6) was obtained in [17] from a holographic bound on the rate of production of D0 branes, and in [9] by considering entropy of reentering quantum fluctuations. We stress that to be useful in analysis of cosmological singularities (6) has to be considered for perturbations that exit the horizon.

We note that if the condition (6) is satisfied, then the cosmological evolution never reaches the nonperturbative region described in [17], allowing a self-consistent analysis using the low energy effective action approach.

It is not apriori clear that the form of GSL and entropy sources remains unchanged when curvature becomes large, in fact, we may expect higher order corrections to appear. For example, the conserved charge of the scaling symmetry of the action will depend in
general on higher order curvature corrections. Nevertheless, in the following we will assume that specific geometric entropy is given by eq. (1), without higher order corrections, and try to verify that, for some reason yet to be understood, there are no higher order corrections to eq. (1). Our results are consistent with this assumption.

We turn now to apply our general analysis to the ‘pre-big-bang’ (PBB) string cosmology scenario [18], in which the universe starts from a state of very small curvature and string coupling and then undergoes a long phase of dilaton-driven inflation (DDI), joining smoothly at later times standard RD cosmology, giving rise to a singularity free inflationary cosmology. The high curvature phase joining DDI and RD phases is identified with the ‘big bang’ of standard cosmology. A key issue confronting this scenario is whether, and under what conditions, can the graceful exit transition from DDI to RD be completed [19]. In particular, it was argued that curvature is bounded by an algebraic fixed point behaviour when both \( H \) and \( \dot{\phi} \) are constants and the universe is in a linear-dilaton deSitter space [15], and coupling is bounded by quantum corrections [20,21]. But it became clear that another general theoretical ingredient is missing, and we propose that GSL is that missing ingredient.

We have studied numerically examples of PBB string cosmologies to verify that the overall picture we suggest is valid in cases that can be analyzed explicitly. We first consider, as in [15,22], \( \alpha' \) corrections to the lowest order string effective action,

\[
S = \frac{1}{16\pi\alpha'} \int d^4x \sqrt{-g} e^{-\phi} \left[ R + (\partial\phi)^2 + \frac{1}{2} \mathcal{L}_{\alpha'} \right],
\]

where

\[
\mathcal{L}_{\alpha'} = k\alpha' \left[ \frac{1}{2} R_{GB} + A (\partial\phi)^4 + D \partial^2 \phi (\partial\phi)^2 
+ C \left( R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) \partial_\mu \phi \partial_\nu \phi \right],
\]

with \( C = -(2A + 2D + 1) \), is the most general form of four derivative corrections that lead to equations of motion with at most second (time) derivatives. The rationale for this choice was explained in [22]. \( k \) is a numerical factor depending on the type of string theory. Action (7) leads to equations of motion, \(-3H^2 + \ddot{\phi} - \ddot{\rho} = 0, \dot{\sigma} - 2\dot{H} + 2\dot{\phi} = 0, \dot{\lambda} - 3H^2 - \ddot{\phi}^2 + 2\ddot{\phi} = 0, \)
FIG. 1. Two lines, separating actions whose generic solutions “turn the right way” at the early stages of evolution (red-dashed), and actions whose generic solutions satisfy classical GSL while close to the (+) branch vacuum (blue-solid). The dots represent \((A, D)\) values whose generic solutions reach a fixed point, and are all in the ”allowed” region.

where \(\bar{\rho}, \bar{\lambda}, \bar{\sigma}\) are effective sources parameterizing the contribution of \(\alpha'\) corrections \[22\]. Parameters \(A\) and \(D\) should have been determined by string theory, however, at the moment, it is not possible to calculate them in general. If \(A, D\) were determined we could just use the results and check whether their generic cosmological solutions are non-singular, but since \(A, D\) are unavailable at the moment, we turn to GSL to restrict them.

First, we look at the initial stages of the evolution when the string coupling and \(H\) are very small. We find that not all the values of the parameters \(A, D\) are allowed by GSL. The condition \(\bar{\sigma} \geq 0\), which is equivalent to GSL on generic solutions at the very early stage of the evolution, if the only relevant form of entropy is geometric entropy, leads to the following condition on \(A, D\) (first obtained by R. Madden \[23\]), \(40.05A + 28.86D \leq 7.253\). The values of \(A, D\) which satisfy this inequality are labeled “allowed”, and the rest are “forbidden”. In \[22\] a condition that \(\alpha'\) corrections are such that solutions start to turn towards a fixed point at the very early stages of their evolution was found \(61.1768A + 40.8475D \leq 16.083\), and such solutions were labeled “turning the right way”. Both conditions are displayed in Fig. 1. They select almost the same region of \((A, D)\) space, a gratifying result, GSL “forbids” actions whose generic solutions are singular and do not reach a fixed point. We further observe that generic solutions which “turn the wrong way” at the early stages of
FIG. 2. Typical solution that “turns the wrong way”. The dashed line is the (+) branch vacuum.

their evolution continue their course in a way similar to the solution presented in Fig. 2. We find numerically that at a certain moment in time $H$ starts to decrease, at that point $\dot{H} = 0$ and particle production effects are still extremely weak, and therefore (3) is the relevant bound, but (3) is certainly violated.

We have scanned the $(A, D)$ plane to check whether a generic solution that reaches a fixed point respects GSL throughout the whole evolution, and conversely, whether a generic solution obeying GSL evolves towards a fixed point. The results are shown in Fig. 1, clearly, the “forbidden” region does not contain actions whose generic solutions go to fixed points. Nevertheless, there are some $(A, D)$ values located in the small wedges near the bounding lines, for which the corresponding solutions always satisfy (3), but do not reach a fixed point, and are singular. This happens because they meet a cusp singularity. Consistency requires adding higher order $\alpha'$ corrections when cusp singularities are approached, which we will not attempt here.

If particle production effects are strong, the quantum part of GSL adds bound (3), which adds another “forbidden” region in the $(H, \dot{\phi})$ plane, the region above a straight line parallel to the $\dot{\phi}$ axis. The quantum part of GSL has therefore a significant impact on corrections to the effective action. On a fixed point $\phi$ is still increasing, and therefore the bounding line described by (3) is moving downwards, and when the critical line moves below the fixed point, GSL is violated. This means that when a certain critical value of the coupling $e^{\phi}$ is
FIG. 3. Graceful exit enforced by GSL on generic solutions. The horizontal line is bound (6) and the curve on the right is bound (5), shaded regions indicate GSL violation.

reached, the solution can no longer stay on the fixed point, and it must move away towards an exit. One way this can happen is if quantum corrections, perhaps of the type discussed in [20,21] exist.

The full GSL therefore forces actions to have generic solutions that are non-singular, classical GSL bounds dilaton kinetic energy and quantum GSL bounds $\dot{H}$ and therefore, at a certain moment of the evolution $\dot{H}$ must vanish (at least asymptotically), and then curvature is bounded. If cusp singularities are removed by adding higher order corrections, as might be expected, we can apply GSL with similar conclusions also in this case. A schematic graceful exit enforced by GSL is shown in Fig. 3. We conclude that the use of thermodynamics and entropy in string cosmology provides model independent tools to analyze cosmological solutions which are not yet under full theoretical control. Our result indicate that if we impose GSL, in addition to equations of motion, non-singular string cosmology is quite generic.

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