Semi-Supervised Off Policy Reinforcement Learning

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Abstract

Reinforcement learning (RL) has shown great success in estimating sequential treatment strategies which account for patient heterogeneity. However, health-outcome information is often not well coded but rather embedded in clinical notes. Extracting precise outcome information is a resource intensive task. This translates into only small well-annotated cohorts available. We propose a semi-supervised learning (SSL) approach that can efficiently leverage a small sized labeled data $L$ with true outcome observed, and a large sized unlabeled data $U$ with outcome surrogates $W$. In particular we propose a theoretically justified SSL approach to Q-learning and develop a robust and efficient SSL approach to estimating the value function of the derived optimal STR, defined as the expected counterfactual outcome under the optimal STR. Generalizing SSL to learning STR brings interesting challenges. First, the feature distribution for predicting $Y_t$ is unknown in the Q-learning procedure, as it includes unknown $Y_{t-1}$ due to the sequential nature. Our methods for estimating optimal STR and its associated value function, carefully adapts to this sequentially missing data structure. Second, we modify the SSL framework to handle the use of surrogate variables $W$ which are predictive of the outcome through the joint law $P_{Y,O,W}$, but are not part of the conditional distribution of interest $P_{Y|O}$. We provide theoretical results to understand when and to what degree efficiency can be gained from $W$ and $O$. Our approach is robust to misspecification of the imputation models. Further, we provide a doubly robust value function estimator for the derived STR. If either the Q functions or the propensity score functions are correctly specified, our value function estimators are consistent for the true value function.

Keywords: Semi-supervised learning, Q-Learning, Reinforcement-learning, Dynamical Treatment Regimes, Double robust value function
1. Introduction

Finding optimal treatment strategies that can incorporate patient heterogeneity is a cornerstone of personalized medicine. When treatment options can change over time, optimal sequential treatment rules (STR) can be learned using longitudinal patient data. With increasing availability of large scale longitudinal data such as electronic health records (EHR) data in recent years, reinforcement learning (RL) has found much success in estimating such optimal STR (Kosorok and Laber, 2019). Existing RL methods include G-estimation (Robins, 2004), Q-learning (Watkins, 1989; Murphy, 2005), and A-learning (Murphy, 2003) and directly maximizing the value function (Zhao et al., 2015). Both G-estimation and A-learning attempt to model only the part of the outcome regression relevant to the treatment contrast, while Q-learning posits complete models for the outcome regression. Although G-estimation and A-learning models can be more efficient and robust to misspecification, Q-learning is widely adopted due to its ease of implement, flexible and interpretable (Watkins, 1989; Chakraborty and Moodie, 2013; Schulte et al., 2014).

Learning STR with EHR data, however, often faces an additional challenge that outcome information is readily available. Outcome information such as development of a clinical event or whether a patient is considered as a responder is often not well coded but rather embedded in clinical notes. Proxy variables such as diagnostic code or mentions of relevant clinical terms in clinical notes via natural language processing (NLP), while predictive of the true outcome, are often not sufficiently accurate to be used directly in replace of the outcome (Hong et al., 2019; Zhang et al., 2019; Cheng et al., 2020). On the other hand, extracting precise outcome information often requires manual chart review, which is resource intensive particularly when aiming to learn STR since the outcome needs to be annotated over time. This signifies the need for semi-supervised learning (SSL) that can efficiently leverage a small sized labeled data $L$ with true outcome observed and a large sized unlabeled data $U$ for predictive modeling. It is worthwhile to note that the SSL setting differs from the standard missing data setting in that the probability of missing tends to 1 asymptotically, which violates the positivity assumption required by the classical missing data methods (Chakrabortty et al., 2018).

While SSL methods have been well developed for prediction, classification and regression tasks (Chapelle et al., 2006; Zhu, 2008; Blitzer and Zhu, 2008; Zhixing and Shaohong, 2011. Qiao et al., 2018; Chakrabortty et al., 2018, e.g.), there is a paucity of literature on SSL methods for estimating optimal treatment rules. Recently, Cheng et al. (2020) and Kallus and Mao (2020) proposed SSL methods for estimating an average causal treatment effect. Finn et al. (2016) proposed a semi-supervised RL method which achieves impressive empirical results and outperforms simple approaches such as direct imputation of the reward. However, there are no theoretical guarantees and the approach lacks causal validity and interpretability within a domain context. Additionally, this method does not leverage available surrogates. In this paper, we fill this gap by proposing a theoretically justified SSL approach to Q-learning using a large unlabeled data $U$ which contains sequential observations on features $O$, treatment assignment $A$, and surrogates $W$ that are imperfect proxies of $Y$ as well as a small set of labeled data $L$ which contains true outcome $Y$ at multiple stages along with $O$, $A$ and $W$. We will also develop robust and efficient SSL ap-
approach to estimating the value function of the derived optimal STR, defined as the expected counterfactual outcome under the optimal STR.

To describe the main contributions of our proposed SSL approach to RL, we first note two important distinctions between the proposed framework from classical SSL methods. First, existing SSL literature often assumes that $U$ is large enough such that the feature distribution is known (Wasserman and Lafferty, 2008). However, under the RL setting, the outcome of the stage $t-1$, denoted by $Y_{t-1}$, becomes a feature of stage $t$ for predicting $Y_t$. As such, the feature distribution for predicting $Y_t$ can not be viewed as known in the $Q$-learning procedure. Our methods for estimating optimal STR and its associated value function, carefully adapts to this sequentially missing data structure. Second, we modify the SSL framework to handle the use of surrogate variables $W$ which are predictive of the outcome through the joint law $P_{Y,O,W}$, but are not part of the conditional distribution of interest $P_{Y|O}$. To address these issues, we propose a two-step fitting procedure for finding optimal STR and for estimating their value function in the SSL setting. Our method consists of using the outcome-surrogates ($W$) and features for non-parametric estimation of the missing outcomes. Then subsequently use these imputations to estimate $Q$ functions, learn the optimal treatment rule and estimate its associated value function. We provide theoretical results to understand when and to what degree efficiency can be gained from $W$ and $O$. We further show that our approach is robust to misspecification of the imputation models. To account for potential misspecification in the models for the $Q$ function, we provide a double robust value function estimator for the derived STR. If either the regression models for the $Q$ function or the propensity score functions are correctly specified, our value function estimators are consistent for the true value function.

We organize the rest of the paper as follows. In Section 2 we formalize the problem mathematically and provide some notation to be used in the development and analysis of the methods. In Section 3 we discuss traditional $Q$-learning and propose an SSL estimation procedure for the optimal STR. Section 4 details an SSL doubly robust estimator of the value function for the derived STR. In Section 5 we provide theoretical guarantees for our approach and discuss implications of our assumptions and results. Section 6 is devoted for numerical experiments as well as real data analysis with an inflammatory bowel disease (IBD) data-set. We end with a discussion of the methods and possible extensions in Section 7. Finally all the technical proofs and supporting lemmas are collected in Appendices C and B.

2. Problem setup

We consider a longitudinal observational study with outcomes, confounders and treatment indices potentially available over multiple stages. Although our method is generalizable for any number of stages, for the ease of presentation we'll use two time points of (binary) treatment allocation as follows. For time point $t \in \{1, 2\}$, let $O_t \in \mathbb{R}^{d_o_t}$ denote the vector of covariates measured prior at stage $t$ of dimension $d_o_t$; $A_t \in \{0, 1\}$ a treatment indicator variable; and $Y_{t+1} \in \mathbb{R}$ the outcome observed at stage $t + 1$, for which higher values of $Y_{t+1}$ are considered beneficial. Additionally we observe surrogates $W_t \in \mathbb{R}^{d_w_t}$, a $d_w_t$-dimensional vector of post-treatment covariates potentially predictive of $Y_{t+1}$. In the labeled data where $Y = (Y_2, Y_3)^T$ is annotated, we observe a random sample of $n$ independent and identically
distributed (iid) random vectors, denoted by
\[
\mathcal{L} = \{ \mathbf{L}_i = (\mathbf{U}_i, Y_i^1)_{i=1}^N \} \text{, where } \mathbf{U}_i = (\mathbf{O}_i^T, A_i, \mathbf{W}_i^T) \text{ and } \mathbf{U}_i = (\mathbf{U}_{1i}, \mathbf{U}_{2i})^T.
\]
We additionally observe an unlabeled set consisting of \( N \) iid random vectors,
\[
\mathcal{U} = \{ \mathbf{U}_j \}_{j=1}^N
\]
with \( N \gg n \). We denote the entire data as \( \mathcal{S} = (\mathcal{L} \cup \mathcal{U}) \). To operationalize our statistical arguments we denote the joint distribution of the observation vector \( \mathbf{L}_i \) in \( \mathcal{L} \) as \( \mathbb{P} \). In order to connect to the unlabeled set, we assume that any observation vector \( \mathbf{U}_j \) in \( \mathcal{U} \) has the distribution induced by \( \mathbb{P} \).

We are interested in finding the optimal STR and estimating its value function to be defined as expected counterfactual outcomes under the optimal regime. To this end, let \( Y^{(a)}_t \) be the potential outcome for a patient at time \( t \) had the patient been treated at time \( t - 1 \) with treatment \( a \in \{0, 1\} \). A dynamic treatment regime is a set of functions \( \mathcal{D} = (d_1, d_2) \), where \( d_t(\cdot) \in \{0, 1\} \), \( t = 1, 2 \) map from the patient’s history up to time \( t \) to the treatment choice \( \{0, 1\} \). We define the patient’s history as \( \mathbf{H}_1 \equiv [\mathbf{H}_{10}^T, \mathbf{H}_{11}^T]^T \) with \( \mathbf{H}_{1k} = \phi_{1k}(\mathbf{O}_1), \mathbf{H}_{2k} = [\mathbf{H}_{20}, \mathbf{H}_{21}]^T \) with \( \mathbf{H}_{2k} = \phi_{2k}(\mathbf{O}_1, A_1, \mathbf{O}_2) \), where \( \{\phi_{ik}(\cdot), t = 1, 2, k = 0, 1\} \) are pre-specified basis functions. We then define features derived from patient history for regression modeling as \( \mathbf{X}_1 = [\mathbf{H}_{10}^T, A_1 \mathbf{H}_{11}^T]^T \) and \( \mathbf{X}_2 = [\mathbf{H}_{20}^T, A_2 \mathbf{H}_{21}^T]^T \). For ease of presentation, we also let \( \bar{\mathbf{H}}_1 = \mathbf{H}_1^T, \bar{\mathbf{H}}_2 = (\mathbf{Y}_2, \mathbf{H}_2^T)^T, \bar{\mathbf{X}}_1 = \mathbf{X}_1, \bar{\mathbf{X}}_2 = (\mathbf{Y}_2, \mathbf{X}_2^T)^T, \) and \( \Sigma_t = \mathbb{E}[\mathbf{X}_t \mathbf{X}_t^T] \).

Let \( \mathbb{E}_\mathcal{D} \) be the expectation with respect to the measure that generated the data under regime \( \mathcal{D} \). Then these sets of rules \( \mathcal{D} \) have an associated value function which we can write as \( V(\mathcal{D}) = \mathbb{E}_\mathcal{D} \left[ Y^{(d_1)}_2 + Y^{(d_2)}_3 \right] \). Thus, an optimal dynamic treatment regime is a rule \( \bar{\mathcal{D}} = (\bar{d}_1, \bar{d}_2) \) such that \( \bar{V} = V(\bar{\mathcal{D}}) \geq V(\mathcal{D}) \) \( \forall \mathcal{D} \), where \( \mathcal{D} \) belongs to a suitable class of admissible decisions (Chakraborty and Moodie, 2013). To identify \( \bar{\mathcal{D}} \) and \( \bar{V} \) from the observed data we will require the following sets of standard assumptions (Robins, 1997; Schulte et al. 2014): (i) consistency \( - Y_{t+1} = Y_{t+1}^{(0)} I_{A_t = 0} + Y_{t+1}^{(1)} I_{A_t = 1} \) for \( t = 1, 2 \), (ii) no unmeasured confounding \( - Y_{t+1}^{(0)}, Y_{t+1}^{(1)} \perp A_t | \mathbf{H}_t \) for \( t = 1, 2 \) and (iii) positivity \( \mathbb{P}(A_t | \mathbf{H}_t) > \nu \) for \( t = 1, 2, A_t \in \{0, 1\} \), for some fixed \( \nu > 0 \).

We will develop SSL inference methods to derive optimal STR \( \bar{\mathcal{D}} \) as well the associated value function \( \bar{V} \) by leveraging the richness of the unlabeled data and the predictive power of surrogate variables which allows us to gain crucial statistical efficiency. Our main contributions in this regard can be described as follows. First, we provide a systematic generalization of the Q-learning framework with theoretical guarantees to the semi-supervised setting with improved efficiency. Second, we provide a doubly robust estimator of the value function in the semi-supervised setup. Third, our Q-learning procedure and value function estimator are flexible enough to allow for standard off-the-shelf machine learning tools and is shown to perform well in finite-sample numerical examples.

### 3. Semi-Supervised Q-learning

In this section we propose a semi-supervised Q-learning approach to deriving an optimal STR. To this end, we first recall the basic mechanism of traditional linear parametric Q-learning Chakraborty and Moodie (2013) and then detail our proposed method. We defer the theoretical guarantees to Section 5.
3.1 Traditional Q-learning

Q-learning is a backward recursive algorithm to identify optimal STR by optimizing two stage Q-functions defined as:

\[ Q_2(\hat{H}_2, A_2) \equiv \mathbb{E}[Y_3|\hat{H}_2, A_2], \quad \text{and} \quad Q_1(\hat{H}_1, A_1) \equiv \mathbb{E}[Y_2 + \max_{a_2} Q_2(\hat{H}_2, a_2)|H_1, A_1] \]

(Sutton, 2018; Murphy, 2005). In order to perform inference one typically proceeds by positing models for the Q functions. In its simplest form one assumes a (working) linear model for some parameters \( \theta \) as the population to the expected normal equations

\[ Q_1(\hat{H}_1, A_1; \theta^0_1) = X_1^T \theta^0_1 + A_1(\hat{H}_1^0 \gamma_1^T), \]

\[ Q_2(\hat{H}_2, A_2; \theta^0_2) = X_2^T \theta^0_2 = Y_2 \beta^1_2 + \hat{H}_2^0 \beta^2_2 + A_2(\hat{H}_2^0 \gamma_2^0). \quad (1) \]

Typical Q-learning consists of performing a least squares regression for the second stage to estimate \( \hat{\theta}_2 \) followed by defining the stage 1 pseudo-outcome as

\[ \hat{Y}_2^* = Y_2 + \max_{a_2} Q_2(\hat{H}_2, a_2; \hat{\theta}_2), \quad \text{for } i = 1, \ldots, n. \]

One then proceeds to estimate \( \hat{\theta}_1 \) using least squares again, using \( \hat{Y}_2^* \) as the outcome variable. Indeed, valid inference on \( \theta \) using the method described above crucially depends on the validity of the model assumed. However as we shall see, even without validity of this model we will be able to provide valid inference on suitable analogues of the Q-function working model parameters, and on the value function using a double robust type estimator. To that end it will be instructive to define instead the least square projections of \( Y_3 \) and \( Y_2^* \) onto \( X_2 \) and \( X_1 \) respectively. The linear regression working models given by equation (1) have \( \theta^0_1, \theta^0_2 \) as unknown regression parameters. To account for the potential misspecification of the working models in (1), we define the target population parameters \( \theta_1, \theta_2 \) as the population to the expected normal equations

\[ \mathbb{E}\{X_1(Y_2^* - X_1^\dagger \theta_1)\} = 0, \quad \text{and} \quad \mathbb{E}\{X_2^\dagger (Y_3 - X_2^\dagger \theta_2)\} = 0, \]

where \( \hat{Y}_2^* = Y_2 + \max_{a_2} Q_2(\hat{H}_2, a_2; \hat{\theta}_2) \). As these are linear in the parameters, uniqueness and existence for \( \theta_1, \theta_2 \) are well defined. In fact, \( Q_1(X_1; \theta_1) = X_1^\dagger \theta_1, Q_2(X_2; \theta_2) = X_2^\dagger \theta_2 \) are the \( L_2 \) projection of \( \mathbb{E}(Y_2^*|X_1) \in L_2(\mathbb{P}_{X_1}) \), \( \mathbb{E}(Y_3|X_2) \in L_2(\mathbb{P}_{X_2}) \) onto the subspace of all linear functions of \( X_1, X_2^\dagger \) respectively. Therefore, Q functions in (1) are the best linear predictors of \( Y_2^* \) conditional on \( X_1 \) and \( Y_3 \) conditional on \( X_2^\dagger \).

Traditionally, one only has access to labeled data \( L \), and hence proceeds by estimating \( (\theta_1, \theta_2) \) in (1) by solving the following sample version set of normal equations:

\[ \mathbb{P}_n \begin{bmatrix} Y_2(Y_3 - X_2^\dagger \theta_2) \\ X_2(Y_3 - X_2^\dagger \theta_2) \end{bmatrix} = \mathbb{P}_n \begin{bmatrix} Y_2(Y_3 - (Y_2, X_2^\dagger) \theta_2) \\ X_2(Y_3 - (Y_2, X_2^\dagger) \theta_2) \end{bmatrix} = 0, \]

\[ \mathbb{P}_n \begin{bmatrix} X_1(Y_2(1 + \beta_{21}) + \hat{H}_2^0 \beta_{22} + [\hat{H}_2^0 \gamma_{21}^T] - X_1 \theta_2) \end{bmatrix} = 0. \quad (2) \]

(Chakraborty and Moodie, 2013), where \( \mathbb{P}_n \) denotes the empirical measure: i.e. for a measurable function \( f : \mathbb{R}^p \rightarrow \mathbb{R} \) and random sample \( \{L_i\}_{i=1}^n \), \( \mathbb{P}_n f = \frac{1}{n} \sum_{i=1}^n f(L_i) \). The asymptotic distribution for the Q function parameters in the fully-supervised setting has been well studied (see Laber et al. (2014)).
3.2 Semi-supervised Q-learning

We next detail our robust imputation based semi-supervised Q-learning that leverages the unlabeled data \( \mathcal{U} \) to replace the unobserved \( Y_t \) in (2) with their properly imputed values for subjects in \( \mathcal{U} \). Our SSL procedure includes three key steps: (i) imputation, (ii) refitting, and (iii) projection to the unlabeled data. In step (i), we develop flexible imputation models for the conditional mean functions \( \{ \mu_t(\cdot), \mu_{2t}(\cdot), t = 2, 3 \} \), where \( \mu_t(\hat{\mathbf{U}}) = \mathbb{E}(Y_t|\hat{\mathbf{U}}) \) and \( \mu_{2t}(\hat{\mathbf{U}}) = \mathbb{E}(Y_2Y_t|\hat{\mathbf{U}}) \). The refitting in step (ii) will ensure the validity of the SSL estimators under potential misspecifications of the imputation models.

**Step I: Imputation.**

Our first imputation step involves weakly parametric or non-parametric prediction modeling to approximate the conditional mean functions \( \{ \mu_t(\cdot), \mu_{2t}(\cdot), t = 2, 3 \} \). Commonly used models such as non-parametric kernel smoothing, basis function expansion or kernel machine regression can be used and we denote the corresponding estimated mean functions as \( \hat{\mu}_t(\cdot), \hat{\mu}_{2t}(\cdot), t = 2, 3 \) under the corresponding imputation models \( \{ m_t(\hat{\mathbf{U}}), m_{2t}(\hat{\mathbf{U}}), t = 2, 3 \} \). Theoretical properties of our proposed SSL estimators on specific choices of the imputation models are provided in section 5. We also provide additional simulation results comparing different imputation models in section 6.

**Step II: Refitting.**

To overcome the potential bias in the fitting from the imputation model, especially under model misspecification, we update the imputation model with an additional refitting step by expanding the imputation model to include linear effects of \( \{ \mathbf{X}_t, t = 1, 2 \} \) with cross-fitting to control overfitting bias. Specifically, to ensure the validity of the SSL algorithm from the refitted imputation model, we note that the final imputation models for \( \{ Y_t, Y_{2t}, t = 2, 3 \} \), denoted by \( \{ \hat{\mu}_t(\hat{\mathbf{U}}), \hat{\mu}_{2t}, t = 2, 3 \} \), need to satisfy

\[
\mathbb{E} \left[ \mathbf{X}_2 \{ Y_3 - \hat{\mu}_3(\hat{\mathbf{U}}) \} \right] = 0, \quad \mathbb{E} \left[ Y_3^2 - \hat{\mu}_{23}(\hat{\mathbf{U}}) \right] = 0.
\]

where \( \mathbf{X} = (1, \mathbf{X}_1^T, \mathbf{X}_2^T)^T \). We thus propose a refitting step that expands \( \{ m_t(\hat{\mathbf{U}}), m_{2t}(\hat{\mathbf{U}}), t = 2, 3 \} \) to additionally adjust for linear effects of \( \mathbf{X}_1 \) and/or \( \mathbf{X}_2 \) to ensure unbiasedness of the subsequent projection step. To this end, let \( \{ \mathcal{I}_k, k = 1, ..., K \} \) denote \( K \) random equal sized partitions of the labeled index set \( \{ 1, ..., n \} \), and let \( \{ \hat{m}_t^{(k)}(\hat{\mathbf{U}}), \hat{m}_{2t}^{(k)}(\hat{\mathbf{U}}), t = 2, 3 \} \) be the counterpart of \( \{ m_t(\hat{\mathbf{U}}), m_{2t}(\hat{\mathbf{U}}), t = 2, 3 \} \) with labeled observations in \( \{ 1, ..., n \} \setminus \mathcal{I}_k \). We then obtain \( \hat{\eta}_2, \hat{\eta}_{22}, \hat{\eta}_3, \hat{\eta}_{23} \) respectively as the solutions to

\[
\sum_{k=1}^{K} \sum_{i \in \mathcal{I}_k} \mathbf{X}_i \left\{ Y_{2i} - \hat{m}_2^{(k)}(\hat{\mathbf{U}}_i) - \eta_2^i \mathbf{X}_i \right\} = 0, \quad \sum_{k=1}^{K} \sum_{i \in \mathcal{I}_k} \left\{ Y_{2i}^2 - \hat{m}_{22}^{(k)}(\hat{\mathbf{U}}_i) - \eta_{22} \right\} = 0,
\]
\[
\sum_{k=1}^{K} \sum_{i \in \mathcal{I}_k} \mathbf{X}_{2i} \left\{ Y_{3i} - \hat{m}_3^{(k)}(\hat{\mathbf{U}}_i) - \eta_3^i \mathbf{X}_{2i} \right\} = 0, \quad \sum_{k=1}^{K} \sum_{i \in \mathcal{I}_k} \left\{ Y_{2i}Y_{3i} - \hat{m}_{23}^{(k)}(\hat{\mathbf{U}}_i) - \eta_{23} \right\} = 0.
\]
Finally, we impute \( Y_2, Y_3, Y_2^2 \) and \( Y_2, Y_3 \) respectively as \( \hat{\mu}_2(\bar{U}) = \sum_{k=1}^{K} \hat{m}_2^{(k)}(\bar{U}) + \hat{\eta}_2^X \), \( \hat{\mu}_3(\bar{U}) = \sum_{k=1}^{K} \hat{m}_3^{(k)}(\bar{U}) + \hat{\eta}_3^X \), \( \hat{\mu}_{22}(\bar{U}) = \sum_{k=1}^{K} \hat{m}_{22}^{(k)}(\bar{U}) + \hat{\eta}_{22} \), and \( \hat{\mu}_{23}(\bar{U}) = \sum_{k=1}^{K} \hat{m}_{23}^{(k)}(\bar{U}) + \hat{\eta}_{23} \).

**Step III: Projection**

In the last step, we proceed to estimate \( \hat{\theta} \) by replacing \( \{Y_t, Y_2Y_t, t = 2, 3\} \) in (2) with their the imputed values \( \{\hat{\mu}_t(\bar{U}), \hat{\mu}_{2t}(\bar{U}), t = 2, 3\} \) and project to the unlabeled data. Specifically, we obtain the final SSL estimators for \( \theta_1 \) and \( \theta_2 \) via the following steps:

1. Stage 2 regression: we obtain the SSL estimator for \( \theta_2 \) as
   \[
   \hat{\theta}_2 = (\hat{\beta}_2, \hat{\gamma}_2) : \text{the solution to} \prod_N \left[ \hat{\mu}_{23}(\bar{U}) - \hat{\mu}_{22}(\bar{U})X_2 \right] \theta_2 = 0
   \]

2. Compute the imputed pseudo-outcome:
   \[
   Y^* = \hat{\mu}_2(\bar{U}) + \max_{a \in \{0,1\}} Q_2(\bar{H}_2, \hat{\mu}_2(\bar{U}), a \hat{\theta}_2),
   \]

3. Stage 1 regression: estimate \( \hat{\theta}_1 = (\hat{\beta}_1, \hat{\gamma}_1) \) as the solution to:
   \[
   \prod_N \left( X_1(Y^*_2 - X_1 \theta_1) \right) = 0.
   \]

Based on the SSL estimator for the Q-learning model parameters, we can then obtain an estimate for the optimal treatment protocol as:

\[
\bar{d}_t \equiv \bar{d}_t(\bar{H}_t) \equiv d_t(\bar{H}_t; \hat{\theta}_t), \quad \text{where} \quad d_t(\bar{H}_t, \theta_t) = \arg\max_{a \in \{0,1\}} Q_t(\bar{H}_t, a; \theta_t), \quad t = 1, 2.
\]

Theorems 2 and 3 of Section 5 demonstrate the consistency and asymptotic normality of the SSL estimators \( \{\hat{\theta}_t, t = 1, 2\} \) for their respective population parameters \( \{\theta_t, t = 1, 2\} \) even possible misspecification of (1). As we explain next, this in turn yields desirable statistical results for evaluating the resulting policy \( \bar{d}_t \equiv d_t(\bar{H}_t) \equiv d_t(\bar{H}_t, \theta_t) = \arg\max_{a \in \{0,1\}} Q_t(\bar{H}_t, a; \theta_t), \) for \( t = 1, 2 \).

**4. Semi Supervised Off-Policy Evaluation of the Policy**

To evaluate the performance of the optimal policy \( \bar{D} = \{\bar{d}_t(\bar{H}_t; \bar{\theta}_t), t = 1, 2\} \) derived under the Q-learning framework, one may estimate the expected population outcome under the policy \( \bar{D} \):

\[
\bar{V} \equiv \bar{V}(\bar{\theta}) = \mathbb{E}(\mathbb{E}[Y_2 + \mathbb{E}[Y_3 | \bar{H}_2, A_2 = \bar{d}_2(\bar{H}_2; \bar{\theta}_2)] | \bar{H}_1, A_1 = \bar{d}_1(\bar{H}_1; \bar{\theta}_1)) \),
\]

where \( \bar{\theta} = (\bar{\theta}_1, \bar{\theta}_2) \). If models in (1) are correctly specified then under standard causal assumptions (consistency, no unmeasured confounding and positivity), an asymptotically consistent supervised estimator for the value function can be obtained as

\[
\bar{V}_Q = \prod_n \left[ Q_1^0(\bar{H}_1; \bar{\theta}_1) \right],
\]
where \( Q_t^\theta(\hat{H}_t; \theta_{t}) \equiv Q_t(\hat{H}_t, d_t(\hat{H}_t; \theta_{t}); \theta_{t}) \). However, \( \hat{V}_Q \) is likely to be biased when the outcome models in (1) are misspecified which arise frequently in practice since \( Q_1(\hat{H}_1, A_1) \) is especially difficult to specify.

To improve the robustness to model misspecification, we propose an SSL doubly robust (SSL-DR) estimator for \( \hat{V} \) by augmenting \( \hat{V}_Q \) via propensity score weighting. To this end, we define propensity scores

\[
\pi_t(\hat{H}_t) = P\{A_t = 1|\hat{H}_t\}, \quad t = 1, 2.
\]

To estimate \( \{\pi_t(\cdot), t = 1, 2\} \), we impose the following generalized linear models (GLM):

\[
\pi_t(\hat{H}_t; \xi_t) = \sigma(\hat{H}_t^T \xi_t), \quad \text{with} \quad \sigma(x) \equiv 1/(1 + e^{-x}) \quad \text{for} \quad t = 1, 2.
\]

We use the logistic model with potentially non-linear basis functions \( \hat{H} \) for simplicity of presentation but one may choose other GLM or alternative basis expansions to incorporate non-linear effects in the propensity model. We estimate \( \xi = (\xi_1^T, \xi_2^T)^T \) based on the standard maximum likelihood estimators using labeled data, denoted by \( \hat{\xi} = (\hat{\xi}_1^T, \hat{\xi}_2^T)^T \). Denote the limit of \( \hat{\xi} \) as \( \hat{\xi} = (\hat{\xi}_1^T, \hat{\xi}_2^T)^T \), which is not necessarily equal to the true model parameter under correct specification of (5), but corresponds to the population solution of the fitted models.

### 4.1 SUP\textsubscript{DR} Value Function Estimation

To derive a SUP\textsubscript{DR} estimator for \( \hat{V} \) overcoming confounding in the observed data, we let \( \Theta = (\theta^T, \xi^T)^T \) and define the inverse probability weights (IPW) using the propensity scores

\[
\omega_1(\hat{H}_1, A_1, \Theta) \equiv \frac{d_1(\hat{H}_1; \theta_1)A_1}{\pi_1(\hat{H}_1; \xi_1)} + \frac{1 - d_1(\hat{H}_1; \theta_1)}{1 - \pi_1(\hat{H}_1; \xi_1)}, \quad \text{and}
\]

\[
\omega_2(\hat{H}_2, A_2, \Theta) \equiv \omega_1(\hat{H}_1, A_1, \Theta) \left( \frac{d_2(\hat{H}_2; \theta_2)A_2}{\pi_2(\hat{H}_2; \xi_2)} + \frac{1 - d_2(\hat{H}_2; \theta_2)}{1 - \pi_2(\hat{H}_2; \xi_2)} \right).
\]

Then we augment \( Q_t^\theta(\hat{H}_t; \theta_{t}) \) based on the estimated propensity scores via

\[
\nu_{\text{SUPDR}}(L; \hat{\Theta}) = Q_t^\theta(\hat{H}_1; \hat{\theta}_1)\omega_1(\hat{H}_1, A_1, \hat{\Theta}) \left[ Y_2 - \left\{ Q_t^\theta(\hat{H}_1; \hat{\theta}_1) - Q_t^\theta(\hat{H}_2; \hat{\theta}_2) \right\} \right]
\]

\[
+ \omega_2(\hat{H}_2, A_2, \hat{\Theta}) \left\{ Y_3 - Q_t^\theta(\hat{H}_2; \hat{\theta}_2) \right\}
\]

and estimate \( \hat{V} \) as

\[
\hat{V}_{\text{SUPDR}} = P_n \left\{ \nu_{\text{SUPDR}}(L; \hat{\Theta}) \right\}. \quad (6)
\]

**Remark 1** The importance sampling estimators previously proposed in Jiang and Li (2016) and Thomas and Brunsfield (2016) for evaluating a given treatment regime also employ similar augmentation strategies but only considers a given treatment regime not an estimated regime. The construction of augmentation in \( \hat{V}_{\text{SUPDR}} \) also differs from the usual augmented IPW estimators (Chakraborty and Moodie, 2013). As we are interested in the value had the population been treated with \( D \) and not a fixed sequence \( (A_1, A_2) \), we augment the weights for a fixed treatment (i.e. \( A_t = 1 \)) with the propensity score weights for the optimal regime \( I(A_t = d_t) \). Finally, we note that this estimator can easily be extended to incorporate non-binary treatments.
In Section 5 we show that $\hat{V}_{\text{SUPDR}}$ is doubly robust in the sense that if either $\|Q_t(\tilde{H}_t, A_t; \tilde{\theta}_t) - Q_t(\tilde{H}_t, A_t)\|_{L_2(P)} \to 0$, or $\|\pi_t(\tilde{H}_t; \tilde{\xi}_t) - \pi_t(\tilde{H}_t)\|_{L_2(P)} \to 0$ for $t = 1, 2$ then $\hat{V}_{\text{SUPDR}} \overset{P}{\to} \bar{V}$ in probability. Moreover under certain reasonable assumptions $\hat{V}_{\text{SUPDR}}$ is asymptotically normal. Theoretical guarantees and proofs for this procedure are shown in Appendix D.2.

### 4.2 SSL\textsubscript{DR} Value Function Estimation

Analogous to the semi-supervised Q-learning, we propose a procedure for adapting the augmented value function estimator to leverage $\mathcal{U}$, by imputing suitable functions of the unobserved outcome in (6). Since $\tilde{H}_2$ involves $Y_2$, both $\omega_2(\tilde{H}_2, A_2; \Theta)$ and $Q^o_2(\tilde{H}_2; \theta_2) = Y_2\beta_2 + Q^o_2(\tilde{H}_2; \theta_2)$ are not available in the unlabeled set, where $Q^o_2(\tilde{H}_2; \theta_2) = \tilde{H}_2^\top \beta_2 + [\tilde{H}_2 \gamma_2]_+$. Writing

$$
V_{\text{SUPDR}}(L; \tilde{\Theta}) = Q^o_1(\tilde{H}_1; \tilde{\theta}_1) + \omega_1(\tilde{H}_1, A_1, \tilde{\theta}_1) \left\{ (1 + \tilde{\beta}_2) Y_2 - Q^o_1(\tilde{H}_1, \tilde{\theta}_1) + Q^o_2(\tilde{H}_2; \tilde{\theta}_2) \right\} \\
+ \omega_2(\tilde{H}_2, A_2, \tilde{\theta}_2) \left\{ Y_3 - \tilde{\beta}_2 Y_2 - Q^o_2(\tilde{H}_2; \tilde{\theta}_2) \right\},
$$

we note that to impute $V_{\text{SUPDR}}(L; \tilde{\Theta})$ for subjects in $\mathcal{U}$, we need to impute $Y_2$, $\omega_2(\tilde{H}_2, A_2; \tilde{\Theta})$, and $Y_t\omega_2(\tilde{H}_2, A_2; \tilde{\Theta})$ for $t = 2, 3$. Define conditional mean functions

$$
\mu_2(\bar{U}) \equiv \mathbb{E}[Y_2(\bar{U})], \quad \mu_{\omega_2}(\bar{U}) \equiv \mathbb{E}[\omega_2(\tilde{H}_2, A_2; \tilde{\Theta})|\bar{U}], \quad \mu_{t\omega_2}(\bar{U}) \equiv \mathbb{E}[Y_t\omega_2(\tilde{H}_2, A_2; \tilde{\Theta})|\bar{U}],
$$

for $t = 2, 3$, where $\tilde{\Theta} = (\tilde{\theta}_1^\top, \tilde{\xi}_2^\top)^\top$. As in Section 3.2 we approximate these expectations by a flexible imputation model followed by a refitting step for bias correction under possible misspecification of the imputation models.

**Step I: Imputation**

We fit flexible weakly parametric or non-parametric models to the labeled data to approximate the functions $\{\mu_2(\bar{U}), \mu_{\omega_2}(\bar{U}), \mu_{t\omega_2}(\bar{U}), t = 2, 3\}$ with unknown parameter $\tilde{\Theta}$ estimated via the SSL Q-learning as in Section 3.2 and the propensity score modeling as discussed above. Denote the respective imputation models as $\{m_2(\bar{U}), m_{\omega_2}(\bar{U}), m_{t\omega_2}(\bar{U}), t = 2, 3\}$ and their fitted values as $\{\hat{m}_2(\bar{U}), \hat{m}_{\omega_2}(\bar{U}), \hat{m}_{t\omega_2}(\bar{U}), t = 2, 3\}$.

**Step II: Refitting**

To correct for potential biases arising from finite sample estimation and model misspecifications, we perform refitting to obtain final imputed models for $\{Y_2, \omega_2(\tilde{H}_2, A_2; \tilde{\Theta}), Y_t\omega_2(\tilde{H}_2, A_2; \tilde{\Theta}), t = 2, 3\}$ as $\{\bar{\mu}_2(\bar{U}) = m_2(\bar{U}) + \eta^o_2, \bar{\mu}_{\omega_2}(\bar{U}) = m_{\omega_2}(\bar{U}) + \eta^{\omega_2}, \bar{\mu}_{t\omega_2}(\bar{U}) = m_{t\omega_2}(\bar{U}) + \eta^{t\omega_2}, t = 2, 3\}$. As for the estimation of $\theta$ for Q-learning training, these refitted models are not required to be correctly specified but need to satisfy the following constraints are satisfied:

$$
\mathbb{E} \left[ \omega_1(\tilde{H}_1, A_1; \tilde{\Theta}) \left\{ Y_2 - \bar{\mu}_2(\bar{U}) \right\} \right] = 0, \\
\mathbb{E} \left[ Q^o_{2-}(\bar{U}; \theta_2) \left\{ \omega_2(\tilde{H}_2, A_2; \tilde{\Theta}) - \bar{\mu}_{\omega_2}(\bar{U}) \right\} \right] = 0, \\
\mathbb{E} \left[ \omega_2(\tilde{H}_2, A_2; \tilde{\Theta}) Y_t - \bar{\mu}_{t\omega_2}(\bar{U}) \right] = 0, \quad t = 2, 3.
$$
To estimate $\eta_{\omega_2}^v$ and $\eta_{\omega_2}^u$ under these constraints, we again employ cross-fitting and obtain $\tilde{\eta}_{\omega_2}^v$ and $\tilde{\eta}_{\omega_2}^u$ as the solution to the following estimating equations:

\[
\sum_{k=1}^{K} \sum_{i \in I_k} \omega_1(\mathbf{H}_{1i}, A_{1i}; \hat{\Theta}) \left\{ Y_2 - \tilde{m}_2^{(-k)}(\tilde{U}_i) - \tilde{\eta}_{\omega_2}^v \right\} = 0, \\
\sum_{k=1}^{K} \sum_{i \in I_k} Q_2^\omega(\tilde{U}_i; \hat{\theta}_2) \left\{ \omega_2(\mathbf{H}_{2i}, A_{2i}; \hat{\Theta}) - \tilde{m}_2^{(-k)}(\tilde{U}_i) - \tilde{\eta}_{\omega_2}^u \right\} = 0, \\
\sum_{k=1}^{K} \sum_{i \in I_k} \omega_2(\mathbf{H}_{2i}, A_{2i}; \hat{\Theta}) Y_{ii} - \tilde{m}_2^{(-k)}(\tilde{U}_i) - \tilde{\eta}_{\omega_2}^u = 0, \quad t = 2, 3.
\]

The resulting imputation functions for $Y_2, \omega_2(\mathbf{H}_2, A_2; \hat{\Theta})$ and $Y_{i\omega_2}(\mathbf{H}_2, A_2; \hat{\Theta})$ are respectively constructed as $\hat{\mu}_2^v(\tilde{U}) = K^{-1} \sum_{k=1}^{K} \tilde{m}_2^{(-k)}(\tilde{U}) + \tilde{\eta}_{\omega_2}^v$, $\hat{\mu}_{\omega_2}(\tilde{U}) = K^{-1} \sum_{k=1}^{K} \tilde{m}_{\omega_2}(\tilde{U}) + \tilde{\eta}_{\omega_2}^u$, and $\hat{\mu}_{\omega_2}(\tilde{U}) = K^{-1} \sum_{k=1}^{K} \tilde{m}_{\omega_2}(\tilde{U}) + \tilde{\eta}_{\omega_2}^u$, for $t = 2, 3$.

**Step III: Semi-supervised augmented value function estimator.**

Finally, we proceed to estimate the value of the policy $\hat{V}$, using the following semi-supervised augmented estimator:

\[
\hat{V}_{ssl}\text{-dr} = \mathbb{P}_N \left\{ \nu_{ssl}\text{-dr}(\tilde{U}; \hat{\Theta}, \hat{\mu}) \right\}, \quad (9)
\]

where $\hat{V}_{ssl}\text{-dr}(\tilde{U})$ is the semi-supervised augmented estimator for observation $\tilde{U}$ defined as:

\[
\nu_{ssl}\text{-dr}(\tilde{U}; \hat{\Theta}, \hat{\mu}) = Q_1^\nu(\mathbf{H}_1; \hat{\theta}_1) + \omega_1(\mathbf{H}_1, A_1; \hat{\Theta}) \left[ (1 + \beta_{21})\hat{\mu}_2^v(\tilde{U}) - Q_1^\nu(\mathbf{H}_1; \hat{\theta}_1) + Q_2^\omega(\mathbf{H}_2; \hat{\theta}_2) \right] + \beta_{3\omega_2}(\tilde{U}) - \beta_{21}\hat{\mu}_{\omega_2}(\tilde{U}) - Q_2^\omega(\mathbf{H}_2; \hat{\theta}_2)\hat{\mu}_{\omega_2}(\tilde{U}).
\]

The above SSL estimator uses both labeled and unlabeled data along with outcome surrogates to estimate the value function, which yields a gain in efficiency as we show in Corollary 9. As its supervised counterpart, $\hat{V}_{ssl}\text{-dr}$ is double robust in the sense that if either the $Q$ functions or the propensity scores are correctly specified, the value function will converge in probability to the true value $\hat{V}$. Additionally, it does not assume that the estimated treatment regime was derived from a different sample. These properties are summarized in Theorem 7 of the following section.

**5. Theoretical Results**

In this section we discuss the assumptions needed followed by our theoretical results for the semi-supervised Q-learning and value function estimators. Throughout, let $g(\cdot)$ be a real valued function, we define the norm $\|g(x)\|_{L_2(\mathbb{P})} \equiv \sqrt{\int g(x)^2d\mathbb{P}(x)}$. Additionally, let $\{U_n\}$, and $\{V_n\}$ be two sequences of random variables, we will use $U_n = O_{\mathbb{P}}(V_n)$ to denote stochastic boundedness of the sequence $\{U_n/V_n\}$, that is, for any $\epsilon > 0$, $\exists M_{\epsilon}, n_\epsilon \in \mathbb{R}$ such that $\mathbb{P}(|U_n/V_n| > M_{\epsilon}) < \epsilon \forall n > n_\epsilon$. We use $U_n = o_{\mathbb{P}}(V_n)$ to denote that $U_n/V_n \xrightarrow{\mathbb{P}} 0$. 

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5.1 Theoretical Results for SSL Q-learning

Assumption 1 (a) Sample size for $\mathcal{U}$, and $\mathcal{L}$, are such that $n/N \to 0$ as $N, n \to \infty$, (b) $\mathbf{H} \in \mathcal{H}_t, \mathbf{X} \in \mathcal{X}_t$ have finite second moments and compact support in $\mathcal{H}_t \subset \mathbb{R}^{p_t}, \mathcal{X}_t \subset \mathbb{R}^{p_t}$ $t = 1, 2$ respectively (c) $\Sigma_1, \Sigma_2$ are nonsingular.

Assumption 2 Functions $m_s, s \in \{2, 3, 22, 23\}$ are such that (i) $\sup |m_s(\mathbf{U})| < \infty$, and (ii) the estimated functions $\hat{m}_s$ satisfy (ii) $\sup |\hat{m}_s(\mathbf{U}) - m_s(\mathbf{U})| = o_P(1), s \in \{2, 3, 22, 23\}$.

Assumption 3 Define the following class of functions:

$$Q_t \equiv \{ Q_t : \mathcal{X}_t \to \mathbb{R}|\theta_1 \in \Theta_1 \subset \mathbb{R}^{p_t} \}, t = 1, 2,$$

with $\Theta_1, \Theta_2$ open bounded sets, and $p_1, p_2$ fixed under (1). Suppose the population equations for the $Q$ functions $E[S_t^0(\theta)] = 0, t = 1, 2$ have solutions $\bar{\theta}_1, \bar{\theta}_2$, where

$$S_t^2(\theta_2) = \frac{\partial}{\partial \theta_2} \| Y_3 - Q_2(\mathbf{X}_2; \theta_2) \|^2_2, S_t^2(\theta_1) = \frac{\partial}{\partial \theta_1} \| Y_2^* - Q_1(\mathbf{X}_1; \theta_1) \|^2_2.$$

The population minimizers satisfy $\bar{\theta}_t \in \Theta_t, t = 1, 2$, We write $\bar{\beta}_t, \gamma_t$ as the components of $\bar{\theta}_t$ according to equation (2).

Assumption 1 (a) distinguishes our setting from the standard missing data context. Theoretical results for the missing completely at random (MCAR) setting generally assume that the missingness probability is bounded away from zero (Tsiatis, 2006), which enable the use of standard semiparametric theory. However, in our setting one can intuitively consider the probability of observing an outcome being $\frac{n}{n/N}$, which converges to 0.

Assumption 2 is fairly standard as it just requires boundedness of the imputation functions – which is natural to expect from the boundedness of the covariates. We also require uniform convergence of the estimated functions to their limit. This allows for the normal equations targeting the imputation residuals in (4) & (8) to be well defined. Moreover, several off-the-shelf flexible imputation models for estimation can satisfy these conditions. For example local polynomial estimators, basis expansion regression like natural cubic splines or wavelets (Tsybakov, 2009, Ch. 1). In particular, it is worth noting that we do not require any specific rate of convergence. As a result, the required condition is typically much easier to verify for many off-the-shelf algorithms. It is likely that other classes of models such as random forests can satisfy Assumption 2, recent work suggests it’s plausible to use the existing point-wise convergence results to show uniform convergence. (see Scornet et al. (2015); Biau et al. (2008)). Similarly, it would be interesting to explore settings which have theoretical results for neural networks, such as random features, to investigate whether uniform convergence can be shown.

Assumption 3 is fairly standard in the literature and ensures well-defined population level solution for $Q$-learning regressions $\bar{\theta}$ exist and belong to the parameter space. In this regard, we differentiate between population solutions $\bar{\theta}$ and true model parameters $\theta_0$ shown in equation (1). If the working models are misspecified Theorems 2 and 3 still guarantee the $\bar{\theta}$ is consistent and asymptotically normal around the population solution $\theta$. However,
in the case that equation (1) is correct $\hat{\theta}$ is asymptotically normal and consistent for true value $\theta^0$. With the assumptions above we are ready to state the theoretical properties of the semi-supervised $Q$-learning procedure described in Section 3.2.

**Theorem 2 (Distribution of $\hat{\theta}_2$)** Under Assumptions 1-3

$$\sqrt{n}(\hat{\theta}_2 - \theta_2) = \Sigma_2^{-1} \sum_{i=1}^{n} \psi_2(\bar{U}_i; \theta_2) + o_p(1) \overset{d}{\to} \mathcal{N} \left( 0, \mathbf{V}_2(\theta_2) \right),$$

where $\Sigma_2 = \mathbb{E}[\mathbf{X}_2 \mathbf{X}_2^T]$ as defined in Section 2, and the influence function $\psi_2$ is

$$\psi_2(\bar{U}; \theta_2) = \left\{ Y_2 \gamma_2 - \bar{\mu}_{22}(\bar{U}) \right\} - \frac{\partial^2 \{ Y_2 t - \bar{\mu}_2(\bar{U}) \}}{\partial \gamma_2} \right\} \mathbb{E}[\mathbf{X}_2 \gamma_2 \{ Y_2 - \bar{\mu}_2(\bar{U}) \}^T],$$

and $\mathbf{V}_2(\theta_2) = \Sigma_2^{-1} \mathbb{E} \left[ \psi_2(\bar{U}; \theta_2) \psi_2(\bar{U}; \theta_2)^T \right] (\Sigma_2^{-1})^T$.

We hold off remarks until the end of the results for the $Q$-learning parameters. Since the first stage regression depends on the second stage regression through a non-smooth maximum function, we make the following standard assumption (Laber et al., 2014) assumption in order to provide valid statistical inference.

**Assumption 4** Non-zero estimable population treatment effects: the population solution to (2), $\gamma_2$ is such that (a) $H^T_{21} \gamma_2 \neq 0 \ \forall H_{21} \neq 0$, and (b) $\gamma_1$ is such that $H^T_{11} \gamma_1 \neq 0 \ \forall H_{11} \neq 0$

Assumption 4 yields regular estimators for the stage 1 regression and value function as they depend on non-smooth components of the form $[x]_+$. This is needed to achieve asymptotic normality of the $Q$-learning parameters for the first stage regression: note that the estimating equation for the stage 1 regression in section 3.2 includes $[H^T_{21} \gamma_2]_+$, thus for asymptotic normality of $\theta_1$, we require $\sqrt{n} \mathbb{P} \left( [H^T_{21} \gamma_2]_+ = [H^T_{21} \gamma_2]_+ \right)$ to be asymptotically normal. Alternatively this is automatically true if $H_{11}$ contains continuous covariates as $\mathbb{P} (H^T_{21} \gamma_2 = 0) = 0$. Violation of Assumption 4 will yield non-regular estimates which translates into poor coverage for the confidence intervals (see Laber et al. (2014) for a thorough discussion on this topic).

**Theorem 3 (Distribution of $\hat{\theta}_1$)** Under Assumptions 1-3, 4 (a)

$$\sqrt{n}(\hat{\theta}_1 - \theta_1) = \Sigma_1^{-1} \sum_{i=1}^{n} \psi_1(\bar{U}_i; \theta_1) + o_p(1) \overset{d}{\to} \mathcal{N} \left( 0, \mathbf{V}_1(\theta_1) \right)$$

where $\Sigma_1^{-1} = \mathbb{E}[\mathbf{X}_1 \mathbf{X}_1^T]$, $\psi_1(\bar{U}; \theta_1) = \mathbb{E} \left[ \mathbf{X}_1 (1 + \beta_{21}) \{ Y_2 - \bar{\mu}_2(\bar{U}) \} \right] + \mathbb{E} \left[ \mathbf{X}_1 (1 + H_{20}) \right] \mathbf{V}_1(\theta_1), \psi_2(\bar{U}; \theta_2) = \mathbb{E} \left[ \mathbf{X}_1 H_{21} \left| \sum_{i=1}^{n} \psi_1(\bar{U}; \theta_1) \psi_1(\bar{U}; \theta_1)^T \right) (\Sigma_1^{-1})^T, \text{ and } \psi_{\beta_2}, \psi_{\gamma_2} \text{ are the elements corresponding to } \beta_2, \gamma_2 \text{ of the influence function } \psi_2 \text{ defined in Theorem 2.}
Remark 4 1) Theorems 2 and 3 establish the \(\sqrt{n}\)-consistency and asymptotic normality (CAN) of \(\hat{\theta}_1, \hat{\theta}_2\) for any \(K \geq 2\). Beyond asymptotic normality at \(\sqrt{n}\) scale, these theorems also provide an asymptotic linear expansion of the estimators with influence functions \(\psi_1\) and \(\psi_2\) respectively.

2) \(V_1(\theta), V_2(\theta)\) reflect an efficiency gain over the fully supervised approach due to sample \(U\) and the surrogates contribution in prediction performance. This gain is formalized in Proposition 5 which quantifies how correlation between surrogates and outcome increases efficiency.

3) Let \(\psi = [\psi_1^T, \psi_2^T]^T\), we collect the vector of estimated Q-learning parameters \(\theta\), then under Assumptions 1-3, 4 (a), we have

\[
\sqrt{n}(\hat{\theta} - \theta) = \Sigma^{-1} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \psi(U_i; \theta) + o_p(1) \xrightarrow{d} N(0, VSSL(\theta))
\]

with \(VSSL(\theta) = \Sigma^{-1} \mathbb{E} \left[ \psi(U; \theta)\psi(U; \theta)^T \right] (\Sigma^{-1})^T\).

4) Theorems 2 and 3 hold even when the Q functions are misspecified, that is, \(\hat{\theta}_1, \hat{\theta}_2\) are CAN for \(\theta_1, \theta_2\). Furthermore, if model (1) is correctly specified then we can simply replace \(\theta\) with \(\theta^0\) in the above result.

3) We can estimate \(VSSL(\theta)\) via sample-splitting as

\[
\tilde{V}_{SSL}(\hat{\theta}) = \tilde{\Sigma}^{-1} \tilde{A}(\hat{\theta}) \left( \tilde{\Sigma}^{-1} \right)^T, \text{ where}
\]

\[
\tilde{A}(\hat{\theta}) = n^{-1} \sum_{k=1}^{K} \sum_{i \in I_k} \psi^{(-k)} \left( \tilde{U}_i; \hat{\theta} \right) \psi^{(-k)} \left( \tilde{U}_i; \hat{\theta} \right)^T,
\]

\[
\tilde{\Sigma}_t = n^{-1} \sum_{k=1}^{K} \sum_{i \in I_k} X_t^{(-k)} \left( X_t^{(-k)} \right)^T, \quad t = 1, 2.
\]

Note that we can decompose \(\psi\) into the influence function for each set of parameters. For example, we have \(\psi_2 = (\psi_{\mu_2}, \psi_{\gamma_2})^T\) where

\[
\psi_{\gamma_2}(\tilde{U}) = H_{22} A_2 \left( \{Y_3 - \tilde{\mu}_3(\tilde{U})\} - \beta_{21} \{Y_2 - \tilde{\mu}_2(\tilde{U})\} \right).
\]

Therefore we can decompose the variance-covariance matrix into a component for each parameter, the variance-covariance for the treatment effect for stage 2 regression \(\gamma_2\) is

\[
\mathbb{E} \left[ \psi_{\gamma_2} \psi_{\gamma_2}^T \right] = \mathbb{E} \left[ H_{21} H_{21}^T A_2^2 \left( \{Y_3 - \tilde{\mu}_3(\tilde{U})\} - \beta_{21} \{Y_2 - \tilde{\mu}_2(\tilde{U})\} \right)^2 \right].
\]

This gives us some insight into how predictive power of surrogates \(W_1, W_2\) decrease parameter standard errors. This is the case for the influence functions for estimating \(\theta_1, \theta_2\) as well. We formalize this result with the following proposition. Let \(\hat{\theta}_{SUP}\) be the estimator for the fully supervised Q-learning procedure (i.e. only using labeled data), with influence function and asymptotic variance denoted as \(\psi_{SUP}\) and \(V_{SUP}\) respectively.
Proposition 5 Under Assumptions 1-3, 4 (a),

$$V_{SSL}(\theta) = V_{SUP}(\theta) - \text{Var} \left[ \mathbb{E} \left[ \psi_{SUP} | \bar{U} \right] \right]$$

Remark 6 Proposition 5 illustrates that the estimates for the semi-supervised $Q$-learning parameters are at least as efficient, if not more, than the supervised ones (see Laber et al. (2014) for the distribution of $\hat{\theta}_{sup}$). This difference in efficiency is explained by how much information is gained by incorporating the surrogates $\hat{W}_1, \hat{W}_2$ into the estimation procedure in other words, the variance of the projected influence function $\psi_{sup}$ onto the space spanned by the surrogates. If there is no new information in the surrogate variables then $\text{Var} \left[ \mathbb{E} \left[ \psi_{sup} | \bar{U} \right] \right] = 0$ and both methods will yield equally efficient parameters. Such an efficiency is relevant especially for the treatment interaction coefficients $\gamma_1, \gamma_2$ to learn the dynamic treatment rules. Finally, note that for Proposition 5, we do not need the correct specification of $Q$ functions.

5.2 Theoretical Results for SSL Estimation of the Value Function

If model (1) is correct, one only needs to add Assumption 4 (b) for $\mathbb{P}_N \{ \hat{Q}^i_t(\mathcal{H}_t; \hat{\theta}_1) \}$ to be a consistent estimator of the value function $\hat{V}$ (Zhu et al., 2019). However, as discussed assuming (1) is likely misspecified, therefore we establish results for our doubly robust semi-supervised value function estimator. For this, define the following class of functions:

$$W_t \equiv \{ \pi_t : \mathcal{H}_t \mapsto \mathbb{R} | \theta_1 \in \Theta_1, \xi_t \in \Omega_t \}, \ t = 1, 2,$$

under propensity score models $\pi_1, \pi_2$ in (5).

Assumption 5 Let the population equations $\mathbb{E} \left[ S^\xi_t(\bar{H}_t; \Theta_t) \right] = 0, t = 1, 2$ have solutions $\bar{\xi}_1, \bar{\xi}_2$, where

$$S^\xi_t(\bar{H}_t; \Theta_t) = \frac{\partial}{\partial \xi_t} \log \left[ \pi_t(\bar{H}_t; \xi_t)^{A_t} \{ 1 - \pi_t(\bar{H}_t; \xi_t) \}^{(1-A_t)} \right], \ t = 1, 2,$$

(i) $\Omega_1, \Omega_2$ are open, bounded sets and the population solutions satisfy $\bar{\xi}_t \in \Omega_t, t = 1, 2,$

(ii) for $\xi_t, t = 1, 2, \inf_{\bar{H}_t \in \mathcal{H}_t} \pi_t(\bar{H}_t; \xi_t) > 0,$

(iii) for $t = 1, 2,$

$$\sup_{\xi_t} \left\| \mathbb{P}_n S^\xi_t(\bar{H}_t; \Theta_t) - \mathbb{E} \left[ S^\xi_t(\bar{H}_t; \Theta_t) \right] \right\|_{L^2(\mathbb{P})} \overset{p}{\to} 0,$$

$$\inf_{\xi_t \in \mathcal{C}(\xi_t, \xi_t) \geq \delta} \left\| \mathbb{E} \left[ S^\xi_t(\bar{H}_t; \Theta_t) \right] \right\|_{L^2(\mathbb{P})} > 0, \forall \delta > 0.$$

Assumption 6 Functions $m_2, m_{\omega_2}, m_{\omega_2} t = 2, 3$ are such that (i) $\sup_{\bar{U}} |m_s(\bar{U})| < \infty, s \in \{ 2, \omega_2, 2\omega_2, 3\omega_2 \}$ and (ii) the estimated functions $\hat{m}_s$ satisfy (ii) $\sup_{\bar{U}} |\hat{m}_s(\bar{U}) - m_s(\bar{U})| = o_p(1), s \in \{ 2, \omega_2, 2\omega_2, 3\omega_2 \}$.
Assumption 5 is standard for empirical process estimation (Kosorok, 2008). In particular (iii) requires that the score converges to its population limit in \( L_2(P) \) norm as defined in Section 3, and a well separated uniqueness condition for \( \hat{\xi} \). Assumption 6 is the propensity score equivalent version of Assumption 2. However note that for this to be satisfied we are relying on the positivity Assumption (ii) made in Section 2. Finally, we use \( \psi^\xi \) to denote the influence function for \( \hat{\xi} \). We are now ready to state our theoretical results for the value function estimator in equation (9), the proof, and exact form of \( \psi^\xi \) can be found in Appendix B.1.

**Theorem 7 (Asymptotic Normality for \( \hat{V}_{SSLDR} \))** Under Assumptions 1-6, \( \hat{V}_{SSLDR} \) as defined in (9) is such that

\[
\sqrt{n} \left\{ \hat{V}_{SSLDR} - \mathbb{E}_\Sigma \left[ V_{SSLDR}(\tilde{U}; \tilde{\Theta}, \tilde{\mu}) \right] \right\} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \psi^v_{SSLDR}(\tilde{U}_i; \tilde{\Theta}) + o_P(1) \xrightarrow{d} N(0, \sigma^2_V). 
\]

where

\[
\psi^v_{SSLDR}(\tilde{U}; \Theta) = \nu_{SSLDR}(\tilde{U}; \Theta) + \psi^\theta(\tilde{U}) \frac{\partial}{\partial \theta} \int \nu_{SUPDR}(L; \Theta) dP_L \bigg|_{\Theta = \Theta} \\
+ \psi^\xi(\tilde{U}) \frac{\partial}{\partial \xi} \int \nu_{SUPDR}(L; \Theta) dP_L \bigg|_{\Theta = \Theta},
\]

\[
\nu_{SSLDR}(\tilde{U}; \Theta) = \omega_1(\tilde{H}_1, A_1; \Theta_1)(1 + \beta_2) \left\{ Y_2 - \bar{\theta}_2(\tilde{U}) \right\} + \omega_2(\tilde{H}_2, A_2; \Theta_2) Y_3 - \bar{\theta}_3(\tilde{U}) \\
- \beta_2 \left\{ \omega_2(\tilde{H}_2, A_2; \Theta_2) Y_2 - \bar{\theta}_2(\tilde{U}) \right\} - Q^{\varphi}_2(\tilde{H}_2; \theta_2) \left\{ \omega_2(\tilde{H}_2, A_2; \Theta_2) - \bar{\theta}_2(\tilde{U}) \right\},
\]

\[
\sigma^2_V = \mathbb{E} \left[ \psi^v_{SSLDR}(\tilde{U}; \Theta)^2 \right], \quad \psi^\theta = [\psi^\varphi_1, \psi^\varphi_2]^T, \text{ and } \nu_{SUPDR}(L; \Theta) \text{ is defined in (6)}.
\]

**Proposition 8 (Double Robustness of \( \hat{V}_{SSLDR} \))** If either (1) or (5) is correctly specified, then under Assumptions 1-6

\[
\sqrt{n} \left( \hat{V}_{SSLDR} - \bar{V} \right) \xrightarrow{d} \mathcal{N} \left( 0, \left( \frac{\psi^v_{SSLDR}}{\sigma_{SSLDR}^V} \right)^2 \right),
\]

with \( \left( \frac{\psi^v_{SSLDR}}{\sigma_{SSLDR}^V} \right)^2 = \mathbb{E} \left[ \psi^v_{SSLDR}(\tilde{U}; \tilde{\Theta})^2 \right] \).

Next we define the supervised influence function for estimator \( \hat{V}_{SUPDR} \). Let \( \psi^\theta_{SUP}, \psi^\xi_{SUP} \), be the influence functions for the supervised estimators \( \hat{\theta}_{SUP}, \hat{\xi}_{SUP} \) for models (1), (5) respectively, the influence function for \( SUP_{DR} \) Value Function Estimation estimator (6), and it’s variance is (see Appendix D.2 for result and proof.):

\[
\psi^v_{SUPDR}(L; \Theta) = \nu_{SUPDR}(L; \Theta) - \mathbb{E}_\Sigma \left[ \nu_{SUPDR}(L; \Theta) \right] \\
+ \psi^\theta_{SUP}(L)^T \frac{\partial}{\partial \theta} \int \nu_{SUPDR}(L; \Theta) dP_L \bigg|_{\Theta = \Theta} + \psi^\xi_{SUP}(L)^T \frac{\partial}{\partial \xi} \int \nu_{SUPDR}(L; \Theta) dP_L \bigg|_{\Theta = \Theta},
\]

\[
\left( \frac{\psi^v_{SUPDR}}{\sigma_{SUPDR}^V} \right)^2 = \mathbb{E} \left[ \psi^v_{SUPDR}(L; \Theta)^2 \right].
\]
Corollary 9 Under Assumptions 1–6,

\[ \text{Var} \left[ \psi_{\text{SSLDR}}^v (\bar{U}; \Theta) \right] = \text{Var} \left[ \psi_{\text{SUPDR}}^v (L; \Theta) \right] - \text{Var} \left[ \mathbb{E} \left[ \psi_{\text{SUPDR}} (L; \Theta) | \bar{U} \right] \right]. \]

Remark 10 1) To estimate standard errors for \( V_{\text{SSLDR}}(\bar{U}; \Theta) \), we will approximate the derivatives of the expectation terms \( \frac{\partial}{\partial \Theta} \int \psi_{\text{SUPDR}}(L; \Theta) dP_L \) using kernel smoothing to replace the indicator functions. In particular, let \( K_h(x) = \frac{1}{h} \sigma(x/h), \sigma \) defined as in (5), we approximate \( d_i(H_t, \theta) = I\{H_{t1}\gamma_t > 0\} \) with \( K_h(H_{t1}\gamma_t) \) \( t = 1, 2 \), and define the smoothed propensity score weights as

\[
\begin{align*}
\tilde{\omega}_1(H_1, A_1, \Theta) & \equiv A_1K_h(H_{11}\gamma_1) \frac{1 - A_1}{1 - \pi_1(H_1; \xi_1)} + \frac{1 - A_1}{1 - \pi_1(H_1; \xi_1)}; \\
\tilde{\omega}_2(H_2, A_2, \Theta) & \equiv \tilde{\omega}_1(H_1, A_1, \Theta) \left[ A_2K_h(H_{21}\gamma_2) \frac{1 - A_2}{1 - \pi_2(H_2; \xi_2)} + \frac{1 - A_2}{1 - \pi_2(H_2; \xi_2)} \right].
\end{align*}
\]

We simply replace the propensity score functions with these smooth versions in \( \psi_{\text{SSLDR}}^v (\bar{U}; \Theta) \), detail is given in Appendix B.1.1. To estimate the variance we simply use sample-splitting:

\[ \left( \hat{\sigma}_{\text{SSLDR}}^v \right)^2 = n^{-1} \sum_{k=1}^{K} \sum_{i \in I_k} \psi_{\text{SSLDR}}^v (\bar{U}_i; \Theta)^2, \quad t = 1, 2. \]

2) Proposition 8 illustrates how \( \hat{V}_{\text{SSLDR}} \) is asymptotically unbiased if either the \( Q \) functions or the propensity scores are correctly specified. 2) An immediate consequence from Corollary 9 is that the semi-supervised estimator is at least as efficient (or more) as its supervised counterpart, that is \( \text{Var} \left[ \psi_{\text{SSLDR}} \right] \leq \text{Var} \left[ \psi_{\text{SUPDR}} \right] \). 3) Corollary 9 also sheds light on how the efficiency is explained by the difference in variance from both estimators is explained by the variance of \( \mathbb{E} \left[ \psi_{\text{SUPDR}} \bar{U} \right] \), the projection of the influence function for supervised value function into the surrogate-augmented covariate space: \( \bar{U} \). If there is no useful information provided from the surrogates \( W_1, W_2 \), we would expect \( \mathbb{E} \left[ \psi_{\text{SUPDR}} | \bar{U} \right] = \mathbb{E} \left[ \psi_{\text{SUPDR}} \right] \), which makes both estimators equally efficient as \( \text{Var} \left[ \mathbb{E} \left[ \psi_{\text{SUPDR}} | \bar{U} \right] \right] = 0 \).

6. Simulations and application to EMR data:

We performed extensive simulations to evaluate the finite sample performance of our method. Additionally we apply our methods to an EHR study of treatment response for patients with inflammatory bowel disease to identify optimal treatment sequence under the challenge of treatment response outcome is labeled for a small subset of patients.

6.1 Simulation results

We compare our SSL Q-learning methods to fully supervised Q-learning using labeled datasets of size \( n \). We focus on exploring the Q-learning parameter estimation under degree of misspecification for model (1). We also compare the SSL value function estimation to
the supervised counterpart under varying degrees of misspecification for the Q models in (1) and the propensity score functions w in (5).

Following a similar set-up as in Schulte et al. (2014), we first consider a simple scenario with a single confounder variable at each stage with $H_{10} = H_{11} = (1, O_1)^T$, $H_{20} = (Y_2, 1, O_1, A_1, O_1A_1, O_2)^T$, and $H_{21} = (1, A_1, O_2)^T$. Specifically, we sequentially generated

$$O_1 \sim \text{Bern}(0.5), \quad A_1 \sim \text{Bern}(\sigma \{H_{10}^{T}\xi_1^{0}\}), \quad Y_2 \sim \mathcal{N}(X_{1}^{T}\theta_1^{0}, 1),$$

$$O_2 \sim \mathcal{N}(H_{20}^{T}\beta_2^{0}, 2), \quad A_2 \sim \text{Bern}(\sigma \{H_{20}^{T}\xi_2^{0} + \xi_2^{0}O_2^{2}\}), \quad Y_3 \sim \mathcal{N}(m_3 \{H_{20}\}, 2).$$

where $m_3(H_{20}) = H_{20}^{T}\beta_2^{0} + A_2(H_{21}^{T}\gamma_2^{0}) + \beta_{27}^{0}O_{2}^{2}Y_{2}$ sin($O_{2}^{-2}/Y_{2}$). In addition to the continuous outcome setting, we also considered a binary outcome setting with $\{O_t, A_t, t = 1, 2\}$ generated from the same models as above but the binary outcomes are generated from $P(Y_2 = 1|H_1) = \sigma(X_1^{T}\theta_1^{0})$, and $P(Y_3 = 1|H_2) = \sigma(H_{20})$. We fit models $Q_1(H_1, A_1) = H_{10}^{T}\beta_1^{0} + A_1(H_{11}^{T}\gamma_1^{0})$, $Q_2(H_2, A_2) = H_{20}^{T}\beta_2^{0} + A_2(H_{21}^{T}\gamma_2^{0})$ for the Q functions, $\pi_1(H_1) = \sigma(H_{10}^{T}\xi_1^{0})$ and $\pi_2(H_2) = \sigma(H_{20}^{T}\xi_2^{0})$ for the propensity scores. Surrogates are generated as $W_t = [Y_{t+1} + z_t^{0}], Z_t \sim \mathcal{N}(0, \sigma_{z_t}^{2}), t = 1, 2$ where $[x]$ corresponds to the integer part of $x \in \mathbb{R}$. Finally we generate For the imputation models, we considered both the random forest with 500 trees and basis expansion with piecewise-cubic splines with 2 equally spaced knots on the quantiles 33 and 67 (Hastie, 1992).

Throughout, we let $\xi_1^{0} = (0.3, -0.5)^T$, $\beta_1^{0} = (3, 0, 0.1, -0.5)^T$, $\delta_0^{0} = (-0.75, 0, 0.5, -0.75, 0.25, 0)^T$, $\gamma_0^{2} = (0, 0.5, 0.1, -1, -0.1, 0, -0.5)^T$, $\beta_2^{0} = (3, 0, 0.1, -0.5, -0.5, 0, 1)^T$, $\gamma_2^{0} = (1, 0.25, 0.5)^T$, $\xi_2^{0} = (0, 0.5, 0.1, -1, -0.1)^T$. The parameters $\xi_2^{0}$ and $\beta_2^{0}$ index the potential degree of misspecifications in the fitted Q-learning outcome model and the propensity score model with a value of 0 corresponding to a correct specification. We let $\xi_2^{0}$ and $\beta_2^{0}$ vary between $(-1,1)$ to allow for different degrees of misspecifications. Under misspecification of the outcome model or propensity score model, the non linear term is highly non linear, in which case the imputation model will be misspecified as well. We note that our method does not need correct specification of the imputation model. We consider two choices of $\{(n, N)\}: (135, 1272)$ which is similar to the sizes of our EHR study and a larger sizes of $\{(500, 10000)\}$. For each configuration, we summarize results based on 500 replications.

In Table 1, we present the results for the estimation of $\bar{\theta}$ under the correct model specification setting with $\beta_{27}^{0} = \xi_{26}^{0} = 0$. Overall, compared to the supervised approach, the proposed semi-supervised Q-learning approach has substantial gains in efficiency while maintaining comparable or even lower bias. This is likely do to the refitting step which helps take care of the finite sample bias, both from the missing outcome imputation and Q function parameter estimation. When the model is correctly specified, imputation with BE yields slightly better estimates than when using RF, both in terms of efficiency and bias.

We summarize in Figures 1 and 2 the results under varying degrees of model misspecifications in either outcome model or the propensity score model, respectively. We compare efficiency of Q-learning parameters and value function estimate measured by empirical mean square error (EMSE). There is substantial efficiency gain from using the proposed SSL methods vs. the supervised methods, both for Q-learning and value function estimation at any misspecification of either, or both models. Additionally, SSL methods yield unbiased estimates, in some cases the bias is even better than the supervised versions thanks to the final sample correction done in the refitting step.
Table 1: Bias, empirical standard error (ESE) of the supervised and the SSL estimators with either random forest imputation or basis expansion imputation strategies for $\bar{\theta}$ when (a) $n = 135$ and $N = 1272$ and (b) $n = 500$ and $N = 10000$. For the SSL estimators, we also obtain the average of the estimated standard errors (ASE) as well as the empirical coverage probabilities (CovP) of the 95% confidence intervals.

Figure 1 shows the relative EMSE and absolute bias difference for the treatment interaction coefficients for the 2-stage $Q$ functions across degrees of model misspecification. Semi-supervised $Q$-learning is more efficient for any degree of misspecification for both small and large finite sample settings. There is a greater gain in efficiency for small sample size as $n$ is insufficient for supervised $Q$ learning to achieve good performance. Regarding bias, the semi-supervised approach is better in all but $\gamma_{23}$, the coefficient for $O_2$. This is highly correlated with the misspecification term: $O_2^2 Y_2 \sin \left( \frac{1}{O_2^2 Y_2^2} \right)$, for which the refitting step is
not enough for bias-correction. We show that this is not the case when the misspecified term is a polynomial in $O_2$ in Appendix A.

Figure 1: Monte Carlo MSE ratios and standardized bias difference for estimation of $\gamma_{11}, \gamma_{12}, \gamma_{21}, \gamma_{22}, \gamma_{23}$ under misspecification of the Q-functions through $\beta_{27}$. Results are shown for the large ($N = 10,000, n = 500$) and small data settings ($N = 1,272, n = 135$). EMSE ratios > 1 & absolute bias difference > 0 favor semi supervised Q-learning.

Figure 2: Monte Carlo estimates for doubly-robust value function estimation: $\hat{V}_{SSLDR}, \hat{V}_{SUPDR}$ under continuous and binary settings. First two columns have EMSE ratio and difference of the absolute value of bias for value function estimation, comparison is done across degree of misspecification of the Q function with $\beta_{27}$ and on the propensity score for $A_2$ with $\xi_{26}$. MSE ratios > 1 & absolute bias difference > 0 favor semi supervised value function estimation.

Finally we analyze performance of the doubly robust value function estimators for both binary and continuous outcomes. Figure 2 shows EMSE ratio and absolute bias difference
across varying levels of misspecification of the $Q$ functions and propensity scores. Bias across simulation settings are relatively similar between $\hat{V}_{SSLDR}$ and $\hat{V}_{SUPDR}$. The low magnitude of bias suggests both estimators are robust to model misspecification. Negative levels of $\beta_0^{27}$ favors $\hat{V}_{SSLDR}$ for both binary and continuous outcomes. The semi-supervised value function estimator, however, is generally more efficient across levels of $\beta_0^{27}$ and $\xi_0^{26}$. Specially for negative $\xi_0^{26}$ values. On the other hand, large $\xi_0^{26}$ values result in efficiency loss for $\hat{V}_{SSLDR}$: there is a price to pay due to the over-fitting bias in the refitting step.

6.2 Application to an EHR Study of Inflammatory Bowl Disease

Anti–tumor necrosis factor (anti-TNF) therapy has greatly changed the management and improved the outcomes of patients with inflammatory bowl disease (IBD) (Peyrin-Biroulet, 2010). However, it remains unclear whether a specific anti-TNF agent has any efficacy advantage over the other agents, especially at the individual level. There has been few randomized clinical trials performed to directly compare anti-TNF agents for treating IBD patients (Sands et al., 2019). Retrospective studies comparing infliximab and adalimumab for treating IBD have found limited and sometimes conflicting evidence of their relative effectiveness (Inokuchi et al., 2019; Lee et al., 2019; Osterman and Lichtenstein, 2017). There is even less evidence regarding optimal STR for choosing these treatments over time (Ananthakrishnan et al., 2016). To explore this, we performed RL using data from a cohort of IBD patients previously identified via machine learning algorithms from the EHR systems of two tertiary referral academic centers in the Greater Boston metropolitan area (Ananthakrishnan et al., 2012). We focused on the subset of $N = 1,272$ patients who initiated either Infliximab ($A_1 = 0$) or Adalimumab ($A_1 = 1$) and continued being treated by either of these two therapies during next 6 months. The observed treatment sequence distribution are shown in Table 2. The outcomes of interest are the binary indicator of treatment response at 6 months ($t = 2$) and at 12 months ($t = 3$), both of which were only available on a subset of $n = 135$ patients whose outcomes were manually annotated via chart review.

To derive STR, we included gender, age, Charlson co-morbidity index (Charlson et al., 1987), prior exposure of anti-TNF agents as well as mentions of clinical terms associated with IBD such as bleeding complications in the clinical notes extracted via natural language processing (NLP) features as confounding variables at both time points. To improve the imputation of $Y_t$, we use 15 relevant NLP features such as mentions of rectal or bowel resection surgery as surrogates at $t = 1, 2$. We transformed all count variables using $x \mapsto \log(1 + x)$ to decrease skewness in the distributions. We used RF with 500 trees to carry out the imputation step, and 10-fold cross-validation (CV) to estimate the value function.

The supervised and semi-supervised estimates are shown in Table 3 for the $Q$-learning models and in Table 4 for the value functions associated with the estimated optimal STR. Similar to those observed in the simulation studies, the semi-supervised $Q$-learning has more power to detect significant predictors of treatment response. Relative efficiency for almost all $Q$ function estimates is near or over 2. The supervised $Q$-learning does not have the power to detect predictors such as prior use of anti-TNF agents, which are clearly relevant to treatment response (Ananthakrishnan et al., 2016). Semi-supervised $Q$-learning is able to detect that patients receiving Adalimumab in the first stage experienced a higher rate of
treatment response, this cannot be detected with supervised Q-learning. Additionally, using Q-learning on $L$ only does not pick up that there is a higher rate of response to Adalimumab among patients which are male, have a high Charlson score, or have experienced an abscess, or fistula. This translates into a far from optimal treatment rule as seen in the value function estimates. Table 4 reflects that using our semi-supervised approach to find the regime and to estimate the value function of such treatment rule not only yields a better treatment regime: $\hat{V}_{\text{supDR}} > \hat{V}_{\text{sslDR}}$, but the estimate is more efficient.

Table 2: Distribution of treatment trajectories for observed sample of size 1407.

| Parameter | Stage 1 regression | Stage 2 regression | Relative Efficiency |
|-----------|--------------------|--------------------|---------------------|
|           | Supervised | Semi-Supervised | Est. | SE  | P-val. | Est. | SE  | P-val. | RE |
| (Intercept) | 0.062  | 0.177  | 0.730 | 0.128 | 0.084 | 0.129 | 2.1 |
| $A_1$ | 0.266  | 0.576  | 0.645 | 0.622 | 0.386 | 0.010 | 2.4 |
| female | -0.292 | 0.124 | 0.020 | -0.190 | 0.058 | 0.002 | 2.1 |
| age | 0.012  | 0.006  | 0.038 | 0.009 | 0.003 | 0.003 | 2.0 |
| Charlson Score | 0.014 | 0.024 | 0.522 | 0.003 | 0.011 | 0.814 | 2.2 |
| prior norm | -0.329 | 0.138 | 0.019 | -0.326 | 0.064 | 0.000 | 2.2 |
| perianal | 0.512  | 0.133  | 0.000 | 0.436 | 0.063 | 0.000 | 2.1 |
| bleeding | 0.005  | 0.006  | 0.486 | 0.007 | 0.003 | 0.006 | 2.5 |
| $A_1 \times$ female | -0.045 | 0.526 | 0.939 | 0.095 | 0.273 | 0.729 | 2.1 |
| $A_1 \times$ age | -0.013 | 0.019 | 0.494 | -0.013 | 0.008 | 0.105 | 2.3 |
| $A_1 \times$ Charlson Score | -0.047 | 0.102 | 0.716 | -0.001 | 0.043 | 0.975 | 2.4 |
| $A_1 \times$ perianal | -0.374 | 0.522 | 0.476 | -0.433 | 0.264 | 0.106 | 2.0 |
| $A_1 \times$ bleeding | 0.039  | 0.035  | 0.267 | 0.010 | 0.011 | 0.366 | 3.2 |

Table 3: Results of Inflammatory Bowel Disease data set, for first and second stage regressions. Fully supervised Q-learning is shown on the left and semi-supervised is shown on the right. Last columns in the panels show relative efficiency (RE) defined as the ratio of standard errors of the semi-supervised vs. supervised method, RE greater than one favors semi-supervised. Significant coefficients at the 0.05 level are in bold.

| Parameter | Stage 1 regression | Stage 2 regression | Relative Efficiency |
|-----------|--------------------|--------------------|---------------------|
|           | Supervised | Semi-Supervised | Est. | SE  | P-val. | Est. | SE  | P-val. | RE |
| (Intercept) | 0.384  | 0.106  | 0.000 | 0.383 | 0.053 | 0.000 | 2.0 |
| $A_1$ | 0.212  | 0.385  | 0.583 | 0.014 | 0.229 | 0.952 | 1.7 |
| female | 0.011  | 0.066  | 0.867 | 0.030 | 0.038 | 0.438 | 1.7 |
| age | 0.003  | 0.084  | 0.376 | 0.003 | 0.002 | 0.164 | 1.9 |
| Charlson Score | 0.005 | 0.015 | 0.728 | 0.002 | 0.008 | 0.798 | 1.9 |
| prior norm | -0.195 | 0.109 | 0.077 | -0.207 | 0.066 | 0.002 | 1.7 |
| perianal | -0.020 | 0.091 | 0.827 | -0.010 | 0.049 | 0.836 | 1.9 |
| bleeding | -0.001 | 0.004 | 0.873 | -0.001 | 0.003 | 0.583 | 1.7 |
| $A_1 \times$ female | 0.066 | 0.291 | 0.823 | -0.104 | 0.169 | 0.541 | 1.7 |
| $A_1 \times$ age | -0.002 | 0.009 | 0.952 | -0.002 | 0.003 | 0.587 | 1.7 |
| $A_1 \times$ Charlson Score | -0.052 | 0.044 | 0.260 | -0.032 | 0.019 | 0.102 | 2.3 |
| $A_1 \times$ perianal | -0.396 | 0.254 | 0.122 | -0.170 | 0.144 | 0.241 | 1.8 |
| $A_1 \times$ bleeding | 0.015 | 0.018 | 0.412 | 0.017 | 0.010 | 0.091 | 1.9 |
| $A_2 \times$ female | -0.197 | 0.270 | 0.468 | -0.054 | 0.165 | 0.760 | 1.6 |
| $A_2 \times$ Charlson Score | 0.007 | 0.005 | 0.133 | 0.004 | 0.002 | 0.855 | 2.2 |
| $A_2 \times$ perianal | 0.004 | 0.007 | 0.564 | 0.007 | 0.003 | 0.172 | 2.2 |
| $A_2 \times$ bleeding | -0.029 | 0.208 | 0.117 | -0.311 | 0.090 | 0.001 | 2.3 |
| $A_2 \times$ age | 0.006 | 0.007 | 0.377 | 0.004 | 0.004 | 0.333 | 1.7 |
| $A_2 \times$ Charlson Score | 0.030 | 0.025 | 0.237 | 0.027 | 0.012 | 0.035 | 2.0 |
| $A_2 \times$ perianal | 0.485 | 0.238 | 0.045 | 0.373 | 0.115 | 0.002 | 2.1 |
| $A_2 \times$ bleeding | 0.008 | 0.006 | 0.999 | -0.003 | 0.003 | 0.426 | 1.9 |
| $A_2 \times$ Charlson Score | -0.022 | 0.013 | 0.102 | -0.012 | 0.006 | 0.046 | 2.4 |
| $A_2 \times$ perianal | -0.005 | 0.007 | 0.499 | -0.006 | 0.003 | 0.080 | 2.1 |

Table 4: Value function estimates for Inflammatory Bowel Disease data set, first row has the estimate for treatment rule learned using $U$ and it’s respective value function second row shows the same for a rule estimated using $L$ and it’s estimated value.
7. Discussion

We have proposed an efficient strategy for the semi-supervised setting for estimation of dynamic treatment rules and their value function. In particular we develop a two step estimation procedure which is amenable to non-parametric imputation of the missing outcomes. This helped us establish $\sqrt{n}$-consistency and asymptotic normality for both the $Q$ function parameters $\hat{\theta}$ and the doubly robust value function estimator $\hat{V}_{SSLDR}$. We additionally provide theoretical results which illustrate if, and when the outcome-surrogates $W$ contribute towards efficiency gain in estimation of $\theta$ and $\hat{V}_{SSLDR}$.

We focus on the 2-time point setting for simplicity but all our theoretical results can be extended to a higher time horizon. In implementation one would need to be careful with the variability of the IPW-value function which increases substantially with time. We believe our semi-supervised framework could be useful for this, as shown in Corollary 9. Additionally, this strategy can be used to estimate DTRs using $A$-learning which can yield efficiency gain under certain scenarios (Schulte et al. 2014).
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Appendix A. Simulation Results for Alternative Settings

In this Section we provide results for data generating scenarios alternative to the ones shown in Section 6.

Let $H_{10} = H_{11} = (1, O_1)$, $H_{20} = (1, Y_2, O_1, A_1, O_1 A_1, O_2, O_2^2)$, $H_{21} = (1, A_1, O_2)$, the data is generated sequentially according to: $O_1, A_1|O_1 \in \{0, 1\}$ with $P(O_1 = 1) = \frac{1}{2}$, $P(A_1 = 1|O_1) = \sigma (H_{10}^T \xi_1)$, $O_2|O_1, A_1, Y_2 \sim N((1, O_1, O_1 A_1, A_1 Y_2) \beta_0^1, 2)$, $A_2|O_1, O_2, A_1, Y_2 \in \{0, 1\}$ with $P(A_2 = 1|O_1, O_2, A_1, Y_2) = \sigma ((1, O_1, A_2, O_2, A_1 O_2, O_2^2, Y_2) \xi_2)$.

Continuous outcome setting: $Y_2|H_1 \sim N(H_{10}^T \theta_0^1 + A_1 (H_{11}^T \gamma_1)^0, 1)$, $Y_3|H_2 \sim N(H_{20}^T \theta_2^0 + A_2 (H_{21}^T \gamma_2)^0, 2)$.

Binary outcome setting:

$P(Y_2 = 1|H_1) = \sigma (H_{10}^T \theta_1^0 + A_1 (H_{11}^T \gamma_1)^0)$, $P(Y_3 = 1|H_2) = (H_{20}^T \theta_2^0 + A_2 (H_{21}^T \gamma_2)^0)$.

| Parameter | Supplied | Semi-supervised |
|-----------|----------|-----------------|
| $\beta_{11}$ | 4.904 | -0.027 | 0.166 | -0.008 | 0.099 | 0.09 | 0.94 | -0.007 | 0.086 | 0.081 | 0.94 |
| $\beta_{12}$ | 1.06 | 0.011 | 0.207 | 0.006 | 0.116 | 0.113 | 0.95 | 0.004 | 0.106 | 0.102 | 0.95 |
| $\gamma_{11}$ | 1.34 | 0 | 0.218 | 0.004 | 0.126 | 0.12 | 0.94 | 0.004 | 0.111 | 0.107 | 0.94 |
| $\beta_{12}$ | 0.609 | 0.008 | 0.279 | 0.002 | 0.173 | 0.155 | 0.94 | 0.002 | 0.161 | 0.141 | 0.94 |
| $\gamma_{21}$ | 0.847 | 0.017 | 0.44 | 0.003 | 0.299 | 0.288 | 0.96 | 0.002 | 0.289 | 0.281 | 0.97 |
| $\beta_{22}$ | 0.003 | 0 | 0.16 | 0.006 | 0.099 | 0.099 | 0.94 | 0 | 0.087 | 0.086 | 0.95 |
| $\beta_{23}$ | -0.135 | 0.001 | 0.208 | 0.003 | 0.134 | 0.127 | 0.93 | 0.004 | 0.119 | 0.112 | 0.93 |
| $\beta_{24}$ | -0.468 | 0.005 | 0.109 | 0.001 | 0.068 | 0.069 | 0.95 | -0.003 | 0.055 | 0.059 | 0.97 |
| $\gamma_{21}$ | 0.847 | -0.003 | 0.229 | -0.002 | 0.135 | 0.139 | 0.95 | -0.008 | 0.118 | 0.121 | 0.96 |
| $\gamma_{22}$ | 0.15 | -0.034 | 0.346 | -0.021 | 0.206 | 0.208 | 0.94 | -0.013 | 0.19 | 0.18 | 0.94 |
| $\gamma_{23}$ | 0.466 | -0.009 | 0.115 | -0.005 | 0.072 | 0.072 | 0.95 | 0 | 0.058 | 0.062 | 0.97 |

Table A.1: Simulation results over 500 datasets, using scenario (1) with $n = 500$, $N = 10,000$

| Parameter | OLS | Random Forests | Basis Expansion |
|-----------|-----|----------------|-----------------|
| $\beta_{11}$ | 4.904 | -0.064 | 0.324 | -0.043 | 0.214 | 0.215 | 0.93 | -0.03 | 0.193 | 0.189 | 0.93 |
| $\beta_{12}$ | 1.061 | 0.017 | 0.4 | 0.011 | 0.252 | 0.256 | 0.95 | 0.01 | 0.233 | 0.225 | 0.92 |
| $\gamma_{11}$ | 1.34 | -0.062 | 0.442 | -0.032 | 0.291 | 0.307 | 0.94 | -0.011 | 0.249 | 0.273 | 0.94 |
| $\gamma_{12}$ | 0.609 | 0.038 | 0.544 | 0.018 | 0.35 | 0.358 | 0.94 | 0.004 | 0.315 | 0.317 | 0.94 |
| $\beta_{21}$ | 0.074 | 0.007 | 0.1 | 0.003 | 0.133 | 0.146 | 0.96 | 0.002 | 0.086 | 0.093 | 0.96 |
| $\beta_{22}$ | 2.953 | 0.003 | 0.404 | 0.003 | 0.299 | 0.328 | 0.96 | 0.01 | 0.238 | 0.291 | 0.97 |
| $\beta_{23}$ | 0.003 | -0.015 | 0.306 | -0.003 | 0.215 | 0.236 | 0.96 | 0.003 | 0.186 | 0.193 | 0.95 |
| $\beta_{24}$ | 0.086 | 0.003 | 0.603 | 0.002 | 0.498 | 0.56 | 0.95 | -0.002 | 0.402 | 0.507 | 0.95 |
| $\beta_{25}$ | -0.351 | 0.02 | 0.396 | -0.004 | 0.289 | 0.299 | 0.96 | -0.007 | 0.246 | 0.249 | 0.95 |
| $\beta_{26}$ | -0.468 | 0.007 | 0.206 | 0.019 | 0.181 | 0.193 | 0.94 | 0.001 | 0.142 | 0.171 | 0.96 |
| $\gamma_{21}$ | 0.847 | 0.017 | 0.44 | -0.008 | 0.337 | 0.364 | 0.95 | -0.016 | 0.274 | 0.314 | 0.95 |
| $\gamma_{22}$ | 0.15 | -0.02 | 0.628 | -0.003 | 0.497 | 0.563 | 0.96 | 0.002 | 0.415 | 0.509 | 0.95 |
| $\gamma_{23}$ | 0.466 | -0.006 | 0.218 | -0.021 | 0.189 | 0.199 | 0.94 | -0.004 | 0.148 | 0.176 | 0.96 |

Table A.2: Simulation results over 500 datasets, using scenario (1) with $n = 135$, $N = 1,272$
Figure A.1: Monte Carlo MSE ratios and standardized bias difference for estimation of $\gamma_{11}$, $\gamma_{12}$, $\gamma_{21}$, $\gamma_{22}$, $\gamma_{23}$ under misspecification of the Q-functions through $\beta_{27}^0$. Results are shown for the large ($N = 10,000$, $n = 500$) and small data settings ($N = 1,272$, $n = 135$). EMSE ratios $> 1$ & bias difference $> 0$ favor semi supervised $Q$-learning.
Figure A.2: Monte Carlo estimates for 500 datasets for simulations under setting (1) and (2). First two columns have Empirical MSE ratio and bias difference for value function estimation, comparison is done across degree of miss-specificaction of the $Q$ function with $\beta_{27}$ and on the propensity score for $A_2$ with $\xi_{26}$. MSE ratios $>1$ & bias difference $>0$ favor semi supervised value function estimation. The third column shows density of $\hat{V}(D^*)$ using regular OLS and semi-supervised estimation, average standard errors are shown on colored dashed lines and true value $V(D^*)$ is shown in black.
Appendix B. Proof of Main Results

B.1 Value Function Results

In this Section we prove the main results for our SSL value function estimator, before the proofs we go over some useful definitions, notation and lemmas. Define the set

$$S(\delta) = \left\{ (\theta, \xi) \bigg| \|\hat{\theta} - \theta\|_2^2 < \delta, \|\hat{\xi} - \xi\|_2^2 < \delta, \theta_t \in \Theta_t, \xi_t \in \Omega_t, t = 1, 2, \pi_1(H_1; \xi_1) > 0, \pi_2(H_2; \xi_2) > 0, \forall H \in \mathcal{H} \right\}.$$ 

We will be using the influence functions for our model parameters $\Theta$, in this regard let $\psi^\theta = (\psi_1, \psi_2)^T$, by Theorems 2 & 3 $\sqrt{n}(\hat{\theta} - \theta) = n^{-1/2} \sum_{i=1}^n \psi^\theta(U_i) + o_p(1)$. Next, from Assumption 5, as we use maximum likelihood estimation for $\xi$, we have the following expansion $\sqrt{n}(\hat{\xi} - \xi) = n^{-1/2} \sum_{i=1}^n \psi^\xi (L_i; \xi) + o_p(1)$, where

$$\psi^\xi (L; \xi) = E\{ \hat{H}^T_l \hat{H}_t \sigma(\hat{H}^T_l \xi_l) [1 - \sigma(\hat{H}^T_l \xi_l)] \}^{-1} \hat{H}_t \{ A_t - \sigma(\hat{H}^T_l \xi_l) \}$$

and $E[\psi^\xi] = 0, E[(\psi^\xi)^T \psi^\xi] < \infty$.

We now introduce a set of definitions used in this section to make the proofs easier to read. Recall from (9) we have $\hat{V}_{\text{SSL-DR}} = \mathbb{P}_Y \left\{ \mathcal{V}_{\text{SSL-DR}}(\hat{U}; \hat{\Theta}, \hat{\mu}) \right\}$, where $\mathcal{V}_{\text{SSL-DR}}(\hat{U}; \hat{\Theta}, \hat{\mu})$ is the semi-supervised augmented estimator for observation $\hat{U}$, we re-write $\mathcal{V}_{\text{SSL-DR}}(\hat{U}; \hat{\Theta}, \hat{\mu})$ as $\mathcal{V}_{\Theta, \mu}(\hat{U})$. Finally, as $\mu_{\omega_2}, \mu_{\omega_2} t = 2, 3$ depend on both the missing outcomes and $\Theta$ through the propensity scores and indicator functions, to make this explicit we’ll write

$$\mu_{\pi_{2}(1)} = E \left[ \frac{1}{\pi_2(H_2; \xi_2)} \bigg| \hat{U} \right], \mu_{\pi_{2}(0)} = E \left[ \frac{1}{1 - \pi_2(H_2; \xi_2)} \bigg| \hat{U} \right],$$

$$\mu_{\pi_{2}(1)} = E \left[ \frac{Y_t}{\pi_2(H_2; \xi_2)} \bigg| \hat{U} \right], \mu_{\pi_{2}(0)} = E \left[ \frac{Y_t}{1 - \pi_2(H_2; \xi_2)} \bigg| \hat{U} \right], t = 2, 3,$$

and therefore we have

$$\mu_{\omega} = \omega_1(\hat{H}, A_1; \Theta) \left\{ d_2(H_2, \theta_2)A_2\mu_{\pi_{2}(1)} + (1 - d_2(H_2, \theta_2))(1 - A_2)\mu_{\pi_{2}(0)} \right\},$$

$$\mu_{\omega} = \omega_1(\hat{H}, A_1; \Theta) \left\{ d_2(H_2, \theta_2)A_2\mu_{\pi_{2}(1)} + (1 - d_2(H_2, \theta_2))(1 - A_2)\mu_{\pi_{2}(0)} \right\} t = 2, 3.$$

Keeping this in mind, we define the following functions:

$$V_{\Theta, \mu}(\hat{U}) = Q_1(\hat{H}_1; \hat{\theta}_1) + \omega_1(\hat{H}_1, A_1, \Theta) \left[ (1 + \beta_{21})\hat{\mu}_{21}^\xi(\hat{U}) - Q_1(H_1; \hat{\theta}_1) + Q_2^\mu(H_2; \hat{\theta}_2) \right]$$

$$+ \omega_1(\hat{H}_1, A_1, \Theta) \left\{ \hat{\mu}_{31}^\xi(\hat{U}) - \beta_{21}^\xi \hat{\mu}_{22}^\xi(\hat{U}) - Q_1^\xi(H_1; \hat{\theta}_1) + Q_2^\xi(H_2; \hat{\theta}_2) \right\}$$

$$+ \omega_1(\hat{H}_1, A_1, \Theta)(1 - d_2)(1 - A_2) \left\{ \hat{\mu}_{31}^\xi(\hat{U}) - \beta_{21}^\xi \hat{\mu}_{22}^\xi(\hat{U}) - Q_1^\xi(H_1; \hat{\theta}_1) + Q_2^\xi(H_2; \hat{\theta}_2) \right\}$$

$$V_{\Theta, \mu}(\hat{U}) \equiv Q_1(\hat{H}_1; \hat{\theta}_1) + \omega_1(\hat{H}_1, A_1, \Theta) \left[ (1 + \beta_{21})^{\hat{\mu}}_2(\hat{U}) - Q_1^1(\hat{H}_1; \hat{\theta}_1) + Q_2^\mu_2(H_2; \hat{\theta}_2) \right]$$

$$+ \omega_1(\hat{H}_1, A_1, \Theta) d_2A_2 \left\{ \hat{\mu}_{31}^\xi(\hat{U}) - \beta_{21}^\xi \hat{\mu}_{22}^\xi(\hat{U}) - Q_1^\xi(H_1; \hat{\theta}_1) + Q_2^\xi(H_2; \hat{\theta}_2) \right\}$$

$$+ \omega_1(\hat{H}_1, A_1, \Theta)(1 - d_2)(1 - A_2) \left\{ \hat{\mu}_{31}^\xi(\hat{U}) - \beta_{21}^\xi \hat{\mu}_{22}^\xi(\hat{U}) - Q_1^\xi(H_1; \hat{\theta}_1) + Q_2^\xi(H_2; \hat{\theta}_2) \right\}.$$
Finally we define the following functions which are weighted sums of the imputation function errors:

\[ \mathcal{V}_{\Theta, \hat{\mu}}(\bar{U}) \equiv \omega_1(\hat{H}_1, A_1; \hat{\Theta}) (1 + \hat{\beta}_2) \left\{ \hat{\mu}_2(\bar{U}) - \mu_2(\bar{U}) \right\} \]

\[ + \omega_1(\hat{H}_1, A_1; \hat{\Theta}) \hat{d}_2 \left[ \hat{\mu}_{3\pi_2}(\bar{U}) - \mu_{3\pi_2}(\bar{U}) \right] + (1 - \hat{d}_2)(1 - A_2) \left\{ \hat{\mu}_{3\pi_2}(\bar{U}) - \mu_{3\pi_2}(\bar{U}) \right\} \]

\[ - \omega_1(\hat{H}_1, A_1; \hat{\Theta}) \hat{d}_2 \left[ \hat{\mu}_{2\pi_2}(\bar{U}) - \mu_{2\pi_2}(\bar{U}) \right] + (1 - \hat{d}_2)(1 - A_2) \left\{ \hat{\mu}_{2\pi_2}(\bar{U}) - \mu_{2\pi_2}(\bar{U}) \right\} \]

\[ - \omega_1(\hat{H}_1, A_1; \hat{\Theta}) Q^0_{\Theta} \left( \hat{H}_2; \hat{\theta}_2 \right) \left[ \hat{\mu}_{\pi_2}(\bar{U}) - \mu_{\pi_2}(\bar{U}) \right] + (1 - \hat{d}_2)(1 - A_2) \left\{ \hat{\mu}_{\pi_2}(\bar{U}) - \mu_{\pi_2}(\bar{U}) \right\} , \]

\[ \mathcal{E}_{\Theta}(\bar{U}) \equiv \omega_1(\hat{H}_1, A_1; \Theta) (1 + \beta_2) \left\{ \hat{\mu}_2(\bar{U}) - \mu_2(\bar{U}) \right\} \]

\[ + \omega_1(\hat{H}_1, A_1; \Theta) \hat{d}_2 \left[ \hat{\mu}_{3\pi_2}(\bar{U}) - \mu_{3\pi_2}(\bar{U}) \right] + (1 - \hat{d}_2)(1 - A_2) \left\{ \hat{\mu}_{3\pi_2}(\bar{U}) - \mu_{3\pi_2}(\bar{U}) \right\} \]

\[ - \omega_1(\hat{H}_1, A_1; \Theta) \hat{d}_2 \left[ \hat{\mu}_{2\pi_2}(\bar{U}) - \mu_{2\pi_2}(\bar{U}) \right] + (1 - \hat{d}_2)(1 - A_2) \left\{ \hat{\mu}_{2\pi_2}(\bar{U}) - \mu_{2\pi_2}(\bar{U}) \right\} \]

\[ - \omega_1(\hat{H}_1, A_1; \Theta) Q^0_{\Theta} \left( \hat{H}_2; \theta_2 \right) \left[ \hat{\mu}_{\pi_2}(\bar{U}) - \mu_{\pi_2}(\bar{U}) \right] + (1 - \hat{d}_2)(1 - A_2) \left\{ \hat{\mu}_{\pi_2}(\bar{U}) - \mu_{\pi_2}(\bar{U}) \right\} . \]

These definitions will come in handy in the following proofs as we can use them to write

\[ \mathcal{V}_{\Theta, \hat{\mu}}(\bar{U}) = \mathcal{V}_{\Theta, \mu}(\bar{U}) + \mathcal{E}_{\Theta}(\bar{U}) \]

\[ \mathcal{V}_{\Theta, \hat{\mu}}(\bar{U}) = \mathcal{V}_{\Theta, \mu}(\bar{U}) + \mathcal{E}_{\Theta}(\bar{U}) . \]

Finally, recalling that \( \mathbb{P}_\bar{U} \) is the underlying distribution of the data, we define function \( g_1 : \Theta \mapsto \mathbb{R} \) as

\[ g_1(\Theta) = \int \mathcal{V}_{\Theta, \hat{\mu}}(\bar{U}) d\mathbb{P}_\bar{U} . \]
With the above definitions we proceed by stating three lemmas that will be used to prove Theorem 7. We defer the proofs of these lemmas for after proving the main Theorem in this section.

**Lemma 11** Under Assumptions 1-6, we have

I) \( \sqrt{n} \left\{ \mathbb{P}_N \left[ V_{\Theta, \mu} - g_1 (\Theta) \right] \right\} = o_P(1) \),

II) \( \sqrt{n} \left\{ g_1 (\hat{\Theta}) - g_1 (\bar{\Theta}) \right\} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ \left( \frac{\partial}{\partial \theta} g_1 (\Theta) \right)^T \psi^\theta (\bar{U}_i) + \left( \frac{\partial}{\partial \xi} g_1 (\Theta) \right)^T \psi^\xi (\bar{U}_i) \right\} + o_P(1) \).

**Lemma 12** Under Assumptions 1-6, the following holds:

\[ \sqrt{n} \left\{ \left( \mathbb{P}_N \left[ V_{\hat{\Theta}, \mu} - g_1 (\hat{\Theta}) \right] \right) - \left( \mathbb{P}_N \left[ V_{\bar{\Theta}, \mu} - g_1 (\bar{\Theta}) \right] \right) \right\} = o_P(1). \]

**Lemma 13** Under Assumptions 1-6, the following assertions hold:

I) \( \sqrt{n} \mathbb{P}_N \left\{ \mathcal{E}_{\Theta} - \mathcal{E}_{\Theta} \right\} = o_P(1) \),

II) \( \sqrt{n} \mathbb{P}_N \left[ \mathcal{E}_{\Theta} \right] = \mathbb{G}_n \left\{ \nu_{\text{SSLDR}} (\bar{U}; \Theta) \right\} \)

\[ + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ \psi^\theta (\bar{U}_i)^T \frac{\partial}{\partial \theta} \int \nu_{\text{SSLDR}} (\bar{U}_i; \Theta) d\mathbb{P}_\bar{U} \bigg|_{\Theta = \Theta} + \psi^\xi (\bar{U}_i)^T \frac{\partial}{\partial \xi} \int \nu_{\text{SSLDR}} (\bar{U}_i; \Theta) d\mathbb{P}_\bar{U} \bigg|_{\Theta = \Theta} \right\} \]

\[ + o_P(1). \]

**Proof** [Proof of Theorem 7] We start by expanding the expression in (9) and using definitions (10), (11), (12):
Next note that which follows from Lemmas 11, 12 & 13 with the influence function \( \psi_{SSLDR}^\nu \) defined as

\[
\psi_{SSLDR}^\nu (\bar{U}; \bar{\Theta}) = \nu_{SSLDR} (\bar{U}; \bar{\Theta}) + \psi^\theta (\bar{U}) \frac{\partial}{\partial \theta} \int \left( \nu_{\Theta, \bar{\mu}} (\bar{U}) + \nu_{SSLDR} (\bar{U}; \Theta) \right) dP \bigg|_{\Theta = \bar{\Theta}}
+ \psi^\xi (\bar{U}) \frac{\partial}{\partial \xi} \int \left( \nu_{\Theta, \bar{\mu}} (\bar{U}) + \nu_{SSLDR} (\bar{U}; \Theta) \right) dP \bigg|_{\Theta = \bar{\Theta}},
\]

\[
\nu_{SSLDR} (\bar{U}; \Theta) = \omega_1 (H_1, A_1; \Theta_1) (1 + \beta_{21}) \left\{ Y_2 - \bar{\mu}_{21} (\bar{U}) \right\} + \omega_2 (H_2, A_2, \Theta_2) Y_3 - \bar{\mu}_{32} (\bar{U})
- \left\{ \omega_2 (H_2, A_2, \Theta_2) Y_2 - \bar{\mu}_{2w2} (\bar{U}) \right\} - Q^\nu_{2-} (H_2; \bar{\Theta}) \left\{ \omega_2 (H_2, A_2, \Theta_2) - \bar{\mu}_{w2} (\bar{U}) \right\}
\]

Next note that

\[
\int \left( \nu_{\Theta, \bar{\mu}} (\bar{U}) + \nu_{SSLDR} (\bar{U}; \Theta) \right) dP \bigg|_{\Theta = \bar{\Theta}} = \int \nu_{SUPDR} (L; \Theta) dP \bigg|_{\Theta = \Theta},
\]

where \( \nu_{SUPDR} (L; \Theta) \) is defined in (6). Finally, all random variables in the expression of \( \psi_{SSLDR}^\nu (\bar{U}; \Theta) \) are bounded by Assumptions 1 and 5 we have \( \mathbb{E} \left[ \psi_{SSLDR}^\nu (\bar{U}; \Theta) \right] < \infty \), the
central limit theorem yields that

$$\sqrt{n}\left\{ \mathbb{P}_N \left[ V_{\vec{\Theta}, \bar{\beta}} - g_1(\vec{\Theta}) \right] \right\} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \psi_{\text{SSTDR}}^{\psi}(\bar{U}_i; \vec{\Theta}) + o_P(1) \xrightarrow{d} N \left( 0, (\sigma_{\text{SSTDR}}^V)^2 \right).$$

\[ \square \]

**Proof** [Proof of Lemma 11] I) We start with $\sqrt{n}\left\{ \mathbb{P}_N \left[ V_{\vec{\Theta}, \bar{\beta}} - g_1(\vec{\Theta}) \right] \right\}$. Note that $V_{\vec{\Theta}, \bar{\beta}}(\bar{U})$ is a deterministic function of random variable $\bar{U}$ as parameters and imputation functions are fixed. We have that $\mathbb{E} \left[ V_{\vec{\Theta}, \bar{\beta}}(\bar{U})^2 \right] < \infty$ holds by Assumption 1 & 5. Thus the central limit theorem yields $\mathbb{G}_N \left\{ V_{\vec{\Theta}, \bar{\beta}} \right\} \xrightarrow{d} \mathcal{N} (0, \text{Var} [V_{\vec{\Theta}, \bar{\beta}}])$, therefore

$$\sqrt{n}\left\{ \mathbb{P}_N \left[ V_{\vec{\Theta}, \bar{\beta}} - g_1(\vec{\Theta}) \right] \right\} = \sqrt{n} \mathbb{G}_N \left\{ V_{\vec{\Theta}, \bar{\beta}} \right\} = O_P \left( \frac{\sqrt{n}}{N} \right) = o_P(1).$$

II) We next consider $\sqrt{n}\left\{ g_1(\vec{\Theta}) - g_1(\vec{\Theta}) \right\}$. Using a Taylor series expansion

$$g_1(\vec{\Theta}) = g_1(\vec{\Theta}) + (\vec{\Theta} - \vec{\Theta})^T \frac{\partial}{\partial \vec{\Theta}} g_1(\vec{\Theta}) + (\vec{\xi} - \vec{\xi})^T \frac{\partial}{\partial \vec{\xi}} g_1(\vec{\Theta}) + O_P \left( n^{-1} \right),$$

as both $\|\vec{\Theta} - \vec{\Theta}\|_2^2 = O_P \left( n^{-1} \right)$ and $\|\vec{\xi} - \vec{\xi}\|_2^2 = O_P \left( n^{-1} \right)$ by Theorems 2, 3 and Assumption 5, therefore

$$\sqrt{n}\left\{ g_1(\vec{\Theta}) - g_1(\vec{\Theta}) \right\} = \sqrt{n}(\vec{\Theta} - \vec{\Theta})^T \frac{\partial}{\partial \vec{\Theta}} g_1(\vec{\Theta}) + \sqrt{n}(\vec{\xi} - \vec{\xi})^T \frac{\partial}{\partial \vec{\xi}} g_1(\vec{\Theta}) + o_P(1).$$

We can write

$$\sqrt{n}\left\{ g_1(\vec{\Theta}) - g_1(\vec{\Theta}) \right\} = \frac{\partial}{\partial \vec{\Theta}} g_1(\vec{\Theta}) \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \psi^\theta(\bar{U}_i) + \frac{\partial}{\partial \vec{\xi}} g_1(\vec{\Theta}) \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \psi^\xi(\bar{U}_i) + o_P(1).$$

\[ \square \]

**Proof** [Proof of Lemma 12] We consider $\sqrt{n}\left\{ \mathbb{P}_N \left[ V_{\vec{\Theta}, \bar{\beta}} - g_1(\vec{\Theta}) \right] \right\} - \left( \mathbb{P}_N \left[ V_{\vec{\Theta}, \bar{\beta}} - g_1(\vec{\Theta}) \right] \right)$, recall that $d_t(\vec{H}_t, \theta_t) = I(\vec{H}_t^\top \gamma_t > 0)$ $t = 1, 2$, thus the inverse probability weight functions are defined as

$$\omega_1(\vec{H}_1, A_1, \Theta) \equiv \frac{I(\vec{H}_1^\top \gamma_1 > 0)A_1}{\pi_1(\vec{H}_1; \xi_1)} + \frac{\{1 - I(\vec{H}_1^\top \gamma_1 > 0)\} \{1 - A_1\}}{1 - \pi_1(\vec{H}_1; \xi_1)},$$

and

$$\omega_2(\vec{H}_2, A_2, \Theta) \equiv \omega_1(\vec{H}_1, A_1, \Theta) \left( \frac{I(\vec{H}_2^\top \gamma_2 > 0)A_2}{\pi_2(\vec{H}_2; \xi_2)} + \frac{\{1 - I(\vec{H}_2^\top \gamma_2 > 0)\} \{1 - A_2\}}{1 - \pi_2(\vec{H}_2; \xi_2)} \right).$$

Define the class

$$\ell_t = \{ I(\vec{H}_t^\top \gamma_t > 0) : \mathcal{H}_{t1}, \gamma \in \mathbb{R}^{q_1} \}, t = 1, 2$$

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and the collection of half spaces $C_\ell \equiv \{ H_t \in \mathbb{R}^{q_t} : H_t^T \gamma_t \geq 0, \gamma \in \mathbb{R}^{q_t}, t \in \{1, 2\} \}$. By Dudley (1979) $C_\ell$ is a VC class of VC dimension $q_t + 1$. Next by van der Vaart and Wellner (1996) we have that as $C_\ell$ is a VC-class $\ell_t$ is a class of the same index. Finally, by Theorem 2.6.7 we have that $\ell_t$ is a $\mathbb{P}$-Donsker class. Next define the following function

$$f_\Theta(\tilde{U}) = Q^0_1(\tilde{H}_1; \theta_1) + \omega_1(\tilde{H}_1, A_1, \Theta) \left[ (1 + \beta_{21})\tilde{\mu}_2^2(\tilde{U}) - Q^0_1(\tilde{H}_1; \theta_1) + Q^2_{-2}(H_2; \theta_2) \right]$$

$$+ \omega_1(\tilde{H}_1, A_1; \Theta) I(\tilde{H}_2^T \gamma_2 > 0) A_2 \left\{ \tilde{\mu}_{3\pi_2}(\tilde{U}) - \beta_{21} \tilde{\mu}_{2\pi_2}(\tilde{U}) - Q^2_{-2}(H_2; \theta_2) \tilde{\mu}_{2\pi_2}(\tilde{U}) \right\}$$

$$+ \omega_1(\tilde{H}_1, A_1; \Theta) \{1 - I(\tilde{H}_2^T \gamma_2 > 0)\} (1 - A_2)$$

$$\times \left\{ \bar{\mu}_{3\pi_2}(\tilde{U}) - \beta_{21} \bar{\mu}_{2\pi_2}(\tilde{U}) - Q^0_2(H_2; \theta_2) \bar{\mu}_{2\pi_2}(\tilde{U}) \right\}.$$

We define the associated class of functions $C_1 = \left\{ f_\Theta(\tilde{U}) | \tilde{U}, \Theta \in S(\delta) \right\}.$

i) By Assumptions 3, 5 and Theorem 19.5 in Vaart (1998), $\ell_t, W_t, Q_t, t = 1, 2$ are $\mathbb{P}$-Donsker classes. Thus it follows that $C_1$ is a Donsker class.

ii) We estimate $\xi_1, \xi_2$ for (5) with their maximum likelihood estimators, $\hat{\xi}_1, \hat{\xi}_2$, solving

$$P_n \left[ S_t(\hat{\xi}_t) = 0, t = 1, 2 \right] = 0,$$

By Assumption (5) and Theorem 5.9 in Vaart (1998) $\hat{\xi}_t \xrightarrow{P} \bar{\xi}_t, t = 1, 2$. Next, by Theorems 2, 3 under Assumptions 1, 2, $\bar{\theta}_t \xrightarrow{P} \bar{\theta}_t, t = 1, 2$. Thus

$$P \left( \Theta \in S(\delta) \right) \rightarrow 1, \forall \delta.$$

iii) We next show

$$\int \left( V_{\Theta, \bar{\mu}} - V_{\Theta, \bar{\mu}} \right)^2 d\mathbb{P}_\tilde{U} \rightarrow 0.$$

By (9), there exists a constant $c \in \mathbb{R}$ such that we can write

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\[
\int (\mathcal{V}_{\Theta, \hat{\mu}} - \mathcal{V}_{\Theta, \hat{\mu}})^2 d\mathbb{P} \hat{\mu}
\]
\[
\leq c \int \left( Q_1^\theta(H_1; \hat{\theta}_1) - Q_1^\theta(H_1; \hat{\theta}_1) \right)^2 d\mathbb{P} \hat{\mu}
\]
\[
+ c \sup_{\hat{\mu}} \left\{ \mu_2^\theta(U) + \hat{\mu}_3^0(0)(\hat{\mu}) \right\}^2 \int \left( \frac{1}{1 - \pi_1(H_1; \xi_1)} - \frac{1}{1 - \pi_1(H_1; \xi_1)} \right)^2 d\mathbb{P} \hat{\mu}
\]
\[
+ c \sup_{\hat{\mu}} \left\{ \mu_2^\theta(U) + \hat{\mu}_3^0(0)(\hat{\mu}) \right\}^2 \int \left( \frac{1}{\pi_1(H_1; \xi_1)} - \frac{1}{\pi_1(H_1; \xi_1)} \right)^2 d\mathbb{P} \hat{\mu}
\]
\[
+ c \sup_{\Theta, \hat{\mu}} Q_1^\theta(H_1, A_1; \theta_1)^2 \int \left( \frac{1}{\pi_1(H_1; \xi_1)} - \frac{1}{\pi_1(H_1; \xi_1)} \right)^2 + \left( \frac{1}{1 - \pi_1(H_1; \xi_1)} - \frac{1}{1 - \pi_1(H_1; \xi_1)} \right)^2 d\mathbb{P} \hat{\mu}
\]
\[
+ c \sup_{\Theta, \hat{\mu}} \frac{1}{\pi_1(H_1; \xi_1)^2(1 - \pi_1(H_1; \xi_1)^2) \int \left( Q_1^\theta(H_1, A_1; \hat{\theta}_1) - Q_1^\theta(H_1, A_1; \hat{\theta}_1) \right)^2 d\mathbb{P} \hat{\mu}
\]
\[
+ c \sup_{\Theta, \hat{\mu}} \frac{1}{\pi_1(H_1; \xi_1)^2(1 - \pi_1(H_1; \xi_1)^2) \int \left( Q_1^\theta(H_2, \hat{\mu}_2^\theta(U); \theta_2) - Q_2^\theta(H_2, \hat{\mu}_2^\theta(U); \theta_2) \right)^2 d\mathbb{P} \hat{\mu}
\]
\[
+ c \sup_{\Theta, \hat{\mu}} \left\{ \beta_2(\hat{\mu}_2^\theta(U) + \hat{\mu}_2^\theta(0)(\hat{\mu})) + (\hat{\mu}_2^\theta(U) + \hat{\mu}_2^\theta(0)(\hat{\mu})) \right\}^2
\]
\[
\times \int \left( \frac{1}{\pi_1(H_1; \xi_1)} - \frac{1}{\pi_1(H_1; \xi_1)} \right)^2 + \left( \frac{1}{1 - \pi_1(H_1; \xi_1)} - \frac{1}{1 - \pi_1(H_1; \xi_1)} \right)^2 d\mathbb{P} \hat{\mu}
\]
\[
+ c \sup_{\Theta, \hat{\mu}} \frac{1}{\pi_1(H_1; \xi_1)^2(1 - \pi_1(H_1; \xi_1)^2) \int \left( \mu_2^\theta(U) + \hat{\mu}_2^\theta(U) \right)^2 (\beta_2 - \beta_2) d\mathbb{P} \hat{\mu}
\]
\[
+ c \sup_{\Theta, \hat{\mu}} \frac{1}{\pi_1(H_1; \xi_1)^2(1 - \pi_1(H_1; \xi_1)^2) \int \left( \mu_2^\theta(U) + \hat{\mu}_2^\theta(U) \right)^2 \left( \beta_2 - \beta_2 \right)^2 d\mathbb{P} \hat{\mu}
\]
\[
\times \int \left( \mu_2^\theta(U) + \hat{\mu}_2^\theta(U) \right)^2 \left( \beta_2 - \beta_2 \right)^2 d\mathbb{P} \hat{\mu}
\]

where we use \((a - b)^2, (a + b)^2 \leq 2a^2 + 2b^2 \forall a, b \in \mathbb{R}, d_1, A_1, \hat{d}_2, A_2 \leq 1 \forall H \in \mathcal{H}\), and boundedness of \(\hat{\theta}_1, t = 1, 2\) by Assumptions 1-3. Next note that all terms outside integrals are bounded by Assumptions 1-3. Finally we consider terms within the integrals with the following example
\[
\int \left( Q_2^o(H_2, \hat{\mu}_2^o(\bar{U}); \bar{\theta}_2) - Q_2^o(H_2, \hat{\mu}_2^o(\bar{U}); \bar{\theta}) \right)^2 d\mathbb{P}_{\bar{U}}
\]
\[
= \int \left( \hat{\beta}_{21} \hat{\mu}_2^o(\bar{U}) + H_{20}^I \hat{\beta}_{22} + [H_{21}^I \hat{\gamma}_2]_+ - \hat{\beta}_{21} \hat{\mu}_2^o(\bar{U}) - H_{20}^I \hat{\beta}_{22} - [H_{21}^I \hat{\gamma}_2]_+ \right)^2 d\mathbb{P}_{\bar{U}}
\]
\[
= 2 \left( \hat{\beta}_{21} - \hat{\beta}_{21} \right)^2 \int \hat{\mu}_2^o(\bar{U})^2 d\mathbb{P}_{\bar{U}} + 4 \| \hat{\beta}_{22} - \hat{\beta}_{22} \|_2^2 \int H_{20}^I H_{20}^I d\mathbb{P}_{\bar{U}}
\]
\[
+ 4 \| \hat{\gamma}_2 - \gamma_2 \|_2^2 \int H_{21}^I H_{21}^I d\mathbb{P}_{\bar{U}} = O_P(n^{-1}),
\]

which follows from Theorem 2 and Lemma 17 (a). All similar terms can be handled accordingly. We get the convergence in probability to 0: \( \int (V_{\hat{\Theta}, \hat{\mu}} - V_{\bar{\Theta}, \bar{\mu}})^2 d\mathbb{P}_{\bar{U}} \rightarrow 0 \) as all other terms within expectation are \( O_P(n^{-1}) \) by the dominating convergence theorem, boundedness conditions as stated in Assumptions 2, 5, and the consistency of \( \hat{\xi} \) and \( \hat{\theta} \) as \( P(\hat{\Theta} \in S(\delta)) \rightarrow 1, \forall \delta > 0. \)

Finally, we have i) \( P(\bar{\Theta} \in S(\delta)) \rightarrow 1, \) ii) \( C_1 \) is a Donsker class, and iii) \( \int (V_{\bar{\Theta}, \bar{\mu}} - V_{\hat{\Theta}, \hat{\mu}})^2 d\mathbb{P}_{\bar{U}} \rightarrow 0, \) then by Theorem 2.1 in Van Der Vaart and Wellner (2007),

\[
\sqrt{\frac{n}{N}} \sqrt{n} \left\{ \left( \mathbb{P}_N \left[ V_{\bar{\Theta}, \hat{\mu}} \right] - g_1(\hat{\Theta}) \right) - \left( \mathbb{P}_N \left[ V_{\hat{\Theta}, \hat{\mu}} \right] - g_1(\bar{\Theta}) \right) \right\} = \sqrt{\frac{n}{N}} o_P(1).
\]
**Proof** [Proof of Lemma 13] I) We show that $\mathbb{P}_N \left[ E_{\hat{\Theta}} - E_{\Theta} \right] = O_P \left( n^{-\frac{1}{2}} \right) o_P \left( 1 \right)$:

First note that from the empirical normal equations (8), we have that the solution $\hat{n}_2$ satisfies $\hat{n}_2 - n_2 = O_P \left( n^{-\frac{1}{2}} \right)$. Therefore

$$\sup_U \left| \hat{\mu}_2^v(\bar{U}) - \mu_2^v(\bar{U}) \right| = \sup_U \left| \frac{1}{K} \hat{m}_2^{(k)}(\bar{U}) + \hat{n}_2^v - m_2(\bar{U}) + \eta_2^v \right|$$

$$\leq \frac{1}{K} \sup_U \left| \hat{m}_2^{(k)}(\bar{U}) + m_2(\bar{U}) \right| + |\hat{n}_2^v - \eta_2^v|$$

$$= o_P(1) + O_P \left( n^{-\frac{1}{2}} \right) = o_P(1),$$

where we additionally use Assumption 6 for the difference of estimated and true imputation models $\hat{m}_2, m_2$. Similarly $\sup_U \left| \hat{\mu}_{\pi_z(a)}(\bar{U}) - \mu_{\pi_z(a)}(\bar{U}) \right| = o_P(1)$, $\sup_U \left| \hat{\mu}_{\pi_z(a)}(\bar{U}) - \mu_{\pi_z(a)}(\bar{U}) \right| = o_P(1)$, $t = 1, 2, a = 0, 1$. Next, using the triangle and Jensen’s inequalities, we have

$$\mathbb{P}_N \left[ E_{\hat{\Theta}} - E_{\Theta} \right]$$

$$\leq \mathbb{P}_N \left[ \omega_1(\bar{H}_1, A_1; \bar{\Theta}_1) \right]$$

$$+ \mathbb{P}_N \left[ \omega_1(\bar{H}_1, A_1; \bar{\Theta}_1) I(\bar{d}_2 = A_2) - \omega_1(\bar{H}_1, A_1; \bar{\Theta}_1) I(\bar{d}_2 = A_2) \right]$$

$$+ \mathbb{P}_N \left[ \omega_1(\bar{H}_1, A_1; \bar{\Theta}_1) I(\bar{d}_2 = A_2) \right]$$

$$+ \mathbb{P}_N \left[ \omega_1(\bar{H}_1, A_1; \bar{\Theta}_1) I(\bar{d}_2 = A_2) Q_{2-}(\bar{H}_2; \bar{\theta}_2) \right]$$

$$\times \sup_U \left| \hat{\mu}_{\pi_z(A_2)}(\bar{U}) - \mu_{\pi_z(A_2)}(\bar{U}) \right|$$

$$\leq \mathbb{P}_N \left[ \omega_1(\bar{H}_1, A_1; \bar{\Theta}_1) \right]$$

$$+ \mathbb{P}_N \left[ \omega_1(\bar{H}_1, A_1; \bar{\Theta}_1) I(\bar{d}_2 = A_2) - \omega_1(\bar{H}_1, A_1; \bar{\Theta}_1) I(\bar{d}_2 = A_2) \right]$$

$$+ \mathbb{P}_N \left[ \omega_1(\bar{H}_1, A_1; \bar{\Theta}_1) I(\bar{d}_2 = A_2) \right]$$

$$+ \mathbb{P}_N \left[ \omega_1(\bar{H}_1, A_1; \bar{\Theta}_1) I(\bar{d}_2 = A_2) Q_{2-}(\bar{H}_2; \bar{\theta}_2) \right]$$

$$\times \sup_U \left| \hat{\mu}_{\pi_z(A_2)}(\bar{U}) - \mu_{\pi_z(A_2)}(\bar{U}) \right|$$

where the last step follows from the above, next we center each term to obtain:
\[
\mathbb{P}_N \left[ \mathcal{E}_{\Theta} - \mathcal{E}_{\hat{\Theta}} \right] \\
\leq \mathbb{P}_N \left[ \frac{A_1}{\pi_1(H_1; \xi_1)} \left\{ \hat{d}_1 (1 + \hat{\beta}_{21}) - \hat{d}_1 (1 + \hat{\beta}_{21}) \right\} \right] o_P(1) \\
+ \mathbb{P}_N \left[ \frac{1 - A_1}{1 - \pi_1(H_1; \xi_1)} \left\{ (1 - \hat{d}_1)(1 + \hat{\beta}_{21}) - (1 - \hat{d}_1)(1 + \hat{\beta}_{21}) \right\} \right] o_P(1) \\
+ \mathbb{P}_N \left[ \frac{1}{\pi_1(H_1; \xi_1)(1 - \pi_1(H_1; \xi_1))} - \frac{1}{\pi_1(H_1; \xi_1)(1 - \pi_1(H_1; \xi_1))} \right] o_P(1) \\
+ \mathbb{P}_N \left[ \frac{A_1 \beta}{\pi_1(H_1; \xi_1)} \left\{ \hat{d}_1 \hat{d}_2 - \hat{d}_1 \hat{d}_2 + \hat{\beta}_{21} - \hat{\beta}_{21} \right\} \right] o_P(1) \\
+ \mathbb{P}_N \left[ \frac{(1 - A_1)(1 - A_2)}{1 - \pi_1(H_1; \xi_1)} \left\{ (1 - \hat{d}_1)(1 - \hat{d}_2) - (1 - \hat{d}_1)(1 - \hat{d}_2) + \hat{\beta}_{21} - \hat{\beta}_{21} \right\} \right] o_P(1) \\
+ \mathbb{P}_N \left[ \frac{1 + \hat{\beta}_{21}}{\pi_1(H_1; \xi_1)(1 - \pi_1(H_1; \xi_1))} - \frac{1 + \hat{\beta}_{21}}{\pi_1(H_1; \xi_1)(1 - \pi_1(H_1; \xi_1))} \right] o_P(1) \\
+ \mathbb{P}_N \left[ \frac{A_1 A_2}{\pi_1(H_1; \xi_1)} \left\{ \hat{d}_1 \hat{d}_2 Q_{2+}^o (H_2 i; \hat{\theta}_2) - \hat{d}_1 \hat{d}_2 Q_{2+}^o (H_2 i; \hat{\theta}_2) \right\} \right] o_P(1) \\
+ \mathbb{P}_N \left[ \frac{(1 - A_1)(1 - A_2)}{1 - \pi_1(H_1; \xi_1)} \left\{ (1 - \hat{d}_1)(1 - \hat{d}_2)Q_{2+}^o (H_2 i; \hat{\theta}_2) - (1 - \hat{d}_1)(1 - \hat{d}_2)Q_{2+}^o (H_2 i; \hat{\theta}_2) \right\} \right] o_P(1) \\
+ \mathbb{P}_N \left[ \frac{Q_{2+}^o (H_2 i; \hat{\theta}_2)}{\pi_1(H_1; \xi_1)(1 - \pi_1(H_1; \xi_1))} - \frac{1}{\pi_1(H_1; \xi_1)(1 - \pi_1(H_1; \xi_1))} \right] o_P(1),
\]

by Lemma 17 (b), we have \(\sup_{H_1} \left| \frac{1}{\pi_1(H_1; \xi_1)} - \frac{1}{\pi_1(H_1; \xi_1)} \right| = o_P \left( n^{-\frac{1}{2}} \right)\), \(\sup_{H_1, A_1} \left| I(\hat{d}_1 = A_1) - I(\hat{d}_1 = A_1) \right| = O_P \left( n^{-\frac{1}{2}} \right)\), \(\sup_{H_1, A_2} \left| I(\hat{d}_1 = A_1)I(\hat{d}_2 = A_2) - I(\hat{d}_1 = A_1)I(\hat{d}_2 = A_2) \right| = O_P \left( n^{-\frac{1}{2}} \right)\), by Theorem 2 we have \(\hat{\beta}_{21} - \bar{\beta}_{21} = o_P \left( n^{-\frac{1}{2}} \right)\), and finally

\[
\left| Q_{2+}^o (H_2 i; \hat{\theta}_2) - Q_{2+}^o (H_2 i; \hat{\theta}_2) \right| \leq \sup_{H_2} \left| H_{20}^\top \hat{\beta}_{22} + A_2(H_{21}^\top \hat{\gamma}_2) - H_{20}^\top \bar{\beta}_{22} + A_2(H_{21}^\top \bar{\gamma}_2) \right| \\
\leq \sup_{H_2} \| H_{20} \|_2 \| \hat{\beta}_{22} - \bar{\beta}_{22} \|_2 + \sup_{H_2, A_2} \| A_2 H_{21} \|_2 \| \hat{\gamma}_2 - \bar{\gamma}_2 \|_2 = O_P \left( n^{-\frac{1}{2}} \right).
\]

Therefore by using Lemma 14, Assumptions 3 and 5, we can show that all functions are uniformly bounded, and we have

\[
\mathbb{P}_N \left[ \mathcal{E}_{\Theta} - \mathcal{E}_{\hat{\Theta}} \right] = O_P \left( n^{-\frac{1}{2}} \right) o_P(1)
\]
II) We next re-write $\sqrt{n}P_N[\mathcal{E}_\Theta]$ by expressing the estimated imputation functions in $\mathcal{E}_\Theta$ in terms of the labeled sample $\mathcal{L}$. Letting

$$c_{n,N}^{(1)} = \frac{P_N\{\omega_1(\hat{H}_1, A_1, \Theta)\}}{P_n\{\omega_1(\hat{H}_1, A_1, \Theta)\}}, \quad c_{n,N}^{(2)} = \frac{P_N\{\omega_1(\hat{H}_1, A_1, \Theta)I(\hat{d}_2 = A_2)\}}{P_n\{\omega_1(\hat{H}_1, A_1, \Theta)I(\hat{d}_2 = A_2)\},}$$

we can calculate:

$$\frac{1}{N} \sum_{j=1}^{N} \omega_1(\hat{H}_{1j}, A_{1j}, \Theta) \left\{ \hat{\mu}_2^v(\bar{U}_j) - \hat{\mu}_2^v(\bar{U}_j) \right\}$$

$$= \frac{1}{N} \sum_{j=1}^{N} \omega_1(\hat{H}_{1j}, A_{1j}, \Theta) \left\{ \frac{1}{K} \sum_{k=1}^{K} \hat{m}_2^{(-k)}(\bar{U}_j) + \hat{\eta}_2^v - m_2(\bar{U}_j) - \eta_2^v \right\}$$

$$= \frac{1}{N} \sum_{j=1}^{N} \omega_1(\hat{H}_{1j}, A_{1j}, \Theta) \left\{ \frac{1}{K} \sum_{k=1}^{K} \hat{m}_2^{(-k)}(\bar{U}_j) - m_2(\bar{U}_j) \right\} + (\hat{\eta}_2^v - \eta_2^v) \frac{1}{N} \sum_{j=1}^{N} \omega_1(\hat{H}_{1j}, A_{1j}, \Theta),$$

where the first step follows from constrains shown in (8) and we simply regroup terms in the second step. Next note that we can use Lemma 15 to replace $\omega_1(\hat{H}_1, A_1, \Theta) \left\{ \frac{1}{K} \sum_{k=1}^{K} \hat{m}_2^{(-k)}(\bar{U}) - m_2(\bar{U}) \right\}$ by

$$\mathbb{E}_\mathcal{L} \left[ \omega_1(\hat{H}_1, A_1, \Theta) \left\{ \frac{1}{K} \sum_{k=1}^{K} \hat{m}_2^{(-k)}(\bar{U}) - m_2(\bar{U}) \right\} \right] + O_p \left( N^{-\frac{1}{2}} \right),$$

where $\mathbb{E}_\mathcal{L}[]$ denotes expectation with respect to $\mathcal{L}$. Additionally, using (8) for the second term we get:

$$\frac{1}{N} \sum_{j=1}^{N} \omega_1(\hat{H}_{1j}, A_{1j}; \Theta) \left\{ \hat{\mu}_2^v(\bar{U}_j) - \hat{\mu}_2^v(\bar{U}_j) \right\}$$

$$= \mathbb{E}_\mathcal{L} \left[ \omega_1(\hat{H}_{1j}, A_{1j}; \Theta) \left\{ \frac{1}{K} \sum_{k=1}^{K} \hat{m}_2^{(-k)}(\bar{U}) - m_2(\bar{U}) \right\} \right] + O_p \left( N^{-\frac{1}{2}} \right)$$

$$+ \hat{C}_{n,N}^{(1)} \frac{1}{n} \sum_{i=1}^{n} \omega_1(\hat{H}_{1i}, A_{1i}; \Theta) \left\{ Y_{2i} - \hat{\mu}_2^v(\bar{U}_i) \right\} - \hat{C}_{n,N}^{(1)} \frac{1}{n} \sum_{i=1}^{n} \omega_1(\hat{H}_{1i}, A_{1i}; \Theta) \left\{ \hat{m}_2^{(-k)}(\bar{U}_i) - m_2(\bar{U}_i) \right\}$$

$$= \hat{C}_{n,N}^{(1)} \frac{1}{n} \sum_{i=1}^{n} \omega_1(\hat{H}_{1i}, A_{1i}; \Theta) \left\{ Y_{2i} - \hat{\mu}_2^v(\bar{U}_i) \right\} + O_p \left( n^{-\frac{1}{2}} \hat{C}_{n,K} \right),$$

where the last step follows from Assumption 6 and Lemma 16 with $\hat{\Delta}_1(\bar{U}) = \hat{m}_2(\bar{U}) - m_2(\bar{U})$, choosing $f$ to be the constant function 1, and with $\hat{C}_{n,N} = \hat{C}_{n,N}^{(1)}$ -which satisfies
\( \hat{C}_{n,N}^{(1)} = 1 + O_P \left( n^{-\frac{1}{2}} \right) \) by Lemma 17 (c). Analogous to the above derivation we get:

\[
P_N \left[ \omega_1(\hat{H}_1, A_1; \Theta) I(\hat{d}_2 = A_2) \{ \tilde{\mu}^\psi_2(\bar{U}) - \tilde{\mu}^\nu_2(\bar{U}) \} \right] \\
= \hat{C}_{n,N}^{(2)} P_n \left[ \omega_1(\hat{H}_1, A_1; \Theta) I(\hat{d}_2 = A_2) \{ \tilde{\mu}^\psi_2(\bar{U}) - \tilde{\mu}^\nu_2(\bar{U}) \} \right] + O_P \left( n^{-\frac{1}{2}} c_{n_K} \right), \\
P_N \left[ \omega_1(\hat{H}_1, A_1; \Theta) I(\hat{d}_2 = A_2) \{ \tilde{\mu}_{t\pi_2}(U) - \tilde{\mu}_{t\pi_2}(\bar{U}) \} \right] \\
= \hat{C}_{n,N}^{(2)} P_n \left[ \omega_1(\hat{H}_1, A_1; \Theta) I(\hat{d}_2 = A_2) \left\{ \frac{Y_i}{\pi_2(H_2; \xi_2)} - \tilde{\mu}_{t\pi_2}(\bar{U}) \right\} \right] + O_P \left( n^{-\frac{1}{2}} c_{n_K} \right), \quad t = 2, 3, \\
P_N \left[ \omega_1(\hat{H}_1, A_1; \Theta) I(\hat{d}_2 = A_2) [\beta_{21}^T, \gamma_{2i}] \xi_2 \{ \tilde{\mu}_{\pi_2}(U) - \tilde{\mu}_{\pi_2}(\bar{U}) \} \right] \\
= \hat{C}_{n,N}^{(2)} P_n \left[ \omega_1(\hat{H}_1, A_1; \Theta) I(\hat{d}_2 = A_2) [\beta_{21}^T, \gamma_{2i}] \xi_2 \left\{ \frac{1}{\pi_2(H_2; \xi_2)} - \tilde{\mu}_{\pi_2}(\bar{U}) \right\} \right] + O_P \left( n^{-\frac{1}{2}} c_{n_K} \right).
\]

Re-writing \( \nu_{SSLDR} \) defined in Theorem 7, using functions \( \tilde{\mu}_{\pi_2}, \tilde{\mu}_{t\pi_2} \) \( t = 2, 3 \) we get:

\[
\nu_{SSLDR}(\tilde{U}_i; \hat{\Theta}) \\
= \omega_1(\hat{H}_1, A_1; \hat{\Theta}) \left[ (1 + \beta_{21}) \left\{ Y_{2i} - \tilde{\mu}_{t\pi_2}(\bar{U}) \right\} I(\hat{d}_2i = A_2) \left\{ \frac{Y_{2i}}{\pi_2(H_2; \xi_2)} - \tilde{\mu}_{t\pi_2}(A_2)(\bar{U}) \right\} \right] \\
- \omega_1(\hat{H}_1, A_1; \hat{\Theta}) I(\hat{d}_2i = A_2) \left\{ \frac{Y_{2i}}{\pi_2(H_2; \xi_2)} - \tilde{\mu}_{t\pi_2}(A_2)(\bar{U}) \right\} - \beta_{21} \left\{ \frac{Y_{2i}}{\pi_2(H_2; \xi_2)} - \tilde{\mu}_{t\pi_2}(A_2)(\bar{U}) \right\} \right\} \\
- \omega_1(\hat{H}_1, A_1; \hat{\Theta}) I(\hat{d}_2i = A_2) Q_{2i}^2(H_2; \hat{\Theta}_2) \left\{ \frac{1}{\pi_2(H_2; \xi_2)} - \tilde{\mu}_{\pi_2}(A_2)(\bar{U}) \right\}.
\]

Using the derivations above, we can write

\[
\frac{\sqrt{n}}{N} \sum_{j=1}^{N} \mathcal{E}_\Theta(\bar{U}_j) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \nu_{SSLDR}(\tilde{U}_i; \hat{\Theta}) + O_P \left( c_{n_K} \right). \tag{13}
\]

We next consider the first term in the right hand side of (13) which can be written as

\[
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \nu_{SSLDR}(\tilde{U}_i; \hat{\Theta}) \\
= \mathcal{G}_n \left\{ \nu_{SSLDR}(\tilde{U}_i; \hat{\Theta}) \right\} + \mathcal{G}_n \left\{ \nu_{SSLDR}(\tilde{U}_i; \hat{\Theta}) - \nu_{SSLDR}(\bar{U}; \hat{\Theta}) \right\} + \sqrt{n} \int \nu_{SSLDR}(\bar{U}; \hat{\Theta}) d\mathbb{P}_{\bar{U}},
\]

we use this expansion to show that:

\[
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \nu_{SSLDR}(\tilde{U}_i; \hat{\Theta}) \\
= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ \nu_{SSLDR}(\tilde{U}_i; \hat{\Theta}) + \left( \frac{\partial}{\partial \Theta} \mathbb{E} \left[ \nu_{SSLDR}(\tilde{U}_i; \hat{\Theta}) \right] \right)^T \psi_\Theta(\tilde{U}_i) + \left( \frac{\partial}{\partial \xi} \mathbb{E} \left[ \nu_{SSLDR}(\tilde{U}_i; \hat{\Theta}) \right] \right)^T \psi_\xi(\tilde{U}_i) \right\} + o_P(1).
\]
1) Using a Taylor series expansion on \( \int \nu_{\text{SSLDR}} (\bar{U}; \hat{\Theta})d\mathbb{P} \) we get
\[
\int \nu_{\text{SSLDR}} (\bar{U}; \hat{\Theta})d\mathbb{P} = \int \nu_{\text{SSLDR}} (\bar{U}; \hat{\Theta})d\mathbb{P} \bigg|_{\Theta = \hat{\Theta}} \left( \Theta - \hat{\Theta} \right)^T \frac{\partial}{\partial \Theta} \int \nu_{\text{SSLDR}} (\bar{U}; \Theta)d\mathbb{P} \bigg|_{\Theta = \hat{\Theta}} + O_{\mathbb{P}} (n^{-1}),
\]
where the remaining terms are of order \( O \left\{ \left( \hat{\Theta} - \Theta \right)^2 \right\} \) which by Theorems 2 & 3 are \( O_{\mathbb{P}} (n^{-1}) \). Next note that from (7) it follows that \( \int \nu_{\text{SSLDR}} (\bar{U}; \Theta)d\mathbb{P} = 0 \), and thus letting \( g_2(\Theta) = \int \nu_{\text{SSLDR}} (\bar{U}; \Theta)d\mathbb{P} \) we have
\[
\sqrt{n}g_2(\Theta) = \sqrt{n}(\hat{\Theta} - \Theta)^T \frac{\partial}{\partial \Theta} g_2(\Theta) \bigg|_{\Theta = \hat{\Theta}} + \sqrt{n}(\hat{\Theta} - \Xi(t))^{\gamma} \frac{\partial}{\partial \Xi(t)} g_2(\Theta) \bigg|_{\Theta = \hat{\Theta}} + o_{\mathbb{P}}(1).
\]
We can write
\[
\sqrt{n}g_2(\Theta) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \psi(\bar{U}_i)^T \frac{\partial}{\partial \Theta} g_2(\Theta) \bigg|_{\Theta = \hat{\Theta}} + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \psi(\bar{U}_i)^T \frac{\partial}{\partial \Xi(t)} g_2(\Theta) \bigg|_{\Theta = \hat{\Theta}} + o_{\mathbb{P}}(1).
\]
2) We next show
\[
\mathbb{G}_n \left\{ \nu_{\text{SSLDR}} (\bar{U}; \hat{\Theta}) - \nu_{\text{SSLDR}} (\bar{U}; \hat{\Theta}) \right\} = o_{\mathbb{P}}(1),
\]
define the class
\[
\mathcal{C}_t = \{ I(\mathbf{H}^T \gamma_t \geq 0) : \mathcal{H}_{t,1}, \gamma \in \mathbb{R}^{q_t} \}, \quad t = 1, 2
\]
and the collection of half spaces \( \mathcal{C}_t = \{ \mathbf{H}_t \in \mathbb{R}^{q_t} : \mathbf{H}^T \gamma_t \geq 0, \gamma \in \mathbb{R}^{q_t}, t \in \{1, 2\} \} \), by Dudley (1979) \( \mathcal{C}_t \) is a VC class of VC dimension \( q_t + 1 \), next by van der Vaart and Wellner (1996) we have that as \( \mathcal{C}_t \) is a VC-class \( \mathcal{C}_t \) is a class of the same index. Finally, by Theorem 2.6.7 we have that \( \mathcal{C}_t \) is a Donsker class.

\[
f_{\Theta}(\bar{U}_i) = \omega_1(\bar{H}_{1i}, A_{1i}; \Theta_1) \times \left[ (1 + \beta_2) \left\{ Y_{2i} - \bar{\mu}_{2i}(\bar{U}_i) \right\} + A_{2i} I(\bar{H}^T_{2i} \gamma_2 \geq 0) \left\{ \frac{Y_{3i}}{\bar{\pi}_{2}(\bar{H}_{2i}; \xi_2)} - \bar{\mu}_{2\pi_2}(\bar{U}_i) \right\} \right]
\]

we define the classes of functions
\[
\mathcal{C}_2 = \left\{ f_{\Theta}(\bar{U}) \bigg| \Theta \in \mathcal{S}(\delta) \right\}
\]

i) By Assumptions 3. 5 and Theorem 19.5 in Vaart (1998), \( W_t, \mathcal{Q}_t, t = 1, 2 \) are a \( \mathbb{P} \)-Donsker class, thus it follows that \( \mathcal{C}_2 \) is a \( \mathbb{P} \)-Donsker class.

ii) We estimate \( \xi_1, \xi_2 \) for (5) with their maximum likelihood estimators, \( \hat{\xi}_1, \hat{\xi}_2 \), solving \( \mathbb{P}_n [S_t(\xi_i)] = 0, t = 1, 2 \), by Assumption 5 and Theorem 5.9 in Vaart (1998) \( \hat{\xi}_t \xrightarrow{p} \xi_2 \).
The document contains mathematical expressions and theorems related to statistics and probability. Here is a transcription of the text:

\[ \xi_t, t = 1, 2. \text{ Next, by Theorems 2, 3, under Assumptions 1, 2, } \hat{\theta}_t \xrightarrow{p} \hat{\theta}_t, t = 1, 2. \text{ Thus } \mathbb{P} \left( \hat{\theta} \in \mathcal{S}(\delta) \right) \xrightarrow{p} 1, \forall \delta. \text{ Therefore, we have } \nu_{\text{SSLDR}}(\hat{U}; \hat{\Theta}) \in C_2 \text{ with high probability.} \]

iii) We then show \( \int \left( \nu_{\text{SSLDR}}(\hat{U}; \hat{\Theta}) - \nu_{\text{SSLDR}}(\overline{\hat{U}}; \overline{\hat{\Theta}}) \right)^2 d\mathbb{P}_{\overline{U}} \rightarrow 0. \) Using simple algebra we have

\[
\int \left( \nu_{\text{SSLDR}}(\hat{U}; \hat{\Theta}) - \nu_{\text{SSLDR}}(\overline{\hat{U}}; \overline{\hat{\Theta}}) \right)^2 d\mathbb{P}_{\overline{U}} \\
\leq (1 + \beta_{21})^2 \sup_{H \in \mathcal{H}} \left\{ Y_2 - \tilde{\mu}_2(\overline{U}) \right\}^2 \int \left( \omega_1(\hat{H}_1, A_1; \hat{\Theta}_1) - \omega_1(\bar{\hat{H}}_1, A_1; \hat{\Theta}_1) \right)^2 d\mathbb{P}_{\overline{U}} \\
+ 4 \sup_{H \in \mathcal{H}} \left\{ Y_2 - \tilde{\beta}_{21} Y_3 \right\}^2 \int \left( \omega_2(\hat{H}_2, A_2; \hat{\Theta}_2) - \omega_2(\bar{\hat{H}}_2, A_2; \hat{\Theta}_2) \right)^2 d\mathbb{P}_{\overline{U}} \\
+ 8 \sup_{H \in \mathcal{H}} \left\{ \tilde{\mu}_{2\omega_2}(\overline{U}) - \tilde{\beta}_{21} \tilde{\mu}_{3\omega_2}(\overline{U}) \right\}^2 \\ \\
\times \int \left( \omega_1(\hat{H}_1, A_1; \hat{\Theta}_1) I(\bar{d}_2 = A_2) - \omega_1(\bar{\hat{H}}_1, A_1; \hat{\Theta}_1) I(\bar{d}_2 = A_2) \right)^2 d\mathbb{P}_{\overline{U}} \\
+ 16 \sup_{H \in \mathcal{H}} \left\{ \tilde{\mu}_{2\omega_2}(\overline{U}) - \tilde{\beta}_{21} \tilde{\mu}_{3\omega_2}(\overline{U}) \right\}^2 \\ \\
\times \int \left( \omega_2(\hat{H}_2, A_2; \hat{\Theta}_2) Q_{2^*}^2(\bar{H}_2; \bar{\theta}_2) - \omega_2(\bar{\hat{H}}_2, A_2; \bar{\theta}_2) Q_{2^*}^2(\bar{H}_2; \bar{\theta}_2) \right)^2 d\mathbb{P}_{\overline{U}} \\
+ 16 \sup_{H \in \mathcal{H}} \tilde{\mu}_{2\omega_2}(\overline{U})^2 \\ \\
\times \int \left( \omega_1(\hat{H}_1, A_1; \hat{\Theta}_1) I(\bar{d}_2 = A_2) Q_{2^*}^2(\bar{H}_2; \bar{\theta}_2) - \omega_1(\bar{\hat{H}}_1, A_1; \bar{\theta}_2) I(\bar{d}_2 = A_2) Q_{2^*}^2(\bar{H}_2; \bar{\theta}_2) \right)^2 d\mathbb{P}_{\overline{U}} \\
\xrightarrow{p} 0
\]

where we use \((a - b)^2, (a + b)^2 \leq 2a^2 + 2b^2 \forall a, b \in \mathbb{R},\) boundedness of \(\hat{\Theta}\) by Assumptions 1, 2 to bound all supremum quantities. Finally, by Assumption 5, Slutsky’s Theorem and the Dominated Convergence Theorem we get the convergence in probability to 0 within the expectation terms.

Therefore we have i) \( \mathbb{P} \left( \hat{\Theta} \in \mathcal{S}(\delta) \right) \xrightarrow{p} 1, \forall \delta, \) ii) \( C_2 \) is a \( \mathbb{P} \)-Donsker class, and iii) \( \int \left( \nu_{\text{SSLDR}}(\overline{U}; \overline{\Theta}) - \nu_{\text{SSLDR}}(\overline{U}; \overline{\Theta}) \right)^2 d\mathbb{P}_{\overline{U}} \rightarrow 0. \) By Theorem 2.1 in Van Der Vaart and Wellner (2007)

\[
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ \left( \nu_{\text{SSLDR}}(\overline{U}_i; \overline{\Theta}) - \mathbb{E}[\nu_{\text{SSLDR}}(\overline{U}_i; \overline{\Theta})] \right) - \left( \nu_{\text{SSLDR}}(\overline{U}_i; \overline{\Theta}) - \mathbb{E}[\nu_{\text{SSLDR}}(\overline{U}_i; \overline{\Theta})] \right) \right\} = o_{\mathbb{P}}(1).
\]

From all the above, we have

\[
\sqrt{n} \left\{ \mathbb{P}_N [\mathcal{E}_\Theta] - \mathbb{E}_\mathcal{H} [\mathcal{E}_\Theta] \right\} \\
= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ \nu_{\text{SSLDR}}(\tilde{U}_i; \tilde{\Theta}) + \psi^\theta(\tilde{U}_i)^\top \frac{\partial}{\partial \Theta} g_2(\Theta) \bigg|_{\Theta = \tilde{\Theta}} + \psi^\xi(\tilde{U}_i)^\top \frac{\partial}{\partial \xi} g_2(\Theta) \bigg|_{\Theta = \tilde{\Theta}} \right\} + o_{\mathbb{P}}(1),
\]
Proof [Proof of Proposition 8]
Recall the definition of $\psi_{\text{SUPDR}}(L; \Theta)$ in (6), using (7) we have
\[ E \left[ \psi_{\text{SSLDR}}(\bar{U}; \Theta, \bar{\mu}) \right] = E \left[ \psi_{\text{SUPDR}}(L; \Theta) \right]. \]

Next by Lemma 18 we have
\[
E \left[ \psi_{\text{SUPDR}}(L; \Theta) \right] - \bar{V} = E \left[ \left\{ 1 - \frac{\pi_1(H_1)}{\pi_1(H_1; \xi_1)} \right\} \{ Q_1^\theta(H_1) - Q_1^\theta(H_1; \bar{\theta}_1) \} \right] \\
+ E \left[ \frac{\pi_1(H_1)}{\pi_1(H_1; \xi_1)} \left\{ 1 - \frac{\pi_2(H_2)}{\pi_2(H_2; \xi_2)} \right\} \{ Q_2^\theta(H_2) - Q_2^\theta(H_2; \bar{\theta}_2) \} \right],
\]
next using Theorem 7
\[
\sqrt{n} \left\{ \bar{V}_{\text{SSLDR}} - \bar{V} \right\} + \sqrt{n} \left\{ \bar{V} - E_S \left[ \psi_{\text{SSLDR}}(\bar{U}; \Theta, \bar{\mu}) \right] \right\} \xrightarrow{d} N \left( 0, (\sigma^2_{\text{SSLDR}}) \right),
\]
if either (1) or (5) are correct $\sqrt{n} \left\{ \bar{V} - E_S \left[ \psi_{\text{SSLDR}}(\bar{U}; \Theta, \bar{\mu}) \right] \right\} = 0$ and we get the required result.

Proof [Proof of Corollary 9]
From Theorem 21 in Appendix D.2 we have that the influence function for the fully-supervised value function estimator (6) is:
\[
\psi_{\text{SUPDR}}^\psi(L; \Theta) = \psi_{\text{SUPDR}}(L; \Theta) - E_S \left[ \psi_{\text{SUPDR}}(L; \Theta) \right] + \psi_{\text{SUP}}^\theta(L)^T \frac{\partial}{\partial \theta} \int \psi_{\text{SUPDR}}(L; \Theta) dP_L \bigg|_{\Theta = \Theta} \\
+ \psi_{\text{SUP}}(L)^T \frac{\partial}{\partial \xi} \int \psi_{\text{SUPDR}}(L; \Theta) dP_L \bigg|_{\Theta = \Theta}.
\]

Simple algebra can be used to show that
\[
\text{Var} \left[ \psi_{\text{SSLDR}}^\psi(\bar{U}; \Theta) \right] = \text{Var} \left[ \omega_1(H_1, A_1; \bar{\Theta}_1) (1 + \bar{\beta}_{21}) \left\{ Y_2 - \bar{\mu}_2^\psi(\bar{U}) \right\} + \omega_2(H_2, A_2, \bar{\Theta}_2) Y_3 - \bar{\mu}_{3w_2}(\bar{U}) \right] \\
- \bar{\beta}_{21} \omega_2(H_2, A_2, \bar{\Theta}_2) Y_2 - \bar{\mu}_{2w_2}(\bar{U}) \right\} \right\} + \psi_{\Theta}^\psi(\bar{U})^T \frac{\partial}{\partial \Theta} \int \psi_{\text{SUPDR}}(L; \Theta) dP_L \bigg|_{\Theta = \Theta} \\
+ \psi_{\Theta}^\psi(\bar{U})^T \frac{\partial}{\partial \Theta} \int \psi_{\text{SUPDR}}(L; \Theta) dP_L \bigg|_{\Theta = \Theta} \\
= \text{Var} \left[ \psi_{\text{SUPDR}}^\psi(L; \Theta) - \mathcal{E}(\bar{U}; \Theta) \right] \\
= \text{Var} \left[ \psi_{\text{SUPDR}}^\psi(L; \Theta) \right] + \text{Var} \left[ \mathcal{E}(\bar{U}; \Theta) \right] - 2 \text{Cov} \left[ \psi_{\text{SUPDR}}^\psi(L; \Theta), \mathcal{E}(\bar{U}; \Theta) \right],
\]
where
\[
\mathcal{E}(\bar{U}; \Theta) = Q_1^\theta(H_1; \bar{\theta}_1) + \omega_1(H_1, A_1, \bar{\Theta}) \left[ (1 + \bar{\beta}_{21}) \bar{\mu}_2^\psi(\bar{U}) - \{ Q_1^\theta(H_1, \bar{\theta}_1) - Q_2^\theta(H_2, \bar{\theta}_2) \} \right] \\
+ \bar{\mu}_{3w_2}(\bar{U}) - \bar{\beta}_{21} \bar{\mu}_{2w_2}(\bar{U}) + Q_2^\theta(H_2; \bar{\theta}_2) \bar{\mu}_{2w_2}(\bar{U}) + \psi_{\Theta}^\psi(\bar{U})^T \frac{\partial}{\partial \Theta} \int \psi_{\text{SUPDR}}(L; \Theta) dP_L \bigg|_{\Theta = \Theta}.
\]
Next note that $E \left[ \psi^v_{\text{SUPDR}}(L; \Theta) | \tilde{U} \right] = \mathcal{E}(\tilde{U}; \Theta)$, therefore using iterated expectations, it is easy to see that

$$
\text{Cov} \left[ \psi^v_{\text{SUPDR}}(L; \Theta), \mathcal{E}(\tilde{U}; \Theta) \right] = E \left[ E \left[ \left( \psi^v_{\text{SUPDR}}(L; \Theta) - E \left[ \psi^v_{\text{SUPDR}}(L; \Theta) \right] \right) \left( \mathcal{E}(\tilde{U}; \Theta) - E \left[ \mathcal{E}(\tilde{U}; \Theta) \right] \right) | \tilde{U} \right] \right]
$$

$$
= E \left[ E \left[ \psi^v_{\text{SUPDR}}(L; \Theta) \left( \mathcal{E}(\tilde{U}; \Theta) - E \left[ \mathcal{E}(\tilde{U}; \Theta) \right] \right) \right] - E \left[ \psi^v_{\text{SUPDR}}(L; \Theta) \right] E \left[ \mathcal{E}(\tilde{U}; \Theta) \right] \right]
$$

$$
= \text{Var} \left[ \mathcal{E}(\tilde{U}; \Theta) \right],
$$

combining the above we get

$$
\text{Var} \left[ \psi^v_{\text{SSLDR}}(\tilde{U}; \Theta) \right] = \text{Var} \left[ \psi^v_{\text{SUPDR}}(L; \Theta) \right] - \text{Var} \left[ \mathcal{E}(\tilde{U}; \Theta) \right].
$$

\begin{flushright}
\text{\textit{\bfseries Proof.}}
\end{flushright}

\section*{B.1.1 Variance Estimation for $\hat{V}_{\text{SUPDR}}$}

As mentioned in Remark 10, to estimate standard errors for $V_{\text{SSLDR}}(\tilde{U}; \Theta)$, we will approximate the derivatives of the expectation terms $\frac{\partial}{\partial \theta} \int V_{\text{SUPDR}}(L; \Theta) d\mathbb{P}_L$ using kernel smoothing to replace the indicator functions. In particular, let $K_h(x) = \frac{1}{h} \sigma(x/h)$, with $\sigma$ defined as in (5), we approximate $d_t(H_t, \theta_2) = I\{H_t^\gamma > 0\}$ with $1 - K_h(H_t^\gamma)$ $t = 1, 2$, and define the smoothed propensity score weights as

$$
\bar{\omega}_1(\bar{H}_1, A_1, \Theta) \equiv \frac{A_1 - K_h(H_{t1}^\gamma)}{\pi_1(H_{t1}; \xi_1)} + \frac{1 - A_1}{1 - \pi_1(H_{t1}; \xi_1)} K_h(H_{t1}^\gamma), \quad \text{and}
$$

$$
\bar{\omega}_2(\bar{H}_2, A_2, \Theta) \equiv \bar{\omega}_1(\bar{H}_1, A_1, \Theta) \left[ \frac{A_2 - K_h(H_{t2}^\gamma)}{\pi_2(H_{t2}; \xi_2)} + \frac{1 - A_2}{1 - \pi_2(H_{t2}; \xi_2)} K_h(H_{t2}^\gamma) \right].
$$

For simplicity we’ll set $h = 1$, the derivatives are as follows:

\begin{align*}
\frac{\partial}{\partial \theta} V_{\text{SUPDR}}(L; \Theta) &= \frac{\partial}{\partial \theta} Q_1(H_1; \theta_1) \bigg\{ Y_2 - \left[ Q_1(H_1; \theta_1) - Q_2(H_2; \theta_2) \right] \bigg\} \\
&\hspace{1cm} + \bar{\omega}_1(\bar{H}_1, A_1, \Theta) \left[ -\frac{\partial}{\partial \theta} Q_1(H_1, \theta) + \frac{\partial}{\partial \theta} Q_2(H_2; \theta_2) \right] \\
&\hspace{1cm} + \left\{ \frac{\partial}{\partial \theta} \bar{\omega}_2(\bar{H}_2, A_2, \Theta) \right\} \left[ Y_3 - Q_2(H_2; \theta_2) \right] \\
&\hspace{1cm} - \bar{\omega}_2(\bar{H}_2, A_2, \Theta) \frac{\partial}{\partial \theta} Q_2(H_2; \theta_2),
\end{align*}
where

\[
\frac{\partial}{\partial \theta} Q_1^*(H_1; \theta) = [H_{10}^T, H_{11}^T I (H_{11}^T \gamma_1 > 0), 0]^T,
\]

\[
\frac{\partial}{\partial \theta} Q_2^*(H_2; \theta_2) = [0^T, H_{20}^T, H_{21}^T I (H_{21}^T \gamma_2 > 0)]^T,
\]

\[
\frac{\partial}{\partial \theta} \tilde{w}_1(H_1, A_1, \Theta) = \left[ 0^T, H_{11}^T \mathbb{K}_h(H_{11}^T \gamma_1) (1 - \mathbb{K}_h(H_{11}^T \gamma_1)) \left\{ \begin{array}{c}
- A_1 \\
1 - A_1
\end{array} \right\} + \left\{ \begin{array}{c}
- A_1 \\
1 - A_1
\end{array} \right\} \right]^T,
\]

\[
\frac{\partial}{\partial \theta} \tilde{w}_2(H_2, A_2, \Theta) = \frac{\partial}{\partial \theta} \tilde{w}_1(H_1, A_1, \Theta) \left\{ \begin{array}{c}
A_2 \left\{ 1 - \mathbb{K}_h(H_{21}^T \gamma_2) \right\} \\
1 - \mathbb{K}_h(H_{21}^T \gamma_2)
\end{array} \right\} + \tilde{w}_1(H_1, A_1, \Theta) \left[ 0^T, H_{21}^T \mathbb{K}_h(H_{21}^T \gamma_2) (1 - \mathbb{K}_h(H_{21}^T \gamma_2)) \left\{ \begin{array}{c}
- A_2 \\
1 - A_2
\end{array} \right\} + \left\{ \begin{array}{c}
- A_2 \\
1 - A_2
\end{array} \right\} \right]^T.
\]

Next we have

\[
\frac{\partial}{\partial \xi} \nu_{\text{supdr}}(L; \Theta) = \left\{ \frac{\partial}{\partial \xi} \tilde{w}_1(H_1, A_1, \Theta) \right\} \left[ Y_2 - \left\{ Q_1^*(H_1, \theta_1) - Q_2^*(H_2; \theta_2) \right\} \right]
\]

\[
+ \left\{ \frac{\partial}{\partial \xi} \tilde{w}_2(H_2, A_2, \Theta) \right\} \left[ Y_3 - Q_2^*(H_2; \theta_2) \right],
\]

where

\[
\frac{\partial}{\partial \xi} \tilde{w}_1(H_1, A_1, \Theta) = \left[ \pi_1(H_1; \xi_1)^T, 0^T \right]^T,
\]

\[
\frac{\partial}{\partial \xi} \tilde{w}_2(H_2, A_2, \Theta) = \left[ \tilde{w}_1(H_1, A_1, \Theta) \pi_2(H_2; \xi_2) + 0^T \right]^T,
\]

\[
\tilde{w}_t(H_t; \xi_t) \equiv H_t \left\{ -d_t(H_t, \theta_t) A_t \frac{1 - \pi_t(H_t; \xi_t)}{\pi_t(H_t; \xi_t)} + \left\{ 1 - d_t(H_t, \theta_t) \right\} \left\{ 1 - A_t \right\} \frac{\pi_t(H_t; \xi_t)}{1 - \pi_t(H_t; \xi_t)} \right\}.
\]

**B.2 Semi-supervised Q-learning asymptotics**

In this section we first show the proofs for the theoretical results on the generalized semi-supervised Q-learning shown in section 5.

**B.2.1 Proofs for theoretical results for Q-learning in section 5**

We first define \( \hat{\Delta}_{sk}(\bar{U}) \equiv \hat{m}_s^{(\bar{U})}(\bar{U}) - m_s(\bar{U}), s \in \{2, 3, 22, 23\} \), and note that from Assumptions 1, 2 & 3 it follows that:

\[
\sum_{k=1}^K \sup_{\bar{U}} \left| \hat{\Delta}_{2k}(\bar{U}) \right| = o_p(1) \text{ for } t = 2, 3,
\]

\[
\sum_{k=1}^K \sup_{\bar{U}, \bar{X}} \| \bar{X} \hat{\Delta}_{2k}(\bar{U}) \| = o_p(1),
\]

\[
\sum_{k=1}^K \sup_{\bar{U}, \bar{X}, \bar{X}} \| \bar{X} \hat{\Delta}_{3k}(\bar{U}) \| = o_p(1),
\]

(14)
Proof [Proof of Theorem 2]
Let
\[
\tilde{\Gamma}_\mathcal{U} = \mathbb{P}_N \begin{bmatrix}
\hat{\mu}_{22}(\bar{U}) & \hat{\mu}_{21}(\bar{U}) & \hat{\mu}_{21}(\bar{U})
\end{bmatrix}
\begin{bmatrix}
H_{20} & H_{20} & 0
\end{bmatrix}.
\]

Recall the estimating equation for stage 2 regression in Section 3.2 is
\[
\mathbb{P}_N \left( \hat{\mu}_{23}(\bar{U}) - \left[ \hat{\mu}_{22}(\bar{U}), \hat{\mu}_{23}(\bar{U}) \right] \theta_2 \right) = 0.
\]

Centering the above at \( \bar{\theta}_2 \) we get
\[
\mathbb{P}_N \left( \left[ \hat{\mu}_{22}(\bar{U}), \hat{\mu}_{23}(\bar{U}) \right] X_2 \left[ \hat{\mu}_{22}(\bar{U}), \hat{\mu}_{23}(\bar{U}) \right] \bar{\theta}_2 \right) = \mathbb{P}_N \left( X_2 \left[ \hat{\mu}_{22}(\bar{U}), \hat{\mu}_{23}(\bar{U}) \right] \bar{\theta}_2 \right). \tag{15}
\]

Define
\[
\mathcal{R}_\mathcal{U} = \mathbb{P}_N \left( \left[ \hat{\mu}_{23}(\bar{U}) - \left[ \hat{\mu}_{22}(\bar{U}), \hat{\mu}_{23}(\bar{U}) \right] \theta_2 \right) \left[ \hat{\mu}_{23}(\bar{U}) - \left[ \hat{\mu}_{22}(\bar{U}), \hat{\mu}_{23}(\bar{U}) \right] \theta_2 \right) \right),
\]
\[
\hat{\mathcal{R}}^{(K)}_{\mathcal{U}, \mathcal{L}} = \mathbb{P}_N \left( \left[ \hat{\mu}_{23}(\bar{U}) - \hat{\mu}_{23}(\bar{U}) \right] - \left[ \hat{\mu}_{22}(\bar{U}) - \hat{\mu}_{22}(\bar{U}) \right] \left[ \hat{\mu}_{23}(\bar{U}) - \hat{\mu}_{23}(\bar{U}) \right] \theta_2 \right).
\]

The right hand side of (15) can be expressed as
\[
\mathbb{P}_N \left( \left[ \hat{\mu}_{23}(\bar{U}) - \left[ \hat{\mu}_{22}(\bar{U}), \hat{\mu}_{23}(\bar{U}) \right] \theta_2 \right) = \mathcal{R}_\mathcal{U} + \hat{\mathcal{R}}^{(K)}_{\mathcal{U}, \mathcal{L}}.
\]

(I) We first consider \( \hat{\mathcal{R}}^{(K)}_{\mathcal{U}, \mathcal{L}} \), let
\[
\hat{\mathcal{S}}^{(n)}_{\mathcal{U}, \mathcal{L}} = \mathbb{P}_N \left( \left[ \hat{\eta}_{23} - \eta_{23} \right] - \left( \left[ \hat{\eta}_{22} - \eta_{22} \right], \left( \hat{\eta}_{23} - \eta_{23} \right) \right) \theta_2 \right) \]
\[
\hat{\mathcal{S}}^{(k)}_{\mathcal{U}, \mathcal{L}} = \frac{1}{K} \sum_{k=1}^{K} \mathbb{P}_N \left( \left[ \hat{\Delta}_{11}^{(k)}(\bar{U}) - \hat{\Delta}_{11}^{(k)}(\bar{U}) \right] \left[ \hat{\Delta}_{11}^{(k)}(\bar{U}) \right] \theta_2 \right) \]
\[
\hat{\mathcal{S}}_{\mathcal{K}} = \mathbb{E}_\mathcal{L} \left[ \left[ \hat{\Delta}_{11}^{(k)}(\bar{U}) \right] \left[ \hat{\Delta}_{11}^{(k)}(\bar{U}) \right] \theta_2 \right] \text{ for } k \in \{1, \ldots, K\}.
\]

From (4) it follows that \( \hat{\mathcal{R}}^{(K)}_{\mathcal{U}, \mathcal{L}} = \hat{\mathcal{S}}^{(n)}_{\mathcal{U}, \mathcal{L}} + \hat{\mathcal{S}}^{(K)}_{\mathcal{U}, \mathcal{L}} \). From (14) and Lemma 15 \( \hat{\mathcal{S}}^{(K)}_{\mathcal{U}, \mathcal{L}} = \hat{\mathcal{S}}_{\mathcal{K}} + O_P\left( N^{-\frac{1}{2}} \right) \).

Now consider \( \hat{\mathcal{S}}^{(n)}_{\mathcal{U}, \mathcal{L}} \), let \( \hat{\Sigma}_2 = \mathbb{P}_N \{ X_2 X_2^\top \}, \hat{\Sigma}_{\mathcal{L}} = \mathbb{P}_n \{ X_2 X_2^\top \} \), note that by CLT
\[
\mathbb{P}_N \left( \left[ \hat{\beta}_{22}, \hat{\gamma}_2 \right] X_2 \right) - \mathbb{E} \left( \left[ \hat{\beta}_{22}, \hat{\gamma}_2 \right] X_2 \right) = O_P\left( N^{-\frac{1}{2}} \right),
\]

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then using (3), (4) we can write

\[
\hat{S}_{u,l}^n = P_N \begin{pmatrix}
    - P_N \left\{ (\hat{\beta}_{22}^T \gamma_2^T) X_2 \right\} \\
    \bar{\beta}_{21} \left( n^{-1} \sum_{k=1}^K \sum_{i \in I_k} \left\{ Y_{2i} - \hat{m}_{i2}^{(k)}(\bar{U}_i) \right\} - \bar{\mu}_{21}(\bar{U}) + m_2(\bar{U}) \right)
\end{pmatrix}
\]

= \bar{T}_L - \bar{T}_{L,K} + O_P \left( n^{-\frac{1}{2}} \right),

where

\[
\bar{T}_L = P_n \left\{ \left( Y_2 Y_3 - \bar{\mu}_{23}(\bar{U}) \right) - \bar{\beta}_{21} \left( Y_2^2 - \bar{\mu}_{22}(\bar{U}) \right) - \bar{\beta}_{21} \left( Y_2 - \bar{\mu}_{22}(\bar{U}) \right) \right\},
\]

\[
\bar{T}_{L,K} = \frac{1}{n} \sum_{k=1}^K \sum_{i \in I_k} \left\{ \left( \hat{\Delta}_{12}^{(k)}(\bar{U}) - \bar{\beta}_{21} \hat{\Delta}_{11}^{(k)}(\bar{U}) - \bar{\beta}_{21} \hat{\Delta}_{2}^{(k)}(\bar{U}) \right) \right\},
\]

note that \( O_P \left( n^{-\frac{1}{2}} \right) \) comes from Assumption 2 in terms like

\[
\left\{ Q_{2-}(H_2; \theta) \right\} \left( Y_2 - \bar{\mu}_{22}(\bar{U}) + \hat{m}_{i2}^{(k)}(\bar{U}_i) - m_2(\bar{U}_i) \right).
\]

Now we can write \( \hat{R}_{u,l}^{(K)} = T_L - \bar{T}_{L,K} + P_K \bar{S}_k + O_P \left( n^{-\frac{1}{2}} c_{K_n} \right) = T_L + O_P \left( c_{K_n} \right), \)

where the last step follows from \( T_{L,K} - \frac{1}{K} \sum_{k=1}^K \bar{S}_k = O_P \left( n^{-\frac{1}{2}} c_{K_n} \right) \) by Assumption (2) and Lemma 16.

(II) Now we consider \( R_u \), from the CLT, assuming working model (1) it follows that

\[
R_u = \mathbb{E} \left( \bar{\beta}_{21} \hat{\Delta}_{2}^{(k)}(\bar{U}) - \bar{\beta}_{21} \hat{\Delta}_{2}^{(k)}(\bar{U}) \left[ H_{20}^T \hat{\beta}_{22} + (H_{21}^T \gamma_2) A_2 \right] \bar{\mu}_{22}(\bar{U}) \right) + O_P \left( N^{-\frac{1}{2}} \right) = 100 \left( N^{-\frac{1}{2}} \right).
\]
(III) Next we focus on \(\hat{\Gamma}_U\) from (15), we use a similar expansion to (I) and write \(\hat{\Gamma}_U = \Gamma_U + \hat{\Gamma}^{(K)}_{U,L}\), where

\[
\Gamma_U = \mathbb{P}_N \begin{pmatrix}
\hat{\mu}_{22}(\hat{U}) & \hat{\mu}_2(\hat{U})H_{20} & \hat{\mu}_2(\hat{U})A_2H_{21}^T \\
\hat{\mu}_2(\hat{U})H_{20} & H_{20}H_{20} & A_2H_{20}H_{21}^T \\
\hat{\mu}_2(\hat{U})A_2H_{21} & A_2H_{21}H_{20}^T & A_2^2H_{21}H_{21}^T
\end{pmatrix},
\]

\[
\hat{\Gamma}^{(K)}_{U,L} = \mathbb{P}_N \begin{pmatrix}
\hat{\mu}_{22}(\hat{U}) - \hat{\mu}_{22}(\hat{U}) & \left[\hat{\mu}_2(\hat{U}) - \hat{\mu}_2(\hat{U})\right]H_{20} & \left[\hat{\mu}_2(\hat{U}) - \hat{\mu}_2(\hat{U})\right]A_2H_{21} \\
\left[\hat{\mu}_2(\hat{U}) - \hat{\mu}_2(\hat{U})\right]H_{20} & \left[\hat{\mu}_2(\hat{U}) - \hat{\mu}_2(\hat{U})\right]H_{20} & 0 \\
\left[\hat{\mu}_2(\hat{U}) - \hat{\mu}_2(\hat{U})\right]A_2H_{21} & 0 & 0
\end{pmatrix}.
\]

Define

\[
\hat{\mathcal{F}}^{(K)}_{U,L} = \frac{1}{K} \sum_{k=1}^{K} \mathbb{P}_N \begin{pmatrix}
\hat{m}_{22}^{(k)}(\hat{U}_i) - m_{22}(\hat{U}_i) & \hat{m}_2^{(k)}(\hat{U}_i) - m_2(\hat{U}_i) \\
\hat{m}_2^{(k)}(\hat{U}_i) - m_2(\hat{U}_i) & \hat{m}_2^{(k)}(\hat{U}_i) - m_2(\hat{U}_i) \\
\end{pmatrix},
\]

\[
\hat{F}_k = \mathbb{E}_L \begin{pmatrix}
\hat{\Delta}_{11k}(\hat{U}) & H_{20}\hat{\Delta}_{1k}(\hat{U}) & H_{21}A_2\hat{\Delta}_{1k}(\hat{U}) \\
H_{20}\hat{\Delta}_{1k}(\hat{U}) & 0 & 0 \\
H_{21}A_2\hat{\Delta}_{1k}(\hat{U}) & 0 & 0
\end{pmatrix} \forall k \in \{1, \ldots, K\},
\]

As in (I), from (3), (4) it follows that \(\hat{\Gamma}^{(K)}_{U,L} = \hat{\mathcal{F}}^{(K)}_{U,L} + \hat{F}_k\). From (14), Assumptions (2) and Lemma 16 \(\hat{\mathcal{F}}_{U,L} - \hat{F}_k = O_P\left(c_{n_K}\right)\), therefore analogous to \(\hat{\mathcal{R}}^{(K)}_{U,L}, \hat{\mathcal{F}}^{(K)}_{U,L} = O_P\left(n^{-\frac{1}{2}} + n^{-\frac{1}{2}}c_{n_K}\right)\).

(IV) By central limit theorem

\[
\Gamma_U = \mathbb{E} \left[ \mathbb{P}_N \begin{pmatrix}
\hat{\mu}_{22}(\hat{U}) & \hat{\mu}_2(\hat{U})H_{20} & \hat{\mu}_2(\hat{U})A_2H_{21}^T \\
\hat{\mu}_2(\hat{U})H_{20} & H_{20}H_{20} & A_2H_{20}H_{21}^T \\
\hat{\mu}_2(\hat{U})A_2H_{21} & A_2H_{21}H_{20}^T & A_2^2H_{21}H_{21}^T
\end{pmatrix} \right] + O_P\left(N^{-\frac{1}{2}}\right) = \mathbb{E}[X_2X_2^T]^{-1} + O_P\left(N^{-\frac{1}{2}}\right),
\]

thus \(\hat{\Gamma}^{-1}_U = \mathbb{E}[X_2X_2^T]^{-1} + O_P\left(n^{-\frac{1}{2}} + n^{-\frac{1}{2}}c_{n_K}\right)\).

From (I)-(IV) we can write (15) as \((\hat{\theta}_2 - \theta_2) = \mathbb{E}[X_2X_2^T]^{-1} T_L + O_P\left(n^{-\frac{1}{2}}c_{n_K}\right)\), it follows that

\[
\sqrt{n}(\hat{\theta}_2 - \theta_2)
\]

\[
= \mathbb{E}[X_2X_2^T]^{-1} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[ (Y_{2i} - \bar{Y}) - \beta_{21}(\bar{Y}) \right] + O_P\left(c_{n_K}\right).
\]

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Proof [Proof of Theorem 3] Recall the solution to stage 1 estimating equation $\theta_1$ in Section 3.2 satisfies
\[
P_N \left\{ X_1 \left( \hat{\mu}_2 (\bar{U}) + \hat{\beta}_{21} \hat{\mu}_2 (\bar{U}) + H_{20}^T \hat{\beta}_{22} + [H_{21}^T \hat{\gamma}_2]_+ - X_1 \bar{\theta}_1 \right) \right\} = 0,
\]
now with the following definitions
\[
\hat{\Sigma}_{u_l} = P_N \{ X_1 X_1^T \},
\]
\[
R^{(1)} = P_N \left\{ X_1 \left( \hat{\mu}_2 (\bar{U}) + \hat{\beta}_{21} \hat{\mu}_2 (\bar{U}) + H_{20}^T \hat{\beta}_{22} + [H_{21}^T \hat{\gamma}_2]_+ - X_1 \bar{\theta}_1 \right) \right\},
\]
\[
R^{(1K)} = P_N \left\{ X_1 \left( \hat{\mu}_2 (\bar{U}) - \hat{\mu}_2 (\bar{U}) \right) \right\},
\]
we can write $\hat{\Sigma}_{u_l} (\bar{\theta}_1 - \bar{\theta}_1) = R^{(1)} + (1 + \hat{\beta}_{21}) R^{(1K)}$. Further define
\[
\hat{\Sigma}^{(1)}_{L, \bar{L}} = P_N \{ X_1 X_1^T (\bar{\eta}_2 - \eta_2) \},
\]
\[
\hat{\Sigma}^{(1K)}_{L, \bar{L}} = \frac{1}{K} \sum_{k=1}^K P_N \{ X_1 \left( \tilde{m}_2^{(k)} (\bar{U}) - m_2 (\bar{U}) \right) \},
\]
\[
\hat{\Sigma}^{(1)}_k = E \left[ X_1 \hat{\Delta}_{1k} (\bar{U}) \right],
\]
from (4) it follows that $R^{(1K)}_{L, \bar{L}} = \hat{\Sigma}^{(1)}_{L, \bar{L}} + \hat{\Sigma}^{(1K)}_{L, \bar{L}}$, next from Assumptions 1, 2, we get
\[
\sum_{k=1}^K \sup_{x, \bar{U}} \| X_1 \hat{\Delta}_{1k} (\bar{U}) \| = o_p(1),
\]
thus by Lemma 15 $\hat{\Sigma}^{(1K)}_{L, \bar{L}} = \hat{\Sigma}^{(1)}_k + \left( N^{-1} \right)$. Using (4) again, and recalling $\tilde{\mu}_2 (\bar{U}) = m_2 (\bar{U}) + X_1 [\eta_2$ and $\tilde{\mu}_2 (\bar{U}) = \frac{1}{n} \sum_{k=1}^K \sum_{i \in \mathcal{I}_k} \tilde{m}_2^{(k)} (\bar{U}_i) + X_1 [\eta_2$, with some algebra we have
\[
\hat{\Sigma}^{(1)}_{L, \bar{L}} = \hat{\Sigma}_{u_l} (\bar{\eta}_1 - \eta_1)
\]
\[
= \hat{\Sigma}_{u_l} \left[ \hat{\Sigma}^{-1} \{ \frac{1}{n} \sum_{k=1}^K \sum_{i \in \mathcal{I}_k} X_1 \left[ Y_{2i} - \tilde{m}_2^{(k)} (\bar{U}_i) - m_2 (\bar{U}_i) + m_2 (\bar{U}_i) \right] \} - \eta_2 \right]
\]
\[
= \hat{\Sigma}_{u_l} \hat{\Sigma}^{-1} \left[ \frac{1}{n} \sum_{i=1}^n X_1 \left[ Y_{2i} - m_2 (\bar{U}_i) - X_1 [\eta_2 - \frac{1}{n} \sum_{k=1}^K \sum_{i \in \mathcal{I}_k} X_1 \tilde{m}_2^{(k)} (\bar{U}_i) - m_2 (\bar{U}_i) \right] \right]
\]
\[
= \hat{\Sigma}_{u_l} \hat{\Sigma}^{-1} \left[ \frac{1}{n} \sum_{i=1}^n X_1 \left[ Y_{2i} - \tilde{\mu}_2 (\bar{U}_i) \right] - \frac{1}{n} \sum_{k=1}^K \sum_{i \in \mathcal{I}_k} X_1 \tilde{m}_2^{(k)} (\bar{U}_i) - m_2 (\bar{U}_i) \right]
\]
\[
= \hat{\Sigma}_{u_l} \hat{\Sigma}^{-1} \left[ T^{(1)}_{L, \bar{L}} - T^{(1)}_{L, K} \right],
\]
where
\[
T^{(1)}_{L} = P_n \{ X_1 [Y_2 - \bar{\mu}_1 (\bar{U})] \},
\]
\[
T^{(1)}_{L, K} = \frac{1}{n} \sum_{k=1}^K \sum_{i \in \mathcal{I}_k} X_1 \left[ \tilde{m}_2^{(k)} (\bar{U}_i) - m_2 (\bar{U}_i) \right].
\]
Now by the CLT we have
\[ \hat{\Sigma}_u \hat{U}^{-1} = \left[ \mathbb{E} [X_1 X_1^T] + O_P \left( N^{-\frac{1}{2}} \right) \right] \left[ \mathbb{E} [X_1 X_1^T]^{-1} + O_P \left( n^{-\frac{1}{2}} \right) \right] = I + O_P \left( n^{-\frac{1}{2}} \right), \]
therefore using Assumptions 1, 2 we obtain
\[ \hat{S}_{u,L}^{(1)} = \hat{\Sigma}_u \hat{U}^{-1} \left( T_{L,k}^{(1)} - T_{L,k}^{(1)} \right) = T_{L,k}^{(1)} - T_{L,k}^{(1)} + O_P \left( n^{-\frac{1}{2}} \right). \]

Now using Lemma 15 and Assumptions 1, 2, it follows that
\[ \hat{S}_{u,L}^{(1K)} = S_k^{(1)} + O_P \left( N^{-\frac{1}{2}} \right), \]
combining the above we can write
\[ R_{u,L}^{(1K)} = \hat{S}_{u,L}^{(1K)} + S_{u,L}^{(1K)} = T_{L,k}^{(1)} - \left( T_{L,k}^{(1)} - \frac{1}{K} \sum_{k=1}^K S_k^{(1)} \right) + O_P \left( n^{-\frac{1}{2}} \right). \]

Next by Assumption 2 and Lemma 16 we have \( T_{L,k}^{(1)} - \frac{1}{K} \sum_{k=1}^K \hat{S}_k \) = \( O_P \left( n^{-\frac{1}{2}} c_n \right) \), also, using (3) and the CLT we get \( T_{L,k}^{(1)} = O_P \left( n^{-\frac{1}{2}} \right) \), therefore \( R_{u,L}^{(1K)} = T_{L,k}^{(1)} + O_P \left( n^{-\frac{1}{2}} c_n \right). \)

Finally using Theorem 2 we can write \( (1 + \hat{\beta}_{21}) R_{u,L}^{(1K)} = (1 + \hat{\beta}_{21}) T_{L,k}^{(1)} + O_P \left( n^{-\frac{1}{2}} c_n \right). \)

Next
\[ R^{(1)} = \mathbb{P}_N \left\{ X_1 \left( \hat{\mu}_2(\bar{U}) + \beta_{21} \hat{\mu}_2(\bar{U}) + H_{20}^T \beta_{22} + [H_{21} \gamma_{22}]_+ - X_1 \theta \right) \right\} + \mathbb{P}_N \left\{ X_1 \left[ \hat{\mu}_2(\bar{U}) [\beta_{21} - \beta_{21}] + H_{20}^T [\beta_{22} - \beta_{22}] + [H_{21} \gamma_{22}]_+ - [H_{21} \gamma_{22}]_+ \right] \right\}, \]

note that under (3) the first term in \( R^{(1)} \) is mean zero, therefore from Assumption 1 and CLT
\[ \sqrt{N} \mathbb{P}_N \left\{ X_1 \left( \hat{\mu}_2(\bar{U}) + \beta_{21} \hat{\mu}_2(\bar{U}) + H_{20}^T \beta_{22} + [H_{21} \gamma_{22}]_+ - X_1 \theta \right) \right\} = O_P \left( 1 \right), \]
using the above we have
\[ \sqrt{n} R^{(1)} = \sqrt{n} \mathbb{P}_N \left\{ X_1 \left( \hat{\mu}_2(\bar{U}) [\hat{\beta}_{21} - \beta_{21}] + H_{20}^T [\hat{\beta}_{22} - \beta_{22}] + [H_{21} \gamma_{22}]_+ - [H_{21} \gamma_{22}]_+ \right) \right\} + O_P \left( \frac{n}{N} \right), \]
\[ = \mathbb{P}_N \left\{ X_1 \left[ \hat{\mu}_2(\bar{U}), H_{20}^T \right] \right\} \sqrt{n} \left( \hat{\beta}_{2} - \beta_{2} \right) + \sqrt{n} \mathbb{P}_N \left\{ X_1 \left( [H_{21} \gamma_{22}]_+ - [H_{21} \gamma_{22}]_+ \right) \right\} + O_P \left( \frac{n}{N} \right), \]
\[ = \mathbb{E} \left\{ X_1 \left[ \hat{\mu}_2(\bar{U}), H_{20}^T \right] \right\} n^{-\frac{1}{2}} \sum_{i=1}^n \psi_{2i} + \sqrt{n} \mathbb{P}_N \left\{ X_1 \left( [H_{21} \gamma_{22}]_+ - [H_{21} \gamma_{22}]_+ \right) \right\} + O_P \left( \frac{n}{N} \right), \]
where the last inequality follows from Lemma 15 and Theorem 2, and \( \psi_{2i} \) is the element corresponding to \( \hat{\beta}_2 \) of the influence function \( \psi_{2i} \) defined in Theorem 2.

Next by Theorem 2 we know that
\[ \sqrt{n} (\gamma_{2} - \gamma_{2}) = O_P \left( 1 \right), \]
using Lemma 17 (a) we have

$$\mathbb{P} \left( \sqrt{n}\mathbb{P}_N \left\{ X_1 (H_{21i}^T \hat{\gamma}_2)_+ - [H_{21i}^T \gamma_2]_+ \right\} \right) = \mathbb{P}_N \left\{ X_1 H_{21i}^T \times I_{\{H_{21i}^T \gamma_2 > 0\}} \right\} \sqrt{n} (\hat{\gamma}_2 - \gamma_2) \rightarrow 1.$$  

Therefore, letting $\psi_{2\gamma}$ be the element corresponding to $\hat{\gamma}_2$ of the influence function $\psi_{2i}$ defined in Theorem 2,

$$\sqrt{n}\mathbb{P}_N \left\{ X_1 (H_{21i}^T \hat{\gamma}_2)_+ - [H_{21i}^T \gamma_2]_+ \right\} = \mathbb{P}_N \left\{ X_1 H_{21i}^T \times I_{\{H_{21i}^T \gamma_2 > 0\}} \right\} \sqrt{n} (\hat{\gamma}_2 - \gamma_2) + \sqrt{n}\mathbb{P}_N \left\{ X_1 (H_{21i}^T \hat{\gamma}_2)_+ - [H_{21i}^T \gamma_2]_+ \right\} I_{\{\hat{\gamma}_2 \notin \mathcal{A}\}}$$

$$= \mathbb{E} [X_1 H_{21i}^T | H_{21i}^T \gamma_2 > 0, \hat{\gamma}_2 \in \mathcal{A}] \mathbb{P} (H_{21i}^T \gamma_2 > 0) \mathbb{P} (\hat{\gamma}_2 \in \mathcal{A}) \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \psi_{2\gamma i} + o_{\mathbb{P}} (1)$$

$$+ O_{\mathbb{P}} \left( c_{n^{-1}} \right) + o_{\mathbb{P}} (1)$$

$$= \mathbb{E} [X_1 H_{21i}^T | H_{21i}^T \gamma_2 > 0] \mathbb{P} (H_{21i}^T \gamma_2 > 0) \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \psi_{2\gamma i} + o_{\mathbb{P}} (1),$$

combining all terms

$$\sqrt{n}\mathcal{R}^{(1)} = \mathbb{E} \left[ X_1 \left( \hat{\mu}_2 (\bar{U}), H_{20}^T \right) \right] n^{-\frac{1}{2}} \sum_{i=1}^{n} \psi_{2i(\beta)}$$

$$+ \mathbb{E} [X_1 H_{21i}^T | H_{21i}^T \gamma_2 > 0] \mathbb{P} (H_{21i}^T \gamma_2 > 0) \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \psi_{2i(\gamma)}$$

$$+ O_{\mathbb{P}} \left( c_{n^{-1}} \right).$$

Finally, using our results for $\mathcal{R}^{(1)}$, $\hat{\mathcal{R}}_{\ell, \mathcal{L}}^{(1K)}$, and the fact that $\hat{\Sigma}_{\ell i} = \mathbb{E} [X_1 X_i]^{-1} + o_{\mathbb{P}} (1)$ by the LLN, we have

$$\sqrt{n}(\hat{\theta}_1 - \theta_1) = \mathbb{E} [X_1 X_i]^{-1} \hat{\Sigma}_{\ell i}^{-1} \mathcal{R}^{(1)} + \mathbb{E} [X_1 X_i]^{-1} (1 + \hat{\beta}_{21}) \hat{\mathcal{R}}_{\ell, \mathcal{L}}^{(1K)} + o_{\mathbb{P}} (1)$$

$$= \mathbb{E} [X_1 X_i]^{-1} (1 + \hat{\beta}_{21}) \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \{Y_{2i} - \hat{\mu}_2 (\bar{U}, i)\}$$

$$+ \mathbb{E} [X_1 X_i]^{-1} \mathbb{E} \left[ X_1 \left( \hat{\mu}_2 (\bar{U}), H_{20}^T \right) \right] \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \psi_{2i(\beta)}$$

$$+ \mathbb{E} [X_1 X_i]^{-1} \mathbb{E} \left[ X_1 H_{21i}^T | H_{21i}^T \gamma_2 > 0 \right] \mathbb{P} (H_{21i}^T \gamma_2 > 0) \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \psi_{2i\gamma}$$

$$+ o_{\mathbb{P}} (1),$$

using (3) we have $\mathbb{E} \left[ X_1 \left( \hat{\mu}_2 (\bar{U}), H_{20}^T \right) \right] = \mathbb{E} [X_1 (Y_2, H_{20}^T)]$ which yields our required result.
Proof [Proof of Proposition 5] Note that the variance-covariance matrices have block diagonals of the form \( \text{Var} [\gamma] \), \( \text{Var} [\beta] \), \( \text{Var} [\gamma_1] \), \( \text{Var} [\beta_1] \), we will show the result is true for block \( \text{Var} [\gamma_2] \), the rest are analogous. Let

\[
\varepsilon_2^n = -H_{22}A_2 \left\{ \bar{\mu}_3(\bar{U}) - [\bar{\mu}_2(\bar{U}), X_2^\top]\bar{\theta}_2 \right\},
\]

note that using (3) it follows that

\[
\mathbb{E} \left[ H_{22}A_2 \left\{ Y_3 - Q_2(\bar{X}_2; \bar{\theta}_2) \right\} | \bar{U} \right] = -\varepsilon_2^n;
\]

next, as \( \theta \) are the population solutions for (2), \( \mathbb{E}[H_{22}A_2 \left\{ Y_3 - Q_2(\bar{X}_2; \bar{\theta}_2) \right\}] = 0 \), using iterated expectations we have

\[
\mathbb{E} [\varepsilon_2^n] = -\mathbb{E} [H_{22}A_2 \left\{ Y_3 - Q_2(\bar{X}_2; \bar{\theta}_2) \right\}] = 0.
\]

Now in light of this,

\[
\text{Cov} \left[ H_{22}A_2 \left\{ Y_3 - Q_2(\bar{X}_2; \bar{\theta}_2) \right\}, \varepsilon_2^n \right] = \mathbb{E} \left[ \left( H_{22}A_2 \left\{ Y_3 - Q_2(\bar{X}_2; \bar{\theta}_2) \right\} \varepsilon_2^n \right) \right]
\]

\[
= \mathbb{E} \left[ \mathbb{E} \left[ H_{22}A_2 \left\{ Y_3 - Q_2(\bar{X}_2; \bar{\theta}_2) \right\} \varepsilon_2^n | \bar{U} \right] \right]
\]

\[
= -\text{Var} [\varepsilon_2^n],
\]

thus, denote by \( \Sigma_{\text{gup}}^{\gamma_2} \) be the block corresponding to \( \gamma_2 \) in \( \Sigma_{\text{gup}} \), for the variance defined by the semi-supervised procedure, from Theorem 2 we have:

\[
\text{Var} [\gamma_2] = \text{Var} \left[ H_{22}A_2 \left\{ Y_3 - \bar{\mu}_3(\bar{U}) \right\} \right] + \text{Var} \left[ H_{22}A_2 \left\{ Y_3 - \bar{\mu}_2(\bar{U}) \right\} \right]
\]

\[
+ 2 \text{Cov} \left[ H_{22}A_2 \left\{ Y_3 - Q_2(\bar{X}_2; \bar{\theta}_2) \right\}, \varepsilon_2^n \right] + \text{Var} [\varepsilon_2^n]
\]

\[
= \Sigma_{\text{gup}}^{\gamma_2} - \text{Var} [\varepsilon_2^n].
\]

\[
\blacksquare
\]

Appendix C. Technical Lemmas

We start with a simple Lemma that will save us some algebra:

Lemma 14 For a fixed \( \ell \), let \( X \in \mathbb{R}^\ell \) be a random bounded vector and functions \( g_1(X), g_2(X) \) be measurable functions of \( X \). Let \( S_n = \{X\}_{i=1}^n \) be an i.i.d. sample, and \( \hat{g}_1(\cdot), \hat{g}_2(\cdot) \) be the estimators for functions \( g_1, g_2 \in \mathbb{R} \) respectively with \( \sup_X |g_1(X)|, \sup_X |g_2(X)|, \sup_X |\hat{g}_1(X)|, \sup_X |\hat{g}_2(X)| < \kappa \) for fixed \( \kappa \in \mathbb{R} \). If \( P_n \{ \hat{g}_k - g_k \} = O_p \left( n^{-\frac{1}{2}} \right) \), for \( k = 1, 2 \), then \( P_n \{ \hat{g}_1\hat{g}_2 - \hat{g}_1g_2 \} = O_p \left( n^{-\frac{1}{2}} \right) \).
Proof [Proof of Lemma 14] By definition, \( P_n\{\hat{g}_1\hat{g}_2 - g_1g_2\} = O_P\left(n^{-\frac{1}{2}}\right) \) if and only if for a given any \( \epsilon > 0 \), \( \exists M > 0 \) such that
\[
P\left(|P_n\{\hat{g}_1\hat{g}_2 - g_1g_2\}| > M\epsilon n^{-\frac{1}{2}}\right) \leq \epsilon \forall n. \]
Let \( M > 0 \),
\[
P\left(|P_n\{\hat{g}_1\hat{g}_2 - g_1g_2\}| > M\epsilon n^{-\frac{1}{2}}\right)
= P\left(|P_n\{\hat{g}_1\hat{g}_2 - g_1g_2 + \hat{g}_1\hat{g}_2 - g_1g_2\}| > M\epsilon n^{-\frac{1}{2}}\right)
\leq P\left(|P_n\{\hat{g}_1\hat{g}_2 - g_1g_2\}| + |P_n\{g_2(\hat{g}_1 - g_1)\}| > M\epsilon n^{-\frac{1}{2}}\right)
\leq P\left(\sup_X |\hat{g}_1(X)||P_n\{\hat{g}_2 - g_2\}| + \sup_X |g_2(X)||P_n\{\hat{g}_1 - g_1\}| > M\epsilon n^{-\frac{1}{2}}\right)
\]
which follows from bounded functions, the union bound, now since \( P_n\{\hat{g}_k(X) - g_k(X)\} = O_P\left(n^{-\frac{1}{2}}\right) \), \( k = 1, 2 \), there exists \( M > 0 \) such that
\[
P\left(|P_n\{\hat{g}_2 - g_2\}| > M\epsilon n^{-\frac{1}{2}}\right) + P\left(|P_n\{\hat{g}_1 - g_1\}| > M\epsilon n^{-\frac{1}{2}}\right) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]

Lemma 15 (Lemma (A.1) (a) in Chakrabortty et al. (2018))
Let \( X \in \mathbb{R}^\ell \) be any random vector and \( g(X) \in \mathbb{R}^\ell \) be any measurable function of \( X \), with \( \ell \) and \( d \) fixed. Let \( S_n = \{X\}_i^{n}, S_N = \{X\}_j^{N} \) be two random samples of \( n \) and \( N \) i.i.d observations of \( X \) respectively, such that \( S_n \perp S_N \). Let \( \hat{g}_n(\cdot) \) be any estimator of \( g(\cdot) \) estimated with \( S_n \) such that the random sequence: \( \hat{T}_n = \sup_{x \in X} \|\hat{g}_n(\cdot)\| = O_P(1) \), where \( X \in X \subseteq \mathbb{R}^\ell \). Further define the following random sequences: \( G_{n,N} = \frac{1}{N} \sum_{j=1}^{N} \hat{g}_n(X_j) \), and \( G_n = \mathbb{E}[G_{n,N}|S_n] = \mathbb{E}_{X}[\hat{g}_n(X)|S_n] \). We assume all expectations involved are finite almost surely (a.s.) \( S_n \forall n \). Then \( G_{n,N} - G_n = O_P\left(N^{-\frac{1}{2}}\right) \).

Proof [Proof of lemma 15]
The following proof follows similar arguments to Chakrabortty et al. (2018).
Let \( G_{n,N}, \tilde{G}_n \) be the \( j \)th element of \( G_{n,N} \) and \( G_n \) respectively, with \( j \in \{1, \ldots, \ell\} \). We show that \( G_{n,N} - \tilde{G}_n = O_P\left(N^{-\frac{1}{2}}\right) \), which implies Lemma 15 for any \( \ell \) dimensional \( G_{n,N}, \tilde{G}_n \). Denote by \( \mathbb{P}_{S_n}, \mathbb{P}_{S_n,S_N} \) denote the joint probability distributions of samples \( S_n \) and \( S_n, S_N \) respectively. Further let \( \mathbb{E}_{S_n}[\cdot] \) denote the expectation with respect to \( S_n \) and \( \mathbb{P}_{S_n|S_N} \) denote the conditional probability of \( S_n \) given \( S_N \).

Since \( S_n \perp S_N \)
\[
\mathbb{P}_{S_N}\left(\left|\hat{G}_{n,N} - \hat{G}_n\right| > N^{-\frac{1}{2}}t\right|S_n) \leq 2 \exp\left(-\frac{2N^2t^2}{4N^2T_n^2}\right) \text{ a.s. } \mathbb{P}_{S_n}.
\]
We have that $\hat{T}_n = O_p(1)$ and non-negative, thus $\forall \epsilon > 0 \exists \delta(\epsilon) > 0$ such that $P_{S_n} \left( \hat{T}_n > \delta(\epsilon) \right) < \epsilon/4$, using the above we have that $\forall n, N$: 

$$
\begin{align*}
P_{S_n,S_N} \left( |\hat{g}_{n,N} - \hat{g}_{n}| > N^{-\frac{1}{2}} t \right) & \leq E_{S_n} \left[ 2 \exp \left( - \frac{t^2}{2T_n^2} \right) \right] \\
& = E_{S_n} \left[ 2 \exp \left( - \frac{t^2}{2T_n^2} \right) \right] \\
& \leq 2P_{S_n} \left( \hat{T}_n < \delta(\epsilon) \right) + 2 \exp \left( - \frac{t^2}{2\delta^2(\epsilon)} \right) \leq 2 \exp \left( - \frac{t^2}{2\delta^2(\epsilon)} \right) + \frac{\epsilon}{2} \leq \frac{2\epsilon}{2} = \epsilon,
\end{align*}
$$

where the last step follows from choosing $t$ large enough such that $\exp \left( - \frac{t^2}{2\delta^2(\epsilon)} \right) \leq \epsilon/4$. ■

For Assumption 7 and Lemma 16 we first define some notation and set up the problem. Let $X = (X_1, X_2) \in \mathbb{R}^{\ell_1+\ell_2}$ be any random vector and $g(X_1) \in \mathbb{R}$ be any measurable function of $X_1 \in \mathbb{R}^{\ell_1}$ with $\ell_1, \ell_2$ fixed. Suppose we’re interested in estimating $m(X_2) = E[g(X_1)|X_2]$. Let $S_n = \{X_i\}_{i=1}^n$ be a random sample of n i.i.d. observations of $X$, and $S_n^K$ denote a random partition of $S_n$ into $K$ disjoint subsets of size $n_K = \frac{n}{K}$ with index sets $\{I_k\}_{k=1}^K$. We will use cross-validation to estimate $m(X_2)$, that is, we use subset $I_k$ to train estimator $\hat{m}_k$ and we estimate $m(X_2)$ with: $\hat{m}(X_2) = K^{-1} \sum_{k=1}^K \sum_{i \in I_k} \hat{m}_k(X_2), K \geq 2$. Denote by $\hat{C}_{n,N} \in \mathbb{R}$ an estimator which depends on both samples $S_n, S_N$. Additionally, let function $\hat{\pi}_n(\cdot) : \mathbb{R}^{\ell_2} \to (0, 1)$ be a random function with limit $\pi(\cdot), \hat{\pi}_n(X_2) : \mathbb{R}^{\ell_2} \to \{0, 1\}$, be a random function with limit $l(X_2)$, and finally function $f : \mathbb{R}^{\ell_2} \to \mathbb{R}^d$, $d \leq \ell_2$ be any deterministic function of $X_2$.

**Assumption 7** Let $X \subset \mathbb{R}^p$ for an arbitrary $p \in \mathbb{N}$ i) function $w : X \to \mathbb{R}$ and estimator $\hat{\pi}_n$ are such that $sup_{X_2} |\hat{\pi}_n(X_2)|^{-1} - \pi(X_2)^{-1} = O_p \left( n^{-\frac{1}{2}} \right)$, ii) function $l : X \to \{0, 1\}$ and estimator $\hat{l}_n$ are such that $sup_{X_2} |\hat{l}_n(X_2) - l(X_2)| = O_p \left( n^{-\frac{1}{2}} \right)$, and iii) function $f : \mathbb{R}^{\ell_2} \to \mathbb{R}^d$, $d \leq \ell_2$ is such that $sup_{X_2} \|f(X_2)\| < \infty$.

**Lemma 16** Define $\hat{G}_k^n(X_2) = \hat{C}_{n,N} \frac{l(X_2)}{\hat{\pi}_n(X_2)} f(X_2) \Delta_k(X_2) - E \left[ \frac{l(X_2)}{\hat{\pi}(X_2)} f(X_2) \Delta_k(X_2) \right]$ for $\Delta_k(X_2) = \hat{m}_k(X_2) - m(X_2)$, and $\hat{C}_{n,N} \in \mathbb{R}$ which satisfies $\hat{C} = 1 + O_p \left( n^{-\frac{1}{2}} \right)$. Under Assumptions 6 and 7, there is $c_{n,n,K}$ such that $sup_{X_2} \|G_k^n(X_2)\| = O_p \left( c_{n,n,K} \right)$.

**Proof** [Proof of Lemma 16] First we define

$$
G_k(n) = n^{-\frac{1}{2}} \sum_{i \in I_k} \frac{l(X_2)}{\hat{\pi}(X_2)} f(X_2) \Delta_k(X_2) - E \left[ \frac{l(X_2)}{\hat{\pi}(X_2)} f(X_2) \Delta_k(X_2) \right],
$$

for any sample subset $S_K \subseteq L$, let $P_{S_K}$ denote the joint probability distribution of $S_K$, and let $E_{S_K}[\cdot]$ denote expectation with respect to $P_{S_K}$, and $G_{n,K} = K^{-\frac{1}{2}} \sum_{k=1}^K G_k(n)$, Next by Assumption 6 we have $\hat{d}_k \equiv sup_{X_2} \Delta_k(X_2) = O_p(1)$. Finally let $B_1 = sup_{X_2} \|f(X_2)\|_2 < \infty, B_2 < \infty$ be the upperbound to $sup_{X_2} |\pi(X_2)^{-1}|, sup_{X_2} |\hat{l}_n(X_2)|, sup_{X_2} |\hat{\pi}(X_2)|$. 55
First note that

\[
\|G_{n,K}\|_2 \leq \left( C_{n,N} - 1 \right) n^{-\frac{1}{2}} \sum_{k=1}^{K} \sum_{i \in I_k} f(X_{2i}) \Delta_k(X_{2i}) \hat{i}_n(X_{2i}) \left( \frac{1}{\hat{\pi}_n(X_{2i})} - \frac{1}{\pi(X_{2i})} \right) + \| \hat{n}(X_{2i}) \Delta_k(X_{2i}) \hat{i}_n(X_{2i}) - E \left[ \frac{l(X_{2i})}{\pi(X_{2i})} f(X_{2i}) \hat{\Delta}_k(X_{2i}) \right] \|_2,
\]

which follows from the triangle inequality, next as \( f(\cdot), \hat{\pi}_n(\cdot)^{-1}, \pi(\cdot)^{-1}, \hat{i}_n(\cdot) \) are bounded \( \forall X_2 \in \mathcal{X} \), and using uniform bounds of \( O_p \left( n^{-\frac{1}{2}} \right) \) for the difference terms we have

\[
\|G_{n,K}\|_2 \leq O_p \left( n^{-\frac{1}{2}} \right) n^{\frac{1}{2}} B_1 B_2 \left| \sum_{k=1}^{K} \hat{d}_k \right| + O_p \left( n^{-\frac{1}{2}} \right) n^{\frac{1}{2}} B_1 B_2 \left| \sum_{k=1}^{K} \hat{d}_k \right|
\]

\[
+ O_p \left( n^{-\frac{1}{2}} \right) n^{\frac{1}{2}} B_1 B_2 \left| \sum_{k=1}^{K} \hat{d}_k \right| + \left| \sum_{k=1}^{K} \frac{1}{K} \sum_{i \in I_k} G_k^{(n)} \right|_2,
\]

\[
\leq \left| \sum_{k=1}^{K} \frac{1}{K} \sum_{i \in I_k} \frac{l(X_{2i})}{\pi(X_{2i})} f(X_{2i}) \Delta_k(X_{2i}) - E \left[ \frac{l(X_{2i})}{\pi(X_{2i})} f(X_{2i}) \hat{\Delta}_k(X_{2i}) \right] \right|_2 + o_p \left( 1 \right).
\]

where the last step follows from \( \hat{d}_k = o_p(1) \). Next we want to bound the first term above by \( c_{n,K} \) in probability, note that \( \forall \varepsilon \exists M > 0 \) such that

\[
\mathbb{P} \left( \left\| \sum_{k=1}^{K} G_k^{(n)} \right\|_2 > M c_{n,K} \right) \leq \mathbb{P} \left( \left| \sum_{k=1}^{K} \frac{G_k^{(n)}}{K} \right|_2 > M c_{n,K} \right)
\]

\[
\leq \sum_{k=1}^{K} \mathbb{P} \left( \left\| G_k^{(n)} \right\|_2 > \frac{M c_{n,K}}{K^{\frac{1}{2}}} \right) \leq \sum_{k=1}^{K} \sum_{j=1}^{d} \mathbb{P} \left( \left\| G_k^{(n)} \right|_{k[j]} > \frac{M c_{n,K}}{(Kd)^{\frac{1}{2}}} \right)
\]

\[
\leq \sum_{k=1}^{K} \sum_{j=1}^{d} \mathbb{E}_{\mathcal{L}_k} \left[ \mathbb{P} \left( \left\| G_k^{(n)} \right|_{k[j]} > \frac{M c_{n,K}}{(Kd)^{\frac{1}{2}}} \right) \right],
\]
where the first 3 steps follow from applying Boole’s inequality and the triangle inequality, the fourth step follows from iterated expectations for the the event \( \{|g_{k[j]}^{(n)}| > \frac{Mc_{n,k}^-}{(Kd)^{1/2}} \} \).

Next, we have \( L_k^- \perp L_k, \forall k \in \{1, \ldots, K\} \), thus conditional on \( L_k^- \), \( n \frac{1}{2} g_{k}^{(n)} \) is a sum of iid centered random vectors \( \{l(X_{2i}) f(X_{2i}) \Delta_k(X_{2i})\}_{i \in I_k} \) which are bounded a.s. \( P_{L_k}^- \), \( \forall k, n \).

Thus we can apply Hoeffding’s inequality to \( g_{k[j]}^{(n)} \forall j \):

\[
P_{L_k} \left( \left| g_{k[j]}^{(n)} \right| > \frac{M c_{n,k}^-}{(Kd)^{1/2}} L_k^- \right) \leq 2 \exp \left\{ - \frac{M^2 c_{n,k}^-^2}{2Kd B^2 d_k^2} \right\}
\]

a.s. \( P_{L_k}^- \forall n \); and for each \( k \in \{1, \ldots, K\}, j \in \{1, \ldots, d\} \). Note that \( \frac{c_{n,k}^-}{d_k} \geq 0 \) is stochastically bounded away from zero as \( d_k = o_P(1) \), therefore \( \forall k \) and given \( \epsilon > 0 \), \( \exists \delta(\epsilon, k) > 0 \) such that

\[
P_{L_k} \left( \frac{c_{n,k}^-}{d_k} \leq \delta(\epsilon, k) \right) \leq \frac{\epsilon}{4Kd}, \text{ let } \delta^*(\epsilon, k) = \min_k \{\delta(\epsilon, k)\}, \text{ we have that}
\]

\[
P_{L_k} \left( \frac{c_{n,k}^-}{d_k} \leq \delta^*(\epsilon, k) \right) \leq \frac{\epsilon}{4Kd}.
\]

Therefore using the bound in (16) and event \( \{\frac{c_{n,k}^-}{d_k} \leq \delta^*(\epsilon, k)\} \):

\[
P \left( \left\| \sum_{k=1}^{K} g_k^{(n)} \right\|_2 > Mc_{n,K}^- \right)
\]

\[
\leq \sum_{k=1}^{K} \sum_{j=1}^{d} \mathbb{E}_{L_k^-} \left[ P_{L_k} \left( \left| g_{k[j]}^{(n)} \right| > \frac{M c_{n,k}^-}{(Kd)^{1/2}} L_k^- \right) \right]
\]

\[
\leq \sum_{k=1}^{K} \sum_{j=1}^{d} \mathbb{E}_{L_k^-} \left[ 2 \exp \left\{ - \frac{M^2 c_{n,k}^-^2}{2Kd B^2 d_k^2} \right\} \left( I \left\{ \frac{c_{n,k}^-}{d_k} \leq \delta^*(\epsilon, k) \right\} + I \left\{ \frac{c_{n,k}^-}{d_k} > \delta^*(\epsilon, k) \right\} \right) \right]
\]

\[
\leq 2Kd \exp \left\{ - \frac{M^2 \delta^*(\epsilon, k)^2}{2Kd B^2} \right\} \mathbb{P}_{L_k^-} \left( \frac{c_{n,k}^-}{d_k} \leq \delta^*(\epsilon, k) \right) + 2Kd \mathbb{P}_{L_k^-} \left( \frac{c_{n,k}^-}{d_k} > \delta^*(\epsilon, k) \right)
\]

\[
\leq 2Kd \frac{\epsilon}{4Kd} + 2Kd \exp \left\{ - \frac{M^2 \delta^*(\epsilon, k)^2}{2Kd B^2} \right\} \mathbb{P}_{L_k^-} \left( \frac{c_{n,k}^-}{d_k} > \delta^*(\epsilon, k) \right),
\]

next note that choosing a large enough \( M \) such that \( \exp \left\{ - \frac{M^2 \delta^*(\epsilon, k)^2}{2Kd B^2} \right\} < \frac{\epsilon}{4Kd} \), since

\[
P_{L_k^-} \left( \frac{c_{n,k}^-}{d_k} > \delta^*(\epsilon, k) \leq 1 \right)
\]

we get \( \mathbb{P} \left( \left\| \sum_{k=1}^{K} g_k^{(n)} \right\|_2 > Mc_{n,K}^- \right) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \).

Finally we have

\[
g_{n,K} = O_P \left( c_{n,K}^- \right) + o_P(1) = O_P \left( c_{n,K}^- \right).
\]
Lemma 17 Let $\hat{\gamma} \in \mathbb{R}^d$ be a random variable such that $\sqrt{n}(\hat{\gamma} - \gamma) = O_P(1)$, then for any fixed vector $a \in \mathbb{R}^d$ we have that (a) $\sqrt{n}\left(a^T[\hat{\gamma}]_+ - a^T[\gamma]_+\right) = \sqrt{n}(\hat{\gamma} - \gamma)I_{\{a^T\gamma > 0\}} + o_p(1)$, 
(b) Functions $\hat{d}_t, t = 1, 2, \ldots$, defined in Section 4 and propensity scores $\pi_1, \pi_2$ in (5) satisfy
\[
\sup_{H_1, A_1} \left| I(\hat{d}_1 = A_1) - I(d^*_1 = A_1) \right| = o_P(1), \\
\sup_{H_2, A_2} \left| I(\hat{d}_1 = A_1)I(A_2 = \hat{d}_2) - I(d^*_1 = A_1)I(d^*_2 = A_2) \right| = o_P(1), \\
\sup_{H_1} \left| \frac{1}{\pi_1(H_1)} - \frac{1}{\pi_1(H_1)} \right| = O_P\left(n^{-\frac{1}{2}}\right).
\]
(c) For $\hat{\theta}, \hat{\xi}$ estimated via our semi-supervised approach, and limits $\bar{\theta}, \bar{\xi}$ defined in Assumptions 3 and 5 respectively
\[
\hat{C}_{n,N}^{(1)} = \mathbb{P}_n \left\{ \frac{I(d_1 = A_1)}{\pi_1(H_1)} \right\}, \\
\hat{C}_{n,N}^{(2)} = \mathbb{P}_n \left\{ \frac{I(d_1 = A_1)I(d_2 = A_2)}{\pi_1(H_1)} \right\}, \\
\bar{C}_{n,N}^{(1)} = 1 + O_P(n^{-\frac{1}{2}}), \\
\bar{C}_{n,N}^{(2)} = 1 + O_P(n^{-\frac{1}{2}}).
\]
Proof [Proof of Lemma 17]
Define set $A_q$ for any $q$ dimensional vector $\hat{\gamma}$ as
\[
A_q = \left\{ \hat{\gamma} \in \mathbb{R}^q \mid \frac{1}{2}a^T\hat{\gamma} < a^T\gamma < 2a^T\gamma, \forall a \in \mathbb{R}^q \right\}.
\]
Now consider $\hat{\gamma} \in A_q$:

- if $\text{sign}(a^T\hat{\gamma}) = 1$, then $0 < \frac{1}{2}a^T\hat{\gamma} < a^T\gamma \implies \text{sign}(a^T\hat{\gamma}) = 1$,
- if $\text{sign}(a^T\hat{\gamma}) = -1$, then $a^T\hat{\gamma} < 2a^T\gamma < 0 \implies \text{sign}(a^T\hat{\gamma}) = -1$.

Assuming $\sqrt{n}(\hat{\gamma} - \gamma) = O_P(1)$, $A_q$ exists and in fact it is such that $\mathbb{P}(\hat{\gamma} \in A_q) \xrightarrow{p} 1$.

(a) Using the above:
\[
\sqrt{n}\left(\{a^T\hat{\gamma}\}_+ - \{a^T\gamma\}_+\right) = \sqrt{n}(\hat{\gamma} - \gamma)I_{\{a^T\gamma > 0\}} + \sqrt{n}\left(\{a^T\hat{\gamma}\}_+ - \{a^T\gamma\}_+\right) I_{\{\hat{\gamma} \notin A_q\}} \\
= \sqrt{n}(\hat{\gamma} - \gamma)I_{\{a^T\gamma > 0\}} + o_P(1).
\]

(b) As $A_{ti} \in \{0, 1\}, t = 1, 2$, we can write
\[
I(\hat{d}_{1i} = A_{1i})I(\hat{d}_{2i} = A_{2i}) = \{A_{1i} = \{H_{1i}^H\hat{\gamma}_1 > 0\}\{A_{2i} = \{H_{2i}^H\hat{\gamma}_2 > 0\}\} \\
= \{A_{1i} = \{H_{1i}^H\hat{\gamma}_1 > 0\}\{A_{2i} = \{H_{2i}^H\hat{\gamma}_2 > 0\}\} \\
= A_{1i}A_{2i}I(\hat{H}_{1i}^H\hat{\gamma}_1 > 0)I(\hat{H}_{2i}^H\hat{\gamma}_2 > 0) \\
+ (1 - A_{1i})(1 - A_{2i})I(\hat{H}_{1i}^H\hat{\gamma}_1 < 0)I(\hat{H}_{2i}^H\hat{\gamma}_2 < 0) \\
+ A_{1i}(1 - A_{2i})I(\hat{H}_{1i}^H\hat{\gamma}_1 > 0)I(\hat{H}_{2i}^H\hat{\gamma}_2 < 0) \\
+ (1 - A_{1i})A_{2i}I(\hat{H}_{1i}^H\hat{\gamma}_1 < 0)I(\hat{H}_{2i}^H\hat{\gamma}_2 > 0),
\]
therefore

\[
\left| I(\hat{d}_{1i} = A_{1i})I(\hat{d}_{2i} = A_{2i}) - I(d^*_{1i} = A_{1i})I(d^*_{2i} = A_{2i}) \right|
\]

\[
= |A_{1i}A_{2i}\{I(H^T_{11}\hat{\gamma}_1 > 0)I(H^T_{21}\hat{\gamma}_2 > 0) - I(H^T_{11}\hat{\gamma}_1 > 0)I(H^T_{21}\hat{\gamma}_2 > 0)\}|
\]

\[
+ (1 - A_{1i})(1 - A_{2i})\{I(H^T_{11}\hat{\gamma}_1 < 0)I(H^T_{21}\hat{\gamma}_2 < 0) - I(H^T_{11}\hat{\gamma}_1 < 0)I(H^T_{21}\hat{\gamma}_2 < 0)\}
\]

\[
+ A_{1i}(1 - A_{2i})\{I(H^T_{11}\hat{\gamma}_1 > 0)I(H^T_{21}\hat{\gamma}_2 < 0) - I(H^T_{11}\hat{\gamma}_1 > 0)I(H^T_{21}\hat{\gamma}_2 < 0)\}
\]

\[
+ (1 - A_{1i})A_{2i}\{I(H^T_{11}\hat{\gamma}_1 < 0)I(H^T_{21}\hat{\gamma}_2 > 0) - I(H^T_{11}\hat{\gamma}_1 < 0)I(H^T_{21}\hat{\gamma}_2 > 0)\}
\]

\[
\leq |A_{1i}A_{2i}\{I(H^T_{11}\hat{\gamma}_1 > 0)I(H^T_{21}\hat{\gamma}_2 > 0) - I(H^T_{11}\hat{\gamma}_1 > 0)I(H^T_{21}\hat{\gamma}_2 > 0)\}|
\]

\[
+ (1 - A_{1i})(1 - A_{2i})\{I(H^T_{11}\hat{\gamma}_1 < 0)I(H^T_{21}\hat{\gamma}_2 < 0) - I(H^T_{11}\hat{\gamma}_1 < 0)I(H^T_{21}\hat{\gamma}_2 < 0)\}
\]

\[
+ A_{1i}(1 - A_{2i})\{I(H^T_{11}\hat{\gamma}_1 > 0)I(H^T_{21}\hat{\gamma}_2 < 0) - I(H^T_{11}\hat{\gamma}_1 > 0)I(H^T_{21}\hat{\gamma}_2 < 0)\}
\]

\[
+ (1 - A_{1i})A_{2i}\{I(H^T_{11}\hat{\gamma}_1 < 0)I(H^T_{21}\hat{\gamma}_2 > 0) - I(H^T_{11}\hat{\gamma}_1 < 0)I(H^T_{21}\hat{\gamma}_2 > 0)\}
\]

where the first step follows from above, the second step from the triangle inequality, now as \(\hat{\gamma}_1, \hat{\gamma}_2\) have dimensions \(q_{12}, q_{22}\) respectively, we use sets \(A_{q_{12}}, A_{q_{22}}\) and have

\[
|\{H^T_{11}\hat{\gamma}_1 < 0\}I(H^T_{21}\hat{\gamma}_2 > 0) - I(H^T_{11}\hat{\gamma}_1 < 0)I(H^T_{21}\hat{\gamma}_2 > 0) | = |\gamma_1 \notin A_{q_{12}}I(\gamma_2 \notin A_{q_{22}}) |
\]

which follows from the fact that for any term within absolute value:

\[
|\{H^T_{11}\hat{\gamma}_1 < 0\}I(H^T_{21}\hat{\gamma}_2 > 0) - I(H^T_{11}\hat{\gamma}_1 < 0)I(H^T_{21}\hat{\gamma}_2 > 0) | = |\gamma_1 \notin A_{q_{12}}I(\gamma_2 \notin A_{q_{22}}) |
\]

since for \(I(H^T_{11}\hat{\gamma}_1 < 0)I(H^T_{21}\hat{\gamma}_2 > 0) \neq I(H^T_{11}\hat{\gamma}_1 < 0)I(H^T_{21}\hat{\gamma}_2 > 0)\) both \(\hat{\gamma}_1, \hat{\gamma}_2\) have to be outside sets \(A_{q_{12}}, A_{q_{22}}\) respectively. Thus \(I(\hat{d}_{1i} = A_{1i})I(\hat{d}_{2i} = A_{2i}) - I(d^*_{1i} = A_{1i})I(d^*_{2i} = A_{2i}) = o_P(1)\), we can analogous show that \(I(\hat{d}_{1i} = A_{1i}) - I(d^*_{1i} = A_{1i}) = o_P(1) \forall i.\)

Next to see \(\sup_H |\frac{1}{\pi_1(H^T_{11})} - \frac{1}{\pi_1(H^T_{11})} | = o_P(n^{-\frac{1}{2}})\), note that as \(H_1, \Omega_1\) are bounded sets we
have

\[
\sup_{H_1} \left| \frac{1}{\bar{\pi}_1(H_1)} - \frac{1}{\hat{\pi}_1(H_1)} \right| = \sup_{H_1 \in H_1} \left| e^{-H_i^1 \hat{\xi}_i} - e^{-H_i^1 \hat{\xi}_i} \right| \\
\leq \sup_{H_1 \in H_1, \xi_i \in \Omega_1} \left| \frac{d}{dx} e^{-x} \bigg|_{x=H_i^1 \xi_i} \right| \sup_{H_1 \in H_1} \left| H_i^1 \hat{\xi}_1 - H_i^1 \hat{\xi}_i \right| \\
\leq \sup_{H_1 \in H_1, \xi_i \in \Omega_1} \left| \frac{d}{dx} e^{-x} \bigg|_{x=H_i^1 \xi_i} \right| \sup_{H_1 \in H_1} \|H_1\| \left\| \hat{\xi}_1 - \xi_1 \right\|_2 = O_p \left(n^{-\frac{1}{2}}\right),
\]

where we use the definition of \( \pi_1 \) in (5), Lipschitz and \( \|\hat{\xi}_1 - \xi_1\|_2 = O_p \left(n^{-\frac{1}{2}}\right) \) from Assumptions (5) and Theorem 5.21 in Vaart (1998) as we are using \( \hat{Z} \)-estimation for \( \xi_1 \).

(c) By definition, \( \frac{1}{n} \sum_{i=1}^{n} \frac{I(\hat{d}_{i1} = A_{1i})I(\hat{d}_{i2} = A_{2i})}{\pi_1(H_{1i})} - P_{d_{i1}, d_{i2}} = O_p \left(n^{-\frac{1}{2}}\right) \) if and only if for a given any \( \epsilon > 0, \exists M_\epsilon > 0 \) such that \( \mathbb{P} \left( \left| \frac{1}{n} \sum_{i=1}^{n} \frac{I(\hat{d}_{i1} = A_{1i})I(\hat{d}_{i2} = A_{2i})}{\pi_1(H_{1i})} - P_{d_{i1}, d_{i2}} \right| > M_\epsilon n^{-\frac{1}{2}} \right) \leq \epsilon \forall n. \)

Let \( M_\epsilon > 0, P_{d_{i1}, d_{i2}} = \mathbb{P} \left( \left| \frac{1}{n} \sum_{i=1}^{n} \frac{I(\hat{d}_{i1} = A_{1i})I(\hat{d}_{i2} = A_{2i})}{\pi_1(H_{1i})} - P_{d_{i1}, d_{i2}} \right| > M_\epsilon n^{-\frac{1}{2}} \right) \)

\[
\leq \mathbb{P} \left( \left| \frac{1}{n} \sum_{i=1}^{n} \frac{I(\hat{d}_{i1} = A_{1i})I(\hat{d}_{i2} = A_{2i})}{\pi_1(H_{1i})} - P_{d_{i1}, d_{i2}} \right| > M_\epsilon n^{-\frac{1}{2}} \right) \\
\leq \mathbb{P} \left( \left| \frac{1}{n} \sum_{i=1}^{n} \frac{I(\hat{d}_{i1} = A_{1i})I(\hat{d}_{i2} = A_{2i})}{\pi_1(H_{1i})} - P_{d_{i1}, d_{i2}} \right| > M_\epsilon n^{-\frac{1}{2}} \right)
\]

We have just shown that \( \sup_{H_1} \left| \frac{1}{\hat{\pi}_1(H_1)} - \frac{1}{\pi_1(H_1)} \right| = O_p \left(n^{-\frac{1}{2}}\right) \). Also, \( \hat{\pi}_1(H_1) \) is bounded away from zero by Assumptions (5), from \( \sqrt{n}(\hat{\gamma}_t - \gamma_t) = O_p(1), t = 1, 2 \) we have \( I\{\hat{\gamma}_1 \notin A_{q_{12}}\}I\{\hat{\gamma}_2 \notin A_{q_{22}}\} = o_p(1) \), therefore

\[
\mathbb{P} \left( \left| \frac{1}{n} \sum_{i=1}^{n} \frac{I(\hat{d}_{i1} = A_{1i})I(\hat{d}_{i2} = A_{2i})}{\pi_1(H_{1i})} - P_{d_{i1}, d_{i2}} \right| > M_\epsilon n^{-\frac{1}{2}} \right)
\]

\[
\leq \mathbb{P} \left( O_p(1) + \frac{1}{n} \sum_{i=1}^{n} \left| \frac{I(\hat{d}_{i1} = A_{1i})I(\hat{d}_{i2} = A_{2i})}{\pi_1(H_{1i})} - P_{d_{i1}, d_{i2}} \right| > M_\epsilon n^{-\frac{1}{2}} \right) \\
\leq \mathbb{P} \left( \frac{P_{d_{i1}, d_{i2}} \left(1 - P_{d_{i1}, d_{i2}}\right)}{n(M_\epsilon n^{-\frac{1}{2}} + o_p(1))} \rightarrow 0, \right)
\]
for any $\epsilon > 0$ as $n \to \infty$. Where the last step follows from Chebyshev’s Inequality. We then have that 
\[
\frac{1}{n} \sum_{i=1}^{n} \frac{I(d_{i1}=A_1)I(d_{i2}=A_2)}{\pi_1(H_{i1})} - P_{\tilde{d}_1,\tilde{d}_2} = O_P(n^{-\frac{1}{2}}) \quad \text{also by CLT}
\]
\[
\frac{1}{n} \sum_{j=1}^{N} \frac{l_j(d_{j1}=A_1)I(d_{j2}=A_2)}{\pi_1(H_{j1})} - P_{\tilde{d}_1,\tilde{d}_2} = O_P(N^{-\frac{1}{2}}),
\]
finally by Slutsky’s theorem $\hat{C}^{(2)}_{n,N} - 1 = O_P(n^{-\frac{1}{2}})$. With similar arguments we can show $\hat{C}^{(1)}_{n,N} - 1 = O_P(n^{-\frac{1}{2}})$.

\[\hfill \square\]

**Lemma 18** Let $Q(t,H_{t};\hat{\theta}_t)$, $\pi_t(H_{t};\hat{\xi}_t)$ $t = 1, 2$ be estimator functions of (1) \:eqref{eq:1} (5) respectively and define the bias as $Bias\left(\mathcal{V}_{\text{SupDR}} (L;\Theta), \overline{V}\right) \equiv E\left[\mathcal{V}_{\text{SupDR}} (L;\Theta) \right] - \overline{V}$, then

\[
Bias\left(\mathcal{V}_{\text{SupDR}} (L;\Theta), \overline{V}\right) \equiv E \left\{ 1 - \frac{\pi_1(H_1)}{\pi_1(H_1;\xi_1)} \left\{ Q_1^{o}(H_1) - Q_1^{o}(H_1;\theta_1) \right\} \right\} \\
+ E \left\{ \frac{\pi_1(H_1)}{\pi_1(H_1;\xi_1)} \left( 1 - \frac{\pi_2(H_2)}{\pi_2(H_2;\xi_2)} \right) \left\{ Q_2^{o}(H_2) - Q_2^{o}(H_2;\theta_2) \right\} \right\},
\]

where $\overline{V} = E[E[Y_2 + E[Y_3|H_2, Y_2, A_2 = \tilde{d}_2]|H_1, A_1 = \tilde{d}_1]]$ is the mean population value under the optimal treatment rule.

**Proof** [Proof of Lemma 18]

\[
Bias\left(\mathcal{V}_{\text{SupDR}} (L;\Theta), \overline{V}\right) = E \left[\mathcal{V}_{\text{SupDR}} (L;\Theta) \right] - E[E[Y_2 + E[Y_3|H_2, Y_2, A_2 = \tilde{d}_2]|H_1, A_1 = \tilde{d}_1]] \\
= E\left[ Q_1^{o}(H_1) - Q_1^{o}(H_1;\theta_1) \right] \\
- E \left[ \frac{I(d_{11}(H_{1};\theta_1) = A_1)}{\pi_1(H_1;\xi_1)} \{ Y_2 - [Q_1^{o}(H_{1};\theta_1) - Q_2^{o}(H_{2};\theta_2)] \} \right] \\
- E \left[ \frac{I(d_{11}(H_{1};\theta_1) = A_1)I(d_{22}(H_{2};\theta_2) = A_2)}{\pi_1(H_1;\xi_1)\pi_2(H_2;\xi_2)} \{ Y_3 - Q_2^{o}(H_{2};\theta_2) \} \right].
\]

Adding and subtracting $Q_2^{o}(H_{2};\theta_2)$,

\[
Bias\left(\mathcal{V}_{\text{SupDR}} (L;\Theta), \overline{V}\right) \\
= E\left[ Q_1^{o}(H_1) - Q_1^{o}(H_1;\theta_1) \right] \\
- E \left[ \frac{I(d_{11}(H_{1};\theta_1) = A_1)}{\pi_1(H_1;\xi_1)} \{ Y_2 + E[Y_3|H_2, d_2(H_{2};\theta_2), Y_2] - Q_1^{o}(H_{1};\theta_1) + Q_2^{o}(H_{2}) - Q_2^{o}(H_{2};\theta_2) \} \right] \\
+ E \left[ \frac{I(d_{11}(H_{1};\theta_1) = A_1)I(d_{22}(H_{2};\theta_2) = A_2)}{\pi_1(H_1;\xi_1)\pi_2(H_2;\xi_2)} \{ Y_3 - Q_2^{o}(H_{2};\theta_2) \} \right],
\]

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using iterated expectations in the second and fourth terms:

\[
\text{Bias} \left( \mathcal{V}_{\text{SUPDR}} (\mathbf{L}; \Theta), \bar{V} \right) \\
= \mathbb{E} \left[ Q_1^\theta (\mathbf{H}_1) - Q_1^\theta (\mathbf{H}_1; \theta_1) \right] \\
- \mathbb{E} \left[ \frac{I(d_1(\bar{H}_1; \theta_1) = A_1)}{\pi_1(\bar{H}_1; \xi_1)} \left\{ Y_2 + \mathbb{E}[Y_3|\mathbf{H}_2, \bar{d}_2, Y_2] - Q_1^\theta (\bar{H}_1; \theta_1) \right\} \right]_{\mathbf{H}_1, A_1} \\
+ \mathbb{E} \left[ \frac{I(d_1(\bar{H}_1; \theta_1) = A_1)}{\pi_1(\bar{H}_1; \xi_1)} \left\{ Q_2^\theta (\bar{H}_2) - Q_2^\theta (\bar{H}_2; \theta_2) \right\} \right]_{\mathbf{H}_1, A_1} \\
+ \mathbb{E} \left[ \frac{I(d_1(\bar{H}_1; \theta_1) = A_1)I_2(d_2(\bar{H}_2; \theta_2) = A_2)}{\pi_1(\bar{H}_1; \xi_1)\pi_2(\bar{H}_2; \xi_2)} \left\{ Y_3 - Q_2^\theta (\bar{H}_2; \theta_2) \right\} \right]_{\mathbf{H}_2, A_2, Y_2} \\
= \mathbb{E} \left[ Q_1^\theta (\mathbf{H}_1) - Q_1^\theta (\mathbf{H}_1; \theta_1) \right] \\
- \mathbb{E} \left[ \frac{I(d_1(\bar{H}_1; \theta_1) = A_1)}{\pi_1(\bar{H}_1; \xi_1)} \left\{ \mathbb{E} \left[ Y_2 + \mathbb{E}[Y_3|\mathbf{H}_2, \bar{d}_2, Y_2]|\mathbf{H}_1, A_1 \right] - Q_1^\theta (\bar{H}_1; \theta_1) \right\} \right] \\
+ \mathbb{E} \left[ \frac{I(d_1(\bar{H}_1; \theta_1) = A_1)}{\pi_1(\bar{H}_1; \xi_1)} \left\{ Q_2^\theta (\bar{H}_2) - Q_2^\theta (\bar{H}_2; \theta_2) \right\} \right] \\
+ \mathbb{E} \left[ \frac{I(d_1(\bar{H}_1; \theta_1) = A_1)I_2(d_2(\bar{H}_2; \theta_2) = A_2)}{\pi_1(\bar{H}_1; \xi_1)\pi_2(\bar{H}_2; \xi_2)} \left\{ \mathbb{E}[Y_3|\mathbf{H}_2, A_2, Y_2] - Q_2^\theta (\bar{H}_2; \theta_2) \right\} \right].
\]

Using the definitions of \(Q_t^\theta, t = 1, 2:\)

\[
\text{Bias} \left( \mathcal{V}_{\text{SUPDR}} (\mathbf{L}; \Theta), \bar{V} \right) \\
= \mathbb{E} \left[ Q_1^\theta (\mathbf{H}_1) - Q_1^\theta (\mathbf{H}_1; \theta_1) \right] \\
- \mathbb{E} \left[ \frac{I(d_1(\bar{H}_1; \theta_1) = A_1)}{\pi_1(\bar{H}_1; \xi_1)} \left\{ Q_1^\theta (\mathbf{H}_1) - Q_1^\theta (\bar{H}_1; \theta_1) \right\} \right] \\
+ \mathbb{E} \left[ \frac{I(d_1(\bar{H}_1; \theta_1) = A_1)}{\pi_1(\bar{H}_1; \xi_1)} \left\{ Q_2^\theta (\bar{H}_2) - Q_2^\theta (\bar{H}_2; \theta_2) \right\} \right] \\
+ \mathbb{E} \left[ \frac{I(d_1(\bar{H}_1; \theta_1) = A_1)I_2(d_2(\bar{H}_2; \theta_2) = A_2)}{\pi_1(\bar{H}_1; \xi_1)\pi_2(\bar{H}_2; \xi_2)} \left\{ Q_2^\theta (\bar{H}_2, A_2) - Q_2^\theta (\bar{H}_2; \theta_2) \right\} \right],
\]

assuming \(A_1 \perp A_2|\mathbf{H}_2, Y_2\) using iterated expectations where we condition on \(A_t = \bar{d}_t:\)

\[
\text{Bias} \left( \mathcal{V}_{\text{SUPDR}} (\mathbf{L}; \Theta), \bar{V} \right) \\
= \mathbb{E} \left[ Q_1^\theta (\mathbf{H}_1) - Q_1^\theta (\mathbf{H}_1; \theta_1) \right] \\
- \mathbb{E} \left[ \frac{w_1(\bar{H}_1)}{\pi_1(\bar{H}_1; \xi_1)} \left\{ Q_1^\theta (\mathbf{H}_1) - Q_1^\theta (\bar{H}_1; \theta_1) \right\} \right] \\
+ \mathbb{E} \left[ \frac{w_1(\bar{H}_1)}{\pi_1(\bar{H}_1; \xi_1)} \left\{ Q_2^\theta (\bar{H}_2) - Q_2^\theta (\bar{H}_2; \theta_2) \right\} \right] \\
+ \mathbb{E} \left[ \frac{w_1(\bar{H}_1)w_2^\theta(\bar{H}_2)}{\pi_1(\bar{H}_1; \xi_1)\pi_2(\bar{H}_2; \xi_2)} \left\{ Q_2^\theta (\bar{H}_2, A_2) - Q_2^\theta (\bar{H}_2; \theta_2) \right\} \right].
\]
finally, factorizing common terms:

\[
\begin{align*}
\text{Bias} \left( \mathcal{V}_{\text{supdr}} \left( \mathbf{L}; \Theta \right), \bar{V} \right) \\
= & \mathbb{E} \left\{ 1 - \frac{\pi_1(\mathbf{H}_1)}{\pi_1(\mathbf{H}_1; \xi_1)} \right\} \left\{ Q_1^\beta(\mathbf{H}_1) - Q_1^\beta(\mathbf{H}_1; \theta_1) \right\} \\
+ & \mathbb{E} \left\{ 1 - \frac{\pi_2(\mathbf{H}_2)}{\pi_2(\mathbf{H}_2; \xi_2)} \right\} \left\{ Q_2^\beta(\mathbf{H}_2) - Q_2^\beta(\mathbf{H}_2; \theta_2) \right\}.
\end{align*}
\]

Appendix D. Additional Theoretical Results

D.1 Basis expansion imputation and theoretical results

In this section we model the conditional expectations needed with basis functions and derive theoretical results for this approach, under the following Assumption, no refitting is needed:

Assumption 8 Let \( \mathbf{Z}_j = \zeta(\mathbf{H}_{2j}, A_{2j}, \mathbf{W}_{1j}, \mathbf{W}_{2j}) \in \mathbb{R}^q \) be a vector of fixed basis expansion functions that can incorporate nonlinear effects, we use a non-parametric model on \( \mathbf{Z} \) to impute the following conditional expectations which will be used for (2): \( m_2(\mathbf{Z}_j) = \mathbb{E}[Y_2|\mathbf{Z} = \mathbf{Z}_j] = \alpha_1^j \mathbf{Z}_j, m_3(\mathbf{Z}_j) = \mathbb{E}[Y_3|\mathbf{Z} = \mathbf{Z}_j] = \alpha_2^j \mathbf{Z}_j, \sigma_1^j(\mathbf{Z}_j) = \mathbb{V}ar[Y_2|\mathbf{Z} = \mathbf{Z}_j] = e^{\alpha_1^j \mathbf{Z}_j}, \sigma_2^j(\mathbf{Z}_j) = \mathbb{C}ov[Y_2, Y_3|\mathbf{Z} = \mathbf{Z}_j] = \alpha_1^j \mathbf{Z}_j, \sigma_{12}(\mathbf{Z}_j) = \mathbb{C}ov[Y_2, Y_3|\mathbf{Z} = \mathbf{Z}_j] = \alpha_1^j \mathbf{Z}_j.

The final imputation we use for the non-linear conditional expectations is: \( \mathbb{E}[Y_2^2|\mathbf{Z} = \mathbf{Z}_j] \equiv m_{22}(\mathbf{Z}_j) = m_2(\mathbf{Z}_j)^2 + \sigma_1^2(\mathbf{Z}_j), \mathbb{E}[Y_2Y_3|\mathbf{Z} = \mathbf{Z}_j] \equiv m_{23}(\mathbf{Z}_j) = m_2(\mathbf{Z}_j)m_3(\mathbf{Z}_j) + \sigma_{12}(\mathbf{Z}_j).

Let

\[
\text{BD}(\mathbb{E}[\mathbf{Z}_i, \mathbf{Z}_i^T])^{-1} \equiv \begin{pmatrix}
\mathbb{E}[\mathbf{Z}_i, \mathbf{Z}_i^T]^{-1} & 0 & 0 & 0 \\
0 & \mathbb{E}[\mathbf{Z}_i, \mathbf{Z}_i^T]^{-1} & 0 & 0 \\
0 & 0 & \mathbb{E}[\mathbf{Z}_i, \mathbf{Z}_i^T]^{-1} & 0 \\
0 & 0 & 0 & \mathbb{E}[\mathbf{Z}_i, \mathbf{Z}_i^T]^{-1}
\end{pmatrix} \in \mathbb{R}^{4q \times 4q},
\]

we fit the imputation models with the following estimating equation using \( \mathcal{L} \) to obtain \( \{\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_{11}, \hat{\alpha}_{12}\} \).

\[
\begin{align*}
\frac{1}{n} \sum_{i=1}^{n} BD(\mathbf{Z}_i) \begin{pmatrix}
Y_{2i} - \alpha_1^i \mathbf{Z}_i \\
Y_{3i} - \alpha_2^i \mathbf{Z}_i \\
(Y_{2i} - \alpha_1^i \mathbf{Z}_i)(Y_{3i} - \alpha_2^i \mathbf{Z}_i) - \alpha_1^i \mathbf{Z}_i
\end{pmatrix}
= & 0.
\end{align*}
\]

Let

\[
\zeta_j \equiv \begin{pmatrix}
-Z_j^T[\mathbf{H}_{20j} \hat{\beta}_{22} + A_{2j} \mathbf{H}_{21j}^T \hat{\gamma}_2] & 0 - \mathbf{\bar{\beta}}_{21} \mathbf{Z}_j^T & \mathbf{Z}_j^T \\
-\hat{\beta}_{21} \mathbf{H}_{20j} \mathbf{Z}_j^T & \mathbf{H}_{20j} \mathbf{Z}_j^T & 0 & 0 \\
-\hat{\beta}_{21} A_{2j} \mathbf{H}_{21j} \mathbf{Z}_j^T & A_{2j} \mathbf{H}_{21j} \mathbf{Z}_j^T & 0 & 0
\end{pmatrix} \in \mathbb{R}^{p \times 4q}
\]
Theorem 19 Distribution of $\hat{\theta}_{2ss}$
Suppose $\frac{n}{N} \rightarrow 0$ as $n, N \rightarrow \infty$. Then under Assumptions (3), (8),

$$n^\frac{1}{2}(\hat{\theta}_{2ss} - \theta_2) = n^{-\frac{1}{2}} \sum_{i=1}^{n} \psi_{2i} + O_p\left(\sqrt{\frac{n}{N}}\right) \overset{d}{\rightarrow} \mathcal{N}\left(0, \Sigma_2(m)\right)$$

where

$$\psi_{2i} = \mathbb{E}[X_{2i}X'_{2i}]^{-1}\mathbb{E}[\alpha_i]BD(\mathbb{E}[\alpha_i])^{-1} \begin{pmatrix} Z_iY_{2i} - Z_iZ_i'\alpha_{1i}^0 \\ Z_iY_{3i} - Z_iZ_i'\alpha_{2i}^0 \\ Z_iY_{1i} - Z_iZ_i'\alpha_{11}^0 \\ Z_iY_{12i} - Z_iZ_i'\alpha_{12i}^0 \end{pmatrix},$$

$$\zeta_j = \begin{pmatrix} -Z_{2j}'[H_{20j}\tilde{\beta}_{22} + A_{2j}H_{21j}\tilde{\gamma}_{2j}] \\ -\beta_{21}H_{20j}Z_j' \\ -\beta_{21}A_{2j}H_{21j}Z_j' \end{pmatrix} 0 - \begin{pmatrix} \tilde{\beta}_{2j}Z_j' \tilde{Z}_j' \end{pmatrix} \in \mathbb{R}^{p_2 \times 4q},$$

and $\Sigma_2(m) = \mathbb{E}[\psi_{2i}\psi'_{2i}]$.

Theorem 20 Distribution of $\hat{\theta}_{1ss}$:
Suppose $\frac{n}{N} \rightarrow 0$ as $n, N \rightarrow \infty$. Then,

$$n^\frac{1}{2}(\hat{\theta}_{1ss} - \theta_1) = n^{-\frac{1}{2}} \sum_{i=1}^{n} \psi_{1i} + O_p\left(\sqrt{\frac{n}{N}} + n^{-\frac{1}{2}}\right) \overset{d}{\rightarrow} \mathcal{N}\left(0, \Sigma_1(m)\right)$$

where

$$\psi_{1i} = \Sigma^{-1}_1\mathbb{E}[X_{1i}Z'\mathbb{E}[ZZ']^{-1}(1 + \tilde{\beta}_{21})Z_i\{Y_{i1} - Z_i\alpha_{1i}^0\}$$

$$+ \Sigma^{-1}_1\mathbb{E}[X_{1i}(Y_{2i}, H_{20i}')]\psi_{2\beta_{2i}}$$

$$+ \Sigma^{-1}_1\mathbb{E}[X_{1i}H_{21i}'H_{21i}\tilde{\gamma}_{2i} > 0]P \{H_{21i}\tilde{\gamma}_{2i} > 0\} \psi_{2\gamma_{2i}},$$

$$\Sigma_1(m) = \mathbb{E}[\psi_{1i}\psi'_{1i}],$$

and $\psi_{2\beta_{2i}}, \psi_{2\gamma_{2i}}$ are the elements corresponding to $\tilde{\beta}_{2i}, \tilde{\gamma}_{2i}$ of the influence function $\psi_{2i}$ defined in Theorem 19.

Proof [Proof of Theorem 19]
We start by replacing functions of the missing outcomes in the stage 2 estimating equation in (2) with the conditional expectations from Assumption (8) which yield the following estimating equation:

$$\mathbb{P}_N \left[ \bar{m}_{23} - (\bar{m}_{22}, \bar{m}_{2X}')(\bar{\theta}_2) \right] = 0.$$ (18)
Let

$$
\hat{\Gamma}_{n,N} = \mathbb{P}_N \begin{pmatrix}
\hat{m}_{22}(\bar{U}) & H_{20}\hat{m}_2(\bar{U}) & H_{21}A_2\hat{m}_2(\bar{U}) \\
H_{20}\hat{m}_2(\bar{U}) & H_{20}H_{20} & H_{20}H_{21}^TA_2 \\
H_{21}A_2\hat{m}_2(\bar{U}) & H_{21}A_2H_{20}^T & A_2^2H_{21}^T
\end{pmatrix},
$$

$$
R_{n,N} = \mathbb{P}_N \begin{pmatrix}
1 & 0 & 0 \\
0 & H_{20} & 0 \\
0 & H_{21}A_2 & 0
\end{pmatrix} \begin{pmatrix}
\hat{m}_{23}(\bar{U}) - \beta_{21}\hat{m}_{22}(\bar{U}) - H_{20}^T\beta_{22}\hat{m}_2(\bar{U}) - (H_{21}^T\gamma_2)A_2\hat{m}_2(\bar{U}) \\
\hat{m}_3(\bar{U}) - \beta_{21}\hat{m}_2(\bar{U}) - H_{20}^T\beta_{22} - (H_{21}^T\gamma_2)A_2 \\
\hat{m}_3(\bar{U}) - \beta_{21}\hat{m}_2(\bar{U}) - H_{20}^T\beta_{22} - (H_{21}^T\gamma_2)A_2
\end{pmatrix},
$$

$$
R_{n} = \mathbb{P}_N \begin{pmatrix}
1 & 0 & 0 \\
0 & H_{20} & 0 \\
0 & H_{21}A_2 & 0
\end{pmatrix} \begin{pmatrix}
m_{23}(\bar{U}) - \beta_{21}m_{22}(\bar{U}) - H_{20}^T\beta_{22}m_2(\bar{U}) - (H_{21}^T\gamma_2)A_2m_2(\bar{U}) \\
m_3(\bar{U}) - \beta_{21}m_2(\bar{U}) - H_{20}^T\beta_{22} - (H_{21}^T\gamma_2)A_2
\end{pmatrix}.
$$

Using a first order Taylor expansion of (18) at $\theta_2$ $(f(x) = 0 \implies f(a) + \nabla f(x)(x - a) = 0)$ we get $\hat{\Gamma}_{n,N}(\theta_2 - \theta_2) = R_{n,N}$. Next, $R_{n,N}$ can be decomposed as

$$
R_{n,N} = R_{n} + \mathbb{P}_N \begin{pmatrix}
1 & 0 & 0 \\
0 & H_{20} & 0 \\
0 & H_{21}A_2 & 0
\end{pmatrix} \times \begin{pmatrix}
\hat{m}_{23}(\bar{U}) - m_{23}(\bar{U}) - \beta_{21}(\hat{m}_{22}(\bar{U}) - m_{22}(\bar{U})) - [H_{20}^T\beta_{22} + H_{21}^T\gamma_2A_2](\hat{m}_{2}(\bar{U}) - m_{2}(\bar{U})) \\
\hat{m}_3(\bar{U}) - m_3(\bar{U}) - (\hat{m}_{2}(\bar{U}) - m_{2}(\bar{U}))
\end{pmatrix},
$$

note that under Assumption (3) term $R_N$ is the centered sum of $N$ iid terms, thus by CLT $R_N = O_P\left(N^{-\frac{1}{2}}\right)$, let $\tilde{U}^\ell, Y^\ell, s \in \{2, 3, 22, 23\}$ be the labeled sample basis expansion feature matrix and outcome vectors respectively, then by Assumption (8)

$$
R_{n,N} = \mathbb{P}_N \begin{pmatrix}
1 & 0 & 0 \\
0 & H_{20} & 0 \\
0 & H_{21}A_2 & 0
\end{pmatrix} \times \begin{pmatrix}
\tilde{U}^\ell\alpha_{12} - \tilde{U}^\ell\alpha_{12}^0 - \beta_{21}(\tilde{U}^\ell\alpha_{11} - \tilde{U}^\ell\alpha_{11}^0) - [H_{20}^T\beta_{22} + H_{21}^T\gamma_2A_2](\tilde{U}^\ell\alpha_1 - \tilde{U}^\ell\alpha_1^0) \\
\tilde{U}^\ell\alpha_2 - \tilde{U}^\ell\alpha_2^0 - \beta_{21}(\tilde{U}^\ell\alpha_1 - \tilde{U}^\ell\alpha_1^0)
\end{pmatrix} + O_P\left(N^{-\frac{1}{2}}\right)
$$

$$
= \mathbb{P}_N \begin{pmatrix}
1 & 0 & 0 \\
0 & H_{20} & 0 \\
0 & H_{21}A_2 & 0
\end{pmatrix} \left(-\tilde{U}^\ell[H_{20}^T\beta_{22} + H_{21}^T\gamma_2A_2] \begin{pmatrix} 0 & -\beta_{21}\tilde{U}^\ell \\ \tilde{U}^\ell & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 - \alpha_1^0 \\ \alpha_2 - \alpha_2^0 \\ \alpha_{11} - \alpha_{11}^0 \\ \alpha_{12} - \alpha_{12}^0
\end{pmatrix} + O_P\left(N^{-\frac{1}{2}}\right)
$$

$$
= \mathbb{P}_N \begin{pmatrix}
-\tilde{U}^\ell[H_{20}^T\beta_{22} + H_{21}^T\gamma_2A_2] & 0 & -\beta_{21}\tilde{U}^\ell & \tilde{U}^\ell \\
-\beta_{21}H_{20}\tilde{U}^\ell & H_{20}\tilde{U}^\ell & 0 & 0
\end{pmatrix} \begin{pmatrix}
\tilde{U}^\ell\tilde{U}^\ell\tilde{U}^\ell & 0 & -\beta_{21}\tilde{U}^\ell & \tilde{U}^\ell \\
\tilde{U}^\ell\tilde{U}^\ell\tilde{U}^\ell & 0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
\tilde{U}^\ell\tilde{U}^\ell\tilde{U}^\ell & 0 & -\beta_{21}\tilde{U}^\ell & \tilde{U}^\ell \\
\tilde{U}^\ell\tilde{U}^\ell\tilde{U}^\ell & 0 & 0 & 0
\end{pmatrix} + O_P\left(N^{-\frac{1}{2}}\right)
$$

$$
= \mathbb{P}_N \begin{pmatrix}
-\tilde{U}^\ell[H_{20}^T\beta_{22} + H_{21}^T\gamma_2A_2] & 0 & -\beta_{21}\tilde{U}^\ell & \tilde{U}^\ell \\
-\beta_{21}H_{20}\tilde{U}^\ell & H_{20}\tilde{U}^\ell & 0 & 0
\end{pmatrix} \begin{pmatrix}
\tilde{U}^\ell\tilde{U}^\ell\tilde{U}^\ell & 0 & -\beta_{21}\tilde{U}^\ell & \tilde{U}^\ell \\
\tilde{U}^\ell\tilde{U}^\ell\tilde{U}^\ell & 0 & 0 & 0
\end{pmatrix} + O_P\left(N^{-\frac{1}{2}}\right)
$$
\[ (P_N \zeta) BD \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \tilde{U}_i \tilde{U}_i^\top \right)^{-1} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \begin{pmatrix} \tilde{U}_i Y_{2i} - \tilde{U}_i \tilde{U}_i^\top \alpha_{0i}^0 \\ \tilde{U}_i Y_{3i} - \tilde{U}_i \tilde{U}_i^\top \alpha_{2i}^0 \\ \tilde{U}_i Y_{2i} - \tilde{U}_i \tilde{U}_i^\top \alpha_{1i}^0 \\ \tilde{U}_i Y_{2i} Y_{3i} - \tilde{U}_i \tilde{U}_i^\top \alpha_{12i}^0 \end{pmatrix} + O_P(N^{-\frac{1}{2}}). \] (19)

Next consider \( \hat{\Gamma}_{n,N} \), by Assumption (8) and Lemma 15 with \( \sup_U \| \tilde{U} (\tilde{U}^\top \tilde{U})^{-1} \tilde{U}^\top Y_r \| = O_P(1) \), for \( r = 1, 2 \) by the boundedness of outcome and covariates from Assumptions (1):

\[
\hat{\Gamma}_{n,N} = \mathbb{E}_U \left[ P_N \begin{pmatrix} \tilde{U}^\top \alpha_{11} & H_{20} \tilde{U}^\top \alpha_1 \\ H_{20} \tilde{U}^\top \alpha_1 & H_{20} H_{20}^\top & H_{20} H_{21}^\top A_2 \\ H_{21} A_2 \tilde{U}^\top \alpha_1 & H_{21} A_2 H_{20} & A_2^2 H_{21}^\top H_{21}\end{pmatrix} \right] + O_P(N^{-\frac{1}{2}})
\]

\[
= \mathbb{E}_U \left[ \begin{pmatrix} Y_2^2 & H_{20} Y_2 \\ H_{20} Y_2 & H_{20} H_{20}^\top & H_{20} H_{21}^\top A_2 \\ H_{21} A_2 Y_2 & H_{21} A_2 H_{20} & A_2^2 H_{21}^\top H_{21}\end{pmatrix} \right] 
+ \mathbb{E} \left[ \begin{pmatrix} \tilde{U}^\top \alpha_{11} - Y_2^2 & H_{20} (\tilde{U}^\top \alpha_1 - Y_2) \\ H_{20} (\tilde{U}^\top \alpha_1 - Y_2) & 0 & 0 \\ H_{21} A_2 (\tilde{U}^\top \alpha_1 - Y_2) & 0 & 0 \end{pmatrix} \right] \right] 
+ O_P(N^{-\frac{1}{2}})
\]

\[
= \mathbb{E} [X_{2i} X_{2i}] + O_P(N^{-\frac{1}{2}}),
\]

where the last step follows from Assumptions (8). Thus we have \( \hat{\Gamma}_{n,N}^{-1} = \mathbb{E}[X_{2i} X_{2i}]^{-1} + O_P(N^{-\frac{1}{2}}) \).

Also note that by CLT: \( (P_N \zeta_j) = \mathbb{E}[\zeta_j] + O_P(N^{-\frac{1}{2}}), ii) (P_N \tilde{U}_i \tilde{U}_i^\top) = \mathbb{E}[\tilde{U}_i \tilde{U}_i^\top] + O_P(n^{-\frac{1}{2}}) \).

Combining (19), our expression for \( \hat{\Gamma}_{n,N}^{-1} \), and i), ii) we can write

\[
\sqrt{n}(\hat{\theta}_2 - \theta_2) = \mathbb{E}[X_{2i} X_{2i}^\top]^{-1} P_N [\zeta_j] BD \left( \left( P_N \tilde{U}_i \tilde{U}_i^\top \right)^{-1} \right) \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \begin{pmatrix} \tilde{U}_i Y_{2i} - \tilde{U}_i \tilde{U}_i^\top \alpha_{0i}^0 \\ \tilde{U}_i Y_{3i} - \tilde{U}_i \tilde{U}_i^\top \alpha_{2i}^0 \\ \tilde{U}_i Y_{2i} - \tilde{U}_i \tilde{U}_i^\top \alpha_{1i}^0 \\ \tilde{U}_i Y_{2i} Y_{3i} - \tilde{U}_i \tilde{U}_i^\top \alpha_{12i}^0 \end{pmatrix} 
+ O_P\left( \left( \frac{n}{N} \right)^{\frac{1}{2}} \right)
\]

\[
= \mathbb{E}[X_{2i} X_{2i}^\top]^{-1} \mathbb{E}[\zeta_j] BD \left( \mathbb{E}[\tilde{U}_i \tilde{U}_i^\top]^{-1} \right) \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \begin{pmatrix} \tilde{U}_i Y_{2i} - \tilde{U}_i \tilde{U}_i^\top \alpha_{0i}^0 \\ \tilde{U}_i Y_{3i} - \tilde{U}_i \tilde{U}_i^\top \alpha_{2i}^0 \\ \tilde{U}_i Y_{2i} - \tilde{U}_i \tilde{U}_i^\top \alpha_{1i}^0 \\ \tilde{U}_i Y_{2i} Y_{3i} - \tilde{U}_i \tilde{U}_i^\top \alpha_{12i}^0 \end{pmatrix} 
+ O_P\left( \left( \frac{n}{N} \right)^{\frac{1}{2}} \right).
\]
which concludes the proof of Theorem 19.

**Proof [Proof of Theorem 20]**

We start by replacing functions of the missing outcomes in the stage 1 estimating equation in (2) with the conditional expectations from Assumption (8), which yield the following estimating equation:

\[
P_N \left\{ X_1 \left( \hat{m}_2(\bar{U}) + \hat{\beta}_{21} \hat{m}_2(\bar{U}) + H_{20}^T \hat{\beta}_{22} + [H_{21}^T \hat{\gamma}_2]_+ - X_1^T \theta_1 \right) \right\} = 0 \tag{20}
\]

Let \( \hat{\Sigma}_1 \equiv P_N X_1 X_1^T \),

\[
R_{n,N}^* = P_N \left\{ X_1 \left( \hat{m}_2(\bar{U}) + \hat{\beta}_{21} \hat{m}_2(\bar{U}) + H_{20}^T \hat{\beta}_{22} + [H_{21}^T \hat{\gamma}_2]_+ - X_1^T \theta_1 \right) \right\},
\]

using a first order Taylor expansion of (20) at \( \bar{\theta}_1 \) we have \( \hat{\Sigma}_1(\bar{\theta}_{1ss} - \bar{\theta}_1) = R_{n,N}^* \).

Next note we can decompose \( R_{n,N}^* \) as \( R_{n,N}^* = R_{1,N}^* + \left( 1 + \hat{\beta}_{21} \right) P_N \left\{ X_1 \left( \hat{m}_2(\bar{U}) - m_2(\bar{U}) \right) \right\} \).

By Theorem 19, \( \hat{\beta}_{21} - \bar{\beta}_{21} = O_P \left( n^{-\frac{1}{2}} \right) \), therefore using Assumptions (8) and by the boundedness of covariates from Assumptions 1:

\[
\left( 1 + \hat{\beta}_{21} \right) P_N \left\{ X_1 \left( \hat{m}_2(\bar{U}) - m_2(\bar{U}) \right) \right\} = \left( 1 + \bar{\beta}_{21} \right) P_N \left\{ X_1 \bar{U}^T (\bar{\alpha}_1 - \alpha_0) \right\} + O_P \left( n^{-\frac{1}{2}} \right)
\]

\[
= \left( 1 + \bar{\beta}_{21} \right) P_N \left\{ X_1 \bar{U}^T \left( [\bar{U}^T \bar{U}^T]^{-1} \bar{U}^T \bar{Y}_1 - \alpha_0 \right) \right\} + O_P \left( n^{-\frac{1}{2}} \right)
\]

\[
= \left( 1 + \bar{\beta}_{21} \right) P_N \left\{ X_1 \bar{U}^T \left\{ \frac{1}{n} \sum_{i=1}^n \bar{U}_i \bar{U}_i^T \right\} \right\} + O_P \left( n^{-\frac{1}{2}} \right).
\]

Next let

\[
S_{\Sigma}^* = P_N \left\{ X_1 \left( m_2(\bar{U}) + \bar{\beta}_{21} m_2(\bar{U}) + H_{20}^T \bar{\beta}_{22} + [H_{21}^T \bar{\gamma}_2]_+ - X_1^T \theta_1 \right) \right\},
\]

\[
\hat{S}_\beta = P_N \left\{ X_1 \left( \left( m_2(\bar{U}), H_{20}^T \right)^T (\bar{\beta}_2 - \beta_2) \right) \right\},
\]

\[
\hat{S}_\gamma = P_N \left\{ X_1 \left( [H_{21}^T \bar{\gamma}_2]_+ - [H_{21}^T \bar{\gamma}_2]_+ \right) \right\},
\]

we write \( R_{n,N}^* = S_{\Sigma}^* + \hat{S}_\beta + \hat{S}_\gamma \), now using iterated expectations, by Assumptions (3), (8) we have

\[
E \left[ S_{\Sigma}^* \right] = E \left[ P_N \left\{ X_1 \left( E[Y_2|\bar{U}] + \bar{\beta}_{21} m_2(\bar{U}) + H_{20}^T \bar{\beta}_{22} + [H_{21}^T \bar{\gamma}_2]_+ - X_1^T \theta_1 \right) \right\} \right]
\]

\[
= E \left[ \left\{ X_1 \left( Y_2 + \bar{\beta}_{21} m_2(\bar{U}) + H_{20}^T \bar{\beta}_{22} + [H_{21}^T \bar{\gamma}_2]_+ - X_1^T \theta_1 \right) \right\} \bar{U} \right]
\]

\[
= E \left[ \left\{ X_1 \left( Y_2 + \bar{\beta}_{21} m_2(\bar{U}) + H_{20}^T \bar{\beta}_{22} + [H_{21}^T \bar{\gamma}_2]_+ - X_1^T \theta_1 \right) \right\} \right] = 0,
\]
therefore $S_N^*$ is a mean zero average of $N$ iid terms, by CLT $S_N^* = O_P\left(N^{-\frac{1}{2}}\right)$.

Next let $\psi_{2\beta_2}$ be the elements corresponding to $\beta_2$ of the influence function $\psi_{2i}$ defined in Theorem 19, by this Theorem we have

$$
\hat{S}_\beta = n^{-\frac{1}{2}}P_N \left\{ X_1 \left( m_2(\hat{U}), H_{20}^T \right) \right\} \sqrt{n}(\hat{\beta_2} - \beta_2)
= n^{-\frac{1}{2}}P_N \left\{ X_1 \left( m_2(\hat{U}), H_{20}^T \right) \right\} \frac{1}{\sqrt{n}} \sum_{i=1}^n \psi_{2\beta_2} + O_P\left(N^{-\frac{1}{2}}\right),
$$

now consider $\hat{S}_\gamma$, by Theorem 19 we know that $\sqrt{n}(\hat{\gamma_2} - \gamma_2) = O_P(1)$, therefore using Lemma 17 (a):

$$
P \left( \sqrt{n}\hat{S}_\gamma = P_N \left\{ X_1 H_{21}^T \times I_{\{H_{21}^T \gamma_2 > 0\}} \right\} \sqrt{n}(\hat{\gamma_2} - \gamma_2) \right) \to 1.
$$

Letting $\psi_{2\gamma_2}$ be the elements corresponding to $\gamma_2$ of the influence function $\psi_{2i}$ defined in Theorem 19, we can then write

$$
\hat{S}_\gamma = n^{-\frac{1}{2}}P_N \left\{ X_1 H_{21}^T \times I_{\{H_{21}^T \gamma_2 > 0\}} \right\} \sqrt{n}(\hat{\gamma_2} - \gamma_2) + n^{-\frac{1}{2}}O_P(1)
= n^{-\frac{1}{2}}E \left[ X_1 H_{21}^T | H_{21}^T \gamma_2 > 0 \right] P \left( H_{21}^T \gamma_2 > 0 \right) \frac{1}{\sqrt{n}} \sum_{i=1}^n \psi_{2\gamma_2} + O_P\left((nN)^{-\frac{1}{2}}\right),
$$

where the last step follows from

$$
P_N \left\{ X_1 H_{21}^T \times I_{\{H_{21}^T \gamma_2 > 0\}} \right\} = E \left[ X_1 H_{21}^T | H_{21}^T \gamma_2 > 0 \right] P \left( H_{21}^T \gamma_2 > 0 \right) + O_P\left(N^{-\frac{1}{2}}\right)
$$

by CLT.

Combining the results above yields

$$
\sqrt{n} R_N^* = O_P \left( \sqrt{\frac{n}{N}} \right)
+ E \left[ X_1 \left( Y_2, H_{20}^T \right) \right] \frac{1}{\sqrt{n}} \sum_{i=1}^n \psi_{2\beta_2i} + O_P\left(\sqrt{\frac{n}{N}}\right)
+ E \left[ X_1 H_{21}^T | H_{21}^T \gamma_2 > 0 \right] P \left( H_{21}^T \gamma_2 > 0 \right) \frac{1}{\sqrt{n}} \sum_{i=1}^n \psi_{2\gamma_2i} + O_P\left(N^{-\frac{1}{2}}\right),
$$

Also, by CLT we have $\hat{S}_1 \equiv P_N \left\{ X_1 X_1^T \right\} = E \left[ X_1 X_1^T \right] + O_P\left(N^{-\frac{1}{2}}\right)$, $P_n \left\{ \bar{U}_i \bar{U}_i^T \right\} = E \left[ \bar{U}_i \bar{U}_i^T \right] + O_P \left( n^{-\frac{1}{2}} \right)$.
Therefore

\[ \sqrt{n}(\hat{\theta}_1 - \theta_1) = (1 + \bar{\beta}_{21}) E \left[ X_1 \bar{U}_1^{\top} \right] E \left[ \bar{U}\bar{U}^{\top} \right]^{-1} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \bar{U}_i \{ Y_i - \bar{U}_i^{\top} \alpha_i \} \]

+ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \psi_{2\beta_{2i}}

+ E \left[ X_1 X_1^{\top} \right]^{-1} E \left[ X_1 (Y_2, H_2^{20}) \right] \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \psi_{2\beta_{2i}}

+ E \left[ X_1 X_1^{\top} \right]^{-1} E \left[ X_1 H_2^{21} | H_2^{21} \gamma_2 > 0 \right] P (H_2^{21} \gamma_2 > 0) \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \psi_{2\gamma_{2i}}

+ O_p \left( n^{-\frac{1}{2}} \right) + O_p \left( \sqrt{\frac{n}{N}} \right).
D.2 Augmented value function estimation

We first re-write Assumption 5 to account for only using sample \( \mathcal{L} \) in estimation of the \( Q \) functions and propensity scores.

**Assumption 9** Define the following class of functions:

\[
Q_1 \equiv \{ Q_1(H_1, A_1; \theta_1) \mid \theta_1 \in \Theta_1 \subset \mathbb{R}^{q_1} \}, \\
Q_2 \equiv \{ Q_2(H_2, A_2, Y_2; \theta_2) \mid \theta_2 \in \Theta_2 \subset \mathbb{R}^{q_2} \}, \\
W_1 \equiv \{ \pi_1(H_1; \xi_1) \mid \xi_1 \in \Omega_1 \subset \mathbb{R}^{p_1} \}, \\
W_2 \equiv \{ \pi_2(H_2; \xi_2) \mid \xi_2 \in \Omega_2 \subset \mathbb{R}^{p_2} \},
\]

with \( p_1, p_2, q_1, q_2 \) fixed under model definitions (1) \( \mathcal{E} \) (5). Let the population equations \( \mathbb{E} \left[ S_{t}^{\xi}(\xi_t) \right] = 0, t = 1, 2 \) have solutions \( \xi_1, \xi_2 \), where

\[
S_{t}^{\xi}(\xi_t) = \frac{\partial}{\partial \xi_t} \log \left[ \pi_1(H_1; \xi_1)^{A_1} (1 - \pi_1(H_1; \xi_1))^{(1 - A_1)} \right], \\
S_{2}^{\xi}(\xi_2) = \frac{\partial}{\partial \xi_2} \log \left[ \pi_2(H_2; \xi_2)^{A_2} (1 - \pi_2(H_2; \xi_2))^{(1 - A_2)} \right],
\]

and the population equations for the \( Q \) functions \( \mathbb{E}[S_{t}^{\theta}(\theta_t)] = 0, t = 1, 2 \) have solutions \( \theta_1, \theta_2 \), where

\[
S_{t}^{\theta}(\theta_t) = \frac{\partial}{\partial \theta_t} \| Y_t - Q_2(H_2, A_2; \theta_2) \|^2_2, \\
S_{2}^{\theta}(\theta_2) = \frac{\partial}{\partial \theta_t} \| Y_t + Q_2(H_2; \theta_2) - Q_1(H_1, A_1; \theta_1) \|^2_2,
\]

(i) \( \xi_1, \xi_2 \) are bounded sets. (ii) \( \Theta_1, \Theta_2 \) are open bounded sets and for some \( r > 0 \) and \( g_t(\cdot) \)

\[
\left| Q_t(\cdot; \theta_t) - Q_t(\cdot; \theta_t') \right| \leq g_t(\cdot) \| \theta_t - \theta_t' \| \ \forall \theta_t, \theta_t' \in \Theta_t, \mathbb{E} \left[ |g_t(\cdot)|^r \right] < \infty, \ t = 1, 2.
\]

(iii) The population minimizers satisfy \( \hat{\theta}_t \in \Theta_t, \tilde{\theta}_t \in \Omega_t, t = 1, 2, \) (iv) For \( \xi_t \), \( t = 1, 2, \)

\( \pi_1(H_1; \xi_1) > 0, \pi_2(H_2; \xi_2) > 0 \ \forall H \in \mathcal{H}. \)

Existence of solutions \( \hat{\theta}_t \in \Theta_t, t = 1, 2 \) is clear as \( \Theta_1, \Theta_2 \) are open and bounded.

**Theorem 21 (Asymptotic Normality for \( \hat{V}_{\text{SSLDR}} \))** Under Assumptions 1, 4, and 9, \( \hat{V}_{\text{SUPDR}} \) as defined in (6) is such that

\[
\sqrt{n} \left\{ \hat{V}_{\text{SUPDR}} - \mathbb{E} \left[ V_{\text{SUPDR}}(L; \Theta) \right] \right\} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \psi_{\text{SUPDR}}^\nu(L_i; \Theta) + o_P(1) \overset{d}{\rightarrow} N \left( 0, (\sigma_{\text{SUPDR}}^\nu)^2 \right).
\]

where

\[
\psi_{\text{SUPDR}}^\nu(L; \Theta) = V_{\text{SUPDR}}(L; \Theta) - \mathbb{E} \left[ V_{\text{SUPDR}}(L; \Theta) \right] + \psi_{\text{SUP}}^\theta(L)^T \frac{\partial}{\partial \theta} \int V_{\text{SUPDR}}(L; \Theta) d\mathbb{P}_L \bigg|_{\Theta = \hat{\Theta}}, \\
\psi_{\text{SUPDR}}^\xi(L)^T \frac{\partial}{\partial \xi} \int V_{\text{SUPDR}}(L; \Theta) d\mathbb{P}_L \bigg|_{\Theta = \hat{\Theta}}, \\
(\sigma_{\text{SUPDR}}^\nu)^2 = \mathbb{E} \left[ \psi_{\text{SUPDR}}^\nu(L; \Theta)^2 \right].
\]
\textbf{Proof} [proof of theorem 21]

Letting \(g(\Theta) = \int V_{\text{SR}}(L; \Theta) d\mathbb{P}_L\), we start with the following expansion of (6):

\begin{align*}
\sqrt{n} \left\{ \mathbb{P}_n \left( V_{\text{SR}}(L; \hat{\Theta}_{\text{SUP}}) \right) - E \left[ V_{\text{SR}}(L; \hat{\Theta}) \right] \right\} \\
\leq G_n \left\{ V_{\text{SR}}(L; \hat{\Theta}_{\text{SUP}}) - V_{\text{SR}}(L; \hat{\Theta}) \right\} + \sqrt{n} \left\{ g(\hat{\Theta}_{\text{SUP}}) - g(\hat{\Theta}) \right\}
\end{align*}

\textit{I) Empirical Process Term}

We first show that under Assumption 9, \(G_n \left\{ V_{\text{SR}}(L; \hat{\Theta}_{\text{SUP}}) - V_{\text{SR}}(L; \hat{\Theta}) \right\} = o_P(1)\), let

\begin{align*}
f_{\Theta}(\tilde{U}) = Q_1^o(\tilde{H}_1; \theta_1) + \omega_1(\tilde{H}_1, A_1, \Theta) \left\{ Y_2 - Q_1^o(\tilde{H}_1; \theta_1) + Q_2^o(\tilde{H}_2; \theta_2) \right\} \\
+ \omega_2(\tilde{H}_1, A_1; \Theta) \left\{ Y_3 - Q_2^o(\tilde{H}_2; \theta_2) \right\},
\end{align*}

we define the class of functions \(\mathcal{C}_3 = \left\{ f_{\Theta}(\tilde{U}) | \tilde{U}, \Theta \in \mathcal{S}(\delta) \right\}\), and

\[\ell = \{l : \{0, 1\}^2 \mapsto \{0, 1\}\}.\]

i) By Assumptions 9 and Theorem 19.5 in Vaart (1998), \(\ell, W_t, Q_t, t = 1, 2\) are a \(\mathbb{P}\)-Donsker class, thus it follows that \(\mathcal{C}_3\) is a Donsker class.

ii) We estimate \(\xi_1, \xi_2\) from (21) with their maximum likelihood estimator \(\hat{\xi}_{1\text{SUP}}, \hat{\xi}_{2\text{SUP}}\), solving \(\mathbb{P}_n [S_t(\xi_t)] = 0, t = 1, 2\) and estimate functions \(\hat{\pi}_1(\tilde{H}_1; \hat{\xi}_{1\text{SUP}}), \hat{\pi}_2(\tilde{H}_2; \hat{\xi}_{2\text{SUP}})\) with \(\hat{\xi}_{1\text{SUP}}, \hat{\xi}_{2\text{SUP}}\). By Assumption 9 and weak law of large numbers \(\hat{\xi}_{t\text{SUP}} \xrightarrow{p} \xi_t, t = 1, 2\).

Analogous, under regularity conditions (18), (20) have unique solutions \(\hat{\theta}_t\) for which \(\hat{\theta}_t \xrightarrow{p} \theta_t, t = 1, 2\) by Assumption 9 and weak law of large numbers. Both regardless of whether models (1) & (5) are correct. Thus \(\mathbb{P} \left( \hat{\Theta}_{\text{SUP}} \in \mathcal{S}(\delta) \right) \rightarrow 1, \forall \delta.\)

iii) We next show \(\int \left( V_{\text{SR}}(L; \hat{\Theta}_{\text{SUP}}) - V_{\text{SR}}(L; \hat{\Theta}) \right)^2 d\mathbb{P}_{\tilde{U}} \rightarrow 0\), using (9) we can write
\[ \int \left( V_{\sup_{\text{DR}}}(L; \hat{\Theta}_{\sup}) - V_{\sup_{\text{DR}}}(L; \Theta) \right)^2 d\mathbb{P}_L \]

\[ \leq 2 \int \left( Q_1^0(H_1; \hat{\theta}_{1\sup}) - Q_1^0(H_1; \theta_1) \right)^2 d\mathbb{P}_L, \]

\[ + 4 \sup_{L, \Theta} \left\{ Y_2 + \frac{Y_3}{\pi_2(H_2; \xi)} \right\} \int \left( \frac{1}{\pi_1(H_1; \xi_{1\sup})} - \frac{1}{\pi_1(H_1; \xi)} \right)^2 d\mathbb{P}_L \]

\[ + 8 \sup_{\Theta} Q_1^0(H_1, A_1; \theta_1)^2 \int \left( \frac{1}{\pi_1(H_1; \xi_{1\sup})} - \frac{1}{\pi_1(H_1; \xi)} \right)^2 d\mathbb{P}_L \]

\[ + 16 \sup_{H_1} \frac{1}{\pi_1(H_1; \xi)} \int \left( Q_1^0(H_1, A_1; \hat{\theta}_{1\sup}) - Q_1^0(H_1, A_1; \hat{\theta}_1) \right)^2 d\mathbb{P}_L \]

\[ + 32 \sup_{\Theta, H_2} Q_2^0(H_2, Y_2; \theta_2)^2 \int \left( \frac{1}{\pi_1(H_1; \xi_{2\sup})} - \frac{1}{\pi_1(H_1; \xi)} \right)^2 d\mathbb{P}_L \]

\[ + 64 \sup_{\Theta, H_1} \frac{1}{\pi_1(H_1; \xi)} \int \left( Q_2^0(H_2, Y_2; \hat{\theta}_{2\sup}) - Q_2^0(H_2, Y_2; \hat{\theta}_2) \right)^2 d\mathbb{P}_L \]

\[ + 128 \sup_{\Theta, L} \frac{1}{\pi_2(H_2; \xi_2)} \left( \frac{Y_2}{\pi_2(H_2; \xi_2)} \right)^2 \int \left( \frac{1}{\pi_1(H_1; \xi_{1\sup})} - \frac{1}{\pi_1(H_1; \xi)} \right)^2 d\mathbb{P}_L \]

\[ + 128 \sup_{H_1} \frac{1}{\pi_1(H_1; \xi)} \int \left( \frac{Y_2}{\pi_2(H_2; \xi_2)} \beta_{21\sup} - \beta_{21} \right)^2 d\mathbb{P}_L \]

\[ \longrightarrow 0 \]

where we again use \((a - b)^2, (a + b)^2 \leq 2a^2 + 2b^2 \forall a, b \in \mathbb{R}\), and that \(\sup_\Theta Q_1^0\) and \(\sup_\Theta \pi_t\), \(t = 1, 2\) are constants with respect to \(\mathbb{P}(\Theta)\). Additionally we use \(I(d_{1\sup} = A_1), I(d_{2\sup} = A_2) \leq 1 \forall H \in \mathcal{H}\), and boundedness of \(\hat{\theta}_{1\sup}, \hat{\theta}_{2\sup}, t = 1, 2\) by Assumptions 1-3.

For the final step, first consider the following

\[ \int \left( Q_2^0(H_2; \hat{\theta}_{2\sup}) - Q_2^0(H_2; \hat{\theta}) \right)^2 d\mathbb{P}_L \]

\[ = \int \left( \hat{\beta}_{21\sup} Y_2 + H_{20}^T \hat{\gamma}_{2\sup} + [H_{21}^T \hat{\gamma}_{2\sup}] + - \hat{\beta}_{21} Y_2 - H_{20}^T \beta_{22} + [H_{21}^T \gamma_2] + \right)^2 d\mathbb{P}_L \]

\[ = 2 \left( \hat{\beta}_{21\sup} - \beta_{21} \right)^2 \int Y_2^2 d\mathbb{P}_L + 4 \| \hat{\beta}_{2\sup} - \beta_2 \|_2^2 \int H_{20}^T H_{20} d\mathbb{P}_L \]

\[ + 4 \| \hat{\gamma}_{2\sup} - \gamma_2 \|_2^2 \int H_{21}^T H_{21} d\mathbb{P}_L = O_P(n^{-1}) \]

which follows from Theorem 2 and Lemma 17 (a). All similar terms can be handled accordingly. We get the convergence in probability to 0 as all other terms within expectation
are $O_p\left(n^{-1}\right)$ by the dominating convergence theorem, boundedness conditions as stated in Assumptions 2, 5, and the consistency of $\hat{\xi}_{\text{SUP}}$ and $\hat{\theta}_{\text{SUP}}$ as $P\left(\hat{\Theta}_{\text{SUP}} \in S(\delta)\right) \to 1$, $\forall \delta > 0$.

Finally we have i) $P\left(\hat{\Theta}_{\text{SUP}} \in S(\delta)\right) \to 1$, $\forall \delta$, ii) $C_1$ is a Donsker class, and iii) $\int (V_{\text{SUP}}(L; \hat{\Theta}_{\text{SUP}}) - V_{\text{SUP}}(L; \bar{\Theta}))^2 \, dP \to 0$, then by Theorem 2.1 in Van Der Vaart and Wellner (2007)

$$\sqrt{n} \left\{ \left( V_{\text{SUP}}(L; \hat{\Theta}_{\text{SUP}}) - g(\hat{\Theta}_{\text{SUP}}) \right) - \left( V_{\text{SUP}}(L; \bar{\Theta}) - g(\bar{\Theta}) \right) \right\} = o_p(1).$$

**Centered Sample Average**

Next we consider $G_n \{ V_{\text{SUP}}(L; \bar{\Theta}) \}$. Note that $V_{\text{SUP}}(L; \bar{\Theta})$ is a deterministic function of random variable $L$ as parameters are fixed. We have that $E \left[ (V_{\text{SUP}}(L; \bar{\Theta})^2 \right] < \infty$ holds by Assumption 1 & 9. Thus the central limit theorem yields

$$G_n \{ V_{\text{SUP}}(L; \bar{\Theta}) \} \overset{d}{\to} \mathcal{N}(0, \text{Var}[V_{\text{SUP}}(L; \bar{\Theta})]).$$

**Bias Term**

We next consider $\sqrt{n} \left\{ g(\hat{\Theta}_{\text{SUP}}) - g(\Theta) \right\}$. Using a Taylor series expansion

$$g(\hat{\Theta}_{\text{SUP}}) = g(\Theta) + (\hat{\Theta}_{\text{SUP}} - \Theta)^T \frac{\partial}{\partial \Theta_{\text{SUP}}} g(\Theta) + (\hat{\xi}_{\text{SUP}} - \bar{\xi})^T \frac{\partial}{\partial \xi_{\text{SUP}}} g(\Theta) + O_p\left(n^{-1}\right),$$

therefore

$$\sqrt{n} \left\{ g(\hat{\Theta}_{\text{SUP}}) - g(\Theta) \right\} = \sqrt{n}(\hat{\Theta}_{\text{SUP}} - \Theta)^T \frac{\partial}{\partial \Theta_{\text{SUP}}} g(\Theta) + \sqrt{n}(\hat{\xi}_{\text{SUP}} - \bar{\xi})^T \frac{\partial}{\partial \xi_{\text{SUP}}} g(\Theta) + o_p(1).$$

Using the $Q$-function and propensity score function influence functions we can write

$$\sqrt{n} \left\{ g(\hat{\Theta}_{\text{SUP}}) - g(\Theta) \right\} = \frac{\partial}{\partial \Theta_{\text{SUP}}} g(\Theta) \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \psi^0_{\text{SUP}}(L_i) + \frac{\partial}{\partial \xi} g(\Theta) \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \psi^\xi_{\text{SUP}}(L_i) + o_p(1).$$