Optically Induced Superconducting-like Properties in the One-Dimensional Hubbard Model

Tatsuya Kaneko, Seiji Yunoki, and Andrew J. Millis

1 Department of Physics, Columbia University, New York, New York 10027, USA
2 Computational Condensed Matter Physics Laboratory, RIKEN Cluster for Pioneering Research (CPR), Wako, Saitama 351-0198, Japan
3 Computational Materials Science Research Team, RIKEN Center for Computational Science (R-CCS), Kobe, Hyogo 650-0047, Japan
4 Computational Quantum Matter Research Team, RIKEN Center for Emergent Matter Science (CEMS), Wako, Saitama 351-0198, Japan
5 Center for Computational Quantum Physics, Flatiron Institute, New York, New York 10010, USA

We show that optical excitation of the Mott insulating phase of the one-dimensional Hubbard model can create a state possessing two of the hallmarks of superconductivity: a nonvanishing charge stiffness and long-ranged pairing correlations. By employing the exact diagonalization method, we show that the optical pulse preferentially creates the \( \eta \)-pairing states; The piece of the superconducting states in the photoinduced state gives rise to nonvanishing charge stiffness and pairing correlation that decay very slowly with system size. Our finding establishes a theoretical proof of principle that optical pumping can lead to superconducting-like properties and demonstrates that \( \eta \)-pairing can provide a new pathway to access superconductivity in Mott insulators.

A fundamental goal of nonequilibrium physics is to use strong light-matter interactions to create new quantum phases [1–4]. Experimental reports of possible light-induced superconductivity [5–9] have created tremendous excitement. While many attempts have been made to model this phenomena [10–17], there has been no unambiguous demonstration within a clearly defined theoretical model. In this paper, we show that optical excitation of the Hubbard model can create a state characterized by two of the hallmarks of superconductivity: a nonvanishing charge stiffness and long-ranged pairing correlations. Optically driven \( \eta \)-pairing [18], characterized by staggered off-diagonal long-range correlation, plays a key role.

We here solve the one-dimensional (1D) Hubbard model subjected to a “pump” electric field using numerical exact diagonalization (ED). The pump drives the system from its ground state to an excited state, which we characterize via the charge stiffness \( D \) and the pairing correlation \( P_{ij} = \langle c_{i\uparrow}^\dagger c_{j\uparrow} c_{i\downarrow} c_{j\downarrow} \rangle \). \( D \) is the coefficient of a delta-function term in the optical response: \( \text{Re} [\sigma(\omega)] = 2\pi D \delta(\omega) + \sigma_{\text{reg}}(\omega) \) [19]. We show that if the system is initially at half-filling in a Mott insulating ground state \( (D = 0, \text{gap in } \sigma_{\text{reg}}) \), then after the pump the system has a \( D > 0 \). The pumped system also exhibits a pairing correlation, whose Fourier transform has a very strong peak at the wave vector \( q = \pi \), indicating that a pair density wave state is created.

We interpret our results by Yang’s \( \eta \)-pairing concept [20]. To describe \( \eta \)-pairing, we introduce a triplet of operators \( \hat{\eta} = (\hat{\eta}_x, \hat{\eta}_y, \hat{\eta}_z) \) which obey the usual \( SU(2) \) commutation relation \( [\hat{\eta}_a, \hat{\eta}_b] = i\epsilon_{abc}\hat{\eta}_c \). Here \( \hat{\eta}_z = \frac{1}{2} \sum_j (\hat{n}_{j\uparrow} - \hat{n}_{j\downarrow} - 1) \) and it is convenient to represent \( \hat{\eta}_x \) and \( \hat{\eta}_y \) in the usual way in terms of the raising and lowering operators \( \hat{\eta}^+ = \sum_j (-1)^j c_{j\uparrow}^\dagger c_{j\downarrow}^\dagger \) and \( \hat{\eta}^- = (\hat{\eta}^+)^\dagger \) [21]. The operator \( \hat{\eta}^+ \) creates in effect a paired state with a staggered pairing amplitude. Yang showed that \( \hat{\eta}_z^2 = \frac{1}{2}(\hat{\eta}^+\hat{\eta}^- + \hat{\eta}^-\hat{\eta}^+) + \hat{\eta}_z^2 \) commutes with the Hubbard Hamiltonian and that Hubbard eigenstates with a nonzero value of \( \langle \hat{\eta}_z^2 \rangle \) have long-ranged pairing correlations \( \langle \eta_i^+\eta_j^- \rangle_{\phi \neq \phi} = (1-\delta_{ij}) \) \( \langle \hat{c}_{i\sigma}^\dagger \hat{c}_{i\sigma} \rangle \) [20]. We show here that, while any thermodynamic average over Hubbard eigenstates at half-filling yields an ensemble with no long-ranged pairing correlations and vanishing charge stiffness [22–25], optical excitation preferentially creates states with a nonvanishing \( \langle \hat{\eta}_z^2 \rangle \) and that even an incoherent superposition of these states gives rise to superconducting-like properties including a \( D \) and long-ranged pairing correlations.

We here study the 1D Hubbard model with the nearest neighbor hopping \( t_h \) and on-site interaction \( U > 0 \):

\[
\hat{H} = -t_h \sum_{j=1}^{L} \left( c_{j\sigma}^\dagger c_{j+1\sigma} + \text{H.c.} \right) + U \sum_{j=1}^{L} \hat{n}_{j\uparrow}^\dagger \hat{n}_{j\uparrow},
\]

where \( \hat{c}_{j\sigma} \) (\( \hat{c}_{j\sigma}^\dagger \)) is the annihilation (creation) operator for an electron at site \( j \) with spin \( \sigma = \uparrow, \downarrow \) and \( \hat{n}_{j\sigma} = \hat{c}_{j\sigma}^\dagger \hat{c}_{j\sigma} \). We specialize to the half-filled case with the number of electrons in each spin channel, \( N_\sigma = L/2 \) (number of sites \( L \) is taken to be even). Since \( [\hat{H}, \hat{\eta}_z^2] = [\hat{H}, \hat{\eta}_z] = 0 \), any eigenstate of \( \hat{H} \) is also the eigenstate \( |\eta, \eta_z \rangle \) of \( \hat{\eta}_z^2 \) and \( \hat{\eta}_z \) with the eigenvalues \( \eta(\eta + 1) \) and \( \eta_z \), respectively. At half-filling, the allowed eigenvalues \( |\eta, \eta_z \rangle \) are \( \eta = 0, 1, 2, \cdots, L/2 \) and \( \eta_z = 0 \).

A time-dependent external field \( A(t) \) is introduced via the Peierls substitution \( t_h c_{j\sigma}^\dagger \hat{c}_{j+1\sigma} \rightarrow t_h e^{iA(t)} c_{j\sigma}^\dagger \hat{c}_{j+1\sigma} \). We use a pump pulse given as \( A(t) = \)
$A_0 e^{-(t-t_0)^2/(2σ_p^2)} \cos[ω_p(t-t_0)]$ with amplitude $A_0$, frequency $ω_p$, and pulse width $σ_p$ centered at time $t_0 (> 0)$ [26]. We assume that for $t = 0$ the system is in the Mott insulating ground state and evolve the state forward in time using $\mathcal{H}(t)$, which is $\mathcal{H}$ with the time-dependent hopping. We employ the time-dependent ED method [27, 28] for a finite-size cluster with the periodic boundary conditions (PBC) [29] and the state at time $t$ is here indicated by $|Ψ(t)\rangle$. For $t-t_0 ≫ σ_p$, the resulting state is projected onto the eigenstates $|ψ_m\rangle$ (eigenenergies $ε_m$) of the unperturbed Hubbard model, obtained by full (exact) diagonalization. For each eigenstate, we directly calculate the $η$-pairing eigenvalue $η(η+1)$. We compute the charge stiffness for each eigenstate $|ψ_m\rangle$ from $D_m = \frac{L}{2} \frac{∂^2 ε_m(Φ)}{∂Φ^2}|_{Φ=0}$, where the twisted boundary conditions (TBC) with the phase $Φ$ is introduced via a vector potential $A_{\text{twist}} = Φ/L$ [29].

Figure 1 shows the calculated stiffnesses $D_m$ for all eigenstates in the half-filled Hubbard chain at a large value of the interaction $U$. The eigenstates are grouped into sectors corresponding to different numbers of doubly occupied sites. Significantly, most of $D_m$ for the $η$-pairing eigenstates ($η > 0$) are positive, but most of $D_m$ for the non-$η$-pairing eigenstates ($η = 0$) are negative. The sum of $D_m$ over all eigenstates is zero, because $S(η) = ∑_m D_m(η)$, the sum of the charge stiffness of all eigenstates with the same $η$, satisfies $S(η = 0) + ∑_{\eta=1}^{L/2} S(η = 0) = 0$ [29]. This implies that the thermal ensemble at infinite temperature cannot have perfect conducting behavior. We find numerically that the sum of $D_m$ over all eigenstates in a given double occupancy sector is also zero, and the sum of $D_m$ over all eigenstates within a given small energy range is close to zero. This strongly suggests that the thermal average of charge stiffness is zero in equilibrium at any temperature as theoretically expected [22–25]. To obtain $D > 0$ in the half-filled Hubbard chain, one must prepare an ensemble in which $η$-pairing ($η > 0$) eigenstates have larger weight than $η = 0$ eigenstates. We next show that photoexcitation produces just such an ensemble.

Before showing $D(t)$, we review the photoinduced state $|Ψ(t)\rangle$ and its weight distribution. As shown in the inset of Fig. 2(a), the external pulse $A(t)$ induces an $η$-pairing correlation $⟨\tilde{η}^2⟩(t) = ⟨Ψ(t)|\tilde{η}^2|Ψ(t)⟩ = ⟨Ψ(t)|\tilde{η}^+\tilde{η}^-|Ψ(t)⟩$ at half-filling, corresponding to the enhancement of the superconducting correlation at momentum $q = π$ shown

![Figure 1](image1.png)

**Figure 1.** Charge stiffness $D_m$ of the eigenstates $|ψ_m\rangle$ (eigenenergies $ε_m$) in the half-filled Hubbard chain calculated by the ED method for $L = 10 \ (N_↑ = N_↓ = 5)$ at $U = 20t_h$. The colors of the points indicate the values of $η$.

![Figure 2](image2.png)

**Figure 2.** (a) All eigenenergies $ε_m$ and eigenvalues $η$ for the eigenstates $|ψ_m\rangle$ of the half-filled Hubbard Hamiltonian $\mathcal{H}$ at $U = 20t_h$ and $L = 10 \ (N_↑ = N_↓ = 5)$ with the PBC. The color of each point indicates the weight $|⟨ψ_m|Ψ(t)⟩|^2$ of the eigenstate $|ψ_m⟩$ in the photoinduced state $|Ψ(t)⟩$ at $t = 40/t_h$ for $A(t) = 0.3, ω_p = 19.36t_h, σ_p = 2/t_h$, and $t_0 = 10/t_h$. The inset shows the time evolution of $⟨\tilde{η}^2⟩ (t)/L = ⟨Ψ(t)|\tilde{η}^2|Ψ(t)⟩ / L$. (b) Total weight $w(η)$ of $|⟨ψ_m|Ψ(t)⟩|^2$ over the states $|ψ_m⟩$ with the same number $η$ in (a). Note that $∑_{\eta=0}^{L/2} w(η) = 1$. (c) Time evolution of the energy $E(t) = ⟨Ψ(t)|\tilde{H}(t)|Ψ(t)⟩$ with the time-dependent flux $Φ(t)$ after the pulse irradiation. The dashed line indicates $ΔE(Φ) = D(t_1)|Φ|^2/L$ with the charge stiffness $D(t)$ at $t = t_1$ evaluated by $D(t) = ∑_m D_m |⟨ψ_m|Ψ(t)⟩|^2$ (see the text). The inset shows the results in the whole energy scale. The results are calculated using the ED method with $δΦ = 0.5 × 10^{-3}$ and $t_1 = 40/t_h$ in $Φ(t)$. 
in Ref. [18]. Figure 2(a) shows the weight distribution of the eigenstates $|\psi_m\rangle$ in the photoinduced state $|\Psi(t)\rangle$, where the color of each point indicates the weight $|\langle \psi_m | \Psi(t) \rangle|^2$ and the total weight is shown as a function of $\eta$ in Fig. 2(b). These results clearly show that photoexcitation preferentially induces eigenstates $|\psi_m\rangle$ with $\eta > 0$ [see Fig. 2(b)], explaining the large value of $\langle \eta^2 \rangle (t)$ observed in the photoinduced state $|\Psi(t)\rangle$. This photoinduced nonthermal distribution implies $D(t) > 0$.

To verify the stiffness $D(t) > 0$ in this photoinduced $\eta$-pairing state $|\Psi(t)\rangle$, we apply the time-dependent flux $A(t) = \Phi(t)/L$, given by $\Phi(t) = \theta(t-t_1) \times [\delta \Phi(t-t_1)]$, beginning at time $t_1$ long after the pump pulse $(t-t_0 \gg \sigma_p)$, where $\theta(t)$ is the Heaviside step function and $\Phi(t)$ increases linearly in time with slope $\delta \Phi$, corresponding to an electric field $\frac{\partial A(t)}{\partial t} \propto \delta \Phi$. To estimate the stiffness in the photoinduced state $|\Psi(t)\rangle$, we compute the energy $E(t) = \langle \Psi(t)| H(t) |\Psi(t)\rangle$ during the time-dependent flux $\Phi(t)$. As shown in Fig. 2(c), the curvature of $E(t)$ is positive with respect to $\Phi(t)$, indicating $D(t) > 0$. To identify the curvature of $E(t)$ at $\Phi = 0$, we should notice that the charge stiffness $D(t) = \sum_m |c_m(t)|^2 D_m$ can also be evaluated directly from the weight $|c_m(t)|^2 = |\langle \psi_m | \Psi(t) \rangle|^2$ in the photoinduced state $|\Psi(t)\rangle$. Comparing with $\Delta \epsilon(\Phi) = D(t_1) \Phi^2/L$, the energy curve $E(t)$ at $\Phi(t) \sim 0$ is perfectly fitted by the stiffness $D(t_1)$ evaluated from the photoinduced weight distribution. Therefore, the photoinduced state $|\Psi(t)\rangle$ has a stiffness $D(t) > 0$.

The above results demonstrate an association between a nonthermal distribution of states with $\langle \eta^2 \rangle \neq 0$ and a nonvanishing charge stiffness. We now show that these two factors are also associated with long-ranged $\eta$-pairing correlation. First, we see this association in Yang’s maximally $\eta$-paired state $|\phi_{N_N}\rangle \propto |\eta^+ N_N \rangle \langle 0|$ generated from the vacuum $|0\rangle$ [20]. For this state, Yang showed that the $\eta$-pairing correlations are distance independent and of infinite range with $|\langle \phi_{N_N} | \eta^+_i \eta^-_j | \phi_{N_N} \rangle|_{i \neq j} = N_N (L-L_{N_N})/L_{N_N}$ [20]. Here we find that the charge stiffness $D_{\eta}$ for Yang’s $\eta$-pairing state $|\phi_{N_N}\rangle$ satisfies $D_{\eta} = 4 J_{\text{ex}} |\langle \phi_{N_N} | \eta^+_i \eta^-_j | \phi_{N_N} \rangle|_{i \neq j} > 0$ with the exchange interaction $J_{\text{ex}} = 2 \hbar^2 L/\omega_p$ [29], which directly indicates the association between the charge stiffness and long-ranged pairing correlations with $|\langle \phi_{N_N} | \eta^2 | \phi_{N_N} \rangle| \neq 0$.

Our numerical evidence strongly suggests that this association is valid beyond Yang’s $\eta$-pairing state. To discuss this, let us review the ingredients of $\langle \eta^2 \rangle$. At half-filling ($\eta_z = 0$), the algebra of $\eta$ operators implies

$$\langle \eta^2 \rangle = L n_d + \sum_{i \neq j} \langle \eta^+_i \eta^-_j \rangle$$

(2)

with the double occupancy $n_d = \frac{1}{L} \sum_j \langle \hat{n}_j^+ \hat{n}_j \rangle$. From the analysis of the eigenstates, we can show $\langle \eta^2 \rangle (n_d) \equiv \frac{1}{N_{n_d}} \sum_m (\psi_m | \eta^2 | \psi_m \rangle) n_d = L n_d$ in each double occupancy ($n_d$) sector, where $N_{n_d}$ is the number of the eigenstates and the suffix $n_d$ indicates the eigenstate within the $n_d$ sector (see the Supplemental Material [29]). We can also show that the average of $\langle \psi_m | \eta^2 | \psi_m \rangle$ over all Hubbard eigenstates at half-filling is $\langle \eta^2 \rangle_{\text{avr.}} / L = 0.25$, which is same with the double occupancy $n_d = 0.25$ at infinite temperature. These relations strongly suggest that a thermal distribution of the eigenstates has no long-range $\eta$-pairing correlation. However, we find for the optically generated state $|\Psi(t)\rangle$ that $\langle \eta^2 \rangle > L n_d$ (see e.g. the inset of Fig. 2(a), where $\langle \eta^2 \rangle (t)/L > 1$, which implies contributions from nonlocal pairing correlations $\langle \eta^+_i \eta^-_j \rangle_{i \neq j}$.

We now consider the spatial correlations and finite size effects associated with the $\eta$-pairing state. One trivial finite size effect is a weak size dependence of the optimal photoexcitation frequency $\omega_p$. For each system size, we calculate $\langle \eta^2 \rangle (t)$ with different $\omega_p$ (see the Supplemental Material [29]). Here we present results obtained at the optimal $\omega_p$ for each size. We represent the amount of optical excitation by the induced double occupancy in Fig. S.6, by plotting $\langle \eta^2 \rangle (t)/L$ as a function of $n_d(t) = \frac{1}{L} \sum_j \langle \Psi(t) | \hat{n}_j^+ \hat{n}_j \rangle |\Psi(t)\rangle$. Note that we here consider a fixed pump strength $A_0$, which produces time-dependent $n_d(t)$ and $\langle \eta^2 \rangle (t)$. Equivalent results could be obtained by $A_0$ dependence considering the long-time limits of $n_d(t)$ and $\langle \eta^2 \rangle (t)$ (see the Supplemental Material [29]). Figure S.6 reveals two important results: $\langle \eta^2 \rangle (t)/L$ under photoexcitation is systematically greater than $n_d$ (dashed line), indicating that the $\langle \eta^+_i \eta^-_j \rangle_{i \neq j}$ term in Eq. (2) is nonzero, and the difference

![Figure 3](image-url)
extrapolation to the calculated results are consistent with either a nonzero limit or is too small to reach a definitive statement, the

The inset of Fig. 4 shows the magnitude is comparable to the value in Yang's maximally paired state. When a long-ranged η-pairing state is formed, \( P_{\hat{v},\hat{j}}^{(\eta)}(t) \) remains nonzero with increasing system size \( L \) corresponding to \( \langle \hat{n}_{i}^{2} \rangle \sim \langle \hat{n}_{i} \hat{n}_{j} \rangle \alpha \propto L^2 \). For Yang's η-pairing state \( |\phi_{\eta}\rangle \), \( P_{\hat{v},\hat{j}}^{(\eta)}(t) = 0.25 \) at \( n_{d} = 0.5 \) regardless of the system size. In Fig. 4, we show \( P_{\hat{v},\hat{j}}^{(\eta)}(t) \) with the different system size \( L \). We see for the optically created state that the magnitude is \( P_{\hat{v},\hat{j}}^{(\eta)}(t) \sim 0.07 \) at \( n_{d}(t) = 0.3 \), which is comparable to the value in Yang’s maximally η-paired state. The value of \( P_{\hat{v},\hat{j}}^{(\eta)}(t) \) varies slowly with system size. The inset of Fig. 4 shows the L dependence of \( P_{\hat{v},\hat{j}}^{(\eta)}(t) \) at \( n_{d}(t) = 0.3 \). While the range of system sizes accessible to us is too small to reach a definitive statement, the calculated results are consistent with either a nonzero extrapolation to the \( L \to \infty \) limit or \( P_{\hat{v},\hat{j}}^{(\eta)} \propto L^{-\alpha} \) with \( \alpha \sim 0.3 \) corresponding to very slow decaying power-law pairing correlations (quasi long-range order).

In conclusion, we have shown that optical excitation of the 1D Hubbard model creates a state possessing two of the hallmarks of superconductivity: a nonvanishing charge stiffness and long-ranged pairing correlations. The fundamental reason is that optical excitation preferentially creates η-pairing states, which have a positive stiffness with typical values of \( D \sim J_{ex} = 2t_h^2/U \). Our results are obtained from finite system numerics but the inferred system size dependence suggests that long-ranged η-pairing correlations can survive even in larger size systems.

While the 1D Hubbard model we used here is in several respects a highly simplified description of real materials, we believe that our results are important because they provide an existence proof of the phenomenon, some new understanding of the qualitative properties of light-induced superconductivity, and perhaps a guide to future research that may extend the work to higher dimensions and richer models. Recent theoretical works also demonstrated long-range η-pairing can be induced by other protocols including injection of doublon-hole pairs [30, 31] and suppression of the competing magnetic correlation [32]. In the experimental side, the 1D Hubbard model may be realized in cold atomic gasses.

The authors thank S. Ejima, D. Golež, and T. Shirakawa for fruitful discussion. This was supported in part by Grants-in-Aid for Scientific Research from JSPS (Projects No. JP18K13509 and No. JP18H01183) of Japan. A.J.M. was supported by the Basic Energy Sciences program of the U.S. Department of Energy under grant DE-SC0018218. T.K. was supported by the JSPS Overseas Research Fellowship.

References

[1] Y. Tokura, J. Phys. Soc. Jpn. 75, 011001 (2006).
[2] D. N. Basov, R. D. Averitt, and D. Hsieh, Nat. Mater. 16, 1077 (2017).
[3] T. Oka and S. Kitamura, Annu. Rev. Condens. Matter Phys. 10, 387 (2019).
[4] S. Ishihara, J. Phys. Soc. Jpn. 88, 072001 (2019).
[5] D. Fausti, R. I. Tobey, N. Dean, S. Kaiser, A. Dienst, M. C. Hoffmann, S. Pyon, T. Takayama, H. Takagi, and A. Cavalleri, Science 331, 189 (2011).
[6] W. Hu, S. Kaiser, D. Nicoletti, C. R. Hunt, I. Gierz, M. C. Hoffmann, M. Le Tacon, T. Loew, B. Keimer, and A. Cavalleri, Nat. Mater. 13, 705 (2014).
[7] S. Kaiser, C. R. Hunt, D. Nicoletti, W. Hu, I. Gierz, H. Y. Liu, M. Le Tacon, T. Loew, D. Haug, B. Keimer, and A. Cavalleri, Phys. Rev. B 89, 184516 (2014).
[8] M. Mitrano, A. Cantaluppi, D. Nicoletti, S. Kaiser, A. Perucchi, S. Lupi, P. Di Pietro, D. Pontiroli, M. Riccò, S. R. Clark, D. Jaksh, and A. Cavalleri, Nature (London) 530, 461 (2016).
A state with a nonzero $D$ also exhibits a divergent imaginary part of the conductivity with $\text{Im}[\sigma(\omega)] \propto 1/\omega$.

See Supplemental Material for details, which includes Refs. [33–38].
Supplemental Material:  
Optically Induced Superconducting-like Properties in the One-Dimensional Hubbard Model

1. Time-dependent exact diagonalization method

To evaluate the state $|\Psi(t)\rangle$ under the time-dependent Hamiltonian $\hat{H}(t)$, we numerically solve the time-dependent Schrödinger equation,

$$\frac{\partial}{\partial t} |\Psi(t)\rangle = \hat{H}(t) |\Psi(t)\rangle,$$  \hspace{1cm} (S1)

with the initial condition that $|\Psi(t=0)\rangle = |\psi_0\rangle$, where $|\psi_0\rangle$ is the ground state of the Hamiltonian $\hat{H}(t=0)$. For this purpose, we employ the time-dependent exact diagonalization (ED) method based on the Lanczos algorithm [S1, S2]. In this method, the time evolution with a short time step $\delta t$ is calculated as

$$|\Psi(t+\delta t)\rangle \approx e^{-i\hat{H}(t)\delta t} |\Psi(t)\rangle \approx \sum_{l=1}^{M_L} e^{-i\xi_l \delta t} |\tilde{\psi}_l\rangle \langle \tilde{\psi}_l |\Psi(t)\rangle,$$  \hspace{1cm} (S2)

where $\xi_l$ and $|\tilde{\psi}_l\rangle$ are eigenenergies and eigenvectors of $\hat{H}(t)$, respectively, in the corresponding Krylov subspace generated with $M_L$ Lanczos iterations [S1–S3]. In our ED calculations, we adopt $\delta t = 0.001/\hbar$ and $M_L = 15$ for the time evolution, which provides results with almost machine precision accuracy.

2. Twisted boundary conditions

In order to estimate the charge stiffness, we consider the one-dimensional (1D) Hubbard model with the flux $\Phi$, described by

$$\hat{H}_\Phi = -t_h \sum_{j,\sigma} \left( e^{i\Phi/2c_j^{\dagger}c_{j+1,\sigma} + \text{H.c.}} \right) + U \sum_j \hat{n}_{j,\uparrow} \hat{n}_{j,\downarrow}. $$  \hspace{1cm} (S3)

Notice that through a transformation $\hat{c}_{j,\sigma} = e^{i\Phi R_j} \hat{c}_{j,\sigma}$, where $R_j$ is the location of site $j$, $\hat{H}_\Phi$ is transformed to $\hat{H}$ defined in Eq. (1) in the main text with a simple substitution $c_j \rightarrow \hat{c}_{j,\sigma}$. However, since the operator $\hat{c}_{j,\sigma}$ satisfies $\hat{c}_{j+1,\sigma} = e^{i\Phi \hat{c}_{j,\sigma}}$, the transformed Hamiltonian $\hat{H}$ has to satisfy twisted boundary conditions (TBC) with the phase $\Phi$, in stead of periodic boundary conditions (PBC).

Even in the presence of the flux $\Phi$, we can still define the $\eta$-pairing operators. With the local pair operators $\hat{n}_{j,\sigma} = (\hat{n}_{j,\sigma})^{\dagger} = (-1)^j \hat{c}_{j,\sigma}^{\dagger} \hat{c}_{j,\sigma}^{\dagger}$ and $\hat{n}_{j,\sigma} = \frac{1}{2} (\hat{n}_{j,\uparrow} + \hat{n}_{j,\downarrow} - 1)$, $\eta$-pairing operators under the flux $\Phi$ are given by $\hat{\eta}_{j,\sigma}^\pm = \sum_j e^{\mp i\Phi R_j} \hat{n}_{j,\sigma}^\pm$ and $\hat{\eta}_z = \sum_j \hat{n}_{j,\sigma}$. These operators also satisfy the $SU(2)$ commutation relations $[\hat{\eta}_z, \hat{\eta}_{j,\sigma}^\pm] = \pm \hat{\eta}_{j,\sigma}^\pm$ and $[\hat{\eta}_{j,\uparrow}, \hat{\eta}_{j,\downarrow}] = 2\hat{\eta}_z$. However, in contrast to the $\eta$-pairing operators at $\Phi = 0$, $[\hat{H}_\Phi, \hat{\eta}_{j,\sigma}^\pm] \neq 0$ for arbitrary $\Phi$. The Hamiltonian $\hat{H}_\Phi$ commutes with $\hat{\eta}_z^2 = \frac{1}{2} (\hat{\eta}_{j,\uparrow}^2 + \hat{\eta}_{j,\downarrow}^2) + \hat{\eta}_z^2$ only at $\Phi = n\pi (n = 0, \pm 1, \pm 2, \ldots)$ [S4].

In Fig. S.1(a), we calculate the eigenenergies $\epsilon_m(\Phi)$ of the half-filled Hubbard Hamiltonian $\hat{H}_\Phi$, only showing the eigenenergies in the vicinity of Yang’s $\eta$-pairing state at $\epsilon_m(\Phi) = U L/2$ and (b) $\epsilon_m(\Phi)$ for the ground state are plotted. The color of each point indicates the $\eta$-pairing correlation $\langle \psi_m(\Phi)|\hat{\eta}_{\downarrow}^2|\psi_m(\Phi)\rangle / L$ for the eigenstate $|\psi_m(\Phi)\rangle$. The inset of (b) is the enlarged plot of (b). Notice that the scale of the vertical axis in the inset of (b) is orders of magnitude smaller.

Figure S.1. Eigenenergies $\epsilon_m(\Phi)$ in the half-filled Hubbard model as a function of the flux $\Phi$ calculated by the ED method for $L = 10$ at $U = 20t_h$. (a) $\epsilon_m(\Phi)$ in the vicinity of Yang’s $\eta$-pairing state at $\epsilon_m(\Phi) = U L/2$ and (b) $\epsilon_m(\Phi)$ for the ground state are plotted. The color of each point indicates the $\eta$-pairing correlation $\langle \psi_m(\Phi)|\hat{\eta}_{\downarrow}^2|\psi_m(\Phi)\rangle / L$ for the eigenstate $|\psi_m(\Phi)\rangle$. The inset of (b) is the enlarged plot of (b). Notice that the scale of the vertical axis in the inset of (b) is orders of magnitude smaller.

Under the TBC, the charge stiffness $D_m$ of the eigen-
state $|\psi_m\rangle$ is given by
\[
D_m = \frac{L}{2} \left. \frac{\partial^2 \varepsilon_m(\Phi)}{\partial \Phi^2} \right|_{\Phi=0} .
\] (S4)

Because $\varepsilon_m(\Phi) = \langle \psi_m(\Phi) | \mathcal{H}_\Phi | \psi_m(\Phi) \rangle$, the second order perturbation analysis with respect to $\Phi$ provides
\[
D_m = -\frac{1}{2L} \langle \psi_m | T | \psi_m \rangle - \frac{1}{L} \sum_{n(\neq m)} \frac{|\langle \psi_n | J | \psi_m \rangle|^2}{\varepsilon_n - \varepsilon_m} ,
\] (S5)

where the kinetic operator $\hat{T}$ and current operator $\hat{J}$ are defined as
\[
\hat{T} = -t_h \sum_{j,\sigma} \left( \hat{c}^\dagger_{j,\sigma} \hat{c}_{j+1,\sigma} + \hat{c}^\dagger_{j+1,\sigma} \hat{c}_{j,\sigma} \right) ,
\] (S6)
\[
\hat{J} = -it_h \sum_{j,\sigma} \left( \hat{c}^\dagger_{j,\sigma} \hat{c}_{j+1,\sigma} - \hat{c}^\dagger_{j+1,\sigma} \hat{c}_{j,\sigma} \right) .
\] (S7)

In this paper, the charge stiffness $D_m$ is computed by $D_m = (L/2) \times [\varepsilon_m(\delta \Phi) + \varepsilon_m(-\delta \Phi) - 2\varepsilon_m(0)]/(\delta \Phi)^2$ with $\delta \Phi = 0.001$.

3. Charge stiffness of Yang’s $\eta$-pairing state

In this section, we derive the charge stiffness of Yang’s maximally $\eta$-paired state
\[
|\phi_{N_\eta}\rangle = \frac{1}{\sqrt{C_{N_\eta}}} (\hat{n}^+)^{N_\eta} |0\rangle ,
\] (S8)

where $N_\eta$ is the number of $\eta$-pairs and $C_{N_\eta} = N_\eta! \prod_{k=1}^{N_\eta} (L-k+1)$, $|\phi_{N_\eta}\rangle$ is an eigenstate of the Hubbard model with the eigenenergy $N_\eta U$ [S5]. Since $\langle \phi_{N_\eta} | \hat{T} | \phi_{N_\eta} \rangle = 0$ in Eq. (S5), the stiffness $D_\eta$ is given by
\[
D_\eta = -\frac{1}{L} \sum_n \frac{|\langle \psi_n | \hat{J} | \phi_{N_\eta} \rangle|^2}{\varepsilon_n - N_\eta U} .
\] (S9)

Because Yang’s $\eta$-pairing state satisfies $\hat{H} \hat{J} |\phi_{N_\eta}\rangle = (N_\eta - 1)U \hat{J} |\phi_{N_\eta}\rangle$, the normalized state
\[
|\tilde{\phi}_{N_\eta-1}\rangle = \frac{1}{\sqrt{\langle \phi_{N_\eta} | J^2 | \phi_{N_\eta} \rangle}} \hat{J} |\phi_{N_\eta}\rangle
\] (S10)
is also the eigenstate of the Hubbard Hamiltonian $\hat{H}$ with the eigenenergy $(N_\eta - 1)U$. Therefore, $D_\eta$ becomes
\[
D_\eta = \frac{1}{LU} \langle \phi_{N_\eta} | J^2 | \phi_{N_\eta} \rangle .
\] (S11)

By using the commutation relations between $\hat{n}^\pm$ and $\hat{J}$ previously derived by two of us [S6], we obtain
\[
\langle \phi_{N_\eta} | J^2 | \phi_{N_\eta} \rangle = 8t_h^2 N_\eta (L - N_\eta) / L - 1 .
\] (S12)

Finally, combining Eqs. (S11) and (S12), we obtain
\[
D_\eta = 4J_{ex} \frac{N_\eta (L - N_\eta)}{L(L - 1)} ,
\] (S13)

where we introduced the exchange interaction $J_{ex} = 2t_h^2 / U$. The stiffness $D_\eta$ of Yang’s $\eta$-pairing state is thus characterized by the exchange interaction $J_{ex}$ and the number of $\eta$-pairs $N_\eta$. At half-filling ($N_\eta = L/2$), $D_\eta = J_{ex} L / (L - 1)$, which becomes $D_\eta = J_{ex}$ in the $L \rightarrow \infty$ limit.

Importantly, since the off-diagonal long-range order in Yang’s $\eta$-pairing state is characterized by $[S5]
\[
\langle \phi_{N_\eta} | \hat{n}_i^+ \hat{n}_{i+j}^- | \phi_{N_\eta} \rangle_{i \neq j} = \frac{N_\eta (L - N_\eta)}{L (L - 1)} ,
\] (S14)

the charge stiffness $D_\eta$ becomes
\[
D_\eta = 4J_{ex} \langle \phi_{N_\eta} | \hat{n}_i^+ \hat{n}_{i+j}^- | \phi_{N_\eta} \rangle_{i \neq j} .
\] (S15)

Therefore, the $\eta$-pairing correlation is directly associated with the stiffness $D_\eta$ and Yang’s state $|\phi_{N_\eta}\rangle$ has the non-vanishing stiffness $D_\eta > 0$.

4. Sum rule for charge stiffness

Here, we show the following relation
\[
S = \sum_m D_m = 0 .
\] (S16)

In this section, to indicate the $\eta$ degrees of freedom explicitly, we describe the eigenstate as $|\psi_m; \eta, \eta_z\rangle$. From Eq. (S5), the charge stiffness of the eigenstate $|\psi_m; \eta, \eta_z\rangle$ at half-filling ($\eta_z = 0$) is given by
\[
D_m(\eta) = -\frac{1}{2L} \langle \psi_m; \eta, 0 | \hat{T} | \psi_m; \eta, 0 \rangle - \frac{1}{L} \sum_n \frac{|\langle \psi_n; \eta' \rangle |^2}{\varepsilon_n(\eta') - \varepsilon_m(\eta)} .
\] (S17)

Because of the selection rule $\langle \psi_n; \eta', 0 | \hat{J} | \psi_m; \eta, 0 \rangle = 0$ when $\eta' \neq \eta \pm 1$ [S6, S7], the charge stiffness is given by
\[
D_m(\eta) = -\frac{1}{2L} \langle \psi_m; \eta, 0 | \hat{T} | \psi_m; \eta, 0 \rangle - \frac{1}{L} \sum_n \frac{|\langle \psi_n; \eta - 1, 0 | \hat{J} | \psi_m; \eta, 0 \rangle|^2}{\varepsilon_n(\eta - 1) - \varepsilon_m(\eta)} - \frac{1}{L} \sum_n \frac{|\langle \psi_n; \eta + 1, 0 | \hat{J} | \psi_m; \eta, 0 \rangle|^2}{\varepsilon_n(\eta + 1) - \varepsilon_m(\eta)} .
\] (S18)

To avoid possible confusion, here we indicate by the prime the sum over all energy eigenstates $|\psi_n; \eta, \eta_z\rangle$ with a particular value of $\eta$ ($\eta_z = 0$ at half-filling).
Here, we define the sum of the charge stiffness for the eigenstates with the same number of \( \eta \) as

\[
S(\eta) = \sum_m' D_m(\eta).
\]  
(S19)

We divide the contribution from \( \hat{T} \) and \( \hat{J} \) operators as

\[
S(\eta) = S_T(\eta) + S_J(\eta)
\]  
(S20)

and discuss \( S_T(\eta) \) and \( S_J(\eta) \) separately. First, \( S_T(\eta) \) is given by

\[
S_T(\eta) = -\frac{1}{2L} \sum_m' \langle \psi_m; \eta, 0 | \hat{T} | \psi_m; \eta, 0 \rangle .
\]  
(S21)

Because we assume the particle-hole symmetric structure with \( \hat{T} = -2t_h \sum_{k,\sigma} \cos(k) c_{k,\sigma}^\dagger c_{k,\sigma} \), the sum for all eigenstates at half-filling satisfies

\[
\sum_{\eta=0}^{L/2} \sum_m' \langle \psi_m; \eta, 0 | \hat{T} | \psi_m; \eta, 0 \rangle = 0,
\]  
(S22)

and we thus obtain

\[
\sum_{\eta=0}^{L/2} S_T(\eta) = 0.
\]  
(S23)

Next, \( S_J(\eta) \) is given by

\[
S_J(\eta) = \frac{1}{L} \sum_{m,n} \frac{\sqrt{\langle \psi_n; \eta, 1 | \hat{J} | \psi_m; \eta - 1, 0 \rangle^2}}{\varepsilon_n(\eta) - \varepsilon_m(\eta - 1)}
- \frac{1}{L} \sum_{m,n} \frac{\sqrt{\langle \psi_n; \eta + 1, 0 | \hat{J} | \psi_m; \eta, 0 \rangle^2}}{\varepsilon_n(\eta + 1) - \varepsilon_m(\eta)},
\]  
(S24)

where the sum \( S_J(\eta) \) is characterized by the transitions \( \eta - 1 \to \eta \) and \( \eta \to \eta + 1 \). Introducing the function

\[
F(\eta + 1, \eta) = \frac{1}{L} \sum_{m,n} \frac{\sqrt{\langle \psi_n; \eta + 1, 0 | \hat{J} | \psi_m; \eta, 0 \rangle^2}}{\varepsilon_n(\eta + 1) - \varepsilon_m(\eta)},
\]  
(S25)

the sum \( S_J(\eta) \) is given by

\[
S_J(\eta) = F(\eta, \eta - 1) - F(\eta + 1, \eta),
\]  
(S26)

where \( F(L/2, L/2 - 1) \) and \( S(0) = -F(1,0) \). Because \( F(\eta + 1, \eta) \) in \( S_T(\eta) \) and \( S_J(\eta + 1) \) cancels each other, the sum of \( S_T(\eta) \) for all \( \eta \) becomes

\[
\sum_{\eta=0}^{L/2} S_J(\eta) = \sum_{\eta=0}^{L/2} [F(\eta, \eta - 1) - F(\eta + 1, \eta)] = 0.
\]  
(S27)

Combining Eqs. (S23) and (S27), we finally obtain

\[
S = \sum_{\eta=0}^{L/2} S(\eta) = \sum_{\eta=0}^{L/2} [S_T(\eta) + S_J(\eta)] = 0.
\]  
(S28)

Therefore, the sum of \( D_m \) over all eigenstate is zero. Because of the derivation shown above, \( S = 0 \) also gives us the following interesting relation:

\[
\sum_{\eta=1}^{L/2} S(\eta) = -S(0).
\]  
(S29)

The sum of the charge stiffness for the \( \eta = 0 \) eigenstates and that for the \( \eta > 0 \) eigenstates have the opposite sign. The calculated \( D_m \) shown in the main text satisfies this relation, and most of \( D_m \) for the \( \eta > 0 \) eigenstates are positive and most of \( D_m \) for the \( \eta = 0 \) eigenstates are negative.

5. Average of \( \langle \hat{n}_d \rangle \)

In this section, we estimate the average of \( \langle \psi_m | \hat{n}_d | \psi_m \rangle \) within each double occupancy \( (n_d) \) sector. In Fig. S.2, we show the value of \( \eta \) and double occupancy

\[
\eta_d^{(m)} = \frac{1}{L} \sum_j \langle \psi_m | \hat{n}_j \hat{n}_{j+1} | \psi_m \rangle
\]  
(S30)

for all eigenstates \( | \psi_m \rangle \) of the ten-site Hubbard ring at half-filling. The large \( U \) means the eigenstates are grouped also into sectors of different double occupancies \( n_d = N_d/L \) \((N_d = 0, 1, \ldots, L/2)\) together with different values of \( \eta \). The average of \( \langle \psi_m | \hat{n}_d^2 | \psi_m \rangle \) belonging to the \( n_d \) sector may be defined as

\[
\langle \hat{n}_d^2 \rangle (n_d) = \frac{1}{N_{n_d}} \sum_{|m|} \langle \psi_m | \hat{n}_d^2 | \psi_m \rangle_{n_d}
\]  
(S31)

where \( N_{n_d} = \sum_{\eta=0}^{N_d} \sum_{|m|<\Delta n_d} \langle \psi_m; \eta, 0 | \hat{n}_d^2 | \psi_m; \eta, 0 \rangle \).
Table I. Number of the eigenstates for the half-filled Hubbard model with $N_{\uparrow} = N_{\downarrow} = L/2 = 5$.

| $\eta$ | 0 | 1 | 2 | 3 | 4 | 5 |
|-------|---|---|---|---|---|---|
| $N_d$ (= $L n_d$) | 252 | 630 | 25200 | 25200 | 6300 | 252 |
| $\eta N$ | 0 | 1 | 2 | 3 | 4 | 5 |
| $n_d$ | 2000 | 6300 | 11340 | 2520 | 90 | 90 |
| $\eta N$ | 0 | 1 | 2 | 3 | 4 | 5 |
| $n_d$ | 2000 | 6300 | 11340 | 2520 | 90 | 90 |

where we sum up $\langle \psi_m | \hat{n}_d^2 | \psi_m \rangle$ for the eigenstates within a range $|n_d^{(m)} - n_d| < \Delta n_d$ and $N_{n_d}$ is the number of the eigenstates within the range. Note that, since we intend to estimate the average of $\langle \psi_m | \hat{n}_d^2 | \psi_m \rangle$ in each $n_d$ sector, we assume an appropriate $\Delta n_d$ (as small as 0.01 in this case) to pick up all eigenstate belonging to the $n_d$ sector. Because $\langle \psi_m ; \eta, 0 | \hat{n}_d^2 | \psi_m ; \eta, 0 \rangle = \eta (\eta + 1)$, we have

$$\langle \hat{n}_d^2 \rangle (n_d) = \frac{\sum_{\eta=0}^{n_d} \eta (\eta + 1) N_{\eta,N_d}}{\sum_{\eta=0}^{n_d} N_{\eta,N_d}}, \quad (S32)$$

where $N_{\eta,N_d}$ is the number of the eigenstates with $\eta$ in the $n_d (= N_{d}/L)$ sector and $N_{n_d} = \sum_{\eta} N_{\eta,N_d}$.

In Table I, we show $N_{\eta,N_d}$, corresponding to Fig. S.2, for the ten-site Hubbard ring at half-filling ($N_{\uparrow} = N_{\downarrow} = L/2$). We can show that $N_{\eta,N_d}$ in Table I is given as

$$N_{\eta,N_d} = \left( \frac{L}{L - 2 N_d} \right) \frac{L - 2 N_d}{L/2 - N_d} \times \left[ \left( \frac{2 N_d}{N_d - \eta} \right) - \left( \frac{2 N_d}{N_d - \eta - 1} \right) \right], \quad (S33)$$

where $\left( \frac{L}{L - 2 N_d} \right) \frac{L - 2 N_d}{L/2 - N_d}$ corresponds to the number of the states on the singly occupied sites and $\left( \frac{2 N_d}{N_d - \eta} \right) - \left( \frac{2 N_d}{N_d - \eta - 1} \right)$ corresponds to the number of the states on the doubly and no occupied sites with the different $\eta$ [S8–S10]. Combining Eqs. (S32) and (S33), we obtain

$$\langle \hat{n}_d^2 \rangle (n_d) = N_d = L n_d. \quad (S34)$$

We can also show that the average of $\langle \psi_m | \hat{n}_d^2 | \psi_m \rangle$ over all eigenstates is $\langle \hat{n}_d^2 \rangle_{\text{avr}} / L = 0.25$, which is the same as the double occupancy $n_d = 0.25$ at infinite temperature.

6. Photoinduced $\eta$-pairing

Here we provide the supplemental data for the photoinduced $\eta$-pairing state. Figure S.3 shows the time-evolution of the double occupancy $n_d(t)$ and the $\eta$-pairing correlation $\langle \hat{n}_d^2 \rangle (t)/L$. Inset: $\langle \hat{n}_d^2 \rangle (t)/L$ as the function of $n_d(t)$. The arrow indicates the time-evolved direction. The results are calculated by the ED method for $L = 10$ under the PBC at $U = 20 t_h$ with $A_0 = 0.3$, $\omega_p = 19.36 t_h$, $\sigma_p = 2/t_h$, and $t_0 = 10/t_h$ in $A(t)$.

Figure S.4. Frequency $\omega_p$ dependence of $\langle \hat{n}_d^2 \rangle (t)/L$ in the half-filled Hubbard chain at $U = 20 t_h$ after the pulse irradiation $(t = 30/t_h)$ with different system size $L$. The results are calculated by the ED method under the PBC with $A_0 = 0.3$, $\sigma_p = 2/t_h$, and $t_0 = 10/t_h$ in $A(t)$.

The results responding to the results in Ref. [S6], the external pulse induces an enhancement of $n_d(t)$ and $\langle \hat{n}_d^2 \rangle (t)$. The inset of Fig. S.3 shows the time-dependent $\langle \hat{n}_d^2 \rangle (t)/L$ as
the function of \( n_d(t) \) at the same time \( t \). The \( \eta \)-pairing correlation \( \langle \hat{\eta}^2 \rangle(t)/L \) increases monotonically with the time-dependent \( n_d(t) \). Figure 3 in the main text corresponds to these results with the systematic \( L \) dependence study.

Because of the finite size effect, the optimal photoexcitation frequency weakly depends on the system size. Figure S.4 shows the frequency \( \omega_p \) dependence of \( \langle \hat{\eta}^2 \rangle(t) \) after pumping with different \( L \). The peaks of \( \langle \hat{\eta}^2 \rangle(t) \) are located at \( \omega_p \sim U \) but they have the system size dependence. With increasing \( L \), the highest peaks of \( \langle \hat{\eta}^2 \rangle(t) \) approach to \( \omega_p = U \). To discuss the finite-size effect systematically, we employ the optimal \( \omega_p \), at which the highest peak of \( \langle \hat{\eta}^2 \rangle(t) \) is located, for each system size \( L \).

In the main text, we show the time-dependent \( \langle \hat{\eta}^2 \rangle(t)/L \) as the function of \( n_d(t) \) at a fixed pump strength \( A_0 \). Equivalent results could be obtained by examining \( A_0 \) dependence of \( \langle \hat{\eta}^2 \rangle \) and \( n_d \) after pumping. Figure S.5 shows the time-averaged \( \langle \hat{\eta}^2 \rangle \) and \( n_d \) after pumping. The \( \eta \)-pairing correlation \( \langle \hat{\eta}^2 \rangle/L \) increases with \( n_d \) as a function of \( A_0 \). Figure S.6 summaries \( \langle \hat{\eta}^2 \rangle/L \) vs. \( n_d \) obtained by varying \( A_0 \), corresponding to the plot shown in the inset of Fig. S.5 but for different system sizes \( L \). Note that here we omit the data at \( n_d > 0.3 \) for better visibility. As in the case studied in Fig 3 in the main text, we find that \( \langle \hat{\eta}^2 \rangle/L \) in the photinduced state is much larger than \( \langle \hat{\eta}^2 \rangle/L = n_d \) (dashed line) expected for a thermal distribution of the eigenstates [see Eq. (S34)] and is enhanced with \( L \). This suggests the long- ranged \( \eta \)-pairing correlation in our optically driven system.

\[ \text{REFERENCES} \]

[S1] T. J. Park and J. Light, J. Chem. Phys. 85, 5870 (1986).
[S2] N. Mohankumar and S. M. Auerbach, Comput. Phys. Commun. 175, 473 (2006).
[S3] H. Hashimoto and S. Ishihara, Phys. Rev. B 93, 165133 (2016).
[S4] A. Nishino and T. Deguchi, Nucl. Phys. B 688, 266 (2004).
[S5] C. N. Yang, Phys. Rev. Lett. 63, 2144 (1989).
[S6] T. Kaneko, T. Shirakawa, S. Sorella, and S. Yunoki, Phys. Rev. Lett. 122, 077002 (2019).
[S7] R. Fujuchi, T. Kaneko, Y. Ohta, and S. Yunoki, Phys. Rev. B 100, 045121 (2019).
[S8] M. Takahashi, Prog. Theor. Phys. 46, 401 (1971).
[S9] F. H. L. Essler, V. E. Korepin, and K. Schoutens, Phys. Rev. Lett. 67, 3848 (1991).
[S10] F. H. Essler, V. E. Korepin, and K. Schoutens, Nucl. Phys. B 372, 559 (1992).