Transition from wakefield generation to soliton formation

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It is well known that when a short laser pulse propagates in an underdense plasma, it induces longitudinal plasma oscillations at the plasma frequency after the pulse, typically referred to as the wakefield. However, for plasma densities approaching the critical density wakefield generation is suppressed, and instead the EM-pulse undergoes nonlinear self-modulation. In this article we have studied the transition from the wakefield generation to formation of quasi-solitons as the plasma density is increased. For this purpose we have applied a one dimensional (1D) relativistic cold fluid model, which has also been compared with particle-in-cell simulations. A key result is that the energy loss of the EM-pulse due to wakefield generation has its maximum for a plasma density of the order 10 percent of the critical density, but that wakefield generation is sharply suppressed when the density is increased further.

I. INTRODUCTION

Wakefield generation is of fundamental interest in plasmas, both from a basic science point of view and when it comes to applications. Of particular interest is the laser wakefield acceleration scheme, which has shown tremendous progress since the pioneering work by Tajima and Dawson [1]. There are numerous experimental demonstration of electron acceleration to GeV of energies [2–4]. Apart from this, a number of different approaches for electron acceleration have been proposed and demonstrated experimentally by different research groups around the globe e.g. bubble regime [5, 6], beat wave acceleration [7, 8], self modulated laser wakefield acceleration [9], and many more. Furthermore, the effects of external magnetic fields [10], effects of modifying the laser chirp [11], and effects of varying the plasma density profile [12] on wakefield generation, have drawn considerable research interest and continue to do so.

When the electromagnetic pulses are long (as compared to the skin depth), wake field generation is suppressed, and instead other nonlinear phenomena becomes more pronounced. Typically in a non-magnetized plasma the nonlinearity is of a focusing type, which can allow for envelope bright solitons [13]. Soliton formation have different features depending on the dimensionality, and lot of interest has been devoted to the 2D [14] and 3D phenomena [15]. However, when the pulses are pancake shaped, the physical scenario can be described by a 1D-model [16] to a good approximation.

In the present paper we will study the competing mechanisms of linear dispersion, soliton formation and wavefield generation, and their dependence on wave amplitude and plasma density. For this purpose we apply an 1D relativistic cold fluid model. One of the main results in the present paper is that the peak in wakefield energy density occurs for densities around \( n \approx 0.1 n_c \), where \( n \) is the plasma density and \( n_c \) is the critical density, but this is followed by a very sharp decrease in the wake field energy for densities \( n \gtrsim 0.2 n_c \). Furthermore, the effect of varying the laser amplitude is studied. The results from the cold relativistic fluid model are compared with particle-in-cell (PIC) simulations, and the agreement is found to be excellent.

The organization of our paper is as follows. In section II the governing equations for 1D wave propagation are derived (based on a cold relativistic fluid model). Next in section III the basic equations are solved numerically, and our main results are presented. In section IV we make a comparison with a simplified theory and with 1D PIC simulations. Finally in Section V we make a summary and present the final conclusions.

II. THE COLD RELATIVISTIC FLUID MODEL

The main purpose of the present article is to study the propagation of an 1D electromagnetic pulse based on the cold relativistic fluid equations for electrons. Apart from treating the ions as immobile, no further approximations are made, and the equations will be solved numerically. The numerical results will be described in the next section. Before implementing the scheme, we first demonstrate that the number of basic equations needed can be reduced somewhat.

We assume all fields to depend on \((z,t)\) and divide the fields in the parallel and perpendicular direction to \(z\) (e.g. the velocity field is written as \( v = v \parallel + v \perp \) with \( v \parallel \equiv v_0 \)). Using the Coulomb gauge, we immediately obtain from Gauss’ law and the perpendicular component of Ampere’s laws

\[
\frac{\partial^2 \phi}{\partial z^2} = \frac{\epsilon (n_e - n_0)}{\epsilon_0}
\]

(1)

and

\[
\nabla^2 A_\perp - \frac{1}{c^2} \frac{\partial^2 A_\_perp}{\partial t^2} = \mu_0 n_e v_\perp.
\]

(2)

Here \( \epsilon \) is the elementary charge, \( n_e \) is the electron number density and \( n_0 \) represents the constant neutralizing ion background, \( \phi \) is the scalar potential and \( A_\perp \) is the perpendicular part of the vector potential. Noting that the Coulomb gauge
and the 1D geometry implies $A_z = 0$, the parallel component of Ampere’s law is written

$$\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t \partial z} = -\mu_0 e n_e v_c$$

(3)

For cold electrons, only the Lorentz force is needed in the electron equation of motion. Dividing the equation in its parallel and perpendicular components gives

$$\frac{d \mathbf{P}_z}{dt} = e \left[ \frac{\partial}{\partial t} + v_c \frac{\partial}{\partial z} \right] A_z = \frac{d}{dt}(eA_z)$$

(4)

and

$$\frac{d \mathbf{P}_\perp}{dt} = e \frac{\partial \phi}{\partial z} \mathbf{e}_z - e \left( v_c \frac{\partial A_z}{\partial z} \right) \mathbf{e}_z$$

(5)

where $\mathbf{P}$ is the momentum and $d/\partial t = \partial/\partial t + v_c \partial/\partial z$ is the total (convective) time derivative. Eq. (4) can be integrated to give

$$\mathbf{P}_\perp = eA_z \implies \mathbf{v}_\perp = eA_z / \gamma m_e$$

(6)

where $m_e$ is the electron mass and $\gamma = \sqrt{1 + P^2 / m_e c^2}$ is the relativistic gamma factor. Given (6) we can write Eq. (5) as

$$\frac{d \mathbf{P}_z}{dt} = \frac{d}{dt}(\gamma m_e v_c) = e \frac{\partial \phi}{\partial z} - \frac{e^2}{2 \gamma m_e} \frac{\partial A_z^2}{\partial z}$$

(7)

which can be further rewritten as

$$\frac{d v_c}{dt} = \frac{e}{\gamma m_e} \frac{\partial \phi}{\partial z} - \frac{e^2}{2 \gamma m_e} \frac{\partial A_z^2}{\partial z} - v_c \frac{d \gamma}{\gamma dt}$$

(8)

Noting that the rate of change of the energy is given by $d \mathcal{E} / dt = -e \mathbf{v} \cdot \mathbf{E}$, where $\mathcal{E} = \gamma m_e c^2$, we find

$$\frac{d \gamma}{dt} = \frac{e}{m_e c^2} \left( v_c \frac{\partial \phi}{\partial z} + \frac{\partial A_z^2}{\partial t} \right)$$

(9)

Combing (9) and (8) we then obtain

$$\frac{d v_c}{dt} = \frac{e}{\gamma m_e} \left( 1 - \frac{v_c^2}{c^2} \right) \frac{\partial \phi}{\partial z} - \frac{\partial A_z^2}{\partial z} \left( v_c \frac{\partial A_z^2}{\partial t} \right)$$

(10)

Next we want want to rewrite $\gamma$ as a function of $v_c$ and $A_z^2$.

Using $\gamma = 1 / \sqrt{1 - (v_c^2 + A_z^2)/c^2}$ and $v_c^2 = (eA_z / \gamma m_e)^2$ we deduce the expression

$$\gamma = \sqrt{\frac{1 + (eA_z / m_e c)^2}{1 - \frac{v_c^2}{c^2}}}$$

(11)

Our system is completed with the electron continuity equation

$$\frac{\partial n_e}{\partial t} + \frac{\partial}{\partial z}(n_e v_c) = 0$$

(12)

Together Eqs. (2), (3), (8), (11) and (12) constitute a closed set.

Next we introduce the normalizations $a = eA_z / m_e c$ and $\varphi = e \phi / m_e c^2$ for the vector and scalar potential respectively. Furthermore, time and space are normalized against the laser frequency and wave number $(\omega t \rightarrow t$ and $kx \rightarrow x$) respectively. The parallel velocity is normalized against the speed of light, $\beta = v_c / c$, and finally the electron density is normalized against the critical density $n_c = n(0) / \omega^2 m_e / e^2$ (from now on $n_c$ represent $n_e / n_c$). By using these normalization our basic equations are written as:

$$\frac{\partial^2 \mathbf{a}}{\partial z^2} - \frac{\partial^2 \mathbf{a}}{\partial t^2} = n_e \frac{\mathbf{a}}{\gamma}$$

(13)

$$\frac{d \beta}{dt} = \frac{(1 - \beta^2)}{\gamma} \frac{\partial \varphi}{\partial z} - \frac{1}{2 \gamma^2} \left( \frac{\partial \mathbf{a}^2}{\partial z} + \beta \frac{\partial \mathbf{a}^2}{\partial t} \right)$$

(14)

$$\frac{\partial n_e}{\partial t} + \frac{\partial}{\partial z} \left( n_e \beta \right) = 0$$

(15)

$$\gamma = \sqrt{1 + \frac{\alpha^2}{1 - \beta^2}}$$

(16)

$$\frac{\partial^2 \mathbf{\varphi}}{\partial t^2} = -n_e \beta$$

(17)

Equations (13)-(17) constitute the basis for the result presented below.

FIG. 1. The spatial profiles of longitudinal field are plotted at $t = 159$ for the case when a 5 cycle laser pulse with $a_0 = 0.01$ interacts with a plasma having density $0.001 n_c$ (a) and $0.1 n_c$ (b). The corresponding spatial profiles of the transverse field (EM driver) is also presented in (c).
The result for the longitudinal fields is quite dramatic, as shown in Fig. 1 (c) fields as evaluated at \( t = 109, 409 \), 749 for the cases when an unperturbed plasma having density 0.1 \( n_c \), and 0.3 \( n_c \) is considered, and the spatial profiles of longitudinal (Fig. 1 a,b) and transverse (Fig. 1 c) fields as evaluated at \( t = 109 \) are shown.

For the lower density we see that the longitudinal field has a pure wakefield nature, i.e. harmonic oscillations are induced after the peak of the EM-pulse. Moreover, since dispersive effects are very small for such a low density, the wakefield generation continues more or less unchanged for a long time. By contrast, for a density of 0.1 \( n_c \), the longitudinal field still have a pronounced wakefield, but there is also a strong peak where the EM-pulse is localized. Furthermore, for the low amplitude \( a_0 = 0.01 \) the wake field amplitude is largest directly after the pulse entrance, and then the wake field amplitude is gradually decreasing. The reason for the diminishing wakefield amplitude is the EM-wave dispersion, which become significant for densities of order 0.1 \( n_c \) and higher. This can be verified by studying Fig 2. In Fig 2(d) we see the EM-wave profile for \( a_0 = 0.01 \) at \( t = 109, 409, 749 \) (note the pulse broadening). As a result of the pulse broadening, the wakefield generation is decreasing, as can be seen from the longitudinal field displayed in Fig 2(a).

When the amplitude is increased, the nonlinearity begins to counteract dispersion, which means that wakefield generation can be sustained. We compare the EM-wave profile for \( a_0 = 0.01 \), \( a_0 = 0.1 \) and \( a_0 = 0.3 \) at \( t = 109, 409, 749 \) in the right panel of Fig 2. While there is some tendency to decreased dispersion for \( a_0 = 0.1 \), the main suppression of dispersion occurs when increasing the amplitude up to \( a_0 = 0.3 \). The result for the longitudinal fields is quite dramatic, as

\[
\begin{align*}
\alpha(0, t) &= a_0 \exp \left( -\frac{4 \log(2) r^2}{\tau_{\text{fwhm}}} \right) \cos(t) \hat{x} \\
n_c(0, t) &= n_0 \\
\beta(0, t) &= \phi'(0, t) = 0 \quad (\phi' = \partial \phi/\partial z) 
\end{align*}
\]
The transition from wake field generation to soliton formation

In Fig. 3, the spatial profile of longitudinal field $E_z = -\phi'$ for different plasma densities calculated at $t = 263$ (≈ 42 cycles) is presented by solving Eqs. (13)-(17) numerically for $a_0 = 0.3$. Varying the density from $n = 0.001n_c$ up to $n = 0.6n_c$ we note that the longitudinal field is a pronounced wakefield for the lower densities, but gradually turns into a driven field that is localized to the same region as the EM field that drives the perturbation. As we increase the initial plasma density, the wakefield amplitude decreases and eventually a soliton like structure propagates in the plasma.

The time evolution of the transverse pulse profile for $a_0 = 0.3$ and $n_c = 0.6n_c$ is presented in Fig. 4(b), along with the corresponding longitudinal field Fig. 4(a). As we can see, the nonlinearity prevents dispersion for most of the energy contained in the pulse, and the central part of the pulse tends to shorten over time. As a result the longitudinal field driven by the transverse part tends to increase somewhat over time. However, we also see that the high- and low-frequency parts of the pulse spectrum tend to irradiate forwards and backwards respectively. Apart from this irradiation, which represent a relatively minor energy loss of the central part of the EM-pulse, the pulse profile approaches a more or less fixed shape. We will refer to these structures as “quasi-solitons”. We cannot exclude that true soliton-formation eventually occur, but for the rather long time-span that we follow, we see a process where parts of the frequency spectrum is irradiated backwards and forwards. To give a more clear view of the longitudinal pulse profile, we present a zoom of the longitudinal fields for three different times in Fig. 4(c).

C. Energy loss as a function of density

For a density much less than the critical density, wakefield generation is the dominant process, as seen e.g. in Fig 3(a). However, due to the limited number of particles, the longitudinal field cannot store a very high energy density, and thus the energy loss for the transverse degrees of freedom (= the electromagnetic pulse) is limited. For a high plasma density (i.e. not much smaller than the critical density), wakefield generation is almost completely suppressed, and there is negligible energy loss to the longitudinal degrees of freedom, see Fig 3(d). These simple observations imply that there is an intermediate plasma density where the energy loss due to wakefield generation has its maximum. The purpose of this subsection is to determine this characteristic value.

As we have seen in Fig 2(a,b), for the higher plasma densities wakefield generation will be suppressed due to dispersion, unless the amplitude is strong enough to keep the pulse short enough, as in Fig 2(c). To avoid dispersive suppression we will consider pulse amplitudes firmly in the nonlinear (= non-dispersive) regime and pick $a_0 = 0.3$. We are interested in the dependence of the wakefield energy as a function of density.

FIG. 3. Fluid simulation: The wakefield for different plasma densities $0.001n_c$, $0.04n_c$, $0.4n_c$, $0.43n_c$ and $0.6n_c$ are presented as calculated at $t = 263$. The laser parameters are $a_0 = 0.3$, 5 cycle (FWHM) Gaussian linearly polarized laser pulse, incident on the plasma slab of $275\lambda$.
(rather than the absolute value), and therefore we have used
the integral \( \int_{z_0}^{z_1} |E_z|^2 \, dz \) as a measure. Here \( z_0 \) and \( z_1 \) are \( 20\lambda \)
and \( 10\lambda \) behind the peak of the laser computed at \( t = 263 \) for
a given plasma density. Note that the wakefield will divide its
energy equally between kinetic energy and electrostatic field
energy, unless we are in an extremely nonlinear regime. The
generated wakefield energy as a function of the plasma den-
sity is presented in Fig. 5. Initially the wakefield energy
increases with density, as expected from standard theory of
wakefield generation. The maximum peak of the wakefield
energy is reached for a density of the order \( 0.1n_e \). However, if
we further increase the plasmas density beyond approximately
\( n_e \sim 0.2n_e \), there is rapid drop in the wakefield energy. The
drop in wakefield energy coincides with the formation of soli-
tary pulses which we label quasi-solitons, as discussed in the
previous sub-section.

IV. COMPARISON WITH THEORY AND
PIC-SIMULATIONS

A first theoretical estimate of the propagation velocity for
the EM-pulses can be found by computing the group velocity
including a nonlinear correction for the relativistic factor in-
verted by the EM field. This suggests that the expression from linearized
theory can be substituted as
\[
\gamma = \frac{1}{\sqrt{1 - \frac{\omega_p^2}{\omega^2}}}
\]
To get a first estimate of the amplitude dependence, we can
replace the unperturbed plasma frequency with the value cor-
rected by the gamma factor due the transverse motion in the
EM-field. This suggests that the expression from linearized
time can be substituted as
\[
ge \rightarrow ne^2/\varepsilon_0m_e - ne^2/\varepsilon_0\gamma m_e
\]
where we let \( \gamma = \sqrt{1 + \alpha_0^2} \), where we use the normalized peak vector
potential. Using the peak potential overestimates the average
gamma factor, but ignoring the contribution from the longi-
tudinal motion underestimates it. Hence this may serve as a
rough estimate for the propagation velocity. The expression
which we will compare with the results from the numerical

FIG. 4. Spatiotemporal evolution of electrostatic (a), electromagnetic
(b) fields with 5 cycle laser with amplitude \( a_0 = 0.3 \) and \( n_e = 0.6n_e \).
Enlarged version of the solitonic structures is also presented sepa-
rately (c).

FIG. 5. The wakefield energy is calculated as \( \int_{z_0}^{z_1} |E_z|^2 \, dz \). Here, \( z_0 \)
and \( z_1 \) are \( 20\lambda \) and \( 10\lambda \) behind the peak of the laser at \( t = 263 \) for
different plasma densities.

FIG. 6. A comparison of the group velocity \( v_g \), the line fitted by
Eq. 22, and the propagation velocity \( v_p \), the dots computed from the
position of the central peak of the EM field, as a function of
plasma density. Here the case of a 5 cycle EM pulse with \( a_0 = 0.3 \) is
considered.
space and time are taken in units of laser wavelength (a of density for the case of
agrees very well with the expression (22), as we can see in
based on the position of the central peak of the EM-field)
octave
the propagation velocity computed by the fluid code (simply
In spite of the rather crude estimates involved, we see that
the wakefield generation. For the results presented here the
[17] is carried out to study the effect of plasma densities on
simulations. The 1D Particle-In-Cell simulation (LPIC++)
[265]. The laser parameters are \( a_0 = 0.3 \), 5 cycle (FWHM) Gaussian linearly polarized laser pulse, incident on the plasma slab of 275\( \lambda \).

\[ v_c = c \left( \frac{1 - \frac{n_c}{n_0}}{n_c \sqrt{1 + a_0^2}} \right) \]  \quad \text{(22)}

In the present paper we have derived a cold, fully relativistic 1D model, that we have solved numerically. The purpose has been to study the competition between linear dispersion, nonlinear self-modulation and wakefield generation for a localized electromagnetic pulse propagating in a homogeneous non-magnetized plasma. First we have deduced that wakefield generation for initially short pulses is suppressed by linear dispersion after a relatively short time, if the plasma density is modest or high (i.e. of the order \( 0.1n_c \) or higher) unless the normalized amplitude is of the order \( a_0 \approx 0.3 \). In case we are firmly in the nonlinear regime, the nonlinear self-modulation keep the pulse short enough to sustain wakefield generation for a long time - until depletion due to the wakefield generation becomes significant. When the plasma density is increased further, wakefield generation is replaced by a formation of quasi-solitons. The central part of the pulse propagates with very little changes of the pulse shape, but a small irradiation backward and forward of the low-frequency and high-frequency part of the spectrum still occur. A key part of the present study is the scaling with the plasma density of the energy loss of the EM-pulse due to wakefield generation. For pulses in the nonlinear regime (not suffering dispersive broadening), we find that the energy loss has a maximum when the plasma density is of the order \( n \approx 0.1n_c \). When the plasma density is increased beyond \( n \approx 0.2n_c \) there is a rapid decrease of the wakefield energy with the plasma density, signalling the onset of (quasi) soliton formation. The results produced by the fluid code has been checked against 1D PIC simulations, and the agreement has been found to be excellent.

Much of the conclusions in the present paper has a rather general nature, as the competition between dispersion, wake-
field generation and self-modulation may take place in a magnetized plasma, and also for different types of wave modes. Qualitatively many of the features of the present study can be assumed to hold in a more general context, as long as the nonlinear self-modulation is of focusing type. In particular wakefield generation is more prominent for short wavepackets (which may be of electromagnetic or electrostatic nature), and are likely to be suppressed for low and high plasma densities (for a given wave frequency). Thus the existence of a plasma density that optimizes wakefield generation is likely a generic feature. To what extent the conclusions reported here holds also for electromagnetic waves propagating in a magnetized plasma remains an issue for further research.

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