THE OPERAD QUAD IS KOSZUL

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ABSTRACT. The purpose of this paper is to prove the koszulity of the operad Quad, governing quadri-algebras. That Quad is Koszul was conjectured by Aguiar and Loday in [1], where it was introduced. The operad Dend, governing dendriform algebras, is known to be Koszul, [4], and Quad is its second black square power. We find a new complex, based on the associahedron, which captures the structure of Dend. This complex behaves well with respect to the black squaring process, and allows us to conclude. Also, this proves koszulity of higher black square powers of Dend.

1. Introduction

Given two quadratic binary operads P and Q, there is a new quadratic binary operad P ■ Q defined by taking pairs of operations from P and Q, and imposing pairs of relations. For properties of this operation, see Ebrahimi-Fard and Guo, [2]. Consider for example the operad Dend, governing dendriform algebras. It was defined by Loday in [5]. We can form

Quad = Dend ■ Dend.

This operad governs quadri-algebras, and was introduced by Aguiar and Loday in [1]. The operation ■ is mimicked on Manin’s black dot operation on quadratic algebras. For operads, the question of how this operation relates to koszulness, is far from being understood. We will show that in the present case, koszulity of Quad can be deduced from the koszulity of Dend. We work over an algebraically closed field k of characteristic zero throughout this paper.

We begin by formulating the koszulity condition in a useful form. We use the operadic bar construction for this.

Then we investigate Dend more closely. It has two binary operations < and >, and three quadratic relations. This data shows that the operation * = < + > is associative. So we have an associative operation that splits in two, and the associativity axiom splits in three. On the next level, we find that the associahedron A_4 splits into four parts. Similarly, the associahedron A_n splits into n parts. We then see that the sum of n copies of the chain complex of the associahedron is a direct summand of the operadic (dual) bar complex for Dend. Both these complexes are acyclic; the chain complex since it comes from a polytope, and the bar complex since Dend is known to be Koszul. We can choose homotopy equivalences between the complexes.
Now the operad Quad has 4 binary operations, and nine quadratic relations. The sum of these four operations is an associative operation; the associativity axiom is exactly the sum of the nine quadratic relations. It then turns out that the associahedron $A_4$ splits into sixteen parts, and in general $A_n$ splits into $n^2$ parts. This proves that the dimension of $Quad^1(n)$ is at least $n^2$, and we know the opposite inequality from [1]. Collecting things together, we can now use pairs of homotopy equivalences from $Dend$ to show that the sum of $n^2$ copies of the chain complex of the associahedron is homotopy equivalent to the operadic bar complex of Quad. Since the chain complex is acyclic, the bar complex is acyclic, and Quad is Koszul. During the proof, we also verify the numerical conjecture from [1].

Instead of $Quad = Dend^{\bullet^2}$, we can of course also consider $Dend^{\bullet^m}$. The case $m = 3$ has been studied by Leroux [3], under the name of octo-algebras. The same proof as for $m = 2$ shows that this operad, for all $m$, is Koszul, modulo a generalization of a lemma from [1]. We prove this lemma in the last section.

Since all operads in this paper come from non-symmetric operads, the symmetric group action will be suppressed throughout. To get the true operads from what is written here, tensor each algebraic construction by $k[\Sigma_n]$ in degree $n$.

2. Koszulity for operads

The operads we consider in this paper are of the following special form: they are generated by a finite number of binary operations. Their relations are quadratic, and take the form

$$(x \circ_1 y) \circ_2 z = x \circ_3 (y \circ_4 z)$$

where $\circ_i$ are binary operations. It will be convenient to think of such a relation as a directed edge between labelled trees

![Diagram of labelled trees](image)

**Remark 2.1.** All trees considered here will have a finite number of leaves at the top (3 in the two trees above), and some vertices below the leaves (2 in the examples above). Each vertex will have a number ($\geq 2$) of incoming edges. The root is the lowest vertex (the roots are labelled by $\circ_2$ and $\circ_3$ above). Each vertex apart from the root has a unique outgoing edge. The trees will have labels at the vertices; these correspond to operations. Labels at the leaves correspond to inputs. Since all the relations we will consider have all the inputs in the same order, the labels at the leaves will be suppressed throughout. An edge between vertices will be referred to as an internal edge, to distinguish it from a leaf.

We dub the space of binary operations $\Omega_P$, and the space of relations $\Lambda_P$, following the notation from [2]. Note that $\Lambda_P \subset \Omega_P^2 \oplus \Omega_P^{\circ2}$. We write $P = P(\Omega_P, \Lambda_P)$. 
Now we can define the squaring operation, still following [2] (this has also been considered in [6]):

**Definition 2.2.** The **black square product** of two operads $\mathcal{P} = \mathcal{P}(\Omega_P, \Lambda_P)$ and $\mathcal{Q} = \mathcal{P}(\Omega_Q, \Lambda_Q)$ is

$$\mathcal{P} \square \mathcal{Q} = \mathcal{P} (\Omega_P \otimes \Omega_Q, S_{23}(\Lambda_P \otimes \Lambda_Q)).$$

The operator $S_{23}$ simply switches tensor factors, so that the relations come at the right place.

We also need the quadratic dual algebra $\mathcal{P}^! = \mathcal{P}(\Omega_P^\vee, \Lambda_P^\perp)$. Here $\Omega_P^\vee$ is the linear dual tensored with the sign representation (since we suppress the symmetric group action, this merely involves a sign in the pairing), and the perpendicular is with respect to a natural pairing; see e.g. Loday [4] for details.

To an operad we can associate its bar complex; this is basically the free operad on the linear dual of the operad. It is a dg operad; for a quadratic operad, its zeroeth homology is the quadratic dual. See Markl, Snider and Stasheff [7] for details, including the grading convention. Since it will be important for us, we will give the explicit structure of this construction for the operad $\text{Dend}$ (actually its dual) later. As part of the proof of the main theorem, we will also find in explicit form the (dual) bar complex of the operad $\text{Quad}$.

**Definition 2.3.** An operad (quadratic, binary) is called **Koszul** if the bar complex is a resolution of the dual operad.

### 3. The Higher Degree Structure of $\text{Dend}$; Splitting the Associahedron

We begin by writing out the dual bar complex of the operad $\text{Dend}$ (i.e. the bar complex of the dual operad $\text{Dias} = \text{Dend}^!$). This operad, introduced in [5], governs dialgebras. Then we make a complex out of the associahedron, and finally we link these two together.

#### 3.1. The dual bar complex of $\text{Dend}$.

**Definition 3.1.** The operad $\text{Dend}$ is generated by two binary operations $\prec$ and $\succ$ satisfying three axioms. In the language of trees, the axioms can be written (let $\ast = \prec + \succ$)
Note that the sum of these three relations is the associativity of $\ast$.

**Lemma 3.2.**

(i) The dual bar complex $\tilde{D} = D(Dend^l) = D(Dias)$ has the following graded parts:

\begin{align*}
D_2^0 & \leftarrow 0 \\
D_3^0 & \leftarrow D_3^{-1} \leftarrow 0 \\
D_4^0 & \leftarrow D_4^{-1} \leftarrow D_4^{-2} \leftarrow 0 \\
& \vdots \\
D_n^0 & \leftarrow \ldots \leftarrow D_n^{-n+3} \leftarrow D_n^{-n+2} \leftarrow 0 \\
& \vdots 
\end{align*}

The piece $D_i^j$ has basis given by labelled trees with $i$ leaves and $i + j - 1$ vertices, with $l$ choices of labels for each vertex with $l$ incoming edges.

(ii) The zeroeth homology of $\tilde{D}$ is the operad $Dend$, the higher homology vanishes.

**Proof.** By definition, $\tilde{D}$ is the free operad construction on the twisted linear dual of $Dias$; this means that the piece $D_i^j$ is given by trees with $i$ leaves and $i + j - 1$ vertices, where each vertex with $l$ incoming edges is labelled by an element of a vector space of the same dimension as $Dias(l)$, see [5]. This space has dimension $l$ (see [5]).

The differential of such a labelled tree $T$ can be understood inductively. First, an unlabelled tree $T'$ with one vertex less than $T$ has a labelling which appears with non-zero coefficient in the differential of $T$ if and only if $T$ is the result of contracting an internal edge of $T'$. In this case, there is a unique labelling with this property. For all vertices except the two vertices of the contracted edge, the labelling is unchanged. Say that the vertex of $T$ that is the image of the contracted edge has $l$ incoming edges. Then the labelling is induced from the map $D_i^{-l+3} \leftarrow D_i^{-l+2}$. The description of this map depends on an explicit description of the basis, and will be given during the proof of Proposition 3.9.

The second part is the definition of koszulity as in Definition[2.3] $Dend$ (and thus $Dias$) is Koszul by [5].

$\square$
**Definition 3.3.** The augmented dual bar complex of $\mathcal{D}_{\text{end}}$ is the complex $\mathcal{D}$ which is equal to $\tilde{\mathcal{D}}$ in non-positive degrees, but is augmented by $\mathcal{D}_{\text{end}}$ in degree $+1$. It is thus exact everywhere.

3.2. **The associahedron.** Let $\mathcal{A}_n$ be the associahedron for $n$ inputs. This is a polytopal cell complex of dimension $n - 2$; in particular, its (augmented) chain complex $\mathcal{C}A_n$ is exact. We grade it by minus the dimension of the cells, so it has the explicit form

\[
\begin{align*}
  n = 2 & : \mathcal{C}A_2^1 \leftarrow \mathcal{C}A_2^0 \leftarrow 0 \\
  n = 3 & : \mathcal{C}A_3^1 \leftarrow \mathcal{C}A_3^0 \leftarrow \mathcal{C}A_3^{-1} \leftarrow 0 \\
  \cdots \\
  n : & \quad \mathcal{C}A_n^1 \leftarrow \mathcal{C}A_n^0 \leftarrow \cdots \leftarrow \mathcal{C}A_n^{-n+2} \leftarrow 0
\end{align*}
\]

A basis for $\mathcal{C}A_i^j$ is given by trees with $j$ leaves and $i + j - 1$ vertices (except for the case $j = 1$, where the generator is represented by the empty cell). The differential of a tree $T$ has nonzero coefficient in a tree $T'$, with one vertex less, if and only if $T$ is the result of contracting an internal edge of $T'$. The coefficient is then $\pm 1$, depending on the choice of orientation of the associahedron.

**Remark 3.4.** $\mathcal{C}A$ is the bar complex of the operad $\mathcal{A}_{\text{ss}}$ governing associative algebras. Its exactness is equivalent to the koszulity of $\mathcal{A}_{\text{ss}}$.

Later on, it will be convenient for us to label each vertex of each tree in the chosen basis for this chain complex by $\ast$.

**Example 3.5 ($\mathcal{A}_4$).** The associahedron for four inputs is a pentagon:
If we use the orientation as shown, and take ordered bases for the chain complex by starting in the upper left corner and going counter-clockwise, we get the following explicit complex $CA_4$:

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 1 \\
-1 & 1 & 0 & 0 & 0 \\
0 & -1 & -1 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 & -1
\end{pmatrix}
\begin{pmatrix}
-1 \\
-1 \\
1 \\
1 \\
1
\end{pmatrix}
$$

3.3. **Splitting the associahedron.** The associahedron for two inputs is just a point, labelled with $\ast$. We will regard the two operations of $Dend$ as a splitting of this associahedron into two parts:

$$\ast = \ast + \ast$$

Similarly, the associahedron $A_3$ is

$$\ast \quad \ast \quad \ast$$

which splits as the sum of the three relations of $Dend$. We dub the three relations of $Dend$

$$1 = \ast + \ast + \ast$$

$$2 = \ast + \ast + \ast$$

$$3 = \ast + \ast + \ast$$

The fact that the sum of these three relations is the associativity condition can be written as

$$1 + 2 + 3 = \ast$$

The chain complex $CA_3$ splits in the same way:
Proposition 3.6. The chain complex $CA_n$ splits as a sum of $n$ chain complexes constructed from $Dend$. These can be labelled from 1 to $n$ by considering the labels of the tree

The first copy has labels $\prec, \cdots, \prec$ on this tree (from top left), the second has $\succ, \prec, \cdots, \prec$, the third has $\ast, \succ, \prec, \cdots, \prec$ and so on. The two last have labels $\ast, \cdots, \ast, \succ$, and $\ast, \cdots, \ast, \succ$.

Remark 3.7. In particular, the binary operations are relabelled as

Proof. We choose once and for all an orientation of each associahedron, and an induced orientation on all cells, which we will use consistently for each copy of it. We do this so that all edges are oriented as

Each edge is exactly represented by this move inside a larger tree.

The proof starts with describing what happens to the labels of each tree for the $n$ copies. The sum over each tree gives the label of the associahedron. Then we do the same for the differentials. This part is inductive. Note first that the proposition is consistent with what we have seen already for $n = 2, 3$. Thus the start of the induction is taken care of.

Let $T$ be a tree. For each leaf, there is a unique path running downwards from the leaf to the root. The leftmost branch of the tree is for instance the path from the leftmost leaf to the root. In the first copy, label all vertices along this leg by 1, and all remaining vertices by $\ast$. Given two neighbouring leaves, say number $i$ and $i + 1$ from the left, there is a unique shortest path connecting them. We go from copy number $i$ to copy number $i + 1$ by changing the label of each vertex along this path by the following rule: If the label at a vertex is an integer $r$, which is less than the number of incoming edges, increase it by one. If the label is equal to the number of incoming edges, replace it by $\ast$. If the label is $\ast$, replace it by 1. In this way we
finally end up with a tree where all the vertices along the rightmost leg is labelled by the number of incoming edges, all other vertices are labelled by $\ast$. For instance,

![Tree Diagram]

It is now obvious that the sum over all these labellings give the labels $\ast$ at each spot, e.g. by induction (remove the root, and look at the forest of smaller trees that remains).

As for the differential, we need to see what happens if the tree $T$ comes from the tree $T'$ by contracting an internal edge, and the labels are as prescribed at the $i$th level for both of them. It is enough to consider this in the case that $T$ has only one vertex, and $T'$ two. If we forget about the labels, this means that $T$ represents the big cell of the associahedron, whereas $T'$ represents a facet. By our choice of orientation, the differential of the tree $T$ has coefficient $\pm 1$ on $T'$ in $CA_n$. In each copy of our new complex, we use the same coefficient. Then, when we sum over the $n$ copies of the associahedron, we get that the differential of $CA_n$ is the sum of the differential of the copies. This concludes the proof. 

\[\square\]

**Definition 3.8.** We use the notation $\mathcal{D}A_n$ for the direct sum of the $n$ copies of the chain complex of the associahedron constructed in the proposition.

**Proposition 3.9.** There is a map from $\mathcal{D}A_n$ to the augmented dual bar complex $\mathcal{D}_n$, which identifies $\mathcal{D}A_n$ with a direct summand of $\mathcal{D}_n$. Each basis element of $\mathcal{D}_n$ appears with non-zero coefficient in the image of exactly one basis element of $\mathcal{D}A_n$, where the coefficient is 1.

**Proof.** The map from $\mathcal{D}A_n$ to $\mathcal{D}_n$ is an isomorphism in degree $-n+2$, where the two parts have the same dimension. This gives us a choice of basis for $\mathcal{D}_n$ as explained in the proof of Lemma 3.2: a tree $T$ with labels on each vertex running from 1 to the number of incoming edges. This labelling corresponds to the ordering from Proposition 3.6. The map from $\mathcal{D}A_n$ to $\mathcal{D}_n$ is given in general non-positive degrees by sending a labelled tree $T$ to the tree with the same labels in $\mathcal{D}$, understood as the sum where we split each label $\ast$ into the sum of the labels from 1 to the number of incoming edges of the vertex. The differential of $T$ splits in the same way; for each labelled tree $T'$ which appears with non-zero coefficient in the differential of $T$, this
coefficient is repeated in the differential as many times as there are summands of $T'$. We extend the map to degree 1 by taking the induced map on cokernels. Since all the coefficient of the inclusion map are 0 or 1, we may choose a projection. Now the statement of the proposition is clear. \hfill \square

Remark 3.10. Note that the description of the differential in $D$, using the explicit basis for $D_n^{n+2}$ given by the isomorphism with $DA_n^{n+2}$, fulfills the remaining part of the proof of Lemma 3.2.

Definition 3.11. We define a few maps relating these two complexes: first, write $d_{D_{end}}$ for the differential in $D$. Let the inclusion of the summand be $f_{D_{end}}$, the projection $p_{D_{end}}$. Then we let $h_{D_{end}}$ be a homotopy between $p_{D_{end}} : D \to D$ and the identity map $I_{D_{end}}$ on $D$; this exists since it is projection on a direct summand, and both complexes are split exact. The homotopy equivalence then takes the form

$$I_{D_{end}} - p_{D_{end}} = d_{D_{end}}h_{D_{end}} + h_{D_{end}}d_{D_{end}}$$

We will use these maps to construct similar maps for $Quad$ in the next section.

Remark 3.12. The chain complex we have constructed is the chain complex of the disjoint union of a number of copies of the associahedron, provided that we include one “empty cell” for each copy.

4. Koszulity of $Quad$

Recall that

$$Quad = D_{end} \boxtimes D_{end}$$

by definition. Using the column notation from [2], this can be written explicitly as follows: There are four binary operations

$$\left[\prec\right], \left[\prec\succ\right], \left[\succ\prec\right] \text{ and } \left[\succ\succ\right].$$

These satisfy nine relations, which are pairs of the relations from $D_{end}$. We label them

$$\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array}$$

where $i$ and $j$ run from 1 to 3. For instance,

$$\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array}$$

This section is devoted to proving the main theorem of this paper:

Theorem 4.1. The operad $Quad$ is Koszul.
The proof proceeds by mimicking the constructions we have made for $Dend$ earlier, but reversing the final implication.

**Proposition 4.2.** The chain complex of the associahedron $CA_n$ splits as a sum of $n^2$ chain complexes constructed from $Quad$. These can be labelled by indexes $i, j$ running from 1 to $n$, where the tree

![Diagram](https://example.com/diagram.png)

has pairs of labels as in Proposition 3.6.

The proof is exactly as for Proposition 3.6.

**Definition 4.3.** The complex constructed from $n^2$ copies of the associahedron is denoted by $QA_n$; it is exact everywhere.

**Definition 4.4.** The dual bar complex of $Quad$ is denoted by $\tilde{E}$, the augmented version is denoted by $E$.

**Proposition 4.5.**

(a) There is a map from $QA_n$ to $E$, identifying $QA_n$ with a direct summand. Each basis element of $\tilde{E}$ appears with non-zero coefficient in the image of exactly one basis element of $QA$, where the coefficient is 1.

(b) The dimension of $Quad'(n)$ is $n^2$.

The basis for $\tilde{E}$ will be constructed in the proof.

**Proof.** We will proceed by induction on $n$, the cases $n = 2, 3$ being trivial (for both parts of the proposition). Now by induction we have the following diagram:

![Diagram](https://example.com/diagram.png)

Each vertical arrow is the inclusion of a direct summand. Now the induced map from $QA_{n}^{-n+2}$ to $E_{n}^{-n+2}$ is clearly injective. In particular, $\dim E_{n}^{-n+2} \geq n^2$. The opposite equality is Lemma 4.6. Thus this map is an isomorphism. We use this isomorphism to choose basis for $E_{n}^{-n+2}$. In particular, it is the inclusion of a direct summand. The induced map on the left hand side is a cokernel of an inclusion of a direct summand; all in all, the complex $QA_n$ is a direct summand of $E_n$.

By the choice of basis, we see that for each tree $T$, the set of labellings for $T$ giving basis elements of $E_n$ is the tensor square of the same for $D_n$. The analogous statement is obviously true for $QA$ and $DA$, and the map from $QA$ to $E$ is locally, i.e. for each tree, the tensor square of the corresponding map from $DA$ to $D$. In particular, each basis element of $E_n$ appears with nonzero coefficient in the image of a unique basis element in $QA$, and the coefficient is 1. \qed
**Lemma 4.6** (Aguiar-Loday). *The dimension of $Quad^1(n)$ is $\leq n^2$.*

This is taken from [1].

**Proof of Theorem 4.1.** Since we know that the basis elements of $QA$ and $E$ are given by pairs of basis elements of $DA$ and $D$, respectively, and that the maps respect this, we can simply form pairs of homotopies as well. Explicitly, using the notation from Definition 3.11, we get that the map from $DA$ to $D$, and the projection onto the summand, and the choice of homotopy are

$$f_{Quad} = \begin{bmatrix} f_{Dend} \\ f_{Dend} \end{bmatrix}, \quad p_{Quad} = \begin{bmatrix} p_{Dend} \\ p_{Dend} \end{bmatrix}, \quad h_{Quad} = \begin{bmatrix} h_{Dend} \\ h_{Dend} \end{bmatrix}$$

Since also

$$d_{Quad} = \begin{bmatrix} d_{Dend} \\ d_{Dend} \end{bmatrix}$$

we get the homotopy relation

$$I_{Quad} - p_{Quad} = d_{Quad}h_{Quad} + h_{Quad}d_{Quad}$$

from the corresponding relation for $Dend$ used twice. Thus the dual bar complex is homotopic to an exact complex, and is therefore itself exact. This proved that $Quad$ is Koszul.

\[\square\]

**Corollary 4.7.** *The dimension of $Quad(n)$ is $d_n$, where*

$$d_n = \frac{1}{n} \sum_{j=n}^{2n-1} \left( \binom{3n}{n+1+j}(j-1) \right)$$

This follows from the numerical data for the dual, and the koszulity; see [1].

5. **Generalization**

If we consider $Dend^m$ for general $m$, the proof of the main theorem goes through modulo the generalization of Lemma 4.6. The aim of this section is to prove this generalization.

**Theorem 5.1.** *For each $m \geq 1$, the operad*

$$Dend^m = Dend \cdot \cdot \cdot Dend$$

*is Koszul.*

Operations in $Dend^m$ are represented by $m$-tuples of operations in $Dend$, relations by $m$-tuples of relations, and so forth. The generalizations of Proposition 4.2 and Proposition 4.5 go through with the same proof, as does the concluding proof of the theorem; we only need to check the generalization of Lemma 4.6. We need a convention about the combinatorial structures that will appear in the proof, summarized as
Notation 5.2. We consider an \( m \)-dimensional hypercube, which is composed of \( 3^m \) unit hypercubes. Each of the constituent unit cubes has coordinates; an \( m \)-tuple of elements from \( \{1, 2, 3\} \). The corners are the unit cubes where no coordinate is equal to 2. \((3, 3, \ldots, 3)\) is the cube with highest coordinates. For each subset of \( \{1, \ldots, m\} \), say with \( j \) elements, there is a subcube of dimension \( j \) where we only use the coordinates in the subset, and set all other coordinates to 1.

Lemma 5.3. The quadratic dual operad \((Dend_\boxdot^m)'\) satisfies

\[
\dim(Dend_\boxdot^m)'(n) \leq n^m
\]

Proof. The proof is also a direct generalization of Aguiar-Loday’s proof of Lemma 4.6, see [1].

We denote the operations in the dual operad by the same symbols as we denote the operations in the original operad, that is as \( m \)-tuples of linear combinations of \( \prec \) and \( \succ \). So this is a space of dimension \( 2^m \). We choose representatives in degree three, one for each relation in \( Dend_\boxdot^m \), that is one for each \( m \)-tuple of relations for \( Dend \). These form a space of dimension \( 3^m \). We label the relations as column vectors, where each element is 1, 2 or 3 (as with the labelling in \( Dend \) from Section 3.3). Then we choose representatives as follows: For each relation with no element equal to three, we use the tree

\[
\begin{array}{c}
 \swarrow \\
 \swarrow \\
 \swarrow
\end{array}
\]

For each element with at least one element equal to three, we use the tree

\[
\begin{array}{c}
 \swarrow \\
 \swarrow \\
 \swarrow \\
 \swarrow
\end{array}
\]

For each element equal to 1, we use the label \( \prec \) at both places, for each element equal to 3 we use the label \( \succ \) at both places, and for each element equal to 2 we use \( \succ \) at the leftmost vertex, \( \prec \) at the rightmost vertex. In particular, whenever no element is equal to 2, the upper and the lower label is the same. For example, with \( m = 3 \) and the two vectors \((1, 2, 1)\) and \((3, 1, 2)\) we get

\[
\begin{array}{c}
 \swarrow \\
 \swarrow \\
 \swarrow \\
 \swarrow
\end{array}, \quad
\begin{array}{c}
 \swarrow \\
 \swarrow \\
 \swarrow \\
 \swarrow
\end{array}
\]

The set of elements where the local patterns (along each edge) is as above, clearly generates the operad as a vector space. Let \( s_n \) be the number of such elements. We will show that \( s_n = n^m \) by recurrence, using these local patterns. Specifically, we will write \( s_n \) as a sum of \( 2^m \) summands corresponding to the label at the root; each of these summands can be written as a sum of terms with lower degree, and this will give the recurrence. Our explicit knowledge about the situation in degree 2 and 3 gives the starting point.
For each vector of labels, there is a unique corner of the $m$th hypercube with that vector at the root. This is true by the choice of labellings.

We let $(≺,≺,\ldots,≺)_n$ be the number of elements of degree $n$ with the label $(≺,\ldots,≺)$ at the root, $(≻,\ldots,≻)_n$, the number of elements with $(≻,\ldots,≻)$ at the root, and a general $m$-vector of $≺$s and $≻$s, subscripted $n$, represents the number of elements of degree $n$ with root labelled by that vector. Obviously, for any vector $(c_1,\cdots,c_m)$ of such labels, we have $(c_1,\cdots,c_m)_2 = 1$. This gives the start of our recurrence.

The box with coordinates $(1,1,\cdots,1)$ has label $(≺,≺,\ldots,≺)$ at the root. This is the same as the label of the root for each box with no coordinate equal to 3, and each possible combination of labels appear exactly once at the upper vertex of a tree in this hypercube (of size $2^m$). So we get

$$(≺,\cdots,≺)_{n+1} = \sum_{c_i \in \{≺,≻\}} (c_1,\cdots,c_m)_n$$

In particular, $s_n = a_{n+1}$.

For each box with exactly one coordinate equal to 3, all the rest being one, the label of the root and the upper are equal, and no other box has either of these labels at any vertex. Thus

$$(c_1,\cdots,c_m)_n = 1$$

if all $c_i$ are equal to $≺$ except for one $≻$.

In general, for each box with $j$ coordinates equal to 3, the rest being 1, there is a subcube of dimension $j$, of size $3^j$, such that the given box is the corner with highest coordinates in this subcube. Now the label at the root of this box also appears on the root of all the boxes in this subcube where all the coordinates are 2 or 3, with at least one 3. So the recurrence relation for this box is equal to the recurrence relation for vector with labels all $≻$s in a hypercube of dimension $j$. We claim that this is

$$(≻,\cdots,≻)_n = \sum_{i=1}^{j} (-1)^{i-1} \binom{j}{i} (n-1)^{j-i} = (n-1)^j - (n-2)^j$$

The case $j = 1$ is the special case considered above, so we get the start of the recurrence relation.

The recurrence relation in general says that

$$(≻,\cdots,≻)_{n+1} = \sum_{c_i \in \{≺,≻\}, \exists t, c_t \neq ≺} (c_1,\cdots,c_j)$$

By induction, the formula above holds for each summand on the right, so we need to show that
\[ n^j - (n - 1)^j = \sum_{k=1}^{j} \left( \begin{array}{c} j \\ k \end{array} \right) ((n - 1)^k - (n - 2)^k) \]

This follows from the Binomial theorem, on writing \( n \) as \((n - 1) + 1\) and \( n - 1 = (n - 2) + 1 \); the two terms with \( k = 0 \) cancel.

Note that there are \( \binom{m}{j} \) vectors with \( j \succ \)s, the rest \( \prec \)s, so our final recurrence relation, for the vector \((\prec, \cdots, \prec)\) takes the form

\[ (\prec, \cdots, \prec)_{n+1} = (\prec, \cdots, \prec)_n + \sum_{j=1}^{m} \binom{m}{j} ((n - 1)^j - (n - 2)^j) \]

This recurrence relation is satisfied by \((\prec, \cdots, \prec)_{n+1} = n^m \); this also follows from the Binomial theorem as above.

So we’ve shown that the vector space \((\mathcal{Dend}m)^!(n)\) is spanned by \( n^m \) elements; this is therefore a bound on the dimension.

\[ \square \]

**Remark 5.4.** The patterns we have chosen in the proof are modelled on the patterns from [1], and for \( m = 2 \) the proof reduces to their proof. The only difference is that we have chosen the other tree in position \((3, 1)\).

**Remark 5.5.** Note that we have computed the dimension

\[ \dim \mathcal{Dend}m(n) = n^m. \]

This follows from the proof of the theorem, exactly as in the case \( m = 2 \).

**References**

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