A SURFACE BIRATIONAL TO AN ENRIQUES SURFACE WITH NON-FINITELY GENERATED AUTOMORPHISM GROUP

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Dedicated to Professor Shigeru Mukai on the occasion of his 65th birthday

Abstract. We will show that there is a smooth complex projective surface, birational to some Enriques surface, such that the automorphism group is discrete but not finitely generated.

1. Introduction

We work over the complex number field \( \mathbb{C} \). A K3 surface is a compact simply connected, in the classical topology, smooth complex surface with nowhere vanishing global holomorphic 2-form. An Enriques surface is a smooth complex surface which is isomorphic to a non-trivial étale quotient of a K3 surface. The quotient map is necessarily of degree two and every Enriques surface is projective.

Our main theorem is the following:

Theorem 1.1. There is a smooth projective surface \( Y \) birational to some Enriques surface such that \( \text{Aut}(Y) \) is not finitely generated.

Remark 1.2. Let \( Y \) be a smooth projective surface birational to an Enriques surface \( S \) and let \( \tilde{S} \) be the universal covering K3 surface of \( S \).

1. \( \text{Aut}^0(S) = \{ \text{id}_S \} \), i.e., \( \text{Aut}(S) \) is discrete. This is because \( H^0(S, T_S) = 0 \) by \( H^0(\tilde{S}, T_{\tilde{S}}) = 0 \). On the other hand, \( \text{Aut}(S) \) itself is finitely generated. This is because, up to finite kernel and cokernel, \( \text{Aut}(S) \) is isomorphic to the quotient group \( \text{O}(\text{NS}(S)/\text{torsion})/W(S) \) of the arithmetic subgroup \( \text{O}(\text{NS}(S)/\text{torsion}) \) by the Weyl group \( W(S) \) generated by the reflections corresponding to the smooth rational curves on \( S \) [Do84, Theorem] and \( \text{O}(\text{NS}(S)/\text{torsion}) \) is finitely generated by a general result on arithmetic subgroups of linear algebraic groups [BH62, Theorem 6.12] (See also Theorem 1.3). So, \( S \) itself is not a candidate surface in Theorem 1.1.

2. \( S \) is the unique minimal model of \( Y \) up to isomorphisms. So, we have a birational morphism \( \nu : Y \to S \), which is a finite composition of blowings up at points. Therefore, we have \( H^0(Y, T_Y) = 0 \) and also an injective group homomorphism

\[
\text{Aut}(Y) \subset \text{Bir}(S) = \text{Aut}(S) ; \ f \mapsto \nu \circ f \circ \nu^{-1},
\]

via \( \nu \). Note that a subgroup of a finitely generated group is not necessarily finitely generated.

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We show Theorem 1.1 by constructing $Y$ explicitly. Our construction is inspired by [Le18] for 6 dimensional example, [DO19], [Og19] for K3 surfaces and also [Mu10] for his new construction of an Enriques surface with numerically trivial involution, which is missed in an earlier paper [MNS84]. As usual in study of Enriques surfaces, our construction is more involved than [DO19], [Og19] for K3 surfaces, whereas the basic strategy of the construction is essentially the same.

As in [DO19] and [Og19], the following purely group theoretical theorem (see eg. [Su82]) will be frequently used in this paper.

**Theorem 1.3.** Let $G$ be a group and $H \subset G$ a subgroup of $G$. Assume that $H$ is of finite index, i.e., $[G : H] < \infty$. Then, the group $H$ is finitely generated if and only if $G$ is finitely generated.

In this paper, for a variety $V$ we denote the group of biregular automorphisms of $V$ and the group of birational automorphisms of $V$ by

$$\text{Aut} \ (V), \ \text{Bir} \ (V),$$

respectively, and for closed subsets $W_1, W_2, \ldots, W_n$ of $V$ the decomposition group and the inertia group by

$$\text{Dec} \ (W_1, \ldots, W_n) := \text{Dec} \ (V, W_1, \ldots, W_n) := \{ f \in \text{Aut} \ (V) \mid f(W_i) = W_i (\forall i) \}$$

$$\text{Ine} \ (W_1, \ldots, W_n) := \text{Ine} \ (V, W_1, \ldots, W_n) := \{ f \in \text{Dec} \ (V, W_1, \ldots, W_n) \mid f_{W_i} = \text{id}_{W_i} (\forall i) \}.$$ 

We believe that large part of our construction should work also in positive characteristic $\geq 3$ if the based field is carefully chosen (see e.g. for some sensitive aspect of the base field in positive characteristic [Og19]). We leave it to the readers who are interested in this generalization.

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### 2. Preliminaries

In this section, first we fix some basic notation concerning a Kummer surface $\text{Km} \ (E \times F)$ of the product of two non-isogenous elliptic curves. Our notation follows [DO19] and [Og19]. Then we recall Mukai’s construction of Enriques surfaces with numerically trivial involution of odd type [Mu10] arizing from $\text{Km} \ (E \times F)$. His construction is very crucial in our construction.

#### 2.1. Kummer surfaces of product type.

Let $E$ be the elliptic curve defined by the Weierstrass equation

$$y^2 = x(x - 1)(x - t) ,$$

and $F$ be the elliptic curve defined by the Weierstrass equation

$$v^2 = u(u - 1)(u - s) .$$

Note that $E/(-1_E) = \mathbb{P}^1$, the associated quotient map $E \to \mathbb{P}^1$ is given by $(x, y) \mapsto x$ and the points 0, 1, $t$ and $\infty$ of $\mathbb{P}^1$ are exactly the branch points of this quotient map. The same holds for $F$ if we replace $t$ by $s$.

Throughout this paper, we assume that
Figure 1. Curves $E_i$, $F_j$ and $C_{ij}$

**Assumption 2.1.** $t$ and $s$ are transcendental over $\mathbb{Q}$ and the two elliptic curves $E$ and $F$ are not isogenous.

Assumption 2.1 is satisfied if $s \in \mathbb{C}$ is generic with respect to a transcendental number $t \in \mathbb{C}$.

Let $X := \text{Km}(E \times F)$ be the Kummer K3 surface associated to the product abelian surface $E \times F$, that is, the minimal resolution of the quotient surface $E \times F / \langle -1_{E \times F} \rangle$. We write $H^0(X, \Omega^2_X) = \mathbb{C}\omega_X$. Since $E$ and $F$ are not isogenous, the Picard number $\rho(X)$ of $X$ is 18 (See eg. [Sh75, Prop. 1 and Appendix]).

Let $\{a_i\}_{i=1}^4$ and $\{b_i\}_{i=1}^4$ be the 2-torsion subgroups of $F$ and $E$ respectively. Then $X$ contains 24 smooth rational curves which form the so called the double Kummer pencil on $X$, as in Figure 1. Here smooth rational curves $E_i \cap C_{ij}$ and $F_i \cap C_{ij}$ are arising from the elliptic curves $E \times \{a_i\}$, $\{b_i\} \times F$ on $E \times F$. Smooth rational curves $C_{ij}$ (1 ≤ $i, j$ ≤ 4) are the exceptional curves over the $A_1$-singular points of the quotient surface $E \times F / \langle -1_{E \times F} \rangle$. Throughout this paper, we will freely use the names of curves in Figure 1.

We denote the unique point $E_j \cap C_{ij}$ by $P_{ij}$ and the unique point $F_i \cap C_{ij}$ by $P'_{ij}$. We may and do adapt $x$ (resp. $u$) the affine coordinate of $E_j$ and $F_i$ and renumbering $i$ and $j$ (if necessary) so that

$P_{1j} = 1, P_{2j} = t, P_{3j} = \infty, P_{4j} = 0$ on $E_j$ with respect to the coordinate $x$ and

$P'_{1i} = 1, P'_{12} = s, P'_{3i} = \infty, P'_{4i} = 0$ on $F_i$ with respect to the coordinate $u$.

Set

$\theta := [(1_E, -1_F)] = [(-1_E, 1_F)] \in \text{Aut}(X)$.

Then $\theta$ is an involution of $X$, i.e., an automorphism of $X$ of order 2. The following lemma was proved in [Og89, Lemmas (1.3), (1.4)] (See also [Og19]).

**Lemma 2.2.**

1. $\theta^* = \text{id}$ on $\text{Pic}(X)$ and $\theta^* \omega_X = -\omega_X$.
2. $f \circ \theta = \theta \circ f$ for all $f \in \text{Aut}(X)$. 
(3) Let $X^g$ be the fixed locus of $\theta$. Then $X^g = \bigcup_{i=1}^4 (E_i \cup F_i)$.
(4) $\text{Aut}(X) = \text{Dec}(X, \bigcup_{i=1}^4 (E_i \cup F_i))$.

2.2. Enriques surfaces with numerically trivial involution of odd type. We employ the same notation as in Subsection 2.1. By Assumption 2.1 the two ordered sets

\[ \{ P'_1, P'_{12}, P'_{13}, P'_{14} \} \subset F_i \cong \mathbb{P}^1, \quad \{ P_{ij}, P_{2j}, P_{3j}, P_{4j} \} \subset E_j \cong \mathbb{P}^1 \]

are not projectively equivalent, i.e., not in the same orbit of the action of $\text{Aut}(\mathbb{P}^1) = \text{PGL}(2, \mathbb{C})$ on $\mathbb{P}^1$.

We recall the construction of Mukai [Mu10] for our $X = \text{Km}(E \times F)$. Let $T := X/(\theta)$ be the quotient surface and

\[ q : X \to T \]

be the quotient morphism. Then $T$ is a smooth projective surface such that $q(C_{ij})$ ($1 \leq i, j \leq 4$) is a $(-1)$-curve, i.e., a smooth rational curve with self intersection number $-1$. Then $T$ is obtained by the blowings up of $\mathbb{P}^1 \times \mathbb{P}^1$ at the 16 points $p_{ij}$ ($1 \leq i, j \leq 4$) of $\mathbb{P}^1 \times \mathbb{P}^1$. We may assume that $p_{ij}$ is the image of $C_{ij}$ under the composite morphism

\[ X \to T \to \mathbb{P}^1 \times \mathbb{P}^1. \]

Let us consider the Segre embedding

\[ \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3, \]

and identify $\mathbb{P}^1 \times \mathbb{P}^1$ as a smooth quadric surface $Q$ in $\mathbb{P}^3$. By Assumption 2.1 the four points $p_{11}, p_{22}, p_{33}, p_{44} \in Q$ are not coplanar in $\mathbb{P}^3$ so that we may adjust coordinates $[x_1 : x_2 : x_3 : x_4]$ of $\mathbb{P}^3$ so that the 4 points are

\[ p_{11} = [1 : 0 : 0 : 0], \quad p_{22} = [0 : 1 : 0 : 0], \quad p_{33} = [0 : 0 : 1 : 0], \quad p_{44} = [0 : 0 : 0 : 1]. \]

Then the equation of $Q$ is of the form

\[ \alpha_1 x_2 x_3 + \alpha_2 x_1 x_3 + \alpha_3 x_1 x_2 + (x_1 + x_2 + x_3) x_4 = 0 \]

for some complex numbers $\alpha_i$ satisfying non-degeneracy condition. Then the Cremona involution of $\mathbb{P}^3$

\[ \tilde{\tau}' : [x_1 : x_2 : x_3 : x_4] \mapsto \frac{\alpha_1}{x_1} : \frac{\alpha_2}{x_2} : \frac{\alpha_3}{x_3} : \frac{\alpha_1 \alpha_2 \alpha_3}{x_4} \]

satisfies $\tilde{\tau}'(Q) = Q$, hence restricts to a birational automorphism of $Q$

\[ \tau' := \tilde{\tau}'|Q \in \text{Bir}(Q). \]

Let $I(\tau')$ be the indeterminacy locus of $\tau'$. By the definition of $\tau'$, we readily check the following ([Mu10, Section 2]):

**Lemma 2.3.** (1) $I(\tau') = \{ p_{ii} \}_{i=1}^4$ and $\tau'$ contracts the (smooth) conic curve $C'_i := Q \cap (x_i = 0)$ to $p_{ii}$.
(2) $\tau'$ interchanges the two lines through $p_{ii}$ for each $i = 1, 2, 3, 4$.
(3) $\mu^{-1} \circ \tau' \circ \mu \in \text{Aut}(B)$, where $\mu : B \to \mathbb{P}^1 \times \mathbb{P}^1$ is the blowing up at the four points $p_{ii}$ ($1 \leq i \leq 4$).
By the property (2), \(\tau'(p_{ij}) = p_{ji}\) if \(1 \leq i \neq j \leq 4\). Therefore \(\tau'\) lifts to 
\[\tau \in \text{Aut}(T).\]

Since \(q : X \to T\) is the finite double cover branched along the unique anti-bicanonical divisor 
\[\sum_{i=1}^{4} (q(E_i) + q(F_i)) \in |-2K_T|,
\]
it follows that \(\tau\) lifts to an involution 
\[\epsilon \in \text{Aut}(X).
\]

Apriori, there are exactly the two choices of the lifting \(\epsilon\); if we denote one lifting by \(\epsilon_0\) then the other is \(\epsilon_0 \circ \theta\). Recall that \(\theta^*\omega_X = -\omega_X\). Thus, we may and do choose the unique lift \(\epsilon\) with 
\[\epsilon^*\omega_X = -\omega_X.
\]
Set 
\[Z := X/\langle \epsilon \rangle.
\]
and denote the quotient morphism by 
\[\pi : X \to Z.
\]

The following discovery due to Mukai [Mu10, Proposition 2] is also crucial for us:

**Proposition 2.4.** The involution \(\epsilon\) acts on \(X\) freely. In particular, \(Z\) is an Enriques surface with a numerically trivial involution \(\theta_Z \in \text{Aut}(Z)\) induced from \(\theta \in \text{Aut}(X)\).

Set 
\[C_i := \epsilon(C_{ii}) \ (i = 1, 2, 3, 4).
\]

**Corollary 2.5.**
1. \(\epsilon(E_i) = F_i, \epsilon(F_i) = E_i\) for all \(i = 1, 2, 3, 4\).
2. \(\epsilon(C_{ij}) = C_{ji}\) for all \(i, j\) such that \(i \neq j\).
3. \((C_i, E_i) = (C_i, F_i) = 1, (C_i, C_{ii}) = 0, (C_i, C_{kj}) = 0\) for all \(i, j, k\) such that \(k \neq j\).
4. \((C_i, E_j) = (C_i, F_j) = 0\) for all \(i, j\) such that \(j \neq i\).

**Proof.** The assertions (1) and (2) follow from the description of \(\tau\). Then the assertions (3) and (4) follow from \(\epsilon(C_{ii}) = C_i\) and the assertions (1) and (2), except possibly \((C_i, C_{ii}) = 0\). The latter follows from the fact that the conic curve \(C_i' \subset Q\) that is contracted to \(p_{ii}\) by \(\tau'\) does not pass through \(p_{ii}\) (See Lemma 2.3 (1)).

\[\square\]

### 3. Construction and proof of Theorem 1.1

We employ the same notations and the assumption (Assumption 2.1) as in Section 2.

For instance, 
\[X = \text{Km}(E \times F), \ Z = X/\langle \epsilon \rangle, \ \pi : X \to Z.
\]

We also use the following notations for curves and points on the Enriques surface \(Z\):
\[H_j := \pi(E_j), \ D_{ij} := \pi(C_{ij}), \ Q_{ij} := \pi(P_{ij}),
\]

and via the isomorphism \(\pi|_{E_j} : E_j \to H_j\), we also regard \(x\) as the affine coordinate of \(H_j\).

Then \(Q_{ij} \in H_j\) and 
\[x(Q_{1j}) = 1, \ x(Q_{2j}) = t, \ x(Q_{3j}) = \infty, \ x(Q_{4j}) = 0.
\]

By Corollary 2.5, we have 
\[\pi^{-1}(H_j) = E_j \cup F_j.
\]
for each \( j = 1, 2, 3, 4 \) and
\[
\pi^{-1}(D_{ij}) = C_{ij} \cup C_{ji}, \quad \pi^{-1}(Q_{ij}) = \{P_{ij}, P'_{ji}\}
\]
if \( i \neq j \), while
\[
\pi^{-1}(D_{ii}) = C_{ii} \cup C_{i}, \quad \pi^{-1}(Q_{ii}) = \{P_{ii} \cup P_{i}\},
\]
again for each \( i \). Here \( P_i \) is the unique intersection point of \( C_i \cap F_i \).

Let \( \mu_1 : Z_1 \to Z \) be the blowing up at the point \( Q_{32} \in H_2 \), i.e., the blowing up at \( \infty \) under the coordinate \( x \) of \( H_2 \). Let
\[
E_\infty := \mathbb{P}(T_{Z,Q_{32}}) \simeq \mathbb{P}^1
\]
be the exceptional divisor of \( \mu_1 \). We then choose three mutually different points on \( \mathbb{P}(T_{Z,Q_{32}}) \), say \( Q_{32k} \) \((k = 1, 2, 3)\). Let \( \mu_2 : Z_2 \to Z_1 \) be the blowings up of \( Z_1 \) at the three points \( Q_{32k} \).

Our main theorem is Theorem 3.1 below. Clearly, Theorem 1.1 follows from Theorem 3.1 by taking \( Y = Z_2 \):

**Theorem 3.1.** \( \text{Aut } (Z_2) \) is not finitely generated.

In the rest of this section, we prove Theorem 3.1.

We denote
\[
\mu := \mu_1 \circ \mu_2 : Z_2 \to Z_1 \to Z.
\]
By \( E_{32k} \), we denote the exceptional curve over \( Q_{32k} \) under \( \mu_2 \) and by \( E'_\infty \) the proper transform of \( E_\infty \) under \( \mu_2 \).

First we reduce the proof to \( Z \). For this, we recall that
\[
\text{Aut } (Z_2) \subset \text{Aut } (Z)
\]
via \( \mu \) (See Remark 1.2). We define
\[
\text{Ine } (Z, Q_{32}, T_{Q_{32}}) := \{ f \in \text{Dec } (Z, Q_{32}) \mid df|_{T_{Z,Q_{32}}} = \text{id}_{T_{Z,Q_{32}}} \}.
\]

**Proposition 3.2.**

1. There is a subgroup \( K \) of \( \text{Aut } (Z_2) \) such that \( [\text{Aut } (Z_2) : K] < \infty \), \( \text{Ine } (Z, Q_{32}, T_{Q_{32}}) \subset K \) via \( \mu \) and \( [K : \text{Ine } (Z, Q_{32}, T_{Q_{32}})] < \infty \).
2. If \( \text{Ine } (Z, Q_{32}, T_{Q_{32}}) \) is not finitely generated, then \( \text{Aut } (Z_2) \) is not finitely generated.

**Proof.** First we show (1). By the canonical bundle formula, we have
\[
|2K_{Z_2}| = \{2E'_\infty + 4(E_{321} + E_{322} + E_{323})\}.
\]
Since \( \text{Aut } (Z_2) \) preserves \( |2K_{Z_2}| \), it follows that
\[
\text{Aut } (Z_2) = \text{Dec } (Z_2, E'_\infty, E_{321} \cup E_{322} \cup E_{323}).
\]
Therefore, via \( \tau_2 \), we have
\[
\text{Aut } (Z_2) = \text{Dec } (Z_1, E_\infty, \{Q_{321}, Q_{322}, Q_{323}\}) \subset \text{Aut } (Z_1).
\]
Thus, the group
\[
K := \text{Dec } (Z_1, E_\infty, \{Q_{321}\}, \{Q_{322}\}, \{Q_{323}\})
\]
is a subgroup of \( \text{Aut } (Z_2) \) with \( [\text{Aut } (Z_2) : K] \leq 6 = |\text{Aut}_{\text{set }} \{Q_{321}, Q_{322}, Q_{323}\}|. \)
We will show that \( K \) satisfies the requirement.
Since only \( \text{id}_{\mathbb{P}^1} \) is the automorphism of \( \mathbb{P}^1 \) pointwisely fixes three points, it follows that
\[
K = \text{Ine } (Z_1, E_\infty).
\]
Since $E_{\infty} = \mathbb{P}(T_{Z,Q_{32}})$, we deduce that
\[ K = \{ f \in \text{Dec}(Z, Q_{32}) \mid df|_{T_{Z,Q_{32}}} = \alpha(f)\text{id}_{T_{Z,Q_{32}}} \ (\exists \alpha(f) \in \mathbb{C}^\times) \} \subset \text{Dec}(Z, Q_{32}) . \]
Observe that if $df|_{T_{Z,Q_{32}}} = \alpha(f)\text{id}_{T_{Z,Q_{32}}}$ for $f \in K$, then
\[ (df \wedge df)^{\otimes 2}|_{(\Lambda^2 T_{Z,Q_{32}})^{\otimes 2}} = \alpha(f)^4\text{id}_{(\Lambda^2 T_{Z,Q_{32}})^{\otimes 2}} . \]

Since the line bundle $(\Omega_Z^2)^{\otimes 2}$ admits a nowhere vanishing global section, it follows that $\alpha(f)^4$ is in the image $\text{Im} r_2$ of the bicanonical representation
\[ r_2 : \text{Aut}(Z) \to \text{GL}(H^0(Z, (\Omega_Z^2)^{\otimes 2})) \simeq \mathbb{C}^\times \]
of $\text{Aut}(Z)$ ([Le75, Section 14]). Since $\text{Im} r_2$ is finite by [Le75, Theorem 14.10], it follows that $\{\alpha(f) \mid f \in K\}$ is also finite. Hence $[K : \text{Ine}(Z, Q_{32}, T_{Q_{32}})] < \infty$ as well.

Let us show (2). Recall Theorem 1.3. Then, if $\text{Ine}(Z, Q_{32}, T_{Q_{32}})$ is not finitely generated, then $K$ is not finitely generated by $[K : \text{Ine}(Z, Q_{32}, T_{Q_{32}})] < \infty$. Hence $\text{Aut}(Z_2)$ is not finitely generated, again by $[\text{Aut}(Z_2) : K] < \infty$. \hfill $\Box$

In what follows, we will show that $\text{Ine}(Z, Q_{32}, T_{Q_{32}})$ is not finitely generated. This is a problem on the Enriques surface $Z$.

**Lemma 3.3.** 1. Let $f \in \text{Dec}(Z, Q_{32})$. Then $f(H_2) = H_2$, i.e., $f \in \text{Dec}(Z, H_2)$.

2. The differential maps $df|_{T_{Z,Q_{32}}}$ for all $f \in \text{Dec}(Z, Q_{32})$ are simultaneously diagonalizable.

3. $f \in \text{Ine}(Z, Q_{32}, T_{Q_{32}})$. Then $f \in \text{Dec}(Z, H_2)$ by (1) and
\[ d(f|_{H_2})|_{T_{H_2,Q_{32}}} = \text{id}_{T_{H_2,Q_{32}}} \]
for the induced action.

**Proof.** Let $f \in \text{Dec}(Z, Q_{32})$. Then, the one of the two lifts of $f$, say $\tilde{f}$, satisfies $\tilde{f}(P_{32}) = P_{32}$. Therefore the result follows from the corresponding result on $X$ (see eg. [DO19]).

For the convenience of the readers, we recall the proof here from [DO19]. Since $\tilde{f} \in \text{Dec}(X, \cup_{j=1}^4 (E_j \cup F_j))$ by Lemma 2.2 (4) and $E_2$ is the unique component of $\cup_{j=1}^4 (E_j \cup F_j)$, containing $P_{32}$, it follows that $\tilde{f} \in \text{Dec}(X, E_2)$. This shows (1).

By Lemma 2.2 (1), (3), one has $\theta(R) = R$ for any smooth rational curve $R$ on $X$ and
\[ d(\theta|_{E_2})P_{32} = 1 , d(\theta|_{C_{32}})P_{32} = -1 . \]
In particular,
\[ T_{X,P_{32}} = T_{E_2,P_{32}} \oplus T_{C_{32},P_{32}} . \]

Note that $\tilde{f}(E_2) = E_2$ as observed above. Let $C_{32}' := \tilde{f}(C_{32})$. Then $P_{32} \in C_{32}' \simeq \mathbb{P}^1$ and the induced action $\theta|_{C_{32}'}$ satisfies
\[ d(\theta|_{C_{32}'})P_{32} = -1 \]
by Lemma 2.2 (1). Thus, $d\tilde{f}|_{T_{X,P_{32}}}$ for all $\tilde{f}$ preserve both $T_{E_2,P_{32}}$ and $T_{C_{32},P_{32}}$. This implies (2).

The assertion (3) is now obvious. \hfill $\Box$

Recall that for $Q \in \mathbb{P}^1$,
\[ \text{Ine}(\mathbb{P}^1, Q, T_{\mathbb{P}^1,Q}) := \{ f \in \text{Ine}(\mathbb{P}^1, Q) \mid df|_{T_{\mathbb{P}^1,Q}} = \text{id}_{T_{\mathbb{P}^1,Q}} \} \simeq (\mathbb{C}, +) . \]
Here \((\mathbb{C}, +)\) is the additive group, in particular, an abelian group. The last isomorphism is given by
\[
\mathbb{C} \ni c \mapsto (z \mapsto z + c) \in \text{Ine}(\mathbb{P}^1, Q, T_{\mathbb{P}^1, Q})
\]
if we choose an affine coordinate \(z\) of \(\mathbb{P}^1\) such that \(z(Q) = \infty\). By Lemma 3.3 (3), we have then a representation
\[
\rho : \text{Ine}(Z, Q_{32}, T_{Q_{32}}) \to \text{Ine}(H_2, Q_{32}, T_{H_2, Q_{32}}) \cong (\mathbb{C}, +)
\]
Here, for the last isomorphism, we can use the affine coordinate \(x\) of \(H_2\) fixed at the beginning of this section.

**Proposition 3.4.**  
(1) There is \(a \in \mathbb{C} \setminus \{0\}\) such that \(t^{-2n}a \in \text{Im} \rho\) for all positive integers \(n\).

(2) \(\text{Ine}(Z, Q_{32}, T_{Q_{32}})\) is not finitely generated.

**Proof.** The assertion (2) follows from the assertion (1). Indeed, the additive subgroup \(M\) generated by \(\{t^{-2n}a | n \in \mathbb{Z}_{\geq 0}\}\) is not finitely generated as \(a \neq 0\) and \(t\) is transcendental over \(\mathbb{Q}\) by our assumption (Assumption 2.1). The assertion (1) says that \(M \subseteq \text{Im} \rho\). Since \(\text{Im} \rho \subset (\mathbb{C}, +)\), the group \(\text{Im} \rho\) is also an abelian group. It follows that the abelian group \(\text{Im} \rho\) is not finitely generated, either, regardless of \([\text{Im} \rho : M]\). Hence \(\text{Ine}(Z, Q_{32}, T_{Q_{32}})\) is not finitely generated as claimed.

In the rest, we will show the assertion (1) by constructing two genus one fibrations on \(Z\) and by considering their Jacobian fibrations.

Consider the following two divisors \(M_1\) and \(M_2\) of Kodaira’s type \(I_8\) and \(IV^*\) on \(Z\):
\[
M_1 := H_2 + D_{32} + H_3 + D_{31} + H_1 + D_{41} + H_4 + D_{42},
\]
\[
M_2 := H_2 + 2D_{32} + H_1 + 2D_{31} + H_4 + 2D_{34} + 3H_3.
\]
Then \(|M_1|\) and \(|M_2|\) define genus one fibrations
\[
\varphi_{M_1} : X \to \mathbb{P}^1, \quad \varphi_{M_2} : X \to \mathbb{P}^1.
\]
\(\varphi_{M_1}\) is the genus one fibration induced from an elliptic fibration \(\Phi_1 : X \to \mathbb{P}^1\) on \(X\) given by the divisor of Kodaira’s type \(I_8\)
\[
N_1 := E_2 + C_{32} + F_3 + C_{31} + E_1 + C_{41} + F_4 + C_{42},
\]
and \(\varphi_{M_2}\) is the genus one fibration induced from an elliptic fibration \(\Phi_2 : X \to \mathbb{P}^1\) on \(X\) given by the divisor of Kodaira’s type \(IV^*\)
\[
N_2 := E_2 + 2C_{32} + F_1 + 2C_{31} + E_4 + 2C_{34} + 3F_3.
\]
By the classification of \([\text{Og89}]\) Theorem 2.1, \(\Phi_1\) then belongs to Type \(J_1\) and \(\Phi_2\) belongs to Type \(J_3\) in \([\text{Og89}]\) Theorem 2.1. By the definition of the action of our Enriques involution \(\epsilon\) on \(X\) and the classification of \([\text{Og89}]\) Theorem 2.1, it follows that the reducible fibers of \(\Phi_1\) are exactly \(N_1\) and \(\epsilon(N_1)\), and the reducible fibers of \(\Phi_2\) are exactly \(N_2\) and \(\epsilon(N_2)\). Thus, \(\varphi_{M_1}\) has no reducible fibers other than \(M_1\) and \(\varphi_{M_2}\) has also no reducible fibers other than \(M_2\).

Let us consider the (proper non-singular, relatively minimal) Jacobian fibration \(\varphi_i : R_i \to \mathbb{P}^1\) of \(\varphi_{M_i}\) for \(i = 1\) and 2. Then the fiber \(R_{i,p}\) of \(\varphi_i\) over general \(p \in \mathbb{P}^1\) is \(\text{Pic}^0(Z_{i,p})\), i.e., the identity component of the Picard group of the corresponding fiber \(Z_{i,t}\) of \(\varphi_{M_i}\). Therefore, the Mordell-Weil group \(\text{MW}(\varphi_i)\) of \(\varphi_i\) acts on \(\varphi_{M_i}\), which is the unique biregular extension of the translation action of \(\text{Pic}^0(Z_{i,p})\) on \(Z_{i,p}\) where \(p \in \mathbb{P}^1\) runs through
has also no reducible fibers other than $M$ to Shioda [Sh90].

Let us consider first the action of MW ($\varphi_2$) on $\varphi_{M_2} : Z \to \mathbb{P}^1$. From the fact that $\varphi_{M_2}$ has also no reducible fibers other than $M_2$, we see that $\varphi_2 : R_2 \to \mathbb{P}^1$ belongs to No. 27 in the classification of [OS91, Main Theorem]. In particular, the narrow Mordell-Weil lattice MWL$^0(\varphi_2)$ of $\varphi_2$ is isomorphic to the positive definite root lattice $A_2$. In particular, there is $P \in$ MW ($\varphi_2$) such that $\langle P, P \rangle = 2$ for the height pairing of MW$^0(\varphi_2)$ [Sh90, Section 8]. For this $P$, we have $(P) \cap (O) = \emptyset$ by [Sh90, Formula 8.19]. Here $(P)$ is the divisor on $R_2$ corresponding to $P$. The action $t_P$ of $P$ on $\varphi_{M_2} : Z \to \mathbb{P}^1$ then preserves each component of $M_2$, particularly the curve $H_2$ and the point $Q_{32} \in H_2$, and the action $t_P|_{H_2}$ is of the form

$$x \mapsto x + a$$

for some $a \neq 0$ under the affine coordinate $x$ of $H_2$. Recall that the action of $d(t_P)$ on $T_{Z,Q_{32}}$ is diagonalizable (Lemma 5.5). Then, by the finiteness of bicanonical representation [Ue75, Theorem 14.10], by replacing $t_P$ by some power $t_P^k$ ($k \neq 0$) and $a$ by $ka$ if necessary, we obtain an element

$$f_2 \in \text{Ine} (Z, Q_{32}, T_{Q_{32}})$$

such that $\rho(f_2) = a \neq 0$.

Next we consider the Jacobian fibration $\varphi_1 : R_1 \to \mathbb{P}^1$. We need an explicit geometric construction of $\varphi_1$ from $\varphi_{M_1}$ explained by [Ko86, Lemma 2.6] and [HS11, Section 3]. Note that $D_{21}$ is a 2-section of $\varphi_{M_1}$ and $\pi^{-1}(D_{21}) = C_{12} \cup C_{21}$. The curves $C_{12}$ and $C_{21}$ are sections of $\Phi_1$. We may set $0 := [C_{21}] \in \text{MW}(\Phi_1)$. Then the element $[C_{12}] \in \text{MW}(\Phi_1)$ is a 2-torsion, because

$$\langle [C_{12}], [C_{12}] \rangle = 2 \cdot 2 + 2 \cdot 2 - \frac{4(8-4)}{8} - \frac{4(8-4)}{8} = 0$$

for the height pairing [Sh90, Theorem 8.6, Formula 8.10] and MW($\Phi_1$) $\simeq \mathbb{Z}^{32} \oplus \mathbb{Z}/2$ by [Og89, Theorem 2.1, Case $\mathcal{J}_1$]. We denote by $T(R) \in \text{Aut}(X)$ the automorphism corresponding to $R \in \text{MW}(\Phi_1)$. Set

$$\iota := T([C_{12}]) \circ \epsilon \in \text{Aut}(X).$$

Then $\iota$ is an involution on $X$ ([Ko86, Lemma 2.6]) such that $X^\epsilon$ consists of two elliptic curves corresponding to the multiple fibers of $\varphi_{M_1}$ by Assumption 2.7. Then, by [Ko86, Lemma 2.6] (see also [HS11, Section 3]), the Jacobian fibration $\varphi_1$ of $\varphi_{M_1}$ is given by

$$\varphi_1 : R_1 = X/\langle \iota \rangle \to \mathbb{P}^1/\langle \epsilon \rangle.$$
i.e., preservation of the zero section under \( \iota \), and that \( Q \in \text{MW}(\Phi_1) \) is induced from some element \( Q' \in \text{MW}(\varphi_1) \) exactly when

\[
\iota \circ T([Q])|_{x_p} = T([Q])|_{x_{\iota(p)}},
\]

for general \( p \in \mathbb{P}^1 \). Here we note that \( \iota \) acts on the base space \( \mathbb{P}^1 \) as an involution.

Note also that if we choose \( C_{11} \) as the zero section of \( \Phi_1 \), then \( C_{22} \) becomes the 2-torsion section for the same reason as above. Hence, \([C_{11}] + [C_{12}] = [C_{22}]\) by the fact that the torsion group of \( \text{MW}(\Phi_1) \) is isomorphic to \( \mathbb{Z}/2 \) [Og89, Theorem 2.1, Case \( J_1 \)]. Combining this with \( \iota = \epsilon \circ T([C_{12}]) \), preservation of the zero section under \( \iota \) and \( \epsilon(C_{22}) = C_2 \), we obtain that

\[
\iota \circ T([C_{11}])|_{x_p} = \epsilon \circ T([C_{22}])|_{x_p} = T([C_2])|_{x_{\iota(p)}}
\]

for general \( p \in \mathbb{P}^1 \). Hence \([C_{11}] + [C_2] \in \text{MW}(\Phi_1)\) is induced from an element of \( Q' \in \text{MW}(\varphi_1) \). Then \( Q' \) induces an automorphism \( \varphi_2 \in \text{Aut}(Z) \) preserving each fiber of \( \varphi_M \).

The action of \( \varphi_2 \) on \( M_1 \setminus \text{Sing} \) is then the same action of \( Q' \) on \( N_1 \setminus \text{Sing} \) and therefore also the same action of \([C_{11}] + [C_2] \) on \( N_1 \setminus \text{Sing} \) under the identifications of these three fibers by \( \pi \) and \( \pi_R \). Thus, representing points on \( M_1 \setminus \text{Sing} = \mathbb{C}^\times \times \mathbb{Z}/8 \) by \((x, m \mod 8)\), we have by [Ko63, Theorem 9.1, Page 604]

\[
f_2 : (x, m \mod 8) \mapsto (tx, m \mod 8) \mapsto (tx, m + 4 \mod 8).
\]

Here we recall that \( C_{22} \cap E_2 = t \) (resp. \( C_2 \cap F_2 = t \)) with respect to the affine coordinate \( x \) on \( E_2 \) (resp. \( u \) on \( F_2 \)). Hence \( f_2^2(H_2) = H_2 \), \( f_2^2(Q_{32}) = Q_{32} \) and

\[
f_2^2(x) = t^2x
\]

on \( H_2 \). Then

\[
(f_2^2)^{-n} \circ f_1 \circ (f_2^2)^n \in \text{Ine}(Z, Q_{32}, T_{Z,Q_{32}})
\]

and

\[
(f_2^2)^{-n} \circ f_1 \circ (f_2^2)^n|_{H_2} : x \mapsto t^{2n}x \mapsto t^{2n}x + a \mapsto x + t^{-2n}a
\]

on \( H_2 \). Thus

\[
t^{-2n}a = \rho((f_2^2)^{-n} \circ f_1 \circ (f_2^2)^n) \in \text{Im} \rho,
\]

as claimed. This completes the proof. \( \square \)

Theorem 3.1, hence Theorem 1.1, now follows from Propositions 3.2 (2) and 3.4 (2).

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