A Wigner Surmise for Hermitian and Non-Hermitian Chiral Random Matrices

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We use the idea of a Wigner surmise to compute approximate distributions of the first eigenvalue in chiral Random Matrix Theory, for both real and complex eigenvalues. Testing against known results for zero and maximal non-Hermiticity in the microscopic large-N limit we find an excellent agreement, valid for a small number of exact zero-eigenvalues. New compact expressions are derived for real eigenvalues in the orthogonal and symplectic classes, and at intermediate non-Hermiticity for the unitary and symplectic classes. Such individual Dirac eigenvalue distributions are a useful tool in Lattice Gauge Theory and we illustrate this by showing that our new results can describe data from two-colour QCD simulations with chemical potential in the symplectic class.

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1. Motivation. Probably one of the most used predictions of Random Matrix Theory (RMT) is the so-called Wigner surmise (WS) describing the universal repulsion of energy levels in many systems in nature, including neutron scattering, quantum billiards and elastomechanical modes in crystals [1]. For large matrices, the nearest-neighbour (nn) spacing distribution \( p^{(\beta)}(s) \) is universal and only depends on the repulsion strength which takes discrete values \( \beta = 1, 2, 4 \) for the three classical Wigner-Dyson (WD) ensembles. It can be computed with surprising accuracy using \( 2 \times 2 \) matrices, which is the WS. Although simple arguments discussed in [2] lead to this rule for \( \beta = 1 \), such an approximation is by no means obvious.

The extension from WD to non-Hermitian RMT introduced long ago by Ginibre [3] has become a very active field in the past decade, in particular due to applications in open quantum systems, see [4] for references and other applications. Here the spacing is known only for the class with broken time-reversal (\( \beta = 2 \)) and has been applied in Lattice Gauge Theory (LGT) [5]. However, a simple surmise based on \( 2 \times 2 \) matrices does not work here.

In this paper we investigate the existence of a surmise for the smallest eigenvalue in chiral RMT and its non-Hermitian extensions. These have become relevant due to applications in Quantum Chromodynamics (QCD) initiated by [6] and extended to non-Hermitian QCD at finite quark chemical potential \( \mu \) [7]. QCD at strong coupling is a notoriously difficult theory, and the chiral RMT approach has become an important tool for LGT with exact chiral fermions [8,9]. For non-Hermitian QCD the complex action hampers a straightforward LGT approach, see [10] for a recent discussion and references. Here RMT predictions remain possible for various quantities [11,12,13].

In this paper we will show that an excellent approximation for the 1st non-zero eigenvalue is possible using a simple \( 2 \times (2 + \nu) \) matrix calculation, capturing the repulsion of a small number \( \nu \) of zero eigenvalues. Being localised and non-oscillatory the 1st eigenvalue is much more suitable for LGT than the spectral density, compare e.g. [9] and [14]. Our surmise fills some gaps in predictions for real eigenvalues in the orthogonal and symplectic classes (\( \beta = 1, 4 \)) [15], where until very recently numerically generated RMT had to be used for comparison [16]. We also provide new predictions for intermediate non-Hermiticity and test them against QCD-like LGT data from [17]. This further completes the picture, compared to previous approximations [14] (\( \beta = 2 \)) based on a Fredholm determinant expansion [16], and exact results at maximal non-Hermiticity [19] (\( \beta = 2, 4 \)).

2. Level spacing in the WD class. We recall here the success of a WS for Hermitian, and its failure for non-Hermitian, WD ensembles. The WD partition function for an \( N \times N \) Hermitian matrix \( H \) with real, complex or quaternion real entries is given terms of eigenvalues by

\[
Z^{(\beta)}_{WD} = \int \mathrm{d} H e^{-\text{Tr} H H^\dagger} \sim \int \mathbb{R}^N \prod_{j=1}^N d\lambda_j e^{-\lambda_j^2} |\Delta_N(\lambda)|^\beta. \tag{1}
\]

The Jacobians of the corresponding ensembles which are called GOE, GUE and GSE (\( \beta = 1, 2 \) and 4) include the Vandermonde determinant, \( \Delta_N(\lambda) \equiv \prod_{k>l}^N (\lambda_k - \lambda_l) \).

The large-\( N \) nn spacing in the bulk of the spectrum can be computed approximately from \( N = 2 \) (WS) by inserting \( \delta(\lambda_1 - \lambda_2 - s) \) into \( Z^{(\beta)}_{WD} \):

\[
p^{(\beta)}_{WS}(s) = a_\beta s^\beta \exp[-b_\beta s^2]. \tag{2}
\]

The constants \( a_\beta, b_\beta \) follow from fixing the norm and first moment to unity (see e.g. in [1]). The latter can always be achieved from \( \int_0^\infty ds s \tilde{p}(s) = m \) by rescaling \( p^{(\beta)}(s) = m p^{(\beta)}(ms) \). This fixes the scale compared with \( N = \infty \).

The exact result \( p^{(\beta)}(s) \) is cumbersome, given in terms of an infinite product of eigenvalues of spheroidal functions (e.g. in [2]), the 5th Painlevé transcendent [2], or...
combining a Taylor series with coefficients given by sums over permutations and Dyson’s asymptotic expansion in a Padé approximation [29]. This is compared to the surmise Eq. (2) in Fig. 1 left. In Table I we give the root gaps in Eqs. (8)–(10) for the first two values of $p$ determined in [23], the chGSE for microscopic limit for $p$ Dirac eigenvalue follows:

$$
m_s \sim \prod_{i=1}^{\nu} \left( \right)^\beta \Delta_N(z) = \Delta_N(z, z^*) \prod_{j=1}^{N} (z_j - z^*_j).$$

For $\beta = 2$ the spacing is obtained from an $N = 2$ surmise by inserting $\delta(|z_1 - z_2| - r)$ in $Z^2_{\text{Gin}}$, and putting one eigenvalue at the origin. The exact spacing for any $N$ obtained in [21] uses translational invariance in the bulk

$$p^{(3)}_{\text{Gin}}(r) = \frac{\partial E^{(1)}_{\text{Gin}}(r)}{\partial r}, \quad E^{(2)}_{\text{Gin}}(r) = \prod_{j=1}^{N} e^{-r^2} \sum_{k=0}^{\nu} \frac{r^{2k}}{k!}. \quad (5)$$

In Fig. 1 right we compare $N = 2$ with increasing $N$, all curves having norm and first moment 1. Clearly a surmise does not work for $\beta = 2$ Ginibre ensemble ($\delta \approx 0.18$), as previously noted in [21]. For $\beta = 4$ and 1 the spacing is currently unknown.

3. First eigenvalue in chiral RMT. The chiral ensembles with real eigenvalues called chGOE, chGUE, and chGSE are defined in terms of $N \times (N + \nu)$ rectangular matrices $W$ with real, complex or quaternion real elements without further symmetry restrictions. Switching to positive eigenvalues $\lambda_j \geq 0$ of the Hermitian Wishart (or covariance) matrix $WW^\dagger$ we obtain

$$Z_{\nu}^{(\beta)} = \int_{0}^{\infty} d\xi_{\nu} \int_{0}^{\infty} d\xi_{\nu} e^{-\lambda_{\nu} |\Delta_N(\lambda)|^\beta}, \quad d \equiv \frac{\nu(n+1)}{2} - 1. \quad (6)$$

Here $N_f$ massless flavours can be added by shifting $d \to d + N_f$. The gap probability $E^{(\beta)}(s)$ that the interval $(0, s)$ is void follows by integrating in Eq. (6) from $s$ to $\infty$. For $N = 2$ we obtain

$$E^{(\beta)}(s) \sim \int_{0}^{\infty} \frac{d\xi_{\nu} d\xi_{\nu}}{d(d + 2)} e^{-2(s^2 + y^2)} y^\beta, \quad (7)$$

after shifting variables. The nested integrals can easily be evaluated. Note that $d = 0$ for $\beta = 2, \nu = 0$, and $\beta = 1, \nu = 0$. These gap probabilities can be computed exactly for any $N$, and our surmise gives the exact result after rescaling.

To compare with Dirac operator eigenvalues we have to switch variables $\lambda_j \to y_j^2$, coming in eigenvalue pairs $\pm y_j$, and thus to $s \to s^2$. The distribution of the first positive Dirac eigenvalue follows: $p^{(3)}(s) = -\partial_s E^{(\beta)}(s^2)$.

We first list all its known $N_f = 0$ results in the universal microscopic limit for $\nu \in \mathbb{N}$ in Eqs. (8)–(10); the chGUE for all $\nu \geq 15$, [22], the chGOE for $\nu = 0$ [23] and odd $\nu \geq 15$, and the chGSE for $\nu = 0$ [24] and $\nu > 0$ [24]. For the latter, only a convergent Taylor series is known with coefficients $a_{\nu}(\nu)$ given by sums over partitions (see Eq. (8) in [24]), much alike for the WS in the WD class,

$$p^{(2)}(s) = s e^{-s^2/4} \frac{d^{i-j+2}}{d^{i-j+2}} \left[I_{i-j+2}(s)\right]/2, \quad (8)$$

$$p^{(1)}_{\nu=0}(s) = \frac{(2 + s) e^{-s^2/8 - s^2/2}}{4}, \quad (9)$$

$$p^{(1)}_{\nu=0,n+1}(s) \sim s^{-2n} \frac{e^{-s^2/8}}{\nu! \left[\frac{1}{2} - \frac{n}{2}\right]}, \quad (10)$$

In Table I in units $10^{-3}$ between approximate $N = 2$ and exact large-$N$ results ($\delta_{\text{WS}}$ from [21]).

| $\delta_{\text{WS}}$ | $\delta_{\text{GUE}}$ | $\delta_{\text{GSE}}$ | $\delta_{\text{GOE}}$ |
|-------------------|-------------------|-------------------|-------------------|
| $\delta_{p^{(2)}}$ | $\delta_{p^{(1)}}$ | $\delta_{p^{(2)}}$ | $\delta_{p^{(1)}}$ |
| $\delta_{p^{(2)}}$ | $\delta_{p^{(1)}}$ | $\delta_{p^{(2)}}$ | $\delta_{p^{(1)}}$ |
| $\delta_{p^{(2)}}$ | $\delta_{p^{(1)}}$ | $\delta_{p^{(2)}}$ | $\delta_{p^{(1)}}$ |
| $\delta_{p^{(2)}}$ | $\delta_{p^{(1)}}$ | $\delta_{p^{(2)}}$ | $\delta_{p^{(1)}}$ |

Next, we give examples following our surmise Eq. (6) where $p^{(3)}(s)$ is not known in elementary form, filling the gaps in Eqs. (10)–(13) for the first two values of $\nu > 0$:

$$p^{(3)}_{\nu=2}(s) \sim 3 s^3 e^{-s^2} + \left(6 s^2 - s^4\right) e^{-s^2/4} \sqrt{2 \pi} \text{Erfc}[s/4], \quad (11)$$

$$p^{(4)}_{\nu=4}(s) \sim \left(s^5 + \frac{s^7}{60} e^{-s^2} + \frac{2 s^9}{20} \sqrt{2 \pi} \text{Erfc}[s/4], \quad (12)$$

$$p^{(4)}_{\nu=1}(s) \sim s^7 \left(13440 + 1440 s^2 + 60 s^4 + s^6 \right) e^{-s^2} \sqrt{2 \pi}, \quad (13)$$

The normalisation constants suppressed above easily follow. However, we cannot set the 1st moment to one as
in the WD class. The position of \( p^{(2)}_{\nu}(s) \) measures the repulsion by \( \nu \) exact zero-eigenvalues, containing important information. Thus we fix the \( N = 2 \) scale by setting the 1st moment equal to the exact one. Without exact (\( \beta = 1 \), even \( \nu \)) or concise (\( \beta = 4 \), \( \nu > 0 \)) results we instead fit to the increasing slope of the known microscopic density \( p^{(2)}_{\nu}(s) \), being the first term in the Fredholm expansion of the 1st eigenvalue [18] (see also Eq. (18)). In Fig. 2 we compare approximate to exact 1st eigenvalues for small topology \( \nu = 0, 1, 2 \) and all \( \beta \). The deviation measured by Eq. \( (3) \) in Table I increases with \( \nu = 2 \) (see Fig. 2). This has to be compared to the statistical error in data, see e.g. Fig. 5.

Note that in chiral RMT the \( n \) spacing also obeys Eq. (16), but does not follow from an \( N = 2 \) surmise [25].

The non-Hermitian chiral ensembles with \( \mu \neq 0 \) are given in terms of a two-matrix model [11, 26]. We only focus on \( \beta = 2, 4 \) here, with their complex eigenvalue representations for \( N_f = 0 \) reading [11, 26]

\[
Z^{(\beta)}_{\nu} = \int_{\mathbb{C}} \prod_{j=1}^{N} d^2 z_j |z_j|^{\beta \nu + 2} K_{\nu} \left( a |z_j|^{2} \right)e^{b |z_j|^{2}} J_\nu (z_j^2). \tag{13}
\]

The weight \( w(z) \) depends on \( a = \frac{1+\mu^2}{2} > b = \frac{1-\mu^2}{2\mu^2} \geq 0 \), with \( \mu \in (0, 1] \). The limit \( \mu \to 0 \) leads back to real eigenvalues, and at \( \mu = 1 \) non-Hermiticity is maximal. The definition of a gap probability on \( \mathbb{C} \) is not unique [14, 19]. For radial ordering it reads

\[
E^{(\beta)}(r) = \sum_{j=1}^{N} \int_{r}^{\infty} d r_j \int_{0}^{2\pi} d \theta_j w(z_j) J_\nu (z_j). \tag{14}
\]

Differentiation yields \( \partial_r E^{(\beta)}(r) = \int_{0}^{2\pi} d \theta p^{(\beta)}(r e^{i \theta}) \), the integrated 1st eigenvalue. For \( \beta = 2 \) (4) the gap probability is given by a Fredholm determinant (Pfaffian) [19]

\[
E^{(2)}(r) \sim \det_{1,...,N} \left[ \int_{r^2}^{\infty} dt t^{k+j+\nu-1} K_\nu(at) I_{k+j-2}(bt) \right], \tag{15}
\]

Its matrix elements \( A^{(\nu)}_{\mu} \) can be computed recursively for any \( \nu \) by differentiating the following matrix element [19]:

\[
A^{(0)}_{11} = \frac{b r^2 I_1(b r^2) K_0(a r^2) + a r^2 I_0(b r^2) K_1(a r^2)}{a^2 - b^2}. \tag{16}
\]

This leads to a \( \beta \times \beta \) determinant (Pfaffian) representation for our \( N = 2 \) surmise valid for any \( \mu \). At \( \mu = 1 \) all Fredholm eigenvalues \( 1 - \lambda^{(\beta)}_{k=0,...,N-1} \) are explicitly known [19], providing an exact result for any \( N \) as in Eq. (6). It contains incomplete Bessel function series \( I^{(\beta)}_{\nu}(x) \) truncated at power \( k (\equiv 0 \text{ for } k < 0) \)

\[
1 - \lambda^{(2)}_{k} = \frac{r^{2k+\nu+1}}{2^{2k+\nu}(k+\nu)!} K_{\nu+1}(r^2) + r^2 \left( I_{\nu+1}^{[k-2]}(r^2) K_{\nu+1}(r^2) \right). \tag{17}
\]

For \( \beta = 4 \) we have the relation \( \lambda^{(4)}_{k} = \lambda^{(2)}_{2k+1} \) with \( \nu \to 2\nu [19] \). In Fig. 3 we compare our surmise to this result, truncated at \( N = 8 \) because of rapid convergence. Here it works better for \( \beta = 4 \) than \( \beta = 2 \), in contrast to \( \mu = 0 \). Due to angular integration only one scale has to be fixed after normalisation, which can be done as in the real case.

\[
\begin{align*}
\rho^{(\beta)}_{\nu}(r e^{i \theta}) & \sim R^{(\beta)}_{\nu}(r, \theta) \\
\rho^{(4)}_{\nu}(r e^{i \theta}) & \sim R^{(4)}_{\nu}(r, \theta)
\end{align*}
\]

\[
\begin{align*}
R^{(2)}_{1,\nu}(z) & = K^{(2)}_{\nu}(z, z^*) = w(z) \sum_{j=0}^{N-1} \left| L^{(\nu)}_{j} \left( \frac{z}{h_j} \right) \right|^2, \tag{19}
\end{align*}
\]

and \( R^{(2)}_{2,\nu}(z, u) = R^{(2)}_{1,\nu}(z) R^{(2)}_{1,\nu}(u) - |K^{(2)}_{\nu}(z, u^*)|^2 \). For \( \beta = 4 \) we have a Pfaffian of a matrix kernel instead [20]. An
example for \( P_{\nu=0}^{(4)}(z) \) is shown in Fig. 3 top right. Here two scales have to be fixed: for \( z \) we fit to the increase of the known microscopic density in the \( x \)-direction, and for rescaling \( 2N\mu^2 = \alpha^2 \) its decrease in the \( y \)-direction. Since \( \alpha \leq 2 \) for \( N = 2 \), we conclude that at large-\( N \) for \( \alpha > 2 \) \( P_{\nu}^{(\beta)}(z) \) must become symmetric wrt rotation \( (\beta = 2) \) or reflections wrt the bisector of each quadrant \( (\beta = 4) \). We have checked this, as well as distributions for \( 0 < \mu < 1 \) by generating ensembles of large random matrices.

4. Lattice data. In \cite{17} two-colour QCD was compared to the \( \beta = 4 \) microscopic spectral density in the complex plane from chiral RMT \cite{20}. We use the same data here but with higher statistics, and refer to \cite{17} for all simulation details. Because unimproved staggered fermions are used we are in the \( \beta = 4 \) class at \( \nu = 0 \). Our \( N_f = 2 \) data are effectively quenched for the smallest eigenvalues due to a large mass. In Fig. 4 we compare to the 1st integrated eigenvalue, with \( \alpha = 1.352 \) being close to maximal non-Hermiticity. No further fits compared to \cite{17} are made. In Fig. 5 we compare LGT data at intermediate \( \mu_{\text{Lat}} = 0.1 \) to the angle-dependent surmise Eq. (18) by taking cuts. Here the two scales are fitted to the data, finding an excellent agreement for \( \alpha = 0.65 \).

An alternative to Eq. (18) is the truncated Fredholm expansion in the microscopic large-\( N \) limit \cite{18} which was successfully applied to the \( \beta = 2 \) class \cite{14}. However, integrals of higher order terms rapidly become cumbersome.

5. Conclusions. Conceptually it is possible within chiral RMT to approximate the 1st eigenvalue distribution using a \( 2 \times (2 + \nu) \) matrix calculation, for both real and complex eigenvalues. It is remarkable that this surmise works and captures the repulsion of \( \nu \)-zero-eigenvalues. We derived new compact expressions for \( \beta = 1 \) and \( 4 \) with real eigenvalues for \( \nu > 0 \). Second, we have shown that our surmise for \( \beta = 4 \) successfully describes \( SU(2) \) Lattice data, in an intermediate regime for \( \mu \neq 0 \) where no results were previously known. It would be very interesting to extend our results to the \( \beta = 1 \) non-Hermitian chiral class, having both real and complex eigenvalues.

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