A Parallel Algorithm for Minimum Cost Submodular Cover

Yingli Ran 1  Zhao Zhang 1 *

1 College of Mathematics and Computer Science, Zhejiang Normal University
Jinhua, Zhejiang, 321004, China

Abstract

In a minimum cost submodular cover problem (MinSMC), given a monotone non-decreasing submodular function $f : 2^V \rightarrow \mathbb{Z}^+$, a cost function $c : V \rightarrow \mathbb{R}^+$, an integer $k \leq f(V)$, the goal is to find a subset $A \subseteq V$ with the minimum cost such that $f(A) \geq k$. MinSMC has a lot of applications in machine learning and data mining. In this paper, we design a parallel algorithm for MinSMC which obtains a solution with approximation ratio at most $\frac{H(\min\{\Delta, k\})}{1-5\varepsilon}$ with probability $1-3\varepsilon$ in $O \left(\frac{\log m \log n \log^2 mn}{\varepsilon^4}\right)$ rounds, where $\Delta = \max_{v \in V} f(v)$, $H(\cdot)$ is the Harmonic number, $n = f(V)$, $m = |V|$ and $\varepsilon$ is a constant in $(0, \frac{1}{5})$. This is the first paper obtaining a parallel algorithm for the weighted version of the MinSMC problem with an approximation ratio arbitrarily close to $H(\min\{\Delta, k\})$.

Keyword: minimum cost submodular cover; approximation algorithm; parallel algorithm.

1 Introduction

Recently, submodular optimization has attracted a lot of interest in machine learning and data mining, where it has been applied to a variety of problems including viral marketing [13], information gathering [12], active learning [8], etc.

In this paper, we study parallel algorithm for the minimum cost submodular cover problem (MinSMC). Given a monotone nondecreasing submodular function $f : 2^V \rightarrow \mathbb{Z}^+$, a cost function $c : V \rightarrow \mathbb{R}^+$, an integer $k \leq f(V)$, the goal of MinSMC is to find a subset $A \subseteq V$ with the minimum cost such that $f(A) \geq k$, where the cost of $A$ is $c(A) = \sum_{v \in A} c(v)$.

MinSMC has numerous applications, including data summarization [15], recommender systems [6], etc. For example, given a set of data, it is desirable to select a cheapest set of data whose utility meets a lower bound of requirement. A lot of commonly used utility functions exhibit submodularity, a natural diminishing returns property, leading MinSMC problems [10]. For MinSMC, a centralized greedy algorithm [10] is known to

*Corresponding author: Zhao Zhang hxhzz@sina.com
have approximation ratio $H(\Delta)$, where $H(\Delta) = \sum_{i=1}^{\Delta} 1/i$ is the $\Delta$th Harmonic number and $\Delta = \max_{v \in V} f(v)$.

However, facing with massive data, sequential and centralized greedy method is impractical. Parallel methods have been proposed recently. The best known parallel algorithm for the unweighted MinSMC problem was presented by Fahrbach et al. in [7], which produces a solution of size at most $O(\log(k)|OPT|)$ in at most $O(\log(n \log k) \log k)$ rounds, where $OPT$ is an optimal solution. Note that the algorithm in [7] only deals with the unweighted MinSMC problem. Furthermore, the approximation ratio is $O(\log k)$ and $k$ might be as large as $\Theta(n)$, while $\Delta$ might be much smaller than $k$. Such an observation motivates us to study NC parallel algorithm for the weighted MinSMC problem, trying to obtain approximation ratio arbitrarily close to $H(\Delta)$.

1.1 Related Works

For the MinSMC problem, Wolsey [16] presented a greedy algorithm with approximation ratio $H(\Delta)$, where $\Delta = \max_{v \in V} f(v)$.

Mirzasoleiman et al. [10] proposed a distributed algorithm for the un-weighted MinSMC problem called DISCOVER, which reduces the problem into a set of cardinality-constrained submodular maximization problems. Employing a greedy algorithm for the cardinality-constrained submodular maximization problem, for any fixed constant $0 < \alpha \leq 1$, DISCOVER can find a solution with size $[2\alpha|OPT| + 72|OPT|\sqrt{\min(m, \alpha |OPT|)} \log k]$ in $[\log(\alpha |OPT|) + 36 \sqrt{\min(m, \alpha |OPT|)} \log k/\alpha + 1]$ rounds of messages, where $m$ denotes the number of machines. As noted in [11], it is strange that in this result, when the number of machines is increased, the number of of rounds will increase (rather than decrease). Then the authors in [11] improved the result to a distributed $[\ln k/(1 - \varepsilon)]$-approximation algorithm in at most $[\log_3/2(n/(m |OPT|)) \log \Delta/\varepsilon + \log k]$ rounds of messages, where $n$ is the number of elements. These algorithms have suboptimal adaptivity complexity because the summarization algorithm of the centralized machine is sequential. The number of rounds in the central machine can be $O(n)$. A parallel algorithm with a low adaptivity complexity was presented in [7] with approximation ratio at most $O(\log k)$ in at most $O(\log(n \log k) \log k)$ rounds.

For some special cases of the submodular cover problem, parallel algorithms have been studied recently. In particular, for the set cover problem (i.e., find a smallest subcollection of sets that covers all elements), Berger et al. [3] provided the first parallel algorithm with an approximation guarantee similar to that of the centralized greedy algorithm. They used bucketing technique to obtain a $(1 + \varepsilon)H(n)$-approximation in $O(\log^3 M)$ rounds, where $M$ is the total sum of the sets’ size. Rajagopalan and Vazirani [14] improved the number of rounds to $O(\log^3(Mn))$ at the cost of a larger approximation ratio of $2(1 + \varepsilon)H(n)$. Blelloch et al. [4] further improved the results by obtaining a $(1 + \varepsilon)H(n)$-approximation algorithm in $O(\log^2 M)$ rounds.

1.2 Our contributions and technical overview

In this paper, we design a parallel algorithm for MinSMC, achieving approximation ratio at most $\frac{H(\min(\Delta, k))}{1 - 5\varepsilon}$ with probability at least $1 - 3\varepsilon$, which runs $O(\frac{\log (m \log n \log^2 mn)}{\varepsilon})$
rounds, where \( n = f(V) \), \( m = |V| \) and \( \varepsilon \) is a constant in \((0, \frac{1}{5})\). This is the first paper studying parallel algorithm for the weighted version of MinSMC. Furthermore, the approximation ratio in this paper is arbitrarily close to \( H(\min\{\Delta, k\}) \), while the \( O(\log k) \)-approximation in [7] only works for the cardinality version and \( \Delta \) might be much smaller than \( k \).

We have tried the following method for MinSMC. Iteratively call the parallel algorithm for the submodular maximization problem with the knapsack constraint in [5] until finding a feasible solution to MinSMC. Note that this method runs in logarithmic number of rounds and the approximation ratio is \( O(\log k) \). To improve the ratio dependence on \( k \) to a ratio dependence on \( \Delta \) needs more effort.

This paper combines the ideas of multi-layer bucket in [3], maximal nearly independent set in [4], and random sample in [7]. Note that [3] and [4] deal with the set cover problem. Since submodular cover structure is much more complicated than set-cover structure, the methods in [3] and [4] cannot be directly used on MinSMC. When applied separately, both of them encounter some structural difficulties. The paper [7] deals with a cardinality budgeted version of the submodular maximization problem. To develop its idea to suit for the weighted version of MinSMC, new ideas have to be explored, especially on how to deal with weights.

The remaining part of this paper is organized as follows. The parallel algorithm for MinSMC and analysis is presented in section 2. Section 3 concludes the paper with some discussions on future work.

## 2 Parallel Algorithm and Analysis for MinSMC

### 2.1 Preliminaries

**Definition 2.1** (submodular and monotone nondecreasing). Given an element set \( V = \{v_1, v_2, \ldots, v_m\} \), and a function \( f : 2^V \rightarrow \mathbb{R}^+ \), \( f \) is submodular if \( f(A) + f(B) \geq f(A \cap B) + f(A \cup B) \) for any \( A, B \subseteq V \); \( f \) is monotone nondecreasing if \( f(A) \geq f(B) \) for any \( B \subseteq A \subseteq V \).

For any set \( A, B \subseteq V \), denote \( f_A(B) = f(A \cup B) - f(A) \) to be the marginal profit of \( B \) over \( A \). Assume \( f(V) = n \). In this paper, \( f \) is always assumed to be an integer-valued, monotone nondecreasing, submodular function. It can be verified that for any \( A \subseteq V \), the marginal profit function \( f_A(\cdot) \) is also a monotone nondecreasing, submodular function.

**Definition 2.2** (Minimum Submodular Cover Problem (MinSMC)). Given a monotone nondecreasing submodular function \( f : 2^V \rightarrow \mathbb{Z}^+ \), a cost function \( c : V \rightarrow \mathbb{R}^+ \), an integer \( k \leq n \), the goal of MinSMC is to find \( A \subseteq V \) satisfying

\[
\min\{c(A) : A \subseteq V, f(A) \geq k\},
\]

where \( c(A) = \sum_{v \in A} c(v) \).

Define a function \( g \) as \( g(A) = \min\{f(A), k\} \) for any subset \( A \subseteq V \). When \( f \) is a monotone nondecreasing submodular function, it can be verified that \( g \) is also a monotone
nondecreasing submodular function. Note that \( \max\{g(A) : A \subseteq V\} = k = g(V) \), and for the modified MinSMC problem

\[
\min\{c(A) : g(A) = g(V)\},
\]

(2)
a set \( A \) is feasible to (2) if and only if \( A \) is feasible to (1). Hence problems (1) and (2) are equivalent in terms of approximability, that is, \( A \) is an \( \alpha \)-approximate solution to problem (2) if and only if \( A \) is an \( \alpha \)-approximate solution to problem (1). In the following, we concentrate on the modified MinSMC problem (2).

The concept of \( \varepsilon \)-maximal nearly independent set (\( \varepsilon \)-MaxNIS) plays a crucial role in the analysis of the parallel algorithm proposed in [4] for the minimum set cover problem. This paper uses a slightly different concept which only needs nearly independent property.

**Definition 2.3** (\( \varepsilon \)-nearly independent set (\( \varepsilon \)-NIS)). For a real number \( \varepsilon > 0 \) and a set \( S \subseteq V \), we say that a set \( J \subseteq V \setminus S \) is an \( \varepsilon \)-NIS with respect to \( S \) and \( \varepsilon \) if \( J \) satisfies the following nearly independent property:

\[
g_S(J) \geq (1 - \varepsilon)^2 \sum_{v \in J} g_S(v).
\]

(3)

### 2.2 Algorithm

The main algorithm is described in Algorithm 1. In line 1 to line 7, the instance is preprocessed, the purpose of which is to ensure that the modified instance satisfies \( c_{\max}/c_{\min} \leq mn^2/\varepsilon \), so that the number of rounds can be bounded by the input size, where \( c_{\max} \) and \( c_{\min} \) are the maximum and the minimum cost of elements, respectively. Sub-procedure MinSMC-Par (described in Algorithm 2) is called in line 8 of Algorithm 1.

**Algorithm 1** MinSMC-Main

**Input:** MinSMC instance \( I = (V, g, c, k) \) and a constant \( 0 < \varepsilon < 1/4 \).

**Output:** A subset \( V' \subseteq V \) such that \( g(V') \geq k \).

1. sort the elements in increasing order of costs as \( c(v_1) \leq c(v_2) \leq \ldots \leq c(v_m) \)
2. \( j \leftarrow \arg \min\{j : g(\{v_1, \ldots, v_j\}) \geq k\} \)
3. \( V_0 \leftarrow \{v \in V : c(v) < \frac{\varepsilon}{c_{\min}}c(v_j)\} \)
4. \( V_1 \leftarrow \{v \in V : c(v) > jc(v_j)\} \)
5. \( V^{\text{mod}} \leftarrow V - (V_0 \cup V_1) \)
6. \( g^{\text{mod}} \leftarrow g_{V_0} \) where \( g_{V_0} \) is the marginal profit function of the set over \( V_0 \)
7. \( k^{\text{mod}} \leftarrow \max\{0, k - g(V_0)\} \)
8. \( B^{\text{mod}} \leftarrow \text{MinSMC-Par}(I^{\text{mod}} = (V^{\text{mod}}, g^{\text{mod}}, c, k^{\text{mod}}, \varepsilon)) \)
9. \( V' \leftarrow B^{\text{mod}} \cup V_0 \)

Algorithm 2 (MinSMC-Par) deals with the modified instance \( I^{\text{mod}} \). It divides the elements into buckets, first by marginal profit-to-cost ratio, then by marginal profit (see line 10 of Algorithm 2). Priority is given to those buckets with higher profit-to-cost ratio. For those buckets with the same profit-to-cost ratio, priority is given to those buckets with
higher marginal profit. Algorithm 2 processes the buckets in decreasing priority. Note
that after some sets are chosen, an element in a bucket of higher priority may drop into
a bucket with lower priority. For each bucket, Algorithm 2 tries to find an \( \varepsilon \)-NIS using
procedure NIS(\( A, B, \varepsilon, \beta, \tau \)) (described in Algorithm 3).

**Algorithm 2** MinSMC-Par(\( V, g, c, k, \varepsilon \))

**Input:** MinSMC-Par instance (\( V, g, c, k, \varepsilon \)).

**Output:** A subset \( B \subseteq V \) with \( g(B) \geq k \).

1: \( t \leftarrow 1 \)
2: \( \beta = \arg \max_{v \in V} g(v)/c(v) \)
3: \( \tau = \arg \max_{v \in V} g(v) \)
4: \( T = \log_{1/(1-\varepsilon)} \frac{k c_{\max}}{c_{\min}} \)
5: \( \ell = \log_{1/(1-\varepsilon)} k \)
6: \( t' \leftarrow 1 \)
7: \( B^1_t = \emptyset \)
8: while \( t \leq T \) do
9:  while \( t' \leq \ell \) do
10:  let bucket \( A'^t_t = \{ v \in V : (1-\varepsilon)^t \beta \leq g_{B'^t_t}(v)/c(v) \leq (1-\varepsilon)^{t-1} \beta, (1-\varepsilon)^t \tau \leq g_{B'^t_t}(v) \leq (1-\varepsilon)^{t-1} \tau \} \)
11:  if \( A'^t_t = \emptyset \) then
12:    break and go to line 18
13:  end if
14:  \( J^t_t = \text{NIS}(A'^t_t, B'^t_t, \varepsilon, \beta, \tau) \)
15:  \( B^t_{t+1} = B'^t_t \cup J^t_t \)
16:  \( t' \leftarrow t' + 1 \)
17: end while
18: if \( g(B'^t_t) \geq k \) then
19:  break and go to line 24
20: end if
21: \( B^1_{t+1} = B'^t_t \)
22: \( t \leftarrow t + 1 \)
23: end while
24: return \( B \leftarrow B'^t_t \)

In the \( p \)th while-loop of Algorithm 3, an \( \varepsilon \)-NIS with respect to \( B_p \) is found. After \( r \)
while-loops, an \( \varepsilon \)-NIS with respect to \( B \) is obtained. For each while-loop of Algorithm 3
a for-loop is used to guess the size \( t_p \) of the \( \varepsilon \)-NIS with respect to \( B_p \). In the for-loop,
a mean operation described in Algorithm 4 is called. As will be shown, if \( t_p \) is correctly
guessed, then \( \bar{\mu}_p \leq 1 - 1.5\varepsilon \), and the random set sampled in line 22 satisfies the property
required by a nearly independent set. A set consisting of \( t \) elements is abbreviated as a \( t \)-set. When we say “select a \( t \)-set \( T \) from \( A \) uniformly and randomly”, it means that
elements are selected sequentially from \( A \) until we have \( t \) elements at hand. So, any
specific \( t \)-set \( T \) appears with probability \( P(T) = \frac{1}{|A|(|A|-1)\cdots(|A|-t+1)} \). Note that viewing \( T \)
as an ordered set will facilitate its selection as well as the probabilistic computations.
Algorithm 3 NIS($A, B, \varepsilon, \ell, \beta, \tau$)

Input: An input set $A$ and $B$, threshold $\beta$ and $\tau$, a constant $0 < \varepsilon < 1/4$, a parameter $\ell$.

Output: A set $J \subseteq A$

1: $J_1 \leftarrow \emptyset$
2: $p \leftarrow 1$
3: $i \leftarrow -1$
4: $A_0 \leftarrow A$
5: $B_1 \leftarrow B$
6: $\bar{\varepsilon} \leftarrow \frac{1}{3}(1 - \frac{1}{2T\ell})\varepsilon$
7: $r \leftarrow \log_{\frac{1}{1-\bar{\varepsilon}}}(2mT\ell)/\varepsilon$
8: $\delta \leftarrow \varepsilon/(2rnT^2\ell)$
9: while $p \leq r$ do
10:   $A_p \leftarrow \{v \in A_{p-1}: (1 - \varepsilon)\beta \leq g_{B_{p}}(v)/c(v) \leq \beta, (1 - \varepsilon)\tau \leq g_{B_{p}}(v) \leq \tau\}$
11:   if $A_p \leftarrow \emptyset$ then
12:      break the while loop
13:   end if
14:   for $i \leq \log_{1+\bar{\varepsilon}} m$ do
15:      $t_p \leftarrow \min\{(1 + \bar{\varepsilon})^i, |A_p|\}$
16:      $\bar{\mu}_p \leftarrow \text{Mean}(B_{p}, A_{p}, t_p, \tau, \bar{\varepsilon}, \delta)$
17:      if $\bar{\mu}_p \leq 1 - 1.5\bar{\varepsilon}$ then
18:         break the for loop
19:      end if
20:      $i \leftarrow i + 1$
21:   end for
22:   select a $t_p$-set $T_p$ from $A_p$ uniformly and randomly, and let $J_{p+1} \leftarrow J_p \cup T_p$
23:   $B_{p+1} \leftarrow B_p \cup J_p$
24:   if $g(B_{p+1}) \geq k$ then
25:      break the while loop
26:  end if
27:  $p \leftarrow p + 1$
28: end while
29: Return $J_p$

Algorithm 4 uses the mean value of function $I_{t, A, B, \tau, \varepsilon}$ to measure the expected quality of a sampled set, where $I_{t, A, B, \tau, \varepsilon}$ is a random indicator function defined as follows. Given two sets $A$, $B$, a parameter $\tau$, and a real number $0 < \varepsilon < 1$, for a random $t$-set $X$ which is selected from $A$ uniformly and randomly, and an element $x$ which is drawn uniformly at random from $A \setminus X$,

$$I_{t, A, B, \tau, \varepsilon}(X, x) = I[g_{B \cup X}(x) \geq (1 - \varepsilon)\tau],$$

that is, $I_{t, A, B, \tau, \varepsilon}(X, x) = 1$ if $g_{B \cup X}(x) \geq (1 - \varepsilon)\tau$, and $I_{t, A, B, \tau, \varepsilon}(X, x) = 0$ otherwise. As a convention,

$$I_{t, A, B, \tau, \varepsilon}(X, x) = 0.$$  \hspace{1cm} (4)
Proof. Let \( Y_m = \sum_{i=1}^{m} I_{t,B,A,\tau,\epsilon}(X_i, x_i) \), and let \( \mu = E[I_{t,B,A,\tau,\epsilon}(X, x)] \). By the Chernoff bound (see \([9]\)), for any \( a > 0 \),

\[
P[|Y_m - m\mu| \geq a] \leq 2e^{-\frac{a^2}{2m\mu}}.
\]  

(5)
For \( a = \frac{\varepsilon_m}{2} \), using \( m = 8\lfloor \log(2/\delta) / \varepsilon^2 \rfloor \) and \( \mu \leq 1 \), we have

\[
\frac{a^2}{2m\mu} \geq \log(2/\delta). \tag{6}
\]

Combining inequalities (5) and (6), we have \( P[|Y_m - m\mu| \geq \frac{\varepsilon_m}{2}] \leq \delta \). That is, \( P[|\mu_p - \mu| \geq \frac{\varepsilon}{2}] \leq \delta \). If \( \mu_p \leq 1 - 1.5\varepsilon \), then \( P(\mu > 1 - \varepsilon) \leq P(|\mu_p - \mu| \geq \varepsilon/2) \leq \delta \), that is, with probability at least \( 1 - \delta \), we have \( E[I_{t_p,B_p,A_p,\tau,\varepsilon}(X,x)] \leq 1 - \varepsilon \). The second half of the lemma can be proved similarly.

\[\Box\]

### 2.3 Performance analysis

The next lemma shows that the expected size of \( A_p \) decreases exponentially, which implies that in Algorithm 3, a bucket will become empty in at most \( \log_{1+\varepsilon} m \) rounds.

**Lemma 2.6.** If the for loop of Algorithm 3 is broken because of line 17, then \( E[|A_{p+1}|] \leq (1 - \varepsilon)|A_p| \) with probability at least \( 1 - \delta \).

**Proof.** The inequality is obvious if \( A_{p+1} = \emptyset \). In the following, assume \( A_{p+1} \neq \emptyset \).

By the assumption of this lemma, we have \( \bar{\mu}_p \leq 1 - 1.5\varepsilon \). Then by Lemma 2.5, with probability at least \( 1 - \delta \),

\[
E[I_{t_p,B_p,A_p,\tau,\varepsilon}(T, x)] \leq 1 - \varepsilon. \tag{7}
\]

Note that after \( T_p \) is picked, \( x \in A_p \) is included into \( A_{p+1} \) only when \( I[g_{B_p \cup T_p}(x) \geq (1 - \varepsilon)\tau, g_{B_p \cup T_p}(x)/c(x) \geq (1 - \varepsilon)\beta] \), also note that this term is zero if \( x \in T_p \), so

\[
|A_{p+1}| = \sum_{x \in A_p \setminus T_p} I[g_{B_p \cup T_p}(x) \geq (1 - \varepsilon)\tau, g_{B_p \cup T_p}(x)/c(x) \geq (1 - \varepsilon)\beta]
\]

\[
\leq \sum_{x \in A_p \setminus T_p} I[g_{B_p \cup T_p}(x) \geq (1 - \varepsilon)\tau].
\]

It follows that

\[
E\left[ \frac{|A_{p+1}|}{|A_p \setminus T_p|} \right] = \sum_{T_p} P[T_p \text{ is picked}] E\left[ \frac{|A_{p+1}|}{|A_p \setminus T_p|} | T_p \right]
\]

\[
\leq \sum_{T_p} P[T_p \text{ is picked}] \left( \sum_{x \in A_p \setminus T_p} P[x \text{ is picked} | T_p] I[g_{B_p \cup T_p}(x) \geq (1 - \varepsilon)\tau] / |A_p \setminus T_p| \right)
\]

\[
= \sum_{T_p, x \in A_p \setminus T_p} P[T_p, x \text{ are picked}] I[g_{B_p \cup T_p}(x) \geq (1 - \varepsilon)\tau] / |A_p \setminus T_p|
\]

\[
\leq \sum_{T_p, x \in A_p \setminus T_p} P[T_p, x \text{ are picked}] I[g_{B_p \cup T_p}(x) \geq (1 - \varepsilon)\tau]
\]

\[
= E[I_{t_p,B_p,A_p,\tau,\varepsilon}(T, x)],
\]

where the second inequality uses the observation \( |A_p \setminus T_p| \geq 1 \) (since \( A_{p+1} \neq \emptyset \)). Combining this with inequality 7, we have \( E\left[ \frac{|A_{p+1}|}{|A_p \setminus T_p|} \right] \leq 1 - \varepsilon \). Thus \( E[|A_{p+1}|] \leq (1 - \varepsilon)E[|A_p \setminus T_p|] \leq (1 - \varepsilon)|A_p| \). The lemma is proved.

\[\Box\]
For clarity of statement, we call the bucket $A_t'$ in line 14 of Algorithm 2 as a subordinate bucket and the bucket $A_t = \{v \in V: (1 - \varepsilon)^i \beta \leq g_B (v) / c(v) \leq (1 - \varepsilon)^{i-1} \beta \}$ as a primary bucket. The following lemma says that for any $t \leq \ell$ and $t' \leq \ell$, when line 14 of Algorithm 2 outputs set $J_t'$, the subordinate bucket $A_t'$ becomes empty with a certain probability.

**Lemma 2.7.** When Algorithm 3 reaches line 20, the set $A_p$ in line 14 is empty with probability at least $1 - \varepsilon / (T^2 \ell)$.

**Proof.** For a $p \leq r$, if the for loop is executed $i = \log_{1+\varepsilon} m$ rounds, then $t_p = |A_p|$, and $A_{p+1}$ becomes empty. Next, consider the case when the for loop breaks because of line 17. Denote $C_p$ to be the event $E[|A_{p+1}|] \leq (1 - \varepsilon)|A_p|$. By Lemma 2.6, $P(C_p) \leq \delta = \varepsilon / (2rnT\ell)$. By the union bound,

$$P[C_1 \cap C_2 \cap \ldots \cap C_r] = 1 - P[\bar{C}_1 \cup \bar{C}_2 \cup \ldots \cup \bar{C}_r] \geq 1 - \sum_{i=1}^{r} P(C_i) \geq 1 - \varepsilon / (2nT^2 \ell).$$

So, with probability at least $1 - \varepsilon / (2nT^2 \ell)$, we have

$$E[|A_r|] \leq (1 - \varepsilon)^r \cdot E[|A_1|] \leq \varepsilon / (2T\ell).$$

Denote the event $D$ to be $E[|A_r|] \leq \varepsilon / (2T\ell)$. We have proved $P(D) \geq 1 - \varepsilon / (2nT^2 \ell)$. Using Markov’s inequality, $P(|A_r| \geq 1 | D) \leq \varepsilon / (2T^2 \ell)$. So, $P(A_r = \emptyset) \geq P(D)P(A_r = \emptyset | D) = P(D) (1 - P(|A_r| \geq 1 | D)) \geq (1 - \varepsilon / (2T^2 \ell)) (1 - \varepsilon / (2T^2 \ell)) \geq 1 - \varepsilon / (T^2 \ell)$. The lemma is proved. \hfill \square

The following corollary shows that when the inner while loop of Algorithm 2 halts, the primary bucket $A_t$ is empty with a certain probability.

**Corollary 2.8.** After $J_t'$ is computed in line 14 of Algorithm 2, the primary bucket $A_t$ is empty with probability at least $1 - \varepsilon / T^2$.

**Proof.** For $1 \leq t' \leq \ell$, let $C_t'$ be the event of $A_t' = \emptyset$. Lemma 2.7 says that $P(C_t') \geq 1 - \varepsilon / (T\ell)$. We now consider the event $D$ computed in line 14 of Algorithm 2. Let $D$ be the event of $A_t = \emptyset$. Then $D = C_{t'} \cap C_{t'} \cap \ldots \cap C_{t'}$. So, after $J_t'$ is computed,

$$P(D) = 1 - P[\bar{C}_1 \cup \bar{C}_2 \cup \ldots \cup \bar{C}_t'] \geq 1 - \sum_{i=1}^{t} P(\bar{C}_i) \geq 1 - \varepsilon / T^2.$$

The lemma is proved. \hfill \square

The next lemma shows that with a certain probability, the set $J_t'$ computed in line 14 of Algorithm 2 satisfies the nearly independent property defined in (3).
Lemma 2.9. $E[g_{B_t^i}(J_{i^*})] \geq (1 - \varepsilon)^2 \sum_{v \in J_{i^*}} g_{B_t^i}(v)$ with probability at least $1 - \delta n r$.

Proof. Consider the call of Algorithm 3 when the input parameter $B$ is $B_t^i$. Note that $B_1 = B_t^i$. We first prove that with probability at least $1 - n \delta$, the set $T_p$ sampled in line 22 of Algorithm 3 satisfies

$$E[g_{B_t}(T_p)] \geq (1 - \varepsilon)^2 \sum_{v \in T_p} g_{B_t}(v),$$

where $B_p$ is the set in line 23. For clarity of statement, denote the size of $T_p$ as $t^*$. Inequality (8) is obviously true if $t^* = 0$ or 1. Next, suppose $t^* \geq 2$. Note that line 22 of Algorithm 3 is executed after the for loop is jumped out. Further note that the jump-out must be because of line 17. In fact, if the number of iterations of the for loop has reached $\log_{1+\varepsilon} m$, then $t_p = |A_p|$, and every $X_i$ in Algorithm 4 is $A_p$, resulting in $\bar{\mu}_p = 0$ (see (11)), at which time the condition of line 17 is satisfied. In the previous round of the for loop, that is, when $t_p$ tries the value $\bar{t} = t^*/(1 + \varepsilon)$, we must have $\bar{\mu}_p > 1 - 1.5 \varepsilon$, and thus

$$E[I_{i,B_p,A_p,\tau,\varepsilon}(X, x)] \geq 1 - 2 \varepsilon$$

by Lemma 2.5. Assume that $T_p = \{v_1, \ldots, v_{t^*}\}$, and for any $i \leq t^*$, denote $T_p^i = \{v_1, \ldots, v_i\}$. By the monotonicity of $g$, we have

$$E[g_{B_p}(T_p)] \geq E[g_{B_p}(T_p^i)] = \sum_{i=1}^{\bar{t}} E[g_{B_p \cup T_p^{i-1}}(v_i)].$$

By the definition of $I_{i,B_p,A_p,\tau,\varepsilon}(X, x)$ and Markov’s inequality,

$$E[I_{i,B_p,A_p,\tau,\varepsilon}(T_p^i, v_{i+1})] = P[g_{B_p \cup T_p^{i}}(v_{i+1}) \geq (1 - \varepsilon)\tau] \leq \frac{E[g_{B_p \cup T_p^{i}}(v_{i+1})]}{(1 - \varepsilon)\tau}. \quad (11)$$

Combining inequalities (10) and (11) we have

$$E[g_{B_p}(T_p)] \geq (1 - \varepsilon)\tau \cdot \sum_{i=1}^{\bar{t}} E[I_{i,B_p,A_p,\tau,\varepsilon}(T_p^i, v_{i+1})]. \quad (12)$$

For any $i \leq \bar{t}$, by Lemma 2.4 and inequality (9), with probability at least $1 - \delta$

$$E[I_{i,B_p,A_p,\tau,\varepsilon}(T_p^i, v_{i+1})] \geq 1 - 2 \varepsilon.$$

Combining inequalities (12), (13) and the union bound, with probability at least $1 - n \delta$

$$E[g_{B_p}(T_p)] \geq (1 - \varepsilon)^2 \bar{t} \bar{t}^* \tau \geq 1 + \varepsilon (1 - 2 \varepsilon)(1 - \varepsilon)\tau \geq (1 - \varepsilon)^2 t^* \tau,$$

where the last inequality comes from the choice of $\varepsilon$. By line 10 and line 14 of Algorithm 2 when the computation enters Algorithm 3 we have $g_{B_t}(v) \leq \tau$ for any $v$ in the input.
set $A$, with respect to the input parameter $\tau$. It follows that $g_{B_i}(v) \leq \tau$ holds for every $v \in T_p$. Combining this with (14), inequality 8 is proved.

Then, by the union bound, and similar to the proof of Corollary 2.8, with probability at least $1 - \delta nr$,

$$
\sum_{p=1}^{r} E[g_{B_p}(T_p)] \geq (1 - \varepsilon)^2 \sum_{i=p}^{r} \sum_{v \in T_p} g_{B_i}(v)
$$

Combining this with $J'_t = \bigcup_{p=1}^{r} T_p$, with probability at least $1 - \delta nr$

$$
E[g_{B'_t}(J'_t)] = \sum_{p=1}^{r} E[g_{B_p}(T_p)] \\
\geq (1 - \varepsilon)^2 \sum_{p=1}^{r} \sum_{v \in T_p} g_{B_p}(v) \\
= (1 - \varepsilon)^2 \sum_{v \in J'_t} g_{B'_t}(v),
$$

where the last inequality comes from $B_1 = B'_t$ and $J'_t = \bigcup_{p=1}^{r} T_p$. □

For simplicity of statement, we assume that every inner while loop is executed $\ell$ times. Denote by $D_t = J'_t \cup J''_t \cup \ldots \cup J''''_t$ for any $t \leq T$. The following corollary shows that the expected cost effectiveness of $\{D_t\}$ decreases geometrically.

**Corollary 2.10.** For any $t \leq T$, $\frac{E[g_{B'_t}(D_t)]}{c(D_t)} \geq (1 - \varepsilon)^{t+2} \beta$ with probability at least $1 - \delta nr \ell$.

**Proof.** By Lemma 2.9 and the union bound, with probability at least $1 - \delta nr \ell$,

$$
\sum_{t'=1}^{\ell} E[g_{B'_t}(J'_t)] \geq (1 - \varepsilon)^2 \sum_{t'=1}^{\ell} \sum_{v \in J'_t} g_{B'_t}(v).
$$

By the definition of $A'_t$ in line 10 of Algorithm 2, for any $t' \leq \ell$ and $v \in J'_t$, we have $\frac{g_{B'_t}(v)}{c(v)} \geq (1 - \varepsilon)^t \beta$. Then by inequality (16), with probability at least $1 - \delta nr \ell$,

$$
\frac{E[g_{B'_t}(D_t)]}{c(D_t)} = \frac{\sum_{t'=1}^{\ell} E[g_{B'_t}(J'_t)]}{c(D_t)} \\
= \frac{(1 - \varepsilon)^2 \sum_{t'=1}^{\ell} \sum_{v \in J'_t} g_{B'_t}(v)}{c(D_t)} \\
= \frac{(1 - \varepsilon)^2 \sum_{t'=1}^{\ell} \sum_{v \in J'_t} g_{B'_t}(v)}{\sum_{t'=1}^{\ell} \sum_{v \in J'_t} c(v)} \\
\geq (1 - \varepsilon)^{t+2} \beta.
$$

The lemma is proved. □
Now, we are ready to analyze the expected performance of Algorithm 2.

**Theorem 2.11.** For any constant $0 < \varepsilon < 1/4$, with probability at least $1 - 3\varepsilon$, Algorithm 2 outputs an $\frac{H(\min(\Delta, k))}{1-4\varepsilon}$-approximate solution to the MinSMC problem in $O\left(\frac{T \log k (\log m + \log kT)}{\varepsilon^3} \log m \right)$ rounds, where $\Delta = \max_{v \in V} f(v)$.

**Proof.** The double while-loops are executed at most $T\ell$ times, and each call of procedure NIS needs at most $r \log_{1+\varepsilon} m$ rounds. Note that $r = O((\log m + \log kT)/\varepsilon)$. So, the total number of rounds is $T\ell r \log_{1+\varepsilon} m = O\left(\frac{T \log k (\log m + \log kT)}{\varepsilon^3} \log m \right)$.

Next we estimate the expected approximation ratio. Let $B$ be the output of Algorithm 2 then $B = D_1 \cup \ldots \cup D_T$, $B^1_t = D_1 \cup \cdots \cup D_{t-1}$ for $t \geq 2$ and $B^1_1 = \emptyset$, where $D_i = J^1_i \cup J^2_i \cup \ldots \cup J^T_i$ for $t \leq T$. The following claim shows that based on $D_1, D_2, \ldots, D_T$, we can construct a sequence of sets whose expected cost-effectiveness is monotone.

**Claim 1.** We can construct a sequence of sets $D'_1, D'_2, \ldots, D'_p$ with $p \leq T$ such that with probability at least $1 - 3\varepsilon/2$,

$$
\frac{E[g_{B'_i}(D'_{i+1})]}{c(D'_{i+1})} \leq \frac{E[g_{B'_i}(D'_i)]}{c(D'_i)}
$$

holds for any $i \leq p$, where $B'_i = \{D'_1, \ldots, D'_i\}$ for $i \leq p$.

If $D_1, D_2, \ldots, D_T$ satisfy \(17\) for all $i = 1, \ldots, T$, then the claim holds by letting $p = T$ and $D'_i = D_i$ for $i = 1, \ldots, T$. Otherwise, let $t$ be the minimum index with

$$
\frac{E[g_{B^1_t}(D_t)]}{c(D_t)} > \frac{E[g_{B^1_{t-1}}(D_{t-1})]}{c(D_{t-1})}.
$$

Let $D'_t = D_1, \ldots, D'_{t-3} = D_{t-3}$. Inequality \(17\) holds for $i = 1, \ldots, t-4$. In the following construction, a $D'_i$ might be the union of a consecutive sets of $\{D_j\}$. Once a set $D'_i$ is constructed, it remains unchanged afterwards.

**Some preparations.** We first prove that with probability at least $1 - \frac{\varepsilon}{2^T}$,

$$
\frac{E[g_{B^1_{t-1}}(D_{t-1} \cup D_t)]}{c(D_{t-1} \cup D_t)} \geq (1 - \varepsilon)^{t+1} \beta,
$$

and with probability at least $1 - \frac{\varepsilon}{2^T}$,

$$
\frac{E[g_{B^1_{t-1}}(D_{t-1} \cup D_t)]}{c(D_{t-1} \cup D_t)} \leq (1 - \varepsilon)^{t-1} \beta.
$$

In fact, by Corollary 2.10 with probability at least $1 - \delta \eta r \ell = 1 - \frac{\varepsilon}{2^T}$,

$$
\frac{E[g_{B^1_{t-1}}(D_{t-1})]}{c(D_{t-1})} \geq (1 - \varepsilon)^{t+1} \beta.
$$

Observe that

$$
\frac{E[g_{B^1_{t-1}}(D_{t-1} \cup D_t)]}{c(D_{t-1} \cup D_t)} = \frac{E[g_{B^1_{t}}(D_{t})]}{c(D_{t})} + \frac{E[g_{B^1_{t-1}}(D_{t-1})]}{c(D_{t-1})}.
$$
Property (19) follows from inequalities (18), (21), (22), and the observation $B^1_i = B^1_{i-1} \cup D_{i-1}$. By Corollary 2.8 after $J_{i-1}'$ is computed, with probability at least $1 - \varepsilon/\mathcal{T}^2$, the primary bucket $A_{i-1}$ is empty. By the union bound, with probability at least $1 - \varepsilon/\mathcal{T}$, all primary buckets $A_i$ with $i \leq t - 1$ are empty, which implies that

$$
\text{every remaining element } v \text{ satisfies } g_{B^1_i}(v)/c(v) < (1 - \varepsilon)^{t-1}\beta,
$$

and thus by the submodularity of function $g_{B^1_i}$, we have $g_{B^1_i}(D_i) \leq \sum_{v \in D_i} g_{B^1_i}(v) < (1 - \varepsilon)^{t-1}\beta c(D_i)$. Combining this with inequality (18), with probability at least $1 - \varepsilon/\mathcal{T}$, both $E[g_{B^1_i}(D_i)]/c(D_i)$ and $E[g_{B^1_{i-1}}(D_{i-1})]/c(D_{i-1})$ are upper bounded by $(1 - \varepsilon)^{t-1}\beta$, and thus by (22), we have property (20).

Similar to the proof of (19) and (20), we can prove that with probability at least $1 - \frac{3\varepsilon}{2\mathcal{T}}$, the following inequalities hold:

If $\frac{E[g_{B^1_i}(D_{i-2})]}{c(D_{i-2})} \geq (1 - \varepsilon)^{t-1}\beta$ and $\frac{E[g_{B^1_{i-2}}(D_{i-1} \cup D_i, D_{i-2})]}{c(D_{i-1} \cup D_i, D_{i-2})} < \frac{E[g_{B^1_{i-1}}(D_{i-1})]}{c(D_{i-1})}$, then

$$
(1 - \varepsilon)^{t+1}\beta \leq \frac{E[g_{B^1_{i-1}}(D_{i-1} \cup D_i, D_{i+1})]}{c(D_{i-1} \cup D_i, D_{i+1})} \leq (1 - \varepsilon)^{t-1}\beta.
$$

If $\frac{E[g_{B^1_i}(D_{i-2})]}{c(D_{i-2})} < (1 - \varepsilon)^{t-1}\beta$ and $\frac{E[g_{B^1_{i-2}}(D_{i-2} \cup D_{i-1} \cup D_i)]}{c(D_{i-2} \cup D_{i-1} \cup D_i)} \geq \frac{E[g_{B^1_{i-1}}(D_{i-1})]}{c(D_{i-1})}$, then

$$
(1 - \varepsilon)^{t+1}\beta \leq \frac{E[g_{B^1_{i-1}}(D_{i-2} \cup D_{i-1} \cup D_i)]}{c(D_{i-2} \cup D_{i-1} \cup D_i)} \leq (1 - \varepsilon)^{t-1}\beta.
$$

If $\frac{E[g_{B^1_i}(D_{i-2})]}{c(D_{i-2})} < (1 - \varepsilon)^{t-1}\beta$ and $\frac{E[g_{B^1_{i-2}}(D_{i-2} \cup D_{i-1} \cup D_i)]}{c(D_{i-2} \cup D_{i-1} \cup D_i)} < \frac{E[g_{B^1_{i-1}}(D_{i-1})]}{c(D_{i-1})}$, then

$$
(1 - \varepsilon)^{t+1}\beta \leq \frac{E[g_{B^1_{i-2}}(D_{i-2} \cup D_{i-1} \cup D_i, D_{i+1})]}{c(D_{i-2} \cup D_{i-1} \cup D_i, D_{i+1})} \leq (1 - \varepsilon)^{t-1}\beta.
$$

In fact, for the above inequalities, the left side holds with probability at least $1 - \frac{\varepsilon}{\mathcal{T}^2}$ and the right side holds with probability at least $1 - \frac{\varepsilon}{\mathcal{T}}$. In order that both sides hold, by the union bound, the probability is at least $1 - (\frac{\varepsilon}{\mathcal{T}^2} + \frac{\varepsilon}{\mathcal{T}}) \geq 1 - \frac{3\varepsilon}{2\mathcal{T}}$.

**Construction of $D_{t-2}'$.** Next, we show how to construct $D_{t-2}'$. Besides property (17), the following property (i) will be satisfied:

(i) If $D_{t-2}' = D_{t-2}$, then $D_{t-1}'$ can be constructed at the same time which contains more than one sets from $\{D_j\}$, and with probability at least $1 - \frac{3\varepsilon}{2\mathcal{T}^2}$,

$$
(1 - \varepsilon)^{t+1}\beta \leq \frac{E[g_{B^1_{t-2}}(D_{t-1}')]}{c(D_{t-1}')} \leq (1 - \varepsilon)^{t-1}\beta.
$$

Else $D_{t-2}'$ contains more than one sets from $\{D_j\}$, and with probability at least $1 - \frac{3\varepsilon}{2\mathcal{T}^2}$,

$$
(1 - \varepsilon)^{t+1}\beta \leq \frac{E[g_{B^1_{t-2}}(D_{t-2}')]}{c(D_{t-2}')} \leq (1 - \varepsilon)^{t-1}\beta.
$$

Furthermore, for both inequalities (27) and (28), the lower bound $(1 - \varepsilon)^{t+1}\beta$ holds with probability at least $1 - \frac{3\varepsilon}{2\mathcal{T}^2}$.
Furthermore, by property (20) and the condition of Case 1, with probability at least 1
\[ \frac{E[g_{B_{i-1}}(D'_i)]}{c(D'_i)} \geq \frac{E[g_{B_{j+1}}(D_{j+1})]}{c(D_{j+1})}. \] (29)

The construction is realized by distinguishing two cases.

**Case 1.** \( \frac{E[g_{B_{i-2}}(D_{i-2})]}{c(D_{i-2})} \geq (1-\epsilon)^{t-1} \beta. \)

In this case, let \( D'_{t-2} = D_{t-2} \). Then (17) holds for \( i = t-3 \) by the choice of \( t \). Furthermore, by property (20) and the condition of Case 1, with probability at least \( 1 - \frac{3\epsilon}{2T} \), we have
\[ \frac{E[g_{B_{i-3}}(D'_{t-2})]}{c(D'_{t-2})} \geq \frac{E[g_{B_{i-2}}(D_{t-1} \cup D_t)]}{c(D_{t-1} \cup D_t)}. \] (30)

In this case, we shall construct \( D'_{t-1} \) by distinguishing two subcases.

**Subcase 1.1** \( \frac{E[g_{B_{i-2}}(D_{t-1} \cup D_t)]}{c(D_{t-1} \cup D_t)} \geq \frac{E[g_{B_{i-2}}(D_{t+1})]}{c(D_{t+1})} \).

In this subcase, let \( D'_{t-1} = D_{t-1} \cup D_t \). By (30), with probability at least \( 1 - \frac{\epsilon}{T} \),
\[ \frac{E[g_{B_{i-3}}(D'_{t-2})]}{c(D'_{t-2})} \geq \frac{E[g_{B_{i-2}}(D'_{t-1})]}{c(D'_{t-1})}, \]
which satisfies (17) for \( i = t-2 \). By the condition of Subcase 1.1,
\[ \frac{E[g_{B_{i-2}}(D'_{t-1})]}{c(D'_{t-1})} \geq \frac{E[g_{B_{i-1}}(D_{t+1})]}{c(D_{t+1})}, \]
which satisfies (29) for \( i = t-1 \). Furthermore, (27) follows from (19) and (20).

**Subcase 1.2** \( \frac{E[g_{B_{i-2}}(D_{t-1} \cup D_t)]}{c(D_{t-1} \cup D_t)} < \frac{E[g_{B_{i-2}}(D_{t+1})]}{c(D_{t+1})} \).

In this subcase, let \( D'_{t-1} = D_{t-1} \cup D_t \cup D_{t+1} \). Then (27) holds by (24). Combining the right side of (24) with the condition of Case 1, with probability at least \( 1 - \frac{3\epsilon}{2T} \),
\[ \frac{E[g_{B_{i-3}}(D'_{t-2})]}{c(D'_{t-2})} \geq \frac{E[g_{B_{i-2}}(D'_{t-1})]}{c(D'_{t-1})}, \]
which satisfies (17) for \( i = t-2 \). Similar to the proof of inequality (20), under the condition of Subcase 1.2, with probability at least \( 1 - \frac{\epsilon}{T} \),
\[ \frac{E[g_{B_{i-2}}(D_{t+2})]}{c(D_{t+2})} \leq (1-\epsilon)^{t+1} \beta. \] (31)

Combining inequality (31) with the left side of (24), by the union bound, with probability at least \( 1 - \frac{3\epsilon}{2T} \),
\[ \frac{E[g_{B_{i-2}}(D'_{t-1})]}{c(D'_{t-1})} \geq \frac{E[g_{B_{i-2}}(D_{t+2})]}{c(D_{t+2})}, \] (32)
which satisfies (29) for \( i = t - 1 \).

**Case 2.** \( \frac{E[g_{B_{t-3}}(D_{t-2})]}{c(D_{t-2})} < (1 - \varepsilon)^{-1} \beta \).

We further distinguish two subcases.

**Subcase 2.1** \( \frac{E[g_{B_{t-3}}(D_{t-2} \cup D_{t-1} \cup D_t)]}{c(D_{t-2} \cup D_{t-1} \cup D_t)} \geq \frac{E[g_{B_{t-3}}^1(D_{t+1})]}{c(D_{t+1})} \).

In this subcase, let \( D'_{t-2} = D_{t-2} \cup D_{t-1} \cup D_t \). Then (29) for \( i \leq t - 2 \) follows from the condition of this subcase, and (28) follows from (25). By Corollary 2.10, with probability at least 1\( - \frac{\varepsilon}{27} \),

\[
\frac{E[g_{B_{t-4}}(D'_{t-3})]}{c(D'_{t-3})} \geq (1 - \varepsilon)^{t-1} \beta.
\]

Combining (33) with the right side of inequalities (25), by the union bound, with probability at least 1\( - \frac{3\varepsilon}{27} \),

\[
\frac{E[g_{B_{t-4}}(D'_{t-3})]}{c(D'_{t-3})} \geq \frac{E[g_{B_{t-3}}(D'_{t-2})]}{c(D'_{t-2})},
\]

and thus (17) holds for \( i = t - 3 \).

**Subcase 2.2** \( \frac{E[g_{B_{t-3}}(D_{t-2} \cup D_{t-1} \cup D_t)]}{c(D_{t-2} \cup D_{t-1} \cup D_t)} < \frac{E[g_{B_{t-3}}^1(D_{t+1})]}{c(D_{t+1})} \).

In this subcase, let \( D'_{t-2} = D_{t-2} \cup D_{t-1} \cup D_t \). Then (28) follows from (26). Combining inequality (31) with the left side of inequality (26), we have (29) for \( i \leq t - 2 \). Combining (33) with the right side of inequality (26), with probability at least 1\( - \frac{3\varepsilon}{27} \), (17) holds for \( i = t - 3 \).

The construction for \( D'_{t-2} \) is completed.

**Construction of the following \( D_t \).** The construction for the following \( D_t \)s continues by iteratively finding the next index \( t \) satisfying inequality (18). Denote by \( D'_{it} \) a constructed set which contains more than one \( D_{j_t} \)s based on an index \( t \) satisfying inequality (18). For example, in the previous construction, \( D'_{it} = D'_{i,t-1} \) in Case 1 and \( D'_{it} = D'_{i,t-2} \) in Case 2. By property (ii), we have

\[
\frac{E[g_{B_{t-3}}^1(D_{t})]}{c(D_{t})} \geq \frac{E[g_{B_{t-3}}^1(D_{t+1})]}{c(D_{t+1})},
\]

where \( D'_{it} = \{D_{j_t}, D_{j_t+1}, \ldots, D_{j_t'}\} \). Consider the next minimum index \( t' \) with

\[
\frac{E[g_{B_{t-3}}^1(D_{t'})]}{c(D_{t'})} \geq \frac{E[g_{B_{t-3}}^1(D_{t'-1})]}{c(D_{t'-1})}.
\]

By (35), we have \( t' > j_t' + 1 \). The next \( D' \) is constructed according to the following cases:

1) \( D_{t'-2} \) is contained in a \( D' \) with more than one \( D \).
2) \( D_{t'-2} \) constitutes a \( D' \) and \( D_{t'-3} \) is contained in a \( D' \) with more than one \( D \).
3) both \( D_{t'-2} \) and \( D_{t'-3} \) are contained in a \( D' \) with exactly one \( D \).

By a similar method, \( D' \) can be constructed satisfying inequality (17) and properties (i), (ii). Furthermore, the newly constructed \( D' \) does not contain any previously constructed \( D' \), that is, once a set \( D' \) is constructed, it remains unchanged afterwards.
Claim 1 is proved.

For a set $B$, denote by $\beta(B) = \max_{v \in V} \frac{g_B(v)}{c_B(v)}$ the maximum marginal profit to cost ratio with respect to $B$. The next claim estimates the loss between the expected cost-effectiveness of $D'_i$ and the ratio $\beta(B'_{i-1})$.

Claim 2. For any $1 \leq i \leq p - 1$, with probability at least $1 - 3\varepsilon/2$,
\[
\frac{E[g_{B'_i}(D'_{i+1})]}{c(D'_{i+1})} \geq (1 - \varepsilon)^i \beta(B'_i).
\]

Assume $D'_i = D_j \cup \cdots \cup D_{j'}$ with $j' \geq j$. Note that $B'_{i-1} = B'_j$. If $D'_i = D_j$, then by Corollary 2.10 with probability at least $1 - \delta \rho \ell = 1 - \frac{\varepsilon}{2T}$,
\[
\frac{E[g_{B'_{i-1}}(D'_i)]}{c(D'_i)} = \frac{E[g_{B'_j}(D_j)]}{c(D_j)} \geq (1 - \varepsilon)^j \beta.
\] (37)

From the above construction in Claim 1, $D'_i$ has the following four cases.
\[
\begin{align*}
D'_{i_t} &= D_{t-1} \cup D_t, \\
D'_{i_t} &= D_{t-2} \cup D_{t-1} \cup D_t, \\
D'_{i_t} &= D_{t-2} \cup D_{t-1} \cup D_t \cup D_{t+1}, \\
D'_{i_t} &= D_{t-1} \cup D_t \cup D_{t+1}.
\end{align*}
\]

So a $D'_i = D_j \cup \cdots \cup D_{j'}$ with $j' > j$ must be of the form $D'_i = D'_{i(j+1)}$, or $D'_i = D'_{i(j+2)}$. By property (i), with probability at least $1 - \frac{\varepsilon}{2T}$,
\[
\frac{E[g_{B'_{i-1}}(D'_i)]}{c(D'_i)} \geq (1 - \varepsilon)^{j+2} \beta \text{ or } (1 - \varepsilon)^{j+3} \beta.
\]

Combining this with inequality (37), no matter whether $D'_i$ contains exactly one $D$ of more than one $D$, with probability at least $1 - \frac{\varepsilon}{2T}$,
\[
\frac{E[g_{B'_{i-1}}(D'_i)]}{c(D'_i)} \geq (1 - \varepsilon)^{j+3} \beta. \quad (38)
\]

Similar to the proof of (33), after $D_1, \ldots, D_{j-1}$ are chosen, for any remaining element $v$, with probability at least $1 - \varepsilon/T$, $E[g_{B'_j}(v)]/c(v) \leq (1 - \varepsilon)^{j-1} \beta$. Hence
\[
\beta(B'_{i-1}) = \beta(B'_j) \leq (1 - \varepsilon)^{j-1} \beta.
\]

Combining this with inequality (38), by the union bound, with probability at least $1 - 3\varepsilon/2T$, a fixed $D'_i$ satisfies Claim 2. Again by the union bound, for every $1 \leq i \leq p - 1$, Claim 2 holds with probability at least $1 - 3\varepsilon/2$.

To complete the estimation of the approximation ratio, we consider an optimal solution $A^*$, and construct an auxiliary weight $w$ as follows. Denote $r_i = E[g_{B'_{i-1}}(D'_i)]$ and $z_{v,i} = E[g_{B'_{i-1}}(v)]$ for $1 \leq i \leq p$ and $v \in A^*$. For any $v \in A^*$, define
\[
w(v) = \sum_{i=1}^{p}(z_{v,i} - z_{v,i+1}) \frac{c(D'_i)}{r_i},
\]
Then, Comparing (39) and (40), to prove Claim 3, it suffices to prove that
\[ A \]
where \( z_{v,p+1} = 0. \)

**Claim 3.** With probability at least \( 1 - 3\varepsilon/2, \) \( c(B'_p) \leq \sum_{v \in A^*} w(v). \)

By the definition of \( w(v), \) it can be calculated that
\[ w(A^*) = \sum_{v \in A^*} \sum_{i=1}^{p} (z_{v,i} - z_{v,i+1}) \frac{c(D'_i)}{r_i} \]
\[ = \frac{c(D'_1)}{r_1} \sum_{v \in A^*} z_{v,1} + \sum_{i=2}^{p} \left( \frac{c(D'_i)}{r_i} - \frac{c(D'_{i-1})}{r_{i-1}} \right) \sum_{v \in A^*} z_{v,i}. \quad (39) \]

Similarly, \( c(B'_p) \) can be rewritten as follows:
\[ c(B'_p) = \sum_{i=1}^{p} c(D'_i) = \sum_{i=1}^{p} \frac{c(D'_i)}{r_i} \]
\[ = \sum_{i=1}^{p} \left( \sum_{j=i}^{p} r_j - \sum_{j=i+1}^{p} r_j \right) \frac{c(D'_i)}{r_i} \]
\[ = \frac{c(D'_1)}{r_1} \sum_{j=i}^{p} r_j + \sum_{i=2}^{p} \left( \frac{c(D'_i)}{r_i} - \frac{c(D'_{i-1})}{r_{i-1}} \right) \sum_{j=i}^{p} r_j \quad (40) \]

By Claim 1, with probability at least \( 1 - 3\varepsilon/2, \)
\[ \frac{c(D'_i)}{r_i} \geq \frac{c(D'_{i-1})}{r_{i-1}} \]
holds for any \( 2 \leq i \leq p. \quad (41) \]

Then, Comparing (39) and (40), to prove Claim 3, it suffices to prove that
\[ \sum_{j=i}^{p} r_j \leq \sum_{v \in A^*} z_{v,i} \text{ for any } 1 \leq i \leq p. \quad (42) \]

The left term of inequality (42) can be written as
\[ \sum_{j=i}^{p} r_j = \sum_{j=i}^{p} \left( E[g(B'_{j,i})] - E[g(B'_{j-1,i})] \right) = E[g(B'_{p,i})] - E[g(B'_{i-1,i})] = k - E[g(B'_{i-1,i})]. \quad (43) \]

Suppose \( A^* = \{v_1, \ldots, v_q\}. \) Denote \( A^*_j = \{v_1, \ldots, v_j\} \) for \( j = 1, \ldots, q, \) and \( A^*_0 = \emptyset. \) The right term of inequality (42) can be bounded by
\[ \sum_{v \in A^*} z_{v,i} = \sum_{j=1}^{q} E[g(B'_{i-1,i}(v_j)] \]
\[ \geq \sum_{j=1}^{q} E[g(B'_{i-1} \cup A^*_j(v_j)] \]
\[ = \sum_{j=1}^{q} \left( E[g(B'_{i-1} \cup A^*_j)] - E[g(B'_{i-1} \cup A^*_j)] \right) \]
\[ = E[g(B'_{i-1} \cup A^*)] - E[g(B'_{i-1})] = k - E[g(B'_{i-1})], \quad (44) \]

17
where the inequality comes from the submodularity of $g$. Inequality (42) follows from (13) and (14), and thus Claim 3 is proved.

**Claim 4.** With probability at least $1 - 3\varepsilon/2$, $w(v) \leq c(v) \cdot \frac{H(\min(\Delta,k))}{1 - 4\varepsilon}$.

By Claim 2, for any $v \in V$ and any $1 \leq i \leq p - 1$, with probability at least $1 - 3\varepsilon/2$,

$$(1 - \varepsilon) \frac{E[g_B'(v)]}{c(v)} \leq \frac{E[g_B'(D_{i+1}')]}{c(D_{i+1}')}.$$  \hspace{1cm} (45)

It follows that

$$w(v) = \sum_{i=1}^{p} (z_{v,i} - z_{v,i+1}) \frac{c(D'_i)}{r_i} \leq \sum_{i=1}^{p} (z_{v,i} - z_{v,i+1}) \frac{c(v)}{(1 - \varepsilon)^4 z_{v,i}} \leq c(v) \cdot H(z_{v,1})/(1 - \varepsilon)^4.$$  \hspace{1cm} (46)

Then Claim 4 follows from $z_{v,1} \leq \max_{v \in V} g(v) \leq \min\{\Delta, k\}$ and $(1 - \varepsilon)^4 \geq 1 - 4\varepsilon$.

Combining Claim 3 and Claim 4, by the union bound, with probability at least $1 - 3\varepsilon$, $c(B'_p) \leq \frac{H(\min(\Delta,k))}{1 - 4\varepsilon} c(A^*)$. The approximation ratio is proved. \hfill \Box

Note that Algorithm 2 (MinSMC-Par) deals with the modified instance $I^{\text{mod}}$. The next theorem analyzes the performance of Algorithm 1 which deals with the original instance.

**Theorem 2.12.** With probability at least $1 - 3\varepsilon$ for any $0 < \varepsilon < 1/5$, Algorithm 1 runs in $O\left(\frac{\log m \log n \log^2 mn}{\varepsilon^4}\right)$ rounds and achieves approximation ratio at most $\frac{H(\min(\Delta,k))}{1 - 5\varepsilon}$.

**Proof.** Suppose $A^*$ is an optimal solution of the MinWSMC instance. Recall the definition of $j$ in line 2 of Algorithm 1 which is $j = \arg\min\{j : g(\{v_1, \ldots, v_j\}) \geq k\}$ for ordered elements $c(v_1) \leq c(v_2) \leq \cdots \leq c(V_m)$. We must have

$$c(v_j) \leq c(A^*).$$  \hspace{1cm} (47)

In face, suppose $c(v_{j'}) = \max\{c(v_i) : v_i \in A^*\}$. If $c(v_{j'}) < c(v_j)$, then $A^* \subseteq \{v_1, \ldots, v_{j'}\}$. By the monotonicity of $g$, we have $k = g(A^*) \leq g(\{v_1, \ldots, v_{j'}\})$, contradicting the choice of $j$. So, $c(v_j) \leq c(v_{j'}) \leq c(A^*)$. On the other hand,

$$c(A^*) \leq jc(v_j).$$  \hspace{1cm} (48)

This is because $g(\{v_1 \cup \ldots \cup v_j\}) \geq k$, and thus $\{v_1, \ldots, v_j\}$ is a feasible solution whose cost is $c(v_1) + \cdots + c(v_j) \leq j c_j$. Since $A^*$ has the minimum cost among all feasible solutions, (48) follows. As a consequence of property (47), the set $V_0$ in line 3 of Algorithm 1 has

$$c(V_0) \leq |V_0| \frac{\varepsilon}{mn} c(v_j) \leq \varepsilon \frac{n}{c(A^*)}.$$  \hspace{1cm} (48)

As a consequence of property (48), no element $v$ with cost $c(v) > jc(v_j)$ could appear in $A^*$. So, the optimal solution for the instance $(V - V_1, g, c, k)$ is the same as $A^*$, where $V_1 = \{v \in V : c(v) > jc(v_j)\}$ is the set defined in line 4 of Algorithm 1.
For instance $I^{\text{mod}}$, we have $\frac{c_{\text{max}}}{c_{\text{min}}} \leq \frac{j_{c(v_j)}}{m_{c(v_j)}} = \frac{j_{mn}}{c(v_j)}$. Substituting this bound into the expression of $T$, by Theorem 2.11, Algorithm 2 produces a solution $B^{\text{mod}}$ in $O\left(\log m \log n \log^2 mn\right)$ rounds such that
\[
c(B^{\text{mod}}) \leq \frac{H(\Delta)}{1 - 4\varepsilon} \text{opt}',
\]
where $\text{opt}'$ is the optimal value for instance $I^{\text{mod}}$. By the submodularity and the monotonicity of $g$, we have $k = g(A^* \cup V_0) \leq g(A^* \setminus V_0) + g(V_0)$, and thus $g(A^* \setminus V_0) \geq k^{\text{mod}}$, that is, $A^* \setminus V_0$ is a feasible solution to $I^{\text{mod}}$. It follows that $\text{opt}' \leq c(A^* \setminus V_0)$, and thus the output of $V'$ in Algorithm 1 has cost
\[
c(V') = c(B^{\text{mod}}) + c(V_0) \leq \left(\frac{H(\Delta)}{1 - 4\varepsilon} + \frac{\varepsilon}{n}\right) c(A^*) \leq \frac{H(\Delta)}{1 - 5\varepsilon} c(A^*).
\]
The theorem is proved.

3 Conclusion and Discussion

In this paper, we present a parallel algorithm for MinSMC to obtain a solution with approximation ratio at most $\frac{H(\min\{\Delta, k\})}{1 - 5\varepsilon}$ in $O\left(\frac{\log m \log n \log^2 mn}{\varepsilon^4}\right)$ rounds with probability at least $1 - 3\varepsilon$, where $0 < \varepsilon < 1/5$ is a constant.

How to obtain a parallel algorithm with approximation ratio arbitrarily close to $H(\min\{\Delta, k\})$ with less number of rounds is a topic deserving further exploration.

Acknowledgment

This research work is supported in part by NSFC (11901533, U20A2068, 11771013), and ZJNSFC (LD19A010001).

References

[1] E. Balkanski, A. Rubinstein, and Y. Singer. An exponential speedup in parallel running time for submodular maximization without loss in approximation. arXiv preprint arXiv:1804.06355, 2018.

[2] E. Balkanski and Y. Singer. The adaptive complexity of maximizing a submodular function. In STOC, 2018.

[3] B. Berger, J. Rompel, and P. W. Shor. Efficient nc algorithms for set cover with applications to learning and geometry. Journal of Computer and System Sciences, 49(3):454–477, 1994.

[4] G. E. Blelloch, R. Peng, and K. Tangwongsan. Linear-work greedy parallel approximate set cover and variants. In SPAA, 2011.
[5] Ch. Chekuri, K. Quanrud. Submodular function maximization in parallel via the multilinear relaxation. SODA, pp. 303–322, 2019.

[6] K. El-Arini and C. Guestrin. Beyond keyword search: discovering relevant scientific literature. In KDD, 2011.

[7] M. Fahrbach, V. S. Mirrokni, and M. Zadimoghaddam. Submodular maximization with optimal approximation, adaptivity and query complexity. CoRR, abs/1807.07889, 2018. URL https://arxiv.org/abs/1807.07889.

[8] D. Golovin and A. Krause. Adaptive submodularity: theory and applications in active learning and stochastic optimization. Journal of Artificial Intelligence Research, 42:427–486, 2011.

[9] M. Mitzenmacher and E. Upfal. Probability and Computing. Cambridge University Press, 2005.

[10] B. Mirzasoleiman, A. Karbasi, A. Badanidiyuru, and A. Krause. Distributed submodular cover: succinctly summarizing massive data. In NIPS, 2015.

[11] B. Mirzasoleiman, M. Zadimoghaddam, and A. Krause. Fast distributed submodular cover: public-private data summarization. NIPS, 2016.

[12] A. Krause and C. Guestrin. Intelligent information gathering and submodular function optimization. Tutorial at the International Joint Conference in Artificial Intelligence, 2009.

[13] D. Kempe, J. Kleinberg, and E. Tardos. Maximizing the spread of influence through a social network. In Proceedings of the ninth ACM SIGKDD, 2003.

[14] S. Rajagopalan and V. V. Vazirani. Primal-dual RNC approximation algorithms for set cover and covering integer programs. SIAM J. Comput., 28(2):525–540, 1998.

[15] S. Tschiatschek, R. Iyer, H. Wei, and J. Bilmes. Learning mixtures of submodular functions for image collection summarization. In NIPS, 2014.

[16] L. A. Wolsey. An analysis of the greedy algorithm for the submodular set covering problem. Combinatorica, 2(4):385–393, 1982.