Non-singular cloaks allow mimesis

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Abstract

We design non-singular cloaks enabling objects to scatter waves like objects with smaller size and very different shapes. We consider the Schrödinger equation, which is valid, for example, in the contexts of geometrical and quantum optics. More precisely, we introduce a generalized non-singular transformation for star domains, and numerically demonstrate that an object of nearly any given shape surrounded by a given cloak scatters waves in exactly the same way as a smaller object of another shape. When a source is located inside the cloak, it scatters waves as if it were located some distance away from a small object. Moreover, the invisibility region actually hosts almost trapped eigenstates. Mimetism is numerically shown to break down for the quantified energies associated with confined modes. If we further allow for non-isomorphic transformations, our approach leads to the design of quantum super-scatterers: a small size object surrounded by a quantum cloak described by a negative anisotropic heterogeneous effective mass and a negative spatially varying potential scatters matter waves like a larger nano-object of different shape. Potential applications might be, for instance, in quantum dots probing. The results in this paper, as well as the corresponding derived constitutive tensors, are valid for cloaks with any arbitrary star-shaped boundary cross sections, although for numerical simulations we use examples with piecewise linear or elliptic boundaries.

Keywords: quantum optics, transformation optics, metamaterials, anisotropic optical materials, invisibility

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Control of electromagnetic waves can be achieved through co-ordinate transformations which introduce exotic material parameters \cite{1–3}. Electromagnetic metamaterials within which negative refraction and focusing effects involving the near field can occur \cite{4–7} can be understood in the light of transformation optics \cite{3}.

Recently, an electron focusing effect across a p–n junction in a graphene film that mimics the Pendry–Veselago lens in optics has been proposed \cite{8}. The subsequent theoretical demonstration of 100\% transmission of cold rubidium atoms through an array of sub-de Broglie wavelength slits brings the original continuous wave phenomenon in contact with the quantum world \cite{9}.

Other types of waves, for example water waves, can be controlled in a similar way using transformation acoustics \cite{10}, leading to invisibility cloaks for pressure waves thanks to the design of two-dimensional \cite{11,12} and three-dimensional cloaks \cite{13,14}. It has been further demonstrated that broadband cloaking of surface water waves can be achieved with a structured cloak \cite{15}. Interestingly, cloaking can be further extended to in-plane elastic waves \cite{10,16} and bending waves in thin plates \cite{17}.

In this paper, we focus our analysis on the cloaking of quantum waves, which involves spatially varying potentials and an anisotropic effective mass of particles, as first proposed by the team of Zhang \cite{18} and further mathematically studied by Greenleaf \textit{et al} \cite{19}. We build upon the former proposal to render a quantum object smaller, larger, or even changed in its shape. Our point here is to apply the versatile tool of transformation physics in an area where the size of the object might have some dramatic implications on the physics: for instance, a quantum super-scatterer might enhance the interactions of quantum dots on the mesoscopic scale, thereby enabling quantum effects in metamaterials.
2. Transformed governing equations for matter waves

Following the proposal by Zhang et al. [18], we consider electrons in a crystal with slowly varying composition: \( V = E_0 + U \) is the spatially varying potential, \( E_0 \) the energy of the local band edge and \( U \) a slowly varying external potential. In cylindrical co-ordinates with \( z \) invariance, and letting the mass density \( m_0 \) be diagonally isotropic in these co-ordinates, the time independent Schrödinger equation takes the form

\[
-\frac{\hbar^2}{2} \nabla \cdot (m_0^{-1} \nabla \Psi) + V \Psi = E \Psi.
\]

(1)

Here, \( \hbar \) is the Planck constant and \( \Psi \) is the wavefunction. Importantly, this equation is supplied with Neumann boundary conditions on the boundary of the object to be cloaked.

Let us consider a map from a co-ordinate system \([u, v, w]\) to the co-ordinate system \([x, y, z]\) given by the transformation characterized by \( x(u, v, w) \), \( y(u, v, w) \) and \( z(u, v, w) \). This change of co-ordinates is characterized by the transformation of the differentials through the Jacobian

\[
\frac{dx}{du} = \mathbf{J}_{\text{uv}} \frac{du}{dw}, \quad \text{with} \quad \mathbf{J}_{\text{uw}} = \frac{\partial (x, y, z)}{\partial (u, v, w)}.
\]

(2)

From a geometrical point of view, the matrix \( \mathbf{T} = \mathbf{J}^T \mathbf{J} / \det(\mathbf{J}) \) is a representation of the metric tensor. The only thing necessary in the transformed co-ordinates is to replace the effective mass (homogeneous and isotropic) and potential by equivalent ones. The effective mass becomes heterogeneous and anisotropic, while the potential has a new expression. Their properties are given by [18]

\[
m' = m_0 T^{-1}, \quad V' = E + T_{zz}^{-1}(V - E),
\]

(3)

where \( T_{zz}^{-1} \) stands for the upper diagonal part of the inverse of \( T \) and \( T_{zz} \) is the third diagonal entry of \( T \).

The transformed equation associated with the quantum mechanical scattering problem (1) reads

\[
-\frac{\hbar^2}{2} \nabla \cdot \left( m' \right) \nabla \Psi + V' \Psi = E \Psi.
\]

(4)

where importantly the energy \( E \) remains unchanged and the wavefunction \( \Psi(x, y) = \exp(i\sqrt{E}(xk_1 + yk_2)) + \Psi_\text{d}(x, y) \) with \( \sqrt{k_1^2 + k_2^2} = 1 \). We note that \( \Psi_\text{d} \) satisfies the usual Sommerfeld radiation condition (also known as the outgoing wave condition in the context of electromagnetic and acoustic waves), which ensures the existence and uniqueness of the solution to (4).

It is indeed the potential \( V' \) and the mass density tensor \( m' \) (e.g., involving ultracold atoms trapped in an optical lattice as proposed in [18]) which play the role of the quantum cloak at a given energy \( E \). However, there is a simple correspondence between the Schrödinger equation and the Helmholtz equation, the energy \( E \) of the former being related to the wave frequency \( \omega \) of the latter via \( \omega = \sqrt{E} \) (up to the normalization \( c = \sqrt{2}/\sqrt{\hbar} \), with \( c \) the wave velocity in the background medium, say vacuum). The present analysis thus covers cloaking of acoustic and electromagnetic waves governed by a Helmholtz equation. The correspondences bridging the current analysis with a model of transverse electric waves in cylindrical metamaterials are \( m' \rightarrow m \) on the one hand and \( \sqrt{V'} - E \rightarrow \omega \) on the other hand.

3. Mathematical setup: generalized cloaks for star domains

This section describes a mathematical model generalizing the blowup of a point to a transformation sending one domain to another, thus making the latter inherit the same electronic, electromagnetic or acoustic behavior as the former, depending upon the physical context. Although in this paper we will restrict ourselves to cloaking regions in the plane or a 3D Euclidean space, the transformation we propose can be readily extended to any star domain in \( \mathbb{R}^n \), that is, domains with a vantage point from which all points are within the line-of-sight. In particular, the transformation still preserves all lines passing through that chosen fixed point.

Here is a description of the transformation in layman’s terms, but this could be formalized mutatis mutandis in a very abstract mathematical setting by working directly with the divergence-form PDE of electrostatics [20] or as the Laplace–Beltrami equation of an associated Riemannian metric [21]. For simplicity, let us consider bounded star domains \( D_1 \) in \( \mathbb{R}^n \), \( n = 2, 3 \) with piecewise smooth arbitrary boundaries \( \partial D_1 \), all sharing the same chosen vantage point, \( p = 0, 1, 2 \). We suppose \( D_2 \) contains \( D_1 \), which in turn contains \( D_0 \). Typically, \( D_1 \) is the domain to be made to mimic \( D_0 \). The new transformation will be the identity outside \( D_2 \), that is, in \( \mathbb{R}^n \setminus D_2 \), but will send the hollow region \( D_2 \setminus D_0 \) to the hollow region \( D_2 \setminus D_1 \), in such a way that the boundary \( \partial D_2 \setminus D_1 \) will stay point-wise fixed, while that of \( D_0 \) will be mapped to \( \partial D_1 \). The hollow region \( D_2 \setminus D_1 \) is meant to be the model for the cloak, endowed with Neumann conditions on its inner boundary \( \partial D_1 \), and in which any type of defect could be concealed, but will still have the same electronic (resp. electromagnetic or acoustic) response as the region \( D_0 \) with a potential wall (resp. infinitely conducting boundary or rigid obstacle) of boundary \( \partial D_0 \). In practice, we may divide the domains \( D_j \) into subdomains, the part of whose boundaries lying inside \( \partial D_j \) is a smooth arbitrary hypersurface.

However, in such an ideal cloaking, there is a dichotomy between generic values of the energy \( E \) (resp. wave frequency \( \omega \)), for which the wavefunction must vanish within the cloaked region \( D_1 \setminus D_0 \), and the discrete set of Neumann eigenvalues of \( D_1 \setminus D_0 \), for which there exist trapped states: waves which are zero outside of \( D_1 \setminus D_0 \) and equal to a Neumann eigenfunction within \( D_1 \setminus D_0 \). Such trapped modes have been discussed in [21] when \( D_0 \) vanishes.

The transformation is constructed as follows. Consider a point \( \mathbf{x} \) of \( D_1 \setminus D_0 \) with \( \mathbf{x} = (x^1, x^2, \ldots) \) relative to a system of co-ordinates centered at the chosen vantage point \( \mathbf{0} \). The line passing through \( \mathbf{x} \) and \( \mathbf{0} \) meets the boundaries \( \partial D_0 \), \( \partial D_1 \), \( \partial D_2 \) at the unique points \( \mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2 \), respectively. We actually need the inverse \( \mathbf{x}' \mapsto \mathbf{x} \) of the transformation, in the co-ordinate
system, it reads \( x' = x_0 + \alpha (x'' - x_1) \), where \( \alpha = \frac{x_0 - x_2}{x_0 - x_1} \). In the three-space with co-ordinates \( (x', x^2, x^3) = (x, y, z) \) we can write this transformation as

\[
\begin{align*}
x &= x_0 + \alpha (x' - x_1), & \text{with } \alpha &= \frac{x_2 - x_0}{x_2 - x_1} \\
y &= y_0 + \beta (y' - y_1), & \text{with } \beta &= \frac{y_2 - y_0}{y_2 - y_1} \\
z &= z_0 + \gamma (z' - z_1), & \text{with } \gamma &= \frac{z_2 - z_0}{z_2 - z_1} \end{align*}
\]

The cases of interest in this paper can all be considered as cylinders over some plane curves (triangular, square, elliptic, sun flower-like cylinders, etc.). Thus, we consider the transformation mapping the region enclosed between the cylinders \( S_0 \) and \( S_2 \) into the space between \( S_1 \) and \( S_2 \) as in figure 1, whose inverse is

\[
\begin{align*}
x &= x_0 + \alpha (x' - x_1), & \text{with } \alpha &= \frac{x_2 - x_0}{x_2 - x_1} \\
y &= y_0 + \beta (y' - y_1), & \text{with } \beta &= \frac{y_2 - y_0}{y_2 - y_1} \\
z &= z_0 + \gamma (z' - z_1), & \text{with } \gamma &= \frac{z_2 - z_0}{z_2 - z_1} \end{align*}
\]

The matrix representation of the tensor \( \mathbf{T}^{-1} \) is thus given by

\[
\mathbf{T}^{-1} = \begin{pmatrix}
T_{11} & T_{12} & 0 \\
T_{12} & T_{22} & 0 \\
0 & 0 & T_{33}
\end{pmatrix}
\]

with

\[
\begin{align*}
T_{11}^{-1} &= \frac{a_{11}}{a_{33}}, & T_{12}^{-1} &= \frac{a_{12}}{a_{33}}, & T_{22}^{-1} &= \frac{a_{22}}{a_{33}}, \\
T_{33}^{-1} &= a_{33},
\end{align*}
\]

where the coefficients \( a_{ij} \) can be expressed as

\[
\begin{align*}
a_{11} &= \left( \frac{\partial x}{\partial y'} \right)^2 + \left( \frac{\partial y}{\partial y'} \right)^2, \\
a_{12} &= \left( \frac{\partial x}{\partial x'} \right)^2 + \left( \frac{\partial y}{\partial y'} \right)^2, \\
a_{22} &= \left( \frac{\partial x}{\partial x'} \right)^2 + \left( \frac{\partial y}{\partial x'} \right)^2, \\
a_{33} &= \frac{\partial x}{\partial y'} \frac{\partial y}{\partial x'} \frac{\partial y}{\partial x'},
\end{align*}
\]

and finally the partial derivatives are as follows

\[
\begin{align*}
\frac{\partial x}{\partial x'} &= \frac{x_2 - x_0 + x_2 - x' \frac{\partial x_0}{\partial x'} - (x_2 - x_0) \frac{\partial x_1}{\partial x'} (x_2 - x_1)^2}{(x_2 - x_1)^2} \\
\frac{\partial x}{\partial y'} &= \frac{(x_2 - x_0)(x_2 - x') \frac{\partial x_0}{\partial y'} - (x_2 - x_0) \frac{\partial x_1}{\partial y'} (x_2 - x_1)^2}{(x_2 - x_1)^2} \\
\frac{\partial y}{\partial x'} &= \frac{y_2 - y_0 + y_2 - y' \frac{\partial y_0}{\partial x'} - (y_2 - y_0) \frac{\partial y_1}{\partial x'} (y_2 - y_1)^2}{(y_2 - y_1)^2} \\
\frac{\partial y}{\partial y'} &= \frac{(y_2 - y_0)(y_2 - y') \frac{\partial y_0}{\partial y'} - (y_2 - y_0) \frac{\partial y_1}{\partial y'} (y_2 - y_1)^2}{(y_2 - y_1)^2}.
\end{align*}
\]

Now after having derived the general formulas for mimesis, we turn to the numerical simulations. From formulas (6) to (14), in order to construct our cloak, we only need to know \( S_0, S_1, S_2 \) and their respective derivatives. The explicit illustrations we have supplied to exemplify the work within this paper have boundaries whose horizontal plane sections are parts of ellipses (sunflower-like petal, cross, circle) or lines (parallelogram, hexagram, triangle).

### 4. Mimesis for non-singular cloaks

In section 3 we presented the theoretical study of the mathematical model underlying our proposal for cloaks with any arbitrary star-shaped boundary cross sections that perform mimesis as well as allowing invisibility. In the present section, we illustrate this by examples with piecewise linear or elliptic boundaries and provide their numerical validation.
that is c_0 = d_0 = 0.195 in equation (17). The inner and outer boundaries of the petals are respectively parts of the ellipses \(x^2/0.7^2 + y^2/0.2^2 = 1\) and \(x^2/0.9^2 + y^2/0.4^2 = 1\) rotated by angle 0 or \(\pi/4\) in (A) or 0, \(\pi/4\) or \(3\pi/4\) in (C). The hexagram (D) is generated from an equilateral triangle with side 1.2. The energy corresponding to the plane matter wave incident from the top is \(E = \sqrt{\omega^2/\omega_0^2 + \omega_0^2} = 4.58, \) where \(\lambda = 0.3\) is the wavelength of a transverse electromagnetic wave in the optics setting, with \(c\) the celerity of light in vacuum, normalized here to one. To enhance the scattering, a flat mirror is located under each quantum cloak and obstacle.

4.2. Formulas for piecewise elliptic boundaries

We suppose here that a piece of at least one of our boundary curves \(S_i\) is part of a nontrivial ellipse \(E_i\) with an equation of the form \((x-a)^2/c_i^2 + (y-b)^2/d_i^2\) where \((a, b)\) is our vantage point. Of course a line \((x(t), y(t)) = (a, b) + t(x' - a, y' - b)\) passing through \((a, b)\) and a different point \((x', y')\) intersects \(E_i\) at two distinct points. For the construction, we need the point \((x_i, y_i)\) of that intersection which is nearer to \((x', y')\) in the sense that \(x_i - a\) and \(y_i - b\) have the same sign as \(x' - a\) and \(y' - b\), respectively.

We have
\[
\begin{align*}
\frac{dx_i}{dx'} &= \frac{1}{\sqrt{(x-a)^2/c_i^2 + (y-b)^2/d_i^2}} \frac{(x' - a)^2}{c_i^2}, \\
\frac{dy_i}{dx'} &= \frac{1}{\sqrt{(x-a)^2/c_i^2 + (y-b)^2/d_i^2}} \frac{(y' - b)^2}{d_i^2}.
\end{align*}
\]

This implies
\[
\begin{align*}
\frac{dx_i}{dx} &= \frac{1}{\sqrt{(x-a)^2/c_i^2 + (y-b)^2/d_i^2}} \frac{(x' - a)(y' - b)}{c_i^2}, \\
\frac{dy_i}{dx} &= \frac{1}{\sqrt{(x-a)^2/c_i^2 + (y-b)^2/d_i^2}} \frac{(x' - a)(y' - b)}{d_i^2}.
\end{align*}
\]
This has been used in figures 2(A), 4(E) and 9(B). We note that outside the cloaks in figures 2(A) and (C), the scattered field is exactly the same as that of the small disc of radius $r_0 = 0.195$. When the radius $r_0$ of the disc (that is, when $c_0 = d_0 = r_0$ in equation (17)) tends to zero, the cloaks become singular and the plane matter wave is unperturbed (invisibility).

4.3. Squaring the circle

In this section, we make a circle have the same (electromagnetic) signature as a virtual small square lying inside its enclosed region and sharing the same center. In particular, its appearance to an observer will look like that of a square. As above, the transformation will map the region enclosed between the small square and the outer circle into the circular annulus bounded by the inner (cloaking surface) and outer circles, where the sides of the square are mapped to the inner circle and the outer circle stays fixed pointwise. To do so, we again use the diagonals of the square to partition those regions into sectors. Indeed, the diagonals provide a natural triangulation by dividing the region enclosed inside the square into four sectors. The natural continuation of such a triangulation gives the needed one. In each sector, we apply the same formulas as above, where $(x_0, y_0)$ are obtained from the small square as in section 4.1 and for both $(x_1, y_1)$ and $(x_2, y_2)$ we use the same formulas as for ellipses in section 4.2. For the numerical simulation, we use a small square of side $L_0 = 0.2$ and two circles with radii $R_1 = c_1 = d_1 = 0.2$, $R_2 = c_2 = d_2 = 0.4$ all centered at $(a, b) = (0, 0)$. So, in both the uppermost and lowermost sectors (see figure 3(E)) the formulas (15) for the square become $(x_0, y_0) = (L_0 \frac{x}{2}, L_0 \frac{y}{2})$, whereas in the leftmost and rightmost sectors we have $(x_0, y_0) = (L_0, L_0 \frac{y}{2})$. For all sectors, formulas (17) now read $(x_i, y_i) = (R_i x (x^2 + y^2)^{-\frac{3}{2}}, R_i y (x^2 + y^2)^{-\frac{3}{2}})$, $i = 1, 2$.

The inner boundary of the cloak corresponds to a potential wall (with transformed Neumann boundary conditions, which also hold for infinite conducting or rigid obstacles depending upon the physical context). We report these results in figures 2 and 4. Some Neumann boundary conditions are set on the ground plane, the inner boundary of the carpet and the rigid obstacle.

4.4. Star-shaped cloaks

In this section, we call star-shaped a region bounded by a star polygon, such as a pentagram, a hexagram, a decagram, etc, as in [38]. Star-shaped regions are particular cases of star regions. The design of star-shaped cloaks requires an adapted triangulation of the corresponding region. That is, a triangulation that takes into account the singularities at the vertices of the boundary of the region so that each vertex belongs to the edge of some triangle. To the resulting triangles, one applies the same maps as in section 3. See e.g. figure 2(D).

4.5. Finite element analysis of the cloak properties

We now turn to specific numerical examples in order to illustrate the efficiency and feasibility of the cloaks we design. For computations with

Figure 3. Upper panel: profile of backward matter wave along the x-axis for $y = 1$ for a plane wave incident from the top, as in figure 2, for a cross-like (solid blue curve, see (A)), a sunflower-like (dashed red curve, see (C)), a hexagram (dotted yellow curve, see (D)) cloak and a circular cylinder of small radius $r_0 = 0.195$ (solid black curve, see (B)); lower panel: ideogram for profile of a forward matter wave for $y = -1$; we note the large amplitude of the forward wave, due to the presence of a mirror below each nano-scatterer at $y = -1.2$. The slight discrepancy between the curves is attributed to the numerical inaccuracy induced by the highly heterogeneous nature of the cloaks, which is further enhanced by the irregularity of the boundary of the star-shaped cloak (dotted yellow curve, see (D)). The amplitude of the wave in both panels can be normalized to one (dividing throughout by $4.5 \times 10^{-3}$ and by $5 \times 10^{-3}$ in the upper and lower panels respectively) for comparison with figure 2.

4.5.1. Comparison of backward and forward scattering of isomorphic cloaks. Let us start with a comparison of both backward and forward scattering for the cloaks shown in figure 2(D). We report in figure 3 the amplitude of the matter wave above and below the scatter (cloak and/or obstacle) along the x-axis respectively for $y = 1$ and $-1.2$. We note the slight discrepancy between the curves, which is a genuine numerical artifact: we have checked that the finer the mesh of the computational domain, the smaller the discrepancy (which is a good test for the convergence of the numerical package COMSOL Multiphysics). The mesh actually needs to be further refined within the heterogeneous anisotropic cloak and the perfectly matched layers compared with the remaining part of the computational domain which is filled with isotropic homogeneous material. We note that the yellow curve (corresponding to the pentagram, see figure 2(D)) is most shifted with respect to the black curve (corresponding to the small obstacle on its own, i.e. the benchmark, see figure 2(B)). This can be attributed to the irregular boundary of the cloak as analyzed in the case of singular star-shaped cloaks in [38]: we considered around 70 000 elements for the mesh in all four computational cases reported in figure 3 in order to exemplify the numerical inaccuracies. For computations with
Figure 4. Left: a hollow parallelogram cylindrical region (A) scatters any incoming plane wave just like a much smaller solid cylinder (B) of the same nature. In (C) the same hollow parallelogram cylinder is designed to have the same response to waves as the small equilateral triangle (D) with side $\sqrt{3}/5$. Right, squaring the circle: metamaterials allow one to make a circular cylindrical hollow region (inner and outer radii 0.2 and 0.4 respectively) and a small solid square cylinder (of side $L_0 = 0.2$ and having the same center) equivalent, as regards their signatures and the way waves see them. In all these cases, the coated regions not only acquire the same electromagnetic signature as any other desired object, but also serve as cloaks with non-singular material properties, in fact the presence of any type of defects hidden inside them has no effect in the way they scatter waves. The energy corresponding to the plane matter wave incident from the top is $E = \sqrt{\omega} = \sqrt{2\pi c/\lambda} = 4.58$, where $\lambda = 0.3$ is the wavelength of a transverse electromagnetic wave in the optics setting with $c$ the celerity of light in vacuum, normalized here to one.

100 000 elements, the yellow curve is shifted downwards and is nearly superimposed on the black curve. Moreover, the strong asymmetry of the yellow curve in the upper panel of figure 3 vanishes for 100 000 elements. We attribute this numerical effect to the artificial anisotropy induced by the triangular finite element mesh of the pentagram.

4.5.2. Analysis of the material parameters within a cloak. Another interesting point in this paper is that the metamaterial in our model is non-singular, therefore enabling the implementation of potentially broadband cloaks over a wide range of wavelengths. Moreover, the cloaked objects display a mimesis phenomenon in that they are designed to acquire any desired quantum or electromagnetic signature.

The result of the numerical exploration of the three eigenvalues $\lambda_1^{-1} \tilde{T}(x, y)$ of the material tensor $\tilde{T}^{-1}$ of figure 4(E) is reported in figure 5. First, using Maple software, we have purposely represented those eigenvalues in a wider domain, including the cloaked region, despite the boundary conditions on the inner boundary of the cloak. One clearly sees that all of the three eigenvalues take finite values away from 0 even inside the cloak itself. The material tensor $\tilde{T}^{-1}$ is hence non-singular. Due to the fourfold symmetric geometry of the cloak (all sectors are obtained by a rotation of any fixed one), we only need to look at those eigenvalues in one sector. We note that $4 > \lambda_3^{-1} > \lambda_1^{-1} > 0.2$ in accordance with the fact that the cloak should display a strong anisotropy in the azimuthal direction in order to detour the wave. For a singular cloak $\lambda_1^{-1}$ tends to zero on the inner boundary of the cloak, while $\lambda_2^{-1}$ tends to infinity.

We also represent the finite element simulation of $\lambda_1^{-1}$, which exemplifies the fourfold symmetry of the isovales within the circular cloak, a fact reminiscent of the fourfold symmetry of the square which the cloak mimics.

5. Generalized mirage effect and almost trapped states

It is known that a point source located inside the coating of a singular cloak (i.e. a cloak such that $x_0 = y_0 = 0$ in (6)) leads to a mirage effect whereby it seems to radiate from a shifted location in accordance with this geometric transformation [22].

5.1. Shifted quantum dot inside the transformed space

This prompts the question as to whether a similar effect can be observed in non-singular cloaks, i.e. when $x_0$ or $y_0$ are nonconstant. As it turns out, the physics is now much richer: we can see in figure 6 that when the source lies inside the coating, it only seems to radiate from a shifted location in accordance with (6), but it is moreover in the presence of a small object of side length $2x_0$. This can be seen as a generalized mirage effect that opens many new possibilities in optical illusions. Indeed, Nicolet et al have proposed extending the concept of mirage effect for point sources located within
Figure 5. (A)–(C) are illustrations of the graphs \{(x, y, \Lambda_1^{-i})\}, \ i = 1–3 of the three eigenvalues \Lambda_1^{-i} of the material tensor \( \mathbf{T}^{-1} \), in the uppermost sector of figure 4(E): (A) \( \Lambda_1^{-1} \); (B) \( \Lambda_3^{-1} \); (C) \( \Lambda_2^{-1} \). We note that each these three surfaces are strictly above the plane \( z = 0 \) even inside the cloak itself. Because all other sectors of the cloak are obtained by a rotation of the uppermost one, it suffices to study the eigenvalues of \( \mathbf{T}^{-1} \) in just one sector. Note that the above surfaces were drawn using the MAPLE software. In (D) we represent the finite element computation of \( \lambda_3 \) (see (B)) for all four sectors of figure 4(E) in the COMSOL MULTIPHYSICS package. We note the fourfold symmetry of the isovalues.

the heterogeneous anisotropic coating of invisibility cloaks to finite size bodies which scatter waves like bodies shaped by the geometric transform [23]. This is in essence an alternative path to our proposal for mimetism. However, this mirage effect can be further generalized to non-singular cloaks for which a finite size body located inside the coating will now create the optical illusion of being another body in the presence of some obstacle; this bears some resemblance with Fata Morgana, a mirage which comprises several inverted (upside down) and erect (right side up) images that are stacked on top of one another. Such a mirage occurs because rays of light are bent when they pass through air layers of different temperature. This creates the optical illusion of levitating castles over seas or lakes, as reported by a number of Italian sailors, hence the name related to Morgana, a fairy central to the Arthurian legend able to make huge objects fly over her lake.

5.2. Field confinement on resonances: anamorphism fall down

Another intriguing feature of singular cloaks is their potential for light confinement associated with almost trapped states which are eigenfields exponentially decreasing outside the invisibility region. Such modes were described in the context of quantum cloaks by Greenleaf et al in [19]. These researchers discovered that such modes are associated with energies for which the Dirichlet to Neumann map is not defined, i.e. on a discrete set of values. Here, we revisit their paper in the light of non-singular quantum cloaks, that is when we consider the blowup to a small region instead of a point. Our findings, reported in figure 7 for a star-shaped and a rabbit-like non-singular cloak mimicking a small disc of radius 0.195, bridge the quantum mechanical spectral problems (panels (a) and (b)) to the scattering problems (panels (c) and (d)) in the following way: we first look for eigenvalues (i.e. quantified energies \( E \)) and associated eigenfunctions \( \Psi \) of equation (4) in the class of square integrable functions on the whole space \( \mathbb{R}^2 \) (note that here, as the metric is non-singular, there is no need to consider a weighted Sobolev space). Note, however, that the set continuity conditions on the inner boundary of the cloak, instead of the Neumann ones. This provides us with a discrete set of complex eigenfrequencies, with a very small imaginary part (also known as leaky modes in the optical waveguide literature). We neglect this imaginary part and launch a plane wave on the non-singular cloak (whereby the invisibility region is also included within the computational domain as we once again set continuity conditions on the inner boundary of the cloak) at the very frequency given by the spectral problem, see panels (C) and (D). We clearly see that the inside of the cloak hosts a quasi-localized eigenstate whose energy is mostly confined inside a cloak shaped like a star (panel (C)) and a rabbit (panel (D)), both of which actually scatter like a small disc.
Figure 6. A non-singular cloak with two square boundaries of side lengths 1 and 1.4 in presence of a quantum dot with energy $E = 4.58$ (resp. an electric current line source of wavelength $\lambda = 0.3$ in the context of transverse electric waves). (A)–(C) When the source is located a distance 0.2 away from the cloak, it seems to emit as if it would be a distance 0.4 away from a square obstacle of side lengths 0.4. (B)–(D) When the source is located a distance 0.1 away from the inner boundary of the cloak (i.e. in the middle of the coating), it seems to emit as if it were a distance 0.25 away from a square obstacle of side length 0.4, in accordance with (6), where $x_0 = 0.2$, $x_1 = 0.5$ and $x_2 = 0.7$.

Figure 7. Left panel: modulus of the fundamental eigenstates associated with quantified normalized energy $E = \sqrt{\omega} = \sqrt{2\pi c/\lambda} = 1.32$ (resp. a transverse electric plane wave of wavelength $\lambda = 3.6$) for a non-singular cloak shaped as a star (A) and a rabbit (B) both of which mimic a small disc (of normalized radius 0.195, e.g. 195 nm); right panel: matter wave incident from the top on the quantum cloaks with a spatially varying potential $V'$ with compact support (i.e. vanishing outside the cloak) and energy $E$; the large amplitude of the field within the cloak is noted.

6. Generalized super-scattering for negatively refracting non-singular cloaks

In this section, we relax the one-to-one feature of the previous transforms and allow for space folding. This means that $|x_0| \geq |x_1|$ and $|x_0| \geq |y_2| \geq |y_1|$ in (6) while the $x_i$ (resp. the $y_i$) $i = 1–3$ all have the same sign, thus making $\alpha$ and $\beta$ strictly negative real-valued functions. It has been known for a while that space folding allows for the design of perfect lenses, corners and checkerboards [2, 24, 25, 27–34, 39–41].
But it is only recently that researchers foresaw the very high potential of space folding when applied to the design of super-scatterers [42–46]. We generalize these concepts to mimetism via space folding.

The mapping leading to the super-scatterer is shown in figure 8. We would like to emphasize that here we get not only a magnification of the scattering cross section of an object, in a way similar to what optical space folding does for a cylindrical perfect lens, but we can also importantly change the shape of the object.

We illustrate our proposal with a numerical simulation for a square obstacle surrounded by an anti-cloak (see figure 9(a)) that mimics a larger square obstacle (see figure 9(b)). We note that the large field amplitude on the anti-cloak’s upper boundary can be attributed to a surface field arising from physical parameters with opposite signs on the cloak outer boundary (an anisotropic mass density in the context of quantum mechanics and an anisotropic permittivity in the context of optics). It is illuminating here to draw some correspondence with electromagnetic waves, as the anisotropic mass density (resp. permittivity) indeed takes opposite values when we cross the outer boundary of the cloak, and this ensures the existence of a surface matter wave (resp. a surface plasmon polariton) clearly responsible for the large field amplitude in the transverse electric wave polarization, i.e. for a magnetic field parallel orthogonal to the computational plane. We further show an example of a small circular obstacle mimicking a large square obstacle in figures 9(a) and (b). Once again, the large field amplitude on the outer cloak boundary comes from the complementary media inside and outside the cloak. We believe such types of mimetism might have tremendous applications in quantum dot probing, bringing the nano-world a step closer to metamaterials.

7. Conclusion

In this paper, we have proposed models of generalized cloaks that create some illusion. We focused here on the Schrödinger equation, which is valid in a number of physical situations, such as matter waves in quantum optics. However, the results in this paper can easily be extended to the Helmholtz equation which governs the propagation of acoustic and electromagnetic waves at any frequency. One simply needs to insert the transformation matrix within the shear modulus and density of an elastic bulk (in the case of anti-plane shear waves), the density and compressional modulus of a fluid (in the case of pressure waves), or the permittivity and permeability of a medium (in the case of electromagnetic waves) [10–15]. In this latter context, this means that in the transverse electric polarization (whereby the magnetic field is parallel to the fiber axis), infinite conducting obstacles dressed with these cloaks display the electromagnetic response of another infinite conducting obstacle. In these cloaks, an electric wire could in fact hide a larger object near it. Actually, any object could mimic the signature of any other one. For instance, we designed a cylindrical cloak so that a circular obstacle behaves like a square obstacle, thereby bringing about one of the oldest enigmas of ancient times: squaring the circle! The ordinary singular cloak then emerges as a particular case, whereby objects appear as an infinitely small infinite conducting object (of vanishing scattering cross section) and hence become invisible. On the contrary, such generalized cloaks are described by a non-singular permittivity and permeability, even at the cloak’s inner surface. We have proposed and discussed some interesting applications in the context of quantum mechanics, such as probing nano-objects.

Obviously, one realizes that, when the inner virtual region \(D_0\) responsible for mimetism tends to zero, one recovers the case in [1] where the material properties are no longer bounded, as one of the eigenvalues of the mass density matrix tends to zero whilst another one recedes to infinity as we approach the inner boundary of the coated region, see also [20].

In this paper, we have played with optical illusions, trying to be as imaginative as possible in order to exhaust the possible geometric transforms we had at hand in Euclidean spaces (non-Euclidean cloaking is a scope for more creative thinking [26]). It should be pointed out that while the emphasis of this paper was on quantum waves, the corresponding non-singular cloaks in electromagnetism that have an inner boundary which is perfectly electric conducting (PEC) and scatter like a reshaped PEC object were investigated in [35–37, 51–56]. However these works focused mostly on the reduction of the scattering cross section of a diffracting object, while the present paper explores the mimetism effect whereby an object scatters like another object of any other scattering cross section (and in particular reduced or enhanced ones).

Metamaterials [47] is a vast field with a variety of composites structured on the subwavelength scale in order to sculpt the electromagnetic wave trajectories, as experimentally...
Figure 9. (A)–(C) Any obstacle surrounded by a square anti-cloak with square boundaries of side lengths 1 and 1.4 scatters a plane matter wave of energy $E = 4.58$ (resp. a transverse electric plane wave of wavelength $\lambda = 0.3$ in the context of optics) which comes from the top like a larger square obstacle of side length 1.6. (b)–(d) Any obstacle surrounded by a circular anti-cloak with circular boundaries of radii 0.2 and 0.4 scatters a plane matter wave of energy $E = 4.58$ (resp. a transverse electric plane wave of wavelength $\lambda = 0.3$) which comes from the top like a larger square obstacle of side length 1. The large field amplitude on the upper boundary of the anti-cloak in (A) and (B) is noted. It can be attributed to some kind of surface matter wave (a surface plasmon polariton in the context of optics).

demonstrated at microwave frequencies by a handful of research groups worldwide [48–50]. Resonant elements within metamaterials are in essence man-made elements allowing one to mimic virtually any electromagnetic response we wish, and this turn allows us to push the frontiers of photonics towards previously unforeseen areas.

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References

[1] Pendry J B, Shurig D and Smith D R 2006 Science 312 1780–2
[2] Leonhardt U 2006 Science 312 1777–80
[3] Leonhardt U and Philbin T G 2006 New J. Phys. 8 247
[4] Veselago V G 1968 Sov. Phys. Usp. 10 509–14
[5] Pendry J B 2000 Phys. Rev. Lett. 86 3966–9
[6] Smith D R, Padilla W J, Vier V C, Nemat-Nasser S C and Schultz S 2000 Phys. Rev. Lett. 84 4184–7
[7] Ramakrishna S A 2005 Rep. Prog. Phys. 68 449–521
[8] Cheianov V V, Fal’ko V and Althaler B L 2007 Science 315 1252–5
[9] Moreno E, Fernández-Domínguez A I, Cirac J I, García-Vidal F J and Martín-Moreno L 2005 Phys. Rev. Lett. 95 170406
[10] Milton G W, Briane M and Willis J R 2006 New J. Phys. 8 248
[11] Cummer S A and Schurig D 2007 New J. Phys. 9 45
[12] Torrent D and Sanchez-Dehesa J 2008 New J. Phys. 10 023004
[13] Cummer S A, Popa B I, Schurig D, Smith D R, Pendry J, Rahm M and Stur A 2008 Phys. Rev. Lett. 100 024301
[14] Chen H and Chan C T 2007 Appl. Phys. Lett. 91 183518
[15] Farhat M, Enoch S, Guenneau S and Movchan A B 2008 Phys. Rev. Lett. 101 134301
[16] Brun M, Guenneau S and Movchan A B 2009 Appl. Phys. Lett. 94 061903
[17] Farhat M, Guenneau S, Enoch S and Movchan A B 2009 Phys. Rev. B 79 033102
[18] Zhang S, Genov D A, Sun C and Zhang X 2008 Phys. Rev. Lett. 100 123002
[19] Greenleaf A, Kurylev Y, Lassas M and Uhlmann G 2008 New J. Phys. 10 110524
[20] Kohn R V, Shen H, Vogelius M S and Weinstein M I 2008 Inverse Problems 24 015016
[21] Greenleaf A, Kurylev Y, Lassas M and Uhlmann G 2007 Commun. Math. Phys. 275 749–89
[22] Zolla F, Guenneau S, Nicolet A and Pendry J B 2007 Opt. Lett. 32 1069–71
[23] Nicolet A, Zolla F and Geuzaine C 2009 arXiv:0909.0848v1
[24] Nicorovici N A, McPhedran R C and Milton G W 1994 Phys. Rev. B 49 8479–82
[25] Greenleaf A, Lassas M and Uhlmann G 2003 Math. Res. Lett. 10 685–93
[26] Leonhardt U and Tyc T 2008 Science 323 110–2
[27] Zhang P, Jin Y and He S 2008 Appl. Phys. Lett. 93 243502
[28] Collins P and McGuirk J 2009 J. Opt. A: Pure Appl. Opt. 11 015104
[29] Liu R, Ji C, Mock J J, Chin J Y, Cui T J and Smith D R 2008 Science 323 366–9
[30] Li J and Pendry J B 2008 Phys. Rev. Lett. 101 203901
[31] Pendry J B and Li J 2008 New J. Phys. 10 115032
[32] Nicorovici N A P, McPhedran R C, Enoch S and Tayeb G 2008 New J. Phys. 10 115020
[33] Milton G and Nicorovici N A 2006 Proc. R. Soc. A 462 3027
[34] Alu A and Engheta N 2005 Phys. Rev. E 95 031623
[35] Greenleaf A, Kurylev Y, Lassas M and Uhlmann G 2008 Commun. Math. Phys. 281 369–85
[36] Gabrielli L H, Cardenas J, Poitras C B and Lipson M 2009 Nat. Photon. 3 461–3
[37] Diatta A, Guenneau S, Dupont G and Enoch S 2010 Opt. Express 18 11537–51
[38] Diatta A, Nicolet A, Guenneau S and Zolla F 2009 Opt. Express 17 13389–94
[39] Pendry J B and Ramakrishna S A 2003 J. Phys.: Condens. Matter 15 6345
[40] Guenneau S, Vutha A C and Ramakrishna S A 2005 New J. Phys. 7 164
[41] Milton G W, Nicorovici N A P, McPhedran R C, Cherednichenko K and Jacob Z 2008 New J. Phys. 10 115021
[42] Zhang J J, Luo Y, Chen S H, Huangfu J, Wu B I, Ran L and Jong J A 2009 Opt. Express 17 6203
[43] Lai Y, Chen H, Zhang Z Q and Chan C T 2009 Phys. Rev. Lett. 102 093901
[44] Lai Y, Ng J, Chen H Y, Han D Z, Xiao J J, Zhang Z Q and Chan C T 2009 Phys. Rev. Lett. 102 253902
[45] Ng J, Chen H Y and Chan C T 2009 Opt. Lett. 34 644
[46] Wee W H and Pendry J B 2010 New J. Phys. 12 033047
[47] Zheludev N I 2010 Science 328 582
[48] Schurig D, Mock J J, Justice B J, Cummer S A, Pendry J B, Starr A F and Smith D R 2006 Science 314 977–80
[49] Kante B, Germain D and de Lustrac A 2009 Phys. Rev. B 80 201104
[50] Tretyakov S, Alitalo P, Luukkonen O and Simovski C 2009 Phys. Rev. Lett. 103 103905
[51] Cummer S A, Liu R and Cui T J 2009 J. Appl. Phys. 105 056102
[52] Hu J, Zhou X and Hu G 2009 Appl. Phys. Lett. 95 011107
[53] Jiang W X, Cui T J, Yang X M, Cheng Q, Liu R and Smith D R 2008 Appl. Phys. Lett. 93 194102
[54] Chen H Y, Zhang X H, Luo X D, Ma H R and Chan C T 2008 New J. Phys. 10 113016
[55] Li C, Yao K and Li F 2008 Opt. Express 16 19366
[56] Jiang W X, Ma H F, Cheng Q A and Cui T J 2010 J. Appl. Phys. 107 034911