Effective mass of \( W \)-boson in a magnetic field

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Simple representation for the average value of the \( W \)-boson one-loop polarization tensor in a magnetic field \( B = \text{const} \), calculated in the ground state of the tree-level spectrum, is derived. It corresponds to Demeur’s formula for electron in QED. The energy of this state, describing effective particle mass, is computed by solving the Schwinger-Dyson equation. As application, we investigate the effective mass squared at the threshold of the tree-level instability, \( B \rightarrow B_c = \frac{m^2}{e} \), and show that it is positive. In this way the stability of the \( W \)-boson spectrum is established. Some peculiarities of the results obtained and other applications are discussed.

1 Introduction

Nowadays, physics of charged vector particles in strong magnetic fields has obtained new stimulus for investigation. This concerns, first of all, \( \rho \)-meson physics where the effective Lagrangian describing electromagnetic interactions has been derived in different approaches [1, 2, 3, 4]. In Refs. [7, 8] on the base of this Lagrangian the properties of the \( \rho \)-meson vacuum in strong magnetic fields of the order \( eB \geq m^2_\rho \), where \( m_\rho \) – particle mass, were investigated and the superconducting state having a structure similar to Abrikosov’s lattice observed. Such type structure was derived already in electroweak theory for the \( W \)-boson vacuum [5], [6], [11]. Other important reasons are the existence of extremely strong magnetic fields in the Universe as well as in collisions of beams of protons and heavy ions at modern colliders. In latter case, they influence characteristics and properties of particles, in particular, \( W \)-bosons that is important for various decay processes which are investigated.

Recently the ground state projection of the \( SU(2) \) polarization tensor for charged gluons in Abelian chromomagnetic field has been calculated and studied at high temperature in Ref.[12]. Simple expression for this function was derived which just corresponds to the Demeur formula for electron in magnetic field in QED. The obtained results (eqs. (17), (22) in Ref. [12]) can be modified to find the ground state energy for charged massive vector particle. Of course, a number of other contributions has to be added in different models.

In what follows, we apply the results of Ref. [12] to calculate the ground state energy, \( \langle t | \Pi(p_\parallel, B) | t \rangle \), for the \( W \)-boson, accounting for the one-loop diagrams.

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Since on the ground state $|t\rangle$ the full $W$-boson polarization tensor is diagonal \[10\], it is possible to write down and solve the Schwinger-Dyson (SD) equation for this state and in this way obtain a nonperturbative effective mass $M(B)$ (or effective ground state energy) of the particle. This mass can be calculated for arbitrary values of the field strength $B$ (at least numerically) that can be useful for various applications.

In the present paper, the calculated expression is investigated in the limit of $B \rightarrow B_c$, where $B_c = m^2/e$ is the critical magnetic field strength for the $W$-boson tree-level spectrum

$$p_0^2 = p^2 + m^2 + (2n + 1)eB - 2e\sigma B,$$

$$(n = 0, 1, \ldots, \quad \sigma = 0, \pm 1) \quad (1)$$

in a homogeneous magnetic background, $B = \text{const}$, described by the potential

$$A_{\mu}^{\text{ext.}} = Bx_1 \delta_{\mu 2}, \quad (2)$$

where $p_\parallel$ is a momentum component along the field direction, $e$ - electric charge, $n$ - Landau level number, $\sigma$ - spin projection. In (1) a tachyon mode is present in the ground state ($|t\rangle = |n = 0, \sigma = +1\rangle$) for the field strength $B \geq B_c = m^2/e$. Considering this limit, we show that the effective energy calculated from the SD (or gap) equation remains real for realistic mass of Higgs particle. Thus, radiation corrections act to prevent the instability of the vacuum. Other obvious applications are in $W$-boson physics for different processes in strong magnetic fields.

The noted problem has been investigated already in one-loop order for the $W$ bosons in the Georgy-Glashow model of electroweak interactions (see review \[11\]). As it was found, the result depends on the value of the Higgs boson mass $m_H$. For heavy Higgs particle, $K = m_H/m_W \geq 1.2$, the spectrum stabilization takes place. For light Higgs particle, $K < 1.2$, the instability was found. However, this problem was not investigated in detail for the standard model. Other note, the absence of an adequate representation for the ground state projection of the $W$-boson polarization tensor, similar to Demeur’s formula \[9\], made investigations of $W$-bosons bulky and complicated.

In the next section, we calculate the $W$- boson polarization tensor and its mean value in the ground state of the spectrum \(\Pi\). In sect. 3, we derive the SD equation for the ground state projection $\langle t|\Pi(B)|t\rangle$ and investigate the limit $B \rightarrow B_c$. General conclusions and discussion are given in the last section.

## 2 $W$- boson polarization tensor

In what follows, we use Euclidean space-time and the representation of the polarization tensor for gluons given in \[13\]. It is reasonable to rewrite color gluon
field, \( V_\mu^a \ (a = 1, 2, 3) \), in terms of charged, \( W_\mu^\pm = \frac{1}{\sqrt{2}}(V_\mu^1 \pm iV_\mu^2) \), and neutral, \( V_\mu^3 = A_\mu \), components. In momentum representation, the initial expression reads

\[
\Pi_{\lambda\lambda'}(p) = \int \frac{dk}{(2\pi)^4} \left\{ \Gamma_{\lambda\nu\rho}G_{\nu\nu'}(p - k)\Gamma_{\lambda'\nu'\rho'}G_{\rho\rho'}(k) \\
+ (p - k)_{\lambda}G(p - k)k_{\lambda'}G(k) \\
+ k_{\lambda'}G(p - k)(p - k)_{\lambda}G(k) \right\} \\
+ \Pi_{\lambda\lambda'}^{\text{tadpol}},
\]

where the second and third lines result from the ghost contribution and the tadpole contribution is given by

\[
\Pi_{\lambda\lambda'}^{\text{tadpol}} = \int \frac{dp}{(2\pi)^4} \left\{ 2G_{\lambda\lambda'}(p) - \delta_{\lambda\lambda'}G_{\rho\rho}(p) - G_{\lambda\lambda'}(p) \right\}.
\]

The contributions of the charged tadpole diagrams are taken into consideration. Only these tadpoles are relevant to the problem of interest. The vertex factor,

\[
\Gamma_{\lambda\nu\rho} = (k - 2p)_{\rho}\delta_{\lambda\nu} + \delta_{\rho\nu}(p - 2k)_{\lambda} + \delta_{\rho\lambda}(p + k)_{\nu},
\]

completes the description of the vector part of the polarization tensor. These formulas hold also in a background field, provided the corresponding expressions for the propagators are used. We take a homogeneous magnetic background field in the representation given in eq. (2). In this case the operator components fulfill the commutation relation

\[
[p_{\mu}, p_{\nu}] = i e F_{\mu\nu} \quad (F_{12} = B).
\]

In what follows, where it will be not misleading, we write \( B \) instead \( eB \) or even put \( B = 1 \), for short. In above formulas (3) and (4) we omitted the coupling factors \( e^2 \). These factors as well as other factors proper to different models of interest can be accounted for in the final expressions. Below, we will also use the notations \( l^2 = l_3^2 + l_4^2 \) and \( h^2 = p_1^2 + p_2^2 \), where we write \( l_3 \) and \( l_4 \) for the momenta in parallel to the background field \( p_{\parallel} = l_3 \) and imaginary time, respectively. Other information relevant to the massless case is given in Refs. [12, 13].

To obtain the results for the electroweak sector, one has to take into account the masses of the \( W^- \), \( Z^- \) and Higgs bosons, and add the contributions of the latter two particles.

First we incorporate the masses in the representation of the polarization tensor as given in eq. (51) of Ref. [13]. It results from the proper time representation of the propagators,

\[
G(p - k) = \int_0^\infty ds \ e^{-sm^2} e^{-s(p-k)^2},
\]
\[ G(k) = \int_0^\infty dt \ e^{-tM^2} e^{-tk^2}, \]
\[ G_{\lambda\lambda'}(p-k) = \int_0^\infty ds \ e^{-sm^2} e^{-s(p-k)^2} E_{\lambda\lambda'}, \]
\[ E_{\lambda\lambda'} = e^{sM^2 s}, \tag{7} \]

for charged and neutral particles, and integration over \( k \) in eq. (3). The mass \( M \) is, \( M = 0, m_Z, m_H \) for photon, \( Z \)- and Higgs boson, correspondingly.

The representation for the \( SU(2) \) sector of the standard model (\( W \)-bosons, massive ghosts and photons) is obtained in terms of the integral over the parameters \( s \) and \( t \),

\[ \Pi_{\lambda\lambda'} = \int_0^\infty ds \int_0^\infty dt e^{-sm^2} \Theta(s, t) \left( \sum_{i,j} M_{i,j}^{\lambda\lambda'} + M_{i,j}^{gh} \right) + \Pi_{\lambda\lambda'}^{\text{tadpol}} \tag{8} \]

with

\[ \Theta(s, t) = \frac{\exp(-H)}{(4\pi)^2(s + t)\sqrt{\Delta}}. \tag{9} \]

Here the following notations are used:

\[ H = \frac{st}{s + t} l^2 + m(s, t)(2n + 1)B, \]
\[ m(s, t) = s + \frac{1}{2} \ln \frac{\mu_-}{\mu_+}, \]
\[ \Delta = \mu_- \mu_+, \]
\[ \mu_{\pm} = t + \sinh(s)e^{\pm s}, \tag{10} \]

which are equivalent to eqs. (23-26) in [13]. The sum over \( i, j \) in (8) follows the subdivision introduced in [13] and the functions \( M_{i,j}^{\lambda\lambda'} \) are given by eq. (53) in [13].

Now we take the tachyonic projection of \( \Pi_{\lambda\lambda'} \), eq. (8). In doing so we note especially \( n = 0 \) (for \( B = 1 \)) and the function \( \Theta \) simplifies,

\[ \Theta(s, t)|_{\mu^2 = 1} = \frac{\exp \left( -\frac{st}{s + t} l^2 - s \right)}{(4\pi)^2(s + t)\mu_-}. \tag{11} \]

For the projection of the functions \( M_{i,j}^{\lambda\lambda'} \) we use representation (55) in [13]. Calculation of these terms is given in the appendix of Ref. [12]. At this place we mention that under the tachyonic projection we get directly a representation suitable for further calculations. The presence of particle masses is reflected in a simple factor in the integrand of eq. (8) and not influenced any computation procedures applied in the massless case.
Note that the expression (8) is calculated in the Feynman-Lorentz-t’Hooft gauge

\[ P_\mu W^-_\mu - m\phi^- = iC^- , \]

in which the mass of charged ghost, \( C^\pm \), and Goldstone, \( \phi^\pm \), fields equals to the \( W \)-boson mass \( m \).

Detailed calculations of \( \langle t | \Pi | t \rangle \) are given in Ref. [12] and not modified for \( m \neq 0 \). Only the contribution from \( M^{33} + M^{gh} \) requires an additional consideration. As it is shown in Ref. [13], eqs. (87) - (89), this part can be written in the form,

\[ M_{33}^{33+gh} = - \int_0^\infty ds dt \, e^{-s m^2} (\delta_{\lambda\lambda'} \frac{\partial \Theta}{\partial s} + E_{\lambda\lambda'} \frac{\partial \Theta}{\partial t}) , \]

where \( \Theta(s, t) \) is the function in eq. (9) and the matrix \( E_{\lambda\lambda'} = e^{-2isF} \). These combining into derivatives allow for carrying out one of the parameter integrations.

Using \( \langle t | \delta_{\lambda\lambda'} | t \rangle = 1 \), \( \langle t | E_{\lambda\lambda'} | t \rangle = e^{2s} \),

\[ \Theta(s = 0, t) = \frac{1}{t^2}, \quad \Theta(s, t = 0) = \frac{1}{s \sinh(s)} , \]

and integrating by part we get in the projection

\[ \int ds dt \, e^{-s m^2} \langle t | M_{\lambda\lambda'}^{33+gh} \Theta(s, t) | t \rangle = \frac{1}{(4\pi)^2} \int dq \left( \frac{1}{q} + \frac{e^{-q m^2} e^{2q}}{\sinh(q)} \right) \]

\[ - \frac{m^2}{(4\pi)^2} \int_0^\infty ds dt \, e^{-s m^2} e^{-s} \exp \left( -\frac{st}{s+t} \right) \]

\( (s+t)\mu_-, \)

where in the last line the function (11) is substituted. To complete this part, we write down the remaining (except \( M_{33+gh}^{33} \)) terms coming from the main diagram, (3),

\[ \langle t | \sum_{ij} \bar{M}^{ij} | t \rangle = \frac{4}{\mu_-} + 4 \frac{s + t e^{2s}}{s + t} \mu_- , \]

and hence

\[ \langle t | \Pi (\sum_{ij} \bar{M}^{ij}) | t \rangle = \frac{1}{(4\pi)^2} \int ds dt \, e^{-s m^2} \]

\[ \times \left[ \frac{4}{\mu_-} + 4 \frac{s + t e^{2s}}{s + t} \mu_- \right] \exp \left( -\frac{st}{s+t} \right) \]

\( (s+t)\mu_- , \)

Here, \( \bar{M}^{ij} \) reminds about the omitted terms. The contributions from the tadpoles, (4), take the form

\[ \langle t | \Pi^{ip} | t \rangle = - \frac{1}{(4\pi)^2} \]

\[ \times \int \frac{dq}{q} e^{-q m^2} \left( 2 + \cosh(2q) + 3 \sinh(2q) \right) \frac{\sinh(q)}{\sinh(q)} . \]
Then we have to add the contributions coming from charged Goldstone bosons. As computations shown, one term coincides up to the sign with the last line in eq. (15), and exactly cancels in the total, as it should be in the renormalizable gauge (12). Other contribution is the tadpole one coming from the contact vertex $\sim e^2 W_+^\mu W_-^\mu \phi^+ \phi^-$. This term up to the factor $-1/4$ coincides with the first term in eq. (18). Thus, the expressions (15) (except the second line), (17) and (18), corrected due to the noted tadpole contribution, represents the electromagnetic part of the $W$- boson polarization tensor in the projection to the lower, the tachyonic state. It corresponds to the Demeur formula for electron in magnetic field in QED (see Ref. [9], eq. (59)).

Now, we turn to the contributions of the $Z$- boson sector, calculated in the gauge

$$\partial_\mu Z_\mu - i m_Z \phi^Z = C^Z,$$

(19)

where $\phi^Z$ and $C^Z$ present the Goldstone and ghost fields having the mass $m_Z$. This part can be expressed by using the obtained expressions eqs. (15), (17). Actually, according to (7), one has to introduce in these formulas the mass factor $e^{-t m_Z^2}$. The contribution of the Goldstone field $\phi^z$ and the term in the second line of (15) are canceled again. But now, after integration by part over the parameter $t$ in eq. (13), new term appears. The sum of contributions from Goldstones and $M^{33+gh}_Z$ looks as follows,

$$\int ds \, dt \, e^{-s m^2 - t m_Z^2} \langle t \mid M^{33+gh}_Z \Theta(s, t) \mid t \rangle = \frac{1}{(4\pi)^2} \int dq \left( \frac{e^{-q m_Z^2}}{q} + \frac{e^{-q m_Z^2} e^{2q}}{\sinh(q)} \right)$$

$$- \frac{m_Z^2}{4 \pi^2} \int_0^{\infty} ds \, dt \, \exp \left( -\frac{st}{s+t} \right) e^s (s + t) \mu_-,$$

(20)

where the factor $e^{2s}$ in the second line appears from $E_{\lambda\lambda}$ in eq. (13). Thus, the contribution from the $Z$- sector is given by eq. (20) and eq. (17) with additional factor $e^{-t m_Z^2}$ in the integrand.

Then, we restore the couplings and the dimensionality in the obtained expressions. Remind that in actual calculations we put $eB = 1$ and therefore the proper-time parameters $s, t, q$ became dimensionless. In fact, this means that we measure them, as well as the masses, in units of $eB$. Thus, to recover the dimensionality one has to substitute $M^2 \rightarrow M^2/(e B), \mu^2 \rightarrow \mu^2/(e B)$ and extra total factor $(e B)$ coming from the $\Theta(s, t)$ in eqs. (10), (11). For the electromagnetic sector, we have to introduce the factor $e^2$ in eqs. (15), (17) and the factor $g^2 = e^2/\sin^2 \theta$ for the tadpole contributions eq. (18). For the $Z$-boson sector, the overall factor in eqs. (15), (17) is $e^2 \cot^2 \theta$. Here $\theta$ is the Weinberg angle.

The contribution of the Higgs boson sector is given by two diagrams and
reads,
\[
\Pi_{\lambda\lambda'}^H(p) = \int \frac{dk}{(2\pi)^4} \left\{ 
(2k-p)_\lambda G(p-k, m^2)(2k-p)_{\lambda'} G(k, m^2_H)
+ 4m^2 G_{\lambda\lambda'}(p-k, m^2) G(k, m^2_H) \right\},
\] (21)

where we marked the mass of the particle. Again, we have to use the representation (7) and then integrate over \( k \). In the ground state projection, the first line simplifies considerably because the condition \( p_{\lambda'} | t \rangle_{\lambda'} = 0 \) holds. So, only the term \( 4k_{\lambda} k_{\lambda'} \) contributes. The corresponding term up to a factor coincides with the term \( \langle t | M_{11} | t \rangle \), eq. (51) in Ref. [12]. The second line equals just to \( \langle t | E_{\lambda\lambda'} | t \rangle \Theta(s, t)_{\hbar^2=1} \) (see eq. (11)).

Thus, for the scalar sector we obtain,
\[
\langle t | \Pi^H(p) | t \rangle = \frac{g^2}{(4\pi)^2} \left\{ 
\begin{array}{c}
\int_0^\infty ds dt e^{-\frac{s^2 + t^2}{eB}} (s + t) \mu \exp \left( - \left[ s \left( \frac{m^2}{eB} + 1 \right) + t \frac{m^2_H}{eB} \right] \right)

+ m^2 \int_0^\infty ds dt e^{-\frac{s^2 + t^2}{eB}} (s + t) \mu \exp \left( - \left[ s \left( \frac{m^2}{eB} - 1 \right) + t \frac{m^2_H}{eB} \right] \right)
\end{array} \right\},
\] (22)

where all the necessary factors are substituted. The expressions (15), (17), (18), (20) with corresponding factors and eq. (22) give the non-renormalized mean value of the \( W^- \) boson polarization tensor in the ground state \( | t \rangle \) in the standard model. Its renormalization is fulfilled in a usual way by subtracting of the terms
\[
c.t.1 = \langle t | \Pi(p^2, B, m^2)_{|B=0} | t \rangle,
\]
\[
c.t.2 = \langle t | \frac{\partial \Pi(p^2, B, m^2)}{\partial p^2} (l^2 = eB - m^2)_{|B=0} | t \rangle
\] (23)
on the mass shell of the spectrum (1) in the ground state \( l^2 = eB - m^2 \) (see Refs. [11, 12] for details). These counter terms are divergent at the lower limit \( s, t = 0 \).

On the base of these formulas, different kind studies can be carried out. In the next section, we investigate the ground state energy in the limit of \( B \to B_c \). For this case, the main contributions come from the upper limit of integrations over the proper time parameters. So, the renormalization is not important.
3 Ground state energy at \( B \sim B_c \)

Let us consider the limit of the field strength \( B \rightarrow m^2/e \) for calculated expressions. In this case, a number of terms is divergent at the upper limit of integration because of the smallness of the "effective tree-level mass", \( \Delta = m^2 - eB \), which enters the cutting factor \( e^{-s\Delta} \) going to unit for \( \Delta \rightarrow 0 \), and integrals diverge. These are the last term in the first line of eq. (15) and similar term in eq. (20), two last terms in eq. (18) and the terms in the second lines in eqs. (20) and (22). The sum of calculated diagrams obtains the overall factor, \( g^2 \), of \( SU(2)_w \) gauge group, and the mass \( m_Z \) has to be substituted by \( m \), due to the relation \( e^2 = g^2 \sin^2 \theta \). They give dominant contributions and should be accounted for.

As a result, when all the relevant terms are gathered together, two types of integrals contribute in this limit,

\[
\epsilon_t^2 = \langle t | \Pi | t \rangle = \frac{g^2}{(4\pi)^2} (I^{(1)} + I^{(2)}).
\]  

First is one parametric,

\[
I^{(1)} = -2eB \int_c^\infty \frac{dq}{q} \exp \left[ -q \left( \frac{m^2}{eB} - 1 \right) \right],
\]  

where \( c \) is a constant of order 1. The second integral is two parametric,

\[
I^{(2)} = m^2 \int_0^\infty dsdt \frac{\exp \left[ - \left( \frac{m^2}{eB} - 1 \right) \frac{s^2}{s+t} \right]}{(s+t)\mu_-} \times \left( e^{-t m^2_z/(eB)} - e^{-t m^2_h/(eB)} \right).
\]

In the last expression, we used the relation \( l^2 = eB - m^2 \). Both of these integrals can be easily estimated. The first is negative and equals,

\[
I^{(1)}(B) \big|_{B \rightarrow B_0} = -2eB \log \left( \frac{1}{m^2/(eB) - 1} \right) + O(1).
\]

The sign of \( I_2 \) depends on the relation between the masses \( m_H \) and \( m_Z \). If \( m_H \geq m_Z \), the second term in eq. (26) is dominant and integral is negative. Otherwise, it is positive. In the special case, \( m_H = m_Z, I_2 = 0 \). We get for the leading term,

\[
I^{(2)}(B) \big|_{B \rightarrow B_0} = m^2 \log \left( \frac{1}{m^2/(eB) - 1} \right) \times \left[ \log \left( \frac{2m^2 + m^2_z}{2m^2 + m^2_h} \right) + \log \frac{m^2_z}{m^2_h} \right] + O(1).
\]
Thus, in the standard model the radiation correction to the ground state energy at the threshold of instability, $B = B_c$, is negative for realistic values of the masses $m_H > m_Z$. Note that in the Georgy-Glashow model $Z$-boson absences and $I_2 > 0$.

On the base of these calculations, we conclude that radiation corrections act to stabilize the tree-level spectrum (1). Really, if one considers the pole of the Schwinger-Dyson operator equation taken in the ground state, 

$$
\langle t | D | t \rangle^{-1} = \langle t | m^2 + l_3^2 - B - \Pi(B, m^2, \Delta \rightarrow 0) | t \rangle,
$$

then the positivity of the "effective mass squared", $m_{\text{eff}}^2(B) = m^2 - eB + \epsilon_t^2$, follows.

This result can be generalized by solving the gap equation for the effective mass. Let us do this, assuming for simplicity that $m_H = m_Z$ when contribution (2) $(B) = 0$.

As it is known [11], the ground state $|t \rangle = |n = 0, \sigma = +1 \rangle$ is the eigenstate for the full polarization tensor. So, all the radiation contributions result in an effective energy or mass. This is key point in deriving the SD equation for this state. In general, the $W$-boson polarization tensor is expressed through four structures:

$$
\hat{l}^2 = p_3^2 + p_3^2,\hat{p}^2 = h^2 = p_3^2 + \hat{p}^2, pp, H = p_3^2 - 2ieF.
$$

Among these, the first and the last operator commutes with all others. For the ground state $|t \rangle$ in addition the Lorentz condition holds: $p|t \rangle = 0$ [12], [11]. So, the polarization tensor is diagonal on this state. This important property can serve as motivation for the choice of the $W$-boson Green function used in the SD equation.

Accounting for noted above, the mean value for the full operator $\langle t | G^{-1} | t \rangle$ can be written in the form:

$$
\langle t | G^{-1} | t \rangle = \hat{l}^2 + M^2(B) - eB.
$$

Here, we introduced the parameter $M^2(B)$ which accounts for all the contributions to ground state energy - particle mass $m^2$ and field dependent radiation corrections giving the effective mass. This definition of the effective mass is more convenient. Expression (29) generalizes the structure of the lower Landau level of the spectrum (1). Taking into account this property, as the exact $W$-boson operator we choice the expression $G = (p^2 - 2ieF + M^2)^{-1}$. We substitute it into the operator SD equation

$$
G^{-1}(p^2, M^2, F) = p^2 - 2ieF + m^2 - \Pi(G(p^2, M^2, F))
$$

and calculated the r.h.s. in one-loop order. As the ansatz for the ghost and Goldstone fields we substitute the expressions used in sect. 2 where we replace the mass $m^2 \rightarrow M^2$. This is because in the gauge [12] used the mass of all the charged fields is the same and coinciding with the $W$-boson mass. In this case the renormalization can be done according to eq. (23).

With these entities used, in the limit of $B \rightarrow B_c$ the SD equation for the $\langle t | G^{-1} | t \rangle$ transforms into gap equation

$$
M^2 - eB = m^2 - eB + 2(eB)\frac{e^2}{(4\pi)^2} \log\left(\frac{1}{M^2/(eB) - 1}\right).
$$
Here, in the r.h.s. the one-loop expression (27) is substituted. This equation can easily be solved graphically by showing the r.h.s. and l.h.s. in one plot. The value of $M^2$ can be determined at a given field strength $B$ as crossing of both curves. In this way a resummation of infinite series of one-loop diagram is fulfilled. In actual calculations, it is convenient to measure all the variables in terms of $m^2$. We denote $x = \frac{M^2}{m^2}$, $y = \frac{eB}{m^2}$. Then eq.(31) takes the form

$$x = 1 + \frac{2y}{(4\pi)^2} \log \left( \frac{1}{x/y - 1} \right).$$  \hspace{1cm} (32)

In Figs.1 and 2 we show the results for some values of $y$. As it is occurred, the effective mass $M^2$ is positive even for field strength $eB$ larger than $m^2$. Thus, the stabilization of the spectrum happens. This is nonperturbative result accounting for the influence of radiation corrections at the threshold of instability.

Fig.1 Effective $W$-boson mass for $y = 0.8$.

Fig.2 Effective $W$-boson mass for $y = 1$.

To complete this part, we note that the fermion contribution in one-loop order does not depend on the unstable mode. So, it does not influence in essential way the effective mass $M^2(B)$. 
4 Discussion and conclusions

We derived simple representation for the mean value of the $W$-boson polarization tensor in external magnetic field calculated in the ground state of the tree-level spectrum \([11]\). It corresponds to the Demeur expression for electron in magnetic field in QED. As an application, we investigated the behavior of the $W$-boson effective mass at the threshold of instability $B \to B_c$. We found, the effective mass determined within the SD equation is positive, that prevents the vacuum instability in strong fields. This result can be compared with the one obtained already in one-loop order for the Georgy-Glashow model \([11]\). In the latter case, however, there exists the range of not heavy Higgs boson mass for which radiation corrections shift the threshold of instability to the weaker than $B_c$ field strengths, and increase instability. In the standard model such type domain absences for realistic values of $m_H > m_Z$.

Note that in one-loop case the applicability of eq.(27) is restricted by the condition $\frac{\alpha}{(4\pi)} \log(1/(m^2/(eB)−1)) << 1$. Therefore, the threshold of instability is not under control. The solution of the SD equation is an effective resummation of infinite series of diagrams. As a result, one can investigate not only the fields $B \sim B_c$ but also the ones $B \geq B_c$ where the spectrum stabilization is observed.

The obtained results need in further discussion. In all previous considerations, the value of the mass $m$ was taken as being fixed. But this is not the case because the $W$-boson mass is determined through the minimum position of the scalar field effective potential: $m^2(B) = \frac{1}{2} g^2 \sin^2 \theta \delta^2(B)$. The minimum position $\delta(B)$ and the behavior of the effective potential depend on the field strength. A detailed picture is discussed in review \([11]\). As it occurs, in strong fields in the effective potential a second minimum appears. It becomes the global one and transition to this state happens for the field strengths close to $B_c$. More details on this point can be found in \([11]\). So, here we restrict ourselves to this remark and refer reader to noted paper. For field strengths not very close to $B_c$ the initial minimum remains global and present consideration is sufficient.

Obtained results on the $W$-boson effective mass can be applied for arbitrary field strengths $B$. In general case, the SD equation in the ground state projection can be solved numerically. It determines the nonperturbative effective mass of particle, that could find applications for studying of various processes with $W$-bosons. For instance, in physics at the LHC where strong magnetic fields have to present.

Acknowledgements

The author is grateful to Michael Bordag for careful reading the manuscript, fruitful discussions and suggestions.
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