COARSE METRIC AND UNIFORM METRIC

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Abstract. We introduce the notion of coarse metric. Every coarse metric induces a coarse structure on the underlying set. Conversely, we observe that all coarse spaces come from a particular type of coarse metric in a unique way. In the case when the coarse structure $\mathcal{E}$ on a set $X$ is defined by a coarse metric that takes values in a meet-complete totally ordered set, we define the associated Hausdorff coarse metric on the set $\mathcal{P}_0(X)$ of non-empty subsets of $X$ and show that it induces the Hausdorff coarse structure on $\mathcal{P}_0(X)$.

On the other hand, we define the notion of pseudo uniform metric. Each pseudo uniform metric induces a uniform structure on the underlying space. In the reverse direction, we show that a uniform structure $\mathcal{U}$ on a set $X$ is induced by a map $d$ from $X \times X$ to a partially ordered set (with no requirement on $d$) if and only if $\mathcal{U}$ admits a base $\mathcal{B}$ such that $\mathcal{B} \cup \{\bigcap \mathcal{U}\}$ is closed under arbitrary intersections. In this case, $\mathcal{U}$ is actually defined by a pseudo uniform metric. We also show that a uniform structures $\mathcal{U}$ comes from a pseudo uniform metric that takes values in a totally ordered set if and only if $\mathcal{U}$ admits a totally ordered base.

Finally, a valuation ring will produce an example of a coarse and pseudo uniform metric that take values in a totally ordered set.

1. Introduction and Notations

The notion of coarse spaces was first introduced in [6]. It can be regarded as an abstract framework for the study of large scale properties of metric spaces. A thorough account for coarse spaces can be found in [12] (see also [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 13, 14, 15, 16] for some information on coarse structure). On the other hand, coarse structure can be deemed as an opposite to uniform structure (see e.g. [5] or [11]).

Obviously, pseudo metric is a common source of examples for both of them. In fact, uniform spaces are generalizations of pseudo metric spaces and coarse structures were first studied for metric spaces (see [6]). It is known that a coarse structure (respectively, a uniform structure) is defined by a pseudo metric if and only if it has a countable base (see e.g. [12, Theorem 2.55] for the case of coarse structures and [11, Corollary I.4.4] for the case of uniform structures). The aim of this article is to introduce two general notions of metric, and to study coarse structures and uniform structures associated respectively with them.

For a partially ordered set $J$ with a smallest element $0_J$, we denote by $J_\infty$ the extension of $J$ by adjoining a new element $\infty$ that is larger than all elements in $J$. The most general form of "metric" on a set $X$ is simply a map
$$d : X \times X \to J_\infty.$$ If the map $d$ satisfies $d(x,y) = d(y,x)$ as well as $d(x,x) = 0_J$ ($x, y \in X$), and it also fulfills certain growth condition as in Definition 2(a) (respectively, descent condition as in Definition 13(b)), then $d$ is called a coarse metric (respectively, uniform metric). It will be obvious that a coarse metric (respectively, uniform metric) will induce a coarse structure (respectively, uniform structure) on the underlying space. A natural question is how to characterize those coarse structures and uniform structures coming from such generalized notions of metric.

In fact, it is not hard to check that any coarse space actually comes from a coarse metric. Furthermore, there is a bijective correspondence between the collection of coarse structures on a set $X$ and the collection of coarse metrics on $X$ that are "saturated" (Theorem 5). Through the correspondence of
coarse structures and coarse metrics, one can rephrase some terminologies in coarse spaces back in metric terms, which makes them easier to understand, and hopefully easier to manipulate. For example, a coarse space is coarsely connected if and only if one (and hence all) of its defining coarse metrics does not take the value $\infty$. A list of other translations can be found in Propositions 8 and 9. Philosophically, the above correspondence tells us that by studying the coarse structure of a metric space, one actually “forgets” the triangle inequality and “remembers” only the growth condition (as in Definition 2(a)).

On the other hand, we will show that a coarse structure has a totally ordered base if and only if it is defined by a coarse metric taking values in a totally ordered set (Example 20). Further, we investigate the relation between the Hausdorff coarse structure on the set of non-empty subsets of a coarse space (Definition 10) and the Hausdorff coarse metric on the same collection of subsets induced by the coarse metric defining the original coarse space (Proposition 11).

In the case of uniform structures, the correspondence is not as perfect. We will show in Section 3 that for a uniform structure $U$, there is a map $d : X \times X \to \mathbb{J}_\infty$ (without any further requirement on $d$) such that

$$\{D_\alpha : \alpha \in \mathbb{J} \setminus \{0\}\}$$

forms a base for $U$, where

$$D_\alpha := \{(x, y) : d(x, y) \leq \alpha\},$$

if and only if $U$ admits a base $B$ with $B \cup \{\bigcap U\}$ being closed under arbitrary intersections. In this case, $U$ is actually defined by a pseudo uniform metric (Theorem 17). In particular, if $U$ admits a totally ordered base, then it is defined by a pseudo uniform metric (Corollary 18).

In Section 4, we give a mild condition, under which a coarse metric will become a pseudo uniform metric (Proposition 19). We will close this short article by giving an example of a coarse and pseudo uniform metric coming from a valuation ring (Example 20).

In the remainder of this section, let us set some notation. Suppose that $J$ and $J_\infty$ are as in the second paragraph of this Introduction.

- The smallest element in $J$, if it exists, is called the zero of $J$ and will always be denoted by $0_I$.
- $J$ is called an upward directed set if for any $\alpha, \beta \in J$, there exists $\gamma \in J$ with $\alpha \leq \gamma$ and $\beta \leq \gamma$.
- If $J$ is another partially ordered set, and $\Lambda : J \to J$ is a map, then we set $\Lambda_\infty : J_\infty \to J_\infty$ to be the extension of $\Lambda$ such that $\Lambda_\infty(\infty) = \infty$. (1.2)

A partially ordered set $J$ is said to be meet-complete if every non-empty subset $S$ of $J$ admits a greatest lower-bound inf $S$ in $J$. Furthermore, $J$ is called a complete lattice if it is meet-complete and each non-empty subset of $J$ has a least upper-bound in $J$.

For a set $X$, we use $P_0(X)$ to denote the collection of all non-empty subsets of $X$ and use $\Delta_X$ to denote the diagonal in $X \times X$; namely,

$$\Delta_X := \{(x, x) \in X \times X : x \in X\}.$$  

Moreover, for any $A, B \subseteq X \times X$, we put $A^{-1} := \{(y, x) : (x, y) \in A\}$ as well as

$$A \circ B := \{(x, z) \in X \times X : \text{there exists } y \in X \text{ such that } (x, y) \in A \text{ and } (y, z) \in B\}.$$  

In this case, $A^{-1}$ is called the inverse of $A$ and $A \circ B$ is called the product of $A$ and $B$. Furthermore, if $S \subseteq X$, we denote

$$B[S] := \{y \in X : (x, y) \in B, \text{ for some } x \in S\},$$

and $B[x] := B[\{x\}]$.

Definition 1. Let $J$ be a partially ordered set with a zero, and $d : X \times X \to J_\infty$ be a map.

(a) We set

$$D(z, \alpha) := D_\alpha[z] \quad (z \in X; \alpha \in J)$$

(where $D_\alpha$ is as in (1.1)), and call $D(z, \alpha)$ the ball of radius $\alpha$ with center $z$. Moreover, we denote

$$\mathcal{B}_d := \{D_\alpha : \alpha \in J \setminus \{0\}\}.$$
(b) Suppose that \( d \) satisfies

\[(D1) \quad d(x, x) = 0, \quad \text{for any} \quad x \in X; \]
\[(D2) \quad d(x, y) = d(y, x) \quad \text{for any} \quad x, y \in X. \]

Then \( d \) is called a **semi-\( I \)-metric**.

### 2. Coarse \( I \)-metric

In this section, we define and study coarse metrics and relate them to coarse structures. Recall that if \( E \subseteq \mathcal{P}_0(X \times X) \) such that \( \Delta_X \in E \), and \( E \) is closed under the formation of subsets, inverses, products and finite unions, then \( E \) is called a **coarse structure** on \( X \). In this case, \( (X, E) \) (or simply \( X \)) is called a **coarse space**. A subcollection \( \mathcal{B} \subseteq E \) is called a base for \( E \) if every element in \( E \) is contained in an element of \( \mathcal{B} \).

**Definition 2.** Let \( I \) be an upward directed set with a zero, and \( d : X \times X \to \mathbb{I}_\infty \) be a semi-\( I \)-metric

(a) Suppose that there is a function \( \Phi : I \to I \) such that \( d(x, z) \leq \Phi(\alpha) \) whenever \( \alpha \in I \) and \( x, y, z \in X \) satisfy \( d(x, y) \leq \alpha \) and \( d(y, z) \leq \alpha \); in other words,

\[\mathcal{D}_\alpha \circ \mathcal{D}_\alpha \subseteq \mathcal{D}_{\Phi(\alpha)} \quad (\alpha \in I).\]

Then \( d \) is called a **coarse \( I \)-metric** on \( X \), and \( (X, I, d) \) is called a **coarse metric space**.

(b) A coarse \( I \)-metric \( d \) is said to be **saturated** if for each \( \alpha, \beta \in I \),

- the inclusion \( \mathcal{D}_\alpha \subseteq \mathcal{D}_\beta \) implies \( \alpha \leq \beta \);
- for any subset \( S \subseteq \mathcal{D}_\alpha \) with \( S^{-1} = S \) and \( \Delta_X \subseteq S \), one can find \( \gamma \in I \) such that \( S = \mathcal{D}_\gamma \).

(c) Suppose that \( J \) is another upward directed set with a zero and \( d' \) is a coarse \( J \)-metric on \( X \). We say that \( d \) is **coarsely dominated by** \( d' \) (and denote this by \( d \preceq d' \)) if there is an increasing map \( \Gamma : J \to I \) such that \( d(x, y) \leq \Gamma(\alpha)(d'(x, y)) \) for every \( x, y \in X \) (see (1.2)). If \( d \preceq d' \) and \( d' \preceq d \), then we say that \( d \) is **coarsely equivalent to** \( d' \) and denote this by \( d \sim d' \).

**Remark 3.** Let \( d \) be a coarse \( I \)-metric.

(a) If \( \Phi \) is a map satisfying the requirement in Definition 2(a), then it is not hard to see that \( d(x, y) \leq \Phi(d(x, y)) \) \( (x, y \in X) \). Moreover, as \( I \) is upward directed, one can always find a map \( \Phi \) with \( \alpha \leq \Phi(\alpha) \) \( (\alpha \in I) \) that satisfies the requirement of Definition 2(a).

(b) Suppose, in addition, that \( J \) is meet-complete. Then clearly, \( \bigcap_{\alpha \in S} \mathcal{D}_\alpha \subseteq \mathcal{D}_{\inf S} \) for \( S \subseteq J \). We set

\[\mathcal{J}(\alpha) := \{ \beta \in \mathcal{J} : \mathcal{D}_\alpha \circ \mathcal{D}_\alpha \subseteq \mathcal{D}_\beta \} \quad (\alpha \in \mathcal{J}),\]

and define \( \Phi(\alpha) := \inf \mathcal{J}(\alpha) \). It is obvious that \( \Phi \) is increasing. If \( x, y, z \in X \) satisfying \( d(x, y) \leq \alpha \) and \( d(y, z) \leq \alpha \), then \( d(x, z) \leq \beta \) for any \( \beta \in \mathcal{J}(\alpha) \), and hence \( d(x, z) \leq \Phi(\alpha) \). Thus, in the case when \( J \) is meet-complete, we can always find an increasing map \( \Phi \) satisfying the requirement in Definition 2(a).

(c) It makes no harm to assume that the directed set \( J \) where a coarse metric \( d \) takes values is meet complete. In fact, for every \( \alpha \in \mathcal{J} \), we set \( \tilde{\alpha} := \{ \beta \in \mathcal{J} : \beta \leq \alpha \} \in \mathcal{P}_0(\mathcal{J}) \). Denote

\[\tilde{\mathcal{J}} := \{ A \in \mathcal{P}_0(\mathcal{J}) : A \subseteq \tilde{\alpha}, \quad \text{for some} \quad \alpha \in \mathcal{J} \},\]

and define \( j : \mathcal{J} \to \tilde{\mathcal{J}} \) to be the map sending \( \alpha \) to \( \tilde{\alpha} \). Then \( \tilde{\mathcal{J}} \) is a meet complete upward directed set with zero (namely, \( \tilde{0} \)) and \( j \) is an order preserving injection. We set \( \tilde{\Phi}(A) := \bigcap\{ j(\Phi(\alpha)) : \alpha \in \mathcal{J} : A \subseteq \tilde{\alpha} \} \quad (A \in \tilde{\mathcal{J}}).\)

If we put \( \tilde{d} := j \circ d \), then \( \tilde{d} \) is a coarse \( \tilde{\mathcal{J}} \)-metric. For every \( \alpha \in \mathcal{J} \), one has \( \mathcal{D}_\alpha = \mathcal{D}_{\tilde{\alpha}} \). On the other hand, for each \( A \in \tilde{\mathcal{J}} \), there exists \( \alpha \in \mathcal{J} \) with \( A \subseteq \tilde{\alpha} \), and hence \( \mathcal{D}_\alpha \subseteq \mathcal{D}_{\tilde{\alpha}} = \mathcal{D}_{\alpha} \). Thus, the coarse structures generated by \{\( \mathcal{D}_A : A \in \tilde{\mathcal{J}} \)\} and by \{\( \mathcal{D}_A : \alpha \in \mathcal{J} \)\} are the same.

Let us begin will the following easy fact.
Lemma 4. Let $\mathcal{T}$ be a subcollection of $\mathcal{P}_0(X \times X)$. We set $\mathcal{T}_s := \{(A \cap A^{-1}) \cup \Delta_X : A \in \mathcal{T}\}$ as well as

$$\overline{\mathcal{T}} := \left\{ \bigcap S : \emptyset \neq S \subseteq \mathcal{T}_s \right\}. $$

The collection $\overline{\mathcal{T}}$ is closed under arbitrary intersections (i.e. the intersection of any subset of $\overline{\mathcal{T}}$ belongs to $\overline{\mathcal{T}}$). Moreover, if $\mathcal{T}$ is totally ordered, then so is $\overline{\mathcal{T}}$.

Proof. The first claim is obvious. Suppose now that $\mathcal{T}$ is totally ordered. Then $\mathcal{T}_s$ is also totally ordered. Consider $\mathcal{E}, \mathcal{D} \subseteq \mathcal{T}_s$. Then either there exists $\mathcal{C}_0 \subseteq \mathcal{C}$ such that $\mathcal{C}_0 \subseteq \mathcal{D}$ for every $\mathcal{D} \subseteq \mathcal{C}$, or for each $\mathcal{C} \subseteq \mathcal{E}$, one can find $\mathcal{D} \subseteq \mathcal{D}$ with $\mathcal{D} \subseteq \mathcal{C}$. In the first case, we have $\bigcap \mathcal{E} \subseteq \mathcal{C}_0 \subseteq \bigcap \mathcal{D}$. In the second case, we know that $\bigcap \mathcal{D} \subseteq \bigcap \mathcal{C}$.

Theorem 5. Let $X$ be a set.

(a) Suppose that $\mathcal{J}$ is a meet complete upward directed set with a zero and $\mathbf{d}$ is a coarse $\mathcal{J}$-metric on $X$. Then $\mathbf{d}_\mathcal{J} \cup \{\mathbf{d}_0\}$ (see Definition 4(a)) is a base for a coarse structure $\mathcal{E}_\mathbf{d}$ on $X$. Moreover, if $\mathcal{J}'$ is another upward directed set with a zero and $\mathbf{d}'$ is a $\mathcal{J}'$-metric on $X$, then $\mathbf{d}$ is coarsely equivalent to $\mathbf{d}'$ if and only if $\mathcal{E}_\mathbf{d} = \mathcal{E}_{\mathbf{d}'}$.

(b) Let $\mathcal{E}$ be a coarse structure on $X$. There is a unique upward directed set $\mathcal{J}^\mathcal{E}$ with a zero such that one can find a (necessarily unique) saturated coarse $\mathcal{J}^\mathcal{E}$-metric $\mathbf{d}^\mathcal{E}$ with $\mathcal{E} = \mathcal{E}_{\mathbf{d}^\mathcal{E}}$. In this case, $\mathcal{J}^\mathcal{E}_\infty$ is a complete lattice.

(c) Let $\mathcal{B}$ be a base for a coarse structure $\mathcal{E}$ on $X$, and $\overline{\mathcal{B}}$ be as in Lemma 4. Then $\overline{\mathcal{B}}$ is a meet-complete upward directed set, and there is a coarse $\overline{\mathcal{B}}$-metric $\mathbf{d}^\overline{\mathcal{B}}$ such that $\mathcal{E} = \mathcal{E}_{\mathbf{d}^\overline{\mathcal{B}}}$.

Proof. (a) For any $\alpha, \beta \in \mathcal{J}$, if $\gamma \in \mathcal{J}$ satisfying $\alpha \leq \gamma$ and $\beta \leq \gamma$, then $\mathbf{D}_\alpha \circ \mathbf{D}_\beta \subseteq \mathbf{D}_{\Phi(\gamma)}$. This gives the first statement. For the second statement, we note $\mathbf{d} \leq \mathbf{d}'$ if and only if $\mathcal{E}_{\mathbf{d}} \subseteq \mathcal{E}_{\mathbf{d}'}$ (for the backward implication, we define $\Gamma(\alpha') := \inf\{\alpha \in \mathcal{J} : \mathbf{D}_\alpha \subseteq \mathbf{D}_\beta\}$ for every $\alpha' \in \mathcal{J}'$).

(b) As $\mathcal{E}$ is closed under the formation of finite unions and subsets, we see that

$$\mathcal{J}^\mathcal{E} := \{E \in \mathcal{E} : E^{-1} = E; \Delta_X \subseteq E\}$$

is a meet-complete lattice (under inclusion), and it contains $\Delta_X$ as its smallest element (i.e. zero). Moreover, since every subset of $\mathcal{J}^\mathcal{E}$ that has an upper bound in $\mathcal{J}^\mathcal{E}$ has a least upper bound in $\mathcal{J}^\mathcal{E}$, we know that $\mathcal{J}^\mathcal{E}_\infty$ is a complete lattice. Let

$$\mathbf{d}^\mathcal{E}(x, y) := \begin{cases} \Delta_X \cup \{(x, y), (y, x)\} & \text{when } (x, y) \in E \text{ for some } E \in \mathcal{E} \\ \infty & \text{otherwise.} \end{cases}$$

Obviously, $\mathbf{d}^\mathcal{E} : X \times X \to \mathcal{J}^\mathcal{E}_\infty$ is a semi-$\mathcal{J}^\mathcal{E}$-metric. If we put $\Phi^\mathcal{E}(E) := E \circ E \in \mathcal{J}^\mathcal{E}$ for each $E \in \mathcal{J}^\mathcal{E}$, then $\Phi^\mathcal{E}$ will satisfy the requirement in Definition 2(a), and $\mathbf{d}^\mathcal{E}$ is a coarse $\mathcal{J}^\mathcal{E}$-metric. Furthermore, as

$$E = \{(x, y) \in X \times X : \mathbf{d}^\mathcal{E}(x, y) \leq E\} \quad (E \in \mathcal{J}^\mathcal{E}),$$

we see that $\mathbf{d}^\mathcal{E}$ is saturated and that $\mathcal{E} = \mathcal{E}_{\mathbf{d}^\mathcal{E}}$.

Suppose now that $J$ is another upward directed set with a zero and $\mathbf{d}$ is a saturated coarse $\mathcal{J}$-metric on $X$ with $\mathcal{E} = \mathcal{E}_{\mathbf{d}}$. Then the saturation assumption of $\mathbf{d}$ implies that $\alpha \mapsto \mathbf{D}_\alpha$ is an order isomorphism from $\mathcal{J}$ onto $\mathcal{J}^\mathcal{E}$. Furthermore, it follows from the definitions and (2.2) that for any $u, v \in X$ and $\beta \in \mathcal{J}$, one has

$$\mathbf{d}(u, v) \leq \beta \quad \text{if and only if} \quad \mathbf{d}^\mathcal{E}(u, v) \leq \mathbf{D}_\beta. \quad (2.3)$$

For every $x, y \in X$, it is not hard to verify, through (2.3), that $\mathbf{d}^\mathcal{E}(x, y) = \mathbf{D}_{\mathbf{d}(x,y)}$. In other words, $\mathbf{d}$ is the same as $\mathbf{d}^\mathcal{E}$ under the order isomorphism $\alpha \mapsto \mathbf{D}_\alpha$.

(c) The meet-completeness of $\overline{\mathcal{B}}$ follows from Lemma 4. Suppose that $S, T \in \mathcal{P}_0(B_\mathcal{E})$, where $B_\mathcal{E}$ is as in Lemma 3. Take any $S \subseteq S$ and $T \subseteq T$. As $B_\mathcal{E}$ is a base for $\mathcal{E}$, there exists $B \in B_\mathcal{E}$ with $S \cup T \subseteq B$. Then $B \in \overline{\mathcal{B}}$ and $(\bigcap S) \cup (\bigcap T) \subseteq B$. This shows that $\overline{\mathcal{B}}$ is upward directed. Let us define

$$\mathbf{d}^\overline{\mathcal{B}}(x, y) := \bigcap\{D \in \overline{\mathcal{B}} : (x, y) \in D\} \quad (x, y \in X);$$

(2.4)
here we use the convention that \( \bigcap \emptyset = \infty \). We also set
\[
\Phi^B(B) := \bigcap \{ D \in \mathbb{F} : B \circ D \in \mathbb{E} \} \quad (B \in \mathbb{E}).
\]
Clearly, \( \Phi^B \) satisfies the requirement in Definition 2(a), and \( d^B \) is a coarse \( \mathbb{F} \)-metric. Moreover, as \( B = \{(x, y) \in X \times X : d^B(x, y) \leq B\} \) for any \( B \in \mathbb{F} \), we know that \( \mathcal{E} = \mathcal{E}_{d^B} \).

Obviously, in the case when \( \mathcal{J} \neq \emptyset \) and \( d \) is a coarse-\( \mathcal{J} \)-metric, then \( \mathcal{B}_d \) is a base for \( \mathcal{E}_d \). On the other hand, although \( \mathcal{J}^\infty \) in part (b) above is a complete lattice, it does not mean that \( \mathcal{J}^\infty \cup \{ \mathcal{J} \} \) is closed under arbitrary unions. In fact, if \( S \subseteq \mathcal{J}^\infty \) such that \( \bigcup S \notin \mathcal{J}^\infty \), then the least upper bound of \( S \) in \( \mathcal{J}^\infty \) is \( \infty \).

**Remark 6.** The maps \( \Phi^E \) and \( \Phi^B \) in the proof of parts (b) and (c) of Theorem 5 are increasing and satisfy the requirement in Definition 2(a) for \( d^E \) and \( d^B \), respectively. Moreover, one has \( E \subseteq \Phi^E(E) \) (respectively, \( B \subseteq \Phi^B(B) \)) for every \( E \in \mathcal{J}^\infty \) (respectively, \( B \in \mathbb{F} \)).

The following example tells us that if \( d \) is a coarse \( \mathcal{J} \)-metric and \( \mathcal{J} \) is an upward directed set containing \( \mathcal{I} \), then the coarse structure induced by \( d \) when \( d \) is considered as a coarse \( \mathcal{J} \)-metric may not be the same as the one when \( d \) is considered as a coarse \( \mathcal{J} \)-metric.

**Example 7.** Let \((\mathbb{R}, d_1)\) be the Euclidean metric space. Clearly, \( d_1 \) is a \( \mathbb{R}^+ \)-metric and the coarse structure generated by this \( \mathbb{R}^+ \)-metric is the usual one.

However, if we set \( \mathcal{J} = \mathbb{R}_+^\infty \), then \( d_1 \) is also a \( \mathcal{J} \)-metric, but the coarse structure generated by this \( \mathcal{J} \)-metric is the “trivial one”, because \( \mathbb{R} = \mathcal{D}_\infty \) is a controlled set.

One can express many concepts concerning coarse structures in terms of metric, which seem easier to understand and handle. Let us list some of them in the following.

**Proposition 8.** Let \((X, \mathcal{E})\) and \((Y, \mathcal{F})\) be two coarse spaces. Suppose that \( d_X \) (respectively, \( d_Y \)) is a coarse \( \mathcal{J} \)-metric on \( X \) (respectively, coarse \( \mathcal{J} \)-metric on \( Y \)) that induces the underlying coarse structure. Let \( f, g : X \to Y \) be two maps.

(a) \((X, \mathcal{E})\) is coarsely connected if and only if the largest element \( \infty \in \mathcal{J}_\infty \) does not belong to \( d_X(X \times X) \).

(b) \( B \subseteq X \) is bounded if and only if one can find \((x, \alpha) \in X \times \mathcal{J} \) with \( B \subseteq d_X(x, \alpha) \).

(c) If \( \mathcal{J} \) is meet complete, then \( f \) is bornologous if and only if \( d_Y \circ (f \times f) \leq d_X \).

(d) \( f \) is proper if and only if there is a map \( \Upsilon : Y \times \mathcal{J} \to X \times \mathcal{J} \) with \( f^{-1}(d_Y(y, \beta)) \subseteq d_X(\Upsilon(y, \beta)) \), for each \((y, \beta) \in Y \times \mathcal{J} \).

(e) If \( \mathcal{J} \) is meet complete, then \( f \) is effectively proper if and only if \( d_X \leq d_Y \circ (f \times f) \).

(f) \( f \) and \( g \) are close if and only if there exists \( \beta \in \mathcal{J} \) with \( d_Y(f(x), g(x)) \leq \beta \), for any \( x \in X \).

**Proof.** (a) Recall that \((X, \mathcal{E})\) is coarsely connected if and only if \( \{(x, y) \in \mathcal{E}, \text{ for every } x, y \in X \}. \) Clearly, this is equivalent to \( d_X(x, y) \in \mathcal{J} \), for every \( x, y \in X \).

(b) Recall that \( B \) is bounded if and only if \( B \subseteq E[x] \) for some \( E \in \mathcal{E} \) and \( x \in X \). The equivalence in the statement is more or less trivial.

(c) Recall that \( f \) is bornologous if and only if \( (f \times f)(\mathcal{J}) \subseteq \mathcal{J} \); equivalently, for every \( \alpha \in \mathcal{J} \), one can find \( \beta \in \mathcal{J} \) such that \( (f \times f)(d_X(x, \alpha)) \subseteq d_X(x, \beta) \). In this case, the map \( \Gamma \) defined by \( \Gamma(\alpha) := \inf \{ \beta \in \mathcal{J} : (f \times f)(d_X(x, \alpha)) \subseteq d_X(x, \beta) \} \) is increasing. Now, for an increasing map \( \Gamma : \mathcal{J} \to \mathcal{J} \), the condition \( (f \times f)(d_X(x, \alpha)) \subseteq d_X(x, \Gamma(\alpha)) \) is equivalent to the requirement as in Definition 2(c) for \( d_Y \circ (f \times f) \leq d_X \).

(d) Recall that \( f \) is proper if and only if \( f^{-1}(B) \) is bounded for any bounded set \( B \subseteq Y \). Thus, this part follows directly from part (b).

(e) Recall that \( f \) is effectively proper if and only if \( (f \times f)^{-1}(\mathcal{E}) \subseteq \mathcal{E} \). This is the same as saying that for every \( \beta \in \mathcal{J} \), one find \( \alpha \in \mathcal{J} \) such that \( (f \times f)^{-1}(d_X(x, \beta)) \subseteq d_X(x, \alpha) \), or equivalently, \( d_X(x, y) \leq \alpha \) whenever \( d_Y(f(x), f(y)) \leq \beta \).
In this case, the map $\Gamma$ defined by $\Gamma(\beta) := \inf\{\alpha \in J : (f \times f)^{-1}(D_\beta) \subseteq D_\alpha\}$ ($\beta \in \bar{J}$) is increasing. Now, an increasing map $\Gamma : \bar{J} \to J$ satisfies the above displayed statement for $\beta = \Gamma(\alpha)$ if and only if it satisfies the requirement in Definition 2(c) for $d_X \preceq d_Y \circ (f \times f)$.

(f) Recall that $f$ and $g$ are close if and only if $\{(f(x), g(x)) : x \in X\} \subseteq F$, which is obviously the same as the requirement in the statement of part (f). □

Notice that in Example 7 although the range of $d_1$ take the value “$\infty$”, it is the largest element in $\mathcal{J} = \mathbb{R}_+^\infty$, but not the largest element of $\mathcal{J}_\infty$. Therefore, the resulting coarse structure is connected.

On the other hand, we can “reverse-engineer” some concepts in coarse structure back to metric space terms. The following is such an example. Let us recall from [12, Definition 3.9] that a coarse space $(X, E)$ is said to have bounded geometry if one can find $E \in \mathcal{E}$ containing $\Delta_X$ such that $E^{-1} = E$ and

$$\sup_{x \in X} \max \{\text{cap}_E((F \circ E)[x]), \text{cap}_E((F^{-1} \circ E)[x])\} < \infty \quad (F \in \mathcal{E}), \quad (2.5)$$

where $\text{cap}_E(S) := \sup\{m \in \mathbb{N} : \text{there exist } y_1, \ldots, y_m \in S \text{ with } (y_i, y_j) \notin E \text{ when } i \neq j\}$.

Clearly, $\text{cap}_{E'}(S) \leq \text{cap}_E(S)$ if $E \subseteq E'$ and $\text{cap}_E(S') \leq \text{cap}_E(S)$ if $S \subseteq S'$. Moreover, one has $F \subseteq F \circ E \subseteq E$. Consequently, one may replace (2.5) with $\sup_{x \in X} \text{cap}_E(F[x]) < \infty$, for every $E \in \mathcal{E}$ with $E^{-1} = F$. Thus, in the case when $\mathcal{E}$ is defined by a coarse $\mathcal{J}$-metric, $(X, \mathcal{E})$ has bounded geometry if and only if one can find $\alpha_1 \in \mathcal{J}$ satisfying

$$\sup_{x \in X} \text{cap}_{D^{\alpha_1}}(D(x, \alpha)) < \infty \quad (\alpha \in \mathcal{J}). \quad (2.6)$$

**Proposition 9.** Let $(X, E)$ be a coarse space and $d_X$ be a coarse $\mathcal{J}$-metric defining $E$. The following statements are equivalent.

B1) $(X, E)$ has bounded geometry.

B2) There is $\alpha_1 \in \mathcal{J}$ satisfying: for any $\alpha \in \mathcal{J}$, there exists $n_1 \in \mathbb{N}$ such that each ball of radius $\alpha$ contains at most $n_1$ points with their pairwise $d_X$-distances not dominated by $\alpha_1$ (i.e. $d_X(x, y) \not\preceq \alpha_1$).

B3) There is $\alpha_2 \in \mathcal{J}$ satisfying: for any $\alpha \in \mathcal{J}$, there exists $n_2 \in \mathbb{N}$ such that for each $x \in X$, the ball $D(x, \alpha)$ contains at most $n_2$ disjoint relative balls of radius $\alpha_2$ (here, relative balls of radius $\alpha_2$ are subsets of the form $D(x, \alpha) \cap D(y, \alpha_2)$ for some $y \in D(x, \alpha)$).

B4) There is $\alpha_3 \in \mathcal{J}$ satisfying: for any $\alpha \in \mathcal{J}$, there exists $n_3 \in \mathbb{N}$ such that each ball of radius $\alpha$ is contained in the union of $n_3$ balls of radius $\alpha_3$.

**Proof.** (B1) $\iff$ (B2) This equivalence is simply a matter of reformulating (2.6).

(B2) $\implies$ (B3) If $D(x, \alpha)$ contains at most $n_1$ points with their mutual $d_X$-distances not dominated by $\alpha_1$, then clearly, it cannot contain more than $n_1$ disjoint relative balls of radius $\alpha_1$.

(B3) $\implies$ (B2) Suppose that $D(x, \alpha)$ contains at most $n_2$ disjoint relative balls of radius $\alpha_2$. Let $\Phi$ be as in Definition 2(a) and $\alpha_1 := \Phi(\alpha_2)$. Then $D(x, \alpha)$ cannot contain more than $n_2$ points with their mutual $d_X$-distance not dominated by $\alpha_1$.

(B1) $\iff$ (B4) Let us recall from [12, Definition 3.1(a)] the following definition:

$$\text{ent}_E(S) := \inf\{n \in \mathbb{N} : \text{there exist } x_1, \ldots, x_n \in X \text{ with } S \subseteq E[x_1] \cup \cdots \cup E[x_n]\}$$

(where $\inf\emptyset := \infty$). It was shown in [12, Proposition 3.2(d)] that

$$\text{cap}_{E\circ E}(S) \leq \text{ent}_E(S) \leq \text{cap}_E(S).$$

Therefore, one can replace “cap” by “ent” in the definition of bounded geometry; namely,

$$\sup_{x \in X} \text{ent}_E(F[x]) < \infty, \quad \text{for every } F \in \mathcal{E} \text{ with } F^{-1} = F.$$

In other words, $(X, E)$ has bounded geometry if and only if there exists $\alpha_3 \in \mathcal{J}$ such that

$$\sup_{x \in X} \text{ent}_{D^{\alpha_3}}(D(x, \alpha)) < \infty \quad (\alpha \in \mathcal{J}).$$

This statement is clearly equivalent to Statement (B4). □
Using the equivalence of Statements (B1) and (B2), one obtains an easy way to see that every bounded geometry space is coarse equivalent to a uniformly discrete space. In fact, consider $\mathcal{B} = \{D(x_i, \alpha_1) : i \in \mathcal{I}\}$ to be a maximal collection of disjoint balls of radius $\alpha_1$. Since $\bigcup_{i \in \mathcal{I}} D(x_i, \Phi(\alpha_1)) = X$, we know that $X$ is coarse equivalent to its subspace $\{x_i : i \in \mathcal{I}\}$ (as in Remark 3(a), we may assume that $\alpha_1 \leq \Phi(\alpha_1)$), and the later is uniformly discrete.

We end this section with a discussion of the coarse structure induced on the collection $\mathcal{P}_0(X)$ of non-empty subsets of a coarse space $X$.

**Definition 10.** Let $(X, \mathcal{E})$ be a coarse space. For any $E \in \mathcal{J}^\mathcal{E}$ (see (2.1)), we set
\[
\tilde{E} := \{(R, S) \in \mathcal{P}_0(X) \times \mathcal{P}_0(X) : R \subseteq E[S] \text{ and } S \subseteq E[R]\}.
\] (2.7)
The coarse structure $\tilde{\mathcal{E}}$ on $\mathcal{P}_0(X)$ generated by $\{\tilde{E} : E \in \mathcal{J}^\mathcal{E}\}$ is called the **Hausdorff coarse structure associated with $\mathcal{E}$**.

Notice that if $\mathcal{B}$ is a base for $\mathcal{E}$, then $\{\tilde{B} : B \in \mathcal{B}\}$ (see Lemma 4) is a base for $\tilde{\mathcal{E}}$.

By Theorem 5(b), both $\mathcal{E}$ and $\tilde{\mathcal{E}}$ are defined by coarse metrics. It is natural to ask the relation between metrics defining $\mathcal{E}$ and those defining $\tilde{\mathcal{E}}$. In the case when $\mathcal{J}$ is meet-complete, one natural guess is the following Hausdorff semi-metric associated with a coarse 3-metric $d$:
\[
d(R, S) := \inf \left\{\alpha \in \mathcal{J} : R \subseteq \bigcup_{s \in S} D(s, \alpha) \text{ and } S \subseteq \bigcup_{r \in R} D(r, \alpha)\right\} \quad (R, S \in \mathcal{P}_0(X));
\] (2.8)
again, we use the convention that $\inf \emptyset = \infty$. Two natural questions are:

1. is $d$ actually a coarse 3-metric?
2. does the coarse structure defined by $d$ coincide with $\tilde{\mathcal{E}}$?

We doubt if these two questions have positive answers in general. However, we will consider a situation when they do.

**Proposition 11.** Let $\mathcal{J}$ be a meet-complete totally ordered set, and $d$ is a coarse 3-metric on $\mathcal{P}_0(X)$.

(a) $\tilde{d}$ is a coarse 3-metric on $\mathcal{P}_0(X)$.

(b) The coarse structure induced by $\tilde{d}$ is precisely $\tilde{\mathcal{E}}_d$.

**Proof.** (a) It is clear that $\tilde{d}(R, R) = 0$ and $\tilde{d}(R, S) = \tilde{d}(S, R)$ for $R, S \in \mathcal{P}_0(X)$. By Remark 3(b), there is an increasing map $\Phi : \mathcal{J} \to \mathcal{J}$ satisfying the requirement in Definition 2(a). Consider $\alpha \in \mathcal{J}$. When $\alpha$ is the largest element of $\mathcal{J}$ (if it exists), then we set $\Phi(\alpha) := \alpha$. When $\alpha$ is not the largest element of $\mathcal{J}$, we fix an element $\beta(\alpha) \in \mathcal{J}$ with $\alpha \leq \beta(\alpha)$ and set $\Phi(\alpha) := \Phi(\beta(\alpha))$. Let $R, S, T \in \mathcal{P}_0(X)$. Consider
\[
\mathcal{J}(R, S) := \left\{\delta \in \mathcal{J} : R \subseteq \bigcup_{s \in S} D(s, \delta) \text{ and } S \subseteq \bigcup_{r \in R} D(r, \delta)\right\}.
\]
Assume that $\tilde{d}(R, S) \leq \alpha$ and $\tilde{d}(S, T) \leq \alpha$. Since $\alpha \in \mathcal{J}$, we know that $\mathcal{J}(R, S) \neq \emptyset$ and $\mathcal{J}(S, T) \neq \emptyset$. If $\alpha$ is the largest element of $\mathcal{J}$, then obviously, $\tilde{d}(R, T) \leq \alpha = \tilde{d}(S, R)$. Otherwise, since $\tilde{d}(R, S) \leq \beta(\alpha)$, there exists $\gamma' \in \mathcal{J}(S, T)$ with $\gamma' \leq \beta(\alpha)$ (because $\mathcal{J}$ is totally ordered). Similarly, one can find $\gamma'' \in \mathcal{J}(S, T)$ with $\gamma'' \leq \beta(\alpha)$. Hence, if $\gamma := \max\{\gamma', \gamma''\}$, then $R \subseteq \bigcup_{t \in T} D(t, \Phi(\gamma))$ and $T \subseteq \bigcup_{r \in R} D(r, \Phi(\gamma))$. This implies that $\tilde{d}(R, T) \leq \Phi(\gamma) \leq \Phi(\alpha)$, because $\Phi$ is increasing.

(b) As said in the above, $\{\tilde{d}_\alpha : \alpha \in \mathcal{J}\}$ is a base for $\tilde{\mathcal{E}}_d$, where $\tilde{d}_\alpha$ is defined as in (2.7). Fix $\alpha_0 \in \mathcal{J}$. When $\alpha_0$ is the largest element in $\mathcal{J}$ (if it exists), one has
\[
\tilde{d}_\alpha = \{(R, S) \in \mathcal{P}_0(X) \times \mathcal{P}_0(X) : R \subseteq \bigcup_{s \in S} D(s, \beta) \text{ and } S \subseteq \bigcup_{r \in R} D(r, \beta), \text{ for some } \beta \in \mathcal{J}\}
\]
Assume that $\alpha_0$ is not the largest element in $\mathcal{J}$. Choose any $\beta \in \mathcal{J}$ with $\alpha_0 \leq \beta$. If $R, S \in \mathcal{P}_0(X)$ satisfying $\tilde{d}(R, S) \leq \alpha_0$, then, as in the argument of part (a), one can find $\gamma \in \mathcal{J}(R, S)$ with $\gamma \leq \beta$. From this, we know that $(R, S) \in \tilde{d}_\alpha$. Conversely, if $(R, S) \in \tilde{d}_\alpha$, then it is clear that $\tilde{d}(R, S) \leq \alpha_0$. □
We end this section with the following result concerning the case when \( E \) has a totally ordered base. Note that part (a) of this result comes from Lemma 4 as well as parts (a) and (c) of Theorem 5. Part (b) is a corollary of Proposition 14 and Theorem 5(c).

**Corollary 12.** Let \( E \) be a coarse structure on a set \( X \).

(a) \( E \) has a totally ordered base \( B \) if and only if there is a meet-complete totally ordered set \( I \) and a coarse \( \beta \)-metric \( d \) with \( E = E_d \). In this case, \( (I, d) \) can be chosen to be \( (E, d^E) \) (see Theorem 5(c)).

(b) Suppose that \( E \) admits a totally ordered base \( B \). Then \( d^B \) is a coarse \( \beta \)-metric on \( \mathcal{P}_0(X) \), and the coarse structure induced by \( d^B \) is precisely the Hausdorff coarse structure associated with \( E \).

### 3. Uniform \( \beta \)-metric

In this section, we consider the uniform structure that comes from some form of metric. Let us recall that a uniform structure on a set \( X \) is a subcollection \( \mathcal{U} \subseteq \mathcal{P}_0(X \times X) \) such that for every \( U, V \in \mathcal{U} \) and \( S \in \mathcal{P}_0(X \times X) \) with \( U \subseteq S \), one has \( \Delta_X \subseteq U, U \cap V, U^{-1}, S \in \mathcal{U} \) and there exists \( W \in \mathcal{U} \) with \( W \circ W \subseteq U \). A subcollection \( \mathcal{B} \subseteq \mathcal{U} \) is called a base for \( \mathcal{U} \) if for every \( U \in \mathcal{U} \), there exist \( V \in \mathcal{B} \) with \( V \subseteq U \).

**Definition 13.** (a) If \( I \) is a partially ordered set with a zero such that \( I \setminus \{0_I\} \) is non-empty and is downward directed, then we say that \( I \) is a D-index set.

(b) Suppose that \( I \) is a D-index set and \( d \) is a semi-\( \beta \)-metric. If there is a function \( \Psi : I \setminus \{0_I\} \rightarrow I \setminus \{0_I\} \) such that for any \( \beta \in I \setminus \{0_I\} \), one has \( d(x, z) \leq \beta \) whenever \( x, y, z \in X \) satisfying \( d(x, y) \leq \Psi(\beta) \) and \( d(y, z) \leq \Psi(\beta) \); i.e.,

\[
D_{\Psi(\beta)} \circ D_{\Psi(\beta)} \subseteq D_\beta,
\]

then \( d \) is called a pseudo uniform-\( \beta \)-metric.

(c) A uniform \( \beta \)-metric is a pseudo uniform-\( \beta \)-metric \( d \) satisfying \( \bigcap_{\alpha \in I \setminus \{0_I\}} D_\alpha = \Delta_X \).

**Remark 14.** If \( I \setminus \{0_I\} \) does not have a smallest element, then \( \inf I \setminus \{0_I\} \) exists and equals \( 0_I \). In this case, a pseudo uniform \( \beta \)-metric is a uniform \( \beta \)-metric if and only if the relation \( d(x, y) = 0_I \) implies \( x = y \).

The following result is more or less trivial.

**Proposition 15.** Let \( X \) be a set, and \( I \) be a D-index set. Suppose that \( d \) is a pseudo uniform \( \beta \)-metric on \( X \). Then \( B_d \) (see Definition 1(a)) is a base for a uniform structure \( U_d \) on \( X \). If, in addition, \( d \) is a uniform \( \beta \)-metric, then the topology induced by \( U_d \) is Hausdorff.

We say that a uniform structure \( \mathcal{U} \) is trivial if it is a principal filter (i.e., there exists \( U_0 \in \mathcal{U} \) with \( \mathcal{U} = \{U \subseteq X \times X : U_0 \subseteq U\} \); otherwise, \( \mathcal{U} \) is said to be non-trivial. It is clear that \( \mathcal{U} \) is non-trivial if and only if

\[
0_{\mathcal{U}} := \bigcap \mathcal{U} \notin \mathcal{U}.
\]

If \( \mathcal{U} \) is a trivial uniform structure and we define \( d : X \times X \rightarrow \mathbb{R}^+ \) by

\[
d(x, y) := \begin{cases} 
0 & \text{when } (x, y) \in 0_{\mathcal{U}} \\
1 & \text{otherwise},
\end{cases}
\]

then \( \mathcal{U} = U_d \) (because \( \bigcap \mathcal{U} = D_{1/2} \)).

**Lemma 16.** Let \( \mathcal{U} \) be a non-trivial uniform structure. Suppose that \( A \) is a base for \( \mathcal{U} \) satisfying:

\[
\bigcap \{A \in \mathcal{A} : (x, y) \in A\} \in \mathcal{U}, \quad \text{for every } (x, y) \in X \times X \setminus 0_{\mathcal{U}}.
\]

If \( \mathcal{B} := \mathcal{A} \setminus \{0_{\mathcal{U}}\} \) (see Lemma 4), then \( \mathcal{B} \) is a base for \( \mathcal{U} \).
Proof. Clearly, the collection \( \mathcal{A}_s \) as in Lemma 3 is a base for \( \mathcal{U} \) and we have \( \mathcal{A}_s \subseteq \mathcal{B} \) (as \( 0_\mathcal{U} \notin \mathcal{A}_s \) by the assumption of non-triviality). Therefore, it remains to show that \( \mathcal{B} \subseteq \mathcal{U} \).

Pick any \( (x, y) \in X \times X \setminus 0_\mathcal{U} \). As \( (y, x) \notin 0_\mathcal{U} \), one has

\[
\{B \in \mathcal{A}_s : (x, y) \in B\} = \{A \cap A^{-1} : A \in \mathcal{A}; (x, y) \in A\} \cap \{A \cap A^{-1} : A \in \mathcal{A}; (y, x) \in A\}.
\]

This shows that \( \{B \in \mathcal{A}_s : (x, y) \in B\} \subseteq \mathcal{U} \).

Suppose now that \( D \in \mathcal{B} \). There exists \( \mathcal{C} \subseteq \mathcal{A}_s \) with \( D = \bigcap \mathcal{C} \). As \( 0_\mathcal{U} \nsubseteq D \), there exists \( (x, y) \in \bigcap \mathcal{C} \setminus 0_\mathcal{U} \). It then follows from \( \mathcal{C} \subseteq \{B \in \mathcal{A}_s : (x, y) \in B\} \) that

\[
\bigcap\{B \in \mathcal{A}_s : (x, y) \in B\} \subseteq D
\]

and the above gives \( D \in \mathcal{U} \).

\[\square\]

**Theorem 17.** Let \( \mathcal{U} \) be a non-trivial uniform structure on a set \( X \).

(a) There exist a partially ordered set \( \mathcal{I} \) with a zero and a map \( d : X \times X \to \mathcal{I}_\infty \) such that \( \mathcal{B}_d \) (see Definition 4(a)) becomes a base for \( \mathcal{U} \) if and only if \( \mathcal{U} \) admits a base \( \mathcal{B} \) such that \( \mathcal{B} \cup \{0_\mathcal{U}\} \) is closed under arbitrary intersections. In this case, one can choose \( \mathcal{I} \) to be the meet-complete \( \mathcal{D} \)-index set \( \mathcal{J}^\mathcal{B} := \mathcal{B}\cup\{0_\mathcal{U}\} \) (see Lemma 4) and \( \mathcal{A}_d \) to be a pseudo uniform \( \mathcal{D} \)-metric.

(b) If \( \mathcal{U} \) admits a base \( \mathcal{B} \) with \( \mathcal{B} \cup \{0_\mathcal{U}\} \) being closed under arbitrary intersections and the topology induced by \( \mathcal{U} \) is Hausdorff, then one can find a uniform \( \mathcal{D} \)-metric \( d \) with \( \mathcal{U} = \mathcal{U}_d \).

**Proof.** (a) Suppose that such a map \( d \) exists. For any \( (x, y) \in X \times X \setminus 0_\mathcal{U} \), as \( 0_\mathcal{U} = \bigcap \mathcal{B}_d \), we have

\[
d(x, y) \neq 0_\mathcal{U} \quad \text{and} \quad D_{d(x, y)} \subseteq \bigcap\{B \in \mathcal{B}_d : (x, y) \in B\}.
\]

Thus, one can apply Lemma 14 to conclude that \( \mathcal{B} := \overline{\mathcal{B}_d \setminus \{0_\mathcal{U}\}} \) is a base for \( \mathcal{U} \). Moreover, we know from Lemma 4 that \( \mathcal{B} \cup \{0_\mathcal{U}\} = \overline{\mathcal{B}_d} \) is closed under arbitrary intersections.

Conversely, suppose that such a base \( \mathcal{B} \) exists. Then \( \mathcal{B}_s \) is a base of \( \mathcal{U} \) such that \( \mathcal{B}_s \cup \{0_\mathcal{U}\} \) is closed under arbitrary intersections. As \( \mathcal{U} \) is non-trivial, \( \mathcal{J}^\mathcal{B} \setminus \{0_\mathcal{U}\} = \mathcal{B}_s \) is downward directed. Define

\[
d_{\mathcal{B}}(x, y) := \bigcap\{B \in \mathcal{B}_s : (x, y) \in B\} \in \mathcal{J}^\mathcal{B}_\infty \quad (x, y \in X)
\]

(we again use the convention that \( \bigcap \emptyset \) := \( \infty \)). It is clear that \( d_{\mathcal{B}} \) is a semi-\( \mathcal{D} \)-metric (observe that \( \bigcap \mathcal{B}_s = 0_\mathcal{U} \)). Moreover, when \( S \in \mathcal{J}^\mathcal{B} \) and \( x, y \in X \), one has \( d_{\mathcal{B}}(x, y) \leq S \) if and only if \( (x, y) \in S \). Thus,

\[
\{(x, y) \in X \times X : d_{\mathcal{B}}(x, y) \leq S\} = S \quad (S \in \mathcal{J}^\mathcal{B}).
\]

(3.1)

Consider any \( S \in \mathcal{B}_s \subseteq \mathcal{U} \). Pick an arbitrary \( B \in \mathcal{B}_s \) with \( B \circ B \subseteq S \). If \( x, y, z \in X \) satisfying \( d_{\mathcal{B}}(x, y) \leq B \) and \( d_{\mathcal{B}}(y, z) \leq B \), then \( (x, z) \in S \), or equivalently, \( d_{\mathcal{B}}(x, y) \leq S \). These show that \( d_{\mathcal{B}} \) is a pseudo uniform \( \mathcal{D} \)-metric. Moreover, (3.1) tells us that \( \mathcal{U} = \mathcal{U}_d \).

(b) It follows from (3.1) that

\[
0_\mathcal{U} = \bigcap_{B \in \mathcal{B}_s}\{(x, y) \in X \times X : d_{\mathcal{B}}(x, y) \leq B\}.
\]

From this, we see that \( d_{\mathcal{B}} \) is a uniform \( \mathcal{D} \)-metric if and only if \( 0_\mathcal{U} = \Delta_X \), or equivalently, the topology induced by \( \mathcal{U} \) is Hausdorff.

\[\square\]

It follows from the proof of part (a) above that we have one more equivalent condition of \( \mathcal{U} \) being defined by a pseudo uniform \( \mathcal{D} \)-metric: \( \mathcal{U} \) admits a base \( \mathcal{A} \) satisfying the requirement in Lemma 16.

**Corollary 18.** Let \( \mathcal{U} \) be a non-trivial uniform structure on a set \( X \). There exist a totally ordered set \( \mathcal{I} \) with a zero and a pseudo uniform \( \mathcal{D} \)-metric \( d \) on \( X \) with \( \mathcal{U} = \mathcal{U}_d \) if and only if there is a totally ordered base \( \mathcal{A} \) of \( \mathcal{U} \). In the case, we can take \( \mathcal{I} = \overline{\mathcal{A}} \).
Proof. Suppose that there is such a pseudo uniform \(3\)-metric \(d\). Then, obviously, \(\{D_\alpha : \alpha \in \mathcal{J} \setminus \{0\}\}\) is a totally ordered base for \(U\). Conversely, suppose that one can find a totally ordered base \(\mathcal{A}\) for \(U\). Let \((x, y) \in X \times X \setminus 0_U\). As \(0_U = \bigcap \mathcal{A}\), there is \(A_0 \in \mathcal{A}\) with \((x, y) \notin A_0\). If \(B \in \mathcal{A}\) contains \((x, y)\), then \(A_0 \subseteq B\) (as \(\mathcal{A}\) is totally ordered), and \(\mathcal{A}\) satisfies the hypothesis of Lemma \([10]\). Hence, \(\mathcal{B} := \mathcal{A} \setminus \{0_U\}\) is a base for \(U\). Furthermore, Lemma \([4]\) tells us that \(\mathcal{A}\) is totally ordered and is closed under arbitrary intersections. The conclusion now follows from Theorem \([17](a)\). \(\square\)

4. An example

Before we give the example concerning valuation rings, we first consider a connection between coarse metrics and pseudo uniform metrics. We recall that a subset \(S\) of a partially ordered set \(\mathcal{J}\) is downw ard cofinal if for every \(\alpha \in \mathcal{J}\), there exists \(\beta \in S\) such that \(\beta \leq \alpha\).

The following result is more or less obvious.

Proposition 19. Let \(J\) be upward directed set which is also a \(D\)-index set. Suppose that \(d\) is a coarse \(3\)-metric on \(X\). If there exists \(\Phi : J \to J\) satisfying the requirement in Definition \([3](a)\) such that \(\Phi(J \setminus \{0\})\) is a downward cofinal subset of \(J \setminus \{0\}\), then \(d\) is also a pseudo uniform \(3\)-metric.

Consequently, if \(J\) is a upward directed set which is also a \(D\)-index set, and \(d\) is a coarse \(3\)-metric on \(X\) such that for every \(\alpha \in J\) and \(x, y, z \in X\), one has \(d(x, z) \leq \alpha\) if \(d(x, y) \leq \alpha\) and \(d(y, z) \leq \alpha\), then \(d\) is also a pseudo uniform \(3\)-metric. In this case, we called \(d\) an pseudo ultra \(3\)-metric.

The following is an example of a pseudo ultra \(3\)-metric with \(J\) being a totally ordered set.

Example 20. Suppose that \(R\) is a (unital) ring and \(\Gamma\) is a totally ordered abelian group. Let \(\Gamma^0\) be the ordered semi-group obtained by adjoining to \(\Gamma\) a new element \(\omega\), such that \(\omega\) is greater than all elements in \(\Gamma\) and that \(\alpha +\beta = \omega\) when either \(\alpha = \omega\) or \(\beta = \omega\). As in \([2]\) Definition VI.3.1, a map \(\nu : R \to \Gamma^0\) is called a valuation if for any \(x, y \in R\), one has

\[
\begin{align*}
(V1) & \nu(xy) = \nu(x) + \nu(y); \\
(V2) & \nu(x + y) \geq \min\{\nu(x), \nu(y)\}; \\
(V3) & \nu(1) = 0; \\
(V4) & \nu(0) = \omega.
\end{align*}
\]

Under the reverse ordering \(\leq_{op}\), the set \(\Gamma^0\) is a upward directed set with a zero (namely, \(\omega\)) and is also a \(D\)-index set (because it is totally ordered). Let us define \(d_\nu : R \times R \to \Gamma^0\) by

\[
d_\nu(x, y) := \nu(x - y) \quad (x, y \in R).
\]

Obviously, Condition \((V4)\) implies Condition \((D1)\) in Definition \([1](b)\). On the other hand, Conditions \((V1)\) and \((V3)\) implies that \(\nu(-x) = \nu(x)\), and this verifies Condition \((D2)\). Using Condition \((V2)\), we know that for every \(x, y, z \in R\), we have

\[
d_\nu(x, z) = \nu(x - z) \leq_{op} \max\{d_\nu(x, y), d_\nu(y, z)\}.
\]

Consequently, \(d_\nu\) is a pseudo ultra \(\Gamma^0\)-metric.

Suppose, furthermore, that \(R\) is a division ring. Then for every \(x \in R \setminus \{0\}\), we learn from Condition \((V1)\) that

\[
0 = \nu(x^{-1}) = \nu(x) + \nu(x^{-1}),
\]

which implies that \(\nu(x) \neq \omega\). Thus, in this case, \(\{x, y \in R \times R : d_\nu(x, y) = \omega\}\) = \(\Delta_R\). Furthermore, since a totally ordered group can never has a greatest element, we know that \((\Gamma, \leq_{op})\) can never has a smallest element. Therefore, if \(R\) is a division ring, then \(d_\nu\) is a uniform \(\Gamma^0\)-metric (see Remark \([14]\)).

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REFERENCES

[1] F. Baudier, G. Lancien and Th. Schlumprecht, The coarse geometry of Tsirelson’s space and applications, J. Amer. Math. Soc. 31 (2018), 699-717.
[2] N. Bourbaki, Commutative algebra, Springer-Verlag, Berlin (1989).
[3] X. Chen, Q. Wang and G. Yu, Guoliang, The coarse Novikov conjecture and Banach spaces with Property (H), J. Funct. Anal. 268 (2015), 2754-2786.
[4] D. Dikranjan and N. Zava, Some categorical aspects of coarse spaces and balleans, Topology Appl. 225 (2017), 164-194.
[5] J.R. Isbell, Uniform spaces, Math. Surveys 12, Amer. Math. Soc. (1964).
[6] N. Higson, Nigel, E.K. Pedersen and J. Roe, C*-algebras and controlled topology, K-Theory 11 (1997), 209-239.
[7] N. Higson and J. Roe, the coarse Baum-Connes conjecture, Novikov conjectures, index theorems and rigidity, Vol. 2 (Oberwolfach, 1993), London Math. Soc. Lecture Note Ser. 227, Cambridge Univ. Press (1995), 227-254.
[8] N. Higson, J. Roe and G. Yu, A coarse Mayer-Vietoris principle, Math. Proc. Cambridge Philos. Soc. 114 (1993), 85-97.
[9] K. Mine and A. Yamashita, Metric compactifications and coarse structures, Canad. J. Math. 67 (2015), 1091-1108.
[10] A. Nicas and D. Rosenthal, Coarse structures on groups, Topology Appl. 159 (2012), 3215-3228.
[11] W. Page, Topological Uniform Structures, Dover Publ., Inc., New York (1988).
[12] J. Roe, Lectures on Coarse Geometry, Univ. Lect. Series 31, Amer. Math. Soc. (2003).
[13] N. Wright, C*-coarse geometry and scalar curvature, J. Funct. Anal. 197 (2003), 469-488.
[14] N. Wright, Simultaneous metrizability of coarse spaces, Proc. Amer. Math. Soc. 139 (2011), 3271-3278.
[15] T. Yamauchi, Straight finite decomposition complexity implies property A for coarse spaces, Topology Appl. 231 (2017), 329-336.
[16] S. Zhang, Coarse quotient mappings between metric spaces, Israel J. Math. 207 (2015), 961-979.

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