OPTIMAL CONTROL OF TWO DIMENSIONAL THIRD GRADE FLUIDS

YASSINE TAHRAOUI AND FERNANDA CIPRIANO

ABSTRACT. The aim of this work is to study the optimal control problems of flows governed by the incompressible third grade fluid equations with Navier-slip boundary conditions. After recalling a result on the well-posedness of the state equations, we study the existence and the uniqueness of solution to the linearized state and adjoint equations. Furthermore, we present a stability result for the state, and show that the solution of the linearized equation coincides with the Gâteaux derivative of the control-to-state mapping. Next, we prove the existence of an optimal solution and establish the first order optimality conditions. Finally, an uniqueness result of the coupled system constituted by the state equation, the adjoint equation and the first order optimality condition is established.

Keywords: Non-Newtonian fluid, Third grade fluid, Navier-slip boundary conditions, Optimal control, Necessary optimality condition.

MSC: 35Q35, 49K20, 76A05, 76D55

1. Introduction

In this work, we are concerned with the optimal control of the velocity field $y$ of a non-Newtonian fluid filling a two-dimensional bounded domain with a smooth boundary. More precisely, we consider a tracking problem and the aim is to minimize the following cost functional

$$J(U,y) = \frac{1}{2} \int_0^T \|y - y_d\|^2 dt + \frac{\lambda}{2} \int_0^T \|U\|^2 dt,$$

where $y_d \in (L^2(D \times (0,T)))^2$ corresponds to a desired target velocity field, $\lambda \geq 0$ sets the intensity of the cost, the control acts through the external force $U$, and the velocity field $y$ is
constrained to satisfy the incompressible third grade fluid equation

\[
\begin{align*}
\partial_t(v(y)) - \nu \Delta y + (y \cdot \nabla) v(y) + \sum_{j=1}^{2} v(y)^j \nabla y^j - (\alpha_1 + \alpha_2) \text{div}(A(y)^2) - \beta \text{div}[\text{tr}(A(y)A(y)^T)A(y)] \\
= -\nabla P + U, \quad v(y) := y - \alpha_1 \Delta y, \quad A(y) := \nabla y + \nabla y^T,
\end{align*}
\]

(1.1)

where the constant \( \nu \) represents the fluid viscosity, \( \alpha_1, \alpha_2, \beta \) are the material moduli, and \( P \) denotes the pressure. The equation will be supplemented with a divergence free initial condition, and a homogeneous Navier-slip boundary condition which allows the slippage of the fluid against the boundary wall (see Section 2 for more details).

Most studies on fluid dynamics have been devoted to Newtonian fluids, which are characterized by the classical Newton’s law of viscosity, establishing a linear relation between the shear stress and the strain rate. However, there exist many real, industrial, or physiological fluids with nonlinear viscoelastic behavior that does not obey Newton’s law of viscosity, and consequently cannot be described by the classical viscous Newtonian fluid model. These fluids include natural biological fluids such as blood, geological flows and others, and arise in polymer processing, coating, colloidal suspensions and emulsions, ink-jet prints, etc. (see e.g [22, 23, 31]). Therefore, it is necessary to consider more general fluid models.

Recently, the class of non-Newtonian fluids of differential type has received a special attention, since it could be related to the viscous Camassa and Holm equation, shallow water models, geodesic motion on the volume-preserving diffeomorphism group for a metric containing the \( H^1 \)-norm of the fluid velocity (see [10, 20, 27]) and it found to be useful in turbulence theory, see [16]. In order to describe the evolution of this special type of fluids, we consider the velocity field \( y \) of the fluid, and introduce the Rivlin-Ericksen kinematic tensors in [33] \( A_n, n \geq 1 \), defined by

\[
\begin{align*}
A_1(y) &= \nabla y + \nabla y^T, \\
A_n(y) &= \frac{d}{dt} A_{n-1}(y) + A_{n-1}(y)(\nabla y) + (\nabla y)^T A_{n-1}(y), \quad n = 2, 3, \ldots
\end{align*}
\]

The constitutive law of fluids of grade \( n \) reads

\[ T = -p I + F(A_1, \ldots, A_n), \]

where \( F \) is an isotropic polynomial function of degree \( n \), subject to the usual requirement of material frame indifference. The constitutive law of third grade fluid is given by the following equation

\[ T = -p I + \nu A_1 + \alpha_1 A_2 + \alpha_2 A_3^2 + \beta_1 A_3 + \beta_2(A_1 A_2 + A_2 A_1) + \beta_3 \text{tr}(A_1^2) A_1, \]

where \( T \) is the shear stress tensor and \((\alpha_i)_{1,2}, (\beta_i)_{1,2,3}\) are material moduli. The momentum equations are given by

\[ \frac{Dy}{Dt} = \frac{dy}{dt} + y \cdot \nabla y = \text{div}(T). \]

If \( \beta_i = 0, i = 1, 2, 3 \), the constitutive equations correspond to a second grade fluid. It has been shown that the Clausius-Duhem inequality and the assumption that the Helmholtz free energy is minimal at equilibrium requires the material moduli to satisfy

\[ \nu \geq 0, \quad \alpha_1 + \alpha_2 = 0, \quad \alpha_1 \geq 0. \]

(1.2)

Although second grade fluids are mathematically more treatable, dealing with several non-Newtonian fluids, the rheologists have not confirmed these restrictions (1.2), thus give the conclusion that the fluids that have been tested are not fluids of second grade but are fluids that are characterized by a different constitutive structure, we refer to [23] and references therein for more details. Moreover, the second grade fluid model does not capture important rheological
properties as for instance the shear thinning and shear thickening effects, so there is a real need to study the more complex third grade fluid model. Following [23], in order to allow the motion of the fluid to be compatible with thermodynamic, it should be imposed that

\[ \nu \geq 0, \quad \alpha_1 \geq 0, \quad |\alpha_1 + \alpha_2| \leq \sqrt{24\nu\beta}, \quad \beta_1 = \beta_2 = 0, \beta_3 = \beta \geq 0. \]

From a practical point of view, recently special attention has been devoted to the study of non-Newtonian viscoelastic fluids of differential type. It is worth to mention that several simulations studies have been performed by using the third grade fluid models, in order to understand and explain the characteristics of several nanofluids (see [25, 30, 32] and references therein). We recall that nanofluids are engineered colloidal suspensions of nanoparticles (typically made of metals, oxides, carbides, or carbon nanotubes) in a base fluid as water, ethylene glycol and oil, which exhibit enhanced thermal conductivity compared to the base fluid, which turns out to be of great potential to be used in technology, including heat transfer, microelectronics, fuel cells, pharmaceutical processes, hybrid-powered engines, engine cooling/vehicle thermal management, etc. Therefore the mathematical analysis of third grade fluids equations should be relevant to predict and control the behavior of these fluids, in order to design optimal flows that can be successfully used and applied in the industry. From mathematical point of view, fluids of grade 3 constitute an hierarchy of fluids with increasing complexity and more nonlinear terms, which is more complex and require more involved analysis.

The study the equation (1.1) requires a boundary condition, and an initial condition in a suitable functional space. Besides the most studies on fluid dynamic equations consider the Dirichlet boundary condition, which assumes that the particles adjacent to the boundary surface have the same velocity as the boundary, there are physical reasons to consider slip boundary conditions. Namely, practical studies (see e.g [31, 38]) show that viscoelastic fluids slip against the boundary, and on the other hand, mathematical studies turn out that the Navier boundary conditions are compatible with the vanishing viscosity transition (see [12, 13, 19, 29]). We recall that for appropriate nondimensionalizations, the Reynolds number \( Re \) is equal to \( 1/\nu \), then the vanishing viscosity corresponds to the transition to turbulent regime, which is associated with high values of \( Re \). In this paper we consider a homogeneous Navier-slip boundary condition. Let us mention that the third grade fluid equation with the Dirichlet boundary conditions was studied in [4, 35], where the authors proved the existence and the uniqueness of local solutions for initial conditions in \( H^3 \) or global in time solution for small initial data when compared with the viscosity (see also [7]). Later on [8, 9], the authors considered the equation with a homogeneous Navier-slip boundary conditions and established the well-posedness of a global solution for initial conditions in \( H^2 \), without any restriction on the size of the data. Recently, the authors in [3, 17] extended the later deterministic result to the stochastic models. It is worth recalling that the question of uniqueness in 3D is always an open problem.

The control problems of Newtonian fluids (fluids of grade 1), where the flows are described by Navier-Stokes equations have been extensively studied in the literature. In general, the issue was the control of the turbulence inside a flow or tracking the velocity of the flows. Without exhaustiveness, let us refer to [1, 21, 26] and the references therein. Directing a velocity field to a desired velocity field over time has a wide range of applications in engineering and science such as combustion, chemical reacting flows and design problems \( \cdots \) etc. (see e.g. [24]). There is a large literature on tracking control problems for Newtonian flows. Let us mention [24], where the authors derived an optimality system for the optimal solutions for an optimal control problem of tracking the velocity for Navier-Stokes flows in bounded two-dimensional domains with bounded distributed controls. Then, a second-order sufficient optimality condition were established in [36]. In [14], the authors considered a boundary optimal control for two dimensional Navier-Stokes equations, where the control acts on the boundary through an injection-suction
device. Recently, the authors in [11] studied an optimal control problem for a two dimensional Navier-Stokes equations with measure valued controls.

On the other hand, despite that there exist many real industrial or physiological fluids cannot be described by the classical linearly viscous Newtonian model, the optimal control of non-Newtonian viscoelastic fluids have been less considered and rarely investigated. To the best of the author’s knowledge, the first result in the theory of optimal control of viscoelastic (non-Newtonian) fluids have been achieved in [28]. The authors studied an optimal control of viscoelastic fluid flow in a 4 to 1 contracting channel, where the control mechanism is based on heating or cooling the fluid along a portion of the boundary of the flow domain. They obtained an optimality system, derived by the use of the Lagrange multipliers with two different cost functionals: one of tracking type, with the viscoelastic flow being tracked to the flow corresponding to the viscous flow, the other penalizing negative contributions of the velocity component in direction of the span of the channel. Some numerical simulations was considered as well. In [37], an optimal control problem for the evolutionary flow for incompressible non-Newtonian fluids in a two-dimensional bounded domain have been studied. The authors proved the existence of optimal controls and established first-order necessary and second-order sufficient optimality conditions, where the cost functional was of tracking type. In [5], the author considered an evolutionary flow of incompressible quasi-Newtonian shear-thickening fluids in two and three dimensional setting. The Newtonian constitutive equation incorporating a shear-rate-dependent viscosity, where the viscosity increases with increasing shear rate such as Carreau model (see [5, (1.1)]). He studied the control of the system through a distributed mechanical force leading the velocity to a given target field and established a necessary optimality conditions.

It is worth mentioning that the previous works considered non-Newtonian fluids but not of differential type. To the best of our knowledge, the control problem for the second grade fluids (differential type) has been addressed for the first time in [6], where the authors proved the existence of an optimal control and deduced the first order optimality conditions, where the cost functional was of tracking type. Recently, the authors in [2] established an uniqueness result for the complete first order optimality system, by assuming enough intensity of the cost. The control problem for stochastic second grade fluid models have been studied in [15, 18]. Since there are many applications contain control mechanisms that one would like to adjust in an optimal way to achieve a given objective as well as possible, this work corresponds to the next step to control complex differential fluids. As far as we know, the optimal control problem for third grade fluids is being addressed here for the first time.

The article is organized as follows: in Section 2, we state the third grade fluid model and define the appropriate functional spaces. Then, we collect some estimates for the state already available in the literature and that are convenient for our analysis. Next, we formulate the control problem and establish the main results of the article. Section 3 is devoted to show the existence and the uniqueness of the solution to the linearized state equation. In Section 4, we prove a stability result for the state equation, which will be a key ingredient in Section 5 to study the differentiability of the control-to-state mapping. In Section 6, we write the adjoint equations and prove the existence and uniqueness of the solution. Finally, in Section 7 we establish a duality relation between the solution of the linearized equation and the adjoint state. Next, we prove the existence of the solution to the control problem, and we deduce the first order optimality condition. Section 8, which deals with the quadratic Lagrangian, is devoted to the proof of the uniqueness of the solution to the coupled system for a large cost intensity.

2. Formulation of the control problem and main results

In this section, we present some results known in the literature that will be convenient for further analysis. Next, we formulate the control problem and establish the main results.
2.1. The state equation. The goal of this work is to study the optimal control of a non-Newtonian third grade fluid, where the control is introduced via the external forces. The fluid fills a bounded and simply connected domain $D \subset \mathbb{R}^2$ with regular (smooth) boundary $\partial D$, and it is governed by the following equations

$$\begin{aligned}
\frac{\partial}{\partial t}(v(y)) - \nu \Delta v + (y \cdot \nabla) v + \sum_{j} v_j \nabla y_j - (\alpha_1 + \alpha_2) \text{div}(A^2) - \beta \text{div}(|A|^2 A) \\
= -\nabla P + U, \quad \text{div}(y) = 0 \quad \text{in } D \times (0, T),
\end{aligned}$$

$$\begin{aligned}
y \cdot \eta = 0, \quad [\eta \cdot D(y)] \cdot \tau = 0 \quad \text{on } \partial D \times (0, T),
\end{aligned}$$

$$y(x, 0) = y_0(x) \quad \text{in } D,$$

where $y := (y_1, y_2)$ denotes the fluid velocity field, $v := v(y) := y - \alpha_1 \Delta y$ and $A = A(y) = \nabla y + \nabla y^T = 2 \mathcal{D}(y)$. $P$ denotes the pressure and $U = (U_1, U_2)$ denotes the external force. The pair $(\eta, \tau)$ stands for the external normal vector and the unitary tangent vector to the boundary $\partial D$ with positive orientation. The constants $\alpha_1, \alpha_2$ and $\beta$ are the material moduli and satisfy the conditions:

$$\nu \geq 0, \quad \alpha_1 \geq 0, \quad |\alpha_1 + \alpha_2| \leq \sqrt{24 \nu \beta}, \quad \beta \geq 0. \quad (2.2)$$

It is worth recalling that $(2.2)$ allows the motion of the fluid to be compatible with thermodynamic laws.

2.2. Functional spaces and notations. For a functional space $E$ and a positive integer $k$, we define

$$(E)^k := \{(f_1, \cdots, f_k) : f_l \in E, \ l = 1, \cdots, k\}.$$ 

Let us introduce the following spaces:

$$H = \{y \in (L^2(D))^2 | \text{div}(y) = 0 \text{ in } D \text{ and } y \cdot \eta = 0 \text{ on } \partial D\},$$

$$V = \{y \in (H^1(D))^2 | \text{div}(y) = 0 \text{ in } D \text{ and } y \cdot \eta = 0 \text{ on } \partial D\},$$

$$W = \{y \in V \cap (H^2(D))^2 | (\eta \cdot D(y)) \cdot \tau = 0 \text{ on } \partial D\},$$

$$\widetilde{W} = (H^3(D))^2 \cap W.$$ 

First, we recall the Leray-Helmholtz projector $\mathbb{P} : (L^2(D))^2 \to H$, which is a linear bounded operator characterized by the following $L^2$-orthogonal decomposition

$$v = \mathbb{P} v + \nabla \varphi, \quad \forall \varphi \in H^1(D).$$

Now, let us introduce the scalar product between two matrices $A : B = tr(AB^T)$ and denote $|A|^2 := A : A$. The divergence of a matrix $A \in \mathcal{M}_{2 \times 2}(E)$ is given by $(\text{div}(A))_{i=1,2} = \sum_{j=1}^{2} \partial_j a_{ij} = 2

The space $H$ is endowed with the $L^2$-inner product $(\cdot, \cdot)$ and the associated norm $\| \cdot \|_2$. We recall that

$$(u, v) = \int_D u \cdot v dx = \int_D u_i v_i dx, \quad \forall u, v \in (L^2(D))^2,$$

$$(A, B) = \int_D A : B dx, \quad \forall A, B \in \mathcal{M}_{2 \times 2}(L^2(D)),$$

On the functional spaces $V$, $W$ and $\widetilde{W}$, we will consider the following inner products

$$\begin{aligned}
(u, z)_V := (v(u), z) = (u, z) + 2\alpha_1 (\mathbb{D} u, \mathbb{D} z),
\end{aligned}$$

$$\begin{aligned}
(u, z)_W := (u, z) + (\mathbb{P} v(u), \mathbb{P} v(z)),
\end{aligned}$$

$$\begin{aligned}
(u, z)_{\widetilde{W}} := (u, z)_V + (\text{curl} u, \text{curl} z),
\end{aligned}$$

and denote by $\| \cdot \|_V$, $\| \cdot \|_W$ and $\| \cdot \|_{\widetilde{W}}$ the corresponding norms.
For the sake of simplicity, we do not distinguish between scalar, vector or matrix-valued notations when it is clear from the context. In particular, \( \| \cdot \|_E \) should be understood as following

- \( \| f \|_E^2 = \| f_1 \|_E^2 + \| f_2 \|_E^2 \) for any \( f = (f_1, f_2) \in (E)^2 \).
- \( \| f \|_E^2 = \sum_{i,j=1}^2 \| f_{ij} \|_E^2 \) for any \( f \in \mathcal{M}_{2 \times 2}(E) \).

Throughout the article, we will denote by \( C, C_i, i \in \mathbb{N}, \) generic constants, which may varies from line to line.

Now, let us recall some results about the solution of (2.1) based on [9].

**Theorem 2.1.** [9, Thm. 1] Let \( y_0 \in W \) and \( U \in L^2_{\text{loc}}([0, \infty[; (L^2(D))^2) \), then there exists a global unique solution \( y \) to (2.1). Moreover, if \( y_0 \in W \) and \( U \in L^2_{\text{loc}}([0, \infty[; (H^1(D))^2) \) the solution \( y \) belongs to \( L^\infty_{\text{loc}}([0, \infty[; \tilde{W}) \).

Following [9], for \( t \geq 0 \) we have the following results.

**Lemma 2.2.** (\( H^1 \) estimates) There exist \( C, C_1, C_2, K_0 > 0 \) such that
\[
\| y(t) \|_{H^1} \leq e^{C_1 t} (\| y_0 \|_{H^1} + \| U \|_{L^2(D \times (0,t))}) := M_0(t);
\]
\[
\| A(y) \|_{L^4(D \times (0,t))} \leq \left( \frac{2}{\beta} \right) \frac{t}{\sqrt{M_0(t)}};
\]
\[
\| y \|_{L^4((0,t);W^{1,4}(D))} \leq K_0 e^{C_3 t} \left( \left( \frac{2}{\beta} \right) \frac{t}{\sqrt{M_0(t)}} + C_1 t \frac{1}{e^{C_1 t}} \right) \left( 1 + \| y_0 \|_{H^1} + \| U \|_{L^2(D \times (0,t))} \right) := M_1(t);
\]
\[
\| y \|_{L^1(0,t;L^\infty(D))} \leq C_2 \frac{t}{\beta} M_1(t) := M_2(t).
\]

**Lemma 2.3.** (\( H^2 \) estimates) There exists \( C_3 > 0 \) such that
\[
\| y(t) \|_{H^2}^2 + \min(\beta, \frac{\alpha_1 \beta}{4}) \int_0^t \| A(y(t)) \|_{H^1}^2 ds + \frac{\alpha_1 \beta}{2} \int_0^t \int_D |A(y(t))|^2 |\nabla A|^2 dx ds \leq e^{C_3 (t + M_2(t) + t M_3^2(t))} \left( \| y_0 \|_{H^2}^2 + 2 \int_0^t \| U(s) \|_{H^1}^2 ds + C_3 M_1^4(t) \right) := M_3(t).
\]

**Lemma 2.4.** (\( H^3 \) estimates) Let \( t \geq 0 \), then there exists \( \epsilon_0(t) := \epsilon_0(t, \| U \|_{L^2(0,t;H^1(D))}) > 0 \) such that
\[
\frac{1}{\epsilon_0(t)} \| y(t) \|_{H^3}^2 \leq 4 \epsilon_0(t) \left( 1 + \| y_0 \|_{H^1}^2 \right) := M_4(t).
\]

**Remark 2.1.** It is worth noting that Theorem 2.1 holds with no smallness conditions on the data, where Navier slip boundary conditions play a crucial role to prove Theorem 2.1. Indeed, (2.1)_3 ensures that \( \Delta y \) is almost tangent to the boundary, in the sense that it can be expressed in terms of derivatives of order 1 of \( y \) (see [10, Prop. 2]). Thanks to the last fact, we can perform some integrations by parts, which yields some boundary terms which can estimated in a satisfactory manner. More precisely, one can prove that the pressure term vanishes and then \( H^2 \)-estimates for \( y \) was obtained. In contrast with homogeneous Dirichlet boundary condition, where \( \Delta y \) does not enjoy such property on the boundary and then Theorem 2.1 does not holds with homogeneous Dirichlet boundary condition. Concerning third grade fluids equation on bounded domain with homogeneous Dirichlet boundary condition, one can prove only local existence and uniqueness for large data and global existence and uniqueness of solutions for small initial data in \( H^3 \) in 2D case (see e.g. [4]); the global existence of solutions for large data in this case remains an open problem. Finally, in the 2D case, the \( H^3 \) regularity is shown to be propagated by the equation by using some Sobolev embedding inequalities, which are true only in the 2D setting, we refer to [9] for more details.
Now we introduce the following modified Stokes problem (see [15, 34] for the properties of the solution)

\[
\begin{cases}
    h - \alpha_1 \Delta h + \nabla \pi = f, & \text{div}(h) = 0 \quad \text{in } D, \\
    h \cdot \eta = 0, & [\eta \cdot D(h)] \cdot \tau = 0 \quad \text{on } \partial D,
\end{cases}
\]  

(2.5)

and denotes its solution \( h \) by \( (I - \alpha_1 \mathbb{P})^{-1} f \). We also consider the trilinear form

\[
b(\phi, z, y) = (\phi \cdot \nabla z, y) = \int_D (\phi \cdot \nabla z) \cdot y dx, \quad \forall \phi, z, y \in (H^1(D))^2,
\]

which verifies \( b(y, z, \phi) = -b(y, \phi, z), \quad \forall y \in V; \forall z, \phi \in H^1(D) \).

Let \( T > 0 \), we assume that the initial data \( y_0 \) and the force \( U \) satisfy

\[
y_0 \in \tilde{W}, \quad U \in L^2(0, T; (H^1(D))^2).
\]

(H-1)

2.3. Control problem. Our main goal is to control the solution of the equation (2.1) by a distributed force \( U \). The control variables \( U \) belong to the set \( \mathcal{U}_{ad} \) of admissible controls, which is defined as a nonempty bounded closed convex subset of \( L^2(0, T; (H^1(D))^2) \). In other words

\[
\mathcal{U}_{ad} := \{ u \in L^2(0, T; (H^1(D))^2) : \| u \|_{L^2(0, T; (H^1(D))^2)} \leq K \}; \quad 0 < K < \infty.
\]

We consider the cost functional given by

\[
J(u, y) = \frac{1}{2} \int_0^T \| y - y_d \|_2^2 dt + \frac{\lambda}{2} \int_0^T \| u \|_2^2 dt,
\]

(2.6)

where \( y_d \in L^2(D \times (0, T)) \) corresponds to a desired target field and \( \lambda \geq 0 \). The control problem reads

\[
\min_{u \in \mathcal{U}_{ad}} \left\{ \frac{1}{2} \int_0^T \| y - y_d \|_2^2 dt + \frac{\lambda}{2} \int_0^T \| u \|_2^2 dt : y \text{ is the solution of (2.1) with force } u \right\}.
\]

(2.7)

Remark 2.2. We wish to draw the reader’s attention that we can consider more general class of cost functional given by

\[
J(U, y) = \int_0^T L(t, U(t), y(t)) dt,
\]

where the Lagrangian \( L : [0, T] \times (H^1(D))^2 \times \tilde{W} \rightarrow \mathbb{R}^+ \) satisfies some properties (see e.g. [15]).

Remark 2.3. (1) In this article, the Lagrangian \( L \) is given by \( L(\cdot, u, y) = \frac{1}{2} \| y - y_d \|_2^2 + \frac{\lambda}{2} \| u \|_2^2 \). Therefore, we have

\[
\int_0^T (\nabla_y L(t, U(t), y(t)), v(t)) dt = \int_0^T (v(t), y(t) - y_d(t)) dt,
\]

\[
\int_0^T (\nabla_u L(t, U(t), y(t)), v(t)) dt = \lambda \int_0^T (v(t), U(t)) dt,
\]

for any \( v \in L^2(0, T; (L^2(D))^2) \).

(2) We wish to draw the reader’s attention to the fact that (2.6) is well defined for \( u \in L^2(0, T; (L^2(D))^2) \) but our aim is to solve (2.7) and establish an optimality condition. The \( H^3 \)-regularity of the solution to the state equation (2.1) play a crucial role in the analysis of the linearized and adjoint equations (see Sect.3 and Sect.6). Moreover, it is also crucial to establish the necessary optimality condition (see Sect.5). For that, we consider \( \mathcal{U}_{ad} \) as a subset of \( L^2(0, T; (H^1(D))^2) \), which ensures the \( H^3 \)-regularity of the solution to (2.1), see Lemma 2.4.

Remark 2.4. Under (H-1), the solution \( y \) of (2.1), belongs to \( L^\infty(0, T; \tilde{W}) \).
2.4. **Main results.** Our first main result shows the existence of a solution to the control problem, and establishes the first order optimality conditions.

**Theorem 2.5.** Assume (H-1). Then the control problem \((2.7)\) admits, at least, one optimal solution

\[
(\tilde{U}, \tilde{y}) \in \mathcal{U}_{ad} \times (L^\infty(0,T;W)) \cap H^1(0,T;V),
\]

where \(\tilde{y}\) is the unique solution of \((2.1)\) with \(U = \tilde{U}\). Moreover, there exists a unique solution \(\tilde{p}\) of \((6.1)\) with \(f = \nabla yL(\cdot, \tilde{U}, \tilde{y})\), such that if \(\tilde{z}\) is the solution of \((3.2)\) for \(y = \tilde{y}\) and \(\psi = \psi - \tilde{U}\), the following duality property

\[
\int_0^T (\psi(t) - \tilde{U}(t), \tilde{p}(t))dt = \int_0^T (\nabla yL(t, \tilde{U}(t), \tilde{y}(t)), \tilde{z}(t))dt,
\]

and the following optimality condition hold

\[
\int_0^T (\psi(t) - \tilde{U}(t), \tilde{p}(t) + \nabla uL(t, \tilde{U}(t), \tilde{y}(t)))dt \geq 0. \tag{2.8}
\]

An additional step in the study of the control problem relies on the analysis of the solutions of the coupled system constituted by the state equation \((2.1)\), the adjoint equation \((6.1)\) and the optimality relation \((2.8)\). Our next result goes in this direction and establishes an uniqueness result for the solutions of the coupled system for \((2.7)\).

**Theorem 2.6.** Assume that \(\lambda > 2\tilde{C}\lambda (\Gamma + 4\kappa(\alpha_1 + \alpha_2) + 12\kappa\beta\gamma)\), where \(\gamma = \sup_{t \in [0,T]} M_4(t)\) (see \((2.4)\)) and \(\tilde{\lambda}, \tilde{C}, \kappa\) are given by \((8.1)\), \((4.3)\), \((8.8)\), respectively. Then the optimal control problem \((2.7)\) has a unique global solution.

**Remark 2.5.**

1. It is important to underline that Theorem 2.6 is achieved under a natural condition \(\lambda > 2\tilde{C}\lambda (\Gamma + 4\kappa(\alpha_1 + \alpha_2) + 12\kappa\beta\gamma)\), which gathers the size of the initial data, the parameters of the model \((\nu, \alpha_1, \alpha_2, \beta)\) and the intensity of the cost \(\lambda\). In other words, if the fluid material is sufficiently viscous and elastic and the initial condition is small enough, or instead if the intensity of the cost is big enough, the solution of the first-order optimality system is unique, and corresponds to the unique solution of the optimal control problem.

2. The standard approach in optimization problem is based on the analysis of the second order sufficient condition, which requires a coercivity of the Lagrange function combined with the first order necessary optimality condition. Generally, the second order optimality condition cannot be expected if \(\lambda = 0\) (see e.g. [36, Rmq. 3.16]). Following a similar analysis as in [2, Subsect. 3.1], one can get a second-order Gâteaux derivative of the control-to-state-mapping, by taking into account the additional terms of the third grade fluids model. Note that an analysis of the second-order Gâteaux derivative of the control-to-state-mapping will require more regularity with respect to the space variable \(x\) of the solution of linearized equation \(z\) (see e.g. [37, Sect. 4] for similar issues). Additionally, one can find, in the literature, that the coercivity of the Lagrange function leads to some condition (see e.g. [2, Th. 3.2]), which mimics \(\lambda > 2\tilde{C}\lambda (\Gamma + 4\kappa(\alpha_1 + \alpha_2) + 12\kappa\beta\gamma)\).

On the other hand, by using \((8.1)\) one can see that the above condition leads to

\[
\lambda > \tilde{C}(\lambda) (\Gamma + 4\kappa(\alpha_1 + \alpha_2) + 12\kappa\beta\gamma)\|y - y_d\|_{L^2(Q)}, \tag{2.9}
\]

\((2.9)\) means that the desired state \(y_d\) could be approximated closely if \(\lambda\) is large enough or \(\|y - y_d\|_{L^2(Q)}\) is small enough, which is similar to the standard second-order optimality condition related to the control theory of Navier-Stokes equations (see e.g. [36, Eqn. 3.30]). Finally, we consider here the uniqueness of global system, since it does not request more regularity for \(z\) and we believe that the analysis of second order optimality condition will lead to a similar condition.
The proof of Theorem 2.5 will be splitted into several steps. First, we prove the solvability of the linearized state equation. Then, we establish a stability result for the state, which is a key ingredient to show that the solution of the linearized state equation corresponds to the Gâteaux derivative of the control-to-state mapping. Next, we will write and study the well-posedness of the adjoint equation. Finally, we prove the existence of an optimal pair and deduce the first order optimality condition. The proof of Theorem 2.6 is presented in Section 8.

3. Linearized state equation

This section is devoted to the study of the linearized state equation. The existence of the solution is based on the Faedo-Galerkin’s approximation method, which relies on a special basis designed according to the structure of the equation, in order to derive the uniform estimates in $H^1$ and in $H^2$.

Let us consider $\psi : D \times [0, T] \rightarrow \mathbb{R}^2$ such that

$$\psi \in (L^2(D \times [0, T]))^2. \quad (3.1)$$

Our goal is to prove an existence and uniqueness result for the following problem

$$\begin{aligned}
\partial_t (v(z)) &- \nu \Delta z + (y \cdot \nabla) v(z) + (z \cdot \nabla) v(y) + \sum_j v(z)^j \nabla y^j \\
+ \sum_j v(y)^j \nabla z^j - (\alpha_1 + \alpha_2) \text{div} \{ A(y) A(z) + A(z) A(y) \} \\
- \beta \text{div} [ |A(y)|^2 A(z) ] - 2\beta \text{div} [ (A(z) : A(y)) A(y) ] &= \psi - \nabla \pi \quad \text{in } D \times (0, T), \\
\text{div}(z) &= 0 \quad \text{in } D \times (0, T), \\
z \cdot \eta &= 0 \quad [\eta \cdot \mathbb{D}(z)] \cdot \tau = 0 \quad \text{on } \partial D \times (0, T), \\
z(x, 0) &= 0 \quad \text{in } D.
\end{aligned} \quad (3.2)$$

**Definition 3.1.** A function $z \in L^\infty(0, T; W)$ with $\partial_t z \in L^2(0, T; V)$ is a solution of (3.2) if $z(0) = 0$ and for any $t \in [0, T]$, the following equality holds

$$(\partial_t v(z), \phi) + 2\nu (\mathbb{D} z, \mathbb{D} \phi) + b(y, v(z), \phi) + b(z, v(y), \phi) + b(\phi, y, v(z)) + b(\phi, z, v(y)) + (\alpha_1 + \alpha_2) (A(y) A(z) + A(z) A(y), \nabla \phi)$$

$$+ 2\beta (A(z) : A(y)) A(y), \nabla \phi) = (\psi, \phi) \quad \text{for all } \phi \in W.$$

**Remark 3.1.** For $u \in L^\infty(0, T; W)$ and $\partial_t u \in L^2(0, T; V)$, $(\partial_t v(u), \phi)$ will be understood in the following sense

$$(\partial_t v(u), \phi) = (\partial_t u, \phi)_V = (\partial_t u, \phi) + 2\alpha_1 (\partial_t \mathbb{D} u, \mathbb{D} \phi).$$

3.1. Approximation. Following the same strategy as in [15], the solution of (3.2) can be obtained as a limit of the finite dimensional Faedo-Galerkin’s approximations. In this way, let us consider an orthonormal basis $\{h_i\}_{i \in \mathbb{N}} \subset H^1(D) \cap W$ in $V$, which satisfies

$$\langle v, h_i \rangle_W = \mu_i \langle v, h_i \rangle_V, \quad \forall v \in W, \quad i \in \mathbb{N}, \quad (3.3)$$

where the sequence $\{\mu_i\}$ of the corresponding eigenvalues fulfils the properties:

$\mu_i > 0, \forall i \in \mathbb{N}$, and $\mu_i \rightarrow \infty$ as $i \rightarrow \infty$. As a consequence of (3.3), the sequence $\{\tilde{h}_i = \frac{1}{\sqrt{\mu_i}} h_i\}$ is an orthonormal basis in $W$. Let us introduce the Galerkin approximations of (3.2).

Consider $W_n = \text{span}\{h_1, \cdots, h_n\}$ and define $z_n(t) = \sum_{i=1}^n c_i(t) h_i$ for each $t \in [0, T]$. The approximated problem for (3.2) reads $z_n(0) = 0$ and

$$(\partial_t v(z_n), \phi) = \left( \psi + \nu \Delta z_n - (y \cdot \nabla) v(z_n) - (z_n \cdot \nabla) v(y) - \sum_j v(z_n)^j \nabla y^j \\
- \sum_j v(y)^j \nabla z^j + (\alpha_1 + \alpha_2) \text{div} \{ A(y) A(z_n) + A(z_n) A(y) \} \\
+ \beta \text{div} [ |A(y)|^2 A(z_n) ] + 2\beta \text{div} [ (A(z_n) : A(y)) A(y) \phi \right), \quad \text{for any } \phi \in W_n. \quad (3.4)$$
Note that (3.4) defines a system of linear ordinary differential equations, which has a unique solution

\[ z_n \in \mathcal{C}([0, T_n], W_n). \]  

(3.5)

3.2. Uniform estimates.

3.2.1. Estimate in the space \( V \) for \( z_n \). Setting \( \phi = h \) in (3.4), we have

\[
\begin{align*}
(\partial_t v(z_n), h) &= \left( \psi + \nu \Delta z_n - (y \cdot \nabla) v(z_n) - (z_n \cdot \nabla) v(y) - \sum_j v(z_n)^j \nabla y^j - \sum_j v(y)^j \nabla z_n^j \right. \\
&\quad \left. + (\alpha_1 + \alpha_2) \text{div}[A(y)A(z_n) + A(z_n)A(y)] + \beta \text{div}\left[ |A(y)|^2 A(z_n) \right] + 2\beta \text{div}[(A(z_n) : A(y)) A(y)] \right) := (f(z_n), h_i).
\end{align*}
\]

Multiplying the equality by \( c_i(t) \) and summing from \( i = 1 \) to \( n \), we deduce

\[
2(\partial_t v(z_n), z_n) = 2(\partial_t (z_n), z_n) + 4\alpha_1 (\partial_t \mathbb{D} z_n, \mathbb{D} z_n) = \frac{d}{dt} \|z_n\|_2^2 + 2\alpha_1 \|\mathbb{D} z_n\|_2^2.
\]

Let us estimate the term \((f(z_n), z_n)\).

\[
I_1 = 2\left( \psi + \nu \Delta z_n - (y \cdot \nabla) v(z_n) - (z_n \cdot \nabla) v(y) - \sum_j v(z_n)^j \nabla y^j - \sum_j v(y)^j \nabla z_n^j \right. \\
&\quad \left. + (\alpha_1 + \alpha_2) \text{div}[A(y)A(z_n) + A(z_n)A(y)] + \beta \text{div}\left[ |A(y)|^2 A(z_n) \right] \\
&\quad + 2\beta \text{div}[(A(z_n) : A(y)) A(y)] , z_n \right) = I_1^0 + I_1^1 + I_1^2 + I_1^3.
\]

Thanks to the free divergence property, we have

\[
I_1^0 = 2(\psi + \nu \Delta z_n, z_n) = 2(\psi, z_n) - 4\nu \|\mathbb{D} z_n\|_2^2.
\]

Notice that

\[
I_1^1 = -2(b(y, v(z_n), z_n) + b(z_n, y, v(z_n)) + b(z_n, v(y), z_n) + b(z_n, z_n, v(y))).
\]

Using [15, Lem. 3.5], we write

\[
I_1^1 = 2(b(y, z_n, v(z_n)) - b(z_n, y, v(z_n))) = -2(c\nabla v(z_n)) \times y, z_n), \\
|I_1^1| = |2(c\nabla v(z_n)) \times y, z_n)| \leq C_1\|y\|_{W} \|z_n\|_V^2.
\]

The Stokes theorem allows to infer that

\[
I_1^2 = 2(\alpha_1 + \alpha_2)(\text{div}[A(y)A(z_n) + A(z_n)A(y)], z_n)
\]

\[
= -2(\alpha_1 + \alpha_2) \int_D [A(y)A(z_n) + A(z_n)A(y)] : \nabla z_n dx + \int_{\partial D} (A(y)A(z_n)\eta + A(z_n)A(y)\eta) \cdot z_n dS.
\]

Since \( y \) and \( z_n \) satisfy the Navier boundary conditions (see (3.2)3), we have

\[
\int_{\partial D} (A(y)A(z_n)\eta + A(z_n)A(y)\eta) \cdot z_n dS = 0.
\]

Hence, Hölder inequality ensures

\[
|I_1^2| \leq C_2\|z_n\|_V^2\|y\|_{W^1}\leq C_2\|z_n\|_V^2\|y\|_{W}.
\]

Applying the Stokes theorem once more and using the symmetry of \( A(z_n) \), we derive

\[
I_1^3 = 2\beta(\text{div}[|A(y)|^2 A(z_n)], z_n) = -2\beta \int_D |A(y)|^2 A(z_n) : \nabla z_n dx + \int_{\partial D} |A(y)|^2 (A(z_n)\eta) \cdot z_n dS
\]
\(-\beta \int_D |A(y)|^2 |A(z_n)|^2 \, dx \leq 0,\)

where we used the Navier boundary conditions to cancel the boundary term. Concerning \(I_4^i\), we have

\[ I_4^i = 4\beta \left( \text{div} \left[ \left( A(z_n) : A(y) \right) A(y) \right], z_n \right) = 4\beta \int_D \text{div} \left[ \left( A(z_n) : A(y) \right) A(y) \right] : z_n \, dx. \]

A similar reasoning yields

\[ I_4^i = -4\beta \int_D \left[ (A(z_n) : A(y)) A(y) \right] : \nabla z_n \, dx + 4\beta \int_{\partial D} \left[ (A(z_n) : A(y)) (A(y) \eta) \right] : z_n \, dS. \]

Again, using the Navier boundary conditions, we deduce

\[ |I_4^i| = \left| 4\beta \int_D \left[ (A(z_n) : A(y)) A(y) \right] : \nabla z_n \, dx \right| \leq C_3 \|z_n\|_{H^1}^2 \|y\|_{W^{1,\infty}}^2 \leq C_3 \|z_n\|_{\bar{W}}^2. \]

Gathering the previous estimates, there exists \(C > 0\) independent of \(n\) such that

\[ |I_1^i + I_2^i + I_3^i + I_4^i| \leq C \|z_n\|_{\bar{W}}^2 \|y\|_{\bar{W}} (1 + \|y\|_{\bar{W}}). \]

For any \(t \in [0, T]\), we integrate over the time variable to obtain

\[ \left( \|z_n(t)\|_{\bar{W}}^2 - 2\alpha_1 \|D z_n(t)\|_{\bar{W}}^2 \right) + 4\nu \int_0^t \|D z_n(s)\|_{\bar{W}}^2 \, ds \leq C \int_0^t \|z_n\|_{\bar{W}}^2 \|y\|_{\bar{W}} (1 + \|y\|_{\bar{W}}) \, ds + C \int_0^t \|\psi(s)\|_{\bar{W}}^2 \, ds. \]

Thanks to Lemma 2.4, for any \(t \in [0, T]\), we have

\[
\sup_{r \in [0, t]} \|z_n(r)\|_{\bar{W}}^2 + 4\nu \int_0^t \|D z_n(s)\|_{\bar{W}}^2 \, ds \leq C \sup_{r \in [0, t]} [M_2(r) + M_4^2(r)] \int_0^t \|z_n(s)\|_{\bar{W}}^2 \, ds + C \int_0^t \|\psi(s)\|_{\bar{W}}^2 \, ds. \quad (3.8)
\]

3.2.2. Estimate in the space \(W\) for \(z_n\). Let \(\tilde{f}_n\) be the solutions of (2.5) for \(f = f(z_n)\). Then

\[ (\tilde{f}_n, h_i)_V = (f(z_n), h_i), \quad \text{for each } i. \quad (3.9) \]

Multiplying (3.6) by \(\mu_i\) and using (3.3), we get

\[ (\partial_t z_n, h_i)_W = (\tilde{f}_n, h_i)_W, \quad (3.10) \]

Multiplying these equalities by \(c_i(t)\) and summing from \(i = 1\) to \(n\), it follows that

\[
(\partial_t z_n, z_n)_V + (\text{div}(\partial_t z_n), \text{div}(z_n)) = (\partial_t z_n, z_n)_W = (\tilde{f}_n, z_n)_V + (\text{div}(\tilde{f}_n), \text{div}(z_n)) = (f(z_n), z_n) + (f(z_n), \text{div}(z_n)).
\]

By using (3.7), the last equality reduces to

\[
\frac{d}{dt} \|\text{div}(z_n)\|_{\bar{W}}^2 = 2(\partial_t \text{div}(z_n), \text{div}(z_n)) = 2(f(z_n), \text{div}(z_n)).
\]

Let us estimate \(2(f(z_n), \text{div}(z_n))\). We write

\[
2(f(z_n), \text{div}(z_n)) = 2(\psi - \nabla \pi_n + \nu \Delta z_n, \text{div}(z_n)) + 2\beta \left( \text{div} \left[ |A(y)|^2 A(z_n) \right], \text{div}(z_n) \right) - 2(\nu \cdot \nabla v(z_n) + (z_n \cdot \nabla) v(y), \text{div}(z_n)) - 2(\nabla v(z_n) \cdot \nabla v(y) + v(y) \nabla^2 z_n, \text{div}(z_n)) + 2(\alpha_1 + \alpha_2) \left( \text{div}[A(y) A(z_n) + A(z_n) A(y)], \text{div}(z_n) \right) + 4\beta \left( \text{div} \left[ (A(z_n) : A(y)) A(y) \right], \text{div}(z_n) \right) = J_0^1 + J_1^1 + J_2^1 + J_3^1 + J_4^1.
\]

The first term verifies

\[ J_0^1 = 2(\psi - \nabla \pi_n + \nu \Delta z_n, \text{div}(z_n)) \leq 2(\psi, \text{div}(z_n)) + C \|z_n\|_{\bar{W}}^2. \]
Taking into account that \( J_1^2 = 0 \), we have
\[
J_1^2 = -2(b(y, v(z_n)) + b(z_n, v(y)) + b(P(y), v(z_n))
\]
\[
+ b(P(z_n), y, v(z_n)) + b(P(y), z_n, v(y))).
\]

Thanks to the Hölder inequality, we deduce
\[
|J_1^2| \leq C\|y\|\|v(z_n) - P(y)|\|_{H^1}\|P(y)|\|_{2} + C\|z_n\|\|v(y)|\|_{H^1}\|P(y)|\|_{2}
\]
\[
+ C\|\nabla v\|\|v(y)\|_{2}\|P(y)|\|_{2} + C\|\nabla z_n\|\|P(y)|\|_{2}\|v(y)|\|_{4}.\]

We recall that \( \tilde{W} \rightarrow W^{1,\infty}(D) \cap W^{2,4}(D) \), \( W \rightarrow W^{1,4}(D) \) \cap \( L^\infty(D) \). Hence
\[
|J_1^2| \leq K_1\|y\|_\tilde{W}\|z_n\|^2_W,
\]
where we used \( \|v(z_n) - P(y)|\|_{H^1} \leq K_1\|z_n\|_{H^2} \) (see [9, Lem. 5]).

Concerning \( J_1^3 \), we have
\[
J_1^3 = 2(\alpha_1 + \alpha_2) \int_D \text{div}[A(y)A(z_n) + A(z_n)A(y)] \cdot P(v(z_n))dx.
\]

We observe that \( \text{div}[A(y)A(z_n) + A(z_n)A(y)] \) can be expressed as the sum of terms of one of the following forms \( \mathcal{D}(y)\mathcal{D}^2(z_n) \) or \( \mathcal{D}(z_n)\mathcal{D}^2(y) \). Therefore,
\[
|\text{div}[A(y)A(z_n) + A(z_n)A(y)]| \leq C(\|\nabla y\|\|\nabla^2 z_n\| + \|\nabla z_n\|\|\nabla^2 y\|).
\]

Hence \( |J_1^3| \leq C \int_D(\|\nabla y\|\|\nabla^2 z_n\| + \|\nabla z_n\|\|\nabla^2 y\|) \cdot |P(v(z_n))|dx. \) The Hölder inequality yields
\[
|J_1^3| \leq C\|\nabla y\|_\infty\|\nabla^2 z_n\|_2\|P(v(z_n))\|_2 + C\|\nabla^2 y\|_4\|\nabla z_n\|_4\|P(v(z_n))\|_2
\]
\[
\leq K_2\|y\|_\tilde{W}\|z_n\|^2_W.
\]

On the other hand, we recall that
\[
J_1^1 + J_1^2 = 2\beta(\text{div}[A(y)^2A(z_n)], P(v(z_n))) + 4\beta(\text{div}[A(z_n) : A(y)]A(y), P(v(z_n)))
\]
\[
= 2\beta \int_D \text{div}[A(y)^2A(z_n)] \cdot P(v(z_n))dx + 4\beta \int_D \text{div}[A(z_n) : A(y)]A(y) \cdot P(v(z_n))dx,
\]
and notice that \( \text{div}[A(y)^2A(z_n)] \) and \( \text{div}[A(z_n) : A(y)]A(y) \) can be expressed as the sum of terms of one of the following forms \( \mathcal{D}(y)\mathcal{D}^2(z_n)\mathcal{D}(y) \) or \( \mathcal{D}(z_n)\mathcal{D}^2(y)\mathcal{D}(y) \). Similar arguments as in the estimation of \( J_1^2 \) give \( |J_1^3 + J_1^1| \leq K_3\|y\|_\tilde{W}\|z_n\|^2_W. \) Thus there exists \( K > 0 \) such that
\[
|J_1^3 + J_1^1 + J_1^2 + J_1^1| \leq K\|y\|_\tilde{W}\|z_n\|^2_W. \quad (3.11)
\]

Therefore, for any \( t \in [0,T] \),
\[
2\int_0^t |(f(z_n), P(v(z_n)))|ds \leq C \sup_{r \in [0,t]} |M_4(r) + M_2^2(r)| \int_0^t \|z_n\|^2_Wds + 2\int_0^t |(\psi, P(v(z_n)))|ds + C \int_0^t \|z_n\|^2_Wds.
\]

Thanks to Lemma 2.4, we obtain
\[
2\int_0^t |(f(z_n), P(v(z_n)))|ds \leq C \int_0^t \|\psi(s)\|^2_Wds + C \int_0^t \|z_n\|^2_Wds
\]
and
\[
\sup_{r \in [0,t]} \|P(v(z_n(r)))\|^2_2 \leq 2\int_0^t |(f(z_n), P(v(z_n)))|ds \leq C \int_0^t \|\psi(s)\|^2_Wds + C \int_0^t \|z_n(s)\|^2_Wds.
\]

Gronwall’s inequality ensures that there exists \( C > 0 \) such that
\[
\sup_{r \in [0,t]} \|P(v(z_n(r)))\|^2_2 \leq C(T) \int_0^t \|\psi(s)\|^2_Wds. \quad (3.12)
\]
Gathering (3.8) and the last inequality, we write
\[ \sup_{r \in [0,t]} \|\psi v(z_n(r))\|_W^2 + \sup_{r \in [0,t]} \|z_n(r)\|_T^2 + 4\nu \int_0^t \|\mathcal{D}z_n(s)\|_W^2 ds \leq C(T) \int_0^t \|\psi(s)\|_W^2 ds. \]

We conclude that there exists \( M_1 > 0 \) such that
\[ \sup_{r \in [0,t]} \|z_n(r)\|_W^2 \leq M_1(T) \int_0^T \|\psi(s)\|_W^2 ds \quad \text{for any } t \in [0,T]. \]  

(3.13)

3.2.3. Estimate in the space \( V \) for \( \partial_t z_n \). Multiplying (3.6) by \( \partial_t c_i(t) \) and summing from \( i = 1 \) to \( n \), we get
\[
\begin{cases}
(\partial_t v(z_n), \partial_t z_n) &= \left( \psi + \nu \Delta z_n - (y \cdot \nabla) v(z_n) - (z_n \cdot \nabla) v(y) - \sum_j v(z_n)^j \nabla y^j \right. \\
&\quad - \sum_j v(y)^j \nabla z_n^j + (\alpha_1 + \alpha_2) \text{div}[A(y)A(z_n)] + A(z_n)A(y)] \\
&\quad + \beta \text{div}[|A(y)|^2 A(z_n)] + 2\beta \text{div}[A(z_n) : A(y)] A(y)], (\partial_t z_n) \\
&=: (f(z_n), \partial_t z_n).
\end{cases}
\]

Notice that
\[
2(\partial_t v(z_n), \partial_t z_n) = 2(\partial_t z_n, \partial_t z_n) + 4\alpha_1 (\partial_t \mathcal{D}z_n, \mathcal{D} \partial_t z_n) = 2\|\partial_t z_n\|_2^2 + 4\alpha_1 \|\mathcal{D} \partial_t z_n\|_2^2.
\]

Let us estimate \( (f(z_n), \partial_t z_n) \).
\[
(f(z_n), \partial_t z_n) = \left( - (y \cdot \nabla) v(z_n) - (z_n \cdot \nabla) v(y) - \sum_j v(z_n)^j \nabla y^j - v(y)^j \nabla z_n^j, \partial_t z_n \right) \\
+ \left( (\alpha_1 + \alpha_2) \text{div}[A(y)A(z_n)] + A(z_n)A(y)] + \beta \text{div}[|A(y)|^2 A(z_n)] \right), (\partial_t z_n) \\
+ 2\beta \left( \text{div}[A(z_n) : A(y)] A(y)], (\partial_t z_n) \right) + \left( \psi + \nu \Delta z_n, \partial_t z_n \right) \\
\leq (\|\psi\|_2 + \nu\|z_n\|_W)\|\partial_t z_n\|_2 + |b(y, v(z_n), \partial_t z_n)| + |b(z_n, v(y), \partial_t z_n)| \\
+ |b(\partial_t z_n, z_n, v(y))| + C(\alpha_1 + \alpha_2) \int_D (|\nabla y||\nabla^2 z_n| + |\nabla z_n||\nabla^2 y|) \cdot |\partial_t z_n| dx \\
+ |b(\partial_t z_n, y, v(y))| + C\beta \int_D (|\nabla y|^2|\nabla^2 z_n| + |\nabla z_n||\nabla^2 y| |\nabla y|) \cdot |\partial_t z_n| dx.
\]

A similar argument to the one used to get (3.8) yields
\[
|b(y, v(z_n), \partial_t z_n)| + |b(z_n, v(y), \partial_t z_n)| + |b(\partial_t z_n, y, v(z_n))| + |b(\partial_t z_n, z_n, v(y))| \\
\leq C\|y(t)\|_{H^1}\|z_n(t)\|_{W} (\|\partial_t z_n(t)\|_2 + \|\mathcal{D} \partial_t z_n(t)\|_2).
\]

On the other hand, we have
\[
\int_D (|\nabla y||\nabla^2 z_n| + |\nabla z_n||\nabla^2 y|) \cdot |\partial_t z_n| dx \leq C\|y\|_{W^{1,\infty}}\|z_n\|_{W} \|\partial_t z_n\|_2 \\
+ C\|z_n\|_{W^{1,4}}\|y\|_{W^{2,1}} \|\partial_t z_n\|_2 \leq C\|y(t)\|_{H^1}\|z_n(t)\|_{W} \|\partial_t z_n(t)\|_2,
\]

and
\[
\int_D (|\nabla y|^2|\nabla^2 z_n| + |\nabla z_n||\nabla^2 y|) \cdot |\partial_t z_n| dx \\
\leq C\|y\|_{W^{1,\infty}}^2 \|z_n\|_{W} \|\partial_t z_n\|_2 + C\|y\|_{W^{1,\infty}} \|z_n\|_{W^{1,4}} \|y\|_{W^{2,1}} \|\partial_t z_n\|_2 \\
\leq C\|y(t)\|_{H^1}^2 \|z_n(t)\|_{W} \|\partial_t z_n(t)\|_2.
\]

Therefore
\[
\|\partial_t z_n\|_2^2 + 2\alpha_1 \|\mathcal{D} \partial_t z_n\|_2^2 \\
\leq C(\|\psi\|_2 + \nu\|z_n\|_W)\|\partial_t z_n\|_2 + C\|y(t)\|_{H^1}\|z_n(t)\|_{W} (\|\partial_t z_n(t)\|_2 + \|\mathcal{D} \partial_t z_n(t)\|_2) \\
+ C\|y(t)\|_{H^1}^2 \|z_n(t)\|_{W} \|\partial_t z_n(t)\|_2 + C\|y(t)\|_{H^1}^2 \|z_n(t)\|_{W} \|\partial_t z_n(t)\|_2.
\]
For any $\delta > 0$, the Young inequality ensures that
\[
\|\partial_t z_n\|_2^2 + 2\alpha_1 \|D\partial_t z_n\|_2^2 \\
\leq C(\|\psi\|_2^2 + 2\|\partial_t z_n\|_2^2 + 2\delta \|\partial_t z_n\|_2^2 + C\|y(t)\|_{H^4} \|z_n(t)\|_{W^4} + \delta \|D\partial_t z_n\|_2^2 \\
+ C\|y(t)\|_{H^3} \|z_n(t)\|_{W^3} + \delta \|\partial_t z_n(t)\|_2^2).
\]

An appropriate choice of $\delta$ and integration with respect to time $t$ give
\[
\int_0^T (\|\partial_t z_n\|_2^2 + \alpha_1 \|D\partial_t z_n\|_2^2)ds \\
\leq C \int_0^T \|\psi(s)\|_{2}^2 ds + C \int_0^T (1 + \|y(s)\|_{H^4} + \|y(s)\|_{H^3}^4) \|z_n(s)\|_{W^4} ds \\
\leq C \int_0^T \|\psi(s)\|_{2}^2 ds + C \sup_{r \in [0,T]} [1 + M_2^4(r) + M_1^4(r)] \int_0^T \|z_n(s)\|_{W^4} ds := C_1(T).
\]

**Remark 3.2.** Before passing to the limit in the approximate problem, let us note that (3.13) guarantees the global in time existence of $z_n$, i.e. $z_n \in C([0,T], W_n)$.

Finally, we derive the following lemma.

**Lemma 3.1.** Assume $\psi$ satisfies (3.1). Then, there exists a unique solution $z_n \in C([0,T], W_n)$ to (3.6). Moreover, there exist positive constants $C(T), C_1(T)$, which are independent on the index $n$, such that the following estimates hold for each $t \in [0,T]$
\[
\sup_{r \in [0,t]} \|z_n(r)\|_{W^2}^2 + 4\nu \int_0^t \|Dz_n(s)\|_{2}^2 ds \leq C(T) \int_0^t \|\psi(s)\|_{2}^2 ds,
\]
\[
\sup_{r \in [0,t]} \|z_n(r)\|_{W^2} \leq C(T) \int_0^t \|\psi(s)\|_{2}^2 ds,
\]
\[
\int_0^T (\|\partial_t z_n\|_2^2 + \alpha_1 \|D\partial_t z_n\|_2^2)ds \leq C_1(T).
\]

**3.3. Transition to the limit.**

**Proposition 3.2.** Let $\psi$ satisfies (3.1), then there exists a unique solution $z$ to (3.2) in the sense of Definition 3.1 and satisfying the following estimates
\[
\sup_{r \in [0,t]} \|z(r)\|_{W^2}^2 + 4\nu \int_0^t \|z(s)\|_{W^2}^2 ds \leq C(T) \int_0^t \|\psi(s)\|_{2}^2 ds \quad \text{for any } t \in [0,T],
\]
\[
\sup_{r \in [0,t]} \|z(r)\|_{W^2} \leq C(T) \int_0^t \|\psi(s)\|_{2}^2 ds,
\]
\[
\int_0^T (\|\partial_t z\|_2^2 + \alpha_1 \|D\partial_t z\|_2^2)ds \leq C_1(T).
\]

**Proof.** By compactness with respect to the weak-* topology in the spaces $L^\infty(0,T; V), L^\infty(0,T; W)$ and the weak topology in the space $L^2(0,T; V)$ there exists $z \in L^\infty(0,T; W)$ such that the following convergences hold, up to sub-sequences (denoted in the same way as the sequences)
\[
z_n \rightharpoonup z \quad \text{in } L^\infty(0,T; V),
\]
\[
z_n \rightharpoonup z \quad \text{in } L^\infty(0,T; W),
\]
\[
\partial_t z_n \rightharpoonup \partial_t z \quad \text{in } L^2(0,T; V).
\]

(3.14)
From (3.14), we deduce that \( z \in C([0,T], V) \) and therefore \( z_N(0) = 0 \) converges to \( z(0) \) in \( V \), i.e. \( z(0) = 0 \). We recall that \( z_N \) solves the equation

\[
(\partial_t v(z_N), \phi) = \left( \psi + \nu \Delta z_N - (y \cdot \nabla) v(z_N) - (z_N \cdot \nabla) v - \sum_j v(z_N) \nabla y^j \right. \\
\left. - \sum_j v(y) \nabla z_N^j + (\alpha_1 + \alpha_2) \text{div}(A(y) A(z_N) + A(z_N) A(y)) \right. \\
+ \beta \text{div}([A(y)^2 A(z_N)]) + 2\beta \text{div}([A(z_N) : A(y)]), \forall \phi \in W_n.
\]

Setting \( \phi = h_i \) and passing to the limit, as \( n \to \infty \), we deduce

\[
(\partial_t v(z), h_i) + 2\nu(\nabla z, \nabla h_i) + b(y, v(z), h_i) + b(z, v(y), h_i) + b(h_i, y, v(z)) \\
+ b(h_i, z, v(y)) + (\alpha_1 + \alpha_2)(A(y) A(z) + A(z) A(y), \nabla h_i) + \beta([A(y)^2 A(z), \nabla h_i]) \\
+ 2\beta((A(z) : A(y)) A(y), \nabla h_i) = (\psi, h_i)
\]

for each \( i \in \mathbb{N} \). Finally, a standard density argument gives the claimed result, namely \( z \in L^2(0,T; W) \) is a solution of (3.2) in the sense of Definition 3.1. \( \square \)

4. A Stability result

The main task of this section is to establish a stability result for the solution of the state equation. This is a crucial step to study the Gâteaux derivative of the control-to-state mapping.

Let us recall the relation (see [8, Appendix])

\[
\frac{1}{2} \partial_t (\alpha_1 | \nabla y |^2 - |y|^2) - (y \cdot \nabla) y + \text{div}(N(y)) \\
= -(y \cdot \nabla) v(y) - \sum_j v(y) \nabla y^j + (\alpha_1 + \alpha_2) \text{div}(A(y)^2),
\]

which allows to write the equations (2.1) in the following form

\[
\begin{align*}
\partial_t (v(y)) &= -\nabla \tilde{P} + \nu \Delta y - (y \cdot \nabla) y + \text{div}(N(y)) + \text{div}(S(y)) + U & \text{in } D \times (0,T), \\
\text{div}(y) &= 0 & \text{in } D \times (0,T), \\
y \cdot \eta = 0, \quad [\eta \cdot \nabla (y)] \cdot \tau &= 0 & \text{on } \partial D \times (0,T), \\
y(x,0) &= y_0(x) & \text{in } D,
\end{align*}
\]

where

\[
S(y) := \beta \left( |A(y)|^2 A(y) \right),
\]

\[
N(y) := \alpha_1 (y \cdot \nabla A(y) + (\nabla y)^T A(y) + A(y) \nabla y) + \alpha_2 (A(y))^2.
\]

\( H_4 \): Assume that \( U_1, U_2 \in L^2(0,T; (H^1(D))^2); \quad y_0^1, y_0^2 \in W \).

**Theorem 4.1.** Let us take \( U_1, U_2 \) and \( y_0^1, y_0^2 \) verifying \( H_4 \), and consider the corresponding solutions of (2.1) \( y_1, y_2 \in L^\infty(0,T;\tilde{W}) \).

Then, there exists a positive constant \( \bar{C} \), which depends only on the data such that the following estimate holds

\[
\sup_{r \in [0,T]} \| y_1(r) - y_2(r) \|^2_W \leq \bar{C} \left[ \| y_0^1 - y_0^2 \|^2_W + \int_0^T \| U_1(s) - U_2(s) \|^2_2 ds \right].
\]

**Proof.** Let \( y_1 \) and \( y_2 \) be two solutions of (2.1) associated with the external forces \( U_1 \) and \( U_2 \) and the initial data \( y_0^1 \) and \( y_0^2 \), respectively.
Denoting \( y = y_1 - y_2 \) and \( y_0 = y_0^1 - y_0^2 \), we can verify that \( y \) solves the system

\[
\begin{aligned}
\partial_t(v(y)) &= -\nabla(\bar{P}_1 - \bar{P}_2) + \nu \Delta y - [(y \cdot \nabla)y_1 + (y_2 \cdot \nabla)y_2] \\
&+ \text{div}(N(y_1) - N(y_2)) + \text{div}(S(y_1) - S(y_2)) + (U_1 - U_2) \\
\text{div}(y) &= 0 \\
y \cdot \eta = 0, \quad [\eta \cdot \nabla(y)] \cdot \tau = 0 \\
(y(x, 0) = y_0(x) \\
\end{aligned}
\]

in \( D \times (0, T) \),

on \( \partial D \times (0, T) \),

in \( D \).

Let us test (4.4) by \( y \). Then we have

\[
\partial_t(\|y\|^{\frac{2}{4}}) + 4\nu \|Dy\|_2^2 = -2 \int_D [(y \cdot \nabla)y_1 + (y_2 \cdot \nabla)y_2] y dx + 2(\text{div}(N(y_1) - N(y_2)), y) \\
+ 2(\text{div}(S(y_1) - S(y_2)), y) + 2 \int_D (U_1 - U_2) \cdot y dx = I_1 + I_2 + I_3 + I_4.
\]

We will estimate \( I_i, i = 1, \cdots, 4 \). Since \( V \to L^4(D) \), the first term verifies

\[
|I_1| = 2 \left| \int_D (y \cdot \nabla)y_1 y dx \right| \leq C\|y\|_2^2\|\nabla y_1\|_2 \leq C\|y\|_2^2\|\nabla y_1\|_2 \leq C\|y\|_2^2\|y\|_{H^3}.
\]

After an integration by parts, the term \( I_3 \), can be treated using the same arguments as in [9, Sect.3], the term on the boundary vanishes and we have

\[
I_3 = 2(\text{div}(S(y_1) - S(y_2)), y_1 - y_2) = -2 \int_D (S(y_1) - S(y_2)) : \nabla y dx
= -\frac{\beta}{2} \left( \int_D (|A(y_1)|^2 - |A(y_2)|^2)^2 dx + \int_D (|A(y_1)|^2 + |A(y_2)|^2)|A(y_1 - y_2)|^2 dx \right) \leq 0.
\]

Concerning \( I_4 \), one has

\[
|I_4| = 2 \left| \int_D (U_1 - U_2) \cdot y dx \right| \leq \|U_1 - U_2\|_2^2 + \|y\|_2^2 \leq \|U_1 - U_2\|_2^2 + \|y\|_V^2.
\]

Let us estimate the term \( I_2 \). Integrating by parts and taking into account that the boundary terms vanish (see [9, Sect.3]), we deduce

\[
I_2 = 2(\text{div}(N(y_1) - N(y_2)), y) = -2 \int_D (N(y_1) - N(y_2)) : \nabla y dx
= -\alpha_2 \int_D (A(y_1)^2 - A(y_2)^2) : A(y) dx - \alpha_1 \int_D (y_1 \cdot \nabla A(y_1) - y_2 \cdot \nabla A(y_2)) : A(y) dx
- \alpha_1 \int_D ((\nabla y_1)^T A(y_1) + A(y_1) \nabla y_1 - (\nabla y_2)^T A(y_2) - A(y_2) \nabla y_2) : A(y) dx
= -\alpha_2 I_2^1 - \alpha_1 I_2^2 - \alpha_1 I_2^3.
\]

Since

\[
I_2^1 = \int_D (A(y_1)^2 - A(y_2)^2) : A(y) dx = \int_D (A(y) A(y_1) + A(y_2) A(y)) : A(y) dx;
\]

\[
I_2^2 = \int_D (y_1 \cdot \nabla A(y_1) - y_2 \cdot \nabla A(y_2)) : A(y) dx
= \int_D (y_1 \cdot \nabla A(y_1) - y_2 \cdot \nabla A(y_2)) : A(y) dx = \int_D (y \cdot \nabla A(y_2)) : A(y) dx;
\]

\[
I_2^3 = \int_D ((\nabla y_1)^T A(y_1) + A(y_1) \nabla y_1 - (\nabla y_2)^T A(y_2) - A(y_2) \nabla y_2) : A(y) dx
= 2 \int_D (A(y_1) A(y)) : \nabla y_1 - (A(y_2) A(y)) : \nabla y_2) dx.
\]
\[ = 2 \int_D ((A(y))^2 : \nabla y_1 + (A(y)A(y)) : \nabla y) \, dx; \]

the Hölder’s inequality and the embedding \( H^1(D) \hookrightarrow L^4(D) \) yield

\[
|I_2|^2 \leq \int_D |(A(y)A(y_1) + A(y_2)A(y_2)) : |A(y)| \, dx \leq C(\|y_1\|_{W^{1,\infty}} + \|y_2\|_{W^{1,\infty}})\|\nabla y\|^2_2; \\
|I_3|^2 \leq \int_D |(y \cdot \nabla A(y_2)) : A(y) | \, dx \leq C\|y\|_1\|y_2\|_{W^{2,4}}\|\nabla y\|_2 \leq C\|y\|_{W^{2,4}}\|\nabla y\|^2_2; \\
|I_4|^2 \leq C \int_D |((A(y))^2 : \nabla y_1 + (A(y_2)A(y)) : \nabla y) | \, dx \leq C(\|y_1\|_{W^{1,\infty}} + \|y_2\|_{W^{1,\infty}})\|\nabla y\|^2_2.
\]

Then the embedding \( H^3(D) \hookrightarrow W^{2,4}(D) \cap W^{1,\infty}(D) \) gives \( |I_2| \leq C(\|y_1\|_{H^3} + \|y_2\|_{H^3})\|y\|^2_1. \) By gathering the previous estimates, we obtain

\[
\|y(t)\|^2_1 + 4\nu \int_0^t \|Dy\|^2_2 \, ds \\
\leq \|y_0\|^2_1 + M_0 \int_0^t (\|y_1\|_{H^3} + \|y_2\|_{H^3} + 1)\|y\|^2_1 \, ds + \int_0^t \|U_1 - U_2\|^2_2 \, ds. \quad (4.5)
\]

Now, multiplying \((4.4)\), by \( \mathbb{P}v(y) \), we write

\[
\partial_t \|\mathbb{P}v(y)\|^2_2 = 2(\nu \cdot A \cdot \mathbb{P}v, \mathbb{P}v(y)) - 2\langle (y \cdot \nabla) y_1 + (y_2 \cdot \nabla) y_1, \mathbb{P}v(y) \rangle + 2\langle (U_1 - U_2), \mathbb{P}v(y) \rangle \\
+ 2\langle \text{div}(N(y_1) - N(y_2)), \mathbb{P}v(y) \rangle + 2\langle \text{div}(S(y_1) - S(y_2)), \mathbb{P}v(y) \rangle.
\]

Using \((4.1)\) and knowing that \( \mathbb{P}v(y) \) is divergence free, we get

\[
\langle -(y_1 \cdot \nabla) y_1 + (y_2 \cdot \nabla) y_2 + \text{div}(N(y_1) - N(y_2)), \mathbb{P}v(y) \rangle \\
= - \langle (y_1 \cdot \nabla) v(y_1) - (y_2 \cdot \nabla) v(y) - \sum_j (v(y))^j \nabla y^j_1 - \sum_j (v(y))^j \nabla y^j_2 \rangle - (\alpha_1 + \alpha_2) \text{div}(A(y_1)^1 - A(y_2)^2), \mathbb{P}v(y) \rangle.
\]

Therefore, we infer that

\[
- 2\langle (y_1 \cdot \nabla) y_1 + (y_2 \cdot \nabla) y_1, \mathbb{P}v(y) \rangle + 2\langle \text{div}(N(y_1) - N(y_2)), \mathbb{P}v(y) \rangle \\
= -2b(y, v(y_1), \mathbb{P}v(y)) - 2\langle (v(y) - \mathbb{P}v(y), \mathbb{P}v(y) \rangle - 2b(\mathbb{P}v(y), y_1, v(y)) \\
- 2b(\mathbb{P}v(y), y_2, v(y_2)) + 2(\alpha_1 + \alpha_2) \text{div}(A(y_1)A(y_1) + A(y_2)A(y_2)), \mathbb{P}v(y) \rangle \\
\leq C\|y\|_{\infty}\|v(y_1)\|_{H^1}\|\mathbb{P}v(y)\|_2 + C\|y_2\|_{\infty}\|v(y) \cdot \mathbb{P}v(y)\|_{H^1}\|\mathbb{P}v(y)\|_2 \\
+ C\|y_1\|_{W^{1,\infty}}\|v(y_1)\|_2\|\mathbb{P}v(y)\|_2 + C\|y_2\|_{W^{1,\infty}}\|v(y_2)\|_2\|\mathbb{P}v(y)\|_2 \\
+ C\|\text{div}[A(y_1)A(y_1) + A(y_2)A(y_2)]\|_2\|\mathbb{P}v(y)\|_2 \leq C(\|y_1\|_{H^3} + \|y_2\|_{H^3})\|y\|^2_1,
\]

where we used that \( H^3(D) \hookrightarrow W^{2,4}(D) \cap W^{1,\infty}(D) \). On the other hand, we have

\[
2\langle \text{div}(S(y_1) - S(y_2)), \mathbb{P}v(y) \rangle = 2\beta \langle \text{div} \{ [A(y_1)]^2 A(y) + [A(y_1) : A(y) + A(y) : A(y_2)] A(y_2) \}, \mathbb{P}v(y) \rangle \\
\leq C(\|y_1\|^2_{H^3} + \|y_2\|^2_{H^3})\|y\|^2_1,
\]

where we used a similar arguments to the one used to get \((3.11)\) to estimate the last two terms. Hence, there exists \( M > 0 \) such that

\[
\|\mathbb{P}v(y(t))\|^2_2 \leq \|\mathbb{P}v(y_0)\|^2_2 + C \int_0^t (1 + \sum_{i=1}^2 \|y_1\|_{H^3} + \|y_2\|_{H^3}))\|y\|^2_1 \, ds + \int_0^t \|U_1 - U_2\|^2_2 \, ds. \quad (4.6)
\]

Consequently, \((4.5)\) and \((4.6)\) give the following relation

\[
\|\mathbb{P}v(y(t))\|^2_2 + \|y(t)\|^2_1 + 4\nu \int_0^t \|Dy\|^2_2 \, ds \\
\leq \|y_0\|^2_1 + \|\mathbb{P}v(y_0)\|^2_2 + M \int_0^t (1 + \|y_1\|_{H^3} + \|y_2\|_{H^3})\|y\|^2_1 \, ds + M \int_0^t \|U_1 - U_2\|^2_2 \, ds.
\]
As a conclusion, we obtain
\[
\sup_{t \in [0,T]} \|y(t)\|_W^2 \leq \|y_0\|_W^2 + M\int_0^T (1 + \|y_1\|_{H^3}^2 + \|y_2\|_{H^3}^2)\|y\|_W^2 \, ds + M\int_0^T \|U_1 - U_2\|_2^2 \, ds,
\]
where \(K\) is a positive constant. Finally, (4.3) is a consequence of Gronwall’s inequality with
\[
C = (M + 1)e^{M(1 + 2K)T}, \quad K = \sup_{t \in [0,T]} M_\rho^2(t).
\]
\(\square\)

5. Gâteaux differentiability of the control-to-state mapping

This section studies the differentiability of the control-to-state mapping. More precisely, with the help of the stability property established in the previous section, we will prove that the Gâteaux derivative of the control-to-state mapping is provided by the solution of the linearized equation.

**Proposition 5.1.** Let us consider \(U\) and \(y_0\) satisfying (H-1) and \(\psi \in L^2(0,T; (H^1(D))^2)\).

Defining
\[
U_\rho = U + \rho \psi, \quad \rho \in (0,1),
\]
let \(y\) and \(y_\rho\) be the solutions of (2.1) associated with \((U, y_0)\) and \((U_\rho, y_0)\), respectively, then the following representation holds
\[
y_\rho = y + \rho z + \rho \delta_\rho \quad \text{with} \quad \lim_{\rho \to 0} \sup_{t \in [0,T]} \|\delta_\rho\|_V^2 = 0,
\]
where \(z \in L^\infty(0,T; W)\) is the solution of (3.2), satisfying the estimates of Proposition 3.2.

**Proof.** We recall that \(y\) verifies the equation
\[
\partial_t (v(y)) = -\nabla \mathbf{P} + \nu \Delta y - (y \cdot \nabla) v - \sum_j v^j \nabla y^j + (\alpha_1 + \alpha_2) \text{div}(A^2) + \beta \text{div}(|A^2|^2 A) + U.
\]

Therefore
\[
\partial_t (v(y_\rho - y)) = -\nabla (\mathbf{P}_\rho - \mathbf{P}) + \nu \Delta (y_\rho - y) - ((y_\rho \cdot \nabla) v_\rho - (y \cdot \nabla) v) - \sum_j (v^j_\rho \nabla y^j_\rho - v^j \nabla y^j) + (\alpha_1 + \alpha_2) \text{div}(A^2_\rho - A^2) + \beta \text{div} \left( |A(y_\rho)|^2 A(y_\rho) - |A(y)|^2 A(y) \right) + \rho \psi,
\]
where \(v_\rho = v(y_\rho), \quad v = v(y), \quad A_\rho = A(y_\rho), \quad A = A(y)\).

Setting \(z_\rho = \frac{y_\rho - y}{\rho}, \quad \pi_\rho = \frac{P_\rho - P}{\rho},\) we notice that \(z_\rho\) is the unique solution for the following equation
\[
\partial_t (v(z_\rho)) = \psi - \nabla \pi_\rho + \nu \Delta z_\rho - [(z_\rho \cdot \nabla) v(y_\rho) + (y \cdot \nabla) v(z_\rho)] - \sum_j [v^j(z_\rho) \nabla y^j_\rho + v^j(y) \nabla z_\rho] + \beta \text{div} \left( |A(y_\rho)|^2 A(z_\rho) + |A(y)|^2 A(z) \right) + \sum_j [v^j(y) A(y_\rho) - v^j(y) A(y)] + (\alpha_1 + \alpha_2) \text{div} \left[ A(z_\rho) A(y_\rho) + A(y_\rho) A(z) \right].
\]

Defining \(\delta_\rho = z_\rho - z\), the following equation holds
\[
\partial_t (v(\delta_\rho)) = -\nabla (\pi_\rho - \pi) + \nu \Delta \delta_\rho - [(y \cdot \nabla) v(\delta_\rho) + (\delta_\rho \cdot \nabla) v(y_\rho)] - (z \cdot \nabla) v(y_\rho - y) - \sum_j [v^j(y) \nabla \delta^j_\rho + v^j(\delta_\rho) \nabla y^j_\rho + v^j(z) \nabla (y_\rho - y)^j] + (\alpha_1 + \alpha_2) \text{div} \left[ A(y_\rho) A(\delta_\rho) + A(\delta_\rho) A(y_\rho) + A(z) A(y_\rho - y) \right] + \beta \text{div} \left\{ A(y) A(\delta_\rho) A(y) + |A(y_\rho)|^2 A(\delta_\rho) \right\} + \beta \text{div} \left[ A(y_\rho - y) A(y_\rho) + A(y) A(y_\rho - y) A(z) \right] + \beta \text{div} \left[ A(\delta_\rho) A(y_\rho) + A(z) A(y_\rho - y) A(y) \right] =: g(\delta_\rho).
\]
Multiplying this equation by \( \delta_p \), we write \( \partial_t \| \delta_p \|_V^2 = 2(g(\delta_p), \delta_p) \). Let us estimate the right hand side.

\[
2(g(\delta_p), \delta_p) = -4\nu \| \mathbb{D} \delta_p \|_2^2 - 2b(y, v(\delta_p), \delta_p) - 2b(\delta_p, v(y_p), \delta_p) - 2b(z, v(y_p - y), \delta_p) - 2b(\delta_p, v(y), \delta_p) - 2b(\delta_p, v(y_p), \delta_p) - 2b(y, \delta_p, v(y_p), \delta_p) - 2b(\delta_p, v(y), \delta_p) - 2b(y, \delta_p, v(y)) - 2b(y, \delta_p, (y_p - y), (v(z))) + 2(\alpha_1 + \alpha_2) (\text{div} [A(y)A(\delta_p) + A(\delta_p)A(y_p) + A(z)A(y_p - y)], \delta_p) + 2\beta (\text{div} [(A(y_p - y) : A(y_p) + A(y) : A(y_p - y)) A(z)], \delta_p) + 2\beta (\text{div} [(A(\delta_p) : A(y_p) + A(z) : A(y_p - y)) A(y)], \delta_p)
\]

\[
= -4\nu \| \mathbb{D} \delta_p \|_2^2 + R_1 + R_2 + R_3.
\]

We have

\[
R_1 = -2b(y, v(\delta_p), \delta_p) - 2b(\delta_p, v(y_p), \delta_p) - 2b(z, v(y_p - y), \delta_p) - 2b(y, \delta_p, v(y_p), \delta_p) - 2b(\delta_p, v(y_p), \delta_p) - 2b(\delta_p, v(y), \delta_p) - 2b(y, \delta_p, v(y)) - 2b(\delta_p, (y_p - y), v(z)) = 2b(y, \delta_p, v(\delta_p)) - 2b(\delta_p, v(y_p), \delta_p) - 2b(\delta_p, v(y), \delta_p) - [2b(\delta_p, v(y_p), \delta_p) + 2b(z, v(y_p - y), \delta_p)] - 2b(\delta_p, (y_p - y), v(z)) \leq C \| y \|_{H^2} \| \delta_p \|_V^2 + C \| y_p \|_{H^3} \| \delta_p \|_V^2 + C \| y_p \|_{H^3} \| \delta_p \|_V^2 + C \| y_p - y \|_{H^2}^2.
\]

Using the Stokes theorem and the boundary conditions for \( \delta_p \), we deduce

\[
R_2 = -2(\alpha_1 + \alpha_2) \int_{D} [A(y)A(\delta_p) + A(\delta_p)A(y_p) + A(z)A(y_p - y)] : \nabla \delta_p \, dx \leq C \| y \|_{W^{1,\infty}} \| \delta_p \|_V^2 + C \| y_p \|_{W^{1,\infty}} \| \delta_p \|_V^2 + C \| z \|_{W^{1,\infty}} \| y_p - y \|_{W^{1,\infty}} \| \delta_p \|_V \leq C (\| y \|_{H^3} + \| y_p \|_{H^3}) \| \delta_p \|_V^2 + C \| z \|_{H^2} \| \delta_p \|_V^2 + C \| y_p - y \|_{H^2}^2.
\]

Analogous arguments give

\[
R_3 = -2\beta \int_{D} \{ A(y) : A(\delta_p), A(y) + |A(y_p)|^2 A(\delta_p) \} : \nabla \delta_p \, dx - 2\beta \int_{D} \{ [A(y_p - y) : A(y_p) + A(y) : A(y_p - y)] A(z) \} : \nabla \delta_p \, dx - 2\beta \int_{D} \{ [A(\delta_p) : A(y_p) + A(z) : A(y_p - y)] A(y) \} : \nabla \delta_p \, dx \leq C (\| y \|_{H^3} + \| y_p \|_{H^3}) \| \delta_p \|_V^2 + C \| y_p - y \|_{H^2}^2,
\]

where we used Hölder and Young inequalities to deduce the last estimate. Summing up, we obtain

\[
\partial_t \| \delta_p \|_V^2 \leq C \bigg( 1 + \| y \|_{H^3}^2 + \| y_p \|_{H^3}^2 \bigg) (1 + \| z \|_{H^2}^2) \| \delta_p \|_V^2 + C (y_p - y) \|_{H^2}^2 \leq \bar{K}_0 \| \delta_p \|_V^2 + C \| y_p - y \|_{H^2}^2,
\]

by Lemma 2.4 and Proposition 3.2. Finally, Gronwall’s inequality yields

\[
\sup_{s \in [0,T]} \| \delta_p(s) \|_V^2 \leq C \int_0^T \| y_p - y \|_{H^2}^2 \, ds.
\]
Applying (4.3) with \( y_1 = y \) and \( y_2 = y_p \), we derive
\[
\sup_{s \in [0,T]} \| \delta_p(s) \|_Y^2 \leq C \rho^2 \int_0^T \| \psi \|_X^2 ds.
\]
Now, taking \( \rho \to 0 \) we infer (5.1), i.e.
\[
y_p = y + \rho z + \rho \delta_p \text{ with } \lim_{\rho \to 0} \sup_{t \in [0,T]} \| \delta_p \|_Y^2 = 0.
\]

5.1. Variation of the cost functional (2.6). As a consequence of Proposition 5.1, we get the following result on the variation for the cost functional (2.6).

**Proposition 5.2.** Let us consider \( U, y_0 \), \( \psi \) and \( U_p = U + \rho \psi \) verifying the hypothesis of the Proposition 5.1. Then
\[
J(U_p, y_p) = J(U, y) + \rho \int_0^T \{(\nabla_u L(t,u(t),y(t)), \psi(t)) + (\nabla_y L(t,u(t),y(t)), z(t))\} dt + o(\rho),
\]
where \( y_p, y \) are the solutions of (2.1), corresponding to \( (U_p, y_0) \) and \( (U, y_0) \), respectively, and \( z \) is the solution of (3.2).

6. ADJOINT EQUATION

Let \( f \in (L^2(D \times [0,T]))^2 \). Our aim is to prove the well posedness of the adjoint equation given by
\[
\begin{cases}
-\partial_t (v(p)) - \nu \Delta p - \text{curl } v(y) \times p + \text{curl } v(y \times p) - (\alpha_1 + \alpha_2) \text{div} [A(y)A(p) + A(p)A(y)] \\
-\beta \text{div} [(A(y))^2 A(p)] - 2\beta \text{div} [(A(y) : A(p)) A(y)] = f - \nabla \pi \quad & \text{in } D \times (0,T), \\
\text{div}(p) = 0 \quad & \text{in } D \times (0,T), \\
p \cdot \eta = 0, \quad [\eta \cdot D(p)] \cdot \tau = 0 \quad & \text{on } \partial D \times (0,T), \\
p(T) = 0 \quad & \text{in } D.
\end{cases}
\]

(6.1)

**Definition 6.1.** A function \( p \in L^\infty(0,T;W) \) with \( \partial_t p \in L^2(0,T;V) \) is a solution of (6.1) if \( p(T) = 0 \) and for any \( t \in [0,T] \), the following equality holds
\[
(-\partial_t v(p), \phi) + 2\nu (Dp, D\phi) - b(\phi, p, v(y)) + b(p, \phi, v(y)) + b(p, y, v(\phi)) \\
- b(y, p, v(\phi)) + (\alpha_1 + \alpha_2) (A(y)A(p) + A(p)A(y), \nabla \phi) + \beta (|A(y)|^2 A(p), \nabla \phi) \\
+ 2\beta (A(y) : A(y)) A(y), \nabla \phi) = (f, \phi), \quad \text{for all } \phi \in W.
\]

Let us state the following result about the solution of (6.1).

**Proposition 6.1.** Let \( f \in (L^2(D \times [0,T]))^2 \), then there exists a unique solution \( p \) to (6.1) in the sense of Definition 6.1 satisfying the following estimates
\[
\sup_{r \in [0,t]} \| p(r) \|_V^2 + 4\nu \int_0^t \| p(s) \|_V^2 ds \leq C(T) \int_0^T \| f(s) \|_X^2 ds \quad \text{for any } t \in [0,T];
\]
\[
\sup_{r \in [0,t]} \| p(r) \|_W^2 \leq C(T) \int_0^T \| f(s) \|_X^2 ds;
\]
\[
\int_0^T (\| \partial_t p \|_X^2 + \alpha_1 \| D\partial_t p \|_X^2) ds \leq C_1(T).
\]
6.1. Proof of Proposition 6.1. Notice that $p$ is the solution of (6.1) if and only if $q(t) = p(T-t)$ is the solution of the following initial value problem with $\bar{y}(t) = \bar{y}(T-t), \bar{f}(t) = f(T-t)$ and $\bar{\pi}(t) = \pi(T-t)$

\[
\begin{aligned}
\partial_t(v) - \nu \Delta q - \text{curl} \, v(\bar{y}) \times q + \text{curl} \, v(\bar{y} \times q) - (\alpha_1 + \alpha_2)\text{div} \left[ A(\bar{y}) A(q) + A(q) A(\bar{y}) \right] \\
- \beta \text{div} \left[ (A(\bar{y}) : A(q)) A(\bar{y}) \right] - 2 \beta \text{div} \left[ (A(\bar{y}) : A(q)) A(\bar{y}) \right] = \bar{f} - \nabla \bar{\pi} & \quad \text{in } D \times (0,T), \\
\text{div}(q) = 0 & \quad \text{in } D \times (0,T), \\
q \cdot \eta = 0, \quad [\eta \cdot \nabla \text{div}(q)] \cdot \tau = 0 & \quad \text{on } \partial D \times (0,T), \\
q(0) = 0 & \quad \text{in } D.
\end{aligned}
\]

(6.2)

According to the Definition 6.1, $q$ is the solution of (6.2) if $q \in L^\infty(0,T;W)$ with $\partial_t q \in L^2(0,T;V)$, $q(0) = 0$ and, in addition, the following equality holds, for all $\phi \in W$,

\[
(\partial_t v(q), \phi) + 2\nu(\text{div}(q), \phi) - b(q, \phi, v(\bar{y})) + b(q, \phi, v(\bar{y}) + b(q, \bar{y}, v(\bar{y}))) - b(\bar{y}, q, v(\phi)) + (\alpha_1 + \alpha_2)(A(\bar{y}) A(q) + A(q) A(\bar{y}), \nabla \phi) + \beta(|A(\bar{y})|^2 A(q), \nabla \phi) + 2\beta((A(q) : A(\bar{y})) A(\bar{y}), \nabla \phi) = (\bar{f}, \phi).
\]

To study the equation (6.2), we will follow closely the analysis used in Section 3 to study the linearized equation. For that, consider $W_n = \text{span}\{h_1, \ldots, h_n\}$ and define the corresponding approximations $q_n(t) = \sum_{i=1}^n a_i(t) h_i$ for each $t \in [0,T]$. The approximated problem for (6.2) can be written as $q_n(0) = 0$ and

\[
(\partial_t v(q_n), \phi) + 2\nu(\text{div}(q_n), \phi) - b(q_n, \phi, v(\bar{y})) + b(q_n, \phi, v(\bar{y})) + b(q_n, \bar{y}, v(\bar{y})) + (\alpha_1 + \alpha_2)(A(\bar{y}) A(q_n) + A(q_n) A(\bar{y}), \nabla \phi) + \beta(|A(\bar{y})|^2 A(q_n), \nabla \phi) + 2\beta((A(q_n) : A(\bar{y})) A(\bar{y}), \nabla \phi) = (\bar{f}, \phi), \quad \text{for any } \phi \in W_n.
\]

Now, we remark that the structure of (6.3) and (3.6) are similar. Therefore, by adapting the arguments used to derive Lemma 3.1, we are able to deduce the following result for (6.3).

Lemma 6.2. Let $f \in (L^2(D \times [0,T]))^2$. Then there exists a unique solution $q_n \in C([0,T], W_n)$ to (6.3). Moreover, there exist positive constants $C(T), C_1(T)$, which are independent on the index $n$, such that the following estimates hold for each $t \in [0,T]$

\[
\sup_{r \in [0,t]} \|q_n(r)\|_W^2 + 4\nu \int_0^t \|\text{div}(q_n(s))\|_2^2 ds \leq C(T) \int_0^t \|f(s)\|_2^2 ds;
\]

\[
\sup_{r \in [0,t]} \|q_n(r)\|_W \leq C(T) \int_0^t \|f(s)\|_2^2 ds;
\]

\[
\int_0^T (\|\partial_t q_n\|_2^2 + \alpha_1 \|\text{div}(q_n)\|_2^2) ds \leq C_1(T).
\]

Consequently, Proposition 6.1 follows by passing to the limit in (6.3).

7. Existence of optimal control and optimality condition

This section starts with the presentation of a duality relation between the solution of the linearized equation and the solution of the adjoint equation and next shows that the control problem has a solution. Taking into account the duality relation, it will be proved that the solution of the control problem satisfies the first order optimality condition.

7.1. Duality property.

Proposition 7.1. Let $y \in L^\infty(0,T;\tilde{W})$ and $f, \psi \in (L^2(D \times [0,T]))^2$. Then we have

\[
\int_0^T (\psi(t), p(t)) dt = \int_0^T (f(t), z(t)) dt,
\]
where \( p \) is the solution of (6.1) and \( z \) is the solution of (3.2).

**Proof.** For any \( t \in [0,T] \), let \( p_n(t) = q_n(T - t) \), where \( q_n \) is the solution of (6.3). Then \( p_n(t) = \sum_{i=1}^n d_i(t) \cdot h_i \) verifies \( p_n(T) = 0 \) and

\[
\begin{cases}
( -\partial_t v(p_n(t), \phi) + 2\nu (\mathbb{D} p_n, \mathbb{D} \phi) - b(\phi, p_n, v(y)) + b(p_n, \phi, v(y)) + b(p_n, y, v(\phi)) \\
\quad - b(y, p_n, v(\phi)) + (\alpha_1 + \alpha_2)(A(y)A(p_n) + A(p_n)A(y), \nabla \phi) \\
\quad + \beta(|A(y)|^2 A(p_n), \nabla \phi) + 2\beta((A(p_n) : A(y))A(y), \nabla \phi) = (f, \phi), \text{ for any } \phi \in W_n.
\end{cases}
\]

(7.1)

On the other hand, let us recall that the solution of the linearized equation \( z_n \) given by \( z_n(t) = \sum_{i=1}^n c_i(t) h_i \) satisfies \( z_n(0) = 0 \) and

\[
\begin{cases}
( \partial_t v(z_n), \phi) + 2\nu (\mathbb{D} z_n, \mathbb{D} \phi) + b(y, v(z_n), \phi) + b(z_n, v(y), \phi) + b(\phi, y, v(z_n)) \\
\quad + b(p_n, y, v(z_n)) - b(z_n, v(y)) + (\alpha_1 + \alpha_2)(A(y)A(z_n) + A(z_n)A(y), \nabla \phi) \\
\quad + \beta(|A(y)|^2 A(z_n), \nabla \phi) + 2\beta((A(z_n) : A(y))A(y), \nabla \phi) = (\psi, \phi), \text{ for all } \phi \in W_n.
\end{cases}
\]

(7.2)

Setting \( \phi = h_i \) in (7.1), multiplying (7.1) by \( c_i(t) \) and summing from \( i = 1 \) to \( n \), we get

\[
\begin{cases}
( -\partial_t v(p_n(0), z_n) + 2\nu (\mathbb{D} p_n, \mathbb{D} z_n) - b(z_n, p_n, v(y)) + b(p_n, z_n, v(y)) \\
\quad + \beta(|A(y)|^2 A(p_n), \nabla z_n) + 2\beta((A(p_n) : A(y))A(y), \nabla z_n) = (f, z_n).
\end{cases}
\]

(7.3)

Similarly, taking \( \phi = h_i \) in (7.2), multiplying (7.2) by \( d_i(t) \) and summing from \( i = 1 \) to \( n \), we obtain

\[
\begin{cases}
( \partial_t v(p_n(t), z_n(t)) , z_n(t)) = (\partial_t v(z_n(t)), p_n(t)) - \frac{d}{dt}(((p_n(t), z_n(t)) + 2\alpha_1 (\mathbb{D} p_n(t), \mathbb{D} z_n(t))).
\end{cases}
\]

Integrating with respect to the time variable on the interval \([0,T]\) and using that \( z_n(0) = 0, \ p_n(T) = 0 \), we derive

\[
\int_0^T ( -\partial_t v(p_n(t), z_n(t)) ) dt = \int_0^T (\partial_t v(z_n(t)), p_n(t)) dt.
\]

Now, combining (7.3) and (7.4), and using the fact that \((A, B) = (A^T, B^T)\), for any \( A, B \in \mathcal{M}_{2 \times 2}(\mathbb{R}) \), we deduce

\[
\int_0^T (\psi(t), p_n(t)) dt = \int_0^T (f(t), z_n(t)) dt.
\]

Therefore, taking the limit as \( n \to \infty \), the result of the Proposition 7.1 holds. \( \square \)

Considering \( f = \nabla_y L(\cdot, U, y) \in (L^2([0,T] \times D))^2 \) in Proposition 7.1, we obtain

**Corollary 7.2.** Under the assumptions of Proposition 7.1, the following duality relation holds

\[
\int_0^T (\psi(t), p(t)) dt = \int_0^T (\nabla_y L(t, U(t), y(t)), z(t)) dt.
\]
7.2. Existence of an optimal control for (2.7). Let \((U_n, y_n)\) be a minimizing sequence, notice that \((U_n)\) is uniformly bounded in the closed convex set \(\mathcal{U}_{ad} \subset L^2(0, T; (H^1(D))^2)\). On the other hand, denoting by \(y_n\) the solution of (2.1), where \(U\) is replaced by \(U_n\), Theorem 2.1 and Lemma 2.4 ensures that \((y_n)\) is uniformly bounded in \(L^\infty(0, T; \tilde{W}) \cap H^1(0, T; V)\).

By compactness with respect to the weak and weak-* topologies in the spaces involved in the following product space
\[
L^2(0, T; (H^1(D))^2) \times (L^\infty(0, T; \tilde{W}) \cap H^1(0, T; V)),
\]
there exists \((U, y) \in L^2(0, T; (H^1(D))^2) \times (L^\infty(0, T; \tilde{W}) \cap H^1(0, T; V))\) such that the following convergences hold, up to sub-sequences (denoted by the sequences)
\[
\begin{align*}
U_n &\to \tilde{U} \quad \text{in } L^2(0, T; (H^1(D))^2), \\
y_n &\rightharpoonup^* \tilde{y} \quad \text{in } L^\infty(0, T; V), \\
y_n &\rightharpoonup \tilde{y} \quad \text{in } L^\infty(0, T; W), \\
\partial_t y_n &\to \partial_t \tilde{y} \quad \text{in } L^2(0, T; V).
\end{align*}
\]
(7.5)

From (7.5), we deduce that \(\tilde{y} \in \mathcal{C}([0, T], V)\) and therefore \(y_n(0) = y(0)\) converges to \(\tilde{y}(0)\) in \(V\), which gives \(\tilde{y}(0) = y_0\). Now, standard arguments (similar reasoning as in [9]) ensure that \((\tilde{U}, \tilde{y})\) solves (2.1).

Recall that \(J : L^2(0, T; (H^1(D))^2) \times L^2(0, T; \tilde{W}) \to \mathbb{R}^+\) given by (2.6) is convex and continuous. From (7.5) we have
\[
U_n \to \tilde{U} \quad \text{in } L^2(0, T; (H^1(D))^2) \quad \text{and} \quad y_n \to \tilde{y} \quad \text{in } L^2(0, T; \tilde{W}).
\]

The (weak) lower semicontinuity of \(J\) ensures
\[
J(\tilde{U}, \tilde{y}) \leq \liminf_n J(U_n, y_n),
\]
which gives that \((\tilde{U}, \tilde{y})\) is an optimal pair.

7.3. A necessary optimality condition for (2.7). Let \((\tilde{U}, \tilde{y})\) be the optimal control pair. Consider \(\psi \in \mathcal{U}_{ad}\) and define \(U_\rho = \tilde{U} + \rho(\psi - \tilde{U})\). Thanks to Proposition 5.1 and Proposition 5.2, we have
\[
\frac{J(U_\rho, y_\rho) - J(\tilde{U}, \tilde{y})}{\rho} = \int_0^T \{\langle \nabla u L(\cdot, \tilde{U}, \tilde{y}), \psi - \tilde{U} \rangle + \langle \nabla y L(\cdot, \tilde{U}, \tilde{y}), z \rangle \} dt + \frac{o(\rho)}{\rho}.
\]
Then, the Gâteaux derivative of the cost functional \(J\) is given by
\[
\lim_{\rho \to 0} \frac{J(U_\rho, y_\rho) - J(\tilde{U}, \tilde{y})}{\rho} = \int_0^T \{\langle \nabla u L(\cdot, \tilde{U}, \tilde{y}), \psi - \tilde{U} \rangle + \langle \nabla y L(\cdot, \tilde{U}, \tilde{y}), z \rangle \} dt \geq 0.
\]

Therefore, we have
\[
\int_0^T \{\langle \nabla u L(\cdot, \tilde{U}, \tilde{y}), \psi - \tilde{U} \rangle + \langle \nabla y L(\cdot, \tilde{U}, \tilde{y}), z \rangle \} dt \geq 0,
\]
where \(z\) is the unique solution to the linearized problem (3.2) with \(\psi\) replaced by \(\psi - \tilde{U}\).

Let \(\tilde{p}\) be the unique solution of (6.1). The application of Proposition 7.1 yields
\[
\int_0^T (\psi(t) - \tilde{U}(t), \tilde{p}(t)) dt = \int_0^T (\nabla y L(t, \tilde{U}(t), \tilde{y}(t)), z(t)) dt.
\]
Finally, we obtain the following optimality condition, for any $\psi \in \mathcal{W}_{ad}$

$$\int_0^T (\psi(t) - \bar{U}(t), \bar{p}(t) + \nabla_u L(t, \bar{U}(t), \bar{y}(t)))dt = \int_0^T (\psi(t) - \bar{U}(t), \bar{p}(t))dt + \int_0^T (\nabla_u L(t, \bar{U}(t), \bar{y}(t)), \psi(t) - \bar{U}(t))dt \geq 0. \quad (7.6)$$

The proof of Theorem 2.5 results from the combination of the previous sections.

8. Uniqueness of the optimal solution

In the previous section, we derived the coupled system constituted by the so-called first order necessary optimality conditions for the control problem. This means that, at this stage, a solution of the coupled system is just a candidate for an optimal solution. Thus, the uniqueness of the solution of the coupled system is a very important issue in determining the optimal solution. This section addresses this uniqueness problem for the cost functional given by the quadratic Lagrangian (2.6), and is devoted to the proof of Theorem 2.6.

Let us consider the problem (2.7) and $f = \nabla_y L(\cdot, U, y)$ in (6.1). Then

**Corollary 8.1.** There exists $\bar{\lambda} > 0$ such that

$$\sup_{r \in [0,T]} \|p(r)\|_W \leq C(T) \int_0^T \|y - y_d\|^2ds \leq C(T) \int_0^T (M_0^2(s) + \|y_d\|^2)ds := \bar{\lambda}^2. \quad (8.1)$$

On the other hand, a standard computation leads to

$$\exists \Gamma > 0 : \|([\text{curl}v(z) \times z, \phi]) \leq \Gamma \|\phi\|_{H^2}^2 \|z\|_{H^2}^2, \quad \forall \phi, z \in W. \quad (8.2)$$

8.1. Proof of Theorem 2.6. Let $U_1, U_2$ be two optimal control variables for (2.7) and $y_1, y_2$ be the corresponding optimal states with the adjoint states $p_1, p_2$.

Now, let us consider $y = y_1 - y_2, \mathbf{P} = \mathbf{P}_1 - \mathbf{P}_2, U = U_1 - U_2$ and notice that $y$ solves the equation

$$\begin{align*}
\partial_t (v(y)) - \nu \Delta y + (y \cdot \nabla)v(y) + (y \cdot \nabla)v(y_2) + (y_2 \cdot \nabla)v(y) \\
+ \sum_j [v(y)^j \nabla y^j + v(y)^j \nabla y_2^j + v(y_2)^j \nabla y^j] \\
- (\alpha_1 + \alpha_2) \text{div} \left[ (A(y))^2 + A(y)A(y_2) + A(y_2)A(y) \right] \\
- \beta \text{div} \left[ |A(y)|^2 A(y) + |A(y)|^2 A(y_2) + (A(y) : A(y_2)) A(y) \right] \\
- \beta \text{div} \left[ (A(y) : A(y_2)) A(y_2) + (A(y_2) : A(y)) A(y) \right] \\
- \beta \text{div} \left[ (A(y_2) : A(y)) A(y_2) \right] - \beta \text{div} \left[ |A(y_2)|^2 A(y) \right] = -\nabla \mathbf{P} + U. \quad (8.3)
\end{align*}$$

Let us multiply (8.3) by $p_2$ and integrate on $D$. Then integrating by parts and taking into account the boundary conditions for the state and adjoint variables, we get the following variational formulation

$$\begin{align*}
(\partial_t y, p_2)_V &= -2\nu(\nabla y, \nabla p_2) - b(y, v(y), p_2) + b(y, v(y_2), p_2) - b(y_2, v(y), p_2) - b(p_2, y, v(y)) \\
- b(p_2, y, v(y)) - b(p_2, y, v(y_2)) - \frac{1}{2} (\alpha_1 + \alpha_2)(|A(y)|^2 + A(y)A(y_2) + A(y_2)A(y), A(p_2)) \\
+ (U, p_2) - \frac{1}{2} \beta(|A(y)|^2 A(y) + |A(y)|^2 A(y_2) + (A(y) : A(y_2)) A(y), A(p_2)) \\
- \frac{1}{2} \beta((A(y) : A(y_2)) A(y_2) + (A(y_2) : A(y)) A(y) + (A(y_2) : A(y)) A(y_2) + |A(y)|^2 A(y), A(p_2)). \quad (8.4)
\end{align*}$$
Considering the adjoint equation for $p_2$, from Definition 6.1, and by setting the test function $\phi = y$, we write
\[
(\partial_t p_2, y)_V = 2\nu(\nabla p_2, \nabla y) - b(y, p_2, v(y_2)) + b(p_2, y, v(y_2)) + b(p_2, y_2, v(y)) - b(y_2, p_2, v(y))
\]
\[-(y_2 - y, y) + (\alpha_1 + \alpha_2)(A(y_2)A(p_2) + A(p_2)A(y_2), \nabla y)
\]+\beta(|A(y_2)|^2A(p_2), \nabla y) + 2\beta((A(p_2) : A(y_2)) A(y_2), \nabla y).
\] (8.5)

Therefore summing the last two equalities (8.4) and (8.5), we deduce
\[
\partial_t (y, p_2)_V = -(\text{curl}(y) \times y, p_2) - (y_2 - y, y) + (U, p_2)
\]
\[-(\alpha_1 + \alpha_2)((A(y))^2, \nabla p_2) - \beta(|A(y)|^2 A(y) + |A(y)|^2 A(y_2), \nabla p_2)
\]+\[2\beta((A(y) : A(y_2)) A(y), \nabla p_2),
\]
where we used the symmetry of $A$ and the bilinear form $b$ to get the last equality. By integrating from $t = 0$ to $t = T$ and taking into account the initial and terminal conditions for $y$ and $p_2$, we have
\[
0 = -\int_0^T (\text{curl}(y) \times y, p_2) - (y_2 - y, y) + (U, p_2) dt - \int_0^T (\alpha_1 + \alpha_2)((A(y))^2, \nabla p_2) dt
\]
\[-\int_0^T \beta(|A(y)|^2 A(y) + |A(y)|^2 A(y_2), \nabla p_2) - 2\beta((A(y) : A(y_2)) A(y), \nabla p_2) dt. \quad (8.6)
\]

Analogously, we can show that $\bar{y} = -y$ verifies the relation
\[
\partial_t (\bar{y}, p_1)_V = -(\text{curl}(\bar{y}) \times \bar{y}, p_1) - (y_1 - y, \bar{y}) - (U, p_1)
\]
\[-(\alpha_1 + \alpha_2)((A(y))^2, \nabla p_1) - \beta(|A(y)|^2 A(y) + |A(y)|^2 A(y_1), \nabla p_1)
\]+\[2\beta((A(y) : A(y_1)) A(y), \nabla p_1).
\]

Integrating from $t = 0$ to $t = T$, taking into account that $\bar{y} = -y$ and using the initial and terminal conditions for $y$ and $p_2$, we have
\[
0 = -\int_0^T (\text{curl}(y) \times y, p_1) + (y_1 - y, \bar{y}) - (U, p_1) dt - \int_0^T (\alpha_1 + \alpha_2)((A(y))^2, \nabla p_1) dt
\]
\[+\int_0^T \beta(|A(y)|^2 A(y) - |A(y)|^2 A(y_1), \nabla p_1) - 2\beta((A(y) : A(y_1)) A(y), \nabla p_1) dt. \quad (8.7)
\]

By summing (8.6) and (8.7), we infer that
\[
\int_0^T ||y||^2 dt - \int_0^T (U, p_1 - p_2) dt
\]
\[= \int_0^T (\text{curl}(y) \times y, p_1 + p_2) dt + \int_0^T (\alpha_1 + \alpha_2)((A(y))^2, \nabla p_1 + \nabla p_2) dt
\]
\[-\int_0^T \beta(|A(y)|^2 A(y) - |A(y)|^2 A(y_1), \nabla p_1) + 2\beta(A(y) : A(y_1)A(y), \nabla p_1) dt
\]
\[+\int_0^T \beta(|A(y)|^2 A(y) + |A(y)|^2 A(y_2), \nabla p_2) + 2\beta((A(y) : A(y_2)) A(y), \nabla p_2) dt
\]
\[:= I_1 + I_2 + I_3.
\]

From (7.6), the following optimality conditions hold
\[
\int_0^T (\psi - U_1, p_1 + \lambda U_1) dt \geq 0, \quad \int_0^T (\psi - U_2, p_2 + \lambda U_2) dt \geq 0, \quad \forall \psi \in \mathcal{W}_{ad}.
\]
Setting $\psi = U_2$ and $\psi = U_1$ in the first and the second optimality conditions, respectively, we deduce \[
\lambda \int_0^T \|U\|_2^2 dt \leq - \int_0^T (U, p_1 - p_2) dt.
\]
On the other hand, (4.3) and (8.1) yield
\[
|I_1| \leq \Gamma \int_0^T (\|p_1\|_{L^2} + \|p_2\|_{L^2}) \|y\|_{L^2}^2 \leq 2\Gamma \tilde{C} \lambda \int_0^T \|U\|_2^2 dt,
\]
\[
|I_2| \leq 4(\alpha_1 + \alpha_2) \int_0^T \|\nabla y\|_{L^2}^2 (\|p_1\|_{L^2} + \|p_2\|_{L^2}) dt \leq 8\kappa \tilde{C} \lambda (\alpha_1 + \alpha_2) \int_0^T \|U\|_2^2 dt,
\]
where $\kappa$ is a positive constant defined by the embedding $H^2(D) \hookrightarrow W^{1,4}(D)$,
\[
\|u\|_{W^{1,4}}^2 \leq \kappa \|u\|_V^2, \quad \forall u \in V. \tag{8.8}
\]
Thanks to (2.4), (4.3) and (8.1), by standard computations we obtain
\[
|I_3| \leq 24\kappa \beta \tilde{C} \lambda \sup_{r \in [0,T]} M_4(r) \int_0^T \|U\|_2^2 dt = 24\kappa \beta \gamma \tilde{C} \lambda \int_0^T \|U\|_2^2 dt.
\]
Consequently, we have
\[
\int_0^T \|y(t)\|_2^2 dt + \lambda \int_0^T \|U(t)\|_2^2 dt \leq 2\tilde{C} \lambda [\Gamma + 4\kappa (\alpha_1 + \alpha_2) + 12\kappa \beta \gamma] \int_0^T \|U(t)\|_2^2 dt,
\]
which gives the claimed result.

ACKNOWLEDGMENT

This work is funded by national funds through the FCT - Fundação para a Ciência e a Tecnologia, I.P., under the scope of the projects UIDB/00297/2020 and UIDP/00297/2020 (Center for Mathematics and Applications).

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