INTERPOLATION WITHOUT SEPARATION IN BERGMAN SPACES

DANIEL H. LUECKING

Abstract. Most characterizations of interpolating sequences for Bergman spaces include the condition that the sequence be uniformly discrete in the hyperbolic metric. We show that if the notion of interpolation is suitably generalized, two of these characterizations remain valid without that condition. The general interpolation we consider here includes the usual simple interpolation and multiple interpolation as special cases.

1. Introduction

Let $dA$ denote area measure and let $G$ be a domain in the complex plane. Let $L^p(G) = L^p(G, dA)$ be the usual Lebesgue space of measurable functions $f$ with $\|f\|_p^p = \int_G |f|^p dA < \infty$. The Bergman space $A^p(G)$ is the closed subspace of $L^p(G)$ consisting of analytic functions. If $1 \leq p < \infty$, these are Banach spaces and if $0 < p < 1$ they are quasi-Banach spaces. We will allow all $0 < p < \infty$ and abuse the terminology by calling $\| \cdot \|_p$ a norm even when $p < 1$. In the case where $G = \mathbb{D}$, the open unit disk, we will abbreviate: $L^p = L^p(\mathbb{D})$, $A^p = A^p(\mathbb{D})$ and $\| \cdot \|_p = \| \cdot \|_{p, \mathbb{D}}$.

Let $\psi(z, \zeta)$ denote the pseudohyperbolic metric:

$$\psi(z, \zeta) = \left| \frac{z - \zeta}{1 - \zeta \overline{z}} \right|.$$  

We will use $D(z, r)$ for the pseudohyperbolic disk of radius $r$ centered at $z$, that is, the ball of radius $r < 1$ in the pseudohyperbolic metric. Let $\rho(z) = 1 - |z|$, denote the distance of a point to the boundary of $\mathbb{D}$, and, for a set $S$ let $\rho(S) = \max \{ \rho(z) : z \in S \}$. Moreover, let $d\lambda(z) = (1 - |z|^2)^{-2}dA(z)$ denote the invariant area measure on $\mathbb{D}$.

We use $|S|$ to denote the Euclidean area of a set $S$ and we recall that $|D(z, r)|$ behaves like $(1 - |z|)^2$ as $|z| \to 1$ for fixed $r < 1$, and like $r^2$ as $r \to 0$ for fixed $z$. We also recall that $\rho(D(z, r))$ behaves like $1 - |z|$ for fixed $r$.

Let $Z = \{ z_k : k = 1, 2, 3, \ldots \}$ be a sequence in $\mathbb{D}$ without limit points in $\mathbb{D}$. The sequence $Z$ is said to be uniformly discrete if there is a lower bound on the pseudohyperbolic distance between entries. That is, there is an $\epsilon > 0$ such that

$$\psi(z_k, z_n) > \epsilon \quad \text{for all } k \neq n.$$  

A uniformly discrete sequence necessarily contains no repeated values.

The usual $A^p$ interpolation problem for $Z$ is the following: given a sequence $w = \{ w_k \}$ of complex numbers satisfying $\sum_{j=1}^{\infty} |w_j|^p (1 - |z_j|^2)^2 < \infty$, find a function $f \in A^p$ such that $f(z_k) = w_k$ for all $k$. The sequence $Z$ is called an interpolating sequence for $A^p$ if every such interpolation problem has a solution. It is well known that interpolating sequences have to be uniformly discrete. K. Seip characterized interpolating sequences for $A^p$ as those that are uniformly discrete and in addition satisfy a certain bound on the upper uniform density (to be defined in section 5): $D^+ (Z) < 1/p$.

M. Krosky and A. Schuster [3] investigated multiple interpolation, in which not only the values of $f$ at $z_k$ but also the values of its derivatives at $z_k$ can be prescribed. The multiple interpolation
problem is the following: given a sequence $w = (w_k)$ of $n$-tuples $w_k = (w_k^{(0)}, w_k^{(1)}, \ldots, w_k^{(n-1)})$ satisfying $\sum_{k=1}^{\infty} \sum_{b=0}^{n-1} |w_k^{(l)}|^p (1 - |z_k|^2)^{p+2}$, find $f \in A^p$ such that $f^{(l)}(z_k) = w_k^{(l)}$ for all $0 \leq l \leq n-1$ and all $k \geq 1$. A sequence $Z$ is called a multiple interpolating sequence of order $n$ for $A^p$ if this interpolation problem has a solution for every such sequence $w$.

Krosky and Schuster showed that multiple interpolating sequences $Z$ are characterized by being uniformly discrete and satisfying the density bound $D^+(Z) < 1/(np)$. However, if we encode the multiplicity of the interpolation by repeating each distinct point $z_k$ in $Z$ the appropriate number of times, then the density condition is the same as Seip’s: $D^+(Z) < 1/p$. The resulting sequence $Z$ is not uniformly discrete, only the sequence of distinct values is.

In this paper we will to remove even this requirement that $Z$ have uniformly discrete values. We will formulate a more general interpolation problem in $A^p$, and show that any sequence (not necessarily uniformly discrete) is an interpolating sequence for this problem if and only if $D^+(Z) < 1/p$. This will cover the case of multiple interpolation at repeated points, but it will also include consideration of distinct points without a positive lower bound on $\psi(z_k, z_n)$.

In [7] this author showed that the usual interpolating sequences could also be characterized in terms of a $\bar{\partial}$-problem in a suitable weighted $L^p$ space. The function

$$ k_\alpha(z) = \sum_{a \in Z} k_\alpha(\zeta), \quad \text{with} \quad k_\alpha(\zeta) = \frac{|\alpha|^2 (1 - |\alpha|^2)^2}{|1 - \alpha z|^2} $$

completely characterizes zero sequences for $A^p$ ([5]) in the following way: $Z$ is a zero sequence for $A^p$ if and only if the weighted space $L^p_Z = L^p(\mathbb{D}, \exp(pk) dA)$ contains nonzero analytic functions. We pose the following weighted $\bar{\partial}$-problem: given $f \in L^p_Z$, find a function $u \in L^p_Z$ satisfying

$$(1 - |z|^2)\bar{\partial}u(z) = f(z)$$

in $\mathbb{D}$. We will say that the $\bar{\partial}$-problem has solutions with bounds in $L^p_Z$ if there is a constant $C$ such that for any $f \in L^p_Z$ there is a solution $u$ to (1.1) in $L^p_Z$ satisfying $\|u\|_{L^p,Z} \leq C \|f\|_{L^p,Z}$. The notation $\|\cdot\|_{L^p,Z}$ denotes the obvious norm for $L^p_Z$.

The main result of [7] is that $Z$ is an interpolating sequence for $A^p$ if and only if it is uniformly discrete and the $\bar{\partial}$-problem has solutions with bounds in $L^p_Z$. In case $p < 1$ one has to modify the space $L^p_Z$ so that its elements are locally integrable in a suitable way, but otherwise the result is valid for all $0 < p < \infty$.

In this paper we will show that for the newly formulated general interpolation problem, any sequence having bounded density is interpolating for $A^p$ if and only if the $\bar{\partial}$-problem has solutions with bounds in $L^p_Z$. A sequence has bounded density if for a given $0 < R < 1$ the cardinality of the set of $k$ with $z_k \in D(z, R)$ has an upper bound independent of $z$.

In the next section we will formulate the general interpolation problem for $A^p$ and define general interpolating sequences for $A^p$ to be those where all such problems have a solution.

In section 3 we will prove some basic properties of general interpolating sequences for $A^p$. This will include two necessary conditions that are so basic to interpolation that we will call sequences that satisfy them admissible sequences. Then in section 4 we show that, in fact, sequences are admissible if and only if they have bounded density.

In section 5 we will show that if a sequence $Z$ is a general interpolating sequences for $A^p$, then it satisfies the mentioned upper density inequality $D^+(Z) < 1/p$. We will complete the proof in section 6, showing that the density condition allows us to construct a solution operator for the $\bar{\partial}$-equation, and solutions of the $\bar{\partial}$-equation allow us to construct solutions of any interpolation problem. Our proof that the density condition implies solutions of the $\bar{\partial}$-equation is more direct and simpler than our proof of a similar implication in [7]. The proof does not appeal to J. Ortega-Cerdà’s result on weighted $\bar{\partial}$-problems ([8]) but does make use of properties of the weight $e^{pkz}$ that Ortega-Cerdà’s work does not assume. Apart from this, once the main properties of general interpolating sequences are established, the arguments of sections 5 and 6 do not differ much from those in [7].

In the final section we supply a few examples of general interpolation problems and the associated interpolating sequences. We will also observe that the methods apply without significant change to the usual weighted Bergman spaces (with the $L^p(dA_n)$ norm, where $dA_n(z) = (1 - |z|^2)^n dA(z)$,
\( \alpha > -1 \). The applicable density condition is then \( D^+(Z) < (\alpha + 1)/p \) and the appropriate space for the \( \bar{\partial} \)-problem is \( L^p(\mathbb{D}, \exp(pkz) \, dA_n) \).

2. The interpolation process

Before formulating the interpolation process, we answer the obvious question: Why are interpolating sequences “always” required to be uniformly discrete? We answer this question via the following proposition. Similar results have been rather vaguely stated in several places, but we have found no convenient reference in the following generality. Let us define \( M_a(z) = (a - z)/(1 - a \bar{z}) \), the Moebius transformation of \( \mathbb{D} \) that takes \( a \) to 0 (and 0 to \( a \)). Let \( e^{(k)} \) denote the sequence having a 1 in position \( k \) and 0 elsewhere. Let \( P_k \) be the operator of projection onto the \( k \)-th component: if \( w = (w_j) \) then \( P_k(w) = w_k e^{(k)} \).

**Proposition 2.1.** Let \( A \) be a quasi-Banach space of analytic functions on \( \mathbb{D} \) and \( Z = \{ z_k : k = 1, 2, \ldots \} \) a sequence of distinct points in \( \mathbb{D} \). Let \( X \) a quasi-Banach sequence space, the sequences being indexed the same as \( Z \). Assume the following:

1. For every \( k \) the sequences \( e^{(k)} \) belongs to \( X \).
2. For every \( k \), \( P_k \) takes \( X \) continuously into \( X \) and \( \sup_k \| P_k \| < \infty \).
3. There is a constant \( C_A \) such that for any \( k \), if \( f \in A \) and \( f(z_k) = 0 \), then \( f/M_{z_k} \in A \) and \( \| f/M_{z_k} \| \leq C_A \| f \|_A \).

If the operator \( \Phi \) taking \( A \) to sequences via \( \Phi(f)_k = f(z_k) \) is continuous and onto \( X \), then \( Z \) is uniformly discrete.

**Proof.** By the open mapping principle there is a constant \( K \) such that for every sequence \( w \in X \) there is a function \( f \in A \) such that \( \Phi(f) = w \) and \( \| f \|_A \leq K \| w \|_X \).

Let \( z_k, z_n \in Z, k \neq n \). The assumptions imply that the sequence \( w = M_{z_n}(z_k)e^{(k)} \) belongs to \( X \). Let \( f \in A \) with \( \Phi(f) = w \) and \( \| f \|_A \leq K \| w \|_X \). Since \( f \) vanishes at \( z_n \) we have \( g = f/M_{z_n} \in A \). Note that \( g(z_k) = 1 \), and so \( P_k(\Phi(g)) = e^{(k)} \). Then we have the following inequalities:

\[
\| e^{(k)} \| \leq C_1 \| \Phi(g) \| \leq C_2 \| g \| \leq C_3 \| f \| \leq C_4 \| w \| = C_4 \| M_{z_n}(z_k) \| e^{(k)} \|
\]

where \( C_1 = \sup_k \| P_k \|, C_2 = C_1 \| \Phi \|, C_3 = C_A C_2, \) and \( C_4 = KC_3 \). We immediately obtain \( \psi(z_k, z_n) = |M_{z_n}(z_k)| \geq 1/C_4 \).

So the keys to uniform discreteness are the lattice properties of the usual sequence spaces and the ability to divide out zeros in a uniform way in the usual function spaces. We note that the sequence space \( X \) in the previous proof could be vector-valued with essentially the same result. For example, the multiple interpolation problem is governed by the above proposition. Partly for this reason it seems impossible to do away completely with some sort of separation condition (see Theorem 3.2).

Now consider the multiple interpolation problem (of which the simple interpolation problem is the special case \( n = 1 \)) from our introduction. In that problem we have associated to each distinct point \( z_k \) in \( Z \) a finite dimensional space, namely \( \mathbb{C}^n \). There is also an associated linear interpolation operator taking analytic functions \( f \) into that space, namely the evaluation of \( f \) and its first \( n - 1 \) derivatives at \( z_k \).

An equivalent way to do multiple interpolation is to replace \( \mathbb{C}^n \) with a quotient space: all functions analytic in a neighborhood \( G_k \) of \( z_k \) divided by the ideal of all such functions that have a zero of order at least \( n \) at \( z_k \). And replace the operator of evaluating \( f \) and its derivatives at \( z_k \) with that of restricting \( f \) to \( G_k \) and applying the quotient mapping. One need only determine a suitable norm on this quotient space and a natural choice (given that we are investigating interpolation in \( A^p \)) would be the quotient norm in \( A^p(G_k) \). This requires us to make a choice of the neighborhood \( G_k \).

It is relatively straightforward (see section 7) to verify that a choice producing a norm equivalent to the norm used in the introduction is to let each \( G_k \) be a pseudohyperbolic disk \( D(z_k, R) \) for some fixed radius \( R < 1 \).

Once it has been shown that a multiple interpolating sequence must be uniformly discrete, it is possible to replace the radius \( R \) chosen with one making the \( G_k \) disjoint, but this is by no means necessary for defining the quotient space and its norm.
For our interpolation process we will start with a sequence of domains $G_k$. If we let $R_k$ denote the diameter of $G_k$ in the pseudohyperbolic metric, we will require an upper bound $\sup_k R_k = R < 1$. We also require, of course, that $Z \subset \bigcup_k G_k$, but do not require that the $G_k$ be disjoint. Then we partition the sequence $Z$ into disjoint finite subsequences $Z_k$ where each $Z_k$ belongs to $G_k$. We will refer to the $Z_k$ as clusters. We require that a repeated point of $Z$ has all its repetitions in the same cluster. Finally, we require that $G_k$ and $Z_k$ be chosen so that $(Z_k)_e \subset G_k$ for some $\epsilon > 0$ independent of $k$, where the notation $(S)_e$ denotes the $\epsilon$-neighborhood of a set $S$ in the pseudohyperbolic metric.

As we have seen, evaluating a function and its derivatives at a point is equivalent to restricting the function to a neighborhood and applying a quotient mapping. In this more general setting, we associate with each cluster $Z_k$ the subspace $N_k = N(Z_k, G_k)$ of $A^p(G_k)$ consisting of all functions whose zero set contains $Z_k$ (counting multiplicity) and the corresponding finite dimensional quotient space $E_k = E(Z_k, G_k) = A^p(G_k)/N_k$. The norm $\| \cdot \|_{E_k}$ on $E_k$ is the quotient norm, that is, for $w_k \in E_k$

$$\|w_k\|_{E_k}^p = \inf \int_{G_k} |g|^p dA$$

where the infimum is taken over all functions $g$ in the equivalence class $w_k$. A function $f$ interpolates $w_k$ if $f|_{G_k}$ belongs to the equivalence class $w_k$.

We could equivalently take $E_k$ to be $\mathbb{C}^{n_k}$ (where $n_k$ is the cardinality of $Z_k$ counting repetitions), and take the operator to be evaluation of functions (and derivatives) at points of $Z_k$, but we would still define the norm of the $n_k$-tuple $w_k$ in the same way, taking the infimum over all $g$ whose values and derivatives produce $w_k$.

One can do interpolation with almost any choice of finite dimensional spaces $E_k$, together with a choice of norms on those spaces and a choice of interpolation operator taking functions in $A^p$ into these spaces.

In this paper, we will always define the spaces as above, always define the norms as above, and always define the interpolation operator as above. Thus, our interpolation process will depend only on the choice of open sets $G_k$ and the choice of subdivision into clusters $Z_k$. We define the term interpolation scheme to mean a particular choice of domains and a particular subdivision into clusters satisfying the following two requirements:

1. there exists $\epsilon > 0$ such that $(Z_k)_e \subset G_k$ for every $k$, and
2. there exists $0 < R < 1$ such that for every $k$ the pseudohyperbolic diameter of $G_k$ is no more than $R$.

It is convenient to assign a name such as $I$ to an interpolation scheme, especially when more than one such scheme is under discussion. We will usually denote by $E_k$ the finite dimensional space defined and normed as above and call them the target spaces of the scheme $I$. We will refer to the $Z_k$ as the clusters of $I$, the $G_k$ as the domains of $I$, the upper bound $R$ as the diameter of $I$ and the largest $\epsilon$ with $(Z_k)_e \subset G_k$ as the inner radius of $I$.

Given an interpolation scheme $I$, the general interpolation problem for $A^p$ is the following: given a sequence $(w_k)$ with $w_k \in E_k$ for each $k$, satisfying $\sum_k \|w_k\|_{E_k}^p < \infty$, find a function $f \in A^p$ such that $f|_{G_k}$ belongs to the equivalence class $w_k$ for every $k$. We call $Z$ an interpolating sequence for $A^p$ relative to the scheme $I$ if every such problem has a solution.

A sequence $Z$ can be general interpolating for some choices of $I$ and not for others. However, we will soon see that if a sequence is interpolating relative to a scheme $I$, then two additional conditions will have to be satisfied. We will call schemes that satisfy these necessary conditions “admissible” and then it will turn out that a sequence is interpolating relative to one admissible scheme if and only if it is interpolating relative to any other admissible scheme. Hence we will sometimes omit the modifier “relative to the scheme $I$”.

We will show that a general interpolating sequence requires that there be a positive lower bound on the pseudohyperbolic distance between distinct clusters, and a finite upper bound on the number of points in each cluster. These two together imply that the sequence $Z$ has bounded density. Conversely, we will see that for any sequence with bounded density there is an admissible interpolation scheme. We will also describe a simple way to select an admissible scheme.
We point out now that the purpose of the condition $(\mathcal{Z}_k)_e \subset G_k$ is to ensure that the norm assigned to $E_k$ is reasonable. Without such a lower bound on the distance to the boundary of $G_k$, there can be functions in $A^p(G_k)$ with arbitrarily small norm but with arbitrarily large values. While this could be a perfectly valid problem, it would make interpolation by functions in $A^p$ impossible.

Given an interpolation scheme $\mathcal{I}$, let us denote by $X^p_\mathcal{I}$ the $l^p$ direct sum of the $E_k$. That is, all sequences $w = (w_k)$ with $w_k \in E_k$ satisfying

$$\|w\|_{X^p_\mathcal{I}} = \sum_{k=1}^{\infty} \|w_k\|_{E_k} < \infty.$$  

Defining a sequence in $X^p_\mathcal{I}$ is equivalent to selecting functions $f_k$ analytic in each $G_k$ with

$$\sum_{k=1}^{\infty} \int_{G_k} |f_k|^p \, dA < \infty.$$  

Of course there always exist minimizing functions such that $\|f_k\|_{p,G_k} = \|w_k\|_{E_k}$. Interpolating by a function $f \in A^p$ then means that $f_k - f|_{G_k} \in N(\mathcal{Z}_k,G_k)$ for all $k$.

### 3. Properties of interpolating sequences

Our first observation is that interpolating sequences are zero sequences. For a nonzero analytic function $f$ we write $\mathcal{Z}(f)$ for the sequence of zeros of $f$, repeated according to multiplicity.

**Proposition 3.1.** Given an interpolation scheme $\mathcal{I}$ with domains $G_k$ and clusters $\mathcal{Z}_k$, if $\mathcal{Z} = \bigcup_k \mathcal{Z}_k$ is general interpolating sequence for $A^p$, then there is a function $f \in A^p$ such that $\mathcal{Z}(f) = \mathcal{Z}$.

**Proof.** Choose a $k_0$ such that $\mathcal{Z}_{k_0}$ is not empty. Define a sequence in $X^p_\mathcal{I}$ by selecting the function $f_{k_0} \equiv 1$ on $G_{k_0}$ and $f_k \equiv 0$ on all other $G_k$, $k \neq k_0$.

If $\mathcal{Z}$ is a general interpolating sequence for $A^p$ then there is a function $g \in A^p$ which has value 1 at all points of $\mathcal{Z}_{k_0}$ and 0 at all other points (with multiplicity at least as great as that of $\mathcal{Z}$). Multiply $g$ by a suitable polynomial to obtain a nontrivial function $f$ in $A^p$ with $\mathcal{Z} \subset \mathcal{Z}(f)$. Since a subsequence of an $A^p$ zero sequence is also an $A^p$ zero sequence (Horowitz [2]) we have the result. □

The following is a crucial property of general interpolating sequences. We could prove it by generalizing the proof of Proposition 2.1, except that we have not yet established that the interpolation mapping from $A^p$ into $X^p_\mathcal{I}$ is bounded.

**Theorem 3.2** (Separation). Given an interpolation scheme $\mathcal{I}$ with clusters $\mathcal{Z}_k$, if $\mathcal{Z} = \bigcup_k \mathcal{Z}_k$ is a general interpolating sequence for $A^p$ then there is a lower bound $\delta > 0$ on the pseudohyperbolic distance between different clusters of $\mathcal{I}$.

**Proof.** Suppose not; let $\{a_j\}$ and $\{a'_j\}$ be subsequences of $\mathcal{Z}$ such that $a_j$ and $a'_j$ lie in different clusters $\mathcal{Z}_j$ and $\mathcal{Z}'_j$ and such that $\psi(a_j,a'_j) = \epsilon_j \to 0$. Functions in $A^p$ satisfy the following inequality: For a fixed positive $r < 1$ there is a constant $C_r$ such that

$$|f'(z)(1-|z|^2)|^p \leq \frac{C_r}{|D(z,r)|} \int_{D(z,r)} |f|^p \, dA.$$  

Suppose $f$ has the value $\gamma_j$ at $a_j$ and 0 at $a'_j$ for each $j$. Then there exists a point $\zeta_j$ on the line segment connecting $a_j$ and $a'_j$ where $|f'(|\zeta_j|)| \geq |\gamma_j|/|a_j - a'_j|$. Combining this with the above inequality gives

$$|\gamma_j|^p(1-|\zeta_j|^2)^2 \leq \frac{|a_j - a'_j|^p}{(1-|\zeta_j|^2)^p} \frac{C_r(1-|\gamma_j|^2)^2}{|D(\zeta_j,r)|} \int_{D(\zeta_j,r)} |f|^p \, dA.$$  

The first fraction above is $O(\epsilon_j^p)$, the second is $O(1)$. Therefore, if $f \in A^p$ and the disks $D(\zeta_j,r)$ are disjoint then

$$\sum_j \epsilon_j^{-p} |\gamma_j|^p(1-|\zeta_j|^2)^2 \leq C\|f\|_p^p < \infty.$$  

(3.1)
The disjointness of $D(\zeta_j, r)$ can easily be arranged by passing to a subsequence if necessary. Select $\gamma_j$ so that $\sum |\gamma_j|^p (1 - |\zeta_j|^2)^2 < \infty$ but $\sum |\gamma_j|^p (1 - |\zeta|^2)^2 = \infty$. Define an element of $X^p_T$ by selecting functions that are constant on each domain $G_k$, equal to $\gamma_j$ on the domain $G_j$ that contains $a_j$ for all $j$, and equal to 0 on every other domain.

The sequence so defined clearly belongs to $X^p_T$, but because of inequality (3.1) it cannot be interpolated by a function in $A^p$. \hfill \Box

If we trim our interpolation scheme $T$ by eliminating those domains $G_k$ where $Z_k$ is empty, it follows from the previous proposition that the domains $G_k$ have a bounded amount of overlap. That is, the function $\sum_k \chi_{G_k}$ is a bounded function. For if there is a point $b \in \mathbb{D}$ contained in $N$ of these sets, there would be at least $N$ points from different $Z_k$ in the disk $D(b, R)$, where $R$ is the diameter of the scheme $T$. By the proposition there is a $\delta > 0$ such that these $N$ points are all separated by $\delta$. These estimates are in contradiction if $N$ can be made arbitrarily large.

For a function $f \in A^p$ we have $\sum f_{G_k} \|f\|^p \leq \|\sum \chi_{G_k}\|^p \|f\|^p$. Thus, the interpolator operator described above (taking $f$ to the sequence of equivalence classes of $f|_{G_k}$) is bounded from $A^p$ into $X^p_T$. By the open mapping theorem, if $Z$ is a general interpolating sequence for $A^p$ then there is an finite interpolation constant $K$. That is, for all $w \in X^p_T$, there is a function $f_w$ in $A^p$ which interpolates $w$ and satisfies the inequality $\|f_w\| \leq K \|w\|_{X^p_T}$.

We continue with the exposition of several properties of general interpolating sequences. For example, they are hereditary, Möbius invariant and finitely extendable.

**Proposition 3.3 (Hereditary).** Given an interpolation scheme $T$ with clusters $Z_k$ and domains $G_k$ let $J$ have clusters $W_k \subset Z_k$ and the same domains. If $Z = \bigcup Z_k$ is a general interpolating sequence for $A^p$ with interpolation constant $K$, relative to $T$, then $W = \bigcup W_k$ is a general interpolating sequence for $A^p$ relative to $J$. Its interpolation constant is at most $K$.

**Proof.** Assume $Z$ is a general interpolating sequence and $u = (u_k) \in X^p_T$. Let $E_k$ be the target spaces of $T$ and $F_k$ denote the target spaces of $J$: $F_k = A^p(G_k)/N(W_k, G_k)$. For each $k$, select a function $g_k$ in the equivalence class $u_k$ such that $\|g_k\|^p_{p, G_k} = \|u_k\|_{F_k}$. The quotient mapping taking $g_k$ into $v_k \in E_k$ then satisfies $\|u_k\|_{E_k} \leq \|g_k\|_{p, G_k} = \|u_k\|_{F_k}$. So $(v_k) \in X^p_T$. If $f \in A^p$ interpolates $(v_k)$ that means that $g_k - f|_{G_k} \in N(Z_k, G_k) \subset N(W_k, G_k)$ and so $f$ interpolates $(u_k)$. \hfill \Box

If we select one point from each cluster, it is easy to see that we get a traditional interpolation problem. This is essentially how we approached the separation between clusters: by examining a point from each cluster.

**Proposition 3.4 (Invariant).** Let $T$ be an interpolation scheme with clusters $Z_k$ and domains $G_k$. If $Z = \bigcup Z_k$ is a general interpolating sequence and $\varphi$ is a conformal map from $\mathbb{D}$ onto itself, then $\varphi(Z)$ is a general interpolating sequence relative to the scheme $\varphi(T)$ which has clusters $\varphi(Z_k)$ and domains $\varphi(G_k)$. Moreover, both sequences have the same interpolation constant.

**Proof.** The map $f \rightarrow f \circ \varphi \cdot (\varphi')^{2/p}$ is an isometry of $A^p$ onto itself, but also an isometry of $A^p(\varphi(G_k))$ onto $A^p(G_k)$ for each $k$. Moreover, it takes $N(\varphi(Z_k), \varphi(G_k))$ to $N(Z_k, G_k)$ so it induces an isometry of the spaces $X^p_T(\mathbb{D})$ and $X^p_T$. Consequently, an interpolation problem in one setting transforms (isometrically) into an equivalent one in the other and the solution in $A^p$ transforms (isometrically) back into a solution in $A^p$. \hfill \Box

The following constants associated with an interpolation scheme $T$ are invariant under conformal maps: the diameter, the inner radius, the separation between clusters, and the interpolation constant. Within the context of $A^p$ interpolation, $p$ may also be considered to be invariant. We will refer to any constant associated to an interpolation scheme that is invariant under conformal maps as an interpolation invariant or simply an invariant of $T$.

**Proposition 3.5 (Extendable).** Let $T$ be an interpolation scheme with clusters $Z_k$ and domains $G_k$. Suppose $Z = \bigcup Z_k$ is a general interpolating sequence and let $z_0 \in \mathbb{D}$. Suppose there is an $\epsilon > 0$ such that $\psi(z_0, Z_k) > \epsilon$ for every $k$. Define a new scheme $J$ whose domains are all the domains of
If \( K \) is the interpolation constant for \( Z \) then the constant for \( W \) is at most \( C_p K / \epsilon \), where \( C_p \) is a positive constant that depends only on \( p \).

We will normally use this with \( \epsilon = 1/4 \) or 1/2 and in that case we see that the interpolation constant of the new sequence can be estimated in terms of invariants of \( \mathcal{I} \).

**Proof.** Because of the previous proposition, we may assume \( z_0 = 0 \). Note that \( E_0 \), the quotient space associated with \( 0 \in G_0 \), is one dimensional. Let \( u = (u_k) \) be a sequence in \( X^p_{p,G_k} \) with \( \|u\|_{X^p_{p,G_k}} = 1 \) Let \( (g_k) \) be a sequence of representative functions satisfying \( \|g_k\|_{p,G_k} = \|u_k\|_{E_k}, k \geq 0 \). We can identify \( u_0 \) with a constant function \( g_0 \) (this minimizes the \( L^p \) norm on a disk) and so it makes sense to consider the functions \( f_k(z) = (g_k(z) - g_0)/z \) on each \( G_k, k \geq 1 \). One easily estimates
\[
\|f_k\|_{p,G_k}^p \leq \frac{C_p}{\epsilon^p} \left( \|g_k\|_{p,G_k}^p + |g_0|^p |G_k| \right), \quad k \geq 1,
\]
and so \( f_k \) represents an element of \( X^p_{p,F_k} \). Let \( f \) interpolate this element, with \( \|f\|_p \leq K^p \sum \|f_k\|_{p,G_k}^p \).

Finally, let \( h(z) = z f(z) + g_0 \) and observe that it interpolates \( u_k \) for \( k \geq 1 \) but also satisfies \( h(0) = g_0 \). Since \( |g_0|^p = 4 |g_0|_{p,G_0}^p \leq 4 \|u\|_{X^p_{p,G_0}}^p \), we get the stated estimate on the interpolation constant. \( \square \)

We will say that the sequence \( Z \) has bounded density if there is a positive constant \( M \) and a fixed radius \( 0 < R < 1 \) such that for each \( z \in \mathbb{D} \), the disk \( D(z,R) \) contains no more than \( M \) terms (counting multiplicity) of the sequence \( Z \). The number \( M \) will depend on \( R \), but the actual value of \( R \) will not affect whether a sequence has bounded density.

The next theorem implies that a general interpolating sequence must have bounded density.

**Theorem 3.6** (Bounded Density). If \( Z \) is a general interpolating sequence for \( A^p \) relative to an interpolation scheme \( \mathcal{I} \) then there is a finite upper bound \( B \) on the number of points (counting multiplicity) in each cluster \( Z_k \) of \( \mathcal{I} \). The bound \( B \) is obviously an invariant of \( \mathcal{I} \).

**Proof.** Let \( B_k \) be the cardinality of \( Z_k \) counting multiplicity and suppose \( B_k \) is not bounded.

For each \( G_k \), either there exists another domain \( G'_k \) with \( \psi(G_k,G'_k) < 1/2 \) or there is a pseudo-hyperbolic disk \( D_k \) of radius \( 1/4 \) disjoint from all \( G_j \) and satisfying \( \psi(G_k,D_k) < 1/2 \). In the first case let \( a_k \) be any point in \( Z_k' \) (the cluster contained in \( G'_k \)); in the second case let \( a_k \) be the center of the disk. Let \( \varphi_k \) be the conformal map which takes \( a_k \) to the origin. Observe that there is a radius \( R' < 1 \) depending only on the diameter \( R \) of \( \mathcal{I} \), such that the number of elements \( \varphi_k(Z) \) inside \( D(0,R') \) is unbounded.

For each \( k \) we have an obvious scheme \( \mathcal{I}_k \) associated with \( \varphi_k(Z) \cup \{0\} \): either \( \varphi_k(\mathcal{I}) \) or \( \varphi_k(\mathcal{I}) \) together with the cluster \( \{0\} \) and the domain \( D(0,1/4) \). We can define an element of \( X^p_{p,\mathcal{I}_k} \) by choosing functions that are 1 in the domain containing the origin and zero in all other domains. The norm of this element is at most one 1 and, by propositions 3.3 through 3.5, there exist a bound depending only on \( p \) and \( K \) (independent of \( k \)) on the \( A^p \) norm of an interpolating function for this element. Thus we obtain a bounded sequence \( f_k \) in \( A^p \) such that \( f_k(0) = 1 \) and the number of zeros in the relatively compact disk \( D(0,R') \) is unbounded. The former condition prevents any subsequence of \( f_k \) from converging to 0 uniformly on compacts, while the latter condition implies that there does exist such a subsequence. This contradiction proves \( B_k \) is bounded. \( \square \)

In fact, given an upper bound on the \( A^p \) norm of a function \( f_k \) and a lower bound on its value at 0 (namely 1), Jensen’s formula provides an upper bound on the number of zeros of the function in \( D(0,R') \). Therefore the bound \( B = \sup_k B_k \) actually depends only on \( p \), \( K \) and \( R \).

To see that a general interpolation scheme has bounded density, let \( \delta > 0 \) be less than half the distance between distinct clusters. If \( D(z,R) \) is some pseudo-hyperbolic disk, the disjointness of \( (Z_k)_{\delta} \) provides an upper bound (depending only on \( \delta \) and \( R \)) on the number of clusters that meet \( D(z,R) \). Since there is an upper bound on the number of points in each cluster, we get an upper bound on the number of points of \( Z \) in \( D(z,R) \). Clearly this upper bound (for fixed \( R \)) is also an interpolation invariant.
4. Admissible interpolation schemes

Let $\mathcal{I}$ be an interpolation scheme with domains $G_k$ and clusters $Z_k$. The following lists the two properties of $\mathcal{I}$ that we required in the definition of an interpolation scheme, together with the properties obtained in Theorems 3.2 and 3.6 for general interpolation sequences.

(P1) There is an $R < 1$ such that the pseudohyperbolic diameter of each $G_k$ is at most $R$.

(P2) There is an $\epsilon > 0$ such that $(Z_k)_\epsilon \subset G_k$.

(P3) There is a $\delta > 0$ such that for all $j \neq k$ the pseudohyperbolic distance from $Z_j$ to $Z_k$ is at least $\delta$.

(P4) There is an upper bound $B$ on the number of points (counting multiplicity) in each cluster $Z_k$.

We will say that an interpolation scheme is admissible if it satisfies, in addition to (P1) and (P2), the conditions (P3) and (P4). We will call a sequence $Z$ admissible if there is an admissible scheme with $Z = \bigcup_k Z_k$. We will refer to $\delta$ as a separation constant of $\mathcal{I}$ and $B$ as a cluster bound.

Verifying that a sequence is admissible would seem to require examining a great many possible choices of interpolation schemes. However, the following shows that the class of admissible sequences coincide with an already very familiar class of sequences.

**Theorem 4.1.** A sequences is admissible if and only if it has bounded density.

**Proof.** We saw earlier that properties (P3) and (P4) imply bounded density. For the converse, assume $Z$ has bounded density.

Pick any $0 < r_0 < 1$ and let $B$ be a bound on the number of points in $D(z, r_0), z \in \mathbb{D}$. We will show that there exists $\epsilon > 0$ such that we can take $G_k$ to be the connected components of $(Z)_\epsilon$ and $Z_k = Z \cap G_k$. Property (P2) is obvious, and so is (P3) with $\delta = 2\epsilon$.

If we are able to show (P1) for these connected components, then (P4) will trivially follow since $Z$ has bounded density. So we are reduced to showing that for sufficiently small $\epsilon > 0$ a connected component of $(Z)_\epsilon$ has diameter at most $R < 1$. We do this by showing that each $G_k$ is contained in some disk $D(z, r_0)$, and then $R$ will be $2r_0/(1 + r_0^2)$, the diameter of $D(z, r_0)$.

Let $G_k$ be some component of $(Z)_\epsilon$. Clearly $G_k = (Z_k)_\epsilon$ for some subset $Z_k$ of $Z$. Select a point $z_0 \in Z_k$ and consider circles $\Gamma_j$, each being the boundary of $D(z_0, \epsilon_j)$ where $\epsilon_1 < \epsilon_2 < \cdots < \epsilon_{B+1}$. Choose these radii so that $\epsilon_1 = \epsilon$ and $\epsilon_{j+1} = (\epsilon_j + \epsilon')/(1 + \epsilon_j \epsilon')$, where $\epsilon'$ is the pseudohyperbolic diameter of $D(z, \epsilon)$. Note that a disk $D(z, \epsilon)$ cannot intersect more than one of these circles.

Now select $\epsilon$ so small that $\epsilon_{B+1} \leq r_0$ and assume, for the purpose of contradiction, that $G_k$ is not contained in $D(z_0, r_0)$. In that case, $G_k$ must intersect all of the $\Gamma_j$ and so there must exist points $z_j, 1 \leq j \leq B$ with the disk $D(z_j, \epsilon)$ intersecting $\Gamma_j$. This forces $B + 1$ points ($z_0$ through $z_B$) of $Z$ to be inside of $D(z_0, r_0)$, where there are assumed to be only $B$ points. This contradiction implies that $G_k$ is contained in $D(z_0, r_0)$. $\square$

Given a sequence $Z$ with bounded density, there are essentially two extreme choices of domains $G_k$ satisfying properties (P1) through (P4). A “minimal one” is to let $G_k$ be the connected components of $(Z)_\epsilon$ for sufficiently small $\epsilon$; a “maximal one” is to let $G_k = D(z_k, r)$ where $D(z_k, r)$ is a disk containing the $k$th connected component of $(Z)_\epsilon$. These differ in the norm imposed on the finite dimensional spaces $E_k$. One suspects that these norms might impose equivalent norms on $X_2^p$ (but we do not have a proof). We will however show that whether or not $Z$ is a general interpolating sequence for $A^p$ is independent of the choice. If $Z$ is a general interpolating sequences then the norms $\| \cdot \|_{X_2^p}$ are equivalent for all admissible $\mathcal{I}$, for in that case they are all equivalent to the quotient space norm in $A^p/N(Z)$ where $N(Z)$ is the subspace of functions vanishing on $Z$.

5. General interpolating sequences, density and the $\bar{\partial}$-problem

The following is one of the main steps in nearly all characterizations of interpolating sequences: some key property is preserved under small perturbations of the sequence. In the present context we have the mild additional complication that a perturbation of a sequence also perturbs the interpolation scheme chosen for it.
Proposition 5.1 (Stability). Let $I$ be an admissible interpolation scheme with domains $G_k$, clusters $Z_k$ and diameter $R$. Assume $Z = \bigcup Z_k$ is a general interpolation sequence for $\mathcal{I}^p$ with interpolation constant $K$. For each $k$ let $\varphi_k$ be defined by $\varphi(z) = r_kz$ and let $J$ be the interpolation scheme with domains $\varphi_k(G_k)$ and clusters $W_k = \varphi_k(Z_k)$. Let $W = \bigcup W_k$.

There exists an $\eta > 0$ depending only on interpolation invariants such that if $\psi(\varphi_k(z), z) < \eta$ for all $z \in G_k$ and for all $k$, then $W$ is a general interpolation sequence for $\mathcal{I}^p$ relative to $J$. Its interpolation constant can be estimated in terms of $\eta$ and interpolation invariants of $I$.

Proof. It is clear that if we choose $\eta$ small enough, then $J$ is admissible since all points of $G_k$ are moved a pseudohyperbolic distance at most $\eta$. We need only take $\eta < \delta/4$, where $\delta$ is a separation constant of $I$, and then $\delta/2$ is a separation constant for $J$.

As usual, let $E_k = \mathcal{I}^p(G_k)/N(Z_k, G_k)$. Let $D_k = \varphi_k(G_k)$ and let $F_k = \mathcal{I}^p(D_k)/N(W_k, D_k)$. Recall $\rho(G_k) = \sup\{1 - |z| : z \in G_k\}$. The hypotheses imply that $1 - r_k$ are small multiples of $\rho(G_k)$.

Let $v = (v_k)$ be any sequence in $X^p_J$. We can transfer this, via $\varphi_k$ to a sequence $u = (u_k)$ in $X^p_I$, let $u_k$ be the element of $E_k$ represented by $f_k \circ \varphi_k$ where $f_k$ represents $v_k \in F_k$. It will be convenient to the omit the factor $(\varphi_k)^{2/p} = r_k^{2/p}$, so this is not an isometry. However, it is still an isomorphism from $X^p_J$ onto $X^p_I$ and there is a constant $C$ with $\|u\|_{X^p_I} \leq C\|v\|_{X^p_J}$.

For $f \in \mathcal{I}^p$, let $\Phi(f)$ denote the element of $X^p_J$ represented by the sequence $(f|_{D_k})$. It suffices to obtain a function $f_v \in \mathcal{I}^p$ such that

\begin{equation}
\|v - \Phi(f_v)\|_{X^p_J} \leq \frac{\|v\|_{X^p_J}}{2}
\end{equation}

and

\begin{equation}
\|f_v\|_p \leq M\|v\|_{X^p_J}
\end{equation}

for some finite $M$. For if this can be done, iterating the process as in [4] produces a sequence $v^{(j)}$ in $X^p_J$ and a corresponding sequence $f^{(j)}$ in $\mathcal{I}^p$ satisfying $\|f^{(j)}\|_p < M\|v\|_{X^p_J}/2^j$ and $\|v - \sum_{j=1}^n \Phi(f^{(j)})\|_{X^p_J} < \|v\|_{X^p_J}/2^n$. Thus, with $f = \sum_j f^{(j)}$, we have $\Phi(f) = v$.

We take $f = f_v$ to be the function that interpolates $(u_k)$ with $\|f\|_p \leq K\|u\| \leq CK\|v\|_{X^p_J}$. Then $(g_k) = (f \circ \varphi_k^{-1})$ represents $v_k$ and it suffices to estimate the difference $\|g_k - f\|_{D_k} \leq D_k$:

\begin{equation}
\|g_k - f\|_{D_k} \leq \int_{D_k} |f(z/r_k) - f(z)|^p \, dA(z).
\end{equation}

If $\eta < 1/4$, we have (see, for example, [4])

\begin{equation}
|f(z/r_k) - f(z)|^p \leq \frac{C_p\eta}{|D(z, 1/2)|} \int_{D(z, 1/2)} |f|^p \, dA,
\end{equation}

Integrating this gives

\begin{equation}
\int_{D_k} |f(z/r_k) - f(z)|^p \, dA(z) \leq C_p\eta \int_{D_k} |f|^p \, dA,
\end{equation}

where $D_k = (D_k)_{1/2}$ is the domain $D_k$ expanded by pseudohyperbolic distance $1/2$. There is a number $B'$ (depending on the diameter and separation constants of $I$) such that at most $B'$ of the domains $D_k$ overlap at any point. Combining equation (5.3) and inequality (5.4) and summing, we have

\begin{equation}
\|v - \Phi(f)\|_{X^p_J} \leq \sum_k \|g_k - f\|_{D_k} \leq C_p\eta \int_{D_k} |f|^p \, dA
\end{equation}

\begin{equation}
\leq C_p\eta \sum_k \int_{D_k} |f|^p \, dA
\end{equation}

\begin{equation}
\leq C_p\eta B'\|f\|_p \leq C_p\eta B'C K\|v\|_{X^p_J}.
\end{equation}

If we take $\eta < 1/(2C_pB'C K)$ we have (5.1) and (5.2) with $M = CK$, as required. □
The definition of upper uniform density can be found in [11]. Here is an equivalent definition. For a sequence $Z$ in the unit disk, let

$$D(Z, r) = \left( \frac{1}{2} \sum_{|z_k| < r} (1 - |z_k|^2) \right) / \left( \log \frac{1}{1 - r^2} \right).$$

and let

$$D^+(Z) = \limsup_{r \to 1} \left[ \sup_{\varphi} D(\varphi(Z), r) \right]$$

where the supremum is taken over all conformal self-maps of $\mathbb{D}$. The quantity $D^+(Z)$ is called the upper uniform density of $Z$.

Because of its connection with the $\bar{\partial}$-problem posed in the introduction, we will in fact use an equivalent version of $D^+(Z)$. Recall that

$$k_Z(\zeta) = \frac{|\zeta|^2}{2} \sum_{k=1}^{\infty} \frac{(1 - |z_k|^2)^2}{|1 - \zeta z_k|^2}.$$

It is easy to compute the average of $k_Z(\zeta)$ over the circle $|\zeta| = r$. It is

$$\hat{k}_Z(r) = \frac{1}{2\pi} \int_0^{2\pi} k_Z(re^{it}) \, dt = \frac{r^2}{2} \sum_{k=1}^{\infty} \frac{(1 - |z_k|^2)^2}{1 - |z_k|^2 r^2}.$$

The following was shown in [7]. There it was assumed that $Z$ was uniformly discrete, but the proof only made use the fact that it had bounded density.

**Theorem A.** For a sequence $Z$ without limit points in $\mathbb{D}$

$$D^+(Z) = S^+(Z) \overset{\text{def}}{=} \limsup_{r \to 1} \left[ \sup_{\varphi} \left( \hat{k}_\varphi(Z)(r) \right) \right] / \left( \log \frac{1}{1 - r^2} \right).$$

Note that the finiteness of either $D^+(Z)$ or $S^+(Z)$ is equivalent to $Z$ having bounded density, so that does not need to be part of the hypotheses.

In order to deal with the case $p < 1$ we define a space $L^p_{q,Z}$ that will (as in [7]) serve as a replacement for $L^p_Z$. Choose some radius $r < 1$, then a measurable function $f$ belongs to $L^p_{q,Z}$ if and only if the function $m_q(f, z)$ belongs to $L^p$, where

$$m_q(f, z)^q = \frac{1}{|D(z, r)|} \int_{D(z, r)} |f|^q \, dA.$$

The radius $r$ should be fixed, but the actual value chosen is not important. Define $L^p_{q,Z}$ to be the set of all $f$ such that $f e^{kz}$ belongs to $L^p_q$. We will always assume $q \geq 1$. Note that $L^p_p = L^p$ and $L^p_{p,Z} = L^p_Z$. Also, the set of analytic functions in $L^p_q$ (resp., $L^p_{q,Z}$) is independent of $q$, being $A^p$ (resp., $A^p_Z$). We will allow $q = \infty$ with $m_\infty(f, z) = \sup_{w \in D(z, r)} |f(w)|$.

Finally, we have the following equivalent conditions for general interpolation.

**Theorem 5.2.** Let $Z$ be a sequence in $\mathbb{D}$ without limit points. The following are equivalent:

1. $Z$ has bounded density and is a general interpolating sequence for $A^p$ relative to any admissible interpolation scheme.
2. $Z$ is a general interpolation sequence for $A^p$ relative to some interpolation scheme.
3. $D^+(Z) < 1/p$ or equivalently $S^+(Z) < 1/p$.
4. $Z$ has bounded density and the $\bar{\partial}$-problem has solutions with bounds in $L^p_{q,Z}$ for all $q \geq 1$.
5. $Z$ has bounded density and the $\bar{\partial}$-problem has solutions with bounds in $L^p_{q,Z}$ for some $q \geq 1$.

It is trivial that (1) implies (2) and (4) implies (5). In this section we will prove that (2) implies (3). In the next section we will show (3) implies (4) and (5) implies (1).
**Proof of** (2) ⇒ (3). Assume (2) and let $I$ be the interpolation scheme with domains $G_k$, clusters $Z_k$ and diameter $R$. We have seen that $I$ is necessarily admissible and $Z$ necessarily has bounded density. Let $K$ be the interpolation constant for $Z$. The proof begins almost exactly as in [7]. We wish to show the existence of a function $g$ in the unit ball of $A^{p/\beta}$, for some $\beta < 1$, that vanishes on “most” of $Z$ and satisfies $|g(0)| > \delta$, where $\delta > 0$ is an interpolation invariant.

Start by removing from $Z$ those clusters $Z_k$ for which $G_k$ intersects $D(0, 1/2)$. The number of points removed is bounded by an amount which is an interpolation invariant. Then append the single point $\{0\}$ to the result. Call this new sequence $W$. By Propositions 3.3 and 3.5, $W$ is a general interpolating sequence relative to a related interpolation scheme. It has an interpolation constant which may be estimated solely in terms of interpolation invariants of $Z$.

Now perturb the sequence $W$ as in Proposition 5.1, with the affine maps $\varphi_k(z) = r_k z$, where each $0 < r_k < 1$ and $1 - r_k$ is a fixed small multiple $\eta$ of $\rho(G_k)$. If $\eta$ is sufficiently small, depending only on interpolation invariants, the result is a general interpolating sequence. Call the new sequence $W'$. If $a$ is a point of $W$ the perturbation $a'$ in $W'$ will satisfy $(1 - |a'|^2) < \beta(1 - |a|^2)$ for some $\beta < 1$ depending only on $\eta$ and $R$. Since this new sequence is a zero set for $A^\theta$ then, according to [6] and [7], perturbing it back to the original $W$ produces a sequence that is a zero sequence for $A^{p/\beta}$. (Strictly speaking, Theorem 4 of [6] assumed $1 - |a|^2 = \beta(1 - |a'|^2)$ for all $a$, but only the inequality is actually needed in the proof.)

Moreover, since $W'$ is interpolating, there is a function $f$ in $A^\theta$ with norm 1 whose zero sequence is $W' \setminus \{0\}$ and satisfies $|f(0)| > \delta$ where $\delta > 0$ depends only on interpolation invariants for $I$. As in [7], there can be constructed a function $g$ in $A^{p/\beta}$ whose zero sequence is $W' \setminus \{0\}$ and which satisfies $\|g\|_{p/\beta} \leq C\|f\|_p$ and $|g(0)| > |f(0)|/C$, where $C$ depends only on interpolation invariants.

Also as in [7], we can produce a function $h$ in $A^{p/\beta}_Z$ ($Z$ the original sequence, which differs from $W'$ by a number of elements that depends only on invariants of $Z$) where $h$ vanishes nowhere and satisfies $\|h\|_{p/\beta, Z} = 1$ and $|h(0)| > \delta$ with some (new) value of $\delta > 0$ depending only on invariants of $I$. Let $h^*$ be that nonvanishing function in the unit ball of $A^{p/\beta}_Z$ which maximizes the absolute value at the origin. As shown in [7], it will satisfy

$$
|h^*(\zeta)|^{p/\beta} e^{(p/\beta) k_Z(\zeta)} (1 - |\zeta|^2) \leq C
$$

with an absolute constant $C$.

Now consider

$$
\frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|r|^\beta} \log |h^*(r e^{i\theta})| + \frac{p}{\beta} k_Z(re^{i\theta}) + \log (1 - |r|^2) \, dt
\leq \log \left( \frac{1}{2\pi} \int_0^{2\pi} |h^*(r e^{i\theta})|^{p/\beta} e^{p/\beta (p/\beta) k_Z(re^{i\theta})} (1 - |r|^2) \, dt \right)
\leq \log C,
$$

valid for all $0 < r < 1$. Divide the above by $\log[1/(1 - r^2)]$ and $p/\beta$ to obtain:

$$
\left( \frac{k_Z(r)}{\log \left( \frac{1}{1 - r^2} \right) \log C} \right) \leq \frac{\beta}{p} \left( 1 + \log \frac{C - \log |h^*(0)|}{\log \frac{1}{1 - r^2}} \right).
$$

Fix a $\kappa > 0$ so that $\beta(1 + \kappa) < 1$ and choose a sufficiently large $r_* < 1$ so that large parenthesis on the right side of the above inequality is less than $1 + \kappa$. Since $-\log |h^*(0)| \leq \log(1/\delta)$, the choice of $r_*$ depends only on interpolation invariants. Applying this to (5.6) with $\varphi(Z)$ in place of $Z$ gives

$$
\left( \frac{k_{\varphi(Z)}(r)}{\log \left( \frac{1}{1 - r^2} \right) \log C} \right) \leq \frac{\beta}{p} (1 + \kappa), \quad \text{all } r > r_*,
$$

for all conformal maps $\varphi$ of the unit disk. Taking the supremum over all such $\varphi$ gives us $S^+(Z) < 1/p$, as required for condition (3). □
6. The \( \bar{\partial} \)-problem and general interpolating sequences

The existence of \( h^* \) satisfying inequality (5.5) is sufficient (when conformal invariance is invoked) to prove condition (4) of Theorem 5.2. So now we want to prove that condition (3) implies the existence of such a function.

We could argue as in the last section of [7] that this density condition implies the hypotheses of a theorem of J. Ortega-Cerdà (8) which implies the weighted \( \bar{\partial} \)-estimates. That, however, turns out to be a much stronger result than we need. Moreover, the paper [8] does not explicitly include the case \( p < 1 \). Therefore, we will prove directly that the density condition (3) implies the existence of an analytic function with the growth property of (5.5).

Define the invariant Laplacian \( \tilde{\Delta} f(z) = (1 - |z|^2)^2 \bar{\partial}\bar{\partial} f(z) \).

**Lemma 6.1.** If \( \tau \) is a real valued \( C^2 \) function in \( \mathbb{D} \) such that \( \tilde{\Delta} \tau \) is negative and bounded, then there exists a harmonic function \( v \) such that \( \tau(z) - v(z) \leq 0 \) and \( \tau(0) - v(0) \geq -C \| \tilde{\Delta} \tau \|_\infty \), where \( C > 0 \).

**Proof.** It suffices to prove that the equation \( \partial\bar{\partial} u = \partial\bar{\partial} \tau \) has a solution \( u \) that is bounded above, with an estimate on the value at 0. Without any loss of generality we can take \( \tau(0) = 0 \). We simply write down a formula for \( u \) and verify its properties:

\[
u(z) = \frac{2}{\pi} \int_{\mathbb{D}} \tilde{\Delta} \tau(w) \left[ \log \left| \frac{w - z}{1 - \bar{w}z} \right| + \text{Re} \frac{1 - |w|^2}{1 - \bar{w}z} \right] \, d\lambda(w),
\]

where \( \lambda \) denotes the invariant area measure on \( \mathbb{D} \). Assuming the integral makes sense, this surely satisfies the equation \( \partial\bar{\partial} u = \partial\bar{\partial} \tau \). We can rewrite the expression in brackets as

\[
\log \left| \frac{w - z}{1 - \bar{w}z} \right| + \text{Re} \frac{1 - |w|^2}{1 - \bar{w}z} = \\
\left[ \log |w| + \frac{1 - |w|^2}{2} \right] + \left[ \log \left| \frac{w - z}{1 - \bar{w}z} \right| + \frac{1}{2} \left( 1 - \left| \frac{w - z}{1 - \bar{w}z} \right|^2 \right) \right] + \frac{|z|^2 (1 - |w|^2)^2}{2 |1 - \bar{w}z|^2}.
\]

Let us examine these three expressions in reverse order. The last one times \( \tilde{\Delta} \tau(w) \) is clearly integrable with respect to \( d\lambda(w) \) and the resulting integral is negative. The first two bracketed expressions times \( \tilde{\Delta} \tau(w) \) are positive, but I claim the integrals are bounded. The second is integrable \( d\lambda(w) \) in each disk \( D(z,R) \) (with the integral independent of \( z \)), and outside such disk it decays like

\[
\left( 1 - \left| \frac{w - z}{1 - \bar{w}z} \right|^2 \right)^2 = \frac{(1 - |z|^2)^2 (1 - |w|^2)^2}{|1 - \bar{w}z|^4}.
\]

whose integral with respect to \( d\lambda(w) \) is a finite constant. Finally, the first bracketed expression is just the second one at \( z = 0 \). Putting these together, we see that \( u(z) \) is the sum of a negative function and a bounded positive one, and the bounded one has absolute value at most \( C \| \tilde{\Delta} \tau \|_\infty \) where \( C \) is an absolute constant. Finally, the value of \( u(0) \) is similarly bounded by a multiple of \( \| \tilde{\Delta} \tau \|_\infty \).

Thus, if we define \( v(z) = \tau(z) - u(z) + C \| \tilde{\Delta} \tau \|_\infty \) for a suitable absolute constant \( C \), then \( v \) satisfies the stated requirements. \( \square \)

We apply this to a function which is almost

\[
(6.1) \quad \tau(\zeta) = \log \frac{1}{1 - |\zeta|^2} - \frac{p}{\beta} k_z(\zeta)
\]

This does not quite satisfy \( \tilde{\Delta} \tau < 0 \), but the argument in [7] shows that a certain smoothing of it does satisfy this. We continue the proof of the main result:
Proof of (3) ⇒ (4) ⇒ (5) ⇒ (1). Let $\tau(z)$ be as in (6.1). If condition (3) is satisfied we have, according to [7], the following inequality for some radius $r_\ast > 0$:

$$\frac{1}{\pi \log[1/(1 - r_\ast^2)]} \int_{D(z, r_\ast)} \tilde{\Delta} \tau(w) \log \frac{r_\ast^2}{1 - \frac{w - z}{1 - w z}}^2 \, d\lambda(w) \leq 0$$

Since the invariant Laplacian commutes with invariant convolution by radially symmetric functions, this says that the function $\tau^*$ defined below has negative Laplacian.

$$\tau^*(z) = \frac{1}{\pi \log[1/(1 - r_\ast^2)]} \int_{D(z, r_\ast)} \tau(w) \log \frac{r_\ast^2}{1 - \frac{w - z}{1 - w z}}^2 \, d\lambda(w)$$

It is straightforward to see that $\tilde{\Delta} \tau$ is bounded and that $\tau^* - \tau$ is bounded. The bounds depend only on $r^*$ and the density of the sequence $\mathcal{Z}$. Condition (3) implies that $\mathcal{Z}$ has bounded density. Moreover, the density of $\mathcal{Z}$ and the value of $r^*$ is clearly the same for all $\varphi(\mathcal{Z})$ where $\varphi$ is any conformal self-map of $\mathbb{D}$. Thus $\tau^*$ satisfies the hypothesis of Lemma 6.1 and therefore the conclusion. Clearly $\tau$ also satisfies the conclusion since it differs from $\tau^*$ by a bounded function.

Given the harmonic upper bound $v(z)$ to $\tau(z)$ we obtain a non vanishing analytic function $h$ such that $\log |h(z)| = -v(z)$. This gives

$$|h(z)e^{kz}|^{p/\beta} (1 - |z|^2) \leq 1 \quad \text{and} \quad |h(0)| \geq \delta > 0,$$

where $\delta = e^{-v(0)}$ depends only on the data provided by condition (3). Moreover, the same is true if $\mathcal{Z}$ is replaced with $\varphi(\mathcal{Z})$ for any conformal $\varphi$, with the same value of $\delta$.

Now the same argument in [7] applies: we can obtain a family $g_\alpha$, $\alpha \in \mathbb{D}$ of analytic functions satisfying the following for some fixed positive numbers $\epsilon, \delta, \eta$ and $C$ (independent of $\alpha$):

$$(6.2) \quad g_\alpha(z)e^{kz(\alpha)} > \delta > 0, \quad z \in D(a, \eta)$$

and

$$(6.3) \quad |g_\alpha e^{kz}|^p \left(1 - \frac{a - z}{1 - \bar{a} z}^2 \right)^{1-\epsilon} \leq C.$$  

Then this family allows one to construct a solution operator with the required $L^p_{\eta, \mathcal{Z}}$ bounds. This yields condition (4).

Clearly condition (4) implies condition (5).

Finally, assume condition (5). Let $\mathcal{I}$ be any admissible interpolation scheme and let $w = (w_k)$ be a sequence in $X^p_{\mathcal{I}}$ with representative functions $g_k$ satisfying $\|g_k\|_{L^p_{\mathcal{I}, \mathcal{Z}}} = \|w_k\|_{E_k}$. Let $\epsilon > 0$ be chosen so that $(\mathcal{Z}_k)_{2\epsilon} \subseteq \mathcal{G}_k$. Without loss of generality, $\epsilon$ is less than the separation of $\mathcal{I}$ so that $(\mathcal{Z}_k)_\epsilon$ are disjoint. It is easy to verify the following for any $q \geq 1$:

$$\left(\frac{1}{|(\mathcal{Z}_k)_\epsilon|} \int_{(\mathcal{Z}_k)_\epsilon} |g_k|^q \right)^{1/q} \leq \sup_{z \in (\mathcal{Z}_k)_\epsilon} |g_k(z)| \leq \left(\frac{C}{|\mathcal{G}_k|} \int_{\mathcal{G}_k} |g_k|^p \right)^{1/p},$$

where $C$ depends only on $\epsilon$ and the diameter of the interpolation scheme $\mathcal{I}$. For each $k$ let $\gamma_k$ denote a positive $C^1$ function supported in $(\mathcal{Z}_k)_\epsilon$ which equals 1 in $(\mathcal{Z}_k)_{\epsilon/2}$ and satisfies $\nabla \gamma_k(z) \leq C/(1 - |z|)$. Then the function $g = \sum_j \gamma_j g_j$ agrees with $g_k$ in a neighborhood of $\mathcal{Z}_k$. We want to correct $g$ by putting $f = g - \Psi_Z$ where $\Psi_Z$ is the function defined in [7] which has zero sequence $\mathcal{Z}$, choosing $u$ so that $f$ is in $A^p$. This requires $\bar{\partial}u = \bar{\partial}g/\Psi_Z$.

The argument of [7] shows that $(1 - |z|^2)\bar{\partial}g/\Psi_Z$ belongs to $L^p_{\mathcal{I}, \mathcal{Z}}$. By condition (5) there is a solution $u$ in $L^p_{\mathcal{I}, \mathcal{Z}}$. It follows that $\Psi_Z$ is in $L^p_{\mathcal{I}}$ and therefore $f = g - u\Psi_Z \in L^p_{\mathcal{I}}$. Being analytic, it must belong to $A^\infty$. Since both $g$ and $u$ are analytic in a neighborhood of each $\mathcal{Z}_k$, the formula for $f$ shows that it agrees with $g$ to order at least that determined by $\mathcal{Z}$ and so $f|_{\mathcal{G}_k}$ is equivalent to $g_k$ for all $k$. That is, $f$ interpolates the sequence $(w_k)$, as required.  

$\square$
7. Remarks and examples

Remark 1. Note that the proofs and results of the previous sections are practically not changed at all for the weighted Bergman space $A^{p,\alpha}$, the measure being $dA_\alpha(z) = (1 - |z|^2)^\alpha dA(z)$ with $\alpha > -1$. The appropriate density inequality is then $D^+(Z) < (\alpha + 1)/p$.

Remark 2. There are other norms than the one described on $X^p_Z$ that can be used without changing the results. For example, let us rewrite our definition of the norm in $X^p_Z$ in terms of averages instead of integrals. That is, define

$$
\operatorname{avg}_q(f, k) = \begin{cases} 
\left( \frac{1}{|G_k|} \int_{G_k} |f|^q \, dA \right)^{1/q} & 0 < q < \infty \\
\sup_{z \in G_k} |f(z)| & q = \infty
\end{cases}
$$

and for $w_k \in E_k$ let

$$
\nu_q(w_k) = \inf \{ \operatorname{avg}_q(f, k) : f \text{ is a representative of } w_k \}.
$$

Then for $w = (w_k) \in X^p_Z$ we have

$$
\|w\|_{X^p_Z}^p = \sum \nu_p(w_k)^p |G_k|.
$$

However, if we examine the proof of the step (5) $\Rightarrow$ (1) of Theorem 5.2, we see that at least for that step we could have used the norm

$$
\|w\| = \sum \nu_p(w_k)^p |G_k|
$$

with any value of $q$. It is not hard to see that the necessary analogues of all the results in sections 3 and 5 go through for this altered norm. This is to be expected for the following reason: one easily sees that $\operatorname{avg}_q(f, k) \leq C_{p, q} \operatorname{avg}_p(f, k)$ (whatever the values of $p$ and $q$) if one uses a slightly smaller domain on the left or a slightly larger one on the right. Thus, since the equivalent condition (3) of Theorem 5.2 is independent of the actual domains chosen, one should get the same result with the above norm for any choice of $q$. The choices $q = 2$ or $q = \infty$ can make calculations of the $\inf f \operatorname{avg}_q(f, k)$ particularly simple in some special cases.

Remark 3. The main result also has a version for $p = \infty$; that is, for the spaces $A^{-\alpha}$, defined for $\alpha > 0$ by

$$
A^{-\alpha} = \left\{ f : f \text{ is analytic and } \|f\|_{\infty, \alpha} = \sup_{z \in \mathbb{D}} |f(z)|(1 - |z|^2)^\alpha < \infty \right\}.
$$

One can define a general interpolation problem in an analogous way, substituting weighted sup norms for $L^p$ norms and suprema for summations. Then appropriate analogues of 3.1 through 3.6 and 5.1 have analogous proofs (for Proposition 3.1 we invoke [5] at the end instead of [2]). The proof of the appropriate analogue of Theorem 5.2 is then nearly the same. If a zero set for $A^{-\alpha}$ with bounded density is perturbed toward the boundary by a factor $\beta < 1$, the resulting sequence is a zero sequences for $A^{-\beta \alpha}$. The construction of functions satisfying the growth in (5.5) is the same and the solution operator constructed in [7] is easily seen to be bounded in the appropriate sup norm.

For the record, we include below the statement of the main theorem for $A^{-\alpha}$. We let $L^{-\alpha}$ denote the set of measurable functions $f$ such that $|f(z)|(1 - |z|^2)^\alpha$ is essentially bounded. Let $L^{-\alpha}_q$ denote those functions where $m_q(f) \in L^{-\alpha}$ and $L^{-\alpha}_q$ are those with $fe^{kz} \in L^{-\alpha}_q$.

Recall that K. Seip’s criterion [11] for simple interpolation in $A^{-\alpha}$ is that the sequence be uniformly discrete and $D^+(Z) < \alpha$.

Theorem 7.1. Let $Z$ be a sequence in $\mathbb{D}$ without limit points. The following are equivalent:

1. $Z$ has bounded density and is a general interpolating sequence for $A^{-\alpha}$ relative to any admissible interpolation scheme.
2. $Z$ is a general interpolation sequence for $A^{-\alpha}$ relative to some interpolation scheme.
3. $D^+(Z) < \alpha$ or equivalently $S^+(Z) < \alpha$. 

(4) $\mathcal{Z}$ has bounded density and the $\mathcal{\bar{D}}$-problem has solutions with bounds in $L_{\alpha}^{-}\gamma$ for all $1 \leq q \leq \infty$.

(5) $\mathcal{Z}$ has bounded density and the $\mathcal{D}$-problem has solutions with bounds in $L_{\alpha}^{-}\gamma$ for some $1 \leq q \leq \infty$.

Remark 4. One can approach Hardy spaces in a similar way. For those, it seems natural to choose the domains $G_k$ to have a bounded perimeter in the hyperbolic metric (rather than just bounded radius), and define the norm on the space $E_k$ to be the infimum of $L^p$ norms on the boundary. If one does that one obtains a notion of general interpolating sequence which turns out to be equivalent to the measure $\sum (1 - |z|^2)^2 \delta_{z_j}$ being a Carleson measure (or any of the equivalent conditions of [1]). This holds for $H^p$ for all $0 < p \leq \infty$. The proof of this will appear elsewhere.

We now turn to a few examples of interpolation schemes.

Example 1. Simple interpolation can be cast as an interpolation scheme $\mathcal{I}$ where each cluster is a single point $\{z_k\}$. If we choose $G_k = D(z_k, R)$ for some $R < 1$, then each $E_k$ is one-dimensional and the quotient map from $A^p(G_k)$ to $E_k$ can be identified with evaluation at $z_k$. If $g \in A^p(G_k)$ and $g(z_k) = w_k$, then

$$|w_k|^p \leq \frac{C_R}{|G_k|} \int_{G_k} |g|^p \, dA,$$

where $C_R$ is a constant depending only on $R$. On taking the infimum, we see $|w_k|^p |G_k| \leq C_R \|w_k\|_{E_k}^p$. On the other hand, considering the constant function identically equal to $w_k$ on $G_k$ shows that $|w_k|^p_{E_k} \leq |w_k|^p |G_k|$. Since $|G_k|$ behaves like $(1 - |z_k|^2)^2$, the norm on the sequence space $X^p_{G_k}$ is equivalent to the norm $\|w\|^p_{p, z} = \sum |w_k|^p (1 - |z|^2)^2$. This scheme is therefore equivalent to simple interpolation.

The main theorem here then contains K. Seip’s density criterion for simple interpolation.

Example 2. For multiple interpolation of order $n$, define $\mathcal{I}$ so that each cluster is a single point $z_k$ repeated $n$ times and the domains $G_k$ are $D(z_k, R)$ as before. Then $E_k$ is isomorphic to $\mathbb{C}^n$. If $f \in A^p(G_k)$ with values $f^{(j)}(z_k) = w_k^{(j)}$ for $0 < j < n - 1$, then it is well known that

$$|w_k^{(j)}(1 - |z_k|^2)^2|^p \leq \frac{C}{|G_k|} \int_{G_k} |g|^p \, dA,$$

where $C$ depends at most on $p$, $j$ and $R$. Summing, and taking the infimum over all $f$ with the same values $w_k = (w_k^{(0)}, \ldots, w_k^{(n-1)})$ gives

$$\sum_{j=0}^{n-1} |w_k^{(j)}|^p (1 - |z_k|^2)^{2j+2} \leq C \|w_k\|_{E_k}.$$

Taking $f = \sum_{j=0}^{n-1} w_k^{(j)} (z - z_k)^j$ easily gives a reverse inequality, so this scheme is equivalent to multiple interpolation. Note, however, that we now can allow the multiplicities to vary. By Theorem 3.6, interpolating sequences for this scheme must have an upper bound on the multiplicities.

The main theorem here essentially contains the equivalent of Krosky and Schuster’s density criterion for multiple interpolation.

Example 3. Finally, we present the simplest example of a sequence which has distinct points, but is not uniformly discrete. Let $\{a_k : k \geq 1\}$ and $\{b_k : k \geq 1\}$ be two sequences without limit points in $D$ and satisfying $a_k \neq b_k$ and $\epsilon_k = \psi(a_k, b_k)$ is bounded away from 1. Select a radius $R \in (0, 1)$ such that $\sup_k \epsilon_k < R$ and choose domains $G_k = D(a_k, R)$ for our interpolation scheme $\mathcal{I}$. Let the clusters be $\mathcal{Z}_k = \{a_k, b_k\}$, and note that they satisfy $(\mathcal{Z}_k)_k \subset G_k$ for some $\epsilon > 0$. $E_k$ can be identified with $\mathbb{C}^2$ where the equivalence class of a function $f$ is mapped to $(f(a_k), f(b_k))$.

Let $w_k = (u_k, v_k) \in E_k$. By inequalities we have already seen many times, if $f \in A^p(G_k)$ satisfies $f(a_k) = u_k$ and $f(b_k) = v_k$, then

$$|u_k|^p + |(v_k - u_k)/\epsilon_k|^p \leq \frac{C}{|G_k|} \int_{G_k} |f|^p \, dA,$$
which gives us \( |u_k|^p + |(v_k - u_k)/\varepsilon_k|^p |G_k| \leq \|w_k\|_{E_k} \). Taking the equivalence class representative

\[
f(z) = u_k + (v_k - u_k) \frac{z - a_k}{1 - \overline{a_k}z} \frac{b_k - a_k}{1 - \overline{a_k}b_k}
\]
gives a reverse inequality. Therefore, in the sequence space \( X^p_\gamma \), the norm \( \|w\|_{X^p_\gamma} \) is equivalent to \( \sum (|u_k|^p + |(v_k - u_k)/\varepsilon_k|^p) (1 - |a_k|^2)^2 \). This gives the following result:

**Proposition 7.2.** Given \( Z = \{a_1, b_1, a_2, b_2, \ldots \} \) let the sequence \( \psi(a_k, b_k) \) be bounded away from 1.

Then the following are equivalent:

1. For every sequence \( \{(u_k, v_k) : k \geq 1\} \) satisfying
   \[
   \sum_{k=1}^{\infty} \left( |u_k|^p + \frac{|v_k - u_k|}{\psi(a_k, b_k)} |\psi(a_k, b_k)|^p \right) (1 - |a_k|^2)^2 < \infty
   \]
   there is a function \( f \in A^p \) satisfying \( f(a_k) = u_k \) and \( f(b_k) = v_k \).

2. There exists \( \delta > 0 \) such that the clusters \( Z_k = \{a_k, b_k\} \) satisfy \( \psi(Z_k, Z_j) > \delta \) for all \( j \neq k \) and \( D^+(Z) < 1/p \).

If the \( \psi(a_k, b_k) \) are bounded away from both 0 and 1, then this is just simple interpolation and the norm is equivalent to the one in example 1. We get something new if \( \psi(a_k, b_k) \) is not bounded away from 0.

8. Addendum

8.1. **Sampling.** A student of mine, N. H. Foster, has recently completed a dissertation in which he proved the complementary result for sampling. That is, given an admissible scheme \( I \) with domains \( G_k \) and clusters \( Z_k \), let \( \Phi(f) \) denote the sequence of cosets of \( f|G_k \) for any \( f \in A^p \). Call \( Z = \bigcup Z_k \) a sampling sequence for \( A^p \) relative to \( I \) if there exists a constant \( C \) such that for all \( f \in A^p \)

\[
\frac{1}{C} \|\Phi(f)\|_{X^p_I} \leq \|f\|_p \leq C \|\Phi(f)\|_{X^p_I}.
\]

The definition of the \( X^p_\gamma \) norm makes the left side inequality automatic, so the main point is the second inequality. His result is that \( Z \) is sampling if and only if a companion density \( D^+(Z) \) (called the lower uniform density) is greater than \( 1/p \).

His result is likely true for the weighted spaces \( A^{p,\alpha} \) as well, with \( D^-(Z) > (\alpha + 1)/p \).

8.2. **O-interpolation.** I was recently made aware of a paper by S. Ostrovsky [9] and one by A. Schuster and T. Wertz [10]. The first is set in spaces of entire functions with certain exponential weights (generalizations of the Fock spaces), the second is set in the unit disk, with weights analogous to those of [9]. Both papers limit the results to \( p = 2 \), but the weights considered are more general than those considered here.

I have extended the general interpolation results obtained in this paper to the weights considered in [10], but those results will appear elsewhere. Those weights have the form

\[
\frac{e^{-\varphi(z)}}{1 - |z|^2}
\]

where \( \varphi \) is \( C^2 \) on \( \mathbb{D} \) and satisfies \( 0 < m \leq \hat{\Delta} \varphi(z) \leq M \) for positive constants \( m \) and \( M \). The standard weights \( (1 - |z|^2)^\alpha, \alpha > -1 \), correspond to

\[
\varphi(z) = (\alpha + 1) \log \left( \frac{1}{1 - |z|^2} \right)
\]

which satisfy \( \hat{\Delta} \varphi = \alpha + 1 \).

I will limit my remarks here to showing that the results in [10], when applied to the standard weighted Bergman spaces \( A^{p,\alpha} \), follow from the results here. In fact, a version for \( p \neq 2 \) also follows.

Suppose \( Z \) is a sequence of distinct points in \( \mathbb{D} \). For each point \( \gamma \in Z \) let \( n_\gamma \) be the number of points in \( Z \cap D(\gamma, 1/2) \) and let \( \delta_\gamma = \inf \{\psi(\gamma, \beta) : \beta \in Z, \beta \neq \gamma\} \) be the pseudohyperbolic distance
to the nearest other point of $Z$. The following is the main theorem of [10], restricted to the standard weights but extended to all $p > 0$.

**Theorem 8.1.** If $p > 0$ and $Z$ is a sequence of distinct points satisfying $D^+(Z) < (\alpha + 1)/p$, then for every sequence $\{c_\gamma : \gamma \in Z\}$ satisfying

$$
\sum_{\gamma \in Z} |c_\gamma|^p \frac{(1 - |\gamma|^2)^{\alpha + 2}}{\delta_\gamma^{(p+1)} < \infty}
$$

(8.1)

there is a function $f \in A^{p,\alpha}$ satisfying $f(\gamma) = c_\gamma$ for all $\gamma \in Z$.

Thus, if a sequence $\mathcal{C}$ for every sequence $\{c_\gamma : \gamma \in Z\}$ satisfies the finiteness condition of the theorem, the $(\gamma)$ determined by these clusters and $G_k$ the domains of this scheme.

We can get an upper bound $B$ on the cardinality of $Z_k$. Therefore each product in this formula contains at most $B$ factors. We wish to estimate the size of this product. If we split it into those factors with $\beta \in D(\gamma, 1/2)$ ($n_\gamma$ factors) and the rest (at most $B$ factors), we can estimate

$$
\left| \prod_{\beta} M_\beta(z) \right| \leq \frac{2B}{\delta_\gamma^\alpha}
$$

Thus

$$
\|w_k\|_{E_k}^p \leq C \sum_{\gamma \in Z_k} |c_\gamma|^p \frac{(1 - |\gamma|^2)^2}{\delta_\gamma^{(p+1)}
$$

Thus, if a sequence $c_\gamma$ satisfies the finiteness condition of the theorem, the $(w_k)$ determined by these $f_k$ belongs to $X_k^p$ and so a function $f \in A^p$ exists that agrees with each $f_k$ on $Z_k$, that is, $f(\gamma) = c_\gamma$, as required.

**Proof.** For simplicity of exposition, let us assume $\alpha = 0$ so that we have the ordinary unweighted Bergman spaces.

Suppose $D^+(Z) < 1/p$. Then certainly $Z$ has bounded density, and so there exists an admissible scheme for $Z$. Let $Z_k$ be the clusters and $G_k$ the domains of this scheme.

The paper [10] calls this mode of interpolation, $O$-interpolation.

The sum in (8.1) defines a norm for a certain weighted sequence space. If we can set up an admissible interpolation scheme where the sequence space $X_k^p$ essentially contains this space, then the implication $(3) \Rightarrow (1)$ of Theorem 5.2 proves the above theorem.

**Proof.** For simplicity of exposition, let us assume $\alpha = 0$ so that we have the ordinary unweighted Bergman spaces.

Suppose $D^+(Z) < 1/p$. Then certainly $Z$ has bounded density, and so there exists an admissible scheme for $Z$. Let $Z_k$ be the clusters and $G_k$ the domains of this scheme.

We can get an estimate on the norm of an equivalence class $w_k \in E_k$: it is less than the norm of any function analytic on $G_k$ belonging to that equivalence class. Since $Z$ consists of distinct points, $w_k$ is determined by the values of such a function at the points in $Z_k$, one such function, which has the values $c_\gamma$ is

$$
f_k(z) = \sum_{\gamma \in Z_k} c_\gamma \prod_{\beta \in Z_k, \beta \neq \gamma} \frac{M_\beta(z)}{M_\beta(\gamma)}
$$

(recalling that $M_\beta(z)$ is the Mobius transformation $(\beta - z)/(1 - \beta \bar{z})$). Note that there is an upper bound $B$ on the cardinality of $Z_k$. Therefore each product in this formula contains at most $B$ factors. We wish to estimate the size of this product. If we split it into those factors with $\beta \in D(\gamma, 1/2)$ ($n_\gamma$ factors) and the rest (at most $B$ factors), we can estimate

$$
\left| \prod_{\beta} M_\beta(z) \right| \leq \frac{2B}{\delta_\gamma^\alpha}
$$

Thus

$$
\|w_k\|_{E_k}^p \leq C \sum_{\gamma \in Z_k} |c_\gamma|^p \frac{(1 - |\gamma|^2)^2}{\delta_\gamma^{(p+1)}
$$

Thus, if a sequence $c_\gamma$ satisfies the finiteness condition of the theorem, the $(w_k)$ determined by these $f_k$ belongs to $X_k^p$ and so a function $f \in A^p$ exists that agrees with each $f_k$ on $Z_k$, that is, $f(\gamma) = c_\gamma$, as required.

**References**

[1] Peter Duren and Alexander P. Schuster, *Finite unions of interpolating sequences*, Proc. Amer. Math. Soc. 130 (2002), 2609–2615.

[2] Charles Horowitz, *Zeros of functions the Bergman spaces*, Duke Math. J. 41 (1974), 693–710.

[3] Mark Kreisky and Alexander Schuster, *Multiple interpolation and extremal functions in the Bergman spaces*, J. Anal. Math. 85 (2001), 141–156.

[4] Daniel H. Luecking, *Forward and reverse Carleson inequalities for functions in Bergman spaces and their derivatives*, Amer. J. Math. 107 (1985), 85–111.

[5] __________, *Zero sequences for Bergman spaces*, Complex Variables Theory Appl. 30 (1996), 345–362.

[6] __________, *Sampling measures for the Bergman spaces on the unit disk*, Math. Ann. 316 (2000), 659–679.

[7] __________, *Interpolating sequences for the Bergman space and the $\bar{\partial}$-equation in weighted $L^p$*, preprint, http://front.math.ucdavis.edu/math.CV/0311360, 2004.

[8] Joaquim Ortega-Cerdà, *Multipliers and weighted $\bar{\partial}$-estimates*, Revista Matemática Iberoamericana 18 (2002), no. 2, 355–377.

[9] Stanislav Ostrovsky, *Weighted-$L^2$ interpolation on non-uniformly separated sequences*, Proc. Amer. Math. Soc. 138, no. 12, (2010), 4413–4422.
[10] Alexander Schuster and Tim Wertz, *Interpolation on non-uniformly separated sequences in a weighted Bergman space*, J. Egyptian Math. Soc., 21 (2013), 97–102.

[11] Kristian Seip, *Beurling type density theorems in the unit disk*, Inventiones Math. 113 (1993), 21–39.

Department of Mathematical Sciences, University of Arkansas, Fayetteville, Arkansas 72701

E-mail address: luecking@uark.edu