SKYRMIONS FROM SU(3) HARMONIC MAPS AND THEIR QUANTIZATION

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Static properties of SU(3) multiskyrmions with baryon numbers up to 6 are estimated. The calculations are based on the recently suggested generalization of the SU(2) rational map ansatz applied to the SU(3) model. Both SU(2) embedded skyrmions and genuine SU(3) solutions are considered and it is shown that although, at the classical level, the energy of the embeddings is lower, the quantum corrections can alter these conclusions. This correction to the energy of the lowest state, depending on the Wess-Zumino term, is presented in the most general case.

1. Topological soliton models, and the Skyrme model among them [1], have recently generated a fair amount of interest because they may be able to describe various properties of low energy baryons. So far, most of such studies have involved the SU(2) Skyrme model, whose solutions were then embedded into the SU(3) model. This is justified as the solutions of the SU(2) model are also solutions of SU(N) models. However, there exist solutions of the SU(N) models which are not embeddings of SU(2) fields and it is important to assess their contribution.

As with solutions of the SU(2) Skyrme models, the solutions of the SU(3) model, with very few exceptions, can only be determined numerically. Like for the SU(2) case [2], one starts with a harmonic map ansatz which gives fields “close” to the genuine solutions [3, 4]. Then one can use these fields as starting configurations of various numerical minimization schemes. For the SU(2) fields such an approach was carried out in [2] where it reproduced the results of the earlier numerical approaches [3, 6] and so showed that the harmonic map approximations are very close to the final fields. The real solutions have energies only few hundreds of MeV lower than the harmonic approximations and the baryonic charge and energy distributions do not look very different etc. This has justified the use of harmonic field approximants in calculating various quantities when estimating quantum corrections to the classical results.

In this paper we take a look at the nonembedding solutions of the SU(3) model. First we recall the results of [3] and use them in a SU(3) numerical minimization program to estimate the values of energies of true solutions. The SU(3) variational minimization program [7] has been rearranged for this purpose to allow the consideration of quite general field ansätze. This is discussed in the next section.

In the following section we discuss various quantum corrections and compare our results with the similar results for the embeddings. The paper ends with a short section presenting our conclusions.

2. The Lagrangian of the SU(3) Skyrme model, in its well known form, depends on the parameters $F_\pi$ and $e$ and is given by [8, 9]:

$$
\mathcal{L} = \frac{F_\pi^2}{16} Tr l_\mu l^\mu + \frac{1}{32e^2} Tr [l_\mu, l_\nu]^2 + \frac{F_\pi^2 m^2}{16} Tr (U + U^\dagger - 2) 
$$

(1)
Here $U \in SU(3)$ is a unitary matrix describing the chiral (meson) fields, and $l_\mu = U^\dagger \partial_\mu U$. In the model $F_\pi$ should be fixed at the physical value: $F_\pi = 186$ Mev. The flavour symmetry breaking (FSB) in the Lagrangian will be considered below.

The Wess-Zumino term, which should to be added to the action, is given by

$$S^{WZ} = \frac{-iN_c}{240\pi^2} \int_\Omega d^5x \epsilon^{\mu
u\lambda\rho\sigma} Tr(\hat{l}_\mu \hat{l}_\nu \hat{l}_\lambda \hat{l}_\rho \hat{l}_\sigma),$$

where $\Omega$ is a 5-dimensional region with the 4-dimensional space-time as its boundary and where $\hat{l}_\mu$ is a 5-dimensional analogue of $l_\mu = U^\dagger \partial_\mu U$. As is well known, this extra term does not contribute to the static masses of classical configurations, but it defines important topological properties of skyrmions [8, 9] and plays an important role in their quantization [10, 11].

In [3] an ansatz was presented which allows us to find approximate solutions of the $SU(3)$ model. This ansatz involves parametrising the static field as

$$U(\vec{x}) = \exp \left\{ f(r) \left( P(\theta, \varphi) - \frac{1}{3} \right) \right\},$$

where $r, \theta$ and $\varphi$ are polar coordinates, $f(r)$ is a radial profile function which has to be determined numerically and $P(\theta, \varphi)$ is a projector involved in the harmonic map ansatz. As shown in [3] this projector is given by

$$P = \frac{FF^\dagger}{|F|^2},$$

where $F$ is a 3-component vector, whose entries are polynomials in $z = \tan(\frac{\theta}{2})e^{i\varphi}$. The largest degree of the polynomial gives the baryon number $B$ of the final $SU(3)$ Skyrme field configuration. For an $SU(2)$ embedding the approach is similar except that this time we put

$$U(\vec{x}) = \begin{pmatrix} U_2 & 0 \\ 0 & 1 \end{pmatrix},$$

where $U_2$ is an $U(2)$ matrix determined in an analogous way as (3) except that the vector $F$ has only two components.

The fields $F$ in Eq. (4) are so chosen that the configuration (3) has the smallest energy; this provides us with the harmonic map approximation to the real minimal energy static solution of the Skyrme model. We have then taken the expressions for $F$ determined in [3] and used the corresponding $U$ in (3) as an initial field in our minimization program. For $B = 2$ $F = (1, \sqrt{2}z, z^2)^T$, for $B = 3$ $F = (1/\sqrt{2}, 1.576z, z^3)^T$, for $B = 4$ $F = (1.27z^2, z^4)^T$, for $B = 5$ $F = (4.5z^2, 2z^4 + 1, z^5 - 2.7z)^T$ and for $B = 6$ $F = (kz^3, 1 - 3z^5, z^6 + 3z)^T$ with $k = 7.06$ [3].

We have performed many minimizations of the field configurations involving baryon numbers 3, 4, 5 and 6. In all cases the initial $SU(3)$ harmonic map fields had energies somewhat higher than the corresponding harmonic map embeddings.

Our $SU(3)$ minimization program, which uses the parametrization of the $SU(3)$ field in terms of two, mutually orthogonal, complex unit vectors was then used to minimize the energy further. The constraints of the orthogonality and of the vectors being of unit length were replaced by the introduction of extra positive contributions to the energy (with large coefficients) - the so called “penalty terms”.

The program itself minimized the energy using a mixture of a finite element and finite step methods, and we varied the coefficients of the penalty terms so as not to be trapped in a local minimum. In each case the final energy of the field configuration was

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1The authors thank W.K. Baskerville for making them aware of this parametrisation.
lower by a few hundreds of MeV from the energy of the harmonic approximant. However, due to the smallness of the lattice the baryon number was also somewhat lower; so our values in the Table were obtained by a linear extrapolation: \( M = M + \Delta M \), where \( \Delta M = \alpha \cdot (B - B_{\text{obs}}) \) and where \( B \) is the baryon number and \( B_{\text{obs}} \) is its value on the lattice. \( \alpha \) was determined by looking at various minimizations (with different values of the coefficients of the penalty terms).

To check the stability of the program we also performed a series of minimizations when the initial field was a mixture of the \( SU(3) \) harmonic map field and of the embedding. When the mixture was close to the embedding or to the \( SU(3) \) harmonic map the minimization program took it down to these fields. When the mixture was far from either of these special fields it evolved to a new configuration of higher energy than either of the special ones. Although some of these new configurations can be numerical artifacts it is clear that the spectrum of static solutions of the \( SU(3) \) model is very complicated, with most states having energies larger than the energies of the embeddings or the fields derived by the harmonic map ansatz. Hence our estimates in the Table have some physical justification.

In the Table we present some of our results. It is difficult to assess their accuracy; it is probably within 1\% for the masses and several \% for other quantities. In all numerical minimizations we have worked on a small tetrahedral lattice and hence we had a small “leakage” of both the energy and of the baryon number.

| \( B \) | \( M_{cl} \) | \( M_{cl}^{SU(2)} \) | \( \Delta E_{M\text{-t.}} \) | \( C_S \) | \( \Theta_{1,2} \) | \( \Theta_{4} \) | \( \Theta_{5} \) | \( \Theta_{6,7} \) | \( \Theta_{8} \) | \( \Theta_{38} \) | \( WZ_{3} \) | \( WZ_{8} \) | \( \Delta E \) |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1   | 1.70 | 1.70 | 46  | --- | 5.56 | 5.56 | 2.04 | 2.04 | 2.04 | 2.04 | 2.04 | 2.04 | 2.04 |
| 2   | 3.38 | 3.26 | 89  | 0.33 | 5.70 | 6.40 | 7.10 | 7.10 | 5.70 | 5.0  | 1.2  | 0.00 | 0.00 |
| 3   | 4.85 | 4.80 | 110 | 0.33 | 7.15 | 7.90 | 8.60 | 8.60 | 9.20 | 5.1  | 1.9  | 0.7  | 0.15 |
| 4   | 6.48 | 6.20 | 174 | 0.31 | 12.2 | 9.80 | 11.6 | 11.6 | 12.1 | 6.2  | 3.0  | 1.0  | 0.6  |
| 5   | 7.90 | 7.78 | 211 | 0.36 | 11.5 | 11.6 | 15.4 | 12.1 | 12.7 | 15.2 | 0.1  | 0.2  | 0.8  |
| 6   | 9.38 | 9.24 | 251 | 0.33 | 15.4 | 14.8 | 15.5 | 15.5 | 14.2 | 16.3 | 1.2  | 0.01 | 0.03 |

Table. The values of the masses \( M_{cl} \) in GeV, the mass term \( \Delta E_{M\text{-t.}} \) (in MeV), the strangeness content \( C_S \) and the moments of inertia (in GeV\(^{-1}\)) for the \( SU(3) \) projector configurations. The quantities for \( B = 1 \) hedgehog and the masses of \( SU(2) \) embeddings \( M_{cl}^{SU(2)} \), in GeV are given for comparison. The components of the Wess-Zumino term \( WZ_{3} \) and \( WZ_{8} \) which are different from zero for the \( SU(3) \) projector ansatz are also shown. \( \Delta E \) in the last column is the quantum correction (in MeV) due to zero modes for the lowest energy state. The parameters of the model are \( F_{\pi} = 186 \text{ MeV}, e = 4.12 \).

As can be seen from this Table, the moments of inertia are pairwise equal, except for the case \( B = 5 \) when \( \Theta_{4} \neq \Theta_{5} \). These equalities are a consequence of the symmetry properties of our multiskyrmion configurations.

It should be noted that the energy of the \( B = 4 \) configuration is very close to that of the \( B = 4 \) toroidal soliton obtained in 1988 [12] within the axially symmetrical generalization of the ansatz by Balachandran et al. [13]. Within the accuracy of our calculations we cannot be certain which configuration has the lowest energy in the chirally symmetrical limit. For \( B = 6 \) the torus-like \( SO(3) \) configuration has energy considerably higher than the energy of the \( SU(3) \) projector ansatz presented in the Table.

Let us add here that the configuration of a double-torus form was considered also within the \( SU(2) \)-version of the model several years ago [14]. It has the energy somewhat higher than the energy of the known minimal energy configuration.

3. Following [10, 11, 13] we consider the contribution of the Wess-Zumino (WZ) term which defines the quantum numbers of the system in the quantization procedure. Its expression is given by (2) above. As usually, we introduce the time-dependent collective coor-
the symmetry of the third and eighth components of the chiral derivatives in the SU

with the substitution

where

\[ \tilde{B} = \frac{N_c}{24\pi^2} \int \sum_{k=1}^{k=8} \omega_k WZ_k d^3x = \sum_{k=1}^{k=8} \omega_k L_k WZ_k, \]

with the angular velocities of rotation in the configuration space defined in the usual way:

\[ A^i A_i = -\frac{1}{2}\omega_k \lambda_k. \]

Summation over repeated indices is assumed here and below. The functions

\( WZ_k \) can be expressed through the chiral derivatives \( L_k \):

\[ WZ_i = WZ_i^R + WZ_i^L = (R_{ik}(U_0) + \delta_{ik})WZ_k^R, \]

\( i, k = 1, \ldots, 8 \), and are given by

\[
\begin{align*}
WZ_1^R &= -(L_1, L_4L_5 + L_6L_7) - (L_2L_3L_8)/\sqrt{3} - 2(L_8, L_4L_7 - L_5L_6)/\sqrt{3} \\
WZ_2^R &= -(L_2, L_4L_5 + L_6L_7) - (L_3L_1L_8)/\sqrt{3} - 2(L_8, L_4L_6 + L_5L_7)/\sqrt{3} \\
WZ_3^R &= -(L_3, L_4L_5 + L_6L_7) - (L_3L_1L_8)/\sqrt{3} - 2(L_8, L_4L_5 - L_6L_7)/\sqrt{3} \\
WZ_4^R &= -(L_4, L_1L_2 - L_6L_7) - (L_3L_5L_8)/2 + 2(\tilde{L}_8, L_1L_7 + L_2L_6)/\sqrt{3} \\
WZ_5^R &= -(L_5, L_1L_2 - L_6L_7) + (L_3L_4L_8)/\sqrt{3} - 2(\tilde{L}_8, L_1L_6 - L_2L_7)/\sqrt{3} \\
WZ_6^R &= (L_6, L_1L_2 + L_4L_5) + (L_3L_7L_8)/\sqrt{3} - 2(\tilde{L}_8, L_1L_5 - L_2L_4)/\sqrt{3} \\
WZ_7^R &= (L_7, L_1L_2 + L_4L_5) - (L_3L_6L_8)/\sqrt{3} + 2(\tilde{L}_8, L_1L_4 + L_2L_5)/\sqrt{3} \\
WZ_8^R &= -\sqrt{3}(L_1L_2L_3) + (L_8L_4L_5) + (L_4L_6L_7),
\end{align*}
\]

where \((L_1, L_2, L_3, \tilde{L}_1, \tilde{L}_2, \tilde{L}_3)\), i.e \((L_1, L_2L_3) = (L_1 \cdot \hat{L}_2 \wedge \hat{L}_3)\) and \(\tilde{L}_3 = (L_3 + \sqrt{3}L_8)/2\), \(\tilde{L}_8 = (\sqrt{3}L_3 - L_8)/2\), \(\tilde{L}_8 = (\sqrt{3}L_3 + L_8)/2\) are the third and eighth components of the chiral derivatives in the \((u, s)\) and \((d, s)\) SU(2)-subgroups. \(R_{ik}(U_0) = \frac{1}{4} Tr \lambda_i U_0 \lambda_k U_0^\dagger\) is a real orthogonal matrix, and \(WZ_i^R\) are defined by the expressions (9) with the substitution \(\tilde{L}_k \rightarrow \tilde{R}_k\). Relations similar to (9) can be obtained for \(WZ_3\) and \(WZ_8\); they are analogs of \(WZ_1\) and \(WZ_2\) for the \((u, s)\) or \((d, s)\) SU(2) subgroups, thus clarifying the symmetry of the WZ-term in the different SU(2) subgroups of SU(3).

The baryon number of the SU(3) skyrmions can be written also in terms of \(L_i\) in a form where its symmetry in the different SU(2) subgroups of SU(3) is more explicit:

\[ B = -\frac{1}{2\pi^2} \int \left( (L_1, L_2L_3) + (L_4, L_5L_3) + (L_6, L_7L_3) + \frac{1}{2}(L_1, L_4L_7 - L_5L_6) + (L_2, L_4L_6 + L_5L_7) \right) d^3r. \]

The contributions of the three SU(2) subgroups enter the baryon number on an equal footing. In addition, mixed terms corresponding to the contribution of the chiral fields from different subgroups are also present.

The results of calculating the WZ-term according to (9) depend on the orientation of the soliton in the SU(3) configuration space. The Guadagnini’s quantization condition [10] was generalized in [15] for configurations of the “molecular” type to

\[ \gamma_{R_{min}}^W = \frac{2}{\sqrt{3}} \frac{\partial L_Z}{\partial \omega_8} \approx \frac{N_c B(1 - 3C_S)}{3}. \]
The function 
\[ \tilde{L}_i = \tilde{\omega}_i \tilde{L}_k - \tilde{\omega}_k \tilde{L}_i, \]
where \( \tilde{\omega}_i \) are the right hypercharge characterizing the \( SU(3) \) irrep under consideration, and the scalar strangeness content \( C_S \) is defined in terms of the real parts of the diagonal matrix elements of the matrix \( U \):
\[ C_S = \frac{<1 - \text{Re} U_{33}>}{<3 - \text{Re}(U_{11} + U_{22} + U_{33})>}, \]
where \(<>\) denotes the averaging or integration over the whole 3-dimensional space. When solitons are located in the \((u,d)\) \( SU(2) \) subgroup of \( SU(3) \) only \( \tilde{L}_1, \tilde{L}_2 \) and \( \tilde{L}_3 \) are different from zero, \( C_S = 0, WZ^R_8 \) and \( WZ^L_8 \) are both proportional to the \( B \)-number density, and the well known quantization condition \( [11] \) takes place
\[ Y_R = N_c B/3. \]

The interpolation (11) does not work so well for configurations we consider here.

The expression for the rotation energy density of the system depending on the angular velocities of rotations in the \( SU(3) \) collective coordinate space defined in Section 2 can be written in the following compact form \( [13] \):
\[ L_{\text{rot}} = \frac{F_2^2}{32} (\tilde{\omega}_1^2 + \tilde{\omega}_2^2 + \ldots + \tilde{\omega}_8^2) + \]
\[ + \frac{1}{16e^2} \left\{ (\tilde{s}_{12} + \tilde{s}_{45})^2 + (\tilde{s}_{45} + \tilde{s}_{67})^2 + (\tilde{s}_{67} - \tilde{s}_{12})^2 + \frac{1}{2} \left( (2\tilde{s}_{13} - \tilde{s}_{46} - \tilde{s}_{57})^2 + (2\tilde{s}_{23} + \tilde{s}_{47} - \tilde{s}_{65})^2 + \right. \right. \]
\[ \left. \left. + (2\tilde{s}_{34} + \tilde{s}_{16} - \tilde{s}_{27})^2 + (2\tilde{s}_{35} + \tilde{s}_{17} + \tilde{s}_{26})^2 + (2\tilde{s}_{36} + \tilde{s}_{14} + \tilde{s}_{25})^2 + (2\tilde{s}_{37} + \tilde{s}_{15} - \tilde{s}_{24})^2 \right) \right\}, \]
(14a)
or
\[ L_{\text{rot}} = V_{ik} \tilde{\omega}_i \tilde{\omega}_k / 2 = V_{ik} g_{il} (g_{mk} \tilde{\omega}_l \tilde{\omega}_m / 2 \]
(14b)
Here \( \tilde{s}_{ik} = \tilde{\omega}_i \tilde{L}_k - \tilde{\omega}_k \tilde{L}_i, \) \( i, k = 1, 2...8 \) are the \( SU(3) \) indices, and \( \tilde{s}_{34} = (\tilde{s}_{34} + \sqrt{3} \tilde{s}_{45})/2, \) \( \tilde{s}_{35} = (\tilde{s}_{35} + \sqrt{3} \tilde{s}_{85})/2, \) \( \tilde{s}_{36} = (\tilde{s}_{36} + \sqrt{3} \tilde{s}_{68})/2, \) \( \tilde{s}_{37} = (-\tilde{s}_{37} + \sqrt{3} \tilde{s}_{78})/2, \) \( \tilde{s}_{38} = (\tilde{s}_{38} + \sqrt{3} \tilde{s}_{87})/2, \) \( \tilde{s}_{45} = (\tilde{s}_{45} + \sqrt{3} \tilde{s}_{58})/2, \) \( \tilde{s}_{46} = (\tilde{s}_{46} + \sqrt{3} \tilde{s}_{67})/2, \) \( \tilde{s}_{47} = (-\tilde{s}_{47} + \sqrt{3} \tilde{s}_{76})/2 \)
\( \tilde{s}_{48} = (\tilde{s}_{48} + \sqrt{3} \tilde{s}_{87})/2, \) \( \tilde{s}_{56} = (\tilde{s}_{56} + \sqrt{3} \tilde{s}_{67})/2, \) \( \tilde{s}_{57} = (-\tilde{s}_{57} + \sqrt{3} \tilde{s}_{76})/2, \) \( \tilde{s}_{58} = (\tilde{s}_{58} + \sqrt{3} \tilde{s}_{87})/2 \).
\( \tilde{s}_{67} = \tilde{s}_{67}, \) \( \tilde{s}_{68} = \tilde{s}_{68}, \) \( \tilde{s}_{69} = \tilde{s}_{69}, \) \( \tilde{s}_{78} = \tilde{s}_{78}, \) \( \tilde{s}_{79} = \tilde{s}_{79}, \) \( \tilde{s}_{87} = \tilde{s}_{87}, \) \( \tilde{s}_{89} = \tilde{s}_{89}, \) \( \tilde{s}_{98} = \tilde{s}_{98}, \) \( \tilde{s}_{99} = \tilde{s}_{99} \).
\( \tilde{s}_{ab} \tilde{s}_{cd} - \tilde{s}_{ac} \tilde{s}_{bd} = \tilde{s}_{ad} \tilde{s}_{bc} \).
The formula (14) possesses remarkable symmetry relative to the different \( SU(2) \) subgroups of \( SU(3) \). The functions \( L_8 \) or \( \tilde{L}_8 \) do not enter (14) nor the expression (10) for the baryon number density. The functions \( \tilde{\omega}_i \) are connected to the body-fixed angular velocities of \( SU(3) \) rotations by the transformations \( \tilde{\omega} = U_0^\dagger \omega U_0 - \tilde{\omega}, \) or, equivalently
\[ \tilde{\omega}_i = (R_{ik}(U_0^\dagger) - \delta_{ik}) \omega_k = g_{ki} \omega_k. \]
(15)
Here \( R_{ik}(V^\dagger) = R_{ik}(V) \) is a real orthogonal matrix, \( i, k = 1...8, \) and \( \tilde{\omega}_i^2 = 2(\omega_i^2 - R_{ik}(U_0)\omega_k \omega_l). \)

The expression for the static energy can be obtained from (14) by means of the substitution \( \tilde{\omega}_i \rightarrow 2L_i \) and \( \tilde{s}_{ik} \rightarrow 2n_{ik}, \) \( [13] \) with \( \bar{n}_{ik} \) being the cross product of \( \tilde{L}_i \) an \( \tilde{L}_k \) \( ie \) \( \bar{n}_{ik} = \tilde{L}_i \wedge \tilde{L}_k. \) From (14) we have then the inequality \( [13] \)
\[ E_{\text{stat}} - M.t. \geq 3\pi^2 B(F_u/e), \]
(16)
which was obtained first by Skyrme \( [1] \) for the \( SU(2) \) model.

Eight diagonal moments of inertia and 28 off-diagonal ones define the rotation energy, a quadratic form in \( \omega_i \omega_k \) as follows from (14) and (15). The analytical expressions for the moments of inertia are too lengthy to be reproduced here. Fortunately, it is possible to perform calculations without explicit analytical formulas, by substituting (15) into (14).
For configurations generated by SU(3) projectors the Lagrangian of the system can be written in terms of angular velocities of rotation and moments of inertia in the form (in the body-fixed system):

\[ L_{\text{rot}} = \frac{\Theta_1}{2}(\omega_1^2 + \omega_2^2) + \frac{\Theta_3}{2} \omega_3^2 + \frac{\Theta_4}{2} \omega_4^2 + \frac{\Theta_5}{2} \omega_5^2 + \frac{\Theta_6}{2} (\omega_6^2 + \omega_7^2) + \frac{\Theta_8}{2} \omega_8^2 + \Theta_{38} \omega_3 \omega_8 + WZ_3 \omega_3 + WZ_8 \omega_8. \]  

(17)

After the standard quantization procedure the Hamiltonian of the system, \( H = \omega_i \partial L / \partial \omega_i - L \), is a bilinear function of the generators \( R_i \) of SU(3) rotations:

\[ H = \frac{R_i^2}{2\Theta_1} + \frac{R_8 (R_3 - WZ_8)^2}{2D_{38}} + \frac{R_4}{2\Theta_4} + \frac{R_3}{2\Theta_5} + \frac{R_2}{2\Theta_6} + \]

\[ + \Theta_3 (R_3 - WZ_8)^2 \frac{2D_{38}}{2D_{38}} - \Theta_{38} (R_3 - WZ_8)(R_3 - WZ_8), \]

(18)

where \( D_{38} = \Theta_3 \Theta_8 - \Theta_{38}^2 \). For the states belonging to a definite SU(3) irrep the rotation energy can be written in terms of the second order Casimir operators of the SU(2) and SU(3) groups:

\[ E_{\text{rot}} = \frac{N(N + 1) - R_3^2}{2\Theta_1} + \frac{U(U + 1) - R_{3,us}^2}{2\Theta_4} + \frac{V(V + 1) - R_{3,ds}^2}{2\Theta_6} + \]

\[ + \frac{\Theta_3 (R_3 - WZ_8)^2}{2D_{38}} + \frac{\Theta_3 (R_3 - WZ_8)^2}{2D_{38}} - \frac{\Theta_{38} (R_3 - WZ_8)(R_3 - WZ_8)}{D_{38}} + \frac{(R_3^2 - R_4^2)(\Theta_5 - \Theta_4)}{4\Theta_4 \Theta_5} \]

(19)

with \( \Theta_4 = 2\Theta_4 \Theta_5 / (\Theta_4 + \Theta_5) \).

\[ R_{3,us} = R_3/2 + \sqrt{3}R_8/2, \quad R_{8,us} = \sqrt{3}R_3/2 - R_8/2, \]

\[ R_{3,ds} = -R_3/2 + \sqrt{3}R_8/2, \quad R_{8,ds} = \sqrt{3}R_3/2 + R_8/2, \]

(20)

with \( R_3 = R_{3,ud}, \quad R_8 = R_{8,ud}; \quad R_{3,ab}, \quad R_{8,ab} \) denote the 3rd and 8th generators of the (a, b) SU(2) subgroup of SU(3).

\( N, U \) and \( V \) are the values of the right isospin in the (u, d) SU(2) subgroup, and so called U-spin and V-spin. They are connected with the second order Casimir operator of the SU(3) group \( C_2(SU_3) = \frac{1}{2}(p^2 + q^2 + pq) + p + q; \quad p, \ q \) being the numbers of the upper and low indices in the tensor describing the SU(3) irrep (p, q) via the symmetric relation \( C_2(SU_3) = N(N + 1) + U(U + 1) + V(V + 1) - (R_3^2 + R_4^2)/2 \). The hypercharge \( Y = 2R_9/\sqrt{3} \). The terms linear in the angular velocities present in the Lagrangian due to the Wess-Zumino term cancel in the Hamiltonian.

For a state with the lowest energy which belongs to the SU(3) singlet with \( (p, q) = 0 \) the quantum correction simplifies to:

\[ \Delta E = \frac{1}{2D_{38}} (\Theta_3 WZ_3^2 + \Theta_3 WZ_8^2 - 2\Theta_{38} WZ_3 WZ_8). \]

(21a)

The quantities \( D_{38}, \Theta_3, \Theta_8, WZ_3^2 + WZ_8^2 \) do not depend on the choice of the particular subgroup of SU(3) group, therefore Eq.(21) also is invariant, as one expects from general arguments, since

\[ \Theta_3 WZ_3^2 + \Theta_3 WZ_8^2 - 2\Theta_{38} WZ_3 WZ_8 = (\Theta_3 + \Theta_4)(WZ_3^2 + WZ_8^2) - \Theta_3 WZ_3^2 - \Theta_8 WZ_8^2 - 2\Theta_{38} WZ_3 WZ_8. \]

The generalization of (21a) to an arbitrary case is

\[ \Delta E^{\text{min}} = \Theta_{ij}^{-1} WZ_i WZ_j / 2, \]

(21b)
where $\Theta^{-1}_{ij}$ is the matrix inverse to the tensor of inertia $\Theta_{ij}$. Eq. (21b) is valid for all cases except the degenerate case when $Det\Theta_{ij} = 0$. Note, that the case of the $SU(2)$ embedding just corresponds to the degenerate case, with $\Theta_8 = 0$, and this leads to the rigorous quantization condition [10].

As can be seen from the Table, for the configurations we consider here and for our choice of the subgroups $\Theta_3^2 \ll \Theta_3^2$, so a small correction due to $\Theta_3^2$ can be neglected, and $\Delta E \approx \frac{3}{2}(\Theta_3^2) + \frac{2}{2}(\Theta_3^2)$. Since the moments of inertia are of the order of magnitude $\sim 10 GeV^{-1}$ and both $WZ_3$ and $WZ_8$ are not greater $\sim 1$, the quantum correction for the lowest states does not exceed several tens of $MeV$: see the Table.

The quantum correction from the $SU(3)$ zero modes which should be added to the classical energy of the soliton located originally in the $(u, d)$ $SU(2)$ subgroup equals to $\Delta E = 3B/4F_Z \sim 365 MeV$ [16].

Note, that for the $SO(3)$ solitons considered in [3, 4] the $WZ$-term is equal to zero, and for this reason the lowest quantized state was an $SU(3)$ singlet without any quantum correction to its mass.

4. The $FSB$ part of the mass terms in the Lagrangian density which defines, in particular, the mass splittings inside $SU(3)$ multiplets is defined, as usual, by

$$L_{FSB} = \frac{F_0^2}{24}m_0^2 - \frac{F_0^2}{24}m_0^2 \sqrt{3} \lambda_8(U + U^\dagger - 2) + \frac{F_0^2}{24}m_0^2 \sqrt{3} \lambda_8(U + U^\dagger - 2)$$

$$\Delta E = 2C_S \Delta E_{M.L} \left( \frac{2C_S \Delta E_{M.L}}{F_0^2 m_0^2} - 1 \right)$$

where $\Delta E_{M.L}$ is the $FS$ mass term contribution shown in the Table. Numerically, it equals 1.13 $GeV$ for $B = 2$ and 3.2 $GeV$ for $B = 6$. Thus it makes sense to include the $FSB$ part of the mass term into the minimization procedure.

We do not consider here the quantum corrections due to the rotations in the ordinary space because they are explicitly zero for the lowest states with even $B$ ($J = 0$), or small [18].

Our investigations have shown that the space of local minima for the $SU(3)$ skyrmions has a rather complicated form. In addition to the known local minima - the $SU(2)$ embeddings, $SO(3)$ solitons and skyrmion molecules - there are also other local minima best approximated by the $SU(3)$ projector ansaetze [3, 4].

It is possible to quantize these configuratons and to estimate the spectrum of states by means of the collective coordinate quantization procedure. The total sum of the classical mass and of the zero modes quantum corrections can be the smallest for some baryon numbers. However, to draw the final conclusions one needs to calculate the Casimir energies of solitons which are essentially different for solitons of different form and size. This is a very complicated problem solved approximately [19 - 21] only for the $B = 1$ hedgehog configurations. For the case of $SU(2)$ embeddings it was, however, possible to draw conclusions of physical relevance under the natural assumption that the unknown Casimir energy, or loop corrections, cancel in the difference of energies of states with different flavours, and that
the states with $u, d$ flavours can be identified with ordinary nuclei [22]. But this assumption may be incorrect for the states generated by the $SU(3)$ projector ansatz.

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