Interacting Fermions on Noncommutative Spaces: Exactly Solvable Quantum Field Theories in $2n + 1$ Dimensions

Edwin Langmann
Interacting fermions on noncommutative spaces: 
Exactly solvable quantum field theories in $2n+1$ 
dimensions:

Edwin Langmann

Mathematical Physics, Department of Physics, Royal Institute of Technology, 
SCFAB, SE-10691 Stockholm, Sweden

Abstract

I present a novel class of exactly solvable quantum field theories. They describe 
non-relativistic fermions on even dimensional flat space, coupled to a constant external 
magnetic field and a four point interaction defined with the Groenewold-Moyal 
star product. Using Hamiltonian quantization and a suitable regularization, I show 
that these models have a dynamical symmetry corresponding to $\mathfrak{gl}_\infty \oplus \mathfrak{gl}_\infty$ at the 
special points $B\theta = I$ and $B\theta = -I$, where $B$ and $\theta$ are the matrices defining the 
magnetic field and the star product, respectively. I construct all eigenvalues and 
eigenstates of the many-body Hamiltonian at these special points. I argue that 
this solution cannot be obtained by any mean-field theory, i.e. the models describe 
correlated fermions. I also mention other possible interpretations of these models 
in solid state physics.
1. Introduction. Exactly solvable models play a special role in quantum field theory. They provide a testing ground for general methods and concepts applicable to other, more complicated, models which can be studied only by approximation methods. Moreover, such approximations are usually based on some exact solution. The majority of the known exactly solvable quantum field theories are free (non-interacting), and most of the few known interacting ones are in low dimensions and/or have supersymmetry (see e.g. [1, 2]).

In this paper I find and study a particular class of interacting field theories without supersymmetry and which are exactly solvable in $2n + 1$ dimensional spacetime with $n = 1, 2, \ldots$ arbitrary. They describe non-relativistic fermions coupled to a constant, external magnetic field and four-point interactions defined with the Groenewold-Moyal star product (a review about the star product including references to earlier work is [3]). These models are specified by two antisymmetric $2n \times 2n$ matrices $B$ and $\theta$ which define the magnetic field and the star product, respectively (precise definitions will be given below). Using the Hamiltonian framework, I will solve these models for the special cases $B\theta = I$ and $B\theta = -I$ and arbitrary interaction constants. A crucial point in the argument is to expand the model in a convenient basis which brings it to a matrix form making manifest a huge dynamical symmetry. It is interesting to note that, in this latter form, this model also has a natural solid-state physics interpretations: it also describes correlated fermions in $2 + 1$ dimensions [4]. Readers interested only in this aspect of this work may skip all the discussion on non-commutative spaces and start immediately with Section 4 below. Another aspect is the mathematical structure involved in the solution: the above-mentioned dynamical symmetry corresponds to the Lie algebra $gl_\infty \oplus gl_\infty$, and, from a mathematical point of view, the solution boils down to decomposing the regular representations of all permutation groups into irreps. The latter is a classical problem in group theory, and we can solve the model by using some beautiful group theory results obtained a long time ago (books containing proofs of the results needed are, e.g. [5, 6, 7]). However, I will first derive the solution by an explicit construction exploiting the dynamical symmetry. That method is perhaps less elegant but is, as I hope, a good starting point for getting an intuitive understanding of the solution. (The former elegant method is, to my opinion, only easier for experts in group theory. One could also read the two methods together as a pedestrian proof of certain group theory results.) The group theory point of view suggests further generalizations of the model with additional (peculiar) $p$-body interactions with $p = 3, 4, \ldots$. These additional interactions correspond to higher order Casimir operators and leave the model solvable.

The idea to use interacting particle models which can be solved exactly using group theory is, of course, not new: This is a standard and very successful approach in nuclear physics (for review see e.g. [8]), but there one is usually interested only in small fermion numbers (less then $10^2$, say). Such models have been used, but not very much, also in solid state physics: interesting work by Richardson on an exact solution of a model for superconductivity in the 60’s [9] received little attention in condensed matter physics until a few years ago [10], and generalizations of this model were proposed only recently.

\footnote{To avoid misunderstanding I stress that the free part of the model is the standard Hamiltonian of fermions in an external magnetic fields: No star product is used there.}
These models are different from ours in that they are related to Lie groups of small dimension [e.g. SU(2) or SO(4)], whereas our model is related to an infinite dimensional Lie group.

As mentioned, the models considered can be naturally interpreted as field theories of fermions on a noncommutative phase space (time remains commutative): using the star product implies that the components of the spatial variables $x = (x^1, \ldots, x^{2n})$ are made noncommutative

$$x^\mu \star x^\nu - x^\nu \star x^\mu = -2i\theta^{\mu\nu}$$

($\mu, \nu = 1, \ldots, 2n$). Thus, generalizing a standard field theory model by replacing the pointwise product of fields by the star product can be interpreted as making space noncommutative. In a similar manner, coupling the fermions to a magnetic field can be interpreted as a further generalization where momentum space is made noncommutative as well: one replaces the commutative momentum operators by noncommutative ones,

$$-i\partial_{\mu} \to -i\partial_{\mu} - B_{\mu\nu} x^\nu.$$

Field theory models with noncommutative space have received much attention recently (for review and a fairly exhaustive list of references I refer to [12, 13, 14]). In particular, these standard noncommutative field theories are known to simplify for the limiting case $\theta = \infty$ [15, 16]. I show here that there are other simple – in fact, exactly solvable – cases in the extended family of field theories with noncommutative phase spaces, namely all those where $B\theta = \pm I$. It is interesting to note that $\theta = \infty$ is precisely the limit where the kinetic energy term is irrelevant, whereas the solutions obtained in this paper allow to study the effect of (a particular kind of) kinetic energy as well.

It is worth noting that the field theories considered here are special in that they look the same in position and in Fourier space and, in particular, for $B\theta = \pm I$ they are (essentially) invariant under Fourier transformation [17]. I believe that it is the latter property which makes this model exactly soluble. In fact, the boson models discussed in [17] are soluble at these special points as well [18]. I also mention that there is a (classical) field theory model of bosons which somewhat similar to the one we solve and whose integrability was noted already in [19]. I believe that the quantum version of this model should be solvable by similar methods as the ones used here.

It is interesting to note that these field theories in 2+1 dimensions provide microscopic models for quantum Hall systems: I add to the standard free quantum Hall Hamiltonian a particular four-point interaction of the fermions, and the resulting model is exactly solvable. In fact, the model I solve is somewhat more general in that there is also a term describing a confining electrical potential. I will elaborate this interpretation slightly in the conclusions, but I stress that my results (i.e. eigenvalues and eigenfunctions) provide only a first step to understand the physics of this model.

The plan of the rest of this paper is as follows. In the next section I first define the models in the context of noncommutative field theories (Section 2). In Sections 3–6 I solve these models in 2+1 dimension. The key is two mathematical facts about Landau eigenfunctions which allow to write the Hamiltonian in a matrix form (Section 3). In Section 4 I give a precise meaning to the model using a natural regularization and constructing its Hilbert space representation. In this Section I also demonstrate the dynamical symmetry
of the model. I then give two methods of solution, one using a somewhat pedestrian approach (Section 5), and another using group theory (Section 6). The generalization of this to $2n + 1$ dimensions in Section 7 is remarkably simple: from an abstract point of view, changing the dimension does not make much of a difference. Section 8 contains a short discussion of generalized models with additional $p$-body interactions ($p = 3, 4 \ldots$). I end in Section 9 with a few remarks on possible applications, generalizations, and open questions. For the convenience of the reader two appendices are added: Appendix A contains an elementary proof of the above-mentioned facts, and Appendix B is to exemplify the general group theory results.

2. Definition of the models. The models we consider are defined by a Hamiltonian $H = H_0 + H_{\text{int}}$ where the free part is $H_0 = \int d^{2n}x \psi^\dagger(x) H_B \psi(x)$ with

$$H_B = (-i\partial_\mu - B_{\mu\nu} x^\nu)^2$$

the (generalized one-particle) Landau Hamiltonian; $B = (B_{\mu\nu})$ is the antisymmetric matrix defining an external magnetic field. (Our conventions for $B$ and $\mu$ differ from the usual ones by factors of 2, in order to simplify some formulas. In particular, the magnetic field is $2B$.) The interaction is

$$H_{\text{int}} = g_0 \int d^{2n}x (\psi^\dagger \star \psi \star \psi^\dagger \star \psi)(x)$$

where $\star$ is the Groenewold-Moyal star product as usual (see e.g. Eq. (9) in [17]) with our conventions such that

$$x^\mu \star x^\nu = x^\mu x^\nu - i\theta^{\mu\nu}$$

and $\theta^{\mu\nu} = -\theta^{\nu\mu}$. The real coupling constant $g_0$ is arbitrary.

Formally, I define the quantum field theory by postulating standard anticommutator relations for the fermion fields:

$$\psi(x)\psi^\dagger(y) + \psi^\dagger(y)\psi(x) = \delta^{2n}(x - y)$$

etc. I will give a precise meaning to this model by introducing a particular cutoff $\Lambda$ taking care of all potential divergences. The limit $\Lambda \to \infty$ turns out to be rather trivial, at least if one is only interested in the eigenfunctions and eigenvalues of the Hamiltonian.\footnote{To make sense of the grand canonical partition function of the model at finite fermion density may require additional renormalizations, but the study of this is beyond the scope of the present paper.} In fact, we will be able to solve a more general class of models with

$$H_0 = \int d^{2n}x \psi^\dagger(x)(aH_B + bH_B - \mu)\psi(x),$$

where we allow for an arbitrary linear combination of the Landau terms with $B$ and $-B$ ($a, b > 0$ are arbitrary constants), and I also find it convenient to inserted a chemical potential $\mu$ (= arbitrary real constant). This generalization is interesting since, as is well-known, the Landau Hamiltonian $H_B$ is highly degenerate, but in this extended family of models we can also study the case where this degeneracy is lifted. It is worth noting
that, in the context of a quantum Hall system, the second term has a natural physical interpretation as a confining electrical potential. As we will see, there is only a single divergence in the model which can be removed by normal ordering or, equivalently, an additive renormalization of $\mu$.

We now rewrite our model by expanding the fermion fields in a convenient basis of one-particle wave functions. The resulting Hamiltonian has a structure such that we can construct a complete set of exact eigenstates and eigenvalues. To simplify the presentation we first concentrate on the special case $2n = 2$ and then give the generalization to arbitrary dimensions $2n$.

**Remarks:** To prepare for the results in the next Section it might be helpful to point out a complimentary interpretation of the ‘noncommutative phase-space’ model above: As known since a long time, the star product corresponds to a representation of Hilbert space operators by functions; see e.g. [3]. Thus the non-commutative field theory amounts to replacing the one-particle states of the theory by Hilbert space operators: plane waves are replaced by operators obeying

$$e^{ik\hat{x}} e^{ik'\hat{x}} = e^{i(k+k')\cdot\hat{x}} e^{i\theta_{\mu}k_{\mu}}$$  \hspace{1cm} (7)

where $k \cdot \theta k' = k_\mu \theta^{\mu\nu} k'_\nu$. The latter relation is equivalent to interpreting the components of $\hat{x}$ as operators obeying the commutator relations $[\hat{x}^\mu, \hat{x}^\nu] = -2i\theta^{\mu\nu}$ (the Fourier variables $k_\mu, k'_\nu$ are real numbers). Thus, instead of fields $\Psi(x)$ multiplied with the star product, we could also use fields

$$\Psi(\hat{x}) = \int_{\mathbb{R}^{2n}} \frac{d^{2n}k}{(2\pi)^n} \tilde{\Psi}(k) e^{ik\cdot\hat{x}}$$  \hspace{1cm} (8)

with ‘standard’ products (and the Fourier transform $\tilde{\Psi}$ an ordinary function). Indeed, for $a = 1$ and $b = 0$ one could also write the Hamiltonian above as

$$\mathcal{H} = \text{Trace}_{\mathcal{L}^2(\mathbb{R}^{2n})} \left( \Psi^\dagger(\hat{x}) \hat{p}^2 \Psi(\hat{x}) + g[\Psi^\dagger(\hat{x})\Psi(\hat{x})]^2 \right)$$  \hspace{1cm} (9)

with

$$\hat{x}^\mu = x^\mu - i\theta^{\mu\nu} \partial_\nu, \quad \hat{p}_\mu = -i\partial_\mu - B_{\mu\lambda} x^\lambda$$  \hspace{1cm} (10)

the ‘Schrödinger representation’ of the non-commutative field theory. Eq. (10) implies

$$[\hat{p}_\mu, \hat{x}^\nu] = -i(\delta^\nu_\mu + B_{\mu\lambda} \theta^{\nu\lambda}),$$  \hspace{1cm} (11)

i.e. if $B\theta = -I$ the ‘non-commutative positions’ $\hat{x}$ commute with the ‘non-commutative Laplacian’ $\hat{p}^2$! This suggests that the model should be special at $B\theta = -I$, and in particular it should have a huge gauge-like symmetry.

We also recall that, in a standard field theory, one can expand the fields $\Psi$ in a basis $|\ell\rangle$ (Dirac-notation) of eigenfunctions parametrized by positive integers $\ell$, and the expansion coefficients are infinite vectors $(A_\ell)_{\ell=1}^\infty$. The corresponding basis $|\ell\rangle\langle m|$ for the Hilbert space operators is labeled by two integers $\ell, m$, and we thus should expect that fields in our model can be represented by infinite matrices $(A_{\ell m})_{\ell,m=1}^{\infty}$. This is a simple argument suggesting a close relation of noncommutative field theories and matrix models.
These remarks will be made precise in the next Section.

3. Matrix form of the $2+1$ dimensional model. We assume $2n=2$. My discussion will be based on two mathematical facts. Both facts are known since many years in the context of phase space quantization; see e.g. [3]. More recently they have been used in the context of noncommutative solitons [15] (for other references see [12, 13, 14]). However, I would like to stress their central importance for noncommutative field theory somewhat more than usually done in this context, not only for the models discussed here but in general. I have formulated them such that they are true, as they stand, also in $2n$ dimensions (this is shown further below). For the convenience of the reader, (elementary) proofs of these facts are given in Appendix A.

**Fact 1:** There is a complete, orthonormal basis of one-particle wave functions $\phi_{\ell m}(x)$, labeled by positive integers $\ell$ and $m$, and which have the following star product relations,

$$\phi_{\ell m} \star \phi_{\ell' m'} = r \delta_{m,\ell'} \phi_{\ell m'}$$

with some positive constant $r$.\(^3\) Moreover, $\phi_{\ell m}^{\dagger} = \phi_{m \ell}$. Explicit expressions for these functions can be found in Appendix A. This fact suggests to expand the fermion fields in this basis,

$$\Psi(x) = \sum_{\ell, m} A_{\ell m} \phi_{\ell m}(x), \quad \Psi^{\dagger}(x) = \sum_{\ell, m} A_{\ell m}^{\dagger} \phi_{\ell m}^{\dagger}(x),$$

where the fermion operators $A_{\ell m}^{(i)}$ obey the usual anticommutator relations,

$$A_{\ell m} A_{\ell' m'}^{\dagger} + A_{\ell' m'}^{\dagger} A_{\ell m} = \delta_{\ell, \ell'} \delta_{m, m'}$$

$$A_{\ell m}^{\dagger} A_{\ell' m'}^{\dagger} + A_{\ell' m'} A_{\ell m} = 0.$$ (14)

A simple computation (using Fact 1 and $\int d^2x \phi_{\ell m}^{\dagger} \star \phi_{\ell' m'}(x) = \delta_{\ell, \ell'} \delta_{m, m'}$) yields,

$$\mathcal{H}_{\text{int}} = g \sum A_{m_1 \ell_1}^{\dagger} A_{m_1 \ell_2} A_{m_2 \ell_2}^{\dagger} A_{m_2 \ell_1} \equiv g \text{Tr}(A^{\dagger} A A^{\dagger} A)$$

where $g = g_0 r^2$. Here we interpreted the $A_{\ell m}$ as components of a (infinite) matrix $A$ with adjoint $A^{\dagger}$ defined as $(A^{\dagger})_{\ell m} = A_{m \ell}$, and Tr is the usual matrix trace (sum of the diagonal). Thus this basis $\phi_{\ell m}$ allows us to write the interaction in a matrix form. We now observe that this basis has a remarkable physical interpretation.

**Fact 2:** The functions $\phi_{\ell m}$ are common eigenfunctions of the Landau Hamiltonians $H_B$ and $H_{-B}$ in (2) for $B = \theta^{-1}$. The corresponding eigenvalues $E_{\ell}$ and $E_{m}$ only depend on $\ell$ and $m$, respectively.

---

\(^3\)This is true in any dimension $2n$. For completeness I quote $r^{-2} = (4\pi)^n \sqrt{\det(\theta)}$. 

---

6
The latter property is the well-known degeneracy of the Landau Hamiltonian. We also recall that the eigenvalues are identical to those of a harmonic oscillator,

\[ E_\ell = 4|B|(\ell - \frac{1}{2}) \quad (16) \]

(recall that we label states by \( \ell = 1, 2, \ldots \)) and similarly for \( E_m \). Thus, if we choose the magnetic field as \( B = \theta^{-1} \), the free part of the Hamiltonian in (6) has the following simple form,

\[ \mathcal{H}_0 = \sum_{m, \ell} (E_m + \tilde{E}_\ell) A_{\ell m}^\dagger A_{\ell m} \quad (17) \]

where we set \( a = 1 \) and defined \( \tilde{E}_\ell = bE_\ell - \mu \). The latter is a useful notation since in our solution below the explicit form of \( E_m \) and \( \tilde{E}_\ell \) is not needed.

Remarks: Fact 1 shows that the functions \( r\phi_{\ell m} \) provide a representation of rank-one Hilbert space operators \(|\ell\rangle \langle m| \) (Dirac bra-ket notation). The existence of such functions should not be surprising (see the remarks at the end of the last Section). What I find remarkable, however, is that these functions are old friends to anybody familiar with the theory of the fractional quantum Hall effect (Fact 2) [20].

4. Regularization and integrability. We consider the eigenstates and eigenvalues of the field theory Hamiltonian \( \mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{int}} \) defined in Eqs. (17) and (15), with fermion operators obeying the usual anticommutator relations (14). We use the regularization defined by restricting the quantum numbers \( \ell, m = 1, 2, \ldots \Lambda \) with \( \Lambda < \infty \). It is interesting to note that \( \Lambda \) provides a UV (= short-distance) and IR (= long-distance) cutoff at the same time (the interested reader can find a more detailed discussion on this in [17]). The regularized model is defined on the fermion Fock space \( \mathcal{F}_\Lambda \) generated by the fermion creation operators \( A_{\ell m}^\dagger \), \( 1 \leq \ell, m < \Lambda \), from a normalized vacuum \( \Omega \) defined by

\[ A_{\ell m}^\dagger \Omega = 0 \quad \text{for all } \ell, m, \quad (18) \]

and such that \( \dagger \) is the Hilbert space adjoint. Since \( \mathcal{F}_\Lambda \) is finite dimensional\(^4\) all potential divergences are taken care of by this regularization.

It is convenient to introduce normal ordering : \( \cdots \dagger \) : in the interaction as usual.\(^5\) We observe that

\[ : \mathcal{H}_{\text{int}} :: = \mathcal{H}_{\text{int}} - g\Lambda \sum_{m, \ell} A_{\ell m}^\dagger A_{m \ell}, \quad (19) \]

i.e., normal ordering amounts to a shift of the chemical potential, \( \mu \rightarrow \mu - g\Lambda \). This shift diverges as \( \Lambda \rightarrow \infty \) and corresponds to a renormalization of the chemical potential. After this renormalization all eigenvalues and eigenstates have a well-defined limit \( \Lambda \rightarrow \infty \). To see that we recall the following natural basis in the fermion Fock space \( \mathcal{F}_\Lambda \),

\[ |N\rangle = A_{\ell_1 m_1}^\dagger A_{\ell_2 m_2}^\dagger \cdots A_{\ell_N m_N}^\dagger \Omega \quad (20) \]

\[^4\mathcal{F}_\Lambda \cong \mathbb{C}^{N'} \text{ with } N' = 2^{N^2}.
\[^5\text{A}_{\ell_1 m_1}^\dagger A_{\ell_2 m_2}^\dagger A_{\ell_3 m_3}^\dagger A_{\ell_4 m_4}^\dagger := A_{\ell_2 m_2}^\dagger A_{\ell_3 m_3}^\dagger A_{\ell_4 m_4}^\dagger A_{\ell_1 m_1}^\dagger.\]
distinguished by the fermion number \( N = 0, 1, 2, \ldots \) and labeled by \( N \) distinct pairs \((\ell_j, m_j)\). As we make explicit below, the Hamiltonian \( \mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{int}} \) : acting on all \(|N\rangle\) is always well-defined even in the limit \( \Lambda \to \infty \). Thus normal ordering is enough to remove all divergences in all the eigenfunctions and eigenvalues of the model.

In the following I write \( \mathcal{H}_{\text{int}} \) short for :\( \mathcal{H}_{\text{int}} :\).

\textit{Remark:} The computation of other quantities of physical interest may require some additional regularization and renormalization. For example, interpreting the model as a quantum Hall system, one would be interested to study the model at fixed fermion density, i.e compute the partition function of the model at finite cut-off \( \Lambda \), and then take the limit \( \Lambda \to \infty \) at fixed expectation value of \( N/\Lambda \).

Before constructing the solution we give an indirect argument showing integrability of the model. For that we define the operators

\[
\rho_{\ell m} = \sum_k A^\dagger_{k\ell} A_{km}, \quad \tilde{\rho}_{\ell m} = \sum_k A^\dagger_{tk} A_{mk}
\]

and observe that they provide two commuting representations of the Lie algebra \( \mathfrak{gl}_\infty \): using the fermion anticommutator relations one can show that the \( \rho \)'s obey the commutator relations

\[
[\rho_{\ell m}, \rho_{\ell' m'}] = \delta_{m, m'} \rho_{\ell m} - \delta_{\ell, \ell'} \rho_{\ell m}
\]

and similarly for the \( \tilde{\rho} \)'s, and \([\rho_{\ell m}, \tilde{\rho}_{\ell' m'}] = 0\) (the latter fact is not obvious but follows from a straightforward computation). Moreover,

\[
\rho^\dagger_{\ell m} = \rho_{m \ell}
\]

and similarly for the \( \tilde{\rho} \)'s. We thus see that the operators \( \rho \) and \( \tilde{\rho} \) represent the Lie algebra \( \mathfrak{gl}_\infty \oplus \mathfrak{gl}_\infty \). We also note

\[
\rho_{\ell m} \Omega = \tilde{\rho}_{\ell m} \Omega = 0 \quad \text{for all } \ell, m.
\]

\textit{Remark:} For the readers appreciating fine points in analysis I note that when I write \( \mathfrak{gl}_\infty \) I mean the inductive limit \( \Lambda \to \infty \) of \( \mathfrak{gl}_\Lambda \) with \( \Lambda \) the matrix cutoff described above: since we only consider the action of operators on finite particle vectors \(|N\rangle\), this limit is trivial. Put differently: some of our Fock space operators become unbounded for \( \Lambda \to \infty \), but we always consider them on a particularly nice common dense invariant dense domain. (One could without difficulty replace everywhere in our discussion ‘\( \infty \)’ by ‘\( \Lambda \)’).

I now discuss the dynamical symmetry of the model. Note that the free part of the model is a linear superposition of Cartan elements of these representations, and the interaction is proportional to a Casimir operator. More specifically, (17) is equivalent to

\[
\mathcal{H}_0 = \sum_m (E_m \rho_{mm} + \tilde{E}_m \tilde{\rho}_{mm}),
\]
and the interaction can be written in the following two equivalent forms,

$$\mathcal{H}_{\text{int}} = g \sum_{\ell,m} : \rho_{\ell m} \rho_{m \ell} : = -g \sum_{\ell,m} : \tilde{\rho}_{\ell m} \tilde{\rho}_{m \ell} :$$  \hspace{1cm} (26)

(the first equality here is obvious from (15), and the second is obtained by interchanging the two \(A\)'s in (15) using that all fermion operators anticommute under the normal ordering symbol). This shows that \(\mathcal{H}\) is a sum of commuting operators. Moreover, even though the \(\rho\)'s and \(\tilde{\rho}\)’s commute with the interaction \(\mathcal{H}_{\text{int}}\), they do not commute with \(\mathcal{H}_0\):

$$[\mathcal{H}_0, \tilde{\rho}_{\ell \ell'}] = (\tilde{E}_\ell - \tilde{E}_{\ell'}) \tilde{\rho}_{\ell \ell'}$$
$$[\mathcal{H}_0, \rho_{mm'}] = (E_m - E_{m'}) \rho_{mm'}.$$  \hspace{1cm} (27)

This shows that the Lie algebra \(\mathfrak{gl}_\infty \oplus \mathfrak{gl}_\infty\) is a dynamical symmetry for our model. This rich symmetry structure suggests that group theory should provide powerful tools to elegantly solve this model. We also note the following commutator relations

$$[\rho_{\ell \ell'}, A_{\ell m}^\dagger] = \delta_{\ell \ell'} A_{\ell m}^\dagger$$
$$[\rho_{m'm''}, A_{\ell m}^\dagger] = \delta_{m'm''} A_{\ell m}^\dagger$$  \hspace{1cm} (28)

which will be useful for us below.

We now compute the action of the Hamiltonian on the vectors \(|N\rangle\) defined in (20). All \(|N\rangle\) obviously are eigenstates of the free Hamiltonian \(\mathcal{H}_0\) in (17) with eigenvalue

$$\mathcal{E}_0 = \sum_{j=1}^{N} \left( \tilde{E}_{j} + E_{m_j} \right).$$  \hspace{1cm} (29)

Moreover, by a straightforward computation using the fermion anticommutator relation we obtain the following equation

$$\mathcal{H}_{\text{int}} |N\rangle = 2g \sum_{1 \leq j < k \leq N} T_{(jk)} |N\rangle$$  \hspace{1cm} (30)

with \(T_{(jk)}\) a transposition operator defined as follows,

$$T_{(jk)} A_{\ell_1 m_1}^\dagger \cdots A_{\ell_N m_N}^\dagger \Omega = A_{\ell_1 m_1}^\dagger \cdots A_{\ell_j m_j}^\dagger \cdots A_{\ell_k m_k}^\dagger \cdots A_{\ell_N m_N}^\dagger \Omega \quad (j < k)$$  \hspace{1cm} (31)

(i.e. \(m_j\) and \(m_k\) are interchanged). Eqs. (30)–(31) will be the key to our solution. Note that the operators \(T_{(jk)}\) generate a (highly reducible) representation \(T\) of the permutation group \(S_N\) which acts on the states \(|N\rangle\) by permuting the quantum numbers \(m_j\). Further below we will show how to use the representation theory of \(S_N\) to construct eigenstates and eigenvalues of the model. However, we first turn to a different approach exploiting the dynamical symmetry.

Remark: A physical interpretation of the relations (27)–(28) is as follows. The one-particle energies of our model are sums of two parts, \(E_{\ell m} = \tilde{E}_\ell + E_m\). Since \([\mathcal{H}_0, A_{\ell m}^\dagger] = (\tilde{E}_\ell + E_m) A_{\ell m}^\dagger\), applying the fermion operator \(A_{\ell m}^\dagger\) adds the corresponding one-particle
energy to the state, as usual. The peculiar feature of our model is that we have operators \( \hat{\rho}_{\ell
u} \) and \( \rho_{\nu m'} \) allowing to change only parts of the one-particle energy.

5. Solution I. Pedestrian approach. We observe that we can generate new eigenstates of \( \mathcal{H} \) from known ones by applying operators \( \hat{\rho}_{\ell
u} \) and \( \rho_{\nu m'} \): according to our discussion above this gives new eigenstates where the eigenvalues are changed by amounts \( E_m - E_{m'} \) and \( \bar{E}_{\ell} - \bar{E}_{\nu} \), respectively (this follows from (27)). We thus can obtain many eigenstates from a sufficiently large number of special ones.

We first construct special states of the form (20) which are eigenstates of all transpositions \( T_{(jk)} \) and thus are trivially also eigenstates of \( \mathcal{H}_{\text{int}} \). Obvious such states are those where all \( m_j \) are the same, e.g.

\[
A_{1,1}^\dagger A_{2,1}^\dagger \cdots A_{N,1}^\dagger \Omega
\]

(note that due to the Pauli principle, i.e. \( (A_{\nu m}^\dagger)^2 = 0 \), such a state is non-zero only if all \( \ell_j \) are different): since all \( m_j = m_k \), applying \( T_{(jk)} \) does not change anything, i.e. this is an eigenstate of all \( T_{(jk)} \) with eigenvalue equal to 1. Thus this state is eigenstate of \( \mathcal{H}_{\text{int}} \) with eigenvalue \( 2gc \) where

\[
c = \text{(number of } T_{(jk)} \text{ with } 1 \leq j < k \leq N) = \frac{1}{2} N(N - 1).
\]

A more general such state is

\[
|\lambda_1, \lambda_2\rangle = A_{1,1}^\dagger A_{2,1}^\dagger \cdots A_{\lambda_1,1}^\dagger A_{\lambda_2,2}^\dagger \cdots A_{1,2}^\dagger A_{2,2}^\dagger \cdots A_{N,1}^\dagger \cdots \Omega,
\]

with \( \lambda_1 + \lambda_2 = N \), i.e. we have two groups of fermion creation operators where the \( m_j \) are equal within each group but different in the other group. This is obviously an eigenstate of all \( T_{(jk)} \) with \( j \) and \( k \) both in the same groups (i.e. \( 1 \leq j < k \leq \lambda_1 \) and \( \lambda_1 + 1 \leq j < k \leq \lambda_1 + \lambda_2 \)), and the eigenvalue of all these is +1. Moreover, it is also in an eigenstate of \( T_{1,\lambda_1+1} \) but with eigenvalue equal to −1: \( T_{1,\lambda_1+1} \) interchanges \( A_{1,1}^\dagger \) and \( A_{1,2}^\dagger \), but due to the fermion anticommutator relations this is the same as multiplying with −1. Similarly, all \( T_{j,\lambda_1+1} \) have eigenvalues −1. However, applying \( T_{1,\lambda_1+2} \) changes \( A_{1,1}^\dagger \) to \( A_{1,2}^\dagger \) and \( A_{1,2}^\dagger \) to \( A_{2,1}^\dagger \). Now at least one of the operators \( A_{1,2}^\dagger \) or \( A_{2,1}^\dagger \) appears twice, and the resulting state thus is zero (Pauli principle). The same is true for all other \( T_{(jk)} \) with \( j \) in the first and \( k \) in the second group and \( \ell_k \neq \ell_j \): all these annihilate the state \(|\lambda_1, \lambda_2\rangle\) due to the Pauli principle. We thus conclude that \(|\lambda_1, \lambda_2\rangle\) is an eigenstate of \( \mathcal{H}_{\text{int}} \) with eigenvalue \( 2gc \) where

\[
c = \frac{1}{2} \lambda_1(\lambda_1 - 1) + \frac{1}{2} \lambda_2(\lambda_2 - 1) - \min(\lambda_1, \lambda_2)
\]

(we counted the number of \( T_{(jk)} \) belonging to the same group and subtracted the number of \( T_{(jk)} \) belonging to different groups but with \( \ell_k = \ell_j \)). It now is obvious how to generalize this to states with an arbitrary number \( L = 1, 2, \ldots N \) different groups: The crucial point in the previous example was not only that the \( m_j \) in each group coincide, but also that a maximal number of the \( \ell_j \) in the different groups are the same. We thus define

\[
|\lambda\rangle = A_{1,1}^\dagger A_{2,1}^\dagger \cdots A_{\lambda_1,1}^\dagger A_{\lambda_2,2}^\dagger \cdots A_{\lambda_L,L}^\dagger \cdots A_{1,L}^\dagger A_{2,L}^\dagger \cdots A_{N,1}^\dagger \cdots \Omega \tag{32}
\]
with a set \([\lambda] = [\lambda_1, \lambda_2, \ldots, \lambda_L]\) of positive integers \(\lambda_i\) such that \(\lambda_1 + \lambda_2 + \cdots + \lambda_L = N\). Similarly as above we check that this is an eigenstate of all \(T_{jk}\) with eigenvalue +1 if both \(j, k\) belong to the same group (i.e., \(m_j = m_k\)), -1 if \(j, k\) belong to different groups but \(\ell_j = \ell_k\), and 0 in all other cases. Thus \([\lambda]\) is an eigenstate of \(\mathcal{H}_{\text{int}}\) with eigenvalue \(2g\epsilon_{[\lambda]}\) where

\[
c_{[\lambda]} = \sum_{i=1}^{L} \frac{1}{2} \lambda_i (\lambda_i - 1) - \sum_{1 \leq i < j \leq L} \min(\lambda_i, \lambda_j)
\]

We thus have obtained various particular eigenstates of \(\mathcal{H}\) with different eigenvalues of the interaction \(\mathcal{H}_{\text{int}}\). To avoid confusion we note that each of these states represents a whole class of states, e.g. using (28) we can write any eigenstate of \(\mathcal{H}\) where all \(m_j\) are the same as follows,

\[
A_{\ell_1, 1}^\dagger A_{\ell_2, 2}^\dagger \cdots A_{\ell_N, N}^\dagger \Omega = \hat{\rho}_{\ell_1, 1} \hat{\rho}_{\ell_2, 2} \cdots \hat{\rho}_{\ell_N, N} (\rho_{m_1, 1})^N [[\lambda]],
\]

where \([[\lambda]] = A_{1, 1}^\dagger A_{2, 2}^\dagger \cdots A_{N, N}^\dagger \Omega\) is the particular state which for now is the only one of this kind taken into account. In fact, at this stage we can further restrict ourselves to states \([[\lambda]]\) which are in one-to-one correspondence to partitions \([\lambda] = [\lambda_1, \lambda_2, \ldots, \lambda_N]\) of \(N\),

\[
\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_L > 0, \quad \lambda_1 + \lambda_2 + \cdots + \lambda_L = N.
\]

This allows us to write a somewhat simpler formula for the possible eigenvalues of \(\mathcal{H}_{\text{int}}\),

\[
c_{[\lambda]} = \sum_{i=1}^{L} \frac{1}{2} \lambda_i (\lambda_i + 1) - \sum_{i=1}^{L} i \lambda_i.
\]

All these states \([[\lambda]]\) are obviously also eigenstates of \(\mathcal{H}_0\) with eigenvalue \(\epsilon_0^0 = \epsilon_0\) as in (29) but with \(\ell_j = \tilde{\ell}_j\) and \(m_j = \tilde{m}_j\) where

\[
(\tilde{\ell}_1, \tilde{\ell}_2, \ldots, \tilde{\ell}_N) = (1, 2, \ldots, 1, 2, \ldots, 1, 2, \ldots, \lambda_L)
\]

\[
(\tilde{m}_1, \tilde{m}_2, \ldots, \tilde{m}_N) = (1, 1, \ldots, 1, 2, 2, \ldots, 2, \ldots, L, L, \ldots, L).
\]

Using this notation we can also write

\[
[[\lambda]] = A_{\tilde{\ell}_1, \tilde{m}_1}^\dagger A_{\tilde{\ell}_2, \tilde{m}_2}^\dagger \cdots A_{\tilde{\ell}_N, \tilde{m}_N}^\dagger \Omega
\]

Thus \([[\lambda]]\) is an eigenstate of \(\mathcal{H}\) with eigenvalue \(\epsilon^0 = \epsilon_0^0 + 2g\epsilon_{[\lambda]}\). As discussed above, we can now generate other eigenstates with different eigenvalues by applying operators \(\hat{\rho}_{\ell\ell'}\) and \(\rho_{mm'}\) to known eigenstates. In particular, the state

\[
\Phi = \hat{\rho}_{\ell_1, \tilde{\ell}_1} \hat{\rho}_{\ell_2, \tilde{\ell}_2} \cdots \hat{\rho}_{\ell_N, \tilde{\ell}_N} \rho_{m_1, \tilde{m}_1} \rho_{m_2, \tilde{m}_2} \cdots \rho_{m_N, \tilde{m}_N} [[\lambda]]
\]

is an eigenstate of \(\mathcal{H}\) but with eigenvalue

\[
\epsilon = \epsilon_0 + 2g\epsilon_{[\lambda]}, \quad \epsilon_0 \text{ in (29) and } c_{[\lambda]} \text{ in (34)}
\]

(to see this, apply \(\mathcal{H}_0\) to \(\Phi\) in (37), move it to the right using repeatedly the relations in (27), and note that the energy differences thus generated add up to \(\epsilon_0 - \epsilon_0^0\).) It is
worth noting that the second of the quantum numbers for the operators $\rho$ and $\tilde{\rho}$ in (37) were forced on us by Eqs. (28) and (24): any other choice would have given zero (this will become obvious further below). I will argue below that we thus have obtained all eigenstates and eigenvalues of our model. In fact, we obtained too many: many of the states in (37) are actually actually zero and thus not all of the eigenvalues in (38) exist. For example, if all $m_j = m$ are identical, $\Phi$ in (37) is nonzero only for $[\lambda] = [N]$, i.e. in this case only the eigenvalue $\mathcal{E}$ (38) exists where $c = N(N - 1)/2$. To characterize all non-zero states is a non-trivial combinatorial problem which we will discuss in more detail further below. In the rest of this Section we derive a more explicit formula of the eigenstates $\Phi$ (37).

It is useful to first consider the special case $[\lambda] = [N]$, i.e.

$$\Phi = \tilde{\rho}_{\ell_1,1} \tilde{\rho}_{\ell_2,2} \cdots \tilde{\rho}_{\ell_N,N} \rho_{m_1,1} \rho_{m_2,1} \cdots \rho_{m_N,1} A^\dagger_{1,1} A^\dagger_{2,1} \cdots A^\dagger_{N,1} \Omega.$$ 

We commute $\rho_{m_N,1}$ to the right of all $A^\dagger$’s using repeatedly (28), until it hits $\Omega$ and disappears (gives zero according to (24)). This produces a sum of $N$ terms, and in each of them one of the $\bar{m}_j = 1$ is changed to $m_N$. Similarly we then remove $\rho_{m_{N-1},1}$, which turns in every term one of the remaining $\bar{m}_j = 1$ to $m_{N-1}$, etc., until we remove $\rho_{m_1,1}$ and the remaining $\bar{m}_j = 1$ in each term is turned to $m_1$. Thus we used (28) and (24) to remove all $\rho$’s, which turns all $\bar{m}_j = 1$ to $m_j$, and we obtain $N$! different terms where we have the $m_j$ in all possible different orders. The result after these manipulations can be written as

$$\Phi = \tilde{\rho}_{\ell_1,1} \tilde{\rho}_{\ell_2,2} \cdots \tilde{\rho}_{\ell_N,N} S_{1,2,\ldots,N} A^\dagger_{1,m_1} A^\dagger_{2,m_2} \cdots A^\dagger_{N,m_N} \Omega$$

where $S_{1,2,\ldots,N}$ means symmetrization over all indices $m_j$. In a similar manner we can remove all $\rho_{\ell_j,j}$, and this now turns the indices $\bar{\ell}_j = j$ to $\ell_j$. Since all $\bar{\ell}_j$ are different and equal to only one of the corresponding indices of the $A^\dagger$, the number of terms does not increase and we obtain

$$\Phi = S_{1,2,\ldots,N} A^\dagger_{\ell_1,m_1} A^\dagger_{\ell_2,m_2} \cdots A^\dagger_{\ell_N,m_N} \Omega = \sum_{P \in S_N} A^\dagger_{\ell_1 m_{P(1)}} A^\dagger_{\ell_2 m_{P(2)}} \cdots A^\dagger_{\ell_N m_{P(N)}} \Omega$$

which is the final formula for the eigenstate in this special case.

It is easy to generalize this to the general case by inserting (35) and (36) into Eq. (37). Again we can remove all the $\hat{\rho}$’s and $\tilde{\rho}$’s using (28) and (24). Removing the $\hat{\rho}$’s turns all $\bar{m}_j$’s to $m_j$’s, but we obtain $\lambda_1! \times \lambda_2! \times \cdots \times \lambda_L!$ terms corresponding to how often the numbers $1, 2, \ldots, L$ appear in $\{\bar{m}_j\}_{j=1}^N$ in (35): 1 appears $\lambda_1$ times and thus one has symmetrization in the first $\lambda_1$ indices $m_j$, 2 appears $\lambda_2$ times and thus one has symmetrization in the next $\lambda_2$ indices $m_j$, etc. Similarly, removing the $\tilde{\rho}$’s turns the $\bar{\ell}_j$’s to $\ell_j$’s, but we further increase the number of terms by factors $\lambda_1! \times \lambda_2! \times \cdots \times \lambda_N!$ where $\bar{\lambda}_i$ is the number of times $i$ appears in $\{\bar{\ell}_j\}_{j=1}^N$. The symmetrizations are now in the indices $\ell_j$ at the positions $j$ where the $\bar{\ell}_j$ coincide. The numbers $\bar{\lambda}_i$ define another partition $[\lambda] = [\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_L]$ of $N$, e.g. if $[\lambda] = [5, 3, 2]$ then $[\bar{\lambda}] = [3, 3, 2, 1, 1]$ (readers familiar with Young tableaux will recognize this as the conjugate partition [6]). The resulting formula
for $\Phi$ (37) can be written conveniently in terms of partial symmetrization operators,

$$
\sum_{P \in \mathcal{S}_K} A^\dagger_{\ell_1m_1} \cdots A^\dagger_{\ell_jm_{jP(1)}} \cdots A^\dagger_{\ell_km_{K(K)}} \cdots A^\dagger_{\ell_Nm_N} \Omega := \mathcal{S}_{j_1j_2 \ldots j_K} A^\dagger_{\ell_1m_1} A^\dagger_{\ell_2m_2} \cdots A^\dagger_{\ell_{K(K)}m_{K(K)}} \cdots A^\dagger_{\ell_Nm_N} \Omega
$$

(39)

(this sum of $K!$ terms corresponds to symmetrization in the $K$ indices $m_j$ at the positions indicated by the subscripts), and another similar symmetrization but with respect to the indices $\ell_j$,

$$
\sum_{P \in \mathcal{S}_K} A^\dagger_{\ell_1m_1} \cdots A^\dagger_{\ell_jm_{jP(1)}} \cdots A^\dagger_{\ell_{K(K)}m_{K(K)}} \cdots A^\dagger_{\ell_Nm_N} \Omega := \tilde{\mathcal{S}}_{j_1j_2 \ldots j_K} A^\dagger_{\ell_1m_1} A^\dagger_{\ell_2m_2} \cdots A^\dagger_{\ell_{K(K)}m_{K(K)}} \cdots A^\dagger_{\ell_Nm_N} \Omega.
$$

(40)

The general formula is

$$
\Phi = S_1,2,\ldots,\lambda_1 \mathcal{S}_{\lambda_1+1,\lambda_1+2,\ldots,\lambda_1+\lambda_2} \cdots \mathcal{S}_{\lambda_1+\ldots+\lambda_{N-1}+1,\ldots,N} \times \tilde{\mathcal{S}}_{\lambda_1+1,\lambda_1+2,\ldots,\lambda_1+\lambda_2+1} \tilde{\mathcal{S}}_{2,\lambda_2+1,\ldots,\lambda_2+\lambda_3+2} \cdots \tilde{\mathcal{S}}_{1,\ldots,\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5+\lambda_6+\lambda_7+\lambda_8} |N\rangle
$$

(41)

with $|N\rangle$ defined in (20). To digest this general formula is is useful to consider few special cases, e.g.

$$
\Phi = S_{1,2,3,4,5} S_{6,7,8} S_{9,10} S_{1,6,9} S_{2,7,10} S_{3,8} |N\rangle \quad \text{for } [\lambda] = [5, 3, 2].
$$

(42)

(we used $\tilde{\mathcal{S}}_4 = \tilde{\mathcal{S}}_5 = I$) etc. A simple method to determine the subscripts in (41) is to draw the Young tableaux corresponding to the partition $[\lambda]$ but write the numbers 1, 2, $\ldots$, $N$ instead of the usual boxes, in increasing order from left to right and from up to down. For our example this gives

| 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| 6 | 7 | 8 |   |   |
| 9 | 10|   |   |   |

for $[\lambda] = [5, 3, 2]$. For each column write one $\tilde{\mathcal{S}}$ and for each row one $\mathcal{S}$, and the subscripts of the $\tilde{\mathcal{S}}$ in (41) are given by the numbers in the columns and the subscripts of the $\mathcal{S}$ by the numbers in the rows. (The order in which the $\tilde{\mathcal{S}}$’s or the $\mathcal{S}$’s are written is irrelevant, of course.)

To make contact with the results obtained in the next Section we note that the interaction can be written also as sum of transpositions $\hat{T}_{(jk)}$ in another representation $\hat{T}$ of the permutation group $S_N$: Defining

$$
\hat{T}_{(jk)} A^\dagger_{\ell_1m_1} \cdots A^\dagger_{\ell_{K(K)}m_{K(K)}} \Omega = A^\dagger_{\ell_1m_1} \cdots A^\dagger_{\ell_km_k} \cdots A^\dagger_{\ell_{K(K)}m_{K(K)}} \cdots A^\dagger_{\ell_Nm_N} \Omega \quad (j < k)
$$

(43)

(i.e. $\ell_j$ and $\ell_k$ are interchanged) we observe that $\hat{T}_{(jk)} T_{(jk)} = -I$ (since this amounts to interchanging $A^\dagger_{\ell jm_j}$ and $A^\dagger_{\ell km_k}$ which is the same as multiplication with $-1$). We thus can also write

$$
\mathcal{H}_{\text{int}} |N\rangle = -2g \sum_{1 \leq j < k \leq N} \hat{T}_{(jk)} |N\rangle.
$$

(44)
We thus have two representations $T$ and $\dot{T}$ of $S_N$ in our model which are, however, not independent but such that their product equals the representation $P \rightarrow (-1)^{|P|}$ (= parity of $P \in S_N$). This fact allows us to somewhat simplify Eq. (41): we can replace the symmetrizers with respect to the indices $\ell_j$ by antisymmetrizers with respect to the indices $m_j$, i.e., the symmetrizers $\mathcal{S}$ are equal to

$$
\mathcal{S}_{\ell_1,\ell_2,\ldots,\ell_K} A_{\ell_1 m_1}^\dagger \cdots A_{\ell_j m_{j_P(1)}}^\dagger \cdots A_{\ell_{j_P(2)} m_{j_P(2)}}^\dagger \cdots A_{\ell_K m_{j_P(K)}}^\dagger A_{\ell_{N m_N}}^\dagger \Omega := \frac{1}{K!} \sum_{P \in S_K} (-1)^{|P|} A_{\ell_1 m_1}^{\dagger} \cdots A_{\ell_j m_{j_P(1)}}^{\dagger} \cdots A_{\ell_{j_P(2)} m_{j_P(2)}}^{\dagger} \cdots A_{\ell_K m_{j_P(K)}}^{\dagger} A_{\ell_{N m_N}}^{\dagger} \Omega.
$$

Inserting that, we write $\Phi = Y^{[\lambda]} |N\rangle$. We now observe that $Y^{[\lambda]}$ is, up to normalization, the Young operator [6] (or Young symmetrizer [7]) associated with $[\lambda]$: this result will be obtained in the next section using group theory.

**Remark:** It is interesting to note that the model remains solvable if we add to the Hamiltonian $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{int}}$ Hartree-Fock type terms of the following form,

$$
\mathcal{H}_{\text{HF}} = \sum_{\ell, m, p} \left( \tilde{V}_{\ell m \ell' \ell'' m m} + \tilde{V}_{\ell m \ell' \ell'' m m} + \tilde{W}_{\ell m \ell' \ell'' m m} \right):
$$

where $\tilde{V}, V$ and $W$ are arbitrary model parameters: the eigenstates $\Phi$ in Eq. (37) remain the same, and the eigenvalues $\mathcal{E}$ are changed by adding $\sum_{i \neq j} (V_{\ell_i \ell_j} + \tilde{V}_{m_i m_j} + W_{\ell_i \ell_j})$. We will come back to that in Section 9.

**6. Solution II. Group theory approach.** As mentioned, there is a representation $T$ of the permutation group $S_N$ on the $N$-particle states of the model,

$$
T_P A_{\ell_1 m_1}^{\dagger} A_{\ell_2 m_2}^{\dagger} \cdots A_{\ell_{N m_N}}^{\dagger} \Omega = A_{\ell_{1 m_{P(1)}}}^{\dagger} A_{\ell_{2 m_{P(2)}}}^{\dagger} \cdots A_{\ell_{N m_{P(N)}}}^{\dagger} \Omega
$$

for all $P \in S_N$.

As we now discuss, diagonalizing $\mathcal{H}$ amounts to decomposing $T$ in irreducible representations (irreps). We first recall (30): the interaction applied to any state $|N\rangle$ defined in (20) is proportional to the sum of all transpositions,

$$
C_{N} = \sum_{1 \leq j < k \leq N} T_{(j k)}
$$

to this state. The crucial fact allowing to solve the model is that the free Hamiltonian $\mathcal{H}_0$ commutes with all permutations $T_P$ in (47) (since the eigenvalues $\mathcal{E}_0$ in (29) are invariant under all permutations of the indices $m_j$). If we therefore make an ansatz

$$
\Phi = \sum_{P \in S_N} a_P T_P |N\rangle
$$

(with $|N\rangle$ short for any of the vectors in (20)) and choose the real coefficients $a_P$ so that

$$
C_{N} \sum_{P \in S_N} a_P T_P = c \sum_{P \in S_N} a_P T_P
$$

(50)
for some real $c$, then $\Phi$ is an eigenvector of $\mathcal{H}$ with eigenvalue $\mathcal{E} = \mathcal{E}_0 + 2gc$.

Eq. (50) can be interpreted as an eigenvalue equation for $C_N$ and can be solved as such. There are two obvious solutions: $a_P = 1$ with $c = N(N - 1)/2$ and $a_P = (-1)^{|P|}$ with $c = -N(N - 1)/2$. They correspond the the eigenstates

$$|N\rangle_\pm = \sum_{P \in S_N} (\pm 1)^{|P|} T_P |N\rangle$$

with the corresponding energy eigenvalues

$$\mathcal{E}_\pm = \mathcal{E}_0 \pm N(N - 1)g.$$  

For fixed $\mathcal{E}_0$ these are the extreme eigenvalues. The total number of solutions is $N!$ generically. To appreciate the problem in Eq. (50) it is instructive to solve it by brute-force as discussed in Appendix B, but this approach is only possible for small $N$.

To see that Eq. (50) is equivalent to a classical group theory problem, note that $C_N = \sum_{i<j} T_{i,j}$ is a class operator, i.e. it commutes with all permutations $T_P$. Thus $C_N$ is equal to a constant $c$ in each irreps. Eq. (50) thus amounts to decomposing the representation $T$ into irreps. The irreps of $S_N$ are well-known. They are in one-to-one with the partitions $[\lambda]$ of $N$, and the value $c_{[\lambda]}$ of $C_N$ in this irreps is well-known and equal to what we found in Eq. (34) before (see e.g. Eq. (4-3) in [21]). If all $\ell_j$ and all $m_j$ are different, then the multiplicity of these eigenvalue equals $(k_{[\lambda]})^2$ where

$$k_{[\lambda]} = \frac{N! \prod_{1 \leq i < j \leq L} (h_i - h_j)}{h_1! h_2! \cdots h_L!}, \quad h_i = \lambda_i + L - i$$

is the dimension the irreps $[\lambda]$ (this follows from the fact that $T$ in this case is equivalent to the regular representation of $S_N$ — see e.g. Theorem 3.25 and Eq. (4-4a) in [21]; the interested reader can find some examples in Appendix B). The corresponding $\sum_P a_P T_P$ is equal to the Young operator $Y^{[\lambda]}$ [6, 7]. We thus have recovered the result in the previous Section. The group theory argument is somewhat more powerful in that it also shows that no eigenfunction was missed. Moreover, the Young operators are projections up to normalizations: these normalizations are known and give the normalizations of the eigenstates.

**Remark:** We thus found all eigenstates and eigenvalues of the model. One remaining problem is that we actually found too many: For a given $|N\rangle$, we obtained $N!$ eigenfunctions $Y^{[\lambda]} |N\rangle$, but all of them are linearly independent only if all $N \ell_j$’s and all $N m_j$’s are different. E.g. if all $m_j = m$ are equal, then only one of these eigenvectors is non-zero (the others will all vanish due to the fermion anticommutator relations) and similarly for states where all $\ell_j$ are equal. If we put identical $\ell_j$’s in groups then the $\ell$-degeneracies can be characterized by the numbers $\nu_j$ of elements in the different groups. This defines a partition $[\nu]$. For example, for $(\ell_j) = (2, 5, 5, 2, 3, 2, 5, 2)$ one gets $[\nu] = [4, 3, 1]$. Similarly the degeneracies of the $m_j$’s can be characterized by another partition $[\mu]$. Thus the degeneracies of the eigenvalues $\mathcal{E}$ depend on three partitions. To find all these multiplicities seems like a rather non-trivial problem in combinatorics. If one knew these multiplicities
mult, one could compute the partition functions \( Z = \text{Trace}_F \exp(-\beta \mathcal{H}) \) of the model as follows,

\[
Z = \sum_{(\nu), (\mu), [\lambda]} e^{-\beta \sum_{t=0}^{\infty} (\nu_t \mathcal{E}_t + \mu \mathcal{E}_t) \text{mult}^{[\lambda]}_{[\nu]_{[\mu]}} e^{-\beta 2 g c_{[\lambda]}}}
\]  

(54)

where \((\nu) = (\nu_1, \nu_2, \ldots)\) with \(\nu_t \geq 0\) and \([\nu]\) is the corresponding partition with the \(\nu_t\)'s ordered (to have this sum finite one might still need a finite cut-off \(\Lambda\)).

7. Generalization to 2n dimensions. We now discuss the generalization to 2\(n\) dimensions. There one has (generalized) Landau eigenfunctions \(\phi_{l,m}\) which are labeled by 2\(n\) positive integers, \(l = (\ell_1, \ldots, \ell_n)\) and similarly for \(m\). This is obvious in the coordinate system where the matrix \(B\) has Jordan normal,

\[
(B_{\mu \lambda}) = \begin{pmatrix}
0 & B_1 \\
-B_1 & 0 \\
& & \ddots \\
& & & 0 & B_n \\
& & & -B_n & 0
\end{pmatrix}, \quad B_{\mu} > 0,
\]  

(55)

so that \(H_B\) in (2) is the sum of \(n\) terms \(H_{B_j}\), depending only on the coordinates \(x_{j-1}\) and \(x_{2j}, j = 1, \ldots, n\). Thus \(\phi_{l,m}(x)\) is just the product of \(n\) two dimensional Landau eigenfunctions \(\phi_{\ell_j,m_j}(x_{2j-1}, x_{2j})\), and the corresponding eigenvalues of \(H_B\) are

\[
E_{l} = \sum_{j=1}^{n} 4B_{j}(\ell_{j} - \frac{1}{2}).
\]  

(56)

Then all what we discussed for two dimensions goes through with \(\ell, m\) replaced by \(l, m\) etc. We now observe that we can map the vectors \(l = (\ell_1, \ldots, \ell_n)\) in a one-to-one way to a single positive integer \(\ell\) (e.g. for \(n = 2\) one such map is \((1, 1) \rightarrow 1, (1, 2) \rightarrow 2, (2, 1) \rightarrow 3, (1, 3) \rightarrow 4, (2, 2) \rightarrow 5, \ldots, (\ell_1, \ell - \ell_1) \rightarrow \ell(\ell - 1)/2 + \ell_1\)). Doing that, all formulas obtained in two dimensions hold true as they stand also in 2\(n\) dimensions with the only difference that the eigenvalues \(E_{\ell}\) and \(E_{m}\) are given by somewhat more complicated expressions. However, our construction of eigenstates and eigenvalues above does not rely on the explicit form of these eigenvalues, and we thus have obtained all eigenstates and corresponding eigenvalues for arbitrary dimensions 2\(n\).

8. More general solvable models: An obvious generalization of our model would be to add a further interactions corresponding to a higher Casimir of \(\mathfrak{gl}_\infty\), for example the term \(\mathcal{H}^{(3)} = g^{(3)} : \text{Tr}(\rho^3) :\), i.e.

\[
\mathcal{H}^{(3)}_{\text{int}} := g^{(3)} \sum_{k,l,m} :\rho_{kl}\rho_{lm}\rho_{mk}:
\]  

(57)

with the colons indicating normal ordering as usual (i.e. move all \(A^\dagger\) to the right of all \(A\)'s). Again we can write this interaction also in terms of the \(\tilde{\rho}\)'s, \(\mathcal{H}^{(3)} = g^{(3)} : \text{Tr}(\tilde{\rho}^3)\). This term comes the following 3-body interaction,

\[
\mathcal{H}^{(3)}_{\text{int}} = g^{(3)}_0 \int d^{2n} x (\Psi^\dagger \star \Psi \star \Psi^\dagger \star \Psi \star \Psi^\dagger \star \Psi)(x)
\]  

(58)
where regularization by normal ordering amounts to a renormalization of the 2-body interaction constant $g$ and the chemical potential $\mu$. Applying this interaction to a state $|N\rangle$ (20) yields

$$\mathcal{H}^{(3)}_{\text{int}} : |N\rangle = g^{(3)} \sum T_{(j\ell)} |N\rangle$$

where the sum on the r.h.s. is over all 3-cycles of the permutation group $S_N$ [5]. The eigenstates of the model which we constructed can be chosen such that they are also eigenstates of this interaction. The possible eigenvalues $c^{(3)}$ of this are known: they are characterized by partitions $[\lambda]$ and are given by (see e.g. Eq. (4-3) in [21])

$$c^{(3)} = \frac{1}{3} \left\{ 2N - \frac{3}{2} N^2 + \sum_{i=1}^{N} \lambda_i \left[ \lambda_i^2 - \left( 3i - \frac{3}{2} \right) \lambda_i + 3i(i - 1) \right] \right\}$$

One can add further interactions $\mathcal{H}_{\text{int}}^{(p)} : \text{Tr}(\rho^p) :$, $p = 4, 5, \ldots$, and still have a solvable model. It is interesting to note that

$$\text{Tr}(\rho^p) : = (-1)^{p-1} : \text{Tr}(\rho^p) :$$

9. Outlook and open questions.

- **More general models.** The model solved in this paper is only the simplest in a class of similar models: one can add a flavor (or spin) index to the fermion operators and thus increase the symmetry from $\text{gl}_\infty$ to $\text{gl}_k \otimes \text{gl}_\infty$. This allows for additional types of interactions (spin-spin-like for $k = 2$, e.g.). Models of this kind can be obtained, e.g., by truncating the 2D Hubbard model: simplifying the 2D Hubbard interaction by keeping only particular terms [4]. In 2D one can keep, in addition to Hartree- and Fock-terms (leading to mean field theory), also particular ‘mixed’ terms and still have a model which is, as I believe, exactly solvable [4].

- **Mean field vs. correlations.** To put my results in perspective, I recall a well-known class of exactly solvable models which can be defined by the Hamiltonian $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{HF}}$ with the free part in (17) and the Hartree-Fock interaction in (46) (this is only a special case [4]). For this model all states $|N\rangle$ in (20) are eigenstates: Hartree-Fock interactions only change the energy eigenvalues but not the energy eigenstates. These eigenstates are Slater determinants which correspond to fermion states without correlations. For the models $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{int}} + \mathcal{H}_{\text{HF}}$ the eigenstates are highly non-trivial linear combinations of Slater determinants (generically: there are exceptions, of course): for fixed $\ell_j$’s and $m_j$’s one has (generically) many different Slater determinants where the $\ell_j$’s and $m_j$’s are distributed over the fermions in all kinds of different ways, and only very particular linear combinations of these are energy eigenstates. For example, if all $\ell_j$’s and all $m_j$’s are different, the 10-particle eigenstate in Eq. (42) is a particular sum of $5!3!2!3!3!2! > 10^5$ terms! I believe that most of these states cannot be written as Slater determinants (a proof of this would be welcome). Thus, different from Hartree-Fock models, our model should describe correlated fermions.
Phase transitions? In this paper I only demonstrate how to solve the models. To explore the physics of the solution is left to future work. However, to give a glimpse in that direction, I now discuss one special case of physical interest in which one can see by simple means that the solution can describe interesting physics: Assume \( n = 2, a = 1 \) and \( b = 0 \):

\[
\mathcal{H} = \sum_{m \geq 1} |B|(m-1)\rho_{mm} + g \sum_{\ell,m,\ell',m'} \rho_{\ell m} \rho_{\ell' m'} : \quad (62)
\]

(I set the chemical potential \( \mu \) to some convenient value). As discussed, this is a toy model for a quantum Hall system [20]. For \( b = 0 \) it is natural to keep the cutoff \( \Lambda \) finite: it has a natural interpretation as a spatial cutoff, and \( \nu = N/\Lambda \) is the fermion density (‘filling factor’) of the QH system. The energy eigenvalues are sums of two terms: the kinetic energy \( \mathcal{E}_0 = |B| \sum_{j=1}^N (m_j - 1) \) and the correlation energy \( \mathcal{E}_{\text{corr}} = 2gc|\lambda| \). Let \( N < \nu \). Then obviously \( \mathcal{E}_0 = 0 \) is minimal if all \( m_j = 1 \) (i.e. all fermions are in the lowest Landau level). Then all \( \ell_j \) need to be different, and for (fixed \( \ell_j \)'s) there is a unique eigenstate \( |N\rangle \) with correlation energy \( \mathcal{E}_{\text{corr}} = gN(N-1) \). This obviously leads to a minimal energy \( \mathcal{E} = gN(N-1) \) if \( g \leq 0 \), but for positive \( g \) it can be preferable have some fermions with \( m_j > 1 \): this increases the kinetic energy but allows to decrease the correlation energy. For example, if \( m_j = j \) (all different: we put each electron in another Landau level) and \( \ell_j = \ell \) (all the same, to have a simple specific example), \( |N\rangle \) is eigenstate with total energy \( \mathcal{E}' = N(N-1)|B|/2 - gN(N-1) \). Obviously, \( \mathcal{E}' \) will be lower than \( \mathcal{E} \) for sufficiently large \( g \). Thus, if one increases the coupling from \( g < 0 \) zero to \( g > |B|/4 \) there is some drastic change of the ground state in between — possibly a quantum phase transition? This QH model becomes more interesting if one adds Hartree-Fock terms as in Eq. (46): this allows to lift the degeneracy (fermions in the same Landau level can repel each other, e.g.).

Meaning of dimension? From an abstract point of view, the model we solved looks the same in all dimensions \( 2n+1 \). This might seem somewhat puzzling, in particular if one recalls that renormalizability of quantum field theory models usually very much depend on dimensions. This is not the case here, and the reason is that the interaction in our model is highly non-local. It is worth noting that this is similar to the above-mentioned Hartree-Fock models which also look the same in all dimensions.

Acknowledgments. I thank H. Grosse, J. Hoppe, J. Mickelsson, R. Szabo, T. Turgut, R. Wulkenhaar, and K. Zarembo for useful discussions. I am grateful to J. Hoppe for comments on the manuscript and to J. Mickelsson for helping me with group theory. I thank R. Szabo for helpful suggestions about the literature. I also acknowledge discussions with G. Semenoff quite some time ago: he suggested to look for new fermion models which can be solved using group theory. I would like to thank the Erwin Schrödinger Institute in Vienna for hospitality. This work was supported in part by the Swedish Science Research Council (VR) and the Göran Gustafsson Foundation.
Appendix A. The star product of the Landau eigenfunctions. Fact 1 and Fact 2 in the main text are known since a long time. However, due to the weight they carry in my discussion I felt I should also include an (elementary) proof. This is the purpose of this Appendix.

We assume $2n = 2$ and write $H_B \equiv H_1 = P_1^2 + P_2^2$ and $H_{-B} \equiv H_2 = \tilde{P}_1^2 + \tilde{P}_2^2$ where

$$
P_1 = -i \partial_1 - |B|x_2, \quad P_2 = -i \partial_2 + |B|x_1
$$

$$
\tilde{P}_1 = -i \partial_1 + |B|x_2, \quad \tilde{P}_2 = -i \partial_2 - |B|x_1
$$

(63)

with $|B| > 0$, $\partial_\mu = \frac{\partial}{\partial x_\mu}$ and $(x^\mu) = (x^1, x^2)$ coordinates on $\mathbb{R}^2$. We observe that

$$
p_1 = \frac{1}{\sqrt{2|B|}} P_1, \quad q_1 = \frac{1}{\sqrt{2|B|}} P_2,
$$

$$
p_2 = \frac{1}{\sqrt{2|B|}} \tilde{P}_2, \quad q_2 = \frac{1}{\sqrt{2|B|}} \tilde{P}_1
$$

(64)

give a representation of the Heisenberg algebra $[p_\mu, q_\nu] = -i \delta_{\mu,\nu}$ etc. Thus $H_1 = 2|B|(p_1^2 + q_1^2)$ is just a harmonic oscillator Hamiltonian which (on the Hilbert space of functions in two variables) is highly degenerate. The operator $H_2 = \tilde{P}_1^2 + \tilde{P}_2^2 = 2|B|(p_2^2 + q_2^2)$ is the ‘complimentary harmonic oscillator’ allowing us to resolve this degeneracy. The Landau eigenfunctions are the common eigenfunctions of $H_1$ and $H_2$. To construct them we introduce creation-and annihilation operators $a_\mu^\pm = \frac{1}{\sqrt{2}}(\mp ip_\mu + q_\mu)$ obeying the usual commutation relations. Then $H_\mu = 4|B|(a_\mu^+ a_\mu^- + \frac{1}{2})$. The common eigenfunctions of $H_1$ and $H_2$ are therefore\(^6\)

$$
|\ell, m\rangle = (-i)^{m-1} \frac{(a_1^+)^{\ell-1} (a_2^+)^{m-1}}{\sqrt{(\ell - 1)!} \sqrt{(m - 1)!}} |0\rangle
$$

(65)

where $a_\mu^- |0\rangle = 0$ and $\ell, m$ positive integers. The eigenvalues are $4|B| |\ell - \frac{1}{2}|$ and $4|B| |m - \frac{1}{2}|$, respectively.

We now compute the normalized Landau eigenfunctions in position space,

$$
\phi_{\ell m}(x) = \langle x | \ell, m \rangle
$$

(66)

where $x = (x^1, x^2)$. It is convenient to define

$$
z = x_1 + ix_2, \quad \bar{z} = x_1 - ix_2
$$

(67)

and to introduce the generating function

$$
F_{s,t}(x) = \sum_{\ell, m = 0}^{\infty} \frac{s^{\ell} t^m}{\ell! m!} \phi_{\ell+1, m+1}(x) = \sum_{\ell, m = 0}^{\infty} \frac{s^{\ell} (-it)^m}{\ell! m!} \langle x | (a_1^+)^\ell (a_2^+)^m |0\rangle = \langle z, \bar{z} | e^{s a_1^- - it a_2^-} |0\rangle.
$$

(68)

\(^6\)The phase factor $(-i)^{m-1}$ is inserted for convenience. I find it also convenient to label these states by positive integers (i.e. what I call $\ell - 1$ is usually called $\ell$).
Since \( z = (a_1^* + ia_2)/\sqrt{|B|} \) and \( \bar{z} = (a_1 - ia_2^*)/\sqrt{|B|} \) one gets

\[
F_{s,t}(x) = \langle x | e^{s a_1^* - i t a_2} e^{t a_1 + i s a_2} | 0 \rangle \equiv \langle x | e^{\sqrt{|B|(sz + tz)}} e^{[s a_1^* - i t a_2^*] a_1 + i s a_2}/2 | 0 \rangle = e^{\sqrt{|B|(sz + tz)}} e^{-st} \phi_{s,t}(x).
\]

The normalized ground state wave function \( \phi_{1,1} \) can be computed by solving \( a_\mu \phi_{1,1} = 0 \) with \( a_1 = (\partial_1 - i \partial_2 + \bar{B}|(x_1 - i x_2))/2\sqrt{|B|} \) and similarly for \( a_2 \). One thus finds

\[
F_{s,t}(x) = \sqrt{|B|/\pi} e^{-st} e^{\sqrt{|B|(sz + tz) - |B||z|^2}/2}.
\]  

(69)

Using this generating function it is easy to check that the Landau eigenfunctions thus defined are a complete orthonormal basis.

To prove Fact 1 we compute \( F_{s_1,t_1} \ast F_{s_2,t_2} \) assuming \( \theta = B^{-1} \). We use

\[
(f \ast g)(x) = (2\pi)^{-2} \int_{\mathbb{R}^2} d^2k \int_{\mathbb{R}^2} d^2q \hat{f}(k) \hat{g}(q) e^{i B^{-1}(k_1 q_2 - k_2 q_1)} e^{i(k+q) \cdot x}
\]

(70)

where \( \hat{f} \) is the Fourier transform of \( f \). We first compute \( (K = k_1 + i k_2) \)

\[
\hat{F}_{s,t}(k) = \int_{\mathbb{R}^2} \frac{d^2x}{2\pi} e^{-ik \cdot x} F_{s,t}(x) = \sqrt{|B|/\pi} e^{st} e^{i(sK + t\bar{K})/\sqrt{|B| - |K|^2/2|B|}}
\]

(we computed a Gaussian integral). Thus

\[
F_{s_1,t_1} \ast F_{s_2,t_2}(x) = (2\pi)^{-2} \int_{\mathbb{R}^2} d^2k \int_{\mathbb{R}^2} d^2q e^{-i\theta(k_1 q_2 - k_2 q_1)} e^{i(k+q) \cdot x} \times \frac{1}{|B|} e^{s_1 t_1 + s_2 t_2} e^{i(s_1 K + t_1 \bar{K} + s_2 Q + t_2 \bar{Q})/\sqrt{|B| - |K|^2/2|B|}} =
\]

\[
= \frac{|B|}{2\pi} e^{-s_1 t_2 + s_2 t_1} e^{(s_1 z + t_2 \bar{z})/\sqrt{|B| - |B||z|^2/2}}
\]

(again a Gaussian integral), i.e.,

\[
F_{s_1,t_1} \ast F_{s_2,t_2}(x) = \frac{|B|}{4\pi} e^{s_2 t_1} F_{s_1,t_2}(x)
\]  

(71)

equivalent to

\[
\phi_{t_1,m_1} \ast \phi_{t_2,m_2} = \frac{|B|}{4\pi} \delta_{m_1,t_2} \phi_{t_1,m_2}.
\]  

(72)

and completing our proof of Fact 1, including the normalization. Our discussion in the beginning of this Appendix also provides a proof of Fact 2.

**Appendix B. Pedestrian solution of Eq. (50).** In the main text we give the general result for all \( N \). To appreciate this result and have a few special cases, I present here a brute-force solution for small \( N \).
One can represent $C_N$ (48) by a $N! \times N!$-matrix in the following way. Define a Hilbert space isomorphic to $\mathbb{R}^{N!}$ by introducing the orthonormal basis $|P\rangle, P \in S_N$. Then $\pi_P(P', P'') := (P', P \cdot P)$ defines a representation $P \to \pi_P$ by $N! \times N!$ matrices (the reader familiar with group theory will recognize that $\pi$ is just the regular representation of $\pi$). We thus can solve (50) by constructing and diagonalizing the $N! \times N!$-matrix $C_N := \sum_{1 \leq j < k \leq N} \pi_{(jk)}$. We now describe the results for $N \leq 4$ and give a few examples for eigenfunctions.

For $N = 0$ and $N = 1$ it is easy to find all eigenstates and eigenvalues of $\mathcal{H}$,

$$\mathcal{H}\Omega = 0 \quad (N = 0)$$

and

$$\mathcal{H}A_{\ell m}^\dagger \Omega = (E_{\ell} + E_m)A_{\ell m}^\dagger \Omega \quad (N = 1).$$

The first non-trivial case is $N = 2$. Labeling the basis $|P\rangle, P \in S_2$, in the following way, $|12\rangle \equiv 1$ and $|21\rangle \equiv 2$, (we use an obvious short-hand notation for permutations on the l.h.s. of these equations) it is easy to see that

$$C_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

which has eigenvectors $(a_1, a_2) = (1, \pm 1)$ corresponding to the eigenvalues

$$c = \pm 1 \quad (N = 2).$$

(From that one obtains the following eigenstates of $\mathcal{H}$,

$$|2\rangle_\pm = (a_1 I + a_2 T_{(12)})|2\rangle = \left(A_{\ell_1 m_1}^\dagger A_{\ell_2 m_2}^\dagger \pm A_{\ell_1 m_2}^\dagger A_{\ell_2 m_1}^\dagger\right) \Omega$$

(we identified $T_1 \equiv T_{12} = I$ and $T_2 = T_{21} = T_{(12)}$) corresponding to the eigenvalues $\mathcal{E} = \mathcal{E}_0 \pm 2g$. Note that if $m_1 = m_2$ then $|N\rangle_+ = 0$ and if $\ell_1 = \ell_2$ then $|N\rangle_- = 0$ (due to the Pauli principle). Thus there are two independent eigenstates only if $m_1 \neq m_2$ and $\ell_1 \neq \ell_2$, otherwise there is only one. Still, the $N = 2$-eigenstates thus constructed provide a complete orthonormal basis in the $N = 2$-subspace of our fermion Hilbert space.

For $N = 3$ we label the basis as $|123\rangle = \hat{1}$, $|312\rangle = \hat{2}$, $|231\rangle = \hat{3}$, $|213\rangle = \hat{4}$, $|132\rangle = \hat{5}$, and $|321\rangle = \hat{6}$, and we obtain

$$C_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

(the matrix elements not written are zero). The eigenvalues of this matrix are

$$c = 3(1), 0(4), -3(1) \quad (N = 3)$$

with the numbers in the parenthesis indicating the multiplicities. It is not difficult to also write down the corresponding eigenstates. For example, $(a_1, \ldots, a_6) = (1, -1, 0, 0, 0, 0)$ is
an eigenvector of $C_3$ with eigenvalue $c = 0$, and the corresponding eigenstate of $\mathcal{H}$ with eigenvalue $\mathcal{E} = \mathcal{E}_0$ is

$$ (T_{123} - T_{312}) |3\rangle = \left( A_{\ell_{1}m_{1}}^{\dagger} A_{\ell_{2}m_{2}}^{\dagger} A_{\ell_{3}m_{3}}^{\dagger} \pm A_{\ell_{1}m_{3}}^{\dagger} A_{\ell_{2}m_{1}}^{\dagger} A_{\ell_{3}m_{2}}^{\dagger} \right) \Omega $$

(78)

eetc. If all three $m_j$’s are different and also the three $\ell_j$’s, these eigenstates all are linearly independent, but otherwise the number of linearly eigenvectors can be less than 6 and not all eigenvalues $\mathcal{E}_0 + 2gc$ are realized. I also constructed and diagonalized the $24 \times 24$-matrix $C_4$ and found the following eigenvalues (and multiplicities),

$$ c = 6(1), 2(9), 0(4), -2(9), -6(1) \quad (N = 4). $$

(79)

Again, the corresponding 24 eigenstates are linearly independent only if the four $m_j$’s are all different and the same for the $\ell_j$’s. At this point this brute-force approach clearly becomes too cumbersome. Fortunately for us, the eigenvalues and eigenvectors of all the matrices $C_N$ are all known from group theory.

References

[1] D. C. Mattis (ed.), *The many-body problem. An encyclopedia of exactly solved models in one dimension.*, World Sci. Publishing, River Edge, NJ, 1993.

[2] W. Lerche, Nucl. Phys. Proc. Suppl. **55B** (1997) 83 [hep-th/9611190].

[3] T. Curtright, T. Uematsu and C.K. Zachos, J. Math. Phys. **42** (2001) 2396 [hep-th/0011137].

[4] E. Langmann, “Exactly solvable models for 2D correlated fermions,” [cond-mat/0206045].

[5] F. D. Murnaghan, *The theory of group representations*, Dover, New York, 1963.

[6] G. B. Robinson, *Representation theory of the symmetric group*, Mathematical Expositions, No. 12. University of Toronto Press, Toronto, 1961.

[7] H. Weyl, *The classical groups*, Princeton Univ. Press, Princeton, NJ, 1997.

[8] A. Klein and E. R. Marshalek, *Rev. Mod. Phys.* **63**, 375 (1991).

[9] R. W. Richardson, *Phys. Lett.* **3**, 277 (1963); *J. Math. Phys.* **6**, 1034 (1965).

[10] G. Sierra, J. Dukelsky, G. G. Dussel, J. von Delft, and F. Braun, *Phys. Rev. B* **61**, R11890 (2000); see also A. Mastellone, G. Falci, and R. Fazio, *Phys. Rev. Lett.* **80**, 4542 (1998); J. Links, H.-Q. Zhou, R. H. McKenzie, M. D. Gould [cond-mat/0110105]; H.-Q. Zhou, J. Links, R. H. McKenzie, and M. D. Gould, *Phys. Rev. B* **65**, 060502(R) (2002).
L. Amico, A. Di Lorenzo, and A. Osterloh, *Phys. Rev. Lett.* **86**, 5759 (2001); J. Dukelsky, C. Esebbag, and P. Schuck, *Phys. Rev. Lett.* **87**, 066403 (2001); X.-W. Guan, A. Foerster, J. Links, H.Q. Zhou [cond-mat/0205124].

A. Konechny and A. Schwarz, Phys. Rept. **360** (2002) 353 [hep-th/0012145].

M. R. Douglas and N. A. Nekrasov, Rev. Mod. Phys. **73** (2002) 977 [hep-th/0106048].

R. J. Szabo, “Quantum field theory on noncommutative spaces,” [hep-th/0109162].

R. Gopakumar, S. Minwalla and A. Strominger, JHEP **0005** (2000) 020 [hep-th/0003160].

G. Mandal, S. J. Rey and S. R. Wadia, Eur. Phys. J. C **24** (2002) 495 [hep-th/0111059].

E. Langmann and R.J. Szabo, Phys. Lett. B **533** (2002) 168 [hep-th/0202039]. The result was stated and proven there only for even dimensional boson models, but it is obvious how to extend it to the fermion models considered here.

E. Langmann, R.J. Szabo and K. Zarembo, work in progress.

J. Hoppe, Phys. Lett. B **250** (1990) 44.

I use some common terminology from quantum Hall physics which is explained, e.g. in R. B. Laughlin’s contribution (Chapter 7) in: R. E. Prange and S. M. Girvin, *The quantum Hall effect*, Springer-Verlag NY Berlin Heidelberg, 1987.

J. Q. Chen, *Group representation theory for physicists*, World Sci. Publishing, Teaneck, NJ, 1989.