Stationary Scalar Clouds Around Rotating Black Holes

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Motivated by novel results in the theory of wave dynamics in black-hole spacetimes, we analyze
the dynamics of a massive scalar field surrounding a rapidly rotating Kerr black hole. In particular,
we report on the existence of stationary (infinitely long-lived) regular field configurations in the
background of maximally rotating black holes. The effective height of these scalar “clouds” above
the central black hole is determined analytically. Our results support the possible existence of
stationary scalar field dark matter distributions surrounding rapidly rotating black holes.

I. Introduction. The ‘no-hair’ conjecture [1, 2] has played a key role in the development of black-hole
physics [3–6]. This conjecture asserts that black holes are fundamental objects — they should be described by
only three externally observable (conserved) parameters: mass, charge, and angular momentum.

The physical idea behind the no-hair conjecture is based on an intuitive (and sometimes oversimplified)
picture according to which all matter fields which are present in the exterior of a black hole would eventually
be swallowed by the black hole or be radiated away to infinity (with the exception of fields which are associated with conserved charges) [4, 7]. In accord with this logic, various no-hair theorems indeed exclude static (time-independent) scalar fields [8], massive vector fields [9], and spinor fields [10] from the exterior of black holes.

It should be stressed, however, that the various no-hair theorems do not rule out the existence of time-dependent field configurations in the black-hole exterior. Indeed, in a very nice work Barranco et. al. [11] have recently found time-decaying regular scalar field configurations surrounding a Schwarzschild black hole that can survive for relatively long times (as compared to the dynamical timescale set by the mass of the black hole).

Ultra-light scalar fields have been invoked in recent years as possible candidates to play the role of the dark matter component of our universe (see [11–14] and references therein). Given the fact that most galaxies seem to contain a super-massive black hole at their centers [15], it was pointed out in [11] that in order to be a viable candidate for the dark matter halo a scalar field configuration must be able to survive for (at least) cosmological timescales. In this context, the important conclusion presented in [11] is very encouraging: time-decaying configurations made of ultra-light scalar fields surrounding a supermassive Schwarzschild black hole can indeed survive for such extremely long periods.

The main aim of the present work is to extend the important results of [11] in three new directions:

1. The analysis of [11] was restricted to a spherically symmetric Schwarzschild black-hole background. However, it is well-known that realistic black holes generally rotate about their axis and are therefore not spherical. Thus, an astrophysically realistic model of wave dynamics in black-hole spacetimes must involve a non-spherical background geometry with angular momentum [16]. In the present work we consider such non-spherical backgrounds and analyze the dynamics of massive scalar fields in realistic rotating (Kerr) black-hole spacetimes.

2. The focus in [11] was mainly on black hole–scalar field configurations characterized by the dimensionless product $M\mu \ll 1$, where $M$ is the mass of the central black hole and $\mu$ is the mass of the surrounding scalar field. In the present work we shall analyze the complementary regime of $M\mu > \frac{1}{2}$.

II. Description of the system. The physical system we consider consists of a massive scalar field coupled to a rotating Kerr black hole of mass $M$ and angular momentum $a$ per unit mass $\alpha$. In order to facilitate a fully analytical study, we shall assume that the black hole is maximally rotating with $a = M$. Similar results can be obtained (with the cost of a more involved analysis) for non-extremal rotating black holes. In the present study we shall restrict ourselves to the case of a test scalar field in the background of the rotating Kerr black-hole spacetime. We shall therefore use the terminology of [11] and talk about scalar “clouds” surrounding the black hole rather than a genuine scalar “hair” [17].

The dynamics of a massive scalar field $\Psi$ in the Kerr spacetime is governed by the Klein-Gordon equation [18]

\[ (\nabla^a \nabla_a - \mu^2) \Psi = 0. \tag{1} \]

(Here $\mu$ stands for $\mathcal{M}/\mathcal{L}$, where $\mathcal{M}$ is the mass of the scalar field. We shall use units in which $G = c = \hbar = 1$. In these units $\mu$ has the dimensions of 1/length.) One may decompose the field as

\[ \Psi_{lm}(t, r, \theta, \phi) = e^{im\theta} S_{lm}(\theta; M\omega) R_{lm}(r; M, \omega) e^{-i\omega t}, \tag{2} \]
where \((t, r, \theta, \phi)\) are the Boyer-Lindquist coordinates, \(\omega\) is the (conserved) frequency of the mode, \(l\) is the spheroidal harmonic index, and \(m\) is the azimuthal harmonic index with \(-l \leq m \leq l\). (We shall henceforth omit the indices \(l\) and \(m\) for brevity.) With the decomposition \([2]\), \(R\) and \(S\) obey radial and angular equations both of confluent Heun type coupled by a separation constant \(K(M\omega)\) \([18,22]\).

The angular functions \(S(\theta; a, \omega)\) are the spheroidal harmonics which are solutions of the angular equation \([18–22]\)

\[
\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial S}{\partial \theta} \right) + \left[ K + M^2 (\mu^2 - \omega^2) \right] - M^2 (\mu^2 - \omega^2) \cos^2 \theta - \frac{m^2}{\sin^2 \theta} S = 0 . \tag{3}
\]

The angular functions are required to be regular at the poles \(\theta = 0\) and \(\theta = \pi\). These boundary conditions pick out a discrete set of eigenvalues \(\{K_{lm}\}\) labeled by the integers \(l\) and \(m\). For \(M^2(\mu^2 - \omega^2) \leq m^2\) one can treat \(M^2(\omega^2 - \mu^2) \cos^2 \theta\) in Eq. (3) as a perturbation term on the generalized Legendre equation and obtain the perturbation expansion \([21]\)

\[
K_{lm} + M^2 (\mu^2 - \omega^2) = l(l+1) + \sum_{k=1}^{\infty} c_k M^{2k} (\mu^2 - \omega^2)^k \tag{4}
\]

for the separation constants \(K_{lm}\). The expansion coefficients \(\{c_k(l,m)\}\) are given in Ref. \([21]\).

The radial Teukolsky equation is given by \([18,22,23]\)

\[
\Delta \frac{d}{dr} \left( \Delta \frac{dR}{dr} + \left[ H^2 + \Delta [2mM\omega - K - \mu^2(r^2+M^2)] \right] \right) R = 0 , \tag{5}
\]

where \(\Delta \equiv (r-M)^2\) and \(H \equiv (r^2 + M^2)\omega - mM\). The degenerate zero of \(\Delta\), \(r_H = M\), is the location of the black-hole horizon.

We are interested in solutions of the radial equation \([5]\) with the physical boundary conditions of purely ingoing waves at the black-hole horizon (as measured by a comoving observer) and a decaying (bounded) solution at spatial infinity \([24,29]\). That is,

\[
R \sim \begin{cases} 
eq & \frac{1}{r} e^{-\sqrt{\mu^2 - \omega^2} y} & \text{as } r \to \infty \ (y \to \infty) ; \\ e^{-i(\omega - m\Omega) y} & \text{as } r \to r_H \ (y \to -\infty) , \end{cases} \tag{6}
\]

where the “tortoise” radial coordinate \(y\) is defined by \(dy = [(r^2 + M^2)/\Delta]dr\). Here \(\Omega = 1/2M\) is the angular velocity of the black-hole horizon.

Note that a bound state (a state decaying exponentially at spatial infinity) is characterized by \(\omega^2 < \mu^2\). For a given mass parameter \(\mu\), the boundary conditions \([9]\) single out a discrete set of resonances \(\{\omega_n(\mu)\}\) which correspond to the bound states of the massive field \([24,29]\). Stationary resonances, which are the solutions we are interested in in this paper, are characterized by \(3\omega = 0\). (We note that, in addition to the bound states of the massive field, the field also has an infinite set of discrete quasinormal resonances \([30,33]\) which are characterized by outgoing waves at spatial infinity.)

**III. The stationary scalar resonances.** As we shall now show, the field \([2]\) with \(\omega = m\Omega\) describes a stationary regular solution of the wave equation \([11]\). It is convenient to define a new dimensionless variable

\[
x \equiv \frac{r - r_H}{r_H} , \tag{7}
\]

in terms of which the radial equation \([5]\) becomes

\[
x^2 \frac{d^2 R}{dx^2} + 2x \frac{dR}{dx} + VR = 0 , \tag{8}
\]

where \(V \equiv (m^2/4 - M^2\mu^2)x^2 + (m^2 - 2M^2\mu^2)x + (\pm K_{lm} + 2m^2 - 2M^2\mu^2)\). Defining

\[
f \equiv xR \quad \text{and} \quad z \equiv 2\sqrt{\mu^2 - m^2/4Mx} , \tag{9}
\]

one obtains the radial equation

\[
\frac{d^2 f}{dz^2} + \left[ - \frac{1}{4} + \frac{k}{z} + \frac{1}{z^2} - \frac{\beta^2}{z^2} \right] f = 0 , \tag{10}
\]

with

\[
k \equiv \frac{m^2 - 2M^2\mu^2}{\sqrt{4M^2\mu^2 - m^2}} \quad \text{and} \quad \beta^2 \equiv K_{lm} + \frac{1}{4} - 2m^2 + 2M^2\mu^2 . \tag{11}
\]

We shall assume without loss of generality that \(\Re \beta \geq 0\). Equation \((10)\) is the familiar Whittaker equation; its solutions can be expressed in terms of the confluent hypergeometric functions \(M(a, b, z)\) \([21,28,34]\)

\[
R = Az^{-\frac{1}{2} + \beta} e^{-\frac{1}{2}z} M\left(\frac{1}{2} + \beta - \kappa, 1 + 2\beta, z\right) + B(\beta \to -\beta) , \tag{12}
\]

where \(A\) and \(B\) are constants. The notation \((\beta \to -\beta)\) means “replace \(\beta\) by \(-\beta\) in the preceding term.”

The near-horizon \((z \to 0)\) limit of Eq. \((12)\) yields \([21,34]\)

\[
R \to Az^{-\frac{1}{2} + \beta} + Bz^{-\frac{1}{2} - \beta} . \tag{13}
\]

Regularity of the solution at the horizon \((R \text{ is bounded for } z \to 0)\) requires

\[
B = 0 \quad \text{and} \quad \Re \beta \geq \frac{1}{2} . \tag{14}
\]

Approximating Eq. \((12)\) for \(z \to \infty\) one gets \([21,34]\)

\[
R \to A\left[ \frac{\Gamma(1 + 2\beta)}{\Gamma(\frac{3}{2} + \beta - \kappa)} z^{-1 - \kappa} e^{\frac{1}{2}z} + \frac{\Gamma(1 + 2\beta)}{\Gamma(\frac{3}{2} + \beta - \kappa + \mu)} z^{-1 - \kappa} (-1)^{-\frac{1}{2} - \beta + \kappa} e^{-\frac{1}{2}z} \right] . \tag{15}
\]

A bound state is characterized by a decaying field at spatial infinity. The coefficient \(1/\Gamma(\frac{3}{2} + \beta - \kappa)\) of the growing
exponent $e^{\pm z}$ in Eq. (15) should therefore vanish. Using the well-known pole structure of the Gamma functions [21], we find the resonance condition for the stationary bound-states of the field:

$$\kappa = \frac{1}{2} + \beta + n,$$  \hspace{1cm} (16)

where the resonance parameter $n$ is a non-negative integer ($n = 0, 1, 2, ...$). Substituting Eqs. (14) and (10) into Eq. (12), one obtains the compact form

$$R = A z^{-\frac{1}{2} + \beta} e^{-\frac{z}{2}} L_n^{(2\beta)}(z)$$ \hspace{1cm} (17)

for the radial solutions, where $L_n^{(2\beta)}(z)$ are the generalized Laguerre Polynomials (see Eq. 13 of [21]).

We note that the r.h.s of the resonance condition (16) is a decreasing function of $\kappa$ for various values of the resonance parameter $n$. Also shown are the effective heights (in units of the black-hole radius), $x_{\text{cloud}}$, of the scalar clouds above the central black hole. In general, the size of a scalar cloud increases with increasing resonance number $n$. Note that all resonances conform to the lower bound (23).

**IV. Effective heights of the scalar clouds.** We shall next evaluate the effective heights of the stationary scalar “clouds” which surround the central rotating black hole. These clouds correspond to the family of wavefunctions (17) satisfying the resonance condition (16). It is worth mentioning that a nice “no short hair theorem” was proved in [22, 23] for static and spherically symmetric hairy black-hole configurations. According to this theorem, the “hairsphere” [the region where the non-trivial (non-asymptotic) behavior of the black-hole hair is present] must extend beyond $\frac{3}{2}$ the horizon radius. This translates into the lower bound

$$x_{\text{hair}} \geq \frac{1}{2},$$  \hspace{1cm} (23)

where $x$ is the dimensionless height defined in (17). A rough estimate of the size of our stationary scalar configurations can be obtained by defining their effective radii as the radii at which the quantity $4\pi r^2 |\Psi|^2$ attains its global maximum. Taking cognizance of Eq. (17) with $L_0^{(2\beta)}(z) \equiv 1$, one finds

$$x_{\text{cloud}}^{(0)} = \frac{2\beta + 1}{2\epsilon}$$ \hspace{1cm} (24)

for the ground-state resonances with $n = 0$. For higher $n$ values one generally obtains larger scalar clouds with the approximated relation $x_{\text{cloud}}^{(n)} \simeq \frac{2n + 1 + 2n}{2n}$, see Eq. (17).

The effective heights of the fundamental scalar clouds above the rotating central black hole are given in Table I from which we learn that $\{x_{\text{cloud}}^{(n)}\}$ are always larger than $\frac{1}{2}$. For $n \gg l$ one finds from (22) $x_{\text{cloud}}^{(n)} \simeq 4n^2/m^2$. It is remarkable that our stationary scalar

| $n$ | $M\mu_{\text{resonance}}$ | $x_{\text{cloud}}$ |
|-----|----------------|-----------------|
| 0   | 0.526          | 8.557           |
| 1   | 0.510          | 23.485          |
| 2   | 0.505          | 46.398          |
| 3   | 0.503          | 77.313          |
clouds respect the lower bound (23) despite the fact that they do not satisfy the conditions of the original theorem (4) – they are neither static nor spherically symmetric.

V. Summary and discussion. In this Letter we have analyzed the dynamics of a massive scalar field in an astrophysically realistic (rotating) Kerr black-hole spacetime. In particular, we have proved the existence of a discrete and infinite family of resonances describing stationary (non-decaying) scalar configurations surrounding maximally rotating black holes. The effective heights of these scalar “clouds” above the central black hole were determined analytically – it was shown that these non-static and non-spherically symmetric configurations conform to the lower bound (24) originally derived in [6] for static and spherically symmetric hairy black-hole configurations. Thus, our analysis provides the first direct evidence for the general validity of the bound (23).

Our results support the possible existence of stationary scalar field dark matter distributions surrounding astrophysically realistic (rotating) black holes. Moreover, while former studies [11, 26] focused on the regime $M\mu \ll 1$, our analysis opens the possibility for the existence of such black hole–scalar field configurations in the new regime $M\mu > 1/2$. Our analytical findings for maximally rotating black holes are in accord with the numerical work of [27] who discussed slowly-decaying bound states of non-extremal black holes.

Probably the most interesting question is the following one: what is the final end-state of gravitational collapse? The no-hair conjecture [1] asserts that the final outcome of rotating gravitational collapse is a bald Kerr black hole. However, our results suggest the following alternative scenario: during the early stages of the evolution some of the fields would indeed be radiated away to infinity while some would be swallowed by the newly-born black hole (see [11] for the spherically symmetric case). However, if the initial matter distribution contains massive scalar fields, then those fields which satisfy the resonance condition (15) may actually survive as infinitely long-lived (stationary) resonances outside the black hole. Thus, the final configuration is expected to be a rotating black hole surrounded by stationary scalar clouds. It would be interesting to verify this prediction using numerical simulations as done in [11] for the spherically symmetric (non-rotating) case.

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[35] For moderate values of $m$ the truncated cubic equation

$$m^4 - 4m^2(2n + 1)\epsilon + 16\left[m^2 - \left(l + \frac{1}{2}\right)^2 + (n + \frac{1}{2})^2\right]\epsilon^2 + 16(2n + 1)\epsilon^3 = 0$$

provides a very good approximation to the exact equation (21).

[36] The resonances are characterized by small values of $\epsilon$, in which case Eq. (21) yields $M\mu(n) \simeq \frac{1}{2} + \left[\frac{2n+1-\sqrt{5}}{8(n+1)-4}\right]^2$.

This analytical expression agrees extremely well with the numerical data presented in Table I.

[37] Note that the regularity condition (14) restricts the allowed values of the azimuthal harmonic index: $m \leq \sqrt{2(l + 1)/3}$ for $n \gg l$ [see Eq. (20)].

[38] See also [7] for a modern version of this theorem, relating the concept of a black-hole “hairosphere” to the more familiar concept of black-hole photosphere.

[39] A lower bound on the effective heights (24) of the stationary scalar clouds can be obtained from the couple of inequalities (14) and (18): $x^{(0)}_{\text{max}} > \frac{\sqrt{2}}{M\mu}$. [Here we have used the inequality $\epsilon < M\mu/\sqrt{2}$ which follows from (18) and (19).]