IRREDUCIBILITY AND STRONG FELLER PROPERTY FOR
STOCHASTIC EVOLUTION EQUATIONS IN BANACH SPACES

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Abstract. The main goal of this paper is to generalize to Banach spaces the well-known results for diffusions on Hilbert spaces obtained in [Peszat, S. and Zabczyk, J. (1995). Strong Feller property and irreducibility for diffusions on Hilbert spaces. Ann. Probab. 23(1): 157-172.]. More precisely, we are aiming to prove the strong Feller property and irreducibility of the solutions to a stochastic evolution equations (SEEs) in Banach spaces. We give sufficient conditions on the path space and the coefficients of the SEEs for these aforementioned properties to hold. We apply our result to investigate the long-time behavior of a stochastic nonlinear heat equations on $L^p$-space with $p > 4$. Our result implies the uniqueness of the invariant measure, if it exists, for the stochastic nonlinear heat equations on $L^p$-space with $p > 4$.

1. Introduction. The uniqueness of an invariant measure for the Markov process associated to the solution of stochastic evolution equations (SEEs) is one of the most important questions in Probability, if not in Mathematics. This question has been very actively investigated in the last thirty years and several important results have been obtained in both finite and infinite dimensional settings. The case of stochastic differential equations (SDEs for short) on the euclidean space $\mathbb{R}^n$ (or a finite dimensional Riemannian manifolds in more generality) is now quite well understood. Several methods used to prove the uniqueness of invariant measure for SDEs has been successfully extended and developed to the case of stochastic system in infinite dimensional (usually Hilbert) spaces, see, for instance, the new edition of an 1992 monograph by Da Prato and Zabczyk [8], the articles [9], [10] [16] [21] [24] [25] [30] and references therein. One of the most well-known and much used approaches to prove the uniqueness of an invariant measure to a Markov semigroup associated to a SEEs in a Hilbert space consists in proving the strong Feller (SF) property and the irreducibility. Roughly speaking a Markov semigroup $\mathcal{P}$

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has the SF property if the image, under $P$, of the space of bounded and measurable real functions defined on the state space is included in the space of bounded and continuous functions. Frequently, for instance if the noise entering the system is non-degenerate, this smoothing property can be proved through the gradient estimate of the semigroup which will follow from the well-known Bismut-Elworthy-Li formula, see, for instance, [8, 30]. The study of a SEEs driven by a degenerate noise is very delicate, however, there are now several results treating successfully this case. Recently, two notions of regularity of Markov process which seem very powerful to treat the ergodic property of SEEs in Hilbert spaces driven by degenerate were introduced in the mathematical literature: the asymptotic feller property (ASF), see [13], [14] and the e-property, see [17]. These notions, not present in the finite dimensional literature, are weaker than the SF property but a irreducible Markov semigroup that is asymptotic SF or has the e-property has at most one invariant measure. Very often, at least in the Hilbert spaces setting, one uses the Malliavin calculus to check ASF and the e-property. However it also possible to prove the uniqueness of invariant measure for SEEs in Hilbert space without making use of the Malliavin calculus. For instance, in [19, 20] uniqueness of invariant measure for various SEEs in Hilbert spaces with degenerate noise was proved by making use of the coupling method. To the best of our knowledge, the results by Hairer and Mattingly are applicable to systems in Hilbert spaces with degenerate but spatially smooth noise. This paper is the beginning of our effort to study the question of uniqueness of an invariant measure for SEEs driven by both degenerate and spatially irregular noise.

In contrast to the SEEs in Hilbert spaces, there are not so many papers treating the uniqueness of invariant measure of stochastic evolution equations (SEEs) in infinite dimensional Banach spaces. However, we refer to [6], [12], [26] and references therein for few relevant results. It is worth noticing that the method developed in [19], see also [20], can be applied to show the uniqueness of invariant measure of a SEE in Banach space with non-degenerate additive noise. Furthermore, the method elaborated in [19, 20] was recently used in [18] to prove that when defined on the space of continuous complex functions the stochastic Complex Ginzburg-Landau with degenerate additive noise has a unique invariant measure. However, the result obtained in [18] relies very much on some good estimates of the solution in the Hilbert-Sobolev spaces $H^m$ with $m > n/2$, where $n$ is the space dimension and as in the work [14], the additive noise is spatially regular.

The difficulties of the study of the uniqueness of an invariant measure for SEEs in the Banach space setting arise mainly due to the fact that the tools frequently used in the Hilbert space framework cannot be extended in a straightforward way to the Banach space setting. In fact these tools rely very much on some nice properties, which might not hold for general Banach spaces, of the underlying Hilbert spaces. For instance, this is the case of the Bismut-Elworthy-Li formula for SEEs with Lipschitz coefficients whose proof relies very much on the approximation of Lipschitz functions by maps of class $C^2$ and with bounded derivatives. In contrast to the Hilbert spaces case, such approximation is not always possible for any Banach space. In fact, it is only recently proved in [15] that such approximation is only possible for some classes of Banach space, for instance, Banach spaces that have a norm of class $C^2$.

In this paper we are mainly interested in studying the uniqueness of invariant measure, if any, associated to a SEE with multiplicative noise in a martingale type
2 Banach space. We are modestly aiming to extend to the Banach setting the results about SF property and irreducibility of diffusions in Hilbert spaces obtained in [30]. One direction in that quest is the derivation of a Bismut-Elworthy-Li gradient formula for the Markovian semigroup associated to a SEE in a martingale type 2 Banach space with norm of class $C^2$. Our main results are stated in Theorem 2.4 and Theorem 2.6 which, roughly speaking, state that the Markov semigroup associated to our SEE is strong Feller and irreducible provided that the coefficient are (in some sense) globally Lipschitz and the state space $E$ is martingale type 2 Banach space with a $C^2$-smooth norm. For the proof of these results we closely follow the scheme of the proofs in [30]. As in the Hilbert space setting, the irreducible property is much easier to be proved than the SF property. However, as mentioned earlier, the SF property is difficult to prove and its proof, which consists of two parts, is the subject of Section 3. In the first part of the proof we assume that the coefficients are twice Fréchet differentiable with bounded derivatives. Under this new assumption, by truncating the noise and approximating the (unbounded) linear part of the SEE by sequence of bounded operators (the Yosida approximation), we can prove a Bismut-Elworthy-Li formula. In the second part we exploit the geometry (mainly the smoothness of the norm) of the state space. The $C^2$-smoothness of the norm enables us to use recent result in [15] about approximation of Lipschitz functions by sequence of twice differentiable functions which are also Lipschitz continuous. By this construction and a careful passage to the limit we dispense of the differentiability of the coefficients and obtain the SF property of the original problem. Thanks to a Girsanov theorem for SEE in Banach space (see [29]), the proof of Theorem 2.6 is quite easy and use the same argument as the proof of [30, Theorem 1.3]. The driving noise of our equation is non-degenerate and the coefficients of the equations are, roughly speaking, globally Lipschitz continuous. In contrast to [12] (resp. [6]) we do not assume that the state space is densely and continuously embedded in a separable Hilbert space (resp. the coefficients are dissipative).

After we have finished the present paper, we learned that Shamarova [33] used the Malliavin calculus in 2-smooth Banach space to prove that any finite dimensional projection of the law of the solution of a SEE with coefficients satisfying the Hörmander conditions is absolutely continuous with respect to the Lebesgue measure. Although her assumptions are a little bit restrictive, it would be interesting to check whether her approach could be used in our framework to prove the strong Feller property. We should note that the development of Malliavin calculus in Banach spaces is still at its infancy, however, we refer to [22], [23],[32] and [33] amongst others to some significant results that have been obtained. With the help of these results, we hope that in near future we will be able to develop and generalize the approach in [13] and [14], so that we will be able to analyze SEE with degenerate noise in Banach setting.

The organization of this paper is as follows. Section 2 is devoted to the introduction of frequently used notations and hypotheses in this paper. We also state in Section 2 our main results. In Section 3 we prove the strong Feller property of our equations. We applied our results to the nonlinear stochastic heat equations in the $L^2$-space setting in Section 4. The appendix is devoted to the proof of some auxiliary results that we could not find in the literature.

2. Notations and the main results. Let $E$ be a separable Banach space with norm $\| \cdot \|$. Let $(H, \| \cdot \|_H)$ be a separable Hilbert space such that the embedding
$E \subset H$ is continuous. Let $\{e_k\}_{k=1}^{\infty}$ be a fixed orthonormal basis of $H$ and $E^*$ be the dual space of the separable Banach space $E$. We use $\langle \varphi, u \rangle$ to denote simultaneously the dual pairing between $\varphi \in E^*$ and $u \in E$ and the scalar product of $\varphi \in H$ and $u \in H$.

The spaces of bounded measurable (resp. continuous) real-valued functions on $E$ are denoted by $B_b(E)$ (resp. $C_b(E)$). The supremum norm on $B_b(E)$ and $C_b(E)$ is denoted by $\|\|_0$. The space of all (bounded or unbounded) linear maps from a Banach space $Y_1$ into another Banach space $Y_2$ is denoted by $\mathcal{L}(Y_1, Y_2)$. The Banach space of bounded linear operators from $Y_1$ into $Y_2$ is denoted by $\mathcal{L}(Y_1, Y_2)$ and the norm on $\mathcal{L}(Y_1, Y_2)$ is $\| \|_{\mathcal{L}(Y_1, Y_2)}$.

Before we proceed further, let us recall from [11] the notion of $C^k$-smoothness of the norm of a Banach space $E$.

**Definition** ($C^k$-smoothness of the norm). A function $f : E \to \mathbb{R}$ is called Fréchet differentiable at $x \in E$ if there exists $\phi \in E^*$, where $E^*$ is the dual space of $E$, such that

$$\lim_{y \to 0} \frac{f(x + y) - f(x) - \phi y}{|y|} = 0.$$  

The functional $\phi$ is called the (Fréchet) derivative of $f$ at $x \in E$ and denoted by $f'(x)$. The higher order Fréchet derivatives $f^{(k)}$ are defined inductively, see, for instance, [7] for more details.

The set of real valued functions on $E$ whose $k$-th Fréchet derivative (resp. all of whose Fréchet derivatives) is continuous on an open set $U \subset E$ is denoted by $C^k(U)$ (resp. $C^\infty(U)$). Following [11, Chapter V, page 184], we say that the Banach space $E$ has a $C^k$-smooth norm if its norm $| \cdot |$ belongs $C^k(E \setminus \{0\})$.

The following concrete examples of Banach spaces are taken from [11, Theorem V.1.1].

**Example** (Bonic-Frampton’s Theorem). For a measure space $(\mathcal{O}, \Sigma, \mu)$ with positive measure $\mu$, we denote by $L^p := L^p(\mathcal{O}, \Sigma, \mu)$, $p \in [1, \infty)$, the space of equivalence of function classes $f : \mathcal{O} \to \mathbb{R}$ such that

$$\|f\|_p := \left( \int_\mathcal{O} |f(x)|^p \mu(dx) \right)^{\frac{1}{p}} < \infty.$$  

(i) If $p$ is even, then the Banach space $L^p$ has a $C^\infty$-smooth norm.
(ii) If $p$ is odd, then $L^p$ has a $C^{p-1}$-smooth norm.
(iii) If $p$ is not an integer, then $L^p$ has a $C^{[p]}$-smooth norm, where $[p]$ is the integer part of $p$.

The notion of $C^k$-smooth bump functions is very important in the approximation of functions defined on a Banach space $E$ by a sequence of $C^k$-smooth functions; we recall its definition below.

**Definition** (Bump functions). A Banach space $E$ admits a $C^k$-smooth bump function on $E$ if there exists a function $\varphi \in C^k(E)$ with non-empty and bounded support.

One the most important facts implied by the $C^k$-smoothness of the norm of a Banach space $E$ is the existence of a $C^k$-smooth bump on $E$. In fact, we have the following lemma.

**Lemma 2.1.** If a Banach space $E$ has a $C^k$-smooth norm, then it admits a $C^k$-smooth bump function.
Proof. The proof is very similar to the proof of [11, Fact I.2.1], hence we just sketch it. Let \( \tau : \mathbb{R} \to [0, 1] \) be a \( C^\infty \)-smooth function such that

\[
\tau(x) = \begin{cases} 
1 & \text{if } x \in [-1, 1], \\
0 & \text{if } x \notin (-2, 2).
\end{cases}
\]

By assumption and definition, \(| \cdot | \) is \( C^k \)-smooth Fréchet differentiable on \( E \setminus \{0\} \), thus the function \( \tau(| \cdot |) \) belongs \( C^k(E) \), and since its support is equal to the closed ball \( B(0, 2) \), it is a \( C^k \)-smooth bump function on \( E \).

Before we continue further, we state the definition of 2-smooth Banach space and Banach space of martingale type 2.

**Definition** (Martingale type 2 Banach space). Following [36, Definition A.3] and the equivalence between results in [31], we say that a Banach space \( E \) is 2-smooth or equivalently a Banach space of martingale type 2 if there exists an equivalent norm defined by the modulus of smoothness of \( (E, | \cdot |) \)

\[
\rho_E(t) = \sup \left\{ \frac{1}{2} (|x + ty| + |x - ty|) - 1 : |x| = |y| = 1 \right\}
\]

satisfying \( \rho_E(t) \leq K t^2 \) for all \( t > 0 \) and some \( K > 0 \).

It is known from [27] and [36, Lemma A.5] that that Banach space \( E \) is 2-smooth or of martingale type 2 if and only if the function \( E \ni x \mapsto |x| \in \mathbb{R} \) is \( C^1 \)-smooth and its Fréchet derivative is globally Lipschitz. Thus, by the same proof of Lemma 2.1, if \( E \) is of martingale type 2, then there exists a \( C^1 \)-smooth bump function whose derivative is globally Lipschitz.

Let \( \gamma \) be a canonical cylindrical Gaussian distribution on \( H \). We say that a bounded linear operator \( L : H \to E \) is \( \gamma \)-radonifying if the image \( \gamma \circ L^{-1} \) of \( \gamma \) by \( L \) has an extension to a \( \sigma \)-additive measure \( \gamma_L \) on \( E \). The set of all \( \gamma \)-radonifying operators from \( H \) into \( E \) is denoted by \( \mathcal{R}(H, E) \). For \( L \in \mathcal{R}(H, E) \), we put

\[
\|L\|_{\mathcal{R}(H, E)} := \left\{ \int_E \left| x \right|^2 d\gamma_L(x) \right\}^{\frac{1}{2}},
\]

which in view of Fernique theorem is a finite number. Then, \( \mathcal{R}(H, E) \) is a separable Banach space under this norm.

Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a complete probability space equipped with a filtration \( \mathbb{F} := \{ \mathcal{F}_t : t \geq 0 \} \) satisfying the usual condition. Let \( \{\beta_k\}_{k=1}^{\infty} \) be a sequence of independent, real-valued \( \mathbb{F} \)-Wiener process. We define a cylindrical Wiener process on \( H \) by the series

\[
W(t) = \sum_{k=1}^{\infty} \beta_k(t) e_k, \quad t \geq 0,
\]

which converges in a Banach space \( U \) such that the embedding \( H \subset U \) is \( \gamma \)-radonifying.

For each \( q \geq 2 \) we denote by \( \mathcal{N}^q(E) \) the set of all \( E \)-valued processes \( u \) defined on \([0, \infty) \times \Omega\) such that \( u_{|[0,t] \times \Omega} \) is \( \mathcal{B}([0,t]) \times \mathcal{F}_t \)-measurable and

\[
\int_0^t |u(s)|^q ds < \infty \quad \mathbb{P}\text{-a.s. } \text{ for all } t \geq 0.
\]

By \( \mathcal{M}^q(E) \) we denote the subset of \( \mathcal{N}^q(E) \) such that

\[
\mathbb{E} \int_0^t |u(s)|^q ds < \infty \quad \text{for all } t \geq 0.
\]
Let $x$ be a $E$-valued random variable and let us consider the following stochastic evolution equation

$$
\begin{cases}
  du(t) = [Au(t) + F(u(t))]dt + B(u(t))dW(t), \\
  u(0) = x,
\end{cases}
$$

where $A$ is the infinitesimal generator of a $C_0$-semigroup $\{S(t) : t \geq 0\}$ on $E$, the maps $F$ and $B$ act from $E$ into $E$ and from $E$ into $\mathcal{L}(H,E)$, respectively.

On $E$, $A$, $F$, $B$ and $\{S(t) : t \geq 0\}$ we make the following standing assumptions.

**Assumptions.**

(I) The Banach space $E$ has an unconditional Schauder basis and a $C^2$-smooth norm.

(II) The Banach space $E$ is martingale type 2.

(III) The operator $A$ is the generator of an analytic semigroup $S(t), t \geq 0$ on $E$.

(IV) The map $F : E \to E$ is Lipschitz continuous, i.e., there exists $\ell_1$ such that for any $u,v \in E$

$$
|F(u) - F(v)| \leq \ell_1 |u - v|.
$$

(V) Let $B : E \to \mathcal{L}(H,H)$ be a map such that there exists $\sigma \in (0, \frac{1}{2})$ for which $A^{-\sigma}B : E \to \mathcal{B}(H,E)$ is Lipschitz, i.e., there exists $\ell_2$ such that for any $u,v \in E$

$$
\|A^{-\sigma}B(u) - A^{-\sigma}B(v)\|_{\mathcal{B}(H,E)} \leq \ell_2 |u - v|.
$$

(VI) There exists a map $G : E \to \mathcal{L}(H,H)$ with the following properties.

(a) $B(z)G(z) = \text{Id}_H$ for any $z \in E$

(b) $\kappa := \sup_{z \in E} \|G(z)\|_{\mathcal{L}(H,H)} < \infty$.

(VII) There exist two sequences of maps $B_n : E \to \mathcal{L}(H,H)$ and $G_n : E \to \mathcal{L}(H,H)$, $n \in \mathbb{N}$, satisfying the following conditions.

(a) For any $n \in \mathbb{N}$ and $z \in E$:

$$
B_n(z)G_n(z) = \text{Id}_H.
$$

Furthermore, there exists $\kappa > 0$ such that for any $n \in \mathbb{N}$

$$
\sup_{z \in E} \|G_n(z)\|_{\mathcal{L}(H,H)} < \kappa.
$$

(b) The sequence $A^{-\sigma}B_n : E \to \mathcal{B}(H,E)$ is uniformly Lipschitz, i.e., there exists $\ell_3$ such that for any $n \in \mathbb{N}$, $x,y \in E$

$$
\|B_n(x) - B_n(y)\|_{\mathcal{B}(H,E)} \leq \ell_3 |x - y|.
$$

(c) The map $A^{-\sigma}B_n$ is $C^2$-smooth and its first and second Fréchet derivatives are continuous and bounded.

(d) We assume that $A^{-\sigma}B_n(\cdot) \to A^{-\sigma}B(\cdot)$ point-wise in $\mathcal{B}(H,E)$.

In the next definition we give the notion of solution to the problem (2).

**Definition 2.2.** We say that an $E$-valued process $u = \{u(t) : t \geq 0\}$ is a mild solution to (2) iff $u \in \mathcal{N}^q(E)$ and the following integral equation is satisfied for any $t \geq 0$ $\mathbb{P}$-a.s.

$$
u(t) = S(t)x + \int_0^t S(t-s)F(u(s))ds + \int_0^t S(t-s)B(u(s))dW(s).
$$

We have the following result.
Theorem 2.3. Let the assumptions (II)-(V) hold. Then, there exists a unique mild solution $u$ to problem (2) satisfying

$$\mathbb{E} \sup_{s \in [0,T]} |u(s)|^q < \infty, \ T > 0.$$ 

The solution has a continuous modification which is still denoted by $u$ and $u \in L^q(\Omega, C([0,T];E))$ for any $q \geq 2, \ T \geq 0$. Moreover, the process $u$ has the Markov property.

Since the proof is almost standard, we just sketch it in Appendix A.

In the present paper we denote by $u(t,x)$ the mild solution to SEE (2) with initial condition $x \in E$. The Markovian semigroup $\mathcal{P}_t := \{\mathcal{P}_t; t \geq 0\}$ associated to SEE (2) is defined by $\mathcal{P}_t \psi(x) := [\mathcal{P}_t \psi](x) = \mathbb{E}[\psi(u(t,x))]$ for any $t \geq 0, \ \psi \in B_0(E)$ and $x \in E$. We recall that $\mathcal{P}_t$ is Feller (resp., strong Feller) if for any $\psi \in C_b(E)$ (resp., $B_b(E)$) and $t > 0$, $\mathcal{P}_t \psi \in C_b(E)$. A probability measure $\nu$ on $E$ is said to be invariant w.r.t. $\mathcal{P}_t$ if

$$\int_E \mathcal{P}_t \psi(x) \nu(dx) = \int_E \psi(x) \nu(dx) \quad \text{for any } \psi \in B_0(E), \ t \geq 0.$$ 

The main theorem of this paper is the following

**Theorem 2.4.** Let the hypotheses (I)-(VII) be satisfied. Then, for all $t > 0$ there exist a constant $c_t > 0$ such that for all $\psi \in B_0(E)$, $x, y \in E$,

$$|\mathcal{P}_t \psi(x) - \mathcal{P}_t \psi(y)| \leq c_t \|\psi\|_0 |x - y|.$$ 

Before we proceed to the statement of the second result, we introduce the following definition.

**Definition 2.5.** The Markov semigroup $\mathcal{P}_t$ is called irreducible on $E$ if for any $t > 0$, $x \in E$ and any open set $\Gamma \subset E$ we have $\mathcal{P}_t 1_{\Gamma}(x) > 0$.

The second result of this paper is the irreducibility of the semigroup $\mathcal{P}_t$.

**Theorem 2.6.** Let the assumptions (II)-(VI) hold. Then, there exists a constant $\kappa > 0$ such that for any $t > 0, x, y \in E$ and $\delta > 0$,

$$\mathbb{P}(|u(t,x) - y| < \delta) > \kappa;$$

that is, the semigroup $\mathcal{P}_t$ is irreducible on $E$.

Since the proof of Theorem 2.6 is almost identical to the proof of [30, Theorem 1.3], we postpone it till Appendix B.

3. **Proof of Theorem 2.4: Strong Feller property.** The proof consists of two parts. In the first part we will assume that the coefficients $F$ and $B$ are of $C^2$ class. In the second part we will use some approximation arguments to remove this restrictive condition. Throughout we fix $T > 0$ and $q \geq 2$ and we denote by $\mathcal{S}_q$ the space of all (equivalence classes of) $E$-valued progressively measurable processes $u$ such that

$$\|u\|_q := \left( \mathbb{E} \sup_{t \in [0,T]} |u(t)|^q \right)^{\frac{1}{q}} < \infty.$$ 

The space $\mathcal{S}_q$ endowed with the norm $\| \cdot \|_q$ is a Banach space.
3.1. **SEE with twice differentiable coefficients.** In addition to Assumptions (III)-(VI), we assume in this part that the coefficients $F$ and $B$ in the problem (2) satisfies the following two sets of conditions.

**Assumptions.**

(A) The map $F : E \to E$ is twice Fréchet differentiable with bounded derivatives. In particular, there exists a positive constant $C_1$ such that

$$|F'(z) \cdot h| \leq C_1|h|,$$

for any $z, h \in E$.

(B) The map $A^{-\sigma}B$ is $C^2$-smooth and there exists a constant $C_3 > 0$ such that

$$\|A^{-\sigma}B'(z) \cdot h\|_{\mathcal{H}(E)} \leq \ell_5|h|,$$

$$\|A^{-\sigma}B''(z) \cdot (h, k)\|_{\mathcal{H}(E)} \leq C_3|h||k|,$$

for any $z, h, k \in E$. Here the constant $\ell_5 > 0$ is the same as in Assumptions (VIIb), respectively.

From here, to simplify notation we set

$$v(t, x) = [P_t\psi](x) \quad \psi \in B_b(E), \quad x \in E, \quad t \geq 0. \quad (3)$$

In the next lemma we give a regularity property of the solution $u(\cdot, x)$ when viewed as a map from $E$ into $\mathcal{S}_p$.

**Lemma 3.1.** Let $E$ be a martingale type 2 Banach space. In addition to Assumptions (III)-(VI) let us assume that the Assumptions (A)-(B) are satisfied and $\psi \in C^2_b(E)$. Then, the map $E \ni x \mapsto u(\cdot, x) \in \mathcal{S}_q$, where $u(\cdot, x)$ is the solution to the problem (2), is Gateaux differentiable, and its derivative $\eta^h := u_x(\cdot, x) \cdot h$ in direction of $h \in E$ is the unique mild solution to

$$d\eta^h(t) = [A\eta^h(t) + F'(u(t))\eta^h(t)]dt + B'(u(t))\eta^h(t)dW(t), \quad \eta^h(0) = h \in E. \quad (4)$$

Moreover, for any $q \geq 2$ there exist positive constants $M_0, M_1 > 0$ and $M_2 > 0$ such that

$$\mathbb{E} \sup_{s \in [0, t]} |\eta^h(s)|^q \geq M_0 e^{qM_1|h|^q} e^{C_1M_1 + \ell_5M_2}, \text{ for every } . \quad (5)$$

The proof of the above lemma will be given later.

The key to the proof of Theorem 2.4 is the following proposition which is the version of the celebrated Bismut-Elworthy-Li lemma for Banach spaces.

**Proposition 1.** Let the assumptions of Lemma 3.1 be satisfied. Let $v(t, x), \quad x \in E, \quad t \geq 0$ be the map defined as in (3). Then, the directional derivative $v_x(t, x) \cdot h$ of $v(t, x)$ in the direction $h \in E$ are given by

$$v_x(t, x) \cdot h = \frac{1}{t} \mathbb{E} \left[ \psi(u(t, x)) \int_0^t \left< G(u(s, x))u_x(s, x) \cdot h, dW(s) \right> \right], \quad (6)$$

for any $t > 0, \quad x \in E, \quad h \in E$.

**Proof of Proposition 1.** For each positive integer $n$ let $A_n = nA(nI - A)^{-1}$ be the Yosida approximation of $A$ and let $Q_n$ be the finite dimensinal projection from $H$ onto $H_n$ where

$$H_n := \text{span}\{e_1, \ldots, e_n\}.$$
Following [1, Section 5] we introduce the $H_n$-valued Wiener process $W_n := Q_n W$ defined by

$$W_n(t) = \sum_{k=1}^{n} \beta_k e_k.$$  

We also set $F_n := n(I + nA)^{-1} F$, and $B_n(x) := n(I + nA)^{-1} B(x) \circ i_n$, $x \in E$, where $i_n$ is the natural embedding $i_n : H_n \rightarrow H$.

Now, let $u_n(\cdot, x)$ be the unique strong solution of the equation

$$du_n(t, x) = (A_n u_n(t, x) + F_n(u_n(t, x)))dt + B_n(u_n(t, x))dW_n(t), \quad u_n(0) = x. \quad (7)$$

Let $t > 0$ be fixed. By applying twice Lemma 3.1 we infer that $v_n(t, \cdot) = E[\psi(u_n(t, \cdot))]$ is twice Gateaux differentiable, and to avoid ambiguity we denote by $\partial_x v_n$ and $\partial^2_{xx} v_n$ its first and second directional derivative. Thanks to the Itô's formula in [3, Theorem A.1] or [28, Theorem 74] we can argue exactly as in the proof of [8, Theorem 9.25] and show that $v_n$ is a strict solution to

$$\begin{cases}
\partial_t v_n(t, x) = \frac{1}{2} \text{Tr} B_n(x) \partial^2_{xx} v_n(t, x) + \mathbb{E}^* \langle \partial_x v_n(t, x), A_n x + F_n(x) \rangle_E,

v_n(0, x) = \psi(x),
\end{cases} \quad (8)$$

where for any $t > 0$ and $x \in E$

$$\text{Tr} B_n(x) Q_n \partial^2_{xx} v_n(t, x) = \sum_{k=1}^{n} \partial^2_{xx} v_n(t, x)[B_n(x)e_k, B_n(x)e_k].$$

We refer to [8, Definition 9.24] for the definition of a strict solution to (8). Now applying Itô's formula, see for e.g. [3, Theorem A.1] or [28, Theorem 74], to the stochastic process $\Psi(s) = v_n(t - s, u_n(s, x))$, $s \in [0, t]$, we obtain that

$$\psi(u_n(t, x)) = v_n(t, x) + \int_0^t \partial_x v_n(t - s, u_n(s, x))(B_n(u_n(s, x)))dW_n(s). \quad (9)$$

Now, let $y_n$ be the $\mathbb{R}$-valued stochastic processes defined by

$$y_n(t) := \int_0^t \langle \partial_x v_n(t - s, u_n(s, x)), B_n(u_n(s, x))dW_n(s) \rangle, \quad t \geq 0.$$

For any $h \in E$ let $z_n$ be $\mathbb{R}$-valued process defined by

$$z_n(t) := \int_0^t \langle Q_n G(u(s, x)) \eta^h(s), dW_n(s) \rangle, \quad t \geq 0$$

where $\eta^h$ is the derivative of $u(\cdot, x)$ at $x \in E$ in the direction $h \in E$ (see Lemma 3.1). Due to the Itô formula for $z_n(t)y_n(t)$ we obtain

$$\mathbb{E}[y_n(t)z_n(t)] = \mathbb{E} \sum_{k=1}^{n} \left( \int_0^t \left[ \langle [B_n(u_n(s, x))]^* \circ [\partial_x v_n(t - s, u_n(s, x))], e_k \rangle \times \langle Q_n G(u(s, x)) \eta^h(s), e_k \rangle \right] ds \right).$$

From this last identity and the fact that

$$\mathbb{E}[v_n(t, x)z_n(t, x)] = v_n(t, x)\mathbb{E}[z_n(t, x)] = 0,$$

we derive that

$$\mathbb{E}[\psi(u_n(t, x))z_n(t)]$$

$$= \mathbb{E} \int_0^t \langle [B_n(u(s, x))]^* \circ [\partial_x v_n(t - s, u_n(s, x))], Q_n G(u(s, x)) \eta^h(s) \rangle ds.$$
Hence it follows from the definition of $B_n$ that
\[
\mathbb{E}[\psi(u_n(t,x))z_n(t)] = \mathbb{E} \int_0^t \langle \partial_x v_n(t - s, u_n(s,x)), n(I + nA)^{-1} B(u_n(s,x))G(u(s,x))\eta^h(s) \rangle ds.
\] (10)

Thanks to [1, Remark 5.4] we derive that for any $q \geq 2$
\[
\lim_{n \to \infty} \mathbb{E} \sup_{t \in [0, T]} |u_n(s,x) - u(s,x)|^q = 0.
\]

Since $\psi \in C^2_0(E)$, we easily infer from the above convergence that there exist a subsequence $u_{n'}$ such that $\mathbb{P}$-almost surely
\[
\lim_{n' \to \infty} \sup_{t \in [0, T]} |u_{n'}(t,x) - u(t,x)| = 0,
\] (11)
\[
\lim_{n' \to \infty} \sup_{t \in [0, T]} |\psi(u_{n'}(t,x)) - \psi(u(t,x))| = 0.
\]

From the last line along with the Lebesgue Dominated Convergence Theorem (DCT) we deduce that
\[
\lim_{n' \to \infty} |v_{n'}(t,x) - v(t,x)| = 0,
\]
\[
\lim_{n' \to \infty} |\partial_x v_{n'}(t,x) - \partial_x v(t,x)|_{E^*} = 0.
\]

It is clear that $\mathbb{P}$-almost surely
\[
\lim_{n' \to \infty} \sup_{t \in [0, T]} |Q_n G(u(s,x))\eta^h(s) - G(u(s,x))\eta^h(s)|_H = 0.
\]

Now, for any $z \in E$ we have
\[
\sup_{s \in [0, T]} |A^{-\sigma}[J_{n'} B(u_{n'}(s,x))z - B(u(s,x)z)]|
\leq \sup_{s \in [0, T]} |J_{n'} [A^{-\sigma}B(u_n(s,x))z - A^{-\sigma}B(u(s,x)z)]|
+ \sup_{s \in [0, T]} |A^{-\sigma}[J_{n'} B(u(s,x)z) - B(u(s,x)z)]|
\]
where $J_{n'} = n'(I + n'A)^{-1}$. Thus, by using the Lipschitz continuity of $A^{-\sigma}B$ and the convergence (11) we infer that $\mathbb{P}$-almost surely
\[
\lim_{n' \to \infty} \sup_{s \in [0, T]} |A^{-\sigma}[J_{n'} B(u_{n'}(s,x))z - B(u(s,x)z)]| = 0,
\]
for any $z \in E$. This last convergence implies that $\mathbb{P}$-almost surely
\[
\lim_{n' \to \infty} \sup_{s \in [0, T]} |A^{-\sigma}[J_{n'} B(u_{n'}(s,x))G(u(s,x))\eta^h(s) - \eta^h(s)]| = 0.
\]

Thanks to these observations and the Lebesgue DCT we conclude, by letting $n'$ tend to $\infty$ in Eq. (10), that
\[
\mathbb{E}[\psi(u(t,x))z(t)] = \mathbb{E} \int_0^t \langle \partial_x v(t - s, u(s,x)), \eta^h(s) \rangle ds.
\] (12)

where $z$ is the $\mathbb{R}$-valued process defined by
\[
z(t) = \int_0^t \langle G(u(s,x))\eta^h(s), dW(s) \rangle, \quad t \geq 0.
\]
Since \( v(t, \cdot) \in C_0^2(E) \) for any \( t > 0 \) we deduce from (10) that

\[
\mathbb{E}[\psi(u(t, x))z(t)] = \int_0^t \partial_x [\mathbb{E}\{v(t - s, u(s, x))\}] \, ds,
\]

where \( \partial_x [\cdot](h) \) indicates the directional derivative in the direction \( h \in E \). To obtain the third line in the chain of equalities above, we used the fact that \( \mathcal{P}_t, t \geq 0 \) is a Markov semigroup and

\[
\mathbb{E}v(t - s, u(s, x)) = \mathbb{E}\mathcal{P}_{t-s} \psi(u(s, x)) = \mathcal{P}_s \mathcal{P}_{t-s} \psi(x) = \mathcal{P}_t \psi(x).
\]

It follows from (13) that

\[
\mathbb{E}[\psi(u(t, x))z(t)] = t \, \partial_x [\mathcal{P}_t \psi(x)](h).
\]

Recalling the definition of \( z(t) \) we deduce from the last identity that

\[
\partial_x [\mathcal{P}_t \psi(x)](h)(t) = \frac{1}{t} \mathbb{E} \left[ \psi(u(t, x)) \int_0^t \left\langle G(u(s, x)) \eta h(s), dW(s) \right\rangle \right],
\]

which ends the proof of Proposition 1.

Now we give the promised proof of Lemma 3.1.

**Proof of Lemma 3.1.** We will check the first part of the lemma for small \( T > 0 \) and repeating the argument given below on \([T, 2T], \ldots\), will yield the general case.

Let us first notice that the problem (4) is a linear stochastic evolution equation with time dependent and random coefficients. We will justify shortly that the problem (4) has a unique mild solution. For this purpose, let

\[
\tilde{F}(t, \omega, \eta) = F'(u(t, \omega)) \cdot \eta
\]

\[
\tilde{B}(t, \omega, \eta) = B'(u(t, \omega)) \cdot \eta,
\]

for any \( t \in [0, T] \), \( \omega \in \Omega \), and \( \eta \in E \). Due to Assumptions (A)-(B) there exists a constant \( C_1 > 0 \) such that

\[
|\tilde{F}(t, \omega, \eta_1) - \tilde{F}(t, \omega, \eta_2)| \leq C_1|\eta_1 - \eta_2|
\]

\[
|A^{-\sigma}\tilde{B}(t, \omega, \eta_1) - \tilde{B}(t, \omega, \eta_2)||_{\mathcal{B}(H, E)} \leq \ell_5|\eta_1 - \eta_2|
\]

for any \( t \in [0, T] \), \( \omega \in \Omega \), and \( \eta_1, \eta_2 \in E \). Since \( F \) and \( A^{-\sigma}B \) are of class \( C^2 \) and the process \( u \) is progressively measurable, then for any \( \eta \in E \) the stochastic processes \( \tilde{F}(\cdot, \cdot, \eta) \) and \( A^{-\sigma}\tilde{B}(\cdot, \cdot, \eta) \) are progressively measurable. Furthermore, the maps \( \tilde{F} \) and \( A^{-\sigma}\tilde{B} \) are linear w.r.t the third variable. Due to these facts we can apply [1, Theorem 4.3] to infer that the problem (4) has a unique mild solution.
Now, we consider the map \( \Lambda : E \times \mathcal{S}_q \to \mathcal{S}_q \) defined for \( x \in E, \ u \in \mathcal{S}_q \) by

\[
\Lambda(x, u)(t) := S(t)x + \int_0^t S(t-s)F(u(s))ds + \int_0^t S(t-s)B(u(s))dW(s),
\]

\[
= \Lambda^0(x)(t) + \Lambda^1(u)(t) + \Lambda^2(u)(t), \ \forall t \in [0, T].
\]

We have the following three properties of \( \Lambda \).

(1) There exists a constant positive \( T \) such that

\[
\|\Lambda(x, u) - \Lambda(x, v)\|_q \leq \frac{1}{2}\|u - v\|_q, \ \text{for any} \ u, v \in \mathcal{S}_q.
\]

The existence of such \( T \) is ensured by Appendix A.

(2) The map \( \Lambda \) is Gateaux differentiable with respect to its variables and for all \( x, y \in E, \ u, v \in \mathcal{S}_q, \) and \( t \in [0, T] \)

\[
(\Lambda_x(x) \cdot y)(t) = S(t)y,
\]

\[
(\Lambda_u(x, u) \cdot v)(t) = \int_0^t S(t-s)F'(u(s)) \cdot v(s)ds + \int_0^t S(t-s)B'(u(s)) \cdot v(s)dW(s),
\]

\( \mathbb{P} \)-almost surely.

To prove this claim it is sufficient to check the following items.

(i) The map \( \Lambda^0 \) and \( \Lambda^1 \) are Gateaux differentiable and for all \( x, y \in E, \ u, v \in \mathcal{S}_q, \) and \( t \in [0, T] \)

\[
(\Lambda^0_x(x) \cdot y)(t) = S(t)y,
\]

\[
(\Lambda^1_u(x, u) \cdot v)(t) = \int_0^t S(t-s)F'(u(s)) \cdot v(s)ds \mathbb{P} - \text{almost surely,}
\]

(ii) The map \( \Lambda^2 \) is Gateaux differentiable and for all \( u, v \in \mathcal{S}_q, \) and \( t \in [0, T] \)

\[
(\Lambda^2_u(u) \cdot v)(t) = \int_0^t S(t-s)B'(u(s)) \cdot v(s)dW(s), \mathbb{P} - \text{almost surely.}
\]

We omit the proof of item (i) since it is quite standard, see for instance [8, Lemma 9.5]. To prove item (ii) we fix a constant \( \delta > 0 \) and define an auxiliary process \( I_\delta \) by

\[
I_\delta(t) := \frac{1}{\delta} \int_0^t S(t-s)\left[B(u(s) + \delta v(s)) - B(u(s)) - \delta B'(u(s)) \cdot v(s)\right]dW(s), \ t \geq 0.
\]

Let us put \( \bar{B} := A^{-\sigma}B \) where \( \sigma \) is given by Assumption (V). By [1, Theorem 3.2] there exists a positive constant \( C \) such that

\[
\|I_\delta\|_q \leq C\mathbb{E} \int_0^T \left\|\frac{1}{\delta} [\bar{B}(u(s) + \delta v(s)) - \bar{B}(u(s))] - \bar{B}'(u(s)) \cdot v(s)\right\|_{\mathcal{S}(H, E)}^q ds,
\]

\[
\leq C\mathbb{E} \int_0^T \left\|\frac{1}{\delta} \int_0^1 \bar{B}'(u(s) + \theta \delta v(s)) \cdot v(s) - \bar{B}'(u(s)) \cdot v(s)d\theta\right\|_{\mathcal{S}(H, E)}^q ds,
\]

where we used the formula (see, for instance [7, Proposition 5.5.1 or Theorem 5.6.1])

\[
\bar{B}(y + \delta h) - \bar{B}(y) = \delta \int_0^1 \bar{B}'(y + \theta \delta h) d\theta \cdot h
\]

to obtain the last line. Since, by assumption, \( \bar{B} \) has bounded Fréchet derivative and \( v \in \mathcal{S}_q \) we easily see that

\[
\|I_\delta\|_q < CT^{1/2}\|v\|_q < \infty.
\]
Thus, thanks to the Fréchet differentiability of $\tilde{B}$ and the Lebesgue Dominated Convergence Theorem we infer that

$$\lim_{\delta \to 0} \|I_\delta\|_q = 0.$$  

This yields the claim (ii).

(3) For $i = 1, 2$ the maps $\Lambda^i_n : \mathcal{G}_q \times \mathcal{G}_q \ni (u, v) \mapsto \Lambda^i_n(u, v) \in \mathcal{G}_q$ is separately continuous.

The separate continuity of $\Lambda^i_n$ wrt to its arguments can be treated as in [8], so we just study the continuity of $\Lambda^2_n$. Fix $u, v \in \mathcal{G}_q$ and let $\{v_n : n \in \mathbb{N}\}$ be a sequence converging to $v$ in $\mathcal{G}_q$. Owing to [1, Theorem 3.2] we have

$$\|\Lambda^2_n(u, v_n) - \Lambda^2_n(u, v)\|_q^q = \mathbb{E} \sup_{t \in [0, T]} \left| \int_0^t S(t - s) \left( B'(u(s)) \cdot v_n(s) - B'(u(s)) \cdot v(s) \right) dW(s) \right|^q \leq C \int_0^T \|A^{-\sigma} B'(u(s)) \cdot (v_n(s) - v(s))\|_q^q \, d\tau,$$

from which we easily derive the continuity of $\Lambda_n(\cdot, \cdot)$ wrt the second variable. Now let $\{u_n : n \in \mathbb{N}\}$ be a sequence converging to $u$ in $\mathcal{G}_q$. Owing again to [1, Theorem 3.2] we obtain

$$\|\Lambda^2_n(u_n, v) - \Lambda^2_n(u, v)\|_q^q = \mathbb{E} \sup_{t \in [0, T]} \left| \int_0^t S(t - s) \left( B'(u(s)) \cdot v_n(s) - B'(u(s)) \cdot v(s) \right) dW(s) \right|^q \leq C \int_0^T \|A^{-\sigma} B'(u_n(s)) \cdot (v_n(s) - v(s))\|_q^q \, d\tau.$$

Thanks to assumption (V) we can now argue as in [8, Proposition 9.6] to show that

$$\lim_{n \to \infty} \|\Lambda^2_n(u_n, v) - \Lambda^2_n(u, v)\|_q^q = 0,$$

which implies the continuity of $\Lambda^2_n$ wrt to its first variable.

The items (1)-(3) and [8, Lemma 9.2] imply that on the small interval $[0, T]$ the solution $u(\cdot, x)$, $x \in E$ to the problem (2) is Gateaux differentiable as a map from $E$ into $\mathcal{G}_q$ and its derivative $\eta^h := u_x(\cdot, x) \cdot h$ in the direction of $h \in E$ is the mild solution to (4). Repeating this argument on $[T, 2T], \ldots$ yields the proof of the first part of the lemma.

Now let us prove the second part of Lemma 3.1. Let $h$ and $\eta^h(\cdot)$ be as in the statement of Lemma 3.1. It is clear that the process $\eta^h(\cdot)$ satisfies

$$\mathbb{E} \sup_{s \in [0, t]} |\eta^h(s)|^q \leq c_2 \sup_{s \in [0, t]} |S(s)h|^2 + c_2 t^{q-1} \mathbb{E} \int_0^t |S(s - r)F'(u(r, x)) \cdot \eta^h(r)|^q dr + c_2 \mathbb{E} \sup_{s \in [0, t]} \left| \int_0^s S(s - r) B'(u(r, x)) \cdot \eta^h(r) dW(r) \right|^q,$$
from which altogether with [1, Theorem 3.2] we infer that
\[
E \sup_{s \in [0,t]} |\eta(s)|^q \leq M_0 e^{qM_1|h|^q} + \tilde{M}_1 E \int_0^t \|S(t-r)\|_{\mathcal{L}(E,E)}^q |F'(u(r,x)) \cdot \eta^h(r)|^q dr
\]
\[
+ \tilde{M}_2 E \int_0^t \|A^{-\sigma}B'(u(r,x)) \cdot \eta^h(r)|^q_{\mathcal{L}(H,E)} dr.
\]
From this last inequality and the boundedness of \( F' \) and \( B' \) (see the assumptions (A)) we derive
\[
E \sup_{s \in [0,t]} |\eta^h(s)|^q \leq M_0 e^{qM_1|h|^q} + C_1 \tilde{M}_1 E \int_0^t \|S(t-r)\|_{\mathcal{L}(E,E)}^q |\eta^h(r)|^q dr
\]
\[
+ \ell_5 \tilde{M}_2 E \int_0^t |\eta^h(r)|^q dr.
\]
Now, we easily conclude from the Gronwall inequality that \( \eta^h(\cdot) \) satisfies (5). This ends the proof of the lemma.

We state and prove the following (important) lemma.

**Lemma 3.2.** In addition to hypotheses (I)-(VI) suppose that the condition (A) is satisfied. Then, for any \( t > 0 \) and \( \psi \in C^2_{\text{b}}(E) \) the function \( \mathcal{P}_t \psi(\cdot) \) satisfies

\[
|\mathcal{P}_t \psi(x) - \mathcal{P}_t \psi(y)| \leq \frac{1}{\sqrt{t}} \kappa \|\psi\|_0 Me^{\frac{1}{2}(M_1C_1 + M_2\ell_5)} |x - y|,
\]

for any \( x \) and \( y \) in \( E \).

**Proof.** Let \( t > 0 \) and \( h \in E \). Arguing as in [30, Lemma 2.5] and using (5) we have

\[
|v_x(t, x) \cdot h| \leq \frac{1}{t} \|\psi\|_0 \kappa \left( E \int_0^t |\eta^h(s)|^2 ds \right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{t}} \kappa \|\psi\|_0 Me^{\frac{1}{2}(M_1C_1 + M_2\ell_5)},
\]

where the constants \( C_1 \) and \( \ell_5 \) are appearing in Assumption (A). Now we can deal with the remainder of the proof by using a similar argument as in [30, Lemma 2.5].

3.2. **SEE with globally Lipschitz coefficients.** In this part we will get rid of the stringent assumptions (A)-(B). The strategy of the proof is to find sequences \( (F_n)_{n \geq 1} \) and \( (B_n)_{n \geq 1} \) approximating \( F \) and \( G \) and satisfying the assumptions (A)-(B).

Thanks to Assumption (VII) it is enough to find a sequence \( (F_n)_{n \geq 1} \) satisfying the properties we mentioned earlier. Since, by Lemma 2.1, any Banach space having \( C^k \)-smooth norm admits a \( C^k \)-smooth bump function, the following lemma, which is given in [15, Theorem H], is sufficient for such aim.

**Lemma 3.3.** Let \( k \) be a positive integer and \( E \) be a Banach space having \( C^k \)-smooth norm and an unconditional Schauder basis. Let \( Y \) be a Banach space. Then, there exists a constant \( C > 0 \) such that for any \( L \)-Lipschitz map \( \Phi : E \to Y \) and positive integer \( n \) there exists a \( CL \)-Lipschitz map \( \Phi_n \in C^k(E,Y) \) for which \( \|\Phi(x) - \Phi_n(x)\|_Y \leq \frac{1}{n} \). Moreover,

\[
\|\Phi_n'(x) \cdot h\|_Y \leq CL|h|, \quad \|\Phi_n''(k,h)\|_Y \leq \tilde{C}L|h||k|,
\]

for any \( x, h, k \in E \) and \( n \in \mathbb{N} \).
Remark 1. Note that the differentiability in the above lemma is taken in the Fréchet sense. The Lipschitz constant of $\Phi_n$ is independent of $n$.

Now, we state the following result.

**Proposition 2.** Let $E$ be a Banach space satisfying the assumption (I). Then, there exists a constant $C_0 > 0$ and a sequence $(F_n)_{n \geq 1}$ approximating $F$ such that

$$|F_n(x) - F_n(y)| \leq C_0 \ell_1 |x - y|, \quad \text{for all } x, y \in E, n \in \mathbb{N}. $$

Moreover, the two estimates in Assumption (A) hold with $C_1 = C_0 \ell_1$ where $\ell_1$ is the positive constant from Assumption (IV).

**Proof.** This is a direct consequence of Lemma 3.3. \hfill $\square$

After stating Proposition 2 we are almost ready to prove our main result (see Theorem 2.4). But before we embark on the proof of Theorem 2.4 we state and prove the following lemma.

**Lemma 3.4.** Let $E$ be a separable Banach space having $C^2$-smooth norm and $\mathcal{P}_t$ be a Markov semigroup on $B_b(E)$. Let $c > 0$ and $t > 0$ be fixed constants. Then, the following two conditions are equivalent

(i) for all $\phi \in C^2_b(E), x, y \in E$, $|\mathcal{P}_t \phi(x) - \mathcal{P}_t \phi(y)| \leq c \|\phi\|_0 |x - y|.$

(ii) for all $\phi \in B_b(E), x, y \in E$, $|\mathcal{P}_t \phi(x) - \mathcal{P}_t \phi(y)| \leq c \|\phi\|_0 |x - y|.$

**Proof.** Since $E$ is separable and has a $C^2$-smooth norm it follows from [11, Theorem VIII.3.2] that bounded continuous function on $E$ can be approximated uniformly by functions of class $C^2$. Now, we can prove the lemma by using the same idea as in [30, Lemma 2.2]. \hfill $\square$

Now, we are ready to prove our main result.

**Proof of Theorem 2.4.** Let $n \in \mathbb{N}$. Let $B_n$ and $G_n$ be the sequences of functions given in Assumption (VII) and $F_n$ be the sequence from Proposition 2. Let $u_n(\cdot, x)$ for $x \in E$ be the unique mild solution to the problem

$$du_n(t, x) = [Au_n(t, x) + F_n(u_n(t, x))] dt + B_n(u_n(t, x)) dW(t), \quad u_n(0) = x,$$

and $\mathcal{P}^{(n)}_t$ be the Markov semigroup associated to $u_n$. Thanks to Proposition 2 and Theorem A.1 we derive that for any $q \geq 2$

$$\lim_{n \to \infty} E \sup_{s \in [0, t]} |u_n((t, x) - u(t, x)|^q = 0.$$ 

Then one can extract a subsequence $n'$ such that a.e. $\omega \in \Omega$

$$\lim_{n' \to \infty} \sup_{s \in [0, t]} |u_{n'}(t, \omega, x) - u(t, \omega, x)|^q = 0.$$ 

Let $\psi \in C^2_b(E)$. Thanks to the boundedness and continuity of $\psi$ we derive that as $n' \to \infty$

$$\mathcal{P}^{(n')}_{\ell}(\psi(x) \to \mathcal{P}_t \psi(x)$$

for all $t \geq 0$. Proposition 1 is applicable to $\mathcal{P}^{(n)}_t$ since $F_n, B_n$ and $G_n$ satisfy all of its assumptions. Hence, we can also apply Lemma 3.2 to $\mathcal{P}^{(n)}_t$. Then, there exist
We fix the following standing assumptions.

**Application:** The nonlinear stochastic heat equations. Let $\mathcal{O} = [0, 1]$, and $W$ be a cylindrical Wiener process on $H := L^2(\mathcal{O})$. Throughout we denote by

$$\langle u, v \rangle := \int_{\mathcal{O}} u(x) \cdot v(x) \, ds, \quad u \in H, \; v \in H$$

the inner product in $H$.

We consider the nonlinear stochastic heat equations

\[
\begin{align*}
\partial_t u(t, \xi) &= \frac{\partial^2}{\partial \xi^2} u(t, \xi) + f(u(t, \xi)) + b(u(t, \xi)) \frac{\partial}{\partial t} W(t, \xi), \\
\quad u(\cdot, 0) &= u(\cdot, 1) = 0, \\
\quad u(0, \cdot) &= x(\cdot) \in L^p(\mathcal{O}).
\end{align*}
\] (15)

We fix the following standing assumptions.

(A) The map $b : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous and there exists two constant $\ell_7 > 0$ and $\ell_8 > 0$ such that

$$\ell_7 \leq |b(x)| \leq \ell_8,$$

for all $x \in \mathbb{R}$.

(B) The map $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous.

For fixed $p > 2$ let $E := L^p(\mathcal{O})$. For $u \in E$ and $v \in H$ we set

$$F(u)(\xi) = (f \circ u)(\xi), \quad B(u)(v)(\xi) = (b \circ u)(\xi)v(\xi), \quad \xi \in \mathcal{O}.$$ 

We denote by $A$ the realization of the Dirichlet Laplacian $\Delta$ in $E$ with domain $D(A) = H^{2,p}(\mathcal{O}) \cap H_0^{1,p}(\mathcal{O})$. With these notations the problem (15) can be recast in the following abstract SEE

\[
\begin{align*}
du(t) &= (Au(t) + F(u(t)))\, dt + B(u(t))\, dW(t), \\
u(0) &= x \in E.
\end{align*}
\] (16)
4.1. Basic auxiliary results. It is well-known that the operator $A$ is the generator of an analytic semigroup $S(t), t \geq 0$ on $E$ which has an extension, denoted by the same symbol, on $H$. This extension is again analytic and its infinitesimal generator, again denoted by $A$, is the realization of the Dirichlet Laplacian on $H$. Note that $H$ has an orthonormal basis \( \{ e_k : k = 1, 2, \ldots \} \) whose elements are eigenfunctions of $A$. We denote by \( \{ \lambda_k : k = 1, 2, \ldots \} \) the corresponding set of eigenvalues.

We prove a result about the mapping $B$.

**Proposition 3.** Suppose $q \in (2, \infty)$. Then, if $\sigma \geq \frac{1}{2q}$

\[
|A^{-\sigma}(bh)|_H \leq C|h|_H|b|_E,
\]

for any $b \in \tilde{E} = L^q(O)$ and $h \in H$.

**Proof.** Let $r = \frac{2q}{q + 2}$. Observe that $A^{-\sigma}$ maps $L^r$ into $H^{2\sigma, r}$ because $A$ has the BIP property and so $D(A^\sigma) = [L^r, D(A)]_\sigma$, and on the other hand $[L^r, D(A)]_\sigma \subset H^{2\sigma, 2}$, see [34]. Thus, since, by assumption, $2\sigma - \frac{1}{r} \geq -\frac{1}{2}$, it follows from the Sobolev embedding $H^{2\sigma, r} \subset H$ and the above observation that

\[
|A^{-\sigma}(bh)|_H \leq C|bh|_{L^r},
\]

from which along with the Hölder inequality we derive that

\[
|A^{-\sigma}(bh)|_H \leq C \int_O |h(x)|^2 dx \left( \int_O |b(x)|^{\frac{2r'}{2r - 1}} dx \right)^{\frac{2r - 1}{r'}},
\]

where $r = \frac{2q}{q + 2}$ and $r' = 1$. Since $r = \frac{2q}{q + 2}$ we easily conclude the proof of the proposition from the last inequality. \(\square\)

As a crucial consequence of this proposition we have the following result.

**Proposition 4.** Let $p > 2$ and assume that assumptions (A) hold. Then, there exists $\sigma \in (0, \frac{1}{2})$ such that $A^{-\sigma} B$ maps $E$ into $\mathcal{B}(H, E)$. Moreover, there exist $\ell > 0$ such that

\[
|A^{-\sigma} (B(u)\cdot) - B(v)\cdot)|_{\mathcal{B}(H, E)} \leq \ell |u - v|_E,
\]

for any $u, v \in E$.

For any $u \in E$ and $h \in H$ let

\[
G(u)[h] := \frac{1}{b(u)}h.
\]

Then $G$ maps $E$ into $\mathcal{L}(E, H)$ and satisfies the following properties

1. For any $z \in E$

\[
B(z)G(z) = \text{Id}_H.
\]

2. There exists a constant $\kappa > 0$ such that

\[
\sup_{z \in E} \|G(z)\|_{\mathcal{L}(H, H)} \leq \kappa.
\]

**Proof.** Since $\lambda_k \leq Ck^2$ we easily derive that

\[
\sum_{k=1}^\infty |A^{-\sigma_1} e_k|^2 = \sum_{k=1}^\infty \lambda_k^{-2\sigma_1} < \infty,
\]

if $\sigma_1 \in \left( \frac{1}{4}, \infty \right)$. Hence, it follows from [4, Theorem 2.3] that $A^{-\sigma_1} \in \mathcal{B}(H, E)$ for any $\sigma_1 \in \left( \frac{1}{4}, \infty \right)$. Now, let $\sigma_2 \geq \frac{1}{2p}$, it follows from Proposition 3 that $A^{-\sigma_2} (B(u)\cdot) \in \mathcal{B}(H, E)$. \(\square\)
$\mathcal{L}(H,H)$ for any $u \in E$. Thus, taking $\sigma = \sigma_1 + \sigma_2$ we find that if $\sigma_2 \geq \frac{1}{2p}$ and $\sigma_1 \in (\frac{1}{4}, \infty)$, the map $A^{-\sigma}(B(u)[\cdot]) \in \mathcal{B}(H,E)$ for any $u \in E$. Moreover,

$$\|A^{-\sigma}(B(u)[\cdot]) - B(v)[\cdot]\|_{\mathcal{B}(H,E)} \leq C\|A^{-\sigma_1}\|_{\mathcal{B}(H,E)}\|A^{-\sigma_2}(B(u)[\cdot]) - B(v)[\cdot]\|_{\mathcal{L}(H,H)},$$

for any $u \in E$, $v \in E$. From Proposition 3 and the Lipschitz continuity of $b$ we derive that

$$\|A^{-\sigma}(B(u)[\cdot]) - B(v)[\cdot]\|_{\mathcal{B}(H,E)} \leq C\|A^{-\sigma_1}\|_{\mathcal{B}(H,E)}\|u - v\|_E, \quad u \in E, \quad v \in E.$$ 

Summing up, taking $\sigma_2 \in \left[\frac{1}{2p}, \frac{1}{4}\right]$ and $\sigma_1 \in \left(\frac{1}{4}, \frac{1}{2} - \sigma_2\right)$, we find $\sigma \in (0, \frac{1}{2})$ such that the map $A^{-\sigma}B$ is Lipschitz continuous from $E$ into $\mathcal{B}(H,E)$.

Due to the boundedness of below of $b(\cdot)$, it is easy to check that $G$ is a well-defined bounded map from $E$ into $\mathcal{L}(H;H)$. Moreover,

$$(B(u) \circ G(u))[h] = b(u) \left(\frac{1}{b(u)}h\right) = h,$$

for any $u \in E$ and $h \in H$. This ends the proof of Proposition 4.

**Theorem 4.1.** Let $q \geq 2$ and $p > 2$. Assume that Assumptions (A) and (B) hold. Then, for any $x \in E$ the problem (16) has a unique mild solution satisfying

$$\mathbb{E} \sup_{s \in [0,T]}|u(s)|_{\mathcal{L}^q}^q < \infty.$$ 

Moreover $u$ has a continuous modification which is still denoted by $u$ and $u \in L^q(\Omega, C([0,T];E))$ for any $q \geq 2$.

**Proof.** It follows from the Bonic-Frampton theorem, Proposition 4 and Assumption (B) that all the assumptions (II)-(V) are satisfied by the problem (16). Thus, Theorem 4.1 is a corollary of Theorem 2.3.

4.2. **Uniqueness of the invariant measure of problem (16).** Let $C^b_b(\mathbb{R}) \subset C^\infty(\mathbb{R})$ be the set of all function $f$ such that it and its derivatives of any order are bounded. Let $(b_n)_{n \in \mathbb{N}} \subset C^b_b(\mathbb{R})$ such that

$$\lim_{n \not\rightarrow \infty} |b_n(x) - b(x)| = 0, \quad \forall x \in \mathbb{R},$$

and there exists a constant $\ell > 0$ for which

$$|b'_n(x) - b_n(y)| \leq \ell|x - y|$$

for any $n \in \mathbb{N}$ and $x, y \in \mathbb{R}$. Furthermore, we require that there exists a constant $\ell_0 > 0$ such that for any $n \in \mathbb{N}$ and $x \in \mathbb{R}$

$$|b_n(x)| \leq \ell_0.$$ 

**Remark 2.** For each $n \in \mathbb{N}$ let $\psi_n : \mathbb{R} \rightarrow [0, \infty)$ be the map defined by

$$\psi_n(x) = \begin{cases} c_1 \ n e^{(n^2 x^2 - 1)^{-1}} & \text{if } |x| < \frac{1}{n}, \\ 0 & \text{otherwise}, \end{cases}$$

where $c_1$ is chosen so that $\int_{\mathbb{R}} \psi_n(x)dx = 1$. The sequence $(b_n)_{n \in \mathbb{N}}$ of maps $b_n : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$b_n(x) = \int_{\mathbb{R}} b(y)\psi_n(x - y)dy, \quad x \in \mathbb{R}, \quad n \in \mathbb{N},$$

satisfies the above conditions.
For \( u \in E \) and \( v \in H \) we set
\[
B_n(u)[v](\xi) = (b_n \circ u)(\xi)v(\xi), \xi \in \mathcal{O}.
\]
We have the following result.

**Proposition 5.** If \( \sigma > \frac{1}{p} \) then the maps, for \( n \in \mathbb{N} \),
\[
E \ni u \mapsto A^{-\sigma}B_n(u) = \{ H \ni h \mapsto A^{-\sigma}B_n(u)[h] := A^{-\sigma}[b_n(u)h]\} \in \mathcal{L}(H, H)
\]
are \( \ell \)-Lipschitz, of class \( C^2 \) in the Fréchet sense and its first and second derivative are bounded. Moreover, \( A^{-\sigma}B_n(\cdot) \) converges in \( \mathcal{L}(H, H) \) to \( A^{-\sigma}B(\cdot) \) point-wise.

**Proof.** The proof of the \( \ell \)-Lipschitz continuity follows the same line as in proof of Proposition 4, so we omit it.

We prove that \( A^{-\sigma}B_n \) is of class \( C^2 \) and have bounded derivatives. For this aim let \( u \in E \), \( v \in E \) and \( h \in H \) be arbitrary. Now, set \( \tilde{B}_n(u)[h] = A^{-\sigma}[b_n(u)h] \) and
\[
\Psi_v = \int_0^1 [b'_n(u + \theta v) - b'_n(u)]d\theta.
\]
It is easy to see that
\[
\tilde{B}_n(u + v)[h] - \tilde{B}_n(u)[h] - A^{-\sigma}[b'_n(u)(vh)] = A^{-\sigma}[v \Psi_v h].
\]
Since \( b'_n \) is bounded, \( \Psi_v \) is an element of \( H \) and it follows from Proposition 3 that
\[
|\tilde{B}_n(u + v)[h] - \tilde{B}_n(u)[h] - A^{-\sigma}[b'_n(u)(vh)]|_H = |A^{-\sigma}[b_n(u)h]|_H \leq c|\Psi_v h|_H|v|_E.
\]
We deduce from this that \( \tilde{B}_n = A^{-\sigma}B_n \) is of class \( C^1 \) and its derivative is defined by
\[
E \ni v \mapsto \tilde{B}'_n(u)(\cdot) = \{ H \ni h \mapsto \tilde{B}'_n(u)[h] := A^{-\sigma}[b'_n(u)(vh)]\} \in \mathcal{L}(H, H).
\]
Furthermore, from the uniform boundedness of \( b'_n \) and Proposition 3 we infer that for any \( u, v \in E \) and \( h \in H \),
\[
|A^{-\sigma}[b'_n(u)(vh)]|_H \leq C\ell|v|H|h|H.
\]
Thus, the derivative of \( \tilde{B}_n \) is uniformly bounded in the sense that there exists a constant \( C > 0 \) such that for any \( n \in \mathbb{N} \), \( u \in E \) and \( v \in E \),
\[
\||\tilde{B}'_n(u)[v]\|_{\mathcal{L}(E, \mathcal{L}(H, H))} \leq C\ell|v|_E.
\]
For any \( u \in E \), \( v \in E \) let
\[
\Phi_v = \int_0^1 [b''_n(u + \theta v) - b''_n(u)]d\theta.
\]
With a similar argument as above, we can show that
\[
\tilde{B}_n(u + v)[y]h - \tilde{B}_n(u)[y]h - A^{-\sigma}[b''_n(u)(yvh)] = A^{-\sigma}[v \Phi_v h],
\]
for any \( u \in E \), \( v \in E \), \( y \in E \) and \( h \in H \). Thus, we infer from Proposition 3 that
\[
|\tilde{B}'_n(u + v)[y]h - \tilde{B}'_n(u)[y]h - A^{-\sigma}[b'_n(u)(yvh)]|_H \leq C|\Phi_v h|_H|y|_E|v|_L^p,
\]
where \( q = \frac{p}{2} \). Cauchy-Schwarz’s inequality yields
\[
|\tilde{B}'_n(u + v)[y]h - \tilde{B}'_n(u)[y]h - A^{-\sigma}[b'_n(u)(yvh)]|_H \leq C|\Phi_v h|_H|y|_E|v|_E,
\]
from which altogether with the boundedness of \( b''_n \) we infer the Fréchet differentiability of \( \tilde{B}'_n \). So we have just proved that \( \tilde{B}_n \) has a second derivative defined by
\[
\tilde{B}''_n(u)[v_1, v_2](\cdot) = A^{-\sigma}[b''(u)(v_1 v_2 \cdot)],
\]
for any \( u, v_1 \) and \( v_2 \in E \). Applying Proposition 3 we deduce that there exists a constant \( C > 0 \) such that for any \( u \in E, v_1 \in E \) and \( v_2 \in E \)

\[
\|B_n^u[v_1, v_2]\|_{L^2(E, \mathcal{L}(H, H))} \leq C|v_1|_E|v_2|_E,
\]

where \( \mathcal{L}^2(E, \mathcal{L}(H, H)) := \mathcal{L}(E, \mathcal{L}(E, \mathcal{L}(H, H))) \).

It remains to prove the point-wise convergence of \( A^{-\sigma}B_n \) to \( A^{-\sigma}B \) in \( \mathcal{L}(H, H) \). If \( \sigma > \frac{1}{p} \), then thanks to Proposition 3 there exists a constant \( C > 0 \) such that

\[
|A^{-\sigma}[(b_n(u) - b(u))h]|_H \leq C|h|_H|b_n(u) - b(u)|_E,
\]

for any \( u \in E, h \in H \) and \( n \in \mathbb{N} \). Since \( b_n(x) \to b(x) \) for any \( x \in \mathbb{R} \) and \( |b_n(u)|_E \leq \ell_0 \) for any \( n \in \mathbb{N} \), it follows from Lebesgue Dominated Convergence Theorem that

\[
|b_n(u) - b(u)|_E \to 0,
\]

as \( n \not\to \infty \). Hence, we easily derive from the estimate (17) that \( A^{-\sigma}B_n \) point-wise converges to \( A^{-\sigma}B \) in \( \mathcal{L}(H, H) \). This ends the proof of our proposition.

The next theorem contains the main result of this section.

**Theorem 4.2.** Let \( p > 4 \) and the assumptions (A) and (B) hold. Then, the unique mild solution \( u \) to the problem (15) is irreducible and strong Feller on \( L^p(0, 1) \).

**Proof.** For each \( n \in \mathbb{N} \) let \( b_n \) be as in Remark 2 and \( G_n : E \to \mathcal{L}(E, H) \) be the map defined by

\[
G_n(u)[h] := \frac{1}{b_n(u)} h,
\]

for any \( u \in E \) and \( h \in H \). Owing to the boundedness from below of \( b_n \), it is easy to check that \( G_n \) is well-defined and bounded. Moreover,

\[
(B_n(u) \circ G_n(u))[h] = b_n(u) \left( \frac{1}{b_n(u)} h \right) = h,
\]

for any \( n \in \mathbb{N}, u \in E \) and \( E \in H \). Thus, if \( p > 4 \) and the assumptions (A) and (B) hold, then in view of the Bony-Frampton Theorem, Propositions 4 and 5 we infer that all the assumptions of Theorem 2.4 and Theorem 2.6 are satisfied. Therefore, the unique mild solution \( u \) to the problem (15) is strong Feller and irreducible.

**Corollary 1.** If there exists an invariant measure \( \mu \) for the corresponding Markov semigroup on \( E \) of (15), see for e.g. [8], then \( \mu \) is unique.

**Remark 3.** It is possible, although has not been done in detail by the authors, to prove similar results to those in Section 4 when the Laplace operator with Dirichlet boundary conditions is replaced by the Laplace operator with general boundary conditions or by a fractional power of the latter.

**Appendix A. Proof of Theorem 2.3 and a convergence result.** The proof of the existence result in Theorem 2.3 is very similar to the proof of [1, Theorem 4.3]. For fixed \( q \geq 2 \) let \( \mathcal{S}_q \) be the space of progressively process \( u : [0, T] \times \Omega \to E \) such that

\[
\|u\|_q^q := \mathbb{E} \sup_{s \in [0, T]} |u(s)|^q < \infty.
\]

It is clear that \( \mathcal{S}_q \) equipped with the norm \( \|\cdot\|_p \) is a Banach space. For \( u \in \mathcal{S}_q, \xi \in L^q(\Omega, \mathcal{F}_0, \mathbb{P}; E) \), and \( t \in [0, T] \) we set

\[
\Lambda(u)(t) := S(t)\xi + \int_0^t S(t - s)F(u(s))ds + \int_0^t S(t - s)B(u(s))dW(s).
\]
By assumption, it is clear that if \( u \in \mathfrak{S}_q \) then \( \Lambda(u) \) is progressively measurable. Now, we check that \( \Lambda \) maps \( \mathfrak{S}_p \) into itself and it is a strict contraction for small \( T \). For this aims let \( u, v \in \mathfrak{S}_q \), \( \xi \in L^q(\Omega, F_0, \mathbb{P}; E) \) and \( Z := \Lambda(u) - \Lambda(v) \). It is easy to check that

\[
\|Z\|^q_q \leq C(q)T^{q-1} \int_0^T \|S(s)\|^q_q \|F(u(s)) - F(v(s))\|_q^q ds
\]

\[+ C(q) \mathbb{E} \sup_{t \in [0, T]} \int_0^t S(t - s)[B(u(s)) - B(v(s))]dW(s)\]

from which altogether with [1, Theorem 3.2] implies that

\[
\|Z\|^q_q \leq CT^{q-\sigma} \mathbb{E} \int_0^T \|A^{-\sigma}[B(u(s)) - B(v(s))]\|^q_q ds
\]

\[+ CT^{q-1}\|u - v\|^q_q \int_0^T \|S(s)\|^q_q d\mathbb{E}_{(E,E)} ds.\]

Thus, it follows from assumption (V) that

\[
\|Z\|^q_q \leq \|u - v\|^q_q \left[ CT^{q-1} \int_0^T \|S(s)\|^q_q d\mathbb{E}_{(E,E)} + CT^{q-\sigma} \right].
\]

This last estimate implies that \( \Lambda(\cdot) \) maps \( \mathfrak{S}_q \) into itself and is a contraction for small \( T \). This fact and standard fixed point argument enable us to conclude the first part of the proof of our theorem.

Owing to assumption (V) and [1, Theorem 3.2], the stochastic process

\[
\phi_A(t) := \int_0^t S(t - s)B(u(s))dW(s), \quad t \in [0, T],
\]

has a continuous modification. Thus, \( u \) has a continuous modification. The Markov property of the process \( u \) follows from the uniqueness result and [8, Theorem 9.14]. This ends the proof of Theorem 2.3.

In the next theorem we give a convergence result which is of great importance in our analysis.

**Theorem A.1.** Let \( F \) and \( B \) two maps satisfying assumptions (IV) and (V). Let \( (B_n)_{n \geq 1} \) be a sequence of maps \( B_n : E \to \mathbf{L}(H, H) \) such the following conditions hold.

1. There exist constant \( \sigma \in (0, \frac{1}{2}) \) and \( \ell > 0 \) such that for any \( n \in \mathbb{N}, x, y \in E \)

\[
\|A^{-\sigma}B_n(x) - B_n(y)\|_{\mathfrak{S}(H, E)} \leq \ell|x - y|,
\]

\[
\|A^{-\sigma}B_n(x)\|_{\mathfrak{S}(H, E)} \leq \ell(1 + |x|).
\]

2. The sequence \( A^{-\sigma}B_n(\cdot) \) converges to \( A^{-\sigma}B(\cdot) \) point-wise in \( \mathfrak{S}(H, E) \).

Let also \( (F_n)_{n \geq 1} \) be a sequence of maps \( F_n : E \to E \) satisfying the following items.

(i) There exists a constant \( \ell > 0 \) such that for any \( n \geq 1, x, y \in E \)

\[
|F_n(x) - F_n(y)| \leq \ell|x - y|,
\]

\[
|F_n(x)| \leq \ell(1 + |x|).
\]

(ii) \( F_n(x) \to F(x) \) in \( E \) for any \( x \in E \).
Let $x \in E$ be fixed and $u$ be the unique mild solution of the problem (2). Let $u_n$ be the unique mild solution to the following problem

$$du_n(t) = (Au_n(t) + F_n(u_n(t)))dt + B_n(u_n(t))dW(t), \quad u_n(0) = x \in E.$$ \hspace{1cm} (18a)

Then,

$$\lim_{n \to \infty} \mathbb{E} \sup_{s \leq t} |u_n(s) - u(s)|^q = 0, \quad (19)$$

for any $q \geq 2$ and $T \in (0, \infty)$.

**Proof.** Notice first that the existence and uniqueness of $u_n$ is ensured by Theorem 2.3. Let $t \in (0, \infty)$, $n \geq 1$ and set

$$\Phi_n^1(t) := \int_0^t S(t-s)[F_n(u_n(s)) - F(u(s))]ds,$$

$$\Phi_n^2(t) := \int_0^t S(t-s)[B_n(u_n(s)) - B(u(s))]dW(s).$$

We have

$$u_n(t) - u(t) = \Phi_n^1(t) + \Phi_n^2(t).$$

It is not difficult to show that

$$|\Phi_n^1(t)|^q \leq c \Phi_n^{1,1}(t) + c \Phi_n^{1,2}(t),$$

where for $t \in [0, T]$,

$$\Phi_n^{1,1}(t) := \left( \int_0^t \|S(t-s)\|_{\mathcal{L}(E,E)}|F_n(u_n(s)) - F_n(u(s))|ds \right)^q,$$

$$\Phi_n^{1,2}(t) := \left( \int_0^t \|S(t-s)\|_{\mathcal{L}(E,E)}|F_n(u(s)) - F(u(s))|ds \right)^q,$$

and $c > 0$ is constant depending only on $q$. From the Hölder inequality and the Lipschitz continuity of $F_n$, we infer the existence of a positive $C(\ell, t)$ such that

$$\Phi_n^{1,1}(t) \leq C(\ell, t) \int_0^t |u_n(s) - u(s)|^q ds.$$  

From the Hölder inequality we deduce that there exists a constant $C(t) > 0$ such that

$$\Phi_n^{1,2}(t) \leq C(t) \int_0^t |F_n(u(s)) - F(u(s))|^q ds.$$  

It follows from [1, Theorem 3.2] that there exists a constant $C_0 = C(t, q, \sigma)$ such that

$$\mathbb{E} \sup_{s \leq t} |\Phi_n^2(s)|^q \leq C_0 \mathbb{E} \int_0^t \|A^{-\sigma}[B_n(u_n(s)) - B_n(u(s))]\|^q_{\mathcal{H}(H,E)}ds$$

$$+ C_0 \mathbb{E} \int_0^t \|A^{-\sigma}[B_n(u(s)) - B(u(s))]\|^q_{\mathcal{H}(H,E)}ds.$$
From the Lipschitz continuity of $A^{-\sigma}B_n$ we infer the existence of a positive constant $c_0 = C(t, q, \ell)$ such that
\[
\mathbb{E} \sup_{s \in [0, t]} |\Phi_n^2(s)|^q \leq C_0 \mathbb{E} \int_0^t \|A^{-\sigma}[B_n(u(s)) - B(u(s))]\|_{\mathfrak{R}(H, E)}^q ds + c_0 \mathbb{E} \int_0^t |u_n(s) - u(s)|^q ds.
\]
Hence, we have just showed that we can find a positive constant $c_0$ such that
\[
\mathbb{E} \sup_{s \in [0, t]} |u_n(s) - u(s)|^q \leq c_0 c_0 \mathbb{E} \int_0^t \|A^{-\sigma}[B_n(u(s)) - B(u(s))]\|_{\mathfrak{R}(H, E)}^q ds + c_0 \mathbb{E} \int_0^t |F_n(u(s)) - F(u(s))|^q ds + c_0 \mathbb{E} \int_0^t |u_n(s) - u(s)|^q ds,
\]
from which along with the Gronwall lemma follows that
\[
\mathbb{E} \sup_{s \in [0, t]} |u_n(s) - u(s)|^q \leq c_0 e^{c_0 t} \mathbb{E} \int_0^t \|A^{-\sigma}[B_n(u(s)) - B(u(s))]\|_{\mathfrak{R}(H, E)}^q ds + c_0 e^{c_0 t} \mathbb{E} \int_0^t |F_n(u(s)) - F(u(s))|^q ds.
\]
(20)

Since, by assumption, there exists a constant $\ell > 0$ such that for any integer $n \geq 1$
\[
\mathbb{E} \int_0^t |F_n(u(s))|^q ds + \mathbb{E} \int_0^t \|A^{-\sigma}B_n(u(s))\|^q_{\mathfrak{R}(H, E)} ds \leq \ell (1 + \mathbb{E} \sup_{s \in [0, t]} |u(s)|^q),
\]
and $F_n \to F$ (resp. $A^{-\sigma}B_n \to B$ ) point-wise in $E$ (resp. in $\mathfrak{R}(H, E)$), we can apply the Lebesgue Dominated Convergence Theorem to infer that as $n \to \infty$
\[
\mathbb{E} \int_0^t |F_n(u(s)) - F(u(s))|^q ds + \mathbb{E} \int_0^t \|A^{-\sigma}[B_n(u(s)) - B(u(s))]\|^q_{\mathfrak{R}(H, E)} ds \to 0.
\]
Therefore, for any $t \in (0, \infty)$
\[
\lim_{n \to \infty} \mathbb{E} \sup_{s \in [0, t]} |u_n(s) - u(s)|^q = 0,
\]
which proves our theorem. \(\square\)

**Appendix B. Proof of Theorem 2.6.** The proof of Theorem 2.6 is very similar to the proof of [30, Theorem 1.3]. It relies very much on the following result. We recall that throughout this section $u$ is the unique solution to the problem (2).

**Lemma B.1.** Let the assumptions of Theorem 2.6 be satisfied. Let $t_1 \in (0, T)$ and let $f : [t_1, T] \times E \to E$ be a bounded and measurable mapping. If $Z$ is the solution of
\[
dZ(s) = (AZ(s) + 1_{(t_1, T)}(s)f(s, Z(s)))ds + B(Z(s))dW(s),
\]
$Z(0) = x \in E$,
then for $t \in [0, T]$ the laws on $(E, \mathcal{B}(E))$ of $u(t, x)$ and $Z(t, x)$ are equivalent.
Proof. Since, by Theorem 2.3, the solution $u(\cdot, x)$ of (2) is continuous in $E$, thus the idea of the proof is very similar to the second part of [30, Proof of Lemma 3.1]. Here we only outline the steps where their proof and ours differ.

First, we assume that $F$ is uniformly bounded. Let $f_1 : [0, T] \times E \to E$ be the map defined by

$$f_1(s, x) := \begin{cases} f(s, x) & \text{if } s \in (t_1, T], \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $f_1$ is a measurable and bounded map. Let $\tilde{F} : [0, T] \times E \to H$ be the map defined by

$$\tilde{F}(s, x) := G(x)i(f_1(s, x) - F(x)), \ s \in [0, T], \ x \in E,$$

where $i$ is the natural embedding $i : E \to H$. It follows from the Assumption (VI) and the boundedness of $f_1$ and $F$ that $\tilde{F}$ is uniformly bounded, i.e., there exists a constant $C > 0$ such that

$$|\tilde{F}(t, x)|_H \leq C, \ \forall t \in [0, T], \ x \in E.$$

Now we set $\alpha(s) := \tilde{F}(s, Z(s)), \ s \in [0, T]$, and

$$M(s) := \exp \left( -\int_0^s \langle \alpha(r), dW(r) \rangle - \frac{1}{2} \int_0^s |\alpha(r)|_H^2 dr \right), \ s \in [0, T].$$

Owing to the uniform boundedness of $\tilde{F}$, the process $\alpha$ satisfies the Novikov condition

$$\mathbb{E} \exp \left( \frac{1}{2} \int_0^T |\alpha(s)|_H^2 ds \right) < \infty.$$

Then, by the Girsanov theorem in the Banach space setting, see for e.g. [29, Section 7], we infer that

1. the process $M(\cdot)$ is a martingale under the measure $\mathbb{P}$,
   $$\mathbb{E}(M(t)) = 1, \ \mathbb{P}(M(T) > 0) = 1,$$
2. the measures $\mathbb{P}$ and $\hat{\mathbb{P}} := M(T) d\mathbb{P}$ are equivalent,
3. the process
   $$\hat{W}(t) := W(t) + \int_0^t \alpha(r) dW(r),$$
   defines a cylindrical Wiener process evolving on $H$, on $(\Omega, \mathcal{F}, \hat{\mathbb{P}})$.
4. for any progressively measurable process $\phi$ with values in $\mathcal{H}(H, E)$ such that
   $$\hat{\mathbb{P}} \left( \int_0^T |\phi(s)|_{\mathcal{H}(H, E)}^2 ds < \infty \right) = 1,$$
   we have
   $$\int_0^t \phi(r) d\hat{W}(r) = \int_0^t \phi(r) \alpha(r) dr + \int_0^t \phi(r) dW(r).$$

Owing to (4), we see that

$$Z(s) = S(s)x + \int_0^s S(s-r) f_1(r, Z(r)) dr + \int_0^s S(s-r) B(Z(r)) dW(r) = S(s)x + \int_0^s S(s-r) \tilde{F}(Z(r)) dr + \int_0^s S(s-r) B(Z(r)) dW(r).$$
This means that $Z$ is the solution to the problem (2) under $\tilde{P}$. Thanks to this remark and the facts (1)-(4), we can follow the exact same lines as in [30, Proof of Lemma 3.1] to prove that the laws on $(E, \mathcal{B}(E))$ of $u(t,x)$ and $Z(t,x)$, $t \in [0,T]$, are equivalent.

Second, we will get rid of the additional assumption on $F$ imposed in the first step. For this purpose, for each real number $\ell > 0$, we set

$$F_\ell(x) = \begin{cases} F(x) & \text{if } |x| \leq \ell, \\ F\left(\frac{x}{|x|}x\right) & \text{otherwise}. \end{cases}$$

It is not difficult to check that for each $\ell > 0$, the map $F_\ell : E \to E$ is Lipschitz continuous and there exists a constant $C_\ell > 0$ such that

$$|F_\ell(x)| \leq C_\ell, \forall x \in E.$$

Let $\ell > 0$ and $u_\ell(\cdot, x)$ be the $E$-valued continuous solution to (2) with $F$ replaced by $F_\ell$. Arguing as above we can show that the laws on $C([0,t]; E)$, $t \in [0,T]$, of $u_\ell(\cdot, x)$ and $Z(\cdot, x)$ are equivalent. Since for each $F = F_\ell$ on the ball $B(0, \ell)$, by the pathwise uniqueness of solution the laws on $B(0, \ell)$ of $u_\ell(t, x)$ and $u(t, x)$, $t \in [0,T]$, are identical. This implies the equivalence of laws on $(E, \mathcal{B})$ of $u(t,x)$ and $Z(t,x)$, $t \in [0,T]$.

Now we are ready to prove Theorem 2.6.

**Proof of Theorem 2.6.** Due to Theorem 2.3 and Lemma B.1, the proof is the same as in [30, Proof of Theorem 1.3] so we omit it.

**Conclusion.** In this paper we have studied the uniqueness of the invariant measure of stochastic evolution equations in Banach spaces. The main difficulty is the proof of the strong Feller property of the Markov solution which was divided in two parts. In the first part we assumed that the coefficients $F$ and $B$ are $C^2$-smooth and established a Bismut-Elworthy-Li (BEL) formula in the context of Banach spaces, see Proposition 1. This BEL type formula along with some nice estimates on the derivative of the solution with respect to the initial condition will imply the strong Feller property. In the second part we used some approximation argument to remove these restrictive conditions. We mainly use the fact that if the state space $E$ has $C^2$-smooth norm then it is possible to approximate Lipschitz functions on $E$ by $C^2$-smooth and Lipschitz functions. In the near future we are aiming to remove this smoothness condition imposed on the norm of $E$. The first step is to consider a 2-smooth Banach space $E$ as in [27].

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