AN AFFINE WEYL GROUP ACTION ON THE BASIC HYPERGEOMETRIC
FUNCTION ARISING FROM THE \textit{q}-GARNIER SYSTEM

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Abstract. Recently, we formulated the \textit{q}-Garnier system and its variations as translations of an extended affine Weyl group of type $A_{2n+1}^{(1)} \times A_1^{(1)} \times A_1^{(1)}$. On the other hand, those systems admit particular solutions in terms of two types of \textit{q}-hypergeometric functions: the basic hypergeometric function and the \textit{q}-Lauricella function. In this article, we investigate an action of the extended affine Weyl group on the basic hypergeometric function.

Key Words: Affine Weyl group, Basic hypergeometric function, Discrete Painlevé equation, Garnier system.

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1. Introduction

The Garnier system is an extension of the sixth Painlevé equation in several variables; see [2, 5]. Its \textit{q}-difference analogue was proposed as the connection preserving deformation of a linear \textit{q}-difference system in [13]. Afterward, the \textit{q}-Garnier system was investigated by the Padé method in [7, 8]. Then we obtained other directions of discrete time evolutions called variations. In recent works [9, 10, 16], we clarified that the \textit{q}-Garnier system and all variations were given as elements called translations of an extended affine Weyl group of type $A_{2n+1}^{(1)} \times A_1^{(1)} \times A_1^{(1)}$.

It is known that the Garnier system admits a particular solution in terms of the Lauricella function $F_D$; see [2, 5]. Particular solutions of the \textit{q}-Garnier system and one variation were investigated in [7, 13, 15]. As a result, two types of \textit{q}-hypergeometric functions appeared: the basic hypergeometric function $\phi_{n+1}^\pm_n$ and the \textit{q}-Lauricella function $\phi_D^{(n)}$.

This fact presents two next problems. One is particular solutions of the other variations and another is the action of the extended affine Weyl group on the \textit{q}-hypergeometric functions. In this article we give an answer to these problems.

We state an outline of this article. In Section 2 we recall the definition of the \textit{q}-Garnier system and the derivation of its particular solution. We first formulate a group of birational transformations on a field of rational functions, which is named $\tilde{G}$. This group is isomorphic to an extended affine Weyl group of type $A_{2n+1}^{(1)} \times A_1^{(1)} \times A_1^{(1)}$ and contains an abelian normal subgroup generated by translations. In the following, we call this subgroup the \textit{q}-Garnier system. We next consider a direction of the \textit{q}-Garnier system. Via a certain specialization, it is reduced to a higher order generalization of the \textit{q}-Riccati equation whose solution is described in terms of the basic hypergeometric function.

The results of this article are given in the following sections. In Section 3 we find elements of $\tilde{G}$ which is consistent with the specialization. Those elements generate a subgroup of $\tilde{G}$, which is named $\tilde{F}$. The group $\tilde{F}$ is isomorphic to an extended affine Weyl group of type $A_1^{(1)} \times A_1^{(1)}$ and provides particular solutions of some directions of the \textit{q}-Garnier system. In Section 4, we formulate $(n + 1) \times (n + 1)$ matrices corresponding to the generators of $\tilde{F}$. Those matrices give
a left action of $\tilde{F}$ on a vector of the basic hypergeometric functions. In Section 5, we formulate translations of $\tilde{F}$ in a framework of $(n+1) \times (n+1)$ matrices. Then we obtain linear $q$-difference systems and ladder operators for the $q$-hypergeometric functions again.

**Remark 1.1.** In [9][16], an extended affine Weyl group of type $A_{m-1}^{(1)} \times A_{m-1}^{(1)} \times A_{m-1}^{(1)}$ was formulated as a generalization of the $q$-Garnier system. Its $q$-hypergeometric solution was also investigated for the case of $(m,n) = (3,2)$. On the other hand, another generalization of the $q$-Garnier system was proposed together with its $q$-hypergeometric solution in [11][12]. We haven’t clarified affine Weyl group symmetries of those $q$-hypergeometric solutions yet. It’s a future problem.

2. The $q$-Garnier system and the basic hypergeometric function

2.1. The $q$-Garnier system. Let $\varphi_{ji} \ (j \in \mathbb{Z}_{2n+2}, i \in \mathbb{Z}_2)$ be dependent variables and $\alpha_j, \beta_i, \beta'_i \ (j \in \mathbb{Z}_{2n+2}, i \in \mathbb{Z}_2)$ parameters defined by

$$\alpha_j = \varphi_{j0} \varphi_{j1}, \quad \beta_i = \prod_{j=0}^{2n+1} \varphi_{ji}, \quad \beta'_i = \prod_{j=0}^{2n+1} \varphi_{ji+j}.$$ 

We also set

$$q = \prod_{j=0}^{2n+1} \varphi_{j0} \varphi_{j1} = \prod_{j=0}^{2n+1} \alpha_j = \beta_0 \beta_1 = \beta'_0 \beta'_1.$$ 

We first define birational transformations $r_j, s_i, s'_i \ (j \in \mathbb{Z}_{2n+2}, i \in \mathbb{Z}_2)$ by

$$r_j(\varphi_{j-1,i}) = \varphi_{j-1,i} \varphi_{j+i+1} \frac{1 + \varphi_{ji}}{1 + \varphi_{ji+1}}, \quad r_j(\varphi_{ji}) = \frac{1}{\varphi_{ji+1}}, \quad r_j(\varphi_{j+1,i}) = \varphi_{j+1,i} \varphi_{j+i,1} \frac{1 + \varphi_{ji+1}}{1 + \varphi_{ji}},$$

$$r_j(\varphi_{k,i}) = \varphi_{k,i} \quad (k \neq j, j \pm 1),$$

$$s_i(\varphi_{ji}) = \frac{Q_{ji}}{Q_{j+1,i}}, \quad s_i(\varphi_{ji+1}) = \varphi_{ji} \varphi_{j+i+1} \frac{Q_{ji+1}}{Q_{ji}},$$

$$s'_i(\varphi'_{ji}) = \frac{Q'_{ji}}{Q'_{j+1,i}}, \quad s'_i(\varphi'_{ji+1}) = \varphi'_{ji} \varphi'_{j+i+1} \frac{Q'_{ji+1}}{Q'_{ji}},$$

where

$$\varphi'_{ji} = \varphi_{j-i,j}, \quad Q_{ji} = \sum_{k=0}^{2n+1} \prod_{l=0}^{k-1} \varphi_{j+l,i}, \quad Q'_{ji} = \sum_{k=0}^{2n+1} \prod_{l=0}^{k-1} \varphi'_{j+l,i}.$$ 

Then they act on the parameters as

$$r_j(\alpha_j) = \frac{1}{\alpha_j}, \quad r_j(\alpha_{j+1}) = \alpha_{j+1} \alpha_j, \quad r_j(\alpha_k) = \alpha_k \quad (k \neq j, j \pm 1), \quad r_j(\beta_i) = \beta_i, \quad r_j(\beta'_i) = \beta'_i,$$

$$s_i(\alpha_j) = \alpha_j, \quad s_i(\beta_i) = \frac{1}{\beta_i}, \quad s_i(\beta_{i+1}) = \beta_{i+1} \beta_i, \quad s_i(\beta'_i) = \beta'_i, \quad s_i(\beta'_{i+1}) = \beta'_{i+1},$$

$$s'_i(\alpha_j) = \alpha_j, \quad s'_i(\beta_i) = \beta_i, \quad s'_i(\beta_{i+1}) = \beta_{i+1}, \quad s'_i(\beta'_i) = \frac{1}{\beta'_i}, \quad s'_i(\beta'_{i+1}) = \beta'_{i+1} \beta'_i.$$ 

**Fact 2.1** ([4][6][10]). If we set

$$G = \langle r_0, \ldots, r_{2n+1} \rangle, \quad H = \langle s_0, s_1 \rangle, \quad H' = \langle s'_0, s'_1 \rangle,$$
then the groups $G$, $H$ and $H'$ are isomorphic to the affine Weyl groups of type $A_{2n+1}^{(1)}$, $A_1^{(1)}$ and $A_1^{(1)}$ respectively. Moreover, any two groups are mutually commutative.

Recall that the affine Weyl group of type $A_{n-1}^{(1)}$ is generated by the generators $r_i$ ($i \in \mathbb{Z}_n$) and the fundamental relations

$$r_i^2 = r_i = 1,$$

for $n = 2$ and

$$r_i^2 = 1, \quad r_i r_j r_i = r_j r_i r_j \quad (|i-j| = 1), \quad r_i r_j = r_j r_i \quad (|i-j| > 1),$$

for $n \geq 3$. Note that we interpret a composition of transformations in terms of automorphisms of the field of rational functions $\mathbb{C}(\varphi_{ji})$. For example, the compositions $r_0 r_1$ act on the parameter $\alpha_0$ as

$$r_0 r_1(\alpha_0) = r_0(\alpha_0 \alpha_1) = r_0(\alpha_0) = \alpha_1.$$

We next define birational transformations $\pi, \pi', \rho$ by

$$\pi(\varphi_{ji}) = \varphi_{j+1,i+1}, \quad \pi'(\varphi_{ji}) = \varphi_{j+1,i}, \quad \rho(\varphi_{ji}) = \varphi'_{ji}.$$

Then they act on the parameters as

$$\pi(\alpha_j) = \alpha_{j+1}, \quad \pi(\beta_i) = \beta_{i+1}, \quad \pi'(\beta_i) = \beta_i', \quad \rho(\alpha_j) = \alpha_{-j}, \quad \rho(\beta_i) = \beta_i', \quad \rho(\beta_i') = \beta_i.$$

**Fact 2.2** ([10]). The transformations $\pi, \pi', \rho$ satisfy relations

$$\pi^{2n+2} = 1, \quad \pi^* = (\pi')^2, \quad \pi' = \pi' \pi, \quad \rho^2 = 1, \quad \pi \rho = \rho (\pi')^{-1},$$

$$r_j \pi = \pi r_{j-1}, \quad s_i \pi = \pi s_{i-1}, \quad s_i' \pi = \pi s_i', \quad r_j \pi' = \pi' r_{j-1}, \quad s_i \pi' = \pi' s_i, \quad s_i' \pi' = \pi' s_{i+1}, \quad r_j \rho = \rho r_{j-1}, \quad s_i \rho = \rho s_i'$$

for $j \in \mathbb{Z}_{2n+2}$ and $i \in \mathbb{Z}_2$.

A semi-direct product $\widetilde{G} = \langle G, H, H' \rangle \rtimes \langle \pi, \pi', \rho \rangle$ can be regarded as an extended affine Weyl group of type $A_{2n+1}^{(1)} \times A_n^{(1)} \times A_1^{(1)}$. Moreover, this group contains an abelian normal subgroup generated by translations; see [10]. In this article we call this subgroup the $q$-Garnier system.

### 2.2. The basic hypergeometric function.

Consider a translation

$$\tau_c = (\pi')^{-1} \pi s_0' s_0,$$

as a direction of the $q$-Garnier system. It acts on the parameters as

$$\tau_c(\alpha_j) = \alpha_j, \quad \tau_c(\beta_0) = q^{-1} \beta_0, \quad \tau_c(\beta_1) = q \beta_1, \quad \tau_c(\beta_0') = q^{-1} \beta_0', \quad \tau_c(\beta_1') = q \beta_1'.$$

**Fact 2.3** ([9] [15]). The action of $\tau_c$ on the dependent variables is consistent with a condition

$$\varphi_{2j+1,0} = -\alpha_{2j+1}, \quad \varphi_{2j+1,1} = -1 \quad (j = 0, \ldots, n),$$

namely

$$\tau_c(\varphi_{2j+1,0}) \big|_{\text{Eq. (2.1)}} = -\alpha_{2j+1}, \quad \tau_c(\varphi_{2j+1,1}) \big|_{\text{Eq. (2.1)}} = -1.$$

Then we obtain a system of non-linear $q$-difference equations

$$\tau_c(\varphi_{2j,0}) = \frac{\varphi_{2j,0}}{\alpha_{2j-1} \alpha_{2j}} \frac{T_{j+1}}{T_j}, \quad (j = 0, \ldots, n),$$

(2.2)
where

\[ T_j = \prod_{l=0}^{n-1} \alpha_{2,j+2l+1} \prod_{l=0}^{n} \alpha_{2l} + \sum_{k=1}^{n} (-1)^k (1 - \alpha_{2,j+2k-1}) \prod_{l=k}^{n-1} \varphi_{2,j+2l+1,0} \prod_{l=0}^{k-1} \varphi_{2l,0} \prod_{l=k}^{n} \alpha_{2l} + (-1)^n \prod_{l=0}^{n} \varphi_{2l,0}. \]

System (2.2) can be regarded as a higher order \( q \)-Riccati system. It has \( n + 1 \) dependent variables \( \varphi_{0,0}, \varphi_{2,0}, \ldots, \varphi_{2n,0} \) and \( 2n + 3 \) parameters \( \alpha_0, \alpha_1, \ldots, \alpha_{2n+1}, \beta_0 \) with a constraint

\[ \prod_{j=0}^{n} \varphi_{2j,0} = \frac{(-1)^{n+1} \beta_0}{\prod_{j=0}^{n} \alpha_{2j+1}}. \]

Note that condition (2.1) implies that between parameters

\[ \beta'_0 = \frac{\beta_0}{\prod_{j=0}^{n} \alpha_{2j+1}}. \] (2.3)

Under condition (2.3), we define a \((2n + 3)\)-tuple of parameters \( \mathbf{c} = (a_1, \ldots, a_{n+1}, b_1, \ldots, b_{n+1}, c) \) by

\[ a_j = \prod_{l=2n-2j+1}^{2n-1} \alpha_l, \quad b_j = \prod_{l=2n-2j}^{2n-1} \alpha_l, \quad c = \frac{\beta_0}{q \prod_{j=0}^{n} \alpha_{2j+1}}. \]

Note that \( b_{n+1} = q \). We also use the notation

\[ a_{j+n+1} = q a_j, \quad b_{j+n+1} = q b_j, \]

for a sake of convenience. The original parameters are expressed in terms of \( \mathbf{c} \) as

\[ \alpha_{2,2j} = \frac{b_{n-j+1}}{a_{n-j+1}}, \quad \alpha_{2j} = \frac{a_{n-j+1}}{b_{n-j}} \quad (j = 0, \ldots, n), \quad \beta_0 = q c \prod_{l=1}^{n+1} \frac{a_l}{b_{l-1}}. \]

The action of \( \tau_c \) on \( \mathbf{c} \) is given by

\[ \tau_c(\mathbf{c}) = (a_1, \ldots, a_{n+1}, b_1, \ldots, b_{n+1}, q^{-1} c). \]

Let \( \mathbf{x} = \mathbf{x}(\mathbf{c}) \) be a vector of functions defined by

\[ \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_{n+1} \end{bmatrix}, \quad x_j = \prod_{l=1}^{n-j+1} (1 - a_l) \prod_{l=n-j+2}^{n} (1 - b_l) \sum_{k=0}^{\infty} q^k c^k \prod_{l=1}^{n-j+1} (q a_l; q)_k \prod_{l=n-j+2}^{n+1} (q b_l; q)_k \prod_{l=n-j+2}^{n+1} (a_l; q)_k \prod_{l=n-j+2}^{n+1} (b_l; q)_k, \]

where \((a; q)_k\) is the \( q \)-shifted factorial defined by

\[ (a; q)_0 = 1, \quad (a; q)_k = (1 - a)(1 - q a) \ldots (1 - q^{k-1} a) \quad (k \geq 1). \]

Each function \( x_j \) is called the basic hypergeometric function \( n+1 \phi_n \) and converges if we set \(|q| < 1\) and \(|c| < q^{-1}\); see [3].

**Fact 2.4** ([9] [15]). The vector \( \mathbf{x} \) satisfies a system of linear \( q \)-difference equations

\[ \mathbf{x}(\tau_c(\mathbf{c})) = M_{\tau_c} \mathbf{x}(\mathbf{c}), \] (2.4)

with an \((n + 1) \times (n + 1)\) matrix

\[ (1 - c) M_{\tau_c} = \sum_{j=1}^{n+1} (b_{n-j+1} - a_{n-j+2} c) E_{j,j} \]
Remark 2.6 (10). \( q \)-P3.1. Results of this section.

The actions of transformations given by (3.1) act on the dependent variables

\[
\sigma = \pi s_0 r_1 r_3 \ldots r_{2n+1}, \quad \sigma' = (\pi')^{-1} s'_0 r_0 r_2 \ldots r_{2n}, \quad \pi_1 = \pi^2, \quad \pi_2 = \rho (\pi')^{-1}\pi.
\]

(3.1)

Theorem 3.1. The actions of transformations given by (3.1) on the dependent variables \( \varphi_{j,i} \) \( (j \in \mathbb{Z}_{2n+2}, i \in \mathbb{Z}_2) \) are consistent with condition (2.1). We will prove this theorem in Section 3.2. Hence we obtain a subgroup of \( \widetilde{G} \) which provides particular solutions of some directions of the \( q \)-Garnier system. As a matter of fact, the translation \( \tau_c = \sigma' \sigma \) is an element of this subgroup and Fact 2.3 follows from Theorem 3.1.

In the following, we always assume that condition (2.1) holds. Then the transformations given by (3.1) act on the dependent variables \( \varphi_{j,0} \) \( (j \in \mathbb{Z}_{n+1}) \) as

\[
p_i(\varphi_{-2i-4,0}) = \frac{\varphi_{-2i-4,0} \{a_{i+1} - b_i + (a_{i+1} - a_i) \varphi_{-2i-2,0}\}}{a_i - b_i},
\]

\[
p_i(\varphi_{-2i-2,0}) = \frac{(a_i - b_i) \varphi_{-2i-2,0}}{a_{i+1} - b_i + (a_{i+1} - a_i) \varphi_{-2i-2,0}},
\]

\[
p_i(\varphi_{2j,0}) = \varphi_{2j,0} \quad (j \neq -2i - 4, -2i - 2),
\]

for \( i = 0, \ldots, n \),

\[
p'_i(\varphi_{-2i-4,0}) = \frac{b_i - b_{i+1} + (b_i - a_{i+1}) \varphi_{-2i-4,0}}{b_{i+1} - a_{i+1}},
\]

\[
p'_i(\varphi_{-2i-2,0}) = \frac{(b_{i+1} - a_{i+1}) \varphi_{-2i-4,0} \varphi_{-2i-2,0}}{b_i - b_{i+1} + (b_i - a_{i+1}) \varphi_{-2i-4,0}},
\]

\[
p'_i(\varphi_{2j,0}) = \varphi_{2j,0} \quad (j \neq -2i - 4, -2i - 2),
\]

for \( i = 0, \ldots, n \) and

\[
\sigma(\varphi_{2j,0}) = \frac{(a_{n-j} - b_{n-j}) \varphi_{2j,0} (b_{n-j-1} + a_{n-j-1}) \varphi_{2j+2,0}}{(a_{n-j-1} - b_{n-j-1})(b_{n-j} + a_{n-j}) \varphi_{2j,0}},
\]
\[ \varphi'(\varphi_{j0}) = \frac{\varphi_{j0} \sum_{k=0}^{n} (-1)^k (b_n - j - k) \prod_{l=1}^{k} \varphi_{j+l,0}}{\sum_{k=0}^{n} (-1)^k (b_n - j - k) \prod_{l=1}^{k} \varphi_{j+l,0}}, \]

\[ \pi_1(\varphi_{j0}) = \varphi_{j+2,0}, \]

\[ \pi_2(\varphi_{j0}) = \frac{b_{j-1}}{a_{j-1} \varphi_{-j0}}. \]

They also act on the parameters \( c \) as

\[ p_i(a_j) = a_{(i,j+1)(j)}, \quad p_i(b_j) = b_j, \quad p_i(c) = c, \]

for \( i = 0, \ldots, n \),

\[ p'_i(a_j) = a_{j}, \quad p'_i(b_j) = \frac{b_{i(i)}(j)}{b_1}, \quad p'_i(c) = c, \]

for \( i = 1, \ldots, n-1 \) and

\[ \sigma(a_j) = a_{j-1}, \quad \sigma(b_j) = b_j, \quad \sigma(c) = c, \]

\[ \sigma'(a_j) = a_{j+1}, \quad \sigma'(b_j) = b_j, \quad \sigma'(c) = q^{-1}c, \]

\[ \pi_1(a_j) = \frac{q a_{j-1}}{b_n}, \quad \pi_1(b_j) = \frac{q b_{j-1}}{b_n}, \quad \pi_1(c) = c, \]

\[ \pi_2(a_j) = \frac{q a_{n-1}}{b_{2n-j}}, \quad \pi_2(b_j) = \frac{q a_{n-1}}{a_{2n-j}}, \quad \pi_2(c) = \frac{\prod_{i=1}^{n} b_i}{q c \prod_{i=1}^{n} a_i}, \]

where \((i_1, i_2)\) is a transposition.

**Proposition 3.2.** If we set

\[ F = (p_0, \ldots, p_n), \quad F' = (p'_0, \ldots, p'_n), \]

then both \( F \) and \( F' \) are isomorphic to the affine Weyl group of type \( A_n^{(1)} \) and \( F F' = F' F \). Moreover, the transformations \( \sigma, \sigma', \pi_1, \pi_2 \) satisfy relations

\[ p_i \sigma = \sigma p_{i+1}, \quad p'_i \sigma = \sigma p'_i, \quad p_i \sigma' = \sigma' p_{i-1}, \quad p'_i \sigma' = \sigma' p'_i, \]

\[ p_i \pi_1 = \pi_1 p_{i+1}, \quad p'_i \pi_1 = \pi_1 p'_{i+1}, \quad p_i \pi_2 = \pi_2 p_{i-3}, \quad p'_i \pi_2 = \pi_2 p'_{i-3}, \]

\[ \sigma \sigma' = \sigma' \sigma, \quad \sigma \pi_1 = \pi_1 \sigma, \quad \sigma' \pi_1 = \pi_1 \sigma', \quad \sigma \pi_2 = \pi_2 (\sigma')^{-1}, \]

\[ \pi'_1 = \pi_2^2 = 1, \quad \pi_1 \pi_2 = \pi_2 \pi_1^{-1}. \]  \hfill (3.2)

**Proof.** Thanks to Theorem 3.1 we can prove this proposition by using only the fundamental relations for \( \tilde{G} \). For example, the relation \( p_i \sigma = \sigma p_{i+1} \) of (3.2) is shown as

\[ p_i \sigma = r_{-2i+3} r_{-2i-2} r_{-2i-3} \pi s_0 r_1 r_3 \ldots r_{2n+1} \]

\[ = \pi s_0 r_{-2i-4} r_{-2i-3} r_{-2i-4} r_1 r_3 \ldots r_{2n+1} \]

\[ = \pi s_0 r_1 \ldots r_{-2i-7} r_{-2i-1} \ldots r_{2n+1} r_{-2i-4} r_{-2i-3} r_{-2i-4} r_{-2i-3} r_{-2i-5} \]

\[ = \pi s_0 r_1 \ldots r_{-2i-7} r_{-2i-1} \ldots r_{2n+1} r_{-2i-3} r_{-2i-5} r_{-2i-4} r_{-2i-5} r_{-2i-4} \]
We can show the other relations of (3.2) and the fundamental relations for $F$ and $F'$ in a similar way. We don’t state its detail here. □

3.2. Proof of Theorem 3.1. We first prove for the transformation $\sigma = \pi s_0 r_1 r_3 \ldots r_{2n+1}$. The action of the composition of the transformations $r_1 r_3 \ldots r_{2n+1}$ on the dependent variables $\varphi_{2j+1,1}$ ($j = 0, \ldots, n$) is described as

$$ r_1 r_3 \ldots r_{2n+1} (\varphi_{2j+1,1}) = \frac{1}{\varphi_{2j+1,0}}. $$

Thus it is enough to show that

$$ \pi s_0 (\varphi_{2j+1,0}) \bigg|_{\text{Eq. (2.1)}} = -1 \quad (j = 0, \ldots, n). \quad (3.3) $$

Note that

$$ \varphi_{2j+1,0} = \frac{\alpha_{2j+1}}{\varphi_{2j+1,1}} \quad (j = 0, \ldots, n). $$

From the definitions of $s_0$ and $\pi$, we obtain

$$ \pi s_0 (\varphi_{2j+1,0}) = \frac{1}{\varphi_{2j+3,1}} \sum_{k=0}^{2n+1} \prod_{l=0}^{k-1} \varphi_{2j+2+l,1}. $$

On the other hand, we have

$$ \sum_{k=0}^{2n+1} \prod_{l=0}^{k-1} \varphi_{2j+l,1} \bigg|_{\text{Eq. (2.1)}} = 1 + (-1)^n \prod_{l=0}^{n} \varphi_{2l,0}, $$

for any $j = 0, \ldots, n$. Combining them, we obtain equation (3.3).

We next prove for the transformation $\sigma'$. The fundamental relations for $\widetilde{G}$ imply

$$ \sigma' = r_1 r_3 \ldots r_{2n+1} (\pi')^{-1}s_0'. $$

The action of $(\pi')^{-1}s_0'$ on the dependent variables is described as

$$ (\pi')^{-1}s_0' (\varphi_{2j+1,1}) = \frac{1}{\varphi_{2j+1,0}} \sum_{k=0}^{2n+1} \prod_{l=0}^{k-1} \varphi_{2j-l,1}. $$

Thus it is enough to show that

$$ r_1 r_3 \ldots r_{2n+1} \left( \frac{1}{\varphi_{2j+1,0}} \sum_{k=0}^{2n+1} \prod_{l=0}^{k-1} \varphi_{2j-l,1} \right) \bigg|_{\text{Eq. (2.1)}} = -1 \quad (j = 0, \ldots, n). \quad (3.4) $$

We focus on a factor

$$ \sum_{k=0}^{2n+1} \prod_{l=0}^{k-1} \varphi_{2j-l,1} + \frac{\beta_1'}{\varphi_{2j+1,0}} + \sum_{k=1}^{n} (1 + \varphi_{2j-2k+1,0}) \prod_{l=0}^{k-2} \varphi_{2j-2l-1,0} \prod_{l=0}^{k-1} \varphi_{2j-2l,1}. $$
By a direct calculation, we obtain
\[
\prod_{i=1}^{n}(1 + \varphi_{2j-2k+1,0}) = \frac{1 + \varphi_{2j-2k+1,0}}{\varphi_{2j-2k+1,0}},
\]
\[
\prod_{i=1}^{n}(1 + \varphi_{2j-2k+1,0}) = \frac{1}{\varphi_{2j-2k+1,0}},
\]
\[
\prod_{i=1}^{n}(1 + \varphi_{2j-2k+1,0}) = \frac{(1 + \varphi_{2j-2k+1,0})(1 + \varphi_{2j-2k+1,0})}{(1 + \varphi_{2j-2k+1,0})} \prod_{i=0}^{k-1} \varphi_{2j-2k+1,0} \varphi_{2j-2k+1,0} \prod_{i=0}^{k-1} \varphi_{2j-2k+1,0}.
\]
It follows that
\[
\prod_{i=1}^{n}(1 + \varphi_{2j-2k+1,0}) = \frac{(1 + \varphi_{2j-2k+1,0})(1 + \varphi_{2j-2k+1,0})}{1 + \varphi_{2j-2k+1,0}} \prod_{i=0}^{k-1} \varphi_{2j-2k+1,0} \varphi_{2j-2k+1,0} \prod_{i=0}^{k-1} \varphi_{2j-2k+1,0} \quad \text{Eq. (2.1)}
\]
\[
= 0.
\]
Moreover, we have
\[
\prod_{i=1}^{n}(1 + \varphi_{2j-2k+1,0}) = \frac{(1 + \varphi_{2j-2k+1,0})(1 + \varphi_{2j-2k+1,0})}{1 + \varphi_{2j-2k+1,0}} \prod_{i=0}^{k-1} \varphi_{2j-2k+1,0} \varphi_{2j-2k+1,0} \prod_{i=0}^{k-1} \varphi_{2j-2k+1,0} \quad \text{Eq. (2.1)}
\]
\[
= 1 - \beta_1.
\]
and
\[
\prod_{i=1}^{n}(1 + \varphi_{2j-2k+1,0}) = \frac{(1 + \varphi_{2j-2k+1,0})(1 + \varphi_{2j-2k+1,0})}{1 + \varphi_{2j-2k+1,0}} \prod_{i=0}^{k-1} \varphi_{2j-2k+1,0} \varphi_{2j-2k+1,0} \prod_{i=0}^{k-1} \varphi_{2j-2k+1,0} \quad \text{Eq. (2.1)}
\]
\[
= -1.
\]
Combining them, we obtain equation (4.1).

We can prove for the other transformations in a similar way. We don’t state its detail here.

### 4. An affine Weyl group action on the basic hypergeometric function

#### 4.1. Results of this section

As is seen in the previous section, the translation \(\tau_c\), or equivalently system (2.2), is an element of the group \(\overline{F}\). Moreover, this group provides a symmetry for system (2.2).

**Proposition 4.1.** Any element of the group \(\langle F, F' \rangle \rtimes (\sigma, \sigma', \pi)\) is commutative with the translation \(\tau_c\). Besides, the transformation \(\pi_2\) satisfies a relation \(\pi_2 \tau_c = \tau_c^{-1} \pi_2\).

**Proof.** The relation \(\pi_2 \tau_c = \tau_c^{-1} \pi_2\) follows from equation (3.2) as
\[
\pi_2 \tau_c = \pi_2 \sigma' \sigma = \sigma^{-1} \pi_2 \sigma = \sigma^{-1} (\sigma')^{-1} \pi_2 = \tau_c^{-1} \pi_2.
\]
Note that the relation \(\sigma \pi_2 = \pi_2 (\sigma')^{-1}\) implies that \(\pi_2 \sigma' = \sigma^{-1} \pi_2\). We can show the other commutation relations in a similar way. We don’t state its detail here. \(\square\)

This proposition means that if \(\varphi_{2j,0} (j \in \mathbb{Z}_{n+1})\) satisfy system (2.2) with the parameters \(c\), then \(\omega(\varphi_{2j,0})\) satisfy system (2.2) with the transformed parameters \(\omega(c)\) for any \(\omega \in \overline{F}\). On the other hand, as is seen in Fact 2.5, the vector of functions \(x\) provides a solution of system (2.2). Hence we expect that
\[
- \frac{x_j + 2(\omega(c))}{x_{j+1}(\omega(c))} = \omega(\varphi_{2j,0}) \quad \text{Eq. (2.5)} \quad (j = 0, \ldots, n - 1),
\]
(4.1)
for some $\omega \in \tilde{F}$. Then the right-hand side of (4.1) is a ratio of two linear combinations of the functions $x_1, \ldots, x_{n+1}$. Based on this conjecture, we define $(n + 1) \times (n + 1)$ matrices corresponding to the generators of $\tilde{F}$ by

$$M_{p_0} = \sum_{j=1}^{n} \frac{a_0 - b_0}{a_1 - b_0} E_{j,j} + E_{n+1,n+1} + \frac{a_0 - a_1}{a_1 - b_0} q c E_{n+1,1},$$

for $i = 1, \ldots, n$,

$$M_{p_i} = \sum_{j=1}^{n-i} E_{j,j} + \frac{a_{i+1} - b_i}{a_i - b_i} E_{n-i+1,n-i+1} + \sum_{j=n-i+2}^{n+1} E_{j,j} + \frac{a_i - a_{i+1}}{a_i - b_i} E_{n-i+1,n-i+2},$$

for $i = 1, \ldots, n-1$ and

$$M_{q_i} = \frac{1}{1 - c} \left( \sum_{j=1}^{n+1} \frac{a_{n-j+1} (a_0 - b_0)}{a_{n-j+1} - b_{n-j+1}} E_{j,j} - \sum_{j=1}^{n} \frac{a_{n-j+1} (a_0 - b_0)}{a_{n-j+1} - b_{n-j+1}} E_{n+1,j+1} - a_{n+1} c E_{n+1,1},

M_{q'} = \frac{1}{1 - c} \left( \sum_{j=1}^{n+1} \sum_{j_2=j}^{n+1} \frac{a_{n-j_2+1} - b_{n-j_2+1}}{a_1 - b_0} E_{j,j_2} + \sum_{j_1=1}^{n+1} \sum_{j_2=j_1}^{n+1} \frac{a_{n-j_2+1} - b_{n-j_2+1}}{a_1 - b_0} c E_{j_1,j_2},

M_{q_1} = c^{\log_q b_1} \left( \sum_{j=1}^{n} E_{j,j+1} + q c E_{n+1,1} \right),

M_{q_2} = c^{\log_q b_{n-1}} \left( E_{1,2} + \frac{b_n}{a_n} E_{2,1} + \sum_{j=3}^{n+1} \frac{b_n}{a_n a_{n+1}} \sum_{l=1}^{j-3} \frac{b_l}{a_l} c^{-1} E_{j,n+4-j} \right).$$

**Theorem 4.2.** For any $\omega \in \langle \sigma, \sigma' \rangle \times \langle \sigma, \sigma' \rangle$, the vector $x$ satisfies

$$x(\omega(c)) = M_{\omega} x(c). \quad (4.2)$$

Here a composition of transformations is lifted to a product of matrices as

$$M_{\omega_2 \omega_1} = \omega_2(M_{\omega_1}) M_{\omega_2}.$$

Besides, for any $\omega = \omega'_0, \omega'_n, \pi_1, \pi_2$, the vector $\omega^{-1}(M_{\omega}) x(\omega^{-1}(c))$ is a solution of system (2.4) and two vectors $x(c)$ and $\omega^{-1}(M_{\omega}) x(\omega^{-1}(c))$ are linearly independent.

We will prove this theorem in Section 4.2. In addition, we give a lemma which will be used to prove Theorem 5.1.
4.2. Proof of Theorem 4.2. Besides, the vector \( x = (x_1, \ldots, x_n) \) is a solution of system (4.2) at \( c = 0 \). Note that this lemma can be proved in a similar way as Theorem 4.2. Although the vectors \( x \) are not convenient. We use this notation for a sake of convenience. Note that

\[
(1 - c) M_{\sigma'} = M_{\sigma',0} + c M_{\sigma',1}, \quad x = \sum_{k=0}^{\infty} x_k c^k.
\]

By using those notations, we can rewrite our goal \( x(\sigma'(c)) = M_{\sigma'} x(c) \) into

\[
x_0(\sigma'(c)) = M_{\sigma',0} x_0(\sigma'(c)),
\]

\[
q^{-k} x_k(\sigma'(c)) = q^{-k+1} x_{k-1}(\sigma'(c)) + M_{\sigma',0} x_k(c) + M_{\sigma',1} x_{k-1}(c) \quad (k \geq 1).
\]

Equation (4.3) is shown as follows. The \( j \)-th row of the vector \( M_{\sigma',0} x_0(\sigma'(c)) \) is rewritten as

\[
\sum_{j' = j}^{n+1} \frac{a_{n-j' + 2} - b_{n-j' + 1}}{a_1 - b_0} x_{0,j'}(c)
\]

\[
= \sum_{j' = j}^{n+1} \frac{a_{n-j' + 2} - b_{n-j' + 1}}{a_1 - b_0} \prod_{l=1}^{n-j' + 2} (1 - a_l) \prod_{l=n-j' + 2}^{n} (1 - b_l)
\]

\[
= \sum_{j' = j}^{n} \prod_{l=1}^{n-j' + 1} (1 - a_{l+1}) \prod_{l=n-j' + 2}^{n} (1 - b_l) - \sum_{j' = j}^{n} \prod_{l=1}^{n-j' + 1} (1 - a_{l+1}) \prod_{l=n-j' + 2}^{n} (1 - b_l) + \prod_{l=1}^{n} (1 - b_l)
\]

\[
= \prod_{l=1}^{n-j+1} (1 - a_{l+1}) \prod_{l=n-j+2}^{n} (1 - b_l)
\]

\[
x_{0,j}(\sigma'(c)).
\]
Equation (4.4) is shown as follows. Its $j$-th row is described as
\[
q^{-k}x_{k,j}(\sigma'(\mathbf{c})) = q^{-k}x_{k-1,j}(\sigma'(\mathbf{c})) + \sum_{j'=1}^{j-1} \frac{a_{n-j'2} - b_{n-j'+1}}{a_1 - b_0} x_{k-1,j'}(\mathbf{c}) + \sum_{j'=j}^{n+1} \frac{a_{n-j'+2} - b_{n-j'+1}}{a_1 - b_0} x_{k,j'}(\mathbf{c}).
\]

Then we have
\[
q^{-k}x_{k-1,j'+1}(\sigma'(\mathbf{c})) + \frac{a_{n-j'+2} - b_{n-j'+1}}{a_1 - b_0} x_{k-1,j'}(\mathbf{c}) = q^{-k}x_{k-1,j'}(\sigma'(\mathbf{c})) \quad (j' = 1, \ldots, j - 1).
\]

It follows that
\[
q^{-k}x_{k-1,j}(\sigma'(\mathbf{c})) + \frac{a_{n-j'+2} - b_{n-j'+1}}{a_1 - b_0} x_{k-1,j'}(\mathbf{c}) = q^{-k}x_{k-1,1}(\sigma'(\mathbf{c})).
\]

Moreover, we have
\[
q^{-k}x_{k-1,1}(\sigma'(\mathbf{c})) + x_{k,n+1}(\mathbf{c}) = q^{-k}x_{k,n+1}(\sigma'(\mathbf{c})),
\]
and
\[
q^{-k}x_{k,j'+1}(\sigma'(\mathbf{c})) + \frac{a_{n-j'+2} - b_{n-j'+1}}{a_1 - b_0} x_{k,j'}(\mathbf{c}) = q^{-k}x_{k,j'}(\sigma'(\mathbf{c})) \quad (j' = j, \ldots, n).
\]

Hence we obtain
\[
q^{-k}x_{k-1,1}(\sigma'(\mathbf{c})) + \sum_{j'=1}^{n+1} \frac{a_{n-j'+2} - b_{n-j'+1}}{a_1 - b_0} x_{k,j'}(\mathbf{c}) = q^{-k}x_{k,j}(\sigma'(\mathbf{c})).
\]

For the other transformations $p_0, \ldots, p_n, p_1', \ldots, p_{n-1}', \sigma$, we can show in a similar way. We don’t state its detail here.

We next prove the latter half of the theorem for the transformation $\pi_2$. We can rewrite our goal
\[
\tau_c \pi_2^{-1}(M_{\pi_2}) x(\tau_c \pi_2^{-1} (\mathbf{c})) = M_{\tau_c} \pi_2^{-1}(M_{\pi_2}) x(\pi_2^{-1}(\mathbf{c})),
\]
into
\[
x(\tau_c^{-1}(\mathbf{c})) = \tau_c^{-1}(M_{\pi_2}^{-1}) \pi_2(M_{\tau_c}) M_{\pi_2} x(\mathbf{c}),
\]
by using the fundamental relation $\pi_2^{-1} = \pi_2$ and Proposition 4.1. Thus it is enough to show that
\[
\tau_c^{-1}(M_{\pi_2}^{-1}) = \tau_c^{-1}(M_{\pi_2}^{-1}) \pi_2(M_{\tau_c}) M_{\pi_2}.
\]

Its right-hand side is expressed as
\[
(1 - \pi_2(c)) \tau_c^{-1}(M_{\pi_2}^{-1}) \pi_2(M_{\tau_c}) M_{\pi_2} = \sum_{j=1}^{n+1} \left( \frac{1}{a_{n-j+2} - b_{n-j+1}} \right) E_{j,j} + \sum_{j_1=1}^{n+1} \sum_{j_2=1}^{n+1} \left( \frac{1}{a_{n-j_2+2} - b_{n-j_2+1}} \right) \prod_{l=n-j_1+2}^{n-j_2+1} b_l E_{j_1,j_2} + \sum_{j_2=1}^{n+1} \sum_{j_1=1}^{n+1} \left( \frac{1}{a_{n-j_2+2} - b_{n-j_2+1}} \right) \prod_{l=n-j_2+2}^{n-j_1+2} a_l \pi_2(c) E_{j_1,j_2}.
\]

By using this expression, we can show
\[
\tau_c^{-1}(M_{\tau_c}) \tau_c^{-1}(M_{\pi_2}^{-1}) \pi_2(M_{\tau_c}) M_{\pi_2} = I.
\]
For the other transformations \( \omega = p'_0, p'_a, \pi_1 \), our goal is rewritten into
\[
\omega(M_{\tau_c}) M_\omega = \tau_c(M_\omega) M_{\tau_c}.
\]

It can be shown by a direct calculation. We don’t state its detail here.

5. \textbf{q-Hypergeometric equations as translations}

5.1. \textbf{Results of this section.} The group \( \overline{F} \) contains three types of translations
\[
\tau_c = \sigma' \sigma, \\
\tau_i = p_i \ldots p_{i+n-1} \sigma \quad (i = 1, \ldots, n+1), \\
\tau_{i,j} = p_i \ldots p_{i+n-1} p'_j \ldots p'_{j-1} \pi_1 \pi'_1 \ldots \pi'_n \quad (i, j = 1, \ldots, n+1).
\]

They act on the parameters \( c \) as
\[
\tau_c(a_k) = a_k, \quad \tau_c(b_k) = b_k, \quad \tau_c(c) = q^{-1} c, \\
\tau_i(a_k) = q^{\delta_{ik}} a_k, \quad \tau_i(b_k) = b_k, \quad \tau_i(c) = c, \\
\tau_{i,j}(a_k) = q^{\delta_{ik}-\delta_{jk}} a_k, \quad \tau_{i,j}(b_k) = q^{\delta_{ik}-\delta_{jk}} b_k, \quad \tau_{i,j}(c) = c,
\]
where \( \delta_{ik} \) is the Kronecker’s delta. Among them, the translation \( \tau_c \) was investigated in the previous sections. As a matter of fact, the corresponding matrix is described as
\[
M_{\tau_c} = \sigma'(M_\sigma) M_{\sigma'},
\]
and Fact 2.4 follows from Theorem 4.2. In this section we investigate the other translations and formulate their corresponding matrices.

Let \( M_{\tau_i} \) \((i = 1, \ldots, n+1)\) and \( M_{\tau_{i,j}} \) \((i, j = 1, \ldots, n+1)\) be \((n+1) \times (n+1)\) matrices defined by
\[
M_{\tau_i} = p_i \ldots p_{i+n-1}(M_\sigma) \times p_i \ldots p_{i+n-2}(M_{p_{i+n-1}}) \times \ldots \times p_i(M_{p_{i+1}}) \times M_{p_i},
\]
and
\[
\Delta_{i,j} M_{\tau_{i,j}} = p_i \ldots p_{i+n-1} p'_j \ldots p'_{j-2}(M_\sigma) \times \ldots \times p_i \ldots p_{i+n-1} p'_j \ldots p'_n \pi_1(M_{p'_j}) \times \ldots \times p_i \ldots p_{i+n-2}(M_{p_{i+n-1}}) \times \ldots \times p_i(M_{p_{i+1}}) \times M_{p_i},
\]
where
\[
\Delta_{i,j} = \frac{1 - b_j}{1 - a_i} \quad (i \neq j \neq n+1), \quad \Delta_{1,j} = \frac{1 - b_j}{1 - q^{-1} a_1} \quad (j \neq n+1),
\]
\[
\Delta_{i,n+1} = q \frac{1 - a_i}{1 - q a_i} \prod_{l=1}^{n} \frac{1 - a_{l+1}}{1 - q a_l} \quad (i \neq 1), \quad \Delta_{1,n+1} = q \prod_{l=1}^{n} \frac{1 - a_{l+1}}{1 - q a_l}.
\]

\textbf{Theorem 5.1.} The vector \( \mathbf{x} \) satisfies systems of \( q \)-difference equations
\[
\mathbf{x}(\tau_i(\mathbf{c})) = M_{\tau_i} \mathbf{x}(\mathbf{c}) \quad (i = 1, \ldots, n+1),
\]
\[
\mathbf{x}(\tau_{i,j}(\mathbf{c})) = M_{\tau_{i,j}} \mathbf{x}(\mathbf{c}) \quad (i, j = 1, \ldots, n+1).
\]

We will prove this theorem in Section 5.2. System 5.1 can be regarded as ladder operators for the basic hypergeometric functions. Besides, a solution of system (5.2) is described in terms of the \( q \)-Lauricella function \( \phi_B^{(n)} \).
Corollary 5.2. Let $y = y(c)$ be a vector of functions defined by

$$y = \begin{bmatrix} y_1 \\ \vdots \\ y_{n+1} \end{bmatrix}, \quad y_j = \sum_{k_1, \ldots, k_n \geq 0} \frac{(q c; q)_{k_1+\ldots+k_n}}{(q a_{n+1} c; q)_{k_1+\ldots+k_n}} \prod_{l=1}^{n} \frac{(b_j; q)_{k_l}}{(q; q)_{k_l}} \prod_{l=n-j+2}^{l=n} q^{k_l} a_i^{k_l} \prod_{l=n-j+2}^{l=n} a_i^{k_l}.$$  

Then the vector $y$ satisfies a system of $q$-difference equations

$$y(\tau_{i,j}(c)) = \frac{1 - a_i}{1 - q b_j} M_{\tau_{i,j}} y(c) \quad (i, j = 1, \ldots, n + 1).$$

We can show this corollary by a direct calculation or formally by using a relation between two vectors

$$y = \frac{(q c, q b_1, \ldots, q b_n; q)_{\infty}}{(q a_{n+1} c, a_1, \ldots, a_n; q)_{\infty}} x,$$

which follows from the transformation formula between $n+1\phi_n$ and $\phi_D^{(n)}$ given in [1].

Remark 5.3. If we regard the translations $\tau_{i,j}$ $(i, j = 1, \ldots, n + 1)$ as elements of $\bar{G}$, then they coincide with the original $q$-Garnier system given in [13]. On the other hand, the original $q$-Garnier system admits a particular solution in terms of the $q$-Lauricella function; see [7, 14]. These facts are consistent with Corollary 5.2.

5.2. Proof of Theorem 5.1. Since system (5.1) follows from Theorem 4.2, we prove system (5.2) here. For a sake of convenience, we set

$$\bar{M}_{\rho_0} = c^{-\log_q b_0} M_{\rho_0}, \quad \bar{M}_{\mu_i} = M_{\mu_i} \quad (i = 1, \ldots, n - 1), \quad \bar{M}_{\rho_n} = c^{-\log_q q^{-1} b_n} M_{\rho_n},$$

$$\bar{M}_{\pi_1} = c^{-\log_q q^{-1} b_1} M_{\pi_1}.$$ 

Note that $\bar{M}_{\pi_1}$ is invariant under the actions of $p_0', \ldots, p_n', \pi_1$. Then Lemma 4.3 implies

$$x(p_n' \pi_1(c)) = \frac{1 - a_0}{1 - b_n} \bar{M}_{\pi_1} \bar{M}_{\rho_0} x(c). \quad (5.3)$$

Moreover, we obtain

$$\bar{M}_{\rho_i}^{-1} = p_i'(\bar{M}_{\rho_i}) \quad (i = 0, \ldots, n), \quad (5.4)$$

$$\bar{M}_{\pi_1}^{n+1} = q c I, \quad (5.5)$$

$$\bar{M}_{\pi_1} M_{\rho_i} = \pi_1(\bar{M}_{\rho_i}) \bar{M}_{\pi_1} \quad (i = 0, \ldots, n), \quad (5.6)$$

where $I$ is the identity matrix, by a direct calculation.

For $j = 1, \ldots, n$, system (5.2) follows from Theorem 4.2 and equation (5.3). For $j = n + 1$, system (5.2) is derived as follows. The fundamental relations for $\bar{F}$ imply

$$(\pi_1^{-1} p_n')^{j+1} \pi_1^{l+1} = (\pi_1^{-1} p_n')^{j-1} \pi_1^{-1} p_n' \pi_1^{l+1} = (\pi_1^{-1} p_n')^{j-1} \pi_1^{l+1} p_n'.$$
Combining them and equation (5.5), we obtain

\[ x(p^{-1}_1 p'_{n}(c)) = \frac{1 - q b'_1}{1 - a'_1} \pi_1^{-1} (\tilde{M}_{p'_n}^{k+1} \cdot \tilde{M}_{p_n} \cdot \tilde{M}_{p_{n-1}} \cdot \cdots \cdot \tilde{M}_{p_1}) x(c). \]

for \( l = 1, \ldots, n \). Equations (5.3) and (5.4) imply

\[ x(p^{-1}_1 p'_{2} \cdots p'_{n}(c)) = x((\pi_1^{-1} p'_n)^n(c)) = \prod_{l=1}^{n} \frac{1 - q b'_1}{1 - a'_1} (\pi_1^{-1} p'_n)^{n-1} (\tilde{M}_{p'_n}^{k+1} \cdot \tilde{M}_{p_n} \cdot \tilde{M}_{p_{n-1}} \cdot \cdots \cdot \tilde{M}_{p_1}) x(c) \]

Equation (5.6) implies

\[ \tilde{M}_{\pi_1}^{k+1} (\pi_1^{-1} p'_n)^{k-1} (\tilde{M}_{p'_n})^{k-1} (\tilde{M}_{p_n})^{k-1} = (\pi_1^{-1} p'_n)^{k-1} (\tilde{M}_{p'_n})^{k-1} (\tilde{M}_{p_n})^{k-1} (\tilde{M}_{p_{n-1}}) \tilde{M}_{\pi_1}^{k-1} (k = 1, \ldots, n). \]

Combining them and equation (5.5), we obtain

\[ x(p_1 p'_2 \cdots p'_n(c)) = x((\pi_1^{-1} p'_n)^n(c)) = x(\pi_1 p'_2 \cdots p'_n(c)). \]

It follows that

\[ x(p_1 p'_2 \cdots p'_n(c)) = q^{-1} \prod_{l=1}^{n} \frac{1 - q b'_1}{1 - a'_1} \pi_1 p'_2 \cdots p'_{n-1}(M_{p'_n}) \cdots \pi_1 p'_1(M_{p'_2}) \pi_1(M_{p'_1}) \tilde{M}_{\pi_1} x(c). \]

Then we obtain system (5.2) from equation (5.7) and Theorem 4.2.

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