Soliton resolution for the focusing Fokas-Lenells equation with weighted Sobolev initial data

Qiaoyuan Cheng, Engui Fan

Abstract

In this paper, we consider the Cauchy problem for the focusing Fokas-Lenells equation

$$u_{tx} + \alpha \beta^2 u - 2i\alpha \beta u_x - \alpha u_{xx} - i\alpha \beta^2 |u|^2 u_x = 0,$$

$$u(x, 0) = u_0(x) \in H^{1,1}(\mathbb{R}),$$

where $H^{1,1}(\mathbb{R})$ is a weighted Sobolev space. Using nonlinear steepest descent method and combining the $\mathcal{F}$-analysis, we show that inside any fixed cone

$$C(x_1, x_2, v_1, v_2) = \{(x, t) \in \mathbb{R}^2 | x = x_0 + vt, x_0 \in [x_1, x_2], v \in [v_1, v_2]\},$$

the long time asymptotic behavior of the solution $u(x, t)$ for the focusing Fokas-Lenells equation can be characterized with an $N(I)$-soliton on discrete spectrum and leading order term $O(|t|^{-1/2})$ on continuous spectrum up to an residual error order $O(|t|^{-3/4}).$

Keywords: focusing Fokas-Lenells equation; Riemann-Hilbert problem, $\mathcal{F}$ steepest descent method, soliton resolution, long time asymptotics.

---

1 School of Mathematical Sciences and Key Laboratory of Mathematics for Nonlinear Science, Fudan University, Shanghai 200433, P.R. China.

* Corresponding author and email address: faneg@fudan.edu.cn
1 Introduction

It was noted that the Camassa-Holm equation can be mathematically derived by utilizing the two Hamiltonian operators associated with the KdV equation \[1\]. Similarly, by utilizing the two Hamiltonian operators associated with the NLS equation, it is possible to derive the following Fokas-Lenells (FL) equation \[1\]

\[ u_{tx} + \alpha \beta^2 u - 2i\alpha \beta u_x - \alpha u_{xx} + \sigma i \alpha \beta^2 |u|^2 u_x = 0. \] \quad (1.1)

In optics, considering suitable higher-order linear and nonlinear optical effects, the FL equation has been derived as a model to describe the femtosecond pulse propagation through
single mode optical silica fiber and several interesting solutions have been constructed. It also belongs to the deformed derivative NLS hierarchy proposed by Kundu.

The various kinds of exact solutions for the FL equation have been constructed by inverse scattering transform method, dressing method and Hirota method. The lattice representation and the n-dark solitons of the FL equation have been presented. Matsuno considered a sophisticated problem on the dark soliton solutions with a plane wave boundary condition using Hirota method. The breather solutions of the FL equation have also been constructed via a dressing-Bäcklund transformation related to the Riemann-Hilbert (RH) problem formulation of the inverse scattering theory. It has been shown that the periodic initial value problem for the FL equation is well-posed in a Sobolev space $H^s(\mathbb{T})$ for $s > 3/2$.

Recently, McLaughlin and Miller have developed a method for the asymptotic analysis of RH problems based on the analysis of $\bar{\partial}$-problems, rather than the asymptotic analysis of singular integrals on contours. Dieng and McLaughlin obtained sharp asymptotics of solutions of the defocusing nonlinear Schrödinger (NLS) equation, based on $\bar{\partial}$ method and under essentially minimal regularity assumptions on initial data. Cussagna and Jenkins study the defocusing NLS equation with finite density initial data. Borghese, Jenkins and McLaughlin have studied the Cauchy problem for the focusing nonlinear Schrödinger equation using the $\bar{\partial}$ generalization of the nonlinear steepest descent method, which has been conjectured for a long time. Jenkins and Liu study the derivative nonlinear Schrödinger equation for generic initial data in a weighted Sobolev space.

It is noted that for defocusing case $\sigma = 1$, the FL equation with zero boundary conditions does not admit soliton solution. For the Schwartz initial value $u(x, t = 0) \in \mathcal{S}(\mathbb{R})$, Xu obtained the long time asymptotics for the FL equation without solitons by using Deift-Zhou method. However, for focusing case $\sigma = -1$, the FL equation takes

$$u_{tx} + \alpha \beta^2 u - 2i \alpha \beta u_x - \alpha u_{xx} - i \alpha \beta^2 |u|^2 u_x = 0,$$  

(1.2)

whose soliton solution will appear for zero boundary conditions. It is a natural question that how to get the asymptotic behavior of the FL equation in the domain of soliton solutions. In this work we apply the $\bar{\partial}$-techniques to obtain the long time asymptotic behavior of solutions to FL equation with weighted Sobolev date $u(x, t = 0) \in H^{1,1}(\mathbb{R})$. The long-time behavior of solutions of FL equation is necessarily more complicated than in the defocusing case due to the presence of solitons which correspond to discrete spectrum.
The structure of the paper is as follows: in section 2, we introduce two kinds of eigenfunctions to formulate the spectral singularity of the Lax pair for the FL equation. The analytical and asymptotics of the eigenfunctions are further studied. In section 3, following the idea in [19], we construct a RH problem for $M(k)$ to formulate the initial value problem of the FL equation in an original space variable $x$. In section 4, we introduce a function $T(k)$ to define a new RH problem for $M^{(1)}(k)$, which admits a regular discrete spectrum and two triangular decompositions of the jump matrix near critical point $\pm k_0$ and $\pm ik_0$. In section 5, by introducing a matrix-valued function $R(k)$, we obtain a mixed $\overline{\partial}$-RH problem for $M^{(2)}(k)$ by continuous extension to $M^{(1)}(k)$. In section 6, we decompose $M^{(2)}(k)$ into a model RH problem for $M^{RHP}(k)$ and a pure $\overline{\partial}$ Problem for $M^{(3)}(k)$. The $M^{RHP}(k)$ can be obtained via an outer model $M^{(out)}(k)$ for the soliton components to be solved in Section 7, and an inner model $M^{(\pm k_0)}$ and $M^{(\pm ik_0)}$ for the stationary phase point $\pm k_0$ and $\pm ik_0$ which are approximated by a solvable model for $M^{FL}(k)$ obtained in [19] in Section 8. In section 9, we compute the error function $E(k)$ with a small-norm RH problem. In Section 10, we analyze the $\overline{\partial}$-problem for $M^{(3)}(k)$. Finally, in Section 11, based on the result obtained above, a relation formula is found

$$M(k) = M^{(3)}(k)E(k)M^{(out)}(k)T(k)^{-\sigma_3},$$

(1.3)

from which we can obtain the soliton resolution and long-time asymptotic behavior for the FL equation (1.2).

2 The spectral analysis

The FL equation (1.2) admits Lax pair

$$\begin{cases}
\psi_x + ik^2 \sigma_3 \psi = kU_x \psi, \\
\psi_t + i\eta^2 \sigma_3 \psi = V \psi,
\end{cases}$$

(2.1)

where

$$U = \begin{pmatrix} 0 & u \\ -\bar{u} & 0 \end{pmatrix}, \quad \eta = \sqrt{\alpha(k - \frac{\beta}{2k})}, \quad V = \alpha k U_x + \frac{i\alpha\beta^2}{2} \sigma_3 \left( \frac{1}{k} U - U^2 \right).$$

Making a transformation

$$\psi = \Phi e^{-i(k^2 x + \eta^2 t)\sigma_3},$$

(2.2)

then $\Phi$ solves

$$\begin{cases}
\Phi_x + ik^2 [\sigma_3, \Phi] = kU_x \Phi, \\
\Phi_t + i\eta^2 [\sigma_3, \Phi] = V \Phi,
\end{cases}$$

(2.3)
which can be written in full derivative form
\[
d(e^{i(k^2x+\eta^2t)\sigma_3}\Phi(x,t,k)) = e^{i(k^2x+\eta^2t)\sigma_3}(kU_xdx + Vdt)\Phi,
\] (2.4)

The Lax pair (2.3) has singularities at \(k = 0\) and \(k = \infty\), which is different from NLS and derivative NLS equations. In order to control the behavior of solutions of (2.3) and construct the solution \(u(x,t)\) of the FL equation (1.2), we need use the \(t\)-part and the expansion of the eigenfunction as spectral parameter \(k \to 0\). So we use two different asymptotic expansions respectively to analyze these two singularities \(k = 0\) and \(k = \infty\).

**Case I:** Control the behavior as \(k \to 0\)

We seek a solution of this equation in the form
\[
\Phi(x,t,k) = D_0 + kD_1 + O(k^2), \quad k \to 0
\]
where \(D_i, i = 0, 1\) are independent of \(k\). Substituting the above expansion in the (2.3) and comparing the coefficients of the same order of \(k\), we can find
\[
D_0 = I, \quad D_1 = \begin{pmatrix} 0 & u \\ -\pi & 0 \end{pmatrix}.
\]

**Case II:** Control the behavior as \(k \to \infty\)

Consider a solution of equation (2.3) of the form
\[
\Phi = \Phi_0 + \frac{\Phi_1}{k} + \frac{\Phi_2}{k^2} + O\left(\frac{1}{k^3}\right), \quad k \to \infty,
\] (2.5)
where \(\Phi_0, \Phi_1, \Phi_2\) are independent of \(k\). Substituting the above expansion into the \(x\)-part and \(t\)-part, we find the following equation
\[
\Phi_0(x,t) = e^{-\frac{i}{2}\int_\omega |u(x',t)|^2dx'\sigma_3},
\] (2.6)
We introduce a new function \(\omega\) by
\[
\Phi(x,t,k) = \Phi_0(x,t)\omega(x,t,k),
\] (2.7)
then \(\omega\) admits the asymptotic
\[
\omega = I + O\left(\frac{1}{k}\right), \quad k \to \infty,
\] (2.8)
and satisfies the Lax pair
\[
\begin{align*}
\omega_x + ik^2 [\sigma_3, \omega] &= V_1\mu, \\
\omega_t + i\eta^2 [\sigma_3, \omega] &= V_2\mu,
\end{align*}
\] (2.9)
which can be written in full derivative form

\[ d \left( e^{i(k^2x+\eta^2t)}\hat{\sigma}_3\omega(x,t,k) \right) = W(x,t,k), \]

where

\[ W(x,t,k) = e^{i(k^2x+\eta^2t)}\hat{\sigma}_3 (V_1(x,t,k)dx + V_2(x,t,k)dt)\omega, \]

and the matrices \( V_1(x,t,k) \) and \( V_2(x,t,k) \) are defined respectively, by

\[ V_1 = e^{\frac{i}{2} \int_{-\infty}^{x} |u_x(x',t)|^2 dx' \hat{\sigma}_3 (k U_x + \frac{i}{2} u_x u_x \hat{\sigma}_3)}, \]

\[ V_2 = e^{\frac{i}{2} \int_{-\infty}^{x} |u_x(x',t)|^2 dx' \hat{\sigma}_3 (\alpha k U_x + \frac{i}{2} \beta^2 \sigma_3 (\frac{1}{k} U - U^2) + \frac{i}{2} (\sigma_x u_x - \beta^2 |u|^2) \sigma_3)}. \]

We now consider the spectral analysis of the \( x\)-part of (2.9). Since the analysis will take place at a fixed time, the \( t\)-dependence will be suppressed. Define two solutions \( \omega_1 \) and \( \omega_2 \) of the \( x\)-part of (2.9) by

\[ \omega_{\pm}(x,k) = I + \int_{-\infty}^{x} e^{ik^2(x'-x)\hat{\sigma}_3} (V_1 \omega_{\pm})(x',k) dx', \]

Similarly, we denote \( \omega_{\pm} = ([\omega_{\pm}]_1, [\omega_{\pm}]_2) \), then we can show that \([\omega_{-}]_1 \) and \([\omega_{+}]_2 \) are analytical in \( D^- \), and \([\omega_{+}]_1 \) and \([\omega_{-}]_2 \) are analytical in \( D^+ \). And the \( \omega_{\pm} \) admit the asymptotics

\[ \omega_{\pm} = I + \frac{\omega_1}{k} + O(k^{-2}), \quad z \to \infty. \]

Again by using transformations (2.2) and (2.7), we have

\[ \omega_{-}(x,k) = \omega_{+}(x,k) e^{-i(k^2x+\eta^2t)\hat{\sigma}_3} S(k), \]

where

\[ S(k) = \begin{pmatrix} \overline{a(k)} & b(k) \\ -\overline{b(k)} & \overline{a(k)} \end{pmatrix}. \]

Evaluation at \( x \to -\infty \) gives

\[ S(k) = I - \int_{-\infty}^{\infty} e^{ik^2x\hat{\sigma}_3} (V_1 \omega_{-})(x,k)dx, \quad \text{Im} k^2 = 0. \]

(2.10) can be written as

\[ ([\omega_{-}]_1, [\omega_{-}]_2) = ([\omega_{+}]_1, [\omega_{+}]_2) e^{-i(k^2x+\eta^2t)\hat{\sigma}_3} \begin{pmatrix} \overline{a(k)} & b(k) \\ -\overline{b(k)} & \overline{a(k)} \end{pmatrix}, \]

which implies that

\[ a(k) = \text{det} ([\omega_{+}]_1, [\omega_{-}]_2). \]
Note that $\omega(x, k) = \omega_\pm(x, k)$ satisfies the symmetry relations
\[
\omega_{11}(x, k) = \omega_{22}(x, k), \quad \omega_{21}(x, k) = -\omega_{12}(x, k),
\]
as well as
\[
\omega_{11}(x, -k) = \omega_{11}(x, k), \quad \omega_{12}(x, -k) = -\omega_{12}(x, k),
\]
\[
\omega_{21}(x, -k) = -\omega_{21}(x, k), \quad \omega_{22}(x, -k) = \omega_{22}(x, k).
\tag{2.21}
\]
From the explicit expression (2.18) for $S(k)$ and the fact that the first column of $\omega_-$ is defined and analytic in $D^+$, we find that $a(k)$ has an analytic continuation to all of $D^+$. The symmetry (2.21) implies that $a(k)$ is an even function of $k$ whereas $b(k)$ is an odd function of $k$, namely
\[
a(-k) = a(k), \quad b(-k) = -b(k). \tag{2.22}
\]
We introduce the reflection coefficient
\[
r(k) = \frac{b(k)}{a(k)}. \tag{2.23}
\]
To deal with our following work, we assume our initial data satisfy this assumption.

Assumption 1. The initial data $u_0(x) \in H^{\frac{1}{1}}(\mathbb{R})$ and it generates generic scattering data which satisfy that
1. $a(k)$ has no zeros on $\mathbb{R} \cup i\mathbb{R}$,
2. $a(k)$ only has finite number of simple zeros,
3. $a(k)$ and $r(k)$ belong $H^{\frac{1}{1}}(\mathbb{R} \cup i\mathbb{R})$. 

Figure 1: Analytical domains $D^+, D^-$. 

7
Since \( a(k) \) is an even function, each zero \( k_j \) of \( a(k) \) is accompanied by another zero at \(-k_j\). We assume that \( a(k) \) has \( 2N \) simple zeros \( \{k_j\}_{j=1}^{2N} \subset D_+ \) such that \( \{k_j\}_{j=1}^{N} \) belong to the first quadrant and \( k_{j+N} = -k_j, j = 1, \ldots, N \) belong to the third quadrant. From (2.13) and (2.20), we obtain the asymptotic of \( a(k) \)

\[
a(k) = 1 + O(k^{-1}), \quad k \to \infty,
\]

Then we have \(|a(k)|^2 + |b(k)|^2 = 1\), which is equivalent to \( 1 + |r(k)|^2 = \frac{1}{|a(k)|^2} \). In the absence of spectral singularities (real zeros of \( a(k) \)), there also exist \( c \in (0,1) \) such that \( c < |a(k)| < 1/c \) for \( k \in \mathbb{R} \), which implies \( 1 + |r(k)| > c^2 > 0 \) for \( k \in \mathbb{R} \).

### 3 The construction of a RH problem

Suppose that \( K = \{\pm k_j, j = 1, \ldots, N\} \) and \( \overline{K} = \{\pm \overline{k_j}, j = 1, \ldots, N\} \) are simple zeros of \( a(k) \) and \( \overline{a(k)} \) respectively, we first calculate residue conditions. According to (2.19), we obtain the relationships

\[
[\omega_-]_1(x, k_j) = b(k_j)[\omega_+]_1(x, k_j)e^{-2it\theta(k_j)}, \\
[\omega_-]_2(x, k_j) = -b(k_j)[\omega_+]_2(x, \overline{k_j})e^{2it\theta(k_j)},
\]

where \( \theta(k) = k^2x + \eta^2(k) \) and \( \eta(k) = \sqrt{\alpha(k) - \beta^2} \). We denote norming constant \( c_j = c(k_j) = b(k_j)/a'(k_j) \) and the collection \( \sigma_d = \{k_j, c_j\}_{j=1}^{N} \) is called the scattering data.

We define a sectionally meromorphic matrix

\[
M(x, t, k) = \begin{cases} 
\left(\frac{\omega_-}{a(k)}, [\omega_+]_2\right), & k \in D^+, \\
\left(\omega_+ - \omega_-, \frac{\omega_-}{a(k)}\right), & k \in D^-,
\end{cases}
\]

which solves the following RHP.

**RHP1.** Find a matrix-valued function \( M(x, t, k) \) which satisfies:

(a) Analyticity: \( M(x, t, k) \) is meromorphic in \( \mathbb{C} \setminus \Sigma, \Sigma = \mathbb{R} \cup i\mathbb{R} \) and has single poles,

(b) Symmetry: \( M(k) = \sigma_2 M(k) \sigma_2 \), where

\[
\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},
\]

(c) Jump condition: \( M \) has continuous boundary values \( M_{\pm} \) on \( \mathbb{R} \) and

\[
M_+(x, t, k) = M_-(x, t, k)V(k), \quad k \in \Sigma,
\]
where
\[
V(k) = \left( 1 + \frac{|r(k)|^2}{r(k)e^{2i\theta(k)}} \frac{r(k)e^{-2i\theta(k)}}{1} \right).
\] (3.6)

(d) Asymptotic behaviors:
\[
M(x, t, k) = I + \mathcal{O}(k^{-1}), \quad k \to \infty,
\] (3.7)

(e) Residue conditions: M has simple poles at each point in K and $\overline{K}$ with:
\[
\text{Res}_{k = \pm k_j} M(k) = \lim_{k \to \pm k_j} M(k) \left( \begin{array}{cc} 0 & 0 \\ \pm \gamma_j e^{2i\theta(\pm k_j)} & 0 \end{array} \right),
\] (3.8)
\[
\text{Res}_{k = \pm \overline{k}_j} M(k) = \lim_{k \to \pm \overline{k}_j} M(k) \left( \begin{array}{cc} 0 & \mp \gamma_j e^{-2i\theta(\pm \overline{k}_j)} \\ 0 & 0 \end{array} \right).
\] (3.9)

The solution of Fokas-Lenells equation (1.2) can be expressed by
\[
u_x(x, t) = m(x, t)e^{-i \int_{-\infty}^{x} |m|^2(x', t) dx'},
\] (3.10)

where
\[
m(x, t) = 2i \lim_{k \to \infty} (kM(x, t, k))_{12}.
\] (3.11)

4 Conjugation

In the jump matrix (3.6), the oscillatory term is
\[
\theta(k) = k^2 \frac{x}{t} + \eta^2(k).
\] (4.1)

It will be found that the long-time asymptotic of RHP1 is affected by the growth and decay of the exponential function $e^{2it\theta}$ appearing in both the jump relation and the residue conditions. In this section, we introduce a new transform $M(k) \to M^{(1)}(k)$, from which we make that the $M^{(1)}(k)$ is well behaved as $|t| \to \infty$ along any characteristic line. We focus on the case $t > 0$, the case $t < 0$ can be dealt with the same way, which will be listed in the appendix.

Let $\alpha > 0$ and we set $k_0 = (\frac{\alpha \beta^2}{4i + 4\alpha})^{\frac{1}{2}}$, where $\pm k_0$ and $\pm ik_0$ are the four critical points of the phase function $\theta(k)$
\[
\theta(k) = \frac{\alpha \beta^2}{4} \left( \frac{k^2}{k_0^2} + \frac{1}{k^2} \right) - \alpha \beta,
\] (4.2)
\[
\text{Re}(2it\theta) = -2t \text{Im}\theta = -\frac{t}{2} \alpha \beta^2 \text{Im}k^2 \left( \frac{1}{k_0^2} - \frac{1}{|k|^2} \right),
\] (4.3)
The partition $\Delta^\pm_{k_0}$ for $k_0 \in \mathbb{R}$ is defined as follows:

\[
\Delta^+_0 = \{ j \in \{1, \ldots, N\} | k_j < k_0 \},
\]

\[
\Delta^-_0 = \{ j \in \{1, \ldots, N\} | k_j > k_0 \}.
\]

This partition splits the residue coefficients $c_j$ in two sets which is shown in Figure. 2.

The intervals on the real and imaginary axes are divided as follows

\[
I_+ = (-\infty, -k_0] \cup [k_0, +\infty), \quad I_- = [-k_0, k_0],
\]

\[
I'_+ = (-\infty, -ik_0] \cup [ik_0, +\infty), \quad I'_- = [-ik_0, ik_0].
\]

Figure 2: In the yellow region, $|e^{2it\theta}| \to \infty$ when $t \to \infty$ respectively. And in white region, $|e^{2it\theta}| \to 0$ when $t \to \infty$ respectively.

We define the following functions in two situations and notation which will used later

\[
T(k) = T(k, k_0) = \prod_{j \in \Delta^-_0} \frac{(k - k_j)(k + k_j)}{(k - k_j)(k + k_j)} \delta(k),
\]
where $\delta(k)$ in (4.6) has been calculated in [19]

$$
\delta(k) = \left(\frac{k - k_0}{k}\right)\left(\frac{k + k_0}{k}\right)^{i\bar{\nu}} e^{\chi_+(k)} e^{\chi_-(k)} \left(\frac{k}{k - ik_0}\right)\left(\frac{k}{k + ik_0}\right)^{i\bar{\nu}} e^{\tilde{\chi}_+(k)} e^{\tilde{\chi}_-(k)},
$$

(4.7)

$$
\nu = -\frac{1}{2\pi} \ln(1 - |r(k_0)|^2), \quad \bar{\nu} = -\frac{1}{2\pi} \ln(1 + |r(ik_0)|^2),
$$

(4.8)

$$
\chi_\pm = \frac{1}{2\pi i} \int_{\pm ik_0}^{\pm k_0} \ln\left(\frac{1 - |r(k')|^2}{|1 - |r(k_0)|^2|}ight) \frac{dk'}{k' - k},
$$

(4.9)

$$
\tilde{\chi}_\pm = \frac{1}{2\pi i} \int_{\pm ik_0}^{\pm k_0} \ln\left(\frac{1 - |r(k')|^2}{|1 + |r(ik_0)|^2|}ight) \frac{dk'}{k' - k}.
$$

(4.10)

In all of the above formulas, we choose the principal branch of power and logarithm functions.

**Proposition 1.** The function defined by (4.6) has following properties:

(a) $T$ is meromorphic in $\mathbb{C} \setminus \mathbb{I}_+ \cup \mathbb{I}'_+$, for each $j \in \Delta_{k_0}$, $T(k)$ has a simple pole at $k_j, -k_j, \bar{k}_j$ and $\bar{k}_j$,

(b) For $\mathbb{C} \setminus \mathbb{I}_+ \cup \mathbb{I}'_+$, $\overline{T(k)} = 1/T(k)$,

(c) For $k \in \mathbb{I}_+ \cup \mathbb{I}'_+$, the boundary values $T_\pm$ satisfy

$$
T_+(k)/T_-(k) = 1 + |r(k)|^2,
$$

(4.11)

(d) $|k| \to \infty$ with $|\arg(k)| \leq c < \pi$

$$
T(k) = 1 + \frac{i}{k} \left[ 2 \sum_{j \in \Delta_{k_0}} \text{Im}(k_j) - \int_{\mathbb{I}_+ \cup \mathbb{I}'_+} k(s) ds \right] + O(k^{-2}),
$$

(4.12)

(e) $k \to \pm k_0$ along any ray $\pm k_0 + e^{i\phi} \mathbb{R}_+$ with $|\phi| \leq c < \pi$

$$
|T(k, k_0) - T_0(\pm k_0)(k \mp k_0)^{ik(\pm k_0)}| \leq C|k \mp k_0|^{1/2},
$$

(4.13)

and $k \to \pm ik_0$ along any ray $\pm ik_0 + e^{i\phi} \mathbb{R}_+$ with $|\phi| \leq c < \pi$

$$
|T(k, ik_0) - T_0(\pm ik_0)(k \mp ik_0)^{ik(\pm ik_0)}| \leq C|k \mp ik_0|^{1/2}.
$$

(4.14)
where
\[
T_0(\pm k_0) = T(\pm k_0, k_0) = \prod_{j \in \Delta k_0} \frac{(\pm k_0 - k_j)(\pm k_0 + k_j)}{(\pm k_0 - k_j)(\pm k_0 + k_j)} e^{i\beta(\pm k_0, \pm k_0)},
\]
and
\[
T_0(\pm ik_0) = T(\pm ik_0, ik_0) = \prod_{j \in \Delta k_0} \frac{(\pm ik_0 - k_j)(\pm ik_0 + k_j)}{(\pm ik_0 - k_j)(\pm ik_0 + k_j)} e^{i\beta(\pm ik_0, \pm ik_0)},
\]
\[
\beta(\pm k, k_0) = -k(\pm k_0) \log(k \mp k_0 + 1) + \int_{\Gamma_c} \frac{k(s) - \chi(\pm k_0)k(\pm k_0)}{s-k} ds,
\]
\[
\beta(\pm ik, ik_0) = -k(\pm ik_0) \log(\mp ik_0 + 1) + \int_{\Gamma_c} \frac{k(s) - \chi'(\pm ik_0)k(\pm ik_0)}{s-k} ds.
\]

Using our partial transmission coefficient 

\[T(k)\]
defined above, we have a new unknown matrix-valued function \(M^{(1)}\)

\[M^{(1)}(k) = M(k)T(k)^{-\sigma_3},\]

and it satisfies the following RH problem:

**RHP3.** Find an analytic function \(M^{(1)} : \mathbb{C} \setminus (\mathbb{R} \cup \mathcal{K} \cup \overline{\mathcal{K}}) \to SL_2(\mathbb{C})\) with the following properties:

(a) \(M^{(1)}(x, t, k)\) is analytic in \(C \setminus \Sigma^{(1)}\), \(\Sigma^{(1)} = \mathbb{R} \cup i\mathbb{R}\),

(b) Symmetry: \(M^{(1)}(\bar{k}) = \sigma_2 M^{(1)}(k) \sigma_2\),

(c) \(M^{(1)}(x, t, k)\) satisfies the jump condition

\[M^{(1)}_+(x, t, k) = M^{(1)}_-(x, t, k) V^{(1)}(x, t, k), \quad k \in \Sigma^{(1)},\]

the jump matrix \(V^{(1)}(k)\) is defined by

\[
V^{(1)} = \begin{cases} 
\begin{pmatrix} \frac{\bar{r}(k)}{1 + |r(k)|^2} T^{-2}(k) e^{2it\theta} & 0 \\
0 & \frac{r(k)T^{'2}(k) e^{-2it\theta}}{1 + |r(k)|^2} \end{pmatrix}, & k \in \Sigma^{(1)}
\end{cases}
\]

(d) Asymptotic condition

\[M^{(1)}(k) = I + \mathcal{O}(k^{-1}), \quad k \to \infty.\]
(e) $M^{(1)}(k)$ has simple poles at each $k_j \in \mathcal{K}$ at which

$$\text{Res}_{k = \pm k_j} M^{(1)} = \begin{cases} \lim_{k \to \pm k_j} M^{(1)} & \begin{bmatrix} 0 & \pm c_j^{-1}(1/T)'(\pm k_j) \cdot e^{-2i\theta(\pm k_j)} \\ 0 & 0 \end{bmatrix} \quad k \in \Delta^-_{k_0}, \\ \lim_{k \to \pm k_j} M^{(1)} & \begin{bmatrix} \pm c_j T(\pm k_j) & 0 \\ 0 & 0 \end{bmatrix} \quad k \in \Delta^+_{k_0}, \end{cases}$$

$$\text{Res}_{k = \pm \bar{k}_j} M^{(1)} = \begin{cases} \lim_{k \to \pm \bar{k}_j} M^{(1)} & \begin{bmatrix} 0 & \mp (\mp c_j)^{-1}T'(\mp k_j) \cdot e^{2i\theta(\mp k_j)} \\ 0 & 0 \end{bmatrix} \quad k \in \Delta^-_{k_0}, \\ \lim_{k \to \pm \bar{k}_j} M^{(1)} & \begin{bmatrix} 0 & \mp \bar{c}_j T(\mp \bar{k}_j) \cdot e^{-2i\theta(\mp \bar{k}_j)} \\ 0 & 0 \end{bmatrix} \quad k \in \Delta^+_{k_0}, \end{cases}$$

Proof. That $M^{(1)}$ is unimodular, analytic in $\mathbb{C} \setminus \mathbb{R} \cup \mathcal{K}$, and approaches identity as $k \to \infty$ follows directly from its definition, Proposition 3.1 and the properties of $M$. Concerning the residues, since $T(k)$ is analytic at each $k_j, \bar{k}_j$ with $k \in \Delta^+_{k_0}$, the residue conditions at these poles are an immediate consequence of (4.10). For $k \in \Delta^-_{k_0}, T(k)$ has zeros at $k_j$ and $-k_j$, and poles at $k_j$ and $-k_j$ so that $M^{(1)}_1 = M_1(k)T(k)^{-1}$ has a removable singularities at $k_j$ and $-k_j$ but acquires poles at $k_j$ and $-k_j$. For $M^{(1)}_2 = M_2(k)T(k)$ the situation is reversed; it has a pole at $k_j$ and a removable singularity at $-k_j$. At $k_j$ we have

$$\text{Res}_{k = k_j} M^{(1)}(k) = \lim_{k \to k_j} M_1(k)T(k)^{-1} = \text{Res}_{k = k_j} M_1(k) \cdot (1/T)'(k_j)$$

$$= c_j e^{-2i\theta(k_j)} M_2(k_j) (1/T)'(k_j),$$

$$\text{Res}_{k = \bar{k}_j} M^{(1)}(k) = \lim_{k \to \bar{k}_j} M_1(k)T(k)^{-1} = \text{Res}_{k = \bar{k}_j} M_1(k) \cdot (1/T)'(k_j)$$

$$= c_j^{-1} [(1/T)'(k_j)]^{-1} e^{2i\theta(k_j)} M^{(1)}_1(k_j).$$

from which the first formula in (4.23) clearly follows. The computation of the residue at $-k_j, \bar{k}_j$ and $-k_j$ for $k \in \Delta^-_{k_0}$ are similar. \[\square\]

5 A mixed $\bar{\partial}$ -RH problem

The main purpose of this section is to reformulate the original RHP3 as an equivalent Riemann-Hilbert problem on the augmented contour $\Sigma^{(2)}$ (see Fig.3),

$$\Sigma^{(2)} = L \cup L_0 \cup \bar{L} \cup \bar{L}_0 \cup \mathbb{R} \cup i\mathbb{R}$$

where $L = L_1 \cup \bar{L}_1 \cup L_2 \cup \bar{L}_2$. Denote the contour

$$L_1 = \{ k = k_0 + k_0 u \frac{\pi}{2}, u \in (-\infty, \frac{1}{\sqrt{2}}) \},$$

$$L_1 = \{ k = k_0 + k_0 u \frac{\pi}{2}, u \in (-\infty, \frac{1}{\sqrt{2}}) \},$$

$$L_1 = \{ k = k_0 + k_0 u \frac{\pi}{2}, u \in (-\infty, \frac{1}{\sqrt{2}}) \}$$

13
\[\tilde{L}_1 = \{k = ik_0 + k_0ue^{-\frac{u}{\sqrt{2}}}, u \in (-\infty, \frac{1}{\sqrt{2}}]\},\]
\[L_2 = \{k = -k_0 + k_0ue^{-\frac{u}{\sqrt{2}}}, u \in (-\infty, \frac{1}{\sqrt{2}}]\},\]
\[\tilde{L}_2 = \{k = -ik_0 + k_0ue^{\frac{u}{\sqrt{2}}}, u \in (-\infty, \frac{1}{\sqrt{2}}]\}.
\]

Denote the contour \(L_0 = \left\{ k_0ue^{\frac{u}{\sqrt{2}}}, \ u \in [-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}] \right\}, \) (5.1)

then the contour \(\Sigma^{(2)}\) numbered \(\Sigma_j, j = 1, \ldots, 20\) and the real axis and the imaginary axis \(\mathbb{R} \cup i\mathbb{R}\) separate complex plane \(\mathbb{C}\) into twenty open sectors denoted by \(D_j, j = 1, \ldots, 20\), starting with sector \(D_1\) between \(I_+\) and \(\Sigma_7\) and numbered consecutively continuing counter clockwise as shown in Fig.3 and Fig.4.

![Figure 3: The jump contour of \(M^{(2)}\)](image)

In addition, we let
\[\rho = \frac{1}{2\lambda, \mu \in \mathbb{K} \cup \mathbb{K}} |\lambda - \mu|, \] (5.2)

Note that, as poles come in conjugate pairs and (by assumption) no pole lies on the real axis, we have \(\rho \leq \text{dist}(\mathbb{K} \cup \overline{\mathbb{K}}, \mathbb{R})\). Let \(\chi_K \in C_0^\infty(\mathbb{C}, [0, 1])\) be supported near the discrete spectrum such that
\[\chi_k(k) = \begin{cases} 1 & \text{dist}(k, K \cup \overline{K}) < \rho/3 \\ 0 & \text{dist}(k, K \cup \overline{K}) > 2\rho/3 \end{cases} \] (5.3)
Lemma 5.1. It is possible to define functions $R_j: \overline{\Omega}_j \rightarrow \mathbb{C}$, $j = 1, 3, 4, 6$, with boundary values satisfying:

$$R_1(k) = \begin{cases} r(k)T(k)^{-2}, & k \in (-\infty, -k_0) \text{ for } -k_0; \ k \in [k_0, +\infty) \text{ for } k_0 \end{cases}$$

$$f_1 := r(k)T_0(k)^{-2}(k-k_0)^{-2i\tilde{\omega}(k_0)}(1-\chi(k)), \quad k \in \Sigma_7,$$

$$= r(-k_0)T_0(-k_0)^{-2}(k+k_0)^{-2i\tilde{\omega}(-k_0)}(1-\chi(k)), \quad k \in \Sigma_2; \quad (5.4)$$

$$R_7(k) = \begin{cases} \frac{r(k)T^2_0}{1+r(k)r(k)}, & k \in (-k_0, 0) \text{ for } -k_0; \ k \in (0, k_0) \text{ for } k_0, \\
\frac{\tilde{r}(k)}{1+r(k)\tilde{r}(k)}, & k \in (-k_0, 0) \text{ for } -k_0; \ k \in (0, k_0) \text{ for } k_0, \\
\tilde{r}(k) \end{cases}$$

$$f_7 := \frac{\tilde{r}(k)}{1+r(k)\tilde{r}(k)}T_0(k)^{-2}(k-k_0)^{-2i\tilde{\omega}(k_0)}(1-\chi(k)), \quad k \in \Sigma_{15},$$

$$= \frac{\tilde{r}(-k_0)}{1+r(-k_0)\tilde{r}(-k_0)}T_0(-k_0)^{-2}(k+k_0)^{-2i\tilde{\omega}(-k_0)}(1-\chi(k)), \quad k \in \Sigma_{10}, \quad (5.5)$$

$$R_{11}(k) = \begin{cases} \frac{r(k)T^2_0}{1+r(k)r(k)}, & k \in (-k_0, 0) \text{ for } -k_0, \ k \in (0, k_0) \text{ for } k_0, \\
r(k) \end{cases}$$

$$f_{11} := \frac{r(k)}{1+r(k)\tilde{r}(k)}T_0(k)^{-2}(k-k_0)^{-2i\tilde{\omega}(k_0)}(1-\chi(k)), \quad k \in \Sigma_{14},$$

$$= \frac{\tilde{r}(-k_0)}{1+r(-k_0)\tilde{r}(-k_0)}T_0(-k_0)^{-2}(k+k_0)^{-2i\tilde{\omega}(-k_0)}(1-\chi(k)), \quad k \in \Sigma_{11}, \quad (5.6)$$

$$R_{20}(k) = \begin{cases} \frac{r(k)T(k)^2}{1+r(k)r(k)}, & k \in (-\infty, -k_0) \text{ for } -k_0, \ k \in [k_0, +\infty) \text{ for } k_0, \\
r(k) \end{cases}$$

$$f_{20} := \frac{r(k)}{1+r(k)\tilde{r}(k)}T_0(k)^{-2}(k-k_0)^{-2i\tilde{\omega}(k_0)}(1-\chi(k)), \quad k \in \Sigma_6,$$

$$= \frac{\tilde{r}(-k_0)}{1+r(-k_0)\tilde{r}(-k_0)}T_0(-k_0)^{-2}(k+k_0)^{-2i\tilde{\omega}(-k_0)}(1-\chi(k)), \quad k \in \Sigma_3, \quad (5.7)$$

$$R_3(k) = \begin{cases} \frac{r(k)T(k)^{-2}}{1+r(k)r(k)}, & k \in (-\infty, -ik_0) \text{ for } -ik_0, \ k \in [ik_0, +\infty) \text{ for } ik_0, \\
r(k) \end{cases}$$

$$f_3 := \frac{r(k)}{1+r(k)\tilde{r}(k)}T_0(ik_0)^{-2}(k-ik_0)^{-2i\tilde{\omega}(ik_0)}(1-\chi(k)), \quad k \in \Sigma_8,$$

$$= \frac{\tilde{r}(-ik_0)}{1+r(-ik_0)\tilde{r}(-ik_0)}T_0(-ik_0)^{-2}(k+ik_0)^{-2i\tilde{\omega}(-ik_0)}(1-\chi(k)), \quad k \in \Sigma_5, \quad (5.8)$$

$$R_8(k) = \begin{cases} \frac{r(k)T(k)^{-2}}{1+r(k)r(k)}, & k \in (-\infty, -ik_0) \text{ for } -ik_0, \ k \in (0, ik_0) \text{ for } ik_0, \\
r(k) \end{cases}$$

$$f_8 := \frac{r(ik_0)}{1+r(ik_0)\tilde{r}(ik_0)}T_0(ik_0)^{-2}(k-ik_0)^{-2i\tilde{\omega}(ik_0)}(1-\chi(k)), \quad k \in \Sigma_{16},$$

$$= \frac{\tilde{r}(-ik_0)}{1+r(-ik_0)\tilde{r}(-ik_0)}T_0(-ik_0)^{-2}(k+ik_0)^{-2i\tilde{\omega}(-ik_0)}(1-\chi(k)), \quad k \in \Sigma_{13}, \quad (5.9)$$

$$R_9(k) = \begin{cases} \frac{r(k)T(k)^{-2}}{1+r(k)r(k)}, & k \in (-\infty, -ik_0) \text{ for } -ik_0, \ k \in (0, ik_0) \text{ for } ik_0, \\
r(k) \end{cases}$$

$$f_9 := \frac{r(ik_0)}{1+r(ik_0)\tilde{r}(ik_0)}T_0(k)^{-2}(k-ik_0)^{-2i\tilde{\omega}(ik_0)}(1-\chi(k)), \quad k \in \Sigma_9,$$

$$= \frac{\tilde{r}(-ik_0)}{1+r(-ik_0)\tilde{r}(-ik_0)}T_0(-ik_0)^{-2}(k+ik_0)^{-2i\tilde{\omega}(-ik_0)}(1-\chi(k)), \quad k \in \Sigma_{12}, \quad (5.10)$$

$$R_4(k) = \begin{cases} \frac{r(k)T(k)^2}{1+r(k)r(k)}, & k \in (-\infty, -ik_0) \text{ for } -ik_0, \ k \in [ik_0, +\infty) \text{ for } ik_0, \\
r(k) \end{cases}$$

$$f_4 := \frac{r(ik_0)}{1+r(ik_0)\tilde{r}(ik_0)}T_0(ik_0)^2(k-ik_0)^{-2i\tilde{\omega}(ik_0)}(1-\chi(k)), \quad k \in \Sigma_1,$$

$$= \frac{\tilde{r}(-ik_0)}{1+r(-ik_0)\tilde{r}(-ik_0)}T_0(-ik_0)^2(k+ik_0)^{-2i\tilde{\omega}(-ik_0)}(1-\chi(k)), \quad k \in \Sigma_4, \quad (5.11)$$
$R_j$ is defined follow the in reverse order. $R_j$ have following property: for $j = 1, 7, 11, 20,$

$$|R_j(k)| \lesssim \sin^2(\arg(k - k_0)) + (\text{Re}(k))^{-1/2}, \quad (5.12)$$

$$|\bar{\partial}R_j(k)| \lesssim |\bar{\partial}\chi(k)| + |p'_j(\text{Re } z)| + |k - k_0|^{-1/2}, \quad (5.13)$$

and for $j = 6, 10, 14, 15,$

$$|R_j(k)| \lesssim \sin^2(\arg(k + k_0)) + (\text{Re}(k))^{-1/2}, \quad (5.14)$$

$$|\bar{\partial}R_j(k)| \lesssim |\bar{\partial}\chi(k)| + |p'_j(\text{Re } k)| + |k + k_0|^{-1/2}, \quad (5.15)$$

and for $j = 3, 4, 8, 9,$

$$|R_j(k)| \lesssim \sin^2(\arg(k - ik_0)) + (\text{Re}(k))^{-1/2}, \quad (5.16)$$

$$|\bar{\partial}R_j(k)| \lesssim |\bar{\partial}\chi(k)| + |p'_j(\text{Re } z)| + |k - ik_0|^{-1/2}, \quad (5.17)$$

and for $j = 12, 13, 17, 18,$

$$|R_j(k)| \lesssim \sin^2(\arg(k + ik_0)) + (\text{Re}(k))^{-1/2}, \quad (5.18)$$

$$|\bar{\partial}R_j(k)| \lesssim |\bar{\partial}\chi(k)| + |p'_j(\text{Re } z)| + |k + ik_0|^{-1/2}, \quad (5.19)$$

where

$$p_1(k) = p_6(k) = p_9(k) = p_{13}(k) = r(k), \quad (5.20)$$

$$p_2(k) = p_{10}(k) = p_{14}(k) = \frac{\bar{r}(k)}{1 + |r(k)|^2}, \quad (5.21)$$

$$p_3(k) = p_8(k) = p_{11}(k) = p_{15}(k) = \frac{r(k)}{1 + |r(k)|^2}, \quad (5.22)$$

$$p_4(k) = p_7(k) = p_{12}(k) = p_{16}(k) = \bar{r}(k). \quad (5.23)$$

and

$$\bar{\partial}R_j(k) = 0, \quad \text{if } k \in D_2 \cup D_5 \cup D_{16} \cup D_{19} \text{ or } \text{dist}(k, K \cup \overline{K}) < \rho/3. \quad (5.24)$$

**Proof.** Let $k - k_0 = s e^{i \phi}$ and $k = u + iv$. Define, for $k \in \overline{D}_2$ the extensions

$$R_1(u, v) = \cos(2\phi)p_1(u) + (1 - \cos(2\phi))f_1(u + iv), \quad (5.25)$$

It follows that $R_1$ satisfies the first two conditions in (5.4) immediately. Note that

$$\bar{\partial}R_1 = (p_1 - f_1)\bar{\partial}\cos(2\phi) + \frac{\cos(2\phi)}{2} p'_1. \quad (5.26)$$
\[
|\bar{\partial}R_1| \leq \frac{c_1}{|k - k_0|}||p_1 - \rho_1(k_0)|| + ||p_1(k_0) - f_1|| + c_2|p'_1|.
\]  

(5.27)

We also have

\[
|p_1(k) - p_1(k_0)| = \left| \int_{k_0}^{k} p'_1 \, ds \right| \leq \int_{k_0}^{k} |p'_1| \, ds \leq ||p_1||_{L^2((k,k_0))} \cdot |k - k_0|^\frac{1}{2} 
\]  

(5.28)

\[
p_1(k_0) - f_1 = p_1(k_0) - p_1(k_0)(\exp[2\nu((k - k_0) \ln(k - k_0) - (k - k_0 + 1) \ln(k - k_0 + 1)) + 2(\beta(k, k_0) - \beta(k_0, k_0))].
\]  

(5.29)

From the estimation of \(\beta(\cdot, k_0)\) we obtain, uniformly in \(\{k : k = ue^{\mu}, u > 0, 0 < \mu < \frac{\pi}{4}\}\)

\[
|\beta(k, k_0) - \beta(k_0, k_0)| = O(\sqrt{k - k_0}),
\]  

(5.30)

\[
|\nu(k_0) \ln(k - k_0)| \leq O(\sqrt{k - k_0}).
\]  

(5.31)

Therefore

\[
|p_1(k_0) - f_1| = p_1(k_0)\{1 - \exp[O(\sqrt{k - k_0})]\} = p_1(k_0)\{O(\sqrt{k - k_0})\}. 
\]  

(5.32)

Combining these estimates yields

\[
|\bar{\partial}R_1| \leq c_1|k - k_0|^{-\frac{1}{2}} + c_2|p'_1|. 
\]  

(5.33)

We introduce a new unknown function \(M^{(2)}\) for deforming the contour \(\Sigma^{(1)}\) to the contour \(\Sigma^{(2)}\):

\[
M^{(2)}(k) = M^{(1)}(k)R^{(2)}(k),
\]  

(5.34)

where \(R^{(2)}(k)\) is chosen to satisfy the following conditions: First, \(M^{(2)}\) has no jump on the real and imaginary axis, so we choose the boundary values of \(R^{(2)}(k)\) through the factorization of \(V^{(1)}(k)\) in [14,20] where the new jumps on \(\Sigma^{(2)}\) match a well known model RH problem; Second, we need to control the norm of \(R^{(2)}(k)\), so that the \(\bar{\partial}\)-contribution to the long-time asymptotics of \(u(x, t)\) can be ignored; Third the residues are unaffected by the transformation.
Figure 4: In the yellow region, $R^{(2)} = I$, in white region, $R^{(2)} \neq I$.

So we choose $R^{(2)}(k)$ as

$$
R^{(2)} = \begin{cases} 
\begin{pmatrix} 1 & 0 \\ (-1)^m R_j(k) e^{2i\theta} & 0 \end{pmatrix}, & k \in D_j, j = 1, 3, 9, 10, 11, 12, 15, 17, \\
\begin{pmatrix} 1 & 0 \\ (-1)^m R_j(k) e^{-2i\theta} & 1 \end{pmatrix}, & k \in D_j, j = 4, 6, 7, 8, 13, 14, 18, 20,
\end{cases}
$$

where $m_1 = m_4 = m_6 = m_7 = m_9 = m_{10} = m_{13} = m_{17} = 1$, $m_3 = m_8 = m_{11} = m_{12} = m_{14} = m_{15} = m_{18} = m_{20} = 0$, and the function $R_j$ is defined in following proposition, where $j = 1, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 18, 20$.

We now use $R^{(2)}$ to define the transformation (5.34), which satisfies the following mixed $\bar{\partial}$-RH problem.

**RH problem 4.** Find a function $M^{(2)}: \mathbb{C} \setminus \Sigma^{(2)} \rightarrow SL_2(\mathbb{C})$ with the following properties.

(a) $M^{(2)}(x,t,k)$ is continuous in $\mathbb{C}$, sectionally continuous first partial derivatives in $\mathbb{C} \setminus (\Sigma^{(2)} \cup K \cup \overline{K})$ and meromorphic in $D_2 \cup D_5 \cup D_{16} \cup D_{19}$,

(b) Symmetry: $M^{(2)}(k) = \sigma_2 M^{(2)}(\sigma_2 k) \sigma_2$,

(c) The boundary value $M^{(2)}(x,t,k)$ at $\Sigma^{(2)}$ satisfies the jump condition

$$
M^+_{(2)}(x,t,k) = M^-_{(2)}(x,t,k) V^{(2)}(x,t,k), \quad k \in \Sigma^{(2)},
$$

where $m_1 = m_4 = m_6 = m_7 = m_9 = m_{10} = m_{13} = m_{17} = 1$, $m_3 = m_8 = m_{11} = m_{12} = m_{14} = m_{15} = m_{18} = m_{20} = 0$, and the function $R_j$ is defined in following proposition, where $j = 1, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 18, 20$.
with

\[
V^{(2)} = \begin{cases}
1 & 0 \\
R_1(k)e^{2it\theta} & 1, \\
R_7(k)e^{-2it\theta} & 0 \\
0 & 1, \\
1 & 0 \\
R_{14}(k)e^{2it\theta} & 1, \\
R_{20}(k)e^{-2it\theta} & 0 \\
0 & 1, \\
1 & 0 \\
R_8(k)e^{-2it\theta} & 0 \\
0 & 1, \\
1 & 0 \\
R_9(k)e^{2it\theta} & 1, \\
1 & 0 \\
R_4(k)e^{-2it\theta} & 0 \\
0 & 1, \\
1 & -R_{10}(k)e^{-2it\theta} \\
R_9(k)e^{2it\theta} & -R_9(k)R_{10}(k) + 1, \\
1 & 0 \\
(R_{14}(k) - R_{12}(k))e^{2it\theta} & 1, \\
1 & -R_{13}(k)e^{-2it\theta} \\
R_{12}(k)e^{2it\theta} & -R_{12}(k)R_{13}(k) + 1, \\
1 & (R_8(k) - R_7(k))e^{-2it\theta} \\
0 & 1,
\end{cases}
\]

\[k \in \Sigma_7 \cup \Sigma_2, \]

\[k \in \Sigma_{15} \cup \Sigma_{10}, \]

\[k \in \Sigma_{14} \cup \Sigma_{11}, \]

\[k \in \Sigma_6 \cup \Sigma_3, \]

\[k \in \Sigma_8 \cup \Sigma_5, \]

\[k \in \Sigma_{16} \cup \Sigma_{13}, \]

\[k \in \Sigma_{1} \cup \Sigma_{4}, \]

\[k \in \Sigma_{17}, \]

\[k \in \Sigma_{18}, \]

\[k \in \Sigma_{19}, \]

\[k \in \Sigma_{20}, \]

(d) Asymptotic condition

\[M^{(2)}(x, t, k) = I + O(k^{-1}), \quad as \quad k \to \infty. \]

(e) Away from \(\Sigma^{(2)}\) we have

\[\overline{\partial}M^{(2)} = M^{(2)}\overline{\partial}R^{(2)}, \]

\[5.38\]

\[5.39\]
holds in $\mathbb{C}\setminus\Sigma(2)$, where

$$
\mathcal{D}R^{(2)} = \begin{cases} 
1 & k \in D_j, j = 1, 3, 9, 11, 15, 17, \\
(1)^m, R_j(k)e^{2it\theta} & k \in D_j, j = 4, 6, 7, 8, 13, 14, 18, 20, \\
\frac{1}{(1)^m, R_j(k)e^{-2it\theta}} & k \in D_j, j = 2, 5, 16, 19.
\end{cases}
$$

(f) Residue conditions: $M^{(2)}$ has simple poles at each point in $K$ with:

$$
\text{Res}_{k = \pm k_j} M^{(2)} = \begin{cases} 
\lim_{k \to \pm k_j} M^{(2)}(0, \pm c_j^{-1}(1/T)'(\pm k_j)^{-2}e^{-2it\theta(\pm k_j)}) & k \in \Delta_-^{k_0}, \\
\lim_{k \to \pm k_j} M^{(2)}(0, 0) & k \in \Delta_0^{+}, \\
\lim_{k \to \pm k_j} M^{(2)}(0, \mp c_j^{-1}T(\pm k_j)^{-2}e^{2it\theta(\pm k_j)}) & k \in \Delta_-^{k_0}, \\
\lim_{k \to \pm k_j} M^{(2)}(0, 0) & k \in \Delta_0^{+}.
\end{cases}
$$

6 Decomposition of the mixed $\bar{\partial}$-RH problem

To solve $\text{RHP4}$, we decompose it into a model RH Problem for $M^{RHP}(x, t, k)$ with $\mathcal{D}R^{(2)} = 0$ and a pure $\mathcal{D}$-Problem with $\mathcal{D}R^{(2)} \neq 0$. For the first step, we establish a RH problem for the $M^{RHP}(x, t, k)$ as follows.

RHP5. Find a matrix-valued function $M^{RHP}(x, t, k)$ with the following properties:

(a) Analyticity: $M^{RHP}(x, t, k)$ is analytical in $\mathbb{C}\setminus(\Sigma(2) \cup K \cup \overline{K})$,

(b) Symmetry: $M^{RHP}(\bar{k}) = \sigma_2 M^{RHP}(k)\sigma_2$,

(c) Jump condition: $M^{RHP}(x, t, k)$ has continuous boundary values $M^{RHP}_\pm$ on $\mathbb{R} \cup i\mathbb{R}$ and

$$
M^{RHP}_+(x, t, k) = M^{RHP}(x, t, k)V^{(2)}(k), \quad k \in \mathbb{R} \cup i\mathbb{R},
$$

(d) $\bar{\partial}$-Derivative: $\mathcal{D}R^{(2)} = 0$, for $k \in \mathbb{C}$,

(e) Asymptotic behaviours:

$$
M^{RHP}(x, t, k) = I + O(k^{-1}), \quad \text{as} \quad k \to \infty.
$$
(f) Residue conditions: $M^{(2)}$ has simple poles at each point in $K \cup \overline{K}$ with:

$$\text{Res}_{k = \pm k_j} M^{RHP} = \begin{cases} \lim_{k \to \pm k_j} M^{RHP} \left( \begin{array}{cc} 0 & \pm e^{-j(1/T)}(\pm k_j)^{-1}T - 1 \pm e^{-2i\theta(\pm k_j)} \\ 0 & 0 \end{array} \right) \quad k \in \Delta_{k_0}^- \\ \lim_{k \to \pm k_j} M^{RHP} \left( \begin{array}{cc} 0 & \pm c_j T(\pm k_j) - 1 \pm e^{-2i\theta(\pm k_j)} \\ 0 & 0 \end{array} \right) \quad k \in \Delta_{k_0}^+ \end{cases}$$

$$\text{Res}_{k = \pm k_j} M^{RHP} = \begin{cases} \lim_{k \to \pm k_j} M^{RHP} \left( \begin{array}{cc} 0 & \mp c_j T(\pm k_j) - 1 \pm e^{-2i\theta(\pm k_j)} \\ 0 & 0 \end{array} \right) \quad k \in \Delta_{k_0}^- \\ \lim_{k \to \pm k_j} M^{RHP} \left( \begin{array}{cc} 0 & \mp e^{-j(1/T)}(\pm k_j)^{-1}T + 1 \pm e^{-2i\theta(\pm k_j)} \\ 0 & 0 \end{array} \right) \quad k \in \Delta_{k_0}^+ \end{cases}$$

The existence and asymptotic of $M^{RHP}(x, t, k)$ will be shown in section 8.

We now use $M^{RHP}(x, t, k)$ to construct a new matrix function

$$M^{(3)}(k) = M^{(2)}(k)M^{RHP}(k)^{-1},$$

which removes analytical component $M^{RHP}(k; x, t)$ to get a pure $\overline{\partial}$-problem.

**RHP6.** Find a matrix-valued function $M^{(3)}(k; x, t)$ with the following properties:

(a) $M^{(3)}(x, t, k)$ is continuous in $\mathbb{C}$, sectionally continuous first partial derivatives in $\mathbb{C} \setminus (\Sigma^{(2)} \cup K \cup \overline{K})$ and meromorphic in $D_2 \cup D_5 \cup D_{16} \cup D_{19}$;

(b) Symmetry: $M^{(3)}(\bar{k}) = \sigma_2 M^{(3)}(k) \sigma_2$;

(c) Asymptotic behaviours:

$$M^{(3)}(x, t, k) = I + O(k^{-1}), \quad \text{as} \quad k \to \infty.$$  

(d) $\overline{\partial}$-Derivative: For $\mathbb{C} \setminus (\Sigma^{(2)} \cup K \cup \overline{K})$ we have $\overline{\partial}M^{(3)} = M^{(3)}W^{(3)}$

$$W^{(3)} = M^{RHP}(k)\overline{\partial}R^{(2)}M^{RHP}(k)^{-1}.$$  

**Proof.** By using properties of the solutions $M^{(2)}$ and $M^{RHP}$ for **RHP4** and **RHP5**, the analyticity and asymptotics are obtained immediately. Since $M^{(2)}$ and $M^{RHP}$ have same jump matrix, we have

$$M^{(3)}_-(k)^{-1}M^{(3)}_+(k) = M^{(2)}_-(k)^{-1}M^{RHP}_+(k)M^{RHP}_+(k)^{-1}M^{(2)}_+(k)$$

$$M^{(3)}_-(k)^{-1}V^{(2)}(k)^{-1}M^{(2)}_+(k) = I,$$

which means $M^{(3)}$ has no jumps and is everywhere continuous. We also can show that $M^{(3)}$ has no pole. For $k_j \in K$, let $N$ denote the nilpotent matrix which appears in the left side of
the corresponding residue condition of \textbf{RHP4} and \textbf{RHP5}, we have the Laurent expansions in \( k - k_j \)

\[
M^{(2)}(k) = a(k_j) \left[ \frac{N}{k - k_j} + I \right] + \mathcal{O}(k - k_j), \tag{6.9}
\]

\[
M^{\text{RHP}}(k) = A(k_j) \left[ \frac{N}{k - k_j} + I \right] + \mathcal{O}(k - k_j), \tag{6.10}
\]

where \( a(k_j) \) and \( A(k_j) \) are the constant terms in their Laurent expansions. Then from

\[
M^{\text{RHP}}(k)^{-1} = \sigma_2 M^{\text{RHP}}(k)^T \sigma_2, \]

it implies that

\[
M^{(3)}(k) = M^{(2)}(k) M^{\text{RHP}}(k)^{-1} = \left\{ a(k_j) \left[ \frac{N}{k - k_j} + I \right] \right\} \left\{ \frac{-N}{k - k_j} + I \right\} A(k_j)^{-1} + \mathcal{O}(k - k_j) \tag{6.11}
\]

\[
= \mathcal{O}(1).
\]

\( M^{(3)} \) has only removable singularities at each \( k_j \). The last property follows immediately from the definition of \( M^{(3)} \), exploiting the fact that \( M^{\text{RHP}} \) has no \( \mathcal{I} \) component.

We construct the solution \( M^{\text{RHP}} \) of the \textbf{RHP5} in the following form

\[
M^{\text{RHP}} = \begin{cases} 
E(k) M^{(\text{out})}(k) & k \notin \{ U_{\pm k_0} \cup U_{\pm ik_0} \}, \\
E(k) M^{(\pm k_0)}(k) & k \in U_{\pm k_0}, \\
E(k) M^{(\pm ik_0)}(k) & k \in U_{\pm ik_0},
\end{cases} \tag{6.12}
\]

where \( U_{\pm k_0} \) and \( U_{\pm ik_0} \) are the neighborhoods of \( \pm k_0 \) and \( \pm ik_0 \), respectively

\[
U_{\pm k_0} = \left\{ k : |k \mp k_0| \leq \min \left\{ \frac{k_0}{2}, \rho/3 \right\} \right\} \triangleq \varepsilon \tag{6.13}
\]

and

\[
U_{\pm ik_0} = \left\{ k : |k \mp ik_0| \leq \min \left\{ \frac{k_0}{2}, \rho/3 \right\} \right\} \triangleq \varepsilon \tag{6.14}
\]

This implies that \( M^{\text{RHP}} \), also \( M^{(\pm k_0)} \) and \( M^{(\pm ik_0)} \) have no poles in \( U_{\pm k_0} \) and \( U_{\pm ik_0} \), since \( \text{dist}(K \cup \overline{K}, \mathbb{R}) > \rho \). This decomposition splits \( M^{\text{RHP}} \) into two parts: \( M^{(\text{out})} \) solves a model RHP obtained by ignoring the jump conditions of \textbf{RHP5}, which will be solved in next Section 7; While \( M^{(\pm k_0)} \) and \( M^{(\pm ik_0)} \), whose solution can be approximated with parabolic cylinder functions if we let \( M^{(\pm k_0)} \) and \( M^{(\pm ik_0)} \) match to the \( M^{(2)} \) and a parabolic cylinder model in \( U_{\pm k_0} \) and \( U_{\pm ik_0} \), these results will given in Section 8. And \( E(k) \) is a error function, which is a solution of a small-norm RH problem and we discuss it in Section 9. And from the \textbf{RHP5}, whose jump matrix admits the following estimates.

22
Proposition 2. For the jump matrix $V^{(2)}(k)$, we have the following estimate

$$
\|V^{(2)} - I\|_{L^\infty(\Sigma \cap U_0)} = O\left(e^{-\frac{\sqrt{2}k_0^2}{k_0}t|k \mp k_0|} (k_0^{-4} - |k|^{-4})\right),
$$

(6.15)

$$
\|V^{(2)} - I\|_{L^\infty(\Sigma \cap U_{k_0})} = O\left(e^{-\frac{\sqrt{2}k_0^2}{k_0}t|k \mp k_0|} (k_0^{-4} - |k|^{-4})\right),
$$

(6.16)

$$
\|V^{(2)} - I\|_{L^\infty(\Sigma_0^{(2)})} = O\left(e^{-\frac{a^2}{4k_0^2}t}\right),
$$

(6.17)

where the contours are defined by

$$
\Sigma^{(2)} = \Sigma_7 \cup \Sigma_{15} \cup \Sigma_{14} \cup \Sigma_6,
$$

(6.18a)

$$
\Sigma^{(2)} = \Sigma_2 \cup \Sigma_{10} \cup \Sigma_{11} \cup \Sigma_3,
$$

(6.18b)

$$
\Sigma^{(2)} = \Sigma_8 \cup \Sigma_{16} \cup \Sigma_9 \cup \Sigma_1,
$$

(6.18c)

$$
\Sigma^{(2)} = \Sigma_{13} \cup \Sigma_5 \cup \Sigma_4 \cup \Sigma_{12},
$$

(6.18d)

$$
\Sigma_0^{(2)} = \Sigma_{17} \cup \Sigma_{18} \cup \Sigma_{19} \cup \Sigma_{20}.
$$

(6.18e)

Proof. We prove (6.17) for $k \in \Sigma_{20}$, other cases can be shown in a similar way. By using definition of $V^{(2)}$ and (5.12), we have

$$
\|V^{(2)} - I\|_{L^\infty(\Sigma_{20})} \leq \|(R_8 - R_7)e^{2it\theta}\|_{L^\infty(\Sigma_{20})}.
$$

(6.19)

Note that $|k| < \sqrt{2}k_0/2$ for $k \in \Sigma_{20}$, together with (4.3), we find that

$$
|e^{2it\theta}| = e^{\text{Re}2it\theta} = e^{-\frac{\alpha^2|k|^2}{2k_0^2}} \leq e^{-\frac{a^2}{4k_0^2}t} \to 0, \quad \text{as } t \to \infty,
$$

(6.20)

which together with (6.19) gives (6.17). The calculation of $\Sigma^{(2)}_{\pm}$ and $\Sigma^{(2)}_{\pm}'$ are similar. \hfill \Box

This proposition means that the jump matrix $V^{(2)}$ uniformly goes to $I$ on both $\Sigma_{\pm}^{(2)} \cap U_{\pm k_0} \cap U_{\pm ik_0}$ and $\Sigma_0^{(2)}$, so outside the $U_{\pm k_0} \cup U_{\pm ik_0}$ there is only exponentially small error (in $t$) by completely ignoring the jump condition of $M^{RHP}$. And note that unlike the neighborhood of $\pm k_0$ and $\pm ik_0$, $V^{(2)} \to I$ as $k \to 0$, it has uniformly property. So we does not need to consider the neighborhood of $k = 0$ alone.

7 Outer model RH problem

In this section, we build a outer model RH problem and show that its solution can approximated with a finite sum of soliton solutions. The key lies in the fact that we need
the property of $M^{\text{out}}(x, k, t)$ as $k \to \infty$, from which we obtain the solution of the following outer model problem.

**RHP7.** Find a matrix-valued function $M^{\text{out}}(x, k, t)$ with following properties:

(a) Analyticity: $M^{\text{out}}(x, k, t)$ is analytical in $\mathbb{C}\backslash(\Sigma^{(2)} \cup \mathcal{K} \cup \overline{\mathcal{K}})$;

(b) Symmetry: $\overline{M^{\text{out}}(\bar{k})} = \sigma_2 M^{\text{out}}(k) \sigma_2$;

(c) Asymptotic behaviours:

$$M^{\text{out}}(x, k, t) \sim I + O(k^{-1}), \quad k \to \infty,$$  

(d) Residue conditions: $M^{\text{out}}$ has simple poles at each point in $\mathcal{K} \cup \overline{\mathcal{K}}$ satisfying the same residue relations with $M^{\text{RHP}}(k)$.

Before showing the existence and uniqueness of solution of the above RHP7, we first consider the reflectionless case of the RHP1. In this case, $M$ has no contour, the RHP1 reduces to the following RH problem.

**RHP8.** Given discrete data $\sigma_d = \{(k_j, c_j)\}_{j=1}^N$ and find a matrix-valued function $m(x, k, t|\sigma_d)$ with following properties:

(a) Analyticity: $m(x, k, t|\sigma_d)$ is analytical in $\mathbb{C}\backslash(\Sigma^{(2)} \cup \mathcal{K} \cup \overline{\mathcal{K}})$;

(b) Symmetry:

$$\overline{m(k|\sigma_d)} = \sigma_2 m(k|\sigma_d) \sigma_2,$$  

(c) Asymptotic behaviours:

$$m(k|\sigma_d) \sim I + O(k^{-1}), \quad k \to \infty,$$  

(d) Residue conditions: $m(x, k, t|\sigma_d)$ has simple poles at each point in $\mathcal{K} \cup \overline{\mathcal{K}}$ satisfying

$$\text{Res}_{k=\pm k_j} m(y, k, t|\sigma_d) = \lim_{k \to \pm k_j} m(y, k, t|\sigma_d) \gamma_j,$$

$$\text{Res}_{k=\pm k_j} \tilde{m}(y, k, t|\sigma_d) = \lim_{k \to \pm k_j} \tilde{m}(y, k, t|\sigma_d) \tilde{\gamma}_j,$$  

where $\tau_j$ is a nilpotent matrix satisfies

$$\tau_j = \begin{pmatrix} 0 & 0 \\ \gamma_j & 0 \end{pmatrix}, \quad \tilde{\tau}_j = \sigma_2 \tau_j \sigma_2, \quad \gamma_j = \pm c_j e^{-2it\theta(\pm k_j)},$$  

Moreover, the solution satisfies

$$\|m(x, k, t|\sigma_d)^{-1}\|_{L^\infty(C\backslash(\mathcal{K} \cup \overline{\mathcal{K}}))} \lesssim 1.$$  

24
Proposition 3. The RHP8 exists an unique solution.

Proof. The uniqueness of solution follows from the Liouville's theorem. The symmetries of $m(k;x,t|\sigma_d)$ means that it admits a partial fraction expansion of following form

$$m(x,k,t|\sigma_d) = I + \sum_{j=1}^{N} \left[ \frac{1}{k-k_j} \begin{pmatrix} \nu_j(x,t) & 0 \\ \zeta_j(x,t) & 0 \end{pmatrix} + \frac{1}{k-k_j} \begin{pmatrix} 0 & -\zeta_j(x,t) \\ \nu_j(x,t) & 0 \end{pmatrix} \right]. \quad (7.8)$$

By using a similar way to Appendix B in [37], we can show the existence of the solution for the RHP8. Since $\det(m(x,k,t|\sigma_d)) = 1$, $\|m(x,k,t|\sigma_d)\|_{L^\infty(\mathbb{C}\setminus(\Sigma(2)\cup \mathbb{K} \cup \overline{\mathbb{K}}))}$ is bounded. And from (7.8), we simply obtain (7.7). $\square$

In reflectionless case, the transmission coefficient admits following trace formula

$$a(k) = \prod_{k=1}^{N} \frac{(k-k_j)(k+k_j)}{(k-k_j)(k+k_j)}, \quad (7.9)$$

whose poles can be split into four parts. Let $\Delta \subseteq \{1, 2, \ldots, N\}$, and define

$$a_\Delta(k) = \prod_{j \in \Delta} \frac{(k-k_j)(k+k_j)}{(k-k_j)(k+k_j)}, \quad (7.10)$$

we make a renormalization transformation

$$m_\Delta(k|\mathcal{D}) = m(k|\sigma_d) a_\Delta(k)\sigma_2, \quad (7.11)$$

where the scattering data are given by

$$\mathcal{D} = \{(k_j,c'_j)\}_{j=1}^{N}, \quad c'_j = c_j a_\Delta(k)^2, \quad (7.12)$$

It is easy to see that the transformation (7.11) splits the poles between the columns of $m_\Delta(k|\mathcal{D})$ according to the choice of $\Delta$, and it satisfies the following modified discrete RH problem.

RHP9. Given discrete data (7.12), find a matrix-valued function $m_\Delta(x,t,k|\mathcal{D})$ with following properties:

(a) Analyticity: $m_\Delta(x,t,k|\mathcal{D})$ is analytical in $\mathbb{C}\setminus(\Sigma(2) \cup \mathbb{K} \cup \overline{\mathbb{K}})$

(b) Symmetry: $m_\Delta(k|\mathcal{D}) = \sigma_2 m_\Delta(k|\mathcal{D})\sigma_2$

(c) Asymptotic behaviours:

$$m_\Delta(x,t,k|\mathcal{D}) \sim I + O(k^{-1}), \quad k \to \infty \quad (7.13)$$

25
Residue conditions: $m^\Delta(x,t,k|D)$ has simple poles at each point in $\mathcal{K} \cup \mathcal{K}$ satisfying

$$\text{Res}_{k=\pm k_j} m^\Delta(x,t,k|D) = \lim_{k \to \pm k_j} m^\Delta(x,k,t|D) \hat{\tau}^\Delta_j, \quad (7.14)$$

$$\text{Res}_{k=\pm k_j} m^\Delta(x,k,t|D) = \lim_{k \to \pm k_j} m^\Delta(x,k,t|D) \gamma_j \hat{\tau}^\Delta, \quad (7.15)$$

where $\hat{\tau}^\Delta_j$ is a nilpotent matrix satisfies

$$\hat{\tau}^\Delta_j = \begin{cases} 0 & \text{for } j \notin \Delta, \\ \gamma_j a^\Delta(\pm k_j)^2 & \text{for } j \in \Delta, \end{cases} \quad (7.16)$$

where

$$\hat{\tau}^\Delta_j = \sigma_2 \hat{\tau}^\Delta_j \sigma_2, \quad (7.17)$$

$$\gamma_j = c_j e^{2i\theta(k_j)}. \quad (7.18)$$

Since (7.11) is an explicit transformation of $m(x,k,t|\sigma_d)$, by Proposition 5, we obtain the existence and uniqueness of the solution of the RHP9.

In the RHP9, take $\Delta = \Delta_{k_0}$ and replace the scattering data $D$ with scattering data

$$\tilde{D} = \{(k_j, \tilde{c}_j)\}_{k=1}^N, \quad \tilde{c}_j = c_j \delta(k_j)^2, \quad (7.19)$$

then we have

**Corollary 1.** There exists and unique solution for the RHP7, moreover,

$$M^{\text{out}}(x,t,k) = m^{\Delta_{k_0}}(x,t,k|\tilde{D}) \quad (7.20)$$

where scattering data $\tilde{D}$ is given by (7.19).

If we choosing $\Delta$ appropriately, the asymptotic limits $t \to \infty$ with $\xi = \tilde{\tau} + \alpha$ and $k_0 = (\frac{\alpha^2}{\xi})^{\frac{1}{2}}$ bounded are under better asymptotic control. Then we consider the long-time behavior of soliton solutions.

Give pairs points $x_1 \leq x_2 \in \mathbb{R}$ and velocities $v_1 \leq v_2 \in \mathbb{R}^-$, we define a cone

$$C(x_1, x_2, v_1, v_2) = \{(x,t) \in \mathbb{R}^2 | x = x_0 + vt, x_0 \in [x_1, x_2], v \in [v_1, v_2]\} \quad (7.21)$$
and denote

\[ I = \{ k : -\frac{1}{4v_1} < |k|^2 < -\frac{1}{4v_2} \} \],  
(7.22)

\[ \mathcal{K}(I) = \{ k_j \in \mathbb{Z} : k_j \in I \}, \quad N(I) = |\mathcal{K}(I)|, \]  
(7.23)

\[ \mathcal{K}^-(I) = \{ k_j \in \mathcal{K} : |k|^2 > -\frac{1}{4v_2} \}, \]  
(7.24)

\[ \mathcal{K}^+(I) = \{ k_j \in \mathcal{K} : |k|^2 < -\frac{1}{4v_1} \}, \]  
(7.25)

\[ c_j(I) = c_j \prod_{\text{Re} k_n \in I} \frac{(k_j - k_n)^2(k_j + k_n)^2}{(k_j - k_n)^2(k_j + k_n)^2} \exp\left[ -\frac{1}{\pi i} \int_{I_+} \log(1 + |r(\zeta)|^2) d\zeta \right]. \]  
(7.26)

\[
\begin{align*}
\begin{array}{c}
\cdot -k_2 \\
\cdot -k_1 \\
I \\
(\frac{\alpha^2}{4\beta^2})^{1/2} \\
(\frac{\alpha^2}{4\beta^2})^{-1/2} \\
\cdot k_1 \\
\cdot k_2 \\
\cdot -k_2 \\
\cdot -k_1 \\
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
x = v_1 t + x_1 \\
x = v_2 t + x_2 \\
C \\
x_1 \\
x_2 \\
x = v_1 t + x_1 \\
x = v_2 t + x_2
\end{array}
\end{align*}
\]

Figure 5: (a) In the example here, the original data has four pairs zero points of discrete spectrum, but inside the cone \( C \) only three pairs points with \( \mathcal{K}(I) = k_1 \); (b) The cone \( C(x_1, x_2, v_1, v_2) \).

We can show the following lemma

**Lemma 1.** Fix reflectionless data \( D = \{ (k_j, c_j') \}_{j=1}^N \), \( D(I) = \{ (k_j, c_j'(I)) | k_j \in \mathbb{Z}(I) \} \). Then as \( |t| \to \infty \) with \( (x, t) \in C(x_1, x_2, v_1, v_2) \), we have

\[
m^\Delta^{k_0}_D(y, t, k|D) = (I + O(e^{-2\nu(I)|t|}))m^\Delta^{k_0}_D(y, t, k|D(I)), \]  
(7.27)
where
\[
\mu(I) = \min_{k_j \in \mathcal{K} \setminus \mathcal{K}(I)} \{ \text{Re} k_j \text{Im} k_j \frac{-v}{|k_j|^4} (|k_j| + (\frac{\alpha^2}{-4v})^{1/4}) (|k_j|^2 + (\frac{\alpha^2}{-4v})^{1/2}) (|k_j| - (\frac{\alpha^2}{-4v})^{1/4}) \text{dist} (k_j, I) \},
\] (7.28)

**Proof.** We denote
\[
\Delta^+(I) = \left\{ j | \text{Re} (k_j) < \frac{2}{\beta} \sqrt{\frac{-v}{\alpha}} \right\}, \quad \Delta^-(I) = \left\{ j | \text{Re} (k_j) > \frac{2}{\beta} \sqrt{\frac{-v}{\alpha}} \right\},
\] (7.29)
and take \( \Delta = \Delta_{\pi_0} \) in the RHP9, then for \( k \in \mathcal{K} \setminus \mathcal{K}(I) \) and \( (x, t) \in C(x_1, x_2, v_1, v_2) \), by using the residue coefficients (7.18), direct calculation shows that
\[
|\gamma_j(x_0 + vt, t)| = |c_j| \left| \exp \left[ -2ix_0 k_j^2 - 2vt(k_j^2 + \alpha \beta^2 - \alpha \beta + \frac{\alpha^2}{4k_j^2}) \right] \right|
= |c_j| |\exp(-4x_0 \text{Re} k_j \text{Im} k_j)| \left| \exp \left[ -2t \text{Re} k_j \text{Im} k_j \frac{-v}{k_j^2} (|k_j| - (\frac{\alpha^2}{-4v})^{1/4}) \right. \right.
\left( |k_j| + (\frac{\alpha^2}{-4v})^{1/4} ) (k_j^2 + (\frac{\alpha^2}{-4v})^{1/2}) \right),
\] (7.30)
We set
\[
\mu(I) = \min_{k_j \in \mathcal{K} \setminus \mathcal{K}(I)} \{ \text{Re} k_j \text{Im} k_j \frac{-v}{|k_j|^4} (|k_j| + (\frac{\alpha^2}{-4v})^{1/4}) (|k_j|^2 + (\frac{\alpha^2}{-4v})^{1/2}) \text{dist} (k_j, I) \},
\] (7.31)
which leads to
\[
\|\tau_j^{\Delta^+(I)}\| = O(e^{-2\mu(I)t}), \quad t \to \infty,
\] (7.32)

Suppose that \( D_j \) is a small disks centrad in each \( k_j \in \mathcal{K} \setminus \mathcal{K}(I) \) with radius smaller than \( \mu \). Denote \( \partial D_j \) is the boundary of \( D_j \). Then we can introduce a new transformation which can remove the poles \( k_j \in \mathcal{K} \setminus \mathcal{K}(I) \) and these residues change to near-identity jumps.

\[
\tilde{m}^{\Delta_{\pi_0}}(k; x, t|D) = \begin{cases} 
\text{m}^{\Delta_{\pi_0}}(x, k, t|D)(I - \frac{\tau_j^{\Delta^+(I)}}{k-j}) & k \in D_j \cup (-D_j), \\
\text{m}^{\Delta_{\pi_0}}(x, k, t|D)(I - \frac{\tau_j^{\Delta^+(I)}}{k-j}) & k \in \overline{D_j} \cup (-\overline{D_j}), \\
m^{\Delta_{\pi_0}}(x, k, t|D) & \text{elsewhere}. 
\end{cases}
\] (7.33)
Comparing with \( m^{\Delta_{\pi_0}} \) the new matrix function \( \tilde{m}^{\Delta_{\pi_0}}(x, k, t|D) \) has new jump in each \( \partial D_j \) which denote by \( \tilde{V}(k) \). Then using (7.32), we have
\[
\| \tilde{V}(k) - I \|_{L^\infty(\mathcal{K})} = O(e^{-2\mu(I)t}), \quad t \to \infty,
\] (7.34)
\[
\Sigma = \cup_{k_j \in \mathcal{K} \setminus \mathcal{K}(I)} (\partial(\pm D_j) \cup \partial(\pm \overline{D_j})).
\] (7.35)
Since $\tilde{m}^{\Delta_0}(x, k, t|D)$ has same poles and residue conditions with $m^{\Delta_0}(x, k, t|D)$, then

$$m_0(k) = \tilde{m}^{\Delta_0}(k; y, t|D)m^{\Delta_0}(x, k, t|D(I))^{-1}$$  \hspace{1cm} (7.36)

has no poles, but it has jump matrix for $k \in \Sigma$

$$m_0^+(k) = m_0^-(k)V_{m_0}(k)$$  \hspace{1cm} (7.37)

where the jump matrix $V_{m_0}(k)$ given by

$$V_{m_0}(k) = m(k|D(I))\tilde{V}(k)m(k|D(I))^{-1},$$  \hspace{1cm} (7.38)

which, by using (7.35), also admits the same decaying estimate

$$\|V_{m_0}(k) - I\|_{L^\infty(\Sigma)} = \|\tilde{V}(k) - I\|_{L^\infty(\Sigma)} = O(e^{-2\mu(t)t}), \quad t \to \infty.$$  \hspace{1cm} (7.39)

Using reconstruction formula to $m^{\Delta_0}(k; x, t|D)$, we immediately obtain the following result.

**Corollary 2.** Let $m_{sol}(x, t; D)$ and $m_{sol}(x, t; D(I))$ denote the N-soliton solution of (1.1) corresponding to discrete scattering data $D$ and $D(I)$ respectively. And $I, C(x_1, x_2, v_1, v_2), D(I)$ is given above. As $|t| \to \infty$ with $(x, t) \in C(x_1, x_2, v_1, v_2)$, we have

$$2i \lim_{k \to \infty} k(m(x, k|\sigma_d))_{12} = m_{sol}(x, t; D) = m_{sol}(x, t; D(I)) + O(e^{-2\mu(t)t})$$  \hspace{1cm} (7.40)

From the outer model we arrive at the following corollary.

**Corollary 3.** The RHP7 exists an unique solution $M^{out}$ with

$$M^{out}(k) = m^{\Delta_0}(k|D^{out})$$

$$= m^{\Delta_0}(x, k, t|D(I)) \prod_{\text{Re } k_j \in I_{k_0}^{\sigma_d}} \left( \frac{(k - k_j)(k + k_j)}{(k - k_j)(k + k_j)}\right)^{-\sigma_1} \delta^3 + O(e^{-2\mu(t)t}),$$  \hspace{1cm} (7.41)

where $D^{out} = \{k_j, c_j(k_0)\}_{j=1}^N$

$$c_j(k_0) = c_j \exp \left[ -\frac{1}{\pi i} \int_{L_{+}} \frac{\log [1 + |r(\zeta)|^2]}{\zeta - k} d\zeta \right]$$  \hspace{1cm} (7.42)

Then substitute (7.41) into (7.4) we immediately have

$$\|M^{(out)}(k)^{-1}\|_{L^\infty(\Sigma \setminus \mathcal{R})} \lesssim 1.$$  \hspace{1cm} (7.43)

Moreover, we have reconstruction formula

$$2i \lim_{k \to \infty} k(M^{(out)})_{12} = m(x, t; D^{(out)}) = m(x, t; D^{(out)}(I)) + O(e^{-2\mu(t)t}),$$  \hspace{1cm} (7.44)

and

$$m_{sol}(x, t; D^{(out)}) = m_{sol}(x, t; D(I)) + O(e^{-\mu(t)t}), \quad t \to \pm \infty.$$  \hspace{1cm} (7.45)
8 A local solvable RH model near phase points

From the Proposition 4, in the neighborhood $U_{\pm k_0}$ of $\pm k_0$ and $U_{\pm i k_0}$ of $\pm i k_0$, we find that $V^{(2)} - I$ doesn’t have a uniformly small jump for large time, so we establish a local model for function $E(k)$ with a uniformly small jump. In this section, we will separate the pure RHP problem from the mixed RHP problem so as to obtain the pure $\partial$ problem in our subsequent discussion.

**RHP10.** Find a matrix-valued function $M^{FL}(x, t, k)$ such that

(a) Analyticity: $M^{FL}(x, t, k)$ is analytical in $\mathbb{C}\setminus \Sigma^{FL}$,

(b) Asymptotic behaviors:

$$M^{FL}(x, t, k) \sim I + \mathcal{O}(k^{-1}), \quad k \to \infty,$$

(8.1)

(c) Jump condition: $M^{FL}(x, t, k)$ has continuous boundary values $M^{FL}_{\pm}(x, t, k)$ on $\Sigma^{FL}$ and

$$M^{FL}_{+}(k) = M^{FL}_{-}(k)V^{FL}(k), \quad k \in \Sigma^{FL},$$

(8.2)
where the jump matrix $V^{FL}(k)$ is given by

$$
V^{(FL)}(k) = \begin{cases}
1 & , k \in \Sigma_7(k_0), \Sigma_2(-k_0), \\
\frac{\bar{r}(\pm k_0)}{1+|\bar{r}(\pm k_0)|^2} & , k \in \Sigma_{15}(k_0), \Sigma_{10}(-k_0), \\
rar(-\pm k_0)^2(\pm k_0)(k \pm k_0)^{2i}e^{2it\theta} & , k \in \Sigma_{14}(k_0), \Sigma_{11}(-k_0), \\
0 & , k \in \Sigma_6(k_0), \Sigma_3(-k_0), \\
1 & , k \in \Sigma_6(i k_0), \Sigma_5(-i k_0), \\
\frac{\bar{r}(\pm k_0)}{1+|\bar{r}(\pm i k_0)|^2} & , k \in \Sigma_{16}(i k_0), \Sigma_{13}(-i k_0), \\
0 & , k \in \Sigma_9(i k_0), \Sigma_{12}(-i k_0), \\
1 & , k \in \Sigma_9(-i k_0), \Sigma_{11}(-i k_0), \\
\frac{\bar{r}(\pm i k_0)}{1+|\bar{r}(\pm i k_0)|^2} & , k \in \Sigma_1(i k_0), \Sigma_4(-i k_0), \\
0 & , k \in \Sigma_{17}, \\
1 & , k \in \Sigma_{18}, \\
\frac{\bar{r}(\pm k_0)}{1+|\bar{r}(\pm k_0)|^2} & , k \in \Sigma_{19}, \\
0 & , k \in \Sigma_{20},
\end{cases}
$$

In order to solve this problem, we need to begin from the PC-model of four stationary-phase points and their jump lines, seeing in Fig.6.
We take $-k_0$ as an example and other three stationary-phase points can be discussed in the same way. The definition of $\Sigma^{(2)}$ in the different domains $\Omega_j$, $j = 1, \ldots, 6$ are shown in Fig.6.

$$
\begin{pmatrix}
1 & \frac{r-k_0}{1+|r-k_0|^2} e^{2i\nu} e^{-i\zeta^2/2} \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
\frac{r_k}{1+|r-k_0|^2} e^{-2i\nu} e^{i\zeta^2/2} & 1
\end{pmatrix}
$$

Figure 7: $\Sigma^{PC}_j$ and domains $\Omega_j$, $j = 1, 2, 7, 11, 19, 20$.

we set

$$r_{-k_0} = r(-k_0)T_0(-k_0)\frac{1}{2}e^{2i\nu(-k_0)\ln\frac{(k_0^2-2)\lambda}{2\pi} e^{i\alpha\beta t}}, \quad (8.4)$$

moreover in the notation of the proposition 3, we have

$$M^{PC}_A(\zeta) = I + \frac{M^{PC}}{\zeta} + O(\zeta^{-2}), \quad (8.5)$$
where
\[ M_{1A}^{PC} = \begin{pmatrix} 0 & -i\beta_{12}(r-k) \\ i\beta_{21}(r-k) & 0 \end{pmatrix}, \quad (8.6) \]
with
\[ \beta_{12}(r-k) = \frac{\sqrt{2\pi}e^{i\pi/4}e^{-\pi\kappa/2}}{r-k}_G(-iv), \quad (8.7) \]
\[ \beta_{21}(r-k) = -\frac{\sqrt{2\pi}e^{-i\pi/4}e^{-\pi\kappa/2}}{r-k}_G(iv) = \frac{\nu(k_0)}{\beta_{12}(r-k)}. \quad (8.8) \]

Expand \( \theta(k) \) to obtain
\[ \theta(k) = \frac{\alpha\beta^2}{4}(k^2 + \frac{1}{k^2}) - \alpha\beta \]
\[ = \frac{\alpha\beta^2}{2k_0^2} - \alpha\beta + \frac{\alpha\beta^2}{k_0^2}(k + k_0)^2 + (\frac{\alpha\beta^2}{4k_0^2k^2} - \frac{3\alpha\beta^2}{4k_0^2k^2})(k + k_0)^3. \quad (8.9) \]

It’s natural to get the following scaling
\[ N_A : f(k) \rightarrow (N_A f)(k) = f\left(\frac{k_0^2}{2\sqrt{\alpha\beta}}\right) \zeta - k_0). \quad (8.10) \]

Considering the action of the operators \( N_A \) on \( \delta(k)e^{-i\theta(k)} \), we find that,
\[ (N_A\delta e^{-i\theta}) (k) = \delta^0(\zeta)\delta^1(\zeta), \quad (8.11) \]
where
\[ \delta^0(\zeta) = \frac{k_0}{(\sqrt{\alpha\beta})}2^{i\nu}e^{i(\alpha\beta-\frac{2i\nu}{2k_0^2})\nu}e^{\chi(k_0)e^{\tilde{\chi}^2(k_0)}}(\zeta-k_0)^{i\nu}, \quad (8.12) \]
and
\[ \delta^1(\zeta) = \zeta^{i\nu}e^{-i\tilde{\chi}^2}e^{-i\frac{k_0^2}{2\sqrt{\alpha\beta}}(\zeta-k_0)\nu} \left( \frac{k_0^2}{2\sqrt{\alpha\beta}}(\zeta-k_0)^{2i\nu} \right) \]
\[ \times \left( \frac{k_0^2}{2\sqrt{\alpha\beta}}(\zeta-k_0 + ik_0)(\zeta-k_0 - ik_0))^{-i\nu} \right) \]
\[ \times e^{\chi(\frac{k_0^2}{2\sqrt{\alpha\beta}}(\zeta-k_0)-\chi^2(k_0))} e^{\tilde{\chi}^2(\frac{k_0^2}{2\sqrt{\alpha\beta}}(\zeta-k_0)-\tilde{\chi}^2(k_0))}. \quad (8.13) \]

The method here is the same as [10]. Instead of using the Taylor expansion of \( \theta(k) \), we explicitly write the coefficients of the power terms in the form of splitting, and the benefits of this method will be very obvious in our subsequent calculations.

**Proposition 4.** As \( t \rightarrow \infty \), then for \( \zeta \in \{ \zeta = \mu k_0e^{\pm t}, -\varepsilon < u < \varepsilon \} \), by observing \( \delta^1(\zeta) \), we have the conclusion about
\[ \delta^1(\zeta) \sim \zeta^{i\nu}e^{-i\tilde{\chi}^2}, \quad (8.14) \]
by using the result of
\[ |e^{-i \frac{k_0^4}{32 \sqrt{\alpha \beta}} \left( \frac{\zeta}{k_0} - \frac{1}{k} \right)}| \rightarrow 1, \quad \text{as } t \rightarrow \infty. \] (8.15)

**Proof.** For \( \zeta = u k_0 e^{\pm i \theta}, \) we have
\[ -i \frac{k_0^4 \zeta^2}{32 \sqrt{\alpha \beta}} \left( \frac{1}{k} - \frac{3}{k_0} \right) = -i \sqrt{2} k_0^4 u + \frac{k_0^4}{32 \sqrt{\alpha \beta}} \left( \frac{\sqrt{2}}{2} u - 3u^2 \right), \] (8.16)
whose real part is
\[ \text{Re} \left\{ -i \frac{k_0^4 \zeta^2}{32 \sqrt{\alpha \beta}} \left( \frac{1}{k} - \frac{3}{k_0} \right) \right\} = \frac{k_0^4}{32 \sqrt{\alpha \beta}} \left( \frac{\sqrt{2}}{2} u - 3u^2 \right) \] (8.17)
\[ \leq \frac{k_0^4}{32 \sqrt{\alpha \beta}} \left( \frac{\sqrt{2}}{2} |u| + 3|u|^2 \right) \leq \frac{k_0^4}{32 \sqrt{\alpha \beta}} \left( \frac{\sqrt{2}}{2} |\zeta| + 3|\zeta|^2 \right) \rightarrow 0, \quad \text{as } t \rightarrow \infty. \]

Therefore,
\[ |e^{-i \frac{k_0^4}{32 \sqrt{\alpha \beta}} \left( \frac{\zeta}{k_0} - \frac{1}{k} \right)}| \rightarrow 1, \quad \text{as } t \rightarrow \infty. \] (8.18)
which the effects of the third power.

Another three points can be calculated in the same way. We note that in the model Riemann-Hilbert problem, the origin is the reference point from which the rays emanate. In the following sections, for convenience, We also use the notation \( \zeta. \) Since \( M^{FL} \) satisfies the asymptotic property
\[ M^{FL} = I + \sum M_{A+B+C+D}^{PC}(\zeta) + \mathcal{O}(\zeta^{-2}) \]
\[ = I + \frac{M_{1A}^{PC} + M_{1B}^{PC} + M_{1C}^{PC} + M_{1D}^{PC}}{\zeta} + \mathcal{O}(\zeta^{-2}), \] (8.19)
where \( M_{1j}^{PC}, j = A, B, C, D \) is the coefficient of \( 1/\zeta. \) Then we substitute (8.10) into (8.19)
\[ M^{FL} = I + \frac{k_0^2 k (M_{1A}^{PC} + M_{1B}^{PC})}{\beta \sqrt{\alpha \ell}(k^2 - k_0^2)} + \frac{k_0^2 k (M_{1C}^{PC} + M_{1D}^{PC})}{\beta \sqrt{\alpha \ell}(k^2 - k_0^2)} + \mathcal{O}(\zeta^{-2}) \]
\[ = I + \frac{k_0^2 k (M_{1A}^{PC} + M_{1B}^{PC})}{\beta \sqrt{\alpha \ell}(k - k_0)(k + k_0)} + \frac{k_0^2 k (M_{1C}^{PC} + M_{1D}^{PC})}{\beta \sqrt{\alpha \ell}(k - i k_0)(k + i k_0)} + \mathcal{O}(\zeta^{-2}) \] (8.20)
\[ = I + \frac{1}{k - k_0} \frac{k_0^2 k M_{1A}^{PC}}{\beta \sqrt{\alpha \ell}(k - k_0)} + \frac{1}{k + k_0} \frac{k_0^2 k M_{1B}^{PC}}{\beta \sqrt{\alpha \ell}(k - k_0)} \]
\[ + \frac{1}{k - i k_0} \frac{k_0^2 k M_{1C}^{PC}}{\beta \sqrt{\alpha \ell}(k - i k_0)} + \frac{1}{k + i k_0} \frac{k_0^2 k M_{1D}^{PC}}{\beta \sqrt{\alpha \ell}(k - i k_0)} + \mathcal{O}(\zeta^{-2}). \]

In the local circular domain of \( k_0 \)
\[ \left| \frac{k}{k + k_0} \right| \leq c, \] (8.21)
where $c$ is independent of $k$. The other three points can be controlled in the same way. We can also reach a consistent conclusion that

$$|M^{FL} - I| \lesssim O(t^{-1/2}).$$  \hfill (8.22)

Besides, it is shown that

$$\|M^{FL}(\zeta)\|_{\infty} \lesssim 1,$$  \hfill (8.23)

We use $M^{FL}(\zeta)$ to define a local model in four circles $k \in U_{\pm k_0}$ and $k \in U_{\pm ik_0}$

$$M^{(\pm k_0, \pm ik_0)}(k) = M^{out}(k)M^{FL}(\zeta),$$  \hfill (8.24)

which is a bounded function in $U_{\pm k_0}$ and $U_{\pm ik_0}$, which have the same jump matrix as $M^{RHP}(k)$.

9 The small norm RH problem for error function

In this section, we consider the error matrix-function $E(k)$. From the definition (6.11) and (8.8), we can obtain a RH problem for the matrix function $E(k)$.

**RHP12.** Find a matrix-valued function $E(k)$ with following properties:

(a) Analyticity: $E(k)$ is analytical in $\mathbb{C}\setminus \Sigma^{(E)}$

$$\Sigma^{(E)} = \partial U_{\pm k_0} \cup \partial U_{\pm ik_0} \cup (\Sigma^{(2)} \setminus (U_{\pm k_0} \cup U_{\pm ik_0})).$$

(b) Symmetry:

$$\overline{E(k)} = \sigma_2 E(k) \sigma_2^{-1},$$  \hfill (9.1)

(c) Asymptotic behaviours:

$$E(k) \sim I + O(k^{-1}), \quad |k| \to \infty,$$  \hfill (9.2)

(d) Jump condition: $E$ has continuous boundary values $E_{\pm}$ on $\Sigma^{(E)}$ satisfying

$$E_+(k) = E_-(k)V^{(E)},$$  \hfill (9.3)

where the jump matrix $V^{(E)}$ is given by

$$V^{(E)}(k) = \begin{cases} M^{(out)}(k)V^{(2)}(k)M^{(out)}(k)^{-1}, & k \in \Sigma^{(2)} \setminus (U_{\pm k_0} \cup U_{\pm ik_0}), \\ M^{(out)}(k)M^{FL}(k)M^{(out)}(k)^{-1}, & k \in \partial U_{\pm k_0} \cup \partial U_{\pm ik_0}, \end{cases}$$  \hfill (9.4)
which is shown in Fig. 7.

We will show that for large times, the error function $E(k)$ solves following small norm RH problem.

By using (7.43) and Proposition 2, we have the following estimates

\[
\left| V^{(E)} - I \right| \lesssim \begin{cases} 
\exp\left\{-\frac{\alpha^2}{4k_0^2} |k| \right\}, & k \in \Sigma^{(2)}_{\pm} \backslash U_{\pm k_0}, \\
\exp\left\{-\frac{\alpha^2}{4k_0^2} |k + ik_0| \right\}, & k \in \Sigma^{(2)}_{\pm} \backslash U_{\pm ik_0}, \\
\exp\left\{-\frac{\alpha^2}{4k_0^2} t \right\}, & k \in \Sigma^{(2)}_{0}.
\end{cases}
\] (9.5)

**Proposition 5.** For $k \in \partial U_{\pm k_0} \cup \partial U_{\pm ik_0}$, $M^{(out)}(k)$ is bounded, we find that

\[
\left| V^{(E)} - I \right| = \left| M^{(out)}(k)^{-1}(M^{FL}(k) - I)M^{(out)}(k) \right| = O(t^{-1/2}),
\] (9.6)

Combing (7.43) with (8.22), (9.6) is obtained. Therefore, the existence and uniqueness
of the RHP12 can shown by using a small-norm RH problem [36, 37], and we have

\[ E(k) = I + \frac{1}{2\pi i} \int_{\Sigma(E)} \frac{(I + \rho(s))(V^{(E)} - I)}{s - k} ds, \]

(9.7)

where \( \rho \in L^2(\Sigma(E)) \) is the unique solution of following equation:

\[ (1 - C_E) \rho = C_E(I), \]

(9.8)

which means \( \|C_E\| < 1 \) for sufficiently large \( t \), therefore \( 1 - C_E \) is invertible, and \( \rho \) exists and is unique. Moreover,

\[ \|\rho\|_{L^2(\Sigma(E))} \lesssim \frac{\|C_E\|}{1 - \|C_E\|} \lesssim t^{-1/2}. \]

(9.9)

Then we have the existence and boundedness of \( E(k) \). In order to reconstruct the solution \( u_{\alpha}(x, t) \) of (1.1), we need the asymptotic behavior of \( E(k) \) as \( k \to \infty \).

**Proposition 6.** As \( k \to \infty \) we have

\[ E(k) = I + \frac{E_1}{k} + O \left( \frac{1}{k^2} \right), \]

(9.10)

where

\[
E_1 = \frac{k^2}{2i\beta\alpha\sigma} \begin{pmatrix}
\sigma_2^{-1} M^{(out)}(k_0) & \begin{pmatrix} 0 & \beta_{12}(r_{k_0}) \\
-\beta_{21}(r_{k_0}) & 0 \end{pmatrix} M^{(out)}(k_0)^{-1} \sigma_2 \\
M^{(out)}(k_0) & \begin{pmatrix} 0 & \beta_{12}(r_{k_0}) \\
-\beta_{21}(r_{k_0}) & 0 \end{pmatrix} M^{(out)}(k_0)^{-1} \sigma_2 \\
-\sigma_2^{-1} M^{(out)}(i k_0) & \begin{pmatrix} 0 & \beta_{12}(r_{i k_0}) \\
-\beta_{21}(r_{i k_0}) & 0 \end{pmatrix} M^{(out)}(i k_0)^{-1} \sigma_2 \\
M^{(out)}(i k_0) & \begin{pmatrix} 0 & \beta_{12}(r_{i k_0}) \\
-\beta_{21}(r_{i k_0}) & 0 \end{pmatrix} M^{(out)}(i k_0)^{-1} \sigma_2 \\
\end{pmatrix} + O(t^{-1}).
\]

(9.11)

**Proof.** Using (9.8)-(9.9) and the bounds on \( V^{(E)} - I \) in (9.6) we have

\[
E_1 = -\frac{1}{2\pi i} \int_{\partial \mathbb{U}_k \cup \partial \mathbb{U}_{-k}} \left( V^{(E)}(s) - I \right) ds + O \left( t^{-1} \right)
= -\frac{1}{2\pi i} \left[ \int_{\partial \mathbb{U}_k} \left( V^{(E)}(s) - I \right) ds + \int_{\partial \mathbb{U}_{-k}} \left( V^{(E)}(s) - I \right) ds \right] + O \left( t^{-1} \right)
= -\frac{1}{2\pi i} (I + II + III + IV) + O \left( t^{-1} \right)
\]

(9.12)
For the integral $I$

$$
I = -\frac{1}{2\pi \beta^{\sqrt{\alpha t}}} \int_{\beta k_0} \frac{k_0^2 s M^{PC}_A}{(s - k_0)(s + k_0)}\left(\begin{array}{cc} 0 & i\beta_{12}(r_{k_0}) \\ -i\beta_{21}(r_{k_0}) & 0 \end{array}\right) M^{\text{out}}(s)^{-1} ds
$$

$$
= \frac{k_0^2}{2i\beta^{\sqrt{\alpha t}}} M^{\text{out}}(k_0) M^{PC}_A \left(\begin{array}{cc} 0 & i\beta_{12}(r_{k_0}) \\ -i\beta_{21}(r_{k_0}) & 0 \end{array}\right) M^{\text{out}}(k_0)^{-1}
$$

(9.13)

In a similar way

$$
II = -\frac{k_0^2}{2i\beta^{\sqrt{\alpha t}}} M^{\text{out}}(-k_0) M^{PC}_B \left(\begin{array}{cc} 0 & \beta_{12}(r_{-k_0}) \\ -\beta_{21}(r_{-k_0}) & 0 \end{array}\right) M^{\text{out}}(-k_0)^{-1}
$$

(9.14)

$$
III = -\frac{k_0^2}{2i\beta^{\sqrt{\alpha t}}} M^{\text{out}}(i k_0) M^{PC}_C \left(\begin{array}{cc} 0 & \beta_{12}(r_{i k_0}) \\ -\beta_{21}(r_{i k_0}) & 0 \end{array}\right) M^{\text{out}}(i k_0)^{-1}
$$

(9.15)

$$
IV = -\frac{k_0^2}{2i\beta^{\sqrt{\alpha t}}} M^{\text{out}}(-i k_0) M^{PC}_D \left(\begin{array}{cc} 0 & \beta_{12}(r_{-i k_0}) \\ -\beta_{21}(r_{-i k_0}) & 0 \end{array}\right) M^{\text{out}}(-i k_0)^{-1}
$$

(9.16)

The four parts integral can be asymptotically computed by

$$
E_1 = -\frac{k_0^2}{2i\beta^{\sqrt{\alpha t}}} \{\sigma_2[\sigma_2^{-1} M^{\text{out}}(k_0) \left(\begin{array}{cc} 0 & \beta_{12}(r_{k_0}) \\ -\beta_{21}(r_{k_0}) & 0 \end{array}\right) M^{\text{out}}(k_0)^{-1}]\sigma_2
$$

$$
-\sigma_2[\sigma_2^{-1} M^{\text{out}}(i k_0) \left(\begin{array}{cc} 0 & \beta_{12}(r_{i k_0}) \\ -\beta_{21}(r_{i k_0}) & 0 \end{array}\right) M^{\text{out}}(i k_0)^{-1}]\sigma_2
$$

$$
+ M^{\text{out}}(i k_0) \left(\begin{array}{cc} 0 & \beta_{12}(r_{i k_0}) \\ -\beta_{21}(r_{i k_0}) & 0 \end{array}\right) M^{\text{out}}(i k_0)^{-1} \sigma_2\} + O(t^{-1}),
$$

(9.17)

Also it has

$$
2i\left(E_1\right)_{12} = |t|^{-1/2} f(x, t) + O(t^{-1}),
$$

(9.18)

where

$$
f(x, t) = \frac{k_0^2}{\beta\alpha} [\beta_{12}(r_{k_0}) M^{\text{out}}_{11}(k_0)^2 + \beta_{21}(r_{k_0}) M^{\text{out}}_{12}(k_0)^2
$$

$$
+ \beta_{12}(r_{-k_0}) M^{\text{out}}_{11}(-k_0)^2 + \beta_{21}(r_{-k_0}) M^{\text{out}}_{12}(-k_0)^2
$$

$$
- \beta_{12}(r_{i k_0}) M^{\text{out}}_{11}(i k_0)^2 - \beta_{21}(r_{i k_0}) M^{\text{out}}_{12}(i k_0)^2
$$

$$
- \beta_{12}(r_{-i k_0}) M^{\text{out}}_{11}(-i k_0)^2 - \beta_{21}(r_{-i k_0}) M^{\text{out}}_{12}(-i k_0)^2].
$$

(9.19)
10 Analysis of the pure $\bar{\partial}$-Problem

Now we consider the proposition and the long time asymptotics behavior of $M^{(3)}$. The RHP of $M^{(3)}$ is equivalent to the integral equation

$$
M^{(3)}(k) = I - \frac{1}{\pi} \int_{C} \frac{\bar{\partial}M^{(3)}(s)}{k-s} dm(s) = I - \frac{1}{\pi} \int_{C} \frac{M^{(3)}(s)W^{(3)}(s)}{k-s} dm(s),
$$

(10.1)

where $m(s)$ is the Lebesgue measure on the $C$. If we denote $C$ is the left Cauchy-Green integral operator,

$$
fC_k(k) = -\frac{1}{\pi} \int_{C} \frac{f(s)W^{(3)}(s)}{k-s} dm(s),
$$

(10.2)

then above equation can be rewritten as

$$
M^{(3)}(k) = I \cdot (I - C_k)^{-1},
$$

(10.3)

To proof the existence of operator $(I - C_k)^{-1}$, we have following Lemma:

**Lemma 2.** The norm of the integral operator $C_k$ decay to zero as $t \to \infty$:

$$
\|C_k\|_{L^\infty} \to L^\infty \lesssim t^{-1/6}
$$

(10.4)

which implies that $(I - C_k)^{-1}$ exists.

**Proof.** For any $f \in L^\infty$

$$
\|fC_k\|_{L^\infty} \leq \|f\|_{L^\infty} \frac{1}{\pi} \int_{C} \frac{|W^{(3)}(s)|}{|k-s|} dm(s)
\lesssim \|f\|_{L^\infty} \frac{1}{\pi} \int_{C} \frac{|\partial R^{(2)}(s)|}{|k-s|} dm(s),
$$

(10.5)

by using

$$
|W^{(3)}(s)| \leq \|M^{RHP}\|_{L^\infty} |\partial R^{(2)}(s)| \|M^{RHP}\|_{L^\infty}^{-1} \lesssim |\partial R^{(2)}(s)|.
$$

(10.6)

So we only need to estimate the integral

$$
\frac{1}{\pi} \int_{C} \frac{|\partial R^{(2)}(s)|}{|k-s|} dm(s).
$$

(10.7)

For $\partial R^{(2)}(s)$ is a piece-wise function, we detail the case in the region $D_1$, the other regions are similar. From (15.13) and we have

$$
\int_{D_1} \frac{|\partial R^{(2)}(s)|}{|k-s|} dm(s) \leq F_1 + F_2 + F_3,
$$

(10.8)
For the first item, note that

\[ F_1 = \int_0^{+\infty} \int_{k_0+v}^{+\infty} |\partial X_\alpha(u)| e^{\frac{\alpha^2 \omega(t)}{k_0} - \frac{1}{(y^2 + v^2)}} dudv, \quad (10.9) \]

\[ F_2 = \int_0^{+\infty} \int_{k_0+v}^{+\infty} |p_\alpha'(u)| e^{\frac{\alpha^2 \omega(t)}{k_0} - \frac{1}{(y^2 + v^2)}} dudv, \quad (10.10) \]

where

\[ F_3 = \int_0^{+\infty} \int_{k_0+v}^{+\infty} \int_{y}^{+\infty} (u - k_0)^2 + v^2 - 1 \frac{\alpha^2 \omega(t)}{k_0} - \frac{1}{(y^2 + v^2)} dudv, \quad (10.11) \]

and we denote \( s = u + iv, k = x + iy \).

To deal with the absolute value sign, we suppose \( y > 0 \). In fact \( y > 0 \) we can directly remove the absolute value sign and use the same way to estimates it.

In the following calculation, we will use the inequality

\[ \|s - k\|_L^2(k_0, +\infty) = \int^{+\infty}_{k_0} \frac{1}{v - y} \frac{\left| u - x \right|^2 + 1}{\left| v - y \right|} d\left( u - x \right) \leq \frac{\pi}{\left| v - y \right|} \quad (10.12) \]

To deal with the absolute value sign, we suppose \( y > 0 \). In fact \( y > 0 \) we can directly remove the absolute value sign and use the same way to estimates it.

For \( F_1 \), noting that \(-\alpha^2 \omega(t)\left( \frac{1}{k_0} - \frac{1}{(y^2 + v^2)} \right)\) is a monotonic decreasing function of \( u \), so

\[ F_1 \leq \int_0^{+\infty} \|s - k\|^{1-1}_L(k_0, +\infty) \|\partial X_\alpha(s)\|_L^2(k_0, +\infty) e^{\frac{\alpha^2 \omega(t)}{k_0} - \frac{1}{(y^2 + v^2)}} d\left( u - x \right) \]

\[ \leq \int_0^{+\infty} |v - y|^{-1/2} \exp(\alpha \omega(t)\left( \frac{1}{k_0} - \frac{1}{(y^2 + v^2)} \right)) dv \]

\[ = \int_0^{y} (y - v)^{-1/2} \exp(\alpha \omega(t)\left( \frac{1}{k_0} - \frac{1}{(y^2 + v^2)} \right)) dv \]

\[ + \int_y^{+\infty} (v - y)^{-1/2} \exp(\frac{-\alpha \omega(t)v^4}{k_0(y^4 + k_0^4)}) dv. \]

For the first item, note that \( e^{-z} \leq z^{-1/6} \) for all \( z > 0 \), then

\[ \int_0^{y} (y - v)^{-1/2} \exp(\alpha \omega(t)\left( \frac{1}{k_0} - \frac{1}{(y^2 + v^2)} \right)) dv \]

\[ \leq \int_0^{y} (y - v)^{-1/2} v^{-1/2} dv \exp(\alpha \omega(t)v^4) \leq \frac{1}{t^{-1/6}}. \]

For the last integral we make the substitution \( w = v - y \) then we get

\[ \int_y^{+\infty} (v - y)^{-1/2} \exp(\frac{-\alpha \omega(t)v^4}{k_0(y^4 + k_0^4)}) dv \]

\[ \leq \int_0^{+\infty} w^{-1/2} \exp(\frac{-\alpha \omega(t)v^4}{k_0(y^4 + k_0^4)}) dv \exp(\frac{-\alpha \omega(t)v^5}{k_0(y^4 + k_0^4)}) \leq \frac{1}{t^{-1/2}}. \]

\[ 40 \]
Substituting (10.15) and (10.14) into (10.13) gives
\[ F_1 \lesssim t^{-1/6}. \] (10.16)

The \( F_2 \) has the same estimate with (10.16). And for \( F_3 \), we first have that
\[
\| (u - k_0)^2 + v^2 \|_{L^p(k_0, +\infty)}^{-1/4} = \left\{ \int_{k_0}^{+\infty} \left( (u - k_0)^2 + v^2 \right)^{-p/4} dv \right\}^{1/p}
\]
\[
= \left\{ \int_{k_0}^{+\infty} \left[ 1 + \left( \frac{u - k_0}{v} \right)^2 \right]^{-p/4} d\left( \frac{u - k_0}{v} \right) \right\}^{1/p} v^{1/p-1/2}
\]
\[
\lesssim v^{1/p-1/2},
\]
and
\[
\| s - k \|_{L^q(k_0, +\infty)}^{-1} = \left\{ \int_{k_0}^{+\infty} \left[ \frac{u - x}{v} \right]^2 + 1 \right\}^{1/q} d(\frac{u - x}{v}) v^{-1/q-1}
\]
\[
\leq |v - y|^{1/q-1},
\]
where \( p > 2 \) and \( \frac{1}{p} + \frac{1}{q} = 1 \). Then we have
\[
F_3 \leq \int_{k_0}^{+\infty} \| s - K \|_{L^q} ((u - k_0)^2 + v^2)^{-1/4} \|_{L^p} \exp\left( -\alpha \beta^2 vt \left( \frac{1}{k_0^3} - \frac{k_0}{(k_0^2 + v^2)^2} \right) \right) dv
\]
\[
\lesssim \int_{k_0}^{+\infty} v^{1/p-1/2} |v - y|^{1/q-1} \exp\left( -\alpha \beta^2 vt \left( \frac{1}{k_0^3} - \frac{k_0}{(k_0^2 + v^2)^2} \right) \right) dv
\]
\[
\lesssim \int_{k_0}^{y} v^{1/p-1/2} (y - v)^{1/q-1} \exp\left( -\alpha \beta^2 vt \left( \frac{1}{k_0^3} - \frac{k_0}{(k_0^2 + v^2)^2} \right) \right) dv
\]
\[
+ \int_{y}^{+\infty} v^{1/p-1/2} (v - y)^{1/q-1} \exp\left( -\alpha \beta^2 vt \left( \frac{1}{k_0^3} - \frac{k_0}{(k_0^2 + v^2)^2} \right) \right) dv.
\]
(10.19)

For the first term, using the inequality \( e^{-z} \leq z^{-1/6} \) for all \( z > 0 \) leads to
\[
\int_{k_0}^{y} v^{1/p-1/2} (y - v)^{1/q-1} \exp\left( -\alpha \beta^2 vt \left( \frac{1}{k_0^3} - \frac{k_0}{(k_0^2 + v^2)^2} \right) \right) dv
\]
\[
\lesssim |t|^{-1/6} \int_{k_0}^{y} v^{1/p-1} (y - v)^{1/q-1} dv \lesssim t^{-1/6}.
\] (10.20)

And for the second term, we estimate similarly as we estimate \( F_1 \). Let \( w = v - y \), then
\[
\int_{y}^{+\infty} v^{1/p-1/2} (v - y)^{1/q-1} \exp\left( -\alpha \beta^2 vt \left( \frac{1}{k_0^3} - \frac{k_0}{(k_0^2 + v^2)^2} \right) \right) dv
\]
\[
\leq \int_{0}^{+\infty} w^{1/q-1} (w + y)^{1/p-1/2} \exp\left( -\alpha \beta^2 wty^4 \right) dw \exp\left( -\frac{-\alpha \beta^2 ty^5}{k_0^4(y^4 + k_0^4)} \right)
\]
\[
\lesssim t^{-1/2} \int_{0}^{+\infty} w^{-1/2} \exp\left( -\frac{-\alpha \beta^2 wty^4}{k_0^4(y^4 + k_0^4)} \right) dw \lesssim t^{-1}.
\] (10.21)
Finally, we have
\[ F_3 \lesssim t^{-1/6}. \tag{10.22} \]

Summary the results obtained above, we obtain the finally consequence. □

Consider the asymptotic expansion of \( M^{(3)}(x,t,k) \) at \( k = \infty \)
\[ M^{(3)}(x,t,k) = I + \frac{M_1^{(3)}(x,t)}{k} + \mathcal{O}(k^{-2}), \quad k \to \infty, \tag{10.23} \]
where
\[ M_1^{(3)}(x,t) = \frac{1}{\pi} \int_{C} M^{(3)}(s)W^{(3)}(s)dm(s), \tag{10.24} \]

To reconstruct the solution \( u(x,t) \) of the FL equation \((1.2)\), we need the asymptotic behavior of \( M_1^{(3)}(x,t) \).

**Lemma 3.** For all \( t \neq 0 \), we have
\[ |M_1^{(3)}(x,t)| \lesssim t^{-3/4}. \tag{10.25} \]

**Proof.** By using \((10.23)\) and \((10.24)\), also noting the boundedness of \( M^{(3)} \) and \( M^{RHP} \), we obtain that
\[
\left\| M_1^{(3)}(x,t) \right\| \leq \frac{1}{\pi} \int_{\Omega_1} |M^{(3)}M^{RHP} \tilde{\partial}R^{(2)}M^{RHP^{-1}}|dm(s)
\leq c \int_{0}^{+\infty} \int_{z_0+v}^{+\infty} \tilde{\partial}R_1(s)e^{-\alpha \beta^2 uvt} \left( \frac{1}{\frac{1}{k^2} - \frac{1}{(u^2+v^2)^2}} \right) dudv
\leq c(I_1 + I_2 + I_3), \tag{10.26} \]
with
\[
I_1 = \int_{0}^{+\infty} \int_{k_0+v}^{+\infty} |\tilde{\partial}X_K(s)|e^{-\alpha \beta^2 uvt} \left( \frac{1}{\frac{1}{k^2} - \frac{1}{(u^2+v^2)^2}} \right) dudv, \tag{10.27} \\
I_2 = \int_{0}^{+\infty} \int_{k_0+v}^{+\infty} |p_1'(u)|e^{-\alpha \beta^2 uvt} \left( \frac{1}{\frac{1}{k^2} - \frac{1}{(u^2+v^2)^2}} \right) dudv, \tag{10.28} \\
I_3 = \int_{0}^{+\infty} \int_{k_0+v}^{+\infty} ((u-k_0)^2 + v^2)^{-1/4} e^{-\alpha \beta^2 uvt} \left( \frac{1}{\frac{1}{k^2} - \frac{1}{(u^2+v^2)^2}} \right) dudv. \tag{10.29} 
\]
We bound $I_1$ by applying the Cauchy-Schwarz inequality:

$$I_1 = \int_0^{+\infty} \int_{k_0+v}^{+\infty} |\tilde{\partial} X_k(s)| e^{-\alpha \beta^2\nu t} \frac{1}{\sqrt{1 + (\nu^2 + v^2)^2}} \, du \, dv$$

$$\leq \int_{k_0}^{+\infty} \|\tilde{\partial} X_k\|_{L^2(v+k_0,\infty)} \left( \int_{k_0+v}^{+\infty} e^{-2\alpha \beta^2\nu t} \frac{1}{\sqrt{1 + (\nu^2 + v^2)^2}} \, du \right)^{1/2} \, dv$$

$$\leq ct^{-1} \int_{k_0}^{+\infty} \exp\left\{ -\frac{12(v+k_0)v}{25k_0^2} \right\} \, dv$$

$$\leq ct^{-1} \int_{k_0}^{+\infty} v^{-1} \exp\{-\alpha \beta^2 t \frac{12v^2}{25k_0^2}\} \, dv$$

$$\lesssim t^{-3/2}.$$  \hfill (10.30)

The bound for $I_2$ follows in the same manner as for $I_1$. For $I_3$ we proceed as with (10.17) applying Hölder’s inequality with $2 < p < 4$

$$I_3 = \int_0^{+\infty} \int_{k_0+v}^{+\infty} (u-k_0^2 + v^2)^{-1/4} e^{-\alpha \beta^2\nu t} \frac{1}{\sqrt{1 + (\nu^2 + v^2)^2}} \, du \, dv$$

$$\leq \int_0^{+\infty} \| (u-k_0^2 + v^2)^{-1/4} \|_{L^p} \left( \int_{k_0+v}^{+\infty} \exp\left( q\alpha \beta^2 (v+k_0)vt \frac{1}{k_0^2} - \frac{1}{((v+k_0)^2 + v^2)^2}) \right)^{1/q} \, dv \right)^{1/p}$$

$$\leq ct^{-1/q} \int_{k_0}^{+\infty} v^{1/p-1/2} \exp\left( q\alpha \beta^2 (v+k_0)vt \frac{1}{k_0^2} - \frac{1}{((v+k_0)^2 + v^2)^2}) \right)^{1/q} \, dv$$

$$\leq ct^{-1/q} \int_{k_0}^{+\infty} v^{2/p-3/2} \exp\{-\alpha \beta^2 t \frac{24v^2}{25k_0^2}\} \, dv \lesssim t^{-3/4}.$$  \hfill (10.31)

where we have used the substitution $w = t^{1/2}v$ and the fact that $-1 < \frac{3}{p} - \frac{3}{2} < -\frac{1}{2}$. So we have

$$I_3 \lesssim t^{-3/4}. \hfill (10.32)$$

\[11 \text{ Soliton resolution for the FL equation}\]

Now we begin to construct the long time asymptotics of the SP equation (1.1). Inverting the sequence of transformations (4.14), (5.9), (6.7) and (6.11), we have

$$M(k) = M^{(3)}(k)E(k)M^{out}(k)R^{(2)}(k)^{-1}T(k)^{\alpha_3}, \quad k \in C \setminus U_{\pm k_0} \hfill (11.1)$$

where $T(k)^{\alpha_3}$ is a diagonal matrix.
To reconstruct the solution \( u_x(x,t) \), we take \( k \to \infty \) along the straight line \( k_0 + R_+i \). Then we have that eventually \( k \in D_2 \), which means \( R^{(2)}(k) = I \). From (11.12), (7.45), (9.10) and (9.14), we have

\[
M = (I + \frac{M^{(3)}_1}{k} + \ldots)(I + \frac{E_1}{k} + \ldots)(I + \frac{M^{(1)}_{out}}{k} + \ldots)(I + \frac{T^{(1)}_{out}}{k} + \ldots), \tag{11.2}
\]

which means the coefficient of the \( k^{-1} \) in the Laurent expansion of \( M \) is

\[
M_1 = M^{(3)}_1 + E_1 + M^{(1)}_{out} + T^{(1)}_{out}, \tag{11.3}
\]

We construct the solution \( u_x(x,t) \) of (11.1) with initial data \( u_0 \) by the transformation and final result is as follows:

**Theorem 1.** Let \( u(x,t) \) be the solution for the initial-value problem (1.1)-(1.2) with generic data \( u_0(x) \in H^{1,1}({\mathbb{R}}) \). For fixed \( x_1, x_2, v_1, v_2 \in {\mathbb{R}} \) with \( x_1 \leq x_2 \) and \( v_1 \leq v_2 \in {\mathbb{R}}^- \), we define two zones for spectral variable \( k \)

\[
I = \{ k : (\frac{\alpha \beta^2}{-4v_1})^{1/2} < |k|^2 < (\frac{\alpha \beta^2}{-4v_2})^{1/2} \}, \quad N(I) = \{ k_j \in {\mathbb{K}} : k_j \in I \} \tag{11.4}
\]

and a cone for variables \( x,t \)

\[
C(x_1, x_2, v_1, v_2) = \{ (x, t) \in R^2 | x = x_0 + vt, \text{ with } x_0 \in [x_1, x_2], v \in [v_1, v_2] \} \tag{11.5}
\]

which are shown in Figure 3. Denote \( u_{sol}(x,t \mid D(I)) \) be the \( N(I) \) soliton solution corresponding to scattering data \( \{ k_j, c_j(I) \}_{j=1}^{N(I)} \) which given in (7.20). Then as \( |t| \to \infty \) with \( (x,t) \in C(x_1, x_2, v_1, v_2) \), from (8.10), (7.45), (9.18) and (10.25) we have

\[
m(x,t) = m_{sol}(x,t; D(I)) + t^{-1/2} f(x,t) + O(t^{-3/4}), \tag{11.6}
\]

Thus

\[
|m(x,t)|^2 = |(m_{sol}(x,t; D(I)) + t^{-1/2} f(x,t) + O(t^{-1}))|^2
\]

\[
= |m_{sol}(x,t; D(I))|^2 + 2t^{-1/2} f(x,t)m_{sol}(x,t; D(I)) + O(t^{-3/4}). \tag{11.7}
\]

Based on the above discussion we can obtain

\[
u_x(x,t) = (m_{sol}(x,t; D(I)) + t^{-1/2} f(x,t) + O(t^{-1}))e^{-i \int_{-\infty}^{t} |m|^2 dx'} \tag{11.8}
\]

where \( f(x,t) \) has been established in (7.19).
A Steepest descent analysis for large negative times

The steps in the steepest descent analysis of RHP1 for \( t \to -\infty \) mirror those presented in Sections 3-10 for \( t \to \infty \). The differences that appear can be traced back to the fact that the regions of growth and decay of the exponential factors \( e^{2it\theta} \) are reversed when one considers \( t \to -\infty \), see Fig. 3.1. Here we briefly sketch those changes, leaving the detailed calculations to the interested reader.

![Figure 9](image.png)

Figure 9: In the yellow region, \( |e^{2it\theta}| \to \infty \) when \( t \to -\infty \) respectively. And in white region, \( |e^{2it\theta}| \to 0 \) when \( t \to -\infty \) respectively.

The first step in the analysis, as in Section 3, is a conjugation to well-condition the problem for large-time analysis. Similar to (4.19) define

\[
M^{(1)}(k) = M(k)T(k)^{-\sigma_3} \tag{A.1}
\]

\( T(k) \) can be defined in proposition 1 with \( t < 0 \). Next, non-analytic extensions of the jump matrices (4.21) are introduced to deform jump matrices onto contours along which they decay to identity as was done in Section 4. The contours and domains are shown in Fig.9.
Accordingly, the definition of $R_j$ is also a little different from $t > 0$, namely we exchange the definition of $R_j$ on the two sides of steady-state phase points on the real and imaginary axis. Once the functions are constructed, the transformation

$$M^{(2)}(k) = M^{(1)}(k)R(k),$$

where $R$ is defined in each sector in Fig. 10, defines a new unknown $M^{(2)}$ which satisfies

**RH problem A.1.** Find a function $M^{(2)}: \mathbb{C} \backslash \Sigma^{(2)} \to SL_2(\mathbb{C})$ with the following properties.

(a) $M^{(2)}(x, t, k)$ is continuous in $\mathbb{C}$, sectionally continuous first partial derivatives in $\mathbb{C} \backslash (\Sigma^{(2)} \cup K \cup \overline{K})$ and meromorphic in $D_2 \cup D_5 \cup D_{16} \cup D_{19}$,

(b) Symmetry: $M^{(2)}(\overline{k}) = \sigma_2 M^{(2)}(k)\sigma_2$,

(c) The boundary value $M^{(2)}(x, t, k)$ at $\Sigma^{(2)}$ satisfies the jump condition

$$M^{(2)}_+(x, t, k) = M^{(2)}_-(x, t, k)V^{(2)}(x, t, k), \quad k \in \Sigma^{(2)},$$

(d) Asymptotic condition

$$M^{(2)}(x, t, k) = I + O(k^{-1}), \quad \text{as} \quad k \to \infty.$$  

(e) Away from $\Sigma^{(2)}$ we have

$$\overline{\partial}M^{(2)} = M^{(2)}\overline{\partial}R^{(2)},$$

Figure 10: In the yellow region, $R^{(2)} = I$, in white region, $R^{(2)} \neq I$.  

46
holds in \( \mathbb{C} \setminus \Sigma^{(2)} \), where

\[
\mathcal{R}^{(2)} = \begin{cases}
1 & k \in D_j, j = 1, 3, 9, 10, 11, 12, 15, 17, \\
-1 & k \in D_j, j = 4, 6, 7, 8, 13, 14, 18, 20, \\
0 & k \in D_j, j = 2, 5, 16, 19.
\end{cases}
\] (A.6)

(f) Residue conditions: \( M^{(2)} \) has simple poles at each point in \( \mathcal{K} \cup \mathcal{K} \) with:

\[
\text{Res } M^{(2)} = \begin{cases}
\lim_{k \to \pm k_j} M^{(2)}(0) & k \in \Delta^-_{k_0}, \\
\lim_{k \to \pm k_j} M^{(2)}(\pm c_j T(\pm k_j) - 2 e^{2i t(\pm k_j)}) & k \in \Delta^+_{k_0}.
\end{cases}

\] (A.7)

\[
\text{Res } M^{(2)} = \begin{cases}
\lim_{k \to \pm k_j} M^{(2)}(0) & k \in \Delta^-_{k_0}, \\
\lim_{k \to \pm k_j} M^{(2)}(\pm c_j T(\pm k_j) - 2 e^{2i t(\pm k_j)}) & k \in \Delta^+_{k_0}.
\end{cases}

\] (A.8)

The final steps of the analysis, mimicking Sections 5–10 are to first construct a solution \( M^{RHP} \) of the Riemann-Hilbert components of RHP A.1, and then to use the solid Cauchy integral operator to prove that the remainder \( M^{(3)} = M^{(2)} M^{RHP}^{-1} \) is uniformly near identity with estimates identical \([10, 25]\). When \( t \to -\infty \) with \( (x, t) \in C(x_1, x_2, v_1, v_2) \) the outer model takes the form

\[
M^{(out)}(k; x, t) = \left[ I + O \left( e^{-2 \mu |t|} \right) \right] m_{\Delta_{k_0}^+}(k; x, t | \sigma_0^- (I))
\] (A.9)

The local model \( M^{(2)} \) is constructed as in Section 8. Define

\[
N_A : f(k) \to (N_A f)(k) = f \left( \frac{k_0^2}{\sqrt{\alpha \beta}} \right)
\] (A.10)

Then the local model \( M^{(E)} \) is given by

\[
M^{(0)}(k) = M^{out}(k)M^{FL}(\sigma_3 M^{FL}(\zeta) | \sigma_3)
\]

where \( M^{FL}(\zeta, r) \) is the solution of RHP10. The residual error \( E(k) \) now satisfies RHP 5.2 but with (5.18) now given by

\[
V^{(E)}(k) = \begin{cases}
M^{(out)}(k) V^{(2)}(k) M^{(out)}(k)^{-1}, & k \in \Sigma^{(2)} \setminus U_{\pm k_0} \cup U_{\pm k_0}, \\
M^{(out)}(k) \sigma_3 M^{FL}(\zeta) | \sigma_3 M^{(out)}(k)^{-1}, & k \in \partial U_{\pm k_0} \cup \partial U_{\pm k_0}.
\end{cases}

\] (A.11)
small-norm theory again can be used to show that \(E\) exists and satisfies
\[
E(k) = I + k^{-1}E_1 + \mathcal{O}(k^{-2})
\]
\[
E_1 = \frac{k^2}{2i\beta \sqrt{-\alpha t}} [M^{(out)}(k_0) \begin{pmatrix} 0 & \beta_{12}(r_{k_0}) \\ -\beta_{21}(r_{k_0}) & 0 \end{pmatrix} M^{(out)}(k_0)^{-1}]
\]
\[
+ M^{(out)}(-k_0) \begin{pmatrix} 0 & \beta_{12}(r_{-k_0}) \\ -\beta_{21}(r_{-k_0}) & 0 \end{pmatrix} M^{(out)}(-k_0)^{-1}
\]
\[
- M^{(out)}(ik_0) \begin{pmatrix} 0 & \beta_{12}(r_{ik_0}) \\ -\beta_{21}(r_{ik_0}) & 0 \end{pmatrix} M^{(out)}(ik_0)^{-1}
\]
\[
- M^{(out)}(-ik_0) \begin{pmatrix} 0 & \beta_{12}(r_{-ik_0}) \\ -\beta_{21}(r_{-ik_0}) & 0 \end{pmatrix} M^{(out)}(-ik_0)^{-1} + \mathcal{O}(t^{-1}).
\] (A.12)

The rest of the results can be given in the same way which is used in section 11.

**B The parabolic cylinder model problem**

**RH problem 11.** Define
\[
r_0 \in \mathbb{R}, \quad \Sigma^{PC} = \bigcup_{j=1}^4 \Sigma^{(j)}, \quad \Sigma^{(j)} = \{\zeta = \mathbb{R}^+ e^{\frac{(2j-1)i\pi}{4}}, j = 1, 2, 3, 4\}.
\]

Find a 2 \(\times\) 2 matrix-valued function \(M^{PC}(\zeta; r_0)\) with the following properties:

(a) \(M^{PC}(\zeta; r_0)\) is analytic for \(\mathbb{C}\setminus\Sigma^{PC}\)

(b) The boundary value \(M^{PC}(\zeta; r_0)\) at \(\Sigma^{PC}\) satisfies the jump condition
\[
M_+^{PC}(\zeta; r_0) = M_-^{PC}(\zeta; r_0) V^{PC}(\zeta), \quad \zeta \in \Sigma^{PC},
\] (B.1)

where
\[
V^{PC}(\zeta) = \begin{cases} 
\begin{pmatrix} 1 & 0 \\
\frac{r_0\zeta^{-2i\nu}e^{\frac{i\pi}{2}}}{1 + |r_0|^2} & 1 
\end{pmatrix}, & \zeta \in \Sigma^{(1)}, \\
\begin{pmatrix} 1 & 0 \\
\frac{r_0\zeta^{-2i\nu}e^{\frac{i\pi}{2}}}{1 + |r_0|^2} & 1 
\end{pmatrix}, & \zeta \in \Sigma^{(2)}, \\
\begin{pmatrix} 1 & 0 \\
\frac{r_0\zeta^{-2i\nu}e^{\frac{i\pi}{2}}}{1 + |r_0|^2} & 1 
\end{pmatrix}, & \zeta \in \Sigma^{(3)}, \\
\begin{pmatrix} 1 & 0 \\
\frac{r_0\zeta^{-2i\nu}e^{\frac{i\pi}{2}}}{1 + |r_0|^2} & 1 
\end{pmatrix}, & \zeta \in \Sigma^{(4)}, 
\end{cases}
\] (B.2)

and
\[
\nu = \nu(r_0) = -\frac{1}{2\pi} \ln(1 + |r_0|^2).
\] (B.3)
Using the well known asymptotic behavior of $D_a(k)$, we obtain the asymptotic behavior of the solution

$$M^{(pC)}(\zeta,r_0) = I + \frac{1}{\zeta} \begin{pmatrix} 0 & -i\beta_{12}(r_0) \\ i\beta_{21}(r_0) & 0 \end{pmatrix} + O(\zeta^{-2}),$$

(B.4)

and $\beta_{12}$ and $\beta_{21}$ are the complex constants

$$\beta_{12} = \beta_{12}(r_0) = \frac{\sqrt{2\pi e^{i\pi/4} e^{-\pi\nu/2}}}{r_0 \Gamma(-i\nu)}, \quad \beta_{21} = \beta_{21}(r_0) = -\frac{\sqrt{2\pi e^{-i\pi/4} e^{-\pi\nu/2}}}{r_0 \Gamma(i\kappa)} = \frac{\nu}{\beta_{12}}$$

(B.5)

References

[1] A. S. Fokas, On a class of physically important integrable equations, Physica D, 87(1995) 145-150.

[2] J. Lenells, Exactly solvable model for nonlinear pulse propagation in optical fibers, Studies in Applied Mathematics, 123 (2009), 215-232.

[3] A. Kundu, Two-fold integrable hierarchy of nonholonomic deformation of the derivative nonlinear Schrödinger and the Lenells"CFokas equation, Journal of Mathematical Physics, 51 (2010), 022901 (18pp).

[4] A. Kundu, Integrable twofold hierarchy of perturbed equations and application to optical soliton dynamics, Theoretical and Mathematical Physics, 167 (2011), 800-810.

[5] J. Lenells, A. S. Fokas, On a novel integrable generalization of the nonlinear Schrödinger equation, Nonlinearity., 22(2009) 11-27.

[6] J. Lenells, Dressing for a novel integrable generalization of the nonlinear Schrödinger Equation, J.Nonline. Sci., 20(2010) 709-722.

[7] Y. Matsuno, A direct method of solution for the Fokas-Lenells derivative nonlinear Schrödinger equation: I. Bright soliton solutions, J. Phys. A: Math. Theor., 45(2011) 235202.

[8] V. E. Vekslerchik, Lattice representation and dark solitons of the Fokas-Lenells equation, Nonlinearity, 24(2011) 1165-1175.

[9] Y. Matsuno, A direct method of solution for the Fokas-Lenells derivative nonlinear Schrödinger equation: II. Dark soliton solutions, J. Phys. A: Math. Theor., 45(2012) 475202.
[10] O.C. Wright, Some homoclinic connections of a novel integrable generalized nonlinear Schrödinger equation, *Nonlinearity*, 22(2009) 2633-2643.

[11] A. S. Fokas, A.A. Himonas, Well-posedness of an integrable generalization of the non-linear Schrödinger equation on the circle, Letters in Mathematical Physics, 96 (2011), 169-189.

[12] K. T. R. McLaughlin, P. D. Miller, The $\bar{\partial}$ steepest descent method and the asymptotic behavior of polynomials orthogonal on the unit circle with fixed and exponentially varying non-analytic weights, *Int. Math. Res. Not.*, (2006), Art. ID 48673.

[13] K. T. R. McLaughlin, P. D. Miller, The $\bar{\partial}$ steepest descent method for orthogonal polynomials on the real line with varying weights, *Int. Math. Res. Not.*, (2008), Art. ID 075.

[14] M. Dieng, K. D. T. McLaughlin, Dispersive asymptotics for linear and integrable equations by the Dbar steepest descent method, Nonlinear dispersive partial differential equations and inverse scattering, 253-291, Fields Inst. Commun., 83, Springer, New York, 2019

[15] S. Cuccagna, R. Jenkins, On asymptotic stability of N-solitons of the defocusing non-linear Schrödinger equation, *Comm. Math. Phys.*, 343(2016), 921-969.

[16] M. Borghese, R. Jenkins, K.D.T-R McLaughlin, Long time asymptotics behavior of the focusing nonlinear Schrödinger equation, *Ann. I. H. Poincaré*, AN 35(2018), 887-920.

[17] V. E. Zakharov, A. B. Shabat, Exact theory of two-dimensional self-focusing and one-dimensional selfmodulation of waves in nonlinear media, *Sov. Phys. JETP*, 34(1972), 62-69.

[18] R. Jenkins, J. Liu, P. Perry, C. Sulem, Global well-posedness for the derivative nonlinear Schrödinger equation, *Math. Phys.*, 363(2018), 1003-1049.

[19] J. Xu, E. G. Fan, Long-time asymptotics for the Fokas-Lenells equation with decaying initial value problem: Without solitons, J. Differential Equations, 259(2015), 1098-1148.

[20] P. Deift, X. Zhou, A steepest descent method for oscillatory Riemann-Hilbert problems, Ann. of Math, 137 (1993), 295-368.
[21] A. Boutet de Monvel, A. Kostenko, D. Shepelsky, G. Teschl, Long-time asymptotics for the Camassa-Holm equation, SIAM J. Math. Anal., 41 (2009), 1559-1588.

[22] K. Grunert, G. Teschl, Long-time asymptotics for the Korteweg-de Vries equation via nonlinear steepest descent, Math. Phys. Anal. Geom, 12 (2009), 287-324.

[23] J. Lenells, A.S. Fokas, On a novel integrable generalization of the nonlinear Schrödinger equation, Nonlinearity, 22 (2009), 11-27.

[24] J. Lenells, A.S. Fokas, An integrable generalization of the nonlinear Schrödinger equation on the half-line and solitons, Inverse Probl, 25 (2009), 115006, 32 pp.

[25] J. Lenells, A.S. Fokas, Dressing for a novel integrable generalization of the nonlinear Schrödinger equation, J. Nonlinear Sci, 20 (2010), 709-722.

[26] O. C. Wright, Some homoclinic connections of a novel integrable generalized nonlinear Schrödinger equation, Nonlinearity, 22 (2009), 2633-2643.

[27] V.E. Vekslerchik, Lattice representation and dark solitons of the Fokas-Lenells equation, Nonlinearity, 24 (2011), 1165-1175.

[28] Y. Matsuno, A direct method of solution for the Fokas-Lenells derivative nonlinear Schrödinger equation: I. Bright soliton solutions, Journal of Mathematical Physics, A 45 (2012), 235202, 19 pp.

[29] Y. Matsuno, A direct method of solution for the Fokas-Lenells derivative nonlinear Schrödinger equation: II. Dark soliton solutions, Journal of Mathematical Physics, A 45 (2012), 475202, 31 pp.

[30] P.A. Deift, A.R. Its, X. Zhou, Long-time asymptotics for integrable nonlinear wave equations, in: Important Developments in Soliton Theory, in: Springer Ser. Nonlin. Dyn., Springer, Berlin, 1993, pp. 181-204.

[31] R. Beals, R. Coifman, Scattering and inverse scattering for first order systems, Comm. Pure Appl. Math, 37 (1984), 39-90.

[32] A.V. Kitaev, A.H. Vartanian, Asymptotics of solutions to the modified nonlinear Schrödinger equation: solution on a nonvanishing continuous background, SIAM J. Math. Anal, 30 (1999), 787-832.
[33] A.V. Kitaev, A.H. Vartanian, Higher order asymptotics of the modified non-linear Schrödinger equation, Comm. Partial Differential Equations, 25 (2000), 1043-1098.

[34] P. Deift, X. Zhou, Perturbation theory for infinite-dimensional integrable systems on the line. A case study, Acta Math., 188(2)(2002), 163-262.

[35] P. Deift, X. Zhou, Long-time asymptotics for solutions of the NLSequation with initial data in aweighted Sobolev space, Comm. Pure and Appl. Math., LVI(2003), 1029-1077.

[36] X. Zhou, P. Deift, Long-time behavior of the non-focusing nonlinear Schrödinger equation-A case study, Lectures in Mathematical Sciences, Graduate School of Mathematical Sciences, University of Tokyo, 1994.

[37] M. Borghese, R. Jenkins, K. D.T.-R. McLaughlin, Long time asymptotic behavior of the focusing nonlinear Schrödinger equation, Ann. I. H. Poincaré ´C AN., 35 (2018) 887-920.

[38] J. He, S. Xu, K. Porsezian, Rogue waves of the Fokas-Lenells equation, J. Phys. Soc. Japan, 81 (2012) 124007 (4pp).

[39] S. Xu, J. He, Y. Cheng, K. Porsezian, The n-order rogue waves of Fokas-Lenells equation, Math. Methods Appl. Sci., 38(2015) 1106-1126.

[40] J. Liu, P. A. Perry, C. Sulem, Long-time behavior of solutions to the derivative nonlinear Schrödinger equation for soliton-free initial data, Ann. I. H. Poincaré , AN35(2018), 217-265.