Sharp Estimate of Fifth Coefficient for Ma Minda Starlike and Convex Functions

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Abstract

In the present investigation, we find the sharp bound of fifth coefficient of analytic normalized function \( f \) satisfying \( \frac{zf'(z)}{f(z)} \prec \varphi(z) \) when coefficients of \( \varphi \) satisfy certain conditions. For an appropriate choice of \( \varphi \), the already known estimates for various other subclasses of starlike functions follow directly from the obtained result.

Keywords: Starlike functions; Convex Functions; Fifth Coefficient.

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1 Introduction

Let \( A \) be the class of analytic functions of the form \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \), in the unit disk \( D = \{ z \in \mathbb{C} : |z| < 1 \} \). Suppose \( \mathbb{B} = \{ z \in \mathbb{C} : |z| \leq 1 \} \) and \( T = \{ z \in \mathbb{C} : |z| = 1 \} \). Further, we denote the subclass of \( A \) containing univalent functions by \( S \). In 1916, Bieberbach conjectured that \( |a_n| \leq n \) for \( f \in S \), which was settled in 1985 by L. D. Branges (see [19, Page 5]). During this time period, the conjecture was verified for various other subclasses of \( S \). Since the growth, covering and distortion theorems for functions \( f \in S \) can be proved using the fact \( |a_2| \leq 2 \), so the coefficient bound plays a major role in identifying the geometric nature of the function.

Undoubtedly, the primary and extensively studied subclasses of \( S \) are the classes of starlike and convex functions, respectively denoted by \( S^* \) and \( C \). Analytically, a function \( f \in S^* \) if and only if \( \text{Re}(zf'(z))/f(z)) > 0 \) and \( f \in C \) if and only if \( \text{Re}(1 + (zf''(z))/f'(z)) > 0 \) for \( z \in D \).

An analytic function \( f \) is subordinate to another analytic function \( g \), if there exists a Schwarz function \( \omega \) such that \( f(z) = g(\omega(z)) \) for all \( z \in D \) simply denoted by \( f \prec g \). By \( B_0 \), we represent the class of Schwarz functions having the form

\[
\omega(z) = \sum_{n=1}^{\infty} c_n z^n.
\]  (1)

Using the concept of subordination, Ma and Minda [12] introduced the classes

\[
S^*(\varphi) = \left\{ f \in A : \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\}
\]

and

\[
C(\varphi) = \left\{ f \in A : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \right\},
\]

where \( \varphi \) is an analytic univalent function in \( \mathbb{D} \) satisfying (i) \( \text{Re} \varphi(z) > 0 \) for \( z \in \mathbb{D} \), (ii) \( \varphi'(0) > 0 \), (iii) \( \varphi(\mathbb{D}) \) is a starlike domain with respect to \( \varphi(0) = 1 \) and (iv) \( \varphi(\mathbb{D}) \) is symmetric about the real axis. Suppose \( \varphi \) has the following series expansion

\[
\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots , \quad B_1 > 0.
\]  (2)
Since \( \varphi(\mathbb{D}) \) is symmetric with respect to the real axis and \( \varphi(0) = 1 \), we have \( \varphi(\bar{z}) = \varphi(z) \), which yields that all \( B_i \)'s are real. For the family \( S^*(\varphi) \) and \( C(\varphi) \) sharp growth, distortion theorems and estimates for the coefficient functional \( |a_3 - ra_2^2| \) are known, where \( r \in \mathbb{R} \) \([12]\). In fact, these classes unify several subfamilies of starlike and convex functions. For instance, if \( \varphi(z) = (1 + (1 - 2\alpha)z)/(1 - z) \), the classes \( S^*(\varphi) \) and \( C(\varphi) \) reduce to the classes of starlike functions of order \( \alpha \) and convex functions of order \( \alpha \), denoted by \( S^*(\alpha) \) and \( C(\alpha) \) respectively \((0 \leq \alpha < 1)\) \([8]\). In case of \( \alpha = 0 \), we simply obtain \( S^*(0) := S^* \) and \( C(0) := C \). If we define \( \varphi(z) = \sqrt{1 + z} \), then the class \( S^*(\varphi) \) coincides with the class \( S^*_1 \) introduced by Sokół and Stankiewicz \([18]\). Geometrically, a function \( f \in S^*_1 \) if and only if \( zf'(z)/f(z) \) lies in the region bounded by the right lemniscate of Bernoulli given by \(|w^2 - 1| < 1\). Therefore \( S^*_L = \{ f \in A : |(zf'(z)/f(z))| < 1 \} \). Mendiratta et al. \([13]\) considered the class \( S^*_{RL} \) of functions \( f \) such that the quantity \( zf'(z)/f(z) \) lies in the interior of the left half of the shifted lemniscate of Bernoulli given by \( \Omega_{RL} = \{ w \in \mathbb{C} : \text{Re } w > 0, |(w - \sqrt{2})^2 - 1| < 1 \} \). Note that the function \( \varphi(z) = \sqrt{2} - (\sqrt{2} - 1)\sqrt{(1 - z)/(1 + 2(\sqrt{2} - 1)z)} \) maps the unit disk onto \( \Omega_{RL} \). Thus

\[
S^*_{RL} = \left\{ f \in A : \frac{zf'(z)}{f(z)} < \sqrt{2} - (\sqrt{2} - 1)\sqrt{\frac{1 - z}{1 + 2(\sqrt{2} - 1)z}} \right\}.
\]

Using this approach, various interesting subclasses of starlike functions by confining the values of \( zf'(z)/f(z) \) to a defined region within the right half-plane were introduced and studied. Some of them are listed in table 1 along with their respective class notations.

| Class      | \( \varphi(z) \) | Reference |
|------------|------------------|-----------|
| \( S^*_{\sin} \) | \( 1 + \sin z \) | \([6]\) Cho et al. |
| \( S^*_{SG} \) | \( 2/(1 + e^{-z}) \) | \([7]\) Goel and Kumar |
| \( S^*_{\phi} \) | \( 1 + ze^z \) | \([9]\) Kumar and Gangania |
| \( S^*_{rb} \) | \( \sqrt{1 + b\bar{z}}, \ b \in (0, 1) \) | \([17]\) Sokół |

Note that the sharp bounds for the initial coefficients \( |a_2|, |a_3| \) and \( |a_4| \) of functions in \( S^*(\varphi) \) for a general choice of \( \varphi \) have been established \((see [2, 10])\), however, for \( |a_5| \), it is still open. Sokół \([16]\) conjectured that \( |a_n| \leq 1/(2(n - 1)) \) for \( f \in S^*_1 \) and Mendiratta et al. \([13]\) conjectured that \( |a_n| \leq (5 - 3\sqrt{2})/(2(n - 1)) \) for \( f \in S^*_{RL} \). Ravichandran and Verma \([15]\) settled these conjectures for \( n = 5 \). For different subclasses of \( S^*(\varphi) \) depending on the different choices of \( \varphi(z) \) in \( S^*(\varphi) \), the bound for \( |a_3| \) is known \([5, 7, 9]\). For \( n \geq 5 \), finding the bound of \( |a_n| \) for functions belonging to the classes \( S^*(\varphi) \) and \( C(\varphi) \) in case of general \( \varphi \) is still an open problem \([1]\). In this study, we obtain sharp bound of fifth coefficient for functions in \( S^*(\varphi) \) and \( C(\varphi) \) under specific conditions on the coefficients of \( \varphi \). The obtained results give several new special cases and some already known results as special cases.

We require the following lemmas to prove our results. Let \( P \) be the Carathéodory class containing analytic functions of the form

\[
p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \quad z \in \mathbb{D}
\]  

(3)
with \( \text{Re}\, p(z) > 0 \). Clearly, the function

\[
L(z) = \frac{1 + z}{1 - z}, \quad z \in \mathbb{D}
\]

(4)
is a member of \( \mathcal{P} \) as it maps the unit disk onto the right half-plane.

**Lemma 1.** [14, Lemma I] If the functions \( 1 + \sum_{n=1}^{\infty} b_n z^n \) and \( 1 + \sum_{n=1}^{\infty} c_n z^n \) are in \( \mathcal{P} \), then the same holds for the function

\[
1 + \frac{1}{2} \sum_{n=1}^{\infty} b_n c_n z^n.
\]

**Lemma 2.** [14, Lemma II] Let \( h(z) = 1 + u_1 z + u_2 z^2 + \cdots \) and \( 1 + G(z) = 1 + d_1 z + d_2 z^2 + \cdots \) be functions in \( \mathcal{P} \), and set

\[
\gamma_n = \frac{1}{2^n} \left[ 1 + \frac{1}{2} \sum_{k=1}^{n} \binom{n}{k} u_k \right], \quad \gamma_0 = 1.
\]

If \( A_n \) is defined by

\[
\sum_{n=1}^{\infty} (-1)^{n+1} \gamma_{n-1} G^n(z) = \sum_{n=1}^{\infty} A_n z^n,
\]

then \( |A_n| \leq 2 \).

It is worth recalling the Möbius function \( \Psi_\zeta \) which maps the unit disk onto the unit disk and given by

\[
\Psi_\zeta(z) = \frac{z - \zeta}{1 - \zeta z}, \quad \zeta \in \mathbb{D}.
\]

(5)

**Lemma 3.** [4, Lemma 2.4] If \( p \in \mathcal{P} \), then for some \( \zeta_i \in \overline{\mathbb{D}}, i \in \{1, 2, 3\}, \)

\[
p_1 = 2\zeta_1,
\]

(6)

\[
p_2 = 2\zeta_1^2 + 2(1 - |\zeta_1|^2)\zeta_2,
\]

(7)

\[
p_3 = 2\zeta_1^3 + 4(1 - |\zeta_1|^2)\zeta_1\zeta_2 - 2(1 - |\zeta_1|^2)\bar{\zeta}_1 \zeta_2 + 2(1 - |\zeta_1|^2)(1 - |\zeta_2|^2)\zeta_3.
\]

(8)

For \( \zeta_1, \zeta_2 \in \mathbb{D} \) and \( \zeta_3 \in \mathbb{T} \), there is a unique function \( p = L \circ \omega \in \mathcal{P} \) with \( p_1, p_2 \) and \( p_3 \) as in (6)-(8), where

\[
\omega(z) = z \Psi_{-\zeta_1}(z \Psi_{-\zeta_2}(\zeta_3 z)), \quad z \in \mathbb{D},
\]

(9)
that is

\[
p(z) = \frac{1 + (\zeta_2 \zeta_3 + \zeta_1 \zeta_3 - \zeta_1 z + (\zeta_1 \zeta_3 + \zeta_1 \bar{\zeta}_2 \zeta_3 + \bar{\zeta}_2) z^2 + \zeta_3 z^3)}{1 + (\zeta_2 \zeta_3 + \zeta_1 \zeta_3 - \zeta_1 z + (\zeta_1 \zeta_3 - \zeta_1 \bar{\zeta}_2 \zeta_3 - \zeta_2) z^2 - \zeta_3 z^3)} z \in \mathbb{D}.
\]

Conversely, if \( \zeta_1, \zeta_2 \in \mathbb{D} \) and \( \zeta_3 \in \overline{\mathbb{D}} \) are given, then we can construct a (unique) function \( p \in \mathcal{P} \) of the form (3) so that \( p_i, i \in \{1, 2, 3\}, \) satisfy the identities in (6)-(8). For this, we define

\[
\omega(z) = \omega_{\zeta_1, \zeta_2, \zeta_3}(z) = z \Psi_{-\zeta_1}(z \Psi_{-\zeta_2}(\zeta_3 z)), \quad z \in \mathbb{D},
\]

(10)
where \( \Psi_\zeta \) is the function given as in (5). Then \( \omega \in B_0 \). Moreover, if we define \( p(z) = (1 + \omega(z))/(1 - \omega(z)), z \in \mathbb{D} \), then \( p \) is represented by (3), where \( p_1, p_2 \) and \( p_3 \) satisfy the identities in (6)-(8) (see the proof of [4, Lemma 2.4]).
2 Estimation of the Fifth Coefficient

We begin with the following lemma:

**Lemma 4.** If $-1 < \sigma < 1$, then $F(z) = (1 + 2\sigma z + z^2)/(1 - z^2)$ belongs to $\mathcal{P}$.

**Proof.** Let us consider

$$\omega(z) = \frac{F(z) - 1}{F(z) + 1} = \frac{z(\sigma + 1)}{1 + \sigma z}.$$

From (5), we have

$$\omega(z) = z\Psi_{-\sigma}(z), \quad z \in \mathbb{D}.$$

Since $\Psi_{-\sigma}(z)$ is a conformal automorphism of $\mathbb{D}$, which gives $|w(z)| < 1$ and $w(0) = 0$. Therefore $w$ is a Schwarz function and $F \in \mathcal{P}$. \hfill \blackslug

To prove our next result, we require the following assumption:

**Assumption.** Let $\varphi(z)$ be as defined in (2), whose coefficients satisfy the following conditions:

- **C1:** $|B_1^2 + 2B_2| < 4B_1^2$.
- **C2:** $|B_1^2 - B_1B_2 + 18B_2^2 - 18B_1B_3| < 3|(B_1^2 + 2B_1 + 2B_2)(2B_2^2 - 3B_1 + 3B_2)|$.
- **C3:**
  $|30B_1^2 - 9B_1^6 - B_1^6(66B_2 - 5) - 648B_2^2 + 324B_4^2 + B_1^6(170B_2 - 126) - 648B_2B_3^2 + B_1^6(-180B_2 + 220B_2^2 + 108B_3 - 360B_3B_2) + B_1(1296B_2B_3 - 720B_2B_3 + 648B_2^2B_4 + B_1^6(108 + 10B_2 - 175B_2^2 + 90B_3 + 162B_4) + B_1^6(-144B_2^2 + 4B_4^2 + 180B_2B_3 - 324B_3^2 - 648B_4 + 648B_4B_2)| < 8|B_1^6 + 9B_1^7 + B_1^7(-27 + 32B_2) + B_1^5(-52 + 63B_2) + 162B_2B_3 + B_1^5(81 - 189B_2 + 164B_2^2 + 9B_3) + B_1^5(18B_2^2 - 9B_2B_3) + B_1(-162B_2^2 + 198B_3^2 - 81B_2^3)|$.
- **C4:**
  $0 < \left(AB_2^2 + 6(B_2 - B_1)/((3B_1^2 + 6(B_2 - B_1)) < 1$.

**Theorem 5.** Let $\varphi(z)$ be as defined in (2), whose coefficients satisfy the above conditions **C1** to **C4.** If $f \in \mathcal{S}^\varphi(\varphi)$, then

$$|a_5| \leq \frac{B_1}{4}.$$

The inequality is sharp.

**Proof.** Suppose $f \in \mathcal{S}^\varphi(\varphi)$, then

$$zf'(z) = \varphi(\omega(z)),$$

where $\omega \in \mathcal{B}_0$. If we choose $\omega(z) = (p(z) - 1)/(p(z) + 1)$, where $p \in \mathcal{P}$ given by (3), then by comparing the coefficients obtained by series expansion of $f(z)$ together with $p(z)$ and $\varphi(z)$ yields

$$a_5 = \frac{B_1}{8}I,$$

where

$$I = p_4 + I_1p_1^4 + I_2p_1^2p_2 + I_3p_1p_3 + I_4p_2^2,$$ (12)
with
\[
I_1 = \frac{1}{48 B_1} \left( B_1^4 - 6 B_1^3 + 11 B_1^2 + 6 B_1^2 B_2 - 6 B_1 + 3 B_2^2 - 22 B_1 B_2 + 18 B_2 - 18 B_3 + 8 B_1 B_3 + 6 B_4 \right),
\]
\[
I_2 = \frac{3 B_1^3 - 11 B_1^2 + 9 B_1 - 18 B_2 + 11 B_1 B_2 + 9 B_3}{12 B_1}, \quad I_3 = \frac{2 B_2^2 - 3 B_1 + 3 B_2}{3 B_1}
\]
and
\[
I_4 = \frac{B_2^2 - 2 B_1 + 2 B_2}{4 B_1}.
\]

Let \( q(z) = 1 + b_1 z + b_2 z^2 + \cdots \) be in \( P \), then by Lemma 1, we have
\[
1 + \frac{1}{2} (p(z) - 1) * (q(z) - 1) = 1 + \frac{1}{2} \sum_{n=1}^{\infty} b_n p_n z^n \in P.
\]

If we assume \( h(z) = 1 + \sum_{n=1}^{\infty} u_n z^n \in P \) and take \( 1 + G(z) := 1 + \frac{1}{2} \sum_{n=1}^{\infty} b_n p_n z^n \), then Lemma 2 gives
\[
|A_4| \leq 2,
\]
where
\[
A_4 = \frac{1}{2} \gamma_0 b_4 p_4 - \frac{1}{4} \gamma_1 b_2^2 p_2 - \frac{1}{2} \gamma_1 b_1 b_3 p_1 p_3 + \frac{3}{8} \gamma_2 b_1^2 b_2 p_1^2 p_2 - \frac{1}{16} \gamma_3 b_1^4 p_1^4
\]
and for \( i \in \{0, 1, 2, 3\} \), \( \gamma_i \)'s are given by \( \gamma_0 = 1 \),
\[
\gamma_1 = \frac{1}{2} \left( 1 + \frac{1}{2} u_1 \right), \quad \gamma_2 = \frac{1}{4} \left( 1 + u_1 + \frac{1}{2} u_2 \right), \quad \gamma_3 = \frac{1}{8} \left( 1 + \frac{3}{2} u_1 + \frac{3}{2} u_2 + \frac{1}{2} u_3 \right).
\]

So from (12) and (16), we can observe that if there exist \( q, h \in P \) such that
\[
b_4 = 2, \quad I_4 = -\frac{1}{128} \left( 1 + \frac{3}{2} u_1 + \frac{3}{2} u_2 + \frac{1}{2} u_3 \right) b_1^4, \quad I_2 = \frac{3}{32} \left( 1 + u_1 + \frac{u_2}{2} \right) b_1^2 b_2,
\]
\[
I_3 = -\frac{1}{4} \left( 1 + \frac{u_1}{2} \right) b_1 b_3 \quad \text{and} \quad I_4 = -\frac{1}{8} \left( 1 + \frac{u_1}{2} \right) b_2^2,
\]
then we have
\[
I = A_4.
\]
The bound for \( |A_4| \) can be obtained from Lemma 2, consequently, we can estimate the bound for \( |I| \) and thus we arrive at the desired bound by using (11). To prove the theorem, we construct the functions \( q \) and \( h \) in such a way that we obtain (18).

From Lemma 3, suppose that the functions \( q \) and \( h \) are constructed by taking \( \zeta_1, \zeta_2 \in \mathbb{D}, \zeta_3 \in \overline{\mathbb{D}} \) and \( \xi_1, \xi_2 \in \mathbb{D}, \xi_3 \in \overline{\mathbb{D}} \) respectively, as follows:
\[
q = L \circ \omega_1 \quad \text{and} \quad h = L \circ \omega_2,
\]
where
\[
\omega_1(z) = z \Psi_{-\zeta_1}(z \Psi_{-\zeta_2}(\zeta_3 z)), \quad \omega_2(z) = z \Psi_{-\xi_1}(z \Psi_{-\xi_2}(\xi_3 z))
\]
and $L(z)$ is given by (4). So again from Lemma 3, the $b_i$'s and $u_i$'s, $i \in \{1, 2, 3\}$ are given by

\begin{align*}
b_1 &= 2\zeta_1, \quad b_2 = 2\zeta_1^3 + 2(1 - |\zeta_1|^2)\zeta_2, \\
b_3 &= 2\zeta_1^3 + 4(1 - |\zeta_1|^2)\zeta_1\zeta_2 - 2(1 - |\zeta_1|^2)|\zeta_1\zeta_2|^2 + 2(1 - |\zeta_1|^2)(1 - |\zeta_2|^2)\zeta_3
\end{align*}

and

\begin{align*}
u_1 &= 2\zeta_1, \quad u_2 = 2\zeta_1^3 + 2(1 - |\zeta_1|^2)\zeta_2, \\
u_3 &= 2\zeta_1^3 + 4(1 - |\zeta_1|^2)\zeta_1\zeta_2 - 2(1 - |\zeta_1|^2)|\zeta_1\zeta_2|^2 + 2(1 - |\zeta_1|^2)(1 - |\zeta_2|^2)\zeta_3.
\end{align*}

There may be many solutions for the above set of equations. For our purpose, we impose some restrictions on the parameters. We take all $\xi_i \in \mathbb{R}$, then

\begin{align*}
u_1 &= 2\xi_1, \quad u_2 = 2\xi_1^3 + 2(1 - \xi_2^2)\xi_2, \\
u_3 &= 2\xi_1^3 + 4(1 - \xi_1^2)\xi_1\xi_2 - 2(1 - \xi_1^2)|\xi_1\xi_2|^2 + 2(1 - \xi_1^2)(1 - \xi_2^2)\xi_3. \quad (21)
\end{align*}

Further, if we define

\begin{align*}
\xi_1 &= -\frac{B_1^2 + 2B_2}{2B_1}, \quad \xi_2 = \frac{B_1^3 - B_1^2B_2 + 18B_2^2 - 18B_1B_3}{3(B_1^2 + B_1 + 2B_2)(2B_1^2 - 3B_1 + 3B_2)}, \\
\xi_3 &= \left(-9B_1^8 + 30B_1^7 - B_1^6(66B_2 - 5) + 2B_1^5(85B_2 - 63) + 4B_1^3(5B_2(11B_2 - 18B_3 - 9) + 27B_3) + 4B_1^2(2B_3 - 36B_2^2 + 81B_3^2 + 45B_2B_3 + 162(B_2 - 1)B_4) - 144B_3^3 (B_2 - 9)B_2B_3 + 324B_2(-2B_3^2 + B_2((B_2 - 2)B_2 + 2B_4)) + 18B_4^2(9B_4 + 5B_3 + 6) - 5B_4^2B_2(35B_2 - 2)\right) \\
&\quad \times \left(8(3B_1^4 + 2B_1^3 + 18B_2^2 + B_1^2(10B_2 - 9) - 9B_1B_3)(B_1(3B_2^2 + B_1 + 11B_2 - 9) + 9B_3)\right),
\end{align*}

then the conditions $\mathbf{C1}$, $\mathbf{C2}$ and $\mathbf{C3}$ on the coefficients $B_1, B_2, B_3$ and $B_4$ yield

\[ |\xi_1| < 1, \quad |\xi_2| < 1, \quad |\xi_3| < 1 \]

respectively. Using these $\xi_i$'s in (21), we can obtain $u_i$'s, which in turn by using (17) gives

\begin{align*}
\gamma_1 &= \frac{1}{4} \left(2 - B_1 - \frac{2B_2}{B_1}\right), \\
\gamma_2 &= \frac{(B_1^2 + 2B_2 - 2B_1)(3B_1^3 - 11B_1^2 + B_1(11B_2 + 9) + 9(B_3 - 2B_2))}{24B_1(2B_1^2 + 3B_2 - 3B_1)}, \\
\gamma_3 &= -\frac{1}{64B_1(2B_1^2 + 3B_2 - 3B_1)} \left(3(B_1^2 + 2B_2 - 2B_1)^2(B_1^4 - 6B_1^3 + B_1^2(6B_2 + 11) + B_1(8B_3 - 22B_2 - 6) + 3(B_2^2 + 6B_2 - 6B_3 + 2B_4)\right). \quad (22)
\end{align*}
Let us consider

\[ q(z) = \frac{1 + 2\sigma z + z^2}{1 - z^2}, \]

with \( \sigma = \sqrt{(4B_1^2 + 6(B_2 - B_1)) / ((3B_1^2 + 6(B_2 - B_1))} \), then

\[ b_1 = b_3 = 2\sigma \quad \text{and} \quad b_2 = b_4 = 2. \quad (23) \]

If we choose \( B_1 \) and \( B_2 \) such that \( 0 < \sigma < 1 \), which is equivalent to condition \( C_4 \). Then by Lemma 4, we have \( q \in P \). On putting the values of \( b_i \) and \( \gamma_i \) obtained from (22) and (23) respectively in (16), we get (18), which together with (11) gives the desired bound for \( |a_5| \).

Let the function \( H : D \rightarrow \mathbb{C} \) be given by

\[ H(z) = z \exp \int_0^z \frac{\varphi(t^4) - 1}{t} dt = z + \frac{B_1}{4} z^5 + \frac{1}{32} (B_1^2 + 4B_2) z^9 + \cdots, \]

where coefficients of \( \varphi(z) \) satisfy the conditions \( C_1 \) to \( C_4 \). Then \( H(0) = 0 \), \( H'(0) = 1 \) and \( zH''(z)/H(z) = \varphi(z^4) \) and hence the function \( H \in S^*(\varphi) \), proving the result to be sharp for the function \( H \).

Choose \( \varphi(z) = 1 + \sin z \), whose coefficients satisfy \( C_1 \) to \( C_4 \), to obtain the following result:

**Corollary 5.1.** If \( f \in S^*_{\sin} \), where \( \varphi(z) = 1 + \sin z \), then

\[ |a_5| \leq \frac{1}{4}. \]

The bound is sharp.

For all below mentioned choices of \( \varphi \) conditions \( C_1 \) to \( C_4 \) are valid. Therefore, the bounds of \( |a_5| \) for some of the known classes are obtained from our result as a special case.

**Remark.**

1. If \( \varphi(z) = 2/(1 + e^{-z}) \), then \( f \in S^*_{SG} \) and \( |a_5| \leq 1/8 \) [7, Theorem 4.1].

2. If \( \varphi(z) = \sqrt{1 + z} \), then \( f \in S^*_L \) and \( |a_5| \leq 1/8 \) [15, Theorem 3.1].

3. If \( \varphi(z) = \sqrt{1 + bz} \), then \( f \in S^*_G \), and \( |a_5| \leq b/8 \), where \( b \in (0,1] \) [5, Theorem 3.1].

4. If \( f \in S^*_HL \), then \( |a_5| \leq (5 - 3\sqrt{2}) / 8 \) [15, Theorem 3.1].

5. If \( \varphi(z) = ((1 + z)/(1 - z))^\delta \), then the conditions of Theorem 5 are satisfied only for \( 0 < \delta \leq \delta_0 \approx 0.350162 \). Therefore, \( |a_5| \leq \delta/2 \) for \( f \in S^*(\varphi) \) where \( 0 < \delta \leq \delta_0 \) [9].

**Theorem 6.** Let \( \varphi(z) \) be as defined in (2), whose coefficients satisfy the conditions \( C_1 \) to \( C_4 \). If \( f \in C(\varphi) \), then

\[ |a_5| \leq \frac{B_1}{20}. \]

The inequality is sharp.
Proof. Since \( f \in \mathcal{C}(\varphi) \), therefore

\[
1 + \frac{zf''(z)}{f'(z)} = \varphi((p(z) - 1)/(p(z) + 1)),
\]

where \( p \in \mathcal{P} \) is given by (3). By comparison of the coefficients of \( z, z^2, z^3 \) in (24) with the series expansion of \( f, \varphi \) and \( p \), we get

\[
a_5 = \frac{B_1}{40} I,
\]

where

\[
I = p_4 + I_1 p_1^4 + I_2 p_1^2 p_2 + I_3 p_1 p_3 + I_4 p_2^2,
\]

with \( I_1, I_2, I_3 \) and \( I_4 \) given as in (13), (14) and (15). Using the same method as in Theorem 5, we obtain

\[|I| \leq 2,\]

when \( B_1, B_2, B_3 \) and \( B_4 \) satisfy all the conditions \( \text{C1}, \text{C2}, \text{C3} \) and \( \text{C4} \). Thus bound of \( |a_5| \) follows from (25).

Let \( H(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in \mathcal{S} \) be given by

\[
1 + \frac{zH''(z)}{H'(z)} = \varphi(z^4),
\]

where coefficients of \( \varphi(z) \) satisfy the conditions \( \text{C1} \) to \( \text{C4} \). Clearly, \( H \in \mathcal{C}(\varphi) \) and for the function \( H \), we have \( a_2 = a_3 = a_4 = 0 \) and \( a_5 = B_1/20 \). Thus bound is sharp for \( H \). \( \square \)

Declarations

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Conflict of interest

The authors declare that they have no conflict of interest.

Author Contribution

Each author contributed equally to the research and preparation of the manuscript.

Data Availability

Not Applicable.
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