ON THE MOTIVE OF INTERSECTIONS OF TWO GRASSMANNIANS IN $\mathbb{P}^9$

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ABSTRACT. Using intersections of two Grassmannians in $\mathbb{P}^9$, Ottem–Rennemo and Borisov–Câldăraru–Perry have exhibited pairs of Calabi-Yau threefolds $X$ and $Y$ that are deformation equivalent, L–equivalent and derived equivalent, but not birational. To complete the picture, we show that $X$ and $Y$ have isomorphic Chow motives.

1. INTRODUCTION

Let $\text{Var}(\mathbb{C})$ denote the category of algebraic varieties over the complex numbers $\mathbb{C}$, and let $K_0(\text{Var}(\mathbb{C}))$ denote the Grothendieck ring. This ring is a rather mysterious object. Its intricacy is highlighted by Borisov [9], who showed that the class of the affine line $L$ is a zero–divisor in $K_0(\text{Var}(\mathbb{C}))$. Following on Borisov’s pioneering result, recent years have seen a flurry of constructions of pairs of Calabi–Yau varieties $X, Y$ that are not birational (and so $[X] \neq [Y]$ in the Grothendieck ring), but

$$([X] − [Y])L' = 0 \quad \text{in} \quad K_0(\text{Var}(\mathbb{C})),$$

i.e., $X$ and $Y$ are “L–equivalent” in the sense of [30]. In most cases, the constructed varieties $X$ and $Y$ are also derived equivalent [19], [20], [37], [28], [41], [10], [30], [15], [36], [27], [26].

According to a conjecture made by Orlov [40, Conjecture 1], derived equivalent smooth projective varieties should have isomorphic Chow motives. This conjecture is true for $K3$ surfaces [17], but is still widely open for Calabi–Yau varieties of dimension $\geq 3$. In [33], [34], I verified Orlov’s conjecture for the Calabi–Yau threefolds of Ito–Miura–Okawa–Ueda [19], resp. the Calabi–Yau threefolds of Kapustka–Rampazzo [27]. The aim of the present note is to check that Orlov’s conjecture is also true for the Calabi–Yau threefolds studied recently by Borisov–Câldăraru–Perry [10], and independently by Ottem–Rennemo [41].

The threefolds of [41], [10] are called GPK$^3$ threefolds. The shorthand “GPK$^3$” stands for Gross–Popescu–Kanazawa–Kapustka–Kapustka, the authors of the papers [13], [23], [24], [22] where they first appeared (the shorthand “GPK$^3$” is coined in [10]). These threefolds are constructed as follows. Given $W$ a 10–dimensional vector space over $\mathbb{C}$, let $\mathbb{P} := \mathbb{P}(W)$. Let $V$ be a 5–dimensional vector space over $\mathbb{C}$, and choose isomorphisms

$$\phi_i : \wedge^2 V \to W, \quad i = 1, 2.$$

Composing the Plücker embedding with the induced isomorphisms $\phi_i : \mathbb{P}(\wedge^2 V) \cong \mathbb{P}$, one obtains two embeddings of the Grassmannian $Gr(2, V)$ in $\mathbb{P}$, whose images are denoted $Gr_i, i = 1, 2$. 

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For $\phi_i$ generic, the intersection

$$X := Gr_1 \cap Gr_2 \subset \mathbb{P}$$

is a smooth Calabi–Yau threefold, called a GPK$^3$ threefold. Let $Gr_i^\vee$ be the projective dual of $Gr_i$. The intersection

$$Y := Gr_1^\vee \cap Gr_2^\vee \subset \mathbb{P}^\vee$$

is again a smooth Calabi–Yau threefold, and it is deformation equivalent to $X$. The pair $X, Y$ are called GPK$^3$ double mirrors, and $X, Y$ are known to be Hodge equivalent, derived equivalent, L–equivalent, and in general not birational [10], [41]. In this note, we prove the following:

**Theorem** (=theorem 6.1). Let $X, Y$ be two GPK$^3$ double mirrors. Then there is an isomorphism of Chow motives

$$h(X) \cong h(Y) \text{ in } M_{\text{rat}}.$$

The proof of theorem 6.1 is an elementary exercise in manipulating Chow groups and correspondences, based on a nice geometric relation between $X$ and $Y$ established in [10] (cf. proposition 2.3 below). The only ingredient in the proof that may perhaps not be completely standard is the use of Bloch’s higher Chow groups ([5], cf. also section 3 below), and some results on higher Chow groups of piecewise trivial fibrations (section 5 below).

**Conventions.** In this note, the word variety will refer to a reduced irreducible scheme of finite type over the field of complex numbers $\mathbb{C}$. All Chow groups will be with $\mathbb{Q}$–coefficients, unless indicated otherwise: For a variety $X$, we will write $A_j(X) := CH_j(X)_{\mathbb{Q}}$ for the Chow group of dimension $j$ cycles on $X$ with rational coefficients. For $X$ smooth of dimension $n$, the notations $A_j(X)$ and $A^{n−j}(X)$ will be used interchangeably.

The notations $A^j_{\text{hom}}(X)$ (and $A^j_{\text{AJ}}(X)$) will be used to indicate the subgroups of homologically trivial (resp. Abel–Jacobi trivial) cycles. For a morphism between smooth varieties $f : X \to Y$, we will write $\Gamma_f \in A^*(X \times Y)$ for the graph of $f$, and $\Gamma^t_f \in A^*(Y \times X)$ for the transpose correspondence.

We will write $M_{\text{rat}}$ for the contravariant category of Chow motives (i.e., pure motives as in [43], [39], with Hom–groups defined using $A^*(X \times Y)$).

We will write $H^j(X) = H^j(X, \mathbb{Q})$ for singular cohomology, and $H_j(X) = H_j^{BM}(X, \mathbb{Q})$ for Borel–Moore homology.

2. The Calabi–Yau threefolds

In this section we consider GPK$^3$ threefolds, as defined in the introduction.

**Proposition 2.1** (Ottem–Rennemo, Kanazawa [41], [22]). The family of GPK$^3$ threefolds is locally complete. A GPK$^3$ threefold $X$ has Hodge numbers

$$h^{1,1}(X) = 1, \quad h^{2,1}(X) = 51.$$

**Proof.** The first statement is [41] Proposition 3.1. The Hodge numbers are computed in [22] Proposition 2.16. □
Theorem 2.2 (Ottem–Rennemo, Borisov–Căldăraşu–Perry [41], [10]). Let $X, Y$ be a general pair of $GPK^3$ double mirrors. Then $X$ and $Y$ are not birational, and so

$$[X] \neq [Y] \text{ in } K_0(\text{Var}(\mathbb{C})).$$

However, one has

$$([X] - [Y]) \mathbb{L}^4 = 0 \text{ in } K_0(\text{Var}(\mathbb{C})).$$

Moreover, $X$ and $Y$ are derived equivalent, i.e. there is an isomorphism of bounded derived categories

$$D^b(X) \cong D^b(Y).$$

In particular, there is an isomorphism of polarized Hodge structures

$$H^3(X, \mathbb{Z}) \cong H^3(Y, \mathbb{Z}).$$

Proof. Non–birationality is [10, Theorem 1.2], and independently [41, Theorem 4.1]. Thanks to the birational invariance of the MRC–fibration, $X$ and $Y$ are not stably birational (cf. [9, Proof of Theorem 2.12]). The celebrated Larsen–Lunts result [31] implies that $[X] \neq [Y]$ in the Grothendieck ring.

The $L$–equivalence is [10, Theorem 1.6]; it is a corollary of the geometric relation of proposition 2.3 below. Derived equivalence is proven in [41, Proposition 1.1], and also in [29, Section 6.1].

The isomorphism of Hodge structures is a corollary of the derived equivalence, in view of [41, Proposition 2.1 and Remark 2.3].

The argument of this note crucially relies on the following (for the notion of piecewise trivial fibration, cf. definition 5.1 below).

Proposition 2.3 (Borisov–Căldăraşu–Perry [10]). Let $X, Y$ be a pair of $GPK^3$ double mirrors. There is a diagram

$$F \xrightarrow{i_F} Q \xleftarrow{G}$$

\begin{equation}
\begin{array}{c}
P_X \vee \quad P \vee \quad q \quad q_Y
\end{array}
\end{equation}

Here,

$$X \xrightarrow{p_X} \ Gr_1 \quad \quad G \xleftarrow{q_Y} \ Gr_2^\vee \quad \xrightarrow{q} \ Y$$

is the intersection of the natural incidence divisor $\sigma \subset \mathbb{P} \times \mathbb{P}^\vee$ with the product $Gr_1 \times Gr_2^\vee \subset \mathbb{P} \times \mathbb{P}^\vee$, the morphisms $p$ and $q$ are induced by the natural projections, the closed subvarieties $F, G$ are defined as $p^{-1}(X)$ resp. $q^{-1}(Y)$, and $p_X, q_Y$ are defined as the restrictions $p|_F$ resp. $q|_G$. The morphisms $p_X, q_Y$ are piecewise trivial fibrations with fibres $F_x$ resp. $G_y$ verifying

$$A_i(F_x) = A_i(G_y) = \begin{cases}
\mathbb{Q} & \text{if } i = 0, 1, 5, \\
\mathbb{Q}^2 & \text{if } i = 2, 3, 4,
\end{cases}$$

$$H_j(F_x) = H_j(G_y) = 0 \text{ for all } j \text{ odd},$$
for all $x \in X, y \in Y$. However, over the open complements

$$U := Gr_1 \setminus X, \quad V := Gr_2 \setminus Y,$$

the restrictions $p_U := p|_{p^{-1}(U)}, q_V := q|_{q^{-1}(V)}$ are piecewise trivial fibrations with fibres $Q_u := p^{-1}(u)$ resp. $Q_v := q^{-1}(v)$ verifying

$$A_i(Q_u) = A_i(Q_v) = \begin{cases} \mathbb{Q} & \text{if } i = 0, 1, 4, 5, \\ \mathbb{Q}^2 & \text{if } i = 2, 3, \end{cases}$$

for all $u \in U, v \in V$.

**Proof.** The diagram is constructed in [10, Section 7]. The computation of homology and Chow groups of the fibres of $p$ easily follows from the explicit description of the fibres as hyperplane sections of the Grassmannian $Gr(2, V)$ [10, Section 7]. Precisely, as explained in [10, Proof of Lemma 7.2], there exists a closed subvariety $Z \subset F_x$ such that $Z \sim \mathbb{P}^2$, and the complement $C := F_x \setminus Z$ is a Zariski locally trivial fibration over $\mathbb{P}^2$, with fibres isomorphic to $\mathbb{P}^3 \setminus \mathbb{P}^1$. Since neither $C$ nor $Z$ have odd–degree Borel–Moore homology, the same holds for $F_x$. As for even–degree homology, there is a commutative diagram with exact rows

$$\begin{array}{cccc}
\to A_i(Z) & \to & A_i(F_x) & \to & A_i(C) & \to & 0 \\
\downarrow \cong & & \downarrow & & \downarrow \cong & & \\
0 & \to & H_{2i}(Z) & \to & H_{2i}(F_x) & \to & H_{2i}(C) & \to & 0,
\end{array}$$

where vertical arrows are cycle class maps. The left and right vertical arrow are isomorphisms, because of the above explicit description of $Z$ and $C$. This implies that the cycle class map induces isomorphisms $A_i(F_x) \cong H_{2i}(F_x)$ for all $i$.

The bottom exact sequence of this diagram shows that

$$H_{2i}(F_x) = \begin{cases} H_{2i}(C) & \text{if } i = 3, 4, 5, \\ H_{2i}(C) \oplus \mathbb{Q} & \text{if } i = 0, 1, 2. \end{cases}$$

Next, one remarks that the open $C$ (being a fibration over $\mathbb{P}^2$ with fibre $T \cong \mathbb{P}^3 \setminus \mathbb{P}^1$) has

$$H_{2i}(C) = \bigoplus_{\ell + m = 2i} H_{\ell}(\mathbb{P}^2) \otimes H_m(T) = \begin{cases} 0 & \text{if } i = 0, 1, \\ \mathbb{Q} & \text{if } i = 2, 5, \\ \mathbb{Q}^2 & \text{if } i = 3, 4. \end{cases}$$

Putting things together, this shows the statement for $H_{2i}(F_x)$.

(A more efficient, if less self–contained, way of determining the Betti numbers of $F_x$ is as follows. One has equality in the Grothendieck ring [10, Lemma 7.2]

$$[F_x] = (\mathbb{L}^2 + \mathbb{L} + 1)(\mathbb{L}^3 + \mathbb{L}^2 + 1) \quad \text{in } K_0(\text{Var}).$$
Let $W^*$ denote Deligne’s weight filtration on Borel–Moore homology $[42]$. The “virtual Betti number”

$$P_{2i}(\cdot) := \sum_j (-1)^j \dim \text{Gr}_{W^2}^{-2i} H_j(\cdot)$$

is a functor on $K_0(\text{Var})$, and so

$$P_{2i}(F_x) = P_{2i}((\mathbb{L}^2 + \mathbb{L} + 1)(\mathbb{L}^3 + \mathbb{L}^2 + 1)) = \begin{cases} 1 & \text{if } i = 0, 1, 5, \\ 2 & \text{if } i = 2, 3, 4. \end{cases}$$

On the other hand, $F_x$ has no odd–degree homology, and the fact that $H_{2i}(F_x)$ is algebraic implies that $H_{2i}(F_x)$ is pure of weight $-2i$. It follows that $P_{2i}(F_x) = \dim H_{2i}(F_x)$.

The homology groups and Chow groups of the fibres $Q_u$ over $u \in U$ are determined similarly: according to loc. cit., there is a closed subvariety $Z \subset Q_u$ such that $Z$ is isomorphic to a smooth quadric in $\mathbb{P}^4$, and the complement $Q_u \setminus W$ is a Zariski locally trivial fibration over $\mathbb{P}^3$, with fibres isomorphic to $\mathbb{P}^2 \setminus \mathbb{P}^1$. □

We record a lemma for later use:

**Lemma 2.4.** The open $U$ (and the open $V$) of proposition 2.3 has trivial Chow groups, i.e. cycle class maps

$$A_i(U) \to H_{2i}(U)$$

are injective.

**Proof.** This is a standard argument. One has a commutative diagram with exact rows

$$
\begin{array}{cccc}
A_i(X) & \to & A_i(Gr_1) & \to & A_i(U) & \to & 0 \\
\downarrow & & \downarrow \cong & & \downarrow & \\
H_{2i}(X) & \to & H_{2i}(Gr_1) & \to & H_{2i}(U) & \to & 0 
\end{array}
$$

(the middle vertical arrow is an isomorphism, as $Gr_1$ is a Grassmannian). Given $a \in A_i(U)$ homologically trivial, there exists $\bar{a} \in A_i(Gr_1)$ such that the homology class of $\bar{a}$ is supported on $X$. Using semisimplicity of polarized Hodge structures, the homology class of $\bar{a}$ is represented by a Hodge class in $H_{2i}(X)$. But $X$ being three–dimensional, the Hodge conjecture is known for $X$, and so $\bar{a} \in H_{2i}(Gr_1)$ is represented by a cycle $d \in A_i(X)$. The cycle $\bar{a} - d \in A_i(Gr_1)$ thus restricts to $a$ and is homologically trivial, hence rationally trivial. □

**Remark 2.5.** We observe in passing that the subvarieties $F, G$ in proposition 2.3 must be singular. Indeed, the fibres $F_x, G_y$ have Picard number 1, but the group of Weil divisors has dimension 2, and so the fibres $F_x, G_y$ are not $\mathbb{Q}$–factorial. By generic smoothness $[14]$ Corollary 10.7, it follows that $F, G$ cannot be smooth.

**Remark 2.6.** As explained in $[41]$ Section 5, the 51–dimensional family of GPK threefolds degenerates to the 50–dimensional family of Calabi–Yau threefolds first studied in $[25], [18]$. Generalized mirror pairs in this 50–dimensional family are also derived equivalent and $L$–equivalent $[27]$, and have isomorphic Chow motives $[34]$. 


3. Higher Chow groups

**Definition 3.1** (Bloch [5], [6]). Let $\Delta^j \cong \mathbb{A}^j(\mathbb{C})$ denote the standard $j$–simplex. For any quasi–projective variety $M$ and any $i \in \mathbb{Z}$, let $z_i^{\mathbf{simp}}(M, \ast)$ denote the simplicial complex where $z_i(X, j)$ is the group of $(i + j)$–dimensional algebraic cycles in $M \times \Delta^j$ that meet the faces properly. Let $z_i(M, \ast)$ denote the single complex associated to $z_i^{\mathbf{simp}}(M, \ast)$. The higher Chow groups of $M$ are defined as

$$A_i(M, j) := H^j(z_i(M, \ast) \otimes \mathbb{Q}) .$$

**Remark 3.2.** Clearly one has $A_i(M, 0) \cong A_i(M)$. For a closed immersion, there is a long exact sequence of higher Chow groups [6], [35], extending the usual “localization exact sequence” of Chow groups. Higher Chow groups are related to higher algebraic $K$–theory: there are isomorphisms

$$(2) \quad \text{Gr}^{n-j}_{\gamma} G_j(M)_{\mathbb{Q}} \cong A_i(M, j) \quad \text{for all } i, j ,$$

where $G_j(M)$ is Quillen’s higher $K$–theory group associated to the category of coherent sheaves on $M$, and $\text{Gr}^{n-j}_{\gamma}$ is the graded for the $\gamma$–filtration [5]. Higher Chow groups are also related to Voevodsky’s motivic cohomology (defined as hypercohomology of a certain complex of Zariski sheaves) [11], [38].

4. Operational Chow cohomology

In what follows, we will rely on the existence of operational Chow cohomology, as constructed by Fulton–MacPherson. The precise definition does not matter here; we merely use the existence of a theory with good formal properties:

**Theorem 4.1** (Fulton [12]). There exists a contravariant functor

$$A^*(\cdot) : \text{Var}_{\mathbb{C}} \to \text{Rings}$$

(from the category of varieties with arbitrary morphisms to that of graded commutative rings), with the following properties:

- for any $X$ and $b \in A^j(X)$ there is a cap–product $b \cap (\cdot) : A_i(X) \to A_{i-j}(X)$, making $A_*(X)$ a graded $A^*(X)$–module;
- for $X$ smooth of dimension $n$, the map $A^j(X) \to A_{n-j}(X)$ given by $b \mapsto b \cap [X] \in A_{n-j}(X)$ is an isomorphism for all $j$;
- for any proper morphism $f : X \to Y$, there is a projection formula:

$$f_* (f^*(b) \cap a) = b \cap f_* (a) \quad \text{in } A_{i-j}(Y) \quad \text{for any } b \in A^j(Y), a \in A_i(X) .$$

**Proof.** This is contained in [12 Chapter 17]. The projection formula is [12 Section 17.3]. \qed
Remark 4.2. For quasi-projective varieties, there is another cohomology theory to pair with Chow groups: the assignment

$$CH^i(X) := \lim_{\to} A^j(Y),$$

where the limit is over all smooth quasi-projective varieties $Y$ with a morphism to $X$. As shown in [3, 12 Example 8.3.13], this theory satisfies the formal properties of theorem 4.1. Since in this note, we are only interested in quasi-projective varieties, we might as well work with this theory rather than operational Chow cohomology.

5. PIECEWISE TRIVIAL FIBRATIONS

This section contains two auxiliary results, propositions 5.2 and 5.4. The first is about Chow groups of the open complement $R := Q \setminus F$ of proposition 2.3. The second concerns the Chow groups of the singular variety $F$.

Definition 5.1 (Section 4.2 in [44]). Let $p: M \to N$ be a projective surjective morphism between quasi-projective varieties. We say that $p$ is a piecewise trivial fibration with fibre $F$ if there is a finite partition $N = \bigcup_j N_j$, where $N_j \subset N$ is locally closed and there is an isomorphism of $N_j$-schemes $p^{-1}(N_j) \cong N_j \times F$ for all $j$.

Proposition 5.2. Let $U := Gr_1 \setminus X$ and $R := Q \setminus F$ and $p_U: R \to U$ be as in proposition 2.3. (i) Let $h \in A^1(R)$ be a hyperplane section, and let $h^j: A_i(R) \to A_{i-j}(R)$ denote the map induced by intersecting with $h^j$. There are isomorphisms

$$\begin{align*}
\Phi_0 &: A_0(U) \xrightarrow{\cong} A_0(R), \\
\Phi_1 &: A_1(U) \oplus A_0(U) \xrightarrow{\cong} A_1(R), \\
\Phi_2 &: A_2(U) \oplus A_1(U) \oplus A_0(U) \xrightarrow{\cong} A_2(R), \\
\Phi_3 &: A_3(U) \oplus A_2(U) \oplus A_1(U) \oplus A_0(U) \xrightarrow{\cong} A_3(R), \\
\Phi_4 &: A_3(U) \oplus A_2(U) \oplus A_1(U) \oplus A_0(U) \xrightarrow{\cong} A_4(R), \\
\Phi_5 &: A_3(U) \oplus A_2(U) \oplus A_1(U) \oplus A_0(U) \xrightarrow{\cong} A_5(R).
\end{align*}$$

The maps are defined as

$$\begin{align*}
\Phi_0 &= h^5 \circ (p_U)^*, \\
\Phi_1 &= \left( h^5 \circ (p_U)^*, h^4 \circ (p_U)^*, (b \cdot h) \circ (p_U)^* \right), \\
\Phi_2 &= \left( h^5 \circ (p_U)^*, h^4 \circ (p_U)^*, h^3 \circ (p_U)^*, (b \cdot h) \circ (p_U)^* \right), \\
\Phi_3 &= \left( h^5 \circ (p_U)^*, h^4 \circ (p_U)^*, h^3 \circ (p_U)^*, (b \cdot h) \circ (p_U)^*, h^2 \circ (p_U)^*, b \circ (p_U)^* \right), \\
\Phi_4 &= \left( h^4 \circ (p_U)^*, h^3 \circ (p_U)^*, h^2 \circ (p_U)^*, h \circ (p_U)^*, b \circ (p_U)^* \right), \\
\Phi_5 &= \left( h^3 \circ (p_U)^*, h^2 \circ (p_U)^*, b \circ (p_U)^*, (p_U)^* \right),
\end{align*}$$
where \( b \in A^2(R) \) is a class made explicit in the proof (and \( b: A_i(R) \to A_{i-2}(R) \) denotes the operation of intersecting with \( b \), and similarly for \( b \cdot h \)).

(ii) \( A^i_{\text{hom}}(R) = 0 \) \( \forall i \).

Proof. (i) As we have seen in proposition 2.3, the morphism \( p_U: R \to U \) is a piecewise trivial fibration, with fibre \( R_u \). Let \( T \) denote the tautological bundle on the Grassmannian \( Gr_2 \), and define

\[
b := (Gr_1 \times c_2(T))|_R \subset A^2(R) .
\]

The (5–dimensional) fibres \( R_u \) of the fibration \( p_U: R \to U \) verify

\[
A^i(R_u) = \begin{cases}
\mathbb{Q} \cdot h^i|_{R_u} & \text{if } i = 0, 1, 4, 5 , \\
\mathbb{Q} \cdot h^2|_{R_u} \oplus \mathbb{Q} \cdot b|_{R_u} & \text{if } i = 2 , \\
\mathbb{Q} \cdot h^3|_{R_u} \oplus \mathbb{Q} \cdot (b \cdot h)|_{R_u} & \text{if } i = 3 .
\end{cases}
\]

To prove the isomorphisms of Chow groups of (i), it is more convenient to prove a more general statement for higher Chow groups. That is, we consider maps

\[
\Phi_i: \bigoplus A_{i-k}(U, j) \to A_i(R, j)
\]

such that \( \Phi_i^0 = \Phi_i \) (the maps \( \Phi_i^k \) are defined just as the \( \Phi_i \), using \( (p_U)^* \) and intersecting with \( h \) and \( b \)). We now claim that the maps \( \Phi_i^j \) are isomorphisms for all \( i = 0, \ldots, 5 \) and all \( j \). The \( j = 0 \) case of this claim proves (i).

To prove the claim, we exploit the piecewise triviality of the fibration \( p_U \). Up to subdividing some more, we may suppose the strata \( U_k \) (and hence also the strata \( R_k \)) are smooth. We will use the notation

\[
U_{\leq s} := \bigcup_{i \leq s} U_i , \quad R_{\leq s} := \bigcup_{i \leq s} R_i .
\]

For any \( s \), the morphism \( p_{U_{\leq s}}: R_{\leq s} \to U_{\leq s} \) is a piecewise trivial fibration (with fibre \( R_u \)). For \( s \) large enough, \( R_{\leq s} = R \). Since \( U \) and \( R \) are smooth, we may suppose the \( U_s \) are ordered in such a way that the \( U_{\leq s} \) (and hence the \( R_{\leq s} \)) are smooth.

The morphism \( p_U \) is flat of relative dimension 5, and so there is a commutative diagram of complexes (where rows are exact triangles)

\[
\begin{array}{cccc}
z_{i+5}(R_{\leq s-1}, *) & \rightarrow & z_{i+5}(R_{\leq s}, *) & \rightarrow & z_{i+5}(R_s, *) & \rightarrow \\
\uparrow (p_{U_{\leq s-1}})^* & & \uparrow p_{U_{\leq s}}^* & & \uparrow (p_U)^* & \\
z_i(U_{\leq s-1}, *) & \rightarrow & z_i(U_{\leq s}, *) & \rightarrow & z_i(U_s, *) & \rightarrow 
\end{array}
\]

Also, given a codimension \( \ell \) subvariety \( M \subset R \), let \( z_i^M(R_{\leq k}, *) \subset z_i(R_{\leq k}, *) \) denote the subcomplex formed by cycles in general position with respect to \( M \). The inclusion \( z_i^M(R_{\leq k}, *) \subset z_i(R_{\leq k}, *) \) is a quasi–isomorphism [5 Lemma 4.2]. The projection formula for higher Chow
groups [5 Exercise 5.8(i)] gives a commutative diagram up to homotopy
\[ z_{i+5-\ell}(R_{\leq s-1}, \ast) \rightarrow z_{i+5-\ell}(R_{\leq s}, \ast) \rightarrow z_{i+5-\ell}(R_s, \ast) \rightarrow \]
\[ \uparrow M|R_{\leq s-1} \uparrow M|R_{\leq s} \uparrow M|R_s \]
\[ z_{i+5}^M(R_{\leq s-1}, \ast) \rightarrow z_{i+5}^M(R_{\leq s}, \ast) \rightarrow z_{i+5}^M(R_s, \ast) \rightarrow . \]
In particular, these diagrams exist for \( M \) being (a representative of) the classes \( h^r, b \in A^*(R) \) that make up the definition of the map \( \Phi_j^i \). The result of the above remarks is a commutative diagram with long exact rows
\[ \rightarrow A_i(R_s, j + 1) \rightarrow A_i(R_{\leq s-1}, j) \rightarrow A_i(R_{\leq s}, j) \rightarrow \]
\[ \uparrow \Phi_{i+1}^i|R_s \uparrow \Phi_{i}^i|R_{\leq s-1} \uparrow \Phi_{i}^i|R_{\leq s} \]
\[ \rightarrow \bigoplus A_{i-k}(U_s, j + 1) \rightarrow \bigoplus A_{i-k}(U_{\leq s-1}, j) \rightarrow \bigoplus A_{i-k}(U_{\leq s}, j) \rightarrow . \]
Applying noetherian induction and the five-lemma, one is reduced to proving the claim for \( R_s \rightarrow U_s \). But \( R_s \) is isomorphic to the product \( U_s \times R_u \) and the fibre \( R_u \) is a linear variety (i.e., \( R_u \) can be written as a finite disjoint union of affine spaces \( \mathbb{A}^r \)). Cutting up the fibre \( R_u \) and using another commutative diagram with long exact rows, one is reduced to proving that \( A_i(U_s, j) \cong A_i(U_u \times \mathbb{A}^r, j) \), which is the homotopy property for higher Chow groups [5 Theorem 2.1]. This proves the claim, and hence (i).

(ii) The point is that there is also a version in homology of (i). That is, for any \( j \in \mathbb{N} \) there are isomorphisms
\[ (3) \quad \Phi_j^h : \bigoplus H_{j-2k}(U) \xrightarrow{\cong} H_j(R) , \]
where \( \Phi_j^h \) is defined as
\[ \Phi_j^h := \left( h^5 \circ (p_U)^*, h^4 \circ (p_U)^*, h^3 \circ (p_U)^*, (b \cdot h) \circ (p_U)^*, h^2 \circ (p_U)^*, b \circ (p_U)^*, h \circ (p_U)^*, (p_U)^* \right) . \]
This is proven just as (i), using homology instead of higher Chow groups. Cycle class maps fit into a commutative diagram
\[ \bigoplus A_{i-k}(U) \xrightarrow{\Phi_i} A_i(R) \]
\[ \downarrow \quad \downarrow \quad \downarrow \]
\[ \bigoplus H_{2i-2k}(U) \xrightarrow{\Phi_{2i}^h} H_{2i}(R) . \]
As the horizontal arrows are isomorphisms, and the left vertical arrow is injective (lemma [2.4]), the right vertical arrow is injective as well. This proves statement (ii). \( \square \)

For later use, we record the following result:
The maps $\Phi_0 \colon A_0(X) \xrightarrow{\cong} A_0(F)$, 
$\Phi_1 \colon A_1(X) \oplus A_0(X) \xrightarrow{\cong} A_1(F)$, 
$\Phi_2 \colon A_2(X) \oplus A_1(X) \oplus A_0(X)^{\oplus 2} \xrightarrow{\cong} A_2(F)$, 
$\Phi_3 \colon A_3(X) \oplus A_2(X) \oplus A_1(X)^{\oplus 2} \oplus A_0(X)^{\oplus 2} \xrightarrow{\cong} A_3(F)$, 
$\Phi_4 \colon A_3(X) \oplus A_2(X)^{\oplus 2} \oplus A_1(X)^{\oplus 2} \oplus A_0(X)^{\oplus 2} \xrightarrow{\cong} A_4(F)$.

The maps $\Phi_j$ are defined as 
$\Phi_0 := h^5 \circ (p_X)^*$, 
$\Phi_1 := \left( h^5 \circ (p_X)^*, h^4 \circ (p_X)^*, h^3 \circ (p_X)^*, (b \cdot h) \circ (p_X)^* \right)$, 
$\Phi_2 := \left( h^5 \circ (p_X)^*, h^4 \circ (p_X)^*, h^3 \circ (p_X)^*, (b \cdot h) \circ (p_X)^*, h^2 \circ (p_X)^*, b \circ (p_X)^* \right)$, 
$\Phi_3 := \left( h^5 \circ (p_X)^*, h^4 \circ (p_X)^*, h^3 \circ (p_X)^*, (b \cdot h) \circ (p_X)^*, h^2 \circ (p_X)^*, b \circ (p_X)^*, h \circ (p_X)^*, (p_X)^*(-) \cap e \right)$, 
$\Phi_4 := \left( h^5 \circ (p_X)^*, h^4 \circ (p_X)^*, (b \cdot h) \circ (p_X)^*, h^2 \circ (p_X)^*, b \circ (p_X)^*, h \circ (p_X)^*, (p_X)^*(-) \cap e \right)$.

where $b \in A^2(F)$ is a class made explicit in the proof (and $b \colon A_1(R) \to A_{1-k}(R)$ denotes the operation of intersecting with $b$, and similarly for $b \cdot h$), and $e \in A_2(F)$ is a class made explicit in the proof (and the last $(p_X)^*(-)$ means pullback of operational Chow cohomology).

(ii) The maps $\Phi_i$ induce isomorphisms of homologically trivial cycles 
$\Phi_i \colon \bigoplus A_{i-k}^{\text{hom}}(X) \xrightarrow{\cong} A_i^{\text{hom}}(F)$.

Proof. (i) The element $b$ is defined just as in proposition 5.2

$b := (t_F)^* \left( (Gr_1 \times c_2(T))_{|Q} \right) \in A^2(F)$,

where $t_F : F \to Q$ denotes the inclusion morphism, and $A^*(F)$ is operational Chow cohomology of the singular variety $F$. 

**Corollary 5.3.** One has 
$H_j(R) = 0$ for $j$ odd.

**Proof.** The threefold $X$ has $H_{j-1}(X) = \mathbb{Q}$ for any $j - 1$ even, and the Grassmannian $Gr_1$ has $H_j(Gr_1) = 0$ for $j$ odd. The exact sequence 
$H_j(Gr_1) \to H_j(U) \to H_{j-1}(X) \to H_{j-1}(Gr_1) \to \cdots$ implies that the open $U := Gr_1 \setminus X$ has no odd–degree cohomology. In view of the isomorphism \(3\), $R$ has no odd–degree homology either. \(\square\)

Let us now turn to the fibration $F \to X$, where $F$ (but not $X$) is singular.

**Proposition 5.4.** Let $p_X : F \to X$ be as in proposition 2.3. Let $h^k : A_i(F) \to A_{i-k}(F)$ denote the operation of intersecting with a hyperplane section.

(i) There are isomorphisms

$\Phi_0 : A_0(X) \cong A_0(F)$, 
$\Phi_1 : A_1(X) \oplus A_0(X) \cong A_1(F)$, 
$\Phi_2 : A_2(X) \oplus A_1(X) \oplus A_0(X)^{\oplus 2} \cong A_2(F)$, 
$\Phi_3 : A_3(X) \oplus A_2(X) \oplus A_1(X)^{\oplus 2} \oplus A_0(X)^{\oplus 2} \cong A_3(F)$, 
$\Phi_4 : A_3(X) \oplus A_2(X)^{\oplus 2} \oplus A_1(X)^{\oplus 2} \oplus A_0(X)^{\oplus 2} \cong A_4(F)$. 

The maps $\Phi_j$ are defined as

$\Phi_0 := h^5 \circ (p_X)^*$, 
$\Phi_1 := \left( h^5 \circ (p_X)^*, h^4 \circ (p_X)^*, (b \cdot h) \circ (p_X)^* \right)$, 
$\Phi_2 := \left( h^5 \circ (p_X)^*, h^4 \circ (p_X)^*, (b \cdot h) \circ (p_X)^*, h^2 \circ (p_X)^*, b \circ (p_X)^* \right)$, 
$\Phi_3 := \left( h^5 \circ (p_X)^*, h^4 \circ (p_X)^*, (b \cdot h) \circ (p_X)^*, h^2 \circ (p_X)^*, b \circ (p_X)^*, h \circ (p_X)^*, (p_X)^*(-) \cap e \right)$, 
$\Phi_4 := \left( h^5 \circ (p_X)^*, h^4 \circ (p_X)^*, (b \cdot h) \circ (p_X)^*, h^2 \circ (p_X)^*, b \circ (p_X)^*, h \circ (p_X)^*, (p_X)^*(-) \cap e \right)$.

where $b \in A^2(F)$ is a class made explicit in the proof (and $b : A_1(R) \to A_{1-k}(R)$ denotes the operation of intersecting with $b$, and similarly for $b \cdot h$), and $e \in A_2(F)$ is a class made explicit in the proof (and the last $(p_X)^*(-)$ means pullback of operational Chow cohomology).

(ii) The maps $\Phi_i$ induce isomorphisms of homologically trivial cycles

$\Phi_i : \bigoplus A_{i-k}^{\text{hom}}(X) \cong A_i^{\text{hom}}(F)$. 

Proof. (i) The element $b$ is defined just as in proposition 5.2

$b := (t_F)^* \left( (Gr_1 \times c_2(T))_{|Q} \right) \in A^2(F)$, 

where $t_F : F \to Q$ denotes the inclusion morphism, and $A^*(F)$ is operational Chow cohomology of the singular variety $F$. 


To define the element $e \in A_7(F)$, we return to the description of the fibres of $p_X : F \to X$ given in [10, Section 7]. By definition of the variety $F$, we have

$$F = \{(x, y) \in Gr_1 \times Gr^\vee_2 \mid x \in X, (x, y) \in \sigma\},$$

where $\sigma \subset \mathbb{P} \times \mathbb{P}^\vee$ is the incidence divisor. As in [10, section 7], given $x \in \mathbb{P}$ (or $y \in \mathbb{P}^\vee$) let $x_i := \phi^\vee_i(x) \in \mathbb{P}(\wedge^2 V)$ (resp. $y_i := \phi_i(y) \in \mathbb{P}(\wedge^2 V^\vee)$). For a point $\omega \in \mathbb{P}(\wedge^2 V)$ (or in $\mathbb{P}(\wedge^2 V^\vee)$), let $rk(\omega)$ denote the rank of $\omega$ considered as a skew–form; the rank of $\omega$ is either 2 or 4. As explained in loc. cit., the expression for $F$ can be rewritten as

$$F = \{(x, y) \in \mathbb{P} \times \mathbb{P}^\vee \mid rk(x_i) = 2, \, rk(y_2) = 2, \, y \in H_{x_2}\}$$

$$= \{(x, y) \in \mathbb{P} \times \mathbb{P}^\vee \mid rk(x_i) = 2, \, rk(y_2) = 2, \, A_y \cap ker(x_2) \neq \emptyset\} \subset \mathbb{P}(\wedge^2 V) \times \mathbb{P}(\wedge^2 V^\vee).$$

(Here, $A_y \subset V^\vee$ denotes the 2–dimensional subspace corresponding to $y$, and $ker(x_2) \subset V^\vee$ denotes the kernel of the skew–form $x_2 \in \mathbb{P}(\wedge^2 V^\vee)$.) The stratification of the fibres $F_x$ as given in loc. cit. can be done relatively over $X$. That is, we define

$$Z := \{(x, y) \in \mathbb{P} \times \mathbb{P}^\vee \mid (x, y) \in F, \, A_y \subset ker(x_2)\} \subset F.$$ The intersection of $Z$ with a fibre $F_x$ is the variety $Z \cong \mathbb{P}^2$ of [10, Lemma 7.2], and so $Z \to X$ is a $\mathbb{P}^2$–fibration.

The complement $F^0 := F \setminus Z$ can be described as

$$F^0 = \{(x, y) \in \mathbb{P} \times \mathbb{P}^\vee \mid rk(x_i) = 2, \, rk(y_2) = 2, \, dim(A_y \cap ker(x_2)) = 1\}.$$

The natural morphism $F^0 \to X$ factors as

$$F^0 \to W \to X,$$

where

$$W := \{(x, s) \in X \times \mathbb{P}(V^\vee) \mid s \in \mathbb{P}(ker(x_2))\},$$

and $W \to X$ is a $\mathbb{P}^2$–fibration. As explained in loc. cit., over each $x \in X$ the morphism from $(F^0)_x$ to $W_x$ is a fibration with fibres isomorphic to $\mathbb{P}^3 \setminus \mathbb{P}^1$. Let $W' \subset W$ be the divisor

$$W' := \{(x, s) \in X \times \mathbb{P}(V^\vee) \mid s \in \mathbb{P}(ker(x_2)) \cap h\},$$

where $h \subset \mathbb{P}(V^\vee)$ is a hyperplane section. The morphism $W' \to X$ is a $\mathbb{P}^1$–fibration. The class $e \in A_7(F)$ is now defined as

$$e := (p_X|_{F^0})^{-1}(W') \in A_7(F^0) \cong A_7(F),$$

where $A_7(F^0) \cong A_7(F)$ for dimension reasons.

We observe that $e \in A_7(F)$ is not proportional to the class of a hyperplane section $h \in A_7(F)$. (Indeed, let $x \in X$ and $w \in (W \setminus W')_x$ and let $\nu \in A^1((F^0)_x)$ be the tautological class with respect to the projective bundle structure of $(F^0)_x \to W_x$. Then $C := ((F^0)_x \nu \nu^2 \cong \mathbb{P}^1$ is an effective curve disjoint from $e$, whereas $h \cap C$ has strictly positive degree.)

Since we know that the fibres $F_x$ have $A_4(F_x) \cong \mathbb{Q}^2$ (proposition 2.3), it follows that

$$h|_{F_x}, \ e|_{F_x} \in A_4(F_x)$$

generate $A_4(F_x)$. (Here, $e|_{F_x} \in A_4(F_x)$ is defined as $\tau^*(e) \in A_4(F_x)$ where $\tau^*$ is the refined Gysin homomorphism [12] associated to the regular morphism $\tau : x \to X.$)
The (5-dimensional) fibres $F_x$ of the fibration $p_X: F \to X$ thus verify

$$A_i(F_x) = \begin{cases} 
\mathbb{Q} \cdot h^{5-i}|_{F_x} & \text{if } i = 0, 1, 5, \\
\mathbb{Q} \cdot h^3|_{F_x} \oplus \mathbb{Q} \cdot (b \cdot h)|_{F_x} & \text{if } i = 2, \\
\mathbb{Q} \cdot h^2|_{F_x} \oplus \mathbb{Q} \cdot b|_{F_x} & \text{if } i = 3, \\
\mathbb{Q} \cdot h|_{F_x} \oplus \mathbb{Q} \cdot e|_{F_x} & \text{if } i = 4.
\end{cases}$$

We would like to prove proposition 5.4 following the strategy of proposition 5.2, i.e. invoking higher Chow groups. The only delicate point is that $F$ is singular, and we need to make sense of the operation of “capping with $h^k$ (or $b$)” on higher Chow groups. Since this seems difficult, we will prove proposition 5.4 without using higher Chow groups.

The piecewise triviality of the fibration $p_X$ means that there exist opens

$$F_0 = F \setminus F_{\geq 1}, \quad X_0 = X \setminus X_{\geq 1}$$

such that $F_0$ is isomorphic to the product $X_0 \times F_x$, and $F_{\geq 1} \to X_{\geq 1}$ is a piecewise trivial fibration (with fibre $F_x$). There is a commutative diagram with long exact rows

$$\to A_i(F_{\geq 1}) \to A_i(F) \to A_i(F_0) \to 0$$

$$\uparrow \Phi_{i|F_{\geq 1}} \quad \uparrow \Phi_i \quad \uparrow \Phi_{i|F_0}$$

$$\to \bigoplus A_{i-k}(X_{\geq 1}) \to \bigoplus A_{i-k}(X) \to \bigoplus A_{i-k}(X_0) \to 0.$$

The arrow $\Phi_{i|F_0}$ is an isomorphism, because the fibre $F_x$ is a linear variety in the sense of [45], which implies (by [45] Proposition 1, cf. also [46] Theorem 4.1) that the natural map

$$\bigoplus_{k+\ell=i} A_k(M) \otimes A_{\ell}(F_x) \to A_i(M \times F_x)$$

is an isomorphism for any variety $M$, and thus in particular for $M = X_0$. By noetherian induction, we may assume that $\Phi_{i|F_{\geq 1}}$ is surjective. Contemplating the diagram, we find that the middle arrow $\Phi_i$ is also surjective.

It remains to prove injectivity. To this end, let us define a map

$$\Psi_i: A_i(F) \to \bigoplus A_{i-k}(X),$$

$$a \mapsto \left( (p_X)_*(a), (p_X)_*(h \cap a), (p_X)_*(h^2 \cap a), (p_X)_*(h^2 \cap a), \ldots, (p_X)_*(h^5 \cap a) \right).$$

Using the above-mentioned isomorphism

$$\bigoplus_{k+\ell=i} A_k(X_0) \otimes A_\ell(F_x) \cong A_i(X_0 \times F_x),$$

one finds that $\Psi_i|_{F_0} \circ \Phi_i|_{F_0}$ is given by an invertible diagonal matrix. Dividing by some appropriate numbers, one can find $\Psi_i$ such that $\Psi_i|_{F_0} \circ \Phi_i|_{F_0}$ is the identity.

---

1 It is not clear whether operational Chow cohomology operates on higher Chow groups of a singular variety, which is a nuisance.
Using the projection formula, we see that there is a commutative diagram
\[
\begin{array}{ccccccc}
\bigoplus A_{i-k}(X_{\geq 1}) & \rightarrow & \bigoplus A_{i-k}(X) & \rightarrow & \bigoplus A_{i-k}(X_0) & \rightarrow & 0 \\
\uparrow \psi_i|_{F_{\geq 1}} & & \uparrow \psi_i & & \uparrow \psi_i|_{F_0} & & \\
A_i(F_{\geq 1}) & \rightarrow & A_i(F) & \rightarrow & A_i(F_0) & \rightarrow & 0 \\
\uparrow \phi_i|_{F_{\geq 1}} & & \uparrow \phi_i & & \uparrow \phi_i|_{F_0} & & \\
\bigoplus A_{i-k}(X_{\geq 1}) & \rightarrow & \bigoplus A_{i-k}(X) & \rightarrow & \bigoplus A_{i-k}(X_0) & \rightarrow & 0.
\end{array}
\]

We now make the following claim:

**Claim 5.5.** For any given \(i\), there exists a polynomial \(p_i(x) \in \mathbb{Q}[x]\) such that
\[
a = p_i(\Psi_i \circ \Phi_i)(a) \quad \forall a \in \bigoplus A_{i-k}(X).
\]

Clearly, the claim implies injectivity of \(\Phi_i\). To prove the claim, we apply noetherian induction. Given \(a \in \bigoplus A_{i-k}(X)\), we know that \(\psi_i|_{F_0} \circ \Phi_i|_{F_0}\) acts as the identity on the restriction \(a|_{X_0} \in \bigoplus A_{i-k}(X_0)\). It follows that we can write
\[
a - (\Psi_i \circ \Phi_i)(a) = b \quad \text{in} \quad \bigoplus A_{i-k}(X),
\]
where \(b\) is in the image of the pushforward map \(\bigoplus A_{i-k}(X_{\geq 1}) \rightarrow \bigoplus A_{i-k}(X)\). By noetherian induction, we may assume the claim is true for the piecewise trivial fibration \(F_{\geq 1} \rightarrow X_{\geq 1}\), and so there is a polynomial \(q_i\) such that
\[
b = q_i(\Psi_i \circ \Phi_i)(b) \quad \text{in} \quad \bigoplus A_{i-k}(X).
\]
Plugging this in (4), we find that
\[
a - (\Psi_i \circ \Phi_i)(a) = q_i(\Psi_i \circ \Phi_i)
\left(a - (\Psi_i \circ \Phi_i)(a)\right) \quad \text{in} \quad \bigoplus A_{i-k}(X).
\]

It follows that
\[
a = p_i(\Psi_i \circ \Phi_i)(a) \quad \text{in} \quad \bigoplus A_{i-k}(X),
\]
where the polynomial \(p_i\) is defined as
\[
p_i(x) := q_i(x) - qx_i(x) + x \quad \in \mathbb{Q}[x].
\]

(ii) As in proposition 5.2, one can also prove a homology version of (i). That is, for any \(j \in \mathbb{N}\) there are isomorphisms
\[
\Phi_j^b: \bigoplus H_{j-2k}(X) \xrightarrow{\cong} H_j(F),
\]

where \(\Phi_j^b\) is now defined as
\[
\Phi_j^b := \left(h^5 \circ (p_X)^*, h^4 \circ (p_X)^*, h^3 \circ (p_X)^*, (b \cdot h) \circ (p_X)^*, h^2 \circ (p_X)^*, b \circ (p_X)^*, h \circ (p_X)^*, (p_X)^*(-) \cap e, (p_X)^*\right).
\]
This is proven just as (i), using homology instead of higher Chow groups. Cycle class maps fit into a commutative diagram

$$\bigoplus A_{i-k}(X) \xrightarrow{\phi_i} A_i(F)$$

$$\downarrow \downarrow$$

$$\bigoplus H_{2i-2k}(X) \xrightarrow{\phi_{2i}} H_{2i}(F).$$

Horizontal arrows being isomorphisms, this proves (ii).

**Remark 5.6.** Comparing propositions [5.2] and [5.4] we observe that the only difference is the class $e \in A_7(F)$ appearing in proposition [5.4] but not in [5.2]. This “extra class” $e$ appears because of the singularities: the fibres of $p : Q \to G$ are smooth over the open $U \subset Gr_1$, but degenerate to singular fibres over $X \subset Gr_1$ (cf. remark [2.5]), and this causes an extra Weil divisor class $e$ to appear in the singular fibres. This observation will be key to the proof of theorem [6.1].

**6. Main result**

**Theorem 6.1.** Let $X, Y$ be a pair of GPK$^3$ double mirrors. Then there is an isomorphism

$$h(X) \cong h(Y) \quad \text{in } M_{\text{rat}}.$$

**Proof.** The proof is a four–step argument, which exploits that the threefolds

$X := Gr_1 \cap Gr_2 \subset \mathbb{P},$

$Y := Gr_1^\vee \cap Gr_2^\vee \subset \mathbb{P}^\vee$

are geometrically related as in proposition [2.3]. In essence, the argument is similar to the proof that $X, Y$ are L–equivalent [10, Section 7], by applying “cut and paste” to the diagram of proposition [2.3]. Here is an overview of the proof. Let

$$Q := \sigma \times_{\mathbb{P} \times \mathbb{P}^\vee} (Gr_1 \times Gr_2^\vee)$$

be the 11–dimensional intersection as in proposition [2.3]. Assuming $Q$ is non–singular, we prove there exist isomorphisms of Chow groups

$$A^i_{\text{hom}}(X) \xrightarrow{\cong} A^{i+4}_{\text{hom}}(Q) \quad \text{for all } i.$$  

(6)

This is done in step 1 (for $i = 2, 3$) and step 2 (for $i = 0, 1$), and relies on the isomorphisms for the piecewise trivial fibrations established in the prior section. In step 3, the isomorphism (6) is upgraded to an isomorphism of Chow motives

$$h^3(X) \cong h^{11}(Q)(-4) \quad \text{in } M_{\text{rat}}.$$

As $X$ and $Y$ are symmetric, this implies an isomorphism of Chow motives $h^3(X) \cong h^3(Y)$, and hence also $h(X) \cong h(Y)$. Finally, in step 4 we show that we may “spread out” this isomorphism to all GPK$^3$ double mirrors.
Step 1: an isomorphism of Chow groups. In this first step, we assume the $Gr_i$ are sufficiently general, so that $Q$ is non–singular (there is no loss in generality; the degenerate case where $Q$ may be singular will be taken care of in step 4 below). The goal of this first step will be to construct an isomorphism between certain Chow groups of $X$ and $Y$:

**Proposition 6.2.** There exist correspondences $\Gamma \in A^7(X \times Q)$, $\Psi \in A^7(Y \times Q)$ inducing isomorphisms

$$\Gamma_* : A^i_{\text{hom}}(X) \xrightarrow{\sim} A^{i+4}_{\text{hom}}(Q) \text{ for } i = 2, 3,$$

$$\Psi_* : A^i_{\text{hom}}(Y) \xrightarrow{\sim} A^{i+4}_{\text{hom}}(Q) \text{ for } i = 2, 3.$$

Before proving this proposition, let us first establish two lemmas (in these lemmas, we continue to assume $X, Y$ are sufficiently general, so that $Q$ is smooth):

**Lemma 6.3.** The pushforward map $(\iota_F)_* : A^i_{\text{hom}}(F) \rightarrow A^i_{\text{hom}}(Q)$ is surjective, for all $i$.

**Proof.** As before, let $R$ denote the open complement $R := Q \setminus F$. There is a commutative diagram with exact rows

$$
\begin{array}{cccccccccccc}
& & A_i(F) & \rightarrow & A_i(Q) & \rightarrow & A_i(R) & \rightarrow & 0 \\
0 & \rightarrow & H_{2i}(F, Q) & \rightarrow & H_{2i}(Q, Q) & \rightarrow & H_{2i}(R, Q) & ,
\end{array}
$$

where vertical arrows are cycle class maps. Here, the lower left entry is 0 because $R$ has no odd–degree homology (corollary 5.3). The lemma follows from the fact that the right vertical arrow is injective, which is proposition 5.2(ii). $\square$

**Lemma 6.4.** Let $e \in A_7(F)$ be as in proposition 5.4. The composition

$$A^i_{\text{hom}}(X) \xrightarrow{(p_X)_*} A^i_{\text{hom}}(F) \xrightarrow{(-) \cap e} A^i_{\text{hom}}(F) \xrightarrow{(\iota_F)_*} A^{i+4}_{\text{hom}}(Q) = A^i_{\text{hom}}(Q)$$

is an isomorphism for $i = 2, 3$. (As before, $A^*(F)$ denotes operational Chow cohomology of the singular variety $F$.)

**Proof.** Let us treat the case $i = 3$ in detail. Proposition 5.4 gives us an isomorphism

$$\Phi : A_3(X) \oplus A_2(X)^{\oplus 2} \oplus A_1(X)^{\oplus 2} \oplus A_0(X)^{\oplus 2} \xrightarrow{\sim} A_4(F),$$

where $\Phi := \Phi_4$ is defined as

$$\Phi = \left( h^4 \circ (p_X)^*, h^3 \circ (p_X)^*, (b \cdot h) \circ (p_X)^*, h^2 \circ (p_X)^*, b \circ (p_X)^*, h \circ (p_X)^*, (p_X)^*(-) \cap e \right).$$

We want to single out the part in $A_4(F)$ coming from the “extra class” $e \in A_7(F)$. That is, we write the isomorphism $\Phi$ as a decomposition

$$A_4(F) = A^\perp \oplus A,$$
where
\[ A := (p_X)^* A_0(X) \cap e , \]
\[ A^\perp := \text{Im} \left( A_3(X) \oplus A_2(X) \oplus A_1(X) \oplus A_0(X) \xrightarrow{\Phi^\perp} A_4(F) \right) , \]
\[ \Phi^\perp := \left( h^4 \circ (p_X)^* , h^3 \circ (p_X)^* , (b \cdot h) \circ (p_X)^* , h^2 \circ (p_X)^* , b \circ (p_X)^* , h \circ (p_X)^* \right) . \]

The decomposition (7) also exists in cohomology, and so there is an induced decomposition
\[ (8) \quad A_4^{\text{hom}}(F) = A_4^\perp \oplus A_4^{\text{hom}} , \]
where we put
\[ A_4^{\text{hom}} := A \cap A_4^{\text{hom}}(F) , \quad A_4^\perp := A^\perp \cap A_4^{\text{hom}}(F) . \]

We now claim that there is a commutative diagram with exact rows
\[ (9) \quad \begin{array}{cccc}
A_4(R, 1) & \xrightarrow{\delta} & A_4(F) & \xrightarrow{(\iota F)_*} & A_4(Q) \\
\uparrow \Phi_4 & & \uparrow \Phi^\perp & & \uparrow \\
\oplus A_k(U, 1) & \xrightarrow{\delta_U} & \oplus A_k(X) & \xrightarrow{\iota_*} & A_0(G) \oplus A_1(G)^{\oplus 2} \\
\end{array} , \]
where \( \Phi_4 \) is the isomorphism of proposition 5.2 (and we use the shorthand \( G := Gr_1 \)).

Granting this claim, let us prove the lemma for \( i = 3 \). The kernel of \( (\iota F)_* \) equals the image of the arrow \( \delta \). Since \( \Phi_4 \) is an isomorphism, the image \( \text{Im} \delta \) is contained in \( \text{Im} \Phi^\perp : A^\perp \). In view of the decomposition (7), this implies injectivity
\[ (10) \quad (\iota F)_* : A \hookrightarrow A_4(Q) , \]
i.e. the composition of lemma 6.4 is injective for \( i = 3 \).

To prove surjectivity, let us consider a class \( b \in A_4^\perp \). We know (from proposition 5.4(ii)) that
\[ b = \Phi^\perp(\beta) , \quad \beta \in A_4^{\text{hom}}(X) \oplus A_0^{\text{hom}}(X) . \]
Referring to diagram (9), we see that \( \iota_* (\beta) \) must be 0 (for the Grassmannian \( G \) has trivial Chow groups). It follows that \( \beta \) is in the image of \( \delta_U \), and so \( b \in \text{Im} \delta \). This shows that
\[ (\iota F)_*(A_4^\perp) = 0 , \]
and hence, in view of the decomposition (8), that
\[ (\iota F)_*(A_4^{\text{hom}}) = (\iota F)_*(A_4^{\text{hom}}(F)) . \]

On the other hand, we know that \( (\iota F)_*(A_4^{\text{hom}}(F)) = A_4^{\text{hom}}(Q) \) (lemma 6.3), and so we get a surjection
\[ (11) \quad (\iota F)_* : A_4^{\text{hom}} \twoheadrightarrow A_4^{\text{hom}}(Q) . \]
Combining (10) and (11), we see that the composition of lemma 6.4 is an isomorphism for \( i = 3 \).
It remains to establish the claimed commutativity of diagram (9). The morphism \( p: Q \to G \) is equidimensional of relative dimension 5, and so there is a commutative diagram of complexes (where rows are exact triangles)

\[
\begin{array}{c}
z_{i+5}(F, \ast) \to z_{i+5}(Q, \ast) \to z_{i+5}(R, \ast) \to \\
\uparrow (p_X)^* \quad \uparrow p^* \quad \uparrow (p_U)^* \\
z_i(X, \ast) \to z_i(G, \ast) \to z_{i+5}(U, \ast) \to
\end{array}
\]

Also, given a codimension \( k \) subvariety \( M \subset Q \), let \( z_i^M(Q, \ast) \subset z_i(Q, \ast) \) denote the subcomplex formed by cycles in general position with respect to \( M \). The inclusion \( z_i^M(Q, \ast) \subset z_i(Q, \ast) \) is a quasi–isomorphism [5, Lemma 4.2]. The diagram

\[
\begin{array}{c}
z_{i+5-k}(F, \ast) \to z_{i+5-k}(Q, \ast) \to z_{i+5-k}(R, \ast) \to \\
\uparrow \cdot M \quad \uparrow \cdot M|_R \\
z_i^M(Q, \ast) \to z_i^M(R, \ast) \to \\
\downarrow \cong \quad \downarrow \cong
\end{array}
\]

(where \( \cong \) indicates quasi–isomorphisms) defines an arrow in the homotopy category

\[ f_M : z_{i+5}(F, \ast) \to z_{i+5-k}(F, \ast). \]

(The arrow \( f_M \) represents “intersecting with \( M \)”). On the other hand, let \( g: \tilde{F} \to F \) be a resolution of singularities, and let \( \tilde{M} := (\iota_F \circ g)^*(M) \in A^k(\tilde{F}) \). The projection formula for higher Chow groups [5, Exercise 5.8(i)] gives a commutative diagram up to homotopy

\[
\begin{array}{c}
z_{i+5-k}(\tilde{F}, \ast) \xrightarrow{\text{(\ref{eq:proj_formula})}} z_{i+5-k}(Q, \ast) \\
\uparrow \cdot \tilde{M} \quad \uparrow \cdot M \\
z_i^M(\tilde{F}, \ast) \xrightarrow{\text{(\ref{eq:proj_formula})}} z_i^M(Q, \ast)
\end{array}
\]

and so there is also a commutative diagram up to homotopy

\[
\begin{array}{c}
z_{i+5-k}(\tilde{F}, \ast) \to z_{i+5-k}(F, \ast) \\
\uparrow \cdot \tilde{M} \quad \uparrow f_M \\
z_{i+5}(\tilde{F}, \ast) \to z_{i+5}(F, \ast)
\end{array}
\]

In particular, this shows that

\[ f_M = (\iota_F)^*(M) \cap (-): A_{i+5}(F) \to A_{i+5-k}(F), \]
where we consider \((\iota_F)^*(M) \in A^k(F)\) as an element in operational Chow cohomology.

Combining the above remarks, one obtains a commutative diagram with long exact rows

\[
\begin{array}{ccccccc}
A_{i+5-k}(R, 1) & \rightarrow & A_{i+5-k}(F) & \xrightarrow{(\iota_F)^*} & A_{i+5-k}(Q) & \rightarrow \\
\uparrow \cdot M_{\text{re}} & & \uparrow (\iota_F)^*(M) \cap (-) & & \uparrow & \\
A_{i+5}(R, 1) & \rightarrow & A_{i+5}(F) & \xrightarrow{(\iota_F)^*} & A_{i+5}(Q) & \rightarrow \\
\uparrow (p_U)^* & & \uparrow (p_X)^* & & \uparrow p^* & \\
A_i(U, 1) & \rightarrow & A_i(X) & \xrightarrow{\iota^*} & A_i(G) & \rightarrow .
\end{array}
\]

In particular, these diagrams exist for \(M\) being (a representative of) the classes \(h^j, b \cdot h, b \in A^*(Q)\) that make up the definition of the map \(\Phi^\perp\). It follows there is a commutative diagram with long exact rows

\[
\begin{array}{ccccccc}
A_i(R, 1) & \rightarrow & A_i(F) & \xrightarrow{(\iota_F)^*} & A_i(Q) & \rightarrow \\
\uparrow & & \uparrow \Phi^\perp & & \uparrow & \\
\bigoplus A_k(U, 1) & \rightarrow & \bigoplus A_k(X) & \rightarrow & \bigoplus A_k(G) & \rightarrow ,
\end{array}
\]

The \(i = 2\) case of the lemma is proven similarly: using proposition 5.2, we can write

\[
(12) \quad A_{5}^{\text{hom}}(F) = A_{5}^{\perp} \oplus (p_X)^* A_{2}^{\text{hom}}(X) \cap e .
\]

Here \(A_{5}^{\perp}\) is

\[
A_{5}^{\perp} = \text{Im} \left( A_{3}^{\text{hom}}(X) \oplus A_{2}^{\text{hom}}(X) \xrightarrow{\Phi^\perp} A_{5}(F) \right),
\]

where \(\Phi^\perp\) is defined as

\[
\Phi^\perp := \left( (p_X)^*(-), h \cap (p_X)^*(-) \right).
\]

As above, there is a commutative diagram with long exact rows

\[
\begin{array}{ccccccc}
A_5(R, 1) & \rightarrow & A_5(F) & \xrightarrow{(\iota_F)^*} & A_5(Q) & \rightarrow \\
\uparrow & & \uparrow \Phi^\perp & & \uparrow & \\
A_0(U, 1) \oplus A_1(U, 2) & \rightarrow & A_0(X) \oplus A_1(X) & \xrightarrow{\iota^*} & A_0(G) \oplus A_1(G) & \rightarrow .
\end{array}
\]

As above, chasing this diagram we conclude that

\[
(\iota_F)_{*} (A_{5}^{\perp}) = (\iota_F)_{*} \Phi^\perp \left( A_{0}^{\text{hom}}(X) \oplus A_{1}^{\text{hom}}(X) \right) = 0 \quad \text{in } A_{5}(Q) .
\]

It follows that the restriction of \((\iota_F)_{*}\) to the second term of the decomposition (12) induces an isomorphism

\[
(\iota_F)_{*} : (p_X)^* A_{2}^{\text{hom}}(X) \cap e \xrightarrow{\cong} A_{5}^{\text{hom}}(Q) .
\]
Let us now proceed to prove proposition 6.2. We will construct the correspondence $\Gamma$ (the construction of $\Psi$ is only notationally different, the roles of $X$ and $Y$ being symmetric). The variety $X$ is smooth, and the variety $Q$ of proposition 2.3 is also smooth, by our generality assumptions. The variety $F$, however, is definitely singular (remark 2.5), and so we need to desingularize. Let $g: \widetilde{F} \to F$ be a resolution of singularities. We let $\bar{e} \subset \widetilde{F}$ denote the strict transform of $e \subset F$, and $\widetilde{\bar{e}} \to \bar{e}$ a resolution of singularities, and we write $\tau: \widetilde{\bar{e}} \to \widetilde{F}$ for the composition of the resolution and the inclusion morphism. The correspondence $\Gamma$ will be defined as

$$
\Gamma := \Gamma_{\bar{e}} \circ \Gamma_g \circ \Theta_g \circ \Theta_{\bar{e}} \circ \Theta_{\tau} \circ \Theta_{\bar{e}} \circ \Theta_{\bar{e}} \in A^7(X \times Q).
$$

By definition, the action of $\Gamma$ decomposes as

$$
\Gamma_*: A^i(X) \xrightarrow{\Theta_{\bar{e}}} A^i(F) \xrightarrow{\tau_*} A^i(\widetilde{F}) \xrightarrow{\bar{e}_*} A^{i+1}(\widetilde{F}) = A_{7-i}(\widetilde{F}) \xrightarrow{g_*} A_{7-i}(Q) = A^{i+4}(Q).
$$

(Here $A^*(F)$ refers to Fulton–MacPherson’s operational Chow cohomology [12].) The projection formula for operational Chow cohomology (theorem 4.1) ensures that for any $b \in A^i(F)$ one has

$$
g_* (g^* (b) \cdot \bar{e}) = g_* (g^* (b) \cap \bar{e}) = b \cap e \quad \text{in} \quad A_{7-i}(F),
$$

and so the action of $\Gamma$ simplifies to

$$
\Gamma_*: A^i(X) \xrightarrow{(\Theta_{\bar{e}})_*} A^i(F) \xrightarrow{(\tau)_*} A_{7-i}(F) \xrightarrow{(\bar{e}_*)_*} A_{7-i}(Q) = A^{i+4}(Q). \quad \text{(13)}
$$

Lemma 6.4 ensures that $\Gamma_*$ induces isomorphisms

$$
\Gamma_*: A^i_{\text{hom}}(X) \xrightarrow{(\Theta_{\bar{e}})_*} A^i_{\text{hom}}(Q) \quad (i = 2, 3),
$$

and so we have proven proposition 6.2.

**Step 2: Trivial Chow groups.** In this step, we study the Chow groups of the incidence variety $Q$. We continue to assume that $Q$ is smooth, just as in step 1. The goal of step 2 will be to show that many Chow groups of $Q$ are trivial:

**Proposition 6.5.** We have

$$
A^i_{\text{hom}}(Q) = 0 \quad \text{for all} \quad i \not\in \{6, 7\}.
$$

(This means that $\text{Niveau}(A^*(Q)) \leq 3$ in the language of [32], i.e. the 11–dimensional variety $Q$ motivically looks like a variety of dimension 3.)

**Proof.** Suppose we can prove that

$$
A^i_{\text{hom}}(Q) = 0 \quad \text{for} \quad j < 4. \quad \text{(13)}
$$

Applying the Bloch–Srinivas argument [8], [32, Remark 1.8.1] to the smooth projective variety $Q$, this implies that

$$
A^i_A(Q) = 0 \quad \text{for all} \quad i \not\in \{6, 7\}.
$$
But $Q$ has no odd–degree cohomology except for degree 11 (proposition 6.6 below), and so there is equality $A^{i}_{A,\ell}(Q) = A^{i}_{\text{hom}}(Q)$ for all $i \neq 6$. That is, to prove proposition 6.5 one is reduced to proving (13).

There is an exact sequence

$$A_{j}(R, 1) \xrightarrow{\delta} A_{j}(F) \xrightarrow{(\ell_{F})_{\ast}} A_{j}(Q) \rightarrow ,$$

and we have seen (lemma 6.3) that $(\ell_{F})_{\ast}A^{i}_{\text{hom}}(F) = A^{i}_{\text{hom}}(Q)$. Thus, to prove the vanishing (13) it only remains to show that

$$A^{i}_{\text{hom}}(F) \subset \text{Im} \delta \quad \text{for } j < 4 \ .$$

The inclusion (14) is proven by the same argument as that of lemma 6.4. That is, we observe that propositions 5.4 and 5.2 give us isomorphisms

$$\Phi_{j} : \bigoplus A_{j-k}(X) \xrightarrow{\cong} A_{j}(F) ,$$

$$\Phi^{1}_{j} : \bigoplus A_{j-k}(U, 1) \xrightarrow{\cong} A_{j}(R, 1) ,$$

where $\Phi_{j}, \Phi^{1}_{j}$ are defined as

$$\Phi_{j} = \begin{cases} h^{5} \circ (p_{X})^{\ast} & \text{if } j = 0 , \\ \sum_{k=0}^{1} h^{5-k} \circ (p_{X})^{\ast} & \text{if } j = 1 , \\ \sum_{k=0}^{2} h^{5-k} \circ (p_{X})^{\ast} + b \circ (p_{X})^{\ast} & \text{if } j = 2 , \\ \sum_{k=0}^{3} h^{5-k} \circ (p_{X})^{\ast} + b \circ (p_{X})^{\ast} + (b \cdot h) \circ (p_{X})^{\ast} & \text{if } j = 3 , \end{cases}$$

$$\Phi^{1}_{j} = \begin{cases} h^{5} \circ (p_{U})^{\ast} & \text{if } j = 0 , \\ \sum_{k=0}^{1} h^{5-k} \circ (p_{U})^{\ast} & \text{if } j = 1 , \\ \sum_{k=0}^{2} h^{5-k} \circ (p_{U})^{\ast} + b \circ (p_{U})^{\ast} & \text{if } j = 2 , \\ \sum_{k=0}^{3} h^{5-k} \circ (p_{X})^{\ast} + b \circ (p_{U})^{\ast} + (b \cdot h) \circ (p_{U})^{\ast} & \text{if } j = 3 . \end{cases}$$

In particular, we observe that for each $j \leq 3$, the isomorphisms $\Phi_{j}, \Phi^{1}_{j}$ of (15) have the same number of direct summands on the left–hand side (i.e., the “extra class” $e \in A_{7}(F)$ does not appear). For each $j \leq 3$, we can construct a commutative diagram with exact rows

$$A_{j}(R, 1) \xrightarrow{\delta} A_{j}(F) \xrightarrow{(\ell_{F})_{\ast}} A_{j}(Q) \rightarrow ,$$

$$\uparrow \Phi^{1}_{j} \quad \uparrow \Phi_{j} \quad \uparrow$$

$$\bigoplus A_{j-k}(U, 1) \xrightarrow{\delta_{U}} \bigoplus A_{j-k}(X) \xrightarrow{\cong} \bigoplus A_{j-k}(Gr_{1})$$

(the commutativity of this diagram is checked as in the proof of lemma 6.4).

Let $a \in A^{j}_{\text{hom}}(F)$, for $j \leq 3$. Then $a = \Phi_{j}(\alpha)$ for some $\alpha \in \bigoplus A^{j}_{\text{hom}}(X)$ (proposition 5.4(ii)). But then $\ell_{j}(\alpha) = 0$, since the Grassmannian $Gr_{1}$ has trivial Chow groups. Using the above diagram, it follows that $\alpha$ is in the image of $\delta_{U}$, and hence $a \in \text{Im} \delta$. This proves (14) and hence proposition 6.5.
(NB: in fact, the above argument does not need that $\Phi_{j}^{1}$ is an isomorphism; we merely need the fact that a map $\Phi_{j}^{1}$ fitting into the above commutative diagram exists.)

To close step 2, it only remains to prove the following proposition:

**Proposition 6.6.** Assume $j$ is odd and $j \neq 11$. Then

$$H_{j}(Q) = 0.$$  

**Proof.** For $j$ odd and different from 11, there is a commutative diagram with exact row

$$
\begin{array}{c}
H_{j+1}(R) \xrightarrow{\delta} H_{j}(F) \xrightarrow{(\iota, F)} H_{j}(Q) \rightarrow 0 \\
\oplus H_{j+1-2k}(U) \rightarrow \oplus H_{j-2k}(X).
\end{array}
$$

Here $\Phi^{h}$ and $\Phi^{h+1}$ are the isomorphisms of (5) resp. (3). The righthand 0 is because $H_{j}(R) = 0$ for $j$ odd (corollary 5.3).

We observe that $j \neq 11$ implies that $k \neq 4$ (the only odd homology of $X$ is $H_{3}(X)$), which means that the “extra class” $e \in A_{7}(F)$ does not intervene in the map $\Phi^{h}_{j}$. It follows that there are the same number of direct summands in the isomorphisms $\Phi^{h}_{j}$, $\Phi^{h+1}_{j}$ (they are both defined in terms of $h^{k}$ and $b$). Observing that a Grassmannian does not have odd–degree cohomology, we thus see that the lower horizontal arrow is surjective. The diagram now shows that $\delta$ is surjective, and hence $(\iota, F)$ is the zero–map. The proposition is proven. □

This ends the proof of proposition 6.5. □

**Step 3: an isomorphism of motives.** We continue to assume (as in steps 1 and 2) that $X, Y$ are general, so that $Q$ is smooth.

The assignment $\pi^{3}_{X} := \Delta_{X} - \pi^{0}_{X} - \pi^{2}_{X} - \pi^{4}_{X} - \pi^{6}_{X}$ (where the Künneth components $\pi^{i}_{X}$, $j \neq 3$ are defined using hyperplane sections) defines a motive $h^{3}(X)$ such that there is a splitting

$$h(X) = 1 \oplus 1(1) \oplus h^{3}(X) \oplus 1(2) \oplus 1(3)$$

in $\mathcal{M}_{rat}$, where 1 is the motive of a point. It follows that

$$A^{i}(h^{3}(X)) = A^{i}_{hom}(X) \forall i.$$  

The variety $Q$ is a smooth ample divisor in the product $P := Gr_{1} \times Gr_{2}$. The product $P$ has trivial Chow groups, and hence in particular verifies the standard conjectures. It follows that $P$ admits a (unique) Chow–Küneth decomposition $\{\pi^{i}_{p}\}$, and that there exist correspondences $C^{j} \in A^{12+j}(P \times P)$ such that

$$(C^{j})_{*} : H^{12+j}(P) \rightarrow H^{12-j}(P)$$

is inverse to

$$\cup Q^{j} : H^{12-j}(P) \rightarrow H^{12+j}(P).$$

(The correspondences $C^{j} \in A^{12+j}(P \times P)$ are well–defined, as rational and homological equivalence coincide on $P \times P.$)
Let \( \tau : Q \to P \) denote the inclusion morphism. One can construct a Chow–Künneth decomposition for \( Q \), by setting
\[
\pi^i_Q := \begin{cases} 
\Gamma_\tau \circ (\Gamma_\tau \circ \Gamma_\tau)^{011-i} \circ C^{12-i} \circ \pi^i_P \circ \Gamma_\tau & \text{if } i < 11, \\
\pi^{22-i} & \text{if } i > 11, \\
\Delta_Q - \sum_{j \neq 11} \pi^j_Q & \text{if } i = 11.
\end{cases}
\]

(To check this is indeed a Chow–Künneth decomposition, one remarks that
\[
(\Gamma_\tau \circ \Gamma_\tau)^{012-i} \circ C^{12-i} \circ \pi^i_P = \pi^i_P \quad \text{in } H^{24}(P \times P),
\]
and because \( P \times P \) has trivial Chow groups one has the same equality modulo rational equivalence:
\[
(\Gamma_\tau \circ \Gamma_\tau)^{012-i} \circ C^{12-i} \circ \pi^i_P = \pi^i_P \quad \text{in } A^{12}(P \times P).
\]

It is now readily checked that \( \{\pi^i_Q\} \) verifies \( \pi^i_Q \circ \pi^j_Q = \delta_{ij} \pi^i_Q \) in \( A^{11}(Q \times Q) \), where \( \delta_{ij} \) is the Kronecker symbol.)

Setting \( h^i(Q) := (Q, \pi^i_Q, 0) \), this induces a decomposition of the motive of \( Q \) as
\[
h(Q) = \bigoplus I(*) \oplus h^{11}(Q) \quad \text{in } \mathcal{M}_{\text{rat}},
\]
and hence one has
\[
(17) \quad A^i(h^{11}(Q)) = A^i_{\text{hom}}(Q) \quad \forall i.
\]

We now consider the homomorphism of motives
\[
\Gamma : \quad h^3(X) \to h^{11}(Q)(-4) \quad \text{in } \mathcal{M}_{\text{rat}},
\]
where \( \Gamma \in A^7(X \times Q) \) is as in step 1. We have seen in steps 1 and 2 that there are isomorphisms
\[
\Gamma_* : \quad A^i_{\text{hom}}(X) \xrightarrow{\cong} A^{i+4}_{\text{hom}}(Q) \quad \forall i.
\]

In view of (16) and (17), this translates into
\[
\Gamma_* : \quad A^i(h^3(X)) \xrightarrow{\cong} A^{i+4}(h^{11}(Q)) = A^i(h^{11}(Q)(-4)) \quad \forall i.
\]

Using that the field \( \mathbb{C} \) is a universal domain, this implies (cf. [17, Lemma 1.1]) there is an isomorphism of motives
\[
\Gamma : \quad h^3(X) \xrightarrow{\cong} h^{11}(Q)(-4) \quad \text{in } \mathcal{M}_{\text{rat}}.
\]
The roles of \( X \) and \( Y \) being symmetric, the same argument also furnishes an isomorphism
\[
\Psi : \quad h^3(Y) \xrightarrow{\cong} h^{11}(Q)(-4) \quad \text{in } \mathcal{M}_{\text{rat}}.
\]
The result is an isomorphism
\[
\Psi^{-1} \circ \Gamma : \quad h^3(X) \xrightarrow{\cong} h^3(Y) \quad \text{in } \mathcal{M}_{\text{rat}}.
\]

Since the difference \( h(X) - h^3(X) \) is just \( 1 \oplus 1(1) \oplus 1(2) \oplus 1(3) \), which is the same as \( h(Y) - h^3(Y) \), there is also an isomorphism
\[
h(X) \xrightarrow{\cong} h(Y) \quad \text{in } \mathcal{M}_{\text{rat}}.
\]
**Step 4: Spreading out.** We have now proven that there is an isomorphism of Chow motives $h^3(X) \cong h^3(Y)$ for a general pair of GPK$^3$ double mirrors. To extend this to all pairs of double mirrors, we reason as follows. Let

$$\pi_X : \mathcal{X} \to B$$

denote the universal family of all GPK$^3$ threefolds (so $B$ is an open in $PGL(\wedge^2 V)$). The double mirror construction corresponds to an involution $\sigma$ on $B$, such that $X_b$ and $Y_b := X_{\sigma(b)}$ are double mirrors. We define the family $\mathcal{Y}$ as the composition

$$\pi_Y := \sigma \circ \pi_X : \mathcal{Y} := \mathcal{X} \to B.$$ 

The above construction of the correspondences $\Gamma, \Psi$ can be done relatively: over the open $B^0 \subset B$ where the incidence variety $Q$ is smooth, one obtains relative correspondences

$$\Gamma \in A^3(\mathcal{X} \times_{B^0} \mathcal{Y}), \quad \Psi \in A^3(\mathcal{Y} \times_{B^0} \mathcal{X}),$$

such that for each $b \in B^0$ the restrictions $\Gamma_b \in A^3(X_b \times Y_b), \Psi_b \in A^3(Y_b \times X_b)$ verify

$$\pi^3_{X_b} = \Psi_b \circ \Gamma_b \quad \text{in} \quad A^3(X_b \times X_b),$$

$$\pi^3_{Y_b} = \Gamma_b \circ \Psi_b \quad \text{in} \quad A^3(Y_b \times Y_b).$$

Taking the closure, one obtains extensions $\bar{\Gamma} \in A^3(\mathcal{X} \times_B \mathcal{Y}), \bar{\Psi} \in A^3(\mathcal{Y} \times_B \mathcal{X})$ to the larger base $B$, that restrict to $\Gamma$ resp. $\Psi$.

The correspondences $\pi^3_{X_b}, \pi^3_{Y_b}$ also exist relatively (note that any GPK$^3$ threefold has Picard number 1, and so $\pi^2, \pi^4$ exist as relative correspondences). The relative correspondences

$$\pi^3_{\mathcal{X}} - \bar{\Psi} \circ \bar{\Gamma} \quad \text{in} \quad A^3(\mathcal{X} \times_B \mathcal{X}), \quad \pi^3_{\mathcal{Y}} - \bar{\Gamma} \circ \bar{\Psi} \quad \text{in} \quad A^3(\mathcal{Y} \times_B \mathcal{Y})$$

have the property that their restriction to a general fibre is rationally trivial. But this implies (cf. [50, Lemma 3.2]) that the restriction to every fibre is rationally trivial, and hence we obtain an isomorphism of motives for all $b \in B$, i.e. for all pairs $(X_b, Y_b)$ of double mirrors. \hfill \qed

**Remark 6.7.** In the proof of theorem 6.1, we have not explicitly determined the inverse to the isomorphism $\Gamma$. With some more work, it is actually possible to show that there exists $m \in \mathbb{Q}$ such that

$$\frac{1}{m} \Gamma : h^{11}(Q) \to h^3(X) \quad \text{in} \quad M_{\text{rat}}$$

is inverse to $\Gamma$.

**Remark 6.8.** It would be interesting to extend theorem 6.1 to the category $M_{\mathbb{Z}_{\text{rat}}}$ of Chow motives with integral coefficients. Let $X, Y$ be as in theorem 6.1. Is it true that $h(X)$ and $h(Y)$ are isomorphic in $M_{\mathbb{Z}_{\text{rat}}}$? Steps 1 and 2 in the above proof probably still work for Chow groups with $\mathbb{Z}$–coefficients (one just needs to upgrade the fibration results of section 5 to $\mathbb{Z}$–coefficients); steps 3 and 4, however, certainly need $\mathbb{Q}$–coefficients.

**Remark 6.9.** In all likelihood, an argument similar to that of theorem 6.1 could also be applied to establish an isomorphism of Chow motives for the Grassmannian–Pfaffian Calabi–Yau varieties of [9], [37], as well as for the Calabi–Yau fivefolds of [36].
7. SOME COROLLARIES

Corollary 7.1. Let $X, Y$ be two $GPK^3$ double mirrors. Let $M$ be any smooth projective variety. Then there are isomorphisms

$$N^j H^i(X \times M, \mathbb{Q}) \cong N^j H^i(Y \times M, \mathbb{Q})$$

for all $i, j$.

(Here, $N^*$ denotes the coniveau filtration \cite{7}.)

Proof. Theorem 6.1 implies there is an isomorphism of Chow motives $h(X \times M) \cong h(Y \times M)$. As the cohomology and the coniveau filtration only depend on the motive \cite{2}, \cite{47}, Proposition 1.2], this proves the corollary.

Remark 7.2. It is worth noting that for any derived equivalent threefolds $X, Y$, there are isomorphisms

$$N^j H^i(X, \mathbb{Q}) \cong N^j H^i(Y, \mathbb{Q})$$

for all $i, j$;

this is proven in \cite{1}.

Corollary 7.3. Let $X, Y$ be two $GPK^3$ double mirrors. Then there are (correspondence–induced) isomorphisms between higher Chow groups

$$A^i(X, j) \cong A^i(Y, j)$$

for all $i, j$.

There are also (correspondence–induced) isomorphisms in higher algebraic $K$–theory

$$G_j(X)_\mathbb{Q} \cong G_j(Y)_\mathbb{Q}$$

for all $j$.

Proof. This is immediate from the isomorphism of Chow motives $h(X) \cong h(Y)$.

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