X₂ series of universal quantum dimensions

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Abstract

The antisymmetric square of the adjoint representation of any simple Lie algebra is equal to the sum of adjoint and X₂ representations. We present universal formulae for quantum dimensions of an arbitrary Cartan power of X₂, and analyze them for singular cases and permuted universal Vogel’s parameters. X₂ has been the only representation in the decomposition of the square of the adjoint with unknown universal series. Application to universal knot polynomials is discussed.

Keywords: simple Lie algebras, universal Lie algebra, Vogel’s plane

1. Introduction

The universal formulae for simple Lie algebras were first derived by Vogel in his universal Lie algebra [1, 2], see also [3]. The main aim was to derive the most general weight system for Vassiliev’s finite knot invariants. This program met difficulties, however, as a byproduct there appeared a uniform parameterization of simple Lie algebras by the values of Casimir operators on three representations, appearing in the decomposition of the symmetric square of the adjoint representations:

\[ S^2 g = 1 + Y_2(\alpha) + Y_2(\beta) + Y_2(\gamma). \] (1)

One denotes the value of the second Casimir operator on the adjoint representation g as 2t, and parameterizes the values of the same operator on representations in (1) as \( 4t - 2\alpha, 4t - 2\beta, 4t - 2\gamma \) correspondingly (see notation of representations in (1)). It appears that \( \alpha + \beta + \gamma = t \). The values of the parameters for all simple Lie algebras are given in table 1, and in table 2 in another form. According to the definitions, the entire theory is invariant with respect to the rescaling of the parameters (which corresponds to the rescaling of the invariant scalar product in algebra), and with respect to the permutations of the universal (=Vogel’s) parameters \( \alpha, \beta, \gamma \). So, effectively they belong to a projective plane, which is factorized w.r.t. its homogeneous coordinates, and is called Vogel’s plane.

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As an example of application of this parameterization universal formulae \[1, 4\] for dimensions of representations from (1) are presented below:

\[
\dim g = \frac{(2t - \alpha)(2t - \beta)(2t - \gamma)}{\alpha \beta \gamma} \tag{2}
\]

\[
\dim Y_2(\alpha) = \frac{(2t - 3\alpha) (\beta - 2t) (\gamma - 2t) t (\beta + t) (\gamma + t)}{\alpha^2 (\alpha - \beta) \beta (\alpha - \gamma) \gamma} \tag{3}
\]

and other two (3) representations which are obtained by permutations of the parameters. These are typical universal formulae for dimensions: ratios of products of linear homogeneous functions of universal parameters.

In works of Deligne et al \[5, 6\] it is shown that all exceptional Lie algebras are located on a line on Vogel’s plane, just like \(\mathfrak{sl}(n)\) and \(\mathfrak{so}(n)/\mathfrak{sp}(n)\) algebras. The idea is that the exceptional line is similar to the ones of special linear and orthogonal algebras, namely, for the exceptional algebras it is also possible to represent the dimensions of all irreps as a ratio of products of linear functions of parameters on the exceptional line. This has been checked for all irreps, appearing in the decomposition of up to the fourth order of the adjoint representation in \[7\], and will be compared with our results below. We shall call this ‘universality on the exceptional line’. Whenever both universal and universal on the exceptional line formulae exist, they coincide.

A number of universal formulae is known for different objects in the theory and applications of simple Lie algebras. E.g. Vogel \[1\] found complete decomposition of third power of the adjoint representation in terms of a self-defined universal Lie algebra and universal dimension formulae for all representations involved. Landsberg and Manivel \[4\] present a method which allows to derive certain universal dimension formulae for simple Lie algebras and derive those for Cartan powers of the adjoint, \(Y_2(\cdot)\), and their Cartan products. Westbury \[8\]

### Table 1. Vogel’s parameters for simple Lie algebras.

| Root system | Lie algebra | \(\alpha\) | \(\beta\) | \(\gamma\) | \(t = \hbar^2\) |
|-------------|-------------|-----------|--------|--------|-------------|
| \(A_n\)    | \(\mathfrak{sl}_{n+1}\)   | -2       | 2      | \(n+1\) | \(n+1\)    |
| \(B_n\)    | \(\mathfrak{so}_{2n+1}\)   | -2       | 4      | \(2n-3\) | \(2n-1\)    |
| \(C_n\)    | \(\mathfrak{sp}_{2n}\)     | -2       | 1      | \(n+2\)  | \(n+1\)    |
| \(D_n\)    | \(\mathfrak{so}_{2n}\)     | -2       | 4      | \(2n-4\) | \(2n-2\)    |
| \(G_2\)    | \(\mathfrak{g}_2\)         | -2       | 10/3   | 8/3     | 4           |
| \(F_4\)    | \(\mathfrak{f}_4\)         | -2       | 5      | 6       | 9           |
| \(E_6\)    | \(\mathfrak{e}_6\)         | -2       | 6      | 8       | 12          |
| \(E_7\)    | \(\mathfrak{e}_7\)         | -2       | 8      | 12      | 18          |
| \(E_8\)    | \(\mathfrak{e}_8\)         | -2       | 12     | 20      | 30          |

### Table 2. Vogel’s parameters for simple Lie algebras: lines.

| Algebra/Parameters | \(\alpha\) | \(\beta\) | \(\gamma\) | \(t\)     | Line          |
|-------------------|-----------|--------|--------|--------|--------------|
| \(\mathfrak{sl}_N\) | -2       | 2      | \(N\)  | \(N\)  | \(\alpha + \beta = 0\) |
| \(\mathfrak{so}_N\) | -2       | 4      | \(N-4\)| \(N-2\)| \(2\alpha + \beta = 0\) |
| \(\mathfrak{sp}_N\) | -2       | 1      | \(N/2+2\)| \(N/2+1\)| \(\alpha + 2\beta = 0\) |
| \(\text{Exc}(n)\) | -2       | \(2n+4\)| \(n+4\)| \(3n+6\)| \(\gamma = 2(\alpha + \beta)\) |

For the exceptional line \(n = -2/3, 0, 1, 2, 4, 8\) for \(\mathfrak{g}_2, \mathfrak{so}_6, \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8\), respectively.
found a universal formula for the quantum dimension of the adjoint representation. Sergeev, Veselov and one of the present authors derived [9] a universal formula for generating function for the eigenvalues of higher Casimir operators on the adjoint representation.

In subsequent works applications to physics were developed, particularly the universality of the partition function of Chern–Simons theory on a sphere [10–12], and its connection with q-dimension of ℬΛ0 representation of affine Kac–Moody algebras [13] were shown, the universal knot polynomials for 2- and 3-strand torus knots [14–17] were calculated.

Another application of universal formulae is the derivation of non-perturbative corrections to Gopakumar-Vafa partition function [18, 19] by gauge/string duality from the universal partition function of Chern–Simons theory. This shows the relevance of the ‘analytical continuation’ of the universal formulæ from the points of Vogel’s table 1 to the entire Vogel’s plane.

A completely different direction of development—the Diophantine classification of simple Lie algebras [20] and its connection with McKay correspondence, [21] is also worth mentioning.

The present paper is part of a larger project, aimed to continue the construction of universal knot polynomials for torus knots [14, 15]. In the context of the present paper ‘universal’ refers to the polynomial, which depends on universal Vogel’s parameters α, β, γ. When these parameters accept values from Vogel’s table, the universal knot polynomials become HOMFLY, or Kauffman, or exceptional groups’ polynomials. So one has truly universal knot polynomials, joining in one expression all previously known types of invariant knot polynomials. In [14, 15] these polynomials are obtained for 2-strand torus knots, and in [15] for 3-strand ones. Such polynomials for arbitrary torus knots are of huge interest.

The works [14, 15] are based on the following Rosso–Jones formula [22] for torus knot polynomials (Wilson loops average) for the representation $R$:

$$P_R^{[m,n]} = \frac{q^{m|X_2}}{D_R(q)} \sum_y \sum_Q q^{-\frac{1}{4} \kappa_Q} \varphi_y(a^{[m,n]})D_y(q).$$

Here $\kappa_Q$ denote the second Casimir operators on $Q$ representations and $D_y(q)$ is the quantum dimensions of the same representation. $Q$ run over the irreps in the decomposition of the $m$th power of representation $R$ (adjoint, in our case) with symmetry of a Young diagram $Y$ with $m$ boxes. Other elements of this formula are not dependent on the group, so we do not consider them here (see e.g. [15] for details, and notations). Evidently, to have a universal output for the Rosso–Jones formula, one needs universal expressions for Casimir eigenvalues and quantum dimensions of representations $Q$. In the present paper we find universal expressions for quantum dimensions for some of the representations $Q$.

Namely, we derive the quantum dimensions for an arbitrary Cartan power of the $X_2$ representation, appearing in the following decomposition

$$\wedge^2 g = g + X_2.$$

$k$th Cartan power of a representation with highest weight $\lambda$ is that with highest weight $k\lambda$. Note that for $sl(n)$ algebras $X_2$ is not an irreducible representation, until one considers the Lie algebra’s semidirect product with the automorphism group of its Dynkin diagram (instead of the algebra itself), as suggested and implemented in [5–7] for the exceptional algebras. Particularly, in $sl(n)$ case one has $Z_2$ as an automorphism group and $X_2$ is the sum of representations with highest weights $2\omega_1 + \omega_{n-2}$ and $\omega_2 + 2\omega_{n-1}$. We consider its Cartan power to be the sum of Cartan powers of these two representations. More generally, any irrep of simple Lie algebras below are considered to be extended by the automorphism group of their Dynkin diagram. We will see, that universal formulæ give answers for irreps of such extended Lie
algebras, i.e. if there appears an irrep which is not invariant under the automorphism, then it appears in the combination with its automorphism-transformed version(s), so that the invariance is recovered.

For $k = 1$ the universal quantum dimension of $X_2$ is given in [23]:

$$D_{Q}^{X_2} = \frac{\sinh \left( \frac{\pi}{4} (2t - \alpha) \right) \sinh \left( \frac{\pi}{4} (2t - \beta) \right) \sinh \left( \frac{\pi}{4} (2t - \gamma) \right)}{\sinh \left( \frac{\pi}{4} \right) \sinh \left( \frac{\pi}{4} \right) \sinh \left( \frac{\pi}{4} \right)} \times \frac{\sinh \left( \frac{\pi}{4} (t + \alpha) \right) \sinh \left( \frac{\pi}{4} (t + \beta) \right) \sinh \left( \frac{\pi}{4} (t + \gamma) \right)}{\sinh \left( \frac{\pi}{4} \right) \sinh \left( \frac{\pi}{4} \right) \sinh \left( \frac{\pi}{4} \right)} \times \frac{\sinh \left( \frac{\pi}{4} (t - \alpha) \right) \sinh \left( \frac{\pi}{4} (t - \beta) \right) \sinh \left( \frac{\pi}{4} (t - \gamma) \right)}{\sinh \left( \frac{\pi}{4} (t - \alpha) \right) \sinh \left( \frac{\pi}{4} (t - \beta) \right) \sinh \left( \frac{\pi}{4} (t - \gamma) \right)}.$$ (6)

The proof and discussion of the properties of this and the general formula for $k > 1$ is given further in the paper.

In the decomposition of the the square of the adjoint representation $X_2$ has been the only representation, with no universal formulae for the (quantum) dimensions of its Cartan powers. For the powers of the other representations, i.e. $Y_2(.)$, both usual and quantum dimensions in universal form are given in [4, 13, 15].

In the next sections we outline the way in which we derive the final formula for the quantum dimension of $k$th Cartan power of $X_2$. The derivation is actually an ‘educated guess’ based on the data from some exceptional algebras and knowledge of existing universal formulae. So, it needs to be proven, which is done in the appendix. It is worth mentioning that the very existence of these formulae is not guaranteed, because of the existence of zero divisors in $\Lambda$ algebra discovered by Vogel [2]. The understanding of the ‘area’ of existence of universal formulae is an important unsolved problem. We assume that some insight may come from physical applications of universality such as in [19].

A highly non-trivial feature of our formula becomes evident when considering it for the permuted parameters. Having in mind, that the universal formulae usually make sense for permuted Vogel’s parameters, we investigate our formula for quantum dimensions of powers of $X_2$ for permuted parameters in section 5. We show that it ‘automatically’ gives quantum dimensions for some other representations of corresponding simple Lie algebras.

2. Technique

There is no regular way of obtaining universal formulae and their very existence is not guaranteed. Vogel’s approach gave unique answers for dimensions, but it was based on the calculation with $\Lambda$ ring, which appears to have [2] divisors of zero, so that approach is not self-consistent if one does not handle that issue carefully. In [4] (and in the present work) the restricted definition of universal formulae is adopted, as defined above—they have to give correct answers for true simple Lie algebras on the corresponding points of Vogel’s table 1.

That allows one to use the Weyl formula for characters, restricted to the Weyl line, i.e. for quantum dimensions (see e.g. [25], 13.170):

$$D_{Q}^{\lambda} = \chi_{\lambda}(xp) = \prod_{\mu > 0} \frac{\sinh (\frac{\pi}{4}(\mu, \lambda + \rho))}{\sinh (\frac{\pi}{4}(\mu, \rho))}$$ (7)
where \( \lambda \) is the highest weight of the given irreducible representation, \( \rho \) is the Weyl’s vector, the sum of the fundamental weights, and the product is over all positive roots \( \mu \). The usual dimensions are obtained in the \( x \to 0 \) limit of the quantum ones. Both sides of this formula are invariant w.r.t. the simultaneous rescaling (in ‘opposite directions’) of the scalar product in algebra and the parameter \( x \). The automorphism of the Dynkin diagram leads to the equality of quantum dimensions of representations corresponding to the highest weights connected by automorphism.

Evidently, only the roots with non-zero scalar product with \( \lambda \) contribute. So, one has to express the scalar product of such roots with \( \lambda \) and \( \rho \) in terms of the universal parameters, and that has to be done in a uniform way for all simple Lie algebras. Then one may hope to get a universal expression for \( D_\lambda \).

To describe the technique, consider, e.g. the case of \( \lambda = \theta \), the highest weight of the adjoint representation. As it is shown in [4], the values of the scalar products of roots with \( \theta \) are either 2 (for root \( \theta \) itself) or 1. These roots can be organized into three ‘segments’ (see definition below) with unit spacing of \( (\rho, \alpha) \) (we normalize the scalar product as in [4] and table 1 by \( \alpha = -2 \), which we present in table 3 for \( E_7 \) as an example. In the first line there are the values of scalar products with \( \rho \), i.e. the heights \( n \) of roots, in the second—\( n_0 \)—the number of roots on that height (remember we consider the roots \( \mu \) with \( (\mu, \theta) \), only). So, we see, that roots with \( (\theta, \mu) = 1 \) can be organized into three sets of roots, which we call ‘segments of roots’, or simply segments. The segment of roots is the finite number of roots with equidistant values of heights including exactly one root for any given height from that equidistant sequence of heights.

The first and the longest segment has length \( t - 2 = 16 \), with heights from 1 to 16, the second, in the center of the first, is of length \( \gamma - 2 = 10 \) (we order universal parameters as \( \gamma \geq \beta > -2 \)), and the third segment, again in the center of the first (and the second) segments, has length \( \beta - 2 \). The same pattern of segments is observed for most of the simple Lie algebras.

With this data it is easy to obtain universal formulae for dimensions [4] and quantum dimensions [13] for \( k \)th Cartan power of the adjoint representation. Namely, the numerators of consecutive roots of the given segment of roots cancel out with the denominators (22), so for each segment there remains a number of the first denominators and the same number of the last numerators, which finally lead to the universal formulae.

These results have been proven in [4] partially by ‘general’ considerations, restricted, however, to the algebras of the rank at least three, and partially by case by case considerations for each algebra separately.

The description above reflects the advantage of the approach—the possibility of using Weyl formula as a basis of calculations, and shortcomings, which come from the use of very restricted sets of truly existing simple Lie algebras (see more on that below). Particularly, one can add an arbitrary polynomial which accepts zero values on the lines of the simple Lie algebras (tables 1 and 2). Such ‘minimal’ symmetric polynomial can be easily written:

\[
(\alpha + \beta)(\beta + \gamma)(\gamma + \alpha)(2\alpha + \beta)(2\beta + \alpha)(2\alpha + \gamma)(2\gamma + \alpha),
\]

\[
(2\beta + \gamma)(2\gamma + \beta)(2\gamma + 2\alpha - \beta)(2\gamma + 2\beta - \alpha)(2\alpha + 2\beta - \gamma).
\]

However, one can require that, first, the formula should be presented as a ratio of products of linear functions over universal parameters (and not the sum of such expressions), and, second, that Deligne hypothesis [23] should be satisfied. Deligne assumes that the standard relations of characters (recall that quantum dimensions are characters on the Weyl line) namely, the product of characters of two representations is equal to the sum of characters of their decomposition, should be satisfied on the entire Vogel’s plane (and not on the points of
Deligne’s hypothesis is checked in some cases [17], particularly for symmetric cube of the adjoint representation. At this time it is not known whether it is possible to satisfy one or both of these requirements, as well as the very existence of universal formulae is not guaranteed. So, we do not approach this problem further in this paper, and present the new universal formulae in the original way we found them.

Below we use this approach to obtain the universal formulae for quantum dimensions of $k$th Cartan powers of $X_2$ representation.

In the next section we present data for $E_n$ algebras and try to rewrite them in the universal form. It appears that because of the ambiguities of rewriting the answers in the universal form the data is not sufficient for derivation of a general formula. So we use two additional ideas: first is that the results should not be singular for $sl(n)$ algebra, and second, that the answer should be invariant w.r.t. the permutation of two parameters. Putting forward these additional requirements, we obtain the final formula (22) below. All this, however, does not combine into formal derivation and altogether should be considered as an educated guess. The formal proof is carried out in the appendix, for all algebras. We nevertheless outline these steps to show how we came to the final, sufficiently complicated formula. The development of a general method for derivation of universal formulae still remains an open problem.

### 3. $E_n$ data

It appears that $E_n$ are the only algebras, which can hint on a universal form of non-trivial contributions to the Weyl formula (22) for $X_2$ representation, the highest weight of which we denote $\lambda$. So below we present the relevant roots and their contributions. From now on we use Dynkin’s labeling [24] of Dynkin diagrams.

#### 3.1. $E_8$

Dimension of $E_8 = 248$, number of positive roots $|\Delta_+| = 120$, Vogel’s parameters $(\alpha, \beta, \gamma) = (-2, 12, 20)$. For $E_8$ the highest weight of $X_2$ is $\lambda = \omega_6$, in Dynkin’s labeling of roots.

The number of positive roots $\mu$ with $(\lambda, \mu) = 0$ is $1 + |\Delta_+|_{E_8} = 1 + 36 = 37$. The number of positive roots $\mu$ with $(\lambda, \mu) = 1$ is 54 and is given in table 4 with corresponding numbers $n_{ht}$ of roots with given scalar product with $\rho$.

So, here we have five segments of roots. The number of positive roots $\mu$ with $(\lambda, \mu) = 2$ is 27 and is given in table 5 with numbers $n_{ht}$.

Here we have three segments of roots. The number of positive roots $\mu$ with $(\lambda, \mu) = 3$ is 2 and is given in table 6.

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**Table 3.** Height $ht = (\rho, \mu)$ and $n_{ht}$ for all roots $\mu$ with $(\theta, \mu) = 1$ for $E_7$

| $ht$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|----------|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|
| $n_{ht}$ | 1 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 3 | 3  | 2  | 2  | 1  | 1  | 1  |    |

**Table 4.** Number $n_{ht}$ versus height $ht = (\rho, \mu)$ for roots $\mu$ with $(\lambda, \mu) = 1$ for $E_8$

| $ht$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
|----------|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|----|
| $n_{ht}$ | 1 | 2 | 2 | 2 | 3 | 4 | 4 | 4 | 5 | 5  | 4  | 4  | 4  | 3  | 2  | 2  | 2  | 1  |

**Table 3.** Height $ht = (\rho, \mu)$ and $n_{ht}$ for all roots $\mu$ with $(\theta, \mu) = 1$ for $E_7$.

**Table 4.** Number $n_{ht}$ versus height $ht = (\rho, \mu)$ for roots $\mu$ with $(\lambda, \mu) = 1$ for $E_8$. 

Vogel’s table, only). Deligne’s hypothesis is checked in some cases [17], particularly for symmetric cube of the adjoint representation. At this time it is not known whether it is possible to satisfy one or both of these requirements, as well as the very existence of universal formulae is not guaranteed. So, we do not approach this problem further in this paper, and present the new universal formulae in the original way we found them.
Now we have one segment, consisting of two roots. Check of the total number of positive roots: 37 + 54 + 27 + 2 = 120.

3.2. E\textsc{7}

Dimension \(E\textsc{7} = 133\), number of positive roots \(|\Delta_+| = 63\), Vogel’s parameters \((\alpha, \beta, \gamma, t) = (-2, 8, 12, 18)\). For \(E\textsc{7} \lambda = \omega_2\), in Dynkin’s labeling of roots.

The number of positive roots \(\mu\) with \((\lambda, \mu) = 0\) is 1 + \(|\Delta_+|\alpha_0 = 1 + 15 = 16\). The number of positive roots \(\mu\) with \((\lambda, \mu) = 1\) is 30 and is given in table 7.

So, here we have five segments of roots. The number of positive roots \(\mu\) with \((\lambda, \mu) = 2\) is 15 and is given in table 8.

Here we have three segments of roots. The number of positive roots \(\mu\) with \((\lambda, \mu) = 3\) is 2 and is given in table 9.

Now we have one segment of roots. Check of the total number of positive roots: 7 + 18 + 9 + 2 = 36.

3.3. E\textsc{6}

\(\dim E\textsc{6} = 78, |\Delta_+| = 36, (\alpha, \beta, \gamma, t) = (-2, 6, 8, 12)\). For \(E\textsc{6} \lambda = \omega_3\), in Dynkin’s labeling of roots.

The number of positive roots \(\mu\) with \((\lambda, \mu) = 0\) is 7. The number of positive roots \(\mu\) with \((\lambda, \mu) = 1\) is 18 and is given in table 10 together with numbers \(n_{0|}\).

So, here we have five segments of roots, i.e. sequences with unit distance between consecutive roots. The number of positive roots \(\mu\) with \((\lambda, \mu) = 2\) is 9 and is given in table 11.

Here we have three segments of roots. The number of positive roots \(\mu\) with \((\lambda, \mu) = 3\) is 2 and is given in table 12.

So, here we have one segment of roots. Check of the total number of positive roots: 7 + 18 + 9 + 2 = 36.

4. Quantum dimensions

Now we calculate the contributions of roots with \((\lambda, \mu) \neq 0\) in the Weyl formula for quantum dimension.
The contribution of roots with \((\lambda, \mu) = 3\) comes from two roots of heights \(t - 1, t - 2\) (recall the normalization \(\alpha = -2\)):

\[
L_3 = \frac{\sinh \left( \frac{x}{4} (t + 3k - 2) \right) \sinh \left( \frac{x}{4} (t + 3k - 1) \right)}{\sinh \left( \frac{x}{4} (t - 2) \right) \sinh \left( \frac{x}{4} (t - 1) \right)}.
\] (9)

Due to the rescaling invariance, mentioned after (7), we can recover the parameter \(\alpha\) in this formula in explicit form by substitution

\[
\beta \rightarrow -\frac{2\beta}{\alpha}, \gamma \rightarrow -2\gamma/\alpha, t \rightarrow -\frac{2t}{\alpha}, x \rightarrow -\frac{x}{2\alpha}.
\] (10)

Then \(L_3\) accepts the form

\[
L_3 = \frac{\sinh \left( \frac{x}{4} (3\alpha(k - 1) - 2(\beta + \gamma)) \right) \sinh \left( \frac{x}{4} (\alpha(3k - 4) - 2(\beta + \gamma)) \right)}{\sinh \left( \frac{x}{4} (2\alpha + \beta + \gamma) \right) \sinh \left( \frac{x}{4} (3\alpha + 2(\beta + \gamma)) \right)}.
\] (11)

Below we skip the intermediate formulae in normalization \(\alpha = -2\), and present the final ones with explicit \(\alpha\) recovered.

Next, consider the roots with \((\lambda, \mu) = 2\). There are three segments, the first (longest) one starts at height \(\beta - 1\) and ends at height \(t - 3\), its contribution in the Weyl formula is

\[
L_{21} = \frac{2k}{\prod_{i=1}^{2k} \sinh \left( \frac{1}{4} x(\alpha(i - 5) - 2(\beta + \gamma)) \right)} {\sinh \left( \frac{1}{4} x(\alpha(i - 2) - 2\beta) \right)}.
\] (12)

The second segment starts at height \(t/2\) and ends at height \((t + \gamma - 4)/2\), the contribution is

\[
L_{22} = \frac{2k}{\prod_{i=1}^{2k} \sinh \left( \frac{1}{4} x(-\alpha(i - 3) + \beta + 2\gamma) \right)} {\sinh \left( \frac{1}{4} x(-\alpha(i - 2) + \beta + \gamma) \right)}.
\] (13)
The third segment includes one root at height \((\gamma + 2\beta - 6)/2\) and its contribution is

\[
L_{23} = \frac{\sinh \left( \frac{1}{4}x(\alpha(3 - 2k) + 2\beta + \gamma) \right)}{\sinh \left( \frac{1}{4}x(3\alpha + 2\beta + \gamma) \right)}.
\]

(14)

Next are the roots with \((\lambda, \mu) = 1\). There are five segments, the first (longest) one starts at height 1 and ends at height \((\gamma + 2\beta - 8)/2\), its contribution in the Weyl formula is

\[
L_{11} = \prod_{i=1}^{k} \frac{\sinh \left( \frac{1}{4}x(-\alpha(i - 4) + 2\beta + \gamma) \right)}{\sinh \left( \frac{a\alpha}{4} \right)}.
\]

(15)

The second segment starts at height 2 and ends at height \(\gamma - 3\), contributing

\[
L_{12} = \prod_{i=1}^{k} \frac{\sinh \left( \frac{1}{4}x(\alpha(i - 3) - 2\beta) \right)}{\sinh \left( \frac{1}{4}x(\alpha(i + 1)x) \right)}.
\]

(16)

The third segment starts at height \((\beta - 2)/2\) and ends at \((\gamma + \beta - 4)/2\), contributing

\[
L_{13} = \prod_{i=1}^{k} \frac{\sinh \left( \frac{1}{4}x(-\alpha(i - 2) + \beta + \gamma) \right)}{\sinh \left( \frac{1}{4}x(\beta - \alpha(i - 2)) \right)}.
\]

(17)

The fourth segment is similar to the third one, but shorter by one element from each end:

\[
L_{14} = \prod_{i=1}^{k} \frac{\sinh \left( \frac{1}{4}x(-\alpha(i - 3) + \beta + \gamma) \right)}{\sinh \left( \frac{1}{4}x(\alpha(i) + \alpha + \beta) \right)}.
\]

(18)

The fifth segment consists of two roots, starting at height \((\gamma - 2)/2\), and contribution will be

\[
\frac{\sinh \left( \frac{1}{4}(\beta - 3 + k) \right) \sinh \left( \frac{1}{4}(\beta - 2 + k) \right)}{\sinh \left( \frac{1}{4}(\beta - 2) \right) \sinh \left( \frac{1}{4}(\beta - 2 + 1) \right)}.
\]

(19)

This contribution is appropriate at \(k = 1\), in a sense that all contributions together—the product of all \(L\)-s—form the corrects answer (6). However, for \(k > 1\) and for \(sl(n)\) algebras (i.e. on the line \(\alpha + \beta = 0\)) one loses the zero of (19) on that line which \(k = 1\) cancels out with the zero in denominator of (17), also on the same line. So, in analogy with other contributions above, we simply change this contribution to other one, namely \(L_{15}\), written below. It cancels the mentioned singularity for an arbitrary \(k\), coincides with (19) on the exceptional line \(\gamma = 2(\alpha + \beta)\), but differs in other points:

\[
L_{15} = \prod_{i=1}^{k} \frac{\sinh \left( \frac{1}{4}x(\alpha(i - 3) - 2\beta) \right) \sinh \left( \frac{1}{4}x(\alpha(i - 2) - 2\beta) \right)}{\sinh \left( \frac{1}{4}x(\gamma - \alpha(i - 2)) \right) \sinh \left( \frac{1}{4}x(\gamma - \alpha(i - 2)) \right)}.
\]

(20)

However, this is not the end of the story. We expect that our final formula should be invariant under switch of the \(\beta\) and \(\gamma\) parameters, in analogy with the universal formula (1) for \(Y_2(\alpha)\). So we add a new multiplier, which in some ‘minimal’ way symmetrizes the product of all \(L\) multipliers above w.r.t. the switch \(\beta \leftrightarrow \gamma\):
The function

\[ X_2(x, k, \alpha, \beta, \gamma) \equiv X_2(k, \alpha) = L_3 L_{21} L_{22} L_{23} L_{13} L_{14} L_{15} L_{\text{corr}} \]  \hspace{1cm} (22)

is equal to the quantum dimensions of \( k \)th Cartan power of above defined \( X_2 \) representation for any given simple Lie algebra on corresponding point of Vogel’s table 1. Exceptions are: \( \mathfrak{sp}(2n) \), for which the formula gives the quantum dimensions of \( X_2 \) at \( k = 1 \), and zero otherwise, and the \( B_2 \) algebra. Exact details are given in tables 13 and 14.

The case by case proof of proposition is given in the appendix.

**Remark 1**, on \( sl(n) \) case. In the case of \( sl(n) \) line denominator of \( L_{13} \) and numerator of \( L_{15} \) both contain a zero multiplier, which however cancel out, i.e. one can continuously extend \( X_2(k, \alpha) \) function on that line. In more details: for the \( \alpha + \beta = 0 \) line the mentioned fraction is

\[ \frac{\sinh((2\beta + 2\alpha)x/4)}{\sinh((\beta + \alpha)x/4)} \]  \hspace{1cm} (23)

and evidently tends to 2 in the limit \( \alpha + \beta \to 0 \) independent on the direction of approaching the given point on the line. Of course, one can simply substitute the expression

\[ \frac{\sinh((2\beta + 2\alpha)x/4)}{\sinh((\beta + \alpha)x/4)} = 2 \cosh((\beta + \alpha)x/4) \]  \hspace{1cm} (24)

in the formula (22) for \( X_2(k, \alpha) \) from the very beginning and avoid the questions about continuity of the function.

**Remark 2**, on tables. The entries of tables 13 and 14 for a given algebra and \( k \) are the representation(s), denoted by highest weight, the quantum dimension of which is given by our main formula (22).

**Remark 3**, on the connection with the dimension formulae [7]. In the \( x \to 0 \) limit \( X_2(x, k, \alpha, \beta, \gamma) \) gives the universal dimension formulae. When considered on the exceptional line by taking \( \alpha = y, \beta = 1 - y, \gamma = 2 \) and for \( k = 2 \), in the \( x \to 0 \) limit the expression for \( X_2(x, k, \alpha, \beta, \gamma) \) gives the following dimension formula

\[ L_{\text{corr}} = \prod_{i=1}^{k} \frac{\sinh((2\beta + 2\alpha)x/4)}{\sinh((\beta + \alpha)x/4)} \]  \hspace{1cm} (21)

Note, that on the exceptional line \( \gamma = 2(\alpha + \beta) \) \( L_{\text{corr}} = 1 \), as it should.

Finally, our main result is

**Proposition.**
\[ \left( y - 1 \right) \left( y - 2 \right) \left( y - 3 \right) \left( y - 4 \right) \left( y - 5 \right) \left( y - 6 \right) \]

which coincides exactly with the universal formula on the exceptional line of [7] for representation \( H \) (which is Cartan square of \( X_2 \) in notations of [7]).

**Remark 4.** on \( sp(2n) \) case. We assume the following interpretation of this case. The point is that Vogel’s parameters for \( sp(2n) \) algebras can be obtained from those of \( so(2n) \) by \((\alpha, \beta, \gamma) \rightarrow (-1/2)(\beta, \alpha, -\gamma)\) transformation, which includes transposition of \( \alpha \) and \( \beta \). And indeed, we see in table 18, that our formula gives quantum dimensions of some sequence of representations of \( sp(2n) \), which however are not the Cartan powers of its \( X_2 \) representation. At the same time, no new representations of \( so \) is represented in table 18. We conclude, that the role of \( X_2 \) sequence of representations for \( sp \) in our formula is played by other series, given in table 18.

**5. Permutations in the main formula**

The common feature of all universal formulae is that they give reasonable results when taken with permuted parameters. That feature is supported by the fact that initially all universal parameters are on an equal footing. By saying ‘reasonable’ for e.g. dimension formulae we mean that they give dimensions for other representations of the same algebra. Sometimes they give dimensions of virtual representations, i.e. dimensions of true representations with minus sign. It happens in all known cases, and we show that for our case of quantum dimensions it also does. We present our formula for all possible permutations of the universal parameters in a sequence of tables. The cases of virtual representations are also denoted by highest weights, but with minus sign.
Our check mainly extends to the level of dimensions of representations, with some random checks on quantum dimensions’ level.

The values of $X_2(k, \gamma)$ for algebras on the exceptional line are presented in table 15. Here we see a new phenomena: the value of $X_2(k, \beta)$ for say point $k = 2$ for $D_4$ algebra (i.e. $\alpha = -2$, $\beta = 4$, $\gamma = 4$) is not defined, since the limit of $0/0$ ambiguity depends on the way of approaching that point. However, if one approaches that point by one of the relevant lines, $Exc$ or $so$, the reasonable results are obtained. They are noted in the table and are in agreement with other cases, e.g. the result for $D_4$ agrees with the results for general orthogonal algebras in table 13.

There are other cases, e.g. $k > 3$ for $D_4$, when the double limit does not exist, however, the limit exists and is unique when restricted to any line, approaching the point $D_4$. We do not mention that cases specially in our tables.

When restricted to the exceptional line, with the identification of parameters as in Remark 3 above, the dimension formula, following from $X_2(2, \beta)$ in the $x \to 0$ limit coincides with that of $H'$ of [7], both are obtained from $H$ (25) by substitution $y \leftrightarrow 1 - y$.  

### Table 16. $X_2(k, \gamma)$ for the exceptional algebras.

| $k$ | 1 | 2 | 3 | $\geq 4$ |
|-----|---|---|---|---------|
| $G_2$ | $3\omega_1$ | $3\omega_1$ | 0 | 0 |
| $F_4$ | $\omega_2$ | $\omega_3$ | 0 | 0 |
| $E_6$ | $\omega_6$ | $\omega_3$ | 0 | 0 |
| $E_7$ | $\omega_6$ | $\omega_2$ | 0 | 0 |
| $E_8$ | $\omega_6$ | $\omega_6$ | 0 | 0 |
| $D_4$ | $\omega_1 + \omega_3 + \omega_4$ | $\omega_1 + \omega_3 + \omega_4$ on the $Exc$ line | 0 on the $so$ line | 0 on the $so$ line |

### Table 17. $X_2(k, \gamma)$ for the classical algebras.

| $k$ | 1 | 2 | 3 | $\geq 4$ |
|-----|---|---|---|---------|
| $A_1$ | 0 | $-2\omega$ on the $sl$ line | 0 | 0 |
| $A_2$ | $3\omega_1 \oplus 3\omega_2$ | $-(\omega_1 + \omega_2)$ | 0 | 0 |
| $A_n, n \geq 3$ | $(2\omega_1 + \omega_{n-1}) \oplus (\omega_2 + 2\omega_n)$ | $-(\omega_1 + \omega_n)$ | 0 | 0 |
| $B_2$ | $\omega_1 + 2\omega_2$ | 0 on the $so$ line | 0 on the $so$ line | 0 |
| $B_3$ | $\omega_1 + 2\omega_3$ | 0 | 0 | 0 |
| $B_n, n \geq 4$ | $\omega_1 + \omega_3$ | 0 | 0 | 0 |
| $C_n, n \geq 3$ | $2\omega_1 + \omega_2$ | 0 | 0 | 0 |
| $D_4$ | $\omega_1 + \omega_3 + \omega_4$ | 0 on the $so$ line | 0 on the $so$ line | 0 |
| $D_5$ | $\omega_1 + \omega_5$ | 0 | 0 | 0 |
| $D_6$ | $\omega_1 + \omega_3$ | 0 on the $so$ line | 0 | 0 |
| $D_n, n \geq 7$ | $\omega_1 + \omega_3$ | 0 | 0 | 0 |

### Table 18. $X_2(k, \beta)$ for the classical algebras for sufficiently large $n$ (depends on $k$).

| $k$ | 1 | 2 | 3 | 4 | $\geq 5$ |
|-----|---|---|---|---|---------|
| $A_n$ | $(2\omega_1 + \omega_{n-1}) \oplus$ | $(2\omega_2 + \omega_{n-3}) \oplus$ | $(2\omega_3 + \omega_{n-5}) \oplus$ | $(2\omega_4 + \omega_{n-7}) \oplus$ | $\cdots$ |
| | $(2\omega_n + \omega_2)$ | $(2\omega_{n-1} + \omega_4)$ | $(2\omega_{n-2} + \omega_6)$ | $(2\omega_{n-3} + \omega_8)$ | $\cdots$ |
| $B_n$ | $\omega_1 + \omega_3$ | 0 | 0 | 0 | 0 |
| $C_n$ | $2\omega_1 + \omega_2$ | $2\omega_2 + \omega_4$ | $2\omega_3 + \omega_6$ | $2\omega_4 + \omega_8$ | $\cdots$ |
| $D_0$ | $\omega_1 + \omega_3$ | 0 | 0 | 0 | 0 |
We see that $X_2(3, \beta)$ for $D_4$ equals 3, independent on $x$. This should be interpreted as quantum dimension of some representation of semidirect product of $D_4$ and its Dynkin diagram’s automorphism group $S_3$. We assume that the corresponding representation is the trivial one for $D_4$ factor and the non-trivial reducible three-dimensional permutation representation of $S_3$ factor. $X_2(k, \gamma)$ for the exceptional algebras are given in table 16.

Again, when restricted to the exceptional line $\alpha = y, \beta = 1 - y, \gamma = 2$ and in the limit $x \to 0$, $X_2(2, \gamma)$ gives the following formula

$$\frac{5(y - 6)(y - 4)(y + 3)(y + 5)}{(y - 1)^2 \gamma^2}$$

which coincides with the dimension formula for $X_2$ from [7], in agreement with table 16.

For the classical algebras $X_2(k, \beta)$ is given in table 18. For small ranked algebras there shows up a complicated picture, so we present the stabilized answers for sufficiently large ranks. The boundary depends on $k$, the larger $k$, the larger the boundary. At least the rank should be large enough to allow the existence of the fundamental weights mentioned in the table. Finally, results for the one remaining permutation of parameters for the classical groups are presented in table 17.

6. Conclusion

The present results are of interest for the representation theory of simple Lie algebras. They hint on the interpretation of the universal formulae as result of some duality transformation between rank $N$ and Cartan power $k$. That transformation is based on a formula used in the appendix, which in its simplest form is the following identity

$$\prod_{i=1}^{N} k + i = \prod_{i=1}^{k} N + i. \quad (26)$$

The analogy of this transformation is that in [26], where for the proof of the $N \leftrightarrow -N$ duality of $SO(2N)$ and $Sp(2N)$ one uses the so called Maya parametrization of Young diagrams, aimed to remove explicit rank $N$ from the range of the products over the running index.

Simultaneously, the very existence of universal formulae for an arbitrary representation in the decomposition of powers of the adjoint is not proven.

As mentioned in the Introduction, the present research is a part of a larger project, aimed to construct universal knot polynomials. In [15] it is shown, that for 2- and 3-strand torus knots adjoint knot polynomials of different types can be presented in the universal form. To do so, one is required to represent the quantum dimensions of all representations in the decomposition of the square and cube of the adjoint representation in the universal form. So, for $k = 2$ our result gives a universal formula for one of the representations in the fourth power of the adjoint. Most of other representations also have a universal dimension formula. We assume that it is possible to obtain universal formulae for all other representations also, so the 4th strand (and higher) torus knots invariant polynomials will be represented in the universal form.

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Appendix. Proof of the main formula

A.1. Notation

Below we omit the numerous sinh signs and use the following notation instead:

\[ a \sinh \left[ x : \frac{A \cdot B}{M \cdot N} \right] \equiv a \frac{\sinh(xA) \sinh(xB)}{\sinh(xM) \sinh(xN)} \]  \hspace{1cm} (A.1)

where \( x, a, A, B, ..., M, N, ... \) are numbers (dots between are not necessary, provided no ambiguity arises). For example

\[ 2 \sinh \left[ \frac{x}{4} : \frac{1}{2} \cdot 4 \right] \equiv 2 \frac{\sinh \left( \frac{x}{4} \right) \sinh \left( \frac{4x}{4} \right)}{\sinh \left( \frac{4}{4} \right)} \]  \hspace{1cm} (A.2)

One can derive simple rules which this notation obeys. E.g.

\[(\sinh \left[ x : A \cdot B \right] \sinh \left[ x : M \cdot N \right]) = \sinh \left[ x : A \cdot B \cdot M \cdot N \right]. \hspace{1cm} (A.3)\]

Of course, our notation belongs to the field of q-calculus, however we did not find this or similar convenient notation, perhaps missed that.

Evidently, one gets the universal dimension formulae for dimensions of corresponding representations just by omitting the front sinh sign for \( L_? \)-s and \( X_2(x, k, \alpha, \beta, \gamma) \) in formulae below.

A.2. \( L_? \)-s in the new notations

The \( L_? \) multipliers in the new notation take the following form

\[ L_{11} = \sinh \left[ \frac{x}{4} : \prod_{i=1}^{k} -\alpha(i - 4) + 2\beta + \gamma \right] \]
\[ L_{12} = \sinh \left[ \frac{x}{4} : \prod_{i=1}^{k} \frac{\alpha(i - 3) - 2\gamma}{\alpha(i + 1)} \right] \]
\[ L_{13} = \sinh \left[ \frac{x}{4} : \prod_{i=1}^{k} \frac{-\alpha(i - 2) + \beta + \gamma}{\beta - \alpha(i - 2)} \right] \]
\[ L_{14} = \sinh \left[ \frac{x}{4} : \prod_{i=1}^{k} \frac{-\alpha(i - 3) + \beta + \gamma}{\alpha(-i) + \alpha + \beta} \right] \]
\[ L_{15} = \sinh \left[ \frac{x}{4} : \prod_{i=1}^{k} \frac{(\alpha(i - 3) - 2\beta)(\alpha(i - 2) - 2\beta)}{(\gamma - \alpha(i - 2))(\alpha(-i) + \alpha + \gamma)} \right] \]
\[ L_{21} = \sinh \left[ \frac{x}{4} : \prod_{i=1}^{2k} \frac{\alpha(i - 5) - 2(\beta + \gamma)}{\alpha(i - 2) - 2\beta} \right] \]
\[ L_{22} = \sinh \left[ \frac{x}{4} : \prod_{i=1}^{2k} \frac{-\alpha(i - 3) + \beta + 2\gamma}{-\alpha(i - 2) + \beta + \gamma} \right] \]
\( L_{23} = \sinh \left[ x \frac{x}{4} \frac{\alpha(3-2k)+2\beta+\gamma}{3\alpha+2\beta+\gamma} \right] \)

\( L_3 = \sinh \left[ x \frac{x}{4} \frac{(3\alpha(k-1)-2(\beta+\gamma))(\alpha(3k-4)-2(\beta+\gamma))}{(4\alpha+2\beta+2\gamma)(3\alpha+2(\beta+\gamma))} \right] \)

\( L_{\text{corr}} = \sinh \left[ x \frac{x}{4} \prod_{i=1}^{k} \frac{\alpha(-(i+k-4))+2\beta+\gamma}{\alpha(i+k-2)-2\gamma} \right]. \)

The proof of the main formula is carried out case by case: for each set of parameters \( \alpha, \beta, \gamma \) from Vogel’s table (except \( C_n \)) we compare the expression (22) with the Weyl quantum dimension (7) for the corresponding algebra.

### A.3. \( A_{N-1} \)

First we carry out the proof for the \( A_{N-1} \) algebra.

Substituting \( \alpha = -2, \beta = 2, \gamma = N \) in the \( L \)-terms, one gets

\( L_{11} = \sinh \left[ x \frac{x}{4} \frac{(N-2) \cdot N \ldots (N+2k-4)}{2 \cdot 4 \ldots 2k} \right] \)

\( L_{12} = \sinh \left[ x \frac{x}{4} \frac{(2N-4) \cdot (2N-2) \ldots (2N+2k-6)}{4 \cdot 6 \ldots (2k+2)} \right] \)

\( L_{13} = \sinh \left[ x \frac{x}{4} \frac{N \cdot (N+2) \ldots (N+2k-2)}{(\alpha+\beta) \cdot 2 \ldots (2k-2)} \right] \)

\( L_{14} = \sinh \left[ x \frac{x}{4} \frac{(N-2) \cdot N \ldots (N+2k-4)}{2 \cdot 4 \ldots 2k} \right] \)

\( L_{15} = \sinh \left[ x \frac{x}{4} \frac{2(\alpha+2\beta) \cdot 2^2 \cdot 4^2 \ldots (2k-2)^2 \cdot 2k}{(N-2) \cdot N^2 \ldots (N+2k-4)^2 \cdot (N+2k-2)} \right] \)

\( L_{21} = \sinh \left[ x \frac{x}{4} \frac{(2N-4) \cdot (2N-2) \ldots (2N+2k-6) \ldots (2N+4k-6)}{2 \cdot 4 \ldots 2k \ldots 4k} \right] \)

\( L_{22} = \sinh \left[ x \frac{x}{4} \frac{(2N-2) \cdot 2N \ldots (2N+2k-4) \ldots (2N+4k-4)}{N \cdot (N+2) \ldots (N+2k-2) \ldots (N+4k-2)} \right] \)

\( L_{23} = \sinh \left[ x \frac{x}{4} \frac{N+4k-2}{N-2} \right] \)

\( L_3 = \sinh \left[ x \frac{x}{4} \frac{(2N+6k-4) \cdot (2N+6k-2)}{(2N-4) \cdot (2N-2)} \right] \)

\( L_{\text{corr}} = \sinh \left[ x \frac{x}{4} \frac{(N+2k-2) \cdot (N+2k) \ldots (N+4k-4)}{(2N+2k-2) \cdot (2N+2k) \ldots (2N+4k-4)} \right]. \)
Notice, that
\[ L_{13} \cdot L_{14} \cdot L_{15} = 2 \cosh \frac{\alpha + \beta}{4} = 2. \]

Next we take the product of \( L_{22}, L_{\text{cont}}, L_{23}, L_3 \):
\[
L_{22} \cdot L_{\text{cont}} \cdot L_{23} \cdot L_3 = \sinh \left[ \frac{x}{4} \cdot \frac{(2N - 2) \cdot 2N \ldots (2N + 2k - 4)}{(N - 2) \cdot (N + 2k - 4) \cdot (N + 4k - 2)} \right] \cdot \frac{N + 4k - 2}{N - 2} \cdot \frac{(2N + 6k - 4) \cdot (2N + 6k - 2)}{(2N - 4) \cdot (2N - 2)}.
\]

Now we take the product of the remaining \( L_{11}, L_{12}, L_{21} \) terms.
\[
L_{11} \cdot L_{12} \cdot L_{21} = \sinh \left[ \frac{x}{4} \cdot \frac{(N - 2)N \ldots (N + 2k - 4)(2N - 4)^2 \ldots (2N + 2k - 6)^2(2N + 2k - 4)(2N + 2k - 4) \ldots (2N + 4k - 6)}{2^4 \cdot 4^2 \ldots (2k)^4 \cdot (2k + 2) \cdot (2k + 6) \ldots 4k} \right]
\]

and finally get
\[
X_2(x, k, -2, 2, N) = \frac{2 \cdot \sinh \left[ \frac{x}{4} \cdot \frac{(2N - 4)(2N - 2)^2(2N)^3 \ldots (2N + 2k - 6)^2(2N + 2k - 4)^2}{2^2 \cdot 4^3 \ldots (2k)^3} \right]}{(2N + 2k - 2) \ldots (2N + 4k - 6)(2N + 6k - 4)(2N + 6k - 2)},
\]

Now we use the Weyl quantum dimension formula for the \( k \)th Cartan power of the \( X_2 \) representation for the \( A_{N-1} \) algebra, which is the sum of the quantum dimensions of two irreps, constituting \( X_2 \) for \( A \) series. Actually, quantum dimensions of these two representations are equal, since they are connected by automorphism of the Dynkin diagram.

Having
\[
D_Q = \prod_{\alpha > 0} \frac{\sinh \left( \frac{x}{4} ((\lambda, \alpha) + (\lambda, \rho)) \right)}{\sinh \left( \frac{((\alpha, \rho)\lambda)}{2} \right)}
\]

and taking into account, that the highest weight of one of the representations is \( \lambda = 2k\omega_1 + k\omega_{N-2} \), one gets
\[
D_Q^{A_{N-1}} = \prod_{i=1}^{N-2} \frac{\sinh \left( \frac{x}{2} (k + i) \right)}{\sinh \left( \frac{x}{2} \right)} \cdot \prod_{i=2}^{N-3} \frac{\sinh \left( \frac{x}{4} (k + i) \right)}{\sinh \left( \frac{x}{4} \right)},
\]
\[
\prod_{i=1}^{N-3} \frac{\sinh \left( \frac{x}{4} (2k + i) \right)}{\sinh \left( \frac{x}{4} \right)} \cdot \prod_{i=N-2}^{N-1} \frac{\sinh \left( \frac{x}{4} (3k + i) \right)}{\sinh \left( \frac{x}{4} \right)}.
\]

In this formula the products have upper limits around \( N \), while the \( X_2(x, k, -2, 2, N) \) is represented by several products with upper limits \( k \). We switch from one product to another by the following identity:
\[
\prod_{i=1}^{N} \frac{\sinh \left( \frac{x}{4} (k + i) \right)}{\sinh \left( \frac{x}{4} \right)} = \prod_{i=1}^{k} \frac{\sinh \left( \frac{x}{4} (N + i) \right)}{\sinh \left( \frac{x}{4} \right)}.
\]

(A.4)

In our notation we can write

\[
D^{x_{k-1}}_Q = \sinh \left[ \frac{x}{4} \prod_{j=1}^{k} \frac{2N - 4 + 2i}{2} \cdot \frac{2}{2k + 2} \prod_{j=1}^{k} \frac{2N - 6 + 2i}{2i} \cdot (6k + 2N - 4)(6k + 2N - 2) \right].
\]

Then, after expanding the product signs and combining the multipliers, we get

\[
\sinh \left[ \frac{x}{4} \cdot \frac{(2N - 4)(2N - 2)^2(2N) \ldots (2N + 2k - 6)(2N + 2k - 4)^2}{2^2 \cdot 4^3 \ldots (2k)^3} \times \frac{(2N + 2k - 2) \ldots (2N + 4k - 6)(2N + 6k - 4)(2N + 6k - 2)}{(2k + 2)^2 \cdot (2k + 4) \ldots 4k} \right]
\]

which is equal to the expression 1/2 \cdot X_2(x, k, -2, 2, N), as expected. So, we have proven the formula (22) for \(A_{N-1} \) algebra.

A.4. \(B_N\)

For this case we should substitute \(\alpha = -2, \beta = 4, \gamma = 2N - 3\), so

\[
L_{11} = \sinh \left[ \frac{x}{4} \cdot \frac{(2N - 1)(2N + 1) \ldots (2N + 2k - 3)}{2 \cdot 4 \ldots 2k} \right],
\]

\[
L_{12} = \sinh \left[ \frac{x}{4} \cdot \frac{(4N - 10)(4N - 8) \ldots (4N + 2k - 12)}{4 \cdot 6 \ldots (2k + 2)} \right],
\]

\[
L_{13} = \sinh \left[ \frac{x}{4} \cdot \frac{(2N - 1)(2N + 1) \ldots (2N + 2k - 3)}{2 \cdot 4 \ldots 2k} \right],
\]

\[
L_{14} = \sinh \left[ \frac{x}{4} \cdot \frac{(2N - 3)(2N - 1) \ldots (2N + 2k - 5)}{4 \cdot 6 \ldots (2k + 2)} \right],
\]

\[
L_{15} = \sinh \left[ \frac{x}{4} \cdot \frac{4 \cdot 6^2 \cdot 8^2 \ldots (2k + 2)^2 \cdot (2k + 4)}{(2N - 5)(2N - 3)^2 \ldots (2N + 2k - 7)^2(2N + 2k - 5)} \right],
\]

\[
L_{21} = \sinh \left[ \frac{x}{4} \cdot \frac{(4N - 6)(4N - 4) \ldots (4N + 4k - 8)}{6 \cdot 8 \ldots (4k + 2)} \right],
\]

\[
L_{22} = \sinh \left[ \frac{x}{4} \cdot \frac{(4N - 6)(4N - 4) \ldots (4N + 4k - 8)}{(2N - 1)(2N + 1) \ldots (2N + 4k - 3)} \right],
\]

\[
L_{23} = \sinh \left[ \frac{x}{4} \cdot \frac{2N + 4k - 1}{2N - 1} \right].
\]
\[ L_3 = \sinh \left[ \frac{x}{4} : \frac{(4N + 6k - 4)(4N + 6k - 6)}{(4N - 6)(4N - 4)} \right] \]

\[ L_{\text{core}} = \sinh \left[ \frac{x}{4} : \frac{(2N + 2k - 1)(2N + 2k + 1) \ldots (2N + 4k - 3)}{(4N + 2k - 8)(4N + 2k - 6) \ldots (4N + 4k - 10)} \right]. \]

The product of \( L_{11}, L_{14}, L_{15} \) gives

\[ L_{11} \cdot L_{14} \cdot L_{15} = \sinh \left[ \frac{x}{4} : \frac{(2k + 2)(2k + 4)(2N + 2k - 5)(2N + 2k - 3)}{2 \cdot 4 \cdot (2N - 5)(2N - 3)} \right]. \]

The product of \( L_{22}, L_{\text{core}}, L_{13} \) gives

\[ L_{22} \cdot L_{\text{core}} \cdot L_{13} = \sinh \left[ \frac{x}{4} : \frac{(4N - 6)(4N - 4) \ldots (4N + 2k - 8)(4N + 4k - 8)}{2 \cdot 4 \ldots 2k} \right]. \]

And the product of remaining four terms is

\[ L_{22} : L_{21} \cdot L_{23} \cdot L_5 \]

\[ = \sinh \left[ \frac{x}{4} : \frac{(4N - 10)(4N - 8)(4N - 6)^2(4N - 4)^5(4N - 2)^7 \ldots (4N + 2k - 12)(4N + 2k - 10)^3(4N + 2k - 8)^2}{(2N - 5)(2N - 3)(2N - 1) \cdot 2^7} \cdot \frac{(4N + 2k - 6) \ldots (4N + 4k - 8)(2N + 4k - 1)(2N + 2k - 5)(2N + 2k - 5)(4N + 6k - 6)(4N + 6k - 4)}{4^9 \ldots (2k)^3 \cdot (2k + 2) \ldots (4k + 4)} \right]. \]

So, the product of all \( L \)-terms is:

\[ X_{2}(x, k, -2, 4, 2N - 3) = L_{11} : L_{12} : L_{21} \cdot L_{22} \cdot L_{\text{core}} : L_{23} : L_5 : L_{13} : L_{14} : L_{15} \]

\[ = \sinh \left[ \frac{x}{4} : \frac{(4N - 10)(4N - 8)(4N - 6)^2(4N - 4)^5(4N - 2)^7 \ldots (4N + 2k - 12)(4N + 2k - 10)^3(4N + 2k - 8)^2}{(2N - 5)(2N - 3)(2N - 1) \cdot 2^7} \cdot \frac{(4N + 2k - 6) \ldots (4N + 4k - 8)(2N + 4k - 1)(2N + 2k - 5)(2N + 2k - 5)(4N + 6k - 6)(4N + 6k - 4)}{4^9 \ldots (2k)^3 \cdot (2k + 2) \ldots (4k + 4)} \right]. \]

(A.5)

Now we turn to the Weyl formula for \( B_N \) algebra:

\[ D_{B_N}^Q = \sinh \left[ \frac{x}{4} : \prod_{i=1}^{N-2} \frac{2k + 2i}{2i} : \prod_{i=1}^{2N - 3} \frac{2k + 2N - 3}{2i} : \prod_{i=N}^{2N-4} \frac{2k + 2i - 2}{2i} : \prod_{i=1}^{N-1} \frac{2N + 2k - 5}{2i} : \prod_{i=N-1}^{2N-5} \frac{2k + 2i - 2}{2i} : \prod_{i=1}^{N-1} \frac{4N + 4k - 8}{2i} : \prod_{i=N+1}^{2N-3} \frac{4k + 2i - 2}{2i} : \prod_{i=1}^{N-1} \frac{4k + 2N - 1}{2i} : \prod_{i=N+1}^{2N-3} \frac{4k + 2i - 2}{2i} : \prod_{i=1}^{N-1} \frac{2N - 5}{2i} : \prod_{i=N+1}^{2N-3} \frac{4k + 2N - 6}{2i} : \prod_{i=1}^{N-1} \frac{2N - 4}{2i} : \prod_{i=N+1}^{2N-3} \frac{4k + 2N - 6}{2i} : \prod_{i=1}^{N-1} \frac{4N - 6}{2i} : \prod_{i=N+1}^{2N-3} \frac{4N - 4}{2i} : \frac{(4N - 8)(4N - 6)(4N - 4)}{(4N - 8)(4N - 6)(4N - 4)} \right]. \]

With the help of (A.4) identity we switch the products limits to \( k \):

\[ \prod_{i=1}^{N-2} \frac{2k + 2i}{2i} = \prod_{i=1}^{k} \frac{2i + 2N - 4}{2i} = \frac{(2N - 2)(2N \ldots (2N + 2k - 4)}{2 \cdot 4 \ldots 2k}, \]

\[ \prod_{i=1}^{N-3} \frac{2k + 2i}{2i} = \prod_{i=1}^{k} \frac{2i + 2N - 6}{2i} = \frac{(2N - 4)(2N - 2) \ldots (2N + 2k - 6)}{2 \cdot 4 \ldots 2k}, \]
And the last term needed to be transformed:

\[
\prod_{i=3}^{N-1} \frac{4k+2i}{2i} = \prod_{i=1}^{2k} \frac{2i+2N-2}{2i} \cdot \frac{2 \cdot 4}{(4k+2)(4k+4)} = \frac{2 \cdot 4}{(4k+2)(4k+4)} \cdot \frac{2N(2N+2) \ldots (2N+4k-2)}{2 \cdot 4 \cdot 4k}.
\]

\[
\prod_{i=N}^{2N-4} \frac{2k+2i-2}{2i-2} = \prod_{i=N-1}^{2N-5} \frac{2k+2i}{2i} = \prod_{i=1}^{k} \frac{4N-10+2i}{2i} = \frac{(4N-8)(4N-6) \ldots (4N+2k-10)}{(2N-2) \cdot (2N-4) \ldots (2N+2k-4)}.
\]

And the last term needed to be transformed:

\[
\prod_{i=N+1}^{2N-3} \frac{4k+2i}{2i} = \prod_{i=N}^{2N-4} \frac{4k+2i}{2i} = \prod_{i=1}^{k} \frac{4N-8+2i}{2i} = \frac{(4N-6)(4N-4) \ldots (4N+4k-8)}{2N \cdot (2N+2) \ldots (2N+4k-2)}.
\]

Next, taking into account the transformations above, we rewrite the expression for \(D^R_{\bar{Q}}\):

\[
D^R_{\bar{Q}} = \sinh \left[ x \left( \frac{2N-2)2N \ldots (2N+2k-4)}{2 \cdot 4 \ldots 2k} \cdot \frac{2k+2N-3(4N-8)(4N-6) \ldots (4N+2k-10)}{2N+3} \cdot \frac{(4N-10)(4N-8) \ldots (4N+2k-12)}{(2N-2) \cdot (2N-4) \ldots (2N+2k-6)} \cdot \frac{2k+4}{2N-5} \cdot \frac{4N-4) \ldots (4N+4k-8)}{2N-1} \cdot \frac{4k+2N-1}{(4N+6)(4N+4k-8)} \right).
\]

Combining similar terms, one gets

\[
D^R_{\bar{Q}} = \sinh \left[ x \left( \frac{(4N-10)(4N-8)4N-6)(4N-2)^2(4N-2) \ldots (4N+2k-12)^3(4N+2k-10)^2(4N+2k-8)^2}{(2N-5)(2N-3)(2N-1)} \cdot \frac{(4N+2k-6) \ldots (4N+4k-8)(2N+4k-1)(2N+2k-5) \cdot (4N+6k-6)(4N+6k-4)}{(4N+2k-6) \ldots (4N+4k-8)(2N+4k-1)(2N+2k-5) \cdot (4N+6k-6)(4N+6k-4)} \right).
\]

which coincides with \(X_2(x,k,2,4,2N-3)\) (A.5).
A.5. \( C_N \)

It was mentioned above that for \( k \)'s greater than 1 our universal formula gives 0 values for \( C_N \) algebra. So now we observe the \( k = 1 \) case. Substituting \( \alpha = -2, \beta = 1, \gamma = N + 2 \) in the \( L \) terms, one has

\[
L_{11} = \sinh \left[ \frac{x}{4} : \frac{N - 2}{2} \right],
\]

\[
L_{12} = \sinh \left[ \frac{x}{4} : \frac{2N}{4} \right],
\]

\[
L_{13} = \sinh \left[ \frac{x}{4} : \frac{N + 1}{1} \right],
\]

\[
L_{14} = \sinh \left[ \frac{x}{4} : \frac{N - 1}{1} \right],
\]

\[
L_{15} = \sinh \left[ \frac{x}{4} : \frac{(-2\beta - \alpha) \cdot 2}{N(N + 2)} \right],
\]

\[
L_{21} = \sinh \left[ \frac{x}{4} : \frac{(2N - 2)2N}{(-2\beta - \alpha) \cdot 2} \right],
\]

\[
L_{22} = \sinh \left[ \frac{x}{4} : \frac{(2N + 1)(2N + 3)}{(N + 1)(N + 3)} \right],
\]

\[
L_{23} = \sinh \left[ \frac{x}{4} : \frac{N + 2}{N - 2} \right],
\]

\[
L_3 = \sinh \left[ \frac{x}{4} : \frac{(2N + 6k - 2)(2N + 6)}{(2N - 2)2N} \right],
\]

\[
L_{\text{corr}} = \sinh \left[ \frac{x}{4} : \frac{N}{2N + 4} \right].
\]

Then we take the product of all these terms

\[
X_2(x, k, -2, 1, N + 2) = L_{11} \cdot L_{12} \cdot L_{13} \cdot L_{14} \cdot L_{15} \cdot L_{21} \cdot L_{22} \cdot L_{23} \cdot L_3 \cdot L_{\text{corr}}
\]

\[
= \sinh \left[ \frac{x}{4} : \frac{2N(N - 1)(2N + 1)(2N + 3)(2N + 6)}{1^2 \cdot 2 \cdot 4 \cdot (N + 3)} \right],
\]

which coincides with the expression one gets after calculating the quantum dimension of the \( 2\omega_1 + \omega_2 \) representation of \( C_N \) algebra using the Weyl formula.

A.6. \( D_N \)

For this case we substitute \( \alpha = -2, \beta = 4, \gamma = 2N - 4 \), so \( L \) terms become

\[
L_{11} = \sinh \left[ \frac{x}{4} : \frac{(2N - 2)2N \ldots (2N + 2k - 4)}{2 \cdot 4 \ldots 2k} \right].
\]
\( L_{12} = \sinh \left[ \frac{x}{4} : \frac{(2N - 6)(2N - 5) \ldots (2N + k - 7)}{2 \cdot 3 \ldots (k + 1)} \right] \),

\( L_{13} = \sinh \left[ \frac{x}{4} : \frac{(N - 1)N \ldots (N + k - 2)}{1 \cdot 2 \ldots k} \right] \),

\( L_{14} = \sinh \left[ \frac{x}{4} : \frac{(N - 2)(N - 1) \ldots (N + k - 3)}{2 \cdot 3 \ldots (k + 1)} \right] \),

\( L_{15} = \sinh \left[ \frac{x}{4} : \frac{2 \cdot 3^2 \cdot 4^2 \ldots (k + 1)^2 \cdot (k + 2)}{(N - 3)(N - 2)^2 \ldots (N + k - 4)^2(N + k - 3)} \right] \),

\( L_{21} = \sinh \left[ \frac{x}{4} : \frac{(2N - 4)(2N - 3) \ldots (2N + 2k - 5)}{3 \cdot 4 \ldots (2k + 2)} \right] \),

\( L_{22} = \sinh \left[ \frac{x}{4} : \frac{(2N - 4)(2N - 3) \ldots (2N + 2k - 5)}{(N - 1)N \ldots (N + 2k - 2)} \right] \),

\( L_{23} = \sinh \left[ \frac{x}{4} : \frac{N + 2k - 1}{N - 1} \right] \),

\( L_3 = \sinh \left[ \frac{x}{4} : \frac{(2N + 3k - 4)(2N + 3k - 3)}{(2N - 4)(2N - 3)} \right] \),

\( L_{\text{const}} = \sinh \left[ \frac{x}{4} : \frac{(N + k - 1)(N + k) \ldots (N + 2k - 2)}{(2N - k + 5)(2N - k + 4) \ldots (2N + 2k - 6)} \right] \).

The product of \( L_{13}, L_{14}, L_{15} \) terms gives

\[
L_{13} \cdot L_{14} \cdot L_{15} = \sinh \left[ \frac{x}{4} : \frac{(k + 1)(k + 2)(N + k - 3)(N + k - 2)}{1 \cdot 2 \cdot (N - 3)(N - 2)} \right].
\]

Then,

\[
L_{22} \cdot L_{\text{const}} \cdot L_{11} \cdot L_3 = \sinh \left[ \frac{x}{4} : \frac{(2N - 2)(2N - 1) \ldots (2N + k - 6) \cdot (2N + 2k - 5)(2N + 3k - 4)(2N + 3k - 3)}{1 \cdot 2 \ldots k} \right].
\]

And the remaining ones

\[
L_{42} \cdot L_{21} \cdot L_{23} = \sinh \left[ \frac{x}{4} : \frac{(2N - 10)(2N - 5)(2N - 4)^2 \ldots (2N + k - 7)^2 \ldots (2N + 2k - 5)(2N + 2k - 1)(N + 2k - 1)}{2 \cdot 3^2 \ldots (k + 1)^2 \cdot (k + 2) \ldots (2k + 2)(N - 1)} \right].
\]

Overall, for \( X_2(x, k, -2, 4, 2N - 4) \) one gets

\[
X_2(x, k, -2, 4, 2N - 4) = \sinh \left[ \frac{x}{4} : \frac{(N + k - 3)(N + k - 2)(N + k - 2)(N + 2k - 6)(2N + 2k - 5)(2N + 2k - 1)(2N + 3k - 4)(2N + 3k - 3)}{(N - 3)(N - 2)(N - 1) \cdot 1^2 \cdot 2^3 \ldots (k + 1)} \times \frac{(2N + k - 6)^2(2N + k - 5) \ldots (2N + 2k - 6)(2N + 2k - 5)^2(2N + 2k - 1)(2N + 3k - 4)(2N + 3k - 3)}{(k + 3)^2(2k + 2)} \right].
\]
The Weyl formula for $D_5$ algebra is

$$
D_{G_5}^{\text{th}} = \sinh \left[ \frac{x}{4} \prod_{i=1}^{2N-6} \frac{2N-6+i}{i} \prod_{i=1}^{2N-7} \frac{k+i}{i} \cdot \frac{(k+2)(k+N-3)(k+N-2)}{2 \cdot (N-3)(N-2)} \cdot \prod_{i=1}^{2N-5} \frac{2k+i}{i} \right.
\cdot \frac{(N+2k-1)(2N-5+2k)}{(N-1)(2N-5)} \cdot \frac{(2N+3k-4)(2N+3k-3)}{(2N-4)(2N-3)} \\
= \sinh \left[ \frac{x}{4} \prod_{i=1}^{2N-6} \frac{2N-6+i}{i} \prod_{i=1}^{2N-7} \frac{k+i}{i} \cdot \frac{(k+2)(k+N-3)(k+N-2)}{2 \cdot (N-3)(N-2)} \cdot \prod_{i=1}^{2N-5} \frac{2k+i}{i} \right.
\cdot \frac{(N+2k-1)(2N-5+2k)}{(N-1)(2N-5)} \cdot \frac{(2N+3k-4)(2N+3k-3)}{(2N-4)(2N-3)} \\
= \sinh \left[ \frac{x}{4} \prod_{i=1}^{2N-6} \frac{2N-6+i}{i} \prod_{i=1}^{2N-7} \frac{k+i}{i} \cdot \frac{(k+2)(k+N-3)(k+N-2)}{2 \cdot (N-3)(N-2)} \cdot \prod_{i=1}^{2N-5} \frac{2k+i}{i} \right.
\cdot \frac{(N+2k-1)(2N-5+2k)}{(N-1)(2N-5)} \cdot \frac{(2N+3k-4)(2N+3k-3)}{(2N-4)(2N-3)} \\
\times \frac{(N+k-3)(N+k-2)(2N-6)(2N-5)(2N-4) \cdots (2N+k-7)}{(N-3)(N-2)(N-1) \cdots 2 \cdot k \cdot (k+1)} \\
\times \frac{(2N+k-6) \cdots (2N+k-5) \cdots (2N+k-6)(2N+k-5) \cdots (2N+k-1)(2N+k-4)(N+k-3)}{(k+3) \cdots (2k+2)}
$$

which coincides with $X_2(x, k, -2, 4, 2N-4)$ (A.6).

A.7 $G_2$

For $G_2$ exceptional algebra Vogel’s parameters take values $\alpha = -2, \beta = 10/3, \gamma = 8/3$. Substituting them in the $L$-terms, one has

$$
L_{G_2}^{G_2} = \sinh \left[ \frac{x}{6} \right. \cdot \frac{5 \cdot 8 \cdots (3k+2)}{3 \cdot 6 \cdots 3k},
L_{G_2}^{G_2} = \sinh \left[ \frac{x}{6} \right. \cdot \frac{2 \cdot 5 \cdots (3k-1)}{6 \cdot 9 \cdots (3k+3)},
L_{G_2}^{G_2} = \sinh \left[ \frac{x}{6} \right. \cdot \frac{6 \cdot 9 \cdots (3k+3)}{2 \cdot 5 \cdots (3k-1)},
L_{G_2}^{G_2} = \sinh \left[ \frac{x}{6} \right. \cdot \frac{3 \cdot 6 \cdots 3k}{5 \cdot 8 \cdots (3k+2)},
L_{G_2}^{G_2} = \sinh \left[ \frac{x}{6} \right. \cdot \frac{7 \cdot 10 \cdots (3k+4)}{1 \cdot 4 \cdots (3k-2)},
L_{G_2}^{G_2} = \sinh \left[ \frac{x}{6} \right. \cdot \frac{6 \cdot 9 \cdots (6k+3)}{7 \cdot 10 \cdots (6k+4)},
L_{G_2}^{G_2} = \sinh \left[ \frac{x}{6} \right. \cdot \frac{7 \cdot 10 \cdots (6k+4)}{6 \cdot 9 \cdots (6k+3)},
L_{G_2}^{G_2} = \sinh \left[ \frac{x}{6} \right. \cdot \frac{6k+5}{5}.
$$
\[ L_{G_2}^{G_2} = \sinh \left( \frac{x}{6} \cdot \frac{(9k + 9)(9k + 6)}{6 \cdot 9} \right) \]

\[ L_{G_2}^{\text{corr}} = 1. \]

First notice, that

\[ L_{G_2}^{G_2} \cdot L_{G_2}^{G_2} = L_{G_2}^{G_2} \cdot L_{G_2}^{G_2} = L_{G_2}^{G_2} \cdot L_{G_2}^{G_2} = 1. \]

Thus, in the expression for \( X_2(x, k, -2, 10/3, 8/3) \) only the \( L_{G_2}^{G_2}, L_{G_2}^{G_2}, L_{G_2}^{G_2} \) terms contribute, so that

\[ X_2(x, k, -2, 10/3, 8/3) = L_{G_2}^{G_2} \cdot L_{G_2}^{G_2} \cdot L_{G_2}^{G_2}, \]

\[ = \sinh \left( \frac{x}{6} \cdot \frac{7 \cdot 10 \ldots (3k + 4)}{1 \cdot 4 \ldots (3k - 2)} \cdot \frac{6k + 5}{5} \cdot \frac{(9k + 9)(9k + 6)}{6 \cdot 9} \right), \]

\[ = \sinh \left( \frac{x}{4} \cdot \frac{(3k + 1)(3k + 4)(6k + 5)(9k + 9)(9k + 6)}{1 \cdot 4 \cdot 5 \cdot 6 \cdot 9} \right), \]

which coincides with the expression the Weyl formula (7) gives for quantum dimension of \( G_2 \) algebra.

**A.8. \( F_4 \)**

In this case we have \( \alpha = -2, \beta = 5, \gamma = 6 \)

\[ L_{F_4}^{F_4} = \sinh \left( \frac{x}{4} \cdot \frac{10 \cdot 12 \ldots (2k + 8)}{2 \cdot 4 \ldots 2k} \right) = \sinh \left( \frac{x}{4} \cdot \frac{(2k + 2)(2k + 4)(2k + 6)(2k + 8)}{2 \cdot 4 \cdot 6 \cdot 8} \right), \]

\[ L_{F_4}^{F_4} = \sinh \left( \frac{x}{4} \cdot \frac{8 \cdot 10 \ldots (2k + 6)}{2 \cdot 4 \ldots (2k + 2)} \right) = \sinh \left( \frac{x}{4} \cdot \frac{(2k + 4)(2k + 6)}{4 \cdot 6} \right), \]

\[ L_{F_4}^{F_4} = \sinh \left( \frac{x}{4} \cdot \frac{9 \cdot 11 \ldots (2k + 7)}{3 \cdot 5 \ldots (2k + 1)} \right) = \sinh \left( \frac{x}{4} \cdot \frac{(2k + 3)(2k + 5)(2k + 7)}{3 \cdot 5 \cdot 7} \right), \]

\[ L_{F_4}^{F_4} = \sinh \left( \frac{x}{4} \cdot \frac{7 \cdot 9 \ldots (2k + 6)}{5 \cdot 7 \ldots (2k + 3)} \right) = \sinh \left( \frac{x}{4} \cdot \frac{2k + 5}{5} \right), \]

\[ L_{F_4}^{F_4} = \sinh \left( \frac{x}{4} \cdot \frac{8 \cdot 10 \ldots (2k + 6)}{4 \cdot 6 \ldots (2k + 2)} \right) = \sinh \left( \frac{x}{4} \cdot \frac{(2k + 4)(2k + 6)}{4 \cdot 6} \right), \]

\[ L_{F_4}^{F_4} = \sinh \left( \frac{x}{4} \cdot \frac{14 \cdot 16 \ldots (4k + 12)}{8 \cdot 10 \ldots (4k + 6)} \right) = \sinh \left( \frac{x}{4} \cdot \frac{(4k + 8)(4k + 10)(4k + 12)}{8 \cdot 10 \cdot 12} \right), \]

\[ L_{F_4}^{F_4} = \sinh \left( \frac{x}{4} \cdot \frac{13 \cdot 15 \ldots (4k + 11)}{9 \cdot 11 \ldots (4k + 7)} \right) = \sinh \left( \frac{x}{4} \cdot \frac{(4k + 9)(4k + 11)}{9 \cdot 11} \right), \]

\[ L_{F_4}^{F_4} = \sinh \left( \frac{x}{4} \cdot \frac{4k + 10}{10} \right). \]
\[ L_{3}^{F_{4}} = \sinh \left( \frac{x}{4} \right) \frac{(6k + 14)(6k + 16)}{14 \cdot 16}, \]

\[ L_{\text{con}}^{F_{4}} = 1. \]

The product of all these terms

\[ X_{2}(x, k, -2, 5, 6) = L_{11}^{F_{4}} \cdot L_{12}^{F_{4}} \cdot L_{13}^{F_{4}} \cdot L_{14}^{F_{4}} \cdot L_{15}^{F_{4}} \cdot L_{21}^{F_{4}} \cdot L_{22}^{F_{4}} \cdot L_{23}^{F_{4}} \cdot L_{3}^{F_{4}} \cdot L_{\text{con}}^{F_{4}} \]

\[ = \sinh \left( \frac{x}{4} \right) \frac{(2k + 2)(2k + 4)^{3}(2k + 5)^{3}(2k + 6)^{3}(2k + 7)(2k + 8)(4k + 8)(4k + 10)^{2}}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10} \]

\[ \times \frac{(4k + 1)(4k + 12)(6k + 14)(6k + 16)}{11 \cdot 12 \cdot 14 \cdot 16} \]

immediately coincides with expression Weyl formula gives for \( F_{4} \) algebra.

\[ A.9. E_{6} \]

For \( E_{6} \) the Vogel parameters are \( \alpha = -2, \beta = 6, \gamma = 8. \)

\[ L_{11}^{E_{6}} = \sinh \left( \frac{x}{4} \right) \frac{7 \cdot 8 \ldots (k + 6)}{1 \cdot 1 \ldots k} = \sinh \left( \frac{x}{4} \right) \frac{(k + 1)(k + 2) \ldots (k + 6)}{1 \cdot 2 \ldots 6}, \]

\[ L_{12}^{E_{6}} = \sinh \left( \frac{x}{4} \right) \frac{6 \cdot 7 \ldots (k + 5)}{2 \cdot 3 \ldots (k + 8)} = \sinh \left( \frac{x}{4} \right) \frac{(k + 2)(k + 3)(k + 4)(k + 5)}{2 \cdot 3 \cdot 4 \cdot 5}, \]

\[ L_{13}^{E_{6}} = \sinh \left( \frac{x}{4} \right) \frac{6 \cdot 7 \ldots (k + 5)}{2 \cdot 3 \ldots (k + 8)} = \sinh \left( \frac{x}{4} \right) \frac{(k + 2)(k + 3)(k + 4)(k + 5)}{2 \cdot 3 \cdot 4 \cdot 5}, \]

\[ L_{14}^{E_{6}} = \sinh \left( \frac{x}{4} \right) \frac{5 \cdot 6 \ldots (k + 4)}{3 \cdot 4 \ldots (k + 2)} = \sinh \left( \frac{x}{4} \right) \frac{(k + 3)(k + 4)}{3 \cdot 4}, \]

\[ L_{15}^{E_{6}} = \sinh \left( \frac{x}{4} \right) \frac{5 \cdot 6 \ldots (k + 4)}{3 \cdot 4 \ldots (k + 2)} = \sinh \left( \frac{x}{4} \right) \frac{(k + 3)(k + 4)}{3 \cdot 4}, \]

\[ L_{21}^{E_{6}} = \sinh \left( \frac{x}{4} \right) \frac{10 \cdot 11 \ldots (2k + 9)}{5 \cdot 6 \ldots (2k + 4)} = \sinh \left( \frac{x}{4} \right) \frac{(2k + 5)(2k + 6) \ldots (2k + 9)}{5 \cdot 6 \ldots 9}, \]

\[ L_{22}^{E_{6}} = \sinh \left( \frac{x}{4} \right) \frac{9 \cdot 10 \ldots (2k + 8)}{6 \cdot 7 \ldots (2k + 5)} = \sinh \left( \frac{x}{4} \right) \frac{(2k + 6)(2k + 7)(2k + 8)}{6 \cdot 7 \cdot 8}, \]

\[ L_{23}^{E_{6}} = \sinh \left( \frac{x}{4} \right) \frac{2k + 7}{7}, \]

\[ L_{3}^{E_{6}} = \sinh \left( \frac{x}{4} \right) \frac{(3k + 10)(3k + 11)}{10 \cdot 11}, \]

\[ L_{\text{con}}^{E_{6}} = 1. \]
The product of all these terms gives

\[ X_2(x, k, -2, 6, 8) = L_{11}^{E_6} \cdot L_{12}^{E_6} \cdot L_{13}^{E_6} \cdot L_{14}^{E_6} \cdot L_{15}^{E_6} \cdot L_{21}^{E_6} \cdot L_{22}^{E_6} \cdot L_{23}^{E_6} \cdot L_3^{E_6} \cdot L_{corr}^{E_6} \]

\[ = \sinh \left[ \frac{x}{4} \cdot \frac{(k + 1)(k + 2)^3(k + 3)^7(k + 4)^4(k + 6)(2k + 5)(2k + 7)^2(2k + 8)^2(2k + 9)}{1 \cdot 2^5 \cdot 3^3 \cdot 4^6 \cdot 5^6 \cdot 6^3 \cdot 7^2 \cdot 8^2 \cdot 9 \cdot 10 \cdot 11} \right] \]

which coincides with quantum dimension \( D_Q^{E_6} (7) \).

A.10. \( E_7 \)

For \( E_7 \) Vogel’s parameters are \( \alpha = -2, \beta = 8, \gamma = 12 \).

\[ L_{11}^{E_7} = \sinh \left[ \frac{x}{4} : 11 \cdot 12 \ldots (k + 10) \frac{1}{1 \cdot 1 \ldots k} \right] = \sinh \left[ \frac{x}{4} : \frac{(k + 1)(k + 2) \ldots (k + 10)}{1 \cdot 2 \ldots 10} \right], \]

\[ L_{12}^{E_7} = \sinh \left[ \frac{x}{4} : 10 \cdot 11 \ldots (k + 9) \frac{2 \cdot 3 \ldots (k + 1)}{2 \cdot 3 \ldots 9} \right] = \sinh \left[ \frac{x}{4} : \frac{(k + 2)(k + 3)(k + 4) \ldots (k + 9)}{2 \cdot 3 \ldots 9} \right], \]

\[ L_{13}^{E_7} = \sinh \left[ \frac{x}{4} : 9 \cdot 10 \ldots (k + 8) \frac{3 \cdot 4 \ldots (k + 2)}{3 \cdot 4 \ldots 8} \right] = \sinh \left[ \frac{x}{4} : \frac{(k + 3)(k + 4)(k + 5)(k + 6)(k + 7)(k + 8)}{3 \cdot 4 \ldots 8} \right], \]

\[ L_{14}^{E_7} = \sinh \left[ \frac{x}{4} : 8 \cdot 9 \ldots (k + 7) \frac{4 \cdot 5 \ldots (k + 3)}{4 \cdot 5 \cdot 6 \cdot 7} \right] = \sinh \left[ \frac{x}{4} : \frac{(k + 4)(k + 5)(k + 6)(k + 7)}{4 \cdot 5 \cdot 6 \cdot 7} \right], \]

\[ L_{15}^{E_7} = \sinh \left[ \frac{x}{4} : 7 \cdot 8 \ldots (k + 6) \frac{5 \cdot 6 \ldots (k + 4)}{5 \cdot 6} \right] = \sinh \left[ \frac{x}{4} : \frac{(k + 5)(k + 6)}{5 \cdot 6} \right], \]

\[ L_{21}^{E_7} = \sinh \left[ \frac{x}{4} : 16 \cdot 17 \ldots (2k + 15) \frac{7 \cdot 8 \ldots (2k + 6)}{7 \cdot 8 \ldots 15} \right] = \sinh \left[ \frac{x}{4} : \frac{(2k + 7)(2k + 8) \ldots (2k + 15)}{7 \cdot 8 \ldots 15} \right], \]

\[ L_{22}^{E_7} = \sinh \left[ \frac{x}{4} : 14 \cdot 15 \ldots (2k + 13) \frac{9 \cdot 10 \ldots (2k + 8)}{9 \cdot 10 \ldots 13} \right] = \sinh \left[ \frac{x}{4} : \frac{(2k + 9) \ldots (2k + 13)}{9 \cdot 10 \ldots 13} \right], \]

\[ L_{23}^{E_7} = \sinh \left[ \frac{x}{4} : \frac{2k + 11}{11} \right] \]

\[ L_3^{E_7} = \sinh \left[ \frac{x}{4} : \frac{(3k + 16)(3k + 17)}{16 \cdot 17} \right]. \]

\[ L_{corr}^{E_7} = 1. \]

The product of all these terms gives

\[ X_2(x, k, -2, 8, 12) = L_{11}^{E_7} \cdot L_{12}^{E_7} \cdot L_{13}^{E_7} \cdot L_{14}^{E_7} \cdot L_{15}^{E_7} \cdot L_{21}^{E_7} \cdot L_{22}^{E_7} \cdot L_{23}^{E_7} \cdot L_3^{E_7} \cdot L_{corr}^{E_7} \]
\begin{align*}
&= \sinh \left( \frac{x}{4} \cdot \frac{(k + 1)(k + 2)(k + 3)(k + 4)(k + 5)(k + 6)(k + 7)(k + 8)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} \cdot \frac{1}{9^9 \cdot 10^9 \cdot 11^9 \cdot 12^9 \cdot 13^9 \cdot 14^9 \cdot 15^9 \cdot 16^9 \cdot 17^9} \right) \\
&= \sinh \left( \frac{x}{4} \cdot \frac{(k + 1)(k + 2) \ldots (k + 18)}{1 \cdot 2 \ldots 18} \right) \\
L_{E_{11}} &= \sinh \left( \frac{x}{4} \cdot \frac{19 \cdot 20 \ldots (k + 18)}{1 \cdot 1 \ldots k} \right) = \sinh \left( \frac{x}{4} \cdot \frac{(k + 1)(k + 2) \ldots (k + 18)}{1 \cdot 2 \ldots 18} \right), \\
L_{E_{12}} &= \sinh \left( \frac{x}{4} \cdot \frac{18 \cdot 19 \ldots (k + 17)}{2 \cdot 3 \ldots (k + 1)} \right) = \sinh \left( \frac{x}{4} \cdot \frac{(k + 2)(k + 3) \ldots (k + 17)}{2 \cdot 3 \ldots 17} \right), \\
L_{E_{13}} &= \sinh \left( \frac{x}{4} \cdot \frac{15 \cdot 16 \ldots (k + 14)}{5 \cdot 6 \ldots (k + 4)} \right) = \sinh \left( \frac{x}{4} \cdot \frac{(k + 5)(k + 6) \ldots (k + 14)}{5 \cdot 6 \ldots 14} \right), \\
L_{E_{14}} &= \sinh \left( \frac{x}{4} \cdot \frac{14 \cdot 15 \ldots (k + 13)}{6 \cdot 7 \ldots (k + 5)} \right) = \sinh \left( \frac{x}{4} \cdot \frac{(k + 6) \ldots (k + 13)}{6 \cdot 7 \ldots 13} \right), \\
L_{E_{15}} &= \sinh \left( \frac{x}{4} \cdot \frac{11 \cdot 12 \ldots (k + 10)}{9 \cdot 10 \ldots (k + 8)} \right) = \sinh \left( \frac{x}{4} \cdot \frac{(k + 9)(k + 10)}{9 \cdot 10} \right), \\
L_{E_{21}} &= \sinh \left( \frac{x}{4} \cdot \frac{28 \cdot 29 \ldots (2k + 27)}{11 \cdot 12 \ldots (2k + 10)} \right) = \sinh \left( \frac{x}{4} \cdot \frac{(2k + 11) \ldots (2k + 27)}{11 \cdot 12 \ldots 27} \right), \\
L_{E_{22}} &= \sinh \left( \frac{x}{4} \cdot \frac{24 \cdot 25 \ldots (2k + 23)}{15 \cdot 16 \ldots (2k + 14)} \right) = \sinh \left( \frac{x}{4} \cdot \frac{(2k + 15) \ldots (2k + 23)}{15 \ldots 23} \right), \\
L_{E_{23}} &= \sinh \left( \frac{x}{4} \cdot \frac{2k + 19}{19} \right), \\
L_{E_{3}} &= \sinh \left( \frac{x}{4} \cdot \frac{(3k + 28)(3k + 29)}{28 \cdot 29} \right). \\
L_{\text{corr}} &= 1.
\end{align*}

which coincides with $D^{\alpha}_{D}$ of (22).

**A.11. $E_8$**

For $E_8$ the Vogel parameters are $\alpha = -2, \beta = 12, \gamma = 20$.

The product of all these terms gives

\[ X_2(x, k, -2, 12, 20) = L_{E_{11}} \cdot L_{E_{12}} \cdot L_{E_{13}} \cdot L_{E_{14}} \cdot L_{E_{15}} \cdot L_{E_{21}} \cdot L_{E_{22}} \cdot L_{E_{23}} \cdot L_{E_{3}} \cdot L_{\text{corr}}. \]
\[
\begin{aligned}
\sinh\left(\sum_{k=1}^{\frac{\sqrt{5}-1}{2}} \left( k+1 \right) (k+2)^2 (k+3)^2 (k+4)^2 (k+5)^2 (k+6)^2 (k+7)^2 (k+8)^2 (k+9)^2 (k+10)^2 (k+11)^2 (k+12)^2 \\
\times \left( k+13 \right)^2 (k+14)^2 (k+15)^2 (k+16)^2 (k+17)^2 (k+18)^2 (k+19)^2 (k+20)^2 (k+21)^2 (k+22)^2 \cdots \right) \frac{1}{1 \cdot 2^4 \cdot 3^4 \cdot 4^4 \cdot 5^4 \cdot 6^4 \cdot 7^4 \cdot 8^4 \\
\times \frac{(2k+16)^2 (2k+17)^2 (2k+18)^2 (2k+19)^2 (2k+20)^2 (2k+21)^2 (2k+22)^2 (2k+23)^2 (2k+24)^2 (2k+25)^2 (2k+26)^2 (2k+27)^2 (2k+28)^2 (2k+29)^2}{180 \cdot 19 \cdot 20 \cdot 21 \cdot 22 \cdot 23 \cdot 24 \cdot 25 \cdot 26 \cdot 27 \cdot 28 \cdot 29}
\right)
\end{aligned}
\]

coinciding with the direct calculation by (7).

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