On the limit of large girth graph sequences*

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Abstract

Let \( d \geq 2 \) be given and let \( \mu \) be an involution-invariant probability measure on the space of trees \( T \in T_d \) with maximum degrees at most \( d \). Then \( \mu \) arises as the local limit of some sequence \( \{G_n\}_{n=1}^\infty \) of graphs with all degrees at most \( d \). This answers Question 6.8 of Bollobás and Riordan [4].

1 Introduction

Let \( \text{Graph}_d \) denote the set of all finite simple graphs \( G \) (up to isomorphism) for which \( \text{deg}(x) \leq d \) for every \( x \in V(G) \). For a graph \( G \) and \( x, y \in V(G) \) let \( d_G(x, y) \) denote the distance of \( x \) and \( y \), that is the length of the shortest path from \( x \) to \( y \). A rooted \((r,d)\)-ball is a graph \( G \in \text{Graph}_d \) with a marked vertex \( x \in V(G) \) called the root such that \( d_G(x, y) \leq r \) for every \( y \in V(G) \). By \( U_{r,d} \) we shall denote the set of rooted \((r,d)\)-balls.

If \( G \in \text{Graph}_d \) is a graph and \( x \in V(G) \) then \( B_r(x) \in U_{r,d} \) shall denote the rooted \((r,d)\)-ball around \( x \) in \( G \). For any \( \alpha \in U_{r,d} \) and \( G \in \text{Graph}_d \) we define the set \( T(G, \alpha) \overset{\text{def}}{=} \{ x \in V(G) : B_r(x) \cong \alpha \} \) and let \( p_G(\alpha) \overset{\text{def}}{=} \frac{|T(G, \alpha)|}{|V(G)|} \).

A graph sequence \( G = \{G_n\}_{n=1}^\infty \subset \text{Graph}_d \) is weakly convergent if \( \lim_{n \to \infty} |V(G_n)| = \infty \) and for every \( r \) and every \( \alpha \in U_{r,d} \) the limit \( \lim_{n \to \infty} p_{G_n}(\alpha) \) exists (see [3]).

Let \( \text{Gr}_d \) denote the set of all countable, connected rooted graphs \( G \) for which \( \text{deg}(x) \leq d \) for every \( x \in V(G) \). If \( G, H \in \text{Gr}_d \) let \( d_g(G, H) = 2^{-r} \), where \( r \) is the maximal number such that the \( r \)-balls around the roots of \( G \) resp. \( H \) are rooted isomorphic. The distance \( d_g \) makes \( \text{Gr}_d \) a compact metric space. Given an \( \alpha \in U_{r,d} \) let \( T(\text{Gr}_d, \alpha) = \{ (G, x) \in \text{Gr}_d : B_r(x) \cong \alpha \} \). The sets \( T(\text{Gr}_d, \alpha) \) are closed-open sets. A convergent graphs sequence \( \{G_n\}_{n=1}^\infty \) define a local limit measure \( \mu_G \) on \( \text{Gr}_d \), where \( \mu_G(T(\text{Gr}_d, \alpha)) = \lim_{n \to \infty} p_{G_n}(\alpha) \). However, not all the probability measures on \( \text{Gr}_d \) arise as local limits. A necessary condition for a measure \( \mu \) being a local limit is its involution invariance (see Section 2). The goal of this paper is to answer a question of Bollobás and Riordan (Question 6.8 [4]):

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Theorem 1 Any involution-invariant measure $\mu$ on $Gr_d$ concentrated on trees arises as a local limit of some convergent graph sequence.

As it was pointed out in [4] such graph sequences are asymptotically treelike, thus $\mu$ must arise as the local limit of a convergent large girth sequence.

2 Involution invariance

Let $\tilde{Gr}_d$ be the compact space of all connected countable rooted graphs $\tilde{G}$ (up to isomorphism) of vertex degree bound $d$ with a distinguished directed edge pointing out from the root. Note that $\tilde{G}$ and $\tilde{H}$ are considered isomorphic if there exists a rooted isomorphism between them mapping distinguished edges into each other. Let $\tilde{U}^{r,d}$ be the isomorphism classes of all rooted $(r,d)$-graphs $\tilde{\alpha}$ with a distinguished edge $e(\tilde{\alpha})$ pointing out from the root. Again, $T(\tilde{Gr}_d, \tilde{\alpha})$ is well-defined for any $\tilde{\alpha} \in \tilde{U}^{r,d}$ and defines a closed-open set in $\tilde{Gr}_d$. Clearly, the forgetting map $F : \tilde{Gr}_d \rightarrow Gr_d$ is continuous. Let $\mu$ be a probability measure on $Gr_d$. Then we define a measure $\tilde{\mu}$ on $\tilde{Gr}_d$ the following way.

Let $\tilde{\alpha} \in \tilde{U}^{r,d}$ and let $F(\tilde{\alpha}) = \alpha \in U^{r,d}$ be the underlying rooted ball. Clearly, $F(T(\tilde{Gr}_d, \tilde{\alpha})) = T(\tilde{Gr}_d, \alpha)$. Let

$$\tilde{\mu}(T(\tilde{Gr}_d, \tilde{\alpha})) := l,$$

where $l$ is the number of edges $e$ pointing out from the root such that there exists a rooted automorphism of $\alpha$ mapping $e(\tilde{\alpha})$ to $e$. Observe that

$$\tilde{\mu}(F^{-1}(T(\tilde{Gr}_d, \alpha))) = \deg(\alpha)\mu(T(\tilde{Gr}_d, \alpha)).$$

We define the map $T : \tilde{Gr}_d \rightarrow \tilde{Gr}_d$ as follows. Let $T(\tilde{G}) = \tilde{H}$, where :

- the underlying graphs of $\tilde{G}$ and $\tilde{H}$ are the same,
- the root of $\tilde{H}$ is the endpoint of $e(\tilde{G})$,
- the distinguished edge of $\tilde{H}$ is pointing to the root of $\tilde{G}$.

Note that $T$ is a continuous involution. Following Aldous and Steele [2], we call $\mu$ involution-invariant if $T_* (\mu) = \mu$. It is important to note [2, 1] that the limit measure of convergent graph sequences are always involution-invariant.

We need to introduce the notion of edge-balls. Let $\tilde{G} \in \tilde{Gr}_d$. The edge-ball $B_r^e(\tilde{G})$ of radius $r$ around the root of $\tilde{G}$ is the following spanned rooted subgraph of $\tilde{G}$:

- The root of $B_r^e(\tilde{G})$ is the same as the root of $\tilde{G}$.
- $y$ is a vertex of $B_r^e(\tilde{G})$ if $d(x,y) \leq r$ or $d(x',y) \leq r$, where $x$ is the root of $\tilde{G}$ and $x'$ is the endpoint of the directed edge $e(\tilde{G})$.
- The distinguished edge of $B_r^e(\tilde{G})$ is $(x,x')$. 

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Let $\vec{E}_r,d$ be the set of all edge-balls of radius $r$ up to isomorphism. Then if $\vec{\phi} \in \vec{E}_r,d$, let $s(\vec{\phi}) \in \vec{U}_r,d$ be the rooted ball around the root of $\vec{\phi}$. Also, let $t(\vec{\phi}) \in \vec{U}_r,d$ be the $r$-ball around $x'$ with distinguished edge $(x',x)$. The involution $T^{r,d}: \vec{E}_r,d \to \vec{E}_r,d$ is defined the obvious way and $t(T^{r,d}(\vec{\phi})) = s(\vec{\phi})$, $s(T^{r,d}(\vec{\phi})) = t(\vec{\phi})$. Since $\vec{\mu}$ is a measure we have

$$\vec{\mu}(T(\vec{G}_d, \vec{\alpha})) = \sum_{\vec{\phi}, s(\vec{\phi}) = \vec{\alpha}} \vec{\mu}(T(\vec{G}_d, \vec{\phi})).$$

(1)

Also, by the involution-invariance

$$\vec{\mu}(T(\vec{G}_d, \vec{\alpha})) = \vec{\mu}(T(\vec{G}_d, T^{r,d}(\vec{\phi}))),$$

(2)

since $T(T(\vec{G}_d, \vec{\phi})) = T(\vec{G}_d, T^{r,d}(\vec{\phi})$. Therefore by (1),

$$\vec{\mu}(T(\vec{G}_d, \vec{\alpha})) = \sum_{\vec{\phi}, t(\vec{\phi}) = \vec{\alpha}} \vec{\mu}(T(\vec{G}_d, \vec{\phi}))$$

(3)

3 Labeled graphs

Let $\vec{G}_d^n$ be the isomorphism classes of

- connected countable rooted graphs with vertex degree bound $d$
- with a distinguished edge pointing out from the root
- with vertex labels from the set $\{1,2,\ldots,n\}$.

Note that if $\vec{G}_*$ and $\vec{H}_*$ are such graphs then they called isomorphic if there exists a map $\rho: V(\vec{G}_*) \to V(\vec{H}_*)$ preserving both the underlying $\vec{G}_d$-structure and the vertex labels. The labeled $r$-balls $\vec{U}_r,d_n$ and the labeled $r$-edge-balls $\vec{E}_r,d_n$ are defined accordingly. Again, $\vec{G}_d^n$ is a compact metric space and $T(\vec{G}_d^n, \vec{\alpha}_*), T(\vec{G}_d^n, \vec{\phi}_*)$ are closed-open sets, where $\vec{\alpha}_* \in \vec{U}_r,d_n, \vec{\phi}_* \in \vec{E}_r,d_n$. Now let $\mu$ be an involution-invariant probability measure on $\vec{G}_d$ with induced measure $\vec{\mu}$. The associated measure $\vec{\mu}_n$ on $\vec{G}_d^n$ is defined the following way.

Let $\vec{\alpha} \in \vec{U}_r,d_n$ and $\kappa_1, \kappa_2$ be vertex labelings of $\vec{\alpha}$ by $\{1,2,\ldots,n\}$. We say that $\kappa_1$ and $\kappa_2$ are equivalent if there exists a rooted automorphism of $\vec{\alpha}$ preserving the distinguished edge and mapping $\kappa_1$ to $\kappa_2$. Let $C(\kappa)$ be the equivalence class of the vertex labeling $\kappa$ of $\vec{\alpha}$. Then we define

$$\vec{\mu}_n(T(\vec{G}_d^n, [\kappa])) := \frac{|C(\kappa)|}{\mu(V(\vec{\alpha}))} \vec{\mu}(T(\vec{G}_d^n, \vec{\alpha})).$$

Lemma 3.1  a) $\vec{\mu}_n$ extends to a Borel-measure.

b) $\vec{\mu}(T(\vec{G}_d, \vec{\alpha})) = \sum_{\vec{\alpha}_* : \vec{\rho}(\vec{\alpha}_*) = \vec{\alpha}} \vec{\mu}_n(T(\vec{G}_d^n, \vec{\alpha}_*))$. 

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Proof. The second equation follows directly from the definition. In order to prove that $\bar{\mu}_n$ extends to a Borel-measure it is enough to prove that

$$\bar{\mu}_n(T(\hat{G}^n_d, \vec{\alpha}_s)) = \sum_{\vec{\beta}_s \in N_{r+1}(\vec{\alpha}_s)} \bar{\mu}_n(T(\hat{G}^n_d, \vec{\beta}_s)),$$

where $\vec{\alpha}_s \in \hat{U}^{r,d}_n$ and $N_{r+1}(\vec{\alpha}_s)$ is the set of elements $\vec{\beta}_s$ in $\hat{U}^{r+1,d}_n$ such that the $r$-ball around the root of $\vec{\beta}_s$ is isomorphic to $\vec{\alpha}_s$. Let $\vec{\alpha} = \mathcal{F}(\vec{\alpha}_s) \in \hat{U}^{r,d}_n$ and let $N_{r+1}(\vec{\alpha}) \subset \hat{U}^{r,d}_n$ be the set of elements $\vec{\beta}$ such that the $r$-ball around the root of $\vec{\beta}$ is isomorphic to $\vec{\alpha}$. Clearly

$$\bar{\mu}(T(\hat{G}^d_d, \vec{\alpha})) = \sum_{\vec{\beta} \in N_{r+1}(\vec{\alpha})} \bar{\mu}(T(\hat{G}^d_d, \vec{\beta})). \tag{4}$$

Let $\kappa$ be a labeling of $\vec{\alpha}$ by $\{1,2,\ldots,n\}$ representing $\vec{\alpha}_s$. For $\vec{\beta} \in N_{r+1}(\vec{\alpha})$ let $L(\vec{\beta})$ be the set of labelings of $\vec{\beta}$ that extends some labeling of $\vec{\alpha}$ that is equivalent to $\kappa$.

Note that

$$\bar{\mu}_n(T(\hat{G}^d_d, \vec{\alpha}_s)) = \bar{\mu}(T(\hat{G}^d_d, \vec{\alpha})) \frac{|C(\kappa)|}{n^{|V(\vec{\beta})|}}.$$

Also,

$$\sum_{\vec{\beta}_s \in N_{r+1}(\vec{\alpha}_s)} \bar{\mu}_n(T(\hat{G}^n_d, \vec{\beta}_s)) = \sum_{\vec{\beta} \in N_{r+1}(\vec{\alpha})} \bar{\mu}(T(\hat{G}^n_d, \vec{\beta})) \frac{|L(\vec{\beta})|}{n^{|V(\vec{\beta})|}}.$$

Observe that $|L(\vec{\beta})| = |C(\kappa)|n^{|V(\vec{\beta})|-|V(\vec{\alpha})|}$. Hence

$$\sum_{\vec{\beta}_s \in N_{r+1}(\vec{\alpha}_s)} \bar{\mu}_n(T(\hat{G}^n_d, \vec{\beta}_s)) = \sum_{\vec{\beta} \in N_{r+1}(\vec{\alpha})} \bar{\mu}(T(\hat{G}^n_d, \vec{\beta})) \frac{|C(\kappa)|}{n^{|V(\vec{\beta})|}}.$$

Therefore using equation (4) our lemma follows.

The following proposition shall be crucial in our construction.

**Proposition 3.1** For any $\vec{\alpha}_s \in \hat{U}^{r,d}_n$ and $\vec{\psi}_s \in \hat{E}^{r,d}_n$

\begin{itemize}
  \item $\bar{\mu}_n(T(\hat{G}_d^n, \vec{\alpha}_s)) = \sum_{\vec{\phi}_s \in \hat{E}^{r,d}_n, s(\vec{\phi}_s) = \vec{\alpha}_s} \bar{\mu}_n(T(\hat{G}_d^n, \vec{\phi}_s))$
  \item $\bar{\mu}_n(T(\hat{G}_d^n, \vec{\alpha}_s)) = \sum_{\vec{\phi}_s \in \hat{E}^{r,d}_n, t(\vec{\phi}_s) = \vec{\alpha}_s} \bar{\mu}_n(T(\hat{G}_d^n, \vec{\phi}_s))$
  \item $\bar{\mu}_n(T(\hat{G}_d^n, \vec{\psi}_s)) = \bar{\mu}_n(T(\hat{G}_d^n, T^{r,d}_n(\vec{\psi}_s))).$
\end{itemize}

**Proof.** The first equation follows from the fact that $\bar{\mu}_n$ is a Borel-measure. Thus the second equation will be an immediate corollary of the third one. So, let us
turn to the third equation. Let $F(\vec{\psi}_s) = \vec{\psi} \in \vec{E}^{r,d}$ and let $\kappa$ be a vertex-labeling of $\vec{\psi}$ representing $\vec{\psi}_s$. It is enough to prove that

$$\bar{\mu}_n(T(\vec{G}_d^n, \vec{\psi}_s)) = \frac{|C(\kappa)|}{n^{|V(\vec{\psi})|}} \bar{\mu}(T(\vec{G}_d^n, \vec{\psi})), $$

where $C(\kappa)$ is the set of labelings of $\vec{\psi}$ equivalent to $\kappa$. Let $\mathcal{N}_{r+1}(\vec{\psi}) \subseteq \vec{U}^{r,d}_n$ be the set of elements $\vec{\beta}$ such that the edge-ball of radius $r$ around the root of $\vec{\beta}$ is isomorphic to $\vec{\psi}$. Then

$$\bar{\mu}(T(\vec{G}_d^n, \vec{\psi})) = \sum_{\vec{\beta} \in \mathcal{N}_{r+1}(\vec{\psi})} \bar{\mu}(T(\vec{G}_d^n, \vec{\beta})). \quad (5)$$

Observe that

$$\bar{\mu}_n(T(\vec{G}_d^n, \vec{\psi}_s)) = \sum_{\vec{\beta} \in \mathcal{N}_{r+1}(\vec{\psi})} \bar{\mu}(T(\vec{G}_d^n, \vec{\beta})) \frac{k(\vec{\beta}, \vec{\psi}_s)}{n^{|V(\vec{\beta})|}},$$

where $k(\vec{\beta}, \vec{\psi}_s)$ is the number of labelings of $\vec{\beta}$ extending an element that is equivalent to $\kappa$. Notice that $k(\vec{\beta}, \vec{\psi}_s) = |C(\kappa)|n^{|V(\vec{\beta})|}$. Hence by (5)

$$\bar{\mu}_n(T(\vec{G}_d^n, \vec{\psi}_s)) = \frac{|C(\kappa)|}{n^{|V(\vec{\psi})|}} \bar{\mu}(T(\vec{G}_d^n, \vec{\psi})), $$

thus our proposition follows.

4 Label-separated balls

Let $\vec{G}_d^n$ be the isomorphism classes of

- connected countable rooted graphs with vertex degree bound $d$
- with vertex labels from the set $\{1, 2, \ldots, n\}$.

Again, we define the space of labeled $r$-balls $\vec{U}^{r,d}_n$. Then $\vec{G}_d^n$ is a compact space with closed-open sets $T(\vec{G}_d^n, M), M \in \vec{U}^{r,d}_n$. Similarly to the previous section we define an associated probability measure $\mu_n$, where $\mu$ is an involution-invariant probability measure on $\vec{G}_d$.

Let $M \in \vec{U}^{r,d}_n$ and let $R(M)$ be the set of elements of $\vec{U}^{r,d}_n$ with underlying graph $M$. If $A \in R(M)$, then the multiplicity of $A$, $l_A$ is the number of edges $e$ pointing out from the root of $A$ such that there is a label-preserving rooted automorphism of $A$ moving the distinguished edge to $e$. Now let

$$\mu_n(M) := \frac{1}{\deg(M)} \sum_{A \in R(M)} l_A \bar{\mu}_n(A).$$

The following lemma is the immediate consequence of Lemma 3.1.

Lemma 4.1 $\mu_n$ is a Borel-measure on $\vec{G}_d^n$ and $\sum_{M \in \mathcal{M}(\alpha)} \mu_n(M) = \mu(A)$ if $\alpha \in \vec{U}^{r,d}$ and $\mathcal{M}(\alpha)$ is the set of labelings of $\alpha$ by $\{1, 2, \ldots, n\}$.
Definition 4.1 $M \in U_{r,d}^n$ is called label-separated if all the labels of $M$ are different.

Lemma 4.2 For any $\alpha \in U_{r,d}$ and $\delta > 0$ there exists an $n > 0$ such that

$$|\sum_{M \in \mathcal{M}(\alpha), M \text{ is label-separated}} \mu_n(T(\mathcal{G}_d, M)) - \mu(T(\mathcal{G}_d, \alpha))| < \delta.$$ 

Proof. Observe that

$$\sum_{M \in \mathcal{M}(\alpha), M \text{ is label-separated}} \mu_n(T(\mathcal{G}_d, M)) = \frac{T(n, \alpha)}{n^{|V(\alpha)|}} \mu(T(\mathcal{G}_d, \alpha)),$$

where $T(n, \alpha)$ is the number of $\{1, 2, \ldots, n\}$-labelings of $\alpha$ with different labels. Clearly, $\frac{T(n, \alpha)}{n^{|V(\alpha)|}} \to 1$ as $n \to \infty$.

5 The proof of Theorem [1]

Let $\mu$ be an involution-invariant probability measure on $\mathcal{G}_d$ supported on trees. It is enough to prove that for any $r \geq 1$ and $\epsilon > 0$ there exists a finite graph $G$ such that for any $\alpha \in U_{r,d}$

$$|p_G(\alpha) - \mu(T(\mathcal{G}_d, \alpha))| < \epsilon.$$ 

The idea we follow is close to the one used by Bowen in [5]. First, let $n > 0$ be a natural number such that

$$|\sum_{M \in \mathcal{M}(\alpha), M \text{ is label-separated}} \mu_n(T(\mathcal{G}_d, M)) - \mu(T(\mathcal{G}_d, \alpha))| < \frac{\epsilon}{10}. \quad (6)$$

Then we define a directed labeled finite graph $H$ to encode some information on $\bar{\mu}_n$. If $A \in \bar{U}_{r+1,d}^n$ then let $L_A$ be the unique element of $\bar{E}_{r+1,d}^n$ contained in $A$. The set of vertices of $H$: $V(H) := \bar{U}_{r+1,d}^n$. If $A, B \in \bar{U}_{r+1,d}^n$ and $L_A = L_B^{-1}$ (we use the inverse notation instead of writing out the involution operator) then there is a directed edge $(A, L_A, B)$ from $A$ to $B$ labeled by $L_A$ and a directed edge $(B, L_B, A)$ from $B$ to $A$ labeled by $L_B = L_A^{-1}$. Note that we might have loops. We define the weight function $w$ on $H$ by

- $w(A) = \bar{\mu}_n(T(\mathcal{G}_d^n, A)).$
- $w(A, L_A, B) = \mu(T(\mathcal{G}_d^n, L_A, B))$, where $L_A, B \in \bar{E}_{r+1,d}^n$ the unique element such that $s(L_A, B) = A, t(L_A, B) = B.$

By Proposition [3] we have the following equation for all $A, B$ that are connected in $H$:

$$w(A, L_A, B) = w(B, L_A^{-1}, A). \quad (7)$$
Also,
\[ w(A) = \sum_{w(A,L_A,B) \in E(H)} w(A,L_A,B) \]  \hspace{1cm} (8)

\[ w(A) = \sum_{w(B,L_A^{-1},A) \in E(H)} w(B,L_A^{-1},A) \]  \hspace{1cm} (9)

Also if \( M \in U_{n}^{r+1,d} \) then
\[ \mu_n(M) = \frac{1}{\deg(M)} \sum_{A \in R(M)} l_A w(A), \]  \hspace{1cm} (10)

where \( l_A \) is the multiplicity of \( w(A) \).

Since the equations (7), (8), (9) have rational coefficients we also have weight functions \( w_\delta \) on \( H \)

- taking only rational values
- satisfying equations (7), (8), (9)
- such that \( |w_\delta(A) - w(A)| < \delta \) for any \( A \in V(H) \), where the exact value of \( \delta \) will be given later.

Now let \( N \) be a natural number such that

- \( \frac{Nw_\delta(A)}{l_A} \in \mathbb{N} \) if \( A \in V(H) \).
- \( Nw_\delta(A,L_A,B) \in \mathbb{N} \) if \( (A,L_A,B) \in E(H) \).

**Step 1.** We construct an edge-less graph \( Q \) such that:

- \( V(Q) = \bigcup_{A \in V(H)} Q(A) \) (disjoint union)
- \( |Q(A)| = Nw_\delta(A) \)
- each \( Q(A) \) is partitioned into \( \bigcup_{(A,L_A,B) \in E(H)} Q(A,L_A,B) \) such that
  \[ |Q(A,L_A,B)| = Nw_\delta(A,L_A,B). \]

Since \( w_\delta \) satisfy our equations such \( Q \) can be constructed.

**Step 2.** We add edges to \( Q \) in order to obtain the graph \( R \). For each pair \( A,B \) that are connected in the graph \( H \) form a bijection \( Z_{A,B} : Q(A,L_A,B) \rightarrow Q(B,L_B,A) \). If there is a loop in \( H \) consider a bijection \( Z_{A,A} \). Then draw an edge between \( x \in Q(A,L_A,B) \) and \( y \in Q(B,L_B,A) \) if \( Z_{A,B}(x) = y \).

**Step 3.** Now we construct our graph \( G \). If \( M \in U_{n}^{r+1,d} \) is a rooted labeled tree such that \( \mu_n(M) \neq 0 \) let \( Q(M) = \bigcup_{A \in R(M)} Q(A) \). We partition \( Q(M) \) into
∪_{i=1}^M Q_i(M) such a way that each Q_i(M) contains exactly l_A elements from the set Q(A). By the definition of N, we can make such partition. The elements of V(G) will be the sets {Q_i(M)}_{M \in U^{r+1,d}, 1 \leq i \leq s_M}. We draw one edge between Q_i(M) and Q_j(M') if there exists x \in Q_i(M), y \in Q_j(M') such that x and y are connected in R. We label the vertex Q_i(M) by the label of the root of M. Let Q_i(M) be a vertex of G such that M is a label-separated tree. Note that if M is not a rooted tree then \mu_n(M) = 0. It is easy to see that the r + 1-ball around Q_i(M) in the graph G is isomorphic to M as rooted labeled balls. Also if M is not label-separated then the r + 1-ball around Q_i(M) can not be a label-separated tree. Therefore

\sum_{L \in U_{n,d}^r, L \text{ is not a label-separated tree}} p_G(L) = (11)

= \sum_{L \in U_{n,d}^r, L \text{ is not a label-separated tree}} \sum_{A \in R(L)} w_\delta(L) \leq \frac{\epsilon}{10} + \delta d |U_{n,d}^r|. (12)

Also, if M is a label-separated tree then

|p_G(M) - \mu_n(T(\text{Gr}_d, M))| \leq |R(M)| \delta \leq d\delta. (13)

Thus by (6), (11), (13) if \delta is choosen small enough then for any \alpha \in U^{r+1,d}

|p_G(\alpha) - \mu(T(\text{Gr}_d, \alpha))| < \epsilon.

Thus our Theorem follows.

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