Shear viscosity from kaon condensation in color-flavor-locked quark matter

Mark G. Alford\textsuperscript{1}, Matt Braby\textsuperscript{2}, Simin Mahmoodifar\textsuperscript{1}

\textsuperscript{1}Department of Physics, Washington University in St. Louis, MO 63130, USA
\textsuperscript{2}Department of Physics, North Carolina State University, Raleigh, NC 27695, USA

(Dated: 6 Jan 2010)

Abstract

We calculate the kaonic contribution to the shear viscosity of quark matter in the kaon-condensed color-flavor locked phase (CFL-K0). This contribution comes from a light pseudo-Goldstone boson which arises from the spontaneous breaking of the flavor symmetry by the kaon condensate. The other contribution, from the exactly massless superfluid "phonon", has been calculated previously. We specialize to a particular form of the interaction lagrangian, parameterized by a single coupling. We find that if we make reasonable guesses for the values of the parameters of the effective theory, the kaons have a much smaller shear viscosity than the superfluid phonons, but also a much shorter mean free path, so they could easily provide the dominant contribution to the shear viscosity of CFL-K0 quark matter in a neutron star in the temperature range 0.01 to 1 MeV ($10^8$ to $10^{10}$ K).
I. INTRODUCTION

In this paper, we explore the shear viscosity of one of the predicted phases of high density quark matter. Transport properties of quark matter, such as the shear viscosity, are of interest because they are the basis for signatures by which we could infer or rule out the presence of exotic phases in the core of neutron stars. Previous work on transport properties has studied the bulk viscosity [1–4], specific heat [5, 6], neutrino emissivity [7–11], and thermal conductivity [12–16].

The shear viscosity is phenomenologically relevant because it damps physically important excitation modes of the star. In particular, a fast-rotating neutron star will spin down rapidly if the internal viscosity is too low, because of the spontaneous growth of r-modes [17]. The observation of rapidly rotating neutron stars can therefore be used to place limits on the internal viscosity. Calculating the viscosity of candidate phases then allows us say whether these phases can be present in the neutron star.

This paper studies the kaon-condensed color-flavor locked phase (CFL-K0) of quark matter. This is a candidate phase of quark matter at the central density of a neutron star. For a review, see Ref. [18]. The CFL-K0 phase has a condensate of kaons [19, 20], which spontaneously breaks the flavor symmetry, producing a Goldstone boson. Since the flavor symmetry is also explicitly broken by the weak interaction, this Goldstone kaon acquires a small mass in the keV range [21]. Our analysis is relevant to temperatures above this value, where one can ignore this small mass. We use the effective theory of the Goldstone kaon, which was worked out in Ref. [5]. The full interaction lagrangian has three independent coupling constants, but we will specialize to a specific ratio of their values, which makes our results dependent on one overall kaon interaction coupling. This enables us to make an estimate of the expected scale of the shear viscosity in this phase. We defer the calculation for the most general interaction lagrangian to future work.

As well as the shear viscosity we calculate the mean free path of the Goldstone kaons, since the concept of shear viscosity will only be applicable to the matter in the neutron star if there is local equilibration on distance scales much smaller than the size of the star.

The rest of the paper is laid out as follows: Section II will discuss the low energy effective theory of the CFL-K0 phase and the interactions among the lowest energy excitations of the theory. Section III will show the results for the contribution of the kaons to the shear
viscosity. Section V will present some conclusions and discuss future directions of this work. In the appendices we cover technical details of the calculation of the mean free path, the treatment of the co-linear regime of the collision integral, and the approximate evaluation of the collision integral.

II. LOW ENERGY EFFECTIVE THEORY

A. Lowest-order lagrangian

The low energy degrees of freedom in color-flavor-locked phases of quark matter are the massless superfluid Goldstone mode, arising from the spontaneous breaking of baryon number, and the light pseudo-Goldstone meson octet, arising from the spontaneous breaking of three-flavor chiral symmetry. The contribution of the superfluid mode to transport properties has been studied previously [3, 22]. We focus on the contribution from the meson octet, described by a meson field $\Sigma$ whose effective lagrangian up to second order is [19, 20]

$$\mathcal{L} = \frac{f_\pi^2}{4} \text{Tr}[D_0 \Sigma D_0 \Sigma^\dagger - v^2 \nabla \Sigma \nabla \Sigma^\dagger] + a \frac{f_\pi^2}{2} \det M \text{Tr}[M^{-1}(\Sigma + \Sigma^\dagger)]$$

(1)

where $D_0 \Sigma = \partial_0 \Sigma - i[A, \Sigma]$. The Bedaque-Shäfer effective chemical potential [19] is $A = -\frac{M M}{2\mu_q}$, $\mu_q$ is the quark chemical potential, and $M = \text{diag}(m_u, m_d, m_s)$ is the quark mass matrix. At asymptotically high density the constants $f_\pi$, $v$, and $a$ can be determined by matching the effective theory to perturbative QCD, [23, 24]

$$f_\pi^2 = \frac{21 - 8 \ln 2}{18} \frac{\mu_q^2}{2\pi^2} \approx (0.21 \mu_q)^2 \quad v \equiv v_H = \frac{1}{\sqrt{3}} \quad a = \frac{3\Delta^2}{\pi^2 f_\pi^2}$$

(2)

where $\Delta$ is the fermionic energy gap at zero temperature. This dependence of $a$ on $\Delta$ is also seen in NJL models [25], so from now on we will work in terms of $f_\pi$ and $\Delta$, assuming that $a$ is given by (2). The meson field $\Sigma$ can be parameterized in terms of fields $\theta_a$,

$$\Sigma = \exp(i\theta/f_\pi)$$

(3)

where $\theta = \theta_a T_a$, and $T_a$ are the Gell-Mann matrices of SU(3) with normalization $\text{tr}(T_a T_b) = 2\delta_{ab}$. The $K^0$ and $K^+$ are the lightest mesonic degrees of freedom [23, 24], and electric neutrality disfavors the presence of charged kaons (since they must be balanced by electrons),
so we focus on the neutral kaons, $K^0$ and $\bar{K}^0$, corresponding to $\theta_6$ and $\theta_7$. The zero-temperature neutral kaon mass and chemical potential can be deduced from the Lagrangian

\[
m_{K^0}^2 = a m_u (m_d + m_s), \quad \mu_K = \frac{m_s^2 - m_d^2}{2 \mu_q}.
\] (4)

We will assume that $\mu_K > m_K$, so there is kaon condensation. The critical temperature for kaon condensation is expected to be of order tens of MeV, well above typical neutron star temperatures, and of the same order as the critical temperature for the CFL condensate itself [5]. We will also assume, following Ref. [5], that the condensate is small, so $\mu_K$ is only a little larger than $m_K$. Given the uncertainty in the effective theory couplings, this is a perfectly conceivable scenario: taking for example $\mu = 400$ MeV, $m_s = 100$ MeV, $\Delta = 80$ MeV and $m_u = 5$ MeV (and ignoring $m_d$ because it makes a negligible contribution) one obtains $\mu_K = 12.5$ MeV and $m_K = 11.0$ MeV. It is then convenient to define, following Ref. [5], an energy gap

\[
\delta m \equiv m_K - \mu_K, \quad \frac{|\delta m|}{m_K} \ll 1.
\] (5)

Note that $\delta m$ is negative in the CFL-K0 phase. Because $|\delta m| \ll 1$ we can usually treat $\mu_K$ and $m_K$ as being identical to leading order in $\delta m$ (an exception is discussed in Sec. IV).

A self-consistent calculation [5] (see also [26]) then yields the excitation energies in the neutral kaon sector,

\[
E_{\pm}^2 = E_p^2 + \mu_K^2 \mp \sqrt{4\mu_K^2 E_p^2 + \delta M^4},
\] (6)

where

\[
E_p^2 = v^2 p^2 + \tilde{M}^2.
\] (7)

In that self-consistent calculation, $\tilde{M}$ and $\delta M$ were thermal masses that depended on temperature and the underlying mass and chemical potential (see Eq. (81) in Ref. [5]). In this paper we are interested in low temperature range applicable for compact stars. In this case, the thermal masses become independent of temperature and are given by

\[
\tilde{M}^2 = 2\mu_K^2 - m_K^2 \approx m_K^2, \quad \delta M^2 = \mu_K^2 - m_K^2 \approx 2m_K |\delta m|.
\] (8)

The mode with energy $E_+$ is massless: this is the Goldstone kaon. We can define a corre-
sponding field $\psi$ using the parameterization

\begin{align}
\theta_6(x) &= (\phi + \rho(x)) \cos \vartheta(x), \\
\theta_7(x) &= (\phi + \rho(x)) \sin \vartheta(x), \\
\psi(x) &= f_\pi \sin(\phi/f_\pi) \vartheta(x),
\end{align}

so $\phi$ is the kaon condensate, $\rho$ is the massive radial mode, and $\vartheta$ is the angular Goldstone mode which we have then rescaled to make a scalar field $\psi$ with a canonically normalized quadratic derivative term and the conventional energy dimension of 1. The mass of the radial modes is given by the value of $E_0(p = 0) = \sqrt{6\mu_K^2 - 2m_K^2} \approx 2m_K$, which is typically on the order of a few MeV, so at the 10 to 100 keV energy scale, which is relevant to neutron stars, it is heavily suppressed and can be ignored. The magnitude of the kaon condensate is [5]

$$\phi^2 = 2f_\pi^2\left(1 - \frac{m_K^2}{\mu_K^2}\right) \approx 4f_\pi^2\frac{\delta m}{m_K}$$

We can then linearize Eq. (6) to obtain a linear dispersion relation for the Goldstone kaon,

$$E(p) = \nu p$$

$$\nu \equiv v \sqrt{\frac{M^2 - \mu_K^2}{M^2 + \mu_K^2}} \approx v \sqrt{\frac{\delta m}{m_K}}.$$ 

The error involved in approximating Eq. (6) by Eq. (11) is less than 5% for $p < 0.6 \sqrt{m_K|\delta m|}/v$.

We will obtain the contribution to the shear viscosity from the Goldstone kaon. For this we need its interaction lagrangian, but it is easy to see that (1) does not contain any interaction terms for the field $\psi$. This follows from the fact that $\psi$, as a Goldstone boson, must couple via derivatives, and (1) only goes to second order in derivatives. We therefore need to write down higher order derivative terms in the effective theory to obtain interactions among the Goldstone modes.

**B. Interaction lagrangian for the Goldstone kaons**

We obtain higher derivative terms in $\psi$ by writing down the leading higher derivative terms in the lagrangian for $\Sigma$, and using (9). We keep only terms with the symmetries of the system, namely rotational symmetry, parity, time-reversal, and the $SU(3)_L \otimes SU(3)_R$
TABLE I: The six interaction terms at fourth order in derivatives for the effective theory (first column), and the interaction terms for ψ that they transform to using (9), when terms involving ρ are dropped. In the effective lagrangian they have coefficients of order $f_\pi^2/\Delta^2$.

chiral flavor symmetry. We also discard terms that, when we substitute (9), will produce interactions that all involve the ρ field; an example is three-derivative terms where Σ enters four times. The allowed terms with no more than four derivatives of Σ are shown in Table I (left column). Since the effective theory breaks down at momenta of order Δ (for example, scattering of Goldstone bosons at that momentum will produce quasiquarks, which are not included in the effective theory) we expect that the momentum expansion will be in powers of $(1/\Delta)\vec{\nabla}$ [19]. We therefore expect the interactions in the left column of Table I to occur in the lagrangian with coefficients $C_i f_\pi^2/\Delta^2$, where the $C_i$ are dimensionless coupling constants.

Using (9) and dropping terms that involve the heavy field ρ, these six terms reduce to the three corresponding interaction terms for ψ shown in the right column. In each case, we find two different interaction terms for Σ reduce to the same interaction term for ψ. This means that the interaction lagrangian for ψ only depends on three linear combinations of couplings. Note that in Table I we have defined a scaled version of the kaon condensate expectation value (9),

$$\varphi \equiv \phi/f_\pi \approx 2\sqrt{|\delta m|/m_K}$$

(12)

The interaction Lagrangian for ψ can then be written out as

$$\mathcal{L} = C_1 \frac{2}{f_\pi^2 \Delta^2} (\partial_0 \psi)^4 + C_3 \frac{2}{f_\pi^2 \Delta^2} (\partial_0 \psi)^2 (\nabla \psi)^2 + C_2 \frac{2}{f_\pi^2 \Delta^2} (\nabla \psi)^4 + C_1 \frac{8\mu_K \sin \varphi}{f_\pi \Delta^2} (\partial_0 \psi)^3 + C_3 \frac{4\mu_K \sin \varphi}{f_\pi \Delta^2} (\partial_0 \psi)(\nabla \psi)^2.$$  

(13)

At this point we specialize to a particular form of the interaction lagrangian, with the
following relationship among the three coupling constants.

\[ C \equiv C_1 = C_2 = -\frac{1}{2}C_3. \] (14)

This reduces the number of coupling constants from three to one. The remainder of our calculation is for this special case, adopted because it leads to a particularly simple interaction lagrangian which is similar to that written down for the superfluid phonon in Refs. [22, 27],

\[ L_{\text{int}} = \frac{\lambda}{4f_\pi^2} (\partial_\mu \psi \partial^\mu \psi)^2 + \frac{g}{2f_\pi^2} (\partial_0 \psi)(\partial_\mu \psi \partial^\mu \psi) \] (15)

where

\[ \lambda = 8C \frac{f_\pi^2}{\Delta^2} \quad \text{and} \quad g = 16 \sin(\varphi)C \frac{K f_\pi}{\Delta^2} \] (16)

We leave the analysis of the fully general interaction lagrangian (13) for future work.

III. SHEAR VISCOSITY

The shear viscosity \( \eta \) is a coefficient in the viscous stress tensor \( \delta T_{ij} \), which is the deviation from equilibrium of the momentum flux tensor \( T_{ij} \) for a fluid with pressure \( P \) and energy density \( \epsilon \),

\[ T_{ij} = T_{ij}^{(eq)} + \delta T_{ij} \]

\[ T_{ij}^{(eq)} = (P + \epsilon)V_i V_j - P\delta_{ij} \]

\[ \delta T_{ij} = -\eta V_{ij} + \cdots \] (17)

where

\[ V_{ij} = \partial_i V_j + \partial_j V_i - \frac{2}{3}\delta_{ij} \nabla \cdot \mathbf{V} \] (18)

and the ellipsis in the equation for \( \delta T_{ij} \) stands for other dissipative terms arising from phenomena such as bulk viscosity and thermal conductivity. \( \mathbf{V}(\mathbf{x}, t) \) is the fluid velocity at a given position and time. We will need to make sure that we only need to keep the first order dissipative terms in Eq. (17) (see discussion at the end of IVB). The stress-energy tensor and the viscosity can be calculated using kinetic theory [28]. For a system of identical bosonic particles with dispersion relation \( E_p \),

\[ T_{ij}(\mathbf{x}, t) = \nu^2 \int \frac{d^3p}{(2\pi)^3} \frac{p_i p_j}{E_p} f_p(\mathbf{x}, t). \] (19)
where $\nu$ is the velocity of the Goldstone kaon (see Eq. 11). The full distribution function is given by

$$f_p(x, t) = \frac{1}{e^{\frac{E_p}{T}} - 1} = f^0_p + \delta f_p(x, t)$$  \hspace{1cm} (20)

where $u^\mu(x, t)$ is the 4-velocity of the fluid, and $\delta f_p$ is a small departure from the equilibrium Bose-Einstein distribution

$$f^0_p = \frac{1}{e^{\frac{E_p}{T}} - 1}.$$  \hspace{1cm} (21)

For shear viscosity we are interested in deviations from equilibrium arising from a shear flow, so we write the deviation from equilibrium as

$$\delta f_p(x, t) = -\chi(p, x, t) f^0_p (1 + f^0_p) \frac{1}{T}$$  \hspace{1cm} (22)

where

$$p_{kl} = p_k p_l - \frac{1}{3} \delta_{kl} p^2.$$  \hspace{1cm} (23)

Substituting (22) into (19) and (17) we find

$$\delta T_{ij}(x, t) = -\nu^2 \int \frac{d^3 p}{(2\pi)^3} \frac{p_i p_j g(p) p_{kl}}{E_p} f^0_p (1 + f^0_p) V_{kl}(x, t).$$  \hspace{1cm} (24)

Using the definition of $V_{ij}$ (see Eq. 18) one can write $\delta T_{ij}$ (Eq. 17) in the following form

$$\delta T_{ij} = -\eta \left[ \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} - \frac{2}{3} \delta_{ij} \delta_{kl} \right] V_{kl},$$  \hspace{1cm} (25)

Comparing this to Eq. (24) gives us

$$\frac{\eta}{2} \left[ \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} - \frac{2}{3} \delta_{ij} \delta_{kl} \right] = \nu^2 \int \frac{d^3 p}{(2\pi)^3} \frac{p_i p_j g(p) p_{kl}}{E_p} f^0_p (1 + f^0_p).$$ \hspace{1cm} (26)

Then, by contracting the tensor on the left hand side with respect to the pairs of indices $i, k$ and $j, l$ we can determine the shear viscosity in terms of the function $g(p)$,

$$\eta = \frac{4 \nu^2}{15 T} \int_p p^4 f^0_p (1 + f^0_p) g(p),$$  \hspace{1cm} (27)

where we have adopted the notation

$$\int_p = \int \frac{d^3 p}{2 E_p (2\pi)^3}.$$  \hspace{1cm} (28)

Using the fact that $p^4 = \frac{3}{2} p_i p_i$ (see Eq. 23) one can write an alternate form of the shear viscosity which will be used later,

$$\eta = \frac{2 \nu^2}{5 T} \int_p f^0_p (1 + f^0_p) g(p) p_i p_i.$$  \hspace{1cm} (29)
To solve for the viscosity, we need to find a form for $g(p)$. To do so, we can use the Boltzmann equation given in the absence of external forces by

$$\frac{df_p}{dt} = \frac{\partial f_p}{\partial t} + \mathbf{V} \cdot \nabla f_p = C[f_p].$$  \hfill (30)

The left-hand side can be written as

$$\frac{df_p}{dt} = \nu f^0_p \frac{p}{2pT} (1 + f^0_p) p_{ij} V_{ij}. \hfill (31)$$

Again, this specific form assumes that we are only interested in shear flows. Thermal gradients and bulk flows would give additional terms on the right-hand side of Eq. (31). It is this form that also helped motivate the structure of the ansatz in Eq. (22).

The collision operator $C[f_p]$ should contain any possible collision terms for the kaons. We will restrict ourselves to the terms lowest order in the coupling constants as more vertices are suppressed because each vertex brings in more powers of $1/f_\pi$ or $1/\Delta$ (see (15) and (16)). Also, we will ignore the $1 \leftrightarrow 2$ processes because for a particle with a linear dispersion relation such processes must be co-linear, so they do not involve momentum transfer that would contribute to the shear viscosity. Finally, we are left with the collision operator for 2-body scattering given by [27]

$$C_{2\leftrightarrow 2} = \frac{1}{2E_p} \int_{k,k',p'} (2\pi)^4 \delta^4(P + K - P' - K') |\mathcal{M}|^2 D_{2\leftrightarrow 2} \hfill (32)$$

Where $\mathcal{M}$ is the $2 \rightarrow 2$ scattering amplitude and $D_{2\leftrightarrow 2}$ contains the distribution functions.

$$D_{2\leftrightarrow 2} = f_{p'} f_{k'} (1 + f_p)(1 + f_k) - f_p f_k (1 + f_{p'})(1 + f_{k'}). \hfill (33)$$

We can also linearize the distributions as $D \approx D^0 + \delta D$ using our definition of $\delta f_p$ in Eq. (22). $D^0$ would make the collision integral vanish by detailed balance. One can then write

$$\delta D_{2\leftrightarrow 2} = \frac{1}{T} f^0_p f^0_{k} (1 + f^0_p)(1 + f^0_k) (\chi(p) + \chi(k) - \chi(p') - \chi(k')) \hfill (34)$$

and the collision integral as

$$C_{2\leftrightarrow 2} \approx \frac{f^0_p}{2E_p T} \int_{k,p',k'} (2\pi)^4 \delta^4(P + K - P' - K') |\mathcal{M}|^2 f^0_k (1 + f^0_k)(1 + f^0_{k'}) \left[ g(p)p_{ij} + g(k)k_{ij} - g(k')k'_{ij} - g(p')p'_{ij} \right] V_{ij}$$

$$\equiv \frac{1}{2E_p T} F_{ij} [g(p)] V_{ij} \hfill (35)$$
Using (31) and the Boltzmann equation, we can conclude that

\[ \nu^2 f_p^0 (1 + f_p^0) p_{ij} = F_{ij}[g(p)] \] (36)

One can then use this equation and (29) to get another expression for the shear viscosity in terms of collision term

\[ \eta = \frac{2}{5T} \int_p g(p) p_{ij} F_{ij}[g(p)] \] (37)

The process now is to evaluate \( \eta \) from (27) and (37), and ensure that they give the same answer. Formally, this is equivalent to solving the Boltzmann equation directly. Ensuring that the two forms are equal is quite non-trivial and is typically done by expanding \( g(p) \) using an orthogonal set of functions, [29–31]

\[ g(p) = p^n \sum_{s=0}^{N} b_s B_s(p) \] (38)

This expansion introduces two new parameters: \( N \), the order of the polynomial approximation; and \( n \), which we call the minimum-exponent parameter, because the lowest power of \( p \) that occurs in the polynomial expansion is \( p^n \). The correct result is obtained in the limit \( N \to \infty \) for any value of \( n \). However, as we will see, the rate of convergence with increasing \( N \) is strongly dependent on the minimum-exponent parameter \( n \).

The polynomials \( B_s(p) \) are defined such that the coefficient of the highest power \( p^s \) is 1 and they obey the orthogonality condition [27]

\[ \int_p f_p^0 (1 + f_p^0) p_{ij} p_{ij} p^n B_r(p) B_s(p) = A_r^{(n)} \delta_{rs} . \] (39)

These conditions uniquely specify the \( B_s(p) \) for all \( s \), starting with \( B_0(p) = 1 \). From the orthogonality condition we find

\[ A_0^{(n)} = \frac{2}{3} \int_p f_p^0 (1 + f_p^0) p^{4+n} = \frac{T^{6+n}}{6 \pi^2 \nu^2 \Gamma(6 + n) \zeta(5 + n)} . \] (40)

Using \( g(p) \) from Eq. (38) in Eq. (27), and using the definition of \( A_r^{(n)} \) from Eq. (39), we get

\[ \eta = \frac{2 \nu^2}{5T} b_0 A_0^{(n)} . \] (41)

An alternative expression for \( \eta \) follows from substituting \( g(p) \) from Eq. (38) into Eq. (37),

\[ \eta = \sum_{s,t=0}^{N=\infty} b_s b_t M_{st} , \] (42)
where
\[ M_{st} = 2 \frac{\nu^2}{5T} \int_{p,k,k',p'} (2\pi)^4 \delta^4(P + K - P' - K') |M|^2 f_p f_k (1 + f_{k'}) (1 + f_{p'}) p^n B_s(p) p_{ij} \Delta_{ij}^t , \]
\[ = \frac{1}{10T} \int_{p,k,k',p'} (2\pi)^4 \delta^4(P + K - P' - K') |M|^2 f_p f_k (1 + f_{k'}) (1 + f_{p'}) \Delta_{ij}^t \]
and
\[ \Delta_{ij}^t = B_t(p) p^n p_{ij} + B_t(k) k^n k_{ij} - B_t(k') k^n k_{ij}' - B_t(p') p'^n p_{ij}' . \] (43)

The second line of Eq. (43) uses the symmetry under relabeling the momenta of the legs in the scattering diagrams \( P \rightarrow K , \) etc., and can be used to demonstrate that the diagonal elements of \( M_{st} \) are positive definite. As we will see below, this ensures that the shear viscosity is also positive.

Requiring the two forms of \( \eta \) to be equal leads to a matrix equation for all the \( b_i \)'s. From that we extract \( b_0 , \)
\[ b_0 = 2 \frac{\nu^2}{5T} A_0^{(n)} (M^{-1})_{00} \] (45)
where \( (M^{-1})_{00} \) means the first entry in the matrix inverse of \( M_{st} . \) Using this in Eq. (41), we find
\[ \eta = \frac{4 \nu^4}{25T^2} (A_0^{(n)})^2 (M^{-1})_{00} . \] (46)

As noted above, this expression becomes accurate in the limit \( N \rightarrow \infty , \) where the matrix \( M \) is of infinite size. It is known \( [22, 27, 29] \) that the result for finite \( N \) rises with \( N , \) so for a matrix \( M_N , \) with finite dimension \( N , \) that obeys (43),
\[ \eta \geq \frac{4 \nu^4}{25T^2} (A_0^{(n)})^2 (M_N^{-1})_{00} \] (47)

We will see below that this expression converges rapidly with \( N \) for the optimal choice of the minimum-exponent parameter \( n . \)

The remaining task is to evaluate the integral in Eq. (43). This requires the matrix elements for the \( 2 \leftrightarrow 2 \) scattering process, \( iM = iM_c + iM_s + iM_t + iM_u , \) see Fig. 1, with the individual channels being given by
\[ iM_c = \frac{\lambda}{f_4^4} [(P \cdot K)(P' \cdot K') + (P \cdot K') (P' \cdot K) + (P \cdot P')(K \cdot K')] \]
\[ iM_s = \frac{\eta^2}{f_4^2} [2(p_0 + k_0)P \cdot K + p_0 k^2 + k_0 P^2] [2(p_0' + k_0')P' \cdot K' + p_0' k'^2 + k_0' P'^2] G(P + K) \]
\[ iM_t = iM_s(P \leftrightarrow K') \]
\[ iM_u = iM_t(P \leftrightarrow K) , \] (48)
where
\[
G(Q) = \frac{1}{(q_0^2 - \nu^2 q^2) + i \text{Im} \Pi(q_0, q)}
\]
is the Goldstone kaon propagator and the last two lines in (48) come from crossing symmetries.

The 12-dimensional integral in Eq. (43) can be simplified by eliminating the \( p' \) integral using the momentum-conserving delta-function. Then one can use the energy-conserving delta-function to eliminate the integral over the magnitude of \( k' \). Three of the remaining 8 integrals can be eliminated by selecting the z-axis to lie along the vector \( p \), and noting that only the difference in the two remaining azimuthal angles matters. This leaves a 5-dimensional integral over the magnitudes of \( p \) and \( k \), two polar angles corresponding to \( k \) and \( k' \) and one azimuthal angle. This can be evaluated numerically (see appendix C) using the \textit{Vegas} Monte Carlo algorithm [32, 33].

The results that we present below are obtained by setting the minimum-exponent parameter \( n \) to \(-1\). This value is expected to give optimal convergence of the calculated shear viscosity to its physical value as a function of \( N \) because, as shown in appendix B, this term most strongly suppresses the co-linear scattering and therefore give the smallest collision term. The shear viscosity is inversely proportional to the collision term and since we have a variational procedure that says the answer we get is a lower bound, we are only interested in the largest value of the shear viscosity that we can calculate.

To check this reasoning we show in Table II results of calculations of the shear viscosity for different values of \( n \) and \( N \). We see that for \( n = -1 \) the value of \( \eta \) at low \( N \) is already close to the maximum (asymptotic) value at \( N = \infty \). For \( n = -2 \) the convergence is almost as good, achieving \( \sim 1\% \) accuracy at \( N = 2 \). For other values of \( n \) the convergence is
TABLE II: Values of shear viscosity in (MeV)$^3$ as a function of the order $N$ of the polynomial approximation to $g(p)$, for different choices of the minimum-exponent parameter $n$. The calculated value rises towards the physical result as $N \to \infty$, and in this limit should be independent of $n$. We see that for $n = -1$ the result converges very rapidly as $N$ rises, but for other values of $n$ the convergence is slower.

dramatically poorer. This behavior was also seen in Refs. [22, 27]. We conclude that we can achieve accuracy of better than 1% by choosing $n = -1$, and only keeping the first polynomial ($N = 0$), i.e. we set $g(p) = 1/p$.

IV. RESULTS

A. Analytic results

We now describe how the shear viscosity depends on the temperature and on the parameters of the effective lagrangian for the Goldstone kaon. The relevant parameters are the speed $\nu$ (11) of the kaon and its interaction couplings $\lambda$ and $g$ (15), which in turn depend on more basic parameters $C$, $f_\pi$, $\Delta$, $m_K$ (16). Recall that we have assumed $|\delta m| \ll 1$, so in the expressions below, $\mu_K$ and $m_K$ are usually interchangeable. One exception is the phonon speed $\nu$, which occurs in the shear viscosity raised to the 11th power (see discussion after (54)) so we use the full expression (the identity in (11)) for it.

Before performing any numerical calculations, we can extract the temperature dependence of the shear viscosity. Because the co-linear scattering will not contribute to the answer, the propagator does not need to be regulated by the self-energy. Since the temperature only appears in the distribution functions and the self-energy, we can now factorize out the
temperature dependence by rescaling all the momenta by the temperature. Doing so we find

\[ M_{st} \sim T^{15+2n+s+t} \]  

(50)

where \( s \) and \( t \) are the indices of the matrix indicating how many terms we are keeping in our expansion for \( g(p) \) and \( n \) is the minimum-exponent parameter (see (38)). We also recall the temperature dependence of \( A^{(n)}_0 \) from Eq. (39),

\[ A^{(n)}_0 \sim T^{n+6} \]  

(51)

Therefore, from Eq. (46), we obtain the temperature dependence of the shear viscosity

\[ \eta \propto T^{-5} \]  

(52)

This dependence on temperature is also seen in other systems where viscosity arises from \( 2 \leftrightarrow 2 \) scattering of Goldstone bosons [22, 27]. The constant of proportionality in (52) has mass dimension 8. In the case of the shear viscosity due to phonons there was only one possible scale, the quark chemical potential \( \mu_q \), so \( \eta_H \propto \mu_q^8/T^5 \). However, we have several scales \( (f_\pi, \Delta, m_K) \) manifesting themselves in two coupling constants \( \lambda \) and \( g \) (16). Which of these is most important depends on whether the scattering is dominated by the contact term or by the exchange of a virtual particle. The dimensionless parameter \( u \) that determines which scattering process dominates is

\[ u = \frac{3g^2}{\lambda} = 96 C \sin^2(\varphi) \left( \frac{\mu_K}{\Delta} \right)^2, \]  

(53)

where \( \varphi = 2 \sqrt{|\delta m|/m_K} \) (12) and the 3 represents the 3 channels for virtual particle exchange. For typical values of \( \delta m, m_K, \) and \( \Delta \), this ratio can be bigger or smaller than 1. When \( u \ll 1 \), the contact term dominates, so the scattering amplitude is proportional to \( \lambda \). When \( u \gg 1 \), the particle-exchange process dominates, so the scattering amplitude is proportional to \( g^2 \).

The shear viscosity is inversely proportional to the scattering cross-section, so

\[ u \ll 1 : \eta_h = h_1(\nu) \frac{1}{C^2} \frac{f_\pi^4 \Delta^4}{T^5}, \]  

\[ u \gg 1 : \eta_h = h_2(\nu) \frac{1}{C^4} \frac{f_\pi^4 \Delta^8}{\sin^4(\varphi) \mu_K^4 T^5}, \]  

(54)

where \( h_1 \) and \( h_2 \) are dimensionless functions that depend only on the Goldstone kaon speed \( \nu \), which depends on \( \delta m \) and \( m_K \). We can obtain their analytic form when \( \nu \ll 1 \). In that
case, the leading-order behavior in both regimes is a $\nu^{11}$ power law (see appendix C). In general they must be calculated numerically, and the result (with the $\nu^{11}$ power law scaled out) is shown in Fig. 4.

Finally, we note that the shear viscosity due to Goldstone kaons will be smaller than that due to the superfluid phonons, since $f_\pi$, $\Delta$, and $m_K$ are much less than $\mu_q$.

B. Numerical results

To begin, we will confirm the temperature dependence predicted in the previous section. To do so, we will fix the mass of the kaon, $m_K = 4$ MeV, $f_\pi = 100$ MeV, $\Delta = 100$ MeV and $C = 1$. In Fig. 2, we show the viscosity as a function of temperature for a few values of $\delta m$. The data points are obtained by numerical evaluation of the 5-dimensional integral, whereas the lines show the fit to a $T^{-5}$ power law (52), which is independent of the regime of coupling constants’ values. On the same plot, we show the contribution from the phonons [22]. As expected, the phonon shear viscosity is much larger. Most of the difference comes from the difference in magnitude of the coupling constants (the kaons coupling constant is larger) and the rest comes from a difference in the speed of the kaon and the phonon (the kaon’s is smaller), which enters the expression for shear viscosity raised to a high power. Using the shear mean free path criterion $l_{\text{shear}} < 1$ km (appendix A) for the validity of hydrodynamics for neutron star oscillations, we expect the phonons to be non-hydrodynamic at $T \lesssim 1$ MeV; with the parameter values given above, the Goldstone kaons become non-hydrodynamic at $T \lesssim 0.03$ MeV.

In Fig. 3 we show the shear viscosity as a function of the gap $\Delta$, with $\delta m = -0.5$ MeV and $T = 1$ MeV. The other parameters have the same values as in Fig. 2. This illustrates the transition between the two regimes given in Eq. (54). The crossover occurs at $u = 1$ which corresponds to $\Delta = 30$ MeV, which is indicated on the graph. As expected, we see that for large $\Delta$ ($u \ll 1$), $\eta \propto \Delta^4$; for small $\Delta$ ($u \gg 1$), $\eta \propto \Delta^8$.

In Fig. 4 we present the results of numerical calculation of $h_{1,2}(\nu)$ (54). We have divided out the dominant behavior $\nu^{11}$ power law behavior (for details see appendix C). We see that the remaining $\nu$ dependence is very mild, so to a good approximation the shear viscosity is
FIG. 2: (Color online) The shear viscosity as a function of temperature for kaons and phonons. For parameter values, see text. In the lower part of the graph, the points are numerical calculations and the straight lines are fits to the power law form given in (52). The phonons’ calculated shear viscosity is many orders of magnitude larger, although using the shear mean free path criterion (appendix A) we expect them to be non-hydrodynamic in neutron stars at $T \lesssim 1$ MeV.

given by (54) with

$$h_1(\nu) \approx 3.44 \times 10^{-4} \nu^{11},$$

$$h_2(\nu) \approx 1.70 \times 10^{-8} \nu^{11}. \quad (55)$$

To show how large the shear viscosity of CFL-K0 quark matter could be, we look at a case where the values of the parameters are pushed in the direction that yields a large value of $\eta$. We take $f_\pi, \Delta \approx 150$ MeV, $m_K \approx 4$ MeV, $\delta m \sim -1.0$ MeV, and $C \approx 0.2$. With these values we find that at $T = 0.1$ MeV ($10^9$ K) the shear mean free path (A1) is 0.26 km, and $\eta = 1.7 \times 10^{13}$ MeV$^3 = 2.3 \times 10^{18}$ erg cm$^{-1}$s$^{-1}$. At this temperature the phonon’s shear mean free path (see appendix A) is larger than the star, so the kaons provide the dominant contribution to the shear viscosity.
FIG. 3: (Color online) The shear viscosity as a function of $\Delta$. (See text for parameter values.) The points are calculated numerically. The straight lines are fits to the power law behaviors of (54).

FIG. 4: (Color online) The functions $h_1(\nu)$ and $h_2(\nu)$ (54). We scale out the $\nu^{11}$ power law behavior (see appendix C). On the $x$-axis we show $\nu$ in units of the kaon velocity $v \approx \sqrt{\delta m/m_K}$ in the non-kaon-condensed phase.
Finally, we check whether the regime of linear hydrodynamics is valid by evaluating the size of the corrections to the equilibrium stress-energy tensor. Linear hydrodynamics is appropriate if $\delta T_{ij} \ll T_{ij}$ (17). (Note that this is different from the criterion of validity for hydrodynamics in general, discussed in appendix A.) This inequality becomes

$$\eta \ll V \ell(P + \epsilon)$$

(56)

where $\ell$ represents the length scale of the velocity gradients, and $V$ is the typical fluid velocity which we assume is of order 1. If we use the energy density of free quark matter $\epsilon \approx 9\mu_q^4/(4\pi^2)$ which is of order $10^{10}$ MeV$^4$ at $\mu_q \approx 500$ MeV, (and $P \lesssim \epsilon$, which is typically the case), and use the length scale $\ell \sim 1$ km which is appropriate for oscillations of neutron stars, we find that linear hydrodynamics is valid as long as

$$\eta \ll 10^{25} \text{MeV}^3$$

(57)

which is easily obeyed by the values of the shear viscosity that we have calculated for the CFL-K0 phase.

V. CONCLUSIONS

In this paper, we have calculated the shear viscosity arising from self-interaction of the Goldstone kaon mode in the CFL-K0 phase of quark matter. The shear viscosity from the other Goldstone mode, the superfluid phonon, has already been explored in Ref. [22]. We find the same $T^{-5}$ temperature dependence that was found for the phonons in the CFL phase and for superfluid modes in a unitary Fermi gas [27]. Our final results are the approximate analytic expressions (54), (55) for the shear viscosity due to Goldstone kaons, and expressions (A1) and (A21) for the “shear mean free path” and “scattering mean free path” of the Goldstone kaons.

Neutron star oscillations have a length scale in the kilometer range, so the phonon and Goldstone kaon fluids in a neutron star can only be described by hydrodynamics when their mean free paths are smaller than this. We argue that the shear mean free path is the appropriate quantity to use for this purpose (see appendix A).

Because the coupling constants for the Goldstone kaons are roughly an order of magnitude larger than those for the superfluid phonon, the shear viscosity and mean free path of the
Goldstone kaon are both several orders of magnitude smaller than for the superfluid phonon (see Fig. 2). Using the shear mean free path (see appendix A), we find that the superfluid phonons in a neutron star are described by hydrodynamics at temperatures above about 1 MeV ($10^{10}$ K). The Goldstone kaons are hydrodynamic down to lower temperatures: the exact threshold depends sensitively on the value of the constants in the effective action, but could easily be lower than the 0.01 to 0.05 MeV range at which our treatment becomes invalid because the weak-interaction mass of the Goldstone kaon [21] must then be taken into account.

We conclude that, in the temperature range 0.01 MeV to 1 MeV, depending on the values of the coupling constants in their effective theory, Goldstone kaons may very well provide the dominant contribution to the shear viscosity in CFL-K0 quark matter.

There are several ways in which this work can be developed further. Firstly, we chose a specific form of the interaction lagrangian (15) which has one coupling constant, rather than the most general form (13) which has three; our calculation should be extended to the most general lagrangian. Secondly, it would be useful to extend our calculation to lower temperatures where, as noted above, one can no longer neglect the effects of weak interactions on the dispersion relation of the Goldstone kaon. It would also be interesting to study shear viscosity from light kaons in the non-kaon-condensed CFL phase. These particles were found to give a large contribution to the bulk viscosity even at temperatures as low as a tenth of their energy gap [1, 2]. Thirdly, we neglected scattering between the Goldstone kaons and the superfluid phonons. It would be interesting to see if these processes shorten the phonon mean free path and make a significant contribution to the shear viscosity. Fourthly, even though our calculation is open to extension and improvement in the ways just described, it would be interesting to perform an analysis along the lines of Ref. [34] to see whether the shear viscosity of the Goldstone kaons can have a significant effect on the development of $r$-modes in a quark star or hybrid neutron star. Fifthly, as discussed in appendix A, we did not consider how the interactions themselves would alter the dispersion relation. This could affect the calculation of the mean free path at leading order in the induced non-linearity, but would provide a subleading correction to the shear viscosity. Finally, even when the superfluid phonons or Goldstone kaons are not in the hydrodynamic regime, they can still transfer momentum over long distances, and it is important to investigate how they could provide ballistic-regime damping (as opposed to hydrodynamic viscous damping) of neutron
star oscillations.

Acknowledgements

We thank Jingyi Chao, Cristina Manuel, Sanjay Reddy, Gautam Rupak, Thomas Schäfer, and Andreas Schmitt for discussions. This research was supported in part by the Offices of Nuclear Physics and High Energy Physics of the U.S. Department of Energy under contracts #DE-FG02-91ER40628, #DE-FG02-05ER41375, and #DE-FG02-03ER41260.
Appendix A: Mean Free Path

In this appendix we discuss the mean free path of the Goldstone kaon in the CFL-K\(^0\) phase. We expect that hydrodynamics will be applicable to neutron star oscillations when the mean free path is well below the kilometer scale, since neutron star radii are about 10 km.

We study two definitions of the mean free path, which we call the “shear mean free path” \(l^{\text{shear}}\) and the “scattering mean free path” \(l^{\text{scat}}\). The shear mean free path is based on the value of the shear viscosity itself, and is probably the physically relevant quantity for deciding when hydrodynamic calculations of shear viscosity are valid. The scattering mean free path is the average distance between collisions of the Goldstone kaons, including co-linear scattering events. Since co-linear scatterings do not contribute to the shear viscosity itself, it seems likely that this quantity is not the relevant one for finding the limits of validity of shear viscosity calculations.

1. Shear mean free path

We take our definition of the shear mean free path from Ref. [27], Eq. (48), where it is referred to as \(\lambda_B\).

\[
l^{\text{shear}} = \frac{\eta}{n \langle p \rangle} \tag{A1}
\]

where \(\eta\) is the shear viscosity (given for kaons by (54) and (55)), \(\langle p \rangle\) is the thermal average momentum, and \(n\) is the boson density,

\[
\langle p \rangle = 2.7 T/\nu , \tag{A2}
\]

\[
n = \int \frac{d^3 p}{(2\pi)^3} f_p = \zeta(3) \frac{T^3}{\pi^2 \nu^3} . \tag{A3}
\]

where \(\nu\) is the speed of the Goldstone bosons (given for kaons by (11)).

This already allows us to make an estimate of the maximum shear viscosity that Goldstone kaons can provide, since it follows that \(\eta \approx 0.3 \nu^{-4} T^4 l^{\text{shear}}\), so the maximum shear viscosity that could possibly occur in a neutron star at temperature \(T\) is when \(l^{\text{shear}} \approx 1\) km, i.e.

\[
\eta_{\text{max}} \approx \frac{T^3}{\nu^4} \frac{T}{7 \times 10^{-16} \text{ MeV}} . \tag{A4}
\]
For Goldstone kaons it is quite possible to get $\nu \approx 0.1$ by using small values of $\delta m$. Using this value we find the following upper limits: at $T = 0.01$ MeV, $\eta_{\text{max}} \sim 10^{11}$ MeV$^3$; at $T = 0.1$ MeV, $\eta_{\text{max}} \sim 10^{15}$ MeV$^3$; at $T = 1$ MeV, $\eta_{\text{max}} \sim 10^{19}$ MeV$^3$.

For superfluid phonons we can make more definite statements because there is less uncertainty about the parameters appearing in these expressions. The shear viscosity is $\eta = 1.3 \times 10^{-4} \mu_q^8/T^5$ and the speed $\nu$ is generally assumed to take its perturbative value $1/\sqrt{3}$ [22], in which case we immediately find that

$$l_{\text{shear}}^H \approx 4 \times 10^{-5} \mu_q^8/T^5$$

(A5)

So phonon hydrodynamics becomes invalid in neutron stars when $l_{\text{shear}}^H \gtrsim 1$ km, i.e. for $\mu \approx 500$ MeV we require $T \gtrsim 1$ MeV ($10^{10}$ K), and at a temperature of 1 MeV the phonon shear viscosity is $5 \times 10^{17}$ MeV$^3$.

2. Scattering mean free path

Our calculation of this quantity closely follows that of Ref. [22]. The scattering mean free path is determined by the 2-body interaction cross section. The relevant scattering amplitude is the sum of four Feynman diagrams, the contact term and the s, t, and u-channel diagrams, see Fig. 1. In the s, t, and u-channels there is a virtual particle which can go on-shell, which means that its self-energy must be included to avoid an unphysical divergence. (The fact that the virtual particle can go on-shell means that the $2 \rightarrow 2$ collision rate already includes the contribution from $1 \rightarrow 2$ splitting process, so these need not be calculated separately [35, 36]. We have performed this separate calculation and verified that the result has the same parametric dependence as the one we obtain below.) The rate (per unit volume) for the $2 \rightarrow 2$ scattering process is

$$\Gamma_{2\rightarrow2} = \frac{1}{2} \int_{p,k,p',k'} (2\pi)^4 \delta^4(P + K - P' - K') |M|^2 f_p f_{k'} (1 + f_{p'}) (1 + f_{k'})$$

(A6)

where

$$\int_p = \int \frac{d^3p}{2\epsilon_p (2\pi)^3} \quad f_p = \frac{1}{e^{E_p/T} - 1}$$

(A7)

and

$$|M|^2 = |M_s|^2 + |M_t|^2 + |M_u|^2 + I$$

(A8)
where \( I \) represents the contact and interference terms, and the matrix elements are given in Eq. (48). The dominant contribution comes from \( |M_s|^2 + |M_t|^2 + |M_u|^2 \), since these each have a large enhancement when the virtual particle is close to being on-shell. The contact term and interference terms have no such enhancement, and so make a much smaller contribution to the rate.

We now give a detailed explanation of the evaluation of \( |M_s|^2 \); the others can be obtained by similar methods. We first define the virtual particle momentum \( Q = (E_q, q) \) and shift the integral over \( k \) to an integral over \( q \). We can then still use the momentum-conserving delta-function to do the integral over \( k' \). We then choose the direction of \( q \) as the \( z \)-axis, so the angular part of the \( q \) integral gives a factor of \( 4\pi \) and the remaining integrand is azimuthally symmetric, so the 2 remaining azimuthal integrals give a factor of \( 2\pi \). This leaves three integrals over the magnitudes of \( p \), \( k' \) and \( q \) as well as over two polar angles, between \( q \) and \( p \) and between \( q \) and \( k' \). We then introduce the auxiliary variable \( \omega \) via the identity

\[
\delta(E_p + E_k - E_{p'} - E_{k'}) = \int_{-\infty}^{\infty} d\omega \delta(\omega - E_p + E_{p'}) \delta(\omega + E_k - E_{k'}). \tag{A9}
\]

The integral over the two polar angles can then be done using these two delta-functions, leaving behind four integrals over \( p, k', \omega \) and \( q \),

\[
\Gamma_s = A \int_0^\infty dq \int_0^\infty \frac{d\omega}{\nu_q} \int_{(\omega-\nu q)/2\nu}^{(\omega+\nu q)/2\nu} dp \int_{(\omega-\nu q)/2\nu}^{(\omega+\nu q)/2\nu} dk' |M_s|^2 f_{\nu p} f_{\omega-\nu p} (1 + f_{\omega-\nu k'}) (1 + f_{\nu k'}) \tag{A10}
\]

To make the temperature dependence explicit we introduce a new set of variables \((x, y, w, \) and \( z)\). For the \( s \)-channel these are

\[
x = \frac{\nu p}{T}, \quad y = \frac{\nu k'}{T}, \quad w = \frac{\omega}{T}, \quad z = \frac{\nu q}{T}, \tag{A11}
\]

and the Mandelstam variable is \( s = w^2 - z^2 \). This leads to the full expression for the \( s \)-channel rate,

\[
\Gamma_s = \frac{g^4 T^{12}}{16\pi^{17}(2\pi)^5 f_\pi^8} \int_0^\infty dz \int_z^\infty dw \int_{(w-z)/2}^{(w+z)/2} dx \int_{(w-z)/2}^{(w+z)/2} dy F(x, y, w) G(x, w, s) G(y, w, s) \frac{s^2 + (\Pi^+/T^2)^2}{s^2 + (\Pi^+/T^2)^2} \tag{A12}
\]

with

\[
F(x, y, w) = f_x f_{w-x} (1 + f_y) (1 + f_{w-y}) \quad G(x, w, s) = w^2 (3x(1 - \nu^2)(w - x) - s)^2 \tag{A13}
\]
The self-energy term has both a real and imaginary part, however the real part is much smaller [22]. We will therefore only consider the imaginary part \( \text{Im} \Pi(\omega, q) \), obtained in Refs. [1, 2, 22],

\[
\text{Im} \Pi(w, z) = \Pi^+ \Theta(w^2 - z^2) + \Pi^- \Theta(z^2 - w^2)
\]

where \( \Pi^+ \) is relevant for the \( s \)-channel where \( w > z \), and \( \Pi^- \) is relevant for the \( t \)- and \( u \)-channels where \( w < z \). We have neglected the tadpole contribution to the self-energy: it only corrects the kaon velocity by term proportional to \( \lambda T^4 \) [22].

In the remaining 4-dimensional integral (A12) we can now use a simple approximation to greatly simplify the integral. In the expression for \( \Gamma_s \) the integral over \( w \) is sharply peaked at the limit of integration where \( w = z \), i.e. \( \omega = \pm \nu q \) (\( s = 0 \)). The integral takes the form

\[
\int_\infty^\omega dw \frac{I(w, z)}{(w^2 - z^2)^2 + (\Pi^+(w, z)/T^2)^2} \approx \frac{\pi T^2}{4 \Pi^+(z, z) z} I(z, z), \tag{A14}
\]

This expression is valid when \( I(w, z) \) is slowly varying near the singular point \( w = z \) and when \( \Pi^+/T^2 \ll 1 \) because \( \Pi \sim g^2 T^6 \) and \( T \ll f_\pi, \Delta, \mu_K \). Applying this approximation to the \( s \)-channel contribution, we find

\[
\Gamma_s = \frac{g^4 T^{14}}{16 \nu^{17}(2\pi)^5 f_\pi^8} \frac{\pi}{4} \int_0^\infty dz \int_0^z dx \int_0^x dy f_x f_{z-x} (1+f_y)(1+f_{z-y}) \frac{G(x, z, 0)G(y, z, 0)}{\Pi^+(z, z) z}. \tag{A15}
\]

where

\[
\Pi^+(z, z) = \frac{g^2 T^6}{16 \nu^7 f_\pi^4 z f_z} \int_0^z dx f_x f_{z-x} G(x, z, 0). \tag{A16}
\]

We can then see that the integral over \( x \) in Eq. (A15) partially cancels the integral contained in \( \Pi^+ \) leaving behind only a double integral. Using \( G(y, z, 0) \) from Eq. (A13), we have

\[
\Gamma_s = \frac{g^2 T^8 (1 - \nu^2)^2}{\nu^{10} f_\pi^4} J, \tag{A17}
\]

where \( J \) is a pure number given by

\[
J = \frac{9}{128 \pi^3} \int_0^\infty dz \int_0^z dy y^2 (z - y)^2 f_z (1+f_y)(1+f_{z-y}) \approx 0.466 \tag{A18}
\]
Applying the same method to the $t$- and $u$-channel integrals, we find that they all give the same rate, so the total rate is
\[ \Gamma_{\text{total}} = 1.40 \frac{g^2 T^8 (1 - \nu^2)^2}{\nu^4 f_\pi^4}. \] (A19)

The scattering mean free path is defined as
\[ l_{\text{scat}}^K = \frac{\nu n}{\Gamma}, \] (A20)
where $n$ is the particle density (A3). So the scattering mean free path of the Goldstone kaons is
\[ l_{\text{scat}}^K = 0.0881 \frac{v^8}{(1 - \nu^2)^2} \frac{f_\pi^4}{g^2 T^5} = 3.44 \times 10^{-4} \frac{v^8}{(1 - \nu^2)^2} \frac{f_\pi^2 \Delta^4}{C^2 \mu_K^2 \sin^2 \varphi T^{-5}}, \] (A21)

For comparison, the scattering mean free path of the phonon is [22]
\[ l_{\text{scat}}^H = 0.181 \frac{v^8}{(1 - \nu^2)^2} \frac{\mu_q^4}{T_5} = 5.02 \times 10^{-3} \frac{\mu_q^4}{T_5} \] (A22)
so the ratio of the two is
\[ \frac{l_{\text{scat}}^K}{l_{\text{scat}}^H} = 17.546 \frac{v^8}{(1 - \nu^2)^2} \frac{f_\pi^4}{g^2 \mu_q^4} \approx 2.14 \times 10^{-4} \frac{f_\pi^2 \Delta^4 |\delta m|^3}{C^2 m_K^5 \mu_q^4}. \] (A23)

Since $\mu_q$ is much larger than any of the other energy scales, this implies that the scattering mean free path of the kaon, like the shear mean free path, is generally much shorter than that of the phonon, giving the Goldstone kaon a much wider range of temperatures where it can be treated hydrodynamically.

We have noted above that we expect the shear mean free path to be a better indicator of the range of validity of hydrodynamics, but for the sake of completeness we now estimate the temperature at which $l_{\text{scat}}^K$ will become greater than 1 km, in the case of very unfavorable parameter choices that lead to a long mean free path. We will use the values used at the end of Sec. IV B to illustrate how high the shear viscosity can be, namely $f_\pi = 150$ MeV, $\Delta = 150$ MeV, $\delta m = -1.0$ MeV, $m_K = 4.0$ MeV and $C = 0.2$. In this case the scattering mean free path is shorter than 1 km for $T \gtrsim 0.006$ MeV. For more favorable choices of the couplings this critical temperature will be much lower. In comparison, from (A22) the scattering mean free path for phonons is shorter than 1 km for $T > 0.04$ MeV.
We have used a linear dispersion for the kaon in calculating the mean free path, which is a requirement for getting a co-linear enhancement. However, there are sources of non-linearities in the dispersion. One comes directly from our expansion of the full kaon dispersion in (11). If we had kept higher order terms, we would get a contribution that behaves as

\[ E = \nu p(1 + \gamma p^2) \]  

(A24)

where \( \gamma > 0 \). This positive curvature would still allow for the co-linear splitting and joining processes. Therefore, keeping this term would provide a subleading contribution to the calculation presented here.

However, we ignored how the higher order derivative interactions themselves could change the kaon dispersion. Something similar has been calculated for the superfluid phonons, [37], where \( \gamma \) was found to be negative and therefore the 1 \( \leftrightarrow \) 2 processes are kinematically forbidden. If the corresponding non-linearity for the kaons were positive, then as above, the calculation presented here would remain the same. However, if the curvature were negative as for the phonons, then the mean free path would be altered at leading order. This is basically because the non-linearity itself would act to regulate the on-shell propagator and the scattering rate would go like \( 1/\gamma \) instead of \( 1/\Pi \) (where \( \Pi \) is the self-energy). The appropriate scales to compare are \( \gamma T^2 \) and \( \Pi/T^2 \) and in the case of the phonons, \( \gamma T^2 \gg \Pi/T^2 \), such that this correction would make the mean free path even larger and affect the validity of hydrodynamics. See [16] for a calculation involving the non-linear phonon dispersion and its affect on regulating the phonon propagator in a calculation of the thermal conductivity. However, it should be noted that including the non-linearity would only provide a subleading correction to the shear viscosity for either sign of \( \gamma \) because the shear viscosity is insensitive to that region of phase space.

**Appendix B: Power counting sharply peaked integrals**

Here we discuss in more detail the evaluation of integrals of the type (A14), having a slowly-varying component \( I \) multiplied by a function with a sharp Lorentzian peak at the edge of the range of integration. In appendix A we assumed that \( I \) was non-zero at the edge of the range and we kept only the leading contribution. Here we include higher-order
corrections by Taylor-expanding the numerator,
\[ \int_0^z \frac{I(w, z)}{(w^2 - z^2)^2 + \epsilon^2} \sim \int_0^z \frac{I(z, z) + (w - z)I'(z, z) + \frac{1}{2}(w - z)^2 I''(z, z) + \ldots}{(w^2 - z^2)^2 + \epsilon^2}, \tag{B1} \]
where \( I'(z, z) \) is the first derivative of \( I \) with respect to \( w \), evaluated at \( w = z \). We then find
\[ J_1 \equiv \int_0^z \frac{1}{(w^2 - z^2)^2 + \epsilon^2} \sim \frac{\pi}{4 \epsilon}, \]
\[ J_2 \equiv \int_0^z \frac{z - w}{(w^2 - z^2)^2 + \epsilon^2} \sim \frac{-\ln \epsilon}{4 \epsilon^2}, \]
\[ J_3 \equiv \int_0^z \frac{(z - w)^2}{(w^2 - z^2)^2 + \epsilon^2} \sim \frac{1}{8 \epsilon}. \tag{B2} \]
This gives us a scheme for power counting any integrals of the form given by (B1). The relevant property is the dependence on \( \epsilon \), since the collision integrals for transport properties take the form (B1) with \( \epsilon = \Pi/T^2 \propto \epsilon^2 \ll 1 \).

We can now justify the statement made in Sec. III that when we calculate the shear viscosity using a polynomial expansion of the function \( g(p) \) (38), the dominant contribution comes from \( g(p) = 1/p \), i.e. choosing the minimum-exponent parameter \( n \) to be \(-1\).

Calculations of the mean free path and the shear viscosity both involve a rate calculation which contains collision integrals. In the mean free path collision integral (A12) there is a sharp peak in the integrand at \( w = z \) corresponding to a co-linear divergence, where two kaons have parallel momenta, and exchange a kaon whose momentum lies in the same direction. In the mean free path calculation this near-divergence is regulated by the self-energy, so the result depends on the self-energy (\( \sim 1/\epsilon \)).

In the case of the shear viscosity, we expect the integral not to have a co-linear divergence, since shear viscosity measures momentum transfer, so processes that do not change the momentum direction of the particles make no contribution. We therefore expect that the collision integrand in the shear viscosity should go to zero at \( w = z \) in such a way that the result does not depend on the self-energy. The true physical \( g(p) \) function will give an integrand that has this property. However, if we make a bad guess at \( g(p) \) (by using inappropriate basis polynomials in the expansion (38)) then each individual term will have a co-linear divergence, which will only cancel out when we add up the contributions from many terms. The best guesses for \( g(p) \) are therefore ones which yield a collision integrand with no co-linear divergence, i.e. no dependence on the self-energy.

In both the mean free path and the shear viscosity calculations the collision integral takes the form (B1). The difference between them lies in behavior of the numerator \( I(w, z) \) in the
The behavior of $\Delta_{ij}^0$ near $w = z$ and the collision integral; $\epsilon$ represents the self-energy. Only the $n = -1$ case has the proper physical suppression of co-linear contributions to the shear viscosity.

Using expansions of $g(p)$ with only one term ($N = 0$) we calculate $\Delta_{ij}^0$ and the shear-viscosity collision integral for different choices of $n$. We summarize the results in Table III. We see that the choice $n = -1$ fully suppresses the co-linear singularity and gives a collision integral that is independent of the self-energy. The choice $n = -2$ partially suppresses co-linear scattering and gives the collision integral a very weak dependence on the self-energy. Other values of $n$ do not suppress the co-linear scattering at all and are akin to the calculation of the mean free path. This explains our finding in Sec. III, Table II that $n = -1$ is the optimal choice for fast convergence of the polynomial approximation for $g(p)$ in the shear viscosity, that $n = -2$ is the next best choice, and other values of $n$ have very poor convergence.

**Appendix C: Approximate evaluation of the collision integral**

Here we describe how the collision integral is reduced to a five-dimensional numerical integral, in which we have factored out the temperature dependence and part of the dependence on the kaon speed $\nu$. We begin with the matrix $M$ (43) that enters into the calculation of the shear viscosity. As described in Sec. III and appendices A and B, we get a
good estimate of the shear viscosity by assuming $g(p) = 1/p$, i.e. we set $N = 0$ and $n = -1$. Then, as described after Eq. (49), we can eliminate seven of these integrals by using the $\delta$-function and spherical symmetry. We rescale the momenta with temperature, and find

$$M_{00} = \frac{1}{10 \cdot 2^8 \pi^6 \nu^6} \left(\frac{T}{\nu}\right)^{13} \int d\Gamma f_x f_y (1 + f_z) (1 + f_w) |\mathcal{M}(\nu, g, \lambda)|^2 \Delta^0_{ij} \bar{\Delta}^0_{ij} \quad (C1)$$

where $f_x \equiv 1/(e^x - 1)$, and

$$\int d\Gamma = \int_0^\infty dx \int_0^\infty dy \int_{-1}^1 d\alpha \int_{-1}^1 d\beta \int_0^{\pi} d\phi \frac{z^2}{1 - \alpha}, \quad (C2)$$

$$x = \frac{\nu p}{T} \quad y = \frac{\nu k}{T} \quad z = \frac{\nu k'}{T} = \frac{xy(1 - \alpha)}{x(1 - \beta) + y(1 - \gamma)} \quad w = \frac{\nu p'}{T} = x + y - z, \quad (C3)$$

$$\alpha = \hat{p} \cdot \hat{k} \quad \beta = \hat{p} \cdot \hat{k}' \quad \gamma = \hat{k} \cdot \hat{k}' = \alpha \beta + \sqrt{(1 - \alpha^2)(1 - \beta^2)} \cos(\phi) \quad (C4)$$

From (48), $\mathcal{M} \equiv (\nu/T)^8 \mathcal{M}$ depends on the speed $\nu$ and the couplings $g$ and $\lambda$ as well as the rescaled momenta $x, y, z$. From (44), $\bar{\Delta}^0_{ij} \equiv (\nu/T) \Delta^0_{ij}$ depends only on the rescaled momenta. The expression for $z$ comes from solving the energy-conserving $\delta$-function, and $\phi$ is the difference in azimuthal angles between the vectors $p, k$ and $p, k'$.

We have scaled out all the temperature dependence of the integrand, but there is still some dependence on $\nu$ and the couplings $g$ and $\lambda$ which comes in via $\mathcal{M}$. The integral can be evaluated numerically using (44) and (48) for given values of $\nu, g, \lambda$.

We now show how to obtain the approximate analytic forms for $\eta$ given in (54), which are valid in the regime $g^2/\lambda \ll 1$ and $g^2/\lambda \gg 1$.

Using (C1) in (46) and evaluating $A_0^{-1}$ from (39), we find that

$$\eta = \text{const} \frac{1}{\nu^8 M_{00}(\nu)} \quad (C5)$$

where const represents a function independent of $\nu$. We can obtain the function $h_1(\nu)$ in (54) by going to large $\lambda$ in which case $\mathcal{M} = \mathcal{M}_c$, and then calculating the shear viscosity with all the dimensionful parameters in the coupling constant $\lambda$ set equal to unity. To calculate $h_2(\nu)$, we go to large $g$ where $\mathcal{M} = \mathcal{M}_s + \mathcal{M}_t + \mathcal{M}_u$ and do the same thing. In both cases we find

$$h_{1,2} = \frac{A_{1,2} \nu^{11}}{\int d\Gamma f(\Gamma)|\mathcal{M}_{1,2}(\nu)|^2} \quad (C6)$$

where $A_{1,2}$ is a pure number and $f(\Gamma)$ represents the parts of the integrand of (C1) that are independent of $\nu$. From (43) we find that

$$\mathcal{M}_{1,2} \sim c_{1,2}^{(0)} + c_{1,2}^{(1)} \nu^2 + c_{1,2}^{(2)} \nu^4 \quad (C7)$$
Therefore, we expect

\[ h_{1,2}(\nu) = C_{1,2} \frac{\nu^{11}}{\sum_{i=0}^{4} a_{1,2}^{(i)} \nu^{2i}}, \]  

(C8)

justifying our statement in the paragraph below Eq. (54) and the resulting scaling in Fig. 4.

Finally, we can explain why \( h_1 \) and \( h_2 \) are so small (see Fig. 4). This is a direct result of \( M_{00} \) being large. Because the all interactions of the Goldstone kaons are derivative interactions, the collision integral involves high powers of momenta. Schematically, it has the form

\[ \int_0^{\infty} dx \ x^d f_x \sim (d + 1)! \]  

(C9)

where \( d = 12 \) in our case. Since \( 1/13! \approx 1.6 \times 10^{-10} \) it is not surprising that \( h_1 \) and \( h_2 \) are of that order.

[1] M. G. Alford, M. Braby, S. Reddy, and T. Schafer, *Bulk viscosity due to kaons in color-flavor-locked quark matter*, Phys. Rev. C75 (2007) 055209, [nucl-th/0701067].

[2] M. G. Alford, M. Braby, and A. Schmitt, *Bulk viscosity in kaon-condensed color-flavor locked quark matter*, J. Phys. G35 (2008) 115007, [arXiv:0806.0285].

[3] C. Manuel and F. J. Llanes-Estrada, *Bulk viscosity in a cold CFL superfluid*, JCAP 0708 (2007) 001, [arXiv:0705.3909].

[4] M. Mannarelli and C. Manuel, *Bulk viscosities of a cold relativistic superfluid: color-flavor locked quark matter*, arXiv:0909.4486.

[5] M. G. Alford, M. Braby, and A. Schmitt, *Critical temperature for kaon condensation in color-flavor locked quark matter*, J. Phys. G35 (2008) 025002, [arXiv:0707.2389].

[6] M. Alford, P. Jotwani, C. Kouvaris, J. Kundu, and K. Rajagopal, *Astrophysical implications of gapless color-flavor locked quark matter: A hot water bottle for aging neutron stars*, Phys. Rev. D71 (2005) 114011, [astro-ph/0411560].

[7] G. W. Carter and S. Reddy, *Neutrino propagation in color superconducting quark matter*, Phys. Rev. D62 (2000) 103002, [hep-ph/0005228].

[8] P. Jaikumar, M. Prakash, and T. Schafer, *Neutrino emission from Goldstone modes in dense quark matter*, Phys. Rev. D66 (2002) 063003, [astro-ph/0203088].

[9] P. Jaikumar, C. D. Roberts, and A. Sedrakian, *Direct Urca neutrino rate in colour superconducting quark matter*, Phys. Rev. C73 (2006) 042801, [nucl-th/0509093].
[10] A. Schmitt, I. A. Shovkovy, and Q. Wang, *Neutrino emission and cooling rates of spin-one color superconductors*, Phys. Rev. **D73** (2006) 034012, [hep-ph/0510347].

[11] Q. Wang, Z.-g. Wang, and J. Wu, *Phase space and quark mass effects in neutrino emissions in a color superconductor*, Phys. Rev. **D74** (2006) 014021, [hep-ph/0605092].

[12] C. J. Pethick, *Cooling of neutron stars*, Rev. Mod. Phys. **64** (1992) 1133–1140.

[13] I. A. Shovkovy and P. J. Ellis, *Thermal conductivity of dense quark matter and cooling of stars*, Phys. Rev. **C66** (2002) 015802, [hep-ph/0204132].

[14] C. J. Horowitz, O. L. Caballero, and D. K. Berry, *Thermal conductivity of the crust of accreting neutron stars*, arXiv:0804.4409.

[15] D. N. Aguilera, V. Cirigliano, J. A. Pons, S. Reddy, and R. Sharma, *Superfluid Heat Conduction and the Cooling of Magnetized Neutron Stars*, Phys. Rev. Lett. **102** (2009) 091101, [arXiv:0807.4754].

[16] M. Braby, J. Chao, and T. Schaefer, *Thermal conductivity of color-flavor locked quark matter*, arXiv:0909.4236.

[17] L. Lindblom, B. J. Owen, and S. M. Morsink, *Gravitational radiation instability in hot young neutron stars*, Phys. Rev. Lett. **80** (1998) 4843–4846, [gr-qc/9803053].

[18] M. G. Alford, A. Schmitt, K. Rajagopal, and T. Schafer, *Color superconductivity in dense quark matter*, Rev. Mod. Phys. **80** (2008) 1455–1515, [arXiv:0709.4635].

[19] P. F. Bedaque and T. Schafer, *High Density Quark Matter under Stress*, Nucl. Phys. **A697** (2002) 802–822, [hep-ph/0105150].

[20] D. B. Kaplan and S. Reddy, *Novel phases and transitions in quark matter*, Phys. Rev. **D65** (2002) 054042, [hep-ph/0107265].

[21] D. T. Son, *Light Goldstone boson and domain walls in the K0-condensed phase of high density quark matter*, hep-ph/0108260.

[22] C. Manuel, A. Dobado, and F. J. Llanes-Estrada, *Shear viscosity in a CFL quark star*, JHEP **09** (2005) 076, [hep-ph/0406058].

[23] D. T. Son and M. A. Stephanov, *Inverse meson mass ordering in color-flavor-locking phase of high density QCD*, Phys. Rev. **D61** (2000) 074012, [hep-ph/9910491].

[24] D. T. Son and M. A. Stephanov, *Inverse meson mass ordering in color-flavor-locking phase of high density QCD: Erratum*, Phys. Rev. **D62** (2000) 059902, [hep-ph/0004095].

[25] V. Kleinhaus, M. Buballa, D. Nickel, and M. Oertel, *Pseudoscalar Goldstone bosons in the
color-flavor locked phase at moderate densities, Phys. Rev. D76 (2007) 074024, [arXiv:0707.0632].

[26] S. Reddy, M. Sadzikowski, and M. Tachibana, Neutrino processes in the K0 condensed phase of color flavor locked quark matter, Phys. Rev. D68 (2003) 053010, [nucl-th/0306015].

[27] G. Rupak and T. Schafer, Shear viscosity of a superfluid Fermi gas in the unitarity limit, Phys. Rev. A76 (2007) 053607, [arXiv:0707.1520].

[28] E. M. Lifshitz and L. P. Pitaevskii, Physical Kinetics. Pergamon, Oxford, 1981.

[29] P. M. V. Rsibois and M. De Leener, Classical kinetic theory of fluids. John Wiley and Sons, New York, 1977.

[30] J.-W. Chen and E. Nakano, Shear Viscosity to Entropy Density Ratio of QCD below the Deconfinement Temperature, Phys. Lett. B647 (2007) 371–375, [hep-ph/0604138].

[31] A. Dobado and S. N. Santalla, Pion gas viscosity at low temperature and density, Phys. Rev. D65 (2002) 096011, [hep-ph/0112299].

[32] G. P. Lepage, A new algorithm for adaptive multidimensional integration, Journal of Computational Physics 27 (1978), no. 2 192 – 203.

[33] G. P. Lepage, VEGAS: An Adaptive Multi-dimensional Integration Program, 1980. Cornell University preprint CLNS 80-447.

[34] P. Jaikumar, G. Rupak, and A. W. Steiner, Viscous damping of r-mode oscillations in compact stars with quark matter, Phys. Rev. D78 (2008) 123007, [arXiv:0806.1005].

[35] P. Arnold, G. D. Moore, and L. G. Yaffe, Transport coefficients in high temperature gauge theories: (I) Leading-log results, JHEP 11 (2000) 001, [hep-ph/0010177].

[36] P. Arnold, G. D. Moore, and L. G. Yaffe, Transport coefficients in high temperature gauge theories. II: Beyond leading log, JHEP 05 (2003) 051, [hep-ph/0302165].

[37] K. Zarembo, Dispersion laws for Goldstone bosons in a color superconductor, Phys. Rev. D62 (2000) 054003, [hep-ph/0002123].