NATURAL TRANSFORMATIONS ASSOCIATED TO A LOCALLY COMPACT GROUP AND UNIVERSALITY OF THE TERRELL LAW

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ABSTRACT. Via the construction of a functor from \( C_\alpha (H) \) to an auxiliary category we associate, to any triplet \((G, F, \rho)\), two natural transformations \( m_\star \) and \( v_\natural \) between functors from the opposite categories of \( C_\alpha (H) \) and \( C_\alpha^0 (H) \) to the category \( \text{Fct}(H, \text{Set}) \) of functors from \( H \) to \( \text{Set} \) and natural transformations. \( G \) and \( F \) are locally compact groups, \( \rho : F \to \text{Aut}(G) \) is a continuous morphism, \( H \) is the external topological semidirect product of \( G \) and \( F \) relative to \( \rho \), a groupoid when seen as a category, \( C_\alpha (H) \) and \( C_\alpha^0 (H) \) are subcategories of the category of \( C^\ast \)-dynamical systems with symmetry group \( H \) and equivariant morphisms. For any object \( \mathfrak{A} \) of \( C_\alpha^0 (H) \) to assemble \( m^\mathfrak{A}_\star \) we exploit the Chern-Connes characters generated by JLO cocycles \( \Phi \) on the unitization of certain \( C^\ast \)-crossed products relative to \( \mathfrak{A} \), while to construct \( v^\mathfrak{A}_\natural \) we exert the states of the \( C^\ast \)-algebra underlying \( \mathfrak{A} \) associated in a convenient manner to the 0−dimensional components of the \( \Phi \)'s. We use \( m^\mathfrak{A}_\star \) and \( v^\mathfrak{A}_\natural \) to define the nucleon phases and the fragment states of the system \( \mathfrak{A} \), and to formulate and generalize in a \( C^\ast \)-algebraic framework the nucleon phase hypothesis advanced by Mouze and Yther. We apply the naturality of \( m_\star \) and \( v_\natural \) to prove the universality of the Terrell law stated as invariance of the mean value of the prompt-neutron yield under the action of \( H \) and the action of suitable equivariant perturbations on the fissioning systems.

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8.2. Universality of the Terrell law

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References

1. Introduction

Let \( H = G \rtimes_{\rho} F \) be the external topological semidirect product of \( G \) and \( F \), two locally compact groups, relative to a strongly continuous morphism \( \rho : F \to \text{Aut}(G) \), moreover let \( \mathcal{C}_u(H) \) be the category of \( C^* \)-dynamical systems with symmetry group \( H \) whose underlying \( C^* \)-algebra is a von Neumann algebra in the canonical standard form, with surjective equivariant morphisms. Firstly we construct a category \( \mathfrak{6}(G, F, \rho) \). Let an equivariant stability consist in a couple of maps \( \overline{m} \) and \( \mathcal{V} \) defined on subsets of \( \text{Obj}(\mathfrak{6}(F, G, \rho)) \) such that \( \overline{m} \) and \( \text{gr} \circ \mathcal{V} \) are \( H \)-equivariant \( [61] \) & \([63]\) and \( \overline{m} \) also \( \mathfrak{6}(F, G, \rho) \)-equivariant \( [72] \). Here \( \text{gr}(f) \) is the graph of any map \( f \) and \( \mathcal{V} \)-equivariance is for \( \text{Mor}_{\mathcal{V}} \)-equivariance for any category \( \mathcal{V} \), moreover in what follows we let equivariant property of \( \overline{m} \) mean equivariant property of \( \text{gr} \circ \mathcal{V} \). Finally for a given category \( \mathcal{C} \) let a \( \mathcal{C} \)-equivariant stability be the couple composed by an equivariant stability and a functor from \( \mathcal{C} \) to \( \mathfrak{6}(G, F, \rho) \), so we can replace \( \mathfrak{6}(G, F, \rho) \) by \( \mathcal{C} \) in the aforementioned equivariant properties.

The main result of this paper is the construction in Thm. \( 6.25 \) of a \( \mathcal{C}_u(H) \)-equivariant stability \( \mathcal{E}_* \) for which \( \mathcal{V} \) is \( \mathcal{C}_u(H) \)-equivariant, where \( \mathcal{C}_u(H) \) is a subcategory of \( \mathcal{C}_u(H) \). As a first application we encode in Thm. \( 7.4 \) the \( H, \mathcal{C}_u(H) \) and \( \mathcal{C}_u(H) \) equivariance properties of the maps \( \overline{m} \) and \( \mathcal{V} \) relative to \( \mathcal{E}_* \) into natural transformations between suitable functors from the categories \( \mathcal{C}_u(H) \) and \( \mathcal{C}_u(H) \) to the category \( \text{Set}^{\text{op}} \) and from the category \( H \) to \( \text{Set} \), where \( H \) has to be understood as the groupoid with its unit as the only object.

Then we apply the main theorem to encode in Thm. \( 7.20 \) the aforementioned equivariance properties into two natural transformations \( \mathfrak{m}_* \) and \( \mathfrak{v}_* \), the first between the functors \( \mathcal{P}_* \) and \( \mathcal{O}_* \) from \( \mathcal{C}_u(H)^{\text{op}} \) to the category \( \text{Fct}(H, \text{Set}) \), the second between the functors \( \mathcal{P}_* \) and \( \mathcal{Z}_* \) from \( \mathcal{C}_u(H)^{\text{op}} \) to \( \text{Fct}(H, \text{Set}) \), provided the hypothesis \( \mathcal{E} \) holds true, ensuring the functoriality of \( \mathcal{P}_* \) and \( \mathcal{Z}_* \). Here \( \text{Fct}(H, \text{Set}) \) is the category of all functors from \( H \) to \( \text{Set} \) whose class of morphisms is the class of natural transformations between them.

Finally the main theorem will be applied to prove the universality of the generalized nucleon phase by showing in Thm. \( 8.9 \) the invariance of the mean value of the prompt-neutron yield under action of the symmetry group \( H \) and under action of suitable equivariant perturbations over the fissioning system. We remark that, satisfied the hypothesis \( \mathcal{E} \), Thm. \( 8.9 \) follows since Thm. \( 7.20 \).

The paper is organized as follows.

In Thm. \( 3.11 \) we prove that the Borelian functional calculus in a Banach space is equivariant under an isometric action. Then we apply this result to ensure in Thm. \( 3.13 \) the same equivariance in case the spectral measure \( \mathcal{E} \) involved is the joint resolution of identity \( (RI) \) constructed by a family of commuting Borel \( RI' \)s in a Hilbert space. Thm. \( 3.8 \) provides a sufficient condition on the Borelian map \( f \) to ensure the selfadjointness of the operator \( f(\mathcal{E}) \). These results are steps in the direction of the construction in Cor. \( 6.14 \) of the object part of the functor \( \mathfrak{6}_H \) from \( \mathcal{C}_u(H) \) to the category \( \mathfrak{6}(G, F, \rho) \), in particular Thm. \( 3.13 \) plays a fundamental rule in showing \( (116) \).
In Thm. 3.18 and Cor. 3.20 we prove the equivariance of the KMS–states under the dual action of surjective equivariant morphisms, then we apply these results in constructing in Thm. 6.24 the functor \( G^H \).

The entire section 3.3 is devoted to show in Thm. 3.46 the \( H \)-equivariance of representations of \( C^\ast \)-crossed products for different symmetry groups, one of the basic result in order to prove Cor. 6.14.

In Lemma 4.2 we prove for any state of a \( C^\ast \)-algebra the existence of a canonical extension to the multiplier algebra, then for a given dynamical system \( \langle A, H, \sigma \rangle \) with \( A \) unital, the result is used in Cor. 4.4 to associate a state of \( A \) to any state of the crossed product \( A \rtimes_\sigma H \). Cor. 4.4 is also exploited via Lemma 5.17 in constructing an equivariant stability in Thm. 5.23. Additional results are Lemmas 4.5 and 4.11 required to prove Thm. 6.13 one of the auxiliary results used in the proof of Cor. 6.14.

In Def. 11 we introduce the category \( G^\ast (F, G, \rho) \), in Def. 30 the equivariant stabilities and in Def. 31 \( C^\ast \)-equivariant stabilities, and provide their physical properties in Prp. 5.29 with the help of the interpretation in Def. 35 and based on the assumption at page 55. Then we prove in Thm. 5.23 that the triplet defined in Def. 34 is an equivariant stability. We remark that the requests (77, 78, 79, 81) in the definition of the category \( G^\ast (G, F, \rho) \), find their justification since they are used in proving Thm. 5.23.

Sec. 6 is devoted to the construction in Thm. 6.24 of the functor \( G^H \), in Cor. 6.14 we construct its object part. In addition to the auxiliary results in the previous sections, the proof of Cor. 6.14 requires Thm. 6.9 Thm. 6.11 and Thm. 6.13. Finally Thm. 5.23 and Thm. 6.24 permit to state in our main Thm. 6.25 the existence of the canonical \( C^\ast_u(H) \)-equivariant stability \( E^\ast \).

In Thm. 7.4 we encode the \( H \) and \( C^\ast_u(H) \) equivariance of the map \( m \), and the \( H \) and \( C^\ast_u(H) \) equivariance of the map \( V \), into natural transformations between functors from \( H \) to \( \text{Set} \) and between functors from the category \( C^\ast_u(H) \) to \( \text{Set}^{op} \) for \( m \) and from \( C^\ast_u(H) \) to \( \text{Set}^{op} \) for \( V \). In Thm. 7.20, one of the main results of this paper, we encode in a unique fashion both the \( H \) and \( C^\ast_u(H) \) equivariance of \( m \) and the \( H \) and \( C^\ast_u(H) \) equivariance of \( V \) by providing modulo a suitable equivalence relation, that these maps realize natural transformations \( m^\ast \) and \( v^\ast \) between functors from the category \( C^\ast_u(H)^{op} \) to the category \( \text{Fct}(H, \text{Set}) \) and from \( C^\ast_u(H)^{op} \) to \( \text{Fct}(H, \text{Set}) \) respectively.

Finally in Sec. 8 we use \( E^\ast \) to prove the universality of the Terrell law and to propose a \( C^\ast \)-algebraic formulation, and in this way a generalization, of the nucleon phase hypothesis initially stated by Mouze and Ythier in [MHY1] quoted in [MHY2] and described by Ricci in [Ric]. Roughly for any object \( \mathcal{A} \) of \( C^\ast_u(H) \) we use the map \( m^\mathcal{A} \) to construct the nucleon phases of the system \( G^H(\mathcal{A}) \) and the map \( V^\mathcal{A} \) to construct the fragment states of the system \( \mathcal{A} \), originated by the nucleon phases. The main result in this section is Thm. 8.9 where we prove the universality of the Terrell law i.e. the invariance of the mean value of the prompt-neutron yield under action, over the fissioning system, of the symmetry group \( H \) and perturbations of \( \mathcal{A} \) implemented by a subset of the morphism class of \( C^\ast_u(H) \). In conclusion we note that under the hypothesis \( E^\ast \), ensuring the functoriality of the functors domain and codomain of \( v^\ast \), the universality of the Terrell law stated in Thm. 8.9 is equivalent to the naturality of the transformations \( m^\ast \) and \( v^\ast \) stated in Thm. 7.20.
2. Terminology and preliminaries

In this section we introduce notations, known results and simple consequences of them used throughout the paper. Given two sets $A, B$ let $\mathcal{P}(A)$ denote the power class of $A$, $B^A$ or $\mathcal{F}(A, B)$ denote the set of all maps on $A$ at values in $B$. Let $\text{ev}_a$ denote the evaluation map, i.e. if $F : A \to B$ is any map and $a \in A$, then $\text{ev}_a(F) := F(a)$. If $x \in \prod_{A \in A} \mathcal{C}_{B_a}$, with $B_a$ and $C_a$ sets for all $a \in A$, often we denote $x(a)(b)$ by $x(a, b)$ where $b \in B_a$. If $f : X \to A$ and $g : X \to B$ then by abuse of the standard language we denote by $f \times g$ the map on $X$ with values in $A \times B$ such that $(f \times g)(x) := (f(x), g(x))$ for all $x \in X$.

Let $\text{Set}$ be the category of sets, maps (and map composition) and $\text{Ab}$ the category of abelian groups and group morphisms. If the contrary is not stated, any diagram involving maps has to be understood in $\text{Set}$. Let $A$, $B$ and $C$ be categories. $A^{op}$ denotes the opposite category of $A$, [MCL, p. 33], often we denote $\text{Obj}(A)$, the class of the objects of $A$, simply by $A$ and for any $x, y$ objects of $A$ remove the index $A$ from the class $\text{Mor}_A(x, y)$ of the morphisms of $A$ from $x$ to $y$. For any $T \in \text{Mor}_A(x, y)$ we set $d(T) = x$ and $c(T) = y$, while the composition on $\text{Mor}_A$ is always denoted by $\circ$. $\text{Fct}(A, B)$ denotes the category of functors from $A$ to $B$ and natural transformations provided by pointwise composition, see [MCL, p. 40] or [Bor, p. 10]. For any functor $F$ let $F_o$ and $F_n$ be the object and morphism map of $F$ respectively, often we let $\hat{F}$ denote $F_o$. Let $\circ : \text{Fct}(A, B) \times \text{Fct}(B, C) \to \text{Fct}(A, C)$ denote the standard composition of functors as defined in [MCL, p. 14], while $\beta * \alpha \in \text{Mor}_{\text{Fct}(A, C)}(H \circ F, K \circ G)$ is the Godement product between the natural transformations $\beta$ and $\alpha$, where $H, K \in \text{Fct}(B, C)$ and $F, G \in \text{Fct}(A, B)$ while $\beta \in \text{Mor}_{\text{Fct}(B, C)}(H, K)$ and $\alpha \in \text{Mor}_{\text{Fct}(A, B)}(F, G)$, see [Bor, Prp. 1.3.4] (or [MCL, p. 42] where it is used the symbol $\circ$ instead of $\ast$). Note that if $Id_F \in \text{Mor}_{\text{Fct}(A, B)}(F, F)$ is the identity morphism then

$$
(1) \quad \beta * Id_F = \beta \circ F_o.
$$

If $S$ is a subset of $\text{Obj}(A)$ then we set $S(S) := \{ u \in \text{Mor}_A(a, b) | a, b \in S \}$.

Set $\mathbb{N}_0 := \mathbb{N} - \{0\}$ and $\mathbb{R}_0 := \mathbb{R} - \{0\}$, while $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ provided with the topology of one-point compactification. If $A$ is any set then $Id_A$ is the identity map on $A$ we often use the convention to remove the index $A$ if it is clear the set involved. Let $T$ be a topological space, $\mathcal{B}(T)$ denotes the set of the Borelian subset of $T$, if $T$ is locally compact and $E$ is a Hausdorff locally convex space, let $\mathcal{C}_c(T, E)$ denote the linear space of continuous $E$-valued maps $f$ on $X$ with compact support $\text{supp}(f)$, where $\text{supp}(f) := f^{-1}(E - \{0\})$, set $\mathcal{C}_c(T) := \mathcal{C}_c(T, \mathbb{C})$.

Let $S$ be a topological space, thus $\text{Cl}(S)$, $\text{Op}(S)$ and $\text{Comp}(S)$ denote the classes of open, closed and compact subsets of $S$, while $\mathcal{B}(S)$ the $\sigma$–field of Borel subsets of $S$. $\mathcal{C}_c(S)$ is the space of continuous maps $f : S \to \mathbb{C}$ with compact support, provided by the inductive limit topology of uniform convergence over compact subsets of $S$.

Let $X$ and $T$ be a locally compact group and locally compact space respectively, then $\mathcal{M}(X)$ is the class of Haar measures on $X$ and $\mathcal{M}(T)$ is the class of Radon measures on $T$. Let $\mu \in \mathcal{M}(T)$ and $S$ be a locally compact subspace of $T$, set $\mu_S : \mathcal{C}_c(S) \to \mathbb{C}$ such that $\mu_S(f) := \mu(\tilde{f})$, where $\tilde{f}$ is the $0$–extension on $T$ of $f \in \mathcal{C}_c(S)$, thus $\mu_S \in \mathcal{M}(S)$. Let $T, S$ be two locally compact spaces, $\mu \in \mathcal{M}(T)$ and $\varepsilon : T \to S$ be $\mu$–proper, thus $\varepsilon(\mu)$ denotes the image of $\mu$ under $\varepsilon$ as defined in [Int Ch. 5, §6, n°1, Def. 1]. By construction $\varepsilon(\mu) \in \mathcal{M}(S)$ such
that for all $f \in C_c(S)$

$$
(2) \quad \int f \, d\varepsilon(\mu) = \int f \circ \varepsilon \, d\mu.
$$

Let $X$ be a locally compact group and $s \in X$, set $L_s, R_s : X \to X$, such that $L_s(x) := s \cdot x$ and $R_s(x) := x \cdot s$, while $L_s^*, R_s^* : C^X \to C^X$ such that $L_s^*(h) := h \circ L_{s^{-1}}$ and $R_s^*(h) := h \circ R_{s^{-1}}$ respectively. Whenever we will deal with different groups, it will be clear by the context to which group the maps $R$ and $L$ are referring to. By definition $\mathcal{H}(X)$ is the class of left-invariant $\mu \in M(X)$, i.e. $\mu \in M(X)$ such that $\mu \circ L_s^* \upharpoonright C_c(X) = \mu$, for all $s \in X$.

If $X$ and $Y$ are two topological linear spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $\mathcal{L}(X, Y)$ denotes the linear space of continuous linear maps from $X$ to $Y$, set $\mathcal{L}(X) := \mathcal{L}(X, X)$ and $X^* := \mathcal{L}(X, \mathbb{K})$. $\mathcal{L}_c(X, Y)$ is the topological linear space whose underlying linear space is $\mathcal{L}(X, Y)$ provided by the topology of pointwise convergence, while $\mathcal{L}_{c, 0}(X, Y)$ is the locally convex linear space whose underlying linear space is $\mathcal{L}(X, Y)$ provided by the topology generated by the following set of seminorms $\{q_{(\phi, x)} | (\phi, x) \in Y^* \times X\}$, where $q_{(\phi, x)}(A) \triangleq |\phi(Ax)|$. In case $X$ is a normed space we assume $\mathcal{L}(X)$ to be provided by the topology generated by the usual sup–norm. If $X$ is any structure including as a substructure the one of normed space say $X_0$, for example the normed space underlying any normed algebra, we let $\mathcal{L}(X)$ denote the normed space $\mathcal{L}(X_0)$ whenever it does not cause conflict of notations. Therefore we never shall use this convention in case $X$ is an Hilbert $C^*$–module, where $\mathcal{L}(X)$ always denotes the set of all adjointable operators on $X$, see later. If $A$ and $B$ are two linear operators in $X$, we set $[A, B] = AB - BA$, where the composition and sum are to be understood in the context of possibly unbounded operators, i.e. defined on the intersection of the corresponding domains. If $X, Y$ are Hilbert spaces and $U \in \mathcal{L}(X, Y)$ is unitary then $\text{ad}(U) \in \mathcal{L}(\mathcal{L}(X), \mathcal{L}(Y))$ denotes the isometry defined by $\text{ad}(U)(a) := UaU^{-1}$, for all $a \in \mathcal{L}(X)$.

20.1. $C^*$–algebras, $C^*$–dynamical systems and their crossed products. For any normed algebra $D$ let $R^D_\mu : D \to \mathcal{L}(D)$ and $L^D_{\pi} : D \to \mathcal{L}(D)$ denote the right and left multiplication map on $D$ respectively, i.e. $R_{\pi}(b) = ba$ and $L_{\pi}(b) = ab$ for any $a, b \in D$, often we remove the index $D$. Let $A$ be a $C^*$–algebra then $E_A$ denotes the set of states of $A$, if $A$ is a von Neumann algebra $\mathcal{N}_A$ denotes the set of its normal states. If $B$ is a $C^*$–algebra $\text{Hom}(A, B)$ is the set of $*$–homomorphisms defined on $A$ and at values in $B$, $\text{End}^*(A)$ and $\text{Aut}^*(A)$ are the sets of $*$–endomorphisms and $*$–automorphisms of $A$ respectively. Denote $\text{Aut}^*_\pi(A)$ the topological group of $*$–automorphisms of $A$ provided by the topology of pointwise convergence. $\text{Rep}(A)$ denotes the class of $*$–representations of $A$, while $\text{Rep}_\pi(A)$ denotes the class of cyclic $*$–representations of $A$. If $\mathfrak{S} = \left\{ \mathfrak{S}_1, \pi_1, \Omega_1 \right\}$ and $\mathfrak{R} = \left\{ \mathfrak{R}_1, \xi_1, \Psi_1 \right\}$ are in $\text{Rep}_\pi(A)$ then we call them unitarily equivalent if their underlying representations of $A$ are unitarily equivalent, say through the unitary operator $U : \mathfrak{S}_1 \to \mathfrak{R}_1$, and $U\Omega_1 = \Psi_1$. If $\pi$ is a nondegenerate representation of $A$ on $\mathfrak{S}_1$, then we denote by $N_{\pi}$ and call $\pi$–normal states its elements, the set of the $\phi \circ \pi$ where $\phi \in N_{\mathcal{L}(\mathfrak{S})}$. Since $\pi$ is nondegenerate $N_{\pi} \subset E_A$. If $\psi$ is a state of $A$ then $\psi$–normal means $\pi$–normal, where $\left\{ \mathfrak{S}_1, \pi_1, \Omega_1 \right\}$ is a cyclic representation associated to $\psi$. If $T \in \text{Hom}^*(A, B)$, set $T^+ : B^* \to A^*$ such that $T^+(w) = \omega \circ T$, if $\lambda \in \text{Hom}^*(A, B)$ bijective set $\lambda^* = (\lambda^{-1})^\dagger$. For any map $f : D \subseteq A \to A$ define the map $\text{ad}(\lambda)(f) : \lambda(D) \to A$ such that for all $b \in \lambda(D)$

$$
(3) \quad \text{ad}(\lambda)(f)(b) := (\lambda \circ f \circ \lambda^{-1})(b).
$$
Let $U$ be any set and $\gamma : U \to Aut^*(A)$, define

$$E^U_A(\gamma) := \{ \psi \in E_A \mid (\forall u \in U)(\psi \circ \gamma(u) = \psi) \}.$$ 

Let $A$ be a $C^*$–algebra and $X$ be a Hilbert $A$–module, [RW, Def. 2.8], and $L_A(X)$ or simply $L(X)$ the $C^*$–algebra of adjointable maps on $X$, [RW] Def. 2.17 and Prp. 2.21. Let $A_A$ denote the Hilbert $A$–module associated to $A$, [RW] Exm. 2.10], then $M(A) := L(A_A)$ and $i^A$ denote the multiplier algebra of $A$ and the canonical embedding of $A$ into $M(A)$ respectively, where $i^A(a)(b) = ab$ for all $a, b \in A$, [RW] Def. 2.48 and Exm. 2.43. Let $B$ be a $C^*$–algebra and $\alpha : B \to L(X)$ be a $*$–morphism, $\alpha$ is said nondegenerate if $span(\alpha(b)x \mid b \in B, x \in X)$ is dense in $X$, [RW, Def. 2.49]. $i^B$ is nondegenerate and injective since any $C^*$–algebra admits an approximate identity. As a consequence of [RW, Prp. 2.50] we have that if $\alpha$ is nondegenerate then there exists a unique $*$–morphism $\alpha^- : M(B) \to L(X)$ such that

$$\alpha^- \circ i^B = \alpha.$$ 

$\alpha^-$ is nondegenerate since it is so $\alpha$, thus $\alpha^-(1) = 1$. Since the double conjugate Hilbert space of any Hilbert space $\mathcal{H}$ equals $\mathcal{H}$, we deduce by [RW, Exm. 2.27] that $\mathcal{H}$ is a $\mathcal{K}(<\mathcal{K})$–Hilbert module with the same norm, where $\overline{\mathcal{H}}$ is the conjugate Hilbert space of $\mathcal{H}$, and $\mathcal{K}(\mathcal{H})$ is the $C^*$–algebra of the compact operators on $\mathcal{H}$, for any Hilbert space $\mathcal{H}$. Therefore since (4) it follows that if $\mathcal{R}$ is a nondegenerate representation of $B$, then $\mathcal{R}^-$ is the unique extension of $\mathcal{R}$ to a representation of $M(B)$ such that

$$\mathcal{R}^- \circ i^B = \mathcal{R}.$$ 

Let $\mathcal{H} = \{ \mathcal{H}, \mathcal{R}, \Omega \}$ be a cyclic representation of $B$, thus for all $c \in M(B)$ and $b \in B$ we deduce by [RW] proof of Prp. 2.50

$$\mathcal{R}^- (c) \mathcal{R}(b) \Omega = \mathcal{R}((i^B)^{-1}(c(i^B)(b))) \Omega.$$ 

$\mathcal{H}^- = \{ \mathcal{H}, \mathcal{R}^-, \Omega \}$ is a cyclic representation of $M(B)$ since (5) and since $\mathcal{H}$ is cyclic. By the uniqueness of the extension to $M(B)$ it follows that if $\{ \mathcal{R}, \mathcal{S}, \Omega \}$ is a cyclic representation of $M(B)$, then $\{ \mathcal{R}, \mathcal{S} \circ i^B, \Omega \}$ is a cyclic one of $B$ such that

$$(\mathcal{S} \circ i^B)^- = \mathcal{S}.$$ 

Let $\mathcal{A} = \{ A, H, \sigma \}$ and $\mathcal{B} = \{ B, H, \theta \}$ be $C^*$–dynamical systems, here called simply dynamical systems or dynamical systems with group symmetry $H$. $T$ is an $(\mathcal{A}, \mathcal{B})$–equivariant morphism, or equivalently, $(\sigma, \theta)$–equivariant morphism if $T \in Hom^*(A, B)$ and $T \circ \sigma(h) = \theta(h) \circ T$ for all $h \in H$. $\{ \mathcal{S}, \pi, W \}$ is a (nondegenerate) covariant representation of $\mathcal{A}$ if $\{ \mathcal{S}, \pi \}$ is a (nondegenerate) $*$–representation of $A$, $W$ is a strongly continuous unitary representation of $H$ on $\mathcal{S}$ such that for all $h \in H$

$$\pi \circ \sigma(h) = \text{ad}(W(h)) \circ \pi.$$ 

We denote by $Cov(\mathcal{A})$ the class of nondegenerate covariant representations of $\mathcal{A}$. $\{ \mathcal{S}, W \}$ is a cyclic covariant representation of $\mathcal{A}$ if $\mathcal{S} = \{ \mathcal{S}, \pi, \Omega \}$ is a cyclic representation of $A$, $\{ \mathcal{S}, \pi, W \}$ is a covariant representation of $\mathcal{A}$ and $W(H)\{\Omega\} = \{\Omega\}$. 


Let \( \varphi \in E^\mu_\mu (\sigma) \) and \( \mathfrak{S} = \{ \mathfrak{S}, \pi, \Omega \} \) be a cyclic representation of \( A \) associated with \( \varphi \). Set \( W^\mu_\Omega : H \to \mathcal{L}(\mathfrak{S}) \) such that for all \( h \in H \) and \( a \in A \)

\[
W^\mu_\Omega (h) \pi(a) \Omega = \pi(\sigma(h)a) \Omega.
\]

Then \( \{ \mathfrak{S}, W^\mu_\Omega \} \) is a cyclic covariant representation of \( \mathfrak{A} \) called the cyclic covariant representation of \( \mathfrak{A} \) induced by \( \mathfrak{S} \). We convene to remove the index \( \sigma \) whenever it does not cause confusion.

Set \( \mathcal{U}(A) := \{ U \in A \mid U^* = U^{-1} \} \) provided by the group structure inherited by the product on \( A \), and \( \mathcal{U}(\mathfrak{S}) := \mathcal{U}(\mathcal{L}(\mathfrak{S})) \) for any Hilbert space \( \mathfrak{S} \). We say that \( \sigma \) is inner if there exists a group morphism \( \nu : H \to \mathcal{U}(A) \) such that \( \sigma = \text{ad} \circ \nu \), in such a case we say that \( \sigma \) is inner implemented by \( \nu \), or that \( \nu \) implements unitarily \( \sigma \). \( \mathfrak{A} \) is said inner implemented by \( \nu \) if \( \sigma \) is so. If \( A \) is a von Neumann algebra, it can be always considered in standard form since [Bla 2, III.2.2.26] and [Tak 2, Def. 9.1.18], called its canonical standard form, then by [Tak 2] Thm 9.1.15 we deduce that \( \sigma \) is inner, moreover said \( \mathcal{U}_\mu (A) \) the subgroup of \( \mathcal{U}(A) \) whose elements \( u \) satisfy \( u \mu u^* = \mu \) and \( uL_\mu (A)_+ = L_\mu (A)_+ \), see [Tak 2, Def. 9.1.18] for the notations, there exists a unique group action \( \nu : H \to \mathcal{U}_\mu (A) \) inner implementing \( \sigma \).

Let \( \mu \in \mathcal{H}(H) \) and let \( \mathfrak{C}^\mu_\mu (H, A) \) denote the \(*\)-algebra whose underlying linear space is \( \mathfrak{C}_\mu (H, A) \), while the product and involution are respectively \(*\mu\) and \(* \) such that for all \( f, g \in \mathfrak{C}_\mu (H, A) \) and \( s \in H \) ([Wil, eqs.2.16 – 2.17])

\[
(f \star^\mu g)(s) := \int f(r)\sigma(r)(g(r^{-1}s))d\mu(r),
\]

\[
f^*(s) := \Lambda_H(s^{-1})\sigma(s)f(s^{-1}),
\]

where the integration is w.r.t. to the norm topology on \( A \). Denote by \( A \rtimes^\mu_\mu H \) the \( C^*\)-crossed product of \( A \) by \( H \) associated to \( \mu \), see [Wil, Lemma 2.27]. It is defined as the \( C^*\)-algebra completion of the normed \(*\)-algebra \( \mathfrak{C}^\mu_\mu (H, A) := \{ \mathfrak{C}^\mu_\mu (H, A), \| \cdot \| \mu \} \), where \( \| \cdot \| \mu \) is the \( \mu \)-universal norm such that for any \( f \in \mathfrak{C}_\mu (H, A) \) ([Wil, eq. 2.22 and Lm. 2.31])

\[
\| f \| \mu = \sup \{ \| (\pi \rtimes^\mu_\mu u)(f) \|_{\mathcal{L}(\mathfrak{S})} \mid \{ \mathfrak{S}, \pi, u \} \in \text{Cov}(\mathfrak{A}) \},
\]

where ([Wil, Prp. 2.23])

\[
(\pi \rtimes^\mu_\mu u)(f) := \int \pi(f(s))u(s)d\mu(s).
\]

Here the integral is valued in the locally convex space \( \mathcal{L}_\mu (\mathfrak{S}) \), i.e. it is an element of \( \mathcal{L}(\mathfrak{S}) \) and the integration is w.r.t. the strong operator topology on \( \mathcal{L}(\mathfrak{S}) \). Its existence is ensured by Rmk. 3.33 and by the fact that the product is continuous as a map on \( \mathcal{L}_\mu (\mathfrak{S}) \times \mathcal{L}_\mu (\mathfrak{S}) \), with values in \( \mathcal{L}_\mu (\mathfrak{S}) \), where \( \mathcal{L}_\mu (\mathfrak{S})_1 \) is the topological subspace of \( \mathcal{L}(\mathfrak{S}) \) of its elements with norm less or equal to 1. We call the following \( L_\mu^\mu \)-norm

\[
\mathfrak{C}_\mu (H, A) \ni f \mapsto \int \| f(s) \| d\mu(s).
\]

It is worthwhile a remark about notations. It is well-known that the Haar measure is unique up to a constant factor [Int 2, Th. 1 §1 n°2], nevertheless in this work, at difference with the standard usage, we prefer to mention expressly which Haar measure we use in (9,10) and a fortiori in \( A \rtimes^\mu_\mu H \).
Since [Wil Prp. 2.39] if \( \langle S, \pi, W \rangle \) is a covariant representation of \( \mathfrak{A} \), then \( \pi \triangleright_t W \) extends uniquely by continuity to a \( * \)-representation of \( \mathfrak{A} \triangleright_t^\mu H \) which is nondegenerate if it is so \( \langle S, \pi, W \rangle \). Vice versa if \( \mathcal{R} \) is a nondegenerate \( * \)-representation of \( \mathfrak{A} \triangleright_t^\mu H \) then there exists a nondegenerate covariant representation \( \langle S, \pi, W \rangle \) of \( \mathfrak{A} \) such that \( \mathcal{R} = \pi \triangleright_t W \). In particular said \( \mathcal{B} = \mathfrak{A} \triangleright_t^\mu H \)

\[
\mathcal{R}^- \circ i_A^\mathcal{B} = \pi,
\]

where \( i_A^\mathcal{B} \) is the canonical embedding of \( \mathfrak{A} \) into \( \text{M}(\mathcal{B}) \) such that \( i_A^\mathcal{B}(a)(f)(l) = af(l) \), for all \( a \in \mathfrak{A}, f \in \mathcal{C}_c(H, \mathcal{A}) \) and \( l \in H \). \( i_A^\mathcal{B} \) is nondegenerate since \( \mathcal{C}_c(H, \mathcal{A}) \) provided by the sup \( * \)-norm equals \( \mathcal{A} \mathcal{A} \mathcal{A} \mathcal{C}_c(H) \) and since \( i_A \) is nondegenerate. We call \( \langle S, \pi, W \rangle \) the covariant representation of \( \mathfrak{A} \) associated to \( \mathcal{R} \).

The set of dynamical systems with symmetry group \( H \), equivariant morphisms and map composition is a category denoted by \( \mathcal{C}_0(H) \), see [GHT pg.26]. For any \((\sigma, \theta)\)-equivariant morphism \( T \) the map \( f \mapsto T \circ f \) defined on \( \mathcal{C}_c^\mu(H, \mathcal{A}) \) extends uniquely by continuity to a \( * \)-homomorphism \( c_u(T) \) from \( \mathfrak{A} \triangleright_t^\mu H \) to \( \mathcal{B} \triangleright_t^\mu H \), moreover \( c_u(S \circ T) = c_u(S) \circ c_u(T) \) for any dynamical system \( \mathcal{C} \) and \( (\mathcal{B}, \mathcal{C}) \)-equivariant morphism \( S \). Hence the functions \( \langle \mathcal{A}, H, \sigma \rangle \mapsto \mathfrak{A} \triangleright_t^\mu H \) and \( T \mapsto c_u(T) \) determine a functor from \( \mathcal{C}_0(H) \) to the category of \( C^* \)-algebras and \( * \)-homomorphisms, see [GHT pg.26] or [Wil Cor. 2.48]. If \( H_0 \) is a locally compact subgroup of \( H \) then any \((\sigma, \theta)\)-equivariant morphism is \((\sigma \uparrow H_0, \theta \uparrow H_0)\)-equivariant, hence the map \( \langle \mathcal{A}, H, \sigma \rangle \mapsto \langle \mathcal{A}, H_0, \sigma \uparrow H_0 \rangle \) and the identity map on \( \text{Mor}\mathcal{C}_0(H) \) determine a functor from \( \mathcal{C}_0(H) \) to \( \mathcal{C}_0(H_0) \). In particular for any \( v \in \mathcal{H}(H_0) \)

\[
c_u(T) \in \text{Hom}^*(\mathfrak{A} \triangleright_t^\gamma H_0, \mathcal{B} \triangleright_t^\gamma H_0).
\]

Let \( G \) and \( F \) be two topological groups, \( \rho : F \to \text{Aut}(G) \) a group homomorphism such that the map \( (g, f) \mapsto \rho_f(g) \) on \( G \times F \) at values in \( G \), is continuous, where \( \text{Aut}(G) \) is the group of the automorphisms of the group underlying \( G \). Thus we denote by \( G \triangleright_t F \) the external topological semi-direct product of \( G \) and \( F \) relative to \( \rho \), see [Top 1 III.19]. By definition for all \((g_1, h_1), (g_2, h_2) \in G \triangleright_t F \)

\[
(g_1, h_1) \cdot (g_2, h_2) := (g_1 \cdot \rho(h_1))(g_2), h_1 \cdot h_2).
\]

Moreover \( j_1 : G \to G \triangleright_t F \) and \( j_2 : F \to G \triangleright_t F \) will be the (continuous) canonical injections.

Since [Top 1 Prp. 14, I.66] \( G \triangleright_t F \) is locally compact if and only if \( G \) and \( F \) are locally compact, so in this case we can consider a dynamical system \( \mathfrak{A} = \langle A, G \triangleright_t F, \sigma \rangle \). In addition let \( F_0 \) and \( G_0 \) be topological subgroups of \( F \) and \( G \) respectively, \( \xi : R \to G \) be a continuous group homomorphism and \( l \in G \triangleright_t F \), set

\[
\tau_{\sigma} := \sigma \circ j_1
\]
\[
\gamma_{\sigma} := \sigma \circ j_2
\]
\[
S^G_{F_0} := G \triangleright_t F_0
\]
\[
\sigma_{F_0} := \sigma \uparrow S^G_{F_0}
\]
\[
\tau_{\sigma(l, \xi)} := \tau_{\sigma} \circ \text{ad}(l) \circ j_1 \circ \xi.
\]

Here and thereafter we use the convention to denote \( \sigma \uparrow F_0 \) simply by \( \rho \) anytime it is clear by the context w.r.t. which subset \( F_0 \) of \( F \) the restriction has been performed. \( \tau_{\sigma(l, \xi)} \) is
well-set, since \( j_1(G) \) is a normal subgroup of \( G \rtimes_\rho F \), moreover we have for all \( h \in G \rtimes_\rho F \)
\[
(13) \quad \tau_{\sigma}^{(l^\rho_F h, \xi)} = \text{ad}(\sigma(l)) \circ \tau_{\sigma}^{(h, \xi)}.
\]

Whenever it is clear by the context which dynamical system is involved we convein to remove the index \( \sigma \). If \( F_0 \) is locally compact, for instance closed in \( F \) by \cite{Top1} Prp. 13, I.66], then \( S^G_{F_0} \) is locally compact since \cite{Top1} Prp. 14, I.66]. Therefore \( \langle \mathcal{A}, S^G_{F_0}, \sigma_{F_0} \rangle \) is a dynamical system, and the crossed product \( \mathcal{A} \rtimes^\rho_{\sigma} S^G_{F_0} \) is well-set for any \( \mu \in \mathcal{H}(S^G_{F_0}) \). For any domain \( D \) in \( C \), i.e. open simply connected subset of \( C \), denote by \( H(D) \) the class of analytic maps on \( D \) with values in \( C \), while by \( H_w(D, \mathcal{A}) \) and \( H_s(D, \mathcal{A}) \) the class of weak and strong analytic maps on \( D \) with values in \( \mathcal{A} \). Let \( f : D \to \mathcal{A} \), then by definition \( f \in H_w(D, \mathcal{A}) \) if \( \phi \circ f \in H(D) \) for all \( \phi \in \mathcal{A}^* \), while \( f \in H_s(D, \mathcal{A}) \) if \( f \) is \( \mathbb{C} \)-derivable, thus differentiable. Thus \( H_s(D, \mathcal{A}) \subseteq H_w(D, \mathcal{A}) \). Let \( \langle \mathcal{A}, \mathbb{R}, \eta \rangle \) be a dynamical system. Set \( \delta_\eta \) to be the infinitesimal generator of the strongly continuous semigroup \( \mathcal{A} \uparrow \mathbb{R}^+ \) acting on \( \mathcal{A} \), i.e. \( a \in \text{Dom}(\delta_\eta) \) iff the following limit exists in the norm topology of \( \mathcal{A} \)
\[
(14) \quad \delta_\eta(a) := \lim_{t \to 0^+, \alpha \to 0} \eta(t)(a) - \frac{t}{a}.
\]

\( \mathcal{A}_\eta \) is the \(*\)-subalgebra of \( \mathcal{A} \) of the entire analytic elements of \( \eta \), see \cite{BR1} Def. 2.5.20.], and \( \overline{\eta} : C \to \mathcal{A}_{\eta} \) denote the unique entire analytic extension of \( \eta \), i.e. for any \( a \in \mathcal{A}_\eta \) the map \( C \ni z \mapsto \overline{\eta}(z)(a) \in \mathcal{A} \) belongs to \( H_w(C, \mathcal{A}) \), so to \( H_s(C, \mathcal{A}) \) since \cite{BR1} Prp. 2.5.21.], and it is the necessarily unique analytic extension of \( \mathbb{R} \ni t \mapsto \eta(t)(a) \in \mathcal{A} \). The uniqueness follows by \cite{BR1} Def. 2.5.20.] and by the uniqueness of entire analytic extension of numerical maps, see \cite{Rud1} Cor. of Thm. 10.18]. If \( \beta \in \mathbb{R} \) by definition \( \omega \in K^n_\beta \) if there exists an \( \eta \)-invariant, norm dense \(*\)-subalgebra \( D \) of \( \mathcal{A}_\eta \) such that \( \omega(a\overline{\eta}(i\beta)b) = \omega(ba) \) for all \( a, b \in D \), see \cite{BR2} Def. 5.3.1. If we replace \( D \) by \( \mathcal{A}_\eta \) we obtain an equivalent statement, see \cite{Ped} Prp. 8.12.3]. \( K_0^n \) is defined in \cite{BR2} Def. 5.3.18.]

2.0.2. \( K_0 \)-theory for \( C^* \)-algebras. In this section \( \mathcal{A} \) denotes a \( C^* \)-algebra.

The \(*\)-algebra \( \mathcal{M}_\omega(\mathcal{A}) \). Denote by \( \mathcal{A}^+ \) the \(*\)-algebra whose underlying linear space is \( \mathcal{A} \times \mathbb{C} \), the involution and product are defined by \( (a, \lambda)^* = (a^*, \overline{\lambda}) \), and \( (a, \lambda) \cdot (b, \mu) := (a \cdot b + \lambda b + \mu a, \lambda \mu) \), for all \( (a, \lambda), (b, \mu) \in \mathcal{A} \times \mathbb{C} \). \cite{Nai3} Ch. 2, §7, n°2, I and §10, n°1, III]. \( (0, 1) \) is the unity of \( \mathcal{A}^+ \), while the map \( \phi : a \mapsto (a, 0) \) is an injective \(*\)-isomorphism of \( \mathcal{A} \) into \( \mathcal{A}^+ \), we often shall identify \( \mathcal{A} \) with its image in \( \mathcal{A}^+ \) under \( \phi \). Under this identification \( \mathcal{A} \) is a two side ideal of \( \mathcal{A}^+ \), so we have \( L_x, R_x \uparrow \mathcal{A} \in \text{End}(\mathcal{A}) \), for any \( x \in \mathcal{A}^+ \). \( \mathcal{A}^+ \) is a \( C^* \)-algebra if provided by the following norm extending the one on \( \mathcal{A} \), \( \| x \| := \| L_x \uparrow \mathcal{A} \| \), well-set since \( L_{(a, \lambda)} \uparrow \mathcal{A} = L_a + \lambda : \mathcal{L}(\mathcal{A}) \) \cite{Nai3} Ch. 3, §16, n°1, III]. Set \( \tilde{\mathcal{A}} \) the smallest unital subalgebra of \( \mathcal{A}^+ \) containing \( \mathcal{A} \), so \( \tilde{\mathcal{A}} = \phi(\mathcal{A}) \), if \( \mathcal{A} \) has the identity, \( \tilde{\mathcal{A}} = \mathcal{A}^+ \) otherwise. Let \( \mathcal{B} \) a \(*\)-algebra and \( \alpha \in \text{Hom}^r(\mathcal{A}, \mathcal{B}) \), set \( \alpha^+ : \mathcal{A}^+ \ni (a, \lambda) \mapsto (\alpha(a), \lambda) \in \mathcal{B}^+ \), thus
\[
(15) \quad \alpha^+ \in \text{Hom}^r_1(\mathcal{A}^+, \mathcal{B}^+),
\]

while in case \( \mathcal{B} \) has the identity, we set
\[
(16) \quad \begin{cases} \hat{\alpha} : \mathcal{A}^+ \to \mathcal{B}, \\ (a, \lambda) \mapsto \alpha(a) + 1_{\mathcal{B}} \lambda, \end{cases}
\]

thus \( \hat{\alpha} \in \text{Hom}^r_1(\mathcal{A}^+, \mathcal{B}) \), called the \(*\)-morphism of \( \mathcal{A}^+ \) induced by \( \alpha \) (\(*\)-representation in case \( \mathcal{B} \) equals \( \mathcal{L}(S) \) for some Hilbert space \( S \)).
Let $n \in \mathbb{N}_0$, set $a^n := a \cdot \ldots \cdot a$ and $\mathcal{P}(A) := \{p \in A \mid p = p^* = p^2\}$. Denote by $\mathcal{M}_n(A)$ the set of the square matrices of order $n$ at elements in $\mathcal{A}$, which is a $\ast$-algebra providing it by the operations such that for all $i, j \in \{1, \ldots, n\}$ and $t \in \mathbb{C}$

1. $(M + N)_{ij} := M_{ij} + N_{ij}$,
2. $(t \cdot M)_{ij} := t \cdot M_{ij}$,
3. $(M \cdot N)_{ij} := \sum_{s=1}^{n} M_{is} \cdot N_{sj}$,
4. $(M^*)_{ij} = M_{ji}$.

Let $\mathcal{H}$ the Hilbert space in which $\mathcal{A}$ acts faithfully, e.g. through the universal representation, see [KR] Rmk. 4.5.8], so we can identify $\mathcal{A}$ as $C^*$-algebra with its faithful image acting on $\mathcal{H}$ provided with the standard operator norm. Next let $\mathcal{H}_k = \mathcal{H}$ for any $k = 1, \ldots, n$, set

\[
\begin{align*}
o : \mathcal{M}_n(A) &\to \mathcal{L}(\bigoplus_{s=1}^{n} \mathcal{H}_s), \\
o(M)(v)_k &:= \sum_{j=1}^{n} M_{kj} v_j, \forall M \in \mathcal{M}_n(A), v \in \bigoplus_{s=1}^{n} \mathcal{H}_s, k = 1, \ldots, n,
\end{align*}
\]

it is possible to show ([KR pg. 147]) that $\mathcal{o}$ is a $\ast$-isomorphism. Hence the following map define a norm

\[
|| \cdot ||_{\mathcal{M}_n(A)} : \mathcal{M}_n(A) \ni M \mapsto ||\mathcal{o}(M)||,
\]
on $\mathcal{M}_n(A)$ for which it is a $C^*$-algebra since isometric to $\mathcal{L}(\bigoplus_{s=1}^{n} \mathcal{H}_s)$.

Let $\mathcal{A} \otimes \mathcal{M}_n(\mathbb{C})$ be the algebraic tensor product of the $\ast$-algebras $\mathcal{A}$ and $\mathcal{M}_n(\mathbb{C})$, which is a $\ast$-algebra providing the product to be defined by $(a \otimes b) \cdot (c \otimes d) := (a \cdot c) \otimes (b \cdot d)$. For any $i, j \in \{1, \ldots, n\}$, set $e_{ij} \in \mathcal{M}_n(\mathbb{C})$ such that $(e_{ij})_{rs} := \delta_{ir} \delta_{js}$, for all $r, s \in \{1, \ldots, n\}$. It is possible to show ([Weg Prp. T.5.20]) that the following map

\[
\begin{align*}
\mathcal{A} \otimes \mathcal{M}_n(\mathbb{C}) &\to \mathcal{M}_n(A), \\
\sum_{i,j=1}^{n} a_{ij} \otimes e_{ij} &\mapsto A, \\
A_{rs} &:= a_{rs}, \forall r, s \in \{1, \ldots, n\},
\end{align*}
\]
is a well-defined $\ast$-isomorphism, called the standard isomorphism between $\mathcal{A} \otimes \mathcal{M}_n(\mathbb{C})$ and $\mathcal{M}_n(A)$. Moreover since [KR Ex. 11.1.5], the fact that $\mathcal{o}$ is an isometry of $\mathcal{M}_n(A)$ onto its image through $\mathcal{o}$ and that the universal representation of $\mathcal{A}$ is faithful, we can state that the previous map is an isometry between $\mathcal{M}_n(A)$, and the spatial tensor product (see [KR pg. 847]) $\mathcal{A} \otimes \mathcal{M}_n(\mathbb{C})$ of the $C^*$ algebras $\mathcal{A}$ and $\mathcal{M}_n(\mathbb{C})$.

Set $\Phi^A_{nm} : \mathcal{M}_n(A) \to \mathcal{M}_m(A)$ such that for any $m \in \mathbb{N}_0$, $m \geq n$ and $A \in \mathcal{M}_n(A)$

\[
\begin{align*}
\Phi^A_{nm}(A)_{ij} &:= A_{ij}, i, j \in \{1, \ldots, n\}, \\
\Phi^A_{nm}(A)_{ij} &:= 0, i, j \in \{n+1, \ldots, m\},
\end{align*}
\]
clearly we have $\Phi^A_{nk} \circ \Phi^A_{nm} = \Phi^A_{mk}$, for all $k, m, n \in \mathbb{N}_0$ such that $k \geq m \geq n$, moreover $\Phi^A_{nm}$ is a $\ast$-isometry into its image. We call

\[
\mathcal{M}_\infty(A) := \varinjlim(\mathcal{M}_n(A), \Phi^A_{rs}),
\]
the normed inductive limit of the system $\{(\mathcal{M}_n, \Phi^A_{nm})\}_{n,m \in \mathbb{N}_0, m \geq n}$ and it is the normed $\ast$-algebra constructed as follows. Let $\mathcal{U} \subseteq \bigcup_{n \in \mathbb{N}_0} \mathcal{M}_n(A)$ and $\lambda : \mathcal{U} \to \mathbb{N}_0$ such that $a \in \mathcal{M}_{\lambda(a)}(A)$, for all $a \in \mathcal{U}$. Define $\simeq$ on $\mathcal{U}$ such that for any $a, b \in \mathcal{U}$

\[
a \simeq b \iff (\exists p \in \mathbb{N}_0)(p \geq \lambda_a, p \geq \lambda_b, \Phi^A_{\lambda_a p}(a) = \Phi^A_{\lambda_b p}(b)),
\]

\[
\]
denote by $\langle a \rangle$ the equivalence class of $a \mod \sim$. Next the following
\[ \|\langle a \rangle\|_\infty := \|a\|_{M_\infty(A)} \]
is a well-defined norm on $U/\sim$, indeed let $a, b \in U$ such that $a \sim b$, then there exists a $p \in \mathbb{N}_0$ such that $\Phi^A_{\lambda p}(a) = \Phi^A_{\lambda p}(b)$, so $\|a\|_{M_\infty(A)} = \|b\|_{M_\infty(A)}$ since $\Phi^A_{\lambda p}$ and $\Phi^A_{\lambda kp}$ are isometries. Set by abuse of language
\[
\begin{cases}
M_\infty(A) := \bigcup_{n \in \mathbb{N}_0} \mathbb{M}_n(A)/\sim, \\
f^n_n : \mathbb{M}_n(A) \to \mathbb{M}_\infty(A), a \mapsto \langle a \rangle, n \in \mathbb{N}_0.
\end{cases}
\]
We obtain
\begin{align}
\mathbb{M}_\infty(A) &= \bigcup_{n \in \mathbb{N}_0} f_n(\mathbb{M}_n(A)), \\
f_m \circ \Phi_{nm} &= f_n, m \geq n.
\end{align}
Set $0^n_A := f_n(0^n_A)$ for some $n \in \mathbb{N}_0$, where $0^n_A$ is the matrix of order $n$ at elements in $\mathcal{A}$ all equal to $0$, well-defined by the second equality in (17). Let $a, b \in \mathbb{M}_\infty(A)$, then since the second equality in (17) there exist $p \in \mathbb{N}_0$ and $a, b \in \mathbb{M}_p(\mathcal{A})$ such that $x = f_p(a)$ and $y = f_p(b)$, the following
\[ x + y := f_p(a + b), \]
\[ x \cdot y := f_p(a \cdot b), \]
\[ t \cdot x := f_p(t \cdot a), t \in \mathbb{C} \]
\[ x^* := f_p(a^*), \]
are well-defined operations making $\left\{ \mathbb{M}_\infty(A), +, \cdot, 0^\infty_A, \cdot, *, \| \cdot \|_\infty \right\}$ a (non unital) normed $\star$-algebra, denoted again by $\mathbb{M}_\infty(A)$. $\| \cdot \|_\infty$ will be called the standard norm in $\mathbb{M}_\infty(A)$. For any $n \in \mathbb{N}_0$, since $\mathcal{A}$ is identifiable with a two side ideal of $\mathcal{A}^+$, we deduce that $\mathbb{M}_n(\mathcal{A})$ and $\mathbb{M}_\infty(\mathcal{A})$ is identifiable with a two side ideal of $\mathbb{M}_n(\mathcal{A}^+)$ and $\mathbb{M}_\infty(\mathcal{A}^+)$ respectively, thus we can consider $a \equiv b \mod \mathbb{M}_\infty(\mathcal{A})$, for any $a, b \in \mathbb{M}_\infty(\mathcal{A}^+)$. Moreover for any $m \in \mathbb{N}_0$, $m \geq n$
\begin{align}
\begin{cases}
f^n_n = f^n_n^* \upharpoonright \mathbb{M}_n(\mathcal{A}), \\
\Phi^A_{nm} = \Phi^A_{nm}^* \upharpoonright \mathbb{M}_n(\mathcal{A}).
\end{cases}
\end{align}
The group $K(\mathcal{A})$. Let $n \in \mathbb{N}_0$, define $\text{diag}^A_n : \mathbb{M}_n(\mathcal{A}) \times \mathbb{M}_n(\mathcal{A}) \to \mathbb{M}_{2n}(\mathcal{A})$ such that for any $a, b \in \mathbb{M}_n(\mathcal{A})$
\[ \text{diag}^A_n(a, b) := \begin{pmatrix} a & 0_n \\ 0_n & b \end{pmatrix}. \]
Define in $\mathbb{P}(\mathcal{A})$ the following equivalences ([Bla 1] Def. 4.2.1, Prp. 4.6.3]). Let $p, q \in \mathbb{P}(\mathcal{A})$ thus
\begin{itemize}
\item **algebraic equivalence**: $p \sim q$ iff there exists $x, y \in \mathcal{A}$ such that $xy = p$ and $yx = q,$
\item **similarity**: $p \sim_s q$ iff there exists an invertible element $z$ in $\mathcal{A}$ such that $zpz^{-1} = q,$
\item **homotopy**: $p \sim_h q$ iff there exists a continuous path of projections in $\mathcal{A}$ from $p$ to $q$, i.e. a norm-continuous map $f : [0, 1] \to \mathcal{A}$ such that $f(t) \in \mathbb{P}(\mathcal{A})$, for all $t \in [0, 1]$ and $f(0) = p$ and $f(1) = q.$
\end{itemize}
Denote by $[e]$ the equivalence class of $e \mod \sim$. For any $e, f \in \mathbb{P}(\mathbb{M}_\infty(\mathcal{A}))$ it follows ([Bla 1] Ch. II, Sez. 4)]
\begin{align}
e \sim f &\iff e \sim_s f \iff e \sim_h f.
\end{align}
Set

\[
\begin{aligned}
V(A) & := \left\{ \mathcal{P}(\mathbb{M}_\infty(A)) / \sim, + \right\}, \\
\left[ e \right] + \left[ f \right] & := \left[ e' + f' \right], \quad \forall e, f \in \mathcal{P}(\mathbb{M}_\infty(A)), \\
e' \in \left[ e \right], f' \in \left[ f \right], e' \perp f', \\
\end{aligned}
\]

(\text{[Bla 1] Def. 5.1.2 and comments following it}), where \( a \perp b \) iff \( a \cdot b = 0 \) for any \( a, b \in \mathcal{P}(A) \). The operation \( + \) is independent by the choice of \( e' \) and \( f' \) which there exist as showed below. Below we convene to denote \( \Phi_n \) by \( \Phi_n^\times \) by \( f_n, \Phi_n^\times \) by \( \Phi_n \) and \( \text{diag}_n^\times \) by \( \text{diag}_n \), for any \( n, p \in \mathbb{N}_0, p \geq n \). Let \( e, f \in \mathcal{P}(\mathbb{M}_\infty(A)) \) then there exist \( m \in \mathbb{N}_0, a, b \in \mathbb{M}_m(A) \) such that \( e = f_m(a) \) and \( f = f_m(b) \). Note that \( \Phi_{m(2n)}(c) = \text{diag}_n(c, 0_m), c \in \{a, b\}, \) thus \( e = f_{2m}(\text{diag}_m(a, 0_m)) \) and \( f = f_{2m}(\text{diag}_m(b, 0_m)) \), since (18,17). Define \( g \in \mathbb{M}_{2m}(A^+) \) as

\[
\begin{aligned}
g & := \begin{pmatrix} 0_m & 1_m \\ 1_m & 0_m \end{pmatrix}, \\
z & := f_{2m}(g), \\
f' & := f_{2m}(\text{diag}_m(0_m, b)).
\end{aligned}
\]

Thus \( z = z^* = z^{-1} \), since \( g = g^* = g^{-1} \), and

\[
z f z^{-1} = zf = f_{2m}(g \text{diag}_m(b, 0_m) g) = f',
\]

moreover \( f' \in \mathcal{P}(\mathbb{M}_{2m}(A^+)) \) thus \( f' \in \left[ f \right] \) since (19). Finally \( f' \cdot e = f_{2m}(\text{diag}_m(0_m, b) \cdot \text{diag}_m(a, 0_m)) = 0_\infty \) and the third sentence of (20) follows. Define the relation \( \approx \) on \( V(A) \times V(A) \) such that for all \( p, q, r, s \in \mathcal{P}(\mathbb{M}_\infty(A)) \)

\[
([p], [q]) \approx ([r], [s]) \Leftrightarrow (\exists z \in \mathcal{P}(\mathbb{M}_\infty(A))) ([p] + [s] + [z] = [q] + [r] + [z]),
\]

\( \approx \) is an equivalence relation, let us denote by \([([p], [q]]) \) the equivalence class of \(([p], [q]) \) mod \( \approx \). So we can define for any \(([p], [q]), ([a], [b]) \in V(A) \times V(A) \) and for some \(([r], [s]) \in V(A) \times V(A) \)

\[
\begin{aligned}
K_{00}(A) & := (V(A) \times V(A)) / \approx, \\
\{(p, q) \} + \{(a, b)\} & := \{(p + [a], q + [b])\}, \\
0 & := \{(\emptyset, \emptyset)\}, \\
K_{00}(A) & := \left\{ K_{00}(A), +, 0 \right\}.
\end{aligned}
\]

It is possible to show that \( K_{00}(A) \) is a well-defined commutative group where if we denote by \( -([p], [q]) \) the inverse of \( ([p], [q]) \), we have \( -([p], [q]) = ([q], [p]) \) ([Weg, Appendix G]). Finally set

\[
\begin{aligned}
K_0(A) & := \left\{ ([p], [q]) \in K_{00}(A^+) \mid p \equiv q \mod \mathbb{M}_\infty(A) \right\}, \\
K_0(A) & := \left\{ K_0(A), + \uparrow (K_0(A) \times K_0(A)), 0 \right\},
\end{aligned}
\]

then \( K_0(A) \) is a subgroup of \( K_{00}(A^+) \) ([Bla 1, Ch. III, Sec. 5.5]). \( V \) is the object part of a functor from the category of \( *- \)algebras and \( *- \)morphisms to the category of commutative semigroups and semigroup morphisms, while \( K_{00} \) and \( K_0 \) are the object part of functors from the category of \( *- \)algebras and \( *- \)morphisms to the category of commutative
groups and group morphisms. More exactly for any couple of $*$-algebras $A$ and $B$ and $*$-morphism $\alpha : A \to B$, we have that

$$
\begin{align*}
\alpha_\# & \in \text{Mor}(V(A), V(B)), \\
\alpha_* & \in \text{Mor}(K_{00}(A), K_{00}(B)), \\
\alpha_* & \in \text{Mor}(K_0(A), K_0(B)),
\end{align*}
$$

and $(\beta \circ \alpha)_* = \beta_* \circ \alpha_*$ for any $*$-algebra $C$, $*$-morphism $\beta : B \to C$ and $\bullet \in \{#, *, \}$. 

Here for any $a \in \bigcup_{n \in \mathbb{N}_0} \mathbb{M}_n(A)$ such that $\langle a \rangle \in \text{IP}(\mathbb{M}_\infty(A))$, and $r, s \in \text{IP}(\mathbb{M}_\infty(A^+))$ such that $r \equiv s \pmod{\mathbb{M}_\infty(A)}$

$$
\begin{align*}
\alpha_\#(\langle a \rangle) &= \langle (a \circ a) \rangle, \\
\alpha_*(\langle [r], [s] \rangle) &= \left[\left( (\alpha^*)_\#([r]), (\alpha^*)_\#([s]) \right) \right],
\end{align*}
$$

while for any $p, q \in \text{IP}(\mathbb{M}_\infty(A))$

$$
\alpha_*(\langle [p], [q] \rangle) = \left[\left( (\alpha_\#([p])), (\alpha_\#([q])) \right) \right],
$$

([Weg] Prp. 6.1.3 and Prp. 6.2.4]). If $A$ is unital then $K_{00}(A)$ is isomorphic as a group to $K_0(A)$, so we shall use the convention to identify $K_0(A)$ with $K_{00}(A)$ whenever $A$ is unital, called here as in [Bla 1] the standard picture of $K_0(A)$ ([Bla 1, Prp. 5.5.5]). Let $H$ be a locally compact group, $\mu \in \mathcal{H}(H), \mathcal{A} = \langle A, H, \sigma \rangle$ and $\mathcal{B} = \langle B, H, \theta \rangle$ be two objects of $C_0(H)$ and $T$ be a $(\mathcal{A}, \mathcal{B})$–equivariant morphism. Set

$$
k_\mu(T) := (c_\mu(T)_*)^\#, $$

thus $k_\mu(T) : K_0((A \rtimes_\eta H^+) \to K_0((B \rtimes_\delta H^+) group morphism, in addition the functions

$$
\langle A, H, \eta \rangle \mapsto K_0((A \rtimes_\eta H^+) and $T \mapsto k_\mu(T)$ define a functor from $C_0(H)$ to the category of abelian groups. In particular for any locally compact subgroup $H_0$ of $H$ and $\nu \in \mathcal{H}(H_0)$ since the discussion prior (12) we have

$$
k_\nu(T) : K_0((A \rtimes_\nu H_0^+) \to K_0((B \rtimes_\delta H_0^+).
$$

2.0.3. The Chern-Connes character. Let $A$ and $B$ be unital $C^*$–algebras. $\langle \cdot, \cdot \rangle_A : K_0(A) \times H^c_\varepsilon(A) \to \mathbb{C}$ denote the pairing as defined in [Con 2, IV.7, Thm. 21]. If $T : A \to B$ is a unit preserving $*$–homomorphism, $\phi = \{\phi_{2n}\}_{n \in \mathbb{N}}$ is an entire even normalized cocycle on $B$ and $e$ an idempotent in $\mathbb{M}_\infty(A)$, then since [Con 2, IV.7, Thm. 21, Lemma 20], we obtain

$$
\langle (T), ([e]), [\phi] \rangle_B = \langle [e], T_\#([\phi]) \rangle_A.
$$

Here $T_\# : H^c_\varepsilon(B) \to H^c_\varepsilon(A)$ denotes the map $\{[\phi_{2n}]_{n \in \mathbb{N}}\} \mapsto \{[\phi_{2n} \circ T^{[2n]}]_{n \in \mathbb{N}}\}, [\phi]$ is the class in $H^c_\varepsilon(B)$ corresponding to $\phi$, and $T^{[N]} : A^N \to B^N$ such that $\text{Pr}_i \circ T^{[N]} = T \circ \text{Pr}_i$ for all $i \in \{1, \ldots N\}$. Let $\text{ch}(A, \pi, D, \Gamma) \in H^c_\varepsilon(A)$ denote the Chern-Connes character, [Con 2, IV.8, Def. 17], of the even $\theta$–summable $K$–cycle $(\mathcal{A}, \pi, D, \Gamma)$, [Con 2, IV.2, Def. 11 and IV.8, Def. 1]. It is well-known that the JLO cocycle [JLO], associated to any even $\theta$–summable $K$-cycle $(\mathcal{A}, \pi, D, \Gamma)$ via [Con 2, IV.8.4, Thm. 21], belongs to $\text{ch}(A, \pi, D, \Gamma), [\text{Con 2, IV.8.4, Thm. 22}].$ Hence one can use the JLO cocycle for computing the pairing $\langle x, \text{ch}(A, \pi, D, \Gamma) \rangle_A$. An important result concerning the Connes character is that $\langle x, \text{ch}(A, \pi, D, \Gamma) \rangle_A \in \mathbb{Z}$ for all $x \in K_0(A)$, [Con 2, IV.8, Thm. 19 and Prp. 18]. In this work whenever we refer to an even $\theta$–summable $K$–cycle $(\mathcal{A}, \pi, D, \Gamma)$, we assume that
$D \neq 0$ which is automatically true in case $\dim \mathcal{H} = \infty$, with $\mathcal{H}$ the Hilbert space where $\pi$ acts.

2.0.4. Borelian functional calculus of possibly unbounded scalar type spectral operators in Banach spaces. Let $S$ be a set, denote by $B(S)$ the Banach space of bounded complex maps on $S$ with the norm $\|f\| = \sup_{s} |f(s)|$. Let $\mathcal{B}$ be a field of subsets of $S$, a complex map defined on $S$ is $\mathcal{B}$–measurable if $f^{-1}(A) \in \mathcal{B}$, for any Borel set $A$ of $\mathbb{C}$. Denote by $TM(\mathcal{B})$ the closure in $B(S)$ of the linear subspace $\mathcal{I}(\mathcal{B})$ generated by the set $\{\chi_{\delta} | \delta \in \mathcal{B}\}$, where $\chi_{\delta}$ is the characteristic map of the set $\delta$. $TM(\mathcal{B})$ as a normed subspace of $B(S)$ is a Banach space, moreover the space of all bounded $\mathcal{B}$–measurable maps on $S$ is contained in $TM(\mathcal{B})$. Let $X$ be a Banach space, a map $F : \mathcal{B} \to X$ is defined to be additive if $F(\emptyset) = 0$ and $F(\bigcup_{i=1}^{n} \sigma_i) = \sum_{i=1}^{n} F(\sigma_i)$, for any $n \in \mathbb{N}$ and any family $\{\sigma_i\}_{i=1}^{n} \subset \mathcal{B}$ of disjoint sets, while $F$ is said to be bounded if $\sup_{\delta \in \mathcal{B}} \|F(\delta)\| < \infty$. Let $F : \mathcal{B} \to X$ be a bounded additive map, thus we can define $l^{F} : \mathcal{I}(\mathcal{B}) \to X$ such that

$$
\begin{equation}
F(\sum_{i=1}^{n} \lambda_i \chi_{\sigma_i}) := \sum_{i=1}^{n} \lambda_i F(\sigma_i),
\end{equation}
$$

for any $n \in \mathbb{N}$, any family $\{\sigma_i\}_{i=1}^{n} \subset \mathcal{B}$ of disjoint sets and $\{\lambda_i\}_{i=1}^{n} \subset \mathbb{C}$. $l^{F}$ is a well defined linear bounded operator, so admits the linear extension by continuity to the space $TM(\mathcal{B})$, we shall denote this extension again by $l^{F}$ ([DS 2, 10.1]). If $Y$ is a Banach space and $\Psi \in \mathcal{L}(X, Y)$, then

$$
\begin{equation}
\Psi \circ l^{F} = l^{\Psi \circ F}.
\end{equation}
$$

Assume now in addition that $\mathcal{B}$ is a $\sigma$–field, let $G$ be a complex Banach space, $Pr(G)$ the subset of $P \in \mathcal{L}(G)$ such that $PP = P$ and $1_{G}$ and $0_{G}$ be the identity and the zero operator on $G$ respectively, we convein to remove the index $G$ whenever it is clear by the context which space is involved. $E$ is defined to be a countably additive spectral measure in $G$ on $\mathcal{B}$ if for all $\{\alpha_n\}_{n \in \mathbb{N}} \subset \mathcal{B}$ disjoint sets and $\delta_1, \delta_2 \in \mathcal{B}$ we have

1. $E(\mathcal{B}) \subseteq Pr(G)$,
2. $E(\delta_1 \cap \delta_2) = E(\delta_1)E(\delta_2)$,
3. $E(\delta_1 \cup \delta_2) = E(\delta_1) + E(\delta_2) - E(\delta_1)E(\delta_2)$,
4. $E(S) = 1$,
5. $E(\emptyset) = 0$,
6. $E(\bigcup_{n \in \mathbb{N}} \alpha_n) = \sum_{n=1}^{\infty} E(\alpha_n)$ w.r.t. the weak operator topology on $\mathcal{L}(G)$.

It results that the convergence in (6) holds w.r.t. the strong operator topology on $G$ and $E$ is bounded ([DS 3, Cor 15.2.4]), hence $l^{E} : \mathcal{B} \to \mathcal{L}(G)$ is well defined. In case $S$ is a topological space and $\mathcal{B}$ is generated as $\sigma$–field by a basis of the topology on $S$ containing $S$, then $\mathcal{B}(S) \subseteq \mathcal{B}$ so we can set ([Ber, Ch. 2, pg. 126])

$$
\text{supp}(E) := \bigcap_{|\delta \in \mathcal{B}| E(\delta) = 1, \delta \in \text{Cl}(S)} \delta,
$$

in particular $\text{supp}(E)$ is closed. By using the same argument to prove [Sil, eq. (1.5)] we obtain

$$
\begin{equation}
E(\text{supp}(E)) = 1.
\end{equation}
$$

We are now able to define the functional calculus (shortly f.c.) associated to a countably additive spectral measure. Let
• $S$ be a set and $\mathcal{B}$ a $\sigma$–field of subsets of $S$;
• $E$ be a countably additive spectral measure in $G$ on $\mathcal{B}$;
• $f$ a $\mathcal{B}$–measurable map;
• $f_\sigma := f_{\chi_\sigma}$, for all $\sigma \in \mathcal{B}$;
• $\delta_n := [-n,n]$ and $f_n := f_{f^{-1}(\delta_n)}$, for all $n \in \mathbb{N}$.

Thus $f_n \in TM(\mathcal{B})$ being $\mathcal{B}$–measurable and bounded, hence we are able to set ([DS 3] Def. 18.2.10) and ([Sil] Def. 1.3.)

$$
\begin{align*}
\text{Dom}(f) &:= \{ x \in G \mid \lim_{n \in \mathbb{N}} I^E(f_n)x \text{ exists} \}, \\
f(E)x &:= \lim_{n \in \mathbb{N}} I^E(f_n)x, \forall x \in \text{Dom}(f(E)).
\end{align*}
$$

The map $f \mapsto f(E)$ is called the functional calculus of the spectral measure $E$ and $f(E)$ is a densely defined closed linear operator in $G$ ([DS 3] Thm. 17.2.11]). It is easy to show that $f(E)E(\sigma) = (f_{\chi_\sigma})(E)$, for all $\sigma \in \mathcal{B}$. In case $S$ is a topological space and $\mathcal{B}$ is generated as $\sigma$–field by a basis of the topology on $S$ containing $S$, then since (27) we obtain for any $f,g : S \to \mathbb{C}$ $\mathcal{B}$–measurable maps

$$
f \uparrow \text{supp}(E) = g \uparrow \text{supp}(E) \Rightarrow f(E) = g(E).
$$

A linear possibly unbounded operator $R$ in $G$ is called a scalar type spectral operator in $G$ if there exists a countably additive spectral measure $F$ in $G$ on $\mathcal{B}(\mathbb{C})$, such that $R = i(F)$, where $i$ is the identity map on $\mathbb{C}$. We call $F$ a resolution of the identity (shortly r.o.i) of $R$. There exists a unique r.o.i of $R$ ([DS 3] Cor. 18.2.14]) denoted by $E_R$, moreover ([DS 3] Lemmas 18.2.13 and 18.2.25])

$$
sp(R) = \text{supp}(E_R).
$$

Denote by $\text{Bor}(\mathbb{C})$ the set of the Borelian maps, i.e. $\mathcal{B}(\mathbb{C})$– measurable maps, thus for any $g \in \text{Bor}(\mathbb{C})$ we convene to denote the operator $g(E_R)$ by $g(R)$ and call the map $\text{Bor}(\mathbb{C}) \ni g \mapsto g(R)$ the Borelian functional calculus of $R$ ([DS 3] Def. 18.2.15]).

If $E$ is a countably additive spectral measure in $G$ on a $\sigma$–field $\mathcal{B}$ of subsets of a set $S$, then for any $\mathcal{B}$–measurable map $f$ the operator $f(E)$ is a scalar type spectral operator in $G$ whose resolution of the identity is ([DS 3] Thm. 18.2.17])

$$
E_{f(E)} = E \circ f^{-1}.
$$

2.0.5. *Joint RI constructed from $\{E_x\}_{x \in X}$*. The material, and with some adaptation the notations, of the present section 2.0.5 arises from [Ber] Ch. 2, §1.3. Let $S$ be a set, $\mathcal{B}$ a $\sigma$–field of subsets of $S$, $\mathcal{G}$ be a Hilbert space, then $E$ is defined to be a RI in $\mathcal{G}$ on $\mathcal{B}$ if it is a countably additive spectral measure in $\mathcal{G}$ on $\mathcal{B}$ such that $E(\sigma)$ is selfadjoint, for all $\sigma \in \mathcal{B}$. If $S$ is a topological space, a RI in $\mathcal{G}$ on $\mathcal{B}(S)$ is called a Borel RI in $\mathcal{G}$ on $S$. Let $X$ be a set, $\mathcal{P}_w(X)$ the class of its finite parts, $\mathcal{R} \equiv \{R_x\}_{x \in X}$, where $R_x$ is a complete separable metric space for all $x \in X$. $\{E_x\}_{x \in X}$ is defined to be a family of commuting Borel RI’s in $\mathcal{G}$ on $\mathcal{R}$, if $E_x$ is a Borel RI in $\mathcal{G}$ on $R_x$, for all $x \in X$, and $[E_y(\sigma_y),E_z(\sigma_z)] = 0$, for all $y,z \in X$ and $\sigma_y,\sigma_z \in \mathcal{B}(R_q), q \in \{y,z\}$. We convene to avoid mentioning $\mathcal{R}$, whenever $R_x = \mathbb{C}$, for all $x \in X$. For any $Y \subseteq X$ let $R_Y$ denote $\prod_{x \in Y} R_x$ and if $Q = \{x_1,\ldots,x_p\} \subseteq X$, $R_{x_1,\ldots,x_p}$ denotes $R_Q$. Set

$$
\begin{align*}
\mathcal{C}(R_X) &:= \bigcup \{ \mathcal{C}(Q,\delta) \mid Q \in \mathcal{P}_w(X), \delta \in \mathcal{B}(R_Q) \}, \\
\mathcal{C}(Q,\delta) &:= \{ \lambda \in R_X \mid \lambda \uparrow Q \in \delta \}, \forall Q \in \mathcal{P}_w(X), \delta \in \mathcal{B}(R_Q).
\end{align*}
$$
$\mathcal{C}(R_X)$ is a field of subsets of $R_X$, called the field of cylindrical sets, $\mathcal{C}_\sigma(R_X)$ denotes the $\sigma$–field generated by $\mathcal{C}(R_X)$. Let $A \in \mathcal{P}(R_X)$, we recall that $\mathcal{B}(R_A)$ is the $\sigma$–field generated by the set $\Psi(\prod_{x \in A} \mathcal{B}(R_x))$, where $\Psi: \prod_{x \in A} \mathcal{B}(R_x) \to \mathcal{P}(R_A)$ is such that $\Psi(\delta) := \prod_{x \in A} \delta_x$, for all $\delta \in \prod_{x \in A} \mathcal{B}(R_x)$. Let $\mathcal{E} := \{E_x\}_{x \in X}$ be a family of commuting Borel RI’s in $\mathcal{S}$ on $R$, then there exists a unique RI in $\mathcal{S}$ on $\mathcal{C}_\sigma(R_X)$, denoted by $\bigotimes_{x \in X} E_x$ and called the joint RI constructed from $\mathcal{E}$, such that for all $Q \in \mathcal{P}_\nu(X)$ and $\delta \in \prod_{y \in Y} \mathcal{B}(R_y)$ we have ($[\text{Ber}, \text{Ch. 2}, \text{§1.3, Thm. 1.3}]$)

$$(32) \quad \bigotimes_{x \in X} E_x(\mathcal{C}(Q, \prod_{y \in Y} \delta_y)) = \prod_{y \in Y} E_y(\delta_y),$$

where $\prod_{y \in Y} E_y(\delta_y) := E_{x_1}(\delta_{x_1}) \circ \cdots \circ E_{x_p}(\delta_{x_p})$, if $Q = \{x_1, \ldots, x_p\}$, well-defined since the commutativity property. Let $f : R_X \to S$ be a $(\mathcal{C}_\sigma(R_X), \mathcal{B})$ measurable map, then $E^f_{\mathcal{E}}$ denotes $\bigotimes_{x \in X} E_x \circ f^{-1}$ defined on $\mathcal{B}$, and $E^f_{\mathcal{E}}$ denotes $E^f_{\mathcal{E}}_{\delta}$ in case $S = R_X$ and $\mathcal{B} = \mathcal{C}_\sigma(R_X)$.

3. Equivariance

3.1. Equivariance of the functional calculus in Banach spaces. We prove in Thm. 3.11 that the Borelian functional calculus in a Banach space is equivariant under an isometric action. We apply this result to ensure in Thm. 3.13 the equivariance when the spectral measure $\mathcal{E}$ involved is the joint RI constructed by a family of commuting Borel RI’s in a Hilbert space. In Thm. 3.13 we provide a sufficient condition concerning the Borelian map $f$ to ensure the selfadjointness of the operator $f(\mathcal{E})$. These results are steps in the direction of the construction in Cor. 6.24 of the object part of a functor from $\mathcal{C}_u(H)$ to the category $\mathcal{B}(G, F, \rho)$. In this section we assume fixed a topological space $S$, a Banach space $G$ and a Hilbert space $\mathcal{S}$.

Lemma 3.1. Let $E$ be a countably spectral measure in $G$ on a $\sigma$–field $\mathcal{R}$ of subsets of a set $Z$, and $\sigma, \delta \in \mathcal{R}$ such that $\delta \supseteq \sigma$ and $E(\sigma) = 1$. Then $E(\delta) = 1$.

Proof. $Z = \sigma \cup \overline{\sigma}$ implies $1 = E(\sigma) + E(\overline{\sigma})$, so $E(\overline{\sigma}) = 0$. Moreover $\delta = \delta \cap (\sigma \cup \overline{\sigma}) = \sigma \cup (\delta \cap \overline{\sigma})$ hence $E(\delta) = E(\sigma) + E(\delta)E(\overline{\sigma}) = 1$. $\Box$

Proposition 3.2. Let $E$ be a countably additive spectral measure in $G$ on a $\sigma$–field $\mathcal{R}$ of subsets of $S$ such that $\mathcal{R}$ is generated as $\sigma$–field by a basis of the topology on $S$ containing $S$. Thus

$$supp(E) = \bigcap_{\sigma \in \mathcal{R}[E(\sigma) = 1]} \overline{\sigma}.$$ 

Proof. The inclusion $\supseteq$ follows since (27) and since $supp(E)$ is closed. Let $\sigma \in \mathcal{R}$ such that $E(\sigma) = 1$, then $E(\overline{\sigma}) = 1$ since Lemma 3.1 and $\overline{\sigma} \in \mathcal{R}$, thus $\{\delta \in \mathcal{R} \mid E(\delta) = 1, \delta \in \overline{\sigma} \subseteq \overline{\sigma} \} \supseteq \overline{\sigma}$ and the inclusion $\subseteq$ follows. $\Box$

Proposition 3.3. Let $Z$ be a set, $\mathcal{R}$ be a $\sigma$–field of subsets of $Z$, $E$ a countably additive spectral measure on $\mathcal{R}$ and $f : Z \to S$ be a $(\mathcal{R}, \mathcal{B}(S))$–measurable map. Thus

1. $supp(E \circ f^{-1}) \subseteq \bigcap_{[\sigma \in \mathcal{R}[E(\sigma) = 1]} \overline{f(\sigma)}$;
2. if $Z$ is a topological space and $\mathcal{R}$ is generated as $\sigma$–field by a basis of the topology on $Z$ containing $Z$, then $supp(E \circ f^{-1}) \subseteq \overline{f(supp(E))}$;
3. if in addition to the conditions in st. (2) $f$ is continuous, then $supp(E \circ f^{-1}) = \overline{f(supp(E))}$. 

Proof. By Prp. 3.2 we have
\[ \text{supp}(E \circ f^{-1}) = \bigcap_{(\delta \in \mathcal{B}(S) \mid E(f^{-1}(\delta)) = 1)} \overline{\delta}. \]

Let \( \sigma \in \mathcal{R} \) thus \( f(\sigma) \in \mathcal{B}(S) \) and \( f^{-1}(f(\sigma)) \supseteq \sigma \), so \( E(\sigma) = 1 \) implies \( E(f^{-1}(f(\sigma))) = 1 \) since Lemma 3.1 and st.(1) follows by (33). St.(2) follows by st.(1) and (27). Let \( f \) be continuous, set \( \mathcal{B}_f = \{ \delta \in \mathcal{B}(S) \mid E(f^{-1}(\delta)) = 1 \} \), \( \mathcal{R}_f = f^{-1}(\mathcal{B}_f) \), and \( A_\sigma = \overline{f(\sigma)} \) and \( B_\delta = \overline{\delta} \), for all \( \sigma \in \mathcal{R}_f \) and \( \delta \in \mathcal{B}_f \). Thus \( A_{f^{-1}(\delta)} \subseteq B_\delta \), since \( f(f^{-1}(\delta)) \subseteq \delta \), for all \( \delta \in \mathcal{B}_f \), therefore \( \bigcap_{(\delta \in \mathcal{B}_f)} A_{f^{-1}(\delta)} \subseteq \bigcap_{(\delta \in \mathcal{B}_f)} B_\delta \), i.e.
\[ \bigcap_{(\sigma \in \mathcal{R}_f)} f(\sigma) \subseteq \bigcap_{(\delta \in \mathcal{B}_f)} \overline{\delta}. \]

Moreover
\[ f(\text{supp}(E)) = f\left( \bigcap_{(\sigma \in \mathcal{R} \mid E(\sigma) = 1, \sigma \in \text{Cl}(Z))} \sigma \right) \subseteq \bigcap_{(\sigma \in \mathcal{R} \mid E(\sigma) = 1, \sigma \in \text{Cl}(Z))} f(\sigma) \subseteq \bigcap_{(\sigma \in \mathcal{R}_f)} f(\sigma) \subseteq \bigcap_{(\sigma \in \mathcal{R}_f)} \overline{f(\sigma)}. \]

Here the second inclusion follows by Lemma 3.1 the forth by the continuity of the map \( f \). Finally (35), (34) and (33) imply \( f(\text{supp}(E)) \subseteq \text{supp}(E \circ f^{-1}) \) and st.(3) follows by st.(2). □

Corollary 3.4. Let \( \mathcal{R} = \{ R_x \}_{x \in X} \) be a family of complete separable metric spaces, \( \mathcal{E} = \{ E_x \}_{x \in X} \) a family of commuting Borel RI’s in \( \mathcal{S} \) on \( \mathcal{R} \), and \( f : R_X \to S \) be \( (\mathcal{C}_\sigma(R_X), \mathcal{B}(S)) \)-measurable. Then for any \( Q \in \mathcal{P}_\sigma(X) \)
\[ \text{supp}(E^f_\mathcal{E}) \subseteq f(\mathcal{E}(Q, \prod_{x \in Q} \text{supp}(E_x))). \]

Proof. By Prp. 3.3(1) and (32). □

Remark 3.5. Let \( \mathcal{B} \) be a \( \sigma \)-field of subsets of a set \( Z \) and \( E \) a RI in \( \mathcal{S} \) on \( \mathcal{B} \). By following the argument in the proof of [DS 2] Thm. 12.2.6.(d)] with the help of [DS 3] Thm. 18.2.11.(i), we obtain \( h(E)^* = h^*(E) \), for all \( \mathcal{B} \)-measurable map \( h \), where \( h^* \) is the complex conjugate map of \( h \).

Lemma 3.6. Let \( \mathcal{B} \) be a \( \sigma \)-field of subsets of a set \( Z \), \( E \) a countably additive spectral measure in \( G \) on \( \mathcal{B} \) and \( \sigma \in \mathcal{B} \) such that \( E(\sigma) = 1 \). Let \( f \) and \( g \) be two \( \mathcal{B} \)-measurable maps, thus \( f \uparrow \sigma = g \uparrow \sigma \) implies \( f(E) = g(E) \).

Proof. Let \( \delta \in \mathcal{B} \), then \( (f \chi_\delta)(E) = f(E)E(\delta) \) since [DS 3] Thm. 18.2.11(f)] and the fact that \( \chi_\delta(E) = E(\delta) \). Moreover \( f \uparrow \delta = g \uparrow \delta \) implies \( f \chi_\delta = g \chi_\delta \) and the statement follows. □
The following result yields sufficient conditions to ensure the selfadjointness of the operator \( f(\mathcal{E}_\xi) \). It will be used in Cor. 6.14 via Rmk. 6.1 to ensure the exactness of the construction of the operator \( D_{\xi}^f \) (\( \mathfrak{g} \)).

**Remark 3.7.** Let \( R \ni [R_x]_{x \in X} \) be a family of complete separable metric spaces, \( \mathcal{E} \ni \{E_x\}_{x \in X} \) a family of commuting Borel RI’s in \( \mathfrak{g} \) on \( R \), and \( f : R_X \to C \) be \( \mathcal{E}_\alpha(R_X) \)–measurable, then since (31) we obtain that \( E_{\xi}^f = E_{f(\mathcal{E}_\xi)} \) i.e. \( E_{\xi}^f \) is the resolution of the identity of \( f(\mathcal{E}_\xi) \).

**Theorem 3.8** (Selfadjointness). Let \( R \ni [R_x]_{x \in X} \) be a family of complete separable metric spaces, \( \mathcal{E} \ni \{E_x\}_{x \in X} \) a family of commuting Borel RI’s in \( \mathfrak{g} \) on \( R \), and \( f : R_X \to C \) be \( \mathcal{E}_\alpha(R_X) \)–measurable. Then

1. for all \( Q \in \mathcal{P}_\omega(X) \) we have
   \[
   \text{sp}(f(\mathcal{E}_\xi)) \subseteq \overline{f(\mathcal{C}(Q, \prod_{x \in Q} \text{supp}(E_x)))},
   \]
   then
   \[
   \text{st.}(1) \text{ follows since Cor. 3.4} \text{ Since Rmk. 3.7} \text{ we have}
   \]
   \[
   \text{sp}(f(\mathcal{E}_\xi)) = \text{supp}(E_{\xi}^f),
   \]
   moreover since (36) and (27)
   \[
   f(\mathcal{E}_\xi) = i(E_{\xi}^f),
   \]
   (38)
   \[
   E_{\xi}^f(\text{sp}(f(\mathcal{E}_\xi))) = 1.
   \]
   Let \( h \) be the 0–extension to \( C \) of \( i \uparrow \text{sp}(f(\mathcal{E}_\xi)) \), then since (38), Lemma 3.6 and (37) we obtain
   \[
   f(\mathcal{E}_\xi) = h(E_{\xi}^f).
   \]
   If there exists an \( A \in \mathcal{P}_\omega(X) \) such that \( f(\mathcal{C}(A, \prod_{x \in A} \text{supp}(E_x))) \subseteq R \text{ then } f(\mathcal{E}_\xi) \text{ is selfadjoint.}
   \]

**Proposition 3.9.** Let \( U_t : \mathbb{R}^+ \to \mathcal{L}(\mathfrak{g}) \) be a strongly continuous semigroup of unitary operators on \( \mathfrak{g} \), for \( r \in \{1, 2\} \), such that \([U_t(1), U_t(2)] = 0\), for all \( s, t \in \mathbb{R}^+ \). Denote by \( A \), the infinitesimal generator of \( U_t \), then \([E_{-iA_t}(\sigma), E_{-iA_t}(\delta)] = 0\), for all \( \sigma, \delta \in \mathcal{B}(\mathbb{C}) \).

**Proof.** In this proof set \( E_r = E_{-iA_r} \), for \( r \in \{1, 2\} \). Since the Stone Thm., see for example [DS 2, Thm. 12.6.1.] and its proof, we have \( U_t(i) = f_i(B_r) \), where \( B_r = -iA_r \), and \( f_i : \mathbb{C} \ni \lambda \mapsto \exp(it\lambda) \), for all \( t \in \mathbb{R}^+ \). Hence, since (51), \( U_t(1) \) is a scalar type spectral operator whose r.o.i. is \( E_{U_t(1)} = E_r \circ f_t^{-1} \). Let \( g_t : \mathbb{C} \setminus \{0\} \ni \lambda \mapsto \frac{1}{t} \ln \lambda \), for any \( t \in \mathbb{R}^+ \setminus \{0\} \), then \( g_t \circ f_t \equiv 1d \) so
   \[
   E_r = E_{U_t(1)} \circ g_t^{-1}.
   \]
   Moreover since [DS 3] Lemma 18.2.13 and Cor. 18.2.4] we deduce for all \( \delta \in \mathcal{B}(\mathbb{C}) \) that \([U_t(1), E_{U_t(1)}(\delta)] = 0\), so by applying again [DS 3] Lemma 18.2.13 and Cor. 18.2.4] we obtain \([E_{U_t(1)}(\sigma), E_{U_t(1)}(\delta)] = 0\), for all \( \sigma, \delta \in \mathcal{B}(\mathbb{C}) \) and the statement follows by (40).
**Proposition 3.10.** Let $E$ be a countably additive spectral measure in $G$ on a $\sigma$-field of subsets of a set $Z$, $F$ a Banach space and $\alpha \in \mathcal{L}(\mathcal{L}_w(G), \mathcal{L}_w(F))$ morphism of unital algebras. Then

1. $\alpha \circ E$ is a countably additive spectral measure in $F$ on $\mathcal{B}$;
2. if $Z$ is a topological space and $\mathcal{B}$ is generated as a $\sigma$-field by a basis for the topology of $Z$ containing $Z$, then $\text{supp}(\alpha \circ E) \subseteq \text{supp}(E)$, in addition $\text{supp}(\alpha \circ E) = \text{supp}(E)$ if $\alpha^{-1}(\{1\}) = \{1\}$.

**Proof.** Trivial. \qed

Now we can state in Thm. 3.11 the equivariance of the general functional calculus, result that shall be applied to ensure in Thm. 3.13 the covariance of the f.c. for of a commuting measure in $G$ on a $\sigma$-field of resolutions of identity.

**Theorem 3.11 (Equivariance of the functional calculus).** Let $E$ be a countably additive spectral measure in $G$ on a $\sigma$-field $\mathcal{B}$ of subsets of a set $Z$, $F$ a Banach space and $U : G \rightarrow F$ be a linear isometry. Then

1. $\text{ad}(U) \circ E$ is a countably additive spectral measure in $F$ on $\mathcal{B}$;
2. if $Z$ is a topological space and $\mathcal{B}$ is generated as a $\sigma$-field by a basis for the topology of $Z$ containing $Z$, then $\text{supp}(E) = \text{supp}(\text{ad}(U) \circ E)$;
3. for any $\mathcal{B}$-measurable map $f$ we have

\[
\begin{align*}
\text{Uf(E)}U^{-1} &= f(\text{ad}(U) \circ E), \\
\text{sp(}\text{Uf(E)}\text{U}^{-1}) &= \text{sp}(f(E)), \\
\text{E}_{\text{ad}(U) \circ E} &= \text{ad}(U) \circ E \circ f^{-1}.
\end{align*}
\]

**Proof.** St. (1) & (2) follow since Prp. 3.10(1) & (2) and since $\text{ad}(U) \in \mathcal{L}(\mathcal{L}_w(G), \mathcal{L}_w(F))$ and it is a morphism of unital algebras. Let $f$ be $\mathcal{B}$-measurable and set $E^U = \text{ad}(U) \circ E$. Thus

\[
\begin{align*}
\{\text{Dom}(f(E^U))\} &= \{y \in G \mid \exists n \in \mathbb{N} \lim_{n \to \infty} E^U(f_n)y\}, \\
\text{f(E^U)}y &= \lim_{n \to \infty} E^U(f_n)y, \quad y \in \text{Dom}(f(E^U)).
\end{align*}
\]

Moreover $E^U = \text{ad}(U) \circ I$, thus since (26), thus since $\text{ad}(U)^{-1} = \text{ad}(U)^{-1}$

\begin{align*}
\text{Dom}(f(E^U)) &= \text{UDom}(f(E)),
\end{align*}

in particular $\text{Dom}(f(E^U)U) = \text{Dom}(f(E))$, and for any $y \in \text{Dom}(f(E))$

\[
\begin{align*}
\text{f(E^U)}Uy &= \lim_{n \to \infty} \text{U1}_E(f_n)y = \text{U} \lim_{n \to \infty} \text{E}(f_n)y = \text{U} f(E)y,
\end{align*}
\]

so $f(E^U)U = Uf(E)$ and the first equality in (41) follows. Let $T$ be a scalar type spectral operator in $G$, since Prp. 3.10(2) we have

\[
\text{supp}(E^U_T) = \text{supp}(E_T).
\]

Moreover by the first equality in (41)

\[
\text{U}T\text{U}^{-1} = \text{U}(E^U_T),
\]

i.e. $\text{U}T\text{U}^{-1}$ is a scalar type spectral operator in $F$ such that

\[
\text{E}^U_T = \text{E}_{\text{UTU}^{-1}},
\]

hence

\[
\text{supp}(E_{\text{UTU}^{-1}}) = \text{supp}(E_T),
\]
therefore by (30) we obtain

\[ sp(UTU^{-1}) = sp(T). \]

Hence the second equality in (41) follows with the position \( T = f(E) \), well-set since \( f(E) \) is a scalar type spectral operator by [DS3, Lemma 18.2.17]. With this position by (43) and the first equality in (41) follows \( E^U_{f(E)} = E_{f(ad(U) \circ E)} \) then the third equality in (41) follows by (31).

Corollary 3.12. Let \( F \) be a Banach space \( U : G \rightarrow F \) a linear isometry, \( T \) a scalar type spectral operator in \( G \) and \( f \) a Borelian map. Thus

1. \( UTU^{-1} \) is a scalar type spectral operator in \( F \) such that \( E_{UTU^{-1}} = ad(U) \circ E_U; \)
2. \( sp(UTU^{-1}) = sp(T); \)
3. \( f(UTU^{-1}) = UF(T)U^{-1}; \)
4. \( E_{f(UTU^{-1})} = ad(U) \circ E_{U \circ f^{-1}}. \)

Proof. St. (1) & (2) follows since (43) & (44), st. (3) & (4) follow by st. (1) and since the first and third equality in (41) applied for the position \( E = E_U. \)

Definition 1. Let \( X \) be a set and \( T = \{ T_x \}_{x \in X} \) such that \( T_x \) is a scalar type spectral operator in \( G \) for all \( x \in X \), set \( E_T := \{ E_{T_x} \}_{x \in X} \).

Now we can state the main result of this section used in Prp. 6.2 to prove the equivariance of the operator \( D^{lf}_{\mathcal{A},\mathcal{B},\mathcal{B}'} \) a fundamental step toward Cor. 6.14

Theorem 3.13 (Equivariance of the f.c. associated to \( E_T \)). Let \( X \) be a set, \( \mathcal{R} \) a Hilbert space, \( U : \mathcal{S} \rightarrow \mathcal{R} \) a unitary operator, and \( f \) a \( \mathcal{C}(\mathcal{C}^\infty) \)-measurable map. Moreover let \( T = \{ T_x \}_{x \in X} \) satisfy the following two properties: \( T_x \) is a scalar type spectral operator in \( \mathcal{S} \) for all \( x \in X \), and \( E_T \) is a family of commuting Borel RI’s in \( \mathcal{S} \). Set \( T(U) = \{ UT_x U^{-1} \}_{x \in X} \), then

1. \( E_{T(U)} = ad(U) \circ E_{E_T}; \)
2. \( f(E_{T(U)}) = UF(E_{E_T})U^{-1}; \)
3. \( E_{f(E_{T(U)})} = ad(U) \circ E_{E_{E_T}} \circ f^{-1}. \)

Proof. St. (1) follows since Cor. 3.12 applied to any \( T_x, x \in X \) and by the uniqueness in [Ben, Thm. 1.3 pg. 122]. St. (2) follows since st. (1) and the first equality in (41), while st. (3) follows since the third equality in (41) and st. (1).

3.2. Equivariance of KMS-states. In Thm. 3.18 and Cor. 3.20 we prove the equivariance of the KMS-states under the dual action of surjective equivariant morphisms. In what follows with approximate identity of a Banach \(*\)-algebra we mean a bounded order preserving approximate identity of positive elements bounded by 1.

Lemma 3.14. Let \( \mathcal{A} \) be a Banach \(*\)-algebra, \( \mathcal{B} \) be \( \mathcal{C}^\infty \)-algebra and \( T \) a \(*\)-homomorphism from \( \mathcal{A} \) to \( \mathcal{B} \) such that \( T(\mathcal{A}) \) is norm dense. If \( \{ e_d \}_{d \in D} \) is an approximate identity of \( \mathcal{A} \) then \( \{ T(e_d) \}_{d \in D} \) is an approximate identity of \( \mathcal{B} \).

Proof. Let \( \{ e_d \}_{d \in D} \) be an approximate identity of \( \mathcal{A} \). Then \( \| T(e_\alpha) \| \leq 1 \) and \( \alpha \leq \beta \Rightarrow T(e_\alpha) \leq T(e_\beta) \) for all \( \alpha, \beta \in D \) since the positivity and the continuity of \( T \) with \( \| T \| \leq 1 \), see [BR1, Prp. 2.3.1]. Next

\[ \| T(e_\alpha)T(a) - T(a) \| = \| T(e_\alpha a - a) \| \leq \| e_\alpha a - a \|. \]
for all $a \in \mathcal{A}$ and $\alpha \in D$, so for all $b \in T(\mathcal{A})$

$$(45) \quad \lim_{a \in D} ||T(e_a) b - b|| = 0.$$ 

Let $\epsilon > 0$ and $B \in \mathcal{B}$ thus there exists $b \in T(\mathcal{A})$ such that $||b - B|| \leq \frac{\epsilon}{2}$, and for all $a \in D$

$$ ||T(e_a) B - B|| \leq ||T(e_a)(B - b)|| \leq ||T(e_a) b - b|| + ||b - B||$$

$$ \leq 2||b - B|| + ||T(e_a) b - b||.$$ 

Hence $\limsup_{a \in D} ||T(e_a) B - B|| \leq \epsilon$ and $\liminf_{a \in D} ||T(e_a) B - B|| \leq \epsilon$ for all $\epsilon > 0$ since [Top 1 IV.24(14), IV.27(33), IV.23(13)] and (45), then $\lim_{a \in D} ||T(e_a) B - B|| = 0$ since [Top 1 IV.23, Cor.1] and the statement follows.$\square$

**Remark 3.15.** Let us assume the hypotheses of Lemma [3.14]. If $\mathcal{A}$ is a C$^*$–algebra $T(\mathcal{A})$ is norm closed [Pal, Thm. 9.5.12.(d)], hence in this case the norm density of $T(\mathcal{A})$ is equivalent to the surjectivity of $T$. While if $\mathcal{A}$ and $\mathcal{B}$ are unital algebras then $T$ is unit preserving since its continuity.

**Lemma 3.16.** Let $\mathcal{A}$, $\mathcal{B}$ be C$^*$–algebras and $T$ be a surjective $*$–homomorphism from $\mathcal{A}$ to $\mathcal{B}$, then $T^*(\mathcal{E}_\mathcal{B}) \subseteq \mathcal{E}_\mathcal{A}$.

**Proof.** Since $T^*(\omega)$ is positive for all $\omega \in \mathcal{E}_\mathcal{B}$ the statement follows by Lemma [3.14] and [BR 7 Prp. 2.3.11], $\square$

**Lemma 3.17.** Let $\langle \mathcal{A}, H, \eta \rangle$ and $\langle \mathcal{B}, H, \theta \rangle$ be dynamical systems and $T$ be a $(\eta, \theta)$–equivariant morphism. Then

1. $T(\mathcal{A}_\eta) \subseteq \mathcal{B}_\theta$ and the for all $z \in \mathcal{C}$ the following diagram is commutative

$$(46) \quad \xymatrix{ \mathcal{B}_\theta \ar[rd]^T \ar[rr]^-{\overline{\eta}(z)} & & \mathcal{B} \\ \mathcal{A}_\eta \ar[ru]_T \ar[rr]_-{\overline{\eta}(z)} & & \mathcal{A} }$$

2. $T \circ \delta_{\eta} \subseteq \delta_{\theta} \circ T$.

**Proof.** Let $a \in \mathcal{A}_\eta$ and $\overline{\eta}'$ denote the map $\mathcal{C} \ni z \mapsto \overline{\eta}'(z)(a) \in \mathcal{A}$, thus $\theta(t)(Ta) = T\eta(t)(a) = (T \circ \overline{\eta}'(t))(a)$ for all $t \in \mathcal{R}$. Next $T \circ \overline{\eta}'$ is analytic being composition of two analytic maps, then $T(a) \in \mathcal{B}_\theta$ and by the uniqueness of the analytic extension, the commutativity of the diagram follows. Let $b \in \text{Dom}(\delta_{\eta})$, so since the continuity of $T$

$$T(\delta_{\eta}b) = \lim_{t \to 0, t \neq 0} \frac{T\eta(t)b - Tb}{t} = \lim_{t \to 0, t \neq 0} \frac{\theta(t)Tb - Tb}{t} = \delta_{\theta}(Tb).$$ \hspace{1cm} \square$

The next result states the equivariance of the KMS–states under the dual of the action of surjective equivariant maps. It will be used via Lemma [6.21] in the proof of Thm. [6.24].

**Theorem 3.18.** Let $\langle \mathcal{A}, H, \eta \rangle$ and $\langle \mathcal{B}, H, \theta \rangle$ be dynamical systems and $T$ an $(\eta, \theta)$–equivariant morphism such that $T$ is surjective or unit preserving in case $\mathcal{A}$ and $\mathcal{B}$ are unital. Then $T^*(K^0_\beta) \subseteq K^0_\beta$ for all $\beta \in \tilde{\mathcal{R}}$. 
In this proof we denote toward the construction of the object part of the functor from consequence of Lemma 3.19. Cor. 3.20 will be used via Cor. 3.26 to prove Thm. 6.9 a step 5, Prp. 2.1, Prop. 5.3.19(3) and Lemma 3.16.

An alternative proof of Thm. 3.18 for β ∈ R follows by Lemma 3.16 by the equivariance of T and by [BR1, Prp. 5.3.7.(2)].

Lemma 3.19. Let (A, R, η) be a dynamical system and λ ∈ Aut*(A). Then

1. \( \lambda(A_η) = A_{\text{ad}(\lambda) \circ \eta} \);
2. \( \text{ad}(\lambda) \circ \eta \circ \eta = \text{ad}(\lambda) \circ \eta \);
3. \( \text{ad}(\lambda)(\delta_η) = \delta_{\text{ad}(\lambda) \circ \eta} \).

Proof. In this proof let \( \eta^λ \) denote \( \text{ad}(\lambda) \circ \eta \), which is clearly a strongly continuous one-parameter group acting on \( A \) by \(*\)-automorphisms. \( \lambda \) is \( C \)-differentiable since it is \( C \)-linear and norm continuous, therefore by the chain rule of differentiable maps and by [BR1, Prp. 2.5.21] we deduce that \( (z \mapsto \lambda \circ \eta^λ(z)(a)) \in H_s(C, A) \), for all \( a \in A_η \). Hence for all \( c \in \lambda(A_η) \)

\[
(z \mapsto \text{ad}(\lambda) \circ \eta^λ(z)(c)) \in H_s(C, A),
\]

(47)

and the inclusion \( \subset \) of st.(1) follows. If we apply this result to the system \( (A, R, \text{ad}(\lambda) \circ \eta) \) and to the \(*\)-automorphism \( \lambda^{-1} \), we deduce the remaining inclusion and st.(1) follows. St.(2) follows by (47), st.(1) and the uniqueness of the entire analytic extension of \( \eta^λ \). Let \( b \in \lambda(Dom(\delta_η)) \) thus by (3), (14) and the norm continuity of \( \lambda \) we deduce that

\[
\text{ad}(\lambda)(\delta_η)(b) = \lim_{t \to 0^+, t \neq 0} \frac{\eta^λ(t)(b) - b}{t},
\]

thus

(48)

\[
\text{ad}(\lambda)(\delta_η) \subset \delta_{\eta^λ}.
\]

By applying (48) we obtain \( \text{ad}(\lambda^{-1})(\delta_{\eta^λ}) \subset \delta_η \), which implies

\[
\delta_{\eta^λ} \subset \text{ad}(\lambda)(\delta_η),
\]

and the statement follows.

If \( (A, R, \eta) \) is a dynamical system and \( \lambda \in Aut^*(A) \) then \( \lambda^{-1} \) is \( (\text{ad}(\lambda) \circ \eta, \eta) \)-equivariant, hence by Thm. 3.18 we obtain the following result, however we prefer to show it as a consequence of Lemma 3.19. Cor. 3.20 will be used via Cor. 3.26 to prove Thm. 6.9 a step toward the construction of the object part of the functor from \( C_\mu(H) \) and \( \theta(G, F, \rho) \).

Corollary 3.20. Let \( (A, R, \eta) \) be a dynamical system, \( \lambda \in Aut^*(A) \) and \( \beta \in R \), then \( \lambda(\text{K}_\beta^η) = \text{K}_{\beta \text{ad}(\lambda) \circ \eta} \).

Proof. In this proof we denote \( \text{ad}(\lambda) \circ \eta \) by \( \eta^λ \). Let \( \beta \in R, \omega \in \text{K}_\beta^η \) and \( D_η \) be a norm dense, \( \eta \)-invariant, \(*\)-subalgebra of \( A_η \) such that \( \omega(a \eta(\omega)(b)) = \omega(b a) \), for all \( a, b \in D_η \). Hence for all \( c, d \in \lambda(D_η) \)

\[
\omega(\lambda^{-1}(\eta^λ(\eta^λ(b)))) = \lambda(\omega)(d c),
\]

(49)
thus
\begin{equation}
\lambda^*(\omega)((\mathfrak{ad}(\lambda) \circ \eta)(i\beta)(d)) = \lambda^*(\omega)(d) c),
\end{equation}

Moreover $\lambda(\mathcal{D}_\eta)$ is a norm dense, since $\lambda$ is surjective and norm continuous, $\eta^\lambda$–invariant, $\ast$–subalgebra of $\lambda(A_\eta)$, hence the inclusion $\subseteq$ of the statement follows since (49) and Lemma 3.19(1) & (2). The remaining inclusion follows by the previous one applied to the system $\left(\mathcal{A}, \mathbb{R}, \mathfrak{ad}(\lambda) \circ \eta \right)$ and to the $\ast$–automorphism $\lambda^{-1}$. Let $\psi \in K^0_\beta$ and $b \in \lambda(Dom(\delta_\eta))$. By definition $i\psi(a\delta_\eta(a)) \geq 0$, for all $a \in Dom(\delta_\eta)$, then
\[
ix^*(\psi)(b \mathfrak{ad}(\lambda)(\delta_\eta)(b)) = i\psi(\lambda^{-1}(b)\delta_\eta(\lambda^{-1}b)) \geq 0,
\]
hence the inclusion $\subseteq$ of the statement follows since Lemma 3.19(3). By applying this inclusion to to the system $\left(\mathcal{A}, \mathbb{R}, \mathfrak{ad}(\lambda) \circ \eta \right)$ and to $\lambda^{-1}$ we obtain $(\lambda^{-1})^*(K_\beta^{\mathfrak{ad}(\lambda)\circ\eta}) \subseteq K^0_\beta$ which is the remaining inclusion and the statement follows.

\section{3.3. Equivariance of representations of $\mathbb{C}^*$–crossed products.}

The main result of this section is Thm. 3.46 important for the proof of Cor. 6.14. In this section we assume fixed two locally compact topological groups $G$ and $F$, a group homomorphism $\rho : F \rightarrow Aut(G)$ such that the map $(g, f) \mapsto \rho_f(g)$ on $G \times F$ at values in $G$, is continuous, moreover let $H$ denote $G \bowtie \rho F$.

\begin{definition}
Let $\mathfrak{A} = \left(\mathcal{A}, H, \sigma \right)$ be a dynamical system and $\omega \in \mathcal{A}^*$ set
\[
F_\omega(\mathfrak{A}) := \{h \in F \mid \omega \circ \gamma_\sigma(h) = \omega\},
\]
and
\[
\widetilde{\mathfrak{A}}_\omega(\mathfrak{A}) := \{F_0 \text{ closed subgroup of } F \mid \omega \in E^F_\mathcal{A}(\gamma_\sigma \uparrow F_0)\}.
\]
We convene to remove $(\mathfrak{A})$ from $F_\omega(\mathfrak{A})$ and $\widetilde{\mathfrak{A}}_\omega(\mathfrak{A})$ whenever it is clear which dynamical system is involved.
\end{definition}

\begin{lemma}
Let $\left(\mathcal{A}, H, \sigma \right)$ be a dynamical system and $\omega \in \mathcal{A}^*$, thus $F_\omega = \max \widetilde{\mathfrak{A}}_\omega$, in particular $F_\omega = \bigcup_{F_0 \in \widetilde{\mathfrak{A}}_\omega} F_0$.
\end{lemma}
\begin{proof}
By construction $F_\omega \supseteq F_0$ for any $F_0 \in \widetilde{\mathfrak{A}}_\omega$, thus it is sufficient to show that $F_\omega \in \widetilde{\mathfrak{A}}_\omega$. Let $h \in F$ such that there exists a net $\{h_\alpha\}_{\alpha \in D}$ in $F$ for which $h = \lim_{\alpha \in D} h_\alpha$ and $\omega \circ \sigma(j_2(h_\alpha)) = \omega$ for all $\alpha \in D$. Since the $\sigma(\mathcal{A}, \mathcal{A}^*)$–continuity of $\sigma$ and the continuity of $j_2$, we have for all $A \in \mathcal{A}$
\[
\omega(\gamma(h)A) = \omega(\sigma(j_2(h))A) = \lim_{\alpha \in D} \omega(\sigma(j_2(h_\alpha)))A = \omega(A),
\]
so $F_\omega$ is closed.
\end{proof}

\begin{remark}
Since Lemma 3.21 the group $S^G_{F_\omega}$ is locally compact, hence for any dynamical system $\left(\mathcal{A}, H, \sigma \right)$ it is so also $\left(\mathcal{A}, S^G_{F_\omega}, \sigma \right)$. In particular it makes sense to consider the $\mathbb{C}^*$–crossed product $\mathcal{A} \bowtie^\mu_s S^G_{F_\omega}$ for any $\mu \in \mathcal{H}(S^G_{F_\omega})$.
\end{remark}

\begin{lemma}
Let $\left(\mathcal{A}, H, \eta \right)$ and $\left(\mathcal{B}, H, \theta \right)$ be dynamical systems, $\omega \in \mathcal{B}^*$ and $T$ a $(\eta, \theta)$–equivariant morphism such that $T(\mathcal{A})$ is $\sigma(\mathcal{B}, \mathcal{B}^*)$–dense. Then $F_\omega = F_{\mathcal{T}(\omega)}$ and $S^G_{F_\omega} = S^G_{F_{\mathcal{T}(\omega)}}$.
\end{lemma}
Proof.

\[ F_{T \omega} = \{ h \in F | \omega \circ T \circ \eta(j_2(h)) = \omega \circ T \} \]
\[ = \{ h \in F | \omega \circ \theta(j_2(h)) \circ T = \omega \circ T \} \]
\[ \supseteq \{ h \in F | \omega \circ \theta(j_2(h)) = \omega \} = F_\omega. \]

The inclusion \( \subseteq \) follows since the density hypothesis and \( \omega \circ \theta(j_2(h)), \omega \in B^\ast \). The second equality follows by the first one.

For the remaining of the present section we assume fixed a \( C^\ast \)-dynamical system \( \mathcal{A} = \langle A, H, \sigma \rangle \).

**Lemma 3.24.** If \( \omega \in E^G_A(\tau) \) and \( F_0 \in \mathcal{G}_\omega \), then \( \omega \in E^G_A(\sigma_{F_0}) \).

**Proof.** The statement follows since \( \sigma \) is a group action and since \( (g, f) = (g, 1) \cdot \rho (1, f) \), for all \( (g, f) \in G \times F \).

Since Lemma 3.21 & 3.24 we can state the following

**Corollary 3.25.** \( \omega \in E^G_A(\tau) \Rightarrow \omega \in E^G_A(\sigma_{F_\omega}) \).

**Corollary 3.26.** Let \( l, h \in H, \beta \in \mathcal{R} \) and \( \xi : \mathcal{R} \rightarrow G \) be a continuous group morphism. Then \( \sigma^\ast(l) \left( \kappa^\ast_{\beta, l} \right) = \kappa^\ast_{\beta, j_2(l)} \).

**Proof.** By Cor. 3.20 and (13).

**Lemma 3.27.** Let \( \psi \in E^G_A(\tau) \) and \( l \in H \), then

1. \( F_{\sigma^\ast(l)(\psi)} = \text{ad}(Pr_2(l))(F_\psi); \)
2. \( S^G_{F_{\sigma^\ast(l)(\psi)}} = \text{ad}(l)(S^G_{F_\psi}). \)

**Proof.** Since \( (g, h) = (1, h) \cdot \rho(h^{-1})g, 1 \) for all \( g \in G \) and \( h \in F \), we have

\[ l = j_2(Pr_2(l)) \cdot \rho j_1 \left( \rho(Pr_1(l^{-1})) Pr(l) \right) , \]

next \( \psi \in E^G_A(\tau) \) by construction, thus

\[ \sigma^\ast(l)(\psi) = \gamma^\prime(Pr_2(l))(\psi). \]

Hence \( \text{ad}(Pr_2(l))(F_\psi) \in \mathcal{G}_{\sigma^\ast(l)(\psi)}, \) so by Lemma 3.21

\[ \text{ad}(Pr_2(l))(F_\psi) \subset F_{\sigma^\ast(l)(\psi)}. \]

Now \( \sigma^\ast(l)(\psi) \in E^G_A(\tau), \) since \( j_1(G) \) is a normal subgroup of \( H \), hence (51) holds if we replace \( \psi \) by \( \sigma^\ast(l)(\psi) \) and \( l \) by \( l^{-1} \) and obtain

\[ \text{ad}(Pr(l^{-1}))(F_{\sigma^\ast(l)(\psi)}) \subset F_\psi, \]

hence \( F_{\sigma^\ast(l)(\psi)} \subset \text{ad}(Pr_2(l))(F_\psi) \) and st. (1) follows by (51), st. (2) follows by \( (g, h) = j_1(g) \cdot \rho j_2(h) \) for all \( g \in G \) and \( h \in F \), by st. (1) and since \( j_1(G) \) is a normal subgroup of \( H \).

**Proposition 3.28.** Let \( V \in \mathcal{U}(A), \psi \in E_A \) and \( \langle H, \pi, \Omega \rangle \) be a cyclic representation of \( A \) associated with \( \psi \). Then \( \langle H, \pi, \pi(V)\Omega \rangle \) is a cyclic representation of \( A \) associated with \( \text{ad}(V)^\ast(\psi) \).
Proof. $\psi \circ \text{ad}(V^{-1}) = \omega_{\pi(V)\Omega} \circ \pi$. Since $AV \subset \mathcal{A}$ and $AV^{-1} \subset \mathcal{A}$ we have $\mathcal{A} = AV^{-1}V \subset AV \subset \mathcal{A}$, so $AV = \mathcal{A}$. Thus $\pi(V)\Omega$ is cyclic for the set $\pi(\mathcal{A})$.

Definition 3. Let $\mathfrak{H} := \langle \mathcal{H}, \pi, \Omega \rangle$ be a cyclic representation of $\mathcal{A}$, $\nu$ a group morphism of $H$ into $\mathcal{U}(\mathcal{A})$, and $l \in H$. Set $\mathfrak{H}_{(\nu,l)} := \langle \mathcal{H}, \pi, \nu(l)\Omega \rangle$.

Remark 3.29. Let $\psi \in E_{\mathcal{A}}$ and $\mathfrak{H}$ be a cyclic representation of $\mathcal{A}$ associated with $\psi$. Since Prp. 3.28 if $\mathfrak{H}$ is inner implemented by $\nu$, then $\mathfrak{H}_{(\nu,l)}$ is a cyclic representation of $\mathcal{A}$ associated with $\sigma^*(l)(\psi)$.

Definition 4. Let $A$ be a nonempty set, $\mathfrak{W} : A \rightarrow E^G(\tau)$ and $\mathfrak{H} : A \rightarrow \text{Rep}_c(\mathcal{A})$ be such that $\mathfrak{H}_{_\alpha} = \langle \mathcal{H}_{_\alpha}, \pi_{_\alpha}, \Omega_{_\alpha} \rangle$ is a cyclic representation of $\mathcal{A}$ associated with $\mathfrak{W}_{_\alpha}$, for all $\alpha \in A$. Thus we can set for all $\alpha \in A$

$$U^\sigma_{\mathfrak{H}_{_\alpha}} := \mathcal{W}^G_{\mathfrak{H}_{_\alpha}}.$$ Whenever it is clear the dynamical system involved, we convene to remove the index $\sigma$, moreover we remove the index $\alpha$ anytime $A$ is a singleton.

Remark 3.30. Since Lemma 3.21 $F_{\mathfrak{W}_{_\alpha}}$ is closed in $F$ for any $\alpha \in A$, so $\langle \mathcal{A}, \mathcal{S}^G_{F_{\mathfrak{W}_{_\alpha}}}, \sigma_{F_{\mathfrak{W}_{_\alpha}}} \rangle$ is a dynamical system, hence Def. 4 is well-set by Cor. 3.25. By construction $U^\sigma_{\mathfrak{H}_{_\alpha}} : \mathcal{S}^G_{F_{\mathfrak{W}_{_\alpha}}} \rightarrow \mathcal{L}(\mathcal{H})$ such that for all $l \in \mathcal{S}^G_{F_{\mathfrak{W}_{_\alpha}}}$ and $a \in \mathcal{A}$ we have

$$U^\sigma_{\mathfrak{H}_{_\alpha}}(l) \pi_{_\alpha}(a)\Omega_{_\alpha} = \pi_{_\alpha}(\sigma(l)a)\Omega_{_\alpha}.$$ Corollary 3.31. Let $\varphi \in E^G(\tau)$ and $l \in H$, thus $\sigma^*(l)(\varphi) \in E^G(\tau)$. Moreover let $\mathfrak{H} = \langle \mathcal{H}, \pi, \Omega \rangle$ be a cyclic representation of $\mathcal{A}$ associated with $\varphi$. If $\mathfrak{H}$ is inner implemented by $\nu$, then $\mathfrak{H}_{(\nu,l)}$ is a cyclic representation of $\mathcal{A}$ associated with $\sigma^*(l)(\varphi)$ and

$$U_{\mathfrak{H}_{(\nu,l)}}(l) = (\text{ad} \circ \pi_{_\nu} \circ \nu)(l) \circ U_{\mathfrak{H}} \circ \text{ad}(l^{-1}) \upharpoonright \mathcal{S}^G_{F_{\sigma^*(\nu)(\varphi)}}.$$ Proof. The first sentence of the statement follows since $l^{-1} : G \rightarrow G$ is a normal subgroup of $H$. If $\mathfrak{H}$ is inner, then the second sentence of the statement follows by Rmk. 3.29 $\mathcal{U}_{\mathfrak{H}_{(\nu,l)}}$ is well-set, since the first two sentences of the statement and Def. 4. Moreover since Lemma 3.27

$$\text{ad}(l)\mathcal{S}^G_{F_{\varphi}} = \mathcal{S}^G_{F_{\sigma^*(\nu)(\varphi)}}.$$ If $\mathfrak{H} = \langle \mathcal{H}, \pi, \Omega \rangle$ thus by $\alpha \in S^G_{F_{\varphi}}$ and $a \in \mathcal{A}$ we have

$$U_{\mathfrak{H}_{(\nu,l)}}(\text{ad}(l)h) \pi(a)\pi(\nu(l))\Omega = \pi(\sigma(\text{ad}(l)h)a)\pi(\nu(l))\Omega$$

$$= \pi(\nu(\text{ad}(l)h)a\nu(\text{ad}(l)h^{-1})\nu(l))\Omega$$

$$= \pi(\nu(l)\nu(h)\nu(l^{-1})a\nu(\nu(l)\nu(h^{-1})))\Omega$$

$$= \pi(\nu(l))\pi(\sigma(h)(\text{ad}(\nu(l^{-1}))a))\Omega$$

$$= \pi(\nu(l))U_{\mathfrak{H}}(h)\pi(\text{ad}(\nu(l^{-1}))a)\Omega$$

$$= \pi(\nu(l))U_{\mathfrak{H}}(h)\pi(\text{ad}(\nu(l^{-1}))a)\pi(\nu(l))\Omega.$$ Moreover $\pi(\nu(l^{-1})) = \pi(\nu(l))^{-1}$, hence (52) follows by (53) and the cyclicity of the representation $\mathfrak{H}_{(\nu,l)}$. □
Remark 3.32. Under the hypotheses of Cor. 3.31 we have for all \( s \in S^G_{\text{F}_1(\psi)} \)
\[
\bigcup_{\underline{\theta}(l)}(s) = \pi(v(l)) \bigcup_{\underline{\theta}}(\text{ad}(l^{-1})s) \pi(v(l^{-1})).
\]

Remark 3.33. Let \( T \) be a locally compact space \( \mu \in \mathcal{M}(T) \), \( \mathfrak{H} \) a Hilbert space. Thus for any map \( f : T \to \mathcal{L}_i(\mathfrak{H}) \) scalarly essentially \( \mu \)-integrable we have \( \int f \, d\mu \in \mathcal{L}(\mathfrak{H}) \), since [Int 1] Ch. 6, §1, n° 4, Cor. 2].

Lemma 3.34. Let \( G_1, G_2 \) be two locally compact groups, \( \eta : G_1 \to G_2 \) be an isomorphism of topological groups and \( \mu \in \mathcal{H}(G_1) \). Then \( \eta(\mu) \) is well-set and \( \eta(\mu) \in \mathcal{H}(G_2) \).

Proof. \( \eta \) is \( \mu \)-measurable since it is continuous. Let \( E \) be a Hausdorff locally convex space and \( g \in \mathcal{C}_c(G_2, E) \), thus

\[
(54) \quad \eta^{-1}(\text{supp}(g)) = \text{supp}(g \circ \eta) \in \text{Comp}(G_1)
\]

Indeed \( \eta^{-1}(\text{supp}(g)) \) is compact \( \eta^{-1} \) being continuous, moreover \( \text{supp}(g \circ \eta) \supseteq \eta^{-1}(\text{supp}(g)) \). Let \( x \in \text{supp}(g \circ \eta) \) there exists a net \( \{x_n\}_{n \in D} \) in \( G_1 \) such that \( \lim_{n \in D} x_n = x \) and \( g(\eta(x_n)) \neq 0 \), for all \( a \in D \). But \( \lim_{n \in D} \eta(x_n) = \eta(x) \) so \( \eta(x) \in \text{supp}(g) \), i.e. \( x \in \eta^{-1}(\text{supp}(g)) \) and (54) follows. Let \( f \in \mathcal{C}_c(G_2) \), then by (54) \( f \circ \eta \in \mathcal{C}_c(G_1) \) and \( \eta(\mu) \) is well-defined. Let \( s \in G_2 \) thus \( L_s \circ \eta = \eta \circ L_{\eta^{-1}(g)} \) so

\[
L_s^*(f) \circ \eta = L_{\eta^{-1}(g)}^*(f \circ \eta),
\]

then

\[
\int L_s^*(f) \, d\eta(\mu) = \int L_{\eta^{-1}(g)}^*(f \circ \eta) \, d\mu = \int f \circ \eta \, d\mu = \int f \, d\eta(\mu),
\]

where the third equality follows by the left invariance of \( \mu \), while the first and forth ones follow by (2). \( \square \)

Remark 3.35. Let \( E \) be a Hausdorff locally convex space, \( T, S \) be two locally compact spaces, \( \mu \in \mathcal{M}(T) \), and \( \varepsilon : T \to S \) be \( \mu \)-proper. Since [Int 1] Ch. 6, §1, n° 1, pg. 4 and Ch. 5, §6, n° 2, Thm. 1] we have for any scalarly essentially \( \varepsilon(\mu) \)-integrable map \( f : S \to E \) that \( f \circ \varepsilon \) is scalarly essentially \( \mu \)-integrable and

\[
\int f \circ \varepsilon \, d\mu = \int f \, d\varepsilon(\mu).
\]

Moreover if \( E = \mathcal{L}_i(\mathfrak{H}) \) for some Hilbert space \( \mathfrak{H} \), then by Rmk. 3.33

\[
\int f \circ \varepsilon \, d\mu \in \mathcal{L}(\mathfrak{H}).
\]

Lemma 3.36. Let \( X \) be a locally compact group, \( \mu \in \mathcal{H}(X) \) and \( Y \) be a locally compact subgroup of \( X \). Then \( \mu_Y \in \mathcal{H}(Y) \) and \( \Delta_Y = \Delta_X \upharpoonright Y \).
Proof. Let \( e \in \mathbb{C}Y \) and \( s \in Y \) then \( s^{-1} \cdot e, e \cdot s^{-1} \in \mathbb{C}Y \), indeed if by absurdum \( s^{-1} \cdot e, e \cdot s^{-1} \in Y \), then \( e = s \cdot (s^{-1} \cdot e) = (e \cdot s^{-1}) \cdot s \in Y \). Let \( f \in C_c(Y) \) then

\[
\widetilde{L_s}(f) = L_s^{\ast}(f) \quad \text{and} \quad \widetilde{R_s}(f) = R_s^{\ast}(f),
\]

so

\[
\mu_Y(L_s^{\ast}(f)) = \mu(L_s^{\ast}(\tilde{f})) = \mu(\tilde{f}) = \mu_Y(f).
\]

Thus \( \mu_Y \in \mathcal{H}(Y) \). Moreover

\[
\Delta_X(s)\mu_Y(R_{s^{-1}}^{\ast}(f)) = \Delta_X(s)\mu(R_{s^{-1}}^{\ast}(\tilde{f})) = \mu(\tilde{f}) = \mu_Y(f),
\]

then since \( \mu_Y \in \mathcal{H}(Y) \) and the independence of the modular function by the Haar measure ([Wil2], Lemma 1.61)], we deduce that \( \Delta_Y = \Delta_X \upharpoonright Y \).

Lemma 3.27(2) and Lemma 3.34 allow to set the following

**Definition 5** (Haar systems). Let \( A \) be a nonempty set and \( \omega : A \to \mathbb{E}_A^G(\tau) \). We define the class of Haar system associated to \( \omega \) and \( \mathcal{U} \), denoted by \( \mathcal{H}(\omega, \mathcal{U}) \), the subset of the

\[
\mu \in \prod_{(\alpha, l) \in A \times \mathcal{H}} \mathcal{H}(\mathbb{S}^G_{F(\rho(\alpha)(\omega \lambda))}),
\]

such that for all \((\alpha, l) \in A \times \mathcal{H}\)

(55)

\[
\mu_{(\alpha, l)} = \text{ad}(l)(\mu_{(\alpha, 1)}).
\]

**Definition 6.** Let \( \nu \in \mathcal{H}(H) \) and \( \omega : A \to \mathbb{E}_A^G(\tau) \), define for all \((\alpha, l) \in A \times \mathcal{H}\)

\[
\nu_{(\alpha, l)} := \text{ad}(l)(\nu^{G}_{F(\rho(\alpha)(\omega \lambda))}).
\]

\( \nu \) will be called the Haar system generated by \( \nu \) and \( \omega \).

\( \mathcal{H}(\omega, \mathcal{U}) \) is nonempty, indeed

**Proposition 3.37.** Let \( \nu \in \mathcal{H}(H) \) and \( \omega : A \to \mathbb{E}_A^G(\tau) \), then the Haar system generated by \( \nu \) and \( \omega \) belongs to \( \mathcal{H}(\omega, \mathcal{U}) \).

**Proof.** Since Lemma 3.36, Lemma 3.27(2) and Lemma 3.34.

**Definition 7.** Let \( A \) be a nonempty set, \( \omega : A \to \mathbb{E}_A^G(\tau) \) and \( \mu \in \mathcal{H}(\omega, \mathcal{U}) \). For any \( l \in H \) and \( \alpha \in A \) set

\[
\mathcal{B}^{\omega, \alpha, l}_{\mu, \alpha, l}(\mathcal{U}) := A \rtimes_{\sigma_r} \mathbb{S}^G_{F(\rho(\alpha)(\omega \lambda))} \quad \text{and} \quad \mathcal{B}^{\omega, \alpha, l}_{\mu, \alpha, l}(\mathcal{U})^+ := (\mathcal{B}^{\omega, \alpha, l}_{\mu, \alpha, l}(\mathcal{U}))^+.
\]

We convene to remove \( l \) when it equals the unity and to remove \( \alpha \) when \( A \) is a singleton. Moreover whenever it is clear which dynamical system is involved, we convene to remove \( \mathcal{U} \) and to denote the map \( \sigma^r(l) \circ \omega \) by \( \omega \) for any \( l \in H \). Let \( \mathfrak{H} : A \to \text{Rep}_c(A) \) be such that \( \mathfrak{H}_\alpha = \left( \mathfrak{H}_\alpha, \tau_{\alpha}, \Omega_{\alpha} \right) \) is a cyclic representation of \( A \) associated with \( \omega_{\alpha} \), for all \( \alpha \in A \). Set for all \( \alpha \in A \)
Proof. Let \((\forall \phi \in \text{ad } H)\) and \((\forall \alpha \in A)\). Clearly \((\forall)\) and \((\forall)\) whenever it is well-defined. Moreover

- \(\mathcal{E}_{\alpha,l}^{\mu,\nu}(\mathcal{U})\) extends uniquely to an \(*\)-representation of \(\mathcal{B}_{\mu,\alpha,l}(\mathcal{V})\) in \(\mathcal{D}_{\alpha,l}\).
- \(\mathcal{E}_{\alpha,l}^{\mu,\nu}(\mathcal{U})\) is the \(*\)-representation of \(\mathcal{B}_{\mu,\alpha,l}(\mathcal{V})\) in \(\mathcal{D}_{\alpha,l}\) induced by \(\mathcal{E}_{\alpha,l}^{\mu,\nu}(\mathcal{U})\) for \(\alpha \in A\) and \(l \in H\).

Lemma 3.21(2) and (54) permit to set the following

**Definition 8.** Define

\[
\sigma \in \prod_{(\psi,l) \in E_A^G(\tau) \times H} \mathcal{F}\left(\mathcal{C}_c(S_{F,\psi}^G, \mathcal{A}), \mathcal{C}_c(S_{F,\sigma(\psi),\phi}^G, \mathcal{A})\right),
\]

such that

\[
\sigma_{(\psi,l)}(f) := \sigma(l) \circ f \circ \text{ad}(l^{-1}) \uparrow \mathcal{S}_{F,\sigma(\psi)}^{G},
\]

for all \((\psi,l) \in E_A^G(\tau) \times H\).

**Proposition 3.39.** \(\text{supp}(\sigma_{(\psi,l)}(f)) = \text{ad}(l)(\text{supp}(f))\), for any \(f \in \mathcal{C}_c(S_{F,\psi}^G, \mathcal{A})\) and \((\psi,l) \in E_A^G(\tau) \times H\), in particular \(\sigma_{(\psi,l)}(f)\) is well-defined.

**Proof.** Let \((\psi,l) \in E_A^G(\tau) \times H\), and \(\phi^l\) and \(H_\psi\) denote \(\sigma^*(l)(\phi)\) and \(S_{F,\psi}^G\) respectively for all \(\phi \in \mathcal{D}_{\alpha,l}\). If \(f \in \mathcal{C}_c(H_\psi, \mathcal{A})\) then \(\sigma_{(\psi,l)}(f)\) is continuous since composition of continuous maps, moreover \(\sigma(l)\) is linear thus

\[
\text{supp}(\sigma_{(\psi,l)}(f)) \subseteq \text{supp}(f \circ \text{ad}(l^{-1}) \uparrow H_\psi).
\]

Clearly \(\text{ad}(l)^{-1}(0) = (f \circ \text{ad}(l^{-1}) \uparrow H_\psi)^{-1}(0)\), then since \(\text{ad}(l)\) is injective it follows

\[
\text{ad}(l)^{-1}(0) = \bigcap (f \circ \text{ad}(l^{-1}) \uparrow H_\psi)^{-1}(0),
\]

where \(\text{ad}(l)\) is a homeomorphism, thus \(\text{supp}(f \circ \text{ad}(l^{-1}) \uparrow H_\psi) = \text{ad}(l)\text{supp}(f)\). Hence

\[
\text{supp}(\sigma_{(\psi,l)}(f)) \subseteq \text{ad}(l)\text{supp}(f),
\]

which applied to the position \(\psi^l, l^{-1}\) and \(\sigma_{(\psi,l)}(f)\), possible according the first sentence in Cor. 3.31 yields

\[
\text{supp}\left(\sigma_{(\psi,l)}^{(l^{-1})}(\sigma_{(\psi,l)}^{(l)}(f))\right) \subseteq \text{ad}(l^{-1})\text{supp}(\sigma_{(\psi,l)}(f)),
\]

for all \((\psi,l) \in E_A^G(\tau) \times H\).
i.e \( \text{ad}(l) \supp(f) \subseteq \supp(\sigma^{(\omega, l)}(f)) \), and the statement follows. \( \square \)

**Lemma 3.40.** Let \( \omega : A \to E^G_A(\tau) \) and \( \mu \) be a Haar system associated to \( \omega \) and \( \mathcal{U} \). Then for any \( (\alpha, l) \in A \times H \) and \( h \in H \) we have

1. \( \sigma^{(\omega, l)} \) is a \(*\)-isomorphism of \(*\)-algebras from \( C_c(\mathfrak{h}_{\omega l}, A) \) onto \( C_c(\mathfrak{h}_{\omega l}, A) \) continuous w.r.t. the inductive limit topologies;
2. \( \mu^h \) is a Haar system associated to \( \omega^h \) and \( \mathcal{U} \), and \( \sigma^{(\omega^h, l)} = (\sigma^{(\omega, l)})^{-1} \);
3. \( \sigma^{(\omega, l)} \) is an isometry of \( C_c(\mathfrak{h}_{\omega l}, A) \) onto \( C_c(\mathfrak{h}_{\omega l}, A) \).

Here \( \mu^h_{(\beta, u)} := \mu_{(\beta, u-h)} \) for all \( (\beta, u) \in A \times H \) and recall that \( \omega^h := \sigma^*(h) \circ \omega \).

**Proof.** Let \( (\alpha, l) \in A \times H \) and \( f, f_1, f_2 \in C_c(\mathfrak{h}_{\omega l}, A) \), then for all \( s \in C_c(\mathfrak{h}_{\omega l}, A) \)

\[
\sigma^{(\omega, l)}(f_1 \ast^{\mu_{(\omega, l)}} f_2)(s) = \sigma(l) \left( \int f_1(r) \sigma(r)(f_2(r^{-1} \text{ad}(l^{-1})(s))) \, d\mu_{(\alpha, l)}(r) \right)
= \int \sigma(l)(f_1(r)) \sigma(r)(f_2(r^{-1} \text{ad}(l^{-1})(s))) \, d\mu_{(\alpha, l)}(r)
= \int \sigma(l)(f_1(\text{ad}(l^{-1})r)) \sigma(r)(f_2(\text{ad}(l^{-1})(r^{-1}s))) \, d\mu_{(\alpha, l)}(r)
= \int \sigma^{(\omega, l)}(f_1)(r) \sigma(r) \sigma^{(\omega, l)}(f_2)(r^{-1}s) \, d\mu_{(\alpha, l)}(r)
= \left( \sigma^{(\omega, l)}(f_1) \ast^{\mu_{(\alpha, l)}} \sigma^{(\omega, l)}(f_2) \right)(s).
\]

Here the second equality follows by the continuity of \( \sigma(l) \) in \( \| \cdot \|_l \)-topology and by the fact the integration is w.r.t. the same topology. The third equality follows by the fact that \( \mu_{(\alpha, l)} = \text{ad}(l^{-1})(\mu_{(\alpha, l)}) \) and by Rmk. 3.35

\[
\sigma^{(\omega, l)}(f)^*(s) = \Delta(s^{-1}) \sigma(s)(\sigma^{(\omega, l)}(f)(s^{-1})^*)
= \Delta(s^{-1}) \sigma(sl)(f(\text{ad}(l^{-1})(s)^{-1})^*)
= \sigma(l) \Delta(\text{ad}(l^{-1})(s)^{-1}) \sigma(\text{ad}(l^{-1})(s^{-1}))
= \sigma^{(\omega, l)}(f)(s).
\]

Here \( \Delta = \Delta_H \). The first and last equality follow by \( \Delta_Y = \Delta \upharpoonright Y \) for \( Y \in \{ C_c(\mathfrak{h}_{\omega l}, A), C_c(\mathfrak{h}_{\omega l}, A) \} \), since Lemma 3.36, while the third one follows since \( \Delta \) is a group morphism. Moreover since Lemma 3.27(2) it is easy to see that \( (\sigma^{(\omega, l)})^{-1} = \sigma^{(\sigma(l)(\omega, l))^{-1}} \), and the first part of st.(1) follows. In the following we let \( H_{1l} = S^G_{\mathfrak{h}_{\omega l}} \) and \( H_{1l} = S^G_{\mathfrak{h}_{\omega l}} \). For any compact subset \( K \) of \( H_{1l} \), let \( \mathcal{C}(H_{1l}, K, A) \) be the space of the \( f \in C_c(H_{1l}, A) \) such that \( \supp(f) \subseteq K \) and \( \iota_K \) be the identity map embedding \( \mathcal{C}(H_{1l}, K, A) \) into \( C_c(H_{1l}, A) \). If we prove that for any compact \( K \) of \( H \) the map \( \sigma^{(\omega, l)} \circ \iota_K \) is continuous w.r.t. the topology of uniform convergence on \( \mathcal{C}(H, K, A) \) and the inductive limit topology on \( C_c(H_{1l}, A) \), then the second part of st.(1) will follow since [Tvs] II.27 Prpf. 5(ii). Next since Prp. 3.39

\[
\supp(\sigma^{(\omega, l)}(\iota_K(f))) \subseteq \text{ad}(l)(K), \forall f \in \mathcal{C}(H_{1l}, K, A)
\]

hence since [Wil] Rmk 1.86 it remains only to show that \( \sigma^{(\omega, l)}(\iota_K(f)) \) converges to \( 0 \) uniformly on \( H_{1l} \) for any net \( \{ f_\beta \} \) in \( \mathcal{C}(H_{1l}, K, A) \) converging uniformly on \( H_{1l} \) to \( 0 \). But this
follows since (56) and since $\sigma(l)$ is an isometry. st.(2) is easy to show. st.(3) follows since st.(1) and [Wil Cor. 2.47].

**Lemma 3.41.** Let $U$ be inner implemented by $v, \psi \in E_{\varphi}^c(\tau)$ and $\langle \mathcal{S}, \pi, U \rangle$ be a covariant representation of $\langle A, S_{\varphi}^G, \sigma_{\varphi} \rangle$. Then $\langle \mathcal{S}, \pi, U' \rangle$ is a covariant representation of $\langle A, S_{\varphi}^G, \sigma_{\varphi} \rangle$, where $U' = \text{ad}(\pi(v(l))) \circ U \circ \text{ad}(l^{-1}) \uparrow S_{\varphi}^G$.

**Proof.** Since Lemma 3.27(2) $U'$ is well-defined, moreover it is a strongly continuous unitary group action since it is so $U$, since $\text{ad}(V)$ is strongly continuous for any unitary operator $V$ on $\mathcal{S}$, and since $\text{ad}(\cdot)$ on $H$ is an isomorphism of topological groups. Let $l \in H$, $h \in S_{\varphi}^G$ and $a \in A$ then

$$U'(h)\pi(a)U'(h^{-1}) = \pi(v(l))U(\text{ad}(l^{-1})h)\pi(v(l^{-1}))\pi(a)\pi(v(l))U(\text{ad}(l^{-1})h^{-1})\pi(v(l^{-1})) = \pi(v(l))U(\text{ad}(l^{-1})h)\pi(\sigma(l^{-1})a)U(\text{ad}(l^{-1})h^{-1})\pi(v(l^{-1})) = \pi(\sigma(\text{ad}(l^{-1})(h^{-1})))\pi(v(l^{-1})) = \pi(\sigma(h)a).$$

**Remark 3.42.** Let us prove directly Lemma 3.40(3) in case $U$ is inner implemented by $v$. Let $\langle \mathcal{S}, \pi, U \rangle$ be a nondegenerate covariant representation of $\langle A, S_{\varphi}^G, \sigma_{\varphi} \rangle$ set

$$U'^{-1} = \text{ad}(\pi(v(l^{-1}))) \circ U \circ \text{ad}(l) \uparrow S_{\varphi}^G,$$

then for any $f \in C_c(S_{\varphi}^G, A)$

$$\left\| (\pi \triangleright^{H_{\varphi,1}} U)(\sigma^{(\omega, l)}(f)) \right\| = \left\| \int \pi(\sigma^{(\omega, l)}(f))(s)U(s)\, d\mu_{\omega, l}(s) \right\| = \left\| \int \pi((\sigma(l) \circ f \circ \text{ad}(l^{-1}))(s))U(s)\, d\mu_{\omega, l}(s) \right\| = \left\| \int \pi(\sigma(l) \circ f)(s)U(\text{ad}(l)s)\, d\mu_{\omega, l}(s) \right\| = \left\| \int \text{ad}(\pi(v(l)))(\pi(f(s))U'^{-1}(s))\, d\mu_{\omega, l}(s) \right\| = \left\| \text{ad}(\pi(v(l)))(\int \pi(f(s))U'^{-1}(s))\, d\mu_{\omega, l}(s) \right\| = \left\| (\pi \triangleright^{H_{\varphi,1}} U'^{-1})(f) \right\| \leq \| f \|^{H_{\varphi,1}}.$$

Here the third equality follows by Rmk. 3.35, the sixth one since $\text{ad}(V) \in L(L_2(\mathcal{S}))$, for any $V \in L(\mathcal{S})$ and the integration is w.r.t. the strong operator topology, the seventh one follows by the fact that $\text{ad}(V)$ is an isometry since $\text{ad}(V) \in Aut(B(\mathcal{S}))$ and the well-known fact
that any \(*\)-morphism between \(C^*\)-algebras is automatically continuous with norm less or equal to 1. Finally the inequality follows since an application of Lemma 3.41 stating that \(\left\langle \bar{\Delta}, \pi, U^{-1} \right\rangle\) is a nondegenerate covariant representation of \(\left\langle A, S^G_{F \omega}, \sigma_{F \omega} \right\rangle\). Therefore

\[
(58) \quad \|\sigma^{(\omega^l,j)}(f)\|_{\mu^{(\alpha,l)}} \leq \|f\|_{\mu^{(\alpha,1)}}.
\]

Let \(h, s \in H\) and \(g \in \mathcal{C}_c(S^G_{F \omega}, A)\), since Lemma 3.40(2) we can apply (58) to \(\omega^h\) and \(\mu^h\) to obtain

\[
\|\sigma^{(\omega^h,s)}(g)\|_{\mu^{(\alpha,1)}} \leq \|g\|_{\mu^{(\alpha,1)}}.
\]

Next \(\mu^l_{(a,l^{-1})} = \mu_{(a,1)}\) and \(\mu^l_{(a,l)} = \mu_{(a,l)}\) thus for any \(g \in \mathcal{C}_c(S^G_{F \sigma^l(\omega^l)}), A\)

\[
(59) \quad \|f\|_{\mu^{(\alpha,1)}} \leq \|\sigma^{(\omega^l,j)}(f)\|_{\mu^{(\alpha,1)}}.
\]

Lemma 3.40(3) follows by (58) & (59).

**Remark 3.43.** Let \(\left\langle \bar{\Delta}, \pi, V^{-1} \right\rangle\) be a nondegenerate covariant representation of \(\left\langle A, S^G_{F \omega}, \sigma_{F \omega} \right\rangle\), and set \(V := \text{ad}(\pi(\nu(l))) \circ V^{-1} \circ \text{ad}(l^{-1}) \upharpoonright S^G_{\sigma^l(\omega^l)}\), then since Lemma 3.41 \(\left\langle \bar{\Delta}, \pi, V \right\rangle\) is a nondegenerate covariant representation of \(\left\langle A, S^G_{F \sigma^l(\omega^l)}, \sigma_{F \sigma^l(\omega^l)} \right\rangle\). Hence we can reload the chain of equalities (57) in the opposite sense by replacing \(U\) by \(V\) and \(U^{-1}\) by \(V^{-1}\), for obtaining (59).

**Corollary 3.44.** Let \(\omega : A \to \mathcal{E}^G_A(\tau)\) and \(\mu\) be a Haar system associated to \(\omega\) and \(\mathfrak{U}\). Then for any \((\alpha, l) \in A \times H\) there exists a unique extension by continuity of \(\sigma^{(\alpha,l)}\) to \(B^\omega_{\alpha,l}\), denoted by the same symbol, such that \(\sigma^{(\omega^l,j)} \in \text{Isom}(B^\omega_{\alpha,l}, B^\omega_{\mu,l})\).

**Proof.** Since Lemma 3.40(3) there exists a unique extension by continuity of \(\sigma^{(\omega^l,j)}\) from \(B^\omega_{\alpha,l}\) to \(B^\omega_{\mu,l}\), while since the first sentence of Lemma 3.40(2) and Lemma 3.40(3) applied to \(\omega^l\) and to \(\mu^l\) there exists a unique extension by continuity of \(\sigma^{(\omega^l,l^{-1})}\) from \(B^\omega_{\mu,l}\) to \(B^\omega_{\mu,l}\). Since the second sentence of Lemma 3.40(2) such two extensions are one the inverse of the other. Finally the extension \(\sigma^{(\omega^l,j)}\) is a \(*\)-morphism since Lemma 3.40(1) and the norm continuity of the product and involution on \(B^\omega_{\mu,l}\).

**Corollary 3.45.** Let \(l, h \in H\) and \(\alpha \in A\), then \(\sigma^{(\sigma^l(h)(\omega^l),j)} \circ \sigma^{(\omega^l,j)} = \sigma^{(\omega^l,h^{-1}l)}\).

**Proof.** The equality holds if restricted to \(\mathcal{C}_c(S^G_{F \omega}, A)\) hence the statement follows by Cor. 3.44.

We are now able to state the third main result of Sec. 3 used in proving Cor. 6.14 were we construct the object part of a functor from \(\mathcal{C}_c(H)\) to \(\mathfrak{G}(G, F, \rho)\).

**Theorem 3.46** (Equivariance of representations). Let \(\mathfrak{U}\) be inner implemented by \(v, \omega : A \to \mathcal{E}^G_A(\tau)\), \(\mu\) be a Haar system associated to \(\omega\) and \(\mathfrak{U}\), while \(\bar{\Delta} : A \to \text{Rep}_c(A)\) such that \(\bar{\Delta}_\alpha := \left(\bar{\Delta}_\alpha, \pi_\alpha, \Omega_\alpha\right)\) is a cyclic representation of \(A\) associated to \(\omega_\alpha\), for all \(\alpha \in A\). Then for all...
\(\alpha \in A\) and \(l \in H\) the following diagram is commutative

\[
\begin{array}{ccc}
B^{\omega_{\alpha,l}}_{\mu} & \overset{g_{\delta,l}}{\longrightarrow} & \mathcal{L}(\mathcal{S}_\alpha) \\
\downarrow \sigma^{(\omega_{\alpha,l})} & & \downarrow \text{ad}(\pi_\alpha(v(l))) \\
B^{\omega_{\alpha,l}}_{\mu} & \overset{g_{\delta,l}}{\longrightarrow} & \mathcal{L}(\mathcal{S}_\alpha)
\end{array}
\]

**Proof.** By construction \(\mathcal{C}_c(S^G_{F_{\omega,\alpha}}, \mathcal{A})\) is \(\| \cdot \|_{\mu_{(a,l)}}\)–dense in \(B^{\omega_{\alpha,l}}_{\mu}\), moreover all the maps involved in the statement are norm continuous and linear, since Cor. 3.44 and the well-known fact that any \(\ast\)-morphism between \(C^*\)-algebras is automatically continuous. Thus it is sufficient to show the statement for the maps \(\sigma^{(\omega_{\alpha,l})}\) and \(g_{\delta,l}\). Let \(f \in \mathcal{C}_c(S^G_{F_{\omega,\alpha}}, \mathcal{A})\), then

\[
\left((\pi_\alpha \triangleright_{\mu_{(a,l)}} U_{\mathcal{S}_\alpha}) \circ \sigma^{(\omega_{\alpha,l})}\right)(f) = \\
\int \pi_\alpha(\sigma^{(\omega_{\alpha,l})}(f(s))) U_{\mathcal{S}_\alpha}(s) d \mu_{(a,l)}(s) = \\
\int \pi_\alpha(\sigma^{(\omega_{\alpha,l})}(f(s))) U_{\mathcal{S}_\alpha}(s) (\text{ad}(l) s) d \mu_{(a,l)}(s) = \\
\int \pi_\alpha((\sigma(l) \circ f)(s)) \pi_\alpha(v(l)) U_{\mathcal{S}_\alpha}(s) \pi_\alpha(v(l^{-1})) d \mu_{(a,l)}(s) = \\
\int \pi_\alpha(v(l)) \pi_\alpha(f(s)) U_{\mathcal{S}_\alpha}(s) \pi_\alpha(v(l^{-1})) d \mu_{(a,l)}(s) = \\
\text{ad}(\pi_\alpha(v(l)))(\int \pi_\alpha(f(s)) U_{\mathcal{S}_\alpha}(s) d \mu_{(a,l)}(s)) = \\
\left((\pi_\alpha \triangleright_{\mu_{(a,l)}} U_{\mathcal{S}_\alpha}) \circ (\pi_\alpha \triangleright_{\mu_{(a,l)}} U_{\mathcal{S}_\alpha})\right)(f).
\]

Here the second equality follows since Rmk. 3.35, the third by Cor. 3.31, the fifth by \(\text{ad}(V) \in \mathcal{L}(\mathcal{L}(\mathcal{S}_\alpha))\), for any \(V \in \mathcal{U}(\mathcal{S}_\alpha)\) and since the integration is w.r.t. the strong operator topology. \(\square\)

### 4. Extensions on the multiplier algebra

We prove in Lemma 4.2 that there is a unique canonical manner to extend a state of a \(C^*\)-algebra to a state of its multiplier algebra. Then we apply this result in Cor. 4.4 to associate a state of \(\mathcal{A}\) to a state of the crossed product \(\mathcal{A} \rtimes_{\sigma}^\mu H\), where \(\langle \mathcal{A}, H, \sigma \rangle\) is a dynamical system such that \(\mathcal{A}\) is unital and \(\mu \in \mathcal{K}(H)\). Cor. 4.4 is used via Lemma 5.17 in constructing an equivariant stability in Thm. 5.23. Lemma 4.5 & 4.11 are used to obtain Thm. 6.13 one of the auxiliary results needed to prove Cor. 6.14. Moreover we prove in a functional analytic setting the convergence formula (66), and the extension results Lemma 4.11 & 4.5. Cor. 4.9 and from Lemma 4.12 to Cor. 4.16. Finally we prove in a different way Lemma 4.7 and the convergence in (70).
In the present section we fix a $C^*$-algebra $\mathcal{B}$ and for any nonzero positive trace class operator $\rho$ on a Hilbert space $\mathcal{H}$, we use the convention of denoting with $\omega_\rho$ the state on $\mathcal{L}(\mathcal{H})$ defined by $\omega_\rho(a) := \frac{\text{tr}(a\rho)}{\text{tr}(\rho)}$, for all $a \in \mathcal{L}(\mathcal{H})$. Moreover by abuse of language for any unit vector $\Omega \in \mathcal{H}$ let $\omega_\Omega$ denote $\omega_{\rho_\Omega}$, where $\rho_\Omega$ is the projector associated to the closed subspace generated by $\Omega$. Let $\psi \in \mathcal{E}_\mathcal{B}$ then $\mathfrak{H} := \left\langle \mathfrak{H}, \mathcal{R}, \rho \right\rangle$ is called a representation of $\mathcal{B}$ relative to $\psi$, if $\left\langle \mathfrak{H}, \mathcal{R} \right\rangle$ is a nondegenerate representation of $\mathcal{B}$ and $\rho$ is a positive trace class operator on $\mathfrak{H}$ such that $\psi = \omega_\rho \circ \mathcal{R}$, define $\mathfrak{H}^- := \left\langle \mathfrak{H}, \mathcal{R}^-, \rho \right\rangle$.

**Lemma 4.2.** Let $\psi \in \mathcal{E}_\mathcal{B}$, then

1. $(\exists !) \psi^- \in \mathcal{E}_{\mathcal{M}(\mathcal{B})}(\psi^- \circ i^\mathcal{B} = \psi)$,
2. if $\mathfrak{H}$ is a representation of $\mathcal{B}$ relative to $\psi$ then $\mathfrak{H}^-$ is a representation of $\mathcal{M}(\mathcal{B})$ relative to $\psi^-$. 

**Remark 4.3.** Since Lemma 4.2(2) if $\mathfrak{H}$ is a cyclic representation of $\mathcal{B}$ associated to $\psi$ then $\mathfrak{H}^-$ is a cyclic representation of $\mathcal{M}(\mathcal{B})$ associated to $\psi^-$. 

**Proof of Lemma 4.2.** In this proof let $i$ denote $i^\mathcal{B}$. Let $\mathfrak{H} = \left\langle \mathfrak{H}, \mathcal{R}, \rho \right\rangle$ be a representation relative to $\psi$, set $\phi = \omega_\rho \circ \mathcal{R}^-$, thus $\phi$ is a state of $\mathcal{M}(\mathcal{B})$ since $\mathcal{R}^-$ is a representation of $\mathcal{M}(\mathcal{B})$ such that $\mathcal{R}^-(1) = 1$. Moreover by construction $\mathfrak{H}^-$ is a representation associated to $\phi$, while $\phi \circ i = \psi$, hence the existence part of st.(1) and st.(2) follow. Let us prove the uniqueness part of st.(1). Let $\psi^- \in \mathcal{E}_{\mathcal{M}(\mathcal{B})}$ such that $\psi^- \circ i = \psi$, and $\left\langle \mathcal{R}, \mathfrak{S}, \Omega \right\rangle$ be a cyclic associated to $\psi^-$, set

\[
\begin{cases} 
\mathcal{R}_0 = \mathcal{S}(i(\mathcal{B}))\Omega, \\
\mathcal{S}_1 : \mathcal{M}(\mathcal{B}) \ni c \mapsto \mathcal{S}(c) \uparrow \mathcal{R}_0,
\end{cases}
\]

where the closure is w.r.t. the norm topology. So $\mathcal{S}_1$ is a representation of $\mathcal{M}(\mathcal{B})$ on $\mathcal{R}_0$, since $\mathcal{S}(c)$ is norm continuous for any $c \in \mathcal{M}(\mathcal{B})$ and $i(\mathcal{B})$ is an ideal of $\mathcal{M}(\mathcal{B})$. Next $\Omega \in \mathcal{R}_0$ since a standard argument, [BR 1, p. 56], applied to the state $\psi = \omega_\Omega \circ \mathfrak{S} \circ i$ and the representation $\mathfrak{S} \circ i$. Hence $\mathcal{R}_1 := \left\langle \mathcal{R}_0, \mathcal{S}_1, \Omega \right\rangle$ is a cyclic representation of $\mathcal{M}(\mathcal{B})$ such that $\omega_\Omega \circ \mathcal{S}_1 = \psi^-$, i.e.

\[
\mathcal{R}_1 \text{ is a cyclic associated to } \psi^-.
\]

Next let

\[
\mathcal{R}_0 := \left\langle \mathcal{R}_0, \mathcal{S}_1 \circ i, \Omega \right\rangle,
\]

then $(\mathcal{S}_1 \circ i)(\mathcal{B})\Omega = \mathcal{S}_1(i(\mathcal{B}))\Omega = \mathcal{R}_0$, hence $\mathcal{R}_0$ is a cyclic representation of $\mathcal{B}$ since $\mathcal{S}_1$ is a representation of $\mathcal{M}(\mathcal{B})$. Moreover $\omega_\Omega \circ \mathcal{S}_1 \circ i = \psi^- \circ i = \psi$ the first equality coming since (60), thus

\[
\mathcal{R}_0 \text{ is a cyclic associated to } \psi.
\]

Next let $\psi^j$ be a state of $\mathcal{M}(\mathcal{B})$ such that $\psi^j \circ i = \psi$ and $\mathcal{R}^j := \left\langle \mathcal{R}^j, \mathfrak{S}^j, \Omega^j \right\rangle$ be a cyclic associated to $\psi^j$, for any $j \in \{1, 2\}$. Thus since (61) and the uniqueness modulo unitary equivalence of the GNS construction for states on a $C^*$-algebra there exists a unique unitary operator $U : \mathcal{R}_0^1 \rightarrow \mathcal{R}_0^2$ such that

\[
\begin{cases} 
U\Omega^1 = \Omega^2, \\
\mathcal{S}_1^2 \circ i = \text{ad}(U) \circ (\mathcal{S}_1^1 \circ i).
\end{cases}
\]

(62)
Corollary 4.4. Let $\psi$ such that $\psi$ is nondegenerate since (7) applied to the cyclic $R_0$, therefore since $S_0 = e$ applied to the cyclic $R_0$ for any $j \in \{1, 2\}$, we obtain for all $c \in M(\mathcal{B})$ and $b \in \mathcal{B}$

$$S_0^2(c)(S_0^2 \circ i)(b)\Omega^2 = (S_0^2 \circ i)(c)(S_0^2 \circ i)(b)\Omega^2$$

$$= (S_0^2 \circ i)(d)\Omega^2$$

$$= U(S_0^2 \circ i)(d)\Omega^1$$

$$= US_0^1(c)(S_0^2 \circ i)(b)\Omega^1$$

$$= US_0^1(c)U^{-1}(S_0^2 \circ i)(b)\Omega^2,$$

where $d = i^{-1}(ci(b))$. Next $R_0^2$ is cyclic so $S_0^2 = ad(U) \circ S_0^1$, therefore

$$\omega_{\Omega^2} \circ S_0^2 = \omega_{\Omega^1} \circ S_0^1,$$

thus $\psi_1 = \psi_2$ since (60).

**Definition 9.** Let $\psi \in E_\mathcal{B}$, then we call the canonical extension of $\psi$ to $M(\mathcal{B})$ the unique state $\psi^-$ such that $\psi^- \circ i_{\mathcal{B}} = \psi$.

**Corollary 4.4.** Let $\mathcal{A} = \langle A, H, \sigma \rangle$ be a dynamical system such that $A$ is unital, $\mu \in \mathcal{H}(H)$ and $\psi$ a state of $\mathcal{B} = A \rtimes^{\mu} H$. Thus $\psi^- \circ i_{\mathcal{A}} \in E_\mathcal{A}$ and

1. if $\langle S_0, R, \rho \rangle$ is a representation relative to $\psi$ then $\langle S_0, \pi, \rho \rangle$ is a representation relative to $\psi^- \circ i_{\mathcal{A}}$

2. if $\langle S_0, R, \Omega \rangle$ is a cyclic representation associated to $\psi$ then $\langle S_0, \pi, \Omega \rangle$ is a cyclic representation associated to $\psi^- \circ i_{\mathcal{A}}$.

Here $\langle S_0, \pi, U \rangle$ is the covariant representation of $\mathcal{A}$ associated to $\mathcal{R}$, $S_0^\pi := \pi(A)\Omega$ and $\pi^\dagger : A \ni a \mapsto \pi(a) \dagger S_0^\pi$.

**Proof.** Let $\mathcal{B} = A \rtimes^{\mu} H$. $\psi^-$ is a state of $M(\mathcal{B})$ since Lemma 4.11, while $i_A^B(1) = 1$ since $i_A^B$ is nondegenerate, thus $(\psi^- \circ i_A^B)(1) = 1$ moreover $\psi^- \circ i_A^B$ is positive hence it is a state of $A$. Let $\langle S_0, R, \rho \rangle$ be a representation relative to $\psi$ which there exists since $\psi$ is a state, thus $\pi$ is nondegenerate since $\mathcal{R}$ it is so, moreover

$$(63) \quad \psi^- \circ i_A^B = \omega_{\rho} \circ \mathcal{R}^- \circ i_A^B$$

$$= \omega_{\rho} \circ \pi,$$

where the first equality follows since Lemma 4.2(2) and the second one since (11), so st.(1) follows. Next if $\langle S_0, R, \Omega \rangle$ is a cyclic representation associated to $\psi$, then $\psi^- \circ i_A^B = \omega_{\Omega} \circ \pi$ since (63) therefore $\Omega \in S_0^\pi$ since the argument in [BR1, p.56] applied to the state $\psi^- \circ i_A^B$ and the representation $\pi$, so st.(2) follows.

The next Lemma 4.5 together Lemma 4.11 are important in showing Thm. 6.13 one of the auxiliary results used in the proof of Cor. 6.14 were we construct the object part of a functor from $C_\eta(H)$ to $\mathcal{E}(G, F, \rho)$. 
Lemma 4.5. Let \( \mathcal{A}, H, \sigma \) be a dynamical system, \( \mu \in \mathcal{H}(H) \), \( B = \mathcal{A} \rtimes_{\sigma} H \). Then for any \( a \in \mathcal{A} \) the following is a commutative diagram

\[
\begin{array}{ccc}
M(B) & \xrightarrow{\text{ev}_{\mathcal{A}}(i_B^B(i_B^B \circ i_A^B))} & M(B) \\
\downarrow i_B^B & & \downarrow i_B^B \\
B & \xrightarrow{\text{ev}_{\mathcal{A}}(i_A^B)} & B
\end{array}
\]

in particular for all \( f \in \mathcal{C}_c(H, \mathcal{A}) \) the following diagram is commutative

\[
\begin{array}{ccc}
B & \xrightarrow{i_A^B} & B \\
\downarrow i_B^B & & \downarrow i_B^B \\
B & & B
\end{array}
\]

Proof. (64) trivially implies (65), moreover all the maps involved in (64) are norm continuous and \( \mathcal{C}_c(H, \mathcal{A}) \) is norm dense in \( B \) hence (65) implies (64), thus let us proof (65). Let \( f \in \mathcal{C}_c(H, \mathcal{A}) \) and let here \( i \) and \( j \) denote \( i_B^B \) and \( i_A^B \) respectively, thus for all \( g \in \mathcal{C}_c(H, \mathcal{A}) \) and \( s \in H \)

\[
(i(a) \circ i(f))(g)(s) = i(a)(f *^\mu g)(s)
\]

\[
= a \int f(r)\sigma(r)(g(r^{-1}s))d\mu(r)
\]

\[
= \int af(r)\sigma(r)(g(r^{-1}s))d\mu(r)
\]

\[
= (i(a)(f) *^\mu g)(s) = i(i(a)(f))(g)(s).
\]

\[\Box\]

Lemma 4.6. Let \( \mathcal{A}, H, \sigma \) be a dynamical system, \( \mu \in \mathcal{H}(H) \) and \( \{E_\beta\}_{\beta \in \mathcal{C}} \) be an approximate identity of \( \mathcal{A} \). Let \( B \) denote \( \mathcal{A} \rtimes_{\sigma} H \) then \( \lim_{\beta \in \mathcal{C}} i_B^B(i(E_\beta)) = 1 \) w.r.t. the topology on \( M(M(B)) \) of simple convergence in \( i_B^B(B) \), i.e. for all \( m \in i_B^B(B) \)

\[
\lim_{\beta \in \mathcal{C}} \|i_B^B(E_\beta)m - m\|_{M(B)} = 0.
\]

Proof. Let \( \mathfrak{U} = \langle \mathcal{A}, H, \sigma \rangle \), \( j = i_B^B \), \( i = i_A^A \) and \( \| \cdot \|_{\mathcal{B}} \) be the universal norm on \( \mathcal{B} \). We have \( \text{supp}(i_A^A(E_\beta) \circ f) \subseteq \text{supp}(f) \) for any \( f \in \mathcal{C}_c(H, \mathcal{A}) \) and \( \beta \in \mathcal{C} \) since \( E_\beta \) is linear and \( \text{supp}(f) \triangleq \bigcup f^{-1}(0) \), hence

\[
\sup_{l \in \text{supp}(f)} \|(E_\beta f - f)(l)\| = \sup_{l \in \text{H}} \{(E_\beta f - f)(l) : l \in \text{supp}(i_A^A(E_\beta) \circ f) \cup \text{supp}(f)\}
\]

\[
= \sup_{l \in \text{H}} \|(E_\beta f - f)(l)\|.
\]
Since the definition of the approximate identity we deduce that \( \lim_\beta i^A(E_\beta) = 1 \) w.r.t. the topology on \( M(\mathcal{A}) \) of simple convergence in \( \mathcal{A} \), moreover \( \|i^A(E_\beta)\|_{M(\mathcal{A})} \leq 1 \) since \( i^A \) is an isometry into its range, next the unit ball of \( M(\mathcal{A}) \) is clearly a bounded subset of \( M(\mathcal{A}) \) w.r.t. the topology of simple convergence in \( \mathcal{A} \), hence it is equicontinuous according [Tvs. III.25 Thm. 1]. Therefore we can apply [Tvs. III.17 Prp. 5(2,3)] and deduce that \( \lim_\beta i^A(E_\beta) = 1 \) w.r.t. the topology on \( M(\mathcal{A}) \) of uniform convergence in compact subsets of \( \mathcal{A} \), i.e. \( \lim_\beta \sup_{a \in K} \|E_\beta a - a\|_A = 0 \), for any compact subset \( K \) of \( \mathcal{A} \). Next \( f(\mathcal{Q}) \) is a compact subset of \( \mathcal{A} \) for any \( f \in \mathcal{C}(H, \mathcal{A}) \) and any compact subset \( Q \) of \( H \), therefore

\[
\lim_\beta \sup\left( \sup_{l \in \mathcal{Q}} \left\| (i(E_\beta)(f) - f)(l) \right\|_A \right) = \lim_\beta \sup\left( \left\| E_\beta f(l) - f(l) \right\|_A \right) = 0.
\]

In particular \( \lim_\beta \sup_{l \in H} \left\| (i(E_\beta)(f) - f)(l) \right\|_A = 0 \) since (67) and \( \text{supp}(f) \) is compact, so \( \lim_\beta i(E_\beta)(f) = f \) w.r.t. the \( L^1_{\mu} \)-norm topology. Next \( \| \cdot \|_B \) is majorized by the \( L^1_{\mu} \)-norm, see [Wil. Lemma 2.27], therefore

\[
\lim_\beta \| i(E_\beta)(f) - f \|_B = 0 \quad \forall f \in \mathcal{C}(H, \mathcal{A}).
\]

Next \( i \) is an isometry onto its range thus by (68) & (64) we obtain

\[
\lim_\beta i^{M(\mathcal{B})}(i(E_\beta)) = 1,
\]

w.r.t. the topology on \( M(M(\mathcal{B})) \) of simple convergence in \( i(\mathcal{C}(H, \mathcal{A})) \). Next by construction, see [Wil. Lemma 2.27], \( \mathcal{C}(H, \mathcal{A}) \) is dense in \( \mathcal{B} \) w.r.t. the \( \| \cdot \|_B \)-topology hence \( i(\mathcal{C}(H, \mathcal{A})) \) is dense in \( i(\mathcal{B}) \), moreover \( \| i^{M(\mathcal{B})}(i(E_\beta)) \|_{M(M(\mathcal{B}))} \leq 1 \) for any \( \beta \in \mathcal{C} \) since \( i^{M(\mathcal{B})} \circ i \) is an isometry. Next the unit ball of \( M(M(\mathcal{B})) \) is clearly a bounded subset w.r.t. the topology on \( M(M(\mathcal{B})) \) of simple convergence in \( M(\mathcal{B}) \) hence it is equicontinuous according [Tvs. III.25 Thm. 1]. Therefore since (69) we can apply [Tvs. III.17 Prp. 5(1,2)] and the st.folows.

**Lemma 4.7.** Let \( \mathcal{A} \) and \( \mathcal{B} \) be \( C^* \)-algebras, \( \mathcal{X} \) be a Hilbert \( \mathcal{B} \)-module and \( \pi: \mathcal{A} \rightarrow \mathcal{L}(\mathcal{X}) \) a \( * \)-homomorphism. Then \( \pi \) maps approximate identities of \( \mathcal{A} \) into approximate identities of \( \pi(\mathcal{A}) \), if in addition \( \pi \) is nondegenerate and \( \{E_\beta\}_{\beta \in \mathcal{C}} \) is an approximate identity of \( \mathcal{A} \), then \( \lim_\mathcal{C} \pi(E_\alpha) = 1 \) w.r.t. the topology on \( \mathcal{L}(\mathcal{X}) \) of simple convergence.

**Proof.** The first sentence of the statement follows since \( \pi \) is norm continuous with norm less or equal to 1 and since it is order preserving. If \( \pi \) is nondegenerate then \( \mathcal{X} = \overline{\text{span}}\{\pi(a)x \mid a \in \mathcal{A}, x \in \mathcal{X}\} \) thus \( \pi(E_\beta) = 1 \) w.r.t. the topology on \( \mathcal{L}(\mathcal{X}) \) of simple convergence in a total subset of \( \mathcal{X} \), hence in the first sentence of the statement. Next for any \( \beta \in \mathcal{C} \) the \( \pi(E_\beta) \) lies in the unit ball of \( \mathcal{L}(\mathcal{X}) \) which is a bounded set w.r.t. the topology of simple convergence in \( \mathcal{X} \) hence equicontinuous by [Tvs. III.25 Thm. 1]. Therefore we can apply [Tvs. III.17 Prp. 5(1,2)] and deduce that \( \lim_\mathcal{C} \pi(E_\beta) = 1 \) w.r.t. the topology on \( \mathcal{L}(\mathcal{X}) \) of simple convergence in \( \mathcal{X} \).

**Remark 4.8.** We can deduce for any \( a \in \mathcal{B} \)

\[
\lim_{\beta \in \mathcal{C}} \|i_\mathcal{A}^\mathcal{B}(E_\beta)a - a\|_\mathcal{B} = 0,
\]

as an application of Lemma 4.7 to the Hilbert \( \mathcal{B} \)-module \( \mathcal{X} = \mathcal{B} \) and to the nondegenerate homomorphism \( \pi = i_\mathcal{B}^\mathcal{B} \). Thus since Lemma 4.5 and since \( i_\mathcal{B}^\mathcal{B} \) is an isometry into its range we obtain (66). Viceversa we can use Lemma 4.5 and (66) to obtain (70).
Corollary 4.9. Let $\mathcal{A}$ be a $C^*$-algebra, $(\mathcal{S}, \mathcal{R})$ a nondegenerate representation of $\mathcal{A}$ and $\rho$ a nonzero positive trace class operator on $\mathcal{S}$. Then $\omega_\rho \circ \mathcal{R} \in E_{\mathcal{L}(\mathcal{S})}$.

Proof. Let $\{E_\beta\}_{\beta \in C}$ be an approximate identity of $\mathcal{A}$ then $\|\mathcal{R}(E_\beta)\| \leq 1$ for all $\beta \in C$ and $\lim_\beta \mathcal{R}(E_\beta) = 1$ weakly, since Lemma 4.7 and the strong operator topology is stronger than the weak operator one. Therefore $\lim_\beta \mathcal{R}(E_\beta) = 1$ $\sigma$-weakly since [BR 1] Prp. 2.4.2, so
\begin{equation}
(71) \quad \lim_\beta \omega_\rho(\mathcal{R}(E_\beta)) = \omega_\rho(1) = 1,
\end{equation}

since $\omega_\rho$ is $\sigma$-weakly continuous, see for example [BR 1] Thm. 2.4.21. Next $\omega_\rho \circ \mathcal{R}$ is positive hence the statement follows since\(^{(71)}\) and [BR 1] Prp. 2.3.11. □

Remark 4.10. Under the hypotheses of Cor. 4.9 and since $Tr(\rho) \omega_\rho = Tr \circ L_\rho$ we have that $\|Tr \circ L_\rho \circ \mathcal{R}\| = Tr(\rho)$.

Lemma 4.11. Let $\mathcal{A}$ be a $C^*$-algebra then the map $M(\mathcal{A}) \ni u \mapsto t^{M(\mathcal{A})}(u) \uparrow \mathcal{K}(\mathcal{A})$ is a $*$-isomorphism of $M(\mathcal{A})$ onto $M(\mathcal{K}(\mathcal{A}))$.

Proof. $(M(\mathcal{A}), Id \uparrow \mathcal{K}(\mathcal{A}))$ is a maximal unitization of $\mathcal{K}(\mathcal{A})$ since [RW] Cor. 2.54, hence by [RW] proof. of Thm. 2.47 there exists a unique $*$-homomorphism $\hat{\Phi} : M(\mathcal{A}) \rightarrow M(\mathcal{K}(\mathcal{A}))$ such that $\hat{\Phi} \circ Id \uparrow \mathcal{K}(\mathcal{A}) = t^{\mathcal{K}(\mathcal{A})}$, moreover $\hat{\Phi}$ is a $*$-isomorphism. Next $t^{M(\mathcal{A})}(u) \uparrow \mathcal{K}(\mathcal{A}) = t^{\mathcal{K}(\mathcal{A})}(u)$ for any $u \in \mathcal{K}(\mathcal{A})$ therefore the statement follows since the above uniqueness. □

Let us end this section by proving useful results concerning the extension of suitable morphisms to multiplier algebras.

Lemma 4.12. Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be $C^*$-algebras, $X$ and $Y$ be a Hilbert $\mathcal{B}$-module and Hilbert $\mathcal{C}$-module respectively, $\beta : \mathcal{L}(X) \rightarrow \mathcal{L}(Y)$ and $\alpha : \mathcal{A} \rightarrow \mathcal{L}(X)$ be $*$-homomorphisms such that $\beta$ is nondegenerate. If $\alpha$ is surjective, or $\alpha(\mathcal{A})$ is strictly dense in $\mathcal{L}(X)$ and $\beta(\mathcal{K}(\mathcal{X})) \supseteq \mathcal{K}(Y)$, then $\beta \circ \alpha$ is nondegenerate and $(\beta \circ \alpha)^- = \beta \circ \alpha^-$.\(^{(4)}\)

Proof. Let $\delta = \beta \circ \alpha$ and $y \in Y$. Case $\alpha$ surjective. Of course $\beta(\mathcal{L}(X)) = \delta(\mathcal{A})$, so
\begin{equation}
(72) \quad \beta(\mathcal{L}(X))Y \subseteq \overline{\text{span}\{\delta(a)z \mid a \in \mathcal{A}, z \in Y\}},
\end{equation}

and the first sentence of the statement follows since $\beta$ is nondegenerate. Remaining case. $\beta$ is strictly continuous since it is norm continuous and $\beta(\mathcal{K}(\mathcal{X})) \supseteq \mathcal{K}(Y)$. Moreover $\beta$ is continuous w.r.t. the strict topology on $\mathcal{L}(X)$ and the strong operator topology on $\mathcal{L}(Y)$ since the strict topology on $\mathcal{L}(Y)$ is stronger than the $*$-strong topology ([RW] Prp. C.7) so stronger than the strong operator topology. Then since the hypothesis it follows that $\beta(\mathcal{L}(X)) \subseteq \delta(\mathcal{A})$ closure w.r.t. the strong operator topology on $\mathcal{L}(Y)$, so (72) follows. Then $\beta \circ \alpha$ is nondegenerate and $(\beta \circ \alpha)^- \circ t^A = \beta \circ \alpha$ since (4). Moreover $\alpha$ is nondegenerate in both cases, indeed $1 \in \mathcal{L}(X)$, while the strict topology on $\mathcal{L}(X)$ is stronger than the strong operator topology. So $\alpha^- \circ t^A = \alpha$, then the equality follows since the uniqueness of which in (4). □

More in general we can state

Proposition 4.13. Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be $C^*$-algebras, $X$ and $Y$ be a Hilbert $\mathcal{B}$-module and Hilbert $\mathcal{C}$-module respectively, $\beta : \mathcal{L}(X) \rightarrow \mathcal{L}(Y)$ and $\alpha : \mathcal{A} \rightarrow \mathcal{L}(X)$ be $*$-homomorphisms. If $1_Y$ belongs to the strong operator closure of the set $(\beta \circ \alpha)(\mathcal{A})$ and $\alpha$ is nondegenerate, then $\beta \circ \alpha$ is nondegenerate and $(\beta \circ \alpha)^- = \beta \circ \alpha^-$.\(^{(4)}\)
Proof. \(\alpha^- \circ i^A = \alpha\) since (4), while since the hypothesis there exists a net \(\{a_i\}\) in \(A\) such that \(y = \lim (\beta \circ \alpha(a_i)y)\) for all \(y \in Y\), thus \(\beta \circ \alpha\) is nondegenerate. Therefore \((\beta \circ \alpha)^- \circ i^A = \beta \circ \alpha\) and the equality in the statement follows since the uniqueness of which in (4). □

Remark 4.14. Under the notations of Prp. 4.13 we obtain the same statement if \(\alpha\) is surjective and \(\beta\) is nondegenerate, indeed in such a case \(\beta \circ \alpha\) is nondegenerate. Clearly we obtain the same statement if we only require \(\alpha\) and \(\beta \circ \alpha\) to be nondegenerate.

Corollary 4.15. Let \(A, B\) and \(\mathcal{C}\) be \(C^*\)-algebras, \(Y\) be a Hilbert \(\mathcal{C}\)-module \(\beta : B \to \mathcal{L}(Y)\) and \(\alpha : A \to B\) be \(*\)-homomorphisms. If \(1_Y\) belongs to the strong operator closure of the set \((\beta \circ \alpha)(A)\) and \(\alpha\) is surjective, then \(\beta \circ \alpha\) is nondegenerate and \((\beta \circ \alpha)^- = \beta^- \circ (i^B \circ \alpha)^-\).

Proof. Since \(i^B\) is nondegenerate and \(\alpha\) is surjective then \((i^B \circ \alpha)^-\) is nondegenerate, moreover \(\beta^- \circ i^B \circ \alpha = \beta \circ \alpha\). Hence we can apply Prp. 4.13 to the maps \(\beta^-\) and \((i^B \circ \alpha)^-\), and the statement follows.

Corollary 4.16. Let \(A, B\) and \(\mathcal{C}\) be \(C^*\)-algebras, \(Y\) be a Hilbert \(\mathcal{C}\)-module \(\beta : B \to \mathcal{L}(Y)\) and \(\alpha : A \to B\) be \(*\)-homomorphisms. If \(\beta \circ \alpha\) is nondegenerate and \(\alpha\) is surjective then and \((\beta \circ \alpha)^- = \beta^- \circ (i^B \circ \alpha)^-\).

Proof. Since \(i^B\) is nondegenerate and \(\alpha\) is surjective then \((i^B \circ \alpha)^-\) is nondegenerate, moreover \(\beta^- \circ i^B \circ \alpha = \beta \circ \alpha\). Thus the statement follows since Rmk. 4.14. □

5. The category of thermal systems

In Def. 11 we introduce the category \(\mathcal{C}(G,F,\rho)\), and in Def. 35 we propose a physical interpretation of the data encoded in any object of the category \(\mathcal{C}(G,F,\rho)\). Then we introduce in Def. 31 the fundamental concept of \(\mathcal{C}\)-equivariant stability for a given category \(\mathcal{C}\), which is the couple of an equivariant stability defined in Def. 30 and a functor from \(\mathcal{C}\) to \(\mathcal{C}(G,F,\rho)\). An equivariant stability essentially consists in a couple of maps \(m\) and \(\mathcal{V}\) defined on subsets of \(\text{Obj}(\mathcal{C}(G,F,\rho))\) such that \(m\) and \(\mathcal{V}\) are both \(H\)-equivariant (91) & (93) and \(m\) also \(\mathcal{C}(G,F,\rho)\)-equivariant (92), where \(g(f)\) is the graph of any map \(f\). Then we describe in Prp. 5.28 & 5.29 the physical properties of a \(\mathcal{C}\)-equivariant stability, finally in Thm. 5.23 we construct an equivariant stability, after having proved the important equality (97). We anticipate that the main result of this work is the construction in Thm. 6.25 of a \(C_u(H)\)-equivariant stability, which will be used to address in a precise setting the concept, and to describe the properties of the nucleon phase, in particular the universality of the Terrell law stated in Thm. 8.9. Thm. 8.9 justifies from the physical point of view the introduction of the category \(\mathcal{C}(G,F,\rho)\), while from the mathematical point of view a justification of this category follows since Thm. 7.20 stating the remarkable existence of nontrivial natural transformations between suitable functors from subcategories of the opposite of the category of \(C^*\)-dynamical systems and equivariant morphisms, to the category \(\text{Fct}(H,\text{Set})\). In this section we assume fixed two locally compact topological groups \(G\) and \(F\), a group homomorphism \(\rho : F \to \text{Aut}(G)\) such that the map \((g,f) \mapsto \rho_f(g)\) on \(G \times F\) at values in \(G\), is continuous, moreover let \(H\) denote \(G \rtimes \rho F\).

5.1. The category \(\mathcal{C}(G,F,\rho)\).

Definition 10 (The category of dynamical systems). For any locally compact group \(V\) we define the category \(\mathcal{C}(V)\) whose object class is the class of the dynamical systems \((A,V,\eta)\) such
that \( \mathcal{A} \) is unital, while for any \( \mathcal{A}, \mathcal{B} \in \text{Obj}(\mathbb{C}(V)) \) we define \( \text{Mor}_{\mathbb{C}(V)}(\mathcal{A}, \mathcal{B}) \) the set of the surjective \((\mathcal{A}, \mathcal{B})\)-equivariant morphisms with law of composition the map composition and if \( \mathcal{A} = \mathcal{B} \) the identity map as the unity. Let \( \mathcal{C}_u(V) \) denote the subcategory of \( \mathbb{C}(V) \) whose object class is the class of the dynamical systems \( \mathcal{A} = (\mathcal{A}, V, \eta) \) such that \( \mathcal{A} \) is a von Neumann algebra in its canonical standard form, and the class of morphisms is the one inherited by \( \mathbb{C}(V) \). Let \( \nu^u \) or \( \nu^v \) denote the unique group morphism of \( H \) into \( \mathcal{U}(\mathcal{A}) \) unitarily implementing and associated to \( \eta \) and to the canonical standard form of \( \mathcal{A} \) according \cite{Tak} Thm 9.1.15.

**Definition 11** (The category of thermal systems). Define \( \text{Obj}(\mathcal{B}(G, F, \rho)) \) the set of the \( \mathcal{S} = \langle \mathbf{I}, l, \beta_c, P, a, e, \varphi, A, \psi, b, m, \varepsilon \rangle \) such that

- \( \mathbf{I} \) is a set;
- \( l : \mathbf{I} \rightarrow \text{Set}; \)
- \( \beta_c \in \prod_{c \in \mathbf{I}} \mathcal{E}_c; \)
- \( P \in \prod_{c \in \mathbf{I}} \mathcal{P}(I^c); \)
- \( a \in \prod_{c \in \mathbf{I}} \prod_{a \in I^c} \mathcal{C}(H); \)
- \( e \in \prod_{c \in \mathbf{I}} \prod_{a \in I^c} \mathcal{C}(\mathbb{R}); \)
- \( \varphi \in \prod_{c \in \mathbf{I}} \prod_{a \in I^c} \mathcal{E}_{A^c}; \)
- \( A \in \text{Ab} \)
- \( \psi : H \rightarrow \text{Aut}_{\text{Ab}}(A); \)
- \( b : H \rightarrow \text{Aut}(\mathbf{I}); \)
- \( m \in \text{Mor}_{\text{Ab}}(A, \prod_{c \in \mathbf{I}} \mathcal{E}^{\rho_c}); \)

Here \( a_{\alpha}^\mathcal{I} = \langle A_{\alpha}^\mathcal{I}, H, \eta_{\alpha} \rangle \), in addition let \( F_{\varphi_{\alpha}^\mathcal{I}} \) denote \( F_{\varphi_{\alpha}^\mathcal{I}}(a_{\alpha}^\mathcal{I}) \) for all \( \mathbf{I} \in \mathbf{I} \) and \( \alpha \in I^\mathcal{I} \), and let \( \mathbf{I}^l \) denote \( b(l)(\mathbf{I}) \) for all \( l \in H \). Then we require for all \( \mathbf{I} \in \mathbf{I} \) and \( \alpha \in I^\mathcal{I} \)

\[
\varphi_{\alpha}^\mathcal{I} \in \mathcal{E}_{A^c}(\tau_{n_{\alpha}^\mathcal{I}}),
\]

that \( |I^\mathcal{I}| = |I^\mathcal{I}^l| \) and

\[
\beta_{c}^{\mathcal{I}^l} = \beta_{c}^{\mathcal{I}}, \quad \text{ad}(\text{Pr}(l))(F_{\varphi_{\alpha}^\mathcal{I}}) = F_{\varphi_{\alpha}^\mathcal{I}^l}.
\]

Moreover

**\(H-\text{actions.} \)** \( b \) and \( \psi \) are group morphisms.

**Phase transition via dynamical symmetry breaking.** For all \( \mathbf{I} \in \mathbf{I} \)

\[
P_{\mathbf{I}} = \{ \alpha \in I^\mathcal{I} | F_{\varphi_{\alpha}^\mathcal{I}} \supseteq F_{\varphi_{\alpha}^\mathcal{I}^l} \}.
\]

**Equivariance.** For any \( l \in H \) and \( f \in A^\mathcal{I} \)

\[
ev_l(m \circ \psi(l)) = \ev_l(m) \circ b(l^{-1}).
\]

**Thermal nature.** For any \( \mathbf{I} \in \mathbf{I} \) and \( \alpha \in I^\mathcal{I} \) we have \( \varepsilon_{\alpha}^\mathcal{I} = \langle A_{\alpha}^\mathcal{I}, \mathbb{R}, \varepsilon_{\alpha}^\mathcal{I} \rangle \) and

\[
\varphi_{\alpha}^\mathcal{I} \in \varepsilon_{\alpha}^\mathcal{I} - \text{KMS}.
\]

**Integrality.** \( m(f)(\mathbf{I}) \) is a \( \mathbb{Z} \)-valued map, for all \( f, \mathbf{I} \) \( A \times \mathbf{I} \).

**Equivariant stability.** \( \varepsilon = \langle \mu, u, \mathcal{H}, D, \Gamma, v, w, \beta \rangle \) such that for all \( \mathbf{I} \in \mathbf{I} \) and \( \alpha \in P_{\mathbf{I}}, \ recount the next line.

\footnote{\( (75) \) is well-set since Rmk. \ref{Rmk1}.}
(1) $\mu^x_\alpha \in \mathcal{H}(S^x_\alpha(S))$,
(2) $u^x_\alpha \in \text{Mor}_{\text{Ab}}(A, K_0(B^x_\alpha(S)^+))$,
(3) $\mathfrak{H}^x_\alpha$ is a cyclic representation of $\mathcal{A}^x_\alpha$ associated to $\varphi^x_\alpha$,
(4) $\left( R^x_\alpha(S), D^x_\alpha, \Gamma^x_\alpha \right)$ is an even $\theta$–summable $K$–cycle,
(5) for all $l \in H$
   a) $v^x_\alpha(l) \in \mathcal{L}(\mathfrak{H}^x_\alpha, \mathfrak{H}^x_\alpha')$ unitary,
   b) $w^x_\alpha(l) \in \text{Hom}^{*}(\mathcal{B}^x_\alpha(S), \mathcal{B}^x_\alpha(S))$,
   c) $z^x_\alpha(l) \in \text{Hom}^{*}(\mathcal{A}^x_\alpha, \mathcal{A}^x_\alpha)$
   d) for all $h \in H$ and $g \in \{ v, w, z \}$

\[
\eta^x_\alpha(h) \circ z^x_\alpha(l) = z^x_\alpha(l) \circ \eta^x_\alpha(h),
\]
(77)
\[
g^x_\alpha(h \cdot l) = g^x_\alpha(h) \circ g^x_\alpha(l)
\]
\[
g^x_\alpha(1) = \text{Id}.
\]
(78)
(79, 80, 81) are commutative diagrams

while the following (79, 80, 81) are commutative diagrams

(79)

(80)
(6) for all \( f \in A \) we have

\[
m(f)(\mathcal{I}, \alpha) = \left( \mu^\mathcal{I}_\alpha(f), \chi\left( \left( \mathcal{R}_\alpha^\mathcal{I}(\mathcal{G}), \mathcal{D}_\alpha^\mathcal{I}, \Gamma_\alpha^\mathcal{I} \right) \right) \right)_{(\mathcal{G}, \mathcal{I}, \alpha)}.
\]

Here for any \( \mathcal{I} \in \mathcal{I} \) and \( \alpha \in \mathbb{P}^\mathcal{I} \) we set

1. \( S^\mathcal{I}_\alpha(\mathcal{G}) := \mathcal{S}_\mathcal{G}^\mathcal{I}_\alpha(a^\mathcal{I}_\alpha), \)
2. \( \mathcal{B}^\mathcal{I}_\alpha(\mathcal{G}) := \mathcal{B}^\mathcal{I}_\alpha(a^\mathcal{I}_\alpha), \)
3. \( i^\mathcal{I}_\alpha := i_{\mathcal{B}^\mathcal{I}_\alpha(a^\mathcal{I}_\alpha)}, \)
4. \( j^\mathcal{I}_\alpha := j_{\mathcal{B}^\mathcal{I}_\alpha(a^\mathcal{I}_\alpha)}, \)
5. \( \mathcal{R}_\alpha^\mathcal{I}(\mathcal{G}) := \mathcal{R}_\alpha^\mathcal{I}(a^\mathcal{I}_\alpha), \)
6. \( \mathcal{R}_\alpha^\mathcal{I}(\mathcal{G}) := \mathcal{R}_\alpha^\mathcal{I}(a^\mathcal{I}_\alpha), \)
7. \( \left\langle \cdot, \cdot \right\rangle_{(\mathcal{G}, \mathcal{I}, \alpha)} = \left\langle \cdot, \cdot \right\rangle_{(\mathcal{B}^\mathcal{I}_\alpha(a^\mathcal{I}_\alpha))}^\mathcal{I}. \)

We call any object \( \mathcal{G} \) of \( \mathcal{G}(H) \) a thermal system of \( H \)-invariants, or a \( H \)-thermal system, or simply a thermal system when it is clear the group \( H \) involved. Moreover we call \( \mathcal{I} \) and \( m \) the class of thermal preparations and the mean value map associated to \( \mathcal{G} \) respectively.

In the following definition and remark let \( \mathcal{G} = \left( \mathcal{I}, l, \beta, P, a, \epsilon, \varphi, A, \psi, b, m, \Xi \right) \) be an object of \( \mathcal{G}(G, F, \rho), l \in H, \mathcal{I} \in \mathcal{I} \) and \( \alpha \in \mathbb{P}^\mathcal{I} \), where \( \Xi = \left( \mu, \nu, \xi, D, \Gamma, \psi, w, z \right) \).

**Definition 12.** Define \( \mathcal{V}(\mathcal{G})^\mathcal{I}_\alpha(l) \in \text{Hom}^\mathcal{G}(A^\mathcal{I}_\alpha, A^\mathcal{I}_\alpha) \) such that \( \mathcal{V}(\mathcal{G})^\mathcal{I}_\alpha(l) := \gamma^\mathcal{I}_\alpha(l) \circ \eta^\mathcal{I}_\alpha(l). \)

**Remark 5.1.** \( \mathbb{P} = \mathbb{P}^\mathcal{I} \) since (73). Let \( g \in \{ v, w, z \} \) then by (77) we deduce that \( g^\mathcal{I}_\alpha(l) \) is bijective and

\[
(\eta^\mathcal{I}_\alpha(l))^{-1} = g^\mathcal{I}_\alpha(l)^{-1},
\]

so \( \text{ad}(\eta^\mathcal{I}_\alpha(l)) \circ \eta^\mathcal{I}_\alpha = \eta^\mathcal{I}_\alpha \), then

\[
\mathcal{V}(\mathcal{G})^\mathcal{I}_\alpha(l \cdot l) = \mathcal{V}(\mathcal{G})^\mathcal{I}_\alpha(l) \circ \mathcal{V}(\mathcal{G})^\mathcal{I}_\alpha(l),
\]

moreover \( \mathcal{V}(\mathcal{G})^\mathcal{I}_\alpha(1) = Id \) thus

\[
(\mathcal{V}(\mathcal{G})^\mathcal{I}_\alpha(l))^{-1} = \mathcal{V}(\mathcal{G})^\mathcal{I}_\alpha(l^{-1}).
\]
Next $w_a^2(l)$ is surjective being bijective and $i_a^{2i}$ is nondegenerate, thus $i_a^{2i} \circ w_a^2(l)$ is nondegenerate since Lemma 4.12. Therefore $(i_a^{2i} \circ w_a^2(l))^-$ in (81) is well set and since (4) satisfies $i_a^{2i} \circ w_a^2(l)^- \circ i_a^{2i} = i_a^{2i} \circ w_a^2(l)$. Finally by an application of (81) we obtain

$$(i_a^{2i} \circ w_a^2(l^{-1}))^- \circ i_a^{2i} = i_a^{2i} \circ \eta_a^2(l^{-1}) \circ i_a^{2i}(l^-).$$

**Definition 13.** For any $i \in \{1, 2, 3\}$ let $G^i = \langle \mathcal{I}, l, \mu, \psi, \eta, \epsilon, \alpha, \beta, \gamma, \delta, \phi, \rho \rangle \in \text{Obj}(\mathcal{S}(G, F, \rho))$. Define $\text{Mor}_{6(G, F, \rho)}(G^1, G^2)$ to be the set of the

$$(g, d) \in \text{Mor}_{A^1}(A^1, A^2) \times \mathcal{I}(\mathcal{I}, \mathcal{I})$$

such that $P_1^{2i} = P_2^r$ for all $\mathcal{I} \in \mathcal{I}_2$, and for all $f \in A^1$ we have

$$\text{ev}_i(m^2 \circ g) = \text{ev}_i(m^1) \circ d.$$

Moreover for any $(g, d) \in \text{Mor}_{6(G, F, \rho)}(G^1, G^2)$ and $(h, s) \in \text{Mor}_{6(G, F, \rho)}(G^2, G^3)$ define

$$(h, s) \circ (g, d) := (h \circ g, d \circ s).$$

Thus we have the following

**Proposition 5.2.** There exists a unique category $(\mathcal{S}(G, F, \rho))$ whose class of objects is $\text{Obj}(\mathcal{S}(G, F, \rho))$ and for all objects $G^1$ and $G^2$ the set of the morphisms from $G^1$ to $G^2$ is $\text{Mor}_{6(G, F, \rho)}(G^1, G^2)$ provided by the law of composition as defined in Def. 13.

**Proof.** For any $i \in \{1, 2, 3\}$ let $G^i \in \mathcal{S}(G, F, \rho)$, moreover $(g, d) \in \text{Mor}(G^1, G^2)$ and $(h, s) \in \text{Mor}(G^2, G^3)$. The identity morphism in $\text{Mor}(G^1, G^1)$ is the couple composed by the identity maps on $A^1$ and on $\mathcal{I}_1$. Let $f \in A^1$, then

$$\text{ev}_i(m^3 \circ h \circ g) = \text{ev}_{6(f)}(m^3 \circ h) = \text{ev}_{6(f)}(m^2) \circ s$$

$$= \text{ev}_i(m^2 \circ g) \circ s = \text{ev}_i(m^1) \circ d \circ s.$$

Hence $(h, s) \circ (g, d) \in \text{Mor}(G^1, G^3)$, it is easy to see that the composition is associative hence the statement follows. □

**Convention 5.3.** Let $G = \langle \mathcal{I}, l, \beta, P, a, \epsilon, \phi, A, \psi, b, m, \Xi \rangle$ be an object of $(\mathcal{S}(G, F, \rho))$, with $\Xi = \langle \mu, u, \delta, D, \Gamma, v, w, \Sigma \rangle$, $\alpha_a^\Xi = \langle A_a^\Xi, H, \eta_a^\Xi \rangle$ and $\epsilon_a^\Xi = \langle A_a^\Xi, R, \epsilon_a^\Xi \rangle$, with $\mathcal{I} \in \mathcal{I}$ and $\alpha \in P_1^\Xi$, then we often shall use the following notation if more than one object of $(\mathcal{S}(G, F, \rho))$ is involved

$$(\mathcal{I}, l, \beta, P, a, \epsilon, \phi, A, \psi, b, m, \Xi) := (\mathcal{I}, l, \beta, P, a, \epsilon, \phi, A, \psi, b, m, \Xi),$$

$$(\mu^G, u^G, \delta^G, D^G, \Gamma^G, a^G, w^G, \Sigma^G) := (\mu, u, \delta, D, \Gamma, v, w, z),$$

$$(A(\alpha)^\Xi, (\eta^\alpha)^\Xi) := (A_a^\Xi, \eta_a^\Xi),$$

$$(\epsilon^\Xi)^\alpha := \epsilon_a^\Xi,$$

$\forall \mathcal{I} \in \mathcal{I}, \forall \alpha \in P_1^\Xi.$

Note that $(\psi^N(l), b^N(l^{-1})) \in \text{Mor}_{6(G, F, \rho)}(N, N)$ for any object $N$ of $(\mathcal{S}(G, F, \rho))$ and $l \in H.$
5.2. **Physical interpretation.** The set $\mathfrak{N}(\epsilon)$ of states originated by a given phase $\epsilon \in \mathcal{A}_N^\ast$ is a primary concept needed in order to provide a reasonable physical interpretation of the structural data of any object $N \in \text{Obj}(\theta(G, F, \rho))$. Let us briefly describe and physically interpret this set, whose precise definition is given in Def. 25&18. Let $\text{Rep}^N(\epsilon)$ be the set of the tuples $\tau = (\mathfrak{I}, \alpha, q, \mathbf{R}, \Phi)$, called representations of $\epsilon$, where $\mathfrak{I} \in \mathfrak{I}_N$, $\alpha \in \mathfrak{P}_N^\ast$, $\mathbf{R} = \left( R, \theta, \Omega \right)$ is a GNS-representation associated to the state $\varphi^\mathbf{R}_a$, $q$ is a group morphism from $\mathcal{A}_N$ to the $K_0$-theory of $\mathcal{B}_r^\ast$, where $\mathcal{B}_r = \mathcal{B}_1^\ast$ and $\Phi$ is an entire normalized even cocycle on $\mathcal{B}_r^\ast$ such that $\epsilon = \nu_r$ with

$$v_r := \mathbf{R}\left( q(\cdot), [\Phi] \right),$$

where $[\Phi]$ is the entire cyclic cohomology class generated by $\Phi$, while $\langle \cdot, \cdot \rangle$ is the standard duality between the entire cyclic cohomology and the $K_0$-theory of $\mathcal{B}_r^\ast$, and $\mathbf{R}$ means real part. We let $\pi_r, \Phi^\mathbf{R}, \mathbf{A}_r^\ast, \mathbf{I}_r$ and $\alpha_r$ denote $\pi^\mathbf{R}_{\mathfrak{I}_r}, \Phi, \mathbf{A}_{\mathfrak{I}_r}^\ast, \theta \rtimes \mathbf{W}, \mathfrak{I}$ and $\alpha$ respectively, where $\mathbf{W}(h)\theta(a)\Omega = \theta(\pi^\mathbf{R}_{\mathfrak{I}_r}(h)(a))\Omega$, for all $h \in \mathfrak{S}_a^\ast$ and $a \in \mathbf{A}_{\mathfrak{I}_r}^\ast$.

For any representation $\tau$ of $\epsilon$ we associate a state $\varrho_r$ of $\mathcal{A}_r$ in the following way: firstly we get the state $(\Phi^\mathbf{R})^\mathbf{R}$ associated to the $0$–dimensional component of $\Phi^\mathbf{R}$: in Def. 17 we construct the state associated to any functional $\phi$ on a unital $C^*$-algebra. Secondly we get the canonical extension $((\Phi^\mathbf{R})^\mathbf{R} \uparrow \mathcal{B}_r)$ of $(\Phi^\mathbf{R})^\mathbf{R} \uparrow \mathcal{B}_r$ to the multiplier algebra $\mathcal{M}(\mathcal{B}_r)$, according the construction provided in Lemma 4.2. Finally we compose the extension so obtained with the canonical injection $i_r$ of $\mathcal{A}_r$ into $\mathcal{M}(\mathcal{B}_r)$, by obtaining a state $\varrho_r$ of $\mathcal{A}_r$, which is required to be $\pi_r$–normal by definition. We are forced to use the multiplier algebra because in general $\mathcal{A}_r$ could not be injectively mapped into $\mathcal{B}_r$. Now the $\pi_r$–normality is required in order to interpret $\varrho_r$ as a state obtained by performing an operation on $\varphi^\mathbf{R}_{\alpha_r}$. Note that if $\alpha_r \in \mathbb{R}_0^+$, then $\varphi^\mathbf{R}_{\alpha_r}$ is an $\alpha_r$–KMS state w.r.t. the dynamics $(\varepsilon^N)^\mathbf{R}_{\alpha_r}(\cdot(\alpha_r^{-1}(\cdot)))$, therefore $\varrho_r$ is a state at the inverse temperature $\alpha_r$ for the physical system evolving in time via the dynamics $(\varepsilon^N)^\mathbf{R}_{\alpha_r}(\cdot(\alpha_r^{-1}(\cdot)))$. By definition $\mathfrak{N}(\epsilon)$ is the set of the states $\varrho_r$ by ranging over all representations $\sigma_s$ so summing up what said we have

$$\varrho_r := ((\Phi^\mathbf{R})^\mathbf{R} \uparrow \mathcal{B}_r) \circ i_r,$$

$$\varrho_r \in \mathbb{N}_{\pi_r},$$

$$\mathfrak{N}(\epsilon) := \left\{ \varrho_s \mid s \in \text{Rep}^N(\epsilon) \right\}.\tag{88}$$

It is worthwhile noting that the presence of cohomology classes in the definition of $\text{Rep}^N(\epsilon)$ it is at the basis of the possible degeneration of $\mathfrak{N}(\epsilon)$. Indeed for any representation $\tau = (\mathfrak{I}, \alpha, q, \mathbf{R}, \Phi)$ of $\epsilon$ it is so also $\tilde{\tau} = (\mathfrak{I}, \alpha, q, \mathbf{R}, \tilde{\Phi})$ whenever $\varrho_r$ is $\pi_r$–normal and the cocycles $\Phi$ and $\tilde{\Phi}$ on $\mathcal{B}_r$ belongs to the same cohomology class, i.e. $[\Phi] = [\tilde{\Phi}]$, hence $\mathfrak{N}(\epsilon)$ will be degenerate as soon as $\varrho_r \neq \varrho_{\tilde{r}}$.

With in mind the characterization of $\mathfrak{N}(\epsilon)$, we will propose in the assumption at page 55 the existence of physical systems $N$ and $\mathcal{O}_a^\mathfrak{I}$ for any $\mathfrak{I} \in \mathfrak{I}_N$ and $\alpha \in \mathfrak{P}_N^\ast$, such that for any $r \in \text{Rep}^N(\epsilon)$ the occurrence of $\varrho_r$ in the state $\varrho_r$ implies the previous occurrence of the system $N$ in the phase $\epsilon$. It is exactly in this sense that has to be understood Def. 35 in which we state that $\mathcal{O}_a^\mathfrak{I}$ is originated by the system $\mathcal{N}$, and that $\epsilon$ originates the state $\varrho_r$ for any $r \in \text{Rep}^N(\epsilon)$. The degeneration of $\mathfrak{N}(\epsilon)$ will be used in the final section to conjecture the relationship between the nucleon phase and the plurality of fragment states it can generate under variation of the fissioning system subject to the reaction.
A natural question arises when $c$ is an integer phase of Chern-Connes type, i.e. $c(A_M) \subseteq \mathbb{Z}$ and there exists a representation $\tau$ of $c$ such that $\Phi^\tau$ is associated to a $\theta$–summable $K$–cycle $\langle B^+_{\tau}, \gamma_{\tau}, D, \Gamma \rangle$; is it the state $\omega_{e^{\eta_{\tau}}} \circ \pi_\tau$ originated by $c$ in the sense above specified, in other words $\omega_{e^{\eta_{\tau}}} \circ \pi_\tau = \eta_\tau$? Lemma 5.17 gives the answer in the positive essentially under the hypothesis that $\Gamma$ is represented via $\gamma_\tau$ by an element with norm less or equal to 1. This is an important result used in Thm. 5.23 in order to prove the $H$–equivariance (93).

Definition 14. Let $M \in \text{Obj}(\mathcal{G}(G, F, \rho))$ set

$$A_M^* := \text{Mor}_{\text{Ab}}(A_M, \mathbb{R}).$$

$c \in A_M^*$ is said to be integer if $c(A_M) \subseteq \mathbb{Z}$. In addition set

$$\overline{m}^M \in \prod_{\mathfrak{S} \in \mathfrak{I}_M} (A_M^*)^{P^\mathfrak{S}_M},$$

$$\overline{m}^M(\mathfrak{S}, \alpha)(f) := m^M(f)(\mathfrak{S}, \alpha),$$

$$\forall \mathfrak{S} \in \mathfrak{I}_M, \alpha \in P^\mathfrak{S}_M, f \in A_M.$$

Definition 15. Let $\Delta_\nu, \mathbb{Z}$ and $\mathbb{V}_\nu$ be maps on $\text{Obj}(\mathcal{G}(G, F, \rho))$ while $\Delta_m$ be the map on $\text{Mor}_{\text{Obj}(\mathcal{G}(G, F, \rho))}$ such that for all $M, N \in \text{Obj}(\mathcal{G}(G, F, \rho))$ and $(b, f) \in \text{Mor}_{\mathcal{G}(G, F, \rho)}(N, M)$, we have

$$\Delta_\nu(M) := \bigcup_{\mathfrak{S} \in \mathfrak{I}_M} (A_M^*)^{P^\mathfrak{S}_M},$$

while

$$\Delta_m(b, f) : \Delta_\nu(M) \rightarrow \Delta_\nu(N), \quad (A_M^*)^{P^\mathfrak{S}_M} \ni f \mapsto \big( (P^\mathfrak{S}_N, \exists \alpha \mapsto f(\alpha) \circ b) \big), \forall \mathfrak{S} \in \mathfrak{I}_M,$$

and

$$\mathbb{Z}(M) := \prod_{\mathfrak{S} \in \mathfrak{I}_M} \prod_{\beta \in P^\mathfrak{S}_M} (A(M)^{\mathfrak{S}})^*,$$

finally $\mathbb{V}_\nu(M) : H \rightarrow \mathbb{Z}(M)^{\mathbb{Z}(M)}$ such that

$$\mathbb{V}_\nu(M)(l) : (\mathfrak{S}, f) \mapsto \big(b^M(l)(\mathfrak{S}), P^{b^M(l)(\mathfrak{S})}_M \ni \alpha \mapsto f(\alpha) \circ \mathbb{V}(M)^{b^M(l)(\mathfrak{S})}(l^{-1})\big).$$

Definition 16. For any map $f$ let us define the map $\text{gr}(f) : \text{Dom}(f) \ni x \mapsto (x, f(x)) \in \text{Graph}(f)$.

Convention 5.4. In order to simplify the notations of what follows, in the remaining of this section $\mathfrak{S} = \langle \mathfrak{I}, \mathfrak{l}, \beta_c, \mathfrak{P}, a, e, \mathfrak{f}, \mathfrak{A}, \mathfrak{y}, b, m, \mathfrak{S} \rangle$ will be an arbitrary object of $\mathcal{G}(G, F, \rho)$, with

$$\mathfrak{S} = \langle \mu, u, \mathfrak{S}_D, \mathfrak{D}, \mathfrak{G}, u, w, \beta \rangle$$

and for all $\mathfrak{S} \in \mathfrak{I}$ and $\alpha \in P^\mathfrak{S}_a$ let $\mathfrak{S}^\mathfrak{S}_a = \langle \delta^\mathfrak{S}_a, \pi^\mathfrak{S}_a, \Omega^\mathfrak{S}_a \rangle$. If it will not cause confusion often we shall remove the index denoting the object of $\mathcal{G}(G, F, \rho)$ in what defined in Def.’s 14 and 15. Moreover for all $l \in H, c \in A^*, f \in A$ and $\mathfrak{S} \in \mathfrak{I}$ let $c^\mathfrak{S}, f^\mathfrak{S}$ and $\mathfrak{S}^\mathfrak{S}$ denote $c \circ \psi(l^{-1}), \psi(l)(f)$ and $b(l)(\mathfrak{S})$ respectively.

Remark 5.5. Since (73) we obtain $(\overline{m}^\mathfrak{S}(\mathfrak{S}, \alpha))^l = \overline{m}^\mathfrak{S}(\mathfrak{S}^\mathfrak{S}, \alpha)$, for all $\mathfrak{S} \in \mathfrak{I}, \alpha \in P^\mathfrak{S}$, moreover since Def. 13 we have $\overline{m}^\mathfrak{S}(f(\mathfrak{S}), \beta) = \overline{m}^\mathfrak{S}(\mathfrak{S}, \beta) \circ \varphi$, for all $\mathfrak{S} \in \text{Obj}(\mathcal{G}(G, F, \rho)), (\varphi, f) \in \text{Mor}_{\mathcal{G}(G, F, \rho)}(\mathfrak{S}, M), \mathfrak{S} \in \mathfrak{I}_M$ and $\beta \in P^\mathfrak{S}_M$. 

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**Definition 17.** Let $\mathcal{B}$ be a unital $C^*$-algebra and $\phi \in \mathcal{B}^* - \{0\}$. We call the state associated to $\phi$ the state defined as follows

$$\phi^\circ := \frac{\phi_1 + \phi_2}{\|\phi_1 + \phi_2\|}.$$ 

Here $(\phi_1, \phi_2)$ is the unique couple such that

1. $\phi_j$ is a positive functional on $\mathcal{B}$, $j \in \{1, 2\}$,
2. $\frac{1}{2} (\phi + \phi^*) = \phi_1 - \phi_2$,
3. $\frac{1}{2} \|(\phi + \phi^*)\| = \|\phi_1\| + \|\phi_2\|,$

where $\phi^* \in \mathcal{B}^*$ such that $\phi^*(a) := \overline{\phi(a)}$, for all $a \in \mathcal{B}$.

**Remark 5.6.** The existence and uniqueness of the couple $(\phi_1, \phi_2)$ in Def. 17 follows since [KR Thm. 4.3.6] applied to the hermitian and bounded functional $\frac{1}{2} (\phi + \phi^*)$, where a functional is hermitian if $\psi^* = \psi$.

**Remark 5.7.** Under the notations of Def. 17 we have that $\|\phi_1 + \phi_2\| = \|\phi_1\| + \|\phi_2\| = \frac{1}{2} \|(\phi + \phi^*)\|$, the first equality coming since [BR 1 Cor. 2.3.12].

Recall that for any unital $C^*$-algebra $\mathcal{B}$ the $\langle \cdot, \cdot \rangle_\mathcal{B}$ denotes the standard duality between the $K_0$-theory of $\mathcal{B}$ and the even entire cyclic cohomology of $\mathcal{B}$. Moreover for any entire normalized even cocycle $\Phi$ on $\mathcal{B}$ [Φ] denotes the element in $H^\lambda_c(\mathcal{B})$ corresponding to $\Phi$, note that $\Phi_0 \in \mathcal{B}^*$, where $\Phi_0$ is the 0-dimension component of $\Phi$, hence if $\Phi_0 \neq 0$ then $\Phi_0$ is the state of $\mathcal{B}$ associated to $\Phi_0$ according Def. 17.

Since Def. 17 and Rmk. 3.30 we have that $\mathcal{B}_{\nu}^{\mathcal{A}_d^\nu}$, shortly $\mathcal{B}_{\nu}^{\mathcal{A}_d^\nu}$, equals the unitarization of $\mathcal{B}_{\nu}^{\mathcal{A}_d^\nu} \simeq \mathcal{A}_d^{\mathcal{A}_d^\nu} \simeq \mathcal{S}_{\mathcal{A}_d}^{\mathcal{A}_d^\nu}$, while $\langle \cdot, \cdot \rangle_{\mathcal{A}_d} = \langle \cdot, \cdot \rangle_{\mathcal{A}_d}$, for any $\alpha \in \mathcal{A}_d$ and $\nu \in H(\mathcal{S}_{\mathcal{A}_d}^{\mathcal{A}_d^\nu})$.

If $\mathcal{R} = \langle \mathcal{R}, \pi, \Omega \rangle$ is a cyclic representation of $\mathcal{A}_d^{\mathcal{A}_d^\nu}$ associated to $\varphi_\mathcal{A}^{\mathcal{A}_d^\nu}$, then $\mathcal{N}_\mathcal{R}$ is the class of all $\pi$-normal states of $\mathcal{A}_d^{\mathcal{A}_d^\nu}$, $\mathcal{R}_\mathcal{R}$, equals $\mathcal{B}_{\nu}^{\mathcal{A}_d^\nu}$, where $\mathcal{R}_\mathcal{R} = \pi \cong \mathcal{R}$ and $\mathcal{U} : \mathcal{S}_{\mathcal{A}_d}^{\mathcal{A}_d^\nu} \rightarrow \mathcal{L}(\mathcal{R})$ such that $\mathcal{U}(l)\pi(a)\Omega = \pi(l_{\mathcal{R}}(a))\Omega$ for all $l \in \mathcal{S}_{\mathcal{A}_d}^{\mathcal{A}_d^\nu}$ and $a \in \mathcal{A}_d^{\mathcal{A}_d^\nu}$.

**Definition 18 (S-representations of a phase).** Let $\epsilon \in \mathcal{A}_d$ define $Rep_S(\epsilon)$ the set of the $\mathcal{R} = \langle \mathcal{X}, \alpha, \nu, \mathcal{R}, \Phi \rangle$ such that

1. $\mathcal{X} \in \mathcal{X}$ and $\alpha \in \mathcal{P}_\mathcal{X}$,
2. $\nu \in H(\mathcal{S}_{\mathcal{A}_d}^{\mathcal{A}_d^\nu})$,
3. $\mathcal{U} \in Mor_{\mathcal{A}_d}(\mathcal{A}, K_0(B_{\mathcal{A}_d}^{\mathcal{A}_d^\nu}))$,
4. $\mathcal{R} = \langle \mathcal{R}, \pi, \Omega \rangle$ is a cyclic representation of $\mathcal{A}_d^{\mathcal{A}_d^\nu}$ associated to $\varphi_\mathcal{A}^{\mathcal{A}_d^\nu}$,
5. $\Phi$ is an entire normalized even cocycle on $\mathcal{B}_{\nu}^{\mathcal{A}_d^\nu}$ such that $\Phi_0 \upharpoonright \mathcal{B}_{\nu}^{\mathcal{A}_d^\nu} \neq 0$,
6. $(\Phi_0^\circ \upharpoonright \mathcal{B}_{\nu}^{\mathcal{A}_d^\nu}) \circ \mathcal{B}_{\nu}^{\mathcal{A}_d^\nu} \in \mathcal{N}_\mathcal{R},$
7. $\epsilon = \nu_\epsilon$.

Here $\Phi_0^\circ$ is the state associated to $\Phi_0$ according Def. 17 and $\nu_\ell \in \mathcal{A}_d^\nu$ such that for all $f \in \mathcal{A}$

$$\nu_\ell(f) := \mathcal{R} \langle \mathcal{U}(f), [\Phi] \rangle_{\nu_\ell},$$

where $\mathcal{R} \lambda$ is the real part of $\lambda \in \mathcal{C}$. We call $\Phi$ the representative of $\epsilon$ relative to $\mathcal{R}$ and call any element of $Rep_S(\epsilon)$ a S-representation of $\epsilon$. 
Remark 5.8. \((\Phi_0^T \downarrow \mathcal{B}^T_{\alpha})^- \circ \mathcal{B}^T_{\alpha})\) is a state of \(\mathcal{A}_{\alpha}\) since Cor. 4.4, thus in Def. 18(6) we require that this state belongs to \(N_{\alpha}\).

Definition 19. Let \(t = \left( T, \alpha, \nu, u, R, L, \Delta \right) \) such that

1. Def. 18(1-4) hold,
2. \(\tilde{R}^y_{\alpha} L, \Delta \) is an even \(\theta\)-summable \(K\)-cycle,

then we set \(w_t \in A^*\) such that for all \(f \in A\)

\[
(90) \quad w_t(f) = \left( u(f), \text{ch}\left( \left( \tilde{R}^y_{\alpha} L, \Delta \right) \right) \right)_{\nu, \Phi^T_{\alpha}}.
\]

Definition 20 (\(C_0\)-representations of an integer phase). Let \(c \in A^*\) be an integer phase, define \(C^5(c)\) the set of the \(t = \left( T, \alpha, \nu, u, R, L, \Delta \right) \) such that

1. Def. 18(1-4) hold,
2. \(\tilde{R}^y_{\alpha} L, \Delta \) is an even \(\theta\)-summable \(K\)-cycle
3. there exists an element \(b \in \mathcal{B}^T_{\alpha} (\tilde{R}^y_{\alpha})^+\) such that \(\|b\| \leq 1\) and \(\tilde{R}^y_{\alpha} (b) = \Delta\),
4. \(c = w_t\).

We call \(C_0\)-representation of \(c\) any element of \(C^5(c)\).

Definition 21 (\(C\)-representations of an integer phase). Let \(c \in A^*\) be an integer phase, define \(C^5(c)\) the set of the \(t = \left( T, \alpha, \nu, u, R, L, \Delta \right) \) such that

1. Def. 18(1-4) hold,
2. \(\tilde{R}^y_{\alpha} L, \Delta \) is an even \(\theta\)-summable \(K\)-cycle
3. Def. 18(5-6) hold where \(\Phi\) is the JLO cocycle associated to \(\tilde{R}^y_{\alpha} L, \Delta\),
4. \(c = w_t\).

We call \(C\)-representation of \(c\) any element of \(C^5(c)\).

Definition 22. Define \(\mathfrak{B}_*(\mathfrak{S})\) the subset of the \(T \in \mathfrak{T}\) such that for any \(\alpha \in \mathfrak{P}^\mathfrak{T}\) there exists an element \(b \in (\mathcal{B}^T_{\alpha} (\mathfrak{S}))^+\) such that \(\|b\| \leq 1\) and \(\tilde{R}^y_{\alpha} (b) = \Gamma^\mathfrak{T}_{\alpha}\).

Definition 23. For any \(T \in \mathfrak{T}\) and \(\alpha \in \mathfrak{P}^\mathfrak{T}\), define \(t^\mathfrak{S}(T, \alpha) := \left( T, \alpha, \upsilon^\mathfrak{T}_{\alpha}, \upsilon^\mathfrak{I}_{\alpha}, \mathfrak{H}^\mathfrak{T}_{\alpha}, \mathfrak{H}^\mathfrak{I}_{\alpha}, \Gamma^\mathfrak{T}_{\alpha}, \Gamma^\mathfrak{I}_{\alpha} \right)\).

We convene to remove the index \(\mathfrak{S}\) whenever it will not cause confusion.

Remark 5.9. If \(\mathfrak{B}_*(\mathfrak{S}) \neq \emptyset\) then \(t(T, \alpha) \in C^5(\mathfrak{M}(T, \alpha))\) for all \(T \in \mathfrak{B}_*(\mathfrak{S})\) and \(\alpha \in \mathfrak{P}^\mathfrak{T}\) since (82).
Definition 24. Let $c \in A^*$ and $r = \langle \mathcal{I}, \alpha, \mu, u, \Phi \rangle$ satisfying Def. \cite{18}, then we set 
$(\mathcal{I}_r, \alpha_r, \mu_r, u_r, \Phi_r) = (\mathcal{I}, \alpha, \mu, u, \Phi)$ and 
\[
\begin{align*}
\mathcal{A}_r & := \mathcal{A}_{\alpha_r} \\
\eta_r & := \eta_{\alpha_r} \\
\delta_r & := \delta_{\alpha_r} \\
\mathcal{B}_r & := \mathcal{B}_{\mu_r} \\
\iota_r & := \iota_{\alpha_r} \\
\Psi_r & := \Phi_0^\mathcal{B}_r \\
\langle \cdot, \cdot \rangle_r & := \langle \cdot, \cdot \rangle_{\mu, \Phi_r} 
\end{align*}
\]
and if $\Phi = \langle \Phi, \pi, \Omega \rangle$ then we set $(\Phi_r, \pi_r, \Omega_r) = (\Phi, \pi, \Omega)$. If $\alpha \in A^*$ is an integer phase and $t \in C^g(c) \cup C_0^g(c)$, whenever $t = \langle \mathcal{I}, \alpha, \mu, u, \Phi, D, \Gamma \rangle$ we set $(\mathcal{I}_r, \alpha_r, \mu_r, u_r, \Phi_r, D_r, \Gamma_r) = (\mathcal{I}, \alpha, \mu, u, \Phi, D, \Gamma)$, moreover if $\Phi = \langle \Phi, \pi, \Omega \rangle$ then we set $(\Phi_r, \pi_r, \Omega_r) = (\Phi, \pi, \Omega)$.

Definition 25 (States of $A^*$ originated by a phase). Let $c \in A^*$ define 
\[
\mathcal{D}^g(c) := \left\{ \Psi_r^* \circ \iota_r \mid r \in \text{Rep}^g(c) \right\},
\]
if in addition $c$ is an integer phase we set 
\[
\mathcal{D}^g(c) := \left\{ \omega_\cdot_{c, r}^* \circ \pi_r \mid t \in C^g_0(c) \right\}.
\]

We convene to remove the index $\mathcal{D}$ whenever it will not cause confusion.

Remark 5.10. According Def. \cite{18}, see also Rmk. \cite{5.8} for any $c \in A^*$ and $r \in \text{Rep}^g(c)$ we have $\Psi_r^* \circ \iota_r \in N_{\pi_r}$, which is at the ground for the physical interpretation constructed in Def. \cite{35}.

In Lemma \cite{5.17} we prove that $C^g_0(c) \subseteq C^g(c)$ for an integer phase $c$, in addition we prove the equality \cite{97}. In order to define a $G$–equivariant stability in Def. \cite{31} we need to introduce the following structures. We use in Def. \cite{29} the set of states originated a phase.

Definition 26. Let $\langle H, \mathcal{U} \rangle$ be defined a $(G, F, \rho)$–map if $\mathcal{U}$ and $H$ are maps on $\text{Dom}(\mathcal{U}) \subseteq \text{Obj}(\mathcal{U}(G, F, \rho))$ such that for all $N \in \text{Dom}(\mathcal{U})$

1. $\mathcal{U}_N \subseteq \mathcal{I}_N$,
2. $H_N$ is a subgroup of $H$,
3. $(\forall M \in \text{Dom}(\mathcal{U}))((\forall (b, \bar{f}) \in \text{Mor}_{\mathcal{U}(G, F, \rho)}(N, M))((\bar{f}(M) \subseteq \mathcal{U}_N))$.

Since Def. \cite{26} we have for any $N \in \text{Dom}(\mathcal{U})$ that $(\forall (b \in H_N)(b^N(l)\mathcal{U}_N) \subseteq U_N)$.

Definition 27. Let $\langle H, D, \mathcal{U} \rangle$ be defined a $(G, F, \rho)$–couple of maps if $\langle H, \mathcal{U} \rangle$ is a $(G, F, \rho)$–map, $D$ is a map on $\text{Dom}(D) \subseteq \text{Dom}(\mathcal{U})$ such that for all $N \in \text{Dom}(D)$

1. $D_N \subseteq U_N$,
2. $(\forall (b \in H_N)(b^N(l)\mathcal{D}_N) \subseteq D_N)$. 

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H is said to be full if it is the constant map equal to H; \( \mathfrak{U} \), resp. \( \langle \mathfrak{D}, \mathfrak{U} \rangle \), is said to be a full \((G, F, \rho)\)-map, resp. full \((G, F, \rho)\)-couple of maps, if \( \langle H, \mathfrak{U} \rangle \) is a \((G, F, \rho)\)-map, resp. if \( \langle H, \mathfrak{D}, \mathfrak{U} \rangle \) is a \((G, F, \rho)\)-couple of maps, and \( H \) is full.

Note that if \( \mathfrak{U} \) is a full \((G, F, \rho)\)-map then \( \langle \mathfrak{U}, T \rangle \) is a full \((G, F, \rho)\)-couple of maps.

**Definition 28** (Equivariant phase). \( \langle H, \mathfrak{U}, m \rangle \) is an equivariant phase if \( \langle H, \mathfrak{U} \rangle \) is a \((G, F, \rho)\)-map and \( m \) is a map defined on \( \text{Dom}(\mathfrak{U}) \) such that for all \( N \in \text{Dom}(\mathfrak{U}) \)

1. \( m^N \in \prod_{\mathfrak{U} \in \text{Dom}(\mathfrak{U})} (\mathfrak{A}^*_N)^{P^N} \),
2. for any \( l \in H_N \)

\[
\begin{array}{c}
\begin{array}{c}
\mathfrak{U}_N \\
\mathfrak{U}_N
\end{array}
\end{array}
\xrightarrow{m^N} \begin{array}{c}
\Delta_N(\mathfrak{U})
\end{array}
\]

\[
\begin{array}{c}
\mathfrak{U}_N \\
\mathfrak{U}_N
\end{array}
\xrightarrow{m^N} \begin{array}{c}
\Delta_N(\mathfrak{U})
\end{array}
\]

(3) for all \( M \in \text{Dom}(\mathfrak{U}) \) and \((h, l) \in \text{Mor}_{6(G, F, \rho)}(N, M)\)

\[
\begin{array}{c}
\mathfrak{U}_N \\
\mathfrak{U}_M
\end{array}
\xrightarrow{m^N} \begin{array}{c}
\Delta_N(\mathfrak{U})
\end{array}
\]

\[
\begin{array}{c}
\mathfrak{U}_M \\
\mathfrak{U}_M
\end{array}
\xrightarrow{m^N} \begin{array}{c}
\Delta_N(\mathfrak{U})
\end{array}
\]

\( m \) is integer if \( m^N(\Xi, \alpha) \) is integer for all \( N \in \text{Obj}(6(G, F, \rho)) \), \( \Xi \in \mathfrak{U}_N \) and \( \alpha \in P^N \), while \( \langle \mathfrak{U}, m \rangle \) is a full equivariant phase if \( \langle H, \mathfrak{U}, m \rangle \) is an equivariant phase and \( H \) is full.

**Definition 29** (Equivariant state associated to an equivariant phase). \( \mathcal{W} \) is a state associated to \( \langle H, \mathfrak{U}, m \rangle \) and equivariant on \( \mathfrak{D} \) if \( \langle H, \mathfrak{U}, m \rangle \) is an equivariant phase, \( \langle H, \mathfrak{D}, \mathfrak{U} \rangle \) is a \((G, F, \rho)\)-couple of maps and \( \mathcal{W} \) is a map on \( \text{Dom}(\mathfrak{D}) \) such that for all \( N \in \text{Dom}(\mathfrak{D}) \)

1. \( \mathcal{W}^N \in \prod_{\mathfrak{D} \in \text{Dom}(\mathfrak{D})} \prod_{\alpha \in P^N} \mathfrak{W}^N(m^N(\mathfrak{D}, \alpha)) \),
2. for all \( \mathfrak{D} \in \mathfrak{D}_N \) and \( \beta \in P^N \) there exists \( t \in \text{Rep}^N(m^N(\mathfrak{D}, \beta)) \) such that \( \mathcal{W}^N(\mathfrak{D}, \beta) = \Psi^ \mathfrak{D} \circ i_t \),
3. for all \( l \in H_N \)

\[
\begin{array}{c}
\mathfrak{D}_N \\
\mathfrak{D}_N
\end{array}
\xrightarrow{\text{gr}(\mathcal{W}^N)} \begin{array}{c}
\mathbb{Z}(N)
\end{array}
\]

\[
\begin{array}{c}
\mathfrak{D}_N \\
\mathfrak{D}_N
\end{array}
\xrightarrow{\text{gr}(\mathcal{W}^N)} \begin{array}{c}
\mathbb{Z}(N)
\end{array}
\]
We call \( W \) a state associated to \( \langle U, m \rangle \) equivariant on \( D \) if it is a state associated to \( \langle H, U, m \rangle \) equivariant on \( D \) such that \( H \) is full.

Remark 5.11. Note that since Def. 29(1) in Def. 29(2) we require just that \( I_I = \emptyset \) and \( \alpha_I = \beta_I \).

**Definition 30 (Equivariant stability on \( D \)).** \( \langle H, U, m, W \rangle \) is an equivariant stability on \( D \) if \( \langle H, U, m \rangle \) is an equivariant phase and \( W \) is a state associated to \( \langle H, U, m \rangle \) and equivariant on \( D \). Moreover \( \langle U, m, W \rangle \) is a full equivariant stability on \( D \) if \( \langle H, U, m, W \rangle \) is an equivariant stability on \( D \) and \( H \) is full, while it is integer if \( m \) it is so.

Next we introduce the main structure of this work, whose properties of invariance will be clarified in Prp. 5.13 the physical consequences discussed in Prp. 5.29 and the property of inducing natural transformations described in Prp. 5.16.

**Definition 31 (\( C \)-equivariant stability on \( D \)).** \( \langle \langle H, U, m, W \rangle, F \rangle \) is a \( C \)-equivariant stability on \( D \) if \( \langle H, U, m, W \rangle \) is an equivariant stability on \( D \), \( C \) is a category and \( F \) is a functor from \( C \) to \( \mathcal{G}(G, F, \rho) \) such that \( \Theta(D, F) \neq \emptyset \), where \( \Theta(L, F) := \{ a \in \text{Obj}(C) \mid F(a) \in \text{Dom}(L) \} \), with \( L \in \{ D, U \} \). \( \langle \langle U, m, W \rangle, F \rangle \) is a full \( C \)-equivariant stability on \( D \) if \( \langle \langle H, U, m, W \rangle, F \rangle \) is a \( C \)-equivariant stability on \( D \) and \( H \) is full.

In Thm. 5.23 we show the existence of a full integer equivariant stability on \( B_a \) and in Thm. 6.24 the existence of a functor from \( C_a(H) \) to \( \mathcal{G}(G, F, \rho) \). So in our main Thm. 6.25 we state the existence of a \( C_a(H) \)-equivariant stability allowing in Thm. 7.4 and Thm. 7.20 to associate natural transformations to \( H \) and to prove in Thm. 8.9 the universality of the Terrell law.

**Convention 5.12.** Let \( F \in \text{Fct}(C, \mathcal{G}(G, F, \rho)) \), and \( \{ F_t \}_{t \in [1,2]} \) be such that for all \( a, b \in \text{Obj}(C) \) the morphism part of \( F \) on \( \text{Mor}_C(a, b) \) equals \( \text{Mor}_C(a, b) \ni t \mapsto (F_1(t), F_2(t)) \in \text{Mor}_{\mathcal{G}(G, F, \rho)}(F(a), F(b)) \).

**Proposition 5.13 (Properties of invariance of a \( C \)-equivariant stability).** Let \( \langle \langle H, U, m, W \rangle, F \rangle \) be a \( C \)-equivariant stability on \( D \), moreover let

- \( a, b \in \Theta(U, F) \) and \( l \in H_{F(a)} \),
- \( I \in U_{F(a)} \) and \( \alpha \in P_{F(a)} \),
- \( I \in U_{F(b)} \) and \( \beta \in P_{F(b)} \),
- \( f \in A_{F(a)} \) and \( a \in A(F(a))_{\alpha} \),
- \( t \in \text{Mor}_C(a, b) \),
- \( c \in \text{Obj}(C) \) and \( l \in \text{Mor}_C(b, c) \).

Thus

\[
\begin{align*}
\phi^{F(a)}(l)(\alpha) &= m^{F(a)}(\alpha)(f), \\
\phi^{F(a)}(F_2(t)(\alpha), \beta) &= m^{F(b)}(\alpha, \beta)(F_1(t)(f)), \\
F_1(l \circ t) &= F_1(l) \circ F_1(t), \\
F_2(l \circ t) &= F_2(l) \circ F_2(t),
\end{align*}
\]

(94)
Definition 32. Let $\mathcal{O}$ and $\mathcal{S}$ maps on $\text{Obj}(\mathcal{O}(G, F, \rho))$ such that for all $N \in \text{Obj}(\mathcal{O}(G, F, \rho))$ we have
\[
\mathcal{S}^N := (1 \mapsto Z(N), V_*(N)),
\]
\[
\mathcal{O}^N := (1 \mapsto \Delta_o(N), H \ni l \mapsto \Delta_m(\psi^N(l^{-1}), b^N(l))).
\]

Let $\langle \mathcal{U}, \mathcal{D} \rangle$ be a full $(G, F, \rho)$–couple of maps then by abuse of language we convein to denote with $\text{Dom}(\mathcal{D})$ the subcategory of $\mathcal{O}(G, F, \rho)$ whose object class is $\text{Dom}(\mathcal{D})$ and whose morphism class is $\mathcal{O}(G, F, \rho)(\text{Dom}(\mathcal{D}))$. Let $\Psi_{\mathcal{D}}$ be the map on $\text{Dom}(\mathcal{D})$ such that for all $\emptyset \in \text{Dom}(\mathcal{D})$ we have
\[
\Psi_{\mathcal{D}}^\emptyset := (1 \mapsto \mathcal{D}_\emptyset, H \ni l \mapsto b^\emptyset(l) \upharpoonright \mathcal{D}_\emptyset).
\]

Define $\text{Dom}(\mathcal{D})^0$ the subcategory of $\mathcal{O}(G, F, \rho)$ such that $\text{Obj}(\text{Dom}(\mathcal{D})^0) := \text{Dom}(\mathcal{D})$, while
\[
\text{Mor}_{\text{Dom}(\mathcal{D})^0} := \bigcup_{N, M \in \text{Dom}(\mathcal{D})} \{ (h, f) \in \text{Mor}_{\mathcal{O}(G, F, \rho)}(N, M) \mid f|_{\mathcal{D}_N} \subseteq \mathcal{D}_M \}.
\]

Moreover let $\Sigma_{\mathcal{D}} := (\mathcal{D}, \Sigma_{\mathcal{D}}^m)$ where $\Sigma_{\mathcal{D}}^m$ is the map on $\text{Mor}_{\text{Dom}(\mathcal{D})^0}$ such that for all $N, M \in \text{Dom}(\mathcal{D})$ and $(h, f) \in \text{Mor}_{\text{Dom}(\mathcal{D})^0}$ such that $f|_{\mathcal{D}_N} \subseteq \mathcal{D}_M$ we have
\[
\Sigma_{\mathcal{D}}^m(h, f) := f \upharpoonright \mathcal{D}_M.
\]

Finally define
\[
\Delta_{\mathcal{U}} := (\Delta_o \upharpoonright \text{Dom}(\mathcal{U}), \Delta_m \upharpoonright \mathcal{O}(G, F, \rho)(\text{Dom}(\mathcal{U}))).
\]

Remark 5.14. Under the assumptions in Def. $\mathcal{S}$ we have that as categories $\text{Dom}(\mathcal{U}) = \text{Dom}(\mathcal{D})^0$, well done since $\langle \mathcal{D}, \mathcal{U} \rangle$ is a full $(G, F, \rho)$–couple of maps. Moreover $\Delta_{\mathcal{U}} \in \text{Fct}(\text{Dom}(\mathcal{U}), \text{Set}^{op})$ and $\Sigma_{\mathcal{D}} \in \text{Fct}(\text{Dom}(\mathcal{D})^0, \text{Set}^{op})$ so $\Sigma_{\mathcal{D}}^m \in \text{Fct}(\text{Dom}(\mathcal{U}), \text{Set}^{op})$, while $\Psi_{\mathcal{D}}^\emptyset, \Psi_{\mathcal{U}}^\emptyset, Z^M, \Sigma^N \in \text{Fct}(\mathcal{H}, \text{Set})$, for any $N \in \text{Dom}(\mathcal{U}), \emptyset \in \text{Dom}(\mathcal{D})$ and $M \in \text{Obj}(\mathcal{O}(G, F, \rho))$.

Remark 5.15. Let $\langle \mathcal{U}, m \rangle$ be a full equivariant phase then it is easy to see that $m \in \text{Mor}_{\text{Fct}(\text{Dom}(\mathcal{U}), \text{Set}^{op})}(\Sigma_{\mathcal{D}}, \Delta_{\mathcal{U}})$, moreover for all $N \in \text{Dom}(\mathcal{U})$ we have $(1 \mapsto m^N) \in \text{Mor}_{\text{Fct}(\mathcal{H}, \text{Set})}(\Sigma_{\mathcal{D}}^m, \Sigma^N)$. Finally if $\mathcal{W}$ is a state associated to $\langle \mathcal{U}, m \rangle$ and equivariant on $\mathcal{D}$ then for all $\emptyset \in \text{Dom}(\mathcal{D})$ we have $(1 \mapsto \text{gr}(\mathcal{W}^\emptyset)) \in \text{Mor}_{\text{Fct}(\mathcal{H}, \text{Set})}(\Psi_{\mathcal{D}}^\emptyset, Z^\emptyset)$.

Definition 33. Let $\langle \langle \mathcal{U}, m, \mathcal{W} \rangle, \mathcal{F} \rangle$ be a full $\mathcal{C}$–equivariant stability on $\mathcal{D}$ where $\mathcal{C}$ is a category. Let $\Theta(\mathcal{U}, \mathcal{F})$ denote by abuse of language the subcategory of $\mathcal{C}$ whose object class is $\Theta(\mathcal{U}, \mathcal{F})$ and whose morphism class is $\mathcal{O}_\mathcal{C}(\Theta(\mathcal{U}, \mathcal{F}))$. Moreover let $\tilde{\mathcal{F}} = (\tilde{\mathcal{F}}_o, \tilde{\mathcal{F}}_m)$, where $\tilde{\mathcal{F}}_o = \mathcal{F}_o \upharpoonright \Theta(\mathcal{U}, \mathcal{F})$ and $\tilde{\mathcal{F}}_m = \mathcal{F}_m \upharpoonright \mathcal{O}_\mathcal{C}(\Theta(\mathcal{U}, \mathcal{F}))$.

Since Rmk. 5.13 and (1) we obtain the following result which will be used in stating Thm. 7.4.
Proposition 5.16 (Natural transformations relative to a full \(C\)-equivariant stability). Let \(\langle \mathfrak{U}, m, \mathcal{W}, \mathcal{F} \rangle\) be a full \(C\)-equivariant stability on \(\mathfrak{D}\) where \(C\) is a category. Then \(\mathcal{F} \in \text{Fct}(\Theta(\mathfrak{U}, \mathcal{F}), \text{Dom}(\mathfrak{U}))\) and

\[
\begin{align*}
(1) \quad & m \circ \mathcal{F}_o \in \text{Mor}_{\text{Fct}(\mathfrak{U}, \mathcal{F}), \text{Set}^p)}(\Sigma^\mathfrak{U} \circ \mathcal{F}, \Delta^\mathfrak{U} \circ \mathcal{F}), \\
(2) \quad & (1 \mapsto m^{\mathcal{F}(a)}) \in \text{Mor}_{\text{Fct}(H, \text{Set})}(\Psi^{\mathcal{F}(a)}, \Sigma^{\mathcal{F}(a)}) \text{ for all } a \in \Theta(\mathfrak{U}, \mathcal{F}), \\
(3) \quad & (1 \mapsto \text{gr}(\mathcal{V}^{\mathcal{F}(b)})) \in \text{Mor}_{\text{Fct}(H, \text{Set})}(\Psi^{\mathcal{F}(b)}, \mathfrak{V}^{\mathcal{F}(b)}) \text{ for all } b \in \Theta(\mathfrak{D}, \mathcal{F}).
\end{align*}
\]

If \(c\) is an integer phase, the next result shows that \(C^G_0(c) \subseteq C^G(c)\), one can associate a \(G\)-representation of \(c\) to any element \(t\) in \(C^G(c)\), and in the \(C^G_0(c)\) case the state in \(\mathfrak{H}(c)\) associated to this representation assumes the simple form \(\omega \circ \pi_t\), which is at the basis of the proof of the \(H\)-equivariance \((23)\) in Thm. \(5.23\). It is in the proof of the next result that we use Cor. \(4.4(1)\).

Lemma 5.17 (States originated by the same integer phase). Let \(c \in A^*\) be integer, then the following map

\[
\begin{align*}
\langle \mathfrak{I}, \alpha, \mu, u, \mathfrak{H}, D, \Gamma \rangle \overset{\theta^G}{\to} \langle \mathfrak{I}, \alpha, \mu, u, \mathfrak{H}, \Phi \rangle,
\end{align*}
\]

where \(\Phi\) is the ILO cocycle associated to \(\langle \mathfrak{R}_{\mathfrak{H}}, D, \Gamma \rangle\), is well-defined on \(C^G(c) \cup C^G_0(c)\) and mapping \(C^G(c)\) into \(\text{Rep}^G(c)\). Moreover for all \(t \in C^G_0(c)\)

\[
\Psi^{-\theta^G(t)} \circ \tilde{i}_{\theta^G(t)} = \omega \circ \pi_t,
\]

in particular

\[
C^G_0(c) \subseteq C^G(c) \text{ and } \mathfrak{I}(c) \subseteq \mathfrak{H}(c).
\]

We convene to remove the index \(\mathfrak{J}\) whenever it will not be cause of confusion.

Remark 5.18. In Assmp. \([1]\) and Def. \(35\) \(\mathfrak{J}\) will be interpreted as a physical system, \(A^*\) as the set of the states of \(\mathfrak{J}\), called phases, and \(\mathfrak{H}(c)\) as the set of the states, of suitable physical systems, originated by the phase \(c\) for any \(c \in A^*\). Thus since Lemma \(5.17\) in case \(c\) is integer, any \(C_0\)-representation of \(c\) induces a state originated by \(c\) whose form is given in \((97)\). In particular since Rmk. \(5.19\) we have that \(\omega \circ \pi_{\tilde{t}_\mathfrak{I}(a)}\) is a state originated by the integral phase \(\tilde{m} (\mathfrak{I}, a)\) for all \(\mathfrak{I} \in \mathfrak{B}_s(\mathfrak{J})\) and \(a \in P^\mathfrak{J}\). This fact is at the basis of the proof in Thm. \(5.23\) that the map \(\mathfrak{V}_\alpha\), defined in Def. \(34\) satisfies the \(H\)-equivariance \((23)\).

Proposition 5.19. Let \(\mathfrak{H}\) be a Hilbert space, \(\Gamma\) a \(Z_2\)-grading on \(\mathfrak{H}\), and \(D\) a possibly unbounded selfadjoint operator in \(\mathfrak{H}\). If \(\Gamma D\Gamma = -D\) and \(D\) is positive, then \(D = 0\).

Proof. If \(D\) is positive then \(D = D^\dagger D\), where \(D^\dagger\) is a positive, in particular selfadjoint, operator in \(\mathfrak{H}\), thus

\[
\begin{align*}
\Gamma D\Gamma &= \Gamma D^\dagger D^\dagger \Gamma \\
&= (D^\dagger \Gamma)^*(D^\dagger \Gamma),
\end{align*}
\]

where the second equality follows since a general rule, see for example [AIT, Prp. 1.2.4.(4)], and since \(\text{Dom}(D^\dagger \Gamma)\) is dense indeed \(\text{Dom}(D^\dagger)\) is dense and \(\Gamma\) is unitary. Therefore \(\Gamma D\Gamma\) is positive, and its spectrum is a subset of \(\mathbb{R}^+\), while the spectrum of \(-D\) is a subset of \(\mathbb{R}^-\). Hence \(\Gamma D\Gamma = -D\) implies that the spectrum of \(-D\) equals \(\{0\}\) hence the statement since \((30)\) and the spectral theorem.
**Proof of Lemma 5.17** The first sentence of the statement is trivial, (98) follows since \( \omega e^{-D^2} \circ \pi_t \in \mathbb{N}_n \) and (97), so let us prove (97). Let \( t = \langle S, \alpha, \mu, u, S, D, \Gamma \rangle \), \( S = \langle S, \pi, \Omega \rangle \), \( \rho = e^{-D^2} \), \( \Phi_0 = \Phi_0 \), \( B^+ = B^+ \), \( B = B_1^+ \) and \( B_1^+ \) the closed unit ball of \( B^+ \), while \( S = S \) and \( S_0 = S \) which is nondegenerate since \( S \) it is so. We deduce that

\[
(99) \quad \Phi_0 = Tr \circ L_{\Gamma \rho} \circ S = Tr \circ L_{\rho} \circ R_\Gamma \circ S,
\]

the first equality follows since [Con2] IV.8.3, the second since the commutativity property of the trace. Next \( \Gamma \) commutes with \( D^2 \) since \( D^2 \Gamma = -D \) by construction, and since \( D^2 \Gamma = (D^2 \Gamma)^2 \Gamma, \) thus \( \Gamma \in E_{D^2}(\mathcal{B}(\mathbb{R}))' \), \( E_{D^2} \) is the resolution of the identity of the selfadjoint operator \( D^2 \) and \( \mathcal{B}(\mathbb{R}) \) is the set of the Borel subsets of \( \mathbb{R} \). Moreover since the construction of the functional calculus and [DS1] Thm. 4.10.8(f), or by [KR] 5.6.26 or [AIT] Thm 1.5.5(vi)], the functional calculus of any possibly unbounded normal operator \( A \) takes values in the algebra of the operators affiliated to the von Neumann algebra \( E_A(\mathcal{B}(\mathbb{R}))'' \), in particular the operator function belongs to this algebra if it is bounded, therefore

\[
(100) \quad [\Gamma, \rho] = 0.
\]

It is worthwhile noting that since the map composition law of the functional calculus \( (D^2)^{1/2} = |D| \) which equals \( D \) if and only if \( D \) is positive, case excluded by Prp. 5.19 and since the trivial case \( D = 0 \) has been excluded from the beginning. Therefore the fact that \( \Gamma \) commutes with \( e^{-D^2} \) does not conflict with \( D^2 \Gamma = -D \). Next \( \Gamma \) and \( \rho \) are selfadjoint then \( \Phi_0 \) is hermitian since (100) since \( Tr \) is hermitian and the commutativity property of the trace, thus

\[
(101) \quad \Phi_0 = \frac{1}{2}(\Phi_0 + \Phi_0^*).
\]

Next let \( b \in B_1^+ \) such that \( S(b) = \Gamma \) which exists by hypothesis, then since \( \Gamma^2 = 1 \), Rmk. 4.10 and (99) we have

\[
||\Phi_0|| \geq \sup\{(Tr \circ L_{\rho} \circ R_\Gamma \circ S)(ab) \mid a \in B_1^+\}
= \sup\{(Tr \circ L_{\rho} \circ S)(a) \mid a \in B_1^+\}
= Tr(\rho).
\]

Next let \( P_\pm \) be the projector associated to the Hilbert subspace \( S^\pm = \{v \in S \mid \Gamma v = \pm v\} \), thus \( P_+ + P_- = 1 \) and \( \Gamma P_j = (-1)^j P_j, j \in \{-1, 1\} \), hence for all \( a \in B_1^+ \)

\[
Tr(\Gamma \rho S(a)) = \sum_{j=-1,1} Tr(\Gamma P_j \rho S(a))
= \sum_{j=-1,1} (-1)^j Tr(P_j \rho S(a)).
\]

Therefore since (99)

\[
(103) \quad \Phi_0 = \sum_{j=-1,1} (-1)^j Tr \circ L_{P_j \rho} \circ S,
\]
in particular by Rmk. 4.10
\[ ||\Phi_0|| \leq \sum_{j=-1,1} ||Tr \circ L_{p,j} \circ S|| \]
\[ = \sum_{j=-1,1} Tr(P_j) \]
\[ = Tr(\rho), \]
which together (102) implies
\[ ||\Phi_0|| = \sum_{j=-1,1} ||Tr \circ L_{p,j} \circ S|| = Tr(\rho). \]
So \( \Phi^\beta_0 = \omega_p \circ S \) since (101,103,104) and Rmk. 5.7 hence
\[ (105) \quad \Phi^\beta_0 \upharpoonright \mathcal{B} = \omega_p \circ S_0, \]
and (97) follows since Cor. 4.4(1).

**Definition 34.** Define
\[ \mathcal{I}_* : Obj(\mathcal{G}(G, F, \rho)) \ni M \mapsto \mathcal{I}_M, \]
\[ \overline{m}_* : Obj(\mathcal{G}(G, F, \rho)) \ni M \mapsto \overline{m}_M, \]
\[ Dom(\mathcal{B}_*) := \{ M \in Obj(\mathcal{G}(G, F, \rho)) \mid \mathcal{B}_*(M) \neq \emptyset \}, \]
moreover set the map \( \mathcal{V}_* \) defined on \( Dom(\mathcal{B}_*) \) such that for all \( M \in Dom(\mathcal{B}_*), \mathcal{I} \in \mathcal{B}_*(M) \) and \( \alpha \in \mathbb{P}^\mathcal{I}_M \)
\[ \mathcal{V}_*(M) \in \coprod_{\mathcal{I} \in \mathcal{B}_*(M)} \prod_{\alpha \in \mathbb{P}^\mathcal{I}_M} \mathcal{R}_M(\overline{m}_M(\mathcal{I}, \alpha)), \]
\[ \mathcal{V}_*(M)(\mathcal{I}, \alpha) := \Psi_{\mathcal{I}}(\mathcal{I}, \alpha) \circ \mathcal{I}_M(\mathcal{I}, \alpha). \]

**Remark 5.20.** \( (e^M \circ \mathcal{I}_M)(\mathcal{I}, \alpha) \) is \( \mathcal{R}_M^\mathcal{I}(\mathcal{I}, \alpha) \) for all \( M \in Dom(\mathcal{B}_*), \mathcal{I} \in \mathcal{B}_*(M) \) and \( \alpha \in \mathbb{P}^\mathcal{I}_M \) since Rmk. 5.9 and Lemma 5.17 so \( \mathcal{V}_* \) is well-defined.

**Lemma 5.21.** Let \( \mathcal{B}, \mathcal{C} \) and \( \mathcal{D} \) be three \(*\)-algebras, where \( \mathcal{D} \) is unital with unit \( 1 \). \( U \in \mathcal{U}(\mathcal{D}), \mathcal{R} \in Hom^\mathcal{I}(\mathcal{C}, \mathcal{D}) \) and \( \eta \in Hom^\mathcal{I}(\mathcal{B}, \mathcal{C}) \). Then \((\mathcal{R} \circ \eta) = \tilde{\mathcal{R}} \circ \eta^+ \) and \((ad(U) \circ \mathcal{R}) = ad(U) \circ \tilde{\mathcal{R}} \).

**Proof.** Let \( b \in \mathcal{B}, c \in \mathcal{C} \) and \( \lambda \in \mathcal{C} \), then
\[ (\mathcal{R} \circ \eta)(b, \lambda) = (\mathcal{R} \circ \eta)(b) + \lambda 1 \]
\[ = \tilde{\mathcal{R}}(\eta)(b, \lambda) = (\tilde{\mathcal{R}} \circ \eta^+)(b, \lambda), \]
and
\[ (ad(U) \circ \mathcal{R})(c, \lambda) = (ad(U) \circ \mathcal{R})(c) + \lambda 1 \]
\[ = ad(U)(\mathcal{R}(c) + \lambda 1) = (ad(U) \circ \tilde{\mathcal{R}})(c, \lambda). \]

**Lemma 5.22.** \( \mathcal{I} \in \mathcal{I} \) and \( \alpha \in \mathbb{P}^\mathcal{I}_a \) such that there exists an element \( b \in (\mathcal{B}_a^\mathcal{I}(\mathcal{I}))^* \) for which \( ||b|| \leq 1 \) and \( \tilde{\mathcal{R}}^\mathcal{I}_a(\mathcal{I})(b) = \Gamma^\mathcal{I}_a \). If \( l \in H \) set \( b^l = (w^l)^* \gamma(l)(b) \) then \( b^l \in (\mathcal{B}_a^\mathcal{I}(\mathcal{I}))^* \) such that \( ||b|| \leq 1 \) and \( \tilde{\mathcal{R}}^\mathcal{I}_a(\mathcal{I})(b^l) = \Gamma^\mathcal{I}_a \). In particular if \( \mathcal{B}_*(\mathcal{I}) \neq \emptyset \) then \( b(l)(\mathcal{B}_*(\mathcal{I})) \subseteq \mathcal{B}_*(\mathcal{I}). \)

\[ \square \]
Proof. The inequality follows since \((w_\alpha^z)^*(l)\) is an isometry being \(w_\alpha^z(l)\) so, while the equality follows since (79) and Lemma 5.21.

The following important result ensures the existence of an equivariant stability; here the requests (77, 78, 79, 81) find their justification since are used in its proof directly and via Lemma 5.22.

**Theorem 5.23 (Existence of a full integer equivariant stability on \(\mathcal{B}_*\)).** \(\langle \mathfrak{I}_*, \overline{\mathfrak{I}}_*, \mathfrak{V}_* \rangle\) is a full integer equivariant stability on \(\mathfrak{V}_*\). Moreover for all \(N \in \text{Dom}(\mathfrak{B}_*), \mathfrak{I} \in \mathfrak{B}_*(N)\) and \(\alpha \in \mathbb{P}_N^z\)

\[
V_*(N)(\mathfrak{I}, \alpha) = \omega_{\exp(-(\mathbb{D}^z)^2)} \circ (\mathbb{N}_N^z)\alpha.
\]

**Proof.** (106) follows since (97). \(\langle \mathfrak{I}_*, \overline{\mathfrak{I}}_*, \mathfrak{V}_* \rangle\) is a full equivariant phase according Rmk. 5.5 while \(V_*\) satisfies Def. 29(1) since construction, \(\mathfrak{B}_*\) satisfies Def. 27(2) since Lemma 5.22 while Def. 29(2) follows by construction. Let us show that \(V_*\) satisfies (93). Let \(\mathfrak{I} \in \mathfrak{B}_*(S), \alpha \in \mathbb{P}^z, l \in H, h \in \{1, l\} \) and set temporarily

\[
\rho_l = \exp(-(\mathbb{D}^z)^2) \quad \eta_l^{-1} = \eta_\alpha^z(l^{-1})
\]

\[
\psi_l = \psi(\alpha(\mathbb{I}^z, \alpha)) \quad \psi_l^{-1} = \psi_\alpha^z(l^{-1})
\]

\[
\mathfrak{R}_l = \mathfrak{R}_\alpha^z(S) \quad \mathfrak{w}_l = \mathfrak{w}_\alpha^z(l^{-1})
\]

\[
\mathfrak{j}_l = \mathfrak{m}_\alpha^z \quad \mathfrak{v}_l = \mathfrak{v}_\alpha^z(l)
\]

\[
\pi = \pi_\alpha^z \quad i = \mathfrak{i}_\alpha^z
\]

\[
\mathfrak{H}_l = \mathfrak{H}_\alpha^z.
\]

We remove the index \(h\) whenever it equals 1.

Since (105) and Lemma 4.22 we deduce (107)

\[
\psi_l^{-1} = \omega_{\rho_l} \circ \mathfrak{R}_l^{-1}.
\]

Next \(\mathfrak{R}_l = \text{ad}(\mathfrak{v}_l) \circ \mathfrak{R} \circ \mathfrak{w}_l^{-1}\) by (83) and (79), hence

\[
\mathfrak{R}_l^{-1} = (\text{ad}(\mathfrak{v}_l) \circ \mathfrak{R})^{-1} \circ (l \circ \mathfrak{w}_l^{-1})^{-1}
\]

\[
= \text{ad}(\mathfrak{v}_l) \circ \mathfrak{R}^{-1} \circ (l \circ \mathfrak{w}_l^{-1})^{-1},
\]

where the first equality follows since \(\mathfrak{w}_l^{-1}\) is surjective, \(\mathfrak{R}_l\) is nondegenerate and Cor. 4.16 while the second one follows since \(\mathfrak{R}\) and \(\text{ad}(\mathfrak{v}_l) \circ \mathfrak{R}\) are nondegenerate and Rmk. 4.14. Next \(\mathfrak{R}_l^{-1} \circ \mathfrak{j}_l = \text{ad}(\mathfrak{v}_l) \circ \mathfrak{R} \circ \mathfrak{j} \circ \mathfrak{w}_l^{-1} \circ \mathfrak{j}_l^{-1}\) since (86) and (108), thus \(\mathfrak{R}_l^{-1} \circ \mathfrak{j}_l = \text{ad}(\mathfrak{v}_l) \circ \pi \circ \mathfrak{w}_l^{-1} \circ \mathfrak{j}_l^{-1}\) since (11), which together (107) yields (109)

\[
\psi_l^{-1} \circ \mathfrak{j}_l = \omega_{\rho_l} \circ \text{ad}(\mathfrak{v}_l) \circ \pi \circ \mathfrak{w}_l^{-1} \circ \mathfrak{j}_l^{-1}.
\]

Next \(\rho_l = \text{ad}(\mathfrak{v}_l)(\rho)\) since (78) and (41), thus for all \(a \in \mathcal{L}(S)\)

\[
\text{Tr}_{\mathfrak{R}_l}(\mathfrak{a}(\rho_l)(\mathfrak{a})) = \text{Tr}_{\mathfrak{R}_l}(\mathfrak{a}(\rho_l)(\rho_a)) = \text{Tr}_{\mathfrak{R}}(\rho_a),
\]

in particular \(\text{Tr}_{\mathfrak{R}_l}(\rho_l) = \text{Tr}_{\mathfrak{R}}(\rho)\) so \(\omega_{\rho_l} \circ \text{ad}(\mathfrak{v}_l) = \omega_{\rho}\), therefore since (109) and (83)

\[
\psi_l^{-1} \circ \mathfrak{j}_l = \omega_{\rho} \circ \pi \circ \mathfrak{w}_l^{-1} \circ \mathfrak{j}_l^{-1}
\]

\[
= \omega_{\rho} \circ \pi \circ (\mathfrak{j}_l^{-1} \circ \mathfrak{w}_l^{-1} \circ \mathfrak{j}_l) \circ \mathfrak{j}_l^{-1},
\]

and (93) follows by (83) and (97).
Remark 5.24. If $\phi$ is a $\beta$–KMS state for some one-parameter dynamics $\tau$ with $\beta > 0$, i.e. a thermal equilibrium state at the inverse temperature $\beta$ of the system whose dynamics is $\tau$, then $\phi$–normal states usually are interpreted as states obtained by performing on $\phi$ small perturbations or operations. Thus, according Def. [18(6), 76] and [BR 2, p. 77], for any $\alpha \in A^*$ and $r \in \text{Rep}^\beta(\alpha)$ such that $\alpha_r \in \mathbb{R}^*_+$, it follows that $\Psi_r^\alpha \circ i_r$ is a state generated by an operation performed on the thermal equilibrium state $\varphi_{\alpha_r}^\tau$ at the inverse temperature $\alpha_r$ of the system whose dynamics is $\varepsilon_{\alpha_r}(-\alpha_r^{-1}(\cdot))$.

Remark 5.25 (Noncommutative geometric nature of the degeneration of $\mathcal{N}^\beta(\cdot)$). Let $\alpha \in A^*$ and $\chi \in \mathcal{N}(\alpha)$, so there exists $r \in \text{Rep}^\beta(\alpha)$ such that

$$\chi = \Psi_r^\alpha \circ i_r.$$  

Next $\Psi_r^\alpha$ is the canonical extension of $\Psi_r$ to $M(B_r)$, where $\Psi_r$ is the restriction to $B_r$ of the state associated to the 0–dimensional component of the entire normalized cyclic even cocycle $\Phi^\tau$ on $B_r$ such that $\alpha = v_r$ i.e.

$$\Psi_r^\alpha \circ i_r = \Psi_r,$$

$$\Psi_r = (\Phi_0^\tau)^r \upharpoonright B_r,$$

$$c(f) = \mathfrak{R}\left\{u_r(f), [\Phi^\tau]\right\}_r \forall f \in A.$$

(110) and (111) justify Assumption [15a] where we propose to consider any element in $\mathcal{N}(\alpha)$ as a state whose occurrence signalels the occurrence of the phase $\alpha$. If we get an entire normalized even cocyle $\tilde{\Phi}$ on $B_r$ such that

$$\tilde{\chi} \doteq ((\tilde{\Phi}_0)^\tau \upharpoonright B_r)^\alpha \circ i_r \in \mathcal{N}_r,$$

$$[\Phi^\tau] = [\tilde{\Phi}],$$

then $\tilde{\chi} \neq \chi$ implies that $\mathcal{N}(\alpha)$ is degenerate and that the nature of this degeneration is of noncommutative geometric nature.

Next we advance a physical assumption by proposing that the elements of $\mathcal{N}^\beta(\cdot)$ are those states, of suitable systems $\mathcal{O}$’s, signaling the occurrence of the phase $\alpha$ of $\mathcal{N}$, i.e. such that whenever one of the systems $\mathcal{O}$’s occurs in one of these states, then the physical system $\mathcal{N}$ previously occurred in the phase $\alpha$. We call the $\mathcal{O}$’s the systems generated by $\mathcal{N}$ and the elements of $\mathcal{N}^\beta(\cdot)$ the states originated by $\alpha$. According Rmk. 5.25 the degeneration of the set of the states originated by the same phase can be of noncommutative geometric nature. Even if we shall not use expressly the following feature except in Def. 76 we remark that we can think of $G$ as the group of spatial translations, mostly $\mathbb{R}^4$, and $F$ the group containing the gauge group $F_0$ and the remaining symmetries of the system, in such a case $\rho$ restricted to $F_0$ needs to be trivial.

Assumption 1. For any $\mathcal{N} \in \text{Obj}(\mathcal{G}(G, F, \rho))$ there exists a unique physical system denoted still by $\mathcal{N}$ such that

1. $A_\mathcal{N}$ is the class of the observables of $\mathcal{N}$;
2. $A_\mathcal{N}$ is the class of the states, said phases of $\mathcal{N}$,
3. $\mathfrak{I}_\mathcal{N}$ is the class of the operations of $\mathcal{N}$,
4. for all $\mathfrak{I} \in \mathfrak{I}_\mathcal{N}$ and $\alpha \in \mathcal{P}_\mathcal{N}$ there exists a physical system $\mathcal{O}_\mathfrak{I}$ such that $A_\mathfrak{I}$ and $E_{A_\mathfrak{I}}$ are the algebra of the observables and the class of the states of $\mathcal{O}_\mathfrak{I}$ respectively,
Definition 35. We call interpretation any map 
\[ r : \mathcal{A}_\alpha \to B \]
that satisfies the following properties hold:

(a) if \( s(\alpha) \) occurs in the state \( \psi_n \), then \( N \) occurred in the phase \( r \).

(b) if the operation \( \mathcal{T}_r \) is performed on the system \( \mathcal{A}_\alpha \), when the state \( \mathcal{P} \) occurs, then \( \mathcal{Q}_r \) will occur in the state \( \psi^n \).

(c) if \( \alpha \in \mathbb{R}^+ \) then \( \mathcal{Q}_r \) evolves in time via \( (\varepsilon)^{\mathcal{T}_r}(\alpha^{-1}(\cdot)) \).

Now we fix those properties of any physical interpretation satisfying Assumption 1 and Rmk. 5.24. In the remaining of this section, except in Proposition 5.29, we fix the following:

1. \( s(\alpha) \) is the system generated by \( \alpha \) whose observable is \( A^\alpha \).
2. \( s(\mathcal{P}) \) is the state \( \chi \) of \( s(\alpha) \).
3. \( s(\mathcal{Q}) \) is the system obtained by transforming \( \mathcal{P} \) through \( l \),
4. \( s(\mathcal{S}) \) is the operation obtained by transforming \( \mathcal{Q} \) through \( d \).

Now we fix those properties of any physical interpretation satisfying Assumption 1 and Rmk. 5.24. In the remaining of this section, except in Proposition 5.29, we fix the following:

- \( s(\alpha) \) is a fixed interpretation.
- \( s(\mathcal{P}) \) is the system generated by \( \alpha \), whose observable is \( A^\alpha \),
- \( s(\mathcal{Q}) \) is the state \( \chi \) of \( s(\alpha) \),
- \( s(\mathcal{S}) \) is the system obtained by transforming \( s(\mathcal{P}) \) through \( l \),
- \( \chi(\alpha) \) equals the mean value of \( s(\mathcal{Q}) \) in \( s(\mathcal{S}) \),
- if \( \alpha \in \mathbb{R}^+ \) then \( s(\mathcal{Q}) \) is the observable of the system \( \mathcal{S} \), originating \( s(\mathcal{Q}) \) through \( l \),
- if \( \mathcal{M} \) is a morphism \( \mathcal{M}(\mathcal{G},\mathcal{F},\mathcal{P}) \), \( (\alpha, \mathcal{F}) \in \mathcal{M}(\mathcal{G},\mathcal{F},\mathcal{P}) \), set for any \( \mathcal{P} \in \mathcal{M} \):
  - \( s(\mathcal{Q}) \) is the observable obtained by transforming \( s(\mathcal{Q}) \) through \( \mathcal{P} \).
Proof.\ (Prop. 5.27); clarify the thermal nature of a phase, state the properties of $H$–equivariance and phase transition of an equivariant stability and the relationship between its phase and state map (Prop. 5.29).

**Proposition 5.26** (Thermal nature of $s(\ell)$). We have

1. $s(\Psi^-_r \circ i_r) \equiv$ the state $\Psi^-_r \circ i_r$ occurring by performing the operation $\mathcal{T}_r$ on the state $\varphi^-_{\alpha_r}$ of the system $\mathcal{G}_{\alpha_r}$ generated by $\mathcal{G}$,
2. if $\alpha_r \in \mathbb{R}^+_0$ then $s(\Psi^-_r \circ i_r)$ occurs by performing the operation $\mathcal{T}_r$ on the thermal equilibrium state $\varphi^-_{\alpha_r}$ at the inverse temperature $\alpha_r$ of the system, generated by $\mathcal{G}$, whose dynamics is $\epsilon^{-1}_{\alpha_r}(\cdot)$,
3. $\ell(f)$ is the mean value of the observable $f$ in the phase, of the system $\mathcal{G}$, originating $s(\Psi^-_r \circ i_r)$.

Next we state in the mean value form the stability formulated in Assump. 1 and Rmk. 5.25 and with the help of Lemma 5.17 we identify a class of states originated by the same integer phase. By recalling the definitions in (89) we have

**Proposition 5.27** (Stability of $s(\ell)$ under variation of the state originated). We have that

1. for all $\mathcal{P}$ in $\text{Rep}^G(\ell)$ the following values are equal
   - $\ell(f)$,
   - the mean value of the observable $f$ in the phase, of the system $\mathcal{G}$, originating $s(\Psi^-_p \circ i_p)$,
   - $v_\mathcal{P}(f)$;
2. if $\ell$ is integer then for all $q$ in $C^G(\ell)$ the following values are equal
   - $\ell(f)$,
   - the mean value of the observable $f$ in the phase, of the system $\mathcal{G}$, originating $s(\Psi^-_{\ell(q)} \circ i_{\ell(q)})$,
   - $w_\mathcal{P}(f)$;
3. if $\ell$ is integer then $\Psi^-_{\ell(q)} \circ i_{\ell(q)} = \omega^{-c}_{\ell(q)} \circ \pi_q$ for all $q$ in $C^G_{\ell(q)}(\ell)$.

*Proof.* Sts. (12) are easy to prove, while st. (3) follows since (97). \(\square\)

**Proposition 5.28** (Equivariance 1 of a $C$–equivariant stability on $\mathfrak{D}$). Under the hypothesis of Prp. 5.13

1. The following values are equal
   - the mean value in $\tilde{s}^{\mathcal{T}(a)}(m^{\mathcal{T}(a)}(\mathfrak{S}, \mathfrak{A}))$ of $\tilde{s}^{\mathcal{T}(a)}(f)$,
   - the mean value in $\tilde{s}^{\mathcal{T}(a)}(m^{\mathcal{T}(a)}(b^{\mathcal{T}(a)}(l)(\mathfrak{S}, \mathfrak{A}))$ of the observable, obtained by transforming $\tilde{s}^{\mathcal{T}(a)}(f)$ through $l$.
2. The following values are equal
   - the mean value in $\tilde{s}^{\mathcal{T}(b)}(m^{\mathcal{T}(b)}(\mathfrak{S}, \mathfrak{B}))$ of the observable, obtained by transforming $\tilde{s}^{\mathcal{T}(a)}(f)$ through $\mathcal{T}_1(f)$,
   - the mean value in $\tilde{s}^{\mathcal{T}(a)}(m^{\mathcal{T}(a)}(\mathcal{T}_2(f)(\mathfrak{S}, \mathfrak{B}))$ of $\tilde{s}^{\mathcal{T}(a)}(f)$.

*Proof.* Since (94). \(\square\)

**Proposition 5.29** (Physical properties 2 of a $C$–equivariant stability on $\mathfrak{D}$). Under the hypothesis of Prp. 5.13 and if $\mathfrak{A} \in \Theta(\mathfrak{S}, \mathfrak{T})$ and $\mathfrak{S} \in \mathfrak{D}_{\mathcal{T}(a)}$, then $b^{\mathcal{T}(a)}(l)(\mathfrak{S}) \in \mathfrak{D}_{\mathcal{T}(a)}$ and

1. $\tilde{s}^{\mathcal{T}(a)}(W^{\mathcal{T}(a)}(\mathfrak{S}, \mathfrak{A})) \equiv$ the state $W^{\mathcal{T}(a)}(\mathfrak{S}, \mathfrak{A})$ occurring by performing the operation $\mathcal{T}_r$ on $\tilde{s}^{\mathcal{T}(a)}((\varphi^{\mathcal{T}(a)})_r)$,
(2) \( s^{T(a)}(W^{T(a)}(b^{T(a)}(l)(\mathcal{Z}), \alpha)) \equiv \) the state \( W^{T(a)}(b^{T(a)}(l)(\mathcal{Z}), \alpha) \) occurring by performing the operation, obtained by transforming \( s^{T(a)}(\mathcal{Z}) \) through \( l, \) on \( s^{T(a)}((\varphi^{T(a)})^{l}(\mathcal{Z})) \).

(3) \( s^{T(b)}(W^{T(b)}(\mathcal{Z}, \beta)) \equiv \) the state \( W^{T(b)}(\mathcal{Z}, \beta) \) occurring by performing the operation \( \mathcal{Z} \) on \( s^{T(b)}((\varphi^{T(b)})^{T(b)}(\mathcal{Z})) \).

(4) \( s^{T(a)}(W^{T(a)}(F_{2}(t)(\mathcal{Z}), \beta)) \equiv \) the state \( W^{T(a)}(F_{2}(t)(\mathcal{Z}), \beta) \) occurring by performing the operation, obtained by transforming \( s^{T(a)}(\mathcal{Z}) \) through \( F_{2}(t), \) on \( s^{T(a)}((\varphi^{T(a)})^{2\tau(t)(\mathcal{Z}))} \).

(5) **Thermal nature of \( m^{T(a)} \).** \( m^{T(a)}(\mathcal{Z}, \alpha)(f) \) equals the mean value of \( s^{T(a)}(f) \) in the phase, of the system \( F(a) \), originating \( s^{T(a)}(W^{T(a)}(\mathcal{Z}, \alpha)) \);

(6) **\( H\text{-invariance of the m.v.} \) in \( s^{T(a)} \circ W^{T(a)} \).** The following values are equal

- the mean value in \( s^{T(a)}(W^{T(a)}(\mathcal{Z}, \alpha)) \) of \( s^{T(a)}(a) \),
- the mean value in \( s^{T(a)}(W^{T(a)}(b^{T(a)}(l)(\mathcal{Z}), \alpha)) \) of the observable, obtained by transforming \( s^{T(a)}(a) \) through \( l \);

(7) **\( H\text{-invariance of the m.v.} \) in \( s^{T(a)} \circ m^{T(a)} \).** The following values are equal

- the mean value of \( s^{T(a)}(f) \) in the phase, of the system \( F(a) \), originating \( s^{T(a)}(W^{T(a)}(\mathcal{Z}, \alpha)) \),
- the mean value of the observable, obtained by transforming \( s^{T(a)}(f) \) through \( l \), in the phase, of the system \( F(a) \), originating \( s^{T(a)}(W^{T(a)}(b^{T(a)}(l)(\mathcal{Z}), \alpha)) \);

(8) **System equivariance of the m.v. \( m \).** The following values are equal

- the mean value of the observable, obtained by transforming \( s^{T(a)}(f) \) through \( F_{1}(t) \), in the phase, of the system \( F(b) \), originating \( s^{T(b)}(W^{T(b)}(\mathcal{Z}, \beta)) \),
- the mean value of the observable \( s^{T(a)}(f) \) in the phase, of the system \( F(a) \), originating \( s^{T(a)}(W^{T(a)}(F_{2}(t)(\mathcal{Z}), \beta)) \).

(9) **Phase transition of \( s^{T(a)} \circ m^{T(a)} \) via symmetry breaking.** \( s^{T(a)}(m^{T(a)}(\mathcal{Z}, \delta)) \) exists only for those \( \delta \in \Gamma^{T(a)} \cap (\mathbb{R}^{+} - \{0\}) \) such that the symmetry \( F(a, \mathcal{Z}, (\beta^{T(a)})^{3}) \) is not broken, i.e. \( F(a, \mathcal{Z}, \delta) \cong F(a, \mathcal{Z}, (\beta^{T(a)})^{3}) \), where we set \( F(a, \mathcal{Z}, \delta) \equiv F_{1}(\varphi^{T(a)})(\alpha_{T(a)}^{3}) \).

(10) **Stability of \( s^{T(a)}(m^{T(a)}(\mathcal{Z}, \alpha)) \).** The statements of **Prop. 5.27** remains true in replacing \( \mathcal{Z} \) by \( m^{T(a)}(\mathcal{Z}, \alpha) \).

**Proof.** Prop. 5.29 \& 6.28 follow since Prop. 5.13, the phase transition statement follows by construction, see (74), the remaining statements follow by **Def. 35** \( \Box \)

**Remark 5.30.** Def. 29 \& 2 are required to ensure Prop. 5.29.

6. **The functor \( (\mathcal{H}, g^{H} \times b^{H}) \)**

In Thm. 6.24 we construct a functor from \( C_{u}(H) \) to \( \mathcal{H}(G, F, \rho) \). Thus since Thm. 5.23 we can state in our main Thm. 6.25 the existence of a full integer \( C_{u}(H) \)-equivariant stability on \( \mathcal{H} \). In the present section we assume fixed two locally compact topological groups \( G \) and \( F \), a group homomorphism \( \rho : F \rightarrow Aut(G) \) such that the map \((g, f) \mapsto \rho_{f}(g) \) on \( G \times F \) at values in \( G \) is continuous. Let \( H \) denote \( G \approx F \).

6.1. **The object part \( \mathcal{H}^{H} \).** The main result in this section is Cor. 6.14 were we construct the object part of a functor from \( C_{u}(H) \) to \( \mathcal{H}(G, F, \rho) \). Auxiliary important results in this directions are Thm. 3.46 and (116), Thm. 6.9, Thm. 6.11 and Thm. 6.13. In the present

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2Thus \( F(a, \mathcal{Z}, \delta) = \{ h \in F | (\varphi^{T(a)})^{3} \circ (\alpha^{T(a)})^{3} \circ j_{2}(h) = (\varphi^{T(a)})^{3} \} \), where \( j_{2} \) is the canonical injection of \( F \) in \( H \).
section 6.1 we fix a dynamical system $\mathcal{A} = \langle A, H, \sigma \rangle$, while starting from Def. 43 and except Thm. 6.13 $\mathcal{A}$ is inner and we fix a group morphism $\nu : H \to \mathcal{U}(A)$ such that $\mathcal{A}$ is implemented by $\nu$. By taking into account (28), Def. 1 and notations after (32) we can give the following

**Definition 36.** Let $\omega \in E^C_A(\tau)$. We say that $\langle \mathcal{S}, \mu, \zeta, f \rangle$ is a $\langle \mathcal{A}, \omega \rangle$--selfadjoint system if

1. $\mathcal{S} := \langle \mathcal{S}, \pi, \Omega \rangle$ is a cyclic representation of $A$ associated with $\omega$;
2. $\mu \in \mathcal{H}(S^C_{F, \omega})$;
3. $\zeta : \mathbb{R}^X \to S^C_{F, \omega}$ is a continuous group morphism, where $X$ is a nonempty set;
4. $f$ is a $C_0(\mathbb{C}^X)$--measurable map such that there exists an $A \in \mathcal{P}_\omega(X)$ satisfying $f(\mathcal{C}(A, \bigcap_{x \in A} \text{supp}(E_{T,x}))) \subseteq \mathbb{R}$.

Here $T = \{ T_x \}_{x \in X}$ is such that $iT_x$ is the infinitesimal generator of the strongly continuous one-parameter semigroup of unitarities $U_\mathcal{S} \circ \zeta \circ i_x \uparrow \mathbb{R}^+$ on $\mathcal{S}$, where $i_x : \mathbb{R} \to \mathbb{R}^X$ is such that $\text{Pr}_y \circ i_x = \delta_{x,y} \text{Id}_{\mathbb{R}}$, for all $x, y \in X$. Set

$$D^C_{\mathcal{S}}(\mathcal{A}) := f(\mathcal{E}_T).$$

We convene to remove $(\mathcal{A})$, whenever it is clear by the context which dynamical system is involved.

**Remark 6.1.** Since $\mathbb{R}^X$ is an abelian group, we deduce by Prp. 3.9 that $\mathcal{E}_T$ is a family of commuting Borel RI’s in $\mathcal{S}$. Then we can apply Thm. 3.8(2) to $\mathcal{E}_T$ and state that $D^C_{\mathcal{S}}(\mathcal{A})$ is a well-defined selfadjoint operator in $\mathcal{S}$.

**Definition 37 (Pre thermal phases).** We say that $\mathcal{T} = \langle h, \xi, \beta_c, I, \omega \rangle$ is a pre thermal phase associated to $\mathcal{A}$, if $h \in H$, $\xi : \mathbb{R} \to G$ is a continuous group morphism, $\beta_c \in \mathbb{R}$, $I$ is a neighbourhood of $\beta_c$, and

$$\omega \in \prod_{\beta \in I} E^C_A(\tau) \cap K^{(A)\beta} \beta,$$

such that for all $\alpha \in I$

$$\begin{cases} \alpha \leq \beta_c \Rightarrow F_{\omega_\alpha} \supseteq F_{\omega_\beta} \\ \beta_c < \alpha \Rightarrow F_{\omega_\beta} \cap \bigcap F_{\omega_\alpha} \neq \emptyset \end{cases}$$

**Definition 38.** Let

- $\mathcal{T} = \langle h, \xi, \beta_c, I, \omega \rangle$ be a pre thermal phase associated to $\mathcal{A}$;
- $\mu$ be a Haar system associated to $\omega$ and $\mathcal{A}$;
- $\mathcal{S} : I \to \text{Rep}_A(A)$ be such that $\mathcal{S}_\beta = \langle \mathcal{S}_\beta, \pi, \Omega_\beta \rangle$ is a cyclic representation of $A$ associated with $\omega_\beta$, for all $\beta \in I$;
- $\zeta : \mathbb{R}^X \to S^C_{F, \omega_\beta}$ is a continuous group morphism, where $X$ is a nonempty set;
- $\Gamma \in \prod_{\alpha \in I \cap [0, \beta_c]} \mathcal{C}(\mathcal{S}_\alpha)$;
- $l \in H, \beta \in I \text{ and } \alpha \in I \cap [0, \beta_c]$.

Define

- $\zeta^l := \text{ad}(l) \circ \zeta$. 

\[ \bullet \mu^I : I \times H \ni (y, h) \mapsto \mu^I_{(y, h)} := \mu_{(y, h, y)}. \]

If there exist \( A \in \mathcal{P}_\omega(X) \) and a \( \mathcal{C}_\omega(C^X) \)-measurable map \( f \) satisfying

\[
(115) \quad f(\mathcal{C}(A, \prod_{x \in A} \text{supp}(E_{T^x}))) \subseteq \mathbb{R},
\]

where \( iT^x_y \) is the infinitesimal generator of the semigroup \( U_{\delta_x} \circ \zeta \circ i_y \uparrow \mathbb{R}^+ \) on \( \delta_a \), for all \( y \in X \), we can set

\[
\bullet D_{\delta_x}^{\zeta,f}(\mathfrak{A}) := D_{\delta_x}^{\zeta,f}(\mathfrak{A}), \quad R_{\delta_x}^{\mu,\zeta,f}(\mathfrak{A}) := (R_{\delta_x}^{\mu,\zeta,f}(\mathfrak{A}), D_{\delta_x}^{\zeta,f}(\mathfrak{A}), \Gamma_a).
\]

If \( \mathfrak{A} \) is inner implemented by \( u \) we can define

\[
\Gamma_{\delta_x,u} : \mathfrak{A} \ni [\infty, \beta, \gamma] \ni \delta \mapsto \Gamma_{\delta_x,u,\beta} := \text{ad}(\pi_u(v(l)))(\Gamma_\delta),
\]

and if in addition \( (115) \) holds we can set

\[
(1) \quad D_{\delta_x,u,\alpha,l}^{\zeta,f}(\mathfrak{A}) := D_{\delta_x}^{\zeta,f}(\mathfrak{A}), \quad (2) \quad R_{\delta_x,u,\alpha,l}^{\mu,\zeta,f}(\mathfrak{A}) := (R_{\delta_x}^{\mu,\zeta,f}(\mathfrak{A}), D_{\delta_x}^{\zeta,f}(\mathfrak{A}), \Gamma_{\delta_x,u,\alpha,l}).
\]

We convene to remove \( \mathfrak{A} \) whenever it is clear the dynamical system involved, in addition to remove both the indices \( u \) and \( l \) whenever \( l \) equals the unit.

The next result shows that Def. \( 38 \) is well-set and the \( H \)-equivariance of the operator \( D_{\delta_x,u,\alpha,l}^{\zeta,f} \) a step toward the construction in Cor. \( 6.14 \) of the object part of a functor from \( \mathcal{C}_\omega(H) \) to \( \mathcal{S}(G, F, \rho) \). Here we use the covariance of the functional calculus relative to a commuting set of resolutions of identity stated in Thm. \( 3.13 \).

**Proposition 6.2.** Def. \( 38(1) \) & \( (2) \) are well-defined and we have

\[
(116) \quad D_{\delta_x,u,\alpha,l}^{\zeta,f} = \pi_u(v(l)) D_{\delta_x}^{\zeta,f} \pi_u(v(l^{-1})).
\]

**Proof.** Since \( (114) \), Lemma \( 3.27(1) \) and the fact that any bijective map on a set \( Y \) induces an isomorphism of the order by inclusion on the power set of \( Y \), we have \( S_{F,\omega}^{\zeta} \subseteq S_{F,\omega}^{\zeta} \), so

\[
\zeta^l : \mathbb{R} \to S_{F,\omega}^{\zeta}.
\]

Therefore \( U_{\delta_x,u,\alpha,l} \circ \zeta^l \) is well-set and \( (52) \) yields

\[
(117) \quad U_{\delta_x,u,\alpha,l} \circ \zeta^l = \text{ad}(\pi_u(v(l))) \circ U_{\delta_x,u} \circ \zeta.
\]

Hence letting \( T_{x}^{\alpha,l} \) be such that \( iT_{x}^{\alpha,l} \) is the infinitesimal generator of the semigroup \( U_{\delta_x,u,\alpha,l} \circ \zeta^l \circ i_x \uparrow \mathbb{R}^+ \) we obtain for all \( x \in X \)

\[
(118) \quad T_{x}^{\alpha,l} = \pi_u(v(l)) T_{x}^{\alpha} \pi_u(v(l^{-1})),
\]

thus by Cor. \( 3.12(2) \), \( (30) \) and \( (115) \),

\[
(119) \quad f(\mathcal{C}(A, \prod_{x \in A} \text{supp}(E_{T^x}))) \subseteq \mathbb{R}.
\]

Therefore according Def. \( 36 \) the objects in Def. \( 38(1) \) & \( (2) \) are well-defined. Finally \( (116) \) follows since \( (118) \) and Thm. \( 3.13(2) \). \( \square \)
Corollary 6.3. Let

1. $\mathcal{T} = \{ h, \xi, \beta, \mathbf{I}, \mathbf{w} \}$ be a pre thermal phase associated to $\mathfrak{A}$;
2. $\mu$ be a Haar system associated to $\mathbf{w}$ and $\mathfrak{A}$;
3. $\mathfrak{S} : I \rightarrow \text{Rep}_c(\mathcal{A})$ be such that $\mathfrak{S}_\beta = \langle \mathfrak{S}_\beta, \pi_\beta, \Omega_\beta \rangle$ is a cyclic representation of $\mathcal{A}$ associated with $\mathbf{w}_\beta$, for all $\beta \in I$;
4. $\zeta : \mathbb{R}^X \rightarrow S^G_{F_{\omega,\mu}}$ is a continuous group morphism, where $X$ is a nonempty set;
5. $\Gamma \in \prod_{\alpha \in (\mathbb{R}^X \setminus \mathbb{R}^\mathfrak{A})} \mathfrak{C}(\mathfrak{A}, \mathfrak{A})$;
6. $l, u \in H$, $\beta \in I$ and $\alpha \in \{ -\infty, \beta, \} \cap I$.

Then

1. $F_{\sigma(l)(\omega)} = \text{ad}(\text{Pr}_2(l))(\omega)$;
2. $S^G_{F_{\sigma(l)(\omega)}} = \text{ad}(\{l\})(S^G_{F_{\omega,\mu}})$;
3. $\zeta : \mathbb{R}^X \rightarrow S^G_{F_{\omega,\mu}}$ is a continuous group morphism;
4. $\mu^l$ is a Haar system associated to $\mathbf{w}^l$;
5. $\langle l \cdot \rho, h, \xi, \beta, I, \mathbf{w} \rangle$ is a pre thermal phase associated to $\mathfrak{A}$;
6. if $\mathfrak{A}$ is inner implemented by $\nu$ then $R^\mu_{\mathfrak{S}(\nu)^{\mathfrak{A},\mathfrak{A},\mathfrak{A}}}(X) = R^\mu_{\mathfrak{S}(\nu)^{\mathfrak{A},\mathfrak{A},\mathfrak{A}}}(X)$, moreover if there exists an $A \in \mathcal{P}_a(X)$ and a $\mathfrak{C}_a(\mathbb{C}^X)$ measurable map $f$ satisfying (115), then

\[
D^{\zeta, f}_{\mathfrak{S}(\nu)^{\mathfrak{A},\mathfrak{A},\mathfrak{A}}} = D^{\zeta, f}_{\mathfrak{S}(\nu)^{\mathfrak{A},\mathfrak{A},\mathfrak{A}}},
\]

in particular

\[
R^\mu_{\mathfrak{S}(\nu)^{\mathfrak{A},\mathfrak{A},\mathfrak{A}}} = R^\mu_{\mathfrak{S}(\nu)^{\mathfrak{A},\mathfrak{A},\mathfrak{A}}},
\]

Proof. St. (1) & (2) follow by $\omega_p \in E^G_{\mathfrak{A}}(\tau)$, and by Lemma 3.27, St. (3) follows by st. (2), while st. (4) since Lemma 3.40. St. (5) follows since Cor. 3.26, st. (1) and since any bijective map on a set induces an isomorphism of the order by inclusion on its power class. $R^\mu_{\mathfrak{S}(\nu)^{\mathfrak{A},\mathfrak{A},\mathfrak{A}}}$ is well-set since st. (4), $D^{\zeta, f}_{\mathfrak{S}(\nu)^{\mathfrak{A},\mathfrak{A},\mathfrak{A}}}$ is well-set since st. (3), Rmk. 3.29 and (119), thus (120) and the first equality in st. (6) follow since

\[
(\zeta^l)^u = \zeta^{u^l}, \quad (\mu^l)^u = \mu^{u^l}, \quad (\mathfrak{S}(\nu)^{\mathfrak{A},\mathfrak{A}})^{u^l} = \mathfrak{S}(\nu^{u^l}).
\]

Definition 39. If $\mathcal{T} = \{ h, \xi, \beta, I, \mathbf{w} \}$ is a pre thermal phase associated to $\mathfrak{A}$ and $l \in H$, then we define $\mathcal{T}^l := \langle l \cdot \rho, h, \xi, \beta, I, \mathbf{w}^l \rangle$ which is a pre thermal phase associated to $\mathfrak{A}$ according Cor. 6.3.

Definition 40. Let $\Xi_{\mathfrak{A}}$ be the set of the $\Xi = \{ \mathcal{T}, \mu, \mathfrak{S}, \zeta, \Gamma \}$ such that

1. $\mathcal{T} = \{ h, \xi, \beta, I, \mathbf{w} \}$ is a pre thermal phase associated to $\mathfrak{A}$;
2. $\mu$ is a Haar system associated to $\mathbf{w}$ and $\mathfrak{A}$;
3. $\mathfrak{S} : I \rightarrow \text{Rep}_c(\mathcal{A})$ is such that $\mathfrak{S}_\beta = \langle \mathfrak{S}_\beta, \pi_\beta, \Omega_\beta \rangle$ is a cyclic representation of $\mathcal{A}$ associated with $\mathbf{w}_\beta$, for all $\beta \in I$;
4. $\zeta : \mathbb{R}^X \rightarrow S^G_{F_{\omega,\mu}}$ is a continuous group morphism, where $X$ is a nonempty set;
Definition 41. 

\( f \) is a \( \mathcal{C}_a(\mathbb{C}^X) \)-measurable map such that there exists an \( A \in \mathcal{P}_\omega(X) \) satisfying (115) for all \( \alpha \in \mathbb{I} \). 

(7) \( \mathcal{R}(\mathcal{I}, \alpha) := R_{\mathcal{I}, \alpha}^{\mu, \omega, f} \) is an even \( \theta \)-summable \( K \)-cycle for all \( \alpha \in \mathbb{I} \). 

In the remaining of this section we convein to denote \( \mathcal{T}_R \) simply by \( \mathcal{T} \).

Remark 6.4. We deduce since [Con 2] IV.2.\( \gamma \) Def. 11 and IV.8 Def. 1 that Def. 40(7) is equivalent to the following two requests:

(1) (126) for \( h = \text{Id} \);
(2) for all \( \alpha \in \mathbb{I} \), \( \Gamma_a \) is a unitary, selfadjoint operator on \( \mathcal{H} \) (a \( \mathbb{Z}_2 \)-grading on \( \mathcal{H} \)), such that for all \( a \in B_{\mu, \omega, +} \)

(a) \( [\Gamma_a, H_{\alpha}] = 0 \),
(b) \( \Gamma_a D_{\mathcal{H}, \alpha}^{\omega, \mu} = -D_{\mathcal{H}, \alpha}^{\omega, \mu} \).

Definition 41. Let \( \mathcal{T} = \langle \mathcal{I}, \mu, \mathcal{H}, \omega, f, \Gamma \rangle \in \mathcal{T} \), where \( \mathcal{I} = \langle h, \xi, \beta, I, \omega \rangle \), define \( \mathcal{P} \) := \( \mathbb{I} \). 

Moreover set

\[ K_{\mathcal{I}}^{\alpha}(\mathcal{I}) := K_0(B_{\mu, \omega, +}^{\mu, \alpha, +}) \]

and

\[ K_{\mathcal{I}}^{\alpha}(\mathcal{I}) := \bigcup_{\xi \in \mathcal{T}} \bigcup_{\alpha \in \mathcal{P}} K_{\mathcal{I}}^{\alpha}(\mathcal{I}) \]

Finally set \( c_{\mathcal{I}}(l) \) as the map defined on \( K_{\mathcal{I}}^{\alpha}(\mathcal{I}) \) such that for any \( \mathcal{I} \in \mathcal{T} \) and \( \alpha \in \mathcal{P} \)

\[ c_{\mathcal{I}}(l)(\mathcal{I}) \] 

Convention 6.5. If \( \omega : A \to E_{\omega}(\tau) \) with \( A \) a nonempty set and \( \mu \) is a Haar system associated to \( \omega \) and \( \mathcal{I} \), then for all \( l \in H \) and \( \alpha \in A \) we convein to denote the pairing \( \langle \cdot, \cdot \rangle_{B_{\mu, \omega, l}^{\mu, \alpha, l}} \) by \( \langle \cdot, \cdot \rangle_{\mu, \omega, l} \). Moreover we remove the index \( l \) if it equals the unity.

Definition 42. Define \( \mathcal{A}^{\mathcal{I}} \) as the group whose underlying set is

\[ \mathcal{A}^{\mathcal{I}} := \bigcup_{\xi \in \mathcal{T}} \bigcup_{\alpha \in \mathcal{P}} K_{\mathcal{I}}^{\alpha}(\mathcal{I}) \]

and provided by the pointwise composition, i.e. \( \cdot \cdot g \in \mathcal{A}^{\mathcal{I}} \) such that \( (f \cdot g)(\mathcal{I})(\alpha) := f(\mathcal{I})(\alpha) \cdot g(\mathcal{I})(\alpha) \), for all \( f, g \in \mathcal{A}^{\mathcal{I}}, \mathcal{I} \in \mathcal{T} \) and \( \alpha \in \mathcal{P} \). Moreover set

\[ m_{\mathcal{I}} \in \mathcal{F}(\mathcal{A}^{\mathcal{I}}, \bigcup_{\xi \in \mathcal{T}} \mathcal{C}^{\mathcal{P}^\mathcal{I}}(\mathcal{I})) \]

such that for any \( f \in \mathcal{A}^{\mathcal{I}} \) and \( \mathcal{I} \in \mathcal{T} \)

\[ m_{\mathcal{I}}(f)(\mathcal{I}) : \mathcal{P}^\mathcal{I} \to \mathcal{C} \]

\[ \alpha \mapsto \langle f(\mathcal{I})(\alpha), \chi(\mathcal{R}(\mathcal{I}, \alpha)) \rangle_{\mu, \omega, \alpha} \]

where \( \mathcal{I} = \langle \mathcal{I}, \mu, \mathcal{H}, \omega, f, \Gamma \rangle \) and \( \mathcal{J} = \langle h, \xi, \beta, I, \omega \rangle \). \( m_{\mathcal{I}} \) is called mean value map associated to \( \mathcal{I} \).
Definition 43. Let $l \in H$, define

$$b^\mathbb{R}_\nu(l) : \mathcal{I} \ni \left( J, \mu, \delta, \zeta, f, \Gamma \right) \mapsto \left( J', \mu', \delta'^{(\nu)}, \zeta', f, \Gamma' \right).$$

Convention 6.6. Often in the remaining of this section we convein to remove $\mathcal{I}$ from $\mathbb{A}_\mathbb{R}$, $m^\mathbb{R}$, $\mathcal{A}$, $\mathcal{K}$ and $\mathcal{K}_n(\mathcal{I})$, for any $\mathcal{I} \in \mathfrak{I}$ and $\alpha \in \mathbb{P}^\mathbb{R}$, moreover we remove $\mathcal{I}$ and $\nu$ from $b^\mathbb{R}_\nu$.

Proposition 6.7. Let $l \in H$, thus $c(l)$ is well-defined and $c(l)(K^\mathbb{Z}_n) \subseteq K_0(B^\omega_{\nu,l,+})$, for any $\mathcal{I} \in \mathfrak{I}$ and $\alpha \in \mathbb{P}^\mathbb{R}$.

Proof. Let $\mathcal{I}, \mathcal{I} \in \mathfrak{I}$ and $\alpha \in \mathbb{P}^\mathbb{R}, \beta \in \mathbb{P}^\mathbb{C}$ such that $K^\mathbb{Z}_n(\mathcal{I}) \cap K^\mathbb{G}_n(\mathcal{I}) \neq \emptyset$ thus $B \doteq B^\omega_{\mu} \cap B^\omega_{\beta} \neq \emptyset$. Now $B^\omega_{\nu}$ is the completion of $\mathcal{E}_c(\mathcal{S}^G_{F_{\mathbb{R}_\nu}}, A)$, thus its underlying set is the set of all minimal Cauchy filters of $\mathcal{E}_c(\mathcal{S}^G_{F_{\mathbb{R}_\nu}}, A)$, see for example [Top 1 II.21], therefore $B \neq \emptyset$ implies $\mathcal{E}_c(\mathcal{S}^G_{F_{\mathbb{R}_\nu}}, A) \cap \mathcal{E}_c(\mathcal{S}^G_{F_{\mathbb{R}_\nu}}, A) \neq \emptyset$. Hence a fortiori $\mathcal{S}^G_{F_{\mathbb{R}_\nu}} = S^G_{F_{\mathbb{R}_\nu}}$ so there exists a constant $C$ such that $\| \cdot \|_{M(\alpha, \beta)} = C \| \cdot \|_{M(\beta, \alpha)}$ then $B^\omega_{\nu,\mu} = B^\omega_{\mu,\beta}$ and $\sigma^\omega_{\nu,\mu} = \sigma^\omega_{\mu,\beta}$ since Cor. 3.44. Thus $K^\mathbb{Z}_n(\mathcal{I}) = K^\mathbb{G}_n(\mathcal{I})$ and $(\sigma^\omega_{\nu,\mu})^* = (\sigma^\omega_{\nu,\mu})^*$, therefore the statement follows since (15) & (21) again Cor. 3.44 and the standard picture of the functor $K_0$.

Remark 6.8 (Integrality). Let $m^\mathbb{R}$ be the mean value map associated to $\mathcal{I}$. Since the general result [Con 2 IV.8.5, Prp. 18, Thm. 19, and IV.8.8 Thm. 22] we deduce that $m^\mathbb{R}(\mathcal{I})$ is a $\mathcal{Z}$–valued map for any $f \in A$ and $\mathcal{I} \in \mathfrak{I}$.

The following Thm. 6.9, Thm. 6.11 and Thm. 6.13 are important steps towards the proof of Cor. 6.14 were we construct the object part of a functor from $C_\alpha(\mathcal{H})$ to $\mathbb{S}(G, F, \rho)$.

Theorem 6.9. $b$ and $c$ are $H$–actions on $\mathcal{I}$ and $\mathcal{K}^\mathbb{R}$ respectively, moreover for any $\mathcal{I} \in \mathfrak{I}$, $\alpha \in \mathbb{P}^\mathbb{R}$ and $l \in H$

$$c(l)(K^\mathbb{Z}_n) = K^b(l(\mathcal{I})).$$

Proof. Let $\mathcal{I} = \left( J, \mu, \delta, \zeta, f, \Gamma \right) \in \mathfrak{I}, l \in H$ and $\alpha \in \mathbb{P}^\mathbb{R}$. In this proof for any $h \in \{ Id, l \}$ we use the following notations

$$
\begin{align*}
\delta_\alpha & \doteq \delta_\alpha, 1 \doteq 1_\delta, \text{ and } Tr \doteq Tr_\delta, \\
D_h & \doteq D^\mathcal{I}_h \delta, \text{ and } R_l(D_h) \doteq (\lambda I - D_h)^{-1}, \forall \lambda \in \rho(D_h), \\
U & \doteq \tau_\alpha(v(l)), \\
B_\alpha & \doteq B^\omega_{\nu,\alpha}, \\
H_h & \doteq H^\mu_{\nu,\alpha} \delta, \\
\sigma_\nu & \doteq \sigma^\omega_{\nu,\mu}, \\
\Xi' & \doteq b(l(\mathcal{I})), \\
\chi' & \doteq c(l)(\alpha), \forall \chi \in K^\alpha_\mathcal{I}, \\
R_h & \doteq R(\Xi', \alpha).
\end{align*}
$$

This irrespectively by the fact that there could be $\delta \in \mathbb{P}^\mathbb{R}$, $\epsilon \in \mathbb{P}^\mathbb{C}$, a $\mathcal{C}$–algebra $\mathcal{D}$ and $*$–embeddings $\lambda_\delta : B^\omega_{\nu} \to \mathcal{D}$ and $\lambda_\epsilon : B^\omega_{\mu} \to \mathcal{D}$ such that $\lambda_\delta(\mathcal{E}_c(\mathcal{S}^G_{F_{\mathbb{R}_\nu}}, A)) \cap \lambda_\epsilon(\mathcal{E}_c(\mathcal{S}^G_{F_{\mathbb{R}_\nu}}, A)) = \emptyset$ although $\lambda_\delta(B^\omega_{\nu}) \cap \lambda_\epsilon(B^\omega_{\mu}) \neq \emptyset$. 

Here \(1 \equiv 1_{\mathcal{H}}\), and \(\rho(T)\) is the resolvent set of any selfadjoint operator \(T\) in \(\mathcal{H}\). We convene to remove the index \(h\) whenever it equals the unit. \([121]\) yields

\[
R_l = R_{\mathcal{H}, \mathcal{L}, \mathcal{F}, \mathcal{H}, \mathcal{F}, \mathcal{F}}^{\mathcal{L}, \mathcal{F}}.
\]

Since Cor. \([6.3]\) and \([119]\) to prove \(i' \in \mathcal{I}\) it is sufficient to show that \(R_l\) is an even \(\theta\)-summable \(K\)-cycle. Since \([116]\) and Cor. \([3.12]\)

\[
\rho(D_l) = \rho(D),
\]

moreover since Lemma \([5.21]\) and Thm. \([3.46]\) we obtain

\[
\bar{R}_l \circ \sigma^+_l = ad(U) \circ \bar{R}.
\]

Let us consider the following set of statements for \(h \in \{Id, l\}\)

\[
\begin{cases}
\{ R(\lambda, D_h) \text{ is a compact operator on } \mathcal{H}, \forall \lambda \in \rho(D_h), \\
\text{Dom}([D_h, \bar{R}_h(a)]) = \text{Dom}(D_h), \forall a \in B^+_h \\
[D_h, \bar{R}_h(a)] \in \mathcal{L}(\text{Dom}(D_h), \mathcal{H}), \forall a \in B^+_h \\
\text{Tr}(\text{exp}(\lambda D_h)) < \infty
\end{cases}
\]

then it holds by hypothesis for \(h = Id\), we claim to show it for \(h = l\). Let \(\lambda \in \rho(D)\), thus since \([116]\), we have \(\lambda 1 - D_l = U(\lambda 1 - D)U^{-1}\), and \(\text{Dom}(D_l) = U\text{Dom}(D)\), moreover \(\text{Dom}(R(\lambda, D)) = \mathcal{H}\), hence \((\lambda 1 - D_l)ad(U)(\lambda 1 - D_l) = 1\) and \(ad(U)(\lambda 1 - D_l)(\lambda 1 - D_l) = Id_{\text{Dom}(D_l)}\).

Therefore

\[
R(\lambda, D_l) = ad(U)(\lambda 1, D_l).
\]

Since \([127]\), \([126]\) for \(h = Id\) and since the class of compact operators on \(\mathcal{H}\) is a two-sided ideal of \(\mathcal{L}(\mathcal{H})\), we obtain

\[
R(\lambda, D_l) \text{ is a compact operator.}
\]

Let \(a \in B^+_l\), thus \(\text{Dom}([D_l, \bar{R}_l(\sigma^+_l(a))]) = U\text{Dom}(D)\) since \([126]\), \([125]\) and \([116]\), moreover

\[
D_l[\bar{R}_l(\sigma^+_l(a)) = \bar{R}_l(\sigma^+_l(a))D_l = U[D_l, \bar{R}_l(a)]U^{-1},
\]

hence for all \(v \in \text{Dom}(D)

\[
||[D_l, \bar{R}_l(\sigma^+_l(a))]Uv|| = ||[D_l, \bar{R}_l(a)]v|| \leq ||[D_l, \bar{R}_l(a)]||\mathcal{L}(\text{Dom}(D), \mathcal{H})||Uv||.
\]

Next \(\sigma^+_l(B^+_l) = B^+_l\) since Cor. \([3.44]\) therefore we can state for all \(b \in B^+_l

\[
\begin{cases}
\text{Dom}([D_l, \bar{R}_l(b)]) = \text{Dom}(D_l), \\
[D_l, \bar{R}_l(b)] \in \mathcal{L}(\text{Dom}(D_l), \mathcal{H}), \\
||[D_l, \bar{R}_l(b)]||\mathcal{L}(\text{Dom}(D_l), \mathcal{H}) = ||[D_l, \bar{R}_l((\sigma^+_l)^{-1}(b))]||\mathcal{L}(\text{Dom}(D_l)).
\end{cases}
\]

For any \(h \in \{ld, l\}\), \(sp(D_h) \subseteq R\), since \(D_h\) is a selfadjoint operator by Rmk. \([6.1]\) therefore \(\exp(-D_h) \in \mathcal{L}(\mathcal{H})\), since the spectral theorem, see for example \([DS,\text{ Thm.}\ 18.2.11(c)]\), and the fact that \(sp(D_l) \ni \lambda \mapsto \exp(-\lambda^2)\), is bounded. Therefore \(\text{Tr}(\exp(-D_h^2))\) is a well-set element of \(\mathcal{R}\). Since Cor. \([3.12]\) and \([116]\)

\[
\exp(-D_h^2) = ad(U)(\exp(-D^2)),
\]

hence \(\text{Tr}(\exp(-D^2)) = \text{Tr}(\exp(-D^2))\), then by \([126]\) for \(h = Id\) we obtain

\[
\text{Tr}(\exp(-D^2)) < \infty.
\]
for $h = l$ follows by (124), (128), (130) and (132). Next $\Gamma_a^l = \Gamma_{\tilde{b} \cdot \alpha}^l = ad(\mathcal{U})(\Gamma_a^l)$ so is a $\mathbb{Z}_2$-grading on $\mathcal{S}$, moreover since (125), (116), the bijectivity of $\sigma^l$ and Rmk. 6.4 we obtain for all $b \in \mathcal{B}$

\begin{equation}
\begin{aligned}
\{ [\Gamma^l_{a, r}, \bar{\mathcal{R}}(b)] & = 0, \\
\Gamma^l_{a, r} & = -D_l.
\end{aligned}
\end{equation}

Thus (126) for $h = l$ and (133) yields that $\mathcal{R}$ is an even $\theta$–summable $K$–cycle, thus $\mathcal{I}^l \in \mathcal{I}$. So $b(l)$ maps $\mathcal{I}$ into $\mathcal{I}$ and $b$ is a $H$–action on $\mathcal{I}$ since (122). Let $x \in K^\mathbb{Z}_a$ thus $x^l \in K^\mathbb{Z}_a$ since $\mathcal{I}^l \in \mathcal{I}$ and Prp. 6.7. Thus $c(l)$ maps $\mathcal{K}^\omega$ into itself since Cor. 3.44 and $c$ is a $H$–action on $\mathcal{K}^\omega$ since Cor. 3.45.

Thm. 6.9 permits the following

**Definition 44.** Define $\bar{c} : H \to \mathcal{F}(\mathcal{K}, \mathcal{K})$ such that $\bar{c}(l)(g) : = c(l) \circ g$ for all $l \in H$ and $g \in \mathcal{K}$.

**Proposition 6.10.** $\bar{c}$ is a $H$–action on $\mathcal{K}$.

**Proof.** Since Thm. 6.9.

Since $A \subset \mathcal{F}(\mathcal{I}, \mathcal{K})$, Thm. 6.9 and Prp. 6.10 permit the following

**Definition 45.** For any $l \in H$ define the map $\psi^\mathcal{A}_\omega^\mathcal{V}(l)$ on $A$ such that for all $f \in A$

\[ \psi^\mathcal{A}_\omega^\mathcal{V}(l)(f) := \bar{c}(l) \circ f \circ b^\mathcal{A}_\omega^\mathcal{V}(l^{-1}). \]

In the remaining of this section, except in Def. 46 we convene to remove $\mathcal{A}$ and $\mathcal{V}$ from $\psi^\mathcal{A}_\omega^\mathcal{V}$.

**Theorem 6.11.** Let $m$ be the mean value map associated to $\mathcal{V}$, then

1. $\psi$ is a $H$–action on $A$ via group morphisms,
2. for all $l \in H$ and $f \in A$ we have

\[ \text{ev}_l(m \circ \psi(l)) \circ b(l) = \text{ev}_l(m). \]

**Proof.** $\psi(l)$ maps $A$ into itself, in addition $\psi$ is a $H$–action since $b$ and $\bar{c}$ are $H$–actions by Thm. 6.9 and Prp. 6.10, finally $\psi(l)$ is a group morphism since the second line in (21) and since the standard picture used for the $K_0$–groups, hence st.(1) follows. Next let us adopt the notations in proof of Thm. 6.9 let $l \in H$, $f \in A$, $\mathcal{I} \in \mathcal{I}$ and $\alpha \in P^\mathbb{Z}$ then

\[ \left( \text{ev}_l(m \circ \psi(l)) \circ b(l) \right)(\mathcal{I})(\alpha) = m(\psi(l)(f))(\mathcal{I}^l)(\alpha) \]

\[ = \left< \psi(l)(f)(\mathcal{I}^l)(\alpha), \chi(R(\mathcal{I}^l, \alpha)) \right>_{\mu^l, \omega^l, \alpha} \]

\[ = \left< \psi(l)(f)(\mathcal{I}(\alpha)), \chi(R_l) \right>_{\mu^l, \omega^l, \alpha} \]

\[ = \left< \left( f(\mathcal{I})(\alpha) \right)^l, \chi(R_l) \right>_{\mu^l, \omega^l, \alpha}. \]

Hence st.(2) follows if we show that for all $x \in K^\mathbb{Z}_a$

\begin{equation}
\left< x', \chi(R_l) \right>_{\mu^l, \omega^l, \alpha} = \left< x, \chi(R) \right>_{\mu, \omega, \alpha}.
\end{equation}

Let us prove (134). By (125), (116) and (131) we obtain for all $s_0, \ldots, s_{2n} \in \mathbb{R}$ and $a_0, \ldots, a_{2n} \in \mathbb{B}^+$

\begin{equation}
\text{Tr}(\Gamma^j_a \tilde{R}_l(\sigma^j_i(a_0))) \exp(-s_0 D^2) [D^i_l, \tilde{R}_l(\sigma^j_i(a_1))] \exp(-s_1 D^2) \ldots
\end{equation}

\begin{equation}
[D^i_l, \tilde{R}_l(\sigma^j_i(a_{2n-1}))] \exp(-s_{2n-1} D^2) [D^i_l, \tilde{R}_l(\sigma^j_i(a_{2n}))] \exp(-s_{2n} D^2) \ldots
\end{equation}

\begin{equation}
(\text{Tr} \circ \text{ad}(U)) \Gamma^j_a \tilde{R}(a_0) \exp(-s_0 D^2) [D, \tilde{R}(a_1)] \exp(-s_1 D^2) \ldots
\end{equation}

\begin{equation}
[D, \tilde{R}(a_{2n-1})] \exp(-s_{2n-1} D^2) [D, \tilde{R}(a_{2n})] \exp(-s_{2n} D^2) \ldots
\end{equation}

(134) and then st. (2) follows since (135), (123) and [Con 2] IV.8.e Thm 22, Thm. 21, and IV.7.δ Thm 21.

\[\square\]

Remark 6.12 (Odd case). Let $\mathcal{I}_{\mathfrak{a}}$ be defined as $\mathcal{I}$ by replacing $K_0$ by $K_1$ and setting $\Gamma_\alpha = 1_{\mathfrak{a}}$ for all $\alpha \in I \cap [-\infty, \beta_0]$. Thus it is easy to show that Thm. 6.11 still holds with $\mathcal{I}_{\mathfrak{a}}$ in place of $\mathcal{I}$ and $K_1$ in place of $K_0$.

By recalling the definition of $v^\mathfrak{a}$ in Def. [10] we set the following.

**Definition 46.** Define $\mathcal{G}^H$ to be the map on $\text{Obj}(C_u(H))$ such that if $\mathfrak{a} \in \text{Obj}(C_u(H))$ then

\begin{equation}
\mathcal{G}^H(\mathfrak{a}) := \left\langle \mathcal{I}_{\mathfrak{a}}, l, \beta_c, P^\mathfrak{a}_c, \alpha, c, \varphi, A_{\mathfrak{a}}, \psi^\mathfrak{a}, b^\mathfrak{a}, m^\mathfrak{a}, \varepsilon^\mathfrak{a} \right\rangle,
\end{equation}

where $\mathcal{I}_{\mathfrak{a}}$ is defined in Def. [40] while if $\mathfrak{a} = \left\langle A, H, \sigma \right\rangle$ then for any $\mathcal{I} \in \mathcal{I}_{\mathfrak{a}}$ with $\mathcal{I} = \left\langle J, \mu, \delta, \xi, f, \Gamma \right\rangle$, $J = \left\langle h, \xi, \beta_c, I, \omega \right\rangle$ and $\mathfrak{H}_a = \left\langle \mathfrak{H}_a, \pi_a, \Omega_a \right\rangle$ with $\alpha \in I$ we set

- $l^\mathfrak{a} = l$;
- $\beta_c^\mathfrak{a} = \beta_c$;
- $P^\mathfrak{a}_c = I \cap [-\infty, \beta_c]$;
- $\alpha^\mathfrak{a} = \forall \alpha \in I$;
- $e^\mathfrak{a}_c = \left\langle A, \mathbb{R}, \tau^{(h,k)}_c(\alpha(\cdot)) \right\rangle$, for all $\alpha \in I$;
- $\varphi^\mathfrak{a}_c = \omega^\mathfrak{a}_c$, for all $\alpha \in I$.

Moreover

- (1) $A_{\mathfrak{a}}$ as defined in Def. [42];
- (2) $\psi^\mathfrak{a} = \psi^\mathfrak{a}_v$ as defined in Def. [45];
- (3) $b^\mathfrak{a} = b^\mathfrak{a}_v$ as defined in Def. [43];
- (4) $m^\mathfrak{a} = m^\mathfrak{a}_v$ as defined in Def. [42];
- (5) $\varepsilon^\mathfrak{a} = \left\langle \nu, u, R, L, \Delta, v, w, z \right\rangle$ such that for all $\alpha \in I$
  - (a) $\nu^\mathfrak{a}_c = \mu^\mathfrak{a}$;
  - (b) $u^\mathfrak{a}_c = \nu^\mathfrak{a} \circ e^\mathfrak{a}_c \uparrow A_{\mathfrak{a}}$;
  - (c) $\mathfrak{R}^\mathfrak{a} = \mathfrak{R}_a$;
  - (d) $L^\mathfrak{a}_c = D^\mathfrak{a}_c(f^\mathfrak{a})$;
  - (e) $\Delta^\mathfrak{a}_c = \Gamma_\alpha$;
  - (f) for all $l \in H$
Next is an important step toward the proof of Cor. 6.14 the main result of this section.

**Theorem 6.13.** Let $\mathcal{I} \in \mathcal{I}_\alpha$, $\alpha \in \mathbb{P}^\text{d}$ and $l \in H$, then

\[
(i^{\mathcal{I}}_\alpha \circ \sigma^{(\omega, \delta)})(i^{\mathcal{I}}_\alpha(a)) \circ (i^{\mathcal{I}}_\alpha \circ \sigma^{(\omega, \delta)})(f) = \quad (137)
\]

\[
(i^{\mathcal{I}}_\alpha \circ \sigma^{(\omega, \delta)})(i^{\mathcal{I}}_\alpha(a)) = (i^{\mathcal{I}}_\alpha \circ \sigma^{(\omega, \delta)})(i^{\mathcal{I}}_\alpha(a)(f)) = \quad (138)
\]

where the first and third equalities follow since (4), the second one by (65), the fourth by construction. Next for all $h \in S^{\mathcal{I}}_\alpha$

\[
\left(\sigma(l) \circ i^{\mathcal{I}}_\alpha(a)(f) \circ \text{ad}(l^{-1}) \uparrow S^{\mathcal{I}}_\alpha\right)(h) = \quad (i^{\mathcal{I}}_\alpha \circ \sigma^{(\omega, \delta)})(f)(h) = (i^{\mathcal{I}}_\alpha(\sigma(l)(a)) \circ \sigma^{(\omega, \delta)}(f)(h),
\]

hence by (137) we have

\[
(i^{\mathcal{I}}_\alpha \circ \sigma^{(\omega, \delta)})(i^{\mathcal{I}}_\alpha(a)) \circ (i^{\mathcal{I}}_\alpha \circ \sigma^{(\omega, \delta)})(f) = \quad (i^{\mathcal{I}}_\alpha \circ \sigma^{(\omega, \delta)})(i^{\mathcal{I}}_\alpha(a)) \circ (i^{\mathcal{I}}_\alpha \circ \sigma^{(\omega, \delta)})(f),
\]

where the last equality follows since (65). Next $\mathcal{C}_c(S^{\mathcal{I}}_\alpha, \mathcal{A})$ is dense in $\mathcal{B}^{\mathcal{I}}_\alpha$ moreover $\sigma^{(\omega, \delta)}$ is an isometry since Cor. 3.44, thus by (138) we deduce

\[
\text{ad}(l^{-1}) \uparrow \mathcal{K}(\mathcal{B}^{\mathcal{I}}_\alpha) = \text{ad}(l^{-1}) \uparrow \mathcal{K}(\mathcal{B}^{\mathcal{I}}_\alpha),
\]

therefore the statement follows since Lemma 4.11. \hfill \Box

**Corollary 6.14.** $\delta^{\mathcal{I}}$ maps $\text{Obj}(\mathcal{C}_\alpha(H))$ into $\text{Obj}(\delta(G, F, \rho))$.

**Proof.** $\delta^{\mathcal{I}}$ and $\psi^{\mathcal{I}}$ are well-defined $H$-actions since Thm. 6.9 and Thm. 6.11[1]. The additivity of $m^{\mathcal{I}}$ follows by the well-known additivity of the Chern-Connes character, (73) follows since Lemma 3.21[1], (74) by (114), (75) since Thm. 6.11[2], (76) by (113), the integrality since Rmk. 6.8, $\sigma^{\mathcal{I}}$ is by construction a group morphism while (82) follows since the construction of $m^{\mathcal{I}}$. Next (77), (78) and (79) since Cor. 3.44[116] and Thm. 3.46 respectively, while (80) follows by the construction of $\psi^{\mathcal{I}}$. Finally (81) follows since Thm. 6.13. \hfill \Box

Notice that $V(\delta^{\mathcal{I}}(\mathcal{U}))^{\mathcal{I}}_\alpha = \sigma$ for all $\mathcal{U} \in \text{Obj}(\mathcal{C}_\alpha(H))$, $\mathcal{I} \in \mathcal{I}_\alpha$ and $\alpha \in \mathbb{P}^{\text{d}}$, where $\sigma$ is the dynamics underlying $\mathcal{U}$.
6.2. The morphism part $\mathfrak{g}^H \times \mathfrak{d}^H$. In Thm. \[\text{6.24}\] we shall prove that $\mathfrak{g}^H$ is the object part of a functor from $\mathcal{C}_u(H)$ to $\mathfrak{h}(G, F, \rho)$.

**Definition 47.** Let $T : \mathcal{A} \to \mathcal{B}$ be a $*$-homomorphism between $C^*$-algebras and $\mathfrak{g} : \mathcal{A} \ni \alpha \mapsto \langle \mathfrak{s}_\alpha, \pi_\alpha, \Omega_\alpha \rangle \in \text{Rep}_c(\mathcal{B})$, define the map $\mathfrak{g}^T$ on $A$ such that $\mathfrak{g}^T_\beta := \langle \mathfrak{s}_\beta, \pi_\beta \circ T, \Omega_\beta \rangle$, for all $\beta \in A$.

**Lemma 6.15.** Let $T : \mathcal{A} \to \mathcal{B}$ be a surjective $*$-homomorphism between $C^*$-algebras, $\omega \in E_\mathcal{B}$ and $\mathfrak{g}$ be a cyclic representation of $\mathcal{B}$ associated with $\omega$. Then $\mathfrak{g}^T$ is a cyclic representation of $\mathcal{A}$ associated with $T^\dagger(\omega)$.

**Proof.** $T^\dagger(\omega) \in E_\mathcal{B}$ since Lemma \[\text{3.16}\] so the statement is well-set, moreover $\mathfrak{g}^T$ is cyclic since $T(A) = \mathcal{B}$. \hfill \Box

In the remaining of this section let $\mathfrak{U} = \langle \mathcal{A}, H, \eta \rangle$, $\mathfrak{B} = \langle \mathcal{B}, H, \theta \rangle$ and $\mathfrak{C} = \langle \mathfrak{C}, H, \delta \rangle$ be objects of $\mathcal{G}(H)$, $T \in \text{Mor}_{\mathcal{G}(H)}(\mathfrak{U}, \mathfrak{B})$ and $S \in \text{Mor}_{\mathcal{G}(H)}(\mathfrak{B}, \mathfrak{C})$. Finally in the proof of Thm. \[\text{6.24}\] we shall assume that $\mathfrak{U}$, $\mathfrak{B}$ and $\mathfrak{C}$ are objects of $\mathcal{C}_u(H)$, hence $T \in \text{Mor}_{\mathcal{C}_u(H)}(\mathfrak{U}, \mathfrak{B})$ and $S \in \text{Mor}_{\mathcal{C}_u(H)}(\mathfrak{B}, \mathfrak{C})$.

**Corollary 6.16.** Let $\phi \in E^G_{\mathcal{A}}(\tau_\theta)$ and $\mathfrak{g}$ be a cyclic representation of $\mathcal{B}$ associated with $\phi$, then $T^\dagger(\phi) \in E^G_{\mathcal{A}}(\tau_\eta)$ and $U^\mathfrak{g}_{\mathfrak{g}^T} = U^\mathfrak{g}_\phi$.

**Remark 6.17.** $U^\mathfrak{g}_{\mathfrak{g}^T}$ makes sense since Def. \[\text{4}\] the first sentence of the statement of Cor. \[\text{6.16}\] and Lemma \[\text{6.15}\]. The equality in Cor. \[\text{6.16}\] is well-set since Lemma \[\text{3.23}\].

**Proof of Cor. \[\text{6.16}\]** $\phi \circ T \circ \eta(j_1(\theta)) = \phi \circ \theta(j_1(\theta)) \circ T = \phi \circ T$ for all $\theta \in G$, so the first sentence of the st. follows since Lemma \[\text{3.16}\]. Let $\mathfrak{s} = \langle \mathfrak{s}, \pi, \Omega \rangle$ and $l \in S^G_{F^\tau_{T^\dagger(\phi)}}$ then for all $a \in \mathcal{A}$

$$U^\mathfrak{g}_{\mathfrak{g}^T}(l)(\pi \circ T(a)) = (\pi \circ T)(\eta(l)a)\Omega = (\pi \circ \theta(l) \circ T)(a)\Omega = U^\mathfrak{g}_\phi(l)(\pi \circ T(a))\Omega,$$

where the last equality follows since Lemma \[\text{3.23}\]. Thus the equality in the st. follows since $\mathfrak{g}^T$ is cyclic by Lemma \[\text{6.15}\]. \hfill \Box

**Proposition 6.18.** Let $\mathcal{A}$ be a nonempty set and $\mathfrak{w} : \mathcal{A} \to E^G_{\mathcal{B}}(\tau_\theta)$. Then $\mathcal{H}(\mathfrak{w}, \mathcal{B}) = \mathcal{H}(T^\dagger \circ \mathfrak{w}, \mathcal{A})$.

**Proof.** Since Cor. \[\text{6.16}\] the st. is well-set. Let $l \in H$ and $\alpha \in \mathcal{A}$ then

$$\eta'(l)(T^\dagger(\mathfrak{w}_\alpha)) = \mathfrak{w}_\alpha \circ \theta(l^{-1}) \circ T = T^\dagger(\theta'(l)(\mathfrak{w}_\alpha)),$$

hence by Lemma \[\text{3.23}\]

\[\text{(139)}\]

$$S^G_{F^\eta'(l)(T^\dagger(\mathfrak{w}_\alpha))}(\mathfrak{U}) = S^G_{F^\theta'(l)(\mathfrak{w}_\alpha)}(\mathfrak{B}),$$

and the st. follows. \hfill \Box

**Remark 6.19.** Let $\mathcal{A}$ be a nonempty set, $\mathfrak{w} : \mathcal{A} \to E^G_{\mathcal{B}}(\tau_\theta)$ and $\mathfrak{m} \in \mathcal{H}(\mathfrak{w}, \mathcal{B})$. Thus for all $\alpha \in \mathcal{A}$ and $l \in H$ let $\text{ad}_{\mathfrak{w}^\theta(l)}(l)$ on $\mathcal{C}_c(S^G_{\mathfrak{w}_\alpha}, \mathcal{A})$ at values in $\mathcal{C}_c(S^G_{\mathfrak{w}_\alpha}, \mathcal{A})$ such that

$$f \mapsto f \circ \text{ad}(l^{-1}) \upharpoonright S^G_{\mathfrak{w}_\alpha}(\mathcal{B}),$$

this map is continuous w.r.t. the inductive limit topology following the line in the proof of Lemma \[\text{3.40}\] hence there exists according \[\text{Wil}\] Cor. 2.47 a unique extension on $B^\mathfrak{w}_{\mathfrak{w}^\theta(l)}(\mathcal{B})$.
at values in $B^\omega_{\mu,\alpha,l}(\mathfrak{B})$ which will be denoted again by the symbol $\text{ad}_{\alpha}(\omega,0)(l)$. Next it is easy to see that

$$
\theta(\omega, l) = c_{\mu(\omega, l)}(\theta(l)) \circ \text{ad}_{\alpha}(\omega,0)(l)
$$

$$
= \text{ad}_{\alpha}(\omega,0)(l) \circ c_{\mu(\alpha,l)}(\theta(l)).
$$

**Remark 6.20.** Let $A$ be a nonempty set, $\omega : A \to E^I_G(\tau_\delta)$ and $\mu \in \mathcal{H}(\omega, \mathfrak{B})$. Thus for all $\alpha \in A$ and $l \in H$ since (12), Lemma 3.16 and Lemma 3.23 we have

$$
c_{\mu(\omega, l)}(T) \in \text{Hom}(B^\omega_{\mu,\alpha,l}(\mathfrak{A}), B^\omega_{\mu,\alpha,l}(\mathfrak{B})),
$$

where $B^\omega_{\mu,\alpha,l}(\mathfrak{A})$ is well-set since Prp. 6.18. In addition since (23) we have

$$
k_{\mu(\omega, l)}(T) : K_0(B^\omega_{\mu,\alpha,l}(\mathfrak{A})) \to K_0(B^\omega_{\mu,\alpha,l}(\mathfrak{B})).
$$

Moreover $T^\dagger \circ \omega : A \to E^I_G(\tau_\delta)$ and $\mu \in \mathcal{H}(T^\dagger \circ \omega, \mathfrak{A})$ since Cor. 6.16 and Prp. 6.18. Finally since (139) we deduce that

$$
\text{ad}_{\alpha}(\omega,0)(l) \circ c_{\mu(\omega, l)}(T) = c_{\mu(\omega, l)}(T) \circ \text{ad}_{\alpha}(\omega,0)(l).
$$

**Definition 48.** Define

$$
b^H(T) : \mathfrak{I}_{\mathfrak{B}} \ni \langle \mathcal{J, \mu, \mathfrak{S}, \zeta, f, \Gamma} \rangle \mapsto \langle \mathcal{J^T, \mu, \mathfrak{S^T, \zeta^T, f, \Gamma}} \rangle,
$$

where if $\mathcal{J} = \langle h, \xi, \beta, c, I, \omega \rangle$ then $\mathcal{J^T} = \langle h, \xi, \beta, c, I, T^\dagger \circ \omega \rangle$.

In the remaining of this section, except Def. 52 and Thm 6.24, we denote $b^H$ by $b$. In the following result we shall use the equivariance of the KMS-states under the action of surjective equivariant maps stated in Thm. 3.18.

**Lemma 6.21.** $b(T) : \mathfrak{I}_{\mathfrak{B}} \to \mathfrak{I}_{\mathfrak{A}}$ and $b(S \circ T) = b(T) \circ b(S)$, i.e. $(\mathfrak{I}_{\mathfrak{A}}, b)$ is a functor from $C(H)^{op}$ to Set.

**Proof.** Let $\mathcal{J} = \langle \mathcal{J, \mu, \mathfrak{S}, \zeta, f, \Gamma} \rangle \in \mathfrak{I}_{\mathfrak{B}}$ with $\mathcal{J} = \langle h, \xi, \beta, c, I, \omega \rangle$, we claim to show that $b(T)(\mathcal{J})$ satisfies the requests in Def. 40. $\mathcal{J^T}$ is a pre thermal phase associated to $\mathfrak{A}$ since Cor. 6.16 and Thm. 3.18. Therefore we obtain Def. 40 (2346) since Prp. 6.18 Lemma 6.15, Lemma 3.23 and Cor. 6.16 respectively. Next let $\alpha \in P^\omega$ and $\mathfrak{S}_\alpha = \langle \pi, \mathfrak{S}_\alpha, \Omega_\alpha \rangle$, then

$$
\mathfrak{R}^{\mu}_{\mathfrak{S}_\alpha, \alpha}(\mathfrak{A}) = (B^\omega_{\mu,\alpha,l}(\mathfrak{A}), \mathfrak{R}^{\mu}_{\mathfrak{S}_\alpha, \alpha}(\mathfrak{A})),
$$

where

$$
\mathfrak{R}^{\mu}_{\mathfrak{S}_\alpha, \alpha}(\mathfrak{A}) = (\pi_\alpha \circ T) \circ \mathbf{U}_\mathfrak{S}_\alpha^\mu \mathfrak{R}^{\mu}_{\mathfrak{S}_\alpha, \alpha}(\mathfrak{A}),
$$

$$
= (\pi_\alpha \circ T) \circ \mathbf{U}_\mathfrak{S}_\alpha^\mu \mathfrak{R}^{\mu}_{\mathfrak{S}_\alpha, \alpha}(\mathfrak{A}),
$$

$$
= (\pi_\alpha \circ \mathbf{U}_\mathfrak{S}_\alpha^\mu \mathfrak{R}^{\mu}_{\mathfrak{S}_\alpha, \alpha}(\mathfrak{A}), c_{\mu(\alpha,l)}(T),
$$

$$
= \mathfrak{R}^{\mu}_{\mathfrak{S}_\alpha, \alpha}(\mathfrak{A}) \circ c_{\mu(\alpha,l)}(T),
$$

where the second equality follows by Cor. 6.16. Next let $X$ be the nonempty set such that $X^1$ is the domain of $\zeta$, thus $D^\omega_{\mathfrak{S}_\alpha, \alpha}(\mathfrak{A}) = D^\omega_{\mathfrak{S}_\alpha, \alpha}(\mathfrak{A}) = f(E_{\mathcal{L}})$, where $L = \{L_x\}_{x \in X}$ such that $iL_x$ is the infinitesimal generator of the strongly continuous one-parameter semigroup $U^\mathfrak{S}_\alpha^\mu \circ \zeta \circ iL \uparrow \mathbb{R}^+$ on $\mathfrak{S}_\alpha$, for all $x \in X$. Thus since Cor. 6.16 we deduce that

$$
D^\omega_{\mathfrak{S}_\alpha, \alpha}(\mathfrak{A}) = D^\omega_{\mathfrak{S}_\alpha, \alpha}(\mathfrak{A}).
$$
Def. (40). (7) follows since (141), (142) and Rmk. 6.4 and our claim is proved so \( \delta(T)(\mathcal{I}) \in \mathcal{I}_{\mathfrak{q}} \).

The remaining part of the statement is easy to show. □

**Definition 49.** Define

\[
K^B_\alpha(T) := \bigcup_{\mathfrak{I} \in \mathcal{I}_B} \bigcup_{\alpha \in \mathcal{P}_T} K^B_{\alpha}(\mathfrak{I}),
\]

and

\[
\mathcal{K}^B_\alpha(T) := \bigcup_{\mathfrak{I} \in \mathcal{I}_B} \prod_{\alpha \in \mathcal{P}_T} K^B_{\alpha}(\mathfrak{I}).
\]

\( \mathcal{K}^B_\alpha(T) \) is a well-defined subset of \( \mathcal{K}^B_\alpha \) since Lemma 6.21.

**Definition 50.** Define \( h^H(T) : K^B_\alpha(T) \to K^B_\alpha \) such that for any \( \mathcal{I} = \{ \mathcal{I}, \mu, \mathfrak{S}, \zeta, f, \Gamma \} \in \mathcal{I}_B \) and \( \alpha \in \mathcal{P}_T \)

\[
h^H(T) \upharpoonright K^B_{\alpha}(\mathfrak{I}) := k_{\mu,\alpha}(T).
\]

In the remaining of this section, except Def. 52, we denote \( h^H \) by \( h \).

**Lemma 6.22.** We have

1. \( h(T) \) is well-defined and \( h(T)(K^B_{\alpha}(\mathfrak{I})) \subseteq K^B_{\alpha}(\mathfrak{I}) \), for any \( \mathfrak{I} \in \mathcal{I}_B \) and \( \alpha \in \mathcal{P}_T \);
2. \( K^B(\circ T) \subseteq \mathcal{K}^B(T) \);
3. Let \( \mathcal{I} \) be the identity map from \( K^B(\circ T) \) to \( \mathcal{K}^B(T) \), then the following diagram is commutative

\[
\begin{array}{ccc}
K^B(\circ T) & \xrightarrow{h(T)} & \mathcal{K}^B(T) \\
\mathcal{I} \downarrow & & \downarrow \mathcal{I} \\
K^B(\circ T) & \xrightarrow{h(T)\circ \mathcal{I}} & \mathcal{K}^B(T)
\end{array}
\]

Proof. \( h(T) \) is well-defined since the same argument used in Lemma 6.7, while the inclusion in st. (1) follows by Rmk 6.20. St. (2) follows by Lemma 6.21. \( h(T) \circ \mathcal{I} \) is a well-set map since st. (2), with values in \( \mathcal{K}^B(S) \) since Lemma 6.21 and st. (1). The diagram is commutative since \( k_{\mu,(\alpha)} \) is a functor from \( C_0(H) \) to \( \text{Ab} \). □

Since Lemma 6.22, we can give the following

**Definition 51.** Define \( \mathcal{K}^B(\circ T) : \mathcal{K}^B(T) \to \mathcal{K}^B(T) \) such that \( \mathcal{K}^B(\circ T)(g) := h(T) \circ g \) for all \( g \in \mathcal{K}^B(T) \).

Moreover set \( \mathcal{I} : \mathcal{K}^B(\circ T) \to \mathcal{K}^B(T) \) such that \( \mathcal{I}(g) = \mathcal{I} \circ g \) for all \( g \in \mathcal{K}^B(\circ T) \).

**Proposition 6.23.** The following diagram is commutative

\[
\begin{array}{ccc}
\mathcal{K}^B(\circ T) & \xrightarrow{\mathcal{I}} & \mathcal{K}^B(T) \\
\mathcal{I} \downarrow & & \downarrow \mathcal{I} \\
\mathcal{K}^B(\circ T) & \xrightarrow{\mathcal{I} \circ \mathcal{I}} & \mathcal{K}^B(T)
\end{array}
\]

Proof. Since (143). □

For any \( f \in \mathcal{A}_B \) the map \( f \circ h(T) \) is well-set since Lemma 6.21, moreover \( f \circ h(T) : \mathcal{I}_B \to \mathcal{K}^B(T) \), thus since Def. 51 and Lemma 6.22(1) we can give the following
Definition 52. Define the map $g^H$ on the class of morphisms of the category $\mathcal{C}_{u}(H)$ such that
\[
g^H(T) : A_\mathcal{H} \to A_\mathcal{B},
\]
and for any $f \in A_\mathcal{H}$
\[
g^H(T)(f) := \tilde{b}^H(T) \circ f \circ \tilde{b}^H(T).
\]

Theorem 6.24. $(\mathfrak{g}^H, g^H \times b^H)$ is a functor from $\mathcal{C}_{u}(H)$ to $\mathfrak{g}(G, F, \rho)$.

Proof. The part of the statement concerning $\mathfrak{g}^H$ follows since Cor. 6.14. Let $g$ denote $g^H$, thus $g(T)$ is a group morphism since the second line in (21) and since the standard picture used for the $K_0$-groups. Next we claim to show that
\[
g(S \circ T) = g(S) \circ g(T),
\]
(145)
\[
\text{ev}_t(m^g \circ g(T)) = \text{ev}_t(m^g) \circ b(T), \forall f \in A_\mathcal{H}.
\]
The first equality follows since (144) and Lemma 6.21 let us prove the second equality of (145). Let $f \in A_\mathcal{H}$, $\mathcal{I} \in \mathcal{I}_G$, and $\alpha \in P^\mathcal{I}$, moreover let $\mathcal{I} = \langle \mathcal{I}, \mu, \delta, \zeta, f, \Gamma \rangle$ with $\mathcal{I} = \langle h, \xi, \beta, c, I, \omega \rangle$, and let $\mathcal{T}^\mathcal{I}$ denote $b(T)(\mathcal{I})$, then
\[
\text{ev}_t(m^g \circ g(T))(\mathcal{I})(\alpha) = m^g(g(T)(f))(\mathcal{I})(\alpha)
\]
(146)
\[
= \langle g(T)(f)(\mathcal{I})(\alpha), \text{ch}(R(\mathcal{I}, \alpha)) \rangle_{\mu, \omega, \alpha}
\]
where the last equality follows since Rmk. 6.20 and (24). Next
\[
\left(\text{ev}_t(m^g) \circ b(T)(\mathcal{I})(\alpha) = m^g(f)(\mathcal{T}^\mathcal{I})(\alpha)
\right.
\]
(147)
\[
= \langle f(\mathcal{T}^\mathcal{I})(\alpha), \text{ch}(R(\mathcal{T}^\mathcal{I}, \alpha)) \rangle_{\mu, T^\alpha, \omega, \alpha}.
\]
Moreover by construction
\[
R(\mathcal{T}^\mathcal{I}, \alpha) = \left(\mathcal{B}^{\mathfrak{g}^H, \omega, \alpha}^+, (\mathfrak{g}), \tilde{\mathcal{B}}^{\mu} \delta, (\mathfrak{g}) \right), \text{D}^{\zeta, f}(\mathfrak{g})_\mathcal{I}, \Gamma_\alpha\right),
\]
thus we obtain since (141) & (142) and Lemma 5.21
\[
R(\mathcal{T}^\mathcal{I}, \alpha) = \left(\mathcal{B}^{\mathfrak{g}^H, \omega, \alpha}^+, (\mathfrak{g}), \tilde{\mathcal{B}}^{\mu} \delta, (\mathfrak{g}) \right) \circ \mathcal{C}^{\mathfrak{g}^H, \omega, \alpha}^+(T), \text{D}^{\zeta, f}(\mathfrak{g})_\mathcal{I}, \Gamma_\alpha\right).
\]
Hence we deduce since [Con 2, Ch IV.8.e, Thm. 22 and Thm. 21] and [Con 2, Ch IV.7.d, Thm. 21 and Lemma 20] that
\[
\text{ch}(R(\mathcal{T}^\mathcal{I}, \alpha)) = (\mathcal{C}^{\mathfrak{g}^H, \omega, \alpha}(T)), \text{ch}(R(\mathcal{I}, \alpha)).
\]
(148)
(146) & (147) & (148) imply the claimed second equality in (145). Next since (145) and Lemma 6.21
\[
(\mathfrak{g}^H \times b^H)(T) \in \text{Mor}_{\mathfrak{g}(G, F, \rho)}(\mathfrak{g}^H(\mathfrak{g}), \mathfrak{g}^H(\mathfrak{g}))
\]
while since the first equality in (145) and Lemma 6.21
\[
(\mathfrak{g}^H \times b^H)(S \circ T) = (\mathfrak{g}^H \times b^H)(S) \circ (\mathfrak{g}^H \times b^H)(T),
\]
where $\circ$ in the right side of the equality is the law of composition in the class of morphisms of $\mathfrak{g}(G, F, \rho)$, and the statement follows. □
According Cor. 6.14 and consistently with Def. 46 and Conv. 5.3, we can set the following

**Definition 53.** Let \( D \in \text{Obj}(C_u(H)) \), set \( \mathfrak{B}(D) := \mathfrak{B}_*(6^H(D)) \), \( \mathfrak{I}_D := \mathfrak{I}_{6^H(D)} \), \( A_D := A_{6^H(D)} \), \( b^D := b_{6^H(D)} \), \( \psi^D := \psi_{6^H(D)} \), \( m^D := m_{6^H(D)} \), \( D^D := D_{6^H(D)} \), \( \pi^D := \pi_{6^H(D)} \), \( \mathfrak{W}^D := \mathfrak{W}_{6^H(D)} \). If in addition \( 6^H(D) \in \text{Dom}(\mathfrak{B}_*) \) then set \( \mathcal{V}^D := \mathcal{V}_*(6^H(D)) \).

Under the notations in (112) and Def. 40 & 41 & 42, we are in the position of stating our

**Main Theorem 6.25** (Canonical \( C_u(H) \)–equivariant stability). We have that

1. \( \left( \left( \mathfrak{T}_*, \overline{m}_*, \mathcal{V}_* \right), \left( 6^H, g^H \times b^H \right) \right) \) is a full integer \( C_u(H) \)–equivariant stability on \( \mathfrak{B}_* \);
2. the statements in Prp. 5.13 & 5.28 & 5.29 hold true in replacing \( \mathfrak{C} \), \( \left( \mathfrak{H}, \mathfrak{U}, m, \mathcal{W} \right) \), \( \mathfrak{D} \) and \( \mathfrak{F} \) by \( C_u(H) \), \( \left( H_*, \mathfrak{T}_*, \overline{m}_*, \mathcal{V}_* \right) \), \( \mathfrak{B}_* \), and \( \left( 6^H, g^H \times b^H \right) \) respectively, where \( H_* \) is full;
3. let \( \mathfrak{A} \in \text{Obj}(C_u(H)) \), \( \mathfrak{T} \in \mathfrak{T}_{\mathfrak{A}} \) and \( \alpha \in \mathfrak{P}^T \), where \( \mathfrak{T} = \left( \mathfrak{T}, \mu, \mathfrak{H}, \mathfrak{E}, f, \Gamma \right) \) and \( \mathfrak{F} = \left( h, \mathfrak{E}, \mathfrak{B}, \mathfrak{W} \right) \), thus
   a. for all \( f \in \mathfrak{A} \)
      \[ \overline{m}^T(\mathfrak{T}, \alpha)(f) = \left( f(\mathfrak{T})(\alpha), \text{ch}(\mathfrak{R}(\mathfrak{T}, \alpha)) \right) \mu, \omega, \alpha \]
   b. if \( 6^H(\mathfrak{A}) \in \text{Dom}(\mathfrak{B}_*) \) and \( \mathfrak{T} \in \mathfrak{B}(\mathfrak{A}) \) then
      \[ \mathcal{V}^D(\mathfrak{T}, \alpha) = \omega \exp \left( -\left( D^\mathfrak{C}_\mathfrak{B}(\mathfrak{A}) \right)^2 \right) \circ \left( \pi^\mathfrak{C} \right)_\alpha, \]
   c. let \( \mathfrak{B} \in \text{Obj}(C_u(H)) \) and \( T \in \text{Mor}_{C_u(H)}(\mathfrak{B}, \mathfrak{A}) \), thus if \( 6^H(\mathfrak{A}) \in \text{Dom}(\mathfrak{B}_*) \), \( \mathfrak{T} \in \mathfrak{B}(\mathfrak{A}) \) and
      i. \( 6^H(\mathfrak{B}) \in \text{Dom}(\mathfrak{B}_*) \) such that \( b^H(T)(\mathfrak{T}) \in \mathfrak{B}(\mathfrak{B}) \) then
      \[ \mathcal{V}^\mathfrak{B} \left( b^H(T)(\mathfrak{T}), \alpha \right) = \mathcal{V}^\mathfrak{B}(\mathfrak{T}, \alpha) \circ T, \]
      ii. if there exists \( b \in \mathbb{B}^T_{\mu, \omega, \alpha, +}(\mathfrak{B}) \) such that \( \| b \| \leq 1 \) and \( b = \mathfrak{C}_{\mu, \alpha, +}(T)(\mathfrak{B}) \), then
      \[ \mathcal{V}^{\mathfrak{B}} \left( b^H(T)(\mathfrak{T}), \alpha \right)(\theta(l)(a)) = \mathcal{V}^{\mathfrak{B}}(\mathfrak{T}, \alpha)(T(a)), \]
      where \( \mathfrak{B} = \left( \mathfrak{B}, H, \theta \right) \) and \( b \in \mathbb{B}^T_{\mu, \omega, \alpha, +}(\mathfrak{B}) \) such that \( \mathfrak{H}^\mu_{\mathfrak{B}, \alpha}(\mathfrak{B})(b) = \Gamma_a \) existing since \( \mathfrak{T} \in \mathfrak{B}(\mathfrak{A}) \).

**Proof.** St. (1) follows since Thm. 5.23 and Thm. 6.24 st. (2) since st. (1) and Prp. 5.13 & 5.28 & 5.29 st. (3a) follows since the construction of \( 6^H \), while st. (3b) since (106). St. (3cii) follows since st. (3b) applied to \( \mathfrak{B} \) and \( \mathfrak{A} \), and by (142) switching \( \mathfrak{A} \) with \( \mathfrak{B} \). The inclusion in st. (3ciii) follows since (141) switching \( \mathfrak{A} \) with \( \mathfrak{B} \) and by Lemma 5.21 the equality follows since this inclusion, st. (1) & (3cii) and the first equality in (95) in replacing \( \mathcal{W}, \mathfrak{A}, \mathfrak{F} \) and \( \mathfrak{I} \) by \( \mathcal{V}_*, \mathfrak{B}, 6^H \) and \( b^H(T)(\mathfrak{T}) \) respectively. \( \square \)

**Definition 54.** Set \( \mathcal{E}_* := \left( \left( \mathfrak{T}_*, \overline{m}_*, \mathcal{V}_* \right), \left( 6^H, g^H \times b^H \right) \right) \) called the canonical \( C_u(H) \)–equivariant stability.
7. Natural transformations associated to \((G,F,\rho)\)

In the present section we assume fixed two locally compact topological groups \(G\) and \(F\), a group homomorphism \(\rho: F \to Aut(G)\) such that the map \((g,f) \mapsto \rho_f(g)\) on \(G \times F\) at values in \(G\), is continuous, let \(H\) denote \(G \rtimes_F F\). As a consequence of the existence of the canonical \(C_\bullet(H)\)–equivariant stability, we encode in Thm. \[7.4\] the \(C_\bullet(H), C_\bullet^0(H)\) and \(H\) equivalence properties of the maps \(\mathcal{M}_\bullet\) and \(\mathcal{V}_\bullet\), into natural transformations between functors from the categories \(C_\bullet(H)\) and \(C_\bullet^0(H)\) to \(\text{Set}^{op}\) and between functors from \(H\) to \(\text{Set}\). Finally we encode in Thm. \[7.20\] in a unique fashion the aforementioned equivariance properties, by providing, modulo a suitable equivalence relation, that the maps \(\mathcal{M}_\bullet\) and \(\mathcal{V}_\bullet\) realize natural transformations between functors from the categories \(C_\bullet(H)^{op}\) and \(C_\bullet^0(H)^{op}\) to the category \(\text{Fct}(H,\text{Set})\). Let us start with the following easy to prove result.

**Proposition 7.1.** There exists a unique subcategory \(C_\bullet^0(H)\) of \(C_\bullet(H)\) such that

\[
\text{Obj}(C_\bullet^0(H)) = \{ \mathfrak{A} \in \text{Obj}(C_\bullet(H)) \mid \mathcal{F}_H(\mathfrak{A}) \in \text{Dom}(\mathfrak{B}_\bullet) \},
\]

and for any \(\mathfrak{A}, \mathfrak{B} \in \text{Obj}(C_\bullet^0(H))\) we have

\[
\text{Mor}_{C_\bullet^0(H)}(\mathfrak{B}, \mathfrak{A}) = \{ T \in \text{Mor}_{C_\bullet(H)}(\mathfrak{B}, \mathfrak{A}) \mid (\forall \Xi \in \mathfrak{B}(\mathfrak{A})) (\mathcal{B}_H(T)(\Xi) \in \mathfrak{B}(\mathfrak{B})) \}.
\]

**Remark 7.2.** From Thm. \[6.25\](\[3(c)ii\]) one obtains a sufficient condition to have \(T \in \text{Mor}_{C_\bullet^0(H)}(\mathfrak{B}, \mathfrak{A})\).

**Definition 55.** Set

\[
\mathcal{F}_H^\mathfrak{A} := (\mathcal{F}_H, \mathcal{F}_H \times \mathcal{B}_H),
\]

\[
\tilde{\mathcal{F}}_H := \mathcal{F}_H \uparrow \text{Obj}(C_\bullet^0(H))
\]

\[
\tilde{\mathcal{F}}_H^\mathfrak{A} := \left(\tilde{\mathcal{F}}_H, (\mathcal{F}_H \times \mathcal{B}_H) \uparrow \text{Mor}_{C_\bullet^0(H)}\right).
\]

By recalling the definition of \(Z\) in Def. \[15\] we can set

**Definition 56.** Define \(\mathcal{V} := (\mathcal{V}_0, \mathcal{V}_m)\) such that

\[
\mathcal{V}_0 : \text{Obj}(C_\bullet^0(H)) \ni \mathfrak{A} \mapsto Z(\tilde{\mathcal{F}}_H(\mathfrak{A}))
\]

while \(\mathcal{V}_m\) is the map on \(\text{Mor}_{C_\bullet^0(H)}\) such that for any \(\mathfrak{B}, \mathfrak{A} \in \text{Obj}(C_\bullet^0(H))\) and \(T \in \text{Mor}_{C_\bullet^0(H)}(\mathfrak{B}, \mathfrak{A})\) we have \(\mathcal{V}_m(T)\) is the map on \(\mathcal{V}_0(\mathfrak{A})\) satisfying for all \(\Xi \in \mathcal{I}_{\mathcal{F}_H(\mathfrak{A})}\) and \(f \in (\mathcal{A}^*\mathfrak{A})^{op}\), where \(\mathcal{A}\) is the \(C^*\)-algebra underlying \(\mathfrak{A}\)

\[
\mathcal{V}_m(T)(\Xi, f) := (\mathcal{B}_H(T)(\Xi), \mathcal{P}_\mathcal{B}_H(T)(\Xi) \ni \alpha \mapsto f(\alpha) \circ T).
\]

We refer to Def. \[32\] for the definition of the category \(\text{Dom}(\mathfrak{B}_\bullet)^0\).

**Proposition 7.3.** We have that

1. \(\tilde{\mathcal{F}}_H^\mathfrak{A} \in \text{Fct}(C_\bullet^0(H), \text{Dom}(\mathfrak{B}_\bullet)^0)\),
2. \(\mathcal{V} \in \text{Fct}(C_\bullet^0(H), \text{Set}^{op})\),
3. \(\mathcal{F}_\mathfrak{A} \circ \tilde{\mathcal{F}}_H^\mathfrak{A} \in \text{Fct}(C_\bullet^0(H), \text{Set}^{op})\).

**Proof.** st.(1) follows since Thm. \[6.24\] st.(2) it is easy to prove, while st.(3) follows since st.(1) and Rmk. \[5.14\] \(\square\)
The following is the first main result of the present section stating that $\overline{m}$ and $\mathcal{V}$, through the functor $\delta^H$ yields natural transformations between functors from the categories $C_u(H)$ and $C_0^u(H)$ to $\text{Set}^{op}$ and between functors from $H$ to $\text{Set}$. The first case encodes the $C_u(H)$ and $C_0^u(H)$ equivariance properties while the second one the $H$ equivariance of the maps $\overline{m}$ and $\mathcal{V}$.

**Theorem 7.4 (\((G, E, \rho)\text{--natural transformations 1})\).** We have for all $\mathcal{U} \in \text{Obj}(C_u(H))$ and $\mathcal{W} \in \text{Obj}(C_0^u(H))$

\begin{enumerate}
  \item $\overline{m} \circ \delta^H \in \text{Mor}_{\text{Fct}(C_u(H), \text{Set}^{op})}(\mathcal{U} \circ \delta^H, \Delta \mathcal{U} \circ \delta^H)$;
  \item $(1 \mapsto \overline{m}^{\mathcal{W}}) \in \text{Mor}_{\text{Fct}(H, \text{Set})}(\mathcal{V} \delta^H(\mathcal{W}), \Sigma \delta^H(\mathcal{W}))$;
  \item $(1 \mapsto \mathcal{V}) \in \text{Mor}_{\text{Fct}(H, \text{Set})}(\mathcal{W} \delta^H(\mathcal{W}), \mathcal{W} \delta^H(\mathcal{W}))$;
  \item $(\mathcal{W} \circ \mathcal{V} \circ \delta^H) \in \text{Mor}_{\text{Fct}(C_0^u(H), \text{Set}^{op})}(\mathcal{W} \circ \delta^H, \mathcal{V})$.
\end{enumerate}

**Proof.** St. (1) & (2) & (3) follow since Thm. 6.25(1) and Prp. 5.16 st. (4) follows since Thm. 6.25(3(c)) and Prp. 7.3 \(\square\)

In Thm. 7.20 we organize in a unique fashion both the $C_u(H)$ and $H$ equivariance properties of the maps $\overline{m}$ and $\mathcal{V}$, but at the cost of rearranging the functors modulo a suitable equivalence relation on a subset of $\mathcal{I}_\mathcal{R}$. More exactly, since $\mathcal{W} \circ \delta^H$ and $\mathcal{I} \circ \delta^H$ are maps from $\text{Obj}(C_u(H))$ to $\text{Obj}(\text{Fct}(H, \text{Set}))$ it is natural to ask if we can arrange them to form the object part of two functors, say $M$ and $N$, from the category $C_u(H)^{op}$ to the category $\text{Fct}(H, \text{Set})$, and then to verify if the following claim holds true:

Let $\mathcal{U} \mapsto (1 \mapsto \overline{m}^{\mathcal{W}})$ realizes a natural transformation between $M$ and $N$. Similarly since $\mathcal{W} \circ \delta^H$ and $\mathcal{I} \circ \delta^H$ are maps from $\text{Obj}(C_0^u(H))$ to $\text{Obj}(\text{Fct}(H, \text{Set}))$ it is natural to ask if we can arrange them to form the object part of two functors, say $M'$ and $N'$ from the category $C_0^u(H)^{op}$ to the category $\text{Fct}(H, \text{Set})$, and then to verify if the following claim holds true: $\mathcal{W} \mapsto (1 \mapsto \mathcal{V})$ realizes a natural transformation between the functors $M'$ and $N'$.

Now the request of the existence of the aforementioned functor $M$ is equivalent to require that $(b^H(T) \circ b_\mathcal{W}(I))(\mathcal{I}) = (b_\mathcal{W}(I) \circ b^H(T))(\mathcal{I})$ for all $\mathcal{U}, \mathcal{W} \in \text{Obj}(C_u(H))$, $T \in \text{Mor}_{C_u(H)}(\mathcal{W}, \mathcal{U})$, $l \in H$ and $\mathcal{I} \in \mathcal{I}_\mathcal{R}$ while we shall prove that $(b^H(T) \circ b_\mathcal{W}(I))(\mathcal{I}) \overset{\mathcal{U}^{\mathcal{W}}}{=} (b_\mathcal{W}(I) \circ b^H(T))(\mathcal{I})$, for all $\mathcal{I} \in \mathcal{I}_\mathcal{R}^{\mathcal{W}}$ where $\overset{\mathcal{U}^{\mathcal{W}}}{=}$ is a suitable equivalence relation on a subset $\mathcal{I}_\mathcal{R}^{\mathcal{W}}$ of $\mathcal{I}_\mathcal{R}$, see Lemma 7.18(1).

Hence it is clear that in order to prove our claim we need to pass in a convenient sense to the quotient all the involved functors w.r.t. the relations $\overset{\mathcal{U}^{\mathcal{W}}}{=}$. The construction culminate in Def. 6.7 in Cor. 7.19 we prove that the constructed structures realize functors from the categories $C_u(H)^{op}$ and $C_0^u(H)^{op}$ to the category $\text{Fct}(H, \text{Set})$, finally we succeed in proving our claim by stating in Thm. 7.20 that $\mathcal{U} \mapsto (1 \mapsto \overline{m}^{\mathcal{W}})$ and $\mathcal{W} \mapsto (1 \mapsto \mathcal{V})$ are natural transformations between the constructed functors where $\overline{m}$ and $\mathcal{V}$ are $\overline{m}$ and $\mathcal{V}$ after passing in a convenient sense to the quotient w.r.t. the respective equivalence relations.

**Convention 7.5.** In the remaining part of this section let $\mathcal{U} = \{A, H, \eta\}$ such that $\mathcal{U} \in \text{Obj}(C(H))$, while by starting from Def. 6.4 let $\mathcal{W} = \{B, H, \theta\}$ and assume that $\mathcal{U}, \mathcal{W} \in C_u(H)$. If $\mathcal{T}, \mathcal{Q} \in \mathcal{T}_\mathcal{R}$ we convein to use the following notation whenever it does not cause confusion $\mathcal{T} = \{T, \mu, \mathcal{S}, \xi, f, \Gamma\}$ and $\mathcal{Q} = \{\mathcal{T}', \mu', \mathcal{S}', \xi', f', \Delta\}$, where $\mathcal{T} = \{h, \xi, \beta, I, \omega\}$ and $\mathcal{T}' = \{h', \xi', \beta', \Gamma, \omega'\}$, $X$ and $X'$ are the sets such that $\mathbb{R}^X$ and $\mathbb{R}^{X'}$ are the domains of the maps.
Let $\zeta$ and $\zeta'$ respectively, while $\mathfrak{S}_a = \left\langle S_a, \pi_a, \Omega_a \right\rangle$ and $\mathfrak{R}_\beta = \left\langle R_\beta, \psi_\beta, \Psi_\beta \right\rangle$ for any $\alpha \in \mathbb{P}^a$ and $\beta \in \mathbb{P}^a$.

**Definition 57.** Let $\mathfrak{A}$ and $\mathfrak{A}$ be the maps on $\text{Obj}(\mathbb{C}(H))$ such that $\mathfrak{A}(\mathfrak{A}), \mathfrak{A}(\mathfrak{A}) \in \prod_{\alpha \in \mathfrak{A}} \prod_{\beta \in \mathfrak{A}} \mathcal{P}(\mathfrak{L}(S_a))$ and for all $\mathfrak{T} \in \mathfrak{I}_{\mathfrak{A}}$ and $\alpha \in \mathbb{P}^a$ we have

$$\mathfrak{A}(\mathfrak{A}) \triangleleft \mathfrak{A} = \pi_\alpha(A) \cup U_{\mathfrak{S}_a}(H), \quad \mathfrak{A}(\mathfrak{A}) \triangleleft (\mathfrak{A}(\mathfrak{A}) \triangleright \alpha).$$

**Definition 58.** Let $\mathfrak{T}$ be the relation such that

$$\mathfrak{T} := \left\{ (\mathfrak{T}, \mathfrak{S}) \in \mathfrak{I}_\mathfrak{A} \times \mathfrak{I}_\mathfrak{A} \mid (\mathfrak{T}, \mathfrak{I}, \zeta, f) = (\mathfrak{T}', \mathfrak{I}', \zeta', f') \land (\forall \alpha \in \mathbb{P}^a)

\begin{align*}
& (\exists \mathfrak{V}_\alpha : \mathfrak{S}_\alpha \to \mathfrak{R}_\alpha \text{ unitary}) (\nu_\alpha = \text{ad}(\mathfrak{V}_\alpha) \circ \pi_\alpha, \Psi_\alpha = \mathfrak{V}_\alpha \Omega_\alpha, \Delta_\alpha = \text{ad}(\mathfrak{V}_\alpha)(\Gamma_\alpha)).
\end{align*}
$$

Set $[\mathfrak{T}]_\zeta = \left\{ \mathfrak{T} \mid \mathfrak{T} \approx \mathfrak{S} \right\}$ and $\mathfrak{I}_\mathfrak{A}^\circ := \left\{ [\mathfrak{T}]_\zeta \mid \mathfrak{T} \in \mathfrak{I}_\mathfrak{A} \right\}$.

**Remark 7.6.** Let $\mathfrak{T} \in \mathfrak{I}_\mathfrak{A}$ and $\alpha \in \mathbb{P}^a$ then since the bicommutant theorem $\mathfrak{A}(\mathfrak{A}) \triangleleft \mathfrak{A}$ is the von Neumann algebra generated by the set $\mathfrak{A}(\mathfrak{A}) \triangleright \alpha$. Thus $\mathfrak{T} \in \mathfrak{B}(\mathfrak{A})$ implies $\Gamma_\alpha \in \mathfrak{A}(\mathfrak{A}) \triangleleft \alpha$, since the integration in (10) is w.r.t. the strong operator topology. Moreover the position $f = f'$ is well-set since agrees with (152).

**Definition 59.** Define

$$\mathfrak{I}_\mathfrak{A}^\circ := \{ \mathfrak{T} \in \mathfrak{I}_\mathfrak{A} \mid (\forall \alpha \in \mathbb{P}^a)(\Gamma_\alpha \in \mathfrak{A}(\mathfrak{A}) \triangleright \alpha) \},$$

$$[\mathfrak{T}]_\zeta := (\approx) \cap (\mathfrak{I}_\mathfrak{A}^\circ \times \mathfrak{I}_\mathfrak{A}^\circ),$$

$$\mathfrak{I}_\mathfrak{A}^\circ := \left\{ \mathfrak{T} \mid [\mathfrak{T}]_\zeta \approx \mathfrak{S} \right\}, \quad \mathfrak{T} \in \mathfrak{I}_\mathfrak{A}^\circ,$

$$[\mathfrak{T}]_\zeta := \left\{ [\mathfrak{T}]_\zeta \mid \mathfrak{T} \in \mathfrak{I}_\mathfrak{A}^\circ \right\},$$

moreover

$$\mathfrak{I}_\mathfrak{A}^\circ := \{ \mathfrak{T} \in \mathfrak{I}_\mathfrak{A} \mid (\forall \alpha \in \mathbb{P}^a)(\Gamma_\alpha \in \mathfrak{A}(\mathfrak{A}) \triangleright \alpha) \},$$

$$[\mathfrak{T}]_\zeta := (\approx) \cap (\mathfrak{I}_\mathfrak{A}^\circ \times \mathfrak{I}_\mathfrak{A}^\circ),$$

$$\mathfrak{I}_\mathfrak{A}^\circ := \left\{ [\mathfrak{T}]_\zeta \mid \mathfrak{T} \in \mathfrak{I}_\mathfrak{A}^\circ \right\},$$

and if $\mathfrak{A} \in \text{Obj}(\mathbb{C}(H))$ set

$$\mathfrak{I}_\mathfrak{B}^\circ := (\approx) \cap (\mathfrak{B}(\mathfrak{A}) \times \mathfrak{B}(\mathfrak{A})),$$

$$[\mathfrak{T}]_\zeta := \left\{ [\mathfrak{T}]_\zeta \mid \mathfrak{T} \approx \mathfrak{S} \right\}, \quad \mathfrak{T} \in \mathfrak{B}(\mathfrak{A})$$

$$\mathfrak{B}_\mathfrak{A}^\circ := \left\{ [\mathfrak{T}]_\zeta \mid \mathfrak{T} \in \mathfrak{B}(\mathfrak{A}) \right\}.$$

**Lemma 7.7.** Let $\mathfrak{T} \approx \mathfrak{S}$ and $\alpha \in \mathbb{P}^a$ then $\mathfrak{A}(\mathfrak{A})^\circ = \text{ad}(\mathfrak{V}_\alpha)(\mathfrak{A}(\mathfrak{A})^\circ)$ and $\mathfrak{A}(\mathfrak{A})^\circ = \text{ad}(\mathfrak{V}_\alpha)(\mathfrak{A}(\mathfrak{A})^\circ)$. 
Proof. Let $\mathcal{I} \cong \mathcal{Q}$ and $\alpha \in \mathcal{P}^\mathcal{Q}$. Thus for any $h \in S^G_{\mathcal{F}_\omega}$ and $a \in A$

\[
U_{\mathcal{R}_a}(h)v_a(a)\psi_a = u_a(\eta(h)a)\psi_a = V_\alpha u_\mathcal{R}_a(h)\psi_a = (\text{ad}(V_\alpha) \circ U_{\mathcal{R}_a})(h)v_a(a)\psi_a,
\]

so we obtain since the cyclicity of $\mathcal{R}_a$

\[(149) \quad U_{\mathcal{R}_a} = \text{ad}(V_\alpha) \circ U_{\mathcal{R}_a},\]

and the first equality of the statement follows. The second equality follows since the first one, since the bicommutant theorem and since the continuity of $\text{ad}(W)$ w.r.t. the weak operator topology on $\mathcal{L}(\mathcal{F}_\omega)$ for any unitary operator $W$ on $\mathcal{F}_\omega$.

\[\square\]

**Proposition 7.8.** $\cong, \approx, \equiv$ and $\sim$ are equivalence relations, moreover $[\mathcal{I}]_\cong = [\mathcal{I}]_\approx$ and $[\mathcal{I}]_\cong = [\mathcal{I}]_\equiv$, for any $\mathcal{I} \in \mathcal{I}_\cong$ and $\mathcal{J} \in \mathcal{I}_\equiv$, in particular $\mathcal{I}_\cong \subseteq \mathcal{I}_\equiv$ and $\mathcal{I}_\equiv \subseteq \mathcal{I}_\equiv$. Finally if $\mathcal{H} \in \text{Obj}(C_0^0(H))$ then $\mathcal{B}(\mathcal{H}) \subseteq \mathcal{I}_\cong$ and $[\mathcal{J}]_\cong = [\mathcal{J}]_\equiv$ for any $\mathcal{J} \in \mathcal{B}(\mathcal{H})$, in particular $\mathcal{B}_\cong \subseteq \mathcal{I}_\equiv$.

**Proof.** The first sentence is easy to show, while the first equality follows since the bicommutant theorem. Next for any $\mathcal{I} \in \mathcal{I}_\cong$ clearly $[\mathcal{I}]_\cong \supseteq [\mathcal{I}]_\equiv$, while $[\mathcal{I}]_\cong \subseteq [\mathcal{I}]_\equiv$ follows since Lemma 7.7 $\mathcal{B}(\mathcal{H}) \subseteq \mathcal{I}_\cong$ follows since Rmk. 7.6 while $[\mathcal{J}]_\cong \subseteq [\mathcal{J}]_\equiv$ for any $\mathcal{J} \in \mathcal{B}(\mathcal{H})$ follows since (150).

\[\square\]

**Convention 7.9.** For any $\mathcal{I} \in \mathcal{I}_\equiv$ we let $[\mathcal{I}]_\equiv$ denote $[\mathcal{I}]_\equiv$ and often when it does not cause confusion we let $\mathcal{I} \cong \approx \mathcal{Q}$.

**Remark 7.10.** By using a line similar to the one in the proof of Thm. 5.9 by (152) and taking into account that two unitary equivalent cyclic representations of a $C^*$–algebra are associated to the same state, we deduce that $[\mathcal{I}]$ holds more than one element for any $\mathcal{I} \in \mathcal{I}_\equiv$.

**Lemma 7.11.** If $\mathcal{H}, \mathcal{B} \in \text{Obj}(C_0^0(H))$, $T \in \text{Mor}_{C_0^0(H)}(\mathcal{B}, \mathcal{H})$, $\mathcal{I} \in \mathcal{I}_\equiv$ and $\alpha \in \mathcal{P}^\mathcal{I}$, then $A(\alpha^\mathcal{H}(\mathcal{I})) = A(\mathcal{H}^\alpha(\mathcal{I}))$.

**Proof.** Since the bicommutant theorem we deduce that $A(\alpha^\mathcal{H}(\mathcal{I})) = \pi_\alpha(T(\mathcal{B})) \cup U_{\mathcal{R}_a}^\mathcal{H}(H)^\alpha$ and $A(\mathcal{H}^\alpha(\mathcal{I})) = \pi_{\alpha}(A) \cup U_{\mathcal{R}_a}(H)^\alpha$, where $S^\alpha$ is the closure w.r.t. the weak operator topology of any subset $S$ of $\mathcal{L}(\mathcal{F}_\omega)$. Thus the statement follows since $T$ is surjective and Cor. 6.16.

\[\square\]

**Proposition 7.12.** If $\mathcal{H}$ is unitarily implemented by $\nu$ and $l \in H$, then $b^\mathcal{H}(l)$ is $(\cong, \sim)$–compatible, moreover $b^\mathcal{H}(l)(\mathcal{I}_\equiv) \subseteq \mathcal{I}_\equiv$ and $b^\mathcal{H}(l)(\mathcal{I}_\equiv) \subseteq \mathcal{I}_\equiv$ hence $b^\mathcal{H}(l) \uparrow \mathcal{I}_\equiv$ is $(\cong, \equiv)$–compatible and $b^\mathcal{H}(l) \uparrow \mathcal{I}_\equiv$ is $(\cong, \equiv)$–compatible. In addition if $\mathcal{H}, \mathcal{B} \in \text{Obj}(C_0^0(H))$ and $T \in \text{Mor}_{C_0^0(H)}(\mathcal{B}, \mathcal{H})$, then $b^T(\mathcal{I}_\equiv) \subseteq \mathcal{I}_\equiv$ and $b^T(\mathcal{I}_\equiv) \subseteq \mathcal{I}_\equiv$ hence $b^T(\mathcal{I}_\equiv) \uparrow \mathcal{I}_\equiv$ is $(\cong, \equiv)$–compatible and $b^T(\mathcal{I}_\equiv) \uparrow \mathcal{I}_\equiv$ is $(\cong, \equiv)$–compatible. If $\mathcal{H} \in \text{Obj}(C_0^0(H))$ then $b^\mathcal{H}(l) \uparrow \mathcal{B}(\mathcal{H})$ is $(\cong, \equiv)$–compatible, while whenever $\mathcal{B} \in \text{Obj}(C_0^0(H))$ and $T \in \text{Mor}_{C_0^0(H)}(\mathcal{B}, \mathcal{H})$ then $b^T(\mathcal{I}_\equiv) \uparrow \mathcal{B}(\mathcal{H})$ is $(\cong, \equiv)$–compatible.
Lemma 7.13. We have

\begin{align*}
\text{(1) } & \overline{m}^\frak{H} \text{ is } (\sim, =)-\text{compatible}, \text{ i.e. } \frak{I} \overset{\text{def}}{=} \frak{Q} \Rightarrow \overline{m}^\frak{H}(\frak{I}) = \overline{m}^\frak{H}(\frak{Q}), \\
\text{(2) } & \text{if } \frak{H} \in \text{Obj}(\frak{C}_a(H)) \text{ then } \nu^\frak{H} \text{ is } (\sim, =)-\text{compatible}.
\end{align*}

Proof. Let \( \frak{I} \overset{\text{def}}{=} \frak{Q} \) and \( a \in \mathbb{P}^\frak{I} \). Since (149) and following the arguments used in the proof of Thm. 3.46 we deduce that the next is a commutative diagram

\[
\begin{array}{ccc}
\mathcal{L}(\frak{S}_a) & \overset{\text{ad}(V_a)}{\longrightarrow} & \mathcal{L}(\frak{R}_a) \\
\downarrow \gamma^\frak{H}_{\frak{S}_a} & & \downarrow \gamma^\frak{H}_{\frak{R}_a} \\
\mathcal{B}_{\mu, \alpha} & & \mathcal{B}_{\mu, \alpha}^\mu,
\end{array}
\]

hence since Lemma 5.21 we obtain

\begin{equation}
\hat{l}_\frak{R}_\frak{R}, \mu = \text{ad}(V_a) \circ \hat{l}_\frak{S}_\frak{S}, \mu.
\end{equation}

Next for any \( x \in X \) if \( S^a_x \) and \( T^a_x \) is such that \( iS^a_x \) is the infinitesimal generator of the semigroup \( \frak{U}_{\frak{R}_a} \circ \zeta \circ i_x \uparrow \mathbb{R}^+ \) and \( \frak{U}_{\frak{S}_a} \circ \zeta \circ i_x \uparrow \mathbb{R}^+ \) respectively then since (149) we obtain

\begin{equation}
S^a_x = V_a T^a_x V^* a,
\end{equation}

thus by Cor 3.12(2), (30) and (115),

\begin{equation}
f(\frak{C}(A, \prod_{x \in A} \text{supp}(E_{S^a})) \subseteq \mathbb{R}.
\end{equation}

Moreover since (151) and Thm. 3.13(2) we deduce that

\begin{equation}
\mathcal{D}^\frak{C}_{\frak{R}, \frak{R}} = V_a \mathcal{D}^\frak{C}_{\frak{S}_a, \frak{S}_a} V^* a.
\end{equation}

By construction we have

\begin{align*}
\textbf{R}(\frak{I}, \alpha) &= \left( (\mathcal{B}_{\mu, \alpha}^{\frak{S}_a}, \hat{l}_{\frak{R}_\frak{R}, \frak{S}_a}); \mathcal{D}^\frak{C}_{\frak{R}, \frak{R}}; \Gamma_a \right), \\
\textbf{R}(\frak{Q}, \alpha) &= \left( (\mathcal{B}_{\mu, \alpha}^{\frak{S}_a}, \hat{l}_{\frak{R}_\frak{R}, \frak{S}_a}); \mathcal{D}^\frak{C}_{\frak{R}, \frak{R}}; \Delta_a \right).
\end{align*}

By (153) and Cor. 3.12(3) we have

\begin{equation}
\exp(-\left(\mathcal{D}^\frak{C}_{\frak{R}, \frak{R}}\right)^2) = \text{ad}(V_a)(\exp(-\left(\mathcal{D}^\frak{C}_{\frak{R}, \frak{R}}\right)^2)).
\end{equation}
Next by abuse of language let $V, \Delta, \Gamma, R_R, R_B, D_R$ and $D_B$ denote $V_{\alpha}, \Delta_{\alpha}, \Gamma_{\alpha}, R_{R_{\alpha}}, R_{B_{\alpha}}, D_{R_{\alpha}}$ and $D_{B_{\alpha}}$, respectively, then since (150) & (153) & (155) we obtain for all $s_0, \ldots, s_{2n} \in R$ and $a_0, \ldots, a_{2n} \in B^\mu_{\mu, a,}^+$

$$\text{Tr}(\Delta R_R(a_0) \exp(-s_0D_R^2)[D_R, \check{R}_R(a_1)] \exp(-s_1D_R^2) \ldots$$

$$[D_R, \check{R}_R(a_{2n-1})] \exp(-s_{2n-1}D_R^2)[D_R, \check{R}_R(a_{2n})] \exp(-s_{2n}D_R^2)) =$$

$$(\text{Tr} \circ \text{ad}(V))(\Gamma R_B(a_0) \exp(-s_0D_B^2)[D_B, \check{R}_B(a_1)] \exp(-s_1D_B^2) \ldots$$

$$[D_B, \check{R}_B(a_{2n-1})] \exp(-s_{2n-1}D_B^2)[D_B, \check{R}_B(a_{2n})] \exp(-s_{2n}D_B^2)) =$$

$$\text{Tr}(\Gamma \check{R}_B(a_0) \exp(-s_0D_B^2)[D_B, \check{R}_B(a_1)] \exp(-s_1D_B^2) \ldots$$

$$[D_B, \check{R}_B(a_{2n-1})] \exp(-s_{2n-1}D_B^2)[D_B, \check{R}_B(a_{2n})] \exp(-s_{2n}D_B^2)).$$

Therefore since (154) and [Con.2, IV.8.6 Thm 22, Thm. 21, and IV.7.δ Thm 21] we have

$$\left\langle \cdot, \text{ch}(R_{\mathfrak{I}, \alpha}) \right\rangle_{\mu, \omega, \alpha} = \left\langle \cdot, \text{ch}(R_{\mathfrak{I}, \alpha}) \right\rangle_{\mu, \omega, \alpha'}$$

and st. (1) follows. St. (2) follows since (155), the hypothesis $v_\alpha = \text{ad}(V_\alpha) \circ \pi_\alpha$ and Thm. 6.25 & (3a).

□

Prp. 7.12 permits to set the following

**Definition 60.** Let $\#, \ast \in \{*, \diamondsuit, \heartsuit\}$, define $b^{H}_{\#, \ast}$ to be the map on $H$ and $b^{H}_{\#, \ast}$ to be the map on $\text{Mor}_{C_{\#}(H)}(\mathfrak{I}, \mathfrak{I})$

$$b^{H}_{\#, \ast}(l) : \mathfrak{I}_{\#} \to \mathfrak{I}_{\#}^*, [\mathfrak{I}]_{\#} \mapsto [b^{H}_{\#, \ast}(l)(\mathfrak{I})]_{\#}, \quad b^{H}_{\#, \ast}(T) : \mathfrak{I}_{\#} \to \mathfrak{I}_{\#}^*, [\mathfrak{I}]_{\#} \mapsto [b^{H}_{\#, \ast}(T)(\mathfrak{I})]_{\#},$$

$$b^{H}_{\#, \ast}(l) : \mathfrak{I}_{\#}^* \to \mathfrak{I}_{\#}^*, [\mathfrak{I}]_{\#} \mapsto [b^{H}_{\#, \ast}(l)(\mathfrak{I})]_{\#}, \quad b^{H}_{\#, \ast}(T) : \mathfrak{I}_{\#}^* \to \mathfrak{I}_{\#}^*, [\mathfrak{I}]_{\#} \mapsto [b^{H}_{\#, \ast}(T)(\mathfrak{I})]_{\#},$$

if in addition $\mathfrak{I} \in \text{Obj}(C_{\#}(H))$ we can set for all $l \in H$ and $T \in \text{Mor}_{C_{\#}(H)}(\mathfrak{I}, \mathfrak{I})$

$$b^{H}_{\#, \ast}(l) : \mathfrak{I}_{\#} \to \mathfrak{I}_{\#}^*, [\mathfrak{I}]_{\#} \mapsto [b^{H}_{\#, \ast}(l)(\mathfrak{I})]_{\#}, \quad b^{H}_{\#, \ast}(T) : \mathfrak{I}_{\#} \to \mathfrak{I}_{\#}^*, [\mathfrak{I}]_{\#} \mapsto [b^{H}_{\#, \ast}(T)(\mathfrak{I})]_{\#}.$$

**Remark 7.14.** Since Prp. 7.8 we have that $b^{H}_{\#, \ast}(l) = b^{H}_{\#, \ast}(l) \uparrow \mathfrak{I}_{\#}^*$ and $b^{H}_{\#, \ast}(T) = b^{H}_{\#, \ast}(T) \uparrow \mathfrak{I}_{\#}^*$ while $b^{H}_{\#, \ast}(l) = b^{H}_{\#, \ast}(l) \uparrow \mathfrak{I}_{\#}^*$ and $b^{H}_{\#, \ast}(T) = b^{H}_{\#, \ast}(T) \uparrow \mathfrak{I}_{\#}^*$, finally if $\mathfrak{I} \in \text{Obj}(C_{\#}(H))$ then $b^{H}_{\#, \ast}(l) = b^{H}_{\#, \ast}(l) \uparrow \mathfrak{I}_{\#}^*$ and $b^{H}_{\#, \ast}(T) = b^{H}_{\#, \ast}(T) \uparrow \mathfrak{I}_{\#}^*$.

**Definition 61.** For any $q \in \mathfrak{I}_{\#}$ set $P^q_{\#} = \text{set} P^q_{\#}$ or simply $P^q$ to be the set $P^q_{\#}$, where $\mathfrak{Q} \in q$. Since $\mathfrak{I} \subseteq \mathfrak{Q} \Rightarrow (\forall \alpha \in P^q)(K^q_{\#}(\mathfrak{I}) = K^q_{\#}(\mathfrak{Q}))$ we can set for all $q \in \mathfrak{I}_{\#}$ and $\alpha \in P^q$

$$K^q_{\#}(\mathfrak{I}) := K^q_{\#}(\mathfrak{Q}), \quad \mathfrak{Q} \in q.$$

Set $s(*) = 0, s(\diamondsuit) = \circ$ and $s(\heartsuit) = \heartsuit$. Let $\# \in \{*, \diamondsuit, \heartsuit\}$, define $A_{\#} : \text{Obj}(C_{\#}(H)) \to \text{Obj}(Ab)$ such that

$$A_{\#, \#} := \prod_{\alpha \in P^q_{\#}} K^q_{\#}(\mathfrak{I}),$$
provided by the pointwise composition. By abuse of language we shall use the same symbol $A_{\#;\mathfrak{M}}$ to denote the set underlying the group $A_{\#;\mathfrak{M}}$. Let $r_\#$ be the map on $\text{Obj}(C_u(H))$ such that $r^{\mathfrak{M}}_\# : A_{\#;\mathfrak{M}} \to A_{\#}$ and for all $f \in A_{\#;\mathfrak{M}}$, $\Sigma \in \mathfrak{I}^{(\#)}_{\mathfrak{M}}$ and $\alpha \in P_{\mathfrak{C}}$ 

$$r^{\mathfrak{M}}_\#(f)(\Sigma)(\alpha) := f([\Sigma]_\#)(\alpha).$$

Define 

$$\psi_\# \in \prod_{\Sigma \in \text{Obj}(C_u(H))} \text{Aut}(A_{\#;\mathfrak{M}})^H,$$

such that for all $l \in H$ and $f \in A_{\#;\mathfrak{M}}$ 

$$\psi_\#^{\mathfrak{M}}(l)(f) := c^{\mathfrak{M}}(l) \circ f \circ b^{\mathfrak{M}}(l^{-1}),$$

where $\text{Aut}(Y)$ is the set of the automorphisms of $Y$, for any group $Y$. Finally 

$$g^{\mathfrak{H}}_\# \in \prod_{\text{LeMor}_{C_u(H)}} \mathcal{F}(A_{\#;\mathfrak{H}}, A_{\#;\mathfrak{L}}),$$

such that for all $W \in \text{Mor}_{C_u(H)}(\mathfrak{U}, \mathfrak{V})$ and $f \in A_{\#;\mathfrak{H}}$ 

$$g^{\mathfrak{H}}_\#(W)(f) := b^{\mathfrak{H}}(W) \circ f \circ b^{\mathfrak{H}}(W).$$

Note that $r_\#$ is well-defined since Prp. 7.8.

Definition 62. Define $Z_\#: \text{Obj}(C^0_u(H)) \to \text{Obj}(\text{Set})$ and $Z^{\mathfrak{M}}_\# \in \prod_{\text{LeMor}_{C^0_u(H)}} Z_\#(c(T), d(T))$, such that 

$$Z_\# : \mathfrak{D} \mapsto \prod_{p \in \mathfrak{P}^1_\#} (E_A)^{P_{\mathfrak{C}}}_p,$$

and if $\mathfrak{U}, \mathfrak{V} \in \text{Obj}(C^0_u(H)$ and $T \in \text{Mor}_{C^0_u(H)}(\mathfrak{V}, \mathfrak{U})$ then 

$$Z^{\mathfrak{M}}_\#(T) : Z_\#(\mathfrak{U}) \to Z_\#(\mathfrak{V}),$$

$$(p, f) \mapsto \left(b^{\mathfrak{H}}_\#(T)(p), P_{\mathfrak{V}}^{b^{\mathfrak{H}}_\#(T)(p)} \ni \alpha \mapsto f(\alpha) \circ T \right).$$

Moreover define 

$$V_\# \in \prod_{\mathfrak{U} \in \text{Obj}(C^0_u(H))} \mathcal{F}(H, Z_\#(\mathfrak{D})^{Z_\#(\mathfrak{E})}),$$

such that if $\mathfrak{U} \in \text{Obj}(C^0_u(H))$ and $l \in H$ then 

$$V_\#(\mathfrak{U})(l) : (p, f) \mapsto \left(b^{\mathfrak{H}}_\#(l)(p), P_{\mathfrak{V}}^{b^{\mathfrak{H}}_\#(l)(p)} \ni \alpha \mapsto f(\alpha) \circ \eta(l^{-1}) \right).$$

For the remaining of the section let $1$ denote the unit element of $H$.

Definition 63. Let $\# \in \{\ast, \ast, \ast\}$. Define $\Psi_\#$ and $\Sigma^{\mathfrak{M}}_\#$ be the maps on $\text{Obj}(C_u(H))$, and $\Sigma^{\mathfrak{M}}_m$ be the map on $\text{Mor}_{C_u(H)}$ such that 

$$\Psi^{\mathfrak{M}}_\# := (1 \mapsto \mathfrak{I}^{\mathfrak{M}}_{\mathfrak{U}}, b^{\mathfrak{M}}_\#),$$

$$\Sigma^{\mathfrak{M}}_\#(\mathfrak{U}) := \mathfrak{I}^{\mathfrak{M}}_{\mathfrak{U}},$$

$$\Sigma^{\mathfrak{M}}_m(T) := b^{\mathfrak{H}}_\#(T), \forall T \in \text{Mor}_{C_u(H)}(\mathfrak{V}, \mathfrak{U}).$$
Moreover define $\Psi$, $\Lambda_\ast$ and $\Sigma^\ast$ be the maps on $\text{Obj}(C_u(H))$, and $\Sigma_m^\ast$ be the map on $\text{Mor}_u(C_u(H))$ such that if $\mathfrak{A}$, $\mathfrak{B}$ be $\text{Obj}(C_u(H))$ then

$$\Psi^\mathfrak{A} := (1 \mapsto \mathfrak{B}_{\mathfrak{A}}^\mathfrak{A}, \mathfrak{B}_{\mathfrak{B}}^\mathfrak{B}),$$

$$\Lambda_{\ast\mathfrak{A}} := (1 \mapsto \mathfrak{Z}_{\mathfrak{A}}(\mathfrak{A}), \mathfrak{V}_{\mathfrak{A}}(\mathfrak{A})),$$

$$\Sigma^\ast = \mathfrak{B}_{\mathfrak{A}}$$

$$\Sigma_m^\ast(\mathfrak{A}) := \mathfrak{B}_{\mathfrak{A}},$$

$$\Sigma_m^\ast(\mathfrak{T}) := \mathfrak{B}_{\mathfrak{A}}(\mathfrak{T}), \forall \mathfrak{T} \in \text{Mor}_u(C_u(H))(\mathfrak{B}, \mathfrak{A}).$$

**Lemma 7.15.** We have that

1. $(\Sigma^\ast, \Sigma_m^\ast) \in \text{Fct}(C_u(H)^{op}, \text{Set})$,
2. $(\mathfrak{Z}_{\mathfrak{A}}, \mathfrak{Z}_{\mathfrak{B}}^\mathfrak{B}) \in \text{Fct}(C_u(H)^{op}, \text{Set})$,
3. $\Psi^\mathfrak{A} \in \text{Fct}(H, \text{Set})$,
4. $\Lambda_{\ast\mathfrak{A}} \in \text{Fct}(H, \text{Set})$, if $\mathfrak{A} \in \text{Obj}(C_u(H))$,
5. $\Psi^\mathfrak{A}$ is a $H$–action on $A_{\ast\mathfrak{A}}$ via group morphisms,
6. $(A_{\ast\mathfrak{A}}, g^H)$ \in \text{Fct}(C_u(H), \text{Ab})$,
7. $\Sigma^\ast$ is a group morphism,
8. $r^\mathfrak{A}_T \circ \psi^\mathfrak{A}(l) = \psi^\mathfrak{A}(l) \circ r^\mathfrak{A}$, for all $l \in H$,
9. $g^H(T) \circ r^\mathfrak{A}_T = r^\mathfrak{A}_T \circ g^H(T)$, for all $T \in \text{Mor}_u(C_u(H))(\mathfrak{B}, \mathfrak{A})$.

**Proof.** st. (1) & (2) follow since Lemma 6.21 st. (3) & (4) by Thm. 6.9. For any $l \in H$ the $\psi^\mathfrak{A}(l)$ maps $A_{\ast\mathfrak{A}}$ into itself since the construction of $b^\mathfrak{A}$ and by Thm. 6.9 $\psi^\mathfrak{A}$ is a $H$–action since $b^\mathfrak{A}$ and $r^\mathfrak{A}$ are $H$–actions by st. (3) and Prp. 6.10 respectively, while $\psi^\mathfrak{A}(l)$ is a group morphism since the second line in (21) and since the standard picture used for the $K_0$–groups. So st. (5) follows. st. (6) follows since (144), st. (1) and the second line in (21). St. (7) is immediate, while st. (8) & (9) follow by direct computation.

Since Rmk. 7.14 and then following the same argument used in the proof of Lemma 7.15 we obtain the following

**Lemma 7.16.** For all $\# \in \{\ast, \#\}$ we obtain

1. $(\Sigma^\#_\ast, \Sigma^\#_m) \in \text{Fct}(C_u(H)^{op}, \text{Set})$,
2. $\Psi^\mathfrak{A} \in \text{Fct}(H, \text{Set})$,
3. $\psi^\mathfrak{A}$ is a $H$–action on $A_{\#\mathfrak{A}}$ via group morphisms,
4. $(A_{\#\mathfrak{A}}, g^H)$ \in \text{Fct}(C_u(H), \text{Ab})$,
5. $\Sigma^\#_\ast$ is a group morphism,
6. $r^\mathfrak{A}_T \circ \psi^\mathfrak{A}(l) = \psi^\mathfrak{A}(l) \circ r^\mathfrak{A}$, for all $l \in H$,
7. $g^H(T) \circ r^\mathfrak{A}_T = r^\mathfrak{A}_T \circ g^H(T)$, for all $T \in \text{Mor}_u(C_u(H))(\mathfrak{B}, \mathfrak{A})$,
8. $(\Sigma^\#_\ast, \Sigma^\#_m) \in \text{Fct}(C_u(H)^{op}, \text{Set})$,
9. $\Psi^\#_\ast \in \text{Fct}(H, \text{Set})$, if $\mathfrak{A} \in \text{Obj}(C_u(H))$.

According Lemma 7.13, Lemmas 7.15 & 7.16, we can set the following
Definition 64. Let $\# \in \{\star, \ast, \diamondsuit\}$, define $\overline{m}_\#$ and $m_\#$ to be the maps on $\text{Obj}(C_u(H))$ such that

$$\overline{m}_\#^\Pi \in \prod_{\pi \in \mathfrak I_u^\#} (A^\#_{\ast, \Pi})_{\Pi}^\pi,$$

$$\overline{m}_\#^\Pi(p)(\beta) := \overline{m}_\#^\Pi(\beta) \circ r_\#^\Pi, \quad p \in \mathfrak I_u^\#, \beta \in \Pi^\#;$$

$$m_\#^\Pi := (1 \mapsto \overline{m}_\#^\Pi).$$

Moreover define $\mathcal V_{\#}$ and $v_\#$ to be the maps on $\text{Obj}(C_u^0(H))$ such that if $\mathfrak A \in \text{Obj}(C_u^0(H))$ then

$$\mathcal V_{\#}^{\mathfrak A} \in \prod_{\pi \in \mathfrak A^\#} \prod_{\beta \in \Pi^\#} \mathfrak A^\#(\beta),$$

$$\mathcal V_{\#}^{\mathfrak A}(p)(\beta) := \mathcal V_{\#}^{\mathfrak A}(\beta), \quad p \in \mathfrak A^\#, \beta \in \Pi^\#;$$

$$v_{\#}^\Pi := (1 \mapsto \text{gr}(\mathcal V_{\#}^\Pi)).$$

According Lemma 7.15[5 & 6] and Lemma 7.16[3 & 4] we can set the following

Definition 65. Let $\# \in \{\star, \ast, \diamondsuit\}$, define $\Delta_{\ast}^\#$ and $\Sigma_{\#}$ be the maps on $\text{Obj}(C_u(H))$, and $\Delta_{m}^\#$ be the map on $\text{Mor}_{C_u(H)}$ such that

$$\Delta_{\ast}^\#(\mathfrak A) := \bigcup_{\pi \in \mathfrak I_u^\#} (A^\#_{\ast, \Pi})_{\Pi}^\pi,$$

while for any $T \in \text{Mor}_{C_u(H)}(\mathfrak B, \mathfrak A)$

$$\Delta_{m}^\#(T) : \Delta_{\ast}^\#(\mathfrak A) \rightarrow \Delta_{\ast}^\#(\mathfrak B),$$

$$(A^\#_{\ast, \Pi})_{\Pi}^\pi \ni g \mapsto \left( P^H_{\mathfrak B}(T) \ni \beta \mapsto g(\beta) \circ g^H(\mathfrak A) \right), \forall p \in \mathfrak I_u^\#.$$

Moreover define

$$\psi_{\#}^\Pi \in \prod_{\mathfrak D \in \text{Obj}(C_u(H))} \mathfrak I(\Delta_{\ast}^\#(\mathfrak D), \Delta_{\ast}^\#(\mathfrak D))^{\mathfrak D},$$

$$\psi_{\#}^\Pi(l) : \Delta_{\ast}^\#(\mathfrak A) \rightarrow \Delta_{\ast}^\#(\mathfrak A),$$

$$(A^\#_{\ast, \Pi})_{\Pi}^\pi \ni g \mapsto \left( P^\Pi_{\mathfrak B} \ni \beta \mapsto g(\beta) \circ \psi_{\#}^\Pi(l^{-1}) \right), \forall p \in \mathfrak I_u^\#, l \in H;$$

finally

$$\Sigma_{\#}^\Pi := (1 \mapsto \Delta_{\ast}^\#(\mathfrak A), \psi_{\#}^\Pi).$$

Lemma 7.17. For all $\# \in \{\star, \ast, \diamondsuit\}$ we obtain

1. $\Sigma_{\#}^\Pi \in \text{Fct}(H, \text{Set})$,
2. $(\Delta_{\ast}^\#, \Delta_{m}^\#) \in \text{Fct}(C_u(H)^{\Pi}, \text{Set}).$

Proof. The case $\# = \star$ follows since Lemma 7.15[5 & 6], while the cases $\# \in \{\ast, \diamondsuit\}$ follow since Lemma 7.16[3 & 4].

Definition 66. Let $T \in \text{Mor}_{C_u(H)}(\mathfrak B, \mathfrak A)$, $l \in H$, $\mathfrak I \in \mathfrak I_u^\#$ and $\alpha \in \Pi^\#$. We say that the hypothesis $E(T, l, \mathfrak I, \alpha)$ holds true if $(d \circ \pi_\alpha \circ T \circ v^\mathfrak A(l)) \upharpoonright U(\mathfrak S_{\mathfrak A}) = (d \circ \pi_\alpha \circ v^\mathfrak A(l)) \upharpoonright U(\mathfrak S_{\mathfrak A})$, while the hypothesis $E(T, l)$ holds true if the hypothesis $E(T, l, \mathfrak I, \beta)$ holds true for all $\mathfrak I \in \mathfrak I_u^\#$ and $\beta \in \Pi^\#$, finally we say that the hypothesis $E$ holds true if the hypothesis $E(T, l)$ holds true for all $l \in H$ and $T \in \text{Mor}_{C_u(H)}$. 
Lemma 7.18. For all \( l \in H \) and \( T \in \text{Mor}_{C,tf}(\mathcal{H}, \mathcal{H}) \) we have

1. \( b^l(H) \circ b^H(T) = b^{Hl}(T) \circ b^l(l) \),
2. \( b^H(T) \circ b^l(l) = b^l(l) \circ b^H(T) \),
3. \( g^H(T) \circ \psi^l(l) = \psi^{Hl}(l) \circ g^H(T) \),
4. \( \psi_{\ast l} \circ \Delta_m(T) = \Delta_m^l(T) \circ \psi_{\ast l}(l) \),
5. if in addition the hypothesis \( \mathbb{E}(T, l) \) holds true then we obtain
   - \( \psi^H(T) \circ b^l(l) = b^l(l) \circ b^H(T) \),
   - \( g^H(T) \circ \psi^l(l) = \psi^{Hl}(l) \circ g^H(T) \),
   - \( \psi_{\ast l} \circ \Delta_m^l(T) = \Delta_m^l(T) \circ \psi_{\ast l}(l) \).

Proof. Let \( l \in H, T \in \text{Mor}_{C,tf}(\mathcal{H}, \mathcal{H}) \) and \( \mathcal{I} \in \mathcal{H}^l \) then

\[
(b^H(T) \circ b^l(l))(\mathcal{I}) = \left\{ (T^l)^T, \mu', (\mathcal{F}^{v^l}(l))^T, \varepsilon^l, f, \Gamma_{\mathcal{F}^{v^l}}^l \right\}
\]

\[
(T^l)^T = \left\{ l \cdot \eta, \xi, \beta, I, T^{l \ast} \circ \eta^l(l) \circ \omega \right\}
\]

where

\[
(b^l(l) \circ b^H(T))(\mathcal{I}) = \left\{ (T^l)^T, \mu', (\mathcal{F}^{v^l}(l))^T, \varepsilon^l, f, \Gamma_{\mathcal{F}^{v^l}}^l \right\}
\]

\[
(T^l)^T = \left\{ l \cdot \eta, \xi, \beta, I, \theta^l(l) \circ T^{l \ast} \circ \omega \right\}
\]

moreover \( T \) is equivariant by hypothesis so \( \theta^l(l) \circ T^{l \ast} = T^{l \ast} \circ \eta^l(l) \), thus

\[
(T^l)^T = (T^l)^T.
\]

Let \( \alpha \in \mathbb{P}^l \), thus \( (\mathcal{F}^{v^l}(l))^T \alpha \) is a cyclic representation associated to \( (T^{l \ast} \circ \eta^l(l))(\omega^l) \), and \( (\mathcal{F}^{v^l}(l))^T \alpha \) is a cyclic representation associated to the state \( (\theta^l(l) \circ T^{l \ast})(\omega^l) = (T^{l \ast} \circ \eta^l(l))(\omega^l) \), so there exists a unitary operator \( V_\alpha \) on \( \mathcal{H}_\alpha \) such that

\[
\pi_\alpha \circ T = \text{ad}(V_\alpha) \circ (\pi_\alpha \circ T),
\]

\[
V_\alpha \pi_\alpha(v^l(l)) \omega_\alpha = (\pi_\alpha \circ T(v^l(l))) \omega_\alpha.
\]

Next for all \( b \in \mathcal{B} \) we have

\[
\text{ad} \left( T(v^l(l)) \right)(T(b)) = (T \circ \theta(l))(b) = (\eta(l) \circ T)(b) = \left( \text{ad}(v^l(l)) \right)(T(b)),
\]

but \( T \) is surjective so

\[
\text{ad} \left( (\pi_\alpha \circ T)(v^l(l)) \right) \uparrow \pi_\alpha(A) = \text{ad} \left( \pi_\alpha(v^l(l)) \right) \uparrow \pi_\alpha(A),
\]
in particular

\[
\text{ad} \left( (\pi_\alpha \circ T)(v^l(l)) \right) \uparrow \pi_\alpha(A) = \text{ad} \left( \pi_\alpha(v^l(l)) \right) \uparrow \pi_\alpha(A),
\]
so since $\text{ad}(W)$ is weakly continuous for any unitary operator $W$ on $\mathcal{F}_*$ and $\pi(A)^\alpha = \pi(A)^\alpha$

since the bicommutant theorem , we obtain

$$\text{ad} \left( (\pi_\alpha \circ T)(v^{\mathcal{B}}(l)) \right) \upharpoonright \pi_\alpha(A)^\alpha = \text{ad} \left( \pi_\alpha(v^{\mathcal{B}}(l)) \right) \upharpoonright \pi_\alpha(A)^\alpha.$$

Moreover $\Gamma_\alpha \in \pi_\alpha(A)^\alpha$ by hypothesis so since (156) & (157)

$$\text{ad} \left( (\pi_\alpha \circ T)(v^{\mathcal{B}}(l)) \right) \upharpoonright \pi_\alpha(A)^\alpha = \text{ad} \left( \pi_\alpha(v^{\mathcal{B}}(l)) \right) \upharpoonright \pi_\alpha(A)^\alpha.$$

Next $V_\alpha \in \pi_\alpha(A)'$ since $T$ is surjective and the first equality of (159), therefore

$$\text{ad} \left( V_\alpha \right) \left( \Gamma_\alpha \right) = \text{ad} \left( V_\alpha \right) \left( \pi_\alpha \circ T \right)(v^{\mathcal{B}}(l))(\mathcal{F}).$$

Finally since (156) & (157) & (158) & (159) & (162) & (163) we obtain

$$\left( b^H(T) \circ v^{\mathcal{B}}(l) \right)(\mathcal{F}) \cong \left( b^H(T) \circ b^H(T) \right)(\mathcal{F}),$$

then $\text{st.} (1)$ follows since Prp. 7.8 & 7.12 $\text{st.} (2)$ follows since Rmk. 6.19, (140) and Prp. 6.18,

and the fact that $(K_0(\cdot), (\cdot), \circ ((\cdot)^+, (\cdot)^+) )$ is a functor form the category of $C^*$--algebras and

$*-$morphisms to the category $\text{Ab}$ we can deduce from Section 2 In conclusion $\text{st.} (3)$ follows since $\text{st.} (1)$ & $\text{st.} (2)$ and, $\text{st.} (4)$ since $\text{st.} (3)$. If the hypothesis $E(T, l)$ holds true then since (161) and the bicommutant theorem we obtain

$$\left( b^H(T) \circ v^{\mathcal{B}}(l) \right)(\mathcal{F}) \cong \left( b^H(T) \circ b^H(T) \right)(\mathcal{F}),$$

then $\text{st.} (5a)$ follows under the same argument used to prove $\text{st.} (1)$. $\text{st.} (5b) \& (5c)$ follow since $\text{st.} (5a) \& (2)$. \hfill $\square$

**Definition 67.** Define

$$P^H := \left( \mathcal{B}_*, \text{Mor}_{C_0(H)^pp} \ni T \mapsto (1 \mapsto \ominus_m^\alpha(T)) \right),$$

$$O^H := \left( \ominus_*, \text{Mor}_{C_0(H)^pp} \ni T \mapsto (1 \mapsto \Delta_m^\alpha(T)) \right),$$

$$P_\mathbb{R}^H := \left( \mathcal{B}_\mathbb{R}^\alpha, \text{Mor}_{C_0(H)^pp} \ni T \mapsto (1 \mapsto \ominus_m^\alpha(T)) \right),$$

$$Z_\mathbb{R}^H := \left( \mathcal{B}_\mathbb{R}^\alpha, \text{Mor}_{C_0(H)^pp} \ni T \mapsto (1 \mapsto \ominus_m^\alpha(T)) \right).$$

**Corollary 7.19.** We have

1. $P^H \in \text{Fct}(C_0(H)^pp, \text{Fct}(H, \text{Set}))$;
2. $O^H \in \text{Fct}(C_0(H)^pp, \text{Fct}(H, \text{Set}))$;
3. If the hypothesis $E$ holds true then
   - $P^H \in \text{Fct}(C_0(H)^pp, \text{Fct}(H, \text{Set}))$;
   - $Z^H_\mathbb{R} \in \text{Fct}(C_0(H)^pp, \text{Fct}(H, \text{Set}))$.

**Proof.** Let $T \in \text{Mor}_{C_0(H)}(\mathcal{B}, \mathcal{B})$. Since Lemma 7.16(1) & (2) to prove $\text{st.} (1)$ it is sufficient that

$$\left( 1 \mapsto \ominus_m^\alpha(T) \right) \in \text{Mor}_{\text{Fct}(H, \text{Set})}(\mathcal{B}_*, \mathcal{B}_*)$$

i.e. $1 \mapsto \ominus_m^\alpha(T)$ is a natural transformation from the functor $\mathcal{B}_*$ to $\mathcal{B}_*$, which is true since Lemma 7.18(1). Since Lemma 7.17(1) & (2) to prove $\text{st.} (2)$ it is sufficient that

$$\left( 1 \mapsto \Delta_m^\alpha(T) \right) \in \text{Mor}_{\text{Fct}(H, \text{Set})}(\mathcal{B}_*, \mathcal{B}_*)$$


which is true since Lemma 7.18. Assume \( u, v \in \text{Obj}(C^0_u(H)) \) and \( T \in \text{Mor}_{C^0_u(H)}(v, u) \).

Since Lemma 7.16 & 9 to prove st. (3a) it is sufficient that

\( (1 \mapsto \Sigma^\star_{\Psi}(T)) \in \text{Mor}_{\text{Fct}(H, \text{Set})}(\Psi^\star_{\Psi}, \Psi^\star_{\Psi}) \),

which is true since Lemma 7.18(5a) and Rmk. 7.14. Since Lemma 7.15(2) & (4) to prove st. (3b) it is sufficient that

\( (1 \mapsto \Sigma^\star_{\Psi}(T)) \in \text{Mor}_{\text{Fct}(H, \text{Set})}(\Psi^\star_{\Psi}, \Psi^\star_{\Psi}) \),

which is true since Lemma 7.18(5a) and Rmk. 7.14, and since \( T \) is equivariant i.e. \( T \circ \theta(l) = \eta(l) \circ T \), for all \( l \in H \). □

Now we are able to state one of the main results of this paper namely that \( m \) is a natural transformation between functors from the category \( C_u(H)^{\text{op}} \) to the category \( \text{Fct}(H, \text{Set}) \), and if in addition the hypothesis \( E \) holds true, \( v \) is a natural transformation between functors from the category \( C^0_u(H)^{\text{op}} \) to the category \( \text{Fct}(H, \text{Set}) \).

**Theorem 7.20** ((\( G, F, \rho \))—natural transformations 2). We have that

1. \( m \in \text{Mor}_{\text{Fct}(C_u(H)^{\text{op}}, \text{Fct}(H, \text{Set}))}(P^H, O^H) \),
2. if hypothesis \( E \) holds true then \( v \in \text{Mor}_{\text{Fct}(C^0_u(H)^{\text{op}}, \text{Fct}(H, \text{Set}))}(P^H, Z^H) \).

**Proof.** St. (1) is well-set since Cor. 7.19 moreover it amounts to be equivalent to the claimed statements (165) & (166), where

\( (1 \mapsto \overline{m}^\star_{\Psi}) \in \text{Mor}_{\text{Fct}(H, \text{Set})}(\Psi^\star_{\Psi}, \Psi^\star_{\Psi}) \),

and for all \( T \in \text{Mor}_{C_u(H)}(v, u) \) the following is a commutative diagram in the category \( \text{Fct}(H, \text{Set}) \)

\[
\begin{array}{ccc}
\Psi^\star_{\Psi} & \xrightarrow{1 \mapsto \overline{m}^\star_{\Psi}} & \Omega^\star_{\Psi} \\
\downarrow & & \downarrow \\
\Psi^\star_{\Psi} & \xrightarrow{1 \mapsto \Delta^\star_{\Psi}(T)} & \Omega^\star_{\Psi} \\
\end{array}
\]

Next (165) equivalently the commutativity in \( \text{Set} \) of the following diagram for all \( l \in H \)

\[
\begin{array}{ccc}
\Psi^\star_{\Psi} & \xrightarrow{\overline{m}^\star_{\Psi}} & \Omega^\star_{\Psi} \\
\downarrow & & \downarrow \\
\Psi^\star_{\Psi} & \xrightarrow{1 \mapsto \Delta^\star_{\Psi}(T)} & \Omega^\star_{\Psi} \\
\end{array}
\]

Next (165) equivalently the commutativity in \( \text{Set} \) of the following diagram for all \( l \in H \)

\[
\begin{array}{ccc}
\Psi^\star_{\Psi} & \xrightarrow{\overline{m}^\star_{\Psi}} & \Omega^\star_{\Psi} \\
\downarrow & & \downarrow \\
\Psi^\star_{\Psi} & \xrightarrow{1 \mapsto \Delta^\star_{\Psi}(T)} & \Omega^\star_{\Psi} \\
\end{array}
\]
Let \( p \in \mathcal{T}_*, \mathcal{I} \in p \) and \( \beta \in P^p \). Then

\[
(\psi_{*,*}(l) \circ \overline{m}_*)(p)(\beta) = \overline{m}_*(p)(\beta) \circ \psi_{*,*}(l^{-1})
\]

\[
= \overline{m}_*(\mathcal{I})(\beta) \circ r_*(\psi_{*,*}(l^{-1}))
\]

\[
= \overline{m}_*(\mathcal{I})(\beta) \circ \psi_{*,*}(l^{-1}) \circ r_*
\]

\[
= \overline{m}_*(b_{*,*}(l)\mathcal{I})(\beta) \circ r_*
\]

\[
= (\overline{m}_* \circ b_{*,*}(l))(p)(\beta),
\]

where the third equality follows since Lemma \( \ref{lem7.16} \(6 \) and the fourth one since Thm. \( \ref{thm6.11} \(2 \). Hence (167) and so also (165) follow. Next (166) equates to the commutativity in \( \text{Set} \) of the following diagram

\[
\begin{array}{ccc}
\mathcal{I}_* & \xrightarrow{\overline{m}_*} & \Delta_*(\Psi) \\
\downarrow{b_{*,*}(T)} & & \downarrow{\Delta_*(T)} \\
\mathcal{I}_* & \xrightarrow{\overline{m}_*} & \Delta_*(\Psi).
\end{array}
\]

Next

\[
(\overline{m}_* \circ b_{*,*}(T))(p)(\beta) = \overline{m}_*(b_{*,*}(T)\mathcal{I} \circ \beta)(p)(\beta)
\]

\[
= \overline{m}_*(b_{*,*}(T)\mathcal{I})(\beta) \circ r_*(\overline{m}_*(p)(\beta))
\]

\[
= \overline{m}_*(b_{*,*}(T)\mathcal{I})(\beta) \circ r_*
\]

\[
= \overline{m}_*(b_{*,*}(T)\mathcal{I})(\beta) \circ \overline{g}_H(T)
\]

\[
= (\Delta_*(T) \circ \overline{m}_*)(p)(\beta),
\]

where the third equality follows since the second equality in (145) (in switching \( \mathcal{U} \) with \( \mathcal{B} \)), while the fourth one since Lemma \( \ref{lem7.16} \(7 \). Therefore (168) and (166) and therefore st.(1) follow. St.(2) is well-set since Cor. \( \ref{cor7.19} \) moreover it ammounts to be equivalent to the claimed statements (169) & (170), where

\[
(1 \mapsto \text{gr}(\mathcal{V}_{*,*}^\mathcal{H})) \in \text{Mor}_{\text{Cat}(\mathcal{H}, \text{Set})}(\mathcal{B}_{*,*}^{\mathcal{H}}, \mathcal{Z}_{*,*}^{\mathcal{H}}),
\]
and for all $T \in \text{Mor}_{C_u(H)}(\mathcal{B}, \mathcal{A})$ the following is a commutative diagram in the category $\text{Fct}(H, \text{Set})$

$$\begin{array}{ccc}
P^\mathcal{B}_\mathcal{A} & \xrightarrow{1\mapsto c_m(T)} & P^\mathcal{B}_\mathcal{A} \\
\downarrow & & \downarrow \\
\mathcal{B} & \xrightarrow{1\mapsto Z_m(T)} & \mathcal{B} \\
\end{array}$$

Next (169) equivales to the commutativity in $\text{Set}$ of the following diagram for all $l \in H$

$$\begin{array}{ccc}
P^\mathcal{B}_\mathcal{A} & \xrightarrow{\text{gr}(\eta)^\mathcal{B}_\mathcal{A}} & Z^\mathcal{A}_\mathcal{B} \\
\downarrow & & \downarrow \\
\mathcal{B} & \xrightarrow{\text{gr}(\eta)^\mathcal{B}_\mathcal{A}} & Z^\mathcal{A}_\mathcal{B} \\
\end{array}$$

i.e. $\eta(l) \circ \eta^\mathcal{B}_\mathcal{A}(p) = (\eta^\mathcal{B}_\mathcal{A} \circ \eta^\mathcal{B}_\mathcal{A}(l))(p)$, for all $p \in \mathcal{B}^\mathcal{B}_\mathcal{A}$, which follows since Thm. 7.4(3), hence (169) is proved. Next (170) equivales to the commutativity in $\text{Set}$ of the following diagram

$$\begin{array}{ccc}
P^\mathcal{B}_\mathcal{A} & \xrightarrow{\text{gr}(\eta)^\mathcal{B}_\mathcal{A}} & Z^\mathcal{A}_\mathcal{B} \\
\downarrow & & \downarrow \\
\mathcal{B} & \xrightarrow{\text{gr}(\eta)^\mathcal{B}_\mathcal{A}} & Z^\mathcal{A}_\mathcal{B} \\
\end{array}$$

which follows since Thm. 6.25(3(c)), therefore (170) as well st. (2) follow.

\[\square\]

Remark 7.21. Thm. 7.20(1) encodes the $H$–equivariance and $C_u(H)$–equivariance of the map $m$, while Thm. 7.20(2) encodes the $H$–equivariance and $C_u(H)$–equivariance of the map $\mathcal{V}$. Indeed according the proof of Thm. 7.20 $m^* \in \text{Mor}_{\text{Fct}(C_u(H)\text{pr},\text{Fct}(H,\text{Set}))}(\mathcal{P}^H, \mathcal{O}^H)$ is equivalent to (167) & (168), for all $\mathcal{B}, \mathcal{A} \in \text{Obj}(C_u(H))$, $l \in H$ and $T \in \text{Mor}_{C_u(H)}(\mathcal{B}, \mathcal{A})$, while, in case the hypothesis $E$ holds true, $m^* \in \text{Mor}_{\text{Fct}(C_u(H)\text{pr},\text{Fct}(H,\text{Set}))}(\mathcal{P}^H, \mathcal{Z}^H)$ is equivalent to (170) & (172) for all $\mathcal{B}, \mathcal{A} \in \text{Obj}(C_u(H))$, $l \in H$ and $T \in \text{Mor}_{C_u(H)}(\mathcal{B}, \mathcal{A})$.

Remark 7.22. Our conjecture is that the hypothesis $E$ holds true. Let $T \in \text{Mor}_{C_u(H)}(\mathcal{B}, \mathcal{A})$, while for the notations of what follows see [Tak 2, Ch. 6 – 9]. Let $\mathcal{A}, S_i, J, \mathcal{B}$ be the canonical standard form of $\mathcal{A}$, $\phi$ a faithful semi-finite normal weight on $\mathcal{A}$, $\pi_\mathcal{A}, S_\phi, J_\phi, \eta_\phi$ the semi-cyclic representation of $\mathcal{A}$ associated to $\phi$, [Tak 2, Def. 7.1.5], and $\pi_\phi(A), S_\phi, J_\phi, \mathcal{P}_\phi$ the standard form associated to $\phi$ realized on $S_i$, thus $\mathcal{A} = \pi_\phi(A)$,
\[
\hat{S} = \hat{S}_\phi, \; J = J_\phi \quad \text{and} \quad \Psi = \Psi_\phi, \quad \text{[Tak2] p. 153} \]; finally let \( \Delta \) denote the modular operator associated to \( \phi \). Since the construction of \( V^g \) and since \([Tak2] \) Lemma 6.1.5\((vi)\), Thm. 7.2.6 and Thm. 9.1.15] we deduce that the following holds

\[
\begin{align*}
\bullet & \; v^g(l)\Psi = \Psi; \\
\bullet & \; v^g(l)Jv^g(l^{-1})\frac{\Delta^\frac{1}{2}}{\eta_\phi(x) = \eta_\phi(x')} \text{, for all } x \in \eta_\phi \cap \eta_\phi^*, \\
\end{align*}
\]

while the hypothesis \( E(T, l) \) holds true if

\( (1) \) \( T(v^g(l))\Psi = \Psi; \)

\( (2) \) \( T(v^g(l))J(T(v^g(l^{-1}))\frac{\Delta^\frac{1}{2}}{\eta_\phi(x) = \eta_\phi(x')} \text{, for all } x \in \eta_\phi \cap \eta_\phi^*, \\
\)

note that the requests \( \text{ad}(v^g(l))\mathcal{A} = \mathcal{A} \) and \( \text{ad}(T(v^g(l)))\mathcal{A} = \mathcal{A} \) are automatically satisfied since \( v^g(l) \in \mathcal{U}(\mathcal{A}) \) and \( v^g(l) \in \mathcal{U}(\mathcal{B}) \) by construction. Now (1) should be not difficult to show, while with the help of (160) it should be possible to prove (2), but until now we did not succeed.

8. Nucleon phases

Any fissioning system \( U \overset{233}{=} n_{lH} + n_{lH}, \; U \overset{235}{=} n_{lH}, \; Pu \overset{239}{=} n_{lH}, \; \text{and} \; Cf \overset{252}{=} \), below referred as \( \text{as} \)-type, exhibits an asymmetric binary fission consisting in obtaining two asymmetric final fragments. Here the asymmetry of the fragments is with respect to their mass numbers in unified atomic mass units. Let \( A_H \) and \( A_L \) be the mass number of the heavy and light fragment respectively. In addition any fissioning system up to \( Cf \overset{252}{=} \) for example \( Fm \overset{258}{=} \) and \( Hg \overset{266}{=} \), below referred as \( \text{s} \)-type, exhibits a symmetric binary fission consisting in obtaining two symmetric final fragments. The nucleon phase hypothesis advanced by Mouze and Ythier states the following, see [MHY1] quoted in [MHY2], or [Ric] for the details. The reaction-time of any binary fission process is \( 1.77 \times 10^{-25} \) s thus occurring at temperatures of the order of \( 10^{13} \) K, according the energy-time uncertainty relation. Hence the distinction between the proton and neutron phase disappears and a new nucleon phase occurs. More exactly whenever a fission reaction involves an \( \text{as} \)-type fissioning system \( \varsigma \), two nucleon cores come into existence, one of mass number 82 and the other of mass number 126. The two final asymmetric fragments consist of these two cores surrounded by their valence nucleons, and the closure of the shells at 82 and 126 explains the following Terrell law \([Ric] \text{ eq. (1)}\]

\[
(173) \quad \overline{\nu} = 0.08(A_L - 82) + 0.1(A_H - 126),
\]

where \( \overline{\nu} \) is the mean value of the prompt-neutron yield in the state describing the fragments occurring next the fission process of \( \varsigma \). Similarly the reaction involving a \( \text{s} \)-type fissioning system generates two nucleon cores each one of mass number 126, and the symmetric final fragments consist of such a cores surrounded by their valence nucleons, \([Ric] \text{ II.a}\]. We can easily deduce the following properties of the two nucleon cores: (stability): their mass number 82 and 126 remain the same under variation of the generating fissioning system; (thermal nature and phase transition): they are generated only for temperatures, of the reaction involving the fissioning system, higher than \( 10^{13} \) K.

The goal of this section is to provide a \( C \)-algebraic setting in which the nucleon phases can be precisely described and most importantly the universality of the Terrell law proved. In particular by using the equivariance of the canonical \( C_n(H) \)-equivariant stability \( E \), constructed in Thm. 6.25 we show in Thm. 8.9 one of the main results of this paper, the
invariance of the Terrell law under the action of the symmetry group $H$ and the action of suitable perturbations in $\text{Mor}_{C^*_u(H)}$ on the fissioning systems, see also Cor. 8.11.

As a consequence of the aforementioned equivariance, provided Conj. 11 is satisfied, we prove in Cor. 8.14 the universality of the mass numbers 82 and 126 in terms of their invariance under the action on the fissioning systems, of $H$ and suitable equivariant perturbations. Finally in such a formulation nucleon phases possess additional properties described in Prp. 5.29 via Thm. 6.25(1), while, provided Conj. 1 is satisfied, the stability property is ascribed to the noncommutative geometric nature of the nucleon phases, Rmk. 8.13. Aim of a future work it is to prove Conj. 1.

The main idea is to consider a suitable object $\mathfrak{A}$ of $C^*_u(H)$, an element $\mathfrak{T} \in \mathfrak{B}(\mathfrak{A})$ and an $\alpha \in P^\mathfrak{T}$ such that $\mathfrak{T}$ is interpreted roughly as the operation which performed at the inverse temperature $\alpha$ produces the reaction of the $as$–type fissioning system denoted by $\nu(\mathfrak{T})$. Thus according the rules in Def. 35 & (11) we can consider the phase $\bar{m}^\mathfrak{A}(\mathfrak{T}, \alpha)$, of the physical system $\mathfrak{G}^H(\mathfrak{A})$, as the nucleon phase originating the state $\mathfrak{V}^\mathfrak{A}(\mathfrak{T}, \alpha)$ of the system $\mathfrak{A}$, describing the asymmetric fragments occurring next the reaction of the fissioning system $\nu(\mathfrak{T})$. As a result we read (174) as the Terrell law relative to $\nu(\mathfrak{T})$, where $f_j$ and $N_j$ are the mass observables of the (light for $j = m$, heavy for $j = w$) nucleon core and the (light for $j = m$, heavy for $j = w$) fragment respectively.

More exactly let $n = \langle \mathfrak{A}, \mathfrak{T}, \alpha, \{f_j, N_j\}_{j \in \{m, w\}} \rangle$ be an asymmetric nucleon scheme relative to $\mathfrak{E}$, where $\mathfrak{A} = \langle A, H, \eta \rangle$ is an object of the category $C^*_u(H)$, and $\mathfrak{T} \in \mathfrak{B}(\mathfrak{A})$, $\alpha \in P^\mathfrak{T}$, $N_j \in A_{\alpha}$, and $f_j \in A_{\alpha}$, satisfying consistent physical requests accurately stated for a general equivariant stability in Def. 88. Thus we have since Rmk. 8.3

- $\mathfrak{V}^\mathfrak{A}(\mathfrak{T}, \alpha)$ is the state of the system $\mathfrak{A}$ occurring by performing at the inverse temperature $\alpha$ the fission process of the fissioning system $\nu(\mathfrak{T})$,
- $\bar{m}^\mathfrak{A}(\mathfrak{T}, \alpha)$ is the (nucleon) phase, of the system $\mathfrak{G}^H(\mathfrak{A})$, originating $\mathfrak{V}^\mathfrak{A}(\mathfrak{T}, \alpha)$,
- $N_j$ is the mass observable of the (light for $j = m$, heavy for $j = w$) fragment,
- $f_j$ is the mass observable of the (light for $j = m$, heavy for $j = w$) nucleon core,
- $\mathfrak{V}^\mathfrak{A}(\mathfrak{T}, \alpha)(N_j)$ is the mean value of $N_j$ in $\mathfrak{V}^\mathfrak{A}(\mathfrak{T}, \alpha)$,
- $\bar{m}^\mathfrak{A}(\mathfrak{T}, \alpha)(f_j)$ is the mean value of $f_j$ in $\bar{m}^\mathfrak{A}(\mathfrak{T}, \alpha)$.

Note that the nucleon phase as well the fragment state is one, while the light and heavy cases are ascribable to the different observables $f_m$ and $f_w$ for the nucleon cores and $N_m$ and $N_w$ for the fragments. Now it is clear that the nucleon phase hypothesis can be reformulated in the following way: for any asymmetric nucleon scheme $n$ relative to $\mathfrak{E}$, the mean value of the prompt-neutron yield in $\mathfrak{V}^\mathfrak{A}(\mathfrak{T}, \alpha)$ equals $\nu^{\mathfrak{E}}(n)$ where

$$\nu^{\mathfrak{E}}(n) = \frac{1}{2} \left( \mathfrak{V}^\mathfrak{A}(\mathfrak{T}, \alpha)(N_m) - \bar{m}^\mathfrak{A}(\mathfrak{T}, \alpha)(f_m) \right) + 0.1 \left( \mathfrak{V}^\mathfrak{A}(\mathfrak{T}, \alpha)(N_w) - \bar{m}^\mathfrak{A}(\mathfrak{T}, \alpha)(f_w) \right).$$

(174)

The power of this description resides in the possibility of implementing the physical transformations, on the set of the fissioning systems, induced by the symmetry group $H$ and by the semigroupoid $\text{Mor}_{C^*_u(H)}$. Indeed if we set $n' = \langle \mathfrak{A}, \mathfrak{T}, \alpha, \{f'_j, N'_j\}_{j \in \{m, w\}} \rangle$, where $\mathfrak{T}' = \mathfrak{V}^\mathfrak{A}(\mathfrak{T})(\mathfrak{T})$, $f'_j = \mathfrak{U}^\mathfrak{A}(\mathfrak{T})(f_j)$ and $N'_j = \eta(l)(N_j)$ for any $l \in H$, while letting $\mathfrak{B}$ be a dynamical system of $C^*_u(H)$, $T \in \text{Mor}_{C^*_u(H)}(\mathfrak{B}, \mathfrak{A})$ such that there exist $N'_j$ and $f'_j$ satisfying $f_j = g^H(T)(f'_j)$
and \( N_j = T(N'_j) \), set \( n^T = \left\langle B, b^T(T)(\mathcal{I}), \alpha, \{ f'_j, N'_j \}_{j \in \{m,w\}} \right\rangle \), we obtain in Thm. \( 8.9 \) the following property of invariance called by us the universality of the Terrell law

\[
\nu^\mathcal{I}_\alpha(n^I) = \nu^\mathcal{I}_\alpha(n), \forall I \in H,
\]

\[
\nu^\mathcal{I}_\alpha(n^T) = \nu^\mathcal{I}_\alpha(n).
\]

Therefore the Terrell law in our setting is invariant under action of the symmetry group \( H \) and under action of suitable equivariant perturbations \( T \) on the fissioning system \( \nu(\mathcal{I}) \). We conclude by remarking that the universality of the Terrell law follows by the properties of invariance called by us the universality of the Terrell law, stated in the main theorem \( 6.25(1) \& 3(c) \), hence since Rmk. \( 7.21 \) in case the hypothesis \( E \) holds true, it follows by Thm. \( 7.20 \) i.e. by the naturality of the transformations \( m_\mathcal{I} \) and \( v_\mathcal{I} \). In the present section we assume fixed two locally compact topological groups \( G \) and \( F \), a group homomorphism \( \rho : F \to Aut(G) \) such that the map \( (g,f) \mapsto \rho_f(g) \) on \( G \times F \) at values in \( G \), is continuous, let \( H \) denote \( G \rtimes_\rho F \).

8.1. **Nucleon schemes relative to a \( \mathcal{C} \)--equivariant stability.** In this section let \( \mathcal{C} \) be a category and \( \mathcal{E} = \left\langle \left\langle \mathcal{U}, m, \mathcal{W} \right\rangle, \mathcal{F} \right\rangle \) be a full \( \mathcal{C} \)--equivariant stability on \( \mathcal{D} \).

**Definition 68** (Nucleon schemes relative to \( \mathcal{E} \)). Let \( N^\mathcal{E}_\text{as} \) be the set of the asymmetric nucleon schemes relative to \( \mathcal{E} \) defined as the set of the tuples

\[
n = \left\langle a, \mathcal{I}, \alpha, \{ f_j, N_j \}_{j \in \{m,w\}} \right\rangle,
\]

such that for all \( j \in \{m,w\} \)

1. \( a \in \Theta(\mathcal{D}, \mathcal{F}) \),
2. \( \mathcal{I} \in \mathcal{D}_{\alpha(a)} \) and \( \alpha \in \mathcal{P}^+ \cap \mathbb{R}_0^+ \),
3. \( f_j \in \mathcal{A}_{\mathcal{F}(a)} \),
4. \( N_j \in \mathcal{A}(\mathcal{F}(a))_{\alpha}^n \) such that \( N_j = N'_j \),
5. we have

\[
(175)
\]

We call \( m^{\mathcal{F}(a)}(\mathcal{I}, \alpha) \) the asymmetric nucleon phase relative to \( n \). Let \( N^\mathcal{E}_\text{n} \) be the set of the nucleon schemes relative to \( \mathcal{E} \) defined as the set of the tuples

\[
q = \left\langle a, \mathcal{I}_{\text{as}}, \alpha_{\text{as}}, \{ f_j, N_j \}_{j \in \{m,w\}} \right\rangle,
\]

such that \( \left\langle a, \mathcal{I}_{\text{as}}, \alpha_{\text{as}}, \{ f_j, N_j \}_{j \in \{m,w\}} \right\rangle \in N^\mathcal{E}_{\text{as}}, \mathcal{I}_s \in \mathcal{D}_{\alpha(a)}, \alpha_s \in \mathcal{P}^+ \) satisfying \( \mathcal{A}^\mathcal{I}_{\text{as}} = \mathcal{A}^\mathcal{I}_s \) and for all \( j \in \{m,w\} \)

\[
(176)
\]

We call \( m^{\mathcal{F}(a)}(\mathcal{I}, \alpha) \) the symmetric nucleon phase relative to \( q \), with \( \# \in \{s,as\} \).
Note that by construction to any nucleon scheme we can associate the asymmetric nucleon scheme extracted from it, while to any asymmetric nucleon scheme we can associate the nucleon scheme where $\mathcal{T}_s = \mathcal{T}_{as}$ and $\alpha_s = \alpha_{as}$.

**Definition 69 (E–Terrell law).** Let $\nu^E : N_{as}^E \to \mathbb{R}$ be the $E$–Terrell law defined as the map such that for all $n = (a, \mathcal{T}, \alpha, \{f_j, N_j\}_{j \in \{m,w\}}) \in N_{as}^E$ we have

$$\nu^E(n) := 0.08 \left( \omega^T(a) (\mathcal{T}, \alpha)(N_m) - m^T(a)(\mathcal{T}, \alpha)(f_m) \right) + 0.1 \left( \omega^T(a) (\mathcal{T}, \alpha)(N_w) - m^T(a)(\mathcal{T}, \alpha)(f_w) \right).$$

**Definition 70.** Define $t^E$ as the map on $H$ such that $t^E(l)$ is a map on $N_{as}^E$ such that for all $n = \langle a, \mathcal{T}, \alpha, \{f_j, N_j\}_{j \in \{m,w\}} \rangle \in N_{as}^E$ we have

$$t^E(l)(n) := \left\langle a, b^T(a)(l)(\mathcal{T}), \alpha, \{ \psi^T(a)(l)(f_j), V(F(a)) (l)(N_j) \}_{j \in \{m,w\}} \right\rangle.$$

**Convention 8.1.** If $a = \langle a, \{ \mathcal{T}, \alpha \}_{\epsilon \in \{s,as\}}, \{ f_j, N_j \}_{j \in \{m,w\}} \rangle \in N^E$ and $n \in N_{as}^E$ extracted by $a$, then for any $\# \in \{ s, as \}$, $j \in \{ m, w \}$ and $l \in H$ whenever it is not cause of confusion, we let $\mathcal{T}_l^l, t^l, N_l^l$ and $n^l$ denote $b^T(a)(l)(\mathcal{T}_\#), \psi^T(a)(l)(f), V(F(a))^2(l)(N_j)$ and $t^E(l)(n)$ respectively.

The following is a simple but fundamental result whose physical interpretation is stated in Rmk. 8.4. In sec. 8.2 we shall apply it to the canonical $C_u(H)$–equivariant stability constructed in the Main Thm. 6.25.

**Proposition 8.2 (H–invariance of the $E$–Terrell law).** We have that

1. $\nu^E$ is a positive map,
2. $t^E$ is an $H$–action on $N_{as}^E$,
3. $H \ni l \mapsto \nu^E \circ t^E(l)$ is a constant map.

**Proof.** Positivity follows by (175), the remaining statements follow since Prp. 5.13 and (84).

**Definition 71.** Let $\mathcal{L}$ be the language, we call a semantics for $E$ any map $\nu \in \prod_{ \mathcal{C} \in \Theta(\mathcal{T}, \mathcal{F}) } \mathcal{U}^{\mathcal{D}(\mathcal{C})}$ such that for all $l \in H$, $a, b \in \Theta(\mathcal{T}, \mathcal{F})$, $\mathcal{I} \in \mathcal{D}_{\mathcal{T}(a)}$, and $T \in \text{Mor}_c(b, a)$ and $\mathcal{T} \in \mathcal{D}_{\mathcal{T}(a)}$ satisfying $\mathcal{F}_c(T)(\mathcal{T}) \in \mathcal{D}_T(b)$ we have

1. $\nu^a(b^T(a)(l)(\mathcal{T})) \equiv \text{the fissioning system obtained by transforming } \nu^a(\mathcal{T}) \text{ through } l$,
2. $\nu^b(\mathcal{F}_c(T)(\mathcal{T})) \equiv \text{the fissioning system obtained by transforming } \nu^a(\mathcal{T}) \text{ through } T$.

In the remaining of this section let $\left\langle a, \{ \mathcal{T}_\#, \alpha_\# \}_{\# \in \{ s, as \}}, \{ f_j, N_j \}_{j \in \{ m, w \}} \right\rangle \in N^E$ and let $n$ be the asymmetric nucleon scheme extracted from it. In addition in the remaining of the paper, except in Conj. 41, let $\nu$ be a semantics: for $E$ in the present section, where we convene to remove the index $a$ in $\nu^a$, and for $E_\#$ in section 8.2. Finally we assume in the remaining of the paper that the interpretation $s$ holds the following additional properties

**Definition 72.** For any $\# \in \{ s, as \}$

1. $s(\mathcal{T}_\#) \equiv \text{the fission process of } \nu(\mathcal{T}_\#),$
2. $s(N_m) \equiv \text{the light fragment mass},$
3. $s(N_w) \equiv \text{the heavy fragment mass},$
4. $s(f_m) \equiv \text{the light nucleon core mass},$
Remark 8.3. According Def. 35 with the positions in Def. 72 we obtain for any \( # \in \{ s, as \} \), \( j \in \{ m, w \} \) and \( l \in H \)

- \( s(W^\tau(a)(\mathcal{I}_#, \alpha_#)) \equiv \) the state \( W^\tau(a)(\mathcal{I}_#, \alpha_#) \) occurring by performing the fission process of \( p(\mathcal{I}_#) \) on the thermal equilibrium state \( (\Phi^\tau(a))^{\mathcal{I}_#}_{\alpha_#} \) at the inverse temperature \( \alpha_# \) of the system, generated by \( \mathcal{F}(a) \), whose dynamics is \( (\mathcal{E}(a)^{\mathcal{I}_#}_{\alpha_#}(-\alpha_#^{-1}(\cdot))) \),

- \( s(W^\tau(a)(\mathcal{I}_#^l, \alpha_#)) \equiv \) the state \( W^\tau(a)(\mathcal{I}_#^l, \alpha_#) \) occurring by performing the fission process, of the fissioning system obtained by transforming \( p(\mathcal{I}_#) \) through \( l \), on the thermal equilibrium state \( (\Phi^\tau(a))^{\mathcal{I}_#^l}_{\alpha_#} \) at the inverse temperature \( \alpha_# \) of the system, generated by \( \mathcal{F}(a) \), whose dynamics is \( (\mathcal{E}(a)^{\mathcal{I}_#^l}_{\alpha_#}(-\alpha_#^{-1}(\cdot))) \),

- \( s(N^l_j) \equiv \) the observable obtained by transforming \( s(N^l_j) \) through \( l \),

- \( W^\tau(a)(\mathcal{I}_#^l, \alpha_#)(N^l_j) \) equals the mean value of \( s(N^l_j) \) in \( s(W^\tau(a)(\mathcal{I}_#^l, \alpha_#)) \),

- \( s(m^\tau(a)(\mathcal{I}_#^l, \alpha_#)) \equiv \) the phase, of the system \( \mathcal{F}(a) \), originating \( s(W^\tau(a)(\mathcal{I}_#^l, \alpha_#)) \),

- \( s(f^l_j) \equiv \) the observable obtained by transforming \( s(f^l_j) \) through \( l \),

- \( m^\tau(a)(\mathcal{I}_#^l, \alpha_#)(f^l_j) \) equals the mean value of \( s(f^l_j) \) in \( s(m^\tau(a)(\mathcal{I}_#^l, \alpha_#)) \).

To simplify the notations we convein to adopt the following abridgment

- \( s(W^\tau(a)(\mathcal{I}_#, \alpha_#)) \equiv \) the state \( W^\tau(a)(\mathcal{I}_#, \alpha_#) \) of the system generated by \( \mathcal{F}(a) \), occurring by performing at the inverse temperature \( \alpha_# \) the fission process of \( p(\mathcal{I}_#) \),

- \( s(W^\tau(a)(\mathcal{I}_#^l, \alpha_#)) \equiv \) the state \( W^\tau(a)(\mathcal{I}_#^l, \alpha_#) \) of the system generated by \( \mathcal{F}(a) \), occurring by performing at the inverse temperature \( \alpha_# \) the fission process of the fissioning system obtained by transforming \( p(\mathcal{I}_#) \) through \( l \).
Lemma 8.8. In addition we show in Thm. 8.9 the invariance of \( v^E(n) \) under action on \( n \) of suitable morphisms of \( C^0_u(H) \). We call universality of the Terrell law the two invariance properties stated in Thm. 8.9 here considered as what claimed in [Ric. IIa] as the universality of the asymmetric nucleon phase. Let us now define the concepts needed to implement the conjecture in the next section.

**Definition 74 (\( \mu, \mathcal{E} \)-compatibility).** Let \( n \in \mathbb{N}^{\mathcal{E}}_n \), then \( Z \) is \( \mu, \mathcal{E} \)-compatible map if

1. \( \mathcal{E}(Z) := \left\langle \left\langle \mathcal{U}, m, Z \right\rangle, \mathcal{T} \right\rangle \) is a \( \mathcal{E} \)-equivariant stability on \( \mathcal{D} \),
2. \( n \in \mathbb{N}^{\mathcal{E}}_n \),
3. there exist \( r, t \in \text{Rep}^{\mathcal{E}}(\mathcal{H}(\alpha, \alpha_{sa})) \) such that
   a. \( \Psi_r \circ \eta_r = \zeta_r^{\mathcal{E}}(\mathcal{T}_{sa}, \alpha_{sa}) \),
   b. \( \Psi_t \circ \eta_t = \zeta_r^{\mathcal{E}}(\mathcal{T}_{sa}, \alpha_{sa}) \),
   c. \( [\Phi^t] = [\Phi^r] \).

**Remark 8.5.** Let \( n \in \mathbb{N}^{\mathcal{E}}_n \), then for all \( \mu, \mathcal{E} \)-compatible map \( Z \) we have

\[
v^E(Z)(n) = 0.08 \left( \zeta_r^{\mathcal{E}}(\mathcal{T}_{sa}, \alpha)(N_m) - m^{\mathcal{E}}(\mathcal{T}_{sa}, \alpha)(f_m) \right) + 0.1 \left( \zeta_r^{\mathcal{E}}(\mathcal{T}_{sa}, \alpha)(N_w) - m^{\mathcal{E}}(\mathcal{T}_{sa}, \alpha)(f_w) \right).
\]

8.2. **Universality of the Terrell law.**

**Convention 8.6.** Let Terrell law denote the \( \mathcal{E}_\bullet \)-Terrell law, where \( \mathcal{E}_\bullet \) is the canonical \( C_u(H) \)-equivariant stability, moreover let \( \Theta(\mathcal{T}) \) denote \( \Theta^{\mathcal{T}}(\mathcal{H}) \), for any \( \mathcal{T} \in \text{Obj}(C_u(H)) \).

Note that \( \Theta(\Omega, \mathcal{E}_\bullet) = \Theta(C^0_u(H)) \).

**Remark 8.7.** Let \( n = \left\langle \mathcal{U}, \mathcal{I}, \alpha, \{f_j, N_j\}_{j \in \{m, w\}} \right\rangle \in \mathbb{N}^{\mathcal{E}_\bullet}_n \), then for all \( l \in H \) we have

\[
(v^E \circ f^E(l)(n)) = 0.08 \left( \zeta_r^{\mathcal{E}}(\mathcal{T}_{sa}, \alpha)(N_m') - m^{\mathcal{E}}(\mathcal{T}_{sa}, \alpha)(f_m') \right) + 0.1 \left( \zeta_r^{\mathcal{E}}(\mathcal{T}_{sa}, \alpha)(N_w') - m^{\mathcal{E}}(\mathcal{T}_{sa}, \alpha)(f_w') \right);
\]

**Definition 75.** We say that \( \mu, \mathcal{B}, T, x \) satisfies the hypothesis \( S \) if \( n = \left\langle \mathcal{U}, \mathcal{I}, \alpha, \{f_j, N_j\}_{j \in \{m, w\}} \right\rangle \in \mathbb{N}^{\mathcal{E}_\bullet}_n \), \( \mathcal{B} \in \text{Obj}(C^0_u(H)) \), \( T \in \text{Mor}_{\text{C}^*(H)}(\mathcal{B}, \mathcal{U}) \), and \( x = \{j', N'_j\}_{j \in \{m, w\}} \) such that \( f_j \in A_{\mathcal{B}}, N'_j \in \mathcal{B} \) with \( (N'_j)^* = N'_j \) and \( f_j = \mathcal{G}^{\mathcal{T}}(T)(f'_j) \) and \( N_j = T(N'_j) \), for any \( j \in \{m, w\} \), where \( \mathcal{B} \) is the \( \mathcal{C}^* \)-algebra underlying \( \mathcal{B} \). Let \( \mu, \mathcal{B}, T, x \) satisfy the hypothesis \( S \), define

\[
n^{(T, x)} := \left\langle \mathcal{B}, \mathcal{G}^{\mathcal{T}}(T)(\mathcal{I}), \alpha, x \right\rangle.
\]

**Lemma 8.8.** Let \( \mu, \mathcal{B}, T, x \) satisfy the hypothesis \( S \) where \( x = \{f'_j, N'_j\}_{j \in \{m, w\}} \), then

\[
(177)
\]

\[
\mathcal{G}^{\mathcal{T}}(T)(\mathcal{I}, \alpha)(N'_j) = \mathcal{V}^{\mathcal{E}}(T)(\mathcal{I}, \alpha)(N_j),
\]

moreover \( n^{(T, x)} \in \mathbb{N}^{\mathcal{E}_\bullet}_n \), and

\[
v^E(n^{(T, x)}) = 0.08 \left( \mathcal{V}^{\mathcal{E}}(T)(\mathcal{I}, \alpha)(N_m') - \mathcal{M}^{\mathcal{E}}(T)(\mathcal{I}, \alpha)(f_m') \right) + 0.1 \left( \mathcal{V}^{\mathcal{E}}(T)(\mathcal{I}, \alpha)(N_w') - \mathcal{M}^{\mathcal{E}}(T)(\mathcal{I}, \alpha)(f_w') \right).
\]
Proof. The first equality in (177) follows by Thm. 6.25(2) and (94), the second equality follows by Thm. 6.25(3(c)). That \( n^{(T,x)} \in N_{\text{as}}^{L} \) follows since the definition of the morphism class of \( C_{\mu}^{L}(H) \) and since (177). □

The physical meaning of the following result is stated in Cor. 8.11.

**Theorem 8.9 (Universality of the Terrell law).** We have

1. \( H \)-invariance of the Terrell law.
   \[
   H \ni l \mapsto v^{\epsilon}(l) \text{ is a constant map.}
   \]

2. System invariance of the Terrell law. Let \( (n, B, T, x) \) satisfy the hypothesis \( S \), then
   \[
   v^{\epsilon}(n^{(T,x)}) = v^{\epsilon}(n) .
   \]

Proof. St. (1) follows since Thm. 6.25(1) and Prp. 8.2 applied to \( \mathcal{E} \), st. (2) follows since Lemma 8.8. □

*Alternative proof of Thm. 8.9.* If hypothesis \( E \) holds true the statements follow since Thm. 7.20 and Rmk. 7.21. □

**Proposition 8.10.** If \( (n, B, T, x) \) satisfies the hypothesis \( S \) where \( n = \left\{ \mathfrak{A}, \mathfrak{T}, \alpha, \{ f_j, N_j \} \right\} \), then \( s(\mathcal{V}^B(b^H(T)(\mathfrak{T}), \alpha)) \equiv \mathcal{V}^B(b^H(T)(\mathfrak{T}), \alpha) \), the state \( \mathcal{V}^B(b^H(T)(\mathfrak{T}), \alpha) \), of the system generated by \( \mathcal{E}(\mathcal{B}) \), occurring by performing at the inverse temperature \( \alpha \) the fission process of the fissioning system obtained by transforming \( \mathcal{V}^B(\mathfrak{T}) \) through \( T \).

Proof. Since \( n^{(T,x)} \in N_{\text{as}}^{L} \) by Lemma 8.8 the statement then follows by Rmk. 8.3 and Def. 71. □

**Corollary 8.11.** Assume \( n \in N_{\text{as}}^{L} \) and in addition in st. (2) assume that \( (n, B, T, x) \) satisfies the hypothesis \( S \), then

1. \( H \)-invariance of the Terrell law. The mean value \( v^{\epsilon}(n) \) of the prompt neutron-yield in the state \( \mathcal{V}^B(\mathfrak{T}, \alpha) \), of the system generated by \( \mathcal{E}(\mathfrak{A}) \), occurring by performing at the inverse temperature \( \alpha \) the fission process of \( \mathcal{V}^B(\mathfrak{T}) \), it is positive and equals the mean value \( v^{\epsilon}(n') \) of the prompt neutron-yield in the state \( \mathcal{V}^B(\mathfrak{T}'(\mathfrak{T}), \alpha) \), of the system generated by \( \mathcal{E}(\mathfrak{A}) \), occurring by performing at the inverse temperature \( \alpha \) the fission process of the fissioning system obtained by transforming \( \mathcal{V}^B(\mathfrak{T}) \) through \( T \);

2. System invariance of the Terrell law. The mean value of the prompt neutron-yield in the state \( \mathcal{V}^B(b^H(T)(\mathfrak{T}), \alpha) \), of the system generated by \( \mathcal{E}(\mathcal{B}) \), occurring by performing at the inverse temperature \( \alpha \) the fission process of the fissioning system obtained by transforming \( \mathcal{V}^B(\mathfrak{T}) \) through \( T \), equals the mean value of the prompt neutron-yield in the state \( \mathcal{V}^B(\mathfrak{T}, \alpha) \), of the system generated by \( \mathcal{E}(\mathfrak{A}) \), occurring by performing at the inverse temperature \( \alpha \) the fission process of \( \mathcal{V}^B(\mathfrak{T}) \).

Proof. St. (1) follows since Thm. 8.9(1) and Rmk. 8.3, st. (2) since Thm. 8.9(2), Rmk. 8.3 and Prp. 8.10. □
Remark 8.12 (Generalized nucleon phase hypothesis). We consider the invariance of the Terrell law in Thm. [8.9] and its physical interpretation stated in Cor. [8.11] as our rendering of the universality of the nucleon phase. Thus we consider the request of the existence of $\eta \in \mathbb{N}^{\text{as}}_{\nu}$ as our generalization w.r.t. $H$, and the $C^*$-formulation, of the nucleon phase hypothesis advanced by G. Mouze and C. Ythier in [MHY2] see also [Ric]. A refinement of this hypothesis is stated in Conj. [1] taking into account the mass numbers 82 and 126, specifying the group $H$ and attributing the stability of the nucleon phase to the noncommutative geometric nature of the phase itself. In such a case we obtain in Cor. [8.14] as a consequence of Thm. [6.25] the universality of the mass numbers 82 and 126.

To any asymmetric nucleon scheme $\eta = \langle \mathfrak{U}, \Sigma, \alpha, \{f_j, N_j\}_{j \in \{m, w\}} \rangle$ relative to $\mathcal{E}_\bullet$ two asymmetric fragments corresponds, they are described by the mean values $\mathcal{V}_n^\eta(\Sigma_{\text{as}}, \alpha_{\text{as}})(N_j)$ of their mass observables, with $j \in \{m, w\}$. However the Terrell law [Ric] eq. (1) or (173) describes the behaviour of a family of couples of fragments each of them arising by a fissioning system belonging in the set $\mathcal{A} := \{U^{233} + n_{\text{th}}, U^{235} + n_{\text{th}}, Pu^{239} + n_{\text{th}}, Cf^{252}\}$. Moreover the values 82 and 126 appears in it. The conjecture below formulated assumes the existence of a nucleon scheme $\eta$ relative to $\mathcal{E}_\bullet$, where $H$ is the symmetry group of the standard model, such that for each $\zeta \in \mathcal{A}$ there exists a $(\eta, \mathcal{E}_\bullet)$-compatible map $\mathcal{Z}_\zeta$ such that $\mathcal{V}_n^\eta(\Sigma, \alpha)(\mathcal{Z}_\zeta(\eta))$ is the Terrell law relative to the fissioning system $\zeta$. This implies that for any fissioning system in the set $\mathcal{A}$, the state describing its asymmetric fragments is originated by a cyclic cocycle cohomologically equivalent to the one originating the state $\mathcal{V}_n^\eta(\Sigma_{\text{as}}, \alpha_{\text{as}})$, see Rmk. [5.25]

We recall that $SU(2, \mathbb{C})$ is the universal covering group of the proper Lorentz group $\mathbb{R}^4_+$ on $\mathbb{R}^4$, see for example [BLOT] p.121, while $F^0 = U(1) \times SU(2) \times SU(3)$ is the gauge group of the standard model. Moreover let the standard action of $SU(2, \mathbb{C})$ on $\mathbb{R}^4$ denote the action defined in [BLOT] eqs. (3.39) – (3.33a), and let $g$ denote the Lorentz metric tensor on $\mathbb{R}^4$. Finally let $\iota_1$ and $\iota_2$ be the canonical injections of $SL(2, \mathbb{C})$ and $F^0$ in $SL(2, \mathbb{C}) \times F^0$ respectively, and $Pr_{\mu}(\lambda) = \lambda_{\mu}$ for any $\lambda \in \mathbb{C}^4$ and $\mu \in \{1, \ldots, 4\}$.

**Definition 76 (Nucleon Systems).** We call $\langle R, \eta, \mathfrak{U} \rangle$ a nucleon system if

1. $R$ is a locally compact group;
2. $SL(2, \mathbb{C}) \times F^0$ is a topological subgroup of $R$;
3. $\eta : R \to \text{Aut}(\mathbb{R}^4)$ is a group homomorphism;
4. the map $(g, f) \mapsto \eta_f(g)$ on $\mathbb{R}^4 \times R$ at values in $\mathbb{R}^4$, is continuous;
5. $\eta \circ \iota_1$ and $\eta \circ \iota_2$ are the standard action of $SL(2, \mathbb{C})$ and the trivial action of $F^0$ on $\mathbb{R}^4$ respectively;
6. $\mathfrak{U} \in \text{Obj}(\mathcal{C}_R(\mathbb{R}^4 \rtimes_{\eta} R))$.

**Definition 77.** $\mathcal{A} := \{U^{233} + n_{\text{th}}, U^{235} + n_{\text{th}}, Pu^{239} + n_{\text{th}}, Cf^{252}\}$.

In what follows we consider that $G = \mathbb{R}^4$, so $H = \mathbb{R}^4 \rtimes_{\rho} F$, and that $SL(2, \mathbb{C}) \times F^0$ is a topological subgroup of $F$.

**Conjecture 1 (Noncommutative geometric nature of the Terrell law stability).** There exist a nucleon system $\langle F, \rho, \mathfrak{U} \rangle$, $\eta \in \mathbb{N}^{\text{as}}_{\nu}$, where $\mathfrak{U}$ is the dynamical system underlying $\eta$, a function $\Sigma$ on $\mathcal{A}$ with values in the set of the $(\eta, \mathcal{E}_\bullet)$–compatible maps and a map $\nu$ on $\mathcal{A}$ such that $\nu_\zeta$ is a semantics for $\mathcal{E}_\bullet(\Sigma_{\zeta})$ for any $\zeta \in \mathcal{A}$, such that said $\eta = \langle \mathfrak{U}, \Sigma, \alpha, \{f_j, N_j\}_{j \in \{m, w\}} \rangle$ we have
(1) \( \overline{m}(\xi, \alpha)(f_m) = 82 \) and \( \overline{m}(\xi, \alpha)(f_{w_1}) = 126 \),
(2) \( v_{\xi}^\alpha(\xi) = \zeta \) for any \( \zeta \in A \), i.e. \( v_{\xi}^\alpha(\xi^\vee n)(n) \) equals the mean value of the prompt neutron-yield in the state, of the system generated by \( G(\mathbb{N}) \), occurring by performing at the inverse temperature \( \alpha \) the fission process of \( \zeta \).

Remark 8.13. If Conj. [111] is satisfied then the noncommutative geometric nature of the stability of the Terrell law, under variation of the fissioning system in the set \( A \), follows since Rmk. 5.25 Def. [74] and Rmk. 8.5.

Corollary 8.14 (Universality of the mass numbers 82 and 126). Let \( n \in \mathbb{N}^* \) satisfying Conj. [111]. Then \( \overline{m}(\xi', \alpha)(f'_m) = 82 \) and \( \overline{m}(\xi', \alpha)(f'_{w_1}) = 126 \), for all \( l \in H \). Moreover \( \overline{m}^\mathbb{R}(b^H(T)(\xi), \alpha)(f'_m) = 82 \) and \( \overline{m}^\mathbb{R}(b^H(T)(\xi), \alpha)(f'_{w_1}) = 126 \), for any \( (n, \mathbb{N}, T, x) \) satisfying the hypothesis \( S \), where \( x = \{f_j', N_j'\}_{j \in \{m, w_1\}} \).

Proof. The first sentence of the statement follows since Thm. 6.25(2) & (94), the second one follows since (177).

Remark 8.15. Let \( r \in \{s, as\} \) then provisionally we can set \( \xi_r = \{T, \mu, \xi_r, \zeta_r, f, \Gamma\} \in \mathbb{I} \) with
\[
\mathcal{T} = \langle h, \xi_r, \beta_r, I, \omega \rangle,
\]
satisfying
\[
\begin{cases}
\text{Dom}(\xi_r) = \mathbb{R}^4, \\
f = \sum_{\mu, \nu = 1}^4 \text{Pr}_\mu g_{\mu, \nu} \text{Pr}_\nu.
\end{cases}
\]
f satisfies (115) for the position \( A = \{1, \ldots, 4\} \) for any \( \alpha \in \mathbb{I}_A \), since \( f \) maps \( \mathbb{I}_A \) into \( \mathbb{R} \) and the support of the resolution of the identity of any selfadjoint operator in a Hilbert space is a subset of \( \mathbb{R} \). Hence the request (178) is well-set. Next \( \zeta_r : \mathbb{R}^4 \rightarrow S^\mathbb{C}_{\mathbb{R}^4} \) since Def. 40. Set \( T^\alpha_r = \{T^\alpha_{v_v} : v \in X\} \), where \( it^\alpha_{v_v} \) is the infinitesimal generator of the one-parameter unitary group \( U_{\xi_r} \circ \zeta_r \circ i_{\nu} \) on \( \xi_r \), and \( i_{\nu} : \mathbb{R} \rightarrow \mathbb{R}^4 \) such that \( \text{Pr}_\nu \circ i_{\nu} = \delta_{\nu, \mu} \), for all \( \mu, \nu \in X \). Since \( \text{Pr}_\mu(E_{\xi_r}) = T^\alpha_{\mu, v_v} \) (178) and an application of the functional calculus, see for example [KR, Thm. 5.6.19] we deduce that
\[
D_{\xi_r, \alpha} = \sum_{\mu, \nu = 1}^4 T^\alpha_{\mu, v_v} g_{\mu, \nu} T^\alpha_{\nu, v_v}
\]
where \( \overline{S} \) is the closure of any closable operator \( S \) in a Hilbert space. Thus in this case the only difference between the two nucleon phases results from the fact that we select via the two group morphisms \( \zeta_{as} \) and \( \zeta_r \) two different sets of infinitesimal generators of commutative subgroups of \( S^\mathbb{C}_{\mathbb{R}^4} \). Finally it is worthwhile noting that the selfadjoint operator \( D_{\xi_r, \alpha} \) in general cannot be considered the mass operator even when \( \zeta_r(x) = (x, 1) \) for any \( x \in \mathbb{R}^4 \), indeed it needs not to be positive.

Remark 8.16. Since (82) it is clear that \( \overline{m}(\xi, \alpha) \) is represented by the Chern-Connes character of an entire cocycle dependent by the element \( (\xi, \alpha) \), while it is desirable to have a unique character associated to the map \( \overline{m}(\xi, \alpha) \). In this line we shall analyse in a future work the possibility of constructing a suitable category and a functor \( \mathcal{L} \) from \( G(\mathbb{G}, F, \rho) \) to it, encoding for any object \( \mathcal{G} \) the data \( \{(\xi, \alpha) : \xi \in \mathbb{I}_\mathcal{G}, \alpha \in \mathbb{P}_\mathcal{G}\} \) into three \( C^* \)-algebras and relating the
map $\overline{m}^g$ to a bivariant Chern-Connes character as defined in [Nis], from which we get the notations of what follows. More precisely we require $\mathcal{L}$ to assign to any object $G$ of $\mathcal{S}(G,F,\rho)$ at least the following data:

- $\mathcal{C}^\star$-algebras $D$ and $\mathcal{R}$ and a smooth subalgebra $\mathcal{R}^\infty$ of $\mathcal{R}$,
- a group morphism $\sigma : A_G \rightarrow K_0(D),$
- $i \in \{0,1\}, p, n \in \mathbb{N}$ such that $(2 - i)n \geq p - 1,$
- $z : \prod_{\alpha \in \mathcal{P}_G} \mathbb{P}_G^\alpha \rightarrow \mathcal{E}'(D;\mathcal{R}^\infty;\mathcal{R}^\infty^+),$
- $\phi \in Ext^{2n+i}_{\Lambda}(\mathcal{R}^\infty^+,\mathbb{C}^\star),$

such that for all $\mathcal{I} \in \mathcal{I}_G$, $\alpha \in \mathbb{P}_G^\mathcal{I}$ and $f \in A_G$ we obtain

$$\overline{m}^g(\mathcal{I},\alpha)(f) = \phi_* (\sigma(f) \otimes D(z(\mathcal{I},\alpha))),$$

which is well-set since its left side is integer by definition, while its right side is so since the index theorem [Nis, Thm. 6.4].

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