Explicit formula of a supersingular polynomial for rank-2 Drinfeld modules and applications

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Abstract

Rank-2 Drinfeld modules are a function-field analogue of elliptic curves, and the purpose of this paper is to investigate similarities and differences between rank-2 Drinfeld modules and elliptic curves in terms of supersingularity. Specifically, we provide an explicit formula of a supersingular polynomial for rank-2 Drinfeld modules and prove several basic properties. As an application, we give a numerical example of an asymptotically optimal tower of Drinfeld modular curves.

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1 Introduction

Throughout this paper, we use the following terminology. We let \( \mathbb{Z} \) denote the set of integers, and \( \mathbb{N} \) denote the set of nonnegative integers. That is,

\[ \mathbb{N} := \{0, 1, 2, 3, \ldots\}. \]

The cardinality of a finite set \( S \) is denoted by \( |S| \). The notation \([x]\) indicates the ceiling of \( x \) and denotes the smallest integer greater than or equal to a real number \( x \), namely, \([x] := \min\{n \in \mathbb{Z} \mid x \leq n\}\). In this paper, \( q \) will be some power of a prime number, and \( \mathbb{F}_q \) will denote the finite field with \( q \) elements. The algebraic closure of a field \( L \) is denoted \( \overline{L} \).

Throughout this paper, we adopt the notation used in the following references: Goss [22] and Thakur [35] for rank-2 Drinfeld modules, Gekeler [17, 20, 21] for Drinfeld modular curves, Silverman [30] and Husemoller [27] for elliptic curves, and Stichtenoth [30] for function fields.

It is known that rank-2 Drinfeld modules are a function-field analogue of elliptic curves. This relation was first discovered by Drinfeld [9, 10] and has been studied since then by many researchers (see, for example, [17, 22, 35]). It is thus natural to investigate similarities and differences between rank-2 Drinfeld modules and elliptic curves. This paper does so in terms of supersingularity.

Before we present the main theorem and its corollary, we recall the analogous results for the elliptic-curve case. Let \( p \geq 3 \) be a prime number. It is well-known that every elliptic curve is isomorphic (over \( \overline{\mathbb{F}_p} \)) to an elliptic curve in Legendre form

\[ E_\lambda: y^2 = x(x-1)(x-\lambda), \]

where \( \lambda \) is an element in \( \overline{\mathbb{F}_p} \) with \( \lambda \neq 0, 1 \) (see, for example, Proposition 1.7 of Chapter III in [30]). Let \( m \geq 1 \) be an integer, and let \( E_\lambda(\overline{\mathbb{F}_p})[m] \) denote the \( m \)-torsion subgroup of \( E_\lambda \). The elliptic curve \( E_\lambda \) is called supersingular when \( E_\lambda(\overline{\mathbb{F}_p})[p] = 0 \).

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From here, \( \lambda \) is regarded as an indeterminate element. We set the Deuring polynomial

\[
H_p(\lambda) := \sum_{i=0}^{(p-1)/2} \binom{(p-1)/2}{i} \lambda^i \in \mathbb{F}_p[\lambda],
\]

which was first defined by Deuring [8]. Then, we know the following facts:

(E1) \( E_\lambda \) is supersingular if and only if \( H_p(\lambda) = 0 \) (e.g., see Theorem 4.1 (b) of Chapter V in [30]).

(E2) \( H_p(\lambda) \) is separable (e.g., see Theorem 4.1 (c) of Chapter V in [30]).

(E3) If \( \lambda \in \bar{\mathbb{F}}_p \) is a root of \( H_p(\lambda^4) = 0 \), then \( \lambda \in \mathbb{F}_p^2 \) (see Theorem of Appendix in [15]).

(E3\text{weak}) If \( \lambda \in \bar{\mathbb{F}}_p \) is a root of \( H_p(\lambda) = 0 \), then \( \lambda \in \mathbb{F}_p^2 \).

(E4) The sequence of elliptic modular curves \( X_0(2^n)/\mathbb{F}_p^2 \) \((n \geq 2)\) is asymptotically optimal, which is expressed as

\[
\frac{N(X_0(2^n)/\mathbb{F}_p^2)}{g(X_0(2^n))} \to p - 1 \quad (n \to \infty),
\]

where \( N(X_0(2^n)/\mathbb{F}_p^2) \) denotes the number of rational points of \( X_0(2^n)/\mathbb{F}_p^2 \), and \( g(X_0(2^n)) \) denotes the genus of \( X_0(2^n) \) (for a more elementary proof, see Theorem 5.2 in [15], and moreover, for a proof of a more general case, see Theorem 3.2 in [21] and Theorem 4.1.52 in [30]).

Note that although a more general case of (E4) is proven, the proof in [15] explicitly describes the set of rational points that split completely in the sequence, in order to apply coding theory.

We know the following function-field analogues of (E1), (E2), (E3) and (E3\text{weak}). Specifically, Gekeler has proven analogues of (E1) and (E3\text{weak}) for rank-2 Drinfeld modules defined by the general type \([4]\) (see Satz (5.3) in [10], Proposition 4.2 in [19]), and has proven analogues of (E2) and (E3) for a rank-2 Drinfeld module defined by the specific type \([4]\) (see Lemmas (5.6) and (5.7) in [16]), where the types \([4]\) and \([5]\) are defined later. Notice that Proposition 4.2 in [19] discusses Drinfeld modules of arbitrary rank. El-Guindy and Papanikolas have computed a formula that corresponds to \( H_p(\lambda) \) for Drinfeld modules of arbitrary rank (see Theorem 8.1, Corollary 8.2 in [11]), but it is not explicit in our sense. Bassa and Beelen have defined polynomials by recursion, and have proven analogues of (E1), (E2) and (E3\text{weak}) for these polynomials (see Corollary 20, Proposition 13, Theorem 18 in [2]). El-Guindy has computed an explicit formula for the polynomials of Bassa and Beelen (see Theorem 3.2 (ii) in [12]). From the viewpoint of the theory of Drinfeld modules, the results of Bassa and Beelen can be regarded as analogous of (E1), (E2) and (E3\text{weak}) for a rank-2 Drinfeld module defined by the specific type \([4]\) (see the remark in Section 2 of this paper), where the type \([4]\) is defined later. Similarly, the result of El-Guindy can be regarded as an explicit formula that corresponds to \( H_p(\lambda) \) for a rank-2 Drinfeld module defined by \([4]\) (see the remark in Section 2 of this paper).

We know the following function-field analogues of (E4). Bassa and Beelen have studied an analogue of (E4) using analogues of (E1), (E2) and (E3\text{weak}) (see Corollary 17 in [2]). An analogue of a more general case of (E4) has also been studied (see Theorem 2.16 in [21], Theorem 4.2.38 in [30]).

An explicit formula that corresponds to \( H_p(\lambda) \) for rank-2 Drinfeld modules defined by the general type \([3]\) has not been before this paper. In the first half of this paper, we provide it, with equivalence proved as Main theorem (1). Using Main theorem (1), we prove analogues of (E2) and (E3) for a rank-2 Drinfeld module defined by \([3]\) (as Main theorem (2)). In the final half of this paper, we prove an analogue of (E4) as a corollary by applying the theorem.

Although Main theorem (2) is similar to Lemmas (5.6) and (5.7) in [16], we prove it here because we need it in the proof of the corollary. Note that our corollary has already been studied by Bassa and Beelen (see Corollary 17 in [2]). However, their proof becomes complicated in showing that the completely splitting points are rational (see Pages 12 and 13), because they do not use a supersingular polynomial in the Gekeler’s sense. In this paper, the proof of this is simple, because we use a supersingular polynomial. Note that our corollary is a special case of Theorem 2.16 in [21] and Theorem 4.2.38 in [30]. However, our result has interesting applications within coding theory (see [31], Chapters 7 and 8 in [33], Parts 3 and 4 in [36], Chapters 3 and 4 in [37]), because our proof is more elementary, and explicitly describes the set of rational points that split completely.
The low-genus curve cases of (E1)–(E4) were studied in [24, 25, 26]. There, we defined explicit polynomials that correspond to \( H_p(\lambda) \) and showed that the sequences of elliptic modular curves \( X_0(3^n), X_0(4^n), X_0(3 \cdot 2^n) \), and \( X_0(2 \cdot 3^n) \) (\( n \geq 2 \)) are asymptotically optimal.

Thakur introduced two distinct hypergeometric functions for function fields (see [33, 34], Subsection 6.5 in [35]). In this paper, we suggest the possibility of another hypergeometric function. As background material, we recall a relation between the polynomial \( H_p(\lambda) \), a hypergeometric function, and a period of an elliptic curve. It is known that a (real) period of an elliptic curve \( E_\lambda \) is equal to the product of the Gauss hypergeometric function

\[
F(1/2, 1/2, 1; \lambda) := \sum_{n=0}^{\infty} \frac{(1/2)_n (1/2)_n}{(1)_n} \frac{\lambda^n}{n!}
\]

and the circular constant \( \pi \) (the ratio of circumference to diameter), that is,

\[
\omega_2(\lambda) = \pi \cdot F(1/2, 1/2, 1; \lambda) \quad (1)
\]

(see, for example, Theorem (6.1) in [27]). For comparison, \( H_p(\lambda) \) is the truncated hypergeometric function

\[
H_p(\lambda) \equiv F_{tr}(1/2, 1/2, 1; \lambda) \quad (mod \ p) \quad (2)
\]

(see Page 261 in [27]). Hence, by applying the equalities (11) and (2), we can regard \( H_p(\lambda) \) as the product of a suitable period \( \omega_{2tr}(\lambda) \) and a suitable constant \( 1/\pi_{tr} \), that is,

\[
H_p(\lambda) \equiv \frac{1}{\pi_{tr}} \cdot \omega_{2tr}(\lambda) \quad (mod \ p).
\]

Surprisingly, this phenomenon also occurs in our Drinfeld-module case (see the last remark of Section 2).

For the main theorem, we introduce a rank-2 Drinfeld module and a partition of a subset of \( \mathbb{N} \).

Let \( A := \mathbb{F}_q[T] \) denote a polynomial ring, and let \( p \) denote its nonzero prime ideal. Then, there is a monic irreducible polynomial \( p(T) \in A \) such that \( p = (p(T)) \). Throughout this paper, we always suppose that \( p(T) \neq T \). Set \( \mathbb{F}_p := A/p \) and \( d := \deg p(T) \). Let \( \mathbb{F}_p^{(2)} \) denote the quadratic extension of \( \mathbb{F}_p \). Let \( \alpha \) be any root of \( p(T) \), and fix this root. Note that \( \alpha \neq 0 \). We see that \( A/p = \mathbb{F}_q^{(d)} = \mathbb{F}_q(\alpha) \). Observe that \( \alpha^q, \ldots, \alpha^{q^{d-1}} \) are the other roots of \( p(T) \), and that \( \alpha^{q^d} = \alpha \).

Let \( K := \mathbb{F}_q(T) \) denote the quotient field of \( A \), and let \( C_K \) denote the completion of an algebraic closure of the completion of \( K \) at the infinite place (see Subsection 4.1 in [22]).

Let \( L \) be an extension of either \( K \) or \( \mathbb{F}_p \), and let \( \iota: A \to L \) denote an \( \mathbb{F}_q \)-algebra homomorphism. Notice that if \( L \) is an extension of \( K \) (resp. \( \mathbb{F}_p \)), then \( \iota(T) = T \) (resp. \( \iota(T) = \alpha \)). Let

\[
\tau : L \to L, \quad \tau(l) = l^q
\]

denote a Frobenius endomorphism, and let \( L\{\tau\} \) denote a polynomial ring in \( \tau \) under addition and composition, that is, \( \tau l = l^q \tau \) for any \( l \in L \) (see Section 1 of Chapter I in [17] or Subsection 1.1 in [22]).

A rank-2 Drinfeld module over \( L \) is an \( \mathbb{F}_q \)-algebra homomorphism

\[
\phi : A \to L\{\tau\}
\]

such that

\[
\phi_T := \phi(T) = \iota(T) + A_1 \tau + A_2 \tau^2 
\]

(3)
(A_1, A_2 \in L, A_2 \neq 0) (see Definition 1.3 of Chapter I in [17], Definition 4.4.2 in [22], Definition 2.2.1 in [33]). The j-invariant of \( \phi \) is defined by \( j(\phi) := A_1^{2 + 1}/A_2 \) (see Example 3.6 of Chapter V in [17], Subsection 2.7 in [25], Subsection 6.1 in [33]). Recall that for any \( a \in A \), the constant term of \( \phi_a := \phi(a) \) is \( i(a) \), and that the degree of \( \phi_a \) in \( \tau \) is \( 2 \deg_T(a) \).

**Ker(\( \phi_a \)) := \{ x \in \bar{L} | \phi_a(x) = 0 \}**

denote the \( a \)-torsion points of \( \phi \), which is a subspace of \( \bar{L} \).

Here, we discuss a normal form for rank-2 Drinfeld modules (see Section 2 in [12]). Suppose that \( L = C_\infty \). In this case, it is known that \( \dim_{\mathbb{F}_q}(\text{Ker}(\phi_T)) = 2 \) (see Proposition 1.6 of Chapter I in [17]). Let \( \delta \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q \) be an element such that \( \delta^q = -\delta \). Observe that \( \mathbb{F}_q \delta \) is a subspace of \( C_\infty \), and \( \dim_{\mathbb{F}_q} \mathbb{F}_q \delta = 1 \). Next, we set

\[
\mathcal{F}_\delta := \left\{ \phi \mid \phi \text{ is a rank-2 Drinfeld module over } C_\infty, \text{ and } \mathbb{F}_q \delta \subseteq \text{Ker}(\phi_T) \right\}.
\]

Assume that \( \phi \) is any Drinfeld module in the set \( \mathcal{F}_\delta \) defined by \( \phi_T = \iota(T) + A_1 \tau + A_2 \tau^2 \). Then, we have the relation \( \lambda := A_2 = A_1 - \iota(T) \). Conversely, any rank-2 Drinfeld module \( \phi \) defined by

\[
\phi_T = \iota(T) + (\iota(T) + \lambda) \tau + \lambda \tau^2
\]

is in the set \( \mathcal{F}_\delta \). In fact, for any \( x \in \mathbb{F}_q \), we have

\[
\phi_T(x\delta) = \iota(T)(x\delta) + (\iota(T) + \lambda)(x\delta)^q + \lambda(x\delta)^q = \iota(T)x\delta - (\iota(T) + \lambda)x\delta + \lambda x\delta = 0.
\]

Hence, we can regard a Drinfeld module defined by (4) as being in a normal form in the above sense. In this paper, we sometimes restrict Drinfeld modules to those defined by (4).

In a 1983 paper, Gekeler regarded a Drinfeld module defined by

\[
\phi_T = \iota(T) + \lambda \tau + \tau^2
\]

as being in a normal form, and using this, proved results similar to Main theorem (2) (see Section 5 in [10]). Notice that Corollary to Main theorem (2) can also be shown by using his results, though an equation used in its proof is different from our equation (13), which is defined in Section 3.

For \( L \) an extension of \( \mathbb{F}_p \), we write

\[
\phi_p := \phi_{p(T)} = \sum_{i=0}^{2d} g_i \tau^i \quad \text{and} \quad H_p^{(d)}(\phi) := g_d.
\]

Then, we know that

\[
g_0 = g_1 = \cdots = g_{d-1} = 0
\]

(see Section 5 in [13], Section 11 in [18]). This fact is often used in this paper. Recall that \( \phi_p(0) = 0 \), since \( \phi_p(x) = \sum_{i=0}^{2d} g_i x^i \). A rank-2 Drinfeld module \( \phi \) over \( L \) is called supersingular at \( \mathfrak{p} \) when \( \text{Ker}(\phi_p) = \{0\} \) (see Remark 2.4 of Chapter VIII in [17], Definition 4.12.16 in [22]). With this, \( \phi \) is supersingular at \( \mathfrak{p} \) if and only if \( g_d = H_p^{(d)}(\phi) = 0 \) (see Satz (5.3) in [10]).

Assume that \( A_1 = \alpha + \lambda \) and \( A_2 = \lambda \), that is, that a Drinfeld module \( \phi \) in the form (4) can be defined. Then, the coefficients \( g_d, g_{d+1}, \ldots, g_{2d} \) are polynomials in \( \lambda \) over \( L \). We set

\[
H_p^{(d)}(\lambda) := H_p^{(d)}(\phi).
\]

Next, we introduce a partition of a subset of \( \mathbb{N} \). For a positive integer \( d > 0 \), we write

\[
\mathbb{N}_{<d} := \{0, 1, \ldots, d - 1\}.
\]

For a finite subset \( S \) of \( \mathbb{N} \) and a positive integer \( j > 0 \), we let \( S + j := \{i + j \mid i \in S\} \). Notice that \( 0 + j = 0 \).

A partition of \( \mathbb{N}_{<d} \) is a collection \( \{S_1, S_2, S_3\} \) of subsets of \( \mathbb{N}_{<d} \) such that

\[
S_1 \cap S_2 = S_2 \cap S_3 = S_3 \cap S_1 = \emptyset \quad \text{and} \quad S_1 \cup S_2 \cup S_3 = \mathbb{N}_{<d}.
\]
For an integer $d$, we define

$$P(d) = P_2(d)$$

$$:= \begin{cases} 
(T^2 - T) & \text{if } L \text{ is an extension of } K; \\
\alpha^q - \alpha & \text{if } L \text{ is an extension of } \mathbb{F}_p, \\
\{ (S_1, S_2) \mid \{ S_1S_2, S_2 + 1 \} \text{ forms a partition of } \mathbb{N}_{< d} \} & \text{if } d > 0; \\
\emptyset & \text{if } d = 0; \\
\emptyset & \text{if } d < 0.
\end{cases}$$

Note that $d = |S_1| + 2|S_2|$ and thus $|S_1| + |S_2| = d - |S_2|$ for $(S_1, S_2) \in P(d)$. It is known that

$$|P(d)| = \begin{cases} 
|P(d - 1)| + |P(d - 2)| & \text{if } d > 0; \\
1 & \text{if } d = 0; \\
0 & \text{if } d < 0
\end{cases} \quad (6)$$

(see Lemma 2.1 (iii) in [11]).

For a nonnegative integer $n$ and a finite subset $S$ of $\mathbb{N}$, set

$$[n] := \begin{cases} 
T^n - T & \text{if } L \text{ is an extension of } K; \\
\alpha^n - \alpha & \text{if } L \text{ is an extension of } \mathbb{F}_p, \\
(-1)^{|S|} \prod_{i \in S} [i] & \text{if } S \neq \emptyset; \\
1 & \text{if } S = \emptyset,
\end{cases}$$

$$L(S) := \begin{cases} 
\sum_{i \in S} q^i & \text{if } S \neq \emptyset; \\
0 & \text{if } S = \emptyset.
\end{cases}$$

The following is our main theorem and a corollary of that theorem.

**Main theorem.** Let $p = (p(T))$ denote a nonzero prime ideal of $A = \mathbb{F}_q[T]$ such that $p(T) \neq T$, and let $\alpha$ be a root of $p(T)$. Further, assume that $L$ is an extension of $\mathbb{F}_p$ and let $\phi$ be any rank-2 Drinfeld module over $L$ defined by $\phi_T = \alpha + A_1 \tau + A_2 \tau^2$. Then, the following hold.

1. The coefficient $H_p^{(d)}(\phi)$ is given by

$$H_p^{(d)}(\phi) = \sum_{(S_1, S_2) \in P(d)} L(S_2 + 1) A_1^{w(S_1)} A_2^{w(S_2)}.$$

2. When $A_1 = \alpha + \lambda$ and $A_2 = \lambda$, the polynomial $H_p^{(d)}(\lambda)$ is separable, and its degree in $\lambda$ is $\deg_\lambda H_p^{(d)}(\lambda) = (q^d - 1)/(q - 1)$. Moreover, if $H_p^{(d)}(\lambda^{q+1}) = 0$, then $\lambda \in \mathbb{F}_p^{(2)}$.

**Corollary.** The sequence of Drinfeld modular curves $X_0(T^n)/\mathbb{F}_p^{(2)}$ $(n \geq 2)$ is asymptotically optimal, that is,

$$\frac{N(X_0(T^n)/\mathbb{F}_p^{(2)})}{g(X_0(T^n))} \rightarrow q^d - 1 \quad (n \rightarrow \infty),$$

where $N(X_0(T^n)/\mathbb{F}_p^{(2)})$ denotes the number of rational points of $X_0(T^n)/\mathbb{F}_p^{(2)}$, and $g(X_0(T^n))$ denotes the genus of $X_0(T^n)$.

Note that we prove the corollary in terms of function fields, rather than curves. The motivation for the corollary comes from coding theory. For applications to the theory, it is essential that the proof is elementary and explicit. Our proof is elementary and explicit. It is well-known that computer science uses fields of characteristic $p = 2$. Then, the base field of (E4) is $\mathbb{F}_4$, which is small. However, with the corollary, we can choose a base field of characteristic $p = 2$ large enough.

The organization of this paper is as follows. In Section[2] we prove Main theorem (1) (Proposition [2.4] (a)). In the proof of Main theorem (1), a polynomial identity (Keylemma) plays a key role. In
2 An explicit formula for a supersingular polynomial

In this section, we prove Main theorem (1) (Proposition 2.3 (a)), which was introduced in Section 1.

In the course of the proof, a polynomial identity (Keylemma) plays a key role. The proof of Main theorem (1) relies on combining our Keylemma with results of El-Guindy and Papanikolas [11].

We recall the setup introduced in Section 1. Let \( p = (p(T)) \) denote a nonzero prime ideal of \( A = \mathbb{F}_q[T] \), where \( p(T) = \sum_{i=0}^{d} \mu_i T^i \in A \) is a monic irreducible polynomial of degree \( d \). We call \( p = (p(T)) \) an ideal of degree \( d \). Let \( \alpha \) be a root of \( p(T) \). Suppose that \( p(T) \neq T \). Note that \( \alpha \neq 0 \).

Let \( L \) be an extension of either \( K \) or \( \mathbb{F}_p \), and let \( \phi \) be a Drinfeld module over \( L \) defined by

\[
\phi_T = \lambda(T) + A_1 \tau + A_2 \tau^2
\]

\((A_1, A_2 \in L, A_2 \neq 0)\).

In order to understand the structure of the proof of Main theorem (1), we consider a relation between the coefficient \( H_p^{(d)}(\phi) \) and the set \( P(d) \). These were introduced in Section 1. In the following examples, we calculate the coefficients \( H_p^{(d)}(\phi) \) by using the rule \( \tau l = l \tau \) \((l \in L)\), and compute the sets \( P(d) \) by using the recursion (3) given in Section 1.

Notice that for \( d = 2, 3, 4, 5 \), our coefficients \( H_p^{(d)}(\phi) \) and the supersingular polynomials \( P_d(j) \) in Examples (2.2) of [4] coincide (cf. Proposition (6.2) (ii) (Deligne’s congruence) in [4]).

Example 2.1. Let \( L \) be an extension of \( \mathbb{F}_p \). Note that \( |n| = \alpha^n - \alpha \) and that \( \alpha \neq 0 \).

(1) Case \( d = 1 \): We compute the coefficient \( H_p^{(1)}(\phi) \). Since \( p(T) = T + \mu_0 = T - \alpha \), we get \( \mu_0 = -\alpha \). So, we have \( \phi_{p(T)} = \phi(T) + \mu_0 = p(\alpha) + A_1 \tau + A_2 \tau^2 = A_1 \tau + A_2 \tau^2 \), and hence

\[
H_p^{(1)}(\phi) = A_1.
\]

The set \( P(1) \) is given by \( P(1) = \{(0, \emptyset)\} \) and \( |P(1)| = 1 \).

Last, we consider the relation between \( H_p^{(1)}(\phi) \) and \( P(1) \). The term \( A_1 \) coincides with the element \( \{(0), \emptyset\} \). In fact, \( A_1 = L(\emptyset + 1)A_1^{w(\emptyset)} A_2^{w(\emptyset)} \), where \( L(\emptyset + 1) = 1 \) and \( w(\emptyset) = 0 \).

(2) Case \( d = 2 \): Recall that \( \alpha \tau^2 = \alpha \). We calculate \( H_p^{(2)}(\phi) \). Since \( p(T) = T^2 + \mu_1 T + \mu_0 = (T - \alpha)(T - \alpha^2) \), we get \( \mu_0 = (\alpha)(-\alpha^2) \) and \( \mu_1 = -\alpha - \alpha^2 \). Then, we have

\[
\begin{align*}
\phi_{p(T)} &= \phi(T)^2 + \mu_1 \phi(T) + \mu_0 \\
&= p(\alpha) + (\alpha^2 + \alpha + \mu_1) A_1 \tau + (A_1^{\tau+1} + \alpha^2 A_2 + \alpha A_2 + \mu_1 A_2) \tau^2 \\
&+ (A_1 A_2^2 + A_1^2 A_2) \tau^3 + A_2^{\tau+1} \tau^4 \\
&= (A_1^{\tau+1} - (\alpha - \alpha^2) A_2) \tau^2 + (A_1 A_2^2 + A_1^2 A_2) \tau^3 + A_2^{\tau+1} \tau^4,
\end{align*}
\]

and hence

\[
H_p^{(2)}(\phi) = A_1^{\tau+1} - [1] A_2 = (j(\phi) - [1]) A_2,
\]

where \( j(\phi) = A_1^{\tau+1} / A_2 \). Notice that \( P_2(j(\phi)) \equiv j(\phi) - [1] = H_p^{(2)}(\phi)/A_2 \pmod p \).

The set \( P(2) \) is given by \( P(2) = \{(0, 1), (\emptyset, \{0\})\} \) and \( |P(2)| = 2 \).

Last, we consider the relation between \( H_p^{(2)}(\phi) \) and \( P(2) \). The first term \( A_1^{\tau+1} \) of \( H_p^{(2)}(\phi) \) coincides with the first element \( \{(0, 1), \emptyset\} \) of \( P(2) \). Indeed, \( A_1^{\tau+1} = L(\emptyset + 1)A_1^{w(\emptyset)} A_2^{w(\emptyset)}. \) The second term \(-[1] A_2 \) corresponds to the second element \( \{\emptyset, \{0\}\} \). In fact, \(-[1] A_2 = L(\{1\}) A_1^{w(\emptyset)} A_2^{w(\emptyset)}).
(3) Case $d = 3$: By the same computation as in the above item (2), we obtain

$$H_p^{(3)}(\phi) = A_1^{q^2+q+1} - [1]A_1^q A_2 - [2]A_1 A_2^q$$

$$= (j(\phi)^q - [1]j(\phi)^{q-1} - [2])A_1 A_2^q$$

$$\equiv P_3(j(\phi))A_1 A_2^q \pmod{p},$$

and $P(3) = \{(\{0, 1, 2\}, \emptyset), (\{2\}, \{0\}), (\{0\}, \{1\})\}$ and $|P(3)| = 3$.

We can check that the $i$th term of $H_p^{(3)}(\phi)$ coincides exactly with the $i$th element of $P(3)$.

(4) Case $d = 4$: By the same computation as in item (2), we obtain

$$H_p^{(4)}(\phi) = A_1^{q^3+q^2+q+1} - [1]A_1^q A_2 - [2]A_1 A_2^q$$

$$= j(\phi)q^2+q+1 - [1]j(\phi)q^2 - [2]j(\phi)^2 - [3]j(\phi) + [1]A_1 A_2^q$$

$$\equiv P_4(j(\phi))A_1 A_2^q \pmod{p},$$

and

$$P(4) = \{(\{0, 1, 2, 3\}, \emptyset), (\{2, 3\}, \{0\}), (\{0, 3\}, \{1\}), (\{0, 1\}, \{2\}), (0, \{0, 2\})\}$$

and $|P(4)| = 5$.

We can check that the $i$th term of $H_p^{(4)}(\phi)$ coincides exactly with the $i$th element of $P(4)$.

(5) Case $d = 5$: By the same computation as in item (2), we obtain

$$H_p^{(5)}(\phi) = A_1^{q^4+q^3+q^2+q+1} - [1]A_1^q A_2 - [2]A_1 A_2^q$$

$$= j(\phi)q^4+q^3+q^2+q+1 - [1]j(\phi)q^4 + [1]A_1 A_2^q$$

$$\equiv P_5(j(\phi))A_1 A_2^q \pmod{p},$$

and

$$P(5) = \{(\{0, 1, 2, 3, 4\}, \emptyset), (\{2, 3, 4\}, \{0\}), (\{0, 3, 4\}, \{1\}), (\{0, 1, 4\}, \{2\}), (\{0, 1, 2\}, \{3\}), (\{4\}, \{0, 2\}), (\{2, 0, 3\}), (\{0\}, \{1, 3\})\}$$

and $|P(5)| = 8$.

We can check that the $i$th term of $H_p^{(5)}(\phi)$ coincides exactly with the $i$th element of $P(5)$.

Let $\mathbb{Z}[X_0, X_1, \ldots, X_{d-1}]$ denote a polynomial ring over $\mathbb{Z}$. Let $s_i \in \mathbb{Z}[X_0, X_1, \ldots, X_{d-1}]$ denote the elementary symmetric polynomial of degree $i$ in $-X_0, -X_1, \ldots, -X_{d-1}$. That is,

$$s_0 = s_0(X_0, X_1, \ldots, X_{d-1}) := 1;$$

$$s_1 = s_1(X_0, X_1, \ldots, X_{d-1}) := \sum_{i=0}^{d-1} (-X_i);$$

$$s_2 = s_2(X_0, X_1, \ldots, X_{d-1}) := \sum_{0 \leq i < j \leq d-1} (-X_i) \cdot (-X_j);$$

$$\vdots$$

$$s_d = s_d(X_0, X_1, \ldots, X_{d-1}) := \prod_{i=0}^{d-1} (-X_i).$$
For a finite subset $S'$ of $\mathbb{N}$ and $n \in \mathbb{Z}$, we define a polynomial $h_n^{S'}$ in $\mathbb{Z}[X_i \mid i \in S']$ as follows:

$$h_n^{S'} = h_n^{S'}(X_i \mid i \in S')$$

$$:= \begin{cases} 
\sum_{(k_i) \in I_n(S')} \prod_{i \in S'} X_i^{k_i} & \text{if } S' \neq \emptyset; \\
1 & \text{if } S' = \emptyset \text{ and } n = 0; \\
0 & \text{if } S' = \emptyset \text{ and } n \neq 0,
\end{cases}$$

where

$$I_n(S') := \{(k_i)_{i \in S'} \mid k_i \in \mathbb{N} \text{ and } \sum_{i \in S'} k_i = n\}.$$  

Recall that if $n < 0$, then $I_n(S') = \emptyset$, which implies $h_n^{S'} = 0$.

The following plays an very important role in the proof of Main theorem (1).

**Keylemma.** Assume that $(S_1, S_2) \in P(d)$, and set $S := S_1 \cup S_2$ and $S' := S \cup \{d\}$. Then

$$\sum_{i=\lfloor d/2 \rfloor}^d s_{d-i} \cdot h_{i-|S|}^{S'} = \prod_{i \in S_2} (X_d - X_{i+1}).$$

**Proof.** Let

$$L := \sum_{i=\lfloor d/2 \rfloor}^d s_{d-i} \cdot h_{i-|S|}^{S'},$$

and expand this sum into a polynomial. Then, each monomial can be uniquely written as

$$X_d^{k_d} \times \prod_{b \in S_2} (-X_{b+1})^{d_b} \times \prod_{c \in S} X_c^{k_c} \times \prod_{c \in S} (-X_c)^{\delta_c},$$

where $k_d$ and $k_c$ are nonnegative integers, and $\delta_b$ and $\delta_c$ are each equal to either 0 or 1. For simplicity, let

$$A := X_d^{k_d};$$

$$B := \prod_{b \in S_2} (-X_{b+1})^{d_b};$$

$$C := D \cdot E;$$

$$D := \prod_{c \in S} X_c^{k_c};$$

$$E := \prod_{c \in S} (-X_c)^{\delta_c}.$$

Since $\{S_1, S_2, S_1 + 1\}$ is a partition, the terms $A$ and $D$ arise from $h_{i-|S|}^{S'}$ in $L$ and not from $s_{d-i}$ in $L$. Conversely, the other terms $B$ and $E$ arise from $s_{d-i}$ and not from $h_{i-|S|}^{S'}$.

For each term $ABC$, define

$$\beta = \beta_{ABC} := \{|b \in S_2 \mid \delta_b \neq 0\};$$

$$\gamma = \gamma_{ABC} := \{|c \in S \mid \delta_c \neq 0\}.$$

From here, we compute the sum of terms $ABC$ in $L$ in two ways: first, where $C = 1$, and then where $C \neq 1$.

First, we consider the sum of terms $ABC$ with $C = 1$ (and then $k_c = \delta_c = 0$ for any $c \in S$). Recall that $k_d + \sum_{c \in S} k_c = i - |S|$, by the definition of $I_{i-|S|}(S')$, and so $k_d = i - d + |S_2|$. Note that $0 \leq \beta \leq |S_2|$. First, the term $AB$ with $\beta = 0$ is exactly the term $X_d^{[S_2]}$. Second, the terms with $\beta = 1$ have the form $(-X_{b+1})X^{[S_2]}_d$ ($b \in S_2$). Third, the terms with $\beta = 2$ have the form $(-X_{b+1})(-X_{b+1})X^{[S_2]}_d$ ($b_1, b_2 \in S_2$, $b_1 \neq b_2$). In general, terms with $\beta = n$ have the form
\((-X_{b_1+1})(-X_{b_2+1}) \cdots (-X_{b_n+1})X_d^{[S_2]-n}\) \((b_1, b_2, \ldots, b_n \in S_2, b_i \neq b_j)\). Last, the term with \(\beta = |S_2|\) is exactly the term \(\prod_{b \in S_2} (-X_{b+1})\). Hence, the sum from \(\beta = 0\) to \(|S_2|\) equals

\[
X_d^{[S_2]} + \sum_{b \in S_2} (-X_{b+1})X_d^{[S_2]-1} \\
+ \sum_{s_1, s_2 \in S_2, b_1 < s_2} (-X_{s_1+1})(-X_{b_2+1})X_d^{[S_2]-2} \\
+ \cdots + \prod_{b \in S_2} (-X_{b+1})
\]

\(= \prod_{i \in S_2} (X_d - X_{i+1})\),

and the computation of the first half is complete.

Next, we consider the sum of the other terms \(ABC\) (that is, where \(C \neq 1\), and show that the sum is equal to 0. By using the notation \(\beta\) and \(\gamma\), each term \(ABC\) can be uniquely rewritten as follows:

\[
ABC = (-1)^{\beta+\gamma} \times X_d^{k_d} \times \prod_{b \in S_2} X_d^{\delta_b} \times \prod_{c \in S} X_c^{k_c+\delta_c};
\]

\[
F := X_d^{k_d} \times \prod_{b \in S_2} X_d^{\delta_b}.
\]

Let

\[
G := \prod_{c \in S} X_c^{k'_c};
\]

\[
N := N_G := |\{c \in S \mid k'_c \neq 0\}|.
\]

Now, fix a term \(F\), which actually exists and is in \(L\), and with this fixed term \(F\), fix a term \(G\) (a sequence \((k'_c)_{c \in S}\)) such that the term \(FG\) actually exists in \(L\). From here, for this fixed sequence \((k'_c)_{c \in S}\), we count the number of terms \(ABC\) such that \((k_c + \delta_c)_{c \in S} = (k'_c)_{c \in S}\) (that is, such that \(\prod_{c \in S} X_c^{k_c+\delta_c} = G\)). Note that \(0 \leq \gamma \leq N\). First, the term \(ABC\) with \(\gamma = 0\) is exactly the term such that \((k_c)_{c \in S} = (k'_c)_{c \in S}\). Second, the terms with \(\gamma = 1\) are terms with a form such that

\[
(\ldots, k_{c_1} + 1, \ldots) = (\ldots, k'_{c_1}, \ldots);
\]

\[
k'_{c_1} \neq 0
\]

\((c \in S)\), where the other components equal each other. Then, the number of such terms is \((N)\).

Third, terms with \(\gamma = 2\) are with a form such that

\[
(\ldots, k_{c_1} + 1, \ldots, k_{c_2} + 1, \ldots) = (\ldots, k'_{c_1}, \ldots, k'_{c_2}, \ldots);
\]

\[
k'_{c_1} \neq 0, \quad k'_{c_2} \neq 0
\]

\((c_1, c_2 \in S, c_1 \neq c_2)\), where the others equal each other. Then, the number of such terms is \((N)\). In general, the number of terms with \(\gamma = n\) is \((N)\). So, the sum from \(\gamma = 0\) to \(N\) equals

\[
\sum_{\gamma=0}^{N} (-1)^{\beta+\gamma} \binom{N}{\gamma} FG = (-1)^{\beta} \sum_{\gamma=0}^{N} (-1)^{\gamma} \binom{N}{\gamma} FG
\]

\[
= (-1)^{\beta} (1 + (-1))^N FG = 0.
\]

From this, the sum of the terms \(ABC\) such that \(C \neq 1\) also equals 0. Keylemma follows from this.

Here, we explain the structure of the proof of Keylemma.
**Example 2.2.** Suppose that \( d = 5 \) and \( (S_1, S_2) = (\{0\}, \{1, 3\}) \in P(5) \). Let \( S = S_1 \cup S_2 = \{0, 1, 3\} \) and \( S' = S \cup \{5\} = \{0, 1, 3, 5\} \). Since the sets \( I_n(S') \) are given by

\[
\begin{align*}
I_0(S') &= \{(0, 0, 0, 0)\}; \\
I_1(S') &= \{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}; \\
I_2(S') &= \{(1, 1, 0, 0), (1, 0, 1, 0), (1, 0, 0, 1), (0, 1, 1, 0), \\
& \quad (0, 1, 0, 1), (0, 0, 1, 1), (2, 0, 0, 0), (0, 2, 0, 0), \\
& \quad (0, 0, 2, 0), (0, 0, 0, 2)\}
\end{align*}
\]

by definition, the polynomials \( h_n^{S'} = h_n^{S'}(X_0, X_1, X_3, X_5) \) can be written as

\[
\begin{align*}
h_0^{S'} &= 1; \\
h_1^{S'} &= X_0 + X_1 + X_3 + X_5; \\
h_2^{S'} &= X_0X_1 + X_0X_3 + X_0X_5 + X_1X_3 \\
& \quad + X_1X_5 + X_3X_5 + X_0^2 + X_1^2 + X_3^2 + X_5^2.
\end{align*}
\]

The polynomials \( s_n = s_n(X_0, X_1, X_2, X_3, X_4) \) are defined by

\[
\begin{align*}
s_0 &= 1; \\
s_1 &= (-X_0) + (-X_1) + (-X_2) + (-X_3) + (-X_4); \\
s_2 &= (-X_0)(-X_1) + (-X_0)X_2 + (-X_0)(-X_3) + (+X_0)(-X_4) \\
& \quad + (-X_1)(-X_2) + (-X_1)(-X_3) + X_1X_4 \\
& \quad + (-X_2)(-X_3) + (-X_2)(-X_4) + (-X_3)(-X_4).
\end{align*}
\]

Therefore, we obtain

\[
\begin{align*}
\sum_{i=\lceil 5/2 \rceil}^{5} s_{5-i} \cdot h_{i-3}^{S'} &= s_2 \cdot h_0^{S'} + s_1 \cdot h_1^{S'} + s_0 \cdot h_2^{S'} \\
&= \left((-X_0)(-X_1) + (-X_0)X_2 + (-X_0)(-X_3) + (+X_0)(-X_4) \\
& \quad + (-X_1)(-X_2) + (-X_1)(-X_3) + X_1X_4 \\
& \quad + (-X_2)(-X_3) + (-X_2)(-X_4) + (-X_3)(-X_4)\right) \cdot 1 \\
& \quad + \left((-X_0) + (-X_1) + (-X_2) + (-X_3) + (-X_4)\right) \\
& \quad \times \left(X_0 + X_1 + X_3 + X_5\right) \\
& \quad + 1 \cdot \left(X_0X_1 + X_0X_3 + X_0X_5 + X_1X_3 + X_1X_5 \\
& \quad + X_3X_5 + X_0^2 + X_1^2 + X_3^2 + X_5^2\right),
\end{align*}
\]

and hence

\[
\begin{align*}
\sum_{i=\lceil 5/2 \rceil}^{5} s_{5-i} \cdot h_{i-3}^{S'} &= X_0^2 + \left((-X_2) + (-X_4)\right)X_5 + (-X_2)(-X_4) \\
& \quad + \left(1 + 2(-1) + (-1)^2\right)X_0X_1 + X_0X_3 + X_1X_3 \\
& \quad + (-1)\left(1 + (-1)\right)X_0X_2 + X_0X_4 + X_1X_2 + X_1X_4 + X_3X_4 \\
& \quad + \left(1 + (-1)\right)X_0X_5 + X_1X_5 + X_3X_5 \\
& \quad + \left(1 + (-1)\right)X_0^2 + X_1^2 + X_3^2 \\
& = (X_5 - X_2)(X_5 - X_4).
\end{align*}
\]

We now consider the structure in detail. The terms \( X_0^2, (-X_2)X_5, (-X_4)X_5, \) and \((-X_2)(-X_4)\) are the terms \( ABC \) such that \( C = 1 \) in the proof of Keylemma. In contrast, the other terms are
Corollary 2.3. Assume that \( \beta \neq 1 \). Now, we fix the terms \( F = 1 \) and \( G = X_0X_1 \) (that is, the sequence \((k_0', k_1', k_2' = (1, 1, 0))\). Note that \( \beta = 0 \) and \( N = 2 \). We count the number of terms such that \((k_0 + \delta_0, k_1 + \delta_1, k_3 + \delta_3) = (1, 1, 0)\). First, the term with \( \gamma = 0 \) is exactly the term such that \((1 + 0, 1 + 0, 0) = (1, 1, 0)\). Second, the terms with \( \gamma = 1 \) are the terms such that either \((0 + 1, 1 + 0, 0) = (1, 1, 0)\) or \((1 + 0, 0 + 1, 0) = (1, 1, 0)\). Last, the term with \( \gamma = 2 \) is exactly the term such that \((0 + 1, 0 + 1, 0) = (1, 1, 0)\). Hence, the sum equals

\[
X_0X_1 + 2(-1)X_0X_1 + (-1)X_0X_1 = 0.
\]

Similarly, the terms \( X_0X_3 \) and \( X_1X_3 \) can be computed.

Next, we fix the terms \( F = X_2 \) and \( G = X_0 \) (that is, the sequence \((k_0', k_1', k_2' = (1, 0, 0))\). Note that \( \beta = 1 \) and \( N = 1 \). We count the number of terms such that \((k_0 + \delta_0, k_1 + \delta_1, k_3 + \delta_3) = (1, 0, 0)\). Then, the term with \( \gamma = 0 \) (resp. \( \gamma = 1 \)) is exactly the term such that \((1 + 0, 0, 0) = (1, 0, 0)\) (resp. \((0 + 1, 0, 0) = (1, 0, 0)\)). Hence, the sum equals

\[
(-1)X_0X_2 + (-1)^{1+1}X_0X_2 = 0.
\]

Similarly, the terms \( X_0X_4, X_1X_2, X_1X_4, X_3X_1, X_0X_5, X_1X_5, \) and \( X_3X_5 \) can be calculated.

Finally, we fix the terms \( F = 1 \) and \( G = X_0^2 \) (that is, the sequence \((k_0', k_1', k_2' = (2, 0, 0))\). Note that \( \beta = 0 \) and \( N = 1 \). We count the number of terms such that \((k_0 + \delta_0, k_1 + \delta_1, k_3 + \delta_3) = (2, 0, 0)\). First, the term with \( \gamma = 0 \) (resp. \( \gamma = 1 \)) is the term such that \((2 + 0, 0, 0) = (2, 0, 0)\) (resp. \((1 + 1, 0, 0) = (2, 0, 0)\)). Hence, the sum equals

\[
X_0^2 + (-1)X_0^2 = 0.
\]

Similarly, the terms \( X_1^2 \) and \( X_2^2 \) can be computed, which finishes the illustration of Keylemma.

Recall that \( \alpha, \alpha^q, \ldots, \alpha^{q^{d-1}} \) are the roots of \( p(T) = \sum_{i=0}^{d} \mu_i T^i \), and that \( \alpha^{q^d} = \alpha \). By the relation between the roots of \( p(T) \) and the coefficients of \( p(T) \), we get

\[
\begin{align*}
\mu_d &:= 1; \\
\mu_{d-1} &= \sum_{i=0}^{d-1} (-\alpha^q) ; \\
\mu_{d-2} &= \sum_{0 \leq i < j \leq d-1} (-\alpha^q) (-\alpha^q) ; \\
& \vdots \\
\mu_0 &= \prod_{i=0}^{d-1} (-\alpha^q).
\end{align*}
\]

Then, it follows from the definitions of \( s_{d-i} \) and \( \mu_i \) that \( s_{d-i}(\alpha, \alpha^q, \ldots, \alpha^{q^{d-1}}) = \mu_i \).

As applications of Keylemma, we have the following corollary.

**Corollary 2.3.** Assume that \((S_1, S_2) \in P(d)\), and let \( S := S_1 \cup S_2 \) and \( S' := S \cup \{d\} \).

(a) Further, suppose that \( L \) is an extension of \( K \). Then

\[
\sum_{i=[d/2]}^{d} s_{d-i}(T, T^q, \ldots, T^{q^{d-1}}) \cdot h_i^{s'}_{i-[S]}(T^{q^i} \mid i \in S) = \prod_{i \in S_2} ([d] - [i + 1]).
\]

(b) Instead, assume that \( L \) is an extension of \( \mathbb{F}_p \). Then

\[
\sum_{i=[d/2]}^{d} \mu_i \cdot h_i^{s'}_{i-[S]}(\alpha^q \mid i \in S) = (-1)^{|S_2|} \prod_{i \in S_2} [i + 1] = L(S_2 + 1).
\]
Proof. (a) Recall that \([i] = T^i - T\). If we substitute \(X_i = T^i\) in Keylemma, then we get:
\[X_d - X_{i+1} = T^d - T^{i+1} = [d] - [i + 1].\]
This proves the first claim.

(b) Recall that \([d] = \alpha^{q^d} - \alpha = 0\). The second claim follows from item (a).

Let \(\phi\) be a rank-2 Drinfeld module over \(L\), and let \(\Lambda_\phi\) be the lattice corresponding to \(\phi\) (see Theorem 2.4 of Chapter I in [17], Theorem 4.6.9 in [22], Theorem 2.4.2 in [35]). Let \(e_\phi(z)\) denote the lattice exponential function of \(\Lambda_\phi\), defined by \(e_\phi(z) := z \prod_{\lambda \notin \Lambda_\phi} (1 - z/\lambda)\) (see Definition 2.1 of Chapter I in [17], Definition 4.2.3 in [22], Subsection 2.4 in [35]). Then, it is known that \(e_\phi(z)\) has the composition inverse function \(\log_\phi(z)\) such that
\[
\log_\phi(z) = \sum_{j \geq 0} \beta_j z^{q^j}
\]
(see Section 2 of Chapter II in [17], Subsection 4.6 in [22], Subsection 2.4 in [35]). Note that \(\beta_0 = 1\) from the definition of \(e_\phi(z)\).

We now collect results of El-Guindy and Papanikolas to use in the proof of Main theorem (1).

**Fact** (Theorem 8.1, Corollary 8.2 and Theorem 3.3 in [11]). Let \(\phi\) be any rank-2 Drinfeld module over a field \(L\) defined by \(\phi_T = i(T) + A_1 \tau + A_2 \tau^2\).

(1) Let \(m \in \mathbb{N}\) and \(n \in \mathbb{Z}\). Now, we define coefficients \(c(n; m) := c(n; m; \phi)\) as follows. For \(0 \leq n \leq 2m\), let
\[
\phi^{2m} = \sum_{n=0}^{2m} c(n; m) \tau^n.
\]
For the other cases \((n < 0 \text{ or } n > 2m)\), set \(c(n; m) = 0\). Then, for any \(m, n \geq 0\), we have that
\[
c(n; m) = \sum_{(S_1, S_2) \in P(n)} A_1^{w(S_1)} A_2^{w(S_2)} \cdot h_{m-|S|},
\]
where \(S := S_1 \cup S_2\) and \(S' := S \cup \{n\}\).

(2) Next, suppose that \(L\) is an extension of \(\mathbb{F}_p\). Then
\[
H_p^{(d)}(\phi) = \sum_{i = [d/2]}^{d} \mu_i \cdot c(d; i).
\]
Moreover, \(\phi\) is supersingular at \(p\) if and only if
\[
\sum_{i = [d/2]}^{d} \mu_i \cdot c(d; i) = 0.
\]

(3) The coefficients \(\beta_j\) of the function \(\log_\phi(z)\) are given by
\[
\beta_j = \sum_{(S_1, S_2) \in P(j)} \frac{A_1^{w(S_1)} A_2^{w(S_2)}}{L(S_1 + 1)L(S_2 + 2)}.
\]

Let \(L\) be an extension of \(K\), and let \(\phi\) be a rank-2 Drinfeld module over \(L\), defined by \(\phi_T = T + A_1 \tau + A_2 \tau^2\). Recall that \(\delta^q = -\delta\) (see Section 1). Set
\[
a(d) := \begin{cases} 
T \cdot \sum_{j=0}^{d} \beta_j \delta^q & \text{if } d \geq 1; \\
T\delta & \text{if } d = 0; \\
0 & \text{if } d < 0
\end{cases}
\]
(see Theorem 6.3 in [11]). It follows from Fact (3) that
\[
a(d) = T \cdot \sum_{j=0}^{d} \beta_j \delta^q = T \delta \cdot \sum_{j=0}^{d} (-1)^j \beta_j
eq T \delta \cdot \sum_{j=0}^{d} (-1)^j \sum_{(S_1, S_2) \in P(j)} \frac{A_1^{w(S_1)} A_2^{w(S_2)}}{L(S_1 + 1)L(S_2 + 2)}.
\]
which is used in proving the proposition below.

For this, we write

\[ L_d := (-1)^d[d][d-1] \cdots [2][1] \quad \text{and} \quad L_0 := 1. \]

We define

\[ b(d) := \begin{cases} \frac{L_d \cdot a(d)}{\delta T^{1+q+\cdots+q^d}} & \text{if } d \geq 1; \\ 1 & \text{if } d = 0; \\ 0 & \text{if } d < 0 \end{cases} \]

(cf. Section 2 in [12]).

Assume that \( A_1 = T + \lambda \) and \( A_2 = \lambda \). Then, the following recursive relations are valid:

\[ \begin{align*}
- [d]a(d) &= -a(d-1)(T^{q^d} + \lambda^{q^d-1}) + \lambda^{q^d-1} a(d-2); \\
\frac{b(d)}{b(d-1)} &= - (1 + D^{q^d-1}) b(d-1) + D^{q^d-1} (T^{1-q^d-1} - 1) b(d-2),
\end{align*} \tag{8} \]

where \( D := \lambda/T^q \). The validity of these recursions can be proven in the same way as for the recursions (12) and (14), respectively, in [12].

The following is Main theorem (1).

**Proposition 2.4.** Let \( p = (p(T)) \) denote a nonzero prime ideal of degree \( d \) such that \( p(T) \neq T \), and let \( \phi \) be any rank-2 Drinfeld module over a field \( L \) defined by \( \phi_T = l(T) = A_1 + A_2 \tau + 2 \).

(a) Suppose that \( L \) is an extension of \( F_p \). Then, the coefficient \( H_p^{(d)}(\phi) \) is given by

\[ H_p^{(d)}(\phi) = \sum_{(S_1, S_2) \in P(d)} L(S_2 + 1) A_1^{w(S_1)} A_2^{w(S_2)}. \]

(b) Assume that \( L \) is an extension of \( K \). Then

\[ H_p^{(d)}(\phi) \equiv \frac{(-1)^d L_d \cdot a(d)}{\delta T^{q^d}} \pmod{p} = (-1)^d T^{1+q+\cdots+q^d-1} b(d). \]

**Proof.** (a) It follows from Fact (1) and the first claim of Fact (2) that

\[ H_p^{(d)}(\phi) = \sum_{i=[d/2]}^d \mu_i \cdot c(d; i) \]

\[ = \sum_{(S_1, S_2) \in P(d)} \left( \sum_{i=[d/2]}^d \mu_i \cdot H_i^{S_i} \right) A_1^{w(S_1)} A_2^{w(S_2)}. \]

By using Corollary 2.3 (b), we have

\[ H_p^{(d)}(\phi) = \sum_{(S_1, S_2) \in P(d)} \left( \sum_{i=[d/2]}^d \mu_i \cdot H_i^{S_i} \right) A_1^{w(S_1)} A_2^{w(S_2)} \]

\[ = \sum_{(S_1, S_2) \in P(d)} L(S_2 + 1) A_1^{w(S_1)} A_2^{w(S_2)}. \]

This completes the proof of this case.

(b) Let \( 0 \leq j \leq d \) be an integer. If \( (S_1, S_2) \in P(j) \), then \( \{S_1, S_2, S_2 + 1\} \) is a partition of \( \mathbb{N} \subset j \), and thus \( \{S_1 + 1, S_2 + 1, S_2 + 2\} \) is a partition of \( \{1, 2, \ldots, j\} \). Hence, for \( 0 \leq j < d \), we obtain

\[ \frac{L_d}{L(S_1 + 1)L(S_2 + 2)} = (-1)^{d-j} [d][d-1] \cdots [j+1]L(S_2 + 1) \equiv 0 \pmod{p}. \]

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Along another line, for \((S_1, S_2) \in P(d)\), we get \(L_d = L(S_1 + 1)L(S_2 + 1)L(S_2 + 2)\). Hence,

\[
\frac{(-1)^d L_d \cdot a(d)}{\delta T^{d+1}} = \frac{(-1)^d L_d \cdot T^j \sum_{j=0}^{d} (-1)^j}{\delta T^{d+1}} \sum_{(S_1, S_2) \in P(j)} \frac{A_1^{w(S_1)} A_2^{w(S_2)}}{L(S_1 + 1)L(S_2 + 2)} = \sum_{(S_1, S_2) \in P(d)} L(S_2 + 1)A_1^{w(S_1)} A_2^{w(S_2)} = H_p^{(d)}(\phi) \pmod{p}
\]

by the equality (10) and item (a). The second claim follows.

We next consider relations between the known polynomials and our polynomials.

**Remark.** Assume that \(L\) is an extension of \(K\), and that \(A_1 = T + \lambda\) and \(A_2 = \lambda\).

1. We explore the relation between the polynomial \(b_d\) in Section 2 of [12] and our polynomial \(b(d)\). First, we easily see that \(b_d = (-1)^d b(d)\). Next, from Theorem 2.2 in [12], we have

\[
b(d) = (-1)^d \sum_{S \subseteq N^d} \frac{(\lambda/T^q)^w(S)}{m(S)},
\]

where \(M(S) := S \setminus (S + 1)\) and \(m(S) := \prod_{i \in M(S)} T^{q} - 1\).

2. We investigate the relation between the polynomial \(p_d(s)\) in Definition 12 of [24] and our polynomial \(H_p^{(d)}(\lambda)\). We see that

\[
H_p^{(d)}(\lambda) \equiv (-1)^d T^{1+q+\cdots+q^{d-1}} p_d(-\lambda/T^q) \pmod{p}
\]

by Proposition 2.4 (b), item (1) of this remark, and Theorem 3.2 (i) in [12].

Last, we study the relation between one period of a lattice \(\Lambda_{\phi}\) and our coefficient \(H_p^{(d)}(\phi)\).

**Remark.** Assume that \(L\) is an extension of \(K\). It is known that the series

\[
f(z) = \sum_{n \geq 0} a(n)z^n
\]

forms a period of \(\Lambda_{\phi}\) (see Theorem 6.3 in [11]). Let \(f^r(z)\) denote the truncated function defined by

\[
f^r(z) := \sum_{n=0}^{d} a(n)z^n.
\]

Recall that \(\delta^q = -\delta\) and \(\delta q^n = (-1)^n \delta\). From Proposition 2.4 (b), we get

\[
L_d \cdot f^r\left(\frac{1}{\delta T}\right) = \sum_{n=0}^{d} L_d \cdot a(n) \left(\frac{1}{\delta T}\right)^n = \sum_{n=0}^{d} L_d \frac{\delta T^{1+q+\cdots+q^n} - b(n)}{\delta T^{q^n} L_n} = (-1)^d \sum_{n=0}^{d} \frac{[d][d-1] \cdots [n+1]}{T^{q^n} L_n} \cdot b(n),
\]

where \(L_d := (-1)^d [d][d-1] \cdots [2][1]\). Note that \([d] \equiv 0 \pmod{p}\). By Proposition 2.4 (b), we obtain

\[
L_d \cdot f^r\left(\frac{1}{\delta T}\right) \equiv (-1)^d T^{1+q+\cdots+q^{d-1}} b(d) \equiv H_p^{(d)}(\phi) \pmod{p}.
\]
Hence, we can regard the constant $1/L_d$ as the constant $\pi \tau$ introduced in Section 1 by the following reasoning.

Let

$$
\xi_* := \prod_{n \geq 0} \left(1 - \frac{[n]}{[n + 1]}\right);
$$

$$
\xi_d := \frac{[1]^{(q^d - 1)/(q-1)}}{(-1)^d L_d} = \prod_{n=0}^{d-1} \left(1 - \frac{[n]}{[n + 1]}\right)
$$

(see Theorem 1.4 of Chapter IV in [17], Subsection 3.2 in [22], Subsection 2.5 in [33]). It is known that the constant $\xi_*$ is a function-field analogue of the circular constant $\pi$ (see Remark 1.5 of Chapter IV in [17]). Hence, we can treat the constant $1/L_d = (-1)^d \xi_d/[1]^{(q^d - 1)/(q-1)}$ as the constant $\pi \tau$.

(2) The Deuring polynomial $H_p(\lambda)$ is related to a hypergeometric function (see Section 1). Thakur defined several hypergeometric functions for function fields (see [33, 34], Subsection 6.5 in [35]). However, it seems that the functions given by Thakur are not related to the coefficient $H_p^{(d)}(\phi)$.

3 Properties for a supersingular polynomial

In this section, we prove Main theorem (2) (Proposition 3.1 (b), (c), (d)), which was introduced in Section 1 in the course of the proof, we often use Main theorem (1).

We recall the following notation. In what follows, $p = (p(T))$ denotes a nonzero prime ideal of degree $d$ in $A = \mathbb{F}_q[T]$ such that $p(T) \neq T$, and $\alpha$ is a root of $p(T)$. Note that $\alpha \neq 0$. Throughout this section, we assume that $L$ is an extension of $\mathbb{F}_p$, and that $\phi$ is a rank-2 Drinfeld module over $L$ defined by

$$
\phi_T = \alpha + (\alpha + \lambda)\tau + \lambda \tau^2.
$$

Set

$$
\phi_p := \phi_{p(T)} = \sum_{i=0}^{2d} g_i \tau^i \quad \text{and} \quad H_p^{(d)}(\lambda) := g_d.
$$

The following proposition is Main theorem (2).

**Proposition 3.1.** (a) The polynomial $H_p^{(d)}(\lambda)$ is a divisor of $g_i$ for all $d \leq i < 2d$. Moreover, if $\phi$ is supersingular at $p$, then $\phi_p = \lambda^{(q^d - 1)/(q-1)}\tau^{2d}$.

(b) $H_p^{(d)}(0) \neq 0$ and $\deg H_p^{(d)}(\lambda) = (q^d - 1)/(q-1)$.

(c) $H_p^{(d)}(\lambda)$ and $H_p^{(d)}(\alpha^q - (s+1)^q - 1)$ are each separable.

(d) If $H_p^{(d)}(\alpha^q)$, then $\lambda \in \mathbb{F}_p^{(2)}$. In particular, all the roots of $H_p^{(d)}(\lambda)$ are in $\mathbb{F}_p^{(2)}$.

**Proof.** (a) Note that $A = \mathbb{F}_q[T]$ is commutative. Recall that $\phi_T = \phi(T) = \alpha + A_1 \tau + A_2 \tau^2$ and $\phi_{p(T)} = \phi(p(T)) = \sum_{i=0}^{2d} g_i \tau^i$. Since $\phi$ is a homomorphism, we have

$$
\phi_T \phi_{p(T)} = \phi_{T \cdot p(T)} = \phi_{p(T) \cdot T} = \phi_{p(T)} \phi_T.
$$

Now, we compute the left-hand side and the right-hand side:

$$
\phi_T \phi_{p(T)} = (\alpha + A_1 \tau + A_2 \tau^2) \cdot \sum_{i=0}^{2d} g_i \tau^i
$$

$$
= \cdots + (g_{i-2}^q A_2 + g_{i-1}^q A_1 + g_i \alpha) \tau^i + \cdots,
$$

and

$$
\phi_{p(T)} \phi_T = \sum_{i=0}^{2d} g_i \tau^i \cdot (\alpha + A_1 \tau + A_2 \tau^2)
$$

$$
= \cdots + (g_{i-2} A_2 + g_{i-1} A_1^q + g_i \alpha^q) \tau^i + \cdots.
$$
Then, the coefficients of $\tau^i$ are the same between sides, as shown by
\[
g_{i-2}^2 A_2 + g_{i-1}^2 A_1 + g_i = g_{i-2} A_{q_{i-2}}^2 + g_{i-1} A_{q_{i-1}}^1 + g_i A_{q_i}^i,
\]
and so
\[
(\alpha q_i - \alpha) g_i = g_{i-2}^2 A_2 - g_{i-2} A_{q_{i-2}}^2 + g_{i-1} A_1 - g_{i-1} A_{q_{i-1}}^1.
\]
(9)

Recall that $g_{d-1} = 0$ and $g_d = H_p^{(d)}(\lambda)$. Note that $\alpha q_i - \alpha \neq 0$ for all $d < i < 2d$. First, suppose that $i = d + 1$ for the recursion \([9\). Since $g_{d-1} = 0$, we get $g_d | g_{d+1}$. Second, assume that $i = d + 2$. Since $g_d | g_{d+1}$, we get $g_d | g_{d+2}$. We obtain $g_d | g_i$ for all $d < i < 2d$ by induction on $i$.

Recall that $p(T) = \sum_{i=0}^d \mu_i T^i$ and $A_2 = \lambda$. Notice that $g_0 = \cdots = g_{d-1} = 0$. Notice that
\[
\phi_{T^2} - \phi_{T^2} = (\alpha + A_1 \tau + A_2 \tau^2)(\alpha + A_1 \tau + A_2 \tau^2) = \cdots + A_2^{q_i+1} \tau^4,
\]
and
\[
\phi_{T^3} = \phi_{T^2} \phi_{T^2} = (\alpha + A_1 \tau + A_2 \tau^2)(\cdots + A_2^{q_i+1} \tau^4) = \cdots + A_2^{q_i+q_i^2+1} \tau^6.
\]

Hence, we have $\phi_{T^i} = \cdots + A_2^{2(i-1)+\cdots+q_i+1} \tau^i$ for all $1 \leq i \leq d$, again by induction on $i$. From this, we obtain $\phi_{p(T)} = \sum_{i=0}^d \mu_i \phi_{T^i} = \cdots + A_2^{2d-1} \cdots + q_i+1 \tau^{2i}$, namely, $g_{2d} = \lambda^{2d-1} \cdots + q_i+1 \tau^{2d}$.

If $\phi$ is supersingular at $p$, then we get $g_d = H_p^{(d)}(\lambda) = 0$ (see Satz (5.3) in \([16\]). It follows from the first claim that $g_{d+1} = \cdots = g_{2d-1} = 0$. Thus, we have $\phi_p = \phi_{p(T)} = \lambda^{2d-1} \cdots + q_i+1 \tau^{2d}$.

(b) By Main theorem (1), we have
\[
H_p^{(d)}(\lambda) = \sum_{(S_1, S_2) \in \Phi(d)} L(S_2 + 1)(\alpha + \lambda)^w(S_1) \lambda^w(S_2).
\]

First, we prove $H_p^{(d)}(0) \neq 0$. Let us consider the constant term of $H_p^{(d)}(\lambda)$. If $S_2 \neq 0$, then $w(S_2) = 0$, and so the terms do not contribute to the constant term. When $S_2 = 0$, we get $w(S_2) = 0$, $L(S_2 + 1) = 1$ and $w(S_1) = (q^d - 1)/(q - 1)$. Hence, $H_p^{(d)}(0) = \alpha^{(q^d - 1)/(q - 1)} \neq 0$.

Next, we show that $\deg_{\lambda} H_p^{(d)}(\lambda) = (q^d - 1)/(q - 1)$. If $S_2 \neq \emptyset$, then
\[
(q^d - 1)/(q - 1) = q^{d-1} + \cdots + q + 1 = w(S_1) + w(S_2) + w(S_2 + 1) > w(S_1) + w(S_2).
\]

When $S_2 = \emptyset$, we get $w(S_1) = (q^d - 1)/(q - 1)$. Hence, $\deg_{\lambda} H_p^{(d)}(\lambda) = (q^d - 1)/(q - 1)$.

(c) We prove the separability of polynomials by using items (a) and (b). Recall that $g_{2d} = \lambda^{q^{2(d-1)}+\cdots+q+1}$ and $g_{2d+1} = 0$. Suppose that $i = 2d + 1$ for the recursion \([9\). Then, we can write
\[
g_{2d-1}^2 \lambda + g_{2d}^2 \cdot (\alpha + \lambda) = g_{2d-1} \lambda^{q^{2d-1}} + g_{2d} \cdot (\alpha + \lambda)^{q^{2d}}.
\]

By differentiating both sides with respect to $\lambda$ and then multiplying both sides by $\lambda$, we have
\[
g_{2d-1}^2 \lambda + g_{2d}^2 \lambda = g_{2d}^2 \lambda^{q^{2d-1}+1} + g_{2d} \cdot (\alpha + \lambda)^{q^{2d}},
\]
where $g_{2d-1}^2$ is the derivative of $g_{2d-1}$. It follows from the equalities \([10\) and \([11\)] that
\[
g_{2d}^2 \alpha = (g_{2d-1} - g_{2d-1}^2 \lambda) \lambda^{q^{2d-1}}.
\]

Now, suppose that $H_p^{(d)}(\lambda)$ has a multiple root $\lambda_0$. Then, from item (a), $g_{2d-1}$ has the same multiple root $\lambda_0$, that is, $g_{2d-1}(\lambda_0) = g_{2d-1}'(\lambda_0) = 0$. Therefore, since the right-hand side of the equality \([12\) has the element $\lambda_0$ as a root, the left-hand side, too, has the element $\lambda_0$ as a root. Since the root of the left-hand side is 0 only, we obtain $\lambda_0 = 0$. This contradicts item (b).
Next, suppose that \( H_p(d)(-\alpha^q s(s+1)^{q-1}) \) has a multiple root \( s_0 \). Set \( \lambda_0 := -\alpha^q s_0(s_0+1)^{q-1} \). Then
\[
\frac{d}{ds} H_p(d)(\lambda_0) = 0, \quad \text{and} \quad \frac{d}{ds} H_p(d)(-\alpha^q s_0(s_0+1)^{q-1}) = \frac{d}{d\lambda} H_p(d)(\lambda_0) \cdot (-\alpha^q(s_0+1)^{q-2}) = 0.
\]
Therefore, we have either \( \frac{d}{d\lambda} H_p(d)(\lambda_0) = 0 \) or \( -\alpha^q(s_0+1)^{q-2} = 0 \). If \( \frac{d}{d\lambda} H_p(d)(\lambda_0) = 0 \), then \( \lambda_0 \) is a multiple root of \( H_p(d)(\lambda) \). However, \( H_p(d)(\lambda) \) does not have a multiple root. So we assume \( -\alpha^q(s_0+1)^{q-2} = 0 \). Then, we obtain \( \lambda_0 = -\alpha^q s_0(s_0+1)^{q-1} = 0 \), which contradicts item (b).

(d) Suppose that \( \lambda_0 \in \overline{\mathbb{F}}_p \) is any element such that \( H_p(d)(\lambda_0^{q+1}) = 0 \), and let \( \lambda := \lambda_0^{q+1} \). Then, \( \phi \) is supersingular at \( p \) since \( H_p(d)(\lambda) = 0 \). It is known that a supersingular Drinfeld module is defined over \( \mathbb{F}_p^{(2)} \), that is, \( \lambda \in \mathbb{F}_p^{(2)} \) (see Remark 9.2 in [20], Proposition 2.15 in [21]). Recall that \( \alpha^{q^d} = \alpha \) and \( \lambda^{q^{2d}} = \lambda \). Assume that \( i = 2d + 1 \) for the recursion \( [9] \). Then
\[
\lambda^{(q^{2d-1} - 1)/(q^2 - 1)}(\alpha + \lambda) - \lambda^{(q^{2d-1} - 1)/(q^2 - 1)}(\alpha + \lambda)^{q^{2d}} = 0,
\]
which can be written as,
\[
\lambda^{(q^{2d-1} - 1)/(q^2 - 1)}(\alpha + \lambda)(\lambda^{(q^{2d-1} - 1)/(q^2 - 1)} - 1) = 0.
\]
Hence, we have either \( \lambda = -\alpha \) or \( \lambda^{(q^{2d-1} - 1)/(q^2 - 1)} - 1 = 0 \). Notice that \( \lambda \neq 0 \) from item (b).

First, consider the case where \( \lambda^{(q^{2d-1} - 1)/(q^2 - 1)} = 1 \). With that, we get \( \lambda_0^{q^{2d} - 1} = \lambda^{(q^{2d-1} - 1)/(q^2 + 1)} = 1 \), and hence \( \lambda_0 \in \mathbb{F}_p^{(2)} \).

Next, consider the case \( \lambda = -\alpha \). When \( d \) is odd, we get
\[
\lambda_0^{q^{2d} - 1} = \lambda_0^{(q^{2d+1} - 1)/(q^2 - 1)} = (\lambda_0^{\frac{q^{2d} - 1}{q^2 - 1}}) \cdot \lambda_0^{(q^{2d+1} - 1)/(q^2 - 1)} = 1,
\]
and hence \( \lambda_0 \in \mathbb{F}_p^{(2)} \). If \( d \) is even, then \( \lambda = -\alpha \) is not a root of \( H_p(d)(\lambda) \). Indeed, we have
\[
H_p(d)(-\alpha) = (-1)^{d/2}[3] \cdots [d-1](-\alpha)^{q^{d-2} + \cdots + q^2 + 1} \neq 0
\]
from Main theorem (1). \( \square \)

**Remark.** (1) Much of the ideas of the above proof are similar to those in Section 5 in [16]. The second claim of item (a) corresponds to the inseparability of Proposition 4.1 (c) in [19].

Gekeler noted the following on reading an earlier version of this paper (personal communication): The item (c) states that the supersingular locus on a moduli scheme is reduced, and this result has already been proven in a more general case.

(2) The background for item (b) is as follows. Let \( \Sigma(p) \) denote the set of supersingular points of \( X(1)/\mathbb{F}_p \), that is, the set of supersingular \( j \)-invariants. It is known that the point \( j = 0 \) is supersingular if and only if \( d \) is odd, and that
\[
|\Sigma(p)| = \begin{cases} 
\frac{q^d - 1}{2} & \text{if } d \text{ is even;} \\
\frac{q^d - q}{q^2 - 1} + 1 & \text{if } d \text{ is odd}
\end{cases}
\]
and \( \Sigma(p) \subseteq \mathbb{F}_p^{(2)} \) (see Satz (5.9) in [16], (2.14) in [21]). Notice that the covering \( X_0(T) \to X(1) \) is given by \( j = (\alpha + \lambda)^{q+1}/\lambda \) (see the last part of Section 4). Since the point \( j = \infty \) is not supersingular for \( X(1) \), the point \( \lambda = 0 \) is also not supersingular for \( X_0(T) \). That is, \( H_p(d)(0) \neq 0 \). Moreover, we can count the supersingular points of \( X_0(T)/\mathbb{F}_p \) as follows:

First, suppose that \( d \) is even. Since all the supersingular points split completely in \( X_0(T) \to X(1) \), the number is equal to
\[
\frac{q^d - 1}{q^2 - 1} \times (q + 1) = \frac{q^d - 1}{q - 1} = \deg(H_p(d)(\lambda)).
\]
Next, assume that $d$ is odd. The supersingular point $j = 0$ is totally ramified in $X_0(T) \to X(1)$ and the other supersingular points split completely in this covering. Hence, the number is equal to

$$\frac{q^d-q}{q^2-1} \times (q+1) + 1 = \frac{q^d-1}{q-1} = \deg_{\lambda} H_p^{(d)}(\lambda).$$

(3) In Proposition 13 of [2], Bassa and Beelen showed that a polynomial $p_d(s)$ is separable. Furthermore, in Theorem 18 of [2], they proved that all roots of $p_d(s)$ are in $F_p^{(2)}$. Although $p_d(s)$ is related to $H_p^{(d)}(\lambda)$ (see the remark in Section 2), their method of proof is completely different from the method used here. Our method of proof is straightforward.

4 An application for a supersingular polynomial

In this section, we prove Corollary (Proposition 13), which was introduced in Section 1. In the course of the proof, a polynomial identity (Proposition 4.1 (b)) plays an important role. By combining this polynomial identity with a generalization of a result by Bezerra and Garcia in [5], the corollary is proven.

Let $\mathfrak{p} = (p(T))$ denote a nonzero prime ideal of degree $d$ in $A = F_q[T]$ such that $p(T) \neq T$. Let $\alpha$ be a root of $p(T)$, and let $F_p^{(2)}$ denote the quadratic extension of $F_p := A/\mathfrak{p} = F_q(\alpha) = F_q^d$.

Consider a tower $E = E^{(d)} := (E_0, E_1, E_2, \ldots)$ that is recursively defined over $F_p^{(2)}$ by the equation

$$Y(Y + 1)^{q-1} = \frac{X^q}{(\alpha(X + 1))^{q-1}}$$

(13)

(see Definition 7.2.12 in [32]). This was first introduced by Bassa and Beelen in [2].

When $d = 1$ (and so, without loss of generality, we can assume that $p(T) = T - 1$ and $\alpha = 1$). The tower $E^{(1)}/F_{q^2}$ was first introduced by Elkies (see the equation (25) in [14]), and was studied by Bezerra and Garcia (see the equation (1) in [5]).

Setting $X = (1 - x)/x$ and $Y = (1 - y)/y$ in the equation (13), we get

$$\frac{y - 1}{y^q} = \frac{x^q - 1}{x},$$

which is the equation (1) in [5]. Bezerra and Garcia showed that the genus of $E^{(1)}/F_{q^2}$ is

$$\gamma(E^{(1)}) := \lim_{n \to \infty} \frac{g(E_n)}{q^n} = \frac{q}{q - 1},$$

where $g(E_n)$ denotes the genus of $E_n$ (see Lemma 4 in [5]).

This result holds for any degree $d$, that is, $\gamma(E^{(d)}) = q/(q - 1)$ (see the remark after Proposition 13 or Theorem 8.1 (iii) in [20], Theorem 2.13 in [21]), and this is used in the proof of Corollary.

The following polynomial identities play an important role in our proof of Corollary.

**Proposition 4.1.** Assume that $A_1 = \iota(T) + \lambda$ and $A_2 = \lambda$. Under this assumption, since the coefficients $b(d)$ are polynomials in $\lambda$, we can write $b(d)(\lambda) := b(d)$.

(a) Further, suppose that $L$ is an extension of $K$. Then

$$b(d)(-T^q s(s + 1)^{q-1}) = (T(s + 1))^{q-1} b(d) \left( \frac{-T^q s^q}{(T(s + 1))^{q-1}} \right)$$

$$+ (T^{q-1} - 1)(s + 1)^{q-1} b(d - 1) \left( \frac{-T^q s^q}{(T(s + 1))^{q-1}} \right).$$

(b) Instead, assume that $L$ is an extension of $F_p$. Then

$$H_p^{(d)}(-\alpha^q s(s + 1)^{q-1}) = (s + 1)^{q-1} H_p^{(d)} \left( \frac{-\alpha^q s^q}{(\alpha(s + 1))^{q-1}} \right).$$
Proof. Since \( \alpha^d - 1 = 1 \), item (b) follows from item (a). So, it is sufficient to prove item (a) only. We show item (a) by induction on \( d \). Notice that although item (b) is dependent on the prime ideal \( p \), item (a) is not.

Recall that \( b(-1)(\lambda) = 0 \), \( b(0)(\lambda) = 1 \), and \( b(1)(\lambda) = (-1)(\lambda/T^q + 1) \) (see the definitions of \( a(d) \) and \( b(d) \) in Section 2). When \( d = 0 \), the claim is obviously true. So, suppose that \( d = 1 \). The left-hand side is then \( s(s + 1)^{q-1} - 1 \). The right-hand side is

\[
(T(s + 1))^{q-1} \left( \frac{s^q}{(T(s + 1))^{q-1}} - 1 \right) + (T^{q-1} - 1)(s + 1)^{q-1} = s(s + 1)^{q-1} - 1.
\]

Thus, the claim is true.

For simplicity, let

\[
L := -T^q s(s + 1)^{q-1} \quad \text{and} \quad R := -\frac{T^q s^q}{(T(s + 1))^{q-1}}.
\]

Assume that the claim holds for \( d - 1 \) and \( d \). That is, assume that

\[
b(d - 1)(L) = (T(s + 1))^{q^{d-1}-1} b(d - 1)(R)
+ (T^{q^{d-1}-1} - 1)(s + 1)^{q^{d-1}-1} b(d - 2)(R);
\]

\[
b(d)(L) = (T(s + 1))^{q^{d-1}} b(d)(R)
+ (T^{q^{d-1}} - 1)(s + 1)^{q^{d-1}} b(d - 1)(R).
\]

It follows from the recursion (8) in Section 2 that

\[
b(d + 1)(L) = - \left( 1 + \left( \frac{L}{T^q} \right)^{q^d} \right) b(d)(L)
+ (T^{1-q^d} - 1) \left( \frac{L}{T^q} \right)^{q^d} b(d - 1)(L); \quad (14)
\]

\[
b(d)(R) = - \left( 1 + \left( \frac{R}{T^q} \right)^{q^{d-1}} \right) b(d - 1)(R)
+ (T^{1-q^{d-1}} - 1) \left( \frac{R}{T^q} \right)^{q^{d-1}} b(d - 2)(R); \quad (15)
\]

\[
b(d + 1)(R) = - \left( 1 + \left( \frac{R}{T^q} \right)^{q^d} \right) b(d)(R)
+ (T^{1-q^d} - 1) \left( \frac{R}{T^q} \right)^{q^d} b(d - 1)(R), \quad (16)
\]

and these relations are used in this proof. First, we can compute the right-hand side of (14) as follows:

\[
b(d + 1)(L) = - \left( 1 + \left( \frac{L}{T^q} \right)^{q^d} \right) b(d)(L)
+ (T^{1-q^d} - 1) \left( \frac{L}{T^q} \right)^{q^d} b(d - 1)(L)
= \left( (s(s + 1)^{q-1})^{q^d} - 1 \right) b(d)(L)
+ (1 - T^{1-q^d}) (s(s + 1)^{q-1})^{q^d} b(d - 1)(L).
\]

Then, it follows from the inductive hypothesis that

\[
b(d + 1)(L) = \left( (s(s + 1)^{q-1})^{q^d} - 1 \right) b(d)(L)
+ (1 - T^{1-q^d}) (s(s + 1)^{q-1})^{q^d} b(d - 1)(L)
= A \cdot b(d)(R) + B \cdot b(d - 1)(R) + C \cdot b(d - 2)(R),
\]
where
\[ A := \left( (s(s + 1)^{q-1})^{q^d-1} - 1 \right) (T(s + 1))^{q^d-1}; \]
\[ B := \left( (s(s + 1)^{q-1})^{q^d-1} - 1 \right) (T_{q^d-1}^d - 1)(s + 1)^{q^d-1} \]
\[ + (1 - T_{1-q^d}^d) \left( (s(s + 1)^{q-1})^{q^d} (T(s + 1))^{q^d-1} \right); \]
\[ C := (1 - T_{1-q^d}^d) \left( (s(s + 1)^{q-1})^{q^d} (T_{q^d-1}^d - 1)(s + 1)^{q^d-1} \right). \]

Next, by using the equality (15), we calculate the right-hand side of the above equality and obtain
\[
b(d + 1)(L) = A \cdot b(d)(R) + B \cdot b(d - 1)(R) + C \cdot b(d - 2)(R) \\
= \left( A + (T_{q^d-1}^d - 1)(s + 1)^{q^d+1-1} \right) b(d)(R) \\
+ \left( B + (T_{q^d-1}^d - 1)(s + 1)^{q^d+1-1} \right) \left( 1 + \left( \frac{R}{T^q} \right)^{q^d-1} \right) b(d - 1)(R) \\
= \left( A + (T_{q^d-1}^d - 1)(s + 1)^{q^d+1-1} \right) b(d)(R) \\
+ (T_{q^d-1}^d - 1)s^{q^d+1}(s + 1)^{q^d-1}b(d - 1)(R). \]

Last, from the equality (15), we have
\[
b(d + 1)(L) = \left( A + (T_{q^d-1}^d - 1)(s + 1)^{q^d+1-1} \right) b(d)(R) \\
+ (T_{q^d-1}^d - 1)s^{q^d+1}(s + 1)^{q^d-1}b(d - 1)(R) \\
= (T(s + 1))^{q^d+1-1}b(d + 1)(R) \\
+ \left( A + (T_{q^d-1}^d - 1)(s + 1)^{q^d+1-1} + (T(s + 1))^{q^d+1-1} \right) \left( 1 + \left( \frac{R}{T^q} \right)^{q^d} \right) b(d)(R) \\
= (T(s + 1))^{q^d+1-1}b(d + 1)(R) + (T_{q^d+1}^d - 1)(s + 1)^{q^d+1-1}b(d)(R), \]
which is the desired result.

**Remark.**
1. The idea for the proof of item (a) is found in Theorem 15 in [2].
2. In the course of the proof, the idea that the reduced polynomial \( H_{p}^{(d)}(\lambda) \) is once-lifted to the unreduced polynomial \( b(d) \) is very important. That is, item (b) can be proven by using item (a). In what follows, item (b) is often used, but item (a) is not used directly.

As an application of Proposition 4.1 (b), we have the following corollary.

**Corollary 4.2.** Let
\[ \Omega = \Omega^{(d)} := \left\{ s \in \bar{\mathbb{F}}_p \mid H_{p}^{(d)}(-\alpha^q s(s + 1)^{q-1}) = 0 \right\}. \]
Then, \( \Omega \subseteq \mathbb{F}_{p^2}^{(2)} \) and \( |\Omega| = q \cdot \deg_{\lambda} H_{p}^{(d)}(\lambda) = q(q^d - 1)/(q - 1) \).

**Proof.** First, we show that \( \mathbb{F}_{p^2}^{(2)} \) contains a primitive \((q + 1)\)th root of unity. If \( \zeta \in \bar{\mathbb{F}}_q \) is a primitive \((q + 1)\)th root of unity, then \( \zeta^{q^2-1} = (\zeta^{q+1})^{q-1} = 1 \), and so \( \zeta \in \mathbb{F}_{p^2} \subseteq \mathbb{F}_{p^2}^{(2)} \).

Next, we prove \(-1 \notin \Omega\). If \(-1 \in \Omega\), then \( H_{p}^{(d)}(\lambda) \) has \( \lambda = 0 \) as a root, which contradicts Proposition 4.1 (b). Hence, \(-1 \notin \Omega\).

For each \( s \in \Omega \), we get
\[
(s + 1)^{q^d-1}H_{p}^{(d)} \left(-\alpha^q \frac{s^q}{(\alpha(s + 1))^{q^d-1}} \right) = H_{p}^{(d)}(-\alpha^q s(s + 1)^{q-1}) \\
= 0
\]
Fix an element $b$ that.

Recall that $\Omega \subseteq X$. Although Corollary is expressed in terms of the curves $X_0(T^2)$ (see, for example, Corollary 6.12 in [23], Remark 2.5 of Chapter II in [30]). Under this correspondence, a function field of the curve $X_0(T^2)$ to-one correspondence with the function fields of one variable (see, for example, Corollary 6.12 in [23], Remark 2.5 of Chapter II in [30]). Hence, $\Omega$ is the set of supersingular points of $X_0(T^2)$. Since all the supersingular points of $X_0(T^2)$ split completely in this covering, and $S \subseteq \mathbb{F}_p^2$ (see Proposition 3.1 (d)), we have $\Omega \subseteq \mathbb{F}_p^2$.

(2) In Corollary 19 in [2], Bass and Beelen proved the same result as the above corollary for another polynomial $p_d(s(s+1)^{q-1})$. Their method of proof is completely different from the method used here.

The following proposition follows as a corollary (which we call Corollary). Here is a justification.

**Proposition 4.3.** The tower $E^d/\mathbb{F}_p^2$ is asymptotically optimal. That is, $\lambda(E) = q^d - 1$.

**Proof.** Recall that $\gamma(E) = q/(q-1)$ (cf. Lemma 4 in [2]).

Now, we compute the limit $\nu(E) := \lim_{n \to \infty} N(E_n/\mathbb{F}_p^2)/q^n$. We write the zero of $x_0 - a$ in the rational function field $E_0 = \mathbb{F}_p^2(x_0)$ as $P_a := P_{x_0 - a}$.

First, we show that for each $a \in \Omega$, the place $P_a$ splits completely in $E/\mathbb{F}_p^2$. Recall that $-1 \notin \Omega$. Fix an element $b \in \mathbb{F}_p^2$ such that $b(b+1)^{q-1} = a^q/(\alpha(a+1))^{q-1}$. It follows from Proposition 4.1 (b) that

$$H_p^{(d)}(-\alpha^q b(b+1)^{q-1}) = H_p^{(d)}(-\alpha^q (\alpha(a+1))^{q-1}) = H_p^{(d)}(-\alpha^q a(a+1)^{q-1}/(a+1)^{q-1}) = 0,$$

and so $b \in \Omega$. Thus, we get $b \in \mathbb{F}_p^2$ by Corollary 4.2. By Kummer’s theorem (see Theorem 3.3.7 in [32]), the place $P_a$ splits completely in $E_1/E_0$. The desired assertion follows by induction.

Next, we calculate $\nu(E)$. By the first claim, we have $N(E_n/\mathbb{F}_p^2) \geq |\Omega| \cdot q^n$, and so

$$\nu(E) = \lim_{n \to \infty} \frac{N(E_n/\mathbb{F}_p^2)}{q^n} = |\Omega| = \frac{q(q^d - 1)}{q-1}$$

from Corollary 4.2.

Combining the above results, we get $\lambda(E) = \nu(E)/\gamma(E) \geq q^d - 1$. It then follows from the Drinfeld-Vlăduţ bound (see Theorem 7.1.3 in [32]) that $q^d - 1 \leq \lambda(E) \leq A(q^{2d}) = q^d - 1$. 

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Remark. (1) Recall that $d = \deg_{\mathfrak{p}} p(T)$ is the degree of $\mathfrak{p} = (p(T))$. Our corollary is a generalization of Theorem 1 of [5] (the Bezerra-Garcia theorem), generalizing that result to allow an arbitrary degree $d$. That is, the case when $d = 1$ corresponds exactly to that theorem. Although our corollary is a special case of Theorem 2.16 in [21] and Theorem 4.2.38 in [36], our proof is more elementary, and explicitly describes the set $\Omega$ of degree-one places. For this reason, our result has applications to coding theory (see [31], Chapters 7 and 8 in [32], Parts 3 and 4 in [36], Chapters 3 and 4 in [37]). The Bezerra-Garcia theorem is a special case of Theorem 10.1 of Gekeler in [20].

First, we compare Corollary 1.2 with the result by Bezerra and Garcia (and a result of Elkies in [14]) in terms of the splitting locus $\text{Split}(E)$ (see Definition 7.2.9 (a) in [32]). Recall that when $d = 1$, we have $p(T) = T - 1$. Bezerra and Garcia showed that the places corresponding to the roots of $x^q + x - 1 = 0$ split completely (see Page 152 in [5]). Setting $s = (1 - x)/x$, we get

$$x^q + x - 1 = 0 \iff -s(s + 1)^{q-1} + 1 = 0 \iff H^{(1)}_s(-s(s + 1)^{q-1}) = 0,$$

and so the special case ($d = 1$) of Corollary 1.2 coincides with the Bezerra-Garcia result. Furthermore, the set of roots for the second equation coincides with the set (26) given by Elkies in [14].

Now, we explain a difference between Corollary 1.2 and the Bezerra-Garcia result. Recall that $\Sigma(p)$ denotes the set of supersingular points of $X(1)/\mathbb{F}_p$. When $d = 1$, we know that $|\Sigma(p)| = 1$ and that the point $j = 0$ is the supersingular point of $X(1)$. Since the covering $X_0(T^2) \to X_0(T)$ is given by $\lambda = -\alpha^3 s(s + 1)^{q-1}$ and the covering $X_0(T) \to X(1)$ is defined by $j = (\alpha + \lambda)^{q+1}/\lambda$ (see the last part of this section), we easily see that all the roots of $-s(s + 1)^{q-1} + 1 = 0$ are above $j = 0$. When $d \geq 3$ is odd, we know that $|\Sigma(p)| \geq q + 1$. Therefore, the completely splitting points are above several supersingular points. In fact, the points $j = 0$ and $j = [1](\alpha^q - \beta)^{q-1}$ ($\beta \in \mathbb{F}_q$) are supersingular points (see Proposition 16 in [29]). Hence, the splitting locus $\text{Split}(E^{(d)})$ cannot be computed by using the approach employed in [5].

Next, we compare our result with the result by Bezerra and Garcia in terms of the ramification locus $\text{Ram}(E)$ (see Definition 7.2.9 (b) in [32]). Bezerra and Garcia showed that the places corresponding to $x = 0, x = 1$ or $x = \infty$ are ramified, and the other places are unramified (see Section 3 in [5]). Setting $s = (1 - x)/x$, we get

$$x = 0, x = 1, x = \infty \iff s = \infty, s = 0, s = -1,$$

respectively. We can see that the points $s = 0, -1, \infty$ are above $j = \infty$. Similarly, in our case, we know that the ramified points are above $j = \infty$. Hence, the ramification locus $\text{Ram}(E^{(d)})$ can be calculated using the method of [5].

(2) In the proof of Proposition 4.3 as a consequence, we see that

$$\nu(E) = \lim_{n \to \infty} \frac{N(E_n/\mathbb{F}_p^{(2)})}{q^n} = |\Omega|,$$

and so $N(E_n/\mathbb{F}_p^{(2)}) = |\Omega| + o(q^n) (n \to \infty)$.

Finally, we consider a background of the tower $E$ in terms of the Drinfeld modular curves $X_0(T^n)$. First, we consider the genus, which yields another proof that $\gamma(E) = q/(q - 1)$. It follows from Theorem 8.1 (iii) in [20] (or Theorem 2.13 in [21]) that the genus of $X_0(T^n)$ is given by

$$g(X_0(T^n)) = \frac{q^{n-1} - q^\lfloor (n-1)/2 \rfloor - q^\lfloor (n-2)/2 \rfloor + 1}{q - 1} = \frac{q}{q - 1} + o(q^{-2}) \quad (n \to \infty).$$

Hence, $g(X(1)) = g(X_0(T)) = g(X_0(T^2)) = 0$ and $g(X_0(T^n)) \geq 1$ for $n \geq 3$. This is used below.

Next, we consider the origin of the equation (13), which is due to Sections 2 and 3 in [2]. Let $\phi$ be a rank-2 Drinfeld module over $K$, defined by $\phi_T = T + (T + \lambda_0)^{q+1}/\lambda_0$. It is known that its $T$-isogenous (rank-2) Drinfeld module $\phi'$ is given by $\phi'_T = T + (T^q + \lambda_0^q)^{q+1}/\lambda_0^q$, and its $j$-invariant is $j_1(T) = (T^q + \lambda_0^{q+1}/\lambda_0^q)$. There is some
modular polynomial $\Phi_T(X, Y) \in A[X, Y]$ such that $\Phi_T(j_0, j_1(T)) = 0$, and this polynomial is very complicated (see any of [29, 1, 6, 3, 4]).

Second basement: Assume that $j_0$ is transcendental over $K$. That is, assume that the function field $K_0 := K(j_0)$ is rational, which is a function field of $X(1)$. The Drinfeld modular curve $X_0(T)$ is defined by the equation $\Phi_T(j_0, j_1(T)) = 0$, and its function field $K(X_0(T))$ is given as $K_1 := K(j_0, j_1(T))$. Recall that the relation $\Phi_T(j_0, j_1(T)) = 0$ provides a plane model for $X_0(T)$, and that

$$[K_1 : K_0] = q^{\deg(T)} \prod_{P \in T} \left(1 + \frac{1}{q^{\deg(P)}}\right) = q + 1.$$

It is known that, for each integer $n \geq 0$, the function field $K(X_0(T^n))$ can be written as

$$K_n := K(j_0, j_1(T), j_1(T^2), \ldots, j_1(T^n)),$$

where

$$\Phi_T(j_0, j_1(T)) = 0 \quad \text{and} \quad \Phi_T(j_1(T^i), j_1(T^{i+1})) = 0$$

for $1 \leq i < n$. Notice that $\Phi_T(j_1(T^i), Y)$ is reducible over $K_i$, and that $[K_n : K_0] = (q + 1)q^{n-1}$.

First basement: Since $X_0(T)$ is also of genus 0, its function field $K(X_0(T))$ is rational. In fact, $K(X_0(T))$ can be given as $K_1 = K(\lambda_0)$ by using the transcendental element $\lambda_0$ (see Proposition 3 in [29]). The function field $K(X_0(T^2))$ is then given as $K_2 = K(\lambda_0, \lambda_1)$, where

$$\frac{(T + \lambda_1)^{q+1}}{\lambda_1} = \frac{(T^q + \lambda_0)^{q+1}}{\lambda_0^q}.$$

This relation is not minimal with respect to degree, in the following sense: Since

$$\frac{(T + \lambda_1)^{q+1}}{\lambda_1} - \frac{(T^q + \lambda_0)^{q+1}}{\lambda_0^q} = \frac{T^{q+1} - \lambda_0\lambda_1}{\lambda_0^q\lambda_1} (T^q + \lambda_0 - (T^{q+1} - \lambda_0\lambda_1)^{q-1} (T + \lambda_1))$$

and $\lambda_0\lambda_1 \neq T^{q+1}$, we can obtain a new relation

$$T^q + \lambda_0 - (T^{q+1} - \lambda_0\lambda_1)^{q-1} (T + \lambda_1) = 0.$$

That is, $T^q + \lambda_0 = (T^{q+1} - \lambda_0\lambda_1)^{q-1} (T + \lambda_1)$, which is a minimal relation (with respect to degree). This minimal relation is used below.

For each integer $n \geq 1$, the function field $K(X_0(T^n))$ is

$$K_n = K(\lambda_0, \lambda_1, \ldots, \lambda_{n-1}),$$

where

$$\frac{(T + \lambda_{i+1})^{q+1}}{\lambda_{i+1}} = \frac{(T^q + \lambda_i)^{q+1}}{\lambda_i^q}$$

for $0 \leq i < n - 1$. Notice that $[K_1 : K_0] = q + 1$ and $[K_n : K_1] = q^n$.

Ground floor: Since $X_0(T^2)$ is also of genus 0, its function field $K(X_0(T^2))$ is rational. In fact, $K(X_0(T^2))$ can be written as $K_2 = K(s_0)$ by using the transcendental element $s_0$ such that

$$-T \cdot (s_0 + 1) = \frac{T^{q+1} - \lambda_0\lambda_1}{T^q + \lambda_0},$$

(see Section 3 in [2]). Since

$$\lambda_0 = -T^q s_0 (s_0 + 1)^{q-1} \quad \text{and} \quad \lambda_1 = -\frac{T^q s_0^q}{(T(s_0 + 1))^{q-1}},$$

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we obtain $K_1 = K(\lambda_0, \lambda_1) = K(s_0)$.

For each integer $n \geq 2$, the function field $K(X_0(T^n))$ is

$$K_n = K(s_0, s_1, \ldots, s_{n-2}),$$

where

$$s_{i+1}(s_{i+1} + 1)^{q-1} = \frac{s_i^q}{(T(s_i+1))^{q-1}}$$

for $0 \leq i < n-2$, which is just the equation [13].

By using the technique of Elkies in [13, 14], the sequence of $X_0(T^n)/\mathbb{F}_p$ $(n \geq 2)$ corresponds to the tower $\mathcal{E}$. In particular, a function field of $X_0(T^{n+2})/\mathbb{F}_p$ corresponds to the function field $E_n$.

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References

[1] Sunghan Bae and Seungjae Lee, On the coefficients of the Drinfeld modular equation, J. Number Theory 66 (1997), no. 1, 85-101.

[2] Alp Bassa and Peter Beelen, Explicit equations for Drinfeld modular towers, arXiv:1110.6076, 2011.

[3] —, A proof of a conjecture by Schweizer on the Drinfeld modular polynomial $\Phi_T(X, Y)$, J. Number Theory 131 (2011), no. 7, 1276-1285.

[4] —, A closed-form expression for the Drinfeld modular polynomial $\Phi_T(X, Y)$, Arch. Math. (Basel) 99 (2012), no. 3, 237-245.

[5] Juscelino Bezerra and Arnaldo Garcia, A tower with non-Galois steps which attains the Drinfeld-Vlăduţ bound, J. Number Theory 106 (2004), no. 1, 142-154.

[6] So Young Choi, Kuk Jin Hong and Daeyeol Jeon, On plane models for Drinfeld modular curves, J. Number Theory 119 (2006), no. 1, 18-27.

[7] Gunther Cornelissen, Zeros of Eisenstein series, quadratic class numbers and supersingularity for rational function fields, Math. Ann. 314 (1999), no. 1, 175-196.

[8] Max F. Deuring, Die Typen der Multiplikatorenringe elliptischer Funktionenkörper, (German) Abh. Math. Sem. Univ. Hamburg 14 (1941), no. 1, 197-272.

[9] Vladimir G. Drinfeld, Elliptic modules, (Russian) Mat. Sb. (N.S.) 94 (136) (1974), 656.

[10] —, Elliptic modules II, (Russian) Mat. Sb. (N.S.) 102 (144) (1977), no. 2, 182194, 325.

[11] Ahmad El-Guindy and Matthew A. Papanikolas, Explicit formulas for Drinfeld modules and their periods, J. Number Theory 133 (2013), no. 6, 1864-1886.

[12] Ahmad El-Guindy, Legendre Drinfeld modules and universal supersingular polynomials, Int. J. Number Theory 10 (2014), no. 5, 1277-1289.

[13] Noam D. Elkies, Explicit modular towers, pages 23-32 in Proceedings of the Thirty-Fifth Annual Allerton Conference on Communication, Control and Computing (1997, T. Basar, A. Vardy, eds.), Univ. of Illinois at Urbana-Champaign 1998 (math.NT/0103107 on the arXiv).

[14] —, Explicit towers of Drinfeld modular curves, European Congress of Mathematics, Vol. II (Barcelona, 2000), 189198, Progr. Math., 202, Birkhauser, Basel, 2001.

[15] Arnaldo Garcia, Henning Stichtenoth and Hans-Georg Rück, On tame towers over finite fields, J. Reine Angew. Math. 557 (2003), 53-80.
[16] Ernst-Ulrich Gekeler, Zur Arithmetik von Drinfeld-Moduln, Math. Ann. 262 (1983), no. 2, 167-182.

[17] —, Drinfeld modular curves, Lecture Notes in Mathematics, 1231, Springer-Verlag, Berlin, 1986.

[18] —, On the coefficients of Drinfeld modular forms, Invent. Math. 93 (1988), no. 3, 667-700.

[19] —, On finite Drinfeld modules, J. Algebra 141 (1991), no. 1, 187-203.

[20] —, Invariants of some algebraic curves related to Drinfeld modular curves, J. Number Theory 90 (2001), no. 1, 166-183.

[21] —, Asymptotically optimal towers of curves over finite fields, Algebra, arithmetic and geometry with applications (West Lafayette, IN, 2000), 325-336, Springer, Berlin, 2004.

[22] David Goss, Basic structures of function field arithmetic, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 35, Springer-Verlag, Berlin, 1996.

[23] Robin Hartshorne, Algebraic geometry, Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977.

[24] Takehiro Hasegawa, On asymptotically optimal towers over quadratic fields related to Gauss hypergeometric functions, Int. J. Number Theory 6 (2010), no. 5, 989-1009.

[25] —, Some remarks on superspecial and ordinary curves of low genus, Math. Nachr. 286 (2013), no. 1, 17-33.

[26] —, A proof of a conjecture by Lotter on the roots of a supersingular polynomial and its application, Mosc. Math. J. 15 (2015), no. 1, 89100, 182.

[27] Dale H. Husemoller, Elliptic curves, Second edition, With appendices by Otto Forster, Ruth Lawrence and Stefan Theisen. Graduate Texts in Mathematics, 111, Springer-Verlag, New York, 2004.

[28] B. Heinrich Matzat, Introduction to Drinfeld modules, Drinfeld modules, modular schemes and applications (Alden-Biesen, 1996), 3-16, World Sci. Publ., River Edge, NJ, 1997.

[29] Andreas Schweizer, On the Drinfeld modular polynomial $\Phi_T(X,Y)$, J. Number Theory 52 (1995), no. 1, 53-68.

[30] Joseph H. Silverman, The arithmetic of elliptic curves, Second edition. Graduate Texts in Mathematics, 106, Springer, Dordrecht, 2009.

[31] Henning Stichtenoth, Transitive and self-dual codes attaining the Tsfasman-Vlăduţ-Zink bound, IEEE Trans. Inform. Theory 52 (2006), no. 5, 2218-2224.

[32] —, Algebraic function fields and codes, Second edition. Graduate Texts in Mathematics, 254, Springer-Verlag, Berlin, 2009.

[33] Dinesh S. Thakur, Hypergeometric functions for function fields, Special issue dedicated to Leonard Carlitz, Finite Fields Appl. 1 (1995), no. 2, 219-231.

[34] —, Hypergeometric functions for function fields II, J. Ramanujan Math. Soc. 15 (2000), no. 1, 43-52.

[35] —, Function field arithmetic, World Scientific Publishing Co., Inc., River Edge, NJ, 2004.

[36] Michael A. Tsfasman, Serge G. Vlăduţ, Algebraic-geometric codes, Translated from the Russian by the authors, Mathematics and its Applications (So viet Series), 58, Kluwer Academic Publishers Group, Dordrecht, 1991.

[37] Michael A. Tsfasman, Serge G. Vlăduţ, Dmitry Yu. Nogin, Algebraic geometric codes: basic notions, Mathematical Surveys and Monographs, 139. American Mathematical Society, Providence, RI, 2007.