Extending Sibgatullin’s ansatz for the Ernst potential to generate a richer family of axially symmetric solutions of Einstein’s equations

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Abstract.
The scope of this talk is to present some preliminary results on an effort, currently in progress, to generate an exact solution of Einstein’s equation, suitable for describing spacetime around a rotating compact object. Specifically, the form of the Ernst potential on the symmetry axis and its connection with the multipole moments is discussed thoroughly. The way to calculate the multipole moments of spacetime directly from the value of the Ernst potential on the symmetry axis is presented. Finally, a mixed ansatz is formed for the Ernst potential including parameters additional to the ones dictated by Sibgatullin. Thus, we believe that this talk can also serve as a comment on choosing the appropriate ansatz for the Ernst potential.

1. Introduction
In order to analytically describe spacetime around a rotating compact object in the framework of general relativity, one needs an exact asymptotically flat solution of Einstein-Maxwell equations. To simplify the problem it is reasonable to assume that a long-lived, quiescent rotating body possesses axial symmetry and reflectional symmetry with respect to the equatorial plane. The solution must depend on a number of arbitrary physical parameters such as the mass, the angular momentum, the mass-quadrupole moment, the magnetic dipole e.t.c. of the central body.

Sibgatullin has proposed a way to generate stationary axisymmetric solutions \[1\], starting from the value of the Ernst potentials \[2,3\] \(E\) and \(\Phi\) on the symmetry axis. His method has been used successfully by many authors (see for example \[4,5\]). However the arbitrary parameters appearing in the proposed ansatz for \(E\) and \(\Phi\) are physically meaningful only in specific and rather special cases (for example the Kerr solution).

On the other hand, it is well known for quite some time that spacetime geometry around an axisymmetric object leads to a specific set of the Geroch-Hansen multipole moments \[6,7\]. Thus, the scalar moments can be used to characterize the spacetime, in the same way that the newtonian gravitational moments characterize the newtonian gravitational field. In this paper we will try to combine multipole moments with Sibgatullin’s arbitrary parameters in order to form an ansatz for the Ernst potential \(E\) on the symmetry axis, suitable to obtain a metric for an

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axisymmetric compact object through Sibgatullin’s method. Due to lack of space and time this study will be restricted to vacuum spacetime. Note however that it can be easily generalized to include electrovacuum spacetimes.

The plan of this paper is the following: In Sec. 2 we give an outline of Sibgatullin’s method. In Sec. 3 we show the way to calculate the moments directly from the value of the Ernst potential on the symmetry axis. We also present a way to generate a solution with specific multipole moments. In Sec. 4 we present a “mixed” ansatz for the Ernst potential including both Sibgatullin arbitrary parameter and a number of new parameters related to the moments. Finally we comment on the restrictions that come up if we additionally impose reflectional symmetry with respect to the equatorial plane.

2. Sibgatullin’s method for generating exact solutions

It is known for quite some time that in the case of a stationary, axisymmetric spacetime the metric functions can be fully determined if the so called Ernst potential \( E \) is known \[2\]. Sibgatullin has proposed a way of generating solutions \[1\] of Einstein’s field equations, that reduces to evaluating the solution of a linear system of algebraic equations. His approach is based on the definition of the Ernst potential on the symmetry axis in terms of rational functions, involving a series of parameters \( a_j \[4\]

\[
E(\rho = 0) = e(z) = \frac{z^n - Mz^{n-1} + \sum_{j=1}^{n} a_j z^{n-j}}{z^n + Mz^{n-1} + \sum_{j=1}^{n} a_j z^{n-j}} \tag{1}
\]

We will merely outline some key points of the method. Sibgatullin has shown that the Ernst potential \( E(\rho, z) \) can be evaluated from the potential \( \mu(\sigma) \) which is given on the symmetry axis, as the integral

\[
E(\rho, \sigma) = \int_{-1}^{1} \frac{\mu(\sigma) e(\xi) d\sigma}{\sqrt{1 - \sigma^2}} \tag{2}
\]

were the function \( \mu(\sigma) \) satisfies the following linear integral equation and normalization condition respectively:

\[
P \int_{-1}^{1} \frac{\mu(\sigma) (e(\xi) + \tilde{e}(\xi))}{(\sigma - \tau)\sqrt{1 - \sigma^2}} d\sigma = 0, \quad \int_{-1}^{1} \frac{\mu(\sigma) d\sigma}{\sqrt{1 - \sigma^2}} = \pi \tag{3}
\]

were the \( P \) denotes the principal value of the integral, \( \xi \equiv z + i\sigma \rho, \quad \sigma, \tau \in [-1, 1] \) and the functions \( e(\xi) \) and \( \tilde{e}(\xi) \) are analytical continuations on the complex plane for the functions \( e(z) \) and \( e^*(z) \). The first step is to express equation \( \mu(\sigma) \) as an expansion of elementary fractions \( e(z) = 1 + \sum_{j=1}^{n} e_j z^{-a_j} \) and the sought for solution of the integral equation as \( \mu(\sigma) = A_0 + \sum_{k=1}^{2n} A_k \xi_k \) were \( \tilde{a}_j \) and \( \xi_k \) are the roots of the polynomials \( z^n + Mz^{n-1} + \sum_{j=1}^{n} a_j z^{n-j} = 0 \) and \( e(z) e^*(z) = 0 \) respectively. Substituting the series expansions of \( e(z) \) and \( \mu(\sigma) \) in the integral equation and the normalization condition we obtain a closed system of \( 2n + 1 \) linear algebraic equations for the \( 2n + 1 \) parameters \( A_k \). These coefficients can be evaluated and they completely determine the Ernst potential that will generate the solution. In particular, it has been shown by Ruiz, Manko and Martin \[8\] that the metric functions can be evaluated with the use of some determinants which are functions of the parameters \( e_j, a_j, \xi_k \).

3. The multipole moments of the solutions

In \[9\] an algorithm is presented for the evaluation of the multipole moments of stationary axisymmetric spacetimes from the series expansion of the function \( \xi \) on the symmetry axis. The
real and the imaginary part of $\tilde{\xi}$ are related to the so-called Hansen potentials respectively \cite{7}. Since the multipole moments are evaluated at infinity, we will use the preferred coordinates

$$\bar{\rho} = \frac{\rho}{\rho^2 + z^2}, \quad \bar{z} = \frac{z}{\rho^2 + z^2},$$

(4)

where $\rho$ and $z$ are Weyl coordinates. As mentioned in \cite{7} $\tilde{\xi}$ can expanded around $\bar{z} = 0$ in the following way.

$$\tilde{\xi}(\bar{\rho} = 0) = \sum_{n=0}^{\infty} m_n \bar{z}^n$$

(5)

The multipole moments can then be evaluated from the $m_n$’s using the relations presented in p. 2255 of \cite{9}.

If we express $\tilde{\xi}$ in terms of the Ernst potential $E$ then

$$\tilde{\xi} = \sqrt{\rho^2 + z^2} \frac{1 - E}{1 + E}, \quad \tilde{\xi}(\rho = 0) = \frac{1 - E(\rho = 0)}{1 + E(\rho = 0)}.$$  

(6)

or in the preferred coordinates

$$\tilde{\xi}(\rho = 0) = \bar{z}^{-1} \frac{1 - E(\rho = 0)}{1 + E(\rho = 0)}.$$  

(7)

It is now obvious that the moments can be evaluated from $E(\rho = 0)$. Using (5) and (11) we get

$$\sum_{i=0}^{\infty} m_i \bar{z}^i = \frac{M}{1 + \sum_{j=1}^{n} a_j \bar{z}^j}.$$  

(8)

Equation (8) can be used to relate $m_i$ with $a_j$ and vice versa. Applying Taylor expansion at $\bar{z} = 0$ we derive the following recursive formulae:

$$m_0 = M, \quad m_n = - \sum_{l=1}^{n} a_l m_{n-l} \hspace{1cm} \text{or} \hspace{1cm} a_0 = 1, \quad a_n = - \sum_{l=1}^{n} \frac{m_l}{M} a_{n-l}.$$  

(9)

To derive the second set of expressions, of course, one has to regard that the number of $m_n$’s is finite and the number of $a_n$’s is infinite. Before we make any comments we present two examples. If we set

$$a_n = 0, \quad n = 1, 2, 3, \ldots$$  

(10)

then it follows that

$$m_0 = M, \quad m_n = 0, \quad n = 1, 2, 3, \ldots$$  

(11)

which will lead to the Schwarzschild solution, while if we set

$$a_1 = -ia, \quad a_n = 0, \quad n = 2, 3, 4, \ldots$$  

(12)

then it follows that

$$M_{2n} = (-1)^n Ma^{2n}, \quad J_{2n+1} = (-1)^n Ma^{2n+1}, \quad n = 0, 1, 2, 3, \ldots$$  

(13)

which are the mass and the mass current moments of the Kerr solution respectively.
We have related $\mathcal{E}(\rho = 0)$ and thus Sibgatullin’s arbitrary parameters with the moments. This is another way to demonstrate that all information regarding spacetime are included in the value of the Ernst potential on the symmetry axis. Using eqs. (9) one can compute the moments without much effort even before generating the solution. One can also get an insight for the number and the nature of the $a_j$’s needed to generate a solution with a certain set of multipoles. However, it is important to say that a physical object is in principle expected to have an infinite number of moments. It is easy to see that a finite number of $a_j$’s imposes a relation between the different order multipoles. The more $a_j$’s one uses the more flexible this relation becomes. However every new $a_j$ included leads to an increase of the order of the polynomials of the Ernst potential, and consequently creates serious difficulties in generating the corresponding metric.

Before we close this section it is important to notice that Sibgatullin’s method can be also used to generate an exact solution with a desired set of multipole moments or equivalently a desired set of $m_i$’s. If we solve (7) for $\mathcal{E}(\rho = 0)$ we get

$$\mathcal{E}(\rho = 0) = \frac{1 - z \sum_{i=0}^{\infty} m_i z^i}{1 + z \sum_{i=0}^{\infty} m_i z^i} = \frac{1 - \sum_{i=0}^{\infty} m_i z^{-(i+1)}}{1 + \sum_{i=0}^{\infty} m_i z^{-(i+1)}}. \quad (14)$$

We can use (14) instead of (1) and follow the procedure described in section 2 to obtain the solution. We could turn the infinite sum of eq. (5) into a finite one, by setting all the moments of order $> n$ equal to zero. This, even though not exact, can most of the times be considered as a good approximation.

4. A mixed ansatz for the Ernst Potential

Sibgatullin’s ansatz for $\mathcal{E}(\rho = 0)$ is rather general, but not the easiest one can use to generate a solution as we will show. Consider the following ansatz for $\mathcal{E}(\rho = 0)$:

$$\mathcal{E}(\rho = 0) = \frac{z^n - M z^{n-1} + \sum_{j=1}^{n} a_j z^{n-j} - \sum_{i=1}^{n} b_i z^{n-1-i}}{z^n + M z^{n-1} + \sum_{j=1}^{n} a_j z^{n-j} + \sum_{i=1}^{n} b_i z^{n-1-i}}. \quad (15)$$

It is easy to see that if $a_j = 0$ for all $j$’s then $b_i = m_i$ and if all $b_i = 0$ the ansatz is reduced to Sibgatullin’s one. It is obvious that it is not possible to define a new finite set of parameters $a_j'$ in order to bring the ansatz in the form of eq. (14). This simply depicts the close relation of the $b_i$’s with the $m_i$’s. Notice that one can use the above ansatz to generate an exact solution with Sibgatullin’s method. No modifications in the generation algorithm are necessary. The merit of using such a mixed ansatz with $l$ $a_i$’s and $n$ $b_i$’s lies in the fact that the degree of its polynomials will be equal either to $l$ or to $n + 1$ (depending on which one is the largest), whereas in the original ansatz it is equal to $k = l + n$ in order for it to have the same number of parameters. This of course can lead to serious simplification of the computations needed to generate the solution. For example a 6 parameter solution would require third degree polynomial instead of fifth.

Before we proceed lets take a moment to focus on the special case in which spacetime has an additional symmetry: reflection symmetry with respect to the equatorial plane. This is a realistic assumption for most astrophysical objects. In this case $m_i$ should be real for even and imaginary for odd $i$’s. By evaluating the $m_i$’s in terms of the $a_i$’s and $b_i$’s one can show that, if there is a finite number of $a_i$’s and $b_i$’s then they should also be real for even $i$’s and imaginary for odd $i$’s.
A simple case for an ansatz is the following:

\[ \mathcal{E}(\rho = 0) = \frac{z^2 - Mz - iaz + bc - icM}{z^2 + Mz - iaz + bc + icM}, \]  

(16)

where all parameters are real. The first five multipole moments of the corresponding spacetime are the following:

\[
\begin{align*}
M_0 &= M, & M_1 &= 0, & M_2 &= -(a^2 + ac + bc)M, & M_3 &= 0, \\
M_4 &= (a^4 + a^3c + 3a^2bc + 2abc^2 + b^2c^2)M - \frac{1}{7} \left[ M \left( (a + c)^2M^2 - (a^2 + ac + bc)M^2 \right) \right] \\
J_0 &= 0, & J_1 &= (a + c)M, & J_2 &= 0, \\
J_3 &= -a^3M - a^2cM - 2abcM - bc^2M, & J_4 &= 0.
\end{align*}
\]

(17)

If we set \( c = 0 \) we get

\[
M_{2n} = (-1)^n Ma^{2n}, \quad J_{2n+1} = (-1)^n Ma^{2n+1}, \quad n = 0, 1, 2, 3, \ldots
\]

(18)

On the other hand if we set \( b = a + c \)

\[
M_{2n} = (-1)^n M(a + c)^{2n}, \quad J_{2n+1} = (-1)^n M(a + c)^{2n+1}, \quad n = 0, 1, 2, 3, \ldots
\]

(19)

Therefore, there are two discrete ways for the metric to be reduced to the Kerr metric, having different angular momentum. The first one is by reducing the ansatz to the known ansatz for the Kerr metric by setting \( c = 0 \) as we showed. The second one which does also reduce to a Kerr metric with angular momentum \( c \) if \( a = 0 \), is by setting \( b = a + c \). From eq. \( (17) \) one can see that by using freely the parameters \( a, b, c \) he can give the desired value to the angular momentum and the mass quadropole and still have two free parameters including the mass. Thus we think that starting from this ansatz it will be possible to generate a solution suitable to describe spacetime around compact rotating objects sufficiently well.

Finally, it is worth mentioning that this whole approach can be generalized to include electrovacuum spacetimes (based on [3, 11, 12]).

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