Structure Formation in Generalized Rastall Gravity

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Recently a modified version of Rastall gravity has been proposed in which a varying coupling parameter could play the role of dark energy (DE) and thus is responsible for the current accelerated expansion of the Universe. Motivated by this modification, we study here the evolution of matter perturbations, in both linear and non-linear regimes, using the top-hat spherical collapse model. The exact solutions in linear regime show that as the Universe evolves, matter perturbations grow and reach a maximum value at a certain redshift after which the perturbations start decreasing at later times. Depending on model parameters, exact oscillatory solutions (in linear regime) can be found representing that matter perturbations could experience either overdense and underrdense regions during the dynamical evolution of the Universe. Numerical solutions in non-linear regime show that the amplitude of perturbations grow much faster than the linear one and diverges at a critical redshift. The growth rate of non-linear perturbations and the critical redshift depend crucially on model parameters. It is found that the running mutual interaction between matter and geometry, encoded in the variable Rastall coupling parameter, could affect on the dynamics of matter perturbations during the evolution of the Universe.

I. INTRODUCTION

Ever since the discovery of the cosmic acceleration found by the observations of type Ia supernovae [1] and later confirmed by cosmological and astrophysical observations [2], several attempts have been made toward understanding the physical mechanism responsible for this phenomenon. In order to make reasonable sense of this acceleration, many cosmological models propose a new and hidden component within the Universe known as DE [3]. Despite of many efforts that have been made so far, the physical nature of DE is still a mystery to us. The simplest candidate for this unknown energy is the cosmological constant Λ, a fluid with equation of state $w = p/\rho = -1$ which is constant throughout the cosmic time $I$. Also, dynamical approach to DE has been introduced in which the observed acceleration of the Universe is caused by the potential energy of a dynamical field, referred to as quintessence field. For recent reviews we consult the reader to [5]. Apart from observing DE as a real ingredient of our Universe, one possible way to address the issue of cosmic speed-up is to modify gravitational theory such that the accelerated expansion could be attributed to this modification. Notable examples in this regard are scalar-tensor theory [6], Kinetic Gravity Braiding (KGB) [7] and $f(R)$ models [8], see also [9] for recent reviews.

In the framework of General Relativity (GR) and most of its modifications the energy-momentum source is described through a divergence-free tensor which couples to the geometry in a minimal way [9]. However, this property of material source which leads to conservation of energy-momentum tensor (EMT), i.e., $\nabla_\nu T^{\mu \nu} = 0$, is not obeyed by the particle production process [10, 11]. It is therefore reasonable to relax the condition on EMT conservation and seek for a new theory of gravitation. Long back in 1972, Rastall proposed such a modified gravity theory wherein, the ordinary conservation law is questioned whether it is reliable generally in curved spacetime or only in special cases, as this law is tested only in the flat Minkowski spacetime or specifically in a gravitational weak field limit [12]. Based on Rastall’s argument, the null covariant derivative of EMT does no longer hold and instead one assumes that the covariant derivative of EMT is proportional to the gradient of the Ricci scalar, i.e., $\nabla_\nu T^{\mu \nu} = \lambda' \nabla^\nu R$, where $\lambda'$ is the Rastall constant parameter. Indeed, this theory provides a phenomenological way for distinguishing aspects of quantum effects in gravitational systems, i.e. the violation of the classical conservation laws [10, 13, 14]. In the past years, many attempts have been made in order to explore the physical features of Rastall gravity among which we can quote the non-minimal coupling between matter fields and geometry [12, 13], compatibility of Rastall gravity with observational data on the Universe age and also on the Hubble parameter [15] and the ability of the theory to explain the cosmological aspects such as the accelerating expansion of the Universe and the inflationary problems [16]. Moreover, Rastall gravity can provide an alternative description for matter dominated era in comparison with GR [17]. It is also supported by observational data from the helium nucleosynthesis [18]. Also, motivated by these evidences, physicists have tried to investigate various cosmic epochs in this framework [19] and in [20], it is argued...
that this theory does not suffer from the entropy and age problems that appear in the context of standard cosmology.

One of the major issues in cosmology is the origin of gravitationally bound objects such as galaxies and galaxy clusters observed on large scales in the spatial distribution of galaxies and in the microwave background radiation. The most widely accepted answer to the question of why galaxies and structures formed during the evolution of the Universe, is provided by the gravitational instability scenario, wherein structure formation occurs due to the gravitational amplification of very small initial density fluctuations. Therefore, given some initial seed fluctuations one would like to be able to give, at least, analytic predictions to describe the gravitational clustering that generates the large-scale structures we observe today [21, 22]. Apart from its unavoidable role in the acceleration of the overall expansion rate of the Universe, the DE component has its own effects on the formation of gravitationally bound objects such as galaxies and galaxy clusters. Hence, searching for the footprints of DE through its influences over the rates of formation and growth of collapsed structures (halos) is of significant importance. A simple, but very powerful tool to investigate the structure formation in the presence of DE is via the Top-Hat Spherical-Collapse (SC) model [24], which was initially employed for Einstein-de Sitter (EdS) background in the standard cold-dark-matter model [24], and later in CDM [26]. The SC model has also been extended to other cosmological settings such as, quintessence fields [27], decaying vacuum models [28], f(R) gravity theories [29], cosmological models with constant equation of state for DE [30, 31] and coupled DE models [32, 33].

Recently, a generalization of Rastall theory has been proposed in which the Rastall parameter is taken to be a variable during the dynamical evolution of the Universe [33]. The authors then established the running non-minimal coupling between pressure-less matter and geometry which can be considered as the DE responsible for the current accelerating cosmic phase. Moreover, a non-singular model of the Universe has been reported in [34] where it is shown that for suitable choice of the varying Rastall parameter along with restriction on the equation of state parameter an emergent scenario can be constructed. The authors have also shown that it is possible to have a complete cosmic scenario from early inflationary era to the current accelerating phase through the matter dominated era of evolution for this model of gravity [34]. Motivated by these considerations, our aim here is to consider the issue of structure formation in generalized Rastall gravity and investigate the possible consequences of a varying Rastall parameter on the growth of density perturbations. The paper is then organized as follows. In Sec. II we briefly review the field equations of generalized Rastall gravity. Utilizing the (SC) model, in Sec. III we derive the evolutionary equations governing pressure-less matter perturbations. Sec. IV deals with exact solutions to the perturbation equation in linear regime. In Sec. V we investigate the evolution of matter perturbations in non-linear regime and finally in Sec. VI we summarize our results.

II. FIELD EQUATIONS OF GENERALIZED RASTALL GRAVITY

In the original theory, Rastall assumed that for all of the spacetimes and energy-momentum sources, the ratio of the covariant divergence of EMT to the covariant derivative of Ricci scalar is a constant. For example, in matter dominated era the energy density of matter decreases as the Universe expands, but the mentioned ratio remains constant in Rastall theory [15, 17]. This means that the coupling between energy-momentum source and geometry is constant and hence the mutual matter-geometry interaction is not affected by the evolution of the cosmic system. In fact, since the cosmic evolution is a continuous process [32], it is a plausible expectation that the mutual coupling between the energy-momentum sources and the geometry varies gradually and smoothly. Therefore, at least, from theoretical viewpoint, one can generalize the Rastall theory to include this assumption, i.e.,

\[ \nabla_{\mu} T^\nu_{\mu} = \nabla_\nu (\lambda R), \]

or equivalently

\[ \nabla^\nu (T_{\mu\nu} - g_{\mu\nu} \lambda R) = 0. \] (2)

Using now the Bianchi identity, \( \nabla^\nu G_{\mu\nu} = 0 \) we get the field equations as

\[ G_{\mu\nu} + \kappa \lambda g_{\mu\nu} R = \kappa T_{\mu\nu}, \] (3)

where \( \kappa \) is a constant and \( \lambda \) is the Rastall dimensionless parameter which is not generally constant. The case of \( \lambda = 0 \) corresponds to GR where the geometry and energy-momentum sources couple to each other in a minimal way.

III. SPHERICAL COLLAPSE

The line element for a spatially flat, homogeneous and isotropic Universe is given by

\[ ds^2 = -dt^2 + a(t)^2 \left[ dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right], \] (4)

where \( a(t) \) is the scale factor of the Universe. The energy-momentum source filling the Universe is assumed to be a perfect fluid with an isotropic EMT given by

\[ T^\nu_{\mu} = \text{diag} [\rho(t), p(t), p(t), p(t)], \] (5)

where \( \rho(t) \) is the energy density and \( p(t) \) is the isotropic pressure. The Friedmann equations can be put into the
form
\[ (12\kappa\lambda - 3)H^2 + 6\kappa\lambda H = -\kappa \rho, \]  
\[ (12\kappa\lambda - 3)H^2 + (6\kappa\lambda - 2)\dot{H} = \kappa \rho, \]
where \( H = \dot{a}/a \) and an over-dot denotes derivative with respect to cosmic time \( t \). Considering a dust fluid as the only constituent of the Universe, i.e., \( p = \rho_m = 0 \), the conservation equation leads to the following continuity equation in generalized Rastall gravity, as
\[ g(\lambda)\rho_m + \rho_m \left[ 3H - \dot{f}(\lambda) \right] = 0, \]
where
\[ g(\lambda) = \frac{3\kappa\lambda - 1}{4\kappa\lambda - 1}, \quad f(\lambda) = \frac{\kappa \lambda}{4\kappa\lambda - 1}. \]

We note that in GR limit where
\[ \lambda \to 0, \quad g(\lambda) \to 1, \quad f(\lambda) \to 0, \]
the usual continuity equation will be recovered. Let us now consider a spherically symmetric region of radius \( r \) filled with a dust cloud of homogeneous density \( \rho_m \). In the framework of SC model, this region is described by a top-hat profile and uniform density so that at time \( t \), \( \rho_m(t) = \rho_m(t) + \delta \rho_m \). In other words, this region initially experiences a small perturbation of the background fluid density, i.e., \( \delta \rho_m \) and is immersed within a homogeneous Universe with energy density \( \rho_m \). If \( \delta \rho_m > 0 \) the spherical region will finally collapse under its own gravitational weight, otherwise, it will expand faster than the average Hubble flow, generating thus, what is known as a void. In analogy with Eq. (5), the continuity equation for the spherical region can be written as
\[ g(\lambda_c)\dot{\rho}_m + \rho_m \left[ 3h - \dot{f}(\lambda_c) \right] = 0, \]
where, \( h = \dot{r}/r \) denotes the local expansion rate inside the spherical perturbed region. In general, the Rastall parameter can be different within the spherical region and outside of it, however, for the sake of simplicity assume that this parameter takes the same value the local and background regions, i.e., \( \lambda_c = \lambda \). A useful quantity through which the evolution of perturbations can be better investigate is the density contrast of a the fluid which is defined as
\[ \delta_m = \frac{\rho_m^c}{\rho_m} - 1 = \frac{\delta \rho_m}{\rho_m}. \]
Indeed this quantity measure the deviation of the local fluid density from the background density. In order to calculate the evolution of density contrast, we take the time derivative of Eq. (12) giving
\[ \dot{\delta}_m = \frac{3}{g}(H - h)(1 + \delta_m), \]  
where use has been made of conservation equations (3) and (11). Differentiating again with respect to time gives
\[ \ddot{\delta}_m = Qg\dot{\delta}_m + \frac{3}{g}(1 + \delta_m)(\dot{H} - h) + \frac{\dot{\delta}_m^2}{1 + \delta_m}, \]
where \( Q = d(g^{-1})/dt \). In order to estimate the second term of the above equation we consider the second Friedmann equation for the local and background regions, given as
\[ \frac{\ddot{a}}{a} = -\kappa(1 - 6\kappa\lambda)/(6(1 - 4\kappa\lambda))\rho_m, \quad \frac{\ddot{r}}{r} = -\kappa(1 - 6\kappa\lambda)/(6(1 - 4\kappa\lambda))\rho_m, \]
whence we have
\[ H - \dot{h} = \kappa(1 - 6\kappa\lambda)/(6(1 - 4\kappa\lambda))\rho_m \delta_m + \dot{h}^2 - H^2. \]
Substituting the above relation into equation (14) along with using the first derivative of density contrast, we finally get
\[ \ddot{\delta}_m + (2H - Qg)\dot{\delta}_m + \frac{(\dot{\delta}_m^2 + \kappa(1 + \delta_m)(1 - 6\kappa\lambda)/2g(1 - 4\kappa\lambda))\rho_m\delta_m}{\rho_m} = 0. \]  
We note that for \( \lambda = 0 \), the above equation reduces to its counterpart for a single dust fluid given in [30]. In order to study the evolution of matter perturbations we need to determine the functionality of the Rastall parameter. To this aim, we note that the behavior of matter density is subject to the ordinary conservation law, that is, \( \rho_m(a) = \rho_{m0}a^{-3} \), where \( \rho_{m0} \) is a constant of integration. Thus, from the conservation equation (8), we obtain the behavior of \( \lambda(a) \) parameter as [33]
\[ \lambda(a) = \frac{1}{\kappa(4 + 6\alpha a^{-3})}, \quad \alpha = C\rho_{m0}, \]
where \( C \) is another constant of integration. Indeed, for \( C = 0 \), we have \( \lambda = 1/4\kappa \) implying that the generalized Rastall parameter could act as a varying cosmological constant. We further note that using the above solution along with Eq. (10) we get the behavior of Hubble parameter as
\[ H(a) = \frac{\kappa\rho_0 a^{1/3}}{3a[\alpha + a^{3}]}^{1/2}, \]
whence we get the deceleration parameter as
\[ q = -1 - \frac{\dot{H}}{H^2} = -1 + \frac{1 + z dH(z)}{H(z)dz} = \frac{\alpha(1 + z)^3 - 2}{2(1 + \alpha(1 + z)^3)}, \]
where \( a = 1/(1 + z) \) and \( z \) being the redshift. To obtain the behavior of density contrast as a function of redshift we first transform the time derivatives in Eq. (17) to
derivatives with respect to the scale factor, bearing in mind the following relations

\[ \delta_m = a^2 H^2 \delta'' + 2a H^2 \left( \frac{2g - 3}{2g} \right) \delta_m' + g \delta_m = aH \delta_m', \quad \delta_m = aH \delta_m', \tag{21} \]

where a prime denotes derivative with respect to the scale factor and use has been made of the second Friedmann equation \[7\]. Substituting for the derivatives into Eq. (17) gives

\[ \delta'' + \frac{2ag' + 6g - 3}{2ag} \delta_m' - \frac{\kappa(1 + \delta_m)(1 - 6\kappa \lambda) \rho_m \delta_m}{2ga^2 H^2 (1 - 4\kappa \lambda)} = 0. \tag{22} \]

IV. SOLUTIONS IN LINEAR REGIME

The linear approximation of cosmological perturbations is valid for all scales during the radiation dominated era and for most scales during the matter dominated era up until very recently. Therefore, the initial stages of structure formation can be adequately investigated within the linear approximation. Hence, in order to extract some physical results from Eqs. (22) we proceed with neglecting the terms containing \(O(\delta^2)\). We then get

\[ \delta'' + \left( \frac{2ag' + 6g - 3}{2ag} \right) \delta_m' - \frac{3\beta \delta_m \Omega_m (1 - 6\beta \lambda)}{2a^2 g (1 - 4\beta \lambda)} = 0, \tag{23} \]

where we have set \(\kappa = \beta \kappa_G\) with \(\kappa_G = 8\pi G\) being the Einstein's gravitational constant. The matter density parameter is given by

\[ \Omega_m = \frac{\kappa G \rho_m}{3H^2} = \frac{\alpha^2 a^{-3}}{\beta^2 \Omega_0^m H_0^2 (1 + a a^{-3})}. \tag{24} \]

where use has been made of the solution for Hubble parameter \[33\] and \(\Omega_0^m\) and \(H_0\) are the present values of density and Hubble parameters, respectively. Now, in order to obtain analytical solutions we consider the above equation within the matter dominated epoch, i.e., \(z \approx 10^3\) where the matter density parameter can be approximated as \(\Omega_m \approx 1\). By changing the independent variable from scale factor to redshift, equation (23) can be recast into the following form

\[ (1 + z)^2 (1 + \alpha (1 + z)^3) \frac{d^2 \delta_m(z)}{dz^2} + \frac{1}{2} (1 + z) (-8 + \alpha (1 + z)^3) \frac{d \delta_m(z)}{dz} - \frac{3\beta}{2} (-2 + \alpha (1 + z)^3) \delta_m(z) = 0. \tag{25} \]

The above differential equation admits an exact solution given by

\[ \delta_m(z) = C_1 (1 + z)^{n_2} F_1 \left[ \ell_+, \ell_-, b_+, -\frac{1}{\alpha (1 + z)^3} \right] + C_2 (1 + z)^{n_2} F_1 \left[ s_+, s_-, b_-, -\frac{1}{\alpha (1 + z)^3} \right], \tag{26} \]

where

\[
\begin{align*}
n_\pm &= \frac{1}{4} \left( 1 \pm \sqrt{1 + 24 \beta} \right), \\
\ell_\pm &= \frac{3}{4} \pm \sqrt{25 - 12 \beta} + \frac{1}{12} \sqrt{1 + 24 \beta}, \\
s_\pm &= \frac{3}{4} \pm \sqrt{25 - 12 \beta} - \frac{1}{12} \sqrt{1 + 24 \beta}, \\
b_\pm &= 1 \pm \frac{1}{6} \sqrt{1 + 24 \beta},
\end{align*}
\]

and \(C_1\) and \(C_2\) are integration constants. To obtain these two constants we note that for large values of the redshift the \(\lambda\) parameter goes to zero where the GR limit is recovered. We therefore consider the adiabatic initial conditions for matter perturbation as

\[ \left. \frac{d \delta_m(z)}{dz} \right|_{z=z_i} = -\frac{\delta_m(z_i)}{1 + z_i}. \tag{28} \]

where \(\delta_m(z_i)\) is the initial value of density contrast at the onset of perturbations, i.e., at \(z = z_i\). In GR limit where \(\lambda \to 0\) and \(\beta \to 1\) Eq. (28) can be re-expressed as

\[ (1 + z)^2 \frac{d^2 \delta_m(z)}{dz^2} + \frac{1}{2} (1 + z) \frac{d \delta_m(z)}{dz} - \frac{3}{2} \delta_m(z) = 0, \tag{29} \]

for which the solution reads

\[ \delta_m(z) = \frac{1 + z_i}{1 + z} \delta_m(z_i). \tag{30} \]

As it is expected, matter perturbations grow as the redshift decreases, how ever, these perturbations have completely different behavior in the framework of generalized Rastall gravity. Figure (1) provides a better demonstration on this issue. The solid curve presents the behavior of perturbations according to Eq. (30). However, considering the solution (26), for \(\beta > 1\), as the dashed, dot-dashed and dotted curves show, the perturbations grow at larger rate in comparison to the GR case and for \(0 < \beta < 1\) these perturbations can proceed with a smaller rate of growth, see the gray curve. More interestingly, the perturbations admit an oscillatory behavior for \(\beta < 0\). Hence, during the evolution of the Universe, formation of structures can occur at the maximum value of the density contrast. Also, negative peaks imply the presence of voids in the matter distribution. Such a behavior occurs during a period of oscillation and at distinct redshifts. The frequency of oscillations depends crucially.
on $\beta$ parameter and the more the absolute value of this parameter the more the frequency of oscillations. We note that the curves in Fig. 1 have been plotted for the $1100 \lesssim z \lesssim 10$ and the solution (20) may not be applied to $z < 10$ till the present epoch or late time. We therefore proceed with considering the dynamical behavior of the density parameter (24) within Eq. (24) and investigate the possible outcomes. By doing so, Eq. (30) can be re-expressed as the following form

$$
(1 + z) \frac{d^2 \delta_m(z)}{dz^2} + \left( \frac{8 + \alpha (1 + z)^3}{2 (1 + \alpha (1 + z)^3)} \right) \frac{d\delta_m(z)}{dz} + \frac{3\alpha^2 (1 + z)^2 (-2 + \alpha (1 + z)^3) \delta_m(z)}{2 (1 + \alpha (1 + z)^3)^2 \Omega_m^0} = 0.
$$

The above differential equation does not admit an exact analytical solution hence we resort to numerical methods. Figure 2 shows the evolution of matter perturbations for redshifts $-1 \lesssim z \lesssim 10$. It is therefore seen that the perturbations (black curves) start growing from their initial values and reach a maximum value at which the rate of growth halts. This maximum occurs at a redshift, $z_{\text{max}}$. For $z < z_{\text{max}}$ the density contrast decreases and reaches a constant nonzero value at late time. As the behavior of deceleration parameter (red curves) shows, the Universe experiences a transition from decelerating, $z < z_{\text{tr}}$, to accelerating $z > z_{\text{tr}}$ phases, where $z_{\text{tr}}$ is the redshift at which the deceleration parameter vanishes. Comparing the two set of curves, we observe that $z_{\text{max}} < z_{\text{tr}}$. This means that as the Universe enters the accelerating phase, the rate of matter density contrast decreases since the accelerated expansion will produce a decrement in matter clustering. We note that the late time value of matter density contrast is greater than its initial value i.e., $\lim_{z_{\text{tr}} \to -z_0} \delta_m(z) > \delta_m(z_0)$. This means that, though the density contrast decays after transition redshift, we still see structure formation, even at late time evolution of the Universe. However, depending on the parameters $(\alpha, \beta)$, it is still possible for a more rapid decay of density contrast towards the formation of voids ($\delta_m < 0$), see the dotted curve. Another possible scenario is the oscillatory behavior of the density contrast which can be achieved by considering negative values of $\beta$ parameter. In this case, overdense and underdense regions within the Universe could form periodically and the redshift, $z_{\text{pr}}$, at which $\delta_m(z_{\text{pr}}) = 0$ depends crucially on the absolute value of $\beta$ parameter. In Fig. 3 we considered three different scenarios (blue curves) for evolution of matter perturbations. The solid curve shows that after a cycle of formation of overdense and underdense regions, the matter perturbation starts growing from an underdense region toward the overdense ones. Such an event occurs at the transition redshift for which $q(z_{\text{tr}}) = 0$, so that, $\delta_m(z) < 0$ for $z > z_{\text{tr}}$ and $\delta_m(z) > 0$ for $z < z_{\text{tr}}$. Hence we could have voids within the decelerated era and matter clustering in accelerating and late time eras. For the second scenario (dashed curve) we have overdense regions just before the transition redshift, i.e., $\delta_m(z) > 0$ for $z > z_{\text{tr}}$ and underdense regions after the transition is passed, i.e., $\delta_m(z) < 0$ for $z < z_{\text{tr}}$. Finally we could have underdense regions even before the transition redshift at which the universe enters an accelerated regime (dot-dashed curve). In this case the perturbations turn into the formation of voids at the redshift $z_1 > z_{\text{tr}}$ for which $\delta_m(z_1) = 0$. As the Universe evolves, the amplitude of perturbations grow in negative direction i.e., $\delta_m(z_{\text{tr}}) < 0$ and reaches a finite negative value at late times.
ate from the linear regime at critical redshift. In the framework of recent generalization of Rastall gravity, it is found that a dynamic coupling parameter could act as an origin for DE, proving then a suitable setting for current accelerated phase of the Universe expansion. In the present work, motivated by this idea, we studied evolution of a pressure-less matter perturbations

V. NON-LINEAR REGIME

In this section we study the evolution of matter density contrast during the non-linear regime of structure formation. We therefore consider Eq. (22) which in terms of redshift can be re-expressed as

\[ (1+z)^4 \frac{d^2 \delta_m}{dz^2} + 2(1+z)^3 \frac{d \delta_m}{dz} - \frac{3}{2} (1+z)^3 x(z) \frac{d^2 \delta_m}{dz^2} \]

\[- \frac{3 \alpha^2}{2 \beta \Omega_m^0 H_0^2} (1+z^5) y(z) \delta_m (1+\delta_m) \]

\[- \left[ 1 + \frac{1 + \alpha(1+z)^3}{3 \alpha(1+z)^3} \right] \left[ \frac{(1+z)^4}{1+\delta_m} \left( \frac{d \delta_m}{dz} \right)^2 \right] = 0, \] (32)

where

\[ x(z) = \frac{\alpha z^3 + 3 \alpha z^2 + 3 \alpha z + \alpha + 4}{\alpha z^3 + 3 \alpha z^2 + 3 \alpha z + \alpha + 1}, \]

\[ y(z) = \frac{-2 + \alpha(1+z)^3}{(1+\alpha(1+z)^3)^2}. \] (33)

In Fig. 4 we have plotted numerical solution (black curves) of Eq. (32) along with the results of linear regime (blue curves). It is seen that, perturbations grow as the Universe evolves and eventually the perturbations deviate from the linear regime at critical redshift \( z_c \) (red vertical lines), where the nonlinear regime starts to dominate the evolution of matter perturbations and gets arbitrary large values. We therefore observe that the collapse of matter occurs when a sufficient amount of density contrast is accumulated, i.e., when \( \delta_m^{NL} \rightarrow \infty \). The larger the values of \( \beta \) parameter the slower the growth of the matter perturbations and the lower the peak in density contrast of linear regime. Moreover, for larger values of \( \beta \) parameter, the critical redshift decreases, hence, it takes longer time for perturbations to accumulate the critical amount of matter in order to trigger the collapse. A useful quantity for describing the spherical collapse model is the critical density contrast, defined as, the value of linear density contrast \( \delta_m^L \) in the limit where the nonlinear one diverges. To compute this quantity, we can run a loop over different values of initial density contrast and then, performing numerical integration for each \( \delta_m(z_i) \) the critical redshift \( z_c \) along with the critical value of linear density contrast \( \delta_c = \delta_m^L(z = z_c) \) are obtained. Figure 5 shows the behavior of critical density contrast as a function of critical redshift. We have set the values of initial density contrast to lie within the interval \( 0.01 \leq \delta_m(z_i) \leq 3 \) for black curves and \( 0.001 \leq \delta_m(z_i) \leq 2 \) for gray curve. The corresponding values of \( z_c \) and \( \delta_c \) then lie within the following intervals

\[ 1.6414 \leq z_c \leq 7.7162, \quad 1.7733 \leq \delta_c \leq 0.0180, \] (34)

for black solid curve,

\[ 1.4905 \leq z_c \leq 7.6512, \quad 1.7671 \leq \delta_c \leq 0.0181, \] (35)

for black dot-dash curve,

\[ 1.2750 \leq z_c \leq 7.5585, \quad 1.7572 \leq \delta_c \leq 0.0181, \] (36)

for black dotted curve,

\[ 0.9538 \leq z_c \leq 7.4262, \quad 1.7416 \leq \delta_c \leq 0.0183, \] (37)

for black dashed curve and

\[ 0.6536 \leq z_c \leq 7.6432, \quad 1.7324 \leq \delta_c \leq 0.0021, \] (38)

for gray curve. It is therefore seen that, as we increase the initial value of density contrast, a smaller value for critical density contrast at higher redshifts is needed for the collapse to occur. Conversely, at lower redshifts, a smaller value of initial density contrast requires larger value for its corresponding density contrast in order that the collapse begins. We then realize that in the later stages of evolution of the Universe, more accumulation of matter within a halo is needed in order that the gravitational collapse commences and vice versa. We also note that the critical density for collapse and the critical redshift depend crucially on the initial value of density contrast and \( (\alpha, \beta) \) parameters.

VI. CONCLUDING REMARKS

In the framework of recent generalization of Rastall gravity, it is found that a dynamic coupling parameter could act as an origin for DE, proving then a suitable setting for current accelerated phase of the Universe expansion. In the present work, motivated by this idea, we studied evolution of a pressure-less matter perturbations
FIG. 4: Evolution of Linear (blue curves, $\delta_m^L$) and non-linear (black curves, $\delta_m^{NL}$) for $\alpha = 3/7$ and different values of $\beta$ parameter. We have set $\kappa_G = 1$, $\Omega_m^0 = 0.25$, $H_0 = 67$ and $\delta_m(z_i) = 0.01$. The same value for parameter $\beta$ has been considered for the blue curves.

FIG. 5: Behavior of critical linear density contrast against the collapse redshift, for $\alpha = 3/7$ (black curves) and $\alpha = 0.435$ gray curve. We have set $\kappa_G = 1$, $\Omega_m^0 = 0.25$ and $H_0 = 67$.

using spherical top-hat collapse model. The exact solutions in linear regime show that matter density contrast grows monotonically until reaching a maximum value at a certain redshift. It then decreases and reaches a finite positive or negative value depending on model parameters. The redshift at which the amplitude of perturbations ceases to increase is smaller than the transition redshift. Such a decrement in the amplitude of perturbations could be due to the accelerated expanding phase the Universe is experiencing after transition redshift. For some choice of model parameters the perturbations show oscillatory behavior so that we could have a period of overdense and underdense regions at different redshifts during the evolution of perturbations. The matter perturbations grow faster in non-linear regime in comparison to the linear one and at a critical redshift ($z_c$), non-linear density contrast starts detaching from linear one and diverges. Hence, the collapse of a massive sphere occurs, when the linearly evolved density contrast exceeds the threshold $\delta_c$. In order to obtain the behavior of threshold density contrast against critical redshift we performed numerical integration over the non-linear differential equation taking different initial values for density contrast. For each initial value $\delta_m(z_i)$, a point in ($\delta_c$, $z_c$) plane is obtained. The collection of these points is plotted in Fig. 5 where we see that the larger the value of critical redshift, the smaller the threshold value of density contrast is required for the collapse process to begin and vice versa. We also observe that, from expression 15, by decreasing the redshift, the coupling between matter and geometry increases and reaches a constant value at late times. As this coupling can play the role of DE, it can assist the formation of structures through boosting the rate of matter density contrast. However, the increasing rate of coupling comes to halt in the later Universe evolution and thus it cannot further assist structure formation, particularly when the Universe passes transition redshift. This behavior along with accelerated expansion phase leads to decay in matter density contrast. We therefore conclude that a running interaction between matter and geometry, represented by a varying coupling parameter in Rastall gravity, could affect the dynamics of matter perturbations and consequently formation of structures during the evolution of Universe.

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