QUASIDUALIZING MODULES

BETHANY KUBIK

Abstract. We introduce and study “quasidualizing” modules. An artinian $R$-module $T$ is quasidualizing if the homothety map $\hat{R} \to \text{Hom}_R(T, T)$ is an isomorphism and $\text{Ext}^i_R(T, T) = 0$ for each integer $i > 0$. Quasidualizing modules are associated to semidualizing modules via Matlis duality. We investigate the associations via Matlis duality between subclasses of the Auslander class and Bass class and subclasses of derived $T$-reflexive modules.

Introduction

Let $R$ be a commutative local noetherian ring with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. The $\mathfrak{m}$-adic completion of $R$ is denoted $\hat{R}$, the injective hull of $k$ is $E = E_R(k)$, and the Matlis duality functor is $(-)^\vee = \text{Hom}_R(-, E)$.

The motivation for this work comes from the study of semidualizing modules. Semidualizing modules were first introduced by Vasconcelos [8]. A finitely generated $R$-module $C$ is semidualizing if the homothety map $R \to \text{Hom}_R(C, C)$ is an isomorphism and $\text{Ext}^i_R(C, C) = 0$ for each integer $i > 0$. For example, $R$ is always a semidualizing $R$-module. Therefore duality with respect to $R$ is a special case of duality with respect to a semidualizing module, as is duality with respect to a dualizing $R$-module when $R$ has one. One the other hand, Matlis duality is not covered in this way. The goal of this paper is to remedy this by introducing and studying the “quasidualizing” modules: An artinian $R$-module $T$ is quasidualizing if the homothety map $\hat{R} \to \text{Hom}_R(T, T)$ is an isomorphism and $\text{Ext}^i_R(T, T) = 0$ for each integer $i > 0$; see Definition 1.14. For example, $E$ is always a quasidualizing module.

This paper is concerned with the properties of quasidualizing modules and how they compare with the properties of semidualizing modules. For instance, the next result gives a direct link between quasidualizing modules and semidualizing modules via Matlis duality; see Theorem 3.1.

Theorem A. If $R$ is complete, then the set of isomorphism classes of semidualizing $R$-modules is in bijection with the set of isomorphism classes of quasidualizing $R$-modules by Matlis duality.

Following the literature on semidualizing modules, we use quasidualizing modules to define other classes of modules. For instance, given an $R$-module $M$, we consider the class $G_M^{\text{full}}(R)$ of “derived $M$-reflexive $R$-modules” and their subclasses $G_M^{\text{noeth}}(R)$ and $G_M^{\text{artin}}(R)$ of noetherian modules and artinian modules respectively. We also consider subclasses of the Auslander class $A_M(R)$ and the Bass class $B_M(R)$.

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Section 1 for definitions. Some relations between these classes are listed in the next result which is proved in Section 3.

**Theorem B.** Assume $R$ is complete, and let $T$ be a quasidualizing $R$-module. Then we have the following inverse equivalences and equalities

(i) $B^{\noeth}(R) \rightsquigarrow \mathcal{A}^{\noeth}(R) = \mathcal{A}^{\noeth}(R)$;

(ii) $B^{\noeth}(R) \rightsquigarrow \mathcal{A}^{\noeth}(R) = \mathcal{A}^{\noeth}(R)$;

(iii) $B^{\noeth}(R) \rightsquigarrow \mathcal{A}^{\noeth}(R) = \mathcal{A}^{\noeth}(R)$; and

(iv) $B^{\noeth}(R) \rightsquigarrow \mathcal{A}^{\noeth}(R) = \mathcal{A}^{\noeth}(R)$.

As a consequence of the previous result, we conclude that the classes $\mathcal{G}^{\noeth}(R)$ and $\mathcal{A}^{\noeth}(R)$ are substantially different. For instance, as we observe next $\mathcal{G}^{\noeth}(R)$ satisfies the two-of-three condition, while the class $\mathcal{G}^{\noeth}(R)$ does not; see Theorem 3.13.

**Theorem C.** Assume that $R$ is complete, and let $T$ be a quasidualizing $R$-module. Then $\mathcal{G}^{\noeth}(R)$ satisfies the two-of-three condition, that is, given an exact sequence of $R$-module homomorphisms $0 \rightarrow L_1 \rightarrow L_2 \rightarrow L_3 \rightarrow 0$ if any two of the modules are in $\mathcal{G}^{\noeth}(R)$, then so is the third.

In Section 1 we provide some definitions and background material. Section 2 describes properties related to quasidualizing modules, and Section 3 describes the relations between the different classes of modules using Matlis duality as well as Theorem C.

1. **Background material**

**Definition 1.1.** We say that an $R$-module $L$ is Matlis reflexive if the natural biduality map $\delta^R_L : L \rightarrow L^\vee^\vee$, given by $l \mapsto [\phi \mapsto \phi(l)]$.

**Fact 1.2.** Let $L$ be an $R$-module. The natural biduality map $\delta_L$ is injective; see [6, Theorem 18.6(i)]. If $L$ is Matlis reflexive, then $L^\vee$ is Matlis reflexive.

**Fact 1.3.** Assume $R$ is complete and let $L$ be an $R$-module. If $L$ is artinian, then $L^\vee$ is noetherian. If $L$ is noetherian, then $L^\vee$ is artinian. Since $R$ is complete, both artinian modules and noetherian modules are Matlis reflexive; see [6, Theorem 18.6(v)].

**Lemma 1.4.** Let $L$ and $L'$ be $R$-modules such that $L$ is Matlis reflexive. Then for all $i \geq 0$ we have the isomorphisms

$\Ext^i_R(L', L) \cong \Ext^i_R(L^\vee, L^\vee)$ and $\Ext^i_R(L', L^\vee) \cong \Ext^i_R(L, L^\vee)$.

**Proof.** For the first isomorphism, since $L$ is Matlis reflexive, by definition the map $\Ext^i_R(L', \delta_L) : \Ext^i_R(L', L) \rightarrow \Ext^i_R(L', \Hom_R(L^\vee, E))$
is an isomorphism. A manifestation of Hom-tensor adjointness yields the following isomorphisms
\[ \text{Ext}_R^i(L', \text{Hom}_R(L^\vee, E)) \cong \text{Ext}_R^i(L' \otimes_R L^\vee, E) \cong \text{Ext}_R^i(L^\vee, L'^\vee). \]
The composition of these maps provides us with the isomorphism \( \text{Ext}_R^i(L', L) \cong \text{Ext}_R^i(L^\vee, L'^\vee). \)

For the second isomorphism, the fact that \( L \) is Matlis reflexive explains the second step in the following sequence \( \text{Ext}_R^i(L', L^\vee) \cong \text{Ext}_R^i(L^\vee, L'^\vee) \cong \text{Ext}_R^i(L, L'^\vee). \)
The first step follows from the first isomorphism since \( L^\vee \) is an isomorphism. A manifestation of Hom-tensor adjointness yields the following

Fact 1.5. Assume \( R \) is complete and let \( A \) and \( A' \) be artinian \( R \)-modules. Then \( \text{Hom}_R(A, A') \) is noetherian. This can be deduced using [5, Theorem 2.11].

Fact 1.6. Let \( L \) be an \( R \)-module. Then \( L \) is artinian over \( R \) if and only if it is artinian over \( \hat{R} \). See [5, Lemma 1.14].

Lemma 1.7. Assume \( R \) is artinian and let \( L \) be an \( R \)-module. Then the following are equivalent

(i) \( L \) is noetherian over \( R \);
(ii) \( L \) is finitely generated over \( R \); and
(iii) \( L \) is artinian.

Proof. The equivalence (i) \( \iff \) (ii) is standard; see [1, Propositions 6.2 and 6.5].

For the implication (ii) \( \implies \) (iii), assume that \( L \) is finitely generated over \( R \). Then there exists an \( n \in \mathbb{N} \) and a surjective map \( R^n \to L \) so that we have \( L \cong \text{Im}(\phi) \cong R^n / \text{Ker}(\phi) \). Since \( R \) is artinian, \( R^n \) is artinian. Thus \( L \) is artinian because the quotient of an artinian module is artinian; see [1, Proposition 6.3].

For the implication (iii) \( \implies \) (i), assume that \( L \) is artinian. Then there exists an \( n \in \mathbb{N} \) such that \( L \to E^n \); see [2, Theorem 3.4.3]. Since \( R \) is artinian, we have \( R^\vee \cong E \) is noetherian over \( \hat{R} \) by Fact 1.3 where the isomorphism follows from [6, Theorem 18.6 (iv)]). Hence we have that \( E^n \) is noetherian over \( \hat{R} = R \) since \( R \) is artinian. Since any submodule of a noetherian module is noetherian, we conclude that \( L \) is noetherian over \( R \); see [1, Proposition 6.3].

Lemma 1.8. Assume \( R \) is complete and let \( A \) be an artinian \( R \)-module. Then there exists an injective resolution \( I \) of \( A \) such that for each \( i \geq 0 \) we have \( I_i \cong E^{b_i} \) for some \( b_i \in \mathbb{N} \). Furthermore, \( I^\vee \) is a free resolution of \( A^\vee \).

Proof. Since \( A \) is artinian, we have the map \( A \to E^{b_0} \) for some \( b_0 \geq 1 \); see [2, Theorem 3.4.3]. Because the finite direct sum of artinian modules is artinian, \( E^{b_0} \) is artinian and we have \( E^{b_0} / A \to E^{b_1} \) for some \( b_1 \geq 0 \). Recursively we can construct an injective resolution of \( A \) such that for each \( i \geq 0 \) we have \( I_i \cong E^{b_i} \) for some \( b_i \in \mathbb{N} \).

Next we show that \( I^\vee \) is a free resolution of \( A^\vee \). The fact that \( I_i \cong E^{b_i} \) explains the first step in the following sequence
\[ I^\vee_i = \text{Hom}_R(I_i, E) \cong \text{Hom}_R(E^{b_i}, E) \cong \text{Hom}_R(E, E)^{b_i} \cong \hat{R}^{b_i} \cong R^{b_i}. \]
The second step is standard. The third step is from [6, Theorem 18.6(iv)], and the last step follows from the assumption that \( R \) is complete. The desired conclusion follows from the fact that \( (-)^\vee \) is exact.
Definition 1.9. Let $L$, $L'$ and $L''$ be $R$-modules. The Hom-evaluation morphism
\[ \theta_{L,L'',L'} : L \otimes_R \text{Hom}_R(L',L'') \to \text{Hom}_R(\text{Hom}_R(L,L'),L'') \]
is given by $a \otimes \phi \mapsto [\beta \mapsto \phi(\beta(a))]$.

Fact 1.10. The Hom-evaluation morphism $\theta_{L,L'',L'}$ is an isomorphism if the modules satisfy one of the following conditions:
(a) $L$ is finitely generated and $L''$ is injective; or
(b) $L$ is finitely generated and projective.
See [4, Lemma 1.6] and [7, Lemma 3.55].

Definition 1.11. An $R$-module $C$ is semidualizing if it satisfies the following
(i) $C$ is finitely generated;
(ii) the homothety morphism $\chi_C^R : R \to \text{Hom}_R(C,C)$, defined by $r \mapsto [c \mapsto rc]$, is an isomorphism; and
(iii) one has $\text{Ext}_R^i(C,C) = 0$ for all $i > 0$.

Remark 1.12. Let $\mathcal{S}_0(R)$ denote the set of isomorphism classes of semidualizing $R$-modules.

Example 1.13. The ring $R$ is always semidualizing.

Definition 1.14. An $R$-module $T$ is quasidualizing if it satisfies the following
(i) $T$ is artinian;
(ii) the homothety morphism $\chi_T^R : \hat{R} \to \text{Hom}_R(T,T)$, defined by $r \mapsto [t \mapsto rt]$, is an isomorphism; and
(iii) one has $\text{Ext}_R^i(T,T) = 0$ for all $i > 0$.

Remark 1.15. The homothety morphism $\chi_T^R$ is well defined since $T$ is artinian implying by Fact 1.6 that $T$ is an $\hat{R}$-module.

Remark 1.16. Let $\mathcal{Q}_0(R)$ denote the set of isomorphism classes of quasidualizing modules.

Example 1.17. The injective hull of the residue field $E$ is always quasidualizing. See [2, Theorem 3.4.1] and [8, Theorem 18.6(iv)] for conditions (i) and (ii) of Definition 1.14. Since $E$ is injective by definition, we have $\text{Ext}_R^i(E,E) = 0$ for all $i > 0$ satisfying the last condition.

Definition 1.18. Let $M$ be an $R$-module. Then an $R$-module $L$ is derived $M$-reflexive if
(i) the natural biduality map $\delta_L^M : L \to \text{Hom}_R(\text{Hom}_R(L,M),M)$ defined by $l \mapsto [\phi \mapsto \phi(l)]$ is an isomorphism; and
(ii) one has $\text{Ext}_R^i(L,M) = 0 = \text{Ext}_R^i(\text{Hom}_R(L,M),M)$ for all $i > 0$.

We write $\mathcal{G}_M^{\text{full}}(R)$ to denote the class of all derived $M$-reflexive $R$-modules, $\mathcal{G}_M^{\text{artin}}(R)$ to denote the class of all Matlis reflexive derived $M$-reflexive $R$-modules, $\mathcal{G}_M^{\text{noeth}}(R)$ to denote the class of all noetherian derived $M$-reflexive $R$-modules.

Remark 1.19. When $M = C$ is a semidualizing $R$-module, the class $\mathcal{G}_M^{\text{noeth}}(R)$ is the class of totally $C$-reflexive $R$-modules, sometimes denoted $\mathcal{G}_C(R)$. 
Definition 1.20. Let $L$ and $L'$ be $R$-modules. We say that $L$ is in the Bass class $B_{L'}(R)$ with respect to $L'$ if it satisfies the following:

(i) the natural evaluation homomorphism $\xi_{L'}^L : \text{Hom}_R(L', L) \otimes_R L' \to L$, defined by $\phi \otimes l \mapsto \phi(l)$, is an isomorphism; and

(ii) one has $\text{Ext}^i_R(L', L) = 0 = \text{Tor}^i_R(L', L)$ for all $i > 0$.

We write $B_{L'}^\text{mat}(R)$ to denote the class of all Matlis reflexive $R$-modules in the Bass class with respect to $L'$.

We write $B_{L'}^\text{artin}(R)$ to denote the class of all artinian $R$-modules in the Bass class with respect to $L'$.

Definition 1.21. Let $L$ and $L'$ be $R$-modules. We say that $L$ is in the Auslander class $A_{L'}(R)$ with respect to $L'$ if it satisfies the following:

(i) the natural homomorphism $\gamma_{L'}^L : L \to \text{Hom}_R(L', L' \otimes_R L)$, which is defined by $l \mapsto [l' \mapsto l' \otimes l]$, is an isomorphism; and

(ii) one has $\text{Tor}^i_R(L', L) = 0 = \text{Ext}^i_R(L', L)$ for all $i > 0$.

We write $A_{L'}^\text{mat}(R)$ to denote the class of all Matlis reflexive $R$-modules in the Auslander class with respect to $L'$. We write $A_{L'}^\text{artin}(R)$ to denote the class of all artinian $R$-modules in the Auslander class with respect to $L'$.

2. Quasidualizing Modules

We begin with a few preliminary results pertaining to quasidualizing modules.

Proposition 2.1. Let $T$ be an $R$-module. Then $T$ is a quasidualizing $R$-module if and only if $T$ is a quasidualizing $\hat{R}$-module.

Proof. We need to check the equivalence of three conditions. For the first condition, $T$ is an artinian $R$-module if and only if $T$ is an artinian $\hat{R}$-module by Fact 1.6. For the rest of the proof we assume without loss of generality that $T$ is artinian.

For the second condition, we have the equality $\text{Hom}_R(T, T) = \text{Hom}_{\hat{R}}(T, T)$ from the fact that $T$ is $\mathfrak{m}$-torsion and [3, Lemma 1.5(a)]. This explains the equality in the following commutative diagram.

$$
\begin{array}{ccc}
\hat{R} & \xrightarrow{\chi_T^R} & \text{Hom}_R(T, T) \\
\downarrow \cong & & \downarrow = \\
\hat{R} & \xrightarrow{\chi_T^\hat{R}} & \text{Hom}_{\hat{R}}(T, T)
\end{array}
$$

Since $\hat{R} \cong \hat{R}$, we have $\chi_T^\hat{R}$ is an isomorphism if and only if $\chi_T^R$ is an isomorphism.

For the last condition, Lemma 1.8 implies that there exists an injective resolution $I$ of $T$ such that for each $i \geq 0$ we have $I_i \cong E^{b_i}$ for some $b_i \in \mathbb{N}$. For all $i \geq 0$, the modules $T$ and $I_i$ are artinian and hence $\mathfrak{m}$-torsion. By [3, Lemma 1.5(a)], we have the equality $\text{Hom}_{\hat{R}}(T, I_i) = \text{Hom}_R(T, I_i)$ and $I$ is an injective resolution of $T$ over $\hat{R}$. This explains the first and second steps in the next display:

$$
\text{Ext}^i_{\hat{R}}(T, T) \cong \text{H}_{-i}(\text{Hom}_{\hat{R}}(T, I_i)) \cong \text{H}_{-i}(\text{Hom}_R(T, I_i)) \cong \text{Ext}^i_R(T, T).
$$

The third step is by definition. Thus we have $\text{Ext}^i_{\hat{R}}(T, T) = 0$ for all $i > 0$ if and only if $\text{Ext}^i_R(T, T) = 0$ for all $i > 0$. \qed
Proposition 2.2. The following conditions are equivalent:

(i) \( E \) is a semidualizing \( R \)-module;
(ii) \( R \) is a quasidualizing \( R \)-module;
(iii) \( E \) is a noetherian \( R \)-module;
(iv) \( R \) is an artinian ring;
(v) \( \Omega_0(R) = \mathcal{S}_0(R) \); and
(vi) \( \Omega_0(R) \cap \mathcal{S}_0(R) \neq 0 \).

Proof. (iii) \( \iff \) (iv) By [6, Theorem 18.6 (ii)] we have \( \text{len}_R(R) = \text{len}_R(E) \), where \( \text{len}_R(L) \) denotes the length of an \( R \)-module \( L \). Since \( R \) is noetherian by assumption, we have \( R \) is artinian if and only if \( R \) has finite length if and only if \( R^\vee = E \) has finite length (by the equalities above), if and only if \( E \) is noetherian over \( R \) (since \( E \) is artinian; see [2, Theorem 3.4.1]). That is, \( R \) is artinian if and only if \( E \) is noetherian over \( R \).

(i) \( \implies \) (iii) If \( E \) is a semidualizing \( R \)-module, then \( E \) is noetherian over \( R \) by definition.

(iv) \( \implies \) (i) Assume that \( R \) is artinian. Then \( E \) is finitely generated by the equivalence (iii) \( \iff \) (iv). We have \( R \cong \hat{R} \) since \( R \) is artinian, and \( \hat{R} \cong \text{Hom}_R(E, E) \) by [6, Theorem 18.6 (iv)] explaining the unspecified isomorphisms in the following commutative diagram.

\[
\begin{array}{ccc}
R & \xrightarrow{\chi_R^E} & \text{Hom}_R(E, E) \\
\cong & & \cong \\
\hat{R} & \downarrow & \\
& \chi_{\hat{R}}^E &
\end{array}
\]

Hence we conclude that the homothety morphism \( \chi_R^E \) is an isomorphism. Since \( E \) is injective, we have that \( \text{Ext}_R^i(E, E) = 0 \) for all \( i > 0 \). Thus \( E \) is a semidualizing \( R \)-module.

(iv) \( \implies \) (v) Assume that \( R \) is artinian, and let \( L \) be an \( R \)-module. We show that \( L \) is a semidualizing module if and only if \( L \) is a quasidualizing module. We need to check the equivalence of three conditions. For the first condition, \( L \) is finitely generated if and only if \( L \) is artinian by Lemma [1.7]. For the second condition, the fact that \( R \) is artinian implies that \( \hat{R} \cong R \). This explains the unlabeled isomorphism in the following commutative diagram.

\[
\begin{array}{ccc}
R & \xrightarrow{\cong} & \hat{R} \\
\chi_R^L & & \chi_{\hat{R}}^L \\
\downarrow & & \downarrow \\
\text{Hom}_R(L, L) & & 
\end{array}
\]

Thus the map \( \chi_L^R \) is an isomorphism if and only if the map \( \chi_{\hat{R}}^L \) is an isomorphism. The Ext vanishing conditions are equivalent by definition.

For the implication (iv) \( \implies \) (i), assume that \( \Omega_0(R) = \mathcal{S}_0(R) \). The \( R \)-module \( R \) is always semidualizing. Then by assumption it is also a quasidualizing \( R \)-module.

The implication (iii) \( \implies \) (iv) is evident since \( R \) is an artinian ring if and only if it is an artinian \( R \)-module. For the implication (iii) \( \implies \) (vi), if \( R \) is a quasidualizing \( R \)-module, then the intersection \( \Omega_0(R) \cap \mathcal{S}_0(R) \) is nonempty since \( R \) is also a semidualizing \( R \)-module.
For the implication (vi) \(\implies\) (ii), assume that the intersection \(\mathcal{Q}_0(R) \cap \mathcal{S}_0(R)\) is nonempty. Let \(L \in \mathcal{Q}_0(R) \cap \mathcal{S}_0(R)\). Then \(L\) is artinian and noetherian, so it has finite length. Since \(L\) is artinian, it is \(m\)-torsion and by [5] Fact 1.2(b) we have \(\text{Supp}_R(L) \subseteq \{m\}\). Since \(L\) is a semidualizing \(R\)-module, the map \(R \to \text{Hom}_R(L, L)\) is an isomorphism so we have \(\text{Ann}_R(L) \subseteq \text{Ann}_R(R) = 0\). This explains the second step in the following sequence

\[
\text{Supp}_R(L) = V(\text{Ann}_R(L)) = V(0) = \text{Spec}(R).
\]

Thus \(\text{Spec}(R) = \text{Supp}_R(L) \subseteq \{m\} \subseteq \text{Spec}(R)\) and we conclude that \(\text{Spec}(R) = \{m\}\). Thus [1] Theorem 8.5 implies that \(R\) is artinian. \(\square\)

3. Classes of Modules and Matlis Duality

This section explores the connections between the class of quasidualizing \(R\)-modules and the class of semidualizing \(R\)-modules as well as connections between different subclasses of \(\mathcal{A}_M(R), \mathcal{B}_M(R), \text{ and } \mathcal{G}_M^{\text{full}}(R)\). The instrument used to detect these connections is Matlis Duality.

**Theorem 3.1.** Assume that \(R\) is complete. Then the maps \(\mathcal{S}_0(R) \xrightarrow{(\cdot)^\vee} \mathcal{Q}_0(R)\) are inverse bijections.

**Proof.** Let \(C \in \mathcal{S}_0(R)\). We show that \(C^\vee \in \mathcal{Q}_0(R)\). Fact [1,3] implies that \(C^\vee\) is artinian. In the following commutative diagram, the unspecified isomorphisms are from Hom-tensor adjointness and the commutativity of tensor product

\[
\begin{array}{ccc}
R & \xrightarrow{\chi_C^{R^\vee}} & \text{Hom}_R(C^\vee, \text{Hom}_R(C, E)) \\
\downarrow{\chi_{R^\vee}} & & \downarrow{\cong} \\
\text{Hom}_R(C, C) & \cong & \text{Hom}_R(C^\vee \otimes_R C, E) \\
\cong & & \cong \\
\text{Hom}_R(C, \text{Hom}_R(C^\vee, E)) & \cong & \text{Hom}_R(C \otimes_R C^\vee, E).
\end{array}
\]

Since \(C \in \mathcal{S}_0(R)\), it follows that \(\chi_C^{R^\vee}\) is an isomorphism. Fact [1,3] implies that the map \(\delta_C^{E_R}\), and by extension the map \(\text{Hom}_R(C, \delta_C^{E_R})\), is an isomorphism. Hence we can conclude from the diagram that \(\chi_C^{R^\vee}\) is an isomorphism.

For the last condition, Lemma [1,4] explains the first step in the following sequence

\[
\text{Ext}_R^1(C^\vee, C^\vee) \cong \text{Ext}_R^1(C, C) = 0.
\]

The second step follows from the fact that \(C\) is a semidualizing module. Thus \(C^\vee\) is a quasidualizing module.

A similar argument shows that given a quasidualizing \(R\)-module \(T\), the module \(T^\vee\) is semidualizing. Fact [1,3] implies that \(C \cong C^\vee\) and \(T \cong T^\vee\), so that the given maps \(\mathcal{S}_0(R) \xrightarrow{(\cdot)^\vee} \mathcal{Q}_0(R)\) and \(\mathcal{Q}_0(R) \xrightarrow{(\cdot)^\vee} \mathcal{S}_0(R)\) are inverse equivalences. \(\square\)

**Example 3.2.** Assume that \(R\) is Cohen-Macaulay and complete and admits a dualizing module \(D\). The fact that \(D\) is dualizing means that \(D\) is semidualizing and has finite injective dimension. Therefore, by Theorem 3.1 we conclude that \(D^\vee\) is quasidualizing.
Proposition 3.3. Assume that $R$ is complete and let $T$ be a quasidualizing $R$-module. Then the maps $\mathcal{B}_T^\text{mr}(R) \xrightarrow{(-)^\vee} \mathcal{G}_T^\text{mr}(R)$ are inverse bijections.

Proof. Let $M$ be a Matlis reflexive $R$-module. We show that if $M \in \mathcal{B}_T^\text{mr}(R)$ then $M^\vee \in \mathcal{G}_T^\text{mr}(R)$. Fact 1.2 implies that $M^\vee$ is Matlis reflexive. There are three remaining conditions to check.

First we show that $\text{Ext}_R^i(M^\vee, T) = 0$ for all $i > 0$. Since $T$ is artinian and $R$ is complete, Fact 3.3 implies that $T$ is Matlis reflexive, so we have

$$\text{(3.3.1)} \quad \text{Ext}_R^i(M^\vee, T) \cong \text{Ext}_T^i(T^\vee, M).$$

by Lemma 1.3. We have $\text{Ext}_R^i(T^\vee, M) = 0$ for all $i > 0$ since $M \in \mathcal{B}_T^\text{mr}(R)$. Thus we conclude $\text{Ext}_R^i(M^\vee, T) = 0$ for all $i > 0$.

Next we show that the map $\delta_M^T$ is an isomorphism. The fact that $M \in \mathcal{B}_T^\text{mr}(R)$ implies the map $\xi_M^T$ is an isomorphism. Therefore the map $\text{Hom}_R(\xi_M^T, E)$ in the following commutative diagram is an isomorphism

$$\begin{array}{ccc}
M^\vee & \xrightarrow{\text{Hom}_R(\xi_M^T, E)} & \text{Hom}_R(\text{Hom}_R(T^\vee, M) \otimes_R T^\vee, E) \\
\downarrow{\delta_M^T} & & \downarrow{=} \\
\text{Hom}_R(\text{Hom}_R(M^\vee, T), T) & \xrightarrow{=} & \text{Hom}_R(\text{Hom}_R(T^\vee, M), T).
\end{array}$$

The unspecified isomorphisms are from Hom-tensor adjointness and the isomorphism (3.3.1). Hence we conclude from the diagram that $\delta_M^T$ is an isomorphism.

For the last condition, let $I$ be an injective resolution of $T$ such that for each $i \geq 0$ we have $I_i \cong E^{b_i}$ for some $b_i \in \mathbb{N}$. Lemma 1.8 implies that $I^\vee$ is a free resolution of $T^\vee$. This explains steps (2) and (6) in the following sequence

$$\text{Ext}_R^i(\text{Hom}_R(M^\vee, T), T) \xrightarrow{(1)} \text{Ext}_R^i(\text{Hom}_R(T^\vee, M), T)$$

$$\xrightarrow{(2)} \text{H}_{-i}(\text{Hom}_R(\text{Hom}_R(T^\vee, M), I))$$

$$\xrightarrow{(3)} \text{H}_{-i}(\text{Hom}_R(\text{Hom}_R(T^\vee, M), I^\vee))$$

$$\xrightarrow{(4)} \text{H}_{-i}(\text{Hom}_R(\text{Hom}_R(T^\vee, M) \otimes_R I^\vee, E))$$

$$\xrightarrow{(5)} \text{Hom}_R(\text{H}_i(I^\vee \otimes_R \text{Hom}_R(T^\vee, M)), E)$$

$$\xrightarrow{(6)} \text{Hom}_R(\text{Tor}_i^R(T^\vee, \text{Hom}_R(T^\vee, M)), E).$$

Step (1) follows from the isomorphism (3.3.1). Step (3) follows from the fact that any finite direct sum of artinian modules is artinian; thus $I_j$ is artinian for all $j$ and we can apply Fact 1.3. Step (4) follows from Hom-tensor adjointness, and step (5) follows from the fact that $E$ is injective and homology commutes with exact functors. Since $M \in \mathcal{B}_T^\text{mr}(R)$, we have $\text{Tor}_i^R(M, \text{Hom}_R(T^\vee, M)) = 0$ for all $i > 0$. Hence we conclude that

$$\text{Ext}_R^i(\text{Hom}_R(M^\vee, T), T) \cong \text{Hom}_R(\text{Tor}_i^R(M, \text{Hom}_R(T^\vee, M)), E) = 0$$

for all $i > 0$. 

Given an $R$-module $M' \in \mathcal{G}_T^{noeth}(R)$, the argument to show that $M' \in \mathcal{B}_T^{noeth}(R)$ is similar. Since $M$ and $M'$ are Matlis reflexive, that is $M \cong M^{\vee \vee}$ and $M' \cong M'^{\vee \vee}$, we conclude that the maps $\mathcal{B}_T^{noeth}(R) \xrightarrow{(\cdot)^{\vee}} \mathcal{G}_T^{noeth}(R)$ and $\mathcal{B}_T^{mr}(R) \xrightarrow{(\cdot)^{\vee}} \mathcal{G}_T^{mr}(R)$ are inverse equivalences.

**Corollary 3.4.** Assume that $R$ is complete and let $T$ be a quasidualizing $R$-module. Then the following maps are inverse bijections

\[
\mathcal{B}_T^{noeth}(R) \xrightarrow{(\cdot)^{\vee}} \mathcal{G}_T^{artin}(R) \quad \text{and} \quad \mathcal{B}_T^{artin}(R) \xrightarrow{(\cdot)^{\vee}} \mathcal{G}_T^{noeth}(R).
\]

**Proof.** Fact 1.3 implies that if $N$ is a noetherian $R$-module, then $N^\vee$ is an artinian $R$-module and $N \cong N^{\vee \vee}$. Furthermore, if $A$ is an artinian $R$-module, then $A^\vee$ is a noetherian $R$-module and $A \cong A^{\vee \vee}$. Together with Proposition 3.3 this implies that the maps $\mathcal{B}_T^{noeth}(R) \xrightarrow{(\cdot)^{\vee}} \mathcal{G}_T^{artin}(R)$ are inverse bijections. The proof for $\mathcal{B}_T^{artin}(R) \xrightarrow{(\cdot)^{\vee}} \mathcal{G}_T^{noeth}(R)$ is similar.

**Proposition 3.5.** Assume that $R$ is complete and let $T$ be a quasidualizing $R$-module. Then the maps $\mathcal{B}_T^{mr}(R) \xrightarrow{(\cdot)^{\vee}} \mathcal{G}_T^{mr}(R)$ are inverse bijections.

**Proof.** Let $M$ be a Matlis reflexive $R$-module. We show that if $M \in \mathcal{G}_T^{mr}(R)$, then $M^\vee \in \mathcal{B}_T^{mr}(R)$. First we show that the map $\delta_M^T$ is an isomorphism. The fact that $M$ is Matlis reflexive implies that the map $\delta_M^T$ in the following commutative diagram is an isomorphism

\[
\begin{array}{ccc}
M & \xrightarrow{\delta_M^T} & M^{\vee \vee} \\
\downarrow & & \downarrow \cong \\
\text{Hom}_R(\text{Hom}_R(M, T^\vee), T^{\vee \vee}) & \cong & \text{Hom}_R(\text{Hom}_R(T, M^\vee) \otimes_R T, E) \\
\end{array}
\]

The unspecified isomorphisms are from Hom-tensor adjointness and Lemma 1.4.

Since $M \in \mathcal{G}_T^{mr}(R)$, we have that the map $\delta_M^T$ is an isomorphism. Hence $(\delta_M^T)^{\vee}$ is an isomorphism. Since $E$ is faithfully injective, this implies that $\xi_{M^\vee}^T$ is an isomorphism.

Next we show that $\text{Ext}_R^i(T, M^\vee) = 0$ for all $i > 0$. Since $M$ is Matlis reflexive, Lemma 1.4 explains the first step in the following sequence $\text{Ext}_R^i(T, M^\vee) \cong \text{Ext}_R^i(M, T^\vee) = 0$. The second step follows from the fact that $M \in \mathcal{G}_T^{mr}(R)$. 

\[
\text{Ext}_R^i(T, M^\vee) = 0.
\]

Since $M$ and $M'$ are Matlis reflexive, that is $M \cong M^{\vee \vee}$ and $M' \cong M'^{\vee \vee}$, we conclude that the maps $\mathcal{B}_T^{noeth}(R) \xrightarrow{(\cdot)^{\vee}} \mathcal{G}_T^{noeth}(R)$ and $\mathcal{B}_T^{mr}(R) \xrightarrow{(\cdot)^{\vee}} \mathcal{G}_T^{mr}(R)$ are inverse equivalences.
Lastly, we show that $\text{Tor}^R_i(T, \text{Hom}_R(T, M^{\vee})) = 0$ for all $i > 0$. The commutativity of tensor product explains the first step in the following sequence

$$\text{Tor}^R_i(T, \text{Hom}_R(T, M^{\vee}))^\vee \cong \text{Tor}^R_i(\text{Hom}_R(T, M^{\vee}), T)^\vee$$

$$\cong \text{Ext}^i_R(\text{Hom}_R(T, M^{\vee}), T^{\vee})$$

$$\cong \text{Ext}^i_R(\text{Hom}_R(M, T^{\vee}), T^{\vee})$$

$$= 0.$$  

The second step follows from [5, Remark 1.9] and the third step follows from Lemma 1.3. The last step follows from the fact that $M \in G_{T^{\vee}}^m(R)$. Given an $R$-module $M' \in B_{T}(R)$, the argument to show that $M^{\vee} \in G_{T^{\vee}}^m(R)$ is similar but easier. Since $M$ and $M'$ are Matlis reflexive, we conclude that the maps $B_{T}(R) \xrightarrow{(-)^\vee} G_{T^{\vee}}^m(R)$ and $G_{T^{\vee}}^m(R) \xrightarrow{(-)^\vee} B_{T}(R)$ are inverse equivalences. $\square$

**Corollary 3.6.** Assume that $R$ is complete and let $T$ be a quasidualizing $R$-module. Then the following maps are inverse bijections

$$B_{T}^{\text{noeth}}(R) \xrightarrow{(-)^\vee} G_{T^{\vee}}^m(R)$$

and

$$B_{T}^{\text{artin}}(R) \xrightarrow{(-)^\vee} G_{T^{\vee}}^m(R).$$

The next proposition establishes the relationship between a subclass of the Auslander class and a subclass of the derived reflexive modules.

**Proposition 3.7.** If $R$ is complete and $T$ is a quasidualizing $R$-module, then

$$G_{T^{\vee}}^m(R) = A_{T}^m(R).$$

**Proof.** Let $M$ be a Matlis reflexive $R$-module. We show that $M$ satisfies the defining conditions of $G_{T^{\vee}}^m(R)$ if and only if $M$ satisfies the defining conditions of $A_{T}^m(R)$. For the isomorphisms, consider the following commutative diagram

$$\begin{array}{ccc}
M & \xrightarrow{\delta_M^{\vee}} & \text{Hom}_R(\text{Hom}_R(M, T^{\vee}), T^{\vee}) \\
\gamma_M^{\vee} & \downarrow & \cong \\
\text{Hom}_R(T, T \otimes_R M) & \equiv & \text{Hom}_R(\text{Hom}_R(T, T \otimes_R M), (T \otimes_R M)^{\vee}) \\
\cong & \downarrow & \equiv \\
\text{Hom}_R(T, \text{Hom}_R((T \otimes_R M)^{\vee}, E)) & \equiv & \text{Hom}_R(T, \text{Hom}_R(M, T^{\vee}), E))
\end{array}$$

The unspecified homomorphisms are Hom-tensor adjointness. The module $T \otimes_R M$ is artinian by [5, Lemma 1.19 and Theorem 3.1]. Fact 1.3 implies that the map $\delta_T^{E \otimes_R M}$, and hence the map $\text{Hom}_R(T, \delta_T^{E \otimes_R M})$, is an isomorphism. Therefore the map $\gamma_M^{\vee}$ is an isomorphism if and only if the map $\delta_M^{T^{\vee}}$ is an isomorphism.

Next we show that for all $i > 0$ we have $\text{Ext}^i_R(M, T^{\vee}) = 0$ if and only if $\text{Tor}^R_i(M, T) = 0$. By [5, Remark 1.9], we have $\text{Ext}^i_R(M, T^{\vee}) \cong \text{Tor}^R_i(M, T)^{\vee}$. Because the Matlis dual of a module is zero if and only if the module is zero, we conclude that $\text{Ext}^i_R(M, T^{\vee}) = 0$ if and only if $\text{Tor}^R_i(M, T) = 0$ for all $i > 0$.

Next we show that for all $i > 0$ we have $\text{Ext}^i_R(\text{Hom}_R(M, T^{\vee}), T^{\vee}) = 0$ if and only if $\text{Ext}^i_R(T, M \otimes_R T) = 0$. Hom-tensor adjointness explains the first step in the
following sequence
\[ \text{Ext}_R^i(Hom_R(M, T^\vee), T^\vee) \cong \text{Ext}_R^i((M \otimes_R T)^\vee, T^\vee) \]
\[ \cong \text{Ext}_R^i(T^\vee, (M \otimes_R T)^\vee) \]
\[ \cong \text{Ext}_R^i(T, M \otimes_R T). \]

The second step follows from Lemma 1.4 and the fact that \( T \) is artinian and thus Matlis reflexive. The third step follows from the fact that \( T \) and \( M \otimes_R T \) are artinian and hence Matlis reflexive; see [5, Corollary 3.9].

**Corollary 3.8.** Assume that \( R \) is complete and let \( T \) be a quasidualizing \( R \)-module. Then \( G_{T^\vee}^{\text{noeth}}(R) = A_{T^\vee}^{\text{noeth}}(R) \) and \( G_{T^\vee}^{\text{artin}}(R) = A_{T^\vee}^{\text{artin}}(R) \).

**Proposition 3.9.** If \( R \) is complete and \( T \) is a quasidualizing \( R \)-module, then \( G_{T^\vee}^{\text{mr}}(R) = A_{T^\vee}^{\text{mr}}(R) \).

**Proof.** Let \( M \) be a Matlis reflexive \( R \)-module. We show that \( M \) satisfies the defining conditions of \( G_{T^\vee}^{\text{mr}}(R) \) if and only if \( M \) satisfies the defining conditions of \( A_{T^\vee}^{\text{mr}}(R) \).

For the isomorphisms, consider the following commutative diagram

\[
\begin{array}{ccc}
M & \xrightarrow{\delta^T_M} & \text{Hom}_R(M, T) \\
\downarrow \gamma_M & & \downarrow \text{Hom}_R(M, T) \\
\text{Hom}_R(T^\vee, T^\vee \otimes_R M) & \cong & \text{Hom}_R(T^\vee, \delta_E^T \otimes_M) \\
\downarrow \text{Hom}_R(T^\vee, \delta_E^T \otimes_M) & & \downarrow \text{Hom}_R(M, T) \\
\text{Hom}_R(T^\vee, (T^\vee \otimes_R M)^{\vee\vee}) & \cong & \text{Hom}_R(Hom_R(M, T), \delta_E^T) \\
\downarrow \cong & & \downarrow \cong \\
\text{Hom}_R(Hom_R(M, T^\vee), T^\vee) & \cong & \text{Hom}_R(Hom_R(M \delta_E^T, T^\vee), T^\vee)
\end{array}
\]

where the unlabeled isomorphisms are Hom-tensor adjointness and Hom-swap. Since \( T \) is artinian and hence Matlis reflexive, both the right hand map and the bottom map are isomorphisms. The module \( T^\vee \otimes_R M \) is Matlis reflexive by [5, Corollary 3.6]. Thus the map \( \delta_E^T \otimes_M \) and hence the map \( \text{Hom}_R(T^\vee, \delta_E^T \otimes_M) \) is an isomorphism. Therefore the map \( \gamma_M^T \) is an isomorphism if and only if the map \( \delta_M^T \) is an isomorphism.

Next we show that for all \( i > 0 \) we have \( \text{Ext}_R^i(M, T) = 0 \) if and only if \( \text{Tor}_i^R(T^\vee, M) = 0 \). The fact that \( T \) is artinian and hence Matlis reflexive explains the first step in the following sequence

\[ \text{Ext}_R^i(M, T) \cong \text{Ext}_R^i(M, T^\vee) \cong \text{Tor}_i^R(M, T^\vee)^\vee \cong \text{Tor}_i^R(T^\vee, M)^\vee. \]

The second step follows from [5, Remark 1.9] and the last step follows from the commutativity of the tensor product. Because the Matlis dual of a module is zero if and only if the module is zero, we conclude that \( \text{Ext}_R^i(M, T) = 0 \) if and only if \( \text{Tor}_i^R(T^\vee, M) = 0 \) for all \( i > 0 \).
Next we show that for all \( i > 0 \) we have \( \text{Ext}^i_R(\text{Hom}_R(M, T), T) = 0 \) if and only if \( \text{Ext}_R(T^\vee, T^\vee \otimes_R M) = 0 \). The fact that \( T \) is artinian and hence Matlis reflexive explains the first and third steps in the following sequence

\[
\text{Ext}^i_R(\text{Hom}_R(M, T), T) 
\cong \text{Ext}^i_R(\text{Hom}_R(M, T^\vee \otimes E), T)
\cong \text{Ext}^i_R(\text{Hom}_R(M \otimes T^\vee, E), T^\vee)
\cong \text{Ext}^i_R(T^\vee, M \otimes T^\vee).
\]

The second step follows from Hom-tensor adjointness and the last step follows from Lemma [14].

**Corollary 3.10.** Assume that \( R \) is complete and let \( T \) be a quasidualizing \( R \)-module. Then \( G^\text{noeth}_T(R) = A^\text{noeth}_T(R) \) and \( G^\text{artin}_T(R) = A^\text{artin}_T(R) \).

The above results show that the classes \( G^\text{mr}_T(R), G^\text{artin}_T(R), \) and \( G^\text{noeth}_T(R) \) do not exhibit some of the same properties as the class \( G^\text{noeth}_C(R) \), where \( C \) is semidualizing. For instance, we consider the following property. We say a class of \( R \)-modules \( \mathcal{C} \) satisfies the two-of-three condition if given an exact sequence of \( R \)-module homomorphisms \( 0 \to L_1 \to L_2 \to L_3 \to 0 \), when any two of the modules are in \( \mathcal{C} \), so is the third. The two-of-three condition holds for some classes of modules and not for others. For example, the class of noetherian modules and the class of artinian modules both satisfy the two-of-three condition. On the other hand, the class \( G^\text{noeth}_C(R) \) does not satisfy the two-of-three condition when \( C \) is semidualizing.

In contrast, the next result shows that the class \( G^\text{full}_T(R) \) satisfies the two-of-three condition when the ring is complete. This is somewhat surprising since the definitions of \( G^\text{noeth}_C(R) \) and \( G^\text{full}_T(R) \) are so similar. First we need a lemma. In the language of [3], is says that quasidualizing implies faithfully quasidualizing.

**Lemma 3.11.** Let \( L \) and \( T \) be \( R \)-modules such that \( T \) is quasidualizing. If one has \( \text{Hom}_R(L, T) = 0 \), then \( L = 0 \).

**Proof.** Assume that \( \text{Hom}_R(L, T) = 0 \).

Case 1: \( T = E \). Because \( \text{Hom}_R(L, E) = 0 \), we have \( L^\vee \otimes = 0 \). Since the map \( \delta^E_L : L \to L^\vee \) is injective by Fact [12], we conclude that \( L = 0 \).

Case 2: \( R \) is complete. Then \( T \) is Matlis reflexive and we have \( 0 = \text{Hom}_R(L, T) \cong \text{Hom}_R(T^\vee, L^\vee) \) from Lemma [13]. Since \( T^\vee \) is semidualizing by Proposition [31], we have \( L^\vee = 0 \) by [3] Proposition 3.6]. By Case 1, we conclude that \( L = 0 \).

Case 3: the general case. The first step in the following sequence is by assumption

\[
0 = \text{Hom}_R(L, T) \cong \text{Hom}_R(L, \text{Hom}_R(T, T)) \cong \text{Hom}_R(\hat{R} \otimes_R L, T).
\]

The second step follows from the fact that \( T \) is artinian and hence has an \( \hat{R} \) structure and the third step is from Hom-tensor adjointness. Since \( T \) is a quasidualizing \( \hat{R} \)-module, we can apply Case 2 to conclude that \( \hat{R} \otimes_R L = 0 \). Then \( L = 0 \) because \( \hat{R} \) is faithfully flat over \( R \). □

**Question 3.12.** Does a version of Lemma [3,11] hold for \( T \otimes_R \) as in [3]?

**Theorem 3.13.** Assume that \( R \) is complete and let \( T \) be a quasidualizing \( R \)-module. Then \( G^\text{full}_T(R) \) satisfies the two-of-three condition.
Proof. Let
(3.13.1) \[ 0 \to L_1 \xrightarrow{f} L_2 \xrightarrow{g} L_3 \to 0 \]
be an exact sequence of \(R\)-module homomorphisms and let \((-)^T = \text{Hom}_R(-, T)\). There are two conditions to check and three cases. We will deal with the case when \(L_1, L_2 \in \mathcal{G}_T^{\text{full}}(R)\). The case where \(L_2, L_3 \in \mathcal{G}_T^{\text{full}}(R)\) is similar. The case where \(L_1, L_3 \in \mathcal{G}_T^{\text{full}}(R)\) is also similar but easier.

Assume that \(L_1, L_2 \in \mathcal{G}_T^{\text{full}}(R)\). Then we have \(\text{Ext}^i_R(L_1, T) = 0 = \text{Ext}^i_R(L_2, T)\) for all \(i > 0\). The following portion of the long exact sequence in \(\text{Ext}^i_R(-, T)\) associated to the short exact sequence (3.13.1) shows that Ext
\[ \text{Ext}^i_R(L_1, T) \to \text{Ext}^i_R(L_2, T) \to \text{Ext}^i_R(L_1, T) \to \cdots \]
shows that Ext
\[ \text{Ext}^i_R(L_3, T) = 0 \] for all \(i > 1\). For the case where \(i = 1\), we apply \((-)^T\) to the following portion of the long exact sequence
\[ 0 \to (L_1)^T \to (L_2)^T \to (L_1)^T \to \text{Ext}^1_R(L_3, T) \to 0 \]
to obtain exactness in the top row of the following commutative diagram
\[ \begin{array}{cccccc}
0 & \to & (\text{Ext}^1_R(L_3, T))^T & \xrightarrow{L_3^T} & (L_1)^T & \xrightarrow{L_2^T} & (L_2)^T \\
0 & \xrightarrow{L_1^T} & L_1 & \xrightarrow{f} & L_2 & \xrightarrow{g} & L_3
\end{array} \]
Since \(f\) is an injective map, the diagram shows that \(L_2^T\) is an injective map. Hence we have \((\text{Ext}^1_R(L_3, T))^T = 0\). From Lemma 3.11 we conclude that \(\text{Ext}^1_R(L_3, T) = 0\).

Next we show that \(\text{Ext}^1_R(\text{Hom}_R(L_3, T), T) = 0\) for all \(i > 0\). From the argument above we have the exact sequence
(3.13.3) \[ 0 \to (L_3)^T \to (L_2)^T \to (L_1)^T \to 0. \]
In a similar, but easier, manner than above, the long exact sequence in \(\text{Ext}^i_R(-, T)\) shows that if \(L_1, L_2 \in \mathcal{G}_T^{\text{full}}(R)\), then \(\text{Ext}^i_R(\text{Hom}_R(L_3, T), T) = 0\) for all \(i > 0\).

Lastly, we show that the map \(L_3^T\) is an isomorphism. From the short exact sequence (3.13.1) and as a consequence of the above argument together with the short exact sequence (3.13.3), we obtain the following commutative diagram with exact rows
\[ \begin{array}{cccccc}
0 & \to & (L_1)^T & \xrightarrow{L_1^T} & L_1 & \xrightarrow{f} & L_2 & \xrightarrow{g} & L_3 & \to 0 \\
0 & \xrightarrow{L_3^T} & (L_1)^T & \xrightarrow{L_2^T} & (L_2)^T & \xrightarrow{L_3^T} & (L_3)^T & \to 0
\end{array} \]
Since \(L_1, L_2 \in \mathcal{G}_T^{\text{full}}(R)\), the maps \(L_1^T\) and \(L_2^T\) are isomorphisms. By the Snake Lemma, we conclude that \(L_3^T\) is an isomorphism. \(\square\)

Corollary 3.14. Assume that \(R\) is complete and let \(T\) be a quasidualizing \(R\)-module. Then \(\mathcal{G}_T^{\text{art}}(R) = \mathcal{A}_T^{\text{art}}(R)\), \(\mathcal{G}_T^{\text{noeth}}(R) = \mathcal{A}_T^{\text{noeth}}(R)\), and \(\mathcal{G}_T^{mrf}(R)\) satisfy the two-of-three condition.

Proof. Apply Theorem 3.13 and Corollary 3.10. \(\square\)
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Bethany Kubik, Department of Mathematical Sciences, 601 Thayer Road #222, West Point, NY 10996 USA
E-mail address: bethany.kubik@usma.edu
URL: http://www.dean.usma.edu/departments/math/people/kubik/