THE \((q,t)\)-CARTAN MATRIX SPECIALIZED AT \(q = 1\)
AND ITS APPLICATIONS

MASAKI KASHIWARA AND SE-JIN OH

Abstract. The \((q,t)\)-Cartan matrix specialized at \(t = 1\), usually called the quantum Cartan matrix, has deep connections with (i) the representation theory of its untwisted quantum affine algebra, and (ii) quantum unipotent coordinate algebra, root system and quantum cluster algebra of skew-symmetric type. In this paper, we study the \((q,t)\)-Cartan matrix specialized at \(q = 1\), called the \(t\)-quantized Cartan matrix, and investigate the relations with (ii’) its corresponding unipotent quantum coordinate algebra, root system and quantum cluster algebra of skew-symmetric type.

Contents

1. Introduction 1
2. Cartan datum and Dynkin quiver 7
3. Quivers 12
4. The \((q,t)\)-Cartan matrix specialized at \(q = 1\) and AR-quivers 22
5. Quantum torus associated with a Dynkin diagram 31
6. Unipotent quantum coordinate algebra and the quantum tori isomorphism 37
7. Compatible pairs 44
Appendix A. \(\tilde{d}_{i,j}(t)\) for \(E_7\) and \(E_8\) 47
References 49

1. Introduction

1.1. \((q,t)\)-Cartan matrix \(C(q,t)\). Let \(\mathfrak{g}\) be a finite-dimensional simple Lie algebra, \(C = (c_{i,j})_{i,j\in I}\) its Cartan matrix, \(\Delta\) its Dynkin Diagram, \(\Phi^+\) its set of positive roots, and \(U'_\nu(\hat{\mathfrak{g}})\) the untwisted quantum affine algebra associated to \(\mathfrak{g}\). When \(\Delta\) is non simply-laced, we sometimes use \(\Delta\) to distinguish with \(\Delta\) of an arbitrary type. In [10], Frenkel-Reshetikhin...
introduced a two-parameter deformation $C(q, t)$ of $C$ to define the deformation of \( \mathcal{W} \)-algebra \( \mathcal{W}_{q,t}(\mathfrak{g}) \) of $\mathfrak{g}$. We call the $C(q, t)$ the $(q, t)$-\textit{Cartan matrix} of $\mathfrak{g}$. Then (i) it is proved in [11] that the limit $t \to 1$ of $\mathcal{W}_{q,t}(\mathfrak{g})$ recovers the commutative Grothendieck ring of the category of finite-dimensional modules $\mathcal{C}_\mathfrak{g}$ over $U'_q(\mathfrak{g})$, and (ii) it is expected in [7] that the limit $q \to \exp(\pi i/r)$ of $\mathcal{W}_{q,t}(\mathfrak{g})$ recovers the one of $\mathcal{C}_l(\mathfrak{g})$ over the Langlands dual $U'_q(L(\mathfrak{g}))$ of $U'_q(\mathfrak{g})$, where $r$ denotes the order of non-trivial Dynkin diagram automorphism $\sigma$ of simply-laced type Dynkin diagram, yielding the Dynkin diagram $\Delta$ of $\mathfrak{g}$:

\[
(1.1) \quad K \left( \mathcal{C}_\mathfrak{g} \right) \xrightarrow{\text{interpolation}} K \left( \mathcal{W}_{q,t}(\mathfrak{g}) \right) \xrightarrow{\text{q-exp}(\pi i/r)} K \left( \mathcal{C}_l(\mathfrak{g}) \right)
\]

In this paper, we study the specialization of $C(q, t)$ at $q = 1$, which is not well-investigated, while the other specialization of $C(q, t)$ at $t = 1$ is intensively investigated since late 1990's, in various view points.

1.2. Quantum Cartan matrix $C(q)$. The specialization of $C(q, t)$ at $t = 1$, denoted by $C(q)$ and called the quantum Cartan matrix, and its inverse $\tilde{C}(q) := C(q)^{-1}$ are known as the controller of the representation theory of $U'_q(\mathfrak{g})$, and has been extensively studied. To name a few, (i) we can read the denominator formulas of the normalized $R$-matrices between fundamental modules from the Laurent expansions of $\tilde{C}(q)_{i,j}$'s at $q = 0$, which determine whether a given tensor product of fundamental modules is simple or not ([4, 28, 43, 12, 46, 44]), (ii) $C(q)$ and $\tilde{C}(q)$ are used as key ingredients for constructing the quantum Grothendieck ring $K_*(\mathcal{C}_\mathfrak{g})$ of $\mathcal{C}_\mathfrak{g}$ ([42, 49, 20, 22]) and the $q$-character theory for $\mathcal{C}_\mathfrak{g}$ ([10, 11, 9]), (iii) the quantum Grothendieck ring $K_*(\mathcal{C}_0^\mathfrak{g})$ is isomorphic to a certain $\mathbb{Z}[t^{\pm 1}]$-algebra inside the quantum torus $\mathcal{Y}(\mathfrak{g})$, which is constructed from $\tilde{C}(q)$ (see [23, 34]). Here $\mathcal{C}_0^\mathfrak{g}$ is the skeleton subcategory of $\mathcal{C}_\mathfrak{g}$, also referred to as the Hernandez-Leclerc category.

Interestingly enough, the computation of $\tilde{C}(q)$ of type $\mathfrak{g}$ is related to the simple Lie algebra $\mathfrak{g}$ of simply-laced type as below:

\[
(\mathfrak{g}, \mathfrak{g}) = (A_n, A_n), \ (D_n, D_n), \ (E_{6,7,8}, E_{6,7,8}), \ (B_n, A_{2n-1}), \ (C_n, D_{n+1}), \ (F_4, E_6), \ (G_2, D_4).
\]

It is proved in [22] that, when $\mathfrak{g} = \mathfrak{g}$ is of simply-laced type, Laurent expansions of $\tilde{C}(q)_{i,j}$'s at $q = 0$ of type $\mathfrak{g}$ can be computed by using any Dynkin quiver $Q$ of the same type and the symmetric bilinear form $(\ , \ )$ on the root lattice $Q_\mathfrak{g}$.

As a generalization of [22], it is proved in [15] that Laurent expansions of $\tilde{C}(q)_{i,j}$'s at $q = 0$ of any finite type $\mathfrak{g}$ can be computed by using any $Q$-datum associated to $\mathfrak{g}$ and $(\ , \ )$ on $Q_\mathfrak{g}$. We remark here that Laurent expansions of $\tilde{C}(q)_{i,j}$'s at $q = 0$ exhibit the remarkable periodicity and the positivity related to the dual Coxeter number $h^\vee$ of $\mathfrak{g}$.

The notion of $Q$-datum is introduced in [15] and can be understood as a generalization of Dynkin quiver. The constituents and properties of $Q$-datum associated to $\mathfrak{g}$ are briefly summarized as follows:
(a) A Q-datum \( \mathcal{Q} = (\Delta, \sigma, \xi) \) consists of (1) the Dynkin diagram \( \Delta_\mathfrak{g} \) of type \( \mathfrak{g} \), (2) the Dynkin diagram automorphism \( \sigma \) yielding \( \Delta_\mathfrak{g} \) as an orbit of \( \Delta_\mathfrak{g} \) via \( \sigma \), and (3) a height function \( \xi \) on \( \Delta \) satisfying certain axioms.

(b) For each Q-datum \( \mathcal{Q} = (\Delta, \sigma, \xi) \), there exists a unique (generalized)-Coxeter element \( \tau_\mathcal{Q} \) in \( \mathcal{W}_\mathfrak{g} \times \langle \sigma \rangle \) satisfying certain properties.

(c) When \( \mathfrak{g} = \hat{\mathfrak{g}} \) and hence \( \sigma = \text{id} \), the notion of Q-datum \( \mathcal{Q} = (\Delta, \sigma, \xi) \) is equivalent to the notion of Dynkin quiver \( Q = (\Delta, \xi) \) of type \( ADE \).

When \( \mathfrak{g} = \hat{\mathfrak{g}} \) is of simply-laced type, we can consider the path algebra \( \mathbb{C}Q \) of a Dynkin quiver \( Q = (\Delta_\mathfrak{g}, \xi) \). Then it is well-known that the Auslander-Reiten(AR) quiver \( \Gamma_Q \) of \( \mathbb{C}Q \) realizes the convex partial order \( \prec_Q \) on \( \Phi_\mathfrak{g}^+ \) coming from the \( Q \)-adapted commutation class \( [Q] \) of the longest element \( w_0 \) of \( \mathcal{W}_\mathfrak{g} \). On the other hand, \( \Gamma_Q \) can be understood as a *heart* of the AR-quiver \( \widehat{\Delta} = (\widehat{\Delta}_0, \widehat{\Delta}_1) \) of the derived category \( D^b(\text{Rep}(\mathbb{C}Q)) \), in which \( \widehat{\Delta} \) is referred to as the repetition quiver. In aspect of combinatorics, (i) the set of vertices \( \widehat{\Delta}_0 \) is in one to one correspondence with \( \Phi_\mathfrak{g}^+ \times \mathbb{Z} \), and (ii) \( \widehat{\Delta} \) satisfies \( \widehat{\Delta} \)-additive property as equations in the root lattice \( Q_\mathfrak{g} \) (see (3.10)). In [22], Hernandez-Leclerc defined the heart subcategory \( \mathcal{E}_Q \) of \( \mathcal{E}_\mathfrak{g} = \mathcal{E}_{\hat{\mathfrak{g}}} \) by using the coordinate system via \( I \times \mathbb{Z} \) for vertices in \( \widehat{\Delta} \) and \( \Gamma_Q \) (see (3.7) and (3.11)). Then they proved that the quantum Grothendieck ring \( K_t(\mathcal{E}_Q) \) of \( \mathcal{E}_Q \) is isomorphic to the unipotent quantum coordinate algebra \( A_\nu(n) \) of the quantum group \( U_\nu(\mathfrak{g}) \). Here \( K_t(\mathcal{E}_Q) \) is contained in a certain sub-torus \( \mathcal{Y}_{t,Q} \) of \( \mathcal{Y}_{t,\mathfrak{g}} \) determined by the coordinate system of \( \Gamma_Q \).

By understanding \( \Gamma_Q \) as the Hasse quiver of \( \prec_Q \), for each Q-datum \( \mathcal{Q} = (\Delta_\mathfrak{g}, \sigma \neq \text{id}, \xi) \) associated to non simply-laced \( \mathfrak{g} \) (\( \neq \hat{\mathfrak{g}} \)), (i) the (combinatorial) AR-quiver \( \Gamma_\mathcal{Q} \), the repetition quiver \( \widehat{\Delta}^\sigma \) and the heart subcategory \( \mathcal{E}_\mathcal{Q} \) of \( \mathcal{E}_\mathfrak{g} \) are defined in [15, 46] by developing \( \mathcal{Q} \)-adapted commutation class \( [\mathcal{Q}] \) of \( w_0 \subset \mathcal{W}_\mathfrak{g} \). (ii) it is proved that \( \widehat{\Delta}^\sigma \) satisfies the \( \widehat{\Delta}^\sigma \)-additive property and the set of vertices of \( \widehat{\Delta}^\sigma \) is also in one to one correspondence with \( \Phi_\mathfrak{g}^+ \times \mathbb{Z} \), and (iii) it is proved in [35, 24, 13] that the quantum Grothendieck ring \( K_t(\mathcal{E}_\mathcal{Q}) \subset \mathcal{Y}_{t,\mathcal{Q}} \) is also isomorphic to \( A_\nu(n) \) of \( U_\nu(\mathfrak{g}) \). Furthermore, it is proved in [13] that the quantum Grothendieck ring \( K_t(\mathcal{E}_\mathfrak{g}^0) \) of \( \mathcal{E}_\mathfrak{g}^0 \) and the one \( K_t(\mathcal{E}_{\hat{\mathfrak{g}}}^0) \) of \( \mathcal{E}_{\hat{\mathfrak{g}}}^0 \) are isomorphic as an algebra, preserving simple \((q,t)\)-characters. Along the proofs in [13], the isomorphism between the quantum torus \( \mathcal{Y}_{t,\mathcal{Q}} \) and \( \mathcal{T}_{t,\mathcal{Q}} \) containing \( A_\nu(n) \) played important roles.

A *cluster algebra* \( \mathcal{A} \), introduced by Fomin-Zelevinsky in [6] is a commutative \( \mathbb{Z} \)-algebra contained in the torus \( \mathbb{Z}[\tilde{X}^\pm_1] \mid k \in K \). By Berenstein and Zelevinsky [2], the notion of cluster algebras is generalized to the non-commutative one, *quantum cluster algebra* \( \mathcal{A}_v \), which is contained in the quantum torus \( \mathbb{Z}[v^{1/2}][\tilde{X}^\pm_1] \mid k \in K \). From their introductions, numerous connections and applications have been discovered in various fields of mathematics including the representation theory of quantum affine algebras.

Note that the Grothendieck ring \( K(\mathcal{E}_\mathfrak{g}^0) \) of \( \mathcal{E}_\mathfrak{g}^0 \) has a \( \Lambda \)-cluster algebra structure (see [32] for its definition). Moreover, for any Q-datum \( \mathcal{Q} \), it is proved in [34, 13] (see also [23]) that the heart subcategory \( \mathcal{E}_\mathcal{Q} \) of \( \mathcal{E}_\mathfrak{g} \) provides a monoidal categorification of the quantum cluster algebra \( K_t(\mathcal{E}_\mathcal{Q}) \simeq A_\nu(n) \).
On the other hand, it is proved in [16, 17] that the unipotent quantum coordinate algebra $A_\nu(n)$ of the quantum group $U_\nu(g)$ ($g$ need not be the same as $\hat{g}$) has a quantum cluster algebra structure of skew-symmetrizable type, whose initial quantum seed arises from the combinatorics of $W_\hat{g}$ and the symmetric bilinear form on the root lattice $Q_\hat{g}$.

1.3. Brief summary and comments on $C(q)$. To sum up, we can conclude that each $Q$-datum $\mathcal{D}$ associated to $g$ having $\Delta$ as a simply-laced type $g$ and its related objects, such as $\tilde{C}(q)$, $\tau_\sigma$, $\Gamma_\sigma$, and $\hat{\Delta}_\sigma$, encode key information of the representation theories of $U'_\nu(\hat{g})$ and $A_\nu(n)$ of the quantum group $U_\nu(g)$. However, for $g$ with $g \neq \hat{g}$, $A_\nu(n)$ of $U_\nu(g)$ looks not related to the representation theory of quantum affine algebras nor known $Q$-data and their related objects, as far as the authors understand at this moment.

1.4. $t$-quantized Cartan matrix $C(t)$. In the definition of $Q$-datum, $\Delta$ of type BCFG does not appear as a constituent, and $\Delta$ do have only the trivial Dynkin diagram automorphism id. In this paper, we mainly consider a Dynkin quiver $Q = (\Delta, \xi)$ as a generalization of the notion of $Q$-datum (see Section 2.4).

From this paper, we start to explore the representation theories which are controlled by the specialization $C(t) := C(1, t)$ and its inverse $\tilde{C}(t)$. We call $C(t)$ $t$-quantized Cartan matrix. As a starting point, we investigate the relations among Dynkin quivers $Q$, $A_\nu(n)$ and $\tilde{C}(t)$ in this paper. As $\tilde{C}(q)$ and objects induced from $\tilde{C}(q)$ were used for representation theories of $U'_q(\hat{g})$ and $A_\nu(n)$, we construct various mathematical objects from $\tilde{C}(t)$ and Dynkin quivers $Q$, and study the newly introduced objects and their applications.

The main achievements of this paper can be summarized as follows:

(A) We introduce quivers $\Gamma_Q$ and $\hat{\Delta}$ for Dynkin quivers $Q$ of any finite type $g$, study their properties and prove that the Laurent expansions of $\tilde{C}(t)_{i,j}$’s at $t = 0$ can be computable by any of them.

(B) Using $\tilde{C}(t)_{i,j}$’s, we construct the quantum torus $X_q$ and prove that its quantum-commutation relation is controlled by the root system of $g$.

(C) For each Dynkin quiver $Q$, we define the quantum sub-torus $X_{q,Q}$ of $X_q$ and show that it is isomorphic to the quantum torus $T_{\nu,[Q]}$ containing $A_\nu(n)$.

(D) For any $Q$-adapted sequence $\tilde{w}$, we prove that the pair $(\Lambda^{\tilde{w}}, B^{\tilde{w}})$ is compatible, which recovers the compatible pair in [22] as a particular case, by using the $\hat{\Delta}$-additive property and $\tilde{C}(t)_{i,j}$’s.

(A) With the notion of Dynkin quiver $Q = (\Delta, \xi)$, including BCFG-types, we can define the notion of $Q$-adaptedness for reduced expressions of elements in $W_\Delta$. Then we can see that the set of all $Q$-adapted reduced expressions of $w_0 \in W_\Delta$ forms a commutation class $[Q]$. In particular, we can see that $[Q]$ of type $BCFG$ enjoys the almost same properties of $[Q]$ in $W$ of type $ADE$ (Theorem 2.5).

Similarly to the combinatorial feature of $\hat{\Delta}$ for ADE-type, we define the repetition quiver $\hat{\Delta} = (\hat{\Delta}_0, \hat{\Delta}_1)$ for every Dynkin quiver $Q = (\Delta, \xi)$ and prove that

(i) $\hat{\Delta}_0$ is in one to one correspondence with $\Phi_\Delta^+ \times \mathbb{Z}$ via the map $\phi_\Delta$ (Theorem 3.10),
(ii) it satisfies the $\hat{\Delta}$-additive property (Theorem 3.14).

Then we define the heart subquiver $\Gamma_Q$ of $\hat{\Delta}$, also called the AR-quiver of $Q$, and prove that it realizes the convex partial order $\prec_{|Q|$ on $\Phi_+^\Delta$ (Proposition 3.13).

Recall that the Laurent expansion of $\tilde{C}(q)_{i,j}$ at $q = 0$ of type $\mathfrak{g}$ can be computed by using an arbitrary $\mathcal{D} = (\Delta, \xi)$ of the same type. Furthermore,

Theorem A (Theorem 4.7, Corollary 4.8, Theorem 4.12, 4.15). The Laurent expansion of $\tilde{C}(t)_{i,j}$ of type $\mathfrak{g}$ at $t = 0$ can be computed by using any Dynkin quiver $Q = (\Delta, \xi)$ or $(\hat{\Delta}, \xi)$ of the same type. Furthermore,

(a) the periodicity and the positivity for $\tilde{C}(t)$ hold,
(b) we obtain the closed formulae for $\tilde{C}(t)_{i,j}$ for all Cartan types.

We remark here that in [14], the inverse of the $C(q,t)$ itself is described in terms of bigraded modules over the generalized preprojective algebras in the sense of Geiß-Leclerc-Schröer [26]. In particular, (a) in Theorem A can be also obtained from [14, Corollary 4.14] by specialization at $q = 1$ (see § 4.4.4).

(B) As we mentioned above, $\tilde{C}(q)$ had used for the constructions of the quantum torus, $(q,t)$-character theory and the quantum Grothendieck ring $K_t(\mathcal{E}_0^0)$ of $\mathcal{E}_0^0$. In [9, 10, 42, 49, 19, 20], the algebraic constructions of $K_t(\mathcal{E}_0^0)$ and its torus $\mathcal{Y}_t^0$ are described in terms of $\tilde{C}(q)$. Moreover, the quantum torus $\mathcal{Y}_t^0$ is constructed from the Laurent expansion of $\tilde{C}(q)_{i,j}$ at $q = 0$ and proved that $K_t(\mathcal{E}_0^0)$ is isomorphic to the intersection of $t$-screening operators $S_{i,t}$ $(i \in I)$ on $\mathcal{Y}_t^0$.

As $\tilde{C}(t)$-analogues, we construct a new quantum torus $\mathcal{X}_q$ (Definition 5.1) and quantum virtual Grothendieck ring $\mathfrak{R}_q$ (Definition 5.9) of $\mathcal{X}_q$, which recovers $\mathcal{Y}_t^0$ and $K_t(\mathcal{E}_0^0)$ respectively, when $\mathfrak{g} = \mathfrak{g}$.

Theorem B (Theorem 5.4, Proposition 5.8). The $q$-commutation relation of $\mathcal{X}_q$ of type $\mathfrak{g}$ is controlled by (i) the symmetric bilinear form on $Q_\mathfrak{g}$ and (ii) the bijection $\phi_Q$ between $\hat{\Delta}_0$ and $\Phi_+^\Delta \times \mathbb{Z}$ for any Dynkin quiver $Q$ of the same type.

(C) Note that, for any $\mathfrak{g}$ and a commutation $[w_0]$ of $w_0 \in \mathcal{W}_\mathfrak{g}$, $A_\nu(n)$ of $U_\nu(\mathfrak{g})$ is contained in the quantum torus $\mathcal{T}_{\nu,[w_0]}$ and generated by the certain set of unipotent quantum minors in $A_\nu(n)$. More precisely, it is isomorphic to the quantum cluster algebra $A_\nu(\Lambda^{[w_0]}, B^{[w_0]})$ contained in the quantum torus $\mathcal{T}_{\nu,[w_0]}$ whose initial quantum seed $\mathcal{S} = \{(\Lambda^{[w_0]}, B^{[w_0]}), \{x_i\}_{1 \leq i \leq \ell([w_0])}\}$ is given by the combinatorics of $\mathcal{W}_\mathfrak{g}$ and $Q_\mathfrak{g}$ [16, 17]. In particular, if $\mathcal{W}$ is of simply-laced type $\mathfrak{g}$ and $[w_0] = [\mathcal{D}]$ for a Q-datum $\mathcal{D} = (\Delta, \sigma, \xi)$, it is proved in [22, 13] that $\mathcal{T}_{\nu,[\mathcal{D}]}$ is isomorphic to the quantum subtorus $\mathcal{Y}_{t,\mathcal{D}}$ of $\mathcal{Y}_t$. We prove the following theorem by considering the Dynkin quiver $Q$ of type BCFG also:

Theorem C (Theorem 6.14). For any $\mathfrak{g}$ and a Dynkin quiver $Q$ of type $\mathfrak{g}$, we have an isomorphism $\Psi_Q$ between $\mathcal{T}_{\nu,[Q]}$ and $\mathcal{X}_{\mathfrak{g},Q}$, where $\mathcal{X}_{\mathfrak{g},Q}$ denotes the quantum subtorus of $\mathcal{X}_\mathfrak{g}$ determined by the coordinate system of $\Gamma_Q$ inside $\Delta$. 
By restricting the map $\Psi_Q$ to the subalgebra $\mathcal{A}_\nu(\Lambda^{|Q|}, B^{|Q|})$ of $\mathcal{T}_\nu|Q|$, we can obtain the subalgebra $\mathfrak{R}_{q,Q}$ of $\mathcal{X}_{q,Q}$. We expect that $\mathfrak{R}_{q,Q}$ is a subalgebra of $\mathfrak{R}_q$ for every Dynkin quiver $Q$ of type $\mathfrak{g}$. Note that $\mathfrak{R}_{q,Q}$ is isomorphic to the quantum Grothendieck ring $K_t(\mathcal{C}_Q)$ of $\mathcal{C}_Q$ when $Q$ is a Dynkin quiver of type $ADE$.

(D) For a Dynkin quiver $Q = (\triangle_q, \xi)$ of simply-laced type and a sequence $\tilde{w}$ of indices of $I_q$ satisfying certain condition in (7.2), we can construct a pair of matrices $(\Lambda^{\tilde{w}}, B^{\tilde{w}})$ by extending the combinatorics of $W_q$ and $Q_q$. In particular, if $\tilde{w}$ is reduced, then the pair $(\Lambda^{\tilde{w}}, B^{\tilde{w}})$ is compatible with an integer $2$, $\mathcal{A}_\nu(\Lambda^{\tilde{w}}, B^{\tilde{w}})$ is isomorphic to $A_\nu(n(\tilde{w})) \subseteq A_\nu(n)$ and hence $K_t(\mathcal{C}_q)$ for a certain subcategory $\mathcal{C}_q^{\tilde{w}}$ of $\mathcal{C}_q$ [2, 22, 29, 33]. Motivated from this, we conjecture for arbitrary $\mathfrak{g}$ that the pair $(\Lambda^{\tilde{w}}, B^{\tilde{w}})$ satisfying (7.2), explicitly can be calculated by using $W_q$ and $Q_q$ ($\mathfrak{g}$ need not to be the same as $\mathfrak{g}$), is compatible (Conjecture 1). Under the $Q$-adapted condition in (7.3), we prove the conjecture:

**Theorem D** (Theorem 7.1). Let $\tilde{w} = (i_k)_{1 \leq k \leq r}, (r \in \mathbb{Z}_{\geq 1} \cup \{\infty\})$ be a $Q$-adapted sequence of $I_q$ for some Dynkin quiver $Q$ of type $\mathfrak{g}$. Then the pair $(\Lambda^{\tilde{w}}, B^{\tilde{w}})$ is compatible; i.e., for $1 \leq k, l \leq r$, we have

$$(\Lambda^{\tilde{w}} B^{\tilde{w}})_{k,l} = 2\delta(k = l)d_k \quad \text{for some } d_k \in \mathbb{Z}_{>0}.\tag{7.3}$$

We expect that the quantum seed associated to the compatible pair give a quantum cluster algebra structure on certain subalgebra of the quantum subtorus $X_{q,\tilde{w}}$ inside $\mathcal{X}_q$.

1.5. **Future works.** As we observe in this paper, $\mathcal{C}(t)$ is closely related to the quantum coordinate ring $A_\nu(n)$ for any $\mathfrak{g}$. Note that $A_\nu(n)$ is also isomorphic to the Grothendieck ring $K(\text{Rep}(R_q))$ of the category $\text{Rep}(R_q)$ of finite dimensional $\mathbb{Z}$-graded modules over the quiver Hecke algebra $R_q$ of type $\mathfrak{g}$ ([37, 38] and [47]). In [36], we will prove that the Laurent expansions of $\mathcal{C}(t)_{i,j}$’s at $t = 0$ determine whether a given convolution product of $[Q]$-cuspidal modules over $R_q$ is simple or not. It is known for Dynkin quivers $Q$ of simply-laced type but not known for the non simply-laced type. This phenomenon tells that $\mathcal{C}(t)$ controls the representation theory of $R_q$. As far as the authors know, there is no Hopf algebra $\mathcal{A}$ whose Grothendieck ring is recovered by the limit $q \to 1$ of $\mathcal{W}_{q,t}(\mathfrak{g})$ ($\mathfrak{g} \neq \mathfrak{g}$) in the sense of (1.1):

$$
\begin{array}{ccc}
K(\mathcal{C}_q) & \xrightarrow{q \to 1} & \mathcal{W}_{q,t}(\mathfrak{g}) \\
\downarrow & & \\
K(\mathcal{C}_{q,t}) & \xrightarrow{t \to 0} & K(\mathcal{C}_{q,\tilde{w}})
\end{array}
\tag{1.2}
$$

The ultimate goal of this project is to construct new algebras $\mathcal{A} = \mathcal{U}_\nu(\mathfrak{g})$ whose representation theory is controlled by $\mathcal{C}(t)$ and recovered by the limit $q \to 1$ of $\mathcal{W}_{q,t}(\mathfrak{g})$ for non simply-laced type $\mathfrak{g}$. We expect that such an algebra might be constructed as a certain subquotient of the untwisted quantum affine algebra $U_\nu(\mathfrak{g})$ of simply-laced type $\mathfrak{g}$ (different from $\mathfrak{g}$ in general), since $\Gamma_Q$ of type $\mathfrak{g}$ Dynkin quiver $Q$ can be obtained from $\Gamma_{\overline{Q}}$ for the
corresponding Dynkin quiver $Q$ of type $g$ via certain folding ([36]). Here the correspondence $g$ and $g$ are given as below:

$$(g, g) = (B_n, D_{n+1}), \quad (C_n, A_{2n-1}), \quad (F_4, E_6), \quad (G_2, D_4).$$

**Remark.** When the authors almost finish the first draft of this paper, Frenkel-Hernandez-Reshetikhin uploaded the paper [8] at Arxiv, which looks deeply related to this present paper. They also consider the refined ring of interpolating $(q, t)$-character $K_{q,t}(g)$ whose specialization at $q = 1$ and $\alpha = d$ (where $d$ is the lacing number of $g$) coincides with the specialization of $K_q$ at $q = 1$ by observing the “folded” phenomenon of $t$-characters (see Remark 5.10 for more discussion). They also raised the similar question related to (1.2) in [8, Remark 3.2].

**Acknowledgments** The second author is grateful to I.-S. Jang, Y.-H. Kim, K.-H. Lee and R. Fujita for helpful discussions.

**Convention.**

(i) For a statement $P$, $\delta(P)$ is 1 or 0 according to whether $P$ is true or not.

(ii) For a finite set $A$, we denote by $|A|$ the number of elements of $A$.

(iii) We denote by $\leq_2$ the partial order on $\mathbb{Z}$ defined by:

$$m \leq_2 n \iff m < n \text{ and } m \equiv n \mod 2.$$  

2. **Cartan datum and Dynkin quiver**

In this section, we first fix the notation on Cartan data and Dynkin quivers. Then we investigate commutation classes of the longest element $w_0$ of Weyl group associated to Dynkin quivers of an arbitrary finite type.

2.1. **Cartan datum.** Let $I$ be an index set. A *Cartan datum* is a quintuple

$$(C, P, \Pi, P^\vee, \Pi^\vee)$$

consisting of

1. a generalized symmetrizable Cartan matrix $C = (c_{i,j})_{i,j \in I}$,
2. a free abelian group $P$, called the *weight lattice*,
3. $\Pi = \{\alpha_i \mid i \in I\} \subset P$, called the set of *simple roots*,
4. $P^\vee := \text{Hom}_\mathbb{Z}(P, \mathbb{Z})$, called the *coweight lattice* and
5. $\Pi^\vee = \{h_i \in P^\vee \mid i \in I\}$, called the set of *simple coroots*,

satisfying

$$\begin{align*}
(a) \quad & \langle h_i, \alpha_j \rangle = c_{i,j} \text{ for } i, j \in I, \\
(b) \quad & \Pi \text{ is linearly independent over } \mathbb{Q}, \\
(c) \quad & \text{for each } i \in I, \text{ there exists } \varpi_i \in P, \text{ called the fundamental weight, such that } \langle h_j, \varpi_i \rangle = \delta(i = j) \text{ for all } j \in I \text{ and} \\
(d) \quad & \text{there exists a } \mathbb{Q}\text{-valued symmetric bilinear form } \langle \cdot, \cdot \rangle \text{ on } P \text{ such that} \\
& \langle h_i, \lambda \rangle = \frac{2(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)} \text{ and } (\alpha_i, \alpha_i) \in \mathbb{Q}_{>0} \text{ for any } i \text{ and } \lambda \in P.
\end{align*}$$

(2.1)
We set $\mathfrak{h} := \mathbb{Q} \otimes \mathbb{Z} P^\vee$, $\mathbb{Q} := \bigoplus_{i \in I} \mathbb{Z} \alpha_i$ and $\mathbb{Q}^+ := \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$, and define $P^+ := \{ \lambda \in P \mid \langle h_i, \lambda \rangle \geq 0 \text{ for any } i \in I \}$, called the set of integral dominant weights. We denote by $\Phi$ the set of roots, by $\Phi^+$ the set of positive roots and by $\Phi^-$ the set of negative roots.

2.2. Finite Cartan datum. For a Cartan datum $(C, P, \Pi, P^\vee, \Pi^\vee)$, we denote by $\Delta$ the corresponding Dynkin diagram. It is a graph with $I$ as the set of vertices and the set of edges between $i, j \in I$ such that $c_{i,j} < 0$. We denote by $\Delta_0$ the set of vertices and $\Delta_1$ the set of edges.

For each finite Cartan datum, we take a symmetric bilinear form $(\cdot, \cdot)$ on $\mathfrak{h}^*$ as in (2.1) (d). We set $d_i := (\alpha_i, \alpha_i)/2 = (\alpha_i, \alpha_i)$ for any $i \in I$.

For $i, j \in I$, $d(i, j)$ denotes the number of edges between $i$ and $j$ in $\Delta$. We have

$$\langle h_i, \alpha_j \rangle = \begin{cases} 2 & \text{if } i = j, \\ -\max(d_j/d_i, 1) & \text{if } d(i, j) = 1, \\ 0 & \text{if } d(i, j) > 1, \end{cases}$$

$$\langle \alpha_i, \alpha_j \rangle = \begin{cases} 2d_i & \text{if } i = j, \\ -\max(d_i, d_j) & \text{if } d(i, j) = 1, \\ 0 & \text{if } d(i, j) > 1. \end{cases}$$

In this paper, we choose the bilinear form $(\cdot, \cdot)$ such that $(\alpha, \alpha) = 2$ for short roots $\alpha$ in $\Phi^+$:

$$A_n, \quad B_n, \quad C_n, \quad D_n, \quad E_n, \quad F_4, \quad G_2.$$

Here $\otimes_k$ means that $(\alpha_k, \alpha_k) = t$.

Note that we have $d_i \in \mathbb{Z}_{>0}$ for any $i \in I$ with our choice of the inner product.

Remark 2.1. In this paper, we use an uncommon convention for Dynkin diagram; i.e., we do not use doubly-laced (triply-laced) arrows but use circles with integers instead of vertices which recover the arrows in the usual convention. We use this convention to describe the Dynkin quiver for non simply-laced types.

Note that the diagonal matrix $D = \text{diag}(d_i \mid i \in I)$ symmetrizes $C$.

The matrices $\overline{B} := DC = ((\alpha_i, \alpha_j))_{i,j \in I}$ and $\overline{B} := CD^{-1} = ((\alpha_i^\vee, \alpha_j^\vee))_{i,j \in I}$ are symmetric, where $\alpha_i^\vee = (d_i)^{-1} \alpha_i$. Note that the entries in $\overline{B}$ are integers, while some entries in $\overline{B}$ are not (see Example 2.2 below).

Example 2.2. Note that, for the Cartan matrix $C$ of finite type ADE, $\overline{B} = C = \overline{B}$. 

(a) For the Cartan matrix $C$ of $B_n$ and $C_n$, $B$ are

$$
B_{B_n} = \begin{pmatrix}
1 & -\frac{1}{2} & 0 & 0 & \cdots & 0 \\
-\frac{1}{2} & 1 & -\frac{1}{2} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & -\frac{1}{2} & 1 & -1 \\
0 & \cdots & \cdots & 0 & -1 & 2
\end{pmatrix}
$$

and

$$
B_{C_n} = \begin{pmatrix}
2 & -1 & 0 & 0 & \cdots & 0 \\
-1 & 2 & -1 & 0 & \cdots & 0 \\
0 & \cdots & \cdots & -1 & 2 & -1 \\
0 & \cdots & \cdots & 0 & -1 & 1
\end{pmatrix}.
$$

(b) For the Cartan matrix $C$ of $F_4$ and $G_2$, $B$ are

$$
B_{F_4} = \begin{pmatrix}
1 & -\frac{1}{2} & 0 & 0 \\
-\frac{1}{2} & 1 & -1 & 0 \\
0 & -1 & 2 & -1 \\
0 & 0 & -1 & 2
\end{pmatrix}
$$

and

$$
B_{G_2} = \begin{pmatrix}
2 & -1 & \frac{1}{2} \\
-1 & 2 & \frac{1}{2}
\end{pmatrix}.
$$

2.3. Weyl group and convex order. We denote by $W$ the Weyl group associated to the finite Cartan datum. It is the subgroup of $\text{Aut}(P)$ generated by simple reflections $\{s_i \mid i \in I\}$:

$$s_i \lambda = \lambda - \langle h_i, \lambda \rangle \alpha_i \quad (\lambda \in \Lambda).
$$

Note that

(i) there exists a unique element $w_0 \in W$ with the biggest length,

(ii) $w_0$ induces the Dynkin diagram automorphism $^* : I \to I$ sending $i \mapsto i^*$, where $w_0(\alpha_i) = -\alpha_i^*$.

Let $w := s_{i_1} \cdots s_{i_l}$ be a reduced expression of $w \in W$, and define

$$\beta_k^w := s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}) \quad \text{for } k = 1, \ldots, l.
$$

Then we have $\Phi^+ \cap w \Phi^- = \{\beta_1^w, \ldots, \beta_l^w\}$ for any reduced expression of $w$ and $|\Phi^+ \cap w \Phi^-|$ coincides with the length $l$ of $w$, denoted by $\ell(w)$. In particular, $\Phi^+ \cap w_0 \Phi^- = \Phi^+$ and $|\Phi^+| = \ell(w_0)$. It is well-known that the total order $<_w$ on $\Phi^+ \cap w \Phi^-$ defined by $\beta_a^w <_w \beta_b^w$ for $a < b$ is convex in the following sense:

$$\beta < w \alpha + \beta < w \beta \quad \text{or} \quad \beta < w \alpha + \beta < w \alpha.
$$

Two reduced expressions $w$ and $w'$ of $w \in W$ are said to be commutation equivalent, denoted by $w \sim w'$, if $w'$ can be obtained from $w$ by applying the commutation relations $s_a s_b = s_b s_a$ ($d(a, b) > 1$). Note that this relation $\sim$ is an equivalence relation and an equivalence class under $\sim$ is called a commutation class. We denote by $[w]$ the commutation class of $w$.

Now let us consider a reduced expression $w_0$ of the longest element $w_0 \in W$. Then, each $w_0$ induces a convex total order $\leq_w$ on $\Phi^+$. For a commutation class $[w]$, we define the convex partial order $\preceq_{[w]}$ on $\Phi^+$ by:

$$\alpha \preceq_{[w]} \beta \quad \text{if and only if} \quad \alpha \leq_w \beta \quad \text{for any } w' \in [w].
$$

Note that (2.3) still holds after replacing $<_w$ with $\preceq_{[w]}$. 

For a commutation class \([w_0]\) of \(w_0\) and \(\alpha \in \Phi^+\), we define the \([w_0]\)-residue of \(\alpha\), denoted by \(\text{res}_{[w_0]}(\alpha)\), to be \(i_k \in I\) if \(\beta_{i_k}^{w_0} = \alpha\) with \(w_0 = s_{i_1} \cdots s_{i_\ell}\). Note that this notion is well-defined; i.e., for any \(w_0' = s_{j_1} \cdots s_{j_\ell} \in [w_0]\) with \(\beta_{j_\ell}^{w_0'} = \alpha\), we have \(j_\ell = i_k\). Note that

\[
(\alpha, \alpha) = (\alpha_{i_k}, \alpha_{i_k}) \quad \text{if} \quad i = \text{res}_{[w_0]}(\alpha).
\]

For a reduced expression \(w_0 = s_{i_1}s_{i_2} \cdots s_{i_\ell}\) of \(w_0\), it is known that the expression \(s_{i_2} \cdots s_{i_\ell}s_{i_1}\) is also a reduced expression of \(w_0\). This operation is sometimes referred to as a *combinatorial reflection functor* and we write \(r_{i_k}w_0 = w_0'\). Also it induces the operation on commutation classes of \(w_0\) (i.e., \(r_{i_k}[w_0] = [r_{i_k}w_0]\) is well-defined if there exists a reduced expression \(w_0' = s_{j_1}s_{j_2} \cdots s_{j_\ell} \in [w_0]\) such that \(j_\ell = i_1\)). The relations \([w] \sim [r_{i_k}w]\) for \(i \in I\) generate an equivalence relation, called the *reflection equivalent relation* \(\sim\), on the set of commutation classes of \(w_0\). For \(w_0\) of \(w_0\), the set of commutation classes \([w_0] := \{w_0' \mid w_0' \sim [w_0]\}\) is called an \(r\)-cluster point.

### 2.4. Dynkin quiver \(Q\)

A *Dynkin quiver* \(Q\) of \(\Delta\) is an oriented graph whose underlying graph is \(\Delta\). To each Dynkin quiver \(Q\) of \(\Delta\), we can associate a function \(\xi : \Delta_0 \to \mathbb{Z}\), called a *height function* of \(Q\), which satisfies the condition:

\[
\xi_i = \xi_j + 1 \quad \text{if} \quad d(i, j) = 1 \quad \text{and} \quad i \to j \quad \text{in} \quad Q.
\]

Note that, since \(\Delta\) is connected, height functions of \(Q\) differ by integers. Conversely, to a Dynkin diagram \(\Delta\) and a function \(\xi : \Delta_0 \to \mathbb{Z}\) satisfying \(|\xi_i - \xi_j| = 1\) for \(i, j \in I\) with \(d(i, j) = 1\), we can associate a Dynkin quiver \(Q\) in a canonical way.

In this paper, we abuse the terminology “Dynkin quiver” for a pair \((\Delta, \xi)\) of a Dynkin diagram \(\Delta\) and a height function \(\xi\) on \(\Delta\).

For a Dynkin quiver \(Q = (\Delta, \xi)\), we call \(i \in \Delta_0\) a *source* of \(Q\) if \(\xi_i > \xi_j\) for all \(j \in \Delta_0\) such that \(d(i, j) = 1\). We also call \(i \in \Delta_0\) a *sink* of \(Q\) if \(\xi_i < \xi_j\) for all \(j \in \Delta_0\) such that \(d(i, j) = 1\).

**Example 2.3.** Here are examples of Dynkin quivers \(Q^\circ\) of non simply-laced types.

1. \[\begin{array}{c}
\circ & 1 & \circ & 2 & \cdots & \circ & n-1 & \circ & 1 \\
\| & \| & \| & \| & \| & \| & \| & \| & \|
\end{array}\] for \(Q^\circ = (\bigtriangleup_{B_n}, \xi)\),

2. \[\begin{array}{c}
\circ & 1 & \circ & 2 & \cdots & \circ & n-1 & \circ & n \circ & 1 \\
\| & \| & \| & \| & \| & \| & \| & \| & \| & \|
\end{array}\] for \(Q^\circ = (\bigtriangleup_{C_n}, \xi)\),

3. \[\begin{array}{c}
\circ & 1 & \circ & 2 & \cdots & \circ & n-1 & \circ & 1 \\
\| & \| & \| & \| & \| & \| & \| & \| & \|
\end{array}\] for \(Q^\circ = (\bigtriangleup_{F_4}, \xi)\),

4. \[\begin{array}{c}
\circ & 1 & \circ & 2 \\
\| & \| & \|
\end{array}\] for \(Q^\circ = (\bigtriangleup_{G_2}, \xi)\).

Here

1. an underlined integer * is the value \(\xi_i\) at each vertex \(i \in \Delta_0\),
2. an arrow \(\circ \to \circ\) means that \(\xi_i = \xi_j + 1\) and \(d(i, j) = 1\).
For a Dynkin quiver $Q = (\Delta, \xi)$ and its source $i$, we denote by $s_i Q$ the Dynkin quiver $(\Delta, s_i \xi)$ where $s_i \xi$ is the height function defined as follows:

$$\tag{2.6} (s_i \xi)_j = \xi_j - \delta(i = j) \times 2.$$ 

Let $Q = (\Delta, \xi)$ be a Dynkin quiver and let $W$ be the Weyl group associated to $\Delta$. For a reduced expression $\underline{w} = s_{i_1} \cdots s_{i_l}$ of $w \in W$, $\underline{w}$ is said to be adapted to $Q$ (or $Q$-adapted) if $i_k$ is a source of $s_{i_k-1} s_{i_k-2} \cdots s_{i_1} Q$ for all $1 \leq k \leq l$.

Definition 2.4. For a Weyl group $W$, a Coxeter element $\tau$ of $W$ is a product of all simple reflections; i.e., there exists a reduced expression $s_{i_1} \cdots s_{i_n}$ of $\tau$ such that $\{i_1, \ldots, i_n\} = \Delta_0$. Here $n = |\Delta_0|$. It is well-known that all of reduced expressions of every Coxeter element $\tau$ form a single commutation class and they are adapted to some Dynkin quiver $Q$. Indeed, for a $Q$-adapted $\tau = s_{i_1} \cdots s_{i_n}$, the height function $\xi$ of $Q$ satisfies $\xi_{i_k} = \xi_{i_l} + 1$ for $1 \leq k < l \leq n$ such that $d(i_k, i_l) = 1$. Conversely, for each Dynkin quiver $Q$, there exists a unique Coxeter element $\tau_Q$, all of whose reduced expressions are adapted to $Q$. Furthermore, when $Q$ is of type $B_n$, $C_n$, $F_4$, or $G_2$, we have

$$\tau_Q^{\frac{|\Phi^+|}{\mid \Delta_0 \mid}} = w_0.$$ 

The following theorem is proved in $[22, 15]$ for a Dynkin quiver $Q = (\Delta, \xi)$ when $\Delta$ is simply-laced, and we omit the proof in the general case since it can be proved by similar arguments.

Theorem 2.5. Let $Q = (\Delta, \xi)$ be a Dynkin quiver and let $h$ be the Coxeter number of $\Delta$.

(i) There exists a unique Coxeter element $\tau_Q \in W_\Delta$ such that it has a reduced expression adapted to $Q$.

(ii) We have $s_i(\tau_Q) s_i = \tau_Q s_i$ for a source $i$ of $Q$ and the order of $\tau_Q$ is $h = \frac{2|\Phi^+|}{|\Delta_0|}$.

(iii) Any reduced expression $s_{i_1} \cdots s_{i_n}$ of $\tau_Q$ is $Q$-adapted and the height function $\xi'$ of the Dynkin quiver $s_{i_1} \cdots s_{i_n} Q$ is given by

$$\xi'_i = \xi_i - 2 \quad \text{for any } i \in \Delta_0.$$ 

(iv) There exists a $Q$-adapted reduced expression of $w_0$ and all $Q$-adapted reduced expressions of $w_0$ form a single commutation class, which is denoted by $[Q]$.

(v) Let $\underline{w}_\Delta = s_{i_1} s_{i_2} \cdots s_{i_k}$ be a $Q$-adapted reduced expression of $w_0$. Then, the height function $\xi'$ of the Dynkin quiver $s_{i_1} \cdots s_{i_k} Q$ is given by

$$\xi'_i = \xi_i - h \quad \text{for any } i \in I_\Delta.$$ 

Moreover, $s_{i_1} \cdots s_{i_k} Q$ is an $(s_i Q)$-adapted reduced expression of $w_0$ where $i_{k+1} = i_1$. 

(vi) Let $Q' = (\Delta, \xi')$ be another Dynkin quiver. Then we have $[Q] \sim [Q']$. Moreover if $[Q] = [Q']$, then there exists $k \in \mathbb{Z}$ such that $\xi_i = \xi'_i + k$ for all $i \in \Delta_0$.

(vii) For a Dynkin diagram $\Delta$, the set $\{(Q) \mid Q = (\Delta, \xi)\}$ forms an $r$-cluster point $[\Delta]$ and $|\{\Delta\}| = 2^{(|\Delta_0|-1)}$. 

3. Quivers

In this section, we first recall the notion of a (combinatorial) Auslander-Reiten (AR) quiver which realizes the convex partial order \( \preceq_{[w]} \) (see (2.3)) for each commutation class \([w]\) of \( w_0 \). Then we introduce quivers related to \([Q]\) and study their properties.

3.1. Hasse quiver. For a reduced expression \( w_n = s_{i_1} \cdots s_{i_\ell} \) of \( w_0 \in W \), we associates a quiver \( Y_{w_n} \) to \( w_0 \) as follows [45]:

\[
(i) \quad \text{The set of vertices is } \Phi = \{ \beta^w_k | 1 \leq k \leq \ell \}.
\]

\[
(ii) \quad \text{We assign } (-\langle h_{i_k}, \alpha_{i_l} \rangle)-\text{many arrows from } \beta^w_k \text{ to } \beta^w_l \text{ if and only if } 1 \leq l < k \leq \ell, d(i_k, i_l) = 1 \text{ and there is no index } j \text{ such that } l < j < k \text{ and } i_j \in \{i_k, i_l\}.
\]

Hence the number of arrows from \( \beta \) to \( \gamma \) in \( Y_{w_n} \) is either 0 or \( \max\left(\frac{\langle \gamma, \gamma \rangle}{\langle \beta, \beta \rangle}, 1\right) \).

We say that a total order \((\beta_1 < \cdots < \beta_\ell)\) of \( \Phi^+ \) is a compatible reading of \( Y_{w_n} \) if

\[
(\beta_1 < \cdots < \beta_\ell) \text{ whenever there is a path from } \beta_l \text{ to } \beta_k \text{ in } Y_{w_n}.
\]

Remark 3.1. In [45], \(-\langle \alpha_{i_k}, \alpha_{i_l} \rangle)-\text{many arrows was assigned so that the assigning rule in (3.1) is slightly different from the one in [45].}

Theorem 3.2 ([45, Lemma 2.19, Proposition 2.20, Theorem 2.21 and 2.22]). The commutation class \([w]\) of a reduced expression \( w_0 \) of \( w_0 \) satisfies the following properties.

\[
(i) \quad A \text{ reduced expression } w'_n \text{ of } w_0 \text{ is commutation equivalent to } w_n \text{ if and only if } Y_{w_n} = Y_{w'_n} \text{ as quivers. Hence } Y_{[w]} \text{ is well-defined.}
\]

\[
(ii) \quad \text{For } \alpha, \beta \in \Phi^+, \alpha \preceq_{[w]} \beta \text{ if and only if there exists a path from } \beta \text{ to } \alpha \text{ in } Y_{[w]} \text{. In other words, the quiver } Y_{[w]} \text{ is the Hasse quiver of the partial ordering } \preceq_{[w]}.
\]

\[
(iii) \quad \text{For } \alpha, \beta \in \Phi^+, \text{ if they are not comparable with respect to } \preceq_{[w]}, \text{ then we have } (\alpha, \beta) = 0.
\]

\[
(iv) \quad \text{If } \Phi^+ = \{ \beta_1 < \cdots < \beta_\ell \} \text{ is a compatible reading of } Y_{[w]}, \text{ then there is a unique reduced expression } w'_n = s_{j_1} \cdots s_{j_\ell} \text{ in } [w] \text{ such that } \beta_k = \beta'^{w'_n}_k \text{ for any } k.
\]

We call \( Y_{[w]} \) the combinatorial AR-quiver of \([w]\). By Theorem 3.2, \( Y_{[w]} \) can be understood as the Hasse quiver of the ordered set \((\Phi^+, \preceq_{[w]})\) (if we forget the number of arrows).

Example 3.3. Let us consider the following reduced expression \( w_n \) of \( w_0 \) of type \( B_3 \):

\[
w_n = s_1s_2s_3s_1s_2s_3s_1s_2s_3.
\]

Then one can easily check that \( w_0 \) is adapted to the following Dynkin quiver \( Q \):

\[
Q : \begin{array}{c}
\cdot \\
1 \\
2 \\
3
\end{array}
\begin{array}{c}
\rightarrow \\
\rightarrow \\
\rightarrow
\end{array}
\begin{array}{c}
\cdot \\
\cdot \\
\cdot
\end{array}
\]
The quiver $\Upsilon_{[w_0]} = \Upsilon_{[Q]}$ can be described as follows (see Proposition 3.12 below):

\[
\begin{array}{cccccc}
(i \setminus p) & -3 & -2 & -1 & 0 & 1 & 2 & 3 \\
\end{array}
\]

\[
\Upsilon_{[Q]} = \begin{array}{c}
1 \quad \bullet \\
2 \quad \circ \quad \circ \quad \circ \\
3 \quad \bullet \\
\end{array}
\]

\[
\begin{array}{ccc}
(1,3) & (2,-3) & (1,-2) \\
(2,3) & (1,2) & (1,-3) \\
(3) & (2) & (1) \\
\end{array}
\]

Here, we use the realization of root system of $B_n$ in $\mathbb{R}^n$

$$\alpha_k = \varepsilon_k - \varepsilon_{k+1} \quad \text{for } k < n, \quad \alpha_n = \varepsilon_n,$$

where $\{\varepsilon_i\}_{1 \leq i \leq n}$ is an orthogonal basis with $(\varepsilon_i, \varepsilon_i) = 2$, and

(a) $\langle a, \pm b \rangle := \varepsilon_a \pm \varepsilon_b$ and $\langle c \rangle := \varepsilon_c$ for $1 \leq a < b \leq 3$ and $1 \leq c \leq 3$,

(b) every positive root in the $i$-th layer has $i$ as its residue with respect to $[w_0]$.

(c) every root with residue $i$ has the same squared length as $(\alpha_i, \alpha_i)$.

(d) the indices $p$ will be defined by using the bijection (3.9) below.

Using the compatible reading on $\Phi^+$

\[
\langle 1, -2 \rangle < \langle 1, -3 \rangle < \langle 1 \rangle < \langle 2, -3 \rangle < \langle 1, 2 \rangle < \langle 2 \rangle < \langle 1, 3 \rangle < \langle 2, 3 \rangle < \langle 3 \rangle,
\]

we obtain the reduced expression $w_0$ of $w_0$ in (3.3).

3.2. Classical quivers. Throughout this subsection, we consider a Dynkin quiver $Q = (\triangle, \xi)$ of type $ADE$.

We denote by $\text{Rep}(Q)$ the category of finite-dimensional modules of $Q$ over $\mathbb{C}$. In this subsection, we recall a description of the Auslander-Reiten(AR) quiver of $\text{Rep}(Q)$ and that of the derived category $\mathcal{D}_Q := D^b(\text{Rep}(Q))$ in the aspect of combinatorics.

By definition, the set of vertices of the AR-quiver of $\text{Rep}(Q)$ (resp. $\mathcal{D}_Q$) is the set of the isomorphism classes of indecomposable objects, denoted by $\text{Ind} \text{Rep}(Q)$ (resp. $\text{Ind} \mathcal{D}_Q$). Considering the bijection $\alpha \mapsto M_Q(\alpha)$ from $\Phi^+$ to $\text{Ind} \text{Rep}(Q) = \{M_k(\alpha) \mid \alpha \in \Phi^+\}$ (resp. $(\alpha, k) \mapsto M_Q(\alpha)[k]$), the set of vertices of AR-quiver of $\text{Rep}(Q)$ (resp. $\text{Ind} \mathcal{D}_Q$) can be labeled by $\Phi^+$ (resp. $\tilde{\Phi}^+ := \Phi^+ \times \mathbb{Z}$). Here $[k]$ denotes the cohomological degree shift by $k$.

We define the repetition quiver $\tilde{\Delta} = (\tilde{\Delta}_0, \tilde{\Delta}_1)$ associated to $Q$ as follows:

\[
\tilde{\Delta}_0 := \{(i, p) \in \Delta_0 \times \mathbb{Z} \mid p - \xi_i \in 2\mathbb{Z}\},
\]

\[
\tilde{\Delta}_1 := \{(i, p) \rightarrow (j, p + 1) \mid (i, p) \in \tilde{\Delta}_0, \ d(i, j) = 1\}.
\]

Note that $\tilde{\Delta}$ depends only on the parity of the height function of $Q$. 
Example 3.4. Here is an example of the repetition quiver $\hat{\Delta}$ of simply-laced type. When $\Delta$ is of type $D_4$, the repetition quiver $\hat{\Delta}$ is depicted as:

![Repetition Quiver Diagram]

It is shown by Happel [18] that the AR-quiver of $D_Q$ is isomorphic to $\hat{\Delta}$ as a quiver by the following bijection. We set

$$\gamma^Q_i := (1 - \tau_Q) \varpi_i \in \Phi^+ \quad \text{for } i \in \Delta_0.$$

Then the bijection between sets of vertices is given by

$$\phi_Q : \hat{\Delta}_0 \rightarrow \text{Ind } D_Q, \quad (i, p) \mapsto \tau^{(\xi_i - p)/2} M_Q(\gamma^Q_i),$$

where $\tau$ denotes the AR-translation of $D_Q$. By exchanging the labels of Ind $D_Q$ to $\hat{\Phi}^+$, the bijection $\phi_Q : \hat{\Delta}_0 \rightarrow \hat{\Phi}^+$ can be described in an inductive way as follows ([22, §2.2]):

$$\phi_Q(i, p \pm 2) = \begin{cases} (\tau_{i}^{\pm 1}(\beta), u) & \text{if } \tau_{i}^{\pm 1}(\beta) \in \Phi^+, \\ (-\tau_{i}^{\pm 1}(\beta), u \pm 1) & \text{if } \tau_{i}^{\pm 1}(\beta) \in \Phi^-. \end{cases}$$

We say that $(i, p)$ with $\phi_Q(i, p) = (\beta, u)$ is the coordinate of $(\beta, u) \in \hat{\Phi}^+$.

The repetition quiver $\hat{\Delta}$ satisfies the $\hat{\Delta}$-additive property $^1$: For $i \in I$, $l \in \mathbb{Z}$ and any Dynkin quiver $Q = (\Delta, \text{id}, \xi)$, we have

$$\tau_Q^l(\gamma^Q_i) + \tau_Q^{l+1}(\gamma^Q_i) = \sum_{j: \, d(i, j) = 1} \tau_Q^{l+\xi_j - \xi_i + 1}(\gamma^Q_j) = \sum_{j \in I \setminus \{i\}} -\langle h_j, \alpha_i \rangle \tau_Q^{l+\xi_j - \xi_i + 1}(\gamma^Q_j)$$

(see [22, Proposition 2.1]).

The following statements are well-known ([1]):

(a) The AR-quiver $\Gamma_Q$ of $\text{Rep}(Q)$ is isomorphic to the full subquiver of $\hat{\Delta}$, whose set of vertices $(\Gamma_Q)_0$ is given as follows:

$$\Gamma_Q)_0 = \phi_Q^{-1}(\Phi^+ \times \{0\}) = \{(i, p) \in \Delta_0 \mid \xi_i \geq p > \xi_i \ast - h\}.$$

(b) $\Gamma_Q \simeq \Upsilon_Q$ as quivers.

$^1$In [15], (3.10) was called the g-additive property.
3.3. Repetition quivers in general case. In this subsection, we consider a Dynkin quiver $Q = (\triangle, \xi)$ of an arbitrary finite type.

In this case, although there is no representation-theoretic interpretation as for the simply-laced case, we can generalize (i) the coordinate system of $Y_{[Q]}$, (ii) the repetition quiver $\widehat{\Delta}$ containing $Y_{[Q]}$, and (iii) the additive property of $\widehat{\Delta}$.

For a Dynkin quiver $Q = (\triangle, \xi)$ with $|\triangle_0| = n$, recall that there exists a unique Coxeter element $\tau_Q = s_{i_1} \cdots s_{i_n}$ with a $Q$-adapted reduced expression.

For $1 \leq k \leq n$, we set
\begin{equation}
\gamma^Q_{ik} := (1 - \gamma^Q_i)w_{ik} = (s_{i_1} \cdots s_{i_{k-1}} - s_{i_1} \cdots s_{i_k})w_{ik} = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}).
\end{equation}

In particular, we have
\begin{equation}
(\gamma^Q_i, \gamma^Q_i) = (\alpha_i, \alpha_i) \quad \text{for any } i \in \triangle_0.
\end{equation}

We define the repetition quiver $\Delta = (\widehat{\Delta}_0, \widehat{\Delta}_1)$ associated to $Q$ similarly to the definition (3.7) in the simply-laced case:
\begin{align*}
\widehat{\Delta}_0 &:= \{(i, p) \in I \times \mathbb{Z} \mid p - \xi_i \in 2\mathbb{Z}\}, \\
\widehat{\Delta}_1 &:= \{(i, p) \xrightarrow{-\langle h_i, \alpha_j \rangle} (j, p+1) \mid (i, p) \in \widehat{\Delta}_0, \; d(i, j) = 1\}.
\end{align*}

Here $(i, p) \xrightarrow{-\langle h_i, \alpha_j \rangle} (j, p+1)$ denotes $(-\langle h_i, \alpha_j \rangle)$-many arrows from $(i, p)$ to $(j, p+1)$.

Example 3.5. Here is an example of the repetition quiver $\widehat{\Delta}$ of non simply-laced type. When $\triangle$ is of type $B_3$, the repetition quiver $\widehat{\Delta}$ is depicted as:

Let us define the map $\phi_Q : \triangle_0 \to \widehat{\Phi}^+$ inductively as in (3.9).

By the definition and (3.13), we have
\begin{equation}
\text{if } \phi_Q(i, p) = (\beta, k), \text{ then } (\beta, \beta) = (\alpha_i, \alpha_i).
\end{equation}

For $w \in \mathcal{W}$, we denote by $\widehat{w}$ the automorphism of $\widehat{\Phi}^+$ defined as follows ([25, Lemma B in §10]):
\begin{equation}
\widehat{w}(\beta, k) := \begin{cases} (w\beta, k) & \text{if } w\beta \in \Phi^+, \\ (-w\beta, k - 1) & \text{if } w\beta \in \Phi^-.
\end{cases}
\end{equation}

Hence we have
\begin{equation}
(\widehat{w})^{-1}(\beta, k) := \begin{cases} (w^{-1}\beta, k) & \text{if } w^{-1}\beta \in \Phi^+, \\ (-w^{-1}\beta, k + 1) & \text{if } w^{-1}\beta \in \Phi^-.
\end{cases}
\end{equation}
Lemma 3.6. Let \((\beta, k) \in \tilde{\Phi}^+\) and let \(s_{i_1} \cdots s_{i_r}\) be a reduced expression of an element \(w\) of \(\mathcal{W}\). Then we have
\[
\hat{w} = \hat{s}_{i_1} \cdots \hat{s}_{i_r}.
\]

Proof. First let us prove the following statement:
for a reduced expression \(s_{i_1} \cdots s_{i_r}\) of an element of \(\mathcal{W}\) and \(\beta \in \Phi^+\), there exists \(a\) such that \(0 \leq a \leq r\) and \(s_{i_a} \cdots s_{i_r} \beta\) is a positive root or a negative root according to that \(a < s \leq r\) or \(1 \leq s < a\).

If \(s_{i_k} \cdots s_{i_r} \beta\) is a positive root for all \(k\), then it is enough to take \(a = 0\). Otherwise, let \(a\) be the largest \(s\) such that \(s_{i_a} \cdots s_{i_r} \beta\) is a negative root. Then \(s_{i_a} \cdots s_{i_r} \beta = -\alpha_i\) and \(s_{i_a} \cdots s_{i_r} \beta = -s_{i_k} \cdots s_{i_{a-1}} \alpha_i \in \Phi^-\) for \(1 \leq s < a\). Thus we obtain (3.16).

Then we can easily see that
\[
\hat{s}_{i_a} \cdots \hat{s}_{i_r}(\beta, k) = \begin{cases} (s_{i_a} \cdots s_{i_r} \beta, k) & \text{if } a < s \leq r, \\ (-s_{i_a} \cdots s_{i_r} \beta, k - 1) & \text{if } 1 \leq s \leq a. \end{cases}
\]
by the descending induction on \(s\).

By this lemma, \(\{\hat{s}_i\}_{i \in \Delta_0}\) satisfies the braid relations. Hence it induces the braid group action on \(\tilde{\Phi}^+\).

The following lemma immediately follows from the above lemma and the definition (3.9).

Lemma 3.7. For any Dynkin quiver \(Q\), \((i, p) \in \tilde{\Delta}_0\) and \(t \in \mathbb{Z}\), we have
\[
\phi_Q(i, p + 2t) = \tilde{\tau}_Q^{-t} \phi_Q(i, p), \\
\phi_Q(i, p) = \tilde{\tau}_Q^{(\xi_i - p)/2} (\gamma_i^Q, 0).
\]

Lemma 3.8. Let \(Q\) be a Dynkin quiver and \((i, p) \in \tilde{\Delta}_0\). If \(\phi_Q(i, p) = (\beta, k)\), then we have \((\ell^*p, p \pm h) \in \tilde{\Delta}_0\) and
\[
\phi_Q(i^*, p \pm h) = (\beta, k \pm 1).
\]
In particular, \(h + \xi_i - \xi_i \in 2\mathbb{Z}\).

Proof. The simply-laced case is already known ([15, Corollary 3.40]). Hence, we assume that \(\Delta\) is of non simply-laced type.
Since \((\tau_Q)^{h/2} = w_0 = -\text{id}\) and \(\ell(\tau_Q) \cdot \frac{h}{2} = \ell(w_0)\), we have \(\tilde{\tau}_Q^{h/2} = \tilde{w}_0\). Hence
\[
\phi_Q(i, p \pm h) = \tilde{w}_0^{\pm 1} (\beta, k) = (\beta, k \pm 1).
\]

Proposition 3.9. Let \(i \in \Delta_0\) be a source of \(Q\).

(i) \(\tau_{s_i} = s_i \tau_Q s_i\) and \(\tilde{\tau}_{s_i} = \tilde{s}_i \tilde{\tau}_Q \tilde{s}_i\).
(ii) \(\gamma_i^Q = \alpha_i\), and
\[
\gamma_j^Q = \begin{cases} s_i \tau_Q \gamma_i^Q & \text{if } j = i, \\ s_i \gamma_j^Q & \text{if } j \in \Delta_0 \setminus \{i\}. \end{cases}
\]
(iii) For any \((j,p) \in \widehat{\Delta}_0\) and \(k \in \mathbb{Z}\), we have
\[
\tau_{s_i}^{((s_i \xi) - p)/2} (\gamma_i^{s_i Q}) = s_i \tau_Q^{(\xi - p)/2} (\gamma_i^Q),
\]
\[
\tilde{\tau}_{s_i}^{((s_i \xi) - p)/2} (\gamma_i^{s_i Q}, k) = \tilde{s}_i^{-1} \tau_Q^{(\xi - p)/2} (\gamma_i^Q, k).
\]

(iv) \(\phi_{s_i Q} = \tilde{s}_i^{-1} \circ \phi_Q\).

Proof. (i) Let us take a reduced expression \(s_{i_1} \cdots s_{i_n}\) of \(\tau_Q\) such that \(i_1 = i\). Then we have \(\tau_{s_i} = s_{i_2} \cdots s_{i_n} \cdot s_{i_1}\). Hence we obtain (i).

(ii) Since a reduced expression of \(s_i \tau_Q\) does not contain \(s_i\), we have \(s_i \tau_Q \varpi_i = \varpi_i\). Hence
\[
\gamma_i^Q = (1 - \tau_Q) \varpi_i = (1 - s_i) \varpi_i = \alpha_i,
\]
and
\[
\gamma_i^{s_i Q} = (1 - \tau_{s_i}) \varpi_i = (s_i \tau_Q - s_i \tau_Q s_i) \varpi_i = s_i \tau_Q (1 - s_i) \varpi_i = s_i \tau_Q \gamma_i^Q.
\]

Now assume that \(j \neq i\). Then \(s_i \varpi_j = \varpi_j\), and we have
\[
\gamma_j^{s_i Q} = (1 - \tau_{s_i}) \varpi_j = (s_i - s_i \tau_Q) \varpi_j = s_i \gamma_j^Q.
\]

(iii) Assume first \(j \neq i\). Then we have
\[
\tilde{\tau}_{s_i}^{((s_i \xi) - p)/2} (\gamma_j^Q, k) = \tilde{s}_i^{-1} \tau_Q^{(\xi - p)/2} \tilde{s}_i (s_i \tau_Q \gamma_i^Q, k) = \tilde{s}_i^{-1} \tau_Q^{(\xi - p)/2} (\gamma_j^Q, k),
\]
as we desired.

Now assume that \(i = j\). Note that \(\tau_Q = \tilde{s}_i \circ (s_i \tau_Q)\). Since \((s_i \xi)_j = \xi_j - 2\), we obtain
\[
\tilde{\tau}_{s_i}^{((s_i \xi) - p)/2} (\gamma_i^{s_i Q}, k) = \tilde{s}_i^{-1} \tau_Q^{(\xi_j - 2 - p)/2} \tilde{s}_i (s_i \tau_Q \gamma_i^Q, k) = \tilde{s}_i^{-1} \tau_Q^{(\xi_j - p)/2} \tilde{s}_i (s_i \tau_Q \gamma_i^Q, k) = \tilde{s}_i^{-1} \tau_Q^{(\xi_j - p)/2} (\gamma_i^Q, k),
\]
as we desired.

(iv) follows immediately from (iii) and Lemma 3.7. \(\square\)

The statements in the next theorem are proved in the simply-laced case (see [22, 15] for instance), but they were not known in the case of type BCFG.

Theorem 3.10. Let \(Q = (\Delta, \xi)\) be a Dynkin quiver.

(i) The map \(\phi_Q\) is a bijection.

(ii) Let \(\Gamma_Q\) be the full subquiver of \(\widehat{\Delta}\) whose set of vertices is \(\phi_Q^{-1}(\Phi^+ \times \{0\})\). Then we have
\[
\phi_Q^{-1}(\Phi^+ \times \{0\}) = \{(i, p) \in \widehat{\Delta}_0 | \xi_i \geq p > \xi_i - h\}.
\]

In particular, we have
\[
\phi_Q(i, p) = (\tau_Q^{(\xi - p)/2} (\gamma_i^Q), 0) \quad \text{for any} \ (i, p) \in (\Gamma_Q)_0.
\]
(iii) If \( i \) is a source of \( Q \), then we have
\[
\phi_{s,Q}(i^*, \xi_i - h) = \phi_Q(i, \xi_i) = (\alpha_i, 0),
\]
and
\[
(\Gamma_{s,Q})_0 = \left( (\Gamma_Q)_0 \setminus \{(i, \xi_i)\} \right) \cup \{(i^*, \xi_i - h)\}.
\]

Note that
\[
(3.18) \quad h \geq |\Delta_0| + 1 \geq \xi_{i^*} - \xi_i + 2.
\]
Note also that \( \xi_{i^*} \equiv \xi_i + h \mod 2 \) as seen in Lemma 3.8.

Hence we have
\[
(\Gamma_Q)_0 = \{(i, p) \in \hat{\Delta}_0 \mid \xi_{i^*} - h + 2 \leq p \leq \xi_i \}
= \left\{(i, \xi_i - 2k) \mid i \in \Delta_0, k \in \mathbb{Z}, 0 \leq k \leq \frac{h + \xi_i - \xi_{i^*}}{2} - 1 \right\}.
\]

(For \( \leq_2 \), see Convention at the end of Introduction.)

We also call the quiver \( \Gamma_Q \) in Theorem 3.10 (ii) the (combinatorial) \( AR \)-quiver of \( Q \).

**Proof of Theorem 3.10.** Since the theorem is already known in the simply-laced case, we shall prove it in the non simply-laced case. Recall that \( h \) is even, \( \tau_Q^{h/2} = w_0 = -1 \) on \( \Phi \), \( \ell(w_0) = \ell(\tau_Q) \cdot (h/2) \) and \( i^* = i \) in this case.

Let us prove first (i) and (ii). Set \( A = \{(i, p) \in \hat{\Delta}_0 \mid \xi_i \geq p > \xi_i - h\} \).

(a) We start from the proof of
\[
(3.19) \quad \phi_Q(A) \subset \Phi^+ \times \{0\}.
\]

It means that
\[
(3.20) \quad \tau_Q^k(\gamma_i^Q) \in \Phi^+ \quad \text{for any } k \text{ such that } 0 \leq k < h/2.
\]

Recall that \( Q^+ := \sum_{i \in \Delta_0} \mathbb{Z}_{\geq 0} \alpha_i. \) Since \( \ell(\tau_Q^{k+1}) = \ell(\tau_Q^k) + \ell(\tau_Q) \), we conclude that
\[
\tau_Q^k(\gamma_i^Q) = \tau_Q^k \xi_i - \tau_Q^{k+1} \xi_i \text{ belongs to } Q^+ \text{ Hence it belongs to } Q^+ \cap \Phi = \Phi^+.
\]

(b) Next we shall show that

if \( i, j \in \Delta_0 \) and \( m \in \mathbb{Z} \) satisfies \( 0 \leq m < h/2 \) and \( \tau_Q^m \gamma_i^Q = \gamma_j^Q \), then
\[
i = j \text{ and } m = 0.
\]

We have \( \tau_Q^{h/2-m} \gamma_j^Q = \tau_Q^{h/2} \gamma_i^Q = w_0 \gamma_i^Q = -\gamma_i^Q \in \Phi^- \). Hence (3.20) implies \( h/2 - m = h/2 \), which implies that \( m = 0 \). Thus, we have \( \gamma_i^Q = \gamma_j^Q \). Hence (3.12) implies \( i = j \).

(c) Now (b) implies that \( \phi_Q|_A : A \to \Phi^+ \times \{0\} \) is injective.

Since \( |A| = \frac{h}{2} |\Delta_0| = |\Phi^+| \), the restriction \( \phi_Q|_A : A \to \Phi^+ \times \{0\} \) is bijective. Then Lemma 3.8 implies the bijectivity of \( \phi_Q \).

It completes the proof of (i) and (ii).

(iii) is a consequence of (i), (ii) and Proposition 3.9. \( \square \)
For a subset \( S \) of \( \Delta_0 \), we say that a sequence \( ((i_k, p_k))_{1 \leq k \leq r} \) in \( S \) is a compatible reading (of \( S \)) if
\[
(3.21) \quad p_a > p_b \text{ if } 1 \leq a < b \leq r \text{ and } d(i_a, i_b) \leq 1
\]
(and \( S = \{(i_k, p_k) \mid 1 \leq k \leq r\} \)). If \( S = (\Gamma_Q)_0 \), this condition is equivalent to the condition:

we have \( a < b \) if there is an arrow from \((i_b, p_b)\) to \((i_a, p_a)\).

We say that a sequence \((i_k)_{1 \leq k \leq r}\) in \( \Delta_0 \) is \( Q \)-adapted if \( i_k \) is a source of \( s_{i_{k-1}} \cdots s_{i_1}Q \) for any \( k \) such that \( 1 \leq k \leq r \).

**Lemma 3.11.** Let \( Q \) be a Dynkin quiver and let \((i_k)_{1 \leq k \leq r}\) be a \( Q \)-adapted sequence in \( \Delta_0 \) with \( r \in \mathbb{Z}_{\geq 1} \). Set
\[
(3.22) \quad p_k = \xi_{i_k} - 2 \times \{s \in \mathbb{Z} \mid 1 \leq s < k, i_s = i_k\}.
\]
Then, we have the followings:

(i) \( ((i_k, p_k))_{1 \leq k \leq r} \) is a compatible reading in \( \Delta_0 \).
(ii) \( \phi_{Q'}(i_k, p_k) = \tilde{s}_{i_1} \cdots \tilde{s}_{i_{k-1}}(\alpha_{i_k}, 0) \) for any \( k \in [1, r] \).
(iii) \( s_{i_1} \cdots s_{i_k}w_{i_k} = \tau_Q(\xi_{i_{k-1}} - p_{k-1})^{2 + 1} w_{i_k} \) for any \( k \in [1, r] \).
(iv) Set \( \xi'_j = s_{i_1} \cdots s_{i_j} \xi \). Then we have
\[
(3.23) \quad \xi'_j = \xi_j - 2 \times \{s \in \mathbb{Z} \mid 1 \leq s < r, i_s = j\} \quad \text{for any } j \in \Delta_0,
\]
and
\[
(3.24) \quad \{(i_k, p_k) \mid 1 \leq k \leq r\} = \{(i, p) \in \Delta_0 \mid \xi'_1 < p \leq \xi_i\}.
\]

**Proof.** Let us first prove (i),(ii),(iii). We shall show that
\[
(3.25)_{k,Q} \begin{cases} 
(a) \quad \phi_{Q'}(i_k, p_k) = \tilde{s}_{i_1} \cdots \tilde{s}_{i_{k-1}}(\alpha_{i_k}, 0), \\
(b) \quad p_s > p_k \text{ for any } s \in \mathbb{Z} \text{ such that } 1 \leq s < k \text{ and } d(i_s, i_k) = 1, \\
(c) \quad s_{i_1} \cdots s_{i_k}w_{i_k} = \tau_Q(\xi_{i_{k-1}} - p_{k-1})^{2 + 1} w_{i_k}
\end{cases}
\]
by an induction on \( k \). If \( k = 1 \), it is evident.

Assume that \( 1 < k \leq r \).

We shall show \((3.25)_{k,Q'}\) assuming that \((3.25)_{k-1,Q'}\) for \( Q' = s_{i_k}Q \). Define a sequence \((i'_s)_{1 \leq s \leq r-1}\) by \( i'_s = i_{s+1} \). Then \((i'_s)_{1 \leq s \leq r-1}\) is a \( Q' \)-adapted sequence. For an integer \( a \) such that \( 1 \leq a < \ell \), we have
\[
p_a' := (s_{i_1} \xi')_{i'_1} - 2 \times \{s \in \mathbb{Z} \mid 1 \leq s < a, i'_s = i'_a\}
= (s_{i_1} \xi)_{i_{a+1}} - 2 \times \{s \in \mathbb{Z} \mid 1 \leq s < a, i_{s+1} = i_{a+1}\}
= \xi_{i_{a+1}} - 2 \delta(i_{a+1} = i_1) - 2 \times \{s \in \mathbb{Z} \mid 2 \leq s < a + 1, i_s = i_{a+1}\}
= \xi_{i_{a+1}} - 2 \times \{s \in \mathbb{Z} \mid 1 \leq s < a + 1, i_s = i_{a+1}\}
= p_{a+1}.
\]

Then the induction hypothesis \((3.25)_{k-1,Q'}\) (a) implies that
\[
\phi_{Q'}(i'_k, p_{k-1}) = \tilde{s}_{i_1} \cdots \tilde{s}_{i_{k-2}}(\alpha_{i_{k-1}}, 0).
\]
Hence we have
\[ \phi_Q(i_k, p_k) = \hat{s}_{i_1} \phi_Q(i_{k-1}', p_{k-1}') = \hat{s}_{i_1} \left( \hat{s}_{i_1} \cdots \hat{s}_{i_k-2}(\alpha_{i_k-1}, 0) \right) = \hat{s}_{i_1} \hat{s}_{i_2} \cdots \hat{s}_{i_k-1}(\alpha_{i_k}, 0) \]
by Proposition 3.9 (iv). Thus we have obtained (3.25)_kQ(a).

Let us show (3.25)_kQ(b). Since \( p_s = p_{s-1}' \) if \( 1 < s < k \), (3.25)_{k-1,Q'}(b) implies (3.25)_kQ(b) for \( s \neq 1 \). When \( s = 1 \) and \( d(i_s, i_k) = 1 \), we have
\[ p_1 = \xi_{i_1} > \xi_{i_k} > p_k. \]

Here the first inequality follows from the fact that \( i_1 \) is a source of \( Q \).

Finally let us show (3.25)_kQ(c). By (3.25)_{k-1,Q'}(c), we have
\[ \tau_Q^z \omega_{i_{k-1}'} = s_{i_1} \cdots s_{i_{k-1}'} \omega_{i_{k-1}'}, \]
where \( z = (s_{i_1} \xi_{i_{k-1}' - p_{i_{k-1}'}})/2 + 1 = (\xi_{i_k} - p_k - 2\delta(i_1 = i_k))/2 + 1 \).

Hence we obtain
\[ s_{i_1} \cdots i_k \omega_{i_k} = s_{i_1} \tau_Q^z \omega_{i_{k-1}'} = \tau_Q^z s_{i_1} \omega_{i_k}. \]
Here the last equality follows from Proposition 3.9 (i).

If \( i_1 \neq i_k \), then \( z = (\xi_{i_k} - p_k)/2 + 1 \) and \( s_{i_k} \omega_{i_k} = \omega_{i_k} \), which implies (c). If \( i_1 = i_k \) then, we have \( s_{i_1} \cdots i_k \omega_{i_k} = \tau_Q^{-1} s_{i_1} \omega_{i_k} \).

Since \( s_{i_1} \tau_Q \) does not contain \( s_{i_1} \), we have \( \tau_Q^{-1} s_{i_1} \omega_{i_k} = \omega_{i_k} \). Thus we obtain (c). This completes the proof of (i), (ii), (iii).

Finally let us prove (iv). We shall argue by induction on \( r \). Set \( \xi'' = s_{i_{r-1}} \cdots s_{i_1} \xi \). Then the induction hypothesis implies \( \xi''_j = \xi_j - 2 \times \{|s_i \in \mathbb{Z} \mid 1 \leq s < r, i_s = j\}| \). Hence, if \( j \neq i_r \), then \( \xi''_j = \xi_j'' - 2 \) and
\[ \{|s_i \in \mathbb{Z} \mid 1 \leq s \leq r, i_s = j\} = \{|s_i \in \mathbb{Z} \mid 1 \leq s < r, i_s = j\}| + 1. \]
Thus we obtain (3.23).

The formula (3.24) is obvious by the definition of \( p_k \). \( \square \)

The following proposition says that the set of compatible readings of \((\Gamma_Q)_0\) has a one-to-one correspondence with the set of \( Q \)-adapted reduced expressions of \( w_0 \) (see [1] for ADE-cases).

**Proposition 3.12.** Let \( Q \) be a Dynkin quiver and let \( \ell = |\Phi^+| \).

(i) Let \((i_k, p_k)_{1 \leq k \leq \ell}\) be a compatible reading of \((\Gamma_Q)_0\). Then we have
\begin{enumerate}
  \item \( w_0 := s_{i_1} \cdots s_{i_r} \) is a \( Q \)-adapted reduced expression of \( w_0 \),
  \item \( \phi_Q(i_k, p_k) = (\beta_k^{w_0}, 0) \) for any \( k \) such that \( 1 \leq k \leq \ell \).
\end{enumerate}

(ii) Conversely, let \( w_0 = s_{i_1} \cdots s_{i_r} \) be a \( Q \)-adapted reduced expression of \( w_0 \), and set
\[ p_k = \xi_{i_k} - 2 \times \{|s_i \in \mathbb{Z} \mid 1 \leq s < k, i_s = i_k\}|. \]

Then, \((i_k, p_k)_{1 \leq k \leq \ell}\) is a compatible reading of \((\Gamma_Q)_0\), and
\[ \phi_Q(i_k, p_k) = (\beta_k^{w_0}, 0). \]

Proof. Let us first prove (i).

By the definition of a compatible reading, \( p_1 = \xi_i \) and \( i_1 \) is a source of \( Q \), and \( \phi_Q(i_1, p_1) = (\alpha_{i_1}, 0) \). Set \( \beta_k = s_{i_1} \cdots s_{i_{k-1}} \alpha_{i_k} \).

We shall show that

\[
(3.26)_{k,Q} \begin{cases}
(a) \quad \beta_k \in \Phi^+,
(b) \quad \phi_Q(i_k, p_k) = (\beta_k, 0),
(c) \quad i_a \text{ is a source of } Q_a := s_{i_{a-1}} \cdots s_i Q \text{ for any } a \leq k,
(d) \quad \ell(s_{i_1} \cdots s_k) = k.
\end{cases}
\]

by induction on \( k \). If \( k = 1 \), it is evident.

Assume that \( 1 < k \leq \ell \).

We shall show (3.26) assuming that (3.26)\(_{k-1,Q'} \) with \( Q' = s_{i_1} Q \).

By Theorem 3.10 (iii), we have

\[
(\Gamma_{Q'})_0 = ((\Gamma_Q)_0 \setminus \{(i_1, p_1)\}) \cup \{(i'_1, \xi_i, -h)\}.
\]

Set

\[
(i'_k, p'_k) = \begin{cases}
(i_{k+1}, p_{k+1}) & \text{if } 1 \leq k < \ell, \\
(i'_1, \xi_i, -h) & \text{if } k = \ell.
\end{cases}
\]

Then we can easily see that \( \{(i'_k, p'_k)\}_{1 \leq k \leq \ell} \) is a compatible reading of \( \Gamma_{Q'} \). Hence, by the induction hypothesis (3.26)\(_{k-1,Q'} \), \( \beta'_{k-1} := s_{i'_1} \cdots s_{i'_{k-2}} \alpha_{i'_{k-1}} \) belongs to \( \Phi^+ \) and \( \phi_Q(i'_{k-1}, p'_{k-1}) = (\beta'_{k-1}, 0) \). Hence \( \phi_Q(i_k, p_k) = \widehat{s}_{i_1}(\beta'_{k-1}, 0) \) by Proposition 3.9. Since it belongs to \( \Phi^+ \times \{0\} \), we have \( \phi_Q(i_k, p_k) = (s_{i_1} \beta'_{k-1}, 0) = (\beta_k, 0) \) and \( \beta_k \in \Phi^+ \). Hence we obtain \( \ell(s_{i_1} \cdots s_{i_k}) = 1 + \ell(s_{i_1} \cdots s_{i_{k-1}}) = 1 + (k - 1) = k \). Thus, the induction proceeds and (3.26)\(_{k,Q} \) holds for any \( k \) such that \( 1 \leq k \leq \ell \). In particular, \( s_{i_1} \cdots s_{i_\ell} \) has length \( \ell \) and \( w_0 \) is a reduced expression of \( w_0 \).

(ii) By the preceding Lemma 3.11, it remains to show that \( (i_k, p_k) \in (\Gamma_Q)_0 \). Since \( \phi_Q(i_k, p_k) = \widehat{s}_{i_1} \cdots \widehat{s}_{i_{k-1}}(\alpha_{i_k}, 0) \) by the preceding lemma and \( \beta_k := s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}) \in \Phi^+ \), Lemma 3.6 implies that \( \phi_Q(i_k, p_k) = (\beta_k, 0) \). Hence \( (i_k, p_k) \in (\Gamma_Q)_0 \).

The following proposition follows immediately from Proposition 3.12.

**Proposition 3.13.** The map \( \phi_Q \) induces a quiver isomorphism \( \Gamma_Q \cong \Upsilon_Q \).

**Theorem 3.14** (\( \Delta \)-additive property). Let \( Q = (\Delta, \xi) \) be a Dynkin quiver. For any \( i \in I \) and \( l \in \mathbb{Z} \), we have

\[
(3.27) \quad \tau^l_Q(\gamma_i^Q) + \tau^{l+1}_Q(\gamma_i^Q) = \sum_{j: d(i,j)=1} -\langle h_j, \alpha_i \rangle \tau^{l+(\xi_j - \xi_i + 1)/2}_Q(\gamma_j^Q).
\]
Proof. By Proposition 3.9 (iii), it suffices to prove the assertion when \( i \) is a source of \( Q \). Then the left hand side of (3.27) is computed as
\[
\tau_Q^l(\gamma^Q_i) + \tau_Q^{l+1}(\gamma^Q_i) = \tau_Q^l(1 - \tau_Q)\varpi_i + \tau_Q^{l+1}(1 - \tau_Q)\varpi_i = \tau_Q^l(1 - \tau_Q)(1 + \tau_Q)\varpi_i = \sum_{j; d(i, j) = 1} -\langle h_j, \alpha_i \rangle \tau_Q^l(1 - \tau_Q)\varpi_j,
\]
which implies our assertion. Here \( \tau \) holds by the fact that
\[
(1 + \tau_Q)\varpi_i = 2\varpi_i - \alpha_i = \sum_{j; d(i, j) = 1} -\langle h_j, \alpha_i \rangle \varpi_j
\]
under the assumption that \( i \) is a source of \( Q \). \( \square \)

Remark 3.15. When \( \Delta \) is of simply-laced type, the \( \hat{\Delta} \)-additive property is closely related to the Auslander-Reiten theory for the path algebra \( CQ \), where \( Q \) is a Dynkin quiver on \( \Delta \) (see [18, 48] for instances). For the non simply-laced type \( \Delta \), the path algebra \( CQ \) has infinitely many indecomposable modules and the Auslander-Reiten theory for \( CQ \) is not well-investigated. Thus the relation between the representation theory of \( CQ \) and \( \hat{\Delta} \) is not clear. On the other hand, \( \hat{\Delta} \)-additive property for simply-laced type \( \Delta \) is also closely related to \( T \)-system in the representation theory of quantum affine algebras [41, 21] and unipotent coordinate algebra [5]. Even in the non simply-laced type \( \Delta \), the (3.27) is related to the \( T \)-system among unipotent quantum minors in (6.2) below by considering weights of the unipotent quantum minors.

4. The \((q, t)\)-Cartan matrix specialized at \( q = 1 \) and AR-quivers

In this section, we first recall the \((q, t)\)-Cartan matrix \( C(q, t) \) introduced by Frenkel-Reshetikhin in [10]. Then we prove the inverse \( \tilde{C}(t) \) of the matrix \( C(t) := C(1, t) \) can be obtained from \( \Gamma_Q \) for any Dynkin quiver \( Q \) whose type is the same as the one of \( C(q, t) \). Finally, we give explicit closed formula for \( \tilde{C}(t) \) for an arbitrary Dynkin diagram.

4.1. The \((q, t)\)-Cartan matrix. For an indeterminate \( x \) and \( k \in \mathbb{Z} \), we set,
\[
[k]_x := \frac{x^k - x^{-k}}{x - x^{-1}}.
\]
For an indeterminate \( q \) and \( i \), we set \( q_i := q^{d_i} \). For instance, when \( \Delta \) is of finite type \( G_2 \), we have \( q_2 = q^3 \).

For a given finite Cartan datum, we set \( I = (I_{i,j})_{1 \leq i, j \leq n} \) the adjacent matrix of \( C \) as follows:
\[
I_{i,j} = -\delta(i \neq j)c_{i,j} = 2\delta(i = j) - c_{i,j} \in \mathbb{Z}_{\geq 0} \quad \text{for } i, j \in I.
\]

In [10], the \((q, t)\)-deformation of Cartan matrix \( C(q, t) = (c_{i,j}(q, t))_{i,j \in I} \) is introduced:
\[
c_{i,j}(q, t) := (qt^{-1} + q^{-1}t) \delta(i = j) - [I_{i,j}]_q.
\]
The specialization of $C(q, t)$ at $t = 1$, denoted by $C(q) := C(q, 1)$, is usually called the quantum Cartan matrix.

4.2. $t$-quantized Cartan matrix. In this paper, we study the specialization of $C(q, t)$ at $q = 1$ and its symmetrizations by $D := \text{diag}(d_i \mid i \in I)$ and $D^{-1}$.

**Definition 4.1.** For each finite Cartan datum, we set

$$C(t) := C(1, t)$$

and call it the $t$-quantized Cartan matrix. We also set

$$\mathcal{B}(t) := C(t)D^{-1} = (b_{i,j}(t))_{i,j \in I} \quad \text{and} \quad \overline{\mathcal{B}}(t) := DC(t) = (\overline{b}_{i,j}(t))_{i,j \in I}.$$ 

Hence we have

$$\mathcal{B}(t)_{B_n} = \begin{pmatrix}
\frac{t+t^{-1}}{2} & -\frac{1}{2} & 0 & 0 & \cdots & 0 \\
-\frac{1}{2} & \frac{t+t^{-1}}{2} & -\frac{1}{2} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & -\frac{1}{2} & \frac{t+t^{-1}}{2} & -1 \\
0 & \cdots & \cdots & 0 & -1 & t+t^{-1}
\end{pmatrix},$$

$$\mathcal{B}(t)_{C_n} = \begin{pmatrix}
t + t^{-1} & -1 & 0 & 0 & \cdots & 0 \\
-1 & t + t^{-1} & -1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & -1 & t + t^{-1} & -1 \\
0 & \cdots & \cdots & 0 & -1 & \frac{t+t^{-1}}{2}
\end{pmatrix},$$

$$\mathcal{B}(t)_{F_4} = \begin{pmatrix}
\frac{(t+t^{-1})}{2} & -\frac{1}{2} & 0 & 0 \\
-\frac{1}{2} & \frac{t+t^{-1}}{2} & -1 & 0 \\
0 & -1 & t + t^{-1} & -1 \\
0 & 0 & -1 & \frac{t+t^{-1}}{3}
\end{pmatrix},$$

$$\mathcal{B}(t)_{G_2} = \begin{pmatrix}
t + t^{-1} & -1 \\
-1 & \frac{t+t^{-1}}{3}
\end{pmatrix}.$$ 

**Example 4.2.** Note that, for simply-laced types, we have $\mathcal{B}(t) = C(t)$. The followings are $\mathcal{B}(t)$ for non simply-laced types:

Note that $\mathcal{B}(t)|_{t=1} = \mathcal{B} \in \text{GL}_I(\mathbb{Q})$. We regard $\mathcal{B}(t)$ as an element of $\text{GL}_I(\mathbb{Q}(t))$ and denotes its inverse by $\overline{\mathcal{B}}(t) = (\overline{b}_{i,j}(t))_{i,j \in I}$. Let

$$\overline{b}_{i,j}(t) = \sum_{u \in \mathbb{Z}} \overline{b}_{i,j}(u)t^u.$$
be the Laurent expansion of $\tilde{B}(t)$ at $t = 0$.

Since $t\tilde{B}(t)$ has no pole at $t = 0$ and $(t\tilde{B}(t))|_{t=0} = D^{-1}$, we obtain the following lemma:

**Lemma 4.3.** For any $i, j \in I$ and $u \in \mathbb{Z}$, we have

(1) $\tilde{b}_{i,j}(u) = 0$ if $u < 1$,
(2) $b_{i,j}(1) = \delta(i = j)\delta_i$.

Since $\tilde{B}(t)$ is symmetric, $\tilde{B}(t)$ is also symmetric: $\tilde{b}_{i,j}(u) = \tilde{b}_{j,i}(u)$ for all $i, j \in I$ and $u \in \mathbb{Z}$.

### 4.3. Computations via AR-quivers

In this subsection, we show that $\tilde{B}(t)$ can be computed by reading the AR-quiver $\Gamma_Q$ for an arbitrary Dynkin quiver $Q$. This result is already proved for Dynkin quivers of type $ADE$ in [22] (see also [14]).

We fix a finite Dynkin diagram $\Delta$. Throughout this subsection, $Q$ denotes a Dynkin quiver on $\Delta$.

**Definition 4.4.** For a Dynkin quiver $Q = (\Delta, \xi)$ and $i, j \in \Delta_0$, we define a function $\eta_{i,j}^Q : \mathbb{Z} \to \mathbb{Z}$ by

$$\eta_{i,j}^Q(u) := \begin{cases} (\varpi_i, \tau_Q^{(u+\xi_j-\xi_i-1)/2}(\gamma_j^Q)) & \text{if } u + \xi_j - \xi_i - 1 \in 2\mathbb{Z}, \\ 0 & \text{otherwise}. \end{cases}$$

Note that $\xi_i - \xi_j \equiv d(i, j) \mod 2$.

**Lemma 4.5.** Let $Q'$ be another Dynkin quiver on $\Delta$. Then we have $\eta_{i,j}^Q = \eta_{i,j}^{Q'}$.

**Proof.** For our assertion, it suffices to prove that

$$\eta_{i,j}^Q(u) = \eta_{i,j}^{s_kQ}(u), \quad \text{if } k \in I \text{ is a source of } Q \text{ and } u + \xi_j - \xi_i - 1 \in 2\mathbb{Z}.$$  

When $k \neq i$, we have $s_k \varpi_i = \varpi_i$ and $(s_k\xi)_i = \xi_i$, and hence Proposition 3.9 tells that

$$\eta_{i,j}^Q(u) = (\varpi_i, s_k\tau_Q^{(u+\xi_j-\xi_i-1)/2}(\gamma_j^Q)) = (\varpi_i, \tau_Q^{(u+(s_k\xi)_i)-(s_k\xi)_i-1/2}(\gamma_j^{s_kQ})) = \eta_{i,j}^{s_kQ}(u).$$

When $k = i$, we have $s_i \tau_Q \varpi_i = \varpi_i$ since $s_i \tau_Q$ has a reduced expression without $s_i$. Using Proposition 3.9 once again, we have

$$\eta_{i,j}^Q(u) = (s_k \tau_Q \varpi_i, s_k \tau_Q^{(u+\xi_j-\xi_i+1)/2}(\gamma_j^Q)) = (\varpi_i, \tau_Q^{(u+(s_k\xi)_i)-(s_k\xi)_i-1/2}(\gamma_j^{s_kQ})) = \eta_{i,j}^{s_kQ}(u),$$

which completes our assertion. Note that $(s_k\xi)_i = \xi_i - 2\delta(k = i)$. □

By the above lemma, the following notation is well-defined:

$$\eta_{i,j} := \eta_{i,j}^Q,$$

where $Q$ is an arbitrary Dynkin quiver on $\Delta$.

**Lemma 4.6.** The function $\eta_{i,j}$ satisfies the following properties: for any $u \in \mathbb{Z}$, we have

(i) $\eta_{i,j}(u + h) = -\eta_{i,j}^*(u),$
(ii) $\eta_{i,j}(u - 1) + \eta_{i,j}(u + 1) = \sum_{k : d(k,j) = 1} -\langle h_k, \alpha_j \rangle \eta_{i,k}(u),$
(iii) $-\eta_{i,j}(-u) = \eta_{j,i}(u)$.
(iv) $\eta_{i,j}(0) = 0$.
(v) $\eta_{i,j}(1) = d_i \delta(i = j)$.

Proof. (i) For a Dynkin quiver $Q$ of simply-laced type, it is proved in [12, Lemma 3.7]. For a Dynkin quiver $Q$ of non simply-laced type, it is a consequence of the fact that $\tau_q^{h/2} = -1$.

(ii) follows from the $\tilde{\Delta}$-additive property in Theorem 3.14.

(iii) Take a Dynkin quiver $Q$ and $u \in \mathbb{Z}$ such that $u + \xi_i - \xi_j - 1 \in 2\mathbb{Z}$. Then we have

$$-\eta_{i,j}(-u) = (\omega_i, \tau_q^{(u+\xi_i-\xi_j-1)/2}(\tau_q - 1)\omega_j) = (\tau_q^{(u+\xi_i-\xi_j+1)/2}(\tau_q - 1)\omega_i, \omega_j)$$

$$= (\omega_j, \tau_q^{(u+\xi_i-\xi_j-1)/2}(1-\tau_q)\omega_i) = \eta_{j,i}(u).$$

(iv), (v) By Lemma 4.5, we may assume that $Q$ satisfies that $\xi_j = 1$ and $\xi_k \in \{0,1\}$ for any $k \in \Delta_q$. Then for any $k \in I$ with $\xi_k = 1$, $k$ is a source and hence $\gamma_k^Q = \alpha_k$ by (3.12). For $i,j \in I$ with $\xi_i = \xi_j = 1$, we have $\eta_{i,j}(1) = (\omega_i, \alpha_j) = \delta(i = j)$. When $i,j \in I$ with $\xi_i \neq \xi_j = 1$, we have $\eta_{i,j}(0) = (\omega_i, \alpha_j) = 0$.

\[\text{Theorem 4.7.} \quad \text{We have } \tilde{b}_{i,j}(u) = \eta_{i,j}(u) \text{ for any } i,j \in I \text{ and } u \in \mathbb{Z}_{\geq 0}. \text{ In other words, we have}
\]

$$\tilde{b}_{i,j}(u) = \begin{cases} (\omega_i, \tau_q^{(u+\xi_j-\xi_i-1)/2}(\gamma_j^Q)) & \text{if } u \in \mathbb{Z}_{\geq 0} \text{ and } u + \xi_j - \xi_i - 1 \in 2\mathbb{Z}, \\
0 & \text{otherwise} \end{cases}$$

for any Dynkin quiver $Q$, $i,j \in I$ any $u \in \mathbb{Z}$. In particular, $\tilde{b}_{i,j}(u) \in \mathbb{Z}$.

Proof. Define

$$H_{i,j}(t) := \sum_{u \geq 0} \eta_{i,j}(u)t^u = \sum_{u \geq 1} \eta_{i,j}(u)t^u \in t\mathbb{Z}[[t]] \quad \text{for } i,j \in I.$$ 

Here, $=$ follows from Lemma 4.6 (iv).

It is enough to show

(4.2) \[\sum_{k \in I} H_{i,k}(t)B_{k,j}(t) = \delta(i = j) \quad \text{for any } i,j \in I.\]

Let $x_{i,j}(u)$ be the coefficient of $t^u$ in the left hand side of (4.2). Since $B_{i,j}(t) \in t^{-1}\mathbb{Q}[[t]]$, it is enough to show $x_{i,j}(u) = \delta(u = 0)\delta(i = j)$ for $u \geq 0$. Now we have

$$d_jx_{i,j}(u) = \delta(u > -1)\eta_{i,j}(u+1) + \delta(u \geq 1)\eta_{i,j}(u-1) + \sum_{k \in I, d(k,j) = 1} \delta(u > 0)\eta_{i,k}(u)$$

for any $u \in \mathbb{Z}$. By Lemma 4.6 (ii), $x_{i,j}(u)$ vanishes for $u \geq 1$. Finally, Lemma 4.6 (v) implies

$$d_jx_{i,j}(0) = \eta_{i,j}(1) = d_i \delta(i = j).$$

\[\text{Corollary 4.8.} \quad \text{The coefficients } \{\tilde{b}_{i,j}(u) \mid i,j \in I, \; u \in \mathbb{Z}\} \text{ enjoy the following properties:}
\]

(i) $\tilde{b}_{i,j}(u) = 0$ unless $u \geq 1$ and $u \equiv d(i,j) + 1 \mod 2$. 

(ii) $\tilde{b}_{i,j}(u + h) = -\tilde{b}_{i,j}(u)$ for $u \geq 0$. In particular, $\tilde{b}_{i,j}(h) = 0$.
(iii) $b_{i,j}(u + 2h) = b_{i,j}(u)$ for $u \geq 0$.
(iv) $b_{i,j}(h - u) = \tilde{b}_{i,j}(u)$ for $0 \leq u \leq h$.
(v) $\tilde{b}_{i,j}(2h - u) = -b_{i,j}(u)$ for $0 \leq u \leq 2h$.
(vi) $\tilde{b}_{i,j}(u) \geq 0$ for $0 \leq u \leq h$.
(vii) $b_{i,j}(u) \leq 0$ for $h \leq u \leq 2h$.
(viii) For any Dynkin quiver $Q$ with a height function $\xi$, $i,j \in I$ and $u \in \mathbb{Z}$, we have

$$
\tilde{b}_{i,j}(u) - \tilde{b}_{i,j}(-u) = \begin{cases} 
(\omega_i, \tau_Q^{(u+\xi_j-\xi_i-1)/2}(\gamma_j^Q)) & \text{if } u + \xi_j - \xi_i - 1 \in 2\mathbb{Z}, \\
0 & \text{otherwise.}
\end{cases}
$$

**Proof.** (i) follows from Theorem 4.7 and the fact that $\xi_j - \xi_i \equiv d(i,j) \mod 2$.
(ii, iii) follow from Lemma 4.6 (i), Theorem 4.7 and (i).
(iv) As we have already seen in Lemma 3.8, we have $\xi_j - \xi_i \equiv h \mod 2$. Hence we may assume that $h - u + \xi_j - \xi_i - 1 \equiv u + \xi_j - \xi_i - 1 \equiv 0 \mod 2$. Then, we have

$$
\tilde{b}_{i,j}(h - u) = \eta_{i,j}(h - u) = -\eta_{i,j}(u) = \tilde{b}_{j,i}(u) = \tilde{b}_{i,j}(u).
$$

(v) is a consequence of (ii) and (iv).
(vi) Let us take a Dynkin quiver $Q$ whose height function $\xi$ satisfies $\xi_i = 0$ and $\xi_k \in \{0, 1\}$ for all $k \in \Delta_0$. It is enough to show that $\eta_{i,j}(u) \geq 0$ if $0 < u < h$ and $u + \xi_j - \xi_i - 1 \in 2\mathbb{Z}$. Then $\beta := \tau_Q^{u+\xi_j-\xi_i-1/2}(\gamma_j^Q)$ is a positive root by Theorem 3.10 (ii) and

$$
0 \leq \frac{u + \xi_j - \xi_i - 1}{2} < \frac{h + \xi_j - \xi_i}{2}.
$$

Hence $\eta_{i,j}(u) = (\omega_i, \beta) \geq 0$.
(vii) follows from (vi) and (ii).
(viii) For $u \geq 0$, the right hand side of (4.3) is equal to $\tilde{b}_{i,j}(u)$ and hence it is nothing but Theorem 4.7. As a function in $u$, the right hand side of (4.3) is 2h-periodic since $\tau_Q^h = 1$. Hence it is enough to show that the left hand side of (4.3) is also 2h-periodic:

$$
\tilde{b}_{i,j}(u + 2h) - \tilde{b}_{i,j}(-u - 2h) = \tilde{b}_{i,j}(u) - \tilde{b}_{i,j}(-u)
$$

for all $u \in \mathbb{Z}$. If $u \geq 0$, then we have $b_{i,j}(u + 2h) - b_{i,j}(-u - 2h) = \tilde{b}_{i,j}(u) - \tilde{b}_{i,j}(-u)$ by (iii). If $u \leq -2h$, then we have

$$
\tilde{b}_{i,j}(u + 2h) - \tilde{b}_{i,j}(-u - 2h) = -\tilde{b}_{i,j}(-u - 2h) = \tilde{b}_{i,j}(u) - \tilde{b}_{i,j}(-u)
$$

also by (iii). Finally if $-2h < u < 0$, then (4.4) holds since $\tilde{b}_{i,j}(u + 2h) = -\tilde{b}_{i,j}(-u)$ and $\tilde{b}_{i,j}(-u - 2h) = \tilde{b}_{i,j}(u)$ by (v).

**Proposition 4.9.** For any $i, j \in I$ and $l \in \mathbb{Z}_{\geq 0}$, we have

$$
\tilde{b}_{i,j}(u) = 0 \quad \text{for } u \leq d(i,j).
$$
**Proof.** Since \( \tilde{b}_{i,j}(u) = 0 \) for \( u \leq 0 \), it suffices to prove (4.5) for \( u = d(i, j) - 2l - 1 \) with an integer \( l \) such that \( 0 \leq 2l < d(i, j) - 1 \). Let us take a height function \( \xi \) such that \( \xi_j - \xi_i = d(i, j) \) and \( \xi_j \geq \xi_j^- \). By (4.3), we have
\[
\tilde{b}_{i,j}(-u) - \tilde{b}_{i,j}(u) = (\tilde{\omega}_i, \tilde{\gamma}_Q^l(\gamma_j^Q)).
\] (4.6)

Since \( -u \leq 0 \), we have \( \tilde{b}_{i,j}(-u) = 0 \) by Lemma 4.3. Hence the left hand side of (4.6) is equal to \( -\tilde{b}_{i,j}(u) \), which is non-positive by Corollary 4.8 (vi) since \( u \leq d(i, j) < d \) (see (3.18)). On the other hand, the right hand side of (4.6) is non-negative since \( \tilde{\gamma}_Q^l(\gamma_j^Q) \in \Phi^+ \) by Theorem 3.10 (ii) and
\[
0 \leq l < \frac{d(i, j) - 1}{2} < \frac{h}{2} \leq \frac{\xi_j - \xi_j^-}{2}.
\]

Thus, we obtain \( \tilde{b}_{i,j}(u) = 0 \), as desired. \( \square \)

Together with Corollary 4.8, we have
\[
\tilde{b}_{i,j}(u) = 0 \text{ unless } d(i, j) + 1 \leq u.
\] (4.7)

Here \( u \leq v \) means that \( u \leq v \) and \( u \equiv v \mod 2 \).

**Corollary 4.10.** For \( i, j \in \triangle_0 \), let us define an even function \( \tilde{\eta}_{i,j} : \mathbb{Z} \to \mathbb{Z} \) as follows:
\[
\tilde{\eta}_{i,j}(u) = \tilde{b}_{i,j}(u) + \tilde{b}_{i,j}(-u) \quad \text{for } u \in \mathbb{Z}.
\] (4.8)

Then we have
\[
\tilde{\eta}_{i,j}(u - 1) + \tilde{\eta}_{i,j}(u + 1) + \sum_{k : d(k, j) = 1} \langle h_k, \alpha_j \rangle \tilde{\eta}_{i,k}(u) = 2d_i \delta(u = 0) \delta(i = j).
\]

**Proof.** It is a direct consequence of Lemma 4.6 and Theorem 4.7. \( \square \)

4.4. **Explicit computation.** With the help of Theorem 4.7, we can obtain \( \tilde{b}_{i,j}(u) \) from any AR-quiver \( \Gamma_Q \). In this subsection, we will explicitly compute \( \tilde{b}_{i,j}(u) \) for non simply-laced type.

**Remark 4.11.** By Corollary 4.8 (ii), it is enough to compute
\[
\tilde{\delta}_{i,j}(t) := (1 + t^b) \tilde{b}_{i,j}(z) = \sum_{u=0}^{h-1} \tilde{b}_{i,j}(u) t^u \quad \text{for each } i, j \in I.
\]

4.4.1. **Simply-laced type.** For each simply-laced type \( \triangle \), \( (\tilde{\delta}_{i,j}(t))_{i,j \in \triangle_0} \) was explicitly calculated as follows:

**Theorem 4.12 ([4, 22, 28, 12, 44]).** Note that \( \tilde{\delta}_{i,j}(t) = \tilde{\delta}_{i,j}(t) \) for \( i, j \in I \).

(1) For \( \triangle \) of type \( A_n \), and \( i, j \in I = \{1, \ldots, n\} \), \( \tilde{\delta}_{i,j}(t) \) is given as follows:
\[
\tilde{\delta}_{i,j}(t) = \sum_{s=1}^{\min(i,j,n+1-i,n+1-j)} t^{\mid i-j \mid + 2s - 1}.
\] (4.9)
(2) For $\Delta$ of type $D_{n+1}$, and $i,j \in I = \{1, \ldots, n, n+1\}$, $\tilde{d}_{i,j}(t)$ is given as follows:

$$
\tilde{d}_{i,j}(t) = \begin{cases} 
\sum_{s=1}^{\min(i,j)} \left( t^{i-j+2s-1} + \delta(\max(i,j) < n) t^{2n-i-j+2s-1} \right) & \text{if } \min(i,j) < n, \\
\sum_{s=1}^{\lfloor (n+\min(i,j))/2 \rfloor} t^{4s-1-2\delta(i,j)} & \text{otherwise}. 
\end{cases}
$$

(4.10)

(3) For $\Delta$ of type $E_6$ and $i \leq j \in I = \{1, \ldots, 6\}$, $\tilde{d}_{i,j}(t)$ is given as follows:

$$
\begin{array}{ll}
\tilde{d}_{1,1}(t) = t + t^7, & \tilde{d}_{1,2}(t) = t + t^8, \\
\tilde{d}_{1,3}(t) = t^2 + t^6 + t^8, & \tilde{d}_{1,4}(t) = t^3 + t^5 + t^7 + t^9, \\
\tilde{d}_{1,5}(t) = t^4 + t^6 + t^{10}, & \tilde{d}_{1,6}(t) = t^5 + t^{11}, \\
\tilde{d}_{2,2}(t) = t^1 + t^5 + t^7 + t^{11}, & \tilde{d}_{2,3}(t) = t^3 + t^5 + t^7 + t^9, \\
\tilde{d}_{2,4}(t) = t^2 + t^4 + 2t^6 + t^8 + t^{10}, & \tilde{d}_{3,3}(t) = t^4 + t^3 + t^5 + 2t^7 + t^9, \\
\tilde{d}_{3,4}(t) = t^2 + 2t^4 + 2t^6 + 2t^8 + t^{10}, & \tilde{d}_{3,5}(t) = t^3 + 2t^5 + t^7 + t^9 + t^{11}, \\
\tilde{d}_{4,4}(t) = t^{1} + 2t^{3} + 3t^{5} + 3t^{7} + 2t^{9} + t^{11}, & \tilde{d}_{i,j}(t) = t^b\tilde{d}_{i,j}(t^{-1}) = \tilde{d}_{j,i}(t) = t^b\tilde{d}_{j,i}(t^{-1}).
\end{array}
$$

(4) For $E_7$ and $E_8$, see Appendix A.

4.4.2. $B_n$ and $C_n$ cases. [$B_n$ case] By using an orthogonal basis $\{\varepsilon_i \mid 1 \leq i \leq n\}$ of $\mathbb{R}^n$ with $(\varepsilon_i, \varepsilon_i) = 2$, the simple roots and $\Phi^+_{B_n}$ can be realized as follows:

$$
\alpha_i = \varepsilon_i - \varepsilon_{i+1} \quad \text{for } i < n, \quad \alpha_n = \varepsilon_n,
$$

$$
\Phi^+_{B_n} = \left\{ \varepsilon_i = \sum_{k=i}^{n} \alpha_k \mid 1 \leq i \leq n \right\} \cup \left\{ \varepsilon_i - \varepsilon_j = \sum_{k=i}^{j-1} \alpha_k \mid 1 \leq i < j \leq n \right\}
$$

$$
\cup \left\{ \varepsilon_i + \varepsilon_j = \sum_{k=i}^{j-1} \alpha_k + 2 \sum_{s=j}^{n} \alpha_s \mid 1 \leq i < j \leq n \right\}.
$$

Recall the notations in Example 3.3: $(a, b) := \varepsilon_a - \varepsilon_b$ and $(c) := \varepsilon_c$ for $1 \leq a < b \leq n$ and $1 \leq c \leq n$.

For the Dynkin quiver $Q^o$ in Example 2.3 (1), we have

$$
\tau_{Q^o} = s_{i_1}s_{i_2}\cdots s_{i_n} = s_1s_2\cdots s_n
$$

and the lemma below, by direct calculation:

Lemma 4.13.

(i) $\gamma_k^o = \gamma_k^o = \langle 1, -(k+1) \rangle$ for $k < n$ and $\gamma_n = \langle 1 \rangle$.

(ii) For $k < n$, we have $\tau_{Q^o}^s(\gamma_k^o) = \begin{cases} 
(1 + s, -(k + 1 + s)) & \text{for } 0 \leq s < n - k, \\
(k + s + 1 - n, 1 + s) & \text{for } n - k \leq s \leq n - 1.
\end{cases}$

(iii) $\tau_{Q^o}^s(\gamma_n^o) = \langle s + 1 \rangle$ for $0 \leq s \leq n - 1$. 
[C_n case] Note that \( W \) of type \( C_n \) is isomorphic to the Weyl group of type \( B_n \). By using the orthonormal basis \( \{ \epsilon_i \mid 1 \leq i \leq n \} \) of \( \mathbb{R}^n \), the simple roots and \( \Phi_{C_n}^+ \) can be realized as follows:

\[
\alpha_i = \epsilon_i - \epsilon_{i+1} \quad \text{for } i < n \quad \text{and} \quad \alpha_n = 2\epsilon_n,
\]

\[
\Phi_{C_n}^+ = \left\{ 2\epsilon_i = \sum_{k=1}^n \alpha_k \mid 1 \leq i \leq n \right\} \bigcup \left\{ \epsilon_i - \epsilon_j = \sum_{k=1}^{j-1} \alpha_k \mid 1 \leq i, j \leq n \right\}
\]

\[
\bigcup \left\{ \epsilon_i + \epsilon_j = \sum_{k=1}^{j-1} \alpha_k + 2 \sum_{s=j+1}^{n-1} \alpha_s + \alpha_n \mid 1 \leq i < j \leq n \right\}.
\]

For the Dynkin quiver \( Q^o \) in Example 2.3 (1), \( \tau_{Q^o} \) is the same as (4.11) and we have the following in a similar way as \( B_n \)-case:

**Lemma 4.14.** Setting \( \langle a, \pm b \rangle := \epsilon_a \pm \epsilon_b \) for \( 1 \leq a < b \leq n \) and \( \langle c, c \rangle := 2\epsilon_c \) for \( 1 \leq c \leq n \), we have

(i) \( \gamma_{Q^o}^{\gamma_k} = \sum_{k=1}^n \alpha_k = (1, -(k+1)) \) for \( k < n \) and \( \gamma_n = (1, 1) \),

(ii) \( \tau_{Q^o}^\gamma(\sum_{k=1}^n \alpha_k) = \begin{cases} (1 + s, -(k + 1 + s)) & \text{for } 0 \leq s < n - k, \\ (k + s + 1 - n, 1 + s) & \text{for } n - k \leq s \leq n - 1. \end{cases} \)

For \( Q \) of type \( B_n \) or \( C_n \), recall

\[
(d_i)_{i \in I} = \begin{cases} (2, 2, \ldots, 2, 1) & \text{if } Q \text{ is of type } B_n, \\ (1, 1, \ldots, 1, 2) & \text{if } Q \text{ is of type } C_n, \end{cases} \quad \text{and} \quad h = 2n.
\]

**Theorem 4.15.** For \( \blacktriangle \) of type \( B_n \) or \( C_n \), and \( i, j \in I = \{1, \ldots, n\} \), the closed formula of \( \tilde{B}_{i,j}(t) \) is given as follows: for any \( i, j \in I \) such that \( i \leq j \), we have \( \tilde{B}_{i,j}(t) = \tilde{B}_{i,j}(t) \) and

\[
(4.12) \quad \tilde{B}_{i,j}(t) = \begin{cases} \max(d_i, d_j) \sum_{s=1}^{i} t^{n-i-1+2s} & \text{if } i \leq j = n, \\ \max(d_i, d_j) \sum_{s=1}^{i} (t^{j-i+2s-1} + t^{2n-j+i+2s-1}) & \text{if } i \leq j < n. \end{cases}
\]

**Proof.** By Theorem 4.7 and Corollary 4.8, (4.12) comes from Lemma 4.13 and Lemma 4.14 which gives explicit computations for \( \tau_{Q^o}^\gamma(\sum_{k=1}^n \alpha_k) \) (0 \( \leq k \) < \( n \), \( j \in I \) in terms of \( \langle a, \pm b \rangle \) (1 \( \leq a \leq b \leq n \)) and \( \langle c \rangle \) (1 \( \leq c \leq n \)).
Example 4.16. For ▲ of types $B_3$ and $C_3$, $\widetilde{B}_{B_3}^B(t)$ and $\widetilde{B}_{C_3}^C(t)$ are given as follows:

\[
\widetilde{B}_{B_3}^B(t) = \frac{1}{1 + t^6} \begin{pmatrix}
2(t^5 + t) & 2(t^4 + t^2) & 2t^3 \\
2(t^4 + t^2) & 2(t^5 + 2t^3 + t) & 2(t^4 + t^2) \\
2t^3 & 2(t^4 + t^2) & t^5 + t^3 + t
\end{pmatrix},
\]

\[
\widetilde{B}_{C_3}^C(t) = \frac{1}{1 + t^6} \begin{pmatrix}
t^5 + t & t^4 + t^2 & 2t^3 \\
t^4 + t^2 & t^5 + 2t^3 + t & 2(t^4 + t^2) \\
2t^3 & 2(t^4 + t^2) & 2(t^5 + t^3 + t)
\end{pmatrix}.
\]

4.4.3. $F_4$ and $G_2$ cases. The simple roots for these cases can be expressed as follows:

(1) For $F_4$-case, we use the notation $(a, b, c, d) := a\varepsilon_1 + b\varepsilon_2 + c\varepsilon_3 + d\varepsilon_4$ where $\{\varepsilon_i\}_{1 \leq i \leq 4}$ is an orthogonal basis with the square length 2. Then

\[
\alpha_1 = (0, 1, -1, 0), \quad \alpha_2 = (0, 0, 1, -1), \quad \alpha_3 = (0, 0, 0, 1) \quad \text{and} \quad \alpha_4 = (1/2, -1/2, -1/2, -1/2).
\]

(2) For $G_2$-case, we use an orthonormal basis $\{\varepsilon_i | 1 \leq i \leq 3\}$ of $\mathbb{R}^3$ and the notation $(a, b, c) := a\varepsilon_1 + b\varepsilon_2 + c\varepsilon_3$. Then

\[
\alpha_1 = (0, 1, -1) \quad \text{and} \quad \alpha_2 = (1, -2, 1).
\]

The AR-quivers $\Gamma_{Q^B}$ for Dynkin quivers in Example 2.3 are depicted as follows:

\[
\begin{array}{cccccccc}
(i \setminus p) & -9 & -8 & -7 & -6 & -5 & -4 & -3 & -2 \quad \text{and} \quad -1 & 0 & 1 & 2 & 3 & 4 \\
\hline
1 & (1, 0, -1, 0) & (0, 0, 1, 1) & (1, 0, 0, -1) & (0, 1, 0, 1) & (0, 0, 1, -1) & (0, 1, -1, 0) \\
2 & (1, -1, 0, 0) & (1, 0, 0, 1) & (1, 0, 1, 0) & (1, 1, 0, 0) & (0, 1, 1, 0) & (0, 1, -1, 0) \\
3 & (\frac{3}{1} - \frac{3}{1} - \frac{3}{1}) & (\frac{3}{1} - \frac{3}{1} - \frac{3}{1}) & (\frac{3}{1} - \frac{3}{1} - \frac{3}{1}) & (\frac{3}{1} - \frac{3}{1} - \frac{3}{1}) & (\frac{3}{1} - \frac{3}{1} - \frac{3}{1}) & (\frac{3}{1} - \frac{3}{1} - \frac{3}{1}) \\
4 & (\frac{3}{1} - \frac{3}{1} - \frac{3}{1}) & (\frac{3}{1} - \frac{3}{1} - \frac{3}{1}) & (\frac{3}{1} - \frac{3}{1} - \frac{3}{1}) & (\frac{3}{1} - \frac{3}{1} - \frac{3}{1}) & (\frac{3}{1} - \frac{3}{1} - \frac{3}{1}) & (\frac{3}{1} - \frac{3}{1} - \frac{3}{1})
\end{array}
\]

and

\[
\begin{array}{cccccccc}
(i \setminus p) & -3 & -2 & -1 & 0 & 1 & 2 \\
\hline
1 & (0, 1, -1) & (1, 0, -1) & (1, -1, 0) \\
2 & (1, 1, -2) & (2, -1, -1) & (1, -2, 1)
\end{array}
\]

Using AR-quivers in (14.13) and (14.14), we can obtain $\left(1 + t^{h}\right)\widetilde{B}_{B,j}(t)$ for these cases:

\[
\left(1 + t^{12}\right)\widetilde{B}_{B,j}(t) = \begin{pmatrix}
2(t_{11} + t^7 + t^5 + t) & 2(t_{10} + t^8 + 2t^6 + t^4 + t^2) & 2(t^9 + t^7 + t^5 + t^3) & 2(t^8 + t^4) \\
2(t_{10} + t^8 + 2t^6 + t^4 + t^2) & 2(t_{11} + 2t^9 + 3t^7 + 3t^5 + 2t^3 + t) & 2(t_{10} + 2t^8 + 2t^6 + 2t^4 + t^2) & 2(t^9 + t^7 + t^5 + t^3) \\
2(t^9 + t^7 + t^5 + t^3) & 2(t^8 + 2t^6 + 2t^4 + t^2) & t_{11} + 2t^9 + 3t^7 + 3t^5 + 2t^3 + t & t_{10} + t^8 + 2t^6 + t^4 + t^2 \\
2(t^8 + t^4) & 2(t^9 + t^7 + t^5 + t^3) & t_{10} + t^8 + 2t^6 + t^4 + t^2 & t_{11} + t^7 + t^5 + t
\end{pmatrix}
\]

and

\[
\left(1 + t^{6}\right)\widetilde{B}_{G,j}^{2}(t) = \begin{pmatrix}
t^5 + 2t^3 + t & 3(t^4 + t^2) \\
3(t^4 + t^2) & 3(t^5 + 2t^3 + t)
\end{pmatrix}.
\]
4.4.4. **Remark on the inverses of** $\mathcal{C}(q)$ and $\mathcal{C}(q,t)$. Note that $\mathbb{B}(t)$ can be obtained from the quantum Cartan matrix $\mathcal{C}(q)$ by just replacing $q$ with $t$, when $\mathfrak{g}$ is of simply-laced type. The inverse $\tilde{\mathcal{C}}(q)$ of quantum Cartan matrix $\mathcal{C}(q)$ also enjoys the similar properties of $\tilde{\mathbb{B}}(t)$ even in the non simply-laced type. We refer [15] for $\tilde{\mathcal{C}}(q)$ of non simply-laced type.

Very recently, in [14], Fujita-Murakami investigated the behavior of the inverse $\tilde{\mathcal{C}}(q,t)$ of $\mathcal{C}(q,t)$, as an matrix with entries in $\mathbb{Z}((q,t))$, which implies the several properties of $\tilde{\mathbb{B}}(t)$ in Corollary 4.8 and an implicit computation of $\tilde{\mathbb{B}}(t)$ via the specialization at $q = 1$. However, as far as the authors understand, they do (i) not use the Coxeter elements and (ii) not give explicit formulas of entries of $\tilde{\mathcal{C}}(q,t)$ and hence $\tilde{\mathbb{B}}(t)$.

5. **Quantum torus associated with a Dynkin diagram**

In this section, we construct a quantum torus related to the $t$-quantized Cartan matrix. Then we investigate the structure of the quantum torus.

**Definition 5.1.** Let $q$ be an indeterminate, and let $\mathcal{X}_q$ be the $\mathbb{Z}[q^{\pm \frac{1}{2}}]$-algebra with the generators $\{\tilde{X}_{i,p}^\pm | (i, p) \in \tilde{\Delta}_0\}$ and the following defining relations:

- $\tilde{X}_{i,p}^{-1}\tilde{X}_{i,p}^{-1} = \tilde{X}_{i,p}^{-1}\tilde{X}_{i,p}^{-1} = 1$ for any $(i, p) \in \tilde{\Delta}_0$,
- $\tilde{X}_{i,p}\tilde{X}_{j,s} = q^{\mathcal{N}(i,p; j,s)}\tilde{X}_{j,s}\tilde{X}_{i,p}$ for $(i, p), (j, s) \in \tilde{\Delta}_0$,

where we set (see (4.8)):

\begin{align}
\mathcal{N}(i, p; j, s) := & \tilde{\eta}_{i,j}(p-s-1) - \tilde{\eta}_{i,j}(p-s+1) \\
& - \tilde{b}_{i,j}(p-s-1) - \tilde{b}_{i,j}(s-p-1) - \tilde{b}_{i,j}(p-s+1) + \tilde{b}_{i,j}(s-p+1).
\end{align}

We call $\mathcal{X}_q$ the quantum torus associated with $\mathcal{C}(t)$.

**Remark 5.2.**

(i) For simply-laced type $\Delta$, the quantum torus $\mathcal{X}_q$ is already defined in [42, 49, 20] in the context of the quantum Grothendieck ring of $\mathbb{C}^0\mathfrak{g}$ of $U_q'(\mathfrak{g})$. More precisely, the $q$-commutation relation in (5.1) coincides with the $t$-commutation relation in [42, 49, 20] when $\Delta$ is of simply-laced type, which is defined by using quantum Cartan matrix $\mathcal{C}(q)$ and its inverse $\tilde{\mathcal{C}}(q)$.

(ii) Note that

\[ \mathcal{N}(i, p; j, s) = \mathcal{N}(j, p; i, s) = -\mathcal{N}(i, s; j, p) = -\mathcal{N}(j, s; i, p) \]

since $\tilde{\mathbb{B}}(t)$ is symmetric. Also we have

\begin{align}
\mathcal{N}(i, p; j, s) := & \tilde{b}_{i,j}(p-s-1) - \tilde{b}_{i,j}(p-s+1) \\
& \quad \text{if } p > s
\end{align}

by Lemma 4.3.

We say that $\tilde{m} \in \mathcal{X}_q$ is a $\mathcal{X}_q$-monomial if it is a product of the generators $\tilde{X}_{i,p}^{\pm 1}, q^{\pm 1/2}$. For a $\mathcal{X}_q$-monomial $\tilde{m}$, writing $\tilde{m} = q^c \prod_{(i,p) \in \tilde{\Delta}_0} \tilde{X}_{i,p}^{\nu_{i,p}}$ with $c \in \mathbb{Z}, \nu_{i,p} \in \mathbb{Z}$ (product is taken in some order), we set $u_{i,p}(\tilde{m}) = \nu_{i,p}$. 
For $\mathcal{X}_q$-monomials $\tilde{m}$ and $\tilde{m}'$, we set
\begin{equation}
\mathcal{N}(\tilde{m}, \tilde{m}') := \sum_{(j,p),(i,s) \in \tilde{\Delta}_0} u_{j,p}(\tilde{m})u_{i,s}(\tilde{m}') \mathcal{N}(j,p; i, s).
\end{equation}
Hence we have
\begin{equation}
\tilde{m} \tilde{m}' = q^{\mathcal{N}(\tilde{m}, \tilde{m}')} \tilde{m}'.
\end{equation}
Note that there exists a $\mathbb{Z}$-algebra anti-involution $\overline{\cdot}$ on $\mathcal{X}_q$ given by
\begin{equation}
q^{1/2} \mapsto q^{-1/2} \quad \text{and} \quad \tilde{X}_{i,p} \mapsto q^{d_i} \tilde{X}_{i,p}.
\end{equation}
Then, for any $\mathcal{X}_q$-monomial $\tilde{m} \in \mathcal{X}_q$, there exists a unique $r \in \frac{1}{2}\mathbb{Z}$ such that $q^r \tilde{m}$ is $\overline{\cdot}$-invariant. An element of this form is called a bar-invariant monomial.

**Definition 5.3** (cf. [13, Definition 5.5]). For a subset $\mathcal{S} \subset \tilde{\Delta}_0$, we denote by $\mathcal{X}_{q,\mathcal{S}}$ the quantum subtorus of $\mathcal{X}_q$ generated by $\tilde{X}_{i,p}^{\pm 1}$ for $(i, p) \in \mathcal{S} \subset \tilde{\Delta}_0$. In particular, for a Dynkin quiver $Q = (\Delta, \xi)$, we denote by $\mathcal{X}_{q,Q}$ the quantum subtorus of $\mathcal{X}_q$ generated by $\tilde{X}_{i,p}^{\pm 1}$ for $(i, p) \in (\Gamma_Q)_0 \subset \tilde{\Delta}_0$.

The following theorem is a generalization of [22, Proposition 3.1] and [15, Proposition 5.21]:

**Theorem 5.4.** Let $(i, p), (j, s) \in \tilde{\Delta}_0$ and let $Q$ be a Dynkin quiver on $\Delta$. Set $\phi_Q(i, p) = (\alpha, k)$ and $\phi_Q(j, s) = (\beta, l)$. Then, we have
\begin{equation}
\mathcal{N}(i, p; j, s) = (-1)^{k+l+\delta(p > s)} \delta((i, p) \neq (j, s)) (\alpha, \beta)
\end{equation}

**Proof.** First we assume that $p > s$. Then by Theorem 4.7 and (5.2), we have
\begin{align*}
\mathcal{N}(i, p; j, s) &= -(\overline{\omega}_i, \tau_Q^{(p-s+\xi_i-\xi_j)/2}(\gamma_j^Q) - \tau_Q^{(p-s+\xi_i-\xi_j)/2-1}(\gamma_j^Q)) \\
&= -(\tau_Q^{(\xi_i-p)/2}(1 - \tau_Q^{(\xi_i-p)/2}(\gamma_i^Q), \tau_Q^{(\xi_i-s)/2}(\gamma_j^Q)) = -(\tau_Q^{(\xi_i-p)/2}(\gamma_i^Q), \tau_Q^{(\xi_i-s)/2}(\gamma_j^Q)) \\
&= (-1)^{1+k+l}(\alpha, \beta).
\end{align*}

By the skew-symmetry, we obtain the assertion for $p < s$. When $p = s$, the left hand side obviously vanished and the right hand side vanishes also since $(\alpha, \beta) = 0$ if $i \neq j$ by Theorem 3.2 (iii).

For $a, b \in \mathbb{Z}$ such that $a \leq b$ and $i \in I$, we define a $\mathcal{X}_q$-monomial $\tilde{m}^{(i)}[a, b]$ by
\begin{equation}
\tilde{m}^{(i)}[a, b] = \prod_{(i,p) \in \tilde{\Delta}_0, a \leq p \leq b} \tilde{X}_{i,p}.
\end{equation}

Note that (see (5.3))
\begin{equation}
\mathcal{N}(\tilde{m}^{(i)}[p, p'], \tilde{m}^{(j)}[s, s']) := \sum_{(i, x), (j, y) \in \tilde{\Delta}_0; \ p \leq x \leq p', \ s \leq y \leq s'} \mathcal{N}(i, x; j, y).
\end{equation}

The following proposition is a generalization of [13, Proposition 8.4]:
Proposition 5.5. Let $Q = (\Delta, \xi)$ be a Dynkin quiver. Let $(i,p), (i,p'), (j,s), (j,s') \in \hat{\Delta}_0$ with $p \leq p'$ and $s \leq s'$. Assuming $p - s \leq d(i,j)$ and $s' - p' \leq d(i,j)$, we have

\[ \mathcal{N}(\tilde{m}^{(i)}[p,p'], \tilde{m}^{(j)}[s,s']) = \left( \tau_Q^{(\xi_i - p)/2 + \eta_i} + \tau_Q^{(\xi_i - p')/2 + \eta_i}, \tau_Q^{(\xi_j - s)/2 + \eta_j} - \tau_Q^{(\xi_j - s')/2 + \eta_j} \right). \]

Proof. Recall that $u \leq v$ means that $u \leq v$ and $u \equiv v \mod 2$.

By the definition of $\mathcal{N}(\tilde{m}^{(i)}[p,p'], \tilde{m}^{(j)}[s,s'])$, we have

\[ \mathcal{N}(\tilde{m}^{(i)}[p,p'], \tilde{m}^{(j)}[s,s']) = \sum_{x : p \leq x \leq p'} \sum_{y : x \leq y \leq x} \mathcal{N}(i, x; j, y) \]

where the last equality follows from Proposition 4.9. On the other hand, we compute

\[ \left( \tau_Q^{(\xi_i - p)/2 + \eta_i} + \tau_Q^{(\xi_i - p')/2 + \eta_i}, \tau_Q^{(\xi_j - s)/2 + \eta_j} - \tau_Q^{(\xi_j - s')/2 + \eta_j} \right) \]

\[ = \sum_{y : s \leq y \leq s'} \left( \tau_Q^{(\xi_i - y)/2 + \eta_i} + \tau_Q^{(\xi_i - y')/2 + \eta_i}, \tau_Q^{(\xi_j - y)/2 + \eta_j} - \tau_Q^{(\xi_j - y')/2 + \eta_j} \right) \]

\[ = \sum_{y : s \leq y \leq s'} \left( \tau_Q^{(\xi_i - p+y-1) + \xi_j - y}/2 + \gamma \right) \]

\[ = \sum_{y : s \leq y \leq s'} \left( \tilde{b}_{i,j}(p - y - 1) - \tilde{b}_{i,j}(y - p + 1) \right) \]

where $= \text{ follows from (4.3)},$ and we again used Proposition 4.9 for $= \text{.}$ From the above two computations, we obtain the conclusion. \hfill \Box

For a $\mathcal{X}_q$-monomial $\tilde{m}$, we define

\[ \text{wt}_Q(\tilde{m}) := \sum_{(i,p) \in \hat{\Delta}_0} u_{i,p}(\tilde{m}) \pi(\phi_q(i,p)) \in \mathbb{Q}. \]

Here we set

\[ \pi((\beta, k)) := (-1)^k \beta \text{ for } (\beta, k) \in \hat{\Phi}^+. \]
We call $\wt_Q(\tilde{m})$ the $Q$-weight of $\tilde{m}$. With this definition, (5.6) reads as
\begin{equation}
\mathcal{N}(i,p;j,s) = (-1)^{\delta(p<s)}\delta((i,p) \neq (j,s)) \left( \wt_Q(\tilde{X}_{i,p}), \wt_Q(\tilde{X}_{j,s}) \right).
\end{equation}

Recall that $C = (c_{i,j})_{i,j}$ is the Cartan matrix. For $(i,p+1) \in \widehat{\Delta}_0$, we set
\begin{equation}
\tilde{B}_{i,p} := q^{n_{i,p}}\tilde{X}_{i,p-1}\tilde{X}_{i,p+1} \prod_{j: d(i,j)=1} \tilde{X}_{j,p}^{c_{j,i}},
\end{equation}
where we choose $n_{i,p} \in \frac{1}{2}\mathbb{Z}$ such that $\tilde{B}_{i,p}$ is a bar-invariant $X_q$-monomial.

The following corollary (cf. [15, (5.3)]) follows from Theorem 3.14.

**Corollary 5.6.** For $(i,p) \in I \times \mathbb{Z}$ with $(i,p+1) \in \widehat{\Delta}_0$ and any Dynkin quiver $Q$, we have
\[ \wt_Q(\tilde{B}_{i,p}) = 0. \]

**Proof.** It is easy to see that
\begin{equation}
\pi(\tilde{w}^m(\beta,k)) = w^m(\pi(\beta,k)) \quad \text{for any } w \in W, m \in \mathbb{Z} \text{ and } (\beta,k) \in \Phi^+.
\end{equation}
Hence, we have
\[
\wt_Q(\tilde{B}_{i,p}) = \pi(\phi_Q(i,p-1)) + \pi(\phi_Q(i,p+1)) + \sum_{j: d(i,j)=1} c_{j,i} \pi(\phi_Q(j,p))
\]
\[
= \pi(\tilde{\tau}_Q(\xi_i-p+1/2(\gamma_i,Q),0)) + \pi(\tilde{\tau}_Q(\xi_i-p-1/2(\gamma_i,Q),0)) + \sum_{j: d(i,j)=1} c_{j,i} \pi(\tilde{\tau}_Q(\xi_j-p)/2(\gamma_j,Q),0)
\]
\[
= \tau_Q(\xi_i-p+1/2(\gamma_i,Q)) + \tau_Q(\xi_i-p-1/2(\gamma_i,Q)) + \sum_{j: d(i,j)=1} c_{j,i} \tau_Q(\xi_j-p)/2(\gamma_j,Q),
\]
which vanishes by Theorem 3.14. \qed

**Example 5.7.** Let $Q$ be a Dynkin quiver of type $B_3$ in (3.4). Then we have
\[ \wt_Q(\tilde{X}_{1,1}) = \langle 2,-3 \rangle, \quad \wt_Q(\tilde{X}_{2,2}) = \langle 1,-3 \rangle, \quad \wt_Q(\tilde{X}_{2,0}) = \langle 1,2 \rangle, \quad \wt_Q(\tilde{X}_{3,1}) = \langle 1 \rangle, \]
by using (3.5), and hence
\[ \wt_Q(\tilde{X}_{2,0}\tilde{X}_{2,2}\tilde{X}_{1,1}^{-1}\tilde{X}_{3,1}^{-2}) = \wt_Q(\tilde{B}_{2,1}) = 0. \]

Also, one can check that all the monomials in the following element in $X_q$ have the same $Q$-weight as $\varepsilon_2 - \varepsilon_3$:
\begin{equation}
q\tilde{X}_{1,1} + q\tilde{X}_{2,2}\tilde{X}_{1,3}^{-1} + q^2\tilde{X}_{2,2}^{-1}\tilde{X}_{2,4}^{-1} + (q^{-1} + q)\tilde{X}_{3,3}\tilde{X}_{3,5}^{-1}
\end{equation}
\[
+ q^2\tilde{X}_{2,4}\tilde{X}_{3,5}^{-2} + q\tilde{X}_{1,5}\tilde{X}_{2,6}^{-1} + q^{-1}\tilde{X}_{1,7}^{-1}.
\]

The following proposition is a generalization of [20, Proposition 3.12]:

**Proposition 5.8.** For $i,j \in I$ and $p,s,t,u \in \mathbb{Z}$ with $(i,p),(j,s+1),(i,t+1),(j,u+1) \in \widehat{\Delta}_0$, we have
\[ \tilde{X}_{i,p}\tilde{B}_{j,s}^{-1} = q^{\beta(i,p;j,s)} \tilde{B}_{j,s}^{-1}\tilde{X}_{i,p} \quad \text{and} \quad \tilde{B}_{i,t}^{-1}\tilde{B}_{j,u}^{-1} = q^{\alpha(i,t;j,u)} \tilde{B}_{j,u}^{-1}\tilde{B}_{i,t}^{-1}. \]
Here,

\begin{equation}
\beta(i, p; j, s) = \delta(i = j)(-\delta(p - s = 1) + \delta(p - s = -1))(\alpha_i, \alpha_i),
\end{equation}

\begin{equation}
\alpha(i, t; j, u) = \begin{cases}
\pm(\alpha_i, \alpha_i) & \text{if } (i, t) = (j, u \pm 2), \\
\pm 2(\alpha_i, \alpha_j) & \text{if } d(i, j) = 1 \text{ and } t = u \pm 1, \\
0 & \text{otherwise}.
\end{cases}
\end{equation}

**Proof.** Let us prove first \((5.13)\).

The indices in \(\hat{\Delta}_0\) appearing in \(\tilde{B}_{j,s}\) can be described as follows:

\[
\begin{array}{c}
\varepsilon_{j,j'} \to (j', s) \\
(j, s - 1) \\
\varepsilon_{j,j'} \to (j, s + 1)
\end{array}
\]

for \(j'\) with \(d(j', j) = 1\).

Let us first assume that (a) \(|p - s| > 1\) or (b) \(d(i, j) > 1\) and \(|p - s| \leq 1\). In these cases, we have \((i, p) \cap \{(j, s + 1), (j, s - 1)\} \cup \{(j', s) \mid i' \in \Delta_0, d(j', j) = 1\} = \emptyset\).

(a) In this case, we have \(p > s + 1\) or \(p < s - 1\). Then \((5.9)\) implies that

\[
\mathcal{N}'(\tilde{X}_{i,p}, \tilde{B}_{j,s}^{-1}) = (-1)^{\delta(p > s + 1)}(\text{wt}_Q(\tilde{X}_{i,p}), \text{wt}_Q(\tilde{B}_{j,s}^{-1})) = 0.
\]

where the last equality follows from Corollary 5.6.

(b) In this case, there are no path between \((i, p)\) and \((j, s \pm 1)\), as well as between \((i, p)\) and \((j', s)\) with \(d(j', j) = 1\). Thus Theorem 3.2 (iii) tells that

\[
(\text{wt}_Q(\tilde{X}_{i,p}), \text{wt}_Q(\tilde{X}_{j,s \pm 1})) = (\text{wt}_Q(\tilde{X}_{i,p}), \text{wt}_Q(\tilde{X}_{j',s})) = 0.
\]

Thus \(\mathcal{N}'(\tilde{X}_{i,p}, \tilde{B}_{j,s}^{-1}) = 0\).

(c) Assume that \(d(i, j) = 1\) and \(p = s\). Then, the relevant terms are

\[
\begin{array}{c}
(j', s) \\
(j, s - 1) \\
(i, s) \\
(j, s + 1)
\end{array}
\]

\(d(j', j) = 1\), \(j' \neq i\).

In this case, Theorem 3.2 (iii) tells that

\[
(\text{wt}_Q(\tilde{X}_{i,s}), \text{wt}_Q(\tilde{X}_{j',s})) = 0.
\]

Also, [45, Proposition 2.15] tells that

\[
(\text{wt}_Q(\tilde{X}_{i,s}), \text{wt}_Q(\tilde{X}_{j,s - 1})) = (\text{wt}_Q(\tilde{X}_{i,s}), \text{wt}_Q(\tilde{X}_{j,s + 1})).\]
Then Theorem 5.4 tells that
\[
\Delta(\tilde{X}_{i,s}, \tilde{B}^{-1}_{j,s}) = - \sum_{j' \neq i,j} c_{j',j} (1)^{s+1} (\text{wt}_Q(\tilde{X}_{i,s}), \text{wt}_Q(\tilde{X}_{j',s})) \\
- (1)^{s+1} (\text{wt}_Q(\tilde{X}_{i,s}), \text{wt}_Q(\tilde{X}_{j,s+1})) \\
= - (\text{wt}_Q(\tilde{X}_{i,s}), \text{wt}_Q(\tilde{X}_{j,s+1}) + (\text{wt}_Q(\tilde{X}_{i,s}), \text{wt}_Q(\tilde{X}_{j,s-1})) = 0.
\]

(d) Now let us assume that \( i = j \) and \( |p - s| = 1 \), which is the only remained case. Then, \((i, p)\) is one of \((j, s \pm 1)\) and
\[
(i, p) = (j, s - 1) \quad (j, s + 1) \quad (j, s - 1) \quad (i, p) = (j, s + 1)
\]

(1) In this case, Theorem 5.4 tells that
\[
\Delta(\tilde{X}_{i,s-1}, \tilde{B}^{-1}_{j,s}) = - \sum_{j' \neq j} c_{j',j} (1)^{s+1} (\text{wt}_Q(\tilde{X}_{j,s-1}), \text{wt}_Q(\tilde{X}_{j',s})) \\
= (\text{wt}_Q(\tilde{X}_{j,s-1}), -\text{wt}_Q(\tilde{B}_{j,s}) + \text{wt}_Q(\tilde{X}_{j,s-1})) \\
= (\text{wt}_Q(\tilde{X}_{j,s-1}), \text{wt}_Q(\tilde{X}_{j,s-1})).
\]

Here the third equality holds by Corollary 5.6. Finally the assertion for this case is completed by (3.14).

(2) In the case \( p = s + 1 \), we have
\[
\Delta(\tilde{X}_{i,s+1}, \tilde{B}^{-1}_{j,s}) = - \sum_{j' \neq j} c_{j',j} (1)^{s+1} (\text{wt}_Q(\tilde{X}_{j,s+1}), \text{wt}_Q(\tilde{X}_{j',s})) \\
- (1)^{s+1} (\text{wt}_Q(\tilde{X}_{j,s+1}), \text{wt}_Q(\tilde{X}_{j,s-1})) \\
= (\text{wt}_Q(\tilde{X}_{j,s+1}), \text{wt}_Q(\tilde{B}_{j,s}) - \text{wt}_Q(\tilde{X}_{j,s-1})) \\
= - (\text{wt}_Q(\tilde{X}_{i,s+1}), \text{wt}_Q(\tilde{X}_{i,s-1})).
\]

Thus we complete the proof of (5.13).

Now, let us prove (5.14). We may assume that \((i, t) \neq (j, u)\).
Assume that
\[
\Delta(\tilde{X}_{i',k}, \tilde{B}^{-1}_{j',\tilde{a}}) \neq 0
\]
for some factor \( \tilde{X}_{i',k} \) of \( \tilde{B}^{-1}_{j',\tilde{a}} \). Then, (5.13) implies that we have either (1) \( i = j \) and \(|t - u| = 2\), or (2) \( d(i, j) = 1 \) and \(|u - t| = 1\).

Hence, we can assume (1) or (2).

(1) Assume that \( i = j \) and \(|t - u| = 2\). Then (5.13) implies that
\[
\alpha(i, t; j, u) = \begin{cases} 
- \beta(i, u - 1; j, u) = -(\alpha_i, \alpha_t) & \text{if } (i, t) = (j, u - 2), \\
- \beta(i, u + 1; j, u) = (\alpha_i, \alpha_t) & \text{if } (i, t) = (j, u + 2).
\end{cases}
\]
(2) Assume that $d(i, j) = 1$ and $|t - u| = 1$. By (5.13), we have
\[
\alpha(i, t; j, u) = \begin{cases} 
-c_{j,i}(j, u - 1; j, u) = -c_{j,i}(\alpha_j, \alpha_j) = -2(\alpha_i, \alpha_j) & \text{if } t - u = -1, \\
-c_{j,i}(j, u + 1; j, u) = c_{j,i}(\alpha_j, \alpha_j) = 2(\alpha_i, \alpha_j) & \text{if } t - u = 1,
\end{cases}
\]
as we desired. \hfill \square

**Definition 5.9.** For $i \in \triangle_0$, we denote by $\mathfrak{R}_{i,q}$ the $\mathbb{Z}[q^{\pm \frac{1}{2}}]$-subalgebra of $\mathcal{X}_q$ generated by
\[(5.15) \quad \tilde{X}_{i,l}(1 + q^{-d_i}B_{i,l+1}^{-1}), \quad \tilde{X}_{j,l} \quad (j \in \triangle_0 \setminus \{i\}, \ l \in \mathbb{Z}).\]
We set
\[\mathfrak{R}_q := \bigcap_{i \in I} \mathfrak{R}_{i,q}\]
and call it the *quantum virtual Grothendieck ring associated to $\mathbb{C}(t)$*.

**Remark 5.10.** When $\mathfrak{g}$ is associated with $\triangle$ of simply-laced type, $\mathfrak{R}_q$ is isomorphic to the quantum Grothendieck ring $\mathfrak{R}_t(\mathfrak{g}) := K_t(\mathcal{O}_q^0)$ invented in [42, 49, 20]. Following the construction of $\mathfrak{R}_t(\mathfrak{g})$ for all $\mathfrak{g}$ in [20], $\mathfrak{R}_{i,t}$ is isomorphic to the kernel of $t$-deformed screening operator $S_{i,t}$ on $\mathfrak{R}_t(\mathfrak{g})$. Then it is proved that (i) $\mathfrak{R}_{i,t}$ is generated by the elements corresponding to (5.15), (ii) $K(\mathcal{O}_q^0) \simeq \mathfrak{R}_t(\mathfrak{g})|_{t=1}$ (see also [9]). In the construction of $\mathfrak{R}_t(\mathfrak{g})$, the quantum Cartan matrix $C(q)$ of type $\mathfrak{g}$ and its inverse $\tilde{C}(q)$ are crucially used. In Definition 5.9, we define the $\mathbb{C}(t)$-analogue $\mathfrak{R}_q$ of $K_t(\mathcal{O}_q^0)$, which is new one for non simply-laced $\mathfrak{g}$, to the best knowledge of the authors. As we mentioned in the introduction, the specialization of $\mathfrak{R}_q$ at $q = 1$ coincides with the the specialization $\mathfrak{K}_t(\mathfrak{g})$ of $\mathfrak{R}_q$ at $q = 1$ and $\alpha = d$, which is commutative and defined in [8] (see [27] also). For non simply-laced $\mathfrak{g}$, it is proved in [8, Theorem 4.3] that $\mathfrak{K}_t(\mathfrak{g})$ is a homomorphic image of $\mathfrak{K}_t(\mathfrak{g})$, where $\mathfrak{g}$ is of simply-laced type and contains $\mathfrak{g}$ as its non-trivial subalgebra. Hence the specialization $B_{i,p}$ of the monomial $\tilde{B}_{i,p}$ in (5.10) at $q = 1$ can be understood as an image of $A_{i,p}$ under the surjection, which is deformation of the simple root $\alpha_i$ in the $q$-character theory ([10, 11, 9]).

In [27], the ring $\mathfrak{R}_q$ will be investigated in more precise way.

Note that one can check that the element in (5.12) is contained in $\mathfrak{R}_q$ of type $B_3$.

### 6. Unipotent Quantum Coordinate Algebra and the Quantum Tori Isomorphism

In this section, we review the unipotent quantum coordinate algebra $A_\nu(n)$ associated to a Dynkin diagram $\triangle$ and the quantum torus $\mathcal{T}_{\nu,|Q|}$ generated by quasi-commuting unipotent quantum minors associated to $|Q|$. Then we shall prove that the quantum torus $\mathcal{T}_{\nu,|Q|}$ and $\mathcal{X}_q|Q|$ are isomorphic.

#### 6.1. Quantum group and its representations

Let $(\mathbb{C}, \mathbb{P}, \Pi, \mathbb{P}^\vee, \Pi^\vee)$ be a finite Cartan datum. Let $\nu$ be an indeterminate. We denote by $U_\nu(\mathfrak{g})$ the quantum group associated to the Cartan datum, which is an algebra over $\mathbb{Q}(\nu)$ generated by $e_i, f_i$ $(i \in I)$ and $\nu^h$ $(h \in \mathbb{P}^\vee)$. For $\beta \in Q$, $U_\nu(\mathfrak{g})_\beta$ denotes the weight space of $U_\nu(\mathfrak{g})$ with weight $\beta$ and we set $\text{wt}(x) = \beta$ for $x \in U_\nu(\mathfrak{g})_\beta$. We denote by $U_\nu^+(\mathfrak{g})$ (resp. $U_\nu^-(\mathfrak{g})$) the subalgebra of $U_\nu(\mathfrak{g})$ generated by $e_i$
(resp. $f_i$) for $i \in I$. For $n \in \mathbb{Z}_{\geq 0}$ and $i \in I$, we set $e_i^{(n)} := e_i/\lbrack n \rbrack !$ and $f_i^{(n)} := f_i/\lbrack n \rbrack !$, where we set

$$\nu_i := \nu^{d_i}, \quad [n]_i := \frac{\nu^n_i - \nu^{-n}_i}{\nu_i - \nu_i^{-1}}, \quad \text{and} \quad [n] ! := \prod_{k=1}^n [k] ! .$$

We set $\mathbb{A} := \mathbb{Z}[\nu^{\pm 1}]$ and denote by $U^\pm_{\mathbb{A}}(\mathfrak{g})$ the $\mathbb{A}$-subalgebra of $U_\nu(\mathfrak{g})^\pm$ generated by $e_i^{(n)}$ (resp. $f_i^{(n)}$) for $i \in I$ and $n \in \mathbb{Z}_{\geq 0}$.

Note that $U_\nu(\mathfrak{g})$ is a Hopf algebra with the comultiplication

$$\Delta: U_\nu(\mathfrak{g}) \to U_\nu(\mathfrak{g}) \otimes U_\nu(\mathfrak{g})$$

given by

$$\Delta(e_i) = e_i \otimes 1 + \nu_i^{d_i} \otimes e_i, \quad \Delta(f_i) = f_i \otimes \nu_i^{-d_i} + 1 \otimes f_i, \quad \Delta(\nu^h) = \nu^h \otimes \nu^h.$$

Let $\varphi$ and $*$ be the $\mathbb{Q}(\nu)$ anti-automorphism of $U_\nu(\mathfrak{g})$ defined by

$$\varphi(e_i) = f_i, \quad \varphi(f_i) = e_i, \quad \varphi(\nu^h) = \nu^{-h} \quad \text{and} \quad e_i^* = e_i, \quad f_i^* = f_i, \quad (\nu^h)^* = \nu^{-h}.$$

There is also a $\mathbb{Q}$-algebra homomorphism $\overline{\varphi}$ of $U_\nu(\mathfrak{g})$ given by

$$\overline{\varphi} = \nu^{-1}, \quad \overline{e_i} = e_i, \quad \overline{f_i} = f_i \quad \text{and} \quad \overline{\nu^h} = \nu^{-h}.$$

For a left $U_\nu(\mathfrak{g})$-module $N$, we denote by $N^r$ the right $U_\nu(\mathfrak{g})$-module $\{m^r \mid m \in M\}$ with the right action induced by $\varphi$:

$$(m^r)x = (\varphi(x)m)^r \quad \text{for} \ m \in M \text{ and } x \in U_\nu(\mathfrak{g}).$$

Let $M$ be a left $U_\nu(\mathfrak{g})$-module. For $\lambda \in \mathcal{P}$, let

$$M_\lambda := \{ m \in M \mid \nu^h m = \nu^{(\lambda, h)} m \text{ for every } h \in \mathcal{P} \}$$

be the corresponding weight space. We say a left $U_\nu(\mathfrak{g})$-module $M$ integrable if (i) $M = \bigoplus_{\eta \in \mathcal{P}} M_\eta$ with $\dim M_\eta < \infty$ and (ii) the actions of $e_i$ and $f_i$ are locally nilpotent for all $i \in I$.

We denote by $\mathcal{O}_{\text{int}}(\mathfrak{g})$ the category of integrable left $U_\nu(\mathfrak{g})$-modules $M$ satisfying that there exist finitely many $\lambda_1, \ldots, \lambda_m$ such that $\text{wt}(M) := \{ \eta \in \mathcal{P} \mid \dim M_\eta \neq 0 \} \subset \bigcup_{j}(\lambda_j + \mathcal{Q}^+)$. The category $\mathcal{O}_{\text{int}}(\mathfrak{g})$ is semisimple with its simple objects being isomorphic to the highest weight modules $V(\lambda)$ with the highest weight vector $u_\lambda$ of the highest weight $\lambda \in \mathcal{P}^+$. We denote by $\mathcal{O}^r_{\text{int}}(\mathfrak{g})$ the category of integrable right $U_\nu(\mathfrak{g})$-modules $M^r$ such that $M \in \mathcal{O}_{\text{int}}(\mathfrak{g})$. Then $\mathcal{O}^r_{\text{int}}(\mathfrak{g})$ is also semisimple with its simple objects being isomorphic to the highest weight modules $V^r(\lambda) := (V(\lambda))^r$ with the highest weight vector $u_\lambda^r$ for some $\lambda \in \mathcal{P}^+$.

Note that the highest weight module $V(\lambda)$ ($\lambda \in \mathcal{P}^+$) has a unique non-degenerate symmetric bilinear form $(\cdot, \cdot)_\lambda$ such that

$$(u_\lambda, u_\lambda)_\lambda = 1 \quad \text{and} \quad (xu, v)_\lambda = (u, \varphi(x)v)_\lambda \quad \text{for} \ u, v \in V(\lambda) \text{ and } x \in U_\nu(\mathfrak{g}).$$

The tensor product of $\mathbb{Q}(\nu)$-spaces $V^r(\lambda) \otimes V(\lambda)$ ($\lambda \in \mathcal{P}^+$) has the natural structure of a $U_\nu(\mathfrak{g})$-bimodule given by

$$x \cdot (u^r \otimes v) \cdot y = (u^r \cdot y) \otimes (x \cdot v).$$
and the bilinear form $(\cdot, \cdot)_\lambda$ on $V(\lambda)$ induces the non-degenerate bilinear form

$\langle \cdot, \cdot \rangle_\lambda: V^r(\lambda) \times V(\lambda) \to \mathbb{Q}(\nu)$

given by $\langle u^r, v \rangle_\lambda = (u, v)_\lambda$.

### 6.2. Unipotent quantum coordinate algebra and quantum minors

Let $U_\nu(g)^*$ be the space

$\text{Hom}_{\mathbb{Q}(\nu)}(U_\nu(g)^*, \mathbb{Q}(\nu))$.

Then the comultiplication $\Delta$ induces the multiplication on $U_\nu(g)^*$ as follows:

$U_\nu(g)^* \otimes U_\nu(g)^* \to (U_\nu(g) \otimes U_\nu(g))^* \xrightarrow{\Delta^*} U_\nu(g)^*$.

Namely, for $f, g \in U_\nu(g)^*$ and $x \in U_\nu(g)$, we have

$(fg)(x) = f(x(1))g(x(2))$,

where $\Delta(x) = x(1) \otimes x(2)$ written by Sweedler’s notation.

Note that $U_\nu(g)^*$ has also a $U_\nu(g)$-bimodule structure given by

$x \cdot (fg) \cdot y = (x(1) \cdot f \cdot y(1))(x(2) \cdot g \cdot y(2))$ for $f, g \in U_\nu(g)^*$ and $x, y \in U_\nu(g)$,

where $(x \cdot f \cdot y)(z) := f(yzx)$ for $z \in U_\nu(g)$.

**Definition 6.1.** We define the quantum coordinate algebra $A_\nu(g)$ as follows:

$A_\nu(g) = \{ f \in U_\nu(g)^* \mid U_\nu(g)f \text{ belongs to } \mathcal{O}_{\text{int}}(g) \text{ and } fU_\nu(g) \text{ belongs to } \mathcal{O}_{\text{int}}(g) \}$.

We sometimes denote by $e_i^*$ and $f_i^*$ the operators on $A_\nu(g)$ acting at the right.

**Proposition 6.2** ([31, Proposition 7.2.2]). We have an isomorphism $\Phi$ of $U_\nu(g)$-bimodules

$\bigoplus_{\lambda \in P^+} V(\lambda) \otimes V^r(\lambda) \xrightarrow{\sim} A_\nu(g)$

given by

$\Phi(u \otimes v^r)(x) = \langle v^r, x \cdot u \rangle_\lambda = \langle v^r \cdot x, u \rangle_\lambda = (v, x \cdot u)_\lambda$

for $u \in V(\lambda)$, $v^r \in V^r(\lambda)$ and $x \in U_\nu(g)$.

By Proposition 6.2, $A_\nu(g)$ admits a biweight space decomposition $A_\nu(g) = \bigoplus_{\eta, \zeta \in P} A_\nu(g)_{\eta, \zeta}$

where

$A_\nu(g)_{\eta, \zeta} := \{ \psi \in A_\nu(g) \mid \nu^{h_1} \cdot \psi \cdot \nu^{h_1} = \nu^{(h_1, \psi) + (h_1, \zeta)} \psi \}$.

For $\phi \in A_\nu(g)_{\eta, \zeta}$, we write $wt_{l}(\phi) = \eta$ and $wt_{r}(\phi) = \zeta$.

**Definition 6.3.** For $w \in P^+$ and $\mu, \zeta \in Ww$, we defined the generalized quantum minor

$\Delta(\mu, \zeta)$ as follows:

$\Delta(\mu, \zeta) := \Phi(u_{\mu} \otimes u_{\zeta}^r) \in A_\nu(g)_{\mu, \zeta}$.

For elements $y, y' \in A_\nu(g)$, we write $y \equiv y'$ if there exists $r \in \mathbb{Z}$ such that $y = v^ry'$.

**Lemma 6.4.** [2, (9.13)] For $u, v \in W$ and $\lambda, \mu \in P^+$, we have

$\Delta(u\lambda, v\lambda)\Delta(u\mu, v\mu) \equiv \Delta(u(\lambda + \mu), v(\lambda + \mu))$
The following proposition is \( \nu \)-analogue of \cite[Theorem 1.17]{kashiwara1990} and can be proved by following the same arguments in \cite[§3.2, 3.3]{lusztig1983}:

**Proposition 6.5.** For \( u, v \in W \), assume that \( \ell(us_i) = \ell(u) + 1 \) and \( \ell(vs_i) = \ell(v) + 1 \). Then we have

\[
\Delta(us_i(w_i), vs_i(w_i)) = \nu^{-d_i} \Delta(us_i(w_i), v(w_i)) \Delta(u(w_i), vs_i(w_i)) + \prod_{i \neq j} \Delta(u(w_j), v(w_j))^{-c_{j,i}}.
\]

(6.1)

Here \( \Delta(u\lambda, v\lambda) = \prod_{i \neq j} \Delta(u(w_j), v(w_j))^{-c_{j,i}} \) for \( \lambda = s_iw_i + w_i \) by the Lemma 6.4.

The tensor product \( U^+_\nu(g) \otimes U^+_\nu(g) \) has the algebra structure defined by

\[
(x_1 \otimes x_2) \cdot (y_1 \otimes y_2) = \nu^{-(\text{wt}(x_2), \text{wt}(y_1))}(x_1 y_1 \otimes x_2 y_2)
\]

for homogeneous elements \( x_1, x_2, y_1, y_2 \in U^+_\nu(g) \). We define the \( \mathbb{Q}(\nu) \)-algebra homomorphism \( \Delta_n: U^+_\nu(g) \rightarrow U^+_\nu(g) \otimes U^+_\nu(g) \) by

\[
\Delta_n(e_i) = e_i \otimes 1 + 1 \otimes e_i \quad \text{for } i \in I.
\]

We set

\[
A_\nu(n) := \bigoplus_{\beta \in \mathbb{Q}^-} A_\nu(n)_\beta \quad \text{where } A_\nu(n)_\beta := \text{Hom}_{\mathbb{Q}(\nu)}(U^+_\nu, g)_{-\beta, \mathbb{Q}(\nu)}.
\]

Defining the bilinear form \( \langle \cdot, \cdot \rangle: (A_\nu(n) \otimes A_\nu(n)) \times (U^+_\nu \otimes U^+_\nu) \rightarrow \mathbb{Q}(\nu) \) by

\[
\langle \psi \otimes \theta, x \otimes y \rangle = \theta(x)\psi(y),
\]

the multiplication on \( A_\nu(n) \) is given by

\[
(\phi \cdot \theta)(x) = \langle \phi \otimes \theta, \Delta_n(x) \rangle \quad \text{for } \phi, \theta \in A_\nu(n), \ x \in U^+_\nu(g).
\]

The algebra \( A_\nu(n) \) is called the unipotent quantum coordinate algebra. We set

\[
A_\nu(n) = \{ \psi \in A_\nu(n) \mid \psi(U^+_\nu(g)) \subseteq A_\nu(n) \}.
\]

We define the bar-involution \( - \) of \( A_\nu(n) \) by

\[
\overline{\psi}(x) = \psi(\overline{x}) \quad \text{for } \psi \in A_\nu(n) \text{ and } x \in U^+_\nu(g).
\]

Note that Lusztig \cite{lusztig1985, lusztig1986} and Kashiwara \cite{kashiwara1988} have constructed a specific \( A \)-basis \( B_{up} \) of \( A_\nu(n) \), which is called the dual canonical/upper global basis.

The homomorphism \( p_n: A_\nu(g) \rightarrow A_\nu(n) \) induced by \( U^+_\nu(g) \rightarrow U^+_\nu(g) \) is given by

\[
\langle p_n(\psi), x \rangle = \psi(x) \quad \text{for any } x \in U^+_\nu(g).
\]

Then we have

\[
\text{wt}(p_n(\psi)) = \text{wt}_1(\psi) - \text{wt}_r(\psi).
\]

**Lemma 6.6** ([\cite{lusztig1985b}, Proposition 8.5.4]). For \( \psi, \theta \in A_\nu(g) \), we have

\[
p_n(\psi\theta) = \nu^{(\zeta_1, \zeta_2 - n_2)} p_n(\psi) p_n(\theta)
\]

where \( \psi \in A_\nu(g)_{n_1, \zeta_1} \) and \( \zeta \in A_\nu(g)_{n_2, \zeta_2} \).
Definition 6.7. For \( \varpi \in \mathbb{P}^+ \) and \( \mu, \zeta \in \mathbb{W}\varpi \), we define the unipotent quantum minor \( D(\mu, \zeta) \) as follows:

\[
D(\mu, \zeta) := p_n(\Delta(\mu, \zeta)) \in A_\nu(n)_{\zeta-\mu}.
\]

By the definition, \( D(\mu, \zeta) \) vanishes if \( \zeta \not\equiv \mu \).

The behavior of \( D(\mu, \zeta) \) is investigated intensively (see [2, 16, 29]):

Proposition 6.8.

1. For \( \Lambda \in \mathbb{P}^+ \) and \( \mu, \zeta \in \mathbb{W}\Lambda \), \( D(\mu, \zeta) \) is a member of \( \mathbb{B}_{up} \) if \( \mu \not\equiv \zeta \), and \( D(\mu, \zeta) = 0 \) otherwise.

2. For \( \lambda, \mu \in \mathbb{P}^+ \) and s, t, s', t' \in \mathbb{W} \) such that
   \( i \ell(s') = \ell(s') + \ell(s) \) and \( \ell(t') = \ell(t) + \ell(t) \),
   \( \lambda \) s's \not\equiv t' \lambda \) and \( s' \mu \not\equiv t' \mu \),
   we have
   \[
   D(s' \mu, t' \mu)D(s' s\lambda, t' \lambda) = v(s' s\lambda + t' \lambda - s' \mu)D(s' s\lambda, t' \lambda)D(s' \mu, t' \mu).
   \]

3. For \( u, v \in \mathbb{W} \) and \( i \in I \) satisfying \( u < us_i \) and \( v < vs_i \), we have

\[
\nu^{(u_\varpi, v_\varpi)}D(\varpi, v_\varpi)D(u_\varpi, \varpi) - D(\varpi, u_\varpi)D(u_\varpi, \varpi) + D(u_\varpi, \varpi)D(u_\varpi, v_\varpi).
\]

where \( \lambda = s_i \varpi + \varpi_2 = 2\varpi_i - \alpha_i \).

The equation (6.2) is referred to as T-system among unipotent quantum minors.

For a sequence of indices \( \varpi = (i_1, \ldots, i_r) \in I^r \), \( 1 \leq k \leq r \) and \( i \in I \), we use the following notations:

\[
w_{\leq k} := s_{i_1} \cdots s_{i_k}, \quad w_{\leq 0} := 1,
\]

\[
k^+ := \min\{\{k < j \leq r \mid i_j = i_k\} \cup \{r + 1\}\},
\]

\[
k^- := \max\{\{1 \leq j < k \mid i_j = i_k\} \cup \{0\}\},
\]

\[
k^-(i) := \max\{\{1 \leq j < k \mid i_j = i\} \cup \{0\}\},
\]

\[
k^+(i) := \min\{\{k < j \leq r \mid i_j = i\} \cup \{r + 1\}\}.
\]

For a reduced expression \( \bar{w}_0 = s_{i_1} \cdots s_{i_j} \) of the longest element \( w_0 \) and \( 0 \leq s \leq t \leq \ell \), we set

\[
D_{\bar{w}_0}[s, t] := \begin{cases} D(w_{\leq t \varpi_i}, w_{\leq s-1 \varpi_i}) & \text{if } i_t = i_s \text{ and } s \geq 1, \\ D(w_{\leq t \varpi_i}, \varpi_i) & \text{if } s = 0, \\ 1 & \text{otherwise}, \end{cases}
\]

by taking \( \bar{w}_0 := (i_1, \ldots, i_\ell) \). Then (6.2) can be written as follows: For \( 1 \leq a < b \leq \ell \) with \( i_a = i_b = i \),

\[
D_{\bar{w}_0}[a^+, b]D_{\bar{w}_0}[a, b^-] = D_{\bar{w}_0}[a, b]D_{\bar{w}_0}[a^+, b^-] + \prod_{j; \epsilon_j < 0} D_{\bar{w}_0}[a^+(j), b^-(j)]^{-\epsilon_j i}.
\]

Here we ignore \( \nu \)-coefficients and understand \( D[x, y] = 1 \) for \( y < x \).
6.3. Quantum torus for unipotent quantum coordinate algebra. Now let us fix a reduced expression \( w_0 = s_{i_1} \cdots s_{i_r} \) of the longest element \( w_0 \in W \). Recall the notation \( \beta_{k}^{w_0} \) in (2.2) and \( k^{\pm} \) in (6.3) for \( k \in [1, \ell] \). For each \( \alpha \in \Phi^+ \), there exists a unique \( k \) such that \( \alpha = \beta_{k}^{w_0} \). Then we define
\[
\alpha^+ := \begin{cases} 
\beta_{k}^{w_0} & \text{if } k^{+} \leq \ell, \\
0 & \text{if } k^{+} = \ell + 1,
\end{cases} \quad \alpha^- := \begin{cases} 
\beta_{k}^{w_0} & \text{if } k^{-} \geq 1, \\
0 & \text{if } k^{-} = 0,
\end{cases}
\]
\[
\lambda_{\alpha} := w_{\leq k} w_{ik}.
\]
By the definition, we have
\[
\lambda_{\alpha^-} = \lambda_{\alpha} + \alpha
\]
for \( \alpha \in \Phi^+ \), where we understand \( \lambda_{\alpha^-} = w_{ik} \) if \( \alpha^- = 0 \).

In the following proposition, we follow the convention that \( \beta^+ \prec_{\{w_0\}} \alpha^+ \) is true if \( \alpha^+ = 0 \).

**Proposition 6.9** ([16, Proposition 10.1, Lemma 11.3], [17, Proposition 10.4]). Set
\[
J := \Phi^+, \quad J_f := \{ \alpha \in \Phi^+ \mid \alpha^+ = 0 \} \quad \text{and} \quad J_c := J \setminus J_f.
\]
Define the \( J \times J_c \)-integer matrix \( \tilde{B}_{w_0} = (b_{\alpha \beta})_{\alpha \in J_c, \beta \in J_c} \) as
\[
b_{\alpha \beta} = \begin{cases} 
1 & \text{if } \beta = \alpha^+, \\
c_{i,j} & \text{if } \alpha \prec_{\{w_0\}} \beta \prec_{\{w_0\}} \alpha^+ \prec_{\{w_0\}} \beta^+ \text{ and } d(i, j) = 1, \\
-1 & \text{if } \beta^+ = \alpha, \\
-c_{i,j} & \text{if } \beta \prec_{\{w_0\}} \alpha \prec_{\{w_0\}} \beta^+ \prec_{\{w_0\}} \alpha^+ \text{ and } d(i, j) = 1, \\
0 & \text{otherwise},
\end{cases}
\]
where \( i = \text{res}_{\{w_0\}}(\alpha) \) and \( j = \text{res}_{\{w_0\}}(\beta) \). Define the \( J \times J \)-skew symmetric matrix \( \Lambda_{w_0} = (\Lambda_{\alpha \beta})_{\alpha, \beta \in J} \) by
\[
\Lambda_{\alpha \beta} = (w_i - \lambda_{\alpha}, w_j + \lambda_{\beta}) \quad \text{for } \beta \not\prec_{\{w_0\}} \alpha.
\]
Then, \( \Lambda_{\alpha \beta}^{w_0} \in \mathbb{Z} \), and \( (\Lambda_{\alpha \beta}^{w_0}, \tilde{B}_{w_0}) \) is a compatible pair, that is
\[
\sum_{\gamma \in J} b_{\gamma \alpha} \Lambda_{\gamma \beta} = (\alpha, \alpha) \delta(\alpha = \beta)
\]
for all \( \alpha \in J_c \) and \( \beta \in J \).

**Remark 6.10.** It is easy to prove that \( \alpha^{\pm} \) depends only on the commutation class of \( w_0 \). Hence, the notations \( \Lambda_{\alpha \beta}^{w_0} \) and \( \tilde{B}_{w_0} \) make sense.

Now we define the following quantum torus by using the matrix \( \Lambda_{\alpha \beta}^{w_0} = (\Lambda_{\alpha \beta}^{w_0})_{\alpha, \beta \in \Phi^+} \):

**Definition 6.11.** The quantum torus \( \mathcal{T}_{\nu_{\{\alpha \beta\}}^{w_0}} \) is the \( \mathbb{Z}[\nu^{\pm \frac{1}{2}}] \)-algebra given by the set of generators \( \{ Y_{\alpha}^{\pm 1} \mid \alpha \in \Phi^+ \} \) and the following relations:
- \( Y_{\alpha} Y_{\alpha}^{-1} = Y_{\alpha}^{-1} Y_{\alpha} = 1 \) for \( \alpha \in \Phi^+ \),
- \( Y_{\alpha} Y_{\beta} = \nu^{\Lambda_{\alpha \beta}} Y_{\beta} Y_{\alpha} = 1 \) for \( \alpha, \beta \in \Phi^+ \).
Let $A_{\nu}(\Lambda_{\w fid}, \widehat{B}_{\w fid})$ be the quantum cluster algebra associated to the compatible pair $(\Lambda_{\w fid}, \widehat{B}_{\w fid})$ in Proposition 6.9. The algebra $A_{\nu}(\Lambda_{\w fid}, \widehat{B}_{\w fid})$ is the $\mathbb{Z}[\nu^{\pm \frac{1}{2}}]$-subalgebra of the quantum torus $\mathcal{T}_{\nu[\Lambda]}$ generated by the union of the elements called quantum cluster variables, which are obtained by all possible sequences of mutations (see [5, 2] for more details).

For $\alpha, \beta \in \Phi^+$ with $\text{res}_{\w fid}(\alpha) = \text{res}_{\w fid}(\beta) = i$, we set

$$D_{\w fid}(\alpha, \beta) := D(\lambda_\alpha, \lambda_\beta), \quad D_{\w fid}(\alpha, 0) := D(\lambda_\alpha, \varpi_i).$$

By the definition, $D_{\w fid}(\alpha, \beta) = 0$ unless $\beta \prec_{\w fid} \alpha$. We set $D_{\w fid}(0,0) := 1$ by convention.

**Theorem 6.12** ([16, Theorem 12.3], [17, Theorem 10.1]). There exists a $\mathbb{Q}(\nu^{\pm \frac{1}{2}})$-algebra isomorphism

$$\text{CL}: \mathbb{Q}(\nu^{\frac{1}{2}}) \otimes_{\mathbb{Z}[\nu^{\pm \frac{1}{2}}]} A_{\nu}(\Lambda_{\w fid}, \widehat{B}_{\w fid}) \cong \mathbb{Q}(\nu^{\frac{1}{2}}) \otimes_{\mathbb{Z}[\nu^{\pm \frac{1}{2}}]} A_{\nu}(\Lambda_{\w fid})$$

sending $Y_\alpha$ to $D_{\w fid}(\alpha, 0)$ for all $\alpha \in \Phi^+$.

**6.4. An isomorphism between the quantum tori.** Recall the quantum torus $\mathcal{X}_{q,Q}$ for a Dynkin quiver $Q = (\Delta, \xi)$ in Definition 5.3. Note that the quantum torus has another presentation given by the set of generators $\{\tilde{m}^{(i)}[p, \xi_i] \pm 1 | (i, p) \in (\Gamma_Q)_0\}$ and the following relations (see (5.7)):

- $\tilde{m}^{(i)}[p, \xi_i]^{-1}$ is the inverse of $\tilde{m}^{(i)}[p, \xi_i]$ for $(i, p) \in (\Gamma_Q)_0$.
- $\tilde{m}^{(i)}[p, \xi_i] \cdot \tilde{m}^{(j)}[s, \xi_j] = \nu^{\kappa(i, p; j, s)} \tilde{m}^{(j)}[s, \xi_j] \cdot \tilde{m}^{(i)}[p, \xi_i]$ for $(i, p), (j, s) \in (\Gamma_Q)_0$.

where

$$\tilde{m}^{(i)}[p, \xi_i] = q^{m_{i,p}} \prod_{x: p \leq t \leq \xi_i} \tilde{X}_{i,t},$$

$$\kappa(i, p; j, s) = \mathcal{N}(\tilde{m}^{(i)}[p, \xi_i], \tilde{m}^{(j)}[s, \xi_j]) = \sum_{x: p \leq x \leq \xi_i, y: s \leq y \leq \xi_j} \mathcal{N}(i, x; j, y).$$

Here we choose $m_{i,p} \in \frac{1}{2} \mathbb{Z}$ such that $\tilde{m}^{(i)}[p, \xi_i]$ is a bar-invariant $\mathcal{X}_q$-monomial.

The following lemma follows from Lemma 3.11.

**Lemma 6.13.** Assume that $\w fid$ is adapted to a Dynkin quiver $Q$. Then, for $\alpha \in \Phi^+$ with $\text{res}_{\w fid}(\alpha) = i$, we have the followings:

(i) We have $\alpha = \tau_Q^{-}\alpha^{-}$ provided that $\alpha^{-} \neq 0$. If $\alpha^{-} = 0$, then $\alpha = \gamma_i^Q = \varpi_i - \tau_Q \varpi_i$.

(ii) We have $\lambda_\alpha = \tau_Q^{\langle \nu(i-p)/2, 1 \rangle} \varpi_i$, where $\phi_Q(i, p) = (\alpha, 0)$.

**Theorem 6.14.** For each Dynkin quiver $Q = (\Delta, \xi)$, there is an algebra isomorphism

$$\Psi_Q: \mathcal{T}_{\nu[Q]} \rightarrow \mathcal{X}_{q,Q}$$

given by

$$\nu^{\pm \frac{1}{2}} \mapsto q^{\pm \frac{1}{2}}, \quad Y_\alpha \mapsto \tilde{m}^{(i)}[p, \xi_i]$$

for all $\alpha \in \Phi^+$, where $\phi_Q(i, p) = (\alpha, 0)$. 

Proof. It suffices to show
\[ \kappa(i, p; j, s) = \Lambda_{\alpha\beta} \]
for \( \alpha, \beta \in \Phi^+ \) such that \( \phi_{\alpha}(i, p) = (\alpha, 0) \) and \( \phi_{\beta}(j, s) = (\beta, 0) \). Since \( \Lambda^{[Q]} \) is skew-symmetric, we may assume \( p \leq s \). Then we have \( \alpha \not\in \Phi_{[Q]} \beta \), and the conditions \( p - s \leq d(i, j) \) and \( \xi_j - \xi_i \leq d(i, j) \) hold obviously. We shall apply Proposition 5.5 with \( p' = \xi_i \) and \( s' = \xi_j \). Then, together with Lemma 6.13 (ii), we obtain
\[
\Delta\big(\tilde{m}^{(i)}[p, \xi_i], \tilde{m}^{(j)}[s, \xi_j]\big) = \left( \tau^{(\xi_i-p)/2+1}_{\xi_i}, \tau^{(\xi_i-p')/2}_{\xi_i}, \tau^{(\xi_j-s)/2+1}_{\xi_j}, \tau^{(\xi_j-s')/2}_{\xi_j} \right)
= (\lambda_{\alpha} + \varpi_i, \lambda_{\beta} - \varpi_j) = -\Lambda_{\alpha\beta}.
\]
\[ \square \]

Definition 6.15. Let \( Q = (\Delta, \xi) \) be a Dynkin quiver. We call the image of \( \mathcal{A}_Q(\Lambda^{[Q]}, \tilde{B}^{[Q]}) \) under \( \Psi_Q \) the quantum virtual Grothendieck ring associated to \( Q \) and denote it by \( \mathcal{R}_{q,Q} \).

Remark 6.16. For a Dynkin quiver \( Q \) of simply-laced type, \( \mathcal{R}_{q,Q} \) and \( \mathcal{R}_{q,Q}|_{q=1} \) are known as the quantum Grothendieck ring and the Grothendieck ring of a certain subcategory \( \mathcal{C}_Q \) of modules over the quantum affine algebra associated to \( Q \). We expect that \( \mathcal{R}_{q,Q} \) (resp. \( \mathcal{R}_{q,Q}|_{q=1} \)) is contained in \( \mathcal{R}_q \) (resp. \( \mathcal{R}_q|_{q=1} \)) when \( Q \) is a Dynkin quiver of type \( BCFG \).

7. Compatible pairs

In this section, we first give a generalization of Proposition 6.9 as a conjecture. Then we prove the conjecture under the condition (7.3) below.

Let \( \tilde{w} = (i_1, \ldots, i_r) \) be a sequence of elements of \( \Delta_0 \). We set \( J := [1, r] \). We define the matrix \( \tilde{B}^{\tilde{w}} \) and \( \Lambda^{\tilde{w}} \) as in Proposition 6.9. Namely, we set
\[
\begin{align*}
    j^+ &:= \max\{s \in J \mid j < s, i_s = i_j\} \cup \{r + 1\} \quad \text{for any } j \in J, \\
    J_f &:= \{j \in J \mid j^+ = r + 1\}, \quad J_c := J \setminus J_f, \\
    w_{s,t} &:= s_{i_1} \cdots s_{i_t} \quad \text{for any } t \in J,
\end{align*}
\]
and the \( J \times J_c \)-matrix \( \tilde{B}^{\tilde{w}} = (b_{s,t})_{s \in J, t \in J_c} \) is defined by
\[
b_{s,t} = \begin{cases} 
    1 & \text{if } t = s^+, \\
    -1 & \text{if } s = t^+, \\
    \c_{i_s,i_t} & \text{if } s < t < s^+ < t^+, \\
    -\c_{i_s,i_t} & \text{if } t < s < t^+ < s^+, \\
    0 & \text{otherwise},
\end{cases}
\]
and the skew-symmetric \( J \times J \)-matrix \( \Lambda^{\tilde{w}} = (\Lambda_{s,t})_{s,t \in J} \) is defined by
\[
\Lambda_{s,t} = (\varpi_{i_s} - w_{s,t}^{\leq s} \varpi_{i_s} + w_{s,t} \varpi_{i_t}) \quad \text{for } s < t.
\]
(7.1)

Note that \( \tilde{B}^{\tilde{w}} \) is skew-symmetrizable by \( \text{diag}(d_{i_k})_{k \in J_c} \).

Consider the condition on \( \tilde{w} \):
\[
\begin{align*}
s_{i_a} \cdots s_{i_b} \text{ has length } b - a + 1 &\text{ for any } a, b \in \mathbb{Z} \text{ such that } 1 \leq a \leq b \leq r \text{ and } 1 + b - a \leq \ell(w_0). 
\end{align*}
\]
(7.2)
Conjecture 1. For any \( \tilde{w} \) satisfying (7.2), the pair \( (\Lambda^{\tilde{w}}, \tilde{B}^{\tilde{w}}) \) is compatible, i.e.,
\[
\Lambda^{\tilde{w}} \tilde{B}^{\tilde{w}} = (-2d_{is}\delta(s=t))_{s\in J, t\in J_c}.
\]

7.1. Q-adapted case. In this subsection, we prove Conjecture 1 when \( \tilde{w} \) is a Q-adapted sequence for some Dynkin quiver \( Q \) but (7.2) is not assumed.

For a sequence \( \tilde{w} \) in (7.2), we assume the following condition:
\[
(7.3) \quad \text{there exists a Dynkin quiver } Q \text{ such that } (i_k)_{1\leq k \leq r} \text{ is Q-adapted, i.e., } i_k \text{ is a source of } s_{i_{k-1}} \cdots s_{i_1} Q \text{ for any } k \text{ with } 1 \leq k \leq r.
\]

Throughout this subsection, we fix \( \tilde{w} \) satisfying (7.3).

For each \( j \in \Delta_0 \), we set
\[
n_j := |\{k \in \mathbb{Z} \mid 1 \leq k \leq r \text{ and } i_k = j\}| \in \mathbb{Z}_{\geq 0} \cup \{\infty\}.
\]

We set
\[
\Gamma^{[\tilde{w}]}_0 := \{(j, p) \in \widehat{\Delta}_0 \mid \xi_j - 2n_j < p \leq \xi_j\}.
\]

For \( k \in [1, r] \), set
\[
p_k = \xi_{i_k} - 2 \times |\{s \in \mathbb{Z} \mid 1 \leq s < k, i_s = i_k\}|.
\]

Then, by Lemma 3.11, \((i_k, p_k)\) is a compatible reading of \( \Gamma^{[\tilde{w}]}_0 \) (3.21). Hence \( k \leftrightarrow (i_k, p_k) \) gives a one-to-one correspondence
\[
\eta : \Gamma^{[\tilde{w}]}_0 \sim J = \{1, 2, \ldots, r\}.
\]

We set
\[
(\Gamma^{[\tilde{w}]}_0)_e := \eta^{-1}(J_e) = \{(j, p) \in \widehat{\Delta}_0 \mid \xi_j - 2n_j - 2 < p \leq \xi_j\}.
\]

Then the exchange matrix \( \tilde{B}^{\tilde{w}} = (b_{(s,t)})_{s\in J, t\in J_c} \) and the skew-symmetric matrix \( J \times J \)-matrix \( \Lambda^{\tilde{w}} \) can be re-written as follows:
\[
(7.4) \quad b_{(i,p),(j,s)} := b_{\eta(i,p),\eta(j,s)} = \begin{cases} (-1)^{\delta(s>p)}c_{i,j} & \text{if } |p-s| = 1 \text{ and } d(i,j) = 1, \\ (-1)^{\delta(s>p)} & \text{if } |p-s| = 2 \text{ and } i = j, \\ 0 & \text{otherwise}, \end{cases}
\]

for \( ((i,p),(j,s)) \in \Gamma^{[\tilde{w}]}_0 \times (\Gamma^{[\tilde{w}]}_0)_e \), and
\[
(7.5) \quad \Lambda'_{(i,p),(j,s)} := \Lambda_{\eta(i,p),\eta(j,s)} = \sum_{x: p \leq x \leq 2 \xi_i \atop y: s \leq y \leq 2 \xi_j} \Lambda(i, x; j, y)
\]

for \( (i,p),(j,s) \in \Gamma^{[\tilde{w}]}_0 \). Here \( u \leq v \) means that \( u \leq v \) and \( u \equiv v \mod 2 \).

Indeed, (7.4) is obvious, and (7.5) is obtained from Lemma 3.11 (iii) and Proposition 5.5 as follows: assuming \( p \leq s \), we have
\[
\Lambda'_{(i,p),(j,s)} = -\Lambda'_{(j,s),(i,p)} = -\left(\varpi_j - \tau_Q^{(\xi_i-s)/2}\varpi_i, \varpi_j, \varpi_i + \tau_Q^{(\xi_i-p)/2}\varpi_i\right) = \sum_{x: p \leq x \leq 2 \xi_i \atop y: s \leq y \leq 2 \xi_j} \Lambda(i, x; j, y)
\]
**Theorem 7.1.** Assume that \( \tilde{w} \) satisfies (7.3). Then, we have
\[
\Lambda^{\tilde{w}} B^{\tilde{w}} = (-2d_i, \delta(s = t))_{(s,t) \in I \times J}.
\]

*Proof.* Let \((i, p) \in \Gamma_0[\tilde{w}]^e\) and \((j, s) \in (\Gamma_0[\tilde{w}])_e^e\). By (7.4), we have
\[
(\Lambda^{\tilde{w}} B^{\tilde{w}})_{(i, p), (j, s)} = \delta(s \neq \xi_j) \Lambda'_{(i, p), (j, s+2)} - \Lambda'_{(i, p), (j, s-2)}
\]
\[+ \sum_{k: d(k, j) = 1} \langle h_k, \alpha_j \rangle \left( \delta(s \neq \xi_k + 1) \Lambda'_{(i, p), (k, s+1)} - \Lambda'_{(i, p), (k, s-1)} \right).
\]

Note that
\[
\delta(s \neq \xi_j) \Lambda'_{(i, p), (j, s+2)} - \Lambda'_{(i, p), (j, s-2)} = \sum_{t: s+2 \leq t \leq 2 \xi_i} \mathcal{N}(\tilde{m}^{(i)}[p, \xi_i], \tilde{X}_{j,t}) - \sum_{t: s-2 \leq t \leq 2 \xi_i} \mathcal{N}(\tilde{m}^{(i)}[p, \xi_i], \tilde{X}_{j,t})
\]
\[= -\mathcal{N}(\tilde{m}^{(i)}[p, \xi_i], \tilde{X}_{j,s-2}) - \mathcal{N}(\tilde{m}^{(i)}[p, \xi_i], \tilde{X}_{j,s})
\]
On the other hand, we have
\[
\mathcal{N}(\tilde{m}^{(i)}[p, \xi_i], \tilde{X}_{j,s}) = \sum_{t: p \leq t \leq 2 \xi_i} \mathcal{N}(i, t; j, s) = \sum_{t: p \leq t \leq 2 \xi_i} (\tilde{h}_{i,j}(t - s - 1) - \tilde{h}_{i,j}(t - s + 1))
\]
\[= \sum_{t: p-1 \leq t \leq 2 \xi_i - 1} \tilde{h}_{i,j}(t - s) - \sum_{t: p+1 \leq t \leq 2 \xi_i + 1} \tilde{h}_{i,j}(t - s)
\]
\[= \tilde{h}_{i,j}(p - s - 1) - \tilde{h}_{i,j}(\xi_i - s + 1).
\]
Hence we obtain
\[
\delta(s \neq \xi_j) \Lambda'_{(i, p), (j, s+2)} - \Lambda'_{(i, p), (j, s-2)} = -\left( \tilde{h}_{i,j}(p - s + 1) - \tilde{h}_{i,j}(\xi_i - s + 3) + \tilde{h}_{i,j}(p - s - 1) - \tilde{h}_{i,j}(\xi_i - s + 1) \right)
\]
\[= -\tilde{h}_{i,j}(p - s - 1) - \tilde{h}_{i,j}(p - s + 1) + \tilde{h}_{i,j}(\xi_i - s + 1) + \tilde{h}_{i,j}(\xi_i - s + 3).
\]
Similarly
\[
\delta(s \neq \xi_k + 1) \Lambda_{(i, p), (k, s+1)} - \Lambda_{(i, p), (k, s-1)} = \sum_{t: s+1 \leq t \leq 2 \xi_k} \mathcal{N}(\tilde{m}^{(i)}[p, \xi_i], \tilde{X}_{k,t}) - \sum_{t: s-1 \leq t \leq 2 \xi_k} \mathcal{N}(\tilde{m}^{(i)}[p, \xi_i], \tilde{X}_{k,t})
\]
\[= -\mathcal{N}(\tilde{m}^{(i)}[p, \xi_i], \tilde{X}_{k,s-1})
\]
\[= -\tilde{h}_{i,k}(p - s) + \tilde{h}_{i,k}(\xi_i - s + 2).
\]
Hence we obtain
\[
(\Lambda^{\tilde{w}} B^{\tilde{w}})_{(i, p), (j, s)} = -\tilde{h}_{i,j}(p - s - 1) - \tilde{h}_{i,j}(p - s + 1) + \tilde{h}_{i,j}(\xi_i - s + 1) + \tilde{h}_{i,j}(\xi_i - s + 3)
\]
\[+ \sum_{k: d(k, j) = 1} \langle h_k, \alpha_j \rangle (-\tilde{h}_{i,k}(p - s) + \tilde{h}_{i,k}(\xi_i - s + 2))
\]
\[= 2d_i \delta(i = j)(-\delta(p - s = 0) + \delta(\xi_i - s + 2 = 0))
\]
\[ -2d(i, \delta((i, p) = (j, s)), \]

where \( \Delta \) follows from Corollary 4.10.

**Remark 7.2.** When (a) \( \Delta \) is simply-laced, (b) \( Q = (\Delta, \xi) \) has a sink-source orientation, that is \( \xi_i \in \{0, 1\} \) for all \( i \in \Delta_0 \), and (c) \( r = \infty \), the above theorem is proved in [3, Proposition 5.1.1] (see also [15, Proposition 5.26 in Arxiv version (arXiv:2007.03159v1)]).

**Appendix A.** \( \tilde{d}_{i,j}(t) \) for \( E_7 \) and \( E_8 \)

**A.1.** \( E_7 \). Here is the list of \( \tilde{d}_{i,j}(t) \) for \( E_7 \).

\[
\begin{align*}
\tilde{d}_{1,1}(t) &= t^1 + t^7 + t^{11} + t^{17}, & \tilde{d}_{1,2}(t) &= t^4 + t^8 + t^{10} + t^{14}, \\
\tilde{d}_{1,3}(t) &= t^2 + t^6 + t^8 + t^{10} + t^{12} + t^{16}, & \tilde{d}_{1,4}(t) &= t^3 + t^5 + t^7 + 2t^9 + t^{11} + t^{13} + t^{15}, \\
\tilde{d}_{1,6}(t) &= t^5 + t^7 + t^{11} + t^{13}, & \tilde{d}_{1,7}(t) &= t^6 + t^{12}, \\
\tilde{d}_{2,2}(t) &= t^1 + t^5 + t^7 + t^9 + t^{11} + t^{13} + t^{17}, & \tilde{d}_{2,3}(t) &= \tilde{d}_{1,4}(t) \\
\tilde{d}_{2,4}(t) &= t^2 + t^4 + 2t^6 + 2t^8 + 2t^{10} + 2t^{12} + t^{14} + t^{16}, & \tilde{d}_{2,5}(t) &= t^3 + t^5 + 2t^7 + t^9 + 2t^{11} + t^{13} + t^{15}, \\
\tilde{d}_{2,6}(t) &= t^4 + t^6 + t^8 + t^{10} + t^{12} + t^{14}, & \tilde{d}_{2,7}(t) &= t^5 + t^9 + t^{13}, \\
\tilde{d}_{3,3}(t) &= t^1 + t^3 + t^5 + 2t^7 + 2t^9 + 2t^{11} + t^{13} + t^{15} + t^{17}, & \tilde{d}_{3,4}(t) &= t^2 + 2t^4 + 2t^6 + 3t^8 + 3t^{10} + 2t^{12} + 2t^{14} + t^{16}, \\
\tilde{d}_{3,5}(t) &= t^3 + 2t^5 + 2t^7 + 2t^9 + 2t^{11} + 2t^{13} + t^{15}, & \tilde{d}_{3,6}(t) &= t^5 + t^7 + t^{11} + t^{13}, \\
\tilde{d}_{3,6}(t) &= t^4 + 2t^6 + t^8 + t^{10} + 2t^{12} + t^{14}, & \tilde{d}_{3,7}(t) &= t^6 + 3t^5 + 4t^7 + 4t^9 + 4t^{11} + 3t^{13} + 2t^{15} + t^{17}, \\
\tilde{d}_{4,4}(t) &= t^1 + 2t^3 + 3t^5 + 4t^7 + 4t^9 + 4t^{11} + 3t^{13} + 2t^{15} + t^{17}, & \tilde{d}_{4,5}(t) &= t^2 + 2t^4 + 3t^6 + 3t^8 + 3t^{10} + 3t^{12} + 2t^{14} + t^{16}, \\
\tilde{d}_{4,6}(t) &= \tilde{d}_{3,5}(t), & \tilde{d}_{4,7}(t) &= t^4 + t^6 + t^8 + t^{10} + t^{12} + t^{14}, \\
\tilde{d}_{5,5}(t) &= t^1 + t^3 + 2t^5 + 2t^7 + 3t^9 + 2t^{11} + 2t^{13} + t^{15} + t^{17}, & \tilde{d}_{5,6}(t) &= t^2 + t^4 + t^6 + 2t^8 + 2t^{10} + t^{12} + t^{14} + t^{16}, \\
\tilde{d}_{5,7}(t) &= t^3 + t^7 + t^{11} + t^{15}, & \tilde{d}_{5,8}(t) &= t^2 + t^8 + t^{10} + t^{16}, \\
\tilde{d}_{6,6}(t) &= t^1 + t^3 + t^7 + 2t^9 + t^{11} + t^{15} + t^{17}, & \tilde{d}_{6,7}(t) &= t^2 + t^8 + t^{10} + t^{16}, \\
\tilde{d}_{7,7}(t) &= t^1 + t^9 + t^{17}, & \text{and} & \tilde{d}_{i,j}(t) &= \tilde{d}_{j,i}(t).
\end{align*}
\]

**A.2.** \( E_8 \). Here is the list of \( \tilde{d}_{i,j}(t) \) for \( E_8 \).

\[
\begin{align*}
\tilde{d}_{1,1}(t) &= t^1 + t^7 + t^{11} + t^{13} + t^{17} + t^{19} + t^{23} + t^{29}, & \tilde{d}_{1,2}(t) &= t^4 + t^8 + t^{10} + t^{12} + t^{14} + t^{16} + t^{18} + t^{20} + t^{22} + t^{26}, \\
\tilde{d}_{1,3}(t) &= t^2 + t^6 + t^8 + t^{10} + 2t^{12} + t^{14} + t^{16} + 2t^{18} + t^{20} + t^{22} + t^{24} + t^{28}, & \tilde{d}_{1,4}(t) &= t^3 + t^5 + t^7 + 2t^9 + 2t^{11} + 2t^{13} + 2t^{15} + 2t^{17} + 2t^{19} + 2t^{21} + t^{23} + t^{25} + t^{27}, \\
\end{align*}
\]
\begin{align*}
\tilde{\sigma}_{1,5}(t) &= t^4 + t^6 + t^8 + 2t^{10} + t^{12} + 2t^{14} + 2t^{16} + t^{18} + 2t^{20} + t^{22} + t^{24} + t^{26}, \\
\tilde{\sigma}_{1,6}(t) &= t^5 + t^7 + t^9 + t^{11} + t^{13} + 2t^{15} + t^{17} + t^{19} + t^{21} + t^{23} + t^{25}, \\
\tilde{\sigma}_{1,7}(t) &= t^6 + t^8 + t^{10} + t^{12} + t^{14} + t^{16} + t^{18} + t^{22} + t^{24}, \\
\tilde{\sigma}_{1,8}(t) &= t^7 + t^{13} + t^{17} + t^{23}, \\
\tilde{\sigma}_{2,2}(t) &= t^1 + t^5 + t^7 + t^9 + 2t^{11} + t^{13} + 2t^{15} + t^{17} + 2t^{19} + t^{21} + t^{23} + t^{25} + t^{29}, \\
\tilde{\sigma}_{2,3}(t) &= t^3 + t^5 + t^7 + 2(t^9 + t^{11} + t^{13} + 2t^{15} + t^{17} + 2t^{19} + t^{21} + t^{23} + t^{25} + t^{27}), \\
\tilde{\sigma}_{2,4}(t) &= t^2 + t^4 + 2t^6 + 2t^8 + 3(t^{10} + t^{12} + t^{14} + t^{16} + t^{18} + t^{20}) + 2t^{22} + 2t^{24} + 2t^{26} + t^{28}, \\
\tilde{\sigma}_{2,5}(t) &= t^3 + t^5 + 2t^7 + 2t^9 + 2t^{11} + 3t^{13} + 3t^{15} + 3t^{17} + 2t^{19} + 2t^{21} + 2t^{23} + t^{25} + t^{27}, \\
\tilde{\sigma}_{2,6}(t) &= t^4 + t^6 + 2t^8 + t^{10} + 2t^{12} + 2t^{14} + 2t^{16} + 2t^{18} + 2t^{20} + 2t^{22} + t^{24} + t^{26}, \\
\tilde{\sigma}_{2,7}(t) &= \tilde{\sigma}_{1,6}(t), \\
\tilde{\sigma}_{2,8}(t) &= t^6 + t^{10} + t^{14} + t^{16} + t^{20} + t^{24}, \\
\tilde{\sigma}_{3,3}(t) &= t^1 + t^3 + t^5 + 2t^7 + 2t^9 + 3t^{11} + 3t^{13} + 2t^{15} + 3t^{17} + 3t^{19} + 2t^{21} + 2t^{23} + t^{25} + t^{27} + t^{29}, \\
\tilde{\sigma}_{3,4}(t) &= t^2 + 2(t^4 + t^6) + 3t^8 + 4(t^{10} + t^{12} + t^{14} + t^{16} + t^{18} + t^{20}) + 3t^{22} + 2(t^{24} + t^{26}) + t^{28}, \\
\tilde{\sigma}_{3,5}(t) &= t^3 + 2t^5 + 2t^7 + 2t^9 + 3t^{11} + 3t^{13} + 4t^{15} + 3t^{17} + 3t^{19} + 3t^{21} + 2t^{23} + 2t^{25} + t^{27}, \\
\tilde{\sigma}_{3,6}(t) &= t^4 + 2t^6 + 2t^8 + t^{10} + 2t^{12} + 3t^{14} + 3t^{16} + 2t^{18} + 2t^{20} + 2t^{22} + 2t^{24} + 2t^{26}, \\
\tilde{\sigma}_{3,7}(t) &= t^5 + 2t^7 + t^9 + t^{11} + 2t^{13} + 2t^{15} + 2t^{17} + t^{19} + t^{21} + t^{23} + t^{25}, \\
\tilde{\sigma}_{3,8}(t) &= t^6 + t^8 + t^{12} + t^{14} + t^{16} + t^{18} + t^{22} + t^{24}, \\
\tilde{\sigma}_{4,4}(t) &= t^1 + 2t^3 + 3t^5 + 4t^7 + 5t^9 + 6t^{11} + 6t^{13} + 6t^{15} + 6t^{17} + 6t^{19} + 5t^{21} + 4t^{23} + 3t^{25} + 2t^{27} + t^{29}, \\
\tilde{\sigma}_{4,5}(t) &= t^2 + 2t^4 + 3t^6 + 4t^8 + 4t^{10} + 5t^{12} + 5t^{14} + 5t^{16} + 5t^{18} + 4t^{20} + 4t^{22} + 3t^{24} + 2t^{26} + t^{28}, \\
\tilde{\sigma}_{4,6}(t) &= t^3 + 2t^5 + 3t^7 + 3t^9 + 3t^{11} + 4t^{13} + 4t^{15} + 4t^{17} + 3t^{19} + 3t^{21} + 3t^{23} + 2t^{25} + t^{27}, \\
\tilde{\sigma}_{4,7}(t) &= \tilde{\sigma}_{3,6}(t), \\
\tilde{\sigma}_{5,5}(t) &= t^1 + t^3 + 2t^5 + 3t^7 + 3t^9 + 4t^{11} + 4t^{13} + 4t^{15} + 4t^{17} + 4t^{19} + 3t^{21} + 3t^{23} + 2t^{25} + t^{27}, \\
\tilde{\sigma}_{5,6}(t) &= t^2 + t^4 + 2t^6 + 2t^8 + 3t^{10} + 3t^{12} + 3t^{14} + 3t^{16} + 3t^{18} + 3t^{20} + 2t^{22} + 2t^{24} + t^{26} + t^{28}, \\
\tilde{\sigma}_{5,7}(t) &= t^3 + t^5 + t^7 + 2t^9 + 2t^{11} + 2t^{13} + 2t^{15} + 2t^{17} + 2t^{19} + 2t^{21} + t^{23} + t^{25} + t^{27}, \\
\tilde{\sigma}_{5,8}(t) &= t^4 + t^6 + t^{10} + t^{12} + t^{14} + t^{16} + t^{20} + t^{22} + t^{26}, \\
\tilde{\sigma}_{6,6}(t) &= t^1 + t^3 + t^5 + t^7 + 2t^9 + 3t^{11} + 2t^{13} + 2t^{15} + 2t^{17} + 3t^{19} + 2t^{21}.
\end{align*}
\[ + t^{23} + t^{25} + t^{27} + t^{29}, \]
\[ \tilde{d}_{6,7}(t) = t^2 + t^4 + t^8 + 2t^{10} + 2t^{12} + t^{14} + t^{16} + 2t^{20} + t^{22} + t^{26} + t^{28}, \]
\[ \tilde{d}_{6,8}(t) = t^3 + t^9 + t^{11} + t^{13} + t^{17} + t^{19} + t^{21} + t^{27}, \]
\[ \tilde{d}_{7,7}(t) = t^1 + t^3 + t^9 + 2t^{11} + t^{13} + t^{17} + 2t^{19} + t^{21} + t^{27} + t^{29}, \]
\[ \tilde{d}_{7,8}(t) = t^2 + t^{10} + t^{12} + t^{18} + t^{20} + t^{28}, \]
\[ \tilde{d}_{8,8}(t) = t^1 + t^{11} + t^{19} + t^{29} \quad \text{and} \quad \tilde{d}_{i,j}(t) = \tilde{d}_{j,i}(t). \]

REFERENCES

[1] R. Bedard, On commutation classes of reduced words in Weyl groups, European J. Combin. 20 (1999), 483–505.
[2] A. Berenstein and A. Zelevinsky, Quantum cluster algebras, Adv. Math. 195 (2005), no. 2, 405–455.
[3] L. Bittmann, A quantum cluster algebra approach to representations of simply laced quantum affine algebras, Math. Z. (2020), 1–37.
[4] E. Date and M. Okado, Calculation of excitation spectra of the spin model related with the vector representation of the quantized affine algebra of type $A^{(1)}_n$, Int. J. Modern Phys. A 9 (1994), 399–417.
[5] S. Fomin and A. Zelevinsky, Double Bruhat cells and total positivity, J. Amer. Math. Soc. 12 (1999), 335–380.
[6] , Cluster algebras I. Foundations, J. Amer. Math. Soc. 15 (2002), no. 2, 497–529.
[7] E. Frenkel and D. Hernandez, Langlands duality for finite-dimensional representations of quantum affine algebras, Lett. Math. Phys. 96, (2011), 217–261.
[8] E. Frenkel, D. Hernandez and N. Reshetikhin, Folded quantum integrable models and deformed W-algebras, Lett. Math. Phys. 112 (2022).
[9] E. Frenkel and E. Mukhin, Combinatorics of q-characters of finite-dimensional representations of quantum affine algebras, Comm. Math. Phys. 216 (1) (2001) 23–57.
[10] E. Frenkel and N. Reshetikhin, Deformations of W-algebras associated to simple Lie algebras, Comm. Math. Phys. 197 (1998), no. 1, 1–32.
[11] , The q-characters of representations of quantum affine algebras, Recent developments in quantum affine algebras and related topics. Contemp. Math. 248, (1999), 163–205.
[12] R. Fujita, Graded quiver varieties and singularities of normalized $R$-matrices for fundamental modules, Selecta Math. (N.S.) 28 (2022), no.1, 1–45.
[13] R. Fujita, D. Hernandez, S-j. Oh and H. Oya, Isomorphisms among quantum Grothendieck rings and propagation of positivity, J. Reine Angew. Math. 785, (2022), 117–185.
[14] R. Fujita and K. Murakami, Deformed Cartan matrices and generalized preprojective algebras I: Finite type, arXiv:2109.07985v3.
[15] R. Fujita and S-j. Oh, Q-datum and Representation theory of untwisted quantum affine algebras, Commun. Math. Phys., 384 (2021), 1351–1407.
[16] C. Geiß, B. Leclerc and J. Schröer, Cluster structures on quantum coordinate rings, Selecta Math. (N.S.) 19 (2013), no.2, 337–397.
[17] K. Goodearl and M. Yakimov, Quantum cluster algebra structures on quantum nilpotent algebras, Mem. Amer. Math. Soc. 247 no. 1169 (2017), vii+119 pp.
[18] D. Happel, On the derived category of a finite-dimensional algebra, Comment. Math. Helv., 62 (3) (1987), 339–389.
[19] D. Hernandez, $t$-analogues des opérateurs d’écrantage associés aux $q$-caractères, Internat. Math. Res. Not. (8) 451–475 (2003).
[20] , Algebraic approach to $q, t$-characters, Adv. Math. 187, no. 1, 1–52 (2004).
The Kirillov–Reshetikhin conjecture and solutions of T-systems, J. Reine Angew. Math., 596 (2006), 63–87.
21 D. Hernandez and B. Leclerc, Quantum Grothendieck rings and derived Hall algebras, J. Reine Angew. Math. 701 (2015), 77–126.
22 D. Hernandez and H. Oya, Quantum Grothendieck ring isomorphisms, cluster algebras and Kazhdan-Lusztig algorithm, Adv. Math. 347, (2019) 192–272.
23 J. E. Humphreys, Introduction to Lie algebras and representation theory, Graduate Texts in Mathematics, vol. 9, Springer-Verlag, New York-Heidelberg-Berlin, 1972
24 A cluster algebra approach to q-characters of Kirillov-Reshetikhin modules, J. Eur. Math. Soc. 18 (2016), no. 5, 1113–1159.
25 D. Hernandez and B. Leclerc, Quantum Grothendieck rings and derived Hall algebras, J. Reine Angew. Math. 701 (2015), 77–126.
26 A diagrammatic approach to q-characters of Kirillov-Reshetikhin modules, J. Reine Angew. Math., 596 (2006), 63–87.
27 I.-S. Jang, K.-H. Lee and S.-j. Oh, Quantization of virtual Grothendieck rings and their structure including quantum cluster algebra, in preparation.
28 S.-J. Kang, M. Kashiwara and M. Kim Symmetric quiver Hecke algebras and R-matrices of quantum affine algebras II, Duke Math. J. 164 (2015), 1549–1602.
29 S.-J. Kang, M. Kashiwara, M. Kim and S.-j. Oh, Monoidal categorification of cluster algebras, J. Amer. Math. Soc. 31 (2018), no. 2, 349–426.
30 M. Kashiwara, On crystal bases of the q-analogue of universal enveloping algebras, Duke Math. J. 63 (1991) 465–516.
31 Global crystal bases of quantum groups, Duke Math. J. 69 (1993), no. 2, 455–485.
32 M. Kashiwara, M. Kim, S.-j. Oh and E. Park, Monoidal categorification and quantum affine algebras, Compos. Math. 156 (2020), no. 2, 1039–1077.
33 PBW theory for quantum affine algebras, arXiv:2011.14253 to appear in the J. Eur. Math. Soc.
34 Monoidal categorification and quantum affine algebras II, arXiv:2103.10067v2.
35 M. Kashiwara and S-j. Oh, Categorical relations between Langlands dual quantum affine algebras: Doubly laced types, J. Algebraic Combin. 49 (2019), 401–435.
36 t-quantized Cartan matrix and R-matrices for cuspidal modules over quiver Hecke algebras, in preparation.
37 M. Khovanov and A. Lauda, A diagrammatic approach to categorification of quantum groups I, Represent. Theory 13 (2009), 309–347.
38 A diagrammatic approach to categorification of quantum groups II, Trans. Amer. Math. Soc. 363 (2011), 2685–2700.
39 Canonical bases arising from quantized enveloping algebras, J. Amer. Math. Soc. 3 (1990), no. 2, 447–498.
40 Quivers, perverse sheaves, and quantized enveloping algebras , J. Amer. Math. Soc. 4 (1991), no. 2, 365–421.
41 H. Nakajima, t-analogs of q-characters of Kirillov-Reshetikhin modules of quantum affine algebras, Represent. Theory 7 (2003), 259–274.
42 Quiver varieties and t-analogs of q-characters of quantum affine algebras, Ann. of Math. (2) 160, 1057–1097 (2004).
43 S.-j. Oh, The denominators of normalized R-matrices of types $A_{2n-1}^{(2)}$, $A_{2n}^{(2)}$, $B_n^{(1)}$, and $D_{n+1}^{(2)}$, Publ. Res. Inst. Math. Sci. 51 (4), 709–744 (2015).
44 S.-j. Oh and T. Scrimshaw, Categorical relations between Langlands dual quantum affine algebras: exceptional cases, Comm. Math. Phys. 368 (2019), no. 1, 295–367.
45 S.-j. Oh and U.R. Suh, Combinatorial Auslander-Reiten quivers and reduced expressions, J. Korean Math. Soc., 56(2) (2019), 353–385.
46 Twisted and folded Auslander-Reiten quiver and applications to the representation theory of quantum affine algebras, J. Algebras 535, 53–132 (2019).
[47] R. Rouquier, 2-Kac-Moody algebras, arXiv:0812.5023v1.
[48] R. Schiffler, Quiver representations, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, Springer, Cham, 2014.
[49] M. Varagnolo and E. Vasserot, Perverse sheaves and quantum Grothendieck rings, in: Studies in memory of Issai Schur, Progr. Math. 210, Birkhäuser-Verlag, Basel, 345–365 (2002).

(M. Kashiwara) Kyoto University Institute for Advanced Study, Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan & Korea Institute for Advanced Study, Seoul 02455, Korea
Email address: masaki@kurims.kyoto-u.ac.jp

(S.-j. Oh) Ewha Womans University Seoul, 52 Ewhayeodae-gil, Daehyeon-dong, Seodaemun-gu, Seoul, South Korea
Email address: sejin092@gmail.com