Iterated Hopf Ore extensions (IHOEs) over an algebraically closed base field $k$ of positive characteristic $p$ are studied. We show that every IHOE over $k$ satisfies a polynomial identity, with PI-degree a power of $p$, and that it is a filtered deformation of a commutative polynomial ring. We classify all 2-step IHOEs over $k$, thus generalising the classification of 2-dimensional connected unipotent algebraic groups over $k$. Further properties of 2-step IHOEs are described: for example, their simple modules are classified, and every 2-step IHOE is shown to possess a large Hopf center and hence an analog of the restricted enveloping algebra of a Lie $k$-algebra. As one of a number of questions listed, we propose that such a restricted Hopf algebra may exist for every IHOE over $k$.

0. Introduction

0.1. Let $p$ be a prime and let $k$ be an algebraically closed field of characteristic $p$. A famous result of Jacobson [Ja1] states that the enveloping algebra $U(g)$ of a finite dimensional Lie algebra $g$ over $k$ is a finitely generated module over its center, denoted by $Z(g)$. In fact it is easy to see from Jacobson’s proof that the PI-degree $d$ of $U(g)$ is a power of $p$, (the PI-degree being by definition the square root of $\dim Q(Z(g)) Q(U(g))$, where $Q(U(g))$ is the central simple quotient division ring of $U(g)$).

Our first aim is to extend Jacobson’s result to certain connected Hopf $k$-algebras, as follows. Hopf Ore extensions $A[x; \sigma, \delta]$ were defined and studied by Panov [Pa] in 2003. His definition was refined and extended in BOZZ and then in Hua, to iterated Hopf Ore extensions (IHOEs) over a field $F$. (The field $F$ may have characteristic zero.) An $n$-step IHOE over $F$ is a Hopf $F$-algebra $H$ with a finite chain of Hopf subalgebras

$$F = H_0 \subset \cdots \subset H_n = H,$$

such that, for $i = 1, \ldots, n$, $H_i = H_{i-1}[x_i; \sigma_i, \delta_i]$, for an algebra automorphism $\sigma_i$ and a $\sigma_i$-derivation $\delta_i$. See Definition [1] for details, and recall that every IHOE is connected, meaning that its coradical is $F$, by BOZZ Proposition 2.5. Our first main result is

**Theorem 0.1.** Let $p$ and $k$ be as above, and let $H$ be an $n$-step $k$-IHOE.
(1) (Corollary 2.4) \(H\) is a finitely generated module over its center \(Z(H)\), which is a finitely generated normal \(k\)-algebra.

(2) (Theorem 3.3) The PI-degree of \(H\) is a power of \(p\).

(3) (Theorem 3.8) The order of the antipode of \(H\) divides \(4p^n - 1\), and the order of the Nakayama automorphism of \(H\) divides \(2p^n\).

The proofs of (1) and (2) of the theorem in Sections 2 and 3 make use of results of [LeM, CL1, CL2] on Ore extensions satisfying a polynomial identity (PI), and ultimately hinge on fundamental results of Kharchenko [Kh]; some aspects of the argument may have relevance for the study of general Ore extensions, that is, beyond the realm of Hopf algebras.

A number of important homological consequences follow by routine arguments from Theorem 0.1 with details included in Corollary 2.4 - namely, \(H\) has global and Gel’fand-Kirillov dimensions equal to \(n\), and is a homologically homogeneous maximal order, and hence is skew Calabi-Yau.

0.2. By contrast with Theorem 0.1, an IHOE over a field \(F\) of characteristic 0 does not satisfy a polynomial identity unless it is commutative, [BOZZ, Theorem 5.4]. Our second main result, however, generalises an aspect of Lie theory where there is no such dichotomy between characteristic 0 and positive characteristic: namely, it is of course fundamental to the study of enveloping algebras that if \(g\) is a Lie algebra of finite dimension \(n\) over a field \(F\), then \(U(g)\) has an ascending filtration whose associated graded algebra is the commutative polynomial algebra in \(n\) variables over \(F\). Zhuang proved in [Zh, Theorem 6.10] that the same conclusion holds for a connected Hopf \(F\)-algebra of finite Gel’fand-Kirillov dimension \(n\), when \(F\) is algebraically closed of characteristic 0. Zhuang’s result does not extend to positive characteristic - for example, the group algebra over \(k\) of the cyclic group of order \(p\) is a connected Hopf algebra. Nevertheless, we prove here:

**Theorem 0.2** (Theorem 4.3). Let \(p\) and \(k\) be as in Theorem 0.1, and let \(H\) be an \(n\)-step \(k\)-IHOE. Then positive integer degrees can be assigned to the defining variables of \(H\) so that the corresponding filtration has associated graded algebra which is the commutative polynomial \(k\)-algebra on \(n\) variables.

One can succinctly state this result as: every \(H\) as in the theorem is a filtered deformation of a polynomial algebra. Etingof asks in [Et, Question 1.1] whether every filtered deformation of a commutative domain in positive characteristic has to satisfy a polynomial identity. Taken together, Theorems 0.1 and 0.2 give some support for a positive answer to this question.

0.3. In Section 5, we classify IHOEs in the lowest dimensions. As noted in Lemma 5.1, there is only one 1-step IHOE, whatever the characteristic of the field, namely \(k[x]\) with \(x\) primitive. But in dimension two the contrast between characteristic 0 and positive characteristic is stark. Zhuang showed in [Zh, Proposition 7.6] that in characteristic 0, there are only two connected Hopf algebras of Gel’fand-Kirillov dimension two, and both are cocommutative IHOEs: namely \(k[x, y]\) with \(x\) and \(y\) primitive, and the enveloping algebra of the 2-dimensional non-abelian Lie algebra. In positive characteristic, however, it’s a different story, as we now explain.
Let \( d_s = \{d_s\}_{s \geq 0}, b_s = \{b_s\}_{s \geq 0} \) and \( c_{s,t} = \{c_{s,t}\}_{0 \leq s < t} \) be sequences of scalars in \( k \) with only finitely many nonzero elements. Let \( H(d_s, b_s, c_{s,t}) \) denote the Ore extension \( k[X_1][X_2; \text{Id}, \delta] \) where

\[
\delta(X_1) = \sum_{s \geq 0} d_s X_1^s.
\]

Define maps \( \Delta, \epsilon, \) and \( S \) on \( \{X_1, X_2\} \) by setting

\[
\Delta(X_1) = X_1 \otimes 1 + 1 \otimes X_1,
\]
\[
\Delta(X_2) = X_2 \otimes 1 + 1 \otimes X_2 + w,
\]
\[
\epsilon(X_1) = \epsilon(X_2) = 0,
\]
\[
S(X_1) = -X_1,
\]
\[
S(X_2) = -X_2 - m(\text{Id} \otimes S)(w),
\]

where \( m \) denotes the multiplication operator, and

\[
w = \sum_{s \geq 0} b_s \left( \sum_{i=1}^{p-1} \frac{(p-1)!}{i!(p-i)!} (X_1^p)^i \otimes (X_1^p)^{p-i} \right) + \sum_{0 \leq s < t} c_{s,t} (X_1^t \otimes X_1^s - X_1^s \otimes X_1^t).
\]

**Theorem 0.3** (Propositions 5.6 and 5.11). Let \( p \) and \( k \) be as in Theorem 0.7. Let \( H \) be a 2-step IHOE over \( k \), so \( H = k[X_1][X_2; \sigma, \delta] \) for \( \sigma \in \text{Aut}_{k\text{-alg}}(k[X_1]) \) and a \( \sigma \)-derivation \( \delta \).

1. (Proposition 5.6(1)) For all choices of the scalars \( d_s, b_s, c_{s,t}, H(d_s, b_s, c_{s,t}) \) is a Hopf algebra.
2. (Proposition 5.6(2)) \( H \) is isomorphic to \( H(d_s, b_s, c_{s,t}) \) for some choice of scalars \( \{d_s\}_{s \geq 0}, \{b_s\}_{s \geq 0} \) and \( \{c_{s,t}\}_{0 \leq s < t} \).
3. (Proposition 5.11) Two such \( H(d_s, b_s, c_{s,t}) \) and \( H(d'_s, b'_s, c'_{s,t}) \) are isomorphic as Hopf algebras if and only if there are nonzero scalars \( \alpha, \beta \) in \( k \) such that, for all the sequences of scalars,

\[
d'_s = d_s \alpha^{-p-1} \beta^{-1}, \quad b'_s = b_s \alpha^{p+1} \beta^{-1}, \quad c'_{s,t} = c_{s,t} \alpha^{p+p'} \beta^{-1}.
\]

If (E0.3.1) holds, there is a Hopf algebra isomorphism \( \phi : H(d_s, b_s, c_{s,t}) \rightarrow H(d'_s, b'_s, c'_{s,t}) \) such that

\[
\phi(X_1) = \alpha X_1, \quad \text{and} \quad \phi(X_2) = \beta X_2.
\]

Restricting the above result to the cases where \( H \) is commutative, that is where \( d_s = 0 \), yields the classification of connected unipotent algebraic groups over \( k \) of dimension 2. This is discussed in 0.4.

0.4. We study further properties of the 2-step IHOEs over \( k \) in 8. Thus we describe their antipodes and determine the Calabi-Yau members of the family in Proposition 8.1, and examine the finite dimensional representation theory of all the 2-step \( k \)-IHOEs in Proposition 8.2, specifying the Azumaya locus, and determining all the simple modules and the extensions between them.

The final subsection, 8.3, is inspired by another seminal paper of Jacobson, [Ja2], where he showed that every restricted Lie algebra \( g \) of finite dimension \( n \) has
a restricted enveloping algebra $u(g)$, a Hopf algebra of dimension $p^n$ which is a Hopf factor of $U(g)$. We show that a similar phenomenon occurs for the 2-step IOHEs. The following is an abbreviated version of Theorem 8.8 and Propositions 8.9-8.10 together.

**Theorem 0.4.** Let $H = H(d_s, b_s, c_s, t)$ be a 2-step IOHE over $k$ and assume that $H$ is not commutative.

1. $H$ contains a unique maximal central Hopf subalgebra $C(H)$, and $H$ is a free $C(H)$-module of rank $p^2$ when $b_0 = 0$, and rank $p^3$ otherwise.
2. Let $C(H)_+$ denote the augmentation ideal of $C(H)$. The resulting Hopf algebras $H/C(H)_+$, of dimensions $p^2$ or $p^3$, are classified.

Finite dimensional connected Hopf $k$-algebras have dimension a power of $p$, [Ma, Proposition 1.1(1)]. Those of dimension at most $p^3$ have been classified, in a series of papers [Wa1, Wa2, NWW1, NWW2] by Xingting Wang and colleagues. We explain in §8.3 how the Hopf algebras of Theorem 0.4 fit into their classification.

0.5. There have been a number of recent papers on infinite dimensional connected Hopf algebras of finite Gel'fand-Kirillov dimension, starting with [Zh] and including the aforementioned [BOZZ, Hua], but also, for example, [BGZ, BZ2, WZZ2, ZSL]. Much of this work has focused on the characteristic $0$ setting (although not, notably, [Hua], on which we rely heavily). Thus the present paper can be viewed as taking some further steps to open the topic in positive characteristic. As befits a gathering of first steps, the paper contains a number of open questions, scattered throughout. We can also point out here an open question of a more general, less precise type. Namely, it is possible that many or even all of the results proved here for positive characteristic IOHEs are valid, when rephrased appropriately, in the wider context of connected Hopf domains over $k$ of finite Gel'fand-Kirillov dimension.

0.6. The paper is organized as follows. Section 1 contains some definitions and preliminaries, leading to Proposition 1.2 where it is shown that the automorphisms used to construct a positive characteristic IOHE have finite order, a key ingredient in the proof of Theorem 1.1. In Section 2, we use the Frobenius map and Noether’s theorem on the finite generation of invariants to obtain the characterisation (Theorem 2.3) of when an iterated Ore extension in positive characteristic satisfies a PI. From this, it is easy to invoke Proposition 1.2 to deduce Theorem 1.1 (Corollary 2.4). Theorem 1.2 is proved in Section 3, using results of Chuang and Lee [CL1], depending ultimately on work of Kharchenko [Kh] to obtain the needed information on the PI-degree of an iterated Ore extension which is known to satisfy a PI. In Section 4 we show that every IOHE has a filtration such that the associated graded ring is isomorphic to the commutative polynomial algebra. The classification of 2-step IOHEs is given in Section 5 including an analysis of their automorphism groups. A list of comments and questions are collected in Section 6. A description of the Hopf center of all 2-step IOHEs is obtained in Section 8 enabling the construction and classification of their restricted Hopf algebra factors. The description of the Hopf center requires a special case of the noncommutative binomial theorem of Jacobson, which we recall in Section 7.
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1. Preliminaries

1.1. Definitions and their consequences. Throughout the paper we shall use the standard notation $\Delta$, $\mu$, $\epsilon$, and $S$ for the comultiplication, multiplication, counit, and antipode of a Hopf algebra $H$, with $\Delta(h) = h_1 \otimes h_2$ for $h \in H$. Unexplained Hopf algebra terminology can be found in [Mo1], for example. All Hopf algebras appearing in this paper will be (factors of) noetherian domains, so $S$ is necessarily bijective by [Sk, Theorem A(ii)]. For an algebra $A$, $Z(A)$ will denote the center of $A$, and if a group $G$ acts on $A$ as automorphisms, $A^G$ will denote the fixed subring of this action; when $G = \langle \sigma \rangle$ generated by single element $\sigma$, $A^\sigma$ will be abbreviated to $A^\sigma$.

The following definition of the algebras in the paper’s title is - on the face of it - significantly weaker than the original definition given in [Pa, Definition 1.0] and the modified definition in [BOZZ, Definition 2.1]. The improvement here is due to recent work of Huang [Hua], as we explain after the definitions. Recall that, given an $F$-algebra automorphism $\sigma$ of an $F$-algebra $R$, a $\sigma$-derivation $\delta$ of $R$ is an $F$-linear endomorphism of $R$ such that $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$ for all $a, b \in R$.

Definition 1.1. Let $F$ be a field of any characteristic.

(1) Let $R$ be a Hopf $F$-algebra. A Hopf Ore extension (abbreviated to HOE) of $R$ is an algebra $H$ such that

(a) $H$ is a Hopf $F$-algebra with Hopf subalgebra $R$;
(b) there exist an algebra automorphism $\sigma$ of $R$ and a $\sigma$-derivation $\delta$ of $R$ such that $H = R[x; \sigma, \delta]$.

(2) An $(n\text{-step})$ iterated Hopf Ore extension of $F$ (abbreviated to $(n\text{-step})$ IHOE (of $F$)) is a Hopf algebra

\[ H = F[X_1][X_2; \sigma_2, \delta_2] \ldots [X_n; \sigma_n, \delta_n], \]

where

(a) $H$ is a Hopf algebra;
(b) $H_{(i)} := F(X_1, \ldots, X_i)$ is a Hopf subalgebra of $H$ for $i = 1, \ldots, n$;
(c) $\sigma_i$ is an algebra automorphism of $H_{(i-1)}$, and $\delta_i$ is a $\sigma_i$-derivation of $H_{(i-1)}$, for $i = 2, \ldots, n$.

Throughout the paper, whenever $H$ is an HOE [respectively, an $n$-step IHOE], we assume that $H$ satisfies the conditions of Definition 1.1 [respectively, (2)], with the same notation.
The definition of an HOE in [BOZZ, Definition 2.1] required, in addition to Definition 1.1(1a,1b), that

\[(1c) \text{ there are } a, b \in R \text{ and } v, w \in R \otimes R \text{ such that} \]
\[
\Delta(x) = a \otimes x + x \otimes b + v(x \otimes x) + w. \tag{E1.1.2}
\]

However, by Huang’s theorem [Hua, Theorem 1.3], if \( R \subseteq T \) are noetherian \( F \)-algebras satisfying hypotheses (1a) and (1b) of Definition 1.1(1), and \( R \otimes R \) is a domain, then, up to a change of variable,

\[(1d) \text{ there are } a \in R \text{ and } w \in R \otimes R \text{ such that} \]
\[
\Delta(x) = a \otimes x + x \otimes 1 + w. \tag{E1.1.3}
\]

This means that under the conditions in [Hua, Theorem 1.3], namely that \( R \otimes R \) is a noetherian domain, Definition 1.1(1) is equivalent to [BOZZ, Definition 2.1]. In turn, this yields the following consequences for IHOEs. Recall that a Hopf \( F \)-algebra is connected if its coradical is \( F \) - see e.g. [Mo1, Definition 5.1.5].

**Proposition 1.2.** Let \( H \) be an IHOE of \( F \).

1. \( H \) is noetherian and \( H \otimes H \) is a domain.
2. After a change of variables (but not of the subalgebras \( H_{(i)} \)),
\[
\Delta(X_i) = 1 \otimes X_i + X_i \otimes 1 + w_i
\]
for some \( w_i \in H_{(i-1)} \otimes H_{(i-1)} \), for \( i = 1, \ldots, n \).
3. \( H \) is a connected Hopf algebra.
4. For \( i = 1, \ldots, n \) there is a character \( \chi_i \) of \( H_{(i-1)} \) such that
\[
\sigma_i = \tau^l_{\chi_i} = \tau^r_{\chi_i}
\]
is a left and right winding automorphism of \( H_{(i-1)} \). In particular for each \( j \) with \( j < i \) there exists \( a_{ji} \in H_{(j-1)} \) such that
\[
\sigma_i(X_j) = X_j + a_{ji}.
\]

**Proof.** (1) This follows by induction on \( n \) using [McR, Theorem 1.2.9].

(2) Fix \( i, 1 \leq i \leq n \) (where we take \( H_{(0)} = F, \sigma_1 = \text{id}_F \) and \( \delta_1 = 0 \)). By (1), \( H_{(i-1)} \) is noetherian and \( H_{(i-1)} \otimes H_{(i-1)} \) is a domain. Hence, by [Hua, Theorem 1.3],
\[
\Delta(X_i) = a_i \otimes X_i + X_i \otimes 1 + w_i
\]
for a group-like element of \( a_i \) of \( H_{(i-1)} \) and \( w_i \in H_{(i-1)} \otimes H_{(i-1)} \). But \( H \) has no nontrivial invertible elements. Therefore, noting that \( \mu \circ (\epsilon \otimes \text{id}) \circ \Delta = \text{id}_{H_{(i-1)}} \), \( a_i = 1 \), yielding (E1.2.1).

(3) This is proved in [BOZZ, Proposition 2.5].

(4) That the map \( \sigma_i \) is a (left and right) winding automorphism (with the same character) of \( H_{(i-1)} \) is proved in [BOZZ, Theorem 1.2] and its improvement [Hua, Theorem 1.3]. From (E1.1.3) applied to \( X_j \) we see that
\[
a_{ji} = \chi_i(X_j) + \sum w_{j(1)}(w_{j(2)}) \in H_{(j-1)}.
\]
\[\square\]
Since it is needed for the proof of Theorem 0.1 we recall a result in classical invariant theory due to E. Noether [No], also see [Mo2, Theorem 1.1] and [Sm, Theorem 2.3.1].

**Theorem 1.3.** Let $A$ be an affine commutative algebra over $F$ and $G$ be a finite subgroup of $\text{Aut}_{F-alg}(A)$. Then the fixed subring $A^G$ is affine over $F$ and $A$ is a finitely generated module over $A^G$.

### 1.2. Positive characteristic aspects

Given Proposition 1.2, the following result is key to our main results.

**Proposition 1.4.** Suppose $k$ has positive characteristic $p$. Let $d$ be a positive integer.

1. Let $R$ be a Hopf $k$-algebra such that the order of every left or right winding automorphism of $R$ divides $d$. Suppose $H := R[x, \sigma, \delta]$ is an HOE of $R$ such that

$$(E1.4.1) \quad \Delta(x) = 1 \otimes x + x \otimes 1 + w$$

for some $w \in R \otimes R$. Then the order of every left or right winding automorphism of $H$ divides $dp$.

2. Let $H$ be an $n$-step IHOE of $k$. Then every left or right winding automorphism of $H$ has order dividing $p^n$.

**Proof.** We prove the results for left winding automorphisms.

(1) Let $\pi : H \to k$ be an algebra map and let $\Xi^l_\pi$ be the corresponding left winding automorphism of $R$. Suppose $H := R[x, \sigma]$ is an HOE of $R$ such that

$$(E1.4.1) \quad \Delta(x) = 1 \otimes x + x \otimes 1 + w$$

where $s = \pi(x) + \mu \circ (\pi \otimes 1)(w) = x + s$

Then

$$(\Xi^l_\pi)^d(x) = x + t$$

for some $t \in R$. Note that $(\Xi^l_\pi)^d(t) = t$ as the order of $\Xi^l_\pi$ restricted to $R$ divides $d$. Then

$$(\Xi^l_\pi)^{dp}(x) = ((\Xi^l_\pi)^d)^p(x) = x + pt = x.$$ 

Therefore $(\Xi^l_\pi)^{dp}$ is the identity map on $H$ as required.

(2) This follows from part (1), Proposition 1.2(2) and induction on $n$. 

Note that there is an anti-monomorphism of groups from the character group $X(H)$ of $H$ (that is, the algebra homomorphisms from $H$ to $k$) to the group of left winding automorphisms of $H$, [BZ1, Section 2.5]. Hence, viewing $H$ in dual language, as a quantum group, Proposition 1.4(2) can be interpreted as giving a bound on the exponent of the maximal classical subgroup of $H$, which is a unipotent algebraic group.
2. Proof of Theorem 0.1

2.1. HOEs and IHOEs in positive characteristic. If \( C \) is a commutative \( k \)-algebra, where \( k \) has positive characteristic \( p \), we denote the Frobenius map \( e \mapsto e^p \) on \( C \) by \( F \). In this section, we always assume the following.

**Convention 2.1.** When \( k \) is a field of positive characteristic and \( C \) is a commutative \( k \)-algebra, we use \( F(C) \) to denote the \( k \)-subalgebra of \( C \) generated by the image of \( C \). Thus, in general, \( F(C) \) is larger than the image of \( C \) under \( F \).

Given an automorphism \( \sigma \) and a \( \sigma \)-derivation \( \delta \) of an algebra \( A \), a subspace \( E \) of \( A \) is called \( (\sigma, \delta) \)-trivial if \( \sigma(e) = e \) and \( \delta(e) = 0 \) for all \( e \in E \).

**Lemma 2.2.** Let \( B \) be an Ore extension \( A[x; \sigma, \delta] \) of an affine \( k \)-algebra \( A \).

1. Suppose that no nonzero element of \( Z(A) \) is a zero divisor on \( A \), and that \( Z(A) \sigma \subseteq Z(A) \). Then
   \[
   zx = xz
   \]
   for all \( z \in Z(A) \sigma \). As a consequence, \( B \) contains the commutative subalgebra \( Z(A) \sigma[x] \).
2. Suppose \( Z(A) = Z(A) \sigma \) and \( \text{char } k = p > 0 \). Then
   \[
   z^p x = xz^p
   \]
   for all \( z \in Z(A) \). As a consequence, \( B \) contains the commutative subalgebra \( F(Z(A))[x] \).
3. Suppose that
   (a) \( A \) is a prime ring that is a finitely generated module over \( Z(A) \);
   (b) \( Z(A) \) is an affine \( k \)-algebra;
   (c) \( \sigma |_{Z(A)} \) has finite order;
   (d) \( \text{char } k = p > 0 \).
   Then \( B \) is a prime noetherian algebra satisfying a polynomial identity, and \( A \) is a finite module over a \( (\sigma, \delta) \)-trivial subalgebra of its center.
4. Suppose, in addition to hypotheses (3)(a)-(3)(d), that
   (e) \( A \) is a maximal order.
   Then \( B \) is a maximal order and is a finitely generated module over \( Z(B) \), which is an affine normal \( k \)-algebra.

**Proof.** (1) Let \( z \in Z(A) \sigma \). Then \( xz = zx + \delta(z) \), so we only need to show that \( \delta(z) = 0 \) for all \( z \in Z(A) \sigma \). Pick any element \( y \in Z(A) \setminus Z(A) \sigma \). Then \( zy = yz \). Applying \( \delta \) to this equation we have
   \[
   \delta(z)y + \sigma(z)\delta(y) = \delta(y)z + \sigma(y)\delta(z).
   \]
   Since \( z \in Z(A) \sigma \), we have \( \sigma(z)\delta(y) = z\delta(y) = \delta(y)z \), which implies that
   \[
   \delta(z)y = \sigma(y)\delta(z).
   \]
   Hence, since \( \sigma(y) \in Z(A) \),
   \[
   \delta(z)(y - \sigma(y)) = 0.
   \]
   But \( y - \sigma(y) \in Z(A) \setminus \{0\} \) and is thus, by hypothesis, not a zero divisor in \( A \). So \( \delta(z) = 0 \) as required, and hence \( Z(A) \sigma[x] \) is commutative.
(2) Suppose that $Z(A) = Z(A)^p$. Then, for every $z \in Z(A)$,
$$xz = \sigma(z)x + \delta(z) = zx + \delta(z).$$
By induction on $n$ and since $z \in Z(A)$, we have, for all $n \geq 1$,
$$xz^n = z^n x + nz^{n-1}\delta(z).$$
The assertion follows because $pz^{p-1}\delta(z) = 0$. Once again, the consequence is clear.

(3) It is clear from [McR, Theorem 1.2.9] and hypotheses (a) and (b) that $B$ is
noetherian, and it is prime by [McR, Theorem 1.2.9(iii)]. By Theorem 1.3, and hypotheses (a) and (c), $A$ is a finitely generated $Z(A)^p$-module. When $Z(A) \neq Z(A)^p$, $B$ is a finitely generated left module over its commutative affine subalgebra $Z(A)^p[x]$ by part (1). When $Z(A) = Z(A)^p$, $B$ is a finitely generated left module over its commutative affine subalgebra $F(Z(A))[x]$ by part (2). In both cases, $B$ thus satisfies a polynomial identity, by [McR, Corollary 13.4.9(i)]. The second claim in the final sentence of (3) is clear from (1), (2) and Theorem 1.3.

(4) Suppose that $A$ satisfies hypothesis (e). Then $B$ is also a maximal order by
[MaR, Proposition V.2.5]. Therefore, since it is a prime noetherian PI ring by part (3), $B$ equals its own trace ring and is thus a finite module over its affine normal center by [McR, Propositions 13.9.8(i) and 13.9.11(ii)].  

Theorem 1.1 is a corollary of the following more general result applying to iterated Ore extensions in positive characteristic.

**Theorem 2.3.** Suppose $k$ has positive characteristic $p$, let $n$ be a non-negative integer, and let $R$ be an $n$-step iterated Ore extension,

$$(E2.3.1) \quad R = k[X_1;|X_2;\sigma_2, \delta_2|] \ldots [X_n;\sigma_n, \delta_n].$$

For $i = 1, \ldots, n-1$, denote $k\langle X_1, \ldots, X_i \rangle$ by $R_i$.

1. $R$ satisfies a polynomial identity if and only if $\sigma_i |Z(R_i)$ has finite order for all $i = 2, \ldots, n$.

2. Suppose that $R$ satisfies a polynomial identity. Then it is a finitely generated module over its center, which is a normal $k$-affine domain. Moreover, $R$ is a homologically homogeneous and GK-Cohen Macaulay domain, with

$$\text{gldim } R = \text{GKdim } R = n.$$ 

**Proof.** (1) Suppose that $R$ satisfies a PI. Then so does $R_{i} = R_{i-1}|X_i; \sigma_i, \delta_i|$ for $i = 2, \ldots, n$. For each such $i$, since $R_{i-1}$ is a domain satisfying a PI, [LeM, Theorem 2.7] then requires that $|\sigma_i |Z(R_{i-1})| < \infty$, as claimed.

Conversely, assume that each $\sigma_i |Z(R_{i-1})$ has finite order. By induction on $n$ we may assume that $A := R_{n-1}$ satisfies a PI, and that part (2) of the theorem has been proved for $A$. In particular, $A$ is a finitely generated $Z(A)$-module. Since $A$ is an iterated Ore extension of a field, it is a domain. By our induction hypothesis and since $\sigma_n |Z(A)$ has finite order, hypotheses (3)(a)-(d) of Lemma 2.2 are satisfied. By [MaR, Proposition V.2.5], $A$ is a maximal order, so (e) of Lemma 2.2(4) holds. Therefore, by Lemma 2.2(4), $R$ is a finitely generated $Z(R)$-module and $Z(R)$ is an affine normal domain.
The statements in the second sentence have already been proved in (1). The
properties listed in the last sentence are also proved by induction on \( n \). By the
induction hypothesis, \( \text{gldim} A = n - 1 \), so \( \text{gldim} R = n - 1 \) or \( n \) by \([\text{McR} \text{ Theorem 7.5.3(i)}]\). Thanks to \([\text{McR} \text{ Corollary 7.1.14}]\), there is a simple (left) \( A \)-module \( V \) with \( \text{projdim} V = n - 1 \). Choose a maximal left ideal \( I \) of \( A \) with \( V \cong A/I \). Let \( W \) be a simple \( R \)-module which is a factor of the nonzero cyclic \( R \)-module \( R/RI \cong R \otimes_A V \), so \( W \cong R/L \) for a left ideal \( L \) of \( R \) containing \( RI \). Since \( R \) is a noetherian affine \( k \)-algebra satisfying a polynomial identity, by part (1),
\( \dim_k W < \infty \) by Kaplansky’s theorem, \([\text{McR} \text{ Theorem 13.10.3(i)}]\). In particular,
\( W \) is a finitely generated \( A \)-module, and \( L \cap A = I \) by maximality of \( I \). Thus \( V \) is an
\( A \)-submodule of \( W \). Hence, choosing an \( A \)-module \( X \) such that \( \text{Ext}_A^{n-1}(V, X) \neq 0 \),
the long exact sequence of \( \text{Ext} \) yields

\[
\text{Ext}_A^{n-1}(W, X) \rightarrow \text{Ext}_A^{n-1}(V, X) \rightarrow 0,
\]
so that \( \text{projdim}_A W = n - 1 \). Therefore \( \text{gldim} R = n \) by \([\text{McR} \text{ Corollary 7.9.18}]\).

Since \( A \) is homologically homogeneous, so is \( R \), by \([\text{Zho} \text{ Theorem 2.3}]\), noting that the hypotheses of Zhong’s theorem are satisfied in view of Lemma \([2.2 \text{(1),(2)}]\), which guarantee that \( A \) is a finite module over a central subalgebra which is \((\sigma_n, \delta_n)\)-trivial. Finally, \( R \), being a homologically homogeneous algebra which is a finite module over an affine central domain, is GK-Cohen Macaulay by \([\text{BM} \text{ Corollary 5.4 and Theorem 4.8}]\). \( \square \)

Theorem \([0.1]\) is now an immediate consequence of the above theorem combined with Proposition \([1.4 \text{(2)}]\):

**Corollary 2.4.** Suppose \( k \) has positive characteristic \( p \) and let \( n \) be a non-negative integer. Then every \( n \)-step IHOE \( H \) of \( k \) is a finitely generated module over its center \( Z(H) \) and \( Z(H) \) is a normal \( k \)-affine domain. Moreover \( H \) is a homologically homogeneous and GK-Cohen Macaulay algebra with

\[
\text{gldim} H = \text{GKdim} H = n.
\]

2.2. First questions around Theorem \([0.1]\) We briefly consider two possible improvements of Theorem \([0.1]\).

**Remarks 2.5.** (1) As a large family of examples to which Corollary \([2.4]\) applies, let \( \mathfrak{g} \) be a finite dimensional completely solvable Lie algebra over a field \( k \) of positive characteristic \( p \). (That is, \( \mathfrak{g} \) has a full flag of ideals.) Thus the enveloping algebra \( U(\mathfrak{g}) \) is an IHOE, so Corollary \([2.4]\) tells us that \( U(\mathfrak{g}) \) is a finite module over its center. This is a special case of a fundamental result due to Jacobson \([Ja1]\):

Let \( \mathfrak{g} \) be a finite dimensional Lie algebra over \( k \). Then \( U(\mathfrak{g}) \) contains a central Hopf subalgebra over which \( U(\mathfrak{g}) \) is a free module of finite rank equal to a power of \( p \).

Note that Jacobson’s theorem does not require \( \mathfrak{g} \) to be restricted. In language not then available to Jacobson, his result implies that the PI-degree of \( U(\mathfrak{g}) \) is a power of \( p \), prefiguring Theorem \([3.3]\) below. Moreover, the Lie algebra case thus suggests a possible strengthened version of Corollary \([2.4]\).

**Question 2.6.** Is every IHOE over a field of positive characteristic a finite module over a central Hopf subalgebra?
(2) Since the enveloping algebra of a finite dimensional Lie algebra \( g \) is not an IHOE when \( g \) contains a non-abelian (classical) simple subalgebra other than \( \mathfrak{sl}(2, k) \), one might wonder, in the light of Jacobson’s theorem discussed in (1), whether every connected affine Hopf algebra of finite Gel’fand-Kirillov dimension is finite over its center, or even finite over a central Hopf subalgebra, when \( k \) has positive characteristic. As a first test case for this, consider an affine connected graded Hopf algebra \( H \), meaning that

\[
H = \bigoplus_{i \geq 0} H_i
\]

is connected graded as both an algebra and as a coalgebra. Two recent papers, [BGZ] and [ZSL], study such Hopf algebras. (In [BGZ] the possibly weaker hypothesis that \( H \) is only connected graded as an algebra is also treated, but we do not discuss that here.) Theorem A of [ZSL] states that if \( k \) has characteristic 0 and \( H \) is an affine connected graded Hopf \( k \)-algebra, then \( H \) is an \( n \)-step IHOE, where \( n \) is the Gel’fand-Kirillov dimension of \( H \). The hypothesis on \( k \) is necessary here - consider \( H = k[x]/(x^p) \) where \( k \) has characteristic \( p \). Nevertheless, Corollary 2.4 and [ZSL, Theorem A] prompt the

**Question 2.7.** Suppose \( H \) is an affine connected graded Hopf \( k \)-algebra of finite Gel’fand-Kirillov dimension, where \( k \) has positive characteristic. Is \( Z(H) \) affine, and is \( H \) a finite \( Z(H) \)-module?

### 3. The PI degree and other invariants

**3.1. PI-degree.** Theorem [3.3] the main result of this subsection, gives information on the PI-degree of an \( n \)-step IHOE in characteristic \( p \). Recall that the PI-degree of a prime PI ring \( R \) is denoted by \( P\text{deg} \ R \) and defined to be \( d \), where \( d^2 \) is the dimension over the field \( Z(Q(R)) \) of the central simple quotient ring \( Q(R) \), see [McR, 13.3.6, 13.6.7] and [BG] p.115. When \( R \) is in addition an affine \( k \)-algebra with the field \( k \) algebraically closed, \( P\text{deg} \ R \) equals the maximum dimension over \( k \) of the simple \( R \)-modules [McR, Theorem 13.10.3], [BG, Theorem 1.13.5(2)]. To prove Theorem 3.3 we need the following definitions and theorem, adapted from [CL1]. Note that the definitions in [CL1] are expressed in terms of the left Martindale quotient ring of a ring \( R \), but when \( R \) is a prime PI ring, as here, this coincides with the total quotient ring \( Q(R) = R[Z(R) \setminus \{0\}]^{-1} \), as follows easily from the definition of the Martindale quotient ring and Posner’s theorem, [McR Sect. 10.3.5 and Theorem 13.6.5].

**Definition 3.1.** [CL1] Let \( R \) be a prime algebra satisfying a polynomial identity, \( \sigma \in \text{Aut}_{k-\text{alg}}(R) \) and \( \delta \) a \( \sigma \)-derivation of \( R \). Let \( Q \) be the total quotient ring of \( R \).

(1) \( \sigma \) is called X-inner if its extension to \( Q \) is inner, that is, there exists a unit \( b \) of \( Q \) such that

\[
\sigma(x) = b^{-1}xb
\]

for all \( x \in Q \). Otherwise, \( \sigma \) is called X-outer.

---

1 See [BOZZ, Example 3.1(iv)], where the characteristic 0 hypothesis is not necessary.
(2) $\sigma$ is called quasi-inner if there exists an integer $n \geq 1$ such that $\sigma^n$ is $X$-inner. The least such integer $n$ is called the outer degree of $\sigma$ and is denoted by $\text{Outdeg}\,\sigma$.

(3) $\delta$ is called $X$-inner if its extension to $Q$ is inner, that is, there exists $b \in Q$ such that
\[
\delta(x) = bx - \sigma(x)b
\]
for all $x \in Q$. Otherwise, $\delta$ is called $X$-outer.

(4) $\delta$ is called quasi-algebraic if there exist a positive integer $n$, an automorphism $g$ of $Q$ and $b_1, \ldots, b_{n-1}, b \in Q$ such that for all $x \in R$,
\[
\delta^n(x) + b_1\delta^{n-1}(x) + \cdots + b_{n-1}\sigma(x) = bx - g(x)b.
\]
The least such integer $n$ is called the quasi-algebraic degree or the outer degree of the $\sigma$-derivation $\delta$ and is denoted by $\text{Outdeg}\,\delta$. In particular, $\text{Outdeg}\,\delta = 1$ if and only if $\delta$ is $X$-inner.

The main result of Chuang-Lee [CL1] is the following.

**Theorem 3.2.** [CL1, Theorem 2.5] Let $R$ be a prime PI algebra, $\sigma \in \text{Aut}_{k-\text{alg}}(R)$ and $\delta$ a $\sigma$-derivation of $R$. Then $R[x; \sigma, \delta]$ is a PI ring if and only if $\delta$ is quasi-algebraic and $\sigma$ is quasi-inner. In this case,

(a) if $\delta$ is $X$-outer, then $\text{Pldeg}\,R[x; \sigma, \delta] = \text{Pldeg}\,R \times \text{Outdeg}\,\delta$;
(b) if $\delta$ is $X$-inner, then $\text{Pldeg}\,R[x; \sigma, \delta] = \text{Pldeg}\,R \times \text{Outdeg}\,\sigma$.

We can now deduce our second main theorem from Corollary 2.4 and Theorem 3.2:

**Theorem 3.3.** Suppose that $k$ is a field of characteristic $p > 0$, and let $H$ be an IHOE over $k$. Then $\text{Pldeg}\,H$ is a power of $p$.

**Proof.** We argue by induction on $n$, where $H$ is an $n$-step IHOE. When $n = 1$ $H = k[x]$ and the result is clear. For the induction step, write $B$ for $H_{(n-1)}$, so, by the induction hypothesis,
\[
\text{Pldeg}\,B = p^t,
\]
for some $t \geq 0$. After relabeling, $H = B[x; \sigma, \delta]$ for $\sigma \in \text{Aut}_{k-\text{alg}}(B)$ and a $\sigma$-derivation $\delta$ of $B$. By Proposition 1.2(1) and Corollary 2.4, $B$ and $H$ are both prime PI-algebras. Thus, by Theorem 3.2 $\sigma$ is quasi-inner and $\delta$ is quasi-algebraic, and there are two cases to consider.

**Case** (a): $\delta$ is $X$-outer. Then, since $B$ is a prime PI ring, a result of Kharchenko, see [CL2] Lemmas 5 and 6, guarantees the existence of a unit $u$ of the quotient ring $Q(B)$ and an $X$-outer derivation $d$ of $Q(B)$ such that, for all $b \in B$,
\[
\sigma(b) = ubu^{-1} \quad \text{and} \quad \delta = ud.
\]
Note that $\sigma$ and $\delta$ extend respectively to an automorphism and a $\sigma$-derivation of $Q(B)$. Then, by [CL1] Lemma 1.3(2),
\[
Q(B)[x; \sigma, \delta] \cong Q(B)[x; d].
\]
Now
\[
\text{Pldeg}\,B = \text{Pldeg}\,Q(B)
\]
and

(E3.3.4) \[ \text{PId} H = \text{PId} Q(B)[x; \sigma, \delta]. \]

To prove the induction step in Case (a) it is therefore enough, by (E3.3.1), (E3.3.2), (E3.3.3) and (E3.3.4), to prove that

(E3.3.5) \[ \text{PId} Q(B)[x; d] = \text{PId} B \times p^\ell \]

for some \( \ell \geq 0 \). But (E3.3.5) follows immediately from Theorem 3.2(a) and a theorem of Kharchenko, stated and proved as [CL2, Theorem, p.60].

Case (b): \( \delta \) is X-inner. Now Theorem 3.2(b) applies. But clearly, from the definition,

\[ \text{Outdeg} \sigma | | \sigma |. \]

Moreover, by [Hua, Theorem 1.3(ii)] and Proposition 1.4(2),

\[ |\sigma| | p^n. \]

Therefore (E3.3.1) is proved in this case also, and so the theorem follows. \( \square \)

Remarks 3.4. (1) A second proof of Theorem 3.3 will be given Section 4, where we obtain it as a corollary of Theorem 4.3 together with a result of Etingof [Et] on filtered deformations of commutative domains in positive characteristic.

(2) Faced with Theorem 3.3 it is natural to ask

Question 3.5. What is the power of \( p \) occurring in Theorem 3.3?

This is a delicate matter, as can be seen from the case of enveloping algebras. Assume that \( k \) is algebraically closed of characteristic \( p > 0 \), and let \( \mathfrak{g} \) be a finite dimensional Lie \( k \)-algebra. For \( f \in \mathfrak{g}^* \) define

\[ \text{stab}_\mathfrak{g}(f) := \{ x \in \mathfrak{g} \mid f([x, -]) = 0 \}, \]

and set

\[ \text{ind}\mathfrak{g} := \min\{\dim_k \text{stab}_\mathfrak{g}(f) \mid f \in \mathfrak{g}^*\}. \]

The first Kac-Weisfeiler conjecture [KW] proposed that when \( \mathfrak{g} \) is restricted,

(E3.5.1) \[ \text{PId} \mathfrak{g}(\mathfrak{g}) = p^{\frac{1}{2}(\dim \mathfrak{g} - \text{ind}\mathfrak{g})}. \]

This was already known, due to Rudakov [Ru], when \( \mathfrak{g} \) is the Lie algebra of a reductive group (in this case \( \text{ind}\mathfrak{g} \) is the rank of \( \mathfrak{g} \)); and Strade [St] proved the conjecture for \( \mathfrak{g} \) solvable in 1978, extending the completely solvable case done in [KW]. Premet and Skryabin [PS] confirmed the conjecture for all restricted \( \mathfrak{g} \) admitting a toral stabilizer, and also showed that \( p^{\frac{1}{2}(\dim \mathfrak{g} - \text{ind}\mathfrak{g})} \) is a lower bound for the PI-degree for all restricted \( \mathfrak{g} \). The general case, however, remains open, although it has recently been confirmed in [MSTT] for all restricted subalgebras \( \mathfrak{g} \) of \( \mathfrak{gl}_n(k) \), for \( p \gg 0 \). Note, however, that the conjecture is not true if the hypothesis that \( \mathfrak{g} \) is restricted is omitted - Topley [To] exhibits pairs of Lie \( k \)-algebras \( \mathfrak{g}_1 \) and \( \mathfrak{g}_2 \), both solvable, with \( \text{ind}\mathfrak{g}_1 \neq \text{ind}\mathfrak{g}_2 \), but with \( U(\mathfrak{g}_1) \cong U(\mathfrak{g}_2) \).

It might be easier to approach Question 3.5 by first seeking an upper bound:

Question 3.6. Is the PI-degree of an \( n \)-step IHOE over a field \( k \) of characteristic \( p \) bounded above by \( p^n \)?
We show in [8.2] that this is true when \( n \leq 2 \).

(3) As recalled at the start of §3.1, when \( k \) is algebraically closed the PI degree is the maximum dimension of a simple \( H \)-module, so Question 3.5 leads one to the broader issue of the possible dimensions of all the simple \( H \)-modules, when \( H \) is an \( n \)-step IHOE over an algebraically closed field of characteristic \( p \). One might hope that every simple module has dimension a power of \( p \), and we confirm that this is the case for all 2-step IHOEs in Proposition 8.2. Further positive evidence is provided by the enveloping algebras of completely solvable Lie algebras. Such a Lie algebra \( \mathfrak{g} \) of dimension \( n \) has a chain of \( n+1 \) ideals from \( \{0\} \) to \( \mathfrak{g} \), so that \( U(\mathfrak{g}) \) is an \( n \)-step IHOE. Kac and Weisfeler showed in [KW] that the dimension of every simple \( U(\mathfrak{g}) \)-module is a power of \( p \). But the naive hope fails for \( \mathfrak{sl}(2, k) \) when \( k \) has characteristic \( p > 2 \), since \( U(\mathfrak{sl}(2, k)) \) is a 3-step IHOE [BOZZ, Example 3.1(iv)], and Rudakov and Shafarevitch showed [RS] that the dimensions of the simple \( U(\mathfrak{sl}(2, k)) \)-modules are 1, \ldots, \( p \). These examples and results suggest the following possibility:

**Question 3.7.** Suppose \( k \) is an algebraically closed field of characteristic \( p > 0 \). Let \( H \) be an \( n \)-step IHOE over \( k \), which has a chain of Hopf subalgebras as in (E1.1.1) with each \( \sigma_i \) being the identity. Is the dimension of every simple \( H \)-module a power of \( p \)?

The above question has a positive answer for all 2-step IHOEs - see Proposition 8.2.

### 3.2. The antipode and Nakayama automorphism.

The antipode \( S \) of a Hopf algebra that is a finite module over its center has finite order up to inner automorphisms by [BZ1, Corollary 0.6(c)]. In the case of IHOEs in positive characteristic, we can easily obtain information about the order, as we show below. The Nakayama automorphism \( \nu \) of an AS-Gorenstein Hopf algebra provides the twist in its rigid dualizing complex, see [BZ1, Sections 0.2, 0.3] for details. It is defined up to an inner automorphism. The **Nakayama order**, that is the lowest power \( \ell \) such \( \nu^\ell \) is inner, is finite in the current setting thanks to [BZ1, Corollary 0.6(c)]. Again, more can be said about its precise value.

**Theorem 3.8.** Let \( n \geq 1 \) and let \( H \) be an \( n \)-step IHOE over a field \( k \) of characteristic \( p > 0 \).

1. The order of the antipode \( S \) divides \( 4p^{n-1} \).
2. The Nakayama order of \( H \) divides \( 2p^n \).

**Proof.** Note that every invertible element in \( H \) is in \( k \), so the Nakayama order of \( H \) coincides with the (well-defined) order of the Nakayama automorphism \( \nu \) of \( H \).

(1) The proof of part (1) is similar to the proof of Proposition 1.4(2). We use induction on \( n \) to show that the order of \( S^4 \) divides \( p^{n-1} \). The assertion holds trivially for \( n = 1 \). Now assume that the assertion holds for \((n - 1)\)-step IHOEs. Let \( H \) an \( n \)-step IHOE and write it as \( B[x; \sigma, \delta] \) where \( B \) is an \((n - 1)\)-step IHOE. Suppose the order of \( S^4 \) restricted to \( B \) is \( p^t \) for some \( t \leq n - 2 \).

Let \( \eta : H \to k \) be the right character of the left homological integral of \( H \) [LWZ]. Let \( \Xi^l_\eta \) [respectively, \( \Xi^r_\eta \)] be the corresponding left [respectively, right] winding
where $w$ begins by describing such a filtration and proving its required properties. We for example in this setting. Thus, for instance, there is often an infinite dimensional space of primitive elements, as in $k$.

4.1. Construction of a filtration. Recall that, by [Z1, Theorem 6.9], if $H$ is an affine connected Hopf algebra of finite Gel’fand-Kirillov dimension $n$ over an algebraically closed field $k$ of characteristic 0, then the associated graded algebra of $H$ with respect to the coradical filtration is a commutative polynomial algebra in $n$ variables. This result does not extend to positive characteristic - in the first place there are nontrivial finite dimensional connected Hopf algebras in characteristic $p$, for example $k[x]/(x^p)$; and, second, the coradical filtration is typically much coarser in this setting. Thus, for instance, there is often an infinite dimensional space of primitive elements, as in $k[x]$ for example. At least for IHOEs there is nevertheless an analogue of Zhuang’s result, provided one uses a more suitable filtration. We begin by describing such a filtration and proving its required properties.

Let $H$ be an $n$-step IHOE with notation as in Definition 1.1. The ordered monomials $x_1^{m_1} \cdots x_n^{m_n}$ with $m_i \in \mathbb{Z}_{\geq 0}$ constitute a $k$-basis $\mathcal{B}(H)$ of $H$, which we
call its PBW basis. (This holds true for all iterated Ore extensions, not just for Hopf algebras.) For $\alpha \in H$, call the presentation of $\alpha$ in terms of the PBW basis the PBW expression of $\alpha$, and let
\[
\text{PBWsupp}(\alpha) := \{ m \in B(H) \mid m \text{ occurs in the PBW expression for } \alpha \}.
\]
We proceed now to define a degree function $\text{deg} : H \rightarrow \mathbb{Z}_{\geq 0}$. Recall from Proposition 1.2(4) the elements $a_{ji} \in H_{(j-1)}$ for $i, j = 1, \ldots, n$ with $i > j$, such that
\[
\sigma_i(x_j) = x_j + a_{ji}
\]
where $a_{ji}$ is given as in (E1.2.2). Moreover we can define elements $c_{ji} \in H_{(i-1)}$ such that, for all $i, j = 1, \ldots, n$ with $i > j$,
\[
\delta_i(x_j) = c_{ji}.
\]
Thus the defining relations of $H$ are
\[
\text{(E4.0.1)} \quad x_i x_j = x_j x_i + a_{ji} x_i + c_{ji}, \quad (1 \leq j < i \leq n).
\]
Recall that $w_i$ is defined in (E1.2.1); fix an expression
\[
w_i = \sum w_i(1) \otimes w_i(2),
\]
where $w_i(1), w_i(2)$ are in $H_{(i-1)} \cap \ker \epsilon$. For each $\ell > i$, let $\chi_\ell$ be the character of $H_{(\ell-1)}$ such that $\sigma_\ell = \tau_\ell^{\chi_\ell}$ as given in Proposition 1.2(4). Then by (E1.2.2),
\[
a_{i\ell} = \chi_\ell(x_i) + \sum w_i(1) \chi_\ell(w_i(2)) \in H_{(i-1)}.
\]
Define the degree $\text{deg}(\lambda) := 0$ for $\lambda \in k$, and inductively define the degree $\text{deg}(x_i) := d_i \in \mathbb{Z}_{>0}$ and $\text{deg}(\alpha)$ for $\alpha \in H_{(i)}$, as follows:
- $d_1 = 1$.
- Suppose that $i > 1$ and that $d_1, \ldots, d_{i-1}$ have been defined. For each PBW basis monomial $m = x_1^{m_1} x_2^{m_2} \cdots x_{i-1}^{m_{i-1}}$, set
\[
\text{(E4.0.2)} \quad \text{deg}(m) := \sum_{j=1}^{i-1} m_j d_j.
\]
- For $\alpha \in H_{(i-1)}$ define
\[
\text{(E4.0.3)} \quad \text{deg}(\alpha) := \max \{ \text{deg}(m) \mid m \in \text{PBWsupp}(\alpha) \}.
\]
- Since $\text{deg}$ is defined on $H_{(i-1)}$, let
\[
D(w_i) = \max_{i(1)} \{ \text{deg}(w_i(1)) \}
\]
which is only dependent on $i$. Then we have
\[
D(w_i) \geq \max_{\ell > i} \{ \text{deg}(a_{i\ell}) \}.
\]
Now set $d_i \in \mathbb{Z}_{>0}$ such that
\[
\text{(E4.0.4)} \quad d_i > D(w_i) \quad (\geq \max_{\ell > i} \{ \text{deg}(a_{i\ell}) \}),
\]
and such that
\[
\text{(E4.0.5)} \quad d_i \geq \max_{j < i} \{ \text{deg}(c_{ji}) \}.
\]
Finally, after the above steps are completed up to \( i = n \), extend (E4.0.2) [resp. (E4.0.3)] to all PBW basis monomials [resp. every element] of \( H \). Clearly this procedure yields a well-defined degree for all \( \alpha \in H \). We define a filtration \( C := \{ C_i \}_{i \geq 0} \) by setting, for all \( i \geq 0 \),

\[
C_i := \{ \alpha \in H \mid \deg(\alpha) \leq i \}.
\]

It is clear that \( C_0 = k \).

**Lemma 4.1.** Keep the above notation.

1. For \( 1 \leq j < i \leq n \), \( x_i x_j = x_j x_i + (\text{lower degree terms}) \).
2. For \( 1 \leq j < i \leq n \), \( \deg(x_i x_j) = \deg(x_j x_i) \).
3. For all monomials \( m = x_{i_1}^{t_1} x_{i_2}^{t_2} \cdots x_{i_m}^{t_m} \), where \( i_\ell \in \{1, \ldots, n\} \) and \( t_\ell \in \mathbb{Z}_{\geq 0} \) for all \( \ell = 1, \ldots, m \),

\[
(E4.1.1) \quad \deg(m) = \sum_{\ell=1}^{m} t_\ell d_{i_\ell}.
\]

**Proof.**

1. Let \( j < i \). In the relation \( a_{ji} x_i + c_{ji} x_i \in H_{(j-1)} \) and \( c_{ji} x_i \in H_{(i-1)} \). Thus

\[
\deg(a_{ji} x_i + c_{ji} x_i) = \max\{\deg(a_{ji} x_i), \deg(c_{ji} x_i)\} < d_j + d_i = \deg(x_j x_i),
\]

where the first equality follows from (E4.0.3) and the fact that there is empty intersection between the PBW expressions for \( a_{ji} x_i \) and \( c_{ji} x_i \), so that no cancellation can occur, and the inequality follows from (E4.0.4) and (E4.0.5), since \( a_{ji} x_i \) is already in PBW form.

2. This follows from part (1) and the definition of the degree of elements of \( H \).

3. Define the weight of a not necessarily PBW-ordered monomial \( m = x_{i_1}^{t_1} \cdots x_{i_m}^{t_m} \) to be

\[
\text{wt}(m) := \sum_{\ell=1}^{m} t_\ell d_{i_\ell}.
\]

Suppose that assertion (3) is false, and choose a counterexample \( m \) of minimal weight, say \( \text{wt}(m) = d > 0 \). Then \( m \) cannot be an ordered monomial by the definition (E4.0.2). So amongst counterexamples to assertion (3) of weight \( d \), choose \( m \) to be one with the minimal number \( \omega \) of badly ordered pairs - that is, pairs of generators occurring in \( m \) as \( m = \cdots x_r \cdots x_s \cdots \) with \( r > s \). Clearly, there must be an adjacent bad pair in \( m \). That is, there exist \( s, r \) with \( 1 \leq s < r \leq n \), with

\[
\begin{align*}
\text{m} & = x_{i_1} \cdots x_r x_s \cdots x_{i_m} \\
& = x_{i_1} \cdots (x_s x_r + a_{sr} x_r + c_{sr}) \cdots x_{i_m} \\
& = (x_{i_1} \cdots x_s x_r \cdots x_{i_m}) + (x_{i_1} \cdots a_{sr} x_r \cdots x_{i_m}) \\
& \quad + (x_{i_1} \cdots c_{sr} \cdots x_{i_m}).
\end{align*}
\]

Here, the first monomial on the right side has fewer than \( \omega \) badly ordered pairs, so assertion (3) is true for it by choice of \( m \). The second and third brackets on the
right consist of monomials of weight strictly less than \(d\), by (E4.0.4) and (E4.0.5). So, again by choice of \(m\), assertion (3) holds for all the monomials in the second and third brackets on the right; in particular, their degree is strictly less than the degree of \(x_{i_1} \cdots x_s x_r \cdots x_{i_m}\). Thus

\[
\deg(m) = \deg(x_{i_1} \cdots x_s x_r \cdots x_{i_m}) = \sum_{\ell=1}^m t_{\ell} d_{i_{\ell}},
\]

and \(m\) is not a counterexample. This proves (3).

Lemma 4.2. Continue with the above notation.

1. The filtration \(C\) defined before Lemma 4.1 is an algebra filtration of \(H\).
2. The associated graded algebra \(\text{gr}_C H\) is a factor ring of the commutative polynomial ring \(k[\overline{x}_1, \ldots, \overline{x}_n]\) where \(\overline{x}_i\) is the principal symbol of \(x_i\).

Proof. (1) We need to prove that, for all \(r, s \in \mathbb{Z}_{\geq 0}\),

\[C_r C_s \subseteq C_{r+s}.\]

For this it is enough to show that if \(r, s\) are ordered monomials in \(C_r\) and \(C_s\) respectively, then \(\deg(rs) \leq r + s\). This is immediate from Lemma 4.1(3).

(2) It is clear from the definitions that \(0 \neq \overline{x}_i \in \text{gr}_C H\) for all \(i = 1, \ldots, n\). From this and (E4.1.1), one sees immediately that \(\text{gr}_C H\) is generated by \(\{\overline{x}_i\}_{i=1}^n\).

Commutativity of \(\text{gr}_C H\) follows from Lemma 4.1(1). Therefore there is an algebra epimorphism \(\phi\) from the commutative polynomial ring \(k[\overline{x}_1, \ldots, \overline{x}_n]\) to \(\text{gr}_C H\).

4.2. The associated graded algebra. We can now prove the main result of this section.

Theorem 4.3. Let \(k\) be an algebraically closed field, and let \(H\) be an \(n\)-step IHOE over \(k\). Then positive integer degrees can be assigned to the defining variables of \(H\) so that the corresponding filtration, constructed from this assignment by using the PBW basis as above, has associated graded algebra which is the commutative polynomial \(k\)-algebra on \(n\) variables.

Proof. Let \(H\) be given by Definition 1.1 and define its filtration \(C\) by the recipe given in §4.1. By Lemma 1.2 \(\text{gr}_C H\) is a factor of the polynomial \(k\)-algebra on \(n\) generators. On the other hand, \(\text{GKdim} (H) = n\) by Corollary 2.4. Since \(C\) is discrete and finite, it follows from [KL, Proposition 6.6] that \(\text{gr}_C H\) grows at the same rate; that is, \(\text{GKdim} (\text{gr}_C H) = n\). Since proper factors of the polynomial algebra on \(n\) generators have GKdimension strictly less than \(n\), the result follows.

Remarks 4.4. (1) An alternative proof that \(\text{gr}_C H\) is a polynomial algebra uses the PBW basis directly. Namely, by the definition of \(C\), \(H\) and \(\text{gr}_C H\) have the same PBW basis, which agrees with the PBW basis of \(k[\overline{x}_1, \ldots, \overline{x}_n]\). Therefore the map \(\phi\) from the proof of Lemma 4.2(2) is an isomorphism.

(2) A theorem of Etingof, [Et, Corollary 3.2(ii)], states that if \(k\) is a field of characteristic \(p > 0\) and \(A\) is any \(k\)-algebra with filtration \(C = \{C_i : i \geq 0\}\) such that \(A\) satisfies a polynomial identity and \(\text{gr}_C A\) is a commutative domain, then the PI-degree of \(A\) is a power of \(p\). In view of Corollary 2.4 and Theorem 4.3 these
hypotheses both are satisfied by any IHOE over the field \(k\), yielding a second proof of Theorem 3.3.

(3) Etingof asks in [Et, Question 1.1] whether every filtered deformation of an affine commutative domain in positive characteristic has to satisfy a polynomial identity. Theorem 4.3 coupled with Corollary 2.4 provide some evidence in favour of a positive answer.

(4) Theorem 4.3 can be restated as: every \(n\)-step IHOE over the field \(k\) is a PBW-deformation of the polynomial ring in \(n\) variables over \(k\) in the sense of [BeG, Section 3]. It follows from Zhuang’s theorem [Zh, Theorem 6.9] that a connected Hopf algebra over a field \(k\) of characteristic zero is a PBW-deformation of the polynomial ring in \(n\) variables if \(n = \text{GKdim} \ H < \infty\). Given these facts, together with the classical PBW theorem for enveloping algebras of finite dimensional Lie algebras, and the fact that the underlying variety of every connected unipotent group of dimension \(n\) is affine \(n\)-space over \(k\), [Se, Chap.VII, sec.6, Corollary and Remark 1, p.170], it is natural to ask:

**Question 4.5.** Is every connected Hopf \(k\)-algebra domain of finite Gel’fand-Kirillov dimension a PBW-deformation of a polynomial algebra over \(k\)?

(5) An \(\infty\)-step IHOE is defined to be

\[
\lim_{n \to \infty} H_n
\]

if there is an infinite sequence of \(n\)-step IHOE

\[
\mathbb{k} = H_0 \subset H_1 \subset H_2 \subset \cdots \subset H_n \subset \cdots
\]

such that each \(H_n\) satisfies the conditions in Definition 1.1(2). If \(\{d_i\}_{i \geq 1}\) is a strictly increasing sequence of integers defined by using the process given before Lemma 4.1, then we can define a locally finite filtration \(C\) of \(H\) such that the associated graded algebra \(\text{gr}_C H\) is isomorphic to \(\mathbb{k}[\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n, \cdots]\) – the polynomial ring of infinitely many variables.

(6) Let \(\chi\) be a character of \(H\). Then the right winding automorphism \(\tau^\chi\) of \(H\) preserves the filtration \(C\) constructed in Section 4.1. It follows that the winding automorphism \(\tau^\chi\) induces an automorphism of \(\text{gr}_C H\), which one can show to be the identity map.

5. **Classification of 1- and 2-step IHOE in positive characteristic**

For the rest of the paper we would like to take the first steps in a project to classify the Hopf algebra domains of Gel’fand-Kirillov dimension at most two in positive characteristic. In characteristic 0 considerable progress has already been made towards this classification, as we shall briefly recall at the start of §5.2. First, we deal with the well-known case of Gel’fand-Kirillov dimension one.

5.1. **Hopf domains of GKdimension one.** Recall that the only connected algebraic groups of dimension 1 over an algebraically closed field \(k\) are the additive and multiplicative groups of \(k\) [Hum, Theorem 20.5]. It is easy to remove the hypothesis of commutativity from this result, as follows. Let \(k^\times\) denote \(k \setminus \{0\}\).
Lemma 5.1. Let $k$ be any algebraically closed field. The only affine Hopf $k$-algebra domains of Gel’fand-Kirillov dimension one are the coordinate rings $k[X]$ and $k[X^\pm 1]$ of $(k, +)$ and $(k^\times, \times)$ respectively. In particular, the only 1-step IHOE over $k$ is $k[X]$ with $X$ primitive.

Proof. By [LWZ, Corollary 7.8 (a)⇒(b)], $H$ is commutative. Therefore $H$ is the coordinate ring of an affine connected algebraic group over $k$, and the result follows from [Hum, Theorem 20.5]. □

By [GZ1, Proposition 2.1], in characteristic 0 the hypothesis that the Hopf algebra is affine can be removed from Lemma 5.1 at the cost of adding the group algebras of non-cyclic subgroups of $\mathbb{Q}$ to the list. But the proof of [GZ1, Proposition 2.1] does not work in positive characteristic. We refer to a recent survey paper [BZ2] for some facts about the prime Hopf algebras of GKdimension one and two.

5.2. Hopf domains of GKdimension two. When $k$ has characteristic 0 the classification of affine Hopf $k$-algebra domains $H$ of Gel’fand-Kirillov dimension 2 was achieved in [GZ1, GZ2] with the imposition of the extra hypothesis that $\mathrm{Ext}^1_H(k, k) \neq 0$ (or equivalently, that the quantum group $H$ contains a non-trivial classical subgroup). Then, in [WZZ1], a family of dimension two $k$-affine Hopf PI domains is constructed which fail to satisfy the extra hypothesis, but it is not yet known whether, with the addition of this family, the list is complete. Staying with $k$ of characteristic 0, if one restricts attention to connected Hopf $k$-algebras of finite Gel’fand-Kirillov dimension, then they are all affine by [Zh, Theorem 6.9], and the classification is complete up dimension at most four [Zh, WZZ3]. In all these classification results in characteristic 0 the outcome takes a similar form - namely, there is a finite number of families in each list, with each family being given by a finite set of discrete or continuously varying parameters.

Turning now to positive characteristic, and restricting attention to 2-step IHOEs, we find that even in this very confined setting the situation is completely different from that pertaining in characteristic 0. This is shown by the main result of this subsection, Proposition 5.5 which lists all the two-step IHOEs in characteristic $p$. Together with Proposition 5.11 in §5.3 which describes the isomorphisms and automorphisms between these algebras, this classifies 2-step IHOEs in positive characteristic. It transpires that their description entails an infinite dimensional space of parameters. The contrast with characteristic 0 could not be starker - when $k$ has characteristic 0, a result of Zhuang [Zh, Proposition 7.4(III)] shows that there are only two connected Hopf $k$-algebras of Gel’fand-Kirillov dimension 2, namely the enveloping algebras of the two 2-dimensional Lie algebras, both of which are obviously IHOEs. Conversely, by [BOZZ, Theorem 1.3], every IHOE is connected, so that [Zh, Proposition 7.4(III)] provides a list of the (two) 2-step IHOEs in characteristic 0.

To understand 2-step IHOEs $k[X_1, X_2; \sigma, \delta]$ in positive characteristic we employ the primitive cohomology studied in [WZZ2], from which we first recall some definitions. Let $(C, \Delta)$ be a coalgebra with a fixed grouplike element $1_C$. Let $T(C)$ be the tensor algebra over the vector space $C$ in cohomological degree 1 with differential
Proposition 5.4. Retain the above notation.

structures on the same or similar spaces, it is convenient to fix some notation. By \[ \text{Lemma 5.1.} \] the divided power Hopf algebra (of one variable) was introduced in \[ \text{Definition 1.2(1)} \], the \( n \)-th primitive cohomology of \( C \) (associated to \( g = \hbar = 1_C \)) is defined to be \[
\mathcal{P}^n_{1_C,1_C}(C) = H^n(T(C), \partial) = \ker \partial^n/\im \partial^{n-1} = Z^n(C)/B^n(C).\]

If \( C \) is \( \mathbb{N} \)-graded locally finite, then \( T(C) \) is \( \mathbb{Z}^2 \)-graded and locally finite. As a consequence, each \( \mathcal{P}^n_{1_C,1_C}(C) \) is \( \mathbb{N} \)-graded and locally finite. We will use following lemma in the computation of primitive cohomology.

Lemma 5.2. Let \( C \) be a connected \( \mathbb{N} \)-graded coalgebra and let \( A \) be the graded dual algebra of \( C \), namely, \( A_i = \operatorname{Hom}_k(C_i, k) \) for all \( i \geq 0 \). Then \( A \) is a connected graded algebra with trivial graded module \( k \) and \[
\dim \mathcal{P}^n_{1_C,1_C}(C)_i = \dim \operatorname{Ext}^n_A(k, k)_i
\]
for all \( n \) and \( i \).

Proof. After we identify \( \dim \operatorname{Ext}^n_A(k, k)_i \) with \( \dim \operatorname{Tor}^A_n(k, k)_{-i} \) for all \( i \), the assertion is a consequence of \[ \text{[WZZ2 Lemma 3.6(2)]}. \]

For the rest of this section we assume that \( k \) has positive characteristic \( p \). We are now ready to compute the primitive cohomology of the coalgebra \( k[X_1] \) from Lemma 5.1. The divided power Hopf algebra (of one variable) was introduced in \[ \text{[Sw], also see [DNR Example 5 in Sect. 4.3]} \]. By definition, the divided power Hopf algebra \( \mathcal{T} \) is a \( k \)-vector space with basis \( \{ t_n \}_{n \geq 0} \) and with its bialgebra structure determined by \[
\Delta(t_n) = \sum_{i=0}^{n} t_i \otimes t_{n-i}, \quad \epsilon(t_n) = \delta_{0,n} \quad \text{and} \quad t_n t_m = \binom{n+m}{n} t_{n+m}
\]
for all \( n, m \geq 0 \). By comparing the structure coefficients of the multiplications and comultiplications of \( \mathcal{T} \) and \( k[X_1] \) (as in Lemma 5.1 respectively, one can easily see that \( \mathcal{T} \) is the graded \( k \)-linear dual of \( k[X_1] \), giving part (1) of Proposition 5.4.

Convention 5.3. Since we are using various different algebra and/or coalgebra structures on the same or similar spaces, it is convenient to fix some notation.

1. Let \( A \) denote the algebra \( \mathcal{T} \) obtained by forgetting the coalgebra structure, namely, \( A = \bigoplus_{d \geq 0} kt_d \) is the divided power algebra of one variable with multiplication determined by \( t_d t_e = \binom{d+e}{d} t_{d+e} \) for all \( d, e \geq 0 \).
2. Let \( C \) be the graded coalgebra \( k[X_1] \) given in Lemma 5.1 by forgetting its algebra structure.

Proposition 5.4. Retain the above notation.

1. \( A \) is isomorphic to the graded dual algebra of the coalgebra \( C \).
(2) As a graded algebra \( A \) is generated by \( \{ t_p, \ s \geq 0 \} \) subject to the relations
\[
(t_p)^p = 0, \quad \text{and} \quad t_p t_p = t_p t_p
\]
for all \( s < t \). As a consequence,
\[
\dim \text{Ext}^1_A(k, k)_i = \begin{cases} 1 & i = p^s \text{ for all } s \geq 0 \\ 0 & \text{otherwise} \end{cases}
\]
and
\[
\dim \text{Ext}^2_A(k, k)_i = \begin{cases} 1 & i = p^{s+1} \text{ for all } s \geq 0 \\ 1 & i = p^s + p^t \text{ for all } 0 \leq s < t \\ 0 & \text{otherwise} \end{cases}
\]

(3) Consider \( C \) as a graded coalgebra with \( 1_C \) being the identity of \( k[X_1] \). Then
\[
\dim \mathcal{P}^1_{1_C, 1_C}(C)_i = \begin{cases} 1 & i = p^s \text{ for all } s \geq 0 \\ 0 & \text{otherwise} \end{cases}
\]
and
\[
\dim \mathcal{P}^2_{1_C, 1_C}(C)_i = \begin{cases} 1 & i = p^{s+1} \text{ for all } s \geq 0 \\ 1 & i = p^s + p^t \text{ for all } 0 \leq s < t \\ 0 & \text{otherwise} \end{cases}
\]

(4) The following elements in \( C \otimes C \) generate a \( k \)-linear basis of \( \mathcal{P}^2_{1_C, 1_C}(C) \):
\[
(E5.4.1) \quad Z_s := \sum_{i=1}^{p-1} \frac{(p-1)!}{i!(p-i)!} (X_1^{p^s})^i \otimes (X_1^{p^s})^{p-i}
\]
for each \( s \geq 0 \), and
\[
(E5.4.2) \quad Y_{s,t} := X_1^{p^s} \otimes X_1^{p^t} - X_1^{p^t} \otimes X_1^{p^s}
\]
for all \( s < t \).

Proof. (2) The assertion follow from the fact that \( \text{Ext}^1_A(k, k) \) can be identified with a minimal set of generators and \( \text{Ext}^2_A(k, k) \) can be identified with a minimal set of relations.

(3) This follows from Lemma 5.2 and parts (1,2).

(4) By part (3), it suffices to show that elements \( Z_s \) and \( Y_{s,t} \) are in \( Z^2(C) \) but not in \( B^2(C) \), which can be verified by some straightforward computations. \( \square \)

In the next lemma \( Z_s \) and \( Y_{s,t} \) are as given in (E5.4.1)-(E5.4.2).

**Lemma 5.5.** Let \( H \) be a 2-step IHOE generated by \( X_1 \) and \( X_2 \) as in Definition 7.4(2); that is, \( H = k[X_1][X_2; \sigma, \delta] \). Retain the notation introduced in Proposition 7.4(4). Then the following hold:

(1) \[
\Delta(X_1) = X_1 \otimes 1 + 1 \otimes X_1
\]
and \[
\Delta(X_2) = X_2 \otimes 1 + 1 \otimes X_2 + w
\]
where

\[(E5.5.1)\quad w = \sum_{s \geq 0} b_s Z_s + \sum_{s < t} c_{s,t} Y_{s,t}\]

for some scalars \(b_s, c_{s,t} \in \mathbb{k}\).

(2) \(\sigma(X_1) = X_1 + \chi\) where \(\chi \in \mathbb{k}\).

(3) \(\delta\) is a \(\sigma\)-derivation of \(\mathbb{k}[X_1]\) such that

\[\Delta(\delta(X_1)) = \delta(X_1) \otimes 1 + 1 \otimes \delta(X_1) - \chi w\]

where \(w\) is given in \((E5.5.1)\).

(4) \(\chi w = 0\).

(5) \(\delta(X_1) = \sum_{s \geq 0} d_s X_1^{p^s}\) for some \(d_s \in \mathbb{k}\).

Proof. Recall that \(C\) is the coalgebra \(\mathbb{k}[X_1]\) and \(1_C\) is the identity 1 in \(\mathbb{k}[X_1]\).

(1) Let \(w\) be \(\partial(X_2) := \Delta(X_2) - (X_2 \otimes 1 + 1 \otimes X_2)\). By [Hua] Theorem 1.3(iii) or [BOZZ] Theorem 2.4(i)(f), \(w \in \mathbb{k}[X_1] \otimes \mathbb{k}[X_1]\) and

\[w \otimes 1 + (\Delta \otimes \text{Id})(w) = 1 \otimes w + (\text{Id} \otimes \Delta)(w)\]

This means that \(w\) is a 2-cocycle in \(Z^2(C)\). Up to a change of variable \(X_2\), one can assume that \(w\) is an element in \(\mathfrak{P}^2_{1_C,1_C}(C)\). By Proposition 5.4(3,4), we can assume that \(w\) is of the form \((E5.5.1)\).

(2) This follows from [BOZZ] Theorem 2.4(i)(d)].

(3) This follows from the fact that \(\mathbb{k}[X_1]\) is commutative and by taking \(r = X_1\) in [BOZZ] Theorem 2.4(i)(d)].

(4) Let \(\partial\) be the differential of the differential graded algebra \(T(C)\) as in \((E5.1.1)\), so \(\partial\) is determined by

\[\partial(f) = \Delta(f) - f \otimes 1 - 1 \otimes f\]

for all \(f \in C = \mathbb{k}[X_1]\). By part (3), \(\partial(\delta(X_1)) = -\chi w\). This means that \(\chi w = 0\)

in \(\mathfrak{P}^2_{1_C,1_C}(C)\). By \((E5.5.1)\) and the fact that \(\{Z_s\}_{s \geq 0} \cup \{Y_{s,t}\}_{s < t}\) form a basis of \(\mathfrak{P}^2_{1_C,1_C}(C)\), it follows that \(\chi w = 0\).

(5) By parts (3) and (4), \(\delta(X_1)\) is primitive. It is well-known that every primitive element in \(\mathbb{k}[X_1]\) is of the form \(\sum_{s \geq 0} d_s X_1^{p^s}\) for some \(d_s \in \mathbb{k}\). This is also a consequence of Proposition 5.4(3). \(\square\)

Recall now the construction of 2-step IHOEs from the introduction. Namely, let \(d_s = \{d_s\}_{s \geq 0}\), \(b_s = \{b_s\}_{s \geq 0}\) and \(c_{s,t} = \{c_{s,t}\}_{0 \leq s < t}\) be sequences of scalars in \(\mathbb{k}\) with only finitely many nonzero elements. Let \(H(d_s, b_s, c_{s,t})\) denote the Ore extension \(\mathbb{k}[X_1][X_2; \text{Id}, \delta]\), where

\[\delta(X_1) = \sum_{s \geq 0} d_s X_1^{p^s}\]

Moreover, the comultiplication \(\Delta: H(d_s, b_s, c_{s,t}) \rightarrow H(d_s, b_s, c_{s,t}) \otimes H(d_s, b_s, c_{s,t})\)

is determined by

\[\Delta(X_1) = X_1 \otimes 1 + 1 \otimes X_1,\]
\[\Delta(X_2) = X_2 \otimes 1 + 1 \otimes X_2 + w,\]
where
\[ w = \sum_{s \geq 0} b_s \left( \sum_{i=1}^{p-1} \frac{(p-1)!}{i!(p-i)!} (X_1^{p^i})^i \otimes (X_1^{p^i})^{p-i} \right) + \sum_{0 \leq s < t} c_{s,t} \left( X_1^{p^s} \otimes X_1^{p^t} - X_1^{p^t} \otimes X_1^{p^s} \right). \]

Similarly, define maps \( \epsilon \) and \( S \) from \( \{X_1, X_2\} \) to \( k \) and \( H(d_s, b_s, c_{s,t}) \) respectively, by

\[ \epsilon(X_1) = \epsilon(X_2) = 0, \]
\[ S(X_1) = -X_1, \]
\[ S(X_2) = -X_2 - m(\text{Id} \otimes S)(w). \]

Now we are ready to prove Theorem 0.3(1),(2).

**Proposition 5.6.** Retain the above definitions and notation.

1. The definitions above of \( \Delta, \epsilon \) and \( S \) extend uniquely to \( H(d_s, b_s, c_{s,t}) \) so that it is a Hopf algebra.
2. Let \( H := k[X_1][X_2; \sigma, \delta] \) be a 2-step IHOE generated by \( X_1 \) and \( X_2 \) as in Definition 1.1(2). Then \( H \) is isomorphic to \( H(d_s, b_s, c_{s,t}) \), for a suitable choice of scalars.

**Proof.** (1) This follows from [BOZZ, Theorem 2.4(ii)] and an easy computation (similar to the proof of Lemma 5.5).

(2) Let \( \chi \) be the scalar given in Lemma 5.5(2). Up to a change of variable, we can assume that \( \chi \) is either 0 or 1.

**Case 1:** \( \chi = 0 \). The assertion follows from Lemma 5.5(1,5).

**Case 2:** \( \chi = 1 \). In this case \( \sigma(X_1) = X_1 + 1 \) by Lemma 5.5(2). By Lemma 5.5(4), \( w = 0 \). By Lemma 5.5(1), both \( X_1 \) and \( X_2 \) are primitive. Since \( \sigma(X_1) = X_1 + 1 \), we have a relation
\[ X_2X_1 = X_1X_2 + X_2 + \sum_{s \geq 0} d_s X_1^{p^s}. \]

Replacing \( X_2 \) by the new primitive generator \( \hat{X}_2 := X_2 + \sum_{s \geq 0} d_s X_1^{p^s} \), the above relation becomes
\[ \hat{X}_2X_1 = X_1\hat{X}_2 + \hat{X}_2. \]

Exchanging \( X_1 \) and \( \hat{X}_2 \) and changing the sign of \( X_1 \), one now sees that \( H \) is isomorphic as a Hopf algebra to \( H(d_0, 0, 0) \) where \( d_0 = \{d_0 = 1, d_s = 0 : s \geq 0\} \). This completes the proof. \( \square \)

We shall see in the next subsection that Proposition 5.6 implies that there is an immense zoo of isomorphism classes of 2-step IHOEs in positive characteristic. But, in contrast to this plethora, the classification up to birational equivalence is very simple. In fact, it exactly parallels the story in characteristic 0, where - in the light of Zhuang’s result [ZH, Proposition 7.4(III)], there are 2 birational equivalence classes, with quotient division rings the field of rational functions \( k(X_1, X_2) \) and the quotient division ring \( Q(A_1(k)) \) of the first Weyl algebra in the commutative and noncommutative cases respectively.
Corollary 5.7. Let \(k\) be algebraically closed of positive characteristic and let \(H\) be a 2-step IHOE over \(k\). If \(H\) is not commutative, that is if \(d_8 \neq 0\), then the quotient division ring of \(H\) is isomorphic to \(Q(A_1(k))\), the first Weyl skew field over \(k\).

Proof. By Proposition 5.6(2) \(H \equiv H(d_8, b_\alpha, c_\alpha)\) for some choice of the parameters, with \(d_8 \neq 0\) since \(H\) is by assumption not commutative. Thus \(H = \mathbb{k}(X_1, X_2)\) with

\[
[X_2, X_1] = \delta(X_1) = \sum_{s \geq 0} d_s X_1^s.
\]

The quotient division ring \(Q\) of \(H\) is therefore generated by \(X_2(\delta(X_1))^{-1}\) and \(X_1\), and these generators satisfy the defining relation of the Weyl skew field, as required. \(\square\)

5.3. Classification of 2-step IHOEs: isomorphisms and automorphisms. This subsection has two interconnected purposes: we complete the classification begun in Proposition 5.6 by determining when any two of the algebras listed there are isomorphic; and in so doing we describe all the automorphisms of these Hopf algebras. The proof of the main result requires three preliminary lemmas.

Lemma 5.8. Let \(H = H(0, b_\alpha, c_\alpha)\) be a commutative 2-step IHOE as described in Proposition 5.6, so

\[
\Delta(X_1) = X_1 \otimes 1 + 1 \otimes X_1,
\]

\[
\Delta(X_2) = X_2 \otimes 1 + 1 \otimes X_2 + w
\]

where \(w\) is given in \((E5.0.22)\). Let \(P(H)\) denote its subspace of primitive elements.

(1) If \(w = 0\), that is if \(b_\alpha = c_\alpha = 0\), then \(P(H)\) has \(\mathbb{k}\)-basis \(\{X_1^{p^i}, X_2^{p^i} : i, j \geq 0\}\), so that \(\mathbb{k}(P(H)) = H\).

(2) If \(w \neq 0\), then \(P(H)\) has \(\mathbb{k}\)-basis \(\{X_1^{p^i} : i \geq 0\}\), and \(\mathbb{k}(P(H)) = \mathbb{k}[X_1]\).

Proof. Let \(f := \sum_{i=0}^n f_i X_2^i \in P(H)\) where \(f_i \in \mathbb{k}[X_1]\). If \(n = 0\), then \(f = f_0 \in \mathbb{k}[X_1]\). It then follows from a direct computation that \(f\) is of the form \(\sum_{s \geq 0} \alpha_s X_1^{s}\) for some finite sequence of scalars \(\alpha_s\). (This fact is also well-known.)

Next assume that the \(X_2\)-degree of \(f\) is \(n\), with \(n \geq 1\), and consider the equation

\[
(E5.8.1) \quad f \otimes 1 + 1 \otimes f = \Delta(f) = \sum_{i=0}^n \Delta(f_i)\Delta(X_2)^i.
\]

For \(i = p^s\), then using commutativity of \(H\) and the fact that \(\text{char } \mathbb{k} = p\),

\[
(E5.8.2) \quad \Delta(X_2)^{p^s} = X_2^{p^s} \otimes 1 + 1 \otimes X_2^{p^s} + w^{p^s},
\]

where \(w^{p^s} \in \mathbb{k}[X_1] \otimes \mathbb{k}[X_1]\).

On the other hand, if \(i \leq n\) and \(i \neq p^s\), then the expansion of \(\Delta(X_2^i)\) has at least one nonzero term of the form \(c X_2^j X_2^{-j}\) where \(0 < j < i\) and \(0 \neq c \in \mathbb{k}\). Suppose \(f_i \neq 0\) for such an integer \(i\). Since \(H \otimes H\) is a domain, this implies that the expansion of \(\Delta(f)\) has a nonzero term of the form \(\Delta(f_i)c X_2^j X_2^{-j}\) where \(0 < j < i\) and \(0 \neq c \in \mathbb{k}\). The total degree in \(X_2\) of this term is \(i\), so it cannot cancel with any other term in \(f \otimes 1 + 1 \otimes f(= \Delta(f))\). This contradicts the hypothesis that \(f\)
is primitive. Therefore \( f = f_0 + \sum_{s \geq 0} g_s X_2^{p^s} \) for some \( f_0, g_s \in k[X_1] \). Since \( f \) is primitive, (E5.8.1) and (E5.8.2) imply that

\[
f \otimes 1 + 1 \otimes f = \Delta(f_0) + \sum_{s \geq 0} \Delta(g_s)[X_2^{p^s} \otimes 1 + 1 \otimes X_2^{p^s} + w^{p^s}],
\]

and hence \( \Delta(g_s) = 1 \otimes g_s = g_s \otimes 1 \) for all \( s \). The counital Hopf algebra axiom now forces \( \beta_s := g_s \in k \) for all \( s \).

1. If \( w = 0 \), then \( \sum_{s \geq 0} \beta_s X_2^{p^s} \) is clearly primitive. Hence \( f_0 \) is primitive and \( f_0 = \sum_{s \geq 0} \alpha_s X_1^{p^s} \) for some \( \alpha_s \in k \), and the claims follow.

2. If \( w \neq 0 \) then it is of the form (E0.2.2). Since \( \text{char } k = p \), for every \( s \geq 0 \), \( w^{p^s} \) is also of the form (E0.2.2), but of higher \( X_1 \)-degree. If \( \beta_s \neq 0 \) for some \( s \), by counting the \( X_1 \)-degree one sees that \( \sum_{s \geq 0} \beta_s w^{p^s} \) is another nonzero element of the form (E0.2.2). In particular, the class of \( \sum_{s \geq 0} \beta_s w^{p^s} \) in \( \mathcal{P}_1^2(k[X_1]) \) is nonzero, by Proposition 5.4(4).

Recall that \( f = f_0 + \sum_{i \geq 1} f_i X_1^i = f_0 + \sum_{s \geq 0} \beta_s X_2^{p^s} \). By (E5.8.2),

\[
0 = \Delta(f) - 1 \otimes f - f \otimes 1
= [\Delta(f_0) - 1 \otimes f_0 - f_0 \otimes 1] + \sum_{s \geq 0} \beta_s w^{p^s}
= \partial(f_0) + \sum_{s \geq 0} \beta_s w^{p^s}.
\]

This implies that the class \( \sum_{s \geq 0} \beta_s w^{p^s} \) in \( \mathcal{P}_1^2(k[X_1]) \) is equal to the class of \(-\partial(f_0)\), which is zero by definition. This yields a contradiction. Therefore all \( \beta_s = 0 \) and the result follows.

**Lemma 5.9.** Let \( H = H(d_s, b_s, c_{s,t}) \) be a noncommutative 2-step IHOE as in Proposition 5.6. That is, \( \delta(X_1) \neq 0 \), equivalently \( d_s \neq 0 \).

1. The commutator ideal \([H, H]\) is a nonzero Hopf ideal of \( H \).

2. Let \( K \) denote the Hopf algebra \( k[t] \), so \( t \) is primitive. Suppose \( \phi : H \to K \) is a Hopf algebra epimorphism. Then \( \ker \phi \) is the principal ideal generated by \( X_1 \).

3. For any Hopf algebra epimorphism \( \phi : H \to K \), \( H^{co \phi} \) is the Hopf subalgebra \( k[X_1] \) of \( H \).

**Proof.**

1. It is well-known that the commutator ideal is a Hopf ideal [CZ1] Lemma 3.7. Since \( H \) is noncommutative, \([H, H]\) \neq 0.

2. Let \( f(X_1) \) denote the polynomial \( \delta(X_1) \in k[X_1] \). Then \( f(X_1) \) is a nonzero element in \([H, H]\). Since \( \text{im} \phi \) is commutative, \([H, H]\) is a subspace of \( \ker \phi \), so \( \phi(f(X_1)) = 0 \). We claim that \( \phi(X_1) = 0 \). Suppose not, and let \( y = \phi(X_1) \). Since \( X_1 \) is primitive, so is \( y \). Then \( 0 \neq y = \sum_{s \geq 0} \alpha_s t^{p^s} \) for a finite sequence of scalars \( \alpha_s \), so that

\[
\phi(f(X_1)) = \phi(\phi(X_1)) = \phi(y) = \phi(\sum_{s \geq 0} \alpha_s t^{p^s})
\]

which is nonzero. This yields a contradiction. Thus \( y = 0 \) as required.
Proposition 5.11. This forces a 2-step IHOEs. In what follows we shall use the adjective trivial maps \( \lambda\) be any element in \( k\). To prove the reverse inclusion, let \( g(\lambda, H) \) be any element in \( k\). Since \( \phi(\lambda) = 0 \) by part (2) and \( \phi \) is an epimorphism, \( \phi(\lambda) \) is an algebra generator of \( k\). Now

\[
\begin{align*}
g(\lambda, X_1 \otimes 1, X_2 \otimes 1) &= g(\lambda, X_1, X_2) \otimes 1 \\
&= (\text{Id} \otimes \phi) \Delta g(\lambda, X_1, X_2) \\
&= g((\text{Id} \otimes \phi) \Delta(\lambda, X_1), (\text{Id} \otimes \phi) \Delta(\lambda, X_2)) \\
&= g(\lambda, X_1 \otimes 1, X_2 \otimes 1 + 1 \otimes \phi(\lambda, X_2) - \sum_{t > 0} c_{0,t} X_1^{p^t} \otimes 1).
\end{align*}
\]

This implies that \( X_2\)-degree of \( g(\lambda) \) is zero, so \( g \in k[X_1] \) as required.

Lemma 5.10. Let \( H = k[X_1, X_2] \) and \( H' = k[X_1', X_2'] \) be two 2-step IHOEs as listed in Proposition 5.6. Let \( \phi : H \to H' \) be a Hopf algebra isomorphism such that \( \phi(k[X_1]) = k[X_1'] \). Then \( \phi(X_2) = cX_2' + v(X_1) \) for some \( 0 \neq c \in k \) and \( v(X_1) \in k[X_1'] \).

Proof. We will not need any Hopf algebra structure for this proof. So up to a change of variable, we can assume that \( \phi(X_1) = X_1' \).

Let \( a \) be the \( X_2\)-degree of \( \phi(X_2) \) and \( b \) be the \( X_2\)-degree of \( \phi^{-1}(X_2') \). Since \( \phi(X_1) = X_1' \) and both \( H \) and \( H' \) are Ore extensions, it is easy to see that \( ab = 1 \). This forces \( a = b = 1 \). Write \( \phi(X_2) = c(X_1')X_2' + v(X_1) \) and \( \phi^{-1}(X_2') = d(X_1)X_2 + u(X_1) \). Then \( c(X_1')d(X_1') = 1 \), so \( 0 \neq c := c(X_1') \in k \) is a nonzero scalar in \( k \).

We can now describe the Hopf algebra isomorphisms and automorphisms between 2-step IHOEs. In what follows we shall use the adjective trivial to specify the IHOE \( H(0, 0, 0) \) - in other words, the trivial 2-step IHOE is the coordinate ring of \((k, +) \times (k, +))\). To describe \( \text{Aut}(H(0, 0, 0)) \), let \( F \) denote the automorphism of \((k, +)\) which maps \( \lambda \in k \) to \( \lambda^p \). Thus \( F \) and \( k \setminus \{0\} \) are in \( \text{Aut}(k) \), where elements of \( k \setminus \{0\} \) act by left multiplication, and together they generate in \( \text{End}(k) \) the skew polynomial subalgebra \( A := k[F; \sigma] \), where \( \sigma(\lambda) = \lambda^{-p} \) for \( \lambda \in k \).

In the next proposition, let \( H \) and \( k[X_1, X_2] \) denote \( H(d_a, b_a, c_{s,t}) \) and let \( H' \) and \( k[X_1', X_2'] \) denote \( H(d'_a, b'_a, c'_{s,t}) \) as listed in Proposition 5.6.

Proposition 5.11. Retain the above notation.

1. Let \( \phi : H \to H' \) be a Hopf algebra isomorphism. Then there are nonzero scalars \( \alpha, \beta \) in \( k \) such that, for all \( s \geq 0 \), \( s < t \),

\[
d'_s = d_s \alpha^{p^s-1} \beta^{-1}, \quad b'_s = b_s \alpha^{p^s+1} \beta^{-1}, \quad c'_{s,t} = c_{s,t} \alpha^{p^s+p^t} \beta^{-1}.
\]

Conversely, if \( \alpha \) and \( \beta \) are nonzero scalars such that \( E5.11.1 \) holds, then there is a Hopf algebra isomorphism \( \phi : H \to H' \) such that \( \phi(X_1) = \alpha X_1' \), and \( \phi(X_2) = \beta X_2' \).

2. Suppose \( H \) is not trivial. Then every Hopf algebra isomorphism from \( H \) to \( H' \) has the form

\[
\phi(X_1) = \alpha X_1', \quad \text{and} \quad \phi(X_2) = \beta X_2' + \sum_{s \geq 0} e_s X_1'^{p^s},
\]
for an arbitrary finite sequence of scalars \( \{e_s : s \geq 0\} \), and nonzero scalars \( \alpha \) and \( \beta \) satisfying \((E5.11.1)\).

(3) If \( H \) is not trivial, then every Hopf algebra automorphism of \( H \) is of the form

\[
\phi(X_1) = \alpha X_1, \\
\phi(X_2) = \beta X_2 + \sum_{s \geq 0} e_s X_1^{p^s}
\]

where \( \{e_s : s \geq 0\} \) is an arbitrary finite sequence of scalars, and \( \alpha, \beta \) are nonzero scalars satisfying

\[
(E5.11.2) \quad d_s = d_s \alpha^{p^s-1} \beta^{-1}, \quad b_s = b_s \alpha^{p^s+1} \beta^{-1}, \quad c_{s,t} = c_{s,t} \alpha^{p^s+p^t} \beta^{-1},
\]
equivalently,

\[
\beta = \begin{cases} 
\alpha^{p^s-1} & \text{if } d_s \neq 0, \\
\alpha^{p^s+1} & \text{if } b_s \neq 0, \\
\alpha^{p^s+p^t} & \text{if } c_{s,t} \neq 0.
\end{cases}
\]

(4) Suppose that \( H \) is trivial, so \( H \cong \mathcal{O}(k(\mathbb{R}^+)^2) \). Then, in the notation introduced before the proposition,

\[
\text{Aut}(H) \cong GL_2(A).
\]

Proof. (1) If both \( H \) and \( H' \) are trivial, then the assertions are obvious. Now we assume that \( H \) is not trivial.

If \( H \) is noncommutative, then \( H' \) is noncommutative, and so \( H' \) is not trivial. If \( H \) is commutative, then \( w \neq 0 \) and, by Lemma 5.8, \( H' \) is not trivial. Thus, in both cases, \( H' \) is not trivial.

By Lemma 5.8(2) and Lemma 5.9(3), the Hopf algebra isomorphism \( \phi : H \to H' \) maps \( k[X_1] \) to \( k[X'_1] \). In particular, \( k[X'_1] = k[\phi(X_1)] \), so \( \phi(X_1) = \alpha X'_1 + a_0 \) for a nonzero scalar \( \alpha \) and a scalar \( a_0 \). Since \( X_1 \) is primitive, \( a_0 = 0 \) and \( \phi(X_1) = \alpha X'_1 \). By Lemma 5.10, \( \phi(X_2) = \beta X'_2 + v(X'_1) \), where \( \beta \) is a nonzero scalar and \( v(X'_1) \in k[X'_1] \). Applying \( \phi \) to the defining relation of \( H \), namely

\[
X_2X_1 - X_1X_2 = \delta(X_1) = \sum_{s \geq 0} d_s X_1^{p^s},
\]

we obtain the following equation in \( H' \):

\[
\alpha \beta (X'_2 X'_1 - X'_1 X'_2) = \sum_{s \geq 0} d_s \alpha^{p^s} X_1^{p^s},
\]

which must agree with

\[
X'_2 X'_1 - X'_1 X'_2 = \delta'(X'_1) = \sum_{s \geq 0} d'_s X_1^{p^s}.
\]

Therefore \( d'_s = d_s \alpha^{p^s-1} \beta^{-1} \) for all \( s \).

Recall that \( \partial \) denotes the map defined in \((E5.1.1)\). Applying \( \phi \circ \phi \) to the following equation in \( H \otimes H \),

\[
\Delta(X_2) = X_2 \otimes 1 + 1 \otimes X_2 + w,
\]

we obtain an equation in \( H' \otimes H' \), namely

\[
\beta \Delta(X'_2) + \partial(v(X'_1)) = \beta (X'_2 \otimes 1 + 1 \otimes X'_2) + (\phi \circ \phi)(w).
\]
With the obvious notation $w' := \partial(X'_2)$, the above equation must agree with

$$\Delta(X'_2) = X'_2 \otimes 1 + 1 \otimes X'_2 + w'.$$

Using the explicit form of $(\phi \otimes \phi)(w)$, one can show that $\partial(v(X'_1)) = 0$ and $w' = \beta^{-1}(\phi \otimes \phi)(w)$. The former implies that $v(X'_1) \in \mathbb{k}[X'_1] \subseteq H'$ is primitive and the latter implies that

$$b'_s = b_s \alpha^p + \beta^{-1}, \quad c'_{s,t} = c_{s,t} \alpha^p + \beta^{-1}$$

for all $s < t$. Therefore (E5.11.1) holds.

The converse is clear.

By the proof of part (1), $\phi(X_1) = \alpha X'_1$ and $\phi(X_2) = \beta X'_2 + v(X'_1)$, with $v(X'_1)$ primitive. Since every primitive element in $\mathbb{k}[X_1]$ is of the form $\sum_{s \geq 0} e_s X'_s$, the assertion follows.

This is a special case of part (2).

Suppose that $H = H(0, 0, 0)$ is trivial. That $\text{Aut}(H) \cong GL_2(A)$ is an immediate consequence of the equivalence of categories between the category of unipotent algebraic groups over $\mathbb{k}$ which are subgroups of $(\mathbb{k}, +)^r$ for some $r$, and the category of finitely generated $A$-modules, under which $(\mathbb{k}, +)$ corresponds to the free $A$-module of rank 1, [Mi, Theorem 14.46 and Example 14.40].

6. Comments and questions

We gather here a number of observations related to the classification results in §5.3.

6.1. Algebraic groups. To recast §§5.2,5.3 in the language of affine algebraic $k$-groups, first consider the overuse in this context of the word “connected”: a Hopf algebra $H$ is by definition connected if its coradical $H_0$ is $\mathbb{k}$, equivalently if it has a unique simple comodule [Mo1, Definition 5.1.5]; an affine algebraic $k$-group $G$ is unipotent if and only if its coordinate ring $O(G) = \mathbb{k}[X]$ is a connected Hopf algebra [Mi, Theorem 14.5]; and an affine algebraic $k$-group $G$ is connected if and only if $G^\circ$, its connected component of 1, equals $G^\circ$, its connected component of 1 that is, $O(G)$ is a domain [2]. In characteristic 0 every unipotent group $U$ is connected - that is, $O(U)$ is always a domain; but the cyclic group of order $p$ shows that this is false in characteristic $p$.

The key part (2) $\Rightarrow$ (1) of the following fundamental result is due to Lazard [La]; a short proof valid in all characteristics can be found in [KP, §4]. For the full statement of the theorem, see for example [Ka, §8].

Theorem 6.1. Let $k$ be an algebraically closed field of arbitrary characteristic, and let $G$ be an affine algebraic $k$-group and let $n$ be a positive integer. Then the following are equivalent.

1. $G$ is a connected unipotent group over $k$ of dimension $n$.
2. $O(G) \cong k[X_1, \ldots, X_n]$ as $k$-algebras.

2In the algebraic groups literature this confusion is sometimes avoided by using coconnected for the first of these usages.
(3) $O(G)$ is an affine connected Hopf domain of Gel’fand-Kirillov dimension $n$.
(4) $O(G)$ is an $n$-step IHoe.
(5) $G$ has a subnormal series of length $n$ with factors isomorphic to $(k, +)$.
(6) $G$ has a central series of length $n$ with factors isomorphic to $(k, +)$.

6.2. Classification of connected unipotent groups of dimension two. Revert to our usual hypothesis that the base field is $k$, algebraically closed of characteristic $p > 0$. The coordinate ring of every 2-dimensional connected unipotent group over $k$ is a 2-step IHoe, by (1) $\iff$ (4) of Theorem 6.1; so the classification of §5.3 incorporates a classification of these groups. Clearly, in the notation of §5.2, $H(d, b, c, t)$ is commutative if and only if $d = 0$, so the 2-dimensional connected unipotent $k$-groups are classified by

$$H(0, b, c, t)/\sim ,$$

where the equivalence is provided by the isomorphisms of Proposition 5.11(1,2). A version of the same classification, over an arbitrary field, in group-theoretic language, is given (“rather formal” in the words of the authors) at [Ka, §3.7].

It is easy to write down the group $G$ such that $H(0, b, c, t) \cong O(G)$. Namely, $G = K^2(k)$, and for $(a, e)$ and $(f, g)$ in $G$

$$(E6.1.1)$$

$$(a, e) * (f, g) = (a + f, e + g + \sum_{s \geq 0} b_s(\sum_{i=1}^{p-1} \lambda_i a^{ip^s} e^{(p-1)p^s}) + \sum_{0 \leq s < t} c_{s,t}(a^{p^s}e^{p^t} - a^{p^t}e^{p^s})).$$

From this it is easy to see that

(•) $G$ is abelian $\iff$ $c_{s,t} = 0$ or $p = 2$.

(•) $G$ has exponent $p$ $\iff$ $b_s = 0$.

Combining these two statements yields the conclusion that the only abelian connected two-dimensional unipotent group of exponent $p$ is $(k, +)^2$, a special case of a result valid for all dimensions, [Se, Proposition VII.11], [Ka, Lemma 1.7.1].

6.3. Ore extensions of dimension two. The Ore extensions of the polynomial ring $k[X_1]$ are classified in [AVV], so one can view Proposition 5.10 as determining which of these Ore extensions admit Hopf algebra structures. Notice that Propositions 5.6 and 5.11 together show that, for every Ore extension $R = k[X_1][X_2; \sigma, \delta]$, the number of distinct Hopf structures which $R$ can carry is either 0 or $\infty$. This is in stark contrast to the situation in characteristic 0, as noted in the opening sentence of §5.2.

A second noteworthy feature is this: every such $R$ admitting a Hopf algebra structure can be presented with $\sigma = Id_{k[X_1]}$. (The enveloping algebra of the non-abelian 2-dimensional Lie algebra can be presented in 2 distinct ways as an Ore extension, only one of which can be written as an Ore extension with $\sigma = Id_{k[X_1]}$.) But this does not extend to higher dimensions: $U(sl(2, k))$ is a 3-step IHoe [BOZ, §3.1, Examples (iii)], but cannot be so presented without using automorphisms.
Another case to consider is \( \text{HOE } k[x_1^\pm 1, x_2, \sigma, \delta] \) where \( x_1 \) is a grouplike element. Let \( C = k[x_1^\pm 1] \). By [Hua] Theorem 1.3(i)], there is a grouplike element \( \alpha = x_1^a \in C \) such that
\[
\Delta(x_2) = \alpha \otimes x_2 + x_2 \otimes 1 + w
\]
where \( w \in C \otimes C \). By [WZZ2] Lemma 2.3(1)], \( w \) is a \((\alpha, 1)\)-2-cocycle. Since \( C \) is cosemisimple, it has primitive cohomology dimension 0 and
\[
\mathcal{H}_{\alpha, 1}(C) = 0.
\]
This implies that \( w \) is a \((\alpha, 1)\)-2-coboundary. Up to a change of variable \( x_2 \), we can assume that \( w = 0 \) by [WZZ2] Lemma 2.3(3)]. By [Hua] Theorem 1.3(ii)], there is a nonzero \( c \in k \) such that \( \sigma(x_1) = c x_1 \). Let \( \delta(x_1) = \sum_n b_n x_1^n \). By [Hua] Theorem 1.3(iii)], we have
\[
\sum_n b_n x_1^n \otimes x_1^n - \left( \sum_n b_n x_1^n \right) \otimes x_1 - x_1^{a+1} \otimes \left( \sum_n b_n x_1^n \right) = 0
\]
which implies that
\[
\delta(x_1) = b_1 (x_1 - x_1^{a+1}).
\]
In summary, we have the following family of Hopf algebras:
\[
K(a, b, c) := k[x_1^\pm 1][x_2; \sigma, \delta]
\]
where \( \sigma(x_1) = c x_1 \) for some nonzero \( c \in k \) and \( \delta(x_1) = b (x_1 - x_1^{a+1}) \) for some \( b \in k \), \( a \in \mathbb{Z} \), and the coalgebra structure of \( K(a, b, c) \) is determined by
\[
\Delta(x_1) = x_1 \otimes x_1, \epsilon(x_1) = 0
\]
and
\[
\Delta(x_2) = x_1^a \otimes x_2 + 1 \otimes x_2, \epsilon(x_2) = 0.
\]

The following lemma is clear.

**Lemma 6.2.** Retain the above notation. Then \( K(a, b, c) \) is commutative if and only if \( b = 0 \) and \( c = 1 \). In this case, there is a unique Hopf ideal \( I \) such that \( K(a, 0, 1)/I \cong k[x_2] \).

### 6.4. Classification of connected algebraic groups of dimension two.

The following is a well-known classification of connected algebraic groups of dimension two. Assume that \( k \) is algebraically closed and that \( G \) is a connected algebraic group of dimension two. Since no quotient group of \( G \) can be semisimple, \( G \) is solvable. Let \( G_u \) be the unipotent radical of \( G \). Then \( G_u \) is normal and unipotent, with \( G/G_u \) connected and with no unipotent elements. By [Hum] Section 19.1], \( G_u \) is connected, and there is a short exact sequence
\[
(\text{E6.2.1}) \quad 1 \to G_u \to G \to T \to 1
\]
where \( T \) is a torus. There are three cases to consider.

Case 1: \( \dim G_u = 0 \). Then \( G = T \) is a torus and \( \mathcal{O}(G) = k[x_1^\pm 1, x_2^\pm 1] \).

Case 2: \( \dim G_u = 2 \). Then \( G \) is unipotent, so \( G \) is classified in (E6.1.1).

Case 3: \( \dim G_u = 1 \). By (E6.2.1), \( G \) is a semidirect product \((k, +) \rtimes (k \setminus \{0\}, \times)\). Dual to (E6.2.1), there is a short exact sequence of commutative Hopf algebras
\[
k \to k[x_1^\pm 1] \to \mathcal{O}(G) \to k[x_2] \to k.
\]
Then \( O(G) \) is one of the Hopf algebras \( K(a, 0, 1) \) given in the last section.

6.5. **Connected affine Hopf algebra domains of GKdimension 2.** We do not know whether Propositions 5.6 and 5.11 give a classification of all connected affine Hopf \( k \)-algebra domains of Gel’fand-Kirillov dimension 2. More precisely, one can ask:

**Question 6.3.** If \( k \) has positive characteristic, is every affine connected Hopf \( k \)-algebra domain \( H \) of Gel’fand-Kirillov dimension 2 an IHOE?

The answer is “yes” when \( H \) is commutative, and not only in dimension 2 - this follows from the structure of connected unipotent groups, Theorem 6.1. If one drops the restriction to dimension 2 in Question 6.3 then the answer is “no” - for example the enveloping algebra of \( \mathfrak{sl}(3, k) \) endowed with its standard cocommutative coproduct is not an IHOE, since \( \mathfrak{sl}(3, k) \) does not contain a full flag of Lie subalgebras.

6.6. **Automorphism groups of 2-step IHOEs.** From Proposition 5.11(3) one can easily read off the structure of the group \( \text{Aut}(H) \) of Hopf algebra automorphisms of \( H := H(\mathfrak{d}_s, \mathfrak{b}_s, \mathfrak{c}_s, t) \), for each non-trivial \( H \). Namely, define subgroups \( T \) and \( N \) of \( \text{Aut}(H) \) by

\[
T = \{ \phi \in \text{Aut}(H) \mid \phi(X_1) = \alpha X_1, \phi(X_2) = \beta X_2 \},
\]
where \( \alpha, \beta \in (k, \times) \) satisfy (E5.11.2); and

\[
N = \{ \phi \in \text{Aut}(H) \mid \phi(X_1) = X_1, \phi(X_2) = X_2 + \sum_{s \geq 0} \epsilon_s X_1^p \},
\]

for an arbitrary sequence \( (\epsilon_s) \in k^N \) with only finitely many nonzero entries. Thus \( N \) is a normal subgroup of \( \text{Aut}(H) \) with \( N \cong (k, +)^N \). The constraints (E5.11.2) mean that \( T \cong (k, \times) \) if \( H \) has only one nonzero parameter, while \( T \) is a finite (possibly trivial) subgroup of \( (k, \times) \) if there are two or more defining parameters for \( H \). It is clear from Proposition 5.11 that \( \text{Aut}(H) \) is the semidirect product of \( N \) by \( T \).

7. **Noncommutative binomial theorem in characteristic \( p \)**

In this short section we derive a corollary of an important theorem due to Jacobson, which is needed in §8. Stated in our notation, Jacobson’s theorem is as follows:

**Theorem 7.1.** [Ja3, pp. 186-7] Let \( F \) be a field of characteristic \( p > 0 \) and let \( R \) be a \( F \)-algebra. For elements \( a, b \) of \( R \),

\[
(a + b)^p = a^p + b^p + \sum_{i=1}^{p-1} s_i(a, b),
\]

where, for \( i = 1, \ldots, p - 1 \) and \( \lambda \in F \setminus \{0\} \), \( s_i(a, b) \) is the coefficient of \( \frac{1}{i!} \lambda^{i-1} \) in \( \text{ad}_{(\lambda a + b)}^{p-1}(a) \).

Here is the required corollary.
Corollary 7.2. In the situation of the theorem, suppose that the elements \( \text{ad}_i(a) \) of \( R \) commute with \( a \) for \( i = 1, \ldots, p - 1 \). Then

\[
(a + b)^p = a^p + b^p + \text{ad}_0^{-1}(a).
\]

Proof. The extra hypothesis of the corollary ensures that, for \( j > 0 \), \( \lambda^j \) does not occur in the expansion of \( \text{ad}_0^{-1}(\lambda a + b)(a) \). So the theorem states that only \( s_1(a, b) = \text{ad}_0^{-1}(a) \), as claimed. \( \square \)

8. Properties of 2-step IHOEs

8.1. The antipode, the center, and the Calabi-Yau property. In this subsection, we study properties of the Hopf algebras listed in Proposition 5.6. The definition and relevant properties of the PI-degree are recalled in §3.1; regarding the Nakayama automorphism and the skew Calabi-Yau property, see §3.2 and [BZ1].

Proposition 8.1. Let \( H \) be a 2-step IHOE \( H(d_s, b_s, c_s, t) \) as in Proposition 5.6.

(1) \( S^2 \) is the identity of \( H \).

(2) \( H \) is commutative if and only if \( d_s = 0 \).

(3) \( H \) is cocommutative if and only if \( c_s, t = 0 \) or \( p = 2 \).

(4) Suppose that \( H \) is not commutative. Then the center of \( H \) is \( k[X_1^p, X_2^p - d_0^{-1}X_2] \). Hence the PI-degree of \( H \) is \( p \).

(5) The Nakayama automorphism \( \mu \) of \( H \) is determined by

\[
\mu(X_1) = X_1, \quad \text{and} \quad \mu(X_2) = X_2 + d_0.
\]

Hence, \( H \) is Calabi-Yau if and only if \( d_0 = 0 \).

(6) The left homological integral \( \int_H^l \) of \( H \) is the one-dimensional \( H \)-bimodule which is trivial as left module and with \( \int_H^l \cong H/(X_1, X_2 - d_0) \) as right module.

Proof. (1) Using the comultiplication formula of \( H(d_s, b_s, c_{s,t}) \) or the definition given before Theorem 0.3, one finds that

\[
S(X_1) = -X_1, \quad \text{and} \quad S(X_2) = -X_2 + f(X_1)
\]

for some polynomial \( f \). It is clear that \( S^2(X_1) = X_1 \). If \( p = 2 \), then

\[
S^2(X_2) = X_2 + 2f(X_1) = X_2
\]

which implies that \( S^2 \) is the identity. (In fact, when \( p = 2 \), \( S(X_2) = X_2 + \sum_{s \geq 0} b_s X_1^{2s+1} \).) Suppose now that \( p \geq 3 \). Again by the comultiplication formula,

\[
S(X_2) = -X_2 - m((S \otimes \text{Id})(w))
\]

where \( w \) is defined in (E5.5.1) and \( m \) denotes multiplication in \( H \). Since \( p \) is odd,

\[
m((S \otimes \text{Id})(Z_s)) = 0 \quad \text{and} \quad m((S \otimes \text{Id})(Y_{s,t})) = 0.
\]

Therefore \( S(X_2) = -X_2 \) and \( S^2 \) is the identity.

(2) This is clear from (E0.2.1).

(3) This is clear from (E0.2.2).
(4) Since \( \sigma = \text{Id} \), \( \delta \) is a derivation of \( \mathbb{k}[X_1] \). Thus \( \delta(X_i^p) = pX_i^{p-1}\delta(X_1) = 0 \). This implies that \( X_2X_i^p = X_i^pX_2 \), so that \( X_i^p \) is central.

One easily checks by induction that
\[
\delta^n(X_1) = d_0^{p-1}\delta(X_1)
\]
for all \( n \geq 2 \). Write the derivation \( \delta = [X_2, -] \) of \( \mathbb{k}[X_1] \) as \( \lambda X_2 - \rho X_2 \) where the symbols \( \lambda X_2 \) [resp. \( \rho X_2 \)] denote left [resp. right] multiplication by \( X_2 \) in \( H \). Note that these linear maps commute in \( \text{End}_k(H) \), so one has
\[
(\lambda X_2 - \rho X_2)^p = \lambda^p X_2 - \rho^p X_2.
\]
In other words,
\[
\delta^p = [X_2^p, -],
\]
so that
\[
X_2^pX_1 = X_1X_2^p + \delta^p(X_1) = X_1X_2^p + d_0^{p-1}\delta(X_1).
\]
Combining this with the relation
\[
X_2X_1 = X_1X_2 + \delta(X_1)
\]
it follows that \( X_2^p - d_0^{p-1}X_2 \) is central. Thus \( H \) is a free module of rank \( p^2 \) over the central polynomial subalgebra \( Z_0 := \mathbb{k}[X_1, X_2 - d_0^{p-1}X_2] \). Hence, using \( Q(\cdot) \) to denote quotient division algebras,
\[
\dim Q(Z_0)Q(H) = p^2.
\]

But \( \dim Q(Z(H))Q(H) \) is an even power of \( p \), by Theorem \( \text{[4.3]} \) and the discussion at the start of \( \text{[3.5]} \). Therefore, since \( H \) is not commutative, the PI-degree of \( H \) is \( p \) and \( Q(Z(H)) = Q(Z_0) \). However, \( Z(H) \) is a finite module over \( Z_0 \) by noetherianity of \( H \) as a \( Z_0 \)-module, and \( Z(H) \) is contained in \( Q(Z_0) \). Since \( Z_0 \) is normal it follows that \( Z(H) = Z_0 \). That the PI-degree of \( H \) is \( p \) is now clear.

(5) By \( \text{[LiM]} \) Theorem 4.2], \( \mu(X_1) = X_1 \) and
\[
\mu(X_2) = X_2 + \frac{d}{dX_1}(\delta(X_1)) = X_2 + d_0.
\]
The consequence is clear (and it also follows from \( \text{[LiM]} \) Corollary 4.3)).

(6) By \( \text{[BZ]} \) Theorem 0.3], \( \mu = S^2 \circ \Xi_q^l \) where \( q \) is the right character of the left homological integral \( l^f \) of \( H \) and \( \Xi_q^l \) is the corresponding left winding automorphism. By (1), \( S^2 = \text{Id} \), so that \( \mu = \Xi_q^l \). Thus (5) implies (6). \( \square \)

8.2. Representation theory of 2-step IHOEs. In this subsection we describe the simple representations of the 2-step IHOEs \( H := H(\mathfrak{d}_s, \mathfrak{b}_s, \mathfrak{c}_s, \mathfrak{t}) \). Recall that if \( k \) is any algebraically closed field and \( A \) is an affine \( k \)-algebra which is a finite module over its center \( Z \), then \( Z \) is also affine by the Artin-Tate lemma \( \text{[McR]} \) Lemma 13.9.10], and a version of Schur's lemma applies \( \text{[McR]} \) Theorem 13.10.3]: namely, if \( V \) is a simple \( A \)-module, then \( \text{End}_A(V) = k \), and so \( V \) is annihilated by a maximal ideal \( m \) of \( Z \), so that \( V \) is a (necessarily finite dimensional) simple module over the finite dimensional algebra \( H/mH \). If \( A \) is prime (as in the current setting, when \( A = H(\mathfrak{d}_s, \mathfrak{b}_s, \mathfrak{c}_s, \mathfrak{t}) \) is a domain), with \( \text{Pdeg}(A) = d \), then for \( m \) in a non-empty (and hence dense) open subset \( \mathcal{A}(A) \) of \( \text{maxspec}(A) \),
\[
A/mA \cong M_d(k).
\]
The set \( \mathcal{A}(A) \) is called the Azumaya locus of \( A \). For convenience, we shall denote the non-Azumaya locus by \( \mathcal{N}(\mathcal{A}(A)) \); that is, \( \mathcal{N}(\mathcal{A}(A)) := \text{maxspec}(Z) \setminus \mathcal{A}(A) \), a proper closed subset of \( \text{maxspec}(Z) \). If the simple \( A \)-module \( V \) has \( \text{Ann}_Z(V) \in \mathcal{N}(\mathcal{A}(A)) \) then \( \dim_k(V) < d \). For more details on this circle of ideas, see for example [BG, Part III].

Suppose now that \( H := H(d_a, b_a, c_{a,t}) = \mathbb{k}[X_1][X_2; \delta] \), so that \( Z := Z(H) = \mathbb{k}[X_1^p, X_2^p - d_0^{-1}X_2] \) by Theorem 8.1(4). In view of the discussion in the previous paragraph, our task is to describe the algebras \( H_{\alpha,\beta} := H/m_{\alpha,\beta}H \), where \( m_{\alpha,\beta} \) denotes the maximal ideal \( \langle X_1^p - \alpha, X_2^p - d_0^{-1}X_2 - \beta \rangle \) of \( Z \) and \( (\alpha, \beta) \) ranges through \( \mathbb{A}^2(\mathbb{k}) \). Notice that, by the PBW theorem for \( H \),

\[
\dim_k(H_{\alpha,\beta}) = p^2
\]

for all \( (\alpha, \beta) \in \mathbb{A}^2(\mathbb{k}) \). Therefore the maximal dimension of a simple \( H \)-module \( V \) is \( p \), and this value is attained by \( V \) if and only if \( V \) is the (unique) simple \( H/m_{\alpha,\beta}H \)-module for some \( (\alpha, \beta) \in \mathcal{A}(H) \).

Recall from (E0.2.1) that \( \delta(X_1) = d_0X_1 + \sum_{s \geq 1} d_sX_1^{p^s} \). It is convenient to define a polynomial \( d(x) = \sum_{s \geq 1} d_sx^{p^{s-1}} \), so that

\[
\delta(X_1)^p = d_0^pX_1^p + \left( \sum_{s \geq 1} d_sX_1^{p^s} \right)^p = d_0^pX_1^p + d(X_1)^p.
\]

**Proposition 8.2.** Suppose \( \mathbb{k} \) is algebraically closed of characteristic \( p > 0 \). Let \( H \) be a 2-step IHoe as given in Proposition 5.6 and fix the notation as above. Suppose that \( H \) is not commutative, that is \( d_a \neq 0 \).

1. The defining ideal of \( \mathcal{N}(\mathcal{A}(H)) \) in \( Z \) is \( \sqrt{(d_0^pX_1^p + d(X_1)^p)} \). That is,

\[
H_{\alpha,\beta} \cong M_p(\mathbb{k}) \iff d_0^p\alpha + d(\alpha)^p \neq 0.
\]

2. Suppose that \( d_0 = 0 \). Let \( m_{\alpha,\beta} \in \mathcal{N}(\mathcal{A}(H)) \); that is, \( d_0^p\alpha + d(\alpha)^p = 0 \), or equivalently,

\[
d(\alpha) = 0.
\]

Then

\[
H_{\alpha,\beta} \cong \mathbb{k}[X, Y]/\langle X^p, Y^p \rangle.
\]

Thus \( H_{\alpha,\beta} \) has a unique simple module, of dimension 1.

3. Suppose that \( d_0 \neq 0 \). Let \( m_{\alpha,\beta} \in \mathcal{N}(\mathcal{A}(H)) \), so \( d_0^p\alpha + d(\alpha)^p = 0 \). Then there are elements \( u, w \in H_{\alpha,\beta} \) such that \( H_{\alpha,\beta} = \mathbb{k}(u, w) \), with relations

(E8.2.1) \[
u^p = 0, \quad w^p - uw = u, \quad w^p - w + \alpha d(\alpha)^{-p} = 0.
\]

In particular, \( H_{\alpha,\beta} \) has a single block of \( p \) simple modules, each of dimension 1.

4. The non-Azumaya locus \( \mathcal{N}(\mathcal{A}(H)) \) is the disjoint union of \( r \) copies of \( \mathbb{A}^1(\mathbb{k}) \), the affine line, where \( r \) is the number of distinct roots of the equation

\[
\sum_{s \geq 0} d_s^p x^{p^s} = 0.
\]
Note that in the setting of part (3) the third equation in \([E8.2.1]\) is equivalent to
\[ w^p - w = \beta d_0^{-p}. \]

**Proof.** (1,2,3) Let \((\alpha, \beta) \in \mathbb{A}^2(\mathbb{k})\). The defining relation of \( H := H(d_s, b_s, c_{a,t}) \), namely \([X_2, X_1] = \delta(X_1)\), yields in \( H_{\alpha,\beta} \) the relation
\[ (E8.2.2) \quad X_2X_1 - X_1X_2 = \delta(X_1) = d_0X_1 + \sum_{s \geq 1} d_s\alpha^{p^{-s-1}} = d_0X_1 + d(\alpha). \]

Suppose first that \( d_0^p + d(\alpha)p \neq 0 \). If \( d_0 = 0 \), then \([E82.2]\) yields the defining relation for the Weyl algebra. Thus \( H_{\alpha,\beta} \) is a factor of the first Weyl algebra over \( \mathbb{k} \), of dimension \( p^2 \). But the Weyl \( \mathbb{k} \)-algebra is Azumaya of PI-degree \( p \) by \([Re\, Theorem\, 2]\), so each of its \( p^2 \)-dimensional factors, in particular \( H_{\alpha,\beta} \), is isomorphic to \( M_p(\mathbb{k}) \).

Next consider the case where \( d_0^p + d(\alpha)p \neq 0 \) and \( d_0 \neq 0 \). Let \( u \) and \( w \) denote the images in \( H_{\alpha,\beta} \) of \( d_0X_1 + d(\alpha) \) and \( d_0^{-1}X_2 \) respectively. These satisfy the following relations in \( H_{\alpha,\beta} \):
\[ wu - uw = u, \]
\[ w^p = d_0^p \alpha + d(\alpha)^p, \]
\[ w^p - w = d_0^{-p}\beta. \]
The second of these relations shows that \( u \) is a unit, so post-multiplying the first relation by \( u^{-1} \) again yields the defining relation of the first Weyl algebra. As before we find that \( H_{\alpha,\beta} \cong M_p(\mathbb{k}) \). We have therefore shown that \( \mathcal{NA}(H) \) is contained in the subvariety of \( \text{maxspec}(\mathbb{Z}) \) defined by \( d_0^p X_1^p + d(X^p)^p \).

Now assume that \( d_0^p + d(\alpha)p = 0 \). Suppose first that \( d_0 = 0 \). Then \( Z = \mathbb{k}[X_1, X_2] \), and so the images in \( H_{\alpha,\beta} \) of \( X_1 - \frac{\alpha}{\beta} \) and \( X_2 - \frac{1}{\beta} \) are mutually commuting nilpotent elements, which together generate \( H_{\alpha,\beta} \) and have index of nilpotency \( p \). Therefore \( H_{\alpha,\beta} \) is the local algebra \( \mathbb{k}[X, Y]/(X^p, Y^p) \) in this case.

Finally, suppose that \( d_0^p + d(\alpha)p = 0 \), with \( d_0 \neq 0 \). Let \( u \) and \( w \) denote the images in \( H_{\alpha,\beta} \) of \( d_0X_1 + d(\alpha) \) and \( d_0^{-1}X_2 \) respectively. Then we obtain the three relations in \([E8.2.1]\). The first two relations of \([E8.2.1]\) now show that \( u \) is a nilpotent normal element, with index of nilpotency at most \( p \). Now \( H_{\alpha,\beta}/uH_{\alpha,\beta} = \mathbb{k}\langle w \rangle \), and the third relation of \([E8.2.1]\) has the form \( f(w) = 0 \) where \( f(x) \) is a polynomial with \( f'(x) = -1 \). Hence \( f(w) = 0 \) has no repeated roots (or \( f(w) = 0 \) has root \( w, w + 1, \ldots, w + p - 1 \) when one root \( w \) is chosen), so that
\[ H_{\alpha,\beta}/uH_{\alpha,\beta} \cong \mathbb{k}^{p^2}. \]
Since \( \dim_{\mathbb{k}} H_{\alpha,\beta} = p^2 \), this shows that \( u^p = 0 \neq u^{p-1} \). Moreover the second relation of \([E8.2.1]\) shows that the \( p \)-dimensional simple \( H_{\alpha,\beta} \)-modules form a single block, the Ext-quiver being a circle. This completes the proof of parts (1), (2) and (3).

(4) From the above proof we see that a maximal ideal \( \mathfrak{m}_{\alpha,\beta} \) of \( Z \) is in \( \mathcal{NA}(H) \) if and only if \( \alpha \) is a root of the equation \( d_0^p x + d(x)^p = 0 \), that is of the equation
\[ d_0^p x + \sum_{s \geq 1} d_s^p x^{p^{-s}} = 0. \]
This proves (4). \( \square \)

8.3. **The Hopf center and restricted Hopf algebras.** Recall the following definition, due to Andruskiewitsch, \([An\, Definition\, 2.2.3]\).
**Definition 8.3.** Let $H$ be a Hopf algebra. The *Hopf center* of $H$, denoted $C(H)$, is the unique largest central Hopf subalgebra of $H$.

It is easy to see that $C(H)$ exists for any Hopf algebra $H$. As already discussed in Remark 2.51, when $k$ has positive characteristic $p$ and $g$ is an $n$-dimensional Lie algebra over $k$, $C(U(g))$ exists and is a polynomial algebra in $n$ variables, with $U(g)$ a free $C(U(g))$-module of rank a power of $p$. In general, even when $H$ is a finite module over its center, $C(H)$ can be very small - consider, for instance, the group algebra $H = FG$ over any field $F$ of the dihedral group

$$G = \langle a, b : b^2 = 1, bab = a^{-1} \rangle,$$

where $C(H) = F$. Nevertheless, current evidence suggests that when $H$ is a connected Hopf $k$-algebra $C(H)$ may always be large, and Question 2.7 proposes that this may be the case for IHOEs over $k$. We shall show in this subsection that this is indeed the case for all 2-step $k$-IHOEs. First, however, we show that whenever a $k$-IHOE is a finite module over its Hopf center, some of the desirable features of the Lie algebra case immediately follow. In this subsection we denote the augmentation ideal of a Hopf algebra $T$ by $T_+$.

**Proposition 8.4.** Let $k$ be algebraically closed of characteristic $p > 0$, let $n$ be a positive integer, and let $H$ be a noncommutative $n$-step IHOE over $k$. Suppose that $H$ is a finite $C(H)$-module. Then $C(H)$ is a polynomial algebra in $n$ variables, and $H$ is a free $C(H)$-module of rank $p^\ell$ for some $\ell \geq 2$.

**Proof.** By [BOZZ, Proposition 2.5], $H$ is connected as a Hopf algebra, so $C(H)$ is also connected. Moreover, $C(H)$ is affine of Gel’fand-Kirillov dimension $n$, by the Artin-Tate lemma and Corollary 2.4. By Theorem 5.1 ((3) $\Rightarrow$ (2)), $C(H)$ is a commutative polynomial ring $k[X_1, \ldots, X_n]$. By [WZ, Theorem 0.3], $H$ is a finitely generated projective module over $C(H)$. Hence $H$ is a free module over $C(H)$ of finite rank. Let $r$ denote the rank of $H$ over $C(H)$. By definition, $C(H)$ is central in $H$, then $\overline{H} := H/(C(H))_+H$ is a Hopf algebra of dimension $r$. By [Mo1, Corollary 5.3.5], $\overline{H}$ is connected. Since $\overline{H}$ is finite dimensional and connected, its dimension is of the form $p^\ell$ for some $\ell > 0$ [Wa1, Proposition 2.2(7)].

Since $H$ is prime, the rank $H$ as a $Z(H)$-module is a square of some integer. This implies that $r$ is not a prime number. Therefore $\ell \geq 2$. \hfill $\Box$

Observe that, in the setting of the proposition, the factor algebra

$$\overline{H} := H/(C(H) + H$$

is a connected Hopf algebra of dimension $p^\ell$. It seems reasonable to call $\overline{H}$ the *restricted Hopf algebra* of $H$.

In the rest of this subsection we confirm that the hypotheses of the proposition are satisfied by all noncommutative 2-step IHOEs, with $\ell$ equal to 2 or 3 in all cases, and we determine their restricted Hopf algebras. The following notation will remain in force throughout the rest of 5.8.

**Notation 8.5.** The field $k$ is algebraically closed of characteristic $p > 0$, and $H$ denotes a 2-step IHOE $H = (d, b, c, r) = k[X_1, X_2]$ as defined in Proposition 5.6, see also [BO]. The element $X_2^p - d_0^{p-1}X_2$ of $H$, which is central by Proposition 8.1(4), will be denoted by $z$. 

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We will always denote the element of $H \otimes H$ defined in \[E0.2.2\] by $w$, so that, writing $\lambda_i$ for $\frac{(p-1)!}{i!(p-1-i)!}$, $(1 \leq i \leq p-1)$,

$$w = \sum_{s \geq 0} b_s \left( \sum_{i=1}^{p-1} \lambda_i (X_i^p)^i \otimes (X_i^p)^{p-i} \right) + \sum_{0 \leq s < t} c_{s,t} (X_s^p \otimes X_t^p - X_t^p \otimes X_s^p).$$

We denote the element $X_2 \otimes 1 + 1 \otimes X_2$ of $H \otimes H$ by $b$.

Let $s$ be a positive integer divisible by $p$, and let $\equiv_s$ denote equality of elements of $H \otimes H$ modulo the subspace $X_1^s H \otimes H + H \otimes X_1^s H$. Since this subspace is preserved by $\text{ad}_b$ it follows that, for elements $f$ and $g$ of $H \otimes H$,

\[f \equiv_s g \implies \text{ad}_b(f) \equiv_s \text{ad}_b(g).\]

Using the fact

$$\text{ad}_{X_2}(X_i^1) = iX_i^{i-1}\delta(X_1)$$

for all $i \geq 1$, one can check that

$$\text{ad}_b(w)$$

$$= [\delta(X_1) \otimes 1] \left( b_0((X_1 \otimes 1 + 1 \otimes X_1)^{p-1} - X_1^{p-1} \otimes 1) + \sum_{t \geq 0} c_{0,t} 1 \otimes X_t^{p} \right)$$

\[\text{(E8.5.2)}\]

$$+ [1 \otimes \delta(X_1)] \left( b_0((X_1 \otimes 1 + 1 \otimes X_1)^{p-1} - 1 \otimes X_1^{p-1}) - \sum_{t \geq 0} c_{0,t} X_t^{p} \otimes 1 \right).$$

**Lemma 8.6.** Retain Notation \[8.5\]

1. $\Delta(z) = z \otimes 1 + 1 \otimes z + w^p - d_b^{p-1}w + \text{ad}_b^{p-1}(w)$. Moreover, the element $w^p - d_b^{p-1}w + \text{ad}_b^{p-1}(w)$ of $H \otimes H$ is in $k[X_1] \otimes k[X_1]$.

2. The subalgebra $k[X_1^p, z]$ of $H$ is a Hopf subalgebra if and only if $-d_b^{p-1}w + \text{ad}_b^{p-1}(w) \in k[X_1^p] \otimes k[X_1^p]$.

3. If $b_0 = 0$, then $k[X_1^p, z]$ is a Hopf subalgebra of $H$.

**Proof.** (1) First of all, for every $i \geq 0$, $\text{ad}_b^i(w) \in k[X_1] \otimes k[X_1]$. Hence $\{\text{ad}_b^i(w)\}_{i \geq 0}$ are in a commutative subalgebra of $H \otimes H$. Thus all the hypotheses of Corollary \[7.2\] are satisfied. Therefore, by that corollary and the definitions of $H$ and $z$,

$$\Delta(z) = \Delta(X_2^p) - d_b^{p-1}\Delta(X_2) = (\Delta(X_2))^p - d_b^{p-1}\Delta(X_2)$$

$$= (w + b)^p - d_b^{p-1}(w + b)$$

$$= w^p + b^p + \text{ad}_b^{p-1}(w) - d_b^{p-1}(w + b)$$

$$= w^p + (X_2 \otimes 1 + 1 \otimes X_2)^p + \text{ad}_b^{p-1}(w) - d_b^{p-1}(X_2 \otimes 1 + 1 \otimes X_2 + w)$$

$$= w^p + (X_2 \otimes 1)^p + (1 \otimes X_2)^p + \text{ad}_b^{p-1}(w) - d_b^{p-1}(X_2 \otimes 1 + 1 \otimes X_2 + w)$$

$$= z \otimes 1 + 1 \otimes z + w^p - d_b^{p-1}w + \text{ad}_b^{p-1}(w).$$

That $w^p - d_b^{p-1}w + \text{ad}_b^{p-1}(w)$ is in $k[X_1] \otimes k[X_1]$ is clear from the definition of $w$.

(2) Denote $k[X_1^p, z]$ by $Z$. Since a connected bialgebra is a Hopf algebra by [HS Corollary 3.5.4(a)], it is enough to check when $\Delta(Z) \subseteq Z \otimes Z$. As $X_1^p$ is primitive,
this amounts to checking whether $\Delta(z) \in Z \otimes Z$. The result now follows from part (1) and the fact that $w^p \in k[X_1^p] \otimes k[X_1^p]$.

(3) If $b_0 = 0$, then, by (E8.5.2),

$$ad_b(w) = [\delta(X_1) \otimes 1] \left( \sum_{t > 0} c_{0,t} X_1^{t'} \right) + [1 \otimes \delta(X_1)] \left( -\sum_{t > 0} c_{0,t} X_1^{t'} \otimes 1 \right).$$

By (E8.1.1) and the fact that $ad_{X_2}(X_1^{p,i}) = 0$ for all $i \geq 1$, we have that

$$ad_b^p(w) = [d_0^{p-1} \delta(X_1) \otimes 1] \left( \sum_{t > 0} c_{0,t} X_1^{t'} \right) + [1 \otimes d_0^{p-1} \delta(X_1)] \left( -\sum_{t > 0} c_{0,t} X_1^{t'} \otimes 1 \right)$$

for all $n \geq 1$. Then

$$ad_b^{p-1}(w) = d_0^{p-2} \left\{ (\delta(X_1) \otimes 1) \left( \sum_{t > 0} c_{0,t} X_1^{t'} \right) - (1 \otimes \delta(X_1)) \left( \sum_{t > 0} c_{0,t} X_1^{t'} \otimes 1 \right) \right\} + \mathcal{T}_1$$

where $\mathcal{T}_1 \in k[X_1^p] \otimes k[X_1^p]$. On the other hand, by definition,

$$d_0^{p-1}w = d_0^{p-1} \left\{ \sum_{t > 0} c_{0,t} (X_1 \otimes X_1^{t'} - X_1^{t'} \otimes X_1) \right\} + \mathcal{T}_2$$

where $\mathcal{T}_2 \in k[X_1^p] \otimes k[X_1^p]$. Hence

$$-d_0^{p-1}w + ad_b^{p-1}(w) \in k[X_1^p] \otimes k[X_1^p].$$

The assertion follows from part (2).

**Lemma 8.7.** Retain Notation 8.5.

(1) For all $i > 0$,

(E8.7.1) \hspace{1cm} ad_b^i(w) \equiv_p 0.

(2) Suppose $b_0 \neq 0$ and $d_0 \neq 0$. Then

$$-d_0^{p-1}w + ad_b^{p-1}(w) \notin k[X_1^p] \otimes k[X_1^p].$$

Hence $k[X_1^p, z]$ is not a Hopf subalgebra of $H$.

(3) Suppose that $d_0 = 0$, that $b_0 \neq 0$ and that $\delta(X_1) \neq 0$. Then $k[X_1^p, z]$ is not a Hopf subalgebra of $H$.

**Proof.** (1) Using (E8.5.2),

$$ad_b(w) \equiv_p (\delta(X_1) \otimes 1)b_0\{(X_1 \otimes 1 + 1 \otimes X_1)^{p-1} - X_1^{p-1} \otimes 1\}$$

$$+ (1 \otimes \delta(X_1))b_0\{(X_1 \otimes 1 + 1 \otimes X_1)^{p-1} - 1 \otimes X_1^{p-1}\}$$

$$\equiv_p db_0(X_1 \otimes 1)\{(X_1 \otimes 1 + 1 \otimes X_1)^{p-1} - X_1^{p-1} \otimes 1\}$$

$$+ db_0(1 \otimes X_1)\{(X_1 \otimes 1 + 1 \otimes X_1)^{p-1} - 1 \otimes X_1^{p-1}\}$$

$$\equiv_p db_0(X_1 \otimes 1 + 1 \otimes X_1)^p - db_0(X_1^p \otimes 1) - db_0(1 \otimes X_1^p) = 0.$$
(2) Assume that \( d_0 \) and \( b_0 \) are non-zero. Then, using Lemma 8.7.1,
\[
- d_0^{p-1} w + ad_b^{p-1}(w) \equiv_p - d_0^{p-1} w \\
\equiv_p - d_0^{p-1} b_0 \left( \sum_{i=1}^{p-1} \lambda_i X_1^i \otimes X_1^{p-i} \right).
\]
This implies that
\[
- d_0^{p-1} w + ad_b^{p-1}(w) \not\in \mathbb{k}[X_1^p] \otimes \mathbb{k}[X_1^p].
\]
The second claim now follows from Lemma 8.6.2.

(3) Using Lemma 8.5.2 and the hypothesis \( d_0 = 0 \) we obtain
\[
ad_b(w) = [\delta(X_1) \otimes 1] \left( b_0 \{(X_1 \otimes 1 + 1 \otimes X_1)^{p-1} - X_1^{p-1} \otimes 1 \} \right) \\
+ [1 \otimes \delta(X_1)] \left( b_0 \{(X_1 \otimes 1 + 1 \otimes X_1)^{p-1} - 1 \otimes X_1^{p-1} \} \right) + T
\]
where \( T \in \mathbb{k}[X_1^p] \otimes \mathbb{k}[X_1^p] \). Since \( d_0 = 0 \), \( \delta(X_1) \in \mathbb{k}[X_1^p] \) and hence is central. Therefore, by induction, we can show that
\[
ad_b^{p-1}(w) = b_0(p-1)! [\delta(X_1) \otimes 1] \left( (\delta(X_1) \otimes 1 + 1 \otimes \delta(X_1))^{p-2} (X_1 \otimes 1 + 1 \otimes X_1) \right) \\
- (\delta(X_1) \otimes 1)^{p-2} (X_1 \otimes 1) \right) \\
+ b_0(p-1)! [1 \otimes \delta(X_1)] \left( (\delta(X_1) \otimes 1 + 1 \otimes \delta(X_1))^{p-2} (X_1 \otimes 1 + 1 \otimes X_1) \right) \\
- (1 \otimes \delta(X_1))^{p-2} (1 \otimes X_1) ) + \delta_{2, \text{char} \mathbb{T}}
\]
\[
= b_0(p-1)! [(\delta(X_1) \otimes 1 + 1 \otimes \delta(X_1))^{p-1} (X_1 \otimes 1 + 1 \otimes X_1) \\
- (\delta(X_1) \otimes 1)^{p-1} (X_1 \otimes 1) - (1 \otimes \delta(X_1))^{p-1} (1 \otimes X_1) ] + \delta_{2, \text{char} \mathbb{T}}.
\]
Observe that this element is not a member of \( \mathbb{k}[X_1^p] \otimes \mathbb{k}[X_1^p] \). Since \( d_0 = 0 \), \( d_0^{p-1} w = 0 \). Therefore \(- d_0^{p-1} w + ad_b^{p-1}(w) \not\in \mathbb{k}[X_1^p] \otimes \mathbb{k}[X_1^p]. \) The assertion thus follows from Lemma 8.6.2.

Lemmas 8.6 and 8.7 can now be applied to determine \( C(H) \) for all 2-step IHOEs:

**Theorem 8.8.** Retain Notation 8.3. Assume that \( H \) is not commutative.

(1) The subalgebra \( \mathbb{k}[X_1^p, z^p] \) is a central Hopf subalgebra of \( H \).

(2) The Hopf center \( C(H) \) of \( H \) is either \( \mathbb{k}[X_1^p, z^p] \) or \( \mathbb{k}[X_1^p, z] \).

(3) \( C(H) = \mathbb{k}[X_1^p, z] \) if and only if \( b_0 = 0 \).

**Proof.** (1) Since \( H \) is connected, it is enough to show that \( \mathbb{k}[X_1^p, z^p] \) is a bialgebra. First, \( X_1^p \) is primitive. To see that \( \Delta(z^p) \in \mathbb{k}[X_1^p, z^p] \otimes \mathbb{k}[X_1^p, z^p] \), note that \( z \) is central and apply Lemma 8.6.1 to determine \( \Delta(z^p) \).

(2) In view of part (1) and Proposition 8.4.4,
\[
\mathbb{k}[X_1^p, z^p] \subseteq C(H) \subseteq \mathbb{k}[X_1^p, z].
\]
Moreover, \( H \) is a free module over each of these subalgebras, with the ranks over the outer two algebras being \( p^3 \) and \( p^2 \). But the rank of \( H \) over \( C(H) \) is also a power of \( p \) by Proposition 8.3. Thus equality must hold at some point in the chain of inclusions.

(3) The assertion follows from Lemmas 8.6.3 and 8.7.2,3.

\[\square\]
The following two propositions examine the restricted Hopf algebra of $H$ in each of
the two cases distinguished by Theorem 8.8.

**Proposition 8.9.** Retain Notation 8.5. Assume that $H$ is not commutative. Suppose that
$b_0 = 0$. In parts (2,3) let $\overline{H}$ denote the quotient Hopf algebra $H/Z_+H$.

1. The center $Z := k[X_1^p, z]$ is a Hopf subalgebra of $H$.
2. Suppose $d_0 = 0$. Then $\overline{H}$ is
   \[ H_{0,0} \cong k[x, y]/(x^p, y^p), \]
   a commutative cocommutative Hopf algebra of dimension $p^2$, with $x$ and $y$
   primitive.
3. Suppose that $d_0 \neq 0$. Then $\overline{H}$ is
   \[ H_{0,0} \cong k(x, y : [y, x] - x, x^p, y^p - y), \]
   a cocommutative Hopf algebra of dimension $p^2$ with $x$ and $y$ primitive, the
   restricted enveloping algebra of the 2-dimensional non-abelian restricted Lie
   algebra.

**Proof.** The center $Z$ is as stated, by Proposition 8.1(4).

1. This is Theorem 8.8(3).
2. By Proposition 8.2(2),
   \[ H/Z_+H = H_{0,0} \cong k[x, y]/(x^p, y^p), \]
   where $x$ and $y$ are the images of $X_1$ and $X_2$ respectively. Since $X_1$ is primitive so
   is $x$, and $y$ is also primitive under the hypotheses.
3. Assume that $d_0 \neq 0$. Then Proposition 8.2(3) applies, yielding
   \[ H/Z_+H = H_{0,0} \cong k(x, y : [y, x] - x, x^p, y^p - y, [y, x] = x), \]
   where $x$ and $y$ are respectively the images of $X_1$ and $d_0^{-1}X_2$. As in (2), $x$ and $y$ are
   both primitive since $b_0 = 0$. \qed

**Proposition 8.10.** Retain Notation 8.5. Assume that $H$ is not commutative. Suppose that $b_0 \neq 0$. In parts (2,3) let $\overline{H}$ denote the quotient Hopf algebra $H/C_+H$.

1. The center $k[X_1^p, z]$ is not a Hopf subalgebra of $H$ and the unique largest
central Hopf subalgebra of $H$ is $C := k[X_1^p, z^p]$.
2. Suppose that $d_0 = 0$. Then
   \[ \overline{H} \cong k[x, y]/(x^p, y^p^3) \]
   with $x$ primitive and
   \[ \Delta(y) = y \otimes 1 + 1 \otimes y + \sum_{i=1}^{p-1} \lambda_i x^i \otimes x^{p-i}. \]
   This is a commutative and cocommutative Hopf algebra of dimension $p^3$. 
(3) Suppose that $d_0 \neq 0$. Then the quotient Hopf algebra $\overline{H}$ is $\mathbb{k}(x, y, z)$, with relations

\[
[y, x] = x, \\
z = y^p - y, \\
[y, z] = [x, z] = 0, \\
x^p = z^p = 0.
\]

This is noncommutative and cocommutative of dimension $p^3$, with $x$ primitive and

\[
\Delta(y) = y \otimes 1 + 1 \otimes y + b_0d_0^{-1} \sum_{i=1}^{p-1} \lambda_i x^i \otimes x^{p-i}
\]
and

\[
\Delta(z) = z \otimes 1 + 1 \otimes z - b_0d_0^{-1} \sum_{i=1}^{p-1} \lambda_i x^i \otimes x^{p-i}
\]

Proof. (1) This follows from Theorem 8.8(2,3).

(2) Assume that $d_0 = 0$. Let $x$ be the image of $X_1$ and $y$ the image of $b_0^{-1}X_2$. The result follows by direct computation.

(3) Assume that $d_0 \neq 0$. Let $x$ be the image of $X_1$, $y$ the image of $b_0^{-1}X_2$, and $z$ the image of $(d_0^{-1}X_2)^p - (d_0^{-1}X_2)$. Most of the assertions follow by routine computations. For example, it is not hard to check that

\[
\Delta(y) = y \otimes 1 + 1 \otimes y + eu
\]
where $u = \sum_{i=1}^{p-1} \lambda_i x^i \otimes x^{p-i}$ and $e := b_0d_0^{-1}$. Here we only prove the formula for $\Delta(z)$. Repeating the computation in Lemma 8.6(1), we have, for $ad_b := ad_{y \otimes 1 + 1 \otimes y}$,

\[
\Delta(z) = z \otimes 1 + 1 \otimes z + (eu)^p - eu + ad_b^{p-1}(eu)
\]
\[
= z \otimes 1 + 1 \otimes z - eu + ad_b^{p-1}(eu)
\]
where the last equation follows from the fact that $u^p = 0$ in $H/C_4H$. Similarly to (ES.5.2), one checks that

\[
ad_b(u) = (x \otimes 1)[(x \otimes 1 + 1 \otimes x)^{p-1} - x^{p-1} \otimes 1]
\]
\[
+ (1 \otimes x)[(x \otimes 1 + 1 \otimes x)^{p-1} - 1 \otimes x^{p-1}]
\]
\[
= (x \otimes 1)(x \otimes 1 + 1 \otimes x)^{p-1} - x^p \otimes 1
\]
\[
+ (1 \otimes x)(x \otimes 1 + 1 \otimes x)^{p-1} - 1 \otimes x^p
\]
\[
= (x \otimes 1 + 1 \otimes x)^p - x^p \otimes 1 - 1 \otimes x^p
\]
\[
= x^p \otimes 1 + 1 \otimes x^p - x^p \otimes 1 - 1 \otimes x^p
\]
\[
= 0.
\]

Therefore $\Delta(z) = z \otimes 1 + 1 \otimes z - eu$. This finishes the proof. □

Remarks 8.11. (1) In the light of Propositions 8.9-8.10 and the known situation for enveloping algebras of finite dimensional Lie algebras over $\mathbb{k}$, we conjecture that the following question has a positive answer:
Question 8.12. Let $k$ be of positive characteristic $p$. Does every $n$-step $k$-IHOE $H$ have a Hopf center $C(H)$ over which $H$ is a finite module?

(2) For an algebraically closed field $k$ of positive characteristic $p$, the connected Hopf $k$-algebras of dimension $p^n$ for $p > 2$ and $n \leq 3$ have been classified in a series of papers by Nguyen, L. Wang and X. Wang [NWW1] [Wa1] [Wa2], culminating in [NWW2]. The Hopf algebra $\mathcal{H}$ in Proposition 8.2(2) is the one in [Wa1] Theorem 7.4(1)], and the Hopf algebra $\overline{\mathcal{H}}$ in Proposition 8.9(3) is isomorphic to the one in [Wa1] Theorem 7.4(5)]. The Hopf algebra $\overline{\mathcal{H}}$ in Proposition 8.10(2) is isomorphic to the one listed in Table 5 with type T1 in the third case in [NWW2, p.858]. The Hopf algebra $\mathcal{H}$ in Proposition 8.10(3) can be written as $k\langle X, Y, Z \rangle$ (where $X = y + z$, $Y = x$ and $Z = -b_{0}^{-1}d_{0}z$), with relations

$$[X, Y] = Y, \quad [X, Z] = 0, \quad [Y, Z] = 0,$$

and

$$X^p = X, \quad Y^p = 0, \quad Z^p = 0,$$

and $X, Y$ being primitive and

$$\Delta(Z) = Z \otimes 1 + 1 \otimes Z + \sum_{i=1}^{p-1} \lambda_i Y^i \otimes Y^{p-i}.$$

Hence this Hopf algebra is isomorphic to the one in [NWW1] Theorem 1.3(B1)].

The following questions are thus very natural:

Question 8.13. (a) Which finite dimensional connected Hopf $k$-algebras can be realised as factors of IHOEs over $k$?

(b) If Question 8.12 has a positive answer, then which connected Hopf $k$-algebras can be realised as the restricted Hopf algebras $H/C(H)_+ H$ of $k$-IHOEs $H$?

Regarding(a), note that a finite dimensional Hopf algebra $T$ which can be realised as a factor of an IHOE necessarily has some rather strict structural constraints: namely, it will have a chain of Hopf subalgebras $T_i$, for $i = 1, \ldots, t$ such that $T_1 = k$ and $T_{i+1} = k(T_i, x_{i+1})$ for elements $x_2, \ldots, x_n$, with a corresponding “PBW-structure”. It is thus clear that not all finite dimensional connected Hopf $k$-algebras can be so realised - for example, the restricted enveloping algebra of $\mathfrak{sl}(3, k)$ presumably cannot be presented as a factor of an IHOE. Conversely, however, Xingting Wang has informed us [Wa3] that he has checked case by case that all connected Hopf $k$-algebras of dimension at most $p^3$ can be realised as Hopf factors of IHOEs over $k$, at least when $p > 2$.

(3) All of the speculations in remarks (1) and (2) can be reformulated with IHOEs over $k$ replaced by all connected Hopf $k$-algebras of finite Gel’fand-Kirillov dimension.

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