QUANTUM LOGIC IN INTUITIONISTIC PERSPECTIVE

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Abstract

In their seminal paper Birkhoff and von Neumann revealed the following dilemma: “... whereas for logicians the orthocomplementation properties of negation were the ones least able to withstand a critical analysis, the study of mechanics points to the distributive identities as the weakest link in the algebra of logic.” In this paper we eliminate this dilemma, providing a way for maintaining both. Via the introduction of the “missing” disjunctions in the lattice of properties of a physical system while inheriting the meet as a conjunction we obtain a complete Heyting algebra of propositions on physical properties. In particular there is a bijective correspondence between property lattices and propositional lattices equipped with a so called operational resolution, an operation that exposes the properties on the level of the propositions. If the property lattice goes equipped with an orthocomplementation, then this bijective correspondence can be refined to one with propositional lattices equipped with an operational complementation, as such establishing the claim made above. Formally one rediscovers via physical and logical considerations as such respectively a specification and a refinement of the purely mathematical result by Bruns and Lakser (1970) on injective hulls of meet-semilattices. From our representation we can derive a truly intuitionistic functional implication on property lattices, as such confronting claims made in previous writings on the matter. We also make a detailed analysis of disjunctivity vs. distributivity and finitary vs. infinitary conjunctivity, we briefly review the Bruns-Lakser construction and indicate some questions which are left open.
In their seminal paper Birkhoff and von Neumann (1936) observe that the lattice of closed subspaces of a Hilbert space retains a number of the familiar features of Boolean algebras (which constitute the semantics of classical propositional logic), namely, it is orthocomplemented and hence satisfies the De Morgan laws. However, the distributive law fails. Confronting the then ongoing tendencies towards intuitionistic logic [Birkhoff and von Neumann (1936) p.839]:

“The models for propositional calculi [of physically significant statements in quantum mechanics] are also interesting from the standpoint of pure logic. Their nature is determined by quasi-physical and technical reasoning, different from the introspective and philosophical considerations which have to guide logicians hitherto [...] whereas logicians have usually assumed that [the orthocomplementation] properties L71-L73 of negation were the ones least able to withstand a critical analysis, the study of mechanics points to the distributive identities L6 as the weakest link in the algebra of logic.”

they point at a fundamental difference between Heyting algebras (the semantics of intuitionistic propositional logic) and orthomodular lattices (the “usual”
semantics of quantum logic) when viewed as generalizations of Boolean algebra. This seems to enforce a dilemma with respect to logical considerations on propositions attributed to physical systems. It is probably fair to say that due to this dilemma, quantum logic became a strictly separated domain of mathematics that had no essential impact on traditional fields of logic. Moreover, most attempts to provide a logical syntax for discussing physical properties, e.g., Hardegree (1979) and Kalmbach (1993), knew serious criticism (and definitely not always unjust), e.g., the arguments in Goldblatt (1984), Malinowski (1990) and Moore (1993). In particular we do want to point in this context to the failure to equip quantum logic with a satisfactory internal implication operation.

However, we will show in this paper that both motivations, i.e., the physical one encoding a non-distributive orthocomplemented lattice and the logical intuitionistic one encoding a distributive pseudocomplemented lattice, are not incompatible but motivate a distinction between the physical properties themselves and logical propositions on physical properties. In particular we will encode all aspects within one mathematical object, namely a complete Heyting algebra equipped with an additional operation, the operational resolution. We will motivate these claims and constructions using the operational methodology\footnote{By some people considered as a doctrine, including one of the fathers, namely Piron himself; dixit Piron: “Les Coeckeries et les Moorisms ne sont pas des Pironeries”. In our view, the operational methodology allows to communicate and refine certain insights, independent on the reader’s personal view on physics, and contributes either in providing an image, an understanding or a model, this depending on the reader’s personal taste.} for quantum logic, which was already implicitly indicated in Birkhoff and von Neumann (1936), but got only truly established in Jauch and Piron (1969) and further developed and refined in Piron (1976), Aerts (1982), Moore (1999) and Coecke, Moore and Smets (2001a,b). This methodology relates properties of a physical system to definite experimental projects in part to provide an answer to [Birkhoff and von Neumann (1936) p.839]:

“What experimental meaning can one attach to the meet and join of two given experimental propositions [on quantum systems]?”

but also to motivate a common framework to discuss both classical and quantum systems, and understand their ontological and epistemological differences. However, we feel that even if one does not fully subscribe to this methodology, most, and in particular all essential aspects of this paper still hold. For exam-
ple, one ingredient of the methodology consists of proving that the lattice of properties of a physical system should be taken complete, i.e., any subset of it has a greatest lower bound and a least upper bound. However, if one chooses to think of a property lattice as having only finite meets, or if one rather has analytical or probabilistic inspirations, as such preferring it to be $\sigma$-complete, the construction and corresponding interpretation of greatest lower bounds for arbitrary (large) sets of properties provides a way to think about completion of this property lattice, and how one should manipulate this extension. In particular is Section 4.2 of this paper devoted to property lattices in which only finite meets are considered as conjunctions.

A striking fact of the mathematics applied in this paper is indeed that the assumption on preservation of finite meets in the considered representation automatically ensures preservation of all infinitary meets as well. About this mathematics, the in this paper proposed representation for the properties of a physical system within a complete Heyting algebra (of logical propositions on these properties) equipped with a particular kind of closure operator (the operational resolution), and which will be motivated by logical reflection on primitive operational physical notions, this representation actually mimics a purely mathematically motivated result of Bruns and Lakser (1970), also independently found by Horn and Kimura (1971), namely proving the existence and characterizing the injective hulls in the category of meet-semilattices. We will specify this result for complete lattices (and show that this specification works), thereby moulding it towards our particular needs.

Concluding this introduction, in order to substantiate our claim at the beginning of the previous paragraph we will specify this representation for complete ortholattices, as such revealing the physical notion of an operational complementation, a pseudo-orthocomplementation that has the operational resolution as its square: This operational complementation will then be the operation that recaptures the orthocomplementation of the properties as an additional operation on the complete Heyting algebra of the logical propositions on these properties. Since in this representation it is the collection of propositions that goes equipped with an internal intuitionistic implication operation, “implication for physical properties” should be envisioned as an external operation that assigns propositions to pairs of properties.
2. FORMAL AND METHODOLOGICAL TOOLS

First we provide and discuss the required formal and methodological tools.

2.1. Logical significance of property lattices

Let us briefly survey the fragment of the above mentioned operational methodology that we will employ in this paper; we refer to Moore (1999) and Coecke, Moore and Smets (2001a,b) for the most recent overview. Any property of a physical system is identified with an equivalence class of definite experimental projects that can be effectuated on that system where:

- A definite experimental project is a precisely defined physical procedure $\alpha$ which includes specification of what should be conceived as the positive outcome when we would effectuate $\alpha$;

- A definite experimental project $\alpha$ is certain for a particular realization of the system, i.e., for the system in a certain state, if we obtain the positive outcome with certainty whenever we would effectuate $\alpha$ on the system in that particular realization;

- Two definite experimental projects are equivalent whenever certainty of one is equivalent to certainty of the other; the underlying preorder (or quasi order) that generates this equivalence then encodes for two definite experimental projects $\alpha$ and $\beta$ as “$\alpha \prec \beta$ if and only if certainty of $\alpha$ implies certainty of $\beta$”; the corresponding equivalence class of $\alpha$ will be denoted as $[\alpha]$ and the physical property to which it corresponds as $a$; the property $a$ is then called actual for a particular realization of the system, or true if one prefers, whenever $\alpha$ is certain for it.

What are the consequences of this operational identification of properties with definite experimental projects? First of all, the partial ordering of the properties induced by the preorder on definite experimental projects can now be understood as an implication relation with respect to actuality (or truth).
also follows that the properties constitute a complete lattice $L$. Indeed, given $a_i \in L$ with corresponding definite experimental projects $\alpha_i$ we can define the product $\prod \alpha_i$ as the definite experimental project that consists of performing one of the $\alpha_i$, chosen in \textit{any possible way}. The property then defined by $\prod \alpha_i$ is true if and only if each of the $\alpha_i$ is true and can therefore be understood as the \textit{conjunction} of $\{a_i\}_i$. It then also obviously follows that with respect to the above discussed preordering of properties, $\prod \alpha_i$ is indeed the \textit{meet} $\bigwedge_i a_i$ of $\{a_i\}_i$ in the complete lattice $L$. By Birkhoff’s theorem all subsets of the lattice then also have a least upper bound given by

$$\bigvee_i a_i := \bigwedge \{b \in L | \forall i : a_i \leq b\}.$$ 

Note here that contra the usual motivation that conjunctions should be finite, our operational methodology motivates arbitrary infinitary ones. Moreover, as already announced above, given a meet-semilattice $L$ in which the elements are all the properties of a physical system ordered by implication with respect to actuality, and in which the meets encode conjunctions, the products $\prod \alpha_i$ and corresponding properties $\bigwedge_i a_i$ then provide an interpretation for the supplementary elements in the canonical or MacNeille completion of $L$.\footnote{For an outline of canonical or MacNeille completion see Banaschewski and Bruns (1967).} We will come back to this point, which is slightly more subtle than it might look at first, in Section 4.2. Conclusively, although in a considerable number of papers a property lattice or its abstract counterpart, somewhat abusively called an \textit{algebraic quantum logic}, is conceived as an orthomodular lattice\footnote{For orthomodular lattices see Kalmbach (1983) and Bruns and Harding (2000). An explicit definition can also be found in Section 3.2 of this paper.} not necessarily complete, we will initially consider property lattices as being general complete lattices. Complete ortholattices are then a particular species.

We will now discuss the join in property lattices. First recall that in the intuitionistic sense, truth of a \textit{disjunction} coincides with truth of one of its members, and it is as such that we will conceive disjunction from now on. Referring to orthodox Hilbert space quantum mechanics, the properties of a physical system are represented by the closed subspaces of a Hilbert space $\mathcal{H}$ and the join encodes as the closed linear span. So the join of two atomic properties $p_1$ and $p_2$ represented by two non-equal rays $\phi_1$ and $\phi_2$ in $\mathcal{H}$ is implied (in the above discussed operational sense) by any atomic property $q$ encoded as a ray $\psi$ in $\mathcal{H}$.\footnote{We will somewhat abusively denote rays in $\mathcal{H}$ by a representative unit vector.}
the plane spanned by $\phi_1$ and $\phi_2$. Thus, if $q \neq p_1, p_2$ then $q \leq p_1 \lor p_2$ although actuality of $q$ excludes that of $p_1$ and it excludes that of $p_2$, i.e., it excludes actuality of $p_1$ or $p_2$. As such, the join is in general not a disjunction, and this observation lies at the base of the construction made in this paper. This fact that the join, which (as we saw above) is from an operational perspective defined in a secondary way via the meet, is not a disjunction has been used by Aerts (1982) to encode the to physicists well-known notion of superposition: 7

If two states of the system are represented by the atomic properties $p_1$ and $p_2$, then all other states $q$ such that $q \leq p_1 \lor p_2$ are called superpositions of $p_1$ and $p_2$. Note however that this notion of superposition has been introduced in Aerts (1982) under the paradigm that states are indeed in one to one correspondence with atomic properties and that these atomic properties are join dense in the property lattice, i.e., the property lattice is atomistic. We stress that we do not fully subscribe to this paradigm; for a counterexample that employs non-atomistic property lattices within the operational methodology we refer to Coecke (2000). Other examples emerge by restriction of the property lattice led by certain topological considerations, for which we refer to Section 4.2 in this paper. Nevertheless, this notion of superposition can be extended to non-atomistic property lattices or more general, any situation where the states are not encoded as properties, in the following way: Actuality of $a \lor b$ does not necessarily imply actuality of $a$ or actuality of $b$, i.e., there exists a state for which $a \lor b$ is actual, but neither $a$ nor $b$ are actual. We can as such define the following for $A \subseteq L$:

- **Superposition states** introduced by the join of $A$ are those states for which $\bigvee A$ is actual while no $a \in A$ is actual.

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7Note here that one of the De Morgan laws, namely $\neg(a \lor b) = \neg a \land \neg b$, indeed still holds in an intuitionistic setting. However, $\neg(a \land b) = \neg a \lor \neg b$ is not valid anymore.

8In many interpretations of orthodox quantum theory, a superposition is understood as a decomposition $\oplus_i c_i \phi_i$ of a ray $\psi \in \mathcal{H}$ that represents the initial state $q$ (so $\psi = \oplus_i c_i \phi_i$), where $\{\phi_i\}_i$ is an orthonormal base of $\mathcal{H}$ that represents the possible outcome states $\{p_i\}_i$ of a measurement, envisioning the states $p_i$ with non-zero $c_i$ as the “possible truths after the measurement” whenever the measurement will have been effectuated. E.g., Schrödinger’s cat gedanken experiment: The cat is neither dead nor alive but in a superposition, say dead$\oplus$alive, as long as the measurement is not completely effectuated, where this effectuation in particular includes observing whether the cat is dead or alive. We don’t subscribe to this perspective but envision a superposition state $q$ of $p_1$ and $p_2$, just as a different possible realization of the system, where $p_1, p_2$ and $q$ are related by the fact that $p_1 \lor p_2 = q \lor p_2 = p_1 \lor q = p_1 \lor p_2 \lor q$. In this view the quantum mechanical measurement described above then induces a change of the state $q$ to a state in $\{p_i\}_i$, for which $c_i \neq 0$. 

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One could as such say that an aspect that characterizes quantum(-like) property lattices is that the join introduces superpositions. This introduction of superpositions by the join should be conceived as the strict counterpart of a join that indeed behaves as a disjunction: Actuality of a join does not necessarily coincide with actuality of its members since its actuality might be implied by one of the superpositions it introduces. Besides superposition states we will also need to consider the following for $A \subseteq L$:

- Superposition properties introduced by the join of $A$ are those $c < \bigvee A$ whose actuality doesn’t imply that at least one $a \in A$ is actual, i.e., for which there exists a state that makes $c$ actual while no $a \in A$ is actual.

Given the so called Cartan map $\mu : L \to \mathcal{P}(\Sigma)$ with $\mathcal{P}(\Sigma)$ the powerset of the state set and which assigns to any property the states in which it is actual, the join of $A$ introduces superposition states if and only if $\mu(\bigvee A) \neq \bigcup \mu[A] := \bigcup_{a \in A} \mu(a)$ and $c < \bigvee A$ is a superposition property if and only if $\mu(c) \not\subseteq \bigcup \mu[A]$.

**Proposition 1.** $\mu : L \to \mathcal{P}(\Sigma)$ is a balanced inf-embedding.

**Proof:** If $\mu(a) = \mu(b)$, then actuality of $\alpha$ coincides with that of $\beta$, i.e., $\alpha \in [\beta]$ so $a = b$. Preservation of infima follows from the construction of infima via products, i.e., they stand for conjunction with respect to actuality. Since the bottom 0 stands for the absurd it cannot be actual in any state so $\mu(0) = \emptyset$. Since the top 1 stands for the trivial it is actual in any state so $\mu(\bigwedge \emptyset) = \mu(1) = \Sigma = \bigcap \emptyset$.

**Example:** Setting $\mu(a) := \{ p \in \Sigma | p \leq a \}$ for a complete atomistic lattice $L$ with atoms $\Sigma$ we have by atomisticity that $a = \bigvee \mu(a)$. Since $\mu(\bigwedge A) = \{ p \in \Sigma | p \leq \bigwedge A \} = \{ p \in \Sigma | \forall a \in A : p \leq a \} = \bigcap_{a \in A} \{ p \in \Sigma | p \leq a \} = \bigcap \mu[A]$, since $\mu(0) = \emptyset$ and

\[ \text{We will use square brackets as a notation for pointwise application of a map throughout the paper, i.e., } f[X] := \{ f(x) | x \in X \}. \]
since $\mu(a) = \mu(b) \Rightarrow \bigvee \mu(a) = \bigvee \mu(b) \Rightarrow a = b$ it follows that $\mu : L \to P(\Sigma) : a \mapsto \mu(a)$ is a balanced \textit{inf}-embedding.

Clearly, the Cartan map captures as such the essence of operational methodology. In particular are conjunctions now encoded as intersections in the state space, i.e., $\mu(\land A) = \bigcap \mu[A]$ and disjunctions $\lor A$ exactly coincide with unions, i.e., $\mu(\lor A) = \bigcup \mu[A]$. Note that $\mu(a) = \emptyset$ implies $a = 0$ by injectivity. From injectivity of $\mu$ it also follows that $a < b$ encodes as $\mu(a) \subset \mu(b)$. Applying all this, superposition properties relate to superposition states in the following way:

**Proposition 2.** If the join of $A \subseteq L$ has superposition properties, then it also has superposition states; the converse is in general not true.

**Proof:** From $c < \lor A$ follows $\mu(c) \subset \mu(\lor A)$ so if $\mu(c) \nsubseteq \bigcup \mu[A] \cup \bigcup \mu[A]$ then $\mu(\lor A) \nsubseteq \bigcup \mu[A]$ which proves the first claim. For $L := \{0, a, a', 1\}$ and $\Sigma := \{p_1, p_2, q\}$ with $\mu(a) = \{p_1\}$, $\mu(a') = \{p_2\}$ and $\mu(1) = \{p_1, p_2, q\}$, the join of $a$ and $a'$ introduces a superposition state $q$ but no superposition property. \hfill \Box

**Example:**
For a complete atomic lattice $L$ with atoms $\Sigma$ and $\mu$ as defined in the example above, existence of a superposition state $p$ does imply existence of a superposition property, namely $p$ itself.

Since we want to use the operational methodology as a motivation for a construction starting from a complete lattice envisioned as a property lattice, we have to make an assumption that the physical essence with respect to superpositions is fully encoded in the property lattice itself and not just in the Cartan map $\mu$. This is indeed necessary, as the counterexample in the proof of Proposition 2 shows: Even for the simplest example of a non-trivial complete Boolean algebra, namely the square $\{0, a, a', 1\}$, the join $a \lor a'$ is not necessarily interpretable as a disjunction if $\mu$ is arbitrarily chosen. Therefore we will assume at this point that the converse of Proposition 2 is also true, i.e., existence of superposition states implies that of superposition properties. Denoting the superposition states introduced by the join of $A$ as $S_{\oplus}(A)$ and the superposition properties as $L_{\oplus}(A)$ this translates as:

- $p \in S_{\oplus}(A) \Rightarrow \exists c_p \in L_{\oplus}(A) : p \in \mu(c_p)$

an axiom to which we will refer as \textit{superpositional faithfulness} of the property lattice (w.r.t. some Cartan map which is not explicitly specified). We
will discuss the interpretation and consequences of posing it at the end of this paper. We also will investigate what happens when we drop it, and why the considerations made in this paper will then still be useful.

Besides this fact that in a property lattice joins do not behave as disjunctions, the emergence of disjunction in measurements is exactly one of the core ingredients of quantum theory. Indeed, any measurement on a system that is not in an eigenstate of that measurement changes the state of the system in a non-deterministic manner. The resulting outcome state will as such be a member in a set of possible outcome states. Put in terms of properties, actuality of a property \( a \) before the measurement guarantees that either \( b_1 \) or \( b_2 \) or \( b_3 \) or \( \ldots \) will be actual after the measurement. We define an actuality set as a set of properties in which at least one member is actual. These actuality sets should then be conceived as the logical propositions that encode disjunction of actuality of properties, where disjunction is now indeed to be understood in the intuitionistic sense, i.e., truth of a disjunction \( A (= A \) is an actuality set) coincides with actuality (= truth) of one of its members. Thus, in other words, actuality sets recapture the notion “actuality” in the passage from properties to propositions. Note that in this setting it is obvious to consider arbitrary infinitary disjunctions. On these actuality sets one can now define an operational resolution (Coecke and Stubbe 1999a,b) as a map that assigns to each actuality set the strongest property of which the actuality is implied by the actuality set \( A \). Formally, given an actuality set \( A \), this property is given by \( \bigvee A \). As such, the operational resolution recaptures the operationally induced logical structure of the properties on the level of actuality sets. But what should be considered as the logical structure of these actuality sets: how do their conjunctions and disjunctions encode? Note for example that in \( \mathcal{P}(L) \), the obvious first candidate to encode actuality sets, for properties \( a \leq b \) we have that \( \{a\} \land_{\mathcal{P}(L)} \{b\} = \{a\} \cap \{b\} = \emptyset \), which clearly doesn’t encode conjunction; thus, \( \mathcal{P}(L) \) is inappropriate as a logic of actuality sets. We will provide a solution to this in Section 3.1.

To conclude this section, if we want to describe a physical system by a

\footnote{In Coecke and Stubbe (1999a,b) operational resolutions are defined in a slightly different fashion, namely as a map \( \mathcal{R} : \mathcal{P}(\Sigma) \rightarrow L \), assigning to a set of states the strongest property that is actual for each of the realizations in this set. Any such operational resolution on the states then canonically induces one on the properties. When substituting \( L \) formally by the isomorphic set of the \( \mathcal{R} \)-closed subsets of \( \Sigma \) (every operational resolution indeed factors in a closure operator \( \mathcal{C} : \mathcal{P}(\Sigma) \rightarrow \mathcal{P}(\Sigma) \) and an isomorphism on its range) then the operational resolution can be seen as the operation that “adds all superposition states”.}
“language” that is closed under all disjunctions of properties, we formally need to introduce those additional propositions that express disjunctions of properties and that do not correspond to a property in the property lattice. Thus we want to embed the property lattice within a larger propositional lattice whose elements are to be interpreted as actuality sets whenever the system is in a state that makes the according proposition true. The next section will provide the mathematical tool that establishes this embedding in an optimal and even universal manner.

2.2. Bruns-Lakser distributive hulls for complete lattices

As discussed in the previous section, infima play in the property lattice a fundamental role having a direct operational and logical interpretation, respectively via products of definite experimental projects and as a conjunction (whereas the join is only secondary defined) encoded in terms of the Cartan map as a balanced $\inf$-embedding.

Whenever an $\inf$-embedding is an inclusion, then we call its domain an $\inf$-subobject of its codomain. An $\inf$-subobject is balanced if the corresponding $\inf$-morphism is. Analogously we define a meet-morphism as a map between meet-semilattices that preserves finite meets and a sup-morphism as a map between complete lattices that preserves all suprema, including the empty join $\bigvee \emptyset = 0$. Recall that a closure (operator) $\mathcal{C} : L \rightarrow L$ on a complete lattice $L$ is isotone, i.e., $a \leq b \Rightarrow \mathcal{C}(a) \leq \mathcal{C}(b)$, increasing, i.e., $a \leq \mathcal{C}(a)$, and idempotent, i.e., $\mathcal{C}(\mathcal{C}(a)) = \mathcal{C}(a)$. It is normalized if moreover $\mathcal{C}(0) = 0$. One then obtains that the range $\mathcal{C}(L)$ of a closure operator $\mathcal{C}$ on a complete lattice $L$ is a $\inf$-subobject of $L$, which is balanced whenever the closure is normalized: We have for all $a \in A \subseteq \mathcal{C}(L)$ by isotonicity that $\mathcal{C}(\bigwedge L A) \leq \mathcal{C}(a) = a$; since moreover $\bigwedge L A \leq \mathcal{C}(\bigwedge L A)$, the fact that $\bigwedge L A$ is an infimum forces $\mathcal{C}(\bigwedge L A) = \bigwedge L A$, and thus $\bigwedge L A = \bigwedge_{\mathcal{C}(L)} A$. Conversely, any balanced $\inf$-subobject $M$ of $L$ defines a normalized closure operator $\mathcal{C}_M : L \rightarrow L : a \mapsto \bigwedge\{b \in M | a \leq b\}$. Codomain restriction of a closure $\mathcal{C}$ to $\mathcal{C}(L)$ turns it into a $\sup$-morphism, so for $A \subseteq \mathcal{C}(L)$ we have $\bigvee_{\mathcal{C}(L)} A = \mathcal{C}(\bigvee L A)$.\footnote{We are aware of the potential objections against this designation, in particular for category theorists. We however couldn’t think of anything more suitable.}

Codomain restriction of a closure $\mathcal{C}$ to $\mathcal{C}(L)$ turns it into a $\sup$-morphism, so for $A \subseteq \mathcal{C}(L)$ we have $\bigvee_{\mathcal{C}(L)} A = \mathcal{C}(\bigvee L A)$.\footnote{See also the appendix at the end of this paper.}

Given a meet-semilattice $H$, i.e., a poset that admits finite meets, which is also bounded, i.e., it has 0 and 1, then we call it a Heyting semialgebra\footnote{Such a Heyting semialgebra exhibits all structural features of a Heyting algebra, i.e., a} if
and only if there exists an additional operation \((- \Rightarrow -) : H \times H \to H\) such that \(a \land b \leq c\) iff \(a \leq (b \Rightarrow c)\). A Heyting semialgebra which is complete (as a lattice) is called a complete Heyting algebra. We say that a subset \(A\) of a meet-semilattice \(L\) has a distributive join if (i) its supremum exists, and (ii) for all \(b \in L\) we have \(b \land \bigvee A = \bigvee\{b \land a | a \in A\}\). We will abbreviate this by saying that \(\bigvee A\) is distributive. One then verifies that complete Heyting algebras are exactly meet-semilattices in which every subset has a distributive join.\(^{14}\) The Heyting implication \((- \Rightarrow -) : H \times H \to H\) then fixes the Heyting negation as \(\neg (-) := (- \Rightarrow 0)\). Algebraically, this Heyting negation is a pseudo-complementation since in general it does not satisfy one of the De Morgan’s laws, namely the excluded middle law \(\neg a \lor a = 1\).

We will now formulate the Bruns-Lakser results. Recall that given a category, i.e., a class of objects equipped with compositionally closed sets of morphisms including identities (e.g., meet-semilattices with meet-morphisms or complete lattices with inf-morphisms), an object \(H\) is injective if for every morphism \(f : L \to H\) with \(L\) a subobject of \(L'\) (e.g., respectively a meet-subobject or inf-subobject) there exists a domain extension \(f' : L' \to H\). Given a subobject \(L\) of \(H\), then we call \(H\) an essential extension of \(L\) whenever injectivity of the domain restriction of a morphism \(f : H \to L'\) to \(L\) implies injectivity of \(f\) itself. An injective hull is then an essential injective extension. Note that such an essential injective extension is actually a minimal inclusion as a subobject in an injective object. Indeed, if given an injective hull \(H\) of \(L\), and if \(H'\) is another injective object that has \(L\) as a subobject, then, by injectivity of \(H\), the inclusion \(L \hookrightarrow H'\) extends to a morphism \(H \to H'\), and since \(H'\) is an essential extension \(L\) this map is injective, so \(H\) is isomorphic to a subobject of \(H'\). In particular it also follows that injective hulls are unique up to an isomorphism. Bruns and Lakser proved that:

- Injective meet-semilattices coincide with complete Heyting algebras;
- Every meet-semilattice has an injective hull.

They also provided an implicit and explicit characterization of these injective hulls of semilattices:\(^{14}\)

\(^{14}\)For proofs we refer to Johnstone (1982) or the appendix at the end of this paper.
A complete Heyting algebra $H$ that has $L$ as a meet-subobject is the injective hull of $L$ if and only if:

1. $L$ is join-dense in $H$, i.e., for all $a \in H$: $a = \bigvee_H \{b \in L \mid b \leq a\}$;
2. If $A \subseteq L$ has a distributive join $\bigvee_L A$ then $\bigvee_H A = \bigvee_L A$.

The injective hull of a meet-semilattice $L$ is isomorphic to its collection of distributive ideals $\mathcal{DI}(L)$ ordered by inclusion, where a distributive ideal $A \in \mathcal{DI}(L)$ is an order ideal, i.e., $a \leq b \in A \Rightarrow a \in A$ and $A \neq \emptyset$, which is also closed under existing distributive joins, i.e., if $B \subseteq A$ has a distributive join then $\bigvee_B \in A$; the inclusion of $L$ in an injective hull $H$ then factors as $L \cong \{\downarrow a \mid a \in L\} \hookrightarrow \mathcal{DI}(L) \cong H$, where the isomorphic correspondence between the distributive hull $H$ of $L$ and the distributive ideals $\mathcal{DI}(L)$ realizes as:

$$\begin{align*}
\theta : H &\rightarrow \mathcal{DI}(L) : a \mapsto \{b \in L \mid b \leq a\} \\
\theta^{-1} : \mathcal{DI}(L) &\rightarrow H : A \mapsto \bigvee_H A
\end{align*}$$

Note for the implicit characterization that the second condition forces $L$ to be a balanced meet-subobject. Indeed, since the join of $\emptyset$ is distributive we have $0_H = \bigvee_H \emptyset = \bigvee_L \emptyset = 0_L$. To illustrate the necessity of this second condition it suffices to consider $L := \{0, a, a', 1\}$ with $a \vee a' = 1$ and $H := \{0, a, a', b, 1\}$ with $a, a' < b$. The explicit construction shows us that all the above can be reformulated for complete lattices and $\inf$-morphisms. Whenever $H$ is the injective hull of a complete semilattice $L$ envisioned as a meet-semilattice, then $L$ is an $\inf$-subobject of $H$ since $\{\downarrow a \mid a \in L\} \hookrightarrow \mathcal{DI}(L)$ also preserves arbitrary infima. Recalling that the MacNeille completion of any poset $L$ consists of closing its principal ideals $\{\downarrow a \mid a \in L\}$ under intersections (Banaschewski and Bruns 1967), one verifies that, up to an isomorphism, the inclusion of a meet-semilattice $L$ in its injective hull factors in (i) the MacNeille completion $\bar{L}$ of $L$ and (ii) the inclusion of $\bar{L}$ in its injective hull with respect to complete lattices and $\inf$-morphisms. For our purpose this $\inf$-restriction suffices. However, for the sceptics concerning the existence of arbitrary infima in property lattices we stress that everything also applies both to meet-semilattices and meet-morphisms and to complete lattices and meet-morphisms. One then gets for free that the embedding of a complete

\[\text{Note:} \quad \text{It was noted in Stubbe (2000), that this fact can be seen as a particular incarnation of the Yoneda embedding which always preserves all limits that happen to exist.}\]
lattice in its injective hull is always an inf-morphism. For obvious reasons we will refer to all these equivalent injective hulls as the distributive hull of a complete lattice. We conclude this section with an example.

Lemma 1. The following are equivalent for \( A \subseteq L \):

(i) \( b \leq \bigvee A \Rightarrow b = \bigvee_{a \in A} (b \wedge a) \);
(ii) \( \forall b \in L : b \wedge \bigvee A = \bigvee_{a \in A} (b \wedge a) \).

Proof: (i) \( \Rightarrow \) (ii): We always have \( b \wedge \bigvee A \geq \bigvee_{a \in A} (b \wedge a) \) and from \( b \wedge \bigvee A \leq \bigvee A \) follows by (i) that \( b \wedge \bigvee A = \bigvee_{a \in A} ((b \wedge \bigvee A) \wedge a) \leq \bigvee_{a \in A} (b \wedge a) \). (ii) \( \Rightarrow \) (i): If \( b \leq \bigvee A \), then \( b = b \wedge \bigvee A \) so \( b = \bigvee_{a \in A} (b \wedge a) \).

Example:

If \( L \) is a complete atomistic lattice with \( \Sigma \) as atoms then \( \bigvee \{ p \in \Sigma | p \leq a \} \) is distributive. Indeed, (formally) setting \( \mu(a) := \{ p \in \Sigma | p \leq a \} \) we have \( a = \bigvee \mu(a) \) and \( a \leq b \iff \mu(a) \subseteq \mu(b) \), so \( b \leq \bigvee \mu(a) \) implies \( b = \bigvee \mu(b) = \bigvee_{p \in \mu(b)} (b \wedge p) = \bigvee_{p \in \mu(a)} (b \wedge p) \). Moreover, for the distributive hull \( H \) of \( L \) we have \( H \cong \mathcal{P}(\Sigma) \), i.e., the distributive hull of a complete atomistic lattice is a complete atomistic Boolean algebra. Indeed, in terms of \( DI(L) \), consider

\[
\theta : DI(L) \to \mathcal{P}(\Sigma) : A \mapsto A \cap \Sigma
\]

\[
\theta^{-1} : \mathcal{P}(\Sigma) \to DI(L) : T \mapsto \{ a \in L | \mu(a) \subseteq T \}
\]

Since \( \mu(a) \subseteq A \) implies \( a = \bigvee \mu(a) \in A \) we have \( a \in A \iff \mu(a) \subseteq A \iff \mu(a) \subseteq A \cap \Sigma \) and thus \( \theta^{-1}(\theta(A)) = \{ a \in L | \mu(a) \subseteq A \cap \Sigma \} = A \). From \( p \in T \iff \mu(p) \subseteq T \iff p \in \{ a \in L | \mu(a) \subseteq T \} \) follows \( \theta(\theta^{-1}(T)) = \Sigma \cap \{ a \in L | \mu(a) \subseteq T \} = T \). Thus \( \theta \) and \( \theta^{-1} \) are inverse, and since they are isotone they define an isomorphism.

3. MAIN RESULTS

This section constitutes the main argument and constructions.

3.1. Complete lattices and operational resolution

Let us denote the subsets of \( L \) that have a distributive join as \( D(L) \). In the next proposition we investigate how the existence of superpositions relates to distributivity.
Proposition 3. For $A \subseteq L$ we have

$$S_{\vec{a}}(A) = \emptyset \Rightarrow A \in \mathcal{D}(L).$$

Assuming superpositional faithfulness of $L$ we moreover have

$$A \in \mathcal{D}(L) \Rightarrow S_{\vec{a}}(A) = \emptyset.$$

Proof: Let $c \leq \bigvee A$. From $c \geq \bigvee_{a \in A} (c \land a)$ it follows that $\mu(c) \supseteq \mu \left( \bigvee_{a \in A} (c \land a) \right)$. Since $p \in \mu(c)$ implies existence of $a \in A$ such that $p \in \mu(a)$ and thus $p \in \mu(c \land a)$ so $p \in \mu \left( \bigvee_{a \in A} (c \land a) \right)$, it follows that $\mu(c) \subseteq \mu \left( \bigvee_{a \in A} (c \land a) \right)$. By injectivity of $\mu$ this results in $c = \bigvee_{a \in A} (c \land a)$, and by Lemma 3 this completes the proof of the first claim. For a proof of the second statement see Proposition 5 in Section 4.1, (i) $\Rightarrow$ (iii). $\square$

Thus, under the assumption of superpositional faithfulness of $L$ disjunctivity and distributivity of properties coincides. This justifies the point of view that, using the Bruns-Lakser results:

- The inclusion of a property lattice $L$ in its distributive hull $H$ adds to the property lattice all propositions that express disjunctions of properties. Indeed, given $A \subseteq L$, then $\bigvee_H A$ expresses this disjunction since all suprema in a complete Heyting algebra are distributive.

- It does this in a non-redundant way. Indeed, by the implicit characterization it follows that (i) existing disjunctions are preserved, and (ii) any other element $a \in H$ indeed expresses a disjunction of properties, namely that of $\{b \in L \mid b \leq a\}$.

- This embedding preserves (i) all infima of properties, i.e., all conjunctions, (ii) the trivial and (iii) the absurd, since $L \hookrightarrow H$ is a balanced $inf$-embedding.

Thus we have embedded $L$ in a logic of propositions that goes equipped with a pseudo-complementation and internal implication arrow that satisfies the same rules of definition and inference of intuitionistic logic. Moreover, as we will see below, conjunctivity and disjunctivity for properties will lift to conjunctivity and disjunctivity for propositions such that we indeed have embedded the property lattice in a true intuitionistic logic which as such goes equipped with an intuitionistic negation and implication. However, the inclusion of $L$ provides $H$
with an additional operation, namely a normalized closure
\[ R : H \to H : a \mapsto \bigwedge H \{ b \in L \mid b \geq a \} = \bigwedge L \{ b \in L \mid b \geq \bigvee H \{ c \in L \mid c \leq a \} \} = \bigwedge L \{ b \in L \mid \forall d \in \{ c \in L \mid c \leq a \} : b \geq d \} = \bigvee L \{ c \in L \mid c \leq a \} \]
referred to as the operational resolution, an operation which recuperates the logical structure of properties on the level of propositions.

This operational resolution can indeed be seen as a domain extension and codomain restriction up to isomorphism of the operational resolution in Coecke and Stubbe (1999a,b) discussed above, in the sense that it assigns the strongest property implied by a proposition, i.e., in terms of distributive ideals, implied by an actuality set. The explicit characterization of distributive ideals will indeed enable us to envision the above in terms of actuality sets. First note that \( A \subseteq L \), as an actuality set, is equivalent both to the implicative closure \( \downarrow[A] \) of \( A \) and disjunctive closure \( \{ \bigvee L B \mid B \subseteq A \cap \mathcal{D}(L) \} \) of \( A \), respectively because of the implicative significance of the \( L \)-ordering and disjunctivity of distributive \( L \)-suprema. It makes as such sense to consider the distributive ideals \( \mathcal{D}(L) \) as the suprema, with respect to inclusion, of equivalence classes of actuality sets for the following relation: Since \( \mathcal{D}(L) \) is closed under intersections we can define the closure
\[ C : \mathcal{P}(L) \to \mathcal{P}(L) : A \mapsto \bigcap \{ B \in \mathcal{D}(L) \mid B \supseteq A \} \]
and an equivalence relation \( \sim \subseteq \mathcal{P}(L) \times \mathcal{P}(L) \) by \( A \sim B \iff C(A) = C(B) \). The logical connectives on propositions then translate into a logic of actuality sets:
\[
\begin{align*}
\bigwedge_{\mathcal{D}(L)} & : \mathcal{P}(\mathcal{D}(L)) \to \mathcal{D}(L) : A \mapsto \bigcap A \\
\bigvee_{\mathcal{D}(L)} & : \mathcal{P}(\mathcal{D}(L)) \to \mathcal{D}(L) : A \mapsto C \left( \bigcup A \right) \\
\Rightarrow_{\mathcal{D}(L)} & : \mathcal{D}(L) \times \mathcal{D}(L) \to \mathcal{D}(L) : (B,C) \mapsto \bigvee_{\mathcal{D}(L)} \{ A \in \mathcal{D}(L) \mid A \cap B \subseteq C \} = \{ a \in L \mid \forall b \in B : a \wedge b \in C \} \\
\neg_{\mathcal{D}(L)} & : \mathcal{D}(L) \to \mathcal{D}(L) : A \mapsto (A \Rightarrow \downarrow 0) \\
R_{\mathcal{D}(L)} & : \mathcal{D}(L) \to \mathcal{D}(L) : A \mapsto \downarrow \left( \bigvee L A \right)
\end{align*}
\]
For $A \subseteq DI(L)$ we then have that $\bigvee_{DI(L)} A = C(\bigcup A)$ is an actuality set if and only if at least one $a \in \bigcup A$ is actual, this since all elements $\bigvee_L B$ in the disjunctive closure are distributive, and as such if and only if at least one $A \in A$ is an actuality set. Thus, $\bigvee_{DI(L)}$ is disjunctive, hence $\bigvee_H$ also. If $\bigwedge_{DI(L)} A$ is an actuality set then at least one $a \in \bigcap A$ is actual so all $A \in A$ are actuality sets. Conversely, if all $A \in A$ are actuality sets then for all $A \in A$ at least one $a_A \in A$ is actual such that $\bigwedge_L \{a_A | A \in A\}$ is actual and thus $\downarrow \left( \bigwedge_L \{a_A | A \in A\} \right) = \bigcap \{ \downarrow a_A | A \in A \} \subseteq \bigcap A = \bigwedge_{DI(L)} A$ is an actuality set. Thus, $\bigwedge_{DI(L)}$ is conjunctive, and as such $\bigwedge_H$ also. This then proves the claim made above that we have embedded $L$ in a true intuitionistic logic.

We are now at the point to understand what an implication arrow on properties should be.\footnote{We refer to the appendix at the end of this paper for other attempts which did not succeed in capturing what we conceive as the true nature of implication.} It canonically turns out to be an external operation

$$\Rightarrow_L : L \times L \to \mathcal{P}(L) : (b, c) \mapsto \{ a \in L | a \land b \leq c \}$$

obtained by restricting $\Rightarrow_{DI(L)}$. If and only $L$ is itself a complete Heyting algebra, then we can represent this external operation faithfully as an internal one by setting $(b \Rightarrow c) := \bigvee_L (b \Rightarrow_L c)$. In particular our external implication arrow can be defined by

$$a \land b \leq c \iff a \in (b \Rightarrow_L c),$$

as such in a more explicit manner expressing that it generalizes the implication that lives on a complete Heyting algebra where $a \in (b \Rightarrow c)$ then coincides with $a \leq (b \Rightarrow c)$. The set $(b \Rightarrow_L c)$ is then indeed the set of properties whose actuality makes the deduction “if $b$ is actual then $c$ is actual” true, and this is exactly the transcription in terms of actuality of the minimal requirement of any functional formal implication with respect to extensional quantification over the state set, i.e., given $a \in (b \Rightarrow_L c)$, then $\forall p \in \mu(a) : p \in \mu(b)$ implies $p \in \mu(c)$.

An inf-subobject $L$ of $H$ is distributive join dense in $H$, denoted as $\mathcal{DJD}$, if it is join dense in $H$ and if $\forall a \in H : \{ b \in L | b \leq a \} \in \mathcal{DIL}(L)$. A closure $\mathcal{F} : H \to H$ is $\mathcal{DJD}$ if $\mathcal{F}(H)$ is $\mathcal{DJD}$ as an inf-subobject of $H$. Note that since $\emptyset \notin \mathcal{DIL}(L)$ the inf-subobject inclusion is balanced so a $\mathcal{DJD}$-closure is always normalized. Referring back to the implicit Bruns-Lakser characterization of distributive hulls of the previous section, the requirement $\forall a \in H : \{ b \in L | b \leq a \} \in \mathcal{DIL}(L)$ is
equivalent to the inclusion $\mathcal{F}(H) \hookrightarrow H$ preserving existing distributive joins. Indeed, given that for $A \in \mathcal{D}(\mathcal{F}(H))$ we have $\bigvee_{\mathcal{F}(H)} A = \bigvee_H A$, or equivalently $\bigvee_{\mathcal{F}(H)} A \leq \bigvee_H A$, then $A \subseteq \{ b \in \mathcal{F}(H) | b \leq a \}$ implies $\bigvee_{\mathcal{F}(H)} A \leq \bigvee_H A \leq a$. Conversely, since $\{ b \in \mathcal{F}(H) | b \leq \bigvee_H A \} \in \mathcal{D}\mathcal{I}(\mathcal{F}(H))$ implies $\bigvee_{\mathcal{F}(H)} A \in \{ b \in \mathcal{F}(H) | b \leq \bigvee_H A \}$ it follows that $\bigvee_{\mathcal{F}(H)} A \leq \bigvee_H A$. We are now in a position to summarize the above within the following definition:

Definition 1. By the “intuitionistic or disjunctive representation of quantum logic” we refer to bijective correspondence between isomorphism classes of
(i) complete lattices, denoted $\text{CLat}$, and,
(ii) complete Heyting algebras equipped with a $\mathcal{D}\mathcal{J}\mathcal{D}$-closure, denoted $\text{DJDHeyt}$, which is realized by the following equivalence:

$$
\begin{align*}
\theta & : \text{CLat} \rightarrow \text{DJDHeyt} : L \mapsto (\mathcal{D}\mathcal{I}(L), \mathcal{R}_{\mathcal{D}\mathcal{I}(L)}) \\
\theta^* & : \text{DJDHeyt} \rightarrow \text{CLat} : (H, \mathcal{F}) \mapsto \mathcal{F}(H)
\end{align*}
$$

Given a complete lattice of “properties”, the $\mathcal{D}\mathcal{J}\mathcal{D}$-closure operator that arises on the complete Heyting algebra of “propositions”, is called the “operational resolution”. It assigns to a proposition the strongest property implied by it.

Example:

Isomorphism classes of complete atomistic lattices and complete atomistic Boolean algebras equipped with a $\mathcal{D}\mathcal{J}\mathcal{D}$ closure are in bijective correspondence. Indeed, if $L$ is atomistic then $\mathcal{D}\mathcal{I}(L) \cong \mathcal{P}(\Sigma)$ and by the Lindenbaum-Tarski theorem these are exactly complete atomistic Boolean algebras. Conversely, if $H \cong \mathcal{P}(\Sigma)$ then $\mathcal{D}\mathcal{J}\mathcal{D}$ requires $p \in \mathcal{F}(H)$, so $\mathcal{F}$ is a $T_1$ closure, i.e., all points are closed, and thus $\mathcal{F}(H)$ is atomistic via the bijective correspondence of isomorphism classes of $T_1$-closure spaces and complete atomistic lattices. Note here that both for orthodox Hilbert space quantum mechanics and phase space classical mechanics atomisticity is an axiom so this example covers essentially the primitive\footnote{\textsuperscript{18}By primitive we refer to the fact that for topological, probabilistic or other reasons one might consider restrictions of this primitive complete atomistic setting that are not complete or not atomistic anymore, as we will discuss in Section 4.2 of this paper.} situations presently encountered in orthodox physical theories. However, since atomisticity cannot be motivated within the operational methodology as

\footnote{An isomorphism between complete Heyting algebras $H_1$ and $H_2$ equipped with respective $\mathcal{D}\mathcal{J}\mathcal{D}$-closures $\mathcal{F}_1$ and $\mathcal{F}_2$ is an order-isomorphism $h : H_1 \rightarrow H_2$ such that $\mathcal{F}_2 \circ h = h \circ \mathcal{F}_1$.}
it is applied in this paper, it shouldn’t play a role in any derivation or construction, and may only be injected at the end as a particular feature of these paradigm examples whenever one wants to consider them explicitly. Moreover, as we mentioned above and will discuss below, there are indeed situations where atomisticity is not the case.

3.2 Complete ortholattices and operational complementation.

We will now go back to our initial goal of merging disjunctive suprema and orthocomplementation within one structure, as such eliminating the Birkhoff-von Neumann dilemma. Note that at this point we do not attribute a particular operational or physical significance to this orthocomplementation. We will just refine the results above in case that there is a given one, as it is the case for orthodox quantum theory.

A complete lattice \( L \) is a complete ortholattice if it goes equipped with an orthocomplementation \( ' : L \to L \), i.e., an operation that satisfies:

- **OC1:** \( a \land a' = 0 \)
- **OC2:** \( a \leq a'' \)
- **OC3:** \( a \leq b \iff b' \leq a' \).

It is a complete pseudo-ortholattice if \( ' : L \to L \) satisfies \( \text{OC1, OC2} \) and

- **OC3:** \( a \leq b \Rightarrow b' \leq a' \).

It is a complete \( DJD \)-pseudo-ortholattice if in addition the range of the orthocomplementation is \( DJD \). Isomorphisms of complete ortholattices and complete pseudo-ortholattices are then obviously those order-isomorphisms that preserve the orthocomplementation.

**Proposition 4.** We have the following for the above axioms:

(i) \([ \text{OC2, OC3} ]\) is equivalent to \([ a = a'' , \ a \leq b \iff b' \leq a' ]\),

(ii) \([ \text{OC2, OC3r} ]\) implies \([ a' = a''' , \ a' \leq b' \iff b'' \leq a'' ]\),

(iii) \([ a' \land a'' = 0 , \ \text{OC2} ]\) implies OC1.

**Proof:**

(i): From \( a' \leq a'' \) follows \( a \geq a'' \) by OC3; \( a \leq b \Rightarrow a'' \leq b' \Rightarrow b' \leq a' \).

(ii): \( a' \leq b' \iff b'' \leq a'' \) by OC2 and OC3r, the rest by applying (i) to \( a' \) and \( b' \).

(iii): \( a \land a' \leq a'' \land a' \) by OC2, so \( a \land a' \leq 0 \).

\( \square \)
Theorem 1. Isomorphism classes of
(i) complete ortholattices, denoted $\text{COLat}$, and,
(ii) complete $\mathcal{DJ}$-pseudo-ortho Heyting algebras, denoted $\text{DJDOHeyt}$,
are in bijective correspondence via the equivalence

$\theta : \text{COLat} \to \text{DJDOHeyt} : (L, \top) \mapsto (\mathcal{D}\mathcal{I}(L), \perp)$

$\theta^* : \text{DJDOHeyt} \to \text{COLat} : (H, \top) \mapsto (H', \top_{H'})$

where the “operational complementation” is defined as

$\perp : \mathcal{D}\mathcal{I}(L) \to \mathcal{D}\mathcal{I}(L) : A \mapsto \downarrow(\bigvee L)'.$

In terms of inclusion of $L$ into its distributive hull $H$ this translates as

$\perp : H \to H : a \mapsto R(a)',$

so the operational complementation has the operational resolution as its square, establishing the operational complementation as a refinement of the latter.

Proof: Consider $(H, \top) \in \text{DJDOHeyt}$. Then $a'' = (a'')''$ via Proposition 3 (ii) and $a \leq b \Rightarrow a'' \leq b''$ via twice OC3, assure $'' : H \to H$ to be a closure. Moreover, since $a' = (a'')'' \in H''$ and $a'' = (a')'$ it follows that $H' = H''$ so $H'$ is a complete lattice. By Proposition 3 (ii) we also have $(a') \leq (b') \Leftrightarrow (b')' \leq (a')'$, so the domain restriction $'_{H'}$ of $'$ to $H'$ defines an orthocomplementation on $H'$, and thus $\theta^{-1}$ is well defined. By $H' = H''$ it also follows that $''$ is a $\mathcal{DJ}$-closure on $H$ which we will denote by $F$ — ref. Definition 1. For $(L, \top) \in \text{COLat}$ we have that $A^{\perp \perp} = \downarrow(V_L(\downarrow(V_L A)'))' = \downarrow(V_L A)'' = R_{\mathcal{D}\mathcal{I}(L)}(A)$ so $\perp \perp$ is a closure and thus $A \subseteq A^{\perp \perp}$. Since moreover $\bigvee_L(-)$ and $\downarrow(-)$ are isotone and $'$ is antitone $A \subseteq B \Rightarrow B^\perp \subseteq A^\perp$. Thirdly, $A^\perp \wedge B^{\perp \perp} = \downarrow(V_L A)' \wedge \downarrow(V_L A) = \downarrow((V_L A)' \wedge (V_L A)) = \downarrow[0]$ by OC1 for $L$, so $\perp$ is OC1 on $\mathcal{D}\mathcal{I}(L)^{\perp}$ and thus by Proposition 3 (iii) on $\mathcal{D}\mathcal{I}(L)$, assuring $\perp$ to be a pseudo-orthocomplementation. Since $\perp \perp = R_{\mathcal{D}\mathcal{I}(L)}$ we have $\mathcal{D}\mathcal{I}(L)^{\perp} \supseteq \downarrow[L]$ so the inclusion of the range of $\perp$ in $\downarrow(-)$ forces $\mathcal{D}\mathcal{I}(L)^{\perp} = \downarrow[L]$. Thus $\perp$ is $\mathcal{DJ}$ so $\theta$ is well defined. By the above it then follows that $\theta^{-1}(\theta(L, \top)) = (\downarrow[L], \perp_{\downarrow[L]}) \cong (L, \top)$. One also straightforwardly verifies that $\theta(\theta^{-1}(H, \top)) = (\mathcal{D}\mathcal{I}(F(L)), \downarrow_{F(H)}(\bigvee_{F(H)} A)') \cong (H, \top)$ what completes the proof.

Note that in terms of the orthogonality relation $a \perp b \Leftrightarrow a \leq b'$ induced by the orthocomplementation we have $A^{\perp} = \{b \in L \mid \forall a \in A : a \perp b\}$. Although in
the strict operational methodology it is possible to motivate the existence of an orthogonality relation, the fact that every property in the property lattice can be written as the supremum of a biorthogonally closed subset of the lattice, i.e., the orthogonality relation realizes an orthocomplementation, is taken as an axiom (Jauch and Piron 1969, Aerts 1982, Moore 1999). One could wonder whether orthocomplementation, and consequently, $DJD$-pseudo-orthocomplementation of the distributive hull, can be obtained in a canonical manner, without having to assume it.

**Question 1.** Given a complete lattice $L$ equipped with an orthogonality relation $\perp \subseteq L \times L$, does there exist an elegant characterization of a minimal extension of $(L, \perp)$ as a $DJD$-pseudo-ortho Heyting algebras sensu the role in this paper of distributive hulls in the category of complete lattices, and can this be translated in terms of a faithful representation sensu Definition \[\text{and Theorem } \Box]?\]

### 4. FURTHER ANALYSIS AND OPEN PROBLEMS

We discuss remaining loose ends and further research.

#### 4.1. Characterization of disjunctivity

We will now characterize the correspondence between disjunctivity and distributivity in the strict operational methodology, i.e., when a Cartan map $\mu : L \to \mathcal{P}(\Sigma)$ is explicitly given. Set $S(p) := \bigwedge \{a \in L | p \in \mu(a)\}$ and write $a < b$ if $a < b$ and $c < b \Rightarrow c \leq a$ for all $c$.

**Proposition 5.** The following are equivalent:

1. $p \in S(A)$ \implies \exists c_p \in L_A : p \in \mu(c_p)$;
2. $L_A = \emptyset \Rightarrow S(A) = \emptyset$;
3. $A \in \mathcal{D}(L) \Rightarrow S_A = \emptyset$;
4. $A \in \mathcal{D}(L) \Rightarrow [S(p) = \bigvee A \Rightarrow p \notin S(A)]$;
5. $S(p) = \bigvee A \Rightarrow S(p) \in A$;
6. For $p \in \Sigma$ we either have: 
   - $\downarrow S(p) \setminus \{S(p)\} = \downarrow a$ for $0 < a \ll S(p)$;
   - $S(p)$ is an atom of $L$.

**Proof:** We proceed by proving (v)\implies(iv)\implies(iii)\implies(i)\implies(ii)\implies(v) and (v)\iff(vi), where (v)\implies(iv) and (i)\implies(ii) are both trivial.
Let $A \in \mathcal{D}(L)$ and $p \in S_\oplus(A)$. Since $S(p) \leq \bigvee A$ it follows that $S(p) = S(p) \land \bigvee A = \bigvee_{a \in A}(S(p) \land a)$. Next, since $b \land \bigvee_{a \in A}(S(p) \land a) = b \land S(p) \land \bigvee A = \bigvee_{a \in A}(b \land S(p) \land a)$ we have $\{S(p) \land a | a \in A\} \in \mathcal{D}(L)$. However, $p \notin \mu(a)$ for $a \in A$, so $p \notin \mu(S(p) \land a)$, what results in $p \in S_\oplus(\{S(p) \land a | a \in A\})$ and this conflicts with (iv).

(iii)$\Rightarrow$(i): There are two possibilities for $p \in S_\oplus(A)$: 1. $S(p) = \bigvee A$: Since $p \in S_\oplus(A)$ we have $S(p) \notin A$ so $A \subseteq \downarrow S(p) \setminus \{S(p)\}$ and thus $S(p) = \bigvee (\downarrow S(p) \setminus \{S(p)\})$. We claim that $\downarrow S(p) \setminus \{S(p)\} \in \mathcal{D}(L)$. Indeed, following Lemma [1] it suffices that $c \leq S(p)$ implies $c = c \land \bigvee (\downarrow S(p) \setminus \{S(p)\}) = \bigvee \{c \land a | a < S(p)\}$ what is the case. By (iii) we obtain $S_\oplus(\downarrow S(p) \setminus \{S(p)\}) \neq \emptyset$ what contradicts with $p \in \mu(S(p))$ and $p \notin \mu(a)$ for $a < S(p)$. 2. $0 < S(p) < \bigvee A$: It suffices to set $c_p := S(p) \in L_\oplus(A)$ since $p \in \mu(S(p))$.

(ii)$\Rightarrow$(v): If for some $p \in \Sigma$ we have $\bigvee (\downarrow S(p) \setminus \{S(p)\}) = S(p)$ then $p \in S_\oplus(\downarrow S(p) \setminus \{S(p)\}) \neq \emptyset$. However, $L_\oplus(\downarrow S(p) \setminus \{S(p)\}) = \emptyset$ since $a < \bigvee (\downarrow S(p) \setminus \{S(p)\})$ implies $a \in \downarrow S(p) \setminus \{S(p)\}$, so $\bigvee (\downarrow S(p) \setminus \{S(p)\}) < S(p)$ by (ii). Thus $A \subseteq \downarrow S(p) \setminus \{S(p)\}$ implies $\bigvee A < S(p)$, so $\bigvee A = S(p)$ implies $A \subseteq \downarrow S(p) \setminus \{S(p)\}$. Since $A \subseteq \downarrow S(p)$ we obtain $S(p) \in A$.

(v)$\Rightarrow$(vi) Above we proved $\bigvee (\downarrow S(p) \setminus \{S(p)\}) < S(p)$. It thus follows that $\downarrow S(p) \setminus \{S(p)\} \Rightarrow \bigvee (\downarrow S(p) \setminus \{S(p)\}) < S(p)$ where $\bigvee (\downarrow S(p) \setminus \{S(p)\}) < S(p)$. The converse is easily verified. □

The first of these conditions is what we defined as superpositional faithfulness, in the sense of: “the property lattice fully reflects the system behavior in terms of superpositions”. Actually, this condition is implicit in everything that has been done up to date in quantum logic since to the current authors’ knowledge no construction that explicitly uses distinct state sets and property lattices as primitive objects have been considered (except then for states being measures on the property lattice, but then the concept of state is not primitive). In the operational methodology one initially takes this into account, but then “kills” the distinction with the axiom that states encode as a join dense set of atoms of the property lattice. The first explicit constructions probably are those that can be found in Coecke and Stubbe (1999a,b) and Coecke (2000). The second condition shows that this superpositional faithfulness can be formulated in a slightly weaker fashion. The third condition identifies distributivity and disjunctivity, and condition four to six constitute a stepwise characterization of the above in terms of the properties $\{S(p) | p \in \Sigma\}$. We already mentioned that when for an atomistic lattice the atoms are envisioned as states we do have superpositional
faithfulness. Since then \( S(p) := p \), this corresponds in the above proposition with the case where (vi).1 is excluded in (vi). We indeed have the following:\[\]

Proposition 6. Given a Cartan map \( \mu : L \to \mathcal{P}(\Sigma) \), the set \( \{S(p) \mid p \in \Sigma\} \) is join dense in \( L \), i.e., \( a = \bigvee \{S(p) \mid p \in \mu(a)\} \) for all \( a \in L \).

Proof: We have \( S(p) \leq a \iff \bigcap\{\mu(c) \mid c \in L, p \in \mu(c)\} = \mu(S(p)) \leq \mu(a) \iff p \in \mu(a) \) since \( \mu \) is an injective \( \inf \)-morphism. Thus we have \( a \geq \bigvee \{S(p) \mid p \in \mu(a)\} \), and less or equal saturates into an equality since \( a > b = \bigvee \{S(p) \mid p \in \mu(a)\} \) both implies \( \mu(a) \supset \mu(b) \) and \( p \in \mu(a) \Rightarrow b \geq S(p) \Rightarrow p \in \mu(b) \), i.e., \( \mu(a) \subseteq \mu(b) \). \( \square \)

An example radically different from the atomistic one is a completely ordered set \( L \), where we set \( \Sigma := L \setminus \{0\} \) and \( \mu : L \to \mathcal{P}(L \setminus \{0\}) \) : \( 0 \mapsto \emptyset \); \( a(\neq 0) \mapsto [0,a] \).

It is clear that not all complete lattices admit a realization as a property lattice equipped with Cartan map that is superpositionally faithful. However, below we will motivate that in view of certain topological considerations a much larger class of complete lattices than the one that one might expect from the results above admits a meaningful distributive hull in the sense of disjunctive completion. Still, even within the setting of this section the distributive hull of any complete lattice provides a “lower bound” for the disjunctive hull, there where an obvious “upper bound” is downset completion, i.e., any extension \( \bar{H} \) of \( L \) isomorphic to

\[\mathcal{I}(L) := \{\downarrow A \mid A \subseteq L\}\]

that makes the following diagram commute

\[
\begin{array}{ccc}
L & \hookrightarrow & \bar{H} \\
\downarrow(-) \searrow & & \nearrow \cong \\
\mathcal{I}(L) & & \\
\end{array}
\]

Indeed, within the context of the strict operational methodology it makes sense to investigate what characterizes the disjunctive hull given an arbitrary \( \mu : L \to \mathcal{P}(\Sigma) \) which not necessarily satisfies superpositional faithfullness. As it is the case for \( D\mathcal{I}(L) \), \( \mathcal{I}(L) \) is a complete Heyting algebra being closed under unions and intersections and as such inheriting distributivity from \( \mathcal{P}(L) \). Since disjunctions are in bijective correspondence with unions of \( \mu[L] \) elements it is

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\[19\]See Moore (1999) for more details on this and related matters.
clear that the disjunctive hull is in general any extension $H_\mu$ of $L$ isomorphic to
\[ D_\mu(L) := \left\{ \bigcup \mu[A] \mid A \subseteq L \right\}. \]
that makes the following diagram commute
\[
\begin{array}{ccc}
L & \hookrightarrow & H_\mu \\
\mu & \searrow & \nearrow \\
& D_\mu(L) &
\end{array}
\]

**Proposition 7.** For every Cartan map $\mu : L \to \mathcal{P}(\Sigma)$ there exist two balanced inf-embeddings $\varphi_\mu : D\mathcal{I}(L) \to D_\mu(L)$ and $\varepsilon_\mu : D_\mu(L) \to \mathcal{I}(L)$. Moreover, given a complete lattice $L$ there exists a Cartan map $\mu : L \to \mathcal{P}(\Sigma)$ that realizes $\mathcal{I}(L)$ as disjunctive hull, i.e., such that $D_\mu(L) \cong \mathcal{I}(L)$. However, this is in general not the case for $D\mathcal{I}(L)$.

**Proof:** First, note that $D_\mu(L)$ is a complete Heyting algebra. Indeed, we have $\bigcup_{i \in I} (\bigcup_{i \in I} A_i) = \bigcup_{i \in I} \mu[A_i]$ and by complete distributivity of $\mathcal{P}(\Sigma)$ we moreover have $\bigcap_{i \in I} (\bigcup_{i \in I} A_i) = \bigcup_{(x_i)_{i \in I} \in X} (\bigcap_{i \in I} \mu(x_i)) = \bigcup_{(x_i)_{i \in I} \in X} \mu(\bigwedge_{i \in I} x_i)$ where $X = \{(a_i)_{i \in I} \mid a_i \in A_i\}$. Thus, it follows that $D_\mu(L)$ is closed under all unions and intersections and as such inherits distributivity from $\mathcal{P}(\Sigma)$. As such, existence of $\varphi_\mu$ is guaranteed since $D\mathcal{I}(L)$ is a distributive hull of $L$ and the codomain restriction of the Cartan map $\mu : L \to \mathcal{P}(\Sigma)$ to $D_\mu(L)$ defines an inf-inclusion of $L$ in an injective object. Next, set $\varepsilon_\mu : T \mapsto \{ a \in L \mid \mu(a) \subseteq T \}$. Since $a \in \varepsilon_\mu(T) \iff \mu(a) \subseteq T$ it follows that $a \in \bigcap_{T \in \mathcal{T}} \varepsilon_\mu(T) \iff \forall T \in \mathcal{T} : a \in \varepsilon_\mu(T) \iff \forall T \in \mathcal{T} : \mu(a) \subseteq T \iff \mu(a) \subseteq \bigcap \mathcal{T} \iff a \in \varepsilon_\mu(\bigcap \mathcal{T})$, so this inclusion $\varepsilon_\mu$ preserves intersections, i.e., infima. Given an arbitrary complete lattice $L$, setting $\mu : L \to \mathcal{P}(L \setminus \{0\}) : 0 \mapsto \emptyset : a(\neq 0) \mapsto ]0,a]$ we clearly realize $\mathcal{I}(L) \cong D_\mu(L)$ via $A \leftrightarrow A \setminus \{0\} : \{0\} \leftrightarrow \emptyset$. However, the lattice of open sets (with respect to the standard topology) of the unit interval cannot realize $D\mathcal{I}(L)$. Indeed, there are no candidates in this lattice to play the role of $S(p)$ in view of condition (vi) of Proposition 2. \[\square\]

**Question 2.** Is there some categorical property that elegantly characterizes $D_\mu(L)$ in some category with as objects Cartan maps, sensu the role in this

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\[20\text{See for example Johnstone (1982) §VII p.278–279.}\]
paper of distributive hulls in the category of complete lattices? How do the different \( D_\mu(L) \) relate for fixed \( L \) and what is the status of \( D_\nu(I) \) and \( I(L) \) for this collection/category? How do the results of Paseka (1994) on covers in generalized frames fit in this picture?

Note that in respect of the second question one can verify that two canonical choices for morphisms between \( D_\mu(L) \) and \( D_\mu'(L) \) present themself, namely an \( \text{inf} \)-morphism \( f : D_\mu(L) \to D_\mu'(L) : T \mapsto \bigcup \{ \mu'(a) \mid \mu(a) \subseteq T \} \), and a \( \text{sup} \)-morphism \( g' : D_\mu'(L) \to D_\mu(L) : T \mapsto \bigcup \{ \mu(S'(p)) \mid p \in T \} \), which prove to be adjointly related.

**Question 3.** In the above, and in particular in the proof of Proposition [?], it seems that for \( D_\nu(I) \) there is a strong connection between complete distributivity and superpositional faithfulness with respect to some Cartan map. Can this be put in a simple picture and could this provide a simplification of the presentation compared to the one in this paper?

In particular complete distributivity seems to arise when at the starting point of the construction we restrict to injective hulls of \( \text{inf} \)-lattices. This setting however requires that all infima are conjunctive, an assumption that we will drop in the next section.

### 4.2. Finitely conjunctive infima

Although in the strict sense of the operational methodology outlined earlier \( S(p) = 0 \) is excluded, this since \( \bigwedge \{ a \in L \mid p \in \mu(a) \} \) is a property that is actual in state \( p \) since all \( a \) with \( p \in \mu(a) \) are actual in \( p \), it does seem to make sense to consider property lattices where only finite meets are conjunctive in view of certain topological motivations, even within an operational setting, as such allowing \( S(p) = 0 \) whenever \( S(p) \) is the infimum of an infinitary set. This finitely conjunctive property lattice should then be envisioned as a restriction of the true property lattice. There are for example arguments in terms of affirmation vs. refutability motivating that so called finitely observational properties are restricted to open sets of states, thus proposing frames as the corresponding property lattices (Vickers 1989). Consider for example

\[
\Sigma := [0,1] \,, \quad (L := \{ T \subseteq [0,1] \mid T \text{ is open} \}, \subseteq) \,, \quad \mu(T) := T
\]

Since all suprema are unions they are all disjunctive. One should then envision this lattice as a restriction of \( \mathcal{P}([0,1]) \) where now only finite infima are to be
seen as conjunctions contrary to $P([0, 1])$ itself where all infima are conjunctions. Consequently, the map $\mu : L \rightarrow P(\Sigma)$ that assigns to properties the states in which they are actual is now a balanced \textit{meet}-embedding that also preserves the top, but which is not necessarily an \textit{inf}-embedding anymore. We will refer to these maps as \textit{weak Cartan maps}. In principle, the domain of such a weak Cartan map should not even be a complete lattice, but only a bounded meet-semilattice.

\textbf{Proposition 8.} Every weak Cartan map $\mu : L \rightarrow P(\Sigma)$ admits a conjunction preserving extension as a Cartan map, namely

$$\bar{\mu} : \bar{L}_{\mu} \rightarrow P(\Sigma)$$

where

$$\bar{L}_{\mu} \cong C_{\mu}(L) := \left\{ \bigcap \mu[A] \mid A \subseteq L \right\} ,$$

is restricted by commutation of

$$L \mu \downarrow \bar{L}_{\mu} \mu \downarrow \cong \bar{C}_{\mu}(L) .$$

\textbf{Proof:} Straightforward verification. \hfill \Box

Since the inclusion $L \hookrightarrow \bar{L}_{\mu}$ is a completion it factors over MacNeille completion $L \hookrightarrow \bar{L}$ (Banaschewski and Bruns 1967), where

$$\bar{L} \cong \left\{ \bigcap \downarrow[A] \mid A \subseteq L \right\} ,$$

again with the obvious commutation property. In general however, $\bar{L}$ and $\bar{L}_{\mu}$ do not coincide: take as a counterexample the standard topology on an interval with as states the points of the interval. Thus, completeness does not imply conjunctivity, although the converse is true.\footnote{Note that via this conjunctive completion we obtain for any weak Cartan map as such $R : P(\Sigma) \rightarrow \bar{L}_{\mu} : T \mapsto \bigcap \{ A \in \mu[L] \mid T \subseteq A \}$ as the operational resolution sensu Coecke and Stubbe (1999a,b). Deriving an operational resolution from a Cartan map indeed requires conjunctivity of all infima in the property lattice.}

One could say that the distributive hull plays the same role for disjunctive completion as MacNeille completion plays for conjunctive completion. By the above it also follows that it makes no essential difference to work either with finitary conjunctive meets or infinitary

\[26]
When evaluating superpositional faithfulness for property lattices with finitely conjunctive infima the conditions of Proposition 3 should then be evaluated on $\bar{L}_\mu$ equipped with the Cartan map $\bar{\mu}$.

As already mentioned above, a canonical interpretation of an arbitrary complete lattice as a property lattice can be realized by taking a copy of $L \setminus \{0\}$ as states with the Cartan map defined by $\mu(a) := [0, a]$, but in general this solution violates superpositional faithfulness. Clearly, besides atomistic lattices there are many examples that do allow a superpositionally faithful interpretation, in particular when generalizing to weak Cartan maps. But can we provide such an interpretation for any complete lattice, or, for any bounded meet-semilattice? So we leave the following questions open:

**Question 4.** Does any complete lattice $L$ admit an interpretation as a property lattice where the distributive hull can be interpreted as the disjunctive hull, i.e., does there exists a weak Cartan map with as extension a Cartan map that is superpositionally faithful such that the restriction of $D_{\bar{\mu}}(\bar{L})$ to $L$-disjunctions is a distributive hull of $L$?

We end by investigating which properties with respect to existing suprema are preserved in this passage from finite to infinitary conjunctions.

**Proposition 9.** For $L$ a bounded meet-semilattice and $\mu : L \to \mathcal{P}(\Sigma)$ a weak Cartan map we have the following:

(i) $\bigvee_L A = \bigvee_{L_\mu} A$ whenever $\bigvee_L A$ exists;

(ii) If $\bigvee_L A$ is disjunctive then it is distributive;

(iii) If $\bigvee_L A$ is disjunctive then $\bigvee_{L_\mu} A$ is disjunctive;

(iv) If $\bigvee_L A$ is disjunctive then $\bigvee_{L_\mu} A$ is distributive.

**Proof:** (i): Clearly, $\bigvee_L A \geq \bigvee_{L_\mu} A$. We moreover have that

$$\bigvee_{L_\mu} A = \bigwedge_{L_\mu} \{ b \in \bar{L}_\mu | \forall a \in A : a \geq b \}$$

$$\cong \bigcap_{L_\mu} \{ B \in \mathcal{C}(L) | \forall C \in \mu[A] : B \supseteq C \}$$

$$\cong \bigcap \{ B \in \bigcap \mathcal{C} | C \subseteq L \bigwedge B \supseteq \bigcup \mu[A] \}$$

$$\cong \bigcap \{ B \in \mu[L] | B \supseteq \bigcup \mu[A] \}$$

Note that the finitary representation expresses the non-primitive nature of suprema in a much stronger sense since in general they even don’t exist.

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22Note that the finitary representation expresses the non-primitive nature of suprema in a much stronger sense since in general they even don’t exist.
and that for $b \in L$, if $\forall a \in A : b \geq a$ then $b \geq \bigvee_{L} A$, so we also have $\bigvee_{L} A \geq \bigvee_{L} A$. (ii): Straightforward verification along the lines of Proposition 3. (iii): $\hat{\mu}(\bigvee_{L} A) = \mu(\bigvee_{L} A) = \mu(\bigvee_{L} A) = \bigcup \mu[A] = \bigcup \hat{\mu}[A]$. (iv): Follows from (iii) and Proposition 3.

**Question 5.** Do their exist and what are the explicit analogues of Proposition 3 and Proposition 7 for weak Cartan maps? What are the answers to the analogues of Question 4 and Question 5 for weak Cartan maps?

In this section we have briefly discussed a situation where we might have infinitary non-conjunctive infima, nor did we assume atomisticity. Their are however other situations considered in physics where with an underlying complete and even atomistic property lattice, e.g., classical or quantum physics, one chooses to consider an incomplete non-atomistic subset, for example for reasons imposed by the very nature of measure theory which forces to restrict to $\sigma$-completeness, e.g., Pták and Pulmannová (1991).

**Question 6.** To which extend do the constructions, representations, interpretations and results of this paper hold, or how should they be modified, when replacing completeness by $\sigma$-completeness or when adopting the settings of any approach within the general field of ordered quantum structures.

5. SUMMARY, CONCLUSION AND PERSPECTIVES

Since any complete lattice can canonically be embedded in a complete Heyting algebra, where this embedding itself equips the complete Heyting algebra with an additional operation, and since that whenever this complete lattice is the lattice of properties of a physical system this complete Heyting algebra encodes the logical propositions on these properties, we are tempted to claim that quantum logic should not be seen as contradicting intuitionism, but entailing a refinement of intuitionism encoded in terms of operational resolution and operational complementation. Complete Heyting algebras saturate this embedding into an isomorphism, encoding exactly those property lattices where all logical expressions involving disjunctions define themself a property of the system,
recalling here that suprema in property lattices are in general not disjunctive but introduce superpositions whereas infima are indeed conjunctive. Since the Bruns-Lakser construction for injective hulls in the category of meet-semilattices turns out to be a distributive hull, it provides a disjunctive hull for superpositionally faithful property lattices (either with respect to an ordinary or a weak Cartan map).

It is our feeling that the need to define actuality sets in order to encode emergence of disjunctions in temporal processes, e.g., measurements, is in a one to one way connected with propositions on the system’s dynamical behavior. This claim is strengthened by the fact that operational resolutions prove to be the mathematical objects that naturally go equipped with state and property transitions as morphisms (Coecke and Stubbe 1999a,b). The fact that the considerations made in this paper haven’t been made before could be connected to intrinsic static nature of what has been conceived as quantum logic. However, since these dynamical considerations formally encode in terms of categories rather than in terms of lattices and require a complementary conceptual discussion than the one in this paper, we have chosen to discuss the dynamical applications of the results of this paper in a separate paper (Coecke 2001).

Finally, the carefull (and probably also the non-carefull) reader has noticed that nowhere in the paper weak modularity plays any role. However, in Coecke and Smets (2001) the claim is made that the transition from either classical or constructive/intuitionistic logic to quantum logic entails besides the introduction of an additional unary connective operational resolution the shift from a binary connective implication to a ternary connective where two of the arguments have an ontological connotation and the third, the new one, an empirical. These ternary connectives have a fundamentally dynamic nature and have the intuitionistic ones introduced in this paper as statical limit. This second aspect of the shift from classical or constructive/intuitionistic to quantum will then be the one that requires orthomodularity of the underlying lattice of properties as a crucial feature.

APPENDIX: IMPLICATION VIA ADJUNCTION

It is the aim of this paragraph to illustrate how one proceeded in previous attempts to equip quantum logic with an implication (Hardegree 1979, Kalmbach 1983). To the present author’s opinion, this can be expressed the best in terms of adjointness between action of conjunction and left action of the implication.
(i) A pair of maps \( f : L \to M \) and \( g : M \to L \) between posets \( L \) and \( M \) are *Galois adjoint*, denoted by \( f \dashv g \), if and only if \( f(a) \leq b \iff a \leq g(b) \).

(ii) Whenever \( f \dashv g \), \( f \) preserves existing suprema and \( g \) existing infima.

(iii) For \( L \) and \( M \) complete lattices, any inf-morphism \( g : M \to L \) has a unique sup-preserving left adjoint \( g_* : a \mapsto \bigwedge \{ b \in M | a \leq g(b) \} \) and any sup-morphism \( f : L \to M \) a unique inf-preserving right adjoint \( f^* : b \mapsto \bigvee \{ a \in L | f(a) \leq b \} \).

Setting \( i : C(L) \hookrightarrow L \) and \( i_* : L \to C(L) : a \mapsto C(a) \) given a closure \( C \) on \( L \), we have for \( a \in L \) and \( b \in C(L) \) that \( a \leq b \Rightarrow C(a) \leq C(b) = b \) and thus \( i_*(a) \leq b \iff C(a) \leq b \leftrightarrow a \leq b \iff a \leq i(b) \) so \( i_* \dashv i \) where \( C = i \circ i_* \), i.e., any closure factors in a sup-endomorphism \( i_* \) and an inf-subobject inclusion \( i \). Thus, the range \( C(L) \) of a closure \( C \) on a complete lattice \( L \) is an inf-subobject of \( L \), and any inf-subobject \( M \) of \( L \) defines a closure \( C_M : L \to L : a \mapsto \bigwedge \{ b \in M | a \leq b \} \). Notice that we have \( (a \land -) \dashv (a \Rightarrow -) \) in any Heyting semialgebra, so \( a \land - \) preserves existing joins what exactly results in saying that the joins of all subsets are distributive. Conversely, if the supremum of every subset of a meet-semilattice \( H \) exists and is distributive, then \( H \) is complete by Birkhoff’s theorem, and for all \( a \in H \) the map \( a \land - : H \to H \) preserves all suprema so it has a unique right adjoint \( a \Rightarrow - : H \to H \), as such encoding \( (- \Rightarrow -) \) when viewing \( a \) as an argument. It then follows that complete Heyting algebras are complete lattices where the suprema of all subsets are distributive. Now, recalling that a complete ortholattice \( L \) is a *complete orthomodular lattice* if it is moreover weakly modular, i.e., if \( a \leq b \) implies \( a \lor (a' \land b) = b \), setting \( \varphi_a : L \to L : b \mapsto a \land (a' \lor b) \) and \( \varphi_a^* : L \to L : b \mapsto a' \lor (a \land b) \) we have \( \varphi_a \dashv \varphi_a^* \). Indeed, if \( a \land (a' \lor b) \leq c \) then \( a' \lor (a \land (a' \lor b)) \leq a' \lor (a \land c) \) where \( b \leq a' \lor b = a' \lor (a \land (a' \lor b)) \) since \( a' \leq a' \lor b \), and analogously one proves the converse. This adjunction embodies why \( \varphi_a^*(-) \) has been interpreted as an implication, since \( \varphi_a \) coincides with \( (a \land -) : L \to L \) in the case that \( L \) is distributive. This view is moreover motivated by the fact that where for a Heyting semialgebra the actions \( \{(a \land -) | a \in L\} \) can be envisioned as projections on \( a \), for orthomodular lattices the *Sasaki projections* \( \{\varphi_a | a \in L\} \) are the closed orthogonal projections in the *Baer *-semigroup of \( L$-hemimorphisms* (Foulis 1960). For the particular case of the lattice of closed subspaces of a Hilbert space the action of these Sasaki projections coincides with that of the projection operators on the corresponding closed subspaces. For details and a more general discussion on the matter we respectively refer to Kalmbach (1983) and Coecke,
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