Curvettes and clusters of infinitely near points over arbitrary fields

J.J. Moyano-Fernández

Abstract

The aim of this paper is to revise the theory of clusters of infinitely near points for arbitrary fields. In particular, we give the intersection matrix in terms of such a cluster, define the notion of curvette over an arbitrary field and prove its main properties.

1 Introduction

The theory of infinitely near points was nicely introduced in the classical treatise of Enriques and Chisini ([5]) from a purely geometrical point of view. Since then, many authors have considered its algebraic counterpart, being remarkable the works of Zariski and Lipman on the theory of complete ideals (see [15], [8], [9]). Recently, these two directions have been compiled by Casas ([3]) and Kiyek and Vicente ([7]).

The aim of this paper is to deal with some topics—not totally covered by the literature—related to the process of embedded resolution of a curve defined over an arbitrary field, in the fashion of [7]. The paper goes as follows. We recall in Section 2 the main concepts and results of the theory of regular local rings of dimension two. In Section 3 we introduce the notions of cluster of infinitely near points and proximity matrix, the latter being a useful tool to encode the proximity relations in the cluster introduced by Du Val in [14]. Such a matrix has to do with intersection relations among components of exceptional divisors created in the resolution process, as we show in Section 4; in particular, we express the intersection matrix in terms of the cluster (Proposition (4.7)). Section 5 is devoted to describe some numerical invariants concerning the resolution (the so-called characteristic

Math. Subject Class.: 13H05, 14H20. Key words: Two-dimensional regular local ring, infinitely near point, proximity, intersection matrix, curvette

Address: Institut für Mathematik, Universität Osnabrück. Albrechtstrasse 28a, D-49076 Osnabrück, Germany. E-mail: jmoyano@math.uni-osnabrueck.de. Supported partially by the grant MEC MTM2007-64704, by Junta de CyL VA065A07, and by the grant DAAD-La Caixa. The author would like to thank Prof. K. Kiyek for teaching him carefully the basics of the theory of two-dimensional regular local rings.
data) as in [13] was done for algebroid curves, and an appropriate machinery to read them off (the Hamburger-Noether tableau). Finally, Sections 6 and 7 are devoted to show the existence and main properties of curvettes (terminology introduced by Deligne in [4, p.13]; it refers to normal-crossing curves at smooth points of the exceptional divisor) by means of the Hamburger-Noether tableau (cf. (6.5), (6.6), (6.7)). The Hamburger-Noether tableau is an instance of the Hamburger-Noether algorithm proposed in [13], well-known in the study of the algebroid curves (see also [1] in case of algebraically closed fields; more general set-up can be founded in [12]). In particular, we show in our more general context that curvettes are basically the same objects as the approximations described by Russell in [13] for algebroid curves (see (7.9), (7.11)).

An important observation for the whole paper is that the ground field of the curve does not play any role in any of the reasonings we do.

2 Generalities of two-dimensional regular local rings

Along this section we will refer to the book of Kiyek and Vicente [7] as a general reference. Let \( R \) be a regular local ring of dimension two with maximal ideal \( \mathfrak{m}_R = \mathfrak{m} \) and residue field \( k_R \). Let \( \{x, y\} \) be a regular system of parameters of \( R \), and let \( K = \text{Quot}(R) \) be the field of fractions of \( R \).

(2.1) For every \( f \in R \setminus \{0\} \) we define the order function of \( f \) as

\[
\text{ord}_R(f) = \text{ord}(f) = m \text{ if } f \in \mathfrak{m}^m, \ f \notin \mathfrak{m}^{m+1}.
\]

If \( m = \text{ord}_R(f) \), then the class of \( f \) in \( \mathfrak{m}^m/\mathfrak{m}^{m+1} \), denoted by \( \text{In}(f) \), is called the initial form or the leading form of \( f \). We can also define the order of a non-zero ideal \( \mathfrak{a} \) of \( R \) to be

\[
\text{ord}_R(\mathfrak{a}) = \text{ord}(\mathfrak{a}) := \min \{\text{ord}(a) \mid a \in \mathfrak{a}\}.
\]

The canonical extension of the order function to \( K \setminus \{0\} \) gives rise to a discrete valuation of rank 1 of \( K \), which we write \( v_R = v \). This valuation is non-negative on \( R \) and has center \( \mathfrak{m} \) in \( R \). The discrete valuation ring of \( v_R \) is denoted by \( V_R = V \).

(2.2) Let \( \mathcal{R}(\mathfrak{m}, R) := \bigoplus_{n \geq 0} \mathfrak{m}^n T^n \subset R[T] \) be the Rees ring of \( R \) with respect to \( \mathfrak{m} \) for an indeterminate \( T \), and let \( \text{gr}_\mathfrak{m}(R) := \bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1} \) be the graded associated ring of \( R \). Consider the homomorphism \( \varphi : \mathcal{R}(\mathfrak{m}, R) \rightarrow \text{gr}_\mathfrak{m}(R) \). We can see immediately that \( \text{gr}_\mathfrak{m}(R) = k_R [\overline{x}, \overline{y}] \), where \( \overline{x} := x \mod \mathfrak{m}^2 \) and \( \overline{y} := y \mod \mathfrak{m}^2 \), and that \( \overline{x}, \overline{y} \) are algebraically independent over \( k_R \).

Let \( \mathbb{P}_R \) be the set of closed points of \( \text{Proj}(\text{gr}_\mathfrak{m}(R)) \) (i.e., homogeneous prime ideals of \( \text{gr}_\mathfrak{m}(R) \) of height 1). For \( p \in \mathbb{P}_R \), the ideal \( p \) is principal and generated by an irreducible homogeneous polynomial \( \overline{f} \in k_R [\overline{x}, \overline{y}] \). We set \( \deg(p) := \deg(\overline{f}) \). Let \( p = (\overline{f}) \in \mathbb{P}_R \), where \( \overline{f} \in \text{gr}_\mathfrak{m}(R) \) is homogeneous of degree \( m \), and choose
Chapter VII, (1.5), the following properties hold:

\[ \text{L} \text{factorial subrings of } \text{length} \text{of the sequence. Note that, if } (2.8) \text{ refer to } [7, \text{Chapter VII, (1.4)}. \]

If \((2.7) \text{ say that } A \text{ contains } B \text{ is the strict transform (or ideal transform) of } R \text{. Let } V \text{ valuation ring} \]

\[ \text{of length } 0. \text{ Notice that, for every } n \text{ (2.6) Let } \Omega(\mathcal{K}) \text{ be the set of all two-dimensional regular local subrings of } \mathcal{K} \text{ having } \]

\[ \text{K as field of fractions. The elements of } \Omega(\mathcal{K}) \text{ shall be called } \text{points.} \]

\[ \text{(2.4) Let } \Omega(\mathcal{K}) \text{ be the set of all two-dimensional regular local subrings of } \mathcal{K} \text{ having } \]

\[ \text{K as field of fractions. The elements of } \Omega(\mathcal{K}) \text{ shall be called } \text{points.} \]

\[ \text{(2.5) Let } R \in \Omega(\mathcal{K}) \text{. If } R' \in \Omega(\mathcal{K}) \text{ and } R' \supset R \text{, then } R' \text{ is said to be infinitely near to } R \text{. In such a case} \]

\[ \text{there exists a uniquely determined strictly increasing sequence} \]

\[ R =: R_0 \subseteq R_1 \subseteq \ldots \subseteq R_n := R', \]

\[ \text{in which } R_i \in \Omega(\mathcal{K}) \text{ and } R_i \text{ is a quadratic transform of } R_{i-1}, \text{ for every } i \in \{1, \ldots, n\}. \]

\[ \text{In particular, } R' \text{ dominates } R \text{ and the degree extension } [R' : R] := [k_{R'} : k_R] \text{ is finite (cf. [7, Chapter VII, (6.4)].} \]

\[ \text{The previous sequence is said to be the quadratic sequence between } R \text{ and } R'. \text{ The integer number } n \text{ is called the} \]

\[ \text{length of the sequence. Note that, if } R = R', \text{ then we have a quadratic sequence of length } 0. \text{ Notice that, for every } n \in \mathbb{N}, N_n(R) \text{ is the set of all } S \in \Omega(R) \text{ such} \]

\[ \text{that the sequence of quadratic transforms between } R \text{ and } S \text{ has length } n. \]

\[ \text{(2.6) Definition: Let } R' \text{ be an infinitely near point to the point } R \text{, and let} \]

\[ R =: R_0 \subseteq \ldots \subseteq R_n := R' \text{ be the quadratic sequence between } R \text{ and } R'. \text{ We say that } R' \text{ is proximate to } R \text{, and we write } R' \succ R \text{, or } R \prec R', \text{ if the discrete} \]

\[ \text{valuation ring } V_R \text{ contains } R'. \]

\[ \text{(2.7) If } A \subset B \text{ are factorial integer rings with } \text{Quot}(A) = \text{Quot}(B), \text{ then we} \]

\[ \text{associate with an ideal } a \text{ of } A \text{ different from } 0 \text{ an ideal } a^B \text{ in } B, \text{ which is called the} \]

\[ \text{strict transform (or ideal transform) of } a \text{ in } B. \text{ For the exact description, we refer to [7, Chapter VII, (1.4)].} \]

\[ \text{(2.8) Remark: Let } A \text{ be a factorial ring with quotient field } L, \text{ and let } B \subset C \text{ be factorial subrings of } L \text{ with } A \subset B. \text{ Let} \]

\[ a \text{ and } b \text{ be non-zero ideals in } A. \text{ By [7, Chapter VII, (1.5)], the following properties hold:} \]

1. \((a^B)^C = a^C; \)
2. \((ab)^B = a^B b^B\);

3. if \(a\) is a principal prime ideal, then either \(ab \cap A = a\), in which case \(a^B\) is a principal prime ideal of \(B\) with \(a^B \cap A = a\), or \(ab \cap A \neq a\), in which case we have \(a^B = B\).

(2.9) Let \(p_1, \ldots, p_s \in \mathbb{P}_R\) be pairwise different points and \(B := S_{p_1} \cap \ldots \cap S_{p_s}\), which is a factorial semilocal ring of dimension 2. For each ideal \(a\) of \(R\), we have

\[
a^{S_{p_i}} = (a^B)^{S_{p_i}} = a^B S_{p_i}, \quad 1 \leq i \leq s.
\]

In particular, \(a^B = B\) if and only if \(a^{S_{p_i}} = S_{p_i}, \quad 1 \leq i \leq s\). If the ideal \(a\) has order \(m\) and \(p \in \mathbb{P}_R\), then \(a^{S_p} = (a/m^m S_p) S_p\). When \(a = (f)\), for \(f \in R \setminus \{0\}\), then \(a^{S_p}\) is principal and any generator of the ideal \(a^{S_p}\) is called the strict transform of \(f\) in \(S_p\). Next lemma will be needed in the sequel (cf. [7, Chapter VII, (2.11)]):

(2.10) Lemma: Let be the ring \(R, p \in \mathbb{P}_R\) and \(S := S_p\). Assume \(mS = xS\). Then we have:

(i) If \(h \in R\) is irreducible and \(m := \text{ord}(h)\), then \((hR)^S = x^{-m} hS\) and then, either \(x^{-m} h\) is irreducible in \(S\) (in this case \(\text{In}(h) \in p\)), or \(x^{-m} h\) is a unit of \(S\) (and \(\text{In}(h) \notin p\)).

(ii) Let \(f, g \in R\) be irreducible and not associated. If \((fR)^S, (gR)^S\) are prime ideals of \(S\), then \((fR)^S \neq (gR)^S\) and \(xS \neq (fR)^S\).

(2.11) Notation: A curve \(E\) in \(R\) is a non-zero principal ideal \(fR\) of \(R\). The element \(f\) is uniquely determined up to units, and every generator of the ideal \(fR\) is called an equation of \(E\). If \(fR = R\), then the curve \(E\) is called empty. A curve \(E\) with equation \(f\) is called irreducible, if \(fR\) is a prime ideal of \(R\). Since \(R\) is factorial, \(fR\) is a prime ideal of \(R\) if and only if \(f\) is an irreducible element of \(R\). Let \(E\) be a non-empty curve with equation \(f\). Let \(f = f_{1}^{e_{1}} \cdots f_{r}^{e_{r}}\) be the prime decomposition of \(f\). For every \(i \in \{1, \ldots, r\}\), let \(E_i\) be the curve with equation \(f_i\). The curves \(E_1, E_2, \ldots, E_r\) are called the irreducible components of \(E\), and for every \(i \in \{1, \ldots, r\}\) \(E_i\) is called irreducible component of multiplicity \(e_i\). An irreducible component of \(E\) is called simple, if it has multiplicity 1.

(2.12) The curve defined by the ideal

\[
E_{R_1} := m_{R_0} R_1
\]

is called the exceptional divisor in \(R_1\). The curve defined by the ideal

\[
E_{R_2} := E_{R_1 R_2} \cdot (E_{R_1})^{R_2}
\]

is called the exceptional divisor in \(R_2\), where \(E_{R_1 R_2} = m_{R_1} R_2\) and \((E_{R_1})^{R_2}\) is the strict transform of \(R_1\) in \(R_2\). Similarly, the curve defined by the ideal

\[
E_{R_3} := E_{R_2 R_3} \cdot (E_{R_2 R_3})^{R_3} \cdot (E_{R_1})^{R_3} = E_{R_2 R_3} \cdot (E_{R_2})^{R_3}
\]
is called the exceptional divisor in $R_3$, where $E_{R_2R_3} = \mathfrak{m}_{R_2} R_3$ and $(E_{R_2})^{R_3}$ is the strict transform of $E_{R_2}$ in $R_3$. In general, the curve defined by the ideal

$$E_{R_i} := E_{R_{i-1}R_i} \cdot (E_{R_{i-1}})^{R_i}$$

is called the exceptional divisor in $R_i$ for every $i \in \mathbb{N}$ with $i > 1$, where $E_{R_{i-1}R_i} = \mathfrak{m}_{R_{i-1}} R_i$ and $(E_{R_{i-1}})^{R_i}$ is the strict transform of $E_{R_{i-1}}$ in $R_i$.

\textbf{(2.13) Lemma:} Let 

$$R =: R_0 \subseteq R_1 \subseteq R_2 \subseteq \ldots \subseteq R_n := S.$$ 

be the quadratic sequence between $R$ and $S$. Then, on every $R_i$ with $i \geq 2$, we have either one or two strict transforms of the exceptional divisor, and when there are two, they meet transversally at the point corresponding to the ideal $\mathfrak{m}_{R_i}$.

\textbf{Proof.} Let $\{x_0, y_0\}$ be a regular system of parameters of $R_0$. Then $\mathfrak{m}_{R_0} = (x_0, y_0)$ and consider the exceptional divisor $\mathfrak{m}_{R_0} R_1 = x_0 R_1$. We have two different cases:

\textbf{Case A:} $R_n = S$ is proximate to $R$.

It means that $R \subset S \subset V_R$ and we can choose a regular system of parameters $\{x_1, y_1\}$ of $R_1$ with $x_1 = x_0$, $v_R(y_1) = 0$ and $\mathfrak{m}_{R_1} = (x_1, y_1)$. Consider the quadratic transform $R_2$ of $R_1$. The exceptional divisor in $R_2$ has two components, namely

$$E_{R_1R_2} = y_1 R_2$$

$$E_{R_1} = y_1 y_2 R_2,$$

which meet transversally at the point corresponding to $\mathfrak{m}_{R_2} = (y_1, \frac{x_1}{y_1})$.

Consider now the transforms of the exceptional divisor $E_{R_3}$ in $R_3$:

$$E_{R_3} = \frac{x_1}{y_1} R_3$$

$$E_{R_3} = y_1 R_3.$$

The transform $(E_{R_2})^{R_3}$ is the whole ring and only the components $(E_{R_2})^{R_3}$ and $E_{R_2R_3}$ survive, and they intersect transversally at the point corresponding to the ideal $\mathfrak{m}_{R_3} = (y_1, \frac{x_1}{y_1})$. We can repeat this reasoning to show that, for $i \geq 2$, the two only components of the exceptional divisor surviving are

$$E_{R_i} = \frac{x_1}{y_1} R_i$$

$$E_{R_iR_{i+1}} = y_1 R_i,$$

which meet transversally at the point given by $\mathfrak{m}_{R_i} = (y_1, \frac{x_1}{y_1})$. 


**Case B:** \( R_n = S \) is not proximate to \( R \).

Assume we have the quadratic sequence

\[
R = R_0 \subset R_1 \subset \ldots \subset R_{h-1} \subset R_h \subset R_{h+1} \subset \ldots \subset R_n,
\]

with \( R \prec R_h \) and \( R \neq R_{h+1} \), for \( h \geq 1 \). If \( h = 1 \), then \( m_{R_0} R_1 = x_1 R_1 \). If \( h \geq 2 \), then the exceptional divisor in \( R_h \) has two components (cf. Case A); namely

\[
\begin{align*}
(E_{R_h})^{R_h} &= \frac{x_1}{y_1} R_h \\
E_{R_{h-1} R_h} &= y_1 R_h,
\end{align*}
\]

which intersect transversally at the point corresponding to \( m_{R_h} = (x_h, y_h) \), where \( x_h = y_1 \) and \( y_h = \frac{y_1}{x_h} \). We now turn to the transforms of the exceptional divisor in \( R_{h+1} \). Since \( R_{h+1} \) is not proximate to \( R \), we have two possibilities:

1) If \( E_{R_h} R_{h+1} = y_h R_{h+1} \),

Then \( m_{R_{h+1}} = \left( y_h, \frac{f(x_h, y_h)}{y_h} \right) \), where \( f \in R, \text{ord}_{R_h}(f) = l \) and \( f \mod m_{R_h}^{l+1} \) is an homogeneous polynomial of degree \( l \). The components of the exceptional divisor in \( R_{h+1} \) are

\[
\begin{align*}
(E_{R_h})^{R_{h+1}} &= (y_h R_h)^{R_{h+1}} = \left( \frac{x_1}{y_1} R_h \right)^{R_{h+1}} = R_{h+1} \\
(E_{R_h})^{R_{h+1}} &= (x_h R_h)^{R_{h+1}} = \frac{y_h}{y_h} R_{h+1} \\
E_{R_h, R_{h+1}} &= y_h R_{h+1}.
\end{align*}
\]

Taking into account the form of \( f(x_h, y_h) \), we have

i) If \( (f(x_h, y_h)) = (x_h) \), then we have the transforms

\[
\begin{align*}
(E_{R_h})^{R_{h+1}} &= \frac{x_h}{y_h} R_{h+1} \\
E_{R_h, R_{h+1}} &= y_h R_{h+1},
\end{align*}
\]

which meet transversally at the point given by \( m_{R_{h+1}} = (y_h, \frac{x_h}{y_h}) \).

ii) If \( (f(x_h, y_h)) \neq (x_h) \), then \( \frac{x_h}{y_h} \) is a unit in \( R_{h+1} \), and only the component \( y_h R_{h+1} \) of the exceptional divisor survives. Hence there is no intersection.

2) If \( E_{R_h} R_{h+1} = x_h R_{h+1} \), then, by the same reasoning as in B.1), \( m_{R_{h+1}} = \left( x_h, \frac{f(x_h, y_h)}{x_h} \right) \) and the components of the exceptional divisor in \( R_{h+1} \) are

\[
\begin{align*}
(E_{R_h})^{R_{h+1}} &= \left( \frac{x_1}{y_1} R_h \right)^{R_{h+1}} = (y_h R_h)^{R_{h+1}} = \frac{y_h}{x_h} R_{h+1} \\
(E_{R_h})^{R_{h+1}} &= (x_h R_h)^{R_{h+1}} = \frac{y_h}{x_h} R_{h+1} \\
E_{R_h, R_{h+1}} &= x_h R_{h+1}.
\end{align*}
\]

We distinguish again the following two cases:
i) If \((f(x_h, y_h)) = (y_h)\), then the components of the exceptional divisor are
\[
\begin{align*}
(E_{R_1})^{R_{h+1}} &= \frac{y_h}{x_h} R_{h+1} \\
E_{R_1, R_{h+1}} &= x_h R_{h+1},
\end{align*}
\]
and they meet transversally at a point given by the maximal ideal 
\(m_{R_{h+1}} = (x_h, \frac{y_h}{x_h})\).

ii) If \((f(x_h, y_h)) \neq (y_h)\), then \((E_{R_1})^{R_{h+1}} = \frac{y_h}{x_h} R_{h+1}\); but \(\frac{y_h}{x_h}\) is a unit in 
\(R_{h+1}\) and therefore there is no intersection. \(\Box\)

\textbf{(2.14) Proposition:} Let \(R, S \in \Omega\) be two points with \(R \neq S\) and \(R \prec S\). Let 
\(S' \in N_1(S)\) with \(R \prec S'\) and \(S'' \in N_1(S')\). Then there is no intersection between 
components of the exceptional divisor if and only if \(S''\) is not proximate to \(R\).

\textit{Proof.} Let \(\{x_0, y_0\}\) be a regular system of parameters of \(R = R_0\) and assume that 
\(\{x_1, y_1\}\) is a regular system of parameters of \(R_1\) with \(x_1 = x_0\) and \(v_R(y_1) = 0\). Let 
be the quadratic sequence:
\[
R = R_0 \subset R_1 \subset \ldots \subset R_{h-1} \subset S \subset R_h = S' \subset R_{h+1} = S'' \subset \ldots
\]
The point \(S'' \in N_1(S')\). Then it may happen:

i) If \(S''\) is proximate to \(R\), the strict transforms of the new exceptional divisors 
in \(R\) and \(S = R_{h-1}\) in \(R_{h+1}\) respectively are:
\[
\begin{align*}
(E_{R_1})^{R_{h+1}} &= \frac{y_1}{y_0} R_{h+1} \\
(E_{R_h})^{R_{h+1}} &= R_{h+1}.
\end{align*}
\]
That is, the intersection between both components of the exceptional divisor 
is empty.

ii) If \(S''\) is not proximate to \(R\), we have two possibilities, namely:

a) Case \(E_{R_h, R_{h+1}} = y_h R_{h+1}\). The strict transform \((E_{R_1})^{R_{h+1}}\) is equal 
to the whole ring \(R_{h+1}\) and therefore there is no intersection between 
components of the exceptional divisor.

b) Case \(E_{R_h, R_{h+1}} = x_h R_{h+1}\). In this case the component of the exceptional 
divisor corresponding to \((E_{R_h})^{R_{h+1}} = R_{h+1}\) never survives, and thus 
the strict transforms of the components of the exceptional divisor in 
\(R_{h+1} = S''\) do not meet. \(\Box\)

\textbf{(2.15) Remark:} Proposition (2.14) is a generalization for a non-algebraically 
closed ground field of [3, Proposition 4.4.2].
(2.16) **Corollary:** If \( x \in R \) with \( \text{ord}_R(x) = 1 \), then \( x \) is a regular parameter of \( R \).

**Proof.** Since \( x \not\in m_R^2 \), the element \( x \mod m_R^2 \) is different from 0 in \( m_R/m_R^2 \). Therefore \( x \mod m_R^2 \) takes part of a basis of the \( k \)-vector space \( m_R/m_R^2 \). By [6, Chapter IV, Korollar 2.4(b)], the assertion follows. \( \diamond \)

(2.17) **Definition:** A curve \( E \) with equation \( f \) is said to have no singularities, if \( \text{ord}_R(f) = 1 \).

(2.18) **Remark:** By Corollary (2.16), a curve \( E \) has no singularities if and only if \( f \) is a regular parameter of \( R \). Consequently, a curve with no singularities is irreducible.

(2.19) **Definition:** Let \( S \in N_n(R) \) with \( n \in \mathbb{N} \), and let 

\[
R =: R_0 \to R_1 \to \ldots \to R_n := S
\]

be the sequence of quadratic transformations between \( R \) and \( S \). The ring \( S \) is said to be free with respect to \( R \) if \( R_{n-1} \) is the unique ring with \( S \triangleright R_{n-1} \); otherwise \( S \) is called satellite with respect to \( R \).

(2.20) **Corollary:** Let \( S \in N_n(R) \) with \( n \in \mathbb{N} \), let 

\[
R =: R_0 \to R_1 \to R_2 \to \ldots \to R_n := S
\]

be the sequence of quadratic transformations between \( R \) and \( S \), and let \( \{x_n, y_n\} \) be the regular system of parameters of \( R_n \) obtained from the above procedure. We have:

1. If \( E_{R_n} = x_nR_n \), then the ring \( R_n \) is free with respect to \( R \); if \( E_{R_n} = x_ny_nR_n \), then the ring \( R_n \) is satellite with respect to \( R \).

2. If \( E_{R_n} \) is a curve with no singularities, then \( R_n \) is free with respect to \( R \); if \( E_{R_n} \) has two irreducible simple components, which are curves with no singularities, then \( R_n \) is satellite with respect to \( R \).

3 **Proximity matrices for clusters of infinitely near points**

(3.1) Let \( R \) be a two-dimensional regular local ring. Let \( \Omega(R) \) be the set of all two-dimensional regular local subrings of \( K \) containing \( R \). Note that if \( S \in \Omega(R) \), then \( m_S \cap R = m_R \). As the set \( \Omega(R) \) consists of infinitely many elements, it would be useful to deal with suitable finite subsets:

(3.2) **Definition:** A **cluster** in \( \Omega(R) \) over \( R \), denoted by \( \mathcal{C}(R) \) (or simply \( \mathcal{C} \), if there is no risk of confusion), is a finite subset of \( \Omega(R) \) such that
(i) the point \( R \in \mathcal{C} \);
(ii) if \( R' \in \mathcal{C} \) and \( R =: R_0 \subset R_1 \subset \ldots \subset R_n := R' \) is the quadratic sequence between \( R \) and \( R' \), then \( R_i \in \mathcal{C} \) for all \( i \in \{1, \ldots, n-1\} \).

\(\textbf{(3.3)}\) Let \( \mathcal{C} \) be a cluster in \( \Omega(R) \) over \( R \). Now we want to define the proximity matrix associated with \( \mathcal{C} \), which is a useful device for the representation of proximity relations. Let \( P_{\mathcal{C}} \) be the matrix with entries \((p_{S,T}), S, T \in \mathcal{C}\), where

\[
p_{S,T} = \begin{cases} 
1, & \text{if } S = T; \\
-1, & \text{if } S \succ T; \\
0, & \text{otherwise.}
\end{cases}
\]

This matrix is called the \textbf{proximity matrix} associated with \( \mathcal{C} \). Consider also the diagonal matrix \( \Delta_{\mathcal{C}} = (d_{S,T}), S, T \in \mathcal{C} \), given by

\[
d_{S,T} = \begin{cases} 
[S : R], & \text{if } S = T; \\
0, & \text{otherwise.}
\end{cases}
\]

The proximity matrix can be slightly turned out to a matrix \( P'_{\mathcal{C}} := \Delta_{\mathcal{C}}^{-1} \cdot P_{\mathcal{C}} \cdot \Delta_{\mathcal{C}} \) with entries \((p'_{S,T}), S, T \in \mathcal{C}\), where

\[
p'_{S,T} = \begin{cases} 
1, & \text{if } S = T; \\
-[S : T], & \text{if } S \succ T; \\
0, & \text{otherwise.}
\end{cases}
\]

Such a matrix was proposed by Lipman in [10] in order to encode the proximity inequalities in a shorter way, and it is called the \textbf{refined proximity matrix} associated with \( \mathcal{C} \). Nevertheless, this matrix does not take into account all possible field extensions from the origin on. To obtain that, we introduce a matrix \( \tilde{P}_{\mathcal{C}} \) with entries \((\tilde{p}_{S,T}), S, T \in \mathcal{C}\), where

\[
\tilde{p}_{S,T} = \begin{cases} 
[S : R], & \text{if } S = T; \\
-[S : R], & \text{if } S \succ T; \\
0, & \text{otherwise.}
\end{cases}
\]

We will call it the \textbf{total proximity matrix} associated with \( \mathcal{C} \).

\(\textbf{(3.4)}\) \textbf{Remark}: From Definition (2.6), it is easy to check that both the matrix \( P_{\mathcal{C}}, P'_{\mathcal{C}} \) and \( \tilde{P}_{\mathcal{C}} \) are column-finite and invertible, and the entries of \( P_{\mathcal{C}}^{-1}, (P'_{\mathcal{C}})^{-1} \) and \( \tilde{P}_{\mathcal{C}}^{-1} \) are non-negative integers (it also follows from [10, Corollary 4.6]).

\(\textbf{4 Intersection matrix in terms of a cluster}\)

\(\textbf{(4.1)}\) Let \( X_0 \) be a two-dimensional regular scheme of finite type over \( k \). Take a closed point \( x_0 \in X_0 \) and blow up at \( x_0 \) to obtain another two-dimensional regular
scheme $X_1$ and repeat the process. We get a finite sequence of blowing ups

$$X = X_s \xrightarrow{\pi_s} X_{s-1} \xrightarrow{\pi_{s-1}} \ldots \xrightarrow{\pi_2} X_2 \xrightarrow{\pi_1} X_1 \xrightarrow{\pi_0} X_0,$$

where $\pi_i$ is the blowing up of a closed point $x_{i-1} \in X_{i-1}$, for $1 \leq i \leq s$. Then $X$ is achieved by blowing up successively closed points $x_i \in X_i$, $0 \leq i \leq s - 1$. Every such a point $x_i$ corresponds to a two-dimensional regular local ring $R_i := O_{X_i,x_i}$, where $R_0 := R$. Sometimes we will speak about points $x_i$ instead of rings $R_i$ and we will apply the notations used for the rings to the points. In particular, if a ring $R_i = O_{X_i,x_i}$ is proximate to a ring $R_j = O_{X_j,x_j}$ for some $i, j \in \mathbb{N}$, then we will write either $R_i \succ R_j$ (as in Definition (3.3)), or $x_i \succ x_j$. Moreover, the ring homomorphism $R \xrightarrow{} R_i$ induces the field extension $k_R \hookrightarrow R_i/\mathfrak{m}_{R_i} =: k_{R_i}$. For convenience, we will denote the residue field $k_{R_i}$ simply by $k_i$. The degree of this field extension is finite, and it will be denoted by $h_i$ or $|R_i : R|$, as we have already seen. The diagonal matrix having the $h_i$'s on the diagonal will be denoted by $\Delta$. Finally, set $\pi := \pi_s \circ \pi_{s-1} \circ \ldots \circ \pi_2 \circ \pi_1$.

(4.2) For every point $x \in X$, we have a local ring $O_{X,x} \in \Omega_*(R)$ which determines a quadratic sequence of length $n \leq s$ between $R$ and $O_{X,x}$. Furthermore, we can associate to the sequence (4.1) a cluster over $R = R_0$, which will be denoted by $C_n$ or $C$, if no risk of confusion arises.

(4.3) Notation: We will write $E_{i,j}$ for the exceptional divisor of $\pi_i$ as divisor of $X_i$, and we denote by $E_{i,j}$ (resp. $E_{i,j}^*$) the strict transform (resp. the total transform) of $E_{i,j}$ in $X_j$ by the morphism $X_j \xrightarrow{} X_i$, for $j \geq i$. We denote by $E_i$ (resp. $E_i^*$) the strict (resp. total) transform $E_{i,s}$ (resp. $E_{i,s}^*$) by the morphism $X \xrightarrow{} X_i$.

Let $E$ be the subgroup of 1-cycles of $X$ of the form $\sum_{i=1}^s n_i E_i$, with $n_i \in \mathbb{Z}$ (i.e., the free $\mathbb{Z}$-module generated by the divisors $E_i$). Both $E = (E_0, \ldots, E_{s-1})$ and $E^* = (E_0^*, \ldots, E_{s-1}^*)$ are $\mathbb{Z}$-basis of $E$. More precisely, the proximity matrix of a cluster $C$ over $R = O_{X_0,x_0}$ is the matrix of the change of basis from $E$ to $E^*$:

(4.4) Lemma: Let $P$ be the proximity matrix of a cluster $C$ over $R$. Then $E = E^* PC$.

Proof. We have to prove that

$$E_i = \sum_j p_{ji} E_j^*,$$

where $p_{ji} = 1$ if $i = j$, $p_{ji} = -1$ if $x_j \succ x_i$ and 0 otherwise. Remember that, if $\pi_1 : X_1 \xrightarrow{} X_0$ denotes the blowing-up of a closed point $x_0$ of a curve given by $f \in R$, we get

$$\pi_1^* C = \bar{C} + \text{ord}_RF_0,$$
where $\pi_1^* C$ is the total transform of $C$ on $X_1$, $\tilde{C}$ denotes its strict transform on $X_1$ and $E_0$ is the exceptional divisor (cf. [11, Proposition 2.23, p.402]).

Now, the point $x_j$ is proximate to $x_i$ if and only if $x_j \in E_{ij}$; in other words, $E_{i,j+1} \cap E_{j,j+1} \neq \emptyset$. We deduce that the multiplicity of $E_{ij}$ at $x_j$ is 1 and also that

$$\pi_j^* E_{ij} = \begin{cases} E_{i,j+1} + E_{j,j+1} & \text{if } x_j > x_i \\ E_{i,j+1} & \text{otherwise.} \end{cases}$$

Thus

$$\pi_j^* E_{ij} = E_{i,j+1} - p_{ji} E_{j,j+1}.$$  

Write $\tilde{E}_{ij}$ for the total transform $(\pi_{j-1}^* \circ \ldots \circ \pi_{i+1}^*)(E_i)$ of $E_i$ in $X_j$. We will show by induction on $j - i$ that, for $i < j$, the following equality holds:

$$\tilde{E}_{ij} + p_{i+1,i} \tilde{E}_{i+1,j} + \ldots + p_{j-1,i} \tilde{E}_{j-1,j} = E_{ij}.$$  

For $j = i + 1$ it is obvious. If it holds for particular values of $j$ and $i$, then, if we apply $\pi_j^*$ to both sides we get

$$\tilde{E}_{i,j+1} + p_{i+1,i} \tilde{E}_{i+1,j} + \ldots + p_{j-1,i} \tilde{E}_{j-1,j} = \pi_j^* E_{ij}.$$ 

Hence the result follows for $i$ and $j + 1$, because

$$p_{ji} \tilde{E}_{j,j+1} + \pi_j^* E_{ij} = E_{i,j+1}.$$ 

Taking $j = k$ in the equation just obtained, one has

$$E_k = E_{i}^* + p_{i+1,i} E_{i+1}^* + \ldots + p_{k,i} E_k^*.$$  

Since $p_{ii} = 1$ and $p_{ji} = 0$ for $i > j$, we are done. ♦

\[ (4.5) \]

Moreover, on every $X_i$ occurring in the sequence of blowing-ups (†) we can define the intersection of cycles. We have a symmetric bilinear intersection form given as follows:

$$E \times E \rightarrow \mathbb{Z}$$

$$(A, B) \mapsto (A \cdot B),$$

which is given by intersecting cycles (cf. [11, §9.1.2 and Proposition 2.5]). If we denote by $h_i$ the degree of the extension $k_R \subset k_i$, by the projection’s formula (see [11, Theorem 9.2.12, p. 398]; see also [2, Proposition 1.10]) we get

$$(E_i^* \cdot E_j^*) = -\delta_{ij} h_i,$$

where $\delta_{ij}$ is the Kronecker’s delta (i.e., $\delta_{ij}$ is equal to 1 if $i = j$ and to 0 if $i \neq j$). Therefore the matrix of the intersection form in the basis $E^*$ is $-\Delta_C$. By Lemma (4.4), the matrix of the intersection form in the basis $E$ is

$$N_C := -P_C \cdot \Delta_C \cdot P_C^t,$$

with $P_C^t$ the transpose of $P_C$. Note that $P_C \cdot \Delta_C = \tilde{P}_C$, the total proximity matrix.
**Definition**: The matrix $N_C$ is called the **intersection matrix** (with respect to the basis $E$) associated with the cluster $C$.

In other words, the intersection matrix associated with the cluster $C$ is $N_C := -\bar{P}_C \cdot P_C^t$.

We can characterise the entries of the intersection matrix of a cluster as follows:

**Proposition**: The entries $n_{S,T}$ of the intersection matrix $N_C = (n_{S,T})$, $(S, T) \in C \times C$ are:

* $- \left( [S : R] + \sum_{U \in C, U \succ S} [U : R] \right)$ if $S = T$.

* $[T : R]$ if $T \succ S$ and the point $T^* \in N_1(T)$ with $S \prec T^*$ does not belong to the cluster $C$.

* $[S : R]$ if $S \succ T$ and the point $S^* \in N_1(S)$ with $T \prec S^*$ does not belong to the cluster $C$.

* $0$ otherwise.

**Proof.** From Definition (3.3), the entries of the matrix $N_C$ are

$$n_{S,T} = - \sum_{U \in C} \bar{p}_{U,S} p_{U,T}.$$

If $S = T$, then we have

$$n_{S,S} = - \sum_{U \in C, U \succ S} \bar{p}_{U,S} p_{U,S}$$
$$= - \sum_{U \in C, U \succ S} \bar{p}_{U,S} p_{U,S} - \bar{p}_{S,S} p_{S,S}$$
$$= - \sum_{U \in C, U \succ S} [U : R] - [S : R].$$

If $S \neq T$, then we have three possibilities, namely:

1) $S \nsubseteq T$ and $T \nsubseteq S$.

$$n_{S,T} = - \sum_{U \in C, U \succ S, U \succ T} \bar{p}_{U,S} p_{U,T}$$
$$= 0.$$

2) $S \subseteq T$. 


a) If \( S \nless T \), then we have:

\[
\begin{align*}
n_{S,T} &= - \sum_{U \in C, \ U \subseteq S \cup T} \tilde{p}_{U,S} p_{U,T} = \\
&= - \sum_{U \in C, \ U \supsetneq S} \tilde{p}_{U,S} p_{U,T} - \tilde{p}_{T,S} p_{T,T} = \\
&= - \sum_{U \in C, \ U \supsetneq T} \tilde{p}_{U,S} p_{U,T} - 0 = \\
&= 0.
\end{align*}
\]

Namely, since \( U \rhd T \), we have \( U \subseteq V_T \); if we assume that \( U \supsetneq S \), then \( U \subseteq V_T \) and so \( U \nless V_T \); then \( U \nless S \) and therefore \( \tilde{p}_{U,S} = 0 \) and \( n_{S,T} = 0 \).

b) If \( S \lhd T \), then there exists a point \( T* \in N_1(T) \) satisfying \( S \lhd T* \). We distinguish two cases:

i) if \( T* \in C \); then

\[
\begin{align*}
n_{S,T} &= - \sum_{U \in C, \ U \supsetneq T} \tilde{p}_{U,S} p_{U,T} = \\
&= - \sum_{U \in C, \ U \supsetneq T} \tilde{p}_{U,S} p_{U,T} - \tilde{p}_{T,S} p_{T,T} = \\
&= - \sum_{U \in C, \ U \supsetneq T} \tilde{p}_{U,S} p_{U,T} + [T : R] = 0.
\end{align*}
\]

To compute \( \sum_{U \in C, \ U \supsetneq T} \tilde{p}_{U,S} p_{U,T} \), let us consider the quadratic sequence

\[
R_0 = R \subset \ldots \subset R_t = S \subset \ldots \subset R_t = T \subset R_{t+1} = T* \subset \ldots
\]

If \( U = T* \), then \( U \rhd T \) and \( U \rhd S \); therefore \( U \) cannot be proximate to any other point of the sequence. Then \( \tilde{p}_{U,S} = \tilde{p}_{T*,S} = -[U : R] = -[T* : R] \) and \( p_{U,T} = p_{T*,T} = -1 \). Whenever \( U \neq T* \), suppose \( U = R_{t+i} \) for some \( i \geq 2 \); then \( U \) is proximate to \( R_{t+i-1} \) and proximate to \( R_t = T \) as well, hence \( U \) cannot be proximate to
$S$ and so $\tilde{p}_{U,S} = 0$. Then

$$n_{S,T} = -\sum_{U \in C, U \succ T} \tilde{p}_{U,S}p_{U,T} + [T : R] =$$

$$= -\tilde{p}_{T^*,S}p_{T^*,T} + [T : R] =$$

$$= -([T^* : R])(-1) + [T : R] =$$

$$= -([T^* : T][T : R]) + [T : R] =$$

$$= [T : R](1 - [T^* : T]).$$

But, by [7, Chapter VII, (7.2)(2)], we have $[T^* : T] = 1$ and therefore $n_{S,T} = [T : R](1 - 1) = 0$.

ii) if $T^* \notin C$, then consider a quadratic sequence as above:

$$R_0 = R \subset \ldots \subset R_s = S \subset \ldots \subset R_t = T \subset R_{t+1} = T^* \subset \ldots$$

Now we have

$$n_{S,T} = -\sum_{U \in C, U \succ T} \tilde{p}_{U,S}p_{U,T} - \tilde{p}_{T,S}p_{T,T}$$

$$= -\sum_{U \in C, U \succ T} \tilde{p}_{U,S}p_{U,T} + [T : R].$$

In this case, the ring $U$ cannot be equal to $T^*$, because $T^* \notin C$ and therefore, by the same reasoning of the case $U \neq T^*$ in B.2.2.i), we have that, for all $U \in C$ with $U \succ T$, then $\tilde{p}_{U,S} = 0$ and $n_{S,T} = [T : R]$.

3) $T \subset S$. This situation is totally symmetric to B.2. $\Diamond$

**Remark:** Of course, from the previous arguments it is now easy to see that the intersection matrix shows whether the components of the exceptional divisors occurring in a blowing-up process intersect. Indeed, the entries $n_{i,j}$ of the intersection matrix $N_{C_s} = (n_{i,j})$, for all $1 \leq i, j \leq s$, are

$$n_{i,j} = \begin{cases} 
-h_i - \sum_{p \succ p_i} h_p & \text{if } i = j \\
+ [k_P : k_R] & \text{if } i \neq j \text{ and } E_i \cap E_j = \{P\} \\
0 & \text{if } i \neq j \text{ and } E_i \cap E_j = \emptyset
\end{cases}$$

where $k_P$ is the residue field of the local ring of $X$ at $P$. 

5 Hamburger-Noether Tableau and characteristic data

Let \( f \in R \) be an analytically irreducible curve. Let \( C \) be the cluster associated with an embedded resolution of \( f \). In this section we introduce the Hamburger-Noether tableau of \( f \) and some important data of the resolution based on the tableau adapting the arguments of [13] (there it is exposed for algebroid curves).

(5.1) Let us consider the sequence of points of length \( n \) defined in (4.1) associated with the resolution of an analytically irreducible curve \( f \in R \). Let \( \tilde{k} \subset R \) be a set of representatives of \( k \in R \) with the additional property that the zero element of \( k \) is represented by the zero element of \( R \). Since \( k = k_1 = \ldots = k_n \) then we see that \( \tilde{k} \) is a set of representatives for every field \( k_i, i \in \{1, \ldots, n\} \) as well.

(5.2) Let \( x, y \in m, x \neq 0 \) and \( y \neq 0 \). We define a matrix

\[
HN(x, y; f) = \begin{pmatrix}
p_i \\
c_i \\
a_i
\end{pmatrix}_{1 \leq i \leq l}
\]

with \( l \in \mathbb{N}, p_i, c_i \in \mathbb{N} \cup \{\infty\}, a_i \in \tilde{k} \setminus \{0\} \) for every \( i \in \{1, \ldots, l-1\} \), and \( a_l = \infty \) by means of the following algorithm (cf. [13]).

If \( x = 0 \), then \( y \neq 0 \) (since \( x \) and \( y \) cannot vanish simultaneously) and set \( p_1 := v(y), c_i = v(x) \) and \( a_i = 0 \) for all \( i \in \mathbb{N} \). If \( x \neq 0 \), then we put \( x_0 := x, y_0 := y \), and we define \( p_1 := v(y_0) \) and \( c_1 := v(x_0) \). If \( n = 0 \), then we define \( l := 1 \) and \( a_1 := \infty \), and we finish. In this case we have \( p_1 = c_1 = 1 \). Otherwise we put \( \eta_0 := y_0 \) and \( \eta_1 := x_0 \), and we define \( \kappa \in \mathbb{N} \), non-zero elements \( \eta_2, \ldots, \eta_{\kappa+1} \in V \) and \( s_1, \ldots, s_\kappa \in \mathbb{N}_0 \) by the requirement that

\[
\eta_{i-1} = \eta_{\kappa}^s \eta_{i+1} \quad \text{for every} \quad i \in \{1, \ldots, \kappa\}
\]

\[
0 < v(\eta_i) < v(\eta_{i-1}) \quad \text{for every} \quad i \in \{2, \ldots, \kappa\} \quad \text{and} \quad v(\eta_{\kappa+1}) = 0.
\]

Note that

\[
v(\eta_{i-1}) = s_i \cdot v(\eta_i) + v(\eta_{i+1}) \quad \text{for} \quad i \in \{1, \ldots, \kappa\}
\]

is the Euclidean algorithm for the natural integers \( v(\eta_0), v(\eta_1) \) hence

\[
v(\eta_\kappa) = \gcd(v(\eta_0), v(\eta_1)).
\]

From \( v(\eta_{\kappa+1}) = v(\eta_{\kappa}^s) \) we see that \( \eta_{\kappa-1}/\eta_{\kappa}^s \) is a unit in the integral closure \( \overline{R} \) of \( R \); then there exists a unique \( a := a(\eta_0, \eta_1) \in \tilde{k} \setminus \{0\} \) such that

\[
v(\eta_{\kappa-1} - a\eta_{\kappa}^s) > v(\eta_{\kappa-1}).
\]
We define also \( m := s_1 + \ldots + s_\kappa \) and \( \kappa_1 := \kappa, m_1 := m, s_0^{(1)} := 0, s_1^{(1)} := s_1 \) and \( s_{\kappa_1}^{(1)} := s_\kappa \).

If \( m_1 - 1 = n \), then we define \( l := 1 \) and \( \rho_1 := \infty \), and we end the algorithm. In this case we get \( \gcd(p_1, c_1) = 1 \). If \( m_1 - 1 < n \), then we define \( \rho_1 := \rho(\eta_0, \eta_1) \) and

\[
\begin{align*}
x_{m_1} & := \eta \kappa \\
y_{m_1} & := \frac{\eta_{\kappa - 1} - \rho_1 \eta \kappa^s}{\eta \kappa^s}
\end{align*}
\]

Note that \( \{x_{m_1}, y_{m_1}\} \) is a regular system of parameters in \( R_{m_1} \). Define \( p_2 := v(y_{m_1}) \) and \( c_2 := v(x_{m_1}) \). If \( m_1 = n \), then we set

\[
\begin{align*}
l & := 2 \\
a_2 & := \infty
\end{align*}
\]

and we end the algorithm. Otherwise we put \( \eta_0 := y_{m_1}, \eta_1 := x_{m_1} \) and define \( \eta_2, \ldots, \eta_\kappa \) as before but for \( R_{m_1} \) instead of \( R \). We get \( s_1, \ldots, s_\kappa \) and \( m = s_1 + \ldots + s_\kappa \). Then we obtain a regular system of parameters \( \{x_j, y_j\} \) in \( R_j \) for \( j \in \{m_1 + 1, \ldots, m_1 + m - 1\} \). In particular, \( \{\eta_\kappa, \eta_{\kappa - 1}/\eta \kappa^s, 1\} \) is a regular system of parameters in \( R_{m_1 + m - 1} \) with \( v(\eta_\kappa) = v(\eta_{\kappa - 1}/\eta \kappa^s, 1) \). Then we define \( \kappa_2 := \kappa, m_2 := m, s_0^{(2)} := 0, s_1^{(2)} := s_1 \), and \( s_{\kappa_2}^{(2)} := s_\kappa \). If \( m_1 + m_2 - 1 = n \) then we put \( l := 2 \) and \( a_2 := \infty \) and end the algorithm. Otherwise we define \( a_2 := \rho(\eta_0, \eta_1) \) and

\[
\begin{align*}
x_{m_1 + m_2} & := \eta \kappa \\
y_{m_1 + m_2} & := \frac{\eta_{\kappa - 1} - \rho_2 \eta \kappa^s}{\eta \kappa^s}
\end{align*}
\]

In this case \( \{x_{m_1 + m_2}, y_{m_1 + m_2}\} \) is a regular system of parameters in \( R_{m_1 + m_2} \) and we define \( p_3 := v(y_{m_1 + m_2}) \) and \( c_3 := v(x_{m_1 + m_2}) \). If \( m_1 + m_2 = n \) then we put \( l := 3 \) and \( a_3 := \infty \) and end the algorithm. Otherwise we put \( \eta_0 := y_{m_1 + m_2}, \eta_1 := x_{m_1 + m_2} \) and get \( \eta_2, \ldots, \eta_\kappa, s_1, \ldots, s_\kappa \) and \( m = s_1 + \ldots + s_\kappa \) as before but for \( R_{m_1 + m_2} \) instead of \( R \).

Proceeding in this way we get natural numbers \( l, m_1, \ldots, m_l, \kappa_1, \ldots, \kappa_l, \) non-negative numbers \( s_1^{(1)}, \ldots, s_\kappa^{(1)} \), elements \( a_1, \ldots, a_{l-1} \in \bar{k} \setminus \{0\} \) and an element \( a_l = \infty \) such that

\[
m_j = s_1^{(j)} + \ldots + s_{\kappa_l}^{(j)} \quad \text{for} \quad j \in \{1, \ldots, l\}.
\]

Note that either \( n = m_1 + \ldots + m_{l-1} + m_l - 1 \) or \( n = m_1 + \ldots + m_{l-1} + m_l \).

(5.3) Definition: The matrix \( \text{HN}(x, y; f) \) will be called the Hamburger-Noether tableau of \( f \) with respect to the regular system of parameters \( \{x, y\} \) in \( R \).

(5.4) Note that
• if \( p_i = \infty \) for some \( i \in \mathbb{N} \), then we have \( p_j = \infty \) and \( c_j = c_i < \infty \) for every \( j \geq i \);

• if \( c_i = \infty \) for some \( i \in \mathbb{N} \), then \( c_j = \infty \) and \( p_j = p_1 < \infty \) for \( j \in \mathbb{N} \). If \( c_1 < \infty \), then \( c_{i+1} = \gcd(c_i, p_i) \) for every \( i \in \{1, \ldots, l-1\} \).

• \( c_1 \geq c_2 \geq \ldots \geq c_l \) and either \( c_l = 1 \) or \( c_l > 1 \) and \( \gcd(c_l, p_l) = 1 \). We define \( c_{l+1} := p_{l+1} := 0 \).

• We have \( a_i = 0 \) if and only if \( p_i = \infty \) or \( c_i = \infty \), for every \( i \in \mathbb{N} \).

\[(5.5) \text{ Definition: Let } l \in \mathbb{N}. \text{ A matrix } HN := \begin{pmatrix} p_i \\ c_i \\ a_i \end{pmatrix}_{1 \leq i \leq l} \text{ with } p_i, c_i \in \mathbb{N} \cup \{\infty\}, a_i \in \bar{k} \setminus \{0\} \text{ for every } i \in \{1, \ldots, l-1\}, \text{ and } a_l = \infty \text{ is called an (abstract) Hamburger-Noether tableau of length } l \text{ if the properties in } (5.4) \text{ hold.} \]

\[(5.6) \text{ An integer } i \in \{1, \ldots, l\} \text{ is called characteristic index of } HN \text{ if } i = 1 \text{ or if } c_{i+1} < c_i. \text{ Let } \]

\[
1 = i_1 < i_2 < \ldots < i_h \leq l, \\
\text{with } h := h(HN) \in \mathbb{N}
\]

be the characteristic indices of \( HN \). It is clear that \( c_j = 1 \) for every \( j \in \{i_h + 1, \ldots, l\} \).

\[(5.7) \text{ Let us define } \]

\[
q_1 := p_1 \\
q_j := p_{i_{j-1}+1} + \ldots + p_{i_j} \text{ for every } j \in \{2, \ldots, h\} \\
d_j := a_{i_j} \text{ for every } j \in \{1, \ldots, h\}.
\]

The sequence \( Ch(HN) := (d_1, q_1, \ldots, q_h) \) is called the characteristic sequence of \( HN \). Notice that

\[
d_i = \gcd(d_{i-1}, q_{i-1}) = \gcd(d_1, q_1, \ldots, q_{i-1}) \forall i \in \{2, \ldots, h\} \\
d_{h+1} = \gcd(d_h, q_h) = \gcd(d_1, q_1, \ldots, q_h) = 1.
\]

The sequence \( d(HN) := (d_1, \ldots, d_{h+1}) \) is called the divisor sequence of \( HN \). Notice that
• if \( h = 1 \), then \( d_1 = c_1 = 1 \) and \( d_2 = 1 \);

• if \( h \geq 2 \), then

\[
d_1 \geq d_2 > d_3 > \ldots > d_h > d_{h+1}
\]

and if \( d_1 = d_2 \) then either \( d_1 \mid q_1 \) or \( q_1 \mid d_1 \).

(5.8) Furthermore, we set

\[
n_i := \frac{d_i}{d_{i+1}} \quad \text{for every } i \in \{1, \ldots, h\};
\]

the sequence \( n(HN) = (n_1, \ldots, n_h) \) is called the \( n \)-sequence of \( HN \). We also set

\[
r_0 := d_1
\]

\[
r_i := \sum_{j=1}^{i} q_j \frac{d_j}{d_i} \quad \text{for every } i \in \{1, \ldots, h\}.
\]

The sequence \( r(HN) := (r_0, \ldots, r_h) \) is called the \textit{semigroup sequence} of \( HN \).

6 Curvettes

Let \( f \in \mathfrak{m}_R \). Every \( S \supset R \) with \( (fR)^S \neq S \) is called a locus point of \( fR \) (or of \( f \)); the set \( L(f) \) of locus points of \( f \) is called the \textit{point locus} of \( f \). Notice that \( L(f) \) is an infinite set.

(6.1) Definition: Set \( S \in \Omega(R) \), \( S \notin L(f) \) and \( S \) not be the intersection of two components of the exceptional divisor. A \textit{curvette} at \( S \) is defined to be a normal-crossing curve \( g \in S \) such that \( (gS \cap R)^S \) is a curve with no singularities at \( S \) and not passing through any other point \( S' \in \Omega(R) \) with \( S \neq S' \).

(6.2) Proposition: Let \( g \in S \) a normal crossing curve, where \( S \) satisfies the conditions of the above definition. Then there exists \( h \in R \) irreducible with \( gS \cap R = hR \) and \((hR)^S\) is normal-crossing at \( S \) with \((hR)^S' = S' \) for every \( S' \in \Omega(R) \) with \( S \neq S' \).

\[\text{Proof.}\] Since \( S \in N_f(R) \), there exists \( p \in \mathbb{P}_R \) generated by an irreducible homogeneous polynomial \( \overline{h} \in k_R[p, \overline{x}, \overline{y}] \) (cf. (1.2)), let us say of degree \( l \). Choose \( h \in \mathfrak{m}^l \) with \( \overline{h} = h \mod \mathfrak{m}^{l+1} \). Without lost of generality, we can assume that \( x \) does not divide \( h \). Then the exceptional divisor has the equation \( xS \) and the strict transform of \( h \) in \( S \) is \( \frac{h}{x}S = (hR)^S \). Thus \( (x, \frac{h}{x}) \) is a regular system of parameters of \( S \). Inductively, it is easy to check this statement for every \( S \in N(R) \). Assume \( xS \) is the equation of the exceptional divisor in \( S \). We have also that \( gS \cap R \neq (0) \)}
and \( gS \cap R \neq m_R \): on the contrary, we would have \( x \in gS \), which is a contradiction. Hence \( gS \cap R \) is a prime ideal of \( R \) different from \( 0 \) of height 1. Since \((gS \cap R)S \cap R = gS \cap R\), the transform \((gS \cap R)^S\) is a principal prime ideal of \( S \) with \((gS \cap R)^S \cap R = gS \cap R\) (by Lemma (2.8) (i)), and therefore \((gS \cap R)^S = gS\). Moreover, for any other subring \( S' \) such that \( S' \) is not infinitely near to \( S \) and \( S \) is not infinitely near to \( S' \), again Lemma (2.8) shows us that \((gS \cap R)^S = S'\). ◇

(6.3) Remark: Notice that by Lemma (2.10) (i) and [7, Chapter VII, (1.1)], the strict transform \((gR)^S\) is irreducible in \( S \) and \( gS \cap R = gR\).

The following result will be used in the sequel (see [7, Chapter VII, (8.8)-(8.9)]):

(6.4) Lemma: (Intersection formula) Let \( R \in \Omega \) and \( \{f, g\} \) be a regular sequence in \( R \). Then we have

\[
\iota_R(Rf, Rg) = \sum_{S \in N(R)} [S : R] \ord_S((fR)^S) \ord_S((gR)^S).
\]

Moreover, we have

\[
\ord_R(f) = \sum_{N_1(R)} [S : R] \iota_S((fR)^S, m_RS).
\]

Let us take now the proximity matrix \( P_C \) with respect to the cluster \( C \) associated with the resolution of \( f \), and its inverse matrix \( Q_C := P_C^{-1} \). We give now an interpretation of the entries of the matrix \( Q_C \) in terms of curvettes.

(6.5) Proposition: Let \( R' \in N_n(R) \).

(i) If we consider the cluster of a resolution of \( f \in R \), we have

\[
\ord_S((fR)^S) = \sum_{T > S} [T : S] \ord_T((fR)^T).
\]

(ii) Furthermore, for any curvette \( g \in T, T \in N(R) \), and any point \( S \in C \), we have

\[
\ord_S((gT \cap R)^S) = \sum_{T' > S} [T' : S] \ord_{T'}((gT \cap R)^{T'}).
\]

Proof. Statement (ii) follows easily from (i), and this is a consequence of (6.4). ◇

(6.6) Corollary: Let \( R' \in N_n(R) \). The following statements hold:

(i) For any \( S \in C \), we have

\[
q_{S,R} = \ord_S((fR)^S).
\]
(ii) For any curvette \( g \in T \), \( T \in N(R) \) and any \( S \in C \), we have
\[
q_{S,T} = \text{ord}_S((gT \cap R)^S).
\]

Proof. First of all, we reformulate the equation (i) in Proposition (6.5) to have
\[
\sum_{T'} [T : T'] \cdot \text{ord}_{T'}((f R)^{T'}) = \delta_{T,R'}
\]
for all \( T \in C \), where
\[
\delta_{T,R'} = \begin{cases} [T : R'] & \text{if } T = R' \\ 0 & \text{if } T \neq R'. \end{cases}
\]
We take now multiplication by \( q_{S,T} \) for \( S \in C \) and sum over all \( T \in C \):
\[
\sum_T \sum_{T'} [T : T'] q_{S,T} \cdot \text{ord}_{T'}((f R)^{T'}) = \sum_T q_{S,T} \cdot \delta_{T,R'}.
\]
Since \( Q \) is the inverse matrix of \( P \) all terms cancel except for those containing \( R' \), and we get
\[
q_{S,R'} = \text{ord}_S((f R)^S).
\]
The same argument works to prove the statement (ii) replacing \( R' \) (resp. \( f R \)) by \( T \) (resp. \( gT \cap R \)). ♦

(6.7) Proposition: Let be the matrix \( M_C := Q_C^t \cdot \Delta_C^{-1} \cdot Q_C \). Let \( T_1, T_2 \) be two points of \( C \). Then the \((T_1, T_2)\)-entry of the matrix \( M_C \) is equal to the intersection number
\[
\iota_R(g_1 T_1 \cap R, g_2 T_2 \cap R)
\]
of two curvettes \( g_1, g_2 \) of \( T_1 \) and \( T_2 \), respectively.

Proof. It is just to consider the equalities \( -N_C^{-1} = (P_C^t \cdot \Delta_C \cdot P_C^t)^{-1} = Q_C^t \cdot \Delta_C^{-1} \cdot Q_C \), and the intersection’s formula (6.4) applied to the cases \( f R = g_1 T_1 \cap R \) and \( g R = g_2 T_2 \cap R \). ♦

7 Curvettes and approximations

(7.1) Let \( f \in R \) be an analytically irreducible curve. Then the ring \( \overline{R} := R/(f) \) is a one-dimensional analytically irreducible local ring with residue field \( k \), hence the integral closure of \( \overline{R} \) is a discrete valuation ring \( W \) which is a finitely generated \( \overline{R} \)-module (cf. [7], II(3.17)). Moreover, we assume \( f \) to be residually rational, i.e., \( W \) has residue field equal to \( k \). Let \( t \) be a uniformizing parameter for \( W \).
(7.2) Let \( m = (x, y) \) be the maximal ideal of \( R \), and let \( \phi : R \rightarrow \overline{R} \) be the canonical homomorphism. Set \( \overline{x} := \phi(x), \overline{y} := \phi(y) \). If \( \overline{x} = 0 \) (resp. \( \overline{y} = 0 \)), then an easy reasoning shows that \( f = ux \) (resp. \( f = u'y \)), for \( u, u' \) units in \( R \). Let us assume that both \( \overline{x} \) and \( \overline{y} \) are non-zero. Take the Hamburger-Noether tableau \( \text{HN}(x, y; f) \) of \( f \) as in Section 5. We have \( \overline{x} = \omega_x t^{c_1} \) and \( \overline{y} = \omega_y t^{p_1} \) for \( \omega_x, \omega_y \in k \setminus \{0\} \). Note that if \( c_1 = p_1 \), then we write \( \omega' = \frac{\omega}{t^{c_1}} \) and \( \text{ord}(f) = c_1 = p_1 \); we set \( \overline{y} := \overline{y} - \omega' \overline{x} \) and \( y' := y - \omega' x \). Then \( v(\overline{x}) = c_1 < v(\overline{y}) \) and there exists \( \lambda \in k \setminus \{0\} \) with \( \text{In}(f) = \lambda (y - \omega' x)^{c_1} \). If \( c_1 < p_1 \), then \( \text{ord}(f) = v(\overline{x}) = c_1 \) and there exists \( \theta \in k \setminus \{0\} \) with \( \text{In}(f) = \theta y^{c_1} \). Also, if \( c_1 > p_1 \), then \( \text{ord}(f) = v(\overline{y}) = p_1 \) and there is \( \theta' \in k \setminus \{0\} \) with \( \text{In}(f) = \theta' x^{p_1} \). By multiplying with an element of \( k \setminus \{0\} \), we can assume in every case that \( \text{In}(f) = (\lambda x + \mu y)^{\min(c_1, p_1)} \), with \( \lambda, \mu \in R \) not vanishing simultaneously. If \( \lambda = 0 \), then \( f \) is said to be \( y \)-regular; if \( \mu = 0 \), then \( f \) is said to be \( x \)-regular.

Let \( f \in R \) be an analytically irreducible residually rational \( y \)-regular curve with \( f \neq uy \) for some unit \( u \in R \). Set \( \text{HN} := \text{HN}(x, y; f) \), \( h := h(\text{HN}(x, y; f)) \), \( r = r(\text{HN}(x, y; f)) \), \( d = d(\text{HN}(x, y; f)) \). We adapt some results proven for algebroid curves in [13] to our more general case. Next lemma corresponds to [13, Lemma 2.10].

(7.3) **Lemma:** Let

\[
\text{HN} = \begin{pmatrix}
P_i & \ldots & P_{i+1} \\
c_1 & \ldots & c_{i+1}
\end{pmatrix}_{1 \leq i \leq l}, \quad \text{HN}' = \begin{pmatrix}
P_i' & \ldots & P_{i+1}' \\
c'_1 & \ldots & c'_{i+1}
\end{pmatrix}_{1 \leq i \leq l}
\]

be two Hamburger-Noether tableaux, and let \( s \in \mathbb{N} \). The following statements are equivalent:

1. We have \( \frac{p_j}{c_1} = \frac{p'_j}{c'_1} \) for every \( j \in \{1, \ldots, s\} \).

2. We have \( \frac{p_j}{c_1} = \frac{p'_j}{c'_1} \) for all \( i, j \in \{1, \ldots, s\} \).

3. We have \( \frac{p_j}{c_1} = \frac{p'_j}{c'_1} \) for every \( j \in \{1, \ldots, s\} \).

Moreover, each of these conditions implies that

\[
\frac{p_j}{c_{s+1}} = \frac{p'_j}{c'_{s+1}} \quad \text{and} \quad \frac{c_j}{c_{s+1}} = \frac{c'_j}{c'_{s+1}} \quad \text{for every} \quad j \in \{1, \ldots, s\}.
\]

(7.4) **Definition:** Let

\[
\text{HN} = \begin{pmatrix}
P_i & \ldots & P_{i+1} \\
c_1 & \ldots & c_{i+1}
\end{pmatrix}_{1 \leq i \leq l}, \quad \text{HN}' = \begin{pmatrix}
P_i' & \ldots & P_{i+1}' \\
c'_1 & \ldots & c'_{i+1}
\end{pmatrix}_{1 \leq i \leq l}
\]
be two Hamburger-Noether tableaux. We set

\[
S(HN, HN') = \{0\} \cup \{j \in \mathbb{N} \mid \frac{p_i}{c_i} = \frac{p_i'}{c_i'}, a_i = a_i' \text{ for } i \geq j\}.
\]

\[
s(HN, HN') = \sup(S(HN, HN')).
\]

Notice that if \(HN = HN'\), then we have \(s(HN, HN') = \infty\).

\textbf{(7.5) Lemma:} Let \(f, g \in R\) be two analytically irreducible and residually rational curves. Let

\[
HN := HN(x, y; f) = \begin{pmatrix} p_i & c_i \\ a_i \end{pmatrix}_{1 \leq i \leq l}, \quad HN' := HN'(x, y; g) = \begin{pmatrix} p'_i & c'_i \\ a'_i \end{pmatrix}_{1 \leq i \leq l}
\]

be the Hamburger-Noether tableaux of \(f\), resp. of \(g\). Set \(s := s(HN, HN')\). Then we have

\[
\iota(f, g) = \sum_{i=1}^{s} p_i c'_i + \min\{\{p_{s+1} c'_{s+1}, p'_{s+1} c_{s+1}\}\}
\]

\[
= \sum_{i=1}^{s} p'_i c_i + \min\{\{p_{s+1} c'_{s+1}, p'_{s+1} c_{s+1}\}\}.
\]

\textbf{Proof.} The reasoning is much more similar as that for algebroid curves in [13, Theorem 3.3]. \(\Box\)

\textbf{(7.6) Definition:} Let \(\mu \in \mathbb{N}\). A Hamburger-Noether tableau \(HN'\) is called a \(\mu\)-th approximation to \(HN\) if

1. \(s(HN, HN') = \mu - 1\);
2. \(c'_\mu = 1\);
3. \(p'_\mu c'_\mu \geq p_\mu\).

\textbf{(7.7) Remark:} Let \(HN'\) be a \(\mu\)-approximation to \(HN\). In this case we have

\[
p'_i = \frac{p_i}{c_\mu} \quad \text{and} \quad c'_i = \frac{c_i}{c_\mu}
\]

for every \(i \in \{1, \ldots, \mu - 1\}\).

\textbf{(7.8) Definition:} A curve \(g \in R\) is called a \(\mu\)-approximation to \(f\) if \(g\) is analytically irreducible, residually rational and if \(HN(x, y; g)\) is a \(\mu\)-th approximation to \(HN(x, y; f)\).
(7.9) Proposition: Let \( g \in R \) be a \( \mu \)-th approximation to \( f \). There exists a curvette \( h \in S \) for some \( S \in N(R) \) with \( hS \cap R = gR \). Conversely, given a curvette \( h \in S \) for some \( S \in N(R) \), there exists a \( \mu \)-th approximation \( g \) to \( f \) such that \( hS \cap R = gR \).

Proof. Let \( g \in R \) be a \( \mu \)-th approximation to \( f \). Assuming \( \text{ord}(g) = l \) and \( \mathfrak{m} \) does not divide \( h \) mod \( \mathfrak{m}^{l+1} \), then the strict transform of \( g \) in \( S \) is \( \frac{g}{y} S = (gR)^S \), and \( \left( \frac{y}{g}, \frac{h}{y} \right) \) is a regular system of parameters of \( S \). Then \( h := \frac{g}{y} \) is a curvette in \( S \) with \( hS \cap R = g \) (by the same reasoning as in the proof of Proposition (6.2)). Conversely, if \( h \in S \) is a curvette for some \( S \in N(R) \), again by Proposition (6.2) there exists an irreducible element \( g \in R \) with \( hS \cap R = gR \). We will see now that \( g \) is a \( \mu \)-th approximation to \( f \). In fact, the curve \( g \in R \) is analytically irreducible and residually rational (see Remark (6.3)). Let us take

\[
\text{HN}' := \text{HN}'(x, y; g) = \left( \frac{p'_i}{c'_i} \right)_{1 \leq i \leq l}
\]

and set \( s = s(\text{HN}, \text{HN}') \). By Lemma (7.5) we have

\[
\iota(f, g) = \sum_{i=1}^{s} p_i c'_i + \delta = \sum_{i=1}^{s} p'_i c_i + \delta,
\]

with \( \delta = \min\{p_{s+1}c'_{s+1}, p'_{s+1}c_{s+1}\} \). By the reasonings in (5.6) - (5.8), we have that \( d_j \mid p_i \) for every \( i \in \{1, \ldots, \mu - 1\} \) and \( d_j \mid c_i \) for every \( i \in \{1, \ldots, \mu\} \). If \( s + 1 < \mu \), then \( d_j \mid \iota(f, g) \); if \( s + 1 = \mu \) and \( \delta = p'_{s+1}c_{s+1} \), then \( d_j \mid \iota(f, g) \). But under our assumptions \( d_j \) is not a divisor of \( \iota(f, g) \); it means that

\[
s + 1 \geq \mu; \quad \text{and, if } s + 1 = \mu \quad \Rightarrow \quad \delta = p_{s+1}c'_{s+1}.
\]

(\*)

By Lemma (7.3) we find that

\[
c'_i = c_i \frac{c'_{s+1}}{c_{s+1}} \geq \frac{c_i}{c_{s+1}} \geq \frac{c_i}{c_\mu} \quad (**)\]

for every \( i \in \{1, \ldots, \mu - 1\} \). By Lemma (7.5) and (\*), we have

\[
\iota(f, g) = \sum_{i=1}^{\mu-1} p_i c'_i + \delta',
\]

with \( \delta' \geq \delta \). Using (**) and the representation of \( r_j \) given above, we see that \( c'_i = \frac{c'_i}{c_\mu} \) for every \( i \in \{1, \ldots, \mu - 1\} \), and \( \delta' = \delta \). Therefore we have \( c'_i = \frac{c_i}{c_\mu} \) and \( c'_{s+1} = 1, \ c_\mu = d_j = c_{s+1} \) by (**).. Since \( \mu \) is a characteristic index of \( \text{HN}(x, y; f) \), we must have \( \mu = s + 1 \). Therefore \( g \) is a \( \mu \)-th approximation to \( f \). \( \diamond \)
Remark: Let \( g \in R \) be a \( \mu \)-th approximation to \( f \). We have

\[
\iota(f, g) = \sum_{i=1}^{\mu} p_i \frac{c_i}{c_\mu}
\]

Proposition: Let \( f \in R \) be an analytically irreducible residually rational \( y \)-regular curve with \( f \neq uy \) for some unit \( u \in R \). Let \( C_s \) be the cluster associated with the minimal resolution \( \pi = \pi_1 \circ \ldots \circ \pi_s \) of \( f \). Let \( S_j \) be a non-singular point of the \( j \)-th component of the exceptional divisor of \( \pi_s \), for \( 1 \leq j \leq s \). Let \( g \in S_j \) be an analytically irreducible and residually rational curve. The following assertions are equivalent:

1. \( g \) is a curvette on \( S_j \);
2. \( \iota(f, gS \cap R) = r_j \).

Proof. By Proposition (7.9), the curve \( g \) is a \( \mu \)-th approximation to \( f \), with \( \mu \) is the \( j \)-th characteristic index of \( \text{HN}(x, y; f) \), i.e., we have \( \mu = i_j \), for \( j \in \{1, \ldots, h\} \). Assume that \( j = 1 \) and \( d_1 \mid r_1 \). Then it is easy to see that (1) and (2) are equivalent (to prove (2) \( \Rightarrow \) (1), use Lemma (7.5)). Assume that \( j > 1 \), or \( j = 1 \) and \( d_1 \nmid r_1 \). Since \( \mu = i_j \), by (5.7) and (5.8) we have

\[
\sum_{i=1}^{\mu} p_i \frac{c_i}{c_\mu} = \sum_{i=1}^{j} q_i \frac{d_i}{d_j} = r_j
\]

and \( d_j = c_\mu \). By Remark (7.7) and (5.6)-(5.8), we see that (2) follows from (1). Conversely, assume (2) holds. Let

\[
\text{HN}' := \text{HN}'(x, y; g) = \begin{pmatrix} p'_i \\ c'_i \\ d'_i \end{pmatrix}_{1 \leq i \leq l}
\]

By setting \( s = s(\text{HN}, \text{HN}') \), we can proceed as in the proof of the previous Proposition (7.9). \( \diamond \)

References

[1] A. Campillo: *Algebroid curves in positive characteristic*. Lecture Notes in Math. **813**, Springer, Berlin-Heidelberg, 1980.

[2] A. Campillo, A.J. Reguera: *Combinatorial aspects of sequences of point blowing-ups*. Manuscripta math. **84** (1994), 29–46.

[3] E. Casas-Alvero: *Singularities of Plane Curves*. London Math. Society, Lecture Note Series **276**, Cambridge U.P., Cambridge, 2000.
[4] P. Deligne: *Intersections sur les surfaces regulieres*. Exposé X in: “Groupes de Monodromie en Géométrie Algébrique”. Lecture Notes in Math. 340. Berlin-Heidelberg, Springer, 1973.

[5] F. Enriques, O. Chisini: *Lezioni sulla teoria geometrica delle equazioni e delle funzioni algebriche* (1915). Reprinted by CM5 Zanichelli, 1985.

[6] E. Kunz: *Einführung in die kommutative Algebra und algebraische Geometrie*. Friedr. Vieweg & Sohn Verlagsgesellschaft mbH, Braunschweig, 1980.

[7] K. Kiyek, J. L. Vicente: *Resolution of Curve and Surface Singularities in Characteristic Zero*. Kluwer, Dordrecht, 2004.

[8] J. Lipman: *Rational singularities with applications to algebraic surfaces and unique factorization*. Publ. Math. IHES 36 (1969), 195–279.

[9] J. Lipman: *On complete ideals in regular local rings*. In *Algebraic Geometry and Commutative Algebra*, vol. I, in honor of Masayoshi Nagata, pp. 203-231. Kinokuniya, 1987/1988.

[10] J. Lipman: *Adjoints and polars of simple complete ideals*. Bull. Soc. Math. Belgique, Sér. A, 45 (1993), 223–244.

[11] Q. Liu: *Algebraic Geometry and Arithmetic Curves*. Oxford U.P., Oxford, 2002.

[12] M. Rybowicz: *Sur le calcul des places et des anneaux d’entiers d’un corps de fonctions algébriques*. Ph.D. Thesis, Limoges, 1990.

[13] P. Russell: *Hamburger-Noether expansions and approximate roots of polynomials*. Manuscripta Math. 31 (1980), 25–95.

[14] P. Du Val: *Reducible exceptional curves*. Amer. J. Math. 58 (1936), 285–289.

[15] O. Zariski: *Polynomial ideals defined by infinitely near base points*. Amer. J. Math. 60 (1938), 151–204.