Supplement B: statistical analysis of two-foil detectors

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The setup of a two-foil detector is discussed in Section V in [1]. This appendix first presents the Fisher information analysis of two-foil detectors assuming non-pixelated detectors, and then discuss the statistical properties of pixelated data.

I. FISHER INFORMATION ANALYSIS

A. Analysis of a single foil

We start the analysis by considering just one phosphor foil, some distance from the image intensifier and illuminated by a stream of charged particles. Due to the strong scattering of light in the phosphor, we can treat the phosphor as an ideal diffuse emitter, which obeys Lambert’s emission law [2] and is referred to as a Lambertian emitter. A Lambertian emitter has the same radiance, which is defined as power per unit solid angle per unit projected source area [2], when viewed from any angle. The distribution of irradiance on the face of the intensifier (which we denote as the plane \( \theta = \theta_{opt} \)), as opposed to \( \cos^3 \theta_{opt} \) from an isotropic point source), where \( \theta_{opt} \) is the angle between the normal to the two planes and the direction of light propagation.

Suppose there are \( K \) particles passing through the phosphor and event \( k \) \( (k = 1, ..., K) \) produces a small spot of light at a point on the phosphor specified by the 2D vector \( \mathbf{r}_k \). The irradiance on the intensifier at point \( \mathbf{r} \) is given by

\[
I(\mathbf{r} | \mathbf{r}_k) \propto \cos^4 \theta_{opt} = \frac{z_p^4}{|\mathbf{r}_k - \mathbf{r}|^2 + z_p^2}, \tag{1}
\]

where \( \mathbf{r} = (x, y) \) and \( |\mathbf{r} - \mathbf{r}_k| \) is the magnitude of the 2D vector \( \mathbf{r} - \mathbf{r}_k \).

Suppose the optical photons generated by the \( k \)th interaction produce a total of \( \mathbf{r}_k \) photoelectrons. The probability density function (PDF) for a particular photoelectron, say the \( j \)th, to be produced at \( \mathbf{r} = \mathbf{r}_{kj} \) is

\[
pr(\mathbf{r}_{kj} | \mathbf{r}_k) = \frac{z_p^2/\pi}{|\mathbf{r}_k - \mathbf{r}_{kj}|^2 + z_p^2}. \tag{2}
\]

In an idealized representation of the data, we know the set of all photoelectron positions, \( \{ \mathbf{r}_{kj} ; j = 1, ..., J_k \} \), as well as \( J_k \) itself. From these data, which we denote as \( \mathbf{G}_k \), we wish to estimate the interaction position \( \mathbf{r}_k \). The maximum-likelihood (ML) estimate [2] of the position of interaction is:

\[
\hat{\mathbf{r}}_k = \text{argmax} \sum_{j=1}^{J_k} \ln \left[ \frac{z_p^2/\pi}{|\mathbf{r}_k - \mathbf{r}_{kj}|^2 + z_p^2} \right], \tag{3}
\]

where the circumflex denotes an estimate.

The \( (p, q) \)th element in a Fisher information matrix [2, 3] is given by

\[
F_{pq} = -\int_{-\infty}^{\infty} d^J \mathbf{G} \frac{\partial^2}{\partial r_p \partial r_q} \ln pr(G | \mathbf{r}), \tag{4}
\]

where

\[
pr(G | \mathbf{r}) = \prod_{j=1}^{J} pr(r_j | \mathbf{r}) = \prod_{j=1}^{J} \frac{z_p^2/\pi}{|\mathbf{r} - \mathbf{r}_j|^2 + z_p^2}. \tag{5}
\]

The integration results are

\[
F_{xx} = F_{yy} = \frac{4N_{pe}}{3z_p^2}; F_{xy} = F_{yx} = 0, \tag{6}
\]

where \( N_{pe} = J \), is the number of detected photoelectrons. The Cramèr-Rao lower bound sets a lower bound on the variance of any unbiased estimator. Therefore, the variation of the estimator satisfy

\[
\sigma(x) = \sigma(y) \geq \frac{\sqrt{3/4}}{\sqrt{N_{pe}}} \cdot z_p \approx 0.866 \sqrt{N_{pe}} \cdot z_p. \tag{7}
\]

The Cramèr-Rao lower bound is compared to simulation results in FIG. 1.

![FIG. 1. Estimation of spatial resolution for a single-foil detector, where the phosphor foil is 10µm away from the face of the intensifier. The Cramèr-Rao bound is plotted in line; the simulation result is plotted in circles. In the simulation, the estimation is repeated 1000 times.](image-url)
B. Analysis of a two-foil detector

Now consider two foils, one in the plane \( z = -z_1 \) and the other in the plane \( z = -z_2 \), where \( z_1 > z_2 > 0 \). Thus the beta particle first encounters foil 1, then foil 2. We denote the two interaction positions, which we will denote as the 2D vectors \( \mathbf{r}_1 \) and \( \mathbf{r}_2 \). The two interaction positions are related by

\[ \mathbf{r}_2 = \mathbf{r}_1 + s_{\perp}d, \]

where \( d = z_1 - z_2 \) and \( s_{\perp} \equiv (s_x, s_y) \) is a 2D vector in the direction of \( \mathbf{r}_2 - \mathbf{r}_1 \).

Our goal is to estimate the position and propagation direction of a charged particle right before the particle enters foil 1. Due to the random deflection of the particle in the first foil, the propagation direction before the particle enters foil 1 is not directly measurable. However, if the particle is only slightly scattered by the first foil, the direction of the particle after passing through foil 1 (described by \( s_{\perp} \)) can be used as an approximation of the propagation direction, so it will suffice to estimate the two 2D vectors \( \mathbf{r}_1 \) and \( s_{\perp} \).

If we have ML estimates of \( \mathbf{r}_{1k} \) and \( \mathbf{r}_{2k} \) for each event \( k \), we can invoke the maximum-likelihood invariance theorem which states that a ML estimate of a function of a parameter is that same function of the ML estimates of the parameter. The ML estimate of \( s_{\perp k} \) is thus

\[ \hat{s}_{\perp k} = \frac{\mathbf{r}_{2k} - \mathbf{r}_{1k}}{d}. \]

Denote \( \hat{\theta}_x \) as the angle between the plane \( y = 0 \) and the direction of particle propagation, in which case \( \tan \hat{\theta}_x = \frac{\hat{s} \perp x}{\hat{s} \perp z} \). The uncertainty of \( \hat{\theta}_x \) is given by

\[ \sigma(\hat{\theta}_x) = \cos^2 \hat{\theta}_x \cdot \frac{\sqrt{\sigma_{\perp x}^2 + \sigma_{\perp z}^2}}{d}. \]

As a preliminary step to study the angular resolution of a two-foil detector, let us assume we can estimate \( \mathbf{r}_{1k} \) and \( \mathbf{r}_{2k} \) respectively with uncertainties given by Equation (7), derived for single foils. Furthermore, let us assume the position shift of the particle passing through one foil is negligible. The angular uncertainty is then found to be

\[ \sigma(\hat{\theta}_z) = \cos^2 \hat{\theta}_z \cdot \frac{0.866}{\sqrt{N_{pc}}} \cdot \frac{\sqrt{\hat{x}_1^2 + \hat{x}_2^2}}{z_1 - z_2}. \]

When \( z_1 \gg z_2 \), the above equation is simplified to

\[ \sigma(\hat{\theta}_z) = \cos^2 \hat{\theta}_z \cdot \frac{0.866}{\sqrt{N_{pc}}}. \]

where the constant 0.866 is from Equation (7).

The Fisher information analysis results are applied to estimate the uncertainty of a simulated detector which is used in a BET system as discussed in [1].

II. STATISTICAL ANALYSIS OF PIXELATED DETECTORS

This section discusses the statistical analysis of CCD/CMOS-based scintillation cameras in the context of a two-foil detector.

If we denote the raw signals on the CMOS camera as \( \mathbf{g} \) and the true attributes of a detected particle as \( \theta = (r_1, r_2, E_1, E_2) \), the probability density function for the signals conditioned on a particular attribute vector, which is denoted as \( \Pr(\mathbf{g}|\theta) \), is referred to as a likelihood function. The likelihood function tells how likely it is that given signals \( \mathbf{g} \) are obtained when some underlying properties \( \theta \) of a detected particle are true.

The scintillation photons are converted to photoelectrons at the continuous photocathode of the image intensifier. If we consider a pixel array at the photocathode defined by the discretization of the final readout, the number of photoelectrons generated in the \( m \)th pixel follows Poisson distribution with a mean given by

\[ \bar{n}_m(\theta) = \eta_{QE} \int_{S_m} d^2\mathbf{r} I(\mathbf{r}|\theta) + \bar{n}_{dark}^m, \]

where \( \eta_{QE} \) is the quantum efficiency of the photocathode, \( S_m \) is the area of the \( m \)th pixel, \( I(\mathbf{r}|\theta) \) is the irradiance on the photocathode of the image intensifier, and \( \bar{n}_{dark}^m \) is the mean number of electrons generated in the dark by thermal excitation. For a two-foil detector, if we assume that the second foil is transparent to the photons produced by the first foil and the second foil does not amplify the signals produced by the first foil, then the irradiance on the intensifier input face at point \( \mathbf{r} \) is

\[ I(\mathbf{r}|\theta) = \sum_{i=1}^{2} \frac{Q(E_i) \cdot z_i^2}{\pi ||\mathbf{r} - \mathbf{r}_i||^2 + z_i^2}, \]

where the subscript \( i \) indicates the \( i \)th phosphor foil and \( Q(E) \) is the radiant flux produced by the phosphor when the energy \( E \) is deposited.

For an image intensifier, there is a random gain process in the Micro-Channel Plate (MCP) following the conversion of scintillation photons to photoelectrons. The mean of the final pixel value \( g_m \) is [4]

\[ \bar{g}_m(\theta) = \bar{A} \bar{n}_m(\theta), \]

where \( \bar{A} \) is the mean amplification, including the effect of all components between the photocathode and the final readout.

The variance of \( g_m \), including the variance of the amplification and the readout noise of the CMOS camera, is given by [4]

\[ \sigma_m^2 = \bar{n}_m [\text{Var}(\bar{A}) + \bar{A}^2] + \sigma_{read}^2, \]

where \( \text{Var}(\bar{A}) \) is the variance of the amplification and \( \sigma_{read}^2 \) is the variance of the readout noise.

If the signal in one pixel does not influence the signal on an adjacent channel, which means that the pixel
size of the CMOS camera is larger than the blur introduced by the optics between the photocathode and the CMOS camera, then the covariance matrix for $g$ is diagonal with diagonal element equals to $\sigma_m^2$. If we further approximate the Poisson distribution for $n_m$ by a Gaussian and assume that both the gain noise and the readout noise are Gaussian, we have

$$p_r(g|\theta) = \prod_{m=1}^{M} \frac{1}{\sqrt{2\pi\sigma_m^2(\theta)}} \exp \left( - \frac{(g_m - \bar{g}_m(\theta))^2}{2\sigma_m^2(\theta)} \right),$$

(15)

When the number of photoelectrons is very small, it is not accurate to approximate the distribution of the number of photoelectrons $n_m$ as a Gaussian distribution and discussion in Section I of this supplement is more appropriate. When Poisson noise and Gaussian noise are combined [5], the likelihood model is fairly complex. Therefore, we relied on the Gaussian approximation for our preliminary demonstration.

[1] Yijun Ding, Luca Caucci, and Harrison H. Barrett, “Charged-particle emission tomography,” (2017).
[2] Harrison H. Barrett and Kyle J. Myers, Foundations of Image Science (John Wiley & Sons, 2004).
[3] Ronald Aylmer Fisher, “Theory of statistical estimation,” in Mathematical Proceedings of the Cambridge Philosophical Society, Vol. 22 (Cambridge University Press, 1925) pp. 700–725.
[4] Brian W. Miller, H. Bradford Barber, Lars R. Furenlid, Stephen K. Moore, and Harrison H. Barrett, “Progress of BazookaSPECT,” in SPIE Optical Engineering + Applications (International Society for Optics and Photonics, 2009) pp. 74500C–74500C.
[5] Harrison H. Barrett, Christopher Dainty, and David Lara, “Maximum-likelihood methods in wavefront sensing: stochastic models and likelihood functions,” JOSA A 24, 391–414 (2007).