Supplementary Materials for

Moiré metasurfaces for dynamic beamforming

Shuo Liu et al.

Corresponding author: Tie Jun Cui, tjcui@seu.edu.cn; Shuang Zhang, shuzhang@hku.hk

Sci. Adv. 8, eabo1511 (2022)
DOI: 10.1126/sciadv.abo1511

This PDF file includes:

Sections S1 to S5
Figs. S1 to S19
Section S1. THEORETICAL, NUMERICAL, AND EXPERIMENTAL RESULTS OF THE TRIANGULAR CASE

Supplementary Figure S1A shows the convolution result of the triangular patterns ($p3m1$ plane symmetry group). Similar to the chessboard case, two groups of moiré impulses (yellow and blue stars) are created by the vectorial sum of six diffraction components. Each group consists of six moiré impulses that respect the $C_6$ rotational symmetry around the origin. These two groups of moiré impulses gradually exchange their locations by increasing the twist angle (Supplementary Fig. S4). Supplementary Figures S1(B and C) show the moiré impulse trajectories for the triangular case, in which the periodicities of patterns 1 and 2 are set as: $p_1=11.4$ mm and $p_2=12$ mm. The trajectories of the two groups of moiré impulses are symmetric to each other along three high symmetry lines ($\phi=30^\circ$, $90^\circ$, $120^\circ$), and they move along opposite directions. The beam trajectories are shown in Supplementary Fig. S1D, in which the elevation angle continuously increases from $8.7^\circ$ to almost $90^\circ$ as pattern 2 rotates from $0^\circ$ to $19.2^\circ$ (red traces), and decreases from almost $90^\circ$ to $8.7^\circ$ as pattern 2 rotates from $40.8^\circ$ to $60^\circ$ (blue traces).

Supplementary Fig. S2A shows the simulation results of the 3D far-field patterns for the triangular pattern at $\psi=0.25^\circ$, $2^\circ$, $4^\circ$, $6^\circ$, $8^\circ$, $10^\circ$, $12^\circ$, and $13^\circ$ (top view in Supplementary S14). Six beams are located symmetrically on the azimuthal plane, with the beam intensity 25% lower in average compared to the chessboard case due to the larger beam number. One may notice from Supplementary Fig. S2A that only three beams are present at $\psi=10^\circ$, which can also be observed from the amplitude difference (blue and orange shaded curves) between the two groups of beams around $10.25^\circ$. The vanishing of three beams for the triangular moiré pattern at certain twist angles is caused by an abrupt change in the surface current distribution, but the basic physical mechanism behind this remains unclear. The six beams sweep from $10^\circ$ to $45^\circ$ in the elevation plane as $\psi$ increases from $0.25^\circ$ to $13^\circ$, as observed from the 2D radiation patterns extracted from the cutting plane of beams #1 and #4 in the linear (Supplementary Fig. S2B) and dB scaling (Supplementary Fig. S15). The beam trajectories follow closely the theoretical predictions, except for a few points at the elevation angles around $42^\circ$ (Supplementary Fig. S2C). The reflected beam is principally left circularly polarized, as confirmed by the beam ellipticity in Supplementary Fig. S16.
Fig. S1. Formation of the moiré impulse, the beam trajectories, and their trajectories for the triangular-type moiré metasurface. (A) Spectrum convolution process for the chessboard-type ($p3m1$ plane symmetry group) moiré patterns at $\psi=10^\circ$. The dominant frequency components for pattern 1 (red square dots) and pattern 2 (green triangular dots) are labeled by two integers, which represent the diffraction orders along the primary reciprocal lattice vectors in the square and triangular lattices. For illustrative purpose, the amplitudes of the primary reciprocal lattice vectors $b_1$ and $b_2$ are set as 3 and 2.5. The spectrum convolution process produces two groups of moiré impulses (yellow and blue stars), labeled with the frequency component indices of patterns 1 and pattern 2. The moiré impulses inside the light cone (red circle) may contribute to real radiations. (B and C) Global view and zoomed view of the moiré impulse trajectories plotted in the polar coordinate for the triangular-type moiré metasurface ($p1=11.4$ mm, $p_2=12$ mm) as pattern 2 rotates from $0^\circ$ to $60^\circ$ (counterclockwise). (D) Beam trajectories of the triangular-type moiré metasurface obtained from the coordinate
transformation of the two groups of the moiré impulses in (B and C). The red (blue) trajectories correspond to the beams trajectories of the first (second) group of the moiré impulses, which varies by the elevation angle from $8.7^\circ$ to $90^\circ$ ($90^\circ$ to $8.7^\circ$) as $\psi$ increases from $0^\circ$ to $19.2^\circ$ ($40.8^\circ$ to $60^\circ$). The arrow indicates the moving direction of the moiré impulses with the increasing of twist angle.

**Fig. S2. Numerical Simulation results for the triangular-type moiré metasurface.** (A) Simulated 3D radiation patterns (LHCP components) for a round-shape triangular-type moiré metasurface with 250mm diameter illuminated by an LHCP plane wave for the twist angles $0.25^\circ$, $2^\circ$, $4^\circ$, $6^\circ$, $8^\circ$, $10^\circ$, $12^\circ$, and $13^\circ$. Six beams with fairly uniform intensities are identified at most rotation angles, except around $\psi=10.5^\circ$, at which only three beams are present, as can be clearly observed from the averaged value (solid line) and fluctuation (colored region) of the two beam groups. For clarity of content, the 3D radiation pattern at $\psi=13^\circ$ is placed at the location of $\psi=14^\circ$. (B) Cutting-plane view of the 3D radiation patterns in (A) extracted from the plane of the maximum radiation of Beams #1 and #4. (C) Trajectories and normalized amplitudes (colormap) of the beam peaks as $\psi$ increases from $0.25^\circ$ to $13^\circ$ with the step of $0.25^\circ$. 
which agrees well with the theoretical prediction (red line) except around the elevation angle of 42°. Color intensity in (C) represents the normalized amplitudes. The beams are indexed in (C). Note that the 3D radiation patterns in (A) have been self-normalized.

The 3D far-field patterns measured for the triangular pattern (Supplementary Figs. S3) also show good agreement with the numerical simulations (Supplementary Fig. S2A, Supplementary Fig. S14). As the two flexible moiré patterns may not be perfectly aligned and contacted, the three-beam radiation pattern that originally occurs at $\psi=10^\circ$ in the numerical simulation (Supplementary Fig. S2A) shifts to $\psi=11^\circ$ in the experiment (Supplementary Fig. S18). The measured results (Supplementary Figs. S15(I to P)) are overall in good agreement with the numerical simulations (Supplementary Figs. S15(A to H)) in both elevation angle and beam efficiency.

Fig. S3. Experimental results of far-field radiations for the triangular-type moiré metasurface. (A to J), Photo of the fabricated sample (A) and measured far-field patterns (B to J). The far-field patterns at twist angles 6°, 8°, 10°, and 12° are measured at six equally spaced elevation planes (4° interval in the azimuthal direction) around the direction of the six beam maxima, as shown in the perspective view (B to E) and top view (G to J), which agree well with the simulations (Supplementary Fig. S2(A), Supplementary Fig. S14). The measured radiation patterns also show precise beam angles in the elevation plane (F) as compared to the simulation results in Supplementary Fig. S2(B).
There are several limitations for the designed moiré metasurfaces. One obvious constraint is that the beams always appear in pairs and are symmetric with respect to the surface normal. Asymmetric beam radiations might offer more flexibility for the wireless communications and radar applications, which can be achieved with low-symmetry patterns. We can also realize multi-path beam trajectories by stacking multiple layers of metasurfaces. We will further try to formulate a generalized theory for multi-layered moiré metasurfaces based on group theory, which summarizes all types of beam trajectories for all possible combinations of pattern symmetries.

Another issue is the absence of accurate and cost-efficient method for calculating the radiation of the physical structure. This is because the divergence of the moiré periodicity and the extremely fine details in the moiré pattern make it almost impossible for analytical modelling. Machine learning will be considered in our future structure design and optimization.

Section S2. ROTATIONAL SYMMETRY OF MOIRÉ PATTERN

There are uncountable number of different moiré patterns by continuously increasing the mutual rotation angle $\psi$. But there exists an angle periodicity for the moiré pattern to repeat with the increasing of mutual rotation. For example, the minimum angle periodicity $\psi_p$ for the configuration with both layers having $p4m$ symmetry group is $45^\circ$, or mathematically as,

$$\psi_p = \frac{90^\circ \cdot (2n-1) - \psi}{\psi}.$$ (S.1)

in which $I(\psi)$ is the moiré pattern at the mutual twist angle $\psi$, $M_{x,y}$ is the mirror symmetry operation with respect to the $x$-$z$ and $y$-$z$ planes, and $n$ is an integer. Eq. (S.1) indicates that the moiré pattern in the minimum angle periodicity exhibits mirror symmetry with respect to the critical twist angle $\psi_{critical} = 90^\circ \cdot n + 45^\circ$. It should be noted that the mutual rotation axis needs to be at the rotation center of each pattern.

A minimum angle periodicity $\psi_p$ of $60^\circ$ exists for the case when both patterns respect the $p3m1$ symmetry group

$$I(\phi) = M_z(\{120^\circ \cdot (2n-1) - \phi\})$$ (S.2)
in which \( M_3 \) is the mirror symmetry operation with respect to the three reflection axes of the hexagonal lattice (along \( \phi=30^\circ, 90^\circ, 120^\circ \)). In this case, \( \psi_{\text{critical}} \) becomes \( 60^\circ \cdot n + 60^\circ \). Due to the different symmetries of the two patterns, the superimposed moiré pattern evolves differently within an angle period for the above two cases, as shown in Figure 2 in the main text. For example, in the triangular case, sweeping the twisting angle through a single period incurs a complete trajectory of the moiré impulses. While for the chessboard case, two angle periods are required to form a complete moiré impulse trajectory, which respect reflection symmetry to the \( x-z \) and \( y-z \) planes. We conclude that the symmetries of both patterns determine the rotational symmetry of the moiré metasurface. This principle is further verified by the case of square patch design shown in Supplementary Fig. S5. While the trajectories of their moiré impulse look different, \( \psi_p \) and \( \psi_{\text{critical}} \) are kept the same with the chessboard case.

**Fig. S4. Moiré metasurface in reciprocal space.** Spectrum convolution process for the (A) chessboard-type moiré pattern (\( p4m \) plane symmetry group) at \( \psi=80^\circ \) and (B) triangular-type moiré pattern (\( p3m1 \) plane symmetry group) at \( \psi=50^\circ \), respectively. The marker representations are the same with those in Fig. 2(A) in the main text.

Supplementary Fig. S6 shows the moiré impulse trajectories and beam traces when we combine the chessboard pattern and triangular pattern, which produces a pair of beams scanning in both elevation and azimuthal directions, and respecting \( C_2 \) rotation symmetry about the origin. The first pair of beams (red) scans from grazing angle to 25° as \( \psi \) increases from 0° to 15°, and
then from 25° to grazing angle as $\psi$ increases from 15° to 30°. The second pair of beams (blue) scans from grazing angle to 25° as $\psi$ increases from 30° to 45°, and then from 25° to grazing angle as $\psi$ increases from 45° to 60°. The two pairs of beams trajectories respect reflection symmetry along the $x$ and $y$ axes.

Fig. S5. Moiré metasurface with regular-square-patch pattern in reciprocal space. (A and B) Spectrum convolution process at $\psi=10^\circ$ and 80°, respectively. For illustrative purpose, the amplitude of primary reciprocal lattice vectors $b_1$ and $b_2$ are set as 3 and 2.5. (C and D) Global view and zoomed view of the moiré impulse trajectories plotted in the polar coordinate as pattern 2 rotates counterclockwise from 0° to 90°. The periodicities of the two patterns are: $p_1=12$ mm, $p_2=11.4$ mm. e, Beam trajectories obtained from coordinate transformation of the two groups of moiré impulses in (C and D). The marker representations are the same with those in Fig. 1(C) and Fig. 2 in the main text. As the square patch pattern respects the same symmetry ($p4m$) as the chessboard pattern, their moiré impulse trajectories and beam trajectories exhibit
the same rotational symmetry during mutual twist.

Fig. S6. Moiré impulse trajectories and beam trajectories for chessboard-triangular-type moiré metasurface. Pattern 1: chessboard pattern, \( p_1 = 11.2 \text{mm} \); Pattern 2: triangular pattern, \( p_2 = 10.5 \text{mm} \). (A and B) Global view and zoomed view of the moiré impulse trajectories plotted in the polar coordinate as pattern 2 rotates counterclockwise from \( 0^\circ \) to \( 60^\circ \). (C) Beam trajectories obtained from coordinate transformation of the two groups of moiré impulses in (A and B). The marker representations are the same with those in Fig. 1(C) and Fig. 2 in the main text.

Section S3. COMMENSURATION CONDITION FOR MOIRÉ METASURFACES

Following the method in Ref. 1, we derive the commensuration condition for more generalized cases when the two lattices have unequal periodicity. The commensuration cell, i.e., the rigorous periodicity of the moiré metasurface, contains the complete information of the moiré pattern and may facilitate the calculation of the radiation.

Let us first consider the commensuration condition of two hexagonal lattices with different periodicities. We start from the equation describing the coordinate transformation between the rotated lattice \( (m_1, m_2)^T \) and unrotated lattice \( (n_1, n_2)^T \),

\[
\begin{pmatrix}
    m_1 \\
    m_2
\end{pmatrix} = \begin{pmatrix}
    \cos \psi - \frac{1}{\sqrt{3}} \sin \psi & -\frac{2}{\sqrt{3}} \sin \psi \\
    \frac{2}{\sqrt{3}} \sin \psi & \cos \psi + \frac{1}{\sqrt{3}} \sin \psi
\end{pmatrix} \begin{pmatrix}
    n_1 \\
    n_2
\end{pmatrix}
\]

(S.3)
in which the matrix is the standard rotation matrix under the basis of hexagonal lattice vectors.
$a_1 = (\sqrt{3},0)$ and $a_2 = (\sqrt{3}/2,3/2)$, $c$ is a scaling factor multiplied to the second pattern. Obviously, $m_1, m_2, n_1, n_2$ must be integers for the two lattices to coincide exactly. This further requires the product of every element of the rotation matrix and the scaling factor $c$ to be a rational number,

$$\frac{1}{\sqrt{3}} \sin \psi \cdot c = \frac{i_1}{i_2}$$  \hspace{1cm} (S.4)

$$\cos \psi \cdot c = \frac{i_3}{i_4}$$  \hspace{1cm} (S.5)

where $i_1, i_2, i_3, i_4 \in \mathbb{Z}$. Combining Eqs. (S.4)-(S.5), we have

$$\frac{3i_1^2 + i_2^2}{i_1^2 + i_2^2} = c^2$$  \hspace{1cm} (S.6)

Let us assume $c^2$, for the moment, as a rational number $i_3/i_4$ ($i_3, i_4 \in \mathbb{Z}$), and substitutes into Eq. (S.6), we have

$$\frac{3i_1^2 + i_2^2}{i_1^2 + i_2^2} = k_1$$  \hspace{1cm} (S.7)

Now we reformulate Eq. (S.7) by redefining the variables in Eq. (S.7) with a new set of integers

$$i_1 i_2 = k_1, \quad i_3 i_4 = k_2, \quad i_1 i_2 = k_3, \quad i_1 i_2 = k_4,$$

$$\frac{3k_1^2 + k_2^2}{k_1^2 + k_2^2} = \frac{k_3}{k_4}$$  \hspace{1cm} (S.7)

in which

$$\frac{1}{\sqrt{3}} \sin \psi \cdot c = \frac{k_1}{k_2}$$  \hspace{1cm} (S.8)

$$\cos \psi \cdot c = \frac{k_3}{k_4}$$  \hspace{1cm} (S.9)

$$c^2 = \frac{k_4}{k_3}$$  \hspace{1cm} (S.10)

As apparent from Eq. (S.8-S.10), $k_1, k_2, k_3,$ and $k_4$ are not independent. Without loss of generality, let us firstly assume $k_1,k_2,k_3$ as independent variables, except for the case when $k_1=k_2=0$ and $k_3=0$. Then we can determine infinitely many unique combinations of $\psi$ and $c$ that satisfy the commensuration condition by,

$$\psi = \cos^{-1} \left( \frac{k_2}{\sqrt{k_4}} \right) = \cos^{-1} \left( \frac{k_2}{\sqrt{3k_1^2 + k_2^2}} \right)$$  \hspace{1cm} (S.11)

$$c = \frac{\sqrt{k_4}}{k_3} = \frac{\sqrt{3k_1^2 + k_2^2}}{k_3}$$  \hspace{1cm} (S.12)

As $\cos^2 \psi$ and $c^2$ are the set of all rational numbers, $\psi$ and $c$ are considered to be dense in
the real number field.

Substitute Eq. (S.9)-(S.10) back to Eq. (S.3), we arrive at the following coupled linear Diophantine equations,

\[ \begin{pmatrix} m_i \\ n_i \end{pmatrix} = \frac{1}{k_i} \begin{pmatrix} k_2 - k_1 \\
2k_1 \end{pmatrix} \begin{pmatrix} n_i \\ n_2 \end{pmatrix} \]  

(S.13)

Obviously, for a given set of \( k_1, k_2, k_3 \) in Eq. (S.13), which correspond to a certain combination of \( \Psi \) and \( c \), there exists an infinite set of integer solutions for \( (m_1, m_2)^T \) and \( (n_1, n_2)^T \). The set of solutions which has the smallest vector length correspond to the primitive vectors of the commensuration lattice. Supplementary Fig. S7 shows the vector length of the commensuration cell as a function of the \( \psi \) and \( c \), which is obtained by numerically finding the integer solution which has the smallest vector length for \( k_1, k_2, k_3 \) ranging from -100 to +100 (\( k_3 \neq 0 \)). The result is only presented for \( 0^\circ < \psi < 60^\circ \), which is a minimum angle period for the overlapped lattices to repeat. As we are interested in the case of two patterns having close periodicities, the scaling factor \( c \) is limited to the range of 0.95~1.05. To provide a clear view for the solutions which depend both \( \psi \) and \( c \), the results are projected on the \( \psi \) (Supplementary Fig. S7(A)) and \( c \) (Supplementary Fig. S7(B)) axes. Each black dot in the plots represents a unique solution for a given set of \( k_1, k_2, k_3 \), which correspond to a specific combination of \( \psi \) and \( c \). Note that the isolated dot in Fig. S4(A) at \( \psi = 0^\circ \) correspond to the case of two perfectly aligned identical lattices, whose commensuration cell is their primary unit cell.

It should be noted that the lower bound of the solutions for a given scaling factor \( c \) corresponds exactly to the vector length of the dominant moiré impulse, as outlined by colored lines for \( c=0.95 \) (red), \( c=1.0 \) (green), \( c=1.05 \) (blue). In other words, the periodicity of the commensuration cell is equal to the moiré periodicity for those solutions that fall on this line. The solutions located inside the red and green lines correspond to the lower bound of the solutions for \( 0.95 < c < 1.05 \) (Supplementary Fig. S7(C)). Based on the above analyses, we can always find a commensuration cell that are close to a given combination of \( \psi \) and \( c \).
**Fig. S7. Commensuration condition for two misoriented hexagonal lattice having different periodicities.** The solutions are presented by the vector length of the commensuration cell as a function of the $\psi$ and $c$, and is obtained by numerically finding the integer solution which has the smallest vector length for $k_1, k_2, k_3$ ranging from -100 to +100 ($k_3 \neq 0$). (A and B) Vector length of the commensuration cell as a function of the $\psi$ and $c$, respectively. (C) Vector length of the commensuration cell for $0^\circ < \psi < 30^\circ$. The colored lines are the vector length of the dominant moiré impulse, which correspond to the lower bounds of the vector length of the commensuration cell for $c=0.95$ (red), $c=1.0$ (green), $c=1.05$ (blue).

Similar approach can be applied to find the rigorous periodicity for the case when both layers have the chessboard pattern, which is equivalent to calculate the commensuration
condition between two misoriented square lattices with different periodicities.

\[
\begin{pmatrix}
  m_1 \\
  m_2
\end{pmatrix} =
\begin{pmatrix}
  \cos \psi \cdot c & -\sin \psi \cdot c \\
  \sin \psi \cdot c & \cos \psi \cdot c
\end{pmatrix}\begin{pmatrix}
  n_1 \\
  n_2
\end{pmatrix}
\]  

(S.14)

All the definition of the variables are kept the same as the previous case. For Eq. (S.14) to have integer solutions, the matrix must be rational, which further lead to the following restrictions on \( \psi \) and \( c \),

\[
\sin \psi \cdot c = \frac{k_1}{k_3}
\]  

(S.15)

\[
\cos \psi \cdot c = \frac{k_2}{k_3}
\]  

(S.16)

\[
c^2 = \frac{k_3}{k_3^2}
\]  

(S.17)

Let \( k_1, k_2, \) and \( k_3 \) to be independently chosen from the integer set, except for the case when \( k_1=k_2=0 \) and \( k_3=0 \). Then we can immediately determine infinitely number of unique combinations of \( \psi \) and \( c \) that satisfy the commensuration condition by,

\[
\psi = \cos^{-1}\left( \frac{k_2}{\sqrt{k_1}} \right) = \cos^{-1}\left( \frac{k_2}{\sqrt{k_1^2 + k_2^2}} \right)
\]  

(S.18)

\[
c = \frac{\sqrt{k_3}}{k_3} = \frac{\sqrt{k_1^2 + k_2^2}}{k_3}
\]  

(S.19)

Same as the previous case, \( \Psi \) and \( c \) are dense in the real number field. By substituting Eq. (S.15) - (S.17) back into Eq. (S.14), we obtain the coupled linear Diophantine equations for solving the primitive lattice vector of the commensuration cell,

\[
\begin{pmatrix}
  m_1 \\
  m_2
\end{pmatrix} = \frac{1}{k_3} \begin{pmatrix}
  1 & -k_1 \\
  k_1 & k_2
\end{pmatrix}\begin{pmatrix}
  n_1 \\
  n_2
\end{pmatrix}
\]  

(S.20)

Supplementary Figure S8 shows the numerically computed periodicity of the commensuration cell for the square lattices as a function of \( \psi \) and \( c \). For clarity of content, \( k_1,k_2,k_3 \) are limited to the range from -100 to +100 (\( k_3 \neq 0 \)). The results are presented only in the minimum angle periodicity \( 0^\circ < \psi < 90^\circ \) and \( 0.95 < c < 1.05 \), along the \( \psi \) (Supplementary Figure S8(A)) and \( c \) (Supplementary Figure S8(B)) axes. The lower bounds of the solutions at \( c=0.95 \) (red), \( c=1.0 \) (green), \( c=1.05 \) (blue) correspond to the moiré periodicities and are outlined with red, green, and blue lines, respectively. Similar to the triangular case, those solutions reside in the red and green lines represent the lower bound of the solutions for \( 0.95 < c < 1.05 \) (Supplementary Figure S8(C)).
**Fig. S8. Commensuration condition for two misoriented square lattice having different periodicities.** The solutions are presented by the vector length of the commensuration cell as a function of the $\psi$ and $c$, and is obtained by numerically finding the integer solution which has the smallest vector length for $k_1, k_2, k_3$ ranging from -100 to +100 ($k_3 \neq 0$). (A and B) Vector length of the commensuration cell as a function of the $\psi$ and $c$, respectively. (C) Vector length of the commensuration cell for $0^\circ < \psi < 45^\circ$. The colored lines are the vector length of the dominant moiré impulse, which correspond to the lower bounds of the vector length of the commensuration cell for $c=0.95$ (red), $c=1.0$ (green), $c=1.05$ (blue).
Section S4. The influence of in-plane misalignment on the \( k \)-space expression

For an infinitely large moiré metasurface, the in-plane alignment error only introduces an additional global phase shift to the entire \( k \)-space expression, and also the radiation pattern which corresponds to the \( k \)-space expression inside the light cone.

The formation of a moiré metasurface from two metallic patterns is equivalent to the logic operation ‘OR’ of two binary images in real space as,

\[
A(r) = A_1(r) \mid A_2(r) = A_1(r) + A_2(r) - A_1(r) \cdot A_2(r)
\]

(S.21)

and in \( k \)-space as,

\[
F_A(k) = F_{A_1}(k) + F_{A_2}(k) - \sum_{k} F_{A_1}(k_i) F_{A_2}(k-k_i)
\]

(S.22)

Now, let us consider an in-plane shift \( d \) of the first pattern,

\[
A(r) = A(r + d)
\]

(S.23)

which transforms into \( k \)-space as,

\[
F_A(k) = \frac{1}{2\pi} \int A(r + d) e^{-i\vec{k} \cdot \vec{r}} dr = e^{i\vec{k} \cdot \vec{d}} \frac{1}{2\pi} \int A(r + d) e^{-i\vec{k} \cdot \vec{r}} dr = e^{i\vec{k} \cdot \vec{d}} F_A(k)
\]

(S.24)

Then the moiré image of the case with in-plane displacement can be written in \( k \)-space as,

\[
F_A(k) = F_A(k) + F_{A_1}(k) - \sum_{k} e^{i\vec{k} \cdot \vec{d}} F_{A_1}(k_i) F_{A_2}(k-k_i)
\]

(S.25)

As each metallic pattern is periodic along the primitive translation vectors \( \vec{a}_1 \) and \( \vec{a}_2 \),

\[
A(r) = A(r + m \vec{a}_1 + n \vec{a}_2) = A(r + \vec{m} \cdot \vec{a}) \quad \text{Def:} \quad \vec{m} \cdot \vec{a} = m \vec{a}_1 + n \vec{a}_2
\]

(S.26)

its \( k \)-space representation is simply delta function sampled at the reciprocal lattice grid,

\[
F_A(k) = B_i(n_i, n) \delta(k - n \vec{b}_1 - n \vec{b}_2) = B_i(n) \delta(k - n \vec{b}) \quad \text{Def:} \quad n \vec{b} = n \vec{b}_1 + n \vec{b}_2
\]

(S.27)

Substituting Eq. (27) into Eq. (25),

\[
F_A(k) = e^{i \vec{k} \cdot \vec{d}} B_i(m_i) \cdot \delta(k - n_1 \vec{b}_1 + n_1 \vec{b}_2) + B_i(m_z) \cdot \delta(k - n_2 \vec{b}_2) - \sum_{k} e^{i \vec{k} \cdot \vec{d}} B_i(m_i) \cdot \delta(k - n_1 \vec{b}_1 + n_1 \vec{b}_2) B_i(m_z) \cdot \delta(k - k_1, -n_2 \vec{b}_2)
\]

(S.28)

Comparing to the original case without in-plane displacement,

\[
F_A(k) = B_i(m_i) \cdot \delta(k - n_1 \vec{b}_1 + n_1 \vec{b}_2) + B_i(m_z) \cdot \delta(k - n_2 \vec{b}_2) - B_i(m_i) B_i(m_z) \delta(k - n_1 \vec{b}_1, -n_2 \vec{b}_2)
\]

(S.29)

the in-plane displacement only introduces an additional global phase shift to the entire \( k \)-space.
Hence, the radiation pattern ($k$-space expression inside the light cone) will not be affected by misalignment between the two infinitely large periodic metallic patterns. We have verified this for the finite-sized sample with numerical simulations.

**Section S5. MORE EXPERIMENTAL RESULTS**

Fig. S9. Photograph of the fabricated chessboard-type moiré metasurface at the selected angles. (A) $0.25^\circ$, (B) $2^\circ$, (C) $4^\circ$, (D) $6^\circ$, (E) $8^\circ$, (F) $10^\circ$, (G) $12^\circ$, (H) $14^\circ$, (I) $16^\circ$. 
Fig. S10. Photograph of the fabricated triangular-type moiré metasurface at the selected angles. (A) 0.25°, (B) 2°, (C) 4°, (D) 6°, (E) 8°, (F) 10°, (G) 12°, (H) 14°, (I) 16°.
Fig. S11. Top view of the simulated 3D radiation patterns (LHCP component) for the chessboard case at different twist angles. (A) $\psi = 0.25^\circ$, (B) $\psi = 2^\circ$, (C) $\psi = 4^\circ$, (D) $\psi = 6^\circ$, (E) $\psi = 8^\circ$, (F) $\psi = 10^\circ$, (G) $\psi = 12^\circ$, (H) $\psi = 14^\circ$. Note that self-normalization has been made to each of the 3D radiation patterns.
Fig. S12. Simulated and measured 2D radiation patterns (dB scaling) for the chessboard case in the plane of maximum radiation at different twist angles. (A to H) Numerical simulation. (I to P) Experiment.
Fig. S13 The ellipticity \( \frac{E_R - E_L}{E_R + E_L} \) in the beam peak direction for the chessboard case for \( 0.25° \leq \psi \leq 4° \). The solid line and colored region represent the averaged value and fluctuation, respectively.
Fig. S14. Top view of the simulated 3D radiation patterns (LHCP component) for the triangular case at different twist angles. (A) $\psi=0.25^\circ$, (B) $\psi=2^\circ$, (C) $\psi=4^\circ$, (D) $\psi=6^\circ$, (E) $\psi=8^\circ$, (F) $\psi=10^\circ$, (G) $\psi=12^\circ$, (H) $\psi=13^\circ$. Note that self-normalization has been made to each of the 3D radiation patterns.
Fig. S15. Simulated and measured 2D radiation patterns (dB scaling) for the triangular case in the plane of maximum radiation at different twist angles. (A to H) Numerical simulation. (I to P) Experiment.
Fig. S16. The ellipticity $(E_R - E_L)/(E_R + E_L)$ in the beam peak direction for the triangular case for $0.25^\circ \leq \psi \leq 3^\circ$. The solid line and colored region represent the averaged value and fluctuation, respectively, of beam #1/#3/#5 (orange) and beam #2/#4/#6 (blue).
Fig. S17. Radiation performance of conventional coding metasurface with regularly shaped patterns. (A) The structure follows exactly from Ref. 2, and the model is cropped into round shape with 250mm diameter. (B and C) 3D and 2D radiation patterns. The beam efficiency -9dB is not better than that of the chessboard-type moiré metasurface (Fig. S12), which is majorly caused by the round boundary truncation that destroys the super unit cell approximation in the conventional metasurface.
Fig. S18. Experimentally measured radiation pattern for the triangular case with $\psi=11^\circ$. The phenomenon of three-beam radiation pattern that originally occurs at $\psi=10^\circ$ in the numerical simulation (Fig. 3(D)) shifts to $\psi=11^\circ$ in the experiment. (A) Perspective view. (B) Top view.

Fig. S19. Experimental setup.