COPRIME EHRHART THEORY AND COUNTING FREE SEGMENTS

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Abstract. A lattice polytope is free (or empty) if its vertices are the only lattice points it contains. In the context of valuation theory, Klain (1999) proposed to study the functions \( \alpha_i(P; n) \) that count the number of free polytopes in \( nP \) with \( i \) vertices. For \( i = 1 \), this is the famous Ehrhart polynomial. For \( i > 3 \), the computation is likely impossible and for \( i = 2, 3 \) computationally challenging.

In this paper, we develop a theory of coprime Ehrhart functions, that count lattice points with relatively prime coordinates, and use it to compute \( \alpha_2(P; n) \) for unimodular simplices. We show that the coprime Ehrhart function can be explicitly determined from the Ehrhart polynomial and we give some applications to combinatorial counting.

1. Introduction

In this paper, we are exclusively concerned with lattice polytopes, that is, convex polytopes \( P \) with vertices in \( \mathbb{Z}^d \), for some \( d \). A lattice polytope \( P \) is called free (or empty) if \( P \cap \mathbb{Z}^d \) are precisely the vertices of \( P \). Free polytopes have been studied in relation to integer programming; see, for example, [20, 19, 22, 14, 18] and they are related to hollow polytopes, whose interior do not contain lattice points. Our interest in free polytopes comes from valuation theory and geometric combinatorics. For a set \( S \subset \mathbb{R}^d \), denote by \([S] : \mathbb{R}^d \to \{0, 1\}\) its indicator function and let \( \alpha_1(S) := |S \cap \mathbb{Z}^d| \). Klain [16] basically proved the following identity for lattice polytopes \( P \):

\[
(1) \quad (-1)^{\dim P}[\text{relint}(P)] = -\sum_Q (-1)^{\alpha_1(Q)}[Q],
\]

where the sum is over all free polytopes \( Q \subseteq P \). Applying the Euler characteristic to both sides of (1) then yields

\[
1 = -\sum_Q (-1)^{\alpha_1(Q)} = \sum_{i \geq 1} (-1)^i \alpha_i(P),
\]

where \( \alpha_i(P) \) to be the number of free polytopes \( Q \subseteq P \) with \( \alpha_1(Q) = i \). Klain [16, Sect. 10] proposed to study the functions

\[
\alpha_i(P, n) := \alpha_i(n \cdot P)
\]

where \( n \cdot P = \{np : p \in P\} \) is the \( n \)-th integer dilate of \( P \). This is motivated by the fact \( \alpha_1(P, n) = |n \cdot P \cap \mathbb{Z}^d| \) is the famous Ehrhart polynomial; see, for example, [4]. For \( i > 1 \), the function \( \alpha_i \) is not a valuation on polytopes. Moreover, in dimensions \( \geq 3 \), there are infinitely many free polytopes up to unimodular equivalence, which probably renders task of computing \( \alpha_i(P; n) \) hopeless in general. However, any two free segments are unimodularly equivalent and there is hope for the computation of \( \alpha_2(P; n) \). Here, \( \alpha_2(P) \) is the number of pairs in \( P \cap \mathbb{Z}^d \) that are visible from each other. If \( P \) is a dilate of the unit cube, then this is related to digital lines [9]. For the unit square the sequence \( \alpha_2([0, 1]^2; n) \) is given in [1] but an explicit description does not seem to be known.

The main goal of our endeavor was to find explicit description of \( \alpha_2(P; n) \) where \( P \) is a unimodular simplex; see Theorem 3 below. In our investigation, it turned out that we need the following number-theoretic variation of Ehrhart theory: The coprime Ehrhart function of a lattice polytope \( P \subseteq \mathbb{R}^d \) is
the function

\[ CE(P; n) := |\{(a_1, \ldots, a_d) \in nP \cap \mathbb{Z}^d : \gcd(a_1, \ldots, a_d, n) = 1\}|. \]

If \( P \) is a half-open free segment (that is, one endpoint removed), then \( CE(P; n) = \phi(n) \). Theorem 1 below gives an explicitly computable description for general lattice polytopes.

Recall that the \textbf{Ehrhart function} of \( P \) is \( E(P; n) := |nP \cap \mathbb{Z}^d| \) for \( n \in \mathbb{Z}_{\geq 0} \). Ehrhart [10] famously proved that the Ehrhart function agrees with polynomial of degree \( r = \dim P \): there are numbers \( e_i(P) \in \mathbb{R} \) for \( i = 0, \ldots, r \) such that

\[ E(P; n) = e_r(P)n^r + e_{r-1}(P)n^{r-1} + \cdots + e_0(P)n^0 \quad \text{for all } n \geq 0. \]

For \( k \geq 0 \), the \textbf{Jordan totient function} is given by

\[ J_k(n) := \left| \left\{ (a_1, \ldots, a_k) \in \mathbb{Z}^k : 1 \leq a_i \leq n \text{ for } i, \gcd(a_1, \ldots, a_k, n) = 1 \right\} \right|. \]

For \( k = 0 \), we have \( J_0(n) = 1 \) if \( n = 1 \) and \( = 0 \) otherwise. For \( k = 1 \), \( J_1(n) = \phi(n) \) is the Euler totient function. See [21] for more on properties of \( J_k(n) \).

\textbf{Theorem 1.} Let \( P \) be an \( r \)-dimensional lattice polytope. Then

\[ CE(P; n) = e_r(P)J_r(n) + e_{r-1}(P)J_{r-1}(n) + \cdots + e_0(P)J_0(n), \]

for all \( n \geq 0 \).

The Jordan totient function can be computed as

\[ J_k(n) = n^k \prod_{p \mid n} \left( 1 - \frac{1}{p^k} \right), \]

where \( p \) ranges over all prime factors of \( n \). This prompts us to define \( J_k(-n) := (-1)^kJ_k(n) \) for all \( n \geq 0 \).

The following is the counterpart to Ehrhart–Macdonald reciprocity (see [4]), which states that the number of lattice points in the relative interior of \( nP \) is given by \(-1)^{\dim P}E(P; -n)\).

\textbf{Theorem 2.} Let \( P \subset \mathbb{R}^d \) be an \( r \)-dimensional lattice polytope and \( n \geq 1 \). Then

\[ CE(\text{relint}(P); n) = (-1)^r CE(P; -n). \]

Theorem 1 allows us to give an easily computable description of \( \alpha_2(P; n) \), where \( P \) is a unimodular simplex. Let us first note that \( \alpha_2(P; n) \) is invariant under \textbf{unimodular transformations}, that is, linear transformations \( T(x) = Ax + b \) with \( A \in \text{SL}(\mathbb{Z}^d) \) and \( b \in \mathbb{Z}^d \). It is therefore sufficient restrict to the \( d \)-dimensional \textbf{standard simplex}

\[ \Delta_d := \text{conv}(e_1, \ldots, e_{d+1}) = \{ x \in \mathbb{R}^{d+1} : x \geq 0, x_1 + \cdots + x_{d+1} = 1 \}. \]

We also define the polytope \( \nabla_d := \Delta_d + (-\Delta_d) \), the \textbf{difference body} [23, Sect. 10.1] of \( \Delta_d \).

\textbf{Theorem 3.} For \( d \geq 1 \), we have

\[ \alpha_2(\Delta_d; n) = \frac{1}{2} \sum_{\ell = 0}^{n} \binom{n - \ell + d}{d} CE(\partial \nabla_d; \ell). \]

The Ehrhart polynomial of \( \nabla_d \) is given in (9) in Section 3. Together with Ehrhart–Macdonald reciprocity this gives \( E(\partial \nabla_d; n) = E(\nabla_d; n) - (-1)^dE(\nabla_d; -n) \) and we can use Theorem 1 to compute \( \alpha_2(\Delta_d; n) \). For
2 ≤ d ≤ 9 this yields the following list:

\[
\begin{align*}
\text{CE}(\partial \nabla_2; n) &= 6J_1(n) \\
\text{CE}(\partial \nabla_3; n) &= 10J_2(n) + 2J_0(n) \\
\text{CE}(\partial \nabla_4; n) &= \frac{35}{3}J_3(n) + \frac{25}{3}J_1(n) \\
\text{CE}(\partial \nabla_5; n) &= \frac{31}{2}J_4(n) + \frac{35}{2}J_2(n) + 2J_0(n) \\
\text{CE}(\partial \nabla_6; n) &= \frac{77}{10}J_5(n) + \frac{49}{2}J_3(n) + \frac{45}{4}J_1(n) \\
\text{CE}(\partial \nabla_7; n) &= \frac{143}{30}J_6(n) + \frac{77}{12}J_4(n) + \frac{707}{90}J_2(n) + 2J_0(n) \\
\text{CE}(\partial \nabla_8; n) &= \frac{143}{36}J_7(n) + \frac{929}{30}J_5(n) + \frac{297}{5}J_3(n) + \frac{701}{70}J_1(n) \\
\text{CE}(\partial \nabla_9; n) &= \frac{2431}{2010}J_8(n) + \frac{715}{48}J_6(n) + \frac{4147}{96}J_4(n) + \frac{14465}{904}J_2(n) + 2J_0(n)
\end{align*}
\]  

(4)

In the plane, there are exactly four free polytopes up to unimodular equivalence. In particular, there are no free polygons with more than four vertices. Using the relations given in [16, Cor. 7.4], this allows us to completely determine all functions α_i(P; n) for unimodular triangles.

**Corollary 4.** Let P ⊂ ℝ^2 be a unimodular triangle.

\[
\begin{align*}
\alpha_1(P; n) &= \left(\frac{n+1}{2}\right), & \alpha_3(P; n) &= \frac{1}{2} \sum_{\ell=0}^{n} \left(\frac{n-\ell+2}{2}\right)J_1(\ell) - 2n^2 - 3n \\
\alpha_2(P; n) &= \frac{3}{2} \sum_{\ell=0}^{n} \left(\frac{n-\ell+2}{2}\right)J_1(\ell), & \alpha_4(P; n) &= \frac{3}{2} \sum_{\ell=0}^{n} \left(\frac{n-\ell+2}{2}\right)J_1(\ell) - \frac{3}{2}(n^2 + n)
\end{align*}
\]

and α_i(P) = 0 for i > 4.

The paper is organized as follows. In Section 2, we briefly develop coprime Ehrhart theory and prove Theorems 1 and 2. Section 3 is devoted to the study of α_2(Δ_d; n). We close with afterthoughts and open questions in Section 4.

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2. **Coprime Ehrhart functions**

A **valuation** on lattice polytopes is a function φ satisfying φ(∅) = 0 and

\[\varphi(P \cup Q) = \varphi(P) + \varphi(Q) - \varphi(P \cap Q)\]

for all lattice polytopes P, Q such that P ∪ Q and P ∩ Q are also lattice polytopes. We call φ **lattice-invariant** if φ(T(P)) = φ(P) for all unimodular transformations T(x).

**Lemma 5.** Let n ≥ 1 be fixed. Then the map P ↦ CE(P; n) is a lattice-invariant valuation.

**Proof.** It is straightforward to see that P ↦ CE(P; n) is a valuation. To see lattice invariance, let a ∈ n · T(P) = A(nP) + nb. That is a = Aa' + nb for some lattice point a' ∈ nP. It is now clear that gcd(a, n) = gcd(Aa' + nb, n) = gcd(Aa', n) = gcd(a', n). Hence CE(T(P); n) = CE(P; n). □

**Proof of Theorem 1.** Betke–Kneser [6] showed that a lattice-invariant valuation is uniquely determined by its values on the unimodular simplices S_k = conv(0, e_1, ..., e_k) for k = 0, ..., d. This implies that the valuations \{e_i(P): i = 0, ..., d\} form a basis for the space of (real-valued) lattice-invariant valuations. Thus, for n ≥ 1 fixed, there are c_{n,i} ∈ ℝ such that

\[
\text{CE}(P; n) = c_{n,d}e_d(P) + c_{n,d-1}e_{d-1}(P) + \cdots + c_{n,0}e_0(P)
\]  

(5)
To determine the coefficients $c_{n,i}$ it suffices evaluate (5) at sufficiently many lattice polytopes or, in fact, half open polytopes; see the methods used in [15]. The $k$-dimensional half-open cube is $H_k := (0, 1]^k$. Its Ehrhart polynomial is readily given by $E(H_k; n) = n^k$ and hence $e_j(H_k) = 1$ if $j = k$ and $= 0$ otherwise. To complete the proof, we simply note that $\text{CE}(H_k; n) = J_k(n)$. □

**Proof of Theorem 2.** In order to prove Theorem 2, we recall the following implication of a classical result due to McMullen [17]. If $\varphi$ is a lattice-invariant valuation and $P$ an $r$-dimensional lattice polytope, then the function $\varphi_P(k) := \varphi(kP)$ agrees with a polynomial of degree at most $r$. Moreover $(-1)^r\varphi_P(-1) = \varphi(\text{relint}(-P))$.

If we set $\varphi(P) := \text{CE}(P; n)$ for $n \geq 1$ fixed, then we obtain
\[
\varphi_P(k) = e_r(P)J_r(n)k^r + e_{r-1}J_{r-1}(n)k^{r-1} + \cdots + e_0(P)J_0(n)k^0
\]
and hence
\[
\text{CE}(\text{relint}(P); n) = \text{CE}(\text{relint}(-P); n) = (-1)^r\varphi_P(-1) = (-1)^r\text{CE}(P; -n)
\]

To conclude this section, let us briefly remark that both results can also be proved with the use of number-theoretic Möbius inversion. For this, we note that
\[
E(P; n) = \sum_{d|n} \text{CE}(P; \frac{n}{d})
\]
and hence
\[
\text{CE}(P; n) = \sum_{d|n} \mu(d)E(P; \frac{n}{d}).
\]
using linearity, we only need to consider
\[
\sum_{d|n} \mu(d) \left(\frac{n}{d}\right)^k = n^k \sum_{d|n} \mu(d) \frac{1}{d^k} = n^k \prod_{p|n} \left(1 - \frac{1}{p^k}\right) = J_k(n).
\]
The Jordan totient functions $J_k(n)$ take the role of the monomial basis $n^k$. In Ehrhart theory and combinatorics, there are two more important bases. For $d \geq 0$, let
\[
S_d = \{ x \in \mathbb{R}^d : x \geq 0, x_1 + \cdots + x_d \leq 1 \},
\]
\[
O_d = \{ x \in \mathbb{R}^d : 0 \leq x_1 \leq x_2 \leq \cdots \leq x_d \leq 1 \}.
\]
Both polytopes are unimodular simplices with Ehrhart polynomial $E(S_d; n) = E(O_d; n) = \binom{n+d}{d}$. Ehrhart–Macdonald reciprocity now states $\binom{n-1}{d} = E(\text{relint}(O_d); n) = (-1)^dE(O_d, -n) = (-1)^k\binom{-n-d}{-d}$. In particular
\[
E(S_d; n) = \sum_{k=0}^d \frac{c(n, k)}{n!} n^k,
\]
where $c(n, k)$ are the unsigned Stirling numbers of the first kind [25, Prop. 1.3.7]. The corresponding coprime Ehrhart function has a nice interpretation.

**Proposition 6.** For $k \geq 1$, $B_k(n) := \text{CE}(S_k; n)$ is the number of compositions $\mu = (\mu_1, \ldots, \mu_l)$ with $\mu_i \geq 1$ and $\mu_1 + \cdots + \mu_l = n$ of length $l \leq k + 1$ with $\gcd(\mu_1, \ldots, \mu_l) = 1$.

The function $(-1)^kB_k((-n + 1))$ was studied by Gould [13] under the name $R_k(n)$ as the number of compositions of $n$ with exactly $k$ relatively prime parts. The functions $B_k(n)$ and $R_k(n)$ take the role of the binomial coefficients in Coprime Ehrhart theory.

We can also consider the fraction of lattice points in $nP$ that get counted by $\text{CE}(P; n)$ as $n$ goes to infinity.
Corollary 7. Let \( P \subset \mathbb{R}^d \) be an \( r \)-dimensional lattice polytope. Then
\[
\limsup_{n \to \infty} \frac{\text{CE}(P; n)}{E(P; n)} = \frac{1}{\zeta(r)},
\]
where \( \zeta \) is the Riemann zeta function.

Proof. \( \square \)

In the case that \( P \) is the unit cube, this seems to be related to [12].

3. Counting free segments in unimodular simplices

In this section, we will determine \( \alpha_2(P; n) \), the number of free segments contained in \( nP \), where \( P \) is a unimodular \( d \)-simplex. We start by some considerations that apply to general lattice polytopes.

Let \( P \subset \mathbb{R}^d \) be a lattice polytope and \( S = [a, b] \) a free segment. For \( n \geq 1 \), we write
\[
E_S(P; n) = | \{ t \in \mathbb{Z}^d : t + S \subseteq nP \} | = | (nP - a) \cap (nP - b) \cap \mathbb{Z}^d |.
\]

Note that \( E_S(P; n) \) is invariant under translation of \( S \) and we may assume that \( a = 0 \). We call a vector \( b \in \mathbb{Z}^d \) primitive if \( \gcd(b) = 1 \) and we write \( E_b(P; n) = E_{[0,b]}(P; n) \). This gives us the representation
\[
\alpha_2(P; n) = \frac{1}{2} \sum_{b \text{ primitive}} E_{[0,b]}(P; n).
\]
The factor \( \frac{1}{2} \) stems from the fact that \([0,-b] = [0,b] - b\).

The functions \( E_S(P; n) \) are vector partition functions [27] and related to multivariate Ehrhart functions. If \( P \) is a unimodular simplex, then the next result states that \( E_S(P; n) \) is in fact an Ehrhart polynomial.

We now consider the standard unimodular simplex \( \Delta_d \subset \mathbb{R}^{d+1} \) and define
\[
P^d := \{ b \in \mathbb{Z}^{d+1} : \gcd(b_1, \ldots, b_{d+1}) = 1, b_1 + \cdots + b_{d+1} = 0 \}.
\]
For \( b \in P^d \), there are unique \( b^+, b^- \in \mathbb{Z}_{\geq 0}^d \) such that \( b = b^+ - b^- \) and \( b^+_i b^-_i = 0 \) for \( i = 1, \ldots, d \). We further define
\[
\ell(b) := \sum_i b^+_i = \sum_i b^-_i.
\]

Lemma 8. Let \( b \in P^d \) and \( n \geq 1 \). Then \( n\Delta_d \cap (n\Delta_d - b) = \emptyset \) if and only if \( \ell(b) > n \). If \( \ell(b) \leq n \), then
\[
n\Delta_d \cap (n\Delta_d - b) = b^- + (n - \ell(b))\Delta_d.
\]
In particular, \( E_b(n) = \binom{n-\ell(b)+d}{d} \) for \( \ell(b) \leq n \) and \( E_b(n) = 0 \) otherwise.

Proof. The polytope \( n\Delta_d \cap (n\Delta_d - b) \) is given by all points \( x \in \mathbb{R}^{d+1} \) such that
\[
x_1 + \cdots + x_{d+1} = n \quad \text{and} \quad x_i \geq \min(0,-b_i) \text{ for all } i = 1, \ldots, d+1.
\]
Summing the inequalities yields \( x_1 + \cdots + x_{d+1} \geq \ell(b) \). Thus, there is no solution if and only if \( \ell(b) > n \). Otherwise, the solutions are given by points of the form \( x = b^- + x' \) with \( \sum_i x'_i = n - \ell(b) \) and \( x' \geq 0 \). That is, \( n\Delta_d \cap (n\Delta_d - b) = b^- + (n - \ell(b))\Delta_d \). The second statement follows from the fact that \( E_b(\Delta_d; n) = E(\Delta_d; n - \ell(b)) = \binom{n-\ell(b)+d}{d} \). \( \square \)
Combining Lemma 8 with (7), yields
\[
\alpha_2(\Delta_d; n) = \frac{1}{2} \sum_{b \in \mathcal{P}^d} E_b(\Delta_d; n) = \frac{1}{2} \sum_{p \in \mathcal{P}^d} \left( \frac{n - \ell(b) + d}{d} \right) = \frac{1}{2} \sum_{\ell=0}^n \left( \frac{n - \ell + d}{d} \right) c_{\ell,d},
\]
where \(c_{\ell,d} := \# \{ b \in \mathcal{P}^d : \ell(S) = \ell \} \).

Let \(\nabla_d := \Delta_d + (-\Delta_d)\). This is a convex polytope with vertices \(e_i - e_j\) for \(i \neq j\). The combinatorial and arithmetic structure of \(\nabla_d\) is easy to understand; see, for example, [8, Sect. 3] and below.

**Proposition 9.** Let \(d, \ell \geq 1\). Then
\[
c_{\ell,d} = \text{CE}(\partial(\Delta_d - \Delta_d); \ell).
\]

**Proof.** Let \(b \in \mathbb{Z}^{d+1}_+\) with \(\sum_i b_i = 0\) and recall that \(b = b^+ - b^-\) where \(b^+, b^- \in \mathbb{Z}_+^{d+1}\) with disjoint supports. It follows that \(b \in \partial(k\nabla_d)\) if and only if \(\ell(b) = k\). Adding the condition \(\gcd(b, n) = 1\) now proves the claim. \(\square\)

For \(c \in \mathbb{R}^{d+1}\), let \(\nabla^c_d := \{ x \in \nabla_d : \langle c, x \rangle \geq \langle c, y \rangle\) for all \(y \in \nabla_d\}\) be the face in direction \(c\). Let \(I_+ := \{ i \in [d+1] : c_i = \max(c) \}\) and \(I_- := \{ i \in [d+1] : c_i = \min(c) \}\). Then
\[
\nabla^c_d = \text{conv}(e_i - e_j : i \in I_+, j \in I_-).
\]
If \(c\) is not a multiple of \((1, \ldots, 1)\), then \(I_+ \cap I_- = \emptyset\) and \(\nabla^c_d\) is unimodularly isomorphic to \(\Delta_{|I_+| - 1} \times \Delta_{|I_-| - 1}\). The number of faces that are isomorphic to \(\Delta_{k-1} \times \Delta_{l-1}\) is \(\binom{d+1}{k,l}\). The boundary of \(\nabla_d\) is the disjoint union of the relative interiors of proper faces. This gives us the following expression
\[
\tag{8}
c_{\ell,d} = \sum_{\substack{k,l \geq 1 \\ k+l \leq d+1}} \binom{d+1}{k,l} \text{CE}(\text{relint}(\Delta_{k-1} \times \Delta_{l-1}); \ell).
\]
Note that \(\text{relint}(\Delta_{k-1} \times \Delta_{l-1}) = \text{relint}(\Delta_{k-1}) \times \text{relint}(\Delta_{l-1})\). Hence
\[
E(\text{relint}(\Delta_{k-1} \times \Delta_{l-1}); n) = E(\text{relint}(\Delta_{k-1}); n) \cdot E(\text{relint}(\Delta_{l-1}); n) = \binom{n-1}{k-1} \binom{n-1}{l-1}
\]
and using Theorem 1 yields explicit expressions for \(c_{\ell,d}\) and subsequently for \(\alpha_2(\Delta_d; n)\).

A different representation of \(c_{\ell,d}\) is as follows. The polytope \(\nabla_d\) is reflexive (cf. [3]). This implies that \(E(\text{relint}(\nabla_d); n) = E(\nabla_d; n - 1)\) and hence \(E(\partial \nabla_d; n) = E(\nabla_d; n) - E(\nabla_d; n - 1)\). Using [8, Cor. 3.16], an explicit description of \(E(\nabla_d; n)\) is given by
\[
\tag{9}
E(\nabla_d; n) = \sum_{j=0}^d \binom{d}{j} \binom{n}{j} \binom{n+d-j}{d-j}.
\]

4. **Afterthoughts and questions**

4.1. **Geometric combinatorics and coprime chromatic functions.** There are a number of counting functions that can be expressed in terms of Ehrhart polynomials; see [4]. Perhaps most prominent is the chromatic polynomial of a graph. Let \(G = (V, E)\) be a simple graph. An \(n\)-**coloring** is a map \(c : V \to \{1, \ldots, n\}\) with \(c(u) \neq c(v)\) for all edges \(uv \in E\). George Birkhoff [7] introduced the function \(\chi_G(n)\) counting the number of \(n\)-colorings of a graph. Birkhoff and Whitney [28] proved that \(\chi_G(n)\) agrees with a polynomial in \(n\) of degree \(d = |V|\). This is known as the chromatic polynomial of \(G\). Beck and Zaslavsky [5] realized \(\chi_G(n)\) as an Ehrhart polynomial of an inside-out polytope. An explicit formula is given by
\[
\chi_G(n) = \sum_{k=0}^d w_k(G)n^{d-k},
\]
where \( w_k(G) \) are the Whitney numbers of the first kind of \( G \); [2].

Now, a **coprime coloring** is an \( n \)-coloring \( c \) with the additional constraint that the set \( c(V) \cup \{n\} \) is coprime. If we denote by \( \chi^c_G(n) \) the number of coprime \( n \)-colorings, then Theorem 1 readily gives us

\[
\chi^c_G(n) = \sum_{k=0}^{d} w_k(G)J_{d-k}(n).
\]

Similarly, we may define **coprime order functions** on posets; see [4, Ch. 1].

4.2. **Rational coprime Ehrhart theory and coprime \( \Pi \)-partitions.** If \( P \subset \mathbb{R}^d \) is a polytope with vertices in \( \mathbb{Q}^d \), then \( E(P; n) \) agrees with a **quasi-polynomial**. That is, there are periodic functions \( c_i(n) \) such that

\[
E(P; n) = c_d(n)n^d + \cdots + c_0(n)n^0 \quad \text{for all } n \geq 1.
\]

**Question 1.** Can the coprime Ehrhart function \( \text{CE}(P; n) \) of a rational polytope be related to its Ehrhart function?

Our construction of coprime Ehrhart functions is in line with the usual approach to Ehrhart theory. For a rational polytope \( P \subset \mathbb{R}^d \), its **homogenization** is the pointed polyhedral cone \( C(P) = \{(x,t) : t \geq 0, x \in tP\} = \text{cone}(P \times \{1\}) \). The set \( M(P) = C(P) \cap \mathbb{Z}^{d+1} \) is a finitely generated monoid and \( E(P; n) = |\{(a,t) \in M(P) : t = n\}| \). If we denote by \( \mathbb{Z}_{\text{prim}}^{d+1} := \{a \in \mathbb{Z}^{d+1}, \gcd(a) = 1\} \), then \( \text{CE}(P; n) = |\{(a,n) \in \mathbb{Z}_{\text{prim}}^{d+1} : (a,n) \in M(P)\}| \). A **grading** of \( M(P) \) is a linear function \( \ell : \mathbb{Z}^{d+1} \rightarrow \mathbb{Z} \) such that \( \ell(p) > 0 \) for all \( p \in M(P) \setminus 0 \). The associated Hilbert function \( H_\ell(P; n) = \{p \in M(P) : \ell(p) = n\} \) is a quasipolynomial for all rational polytopes \( P \) [4, Sect. 4.7]. This rests on the rationality of the **integer point transform**

\[
\sum_{p \in M(P)} z^p \in \mathbb{Z}_{[1^\pm_1, \ldots, 1^\pm_{d+1}]}.\]

**Question 2.** Is there a coprime version of the rational integer point transform?

A nice combinatorial consequence would be a coprime theory of \( \Pi \)-partitions. Let \( \Pi \) be a finite set partially ordered by \( \preceq \). A \( \Pi \)-partition of \( n \geq 0 \) is a map \( f : \Pi \rightarrow \mathbb{Z}_{\geq 0} \) such that \( \sum_{a \in \Pi} f(a) = n \) and \( f(a) \leq f(b) \) whenever \( a \preceq b \). This setup was introduced by Stanley (see [25]) as a generalization of usual partitions and plane partitions. It can be shown that the function \( c_\Pi(n) \) counting the \( \Pi \)-partitions of \( n \) is of the form \( H_\ell(P; n) \) for some rational polytope \( P \) and linear function \( \ell \). A \( \Pi \)-partition is **strict** if \( f(a) > 0 \) and \( f(a) < f(b) \) when \( a < b \). It would be desirable to obtain explicit formulas for counting coprime (strict) \( \Pi \)-partitions.

Work in this direction was done by El Bachraoui [11]. A **relatively prime partition** of \( n \in \mathbb{Z}_{\geq 0} \) is a sequence of natural numbers \( \lambda_1 > \lambda_2 > \cdots > \lambda_k > 0 \) such that \( n = \lambda_1 + \lambda_2 + \cdots + \lambda_k \) and the \( \lambda_i \) are coprime. The number of parts of the partition is \( k \). We write \( \text{rpp}_k(n) \) for the number of relatively prime partitions of \( n \) with exactly \( k \) parts. Note that \( \text{rpp}_2(n) \) is the number of coprime \( 0 < a < b \) with \( n = a + b \) and hence \( \text{rpp}_2(n) = \frac{1}{2} \varphi(n) \). For the number of relatively prime partitions with 3 parts El Bachraoui [11] showed that for \( n \geq 4 \)

\[
\text{rpp}_3(n) = \frac{1}{12} J_2(n).
\]

4.3. **Mixed versions.** Upon closer inspection of the proof of Theorem 3, it can be seen that

\[
\alpha_2(\Delta_d; n) = |\{(t,b) \in (\mathbb{Z}^d+1)^2 : b \in n\nabla_d, t \in b^+ + (n - \ell(b))\Delta_d, \gcd(b,n) = 1\}|.
\]

This prompts the question of a **mixed** version of \( \text{CE}(P; n) \). For a lattice polytope \( P \subset \mathbb{R}^d \) and \( I \subseteq [d] \), define

\[
\text{CE}_I(P; n) := |\{p \in nP \cap \mathbb{Z}^d : \gcd(\{p_i : i \in I\} \cup \{n\}) = 1\}|.
\]

It would be interesting if a reasonable expression for \( \text{CE}_I(P; n) \) could be found in general.
4.4. **Counting free triangles.** In every dimension $\geq 2$, the unimodular triangle is the unique free polytope with 3 vertices, up to unimodular equivalence.

**Question 3.** Is there a closed expression for $\alpha_3(\Delta_d; n)$?

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