Alfes-Neumann, Claudia; Bringmann, Kathrin; Schwagenscheidt, Markus
On the rationality of cycle integrals of meromorphic modular forms. (English) Zbl 1464.11050
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For an integer $k$, the Shintani lift is known to take modular forms of weight $2k$ to (analytic) modular forms of weight $k + \frac{1}{2}$, but cusp forms to cusp forms. The most useful way to analyze it is by realizing it as a theta lift, for an appropriate theta function of two variables from the upper half-plane. One important reason why these lifts are interesting is because in many cases, their Fourier coefficients of positive indices are the traces of the lifted modular form, namely appropriate combinations of their integrals along geodesics that are arithmetically meaningful.

Recently several papers consider generalizing the Shintani lift to cases where the theta integral no longer converges. This is done by appropriately regularizing this integral, and the resulting lifts still yield interesting results. The paper in question does this for meromorphic cusp forms, where regularizations at internal points of the upper half-plane are required. It follows that for a special class of meromorphic cusp forms, the corresponding traces are rational.

In more detail, for every positive even $k$ and an $\text{SL}_2(\mathbb{Z})$-class $A$ of integral quadratic forms of discriminant $d$, there is an appropriate modular form $f_{k,A}$ of weight $2k$. It is a cusp form when $d > 0$, and a meromorphic cusp form for $d < 0$. For every discriminant $D > 0$ that is not a square there is an $\text{SL}_2(\mathbb{Z})$-class of geodesics $C_Q$, with infinite cyclic stabilizers $\Gamma_Q$, and the index $D$ trace of a modular form $f$ is the sum over the finitely many classes of the integrals of $f$ along these cycles with respect to the weight $2k$ measure. When $f$ is cuspidal, the resulting power series is the $q$-expansion of the Shintani lift of $f$, of weight $k + \frac{1}{2}$.

If $f$ is a meromorphic cusp form, both the Shintani lift and the cycle integrals have to be regularized (the latter in case a pole of $f$ lies on the geodesic itself). This is done by using the elliptic expansion of $f$ and the theta kernel at the poles of $f$, and principal values for the cycle integrals. All this process is valid equally well for lifts that are twisted by a genus character, and are shown to produce explicit completions of the generating series of the (regularized) twisted traces. In the non-twisted case, linear combinations of the traces themselves with coefficients that come from principal parts of weakly holomorphic modular forms can be expressed, when $f = f_{k,A}$ from above, in more explicit terms, which are then viewed to be rational if the principal part in question has that property.

The paper is divided into 6 sections. Section 1 is the Introduction, with the statement of results, some examples, and a description of the basic ideas. Then Section 2 gives the basic definitions, and Section 3 proves the convergence of the regularized Shintani lift. Section 4 carries out the evaluation of the twisted Shintani lift, and Section 5 evaluates the combination of the non-twisted traces with respect to principal parts of weakly holomorphic modular forms. Finally, Section 6 explains how to evaluate these expressions numerically, complementing the example from the introduction.

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11F37 Forms of half-integer weight; nonholomorphic modular forms
11F27 Theta series; Weil representation; theta correspondences
11F25 Hecke-Petersson operators, differential operators (one variable)

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