UNSTEADY SEPARATION FOR THE INVISCID TWO-DIMENSIONAL PRANDTL'S SYSTEM

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Abstract. We consider the inviscid unsteady Prandtl’s system in two dimensions, motivated by the fact that it should model to leading order separation for the original viscous system. We give a sharp expression for the maximal time of existence of regular solutions, showing singularities only happen at the boundary or on the set of zero vorticity. We then exhibit new Lagrangian formulae for backward self-similar profiles, and study them also on the Eulerian side with a different approach that was initiated by Elliott-Smith-Cowley and Cassel-Smith-Walker. One particular profile is the fundamental one, which at the heart of the so-called Van-Dommelen and Shen singularity, and we prove its generic appearance for any prescribed Eulerian outer flow. We comment on the connexion between these results and the full viscous Prandtl’s system. This paper designs a new way to study singularities for quasilinear transport equations.

1. Introduction

We consider the inviscid Prandtl’s equations on the upper half-plane

\[
\begin{align*}
    u_t + uu_x + vu_y &= -p_x^E & (t, x, y) &\in [0, T) \times \mathbb{R} \times \mathbb{R}_+, \\
    u_x + v_y &= 0, \\
    v|_{y=0} &= 0, \\
    \lim_{y \to \infty} u(t, x, y) &= u^E(t, x),
\end{align*}
\]

(1.1)

where \( p^E \) and \( u^E \) are the trace of the Eulerian pressure and tangential flow at the boundary \( \mathbb{R} \times \{0\} \) induced by the Eulerian flow at infinity. \( p^E \) and \( u^E \) are prescribed, and then act as forcing terms for \( u \). They are linked through Bernoulli’s equation:

\[
    u_t^E + u^E u_x^E = -p_x^E,
\]

(1.2)

whose solution have to be global in two dimensions.

1.1. Historic

Prandtl’s system comes from the vanishing viscosity limit of Navier-Stokes equations with Dirichlet boundary condition. It describes the formation of a boundary layer, where the solution does a sharp transition from the vanishing at the boundary induced by the Dirichlet condition to a solution of the Euler system inside the domain. The first rigorous mathematical justification of the Prandtl’s boundary layer is due to Oleinik [23]. She proved the local wellposedness by imposing a monotonicity condition on the tangential velocity in order to use the Crocco transform. Xin and Zhang [28] obtained global existence of weak solutions by imposing monotonicity and an extra condition on the pressure. The monotonicity condition allows Masmoudi and Wong in [22], and Alexandre, Wang, Xu and Yang in [1] to prove wellposedness in Sobolev regularity. Without the monotocity condition, the equation can be ill-posed in Sobolev regularity [14]. The
authors in [15] constructed instabilities that can prevent the Prandtl’s system to be a good approximation of the Navier-Stokes system in the vanishing viscosity limit. We refer to [24, 13, 21] and the references therein for further informations. Otherwise, in the general case the system is locally well-posed in the analytical setting [25, 20, 18, 9]. In the steady case, Dalibard and Masmoudi [7] gave recently a complete description of the so-called Goldstein singularity. This singularity is of a different type than the one of the unsteady case.

The formation of singularity for the Prandtl system is linked to a physical phenomenon that is called the unsteady separation. The first reliable numerical result explaining how the separation is linked to the formation of singularity was obtained by Van Dommelen and Shen [26]. They characterized the singularity as a result of particles squashed in the streamwise direction, with a compensating expansion in the normal direction of the boundary. We refer to [12, 8] for recent numerical simulations and references therein for previous ones.

Up to our knowledge, the only local existence with general smooth initial data for the Inviscid Prandlt’s system is due to Hong and Hunter, in [17] (see also [2, 16]). They find a lower bound for the maximal time existence which corresponds to that of the Burger’s equation

\[ T = \left( -\inf_{x \in \mathbb{R}, y \geq 0} \partial_y u_0 \right)^{-1} \]

in the case of a trivial outer Eulerian flow. We find in this current paper that the sharp maximal existence time for the homogeneous inviscid Prandlt’s system is

\[ T = -\left( \inf \{ \partial_y u_0 = 0 \} \cup \{ y = 0 \} \right) \partial_x u_0 \]^{-1} \]

which is larger. We prove that this time is sharp which clarifies why is the monotonicity condition important to guarantee global wellposedeness for the inviscid Prandtl’s system. Let us also mention that \( \partial_y u \) is the vorticity of the solution, when considering the approximation of the Navier-Stokes equations with the Prandtl’s system. We obtain more generally a sharp maximal time of existence in the case of nontrivial Eulerian flows.

The first rigorous result on singularity formation is due to E and Enquist [10] (see [19] for nontrivial outer flows), where they considered the trace of the tangential derivative of odd solutions in \( x \) along the transversal axis

\[ \xi(t, y) = -u_x(t, 0, y), \]

which obey the following equation for \( y \in [0, +\infty) \):

\[
\begin{cases}
\xi_t - \xi_{yy} - \xi^2 + \left( \int_0^y \xi \right) \xi_y = \partial_y E_x(0), \\
\xi(t, 0) = 0, \quad \xi(0, y) = \xi_0(y), \quad \lim_{y \to \infty} \xi(t, y) = -u_x E(0).
\end{cases}
\]

In a first paper [4] we have been able to give a precise description of the singular dynamic of the equation above, where we found a stable profile (and instable ones) and proved that the blow point is ejected to infinity in the transversal direction because of the incompressibility condition. This result can be interpreted as a partial stability result for one of the profiles studied in the present paper, see Remark 7. In order to understand the effect of the transversal viscosity on the horizontal transport we considered in [5] a two dimensional Burger’s system:

\[ u_t - u_{yy} + uu_x = 0 \quad (t, x, y) \in [0, T) \times \mathbb{R}^2, \]

where we found infinitely many different profiles, one being stable under suitable perturbations.

In a first part of the paper [5], we treated the inviscid Burger’s equation \( u_t + uu_x = 0 \) and proved that the Taylor expansion of the initial data around the blowup point will decide which profile and scaling the flow will select to form the singularity. That is, the solution will be of the form \( u = (T - t)^\alpha F_\alpha(X/(T - t)^\beta(\alpha)) \) where \( F_\alpha \) is the profile and \( \alpha, \beta \) the scaling exponents.
We proved that the fundamental profile

$$\Psi_1(X) := \left( -\frac{X}{2} + \left( \frac{1}{27} + \frac{X^2}{4} \right)^{\frac{1}{2}} \right)^{\frac{2}{3}} + \left( -\frac{X}{2} - \left( \frac{1}{27} + \frac{X^2}{4} \right)^{\frac{1}{2}} \right)^{\frac{2}{3}}, \quad (1.6)$$

which is the inverse of the polynomial function $-X - X^3$ (i.e. $-\Psi_1 - \Psi_1^2 = x$) appears generically during blow-up. Surprisingly the above fundamental profile is going also to play a role in the generic separation phenomenon for the Prandlt’s system. Note that in many nonlinear transport problem coming from fluid mechanics, there exists a group of scaling transformations leaving the equation invariant which is of dimension greater than 2. Compared to other semi-linear models like the semi-linear heat, Schrödinger and wave equations, the KDV equation and many others, where this transformation group is one-dimensional, here this degeneracy is a real difficulty since one does not know in advance which scaling law the flow will select.

In a second part of the paper [5] we treated the Burgers equation with transverse viscosity (1.5), where we found that disregard the infinite speed of propagation induced by the transverse viscosity, the Taylor expansion of the initial data around the blowup point will still decide the profile and the scaling. We proved that the vertical viscosity affects the shock formation of Burgers equation, in the sense that the solutions are now anisotropic and of the form $u(x, y, t) \sim \lambda^{1/(2i)}(t, y)\Psi_i(x/\lambda^{1-1/(2i)}(t, y))$ where $\Psi_i$ is a profile of the Burger’s equation and $\lambda \to 0$ depends on the solutions of a parabolic system similar to (1.4) without the nonlocal term.

Inspired by [27, 26, 11, 3, 6] where it is suggested based on numerics and formal calculations that the viscosity is asymptotically negligible, and that Lagrangian variables should provide a suitable framework to study separation for the full viscous Prandtl’s system, we treated in this paper the singularity formation for the inviscid problem. In a forthcoming paper we treat the viscous case.

In this paper, we provide a complete description of the mechanism that leads to the singularity, including the case of nontrivial inviscid flows in the outer region. The new theory developed here can be used to study singularity formation for other quasilinear transport equations. We obtain mainly three results. The first result is a sharp local wellposedness result with a necessary and sufficient condition on the initial data to have separation. The second result is about the existence and a complete description of self-similar profiles. The focus is on the generic profile that has been observed numerically by Van Dommelen and Shen [26] and on other degenerate profiles. Actually, we prove the existence of the self-similar solutions in two different ways. The first approach is in the Eulerian coordinates and is based on the Crocco transform in order to reduce the nonlocal partial differential equation to a local one. The second is in the Lagrangian coordinates where we exhibit an explicit formula relating the profile to a specific volume preserving diffeomorphism. This special structure is due to the fact near the singularity the characteristics themselves become self-similar. Other degenerate profiles can be studied similarly with these two approaches. In particular we describe completely a degenerate one that corresponds to the stable singularity formation in the case of solutions that are odd in $x$ having their singularity on the transversal axis.

The third result is on the genericity of the fundamental profile during separation. Genericity here means that the basin of attraction is open and dense among solutions becoming singular outside the boundary. The approach we present here to prove the genericity is new and based on the precise use of characteristics. There is indeed a geometrical interpretation for this genericity.
Separation being a consequence of a tangential Burgers type compression happening on the set of zero vorticity, the generic case corresponds to the existence of a point near which the set of zero vorticity is nondegenerate (a curve), at which the tangential derivative of the Lagrangian to Eulerian characteristics map attains a nondegenerate minimum (restricted to that set, the differential of this function is zero and its hessian is positive), see Lemma 8. Other degenerate singularities happen when the set of zero vorticity is degenerate and/or the minimum of this function is degenerate.

1.2. Statement of the results

We denote by \((X, Y)\) the Lagrangian variables and \((x, y)\) the Eulerian ones for equation (1.1). Using dots for differentiation with time \(t\), they are related by the characteristics ODE:

\[
\dot{x} = u(x, y), \quad \dot{y} = v(x, y), \quad (x(0), y(0)) = (X, Y).
\]

From (1.1), along the characteristics \(u\) solves the ODE:

\[
\dot{u} = -p^E_x(x).
\]

Therefore, the tangential position of the particle can be retrieved without any knowledge about the normal one, by solving the following ODE for each triple \((X, Y, u_0(X, Y))\):

\[
\begin{aligned}
\dot{x} &= u, \\
\dot{u} &= -p^E_x(x), \\
(x(0), u(0)) &= (X, u_0(X, Y)).
\end{aligned}
\]

The above equation is that of one dimensional particle moving in a force field \(-p^E_x\). The corresponding change of variables \((X, Y) \mapsto (X, u)\) is called the Crocco transform and has been used extensively in the study of the Prandtl’s system. One key fact about (1.1) is that the vorticity \(u_y\) is preserved along the characteristics, as differentiating (1.1) with respect to \(y\) yields:

\[ u_{yt} + uu_{yx} + vu_{yy} = 0 \]

because \(p^E\) does not depend on \(y\), and using incompressibility. Hence the set \(\{u_y = 0\}\) is preserved by the characteristics. The boundary \(\{y = 0\}\) is also preserved as \(v_{|y=0} = 0\). Hence for any \((x, y)\) either in the set \(\{u_y = 0\}\) or at the boundary \(\{y = 0\}\), differentiating (1.1) with respect to \(x\) and injecting \(u_y = 0\) yields:

\[ u_{xt} + uu_{xy} + vu_{xy} = -(u_x)^2 - p^E_{xx}. \]

It follows from the above equation and the previous discussion that the transport along the tangential variable and the tangential compression, when restricted to these two sets, are given by the previous ODE completed by an inhomogeneous Riccati equation:

\[
\begin{aligned}
\dot{x} &= u, \\
\dot{u} &= -p^E_x(x), \\
\dot{u}_x &= -(u_x)^2 - p^E_{xx}(x) \\
(x, u, u_x)(0) &= (X, u_0(X, Y), u_{0x}(X, Y)).
\end{aligned}
\]

Given a global in time pressure field \(p^E \in C^k([0, \infty) \times \mathbb{R})\) with \(k \geq 2\), the solution to the above system might not exist for all time due to the nonlinearity in the last equation, and we denote by \(T(X, Y)\) the corresponding maximal time of existence. We will distinguish later on between singularities happening at the boundary or away from it, and define to this aim:

\[
T := \min(T_a, T_b), \quad T_a := \min\{T(X, Y), \ u_0Y(X, Y) = 0, \ Y > 0\}, \quad T_b := \min\{T(X, Y), \ Y = 0\}.
\]

The time \(T\) defined above is a natural upper bound for the maximal existence of a solution to (1.1) with \(u_x \in L^\infty\). In fact, this time is sharp. Indeed, outside of the set \(\{u_y = 0\}\), the condition \(u_y \neq 0\) is propagated by the flow. Monotonicity in the normal variable is believed to
Theorem 1 (Local well-posedness and sharp maximal time). Let \( u_0 \in C^2(\mathbb{R} \times [0, \infty)) \) such that \( \nabla u_0 \in L^\infty(\mathbb{R} \times [0, \infty)) \). Let \((u^E, p^E) \in C^2([0, \infty) \times \mathbb{R})\). Then there exists a unique solution \( u \in C^1([0, T) \times \mathbb{R} \times [0, \infty))\) of (1.1) where \( T \) is defined by (1.9), which satisfies moreover \( \|\nabla u\|_{L^\infty([0,T] \times \mathbb{R} \times [0,\infty))} < \infty \) for any \( \hat{T} < T \). If \( T \) is finite then the solution satisfies:

\[
\lim_{t \uparrow T} \|u_x\|_{L^\infty(\mathbb{R} \times [0,\infty))} = \infty.
\]

If in addition \( u_0 \in C^k(\mathbb{R} \times [0, \infty)) \) and \((u^E, p^E) \in C^k([0, \infty) \times \mathbb{R})\) for some \( k \geq 3 \), then \( u \in C^{k-1}([0, T) \times \mathbb{R} \times [0, \infty))\). The mapping which to \( u_0 \) assigns the solution \( u \) is strongly continuous from \( C^k(\mathbb{R} \times [0, \infty)) \) into \( C^{k-1}([0, T') \times \mathbb{R} \times [0, \infty)) \) for any \( T' < T \).

Remark 2. We use the word singularity as it is usual, but it can be misleading: the solution might remain smooth at time \( T \). Indeed, the points where \( u_x \) becomes large can be sent to infinity in the normal direction as the study below shows. Also, from Theorem 1 and the ODE (1.8) one easily derives criteria for global-well posedness or finite time blow-up:

- In the case of normal monotonicity \( u_y > 0 \) or \( u_y < 0 \), the solution is global if and only if the solution to the Burgers equation at the boundary

\[
\partial_t u|_{y=0} + u|_{y=0} \partial_x u|_{y=0} = -p^E_x
\]

is global.

- The solution is global in the case of tangential growth \( u_\alpha x \geq 0 \) on the set of zero vorticity \( \{u_{0y} = 0\} \) and at the boundary \( \{y = 0\} \), and of concave pressure \( p^E_x \leq 0 \).

- The maximal time of existence is \( T = \left(-\min\{u_{0y}=0\} \cup \{y=0\}\right)^{-1} \) in the pressureless case \( p^E = 0 \) (with the convention \( T = \infty \) if the min is nonnegative).

Considering the nonlinearity in the ODE (1.8), Theorem 1 and a standard convexity argument shows that given any prescribed Eulerian flow \( u^E \) and \( p^E \), there exist initial data \( u_0 \) such that the corresponding solution becomes singular in finite time. Theorem 1 also indicates where singularities of (1.1) form: either at the boundary \( \{y = 0\} \), or away from the boundary on the set of zero vorticity \( \{y > 0, u_y = 0\} \). We now focus on the description of this phenomenon. Motivated by the full original viscous Prandtl’s system, where the Dirichlet boundary condition forces \( u_{y=0} = 0 \), we will only study what happens when the singularity forms away from the boundary, i.e. \( T_a < T_b \) and \( T_a < \infty \). It is moreover straightforward to adapt the study done in this document to the case of a blow-up at the boundary: the patterns and the description are similar.

We are interested first in a leading order description of the singularity. Since as \( \nabla u \) becomes large, the pressure \( p^E \) and the boundary conditions become of lower order, we start by dropping them and investigate the homogeneous inviscid Prandtl’s system

\[
\begin{aligned}
&u_t + uu_x + vu_y = 0 & (t, x, y) \in [0, T) \times \mathbb{R} \times \mathbb{R}_+,
&u_x + v_y = 0, & v|_{y=0} = 0.
\end{aligned}
\]

(1.10)

This equation has the following invariances. If \( u \) is a solution then so is

\[
\frac{\mu}{\lambda} tu \left( \frac{t}{\lambda}, \frac{x - ct}{\mu}, \frac{y}{\nu} \right) + c
\]

for \((\nu, \lambda, \mu, \nu, c) \in \{-1, 1\} \times (0, \infty)^3 \times \mathbb{R}\). Backward self-similar solutions are special solutions living in the orbit of the initial datum under the action of the scaling subgroup. The fundamental one is the one related to the so-called Van-Dommelen and Shen singularity [26]. We obtain here
Proposition 3 (Fundamental self-similar profile - Lagrangian side). For any \( p > 0 \), the mapping \((a, b) \mapsto (\mathcal{X}, \mathcal{Y})\) given by:

\[
\Phi(a, b) = \left( a + b^2 + p^2 a^3, \int_{-\infty}^{b} \frac{db}{1 + 3\Psi_1^2 \left( p \left( a + p^2 a^3 + b^2 - \bar{b}^2 \right) \right)} \right),
\]

where \( \Psi_1 \) is defined by (1.6), defines a volume preserving diffeomorphism between \( \mathbb{R}^2 \) and the subset of the upper half plane \( \{ 0 < \mathcal{Y} < 2\mathcal{Y}^*(\mathcal{X}) \} \) where \( \mathcal{Y}^* \) is defined by (1.15). The opposite of the tangential component of its inverse:

\[
\Theta := -\Phi^{-1}_1 : (\mathcal{X}, \mathcal{Y}) \mapsto -a,
\]

is a self-similar profile, that is, the following is a solution of (1.10):

\[
u(t, x, y) = (T - t)^{1/4} \Theta \left( \frac{x}{(T - t)^{1/4}}, \frac{y}{(T - t)^{1/4}} \right).
\]

In this paper we fix \( \Theta \) as corresponding to the value (\( \Gamma \) being the Gamma function)

\[
p = p^* := \frac{4}{9\pi^3} \Gamma \left( \frac{1}{4} \right)^4
\]

to ensure the square in the expansion (1.19) and the 1 in the expansion (1.16). Another value of \( p \) would simply correspond to a rescaling of \( \Theta \). Note that from the invariances of the equation, \( \Theta \) generates the full family of fundamental profiles \( (\Theta_{\mu, \nu, \iota})_{(\mu, \nu, \iota) \in (0, \infty)^2 \times \{-1, 1\}} \) where

\[
\Theta_{\mu, \nu, \iota}(\mathcal{X}, \mathcal{Y}) = \mu \Theta \left( \frac{\mathcal{X}}{\mu}, \frac{\mathcal{Y}}{\nu} \right).
\]

Proposition 3 gives a new elegant and insightful formula which will prove to be useful in the proof of the next genericity Theorem 5. However, there exists another complementary approach to obtain information on \( \Theta \) in Eulerian variables. In [11, 3], the Eulerian study of \( \Theta \) was performed. We perform again that analysis here, obtaining additional results and with full rigour.

Proposition 4 (Fundamental self-similar profile - Eulerian side). For \( p = p^* \),

\[
\mathcal{X} \mapsto \mathcal{Y}^*(\mathcal{X}) = \frac{1}{2} \int_{T_p(\mathcal{X})}^{\infty} \frac{d\Theta}{\sqrt{\Theta + p^2 \Theta^3 + \mathcal{X}^2}} = \int_{0}^{\infty} \frac{db}{1 + 3\Psi_1^2(p(\mathcal{X} - b^2))},
\]

defines an analytic curve on \( \mathbb{R} \) that has the following expansion:

\[
\mathcal{Y}^*(\mathcal{X}) = \frac{3\pi}{8} + \mathcal{X} + \frac{c_2}{2} \mathcal{X}^2 + \text{hot}, \quad c_2 = -\frac{5\Gamma \left( \frac{1}{4} \right)}{72\pi^3}, \quad \mathcal{X} \to \infty,
\]

\[
\mathcal{Y}^*(\mathcal{X}) = C_{\pm} |\mathcal{X}|^{-\frac{5}{6}} + O(|\mathcal{X}|^{-\frac{7}{6}}) \quad \text{as} \quad \mathcal{X} \to \pm\infty,
\]

where \( \Gamma \) is the Gamma function, and \( C_{\pm} = 2^{-1} p^{-2/3} \int_{\pm1}^\infty (z^3 \pm 1)^{-1/2} dz \) (see (3.7) for a formula). The profile \( \Theta \) defined by (1.12) has domain \( \{ 0 < \mathcal{Y} < 2\mathcal{Y}^*(\mathcal{X}) \} \) where it is analytic and solves:

\[
\begin{align*}
\frac{\partial \Theta}{\partial \mathcal{X}} &= -2\sqrt{\mathcal{X} + \Theta + p^2 \Theta^3}, \quad \text{for} \quad 0 < \mathcal{Y} \leq \mathcal{Y}^*(\mathcal{X}), \\
\frac{\partial \Theta}{\partial \mathcal{Y}} &= 2\sqrt{\mathcal{X} + \Theta + p^2 \Theta^3}, \quad \text{for} \quad \mathcal{Y}^* \leq \mathcal{Y} < 2\mathcal{Y}^*(\mathcal{X}).
\end{align*}
\]
It moreover satisfies the following properties:

(i) $\Theta_Y$ is zero on the curve \( \{ Y = Y^*(X) \} \), where $\Theta(X, Y^*(X)) = \Psi_1(pX)/p$. On this curve, $\partial X \Theta$ is minimal at $0, 3\pi/8$ with the following expansion:

\[
\Theta \left( X, \frac{3\pi}{8} + Y \right) = -X + (X - Y)^2 + (p^2 + c_2) X^3 - c_2 X^2 Y + \text{hot} \tag{1.19} \]

(ii) Behaviour near the boundary of its domain:

\[
\Theta(X, Y) = p^{-2} Y^{-2} + O(1) \quad \text{as} \quad Y \to 0, \tag{1.20} \]

\[
\Theta(X, Y) = p^{-2}(2Y^* - Y)^{-2} + O(1) \quad \text{as} \quad Y \to 2Y^*. \tag{1.21} \]

(iii) Behaviour at infinity\(^1\) in $X$, for any $\epsilon > 0$, for $0 < Y < (2 - \epsilon) Y^*(X)$:

\[
\Theta(X, Y) \xrightarrow{X \to -\infty} |X|^\frac{1}{2} \varphi_-(Y|X|^{1/6}) + O \left( |X|^\frac{1}{2} Y^{-2} + |X|^{-1}(2Y^*(X) - Y)^4 \right), \tag{1.22} \]

\[
\Theta(X, Y) \xrightarrow{X \to \infty} |X|^\frac{1}{2} \varphi_+(Y|X|^{1/6}) + O \left( |X|^\frac{1}{2} Y^{-2} + |X|^{-1}(2Y^*(X) - Y)^4 \right), \tag{1.23} \]

$\varphi_\pm \in C^\infty((0, 2C_\pm), \mathbb{R})$ being decreasing on $(0, C_\pm)$, increasing on $[C_\pm, 2C_\pm)$, with $\varphi_\pm(z) \sim p^{-2}z^{-2}$ and $\varphi_+(2C_\pm - z) \sim p^{-2}z^{-2}$ as $z \to 0$, $\varphi_-(C_-) = p^{-2/3}$ and $\varphi_+(C_+) = -p^{-2/3}$.

The above self-similar profile $\Theta$ is at the heart of the singularity formation for the inviscid Prandtl’s equations. The fact that it is singular both at infinity in space in the tangential variable, and as approaching the boundary of its support, indicates that it fails to describe the solution there. In this paper we show the appearance of the fundamental self-similar profile from generic initial data, and we solve this reconnection problem. This shows the effects of the outer Eulerian flow are lower order and does not prevent nor alter the singularity to leading order. The generic singularity is a consequence of a tangential generic Burgers-type compression occuring on a line of zero vorticity, and of an expansion induced by volume preservation.

**Theorem 5** (Generic Separation). Let $p^E_x, u^E$ be of class $C^4$. In the subset of $C^4(\mathbb{R} \times [0, \infty))$ of initial data $u_0$ with $\|\nabla u_0\|_\infty < \infty$, such that $T < T_0$ and $T < T_0$, there exists a dense open set for which the corresponding solution satisfies the following. There exist parameters ($\mu, \nu, \iota$) in $\{ 0, \infty \}^2 \times \{-1, 1\}$ and a constant $\eta > 0$ such that:

- Location of the singularity. There exists $x^*(t) \in C^4([0, T], \mathbb{R})$, regular up to time $T$ such that $\nabla u(t)$ remains bounded in $\{(x, y, t), 0 \leq t \leq T, |x - x^*| \geq \epsilon \}$ for any $\epsilon > 0$.
- Displacement line. There exists $y^* \in C^3([0, T])$ with:

\[
y^*(t, x) = \frac{2}{(T - t)^{\frac{1}{2}}} Y^*_{\mu, \nu, \iota} \left( \frac{x - x^*}{(T - t)^{\frac{3}{2}}} \right) \left( 1 + O \left( (1 - t)^\eta + |x - x^*|^{\eta} \right) \right) \tag{1.24} \]

- Self-similarity. Let $u^* = x^*$ and $\eta(t, x, y) := ((T - t)^\eta + |x - x^*|^\eta + y^{-\eta} + |y^* - y|^{-\eta})$. For any $\epsilon > 0$, below the displacement line $y \leq (1 - \epsilon)y^*(t, x)$ and for $|x - x^*|$ small enough:

\[
u(t, x, y) = u^*(t) + (T - t)^{\frac{1}{4}} \left( \Theta_{\mu, \nu, \iota} + \tilde{u} \right) \left( \frac{x - x^*}{(T - t)^{\frac{3}{4}}}, \frac{y}{(T - t)^{-\frac{1}{4}}} \right), \]

\(^1\)The condition $Y < (2 - \epsilon) Y^*$ ensures both functions are defined. Note that on $[(2 - \epsilon)Y^*, 2Y^*]$ the asymptotics (1.20) prevails.
where \( \tilde{u} \) satisfies, for some uniform constant \( C > 0 \):

\[
|\tilde{u}(t, \mathcal{X}, \mathcal{Y})| \leq C \left( |\mathcal{X}|^4 + |\mathcal{Y}|^{-2} + \left( (T-t)^{\frac{3}{2}} y^*(x^* + (T-t)^{\frac{3}{2}} \mathcal{X}) - \mathcal{Y} \right)^{-2} \right) \eta(t, x, y) \quad (1.25)
\]

\[
|\tilde{u}_x(t, \mathcal{X}, \mathcal{Y})| \leq C \left( |\mathcal{Y}|^4 + (1 + |\mathcal{X}|)^{-\frac{7}{6}} \left( (T-t)^{\frac{3}{2}} y^*(x^* + (T-t)^{\frac{3}{2}} \mathcal{X}) - \mathcal{Y} \right)^{-3} \right) \eta(t, x, y) \quad (1.26)
\]

\[
|\tilde{u}_y(t, \mathcal{X}, \mathcal{Y})| \leq C \left( |\mathcal{Y}|^{-3} + \left( (T-t)^{\frac{3}{2}} y^*(x^* + (T-t)^{\frac{3}{2}} \mathcal{X}) - \mathcal{Y} \right)^{-3} \right) \eta(t, x, y). \quad (1.27)
\]

Closer to the displacement line, given any \( K > 0 \), for \((1 - \epsilon)y^*(t, x) \leq y \leq y^*(t, x) - K \) and \(|x - x^*| \) small enough:

\[
u(t, x, y) = u^*(t) + \frac{\mu \nu^2}{p^2(y - y^*(t, x))} + \tilde{v}(t, x, y), \quad (1.28)
\]

where, for \( \tilde{\eta}(t, x, y) := ((T-t)^{\eta} + |x - x^*|^{\eta} + (y^* - y)^{-\eta} + (|y^* - y|/y^*)^{\eta}) \):

\[
|\tilde{v}| \leq \frac{C\tilde{\eta}(t, x, y)}{|y - y^*(t, x)|^2}, \quad |\partial_{\tilde{v}}| \leq \frac{C\tilde{\eta}(t, x, y)}{(T-t)^{\frac{3}{2}} + |x|^{\frac{1}{6}} |y - y^*(t, x)|^3}, \quad |\partial_y \tilde{v}| \leq \frac{C\tilde{\eta}(t, x, y)}{|y - y^*(t, x)|^{\frac{1}{2}}} \quad (1.29)
\]

- Reconnections below and above. There exist two functions \( f \in C^3([0, \infty), \mathbb{R}) \) and \( g \in C^3(\mathbb{R}, \mathbb{R}) \) depending on \( u_0, v^E \) and \( p^E \) such that for any \( K > 0 \), as \((t, x) \to (T, x^*(T))\):

\[
u(t, x, y) \to f(y) \quad \text{for} \ y \leq K, \quad f(y) - u^* \sim \frac{\mu \nu^2}{p^2 y^{\gamma - 2}} \quad \text{as} \ y \to \infty, \quad (1.30)
\]

\[
u(t, x, y) \to g(y - y^*(t, x)) \quad \text{for} \ y^* - K \leq y \leq y^* + K, \quad g(y) - u^* \sim \frac{\mu \nu^2}{p^2 y^{\gamma - 2}} \quad \text{as} \ y \to -\infty. \quad (1.31)
\]

Let us make the following comments on the results of Theorem 5.

1. The set considered in the above Theorem is nonempty. Indeed, given any outer Eulerian flow, there exist solutions blowing up outside the boundary, as negative enough initial data for the third equation in (1.8) will tend to \(-\infty\) in finite time. However, the separating structure described in this Theorem is not the only one occurring, and degenerate unstable singularities also exist, see Proposition 6.

2. The estimates for the error in (1.25), (1.26), (1.27) and (1.29) should be interpreted as follows. The first term in the right hand side is the typical size of \( \Theta \), \( \partial_x \Theta \) and \( \partial_y \Theta \) respectively. The \( \eta(t, x, y) \) small term then quantifies a gain. For example, for \((\mathcal{X}, \mathcal{Y})\) in a compact set \( K \) in the support of \( \Theta_{\mu, \nu, \epsilon} \), these estimates imply \( \|\tilde{u}\|_{C^1(K)} = O(\|(T-t)C^0(\eta)\|) \to 0 \). We have to distinguish between below the displacement curve and near it as in this latter region the solution is close to a displaced version of \( \Theta \). The identities (1.28) and (1.29) show the solution is close to the asymptotic expansion (1.20) of \( \Theta \) near the top part of its support, with \( y^* \) replacing \( 2y^* \). From the asymptotic behaviour of \( \Theta \) in Proposition 4, \( \tilde{u} \) is of lower order compared to \( \Theta \) precisely in a size one zone in \( x \) around \( x^* \), and until a size one distance to the boundary and the displacement curve. These estimates are then sharp since they precisely fail when the solution reconnect to another nonsingular behaviour.

3. Note that the asymptotic behaviour of \( \Theta \) and that of the reconnection functions \( f \) and \( g \), in (1.30) and (1.31) respectively, are compatible from Proposition 4.
4. This convergence result also holds for higher order derivatives, which is a direct consequence of the proof of the Theorem. In particular the weighted estimates adapt naturally.

Other degenerate singular behaviours are also possible. The degeneracy can come from two distinct aspects: at the singular point, the set of zero vorticity can locally not be a line, and the tangential compression can be induced by a degenerate shock formation for Burgers. There exist a large range of self-similar profiles corresponding to these (infinitely many) degenerate cases. Their properties can also be studied with the same strategy used in the proofs of Propositions 3 and 4, and their stability similarly as in Theorem 5. As a particular interesting example, we study in this paper one of the least degenerate cases, corresponding to a Burgers generic shock happening at the crossing of two lines of zero vorticity. A particular self-similar profile corresponding to this case enjoys remarkable properties: it is odd in $x$, admits an analytic expansion beyond its support, and explicit formulas can be obtained on the vertical axis. In a forthcoming paper we shall show its stability for the full Prandtl’s system.

**Proposition 6** (Degenerate symmetric profile). The following mapping $(a,b) \mapsto (X,Y)$ is a volume preserving diffeomorphism,

$$\tilde{\Phi}(a,b) = \left( a + a^3 + \frac{b^2 a}{4}, 2 \int_{-\infty}^{\frac{b}{2}} \frac{db}{1 + b^2} \right),$$

between $\mathbb{R}^2$ and the subset $\{0 < Y < 2\tilde{Y}^*(X)\}$ of the upper half plane, where

$$\tilde{Y}^*(X) = \int_{0}^{\psi_1(X)} \frac{dz}{\sqrt{z} \sqrt{X - z - z^3}},$$

is an analytic curve with asymptotic expansion

$$\tilde{Y}^*(X) \approx \pi - \frac{15}{16} \pi X^2 + \text{h.o.t.}, \quad Y^*(X) \approx \frac{B(\frac{1}{6}, \frac{1}{2})}{3} |X|^{-\frac{5}{2}} + O(|X|^{-1}),$$

with $B$ the Euler integral of the first kind. The opposite of the first component of its inverse

$$\Theta = -\tilde{\Phi}^{-1} : (X,Y) \mapsto -a$$

is a self-similar profile. Namely,

$$u(t,x,y) = (T-t)^{\frac{1}{2}} \Theta \left( \frac{x}{(T-t)^{\frac{1}{2}}}, \frac{y}{(T-t)^{\frac{1}{2}}} \right)$$

solves (1.10). The function $\tilde{\Phi}$ moreover enjoys the following properties.

(i) $\tilde{\Phi}$ is odd in $X$, it is positive on the set $\{X < 0, 0 < Y < 2\tilde{Y}^*(X)\}$ and negative on $\{X > 0, 0 < Y < 2\tilde{Y}^*(X)\}$.

(ii) The set $\{\partial_\gamma \tilde{\Phi} = 0\}$ corresponds to $\{X = 0\} \cup \{Y = \tilde{Y}^*(X)\}$ where $\tilde{\Phi}(X, \tilde{Y}^*(X)) = \psi_1(X)$.

The minimum of $\partial_X \tilde{\Phi}$ on this set is attained at $(0,\pi)$ where one has the expansion:

$$\Theta(X, \pi + Y) = -X + X^3 + \frac{Y^2}{4} X + O(|X|^5 + |Y|^4 |X|) \quad \text{as} \ (X,Y) \to (0,0).$$
Remark 7. 

(iii) $\tilde{\Theta}$ solves the following ODEs:

\[
\begin{align*}
\partial_\gamma \tilde{\Theta} &= \sqrt{\tilde{\Theta} - \Theta^3 - \Theta - \gamma} & \text{for } \gamma < 0 \text{ and } 0 \leq \gamma \leq \gamma^*(\gamma) \\
\partial_\gamma \tilde{\Theta} &= -\sqrt{\tilde{\Theta} - \Theta^3 - \Theta - \gamma} & \text{for } \gamma < 0 \text{ and } \gamma^*(\gamma) \leq \gamma \leq 2\gamma^*(\gamma) \\
\partial_\gamma \tilde{\Theta} &= -\sqrt{-\Theta^3 + \Theta + \gamma} & \text{for } \gamma > 0 \text{ and } 0 \leq \gamma \leq \gamma^*(\gamma) \\
\partial_\gamma \tilde{\Theta} &= \sqrt{\Theta^3 + \Theta + \gamma} & \text{for } \gamma > 0 \text{ and } \gamma^*(\gamma) \leq \gamma \leq 2\gamma^*(\gamma)
\end{align*}
\]  

(iv) $\tilde{\Theta}$ is analytic, including at the boundary, when restricted to its support $\{(\gamma, \gamma) \in \mathbb{R} \times [0, +\infty), \ 0 \leq \gamma \leq 2\gamma^*(\gamma)\}$. Extending $\tilde{\Theta}$ by solving the above ODE in a periodic way in $\gamma$ at each fixed $\gamma$ yields a global analytic self-similar profile.

(v) The trace of the first order derivative on the vertical axis is given by:

\[
\partial_\gamma \tilde{\Theta}(0,0) = -\sin^2 \left( \frac{\gamma}{2} \right) \mathbb{I}_{0 \leq \gamma \leq 2\pi}.
\]

(vi) The trace of the third order derivative on the vertical axis is given by:

\[
\partial_{\gamma^3} \tilde{\Theta}(0,0) = \frac{1}{576} \left[ 96 \cos^8 \left( \frac{\gamma}{2} \right) \sin^2 \left( \frac{\gamma}{2} \right) + \frac{1}{6} - \sin(\gamma) \left( 270 \gamma - 80 \sin(\gamma) + 3 \sin(2\gamma) - \frac{683 \sin(\gamma)}{3 \sin^2 \left( \frac{\gamma}{2} \right) + \frac{1}{2}} \right) \right] \mathbb{I}_{0 \leq \gamma \leq 2\pi}.
\]

The above function is strictly positive on $(0, 2\pi)$. It admits the expansions as $\gamma \to 0$:

\[
\partial_{\gamma^3} \tilde{\Theta}(0,0) = c\gamma^8 + O(\gamma^{10}), \quad \partial_{\gamma^3} \tilde{\Theta}(0,2\pi-\gamma) = c'\gamma + O(\gamma^2), \quad c, c' > 0.
\]

Remark 7. 

- Note the difference in the scaling exponents when comparing (1.36) with the fundamental profile (1.13). The above degenerate profile has a quicker expansion as its symmetry maximises the strength of the expulsion by nonlocal transport.

- In [4], we show that there is a stable blow-up pattern for equation 1.4, for which solutions converge to $(T-t)^{-1} \sin^2 \left( \frac{y}{(2(T-t)^{-1/2})} \right) \mathbb{I}_{0 \leq y \leq 2\pi(T-t)^{-1/2}}$. This partially shows that the profile of Proposition 6 is the stable attractor for solutions that are odd in $x$ when the singularity is located on the transversal axis.

1.3. Strategy of the proof and organisation of the paper

The proof relies on a careful study of the characteristics and of the ODEs underlying them. The sharp expression we find for the blow-up time is a consequence two volume preservations: that of the characteristics map, and that of the aforementioned ODE (1.7). In the study of the self-similar profiles, the explicit formulas are found by computing the expression of the characteristics for specific well-chosen Taylor expansions in Lagrangian variables. The Eulerian study is then permitted by the use of the Crocco transformation which links solutions of the stationary self-similar equation to solutions of a local equation. Some unknown functions corresponding to integration constants are then found by compatibility with Taylor expansions near the point at which the shock will form. The original solution is then showed to solve a local ODE in one variable only, which allows for an almost explicit study. To obtain the generic appearance of the fundamental profile during separation, we first show using a soft control argument that the initial datum $u_0$ can always be perturbed to produce a necessary generic condition for the characteristics map. We then reconstruct the solution around the point at which the shock is forming, by proving stability of the leading order term for the characteristics map. For this we show uniform invertibility around a specific diffeomorphism in an unbounded zone. This is attained through sharp estimates for the errors and a renormalisation procedure at various locations to ensure uniformity in solving the perturbation problems.
The paper is organised as follows. Section 2 is devoted to the proof of Theorem 1, showing local existence of solutions and the formula giving the maximal time of existence. Self-similar profiles are then studied in Section 3. The fundamental one is studied first, with the proof of Proposition 3 followed by that of Proposition 4 in Subsection 3.1. The degenerate presented in Proposition 6 is studied next in Subsection 3.2. The last part, Section 4, focuses on the proof the genericity Theorem 5. A generic condition for the characteristics map is established in Lemma 8, then various estimates around the points where the shock is forming are obtained in Lemma 10, which allows to parametrise some level curves in Lemma 11, before performing the full stability analysis in the end of this section.

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Notations

We write \( x \lesssim y \) if there exists a constant \( C > 0 \) independent of the context such that \( x \leq Cy \). We write \( x \approx y \) if \( x \lesssim y \) and \( y \lesssim x \). We use Lagrangian variables \((X,Y)\) and Eulerian variables \((x,y)\). As they are equal at the initial time, we might use one notation or the other in several places, but in this context only. We use the notations \( \partial_x \), \( \partial / \partial_x \) or the subscript \( {}_x \) to indicate partial differentiation.

2. Local well-posedness and time of existence

This section is devoted to the proof of Theorem 1. We establish here local-existence of solutions to the inviscid Prandtl’s system (1.1) and prove that \( T \) given by (1.9) is the maximal time of existence.

Proof of Theorem 1. The proof relies on the special structure of the characteristics and uses the Crocco transformation. The existence follows from their nondegeneracy until time \( T \), while the regularity follows from standard regularity theory for level sets of functions.

Step 1 Existence. We first solve for the tangential displacement, which we denote by:

\[
x(t,X,Y) = \phi_1[t](X,Y),
\]

where \( x \) above is the solution of (1.7). Notice that (1.7) can always be solved globally in time and the above function \( x \) is well-defined at any time \( t > 0 \). We next study the level sets \( x = Ct \) in Lagrangian variables. Let us show first that they are non-degenerate. In the first case, assume that \((X_0,Y_0)\) is such that \( u_{0Y}(X_0,Y_0) \neq 0 \). Then the Crocco transformation \((X,Y) \mapsto (X,u)\) is a well defined local diffeomorphism in a neighbourhood of \((X_0,Y_0)\). The ODE solved by (1.7) is divergence free in the \((x,u)\) phase space. Therefore, at any time \( t > 0 \), the mapping \((x,u) \mapsto (x(t),u(t))\) is volume preserving, and is in particular a diffeomorphism. Hence, the mapping \((X,Y) \mapsto (x(t),u(t))\) is a local diffeomorphism near \((X_0,Y_0)\). It follows that \( \nabla x(t,X_0,Y_0) \neq 0 \) for any \( t > 0 \) in this first case.

In the second case, we assume that \( u_{0Y}(X_0,Y_0) = 0 \) or \( Y_0 = 0 \). Let us consider the set \( Y = Y_0 \) in Lagrangian variables. At each \( X \) close to \( X_0 \), the couple \((x,u)\) solves (1.7), so that in particular:

\[
\partial_t(\partial_X x) = \partial_X u,
\quad \text{implying} \quad \partial_t(\partial_X x) = \partial_X (-p^E_x(t,x)) = -(\partial_x x)p^E_{xx}(t,x).
\]
This shows that at each fixed $X$:
\[
\frac{d}{dt} \left( \frac{\partial \phi_x}{\partial x} \right) = - \left( \frac{\partial \theta_y x}{\partial x} \right)^2 - p_x(t, x), \quad \frac{\partial \phi_x}{\partial x} (0) = \partial_x u_0.
\]
In particular, at the point $(X_0, Y_0)$, the quantity $\partial_\theta y x / \partial_\theta x$ is precisely the third component of the ODE system (1.8). Because of the definition of $T (1.9)$, one obtains that the solution to the above differential equation is well defined for $t < T$. Hence $\partial_t \log(\partial_\theta x)$ is well-defined for $t < T$ which after integration gives that $\partial_\theta x (X_0, Y_0) > 0$. Hence, $\nabla \theta x (t, X_0, Y_0) \neq 0$ in this second case as well.

We just showed that $\nabla_{X,Y} x \neq 0$ everywhere as long as $t < T$. Hence, in Lagrangian variables, the level sets $x = Cte$ are non-degenerate. At the boundary, the previous discussion implies that $\partial_\theta x |_{Y=0} \neq 0$. Therefore, the upper half plane is foliated by curves $\Gamma[x]$ corresponding to the level sets $\{x(X, Y) = x\}$. Since $u_0, u^x, p^x$ are $C^2$, solving the ODE (1.7) produces a solution map that is also of class $C^2$, and $x(t, X, Y)$ is a $C^2$ function. Hence the curves $\Gamma[x]$ are $C^1$.

This allows us to define an arclength parametrisation $s$ for each of these curves, where $s = 0$ corresponds to the point at the boundary $Y = 0$.

The change of coordinates $(t, X, Y) \mapsto (t, x, s[t, x](X, Y))$ is a $C^1$ diffeomorphism from $[0, T) \times \mathbb{R} \times [0, \infty)$ onto itself. At a point $(X, Y)$, considering the orthonormal base $(v_1, v_2)$ with $v_1 = \frac{\nabla \phi_1[t](X, Y)}{|\nabla \phi_1[t](X, Y)|}$ and $v_2 = \frac{\nabla \phi_1[t](X, Y)}{|\nabla \phi_1[t](X, Y)|}$ where $(z_1, z_2) = (z_2, z_1)$ one sees that
\[
\frac{\partial x}{\partial v_1[t,v_2]} = |\nabla \phi_1[t](X, Y)|, \quad \frac{\partial x}{\partial v_2[t,v_2]} = 0, \quad \frac{\partial s}{\partial v_2[t,v_2]} = 1.
\]

This shows the following value for the determinant of the change of variables:
\[
|\text{Det} \left( \frac{\partial(x, s)}{\partial X}, \frac{\partial(x, s)}{\partial Y} \right) | = |\nabla \phi_1[t](X, Y)|.
\]

To find the second component of the characteristics, we look for a $C^1$ mapping $(x, s) \mapsto (x, y)$. It satisfies:
\[
\frac{\partial x}{\partial s} = 0, \quad \frac{\partial x}{\partial x} = 1,
\]
and hence its determinant is
\[
|\text{Det} \left( \frac{\partial(x, y)}{\partial x} \big|_s, \frac{\partial(x, y)}{\partial s} \big|_x \right) | = |\frac{\partial y}{\partial s} \big|_x |.
\]

Since the mapping $(X, Y) \mapsto (x, y)$ has to preserve volume, from the two determinants above we infer that:
\[
1 = |\text{Det} \left( \frac{\partial(x, y)}{\partial X} \big|_Y, \frac{\partial(x, y)}{\partial Y} \big|_X \right) | = |\text{Det} \left( \frac{\partial(x, y)}{\partial X} \big|_Y, \frac{\partial(x, y)}{\partial Y} \big|_X \right) ||\text{Det} \left( \frac{\partial(x, y)}{\partial X} \big|_Y, \frac{\partial(x, y)}{\partial Y} \big|_X \right) |
\]
\[
= |\nabla \phi[t](X, Y)| \left| \frac{\partial y}{\partial s} \big|_x \right|.
\]

This and the boundary condition forces the choice
\[
\frac{\partial y}{\partial s} \big|_x = \frac{1}{|\nabla \phi[t](X(x, s), Y(x, s))|},
\]
yielding the formula for $y$:
\[
y(t, X, Y) = \phi_2[t](X, Y) = \int_0^{s[t,x](X,Y)} \frac{d\tilde{s}}{|\nabla \phi_1[t](\gamma(t, x)[\tilde{s}])|}.
\]
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Note that before $T$, the denominator in the above integral is uniformly away from 0. The function $y$ above is of class $C^1$ because $\gamma$, $s$ and $\nabla \phi_1$ are. The mapping $(t, X, Y) \mapsto (t, x, y)$ is thus a $C^1$ diffeomorphism from $[0, T) \times \mathbb{R} \times [0, \infty)$ onto itself. We finally define the solution as $u(t, x, y) = u_0(X, Y)$. Clearly,

$$\frac{\partial x}{\partial t}|_{X,Y} = u_0(X, Y) = u(t, x, y).$$

Since the mapping $(X, Y) \mapsto (x, y)$ is $C^1$ and preserves the measure, $\partial_x \frac{\partial x}{\partial t}|_{X,Y} + \partial_y \frac{\partial y}{\partial t}|_{X,Y} = 0$, yielding:

$$\frac{\partial y}{\partial t}|_{X,Y} = -\int_0^u \partial_x u(t, x, \tilde{y})d\tilde{y}.$$ 

And since $\partial_t u(t, x(t), y(t)) = -p^E_x(x(t))$ and $u$ is $C^1$, one deduces that $u$ solves the inviscid Prandtl’s equations. Note that the matching condition at infinity in (1.1) are indeed satisfied for the following reason. Initially as $y \to \infty$, $u_0 \to u^E$. $u^E$ solves the Bernouilli equation (1.2) that has a global solution, and whose characteristics correspond to the tangential displacement (1.7) of the characteristics for $u$. This gives the desired compatibility.

**Step 2** Regularity. Assume $u_0 \in C^k$. The formula (2.1) for $t \in [0, T)$ defines a $C^k$ function since $x$ is obtained as the solution of the ODE (1.7) with a $C^k$ vector field. In the formula (2.2), $\nabla \phi_1[\tilde{t}]$ is $C^{k-1}$, and $s$ and $\gamma$ come from the parametrisation of the level sets of a $C^k$ function, hence are also $C^{k-1}$. Therefore, $u$ is of class $C^{k-1}$. The continuity of the flow follows from similar arguments.

**Step 3** Uniqueness. If $u$ is a $C^2$ solution then uniqueness is straightforward as the characteristics are well defined and have to produce the diffeomorphism constructed above. In the case where $u \in C^1$ only, let us detail how the normal component of the characteristics and the volume preservation can be obtained. Define the characteristics $(x(t), y(t))$ through:

$$\frac{\partial x}{\partial \tilde{t}} = u(t, x, y), \quad x(0) = X, \quad \frac{\partial y}{\partial \tilde{t}} = -\int_0^{y(t)} u_x(t, x, y), \quad y(0) = Y.$$ 

One can indeed solve the second equation because the function $\int_0^y u_x(t, x, y)$ is $C^1$ in the third variable. One obtains characteristics $(x, y)$ such that $x$ is $C^1$ in $(X, Y)$ and $y$ is only $C^1$ in $t$ and continuous in the other variables. $u$ then solves $\dot{u} = -p^E_x(x)$ along the characteristics, implying that it is given by the formula (2.1). Moreover, since $x$ is a $C^1$ function, and $y$ is a $C^1$ function in $t$, with $\partial_y$ being $C^1$ in $y$, such that $\partial_y(\partial_t y(t)) = -\partial_x (\partial_t (x(t)))$, an approximation argument using a regularisation procedure gives that the characteristics must preserve volume. The mapping $(X, Y) \mapsto (x, y)$ is then a bijection preserving volume with $x \in C^1$, which can be showed to be necessarily of the form described in Step 1.

**Step 4** Blow-up. Assume that $T < \infty$. Then by definition $T$ the solutions to the ODEs (1.8) must blow up at time $T$, which is only possible if $u_x \to -\infty$ as $t \to T$.

3. Construction of self-similar profiles

In this section we study first the fundamental self-similar profile and prove Propositions 3 and 4. We then study a degenerate one, which corresponds to the stable blow-up in the case of odd in $x$ symmetry, and prove Proposition 6.
3.1. Generic self-similar profile

It is convenient to decompose the proof of Propositions 3 and 4 in three parts. First we study the curve \( X ↦→ Y^*(X) \) and prove (1.16) and (1.17). Then to show that \( \Theta \) defined by Proposition 3 is a self-similar profile, satisfying the Taylor expansion (1.19) for the special choice of \( p^* = \frac{4}{9\pi^3} \Gamma \left( \frac{1}{4} \right)^4 \). This allows then to complete the proof of Proposition 4 as the Taylor expansion fixes integration constants that appear.

A solution under the form (1.13) solves (1.10) if and only if \( \Theta \) solves the corresponding stationary self-similar equation (where \( \partial^2_Y − \frac{1}{2} \partial_X \Theta + \left( \frac{3}{2} X + \Theta \right) \partial_X \Theta + \left( -\frac{1}{4} Y - \partial_Y^{-1} \partial_X \Theta \right) \partial_Y \Theta = 0 \), (3.1) eq:theta

which is a stationary nonlinear nonlocal transport equation. While usually self-similar solutions are found by solving the above equation, it is remarkable that we can prove here that \( \Theta \) defined by (1.12) is a self-similar solution without proving that it solves the above equation. Note that although this profile diverges to infinity as \( Y \to 0 \) or \( 2Y^*(X) \), it still makes sense to speak of (1.13) as a solution of Prandtl’s equations: the characteristics are well defined, and the quantity \( \int_{\gamma_t} \partial_x u(x, y) \, dy \) is well defined on the support of \( u(t) \). This is our first Lagrangian approach.

The second approach is based on solving the above equation, and we proceed as if we did not know the formula (1.13) (indeed, the formula (1.11) would directly imply the ODEs (1.18)). The reason is the following: this approach to solve the stationary self-similar equation is interesting and robust and can be used to study other degenerate profiles, for which the formula is no longer valid but analogues of the properties listed in Proposition 4 hold.

We first prove the properties of the displacement curve \( Y^* \) of the fundamental profile.

**Proof of (1.16) and (1.17).** These identities are obtained through direct computations. First of all notice that \( X + \Theta + p^2 \Theta^3 = 0 \) for \( \Theta = \Psi_1(pX) / p \). Set \( \Theta = z + \frac{\Psi_1(pX)}{p} \) in the integral (1.15) and use the previous equation to get that

\[
Y^*(X) = \frac{1}{2} \int_0^\infty \frac{dz}{\sqrt{z + 3\Psi_1^2(pX)z + 3p\Psi_1(pX)z^2 + p^2 z^3}} = \frac{1}{2} \int_0^\infty \frac{dz}{\sqrt{z}\sqrt{1 + p^2 z^2 + 3pz\Psi_1(pX) + 3\Psi_1^2(pX)}}. \tag{3.2} \]

In addition, we have that for the specific value \( p = p^* \) given by (1.14):

\[
\frac{1}{2} \int_0^\infty \frac{dz}{\sqrt{z + p^2 z^3}} = \frac{4\Gamma \left( \frac{5}{4} \right)^2}{\sqrt{\pi}p} = \frac{3\pi}{8}. \tag{3.3}\]

A direct computation using (3.2) then yields that:

\[
\partial_X Y^*(0) = \sqrt{\frac{3}{4}} \int_0^\infty \frac{\sqrt{z}dz}{(1 + z^2)^{\frac{3}{2}}} = \frac{3\sqrt{p}}{4\sqrt{\pi}} \Gamma \left( \frac{3}{4} \right)^2 = 1.
\]

\[
\partial_X^2 Y^*(0) = -\frac{3}{2p^2} \int_0^\infty \frac{dz}{\sqrt{z}(1 + z^2)^{\frac{3}{2}}} + \frac{27}{8} p \int_0^\infty \frac{z^3 dz}{(1 + z^2)^{\frac{5}{2}}} = -\frac{5\Gamma \left( \frac{1}{4} \right)}{72\pi^4},
\]

\[
\partial_Y Y^*(0) = \frac{1}{2p(1 + p^2)} \int_0^\infty \frac{dz}{\sqrt{z}(1 + z^2)^{\frac{3}{2}}} = \frac{1}{2p(1 + p^2)} \Gamma \left( \frac{1}{4} \right)^2 = 1.
\]
which concludes the proof of (1.16). We turn to the asymptotic behaviours. If $\mathcal{X} > 0$ then $\Psi_1(p\mathcal{X}) < 0$ and we change variables in (3.2) setting $\tilde{z} = \frac{z}{-\Psi_1(p\mathcal{X})}$:

$$\mathcal{Y}^*(\mathcal{X}) = \frac{1}{2\sqrt{7\Psi_1(p\mathcal{X})}} \int_0^\infty \frac{d\tilde{z}}{\sqrt{\tilde{z}^2 + 3 + \Psi_1^{-2}(p\mathcal{X}) - 3\tilde{z} + \tilde{z}^2}}$$

(3.4)

Since, $\Psi_1(\mathcal{X}) = -\mathcal{X}^\frac{3}{4} + \frac{1}{2}\mathcal{X}^{-\frac{1}{4}} + O(\mathcal{X}^{-1})$ as $\mathcal{X} \to \infty$, we deduce that

$$\mathcal{Y}^*(\mathcal{X}) = C_+\mathcal{X}^{-\frac{1}{2}} + O(\mathcal{X}^{-\frac{1}{2}}) = C_+\mathcal{X}^{-\frac{1}{2}} + O(\mathcal{X}^{-\frac{1}{2}})$$

(3.5)

where $C_+ = 2^{-1}p^{-2/3}\int_{-1}^\infty (z^3 + 1)^{-1/2}dz$. A very similar computation in the case $\mathcal{X} \to -\infty$ gives:

$$\mathcal{Y}^*(\mathcal{X}) = C_-|\mathcal{X}|^{-\frac{1}{2}} + O(|\mathcal{X}|^{-\frac{1}{2}})$$

as $\mathcal{X} \to -\infty$,

(3.6)

where $C_- = 2^{-1}p^{-2/3}\int_1^\infty (z^3 - 1)^{-1/2}dz$. The constants $C_+$ can be computed explicitly:

$$C_+ = \frac{3(\frac{3}{2})^{\frac{1}{4}}\pi^{\frac{3}{4}}\Gamma(\frac{3}{2})}{4\Gamma(\frac{1}{4})\pi^{\frac{1}{2}}\Gamma(\frac{5}{4})}$$

(3.7)

This ends the proof of (1.17).

We now show Proposition 3, and that $\Theta$ defined by (1.13) has the desired Taylor expansion (1.19) thanks to the asymptotic behaviour of $\mathcal{Y}^*$ established above.

Proof of Proposition 3 and of (1.19) in Proposition 4: Step 1 Self-similarity. This is a consequence of the volume preserving property of $\Phi$ and of the formula for the characteristics. The fact that $\Phi$ is volume preserving diffeomorphism is a direct computation. We now study its image. Let us fix $\mathcal{X} \in \mathbb{R}$ and study the curve of equation $a + b^2 + p^2a^3 = \mathcal{X}$, that we parametrise with the variable $b$. The vertical component of the image is:

$$\mathcal{Y} = \int_{-\infty}^b \frac{db}{1 + 3\Psi_1^2(p(\mathcal{X} - b^2))},$$

which is an increasing function of $b$, hence, the set $\{\Phi(a,b), \Phi_1 = \mathcal{X}\}$ consists of the interval $(0 < \mathcal{Y} < 2\mathcal{Y}^*(\mathcal{X}))$ where

$$2\mathcal{Y}^*(\mathcal{X}) = \int_\mathbb{R} \frac{db}{1 + 3\Psi_1^2(p(\mathcal{X} - b^2))} = \frac{1}{2} \int_{\Psi_1^{-1}(p\mathcal{X})}^\infty \frac{d\Theta}{\sqrt{\Theta + p^2\Theta^3 + \mathcal{X}}}.$$

We now prove that $\Theta$ is a self-similar profile. Let us denote by $\tilde{\Theta}$ the vertical component of the inverse:

$$\Phi^{-1} = (-\Theta, \tilde{\Theta}).$$

Let us solve the inviscid Prandtl’s system with initial datum $u_0 = \Theta$. The corresponding characteristics $(X,Y) \mapsto (x,y)$ is a volume preserving diffeomorphism whose tangential component is given by:

$$x = X + t\Theta(X,Y).$$

Let us perform the change of variables $(a,b) \mapsto (X,Y)$, with $(a,b) = (-\Theta(X,Y), \tilde{\Theta}(X,Y))$, so that $(X,Y) = \Phi(a,b)$ by definition. The definition of $\Phi$ gives:

$$X = a + b^2 + p^2a^3$$

so that $x = a + b^2 + p^2a^3 + t\Theta(X,Y) = a(1 - t) + b^2 + p^2a^3$. 
Moreover the mapping \((a, b) \mapsto (x, y)\) is volume preserving. Hence one can retrieve from the above formula for \(x(a, b)\) the formula for \(y(a, b)\). Indeed, the set \(x(\tilde{a}, \tilde{b}) = x(a, b)\) is given by the equation
\[
a(1 - t) + b^2 + p^2a^3 = \tilde{a}(1 - t) + \tilde{b}^2 + p^2\tilde{a}^3
\]
and corresponds to a curve which, parametrised by \(\tilde{b}\), is given by:
\[
\left\{ \left( \frac{-(1-t)^{\frac{1}{2}}}{p} \Psi_1 \left( \frac{p}{(1-t)^{\frac{1}{2}}} \left( a(1 - t) + p^2a^3 + b^2 - \tilde{b}^2 \right) \right), \tilde{b} \right) : \tilde{b} \in \mathbb{R} \right\}.
\]
As \(\partial_a x = (1 - t) + 3p^2a^2\), one deduces from (2.2) (which can be rewritten under the form (4.40)) and performing a change of variables that
\[
y(a, b) = \int_{-\infty}^{b} \frac{d\tilde{b}}{(1-t)^{\frac{1}{2}} - p \Psi_1 \left( \frac{p}{(1-t)^{\frac{1}{2}}} \left( a(1 - t) + p^2a^3 + b^2 - \tilde{b}^2 \right) \right)} = \frac{1}{1 - t} \int_{-\infty}^{b} \frac{d\tilde{b}}{1 + 3\Psi_1 \left( \frac{p}{(1-t)^{\frac{1}{2}}} \left( a(1 - t) + p^2a^3 + b^2 - \tilde{b}^2 \right) \right)}.
\]
Therefore, the mapping \((a, b) \mapsto (x, y)\) is given by:
\[
(x, y) = \left\{ a(1 - t) + b^2 + p^2a^3, \frac{1}{1 - t} \int_{-\infty}^{b} \frac{1}{1 + 3\Psi_1 \left( \frac{p}{(1-t)^{\frac{1}{2}}} \left( a(1 - t) + p^2a^3 + b^2 - \tilde{b}^2 \right) \right)} \right\}.
\]
A direct consequence of the formula (1.11) for \(\Phi\) and the fact that \((-\Theta, \tilde{\Theta}) = \Phi^{-1}\) is that the inverse of the above mapping is:
\[
(a, b) = \left\{ -(1-t)^{\frac{1}{2}} \Theta \left( \frac{x}{(1-t)^{\frac{1}{2}}}, \frac{y}{(1-t)^{-\frac{1}{4}}} \right), (1-t)^{\frac{1}{2}} \tilde{\Theta} \left( \frac{x}{(1-t)^{\frac{1}{2}}}, \frac{y}{(1-t)^{-\frac{1}{4}}} \right) \right\}.
\]
As \(a = -\Theta(X, Y)\) one gets that:
\[
\Theta(X, Y) = (1-t)^{\frac{1}{2}} \Theta \left( \frac{x}{(1-t)^{\frac{1}{2}}}, \frac{y}{(1-t)^{-\frac{1}{4}}} \right).
\]
As along the characteristics \(u(t, x, y) = u_0(X, Y) = \Theta(X, Y)\), this gives the desired formula (1.13).

**Step 2 Taylor expansion.** This is a direct consequence of the Taylor expansion of the fonction \(\Phi\) near \((0, 0)\). Let us write \(\Theta_{ij} = \partial_i \partial_j \Theta(0, \Psi^*(0))\). Let us first look at the set \(X = 0 = a + b^2 + p^2a^3\), or equivalently \(a = \Psi_1(pb^2)/p\). One has since \(\Psi_1\) is odd and \(\Psi_1(X) = -X + X^3 + O(|X|^5|)\) as \(X \to 0\):
\[
\mathcal{Y} = \int_{-\infty}^{b} \frac{d\tilde{b}}{1 + 3\Psi_1 \left( \frac{p}{(1-t)^{\frac{1}{2}}} \right)} = \mathcal{Y}^*(0) + b + O(|b|^5).
\]
So that \(b(0, \mathcal{Y}^*(0) + \mathcal{Y}) = \mathcal{Y} + O(|\mathcal{Y}|^5)\) as \(\mathcal{Y} \to 0\). As \(a = \Psi_1(pb^2)/p = -b^2 + O(b^6)\), one deduces from that information on the vertical derivatives of \(\Theta\):
\[
\Theta(0, \mathcal{Y}^*(0)) = 0, \quad \Theta_{01} = 0, \quad \Theta_{02} = 2, \quad \Theta_{03} = 0.
\]
Since the mapping \((a, b) \mapsto (\mathcal{X}, \mathcal{Y})\) is volume preserving, one inverts the Jacobian matrix to find:

\[
\begin{pmatrix}
\frac{\partial \mathcal{Y}}{\partial \mathcal{X}} & -\frac{\partial \mathcal{Y}}{\partial \mathcal{Y}} \\
-\frac{\partial \mathcal{Y}}{\partial \mathcal{X}} & \frac{\partial \mathcal{Y}}{\partial \mathcal{Y}}
\end{pmatrix} = \begin{pmatrix}
\frac{\partial \mathcal{Y}}{\partial \mathcal{X}} & \frac{\partial \mathcal{Y}}{\partial \mathcal{Y}} \\
\frac{\partial \mathcal{Y}}{\partial \mathcal{X}} & \frac{\partial \mathcal{Y}}{\partial \mathcal{Y}}
\end{pmatrix}.
\]

Let us secondly look at the set \(b = 0\), corresponding to \(\mathcal{Y} = \mathcal{Y}(\mathcal{X})\). From the above identity, \(\partial_2 a = -\partial_3 b = 0\) on this set. Hence \(\partial_2 \Theta(\mathcal{X}, \mathcal{Y}(\mathcal{X})) = 0\). Differentiating with respect to \(\mathcal{X}\) once and twice this identity, setting \(\mathcal{X} = 0\), using the Taylor expansion (1.16) of \(\mathcal{Y}(\mathcal{X})\) gives:

\[
\Theta_{11} = -2, \quad \Theta_{21} + 2c_2 + 2\Theta_{12} = 0.
\]

Still on the set \(b = 0\), one has \(\Theta = -p\Psi_1(\mathcal{X})/p\). We differentiate once, twice and three times this identity with respect to \(\mathcal{X}\), and set \(\mathcal{X} = 0\). Using \(\Psi_1(\mathcal{X}) = -\mathcal{X} + \mathcal{X}^3 + O(|\mathcal{X}|^5)\), (3.8), (3.9) and (1.16) this gives:

\[
\Theta_{10} = -1, \Theta_{20} = 2, \quad \Theta_{30} + 3\Theta_{21} + 3\Theta_{12} = 6p^2.
\]

We need one last information. We take the identity \(\partial_2 a = \partial_3 b\). Consider the set \(\mathcal{X} = a + b^2 + p^2 a^3 = 0\). Then on this set from (1.11), since \(\Psi_1(0) = 0\) and \(\Psi_1' = -1/(1 + 3\Psi_1^2)\):

\[
\begin{align*}
\frac{\partial \mathcal{Y}}{\partial b} &= \frac{1}{1 + 3p^2 a^2} - b \int_{-\infty}^{b} \frac{12p\Psi_1(p\tilde{b}^2)}{(1 + 3\Psi_1^2(p\tilde{b}^2))^3} \tilde{b} \, d\tilde{b} \\
&= 1 + O(b^4) - b \int_{-\infty}^{0} \frac{12p\Psi_1(p\tilde{b}^2)}{(1 + 3\Psi_1^2(p\tilde{b}^2))^3} \tilde{b} + O(b^3) \\
&= 1 - b\sqrt{p} \int_{0}^{\infty} \frac{1}{\left(\sqrt{\frac{x}{3}} + \sqrt{\frac{\tilde{x}}{3}}\right)^2 (1 + x)^2} \tilde{b} + O(b^3)
\end{align*}
\]

where we changed variables \(z = 3p\Psi_1(p\tilde{b}^2)\). Hence as \(b(0, \mathcal{Y}^*(0) + \mathcal{Y}) = \mathcal{Y} + O(|\mathcal{Y}|^5)\) on this set, \(\partial_2 \Theta(0, \mathcal{Y}^*(0) + \mathcal{Y}) = -1 + C\mathcal{Y} + O(|\mathcal{Y}|^3)\). This implies \(\Theta_{12} = 0\), and we obtain the desired Taylor expansion (1.19) for \(\Theta\) using the previous information.

We now perform the study of the stationary self-similar equation (3.1). The first part of the proof proves that any solution satisfies a nonexplicit local ODE, this part is valid for any solution, and adapts to the equations corresponding to degenerate self-similar profiles. The second part gives an explicit formula for this ODE, under the hypothesis that a certain Taylor expansion is satisfied. In particular, we proved above that \(\Theta\) defined by (1.13) satisfies this Taylor expansion. All remaining properties of Proposition 4 are then consequences of the ODEs (1.18), which ends the proof of Proposition 4.

**Proof of (1.18), (i), (ii) and (iii) in Proposition 4.** **Step 1 Obtention of a local ODE: proof of (1.18).** Assume \(\Theta\) is \(C^1\) and solves (3.1). We use the Crocco transformation where \(\partial_2 \Theta\) is non zero and has constant sign:

\[
(\mathcal{X}, \mathcal{Y}) \mapsto (\mathcal{X}, \Theta), \quad \frac{\partial \mathcal{Y}}{\partial \mathcal{X}}|_{\Theta} = -\frac{\partial \mathcal{X} \Theta}{\partial \mathcal{Y} \Theta}, \quad \frac{\partial \mathcal{Y}}{\partial \mathcal{X}}|_{\Theta} = \frac{1}{\partial_2 \Theta},
\]

and study the new unknown \(\tau = \partial_2 \Theta\mathcal{X}\) (this notation means \(\mathcal{X}\) is kept fixed). Let’s divide by \(-\tau\) the right hand side of Equation (3.1) using the above identities:

\[
\frac{-\frac{1}{2} \Theta + \left(\frac{3}{2} \mathcal{X} + \Theta\right) \partial_2 \mathcal{X} + (-\frac{1}{2} \mathcal{Y} - \partial_2^{-1} \partial_2 \Theta) \partial_2 \Theta}{-\tau} = \frac{1}{2} \tau + \left(\frac{3}{2} \mathcal{X} + \Theta\right) \frac{\partial \mathcal{Y}}{\partial \mathcal{X}}|_{\Theta} + \frac{1}{4} \mathcal{Y} + \partial_2^{-1} \mathcal{X} \frac{\partial \Theta}{\partial \mathcal{X}}|_{\mathcal{Y}}.
\]
Now we apply $\partial/\partial Y$ to the right hand side and the equation will become local from a cancellation due to the Crocco transform:

$$\frac{\partial}{\partial Y} \left( \frac{1}{2} \Theta \frac{\partial \tau}{\partial \Theta} + \left( \frac{3}{2} \mathcal{X} + \Theta \right) \frac{\partial Y}{\partial X} \bigg|_{\Theta} + \frac{1}{4} Y + \partial_{\mathcal{X}}^{-1} X \frac{\partial \Theta}{\partial X} \right) = \frac{1}{2} - \frac{1}{2} \frac{\Theta}{\partial \Theta} \bigg|_{\Theta} + \frac{1}{3} Y - \frac{1}{3} \frac{\partial Y}{\partial X} \bigg|_{\Theta}.$$

Hence, in coordinates $(\mathcal{X}, \Theta)$, $\tau$ solves the following local equation:

$$-\frac{3}{2} \tau + \Theta \frac{\partial \tau}{\partial \Theta} - (3 \mathcal{X} + 2 \Theta) \frac{\partial \tau}{\partial X} = 0.$$

On the characteristic curves one can rewrite the above transport equation as the following

$$\frac{d \Theta}{\Theta} = \frac{d \mathcal{X}}{3 \mathcal{X} + 2 \Theta} = \frac{2 d \tau}{3 \tau}.$$

Hence,

$$\frac{2 d \Theta}{\Theta} = \frac{d \mathcal{X}}{\frac{3}{2} \mathcal{X} + \Theta}, \quad (3.10)$$

by dividing by $\Theta^4$ one gets that

$$\frac{3 \mathcal{X} + 2 \Theta}{\Theta^4} d \Theta - \Theta^{-3} d \mathcal{X} = 0. \quad (3.11)$$

By integrating the above equation one gets that

$$\Theta^{-3} (\mathcal{X} + \Theta) = C,$$

$$\mathcal{X} + \Theta = (1 - C) \Theta^3 - \Theta^3,$$

$$\Theta^{-3} (\mathcal{X} + \Theta + \Theta^3) = 1 - C.$$

In addition, one gets by integrating that

$$\frac{2 d \Theta}{\Theta} = \frac{d \tau}{\tau},$$

$$\tau = \partial_Y \Theta = \left| \Theta \right|^2 \tilde{C}.$$

It follows that there exists a function $g$ such that,

$$\tau = \left| \Theta \right|^2 g(\Theta^{-3} (\mathcal{X} + \Theta + \Theta^3)), \quad (3.12)$$

where $g$ can for the moment be any function, and which depends on the domain on which we performed the Crocco transform. Note that in a neighbourhood of a point $(0, \mathcal{Y}_0)$ where $\Theta(0, \mathcal{Y}_0) = 0$, the function $\Theta^{-3} (\mathcal{X} + \Theta + \Theta^3)$ can take all possible positive values. It follows that $g$ will be entirely determined through the following use of the Taylor expansion near that point. Indeed, assume one has the expansion (which is satisfied by $\Theta$ defined by (1.13)):

$$\Theta(\mathcal{X}, \mathcal{Y}_0 + \mathcal{Y}) = -\mathcal{X} + (\mathcal{X} - \mathcal{Y})^2 + c_1 \mathcal{X}^3 + c_2 \mathcal{X}^2 \mathcal{Y} + c_3 \mathcal{X} \mathcal{Y}^2 + c_4 \mathcal{Y}^3 + \ldots, \quad c_1 + c_2 + c_3 + c_4 = p^2 > 0. \quad (3.13)$$

Let $\alpha > 0$ and consider the curve $(\mathcal{X} + \alpha \mathcal{X}^{3/2}, \mathcal{X})$ as $\mathcal{X} \to 0^+$. On this curve from the above Taylor expansion we obtain that

$$\partial_Y \Theta = -2 \alpha \mathcal{X}^2 + O(\mathcal{X}^2), \quad \Theta = -\mathcal{X} + O(\mathcal{X}^2), \quad \mathcal{X} + \Theta + \Theta^3 = (\alpha^2 + p^2 - 1) \mathcal{X}^3,$$
and by assuming that \( g \) is continuous, one obtains that \( g(-\alpha^2 - p^2 + 1) = -2\alpha \) by letting \( X \) go to zero in (3.12). Inverting this equation, we deduce that

\[
g(z) = -2\sqrt{-z - p^2 + 1}, \quad \text{for} \quad z \leq 1 - p^2,
\]

which implies that \( \Theta \) solves

\[
\frac{\partial \Theta}{\partial Y} = -2\sqrt{X + \Theta + p^2\Theta^3}.
\]  \( \text{eq:Thetabis1} \)

There exists a unique solution to the above equation. It can be solved by writing

\[
\frac{\partial \Theta}{\partial Y} = \frac{1}{2\sqrt{X + \Theta + p^2\Theta^3}}
\]  \( \text{eq:Thetabis2} \)
as a function of \((X, \Theta)\). The maximal value or \( Y \) obtained when solving this ODE is precisely \( Y^* \) given by (1.15). Hence, for \( X \in \mathbb{R} \) and \( 0 < Y < Y^* \) there exists a unique solution to (3.14) for \( Y \) immediately after 0 with \( \lim_{Y \to 0} \Theta = +\infty \) and for larger \( Y \), the function \( \Theta \) is strictly decreasing until it reaches the value \( \frac{\Psi_1(pX)}{p} \) for which \( X + \Theta + p^2\Theta^3 = 0 \). For \( Y^* < Y < 2Y^* \), \( \Theta \) solves \( \frac{\partial \Theta}{\partial Y} = 2\sqrt{X + \Theta + p^2\Theta^3} \), where we obtain that \( \Theta \) is increasing from \( \Theta = \frac{\Psi_1(pX)}{p} \) at \( Y = Y^* \) till infinity at \( Y = 2Y^* \) with the symmetry

\[
\Theta(Y^* - Y) = \Theta(Y^* + Y),
\]

which closes the proof of (i).

**Step 2** Proof of (i). From the ODEs (1.18), one obtains that \( \partial_Y \Theta \neq 0 \) unless \( Y = Y^*(X) \). From (3.15) and the definition of \( Y^*(X) \), one obtains that solving (3.15) from the boundary until \( Y^*(X) \) precisely gives the formula \( \Theta(X, Y^*(X)) = -\Psi_1(pX)/p \). From this formula and the fact that \( \partial_Y \Theta = 0 \) on this curve, one deduces that:

\[
\partial_X \Theta(X, Y^*(X)) = -\Psi'_1(pX).
\]

The function \( \Psi'_1 \) attains its minimum at the origin. Hence, on the set \( \{ \partial_Y \Theta = 0 \} \), the function \( \partial_X \Theta \) attains its minimum at \((0, Y^*(0))\). Finally, the Taylor expansion (1.19) has already been proved previously in Step 2 of the previous proof.

**Step 3** Proof of (ii). Since we expect \( \Theta \) to go to infinity as \( Y \) goes to zero, we do the following taylor expansion in (3.15) for \( X \) fixed and \( Y \) close to zero

\[
Y = \frac{1}{2p} \int_{\Theta}^{\infty} \frac{du}{u^{3/2}} + O(p^{-3}\Theta^{-3/2}),
\]  \( \text{eq:TaylorExp1} \)

which implies (1.20). Similarly when \( Y \) approaches \( 2Y^* \), we obtain from (3.15),

\[
2Y^* - Y = \frac{1}{2p} \int_{\Theta}^{\infty} \frac{du}{u^{3/2}} + O(p^{-3}\Theta^{-3/2}).
\]  \( \text{eq:TaylorExp2} \)

**Step 4** Proof of (iii). We will now look at the asymptotic behaviour as \( X \to \pm \). We only treat the case \( X \to -\infty \), as the other case can be handled with similar ideas. Let \( \psi_-(p^{-2/3}, \infty) \) be the following function:

\[
\psi_-(z) = \frac{1}{2} \int_{-\infty}^{z} \frac{dz}{\sqrt{-1 + p^2z^2}}.
\]

Note that by definition (3.7) of \( C_- \), the range of \( \psi_- \) is \((0, C_-] \). Denote by \( \psi_-^{-1} : (0, C_-] \to [p^{-2/3}, \infty) \) its inverse, and define \( \varphi_- \) to be its extension on \((0, 2C_-)\) by even symmetry about
\[ C_-: \]
\[ \varphi_-(\hat{Y}) := \begin{cases} 
\psi^{-1}(Y) & \text{if } 0 < \hat{Y} \leq C_- , \\
\psi^{-1}(2C_- - \hat{Y}) & \text{if } C_- \leq \hat{Y} < 2C_. 
\end{cases} \]

The properties of \( \varphi_- \) listed in (iii) of Proposition 4 are verified by a direct check. It remains to prove the convergence. For this we use (3.15). Let first \( 0 < Y \leq Y^* (X) \), then:

\[ \mathcal{Y}(\Theta) = \frac{1}{2} \int_\Theta^\infty \frac{d\tilde{\Theta}}{\sqrt{X + \Theta + p^2 \tilde{\Theta}^3}}. \]

We change variables and set \( \tilde{\Theta} = zp^{-1/3} \Psi_1(p\lambda) \), using \( X = -\Psi_1(p\lambda)/p - \Psi_1^2(p\lambda)/p \) so that:

\[ \frac{\Psi_1^\frac{1}{p} (p\lambda)}{p^\frac{1}{p}} Y(\Theta) = \frac{1}{2} \int_{\frac{p^\frac{1}{3} \Theta}{\Psi_1(p\lambda)}}^\infty \frac{dz}{\sqrt{-1 + p^2 z^3}} \left( 1 + \frac{1}{z^2 p^\frac{1}{3} + z p^\frac{2}{3} + 1} \Psi_1^{-2}(p\lambda) \right)^{-\frac{1}{2}} \]

\[ = \psi_-(\frac{p^\frac{1}{3} \Theta}{\Psi_1(p\lambda)}) + 1 \int_{\frac{p^\frac{1}{3} \Theta}{\Psi_1(p\lambda)}}^\infty \frac{dz}{\sqrt{-1 + p^2 z^3}} \left( 1 - \frac{1}{z^2 p^\frac{1}{3} + z p^\frac{2}{3} + 1} \Psi_1^{-2}(p\lambda) \right)^{-\frac{1}{2}} \]

\[ = \psi_-(\frac{p^\frac{1}{3} \Theta}{\Psi_1(p\lambda)}) + O \left( |X|^{-\frac{2}{3}} \left( 1 + \frac{\Theta}{\Psi_1(p\lambda)} \right)^{-\frac{2}{3}} \right) \]

where the quantity in the \( O \) is positive. Given the sign of \( \psi'_- \), the asymptotic behaviour of the \( O \) and \( \psi_- \), from the above equation one deduces that:

\[ \frac{p^\frac{1}{3} \Theta}{\Psi_1(p\lambda)} = \psi^{-1} \left( \frac{\Psi_1^\frac{1}{p} (p\lambda)}{p^\frac{1}{p}} Y(\Theta) \right) + O \left( |X|^{-\frac{2}{3}} \left( \frac{\Theta}{\Psi_1(p\lambda)} \right)^{-\frac{2}{3}} \right) \]

which, given that \( \psi^{-1} (\tilde{\Theta}) \approx \tilde{\Theta}^{-2} \) uniformly on \((0, C_-)\), and that \( \Psi_1^2(X) = |X|^{1/3} + O(|X|^{-1/3}) \) gives finally:

\[ \Theta = p^\frac{1}{3} \Psi_1(p\lambda) \psi^{-1} \left( \frac{\Psi_1^\frac{1}{p} (p\lambda)}{p^\frac{1}{p}} Y \right) + p^\frac{1}{3} \Psi_1(p\lambda) O \left( |X|^{-\frac{2}{3}} |Y|^\frac{4}{3} \right) \]

\[ = |X|^{-\frac{1}{3}} \varphi_- \left( |X|^\frac{1}{3} Y \right) + O \left( |X|^{-\frac{2}{3}} Y^{-2} \right) \]

where we used the fact that \( |Y| \lesssim |X|^{-1/6} \) as \( X \to -\infty \) and \( Y \leq Y^* \). For \( Y^* < Y \leq (2 - \epsilon)Y^* \) we write:

\[ Y = Y^* + 2 \int_{p^\frac{1}{3} \Psi_1(p\lambda)}^\Theta \frac{d\tilde{\Theta}}{\Psi_1^\frac{1}{p} (p\lambda) 2 \sqrt{X + \Theta + \tilde{\Theta}^3}} \]

so that:

\[ \frac{\Psi_1^\frac{1}{p} (p\lambda)}{p^\frac{1}{p}} Y = \frac{\Psi_1^\frac{1}{p} (p\lambda)}{p^\frac{1}{p}} Y^* + 1 \int_{\frac{p^\frac{1}{3} \Theta}{\Psi_1(p\lambda)}}^{p^\frac{1}{3} \Theta} \frac{dz}{\sqrt{-1 + p^2 z^3}} \left( 1 + \frac{1}{z^2 p^\frac{1}{3} + z p^\frac{2}{3} + 1} \Psi_1^{-2}(p\lambda) \right)^{-\frac{1}{2}} \]

\[ = C_- + 1 \int_{\frac{p^\frac{1}{3} \Theta}{\Psi_1(p\lambda)}}^{p^\frac{1}{3} \Theta} \frac{dz}{\sqrt{-1 + p^2 z^3}} + O(|X|^{-\frac{2}{3}}). \]
As we are restricting to the range $Y^* < Y \leq (2 - \epsilon)Y^*$ the above equation, using the asymptotic behaviour of $\varphi_-$, gives:

$$
\Theta = \frac{\Psi_1(p\mathcal{X})}{p\mathcal{X}^3} \varphi_+ \left( \frac{\Psi_1^2(p\mathcal{X})}{p\mathcal{X}^3} Y \right) + O(|X|^{-1}\Theta^2)
$$

This shows the desired asymptotic behaviour (iii) in Proposition 4 at $-\infty$. The behaviour at $\infty$ can be proved along similar lines.

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### 3.2. Degenerate self-similar profiles

In this subsection, we prove Proposition 6. We construct an odd in $\mathcal{X}$ self-similar profile that is the two dimensional version of the profile found for the full viscous Prandtl’s system in [4] on the transversal axis. In order to simplify notations, we drop the tilde for $\tilde{\Theta}$ and $\tilde{Y}$ used in Proposition 6 to distinguish the profile from the one studied in Propositions 3 and 4. Again, we proceed in three times. First, we study the displacement line. Then, we show that $\Theta$ defined by (1.32) is a self-similar profile (without showing that it solves the equation below). A self-similar solution is of the form (1.36) if and only if $\Theta$ solves the corresponding stationary self-similar equation

$$
-\frac{1}{2} \Theta + \left( \frac{3}{2} \mathcal{X} + \Theta \right) \partial_{\mathcal{X}} \Theta + \left( -\frac{1}{2} \mathcal{Y} - \partial_{\mathcal{Y}}^{-1} \partial_{\mathcal{X}} \Theta \right) \partial_{\mathcal{Y}} \Theta = 0,
$$

(3.18) **eq:tildetheta**

(notice that only constants changed when comparing with the equation for the fundamental profile (3.1)). In the last part, we study the equation above with a general approach. The strategy used here shows how the two studies, in Lagrangian and Eulerian variables respectively, are robust, independent, and complimentary in the understanding of the self-similar solutions to the Prandtl’s system.

First we study the displacement line.

**Proof of (1.34).** The function $Y^*$ given by (1.33) can be rewritten, after a change of variables, as:

$$
Y^*(\mathcal{X}) = 2 \int_{-\infty}^{\infty} \frac{d\bar{b}}{(1 + \bar{b}^2)^2 \left( 1 + 3\Psi_1^2 \left( \frac{\mathcal{X}}{1 + \bar{b}^2} \right)^2 \right)},
$$

(3.19) **eq:tildemY*cl**

which shows directly that this curve is analytic as $\Psi_1$ is. Next, for $\mathcal{X} > 0$, we write:

$$
Y^*(\mathcal{X}) = \sqrt{-\Psi_1(\mathcal{X})} \mathcal{X} \int_0^1 \frac{d\tilde{z}}{\sqrt{\tilde{z} \left( 1 + \mathcal{X}^3 \Psi_1(\mathcal{X}) \tilde{z} + \tilde{z}^3 \Psi_1(\mathcal{X}) \right)}}
$$

(3.20) **eq:tildemY*cl**

In particular, using that $\Psi_1(X) = -X + X^3 + O(X^5)$ as $X \to 0$ one obtains the expansion of $Y^*$ near the origin in (1.34) from the above identity and the computations:

$$
\int_0^1 \frac{d\tilde{z}}{\sqrt{1 - \tilde{z}}} = B \left( \frac{1}{2}, \frac{1}{2} \right) = \pi, \quad \int_0^1 \frac{\sqrt{\tilde{z}}(1 + \tilde{z})}{\sqrt{1 - \tilde{z}}} d\tilde{z} = \frac{7}{8} \pi.
$$
We now turn to the expansion at infinity. One has $\Psi_1(\mathcal{X}) = -\mathcal{X}^{1/3} + \frac{1}{3} \mathcal{X}^{-1/3} + O(\mathcal{X}^{-1})$ as $\mathcal{X} \to +\infty$. We write:

\[ Y^*(\mathcal{X}) = \sqrt{-\Psi_1(\mathcal{X})} \int_0^1 \frac{d\tilde{z}}{\sqrt{\frac{1 + \tilde{z}^3}{1 + \tilde{z}^3} \Psi_1(\mathcal{X})} + \tilde{z}^3 \frac{\Psi_1(\mathcal{X})}{\mathcal{X}}} \]

\[ = \sqrt{-\Psi_1(\mathcal{X})} \int_0^1 \frac{d\tilde{z}}{\sqrt{\frac{1}{1 + \tilde{z}^3} (1 + g(\mathcal{X}, \tilde{z}))^{-1/2}} \mathcal{X}} \]

where using that $\Psi_1(\mathcal{X}) + \Psi_1^3(\mathcal{X}) + X = 0$:

\[ g(\mathcal{X}, \tilde{z}) = \frac{\tilde{z} \Psi_1(\mathcal{X}) + \tilde{z}^3 \left(\frac{\Psi_1^3(\mathcal{X})}{\mathcal{X}} + 1\right)}{1 - \tilde{z}^3} = \frac{\tilde{z} - \tilde{z}^3 \Psi_1(\mathcal{X})}{1 - \tilde{z}^3} = \frac{\tilde{z}(1 + \tilde{z})}{1 + \tilde{z} + \tilde{z}^2} \frac{\Psi_1(\mathcal{X})}{\mathcal{X}}. \]

The integral values

\[ \int_0^1 \frac{d\tilde{z}}{\sqrt{\frac{1}{1 - \tilde{z}^3} (1 + \tilde{z})^{-1/2}}} = \frac{2\sqrt{\pi} \Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{2}{3}\right)} \]

then gives the expansion (1.17) and finishes the proof of (1.34).

We now show that the formula (1.35) defines a self-similar profile, i.e. that (1.36) is a solution of (1.10). Note that the function defined by (1.35), when extended by $0$ for $Y \geq 2Y^*(\mathcal{X})$, is $C^1$, hence it makes sense to speak of (1.36) as a $C^1$ solution to (1.10). We also show that it satisfies the Taylor expansion (1.37), so that in the strategy employed right after to study equation (3.18) we will be able to determine the integration constants.

**Proof of (1.36) and (1.37).** The proof is based on similar computations as the one performed in the previous subsection for the fundamental profile.

**Step 1 Self-similarity.** The fact that $\Phi : (a, b) \mapsto (\mathcal{X}, \mathcal{Y})$ defined by (1.32) preserves is a direct computation. By fixing $\mathcal{X} \in \mathbb{R}$, and so fixing the relation $a + a^3 + b^2a/4 = \mathcal{X}$, the vertical component of the image is:

\[ \mathcal{Y}(a, b) = 2 \int_{-\infty}^b \frac{db}{1 + \tilde{b}^2} \left(1 + 3 \Psi_1\left(\frac{\mathcal{X}}{(1 + \tilde{b}^2)^{3/2}}\right)\right). \]

Hence, using the formula (3.19) obtained after a change of variables, the set $\{ \Phi(a, b), \Phi_1 = \mathcal{X} \}$ consists of the interval $(0 < \mathcal{Y} < 2\mathcal{Y}^*(\mathcal{X}))$ where $\mathcal{Y}^*$ is indeed defined by (1.33). We now prove that $\Theta$ is a self-similar profile. Let us denote by $\tilde{\Theta}$ the vertical component of the inverse:

$\Phi^{-1} = (-\Theta, \tilde{\Theta})$.

We solve (1.10) with initial datum $u_0 = \Theta$. The corresponding characteristics $(X, Y) \mapsto (x, y)$ is a volume preserving mapping, and its tangential component is given by:

\[ x = X + t\Theta(X, Y). \]

Let us perform the change of variables $(a, b) \mapsto (X, Y)$, with $(a, b) = (-\Theta(X, Y), \tilde{\Theta}(X, Y))$, so that $(X, Y) = \Phi(a, b)$ by definition. The definition of $\Phi$ gives:

\[ X = a + a^3 + \frac{b^2a}{4} \text{ so that } x = a(1 - t) + a^3 + \frac{b^2a}{4}. \]
From the above formula and the fact that the mapping \((a, b) \mapsto (x, y)\) is volume preserving one can retrieve the formula for \(y(a, b)\). Indeed, the set \(x(\tilde{a}, \tilde{b}) = x(a, b)\) is given by the equation
\[
a(1 - t) + a^3 + \frac{b^2a}{4} = \tilde{a}(1 - t) + \tilde{a}^3 + \frac{\tilde{b}^2\tilde{a}}{4}
\]
and corresponds to a curve which, parametrised by \(\tilde{b}\), is given by:
\[
\left\{ \left( -\left( 1 - t + \frac{\tilde{b}^2}{4} \right)^{\frac{1}{2}} \Psi_1 \left( \frac{a(1 - t) + a^3 + \frac{b^2a}{4}}{(1 - t + \frac{\tilde{b}^2}{4})^{\frac{3}{2}}} \cdot \tilde{b} \right), \tilde{b} \in \mathbb{R} \right) \right\}.
\]

As \(\partial_a x = (1 - t) + \frac{\tilde{b}^2}{4} + 3a^2\), one deduces from (2.2) (which can be rewritten under the form (4.40)) and performing a change of variables that
\[
y(a, b) = \int_{-\infty}^b \frac{db}{(1 - t) + \frac{\tilde{b}^2}{4} + 3\tilde{a}^2(x, \tilde{b})} = \int_{-\infty}^b \frac{db}{(1 - t) + \frac{\tilde{b}^2}{4} + 3\Psi_1^2 \left( \frac{a(1 - t) + a^3 + \frac{\tilde{b}^2a}{4}}{(1 - t + \frac{\tilde{b}^2}{4})^{\frac{3}{2}}} \right)}.
\]

Therefore, the mapping \((a, b) \mapsto (x, y)\) is given by:
\[
(x, y) = \left( a(1 - t) + \frac{\tilde{b}^2}{4}a + a^3, \frac{1}{(1 - t)} \int_{-\infty}^b \frac{db}{(1 + \frac{\tilde{b}^2}{4(1 - t)})} \left( 1 + 3\Psi_1^2 \left( \frac{a(1 - t) + a^3 + \frac{\tilde{b}^2a}{4}}{(1 - t + \frac{\tilde{b}^2}{4})^{\frac{3}{2}}} \right) \right) \right).
\]

A direct consequence of the formula (1.32) for \(\Phi\) is that the above mapping can be written as:
\[
(x, y) = \left( \begin{array}{cc} (1 - t)^{\frac{3}{2}} & 0 \\ 0 & (1 - t)^{-\frac{3}{2}} \end{array} \right) \circ \Phi \circ \left( \begin{array}{cc} (1 - t)^{-\frac{1}{2}} & 0 \\ 0 & (1 - t)^{-\frac{1}{2}} \end{array} \right).
\]

From the fact that \((-\Theta, \tilde{\Theta}) = \Phi^{-1}\), the inverse of the above mapping is:
\[
(a, b) = \left( -(1 - t)^{\frac{3}{2}} \Theta \left( \frac{x}{(1 - t)^{\frac{3}{2}}}, \frac{y}{(1 - t)^{-\frac{3}{2}}} \right), (1 - t)^{\frac{3}{2}} \tilde{\Theta} \left( \frac{x}{(1 - t)^{\frac{3}{2}}}, \frac{y}{(1 - t)^{-\frac{3}{2}}} \right) \right).
\]

As \(a = -\Theta(X, Y)\) one gets that:
\[
\Theta(X, Y) = (1 - t)^{\frac{3}{2}} \Theta \left( \frac{x}{(1 - t)^{\frac{3}{2}}}, \frac{y}{(1 - t)^{-\frac{3}{2}}} \right).
\]

As along the characteristics \(u(t, x, y) = u_0(X, Y) = \Theta(X, Y)\), this shows (1.36).

**Step 2 Taylor expansion.** We write \(\Theta_{ij} = \partial_{x^i} \partial_{x^j} \Theta(0, \pi)\). As \(\Theta\) is odd in \(X\), we obtain that:
\[
\Theta_{00} = \Theta_{01} = \Theta_{02} = \Theta_{03} = 0, \quad \Theta_{20} = \Theta_{21} = 0
\]
Since the mapping \((a, b) \mapsto (X, Y)\) is volume preserving, one inverts the Jacobian matrix to find:
\[
\begin{pmatrix}
\frac{\partial Y}{\partial b} & \frac{\partial X}{\partial b} \\
-\frac{\partial Y}{\partial a} & \frac{\partial X}{\partial a}
\end{pmatrix} = \begin{pmatrix}
\frac{\partial a}{\partial X} & \frac{\partial a}{\partial Y} \\
\frac{\partial b}{\partial X} & \frac{\partial b}{\partial Y}
\end{pmatrix}.
\]
Let us secondly look at the set \(b = 0\), corresponding to \(Y = Y^*(X)\). One has \(\partial_b X = ba/2\). Hence from the above identity, \(\partial_2 a = -\partial_1 b = 0\) on this set. Hence \(\partial_2 Y = \partial_1 b = 0\) on this set. Hence \(\partial_2 Y \Theta(X, Y^*(X)) = 0\).

Differentiating with respect to \(X\) this identity, setting \(X = 0\), using the Taylor expansion \((1.34)\) of \(Y^*\) and the coefficients computed above:
\[
\Theta_{11} = 0.
\]

Still on the set \(b = 0\), one has \(\Theta = -\Psi_1(X)\). We differentiate once and three times this identity with respect to \(X\), and set \(X = 0\). Using \(\Psi_1(X) = -X + X^3 + O(|X|^5)\), the coefficients computed above and the Taylor expansion of \(Y^*\) at 0 \((1.34)\) this gives:
\[
\Theta_{10} = -1, \quad \Theta_{30} = 6.
\]
We now set \(a = 0\). Then:
\[
Y(0, b) = \int_{-\infty}^{b} \frac{d\tilde{b}}{1 + \frac{\tilde{b}^2}{\tau}}
\]
so that \(\partial_3 Y(0, 0) = 1, \partial_4 Y(0, 0) = 0\) and \(\partial_{bb} Y(0) = -\frac{1}{2}\). Inverting this relation one gets \(\partial_3 b(0, \pi) = 1, \partial_4 b(0, \pi) = 0\). We now use the relation obtained from the identity above \(\partial_2 \gamma \gamma a(0, \pi) = \partial_2 \gamma \gamma_X (\partial_3 b Y)(0, 0)\), which, injecting all values of the coefficients already found, gives:
\[
\Theta_{12} = 2
\]
and ends the proof of \((1.37)\).

We can now end the proof of Proposition 6 which we will do by studying equation \((3.18)\). Again, we do it to show the robustness of the approach, while formula \((1.32)\) would directly imply the ODEs \((1.38)\).

**Proof of (i) to (vi) in Proposition 6.** **Step 1** **Obtention of a local equation, proof of (iii).** We use the Crocco transform to solve equation \((3.18)\) as we did for the generic profile. In zones where \(\partial_2 \gamma \Theta\) does not change sign, we switch to variables \((X, \Theta)\), and study \(\partial_2 \gamma \gamma \Theta\). A similar calculation as in the step 1 of the proof of Proposition 4 using the Crocco transform gives that
\[
\partial_2 \gamma \Theta = \Theta^2 g(\Theta^{-3}(X + \Theta + \Theta^3))
\]
where \(g\) is some function which depends on the zone for which the transformation was done. It will be decided with the Taylor expansion of \(\Theta\) near the blow-up point. We solve first for \(\tau > 0\) hence \(g > 0\). In addition, we suppose that around the blowup point \((0, Y^*(0))\) we have the following Taylor expansion (which we already know for the profile defined by \((1.35)\)):
\[
\Theta(X, Y + Y^*(0)) = -X + X^3 + \frac{X Y^2}{4} + h.o.t.
\]
Let \(\alpha > 0\) and consider the curve \((X, \alpha X)\) as \(X \to 0^+\). On this curve from the above Taylor expansion we obtain that
\[
\partial_2 \gamma \Theta = \frac{\alpha}{2} X^2 + O(X^3), \quad X + \Theta + \Theta^3 = \frac{\alpha^2}{4} X^3 + O(X^4),
\]
and by assuming that \( g \) is continuous, one obtains as \( X \) goes to zero in (3.21) that
\[
g\left(\frac{\alpha}{2}\right) = \frac{\alpha^2}{4}. \tag{3.22}
\]
It follows that \( g(\cdot) = \sqrt{\cdot} \), which implies
\[
\frac{\partial \Theta}{\partial Y} = -\frac{1}{\sqrt{\Theta X + \Theta + \Theta^3}}, \tag{3.23}
\]
with the corresponding analogues in the other zones. This gives that the value of \( \gamma^* \) given by (1.33) is the one until which the above ODEs (1.38) can be solved. By a continuity argument, the zones in which \( \Theta \) solve (1.38) can be extended to show that the equations (1.38) are indeed satisfied on the domains presented in this identity.

**Step 2 Proof of (i)** This is a direct consequence of the signs in the ODEs (1.38).

**Step 3 Proof of (v)** Let us consider the zone \( \mathcal{X} > 0 \) and \( 0 < \gamma < \gamma^* \). One has the following convergence result
\[
\gamma(\mathcal{X}, \Theta) = \int_0^{-\Theta} \frac{dz}{\sqrt{\gamma - z - z^3}} = \int_0^{-\Theta} \frac{du}{\sqrt{u^2 + 1 - u - \mathcal{X}^2 u^2}} \to \int_0^{-\partial_x \Theta(0, \gamma)} \frac{du}{\sqrt{u^2 + 1 - u}}
\]
as \( \mathcal{X} \to 0 \). Hence on the vertical axis \( \gamma \) and \( \partial_x \Theta(0, \gamma) \) are linked by
\[
\gamma = \int_0^{-\partial_x \Theta(0, \gamma)} \frac{du}{\sqrt{u^2 + 1 - u}} = 2 \arcsin\left(\sqrt{-\partial_x \Theta(0, \gamma)}\right)
\]
which yields the desired formula.

**Step 4 Proof of (ii)** We recall that \( \Theta + \Theta^3 + \mathcal{X} = 0 \) for \( \Theta = \Psi_1(\mathcal{X}) \). From the sign condition (i) proved above and the ODEs (1.38), one obtains that \( \partial_y \Theta = 0 \) when \( \Theta = \Psi_1(\mathcal{X}) \). From the formula (1.33) giving \( \gamma^* \), this happens precisely only at \( \gamma = \gamma^*(\mathcal{X}) \) or \( \mathcal{X} = 0 \). On the line \( \gamma = \gamma^*(\mathcal{X}) \), as \( \Theta(\mathcal{X}, \gamma^*(\mathcal{X})) = \Psi_1(\mathcal{X}) \) and \( \partial_y \Psi_1(\mathcal{X}, \gamma^*(\mathcal{X})) = 0 \), one gets that \( \partial_x \Theta(\mathcal{X}, \gamma^*(\mathcal{X})) = \Psi_1'(\mathcal{X}) \). Hence, as \( \Psi_1 \) attains its minimum at \( \gamma = \gamma^* \), the minimum on the vertical axis is also attained at \( (0, \pi) \). The Taylor expansion (1.37) has already been proved, which ends the proof of (ii).

**Step 5 Proof of (iv)** The analyticity of \( \Theta \) in the interior of its domain could be proved by studying the ODEs (1.38), but note that it is a direct consequence of the formula (1.35), as the diffeomorphism defined by (1.32) is analytic. We now prove the existence of an analytic extension by solving periodic manner in \( \gamma \) the ODEs (1.38). To do this, we will show that these ODEs can be used to prove the analyticity at the boundary \( \{ \gamma = 0 \} \) (where the natural extension of \( \Theta \) is 0).

We will establish that an extension of the mapping \( (\mathcal{X}, \sqrt{-\Theta}) \mapsto (\mathcal{X}, \gamma) \) is an analytic diffeomorphism, implying the result by taking the inverse transformation. Without loss of generality we consider the case \( \mathcal{X} > 0 \). From (3.21) we infer that there is a one-to-one relation between
(X, Y) and (X, Θ(X, Y)) between the sets \{X > 0, 0 ≤ Y ≤ Y^*\} and \{X > 0, Ψ_1(X) ≤ Θ ≤ 0\} with the formula

\[ Y(X, Θ(X, Y)) = \int_0^{-Θ(X, Y)} \frac{dz}{\sqrt{zX - z - z^3}} \]

We change variables and define U = \sqrt{-Θ}. Then the above formula becomes

\[ Y(X, U) = 2 \int_0^U \frac{du}{\sqrt{X - u^2 - u^6}} \]

This formula also makes sense for \(-\sqrt{-Ψ_1(X)} ≤ U < 0\), since \(X - u^2 - u^6 > 0\) for \(|u| ≤ \sqrt{-Ψ_1(X)}\), and the mapping \((X, U) \mapsto (X, Y)\) is one-to-one from \(\{X > 0, |U| < \sqrt{-Ψ_1(X)}\}\) onto \(\{X > 0, |Y| < Y^*\}\), such that its restriction to nonnegative Y satisfies \(U(X, Y) = -\sqrt{-Θ(X, Y)}\). Let \(X_0 > 0\). For \((X, u)\) close to \((X_0, 0)\), the function under the integral sign is analytic

\[ \frac{1}{\sqrt{X - u^2 - u^6}} = \sum_{α ∈ N^2} a_α (X - X_0)^{α_1} u^{2α_2} \]

with \(a_{(0,0)} = 1/\sqrt{X_0} > 0\) and \(a_{(0,1)} = 1/(2X_0^{3/2}) > 0\), and hence the integral is also analytic

\[ Y(X, U) = 2 \sum_{α ∈ N^2} \frac{a_α}{2α_2 + 1} (X - X_0)^{α_1} U^{2α_2 + 1}. \]

Therefore, the mapping \((X, U) \mapsto (X, Y)\) is an analytic diffeomorphism near \((X_0, 0)\). Its inverse is then analytic of the form (since odd in Y):

\[ U(X, Y) = \sum_{α ∈ N^2} b_α (X - X_0)^{α_1} Y^{1 + 2α_2} \]

Since \(U\) coincides with \(-\sqrt{Θ}\) for \(Y ≥ 0\) this means that \(Θ\) is analytic of the form:

\[ Θ = -U^2 = \sum_{α ∈ N^2} c_α (X - X_0)^{α_1} Y^{2α_2 + 2}. \]

**Step 6** Proof of (vi) Let \(ϕ(Y) = -∂_X Θ(0, Y)\) and \(ψ(Y) = δ^2_X Θ(0, Y)\). Since \(Θ\) solves (3.1) and vanishes on the vertical axis \(X = 0\), differentiating 3 times yields the following equation for \(ψ\):

\[ 4(1 - ϕ)ψ + \left( ∂^{-1}_Y ϕ - \frac{1}{2} Y \right) ∂_Y ψ + 3∂^{-1}_Y ψ ∂_Y ϕ = 0, \quad ψ(0) = 0 \]

One computes from the formula of \(ϕ\):

\[ 0 = 4ψ \left( \frac{1}{2} - \frac{1}{2} \cos(Y - π) \right) + \frac{1}{2} \sin(Y - π) ∂_Y ψ - \frac{3}{2} ∂^{-1}_Y ψ \sin(Y - π) \]

\[ ↔ 0 = 4ψ \sin^2 \left( \frac{Y - π}{2} \right) + \frac{1}{2} \sin(Y - π) ∂_Y ψ - \frac{3}{2} ∂^{-1}_Y ψ \sin(Y - π) \]

\[ ↔ 0 = 4ψ \tan \left( \frac{Y - π}{2} \right) + ∂_Y ψ - 3∂^{-1}_Y ψ. \]

We change variables and set \(ψ(Y) = \tilde{ψ}((Y - π)/2)\), with \(x = (Y - π)/2\). One then has:

\[ 8\tilde{ψ} \tan(x) + ∂_x \tilde{ψ} - 12 \int_{-π/2}^x \tilde{ψ} = 0. \]
We finally change again variables by setting
\[ \int_{-\frac{\pi}{2}}^{x} \tilde{\psi} = f(x) \]
and \( f \) finally solves
\[ f'' + 8 \tan(x) f' - 12 f = 0, \quad f\left(-\frac{\pi}{2}\right) = f'\left(-\frac{\pi}{2}\right) = 0. \] (3.25)  

A first solution (forgetting about the boundary conditions) is
\[ f_1 = \sin^2 x + \frac{1}{6} \]
The wronskian \( W = f_1' f_2 - f_1 f_2' \) between solutions solves
\[ W' = -8W \tan(x) \]
and therefore is equal up to renormalisation to
\[ W = (\cos(x))^8 \]

From the Wronskian relation we deduce that a second solution is given by
\[ f_2(x) = \left( \sin^2(x) + \frac{1}{6} \right) \left( 540x + 80 \sin(2x) + 3 \sin(4x) + \frac{686}{3} \sin(2x) \right) \]
The set of all solutions satisfying (3.25) with the boundary conditions is spanned by
\[ f = \left( \sin^2(x) + \frac{1}{6} \right) U(x) \]
where
\[ U(x) := \left( 540 \left( x + \frac{\pi}{2} \right) + 80 \sin(2x) + 3 \sin(4x) + \frac{686}{3} \sin(2x) \right) \]
satisfies
\[ \frac{d}{dx} (U) = 96 \frac{\cos^8(x)}{(\sin^2(x) + \frac{1}{6})^2} \]
so that the function \( f \) and its derivatives up to order 8 vanish at \(-\pi/2\). The solution to (3.24) is then
\[ \tilde{\psi} = \frac{d}{dx} f = 96 \frac{\cos^8(x)}{\sin^2(x) + \frac{1}{6}} + \sin(2x) \left( 540 \left( x + \frac{\pi}{2} \right) + 80 \sin(2x) + 3 \sin(4x) + \frac{686 \sin(2x)}{3 \sin^2(x) + \frac{1}{2}} \right) \]
The original solution is then:
\[ \psi(Y) = 96 \frac{\cos^8 \left( \frac{Y-\pi}{2} \right)}{\sin^2 \left( \frac{Y-\pi}{2} \right) + \frac{1}{6}} - \sin(Y) \left( 270Y - 80 \sin(Y) + 3 \sin(2Y) - \frac{683 \sin(Y)}{3 \sin^2 \left( \frac{Y-\pi}{2} \right) + \frac{1}{2}} \right) \]
where
\[ V(Y) := \left( 270Y - 80 \sin(Y) + 3 \sin(2Y) - \frac{686 \sin(Y)}{3 \sin^2 \left( \frac{Y-\pi}{2} \right) + \frac{1}{6}} \right) \]
The strict positivity of \( \psi \) comes from the equation. Indeed, assume that \( \psi(Y_0) = 0 \) for some \( 0 < Y_0 < 2\pi \), and that \( \psi > 0 \) on \((0, Y_0)\). Then at \( Y_0 \) there holds
\[
\partial_Y \psi(Y_0) = 3 \int_0^{Y_0} \psi > 0
\]
which is a contradiction. The regularity properties and the limited development come from direct computations, namely
\[
\partial^j_Y \psi(0) = 0, \quad j = 0, \ldots, 8, \quad \psi(2\pi) = 0, \quad \partial_Y \psi(2\pi) = -V(2\pi) = -\frac{540}{\pi} < 0.
\]
This ends the proof of (vi). \( \square \)

4. Generic self-similar separation

This section is devoted to the proof of Theorem 1. We need several Lemmas to perform its proof. First, in Lemma 8 below we state a generic condition for the characteristics mapping relating Lagrangian and Eulerian coordinates, and prove that it happens for a dense open set of initial data blowing up outside the boundary. This generic condition allows us to obtain a Taylor expansion for the \( x \) component of the characteristics near the point \((T, X_0, Y_0)\) at which the shock will form in Lagrangian variables. This Taylor expansion is intimately related to the fundamental profile. However it is not precise enough to allows directly for computations everywhere around \((X_0, Y_0)\). The next Lemma 10 then provides a useful splitting in several zones in which, after a change of variables, some key quantities are non-degenerate. This allows to parametrise the level curves \( \{x = \text{Cte}\} \) near \((X_0, Y_0)\) for \( t \) close to \( T \) in Lemma 11.

With the previous estimates in hand, and the level curves of \( x \) being parametrised, we can compute the vertical component \( y \). Once the characteristics \((X, Y) \mapsto (x, y)\) are computed and showed to be close to a renormalised version of \( \Phi \) defined by (1.11), we show the stability of this diffeomorphism. This allows to retrieve the solution and to end the proof of Theorem 1.

In what follows, \( \nabla \) is the gradient in Lagrangian \((X, Y)\) variables. We denote by \( \nabla^\perp \) the orthogonal gradient, \( J \) the Jacobian matrix and \( v^t \) the transposition:
\[
\nabla^\perp f = (-\partial_y f, \partial_x f), \quad J f = \left( \begin{array}{cc}
\partial_{xy} f & \partial_{yy} f \\
\partial_{yx} f & \partial_{yy} f
\end{array} \right), \quad (v_1, v_2)^t = (v_2, v_1).
\]
The generic condition is the following.

**Lemma 8** (Generic condition). Let \( p^E_x, u^E \) be of class \( C^4 \). In the subset of \( C^4 \cap W^{1,\infty}(\mathbb{R} \times [0, \infty)) \) of initial data \( u_0 \) such that \( T < \infty \) and \( T < T_b \), there exists a dense open set such that the mapping \( x : (X, Y) \mapsto x(X, Y) \) satisfies the following nondegeneracy properties:

(i) Uniqueness of the singular point. At time \( T \), there exists a unique point \((X_0, Y_0) \in \mathbb{R} \times (0, \infty)\) such that the mapping \( x : (X, Y) \mapsto x(X, Y) \) satisfies \( \nabla x(T, X_0, Y_0) = 0 \). Everywhere else on the set \( x_Y = 0 \) there holds \( x_X > 0 \).

(ii) Local nondegeneracy of the set of zero vorticity. For all \( t \in [0, T] \) there holds:
\[
x_Y(t, X_0, T_0) = 0, \quad \nabla x_Y(T, X_0, Y_0) \neq 0. \tag{4.1}
\]

(iii) Nondegenerate minimality for \( x_X \). There holds at the point \((T, X_0, Y_0)\):
\[
(\nabla x_Y)^t \nabla^\perp x_X = 0 \quad \text{and} \quad p^2_x = (\nabla^\perp x_Y)^t \left( J x_Y - \frac{(\nabla x_Y)^t \nabla x_X}{|\nabla x_Y|^2} J x_Y \right) \nabla^\perp x_Y > 0. \tag{4.2}
\]
(iv) Taylor expansion of $u$. There holds:

$$u_X(T, X_0, Y_0) < 0, \quad u_Y(t, X_0, Y_0) = 0 \quad \text{for all } t \in [0, T].$$

Remark 9. In the case of a trivial outer Eulerian flow $p^E = u^E = 0$, the above generic condition can be read on the initial datum $u_0$. It requires that at a point $(X_0, Y_0)$ with $Y_0 > 0$ and $u_{0y} = 0$, the restriction of $u_{0x}$ to the set $\{u_{0y} = 0\}$ attains a negative global minimum, that moreover $\nabla u_{0y}(X_0, Y_0) \neq 0$ so that the set $\{u_{0y} = 0\}$ is locally nondegenerate, and that:

$$p_0^2 = (\nabla u_{0y})^t \left( J_u_{0y} - \frac{(\nabla u_{0y})^t \nabla u_{0x}}{|\nabla u_{0y}|^2} J_u_{0y} \right) \nabla u_{0y} > 0,$$

when evaluated at $(X_0, Y_0)$, so that this minimum is nondegenerate. The quantity $p_0^2$ is natural in this case, as along the characteristics starting from $(X_0, Y_0)$ a computation shows:

$$p_0^2(t) = \frac{p_0^2}{(1 - \frac{1}{T})^2},$$

where $T$ is the blow-up time, making $p_0^2$ an almost conserved quantity along the flow in that case.

Proof of Lemma 8. Step 1 The control argument and universal properties. Let $(\overline{x}, \overline{u})$ denote the original solution to the ODE (1.7) corresponding to $u(0, X, Y) = u_0(X, Y)$. Let $(x, u)$ be the one corresponding to $u(0, X, Y) = (u_0 + v)(X, Y)$. The linearisation of the solution of the ODE is:

$$\begin{pmatrix} x(t) \\ u(t) \end{pmatrix} = \begin{pmatrix} \pi(t) \\ \overline{u}(t) \end{pmatrix} + M \begin{pmatrix} 0 \\ v \end{pmatrix} + f v^2,$$

where $f$ is a $C^3$ function and:

$$\dot{M} = A(t)M, \quad M(0) = Id, \quad A(t) := \begin{pmatrix} 0 & 1 \\ -p^E_{xx}(x(t)) & 0 \end{pmatrix}.$$ (4.4)

From the above equation, as a direct consequence of the fact that the ODE (1.7) is volume preserving, one obtains that $M(t)$ is of determinant 1, and in particular invertible. From the ODE (1.7), one deduces that at any $(t, X, Y)$

$$\begin{pmatrix} x_X & x_U \\ u_X & u_U \end{pmatrix} = \begin{pmatrix} x_{X} & x_{Y} \\ u_{X} & u_{Y} \end{pmatrix},$$

and therefore:

$$\begin{pmatrix} x_X & x_{U} \\ u_X & u_U \end{pmatrix} = M \begin{pmatrix} 1 \\ u_{0X} \end{pmatrix}, \quad \begin{pmatrix} x_{Y} & u_{Y} \end{pmatrix} = M \begin{pmatrix} 0 \\ u_{0Y} \end{pmatrix}.$$ (4.5)

As $T < T_b$, the singularity happens away from the boundary, and $T$ is characterised as being the first time at which $x_X$ touches zero on the set $\{u_{0Y} = 0\}$. We recall that the set $\{u_{0Y} = 0\}$ is preserved by the flow, which can be seen directly from the above equation. We recall from the proof of Theorem 1 in Section 2 that $\nabla x \neq 0$ for all times in the complement of the set $\{u_{0Y} = 0\}$. At a point $(X_0, Y_0)$ such that $x_X(T, X_0, Y_0) = u_{0Y}(X_0, Y_0) = 0$, one then has that the first identity in (4.1) always hold true for any solution. One also deduces from the above equation that:

$$\begin{pmatrix} 0 \\ u_X \end{pmatrix} = M(T, X_0, Y_0) \begin{pmatrix} 1 \\ u_{0X} \end{pmatrix}, \quad \begin{pmatrix} x_Y \\ u_Y \end{pmatrix} = M \begin{pmatrix} 0 \\ u_{0Y} \end{pmatrix}.$$ (4.5)

In particular, since $M$ is invertible, $u_X(T, X_0, Y_0) \neq 0$, and as $\dot{x}_X = u_X$ and $T$ is the first time for which $x_X$ touches zero one infers that $u_X(T, X_0, Y_0) < 0$. This proves that (4.3) always holds.
true for any solution. As $M$ is invertible and sends the vector $(1, u_{0X})$ onto the vector $(0, u_X)$, one necessarily has that
\[
M \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} m \\ n \end{pmatrix} \quad \text{with} \quad m = m(T, X_0, Y_0) \neq 0. \quad (4.6)
\]
Hence, we can rewrite the solution at a time $t$ close to $T$ as:
\[
x(t) = \pi(t) + mv + f v^2 
\]
where $m(T, X_0, Y_0) \neq 0$, $\pi$ is $C^4$, and $m$ and $f$ being $C^3$ functions. We will use this fact to modify the function at time $T$ by changing the initial velocity field $u_0$. We end this first step by writing the ODEs for higher order derivatives:
\[
\begin{pmatrix} x_{YY} \\ u_{YY} \end{pmatrix} = A \begin{pmatrix} x_{YY} \\ u_{YY} \end{pmatrix} + \begin{pmatrix} 0 \\ -(x_Y)^2 p_{xxx}^E(x) \end{pmatrix}, \quad \begin{pmatrix} x_{XY} \\ u_{XY} \end{pmatrix} = A \begin{pmatrix} x_{XY} \\ u_{XY} \end{pmatrix} + \begin{pmatrix} 0 \\ -x_X x_Y p_{xxx}^E(x) \end{pmatrix}. \quad (4.8)
\]

**Step 2** The nondegeneracy condition defines an open set. We show in this step that if $u_0$ is such that the properties (i) to (iv) are satisfied, then this is also the case for solutions with initial data nearby $u_0$ in $C^4$. We use bars to denote the unperturbed solution. First, at $(\overline{X}_0, \overline{Y}_0)$ as $\overline{u}_Y = \overline{Y}_0 = 0$ for all times from Step 1, one gets from (4.5), (4.4) and (4.6) that:
\[
\overline{Y}_Y(t, \overline{X}_0, \overline{Y}_0) = m(t, \overline{X}_0, \overline{Y}_0) \overline{u}_{0Y}(\overline{X}_0, \overline{Y}_0).
\]
Therefore, locally near $(T, X_0, Y_0)$, as $\overline{m}(T, X_0, Y_0) \neq 0$, one gets that the sets $\overline{Y}_Y = 0$ and $\overline{u}_{0Y} = 0$ coincides. Moreover, near $(T, X_0, Y_0)$ as $\nabla \overline{Y}_Y(T, X_0, Y_0) \neq 0$ from (4.1) and $\overline{Y}_Y(t, \overline{X}_0, \overline{Y}_0) = 0$ for all times, at a fixed $t$ the set $\{\overline{Y}_Y = 0\}$ is a curve $\Gamma$ that can be parametrised as (the derivatives being computed at $(\overline{X}_0, \overline{Y}_0)$):
\[
s \mapsto (\overline{X}_0, \overline{Y}_0) + s \nabla \overline{Y}_Y + c s^2 \nabla \overline{Y}_Y + O(s^3), \quad c = -\frac{1}{2|\nabla \overline{Y}_Y|^2}(\nabla \overline{Y}_Y)^t J \overline{u}_{0Y} \nabla \overline{Y}_Y,
\]
for $s \in \mathbb{R}$ close enough to 0. A direct consequence of the above parametrisation is that on $\Gamma$, close to $(T, X_0, Y_0)$:
\[
\begin{align*}
\overline{Y}_X(t) &= -(\overline{T} - t) \overline{u}_X(\overline{T}, \overline{X}_0, \overline{Y}_0)(1 + O(\overline{T} - t)) + \frac{1}{2} s^2 \overline{p}^2(1 + O(\overline{T} - t)) + O(s^3), \\
\frac{d^2}{ds^2} \overline{Y}_X(t) &= \overline{p}^2(1 + O(\overline{T} - t) + O(s)),
\end{align*}
\]
where $\overline{p}^2 > 0$ is defined by (4.2). We now study a perturbation, where the initial datum is $u_0 + v$, with $\|v\|_{C^4} = O(\epsilon)$ and $0 < \epsilon \ll 1$. First, note that for the unperturbed solution, outside any neighbourhood of $(\overline{T}, \overline{X}_0, \overline{Y}_0)$, there exists $c > 0$ such that $|\nabla \pi| > c$ is uniformly far away from 0. Hence, for $\epsilon$ small enough, any potential time $T$ such that the solution satisfies $\nabla x = 0$ for the first time is close to $\overline{T}$ and this happens at a point near $(\overline{X}_0, \overline{Y}_0)$. The set $\{x_Y = 0\}$ being invariant with time near $(\overline{X}_0, \overline{Y}_0)$, we use the time $\overline{T}$ to parametrise it. From all the nondegeneracy conditions, one obtains for $x$ that near $(\overline{T}, \overline{X}_0, \overline{Y}_0)$, the set $\{x_Y = 0\}$ is a curve $\Gamma$ that can be parametrised by:
\[
s \mapsto (X_0, Y_0) + s \nabla x^Y + c s^2 \nabla x^Y + O(s^3), \quad c = -\frac{1}{2|\nabla x^Y|^2}(\nabla x^Y)^t J x_{0Y} \nabla x^Y,
\]
where $(X_0, Y_0)$ is defined as the point near $(\overline{X}_0, \overline{Y}_0)$ at which $x^Y$ is minimal at time $\overline{T}$ (non necessarily 0). A direct consequence of the above parametrisation is that on $\Gamma$, close to $(\overline{T}, X_0, Y_0)$:
\[
x^Y(t) = x^Y(\overline{T}, X_0, Y_0) - (\overline{T} - t) \overline{u}_X(\overline{T}, \overline{X}_0, \overline{Y}_0)(1 + O((\overline{T} - t) + |s| + \epsilon)) + \frac{1}{2} s^2 \overline{p}^2(1 + O((\overline{T} - t) + |s| + \epsilon)).
\]
As \( x(T, X_0, Y_0) = O(\epsilon) \) and \( \pi_x(T, X_0, Y_0) < 0 \), we infer from the above expansion that the first time \( x_X \) touches zero on the curve \( \{x_Y = 0\} \) is \( T = \overline{T} + O(\epsilon) \), at least at one location \( s = O(\epsilon) \). The condition
\[
\frac{d^2}{ds^2} x_X(t) = \overline{\pi}^2 (1 + O(\overline{T} - t) + O(s) + O(\epsilon)),
\]
ensures that this happens at a unique location. Hence (i) is satisfied. The equality in (ii) is satisfied by definition, and the strict inequality is satisfied as at this point \( \nabla x_Y = \nabla \pi_Y + O(\epsilon) \neq 0 \). The equality in (iii) is satisfied as \( \nabla x_Y \neq 0 \) and \( x_X \) attains a minimum, and the strict inequality also holds since it holds for \( \pi \). Finally, the conditions in (iv) were showed to be true for any singular solutions in Step 1. Hence any solution with initial data near \( u_0 \) satisfies the nondegeneracy condition.

**Step 3 Density of nondegenerate singularities.** We now assume that the singularity is degenerate, and will find nearby initial data leading to a nondegenerate one. As the nondegeneracy involves several information on derivatives, there are several cases to consider. We start with the first one: we assume that the condition (ii) holds true (the equality always holds true, so only \( \nabla \pi_Y (T, X_0, Y_0) \neq 0 \) is real assumption here), and that (iv) fails (again, the first equality always holds so the real assumption is that \( \overline{\pi}^2 \) is nonpositive here). We do not assume anything regarding (i), and we recall that (iv) is always true.

From \( \nabla \pi_Y (T, X_0, Y_0) \neq 0 \), as in Step 2 we infer that near \( (T, X_0, Y_0) \), the set \( \{x_Y = 0\} \) is a curve \( \Gamma \) that we can still parametrise with
\[
s \mapsto (X_0, Y_0) + s \pi_Y^\perp + cs^2 \nabla \pi_Y + O(s^3), \quad c = -\frac{1}{2|\nabla \pi_Y|^2} (\nabla \pi_Y^\perp)^t J_{0Y} \nabla \pi_Y^\perp, \quad (4.9)
\]
but this time:
\[
\frac{d^2}{ds^2} x_X(T, X_0, Y_0) = \overline{\pi}^2 = 0,
\]
this quantity cannot be negative by minimality of \( \pi_X \) at \( (X_0, Y_0) \), so indeed \( \overline{\pi}^2 = 0 \). We assume without loss of generality that the control function \( m \) defined by (4.6) satisfies \( m(T, X_0, Y_0) > 0 \). We then perturb the initial datum \( u_0 \) by
\[
v(X_0 + X, Y_0 + Y) = \epsilon^4 \Psi \left( \frac{K(X - X_0)}{\epsilon} \right) \chi \left( \frac{K'(Y - Y_0)}{\epsilon} \right), \quad 0 < \epsilon \ll 1, \quad K \gg K' \gg 1,
\]
where
- \( \Psi \) is odd in \( X \) and compactly supported in \([-4, 4]\).
- \( \Psi' \geq 0 \) on \([-4, -2] \), \( \Psi' \leq 0 \) on \([-2, 0] \), \( \Psi' < 0 \) on \([-1, 0] \) with \( \Psi' \) attaining its minimum at the origin where \( \Psi'(0) = -1 \).
- \( \Psi'' \geq 1 \) on \([-1, 0] \) with \( \Psi''(0) = 1 \).
- \( \chi \) is a standard smooth cutoff function \( \chi = 1 \) on \([-1, 1] \) and \( \chi = 0 \) outside \([-2, 2] \).

Note that for any value of \( K \) and \( K' \), \( v \) is indeed small in \( C^3 \) for \( \epsilon \) small enough. Note also that as \( v \) vanishes for \( |X - X_0| \geq 4\epsilon/K \) or \( |Y - Y_0| \geq 2\epsilon/K' \) the solution \((x, u)\) remains unchanged outside the modification zone. In the modification zone, note first that one has the formula from (4.7):
\[
x(t, X, Y) = \overline{\pi}(T) - (T - t)\overline{\pi}(T) + m(T) v + f v^2 + g v (T - t) + h(T - t)^2,
\]
\[
= \overline{\pi}(T, X, Y) - (T - t)\overline{\pi}(T, X, Y) + m(T, X_0, Y_0) v + v^2 f_1 + v(T - t) f_2 + v(X - X_0) f_3 + v(Y - Y_0) f_4 + (T - t)^2 f_5, \quad (4.10)
\]
where all the $f_i$’s are $C^3$ functions with $O(1)$ size uniformly in $\epsilon$. Note that the curve $\{x_Y = 0\}$ is unchanged for $|Y - Y_0| \leq \epsilon/K'$. Hence it is still parametrised by (4.9), and there holds on this part of the curve for $t = T + O(K^3)$ and $|s| \lesssim \epsilon/K$:

$$\begin{align*}
x_X &= | \bar{\pi}_X - (T - t)\bar{\pi}_X(T, \bar{X}_0, \bar{Y}_0) (1 + O(|X - \bar{X}_0| + |Y - \bar{Y}_0|)) \\
&+ K\epsilon^3 \Psi' \left( \frac{K(X - \bar{X}_0)}{\epsilon} \right) \chi \left( \frac{K'(Y - \bar{Y}_0)}{\epsilon} \right) \bar{m}(T, \bar{X}_0, \bar{Y}_0) + O(\epsilon^4) \\
&= O(|s|) - (T - t)\bar{\pi}_X(T, \bar{X}_0, \bar{Y}_0) (1 + O(|X - \bar{X}_0| + |Y - \bar{Y}_0|)) \\
&+ K\epsilon^3 \Psi' \left( \frac{K(X - \bar{X}_0)}{\epsilon} \right) \chi \left( \frac{K'(Y - \bar{Y}_0)}{\epsilon} \right) \bar{m}(T, \bar{X}_0, \bar{Y}_0) + O(\epsilon^4) \\
&= O \left( \frac{3}{K^3} \right) - (T - t)\bar{\pi}_X(T, \bar{X}_0, \bar{Y}_0) + K\epsilon^3 \Psi' \left( \frac{K(X - \bar{X}_0)}{\epsilon} \right) \chi \left( \frac{K'(Y - \bar{Y}_0)}{\epsilon} \right) \bar{m}(T, \bar{X}_0, \bar{Y}_0)
\end{align*}$$

We now restrict further to the zone of this curve for which $|X - \bar{X}_0| \ll \epsilon/K$, hence $\chi = 1$ and we Taylor expand $\Psi$:

$$x_X = \left( -1 + \frac{t - T}{\epsilon^3 K} \bar{\pi}_X + \left( \frac{K(X - X_0)}{\epsilon} \right)^2 + O \left( \left| \frac{K(X - X_0)}{\epsilon} \right|^3 \right) + O \left( K^{-4} \right) \right) \epsilon^3 K \bar{m}.$$ 

where we wrote $\bar{\pi}_X < 0$ and $\bar{m} > 0$ for $\bar{m}(T, \bar{X}_0, \bar{Y}_0)$ and $\bar{m}(T, \bar{X}_0, \bar{Y}_0)$ to ease notation. The above identity implies that the first time $\partial_X x$ touches 0 in the zone of the curve $\Gamma$ for which $|X - X_0| \ll \epsilon/K$ is at a spacetime point $T$ with:

$$T = T + \frac{\epsilon^3 KM_1(T, X_0)}{\partial_X u(X_0, T)} + O \left( \epsilon^3 K K^{-4} \right), \quad X = X_0 + O \left( \frac{\epsilon}{K} K^{-2} \right).$$

At such a time, on the part of the curve for which $|X - X_0| \geq \epsilon/K$ there holds for some $c > 0$
from the condition on $\Psi$:

$$\partial_X \bar{x}(T) \geq (T - T)\bar{\pi}_X + \epsilon^3 K \bar{m} (-1 + c) + O \left( \frac{\epsilon^3}{K^3} \right) \geq c' \epsilon^3 K, \quad c' > 0.$$ 

Hence at time $T$, $x_X$ did not touch zero outside the part of $\Gamma$ in the zone $|X - X_0| \leq \epsilon/K$. We recall that from Step 2 at any point of $\Gamma$, the vectors $\nabla x_Y$ and $\nabla x_X$ are collinear, and that the second derivative of $x_X$ with respect to $s$ is positively proportional to $p^2$ computed at this point. The collinearity implies the simplification:

$$\frac{p^2}{x_{YY}} = x_{XXX} x_{YY} - 3 x_{XXX} x_{XY} + 3 x_{XXY} x_{XY} - \frac{x_{YY}^3}{x_{YY}} x_{XXX} \quad (4.11) \quad \text{id:p}$$

From (4.10) we infer that:

$$\begin{align*}
x_{XXX} &= \bar{\pi}_{XXX} + O(|T - T|) + \bar{m} K^3 \epsilon \Psi''' \left( \frac{K(X - \bar{X}_0)}{\epsilon} \right) + O(\epsilon^2) \\
&= \bar{\pi}_{XXX}(T, \bar{X}_0, \bar{Y}_0) + \bar{m} K^3 \epsilon \Psi''' \left( \frac{K(X - \bar{X}_0)}{\epsilon} \right) + O(\epsilon/K) \\
x_{YY} &= \bar{\pi}_{YY}(T, \bar{X}_0, \bar{Y}_0) + O(\frac{\epsilon}{K}), \quad x_{XYY} = \bar{\pi}_{XYY}(T, \bar{X}_0, \bar{Y}_0) + O(\frac{\epsilon}{K}), \quad x_{YY} = \bar{\pi}_{YY}(T, \bar{X}_0, \bar{Y}_0) + O(\frac{\epsilon}{K}) \\
x_{XY} &= \bar{\pi}_{XY}(T, \bar{X}_0, \bar{Y}_0) + O(\frac{\epsilon}{K}), \quad x_{XYY} = \bar{\pi}_{XYY}(T, \bar{X}_0, \bar{Y}_0) + O(\frac{\epsilon}{K}), \quad x_{XX} = \bar{\pi}_{XX}(T, \bar{X}_0, \bar{Y}_0) + O(\frac{\epsilon}{K}).
\end{align*}$$
The condition $\nabla \bar{x}_Y \neq 0$ and the fact that $\nabla \bar{x}_Y \cdot \nabla x = 0$ ensures $\bar{x}_{YY} \neq 0$ and therefore:

$$x_{XXX} = p(T, X_0, Y_0) + (\bar{x}_{YY})^2 mK^3 \epsilon \Psi''' \left( \frac{K(X - X_0)}{\epsilon} \right) + O(\epsilon/K)$$

$$= (\bar{x}_{YY})^2 mK^3 \epsilon \Psi''' \left( \frac{K(X - X_0)}{\epsilon} \right) + O(\epsilon/K) \geq cK^3 \epsilon$$

for some $c > 0$. This ensures that the zero we found for $x_X$ on the curve was unique in the part $|X - X_0| \leq \epsilon/K$, and hence is globally unique. This also ensures that $p^2 > 0$ at this zero. Therefore, the initial datum $u_0 + v$ leads to a non-degenerate singularity.

**Step 4 Other cases** Step 3 does not cover all cases. One also has to treat the case $\nabla \bar{x}_Y = 0$, for which (ii) fails. The set $\{\bar{x}_Y = 0\}$ is degenerate. As the parameter $p$ depends on third order derivatives, one has to consider three subcases, whether the symmetric Jacobian matrix $J\bar{x}_Y(T, X_0, Y_0)$ has eigenvalues with the same sign, different signs, or if it is degenerate. In each of these cases we can perform a similar analysis as in Step 3, so we only explain the strategy and leave the details to the reader.

In the case for which $J\bar{x}_Y(T, X_0, Y_0)$ has two eigenvalues with different signs, the set $\{\bar{x}_Y = 0\}$ is locally two crossing curves. Hence as at time $T$, on the set $\{\bar{x}_Y = 0\}$, the quantity $\bar{x}_X(T)$ has to be minimal at $(X_0, Y_0)$, one gets that $Ju_X(T, X_0, Y_0)$ must be a nonnegative matrix. We can thus perturb the initial datum to separate the two crossing curves in two non-crossing curves, while making a minimum for $\bar{x}_X$ appear on one of these curves.

In the case for which $J\bar{x}_Y(T, X_0, Y_0)$ has two eigenvalues with the same sign, the set $\{\bar{x}_Y = 0\}$ is locally a point. As $u_X(X_0, Y_0) < 0$, perturbing the initial datum one can transform this point into a circle, which makes us go back to Step 3. In the case of a degenerate matrix $J\bar{x}_Y(T, X_0, Y_0)$ whose eigenvalues are possibly zero, one perturbs the initial data to make these eigenvalues nonzero, which makes us go back to the two previous cases.

Assume from now on that the solution $u$ is nondegenerate in the sense that it satisfies the properties of Lemma 8. From the first equality in (4.2), we obtain at $(T, X_0, Y_0)$ that $x_{XX} = x_{XX} x_{YY}$. There are four cases to consider, depending on the sign of $x_{XX}$ (which is the same as that of $x_{YY}$), and the sign of $x_{XY}$. We treat here the case $x_{XX} > 0$ and $x_{XY} < 0$. It is clear from the proof that the sign of $x_{XY}$ does not matter for the result (i.e. does not change the profile), and that the case $x_{XX} < 0$ would lead to a singularity with profile $-\Theta(-X', Y')$. Moreover, we also assume that near $(T, X_0, Y_0)$, the lines $x = Cte$ coming from the boundary of the upper half plane enter via the bottom a small neighbourhood of $(X_0, Y_0)$ and not from the top (which would yield the very same result).

For $t$ close to $T$ we change variables in both Lagrangian and Eulerian sides, near $(X_0, Y_0)$ and $(x(t, X_0, Y_0), y(t, X_0, Y_0))$ respectively. Define the following Lagrangian self-similar variables $(a, b)$ (where all derivatives are taken at $(T, X_0, Y_0)$):

$$\left( \begin{array}{c} a \\ b \end{array} \right) = \left( \begin{array}{cc} -u_X \frac{x_{YY}}{2\sqrt{2}} (T - t)^{-\frac{1}{2}} & -u_X \frac{x_{XX} x_{YY}}{2\sqrt{2}} (T - t)^{-\frac{3}{2}} \\ \frac{x_{XXX}}{2\sqrt{2}} (T - t)^{-\frac{3}{2}} & \frac{x_{YY}}{2\sqrt{2}} (T - t)^{-\frac{3}{2}} \end{array} \right) \left( \begin{array}{c} X - X_0 \\ Y - Y_0 \end{array} \right),$$
noticing that the two line vectors in the above matrix are orthogonal, so that:

\[
X = X_0 - \frac{1}{u_X}(T-t)^{3/2}a - \frac{2\sqrt{2}\sqrt{x_{XX}}}{x_{XX} + x_{YY}}(T-t)^{3/4}b, \\
Y = Y_0 - \frac{1}{u_X}\sqrt{\frac{x_{XX}}{x_{YY}}}(T-t)^{3/2}a + \frac{2\sqrt{2}\sqrt{x_{YY}}}{x_{XX} + x_{YY}}(T-t)^{3/4}b.
\]

We renormalise the Eulerian side and use Eulerian self-similar variables \((X, Y)\) according to

\[
x^*(t) = x(t, X_0, Y_0), \quad \mathcal{X} = \frac{x - x^*(t)}{(T-t)^{3/2}}, \quad Y = \frac{-u_X\sqrt{X_{YY}}}{2\sqrt{2}}(T-t)^{-\frac{1}{4}},
\]

so that the mapping \((a, b) \mapsto (X, Y)\) preserves volume. Finally, define the parameter:

\[
q^2 = \frac{p^2}{6(-u_X)^2x_{YY}^2} > 0.
\]

From the assumption, \(x\) is of class \(C^4\). A Taylor expansion then gives, roughly, that after the change of variables:

\[
x(t, X, Y) = x^* + (T-t)^{3/2}(a + q^2a^3 + 4b^2) + \text{hot}.
\]

This is made precise is the next Lemma, and this is the starting point of the analysis. The first step is to describe the zones near \((X_0, Y_0)\), in \((a, b)\) variables, for which the curve \(\mathcal{X} = Cte\) can be parametrised either with the variable \(a\) or the variable \(b\) (the Taylor expansion is not precise enough to allow for a single parametrisation). For this purpose, define the zones for a small parameter \(0 < \delta < 1\) and a large constant \(L \gg 1\):

\[
Z_0 := (\mathbb{R} \times [0, \infty)) \setminus \left\{ |a| \geq \delta (T-t)^{-\frac{1}{2}} \text{ or } |b| \geq \delta (T-t)^{-\frac{3}{4}} \right\},
\]

\[
Z_1 = Z_1[L] := Z_0^c \cap \{ |b| \leq L(1 + |a|^{\frac{3}{2}}) \},
\]

\[
Z_2 = Z_2[L] := Z_0^c \cap Z_1^c = Z_0^c \cap \{ |b| > L(1 + |a|^{\frac{3}{2}}) \}.
\]

The zone \(Z_0^c\) defines a size one neighbourhood of \((X_0, Y_0)\) in Lagrangian variables, and so \(Z_0\) is an exterior zone where the dynamics remains regular. In \(Z_1\) one can use \(a\) as a dominant parameter, and in \(Z_2\) one can use \(b\) as a dominant parameter. The following Lemma gives all estimates we will use later on. The precise value of the constants \(L\) above do not matter, we simply will take them large enough to ensure a dichotomy for the level curves \(\{x = Cte\}\), see next Lemma 11.

\[\text{Lemma 10.}\]

Let \(r = u_X\frac{2\sqrt{T\sqrt{x_{XX}x_{YY}}}}{x_{XX} + x_{YY}}, L \gg 1\) be fixed. For \((X, Y) \in Z_0^c\), there holds:

\[
\mathcal{X} = a + q^2a^3 \left( 1 + O \left( (T-t)^{1/2} |a| \right) \right) + 4b^2 \left( 1 + O \left( (T-t)^{3/4} |b| \right) \right)
+ (T-t)^{3/4}rb + O \left( a^2b^2 |1 - t|^{-3/4} + (T-t)^{1/4} |b|^2 \right),
\]

\[
\partial_a \mathcal{X} = 1 + 3q^2a^2 + O \left( |a||b|(T-t)^{1/4} + |b^2|(T-t)^{1/4} + |a^3|(T-t)^{1/4} + (T-t)^{3/4} |a| + (T-t)^{3/4} |b| \right).
\]

\[
\partial_b \mathcal{X} = 8b + r(T-t)^{1/4} + O \left( (T-t)^{1/4} |b^2| + (T-t)^{3/4} |b| a + (T-t)^{3/4} |a|^2 + (T-t)^{3/4} |b| + (T-t)^{3/4} |a| \right).
\]

For \((X, Y) \in Z_1\) there holds:

\[
\mathcal{X} = (a + q^2a^3) \left( 1 + O \left( (T-t)^{3/4} + |a^{3/2} T-t^{3/4} \right) \right) + 4b^2 + O((T-t)^{1/4}).
\]
\[ a = -\frac{1}{q} \Psi_1 (q (X - 4b^2)) \left( 1 + O \left( (T - t)^\frac{1}{2} + |X|^{\frac{1}{2}} (T - t)^\frac{1}{2} + |b|^{\frac{1}{2}} (T - t)^\frac{1}{2} \right) \right) + O((T - t)^\frac{3}{2}) \]  

(4.19)  

\[ \partial_a X = (1 + 3q^2a^2)(1 + O((T - t)^{\frac{1}{2}} + |a|^{\frac{1}{2}} (T - t)^{\frac{1}{2}})) \]  

(4.20)

\[ = (1 + 3q^2a^2 (q(X - 4b^2))) \left( 1 + O((T - t)^{\frac{1}{2}} + |X|^{\frac{1}{2}} (T - t)^{\frac{1}{2}} + |b|^{\frac{1}{2}} (T - t)^{\frac{1}{2}} \right) + O((T - t)^{\frac{3}{2}}) \]  

\[ \partial_{aa} X = 6q^2 a \left( 1 + O \left( (|a|^{\frac{1}{2}} (T - t)^{\frac{1}{2}}) \right) \right) + O \left( (T - t)^{\frac{1}{2}} \right) \]  

(4.21)

\[ = -6q \Psi_1 (q(X - 4b^2)) \left( 1 + O \left( (T - t)^{\frac{1}{2}} + |X|^{\frac{1}{2}} (T - t)^{\frac{1}{2}} + |b|^{\frac{1}{2}} (T - t)^{\frac{1}{2}} \right) \right) + O((T - t)^{\frac{3}{2}}) \]  

\[ \partial_b X = 8b \left( 1 + O \left( (T - t)^{\frac{1}{2}} + a |(T - t)^{\frac{1}{2}} \right) \right) + O \left( (T - t)^{\frac{1}{2}} + (T - t)|a|^2 \right) \]  

(4.22)

For \((X, Y) \in Z_2\) there holds:

\[ X = (a + q^2a^3 + 4b^2) \left( 1 + O \left( (T - t)^{\frac{1}{2}} |b|^{\frac{1}{2}} \right) \right) \]  

(4.23)

\[ a = -\frac{1}{q} \Psi_1 (q(X - 4b^2)) + O \left( |X|^{\frac{1}{2}} + |b|^{\frac{1}{2}}(T - t)^{\frac{1}{2}} \right) \]  

(4.24)

\[ b = \pm \frac{1}{2} \sqrt{X - a - p^2a^3} \left( 1 + O \left( |b|^{\frac{1}{2}} (1 - t)^{\frac{1}{2}} \right) \right) = \pm \frac{1}{2} \sqrt{X - a - q^2a^3(1 + O(|X|^{\frac{1}{2}}(T - t)^{\frac{1}{2}})) \right) \]  

(4.25)

\[ \partial_b X = 8b \left( 1 + O \left( |b|^{\frac{1}{2}} (1 - t)^\frac{1}{2} \right) \right) \]  

(4.26)

\[ \partial_{bb} X = 8 \left( 1 + O \left( |b|^{\frac{1}{2}} (T - t)^\frac{1}{2} \right) \right) = 8 \left( 1 + O \left( |X|^{\frac{1}{2}} (T - t)^\frac{1}{2} \right) \right) \]  

(4.27)

\[ \partial_a X = 1 + 3q^2a^2 + O \left( |b|^{\frac{1}{2}} (T - t)^\frac{1}{2} \right) = 1 + 3q^2a^2 + O \left( |X|^{\frac{1}{2}} (T - t)^\frac{1}{2} \right) \]  

(4.28)

**Proof.** The proof is a standard computation using the Taylor expansion of \( x \) near \((T, X_0, Y_0)\) and some manipulations. As this is very standard, we only prove the estimates for \( X, a \) and \( b \), which already highlight all the main algebraic relations. From Lemma 8, for all \( t \) there holds \( x_Y(t, X_0, Y_0) = 0 \) and at the point \((T, X_0, Y_0)\) there holds \( x_X = 0, \) \( x_{Xt} = u_X \), and \( x_{XX}x_{YY} = x_{XY}^2 \) with the assumption \( x_{XX} > 0 \) and \( x_{XY} < 0 \). Thus for \( x \) a Taylor expansion gives:

\[ x = x^*(t) - u_X(T - t)(X - X_0) + \left( \sqrt{\frac{x_{XX}}{2}} (X - X_0) - \sqrt{\frac{x_{YY}}{2}} (Y - Y_0) \right)^2 + \frac{x_{XXX}}{6} (X - X_0)^3 + \frac{x_{XYY}}{2} (X - X_0)^2 (Y - Y_0) + \frac{x_{XY}}{2} (X - X_0)(Y - Y_0)^2 + \frac{x_{YY}}{6} (Y - Y_0)^3 + O \left( |X - X_0|^4 + |Y - Y_0|^4 + (T - t)|X - X_0|^2 \right. \]  

\[ + (T - t)|Y - Y_0|^2 + (T - t)^2 |X - X_0| \right) \]  

(4.29)
We use now the variables \((a, b)\), in a neighbourhood of \((X_0, Y_0)\), which means that \(|a| \ll (T - t)^{-1/2}\) and \(|b| \ll (T - t)^{-3/4}\). From the identity (4.11), the above expression becomes:

\[
x - x^e(t) = (T - t)^{\frac{3}{4}} a + (T - t)^{\frac{3}{2}} q^2 a^3 \left( 1 + O \left( (T - t) + |a||T - t|^{\frac{1}{2}} \right) \right) + (T - t)^{\frac{3}{4}} 4b^2 \left( 1 + O \left( (T - t)^{\frac{3}{2}} + (T - t)^{\frac{3}{2}}|a| \right) \right) + (T - t)^{\frac{3}{4}} rb + O \left( a^2|b|(1 - t) + (T - t)^2|a|^2 + (T - t)^{2 + \frac{3}{4}}|b| \right)
\]

which shows (4.15). Consider now (4.15) in the zone \(Z_1\) where \(|b| \lesssim 1 + |a|^{3/2}\). As \(a\) and \(a^3\) share the same, and since \(|a|(T - t)^{1/2} \ll 1\) and \(|b|(T - t)^{3/4} \ll 1\) sign we can write:

\[
X = (a + q^2 a^3) \left( 1 + O \left( (T - t) + |a|(T - t)^{\frac{1}{2}} \right) \right) + 4b^2 \left( 1 + O \left( (T - t)^{\frac{3}{2}} + (T - t)^{\frac{1}{2}}|a| \right) \right) + (T - t)^{\frac{3}{4}} rb + O \left( a^2|b|(1 - t) + (T - t)^2|a|^2 + (T - t)^{2 + \frac{3}{4}}|b| \right)
\]

where we used \(|a|^2 + |a|^{3/2} \lesssim |a| + a^3\). This proves (4.18). We rewrite the above identity as:

\[
\left( X - 4b^2 + O((T - t)^{\frac{3}{4}}) \right) \left( 1 + O \left( (T - t)^{\frac{3}{4}} + |a|^{\frac{1}{2}}(T - t)^{\frac{1}{2}} \right) \right) = a + q^2 a^3
\]

Note that the solution to \(a + q^2 a^3 = f\) is \(a = -\Psi_1(qf)/q\). Hence:

\[
a = -\frac{1}{q} \Psi_1 \left( q \left( X - 4b^2 + O((T - t)^{\frac{3}{4}}) \right) \left( 1 + O \left( (T - t)^{\frac{3}{4}} + |a|^{\frac{1}{2}}(T - t)^{\frac{1}{2}} \right) \right) \right)
\]

Now, in the zone under consideration \(O \left( (T - t)^{\frac{3}{4}} + |a|^{\frac{1}{2}}(T - t)^{\frac{1}{2}} \right) = o(1)\), hence using (4.38) the above identity gives:

\[
a = -\frac{1}{q} \Psi_1 \left( q \left( X - 4b^2 + O((T - t)^{\frac{3}{4}}) \right) \left( 1 + O \left( (T - t)^{\frac{3}{4}} + |a|^{\frac{1}{2}}(T - t)^{\frac{1}{2}} \right) \right) \right).
\]

Hence, from (4.38) \(|a| \lesssim |\Psi_1 \left( q \left( X - 4b^2 + O((T - t)^{\frac{3}{4}}) \right) \right)| \lesssim |X|^{1/3} + |b|^{2/3} + (T - t)^{1/12}\). This, reinjected in the above identity and using (4.38), gives (4.19). Consider now (4.15) in the zone \(Z_2\) where \(|a| \ll |b|^{2/3}\) and \(|b| \gg 1\). Then the dominant term in the right hand side of (4.15) is \(b_2\) and we infer that \(X \approx 4b^2 \gg 1, |a|^3\). Injecting these bounds in (4.15) gives the desired identity (4.23) for \(X\) in \(Z_2\). Then we rewrite (4.23) as:

\[
4b^2 = X(1 + O(|b|^{\frac{1}{3}}(T - t)^{\frac{1}{2}))) - a - q^2 a^3 = (X - a - q^2 a^3)(1 + O(|b|^{\frac{1}{3}}(T - t)^{\frac{1}{2})))
\]

as \(X^{1/3} \gg |a|\). Hence the solution is:

\[
b = \pm \frac{1}{2} \sqrt{(X - a - q^2 a^3)(1 + O(|b|^{\frac{1}{3}}(T - t)^{\frac{1}{2})))}
\]

\[
= \pm \frac{1}{2} \sqrt{X - a - q^2 a^3(1 + O(|b|^{\frac{1}{3}}(T - t)^{\frac{1}{2})))}
\]

\[
= \pm \frac{1}{2} \sqrt{X - a - q^2 a^3(1 + O(|X|^{\frac{1}{3}}(T - t)^{\frac{1}{2})))}
\]

which shows (4.25). We also rewrite (4.23) as:

\[
a + q^2 a^3 = X - 4b^2 + O \left( |X|^{1 + \frac{1}{3}}(T - t)^{\frac{1}{2}} \right).
\]
So that, using (4.38):
\[
a = -\frac{1}{q} \left( q \left( |x|^{1+\frac{3}{4}} - 4b^2 + O \left( |x|^{1+\frac{3}{4}}(T-t)^{\frac{1}{4}} \right) \right) \right) = -\frac{1}{q} \Psi_1 \left( q \left( X - 4b^2 \right) \right) + O \left( |X|^{1+\frac{3}{4}}(T-t)^{\frac{1}{4}} \right)
\]
This shows (4.24). The other estimates, for the derivatives, can be proved along the very same lines.

With the control of the behaviour of $X$ in the zones $Z_1$ and $Z_2$ obtained in Lemma 10, we can now establish the properties of the level lines $X = Cte$ near $(T, X_0, Y_0)$. As a result, a parametrisation of these level curves with only the variable $a$ is possible most of the time, but there is a case for which the curve cannot be parametrised efficiently as a graph, for which we cut in several zones, each parametrised with either the variable $a$ or the variable $b$.

**Lemma 11** (Lagrangian structure of the curves $\{x = Cte\}$). For $\delta, \epsilon > 0$ small enough, $L_2 \gg L_1 \gg 1$ and $K \gg 1$ large enough and $t$ close enough to $T$ the following holds.\footnote{As is clear from the proof, $\delta$ is chosen first, then $\epsilon$, $L_1$, $K$ and finally $L_2$.} Let $\Gamma[x]$ denote the curve $\{x(X, Y) = x\}$ in Lagrangian variables, with starting point at the boundary of the upper half plane. For any $|x - x^*| \leq \epsilon$ the set $\Gamma \cap Z_0^c$ consists of a curve entering and exiting at the points $(a_{in}, b_{in})$ and $(a_{out}, b_{out})$ with $b_{in} = -\delta(T-t)^{-3/4}$ and $b_{out} = \delta(T-t)^{-3/4}$. In addition,

(i) For $-\epsilon \leq x - x^* \leq K(T-t)^{3/2}$ this set can be parametrised with the variable $b$, via $a = a(x, b)$, and is contained in $Z_1L_2$.

(ii) For $K(T-t)^{3/2} \leq x - x^* \leq \epsilon$ this set consists on the concatenation of five curves $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5$ with starting and end points $(a_{in}, b_{out}), (a_1, b_1), (a_2, b_2), (a_3, b_3), (a_4, b_4)$, and $(a_{out}, b_{out})$ where\[
b_1 = -L_1(1 + |a_1|^\frac{2}{3}), \quad b_2 = -L_1(1 + |a_2|^\frac{2}{3}), \quad b_3 = L_1(1 + |a_3|^\frac{2}{3}), \quad b_4 = L_1(1 + |a_4|^\frac{2}{3}).
\]
$\Gamma_1$ lies in $Z_1L_1$ with $a, b \ll -1$, and can be parametrised as a curve $a = a(x, b)$. $\Gamma_2$ lies in $Z_2L_1$ with $b < 0$, and can be parametrised as a curve $b = b(x, a)$. $\Gamma_3$ lies in $Z_1L_1$ with $a > 0$, and can be parametrised as a curve $a = a(x, b)$. $\Gamma_4$ lies in $Z_2L_1$ with $b > 0$, and can be parametrised as a curve $b = b(x, a)$. $\Gamma_5$ lies in $Z_1L_1$ with $a < 0$ and $b > 0$, and can be parametrised as a curve $a = a(x, b)$.

**Proof.**\footnote{Step 1} The in and out points Fix $|x| \leq \epsilon$, we show the existence of the in and out points and show that they are the only points at the boundary of $Z_0^c$. The in point is the point of the curve belonging to the south boundary $\{b = -\delta(T-t)^{-3/4}\}$ of $Z_0^c$. We fix $b_{in} = -\delta(T-t)^{-3/4}$ and look for the corresponding parameter $a_{in}$. For $(a, b) \in Z_2L_2$ one has $|b| \gtrsim L_2, |a| \ll |b|^{2/3}$ and $(T-t)^{\frac{2}{3}}|b| = o(1)$ as $\delta \to 0$. Hence from (4.23):
\[
X \gtrsim |b|^2.
\]
In particular, if $(a, b_{in}) \in Z_2$ then $x(a, b_{in}) - x^* \gtrsim \delta^2 > \epsilon \geq x - x^*$. Hence there are no solutions in $Z_2$. As $(0, b_{in}) \in Z_2$, one has from the inequality above $x(0, b_{in}) - x^* > x - x^*$ and as $(-\delta^{1/3}(T-t)^{-1/2}, b_{in}) \in Z_1$ one has from (4.18)
\[
x(-\delta^{1/3}(T-t)^{-1/2}, b_{in}) - x^* = -\delta^2(1 + o(1)) < -\epsilon \leq x - x^*
\]
Hence by the intermediate value Theorem there exists a solution $a_{in}$ to $x(a_{in}, b_{in}) = x$, which is unique as $\partial_a x > 0$ on $Z_1$ from (4.20). One can show the existence and uniqueness of the out point $(a_{out}, b_{out})$ at the north boundary $\{b = \delta(T-t)^{-3/4}\}$ with the same argument. The computation (4.30) holds true at the left boundary $\{a = -\delta^{1/3}(T-t)^{-1/2}\}$ of $Z_0^c$, and a similar
computation shows that \( x - x^* > \epsilon \) at the right boundary \( \{ a = \delta^{1/3}(T-t)^{-1/2} \} \). Hence the two points \((a_{in}, b_{in})\) and \((a_{out}, b_{out})\) are the only points of the curve at the boundary of \( Z_0^* \).

**Step 2** Proof of (i). From the first step, the strict monotonicity \( \partial_a x > 0 \) in \( Z_1[L_2] \) from (4.20), and since \( \Gamma \cap Z_0^* \subset Z_1[L_2] \), a standard application of the implicit function theorem gives that the curve \( x(a, b) = x \) can be parametrised with the variable \( b \) as \( a = a(x, b) \) for \( b \in [-\delta(T-t)^{-3/4}, \delta(T-t)^{-3/4}] \).

**Step 3** Proof of (ii). Fix \( K(T-t)^{3/2} \leq x - x^* \leq \epsilon \). We first prove the existence and uniqueness of the point \((a_1, b_1)\) on the curve \( b_1 = -L_1(|a_1|^{3/2} + 1) \) with \( a_1 < 0 \). On this curve, from (4.23), using the fact that \( |a_1| \ll |b_1| \):

\[
\mathcal{X}(a_1, b_1) = 4b_1^2(1 + o(1))
\]

where the \( o(1) \) is as \( L_1 \to \infty \). Hence, as for \( a_1 = 0 \), \( \mathcal{X}(0, -L_1) = O(L_1^2) < K \leq \mathcal{X} \) and for \( b_1 = -\delta(T-t)^{-3/4} \):

\[
x \left( -\left( \frac{\delta}{(T-t)^{3/4}L_1} - 1 \right)^{2/3} \right) - x^* = 4\delta^2(1 + o(1)) > \epsilon \geq x - x^*,
\]

there exists a solution \((a_1, b_1)\) with \( b_1 = -L_1(|a_1|^{3/2} + 1) \) and \( a_1 < 0 \) to \( x(a_1, b_1) = x \). One can also show that \( \partial_{b_1}\mathcal{X} = 8b_1(1 + o(1)) \) on this curve which implies the uniqueness of the point \((a_1, b_1)\). The existence and uniqueness of \((a_2, b_2)\), \((a_3, b_3)\) and \((a_4, b_4)\) can be done similarly.

We now show that \((a_{in}, b_{in})\) is connected to \((a_1, b_1)\) by the part of the curve staying in \( Z_1[L_1] \) with \( a, b < 0 \). Indeed, note that for \( b = 0 \) and \( a < 0 \), on has \( x < 0 \) from (4.18). Also, for \( |a|, |b| \leq L_1 \) one has from (4.18) that \( x(a, b) \leq C(L_1)(T-t)^{3/2} \) so \( x(a, b) < x \) for \( K \) large enough. Consider the part of \( \Gamma \) which is in the zone \( Z_1 \) with \( a, b < 0 \). This proves that the only points of this set at the boundary of \( Z_1 \cap \{ a, b < 0 \} \) are precisely \((a_{in}, b_{in})\) and \((a_1, b_1)\). From a direct check on the gradient of \( x \), the curve \( \Gamma \) indeed penetrates the zone \( Z_1 \cap \{ a, b < 0 \} \) at these two points, and so the part of \( \Gamma \) in the zone \( Z_1 \cap \{ a, b < 0 \} \) consists of a curve joining these two points. Moreover, as in \( Z_1 \) there holds \( \partial_a x > 0 \) from (4.20), it can be parametrised with the variable \( b \).

The proof of the properties for \( \Gamma_i \) for \( i \geq 2 \) can be proved following the same ideas. We just mention that in \( Z_2[L_1] \) there holds \( \partial_b \mathcal{X} > 0 \) for \( b > 0 \) and \( \partial_b \mathcal{X} < 0 \) for \( b < 0 \) from (4.26), which shows that the curves \( \Gamma_2 \) and \( \Gamma_4 \) can indeed be parametrised with the variable \( a \).

\[
\square
\]

We are now able to compute the renormalised vertical component \( \mathcal{Y} \). In particular, we will show that the mapping \((a, b) \mapsto (\mathcal{X}, \mathcal{Y})\) is close to the mapping \( \Phi \) defined by (1.11). We will invert the characteristics and use a perturbative argument to show the inverse is close to \( \Phi^{-1} \). However, this inversion is done in an unbounded zone, which forces us to renormalise the perturbation problems when approaching infinity or the support of \( \Theta \). Hence we need precise asymptotic estimates for \( \Theta \). The Lemma below provides us with all the estimates that will be used to conclude the proof of Theorem 1.
In the \((a, b, \mathcal{X}, \mathcal{Y})\) variables the leading order term will be:

\[
\mathcal{X}^\Theta(a, b) = a + q^2 a^3 + 4b^2, \quad \mathcal{Y}^\Theta(a, b) = \int_{-\infty}^{b} \frac{db}{1 + 3\Psi_1^2 \left( q \left( \mathcal{X}^\Theta(a, b) - 4b^2 \right) \right)}, \quad (a^\Theta, b^\Theta) = (\mathcal{X}^\Theta, \mathcal{Y}^\Theta)^{-1}
\]  

(4.31)

One has in particular \(a^\Theta(\mathcal{X}, \mathcal{Y}) = -\frac{\mathcal{X}}{q} \frac{\partial}{\partial \mathcal{X}} \left( \frac{\mathcal{Y}}{\sqrt{q}} \right) \) (hence we are making a small abuse of notation writing \(a^\Theta\) instead of \(a^\Theta_{\mu, \nu}\)).

**Lemma 12.** One has the formulae for the mappings \((\mathcal{X}^\Theta, \mathcal{Y}^T)\) and \((a^\Theta, b^T)\) defined by (4.31):

\[
\partial_a \mathcal{Y}^\Theta(a, b) = 6q \left( 1 + 3q^2 a^2 \right) \int_{-\infty}^{b} \frac{\Psi_1 \left( p \left( \mathcal{X}^\Theta(a, b) - 4b^2 \right) \right)}{\left( 1 + 3\Psi_1^2 \left( p \left( \mathcal{X}^\Theta(a, b) - 4b^2 \right) \right) \right)^3} \frac{db}{\sqrt{b}},
\]  

(4.32)

\[
\partial_b \mathcal{Y}^\Theta(a, b) = \frac{1}{1 + 3q^2 a^2} + 48qb \int_{-\infty}^{b} \frac{\Psi_1 \left( p \left( \mathcal{X}^\Theta(a, b) - 4b^2 \right) \right)}{\left( 1 + 3\Psi_1^2 \left( p \left( \mathcal{X}^\Theta(a, b) - 4b^2 \right) \right) \right)^3} \frac{db}{\sqrt{b}},
\]  

(4.33)

(i) Bottom of the self similar zone. For \(|\mathcal{Y}| \leq \frac{\kappa}{1 + |\mathcal{X}|^2}\) one has, where \(c_i \neq 0\) for \(i = 1, ..., 4\):

\[
|p^2 a^3 + 4b^2| = o(b^2) \quad \text{as} \quad \kappa \to 0, \quad |b| \approx \frac{1}{y^3}, \quad |a| \approx \frac{1}{y^2}, \quad \left| \int_{-\infty}^{b} \frac{\Psi_1(x - 4b^2)}{\left( 1 + 3\Psi_1^2((x - 4b^2))^3 \right) \frac{db}{\sqrt{b}}} \right| \approx \frac{|b|^{-\frac{7}{3}}}{\sqrt{b}}.
\]  

(ii) Sides of the self similar zone. For \(|\mathcal{X}| \geq M\) and \(\epsilon |\mathcal{X}|^{-1/6} \leq \mathcal{Y} \leq (2 - \epsilon) C_+ |\mathcal{X}|^{-1/6}\) where \(\pm = sgn(\mathcal{X})\) one has, as \(M \to \infty\) and \(\epsilon\) is kept fixed:

\[
|b^\Theta| \lesssim \left| q^2 a^\Theta_3 + 4b^\Theta_2 \right|, \quad |q^2 a^\Theta_3 + 4b^\Theta_2| \gg 1, \quad |b^\Theta| \approx |\mathcal{X}|^{1/2}, \quad |a^\Theta| \approx |\mathcal{X}|^{1/2},
\]  

\[
|\partial_a \mathcal{X}^\Theta| \sim c_1 |b|^{\frac{1}{3}}, \quad |\partial_b \mathcal{X}^\Theta| \sim c_2 |b|, \quad |\partial_a \mathcal{Y}^\Theta| \sim c_3 |b|^{-1}, \quad |\partial_b \mathcal{Y}^\Theta| \sim c_4 |b|^{-\frac{7}{12}}, \quad \mathcal{Y}^\Theta \approx |b|^{-1/3}. \quad (4.34)
\]

for some functions \(\varphi_{\pm} \in C^\infty(0, 2C_{-})\) where \(\pm = sgn(\mathcal{X})\), such that \(((\varphi_{\pm}^1, \varphi_{\pm}^2), (\varphi_{\pm}^3, \varphi_{\pm}^4))\) is volume preserving and:

\[
\mathcal{Y}^\Theta(a, b) \approx |\mathcal{X}^\Theta(a, b)|^{-\frac{1}{3}}, \quad \int_{-\infty}^{b} \frac{\left| \frac{1}{\Psi_1^2 \left( p \left( \mathcal{X}^\Theta(a, b) - 4b^2 \right) \right) \right| db}{\left| p \left( \mathcal{X}^\Theta(a, b) - 4b^2 \right) \right|^{\frac{1}{3}}} \lesssim |\mathcal{X}^\Theta(a, b)|^{-\frac{1}{3} + \epsilon}, \quad (4.36)
\]

Proof of Lemma 12. All identities in Lemma 12 are direct consequences of the formula for the mapping \(\Phi\) defined by (1.11), and of the other properties of \(\Theta\) listed in Propositions 3 and 4. To avoid repetition, we do not give a proof and refer to Section 3 for the study of these latter functions.
We are now in position to end the proof of Theorem 1. The proof is lengthy and a bit repetitive. The computations are however detailed for clarity.

**Proof of Theorem 1.** At several moments in the proof, we will use the following. The function $\Psi_1$ enjoys:

$$
\left| X \frac{\partial X}{\partial \Psi_1}(X) \right| \leq \frac{1}{C} \varepsilon |X| \left| \frac{\partial \Psi_1(X)}{\partial X} \right| \leq \frac{c}{C} \varepsilon \left| X \right|^{\frac{1}{2}} \rho \varepsilon \left( X + X' \right) = \Psi_1(X) + O(|X|^{\frac{1}{2}}) \tag{4.38} \bd\text{: Psi11}
$$

uniformly in $X, X'$. From the first bound above, given a small quantity $O(|z|)$ that has size $|z|$, as an application of the implicit function theorem it is true that:

$$
\Psi_1(X(1 + O(|z|))) = \Psi_1(X)(1 + O(|z|)). \tag{4.39} \bd\text{: Psi12}
$$

where the "new" $O(|z|)$ in the right hand side enjoys the same differentiability properties and similar bounds, as the one in the left hand side. To rearrange the $O()$’s in what follows, we shall simplify $O(|z_1|) + O(|z_2|) = O(|z_2|)$ in zones where $|z_1| \lesssim |z_2|$, and $O(|z|^\alpha) = O(|z|^\beta)$ if $0 < \beta \leq \alpha$ as all quantities in the $O()$’s will be small.

To compute the vertical component, we will use the following result. Assume that $(a, b)$ and $(a', b')$ belong to the same level set curve $\Gamma(x) = \{x(a, b) = x\}$. Assume that, when ordering points on $\Gamma$ with their distance to the boundary, $(a', b')$ is after $(a, b)$, and that either $b \leq b'$ or $a \leq a'$. Assume moreover that $\Gamma(a', b')$, the part of $\Gamma$ between $(a, b)$ and $(a', b')$, can be either parametrised with the variable $\tilde{b}$ as $\tilde{a} = \tilde{a}(X(\tilde{b}))$ for $b \leq \tilde{b} \leq b'$ or with the variable $\tilde{a}$ as $\tilde{b} = \tilde{b}(X(\tilde{a}))$ for $a \leq \tilde{a} \leq a'$. Then:

$$
\int_{\Gamma(a', b')} \frac{ds}{|\nabla x|} = \frac{1}{C(T-t)^{\frac{1}{2}}} \int_0^{b'} \frac{d\tilde{b}}{|\partial a X|} \text{ or } \int_{\Gamma(a', b')} \frac{ds}{|\nabla x|} = \frac{1}{C(T-t)^{\frac{1}{2}}} \int_a^{a'} \frac{d\tilde{a}}{|\partial b X|}, \tag{4.40} \id\text{: int change}
$$

$C = \frac{-u_x \sqrt{x_1 y_1}}{2\nu}$. This is indeed a direct consequence of the identities

$$
ds = \frac{\sqrt{|e_b|^2 |\partial_a x|^2 + |e_a|^2 |\partial_b x|^2}}{|\partial_a x|} d\tilde{b}, \quad |\nabla x| = \sqrt{|e_b|^2 |\partial_a x|^2 + |e_a|^2 |\partial_b x|^2}.
$$

where $e_a = \frac{\left((T-t)^\frac{1}{2} - u_x\right)}{\sqrt{x_1 y_1}}$ and $e_b = \frac{2\sqrt{(T-t)^\frac{1}{2}}}{\sqrt{x_1 y_1} + \sqrt{x_2 y_2}}$. The vectors of change of variables. We will invert the characteristics, bearing in mind that the leading order term is $(a^\Theta, b^\Theta)$ as defined in (4.31). The change of variables preserving volume, one has:

$$(\partial_{X}a \partial_Ya \partial_{X}b \partial_Yb) = (\partial_{b}Y - \partial_{a}X \partial_{b}X - \partial_{a}Y). \tag{4.41} \text{id: taylor u}
$$

Finally, once we invert the characteristics, we will use the Taylor expansion of $u$ in Lagrangian variables near $(X_0, Y_0)$:

$$
u(t, X, Y) = u^* - a(T-t)^\frac{1}{2} + O \left( (T-t)^\frac{1}{2} + (T-t)^\frac{1}{2} \right) \tag{4.42} \text{id: taylor u}
$$

and

$$
\begin{align*}
\frac{\partial a u}{\partial a} &= -(T-t)^\frac{1}{2} \left( 1 + O \left( T - t + (T-t)^\frac{1}{2} a \right) \right), \quad \frac{\partial b u}{\partial b} = O \left( (T-t)^\frac{1}{2} \right), \\
\frac{\partial X u}{\partial X} &= \frac{\partial a u}{\partial a} \partial_Y - \partial_{b} u \partial_{a} Y, \quad \frac{\partial Y u}{\partial Y} = -\frac{\partial a u}{\partial a} \partial_X + \partial_{b} u \partial_{a} X. \tag{4.43} \text{id: derive}
\end{align*}
$$

The parameters associated to the profile are:

$$
\mu = \frac{p^s}{q}, \quad \nu = \frac{2}{C} \sqrt{\frac{q}{p^s}}.
$$
Step 1 The Eulerian vertical component for left part, the central part, and the bottom part of the self-similar zone. We first derive rough estimates that will be used in the next steps. For $K \gg 1$ a fixed large constant, fix $b \in [-\delta(T-t)^{-3/4}, \delta(T-t)^{-3/4}]$, and $a \in [-\delta^{1/3}(T-t)^{-1/2}, \delta(T-t)^{-1/2}]$ such that either $-\varepsilon \leq x(a,b) - x_s \leq K(T-t)^{3/2}$, or $K(T-t)^{3/2} \leq x - x_s \leq \varepsilon$ and $b < 0$ and $|X| \ll |b|^2$. Note that in the second case, from (4.15), one has necessarily $|a| \approx |b|^{2/3} \gg 1$. Let $\Gamma$ denote the part of the curve $\{x(\tilde{X},Y) = x(X(a,b),Y(a,b))\}$ which joins the boundary of the upper half plane and the point $(X(a,b),Y(a,b))$. We decompose it in two parts:

$$\Gamma_1 := \Gamma \cap Z_0, \quad \Gamma_2 = \Gamma \cap Z_0^c, \quad \mathcal{Y} = C(T-t)^{\frac{3}{2}} \left( \int_{\Gamma_1} \frac{ds}{|\nabla x|} + \int_{\Gamma_2} \frac{ds}{|\nabla x|} \right), \quad (4.44) \quad \text{id:mystep1}$$

The integral in $Z_0$ is at distance one to $(X_0,Y_0)$, and everything then remains regular:

$$\int_{\Gamma_1} \frac{ds}{|\nabla x|} = O(1), \quad \partial_X \left( \int_{\Gamma_1} \frac{ds}{|\nabla x|} \right) = O((T-t)^{\frac{3}{2}}). \quad (4.45) \quad \text{id:mystep1}$$

In $Z_0^c$, from Lemma 11 the curve $\Gamma = \{x(\tilde{a},\tilde{b}) = x(a,b)\}$ lies in $Z_1$, so it can be parametrised with the variable $\tilde{b}$. Also, from (4.40), (4.20) and as $|\tilde{b}|(T-t)^{3/4} \ll 1$ for $-\delta(T-t)^{-3/4} \leq \tilde{b} \leq b$:

$$C(T-t)^{\frac{3}{4}} \int_{\Gamma_2} \frac{ds}{|\nabla x|} = 2 \int_{-\frac{\delta}{(T-t)^{-\frac{3}{4}}}}^{\frac{b}{(T-t)^{\frac{3}{4}}}} \frac{d\tilde{b}}{\partial_a X}$$

$$= \int_{-\frac{\delta}{(T-t)^{-\frac{3}{4}}}}^{\frac{b}{(T-t)^{\frac{3}{4}}}} \frac{1 + O \left( (T-t)^{\frac{1}{12}} + |\tilde{b}|^\frac{1}{3}(T-t)^{\frac{3}{16}} + |X|^\frac{1}{3}(T-t)^{\frac{1}{3}} \right)}{1 + 3 \Psi_1^2 \left( q \left( X - 4\tilde{b}^2 \right) \right)} d\tilde{b} + O \left( \int_{-\infty}^{-\frac{\delta}{(T-t)^{-\frac{3}{4}}}} \frac{d\tilde{b}}{|\tilde{b}|^{\frac{3}{4}}} \right)$$

$$= \int_{-\infty}^{b} \frac{1 + O \left( (T-t)^{\frac{1}{12}} + |\tilde{b}|^\frac{1}{3}(T-t)^{\frac{3}{16}} + |X|^\frac{1}{3}(T-t)^{\frac{1}{3}} \right)}{1 + 3 \Psi_1^2 \left( q \left( X - 4\tilde{b}^2 \right) \right)} d\tilde{b} + O \left( (T-t)^{\frac{3}{4}} \right) \quad (4.46)$$

where we used the fact that $|X| \leq \varepsilon(T-t)^{-3/2}$, and that for $\tilde{b} \leq -\delta(T-t)^{-3/4}$, $\tilde{b}^2 \gg X$ if $\varepsilon$ is small enough, implying $\Psi_1^2(q(X - 4\tilde{b}^2)) \approx |\tilde{b}|^{4/3}$. Hence, injecting (4.45) and (4.46) in (4.44):

$$\mathcal{Y}(a,b) = \int_{-\infty}^{b} \frac{1 + O \left( (T-t)^{\frac{1}{12}} + |\tilde{b}|^\frac{1}{3}(T-t)^{\frac{3}{16}} + |X|^\frac{1}{3}(T-t)^{\frac{1}{3}} \right)}{1 + 3 \Psi_1^2 \left( q \left( X - 4\tilde{b}^2 \right) \right)} d\tilde{b} + O((T-t)^{\frac{3}{4}}). \quad (4.47) \quad \text{id:myleftcenter}$$

We now study the derivative of $\mathcal{Y}$. As $X(\tilde{a},\tilde{b}) = X$ is inverted through $\tilde{a} = a(X,\tilde{b})$ one has:

$$\frac{\partial}{\partial X} \left( \frac{1}{\partial_a X(\tilde{a}(X,\tilde{b}),\tilde{b})} \right) = - \frac{\partial_a X(\tilde{a}(X,\tilde{b}),\tilde{b})}{(\partial_a X(\tilde{a}(X,\tilde{b}),\tilde{b}))^3} \frac{1}{\partial_a X(\tilde{a}(X,\tilde{b}),\tilde{b})}.$$
One deduces from (4.40), (4.20), (4.21) that:

$$\frac{\partial}{\partial \mathcal{X}} \left( C(T-t)^{\frac{1}{3}} \int_{\Gamma_{2}} \frac{ds}{|\nabla x|} \right) = \frac{\partial}{\partial \mathcal{X}} \left( \int_{-\delta}^{b} \frac{d\tilde{b}}{\partial_{a} \mathcal{X}(\tilde{a}, \tilde{b})} \right) = - \int_{-\delta}^{b} \frac{\partial_{a} \mathcal{X}(\tilde{a}, \tilde{b}) d\tilde{b}}{(\partial_{a} \mathcal{X}(\tilde{a}, \tilde{b}))^{3}}$$

$$= 6q \int_{-\delta}^{b} \frac{\Psi_{1}(q(\mathcal{X} - 4\tilde{b}^{2}))}{(1 + 3\Psi_{1}^{2}(q(\mathcal{X} - 4\tilde{b}^{2})))^{3}} \left( 1 + O \left( (T-t)^{\frac{1}{6}} + |\mathcal{X}|^{\frac{1}{6}}(T-t)^{\frac{1}{6}} + |\tilde{b}|^{\frac{1}{6}}(T-t)^{\frac{1}{6}} \right) \right) \tilde{b}\right)$$

Note that in the integrals above, for $\tilde{b} \leq -\delta(T-t)^{-3/4}$ there holds $\Psi_{1}(q(\mathcal{X} - 4\tilde{b}^{2})) \geq \tilde{b}^{2/3}$. Hence, integrating from infinity instead of $-\delta(T-t)^{-3/4}$ produces an error which is $O((t-T)^{7/4})$ and:

$$\frac{\partial}{\partial \mathcal{X}} \left( C(T-t)^{\frac{1}{3}} \int_{\Gamma_{2}} \frac{ds}{|\nabla x|} \right)$$

$$= 6q \int_{-\infty}^{b} \frac{\Psi_{1}(q(\mathcal{X} - 4\tilde{b}^{2}))}{(1 + 3\Psi_{1}^{2}(q(\mathcal{X} - 4\tilde{b}^{2})))^{3}} \left( 1 + O \left( (T-t)^{\frac{1}{6}} + |\mathcal{X}|^{\frac{1}{6}}(T-t)^{\frac{1}{6}} + |\tilde{b}|^{\frac{1}{6}}(T-t)^{\frac{1}{6}} \right) \right) \tilde{b}\right)$$

Therefore, injecting (4.20), (4.45) and the above identity in (4.44):

$$\partial_{a} \mathcal{Y}(a, b) = \frac{\partial_{a} \mathcal{X}}{6q} \int_{\Gamma_{1}} \frac{ds}{|\nabla x|} + \frac{\partial}{\partial \mathcal{X}} \left( C(T-t)^{\frac{1}{3}} \int_{\Gamma_{2}} \frac{ds}{|\nabla x|} \right)$$

$$= \frac{\partial_{a} \mathcal{X}}{6q} \int_{-\infty}^{b} \frac{\Psi_{1}(q(\mathcal{X} - 4\tilde{b}^{2}))}{(1 + 3\Psi_{1}^{2}(q(\mathcal{X} - 4\tilde{b}^{2})))^{3}} \left( 1 + O \left( (T-t)^{\frac{1}{6}} + |\mathcal{X}|^{\frac{1}{6}}(T-t)^{\frac{1}{6}} + |\tilde{b}|^{\frac{1}{6}}(T-t)^{\frac{1}{6}} \right) \right) \tilde{b}\right)$$

$$+ \int_{-\infty}^{b} \frac{O \left( (T-t)^{\frac{1}{6}} \right)}{(1 + 3\Psi_{1}^{2}(q(\mathcal{X} - 4\tilde{b}^{2})))^{3}} \tilde{b} + O \left( (T-t)^{\frac{7}{6}} \right)$$

$$= 6q \left( 1 + 3\Psi_{1}^{2}(q(\mathcal{X} - 4\tilde{b}^{2})) \right) \left( \int_{-\infty}^{b} \frac{\Psi_{1}(q(\mathcal{X} - 4\tilde{b}^{2}))}{(1 + 3\Psi_{1}^{2}(q(\mathcal{X} - 4\tilde{b}^{2})))^{3}} x \right) \left( 1 + O \left( (T-t)^{\frac{1}{6}} + |\mathcal{X}|^{\frac{1}{6}}(T-t)^{\frac{1}{6}} + |\tilde{b}|^{\frac{1}{6}}(T-t)^{\frac{1}{6}} + (T-t)^{\frac{1}{6}} |\tilde{b}|^{\frac{1}{6}} \right) \right) \tilde{b}\right)$$

$$+ \int_{-\infty}^{b} \frac{O \left( (T-t)^{\frac{1}{6}} \right)}{(1 + 3\Psi_{1}^{2}(q(\mathcal{X} - 4\tilde{b}^{2})))^{3}} \tilde{b} + O((T-t)^{\frac{7}{6}})$$

(4.48)
**Step 2** The core of the self-similar zone. Let \( M \gg 1 \) be large and then \( 0 < \kappa \ll 1 \) be small. Fix \((\hat{x}, \hat{y})\) in the zone:

\[
|\hat{x} - x^*| \leq M(T-t)^{\frac{1}{2}}, \quad \kappa(T-t)^{-\frac{1}{2}} \lesssim \hat{y} \lesssim \frac{2\kappa}{(T-t)^{\frac{1}{2}}} (\hat{x} - x^*) - \kappa.
\]

(4.49)

We want to invert the characteristics and to find the corresponding \( \hat{a} \) and \( \hat{b} \) such that \((x, y)(\hat{a}, \hat{b}) = (\hat{x}, \hat{y})\). Let \( N \gg 1 \) be large depending on \( M \) and \( \kappa \) and consider the zone

\[
|a|, |b| \leq N.
\]

(4.50)

This zone lies clearly in the zone \( Z_1 \), so the corresponding estimates in Lemma 10 apply. Moreover, as \( |a|, |b|, |\mathcal{X}| \lesssim 1 \) one obtains from (4.15), (4.20) and (4.22):

\[
\mathcal{X} = a + q^2a^3 + 4b^2 + O\left((T-t)^{\frac{1}{2}}\right) = \mathcal{X}^\Theta(a, b) + O\left((T-t)^{\frac{1}{2}}\right),
\]

and similarly:

\[
\partial_a \mathcal{X} = 1 + 3q^2a^2 + O((T-t)^{\frac{1}{2}}) = \partial_a \mathcal{X}^\Theta + O((T-t)^{\frac{1}{2}}), \quad \partial_b \mathcal{X} = 8b + O((T-t)^{\frac{1}{2}}) = \partial_b \mathcal{X}^\Theta + O((T-t)^{\frac{1}{2}}).
\]

(4.51)

Since \(|\mathcal{X}| \lesssim 1\), one can use the identities derived in Step 1. Injecting that \(|\mathcal{X}|, |b| \lesssim 1\) and that \(1 + 3q_1^2\left(q\left(\mathcal{X} - 4b^2\right)\right) \gtrsim 1 + |\hat{b}|^{4/3}\) for \(\hat{b} \leq b\) in (4.47) gives:

\[
\mathcal{Y}(a, b) = 2 \int_{-\infty}^{b} \frac{1}{1 + 3q_1^2(p(\mathcal{X} - 4b^2))} d\hat{b} + O\left(\int_{-\infty}^{b} \frac{|\hat{b}|^{\frac{1}{2}}(T-t)^{\frac{3}{2}}}{1 + |\hat{b}|^{\frac{3}{2}}} d\hat{b}\right) + O((T-t)^{\frac{1}{2}})
\]

\[
= \mathcal{Y}^\Theta(a, b) + O((T-t)^{\frac{1}{2}}).
\]

Injecting that \(|\mathcal{X}|, |b| \lesssim 1\) and \(1 + 3q_1^2\left(q\left(\mathcal{X} - 4b^2\right)\right) \approx 1 + |\hat{b}|^{4/3}\) in (4.48) gives similarly:

\[
\partial_a \mathcal{Y}(a, b) = 6q \left(1 + 3q_1^2\left(q\left(\mathcal{X} - 4b^2\right)\right)\right) \int_{-\infty}^{b} \frac{\Psi_1\left(q\left(\mathcal{X} - 4b^2\right)\right)}{1 + 3q_1^2\left(q\left(\mathcal{X} - 4b^2\right)\right)^{3/2}} d\hat{b}
\]

\[
\quad + O\left(\int_{-\infty}^{b} \frac{(T-t)^{\frac{3}{2}} + |\hat{b}|^{\frac{1}{2}}(T-t)^{\frac{3}{2}}}{1 + |\hat{b}|^{\frac{3}{2}}} d\hat{b}\right) + O((T-t)^{\frac{1}{2}})
\]

\[
= \partial_a \mathcal{Y}^\Theta(a, b) + O((T-t)^{\frac{1}{2}}).
\]

We retrieve the last partial derivative by incompressibility, using that \(\partial_a \mathcal{X} \approx 1\), (4.51) and the above identity:

\[
\frac{\partial_b \mathcal{Y}^\Theta}{\partial_a \mathcal{X}^\Theta} = 1 + \frac{\partial_b \mathcal{Y}^\Theta}{\partial_a \mathcal{X}^\Theta} + O((T-t)^{\frac{1}{2}}))\left(\partial_a \mathcal{X}^\Theta + O((T-t)^{\frac{1}{2}})\right) = \partial_b \mathcal{Y}^\Theta + O\left((T-t)^{\frac{1}{2}}\right).
\]

We are now ready to invert the characteristics. We look for a solution of the form \((a, b) = (\overline{a} + h, \overline{b} + h)\) to \((x, y)(a, b) = (x, y)\), where

\[
(\overline{a}, \overline{b}) = \left(a^\Theta \left(\frac{x - x^*}{(T-t)^{\frac{1}{2}}}, \frac{y}{(T-t)^{-\frac{1}{2}}}\right), b^\Theta \left(\frac{x - x^*}{(T-t)^{\frac{1}{2}}}, \frac{y}{(T-t)^{-\frac{1}{2}}}\right)\right).
\]

As in the zone under consideration, \((\hat{X}, \hat{Y})\) is in a compact zone inside the support of \(\Theta\), then \((\overline{a}, \overline{b})\) belongs also to a compact zone, included in \([-N, N]^2\) for \(N\) large enough so that our previous computations apply. Consider then the mapping:

\[
\Phi : (h_1, h_2) \mapsto (\mathcal{X}(\overline{a} + h_1, \overline{b} + h_2), \mathcal{Y}(\overline{a} + h_1, \overline{b} + h_2))
\]
From the estimates on the derivatives done above, there holds for $|h_1|, |h_2| = O((T - t)^{1/12})$:

$$\begin{pmatrix}
\partial_{h_1} \Phi_1 & \partial_{h_2} \Phi_1 \\
\partial_{h_1} \Phi_2 & \partial_{h_2} \Phi_2
\end{pmatrix} = \left(\begin{array}{c}
\partial_u \chi^\theta(\pi, \overline{b}) \\
\partial_u \chi^\theta(\pi, \overline{b})
\end{array}\right) + O((T - t)^{1/12}).$$  

(4.52)  

\[\text{id:jacobian}\]

Also, again from the computations performed above:

$$\Phi(0,0) = \left(\begin{array}{c}
\hat{X} \\
\hat{Y}
\end{array}\right) = O((T - t)^{1/12}).$$

Note that, as $\pi$ and $\overline{b}$ vary in the compact zone $[-N, N]^2$, the leading order term matrices in (4.52) belong to a compact set of invertible matrices. Hence we can invert the above equation, uniformly as $t \to T$ in the zone that we consider: there exists $(h_1, h_2) = O((T - t)^{1/12})$ such that:

$$\Phi(h_1, h_2) = \left(\begin{array}{c}
\hat{X} \\
\hat{Y}
\end{array}\right).$$

Hence we inverted the characteristics and:

$$\hat{a} = \pi + O((T - t)^{1/12}), \quad \hat{b} = \overline{b} + O((T - t)^{1/12}).$$

Using the Taylor expansion (4.41) of $u$ and the fact that $|\hat{a}|, |\hat{b}| \lesssim 1$:

$$u(t, \hat{x}, \hat{y}) = u(t, \hat{X}, \hat{Y}) = u(t, X_0, Y_0) - (T - t)^{1/2}(\hat{a} + O((T - t)^{1/12})), $$

so we infer that:

$$u(t, \hat{x}, \hat{y}) = u^* + (T - t)^{1/2}\left(\Theta_{\mu, \nu}\left(\frac{\hat{x} - x^*}{(T - t)^{1/2}}, \frac{\hat{y}}{(T - t)^{-1/2}}\right) + O((T - t)^{1/12})\right).$$

(4.53)  

\[\text{id:uthcorese}2\]

Once the inversion is done, the estimates for the derivatives follow naturally, as from (4.42), (4.43), (4.33) and the above estimates:

$$\partial_X u(t, \hat{x}, \hat{y}) = -(T - t)^{1/2}\partial_b \chi^\theta\left(1 + O((T - t)^{1/12})\right) + O((T - t)^{1/12}) = -(T - t)^{1/2}\partial_X \Theta_{\mu, \nu} + O((T - t)^{1/12})$$

(4.54)  

\[\text{id:Xuthcorese}\]

$$\partial_Y u(t, \hat{x}, \hat{y}) = (T - t)^{1/2}\partial_b \chi^\theta\left(1 + O((T - t)^{1/12})\right) + O((T - t)^{1/12}) = -(T - t)^{1/2}\partial_Y \Theta_{\mu, \nu} + O((T - t)^{1/12})$$

(4.55)  

\[\text{id:Yuthcorese}\]

**Step 3** The bottom of the self-similar zone. We apply a similar strategy as in Step 2. However the leading order term of the characteristics, which is still $(\chi^\theta, \chi^\theta)$, becomes degenerate in this zone. Hence we will prove specific renormalised stability estimates, and will renormalise the characteristics to prove that the invertibility is uniformly possible in the zone at hand. Let $K \gg 1$ be large, and then $0 < \kappa, \epsilon \ll 1$ be small, and fix $(\hat{x}, \hat{y})$ in the zone:

$$\begin{align*}
|\hat{x}| \leq \epsilon, & \quad K \leq \hat{y} \leq \frac{\kappa}{|\hat{x} - x^*|^{1/5} + (T - t)^{-1/5}} \tag{4.56} \\
\text{corresponding to } |\hat{X}| \leq \epsilon(T - t)^{-3/2} & \quad \text{and } K(T - t)^{1/4} \leq \hat{Y} \leq \kappa/(|X|^{1/6} + 1). \quad \text{Let } N_1 \gg N_2 \gg 1 \text{ be large, depending on } K, \epsilon \text{ and } \kappa \text{ and consider } (a,b) \text{ in the zone:}
\end{align*}$$

$$|q^2 a^3 + 4b^2| \leq \frac{b^2}{N_2}, \quad -\delta(T - t)^{-3/4} \leq b \leq -N_1. \tag{4.57}$$

\[\text{autosimabbottom}\]

One has in particular from the above bounds and from (4.15) that:

$$a < 0, \quad |b| \approx |a|^{3/2} \gg 1 \quad \text{and} \quad |X| \ll |b|^2$$

(4.58)  

\[\text{bd:amXautosim}\]

hence $(a, b) \in Z_1$. Consequently all the computations corresponding to the zone $Z_1$ in Lemma 10 apply. The identities (4.18), (4.20) and (4.22) give, when injecting the bounds (4.58):

$$X = (a + q^2 a^3 + 4b^2) + O((T - t)^{1/2}|b^{2+3/4}|) = X^\theta(a, b) + O((T - t)^{1/2}|b|^{2+3/4}).$$

(4.59)  

\[\text{bd:bottom:x}\]
\[ \partial_a \mathcal{X} = (1 + 3q^2 a^2) \left( 1 + O \left( \frac{1}{b} \left( T - t \right)^{\frac{1}{3}} \right) \right) \partial_a \mathcal{X} (a, b) \left( 1 + O \left( \frac{1}{b} \left( T - t \right)^{\frac{1}{3}} \right) \right), \]  
\[ \partial_b \mathcal{X} = 8b \left( 1 + O \left( \frac{1}{b} \left( T - t \right)^{\frac{1}{3}} \right) \right) \partial_b \mathcal{X} (a, b) \left( 1 + O \left( \frac{1}{b} \left( T - t \right)^{\frac{1}{3}} \right) \right). \]  
We now compute \( \mathcal{Y} \). Since \( b < 0 \) and \( |\mathcal{X}| \ll |b|^2 \), Step 1 applies. As \( |\mathcal{X}| \ll |b|^2 \) and \( b < 0 \), then for \( b \leq b \) one has from \((4.38)\) and \((4.58)\):

\[ |b| \gg 1, \quad \Psi_1^2 (q (\mathcal{X} - 4 \hat{b}^2)) \approx |b|^{4/3} \quad \text{and} \quad |\mathcal{X}| \ll |\hat{b}|. \]

Injecting the above bound and \((4.58)\) in the identity \((4.47)\), and using \((4.34)\), gives:

\[ \mathcal{Y} = 2 \int_{-\infty}^{b} \frac{db}{1 + 3 \Psi_1^2 \left( q \left( \mathcal{X} - 4 \hat{b}^2 \right) \right)} + O \left( \int_{-\infty}^{b} \frac{(T - t)^{\frac{1}{3}} |b|^{\frac{1}{3}} db}{|\hat{b}|^{\frac{10}{3}}} \right) + O((T - t)^{\frac{1}{3}}) \]

\[ = \mathcal{Y} (a, b) \left( 1 + O \left( (T - t)^{\frac{1}{3}} |b|^{\frac{1}{3}} \right) \right). \]  
Similarly, injecting \((4.62)\) and \((4.58)\) in the integral \((4.48)\) giving \( \partial_a \mathcal{Y} \), and using \((4.32)\) gives:

\[ \partial_a \mathcal{Y} (a, b) = 6q \left( 1 + 3 \Psi_1^2 \left( q \left( \mathcal{X} - 4 \hat{b}^2 \right) \right) \right) \left( \int_{-\infty}^{b} \frac{\Psi_1 \left( q \left( \mathcal{X} - 4 \hat{b}^2 \right) \right)}{1 + 3 \Psi_1^2 \left( q \left( \mathcal{X} - 4 \hat{b}^2 \right) \right)} db \right) + O((T - t)^{\frac{1}{3}}) \]

\[ = \partial_a \mathcal{Y} (a, b) \left( 1 + O \left( (T - t)^{\frac{1}{3}} |b|^{\frac{1}{3}} \right) \right). \]

We compute the last partial derivative via the incompressibility, using \((4.60)\), \((4.61)\) and the above bound, and \((4.34)\):

\[ \partial_b \mathcal{Y} = \frac{1 + \partial_a \mathcal{Y} \partial_b \mathcal{X}}{\partial_a \mathcal{X}} = \frac{1 + \partial_a \mathcal{Y} \partial_b \mathcal{X}}{\partial_a \mathcal{X}} \left( 1 + O \left( \frac{|b|}{T - t} \right)^{\frac{1}{3}} \right) \]

\[ = \partial_b \mathcal{Y} \left( 1 + O \left( (T - t)^{\frac{1}{3}} |b|^{\frac{1}{3}} \right) \right). \]

We can now invert the characteristics. We look for a solution of \((x, y)(\hat{a}, \hat{b}) = (\hat{x}, \hat{y})\) of the form \((a, b) = (\hat{a} + h_1 \hat{b}, \hat{b}(1 + h_2))\) close to

\[ (\bar{u}, \bar{b}) = \left( a \Theta \left( \frac{\hat{x} - x^*}{(T - t)^{\frac{1}{3}}} \right), \hat{y} \Theta \left( \frac{\hat{y}}{(T - t)^{-\frac{1}{3}}} \right) \right). \]

and with a priori bound \(|h_1|, |h_2| \lesssim |\bar{b}|^{1/12} (T - t)^{1/16}\). From (ii) in Lemma 12, and given the conditions on \(x\) and \(y\):

\[ \bar{b} \approx \frac{1}{(T - t)^{\frac{1}{5}} \hat{y}^3}, \quad |\bar{u}| \approx \frac{1}{(T - t)^{\frac{1}{2}} \hat{y}^2}, \quad \bar{b} < 0, \]

and the hypothesis \(|q^2 \bar{u}^3 + \bar{b}^2| \leq |\bar{b}^2|/N_2\) is indeed satisfied if \(N_2\) is chosen large but not too large depending on \(\kappa\). Hence all computations performed earlier in this step are valid near \((\bar{u}, \bar{b})\). Consider now the mapping:

\[ \Phi : (h_1, h_2) \mapsto \left( \frac{\mathcal{X} (\bar{u} + h_1 \bar{b}^{\frac{1}{2}}, \bar{b}(1 + h_2))}{\bar{b}^{\frac{1}{2}}}, \mathcal{Y} (\bar{u} + h_1 \bar{b}^{\frac{1}{2}}, \bar{b}(1 + h_2)) \right). \]
From the computations on the derivatives above, (4.34) and (4.64):

\[
\begin{pmatrix}
\partial_{h_1} \Phi_1 & \partial_{h_2} \Phi_1 \\
\partial_{h_1} \Phi_2 & \partial_{h_2} \Phi_2
\end{pmatrix} = \begin{pmatrix}
\bar{b}^{-\frac{1}{2}} \partial_a \mathcal{X}^\Theta(\bar{a}, \bar{b}) & \bar{b}^{-1} \partial_b \mathcal{X}^\Theta(\bar{a}, \bar{b}) \\
\bar{b} \partial_a \mathcal{Y}^\Theta(\bar{a}, \bar{b}) & \bar{b}^{\frac{1}{2}} \partial_b \mathcal{Y}^\Theta(\bar{a}, \bar{b})
\end{pmatrix} + O(\hat{y}^{-\frac{1}{3}}).
\]

From (4.34) the above matrix is bounded, and the leading order term in the right hand side is close to a fixed invertible matrix in the whole zone under consideration. From (4.59) and (4.63):

\[
\Phi(0,0) - \left(\frac{\dot{X}}{|\bar{b}|^\frac{1}{2}}, \frac{\dot{Y}}{|\bar{b}|^\frac{1}{2}}\right) = O(\bar{b}^\frac{1}{3} (T-t)^\frac{1}{2}) = O(\hat{y}^{-\frac{1}{3}})
\]

Therefore, one can invert the above equation: there exists \((h_1, h_2) = O(\hat{y}^{-1/3})\) such that such that \(\Phi(h_1, h_2) = \left(\frac{\dot{X}}{|\bar{b}|^\frac{1}{2}}, \frac{\dot{Y}}{|\bar{b}|^\frac{1}{2}}\right)\) or equivalently for the characteristics:

\[
\hat{a} = \bar{a} + h_1 |\bar{b}|^\frac{1}{2} = \bar{a} \left(1 + O(\hat{y}^{-\frac{1}{3}})\right), \quad \hat{b} = \bar{b}(1 + h_2) = \bar{b} \left(1 + O(\hat{y}^{-\frac{1}{3}})\right),
\]

where we used that \(|\bar{a}| \approx |\bar{b}|^{2/3}\). Hence, since, using the Taylor expansion (4.41) of \(u\) and the fact that \(|\hat{a}|^3 \approx |\hat{b}|^2 \gg 1\):

\[
u(t, \hat{x}, \hat{y}) = u(t, \hat{X}, \hat{Y}) = u(t, X_0, Y_0) - (T-t)^\frac{1}{2} \hat{a}(1 + O(|\hat{a}|(T-t)^{1/2}))
\]

we infer that:

\[
u(t, \hat{x}, \hat{y}) = u^* + (T-t)^\frac{1}{2} \Theta_{\mu, \nu} \left(\frac{\hat{x} - x^*}{(T-t)^\frac{1}{2}}, \frac{\hat{y}}{(T-t)^\frac{1}{2}}\right) \left(1 + O\left(\frac{1}{\hat{y}^\frac{1}{2}}\right)\right) + O\left((T-t)^\frac{3}{2} |\hat{b}|^{-1}\right)
\]

Again, the estimates for the derivatives follow from (4.42), (4.43), (4.33), (4.34) and the above estimates:

\[
\partial_X u = \partial_a u \partial_b \mathcal{Y} - \partial_b u \partial_a \mathcal{Y} = -(T-t)^\frac{1}{2} \partial_b \mathcal{Y}^\Theta \left(1 + O\left(\frac{1}{|\hat{y}|^\frac{1}{2}}\right)\right) + O\left((T-t)^\frac{3}{2} |\hat{b}|^{-1}\right)
\]

\[
\partial_Y u = -\partial_a u \partial_b \mathcal{X} + \partial_b u \partial_a \mathcal{X} = (T-t)^\frac{1}{2} \partial_b \mathcal{X}^\Theta \left(1 + O\left(\frac{1}{|\hat{y}|^\frac{1}{2}}\right)\right) + O\left((T-t)^\frac{3}{2} |\hat{b}|\right)
\]

**Step 4 The sides of the self-similar zone.** Let \(K \gg 1\) be large, and \(0 < \kappa, \epsilon \ll 1\) be small, and consider the zone:

\[
K(T-t)^\frac{3}{2} \leq |\hat{x} - x^*| \leq \epsilon, \quad \frac{\kappa}{|\hat{x} - x^*|} \leq \hat{y} \leq \frac{C_{sgn(\hat{x} - x^*)} - \kappa}{|\hat{x} - x^*|^{\frac{3}{2}}} \tag{4.68}
\]

corresponding to \(\epsilon(T-t)^{-3/2} \leq \hat{X} \leq -K\) and \(\kappa|\hat{X}|^{-1/6} \leq \hat{Y} \leq (C_+ - \kappa)|\hat{X}|^{-1/6}\) on the left side, or to \(K \leq \hat{X} \leq \epsilon(T-t)^{-3/2}\) and \(\kappa|\hat{X}|^{-1/6} \leq \hat{Y} \leq (C_- - \kappa)|\hat{X}|^{-1/6}\) on the right side. Let \(N_1 \gg N_2 \gg 1\) be large and consider the zone

\[
b^2 \leq N_2 |q^2 a^3 + 4b^2|, \quad |q^2 a^3 + 4b^2| \geq N_1. \tag{4.69}
\]
In this zone, from the above bound (4.69), \(|a| \lesssim (|q^2a^3 + 4b^2| + b^2)^{1/3} \lesssim N_2^{1/3}|q^2a^3 + 4b^2|\), so one infers from (4.15) that:

\[
\mathcal{X} = (q^2a^3 + 4b^2)^2 \left( 1 + \mathcal{O} \left( \frac{|a|}{|q^2a^3 + 4b^2|} + \frac{|b|(T - t)^{1/4}}{|q^2a^3 + 4b^2|} + |q^2a^3 + 4b^2| \frac{1}{(T - t)^{1/4}} \right) \right)
\]

\[
= (q^2a^3 + 4b^2)^2 \left( 1 + \mathcal{O} \left( N_2^{1/3}N_1^{-1/3} + |q^2a^3 + 4b^2| \frac{1}{(T - t)^{1/6}} \right) \right)
\]

which implies that \(|\mathcal{X}| \approx |q^2a^3 + 4b^2|\). Hence the bounds in this zone:

\[
|\mathcal{X}| \gtrsim N_1 \gg 1, \quad |b| \lesssim |\mathcal{X}|^{1/3}, \quad |a| \lesssim |\mathcal{X}|^{1/4}.
\]

Injecting these bounds in (4.15), (4.16) and (4.17), one gets:

\[
\partial_a \mathcal{X} = 1 + 3q^2a^2 + \mathcal{O}(\mathcal{X} \frac{1}{(T - t)^{1/4}}), \quad \partial_b \mathcal{X} = \mathcal{O}(\mathcal{X} \frac{1}{(T - t)^{1/4}}),
\]

\[
\partial_a Y = \partial_b Y = \mathcal{O}(\mathcal{X}^{1/6}), \quad \partial_a Y = \partial_b Y = \mathcal{O}(\mathcal{X}^{1/6}).
\]

We claim in addition that the following estimates are true for \(Y\):

\[
\mathcal{Y} = \mathcal{Y} (a, b) \left( 1 + O((T - t)^{1/4}) \right), \quad \partial_a \mathcal{Y} = \partial_a \mathcal{Y} + \mathcal{O}(T - t)^{1/4},
\]

\[
\partial_b \mathcal{Y} = \partial_b \mathcal{Y} + \mathcal{O}(T - t)^{1/4}.
\]

whose proofs are relegated to the next step. With these identities we can now invert the characteristics, and again, will renormalise in a suitable way the perturbation problem. We look for a solution \((\hat{a}, \hat{b})\) of the form \((a, b) = (\xi + h_1|\mathcal{X}|^{1/3}, \Theta + h_2|\mathcal{X}|^{1/2})\) to \((x, y)(\hat{a}, \hat{b}) = (\bar{x}, \bar{y})\), where

\[
(a, b) = \left( a^\Theta \left( \frac{\ddot{x} - \bar{x}^*}{(T - t)^{1/2}}, \frac{\ddot{y}}{(T - t)^{-1/2}} \right), \quad b^\Theta \left( \frac{\ddot{x} - \bar{x}^*}{(T - t)^{1/2}}, \frac{\ddot{y}}{(T - t)^{-1/2}} \right) \right).
\]

As \(x, y\) is in the zone (4.56) this produces thanks to (ii) in Lemma 12:

\[
|\ddot{b}|^2 \lesssim |q^2a^3 + 4b^2|, \quad |q^2a^3 + 4b^2| \gg 1,
\]

which is compatible with all the computations done so far. From the above computations one infers that:

\[
\mathcal{X}(\pi, \bar{b}) = \mathcal{X} \left( 1 + O(|\ddot{x} - \bar{x}^*|^{1/6}) \right), \quad \mathcal{Y}(\pi, \bar{b}) = \mathcal{Y} \left( 1 + O(|\ddot{x} - \bar{x}^*|^{1/6}) \right).
\]

Consider the mapping:

\[
\Phi : (h_1, h_2) \mapsto \left( \mathcal{X} \left( \pi + h_1|\mathcal{X}|^{1/3}, \bar{b} + h_2|\mathcal{X}|^{1/2} \right) |\mathcal{X}|^{1/6} \right), \quad \mathcal{Y} \left( \pi + h_1|\mathcal{X}|^{1/3}, \bar{b} + h_2|\mathcal{X}|^{1/2} \right) |\mathcal{X}|^{1/6} \right)
\]

From the estimates on the derivatives done above, there holds for \(|h_1|, |h_2| = O(|\ddot{x}|^{1/18}):\)

\[
\begin{pmatrix} \partial_{h_1} \Phi_1 & \partial_{h_2} \Phi_1 \\ \partial_{h_1} \Phi_2 & \partial_{h_2} \Phi_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{|\mathcal{X}|^{1/6}} \partial_a \mathcal{X} \left( \pi, \bar{b} \right) \frac{1}{|\mathcal{X}|^{1/6}} \partial_b \mathcal{X} \left( \pi, \bar{b} \right) + O(|x|^{1/18}) \end{pmatrix}
\]

and the Jacobian matrix of \(\Phi\) must preserve volume. From the computations above,

\[
\Phi(0, 0) - \left( \frac{\mathcal{X}}{|\mathcal{X}|^{1/6}}, \mathcal{Y} |\mathcal{X}|^{1/6} \right) = O(|\ddot{x}|^{1/18}).
\]
Note that, due to (4.35), as $\mathcal{X} \to +\infty$, in the zone that we consider (4.69), the leading order matrix in the identity giving the Jacobian of $\Phi$ in the right hand side belongs to a compact set of matrices with determinant 1. Hence we can invert the above equation, uniformly as $\epsilon \to 0$ in the zone that we consider: there exists $(h_1, h_2) = O(|x|^{18})$ such that $\Phi(h_1, h_2) = \left(\frac{x}{|x|^{18}}, \frac{\hat{y}}{|x|^{18}}\right)$.

Hence we inverted the characteristics and:

$$\hat{a} = \alpha \left(1 + O \left(|\hat{x}|^{18}\right)\right), \quad \hat{b} = \beta \left(1 + O \left(|\hat{x}|^{18}\right)\right).$$

Hence, since, using the Taylor expansion (4.41) of $u$ and the fact that $|\hat{a}|^3, |\hat{b}|^2 \lesssim |\hat{X}|$:

$$u(t, \hat{x}, \hat{y}) = u(t, X, Y) = u^* - (T - t)^{\frac{3}{2}} \hat{a} + O \left(|\hat{b}|(T - t)^{\frac{3}{2}} + |\hat{a}|^3(T - t)^{\frac{3}{2}}\right)$$

$$= -(T - t)^{\frac{3}{2}} \left(\hat{a} + O \left(|\hat{X}|^{\frac{3}{2} + \frac{1}{18}}(T - t)^{\frac{1}{2}}\right)\right)$$

we infer that:

$$u(t, \hat{x}, \hat{y}) = u^* + (T - t)^{\frac{3}{2}} \left(\Theta_{\mu, \nu} \left(\frac{\hat{x} - x^*}{(T - t)^{\frac{3}{2}}}, \frac{\hat{y}}{(T - t)^{\frac{3}{2}} - \frac{\nu}{\mu}}\right) + O \left(\frac{|\hat{x} - x^*|^{\frac{3}{2} + \frac{1}{18}}}{(T - t)^{\frac{3}{2}}}\right)\right) \quad (4.75) \quad \text{id:uthside}$$

The estimates for the derivatives follow from (4.42), (4.43), (4.33), (4.34) and the above estimates:

$$\partial_X u = \partial_a u \partial_b Y - \partial_b u \partial_a Y$$

$$= -(T - t)^{\frac{3}{2}} \partial_Y \Theta + O \left((T - t)^{\frac{3}{2} + \frac{1}{18}}|X|^{\frac{3}{2} + \frac{1}{18}}\right) \left(1 + O \left(\frac{1}{|\hat{y}|^{\frac{3}{2}}}\right)\right) + O \left((T - t)^{\frac{3}{2}}|X|^{\frac{3}{2}}\right)$$

$$= -(T - t)^{\frac{3}{2}} \partial_Y \Theta + O \left((T - t)^{\frac{3}{2}}|x|^{\frac{3}{2} + \frac{1}{18}}\right) \quad (4.76)$$

$$\partial_Y u = -\partial_a u \partial_b X + \partial_b u \partial_a X = (T - t)^{\frac{3}{2}} \partial_X \Theta + O \left((T - t)^{\frac{3}{2}}|X|^{\frac{3}{2}}\right) + O \left((T - t)^{\frac{3}{2}}|X|^{\frac{3}{2}}\right)$$

$$= -(T - t)^{\frac{3}{2}} \partial_X \Theta + O \left((T - t)^{-\frac{1}{2}}|x|^{\frac{3}{2} + \frac{1}{18}}\right) \quad (4.77)$$

**Step 5** *End of the proof of (1.25), (1.26) and (1.27).* These stability estimates concerning the self-similar zone and the bottom, are direct consequences of the identities (4.53), (4.54) and (4.55) found in the core of the self-similar zone, the identities (4.65), (4.66) and (4.67) found for the bottom of the self-similar zone, and (4.75), (4.76) and (4.77) found for the sides of the self-similar zone.

**Step 6** *Computations for $Y$ in the sides of the self-similar zone.* We stay in the zone described at the beginning of Step 4, and prove the estimates (4.73) and (4.74) concerning $Y$ that were used there. A new issue with respect to the previous steps is that the parametrisation of the curve $\Gamma = \{(a, \hat{b}), \ X(a, \hat{b}) = X(a, b)\}$ has to be done more carefully. Indeed, in the case of the left side of the self-similar zone, i.e. $x < 0$ in (4.68), we are in case (i) of Lemma 11, and then inside $Z_0^x$ the curve $\Gamma$ can be parametrised with the variable $\hat{b}$. In the case of the right side, i.e. $x > 0$ in (4.68), we are in case (ii) of Lemma 11, and then inside $Z_0^x$, the curve $\Gamma$ can be decomposed in five curves, $\Gamma_i$ for $i = 1, ..., 5$ that enjoy the properties described there. We claim that the case of the right side is harder to treat than the case of the left side as the parametrisation is more involved. Therefore we only treat the case of the right side, as the case of the left side could be done the very same way.
We want to invert the characteristics, and look for a solution \((a, b)\) close to:

\[
(\pi, \theta) = \left( a \Theta \left( \frac{x - x^*}{(T - t)^{\frac{1}{2}}} \right), b \Theta \left( \frac{x - x^*}{(T - t)^{\frac{1}{2}}} \right) \right).
\]

In the zone \((4.68)\) under consideration, \(\pi\) and \(\theta\) above indeed satisfy the condition \((4.69)\) from (iii) in Lemma 12. To compute the vertical component \(\gamma\) near \((\pi, \theta)\), we integrate \(ds/|\nabla x|\) on \(\Gamma\) from the boundary of the upper half plane until \((a, b)\). We use different variables to parametrise the curves \(\Gamma_i\), applying Lemma 11. On \(\Gamma_1\) we use the variable \(\tilde{b}\), on \(\Gamma_2 \tilde{a}\), on \(\Gamma_3 \tilde{b}\), on \(\Gamma_4 \tilde{a}\) and on \(\Gamma_5 \tilde{b}\). Without loss of generality for the argument, we assume that \((\pi, \theta)\) is located on \(\Gamma_2\).

Indeed, treating the case of three or more different parameters can be done the very same way. We consider a point \((a, b)\) close to \((\pi, \theta)\), which still belongs to \(\Gamma_2\) (up to changing slightly the constants in the definition of \(Z_1\) and \(Z_2\)). We denote by \(\Gamma_0\) the part of the curve outside \(Z_0\). Hence from \((4.40)\):

\[
\gamma = C(T - t)^{\frac{1}{2}} \int_{\Gamma_0} \frac{ds}{|\nabla x|} + \int_{\Gamma_0} \frac{db}{\partial a \chi} + \int_{a_1}^{a} \frac{d\tilde{a}}{\partial b \chi},
\]

where we recall that \((a_1, b_1)\) is the endpoint of \(\Gamma_1\) and the starting point of \(\Gamma_2\), defined in Lemma 11. The integral over \(\Gamma_0\) is at distance one to \((X_0, Y_0)\) and hence in a zone where everything remains regular:

\[
\int_{\Gamma_0} \frac{ds}{|\nabla x|} = O(1), \quad \partial \chi \left( \int_{\Gamma_0} \frac{ds}{|\nabla x|} \right) = O((T - t)^{\frac{1}{2}}).
\]

We now consider the second integral, corresponding to the part \(\Gamma_1\) of the curve joining the points \((a_m, -\delta(T - t)^{-3/4})\) and \((a_1, b_1)\). Since this part is in \(Z_1\), one obtains from \((4.20)\), injecting \((4.70)\):

\[
\int_{\Gamma_1} \frac{db}{\partial a \chi} = \int_{\Gamma_1} \frac{db}{\partial a \chi} \left( \frac{1 + 3 \Psi_1^2 \left( q \left( \chi - 4b^2 \right) \right)}{1 + O \left( (T - t)^{\frac{1}{2}} |\chi|^{\frac{1}{2}} + (T - t)^{\frac{1}{2}} |\tilde{b}|^{\frac{1}{2}} \right)} \right)
\]

We turn to the third integral, corresponding to the part \(\Gamma_2\) of the curve joining the points \((a_1, b_1)\) and \((a, b)\). There, since this part is in \(Z_2\), one obtains from \((4.26)\):

\[
\int_{a_1}^{a} \frac{d\tilde{a}}{\partial b \chi} = \int_{a_1}^{a} \frac{d\tilde{a}}{\partial b \chi} \left( 1 + O \left( |\chi|^{\frac{1}{2}} (T - t)^{\frac{1}{2}} \right) \right).
\]

In \(Z_2\), one has from \((4.14)\) that \(|\tilde{a}| \ll |\tilde{b}|^{2/3}\) and \(|\tilde{b}| \gg 1\), so that from \((4.15)\) one has

\[
|\chi| \approx |\tilde{b}|^2 \gg |\tilde{a}|^3 \quad \text{and} \quad \sqrt{\chi - \tilde{a} - q^2 \tilde{a}^3} \approx \chi^{\frac{1}{2}}
\]

uniformly for \(\tilde{a}\) in \(Z_2\), as well as \(|a_1|, |a| \ll \chi^{1/3}\). Using these bounds one infers from the above identity that:

\[
\int_{a_1}^{a} \frac{d\tilde{a}}{\partial b \chi} = \int_{a_1}^{a} \frac{d\tilde{a}}{\partial b \chi} + O \left( \frac{|a - a_1|}{|\chi|^{\frac{1}{2}}} (T - t)^{\frac{1}{2}} \right)
\]

Also, from the identity \((4.24)\), using the above bound \((4.81)\):

\[
|a + \frac{1}{p} \Psi_1 \left( q \left( \chi - 4b^2 \right) \right) | - |a + \frac{1}{p} \Psi_1 \left( q \left( \chi - 4b^2 \right) \right) | \lesssim |\chi|^{\frac{1}{2}} (T - t)^{\frac{1}{2}},
\]
\[ \left| \int_a^1 \frac{\frac{1}{p} \Phi_1(q(X - 4t^2))}{\sqrt{X - \hat{a} - q^2 \hat{a}^3}} \right| \lesssim \left| \frac{a + \frac{1}{p} \Phi_1(q(X - 4t^2))}{|X|^{\frac{1}{2}}} \right| \lesssim |X|^{-\frac{1}{2} + \frac{1}{2p}} (T - t)^{\frac{1}{2p}}, \]

\[ \left| \int_a^1 \frac{\frac{1}{p} \Phi_1(q(X - 4t^2))}{\sqrt{X - \hat{a} - q^2 \hat{a}^3}} \right| \lesssim \left| \frac{a_1 + \frac{1}{p} \Phi_1(q(X - 4t^2))}{|X|^{\frac{1}{2}}} \right| \lesssim |X|^{-\frac{1}{2} + \frac{1}{2p}} (T - t)^{\frac{1}{2p}}. \]

Therefore:

\[ \int_a^1 \frac{d\hat{a}}{\partial bX} = \int_{\frac{b_1}{1 + 3 \Phi_1^2(q(X - 4t^2))}}^{b} d\hat{b} + O \left( |X|^{-\frac{1}{2} + \frac{1}{2p}} (T - t)^{\frac{1}{2p}} \right). \]

We inject the identities (4.79), (4.80) and the above identity in the expression (4.78), giving the following expression for \( \Psi \):

\[ \Psi = \int_{\frac{b_1}{1 + 3 \Phi_1^2(q(X - 4t^2))}}^{b} d\hat{b} \left( 1 + O \left( \frac{|X|^\frac{1}{2} (T - t)^{\frac{1}{2p}} + \hat{b} |X|^\frac{1}{2} (T - t)^{\frac{1}{2p}} \right) \right) + O \left( |X|^{-\frac{1}{2} + \frac{1}{2p}} (T - t)^{\frac{1}{2p}} \right) \]

\[ = \Psi(a, b) \left( 1 + O((T - t)^{\frac{1}{2p}} |X|^{\frac{1}{2p}} \right), \quad (4.83) \]

where we used (4.36). This shows the first desired bound in (4.73). We now consider the derivatives of \( \Psi \). The point \((a_1, b_1)\) changes as \( X \) changes, but the identity \( X(a_1, b_1) = X(a, b) \) ensures that

\[ \partial X a_1 \partial aX(a_1, b_1) + \partial X b_1 \partial bX(a_1, b_1) = 1. \quad (4.84) \]

Hence, differentiating the sum of the two leading order integrals in (4.78) one obtains:

\[ (T - t)^{\frac{1}{2}} C \partial a \left( \int_{\Gamma_1} \frac{ds}{|VX|} + \int_{\Gamma_2} \frac{ds}{|VX|} \right) = \partial a \left( \int_{\frac{b_1}{1 + 3 \Phi_1^2(q(X - 4t^2))}}^{b} \frac{d\hat{b}}{\partial aX(\hat{a}(X, \hat{b}), \hat{b})} + \int_{\frac{a}{1 + 3 \Phi_1^2(q(X - 4t^2))}}^{a} \frac{d\hat{a}}{\partial bX(\hat{a}, \hat{b}(X, \hat{a}))} \right) \]

\[ = \partial aX(a, b) \left( \int_{\frac{b_1}{1 + 3 \Phi_1^2(q(X - 4t^2))}}^{b} \frac{1}{\partial aX(\hat{a}(X, \hat{b}), \hat{b})} d\hat{b} + \int_{\frac{a}{1 + 3 \Phi_1^2(q(X - 4t^2))}}^{a} \frac{d\hat{a}}{\partial bX(\hat{a}, \hat{b}(X, \hat{a}))} \right) \]

\[ + \frac{\partial X b_1}{\partial aX(a_1, b_1) + \partial bX(a_1, b_1)} - \frac{1}{\partial bX(a, b)} \]

\[ = \partial aX(a, b) \left( \int_{\frac{b_1}{1 + 3 \Phi_1^2(q(X - 4t^2))}}^{b} \frac{-\partial aX(\hat{a}(X, \hat{b}), \hat{b})}{\partial aX(\hat{a}(X, \hat{b}), \hat{b})^2} d\hat{b} + \int_{\frac{a}{1 + 3 \Phi_1^2(q(X - 4t^2))}}^{a} \frac{\partial bX(\hat{a}, \hat{b}(X, \hat{a}))}{\partial bX(\hat{a}, \hat{b}(X, \hat{a}))^3} d\hat{a} \right) \]

\[ + \frac{1}{\partial aX(a_1, b_1) \partial bX(a_1, b_1)} - \frac{1}{\partial bX(a, b)} \quad (4.85) \]
The first integral is located in $Z_1$ with $|a| \gg 1$, hence from (4.20) and (4.21):
\[
\int_{b_1}^{b_2} -\frac{\partial_{bb}X}{\partial aX} \left( a(X, b), \tilde{b} \right) \tilde{b} d\tilde{b} = -\frac{1}{8} \int_{b_1}^{b_2} \frac{\partial_{bb}X}{\partial aX} \left( a(X, \tilde{b}), \tilde{b} \right) \tilde{b} d\tilde{b} + O \left( |a|^{-\frac{3}{2}} |T-t|^\frac{1}{2} \right) \]
\[
\int_{a_1}^{a_2} \frac{\partial_{bb}X}{\partial aX} \left( a, b(X, \tilde{a}) \right) \tilde{a} d\tilde{a} = -\frac{1}{8} \int_{a_1}^{a_2} \frac{\partial_{bb}X}{\partial aX} \left( a(X, \tilde{a}), \tilde{a} \right) \tilde{a} d\tilde{a} + O \left( |a|^{-\frac{3}{2}} |T-t|^\frac{1}{2} \right). \tag{4.86}
\]

The second integral is located in $Z_2$, hence from (4.26) and (4.27):
\[
\int_{a_1}^{a_2} \frac{\partial_{bb}X}{\partial aX} \left( a, b(X, \tilde{a}) \right) \tilde{a} d\tilde{a} = -\frac{1}{8} \int_{a_1}^{a_2} 1 + O \left( |X|^{-\frac{3}{2}} |T-t|^\frac{1}{2} \right) \]
\[
\int_{a_1}^{a_2} \frac{\partial_{bb}X}{\partial aX} \left( a, b(X, \tilde{a}) \right) \tilde{a} d\tilde{a} = -\frac{1}{8} \int_{a_1}^{a_2} 1 + O \left( |X|^{-\frac{3}{2}} |T-t|^\frac{1}{2} \right) \]

We now change variables, taking $2\tilde{b} = -\sqrt{X - a - q^2 \tilde{a}^3}$. Note that this change of variables ensures $X^\Theta (a, \tilde{b}) = \tilde{a} + q^2 \tilde{a}^3 + 4\tilde{b}^2 = Cte = X$, and
\[
\frac{1}{8} \frac{1}{(X - a - q^2 \tilde{a}^3)^\frac{3}{2}} = \frac{\partial_{bb}X^\Theta}{\partial aX^\Theta} \]

There holds in this case a general formula, obtained by performing a change of variables and an integration by parts (note the signs $\partial_{aX} > 0$ and $\partial_{bX} < 0$ in the present case):
\[
\int_{a_1}^{a_2} \frac{\partial_{bb}X^\Theta}{\partial aX^\Theta} da = -\int_{b_1}^{b_2} \frac{\partial_{bb}X^\Theta}{\partial aX^\Theta} db
\]
\[
= -\int_{b_1}^{b_2} \left( \frac{d}{db} \partial_{bX}^\Theta + \frac{\partial_{bb}X^\Theta}{\partial aX^\Theta} \partial_{ba}X^\Theta \right) db
\]
\[
= -\int_{b_1}^{b_2} \frac{d}{db} \left( \partial_{bX}^\Theta \right) db + \frac{1}{\partial_{bX}^\Theta} \frac{1}{\partial_{aX}^\Theta} db
\]
\[
= -\int_{b_1}^{b_2} \frac{d}{db} \left( \partial_{bX}^\Theta \frac{1}{\partial_{bX}^\Theta} \partial_{aX}^\Theta \right) db + \int_{b_1}^{b_2} \frac{d}{db} \partial_{bX}^\Theta \left( \frac{1}{\partial_{bX}^\Theta} \partial_{aX}^\Theta \right) db
\]
\[
= \int_{a_1}^{a_2} \frac{1}{\partial_{aX}^\Theta(a_2, b_2) \partial_{bX}^\Theta(a_2, b_2)} + \frac{1}{\partial_{aX}^\Theta(a_1, b_1) \partial_{bX}^\Theta(a_1, b_1)} - 2 \int_{b_1}^{b_2} \partial_{bb}X^\Theta \frac{1}{\partial_{bX}^\Theta} db
\]
\[
= -\int_{a_1}^{a_2} \frac{1}{\partial_{aX}^\Theta(a_2, b_2) \partial_{bX}^\Theta(a_2, b_2)} + \frac{1}{\partial_{aX}^\Theta(a_1, b_1) \partial_{bX}^\Theta(a_1, b_1)} + 2 \int_{a_1}^{a_2} \partial_{bb}X^\Theta \frac{1}{\partial_{bX}^\Theta} db + \int_{b_1}^{b_2} \frac{\partial_{aa}X^\Theta}{\partial aX^\Theta} db.
from what one deduces the change of parametrisation identity:
\[
\int_{a_1}^{a_2} \frac{\partial_{bb} \chi^\Theta}{(\partial_b \chi^\Theta)^3} \, da = \frac{1}{\partial_a \chi^\Theta(\phi_1, x_2) \partial_{bb} \chi^\Theta(a_2, b_2) - \partial_a \chi^\Theta(a_1, b_1) \partial_{bb} \chi^\Theta(a_1, b_1)} - \int_{b_1}^{b_2} \frac{\partial_{aa} \chi^\Theta}{(\partial_a \chi^\Theta)^3} \, db.
\]
Applied to our case this produces, noticing that the endpoint are \((a_1, b_1)\) and \((a, b)\):
\[
- \frac{1}{8} \int_{\tilde{\mathcal{H}}(\phi_1, x_2)}^{\tilde{\mathcal{H}}(\phi_1, x_2)} \frac{1}{(\mathcal{X} - \tilde{a} - q^2 \tilde{a}^3)^3} \, d\tilde{a} = \frac{1}{\partial_a \chi^\Theta(a, b) \partial_{bb} \chi^\Theta(a, b)} - \frac{1}{\partial_a \chi^\Theta(a_1, b_1) \partial_{bb} \chi^\Theta(a_1, b_1)} - \int_{b_1}^{b} \frac{\partial_{aa} \chi^\Theta}{(\partial_a \chi^\Theta)^3} \, db.
\]
Note that since \((a_1, b_1)\) belong to both \(Z_1\) and \(Z_2\), from (4.81), (4.26) and (4.20):
\[
\frac{1}{\partial_a \chi(a_1, b_1) \partial_{bb} \chi(a_1, b_1)} = \frac{1 + O \left( |X|^\frac{1}{6} (T - t)^\frac{1}{2} \right)}{(1 + 3q^2 \sigma^2) a_1 b_1} = \frac{1}{(1 + 3q^2 \sigma^2) b_1} + O \left( |X|^{-\frac{7}{6}} + \frac{1}{18} (T - t)^\frac{1}{2} \right)
\]
so that:
\[
\int_{a_1}^{a_2} \frac{\partial_{bb} \chi^\tilde{a}(a, \tilde{b}(\mathcal{X}, \tilde{a}))}{(\partial_b \chi^\tilde{a}(\tilde{a}, \tilde{b}(\mathcal{X}, \tilde{a})))^3} \, d\tilde{a} = \frac{1}{(1 + 3q^2 \sigma^2) b_1} - \frac{1}{\partial_a \chi(a_1, b_1) \partial_{bb} \chi(a_1, b_1)} + 6q \int_{b_1}^{b} \frac{\Psi_1 \left( q \left( \mathcal{X} - \tilde{b}^2 \right) \right)}{\left( 1 + 3 \Psi_1^2 \left( q \left( \mathcal{X} - 4\tilde{b}^2 \right) \right) \right)^3} \, db + O \left( |X|^{-\frac{7}{6}} + \frac{1}{18} (T - t)^\frac{1}{2} \right)
\]
From the above identity and (4.86) one concludes that:
\[
\int_{\tilde{\mathcal{H}}(T - t)^{\frac{3}{4}}}^{\tilde{\mathcal{H}}(T - t)^{\frac{3}{4}}} \left( \partial_a \chi^\tilde{a}(a, \tilde{b}(\mathcal{X}, \tilde{a})) \right) \left( \partial_{bb} \chi^\tilde{a}(a, \tilde{b}(\mathcal{X}, \til{a})) \right) \frac{1}{\partial_a \chi(a_1, b_1) \partial_{bb} \chi(a_1, b_1)} \, db + 6q \int_{b_1}^{b} \frac{\Psi_1 \left( q \left( \mathcal{X} - \tilde{b}^2 \right) \right) \left( 1 + O \left( (T - t)^\frac{1}{12} \right) \right)}{\left( 1 + 3 \Psi_1^2 \left( q \left( \mathcal{X} - 4\tilde{b}^2 \right) \right) \right)^3} \, db + O \left( |X|^{-\frac{7}{6}} + \frac{1}{18} (T - t)^\frac{1}{2} \right)
\]
From (4.37), and using the fact that for \(\tilde{b} \leq -\delta(T - t)^{3/4}\) one has \(\Psi_1 \left( q \left( \mathcal{X} - \tilde{b}^2 \right) \right) | \approx | \tilde{b}|^{2/3}:
\[
\int_{\tilde{\mathcal{H}}(T - t)^{\frac{3}{4}}}^{\tilde{\mathcal{H}}(T - t)^{\frac{3}{4}}} \frac{\Psi_1 \left( q \left( \mathcal{X} - \tilde{b}^2 \right) \right) \left( 1 + O \left( (T - t)^\frac{1}{12} \right) \right)}{\left( 1 + 3 \Psi_1^2 \left( q \left( \mathcal{X} - 4\tilde{b}^2 \right) \right) \right)^3} \, db = \int_{-\infty}^{b} \frac{\Psi_1 \left( q \left( \mathcal{X} - \tilde{b}^2 \right) \right)}{\left( 1 + 3 \Psi_1^2 \left( q \left( \mathcal{X} - 4\tilde{b}^2 \right) \right) \right)^3} \, db + O \left( |X|^{-\frac{7}{6}} + \frac{1}{18} (T - t)^\frac{1}{2} \right).
\]
Therefore:
\[
\int_{t-\frac{s}{(T-t)^{\frac{1}{2}}}}^{b_1} \frac{-\partial_{aa}\chi\left(\tilde{a}(\chi, \tilde{b})\right)}{\partial_{a}\chi\left(\tilde{a}(X, \tilde{b})\right)} \, d\tilde{b}_1 + \int_{a_1}^{a} \frac{\partial_{bb}\chi\left(\tilde{a}, \tilde{b}(X, \tilde{a})\right)}{\partial_{b}\chi\left(\tilde{a}, \tilde{b}(X, \tilde{a})\right)} \, d\tilde{a}_1 + \frac{1}{\partial_{a}\chi(a_1, b_1)\partial_{b}\chi(a_1, b_1)}
\]
\[= \int_{-\infty}^{b} \frac{\Psi_1\left(q \left(\chi - \tilde{b}^2\right)\right)}{\left(1 + 3\Psi_1^2\left(q \left(\chi - 4\tilde{b}^2\right)\right)\right)} \, d\tilde{b} + \frac{1}{\left(1 + 3q^2a^2\right)b} + O\left(|\chi|^{-\frac{1}{6} + \frac{1}{12}}(T-t)^{-\frac{1}{12}}\right)
\] (4.77)

Since \((a, b)\) is in \(Z_2\), from (4.81), (4.26) and (4.20):
\[
\frac{\partial_a\chi(a, b)}{(1 + 3q^2a^2)b} = \frac{1}{\partial_b\chi(a, b)} + O\left(|\chi|^{-\frac{1}{6} + \frac{1}{12}}(T-t)^{-\frac{1}{12}}\right).
\]

Injecting the two identities above, (4.86) and \(|\partial_a\chi| < |\chi|^{2/3}\) in the identity (4.85) gives:
\[
(T-t)^{\frac{1}{2}}C_{\partial_b}\left(\int_{\Gamma_1}^{\frac{ds}{\nabla x}} + \int_{\Gamma_2}^{\frac{ds}{\nabla x}}\right) = 6q\partial_a\chi(a, b) \int_{-\infty}^{b} \frac{\Psi_1\left(q \left(\chi - \tilde{b}^2\right)\right)}{\left(1 + 3\Psi_1^2\left(q \left(\chi - 4\tilde{b}^2\right)\right)\right)} \, d\tilde{b} + O\left(|\chi|^{-\frac{1}{6} + \frac{1}{12}}(T-t)^{-\frac{1}{12}}\right)
\]

From this identity, (4.78), (4.79) and (4.71) we have proved that:
\[
\partial_a\mathcal{Y} = \partial_a\mathcal{Y}\Theta + O\left(|\chi|^{-\frac{1}{6} + \frac{1}{12}}(T-t)^{-\frac{1}{12}}\right),
\]
which is the second identity in (4.73) that we had to show. We now turn to the partial derivative with respect to \(b\). From (4.78) and (4.84) and then injecting (4.87), (4.71), (4.72) and (4.70):
\[
(T-t)^{\frac{1}{2}}C_{\partial_b}\left(\int_{\Gamma_1}^{\frac{ds}{\nabla x}} + \int_{\Gamma_2}^{\frac{ds}{\nabla x}}\right) = \partial_b\left(\int_{-\infty}^{b_{1\left(a,b\right)}} \frac{db}{\partial_a\chi(a, b)} + \int_{a_1}^{a} \frac{d\tilde{a}}{\partial_{a}\chi(\tilde{a}, \tilde{b}(X, \tilde{a}))}\right)
\]
\[= \partial_b\chi(a, b) \left\{ \int_{-\infty}^{b} \frac{\Psi_1\left(q \left(\chi - \tilde{b}^2\right)\right)}{\left(1 + 3\Psi_1^2\left(q \left(\chi - 4\tilde{b}^2\right)\right)\right)} \, d\tilde{b} + \frac{1}{\left(1 + 3q^2a^2\right)b} + O\left(|\chi|^{-\frac{1}{6} + \frac{1}{12}}(T-t)^{-\frac{1}{12}}\right)\right\}
\]
which was the last estimate (4.74) we had to show. We claim that the computations we performed for this right side of the self-similar zone can be adapted in a straightforward way in the case where one has to consider more parts of the curve \(\Gamma\) inside \(Z_0\) to parametrise: the integral over \(\Gamma_3\), \(\Gamma_4\) and \(\Gamma_5\) can be treated the very same way, leading to the same result. For the left side of the self-similar zone, there is only one parametrisation, making this case even easier.

**Step 7 The displacement line, proof of (1.24).** Let \(\Gamma_{\text{out}}(X_{\text{out}}, Y_{\text{out}})\) be the part of the curve \(\Gamma\) joining the boundary of the upper half plane and the point \((a_{\text{out}}, b_{\text{out}})\) defined in Lemma 11. For \(|x| \leq \epsilon\)
we define:
\[
y^*(x) = \int_{\Gamma(x_{out},y_{out})} ds \frac{1}{|\nabla x|}.
\]  
(4.88)  
\[\text{def: } y^*\]

Using the computations done in Step 5, the analogue of (4.83) adapts, namely:
\[
y^*(x) = \frac{2y^{\Theta}(a_{out}, b_{out})}{(T-t)^{\frac{1}{2}}} \left( 1 + O \left( (T-t)^{\frac{1}{4}} + |\mathcal{X}|^{\frac{1}{4}}(T-t)^{\frac{1}{4}} \right) \right).
\]

By definition, as \(b_{out} = \delta(T-t)^{-3/4}\):
\[
y^{\Theta}(a_{out}, b_{out}) = \int_{-\infty}^{\delta(T-t)^{-3/4}} \frac{db}{1 + 3\Psi_1^2 \left( q \left( \mathcal{X} - 4b^2 \right) \right)}
= 2y^*(\mathcal{X}) + O((T-t)^{\frac{1}{2}}),
\]
as for \(b \geq \delta(T-t)^{-3/4}\) there holds \(|\mathcal{X}| \ll b\) for \(\epsilon\) small enough, so that \(\Psi_1^2 \left( q \left( \mathcal{X} - 4b^2 \right) \right) \approx b^{4/3}\).
Therefore:
\[
y^*(x) = \frac{2y^*(x)}{(T-t)^{\frac{1}{2}}} \left( 1 + O \left( (T-t)^{\frac{1}{4}} + |x|^{\frac{1}{4}} \right) \right).
\]

This shows (1.24).

**Step 8 Reconnection functions.** The reconnection functions \(f\) and \(g\) are defined the following way. Fix \(t = T\), and consider the curve \(\Gamma := \{x(T, X, Y) = x^*(T)\}\). We split it in two parts. \(\Gamma_b\) is the bottom part which goes from the boundary of the upper half plane to \((X_0, Y_0)\), and \(\Gamma_t\) which is the top part from \((X_0, Y_0)\) and beyond. We change variables:
\[
\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -u_X \frac{x_{yy}}{2^{\frac{3}{2}}} - u_X \frac{\sqrt{x_{xx}x_{yy}}}{2^{\frac{3}{2}}} \\ \frac{\sqrt{x_{xx}x_{yy}}}{2^{\frac{3}{2}}} \end{pmatrix} \begin{pmatrix} X - X_0 \\ Y - Y_0 \end{pmatrix},
\]
so that from the Taylor expansion of \(x\) (4.29), for \(|A|, |B| \ll 1\):
\[
x(T, X, Y) = x^*(T) + q^2 A^3 + 4B^2 + O(|A|^4 + |A||B|^2 + |B|^3)
\]
which extends naturally to higher order derivatives. Therefore, near \((X_0, Y_0)\), on \(\Gamma\) one has:
\[
A = -\frac{4}{q^3} B^{\frac{3}{2}} \left( 1 + O(|B|^{\frac{2}{3}}) \right).
\]  
(4.89)  
\[\text{id:A:bottomreconnect}\]

Denote by \((X_0^*, 0)\) the point of the boundary such that \(x(T, X_0^*, 0) = x^*(T)\), \((X_{in}, Y_{in})\) that for which \(B = -\delta\), and \((X_{out}, Y_{out})\) that for which \(B = \delta\). On \(\Gamma_B\) for \(|B| \ll 1\) and \(B < 0\) we split:
\[
y = \int_{\Gamma(x_1, y_1)} ds \frac{1}{|\nabla x(T)|} + C \int_{-\delta}^{B} \frac{d\tilde{B}}{\partial A x}
= O(1) + \frac{1}{C3(4q)^{\frac{3}{2}}} \int_{-\delta}^{B} (1 + O(|\tilde{B}|^{\frac{2}{3}})) d\tilde{B} \frac{1}{\tilde{B}^{\frac{3}{2}}}
= O(1) - \frac{1}{C(4q)^{\frac{3}{2}}} \frac{1}{B^{\frac{3}{2}}}.
\]
This shows that \( y \to \infty \) as \((X,Y)\) approaches \((X_0,Y_0)\) on \(\Gamma_b\). Given any \(y \geq 0\), there exists a unique point \(\phi(y)\) on \(\Gamma_b\) between the boundary and \((X_0,Y_0)\) such that \(y(X,Y) = y\) and we thus define:

\[
f(y) = u(T,\phi(y)).
\]

From the above expansion for \(y\) one obtains that

\[
B(\phi(y)) \sim -\frac{1}{y^3C^3(4q)^2} \quad \text{as} \quad y \to \infty,
\]

which, from (4.41) and (4.89) gives:

\[
f(y) - u^* = u(T,\phi(y)) - u^* \sim \frac{1}{y^2(2qC)^2} = \frac{1}{y^2} \frac{\mu^2}{p^2} \quad \text{as} \quad y \to \infty.
\]

On \(\Gamma_t\) now, we define:

\[
\tilde{y}(T,X,Y) = \int_{(X_{out},Y_{out})}^{(X,Y)} \frac{ds}{|\nabla x(T)|}
\]

where this integral is taken with positive sign if \((X,Y)\) is after \((A_{out},B_{out})\) on \(\Gamma\), and with negative sign if it is between \((A_{out},B_{out})\) and \((X_0,Y_0)\). It follows from (4.89) that:

\[
\tilde{y}(A_{out},B_{out}) = 0, \quad \lim_{|(X,Y)| \to \infty} \tilde{y}(A_{out},B_{out}) = \infty, \quad \tilde{y} \sim -\frac{1}{C(4q)^2} \frac{1}{B^2} \quad \text{as} \quad B \to 0^+.
\]

Hence \((X,Y) \mapsto \tilde{y}\) defines a \(C^3\) diffeomorphism between \(\Gamma_t\) and \(\mathbb{R}\). Given any \(\tilde{y} \in \mathbb{R}\), there exists a unique point \(\tilde{\phi}(\tilde{y})\) on \(\Gamma_t\) such that \(y(X,Y) = y\) and we thus define:

\[
g(\tilde{y}) = u(T,\tilde{\phi}(\tilde{y})). \tag{4.90}
\]

From (4.41) and (4.89) one obtains:

\[
g(\tilde{y}) - u^* \sim \frac{1}{y^2(2qC)^2} = \frac{1}{y^2} \frac{\mu^2}{p^2} \quad \text{as} \quad y \to -\infty.
\]

**Step 9** Reconnection: proof of (1.30), (1.31), (1.28) and (1.29) For \(K > 0\) and let \((t,x) \to (T,x^*)\), the characteristics map \((X,Y) \mapsto (x,y)\) still defines a diffeomorphism when restricted to the preimage of \(y < K\), since it avoids a size one zone near \((X_0,Y_0)\). Hence the convergence of \(u\) to \(f\) for \(y < K\) follows from a direct continuity argument. This shows the reconnection at the bottom (1.30).

We now turn to the reconnection at the top. For any \(K > 0\), the zone \(y^*-K \leq y \leq y^*+K\) corresponds in Lagrangian variable to a zone which stays at a distance 1 away from \((X_0,Y_0)\), and hence where the parametrisation of the curves \(\Gamma := \{x(X,Y) = Cte\}\) and \(\nabla x\) remain uniformly regular as \(t \to T\). From this fact and the definition (4.88) of \(y^*\), as \((x,t) \to (T,x^*)\), the inverse of the characteristics map \((x,y) \mapsto (X,Y)\) is such that:

\[
(X,Y)(x,y) \to \tilde{\phi}(y - y^*)
\]

where \(\tilde{\phi}\) is defined by (4.90). Hence from (4.90) one has \(u(t,X,Y) \to g(y - y^*(x))\) in this zone which proves (1.31).

We now turn to the convergence below the displacement line. Let \(K \gg 1\) be large, and then \(0 < \kappa, \epsilon \ll 1\) be small, and consider the zone:

\[
|x - x^*| \leq \epsilon, \quad y^*(x)(1 - \kappa) \leq y \leq y^*(x) - K,
\]
corresponding to
\[ |\mathcal{X}| \leq c(T - t)^{-\frac{3}{2}}, \quad C(T - t)^{\frac{1}{3}}y^*(x)(1 - \kappa) \leq \mathcal{Y} \leq C(T - t)^{\frac{1}{4}}(y^*(x) - K), \] (4.91)

For \( |\mathcal{X}| \leq c(T - t)^{-3/2} \) define \( \overline{b} \) the following way:
\[ \overline{b} = M(1 + |\mathcal{X}|^{\frac{1}{2}}) \]
where \( M \gg 1 \) will be chosen later on. One has for \( \epsilon \) small enough depending on \( \delta \) that \( |\overline{b}| \gg |\mathcal{X}|^{1/2} \). Fix \( \overline{b} = \overline{b}_0 \) we now look for the solution to \( \mathcal{X}(\overline{a}, \overline{b}) = \mathcal{X} \). In the zone \( a < 0 \) with \( |a| \approx |\overline{b}|^{2/3} \) one is in \( Z_1 \) and therefore from (4.20) one has \( \partial_a \mathcal{X} \approx |\overline{b}|^{4/3} \). Hence, as \( |\mathcal{X}| \ll |b|^2 \) this condition on the derivative indeed implies that there exists \( \overline{a} < 0 \) with \( |\overline{a}| \approx |\overline{b}|^{2/3} \) such that \( \mathcal{X}(\overline{a}, \overline{b}) = \mathcal{X} \). From Lemma 11, for any \( |\mathcal{X}| \leq c(T - t)^{-3/2} \), inside \( Z_0 \) the curve \( \Gamma \), before leaving by the point \((a_{out}, b_{out})\), is in \( Z_1 \) where it can be parametrised with the variable \( \overline{b} \) as \( \overline{a} = \overline{a}(\mathcal{X}, \overline{b}) \). Either in case (i) of Lemma 11 it always stayed in \( Z_1 \), or in case (ii) before that stage the curve was in \( Z_2 \).

As \((\overline{a}, \overline{b})\) belongs to \( Z_1 \), this means that in any case, one can parametrise with the variable \( \overline{b} \) the last part of \( \Gamma \) before exiting \( Z_0 \), as a curve \( \overline{a} = \overline{a}(\mathcal{X}, \overline{b}) \) for \( \overline{b} \leq \overline{b} \leq \delta(T - t)^{-3/4} \). For \((a, b)\) along this curve there holds by definition of \( y^* \):
\[ \mathcal{X}(a, b) = \mathcal{X}, \quad \mathcal{Y}(a, b) = C(T - t)^{\frac{1}{4}}y^*(x) - \int_b^{\frac{\mathcal{X}}{b}} \frac{d\overline{b}}{\partial_a \mathcal{X}(\overline{a}(\mathcal{X}, b), \overline{b})}. \]

As this curve lies in \( Z_1 \) from (4.20), and from the inequality \( |\mathcal{X}| \ll |\overline{b}| \) for \( b \leq \overline{b} \leq \delta(T - t)^{-3/4} \):
\[ \mathcal{Y}(a, b) = C(T - t)^{\frac{1}{4}}y^*(x) - \int_b^{\frac{\mathcal{X}}{b}} \frac{d\overline{b}}{1 + 3\Psi_1^2 \left(q(\mathcal{X} - 4\overline{b})^2\right)} \left(1 + O\left(|\overline{b}|^{\frac{1}{4}}(T - t)^{\frac{1}{12}}\right)\right). \]

An expansion at infinity of \( \Psi_1 \) gives that:
\[ \frac{1}{1 + 3\Psi_1^2 \left(q(\mathcal{X} - 4\overline{b})^2\right)} \left(1 + O\left(|\overline{b}|^{\frac{1}{4}}(T - t)^{\frac{1}{12}}\right)\right) = \frac{1}{(4q)^{\frac{1}{4}}\overline{b}^{\frac{1}{4}}} + O\left(|\overline{b}|^{-\frac{1}{4}}\frac{|\mathcal{X}||\overline{b}|^{-2}}{|\overline{b}|^{\frac{1}{4}}\overline{b}^{\frac{1}{4}}}\right), \]
so that:
\[ \mathcal{Y}(a, b) = C(T - t)^{\frac{1}{4}}y^*(t, x) - \frac{1}{(4q)^{\frac{1}{4}}\overline{b}^{\frac{1}{4}}} + O\left(|b|^{-\frac{1}{4}}\overline{b}^{\frac{1}{4}}(T - t)^{\frac{1}{12}} + |b|^{-\frac{1}{4}}|\mathcal{X}|\right). \]

We write \( \mathcal{Y} = C(T - t)^{1/4}y^*(t, x) - \Delta \mathcal{Y} \), and from (4.91) and (1.24):
\[ C(T - t)^{\frac{1}{4}}K \leq \Delta \mathcal{Y} \leq \frac{C \kappa}{1 + |\mathcal{X}|^{\frac{1}{4}}}. \]

Solving \( \mathcal{Y}(a, b) = \mathcal{Y} \) then amount to solve:
\[ -\frac{1}{(4q)^{\frac{1}{4}}\overline{b}^{\frac{1}{4}}} + O\left(|b|^{-\frac{1}{4}}\overline{b}^{\frac{1}{4}}(T - t)^{\frac{1}{12}} + |b|^{-\frac{1}{4}}|\mathcal{X}|\right) = -\Delta \mathcal{Y}. \]

For \( b = \delta(T - t)^{-3/4} \) one has \( \frac{1}{\overline{b}^{\frac{1}{4}}} \geq (T - t)^{1/4}\delta^{-1/3} \gg (T - t)^{\frac{1}{12}}K \) for \( \delta \) small enough, and for \( b = \overline{b} \) one has \( \frac{1}{\overline{b}^{\frac{1}{4}}} \geq M^{-1/3}(1 + |\mathcal{X}|^{1/6})^{-1} \gg \kappa(1 + |\mathcal{X}|^{1/6})^{-1} \) for \( M \) large enough. Hence, there exists
Indeed a solution to the above equation by the intermediate value theorem, and bootstrapping information from the above equation and using that $|\mathcal{X}| \approx |b|^2$ one obtains:

$$b = \frac{1}{(4q)^2C^3(y^*(t,x) - y)^3(T-t)\frac{3}{4}} \left(1 + O \left( \frac{1}{(y^*(x) - y)^{\frac{1}{2}}} \right) \right).$$

(4.92)

Bootstrapping information from the equation (4.18) gives finally:

$$a = -\frac{4\frac{3}{4}}{q^2} b^\frac{3}{4} \left(1 + O \left( \frac{|\mathcal{X}|}{b^{\frac{1}{2}}} + (T-t)^{\frac{1}{12}} |b|^{\frac{1}{3}} \right) \right)
\begin{align*}
= -\frac{1}{4C^2q^2(y^*(t,x) - y)^2(T-t)^{\frac{3}{4}}} \left(1 + O \left( \frac{|y - y^*(t,x)|}{y^*} + \frac{1}{(y - y^*(t,x))^{\frac{1}{2}}} \right) \right).
\end{align*}

And hence from (4.41):

$$u(t,x,y) = \frac{1}{4q^2C^2A(y^*(t,x) - y)^2} \left(1 + O \left( \frac{|y - y^*|}{y^*} + \frac{1}{(y - y^*)^{\frac{1}{2}}} \right) \right).$$

The estimates for the derivative proved in Step 5 adapt, namely at the point $(a,b)$ such that $(x,y)(a,b) = (x,y)$, using (4.92):

$$\partial_a \mathcal{X} = 1 + 3q^2a^2 + O(|b|^\frac{3}{4}(T-t)^\frac{1}{2}) = \partial_a \mathcal{X}^{\Theta} + O \left( \frac{(y^* - y)}{(y^* - y)^4(T-t)} \right),$$

$$\partial_b \mathcal{X} = 8b + O((T-t)^\frac{3}{4}|b|^{\frac{1}{2}}) = \partial_b \mathcal{X}^{\Theta} + O \left( \frac{(y^* - y)^{-1}}{(y^* - y)^3(T-t)^{\frac{3}{4}}} \right),$$

$$\partial_a \mathcal{Y} = \partial_a^{\Theta} \mathcal{Y} + O((T-t)^{\frac{1}{12}} |b|^{-1 + \frac{1}{8}}) = \partial_a^{\Theta} \mathcal{Y} + O \left( \frac{(y^* - y)^{3-\frac{1}{8}}(T-t)^{\frac{2}{3}}} \right),$$

$$\partial_b \mathcal{Y} = \partial_b \mathcal{Y}^{\Theta} + O \left( |1 + |\mathcal{X}|^{\frac{7}{6}} |b|^{1 + \frac{1}{8}}(T-t)^{\frac{1}{12}} \right) = \partial_b^{\Theta} \mathcal{Y} + O \left( (y^* - y)^{4-\frac{1}{8}}(T-t) \right),$$

so that from (4.42), (4.43), (4.33), (4.34) and the above estimates:

$$\partial \mathcal{X} u = \partial_a u \partial_b \mathcal{Y} + \partial_b u \partial_a \mathcal{Y} = -(T-t)^{\frac{3}{4}} \partial_a \mathcal{X}^{\Theta} + O \left( (T-t)^{\frac{3}{4}}((T-t)^{\frac{3}{4}} + |x - x^*|^{-\frac{7}{6}} |y^* - y|^{3-\frac{1}{4}}) \right),$$

$$\partial \mathcal{Y} u = -\partial_a u \partial_b \mathcal{X} + \partial_b u \partial_a \mathcal{X} = (T-t)^{\frac{3}{4}} \partial_b \mathcal{X}^{\Theta} + O \left( \frac{(y^* - y)}{(y^* - y)^3(T-t)^{\frac{3}{4}}} \right),$$

which completes the proof of (1.29).

\[\square\]

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