Blazhko RR Lyrae light curves as modulated signals

J. M. Benkő, R. Szabó and M. Paparó

Konkoly Observatory, Konkoly Thege M. u. 15-17, H-1121 Budapest, Hungary

Accepted 2011 June 23. Received 2011 June 23; in original form 2011 April 28

ABSTRACT
We present an analytical formalism for the description of Blazhko RR Lyrae light curves. In this formalism the amplitude and frequency modulations are treated in a manner similar to the theory of electronic signal transmission. We consider monoperiodic RR Lyrae light curves to be carrier waves, and modulate their amplitude (AM), frequency (FM) and phase (PM); as a general case we discuss simultaneous AM and FM. The main advantages of this method are the following: (i) the mathematical formalism naturally explains numerous light-curve characteristics found in the Blazhko RR Lyrae stars such as mean brightness variations, complicated envelope curves and non-sinusoidal frequency variations; (ii) our elucidation also explains the properties of the Fourier spectra such as apparent higher order multiplets, amplitude distribution of the sidepeaks, the appearance of the modulation frequency itself and its harmonics. In addition, compared to the traditional methods, our light-curve solutions reduce the number of necessary parameters. This formalism can be applied to any type of modulated light curves, not just to Blazhko RR Lyrae star light curves.

Key words: methods: analytical – methods: data analysis – stars: oscillations – stars: variables: general – stars: variables: RR Lyrae.

1 INTRODUCTION
The Blazhko effect (Blazhko 1907) is a periodic amplitude and phase variation in the light curves of RR Lyrae variable stars. The typical cycle length of these variations is about 10–100 times the main pulsation period (0.3–0.7 d). Almost half of the RR Lyrae stars pulsating in their fundamental mode (type RRab) and a smaller but non-negligible fraction of the first overtone mode pulsating stars (type RRc) show this effect (Chadid et al. 2009; Jurcsik et al. 2009c; Benkő et al. 2010; Kolenberg et al. 2010). It is usually interpreted either as a modulation phenomenon or as a beating phenomenon, but both of these hypotheses have their own problems. The beating picture describes the main feature of the light curves and the Fourier spectra well (see Breger & Kolenberg 2006; Kolenberg et al. 2006), but in this framework it is not possible to reproduce phase variations or determine multiplet structures found in the Fourier spectra of certain stars (Jurcsik et al. 2008; Chadid et al. 2010). On the other hand, stars showing doublet structures in their Fourier spectra (Alcock et al. 2000, 2003; Moskalik & Poretti 2003) seemed to contradict the modulation picture.

In this paper we adopt the modulation hypothesis for the Blazhko effect, and derive the mathematical consequences of this assumption by developing a consistent analytical framework. Using this framework we demonstrate that many light-curve characteristics are identified out of the mathematical consequences of the modulation assumption. By disentangling these features we get closer to the physics of the Blazhko effect.

The question of the possibility of the modulation/Blazhko effect has been raised in regard to many types of pulsating stars, from Cepheids to δ Scuti stars (see, e.g. Koen 2001; Henry, Fekel & Henry 2005; Moskalik & Kołaczkowski 2009; Breger 2010; Poretti et al. 2011). The main objective of this paper is to derive a mathematical description of the Blazhko RR Lyrae star light curves and to investigate their properties. Most of our results can be applied directly to any other type of variable stars where modulation is suspected to be in operation. Our deduced formulae and the related phenomena may help one to either prove or reject the modulation hypothesis.

The basic idea of this paper is presented in Benkő et al. (2009). Modulation is a technique that has been used in electronic communications for a long time, mostly for transmitting information signals via a radio carrier wave. In those cases, the carrier wave is a sinusoidal electromagnetic (radio) wave that is modulated by a generally non-periodic information signal (e.g. speech, music). In this work the formalism developed by engineers for broadcasting radio signals has been modified in a way that has made it possible for us to assume a monoperiodic non-modulated RR Lyrae light variation as a carrier wave. While for communications usually only one type of modulation is applied at a time, we allow two types of modulations (amplitude and angle) to happen together.

In Section 2 we present a collection of classical formulae that are well known in the physics of telecommunications (Carson 1922; van der Pol 1930; Roder 1931). Some of the more complicated cases
is defined as
\[
\sin(2\theta_1 + \tau_2) = \sin(\theta_1 + \tau_2) + \sin(\theta_1 - \tau_2).
\]

In this section we briefly review some classical definitions and formulae (see, e.g. Schottstaedt 2003; Newkirk & Karlquist 2004) that will be used through the following sections. The simplest periodic signal is a sinusoidal function. It has three basic parameters: amplitude, frequency and phase, and any of these can be modulated.

### 2.1 Amplitude modulation

Amplitude modulation (AM) is the simplest of the three cases. Let the carrier wave \( c(t) = U_c \sin(2\pi f_c t + \varphi_c) \),
where the constant parameters \( U_c, f_c \) and \( \varphi_c \) are the amplitude, frequency and initial phase of the carrier wave, respectively.

Let \( U_m(t) \) represent an arbitrary waveform that is the message to be transmitted. The transmitter uses the information signal \( U_m(t) \) to vary the amplitude of the carrier, \( U_c \), to produce a modulated signal:

\[
U_{AM}(t) = [U_c + U_m(t)] \sin(2\pi f_c t + \varphi_c).
\]

In the simplest case, where the modulation is also sinusoidal,
\[
U_m(t) = U_m^A \sin(2\pi f_m t + \varphi_m^A).
\]

Substituting the equation (3) into (2) and using basic trigonometrical identities, expression (2) can be rewritten as

\[
U_{AM}(t) = U_c \sin(2\pi f_c t + \varphi_c) \]
\[
+ \frac{U_m^A}{2} \left[ \sin(2\pi f_c - f_m t + \varphi^-) + \sin(2\pi (f_c + f_m) t + \varphi^+) \right],
\]

where \( \varphi^- = \varphi_c - \varphi_m^A + \pi/2 \) and \( \varphi^+ = \varphi_c + \varphi_m^A - \pi/2 \). The initial shifts (±\( \pi/2 \)) appear because throughout this paper we have used a sinusoidal representation instead of sin and cos functions.

The exact analytical Fourier transformation of (4) is given in Appendix A; however, the basic structure of the frequency spectrum can be easily approximated from equation (4). Since the Fourier spectrum of a single sinusoidal function shows a peak at the frequency of the sinusoid, from expression (4) the well-known triplet structure composed of the peaks \( f_c \) and \( f_c + f_m \) can be seen. The amplitudes of the sidepeaks \( f_c \pm f_m \) are always equal. The Fourier amplitude of the carrier wave \((\pi \sqrt{2} PU_c)\), which represents the energy at the carrier frequency, is constant.

The ratio of the carrier wave amplitude \( A(f_c) \) to the sidepeaks \( A(f_c \pm f_m) \) has a connection with the modulation depth. We rewrite equation (2) as

\[
U_{AM}(t) = \left[ 1 + \frac{U_m(t)}{U_c} \right] c(t).
\]

If \( U_m(t) \) is a bounded function, and if we let \( U_m^\text{max} \) represent the maximum value of this modulation function, then modulation depth is defined as \( h = U_m^\text{max}/U_c \). In the above-discussed sinusoidal case \( h = U_m^A/U_c \) and \( A(f_c \pm f_m)/A(f_c) = \frac{1}{2} h \). In other words, the amplitude of the central peak is twice as high as the sidepeaks.

### 2.2 Angle modulations

The phase and frequency modulations together are called angle modulations. Since we assume the sinusoidal carrier wave equation (1) to be \( c(t) = U_c \sin(\theta(t)) \), the \( \theta(t) = 2\pi f_c t + \varphi_c \) denotes the angle part of the function.

Phase modulation (PM) changes the phase angle of the carrier signal. Suppose that the modulating or message signal is \( U_m(t) \), then \( \theta(t) = 2\pi f_c t + [\varphi_c + U_m(t)] \). Let \( U_m(t) \) again be a bounded function. In this case we can define a constant as \( k_{PM} = |U_m^\text{max}(t)|/2 \). This transforms the modulated signal:

\[
U_{PM}(t) = U_c \sin \left[ 2\pi f_c t + k_{PM} U_m^P(t) + \varphi_c \right],
\]

where \( |U_m^P(t)| \leq 1 \). The instantaneous frequency of the modulated signal is

\[
f(t) = f_c + k_{PM} \frac{dU_m^P(t)}{dt}.
\]

Frequency modulation (FM) uses the modulation signal \( U_m(t) \) to vary the carrier frequency, \( \theta(t) = 2\pi f_c t + \varphi_c \) and here the instantaneous frequency \( f(t) \) is modulated by the signal of \( k_{FM} U_m^P(t) \) as

\[
f(t) = f_c + k_{FM} U_m^P(t).
\]

In this equation \( k_{FM} \) is the frequency deviation, which represents the maximum shift from \( f_c \) in one direction, assuming that \( U_m^P(t) \) is limited to the range \((-1, \ldots, +1)\). Using the definitions of the instantaneous frequency and phase, equation (8) can be rewritten as \( \theta(t) = 2\pi f_c t + 2\pi k_{FM} \int_0^t U_m^P(\tau) d\tau + \varphi_c \). The modulated signal is

\[
U_{FM}(t) = U_c \sin \left[ 2\pi f_c t + 2\pi k_{FM} \int_0^t U_m^P(\tau) d\tau + \varphi_c \right].
\]

This definition of FM is the least intuitive of the three equations (2, 6 and 9). If we compare equations (6) and (9) we realize that the modulation signals are in a derivative–integral connection with each other. In practice, we have modulating signals that can be represented by analytical functions. This is the reason why when we detect an FM or a PM signal without the knowledge of whether it is FM or PM, we find it impossible to identify it as an FM signal or a PM signal.

First, let the modulating signal be represented by a sinusoidal wave with a frequency \( f_m \). The integral of such a signal is

\[
U_m^P(t) = \frac{U_m^F}{2\pi f_m} \sin \left( 2\pi f_m t + \varphi_m^F \right).
\]

Thus, in this case equation (9) gives

\[
U_{PM}(t) = U_c \sin \left[ 2\pi f_c t + \eta \sin \left( 2\pi f_m t + \varphi_m^F + \varphi_c \right) \right],
\]

where the modulation index \( \eta \) is defined as \( \eta = (k_{PM} U_m^F) f_m \). Equation (11) can be deduced from equation (6) as well. The only difference is the value of \( \eta = k_{PM} U_m^F \) which is independent of the modulation frequency \( f_m \). Let us transcribe equation (11) using relations for trigonometrical and Bessel functions (Abramowitz & Stegun 1972):

\[
U_{PM}(t) = U_c \sum_{k=-\infty}^{\infty} J_k(\eta) \sin \left[ 2\pi (f_c + k f_m) t + k \varphi_m^F + \varphi_c \right],
\]

where \( J_k(\eta) \) is the Bessel function of the first kind with integer order \( k \) for the value of \( \eta \) (Fig. 1), and \( \varphi_m^F \) denotes either \( \varphi_m^F \) or \( \varphi_m^P \). This formula is known as Chowning relation (Chowning 1973). Although it was deduced previously by many authors, Chowning
first recognized its key role in the electronic sound-creation method called FM synthesis.

Similarly to equation (4), expression (12) also helps us to conceptualize the spectrum. It is made up of a carrier at \( f_c \), and symmetrically placed sidepeaks separated by \( f_m \). The amplitudes follow the Bessel functions. The behaviour of the Bessel functions is well known: except for small arguments \((x < k)\), they behave like damped sine functions (see also, Fig. 1). For higher indices the higher order sidepeaks gradually become more and more important. As a consequence, the amplitude of the central peak gets reduced. The frequency spectrum of an actual FM signal has an infinite number of sidepeak components, though they become negligibly small beyond a point.

If \( |\eta| \ll 1 \), we find that \( J_0(\eta) \approx 1, |J_{\pm 1}| = \eta/2 \) and \( J_2 \approx 0 \) for \( k > 1 \). That is, the spectrum can be approximated by an equidistant triplet similarly to AM, but the characteristics of the signal differ from those of AM: the total amplitude of the modulated wave remains constant. When \( \eta \) increases, the amplitude of the sidepeaks also increases, but the Fourier amplitude of the carrier remains the same. In other words, the sidepeaks could be larger than the central peak; on the other hand, higher order side frequencies could also be of a larger amplitude than the lower order ones.

A more general case is formulated by Schottstaedt (1977):

\[
U_{\text{FM}}(t) = U_c \sin \left[ 2\pi f_c t + \sum_{p=1}^{q} U_m^{(p)} \sin \left( 2\pi f_m^{(p)} t + \varphi_m^{(p)} \right) + \varphi_c \right]
\]

\[
= U_c \sum_{k_p = -\infty}^{\infty} \sum_{k_m = -\infty}^{\infty} \prod_{p=1}^{q} J_{k_p} \left( U_m^{(p)} \right) \sin \left[ 2\pi f_c t + \sum_{p=1}^{q} k_p \left( 2\pi f_m^{(p)} t + \varphi_m^{(p)} \right) + \varphi_c \right].
\]  

(13)

This expression results in an infinite number of amplitude-modulated waves. After trigonometrical transformations, we get

\[
U_{\text{Comb}}(t) = U_c \left( 1 + h \sin \left( 2\pi f_c t + k \phi_m \right) \right)
\]

\[
\times \sum_{k=-\infty}^{\infty} J_k(\eta) \sin \left[ 2\pi \left( f_c + k f_m \right) t + k \phi_m \right].
\]  

(15)

It can be seen that each term consists of three sinusoidal functions with different phases. On the basis of expression (16), the spectrum of combined modulation (14) is comprehensible as a combination of three FM spectra. The peaks are at the same places as the frequencies of the spectrum of (12), but the amplitudes of a pair of sidepeaks are generally asymmetrical. Using some trigonometrical identities, the rules of summation of parallel harmonic oscillations and the relations for Bessel functions, we arrive at the expression for the Fourier amplitudes of a certain frequency:

\[
A(f_c + k f_m) \sim U_c \left\{ J_k^2(\eta) \left( 1 - \frac{hk}{\eta} \sin \phi_m \right)^2 \right. 
\]

\[
+ \frac{h^2}{4} \cos^2 \phi_m \left[ J_{k+1}(\eta) - J_{k-1}(\eta) \right]^2 \left. \right\}^{1/2},
\]

(17)

\( k = 0, \pm 1, \pm 2, \ldots \). Introducing the power difference of the sidepeaks as it was done by Szeidl & Jurcsik (2009), \( \Delta_1 : A^2(f_c + kf_m) - A^2(f_c - kf_m) \), where \( l = 1, 2, 3, \ldots \), and taking into account formula (17), we get

\[
\Delta_1 = -4 \frac{hl}{\eta} U_c^2 J_1^2(\eta) \sin \phi_m.
\]  

(18)

This formula is a direct generalization of the formulae given by Szeidl & Jurcsik (2009) for \( l = 1 \) and 2. It is evident that this asymmetry parameter depends only on \( \phi_m \), the relative phase of AM and FM. The left-hand side peaks are higher than the right-hand side ones (\( \Delta_1 < 0 \)) if \( 0 < \phi_m < \pi \), otherwise the situation is

---

**2.3 Combined modulation**

In practice, the electronic circuits that generate modulated signals generally produce a mixture of amplitude and angle modulations. This combined modulation is not preferred in radio techniques but welcomed in sound synthesis and, as we will see, they appear in the case of Blazhko RR Lyrae stars as well. Let us have an overview of the basic phenomena of combined modulations following Cartianu (1966). We start with the simplest case – both AM and FM are sinusoidal and their frequencies are the same:

\[
U_{\text{Comb}}(t) = U_c \left( 1 + h \sin \left( 2\pi f_m t \right) \sin \left( 2\pi f_c t \right) \right)
\]

\[+ \eta \sin (2\pi f_m t + \phi_m + \varphi_c).\]

(14)

By a suitable choice of the starting epoch, without any restriction on the general validity, we can set \( \varphi_c = 0 \). Here \( \phi_m \) is the relative phase difference between the modulating FM and AM signals. Other designations are the same as before. The third term of the product (14) is the same as in equation (11), therefore, after applying the Chowning relation (12),

\[
U_{\text{Comb}}(t) = U_c \left( 1 + h \sin \left( 2\pi f_m t \right) \right)
\]

\[\times \sum_{k=-\infty}^{\infty} J_k(\eta) \sin \left[ 2\pi \left( f_c + k f_m \right) t + k \phi_m \right].
\]  

(15)

---

**Figure 1.** The graph of the first three and the 10th Bessel functions of the first kind with integer order \( k \) for the value of \( x \). \( J_0(x) \) – (red) continuous line; \( J_1(x) \) – (green) dashed line; \( J_2(x) \) – (blue) dotted line; \( J_{10}(x) \) – (purple) dash–dotted line.
opposite: $\pi < \phi_m < 2\pi$ and $\Delta \eta > 0$. In those very special cases, where $\phi_m = 0$ or $\phi_m = \pi$, the sidepeaks' amplitudes are equal. Note that even if one of the modulations – AM or FM – dominates in the combined case ($\eta < \phi$ or $\eta \ll 1$), the amplitudes of the sidepeaks remain almost the same.

3 BLAZHKO MODULATION

RR Lyrae light curves are traditionally described by a Fourier series of a limited number of terms. In Blazhko-modulated RR Lyrae stars, the sum builds up from the terms of harmonics of the main pulsation frequency and the sidepeaks due to the modulation and the modulation frequency and even its harmonics:

$$m(t) = A_0 + \sum_{i=1}^{N} A_i \sin[2\pi F_i t + \Phi_i],$$  

(19)

where $F_i = jf_0$ ($j = 1, 2, \ldots, n$) or $F_i = kf_m$ ($k = 1, 2, \ldots, m$) or $F_i = jf_0 + kf_m$ ($j = 1, 2, \ldots, n', k = 1, 2, \ldots, m'$); $F_i = j^*f_0 - k^*f_m$ ($j^* = 1, 2, \ldots, n^*$, $k^* = 1, 2, \ldots, m^*$); $f_0$ and $f_m$ are the main pulsation frequency and the modulation one, respectively. The amplitudes $A_i$ and phases $\Phi_i$ are considered as independent quantities and are determined by a non-linear fit. The necessary number of parameters for a complete light-curve solution is $2N + 3$ (amplitudes and phases and two frequencies and the zero-point $A_0$). The number of parameters can be as large as 500–600 for a long time series of good quality (see e.g. Chadid et al. 2010).

In the following subsections we show how the modulation paradigm can be applied and what advantages it offers compared to the traditional handling (equation 19).

3.1 Blazhko stars with AM

To start with, we discuss Blazhko star light curves with the pure AM effect, although the recent space-borne data suggest that all the Blazhko RR Lyrae stars show amplitude modulation and simultaneous period changes (Chadid et al. 2010; Benkő et al. 2010; Poretti et al. 2010). We follow a step-by-step generalization process that allows us to separate the effects more clearly. We note that the most striking feature of a Blazhko RR Lyrae light curve is the amplitude variation, which is generally easy to find and is in many cases the only detectable modulation (see Stothers 2010 and references therein).

To apply the framework described in Section 2.1 to an RR Lyrae light curve, the course-book formule need some extensions. We choose a continuous, infinite, periodic function with a non-modulated RR Lyrae shape as a ‘carrier wave’. This function is described by the frequency $f_0$ and its harmonics, that is $c^*(t) := m(t)$ if $F_i = jf_0$ in equation (19).

Although the exact analytical Fourier spectra of any of the modulated signals discussed in this paper can be calculated without any problems, at least in theory (see also Appendix A), to illustrate the different formulæ synthetic light curves and their Fourier spectra were also generated and plotted. An artificial light curve was constructed as a carrier wave with typical RR Lyrae parameters ($f_0 = 24$ d$^{-1}$ and its nine harmonics) on a 100 d long time-span sampled by 5 min (insert in Fig. 2). The Fourier transform of such a signal is well known (Fig. 2): it consists of the transformed sinusoidal components given in equation (A2). (More precisely, due to the finite length of the data set and its sampling, the Fourier transformations should always be multiplied by the Fourier transform of the appropriate window function.)

![Figure 2. Fourier amplitude spectrum of the artificial RR Lyrae light curve acting as a carrier wave of all the modulated light curves constructed in this paper (main panel), and a part of the light curve itself (insert).](https://academic.oup.com/mnras/article-abstract/417/2/974/983753)

Substituting the $c^*(t)$ carrier wave into the definition of AM in equation (5),

$$m_{AM}^*(t) = \left[1 + \frac{U_m^*(t)}{U_c^*}\right] c^*(t)$$

$$= \left[1 + m_m^*(t)\right] \left[a_0 + \sum_{j=1}^{\pi} a_j \sin(2\pi j f_0 t + \phi_j)\right].$$  

(20)

Expression (20) describes a general amplitude-modulated RR Lyrae light curve, $U_m^*(t)$ is the modulation signal, $U_c^*$ is the amplitude of the non-modulated light curve. On the one hand, the non-zero constant term of $a_0$ is obligatory from the mathematical point of view, otherwise the Fourier sum would not comprise a complete set of functions. On the other hand, this value represents the difference between the magnitude and intensity means. More precisely, either we use physical quantities (viz. positive definite fluxes) or we transform normalized fluxes into a magnitude scale. In this latter case the average of the transformed light curve differs from zero. In this paper, for traditional purposes we use the second approach. For RR Lyrae stars the typical value of this difference is about one hundredths of a magnitude ($a_0 \leq 1$). It is evident that this constant differs from the zero-point of the light curve $A_0$ given in the apparent magnitude scale.

3.1.1 Sinusoidal amplitude modulation

In the simplest case the modulation is sinusoidal:

$$U_m^*(t) = a_m \sin(2\pi f_m t + \phi_m).$$  

(21)

Sample light curves obtained with this assumption from equation (20) are shown in Fig. 3. Introducing the modulation depth as $\eta = a_m/U_c^*$, the parameters were chosen as $a_0 \leq U_c^*$ and $a_m \leq a_0$, resulting in modulations symmetrical to an averaged value, viz. a horizontal line (left-hand panels). The right-hand panels show cases with higher modulation depths ($a_m > a_0$), where this symmetry is broken. A common feature of these light curves is that the maxima and minima of the envelope curves coincide in time. Furthermore, the average brightness of all light curves varies with $f_m$. It can be seen directly from equation (20): the $m_{AM}^*(t)$ term is responsible for this behaviour. That is, the mean brightness ($\bar{V}$) variations found to occur during the Blazhko cycle ( Jurcsik et al. 2005) are a natural consequence of the AM.
Figure 3. Artificial light curves with a sinusoidal AM computed with the formula equation (20). Left-hand panels show symmetrical modulation \((a_m < a_0); a_0 = 0.2\), the right-hand ones are asymmetrical \((a_m > a_0); a_0 = 0.005\). The modulation depth \(h\) is increasing from the top to bottom as \(h = 0.1, 0.2, 0.4, f_m = 0.05\) d\(^{-1}\) and \(\varphi_m = 270^\circ\) are fixed.

Figure 4. Bottom panel: artificial light curve with a sinusoidal AM computed with the formula (20). The modulation depth is \(h = 1.2\). Other parameters are \(a_0 = 0.01, f_m = 0.05\) d\(^{-1}\) and \(\varphi_m = 270^\circ\). Top panels: 2-d zooms around a maximum (top left) and a minimum (top right) of the modulation cycle.

There is a fascinating case when the modulation is very strong, i.e. when the modulation depth is \(h > 1\). Besides the strong light-curve changes (Fig. 4), in some Blazhko phases the shape of the light curve looks very unfamiliar (see top-right panel in Fig. 4). The relevance of this mathematical case is corroborated by the Kepler observation of V445 Lyr which shows similar characteristics (fig. 2 in Benkő et al. 2010).

Using some trigonometrical relations, equation (20) with (21) can be converted to a handy sinusoidal decomposition form from where the Fourier spectrum can be seen easily:

\[
m_{AM}(t) = a_0 + h a_0 \sin(2\pi f_0 t + \varphi_m) + \sum_{j=1}^{n} a_j \sin(2\pi j f_0 t + \varphi_j) + \frac{h}{2} \sum_{j=1}^{n} a_j \left\{ \sin \left[ 2\pi (j f_0 - f_m) t + \varphi_j^+ \right] + \sin \left[ 2\pi (j f_0 + f_m) t + \varphi_j^- \right] \right\},
\]

where \(\varphi_j^- = \varphi_j - \varphi_m + \pi/2, \varphi_j^+ = \varphi_j + \varphi_m - \pi/2\). The Fourier spectrum of such an AM signal is familiar for RR Lyrae star experts (see also Fig. 5). It consists of the spectrum of the non-modulated star (third term) as in Fig. 2 and two equidistant sidepeaks around each harmonic (last term). The amplitudes of the pairs of sidepeaks are always equal: \(A(j f_0 \pm f_m) \sim a_j h/2\). The second term in equation (22), which causes the average brightness variation, produces the frequency \(f_m\) itself in the spectrum. The Blazhko modulation frequency is always found in observed data sets that are long enough (see, e.g. Kovács 1995; Nagy 1998; Jurcsik et al. 2005, 2008; Chadid et al. 2010; Poretti et al. 2010; Kolenberg et al. 2011).

It is a long-standing question as to whether there is any Blazhko phase where the modulated light curve looks identical to the unmodulated one (see Jurcsik, Benkő & Szeidl 2002, and further references therein). In this simplest case the answer is easy to give. It happens if the second and fourth terms in (22) disappear simultaneously, namely, in the zero-points of the modulated sinusoidal function at \(t = (k \pi - \varphi_m)/(2\pi f_m), k\) is an arbitrary integer.

The number of used parameters for solving such a light curve (Fig. 3) in the traditional way (according to equation 19) is \(6n + 5\), where \(n\) denotes the number of detected harmonicons including the main frequency. The necessary number of parameters in our handling is \(2n + 5\). The modulation is described by three parameters \((f_m, a_m, \varphi_m)\) as opposed to the traditional framework where this number is \(4n + 3\).

3.1.2 Non-sinusoidal AM

As a next step, we assume the modulation function \(m^*_m(t)\) to be an arbitrary periodic signal represented by a Fourier sum with a constant frequency \(f_m\). Substituting it into equation (20) we get

\[
m^*_m(t) = \left[ a_0^* + \sum_{p=1}^{q} a_p^* \sin \left( 2\pi p f_0 t + \varphi_p^* \right) \right] c^*(t),
\]

where constants are defined by \(a_0^* = 1 + (a_0^* U^*_0), a_p^* = a_p^* U^*_p\). From now on, the superscript \(A\) will denote the AM parameters. Some typical light curves are shown in Fig. 6. It is evident that their envelope curves are non-sinusoidal and their shapes depend on the actual values of \(a_p^*\) and \(\varphi_p^*\). The maxima and minima of
the same Blazhko phase as in the previous sinusoidal cases. Rewriting of (23) similarly to equation (22), but in a more compact form, yields

\[ m_{\text{AM}}^*(t) = \sum_{p=0}^{q} \sum_{j=0}^{n} \frac{a_j^p}{a_1^p} \sin \left[ 2\pi (j f_0 \pm p f_m) t + \psi_j^p \right]. \]  

(24)

where the two sinusoidal terms appearing analogously to equation (22) are formally unified into one formula and denoted by ± signs: \( \psi_j^+ = \psi_j + \psi_j^0 - \pi/2; \psi_j^- = \psi_j - \psi_j^0 + \pi/2. \) The arbitrary constants are chosen to be \( \psi_j^0 := -\pi/2. \)

By investigating the Fourier amplitude of the sidepeaks we found that \( A(f_0 \pm p f_m)/A(f_0) = a_j^p \). (i) The amplitude ratio between the sidepeaks of a given order and the central peak is constant. (ii) The commonly used amplitude ratio \( A(f_0 \pm p f_m)/A(f_0) \sim a_j/a_1 \) versus harmonic order is the same as the amplitude ratio of the main frequency \( A(f_0)/A(f_0) \sim a_j/a_1 \) versus harmonic order. (iii) Since the same coefficient \( a_j^p \) belongs to both sidepeaks (at ±p), the amplitudes of left-hand side and right-hand side peaks are the same. According to this, the generated Fourier spectrum (Fig. 7) now shows a symmetrical multiplet structure of peaks around the main frequency and its harmonics \( (f_0 \pm p f_m) \). Each multiplet structure is the same at each harmonic order, that is, the number of the sidepeaks, their frequency differences and amplitude ratios to their central peaks are the same. It is important to note that the number of sidepeaks (on one side) is equal to \( p \). In addition, the harmonic components of the modulation frequency \( pf_m \) also appear [this can be obtained from equation (24) if \( j = 0 \)]

Such a phenomenon was not detected in the observed data of Blazhko stars until recently. Hurta et al. (2008) found equidistant quintuplets in the spectrum of RV UMa for the first time. Besides triplets and quintuplets, sextuplet structures were also found by Jurcsik et al. (2008) in the spectrum of MW Lyr, while Chadid et al. (2010) detected even eighth order (septdecaplet) multiplet frequencies in the spectrum of CoRoT data of V1127 Aql. According to Section 2.2, the angle modulations cause an infinite number of sidepeaks around each harmonic, therefore the origin of the observed multiplets as a non-sinusoidal amplitude modulation is confirmed for those cited cases (e.g. MW Lyr and V1127 Aql) where the harmonics of the modulation frequency are also detected.

In search of a Blazhko phase where the modulated and carrier waves are identical, we concluded that the modulation terms can only entirely disappear from formula (24) if \( a_j^0 = 1 \) \( (a_j^0 = 0) \) is true, otherwise no such Blazhko phase exists. This necessary condition is complemented by an additional one: the modulation virtually disappears in the moments when \( \sum_{p=0}^{q} a_j a_p \sin[2\pi (j f_0 \pm p f_m) t + \psi_j^p] = 0. \) The sum has either zero or infinite numbers of zero-points depending on the values of the parameters \( a_j^p, \psi_j^p \). That is, generally there are no such phases where a non-sinusoidal AM light curve and its carrier wave are identical.

The necessary number of parameters for a light-curve fit of (19) and (23) is \( (2q + 1)n + 2q + 3 \) and \( 2n + 2q + 3 \), respectively. Here \( n \) denotes the total number of used harmonics including the main frequency and \( q \) is the order of sidepeak structures as above, (i.e. \( q = 1 \) means triplets, \( q = 2 \) means quintuplets, etc.). In the traditional description each additional sidepeak order increased the number of terms by \( 4n + 2 \) as opposed to our method, where this increment is only 2.

3.1.3 Parallel AM modulation

Multiperiodic modulation was suspected in XZ Cyg (LaCluyzé et al. 2004), UZ UMa (Sódlor et al. 2006), SU Col (Szczygiel & Fabrycky 2007), and LS Her (Wils, Kleidis & Brooks 2008). The Blazhko
RR Lyrae stars of the MACHO and OGLE surveys (Alcock et al. 2000; Moskalik & Poretti 2003), which have unequally spaced triplet structures in their Fourier spectra, are possibly also multiperiodically modulated variables. CZLac (Sórdor et al. 2011) is the first Blazhko star with multiperiodic modulation where both the modulation periods are identified. Not only modulation sidepeaks but also linear combinations of the modulation frequencies appear. Signs of multiple modulation were discovered in Kepler data of V445 Lyr (Benkő et al. 2010). There are numerous possibilities of creating a multiply modulated light curve. Let us review some of them.

The most simple case is a natural generalization of equation (23) when the modulation signal is assumed to be a sum of signals with different \( f_m^2 \), where \( r = 1, 2, \ldots \) denotes constant frequencies. Let signals be independent, i.e., the modulation signal consists of linearly superimposed waves. In this case, equation (23) reads

\[
m_{\text{AM}}(t) = \left[ a_0 + \sum_{q=1}^c \sum_{r=1}^q \hat{a}_{q,r} \sin(2\pi \hat{f}_{m} t + \phi_{q,r}) \right] \sin(\hat{f}_{0} t + \phi_0)
\]

where \( \hat{a}_0 = 1 + \sum_{q=1}^c a_{m}^q / U_{C}^r \), and \( \hat{a}_{q,r} = a_{m}^{q,r} / U_{C}^r \). This formula is demonstrated in Fig. 8. In this figure, only two modulation waves are taken into account and the only varied parameter is the frequency of the second modulation \( f_m^2 \). When the two modulation frequencies are comparable \( (\hat{f}_m^2 = 0.3 \text{ d}^{-1} \) and \( \hat{f}_m^2 = 0.09 \text{ d}^{-1} \) in panel a) the envelope shape of the light curve shows the well-known beating phenomenon. Here the beating period is 200 d, even though the modulation periods are close to the shortest known ones.

![Figure 8](https://example.com/figure8.png)

**Figure 8.** Artificial light curves calculated with two independent sinusoidal AM modulations according to the formula of (25). The fixed parameters were \( a_0 = 0.01, \hat{a}_1 = 0.3, \hat{a}_{12} = 0.2 \text{ mag}, f_m^1 = 0.1 \text{ d}^{-1}, \hat{f}_m = 270^\circ, \phi_1 = 120^\circ \), where \( f_m^1 \) changes from top to bottom as 0.09, 0.075, 0.05 and 0.01 d\(^{-1} \), respectively. It is easy to understand that observations taken over a moderate time-span often detect only the gradual increase or decrease of the amplitude of the Blazhko cycles. In panel (b) of Fig. 8, \( f_m^2 \) was set to 0.075 d\(^{-1} \), that is the ratio of the modulation frequencies is 4:3, similarly to the case of CZLac during its second observed season in Sórdor et al. (2011). The amplitude changes of the consecutive Blazhko cycles need well-covered long-term time series observations, otherwise the interpretation becomes difficult. Panel (c) in Fig. 8 shows the case where the frequency of the second modulation is half of the first one \( (\hat{f}_m^2 = 0.05 \text{ d}^{-1}) \). These specially selected values cause alternating higher and lower Blazhko cycles. The exact 2:1 ratio between the two modulation frequencies leads to the same result as a two-term non-sinusoidal modulation in equation (23) (see also top panel in Fig. 6). The bottom panel in Fig. 8 shows a case where the second modulation has a much longer period than the primary Blazhko cycle. In a first inspection the top and bottom panels are very similar apart from a phase shift.

To reveal the real situation we need to compare their Fourier spectra starting with

\[
m_{\text{AM}}(t) = \sum_{s=1}^{c} \sum_{r=1}^q \hat{a}_{q,r} \sin(2\pi \hat{f}_s t + \phi_{q,r})
\]

where the constants are chosen similarly to (24): \( \hat{f}_{p,s} = \hat{f}_s + \pi / 2; \hat{f}_{r,s} = \hat{f}_s + \pi / 2; \phi_0 = \pi / 2 \). It is easy to see that the Fourier spectrum of expression (26) contains \( s \) sets of sidepeaks shown in Fig. 7. The qualitative structure of these sets is the same. It consists of the carrier’s spectrum \( (\hat{f}_0)^s \), the peaks of the different modulation frequencies and their harmonics \( (p \hat{f}_m^2)^s \), and the sidepeaks around the main frequency and its harmonics:\n
\[
\hat{f}_0 \pm p f_m^2, \quad (p \hat{f}_m^2)^s, \quad (p \hat{f}_r^s)^s
\]

Due to the independence of the modulation waves no further sidepeaks appear.

3.1.4 Modulated modulation – the AM cascade

It is hard to imagine, however, that in the case of a real star, the different modulating waves are independently superimposed without any interactions. Let us investigate the possibility of the modulated modulation: the cascade. In other words, the modulation signal is composed of recursively modulated waves as

\[
c^{(1)}(t) := c^0(t), \quad m_{\text{AM}}^{(1)}(t) = m_{\text{AM}}^2(t),
\]

\[
c^{(2)}(t) := m_{\text{AM}}^{(1)}(t) = m_{\text{AM}}^2(t) c^{(1)}(t), \quad m_{\text{AM}}^{(2)}(t) = \left[ 1 + m_{\text{AM}}^2(t) \right] c^{(2)}(t), \ldots,
\]

\[
m_{\text{AM}}^{(s)}(t) = \left[ 1 + m_{\text{AM}}^2(t) \right] c^{(s)}(t).
\]

\[
m_{\text{AM}}^{(r)}(t) = \prod_{r=1}^{c} \left[ a_0 + \sum_{r=1}^q \hat{a}_{q,r} \sin(2\pi \hat{f}_s t + \phi_{q,r}) \right] c^r(t),
\]

where \( \hat{a}_0 = 1 + a_{m}^q / U_{C}^r, \hat{a}_{q,r} = a_{m}^{q,r} / U_{C}^r \). \( U_{C}^r \) denotes the amplitude of the \( r \)-th carrier wave \( c^{(r)}(t) \). On the basis of a visual inspection it can be said that there are imperceptible differences among the light curves produced by this expression (28) and those that can be seen in Fig. 8. The Fourier spectrum, however, contains additional peaks at the linear combinations of \( f_0 \) and \( f_m^2 \) as shown in Fig. 9. To understand this spectrum we generate equation (28) in a form...
similar to equation (26):
\[
m_{AM}^r(t) = \sum_{r=1}^{s} \sum_{j=1}^{m_r} \sum_{n=1}^{m_{rj}} \left( \prod_{r'=1}^{r} \tilde{a}_{jr} \right) \tilde{\alpha}_{jr} t + \tilde{\beta}_{jr}
\times \sin \left[ 2\pi \left( j f_0 \pm \frac{r}{m_r} \right) t + \tilde{\phi}_{rjr}^\pm \right] + \sum_{r=1}^{s} \sum_{j=1}^{m_r} \sum_{n=1}^{m_{rj}} \left( \prod_{r'=1}^{r} \tilde{a}_{jr} \right) \tilde{S}_r(\alpha) + \sum_{r=1}^{s} \sum_{j=1}^{m_r} \sum_{n=1}^{m_{rj}} \left( \prod_{r'=1}^{r} \tilde{a}_{jr} \right) S_{r+1}(\beta),
\]
(29)

where

- \( \tilde{\phi}_{rjr}^\pm = \phi_j - \phi_{jr} \pm \pi/2 \); \( \tilde{\phi}_{rjr}^\pm = \phi_j + \phi_{jr} - \pi/2 \); \( \tilde{\phi}_{0r} = \pi/2 \). An index set \( R_r \) is defined so that it contains all the \( r \) indices from \( r = 1, 2, \ldots, s \). Each index set \( R_r \) means all subsets of \( R_r \) that contain \( k \) elements. Therefore, the total number of \( R_r \) sets is \( \left( \begin{array}{c} s \end{array} \right) \). The sums over \( r \in R_r \) mean over all the possible combinations of the \( k \) number of different indices \( r \). Similarly, \( r' \) always runs over the complement of a set \( R_r^c \) of the actual \( R_r \). The functions \( S_k \) build up sums of sinusoidal functions (see definition and further details in Appendix B) of the linear combinations of \( k \) angles. Here the components of \( \alpha \) and \( \beta \) vectors are

\[
\alpha_r = 2\pi p f_d^m t + \tilde{\phi}_{0r}, \quad r \in R_r, \quad \beta_{k+1} = 2\pi j f_0 t + \phi_j, \quad r = 1, 2, \ldots, s.
\]

For a better comparison we present the formula in the form of equation (29) instead of the most possible compact one. Although equation (29) seems to be complicated, the meaning of each term is simple: the first term can be directly compared to the linearly superimposed case, equation (26). It produces all the peaks in the Fourier spectrum appearing in that case: the main pulsation frequency and its harmonics \( j f_0 \); modulation frequencies and their harmonics \( p f_d^m \); and the side frequencies around the main frequency and its harmonics \( (j f_0 \pm p f_d^m) \). The second sum in equation (29) belongs to the peaks of all the possible linear combinations of \( p f_d^m \), whilst the last term is responsible for the many linear combination frequencies around the main frequency and its harmonics (Fig. 9). The latter two types of combination terms were detected by Söder et al. (2011) in the Fourier spectrum of CZ Lac, the only well-studied multiply modulated RR Lyrae star.

Long-term secondary changes in Blazhko cycles can be explained by the variable strength of the modulation. To formulate this assumption we arrived at

\[
m_{AM}^r(t) = \left\{ 1 + [1 + m_{AM}^r(t)] m_{AM}^r(t) \right\} c'(t).
\]
(30)

The formula (30) can be considered as a special case of (28) when \( s = 2 \) and \( a_{01} = 0 \).

3.2 Blazhko stars with FM

We remind the reader of the possible absence of real Blazhko stars with pure AM that was mentioned in the introductory paragraphs of Section 3.1. The only difference between pure AM and FM cases is that RR Lyrae stars showing pure PM/FM are much more rarely reported than pure AM ones, but some examples are there (e.g. Kurtz et al. 2000; Derekas et al. 2004).

How can the formalism discussed in Section 2.2 be applied to RR Lyrae stars? Let us assume the same carrier wave as in the case of AM, but here the instantaneous frequency \( f(t) \) is denoted by \( f_0 + m_{AM}^r(t) \), and \( m_{AM}^r(t) \) is an arbitrary (bounded) modulation signal:

\[
m_{AM}^r(t) = a_0 + \sum_{j=1}^{n} a_j \sin \left( 2\pi j f_0 t + \phi_j \right). \quad \sin \left( 2\pi j f_0 t + \phi_j \right).
\]
(31)

Expression (31) describes a general frequency modulated RR Lyrae light curve.

3.2.1 The sinusoidal FM

When the modulating function is sinusoidal and expressed in the same form as (21), equation (31) becomes

\[
m_{FM}^r(t) = a_0 + \sum_{j=1}^{n} a_j \sin \left( 2\pi j f_0 t + \phi_j \right) \left( 2\pi j f_0 + \phi_j \right) \sin \left( 2\pi j f_0 t + \phi_j \right),
\]
(32)

where \( d^\alpha = a_{0d} f_d^m, \phi^\alpha = \phi_{d} + \pi/2 \) and the superscript F marks the parameters of FM. The amplitude of this signal is determined by the Fourier amplitudes \( a_j \) of the carrier signal, hence no amplitude changes are present. In the bottom panel of Fig. 10 a simulated light curve is shown. It is evident that there is no amplitude change. Two-day zooms from two different phases of the modulation cycle are shown in the top panels. The periodic phase shift caused by FM can be identified well: in the left-hand side panel the non-modulated light curve is to the left from the FM light curve whilst in the right-hand side panel the situation is opposite.

Using Chowning’s relation (12) we get from (32)

\[
m_{FM}^r(t) = a_0 + \sum_{j=1}^{n} a_j \sin \left( 2\pi j f_0 t + k f_m t + \phi_j \right) \sin \left( 2\pi j f_0 t + \phi_j \right).
\]
(33)

This equation shows the main characteristics of the Fourier spectrum (Fig. 11). For a good modulation frequency and its harmonics \( j f_0 \) and each of them is surrounded by sidepeaks at \( (j f_0 \pm k f_m) \) with symmetrical amplitudes at the two sides. This symmetry of the amplitudes can be seen from the expression of amplitude ratio \( A(j f_0 \pm k f_m)/(A(j f_0) \sim \lambda_{J_1}(d^\alpha)) \), and it is known that \( J_{-1}(z) = (-1)^J J_{1}(z) \). It is interesting to compare the AM spectra in Figs 5 and 7 with this FM spectrum. The Fourier amplitudes of the sidepeaks are proportional to the Bessel function, and an immediate consequence can be
such a type. Looking at formula (32) it can be realized that the modulation disappears at the moments of time $t = ((t - \varphi p)/(2\pi f_m))$, where $i$ is an arbitrary integer.

To estimate the number of necessary parameters for a light-curve fit, the traditional description equation (19) needs $2n + 3 + 4\sum_{n=1}^{\infty} n \left\lfloor j(n^p + 1) \right\rfloor$, where ‘int’ means the integer function, and $n$ is the number of all harmonics including the main frequency as well. At the same time, equation (32) requires only $2n + 5$ parameters, no more than that in the sinusoidal AM case. For a typical case as the one plotted in Fig. 10 ($n = 10$ and $d^2 = 0.27$), the difference is 143 parameters versus 25.

### 3.2.2 The case of non-sinusoidal FM

Assuming an arbitrary periodic modulation with a fixed frequency we substitute a Fourier sum representing this modulation signal into equation (31) and get

$$m_{\text{FM}}(t) = a_0 + \sum_{j=1}^{n} a_j \sin\left[2\pi j f_0 t - \sum_{p=1}^{q} a_p^E \sin\left(2\pi p f_m t + \varphi_p^E + \varphi_j\right)\right].$$

where the constant terms are contracted as $\varphi_j = j\varphi_0^C + \varphi$. As in the previous sinusoidal case the equation can be rewritten as

$$m_{\text{FM}}(t) = a_0 + \sum_{j=1}^{n} \sum_{k_1, k_2, \ldots, k_p=-\infty}^{\infty} a_j \left[\prod_{p=1}^{q} J_{k_p}(ja_p^E)\right] \times \sin\left[2\pi \left(j f_0 + \sum_{p=1}^{q} k_p f_m^p\right) t + \sum_{p=1}^{q} k_p \varphi_p^E + \varphi_j\right].$$

(35)

In one sense, this formula is a generalization of (13) to the case for a non-sinusoidal carrier wave; in the other sense, however, the modulation frequencies are chosen specially as $f_m^p := pf_m$.

Comparing equation (35) with (33) it can be seen that the structure of the two Fourier amplitude spectra should be similar (cf. also Figs 11 and 12), although there are significant differences as well. First of all, in spite of the values of the common Fourier parameters of a sinusoidal and a non-sinusoidal case being the same, the detectable sidepeaks in the sinusoidal case are more numerous than in the non-sinusoidal case. The reason is simple: the higher order terms in the modulation signals’ sum increase the ‘effective modulation index’. The most noteworthy difference is the disappearance of the symmetry between the amplitudes of the sidepeaks from the lower and higher frequency parts.

To understand this let us investigate the simplest non-sinusoidal case, if $q = 2$, and concentrate only on the sidepeaks around the main pulsation frequency ($j = 1$). Then the above expression (35) is simplified to

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} a_1 J_{k_1}(a_1^E) J_{k_2}(a_2^E) \times \sin\left[2\pi \left(f_0 + (k_1 + 2k_2) f_m\right) t + k_1 \varphi_1^E + k_2 \varphi_2^E + \varphi_1\right].$$

(36)

For calculating the amplitude of the triplet’s sidepeaks $Af_0 \pm f_m$, we have to work out the corresponding terms from the above infinite sum such as $k_1 = 1 - 2k_2$ and $k_1 = -2(k_2 + 1)$, for the right-hand side and left-hand side peaks, respectively ($k_2$ is an arbitrary integer). It can be seen that both sums include the same elements, because $J_{-2}(a_1^E) J_{-a_2^E}(a_2^E) = J_{a_1^E}(a_2^E) J_{a_2^E}(a_2^E) = J_{2a_1^E}(a_2^E) J_{2a_2^E}(a_2^E)$.

Figure 10. Bottom: artificial FM light curve produced with sinusoidal modulation according to equation (32). Fourier parameters of the ‘carrier light curve’ are the same as before, $d^2 = 0.279$, $\varphi_0 = 0$. Boxes show the locations of the top panels. Top panels: 1-d zooms from two different phases of the modulation cycle. The non-modulated ‘carrier’ light curve is shown by a (red) continuous line while the (blue) dotted line denotes the FM signal. The periodic phase shift (viz. FM) caused by FM can be clearly identified.

Figure 11. Bottom: Fourier amplitude spectrum of the artificial sinusoidal FM light curve in Fig. 10 after the data were pre-whitened with the main frequency and its harmonics. Top panels are zooms around the positions of the main frequency $f_0 = 2d^{-1}$ (top left), and its seventh harmonics $8f_0 = 16d^{-1}$ (top right), respectively. The modulation frequency $f_m$ is missing from the spectrum (insert).

seen in the figure: the amplitudes of the triplet peaks are higher at $3f_1$ than at $2f_0$. (Although it is not shown in the figure, the higher order $j > 5$ harmonics have smaller amplitudes than their side frequencies’ amplitudes.) Since the argument of the Bessel functions depends on the order of harmonics $j$, higher order harmonics ‘feel’ a larger modulation index, which results in more sidepeaks around the higher order harmonics (see inserts in Fig. 11). This effect was found for V1127 Aql from its CoRoT data by Chadid et al. (2010). A further remarkable flavour of this Fourier spectrum is that it does not include $f_m$ as opposed to any of the AM spectra (see insert in Fig. 11).

Let us return to the question of whether there is any phase in the modulation cycle where the light curve is identical to the mono-periodic light curve (carrier signal). As has been shown in the case of sinusoidal AM modulation, it is possible that some phases are of
Amplitude ratios of the harmonic components of the main pulsation are defined in (31). The expression is the same as in the top panel, but filled symbols + and • represent frequencies
\[ f = 0.27 \] and \[ f = 0.1 \], these values are
\[ \hat{f} = \sin j \theta \]
27 and 0.1 mag, \[ \hat{f} = \sin j \theta \] values (lower frequency sidepeaks).

\[ \hat{C} \]
Bottom: Fourier amplitude spectrum of an artificial non-sinusoidal FM light curve calculated by the formula of (34) after the data were pre-whitened with the main frequency and its harmonics. Parameters of the generated light curve were the same as for the light curve in Fig. 10, and \( p = 2 \), \( a_j^0 = -0.1 \) mag, \( \phi_j^0 = \pi/4 \). Top panels are zooms around the positions of the main frequency \( f_0 = 2 \) d\(^{-1} \) (top left), and its seventh harmonic \( 8f_0 = 16 \) d\(^{-1} \) (top right), respectively.

\[ J_5(a_j^0 J_{-2}(a_j^0), \ldots \text{for each pair and the relative phase differences have the same values with an opposite sign. The only differing terms contain } J_1(a_j^0 J_0(a_j^0) \text{ in the sum of } A(f_0 + f_m) \text{ and } J_{-1}(a_j^0 J_0(a_j^0) \text{ in the sum of } A(f_0 - f_m). \text{ These terms are responsible for the asymmetry of the sidepeaks. Introducing the power difference of the sidepeaks as in Section 2.3, we get}
\]
\[ \Delta_1 = 4 \hat{A}_1 J_1(a_j^0) J_0(a_j^0) \cos(\hat{\phi}_1 - \phi_j^0). \]

Here \( \hat{A}_1 \) and \( \hat{\phi}_1 \) indicate the amplitude and phase of a sinusoidal oscillation obtained by summing all the terms in (36) except the different ones. The asymmetry of the higher order sidepeaks \( |k_1 + 2k_2| > 1 \) can be verified in a similar manner.

This asymmetry has a further consequence. The functions of amplitude ratio versus harmonic orders belonging to a given pair of sidepeaks are divergent from each other (Fig. 13). This behaviour is well known from the similar diagrams of observed Blazhko RR Lyrae stars (Jurcsik et al. 2009b; Chadid et al. 2010; Kolenberg et al. 2011). It can also be seen that the actual characteristics of the asymmetry can change with the harmonic order \( j \) or even within a given order with a different \( p \). For example, in Fig. 13 if \( p = 3 \) (triplets), the right-hand side peaks are always higher than the left-hand side ones and the difference between the pairs is increasing with harmonic orders. Meanwhile, if \( p = 3 \) (septuplets), the situation is the opposite. In the case of \( p = 2 \) (quintuplets), while the lower frequency peaks have higher amplitude around the lower order \( (j < 5) \) harmonics, for higher harmonics \( (j > 7) \) the amplitude ratios of the pairs of sidepeaks are the opposite.

As was discussed in the introductory Section 2.3, simultaneous and sinusoidal amplitude and phase modulations result in an asymmetrical spectrum, therefore the asymmetry of the amplitude spectrum alone is not a good criterion for detecting a non-sinusoidal FM. The classical O – C diagram is an ideal tool for this purpose (see, e.g. Sterken 2005). Fig. 14 illustrates the O – C diagrams of the maxima for two artificial FM light curves: a sinusoidal and a simple non-sinusoidal.

\[ \hat{m}_{FM}(t) = a_0 + \sum_{j=1}^{n} a_j \sin \left( 2\pi j f_0 t \right) \]
\[ + j \left[ \sum_{r=1}^{s} \sum_{p=1}^{\Phi} \hat{a}_{pr} \sin \left( 2\pi p f_0 t + \phi_{pr} \right) \right] + \hat{\phi}_j. \]

At the end of this section we compare the necessary parameters of a potential fit based on the classical description (19) and the present (31) one. In the latter case this value is \( 2n + 2q + 3 \), where \( n \) and \( q \) are defined in (31). The expression is the same as in the case of a non-sinusoidal AM. The traditional formula needs \( \approx 2n + 3 + \{4\sum_{j=1}^{n} \text{int}(j\sum_{p=1}^{\Phi} a_{pr}) + 1\} \) parameters. For the case shown in Fig. 12 \( n = 10 \), \( a_l^2 = 0.27 \), \( a_2^2 = 0.1 \), these values are 27 and 163, respectively.

3.2.2 Parallel FM

We continue the discussion of the case of AM. The next step is the multiply modulated FM with independently superimposed modulation signals (parallel modulation). As has already been noted, the chance of such a scenario happening is very low for stars, but this case shows a new phenomenon, which is why it is worth having a look at it:

\[ \hat{m}_{FM}(t) = a_0 + \sum_{j=1}^{n} a_j \sin \left( 2\pi j f_0 t \right) \]
\[ + j \left[ \sum_{r=1}^{s} \sum_{p=1}^{\Phi} \hat{a}_{pr} \sin \left( 2\pi p f_0 t + \phi_{pr} \right) \right] + \hat{\phi}_j. \]
Here \( \tilde{\phi}_j := j \tilde{a}^F_n + \varphi_j \). The formula (38) can easily be transformed with the help of equations (13) and (35) to

\[
\tilde{m}^*_{\text{FM}}(t) = a_0 + \sum_{j=1}^{n} \sum_{k_1, k_2, \ldots, k_m = -\infty}^{\infty} a_j \left[ \prod_{p=1}^{q_1} J_{k_p} \left( j \tilde{a}^F_{k_p} \right) \right] \sin \left[ 2\pi \left( j f_0 + \sum_{r=1}^{s} \sum_{p=1}^{q_r} k_p p \tilde{f}^m_r \right) t + \sum_{r=1}^{s} \sum_{p=1}^{q_r} k_p \tilde{\varphi}_{k_p} + \tilde{\phi}_j \right].
\]

(39)

There is a fundamental difference in the construction of Fourier spectra between parallel AM and FM signals. While the AM spectra build up from a simple sum of the component spectra belonging to a given modulation frequency \( \tilde{f}^m \), FM spectra contain all the possible linear combinations of \( \tilde{f}^m \) and the harmonics of the main pulsation frequency \( j f_0 \). This is illustrated in Fig. 15. In practice, this effect complicates the distinction between the Fourier spectra from the cascade AM and those from the parallel FM.

### 3.2.4 The FM cascade

Although a parallel FM modulation results in a more complex Fourier spectrum than either a parallel or even a cascade AM, our former statement is still true. There is a very small chance for independently superimposed modulation signals occurring in real stars. Let us turn to the FM cascade (viz. the modulated modulation) case now:

\[
\tilde{m}^{*}_{\text{FM}}(t) = a_0 + \sum_{j=1}^{n} a_j \sin[2\pi j f_0 t + j \tilde{C}_{\text{FM}}(t) + \tilde{\phi}_j],
\]

(40)

where

\[
\tilde{C}_{\text{FM}}(t) := m^{(1)}_{\text{FM}}(t) = \sum_{p=1}^{q_1} \tilde{a}^F_{k_p} \sin \left[ 2\pi p \tilde{f}^m_1 t + p m \tilde{f}^m_1(t) + \tilde{\varphi}_{k_p} \right],
\]

\[
m^{(0)}_{\text{FM}}(t) = \sum_{p=1}^{q_2} \tilde{a}^F_{k_p} \sin \left[ 2\pi p \tilde{f}^m_2 t + \tilde{\varphi}_{k_p} \right],
\]

\[
m^{(1)}_{\text{FM}}(t) = \sum_{p=1}^{q_3} \tilde{a}^F_{k_p} \sin \left[ 2\pi p \tilde{f}^m_3 t + \tilde{\varphi}_{k_p} \right],
\]

(41)

and \( \tilde{\phi}_j = j \tilde{a}^F_{k_1} + \varphi_j \). As we can see, the function \( \tilde{C}_{\text{FM}}(t) \) consists of a modulation cascade with \( s \) elements, where all elemental modulation functions \( m^{(s)}_{\text{FM}}(t) \) are represented by finite Fourier sums. That is, they are assumed to be independent periodic signals with the frequencies \( \tilde{f}^m \). Since an FM modulation can be reproduced by an infinite series of sinusoidal functions (see Chowning relation), it is not a surprise that the sinusoidal decomposition of the expression (40) is very similar to the parallel case (39). Namely

\[
\tilde{m}^{*}_{\text{FM}}(t) = a_0 + \sum_{j=1}^{n} \sum_{k_1, k_2, \ldots, k_m = -\infty}^{\infty} a_j \times \left[ \prod_{p=1}^{q_1} J_{k_p} \left( j \tilde{a}^F_{k_p} \right) \prod_{r=2}^{s} \sum_{p=1}^{q_r} J_{k_p} \left( k_{p-1} \tilde{\varphi}_{k_p} \right) \right] \sin \left[ 2\pi \left( j f_0 + \sum_{r=1}^{s} \sum_{p=1}^{q_r} k_p p \tilde{f}^m_r \right) t + \sum_{r=1}^{s} \sum_{p=1}^{q_r} k_p \tilde{\varphi}_{k_p} + \tilde{\phi}_j \right].
\]

(42)

The frequency content is exactly the same as in the parallel case, only the values of amplitudes and phases are different.

### 3.3 The case of PM

Here we discuss the phase modulation. As we have stated in Section 2, there is no chance of distinguishing between the FM and PM phenomena on the basis of their measured signals (inverse problem), if the modulation function \( m^*_n(t) \) in equation (31) is allowed to be arbitrary. At the same time, if the basic physical parameters such as effective temperature, radius and \( \log g \) are changing during the Blazhko cycle as was found recently (Jurcsik et al. 2009a,b; Sódor, Jurcsik & Szeidl 2009), then the cyclic variation in the fundamental pulsation period (viz. frequency) that results in FM would be a plausible explanation for the observed effects. There is an additional possible argument against the existence of PM in RR Lyrae stars.
If we assume that the modulating function $m^\ast_m$ contains no explicit time variation – as in the usual definition for PM in electronics – equation (31) reads

$$m^\ast_{PM}(t) = a_0 + \sum_{j=1}^n a_j \sin \left[ 2\pi j f_0 t + m^\ast_m(t) + \phi_j \right].$$

(43)

When this formula is expressed as equation (33) or (35) according to a sinusoidal or an arbitrary periodic modulating function, respectively, the arguments of Bessel functions are independent of the harmonic order $j$ as opposed to the case of FM. It causes a systematic difference between Fourier spectra of FM and PM. While the number of detectable sidepeaks in FM increases with the order of harmonics, for PM the number of sidepeaks is the same for all harmonics.

There are two Blazhko RR Lyrae stars, both of which show strong phase variations and have precise enough data: these are the CoRoT targets V1127 Aql and CoRoT 105288363 (Chadid et al. 2010; Guggenberger et al. 2011). The spectrum of V1127 Aql clearly shows the existence of FM: third-order side frequencies are detected around the main pulsation frequency while the eighth order is around the 19th harmonic. The Fourier analysis of separate Blazhko cycles of CoRoT 105288363 showed that with the increasing strength of phase variation, the number of detected sidepeaks around higher order harmonics also increased (Guggenberger et al. 2011). This is an evidence of (changing) FM.

### 3.4 Real Blazhko stars with simultaneous AM and FM

In this section we discuss the general combined case, when both types of modulations occur simultaneously. As was mentioned before, both AM- and FM-type modulations were detected for all the observed Blazhko RR Lyrae stars if the observed data sets were precise and long enough. This is the situation for ground-based (Jurcsík et al. 2009c and references therein) and space-borne observations of CoRoT and Kepler (Benkő et al. 2010; Chadid et al. 2010; Poretti et al. 2010; Kolenberg et al. 2011) as well.

Generalizing the sinusoidal case of combined modulation, equation (14), discussed in Section 2.3, we get

$$m^\ast_{\text{Comb}}(t) = [1 + m^\ast_m(t)] m^\ast_{PM}(t),$$

(44)

where $m^\ast_{PM}(t)$ is the general modulated FM function defined by equation (31). Since all the observed Blazhko stars show AM and FM with the same frequency, we have investigated only those cases where this assumption is fulfilled.

#### 3.4.1 Combined modulations with sinusoidal functions

The simplest case similar to the pure AM and FM cases is the simultaneous but sinusoidal modulations:

$$m^\ast_{\text{Comb}}(t) = (1 + h \sin 2\pi f_{mt}) \times \left\{ a_0 + \sum_{j=1}^n a_j \sin[2\pi j f_0 t + j a^e \sin(2\pi f_{mt} + \phi_m) + \phi_j] \right\},$$

(45)

where the notations are the same as, or directly analogous to, the previously defined ones: $h = a_0 U^\ast_m U^\ast_{PM}$ and $U^\ast_m U^\ast_{PM}$ is the amplitude of the second term (the frequency-modulated ‘carrier wave’). The relative phase between AM and FM signals is $\phi_m = \phi^e - \phi_m$.

According to the schema of (16), expression (45) can be reformulated into

$$m^\ast_{\text{Comb}}(t) = a_0 + a_0 h \sin 2\pi f_{mt}$$

$$+ \sum_{j=1}^n \sum_{k=-\infty}^\infty a_j \left\{ J_k(j a^e) \sin[2\pi(f_0 + k f_{mt} + k \phi_m + \phi_j)] \right\}$$

$$+ \sum_{j=1}^n \sum_{k=-\infty}^\infty a_j \left\{ J_{k+1}(j a^e) \sin[2\pi(f_0 + k f_{mt} + (k + 1) \phi_m + \phi_j)] \right\},$$

(46)

where $\phi^e_j = \phi_j + \pi/2$. Based on equation (45) the Fourier spectrum in Fig. 16 can be interpreted as a sum of the combined modulation with sinusoidal carrier wave (16) with an additional term describing the modulation frequency itself (insert in bottom panel). Each harmonic is surrounded by a multiplet structure of peaks just like the main frequency. The number of sidepeaks increases with the harmonic order $j$ similarly for FM (Section 3.2.1).

The asymmetrical amplitudes of pairs of side frequencies belonging to a given harmonic can be characterized similarly to the sinusoidal carrier wave case (18):

$$\Delta_{jl} = -4 \frac{hl}{ja^e} J_j(j a^e) \sin \phi_m.$$  

(47)

Here $\Delta_{jl} = A_j^2(f_0 + j f_{mt}) - A_j^2(f_0 - j f_{mt})$ is the power difference of the $jl$th sidepeaks at the $jl$th harmonics ($l = 1, 2, \ldots$). Similarly to the course-book case discussed in Section 2.3, the asymmetry depends on the actual value of $h$ and $a^e$ (viz. the relative strengths of AM and FM) and the relative initial phase angle $\phi_m$. The most extreme possibility is when one of the sidepeaks completely disappears. The necessary conditions are $\phi_m = \pm \pi/2$ and $ja^e = hl$. The asymmetry decreases with the increasing harmonic order $j$ (see also top panels in Fig. 16), because all the Bessel functions quickly converge to zero with increasing arguments, therefore dominate the right-hand side of expression (47).

Non-equidistant sampling and large gaps in the observed time series can cause significant differences between sidepeak amplitudes (see Jurcsík et al. 2005). Such sampling effects, however, cannot explain huge differences, such as when sidepeaks completely

Figure 16. Bottom: Fourier amplitude spectrum of an artificial combined (AM and FM) light curve computed from equation (45) after the data were pre-whitened with the main frequency and its harmonics. Top panels are zooms around the positions of the main frequency $f_0 = 2 \text{d}^{-1}$ (top left), and its seventh harmonics $8 f_0 = 16 \text{d}^{-1}$ (top right), respectively. The relative phase between AM and FM is set to $\phi_m = 270^\circ$.

Non-equidistant sampling and large gaps in the observed time series can cause significant differences between sidepeak amplitudes (see Jurcsík et al. 2005). Such sampling effects, however, cannot explain huge differences, such as when sidepeaks completely
disappear on one side and the spectra show doublets, though numerous examples were found by large surveys such as MACHO and OGLE (Alcock et al. 2000, 2003; Moskalik & Poretti 2003). But as illustrated in Fig. 16, highly asymmetrical sidepeaks can easily be generated by (45). This asymmetry effect can be a possible explanation for the observed doublets (RR-\(v1\) stars) and even for triplets (RR-\(v2\) stars). In the latter case the two side frequencies can originate from a quintuplet structure (equidistant triplet on one side) or from a multifrequency modulation (non-equidistant triplet on one side).

Searching for phases where the modulated and non-modulated light curves are identical, we conclude that such phases exist only if \(\phi_m = (k_1 - k_2)\pi\); \((k_1, k_2\) are integers) and then the moments of the coincidences are \(t = k_3/2f_m\). Assuming that all the Blazhko stars are showing both AM and FM, this conclusion supports the finding of Jurcsik et al. (2002), who studied light and radial velocity curves of Blazhko RR Lyrae stars.

The amplitude ratio versus harmonic order diagrams show similar shapes and relative positions as were discussed in Section 3.2.2 in connection with Fig. 13. Let us look at the maximum brightness versus maximum phase diagrams, a classical tool for analysing Blazhko RR Lyrae stars. Such diagrams are plotted in Fig. 17 for synthetic light curves generated from the formula of (45). All the diagrams have a simple round shape. They reflect the relative strength of AM and FM components. In panels A and B the relative strengths are opposite \(2h = a^2\) and \(h = 2a^3\), respectively. As a consequence, the loop is deformed vertically or horizontally. When the angle \(\phi_m\) differs from the special values of \(\pi/2\) \((l = 0, 1, 2, 3, 4)\), the axes of the loops are inclined to the vertical horizontal position. This angle also determines the direction of motion. If \(0 < \phi_m < \pi\) it is clockwise, whilst if \(\pi < \phi_m < 2\pi\) it is anticlockwise. [These conditions are the same as found by Szeidl & Jurcsik (2009) for sinusoidal carrier waves.] It is noteworthy that the same ranges of \(\phi_m\) also determine the character of the power difference of the sidepeaks: if the right-hand side peaks are higher than the left-hand side ones then the direction of motion is anticlockwise and vice versa.

### 3.4.2 Non-sinusoidal combined modulation

On the basis of the previous sections it is easy to define the light curves which are modulated by general periodic signals simultaneously in both their amplitudes and phases:

\[
m_{\text{Comb}}^A(t) = \frac{m_{\text{AM}}^A(t)}{c^A(t)} m_{\text{FM}}^A(t),
\]

where the functions \(m_{\text{AM}}^A(t)\) and \(m_{\text{FM}}^A(t)\) are defined by equations (23) and (34), respectively. Since the two modulations are described by different functions (they are represented by different order of Fourier sums), no such simple relative phase can be defined as \(\phi_m\) for the sinusoidal case given in Section 3.4.1. Therefore, we obtain for the mathematical form of such a generally modulated light curve:

\[
m_{\text{Comb}}^A(t) = \left[ a_0^A + \sum_{p=1}^{q'} a_{p}^A \sin \left( 2\pi p f_m t + \psi_{p}^A \right) \right] \times \left\{ a_0 + \sum_{j=1}^{n} a_j \sin \left[ 2\pi j f_{jd} \right] + j \sum_{p=1}^{q} a_p^B \sin \left( 2\pi p f_m t + \psi_p^B + \phi_j \right) \right\},
\]

where the notations are the same as in equations (23) and (34). This expression describes all the discussed phenomena of a light curve modulated regularly with a single frequency \(f_m\). The envelopes of these light curves are very similar to the envelopes of non-sinusoidal AM light curves shown in Fig. 6. The light curves show a non-sinusoidal phase variation as well (see also Figs 10 and 14).

As in the former simpler cases, the Fourier spectrum can also be constructed analytically with the help of the sinusoidal decomposition of (49):

\[
m_{\text{Comb}}^A(t) = a_0 a_0^A + \sum_{p=1}^{q'} a_0 a_p^A \sin \left( 2\pi p f_m t + \psi_{p}^A \right) + \sum_{j=1}^{n} \sum_{p'=0}^{q'-1} \sum_{k_1+2k_2+\ldots+k_{s-1}=-\infty} a_{p'}^B \left[ \prod_{p=1}^{s} I_p \left( j a_p^B \right) \right] \times \sin \left\{ 2\pi \left[ f_{jd} + \sum_{p=1}^{q} k_p p \pm p' \right] t + \psi_{p'}^+ \right\}.
\]

Here \(\psi_{p'}^+ := \sum_{p=1}^{s} k_p p \pm p' + \phi_j + \pi/2\) and the arbitrary constant is chosen as \(\psi_{p'}^+ := \pi/2\). The qualitative structure of this spectrum is simple and well understandable on the basis of the previously discussed cases. The second term is responsible for the appearance of the modulation frequency and its higher harmonics.
The resulting light curves and Fourier spectra can be interpreted as equation (25) (parallel AM) or equation (28) (AM cascade). A combined modulation with multiperiodic AM or FM or both can be handled analogously to the simpler presented cases. We can substitute $m_0(t)$ into the general expression (44) as defined by equation (25) (parallel AM) or equation (28) (AM cascade). Writing $m_{\nu x}(t)$ as equation (38) (parallel FM) or equation (40) (FM cascade) in principle is straightforward. In practice, however, calculating coefficients (amplitudes, phases) is more complicated. The resulting light curves and Fourier spectra can be interpreted on the basis of their constituents. They do not show new features except their maximum brightness versus maximum phase diagrams which show time-dependent and generally non-closed curves as opposed to those in Fig. 18. The direction of the motion can arbitrarily change on tuning of the initial phases.

3.4.3 Combined multifrequency modulations

A combined modulation with multiperiodic AM or FM or both can be handled analogously to the simpler presented cases. We can substitute $m_0(t)$ into the general expression (44) as defined by equation (25) (parallel AM) or equation (28) (AM cascade). Writing $m_{\nu x}(t)$ as equation (38) (parallel FM) or equation (40) (FM cascade) in principle is straightforward. In practice, however, calculating coefficients (amplitudes, phases) is more complicated. The resulting light curves and Fourier spectra can be interpreted on the basis of their constituents. They do not show new features except their maximum brightness versus maximum phase diagrams which show time-dependent and generally non-closed curves as opposed to those in Fig. 18. The direction of the motion can arbitrarily change on tuning of the initial phases.

4 PRACTICAL APPLICATION – A CASE STUDY

To demonstrate how our formalism works in practice, we generated two artificial light curves with simultaneous non-sinusoidal AM (with two harmonics, $p' = 2$) and FM (with three harmonics, $p = 3$). The light curves are 100 d long and sampled by 5 min in the same manner as all other synthetic light curves have been in the paper. We added Gaussian noise to the light curves with either rms = 0.01 mag (model A) or $10^{-4}$ mag (model B), respectively. Model A is similar to a good-quality ground-based observation, while model B simulates a typical space-borne data set. These two artificial light curves (top panels in Fig. 19) were analysed with a blind test (i.e. without knowledge of frequencies, amplitudes and phases) both by the traditional way and by our method.

4.1 Classical light-curve analysis

Constructing the mathematical model equation (19) of the light curves in the traditional analysis is a successive pre-whitening process. It consists of Fourier spectra building, fitting the data with the parameters of the highest peak(s) in the spectrum by a non-linear algorithm, and subtracting the fitted function from the data and so on. The process continues as long as significant peaks are detected. At the end of this analysis the noise of the residual data reaches the observational scatter.

In the case of model A the highest peaks belong to the main frequency $f_0$ and its harmonics. In the further pre-whitening steps the triplets $(f_0 \pm f_m)$, quintuplets $(f_0 \pm 2f_m)$, septuplets $(f_0 \pm 3f_m)$, nonuplets $(f_0 \pm 4f_m)$ and the modulation frequency $(f_m)$ were found and fitted. The significance level was chosen at $S/N = 4$, while the signal-to-noise ratio $(S/N)$ was estimated following Breger et al. (1993). To remove all the significant peaks from the spectrum, five from the undecaplet peaks $(f_0 \pm 5f_m)$ and two aliases had to be fitted and subtracted. (We note that the frequency $2f_m$ was detectable but, given the significance level, it was not fitted.) The residual light curve of this process and its Fourier spectrum are plotted in the middle panels of Fig. 19. The rms of the residual light curve is $10^{-2}$, so we got back the input noise value. The number of fitted frequencies is 103 and the number of parameters used in this successive fit is 201. If the frequencies of the sidepeaks are also fitted independently (‘let it free approach’) this count increases to 286.

The process works similarly in the case of model B as well. Naturally, many more significant peaks are detectable. From the highest peaks to the lowest: $f_0$, its harmonics, the sidepeaks up to the orders of six $(f_0 \pm 6f_m)$, the modulation frequency and its harmonic $(f_m, 2f_m)$ are significant. Many alias peaks originating from the finite data length are also detectable: 46 such frequencies were removed up to the significance level of the sixth order of sidepeaks. We stopped the analysis here at $S/N \approx 40$ because we already have found 168 frequencies and used 368 parameters (or 478 if we fit each frequency independently). The resulting light curve and its Fourier spectrum are shown in the middle panels of Fig. 19. The rms of this light curve is $10^{-3}$, an order of magnitude higher than the input noise parameter.

4.2 Light-curve analysis in our framework

When we apply the approach of this work we need to calculate only one Fourier spectrum and a single non-linear fit for each light curve. The Fourier spectrum and the characteristics of the light curve help us to choose the proper fitting formula and to determine the initial values of the fit.

The Fourier spectrum for any of the light curves A and B provides us with the necessary parameters $(f_m, a_i, a_j, g_i)$ of the carrier wave. The amplitude modulation and its non-sinusoidal nature in both light curves are apparent. Searching for peaks in the low-frequency

Figure 18. Some typical maximum brightness versus maximum phase diagrams for synthetic light curves with combined non-sinusoidal modulations. The relative strength, the initial phases and the number of used harmonics of AM and FM have been varied.
range of the Fourier spectra results in good initial values for $f'_m$, $p' = 1, 2, a^p_m$ and $\varphi^p_m$. While more sidepeaks are detectable around the harmonics of the main frequency than the number of harmonics of $f_m$, and while the sidepeaks are asymmetrical, an FM has to be assumed. To check its non-sinusoidal nature we may prepare e.g. an O–C or maximum brightness versus maximum phase diagram. At the end of this preparatory work we can choose the fitting formula (49). We note that the determination of the correct initial values of $a^p_m$ and $\varphi^p_m$ depends on the tool used for finding the frequency variations. In the worst case, they can be estimated by some numerical trials with the non-linear fit. Using initial values that are good enough, the non-linear fit converged fast for models A and B. The algorithm reached the noise levels $10^{-3}$ and $10^{-4}$ automatically within a few (less than 10) iterations. The residual light curves and their Fourier spectra are plotted in the bottom panels of Fig. 19. The number of fitted parameters for both the models is only 33. Due to the finite numerical accuracy the residual spectra always show a structure reflecting the original spectra at very low levels.

In conclusion, our method fits the light curves in a single step with many fewer parameters than the traditional method requires. In addition we avoid the time-consuming alias fitting and subtracting processes. The difference in the number of used parameters increases with the increasing accuracy of the observed data sets. In our example the number of regressed parameters is reduced, from six times (model A) to more than 10 times (model B). Our description has advantages in the numerical fit of the ground-based observations as well, but its advantages are outstanding in the analysis of the space-borne time series.

5 DISCUSSION AND SUMMARY

In this paper we have investigated the mathematical representation of artificial light curves. These light curves are defined as modulated signals where their carrier wave is a monoperiodic RR Lyrae light curve defined by its finite Fourier sum. Different types of periodic functions are taken into account as modulation functions from the simple sinusoidal to the multiperiodic and the general ones. The consequences of these modulation functions and modulation types (AM, FM, combined) are reviewed and presented.

We followed a step-by-step analysis from the simplest case to the more complicated ones. The results of this process can be summarized as follows.

(i) (a) Tuning AM by the used modulation function, namely the number of harmonic terms and their amplitudes and phases, the synthetic light curves reproduce well the observed shape of the Blazhko envelope curves. This is always true for the envelopes of maxima; however, the envelopes of minima are affected by the fine structure of the bump and its phase shift along the light curve and the changing shapes. These variations in the shock are associated with the Blazhko modulation in such a way that a displacement of the shock-forming region occurs over the Blazhko cycle. This induces an earlier or later occurrence of the bump in the light curves (see Preston, Smak & Paczyński 1965). If these variations are strong enough, the envelope curve of minima could be different from our simple modulated ones. (b) We showed that AM with an extremely high modulation depth ($h > 1$) might be an explanation for the strange light curve shape of V445 Lyrae observed by Kepler. (c) When the AM function depends on more than one frequency (in a parallel or cascade modulation), we can generate envelope curves with various observed phenomena such as the beating effect; alternating maxima; long-term, periodic amplitude changes etc.

(ii) Fourier spectra of AM cases can be easily classified (see also Table 1). If the modulation signal is represented by a finite Fourier sum the spectrum of the synthetic light curve shows the modulation frequency, multiplets around the pulsation frequency and its harmonics, as well. The order of multiplets (the number of peaks on one side) is the same as the number of terms used for describing the modulation function. As a special case, the sinusoidal modulation results in triplet structures and the appearance of $f_m$ only. For a multiperiodic parallel modulation, the spectrum consists of a sum of the frequencies of each component modulation, while in the case of modulated, the modulation (AM cascade) spectra include additional peaks at all the possible linear combination frequencies as well.

(iii) For all the AM cases the Fourier amplitudes of the side frequencies are proportional to the amplitude of the given harmonics, therefore the numbers and the sidepeak amplitude ratios compared to the central frequencies are the same for all the orders of harmonics. Pure AM results in symmetrical multiplets: side frequencies of the same order have the same amplitudes.

(iv) The variation in the mean brightness through the Blazhko cycle is a consequence of the AM modulation. The effect appears even for the simplest sinusoidal modulation function, and is closely
related to the appearance of the modulation frequency itself in the Fourier spectrum.

(v) Pure FM light curves show no amplitude changes but they show significant phase variations. Their Fourier spectra show a multiplet structure around the pulsation frequency and its harmonics even for the simplest sinusoidal modulation. As opposed to AM the detectable number of sidepeaks increases with increasing harmonic orders. The modulation frequencies (and their harmonics) are absent in the spectra. The sidepeak structure can be symmetrical (for sinusoidal modulation) or asymmetrical (most of the non-sinusoidal cases). The non-sinusoidal FM can be characterized by the classical O – C tool. The multifrequency FM spectra include all the possible linear combination frequencies in the parallel case.

(vi) The Fourier amplitude ratio of the sidepeaks versus harmonic orders is determined by the Bessel functions of the first kind (for the sinusoidal case) or their product (for the non-sinusoidal case). As a consequence, these relationships are generally non-strictly decreasing with increasing harmonic orders. Therefore, we can find larger amplitudes for the higher order sidepeaks than for the lower order ones for comparing the same order of sidepeaks around different harmonics and different orders of sidepeaks around a given harmonic. The latter can be true also for the amplitude of the sidepeaks as well as of the central peaks (the harmonics).

(vii) The simultaneous AM and FM show all the above-mentioned effects. The sidepeak amplitudes in the Fourier spectra of the combined modulations are generally asymmetrical already for sinusoidal AM and FM. The asymmetry can totally remove sidepeak(s) on one side. The strength of asymmetry depends on the initial phase difference of the AM and FM modulation functions. This phase difference determines both the inclination and direction of motion of the loop in the maximum brightness versus maximum phase diagram.

(viii) In the case of the pure sinusoidal AM and FM (and with very special parameters for non-sinusoidal and combined cases as well), there are well-defined phases where the modulated and non-modulated light curves are identical. However, we have pointed out that, when a star shows general combined modulation (even a sinusoidal one), its light curve is always different from any monoperiodic (non-modulated) stars.

(ix) We showed that our modulation description presented here needs many (typically 3–10 times) fewer parameters for fitting a light curve than the classical solution, where all the detected frequencies have their own independent amplitude and phase. The price to pay for the relatively small number of parameters is a more complicated fitting formula. Our approach fits the light curves only once, instead of the traditional successive pre-whitening process, and is not affected by most aliasing problems.

It is equally important to list those observed features that do not follow this picture. One of these effects is the phase lag, a difference between the Blazhko phases of maxima of maxima and minima of minima of the modulated light curves. These phase lags can be several pulsation cycles long. In our study, however, all envelopes are highly symmetrical: their maxima and minima are practically at the same Blazhko phases. If we see the well-sampled observed light curves (e.g. CoRoT or Kepler data) showing this phase lag, we can realize that the effect builds up from the systematic motions and changing the shape of the bump caused by the hydrodynamical shock in the RR Lyrae atmospheres.

An additional unsolved problem is the distribution of the sidepeaks’ asymmetry. Large surveys (OGLE, MACHO) found that the spectra of Blazhko RR Lyrae stars more frequently contain symmetrical side-frequency patterns, whereas the higher frequency sidepeaks have a higher amplitude than vice versa [e.g. 74 versus 26 per cent for the Large Magellanic Cloud reported by Alcock et al. (2000)]. All the formulae suggest 50–50 per cent probabilities, if the distribution of the relative phases between AM and FM is uniform. The detected asymmetrical distribution might have a deeper physical origin.

Our description does not explain additional frequencies such as half-integer frequencies belonging to period doubling, temporary overtone frequencies and further exotic frequencies in the Fourier spectra, recently discovered from the space data (Gruberbauer et al. 2007; Benkő et al. 2010; Chadid et al. 2010; Kolenberg et al. 2010; Poretti et al. 2010; Szabó et al. 2010). Their connection with the physics of the Blazhko effect is poorly understood and their modelling is beyond the scope of this paper.

The main result of this analysis is the demonstration that the modulation paradigm successfully accounts for most of the observed properties of Blazhko RR Lyrae light curves. Moreover, this work may help us to distinguish between those features that have a deeper physical origin and those that appear simply due to the modulation. This framework seems to be very flexible and can easily be generalized to a larger fraction of variable stars taking into account, e.g. multifrequency carrier waves or stochastic modulations.

### ACKNOWLEDGMENTS

This project has been partially supported by the ESA PECS projects No. 98022 and 98114, and the Hungarian OTKA grant K83790. RSz acknowledges the support of the János Bolyai Research Scholarship.
REFERENCES

Abramowitz M., Stegun I. A., 1972, Handbook of Mathematical Functions, 10th issue. Natl. Bureau Standards, Washington DC
Alcock C. et al., 2000, ApJ, 542, 257
Alcock C. et al., 2003, ApJ, 598, 597
Benkő J. M., Paparó M., Szabó R., Chadid M., Kolenberg K., Poretti E., 2009, in Guzik J. A., Bradley P. A., eds, AIP Conf. Proc. Vol. 1170, Stellar Pulsation: Challenges for Theory and Observation. Am. Inst. Phys., New York, p. 273
Benkő J. M. et al., 2010, MNRAS, 409, 1585
Blazhko S. N., 1907, Astron. Nachr., 175, 325
Breger M., 2010, in Sterken C., Samus' N. N., Szabados L., eds, Variable Stars, the Galactic Halo and Galaxy Formation. Sternberg Astron. Inst., Moscow, p. 95
Breger M., Kolenberg K., 2006, A&A, 460, 167
Breger M. et al., 1993, A&A, 271, 482
Carson J. R., 1922, Proc. IRE, 10, 57
Cartianu Gh., 1966, Frequency Modulation. Academic Press, Bucharest
Chadid M. et al., 2009, in Guzik J. A., Bradley P. A., eds, AIP Conf. Proc. Vol. 1170, Stellar Pulsation: Challenges for Theory and Observation. Am. Inst. Phys., New York, p. 235
Chadid M. et al., 2010, A&A, 510, A39
Chowning J. M., 1973, J. Audio Eng. Soc., 21, 526
Dereks A., Kiss L. L., Událski A., Bedding T. R., Szematyó K., 2004, MNRAS, 354, 821
Gruenbergaer E., Kolenberg K., Chapelleraer E., Poretti E., Szabó R., Benkő J. M., Paparó M., 2007, MNRAS, 379, 1498
Henry G. W., Fekel F. C., Henry S. M., 2005, AJ, 129, 2815
Jurcsik J., Szabó K., Sódor Á., 2008, AJ, 135, 957
Jurcsik J., Benkő J. M., Szeidl B., 2002, A&A, 390, 133
Jurcsik J. et al., 2005, A&A, 430, 1049
Jurcsik J. et al., 2006, MNRAS, 391, 164
Jurcsik J. et al., 2009a, MNRAS, 393, 1555
Jurcsik J. et al., 2009b, MNRAS, 397, 350
Jurcsik J. et al., 2009c, MNRAS, 400, 1006
Koen C., 2001, MNRAS, 322, 97
Kolenberg K. et al., 2006, A&A, 459, 577
Kolenberg K. et al., 2010, AJ, 713, L198
Kolenberg K. et al., 2011, MNRAS, 411, 878
Kovács G., 1995, A&A, 295, 693
Kurtz D. W. et al., 2000, in Szabados L., Kurtz D. W., eds, ASP Conf. Ser. Vol. 203, The Impact of Large-Scale Surveys on Pulsating Star Research. Astron. Soc. Pac., San Francisco, p. 291
LaCluyzé A. et al., 2004, AJ, 127, 1653
Moskalik P., Kołaczkowski Z., 2009, in Goupil M., Kolláth Z., eds, ASP Conf. Ser. Vol. 335, The Light-Time Effect in Astrophysics. Astron. Soc. Pac., San Francisco, p. 3
Mºthner R. B., 2010, PASP, 122, 536
Szaãbro R. et al., 2010, MNRAS, 409, 1244
Szczygiel D. M., Fabrycky D. C., 2007, MNRAS, 377, 1263
Szeidl B., Jurcsik J., 2009, Commun. Asteroseismol., 160, 17
van der Pol B., 1930, Proc. IRE, 18, 1194
Wils P., Kleidis S., Broens E., 2008, MNRAS, 387, 783

APPENDIX A: ABOUT THE EXACT FOURIER TRANSFORMS

There are many definitions of the Fourier transform, differing from each other by their normalizations. In this paper we adhere to the following definition:

\[
\mathcal{F}(f) = \int_{-\infty}^{\infty} g(t) e^{-i2\pi ft} \, dt. \tag{A1}
\]

Since we always transform our equations describing the different types of modulated signals to linear combinations of sinusoidal functions, it is necessary to know the exact Fourier transformation of these elements.

The Fourier transform of the sinusoidal carrier wave equation (1) itself is

\[
\mathcal{F} [c(t)] = \sqrt{2 \pi} U_c [i \cos \varphi_c \delta (f - f_c) - \delta (f + f_c)] + i \sin \varphi_c \delta (f - f_c) + \delta (f + f_c)], \tag{A2}
\]

where \(i\) is the imaginary unit, \(\delta\) is the Dirac function. As we see, the Fourier amplitude spectrum belonging to \(c(t)\) has two components: one at positive frequency (centred on \(+f_c\)) and one at negative frequency (centred on \(-f_c\)). We are concerned with only the positive frequency, since the negative ones have no physical meaning.

Demonstrating the use of the above formulae, the positive part of the Fourier transformation of the sinusoidal AM signal (4) is given:

\[
\mathcal{F}^+[U_{AM} (t)] = \mathcal{F}^+[c(t)] + \sqrt{\frac{\pi}{2}} U_m [i \cos (\varphi_c - \varphi_m) + i \sin (\varphi_c - \varphi_m)] \delta (f + f_c - f_m)
\]

\[- \cos (\varphi_c + \varphi_m) + i \sin (\varphi_c + \varphi_m)] \delta (f + f_c + f_m)] \tag{A3}.
\]

According to this schema, the investigated signals describing the more complicated expressions can also be calculated in a straightforward way.

APPENDIX B: GENERALIZED PRODUCT-TO-SUM FORMULAE

Let us calculate the generalized form of the product-to-sum expression from the well-known identity of trigonometry:

\[
\sin(\alpha_1) \sin(\alpha_2) = \frac{1}{2} [\cos(\alpha_1 - \alpha_2) - \cos(\alpha_1 + \alpha_2)]. \tag{B1}
\]

Assuming an \(n\) factor on the left-hand side, it can be formally written as \(\prod_{i=1}^{n} \sin \alpha_i\). Introducing the vector–scalar function \(S_n(\alpha) := \prod_{i=1}^{n} \sin \alpha_i \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)^2\), where \(T\) indicates transposition, and applying formula (B1) \(n\) times recursively, we arrive at

\[
\sum_{j=1}^{n} \left| S_n(\alpha) \right| \sin \alpha_j = \frac{1}{2} \left| \prod_{i=1}^{n} \cos \alpha_i \right| \sin \alpha_j \tag{B2}.
\]

© 2011 The Authors, MNRAS 417, 974–991

Monthly Notices of the Royal Astronomical Society © 2011 RAS

990 \textit{J. M. Benkő, R. Szabó and M. Paparó}

of the HAS. The authors thank the referee Dr Katrín Kolenberg for her helpful comments and close reading of the manuscript.
where
\[ N = \frac{3n}{2} - \sum_{i=1}^{n} Q_{li} \quad \text{and} \quad N' = \frac{3(n - 1)}{2} - \sum_{i=1}^{n} Q_{li}. \]

Each sin/cos term includes a sum of \( n \) terms of the \( \alpha_i \) angles inside their arguments as \( \alpha_1 \pm \alpha_2 \pm \alpha_3 \pm \cdots \pm \alpha_n \). One combination from these sets contains \( l \) positive and \( n - l \) negative angles. The total number of such a combination is \( 2^{n-1} \). Each row of the matrix \( Q \) contains the signs of the angles of a possible set from the total number of \( 2^{n-1} \). Namely, the elements of the matrix \( Q \) are either +1 or -1, but \( Q_{11} = 1 \), i.e. the first columns contain +1 for all the \( l \) values.

This paper has been typeset from a \TeX/LaTeX file prepared by the author.