SHARP BOUNDS FOR THE ANISOTROPIC p-CAPACITY OF EUCLIDEAN COMPACT SETS

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Abstract. We prove various sharp bounds for the anisotropic p-capacity $\text{Cap}_{F,p}(K)$ ($1 < p < n$) of compact sets $K$ in the Euclidean space $\mathbb{R}^n$ ($n \geq 3$). For example, using the inverse anisotropic mean curvature flow (IAMCF), we get an upper bound of Szegö type (1931) for $\text{Cap}_{F,p}(K)$ when $\partial K$ is a smooth, star-shaped and $F$-mean convex hypersurface in $\mathbb{R}^n$ ($n \geq 3$). Moreover, for such a surface $\partial K$ in $\mathbb{R}^3$, by introducing the anisotropic Hawking mass and studying its monotonicity property along IAMCF, we obtain an upper bound of Bray–Miao type (2008) for $\text{Cap}_{F,p}(K)$.

1. Introduction

The capacity problem is one of the most extensively-investigated topics in the potential theory, the mathematical physics, the partial differential equations, the convex geometry and other fields. The classical (electrostatic) capacity of a compact set $K$ in the Euclidean space $\mathbb{R}^3$ admits the physical interpretation that, it represents the maximal charge that can be put on $K$ while the electrical potential of the vector field created by this charge is no greater than one. See e.g. [17,19,21,23,26] for some background information. In this paper we are concerned with sharp bounds of the anisotropic p-capacity for compact sets in the Euclidean space in terms of their various geometric quantities.

Let $F \in C^\infty(\mathbb{R}^n \setminus \{0\}) \cap C(\mathbb{R}^n)$ be a Minkowski norm on $\mathbb{R}^n$ and $K \subset \mathbb{R}^n$ be a compact set. Throughout the paper we consider $n \geq 3$. For $1 < p < n$, the anisotropic p-capacity of $K$ is defined as

$$\text{Cap}_{F,p}(K) = \inf \{ \int_{\mathbb{R}^n} F^p(Dv)dx : v \in C^\infty_c(\mathbb{R}^n), v \geq 1 \text{ on } K \},$$

(1.1)

where $C^\infty_c(\mathbb{R}^n)$ is the set of smooth functions with compact support in $\mathbb{R}^n$. See Section 2.1 for more details on the anisotropic p-capacity. In particular, by (2) in Proposition 16, $\text{Cap}_{F,p}(K) = \text{Cap}_{F,p}(\partial K)$ for a compact set. In view of this property some works in the literature are only concerned with the capacity for hypersurfaces that are the boundaries of compact sets.

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Since the bounds we obtain for $\text{Cap}_{F,p}(K)$ involve some anisotropic geometric quantities, let us first introduce them. Let $M$ be an immersed oriented hypersurface in $\mathbb{R}^n$ with the unit normal vector field $\nu$ and $d\mu$ be the area element of the induced metric on $M$. We can define the anisotropic area element on $M$ as $d\mu_F := F(\nu)d\mu$ and the anisotropic area of $M$ as $|M|_F := \int_M d\mu_F = \int_M F(\nu)d\mu$.

Meanwhile we define the anisotropic unit normal $\nu_F$ along $M$ as $\nu_F := DF(\nu)$.

We can check that $\nu_F$ lies on the boundary $\partial W$ (named the Wulff shape) of the so-called Wulff ball $W$, which is by definition the set $W := \{x \in \mathbb{R}^n : F^0(x) < 1\}$.

Here $F^0$ is the dual norm of $F$, given as $F^0(x) := \sup_{\xi \neq 0} \frac{\langle x, \xi \rangle}{F(\xi)}$.

Moreover, for any point $x \in M$, the tangent hyperplane $T_x M$ is parallel to the tangent hyperplane $T_{\nu_F(x)} \partial W$. So we may define the anisotropic Weingarten map of $M$ as $d\nu_F : T_x M \rightarrow T_{\nu_F(x)} \partial W$.

The anisotropic Weingarten map $d\nu_F$ has $n - 1$ real eigenvalues $\kappa_i^F$, $1 \leq i \leq n - 1$, which are called the anisotropic principal curvatures of the hypersurface. From them, for $1 \leq k \leq n - 1$ we can define the $k$th anisotropic mean curvature $\sigma_k(\kappa^F)$ of the hypersurface as $\sigma_k(\kappa^F) := \sum_{1 \leq i_1 < \cdots < i_k \leq n-1} \kappa_{i_1}^F \cdots \kappa_{i_k}^F$.

We make a convention $\sigma_0(\kappa^F) = 1$. The special cases $H_F := \sigma_1(\kappa^F)$ and $K_F := \sigma_{n-1}(\kappa^F)$ are called the anisotropic mean curvature and the anisotropic Gaussian curvature of $M$, respectively. The hypersurface $M$ is called $F$-mean convex if $H_F > 0$ on the hypersurface.

**Remark 1.** For the isotropic case $F(\xi) = |\xi|$, all the geometric quantities above reduce to the ordinary ones in the Euclidean space.

**Remark 2.** The Euclidean space $\mathbb{R}^n$ equipped with a Minkowski norm $F$ as the anisotropy makes a nice model for applications and produces fruitful results in the geometry and analysis. For instance, it may model an anisotropic medium where the growth of crystals, the noise-removal procedures in the digital image processing, the crystalline fracture theory, etc. can be studied. See e.g. [6, 7] and references therein for some introductions.

Now we are ready to state our main results in this paper. First we aim at the following sharp upper bounds for $\text{Cap}_{F,p}(K)$. 
**Theorem 3.** Let $K \subset \mathbb{R}^n$ ($n \geq 3$) be a compact set with non-empty interior. Suppose its boundary $\partial K$ is smooth, star-shaped and $F$-mean convex.

If $2 \leq p < n$, then

$$\text{Cap}_{F,p}(K) \leq \left( \frac{(p-1)(n-1)}{n-p} \right)^{1-p} \int_{\partial K} H_F^{p-1} d\mu_F. \quad (1.2)$$

If $1 < p \leq 2 \leq q < n$, then

$$\text{Cap}_{F,p}(K) \leq \left( \frac{(p-1)(n-1)}{n-p} \right)^{1-p} \left( \int_{\partial K} H_F^{q-1} d\mu_F \right)^{\frac{p-1}{q-1}} |\partial K|_F^\frac{q-1}{q}. \quad (1.3)$$

Moreover, the equality holds in (1.2) or (1.3) if and only if $\partial K$ is a translated scaled Wulff shape.

**Remark 4.** For the isotropic case $F(\xi) = |\xi|$, the first result in the spirit of Theorem 3 may go back to Szegö’s [27] (see also [23]), where he proved the upper bound in the case $n = 3$ and $p = 2$ for smooth convex compact sets. In [12], Freire and Schwartz obtained the bound in Theorem 3 for the isotropic case with $n \geq 3$ and $p = 2$ for smooth compact sets with mean-convex and outer-minimizing boundary. Last, Xiao in [33] proved exactly Theorem 3 for the isotropic case.

**Remark 5.** Recently, Xia and Yin in [31] considered the anisotropic case and derived an interesting related upper bound of $\text{Cap}_{F,p}(K)$ for smooth compact connected sets in $\mathbb{R}^n$ ($n \geq 3$). Their result [31] and our Theorem 3 are not mutually inclusive. Besides, we use a different method from theirs.

Before stating our next result, we define the anisotropic Hawking mass $m_F^H(\Sigma)$ for an immersed smooth oriented compact surface $\Sigma \subset \mathbb{R}^3$ without boundary as

$$m_F^H(\Sigma) := \sqrt{\frac{|\Sigma|_F}{4|\partial \Sigma|_F}} \left( 1 - \frac{\int_{\Sigma} H_F^2 d\mu_F}{4|\partial \Sigma|_F} \right). \quad (1.4)$$

Note that if $\Sigma$ is embedded, then $m_F^H(\Sigma) \leq 0$ with the equality if and only if $\Sigma$ is a translated scaled Wulff shape; see Corollary 27 below.

The classical (isotropic) Hawking mass was introduced by S. W. Hawking in [15] and is one of the most important concepts in the mathematical physics. In [14] Geroch discovered the significant monotonicity property of the Hawking mass along the smooth inverse mean curvature flow (IMCF) in the manifold setting. This monotonicity property was extended by Huisken and Ilmanen [18] to the weak IMCF to prove the famous Riemannian Penrose inequality. Later Bray and Miao [9] applied the monotonicity property of the Hawking mass from [18] to obtain a new sharp upper bound for the capacity in the manifold setting. See [32] for the $p$-capacity generalization of [9].

Here we find an application of the anisotropic analogue of the Hawking mass in the Euclidean space as in the second main result of our paper, and
we hope this new concept would be of use in other problems in the future. Our result is the anisotropic version of some results from [9] and [32] in \( \mathbb{R}^3 \).

**Theorem 6.** Let \( K \subset \mathbb{R}^3 \) be a compact set with non-empty interior. Suppose its boundary \( \Sigma = \partial K \) is smooth, star-shaped and \( F \)-mean convex. Let \( 1 < p < 3 \).

If the anisotropic Hawking mass \( m^F_H(\Sigma) = 0 \), then \( K = r_0 \overline{W} + x_0 \) for some \( r_0 > 0 \) and \( x_0 \in \mathbb{R}^3 \), and

\[
\text{Cap}_{F,p}(K) = \left( \frac{3 - p}{p - 1} \right)^{p-1} |\partial W|_F r_0^{3-p}.
\tag{1.5}
\]

If \( m^F_H(\Sigma) < 0 \), then

\[
\text{Cap}_{F,p}(K) \leq \left( \frac{3 - p}{p - 1} \right)^{p-1} |\partial W|_F^{\frac{p-1}{p}} |\Sigma|_F^{\frac{3-p}{p}}
\times \left( \frac{\int_{\Sigma} H^2_F d\mu_F}{4|\partial W|_F} - 1 \right)^{3-p} \theta^{1-p},
\tag{1.6}
\]

where \( \theta \) is defined as

\[
\theta := \int_0^{\infty} \left( \frac{\int_{\Sigma} H^2_F d\mu_F}{4|\partial W|_F} - 1 \right)^{\frac{3-p}{p-1}} \left( 1 + \frac{p-1}{p-3} \right)^{-\frac{1}{p}} dr.
\]

**Remark 7.** The surface \( \Sigma \) in Theorem 6 is embedded. So by the remark after (1.4), we know \( m^F_H(\Sigma) \leq 0 \). In addition, the bound (1.6) is sharp in the sense that as \( m^F_H(\Sigma) \to 0^- \), it reduces to the equality case (1.5).

When \( p = 2 \), we obtain the following partial anisotropic generalization of a result in [9].

**Corollary 8.** Let \( K \subset \mathbb{R}^3 \) be a compact set with non-empty interior. Suppose its boundary \( \Sigma = \partial K \) is smooth, star-shaped and \( F \)-mean convex. Then we have

\[
\text{Cap}_{F,2}(K) \leq \frac{1}{2} \sqrt{|\partial W|_F |\Sigma|_F} \left( 1 + \sqrt{\frac{1}{|\partial W|_F} \int_{\Sigma} H^2_F d\mu_F} \right).
\tag{1.7}
\]

The equality holds if and only if \( \Sigma \) is a translated scaled Wulff Shape.

Third, we obtain the following result, which is motivated by Sections 3.4 and 3.5 of Pólya and Szegő’s book [23].

**Theorem 9.** Let \( K \subset \mathbb{R}^n (n \geq 3) \) be a compact set with non-empty interior and with smooth boundary and let \( 1 < p < n \).

1. Assume that \( \partial K \) is convex. Then we have

\[
\text{Cap}_{F,p}(K) \leq \left( \int_0^\infty \left( \sum_{i=0}^{n-1} \int_{\partial K} \sigma_i(\kappa^F) d\mu_F \cdot \hat{t}^i \right) dt \right)^{1/(1-p)},
\tag{1.8}
\]
where $\sigma_i(\kappa^F)$ ($0 \leq i \leq n-1$) is the $i$th anisotropic mean curvature of $\partial K$. Moreover, the equality holds if and only if $\partial K$ is a translated scaled Wulff shape.

(2) Assume that $\partial K$ is star-shaped with respect to the origin. Then we have

$$\text{Cap}_{F,p}(K) \leq \left( \frac{n-p}{p-1} \right)^{p-1} \int_{\partial K} h_F^{1-p} d\mu_F,$$

where $h_F = \langle X, \nu \rangle / F(\nu)$ is the anisotropic support function of the hypersurface $\partial K$. Moreover, the equality holds if and only if $\partial K$ is a scaled Wulff shape centered at the origin.

**Remark 10.** For Case (1), when $n = 3$ and $p = 2$, we obtain

$$\int_0^\infty \left( \sum_{i=0}^{n-1} \int_{\partial K} \sigma_i(\kappa^F) d\mu_F \cdot t^i \right)^{1/(1-p)} dt = \int_0^\infty \frac{1}{|\partial K|_F + \int_{\partial K} \sigma_1(\kappa^F) d\mu_F \cdot t + |\partial W|_F \cdot t^2} dt$$

$$= \frac{1}{\int_{\partial K} \sigma_1(\kappa^F) d\mu_F \cdot \varepsilon} \log \frac{1 + \varepsilon}{1 - \varepsilon},$$

where

$$\varepsilon = \sqrt{1 - \frac{4|\partial W|_F|\partial K|_F}{(\int_{\partial K} \sigma_1(\kappa^F) d\mu_F)^2}} \in [0,1).$$

Consequently,

$$\text{Cap}_{F,2}(K) \leq \int_{\partial K} \sigma_1(\kappa^F) d\mu_F \left( \frac{\varepsilon}{\log((1 + \varepsilon)/(1 - \varepsilon))} \right) \leq \frac{\int_{\partial K} \sigma_1(\kappa^F) d\mu_F}{2}.$$  

This is the anisotropic analogue of the original result due to Szegö [27] in 1931.

**Remark 11.** In [23], Pólya and Szegö did not consider the equality case of (1.8) or (1.9). Here for the equality case of (1.8), we need the rigidity result Theorem 1.2 of [6]; while for that of (1.9), we are inspired by Theorem 1.2 of [8].

For the proofs of Theorems 3, 6 and 9, the idea may originate from Szegö’s work [27]. Let $U = \mathbb{R}^n \setminus K$. In each case we use a smooth family $\{M_t\}_{t \geq 0}$ with $M_0 = \partial K$ of hypersurfaces to foliate $U$ and construct a suitable test function $f(x)$ with level sets being $M_t$. Then the problem to find an upper bound of $\text{Cap}_{F,p}(K)$ is reduced to the estimate for certain geometric quantity on the hypersurface $M_t$. Employing different techniques for such an estimate gives rise to different results as in Theorems 3, 6 and 9. More precisely, for Theorem 3, we need the inverse anisotropic mean curvature flow (IAMCF) [30]. For Theorem 6, we rely further on the monotonicity property of the anisotropic Hawking mass along the IAMCF. And for Theorem 9, we construct a natural flow in each case.
Last, we derive the following sharp lower bound of \( \text{Cap}_{F,p}(K) \), which is a generalization of [33, Theorem 2.1].

**Theorem 12.** Let \( K \subset \mathbb{R}^n \) \((n \geq 3)\) be a compact convex set with non-empty interior and \( 1 < p < n \). Then

\[
\frac{p(n-1)}{n(n-p)} |\partial K|^p/(n-1) |\partial \mathcal{W}|^{1/(1-n)} F \geq |K| + \frac{|\partial K|^p/(p-1)}{\text{Cap}_{F,p}(K)^{1/(p-1)}}.
\]

(1.10)

The equality holds if and only if \( \partial K \) is a translated scaled Wulff shape.

**Remark 13.** For the set \( K \) in Theorem 12, it is well-known that the outward unit normal \( \nu \) along \( \partial K \) is well-defined almost everywhere; see e.g. [25]. So \( |\partial K|_{F} \) is well-defined. In addition, see (6.1) below for its expression as a mixed volume.

**Remark 14.** If \( K \subset \mathbb{R}^n \) is an arbitrary compact convex set (possibly with empty interior), we may consider the compact convex set \( K_\varepsilon := \{ z \in \mathbb{R}^n : z = x + \varepsilon y, x \in K, y \in B_1 \} \) for \( \varepsilon > 0 \). Applying first Theorem 12 to \( K_\varepsilon \) and then taking \( \varepsilon \to 0^+ \), we get the result for \( K \) itself (in light of Proposition 16 below).

**Remark 15.** For the isotropic case \( F(\xi) = |\xi| \), Xiao [33] proved the inequality (1.10), which leads to a crucial step towards the Pólya–Szegő conjecture. See e.g. [22, 23, 33] for information on this important conjecture. On the other hand, in [33], Xiao did not handle the “only if” part of the equality case. Here we are able to do it because we have a key observation (6.2) in Section 6.

The proof of Theorem 12 follows closely that of [33, Theorem 2.1]. In the proof we study the super-level set \( K_t := \{ x \mid u(x) \geq t \} \) \((0 < t < 1)\) of the anisotropic \( p \)-capacitary potential \( u(x) \) associated with \( K \). The main tools we use include the relationship between \( \text{Cap}_{F,p}(K) \) and \( \text{Cap}_{F,p}(K_t) \) and the anisotropic isocapacitary inequality. See Section 6 for details.

Finally, we like to mention that some related works on the estimates of the capacity can be found in [1–3, 9, 12, 17, 21, 23, 31–33] and references therein. Meanwhile, as a concluding remark for the Introduction, it is worth highlighting that although our methods are from the literature, our results are consequences of highly non-trivial works [4–6, 30, 31] etc. and we have got some completely new results, e.g., the equality cases in Theorems 9 and 12. Besides, to the best of our knowledge, so far there have been few estimates on the anisotropic \( p \)-capacity in addition to the classical ones in [17, 21] and the recent ones in [31]. We hope our results would be a nice stimulation in the field of estimates on the anisotropic \( p \)-capacity.

The paper is structured as follows. In Section 2 we review some basic facts on the anisotropic \( p \)-capacity, the anisotropic geometry of hypersurfaces in the Euclidean space and the inverse anisotropic mean curvature flow. In Section 3 we first introduce a general approach to attack Theorems 3, 6, 9,
and then prove Theorem 3. In the next Sections 4 and 5 we prove Theorems 6 and 9 respectively along this general approach. In the final Section 6 we prove Theorem 12 following the method in [33]. Throughout the paper, the Einstein convention for the summation of indices is used unless otherwise stated, and we usually use $M$ to denote a hypersurface in $\mathbb{R}^n$ while $\Sigma$ to denote a surface in $\mathbb{R}^3$.

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2. Preliminaries

This section is devoted to a brief overview of some preliminary materials required in this paper, including the anisotropic $p$-capacity for sets in the Euclidean space, the classical anisotropic geometry of Euclidean hypersurfaces in the differential geometry, and the relatively new anisotropic geometry of Euclidean hypersurfaces in the geometric analysis together with the resulting inverse anisotropic mean curvature flow.

2.1. Anisotropic $p$-capacity. For this subsection nice references include [21, Section 2.2] and [17, Chapters 2 and 5]. First we introduce the Minkowski norm on $\mathbb{R}^n$.

Definition 2.1. A function $F \in C^\infty(\mathbb{R}^n \setminus \{0\}) \cap C(\mathbb{R}^n)$ is called a Minkowski norm if

1. $F$ is a convex, even, 1-homogeneous function, and $F(\xi) > 0$ if $\xi \neq 0$;
2. $F$ satisfies the uniformly elliptic condition, i.e., $\text{Hess}_{\mathbb{R}^n}(F^2)$ is positive definite in $\mathbb{R}^n \setminus \{0\}$.

Let $K \subset \mathbb{R}^n$ be a compact set. For $n \geq 3$ and $1 < p < n$, the anisotropic $p$-capacity of $K$ is defined as

$$\text{Cap}_{F,p}(K) = \inf \left\{ \int_{\mathbb{R}^n} F^p(Dv)dx : v \in C^\infty_c(\mathbb{R}^n), v \geq 1 \text{ on } K \right\},$$

where $C^\infty_c(\mathbb{R}^n)$ is the set of smooth functions with compact support in $\mathbb{R}^n$.

For completeness, let us also introduce the anisotropic $p$-capacity for a general set in $\mathbb{R}^n$ as follows. For an open set $U \subset \mathbb{R}^n$, define

$$\text{Cap}_{F,p}(U) = \sup_{K \subset U \text{ compact}} \text{Cap}_{F,p}(K),$$
Then for any set \( E \subset \mathbb{R}^n \), define
\[
\text{Cap}_{F,p}(E) = \inf_{E \subset \mathcal{U} \text{ open}} \text{Cap}_{F,p}(\mathcal{U}).
\]
We remark that in this paper we are mainly concerned with the anisotropic \( p \)-capacity for a compact set \( K \).

Next we recall some basic facts on the anisotropic \( p \)-capacity.

**Proposition 16** ([17, 21]). The set function \( E \mapsto \text{Cap}_{F,p}(E) \) for \( E \subset \mathbb{R}^n \) enjoys the following properties.

1. If \( E_1 \subset E_2 \), then \( \text{Cap}_{F,p}(E_1) \leq \text{Cap}_{F,p}(E_2) \).
2. For a compact set \( K \), we have \( \text{Cap}_{F,p}(K) = \text{Cap}_{F,p}(\partial K) \).
3. If \( \{K_i\}_{i \geq 1} \) is a decreasing sequence of compact sets in \( \mathbb{R}^n \) with \( K = \bigcap_{i \geq 1} K_i \), then
\[
\text{Cap}_{F,p}(K) = \lim_{i \to \infty} \text{Cap}_{F,p}(K_i).
\]

Under some regularity assumptions on a compact set \( K \), there exists a unique weak solution to the following partial differential equation
\[
\begin{cases}
\Delta_{F,p} u = 0 & \text{in } \mathbb{R}^n \setminus K, \\
u = 1 & \text{on } K, \\
u(x) \to 0 & \text{as } |x| \to \infty.
\end{cases}
\tag{2.1}
\]
The weak solution \( u \) is called the anisotropic \( p \)-capacitary potential of \( K \).

Here a function \( u \in W^{1,p}_{\text{loc}}(U) \) for an open set \( U \subset \mathbb{R}^n \) is called a weak solution of \( \Delta_{F,p} u = f \) on \( U \) for \( f \in L^q_{\text{loc}}(U) \) with \( q = p/(p-1) \), if
\[
-\int_U \langle F^{p-1} DF(Du), Dv \rangle \, dx = \int_U f v \, dx
\]
for any \( v \in C_c^\infty(U) \). In the literature \( \Delta_{F,p} u \) is called the anisotropic \( p \)-Laplacian of \( u \in W^{1,p}_{\text{loc}}(U) \). For a \( C^2 \) function \( u \), at its regular points (where \( Du \neq 0 \)) we have
\[
\Delta_{F,p} u := \frac{1}{p} \text{div}(D(F^p)(Du)) = F^{p-2}(FF_{ij} + (p-1)F_iF_j)u_{ij}.
\]
When \( p = 2 \), the operator \( \Delta_{F,2} u = \Delta_{F,2} u \) is called the anisotropic Laplacian of \( u \in W^{1,2}_{\text{loc}}(U) \).

For later use, we collect some results from [4] on the anisotropic \( p \)-capacitary potential \( u \) of a compact convex set.

**Proposition 17** (Lemmas 4.1, 4.3 and 4.4 in [4]). Let \( K \subset \mathbb{R}^n \) be a compact convex set with \( \text{Cap}_{F,p}(K) > 0 \). Then there exists a unique locally H"older continuous function \( 0 < u \leq 1 \) on \( \mathbb{R}^n \) satisfying (2.1) with \( u \in L^{np/(n-p)}(\mathbb{R}^n) \) and \( |Du| \in L^p(\mathbb{R}^n) \). For each \( t \in (0, 1) \), the set \( \{x \in \mathbb{R}^n : u(x) > t\} \) is convex. Moreover, if the interior of \( K \) is non-empty, then \( Du(x) \neq 0 \) for \( x \in \mathbb{R}^n \setminus K \).
Remark 18. When $K$ is of non-empty interior, we see $\text{Cap}_{F,p}(K) > 0$ in view of (1) in Proposition 16, since we can find a small closed ball in $K$ with positive anisotropic $p$-capacity.

2.2. Anisotropic geometry of hypersurfaces. In this subsection we re-
view the anisotropic geometry of hypersurfaces in the Euclidean space which is classical in the differential geometry. In contrast, in Section 2.3 we will int-
roduce the relatively new anisotropic geometry of Euclidean hypersurfaces
in the geometric analysis, especially where an anisotropic curvature flow is
considered.

Let $F$ be a Minkowski norm on $\mathbb{R}^n$. We can define its dual norm $F^0$ as follows.

**Definition 2.2.** The dual norm $F^0$ of $F$ is defined as 

$$ F^0(x) = \sup_{\xi \neq 0} \frac{\langle \xi, x \rangle}{F(\xi)}. $$

It is known that $F^0$ is also a Minkowski norm.

Recall that $F$ and $F^0$ satisfy the following properties, which are very
useful when we want to understand the relationship between the unit normal $\nu$ and the anisotropic unit normal $\nu_F$ of a hypersurface below.

**Proposition 19.**

1. $F(DF^0(x)) = 1$, $F^0(DF(\xi)) = 1$.
2. $F^0(x)DF(DF^0(x)) = x$, $F(\xi)DF^0(DF(\xi)) = \xi$.

Next we define the Wulff ball and the Wulff shape determined by $F$.

**Definition 2.3.** The Wulff ball $W$ centered at the origin is defined as

$$ W := \{ x \in \mathbb{R}^n : F^0(x) < 1 \}. $$

Its boundary $\partial W$ is called the Wulff shape.

Given a Wulff ball $W$, we can recover $F$ as the support function of $W$, namely,

$$ F(\xi) = \sup_{X \in W} \langle \xi, X \rangle, \quad \xi \in S^{n-1}. $$

Now we introduce the anisotropic area of a smooth oriented hypersurface
$X : N \to M \subset \mathbb{R}^n$.

**Definition 2.4.** Let $M \subset \mathbb{R}^n$ be a smooth oriented hypersurface and $\nu$ be its unit normal vector. We define the anisotropic area of $M$ as $|M|_F := \int_M F(\nu) \, d\mu$. Denote by $d\mu_F = F(\nu) \, d\mu$ the anisotropic area element of $M$.

**Remark 20.** For $M = \partial W$, we can check by the divergence theorem that

$$ |\partial W|_F := \int_{\partial W} F(\nu) \, d\mu = \int_{\partial W} \langle X, \nu \rangle \, d\mu = \int_{\mathcal{W}} \text{div} X \, dx = n|\mathcal{W}|. $$

Next we introduce the anisotropic Gauss map for an oriented hypersurface
in $\mathbb{R}^n$. 

Definition 2.5. The anisotropic Gauss map \( \nu_F : M \to \partial W \) from an oriented hypersurface \( M \) in \( \mathbb{R}^n \) to the Wulff shape \( \partial W \) is defined by

\[
\nu_F : M \to \partial W, \\
X \mapsto DF(\nu(X)) = F(\nu(X))\nu(X) + \nabla^{S^{n-1}} F(\nu(X)),
\]
where \( \nu \) is the unit normal vector of \( M \).

Remark 21. The vector \( \nu_F \) is also called the anisotropic unit normal of the hypersurface.

Let \( A_F \) be the 2-tensor on \( S^{n-1} \) defined by

\[
A_F(\xi) = (\nabla^{S^{n-1}})^2 F(\xi) + F(\xi)\sigma, \quad \xi \in S^{n-1},
\]
where \( \sigma \) is the standard metric on \( S^{n-1} \).

Definition 2.6. The anisotropic principal curvatures \( \kappa_1^F, \ldots, \kappa_{n-1}^F \) of a smooth oriented hypersurface \( M \) in \( \mathbb{R}^n \) are defined as the eigenvalues of the tangent map

\[
d\nu_F : T_X M \to T_{\nu_F(X)}\partial W \cong T_X M.
\]

The anisotropic mean curvature is defined as

\[
H_F := \text{tr}(d\nu_F) = \sum_i \kappa_i^F = \sum_{i,j,k} (A_F)_{i}^{j} (\nu(X)) g^{ik}(X) h_{kj}(X),
\]
where \( g \) and \( h \) are the first and second fundamental forms of the hypersurface respectively. The anisotropic Gaussian curvature is defined as

\[
K_F := \det(d\nu_F) = \prod_i \kappa_i^F = \det(A_F) \det(g^{-1}h).
\]

Given an integer \( 1 \leq k \leq n-1 \), the \( k \)th anisotropic mean curvature is defined as

\[
\sigma_k(\kappa^F) := \sum_{1 \leq i_1 < \cdots < i_k \leq n-1} \kappa_{i_1}^F \cdots \kappa_{i_k}^F.
\]

Note that \( H_F = \sigma_1(\kappa^F) \) and \( K_F = \sigma_{n-1}(\kappa^F) \).

### 2.3. Anisotropic Riemannian metric on hypersurfaces and inverse anisotropic mean curvature flow.

The materials presented in Section 2.2 mostly suffice for anisotropic problems in the differential geometry. However, when we come to the field of the geometric analysis, especially when we consider anisotropic curvature flows, those seem insufficient.

In [5], to study the volume-preserving anisotropic curvature flow in the Euclidean space, Ben Andrews introduced a new Riemannian metric on hypersurfaces in \( \mathbb{R}^n \) associated with the anisotropy \( F \). Later Chao Xia [29] reformulated Andrews’ setting and used it to study the inverse anisotropic mean curvature flow [30]. It turns out that Andrews’ new metric is very
useful in the geometric analysis concerning the anisotropic geometry of Euclidean hypersurfaces. In this subsection, we shall review the settings in [5] and [29] and also the inverse anisotropic mean curvature flow in [30].

Using the dual norm \( F^0 \), for any non-vanishing vector field \( z(x) \) on \( \mathbb{R}^n \) we can define a Riemannian metric \( G \) on \( T \mathbb{R}^n \) by
\[
G(z)(V, W) := \sum_{\alpha, \beta=1}^{n} \frac{\partial^2 ((F^0)^2(z)/2)}{\partial z^\alpha \partial z^\beta} V^\alpha W^\beta,
\]
\( \forall 0 \neq z(x) \in \mathbb{R}^n, \ V, W \in T_x \mathbb{R}^n \).

In general, the third derivatives of \( (F^0)^2(z) \) do not vanish. So we can define a 3-tensor \( Q \) as
\[
Q(z)(U, V, W) := \sum_{\alpha, \beta, \gamma=1}^{n} \frac{\partial^3 ((F^0)^2(z)/2)}{\partial z^\alpha \partial z^\beta \partial z^\gamma} U^\alpha V^\beta W^\gamma,
\]
\( \forall 0 \neq z(x) \in \mathbb{R}^n, \ U, V, W \in T_x \mathbb{R}^n \).

By virtue of the 1-homogeneity of \( F^0 \), we can derive
\[
G(z)(z, z) = 1, \ G(z)(z, V) = 0 \text{ for } z \in \partial \mathcal{W}, \ V \in T_z \partial \mathcal{W},
\]
and
\[
Q(z)(z, V, W) = 0 \text{ for } z \in \partial \mathcal{W}, \ V, W \in T_z \partial \mathcal{W}.
\]

Let \( M = X(N) \) be a smooth oriented hypersurface \( X : N \to \mathbb{R}^n \) from a manifold \( N \). Setting \( z(X) = \nu_F(X) \), we get along the hypersurface
\[
G(\nu_F(X))(\nu_F(X), \nu_F(X)) = 1,
\]
\[
G(\nu_F(X))(\nu_F(X), V) = 0 \text{ for } V \in T_X M.
\]

Motivated by the above identities, we define a Riemannian metric \( \hat{g} \) on the hypersurface \( M \) by restricting \( G(\nu_F(X)) \) on its tangent space,
\[
\hat{g}(X) = G(\nu_F(X))|_{T_X M}, \ X \in M.
\]

Let \( \hat{\nabla} \) be the Levi-Civita connection of \( \hat{g} \) on \( M \). Then we have the following expressions for the anisotropic Riemannian metric \( \hat{g} \), the anisotropic second fundamental form \( \hat{h} \) and the 3-tensor \( Q \):
\[
\hat{g}_{ij} := G(\nu_F(X))(\partial_i X, \partial_j X),
\]
\[
\hat{h}_{ij} := -G(\nu_F(X))(\nu_F, \partial_i \partial_j X),
\]
\[
Q_{ijk} := Q(\nu_F)(\partial_i X, \partial_j X, \partial_k X),
\]
where \( \{\partial_i\}_{i=1}^{n-1} \) are local coordinate vectors on \( N \). For more details on \( \hat{g} \), \( \hat{h} \) and \( Q \), see [5] and [30].

In [30], following the (isotropic) works [13, 28], Chao Xia considered the inverse anisotropic mean curvature flow for a star-shaped \( F \)-mean convex
The general approach presented here may date back to [23, 27] and has been applied successfully in some works, e.g., in [9, 12, 32, 33]. Let $U = \mathbb{R}^n \setminus K$. For the proofs of Theorems 3, 6 and 9, we use the fact that $\bar{U}$ can be foliated by a family of hypersurfaces $M_t (t \geq 0)$ such that $M_0 = \partial U = \partial K$ and $M_t \to \infty$ as $t \to \infty$. Later we shall specify which foliation we choose in each case. For the moment we note that $M_t$ defines a function $\psi$ on $\bar{U}$ with level sets being $M_t$, i.e.,

$$\psi(x) = t, \text{ when } x \in M_t.$$
Define two functions
\[ T_p(t) := \int_{\psi(x)=t} \frac{F_p(D\psi)}{|D\psi|} d\mu, \]
\[ \lambda(t) := \frac{\int_t^\infty T_p^{-1/(p-1)}(s)ds}{\int_0^\infty T_p^{-1/(p-1)}(s)ds}. \]
Note that \( \lambda \in C^1([0, \infty)) \) is a non-increasing function satisfying \( \lambda(0) = 1 \) and \( \lambda(\infty) = 0 \). Moreover, set
\[ f(x) = \lambda(\psi(x)), \quad x \in \overline{U}. \]
Now we point out that in the definition of \( \text{Cap}_{F,p}(K) \), we can actually choose test functions as Lipschitz or even locally Lipschitz functions \( f(x) \) satisfying \( f = 1 \) on \( \partial K \) and \( f(x) \to 0 \) as \( |x| \to \infty \). See e.g. [9, Definition 1] and the remark below [26, Definition 1.2].

Back to our setting we can check in each case that the function \( f \) we choose later is a Lipschitz function satisfying \( f = 1 \) on \( \partial K \) and \( f(x) \to 0 \) as \( |x| \to \infty \). So \( f \) is admissible for the definition of \( \text{Cap}_{F,p}(K) \) and we get
\[ \text{Cap}_{F,p}(K) \leq \int_U F_p(Df)dx. \]
Next we estimate \( \int_U F_p(Df)dx \). By the co-area formula we obtain
\[ \int_U F_p(Df)dx = \int_U |X'(\psi(x))|^p F_p(D\psi)dx \]
\[ = \int_0^\infty |X'(t)|^p \int_{\psi(x)=t} \frac{F_p(D\psi)}{|D\psi|} d\mu dt \]
\[ = \int_0^\infty |X'(t)|^p T_p(t)dt. \]
On the other hand, by the H"older inequality we have
\[ 1 = (\lambda(0))^p = \left( -\int_0^\infty X'(t)dt \right)^p \]
\[ = \left( \int_0^\infty (-X'(t))T_p^{1/p}(t) \cdot T_p^{-1/(p-1)}(t)dt \right)^p \]
\[ \leq \int_0^\infty |X'(t)|^p T_p(t)dt \cdot \left( \int_0^\infty T_p^{-1/(p-1)}(t)dt \right)^{p-1}, \]
with the equality if \( X'(t) = cT_p^{-1/(p-1)}(t) \). In view of the definition of \( \lambda \), the above inequality is indeed an equality. As a consequence, we obtain
\[ \int_U F_p(Df)dx = \left( \int_0^\infty T_p^{-1/(p-1)}(t)dt \right)^{1-p}, \]
and then
\[ \text{Cap}_{F,p}(K) \leq \left( \int_0^\infty T_p^{-1/(p-1)}(t)dt \right)^{1-p}. \quad (3.1) \]
3.2. Proof of Theorem 3. Now we consider the inverse anisotropic mean curvature flow
\[
\begin{aligned}
X : N \times [0, T) &\to \mathbb{R}^n, \\
\partial_t X &= \frac{1}{H_F} \nu_F,
\end{aligned}
\]
with the initial condition \(X(N, 0) = \partial K\).

In this case we have \(\psi(X(p, t)) = t\). Taking the derivative with respect to \(t\) yields
\[
\langle D\psi, \frac{1}{H_F} \nu_F \rangle = 1.
\]
Noting \(\nu_F = F(\nu) \nu + \nabla^{S^n-1} F(\nu)\) and \(D\psi = |D\psi| \nu\), we obtain
\[
D\psi = \frac{H_F}{F(\nu)} \nu.
\]
So we get
\[
T_p(t) = \int_{\psi(x)=t} \frac{F_p(D\psi)}{|D\psi|} d\mu = \int_{\psi(x)=t} H_F^{p-1} F(\nu) d\mu = \int_{\psi(x)=t} H_F^{p-1} d\mu_F.
\]
Using the evolution equations (2.3) and (2.4) along the inverse anisotropic mean curvature flow, we derive
\[
\frac{d}{dt} T_p(t) = \int_{\psi(x)=t} \left( (p-1) H_F^{p-2} \left( \hat{\Delta} H_F + \hat{g}^ik A_{pik} \hat{\nabla}^p H_F \right) - 2 \frac{\nabla H_F^{2}_{\hat{\nu}}}{H_F} - \frac{\hat{h}_{\hat{\nu}}^{2}}{H_F} + H_F^{p-1} \right) d\mu_F = \int_{\psi(x)=t} \left( (p-1)(p-2) H_F^{p-5} \hat{\nabla} H_F^{2}_{\hat{\nu}} - (p-1) H_F^{p-3} \hat{h}_{\hat{\nu}}^{2} + H_F^{p-1} \right) d\mu_F \leq \int_{\psi(x)=t} \left( (p-1)(p-2) H_F^{p-5} \hat{\nabla} H_F^{2}_{\hat{\nu}} + \frac{n-p}{n-1} H_F^{p-1} \right) d\mu_F,
\]
where in the second equality we used (2.5) for the integration by parts and in the inequality we used \(|\hat{h}_{\hat{\nu}}^{2} \geq H_F^{2}/(n-1)\).

First assume \(p \in [2, n)\). We get
\[
\frac{d}{dt} T_p(t) \leq \int_{\psi(x)=t} \frac{n-p}{n-1} H_F^{p-1} d\mu_F = \frac{n-p}{n-1} T_p(t).
\]
Solving the above ordinary differential inequality we get
\[
T_p(t) \leq e^{\frac{n-p}{n-1} t} \int_{\partial K} H_F^{p-1} d\mu_F, \quad \forall p \in [2, n).
\]
Now we consider the case $2 \leq p < n$ in Theorem 3. Plugging (3.4) into Inequality (3.1) leads to

$$\text{Cap}_{F,p}(K) \leq \left( \frac{(p-1)(n-1)}{n-p} \right)^{1-p} \int_{\partial K} H_F^{p-1} d\mu_F.$$  

Next we consider the case $1 < p \leq 2 \leq q < n$. We obtain by the Hölder inequality

$$T_p(t) = \int \psi(x)=t H_F^{p-1} d\mu_F$$

$$\leq \left( \int \psi(x)=t H_F^{q-1} d\mu_F \right)^{\frac{p}{q-1}} |M_t|^{\frac{q-p}{q-1}}$$

$$= T_q^{\frac{p}{q-1}}(t)|M_t|^{\frac{q-p}{q-1}}.$$  

Noticing $|M_t|_F = |\partial K|_F e^t$ and using (3.4) for $T_q(t)$, we get

$$T_p(t) \leq \left( \int \partial K H_F^{q-1} d\mu_F \right)^{\frac{p}{q-1}} |\partial K|^{\frac{q-p}{q-1}} e^{(n-q)(p-1)t + \frac{q-p}{q-1} t}$$

$$= \left( \int \partial K H_F^{q-1} d\mu_F \right)^{\frac{p}{q-1}} |\partial K|^{\frac{q-p}{q-1}} e^{n-p t}.$$  

Therefore in light of Inequality (3.1) we conclude

$$\text{Cap}_{F,p}(K) \leq \left( \int \partial K H_F^{q-1} d\mu_F \right)^{\frac{p}{q-1}} |\partial K|^{\frac{q-p}{q-1}} \left( \int_0^\infty \frac{(n-q)(p-1)}{e^{(n-1)(1-p)t}} dt \right)^{1-p}$$

$$= \left( \frac{(p-1)(n-1)}{n-p} \right)^{1-p} \left( \int \partial K H_F^{q-1} d\mu_F \right)^{\frac{p}{q-1}} |\partial K|^{\frac{q-p}{q-1}}.$$  

If the equality holds, then (3.3) must be an equality. So $M_t$ is anisotropically umbilical. By the anisotropic Codazzi equation (see e.g. [5, 29, 30]), $H_F$ is constant, which implies that $M_t$ is a translated scaled Wulff shape ([16]). Hence $M$ is a translated scaled Wulff shape.

4. PROOF OF THEOREM 6

Given a compact oriented surface $\Sigma$ without boundary in $\mathbb{R}^3$, we define the anisotropic Hawking mass $m^F_H(\Sigma)$ of $\Sigma$ by

$$m^F_H(\Sigma) := \sqrt{\frac{|\Sigma|_F}{4|\partial \Sigma|_F}} \left( 1 - \frac{\int_{\Sigma} H_F^2 d\mu_F}{4|\partial \Sigma|_F} \right),$$  

where as before $|\cdot|_F$ denotes the anisotropic area, $H_F$ is the anisotropic mean curvature, and $d\mu_F = F(\nu) d\mu$ is the anisotropic area element. As mentioned in the Introduction, the classical (isotropic) Hawking mass $m_H(\Sigma)$ (corresponding to $F(\xi) = |\xi|$) was introduced by S. W. Hawking [15].

First we have the following observation.
**Lemma 25.** For any smooth compact star-shaped hypersurface $M$ without boundary in $\mathbb{R}^n$ with $K_F$ denoting its anisotropic Gaussian curvature, we have

$$\int_M K_F d\mu_F = |\partial W|_F.$$

**Proof.** Since both $M$ and the Wulff shape $\partial W$ are star-shaped, they can be represented by using the polar coordinates as

$$M = \{(r_1(p), p) : p \in \mathbb{S}^{n-1}\},$$

$$\partial W = \{(r_2(p), p) : p \in \mathbb{S}^{n-1}\},$$

where $r_i \in C^\infty(\mathbb{S}^{n-1})$ ($i = 1, 2$) are two smooth positive functions on $\mathbb{S}^{n-1}$.

Now consider the smooth variation $X : \mathbb{S}^{n-1} \times [0, 1]$ connecting $M$ and $\partial W$,

$$X(p, t) = ((1 - t)r_1(p) + tr_2(p), p) \in \mathbb{R}^n, \quad p \in \mathbb{S}^{n-1}, \quad t \in [0, 1],$$

and let $M_t = X(\mathbb{S}^{n-1}, t)$ for $t \in [0, 1]$.

Since the anisotropic integration of the anisotropic Gaussian curvature over a compact smooth hypersurface without boundary is invariant under any smooth variation ($[24$, Theorem 4$]$), we get

$$\int_M K_F d\mu_F = \int_{M_t} K_F d\mu_F = \int_{\partial W} K_F d\mu_F = |\partial W|_F.$$

□

The next result we will use later is a sharp lower bound for the anisotropic Willmore energy of an embedded compact hypersurface without boundary. It may be well-known to specialists. See e.g. $[31$, Theorem 1.2$]$ as a corollary of the main result there. Here we present a direct proof for the readers’ convenience.

**Proposition 26.** Let $M$ be an embedded smooth compact oriented hypersurface without boundary in $\mathbb{R}^n$. We have

$$\int_M |H_F|^{n-1} d\mu_F \geq (n - 1)^{n-1} |\partial W|_F,$$

with the equality holding if and only if $M$ is a translated scaled Wulff shape.

**Proof.** Fix any $p \in \partial W$. Consider the tangent hyperplane $T_p \partial W$. By moving this hyperplane from the infinity to $M$, we can find a point $x \in M$ such that $T_x M$ is parallel to $T_p \partial W$ and a neighbourhood of $x$ on $M$ lies on the same side of $T_x M$ from which $\nu_F(x) = p$ points. Note that at $x$, all the anisotropic principal curvatures are nonnegative. Let $\tilde{M}$ be the set of points on $M$ where all the anisotropic principal curvatures are nonnegative. Then in view of the
above observation and the Sard’s theorem, we can readily derive
\[
\int_M |H_F|^{n-1} d\mu_F \geq \int_M |H_F|^{n-1} d\mu_F \\
\geq (n-1)^{n-1} \int_M K_F d\mu_F \\
\geq (n-1)^{n-1} |\partial W|_F.
\]

Next assume that the equality holds. By applying the inequality to each connected component of \(M\), we know that \(M\) itself must be connected. Then analyzing the equality cases of the above sequence of inequalities, we see that all points on \(M\) are anisotropically umbilical. Then the same argument as at the end of Section 3 implies that \(M\) is a translated scaled Wulff shape. The proof is complete. \(\square\)

From the above result, we have an immediate corollary.

**Corollary 27.** Let \(\Sigma\) be an embedded smooth oriented compact surface without boundary in \(\mathbb{R}^3\). Then \(m^F_H(\Sigma) \leq 0\), with the equality holding if and only if \(\Sigma\) is a translated scaled Wulff shape.

Next we prove a monotonicity result for the anisotropic Hawking mass of a star-shaped \(F\)-mean convex surface in \(\mathbb{R}^3\) along the inverse anisotropic mean curvature flow. The corresponding and more general result in the isotropic case can be found in [14, 18].

**Proposition 28.** Let \(\Sigma\) be a compact star-shaped \(F\)-mean convex surface without boundary in \(\mathbb{R}^3\). Along the inverse anisotropic mean curvature flow (2.2) starting from \(\Sigma\), the anisotropic Hawking mass \(m^F_H(\Sigma_t)\) is non-decreasing in \(t\). Moreover, if \((d/dt)m^F_H(\Sigma_t) = 0\) at some time \(t > 0\), then \(\Sigma\) is a translated scaled Wulff shape.

**Proof.** First recalling the computation in (3.2), we get (let \(p = 3\) there)
\[
\frac{d}{dt} \int_{\Sigma_t} H_F^2 d\mu_F = \int_{\Sigma_t} \left( -2 \frac{\tilde{\nabla} H_F^2}{H_F^2} - 2 |\hat{h}|_g^2 + H_F^2 \right) d\mu_F.
\]

Using \(H_F^2 = 2K_F + |\hat{h}|_g^2\) and Lemma 25, we obtain
\[
\frac{d}{dt} \int_{\Sigma_t} H_F^2 d\mu_F = \int_{\Sigma_t} \left( -2 \frac{\tilde{\nabla} H_F^2}{H_F^2} - |\hat{h}|_g^2 + 2K_F \right) d\mu_F \\
\leq 2 |\partial W|_F - \int_{\Sigma_t} |\hat{h}|_g^2 d\mu_F.
\]

Noting \( |\hat{h}|_g^2 \geq H_F^2 / 2\), we get
\[
\frac{d}{dt} \int_{\Sigma_t} H_F^2 d\mu_F \leq \frac{1}{2} \left( 4 |\partial W|_F - \int_{\Sigma_t} H_F^2 d\mu_F \right).
\]
Therefore in view of \((d/dt)|\Sigma_t|_F = |\Sigma_t|_F\) we conclude
\[
\frac{d}{dt} \left( |\Sigma_t|_F \left( 4|\partial W|_F - \int_{\Sigma_t} H_F^2 \, d\mu_F \right) \right) \\
\geq \frac{1}{2} |\Sigma_t|_F \left( 4|\partial W|_F - \int_{\Sigma_t} H_F^2 \, d\mu_F \right) - \frac{1}{2} |\Sigma_t|_F \left( 4|\partial W|_F - \int_{\Sigma_t} H_F^2 \, d\mu_F \right) \\
= 0.
\]
So the anisotropic Hawking mass is non-decreasing in \(t\).

Now assume \((d/dt)m_H^F(\Sigma_t) = 0\) at some time \(t > 0\). Then checking the above argument we see that \(H_F\) is constant on \(\Sigma_t\), which implies that \(\Sigma_t\) is a translated scaled Wulff shape (\([16]\)). So the initial surface \(\Sigma\) is a translated scaled Wulff shape. The proof is complete.

\[\square\]

Proof of Theorem 6. For the proof we still use the inverse anisotropic mean curvature flow. Recall along this flow we have proved
\[
\text{Cap}_{F,p}(K) \leq \left( \int_0^\infty T_p^{\frac{1}{p-1}}(t) \, dt \right)^{1-p},
\]
where
\[
T_p(t) = \int_{\Sigma_t} H_F^{p-1} \, d\mu_F \leq \left( \int_{\Sigma_t} H_F^2 \, d\mu_F \right)^{\frac{p-1}{2}} (|\Sigma_t|_F)^{\frac{3-p}{2}},
\]
(4.1)
after using the H\ölder inequality.

Note that \(m_H^F(\Sigma_t)\) is monotone non-decreasing in \(t\). So
\[
m_H^F(\Sigma) \leq m_H^F(\Sigma_t) = \sqrt{|\Sigma_t|_F} \left( 1 - \frac{\int_{\Sigma_t} H_F^2 \, d\mu_F}{4|\partial W|_F} \right),
\]
which means
\[
\int_{\Sigma_t} H_F^2 \, d\mu_F \leq 4|\partial W|_F \left( 1 - \frac{4|\partial W|_F m_H^F(\Sigma)}{|\Sigma_t|_F} \right).
\]
Consequently, we get
\[
T_p(t) \leq (4|\partial W|_F)^{\frac{p-1}{2}} \left( 1 - \sqrt{\frac{4|\partial W|_F}{|\Sigma_t|_F} m_H^F(\Sigma)} \right)^{\frac{p-1}{2}} (|\Sigma_t|_F)^{\frac{3-p}{2}} \\
= (4|\partial W|_F)^{\frac{p-1}{2}} \left( 1 - \sqrt{\frac{4|\partial W|_F}{|\Sigma|_F e^{t/2}} m_H^F(\Sigma)} \right)^{\frac{p-1}{2}} (|\Sigma|_F e^{t/2})^{\frac{3-p}{2}} \\
= (4|\partial W|_F)^{\frac{p-1}{2}} |\Sigma|_F^{\frac{3-p}{2}} \left( 1 - \sqrt{\frac{4|\partial W|_F}{|\Sigma|_F} m_H^F(\Sigma) e^{-t/2}} \right)^{\frac{p-1}{2}} e^{-\frac{3-p}{2} t},
\]
where we used \(|\Sigma_t|_F = |\Sigma|_F e^{t/2}|. \]
So we get
\[
\text{Cap}_{F,p}(K) \leq \left( \int_0^\infty T_p^{-1/p} (t) \, dt \right)^{1-p} \\
\leq (4|\partial W|_F)^{\frac{p+1}{2}} |\Sigma|^{\frac{3-p}{2}} \left( \int_0^\infty \left( 1 - \sqrt{\frac{4|\partial W|_F}{|\Sigma|_F}} m_H^F(\Sigma) e^{-t/2} \right)^{-1/2} e^{-\frac{3-p}{2} t} \, dt \right)^{1-p}.
\]

If \( m_H^F(\Sigma) = 0 \), then by Corollary 27 the surface \( \Sigma \) is a translated scaled Wulff shape \( r_0 \partial W + x_0 \) \( (r_0 > 0, x_0 \in \mathbb{R}^3) \) and we can compute directly to get
\[
\text{Cap}_{F,p}(K) = \left( \frac{3-p}{p-1} \right)^{p-1} |\partial W|_F r_0^{3-p}.
\]

If \( m_H^F(\Sigma) < 0 \), then using the change of variables
\[- \sqrt{\frac{4|\partial W|_F}{|\Sigma|_F}} m_H^F(\Sigma) e^{-t/2} = \frac{p-1}{2} r^{3-p},\]
we get by direct computation
\[
\text{Cap}_{F,p}(K) \leq \left( \frac{3-p}{p-1} \right)^{p-1} |\partial W|_F^{\frac{p+1}{2}} |\Sigma|^{\frac{3-p}{2}} \left( \int_\Sigma \frac{H^2_F}{4|\partial W|_F} \, d\mu_F - 1 \right)^{3-p} \theta^{1-p},
\]
where
\[
\theta := \int_0^1 \left( \frac{\int_\Sigma \frac{H^2_F}{4|\partial W|_F} \, d\mu_F - 1}{4|\partial W|_F} \right)^{\frac{3-p}{2}} \left( 1 + r^{\frac{p-1}{2}} \right)^{-\frac{1}{2}} \, dr.
\]

Should the equality hold, then (4.1) must be an equality. The Hölder inequality becomes an equality, which implies that \( H_F \) is constant. So \( \Sigma_t \) and then \( \Sigma \) are translated scaled Wulff shapes (16), which is impossible. Thus we cannot have the equality in this case. The proof is now complete.

\[\square\]

**Proof of Corollary 8.** Let \( p = 2 \) in Theorem 6. If \( m_H^F(\Sigma) = 0 \), the conclusion follows immediately.

If \( m_H^F(\Sigma) < 0 \), we obtain
\[
\text{Cap}_{F,2}(K) \leq \sqrt{|\partial W|_F |\Sigma|_F} \left( \frac{1}{4|\partial W|_F} \int_\Sigma H^2_F \, d\mu_F - 1 \right) \cdot \theta^{-1},
\]
where
\[
\theta = \int_0^1 \left( \frac{\int_\Sigma \frac{H^2_F}{4|\partial W|_F} \, d\mu_F - 1}{4|\partial W|_F} \right) \, (1 + r)^{-\frac{1}{2}} \, dr = 2 \left( \sqrt{\frac{1}{4|\partial W|_F} \int_\Sigma H^2_F \, d\mu_F - 1} \right).
\]
So we get
\[ \text{Cap}_{F,2}(K) < \frac{1}{2} \sqrt{\frac{1}{4\|\partial W\|_F}} \left( 1 + \frac{1}{\|\partial W\|_F} \int_{\Sigma} H_F^2 \, d\mu_F \right). \]
Combining both cases we finish the proof of Corollary 8. □

5. PROOF OF THEOREM 9

5.1. The case that $M = \partial K$ is convex. In this case we use the anisotropic unit speed flow
\[
\begin{cases}
X : N \times [0, T) \to \mathbb{R}^n, \\
\partial_t X = \nu_F, \\
X(N, 0) = M.
\end{cases}
\]
So we can derive $D\psi = F^{-1}(\nu)\nu$ and then
\[
T_p(t) = \int_{\psi(x) = t} \frac{F_p(D\psi)}{|D\psi|} \, d\mu = \int_{\psi(x) = t} F(\nu) \, d\mu.
\]
Note that on $M_t = \{x|\psi(x) = t\}$ we have
\[
X(z, t) = X(z, 0) + t\nu_F(z, 0), \quad z \in N.
\]
Let $\Omega$ be the volume form on $\mathbb{R}^n$ and choose a local coordinate system \{y\}^{n-1}_{i=1} on $N$ such that \{\partial X(z, 0)/\partial y_i\}^{n-1}_{i=1} correspond to the anisotropic principal directions of $M = X(N, 0)$ at the point $X(z, 0)$. Then we get
\[
\begin{align*}
    d\mu(X(z, t)) &= \Omega \left( \nu, \frac{\partial X(z, t)}{\partial y^1}, \ldots, \frac{\partial X(z, t)}{\partial y^{n-1}} \right) \, dy^1 \wedge \cdots \wedge dy^{n-1} \\
    &= \left( \prod_{i=1}^{n-1} (1 + \kappa_i^F t) \right) \Omega \left( \nu, \frac{\partial X(z, 0)}{\partial y^1}, \ldots, \frac{\partial X(z, 0)}{\partial y^{n-1}} \right) \, dy^1 \wedge \cdots \wedge dy^{n-1} \\
    &= \left( \prod_{i=1}^{n-1} (1 + \kappa_i^F t) \right) d\mu(X(z, 0)).
\end{align*}
\]
As a consequence, we obtain (note that at $X(z, 0)$ and $X(z, t)$ the outward unit normal is the same $\nu$)
\[
T_p(t) = \int_M \prod_{i=1}^{n-1} (1 + \kappa_i^F t) F(\nu) \, d\mu = \sum_{i=0}^{n-1} \int_M \sigma_i(\kappa_i^F) d\mu_F \cdot t^i.
\]
Therefore we get
\[
\begin{align*}
    \text{Cap}_{F,p}(K) &\leq \left( \int_0^{\infty} T_p^{-\frac{1}{p-1}}(t) \, dt \right)^{1-p} \\
    &= \left( \int_0^{\infty} \left( \sum_{i=0}^{n-1} \int_M \sigma_i(\kappa_i^F) d\mu_F \cdot t^i \right)^{\frac{1}{1-p}} \, dt \right)^{1-p}.
\end{align*}
\]
Next assume that the equality holds. Then the hypersurfaces $M_t$ ($t \geq 0$) constructed above are exactly the level sets of the $p$-capacitary potential $u$ of the set $K$ and $\lambda(\psi(x)) = u(x)$. In particular, on $M$ we obtain

$$F(Du) = |\lambda'(0)|F(D\psi) = |\lambda'(0)|F(F^{-1}(\nu)\nu) = |\lambda'(0)|,$$

a constant. Then by [6, Theorem 1.2], we conclude that $M$ is a translated scaled Wulff shape. So we finish the proof in this case.

5.2. The case that $M$ is star-shaped with respect to the origin. In this case we expand $M$ by homothety so that $M_t = (1 + t)M$ for $t \geq 0$. So we have

$$\psi((1 + t)X) = t, \quad X \in M.$$

Let $X_t := (1 + t)X$. So taking the derivative with respect to $t$ yields

$$\langle D\psi(X_t), X \rangle = 1.$$

Note that at points $X_t$ and $X$ the outward unit normal is the same $\nu$, and $D\psi(X_t) = |D\psi(X_t)|\nu$. Hence we get

$$|D\psi(X_t)| = \frac{1}{\langle X, \nu \rangle}.$$

Then we have (note that $d\mu(X_t) = (1 + t)^{n-1}d\mu(X)$)

$$T_p(t) = \int_{M_t} \frac{F^p(D\psi)}{|D\psi|}d\mu = \int_{M_t} \frac{F^p(\nu)}{\langle X, \nu \rangle^{p-1}}d\mu = (1 + t)^{n-1} \int_M h_F^{1-p}d\mu_F,$$

where $h_F = \langle X, \nu \rangle / F(\nu)$ is the anisotropic support function. So we get

$$\text{Cap}_{F,p}(K) \leq \left( \int_0^\infty T_p^{-1}(t)dt \right)^{1-p} (n-p)^{p-1} \int_M h_F^{1-p}d\mu_F.$$

Next assume that the equality holds. Then again the hypersurfaces $M_t$ ($t \geq 0$) constructed above are the level sets of the $p$-capacitary potential $u$ of the set $K$. Let $\rho(t) = u(X)$ for $X \in M_t$, $t \geq 0$.

Now consider any two points $X, Y \in M$. Then $(1 + t)X$ and $(1 + t)Y$ belong to $M_t$ for $t \geq 0$. We apply Proposition 3.1 in [31] (cf. Lemma 5.2 in [4]) to deduce

$$\text{Cap}_{F,p}(K)^{1/(p-1)} = \lim_{t \to +\infty} \frac{u((1 + t)X)}{c(n, p)(F^0((1 + t)X))^{(p-n)/(p-1)}}$$

$$= (F^0(X))^{(n-p)/(p-1)} \lim_{t \to +\infty} \frac{\rho(t)}{c(n, p)(1 + t)^{(p-n)/(p-1)}},$$

where $c(n, p)$ is a constant depending only on $n$ and $p$. The same holds for $Y$. Thus we have $F^0(X) = F^0(Y)$, which means that $M$ is a scaled Wulff shape centered at the origin. So we finish the proof in this case as well.
6. Proof of Theorem 12

Proof. Let \( u \) be the anisotropic \( p \)-capacitary potential for \( K \), i.e., \( u \) solves (2.1). Moreover, thanks to Proposition 17, we know that the set \( K_t := \{ x | u(x) \geq t \} \) is convex with smooth boundary for any \( 0 < t < 1 \).

Now recall that for a convex body \( K \) (i.e., a compact convex set with non-empty interior), its anisotropic perimeter \( |\partial K|_F \) can be expressed as a mixed volume of \( K \) and \( \overline{W} \). More precisely, we have

\[
|\partial K|_F = \frac{1}{n} V(1)(K, \overline{W}),
\]

where \( V(i)(K, \overline{W}) \) \((0 \leq i \leq n)\) is the \( i \)th mixed volume of the convex bodies \( K \) and \( \overline{W} \) defined as in the expression

\[
|K + t\overline{W}| = \sum_{i=0}^{n} \binom{n}{i} t^i V(i)(K, \overline{W}), \quad t \geq 0.
\]

See the classical book [25] for the definition and properties on mixed volumes. In particular, it follows from [25, (5.25)] that for any two convex bodies \( K_1 \) and \( K_2 \) with \( K_1 \subset K_2 \) we have \( V(i)(K_1, \overline{W}) \leq V(i)(K_2, \overline{W}) \), \( 0 \leq i \leq n \).

In our case we know that \( K \) and \( K_t \) \((0 < t < 1)\) are convex bodies with \( K \subset K_t \), which implies \( |\partial K|_F \leq |\partial K_t|_F \) for any \( t \in (0, 1) \). Furthermore, we make a key observation that in fact,

\[
|\partial K|_F < |\partial K_t|_F, \quad \forall t \in (0, 1).
\]

This observation will be used in the argument for the equality case at the end of the proof and can be checked based on the fact that the Hausdorff distance of \( K \) and \( K_t \) is positive for a fixed \( t \in (0, 1) \).

Next applying the Hölder inequality, the co-area formula and the fact

\[
\text{Cap}_{F,p}(K) = \int_{u(x) = t} \frac{F^p(Du)}{|Du|^p} \, d\mu \quad \text{for any} \quad t \in (0, 1),
\]

which can be proved straightforwardly (see Lemma 4.2 in [4]; cf. Lemma 2.16 in [10]), we derive (note that \( \nu = -Du/|Du| \) on \( \{ x | u(x) = t \} \))

\[
|\partial K|_F < \int_{u(x) = t} \left( |Du|^{\frac{p-1}{p}} F(\nu) \right) |Du|^{\frac{1}{p}} \, d\mu
\leq \left( \int_{u(x) = t} \frac{F^p(Du)}{|Du|^p} \, d\mu \right)^{\frac{1}{p}} \left( \int_{u(x) = t} |Du|^{-1} \, d\mu \right)^{\frac{p-1}{p}} = \text{Cap}_{F,p}(K)^{\frac{1}{p}} \left( -\frac{d}{dt}[K_t] \right)^{\frac{p-1}{p}}.
\]

It follows that

\[
\left( \frac{|\partial K|_F}{\text{Cap}_{F,p}(K)^{\frac{1}{p}}} \right)^{\frac{p}{p-1}} < -\frac{d}{dt}[K_t],
\]
Integrating the above inequality over \((t, 1)\) yields

\[
(1 - t) \left( \frac{|\partial K|_F}{\text{Cap}_{F,p}(K)^{\frac{1}{p}}}} \right)^{\frac{p}{p-1}} < |K_t| - |K|. \tag{6.3}
\]

Now we recall the following isocapacitary inequality (see, e.g., (2.2.8) in [21])

\[
\text{Cap}_{F,p}(K) \geq n|\mathcal{W}|^{\frac{n}{np}} \left\{ \frac{\text{Cap}_{F,p}(K_t)}{((n - p)/(p - 1))^{p-1}} \right\}^{\frac{n}{np}}.
\]

Applying the above inequality to \(K_t\) gives us

\[
|K_t| \leq n^{-\frac{n}{np}} |\mathcal{W}|^{-\frac{n}{np}} \left\{ \frac{\text{Cap}_{F,p}(K_t)}{((n - p)/(p - 1))^{p-1}} \right\}^{\frac{n}{np}} |K|.
\]

Combining it with (6.3), we get

\[
(1 - t) \left( \frac{|\partial K|_F}{\text{Cap}_{F,p}(K)^{\frac{1}{p}}}} \right)^{\frac{p}{p-1}} < n^{-\frac{n}{np}} |\mathcal{W}|^{-\frac{n}{np}} \left\{ \frac{\text{Cap}_{F,p}(K_t)}{((n - p)/(p - 1))^{p-1}} \right\}^{\frac{n}{np}} - |K|.
\]

Notice that \(u/t\) is the anisotropic \(p\)-capacitary potential for the set \(K_t\). So we have

\[
\text{Cap}_{F,p}(K_t) = \int_{u(x) = t} \frac{F_p(D(u/t))}{|D(u/t)|} d\mu = t^{1-p}\text{Cap}_{F,p}(K), \quad 0 < t < 1.
\]

Then we get

\[
(1 - t) \left( \frac{|\partial K|_F}{\text{Cap}_{F,p}(K)^{\frac{1}{p}}}} \right)^{\frac{p}{p-1}} < n^{-\frac{n}{np}} |\mathcal{W}|^{-\frac{n}{np}} \left\{ \frac{t^{1-p}\text{Cap}_{F,p}(K)}{((n - p)/(p - 1))^{p-1}} \right\}^{\frac{n}{np}} - |K|, \quad t \in (0, 1). \tag{6.4}
\]

Next define

\[
\varphi(t) := (1 - t) \left( \frac{|\partial K|_F}{\text{Cap}_{F,p}(K)^{\frac{1}{p}}}} \right)^{\frac{p}{p-1}} - n^{-\frac{n}{np}} |\mathcal{W}|^{-\frac{n}{np}} \left\{ \frac{t^{1-p}\text{Cap}_{F,p}(K)}{((n - p)/(p - 1))^{p-1}} \right\}^{\frac{n}{np}} - |K|,
\]

\[
\bar{t} := \frac{p - 1}{n - p} n^{\frac{1}{p-1}} |\mathcal{W}|^{\frac{1}{p-1}} |K|^{\frac{1}{p-1}} \text{Cap}_{F,p}(K)^{\frac{1}{p-1}}.
\]

Note that by the isocapacitary inequality, we have \(\bar{t} \geq 1\).
We claim that $\varphi(t) \leq 0$ on the interval $(0, \bar{t})$ and $\varphi(t) < 0$ on $(0, 1)$. If $t \in (0, 1)$, the claim follows from (6.4). If $1 \leq t \leq \bar{t}$, it follows from the fact

$$(1 - t) \left( \frac{|\partial K|_F}{\text{Cap}_{F,p}(K)^{\frac{1}{p}}} \right) \leq 0$$

$$\leq n^{-\frac{n}{p-p}} |\mathcal{W}|^{-\frac{n}{p-p}} \frac{t}{(n-p)/(p-1)} \left( \frac{\text{Cap}_{F,p}(K)}{((n-p)/(p-1))^{p-1}} \right)^{\frac{n}{p-p}} - |K|.$$ 

So we obtain the claim. By the claim we have $\sup_{t \in (0, \bar{t})} \varphi(t) \leq 0$.

Next we compute

$$\varphi'(t) = - \left( \frac{|\partial K|_F}{\text{Cap}_{F,p}(K)^{\frac{1}{p}}} \right)^{\frac{p}{p-1}}$$

$$- n^{-\frac{n}{p-p}} |\mathcal{W}|^{-\frac{n}{p-p}} \frac{n(1-p)}{n-p} \left( \frac{\text{Cap}_{F,p}(K)}{((n-p)/(p-1))^{p-1}} \right)^{\frac{n}{p-p}} t^{-\frac{p(n-1)}{n-p}}.$$ 

So the unique critical point $t_0$ of $\varphi(t)$ reads

$$t_0 = (n|\mathcal{W}|)^{-\frac{1}{n-1}} \frac{p-1}{n-p} |\partial K|_F^{-\frac{n}{p-1}} \text{Cap}_{F,p}(K)^{1/(p-1)}.$$ 

Now recall the anisotropic isoperimetric inequality (see e.g. Theorem 20.8 in [20] or Corollary 2.8 in [11])

$$|\partial K|_F \geq n|\mathcal{W}|^{1/n} |K|^{(n-1)/n},$$

with the equality holding if and only if $\partial K$ is a translated scaled Wulff shape. Using it we know $t_0 \leq \bar{t}$. Note that $\varphi(0^+) = -\infty$, $\varphi(t)$ is non-decreasing on $(0, t_0]$ and non-increasing on $[t_0, \bar{t}]$. So $\varphi(t_0) \leq 0$, which by direct computation is equivalent to

$$\frac{p(n-1)}{n(n-p)} |\partial K|_F^{\frac{n}{p-1}} |\partial \mathcal{W}|_{F}^{\frac{1}{n-1}} \geq |K| + \frac{|\partial K|_F^{p/(p-1)}}{\text{Cap}_{F,p}(K)^{1/(p-1)}}.$$ 

the desired inequality.

Last, if $\partial K$ is a translated scaled Wulff shape, we can check directly that the equality in the above inequality holds. Conversely, assume that we have the equality $\varphi(t_0) = 0$, i.e.,

$$\frac{p(n-1)}{n(n-p)} |\partial K|_F^{\frac{n}{p-1}} |\partial \mathcal{W}|_{F}^{\frac{1}{n-1}} = |K| + \frac{|\partial K|_F^{p/(p-1)}}{\text{Cap}_{F,p}(K)^{1/(p-1)}}.$$ 

Using the anisotropic isoperimetric inequality (6.5) to replace $|K|$ leads to

$$\frac{p-1}{n-p} |\partial \mathcal{W}|_{F}^{\frac{1}{n-1}} |\partial K|_F^{\frac{n}{p-1}(n-1)} \text{Cap}_{F,p}(K)^{\frac{1}{p-1}} \leq 1.$$ 

This is nothing but $t_0 \leq 1$. However, $t_0 < 1$ can not occur, since $\varphi(t_0) = 0$ and by the previous claim we know $\varphi(t) < 0$ for $t \in (0, 1)$. (This is where
we use our key observation (6.2).) Thus we must have $t_0 = 1$ and then we conclude that $\partial K$ is a translated scaled Wulff shape from the rigidity part of the anisotropic isoperimetric inequality.

Now the proof of Theorem 12 is complete. □

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