THE SINGULARITY CATEGORY OF A NAKAYAMA ALGEBRA

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Abstract. Let $A$ be a Nakayama algebra. We give a description of the singularity category of $A$ inside its stable module category. We prove that there is a duality between the singularity category of $A$ and the singularity category of its opposite algebra. As a consequence, the resolution quiver of $A$ and the resolution quiver of its opposite algebra have the same number of cycles and the same number of cyclic vertices.

1. Introduction

Let $A$ be an artin algebra. Denote by $A$-mod the category of finitely generated left $A$-modules, and by $D^b(A$-mod) the bounded derived category of $A$-mod. Recall that a complex in $D^b(A$-mod) is perfect provided that it is isomorphic to a bounded complex of finitely generated projective $A$-modules. Following [3, 11, 16], the singularity category $D_{sg}(A)$ of $A$ is the quotient triangulated category of $D^b(A$-mod) with respect to the full subcategory consisting of perfect complexes. Recently, the singularity category of a Nakayama algebra is described in [7].

Let $A$ be a connected Nakayama algebra without simple projective modules. Following [19], the resolution quiver $R(A)$ of $A$ is defined as follows: the vertex set is the set of isomorphism classes of simple $A$-modules, and there is an arrow from $S$ to $\tau \text{soc} P(S)$ for each simple $A$-module $S$; see also [9]. Here, $P(S)$ is the projective cover of $S$, ‘soc’ denotes the socle of a module, and $\tau = D \text{Tr}$ is the Auslander-Reiten translation [1]. A simple $A$-module is called cyclic provided that it lies in a cycle of $R(A)$.

The following consideration is inspired by [19]. Let $A$ be a connected Nakayama algebra of infinite global dimension. Let $\mathcal{S}_c$ be a complete set of pairwise non-isomorphic cyclic simple $A$-modules. Let $\mathcal{X}_c$ be the set formed by indecomposable $A$-modules $X$ such that $\text{top} X$ and $\tau \text{soc} X$ both belong to $\mathcal{S}_c$. Here, ‘top’ denotes the top of a module. Denote by $\mathcal{F}$ the full subcategory of $A$-mod whose objects are finite direct sums of objects in $\mathcal{X}_c$. It turns out that $\mathcal{F}$ is a Frobenius abelian category, and it is equivalent to $A'$-mod with $A'$ a connected selfinjective Nakayama algebra. Denote by $\mathcal{F}'$ the stable category of $\mathcal{F}$ modulo projective objects; it is a triangulated category by [10]. We emphasize that the stable category $\mathcal{F}'$ is a full subcategory of the stable module category $A$-mod of $A$.

The well-known result of [3, 11] describes the singularity category of a Gorenstein algebra $A$ via the subcategory of $A$-mod formed by Gorenstein projective modules. Here, we recall that an artin algebra is Gorenstein if the injective dimension of the regular module is finite on both sides. In general, a Nakayama algebra is not Gorenstein [19, 7]. The following result describes the singularity category of a Nakayama algebra via the subcategory $\mathcal{F}'$ of $A$-mod. For a Gorenstein Nakayama algebra, these two descriptions coincide; compare [19].
Theorem 1.1. Let $A$ be a connected Nakayama algebra of infinite global dimension. Then the singularity category $D_{sg}(A)$ and the stable category $F$ are triangle equivalent.

Denote by $A$-inj the category of finitely generated injective $A$-modules, and by $K^b(A$-inj) the bounded homotopy category of $A$-inj. We view $K^b(A$-inj) as a thick subcategory of $D^b(A$-mod) via the canonical functor. Then the quotient triangulated category $D^b(A$-mod)/$K^b(A$-inj) is triangle equivalent to the opposite category of the singularity category $D_{sg}(A^{\text{op}})$ of $A^{\text{op}}$. Here, $A^{\text{op}}$ is the opposite algebra of $A$. In general, it seems that for an arbitrary artin algebra $A$, there is no obvious relation between $D_{sg}(A)$ and $D_{sg}(A^{\text{op}})$. However, we have the following result for a Nakayama algebra.

Proposition 1.2. Let $A$ be a Nakayama algebra. Then the singularity category $D_{sg}(A)$ is triangle equivalent to $D^b(A$-mod)/$K^b(A$-inj). Equivalently, there is a triangle duality between $D_{sg}(A)$ and $D_{sg}(A^{\text{op}})$.

Let $A$ be a connected Nakayama algebra of infinite global dimension. Recall from [20] that the resolution quivers $R(A)$ and $R(A^{\text{op}})$ have the same number of cyclic vertices. The following result strengthens the previous one by a different method.

Proposition 1.3. Let $A$ be a connected Nakayama algebra of infinite global dimension. Then the resolution quivers $R(A)$ and $R(A^{\text{op}})$ have the same number of cycles and the same number of cyclic vertices.

The paper is organized as follows. In Section 2, we recall some facts on singularity categories of artin algebras and the simplification in the sense of [18]. In Section 3, we introduce the Frobenius subcategory $F$ and prove Theorem 1.1. The proofs of Propositions 1.2 and 1.3 are given in Sections 4 and 5, respectively.

Throughout this paper, we fix a commutative artinian ring $R$. All categories, morphisms and functors are $R$-linear.

2. Preliminaries

We first recall some facts on the singularity category of an artin algebra.

Let $A$ be an artin algebra over $R$. Recall that $A$-mod denotes the category of finitely generated left $A$-modules. Let $A$-proj denote the full subcategory consisting of projective $A$-modules, and $A$-inj the full subcategory consisting of injective $A$-modules. Denote by $A$-mod the projectively stable category of finitely generated $A$-modules; it is obtained from $A$-mod by factoring out the ideal of all maps which factor through projective $A$-modules; see [11 IV.1].

Recall that for an $A$-module $M$, its syzygy $\Omega(M)$ is the kernel of its projective cover $P(M) \to M$. This gives rise to the syzygy functor $\Omega : A$-mod $\to A$-mod. Let $\Omega^{i}(M) = M$ and $\Omega^{i+1}(M) = \Omega(\Omega^{i}(M))$ for $i \geq 0$. Denote by $\Omega^{i}(A$-mod) the full subcategory of $A$-mod formed by modules $M$ such that there is an exact sequence $0 \to M \to P_{n-1} \to \cdots \to P_{1} \to P_{0}$ with each $P_{j}$ projective. We also denote by $\Omega^{i}_{0}(A$-mod) the full subcategory of $\Omega^{i}(A$-mod) formed by modules without indecomposable projective direct summands.

Recall that $D^{b}(A$-mod) denotes the bounded derived category of $A$-mod, whose translation functor is denoted by $[1]$. For each integer $n$, let $[n]$ denote the $n$-th power of $[1]$. The category $A$-mod is viewed as a full subcategory of $D^{b}(A$-mod) by identifying an $A$-module with the corresponding stalk complex concentrated at degree zero. Recall that a complex in $D^{b}(A$-mod) is perfect provided that it is isomorphic to a bounded complex of finitely generated projective $A$-modules. Perfect complexes form a thick subcategory of $D^{b}(A$-mod), which is denoted by
perf(A). Here, we recall that a triangulated subcategory is thick if it is closed under direct summands.

Following [5, 11, 16], the quotient triangulated category

$$\mathcal{D}_{sg}(A) = \mathcal{D}^b(A\text{-mod})/\text{perf}(A)$$

is called the singularity category of $A$. Denote by $q : \mathcal{D}^b(A\text{-mod}) \to \mathcal{D}_{sg}(A)$ the quotient functor. We recall that the objects in $\mathcal{D}_{sg}(A)$ are bounded complexes of finitely generated $A$-modules. The translation functor of $\mathcal{D}_{sg}(A)$ is also denoted by $[1]$.

The following results are well known.

**Lemma 2.1** ([5, Lemma 2.1]). Let $X$ be a complex in $\mathcal{D}_{sg}(A)$ and $n_0 > 0$. Then for any $n$ sufficiently large, there exists a module $M$ in $\Omega^{n_0}(A\text{-mod})$ such that $X \simeq q(M)[n]$.

**Lemma 2.2** ([5, Lemma 2.2]). Let $0 \to N \to P_{n-1} \to \cdots \to P_0 \to M \to 0$ be an exact sequence in $A$-mod with each $P_i$ projective. Then there is an isomorphism $q(M) \simeq q(N)[n]$ in $\mathcal{D}_{sg}(A)$. In particular, there is a natural isomorphism $\theta^n_M : q(M) \simeq q(\Omega^n(M))[n]$ for any $M$ in $A$-mod and $n \geq 0$.

Observe that the composition $A$-mod $\to \mathcal{D}^b(A\text{-mod}) \overset{q}{\to} \mathcal{D}_{sg}(A)$ vanishes on projective modules. Then it induces a unique functor $q' : A\text{-mod} \to \mathcal{D}_{sg}(A)$. It follows from Lemma 2.2 that for each $n \geq 0$, the following diagram of functors

$$\begin{array}{ccc}
A\text{-mod} & \xrightarrow{\Omega^n} & A\text{-mod} \\
q' & & q'
\end{array}$$

$$\begin{array}{ccc}
\mathcal{D}_{sg}(A) & \xrightarrow{[-n]} & \mathcal{D}_{sg}(A)
\end{array}$$

is commutative. Let $M$ and $N$ be in $A$-mod and $n \geq 0$. Lemma 2.2 yields a natural map

$$\Phi^n : \text{Hom}_A(\Omega^n(M), \Omega^n(N)) \to \text{Hom}_{\mathcal{D}_{sg}(A)}(q(M), q(N)).$$

Here, $\Phi^0$ is induced by $q'$ and $\Phi^n(f) = (\theta^n_M)^{-1} \circ (q'(f)[n]) \circ \theta^n_M$ for $n \geq 1$.

Consider the following chain of maps $\{G^{n,n+1}\}_{n \geq 0}$ such that

$$G^{n,n+1} : \text{Hom}_A(\Omega^n(M), \Omega^n(N)) \to \text{Hom}_A(\Omega^{n+1}(M), \Omega^{n+1}(N))$$

is induced by the syzygy functor $\Omega$. The sequence $\{\Phi^n\}_{n \geq 0}$ is compatible with $\{G^{n,n+1}\}_{n \geq 0}$, that is, $\Phi^{n+1} \circ G^{n,n+1} = \Phi^n$ for each $n \geq 0$. Then we obtain an induced map

$$\Phi : \lim_{n \geq 0} \text{Hom}_A(\Omega^n(M), \Omega^n(N)) \to \text{Hom}_{\mathcal{D}_{sg}(A)}(q(M), q(N)).$$

**Lemma 2.3** ([13, Exemple 2.3]). Let $M$ and $N$ be in $A$-mod. Then there is a natural isomorphism

$$\Phi : \lim_{n \geq 0} \text{Hom}_A(\Omega^n(M), \Omega^n(N)) \simeq \text{Hom}_{\mathcal{D}_{sg}(A)}(q(M), q(N)).$$

Next we recall the simplification in the sense of [13].

Let $\mathcal{A}$ be an abelian category. Recall that an object $X$ in $\mathcal{A}$ is a brick if $\text{End}_\mathcal{A}(X)$ is a division ring. Two objects $X$ and $Y$ are orthogonal if $\text{Hom}_\mathcal{A}(X,Y) = 0$ and $\text{Hom}_\mathcal{A}(Y,X) = 0$. A full subcategory $\mathcal{W}$ of $\mathcal{A}$ is called a wide subcategory if it is closed under kernels, cokernels and extensions. In particular, $\mathcal{W}$ is an abelian category and the inclusion functor is exact. Recall that an abelian category $\mathcal{A}$ is called a length category provided that each object in $\mathcal{A}$ has a composition series.
Let \( \mathcal{E} \) be a set of objects in an abelian category \( \mathcal{A} \). For an object \( C \) in \( \mathcal{A} \), an \( \mathcal{E} \)-filtration of \( C \) is given by a sequence of subobjects
\[
0 = C_0 \subseteq C_1 \subseteq C_2 \subseteq \cdots \subseteq C_m = C,
\]
such that each factor \( C_i/C_{i-1} \) belongs to \( \mathcal{E} \) for \( 1 \leq i \leq m \). Denote by \( \mathcal{F}(\mathcal{E}) \) the full subcategory of \( \mathcal{A} \) formed by objects in \( \mathcal{A} \) with an \( \mathcal{E} \)-filtration.

**Lemma 2.4** ([18 Theorem 1.2]). \( \mathcal{E} \) be a set of pairwise orthogonal bricks in \( \mathcal{A} \). Then \( \mathcal{F}(\mathcal{E}) \) is a wide subcategory of \( \mathcal{A} \); moreover, \( \mathcal{F}(\mathcal{E}) \) is a length category and \( \mathcal{E} \) is a complete set of pairwise non-isomorphic simple objects in \( \mathcal{F}(\mathcal{E}) \).

Let \( A \) be a connected Nakayama algebra without simple projective modules. Recall that the vertex set of the resolution quiver \( R(A) \) of \( A \) is the set of isomorphism classes of simple \( A \)-modules, and there is an arrow from \( S \) to \( \gamma(S) = \tau \text{soc} P(S) \) for each simple \( A \)-module \( S \). Since each vertex in \( R(A) \) is the start of a unique arrow, each connected component of \( R(A) \) contains precisely one cycle. A simple \( A \)-module is called cyclic provided that it lies in a cycle of \( R(A) \).

Let \( A \) be a connected Nakayama algebra of infinite global dimension. In particular, \( A \) has no simple projective modules. Let \( \mathcal{S} \) be a complete set of pairwise non-isomorphic simple \( A \)-modules. Denote by \( \mathcal{S}_c \) the subset of all cyclic simple \( A \)-modules, and by \( \mathcal{S}_{nc} \) the subset of all noncyclic simple \( A \)-modules.

**Lemma 2.5.** Let \( A \) be a connected Nakayama algebra of infinite global dimension. Then \( \mathcal{S}_c \) is a complete set of pairwise non-isomorphic simple \( A \)-modules of infinite injective dimension, and \( \mathcal{S}_{nc} \) is a complete set of pairwise non-isomorphic simple \( A \)-modules of finite injective dimension.

**Proof.** This is dual to [14 Corollaries 3.6 and 3.7]. \( \square \)

We will need the following fact. Recall that ‘top’ denotes the top a module.

**Lemma 2.6** ([14 Corollary to Lemma 2]). Let \( A \) be a connected Nakayama algebra without simple projective modules. Assume that \( M \) is an indecomposable \( A \)-module and \( m \geq 0 \). Then either \( \Omega^{2m}(M) = 0 \) or else \( \text{top} \Omega^{2m}(M) = \gamma^m(\text{top} M) \).

3. A Frobenius Subcategory

In this section, we introduce a Frobenius subcategory in the module category of a Nakayama algebra, whose stable category is triangle equivalent to the singularity category of the given algebra.

Throughout this section, \( A \) is a connected Nakayama algebra of infinite global dimension. Denote by \( n(A) \) the number of the isomorphism classes of simple \( A \)-modules. Denote by \( l(M) \) the composition length of an \( A \)-module \( M \). Recall that \( \mathcal{S} \) denotes a complete set of pairwise non-isomorphic simple \( A \)-modules, \( \mathcal{S}_c \) the subset of all cyclic simple \( A \)-modules and \( \mathcal{S}_{nc} \) the subset of all noncyclic simple \( A \)-modules. Observe that the map \( \gamma \) restricts to a permutation on \( \mathcal{S}_c \). Let \( \mathcal{X}_c \) be the set formed by indecomposable \( A \)-modules \( X \) such that both top \( X \) and \( \tau \text{soc} X \) belong to \( \mathcal{S}_c \).

The proof of the following well-known lemma uses the structure of indecomposable modules over Nakayama algebras; see [1] IV.3 and VI.2. Each indecomposable \( A \)-module \( X \) is uniserial, and it is uniquely determined by its top and its composition length. Its composition factors from the top are \( S, \tau S, \ldots, \tau^{l-1} S \), where \( S = \text{top} X \) and \( l = l(X) \). In particular, the projective cover \( P(X) \) of \( X \) is indecomposable.

**Lemma 3.1.** Let \( M \) be an indecomposable \( A \)-module which contains a nonzero projective submodule \( P \). Then \( M \) is projective.
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**Proof.** Suppose that, on the contrary, $M$ is nonprojective. Then there is a proper surjective map $\pi : P(M) \to M$, where $P(M)$ is the projective cover of $M$. We have a proper surjective map $\pi^{-1}(P) \to P$; it splits, since $P$ is projective. This is impossible, since $P(M)$ is indecomposable, and thus its submodule $\pi^{-1}(P)$ is indecomposable. \hfill \Box

**Lemma 3.2.** Let $f : X \to Y$ be a morphism in $\mathcal{X}_c$. Then $\text{Ker } f$, $\text{Coker } f$ and $\text{Im } f$ belong to $\mathcal{X}_c \cup \{0\}$.

**Proof.** We may assume that $f$ is nonzero. Then we have $\text{top}(\text{Im } f) = \text{top } X$ and $\tau \text{ soc}(\text{Im } f) = \tau \text{ soc } Y$, both of which belong to $\mathcal{S}_c$. Thus, $\text{Im } f$ belongs to $\mathcal{X}_c$.

If $f$ is not a monomorphism, then $\text{top}(\text{Ker } f) = \tau \text{ soc}(\text{Im } f) = \tau \text{ soc } Y$ and $\tau \text{ soc}(\text{Ker } f) = \tau \text{ soc } X$. Thus, $\text{Ker } f$ belongs to $\mathcal{X}_c$.

If $f$ is not an epimorphism, then $\text{top}(\text{Coker } f) = \text{top } Y$ and $\tau \text{ soc}(\text{Coker } f) = \text{top}(\text{Im } f) = \text{top } X$. Thus, $\text{Coker } f$ belongs to $\mathcal{X}_c$. \hfill \Box

**Lemma 3.3.** Let $X$ be an object in $\mathcal{X}_c$. If $0 \subseteq X'' \subseteq X' \subseteq X$ are subobjects of $X$ such that $X'/X''$ belongs to $\mathcal{X}_c$, then $X''$ and $X/X'$ belong to $\mathcal{X}_c$.

**Proof.** Since both $\text{top } X'' = \tau \text{ soc } X'/X''$ and $\tau \text{ soc } X'' = \tau \text{ soc } X$ belong to $\mathcal{S}_c$, it follows from the definition that $X''$ belongs to $\mathcal{X}_c$. Similarly, since both $\text{top } X'/X' = \text{top } X$ and $\tau \text{ soc } X/X' = \tau \text{ soc } X'/X''$ belong to $\mathcal{S}_c$, it follows from the definition that $X'/X''$ belongs to $\mathcal{X}_c$. \hfill \Box

Denote by $\mathcal{P}_c$ a complete set of projective covers of modules in $\mathcal{S}_c$. We claim that $\mathcal{P}_c$ is a subset of $\mathcal{X}_c$. Indeed, we have $\text{top } P(S) = S$ and $\tau \text{ soc } P(S) = \gamma(S)$, both of which belong to $\mathcal{S}_c$. It follows that $\mathcal{X}_c$ is closed under projective covers.

For each $S$ in $\mathcal{S}_c$, let $E(S)$ denote the indecomposable $A$-module of the least composition length among those objects $X$ in $\mathcal{X}_c$ with $\text{top } X = S$. Inspired by [19] Section 4, we call $E(S)$ the elementary module associated to $S$. Denote by $\mathcal{E}_c$ the set of elementary modules. Recall that $F(\mathcal{E}_c)$ is the full subcategory of $A$-mod formed by $A$-modules with an $\mathcal{E}_c$-filtration.

The support of an $A$-module $M$ is the subset of $\mathcal{S}$ consisting of those simple $A$-modules appearing as a composition factor of $M$. For a set $\mathcal{X}$ of $A$-modules, we denote by $\text{add } \mathcal{X}$ the full subcategory of $A$-mod whose objects are direct summands of finite direct sums of objects in $\mathcal{X}$.

The following result is in spirit close to [19] Proposition 2. In particular, we prove that each elementary module $E$ is a brick and thus $l(E) \leq n(A)$.

**Proposition 3.4.** Let $A$ be a connected Nakayama algebra of infinite global dimension. Then the following statements hold.

1. The set $\mathcal{E}_c$ of elementary modules is a set of pairwise orthogonal bricks, and thus $F(\mathcal{E}_c)$ is a wide subcategory of $A$-mod.
2. $F(\mathcal{E}_c) = \text{add } \mathcal{X}_c$, which is closed under projective covers.
3. Let $E$ and $E'$ be elementary modules. Then $E = E'$ if and only if their supports have nonempty intersection.

**Proof.** (1) Let $f : E \to E'$ be a nonzero map between elementary modules. By Lemma 3.2 $\text{Im } f$ belongs to $\mathcal{X}_c$. However, $\text{Im } f$ is a factor module of $E$. By the definition of the elementary module $E$ we have $E = \text{Im } f$. Then $f$ is an injective map. Similarly, $f$ is a surjective map and thus an isomorphism. Therefore $\mathcal{E}_c$ is a set of pairwise orthogonal bricks.

By Lemma 2.4 $F(\mathcal{E}_c)$ is a wide subcategory of $A$-mod. In particular, it is closed under direct sums and direct summands.

(2) We prove that any module $X$ in $\mathcal{X}_c$ belongs to $F(\mathcal{E}_c)$, and thus $\text{add } \mathcal{X}_c \subseteq F(\mathcal{E}_c)$. We use induction on $l(X)$. Set $S = \text{top } X \in \mathcal{S}_c$. If $X = E(S) \in \mathcal{E}_c$, we are
that top $X$

Recall that each elementary module $E$ satisfies that top $E \in \mathcal{S}_e$ and $\tau \text{soc } E \in \mathcal{S}_e$. It follows from its $\mathcal{E}_e$-filtration that each indecomposable object $X$ in $\mathcal{F}(\mathcal{E}_e)$ satisfies that top $X \in \mathcal{S}_e$ and $\tau \text{soc } X \in \mathcal{S}_e$. Then by definition $X$ belongs to $\mathcal{X}_e$. Therefore add $\mathcal{X}_e \supseteq \mathcal{F}(\mathcal{E}_e)$, and thus $\mathcal{F}(\mathcal{E}_e) = \text{add } \mathcal{X}_e$. Since $\mathcal{X}_e$ is closed under projective covers, we infer that add $\mathcal{X}_e$ is closed under projective covers.

(3) Suppose that $E \neq E'$ have a common composition factor. We may assume that $\ell(E) \leq \ell(E')$. Since $E$ and $E'$ are orthogonal, there exists a chain $0 \subseteq E_1 \subseteq E_2 \subseteq E'$ of $A$-modules such that $E_2/E_1 = E$. By Lemma $3.3$ we have that $E'/E_2$ belongs to $\mathcal{X}_e$. This contradicts to the definition of the elementary module $E'$.

Lemma 3.5. Let $S$ be a cyclic simple $A$-module. Then the following statements hold.

(1) The injective dimension of $E(S)$ is infinite, and the injective dimension of $P(S)$ is finite.

(2) There is a unique simple $A$-module $S'$ in $\mathcal{S}_e$ such that top $E(S') = \tau \text{soc } E(S)$ and $\text{Ext}^1_A(E(S), E(S')) \neq 0$.

Proof. (1) We recall from Lemma $2.5$ that $\mathcal{S}_e$ is a complete set of pairwise non-isomorphic simple $A$-modules of infinite injective dimension. Since the elementary modules have pairwise disjoint supports, for each $S$ in $\mathcal{S}_e$, the support of $E(S)$ contains precisely one cyclic simple $A$-module, that is, $S$. In other words, each composition factor of $E(S)$ different from $S$ is a noncyclic simple $A$-module, and thus has finite injective dimension. It follows that $E(S)$ has infinite injective dimension.

Let $h : P(S) \to I$ be an injective envelope of the $A$-module $P(S)$. We claim that each composition factor $S'$ of Coker $h$ is a noncyclic simple $A$-module, and thus has finite injective dimension. Consequently, the injective dimension of Coker $h$ is finite. Therefore the injective dimension of $P(S)$ is finite.

For the claim, we observe by Lemma $3.1$ $P(S) \subseteq P(S') \subseteq I$. Then we have $\gamma(S') = \gamma(S)$. Recall that the restriction of $\gamma$ on cyclic simple $A$-modules is injective. Therefore $S'$ is a noncyclic simple $A$-module, since $S$ is a cyclic simple $A$-module and $S' \neq S$.

(2) Let $E = E(S)$. Recall that $P(S)$ lies in $\mathcal{X}_e$ and thus in $\mathcal{F}(\mathcal{E}_e)$. Consider the $\mathcal{E}_e$-filtration of $P(S)$, say $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{t-1} \subseteq M_t = P(S)$,

such that $M_i/M_{i-1}$ is elementary for $1 \leq i \leq t$. We observe that $M_i/M_{i-1} = E$ and $t \geq 2$, since by (1) we have $E(S) \neq P(S)$. Set $E' = M_{t-1}/M_{t-2}$. Note that $E' = E(S')$ for some cyclic simple $A$-module $S'$. Then $\text{top } E' = \text{top}(M_{t-1}/M_{t-2}) = \tau \text{soc } (M_t/M_{t-1}) = \tau \text{soc } E$.

Since $M_t/M_{t-2} = P(S)/M_{t-2}$ is indecomposable, the exact sequence $0 \to M_{t-1}/M_{t-2} \to M_t/M_{t-1} \to M_t/M_{t-1} \to 0$ does not split. Then we have $\text{Ext}^1_A(E, E') \neq 0$. The uniqueness of $S'$ is obvious, since $S' = \tau \text{soc } E(S)$.

Recall that by definition $\tau \text{soc } E$ lies in $\mathcal{S}_e$ for each elementary module $E$. We have a map $\delta : \mathcal{S}_e \to \mathcal{S}_e$, which sends a cyclic simple $A$-module $S$ to $\delta(S) = \tau \text{soc } E(S)$. We claim that $\delta$ is injective and thus bijective. Indeed, if $\delta(S) = \delta(\tilde{S})$, then $\text{soc } E(S) = \text{soc } E(\tilde{S})$. It follows from Lemma $3.4(3)$ that $S = \tilde{S}$.
Corollary 3.6. Let \( S \) be a cyclic simple \( A \)-module and \( t \) the minimal positive integer such that \( \delta^t(S) = S \). Then \( \mathcal{S}_c = \{ S, \delta(S), \cdots, \delta^{t-1}(S) \} \) and \( S \) is the disjoint union of the supports of all elementary modules.

Proof. Since \( A \) is a connected Nakayama algebra without simple projective modules, any nonempty subset of \( \mathcal{S} \) which is closed under \( \tau \) must be \( \mathcal{S} \). By Lemma 3.5(2) the union of the supports of all \( E(\delta^i(S)) \) is closed under \( \tau \), we infer that this union is \( \mathcal{S} \). Let \( S' \) be a cyclic simple \( A \)-module. Then there exists an integer \( 0 \leq i \leq t-1 \) such that the support of \( E(\delta^i(S)) \) and the support of \( E(S) \) have nonempty intersection. It follows from Lemma 3.4(3) that \( S' = \delta^i(S) \).

Proposition 3.7. Let \( A \) be a connected Nakayama algebra of infinite global dimension. Then \( \mathcal{F}(\mathcal{E}_c) \) is equivalent to \( A' \)-mod, where \( A' \) is a connected selfinjective Nakayama algebra.

Proof. Let \( P = \bigoplus_{S \in \mathcal{S}_c} P(S) \) and \( A' = \text{End}_A(P)^{\text{op}} \). Then \( P \) is a projective object in \( \mathcal{F}(\mathcal{E}_c) \), since \( \mathcal{F}(\mathcal{E}_c) \) is a wide subcategory of \( A\text{-mod} \). The natural projection \( P(S) \to E(S) \) is a projective cover in the category \( \mathcal{F}(\mathcal{E}_c) \). Recall from Lemma 2.4 that \( \mathcal{F}(\mathcal{E}_c) \) is a length category with \( \mathcal{E}_c = \{ E(S) \mid S \in \mathcal{S}_c \} \) a complete set of pairwise non-isomorphic simple objects. We infer that for each object \( X \) in \( \mathcal{F}(\mathcal{E}_c) \), there is an epimorphism \( P' \to X \) with \( P' \) in \( \mathcal{P} \). Then \( P' \) is a projective generator for \( \mathcal{F}(\mathcal{E}_c) \). We have an equivalence \( \mathcal{F}(\mathcal{E}_c) \sim A'\text{-mod} \); compare [13, Chapter IV, Theorem 5.3].

Since each indecomposable object in \( \mathcal{F}(\mathcal{E}_c) \) is uniserial, we infer that \( A' \) is a Nakayama algebra. Denote by \( \tau' \) the Auslander-Reiten translation of \( A' \). Then we have \( \tau' E(S) = E(\delta(S)) \) by Lemma 3.5(2). It follows from Corollary 3.6 that all simple \( A' \)-modules are in the same \( \tau' \)-orbit. Therefore, the Nakayama algebra \( A' \) is connected.

It remains to show that \( A' \) is selfinjective. Since \( \gamma \) restricts to a permutation on \( \mathcal{S}_c \), the modules in \( \mathcal{P}_c \) have pairwise distinct socles. Therefore, we have \( \mathcal{S}_c = \{ \tau \text{soc} P \mid P \in \mathcal{P}_c \} \). Let \( E \) be an elementary module. Since \( \tau \text{soc} E \) lies in \( \mathcal{S}_c \), there exists \( P \) in \( \mathcal{P}_c \) with \( \text{soc} P = \text{soc} E \) and \( \text{soc} P \neq \text{soc} E' \) for any elementary module \( E' \neq E \). It follows that the socle of \( P \) in the category \( \mathcal{F}(\mathcal{E}_c) \) is \( E \). We have proven that every simple \( A' \)-module can embed into a projective \( A' \)-module. Therefore, \( A' \) is selfinjective.

The following result is analogous to [13, Proposition 4].

Lemma 3.8. The following statements are equivalent for an indecomposable non-projective \( A \)-module \( M \).

1. \( M \) belongs to \( \mathcal{F}(\mathcal{E}_c) \).
2. There is an exact sequence
   \[
   0 \to M \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0
   \]
   for some \( n \geq 1 \) such that each \( P_i \) belongs to \( \mathcal{P}_c \).
3. There is an exact sequence
   \[
   P_1 \to P_0 \to M \to 0
   \]
   such that \( P_i \) belongs to \( \mathcal{P}_c \) for \( i = 0, 1 \).

Proof. “(1) \Rightarrow (2)” Recall that for an indecomposable nonprojective module \( M' \) over a selfinjective Nakayama algebra \( A' \), there exists an exact sequence
\[
0 \to M' \to P_n' \to \cdots \to P_1' \to P_0' \to M' \to 0
\]
for some \( n \geq 1 \) such that each \( P_i' \) is indecomposable. Then (2) follows from Proposition 3.7.

“(2) \Rightarrow (3)” This is obvious.

“(3) \Rightarrow (1)” Observe that top \( M = \text{top} P_0 \) and \( \tau \text{soc} M = \top \Omega M = \text{top} P_1 \), both of which belong to \( \mathcal{S}_c \). Then by definition \( M \) belongs to \( \mathcal{X}_c \).

Recall that each component of the resolution quiver \( R(A) \) has a unique cycle. For each noncyclic vertex \( S \) in \( R(A) \), there exists a unique path of minimal length starting with \( S \) and ending in a cycle. We call the length of this path the distance
between $S$ and the cycle. Let $d(A)$ be the maximal distance between noncyclic vertices and cycles. Observe that $\gamma^d(S)$ is cyclic for each simple $A$-module $S$.

**Lemma 3.9.** Let $d = d(A)$ be as above. Then the following statements hold.

1. $\Omega^{2d}(M)$ belongs to $F(E_c)$ for any $M$ in $A$-mod.
2. $\Omega_0^{2d}(A\text{-mod}) \subseteq F(E_c) \subseteq \Omega^{2d}(A\text{-mod})$.

**Proof.** (1) We may assume that $M$ is indecomposable. It follows from Lemma 2.4 that either $\Omega^{2d}(M)$ is zero or $\Omega^{2d}(M) = \gamma^d(\text{top } M)$. If $\Omega^{2d}(M)$ is indecomposable, then $\Omega^{2d}(M)$ belongs to $\mathcal{P}_c$. If $\Omega^{2d}(M)$ is indecomposable non-projective, then $\text{top } \Omega^{2d}(M) = \gamma^d(\text{top } M)$ and $\mathcal{S}\mathcal{O}\mathcal{C}\Omega^{2d}(M) = \text{top } \Omega^{2d+1}(M) = \gamma^d(\text{top } M)$. Then by definition $M$ belongs to $\mathcal{X}_c$.

(2) The first inclusion follows from (1), and the second one follows from Lemma 3.8(2).

By Proposition 3.7, $F(E_c)$ is a Frobenius category whose projective objects are precisely add $\mathcal{P}_c$. Denote by $\mathcal{F}(E_c)$ the stable category of $F(E_c)$ modulo projective objects. It is a triangulated category; see [10].

Recall from Proposition 3.4 that $F(E_c)$ is a wide subcategory of $A\text{-mod}$ which is closed under projective covers. Consider the inclusion functor $i : F(E_c) \rightarrow A\text{-mod}$. It induces uniquely a fully-faithful functor $i' : F(E_c) \rightarrow A\text{-mod}$. We recall the induced functor $q' : A\text{-mod} \rightarrow D_{\mathcal{S}\mathcal{O}\mathcal{C}}(A)$ in Section 2.

The following is the main result of this section, which describes the singularity category of $A$ as a subcategory of the stable module category of $A$.

**Theorem 3.10.** Let $A$ be a connected Nakayama algebra of infinite global dimension. Then the composite functor $q' \circ i' : F(E_c) \rightarrow A\text{-mod} \rightarrow D_{\mathcal{S}\mathcal{O}\mathcal{C}}(A)$ is a triangle equivalence.

**Proof.** Observe that the composite functor $F(E_c) \xrightarrow{i} A\text{-mod} \rightarrow D^b(A\text{-mod}) \xrightarrow{2\phi} D_{\mathcal{S}\mathcal{O}\mathcal{C}}(A)$ is a $\mathcal{P}$-functor in the sense of [12, Section 1]; compare [4, Lemma 2.4]. Then the functor $q' \circ i'$ is a triangle functor; see [4, Lemma 2.5].

Recall that the subcategory $F(E_c)$ of $A\text{-mod}$ is wide and closed under projective covers; moreover, $F(E_c)$ is a Frobenius category. Then the restriction of the syzygy functor $\Omega : A\text{-mod} \rightarrow A\text{-mod}$ on $F(E_c)$ is an autoequivalence, in particular, it is fully faithful. Then the functor $q' \circ i'$ is fully faithful by the natural isomorphism in Lemma 2.3.

It remains to show that the functor $q' \circ i'$ is also dense. Let $X$ be an object in $D_{\mathcal{S}\mathcal{O}\mathcal{C}}(A)$. It follows from Lemmas 2.1 and 3.9(1) that there exists a module $M$ in $F(E_c)$ and $n$ sufficiently large such that $X \simeq q(M)[n]$ in $D_{\mathcal{S}\mathcal{O}\mathcal{C}}(A)$. By above, the image $\text{Im}(q' \circ i')$ is a triangulated subcategory of $D_{\mathcal{S}\mathcal{O}\mathcal{C}}(A)$, in particular, it is closed under $[m]$ for all $m \in \mathbb{Z}$. It follows from $X \simeq q(M)[n]$ that $X$ lies in $\text{Im}(q' \circ i')$. This finishes our proof.

We observe the following immediate consequence of Proposition 3.7 and Theorem 3.10.

**Corollary 3.11.** (compare [7, Corollary 3.11]). Let $A$ be a connected Nakayama algebra of infinite global dimension. Then there is a triangle equivalence between $D_{\mathcal{S}\mathcal{O}\mathcal{C}}(A)$ and $A'\text{-mod}$ for a connected selfinjective Nakayama algebra $A'$.

4. A DUALITY BETWEEN SINGULARITY CATEGORIES

In this section, we prove that there is a triangle duality between the singularity category of a Nakayama algebra and the singularity category of its opposite algebra. The proof uses the Frobenius subcategory in the previous section.
Let \( A \) be a connected Nakayama algebra of infinite global dimension. We recall from Propositions 5.3 and 5.7 that the category \( \mathcal{F} = \mathcal{F}(\mathcal{E}_c) \) is a wide subcategory of \( A\text{-mod} \) closed under projective covers; it is equivalent to \( A'\text{-mod} \) for a connected selfinjective Nakayama algebra \( A' \).

Consider the inclusion functor \( i : \mathcal{F} \to A\text{-mod} \). We claim that it admits an exact right adjoint \( i_p : A\text{-mod} \to \mathcal{F} \).

For the claim, recall from the proof of Proposition 5.7 that \( A' = \text{End}_A(P)^{\text{op}} \) with \( P = \bigoplus_{S \in \mathcal{S}_P} P(S) \). We identify \( \mathcal{F} \) with \( A'\text{-mod} \). Then the inclusion \( i \) is identified with \( \bigoplus_{S \in \mathcal{S}_P} P \otimes_A - \). The right adjoint is given by \( i_p = \text{Hom}_A(P, -) \). It is exact since \( _AP \) is projective.

The adjoint pair \((i, i_p)\) induces an adjoint pair \((i^*, i_p^*)\) of triangle functors between bounded derived categories. Here, for an exact functor \( F \) between abelian categories, \( F^* \) denotes its extension on bounded derived categories.

Recall that \( \mathbf{K}^b(\text{A-inj}) \) denotes the bounded homotopy category of \( A\text{-inj} \). We view \( \mathbf{K}^b(\text{A-inj}) \) as a thick subcategory of \( \mathbf{D}^b(\text{A-mod}) \) via the canonical functor. We mention that by the usual duality on module categories, the quotient triangulated category \( \mathbf{D}^b(\text{A-mod})/\mathbf{K}^b(\text{A-inj}) \) is triangle equivalent to the opposite category of the singularity category \( \mathbf{D}_{sg}(A^{\text{op}}) \) of \( A^{\text{op}} \), Here, \( A^{\text{op}} \) is the opposite algebra of \( A \).

The proof of the following result is similar to [7, Proposition 2.13]. Recall that \( \mathcal{P}_c \) is a complete set of pairwise non-isomorphic indecomposable projective objects in \( \mathcal{F} = \mathcal{F}(\mathcal{E}_c) \).

**Lemma 4.1.** Let \( A \) be a connected Nakayama algebra of infinite global dimension. Then the above functors \( i_p^* \) and \( i^* \) induce mutually inverse triangle equivalences between \( \mathbf{D}^b(\text{A-mod})/\mathbf{K}^b(\text{A-inj}) \) and \( \mathbf{D}^b(\mathcal{F})/\mathbf{K}^b(\text{add } \mathcal{P}_c) \).

**Proof.** Observe by [6, Lemma 3.3.1] that \( i^* : \mathbf{D}^b(\mathcal{F}) \to \mathbf{D}^b(\text{A-mod}) \) is fully faithful. It follows that its right adjoint \( i_p^* \) induces a triangle equivalence

\[
\overleftarrow{i_p^*} : \mathbf{D}^b(\text{A-mod})/\ker i_p^* \cong \mathbf{D}^b(\mathcal{F});
\]

see [8, Chapter I, Section 1, 1.3 Proposition]. Here, \( \ker F \) denotes the essential kernel of an additive functor \( F \).

We claim that \( \ker i_p^* = \text{thick}(\mathcal{S}_{nc}) \), the smallest thick subcategory of \( \mathbf{D}^b(\text{A-mod}) \) containing \( \mathcal{S}_{nc} \). Here, we recall that \( \mathcal{S}_{nc} \) denotes the set of noncyclic simple \( A \)-modules. By Lemma 2.5 each noncyclic simple \( A \)-module has finite injective dimension. It follows from the claim that \( \ker i_p^* \subseteq \mathbf{K}^b(\text{A-inj}) \).

For the claim, we observe that \( \ker i_p = \mathcal{F}(\mathcal{S}_{nc}) \), the full subcategory of \( \text{A-mod} \) formed by \( \mathcal{A} \)-modules with an \( \mathcal{S}_{nc} \)-filtration. The claim follows from the fact that a complex \( X \) is in \( \ker i_p^* \) if and only if its first cohomology \( H^1(X) \) is in \( \ker i_p \).

We observe that \( i_p \) preserves injective objects since it has an exact left adjoint. It follows that \( i_p^*(\mathbf{K}^b(\text{A-inj})) \subseteq \mathbf{K}^b(\text{add } \mathcal{P}_c) \). By Lemma 3.5(1) each module in \( \mathcal{Q}_c \) has finite injective dimension. Note that \( i_p^*Q = Q \). Therefore \( i_p^*(\mathbf{K}^b(\text{A-inj})) \supseteq \mathbf{K}^b(\text{add } \mathcal{P}_c) \), and thus \( i_p^*(\mathbf{K}^b(\text{A-inj})) = \mathbf{K}^b(\text{add } \mathcal{P}_c) \). From this equality, the triangle equivalence \( \overleftarrow{i_p^*} \) restricts to a triangle equivalence

\[
\mathbf{K}^b(\text{A-inj})/\ker i_p^* \cong \mathbf{K}^b(\text{add } \mathcal{P}_c).
\]

The desired equivalence follows from [21, Chapitre I, §2, 4-3 Corollaire]. □

**Proposition 4.2.** Let \( A \) be a Nakayama algebra. Then the singularity category \( \mathbf{D}_{sg}(A) \) is triangle equivalent to \( \mathbf{D}^b(\text{A-mod})/\mathbf{K}^b(\text{A-inj}) \). Equivalently, there is a triangle duality between \( \mathbf{D}_{sg}(A) \) and \( \mathbf{D}_{sg}(A^{\text{op}}) \).

**Proof.** Without loss of generality, we may assume that \( A \) is a connected Nakayama algebra of infinite global dimension. Then the singularity category \( \mathbf{D}_{sg}(A) \) is triangle equivalent to the stable category \( \overleftarrow{\mathcal{F}} \) by Theorem 3.10.
Since \( \mathcal{F} \) is a Frobenius abelian category, it follows from [17, Theorem 2.1] that the stable category \( \mathcal{F} \) is triangle equivalent to \( D^b(\mathcal{F})/K^b(\text{add} \mathcal{P}) \). Then the conclusion follows from Lemma 4.3.

5. THE RESOLUTION QUIVERS

Let \( A \) be a connected Nakayama algebra of infinite global dimension. Recall that \( n(A) \) denotes the number of isomorphism classes of simple \( A \)-modules. By Corollary 3.11 the Auslander-Reiten quiver of the singularity category \( D_{sg}(A) \) is isomorphic to a truncated tube \( \mathbb{Z}_m / (\tau^r) \), where \( m = m(A) \) denotes its height and \( t = t(A) \) denotes its rank. Here, we use the fact that the Auslander-Reiten quiver of the stable module category of a connected selfinjective Nakayama algebra is a truncated tube; compare [11, VI.2].

Recall that \( R(A) \) denotes the resolution quiver of \( A \). We denote by \( c(A) \) the number of cycles in \( R(A) \). Let \( C \) be a cycle in \( R(A) \). Then the size \( s(C) \) of \( C \) is the number of vertices in \( C \), and the weight \( w(C) \) of \( C \) is \( \sum_{i} |l(P(S_i))| \), where \( S \) runs through all vertices of \( C \). Here, \( l(P(S)) \) is the composition length the projective cover \( P(S) \) of a simple \( A \)-module \( S \). Recall from [20] that all cycles in the resolution quiver \( R(A) \) have the same size and the same weight. We denote \( s(A) = s(C) \) and \( w(A) = w(C) \) for an arbitrary cycle \( C \) in \( R(A) \).

For two positive integers \( a \) and \( b \), we denote their greatest common divisor by \( (a, b) \).

**Lemma 5.1.** Let \( m = m(A) \) and \( t = t(A) \) be as above. Then \( c(A) = (m + 1, t) \), \( s(A) = \frac{t}{(m + 1, t)} \) and \( w(A) = \frac{m + 1}{(m + 1, t)} \).

**Proof.** Recall from [7, Theorem 3.8] that there exists a sequence of algebra homomorphisms

\[
A = A_0 \xrightarrow{\eta_0} A_1 \xrightarrow{\eta_1} A_2 \rightarrow \cdots \rightarrow A_{r-1} \xrightarrow{\eta_{r-1}} A_r,
\]

such that each \( A_i \) is a connected Nakayama algebra and \( A_r \) is selfinjective; moreover, each \( \eta_i \) induces a triangle equivalence between \( D_{sg}(A_i) \) and \( D_{sg}(A_{i+1}) \). Following [20, Lemma 2.2], each \( \eta_i \) induces a bijection between the set of cycles in \( R(A_i) \) and the set of cycles in \( R(A_{i+1}) \), which preserves sizes and weights. Thus we have \( m(A_i) = m(A_{i+1}) \), \( t(A_i) = t(A_{i+1}) \), \( c(A_i) = c(A_{i+1}) \), \( s(A_i) = s(A_{i+1}) \) and \( w(A_i) = w(A_{i+1}) \) for \( 0 \leq i \leq r - 1 \). Therefore it is enough to prove the equations for selfinjective Nakayama algebras.

Let \( A \) be a connected selfinjective Nakayama algebra. Then \( t \) equals the number of isomorphism classes of simple \( A \)-modules, and \( m + 1 \) equals the radical length of \( A \). We claim that \( s(A) = \frac{t}{(m + 1, t)} \). Therefore, we have \( c(A) = \frac{t}{s(A)} = (m + 1, t) \) and \( w(A) = \frac{m + 1}{(m + 1, t)} \).

For the claim, let \( \{S_1, \ldots, S_t\} \) be a complete set of pairwise non-isomorphic simple \( A \)-modules such that \( \tau S_i = S_{i+1} \) for \( 1 \leq i \leq t \). Here, we let \( S_{t+j} = S_j \) for each \( j > 0 \). Then we have \( \gamma(S_i) = S_{i+m+1} \), and thus \( \gamma(S_i) = S_i \) if and only if \( t \) divides \( d(m + 1) \). It follows that \( R(A) \) consists of cycles of size \( \frac{t}{(m + 1, t)} \).

The following result establishes the relationship between the resolution quiver of a Nakayama algebra and the resolution quiver of its opposite algebra.

**Proposition 5.2.** Let \( A \) be a connected Nakayama algebra of infinite global dimension. Then the following statements hold.

1. The resolution quivers \( R(A) \) and \( R(A^{op}) \) have the same number of cycles and the same number of cyclic vertices.
2. All cycles in \( R(A) \) and \( R(A^{op}) \) have the same weight.
Proof. By Proposition \([\text{4.2}}\) there is a triangle duality between \(D_{\text{sg}}(A)\) and \(D_{\text{sg}}(A^{\text{op}})\). Then the Auslander–Reiten quiver of \(D_{\text{sg}}(A^{\text{op}})\) is isomorphic to the opposite quiver of the Auslander–Reiten quiver of \(D_{\text{sg}}(A)\). Therefore, we have \(m(A) = m(A^{\text{op}})\) and \(t(A) = t(A^{\text{op}})\). It follows from Lemma \([6.1}\) that \(c(A) = c(A^{\text{op}})\), \(s(A) = s(A^{\text{op}})\) and \(w(A) = w(A^{\text{op}})\). \(\Box\)

The following example shows that these two resolution quivers \(R(A)\) and \(R(A^{\text{op}})\) may not be isomorphic in general.

**Example 5.3.** Let \(A\) be a connected Nakayama algebra with admissible sequence \((7, 6, 6, 5)\). Assume that \(\{S_1, S_2, S_3, S_4\}\) is a complete set of pairwise non-isomorphic simple \(A\)-modules such that \(\tau S_i = S_{i+1}\) for \(1 \leq i \leq 4\). Then we have \(l(P_1) = 7\), \(l(P_2) = 6\), \(l(P_3) = 6\) and \(l(P_4) = 5\). There is an arrow from \(S_i\) to \(S_j\) in \(R(A)\) if and only if \(4\) divides \(i - j + l(P_i)\).

Denote by \(D\) the usual duality, and by \((-)^*\) the duality on projectives. Then \(\{DS_4, DS_3, DS_2, DS_1\}\) is a complete set of pairwise non-isomorphic simple \(A^{\text{op}}\)-modules such that \(\tau DS_i = DS_{i+1}\). Here, \(\tau\) is the Auslander–Reiten translation of \(A^{\text{op}}\). We observe that \(l(P_1^\ast) = 6\), \(l(P_2^\ast) = 6\), \(l(P_3^\ast) = 7\) and \(l(P_4^\ast) = 5\). Therefore the admissible sequence of \(A^{\text{op}}\) is \((6, 7, 6, 5)\). There is an arrow from \(DS_i\) to \(DS_j\) in \(R(A^{\text{op}})\) if and only if \(4\) divides \(i - j + l(P_i^\ast)\).

The resolution quivers \(R(A)\) and \(R(A^{\text{op}})\) are shown as follows.

\[
\begin{align*}
DS_4 & \xrightarrow{\tau} DS_3 & & \xrightarrow{\tau} DS_2 & & \xrightarrow{\tau} DS_1 & & S_3 \\
S_1 & & S_4 & & S_2 & & S_1 \\
DS_3 & \xrightarrow{\tau} DS_4 & & \xrightarrow{\tau} DS_2 & & \xrightarrow{\tau} DS_1 & & S_3 \\
& & & & & & & & \text{DS}_5
\end{align*}
\]

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