Non-symmetric Riemannian gravity and Sasaki–Einstein 5-manifolds

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Abstract

We show that a connection with skew-symmetric torsion satisfying the Einstein metricity condition exists on an almost contact metric manifold exactly when it is D-homothetic to a cosymplectic manifold. In dimension five, we get that the existence of a connection with skew torsion satisfying the Einstein metricity condition is equivalent to the existence of a Sasaki–Einstein 5-manifold and vice versa, any Sasaki–Einstein 5-manifold generates a two-parametric family of connections with skew torsion satisfying the Einstein metricity condition. Formulas for the curvature and the Ricci tensors of these connections are presented in terms of the Sasaki–Einstein SU(2) structures.

Keywords: Einstein metricity condition, skew-symmetric torsion, Sasaki–Einstein 5-manifolds, gravity

1. Introduction

In this note we consider the geometry arising on an odd-dimensional manifold from the Einstein gravitational theory on a non-symmetric (generalized) Riemannian manifold \((M^{2n+1}, G = g + F)\), where the generalized metric \(G\) has non-degenerate symmetric part \(g\) and a skew-symmetric part \(F\) of rank \(= 2n\).

1.1. Motivation from general relativity

General relativity (GR) was developed by Albert Einstein in 1916 [15] and contributed to by many others after 1916. In GR the equation \(ds^2 = g_{\mu\nu}dx^\mu dx^\nu\), \((g_{\mu\nu} = g_{\nu\mu})\) is valid, where \(g_{\mu\nu}\) are
functions of a point. In GR, which is a four dimensional space-time continuum, metric properties depend on mass distribution. The magnitudes $g_{ij}$ are known as gravitational potential. Christoffel symbols, commonly expressed by $\Gamma^k_{ij}$, play the role of magnitudes which determine the gravitational force field. GR explains gravity as the curvature of space-time.

In GR the metric tensor obeys the Einstein equations $R_{ij} - \frac{1}{2} R g_{ij} = T_{ij}$, where $R_{ij}$ is the Ricci tensor of the metric of space-time, $R$ is the scalar curvature of the metric, and $T_{ij}$ is the energy-momentum tensor of matter. In 1922, Friedmann [24] found a solution in which the universe may expand or contract, and later Lemaître [38] derived a solution for an expanding universe. However, Einstein believed that the universe was apparently static, and since static cosmology was not supported by the general relativistic field equations, he added the cosmological constant $\Lambda$ to the field equations, which became $R_{ij} - \frac{1}{2} R g_{ij} + \Lambda g_{ij} = T_{ij}$. From 1923 to the end of his life Einstein worked on various variants of unified field theory [16]. This theory had the aim to unite the theory of gravitation and the theory of electromagnetism.

Starting from 1950, Einstein used the real non-symmetric basic tensor $G$, sometimes called generalized Riemannian metric manifold. In this theory the symmetric part $g_{ij}$ of the basic tensor $G_{ij}(G_{ij} = g_{ij} + F_{ij})$ is related to gravitation, and the skew-symmetric one $F_{ij}$ to electromagnetism.

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More recently the idea of a non-symmetric metric tensor appears in Moffat’s non-symmetric gravitational theory [41].

In Moffat’s theory the skew-symmetric part of the metric tensor represents a Proca field (massive Maxwell field) which is a part of the gravitational interaction, contributing to the rotation of galaxies.

While on a Riemannian space the connection coefficients are expressed as functions of the metric, $g_{ij}$, in Einstein’s works the connection between these magnitudes is determined by the so called Einstein metricity condition, i.e. the non-symmetric metric tensor $G$ and the connection components $\Gamma^k_{ij}$ are connected by

$$\frac{\partial G_{ij}}{\partial x^m} - \Gamma^p_{im} G_{pj} - \Gamma^p_{jm} G_{ip} = 0.$$ (1.1)

A generalized Riemannian manifold satisfying the Einstein metricity condition (1.1) is called an NGT-space [16, 39, 41].

The choice of a connection in NGT is uniquely determined in terms of the structure tensors [35]. Special attention was paid when the torsion of the NGT connection is totally skew-symmetric with respect to the symmetric part $g$ of $G$ [17] (NGTS connection for short). One reason for that comes from supersymmetric string theories and non-linear $\sigma$-models (see e.g. [1, 2, 26, 28, 49] and references therein), as well as from the theory of gravity itself [33].

In even dimensions with nondegenerate skew-symmetric part $F$ one arrives at Nearly Kähler manifolds, namely, an almost Hermitian manifold is NGTS exactly when it is a nearly Kähler manifold [35, theorem 3.3]. In this case the NGTS connection coincides with the Gray connection [29–31], which is the unique connection with skew-symmetric torsion preserving the Nearly Kähler structure [22]. Nearly Kähler manifolds (called almost Tachibana spaces in [52]) were developed by Gray [29–31] and have been intensively studied since then in [10, 20, 36, 42–44]. Nearly Kähler manifolds also appear in supersymmetric string theories (see e.g. [32, 37, 46, 47]). The first complete and therefore compact inhomogeneous examples of 6-dimensional Nearly Kähler manifolds were presented recently in [21].
In odd dimensions $2n + 1$ with rank $F = 2n$ one gets an almost contact metric manifold. It is shown in [35] that a connection with skew-symmetric torsion satisfying the Einstein metricity condition exists on an almost contact metric manifold exactly when it is \textit{almost nearly cosymplectic} (the precise definition is given in definition 2.1 below).

The aim of this note is to investigate the geometry of almost-nearly cosymplectic spaces. It is well known that nearly cosymplectic manifolds are the odd dimensional analog of nearly Kähler spaces and that the trivial circle bundle over a nearly Kähler space is nearly cosymplectic.

We establish a relation between almost nearly cosymplectic spaces and nearly cosymplectic spaces, namely we present a special D-homothetic deformation relating these two objects. Applying the structure theorem for nearly cosymplectic structures established by de Nicola–Dileo–Yudin in [14] we present a structure theorem for almost nearly cosymplectic structures (theorem 3.4) and give a formula relating the curvature and the Ricci tensors of the almost nearly cosymplectic structure to the corresponding D-homothetic nearly cosymplectic structure.

In dimension five we found a closed relation between almost nearly cosymplectic structures and the 5-dimensional (5D) Sasaki–Einstein spaces. Namely, using the fundamental observation connecting nearly cosymplectic 5-manifolds with the SU(2) structures and Sasaki–Einstein 5-manifolds due to Cappelletti-Montano–Dileo in [11], we show in theorem 4.5 that any almost nearly cosymplectic 5-manifold is D-homothetically equivalent to a 5D Sasaki–Einstein space. Moreover, we show that a Sasaki–Einstein 5-manifold generates a two-parametric family of almost nearly cosymplectic structures and therefore a two-parametric family of NGTS connections. We give an explicit formula connecting these NGTS connections with the Levi-Civita connection of the corresponding Sasaki–Einstein structure. We express the curvature and Ricci tensor of these NGTS connections in terms of the curvature of the Sasaki–Einstein metric and the corresponding Sasaki–Einstein SU(2) structure.

On the other hand, by virtue of admitting real Killing spinors [23], Sasakian–Einstein 5-manifolds admit supersymmetry and have received a lot of attention in physics from the point of view of AdS/CFT correspondence (see e.g. [3, 4, 25, 27, 40, 48]). The AdS/CFT correspondence relates string theory on the product of anti-deSitter space with a compact Einstein space to quantum field theory on the conformal boundary. The renewed interest in these manifolds has to do with the so-called p-brane solutions in superstring theory. For example, the case of D3-branes of string theory the relevant near horizon geometry is that of a product of anti-deSitter space with a Sasakian–Einstein 5-manifold. This led to a construction of a number of examples of irregular compact Sasakian–Einstein 5-manifolds [25, 27, 48].

In this spirit, our results may help to establish a possible relation between the NGT with skew-symmetric torsion, supersymmetric string theories and quantum field theory.

2. Preliminaries

2.1. Einstein metricity condition (NGT)

In his attempt to construct a unified field theory (non-symmetric gravitational theory, briefly NGT) Einstein [16] considered a generalized Riemannian manifold $(G = g + F)$ with non-degenerate symmetric part $g$ and skew-symmetric part $F$ and used the so-called metricity condition (1.1), which can be written as follows $XG(Y, Z) - G(\nabla^g Y, Z) - G(Y, \nabla^g Z) = 0$.

For the non-degenerate symmetric part $g$ we have a (1,1) tensor $A$ given by $F(X, Y) = g(AX, Y)$. 
The metricity condition (1.1) can be written in terms of the torsion $T(X, Y) = \nabla^\text{ngt} X - \nabla^\text{ngt} Y - [X, Y]$ of the NGT connection $\nabla^\text{ngt}$ and the endomorphism $A$ in the form

$$(\nabla^\text{ngt}_X G)(Y, Z) = -G(T(X, Y), Z) \iff (\nabla^\text{ngt}_X (g + F))(Y, Z) = -T(X, Y, Z) + T(X, Y, AZ).$$

(2.1)

A general solution for the connection $\nabla^\text{ngt}$ satisfying (1.1) is given in terms of $g, F, T$ [34] (see also [41]) and in terms of $dF$ in [35].

### 2.2. Almost contact metric and almost nearly cosymplectic structures

In the case of odd dimension, we consider an almost contact metric manifold $(M^{2n+1}, g, \phi, \eta, \xi)$, i.e. a $(2n + 1)$-dimensional Riemannian manifold equipped with a 1-form $\eta$, a (1,1)-tensor $\phi$ and the vector field $\xi$ dual to $\eta$ with respect to the metric $g$, $\eta(\xi) = 1$, $\eta(X) = g(X, \xi)$ such that the following compatibility conditions are satisfied (see e.g. [5])

$$\phi^2 = -id + \eta \otimes \xi, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \phi \xi = \eta \phi = 0.$$  

(2.2)

The fundamental 2-form is defined by $F(X, Y) = g(\phi X, Y)$.

Such a space can be considered as a generalized Riemannian manifold with $G = g + F$, $A = \phi$ and in this case the skew-symmetric part $F$ is degenerate $F(\xi, \cdot) = 0$ and has rank $F = 2n$.

**Definition 2.1 ([35]).** An almost contact metric manifold $(M^{2n+1}, g, \phi, \eta, \xi)$ is said to be almost-nearly cosymplectic if the covariant derivative of the fundamental tensor $\phi$ with respect to the Levi-Civita connection $\nabla$ of the Riemannian metric $g$ satisfies the following condition

$$g((\nabla_X \phi)Y, Z) = \frac{1}{3}dF(X, Y, Z) + \frac{1}{3}\eta(X)d\eta(Y, \phi Z) - \frac{1}{6}\eta(Y)d\eta(Z, \phi X) - \frac{1}{6}\eta(Z)d\eta(\phi X, Y).$$

(2.3)

The following relations are also valid [35, section 3.5]

$$d\eta(X, Z) = dF(X, \phi Z, \xi) = dF(\phi X, Z, \xi), \quad d\eta(X, \xi) = 0,$$

$$d\eta(\phi X, Z) = d\eta(X, \phi Z) = dF(\phi X, \phi Z, \xi) = -dF(X, Z, \xi),$$

$$dF(\phi X, \phi Y, \phi Z) + dF(X, Y, \phi Z) = \eta(X)d\eta(Y, Z) + \eta(Y)d\eta(Z, X).$$

(2.4)

and the vector field $\xi$ is a Killing vector field

$$(\nabla_X \eta)Y = g(\nabla_X \xi, Y) = \frac{1}{2}d\eta(X, Y).$$

(2.5)

Almost nearly cosymplectic manifolds arise in a natural way from NGT. Namely, we have the following

**Theorem 2.2 ([35, theorem 3.8]).** Let $(M, \phi, g, F, \eta, \xi)$ be an almost contact metric manifold with a fundamental 2-form $F$ considered as a generalized Riemannian manifold $(M, G)$ with a generalized Riemannian metric $G = g + F$. Then $(M, G)$ satisfies the Einstein metricity condition (1.1) with a totally skew-symmetric torsion $T$ if and only if it is almost-nearly cosymplectic, i.e. (2.3) holds.

The skew-symmetric torsion is determined by the condition

$$T(X, Y, Z) = -\frac{1}{3}dF(X, Y, Z).$$

(2.6)

The connection is uniquely determined by the formula
\[ g(\nabla^g_X Y, Z) = g(\nabla_Y Z, X) - \frac{1}{6} dF(X, Y, Z) + \frac{1}{6} \left[ \eta(X) d\eta(Y, Z) + \eta(Y) d\eta(X, Z) \right]. \] (2.7)

*The Einstein metricity condition has the form*

\[ (\nabla^g_X G)(Y, Z) = \frac{1}{3} \left[ dF(X, Y, Z) - dF(X, Y, \phi Z) \right]. \]

*The covariant derivative of \( g \) and \( F \) are given by*

\[ (\nabla^g_X g)(Y, Z) = \frac{1}{6} \left[ \eta(Y) d\eta(Z, X) + \eta(Z) d\eta(Y, X) \right]; \]

\[ (\nabla^g_X F)(Y, Z) = \frac{1}{3} \left[ dF(X, Y, Z) - dF(X, Y, \phi Z) \right] - \frac{1}{6} \left[ \eta(Y) d\eta(Z, X) + \eta(Z) d\eta(Y, X) \right]. \]

### 2.3. Nearly cosymplectic manifolds

We recall here the definition of nearly cosymplectic structures together with their basic properties from [11, 14].

An almost contact metric structure is cosymplectic if the endomorphism \( \phi \) is parallel with respect to the Levi-Civita connection, \( \nabla \phi = 0 \) and it is Sasakian if it is normal and contact, \( [\phi, \phi] + \eta \otimes \xi = 0, d\eta = -2F \). In terms of the Levi-Civita connection the Sasakian condition has the form

\[ (\nabla_X \phi)Y = g(X, Y) \xi - \eta(Y)X. \] For further details on Sasakian and cosymplectic manifolds, we refer the reader to [5, 8, 12].

An almost contact metric manifold is called nearly cosymplectic if \( [6, 7] \) \( (\nabla_X \phi)X = 0 \) which is also equivalent to the condition

\[ g\left((\nabla_X \phi)Y, Z\right) = \frac{1}{3} dF(X, Y, Z). \] (2.8)

In this case the vector field \( \xi \) is Killing, \( \nabla \xi = \nabla \xi \eta = 0 \) and

\[ d\eta(\phi X, Y) = d\eta(X, \phi Y). \] (2.9)

The tensor field \( h \) of type \((1,1)\) defined by

\[ \nabla_X \xi = hX \] (2.10)

has the following properties

\[ g(hX, Y) = -g(X, hY) = \frac{1}{2} d\eta(X, Y); \quad Ah + hA = 0, \] (2.11)

i.e. it is skew-symmetric, anticommutes with \( \phi \) and satisfies \( h\xi = 0, \eta \circ h = 0 \).

The following formulas also hold \([18, 19]\):

\[ g((\nabla_X \phi)Y, hZ) = \eta(Y) g(h^2 X, \phi Z) - \eta(X) g(h^2 Y, \phi Z). \] (2.12)

\[ (\nabla_X h)Y = g(h^2 X, Y) \xi - \eta(Y) h^2 X. \] (2.13)

\[ \text{tr}(h^2) = \text{constant.} \] (2.14)
A consequence of (2.13) and (2.14) is that the eigenvalues of symmetric operator $h^2$ are constants [11, 14]. A fundamental observation that if 0 is a simple eigenvalue of $h^2$ then the nearly cosymplectic manifold is of dimension five was made by de Nicola et al in [14] which lead to their structure theorem:

**Theorem 2.3 ([14, theorem 4.5])**. Let $(M, \phi, \xi, \eta, g)$ be a nearly cosymplectic non-cosymplectic manifold of dimension $2n + 1 > 5$. Then $M$ is locally isometric to one of the following Riemannian products:

$$R \times N^{2n}, \quad M^5 \times N^{2n-4},$$

where $N^{2n}$ is a nearly Kähler non-Kähler manifold, $N^{2n-4}$ is a nearly Kähler manifold and $M^5$ is a nearly cosymplectic non-cosymplectic manifold of dimension five.

If the manifold $M$ is complete and simply connected the above isometry is global.

Note also that the nearly Kähler factor can be further decomposed according to [43, theorem 1.1 and proposition 2.1].

### 2.4. Nearly cosymplectic manifolds in dimension 5

According to theorem 2.3, the non-trivial case of nearly cosymplectic manifolds is the 5D case. In this case Cappelletti-Montano and Dileo show in [11] that any 5D nearly cosymplectic manifold carries a Sasaki–Einstein structure and vice-versa. In order to describe the construction in [11] we recall below the notion of an SU(2) structure developed by Conti and Salamon in [13].

An SU(2) structure in dimension 5 is an SU(2)-reduction of the bundle of linear frames and it is equivalent to the existence of three almost contact metric structures $(\phi_i, \xi, \eta, g)$, $i = 1, 2, 3$ related to each other through $\phi_i \phi_j = -\phi_j \phi_i = \phi_k$ for any even permutation $\{i,j,k\}$ of $\{1,2,3\}$. In [13] Conti and Salamon proved that, in the spirit of special geometries, such a structure is equivalently determined by a quadruplet $(\eta, \omega_1, \omega_2, \omega_3)$, where $\eta$ is a 1-form and $\omega_i$ are 3 2-forms satisfying $\omega_i \wedge \omega_j = \delta_{ij} \eta$ for some 4-form $\eta$ with $\eta \neq 0$ and $X.\omega_i = Y.\omega_j \Rightarrow \omega_k(X,Y) \geq 0$. The endomorphisms $\phi_i$, the Riemannian metric $g$ and the 2-forms $\omega_i$ are related to each other through $\omega_i(X,Y) = g(\phi_iX,Y)$. The class of Sasaki–Einstein structures in dimension 5 is characterized by the following differential equations

$$d\eta = -2\omega_3, \quad d\omega_1 = 3\eta \wedge \omega_2, \quad d\omega_2 = -3\eta \wedge \omega_1. \quad (2.15)$$

For such a manifold the almost contact metric structure $(\phi_3, \xi, \eta, g)$ is Sasakian, with Einstein Riemannian metric $g$. A Sasaki–Einstein 5-manifold may equivalently be defined as a Riemannian manifold for which the cone metric is Kähler Ricci flat [8].

There are several generalizations of Sasaki–Einstein structures in dimension five. We only recall that the hypo structures introduced in [13] are defined by

$$d\omega_3 = 0, \quad d(\eta \wedge \omega_1) = 0, \quad d(\eta \wedge \omega_2) = 0. \quad (2.16)$$

These structures arise naturally on hypersurfaces of a 6-manifold endowed with an integrable SU(3) structure, i.e. a hypersurface of a Kähler Ricci flat 6-manifold.

Starting with a 5D nearly cosymplectic manifold Cappelletti-Montano and Dileo show in [11, theorem 5.1] that

$$h^2 = -\lambda^2(I - \eta \otimes \xi), \quad \lambda = \text{const.} \neq 0 \quad (2.17)$$

induces an SU(2) structure $(\eta, \omega_1, \omega_2, \omega_3)$ determined by
\[
\phi_1 = -\frac{1}{\lambda} \phi h, \quad \phi_2 = \phi, \quad \phi_3 = -\frac{1}{\lambda} h; \quad \omega_i(X, Y) = g(\phi_i X, Y)
\]  
(2.18)
which satisfies the relations
\[
d\eta = -2\lambda \omega_3; \quad d\omega_1 = 3\lambda \eta \wedge \omega_2; \quad d\omega_2 = -3\lambda \eta \wedge \omega_1.
\]  
(2.19)
In particular these SU(2) structures are hypo.

Consider the homothetic SU(2) structures determined by
\[
\bar{\eta} = \lambda \eta, \quad \bar{\omega}_i = \lambda^2 \omega_i
\]
It follows from (2.19) that these new SU(2) structures satisfy (2.15) and therefore \(\bar{\omega}_3\) is a Sasaki–Einstein structure while \(\bar{\omega}_1\) and \(\bar{\omega}_2\) are nearly cosymplectic structures [11]. This shows

**Theorem 2.4 ([11]).** A nearly cosymplectic non cosymplectic 5-manifold carries a Sasaki–Einstein structure and vice versa, any Sasaki–Einstein 5 manifold supports 2 nearly cosymplectic structures. In particular, a nearly cosymplectic non cosymplectic 5-manifold is Einstein with positive scalar curvature.

In terms of nearly cosymplectic structures, the attached Sasaki–Einstein structure is given by
\[
\bar{\phi} = -\frac{1}{\lambda} \phi h, \quad \bar{\xi} = \frac{1}{\lambda} \xi, \quad \bar{\eta} = \lambda \eta, \quad \bar{g} = \lambda^2 g, \quad \text{Scal} = \lambda^2 \text{Scal} = 20\lambda^2
\]
while \((\phi, \bar{\eta}, \bar{g})\) and \((-\frac{1}{\lambda} \phi h, \bar{\eta}, \bar{g})\) are nearly cosymplectic structures.

### 3. Almost nearly cosymplectic manifolds

Let \((M, \phi, \xi, \eta, g)\) be an almost nearly cosymplectic manifold of dimension \(2n + 1\). For the \((1,1)\) tensor \(h\) defined by (2.10) we have

**Lemma 3.1.** On an almost nearly cosymplectic manifolds the following relations are valid:
\[
g(hX, Y) = -g(X, hY) = \frac{1}{2} d\eta(X, Y); \quad h\xi = 0, \quad \phi h + h\phi = 0; \quad (\nabla_X \phi) \xi = -\phi hX.
\]  
(3.1)

**Proof.** The Killing condition (2.5) together with (2.10) yield
\[
g(\nabla_X \xi, Y) = g(hX, Y) = \frac{1}{2} d\eta(X, Y) = -g(hY, X) \quad \text{which proves the first equality. The second and the third are consequences of the first one and the first two lines in (2.4). Indeed, for the third one we have}
\[
g(h\phi X, Y) = \frac{1}{2} d\eta(\phi X, Y) = \frac{1}{2} d\eta(X, \phi Y) = g(hX, \phi Y) = g(hhX, Y).
\]
The last equation follows directly from (2.3).

#### 3.1. D-homothetic transformations

We recall [50, 51] that the almost contact metric structure \((\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})\) defined by
\[
\bar{\eta} = a\eta, \quad \bar{\xi} = \frac{1}{a} \xi, \quad \bar{\phi} = \phi, \quad \bar{g} = ag + (a^2 - a)\eta \otimes \eta, \quad a > 0 \quad \text{constant}
\]  
(3.2)
is called D-homothetic to $(\phi, \xi, \eta, g)$.

We have $F = aF$.

Our main result follows

**Theorem 3.2.** Any almost nearly cosymplectic structure is D-homothetic to a nearly cosymplectic structure and vice versa.

The corresponding $(1,1)$ tensors $\bar{h}$ and $h$ coincide.

**Proof.** Let $\{X_1, \ldots, X_{2n}, X_{2n+1} = \xi\}$ be an orthonormal basis. The Koszul formula and (3.2) give

\[
2g(\nabla_X X_j, X_k) = X_i(g(X_j, X_k) + X_j(g(X_i, X_k) - X_k(g(X_i, X_j)) + \bar{g}(X_i, X_k) - \bar{g}(X_k, X_i))
\]

\[
= 2ag(\nabla_X X_j, X_k) + (a^2 - \alpha) \left[ X_i[\eta(X_j)\eta(X_k)] + X_j[\eta(X_i)\eta(X_k)] - X_k[\eta(X_i)\eta(X_j)] \right]
\]

\[
+ \left( a^2 - \alpha \right) \left[ \eta([X_i, X_j])\eta(X_k) + \eta([X_i, X_k])\eta(X_j) - \eta([X_j, X_k])\eta(X_i) \right]
\]

\[
= 2ag(\nabla_X X_j, X_k) + (a^2 - \alpha) \left[ (\nabla_X \eta)X_j + (\nabla_X \eta)X_k + 2\eta(\nabla_X X_j)\eta(X_k) \right]
\]

\[
+ (a^2 - \alpha) \left[ (\nabla_X \eta)X_k - (\nabla_X \eta)X_j + (\nabla_X \eta)X_k - (\nabla_X \eta)X_j \right]
\]

\[
= 2ag(\nabla_X X_j, X_k) + (a^2 - \alpha) \left[ (\nabla_X \eta)X_j + (\nabla_X \eta)X_k + 2\eta(\nabla_X X_j)\eta(X_k) \right]
\]

\[
+ (a^2 - \alpha) \left[ d\eta(X_i, X_k)\eta(X_j) + d\eta(X_j, X_k)\eta(X_i) \right].
\]

(3.3)

The Killing condition applied to (3.3) yields

\[
\bar{g}(\nabla_X X_j, X_k) = ag(\nabla_X X_j, X_k)
\]

\[
+ \frac{a^2 - \alpha}{2} \left[ d\eta(X_i, X_k)\eta(X_j) + d\eta(X_j, X_k)\eta(X_i) + 2\eta(\nabla_X X_j)\eta(X_k) \right].
\]

(3.4)

Using the Killing condition, we evaluate the last term in (3.4) as follows,

\[
2\eta(\nabla_X X_j) = 2g(\nabla_X X_j, \xi) = -2g(\nabla_X \xi, X_j) = -d\eta(X_i, X_j).
\]

(3.5)

For a D-homothetic transformation with Killing vector field $\xi$ we obtain substituting (3.5) into (3.4) that

\[
\bar{g}(\nabla_X X_j, X_k) = ag(\nabla_X X_j, X_k)
\]

\[
+ \frac{a^2 - \alpha}{2} \left[ d\eta(X_i, X_k)\eta(X_j) + d\eta(X_j, X_k)\eta(X_i) - d\eta(X_i, X_k)\eta(X_j) \right].
\]

(3.6)

We have, applying (3.6), that the following tensor equality holds for all vector fields $X, Y, Z$

\[
\bar{g}((\nabla_X \phi)Y, Z) = \bar{g}(\nabla_X \phi Y, Z) + \bar{g}(\nabla_X \phi, Z)
\]

\[
= ag((\nabla_X \phi)Y, Z) + \frac{a^2 - \alpha}{2} \left[ d\eta(\phi Y, Z)\eta(X) - d\eta(X, \phi Y)\eta(Z) + d\eta(X, \phi Z)\eta(Y) + d\eta(Y, \phi Z)\eta(X) \right]
\]

\[
= ag((\nabla_X \phi)Y, Z) + \frac{a^2 - \alpha}{2} \left[ 2d\eta(\phi Y, Z)\eta(X) + d\eta(X, \phi Z)\eta(Y) - d\eta(X, \phi Y)\eta(Z) \right],
\]

(3.7)

where the last equality follows from (2.9).
Suppose \((\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})\) is a nearly cosymplectic structure, i.e. \((2.8)\) holds. Then, taking \(a = \frac{1}{2}\), the equalities \((3.7), (2.8)\) and \((2.9)\) imply
\[
\bar{g}((\bar{\nabla}_X \bar{\phi}) Y, Z) = \frac{3}{2} \bar{g}((\bar{\nabla}_X \bar{\phi}) Y, Z) + \frac{3}{8} \left[ 2d\bar{\eta}(\bar{\phi} Y, Z) \eta(X) + d\bar{\eta}(X, \bar{\phi} Z) \eta(Y) - d\bar{\eta}(X, \bar{\phi} Y) \eta(Z) \right]
= \frac{1}{3} dF(X, Y, Z) + \frac{1}{6} \left[ 2d\bar{\eta}(Y, \bar{\phi} Z) \eta(X) - d\bar{\eta}(Z, \bar{\phi} X) \eta(Y) - d\bar{\eta}(\bar{\phi} X, Y) \eta(Z) \right]
\]
i.e. the structure \((\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})\) is an almost nearly cosymplectic. Vice versa, starting from an almost nearly cosymplectic structure and making a D-homothetic deformation with constant \(a = \frac{1}{2}\) we get a nearly cosymplectic one which proves the first claim.

To show that \(\bar{h} = h\), we put \(X_j = \xi = \frac{1}{2} \xi\) in \((3.6)\) taken with \(a = \frac{1}{2}\) and use \((3.1)\) to get
\[
\bar{g}((\bar{\nabla}_X \xi, X_k) = g((\nabla_X \xi, X_k) + \frac{1}{4} d\eta(X_i, X_k) = \frac{3}{2} g((\nabla_X \xi, X_k).
\]
On the other hand, from \((3.2)\), we have
\[
\bar{g}((\nabla_X \xi, X_k) = \frac{3}{2} g((\nabla_X \xi, X_k) + \frac{3}{4} \eta((\nabla_X \xi) \eta(X_k) = \frac{3}{2} g((\nabla_X \xi, X_k).
\]
The last two equalities imply \(\bar{h}X_i = hX_i\) which completes the proof. \(\square\)

**Corollary 3.3.** Let \((M, \phi, \xi, \eta, g)\) be an almost contact metric manifold with Killing vector field \(\xi\). The D-homothetic almost contact metric manifold \((M, \bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})\) has a Killing vector field \(\bar{\xi}\) and the two corresponding Levi-Civita connections \(\nabla\) and \(\bar{\nabla}\) of these manifolds are related by
\[
g((\bar{\nabla}_Y X, Z) = g(\nabla_Y X, Z) + \frac{a^2 - a}{2a} \left[ d\eta(X, Y) \eta(Z) + d\eta(Y, Z) \eta(X) \right].
\]

**Proof.** Observe that any orthonormal basis for \(g\) is an orthogonal basis for \(\bar{g}\). We obtain from \((3.6)\) and \((3.2)\)
\[
\bar{g}((\bar{\nabla}_X \xi, X_k) = g((\nabla_X \xi, X_k) + \frac{a^2 - a}{2a} d\eta(X_i, X_k)
\]
which shows that \(\bar{\xi}\) is a Killing vector field for \(\bar{g}\).

Using \((3.2)\) together with the Killing condition, we have
\[
\bar{g}((\bar{\nabla}_X \xi, X_k) = a g((\nabla_X \xi, X_k) + (a^2 - a) \eta((\nabla_X \xi) \eta(X_k)
= a g((\nabla_X \xi, X_k) + \frac{a^2 - a}{a} \eta((\nabla_X \xi) \eta(X_k) = a g((\nabla_X \xi, X_k) - \frac{a^2 - a}{2a} d\eta(X_i, X_k) \eta(X_k)
= a g((\nabla_X \xi, X_k) - \frac{a^2 - a}{2a} d\eta(X_i, X_k) \eta(X_k).
\]

Compare \((3.10)\) with \((3.6)\) to get
\[
ag((\nabla_X \xi - \nabla_X \xi, X_k) = \frac{a^2 - a}{2} \left[ d\eta(X_i, X_k) \eta(X_k) + d\eta(X_i, X_k) \eta(X_k) \right]
\]
which implies (3.3) since the difference between these two connections is a tensor field.

Combine theorems 3.2 with 2.3 to get the structure theorem for almost nearly cosymplectic manifold.

**Theorem 3.4.** Let \((M, \phi, \xi, \eta, g)\) be an almost nearly cosymplectic non-cosymplectic manifold of dimension \(2n + 1 > 5\). Then \(M\) is locally D-homothetic with the constant \(a = \frac{3}{2}\) to one of the following Riemannian products:

\[ R \times N^{2n}, \quad M^\delta \times N^{2n-4}, \]

where \(N^{2n}\) is a nearly Kähler manifold, \(N^{2n-4}\) is a nearly Kähler manifold and \(M^\delta\) is a nearly cosymplectic non-cosymplectic manifold of dimension five.

If the manifold \(M\) is complete and simply connected the above D-homothety is global.

We also have

**Proposition 3.5.** On an almost nearly cosymplectic manifolds \((M, \phi, \bar{\eta}, \xi, \bar{g})\) the following relations hold.

\[ \bar{g}((\nabla_X \phi)Y, \bar{h}Z) = \bar{\eta}(Y)\bar{g}(\bar{h}^2X, \phi Z), \tag{3.11} \]

\[ -\bar{R}(\bar{\xi}, X, Y, Z) = \bar{g}((\nabla_X \bar{h})Y, Z) = -\bar{\eta}(Y)\bar{g}(\bar{h}^2X, Z) + \bar{\eta}(Z)\bar{g}(\bar{h}^2X, Y). \tag{3.12} \]

**Proof.** We apply theorem 3.2. Using (3.8) and \(\bar{h} = h\), we have

\[ \bar{g}((\nabla_X \phi)Y, \bar{h}Z) = \frac{3}{2} \bar{g}((\nabla_X \phi)Y, hZ) + \frac{3}{8} \left[ 2d\eta(Y, hZ)\eta(X) + d\eta(X, \phi hZ)\eta(Y) \right]. \]

For this D-homothetic transformation, and formula (2.12), we obtain

\[ \bar{g}((\nabla_X \phi)Y, \bar{h}Z) = \frac{3}{2} \left[ \eta(Y)\bar{g}(\bar{h}^2X, \phi Z) - \eta(X)\bar{g}(\bar{h}^2Y, \phi Z) \right] + \frac{3}{8} \left[ 4\eta(\bar{h}^2Y, \phi Z)\eta(X) + 2\bar{g}(\bar{h}^2X, \phi Z)\eta(Y) \right] \]

\[ = \frac{9}{4} \eta(Y)\bar{g}(\bar{h}^2X, \phi Z) = \bar{\eta}(Y)\bar{g}(\bar{h}^2X, \phi Z) \]

which proves (3.11).

To prove the second line we first note that since \(\xi\) is a Killing vector field we have the well known relation \(g((\nabla_X \bar{h})Y, Z) = -\bar{R}(\xi, X, Y, Z)\) (see e.g. [53]). Further, starting from (3.9) taken for \(a = \frac{3}{2}\), we obtain

\[ \bar{g}((\nabla_X \bar{h})Y, Z) = \frac{3}{2} \bar{g}((\nabla_X \bar{h})Y, Z) + \frac{3}{8} \left[ d\eta(X, \bar{h}Z)\eta(Y) - d\eta(X, hY)\eta(Z) \right], \]

which, in view of (2.13) and (2.11), takes the form

\[ \bar{g}((\nabla_X \bar{h})Y, Z) = \frac{3}{2} \left[ -\eta(Y)\bar{g}(\bar{h}^2X, Z) + \eta(Z)\bar{g}(\bar{h}^2Y, \bar{h}X) \right] + \frac{3}{8} \left[ -d\eta(hX, Z)\eta(Y) + d\eta(hX, Y)\eta(Z) \right] \]

\[ = -\frac{9}{4} \eta(Y)\bar{g}(\bar{h}^2X, Z) + \frac{9}{4} \eta(Z)\bar{g}(\bar{h}^2X, \bar{h}X) = -\bar{\eta}(Y)\bar{g}(\bar{h}^2X, Z) + \bar{\eta}(Z)\bar{g}(\bar{h}^2X, Y). \]

This completes the proof.
3.2. The curvature of almost nearly cosymplectic manifold

Let \((M, \phi, \eta, \xi, g)\) be an almost nearly cosymplectic manifold \(D\)-homotetically related with \(a = \frac{3}{2}\) to the nearly cosymplectic manifold \((M, \phi, \eta, \xi, g)\) in the sense of theorem 3.2.

Put \(a = \frac{3}{2}\) into (3.9) and use (2.11) to get the following relations between the Levi-Civita connections

\[
\nabla_X Y = \nabla_X Y + \frac{1}{2} \eta(Y)hX + \frac{1}{2} \eta(X)hY. \tag{3.13}
\]

For the curvatures, we calculate using (3.13), (3.1), (2.11) and \(\bar{h} = h\) that

\[
R(Z, X)Y = \nabla_Z \nabla_X Y - \nabla_X \nabla_Z Y - \nabla_{[Z, X]} Y
\]

\[= R(Z, X)Y + \frac{1}{2}(\nabla_Z \eta)Y \cdot hX - \frac{1}{2}(\nabla_X \eta)Y \cdot hZ + \frac{1}{2} \eta([Z, X])hY - \frac{1}{2} \eta(Z)(\nabla_X h)Y
\]

\[+ \frac{1}{2} \eta(X)(\nabla_Z h)Y + \frac{1}{2} \eta(Y)\left(\nabla_Z hX - (\nabla_X h)Z\right) + \frac{1}{4} \eta(Y)\eta(Z)h^2X - \frac{1}{4} \eta(Y)\eta(X)h^2Z.
\]

Applying (2.13) to (3.14), we get

\[
R(Z, X)Y = R(Z, X)Y + \frac{1}{2} g(hZ, Y)hX - \frac{1}{2} g(hX, Y)hZ + g(hZ, X)hY
\]

\[+ \frac{5}{4} \eta(Y)\eta(Z)h^2X - \frac{5}{4} \eta(Y)\eta(X)h^2Z + \frac{1}{2} \eta(X)g(h^2Z, Y) - \eta(Z)g(h^2X, Y)\xi.
\]

(3.15)

For the Ricci tensors \(\overline{\text{Ric}}\) and \(\text{Ric}\), we get

\[
\overline{\text{Ric}}(X, Y) = \text{Ric}(X, Y) + g(h^2X, Y) - \frac{5}{4} \text{tr}(h^2)\eta(X)\eta(Y). \tag{3.16}
\]

4. Almost nearly cosymplectic manifolds in dimension 5

In view of theorem 3.4 we restrict our attention to dimension five. Let \((M, \phi, \xi, \eta, g)\) be a 5D almost nearly cosymplectic manifold, \((M, \phi, \xi, \eta, g)\) be the \(D\)-homothetically corresponding nearly cosymplectic manifold according to theorem 3.2 and \((M, \phi, \xi, \eta, \tilde{g})\) be the Sasaki–Einstein manifold homothetically attached to the nearly cosymplectic structure \((M, \phi, \xi, \eta, g)\), i.e. we have

\[
\tilde{\eta} = \frac{3}{2} \eta, \quad \tilde{\xi} = \frac{2}{3} \xi, \quad \tilde{\phi} = \phi, \quad \tilde{g} = \frac{3}{2} g + \frac{3}{4} \eta \otimes \eta, \quad g = \frac{2}{3} \tilde{g} - \frac{2}{9} \tilde{\eta} \otimes \tilde{\eta}.
\]

\[
\tilde{\phi} = -\frac{1}{\lambda} h, \quad \tilde{\eta} = \lambda \eta, \quad \tilde{\xi} = \frac{1}{\lambda} \xi, \quad \tilde{g} = \lambda^2 g = \frac{2\lambda}{3} \tilde{g} - \frac{2\lambda^2}{9} \tilde{\eta} \otimes \tilde{\eta}.
\]

(4.1)

We recall that an almost contact metric manifold \((M, \phi, \xi, \eta, g)\) is called \(\eta\)-Einstein if its Ricci tensor satisfies

\[
\text{Ric}(X, Y) = a g(X, Y) + b \eta(Y)\eta(Y),
\]

where \(a\) and \(b\) are smooth functions on \(M\). \(\eta\)-Einstein metrics were introduced and studied by Okumura [45]. In particular, he studied the relation between the existence of \(\eta\)-Einstein metrics and certain harmonic forms and showed that a \(K\)-contact \(\eta\)-Einstein manifold of dimension \(2n + 1\) bigger than 3 the functions \(a\) and \(b\) are constants satisfying \(a + b = 2n\). Sasakian \(\eta\)-Einstein spaces are studied in detail in [9].
Proposition 4.1. Let \((M, \phi, \xi, \eta, \mathring{g})\) be a 5D almost nearly cosymplectic manifold. Then it is \(\eta\)-Einstein and the Ricci tensor is given by

\[
\mathring{\text{Ric}}(X, Y) = 2\lambda^2 \mathring{g}(X, Y) + 2\lambda^2 \eta(X)\eta(Y).
\] (4.2)

Proof. We get from (3.15) and (2.17)

\[
R(Z, X)Y = R(Z, X)Y + \frac{1}{2} \mathring{g}(hZ, Y)hX - \frac{1}{2} \mathring{g}(hX, Y)hZ + \mathring{g}(hZ, X)hY
- \frac{5\lambda^2}{4} \eta(Y)\eta(X)X
+ \frac{5\lambda^2}{4} \eta(X)\eta(Z) - \frac{\lambda^2}{2} \eta(\mathring{g}(Z, Y) - \eta(Z)\mathring{g}(X, Y))\xi.
\] (4.3)

For the Ricci tensors \(\mathring{\text{Ric}}\) and \(\text{Ric}\), we obtain from (3.16)

\[
\mathring{\text{Ric}}(X, Y) = \text{Ric}(X, Y) + g(h^2X, Y) - \frac{5}{4} \mathring{g}(h^2\eta)(X)\eta(Y)
= \text{Ric}(X, Y) - \lambda^2 \mathring{g}(X, Y) + 6\lambda^2 \eta(X)\eta(Y)
= 3\lambda^2 \mathring{g}(X, Y) - \frac{2}{3} \mathring{g}(X)\eta(Y)) + 6\lambda^2 \eta(X)\eta(Y)
= 2\lambda^2 \mathring{g}(X, Y) + 2\lambda^2 \eta(X)\eta(Y)
\]

since \(\text{Ric} = \frac{\text{scal}}{5} \mathring{g} = 4\lambda^2 \mathring{g}\) is an Einstein space according to theorem 2.4. This completes the proof.

Let \((\eta, \omega_1, \omega_2, \omega_3)\) be the SU(2) structure induced by the nearly cosymplectic structure determined by (2.18) and satisfying (2.19) [11]. Since \(\phi = \phi\) and \(h = h\) we get an SU(2) structure \((\mathring{\eta}, \mathring{\omega}_1, \mathring{\omega}_2, \mathring{\omega}_3) \mathring{=} g(\phi, \ldots))\) induced by the almost nearly cosymplectic structure. These two SU(2) structures are related by

\[
\mathring{\eta} = \frac{3}{2} \eta, \quad \mathring{\omega}_i = \frac{3}{2} \omega_i.
\] (4.5)

We obtain from (2.19), theorem 2.4, (4.5) and proposition 4.1 the following

Proposition 4.2. An almost nearly cosymplectic structure on a 5D manifold is equivalent to an SU(2)-structure \((\mathring{\eta}, \mathring{\omega}_1, \mathring{\omega}_2, \mathring{\omega}_3)\) satisfying

\[
d\mathring{\eta} = -2\lambda \mathring{\omega}_3, \quad d\mathring{\omega}_1 = 2\lambda \mathring{\eta} \wedge \mathring{\omega}_2, \quad d\mathring{\omega}_2 = -2\lambda \mathring{\eta} \wedge \mathring{\omega}_1
\] (4.6)

for some real number \(\lambda \neq 0\). These structures are hypo and the structures \((\eta, \omega_1, \omega_2)\) and \((\mathring{\eta}, \mathring{\omega}_1, \mathring{\omega}_2)\) are almost nearly cosymplectic \(\eta\)-Einstein structures.

The homothetic SU(2) structure \(\eta^* = \lambda^2 \mathring{\eta}, \omega_i^* = \lambda^2 \mathring{\omega}_i, i = 1, 2, 3\) satisfies (4.6) with \(\lambda = 1\). The structure \((\eta^*, \omega^*_1)\) is a Sasaki \(\eta^*\)-Einstein structure while the structures \((\eta^*, \omega^*_i)\) and \((\eta^*, \omega^*_2)\) are almost nearly cosymplectic \(\eta^*\)-Einstein structures. We have

Corollary 4.3. An SU(2) structure \((\eta, \omega_1, \omega_2, \omega_3)\) on a 5-manifold is Sasaki \(\eta\)-Einstein if and only if it satisfies the relations

\[
d\eta = -2\omega_3, \quad d\omega_1 = 2\eta \wedge \omega_2, \quad d\omega_2 = -2\eta \wedge \omega_1.
\] (4.7)

Consider the structures \((\mathring{\eta}, \mathring{\Omega}_1, \mathring{\Omega}_2, \mathring{\omega}_3)\), where
\[
\Omega_1' = \cos t \omega_1 + \sin t \omega_2, \quad \phi_1' = -\frac{1}{\lambda} \phi h \cos t + \phi \sin t \\
\Omega_2' = -\sin t \omega_1 + \cos t \omega_2, \quad \phi_2' = \frac{1}{\lambda} \phi h \sin t + \phi \cos t.
\] (4.8)

It is easy to check that these structures are SU(2) structures satisfying (4.6) and \((\bar{\eta}, \Omega_1')\) and \((\bar{\eta}, \Omega_2')\) are almost nearly cosymplectic \(\bar{\eta}\)-Einstein structures.

**Remark 4.4.** Starting with an SU(2) structure \((\eta, \omega_1, \omega_2, \omega_3)\) on a 5-manifold which is Sasaki Einstein one obtains an \(\mathbb{R}^5\)-famil of nearly cosymplectic structures \((\eta, \Omega_1', \Omega_2', \omega_3)\) given by

\[
\Omega_1' = \cos t \omega_1 + \sin t \omega_2, \quad \phi_1' = -\frac{1}{\lambda} \phi h \cos t + \phi \sin t \\
\Omega_2' = -\sin t \omega_1 + \cos t \omega_2, \quad \phi_2' = \frac{1}{\lambda} \phi h \sin t + \phi \cos t
\] (4.9)

which satisfy (2.19).

Applying theorems 2.4, 3.2 and proposition 4.1, we obtain

**Theorem 4.5.** An almost nearly cosymplectic non cosymplectic 5-manifold \((M, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})\) carries a D-homothetic Sasaki–Einstein structure \((M, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})\) and vice versa, any Sasaki–Einstein 5 manifold supports a \(\mathbb{R}^5\)-famil of almost nearly cosymplectic structures.

In particular, an almost nearly cosymplectic non cosymplectic 5-manifold is \(\eta\)-Einstein, \(\text{Ric} = 2\lambda^2 (\bar{\tilde{g}} + \tilde{\eta} \otimes \tilde{\eta})\) with positive scalar curvature \(\text{Scal} = 12\lambda^2\).

In terms of almost nearly cosymplectic structures, the attached Sasaki–Einstein structure is given by

\[
\tilde{\phi} = -\frac{1}{\lambda} \tilde{h}, \quad \bar{\tilde{\eta}} = \frac{2}{3} \tilde{\eta}, \quad \bar{\tilde{\xi}} = \frac{3}{2\lambda} \tilde{\xi}, \quad \bar{\tilde{g}} = \frac{2}{3} \tilde{g} - \frac{2\lambda^2}{9} \tilde{\eta} \otimes \tilde{\eta}.
\]

The structures \(-\frac{1}{\lambda} \phi h, \bar{\eta}, \bar{g}\) and \((\phi, \tilde{\eta}, \tilde{g})\) are almost nearly cosymplectic generating the circle-famil \((\phi_1', \bar{\eta}, \bar{g}), (\phi_2', \bar{\eta}, \bar{g})\) of almost nearly cosymplectic structures defined by (4.8).

Since the nearly cosymplectic structure is homothetic to the Sasaki–Einstein structure then the corresponding Levi-Civita connections coincide, \(\nabla = \bar{\nabla}\) and (3.13) takes the form

\[
\bar{\nabla}_X Y = \bar{\nabla}_X Y + \frac{1}{2} \bar{\eta}(Y) hX + \frac{1}{2} \eta(X) hY, \quad R(X, Y) Z = \bar{R}(X, Y) Z.
\] (4.10)

We have

**Proposition 4.6.** The curvature of an almost nearly cosymplectic manifold is connected with the curvature of the associated Sasaki–Einstein manifold by

\[
R(Z, X) Y = \bar{R}(Z, X) Y + \frac{1}{2} \bar{g}(\phi Z, Y) \phi X - \frac{1}{2} \bar{g}(\phi X, Y) \phi Z + \bar{g}(Z, X) \phi Y - \frac{5}{4} \bar{\eta}(Y) \bar{\eta}(Z) X + \frac{5}{4} \bar{\eta}(Y) \bar{\eta}(X) Z - \frac{1}{2} \left[ \bar{\eta}(X) \bar{g}(Z, Y) - \bar{\eta}(Z) \bar{g}(X, Y) \right] \bar{\xi}.
\] (4.11)
In particular, we have
\[ R(Z, X)\xi = \lambda^2 \left[ \eta(X)Z - \bar{\eta}(Z)X \right]. \] (4.12)

**Proof.** Applying (4.1)–(4.3) we obtain (4.11). Using the well known identity
\[ R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \] (4.13)
valid for any Sasakian manifold (see e.g. [5]), we obtain (4.12) from (4.11) and (4.1). \qed

5. The NGT-connections in dimension 5

Due to theorem 4.5, any Sasaki Einstein 5-manifold generates an \((R^2 - \{0\})\)-family of almost nearly cosymplectic structures which are D-homothetic to the Sasaki–Einstein structure. These almost nearly cosymplectic structures have the same Levi-Civita connection. According to theorem 2.2 each almost nearly cosymplectic structure generates a unique NGT-connection with skew-symmetric torsion. The NGTS connections may be different for different almost nearly cosymplectic structures from the family because the skew-symmetric torsions may be different. We describe these NGTS connections explicitly and express their curvature and the Ricci tensors in terms of the Sasaki–Einstein curvature and the generated SU(2)-structures.

According to theorem 2.2 and taking into account proposition 4.2 and (4.8), the torsions \(T_1\), resp \(T_2\), of the NGTS connection \((\bar{\eta}, \bar{\Omega}_2)\), resp. \((\bar{\eta}, \bar{\Omega}_1)\) are given respectively by
\[ T_2(X, Y, Z) = -\frac{1}{3} d\bar{\Omega}_2(X, Y, Z) = \frac{2}{3} \lambda (\bar{\eta} \wedge \bar{\Omega}_1)(X, Y, Z) \]
\[ = -\frac{2}{3} \cos t \left( \bar{\eta}(X)g(\phi hY, Z) + \bar{\eta}(Y)g(\phi hZ, X) + \bar{\eta}(Z)g(\phi hX, Y) \right) \]
\[ + \frac{2}{3} \lambda \sin t \left( \bar{\eta}(X)g(\phi Y, Z) + \bar{\eta}(Y)g(\phi Z, X) + \bar{\eta}(Z)g(\phi X, Y) \right); \] (5.1)
\[ T_1(X, Y, Z) = -\frac{1}{3} d\bar{\Omega}_1(X, Y, Z) = -\frac{2}{3} \lambda \eta \wedge \bar{\Omega}_2(X, Y, Z) \]
\[ = -\frac{2}{3} \sin t \left( \eta(X)g(\phi hY, Z) + \eta(Y)g(\phi hZ, X) + \eta(Z)g(\phi hX, Y) \right) \]
\[ - \frac{2}{3} \lambda \cos t \left( \eta(X)g(\phi Y, Z) + \eta(Y)g(\phi Z, X) + \eta(Z)g(\phi X, Y) \right). \] (5.2)

Letting \( t = t + \frac{\pi}{3} \) we can write all the torsions with one relation as follows
\[ T_1(X, Y, Z) = -\frac{2}{3} \sin t \left( \eta(X)g(\phi hY, Z) + \eta(Y)g(\phi hZ, X) + \eta(Z)g(\phi hX, Y) \right) \]
\[ - \frac{2}{3} \lambda \cos t \left( \eta(X)g(\phi Y, Z) + \eta(Y)g(\phi Z, X) + \eta(Z)g(\phi X, Y) \right). \] (5.3)
5.1. The NGTS connections $\nabla_{\tilde{\tau}}^{\text{ngt}}$ and its curvature

Insert (5.3) into (2.7) to get for the NGTS connections $\nabla_{\tilde{\tau}}^{\text{ngt}}$ the following expression

$$\bar{g}(\nabla_{\tilde{\tau}}^{\text{ngt}} X, Y, Z) = \bar{g}(\nabla_{X} Y, Z) + \frac{1}{3} \left[ \bar{g}(\nabla_{X} Y, Z) + \bar{g}(\nabla_{Y} X, Z) + \bar{g}(\nabla_{Z} X, Y) \right]$$

$$- \frac{1}{3} \sin t \left[ \bar{g}(\nabla_{X} Y, Z) + \bar{g}(\nabla_{Y} X, Z) + \bar{g}(\nabla_{Z} X, Y) \right]$$

$$- \frac{1}{3} \lambda \cos t \left[ \bar{g}(\nabla_{X} Y, Z) + \bar{g}(\nabla_{Y} X, Z) + \bar{g}(\nabla_{Z} X, Y) \right]$$

(5.4)

which, using (4.1) and (3.13) yields

$$\nabla_{\tilde{\tau}}^{\text{ngt}} X = \nabla_{X} Y - \frac{\lambda}{2} \left[ \eta(X)(\phi'_{2} + 2\phi_{3}) Y - \eta(Y)(\phi'_{2} - 2\phi_{3}) X + \frac{2}{3} \bar{g}(\phi'_{2} X, Y) \xi \right].$$

(5.5)

The equation (5.5) implies

**Proposition 5.1.** The NGTS connections $\nabla_{\tilde{\tau}}^{\text{ngt}}$ with totally skew-symmetric torsion on a Sasaki–Einstein 5-manifold $(M, \tilde{\phi}, g, \tilde{\xi})$ are related to the Levi-Civita connection $\nabla$ of $\bar{g}$ by

$$\nabla_{\tilde{\tau}}^{\text{ngt}} X = \tilde{\nabla}_{X} Y - \frac{\lambda}{2} \left[ \eta(X)(\phi'_{2} + 2\phi_{3}) Y - \eta(Y)(\phi'_{2} - 2\phi_{3}) X + \frac{2}{3} \bar{g}(\phi'_{2} X, Y) \xi \right].$$

(5.6)

Further, to calculate the curvature of the NGTS connection $\nabla_{\tilde{\tau}}^{\text{ngt}}$ we use (5.5). We have

$$\nabla_{\tilde{\tau}}^{\text{ngt}} \nabla_{\tilde{\tau}}^{\text{ngt}} X = \nabla_{Z} \nabla_{X} Y - \frac{\lambda}{2} \left[ \eta(Z)(\phi'_{2} + 2\phi_{3}) \nabla_{X} Y - \eta(\nabla_{X} Y)(\phi'_{2} - 2\phi_{3}) Z + \frac{2}{3} \bar{g}(\phi'_{2} Z, \nabla_{X} Y) \xi \right]$$

$$- \frac{\lambda}{2} \left[ Z\eta(X)\phi'_{2} Y + \eta(X)\nabla_{Z} \phi'_{2} Y - Z\eta(Y)\phi'_{2} X - \eta(Y)\nabla_{Z} \phi'_{2} X + \frac{2}{3} \bar{g}(\phi'_{2} Y, \eta(X) \xi) \right]$$

$$- \frac{\lambda}{2} \left[ Z\eta(Y)\phi'_{2} Y + \eta(Y)\nabla_{Z} \phi'_{2} Y + Z\eta(Y)\phi'_{2} X + \eta(Y)\nabla_{Z} \phi'_{2} X \right].$$

(5.7)

Applying the fact that $\phi'_{2}$ and $\phi'_{1}$ are nearly cosymplectic structures, we get from (5.7) and (5.5) that

$$\frac{2}{\lambda} R_{\text{ngt}}^{\text{ngt}}(Z, X) Y = \frac{2}{\lambda} R(Z, X) Y - d\eta(Z, X)[\phi'_{2} + 2\phi_{3}] Y$$

$$+ (\nabla_{Y})[\phi'_{2} - 2\phi_{3}] X - (\nabla_{X})[\phi'_{2} - 2\phi_{3}] Z$$

$$- \eta(X)[(\nabla_{Z} \phi'_{2}) Y + 2(\nabla_{\phi'_{2}} Y) + \eta(Z)[(\nabla_{X} \phi'_{2}) Y + 2(\nabla_{\phi'_{2}} Y) Y$$

$$- 2\eta(Y)[(\nabla_{\phi'_{2}}) X - (\nabla_{X} \phi'_{2}) Z - (\nabla_{Z} \phi'_{2}) X] + 3 \left[ \eta(Y) \eta(Z) \right] \left[ \frac{5}{2} - \frac{1}{2} \phi'_{2} - \frac{1}{2} \phi'_{1} \right]$$

$$+ \frac{4\lambda}{3} [g(\phi'_{2} X, Y) \eta(Z) - g(\phi'_{2} Z, Y) \phi'_{2} X] - \frac{\lambda}{3} [g(\phi'_{2} X, Y) \phi'_{2} Z - g(\phi'_{2} Z, Y) \phi'_{2} X]$$

$$- \frac{2}{3} [g(\nabla_{Z} \phi'_{2}) Y - g(\nabla_{X} \phi'_{2}) Z)] \xi + \frac{\lambda}{3} [\eta(X) g(Z, Y) - \eta(Z) g(X, Y)] \xi$$

$$- \frac{2\lambda}{3} [g(\phi'_{2} X, Y) - \eta(Z) g(\phi'_{2} X, Y) + 2\eta(Y) g(\phi'_{2} Z, Y)] \xi$$

(5.8)

which, due to the Killing condition, (2.19), (4.9) and (5.10), is equivalent to
\[
\frac{2}{\lambda} R_{\text{NS}}^{\text{ep}}(Z, X) Y = \frac{2}{\lambda} R(Z, X) Y + 2\omega_3(Z, X) \left[ \phi_2 + 2\phi_3 \right] Y \\
- \lambda\omega_3(Z, Y) [\phi_2 - 2\phi_3] X + \lambda\omega_3(X, Y) [\phi_2 - 2\phi_3] Z \\
- \eta(X)[(\nabla Z)\phi_2] Y + 2(\nabla Z)\phi_3 Y + \eta(Z)[(\nabla X)\phi_2] Y + 2(\nabla X)\phi_3 Y \\
- 2\eta(Y) [(\nabla Z)\phi_3] X - (\nabla Z)\phi_3 Z - (\nabla Z)\phi_3 X + \lambda \left[ \eta(Y) \eta(Z) \frac{5}{2} - \frac{1}{2} \phi_1 |X| - \lambda \eta(Y) \eta|X| \frac{5}{2} - \frac{1}{2} \phi_1 |Z| \right] \\
+ \frac{4\lambda}{3} [\Omega_2(Y, X) \phi_2 Z - \Omega_2(Z, Y) \phi_2 X] - \frac{\lambda}{3} [\Omega_2(Y, X) \phi_2 Z - \Omega_2(Z, Y) \phi_2 X] \\
+ \frac{\lambda}{3} [\eta(X) g(Z, Y) - \eta(Z) g(X, Y)] \xi - 2\lambda [\eta(Y) \Omega_1(Z, Y) - \eta(Z) \Omega_1(X, Y)] \xi.
\]

(5.9)

At this point we need (2.8), remark 4.4 and (2.19) yielding to

\[
g((\nabla Z)\phi_2) Y, Z) = \frac{1}{3} d\Omega_2(X, Y, Z) = -\lambda \eta \wedge \Omega_1(X, Y, Z)
\]

\[
= -\lambda \left[ \eta(X) g(\phi_2 Y, Z) + \eta(Y) g(\phi_2 Z, X) + \eta(Z) g(\phi_2 X, Y) \right];
\]

(5.10)

\[
g((\nabla Z)\phi_2) Y, Z) = \frac{1}{3} d\Omega_2(X, Y, Z) = \lambda \eta \wedge \Omega_1(X, Y, Z)
\]

\[
= \lambda \left[ \eta(X) g(\phi_2 Y, Z) + \eta(Y) g(\phi_2 Z, X) + \eta(Z) g(\phi_2 X, Y) \right]
\]

(5.11)

\[
g((\nabla Z)\phi_2) Y, Z) = \frac{1}{3} g((\nabla Z) h) Y, Z) = \lambda \left[ g(X, Y) \eta(Z) - g(X, Z) \eta(Y) \right].
\]

(5.12)

Applying (5.10)–(5.11) to (5.9), we obtain

\[
\frac{2}{\lambda} R_{\text{NS}}^{\text{ep}}(Z, X) Y = \frac{2}{\lambda} R(Z, X) Y + 2\omega_3(Z, X) \left[ \phi_2 + 2\phi_3 \right] Y \\
+ \frac{3}{2} \eta(X) \eta(Y) [Z + \phi_1 Z] + \left[ \omega_3(X, Y) - \frac{1}{3} \Omega_2(X, Y) \right] \phi_2 Z - 2 \left[ \omega_3(X, Y) - \frac{2}{3} \Omega_2(X, Y) \right] \phi_3 Z \\
- \frac{3}{2} \eta(Z) \eta(Y) [X + \phi_1 X] - \left[ \omega_3(Y, Z) - \frac{1}{3} \Omega_2(Y, Z) \right] \phi_2 X + 2 \left[ \omega_3(Y, Z) - \frac{2}{3} \Omega_2(Y, Z) \right] \phi_3 X
\]

\[
+ \frac{5}{3} \left( \eta(Z) g(X, Y) - \eta(X) g(Z, Y) \right) + \eta(Y) \Omega_1(X, Y) - \eta(Z) \Omega_1(Z, Y) - 2\eta(Y) \Omega_1(Z, X) \xi.
\]

(5.13)

The equations (5.13), (4.13) and the Sasaki–Einstein condition $Ric = 4\hat{g}$ imply

**Proposition 5.2.** The curvatures $R_{\text{NS}}^{\text{ep}}$ of the NGTS connections with totally skew-symmetric torsion on a Sasaki–Einstein 5-manifold $(M, \eta, \phi, \tilde{g}, \tilde{\xi})$ are related to the Sasaki–Einstein curvature $\tilde{R}$ and the corresponding SU(2) structure $(\tilde{\eta}, \Omega_1, \Omega_2, \tilde{\omega}_3)$ by

\[
R_{\text{NS}}^{\text{ep}}(Z, X) Y = \tilde{R}(Z, X) Y + \tilde{\omega}_3(Z, X) \left[ \phi_2 + 2\phi_3 \right] Y
\]

\[
+ \frac{3}{4} \tilde{\eta}(X) \tilde{\eta}(Y) \left[ Z + \phi_1 Z \right] + \frac{1}{2} \tilde{\omega}_3(X, Y) - \frac{1}{3} \tilde{\Omega}_2(X, Y) \right] \phi_2 Z - \left[ \tilde{\omega}_3(X, Y) - \frac{2}{3} \tilde{\Omega}_2(X, Y) \right] \phi_3 Z
\]

\[
- \frac{3}{4} \tilde{\eta}(Z) \tilde{\eta}(Y) \left[ X + \phi_1 X \right] - \frac{1}{2} \tilde{\omega}_3(Z, Y) - \frac{1}{3} \tilde{\Omega}_2(Z, Y) \right] \phi_2 X + \left[ \tilde{\omega}_3(Y, Z) - \frac{2}{3} \tilde{\Omega}_2(Y, Z) \right] \phi_3 X
\]

\[
+ \frac{5}{6} \tilde{\eta}(Z) \tilde{g}(X, Y) - \frac{5}{6} \tilde{\eta}(X) \tilde{g}(Z, Y) + \frac{1}{2} \tilde{\eta}(Z) \tilde{\Omega}_1(X, Y) - \frac{1}{2} \tilde{\eta}(X) \tilde{\Omega}_1(Z, Y) - \tilde{\eta}(Y) \tilde{\Omega}_1(Z, X) \xi.
\]

(5.14)
In particular,
\[
R_{ij}^{\text{ngt}}(Z,X) \bar{\xi} = \frac{7}{4} \tilde{\eta}(X)Z - \frac{7}{4} \tilde{\eta}(Z)X + \frac{3}{4} \tilde{\eta}(X) \phi_1 Z - \frac{3}{4} \tilde{\eta}(Z) \phi_1 X - \tilde{\Omega}_1(Z,X) \bar{\xi}.
\]
(5.15)

The Ricci tensors of the NGTS connections are given by
\[
\text{Ric}_{ij}^{\text{ngt}}(X,Y) = \frac{5}{3} \tilde{g}(X,Y) + \frac{16}{3} \tilde{\eta}(X) \tilde{\eta}(Y) + \frac{4}{3} \tilde{\Omega}_1(X,Y).
\]
(5.16)

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