LOW–TEMPERATURE REGIMES AND FINITE SIZE–SCALING IN A QUANTUM SPHERICAL MODEL

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Abstract

A $d$–dimensional quantum model in the spherical approximation confined to a general geometry of the form $L^{d-d'}\times\infty^{d'}\times L_{\tau}^z$ ($L$–linear space size and $L_{\tau}$–temporal size) and subjected to periodic boundary conditions is considered. Because of its close relation with the quantum rotors model it can be regarded as an effective model for studying the low–temperature behavior of the quantum Heisenberg antiferromagnets. Due to the remarkable opportunity it offers for rigorous study of finite–size effects at arbitrary dimensionality this model may play the same role in quantum critical phenomena as the popular Berlin–Kac spherical model in classical critical phenomena. Close to the zero–temperature quantum critical point, the ideas of finite–size scaling are utilized to the fullest extent for studying the critical behavior of the model. For different dimensions $1 < d < 3$ and $0 \leq d' \leq d$ a detailed analysis, in terms of the special functions of classical mathematics, for the free energy, the susceptibility and the equation of state is given. Particular attention is paid to the two–dimensional case.

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I. INTRODUCTION

In recent years there has been a renewed interest in the theory of zero-temperature quantum phase transitions initiated in 1976 by Hertz’s quantum dynamic renormalization group for itinerant ferromagnets. Distinctively from temperature driven critical phenomena, these phase transitions occur at zero temperature as a function of some non-thermal control parameter (or a competition between different parameters describing the basic interaction of the system), and the relevant fluctuations are of quantum rather than thermal nature.

It is well known from the theory of critical phenomena that for the temperature driven phase transitions quantum effects are unimportant near critical points with $T_c > 0$. It could be expected, however, that at rather small (as compared to characteristic excitation in the system) temperature, the leading $T$ dependence of all observables is specified by the properties of the zero-temperature critical points, which take place in quantum systems. The dimensional crossover rule asserts that the critical singularities of such a quantum system at $T = 0$ with dimensionality $d$ are formally equivalent to those of a classical system with dimensionality $d + z$ ($z$ is the dynamical critical exponent) and critical temperature $T_c > 0$. This makes it possible to investigate low-temperature effects (considering an effective system with $d$ infinite space and $z$ finite time dimensions) in the framework of the theory of finite-size scaling (FSS). The idea of this theory has been applied to explore the low-temperature regime in quantum systems (see Refs. 6–9), when the properties of the thermodynamic observables in the finite-temperature quantum critical region have been the main focus of interest. The very quantum critical region was introduced and studied first by Chakravarty et al. using the renormalization group methods. The most famous model for discussing these properties is the quantum nonlinear $O(n)$ sigma model (QNLσM). 6–16

Recently an equivalence between the QNLσM in the limit $n \to \infty$ and a quantum version of the spherical model or more precisely the “spherical quantum rotors” model (SQRM) was announced. 17 The SQRM is an interesting model on its own. Due to the remarkable opportunity it offers for a rigorous study of finite-size effects at arbitrary dimensionality SQRM may play the same role in quantum critical phenomena as the popular Berlin–Kac spherical model in classical critical phenomena. The last one became a touchstone for various scaling hypotheses and a source of new ideas in the general theory of finite size scaling (see for example Refs. 18–27 and references therein). Let us note that an increasing interest related with the spherical approximation (or large $n$–limit) generating tractable models in quantum critical phenomena has been observed in the last few years. 17, 28–30

In Ref. 17, the critical exponents for the zero-temperature quantum fixed point and the finite-temperature classical one as a function of dimensionality was obtained. What remains beyond the scope of Ref. 17 is to study in an exact manner the scaling properties of the model in different regions of the phase diagram including the quantum critical region as a function of the dimensionality of the system. In the context of the finite–size scaling theory both cases: (i) The infinite $d$–dimensional quantum system at low-temperatures $\infty^d \times L_{\tau}^z$ ($L_{\tau} \sim \left(\frac{\hbar}{k_B T}\right)^{1/z}$ is the finite–size in the imaginary time direction) and (ii) the finite system confined to the geometry $L^{d-d} \times \infty^d \times L_{\tau}^z$ ($L$–linear space size) are of crucial interest.

In this paper a detailed theory of the scaling properties of the quantum spherical model with nearest–neighbor interaction is presented. The plan of the paper is as follows: we start
with a brief review of the model and the basic equations for the free energy and the quantum spherical field in the case of periodic boundary conditions (Section II). The relation of this model with the model due to Schneider, Stoll and Beck has been briefly commented in Section III. Since we would like to exploit the ideas of the FSS theory, the bulk system in the low–temperature region is considered like an effective \((d + 1)\)–dimensional classical system with one finite (temporal) dimension. This is done to enable contact to be made with other results based on the spherical type approximation e.g. in the framework of the spherical model and the QNL\(\sigma\)M in the limit \(n \to \infty\). The scaling forms of the free energy and of the spherical field equation are derived for the infinite (Section IV A) and finite (Section IV B) system confined to the general geometry \(L^d \times \cdots \times L^d\). In Section V we analyze in detail the equation for the spherical field. This equation turns out to allow for analytic studies of the finite–size and low–temperature asymptotes for different \(d\) and \(d'\). Special attention is laid on the two dimensional system. The remainder of the paper contains the details of the calculations: Appendices A and B.

II. THE MODEL

The model we will consider here describes a magnetic ordering due to the interaction of quantum spins. This has the following form:

\[
\mathcal{H} = \frac{1}{2} g \sum_{\ell} \mathcal{P}_\ell^2 - \frac{1}{2} \sum_{\ell \ell'} J_{\ell \ell'} S_\ell S_{\ell'} + \frac{\mu}{2} \sum_{\ell} S_\ell^2 - H \sum_{\ell} S_\ell, \tag{2.1}
\]

where \(S_\ell\) are spin operators at site \(\ell\), the operators \(\mathcal{P}_\ell\) are “conjugated” momenta (i.e. \([S_\ell, S_{\ell'}] = 0, [\mathcal{P}_\ell, \mathcal{P}_{\ell'}] = 0,\) and \([\mathcal{P}_\ell, S_{\ell'}] = i\delta_{\ell \ell'}\), with \(\hbar = 1\), the coupling constants \(J_{\ell \ell'} = J\) are between nearest neighbors only, the coupling constant \(g\) is introduced so as to measure the strength of the quantum fluctuations (below it will be called quantum parameter), \(H\) is an ordering magnetic field, and finally the spherical field \(\mu\) is introduced so as to ensure the constraint

\[
\sum_{\ell} \langle S_\ell^2 \rangle = N. \tag{2.2}
\]

Here \(N\) is the total number of the quantum spins located at sites “\(\ell\)’ of a finite hypercubical lattice \(\Lambda\) of size \(L_1 \times L_2 \times \cdots \times L_d = N\) and \(\langle \cdots \rangle\) denotes the standard thermodynamic average taken with \(\mathcal{H}\).

Let us note that the commutation relations for the operators \(S_\ell\) and \(\mathcal{P}_\ell\) together with the quadratic kinetic term in the Hamiltonian (2.1) do not describe quantum Heisenberg–Dirac spins but quantum rotors as it was pointed out in Ref. [17].

The free energy of the model in a finite region \(\Lambda\) is given by the Legendre transformation

\[
f_\Lambda (\beta, g, H) := \sup_{\mu} \left\{ -\frac{1}{N\beta} \ln Z_\Lambda (\beta, g, H; \mu) - \frac{1}{2} \mu \right\}, \tag{2.3}
\]

where \(Z_\Lambda (\beta, g, H; \mu) = \text{Tr} (\exp (-\beta \mathcal{H}))\) is the partition function of the model and \(\beta\) is the inverse temperature with the Boltzmann constant \(K_B = 1\). Under periodic boundary conditions applied across the finite dimensions the free energy takes the form
\[ \beta f_\Lambda (\beta, g, H) = \sup_{\mu} \left\{ \frac{1}{N} \sum_q \ln \left[ 2 \sinh \left( \frac{1}{2} \beta \omega (q; \mu) \right) \right] - \frac{\beta gJ}{2\omega^2 (0; \mu)} H^2 \right\}. \]  

Here the vector \( q \) is a collective symbol, which has for \( L_j \) odd integers, the components:

\[ \left\{ \frac{2\pi n_1}{L_1}, \ldots, \frac{2\pi n_d}{L_d} \right\}, \quad n_j \in \left\{ \frac{-L_j - 1}{2}, \ldots, \frac{L_j - 1}{2} \right\}, \]

and

\[ \omega^2 (q; \mu) = g \left( \mu - 2J \sum_{i=1}^d \cos q_i \right). \]

The supremum of the free energy is attained at the solutions of the mean–spherical constraint, Eq. (2.2), that reads

\[ 1 = \lambda \frac{1}{2N} \sum_q \frac{1}{\sqrt{\phi + 2 \sum_{i=1}^d (1 - \cos q_i)}} \coth \left( \frac{\lambda}{2t} \sqrt{\phi + 2 \sum_{i=1}^d (1 - \cos q_i)} \right) + \frac{h^2}{\phi^2}, \]  

which is equivalent to the following

\[ 1 = \frac{t}{N} \sum_{m=-\infty}^{\infty} \sum_q \frac{1}{\phi + 2 \sum_{i=1}^d (1 - \cos q_i) + b^2 m^2} + \frac{h^2}{\phi^2}, \]

where we have introduced the notations: \( \lambda = \frac{g}{\sqrt{J}} \) is the normalized quantum parameter, \( t = \frac{T}{J} \) the normalized temperature, \( h = \frac{H}{\sqrt{J}} \) the normalized magnetic field, \( b = \frac{2\pi t}{\lambda} \), and \( \phi = \frac{t}{J} - 2d \) is the shifted spherical field.

Eqs. (2.4) and (2.5) provide the basis of the study of the critical behavior of the model under consideration.

A previous direct analysis \cite{17} of Eq. (2.5) in the thermodynamic limit shows that there can be no long–range order at finite temperature for \( d \leq 2 \) (in accordance with the Mermin–Wagner theorem). For \( d > 2 \) one can find long–range order at finite temperature up to a critical temperature \( t_c (\lambda) \). Here we shall consider the low–temperature region for \( 1 < d < 3 \). Before passing to the corresponding analysis we would like to make some comments on the relations of the model considered here with other models known in the literature.

**III. RELATIONS TO OTHER MODELS**

Recently \cite{31–36} another model, suitable to handle the joint description of classical and quantum fluctuations in an exact manner, was considered. If we consider \( P_\ell \) and \( S_\ell \) as canonically conjugated momentum and coordinate, respectively, of the \( \ell \) atom with mass \( g^{-1} \) at each lattice point \( \ell \in \mathbb{Z}^d \), then the first three terms in Eq. (2.1) describe a harmonic interaction of the \( \ell \)–th and \( \ell' \)–th atoms. One can see that in the absence of a spherical constraint, if \( A = \frac{a_1}{4} + \frac{a_2}{2} < 0 \), such a lattice is unstable i.e. the parameter \( A < 0 \) defines the frequency of a mode unstable in the harmonic approximation suggesting that an appropriate
stabilization of the lattice can create a gap in the phonon spectrum. In the spirit of the self–consistent phonon approximation, it is possible to add the term
\[ \frac{B}{4N} \left( \sum \ell S^2_{\ell} \right)^2, \quad B > 0, \] (3.1)

“switching on” an anharmonic interaction which stabilizes the lattice. Because the term is inversely proportional to the particle number, the model under consideration turns out to be exactly soluble in the thermodynamic limit.

In normal coordinates the “new” model Hamiltonian reads
\[ H_{\text{anh.}} = \frac{1}{2} \sum_q \left( P_q P_{-q} + \omega^2_{h}(q) S_q S_{-q} \right) + \frac{b}{4N} \sum_{qq'} S_q S_{-q} S_{q'} S_{-q'} \] (3.2)

where \( P_q \) and \( S_q \) are Fourier components of the operators \( P_\ell \) and \( S_\ell \), respectively. The frequency of phonons is given by
\[ \omega^2_{h}(q) = -\nu_0^2 + 2gJ d \sum_{i=1}^{d} (1 - \cos q_i). \] (3.3)

The anharmonicity constant \( b = g^2B = \frac{\nu_0^4}{4E_0} \), where \( E_0 = \frac{A^2}{4B} \) is the barrier height of the double–well potential in (3.2) at a uniform displacement of all particles: \( S_\ell \to x; \)
\[ U(x) = -\frac{A}{2} x^2 + \frac{B}{4} x^4. \]

In Ref. 31 the following pseudoharmonic approximating Hamiltonian has been proposed
\[ H_{\text{appr.}}(\Delta) = \frac{1}{2} \sum_q \left( P_q P_{-q} + \Omega^2_{h}(q, \Delta) S_q S_{-q} \right) - N E_0 (1 + \Delta)^2, \] (3.4)

where the trial harmonic frequency
\[ \Omega^2_{h}(q, \Delta) = \omega^2_{h}(q) + \nu_0^2 (1 + \Delta) \] (3.5)
is defined by the equation for the temperature–dependent gap \( \Delta \) (c.f. with Eq. (2.5)) :
\[ 1 + \Delta = \frac{b}{N\nu_0^2} \sum_q \langle S_q S_{-q} \rangle_{\text{appr.}} = \frac{b}{2N\nu_0^2} \sum_q \frac{1}{\Omega_{\text{anh}}(q, \Delta)} \coth \frac{\Omega_{\text{anh}}(q, \Delta)}{2T}. \] (3.6)

Eq. (3.6) plays the role of the “soft spherical constraint”. If \( B \) goes to zero, the quartic self–interaction disappears and our model becomes a pure harmonic model. One can see that this limit is singular and that \( B \), for \( d \) more than the upper critical dimension, will be a dangerous irrelevant variable. The model (3.2) is a quantum counterpart of the “soft” classical mean spherical model studied in Ref. 38 in the context of FSS theory.

The inspection of Eq. (3.6) shows that it is similar to Eqs. (2.5) up to a linear term in the l.h.s of (3.6) (This term appears to be essential only above the upper critical dimension.)
Thus there is a clear mathematical analogy between the model defined in Section II and the model presented here.

One of the main problems in studying the FSS effects for this type of models is to answer the question: how correct is the finite–size description of the initial model (3.2) on the base of the approximating Hamiltonian (3.4). Evidently the answer to this question will clarify the more general and subtle problem of the status of finite–size results obtained by approximating methods. A successful step in this direction is presented in Ref. 39 when the classical mean spherical model in the Husimi–Temperely limit has been considered. In Ref. 39 an appropriate modification of the approximating Hamiltonian approach, which reproduced some exact FSS results, has been suggested. Let us note that, if we parallel Ref. 39, for studying quantum systems a more refined mathematical problems must be solved and that up to now this is an open problem.

IV. SCALING FORM OF THE FREE ENERGY AT LOW TEMPERATURES

Let us denote by $\tilde{f}_\Lambda (\beta, g, H)$ the reduced free energy $f_\Lambda (\beta, g, H) / J$. Then, from Eq. (2.4) one obtains immediately

$$\tilde{f}_\Lambda (t, \lambda, h) = \sup_\phi \left\{ \frac{t}{N} \sum_q \ln \left[ 2 \sinh \left( \frac{\lambda}{2t} \sqrt{\phi + 2 \sum_{i=1}^d (1 - \cos q_i)} \right) \right] - \frac{1}{2} \frac{h^2}{\phi} - \frac{1}{2} \phi \right\} - d.\quad (4.1)$$

Using the identities

$$\ln \frac{\sinh b}{\sinh a} = \frac{1}{2} \sum_{m=-\infty}^{\infty} \ln \frac{b^2 + \pi^2 m^2}{a^2 + \pi^2 m^2},\quad (4.2)$$

where $ab > 0$, $a, b$ are arbitrary real numbers and

$$\ln (a + b) = \ln a + \int_0^\infty \exp (-ax) (1 - \exp (-bx)) \frac{dx}{x},\quad (4.3)$$

where $a > 0$, $a + b > 0$ the above expression for the free energy can be rewritten in the form

$$\tilde{f}_\Lambda (t, \lambda, h) = \frac{t}{2} \sum_{m=-\infty}^{\infty} \left\{ \int_0^\infty \frac{dx}{x} \exp \left[ -x \left( \phi + b^2 m^2 \right) \right] \left[ 1 - \frac{1}{L_1} S_{L_1} (x) \times \cdots \times \frac{1}{L_d} S_{L_d} (x) \right] \right\}$$

$$+ t \ln \left[ 2 \sinh \left( \frac{\lambda}{2t} \sqrt{\phi} \right) \right] - \frac{1}{2} \frac{h^2}{\phi} - \frac{1}{2} \phi - d.\quad (4.4)$$

Here

$$S_L (x) = \sum_{m=0}^{L-1} \exp \left[ -2x \left( 1 - \cos \frac{2\pi m}{L} \right) \right].\quad (4.5)$$
and $\phi$ in (4.4) is the solution of the corresponding spherical field equation \((2.3)\). In the remainder of this article we will be interested only in systems with geometry $L^{d-d'} \times \infty^{d'} \times L_\tau$.

With the help of the Jacobi identity

$$\sum_{m=-\infty}^{\infty} \exp(-um^2) = \left(\frac{\pi}{u}\right)^{1/2} \sum_{m=-\infty}^{\infty} \exp\left(-m^2 \frac{\pi^2}{u}\right),$$

(4.6)

the above expression for the free energy can be rewritten in the following equivalent form

$$2\tilde{f}_\Lambda (t, \lambda, h) = \frac{\lambda}{\sqrt{4\pi}} \int_0^\infty \frac{dx}{x^{3/2}} \exp(-x \phi) \left\{ 1 - \left(\frac{1}{L} S_L(x)\right)^{d-d'} \right\} \left[ \exp(-2x) I_0(2x) \right]^{d'}$$

$$+ \frac{\lambda}{\sqrt{\pi}} \int_0^\infty \frac{dx}{x^{3/2}} \exp(-x \phi) R\left(\frac{\pi^2}{xb^2}\right) \left\{ 1 - \left(\frac{1}{L} S_L(x)\right)^{d-d'} \right\}$$

$$\times \left[ \exp(-2x) I_0(2x) \right]^{d'} + 2t \ln \left[ 2 \sinh\left(\frac{\lambda}{2t} \sqrt{\phi}\right) \right] - (\phi + 2d) - \frac{h^2}{\phi},$$

(4.7)

where

$$R(x) = \sum_{m=1}^{\infty} \exp(-xm^2),$$

(4.8)

and $I_0(x)$ is a modified Bessel function.

Dividing the integrals from 0 to $\infty$ in the r.h.s of (4.7) into integral from 0 to $L^2$ and from $L^2$ to $\infty$, after using for these intervals the corresponding asymptotics of $S_L(x)$

$$S_L(x) = 1 + 2R\left(\frac{4\pi^2}{L^2} x\right) - Lv(x) + O(\exp(-x \text{ const}))$$

(4.9)

for $x > L^2$, where

$$v(x) = (4\pi x)^{-1/2} \left[ 1 - \text{erf} \left(\frac{\pi}{\sqrt{x}}\right) \right],$$

and

$$S_L(x) = \frac{L}{\sqrt{\pi} x} \exp(-2x) I_0(2x) + \left(\frac{L}{\sqrt{\pi} x}\right) R\left(\frac{L^2}{4x}\right)$$

(4.10)

for $0 < x \leq L^2$, one ends up with the following expression for the free energy at low temperatures ($\frac{L}{t} \gg 1$)

$$2\tilde{f}_\Lambda (t, \lambda, h) = \lambda f_d (\phi) + \lambda \sqrt{\phi} - (\phi + 2d) - \frac{h^2}{\phi}$$

$$+ \lambda \int_0^\infty \frac{dx}{x} \frac{(4\pi x)^{-(d+1)/2}}{x^{3/2}} \exp(-x \phi)$$

$$\times \left[ 1 - \left(1 + 2R\left(\frac{\pi^2}{xb^2}\right)\right) \left(1 + 2R\left(\frac{L^2}{4x}\right)\right)^{d-d'} \right],$$

(4.11)

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where
\[ f_d(\phi) = \frac{1}{\sqrt{4\pi}} \int_0^\infty \frac{dx}{x^{3/2}} \exp\left( -x\phi \right) \left[ 1 - \left( \exp\left( -2x \right) I_0(2x) \right)^d \right]. \] (4.12)

The above expressions are valid for any \( d \). For \( 1 < d < 3 \) it is easy to show that
\[ f_d(\phi) = f_d(0) + W_d(0) \phi - 4\Gamma \left( \frac{3-d}{2} \right) \frac{\lambda}{(d^2-1)(4\pi)^{(d+1)/2}} \frac{\phi^{(d+1)/2}}{\phi} - \sqrt{\phi}, \] (4.13)

where
\[ W_d(\phi) = \frac{1}{2(2\pi)^d} \int_{-\pi}^{\pi} dq_1 \cdots \int_{-\pi}^{\pi} dq_d \left( \phi + 2 \sum_{i=1}^{d} (1 - \cos q_i) \right)^{-1/2}. \] (4.14)

is the well–known Watson type integral.

Combining (4.11) and (4.13) and denoting \( \lambda_c := 1/W_d(0) \) we obtain
\[ 2\tilde{f}_L(t, \lambda, h) = \lambda f_d(0) - 2d + \left( \frac{\lambda}{\lambda_c} - 1 \right) \phi - \frac{4\Gamma \left( \frac{3-d}{2} \right) \lambda}{(d^2-1)(4\pi)^{(d+1)/2}} \phi^{(d+1)/2} - \frac{h^2}{\phi} \]
\[ + \lambda \int_0^\infty \frac{dx}{x} \left( 4\pi x \right)^{-1/2} \exp\left( -x\phi \right) \]
\[ \times \left[ 1 - \left( 1 + 2R \left( \frac{\pi^2}{xb^2} \right) \right) \left( 1 + 2R \left( \frac{L^2}{4x} \right) \right)^{d-d'} \right]. \] (4.15)

**A. The infinite system**

Here we consider the behavior of the free energy of the bulk system for low temperatures. From Eq. (4.13), taking the limit \( L \to \infty \) one derives immediately
\[ 2\tilde{f}_\infty(t, \lambda, h) = \lambda f_d(0) - 2d + \left( \frac{\lambda}{\lambda_c} - 1 \right) \phi - \frac{4\Gamma \left( \frac{3-d}{2} \right) \lambda}{(d^2-1)(4\pi)^{(d+1)/2}} \phi^{(d+1)/2} \]
\[ + \frac{h^2}{\phi} - 2\lambda \int_0^\infty \frac{dx}{x} \left( 4\pi x \right)^{-1/2} \exp\left( -x\phi \right) R \left( \frac{\pi^2}{xb^2} \right), \] (4.16)

where \( \phi \) is the solution of the corresponding equation for the bulk system (see Eq. (5.1) below). Formally, one can obtain the corresponding equation by requiring the first derivative, with respect to \( \phi \), of the r.h.s of (4.16) to be zero. Therefore, the singular (the \( \phi \) dependent) part of the free energy can be rewritten in the form
\[ \tilde{f}_{\text{sing, bulk}}(t, \lambda, h) = \frac{1}{2} \lambda b^{d+1} X_b(x, h, \lambda), \] (4.17)

where
$$x_\lambda = \left( \frac{1}{\lambda} - \frac{1}{\lambda_c} \right) b^{-(d-1)} \quad (4.18)$$

and

$$h_\lambda = h b^{-(d+3)/2} \lambda^{-1/2} \quad (4.19)$$

are the scaling variables and

$$X_b(x_\lambda, h_\lambda) = -x_\lambda y_\lambda - \frac{4\Gamma \left( \frac{d-2}{2} \right)}{(d^2 - 1) (4\pi)^{(d+1)/2}} y_\lambda^{(d+1)/2} \frac{h_\lambda^2}{y_\lambda}$$

$$-2 \int_0^\infty dx \left( 4\pi x \right)^{-d+1/2} \exp (-x y_\lambda) R \left( \frac{\pi^2}{x} \right) \quad (4.20)$$

is the universal scaling function for the free energy. In the last expression $y_\lambda := \phi/b^2$ is the solution of the spherical field equation

$$0 = -x_\lambda + \frac{\Gamma \left( \frac{1-d}{2} \right)}{(4\pi)^{(d+1)/2}} y_\lambda^{(d-1)/2} \frac{h_\lambda^2}{y_\lambda} - \frac{1}{2\pi} \int_0^\infty dx \left( 4\pi x \right)^{-d+1/2} \exp (-x y_\lambda) R \left( \frac{\pi^2}{x} \right) \quad (4.21)$$

**B. The finite system**

For the finite system from Eq. (4.13), in a full analogy with the previous results for the infinite system, we derive

$$\tilde{f}_{\text{sing, finite}}(t, \lambda, h, L) = \frac{1}{2} \lambda b^{d+1} X_{\text{finite}}(x_\lambda, h_\lambda, a) \quad (4.22)$$

where the new scaling variable

$$a = Lb \quad (4.23)$$

is introduced, whereas the scaling function now is

$$X_{\text{finite}}(x_\lambda, h_\lambda, a) = -x_\lambda y_\lambda - \frac{4\Gamma \left( \frac{3-d}{2} \right)}{(d^2 - 1) (4\pi)^{(d+1)/2}} y_\lambda^{(d+1)/2} \frac{h_\lambda^2}{y_\lambda}$$

$$+ \int_0^\infty dx \left( 4\pi x \right)^{-d+1/2} \exp (-x y_\lambda)$$

$$\times \left[ 1 - \left( 1 + 2R \left( \frac{\pi^2}{x} \right) \right) \left( 1 + 2R \left( \frac{a^2}{4x} \right) \right)^{d-d'} \right]. \quad (4.24)$$

Here $y_\lambda$ is the solution of the corresponding spherical field equation for the finite system which reads
The expressions (4.20) and (4.21) for the infinite system, and (4.24) and (4.25) for the finite one, represent actually the verification of the analog of the Privman–Fisher hypothesis for the finite–size scaling form of the free energy, formulated for classical systems, for the case when the quantum fluctuations are essential. Note, that in that case one has both finite space dimensions and one additional finite dimension that is proportional to the inverse temperature. According to the finite–size scaling hypothesis one has to expect that the temperature dependent scaling field multiplying the universal scaling function will be with exponent \( p = 1 + d/z \), where the dynamic–critical exponent \( z \) expresses the anisotropic scaling between space and “temperature” (“imaginary–time”) directions. For the system considered here \( z = 1 \), in full conformity with our results. It seems important enough to emphasize that in the quantum low–temperature case there is one nonuniversal pre-factor (for the system considered here it is \( J_\lambda \)) that multiplies the universal finite–size scaling function of the free energy, which is not the case for the classical systems (provided their free–energy density is normalized per \( k_B T \)).

Now we pass to the analysis of the spherical field equations for both the finite and infinite systems following Ref. [41]

V. ANALYSIS OF THE SPHERICAL FIELD EQUATIONS

A. The infinite system

After performing the integration in the right–hand side of the equation for the spherical field (4.21) and using the integral representation (A2) we get

\[
0 = -x_\lambda + \frac{\Gamma\left(\frac{1-d}{2}\right)}{(4\pi)^{(d+1)/2}} y_\lambda^{(d-1)/2} + \frac{h_\lambda^2}{y_\lambda} + \frac{1}{4\pi} \int_0^\infty \frac{dx}{x} (4\pi x)^{-(d-1)/2} \exp(-xy_\lambda) \times \left[ 1 - \left(1 + 2R\left(\frac{\pi^2}{x}\right)\right) \left(1 + 2R\left(\frac{a^2}{4x}\right)\right)^{d-d'} \right].
\]  

(4.25)

\[
\frac{1}{\lambda} - \frac{1}{\lambda_c} = -\frac{1}{(4\pi)^{(d+1)/2}} \left[ \Gamma\left(\frac{1-d}{2}\right) \right] \phi^{(d-1)/2} + \frac{2}{(4\pi)^{(d+1)/2}} \phi^{(d-1)/2} K\left(\frac{d-1}{2}, \frac{\lambda}{2t} \phi^{1/2}\right) + \frac{h^2}{\phi^2},
\]  

(5.1)

where

\[
K(\nu, y) = 2 \sum_{m=1}^\infty (ym)^{-\nu} K_{\nu}(2my)
\]  

(5.2)

and \( K_{\nu}(x) \) is the other modified Bessel function.

Eq. (5.1) was derived in Ref. [41] by a different technique. The approach used here is based on the estimations (4.9,4.10) and has the advantage that it can easily be generalized for other boundary conditions.

In the remainder of this section we will study the effect of the temperature on the susceptibility and the equation of state near the quantum critical fixed point.
1. Zero-field susceptibility

After making the field \( h \) vanishes, from Eq. (5.1) we find that the normalized zero-field susceptibility \( \chi = \phi^{-1} \) on the line \( \lambda = \lambda_c \) \((t \to 0^+)\) is given by

\[
\chi = \frac{\lambda_c^2}{4y_0^2} t^{-2},
\]

(5.3)

where \( y_0 \) is the universal solution of

\[
\left| \Gamma \left( \frac{1-d}{2} \right) \right| = 2 \mathcal{K} \left( \frac{d-1}{2}, y \right).
\]

(5.4)

The behavior of the universal constant \( y_0 \) as a function of the dimensionality \( d \) of the system is shown in FIG. 1.

Eq. (5.3) tells us that at low-temperature the susceptibility increases as the inverse of the square of the temperature above the quantum critical point.

In what follows we will try to investigate Eq. (5.1) for different dimensions \((1 < d < 3)\) of the system and in different regions of the \((t, \lambda)\) phase diagram. To this end we will consider first the function \( \mathcal{K}(\nu, y) \). For \( d \neq 2 \) its asymptotic form is \((y \ll 1)\) \(\mathcal{K}(\nu, y) \sim y^{-1}\) (see also Appendix [?])

\[
\mathcal{K} \left( \frac{d-1}{2}, y \right) = \mathcal{K}_1 \left( \frac{d-1}{2} \right) 1, y \right)
\]

\[
\approx \frac{\pi^{1/2}}{2} \left( 1 - \frac{d}{2} \right) y^{-1} + \Gamma \left( \frac{d-1}{2} \right) \zeta(d-1)y^{1-d} - \frac{1}{2} \Gamma \left( \frac{1-d}{2} \right),
\]

(5.5)

where \( \zeta(x) \) is the Riemann zeta function. Introducing the “shifted” critical value of the quantum parameter due to the temperature by

\[
\frac{1}{\lambda_c(t)} = \frac{1}{\lambda_c} + \frac{1}{2\pi^{(d+1)/2}} \left( \frac{t}{\lambda_c(t)} \right)^{d-1} \Gamma \left( \frac{d-1}{2} \right) \zeta(d-1),
\]

(5.6a)

or

\[
\frac{1}{\lambda_c(t)} \approx \frac{1}{\lambda_c} + \frac{1}{2\pi^{(d+1)/2}} \left( \frac{t}{\lambda_c} \right)^{d-1} \Gamma \left( \frac{d-1}{2} \right) |\zeta(d-1)|,
\]

(5.6b)

one has to make a difference between the two cases \( d < 2 \) “sign −” and \( d > 2 \) “sign +”.

In the first case \((1 < d < 2)\), it is possible to define the quantum critical region by the inequality

\[
\left| \frac{1}{\lambda} - \frac{1}{\lambda_c} \right| \ll \frac{1}{2\pi^{(d+1)/2}} \left( \frac{t}{\lambda_c(t)} \right)^{d-1} \Gamma \left( \frac{d-1}{2} \right) |\zeta(d-1)|.
\]

(5.7)

For \( d < 2 \) the function \( \mathcal{K}(\nu, y) \sim y^{-1} \), by substitution in (5.1) we obtain for \( \lambda < \lambda_c \) (outside of the quantum critical region)
\[
\chi \approx \left[ \frac{\Gamma \left( 1 - \frac{d}{2} \right)}{(4\pi)^{d/2}} \frac{\lambda_c}{\lambda_c - \lambda} \right]^{2/(d-2)} t^{2/(d-2)},
\]

(5.8)

We see that the susceptibility is going to infinity with power law degree when the quantum fluctuations become important \((t \to 0^+)) and there is no phase transition driven by \(\lambda\) in the system for dimensions between 1 and 2.

In the second case \((2 < d < 3)\), after the insertion of Eq. (5.5) in Eq. (5.1) one has

\[
\chi \approx \left[ \frac{\Gamma \left( 1 - \frac{d}{2} \right)}{\lambda - \lambda_c(t)} \right]^{2/(d-2)} t^{2/(d-2)},
\]

(5.9)

as a solution for \(\lambda\) less than \(\lambda_c\) and greater than the critical value \(\lambda_c(t)\) of the quantum parameter. Here for finite temperatures there is a phase transition driven by the quantum parameter \(\lambda\) with critical exponent of the \(d\)-dimensional classical spherical model \(\gamma = \frac{2}{d-2}\). This however is valid only for very close values of \(\lambda\) to \(\lambda_c(t)\). For \(\lambda < \lambda_c(t)\) the susceptibility is infinite.

In the region where \(\lambda > \lambda_c\) the zero–field susceptibility is given by

\[
\chi \approx \left[ \frac{(4\pi)^{(d+1)/2}}{\Gamma \left( \frac{1-d}{2} \right)} \left( \frac{1}{\lambda - \lambda_c} \right) \right]^{\frac{2}{d-2}}.
\]

(5.10)

This result is valid for every \(d\) between the lower and the upper quantum critical dimensions.

Eq. (5.5) is derived for dimensionalities \(d \neq 2\). The important case \(d = 2\) requires special care. In this case Eq. (5.1) takes the form \((h = 0)\)

\[
\frac{\lambda}{\lambda_c} - 1 = \frac{\lambda \phi^{1/2}}{4\pi} + \frac{t}{2\pi} \ln \left[ 1 - \exp \left( -\frac{\lambda}{t} \phi^{1/2} \right) \right].
\]

This equation can be solved easily and one gets

\[
\phi^{1/2} = \frac{2t}{\lambda} \arcsinh \left\{ \frac{1}{2} \exp \left[ \frac{2\pi}{t} \left( \frac{\lambda}{\lambda_c} - 1 \right) \right] \right\}
\]

(5.11)

For the susceptibility, Eq. (5.11) yields

\[
\chi \approx \frac{\lambda^2}{t^2} \exp \left( 4\pi \frac{\lambda_c - \lambda}{t\lambda_c} \right)
\]

(5.12a)

for \(\frac{2\pi}{t} \left| \frac{\lambda}{\lambda_c} - 1 \right| \gg 1\) and \(\lambda < \lambda_c\) i.e. in the renormalized classical region. For \(\lambda = \lambda_c = 3.1114...\)

\[
\chi = \frac{1}{\Theta^2} \left( \frac{\lambda_c}{t} \right)^2,
\]

(5.12b)

where the universal constant
\[
\Theta = 2y_0 = 2 \ln \left( \frac{\sqrt{5} + 1}{2} \right) = -2 \ln \left( \frac{\sqrt{5} - 1}{2} \right) = .962424...
\] (5.12c)

was first obtained in the framework of the 3-dimensional classical mean spherical model with one finite dimension.\[^3\] Finally for \( \frac{2\pi}{t} \left| \frac{\lambda}{\lambda_c} - 1 \right| \gg 1 \) and \( \lambda > \lambda_c \) i.e. in the quantum disordered region

\[
\chi \approx \left[ \frac{\lambda \lambda_c}{4\pi (\lambda - \lambda_c)} \right]^2 \left\{ 1 + \frac{2t\lambda_c}{\pi (\lambda - \lambda_c)} \exp \left[ -\frac{4\pi}{t\lambda_c} (\lambda - \lambda_c) \right] \right\}. 
\] (5.12d)

The first term of Eq. (5.12d) is a particular case of Eq. (5.10) for \( d = 2 \).

From Eqs. (5.12) one can transparently see the different behaviors of \( \chi(T) \) in tree regions: a) renormalized classical region with exponentially divergence as \( T \to 0 \), b) quantum critical region with \( \chi(T) \sim T^{-2} \) and crossover lines \( T \sim |\lambda - \lambda_c| \), and c) quantum disordered region with temperature independent susceptibility (up to exponentially small corrections) as \( T \to 0 \). The above results (5.11) and (5.12) coincide in form with those obtained in Refs. 8,10 for the two dimensional QNL\(\sigma\)M in the \( n \to \infty \) limit. The only differences are: in Eq. (5.11) the temperature is scaled by \( \lambda \), and the critical value \( \lambda_c \) is given by \( \lambda_c = 1/Wd(0) \) while for the QNL\(\sigma\)M it depends upon the regularization scheme. It will be useful to clarify the bulk critical behavior of \( \chi \) found above in the context of FSS theory. First, from Eqs. (5.6) one can see that the exponent \( \psi \) characterizing the shift in the quantum parameter \( \lambda_c \) by the temperature \( t \) is equal to \( \nu^{-1} = d - 1 \) in accordance with FSS prediction. Second, let us consider the critical behavior of \((d + 1)\)-dimensional classical model with \( L_\tau \sim 1/T \) playing the role of a finite–size in the imaginary time direction, i.e. with slab geometry \( \infty \times L_\tau \). FSS calculation (see for example Ref. 20) for the susceptibility at the critical point (in the case under consideration this is \( \lambda = \lambda_c \)) gives: (i) for \( 1 < d < 2 \),

\[
\chi(\lambda_c) \sim L_\tau^{\gamma_\lambda/\nu_\lambda},
\]

where \( \gamma_\lambda = 2\nu_\lambda = \frac{2}{d-1} \) and \( \chi(\lambda_c) = T^{-\gamma_T} \); \( \gamma_T = 2 \), (ii) for \( 2 < d < 3 \),

\[
\chi(\lambda) \sim L_\tau^{(\gamma_\lambda-\hat{\gamma}_\lambda)/\nu_\lambda} |\lambda - \lambda(L_\tau)|^{-\hat{\gamma}_\lambda},
\]

where the exponent \( \hat{\gamma}_\lambda = \frac{2}{d-2} \) is pertaining to the infinite size (space) dimensions \( d \). In the last case for the shift of \( \lambda_c \) we take from Eqs. (5.6) \( \lambda_c - \lambda(L_\tau) \sim t^{d-1} \) and again \( \chi(\lambda_c) \sim t^{-\gamma_T} \). So the different types of critical exponent depending on \( \lambda \) and \( T \) are related by \( \gamma_T = \gamma_\lambda/\nu_\lambda \) or \( \gamma_T = \psi\gamma_\lambda \) (\( \psi = 1/\nu_\lambda \) is the crossover exponent).

2. Equation of state

The equation of state of the Hamiltonian (2.1) near the quantum critical point is obtained after substituting the shifted spherical field \( \phi \) by the magnetization \( M \) through the relation

\[
M = \frac{h}{\phi},
\] (5.13)
in Eq. (5.1), which allows us to write the equation of state in a scaling form

\[ 1 + \frac{\delta \lambda}{\mathcal{M}^{1/\beta}} = (4\pi)^{-(d+1)/2} \left[ \frac{h}{\mathcal{M}^\delta} \right]^{1/\gamma} \left\{ \left| \Gamma \left( \frac{1-d}{2} \right) \right| - 2\kappa \left( \frac{d-1}{2}, \frac{\lambda}{2} \left( \frac{\mathcal{M}^{\nu/\beta}}{t} \right) \left( \frac{h}{\mathcal{M}^\delta} \right)^{\beta} \right) \right\}, \]

(5.14)

where

\[ \delta \lambda = \frac{1}{\lambda_c} - \frac{1}{\lambda}. \]

We conclude that near the quantum critical point Eq. (5.14) may be written in general forms as

\[ h = \mathcal{M}^\delta f_h(\delta \lambda \mathcal{M}^{-1/\beta}, (t/\lambda)^{1/\nu} \mathcal{M}^{-1/\beta}), \]

(5.15a)

or

\[ \mathcal{M} = \left( \frac{t}{\lambda} \right)^{-\beta/\nu} f_M(\delta \lambda \mathcal{M}^{-1/\beta}, h \mathcal{M}^{-\delta}). \]

(5.15b)

In Eqs. (5.15) \( f_h(x, y) \) and \( f_M(x, y) \) are some scaling functions, furthermore \( \gamma = \frac{2}{d-1} \), \( \nu = \frac{1}{d-1} \), \( \beta = \frac{1}{2} \) and \( \delta = \frac{d+3}{d-1} \) are the familiar bulk critical exponents for the \((d+1)\)-dimensional classical spherical model. Eqs. (5.15) are a direct verification of FSS hypothesis in conjunction with classical to quantum critical dimensional crossover. They can be easily transformed into the scaling form (Eq. (21)) obtained in Ref. [17], however here they are verified for \( 1 < d < 3 \) instead of \( 2 < d < 3 \) (c.f. Ref. [17]), i.e. the non-critical case is included.

Hereafter we will try to give an explicit expression of the scaling function \( f_h(x, y) \) \( (x \equiv \delta \lambda \mathcal{M}^{-2}, y \equiv (t/\lambda)^{d-1} \mathcal{M}^{-2}) \) in the neighborhood of the quantum critical fixed point. This may be performed, in the case \( \frac{t}{\lambda} \sqrt{h/\mathcal{M}} \ll 1 \), with the use of the asymptotic form from Eq. (5.5) to get (for \( d \neq 2 \)),

\[ \delta \lambda + \mathcal{M}^2 + \frac{t/\lambda}{(4\pi)^{d/2} \Gamma} \left( 1 - \frac{d}{2} \right) \left( \frac{h}{\mathcal{M}} \right)^{(d-2)/2} + \frac{1}{2\pi^{(d+1)/2}} \left( \frac{t}{\lambda} \right)^{d-1} \Gamma \left( \frac{d-1}{2} \right) \zeta(d-1) = 0. \]

(5.16)

From this equation we find for the scaling function the result

\[ f_h(x, y) = \left[ \frac{(4\pi)^{d/2} y^{-\nu}}{\Gamma (1 - \frac{d}{2})} \left( 1 + x + \frac{1}{2\pi^{(d+1)/2}} \Gamma \left( \frac{d-1}{2} \right) \zeta(d-1)y \right) \right]^{2/(d-2)}. \]

(5.17)

For the special case \( d = 2 \) Eq. (5.14) reads

\[ \delta \lambda = \frac{h^{1/2}}{4\pi \mathcal{M}^{1/2}} + \frac{t}{2\pi \lambda} \ln \left[ 1 - \exp \left( -\frac{\lambda}{t} \frac{h^{1/2}}{\mathcal{M}^{1/2}} \right) \right] - \mathcal{M}^2, \]

(5.18)

which yields
\[ h = M^5 f_h(x, y) \]  

where the scaling function is given by the expression

\[ f_h(x, y) = 4y^2 \left[ \arcsinh \frac{1}{2} \exp \left( \frac{2\pi}{y} \frac{1 + x}{y} \right) \right]^2. \]  

At \( x = 0 \) and \( y \gg 1 \) (fixed low-temperature and \( h \to 0^+ \)) the scaling function (5.20) reduces to

\[ f_h(0, y) \approx y^2 \exp \left( \frac{4\pi}{y} \right). \]  

In the region \( x < -1 \), and for \( y \ll 1 \) (fixed weak field and \( t \to 0^+ \)) the corresponding scaling function is

\[ f_h(x, y) \approx y^2 \exp \left( 4\pi \frac{x + 1}{y} \right), \]  

and in region \( x > -1 \) and \( y \ll 1 \) we have

\[ f_h(x, y) \approx 16\pi^2(x + 1)^2 \left[ 1 + \frac{y}{\pi(1 + x)} \exp \left( -4\pi \frac{1 + x}{y} \right) \right]. \]  

This identifies the zero temperature \((y = 0)\) form of the scaling function (5.23) with those of the 3-dimensional classical spherical model.

B. System confined to a finite geometry

When the model Hamiltonian (2.1) is confined to the general geometry \( L_{d-d'}^d \times L_{d'} \), with \( 0 \leq d' \leq d \), equation (4.25) of the spherical field \( \phi \) takes the form (after some algebra)

\[ \frac{1}{\lambda} = \frac{1}{\lambda_c} - (4\pi)^{-d(d+1)/2} \Gamma \left( \frac{1-d}{2} \right) \phi^{(d-1)/2} \]

\[ + \frac{\phi^{(d-1)/2}}{(2\pi)^{(d+1)/2}} \sum_{m,l(d-d')} K_{(d-1)/2} \left[ \phi^{1/2} \left( \frac{\lambda m/t}{(L|u|)^2} + (L|u|)^2 \right)^{1/2} \right]^{(d-1)/2} + h^2 \phi^2, \]  

where

\[ |u| = (l_1^2 + l_2^2 + \cdots + l_{d-d'}^2)^{1/2} \]

and the primed summation indicates that the vector with components \( m = l_1 = l_2 = \cdots = l_{d-d'} = 0 \) is excluded.
1. Shift of the critical quantum parameter

The finite-size scaling theory (for a review see Ref. [44]) asserts, for the temperature driven phase transition, that the phase transition occurring in the system at the thermodynamic limit persists, if the dimension \( d' \) of infinite sizes is greater than the lower critical dimension of the system. In this case the value of the critical temperature \( T_c(\infty) \) at which some thermodynamic functions exhibit a singularity is shifted to \( T_c(L) \) critical temperature for a system confined to the general geometry \( L^{d-d'}\times\infty^{d'} \), when the system is infinite in \( d' \) dimensions and finite in \((d-d')\)-dimensions. In the case when the number of infinite dimensions is less than the lower critical dimension, there is no phase transition in the system and the singularities of the thermodynamic functions are altered. The critical temperature \( T_c(\infty) \) in this case is shifted to a pseudocritical temperature, corresponding to the center of the rounding of the singularities of the thermodynamic functions, holding in the thermodynamic limit.

In our quantum case, having in mind that we have considered the low-temperature behavior of model (2.1) in the context of the FSS theory it is convenient to choose the quantum parameter \( \lambda \) like a critical instead of the temperature \( t \) and to consider our system confined to the geometry \( L^{d-d'}\times\infty^{d'}\times\Lambda L \). So the shifted critical quantum parameter \( \lambda_c(t, L) \equiv \lambda_{tL} \) is obtained by setting \( \phi = 0 \) in Eq. (5.24). This gives

\[
\frac{1}{\lambda_{tL}} - \frac{1}{\lambda_c} = \frac{\Gamma\left(\frac{d-1}{2}\right)}{4\pi^{(d+1)/2}} \sum'_{m.t(d-d')} \left( (\lambda_{tL} m/t)^2 + (L|t|^d/2)^{(1-d)/2} \right). \tag{5.25}
\]

The sum in the r.h.s. of Eq. (5.25) is convergent for \( d' > 2 \), however it can be expressed in terms of the Epstein zeta function (see Ref. [45])

\[
\zeta\left|\begin{array}{l}0 \\ 0\end{array}\right| \left( L^2 l^2 + \left( \frac{\lambda}{t} \right)^2 m^2, d-1 \right) = \sum'_{m.t(d-d')} \left( L^2 l^2 + \left( \frac{\lambda}{t} \right)^2 m^2 \right)^{\frac{1-d}{2}}, \tag{5.26}
\]

which can be regarded as the generalized \((d-d'+1)\)-dimensional analog of the Riemann zeta function \( \zeta\left(\frac{d-1}{2}\right) \). In the case under consideration the Epstein zeta function has only a simple pole at \( d' = 2 \) and may be analytically continued for \( 0 \leq d' < 2 \) to give a meaning to Eq. (5.25) for \( d' < 2 \) as well. It is hard to investigate the sum appearing in Eq. (5.26). The anisotropy of the sum \( L^2 l_1^2 + \cdots + L^2 l_{d-d'}^2 + \left( \frac{\lambda}{t} \right)^2 m^2 \) is an additional problem. For these reasons we will try to solve it asymptotically, considering different regimes of the temperature, depending on that whether \( L \ll \frac{\lambda}{t} \) or \( L \gg \frac{\lambda}{t} \) which will be called, respectively, very low-temperature regime and low-temperature regime. To do that it is more convenient to write Eq. (5.25) in the following form

\[
\frac{1}{\lambda_{tL}} - \frac{1}{\lambda_c} = \frac{1}{4\pi^{(d+1)/2}} \sum'_{m.t(d-d')} \int_0^\infty dx x^{\frac{1}{d}+\frac{1}{2}} \exp \left\{ -x L^2 l^2 + \left( \frac{\lambda_{tL}}{t} \right)^2 m^2 \right\}. \tag{5.27}
\]

This will be the starting equation for the two cases we will consider below.
a. Low-temperature regime $\frac{\lambda t}{t} \ll L$. In this case we single out, from the sum in Eq. (5.27), the term with $l = 0$ to obtain

$$
\frac{1}{\lambda_{tL}} - \frac{1}{\lambda_c} = \frac{1}{2\pi^{d+1/2}} \left( \frac{t}{\lambda_{tL}} \right)^{d-1} \Gamma \left( \frac{d-1}{2} \right) \zeta(d-1)
$$

$$
+ \frac{1}{4\pi^{(d+1)/2}} \sum'_{l(d-d')} \sum'_{m=-\infty} \int_0^\infty dx x^{\frac{d-3}{2}} \exp \left\{ -x \left[ L^2 t^2 + \left( \frac{\lambda_{tL}}{t} \right)^2 m^2 \right] \right\}.
$$

(5.28)

After the application of the Jacobi identity \[(16)\] to the sum over $m$ and the calculation of the arising integrals in the resulting expression we obtain the final result

$$
\frac{1}{\lambda_{tL}} - \frac{1}{\lambda_c} = \frac{1}{2\pi^{d+1/2}} \left( \frac{t}{\lambda_{tL}} \right)^{d-1} \Gamma \left( \frac{d-1}{2} \right) \zeta(d-1) + \frac{t}{\lambda_{tL}} \frac{L^{2-d}}{4\pi^{d/2}} \Gamma \left( \frac{d}{2} - 1 \right) \sum'_{l(d-d')} |u|^{2-d}
$$

$$
+ \left( \frac{t}{\lambda_{tL}} \right)^{d/2} \frac{L^{1-d/2}}{\pi} \sum'_{l(d-d')} \sum_{m=1}^\infty \left( \frac{m}{|L|} \right)^{d/2-1} K_{\frac{d}{2}-1} \left( 2\pi \frac{t}{\lambda_{tL}} Lm |u| \right).
$$

(5.29)

The first term of the r.h.s of Eq. (5.29) is the shift of the critical quantum parameter (see Eq. (5.30)) due to the presence of the quantum effects in the system. The second term is a correction resulting from the finite sizes. It is just the shift due the finite-size effects in the $d$–dimensional spherical model \[2\] multiplied by the temperature scaled to the quantum parameter. Here the $(d - d')$–fold sum may be continued analytically beyond its domain of convergence with respect to “$d$” and “$d'$” (which is $2 < d' < d$). The last term is exponentially small in the considered limit i.e. $\frac{\lambda t}{t} \ll L$.

In the borderline case $d = 2$, one can see that in the first and second terms in the r.h.s of Eq. (5.29) singularities take place. In Appendix B we can show that they cancel and Eq. (5.29) then yields

$$
\frac{1}{\lambda_{tL}} - \frac{1}{\lambda_c} = \frac{t}{2\pi \lambda_{tL}} \left\{ B_0 + \gamma_E + \ln \frac{tL}{2\lambda_{tL}} + 2 \sum'_{l(d-d')} \sum_{m=1}^\infty K_0 \left( 2\pi \frac{t}{\lambda_{tL}} Lm |u| \right) \right\},
$$

(5.30)

where $\gamma_E = 0.577...$ is the Euler constant.

In the particular case of strip geometry $d' = 1$, the shift is given by (see Appendix B)

$$
\frac{1}{\lambda_{tL}} - \frac{1}{\lambda_c} = \frac{t}{2\pi \lambda_{tL}} \left\{ \gamma_E + \ln \frac{tL}{4\pi \lambda_{tL}} + 4 \sum_{l=1}^\infty \sum_{m=1}^\infty K_0 \left( 2\pi \frac{t}{\lambda_{tL}} Lml \right) \right\}.
$$

(5.31)

For the fully finite geometry case $(d' = 0)$ we obtain for the shift (see Appendix B)

$$
\frac{1}{\lambda_{tL}} - \frac{1}{\lambda_c} = \frac{t}{2\pi \lambda_{tL}} \left\{ \gamma_E + \ln \frac{tL}{\lambda_{tL}} - \ln \left[ \frac{\Gamma \left( \frac{1}{4} \right)^2}{\sqrt{\pi}} \right] + 2 \sum'_{l_1, l_2} \sum_{m=1}^\infty K_0 \left( 2\pi \frac{t}{\lambda_{tL}} Lm \sqrt{l_1^2 + l_2^2} \right) \right\}.
$$

(5.32)

Let us note that the last term in the r.h.s of Eqs. (5.31) and (5.32) gives exponentially small corrections and in the considered limit they can be omitted.
b. Very low–temperature regime $\frac{\lambda}{L} \gg 1$  

To get an appropriate expression now, we single out the term corresponding to $m = 0$ from the sum in Eq. (5.27). This leads to

\[
\frac{1}{\lambda_{tL}} - \frac{1}{\lambda_c} = \frac{L^{1-d}}{4\pi^{(d+1)/2}} \Gamma \left( \frac{d-1}{2} \right) \sum_{l(d-d')} |l|^{-d} \\
+ \frac{1}{2\pi^{(d+1)/2}} \sum_{l(d-d')} \sum_{m=1}^{\infty} \int_0^\infty dx \, x^{d'-1} \exp \left\{ -x \left[ L^2 t^2 + \left( \frac{\lambda_{tL}}{t} \right)^2 m^2 \right] \right\}. \tag{5.33}
\]

The next step consists in the application of the Jacobi identity to the $(d - d')$–dimensional sum and the calculation of the arising integrals to obtain

\[
\frac{1}{\lambda_{tL}} - \frac{1}{\lambda_c} = \frac{L^{1-d}}{4\pi^{(d+1)/2}} \Gamma \left( \frac{d-1}{2} \right) \sum_{l(d-d')} |l|^{-d} + \frac{L^{d-d}}{2\pi^{(d'+1)/2}} \Gamma \left( \frac{d' - 1}{2} \right) \left( \frac{t}{\lambda_{tL}} \right)^{d'-1} \zeta(d' - 1) \\
+ \frac{L^{1/2+d'/2-d}}{\pi} \left( \frac{t}{\lambda_{tL}} \right)^{d'-1} \sum_{l(d-d')} \sum_{m=1}^{\infty} \left( \frac{|l|}{m} \right)^{d'-1} K_{d'-1} \left( 2\pi \frac{\lambda_{tL}}{tL} m |l| \right). \tag{5.34}
\]

Here, in the r.h.s, the first term is the expression of the shift of the critical quantum parameter, at zero temperature due to the finite sizes of the system. This is equivalent to the shift of a $(d + 1)$–dimensional spherical model confined to the geometry $L^{d+1-d} \times \infty^{d'}$. The second term gives corrections due to the quantum effects. This is the shift of critical quantum parameter of a $d'$–dimensional infinite system multiplied by the volume of a $(d - d')$–dimensional hypercube. The third term is exponentially small in the limit of very low temperatures. The singularity for $d' = 1$ of the r.h.s of Eq. (5.34) is fictitious. So for $d' = 1$ Eq. (5.34) yields (see Appendix B)

\[
\frac{1}{\lambda_{tL}} - \frac{1}{\lambda_c} = \frac{L^{1-d}}{2\pi} \left\{ \ln \frac{\lambda_{tL}}{2\sqrt{\pi} tL} + \frac{\gamma_E}{2} + \tilde{C}_0 + 2 \sum_{l(d-1)} \sum_{m=1}^{\infty} K_0 \left( 2\pi \frac{\lambda_{tL}}{tL} m |l| \right) \right\}, \tag{5.35}
\]

where

\[ \tilde{C}_0 = \left[ 2\pi^{(d-1)/2} \right]^{-1} \Gamma \left( \frac{d-1}{2} \right) C_0, \]

In the particular case $d' = 1$ and $d = 2$ the constant $C_0$ is given by $C_0 = \gamma_E - \ln 4\pi$ and

\[
\frac{1}{\lambda_{tL}} - \frac{1}{\lambda_c} = \frac{1}{2\pi L} \left\{ \gamma_E + \ln \frac{\lambda_{tL}}{4\pi tL} + 4 \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} K_0 \left( 2\pi \frac{\lambda_{tL}}{tL} m l \right) \right\}. \tag{5.36}
\]

Comparing between Eqs. (5.31) and (5.36) one can see the crucial role (in symmetric form) of $L$ or $\lambda_{tL}/t$ in the low–temperature regime and the very low–temperature one, respectively.

In another particular case of a two–dimensional bloc geometry $d' = 0$ and $d = 2$, using the Hardy formula (see Appendix B) one gets
\[
\frac{1}{\lambda_t} - \frac{1}{\lambda_c} = \frac{L^{-1}}{\pi} \zeta \left( \frac{1}{2} \right) \beta \left( \frac{1}{2} \right) - \frac{\lambda_t}{tL^2}\zeta(-1) + \frac{L^{-1}}{2\pi} \sum_{m=1}^{\infty} \sum_{l_1, l_2} \exp \left[ \frac{-2\pi\lambda_t}{tL} m (l_1^2 + l_2^2)^{1/2} \right] \frac{1}{\sqrt{l_1^2 + l_2^2}}.
\]

Instead of the previous case of low-temperature regime here the lower quantum critical dimension \(d' = 1\) is responsible for the logarithmic dependence in Eq. (5.35). This is the reason for the significant difference between Eqs. (5.36) and (5.37).

The above obtained equations for \(\lambda_tL\) will be exploited later for the study of the two dimensional case.

2. Zero-field susceptibility

It is possible to transform Eq. (5.24) in the following equivalent forms

\[
L^{d-1} \delta \lambda + \left( \frac{hL^{(d+3)/2}}{L^2 \phi} \right)^2 = \left( \frac{L\phi^{1/2} t}{(4\pi)(d+1/2)} \right)^{d-1} \left[ \Gamma \left( \frac{1-d}{2} \right) \right] - 2K_{\frac{d}{2}} \left( \frac{d-1}{2} \right) - 2K_{\frac{d}{2}} \left( d - d' + 1, \frac{L\phi^{1/2}}{2t} \right),
\]

where

\[
K_\nu(p, y) = \sum_{m, l(p-1)} \frac{K_\nu \left( 2y \sqrt{a^2 m^2 + l^2} \right)}{y \sqrt{a^2 m^2 + l^2}}, \quad y > 0, \quad \nu > 0, \quad L^2 = l_1^2 + l_2^2 + \cdots + l_{p-1}^2.
\]

or

\[
\left( \frac{t}{\lambda} \right)^{1-d} \delta \lambda + \left[ \frac{h}{\phi} \left( \frac{t}{\lambda} \right)^2 \left( \frac{\lambda}{t} \right)^{(d+3)/2} \right]^2 = \left( \frac{hL^{(d+3)/2}}{(4\pi)(d+1/2)} \right)^{d-1} \left[ \Gamma \left( \frac{1-d}{2} \right) \right] - 2\K_{\frac{d}{2}} \left( \frac{d-1}{2} \right) - 2\K_{\frac{d}{2}} \left( d - d' + 1, \frac{h\phi^{1/2}}{2t} \right),
\]

where

\[
\K_{\nu}(p, y) = \K_{\nu}(p; ay) = \sum_{m, l(p-1)} \frac{K_\nu \left( 2y \sqrt{a^2 l^2 + m^2} \right)}{y \sqrt{a^2 l^2 + m^2}}, \quad y > 0.
\]

The functions \(K_\nu(p; y)\) and \(\K_{\nu}(p; y)\) are anisotropic generalizations of the \(K\)-function introduced in Ref. 18.

Eqs. (5.38) and (5.39) show that the correlation length \(\xi = \phi^{-1/2}\) will scale like

\[
\xi = Lf_\nu \left\{ \delta \lambda L^{1/\nu}, \frac{tL}{\lambda}, \frac{hL^{\Delta/\nu}}{\nu} \right\},
\]

or like

\[
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\[ \xi = \frac{\lambda}{t} f_\xi^2 \left\{ \delta \lambda \left( \frac{t}{\lambda} \right)^{-1/\nu} , \frac{tL}{\lambda}, h \left( \frac{t}{\lambda} \right)^{-\Delta/\nu} \right\} , \quad (5.40b) \]

which suggests also that there will be some kind of interplay (competition) between the finite-size and the quantum effects.

Hereafter we will try to find the behavior of the susceptibility \( \chi = \phi^{-1} \) as a function of the temperature \( t \) and the size \( L \) of the system. For simplicity, in the remainder of this section, we will investigate the free field case \( (h = 0) \).

1. For \( \frac{\lambda t \phi^{1/2}}{2} \ll 1 \), after using the asymptotic form of the function defined in (5.38b) (see Appendix A)

\[ K_a(\nu)p, y \approx \frac{\pi p/2}{2a} - \frac{\pi^2 - 2y C_a(p, y)}{2a} \]

Eq. (5.24) reads \( (d' \neq 2, 1 < d < 3) \)

\[ \delta \lambda + \frac{t}{\lambda} \frac{L^{d-d}}{(4\pi)^{d/2}} \Gamma \left( 1 - \frac{d'}{2} \right) \phi^{(d'-2)/2} + \frac{1}{4\pi^{(d-1)/2}} \Gamma \left( \frac{d-1}{2} \right) \sum_{m, l(d-d')} \left( \frac{\lambda m}{t} \right)^2 + (Ll)^2 \right\}^{1-d'} = 0. \quad (5.42) \]

Now we will examine Eq. (5.42) in different regimes of \( t \) and \( L \) and for different geometries of the lattice:

1.a. \( \frac{\lambda t \phi^{1/2}}{2} \ll 1 \) and \( \frac{tL}{\lambda} \gg 1 \): In this case Eq. (5.42) transforms into (up to exponentially small corrections c.f. with Eq. (5.29))

\[ \delta \lambda + \frac{t}{\lambda} \frac{L^{d-d}}{(4\pi)^{d/2}} \Gamma \left( 1 - \frac{d'}{2} \right) \phi^{(d'-2)/2} + \frac{1}{2\pi^{(d-1)/2}} \Gamma \left( \frac{d-1}{2} \right) \zeta(d-1) \]

\[ + \frac{t}{\lambda} \frac{L^{2-d}}{4\pi^{d/2}} \Gamma \left( \frac{d}{2} - 1 \right) \sum_{l(d-d')} |l|^{2-d}. \quad (5.43) \]

This equation has different types of solutions depending on whether the dimensionality \( d \) is above or below the classical critical dimension 2.

At \( \lambda = \lambda_c \) and when \( d' < 2 < d < 3 \) (i.e. when there is no phase transition in the system) we obtain for the zero-field susceptibility

\[ \chi = \left( \frac{t}{\lambda_c} \right)^{-2} \left( \frac{tL}{\lambda_c} \right)^{2(d-d')(2-d')} \left[ \frac{2^{d-1}}{\pi^{(d-1)/2}} \Gamma \left( \frac{d-1}{2} \right) \zeta(d-1) \right]^{2-d'} . \quad (5.44) \]

However, for \( 1 < d < 2 \), Eq. (5.43) has no solution at \( \lambda = \lambda_c \) obeying the initial condition \( \frac{\lambda t \phi^{1/2}}{2} \ll 1 \).

Eq. (5.44) generalizes the bulk result (5.3) for \( d \) close to the upper quantum critical dimension i.e. \( d = 3 \).

At the shifted critical quantum parameter \( \lambda_c(t) \) given by Eq. (5.6) we get
where $\chi = L^2 \left[ \frac{2^{d-2}}{\pi^{(d-d')/2}} \frac{\Gamma \left( \frac{d}{2} - 1 \right)}{\Gamma \left( 1 - \frac{d}{2} \right)} \sum' |t|^{2-d} \right]^{\frac{2}{2-d'}}$. (5.45)

However this solution is valid only for $3 > d > 2 > d'$ i.e. here again there is no phase transition in the system.

1.b. $\frac{\lambda}{t} \phi^{1/2} \ll 1$ and $\frac{L}{t} \ll 1$: In this case, Eq. (5.42) gives (up to exponentially small corrections c.f. with Eq. (5.34))

\[
0 = \delta \lambda + \frac{t}{\lambda} \frac{L^{d'-d}}{(4\pi)^{d'/2}} \Gamma \left( 1 - \frac{d'}{2} \right) \phi^{(d'-2)/2} \frac{L^{1-d}}{4\pi^{(d+1)/2}} \Gamma \left( \frac{d-1}{2} \right) \sum' |t|^{1-d} \\
+ \frac{L^{d'-d}}{2\pi^{(d+1)/2}} \Gamma \left( \frac{d'}{2} - 1 \right) \left( \frac{t}{\lambda} \right)^{d'-1} \zeta(d' - 1). \tag{5.46}
\]

Here we find that the solutions of Eq. (5.46) depend upon whether the dimensionality $d' < 1$ or $d' > 1$.

At $\lambda = \lambda_c$ and for $1 < d' < 2$, Eq. (5.46) has

\[
\chi = L^2 \left( \frac{\lambda_c}{tL} \right)^{2/(2-d')} \left[ \frac{2^{d-2}}{\pi^{(d-d')/2}} \frac{\Gamma \left( \frac{d}{2} - 1 \right)}{\Gamma \left( 1 - \frac{d}{2} \right)} \sum' |t|^{1-d} \right]^{\frac{2}{2-d'}}. \tag{5.47}
\]

as a solution. For $0 \leq d' < 1$, however, it has no solution obeying the initially imposed restriction $\frac{\lambda}{t} \phi^{1/2} \ll 1$.

At the shifted critical quantum parameter $\lambda_c(L)$ given by\footnote{Eq. (5.46) has a solution obeying the initial condition $\frac{\lambda_c}{t} \phi^{1/2} \ll 1$ only for $d' = 1 + \varepsilon$ and in this case the susceptibility behaves like

\[
\chi = \frac{1}{(\pi \varepsilon)^2} \frac{\lambda_c^2}{t^2} \left[ 1 - \varepsilon \left( \gamma_E + \ln \frac{\varepsilon}{2} \right) \right]^2. \tag{5.48}
\]

2. For $L \phi^{1/2} \ll 1$ from Eqs. (5.38) and Eq. (A3) we get once again Eq. (5.43). In spite of the fact that we have the same equation as in the case $\frac{\lambda}{t} \phi^{1/2} \ll 1$, the expected solutions for the susceptibility may be different because of the new imposed condition. Here also we will consider the two limiting cases of low–temperature and very low–temperature regimes.

2.a. $L \phi^{1/2} \ll 1$ and $\frac{L}{\lambda} \gg 1$: In this case Eq. (5.42) again is transformed into Eq. (5.43) and we obtain at $\lambda = \lambda_c$ the solution given by Eq. (5.44), which is valid only for $d' < 2 < d < 3$, i.e. we have the same solution as in the previous case i.e. $\frac{\lambda}{t} \phi^{1/2} \ll 1$.

At $\lambda = \lambda_c(t)$, we formally obtain Eq. (5.45) which, however, may be considered like a solution only in the neighborhood of the lower classical critical dimension $d = 2$. For the cylindric geometry ($d' = 1$ and $d = 2 + \varepsilon$) we get
\[ \chi = \frac{L^2}{(\pi \varepsilon)^2} \left[ 1 - \frac{\varepsilon}{2} \left( \gamma_E - \ln 4\pi \right) \right]^2. \tag{5.49} \]

This result is contained in Eq. (30.109) of Ref. 46 in the large \( n \)-limit case for the NLQ\( \sigma \)M.

In the case of slab geometry \( d - d' = 1 \) \( (d = 2 + \varepsilon, d' = 1 + \varepsilon) \) instead of (5.49) we obtain

\[ \chi = \frac{L^2}{(\pi \varepsilon)^2} \left[ 1 - \varepsilon (\gamma_E - \ln 2) - \varepsilon \ln \varepsilon \right]^2. \tag{5.50} \]

In the case of a bloc geometry \( (d = 2 + \varepsilon \) and \( d' = 0) \) we find the following behavior for the susceptibility

\[ \chi = \frac{L^2}{2\pi \varepsilon} \left[ 1 - \frac{\varepsilon}{4} \left( 2\gamma_E + \ln \frac{\pi \varepsilon}{2} - 2 \ln \left[ \frac{\Gamma (\frac{1}{4})}{2\sqrt{\pi}} \right]^2 \right) \right]^2. \tag{5.51} \]

For the case of “quasi–bloc geometry” \( (d = 2 + \varepsilon \) and \( d' = \varepsilon) \) we get

\[ \chi = \left( \frac{\lambda_c}{2t} \right)^2 \left[ \frac{2}{\pi^{1/2} \Gamma (1 - \frac{d'}{2})} \zeta(d' - 1) \right]^{\frac{2}{2-d'}}. \tag{5.52} \]

Here the appearance of \( \varepsilon \) in the denominator in formulas (5.48)–(5.52) signalizes that the scaling in its simple form will fail at \( \varepsilon = 0 \).

2.b. \( L \phi^{1/2} \ll 1 \) and \( \frac{L}{\Lambda} \ll 1: \) Here we find that Eq. (5.46) is valid, and it has Eq. (5.47) as a solution at \( \lambda = \lambda_c \) and for \( 0 \leq d' < 1 \). For \( 1 < d' < 2 \) the susceptibility is given by

\[ \chi = \left[ \frac{(4\pi)^{d'/2}}{\Gamma (1 - \frac{d'}{2})} \left( 1 - \frac{\lambda}{\lambda_c} \right) \right]^{\frac{2}{2-d'}} t^{-2/(2-d')} L^{2(d-d')(2-d')}. \tag{5.53} \]

At the shifted critical point \( \lambda_c(L) \), for the susceptibility we obtain Eq. (5.53) under the restriction \( 2 > d' > 1 \), which guarantees the positiveness of the quantity under brackets.

When \( \lambda < \lambda_c \) for \( 1 < d < 3 \) and \( d' < 2 \), i.e. when there is no phase transition in the system, we obtain

\[ \chi = \frac{L^2}{(\pi \varepsilon)^2} \left[ 1 - \frac{\varepsilon}{2} \left( \gamma_E - \ln \frac{\Gamma (\frac{1}{4})}{2\sqrt{\pi}} \right) \right]^2. \tag{5.54} \]

If \( d' > 2 \) there is a phase transition in the system at the shifted value of the critical quantum parameter \( \lambda_{tL} \) (the shift in this case is due to the quantum and finite–size effects) and Eq. (5.42) transforms to

\[ 1 - \frac{\lambda}{\lambda_{tL}} = t \frac{L^{d-d}}{(4\pi)^{d/2}} \frac{\Gamma (1 - \frac{d'}{2})}{\phi^{(d'-2)/2}}. \tag{5.55} \]

which has the following solutions

\[ \chi = \begin{cases} \left[ \frac{(4\pi)^{d'/2}}{\Gamma (1 - \frac{d'}{2})} \left( 1 - \frac{\lambda}{\lambda_{tL}} \right) \right]^{\frac{2}{2-d'}} t^{-2/(2-d')} L^{2(d-d')(2-d')}, \lambda > \lambda_{tL} \\ \infty, \lambda \leq \lambda_{tL} \end{cases} \tag{5.56} \]

Let us notice that Eqs. (5.54) and (5.56) are the finite–size forms, for the susceptibility, of Eqs. (5.8) and (5.9), respectively, found for the bulk system.
3. Two–dimensional case

The two–dimensional case needs special treatment because of its physical reasonability and the increasing interest in the context of the quantum critical phenomena. From Eq. (5.24) for \( d = 2 \) and in the absence of a magnetic field \( h = 0 \) we get

\[
\delta \lambda = \frac{\phi^{1/2}}{4\pi} - \frac{1}{4\pi \lambda} \sum_{m,l(2-d')} \exp \left[ -\frac{\phi^{1/2}}{2\pi} \left( \frac{m^2}{2} + L^2l^2 \right)^{1/2} \right].
\]  
(5.57)

Introducing the scaling functions \( Y_{t}^{d'} = A_{t}^{1/2} / \phi \) and \( Y_{L}^{d'} = L \phi^{1/2} \), where the superscript \( d' \) denotes the number of infinite dimensions in the system, and the scaling variable \( a = \frac{L}{\phi} \) it is easy to write Eq. (5.57) in the scaling forms given in Eqs. (5.38) and (5.39). The solutions of the obtained scaling equations will depend on the number of the infinite dimensions in the system. Here we will consider the two most important particular cases: strip geometry \( d' = 1 \) and bloc geometry \( d' = 0 \). Our analysis will be confined to the study of the behavior of the scaling functions at the critical value of the quantum parameter \( \lambda_c \), and at the shifted critical quantum parameter \( \lambda tL \) (see Section V B 1). It is difficult to solve Eq (5.57) by using an analytic approach that is for what we will give a numerical treatment of the problem. It is, however, possible to consider the two limits: \( a \gg 1 \) i.e. the low–temperature regime and \( a \ll 1 \) i.e. the very low–temperature regime.

Strip geometry (\( d' = 1 \)): In this case in the r.h.s of Eq. (5.57) we have a two–fold sum which permits a numerical analysis of the geometry under consideration. FIG. 2 graphs the variation of the scaling functions \( Y_{t}^{1} \) and \( Y_{L}^{1} \) against the variable \( a \) at \( \lambda = \lambda_c \). This shows that for a comparatively small value of the scaling variable \( a \sim 5 \) the finite–size behavior (see the curve of the function \( Y_{t}^{1} (a) \)) merges in the low temperature bulk one, while the behavior of \( Y_{L}^{1} (a) \) shows that for relatively not very low–temperatures (\( a \sim \frac{1}{5} \), \( L \)-fixed) the system simulates the behavior of a three–dimensional classical spherical model with one finite dimensions. The mathematical reasons for this are the exponentially small values of the corrections, as we will show below.

Bloc geometry (\( d' = 0 \)): In this case the three–fold sum in the r.h.s of Eq. (5.57) is not an obstacle to analyze it numerically. For \( \lambda = \lambda_c \) the behavior of the scaling functions \( Y_{L}^{0} (a) \) and \( Y_{t}^{0} (a) \) is presented in FIG. 4 They have the same qualitative behavior as in the strip geometry the only difference is the appearance of a new universal number for \( t = 0 \), i.e. \( \Omega \), instead of the constant \( \Theta \) as a consequence of the asymmetry of the sum in the low–temperature and the very low–temperature regimes.

Analytically for arbitrary values of the number of infinite dimensions \( d' \) we can treat first the problem in the low–temperature regime (\( a \gg 1 \)). In this limit Eq. (5.57) can be transformed into (up to small corrections \( O(e^{-2\pi a}) \))

\[
\delta \lambda = \frac{1}{2\pi \lambda} \ln 2 \sinh \frac{\lambda}{2t} \phi^{1/2} - \frac{1}{2\pi \lambda} \sum_{l(2-d')} K_0 \left( L \phi^{1/2} |l| \right).
\]  
(5.58)

For \( \lambda = \lambda_c \) Eq. (5.58) has the solution

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\[
\chi^{-1/2} \approx \frac{t}{\lambda_c} \Theta + (2 - d') \sqrt{\frac{2\pi}{5\Theta}} \left( \frac{t}{L\lambda_c} \right)^{1/2} \exp \left( -\frac{tL}{\lambda_c} \Theta \right) \tag{5.59}
\]
i.e. the finite-size corrections to the bulk behavior are exponentially small.

In the very low-temperature regime \(a \ll 1\), Eq. (5.57) reads (up to \(O(e^{-2\pi/a})\))

\[
\delta \lambda = \frac{\phi^{1/2}}{4\pi} - \frac{L^{-1}}{4\pi} \sum_{l=1}^{d'-1} \delta \lambda \left( \frac{L\phi^{1/2}}{|l|} \right) + \frac{L^{d'-2}}{\pi^{(d'+1)/2}} \left( \frac{2\lambda}{t} \right)^{1/d'} \sum_{m=0}^{\infty} K_{2|l|} \left( \frac{\lambda t\phi^{1/2}}{m} \right), \tag{5.60}
\]
which has the solutions:

\[
\chi^{-1/2} \approx \frac{1}{L} \Theta + \sqrt{\frac{2\pi}{5\Theta}} \left( \frac{tL\lambda_c}{\lambda_c} \right)^{1/2} \exp \left( -\frac{\lambda c tL}{\lambda_c} \Theta \right) \tag{5.61}
\]
for \(d' = 1\), and

\[
\chi^{-1/2} \approx \frac{1}{L} \Omega + \frac{1}{L} \left\{ \frac{1}{2\Omega} + \frac{\Omega}{2} \sum_{l=1}^{d'-1} \left( \Omega^2 + 4\pi^2 l^2 \right)^{-3/2} \right\}^{-1} \exp \left( -\frac{\lambda_c}{tL} \right) \tag{5.62}
\]
for \(d' = 0\).

In Section IV B 1 an analytic continuation of the shift of the critical quantum parameter for \(d = 2\) was presented. It is possible to consider the solutions of Eq. (5.57) at \(\lambda = \lambda_{tL}\) (from Eqs. (5.31), (5.32), (5.36) and (5.37)) and for different geometries. In this case the scaling functions \(Y_1, Y_1^L, Y_0, Y_0^L\) are graphed in FIG. 3. For \(d' = 1\) again we see that a symmetry between the two limits \(a \ll 1\) and \(a \gg 1\) take place, since the scaling functions \(Y_1^t\) and \(Y_1^L\) are limited by the universal constant \(\Xi\). The asymmetric case \(d' = 0\) has two different constants \(\Sigma_t\) and \(\Sigma_L\), limiting the solutions of \(Y_0^t\) and \(Y_0^L\) from above.

The constants \(\Xi, \Sigma_t\) and \(\Sigma_L\) are obtained from the asymptotic analysis (with respect to \(a\)) of Eq. (5.57) for \(\lambda = \lambda_{tL}\).

In the limit \(a \gg 1\) for arbitrary values of \(d'\) we get (from Eq. (5.58))

\[
B_0 + \gamma_E + \ln \frac{L\phi^{1/2}}{2} = \sum_{l=2}^{d'-1} K_0 \left( L\phi^{1/2} |l| \right), \tag{5.63}
\]
where the equation of \(\lambda_{tL}\) from Eq. (5.30) is used. Eq. (5.63) has the solutions:

\[
L\chi^{-1/2} = \begin{cases} \Xi & \text{for } d' = 1, \\ \Sigma_L & \text{for } d' = 0, \end{cases} \tag{5.64}
\]
where the universal numbers \(\Xi = 7.061\) and \(\Sigma_L = 4.317\) are the solutions of the scaling equation (5.63) for \(d' = 1\) and \(d' = 0\), respectively.

In the opposite limit \(a \ll 1\), for \(d' = 1\), we get from Eqs. (5.36) and (5.64) the equation

\[
\gamma_E + \ln \frac{\lambda_{tL}\phi^{1/2}}{4\pi t} = 2 \sum_{m=1}^{\infty} K_0 \left( \frac{\lambda_{tL}}{t} \phi^{1/2} m \right), \tag{5.65}
\]
which has
\[ \frac{\lambda t}{L} \chi^{-1/2} = \Xi, \] (5.66)
as a universal solution. For \( d' = 0 \) we have
\[ \left( \frac{\lambda t}{L} \chi^{-1/2} - 6 \right) \exp \left( \frac{\lambda t}{L} \phi^{1/2} \right) - \frac{\lambda t}{L} \phi^{1/2} - 6 = 0 \] (5.67)
obtained from Eqs. (5.37) and (5.60), where we have used the identity (A11).

From Eq. (5.67) we obtain the universal result
\[ \frac{\lambda t}{L} \chi^{-1/2} = \Sigma_t = 6.028966.... \] (5.68)

We finally conclude that if we take \( \lambda = \lambda_c \) the scaling functions \( Y_t^{d'} \) and \( Y_L^{d'} \) have a similar qualitative behavior weakly depending on the geometry (i.e. block \( d' = 0 \) or strip \( d' = 1 \)) of the system. However, for a given geometry one distinguishes quite different quantitative behavior of the scaling functions depending on whether the quantum parameter \( \lambda \) is fixed at its critical value, i.e. \( \lambda = \lambda_c \), or takes “running” values \( \lambda t/L \) obtained from the “shift equations” (5.31), (5.32), (5.36) or (5.37).

4. Equation of state

The equation of state of the Hamiltonian (2.1) for dimensionalities \( 1 < d < 3 \) is given by (see Eqs. (5.13) and (5.24))
\[ 0 = \delta \lambda - (4\pi)^{-(d+1)/2} \int \left( \frac{1 - d'}{2} \right) \left( \frac{h}{M} \right)^{(d-1)/2} \]
\[ + \left( \frac{h}{M} \right)^{(d-1)/2} \sum_{m,l(d-d')} K_{(d-1)/2} \left[ \left( \frac{h}{M} \right)^{1/2} ((\lambda m/t)^2 + (L|l|)^2)^{1/2} \right]^{(d-1)/2} + M^2. \] (5.69)
It is straightforward to write this equation in a similar form as in Eq. (5.38) or Eq. (5.39), i.e.
\[ h = M^{1/2} f_h \left\{ \delta \lambda M^{-1/\beta} \left( tL \right)^{1/\nu} M^{-1/\beta} \right\}, \] (5.70a)
or
\[ h = M^{1/2} f_h \left\{ \delta \lambda M^{-1/\beta} \left( tL \right)^{1/\nu} M^{-1/\beta} \right\}. \] (5.70b)

Eqs. (5.70) are generalizations of Eqs. (5.15) in the case of systems confined to a finite geometry. The appearance of an additional variable \( tL \) is a consequence of the fact that the system under consideration may be regarded as an “hyperparallelepiped” (in not necessary an Euclidean space) of linear size \( L \) in \( d-d' \) directions and of linear size \( L_\tau \) in one direction with periodic boundary conditions.
VI. SUMMARY

Since exact solvability is a rare event in statistical physics, the model under consideration yields a conspicuous possibility to investigate the interplay of quantum and classical fluctuations as a function of the dimensionality and the geometry of the system in an exact manner. Its relation with the QNL\(\sigma\)M in the \(n \rightarrow \infty\) limit may serve as an illustration of Stanley’s arguments of the relevance of the spherical approximations in the quantum case. Let us note, however, that the use of such types of arguments needs an additional more subtle treatment in the finite size case. For this reason we gave a brief overview of the bulk low–temperature properties, which are similar to those obtained by saddle–point calculation for the QNL\(\sigma\)M (see Section V A).

The Hamiltonian (2.1) can be obtained also in the “hard–coupling limit” from a more realistic model that takes into account the quartic self–interaction term \(Q^4\), by the ansatz \(Q^2 \Rightarrow \frac{1}{N} \sum_i Q^2_i\) frequently used in the theory of structural phase transitions (see Refs. 33–36 and Section III).

An analog of the Privman–Fisher hypothesis\(^9\),\(^40\) for the FSS form of the free energy in the quantum case was shown to be consistent with the exact results obtained in Section IV. Furthermore, the explicit expression for the universal scaling function is presented and as a consequence the scaling form of the equation for the spherical field.

So the Hamiltonian (2.1) can be thought of as a simple but rather general model to test some analytical and numerical techniques in the theory of magnetic and structural phase transitions. The discussion of the results obtained, in Section V A serves as a basis for further FSS investigations. Identifying the temperature, which governs the crossover between the classical and the quantum fluctuations as an additional temporal dimension one makes possible the use of the methods of FSS theory in a very effective way.

In Subsection V B 1 the shift of the critical quantum parameter \(\lambda\) as a consequence of the quantum and finite–size effects is obtained. In comparison with the classical case (for details see Ref. 27 and references therein) here the problem is rather complicated by the presence of two finite characteristic lengths \(L\) and \(L_\tau\). We observe a competition between finite size and quantum effects which reflects the appearance of two regimes: low–temperature \((L \gg L_\tau)\) and very low–temperature \((L \ll L_\tau)\). The behavior of the shift is analyzed in some actual cases of concrete geometries e.g. strip and bloc.

In the parameter space (temperature \(t\) and quantum parameter \(\lambda\)) where quantum zero point fluctuations are relevant, there are three distinct regions named “renormalized classical”, “quantum critical”, and “quantum disordered”. The existence of these regions in conjunction with both regimes: low–temperature and very low–temperature make the model a useful tool for the exploration of the qualitative behavior of an important class of systems.

In Subsection V B 2 the susceptibility (or the correlation length) is calculated and the critical behavior of the system in different regimes and geometries is analyzed. We have studied the model (2.1) via \(\varepsilon\)–expansion in order to illustrate the effects of the dimensionality \(d\) on the existence and properties of the ordered phase. An indicative example is given by Eqs. (5.49) and (5.50), while the former is known (see Ref. 46) the last one is quite different and new. These shows that one must be accurate in taking the limit \(\varepsilon \rightarrow 0^+\) in the context of the FSS theory.
In Subsection V B 3, special attention is paid to the two–dimensional case. The two important cases of strip and bloc geometries are considered. The universal constant \( \Theta \) given by Eq. (5.12c), which characterizes the bulk system, is changed to a set of new universal constants: \( \Omega \) (see Eq. (5.62)), \( \Xi \) and \( \Sigma_L \) (see Eq. (5.64)), and \( \Sigma_t \) (see Eq. (5.68)). The appearance of new universal constants reflects the new situation, when there are two relevant values of the quantum parameter \( \lambda \): \( \lambda = \lambda_c \) in the bulk case and \( \lambda = \lambda_{tL} \) in the case of finite geometries. Due to their universality these constants may play an important role even in studying more complicated models. The behaviors of the scaling functions at the bulk critical quantum parameter \( \lambda_c \) and the shifted critical quantum parameter \( \lambda_{tL} \) are given in FIGs. 2 and 3. \( L_\tau \) is the main characteristic length and the \( \frac{1}{L} \) corrections are exponentially small in the case of low–temperature regime, and vice versa in the case of very low–temperature regime.

The equation of state, for the system confined to the general geometry \( L^{d-d'}\times\infty^{d'}\times L_\tau \), is obtained in Subsection V B 4. This reflects the modifications of the scaling functions as a consequence of the finite sizes and the temperature.

Finally let us note that this treatment is not restricted to the Hamiltonian (2.1), but it can be applied to a wide class of finite lattice models (e.g. directly to the anharmonic crystal model see Refs. 33–36) and it can also provide a methodology for seeking different quantum finite–size effects in such systems.

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**APPENDIX A: ASYMPTOTICS OF THE FUNCTIONS \( K_a(\nu|p,y) \)**

In this appendix we will sketch a way to find the asymptotic behavior of the functions \( K_a(\nu|p,y) \) defined in section V B 3 (see Eq. (5.38)). They have the following form

\[
K_a(\nu|p,y) = \sum_{m,l(p-1)} K_\nu \left( \frac{2y\sqrt{l^2 + a^2m^2}}{y\sqrt{l^2 + a^2m^2}} \right)^{\nu}, \quad y > 0, \quad (A1a)
\]

where

\[
l^2 = l_1^2 + l_2^2 + \cdots + l_{p-1}^2. \quad (A1b)
\]

By the use of the integral representation of the modified Bessel function

\[
K_\nu \left( 2\sqrt{zt} \right) = K_{-\nu} \left( 2\sqrt{zt} \right) = \frac{1}{2} \left( \frac{z}{t} \right)^{\nu/2} \int_0^{\infty} x^{-\nu-1}e^{-tx-z/x}dx \quad (A2)
\]
and the Jacobi identity for a $p$-dimensional lattice sum
\[ \sum_{m,l,p-1} e^{-(l^2+a^2m^2)t} = \frac{1}{a} \left( \frac{\pi}{t} \right)^{p/2} \sum_{m,l,p-1} e^{-\pi^2(l^2+m^2/a^2)/t}, \tag{A3} \]
we may write (A1) as
\[ K_\alpha(p|\nu) = \frac{\pi^{p/2}}{2a} \Gamma \left( \frac{p}{2} - \nu \right) y^{-p} \]
\[ + \frac{\pi^{2\nu-p/2}}{2a} y^{2\nu} \int_0^\infty dx x^{\nu-1} e^{-xy^2/\pi^2} \left[ \sum_{m,l,p-1} e^{-x(l^2+m^2/a^2)} - a \left( \frac{\pi}{x} \right)^{p/2} \right]. \tag{A4} \]
Let us notice that the two terms in the square brackets in the last equality cannot be integrated separately, since they diverge. Nevertheless, in order to encounter this divergence, we can transform further (A4) by adding and subtracting the unity from $\exp(-xy^2/\pi^2)$, which enables us to write down (after some algebra) the result
\[ K_\alpha(p|\nu) = \frac{\pi^{p/2}}{2a} \Gamma \left( \frac{p}{2} - \nu \right) y^{-p} + \frac{\pi^{2\nu-p/2}}{2a} y^{2\nu} C_a(p|\nu) - \frac{1}{2} \Gamma(-\nu) \]
\[ + \frac{\pi^{2\nu-p/2}}{2a} \Gamma \left( \frac{p}{2} - \nu \right) y^{2\nu} \sum_{m,l,p-1} \left[ \left( l^2 + \frac{m^2}{a^2} \right)^{\nu-p/2} - \left( l^2 + \frac{m^2}{a^2} \right)^{\nu-p/2} \right], \tag{A5} \]
where
\[ C_a(p|\nu) = \lim_{\delta \to 0} \int_0^\infty dx x^{\nu-1} \left[ \sum_{m,l,p-1} e^{-x(l^2+m^2/a^2)} - a \left( \frac{\pi}{x} \right)^{p/2} \right], \tag{A6a} \]
\[ = \lim_{\delta \to 0} \left\{ \sum_{m,l,p-1} \frac{\Gamma \left[ \frac{p}{2} - \nu, \delta \left( l^2 + \frac{m^2}{a^2} \right) \right]}{\left( l^2 + \frac{m^2}{a^2} \right)^{\nu-p/2}} \right. \]
\[ - \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty dm dp^{-1} \left[ \frac{\Gamma \left[ \frac{p}{2} - \nu, \delta \left( l^2 + \frac{m^2}{a^2} \right) \right]}{\left( l^2 + \frac{m^2}{a^2} \right)^{\nu-p/2}} \right] \right\} \tag{A6b} \]
is the Madelung–type constant and $\Gamma[\alpha, x]$ is the incomplete gamma function.

We see from Eq. (A3) that the shift of the critical quantum parameter is given by the Madelung type constant (A6) instead of the sum in Eq. (5.25). Indeed it is possible to show that these two representations are equivalent. This may be done, following Ref. 12, by starting from the Jacobi identity Eq. (A3), where we multiply the two sides by $\delta^{p/2-\nu-1}$ and integrating over $\delta$ to obtain the key equation
\[ C_a(p|\nu) = \sum_{m,l,p-1} \frac{\Gamma \left[ \frac{p}{2} - \nu, \delta \left( l^2 + \frac{m^2}{a^2} \right) \right]}{\left( l^2 + \frac{m^2}{a^2} \right)^{p/2-\nu}} \]
\[ - \frac{\delta^{p/2-\nu}}{\frac{p}{2} - \nu} \]
\[ + a \pi^{p/2-\nu} \sum_{m,l,p-1} \frac{\Gamma \left[ \frac{\nu}{\delta}, \frac{\pi^2(l^2+a^2m^2)}{\delta} \right]}{\left( l^2 + a^2m^2 \right)^{\nu\delta}} - a \frac{\pi^{p/2}}{\nu\delta^\nu}. \tag{A7} \]
Finally from Eq. (A7) we easily see that the integration constant $C_a(p|\nu)$ may be written in two different forms. In the first case we take the limit $\delta \to \infty$ and obtain

$$C_a(p|\nu) = a\pi^{p/2-2\nu}\Gamma(\nu) \sum'_{m,l(p-1)} \frac{1}{(l^2 + a^2m^2)^\nu}. \quad (A8)$$

In the other case we take the limit $\delta \to 0$, and then both first and last terms in the r.h.s of Eq. (A7) yields Eq. (A6).

Using a similar procedure we find, for the functions $\tilde{K}_a(\nu|p,y)$ defined in (5.39b), the following expression

$$\tilde{K}_a(\nu|p,y) = \frac{\pi^{p/2}}{2a^{p-1}} \Gamma\left(\frac{p}{2} - \nu\right) y^{-p} + \frac{\pi^{2\nu-p/2}}{2a^{p-1}} y^{-2\nu} C_a(p|\nu) - \frac{1}{2} \Gamma(-\nu)$$

$$+ \frac{\pi^{2\nu-p/2}}{2a^{p-1}} \frac{\Gamma\left(\frac{p}{2} - \nu\right)}{y^{2\nu}} \sum'_{m,l(p-1)} \left[ \left(\frac{l^2}{a^2} + m^2 + \frac{y^2}{\pi^2}\right)^{\nu-p/2} - \left(\frac{l^2}{a^2} + m^2\right)^{\nu-p/2}\right]. \quad (A9)$$

Here the Madelung type constant is given by

$$\tilde{C}_a(p|\nu|) = \lim_{\delta \to 0} \int_\delta^\infty d\delta x^{\frac{p}{2} - \nu - 1} \left[ \sum'_{m,l(p-1)} e^{-x(l^2/a^2+m^2)} - a^{\nu-1} \left(\frac{\pi^2}{x}\right)^{p/2} \right], \quad (A10a)$$

$$= \lim_{\delta \to 0} \left\{ \sum'_{m,l(p-1)} \Gamma\left[\frac{p}{2} - \nu, \delta \left(\frac{l^2}{a^2} + m^2\right)\right] \right\}$$

$$- \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty dmd^{p-1}l \left[ \Gamma\left[\frac{p}{2} - \nu, \frac{l^2}{a^2} + m^2\right] \right] \right\} \quad (A10b)$$

$$= a^{\nu-1} \pi^{p/2-2\nu} \Gamma(\nu) \sum'_{m,l(p-1)} \frac{1}{(a^2l^2 + m^2)^\nu}. \quad (A10c)$$

Eqs. (A5) and (A9) are slight generalizations (for the anisotropic case $a \neq 1$) of the result obtained in Ref. 42 from one side, and get in touch with the Watson type sums proposed earlier in Ref. 19 from the other (see also Ref. 27).

If we put in Eqs. (A5) or (A9) $d = 2, d' = 0$ and $a = 1$ we obtain the identity

$$\sum'_{l_1,l_2} \exp\left(-y\sqrt{l_1^2 + l_2^2}\right) = 2\pi \frac{y}{\sqrt{l_1^2 + l_2^2}} + 4\zeta\left(\frac{1}{2}\right) \beta\left(\frac{1}{2}\right) + y$$

$$+ 2\pi \sum'_{l_1,l_2} \left\{ \frac{1}{\sqrt{y + 4\pi^2(l_1^2 + l_2^2)}} - \frac{1}{2\pi\sqrt{l_1^2 + l_2^2}} \right\}. \quad (A11)$$
APPENDIX B: SHIFT OF THE CRITICAL QUANTUM PARAMETER FOR SOME PARTICULAR GEOMETRIES

Our task in this appendix is to explain how the shift of the critical quantum parameter in the special cases \( d = 2 \) (for the low–temperature regime) Eq. (5.30) and \( d' = 1 \) (for the very low–temperature regime) Eq. (5.33) are obtained.

1. Low–temperature regime, \( d = 2 \)

The expression for the shift is given by Eq. (5.29) for an arbitrary dimensionality of the system. To calculate the shift for \( d = 2 \) we proceed as follows: we first put \( d = 2 + \varepsilon \) and expand the obtained expressions around \( \varepsilon = 0 \). So, for the functions in the first term of Eq. (5.29) we obtain

\[
\pi^{-(3+\varepsilon)/2} = \pi^{-3/2} \left( 1 - \frac{\varepsilon}{2} \ln \pi \right) + \mathcal{O}(\varepsilon^2),
\]

\[
\left( \frac{t}{\lambda L} \right)^{1+\varepsilon} = \frac{t}{\lambda L} \left( 1 + \varepsilon \ln \frac{t}{\lambda L} \right) + \mathcal{O}(\varepsilon^2),
\]

\[
\Gamma \left( \frac{1+\varepsilon}{2} \right) = \sqrt{\pi} \left[ 1 - \frac{\varepsilon}{2} (2 \ln 2 + \gamma_E) \right] + \mathcal{O}(\varepsilon^2),
\]

\[
\zeta(1+\varepsilon) = \frac{1}{\varepsilon} + \gamma_E + \mathcal{O}(\varepsilon),
\]

where \( \gamma_E \) is Euler’s constant. For the second term we obtain

\[
L^{-\varepsilon} = 1 - \varepsilon \ln L + \mathcal{O}(\varepsilon^2),
\]

\[
\Gamma \left( \frac{\varepsilon}{2} \right) = \frac{2}{\varepsilon} \left[ 1 - \frac{\varepsilon}{2} \gamma_E \right] + \mathcal{O}(\varepsilon),
\]

and

\[
\lim_{d \to 2+\varepsilon} \sum_{l(d-d')}^\prime |l|^{2-d} \equiv Z \begin{bmatrix} 0 \\ 0 \end{bmatrix} (l, \varepsilon) = -1 + B_0 \varepsilon + \mathcal{O}(\varepsilon^2). \tag{B1}
\]

Finally after the substitution of the above expansions in Eq. (5.29) and then taking the limit \( \varepsilon \to 0 \) we obtain Eq. (5.30).

Let us now consider the particular cases \( d' = 1 \) and \( d' = 0 \)

The case \( d' = 1 \): Here the \((d - d')\)–dimensional sum in Eq. (B1) reduces to

\[
\sum_{l}^\prime l^{-\varepsilon} = 2 \zeta(\varepsilon) = -1 - \varepsilon \ln(2\pi) + \mathcal{O}(\varepsilon^2), \tag{B2}
\]

hence the constant \( B_0 = -\ln 2\pi \) and the result (5.31) emerges.

The case \( d' = 0 \): Now the sum in Eq. (B1) is two–dimensional. Then we obtain the following result due to Hardy:

30
\[
\sum_{l_1,l_2}^\prime (l_1^2 + l_2^2)^{-\varepsilon/2} = 4\zeta(\varepsilon/2)\beta(\varepsilon/2),
\] 
where
\[
\beta(s) = \sum_{l=1}^{\infty} \frac{(-1)^l}{(2l+1)^s}.
\]

To obtain the expansion of the functions in Eq. (B3) we use (B2) and
\[
\beta(\varepsilon) = 4^{-\varepsilon} \left\{ \zeta\left(\frac{\varepsilon}{4}, \frac{1}{4}\right) - \zeta\left(\frac{\varepsilon}{4}, \frac{3}{4}\right) \right\} = \frac{1}{2} + \frac{\varepsilon}{2} \ln \frac{\Gamma\left(\frac{1}{4}\right)}{2\sqrt{2\pi}} + \mathcal{O}(\varepsilon^2),
\]
obtained with the aid of the expansion
\[
\zeta(s,a) = \frac{1}{2} - a + \varepsilon \left[ \ln \Gamma(a) - \frac{1}{2} \ln 2\pi \right] + \mathcal{O}(\varepsilon^2),
\]
for the incomplete zeta function \(\zeta(s,a)\). Finally we get
\[
\sum_{l_1,l_2}^\prime (l_1^2 + l_2^2)^{-\varepsilon/2} = -1 - \varepsilon \ln \frac{\Gamma\left(\frac{1}{4}\right)^2}{2\sqrt{\pi}} + \mathcal{O}(\varepsilon^2),
\]
which makes possible the calculation of the shift, Eq. (5.32), of the critical quantum parameter for the fully finite geometry.

2. Very low–temperature regime, \(d' = 1\)

In this limit the shift is given by Eq. (5.34). Here we will set \(d' = 1 + \varepsilon\) and expand around \(\varepsilon = 0\). In the first term of Eq. (5.34) we have a \((d - d')\)–dimensional sum, which can be replaced by the corresponding Epstein zeta function. Then we can use the following asymptotic form
\[
\sum_{l_{d-d'}} |l|^{1-d} \equiv \zeta\left|_0^{d'} \right. (l^2, d-1) = \frac{2\pi^{(d-1)/2}}{\Gamma\left(\frac{d}{2}\right)} \varepsilon + C_0 + \mathcal{O}(\varepsilon), \quad d' = 1 + \varepsilon,
\]
where the expressions for the constants \(C_0\) are quite complicated except for some special cases (see Refs. 42, e.g. for \(d = 2, d' = 1 - \varepsilon\) we have \(C_0 = \gamma_E - \ln(4\pi)\) (c.f. Ref. 46, Eq. (30.104)).

Further for the function in the second term of Eq. (5.34) we use the expansions given in the preceding subsection. After substituting the obtained expansions in the basic expression (Eq.(5.34)) for the shift, we obtain Eq. (5.35).
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FIG. 1. The dependence of the universal constant $y_0$ upon the dimensionality $d$. The constant $\Theta = 0.962424\ldots$ is obtained for the two-dimensional system (see Eq. (5.12c)).
FIG. 2. The effects of the finite-size geometry on the bulk behavior of $\phi^{1/2}$ for the two dimensional case at $\lambda = \lambda_c$. The superscript $d'$ in $Y^{d'}_L = L^{d'}\phi^{1/2}$ and $Y^{d''}_t = \frac{\lambda}{t}\phi^{1/2}$ indicates the number of infinite dimensions in the system. The scaling variable $a = \frac{tL}{\lambda_c}$. The universal numbers are $\Theta = 0.962424...$ (see Eq. (5.12c)) and $\Omega = 1.511955...$
FIG. 3. The same as in FIG. [3] but for $\lambda = \lambda_{tL}$ and $a = \frac{tL}{\lambda_{tL}}$. The universal numbers are: $\Xi = 7.061132...$, $\Sigma_t = 6.028966...$ and $\Sigma_L = 4.317795...$. 