Projective geometry of homogeneous second order Hamiltonian operators

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Abstract
We prove the invariance of homogeneous second-order Hamiltonian operators under the action of projective reciprocal transformations. We establish a correspondence between such operators in dimension $n$ and 3-forms in dimension $n + 1$. In this way we classify second order Hamiltonian operators using the known classification of 3-forms in dimensions $\leq 9$. Systems of first-order conservation laws that are Hamiltonian with respect to such operators are also explicitly found. The integrability of the systems is discussed in detail.
1 Introduction

Hamiltonian operators play a fundamental role in the theory of integrability of partial differential equations (PDEs). A distinguished class of Hamiltonian operators was introduced in 1983 by Dubrovin and Novikov [9]. One of the main features of the new class was that it was invariant under diffeomorphisms of the underlying space, thus bringing geometry into the theory of integrable PDEs.

More precisely, let us denote by $u^i = u^i(t, x)$ $n$ unknown functions of two independent variables $t$ and $x$, $i = 1, \ldots, n$, and denote by $u^i_\sigma$ the $x$-derivative of $u^i$ $\sigma$ times. An element of the class has the form of a matrix of first-order differential operators where each summand contains the same number of $x$-derivatives (1 in this case):

$$P^{ij} = g^{ij} \partial_x + \Gamma^{ij}_k u^k_x. \tag{1}$$

Then, a partial differential equation of the form $u^i_t = f^i(u^j, u^j_\sigma)$ is Hamiltonian if there exists a density $H = \int h(u^j)dx$ such that

$$u^i_t = f^i = P^{ij} \frac{\delta H}{\delta u^j}. \tag{2}$$

The typical situation is when $f^i = V^i_j(u^k)u^j_x$, i.e. the system of PDEs is quasilinear and of the first order. In that case, if the system of PDEs is
Hamiltonian, it is straightforward to realize that both the system and its Hamiltonian formulation are form-invariant with respect to a transformation involving the dependent variables only: \( \bar{u}^i = \bar{u}^i(w^j) \). We recall that the Hamiltonian property of \( P \) is the fact that it induces a Poisson bracket on the space of densities:

\[
\{ F, G \}_P = \int \frac{\delta f}{\delta u^i} P^{ij} \frac{\delta g}{\delta w^j} dx,
\]

where \( F = \int f dx \) and \( G = \int g dx \). The Poisson bracket property is also invariant with respect to the above transformations.

The first order case was soon generalized to the higher order case in [8], the homogeneity degree being equal to the order of the operators. Numerous examples shown that homogeneous Hamiltonian operators (HHOs) are ubiquitous, either as stand-alone operators or in linear combinations with operators of different homogeneity degrees. See [20] for many examples of the latter kind.

Recently, it was observed that third-order homogeneous Hamiltonian operators are invariant (when in their canonical form, see [11, eq. (2)]) with respect to a non-obvious class of transformations, namely projective reciprocal transformations [11]. They have the form of a projective transformation of dependent variables coupled with a non-local transformation of \( x \):

\[
\bar{u}^i = A_{ij}^0 w^j + A_{ij}^0 , \quad d\bar{x} = (A_i^0 w^i + A_0^0) dx.
\]

Third-order homogeneous Hamiltonian operators have been classified in low dimensions \( n \leq 4 \) ([11, 12]) with respect to the action of the above group. It was also proved that there exists a multiparametric family of quasilinear systems of first-order PDEs that are Hamiltonian with respect to any such operator [13].

The goal of the current paper is to prove that second-order homogeneous Hamiltonian operators have the same invariance properties of third-order homogeneous Hamiltonian operators.

The interest in such a result is that projective-geometric invariance is not just an ‘isolated’ feature of third-order operators: being also a property of second-order operators it is reasonable to think that it is something bound to all homogeneous operators.

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A classification then follows from the above invariance result; the algebraic variety that is identified with a second-order operator has been extensively studied in [23] (linear line congruence), and has a different nature with respect to that which is associated with a third-order operator (quadratic line complex). We also obtain similar (but not identical!) results concerning associated systems of first-order PDEs.

More precisely, second order homogeneous Hamiltonian operators have the general form

$$P^{ij} = g^{ij} \partial_x^2 + b_k^{ij} u^k \partial_x + c_k^{ij} u^k u_x^k + c_{kh}^{ij} u_x^k u_x^h. \quad (5)$$

We will always consider the non-degenerate case $\det(g^{ij}) \neq 0$. Under a coordinate transformation of the type $\bar{u}^i = \bar{u}^i(u^j)$ the symbols $\Gamma^k_{ij} = -g_{ip}c_{jk}^p$ transform as a linear connection. It is proved in [24, 6] (but see also [21, 14]) that the Hamiltonian property of the above operator implies that $\Gamma^k_{ij}$ is symmetric and flat. With respect to flat coordinates the operator can be rewritten as

$$P^{ij} = \partial_x g^{ij} \partial_x. \quad (6)$$

The Hamiltonian property in flat coordinates is then equivalent to the fact that

$$g_{ij} = T_{ijk} u^k + g_{ij}^0, \quad (7)$$

where $T_{ijk}$ are constant and skew-symmetric with respect to $i, j, k$ and $g_{ij}^0$ is constant and skew-symmetric with respect to $i, j$. The above equations (6), (7) have been independently found in [24, 6]. See also [21] for a thorough review on homogeneous Hamiltonian operators, and see [14] for a further differential-geometric analysis of the properties of homogeneous second-order Hamiltonian operators and their pencils.

Here, we will prove the following theorem (Theorem 6 on page 9).

**Theorem 1** Second-order homogeneous Hamiltonian operators are invariant under projective reciprocal transformations (4).

Then, we will prove a result that enables us to classify second-order homogeneous Hamiltonian operators in low dimensions ($n \leq 8$) (Theorem 8 on page 12).

**Theorem 2** Second-order homogeneous Hamiltonian operators in dimension $n$ can be put in bijection with 3-forms in the $n + 1$-dimensional space $\mathbb{C}^{n+1}$. 

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A projective reciprocal transformation induces an SL($n+1$)-transformation on the corresponding 3-form which commutes with the action on the corresponding second-order homogeneous Hamiltonian operator.

It is interesting to observe that, in a generic situation, 3-forms define linear line congruences (see [23] for the algebraic geometric description and properties of that correspondence), hence second-order homogeneous Hamiltonian operators are in correspondence with algebraic varieties, as it happened in the third-order case (for different algebraic varieties, i.e. quadratic line complexes).

Let us consider a quasilinear first-order system of PDEs in conservative form
\[ u^i_t = (V^i)_x, \quad (8) \]
where $V^i = V^i(u^j)$. We will prove the following Theorem (which is the union of the statements of Theorem 12, Corollary 18, Proposition 22 and Theorem 24).

**Theorem 3** Every second-order homogeneous Hamiltonian operator is the Hamiltonian operator of a multiparameter family of systems of conservation laws as in (8). The systems are linearly degenerate, possess a non-local Hamiltonian and have identically vanishing Haantjes tensor.

Strictly speaking, we do not have a general proof of integrability of the systems, but it turns out that in many examples they are indeed integrable. Indeed, in many computational experiments where we randomly chose the parameters it turned out that the systems are diagonalizable, and, by known results [25], they have a maximal set of conservation laws. Moreover, until now we did not find any counterexamples to integrability in our experiments. See the discussion at the end of Subsection 3.4.

We would like to stress that homogeneous Hamiltonian operators are important building blocks in the theory of integrable systems. We can mention several ways in which they are involved:

- Many bi-Hamiltonian systems have a bi-Hamiltonian pair of the form $P = P_1 + R$ and $Q = Q_1$, where $P_1$, $Q_1$ are compatible homogeneous first-order Hamiltonian operators and $R$ is a homogeneous second-order or third-order Hamiltonian operator which is compatible with $P_1$ and $Q_1$. We call such systems bi-Hamiltonian systems of KdV type [20]. Examples include the AKNS (or two-boson) hierarchy, the two-component
Camassa-Holm hierarchy \[ \text{[10]}, \text{a multiparameter family defined in [26]} \text{(see [20]) and the Kaup–Broer system [19] when}\]

\[
R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \partial_x^2. \tag{9}
\]

Other examples with \(R\) a third-order HHO are the KdV equation, the Camassa-Holm equation \[\text{[20]}, \text{a dispersive water waves equation [4] and a coupled Harry–Dym hierarchy [3]. See also the recent papers [5, 18], with a differential-geometric focus on the same construction.}\]

- Fewer systems are determined by a pair of Hamiltonian operators of the form \(P = P_1\) and \(R\); here we mention the WDVV systems \[\text{[29]}, \text{where } R \text{ is of the third order. No instances of systems that we determined in this paper were previously known to our knowledge. A probable explanation is that the first non-trivial systems (although linearizable) appear in dimension 4, and non-linearizable ones in dimension 6 and greater, and that makes their investigation quite complicated.}\]

- Homogeneous operators play a central role in Dubrovin–Zhang’s perturbative approach to the classification of integrable systems under the action of the group of Miura transformations \[\text{[7]. Since deformations of a first-order Poisson pencil are given as a formal series of homogeneous operators, one might expect that projective transformations and invariance can play a role.}\]

The results obtained so far show that the group of projective reciprocal transformations act on hierarchies defined by trios of compatible operators \(P_1, Q_1, R\) or by pairs of compatible operators \(P_1, R\). The action preserves the locality of second-order or third-order HHOs in canonical form, even if it does not preserve the locality of \(P_1, Q_1\). So, a projective geometric study of the above hierarchies makes sense and is potentially interesting.

The chances that the projective invariance properties that are shared by second-order and third-order HHOs might be generalized to HHOs of arbitrary order are high enough to consider that possibility in the framework of the perturbative approach in \[\text{[7].}\]

More generally, our results might indicate that a projective-geometric approach to integrable systems is starting to emerge in the field. We will pursue that research line in the future.
2 Projective geometry and Hamiltonian operators

Let us consider a projective transformation \( T: \mathbb{P}^n \rightarrow \mathbb{P}^n \). We will treat \((u^i)\) as an affine chart of the homogeneous coordinates \([v] = [v^0, \ldots, v^n]\), where \( v^i = u^i/u^{n+1} \). In homogeneous coordinates we have \( T(v) = [a^\lambda \mu v^\mu] \), where \((a^\lambda \mu) \in SL(n+1)\). In this section latin indices \( i, j, \ldots \) will run from 1 to \( n \) and greek indices \( \lambda, \mu, \ldots \) will run from 1 to \( n+1 \). A projective transformation in the affine chart has the form:

\[
\tilde{u}^i = T^i(u^j) = \frac{a^i_j u^j + a^i_{n+1}}{a^{n+1}_j u^j + a^{n+1}_{n+1}}.
\] (10)

In this section we will calculate the action of a projective transformation on a second-order homogeneous Hamiltonian operator. We will realize that the transformation (10) alone is not enough to yield invariance, while reciprocal projective transformations guarantee the invariance of the form (6) of our operators.

2.1 Projective invariance of the Hamiltonian operators

We would like to find the change of coordinates formula on the leading coefficient \( g \) of a second-order homogeneous Hamiltonian operator; in other words, we are looking for a formula connecting

\[
g = (T_{ijk} u^k + g^0_{ij}) du^i \wedge du^j \quad \text{and} \quad \tilde{g} = (\tilde{T}_{ijk} \tilde{u}^k + \tilde{g}^0_{ij}) d\tilde{u}^i \wedge d\tilde{u}^j.
\](11)

Note that we will work with the covariant version of the leading coefficient; this is possible due to our assumption \( \det(g) \neq 0 \). As a preliminary remark, note that

\[
d (\tilde{u}^i) = d \left( \frac{a^i_s u^s + a^i_{n+1}}{a^{n+1}_s u^s + a^{n+1}_{n+1}} \right) = \frac{A a^i_s du^s - (a^i_s u^s + a^i_{n+1}) a^{n+1}_i du^l}{A^2}
\] (12)

where \( A = a^{n+1}_s u^s + a^{n+1}_{n+1} \).

\(^1\)In skew-symmetric tensors indices are summed on all their ranges, according to Einstein’s convention.
Theorem 4  Under the transformation (10) we obtain

\[ T_{ics} = \frac{1}{2A^3} \left( \tilde{T}_{ijk} (a^i_c a^j_c - a^i_c a^j_i) a_s^k + \tilde{g}^0_{ij} (a^i_c a^j_c - a^i_c a^j_i) a_s^{n+1} \\
- \tilde{g}^0_{ij} (a^i_c a_s^{n+1} - a^i_c a_t^{n+1}) a_j^i - \tilde{g}^0_{ij} (a_t^{n+1} a^j_c - a^i_c a_t^{n+1}) a_s^i \right) \quad (13a) \]

\[ g^0_{ic} = \frac{1}{2A^3} \left( \tilde{T}_{ijk} (a^i_c a^j_c - a^i_c a^j_i) a_s^k + \tilde{g}^0_{ij} (a^i_c a^j_c - a^i_c a^j_i) a_s^{n+1} \\
- \tilde{g}^0_{ij} (a^i_c a_c^{n+1} - a^i_c a_t^{n+1}) a_j^i - \tilde{g}^0_{ij} (a_t^{n+1} a^j_c - a^i_c a_t^{n+1}) a_s^i \right) \quad (13b) \]

Proof. Applying the transformation to \( \tilde{g}^0_{ij} d\bar{u}^i \wedge d\bar{u}^j = \tilde{T}_{ijk} \bar{u}^k d\bar{u}^i \wedge d\bar{u}^j + \tilde{g}^0_{ij} d\bar{u}^i \wedge d\bar{u}^j \) we obtain

\[
\tilde{g}^0_{ij} d\bar{u}^i \wedge d\bar{u}^j = \\
= \frac{\tilde{g}^0_{ij}}{A^4} \left( \frac{Aa^j_i du^s - (a_s^j u^s + a^j_{n+1}) a^j_{n+1} du^l}{A^2} \right) \wedge \left( \frac{Aa^j_i du^s - (a_s^j u^s + a^j_{n+1}) a^j_{n+1} du^l}{A^2} \right) \\
= \frac{\tilde{g}^0_{ij}}{A^4} \left( A^2 a^j_i du^s \wedge du^l - Aa^j_i (a^j_b u^b + a^j_{n+1}) a^j_{n+1} du^s \wedge du^c \right. \\
- A(a^i_m u^m + a^i_{n+1}) a^j_{n+1} a^j_i du^l \wedge du^s + \left. (a^i_m u^m + a^i_{n+1}) a^j_{n+1} (a^j_b u^b + a^j_{n+1}) a^j_{n+1} du^l \wedge du^c \right) \\
= \frac{\tilde{g}^0_{ij}}{A^4} \left( A^2 a^j_i du^s \wedge du^l - Aa^j_i (a^j_b u^b + a^j_{n+1}) a^j_{n+1} du^s \wedge du^c \right. \\
- A(a^i_m u^m + a^i_{n+1}) a^j_{n+1} a^j_i du^l \wedge du^s \left. \right) \\
= \frac{\tilde{g}^0_{ij}}{A^4} \left( Aa^j_i du^s \wedge du^l - a_s^i (a^j_b u^b + a^j_{n+1}) a^j_{n+1} du^s \wedge du^c \right. \\
- (a^i_m u^m + a^i_{n+1}) a^j_{n+1} a^j_i du^l \wedge du^s \left. \right)
\]

Analogously,

\[
\tilde{T}_{ijk} \bar{u}^k d\bar{u}^i \wedge d\bar{u}^j = \tilde{T}_{ijk} \frac{a^i_j u^s + a^i_{n+1}}{A^5} A^2 a^j_i du^l \wedge du^c \\
= \frac{\tilde{T}_{ijk}}{A^3} (a^j_b u^s + a^j_{n+1}) a^j_i du^l \wedge du^c,
\]

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where three terms cancel due to the skew-symmetry of $\tilde{T}_{ijk}$. We finally obtain

$$\tilde{g}_{ij}d\tilde{u}^i \wedge d\tilde{u}^j = \frac{1}{A^3} \left[ \tilde{T}_{ijk}(a^k_su^s + a^k_{n+1})a^l_i a^l_j + \tilde{g}_{ij}^0 a^l_i (a^{n+1}_su^s + a^{n+1}_{n+1})a^l_c 
- \tilde{g}_{ij}^0 a^l_i (a^l_su^s + a^l_{n+1})a^c_c + \tilde{g}_{ij}^0 a^l_i (a^l_su^s + a^l_{n+1})a^l_{n+1} \right] du^l \wedge du^c \quad (14)$$

Collecting $u^s$, and comparing the left-hand side with $g_{lc} dul \wedge du^c$ with respect to a basis (i.e., $l < c$) we obtain the change of coordinates formula (13) □

**Corollary 5** The indexed families $T_{lcs}$ and $g^0_{lc}$ as obtained from $\tilde{T}_{ijk}$ and $\tilde{g}_{ij}$ by means of the above transformation are skew-symmetric with respect to all of their indices. Hence, a projective transformation of the leading coefficient of a second-order HHO preserves its form up to a conformal factor:

$$\tilde{g}_{ij}d\tilde{u}^i \wedge d\tilde{u}^j = \frac{1}{A^3} g_{lc} dul \wedge du^c. \quad (15)$$

**Proof.** The skew-symmetry of $g^0_{lc}$ is evident, and it is easy to show that $T_{lcs} = -T_{lsc}$ by observing that the skew-symmetry holds separately in the summand $\tilde{T}_{ijk}(a^l_i a^l_c - a^l_i a^l_j) a^k_s$ and in the remaining three summands. □

We recall that a reciprocal transformation is a nonlocal change of the independent variables $t, x$ defined as

$$d\tilde{t} = B(u)dt, \quad d\tilde{x} = A(u)dx \quad (16)$$

where $A(u), B(u)$ are functions depending on $u$. Projective reciprocal transformation were introduced in [11] as invariance transformations for the canonical form of third-order HHOs. They are reciprocal transformations of the form

$$d\tilde{t} = dt, \quad d\tilde{x} = A dx = (a^{n+1}_k u^k + a^{n+1}_{n+1}) dx \quad (17)$$

coupled with a projective transformation $T$ as in (10). We are going to prove that projective reciprocal transformations preserve the canonical form (6) of second-order HHOs. The proof follows the lines of the proof of the analogous result for third-order HHOs [11].

**Theorem 6** Projective reciprocal transformations preserve the canonical form (6) of second-order HHOs.
Proof. It is enough to prove the result for a transformation of the type $\tilde{u}^i = u^i/A$, where $A = a_k^{n+1}u^k + a_{n+1}^{n+1}$. It is easy to see that $\int u^i dx$ transform as $\int \frac{u^i}{A} d\tilde{x} = \int \tilde{u}^i d\tilde{x}$, and, more generally, two densities $F = \int f(u) dx$ and $H = \int h(u) dx$ transform as $f = A\tilde{f}$ and $h = A\tilde{h}$. Moreover, we have:

$$f_j = \frac{\partial f}{\partial u^j} = \frac{\partial A}{\partial u^j} \tilde{f} + A \frac{\partial \tilde{f}}{\partial u^j} = a_j^{n+1} \tilde{f} + A\tilde{f}_j$$  \hspace{1cm} (18)

and analogously $h_j = a_j^{n+1} \tilde{h} + A\tilde{h}_j$, then:

$$\{F, H\} = \int f_i P^{ij} h_j dx = \int (A \frac{\partial \tilde{f}}{\partial u^j} + a_k^{n+1} \tilde{f}) A\partial_x \left( g^{ij} A\partial_x (a_k^{n+1} \tilde{h} + A\tilde{h}_j) \right) \frac{1}{A} d\tilde{x},$$  \hspace{1cm} (19)

where we used $\partial_x = A\partial_\tilde{x}$. We can cancel $A$ once and obtain a new second order HHO with leading term $A^3 g^{ij}$. Let us first observe that $a_{n+1}^{n+1} + a_k^{n+1} u^k = A = a_{n+1}^{n+1} \frac{1}{1-a_{n+1}^{n+1} u^l}$. Then, we have

$$\frac{\partial \tilde{u}^i}{\partial u^j} = \frac{\delta^i_j A - a_j^{n+1} u^i}{A^2} = \frac{\delta^i_j - a_j^{n+1} \tilde{u}^i}{A}$$  \hspace{1cm} (20)

$$A \frac{\partial \tilde{f}}{\partial u^j} = A \frac{\partial \tilde{u}^k}{\partial u^j} \frac{\partial \tilde{f}}{\partial \tilde{u}^k} = (\delta^k_j - a_j^{n+1} \tilde{u}^k) \frac{\partial \tilde{f}}{\partial \tilde{u}^k}$$  \hspace{1cm} (21)

Now, let us consider again the bracket in (19) and carry out the coordinate change:

$$\{F, H\} = \int \left( A \frac{\partial \tilde{f}}{\partial u^j} + a_j^{n+1} \tilde{f} \right) \partial_x \left( g^{ij} A\partial_x \left( \frac{\partial \tilde{h}}{\partial u^j} + a_j^{n+1} \tilde{h} \right) \right) d\tilde{x}$$

$$= \int \left( (\delta^i_k - a_i^{n+1} \tilde{u}^k) \frac{\partial \tilde{f}}{\partial \tilde{u}^k} + a_i^{n+1} \tilde{f} \right) \partial_x \left( g^{ij} A\partial_x \left( (\delta^l_j - a_j^{n+1} \tilde{u}^l) \frac{\partial \tilde{h}}{\partial \tilde{u}^l} + a_j^{n+1} \tilde{h} \right) \right) d\tilde{x}$$

Using the identity:

$$\partial_x \left( (\delta^l_j - a_j^{n+1} \tilde{u}^l) \frac{\partial \tilde{h}}{\partial \tilde{u}^l} + a_j^{n+1} \tilde{h} \right) = (\delta^l_j - a_j^{n+1} \tilde{u}^l) \partial_x \frac{\partial \tilde{h}}{\partial \tilde{u}^l}$$
we obtain
\[
\{F, H\} = \int \left( \left( \delta^k_i - a_i^{n+1} \bar{u}^k \right) \frac{\partial \tilde{f}}{\partial \bar{u}^k} + a_i^{n+1} \bar{f} \right) \partial \tilde{\varepsilon} \left( g^{ij} A(\delta^l_j - a_j^{n+1} \bar{u}^l) \partial \tilde{\varepsilon} \frac{\partial \bar{h}}{\partial \bar{u}^l} \right) d\tilde{x}
\]
\[
= \int \left( \left( \delta^k_i - a_i^{n+1} \bar{u}^k \right) \frac{\partial \tilde{f}}{\partial \bar{u}^k} \right) \partial \tilde{\varepsilon} \left( g^{ij} A(\delta^l_j - a_j^{n+1} \bar{u}^l) \partial \tilde{\varepsilon} \frac{\partial \bar{h}}{\partial \bar{u}^l} \right) d\tilde{x}
\]
\[
- \int a_i^{n+1} \partial \tilde{\varepsilon} \tilde{f} \cdot \left( g^{ij} A(\delta^l_j - a_j^{n+1} \bar{u}^l) \partial \tilde{\varepsilon} \frac{\partial \bar{h}}{\partial \bar{u}^l} \right) d\tilde{x}.
\]
Finally, observing that \(\partial \tilde{\varepsilon} \tilde{f} = \tilde{f}_m \bar{u}^m \) and by using the identity
\[
(\delta^k_i - a_i^{n+1} \bar{u}^k) \frac{\partial \tilde{f}}{\partial \bar{u}^k} \partial \tilde{\varepsilon} - a_i^{n+1} \frac{\partial \tilde{f}}{\partial \bar{u}^k} \bar{u}^k = \frac{\partial \tilde{f}}{\partial \bar{u}^k} \partial \tilde{\varepsilon} (\delta^k_i - a_i^{n+1} \bar{u}^k)
\]
we have
\[
\{F, H\} = \int \frac{\partial \tilde{f}}{\partial \bar{u}^k} \tilde{P}^{kl} \frac{\partial \bar{h}}{\partial \bar{u}^l} d\tilde{x}
\]  
(22)
with
\[
\tilde{P}^{kl} = \partial \tilde{\varepsilon} (\delta^k_i - a_i^{n+1} \bar{u}^k) g^{ij} A(\delta^l_j - a_j^{n+1} \bar{u}^l) \partial \tilde{\varepsilon} = \partial \tilde{\varepsilon} \tilde{g}^{ij} \partial \tilde{\varepsilon}.
\]  
(23)
where \(\tilde{P}\) is again a local homogeneous operator of second order in view of Corollary 5. ■

### 2.2 Projective interpretation of the Hamiltonian operators

The action of the projective group on second-order HHOs allows us to classify such operators. Indeed, we exhibit a bijective correspondence of the leading term of the operator (in dimension \(n\)) with a projective 3-form (in dimension \(n+1\)). Such geometric objects are well-known in algebraic geometry [23] and there exist a classification in dimensions up to \(n+1 = 9\). Of course, we are interested in the even cases \(n = 2, 4, 6, 8\) due to the assumption \(\det(g) \neq 0\).

Let us set
\[
T_{n+1,jk} = -T_{jn+1,k} = T_{jk,n+1} = g_{jk}^0.
\]  
(24)
Then, we have a skewsymmetric indexed family \(T_{\lambda \mu \nu}\) with (greek) indices running from 1 to \(n+1\), extending \(T_{ijk}\) (recall that latin indices run from 1 to \(n\)). We have the following statement.
Lemma 7  A projective reciprocal transformation induces the transformation
\[ T_{\lambda \mu \nu} = \frac{1}{A^3} \tilde{T}_{\alpha \beta \gamma} a_{\lambda}^\alpha a_{\mu}^\beta a_{\nu}^\gamma. \]  

(25)

Thus, \( T_{\lambda \mu \nu} \) transforms as a tensor in \( \mathbb{C}^{n+1} \) up to a conformal factor.

Proof. It follows from
\[ T_{\text{lc}s} = \frac{1}{2A^3} \left( \tilde{T}_{ijk}(a_i^c a_j^e - a_i^c a_j^d) a_s^k + \tilde{T}_{ij\, n+1}(a_i^c a_j^e - a_i^c a_j^d) a_s^{n+1} \right. \\
- \left. \tilde{T}_{ij\, n+1}(a_i^c a_j^{n+1} - a_i^c a_j^{n+1}) a_s^j - \tilde{T}_{ij\, n+1}(a_i^{n+1} a_j^c - a_i^{n+1} a_j^c) a_s^j \right) \\
= \frac{1}{2A^3} \left( \tilde{T}_{i\nu}(a_i^c a_i^\nu) a_s^\nu + \tilde{T}_{i\, n+1}(a_i^{n+1} - a_i^{n+1}) a_s^k \right. \\
+ \left. \tilde{T}_{n+1\, jk}(a_i^{n+1} a_j^c - a_i^{n+1} a_j^c) a_s^k \right) \\
= \frac{1}{2A^3} \tilde{T}_{\lambda \mu \nu}(a_i^\lambda a_i^\mu - a_i^\lambda a_i^\mu) a_s^\nu \\
= \frac{1}{A^3} \tilde{T}_{\lambda \mu \nu} a_i^\lambda a_i^\mu a_s^\nu. \]

A similar proof holds for \( T_{\text{lc}\, n+1} = g_{lc}^0 \). □

In what follows we will identify three-forms \( \omega \in \wedge^3(\mathbb{C}^{n+1})^* \) on a vector space \( \mathbb{C}^{n+1} \) with maps of the form (see also [23] for more details)
\[ i(\omega) : \mathbb{C}^{n+1} \to \wedge^2(\mathbb{C}^{n+1})^*, \; v \mapsto \frac{1}{3} i_v(\omega). \]  

(26)

Clearly, the map \( \omega \mapsto i(\omega) \) is an isomorphism onto its image. If \( (v^i) \) are coordinates on \( \mathbb{C}^{n+1} \), then \( (dv^i) \) is a basis of \( (\mathbb{C}^{n+1})^* \) and the above isomorphism reads as
\[ \omega_{ijk} dv^i \wedge dv^j \wedge dv^k \mapsto \omega_{ijk} v^k dv^i \wedge dv^j. \]

(27)

Theorem 8  There is a bijective correspondence between leading coefficients of second order HHOs \( g = (T_{ijk} v^k + g_{ij}^0) dv^i \wedge dv^j \) as in (7), and three-forms \( \omega = \omega_{\lambda \mu \nu} dv^\lambda \wedge dv^\mu \wedge dv^\nu \). The bijective correspondence is preserved by projective reciprocal transformations up to a conformal factor.
Proof. Let us consider a three-form \( \omega = \omega_{\lambda \mu \nu} dv^\lambda \wedge dv^\mu \wedge dv^\nu \). Using the isomorphism (27) we can rewrite the form as

\[
i(\omega) = \omega_{\lambda \mu \nu} v^\nu dv^\lambda \wedge dv^\mu
\]

\[
= \omega_{\mu \nu} v^\nu dv^\mu + \omega_{n+1 \mu \nu} v^\nu dv^{n+1} \wedge dv^\mu
\]

\[
+ \omega_{\lambda \nu} v^\nu dv^\lambda \wedge dv^\nu + \omega_{\lambda n+1 \nu} v^\nu dv^{n+1} \wedge dv^{n+1}
\]

\[
+ \omega_{\lambda \mu} v^\lambda dv^\mu + \omega_{\lambda \mu n+1} v^{n+1} dv^\lambda \wedge dv^\mu
\]

\[
= \omega_{ij \nu} v^\nu dv^i \wedge dv^j + \omega_{i n+1 j} v^j dv^i \wedge dv^{n+1} + \omega_{i j n+1} v^{n+1} dv^i \wedge dv^j
\]

\[
+ \omega_{ij \mu} v^\mu dv^i \wedge dv^j + \omega_{i n+1 j} v^{n+1} dv^i \wedge dv^{n+1} + \omega_{i j n+1} v^{n+1} dv^i \wedge dv^j
\]

\[
+ \omega_{j \mu i} v^\mu dv^j \wedge dv^i + \omega_{j n+1 i} v^{n+1} dv^j \wedge dv^{n+1}
\]

\[
= 3 \omega_{ijk} v^k dv^i \wedge dv^j + 3 \omega_{ij n+1} v^{n+1} dv^i \wedge dv^j + 6 \omega_{ij n+1 i} v^i dv^j \wedge dv^{n+1}.
\]

Using the affine chart restriction \( v^{n+1} = 1, dv^{n+1} = 0 \) we obtain a second-order HHO by setting

\[
T_{ijk} = 3 \omega_{ijk} \quad \text{and} \quad g^0_{ij} = T_{ij n+1} = 3 \omega_{ij n+1}.
\] (28)

On the other hand, from a second-order HHO \( g \) as in the statement one can define the form in homogeneous coordinates

\[
G = (T_{ijk} v^k + g^0_{ij} v^{n+1}) dv^i \wedge dv^j.
\] (29)

Reversing the steps of the first part of the proof we get the desired three-form \( \omega \).

The fact that the correspondence is preserved by projective reciprocal transformation up to the conformal factor \( 1/A^3 \) follows from Lemma 7. □

There is an immediate and important consequence of the above Theorem.

Corollary 9 There is a bijective correspondence between homogeneous second order Hamiltonian operators in dimension \( n \) and three-forms in dimension \( n + 1 \). The bijective correspondence is preserved by projective reciprocal transformations.

At this point we observe two important facts:

- from a geometric viewpoint, second-order HHOs yield algebraic varieties using the corresponding three-forms and the mechanism explained in [23].
• from an algebraic viewpoint, second-order HHOs can be classified under the action of the projective reciprocal transformations by means of the classification of three-forms under the action of $SL(n + 1, \mathbb{C})$.

Let us first summarize the main features of the geometric properties of second-order HHOs. Our main source is [23]. Let $\omega$ be a three-form as above. A line $L$ in $\mathbb{C}^{n+1}$ can be identified with the skew-symmetric tensor $L = p^\lambda \partial/\partial v^\lambda \wedge \partial/\partial v^\mu$; $(p^\lambda)$ are the Plücker coordinates. The system

$$i_L \omega = 0 \iff \omega_{\lambda \mu \nu} p^{\mu \nu} = 0$$

(30)

is a system of $n + 1$ linear equations whose solutions constitute a linear subspace $\Lambda_\omega \subset \mathbb{P}(\wedge^2 \mathbb{C}^{n+1})$. If $\omega$ is a generic 3-form, then the intersection of $\Lambda_\omega$ with the Grassmannian $G$, $X_\omega = G \cap \Lambda_\omega$ is an $n - 1$-dimensional variety, i.e. it is a linear line congruence.

As a direct consequence of Theorem 8, the problem of classifying non-degenerate $n$-components second-order HHOs under the action of projective reciprocal transformations is solved by means of the classification of 3-forms in $\mathbb{C}^{n+1}$ under the action of the group $SL(n + 1, \mathbb{C})$. This is what will be exposed in next Section.

### 2.3 Projective classification of Hamiltonian operators

The following results are a direct consequence of the classification of 3-forms in $\mathbb{C}^{n+1}$ under the action of the group $SL(n + 1, \mathbb{C})$. Such a classification can be found in the book [15] for $n \leq 7$, while the case $n = 8$ is covered in [31]. It should be remarked that the latter paper presents the classification of trivectors in dimension 9, i.e. elements of $\wedge^3 \mathbb{C}^9$, under the natural action of $SL(n + 1, \mathbb{C})$. It is easy to realize that the classification of 3-forms (i.e. the set of orbits) is put in bijective correspondence with the classification of trivectors by any isomorphism, for example, the correspondence defined by the passage from a basis to its dual $e_i \mapsto e^i$.

**The case $n = 2$.** There is only one (nontrivial) 3-form, namely $\omega = dv^1 \wedge dv^2 \wedge dv^3$. We can rewrite it as

$$i(\omega) = \frac{1}{3} (v^1 dv^2 \wedge dv^3 - v^2 dv^3 \wedge dv^1 + v^3 dv^1 \wedge dv^2)$$

(31)
The affine projection \( v^3 = 1, dv^3 = 0 \), yields, up to a factor, the leading coefficient \( du^1 \wedge du^2 \) of the second-order HHO
\[
R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \partial_x^2. \tag{32}
\]

The case \( n = 4 \). There are two (nontrivial) orbits. The open orbit is generated by
\[
\omega = dv^5 \wedge (dv^1 \wedge dv^2 + dv^3 \wedge dv^4), \tag{33}
\]
that corresponds to the leading coefficient \( du^1 \wedge du^2 + du^3 \wedge du^4 \) of the second-order HHO
\[
R = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \partial_x^2. \tag{34}
\]
the closed orbit is totally decomposable and generated by
\[
\omega = du^1 \wedge dv^2 \wedge dv^3; \tag{35}
\]
the corresponding leading coefficient is degenerate: \( \det(g_{ij}) = 0 \).

The case \( n = 6 \). The classification in this case is due to Schouten (see [15]). There are nine nontrivial orbits. We list below the generators of the orbits which lead to a non-degenerate 2-form \( i(\omega) \).

1. The open orbit is generated by
\[
\omega = dv^1 \wedge dv^2 \wedge dv^3 + dv^4 \wedge dv^5 \wedge dv^6 \\
+ dv^7 \wedge (dv^1 \wedge dv^4 + dv^2 \wedge dv^5 + dv^3 \wedge dv^6). \tag{36}
\]
(case X in [15]). By using the map \( i(\omega) \):

\[
i(\omega) = \frac{1}{3}(v^1 dv^2 \wedge dv^3 - v^2 dv^1 \wedge dv^3 + v^3 dv^1 \wedge dv^2 + v^4 dv^5 \wedge dv^6 - v^5 dv^4 \wedge dv^6 + v^6 dv^4 \wedge dv^5 + v^7 dv^1 \wedge dv^4 - v^1 dv^7 \wedge dv^4 + v^4 dv^7 \wedge dv^1 + v^7 dv^2 \wedge dv^5 - v^2 dv^7 \wedge dv^5 + v^5 dv^7 \wedge dv^2 + v^7 dv^3 \wedge dv^6 - v^3 dv^7 \wedge dv^6 + v^6 dv^7 \wedge dv^3) \tag{37}
\]
Then with the affine projection \( v^7 = 1, dv^7 = 0 \):

\[
\begin{align*}
\omega &= \frac{1}{3}(v^3 dv^1 \wedge dv^2 - v^2 dv^1 \wedge dv^3 + v^3 dv^1 \wedge dv^2 + v^4 dv^5 \wedge dv^6 - v^5 dv^4 \wedge dv^6 + v^6 dv^4 \wedge dv^5 + dv^1 \wedge dv^4 + dv^2 \wedge dv^5 + dv^3 \wedge dv^6) \quad (42) \\
&= \frac{1}{3}(v^3 dv^1 \wedge dv^2 - v^2 dv^1 \wedge dv^3 + v^3 dv^1 \wedge dv^2) \quad (43) \\
&= (42) + (43) + (44)
\end{align*}
\]

Then, the associated 2-form is (up to a factor)

\[
g^1_{ij} = 
\begin{pmatrix}
0 & v^3 & -v^2 & 1 & 0 & 0 \\
-v^3 & 0 & v^1 & 0 & 1 & 0 \\
v^2 & -v^1 & 0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 & v^6 & -v^5 \\
0 & -1 & 0 & -v^6 & 0 & v^4 \\
0 & 0 & -1 & v^5 & -v^4 & 0
\end{pmatrix} \quad (45)
\]

and \( \det(g^1_{ij}) = (v^1 v^4 + v^2 v^5 + v^3 v^6 - 1)^2 \).

2. We have the 3-form

\[
\omega = dv^1 \wedge dv^2 \wedge dv^3 + dv^4 \wedge dv^5 \wedge dv^6 + (dv^1 \wedge dv^4 + dv^2 \wedge dv^5) \wedge dv^7 \quad (46)
\]

(case IX in [15]). In the affine chart (removing the factor 1/3),

\[
\begin{align*}
3i(\omega) &= v^1 dv^2 \wedge dv^3 - v^2 dv^1 \wedge dv^3 + v^3 dv^1 \wedge dv^2 + v^4 dv^5 \wedge dv^6 - v^5 dv^4 \wedge dv^6 + v^6 dv^4 \wedge dv^5 + dv^1 \wedge dv^4 + dv^2 \wedge dv^5 \quad (47) \\
&= (48) + (49)
\end{align*}
\]

The leading coefficient of the associated operator is

\[
g^2_{ij} = 
\begin{pmatrix}
0 & v^3 & -v^2 & 1 & 0 & 0 \\
-v^3 & 0 & v^1 & 0 & 1 & 0 \\
v^2 & -v^1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & v^6 & -v^5 \\
0 & -1 & 0 & -v^6 & 0 & v^4 \\
0 & 0 & 0 & v^5 & -v^4 & 0
\end{pmatrix} \quad (50)
\]

we have \( \det(g^2_{ij}) = (v^1 v^4 + v^2 v^5)^2 \).
3. We have the 3-form

$$\omega = dv^1 \wedge dv^2 \wedge dv^3 + dv^4 \wedge dv^5 \wedge dv^6 + dv^1 \wedge dv^4 \wedge dv^7$$  \hfill (51)

(case VIII in [15]). In the affine chart,

$$3i(\omega) = v^1 dv^2 \wedge dv^3 - v^2 dv^1 \wedge dv^3 + v^3 dv^1 \wedge dv^2$$
$$+ v^4 dv^5 \wedge dv^6 - v^5 dv^4 \wedge dv^6 + v^6 dv^4 \wedge dv^5 + dv^1 \wedge dv^4 \wedge dv^7$$  \hfill (52)

The leading coefficient of the associated operator is

$$g^3_{ij} = \begin{pmatrix}
0 & v^3 & -v^2 & 1 & 0 & 0 \\
-v^3 & 0 & v^1 & 0 & 0 & 0 \\
v^2 & -v^1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & v^6 & -v^5 & 0 \\
0 & 0 & 0 & -v^6 & 0 & v^4 \\
0 & 0 & 0 & v^5 & -v^4 & 0
\end{pmatrix}$$  \hfill (53)

we have $$\det(g^3_{ij}) = (v^1 v^4)^2$$.

4. We have the 3-form

$$\omega = dv^4 \wedge dv^5 \wedge dv^6 + dv^7 (du^1 \wedge dv^4 + dv^2 \wedge dv^5 + dv^3 \wedge dv^6)$$  \hfill (54)

(case VII in [15]). In the affine chart,

$$3i(\omega) = v^4 dv^5 \wedge dv^6 - v^5 dv^4 \wedge dv^6 + v^6 dv^4 \wedge dv^5$$
$$+ du^1 \wedge dv^4 + dv^2 \wedge dv^5 + dv^3 \wedge dv^6$$  \hfill (55)

The leading coefficient of the associated operator is

$$g^4_{ij} = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
-1 & 0 & 0 & v^6 & -v^5 & 0 \\
0 & -1 & 0 & -v^6 & 0 & v^4 \\
0 & 0 & -1 & v^5 & -v^4 & 0
\end{pmatrix}$$  \hfill (56)

we have $$\det(g^4_{ij}) = 1$$.  

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5. We have the 3-form
\[ \omega = dv^7 \wedge (dv^1 \wedge dv^4 + dv^2 \wedge dv^5 + dv^3 \wedge dv^6) \] (57)

(case VI in [15]). In the affine chart we have,
\[ 3i(\omega) = dv^1 \wedge dv^4 + dv^2 \wedge dv^5 + dv^3 \wedge dv^6 \] (58)

The leading coefficient of the associated operator is
\[
g^6_{ij} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} \] (59)

we have \( \det(g^6_{ij}) = 1. \)

The case \( n = 8. \) We will follow the classification of trivectors in dimension 9 [31]. We will use the isomorphism between \( \mathbb{C}^{n+1} \) and \( (\mathbb{C}^{n+1})^* \) defined by a basis and its dual in order to put trivectors and 3-forms into correspondence. We recall that a trivector is said to be *semisimple* if its equivalence class is closed in the space of all trivectors, whereas it is said to be *nilpotent* if the closure of this class contains the zero form. Every trivector \( u \) can be uniquely written as the sum \( u = p + e \), where \( p \) is a semisimple trivector and \( e \) is a nilpotent trivector such that \( p \wedge e = 0. \)

Semisimple trivectors \( p \) are divided into seven different families for each of which all possible nilpotent parts are provided. Let us introduce the following 3-forms:
\[
p_1 = dv^1 \wedge dv^2 \wedge dv^3 + dv^4 \wedge dv^5 \wedge dv^6 + dv^7 \wedge dv^8 \wedge dv^9 \] (60)
\[
p_2 = dv^1 \wedge dv^4 \wedge dv^7 + dv^2 \wedge dv^5 \wedge dv^8 + dv^3 \wedge dv^6 \wedge dv^9 \] (61)
\[
p_3 = dv^1 \wedge dv^5 \wedge dv^9 + dv^2 \wedge dv^6 \wedge dv^7 + dv^3 \wedge dv^4 \wedge dv^8 \] (62)
\[
p_4 = dv^1 \wedge dv^6 \wedge dv^8 + dv^2 \wedge dv^4 \wedge dv^9 + dv^3 \wedge dv^5 \wedge dv^7 \] (63)

Every semisimple trivector is equivalent to a trivector whose corresponding 3-form is of the type
\[
p = \lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \lambda_4 p_4, \] (64)
where the coefficients are determined up to a linear transformation from a group generated by complex reflections of order 3 \([31]\).

The first family of 3-forms is generated by \(p\) only as in \((64)\); more precisely, it consists only of semisimple trivectors \((e = 0)\). The coefficients \(\lambda_i\) must satisfy a complicated system of algebraic inequalities \([31]\). The stabilizer subgroup \(S\) of this class is a cyclic Abelian group of order 81. The corresponding non-degenerate 2-form in this class is

\[
g^{(1)}_{ij} = \begin{pmatrix}
0 & \lambda_1 v^3 & -\lambda_1 v^2 & \lambda_2 v^7 & \lambda_3 & \lambda_4 v^8 & -\lambda_2 v^4 & -\lambda_4 v^6 \\
-\lambda_1 v^3 & 0 & \lambda_1 v^1 & \lambda_4 & \lambda_2 v^8 & \lambda_3 v^7 & -\lambda_3 v^6 & -\lambda_2 v^5 \\
\lambda_1 v^2 & -\lambda_1 v^1 & 0 & \lambda_3 v^8 & \lambda_4 v^7 & \lambda_2 & -\lambda_4 v^5 & -\lambda_3 v^4 \\
-\lambda_2 v^7 & -\lambda_4 & -\lambda_3 v^8 & 0 & \lambda_1 v^6 & -\lambda_1 v^5 & \lambda_2 v^3 & \lambda_3 v^3 \\
-\lambda_3 & -\lambda_2 v^8 & -\lambda_4 v^7 & -\lambda_1 v^6 & 0 & \lambda_1 v^4 & \lambda_4 v^3 & \lambda_2 v^2 \\
-\lambda_4 v^8 & -\lambda_3 v^7 & -\lambda_2 & \lambda_1 v^5 & -\lambda_1 v^4 & 0 & \lambda_3 v^2 & \lambda_4 v^1 \\
\lambda_2 v^4 & \lambda_3 v^5 & \lambda_4 v^5 & -\lambda_2 v^1 & -\lambda_4 v^3 & -\lambda_3 v^2 & 0 & \lambda_1 \\
\lambda_4 v^6 & \lambda_2 v^5 & \lambda_3 v^4 & -\lambda_3 v^3 & -\lambda_2 v^2 & -\lambda_4 v^1 & -\lambda_1 & 0
\end{pmatrix}
\]

\((65)\)

The second family is generated by the semisimple trivector

\[p = \lambda_1 p_1 + \lambda_2 p_2 - \lambda_3 p_3,\]

again with \(\lambda_i\) fulfilling an algebraic constraint. The coefficients are determined up to a linear transformation generated by complex reflections. The possible nontrivial nilpotent parts are two:

\[e_1 = dv^1 \wedge dv^6 \wedge dv^8 + dv^2 \wedge dv^4 \wedge dv^9\]
\[e_2 = dv^1 \wedge dv^6 \wedge dv^8\]

Here, the dimension of the stabilizer \(S\) is 0 for \(e_1\) and 1 for \(e_2\). By summing \(p + e_i\) and applying the correspondence, we finally obtain the following two 2-forms:

\[
g^{(2)}_{ij} = \begin{pmatrix}
0 & \lambda_1 u^3 & -\lambda_1 u^2 & \lambda_2 u^7 & -\lambda_3 & u^8 & -\lambda_2 u^4 & -u^6 \\
-\lambda_1 u^3 & 0 & \lambda_1 u^1 & 1 & \lambda_2 u^8 & -\lambda_3 v^7 & \lambda_3 u^6 & -\lambda_2 u^5 \\
\lambda_1 u^2 & -\lambda_1 u^1 & 0 & -\lambda_3 u^8 & 0 & \lambda_2 & 0 & \lambda_3 u^4 \\
-\lambda_2 v^7 & -1 & \lambda_3 u^8 & 0 & \lambda_1 u^6 & -\lambda_1 u^5 & \lambda_2 u^4 & -\lambda_3 u^3 \\
\lambda_3 & -\lambda_2 u^8 & 0 & -\lambda_1 u^6 & 0 & \lambda_1 u^4 & 0 & \lambda_2 u^2 \\
-u^8 & \lambda_3 u^7 & -\lambda_2 & \lambda_1 u^5 & -\lambda_1 u^4 & 0 & -\lambda_3 u^2 & u^1 \\
\lambda_2 u^4 & -\lambda_3 u^6 & 0 & -\lambda_2 u^1 & 0 & \lambda_3 u^2 & 0 & \lambda_1 \\
u^6 & \lambda_3 u^5 & -\lambda_3 u^4 & \lambda_3 u^3 & -\lambda_2 u^2 & -u^1 & -\lambda_1 & 0
\end{pmatrix}
\]

\((69)\)
In both cases the determinants are non-zero.

The total number of non-degenerate two-forms in the classification is 132.

For reasons of space, we will not list elements in the families 3–7; however, we are ready to privately provide the list of non-degenerate two-forms to the interested reader.

**Remark 10** The dimension of the space of 3-forms $\wedge^3(\mathbb{C}^{n+1})^*$ grows with the dimension $n$ in a much faster way than the dimension of $SL(\mathbb{C}^{n+1})$. However, for small values of $n$ the dimension of the group is prevailing: this is the reason for the lack of non-trivial classes when $n \leq 4$. The same argument shows that a classification for higher values of $n$ does not make sense, in view of the huge number of free parameters that the generic element would depend on.

## 3 Systems of PDEs with second-order Hamiltonian structure

In [30, Theorem 10] it was proved that the necessary conditions for a second-order HHO $P \ (6)$ to be a Hamiltonian operator for a quasilinear system of first-order conservation laws (8) are

\[
\begin{align*}
g_{qj} V^j_p + g_{pj} V^j_q &= 0, \\
g_{qk} V^k_{pl} + g_{pq,k} V^k_j + g_{qk,l} V^k_p &= 0.
\end{align*}
\]

This result is analogue to the results in [27] concerning first-order HHOs and quasilinear systems of first-order PDEs, and is obtained by a method introduced in [17] and later adapted to HHOs [13].

We observe that the above conditions have no direct differential-geometric interpretation as they are derived in flat coordinates of the connection $\Gamma^i_{jk}$ (see the Introduction). However, we will be able to parametrize the space of
solutions of the above equations, thus exhibiting large families of systems of PDEs that are Hamiltonian with respect to second-order Hamiltonian operators. Interesting properties of such systems will be thoroughly investigated.

3.1 Solution of the compatibility conditions

We will now solve completely the system of compatibility conditions between a quasilinear system of first-order PDEs (71). We will first prove that the system is in involution, then we will parametrize its solutions.

Proposition 11 The system (71) is in involution.

Proof. Let us derive (11):

\[ g_{qj,l}V_{pj} + g_{qj}V_{pj,l} + g_{pq,j}V_{pj} = 0 \]  

then, by using condition (71b) we can substitute

\[ g_{qj}V_{pj} + g_{pq,j}V_{pj} = -g_{pq,j}V_{j} \]  
\[ g_{pq,k}V_{j} + g_{pq,j}V_{k} = -g_{pq,j}V_{k} \]  

in (72), which yields

\[ -g_{pq,j}V_{j} - g_{pq,j}V_{j} = 0, \]

which is an identity.

The condition (72) can be rewritten as

\[ (g_{qk}V_{jk})_{t} + g_{pq,k}V_{jk} = 0. \]

From the consistency condition \( V_{plm} = V_{pml} \) we obtain

\[ (g_{qk}V_{jk})_{tm} + (g_{pq,k}V_{jk})_{tm} = (g_{qk}V_{jk})_{tm} + (g_{pq,k}V_{jm})_{t} \]

which yields the identity \( g_{pq,k}V_{jkm} = g_{pq,k}V_{jml} \) in view of \( g_{pq,k} = 0 \).

The above Proposition shows that, since (72) expresses all second-order derivatives, the general solution of the system depends on no more than \( n + n^2 \) parameters. The equations (11) impose further \( n(n - 1)/2 \) additional restrictions, so that the total number of arbitrary constants in the general solution is

\[ n + n^2 - \frac{n(n - 1)}{2} = n(n + 3)/2. \]

Now, we will solve the system (71).
Theorem 12 Let \( C \) be a second-order HHO. Then, the (explicit) solution of the system (71) is the vector \( V^i \) given by

\[
V^i = g^{ij}W_j, \tag{79}
\]

where \( W_j \) is the covector

\[
W_j = A_j u^l + B_j \tag{80}
\]

where \( A_{ij} = -A_{ji}, \) \( B_i \) are arbitrary constants.

Proof. 

\[
(W_j)_{ab} = g_{jk}V^k_{,ab} + g_{jk,b}V^k_a + g_{jk,a}V^k_b + g_{jk,ab}V^k \tag{81}
\]

Since \( g_{jk,ab} = 0 \) and \( g_{bj,k} = g_{j,k,b} \) the above equation becomes the right-hand side of (71b). Then \( W_j = A_j u^l + B_j. \) Moreover, we have the following identity:

\[
W_{j,p} + W_{p,j} = g_{jl}V^l_{,p} + g_{pl}V^l_j \tag{82}
\]

which is the right-hand side of (11). But we have

\[
W_{j,p} + W_{p,j} = A_{jp} + A_{pj}, \tag{83}
\]

which completes the proof. ■

Remark 13 The above solution of the system (71) is the most general: indeed, \( W_i \) depends on \( n(n + 1)/2 \) arbitrary constants, and \( V^i \) is defined up to \( n \) arbitrary constants (as it enters the right-hand side of (8)). The total figure is equal to the dimension count following the proof of Proposition 11.

Corollary 14 The fluxes \( V^i \) are rational functions of the form:

\[
V^i = \frac{Q}{\text{Pf}(g)} \tag{84}
\]

where \( Q \) is a polynomial of degree \( n/2 \) and the denominator is \( \text{Pf}(g). \)

Proof. We have \( V^i = g^{is}W_s, \) where \( g^{ij} \) is the inverse matrix of \( g_{ij}. \) By means of properties of the determinant of skew-symmetric matrices [22] the inverse matrix has rational functions entries where the numerator has degree \( (n - 2)/2 \) and the denominator is the Pfaffian of \( g_{ij}, \) whose degree is at most \( n/2. \) Since \( W_s \) are linear functions, the statement follows. ■
**Corollary 15** The eigenvalues of the matrix $V_j^i$ have algebraic multiplicity 2.

**Proof.** The eigenvalues are computed by the characteristic polynomial: $\det(V_j^i - \lambda \delta_j^i)$. Lowering one index we obtain a skew-symmetric matrix:

$$g_{hi}(V_j^i - \lambda \delta_j^i) = g_{hi}(g^{ik}W_k)_j - \lambda g_{hj}$$

$$= -g_{hi,j}g^{ik}W_k + g_{hi}g^{ik}A_{kj} - \lambda g_{hj}$$

$$= T_{hij}g^{ik}W_k + A_{hj} - \lambda g_{hj}$$

whose determinant is the square of its Pfaffian. Since $\det(g_{ij})$ is also a perfect square we get the result. ■

**Remark 16** The above result has important consequences on the integrability of the system (8). Indeed, in [28] the generalized hodograph method for the solution of semi-Hamiltonian quasilinear first-order systems is developed. However, one of the hypothesis in the above paper is that all eigenvalues of $V_j^i$ are distinct. We will see in Subsection 3.4 that, at least in some cases, this does not prevent the semi-Hamiltonianity of the systems determined by second-order HHOs.

### 3.2 Projective geometry of the systems of PDEs

Let us recall that for every hydrodynamic-type system

$$u_t^i = V_j^i(u)u_x^j$$

it is possible to associate a congruence

$$y^i = u^iy^{n+1} + V^iy^{n+2}$$

in auxiliary projective space $\mathbb{P}^{n+1}$ with homogeneous coordinates $(y^1 : \cdots : y^{n+2})$.

**Proposition 17** Let $u_t^i = V_j^i u_x^j$ be a system compatible with a second order HHO. Then the associated congruence is linear.
Proof. By the previous Lemma we obtain that
\[ A_{jl}u^l + B_j = (T_{jkl}u^l + g^0_{jk})V^k \] (87)
then
\[ \frac{1}{2}T_{jkl}(u^lV^k - u^kV^l) + g^0_{jk}V^k = A_{jl}u^l + B_j \] (88)
This yields a system of \( n \) linear relations between Plücker’s coordinates describing the line congruence, hence the statement is proved. ■

Corollary 18 The first-order quasilinear systems of PDEs compatible with second-order HHOs are linearly degenerate.

Proof. It is a straightforward consequence of the linearity of the corresponding congruence of lines [1]. ■

Remark 19 The fact that the associated congruence is linear does not imply that the system is non-diagonalizable (see the Remark in [2, p. 1771]).

We can have a look at the case \( n = 2 \) (already considered in [30]) using the theory of congruences.

Proposition 20 For \( n = 2 \) the system \( V^i_j \) compatible with a second order HHO is linearisable.

Proof. Let \( n = 2 \). Then in \( \mathbb{P}^3 \) linear congruences can be brought (modulo projective transformations) to the form
\[ y^1 = u^2y^3 + u^2y^3 \quad y^2 = u^2y^3 + u^1y^4 \] (89)
By using the affine chart \( y^4 = 1 \) we have only two cases \((y^1 = y^2, y^3 = 1)\) or \((y^1 = -y^2, y^3 = -1)\). But by condition \((71a)\) we obtain that the system is skew symmetric, then
\[ \begin{cases} u^1_t = u^2_x \\ u^2_t = -u^1_x \end{cases} \] (90)
In particular, every system \( u^i_t = (V^i)_x \) compatible with a second order operator is linearisable. ■

We stress that an obvious alternative proof immediately follows from the classification of second-order HHOs in Subsection 2.3. The same argument yields the following Proposition.

Proposition 21 In the case \( n = 4 \) systems of quasilinear first-order conservation laws that admit a second-order HHO are linearisable by projective reciprocal transformations.
3.3 Hamiltonian systems

We would like to find an Hamiltonian for the systems that we found in Section 3.1. To this aim we observe that, in potential coordinates $u^i = b^i_x$, the second-order HHOs (6) in potential coordinates becomes the ultralocal operator $P^{ij} = -g^{ij}(b_k^x)$, and the system of first-order conservation laws (8) becomes $b^i_t = V^i(b_k^x)$. Solving with respect to $H$ the equation:

$$-g^{ik} \frac{\delta H}{\delta b_k} = V^i,$$

or

$$\frac{\delta H}{\delta b_k} = -A_{ks} b_s^x - B_k,$$

yields the following result.

Proposition 22 We have

$$H = -\int \left( \frac{1}{2} A_{sl} b_s^l + B_s \right) b^s dx$$

(93)

Remark 23 At difference with the third-order case [13] we observe that there are no non-trivial nonlocal Casimirs, as the equation

$$-g^{ik} \frac{\delta F^j}{\delta b_k} = 0$$

(94)

has no non-trivial solutions.

3.4 The Haantjes tensor and integrability

For general quasilinear first-order systems of the type $u^i_t = V^i_j(u^k)u^j_x$ it is known [16] that the Haantjes tensor

$$H^i_{jk} = N^i_{pr} V^p_j V^r_k - N^p_{jr} V^i_p V^r_k - N^p_{rk} V^i_p V^r_j + N^p_{jk} V^i_r V^r_p,$$

(95)

where $N^i_{jk}$ is the Nijenhuis tensor

$$N^i_{jk} = V^p_j V^i_{kp} - V^p_k V^i_{jp} - V^i_p (V^p_k - V^p_j),$$

(96)

is related with the diagonalizability of the quasilinear system of PDEs (8). More precisely, if $H^i_{jk} = 0$ and the eigenvalues of the velocity matrix $V^i_j$
have the same algebraic multiplicity as their geometric multiplicity, then the system of PDEs is diagonalizable.

It is possible to compute the Haantjes tensor for our conservative systems (8). We have the following result.

**Theorem 24** The Haantjes tensor of a conservative quasilinear system (8) that admits a second-order homogeneous Hamiltonian operator is identically vanishing.

**Proof.** It is easy to prove the following identity

\[ g^i_k g^j_l = -g^i_l T_{ajk} \]  \hspace{1cm} (97)

Then, from (71b) we have

\[ V^{a}_{pl} = -g^{aq}(T_{pqk}V^k_l + T_{qkl}V^k_p) \]

hence the Nijenhuis tensor can be written as

\[ N^i_{jk} = g^i (T_{jal}V^l_p V^p_k - T_{kal}V^l_k V^p_l - 2T_{alp}V^l_l V^p_p) . \] \hspace{1cm} (98)

Using (98) we obtain

\[ H^i_{kj} = g^i (T_{pal}V^l_s V^p_r - T_{ral}V^l_s V^p_r - 2T_{als}V^l_p V^p_s) V^p_l V^r_j + \]

\[ -g^{pa} (T_{kal}V^l_v V^p_s - T_{ral}V^l_v V^p_k - 2T_{als}V^l_s V^p_k) V^p_l V^r_j + \]

\[ -g^{pa} (T_{ral}V^l_s V^p_j - T_{jal}V^l_s V^p_r - 2T_{als}V^l_j V^p_r) V^p_l V^r_k + \]

\[ +g^{pa} (T_{kal}V^l_v V^p_s - T_{jal}V^l_v V^p_k - 2T_{als}V^l_j V^p_k) V^p_l V^r_p \] \hspace{1cm} (99)

Let us consider, for example, the summand

\[ S = -2g^i T_{als}V^l_s V^p_k V^p_l V^r_j + 2g^{pa} T_{als}V^l_s V^p_k V^p_l V^r_p . \] \hspace{1cm} (100)

It is clear that \( S = 0 \) if \(-2g^i T_{als}V^l_s V^p_k V^p_l V^r_j + 2g^{pa} T_{als}V^l_s V^p_k V^p_l V^r_p = 0\). Now, we use the identities (97), (11) and the upper indices version \( g^{il} V^j_l + g^{jl} V^i_l = 0 \) to prove that \( g^i T_{als}V^l_s V^p_k V^p_l V^r_j = 0 \), so that \( S = 0 \).

With similar algebraic manipulations it is easy to prove that the following pairs of summands annihilate:

\[ + 2g^{pa} T_{als}V^l_s V^p_k V^p_l V^r_j - 2g^i T_{als}V^l_s V^p_k V^p_l V^r_p = 0, \] \hspace{1cm} (101)

\[ + g^{ia} T_{pal}V^l_s V^p_k V^p_l V^r_j - g^{pa} T_{kal}V^l_s V^p_k V^p_l V^r_j = 0, \] \hspace{1cm} (102)

\[ - g^{ia} T_{pal}V^l_s V^p_k V^p_l V^r_j + g^{pa} T_{kal}V^l_s V^p_k V^p_l V^r_j = 0, \] \hspace{1cm} (103)

\[ + g^{pa} T_{kal}V^l_s V^p_k V^p_l V^r_j - g^{pa} T_{ral}V^l_s V^p_k V^p_l V^r_j = 0, \] \hspace{1cm} (104)

\[ - g^{pa} T_{jal}V^l_s V^p_k V^p_l V^r_j + g^{pa} T_{ral}V^l_s V^p_k V^p_l V^r_j = 0. \] \hspace{1cm} (105)
Summarizing, we have obtained that each second-order HHO defines a multiparametric family of systems of conservation laws which is completely characterized by Theorem 12. Since the velocity matrix of the systems always has double eigenvalues in a generic situation (Corollary 15), according to Haantjes’ Theorem [16] the systems are diagonalizable if and only if there exist a two-dimensional subspace of eigenvectors corresponding to every eigenvalue.

Unfortunately, we were not able to provide a generic proof on the diagonalizability of the above systems. However, we can summarize the situation in the following list.

\( n = 2 \) All systems are linear: there is only one (non-degenerate) second-order HHO.

\( n = 4 \) All systems are linearizable by a projective reciprocal transformation, as the only nondegenerate representative of the second-order HHO in the classification is a constant coefficient 2-form.

\( n = 6 \) Computational experiments show that for every equivalence class in the classification an explicit but random choice of the coefficients that determine the system leads to a diagonalizable velocity matrix.

\( n = 8 \) Again, computational experiments performed on randomly chosen members of the third family show that they are diagonalizable. More generic families need a more sophisticated computational approach in order to tame their complexity.

Given the results of the above computational experiments, we can conjecture that all systems corresponding to second-order HHOs are diagonalizable.

If the above conjecture is true, it turns out that all the above systems are semi-Hamiltonians, as they are conservative and diagonalizable, by a theorem in [25]. This already implies that such systems are integrable, as they have infinitely many commuting flows (symmetries).

However, strictly speaking we cannot use the results of [28] and conclude that the systems can be solved by the generalized hodograph method, as it was developed for semi-Hamiltonian systems whose velocity matrix has distinct eigenvalues.

Despite the lack of general results on systems of our type, again computational experiments show that we can solve randomly chosen systems
corresponding to second-order HHOs by the generalized hodograph method. So, it seems very reasonable to make a stronger conjecture that systems corresponding to second-order HHOs are integrable.

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