Demiański–Newman Solution Revisited

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Abstract

The derivation of the Demiański–Newman solution within the framework of the Ernst complex formalism is considered. We show that this solution naturally arises as a two–soliton specialization of the axisymmetric multi–soliton electrovacuum metric, and we work out the full set of the corresponding metrical fields and electromagnetic potentials. Some limits and physical characteristics of the DN space–time are briefly discussed.

1 Introduction

The Demiański–Newman (DN) metric [1] is a five–parameter stationary axisymmetric solution of the Einstein–Maxwell equations which generalizes the well–known Kerr–Newman spacetime [2], its two additional parameters being the gravitomagnetic and magnetic monopoles; it is in turn a special case of some more general metrics [3] widely discussed in the literature. The DN metric was originally derived with the aid of the complex coordinate transformation procedure, so it seems likely to have its representation within the framework of the Ernst formalism [4] because this solution turns out to be the most general two–soliton specialization of the electrovacuum multi–soliton metric [5], constituting a basic element of the stationary systems of aligned charged, magnetized, spinning particles. In Section 2 we shall obtain the ‘canonical’ form of the Ernst complex
potentials for the DN solution involving the analytically extended parameter set, the corresponding metric functions and electromagnetic potentials. In Section 3 we shall discuss some properties of the DN spacetime. Section 4 contains concluding remarks.

2 The Ernst potentials of the DN solution, the corresponding metric functions and electromagnetic potentials

The Ernst complex potentials $\mathcal{E}$ and $\Phi$ [4] defining the DN solution arise from the axis data of the form

$$
\mathcal{E}(\rho = 0, z) = \frac{z - m - i(a + \nu)}{z + m - i(a - \nu)}, \quad \Phi(\rho = 0, z) = \frac{q + ib}{z + m - i(a - \nu)},
$$

(1)

where $\rho$ and $z$ are the Weyl–Papapetrou cylindrical coordinates, and the five arbitrary parameters $m, a, \nu, q, b$ are the total mass, total angular momentum per unit mass, gravitomagnetic monopole (NUT parameter [6]), electric and magnetic charges, respectively. This interpretation of the parameters follows from the consideration of the Simon multipole moments [7] corresponding to the axis data (1) with the aid of the Hoenselaers–Perjes procedure [8] which gives the following expressions for the first four moments ($M_i, J_i, Q_i, B_i$ stand, respectively, for the mass, angular momentum, electric and magnetic moments):

$$
M_0 = m, \quad M_1 = -\nu a, \quad M_2 = -ma^2, \quad M_3 = \nu a^3,
$$

$$
J_0 = \nu, \quad J_1 = ma, \quad J_2 = -\nu a^2, \quad J_3 = -ma^3,
$$

$$
Q_0 = q, \quad Q_1 = -ba, \quad Q_2 = -qa^2, \quad Q_3 = ba^3,
$$

$$
B_0 = b, \quad B_1 = qa, \quad B_2 = -ba^2, \quad B_3 = -qa^3.
$$

(2)

The axis data (1) is the $N = 1$ specialization of the axis expressions of the electrovacuum axisymmetric solution [3] obtained with the aid of Sibgatullin’s method [3], so we can use the results of the paper [4] for writing out the potentials
\( \mathcal{E}(\rho, z), \Phi(\rho, z) \) and all the metric fields of the DN solution. Then for the potentials \( \mathcal{E} \) and \( \Phi \) we have

\[
\mathcal{E} = E_+/E_-, \quad \Phi = F/E_-,
\]

\[
E_\pm = \begin{vmatrix}
1 & 1 & 1 \\
\pm 1 & \frac{r_1}{\alpha_1 - \beta} & \frac{r_2}{\alpha_2 - \beta} \\
0 & \frac{h(\alpha_1)}{\alpha_1 - \beta} & \frac{h(\alpha_2)}{\alpha_2 - \beta}
\end{vmatrix}, \quad F = \begin{vmatrix}
f(\alpha_1) & f(\alpha_2) \\
h(\alpha_1) & h(\alpha_2)
\end{vmatrix}, \quad (3)
\]

where \( r_n \equiv \sqrt{\rho^2 + (z - \alpha_n)^2} \) (a bar over a symbol means complex conjugation), and for the definitions of \( \alpha_n, \beta, h(\alpha_n) \) and \( f(\alpha_n) \) we refer to [5]. The corresponding metric functions \( f, \gamma \) and \( \omega \) entering the axisymmetric ‘canonical’ line element

\[
ds^2 = f^{-1} \left[ e^{2\gamma} (d\rho^2 + dz^2) + \rho^2 d\varphi^2 \right] - f(dt - \omega d\varphi)^2, \quad (4)
\]

are given by the expressions

\[
f = \frac{E_+ E_- + \tilde{E}_+ E_- + 2F\tilde{F}}{2E_+ E_-}, \quad e^{2\gamma} = \frac{E_+ \tilde{E}_- + \tilde{E}_+ E_- + 2F\tilde{F}}{2K_0 K_0 r_1 r_2},
\]

\[
\omega = \frac{2\text{Im}\{E_- \tilde{H} - \tilde{E}_- G - F\tilde{F}\}}{E_+ E_- + \tilde{E}_+ E_- + 2F\tilde{F}},
\]

\[
G = \begin{vmatrix}
r_1 + \alpha_1 - z & r_2 + \alpha_2 - z \\
h(\alpha_1) & h(\alpha_2)
\end{vmatrix}, \quad H = \begin{vmatrix}
z & 1 & 1 \\
-\beta & \frac{r_1}{\alpha_1 - \beta} & \frac{r_2}{\alpha_2 - \beta}
\end{vmatrix}, \quad (5)
\]

\[
I = \begin{vmatrix}
f_1 & 0 & f(\alpha_1) & f(\alpha_2) \\
z & 1 & 1 & 1 \\
-\beta & -1 & \frac{r_1}{\alpha_1 - \beta} & \frac{r_2}{\alpha_2 - \beta} \\
\bar{e} & 0 & \frac{h(\alpha_1)}{\alpha_1 - \beta} & \frac{h(\alpha_2)}{\alpha_2 - \beta}
\end{vmatrix}, \quad K_0 = \begin{vmatrix}
1 & 1 \frac{1}{\alpha_1 - \beta} & \frac{1}{\alpha_2 - \beta} \\
h(\alpha_1) & h(\alpha_2)
\end{vmatrix}.
\]

The formulae (1), (3) and (5) permit one to work out a very concise ‘canonical’ representation of the DN metric. Indeed, using the results of Ref. [5], we have

\[
e = -2(m + iv), \quad \beta = -m + i(a - v),
\]

3
\[ f_1 = q + ib, \quad f(\alpha_i) = \frac{q + ib}{\alpha_i + m - i(a - \nu)}, \]
\[ \alpha_1 = -\alpha_2 = \kappa = \sqrt{m^2 + \nu^2 - a^2 - q^2 - b^2}. \] (6)

Now, expanding the determinants in (3), (5) and introducing the generalized spheroidal coordinates \( x, y \) via the formulae
\[ 2\kappa x = r_2 + r_1, \quad 2\kappa y = r_2 - r_1, \] (7)
so that \( r_1 \) and \( r_2 \) can be substituted by
\[ r_1 = \kappa(x - y), \quad r_2 = \kappa(x + y), \] (8)
we arrive, after performing some computer algebra and getting rid of common factors, at the final expressions for \( E, \Phi, f, \gamma \) and \( \omega \):
\[ E = \frac{\kappa x - m - i(ay + \nu)}{\kappa x + m - i(ay - \nu)}, \quad \Phi = \frac{q + ib}{\kappa x + m - i(ay + \nu)}, \]
\[ f = \frac{\kappa^2(x^2 - 1) - a^2(1 - y^2)}{(\kappa x + m)^2 + (ay - \nu)^2}, \quad e^{2\gamma} = \frac{\kappa^2(x^2 - 1) - a^2(1 - y^2)}{\kappa^2(x^2 - y^2)}, \]
\[ \omega = 2\nu(y - 1) - \frac{a(1 - y^2)[2(m\kappa x - \nu ay + m^2 + \nu^2) - q^2 - b^2]}{\kappa^2(x^2 - 1) - a^2(1 - y^2)}. \] (9)

Note that in the new coordinates the line element (4) assumes the form
\[ ds^2 = \kappa^2 f^{-1} \left[ e^{2\gamma}(x^2 - y^2) \left( \frac{dx^2}{x^2 - 1} + \frac{dy^2}{1 - y^2} \right) + (x^2 - 1)(1 - y^2)d\varphi^2 \right] - f(dt - \omega d\varphi)^2. \] (10)

To complete the description of the DN solution in the ‘canonical’ formalism, it only remains to write out the respective components of the electromagnetic four–potential. As is well known [4], the electric component \( A_4 \) is defined by the real part of the potential \( \Phi \), while the magnetic component \( A_3 \) is defined by the real part of Kinnersley’s potential \( \Phi_2 \) [10] for which Sibgatullin’s method gives the expression [3]
\[ \Phi_2 = -iI/E_. \] (11)
From (3), (5) and (10) we then obtain

\[
A_4 = \frac{q(\kappa x + m) + b(\nu - ay)}{(\kappa x + m)^2 + (ay - \nu)^2},
\]
\[
A_3 = b(1 - y) + \frac{(1 - y)(ay + a - 2\nu)(q(\kappa x + m) + b(\nu - ay))}{(\kappa x + m)^2 + (ay - \nu)^2}.
\] (12)

Therefore, formulae (9), (12) and (7) fully describe the gravitational and electromagnetic fields in the DN spacetime. It should be stressed that the parameters \(m, a, \nu, q, b\) represent an analytically extended parameter set which covers all the possibilities for a DN massive source to be either a subextreme object (real \(\kappa\), or a superextreme object (pure imaginary \(\kappa\)) or an extreme object (\(\kappa = 0\)). This is due to the fact that the parameters in the axis data (1) represent five arbitrary and independent multipole moments, as can be easily seen from (2). Hence, the analysis of the physical properties of the DN metric does not require in principle the introduction of the Boyer–Lindquist–like coordinates, and can be carried out either in the generalized spheroidal coordinates, taking into account that the product \(\kappa x\) is always a real non–negative quantity, or directly in the Weyl–Papapetrou cylindrical coordinates \((\rho, z)\). Since, however, the Boyer–Lindquist–like coordinates \((r, \vartheta)\) which are introduced via the formulae

\[
r = \kappa x + m, \quad \cos \vartheta = y
\] (13)

are advantageous for treating the extreme case, below we write out the DN metric in these coordinates:

\[
ds^2 = \frac{D}{N} \left[ N \left( \frac{dr^2}{\Delta} + d\vartheta^2 \right) + \Delta \sin^2 \vartheta \, d\varphi^2 \right] - \frac{N}{D} \left[ dt + \left( 2\nu(1 - \cos \vartheta) + \frac{aW \sin^2 \vartheta}{N} \right) d\varphi \right]^2,
\]

\[
D = r^2 + (a \cos \vartheta - \nu)^2, \quad N = r^2 - 2mr + a^2 \cos^2 \vartheta - \nu^2 + q^2 + b^2,
\]
\[
W = 2mr + 2\nu(\nu - a \cos \vartheta) - q^2 - b^2,
\]
\[
\Delta = r^2 - 2mr - \nu^2 + a^2 + q^2 + b^2.
\] (14)
3 Some remarks on the limits and physical properties of the DN solution

Since the DN metric is a particular case of the Plebański–Demiański solution [3] the physical and geometric characteristics of which have been widely discussed in the literature, in what follows we shall restrict ourselves to only making some remarks on the properties of the DN metric which looked to us interesting to be mentioned here.

In the absence of the NUT parameter $\nu$, the DN metric is asymptotically flat and represents the four–parameter black hole spacetime involving magnetic charge. This black hole limit, together with all further possible reductions, was analyzed in detail by Carter [11].

The stationary pure vacuum limit ($q = b = 0$) is the combined Kerr–NUT solution. Many authors (see, e.g., [12, 13]) associate the DN metric exclusively with the latter three–parameter vacuum solution, possibly not being aware of the second part of the paper [1] where Demiański and Newman presented their five–parameter electrovacuum metric. A good discussion of different interpretations of the NUT parameter and of physical consequences its presence causes in exact solutions can be found in the review article [14]. Here we would only like to notice that although in a single Kerr–NUT solution the NUT parameter is non–physical, already in a non–linear superposition of two Kerr–NUT solutions of Kramer and Neugebauer [14] the formal NUT parameters associated with the two constituents may give rise to the physical quantities defined via the multipole moments of the system and, moreover, are necessary for achieving gravitational equilibrium of the constituents [16]. Let us illustrate this with the axis data of the form

$$e(z) = \frac{z - k - m - i(a + \nu)}{z - k + m - i(a - \nu)} \cdot \frac{z + k - m - i(a - \nu)}{z + k + m - i(a + \nu)}, \quad f(z) = 0 \quad (15)$$

which can be formally interpreted, taking into account (1), as defining a double Kerr–NUT solution whose constituents have equal masses $m$, angular momenta $a$ and opposite NUT parameters $\nu$ and $-\nu$, the constant $k$ playing a role of the separation parameter. As a matter of fact, the above axis data represents
a system of identical particles possessing parallel angular momenta. This can be readily seen from the expressions of the multipole moments corresponding to (15):

\[ \begin{align*}
M_0 &= 2m, & M_1 &= 0, & M_2 &= -2m(m^2 + \nu^2 - k^2) - 2a(ma + 2k\nu), \\
J_0 &= 0, & J_1 &= 2(ma + k\nu), & J_2 &= 0, \\
J_3 &= -2a^2(ma + 3k\nu) - 2(3ma + k\nu)(m^2 + \nu^2 - k^2).
\end{align*} \] (16)

The total NUT parameter of the system is zero \((J_0 = 0)\) and, therefore, the genuine individual NUT parameters of the constituents are equal to zero too because of the additional equatorial symmetry of the system characterized by zero odd mass–multipoles \(M_{2n+1}\) and even angular momentum multipoles \(J_{2n}\), \(n = 1, 2...\). It is important to underline that the formal NUT parameter \(\nu\) in (15) has already nothing to do with either the physical NUT parameter of the system or with the individual physical NUT parameters of the constituents; together with the separation parameter \(k\) it defines two physical arbitrary multipole moments in (16) which are the mass–quadrupole moment \(M_2\) and the angular momentum octupole moment \(J_3\).

In general, the DN metric has another non–physical parameter, the magnetic charge \(b\), but like in the case of the Kerr–NUT solution, this parameter can give rise to physical multipole moments in the two and many–body systems of aligned, charged, magnetized, spinning particles. Therefore, the importance of the DN solution consists in that it provides a single constituent with the whole set of the parameters which may have physical sense in the axisymmetric many–body systems of aligned sources.

In view of the above said it would be logic to envisage the electrovacuum metric \([5]\) as describing the non–linear superposition of \(N\) Demiański–Newman solutions.
4 Conclusion

The formulae obtained in the present paper give a complete description of the DN metric and corresponding electromagnetic potentials within the framework of the Ernst formalism. They can be used for further investigation of the properties of the DN spacetime, for example following the line of the recent paper [13]. We have also illustrated that the non–physical parameters of a single DN solution can acquire physical sense in the many–body systems.

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