The Geometric Block Model

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Abstract
To capture the inherent geometric features of many community detection problems, we propose to use a new random graph model of communities that we call a Geometric Block Model. The geometric block model generalizes the random geometric graphs in the same way that the well-studied stochastic block model generalizes the Erdős-Rényi random graphs. It is also a natural extension of random community models inspired by the recent theoretical and practical advancement in community detection. While being a topic of fundamental theoretical interest, our main contribution is to show that many practical community structures are better explained by the geometric block model. We also show that a simple triangle-counting algorithm to detect communities in the geometric block model is near-optimal. Indeed, even in the regime where the average degree of the graph grows only logarithmically with the number of vertices (sparse-graph), we show that this algorithm performs extremely well, both theoretically and practically. In contrast, the triangle-counting algorithm is far from being optimum for the stochastic block model. We simulate our results on both real and synthetic datasets to show superior performance of both the new model as well as our algorithm.

1 Introduction
The planted-partition model or the stochastic block model (SBM) is a random graph model for community detection that generalizes the well-known Erdős-Rényi graphs \[2,3,11,12,14,19,20,23\]. Consider a graph \(G(V,E)\), where \(V = V_1 \cup V_2 \cup \cdots \cup V_k\) is a disjoint union of \(k\) clusters denoted by \(V_1, \ldots, V_k\). The edges of the graph are drawn randomly: there is an edge between \(u \in V_i\) and \(v \in V_j\) with probability \(q_{i,j}\), \(1 \leq i,j \leq k\). Given the adjacency matrix of such a graph, the task is to find exactly (or approximately) the partition \(V_1 \cup V_2 \cup \cdots \cup V_k\) of \(V\).

This model has been incredibly popular both in theoretical and practical domains of community detection, and the aforementioned references are just a small sample. Recent theoretical works focus on characterizing sharp threshold of recovering the partition in the SBM. For example, when there are only two communities of exactly equal sizes, and the inter-cluster edge probability is \(\frac{b \log n}{n}\) and intra-cluster edge probability is \(\frac{a \log n}{n}\), it is known that perfect recovery is possible if and only if \(\sqrt{a} - \sqrt{b} > \sqrt{2}\).\[2,23\]

The regime of the probabilities being \(\Theta\left(\frac{\log n}{n}\right)\) has been put forward as one of most interesting ones, because in an Erdős-Rényi random graph, this is the threshold for graph connectivity.\[9\]. This result has been subsequently generalized for \(k\) communities \(\frac{3,4,18}{2}\) (for constant \(k\) or when \(k = o(\log n)\)), and under the assumption that the communities are generated according to a probabilistic generative...
motif-counting algorithms are extensively tested on both synthetic and real-world datasets. They
are quite well studied theoretically (though not nearly as much as the Erdős-Rényi graphs), and
very precise results exist regarding their connectivity, clique numbers and other structural properties
of an edge between $x$ and $z$ there is independent of the fact that there exist edges between $x$ and $y$ and
$y$ and $z$. However, one needs to be careful such that by allowing such “transitivity”, the simplicity and
elegance of the SBM is not lost.

Inspired by the above question, we propose a random graph community detection model analogous
to the stochastic block model, that we call the geometric block model (GBM). The GBM depends on
the basic definition of the random geometric graph that has found a lot of practical use in wireless
networking because of its inclusion of the notion of proximity between nodes [24].

Definition. A random geometric graph (RGG) on $n$ vertices has parameters $n$, an integer $t > 1$ and
a real number $\beta \in [-1, 1]$. It is defined by assigning a vector $Z_i \in \mathbb{R}^t$ to vertex $i, 1 \leq i \leq n$, where
$Z_i, 1 \leq i \leq n$ are independent and identical random vectors uniformly distributed in the Euclidean
sphere $S^{t-1} = \{x \in \mathbb{R}^t : \|x\|_2 = 1\}$. There will be an edge between vertices $i$ and $j$ if and only if
$\langle Z_i, Z_j \rangle \geq \beta$.

Note that, the definition can be further generalized by considering $Z_i$s to have a sample space other
than $S^{t-1}$, and by using a different notion of distance than inner product (i.e., the Euclidean distance).
We simply stated one of the many equivalent definitions [10].

Random geometric graphs are often proposed as an alternative to Erdős-Rényi random graphs.
They are quite well studied theoretically (though not nearly as much as the Erdős-Rényi graphs), and
very precise results exist regarding their connectivity, clique numbers and other structural properties.
For a survey of early results on geometric graphs and the analogy to results in Erdős-Rényi graphs, we refer the reader to [24]. A very interesting question of distinguishing an
Erdős-Rényi graph from a geometric random graph has also recently been studied [10]. This will provide
a way to test between the models which better fits a scenario, a potentially great practical use.

As mentioned earlier, the “transitivity” feature led to random geometric graphs being used extensively
to model wireless networks (for example, see [6][17]). Surprisingly, however, to the best of our knowledge,
random geometric graphs are never used to model community detection problems. In this paper we
take the first step towards this direction. Our main contributions can be classified as follows.

- We define a random generative model to study canonical problems of community detection, called the
  geometric block model (GBM). This model takes into account a measure of proximity between nodes
  and this proximity measure characterizes the likelihood of two nodes being connected when they are in
  same or different communities. The geometric block model inherits the connectivity properties of the
  random geometric graphs, in particular the likelihood of “transitivity” in triplet of nodes (or more).
- We experimentally validate the GBM on various real-world datasets. We show that many practical
  community structures exhibit properties of the GBM. We also compare these features with the
  corresponding notions in SBM to show how GBM better models data in many practical situations.
- We propose a simple motif-based efficient algorithm for community detection on the GBM. We
  rigorously show that this algorithm is optimal up to a constant fraction (to be properly defined later)
even in the regime of sparse graphs (average degree $\sim \log n$).
- The motif-counting algorithms are extensively tested on both synthetic and real-world datasets. They
  exhibit very good performance in three real datasets, compared to the spectral-clustering algorithm
(see Section 3). Since simple motif-counting is known to be far from optimum in stochastic block model (see Section 4), these experiments give further validation to GBM as a real-world model.

Given any simple random graph model, it is possible to generalize it to a random block model of communities much in line with the SBM. We however stress that the geometric block model is perhaps the simplest possible model of real-world communities that also captures the transitive/geometric features of communities. Moreover, the GBM explains behaviors of many real world networks as we will exemplify subsequently.

**Independent work:** In a parallel and independent work that is yet to appear, Emmanuel Abbe also proposed the model that we consider [1]. We were made aware of this parallel effort recently after the writing of this paper.

## 2 The Geometric Block Model and its Validation

Let \( V \equiv V_1 \sqcup V_2 \sqcup \cdots \sqcup V_k \) be the set of vertices that is a disjoint union of \( k \) clusters, denoted by \( V_1, \ldots, V_k \). Given an integer \( t \geq 2 \), for each vertex \( u \in V \), define a random vector \( Z_u \in \mathbb{R}^t \) that is uniformly distributed in \( S^{t-1} \subset \mathbb{R}^t \), the \( t-1 \)-dimensional sphere.

**Definition (Geometric Block Model \((V,t,\beta_{i,j}, 1 \leq i \leq j \leq k)\).** Given \( V, d \) and a set of real numbers \( \beta_{i,j} \in [-1,1], 1 \leq i \leq j \leq k \), the geometric block model is a random graph with vertices \( V \) and an edge exists between \( v \in V_i \) and \( u \in V_j \) if and only if \( \langle Z_u, Z_v \rangle \geq \beta_{i,j} \).

**The case of \( t = 2 \):** In this paper we particularly analyze our algorithm for \( t = 2 \). In this special case, the above definition is equivalent to choosing random variable \( \theta_u \) uniformly distributed in \([0, 2\pi]\), for all \( u \in V \). Then there will be an edge between two vertices \( u \in V_i, v \in V_j \) if and only if \( \cos \theta_u \cos \theta_v + \sin \theta_u \sin \theta_v = \cos(\theta_u - \theta_v) \geq \beta_{i,j} \) or \( \min\{|\theta_u - \theta_v|, 2\pi - |\theta_u - \theta_v|\} \leq \arccos \beta_{i,j} \). This in turn, is equivalent to choosing a random variable \( X_u \) uniformly distributed in \([0, 1]\) for all \( u \in V \), and there exists an edge between two vertices \( u \in V_i, v \in V_j \) if and only if

\[
\min(|X_u - X_v|, 1 - |X_u - X_v|) \leq r_{i,j},
\]

where \( r_{i,j} \in [0, \frac{1}{2}], 0 \leq i, j \leq k \), are a set of real numbers.

For the rest of this paper, we concentrate on the case when \( r_{i,i} = r_s \) for all \( i \in \{1, \ldots, k\} \), which we call the “intra-cluster distance” and \( r_{i,j} = r_d \) for all \( i, j \in \{1, \ldots, k\}, i \neq j \), which we call the “inter-cluster distance,” mainly for the clarity of exposition. To allow for edge density to be higher inside the clusters than across the clusters, assume \( r_s \geq r_d \).

| Area 1 | Area 2 | same | different |
|--------|--------|------|----------|
| MOD    | AI     | 10   | 2        |
| ARCH   | MOD    | 6    | 1        |
| ROB    | ARCH   | 3    | 0        |
| MOD    | ROB    | 4    | 0        |
| ML     | MOD    | 7    | 1        |

| Area   | same | different |
|--------|------|----------|
| MOD    | 19   | 35       |
| ARCH   | 13   | 15       |
| ROB    | 24   | 16       |
| AI     | 39   | 32       |
| ML     | 14   | 42       |

Table 1: On the left we count the number of inter-cluster edges when authors shared same affiliation and different affiliations. On the right, we count the same for intra-cluster edges.

The main problem that we seek to address is following. Given the adjacency matrix of a geometric block model with \( k \) clusters, and \( t, r_d, r_s, r_s \geq r_d \), find the partition \( V_1, V_2, \ldots, V_k \).

We next give two examples of real datasets that motivate the GBM. In particular, we experiment with two different types of real world datasets in order to verify our hypothesis about geometric block...
2.1 Motivation of GBM: Academic Collaboration

We consider the collaboration network of academicians in Computer Science in 2016 (data obtained from csrankings.org). According to area of expertise of the authors, we consider five different communities: Data Management (MOD), Machine Learning and Data Mining (ML), Artificial Intelligence (AI), Robotics (ROB), Architecture (ARCH). If two authors share the same affiliation, or shared affiliation in the past, we assume that they are geographically close. We would like to hypothesize that, two authors in the same communities might collaborate even when they are geographically far. However, two authors in different communities are more likely to collaborate only if they share the same affiliation (or are geographically close). Table 1 describes the number of edges across the communities. It is evident that the authors from same community are likely to collaborate irrespective of the affiliations and the authors of different communities collaborate much frequently when they share affiliations or are close geographically. This clearly indicates that the inter cluster edges are likely to form if the distance between the nodes is quite small, motivating the fact $r_d < r_s$ in the GBM.

2.2 Motivation of GBM: Amazon Metadata

The next dataset that we use in our experiments is the Amazon product metadata on SNAP (https://snap.stanford.edu/data/amazon-meta.html), that has 548552 products and each product is one of the following types {Books, Music CD’s, DVD’s, Videos}. Moreover, each product has a list of attributes, for example, a book may have attributes like ⟨“General”, “Sermon”, “Preaching”⟩. We consider the co-purchase network over these products. We make two observations here: (1) edges get formed (that is items are co-purchased) more frequently if they are similar, where we measure similarity by the number of common attributes between products, and (2) two products that share an edge have more common neighbors than two products with no edge in between.

Figures 1 and 2 show respectively average similarity of products that were bought together, and not bought together. From the distribution, it is quite evident that edges in a co-purchase network gets formed according to distance, a salient feature of random geometric graphs, and the GBM.

![Figure 1: Histogram: similarity of products bought together (mean ≈ 6)](image1)

![Figure 2: Histogram: similarity of products not bought together (mean≈ 1.5)](image2)

We next take equal number of product pairs inside Book (also inside DVD, and across Book and DVD) that have an edge in-between and do not have an edge respectively. Figure 3 shows that the number of common neighbors when two products share an edge is much higher than when they do not—in fact, almost all product pairs that do not have an edge in between also do not share any common neighbor. This again strongly suggests towards GBM due to its transitivity property. On the other
hand, this also suggests that SBM is not a good model for this network, as in SBM, two nodes having common neighbors is independent of whether they share an edge or not.

**Difference between SBM and GBM.** It is important to stress that the network structures generated by the SBM and the GBM are quite different, and it is significantly difficult to analyze any algorithm or lower bound on GBM compared to SBM. This difficulty stems from the highly correlated edge generation in GBM (while edges are independent in SBM). For this reason, analyses of the sphere-comparison algorithm and spectral methods for clustering on GBM cannot be derived as straightforward adaptations. Whereas, even for simple algorithms, a property that can be immediately seen for SBM, will still require a proof for GBM.

### 3 The Motif-Counting Algorithm

Suppose, we are given a graph $G = (V, E)$ with $k$ disjoint clusters, $V_1, V_2, ..., V_k \subseteq V$ generated according to $GBM(V, t, r_s, r_d, k)$. Our clustering algorithm is based on counting motifs, where a motif is simply defined as a configuration of triplets in the graph. Let us explain this principle by one particular motif, a triangle. For any two vertices $u$ and $v$ in $V$, where $(u, v)$ is an edge, we count the total number of common neighbors of $u$ and $v$. We show that this count is different when $u$ and $v$ belong to the same cluster, and when they belong to different clusters. We assume $G$ is connected, because otherwise it is impossible to recover the clusters. For every pair of vertices in the graph that share an edge, we decide whether they are in the same cluster or not by this count of triangles. In reality, we do not have to check every such pair, instead we can stop when we form a spanning tree. At this point, we can transitively deduce the partition of nodes into clusters.

The main new idea of this algorithm is to use this triangle-count (or motif-count in general), since they carry significantly more information regarding the connectivity of the graph than an edge count. However, we can go to statistics of higher order (such as the two-hop common neighbors) at the expense of increased complexity. Surprisingly, the simple greedy algorithm that rely on triplets can separate clusters when $r_d$ and $r_s$ are $\Omega(\log n/n)$, which is also a minimal requirement for connectivity of random geometric graphs [24]. Therefore this algorithm is optimal up to a constant factor. It is interesting to note that this motif-counting algorithm is not optimal for SBM (as we observe), in particular, it will not detect the clusters in the sparse threshold region of $\log^2 n/n$, however, it does so for GBM.

The pseudocode of the algorithm is described in Algorithm 1. The algorithm looks at individual pairs of vertices to decide whether they belong to the same cluster or not. We go over pairs of vertices and label them same/different, till we have enough labels to partition the graphs into clusters.

At any stage, the algorithm picks up an unassigned node $v$ and queries it with a node each from the already formed clusters. To decide whether a point belongs to a cluster, it calls a subroutine called process. The process function tries to infer if the node $v$ belongs to the cluster $V_i$ by first identifying
a vertex \( u \in V_i \) that has an edge with \( v \), and then by counting the number of common neighbors of \( u \) and \( v \) to make a decision. This procedure is continued till all nodes in \( V \) are processed.

\[ \text{Algorithm 1: Graph recovery from GBM} \]

**Require:** GBM \( G = (V, E) \), \( r_s, r_d, k \)

**Ensure:** \( V = V_1 \cup \ldots \cup V_k \)

1. \( V_1, \ldots, V_k \leftarrow \emptyset \)
2. **for** \( v \in V \) **do**
   3. **for** \( i \in \{1, 2, \ldots, k - 1\} \) **do**
      4. **if** \( \text{process}(V_i, v, r_s, r_d) \) **then**
         5. \( V_i \leftarrow V_i \cup \{v\} \)
         6. \( \text{added} \leftarrow \text{true} \)
      7. **end if**
   8. **end for**
   9. **if** \( \neg \text{added} \) **then**
      10. \( V_k \leftarrow V_k \cup \{v\} \)
      11. **end if**
  12. **end for**

**Algorithm 2: process**

**Require:** \( C, v, r_s, r_d \)

**Ensure:** true/false

1. Choose \( u \in C \mid (u, v) \in E \)
2. \( \text{count} \leftarrow |\{z : (z, u) \in E, (z, v) \in E\}| \)
3. **if** \( |\frac{\text{count}}{n} - E_S(r_d, r_s)| < |\frac{\text{count}}{n} - E_D(r_d, r_s)| \) **then**
    4. **return** true
   5. **end if**
6. **return** false

The \( \text{process} \) function counts the number of common neighbors of two nodes and then compares the difference of the count with two functions of \( r_d \) and \( r_s \), called \( E_D \) and \( E_S \).

Formulae for \( E_D \) and \( E_S \) are different when \( r_s < 2r_d \) to \( r_s \geq 2r_d \). We have compiled this in Table 2.

In this table we have assumed that there are only two clusters of equal size. The functions change when the cluster sizes are different. Our analysis described in later sections can be used to calculate new function values. In the table, \( u \sim v \) means \( u \) and \( v \) are in the same cluster.

Similarly, the \( \text{process} \) function can be run on other set of motifs (other patterns of triplets) by fixing two nodes. On considering a larger set of motifs, the \( \text{process} \) function can take a majority vote over the decisions received from different motifs. This is helpful to resolve the clusters even when the gap between \( r_s \) and \( r_d \) is small (by a constant factor than compared to just triangle motif).

Note that, our algorithm counts motifs only for edges, and does not count motifs for more than \( n - 1 \) edges, as that many edges are sufficient to construct a spanning tree of the graph.

4 Analysis of the Algorithm

The critical observation that we have to make to analyze the motif-counting algorithm is the fact that given a GBM graph \( G(V, E) \) with two clusters \( V = V_1 \cup V_2 \), and a pair of vertices \( u, v \in V : (u, v) \in E \), the events \( E_z^{u,v}, z \in V \) of any other vertex \( z \) being a common neighbor of both \( u \) and \( v \) are independent (this is obvious in SBM, but does not lead to the same result). However, the probabilities of \( E_z^{u,v} \) are different when \( u \) and \( v \) are in the same cluster and when they are in different clusters. Therefore the count of the common neighbors are going to be different, and substantially separated with high
Table 2: $E_S, E_D$ values for different motifs considering different values of $r_s$ and $r_d$, when there are two equal sized clusters. Here $\text{Bin}(n, p)$ denotes a binomial random variable with mean $np$.

| Motif | Distribution of count ($r_s > 2r_d$) | Distribution of count ($r_s \leq 2r_d$) |
|-------|-----------------------------------------|-----------------------------------------|
| $z \mid (z, u) \in E, (z, v) \in E$ | $\text{Bin}(\frac{n}{2} - 2, \frac{3r_s}{2}) + \text{Bin}(\frac{r_d}{2}, \frac{r_d}{r_s})$; $E_S = \frac{n}{2} + \frac{r_d}{2}$ | $\text{Bin}(n - 2, 2r_d); E_D = 2r_d$ |
| $z \mid (z, v) \notin E, (z, v) \in E$ | $\text{Bin}(\frac{n}{2} - 2, \frac{3r_s}{2}) + \text{Bin}(\frac{r_d}{2}, \frac{r_d}{r_s})$; $E_S = \frac{n}{2} + \frac{r_d}{2}$ | $\text{Bin}(n-2, \frac{r_s}{2})$; $E_S = \frac{n}{2}$ |
| $z \mid (z, u) \notin E, (z, v) \in E$ | $\text{Bin}(\frac{n}{2} - 2, \frac{3r_s}{2}) + \text{Bin}(\frac{r_d}{2}, \frac{r_d}{r_s})$; $E_S = \frac{n}{2} + \frac{r_d}{2}$ | $\text{Bin}(n-2, \frac{r_s}{2})$; $E_S = \frac{n}{2}$ |

Lemma 1. For any two vertices $u, v \in V_i : (u, v) \in E, i = 1, 2$ belonging to the same cluster, the count of common neighbors $C_{u,v} \equiv \{z \in V : (z, u), (z, v) \in E\}$ is a random variable distributed according to $\text{Bin}(\frac{n}{2} - 2, \frac{3r_s}{2}) + \text{Bin}(\frac{r_d}{2}, \frac{r_d}{r_s})$ when $r_s > 2r_d$ and according to $\text{Bin}(\frac{n}{2} - 2, \frac{3r_s}{2}) + \text{Bin}(\frac{r_d}{2}, 2r_d - \frac{3}{2})$ when $r_s \leq 2r_d$, where $\text{Bin}(n, p)$ is a binomial random variable with mean $np$.

Lemma 2. For any two vertices $u \in V_1, v \in V_2 : (u, v) \in E$ belonging to different clusters, the count of common neighbors $C_{u,v} \equiv \{z \in V : (z, u), (z, v) \in E\}$ is a random variable distributed according to $\text{Bin}(n-2, 2r_d)$ when $r_s > 2r_d$ and according to $\text{Bin}(n-2, 2r_s - \frac{r_d^2}{2r_d})$ when $r_s \leq 2r_d$.

Similar lemmas exist for other motifs as well. We describe those in detail in the appendix. Here let us give the proof of Lemma 1. The proof of Lemma 2 will follow similarly. These expressions can also be generalized straightforwardly when the clusters are of unequal sizes, but we omit those for clarity of exposition.

Proof of Lemma 1. Let $X_w \in [0, 1]$ be the uniform random variable associated with $w \in V$. Let us also denote by $d_L(X, Y) \equiv \min\{|X - Y|, 1 - |X - Y|\}, X, Y \in \mathbb{R}$. Without loss of generality, assume $u, v \in V_1$. For any vertex $z \in V$, let $\mathcal{E}_{z,v}^{u,v} \equiv \{(u, z), (v, z) \in E\}$ be the event that $z$ is a common neighbor. For $z \in V_1$,

$$
\Pr(\mathcal{E}_{z,v}^{u,v}) = \Pr((z, u) \in E, (z, v) \in E \mid (u, v) \in E) = \int_0^{r_s} \frac{1}{r_s} \Pr((z, u) \in E, (z, v) \in E \mid d_L(X_u, X_v) = x)dx
$$

$$
= \int_0^{r_s} \frac{1}{r_s} (2r_s - x)dx = \frac{3r_s}{2}.
$$

For $z \in V_2$, we have,

$$
\Pr(\mathcal{E}_{z,v}^{u,v}) = \Pr((z, u) \in E, (z, v) \in E \mid (u, v) \in E)
$$
\[
\int_{0}^{\min(r_s,2r_d)} \frac{1}{r_s} \Pr((z,u),(z,v) \in E \mid \text{d}_L(X_u,X_v)=x) dx \\
= \int_{0}^{\min(r_s,2r_d)} \frac{1}{r_s}(2r_d-x)dx = \begin{cases} \frac{2r_d^2}{r_s} & \text{if } 2r_d < r_s \\ 2r_d - \frac{r_s}{2} & \text{otherwise} \end{cases}.
\]

Now since there are \(\frac{n}{2} \) points in \(V_1 \setminus \{u,v\}\) and \(\frac{n}{2}\) points in \(V_2\), we have the statement of the lemma.

The proof of Lemma 2 is similar and we delegate it to the appendix in the supplementary material. By leveraging the concentration of binomial random variables, in our algorithm we just check whether the count of common neighbors is closer to the average value of the random variable described in Lemma 1 or in Lemma 2. While more general statements are possible, we give a theorem concentrating on the special case when \(r_s, r_d \sim \frac{\log n}{n}\).

**Theorem 1.** If \(r_s = \frac{a \log n}{n}\) and \(r_d = \frac{b \log n}{n}\), \(r_s > r_d\), Algorithm 1 can recover the clusters \(V_1, V_2\) accurately with a probability of \(1 - \frac{3}{n}\) if

\[
\begin{cases}
\frac{3(a-2b)(a-2b)}{4a} \geq \left(\sqrt{\frac{3a}{4}} + \sqrt{\frac{b^2}{n}} + \sqrt{2b}\right)\sqrt{6}, \text{when } a \geq 2b \\
\frac{3(b-a)(b-2b)}{4a} \geq \left(\sqrt{\frac{3a}{4}} + \sqrt{\frac{b^2}{n}} + \sqrt{2a - \frac{a^2}{2b}}\right)\sqrt{6}, \text{else}.
\end{cases}
\]

**Proof.** Let us consider the case \(r_s > 2r_d\) first. Let \(Z\) denote the random variable that equals the number of common neighbors of two nodes \(u, v \in V\) : \((u,v) \in E\). Let us also denote \(\mu_s = \mathbb{E}(Z|u \sim v)\) and \(\mu_d = \mathbb{E}(Z|u \sim v)\), where \(u \sim v\) means \(u\) and \(v\) are in the same cluster. We can easily find \(\mu_s\) and \(\mu_d\) from Lemmas 1 and 2. We see that,

\[
\mu_s - \mu_d = \frac{(n - O(1))(3r_s - 2r_d)(r_s - 2r_d)}{4r_s} = \frac{(3a - 2b)(a - 2b) \log n}{4a} - O\left(\frac{\log n}{n}\right).
\]

Now, since \(Z\) is a sum of independent binary random variables, using the Chernoff bound, \(\Pr(Z < (1 + \delta)\mathbb{E}(Z)) \leq \Pr(Z > (1 + \delta)\mathbb{E}(Z)) \leq e^{-\delta^2 \mathbb{E}(Z)/3} = \frac{1}{n^2}\), when \(\delta = \sqrt{\frac{6 \log n}{\mathbb{E}(Z)}}\). Now with probability at least \(1 - \frac{3}{n^2}\) (since there are three binomial terms involved and they can have deviations more than \(\sqrt{3a/2 \log n}, \sqrt{6b^2/4 \log n}, \text{and} \sqrt{12b \log n}\) with probability \(\frac{1}{n^2}\) each), the algorithm will be successful to label correctly as long as \(\frac{(3a-2b)(a-2b)\log n}{4a} \geq \left(\sqrt{\frac{3a}{4}} + \sqrt{\frac{b^2}{n}} + \sqrt{2b}\right)\sqrt{6}\log n\). The case of \(r_s \leq 2r_d\) will follow similarly. Now we need the labeling to be successful for \(n\) pairs of vertices (so that a spanning tree can be formed). Applying union bound over \(n\) distinct pairs guarantees the probability of recovery as \(1 - 3/n\).

| Dataset       | Total no. of nodes | \(T_1\) | \(T_2\) | \(T_3\) | Accuracy Motif-Counting | Spectral clustering | Running Time (sec) | Motif-Counting | Spectral clustering |
|---------------|--------------------|--------|--------|--------|------------------------|---------------------|-------------------|-----------------|------------------|
| Political Blogs | 1222               | 20     | 2      | 1      | 0.788                  | 0.53                | 1.62              | 0.29            |
| DBLP          | 12138              | 10     | 1      | 2      | 0.675                  | 0.63                | 3.93              | 18.077          |
| LiveJournal   | 2366               | 20     | 1      | 1      | 0.7768                 | 0.64                | 0.49              | 1.54            |

Table 3: Performance on real world networks

Now instead of relying only on the triangle motif, if we consider all different motifs (as defined in Table 2), and then take the aggregate (majority vote) decision, we can improve the above theorem slightly.
Theorem 2. If \( r_s = \frac{a \log n}{n} \) and \( r_d = \frac{b \log n}{n} \), the algorithm considering all three motifs (see Table 2) for a pair of nodes can recover the clusters \( V_1, V_2 \) correctly with probability \( 1 - O(\frac{1}{n}) \) if \( \left( \frac{3a-b}{4a} \right)(a-2b) \geq \sqrt{3} \min \left\{ \sqrt{\frac{3a^4}{4} + \sqrt{2a^2 + \sqrt{2a^2 + 2ab}}} \right\} \) when \( a \geq 2b \), and \( \frac{(a-b)(a-2b)}{2a} \geq \sqrt{3} \min \left\{ \sqrt{\frac{3a^4}{4} + \sqrt{2a^2 + 2ab}} \right\} \) when \( a < 2b \).

The proof of this theorem is delegated to the appendix.

Remark 1. Instead of using Chernoff bound we could have used better concentration inequality (such as Poisson approximation) in the above analysis, to get tighter condition on the constants. We again preferred to keep things simple.

Remark 2 (GBM for \( t = 3 \) and above). For GBM with \( t = 3 \), to find the number of common neighbors of two vertices, we need to find out the area of intersection of two spherical caps on the sphere. It is possible to do that. It can be seen that, our algorithm will successfully identify the clusters as long as \( r_s, r_d \sim \sqrt{\frac{\log n}{n}} \) again when the constant terms satisfy some conditions. However tight characterization becomes increasingly difficult. For general \( t \), our algorithm should be successful when \( r_s, r_d \sim \left( \frac{\log n}{n} \right)^{\frac{1}{t-1}} \), which is also the regime of connectivity threshold.

Remark 3 (More than two clusters). When there are more than two clusters, the same analysis technique is applicable and we can estimate the expected number of common neighbors. This generalization is straightforward but tedious.

Motif counting algorithm for SBM. While our algorithm is near optimal for GBM in the regime of \( r_s, r_d \sim \frac{\log n}{n} \), it is far from optimal for the SBM in the same regime of average degree. Indeed, by using simple Chernoff bounds again, we see that the motif counting algorithm is successful for SBM with inter-cluster edge probability \( q \) and intra-cluster probability \( p \), when \( p, q \sim \sqrt{\frac{\log n}{n}} \). The experimental success of our algorithm in real sparse networks therefore somewhat enforce the fact that GBM is a better model for those network structures, than SBM.

5 Experimental Results

In addition to validation experiments in Section 2.1 and 2.2, we also conduct an in-depth experimentation of our proposed model and techniques over a set of synthetic and real world networks. Additionally, we
compare the efficacy and efficiency of our motif-counting algorithm with the popular spectral clustering algorithm using normalized cuts\(^1\) and correlation clustering algorithm \([7]\).

**Real Datasets.** We use three real datasets described below.

- **Political Blogs.** \([5]\) It contains a list of political blogs from 2004 US Election classified as liberal or conservative, and links between the blogs. The clusters are of roughly the same size with a total of 1200 nodes and 20K edges.

- **DBLP.** \([26]\) The DBLP dataset is a collaboration network where the ground truth communities are defined by the research community. The original graph consists of roughly 0.3 million nodes. We process it to extract the top two communities of size \(\sim 4500\) and \(7500\) respectively. This is given as input to our algorithm.

- **LiveJournal.** \([21]\) The LiveJournal dataset is a free online blogging social network of around 4 million users. Similar to DBLP, we extract the top two clusters of sizes 930 and 1400 which consist of around 11.5K edges.

**Synthetic Datasets.** We generate synthetic datasets of different sizes according to the GBM with \(t = 2, k = 2\) and for a wide spectrum of values of \(r_s\) and \(r_d\), specifically we focus on the sparse region where \(r_s = \frac{a \log n}{n}\) and \(r_d = \frac{b \log n}{n}\) with variable values of \(a\) and \(b\).

**Experimental Setting.** For real networks, it is difficult to calculate an exact threshold as the exact values of \(r_s\) and \(r_d\) are not known. Hence, we follow a three step approach. Using a somewhat large threshold \(T_1\) we sample a subgraph \(S\) such that \(u, v\) will be in \(S\) if there is an edge between \(u\) and \(v\), and they have at least \(T_1\) common neighbors. We now attempt to recover the subclusters inside this subgraph by following our algorithm with a small threshold \(T_2\). Finally, for nodes that are not part of \(S\), say \(x \in V \setminus S\), we select each \(u \in S\) that \(x\) has an edge with and use a threshold of \(T_3\) to decide if \(u\) and \(x\) should be in the same cluster. The final decision is made by taking a majority vote. We can employ sophisticated methods over this algorithm to improve the results further, which is beyond the scope of this work.

We use the popular f-score metric which is the harmonic mean of precision (fraction of number of pairs correctly classified to total number of pairs classified into clusters) and recall (fraction of number of pairs correctly classified to the total number of pairs in the same cluster for ground truth), as well as the node error rate for performance evaluation. A node is said to be misclassified if it belongs to a cluster where the majority comes from a different ground truth cluster (breaking ties arbitrarily). Following this, we use the above described metrics to compare the performance of different techniques on various datasets.

**Results.** Table 3 shows that the motif-counting algorithm gives an accuracy as high as 78%. The spectral clustering performed worse as compared to our algorithm for all real world datasets. It obtained the worst accuracy of 53% on political blogs dataset. The correlation clustering algorithm generates various small sized clusters leading to a very low recall, performing much worse than the motif-counting algorithm for the whole spectrum of parameter values.

Additionally, we observe in Table 3 that our algorithm is much faster than the spectral clustering algorithm for larger datasets (LiveJournal and DBLP). This confirms that motif-counting algorithm is more scalable than the spectral clustering algorithm. Correlation clustering takes 8-10 times longer as compared to motif-counting algorithm for the various range of its parameters. Moreover, our algorithm is easily parallelizable to achieve better improvements. This clearly establishes the efficiency and effectiveness of motif-counting.

We observe similar gains on synthetic datasets. Figures 4a, 4b, and 4c report results on the synthetic datasets with 5000 nodes. Figure 4a plots the minimum gap between \(a\) and \(b\) that guarantees exact recovery according to Theorem 1 (only triangle motif) and Theorem 2 (all three motifs) vs minimum value of \(a\) for varying \(b\) for which experimentally (with only triangle motif, and average of three runs)

\(^1\)http://scikit-learn.org/stable/modules/clustering.html#spectral-clustering
we were able to recover the clusters exactly. Empirically, our results demonstrate the near-optimal performance of motif-counting algorithm, confirming the theoretical bound. We also see a clear threshold behavior on both f-score and node error rate in Figures 4b and 4c. We have also performed spectral clustering on this 5000-node synthetic dataset (Figures 5a and 5b). Compared to the plots of figures 4b and 4c, they show suboptimal performance, indicating the relative ineffectiveness of spectral clustering in GBM compared to the motif counting algorithm.

Figure 5: Results of the spectral clustering on a synthetic dataset with 5000 nodes.
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A Appendix

For the analysis, let $X_w \in [0,1]$ be the uniform random variable associated with $w \in V$. Let us also denote by $d_L(X,Y) = \min\{|X-Y|,1-|X-Y|\}$, $X,Y \in \mathbb{R}$.

A.1 Proof of Lemma 2

Proof of Lemma 2  For any vertex $z \in V$, let $E_{z}^{u,v} \equiv \{(u,z),(v,z) \in E\}$ be the event that $z$ is a common neighbor. For $z \in V \setminus \{u,v\}$,

$$\Pr(E_{z}^{u,v}) = \Pr((z,u) \in E, (z,v) \in E | (u,v) \in E)$$

$$= \int_{0}^{r_d} \frac{1}{r_d} \Pr((z,u) \in E, (z,v) \in E | d_L(X_u,X_v) = x) dx$$

$$= \int_{0}^{r_d} \frac{1}{r_d} (\min\{2r_d,r_d + r_s - x\}) dx$$

$$= \begin{cases} 2r_d & \text{if } 2r_d < r_s \\ 2r_s - \frac{x^2}{2r_s^2} & \text{otherwise} \end{cases}.$$ 

Now since there are $n - 2$ points in $V \setminus \{u,v\}$, we have the statement of the lemma.

A.2 Lemmas for other motifs

Next, we describe two lemmas for a GBM graph $G(V,E)$ with two unknown clusters $V = V_1 \cup V_2$, and parameters $r_s, r_d$ on considering other motifs than triangles. These results are used to populate Table 2.

Lemma 3. For any two vertices $u,v \in V_i : (u,v) \in E, i = 1,2$ belonging to the same cluster, the count of neighbors of $u$ and non neighbors of $v$, $C_{u,v} \equiv |\{z \in V : (z,u) \in E, (z,v) \notin E\}|$ is a random variable distributed according to Bin($\frac{n}{2} - 2, \frac{r_s^2}{r_s^2}$) + Bin($\frac{n}{2}, \frac{2r_d(r_s-r_d)}{r_s^2}$) when $r_s > 2r_d$ and according to Bin($n-2, \frac{r_s^2}{2}$) + Bin($\frac{n}{2}, 2r_d - \frac{r_s^2}{r_s^2}$) when $r_s \leq 2r_d$, where Bin($n,p$) is a binomial random variable with mean np.

Proof. Without loss of generality, assume $u,v \in V_1$. For any vertex $z \in V$, let $E_{z}^{u,v} \equiv \{(u,z) \in E,(v,z) \notin E\}$ be the event that $z$ is a neighbor of $u$ and non neighbor of $v$. For $z \in V_1$,

$$\Pr(E_{z}^{u,v}) = \Pr((z,u) \in E, (z,v) \notin E | (u,v) \in E)$$

$$= \int_{0}^{r_s} \frac{1}{r_s} \Pr((z,u) \in E, (z,v) \notin E | d_L(X_u,X_v) = x) dx$$

$$= \int_{0}^{r_s} \frac{1}{r_s} (x) dx = \frac{r_s}{2}.$$ 

For $z \in V_2$, we have,

$$\Pr(E_{z}^{u,v}) = \Pr((z,u) \in E, (z,v) \notin E | (u,v) \in E)$$

$$= \int_{0}^{r_s} \frac{1}{r_s} \Pr((z,u) \in E, (z,v) \notin E | d_L(X_u,X_v) = x) dx$$

$$= \int_{0}^{r_s} \frac{1}{r_s} \min\{x,2r_d\} dx$$

$$= \begin{cases} \frac{2r_d(r_s-r_d)}{r_s} & \text{if } 2r_d < r_s \\ \frac{r_s}{2} & \text{otherwise} \end{cases}.$$
Now since there are $\frac{n}{2} - 2$ points in $V_1 \setminus \{u, v\}$ and $\frac{n}{2}$ points in $V_2$, we have the statement of the lemma.

**Lemma 4.** For any two vertices $u \in V_1, v \in V_2 : (u, v) \in E$ belonging to different clusters, the count of number of nodes forming motif of second type with $u$ and $v$ (i.e. neighbor of $u$ and non neighbor of $v$), $C_{u,v} \equiv \{|z \in V : (z, u) \in E, (v, z) \notin E\}$ is a random variable distributed according to $\text{Bin}(n-2, r_d)$ when $r_d > 2r_d$ and according to $\text{Bin}(n-2, \frac{r_d^2+2r_s^2-2r_dr_s}{2r_d})$ when $r_s \leq 2r_d$.

**Proof.** For any vertex $z \in V_1$, let $\mathcal{E}_{z,u}^v = \{(u, z) \in E, (v, z) \notin E\}$ be the event that $z$ is a neighbor of $u$ and a non neighbor of $v$. For $z \in V_1 \setminus \{u\}$

$$\Pr(\mathcal{E}_{z,u}^v) = \Pr((z, u) \in E, (z, v) \notin E \mid (u, v) \in E)$$

$$= \int_0^{r_d} \frac{1}{r_d} \Pr((z, u) \in E, (z, v) \notin E \mid d_L(X_u, X_v) = x)dx$$

$$= \int_0^{r_d} \frac{1}{r_d} (\max\{x + r_d - r_s, 0\})dx$$

$$= \begin{cases} 0 & \text{if } 2r_d < r_s \\ \frac{(2r_d - r_s)^2}{2r_d} & \text{otherwise} \end{cases}.$$  

Now for $z \in V_2 \setminus \{v\}$

$$\Pr(\mathcal{E}_{z,u}^v) = \Pr((z, u) \in E, (z, v) \notin E \mid (u, v) \in E)$$

$$= \int_0^{r_d} \frac{1}{r_d} \Pr((z, u) \in E, (z, v) \notin E \mid d_L(X_u, X_v) = x)dx$$

$$= \int_0^{r_d} \frac{1}{r_d} (\max\{0, r_s - x - r_d\} + (x - r_d + r_s))dx$$

$$= \begin{cases} 2r_s - 2r_d & \text{if } 2r_d < r_s \\ \frac{2(2r_d - r_s)^2}{r_d} + \frac{(2r_d - r_s)(3r_d - 2r_s)}{2r_d} & \text{otherwise} \end{cases}.$$  

Now since there are $\frac{n}{2} - 1$ points in $V_1 \setminus \{u\}$, and $\frac{n}{2} - 1$ points in $V_2 \setminus \{v\}$, we have the statement of the lemma after some simplification.

**A.3 Proof of Theorem 2**

**Proof of Theorem 2.** Let us consider the case $r_s > 2r_d$ first. Let $Z_1$ denote the random variable that equals the number of common neighbors (first motif) of two nodes $u, v \in V : (u, v) \in E$. Let us also denote $\mu_1^u = \mathbb{E}(Z_1 | u \sim v)$ and $\mu_1^d = \mathbb{E}(Z_1 | u \sim v)$, where $u \sim v$ means $u$ and $v$ are in the same cluster. We can easily find $\mu_1^u$ and $\mu_1^d$ from Lemmas 1, 2. We see that,

$$\mu_s - \mu_d = \frac{(n - O(1))(3r_s - 2r_d)(r_s - 2r_d)}{4r_s}$$

$$= \frac{(3a - 2b)(a - 2b) \log n}{4a} - O\left(\frac{\log n}{n}\right).$$  

Now, since $Z_1$ is a sum of independent binary random variables, using Chernoff bound, $\Pr(Z_1 < (1 + \delta)\mathbb{E}(Z_1)) \leq \Pr(Z_1 > (1 + \delta)\mathbb{E}(Z_1)) \leq e^{-\delta^2\mathbb{E}(Z_1)/3} = \frac{1}{n}$, when $\delta = \sqrt{\frac{3\log n}{\mathbb{E}(Z_1)}}$. Now with probability at least $1 - \frac{3}{n}$ (since there are three binomial terms involved), the algorithm will be successful to label
We can easily find $\mu_{ij} - \mu_{ij}'$ with a probability of at least $1$ when $\frac{\mu}{n} - \frac{k}{n} - \frac{\delta}{n}\log n$. The case of $\frac{\mu}{n} - \frac{k}{n}$ will follow similarly. Now let $Z_2$ denote the random variable that equals the number of motif of second type (neighbors of $u$ and non neighbors of $v$) i.e. $\{z \mid (z, u) \in E, (z, v) \in E\}$ of two nodes $u, v \in V : (u, v) \in E$. Let us also denote $\mu_s^2 = \mathbb{E}(Z_2 | u \sim v)$ and $\mu_d^2 = \mathbb{E}(Z_2 | u \sim v)$, where $u \sim v$ means $u$ and $v$ are in the same cluster. We can easily find $\mu_s^2$ and $\mu_d^2$ from Lemmas 1, 2. We see that,

$$\mu_s^2 - \mu_d^2 = \frac{(n - O(1))(3r_s - 2r_d)(r_s - 2r_d)}{4r_s} = \frac{(3a - 2b)(a - 2b)\log n}{4a} - O\left(\frac{\log n}{n}\right).$$

Now, since $Z_2$ is a sum of independent binary random variables, using Chernoff bound, $\Pr(Z_2 < (1 + \delta)\mathbb{E}(Z_2)) \leq \Pr(Z_2 > (1 + \delta)\mathbb{E}(Z_2)) \leq e^{-\delta^2\mathbb{E}(Z_2)/3} = \frac{1}{n}$, when $\delta = \sqrt{\frac{3\log n}{\mathbb{E}(Z_2)}}$. Now with probability at least $1 - \frac{2}{n}$ (since there are three binomial terms involved), the algorithm will be successful to label correctly as long as,

$$\frac{(3a - 2b)(a - 2b)\log n}{4a} \geq \left(\sqrt{\frac{a}{2}} + \frac{b(a - b)}{a} + \sqrt{a - b}\right)\sqrt{3\log n}.$$

The case of $r_s \leq 2r_d$ will follow similarly. Also, the expression for the third motif ($Z_3$) which considers the neighbors of $v$ and non neighbors of $u$ is same as the second motif. Using all three motifs ($Z_1, Z_2, Z_3$), and taking a majority vote guarantees us the resolution of a pair with a probability of at least $1 - \frac{2}{n}$ if both the inequalities above hold. In order to successfully label $n$ pairs of vertices we apply union bound over $n$ distinct pairs of nodes which guarantees the accuracy of recovery as $1 - O(1/n)$.

Now we generalize the analysis for a GBM graph $G(V, E)$ with $k$ equal sized (unknown) clusters $V = V_1 \sqcup \ldots \sqcup V_k$, and parameters $r_s$, $r_d$ for a common neighbor motif.

**Lemma 5.** For any two vertices $u,v \in V_i : (u, v) \in E, i \in [1, k]$ belonging to the same cluster, the count of common neighbors $C_{u,v} \equiv \{|z \in V : (z,u), (z,v) \in E\}$ is a random variable distributed according to $\text{Bin}(\frac{n}{k} - 2, \frac{3r_s}{2}) + \text{Bin}(n - \frac{n}{k}, \frac{2r_d}{r_s})$ when $r_s > 2r_d$ and according to $\text{Bin}(\frac{n}{k} - 2, \frac{3r_s}{2}) + \text{Bin}(n - \frac{n}{k}, 2r_d - r_s)$ when $r_s \leq 2r_d$, where $\text{Bin}(n, p)$ is a binomial random variable with mean $np$.

**Proof of Lemma 5.** Without loss of generality, assume $u, v \in V_1$. For any vertex $z \in V$, let $E_{u,v} \equiv \{(u,z), (v,z) \in E\}$ be the event that $z$ is a common neighbor. For $z \in V_1$,

$$\Pr(E_{u,v}) = \Pr((z,u) \in E, (z,v) \in E \mid (u,v) \in E) = \int_0^{r_s} \frac{1}{r_s} \Pr((z,u) \in E, (z,v) \in E \mid d_L(X_u, X_v) = x)dx = \int_0^{r_s} \frac{1}{r_s}(2r_s - x)dx = \frac{3r_s}{2}.$$ 

For $z \in V \setminus V_1$, we have,

$$\Pr(E_{u,v}) = \Pr((z,u) \in E, (z,v) \in E \mid (u,v) \in E)$$

$$= \frac{1}{r_s} \Pr((z,u) \in E, (z,v) \in E \mid d_L(X_u, X_v) = x)dx = \int_0^{r_s} \frac{1}{r_s}(2r_s - x)dx = \frac{3r_s}{2}.$$
clusters, the count of common neighbors accurately with a probability of 1. If

For any vertex \( v \) in \( V \) belonging to different clusters, the count of common neighbors \( C_{u,v} \) is a random variable distributed according to \( \text{Bin}(n - \frac{2n}{k}, 3d_r/2) \) when \( r_s > 2d_r \) and according to \( \text{Bin}(\frac{2n}{k} - 2, 2r_d, -r_s^2) + \text{Bin}(n - \frac{2n}{k}, 3d_r/2) \) when \( r_s \leq 2d_r \).

**Proof of Lemma 6.** For any vertex \( z \) in \( V \), let \( E_z^{u,v} \) be the event that \( z \) is a common neighbor. For \( z \) in \( V \setminus (V_i \cup V_j) \),

\[
\Pr(E_z^{u,v}) = \Pr((z, u) \in E, (z, v) \in E \mid (u, v) \in E) = \int_0^{2r_d} \frac{1}{r_d} \Pr((z, u) \in E, (z, v) \in E \mid d_L(X_u, X_v) = x) dx.
\]

Similarly for \( z \) in \( V \setminus (V_i \cup V_j) \),

\[
\Pr(E_z^{u,v}) = \Pr((z, u) \in E, (z, v) \in E \mid (u, v) \in E) = \int_0^{r_d} \frac{1}{r_d} \Pr((z, u) \in E, (z, v) \in E \mid d_L(X_u, X_v) = x) dx = \frac{3r_d}{2}.
\]

Now since there are \( \frac{n}{k} - 2 \) points in \( V \setminus (V_i \cup V_j) \), we have the statement of the lemma.

**Theorem 3.** If \( r_s = \frac{a \log n}{n} \) and \( r_d = \frac{b \log n}{n} \), \( r_s > r_d \), Algorithm 1 can recover the clusters \( V_1, \ldots, V_k \) accurately with a probability of \( 1 - \frac{\Delta^2}{n} \) if

\[
\begin{cases}
\frac{(1.5a^2 - ab - 2b^2) \log n}{ak} - \frac{3b \log n}{2} \geq \sqrt{\frac{9a}{k} + \sqrt{\frac{12}{k} \left( 1 - \frac{1}{k} \right) \frac{b^2}{a} + \sqrt{\frac{24b}{k} + \sqrt{3b/2 \left( 1 - \frac{2}{k} \right)}}} \log n, \quad \text{when } a \geq 2b \\
\frac{(a-b)(a-b-2b/2)}{12} \geq \sqrt{\frac{9a}{k} + \sqrt{\frac{6}{2} \left( 2b - \frac{a^2}{2} \right) \left( 1 - \frac{1}{k} \right) + \sqrt{12/k \left( 2a - \frac{a^2}{2} \right) + 9b(1-2/k)}}}, \quad \text{otherwise.}
\end{cases}
\]

**Proof.** Let us consider the case \( r_s > 2r_d \) first. Let \( Z \) denote the random variable that equals the number of common neighbors of two nodes \( u, v \in V : (u, v) \in E \). Let us also denote \( \mu_s = \mathbb{E}(Z | u \sim v) \) and \( \mu_d = \mathbb{E}(Z | u \sim v) \), where \( u \sim v \) means \( u \) and \( v \) are in the same cluster. We can easily find \( \mu_s \) and \( \mu_d \) from Lemmas 1 and 2. We see that,

\[
\mu_s - \mu_d = \frac{(n/k - O(1))(1.5r_s^2 - r_d r_s - 2r_d^2)}{r_s} - \frac{3nr_d}{2}.
\]
\[ \left( \frac{1.5a^2 - ab - 2b^2}{ak} \right) \log n - \frac{3b \log n}{2} - O\left( \frac{\log n}{n} \right). \]

Now, since \( Z \) is a sum of independent binary random variables, using the Chernoff bound, \( \Pr(Z < (1 + \delta)\mathbb{E}(Z)) \leq \Pr(Z > (1 + \delta)\mathbb{E}(Z)) \leq e^{-\delta^2 \mathbb{E}(Z)/3} = \frac{1}{n^2} \), when \( \delta = \sqrt{\frac{\log n}{\mathbb{E}(Z)}} \). Now with probability at least \( 1 - \frac{1}{n^2} \) (since there are four binomial terms involved and they can have deviations more than \( \sqrt{\frac{9a}{k}} \log n, \sqrt{12(1 - \frac{1}{k})b^2/a} \log n, \sqrt{\frac{2b}{k}} \log n, \) and \( \sqrt{\frac{2b}{k}}(1 - \frac{2}{k}) \log n \) with probability \( \frac{1}{n^2} \) each), the algorithm will be successful to label correctly as long as,

\[ \frac{(1.5a^2 - ab - 2b^2) \log n}{ak} - \frac{3b \log n}{2} \geq \left( \sqrt{\frac{9a}{k}} + \sqrt{12 \left( 1 - \frac{1}{k} \right) \frac{b^2}{a}} + \sqrt{\frac{2b}{k}} + \sqrt{\frac{3b}{2k} \left( 1 - \frac{2}{k} \right)} \right) \log n. \]

The case of \( r_s \leq 2r_d \) will follow similarly. Now we need the labeling to be successful for \( n \) pairs of vertices (so that a spanning tree can be formed). Applying union bound over \( n \) distinct pairs guarantees the probability of recovery as \( 1 - 4/n \). \( \square \)