Dynamics of black holes

Sean A. Hayward

Center for Astrophysics, Shanghai Normal University, 100 Guilin Road, Shanghai 200234, China

This is a review of current theory of black-hole dynamics, concentrating on the framework in terms of trapping horizons. Summaries are given of the history, the classical theory of black holes, the defining ideas of dynamical black holes, the basic laws, conservation laws for energy and angular momentum, other physical quantities and the limit of local equilibrium. Some new material concerns how processes such as black-hole evaporation and coalescence might be described by a single trapping horizon which manifests temporally as separate horizons.

Keywords: Black holes, gravitational radiation, energy, angular momentum

I. INTRODUCTION

Black holes are now generally regarded as astrophysical realities, which are expected to be major sources of gravitational radiation, prompting extensive studies of dynamical, strong-field processes such as binary mergers. The textbook theory of black holes, however, mostly concerns stationary black holes or physically unlocatable event horizons. In recent years, a new paradigm for dynamical black holes has been developed in terms of trapping horizons, generated from surfaces where light is momentarily caught by the gravitational field. They locate the black hole in a practical, local way.

The original idea due to Penrose is that of a trapped surface, which compose the interior of a Schwarzschild black hole. Such surfaces played a key role in the singularity theorems of Penrose and Hawking. The boundary of the black hole is composed of marginally trapped surfaces, here called marginal surfaces. They are also commonly called apparent horizons, though the textbook definition of the latter differs. By itself, this does not capture the idea of a black hole, as such surfaces also exist in cosmological models. Presumably for this reason, such quasi-local ideas were not much developed at the time. Instead history took a different turn, with the introduction of global methods and the near-universal acceptance of an event horizon as the definition of black hole.

The author’s interest in this area began with a rather simple observation: that it is outgoing (rather than ingoing) wavefronts which are marginally trapped, that ingoing wavefronts are converging, and that the trapped surfaces are inside (rather than outside). This led to a definition of a black hole by a future (rather than past) outer (rather than inner) trapping horizon. Here trapping horizon is used to mean any hypersurface foliated by marginal surfaces, a change from the original definition.

Remarkably, this refinement sufficed to derive some key expected properties of black holes, assuming positive-energy conditions: the marginal surfaces have spherical topology; the horizon is spatial or null, and null only when a certain energy density vanishes, so that the horizon is one-way traversable; and the area $A$ is constant when null and increasing when spatial. Physically, if something falls into a black hole, the horizon moves out and the area increases.

In the case of spherical symmetry, a rather complete picture emerged. There is a standard definition of active gravitational mass $M$. It satisfies an energy conservation law akin to the first law of thermodynamics, with energy-supply and work terms, reducing to the Bondi energy equation at null infinity. There is also a natural definition of surface gravity $\kappa$, taking a quasi-Newtonian form, just $M/r^2$ in vacuo, in units $G = 1$, where $A = 4\pi r^2$. It has various desirable properties, including that $\kappa > 0$, $\kappa = 0$ or $\kappa < 0$ on outer, degenerate or inner trapping horizons respectively. Then projecting the energy conservation law along a trapping horizon, denoted by a prime, yields $M' \equiv \kappa A'/8\pi$ plus work term, where $\equiv$ denotes evaluation on a trapping horizon. This is a dynamic version of the so-called first law of black-hole mechanics.

Versions of these definitions and results also exist in cylindrical symmetry, which is the simplest situation allowing gravitational radiation, in plane symmetry and in a quasi-spherical approximation, which allows the interaction of roughly spherical black holes with gravitational radiation. In all these cases, there is an effective energy tensor $\Theta_{\alpha\beta}$ for the gravitational radiation, entering the energy conservation law additively with the matter energy tensor $T_{\alpha\beta}$.

Meanwhile, Ashtekar and others developed a theory of isolated horizons, which are null trapping horizons with a hierarchy of additional conditions, each intended to describe a black hole in local equilibrium to some degree, generalizing Killing horizons. In particular, they defined angular momentum $J$, entering a first law in the expected way.

A breakthrough came with the work of Ashtekar & Krishnan on what they called dynamical horizons, which are spatial future trapping horizons. They derived flux laws for both energy and angular momentum. In the versions subsequently developed by the author, they hold for any trapping horizon and take the form of conservation laws

$$ L_\xi M \equiv \oint_S \star (T_{\alpha\beta} + \Theta_{\alpha\beta}) k^\alpha \tau^\beta $$

$$ L_\xi J \equiv -\oint_S \star (T_{\alpha\beta} + \Theta_{\alpha\beta}) \psi^\alpha \tau^\beta $$

(1)
where $S$ is a marginal surface, *1 its area form, $L$ is the Lie derivative, $\xi$ is a generating vector of the trapping horizon, $\tau$ its normal dual, and $k$ and $\psi$ are certain vectors which play the role normally played by Killing vectors. Here $\Theta_{\alpha\beta}$ is again interpreted as an effective energy tensor for gravitational radiation. Apart from the inclusion of $\Theta_{\alpha\beta}$, these have the same form as corresponding flat-space conservation laws in surface-integral form.

Some subtleties remained. Firstly, in what one might call the bottom-up approach of Ashtekar and co-authors, the formalisms for isolated and dynamical horizons were quite different, and it was not clear how to relate them. Some progress was made by Booth & Fairhurst for slowly evolving horizons. Conversely, in the author’s top-down approach, whereby local equilibrium is given simply by the trapping horizon being null, there is a degeneracy in taking the null limit. Remarkably, consideration of an angular momentum closed this gap by suggesting a natural way to fix the degeneracy. The resulting gauge-fixed null trapping horizon is then equivalent, modulo further gauge-fixing, to a weakly isolated horizon. Thus the top-down and bottom-up approaches have converged on the same definition of a black hole in local equilibrium.

For this reason, combined with the existence of the reviews of Ashtekar & Krishnan, Booth, Krishnan, and Gourgoulhon & Jaramillo, this review will concentrate on the paradigm in terms of trapping horizons, without attempting to make detailed comparisons with the other approaches. Indeed, it is intended to be largely complementary to those reviews, apart from inevitable overlap on core issues. No attempt will be made at a comprehensive list of references, since the area and related areas are rapidly evolving. With a broad audience in mind, detailed calculations and proofs will be deferred to references. General Relativity, including the Einstein equation, will be assumed throughout, though the essential ideas can be generalized to other metric theories or dimensions.

II. CLASSICAL THEORY OF BLACK HOLES

What one might nowadays call Newtonian black holes were described long ago by Mitchell and Laplace, as objects for which the escape speed $\sqrt{2GM/r}$ is greater than the speed $c$ of light. Any light leaving such an object would be trapped by the gravitational field and fall back, rendering it invisible. Curiously, the Newtonian relation $r < 2GM/c^2$ will survive quite generally in a sense to be revealed, with units $c = 1 = G$ henceforth.

Almost immediately after Einstein formulated General Relativity in 1915, Schwarzschild derived the solution for the gravitational field of a point mass $M$. Charge $Q$ was soon added by Reissner and Nordström. However, the geometry of the solutions was not understood for decades. Einstein & Rosen deciphered the ($Q=0$) spatial geometry in 1935, with a wormhole connecting two asymptotically flat spaces, but it was not until around 1960 that the maximally extended space-time geometry was understood, with work of Wheeler, Kruskal and Fronsdal. The space-time diagram clearly shows a region $r < 2M$ from which light cannot escape. It had already been shown by Oppenheimer & Snyder in 1939 that gravitational collapse could indeed produce such a region. According to relativistic causality, nothing else can escape either.

Angular momentum $J$ was added by Kerr and (for $Q \neq 0$) Newman et al. and finally around 1968, the term “black hole” entered the lexicon, attributed to Wheeler. A few years of rapid progress followed, culminating in the classical paradigm. Black holes were defined in general by event horizons as defined by Penrose, which satisfy the area-increase theorem of Hawking, $A' \geq 0$. This became known as a second law, part of the four laws of black-hole mechanics formulated by Bardeen, Carter & Hawking, which seemed analogous to the four laws of thermodynamics. The zeroth law is that surface gravity $\kappa$ is constant on stationary black holes. The first law is

$$\delta E = \kappa \delta A/8\pi + \Omega \delta J + \Phi \delta Q$$

for perturbations of stationary black holes, where $E$ is the ADM energy at spatial infinity, $\Omega$ is the angular speed and $\Phi$ is the electric potential. The third law is $\kappa \neq 0$, by perturbations of stationary black holes. The discovery of quantum black-hole radiance by Hawking, with temperature $\kappa/2\pi$ in units $k = 1$, led to much research on black holes as a key area to generate and test ideas concerning the interface of gravity, quantum theory and thermodynamics.

Once a theoretical curiosity, observational evidence for black holes has by now accumulated past the point where
null normal directions, along null normal vectors \( l_\pm \):

\[
g(l_\pm, l_\pm) = 0, \quad \perp l_\pm = 0
\]  

where \( g \) is the space-time metric and \( \perp \) denotes projection onto \( S \). The null expansions are

\[
\theta_\pm = L_\pm \log \ast 1
\]

where \( L_\pm = L_{l_\pm} \).

Then \( S \) is said to be trapped if \( \theta_+ \theta_- > 0 \), marginal if \( \theta_+ = 0 \) or \( \theta_- = 0 \), and untrapped (or mean convex) if \( \theta_+ \theta_- < 0 \). In terms of the expansion vector or mean-curvature vector

\[
H = g^{-1}(d \log \ast 1)
\]

this is equivalent to \( H \) being temporal, null or spatial, respectively. One finds

\[
H = -e^f (\theta_- l_+ + \theta_+ l_-)
\]

where \( f \) is a normalization function,

\[
e^{-f} = -g(l_+, l_-).
\]

For any normal vector \( \eta \), \( \perp \eta = 0 \), the expansion \( \theta_\eta = L_\eta \log \ast 1 \) is given by \( \theta_\eta = \eta(H, \eta) \).

Untrapped surfaces have a local spatial orientation: an achronal (meaning spatial or null) normal vector \( \eta \) is outward or inward if \( \theta_\eta > 0 \) or \( \theta_\eta < 0 \) respectively. Locally one can conventionally fix \( \theta_+ > 0 \), \( \theta_- < 0 \) in an untrapped region; then \( l_+ \) is outward and \( l_- \) is inward. Conversely, trapped surfaces have a local causal orientation: if \( H \) is future or past causal (meaning temporal or null), the surface is future or past trapped respectively. Then future or past trapped surfaces have \( \theta_\pm < 0 \) or \( \theta_\pm > 0 \) respectively. Marginal surfaces have both orientations, if the other null expansion has fixed non-zero sign.

A trapping horizon is a hypersurface foliated by marginal surfaces. It is said to be outer or inner if \( L_- \theta_+ < 0 \) or \( L_+ \theta_- > 0 \) respectively, for the case \( \theta_+ = 0 \), where the null normals \( l_\pm \) are extended off the horizon to generate ingoing and outgoing wavefronts from the marginal surfaces, i.e. two families of null hypersurfaces labelled by \( x^\pm \), intersecting in the marginal surfaces, such that \( l_A(dx^B) = \delta_A^B \). This is locally unique unless the trapping horizon is itself null. Equivalently, the horizon is outer or inner if \( \nabla \cdot (H - H^*) > 0 \) or \( \nabla \cdot (H - H^*) < 0 \) respectively, where \( H = \eta^+ l_+ + \eta^- l_- \) is the normal dual vector \(36,37,38\) to a normal vector \( \eta = \eta^+ l_+ + \eta^- l_- \):

\[
\perp \eta^* = 0, \quad g(\eta^*, \eta) = 0, \quad g(\eta^*, \eta^*) = -g(\eta, \eta).
\]

This is a vector version of the normal Hodge dual for 1-forms.

Then a future (respectively past) outer trapping horizon provides a local definition of a generic black (respectively white) holes\(\text{\sloppy}^8\). More precisely, the idea is that a non-degenerate black hole exists only if such a horizon exists. As to the converse, see the Concluding Remarks.
Note that outer versus inner has been defined with respect to the ingoing null direction, as this is invariant, rather than in some ingoing spatial direction. Also, demanding a strict sign for $L_{-} \Theta_{+}$ has excluded extremal black holes, for which $L_{-} \Theta_{+}$ vanishes. This is for simplicity only, as they can be treated as special cases. Relaxing the definition to $L_{-} \Theta_{+} \leq 0$ would allow cases which are not black holes.

IV. BASIC LAWS

Summarized here are some basic properties of trapping horizons. Where not obvious, the proofs involve either (for $\theta_{+} = 0$) the $T_{++}$ component of the Einstein equation, where the null energy condition (NEC) is cited, or the $T_{+-}$ component, where the dominant energy condition (DEC) is cited. One introduces a normal generating vector $\xi$ of the marginal surfaces in the horizon, which therefore satisfies $L_{\xi} \theta_{+} = 0$ on the horizon. The area of the marginal surfaces, if compact, is

$$A = \int_{S} *1.$$

Trapping: for a future/past, outer/inner trapping horizon, there are trapped surfaces to one side and untrapped surfaces to the other side. For a future outer trapping horizon, this reflects the defining idea that outgoing light rays are momentarily parallel, $\theta_{+} = 0$, diverging just outside, $\theta_{+} > 0$, and converging just inside, $\theta_{+} < 0$, while ingoing light rays are converging, $\theta_{-} < 0$.

Signature: assuming NEC, an outer or inner trapping horizon is achronal or causal respectively, and null if and only if the effective ingoing energy density $T_{++} + \Theta_{++}$ vanishes, where the explicit expressions for $\Theta_{a\beta}$ are given later. In particular, this means that black-hole horizons are one-way traversable: one can fall into a black hole but not escape, at least through the outer horizon.

Area: assuming NEC, future outer or past inner trapping horizons have non-decreasing area form, $\theta_{\xi} \geq 0$, and therefore (if compact) non-decreasing area, $L_{\xi} A \geq 0$, instantaneously constant ($\theta_{\xi} = 0$) if and only if the horizon is null. For past outer or future inner trapping horizons, all signs reverse. Here the orientation of $\xi$ is such that, in the null limit, it is future-null, which means that it is future causal for inner horizons and outward achronal for outer horizons. In particular, this means that black holes grow if they absorb any matter or gravitational radiation, and otherwise remain the same size.

Topology: assuming DEC, a future/past outer trapping horizon has marginal surfaces of spherical topology (if compact). The proof uses the Gauss-Bonnet and Gauss divergence theorems. Thus realistic black holes are topologically spherical. If degenerate horizons are considered, then toroidal topology is just allowed, but highly non-generic, in particular Gaussian flat, and so presumably unstable.

Area limit: assuming DEC and a positive cosmological constant $\Lambda$, outer trapping horizons satisfy $A \leq 4\pi/\Lambda$. Thus black holes are smaller than the cosmological horizon scale, corresponding to an area $12\pi/\Lambda$.

Pedagogically, this would be the place to give details in spherical symmetry, plane symmetry, and the quasi-spherical approximation, but they are omitted here due to limitations of space, apart from as explained in the Introduction.

V. CONSERVATION OF ENERGY

The simplest generalization of the Schwarzschild relation $1 - 2M/r = g^{rr}$ to a general surface is the Hawking mass:

$$M = \frac{r}{2} \left( 1 - \frac{1}{8\pi} \oint_{S} *g(H,H) \right),$$

$$= \frac{r}{2} \left( 1 + \frac{1}{8\pi} \oint_{S} e^{\xi_{+}\theta_{+}} \right),$$

where

$$r = \sqrt{A/4\pi}$$

is the area radius. It has various useful properties. Large spheres: in an asymptotically flat space-time, $M$ tends to the Bondi or ADM energy at null or spatial infinity, respectively. Small spheres: $M$/volume → density at a regular centre. Trapping: a surface is trapped, marginal or untrapped if $r < 2M$, $r = 2M$ or $r > 2M$ respectively.

In particular, this means that $M$ is the irreducible mass of a future outer trapping horizon, $L_{\xi} M \geq 0$, assuming NEC. This follows directly from the area law, since $A \approx 4\pi(2M)^2$, where $\approx$ henceforth denotes evaluation on a marginal surface. Recall the irreducible mass for stationary black holes: $M \approx (\frac{1}{16\pi} M(m + (m^2 - a^2)^{1/2})^2)^{1/2}$ for Kerr black holes, the mass which remains even if rotational energy is removed by the Penrose process. The original concept arose due to quasi-stationary arguments, but here it is exact and gives a physical meaning to $M$.

The simplest generalization of the Schwarzschild stationary Killing vector $k = \partial_{r}$ is the canonical time vector:

$$k = (g^{-1}(dr))^{*} = e^{\xi} (L_{+} r l_{-} - L_{-} r l_{+})$$

or equivalently

$$\perp k = 0, \quad k \cdot dr = 0, \quad g(k,k) = -g^{-1}(dr,dr).$$

In particular, $g(k,k) \cong 0$ and $k \cong \pm g^{-1}(dr,dr)$ on a trapping horizon $\theta_{\perp} \cong 0$. So trapping horizons are characterized by $k$ being null, just as Killing horizons are characterized by a stationary Killing vector being null (Fig.2). In spherical symmetry, $k$ reduces to the Kodama vector, which is a Noether current with Noether
charge $M$, and is related\textsuperscript{1} to $k^\beta\nabla_\beta k_\alpha \equiv \pm \kappa k_\alpha$ on a trapping horizon $\theta_\pm \equiv 0$, analogously to the usual definition of surface gravity for stationary black holes.

Lastly, one needs the dual normal vector to the trapping horizon:

$$\tau = \xi^* = \xi^+ l_+ - \xi^- l_-$$

which can be locally chosen to be future-pointing for outward-pointing $\xi$. For a spatial trapping horizon, $\xi$ is spatial and $\tau$ is temporal, while $\tau \to \xi$ as a trapping horizon becomes null (Fig. 3).

With these ingredients, conservation of energy takes the surface-integral form\textsuperscript{32,33}

$$L_\xi M \cong \oint_S \star(T_{AB} + \Theta_{AB})k^A \tau^B$$

where the components $\Theta_{AB}$ are given below. The proof is a calculation using the $T_{++}$ and $T_{+-}$ components of the Einstein equation and the Gauss-Bonnet and Gauss divergence theorems. It may be noted that evaluation on a trapping horizon yields remarkable cancellations in what would otherwise be lengthy expressions on the right-hand side. The conservation law can equivalently be written in volume-integral form

$$[M] \cong \int_H \star(T_{AB} + \Theta_{AB})k^A \tau^B \wedge dx$$

where $x$ labels the marginal surfaces, $\xi = \partial_x$, which expresses the change $[M]$ in $M$ between two marginal surfaces in the horizon $\hat{H}$. For a spatial trapping horizon with unit normal $\hat{\tau} = \tau/\sqrt{g_{xx}}$ and proper volume element $\hat{*}1 = \star\sqrt{g_{xx}} \wedge dx$, there is the proper-volume form

$$[M] \cong \int_H \hat{*}(T_{AB} + \Theta_{AB})k^A \hat{\tau}^B$$

which has the same form as the usual expression for energy, with $k$ replacing a stationary Killing vector. However $\hat{\tau}$ is ill defined and $\hat{*}1 \to 0$ if the trapping horizon becomes null, $g_{xx} \to 0$, which is the physically important limit where a growing black hole ceases to grow, so the above surface-integral form is preferred.

To spell out the components $\Theta_{AB}$, introduce the transverse metric $h_{ab}$, i.e. the induced metric of $S$, the null shears

$$\sigma_{\pm ab} = h_{a}^b h_{b}^d L_{+} h_{\gamma \delta} - \theta_{\pm} h_{ab}$$

and the normal fundamental forms

$$\zeta_{\pm a} = c^f h_{a}^\gamma (l_{\pm} \nabla_\gamma l_{\mp}^\beta).$$

Note here that indices $\alpha, \beta \ldots$ are general, $A, B \ldots$ normal, and $a, b \ldots$ transverse. Then the symmetric bilinear forms $\sigma_{\pm}$ are transverse, $\sigma_{\pm} = \pm \sigma_{\pm}$, and traceless, $h_{ab}\sigma_{\pm ab} = 0$, while $\zeta_{\pm}$ are transverse 1-forms, $\zeta_+ = \nabla \zeta_\pm$. Then the components of $\Theta_{\alpha\beta}$ with respect to $l_{\pm}$ are

$$\Theta_{\pm} = ||\sigma_{\pm}||^2/32\pi, \quad \Theta_{\pm \mp} = -l_{\pm}^{-1}||\zeta_{\pm}||^2/8\pi$$

where $||\zeta||^2 = h_{ab} \zeta_a \zeta_b \geq 0, ||\sigma||^2 = h_{ab} h_{cd} \sigma_{abc} \sigma_{bd} \geq 0$ are transverse norms. Here the signs indicate that $\Theta_{\alpha\beta}$ satisfies DEC, so that gravitational radiation carries positive energy.

The components may be interpreted as gravitational energy densities, by geodesic deviation of test particles\textsuperscript{44}: $\Theta_{++}$ is the ingoing transverse mode, reducing to the Bondi energy density at past null infinity; $\Theta_{--}$ is the outgoing transverse mode, reducing to the Bondi energy density at future null infinity; $\Theta_{+-}$ is the ingoing longitudinal mode, with $r^2\Theta_{+-} \to 0$ at null infinity; $\Theta_{-+}$ is the outgoing longitudinal mode, with $r^2\Theta_{-+} \to 0$ at null infinity. The $\Theta_{\pm \pm}$ components also recover expressions for energy density of gravitational radiation, with $\Theta_{\pm \mp}$ vanishing, in the high-frequency linearized approximation\textsuperscript{45}, in cylindrical symmetry\textsuperscript{46}, plane symmetry\textsuperscript{47,15} and in the quasi-spherical approximation\textsuperscript{48,49,50,19}.

Physically, the energy conservation law expresses the increase in irreducible black-hole mass $M$ in terms of the energy densities of the infalling matter and gravitational radiation. Since $A \cong 4\pi (2M)^2$, it also describes how a black hole grows.

### VI. CONSERVATION OF ANGULAR MOMENTUM

The standard definition of angular momentum for an axial Killing vector $\psi$ and at spatial infinity is the Komar momenta

$$\int \frac{\hat{*}k_{a} \hat{\tau}^a \psi}{\sqrt{g_{xx}}} dx$$

with $\hat{*}k_{a} \hat{\tau}^a \psi$ the angular momentum flow

$$\int \frac{\hat{*}k_{a} \hat{\tau}^a \psi}{\sqrt{g_{xx}}} dx = \int \frac{\hat{*}k_{a} \hat{\tau}^a \psi}{\sqrt{g_{xx}}} dx + \int \frac{\hat{*}k_{a} \hat{\tau}^a \psi}{\sqrt{g_{xx}}} dx$$

for Killing vectors $\hat{k}_{a}$.

### VII. NON-SPAN TRIQUETRA

A non-null hypersurface $H$ foliated by spatial surfaces $S$, with generating vector $\xi$ and its normal dual $\tau = \xi^*$.

![Non-null hypersurface](image)
\[ J[\psi] = -\frac{1}{16\pi} \oint_S *\epsilon_{\alpha\beta}\nabla^\alpha \psi^\beta \]  

(22)

where \( \epsilon_{AB} \) is the binormal. For a general transverse vector \( \psi \), \( \perp \psi = \bar{\psi} \), it can be rewritten as:

\[ J[\psi] = \frac{1}{8\pi} \oint_S *\psi^a \omega_a \]  

(23)

where the twist

\[ \omega_a = \frac{1}{2} \epsilon^I h_{a\beta}[l_-, l_+]^\beta \]  

(24)

is a transverse 1-form, \( \perp \omega = \omega \), measuring the non-integrability of the normal space.

For the weak-field metric in spherical polar coordinates \( (r, \theta, \varphi) \), \( J[\partial_\varphi] \) recovers the standard definition of angular momentum. Also the precessional angular velocity \( (\tilde{\Omega} \cdot \hat{r}) \tilde{r} - \frac{1}{4} \Omega \) of a gyroscope in the unit direction \( \hat{r} \), due to the Lense-Thirring effect, is directly related to the twist by \( \omega \sim \Omega \times \hat{r} \). Thus the twist indeed encodes the twisting around of space-time due to a rotating mass.

The twist is an invariant of a non-null foliated hypersurface \( H \), so the twist expression for \( J[\psi] \) is also an invariant of \( H \). It coincides with the 1-form used to define angular momentum for dynamical horizons by Ashtekar & Krishnan, but not that used for isolated horizons by Ashtekar et al.\(^{23,24} \), which is \( \omega = \frac{1}{2} Df \), where \( D \) is the covariant derivative of \( h \). They will give compatible \( J[\psi] \), by the Gauss divergence theorem, if the axial vector has vanishing transverse divergence,

\[ D_\alpha \psi^\alpha \cong 0. \]  

(25)

If there exist angular coordinates \( (\theta, \varphi) \) on \( S \), completing coordinates \( (x, \theta, \varphi) \) on \( H \), such that \( \psi = \partial_x \), then recalling that \( \xi = \partial_x \) and that coordinate vectors commute,

\[ L_\xi \psi \cong 0 \]  

(26)

which was previously proposed as a natural way to propagate \( \psi \) along \( \xi \) by Gourgoulhon. There have been various suggestions for specifying \( \psi \) more uniquely.\(^{34,35,68,69} \)

Then conservation of angular momentum takes the form\(^{34,35} \):

\[ L_\xi J \cong -\oint_S (T_{aB} + \Theta_{aB}) \psi^\alpha \tau^B. \]  

(27)

The proof is a calculation using the \( T_{aB} \) components of the Einstein equation, and requires various cancellations due to evaluation on a trapping horizon and the conditions. Here

\[ \Theta_{a\pm} = -\frac{1}{16\pi} h^{cd} D_d \sigma_{a\pm c} \]  

(28)

is the transverse-normal block of the effective energy tensor for gravitational radiation. Recalling the energy densities \( \Theta_{\pm} = ||\sigma_{\pm}||^2/32\pi \)\(^{24} \), indicating that transverse gravitational radiation is encoded in null shear \( \sigma_{\perp} \), it seems that differential gravitational radiation has angular momentum density. So this describes how a black hole spins up or down, due to infall of co-rotating or counter-rotating matter or gravitational radiation.

Thus conservation of energy and angular momentum take a similar form\(^{11, 12} \). Both take the same form as standard expressions in flat space-time for a stationary Killing vector \( k \) and an axial Killing vector \( \psi \), except for the inclusion of gravitational radiation in \( \Theta_{a\beta} \). They are the independent conservation laws expected for an astrophysical black hole, which defines its own centre-of-mass frame and its own axis of rotation.

Higher source multipoles have been defined for isolated horizons by Ashtekar et al.\(^{70} \) and for dynamical horizons by Schnetter et al.\(^{22} \). A gauge-dependent definition of linear momentum has been proposed by Krishnan et al.\(^{22} \), which may be useful in studying the recoil or kick effect in asymmetric binary mergers.

### VII. QUASI-LOCAL CONSERVATION LAWS

To compare with the classical paradigm, one needs to include charge \( Q \), which is defined in terms of charge-current density \( j \) as

\[ Q = -\int_H *g(j, \tau) \wedge dx = -\int_H *g(j, \hat{\tau}) \]  

(29)

where the more usual second expression holds only for a spatial hypersurface \( H \). As above, the surface-integral form is

\[ L_\xi Q = -\oint_S *g(j, \tau). \]  

(30)

The above conservation laws can be written in the same form

\[ L_\xi M \cong -\oint_S *g(\bar{j}, \tau), \quad L_\xi J \cong -\oint_S *g(\bar{j}, \tau) \]  

(31)

by identifying current vectors

\[ j^B = -k_A(T^{AB} + \Theta^{AB}), \quad \bar{j}^B = \psi_a(T^{aB} + \Theta^{aB}). \]  

(32)

The standard physical interpretation of the conserved vectors is \( j \) = (energy density, energy flux), \( \bar{j} \) = (angular momentum density, angular stress), \( j \) = (charge density, current density). For spatial \( \xi \): \( \oint_S *g(\bar{j}, \xi) = \) power; \( \oint_S *g(\bar{j}, \xi) = \) torque; \( \oint_S *g(\bar{j}, \xi) = \) current; \( -\oint_S *g(\bar{j}, \tau) = \) energy gradient; \( -\oint_S *g(\bar{j}, \tau) = \) angular momentum gradient; \( -\oint_S *g(\bar{j}, \tau) = \) charge gradient.

Charge conservation in local differential form is

\[ \nabla_\alpha \bar{j}^\alpha = 0. \]  

(33)

However, for energy and angular momentum, one has only quasi-local conservation laws holding on a trapping horizon:

\[ \oint_S *\nabla_\alpha \bar{j}^\alpha \cong -\oint_S *\nabla_\alpha \bar{j}^\alpha \cong 0. \]  

(34)
This subtly confirms the view that energy and angular momentum in General Relativity cannot be localized, but might be quasi-localized as surface integrals, as long ago argued by Penrose. The corresponding conservation laws have indeed been obtained in surface-integral but not local form.

VIII. STATE SPACE

There are now three conserved quantities \((M, J, Q)\), forming a state space for dynamical black holes. Following various authors, related quantities may then be defined by formulas satisfied by Kerr-Newman black holes, specifically those for the ADM energy

\[
E \simeq \frac{\sqrt{(2M)^2 + Q^2)^2 + (2J)^2}}{4M}
\]

(35)

the surface gravity

\[
\kappa \equiv \frac{(2M)^4 - (2J)^2 - Q^4}{2(2M)^4 \sqrt{((2M)^2 + Q^2)^2 + (2J)^2}}
\]

(36)

the angular speed

\[
\Omega \equiv \frac{J}{M \sqrt{((2M)^2 + Q^2)^2 + (2J)^2}}
\]

(37)

and the electric potential

\[
\Phi \equiv \frac{(2M)^2 + Q^2)Q}{2M \sqrt{((2M)^2 + Q^2)^2 + (2J)^2}}
\]

(38)

These formulas may not be so familiar, since the classical theory uses \((E, J, Q)\) as parameters with \(M\) dependent, but are easily obtained by inverting standard formulas.

In the dynamical context, \(E \geq M\) is not the ADM energy, but can be interpreted as the effective energy of the black hole. Expanding for \(J \ll M^2\) and \(Q \ll M\), \(E \approx M\) to leading order, then to next order,

\[
E \approx M + \frac{1}{2} I \Omega^2 + \frac{1}{2} Q^2 / r
\]

(39)

where \(J = I \Omega\) defines the moment of inertia \(I \equiv M \sqrt{((2M)^2 + Q^2)^2 + (2J)^2} \approx E r^2\). Thus \(E\) includes irreducible mass \(M\), rotational kinetic energy \(\approx \frac{1}{2} I \Omega^2\) and electrostatic energy \(\approx \frac{1}{2} Q^2 / r\), the latter being standard Newtonian expressions.

The state-space formulas

\[
\kappa \approx 8\pi \frac{\partial E}{\partial A} \approx \frac{1}{4M} \frac{\partial E}{\partial J}, \quad \Omega \approx \frac{\partial E}{\partial J}, \quad \Phi \approx \frac{\partial E}{\partial Q}
\]

(40)

then yield a dynamic version of the so-called first law of black-hole mechanics:

\[
L_\xi E \approx \frac{\kappa}{8\pi} L_\xi A + \Omega L_\xi J + \Phi L_\xi Q.
\]

(41)

Here the state-space perturbations in the classical law for Killing horizons, or the versions for isolated horizons, have been replaced by derivatives along the trapping horizon, thereby promoting it to a dynamical law.

IX. EQUILIBRIUM: NULL TRAPPING HORIZONS

When a growing black hole ceases to grow, the generically spatial trapping horizon becomes null. The central result here is that, assuming DEC,

\[
g(\bar{\omega}, \tau) \approx g(\bar{\omega}, \tau) \approx g(\bar{\omega}, \tau) \approx 0
\]

(42)

on a null trapping horizon. Therefore the conserved quantities are actually preserved:

\[
L_\xi M \equiv L_\xi J \equiv L_\xi Q \approx 0.
\]

(43)

This indicates that local equilibrium is attained when a trapping horizon becomes null.

It follows that

\[
L_\xi E \equiv L_\xi \kappa \equiv L_\xi \Omega \equiv L_\xi \Phi \approx 0.
\]

(44)

In particular, the surface gravity, which already satisfies \(D\kappa \approx 0\) by definition, is constant where a trapping horizon becomes null. This is a quite general zeroth law.

The above result is stronger still, since it expresses complete local equilibrium, not just thermal equilibrium.

By the area law, which includes \(L_\xi A \approx 0 \Rightarrow H\) null, this also shows that a black hole cannot change its angular momentum or charge without increasing its area.

On a null trapping horizon, one may take \(\xi \approx \tau \approx \ell_+\), but the other null vector \(l_-\) is non-uniquely, leading to non-uniqueness in \(\omega\), used to define angular momentum here and for dynamical horizons by Ashtekar & Krishnan. However, \(\omega + \frac{1}{2} Df\) is unique, an intrinsic normal fundamental form of a null hypersurface, therefore used to define angular momentum for isolated horizons by Ashtekar et al.

On the other hand, the extrinsic normal fundamental form \(\omega - \frac{1}{2} Df\) is preserved. DEC \(\Rightarrow L_+ (\omega - \frac{1}{2} Df) \approx 0\). The non-uniqueness is therefore naturally fixed by

\[
Df \approx 0.
\]

(45)

Recalling the definition of the normalization function \(f\), this is a legitimate choice of gauge. Then all three normal fundamental forms coincide. Also \(L_\xi \omega \approx 0\), so that \(L_\xi J \approx 0\) assuming only that \(\psi\) is a coordinate vector.

Comparing with energy, NEC \(\Rightarrow L_\xi M \approx 0\) automatically on a null trapping horizon.

Thus consideration of angular momentum resolves the ambiguity in taking the null limit. The general formalism for trapping horizons then applies in all cases, describing transitions between growing and non-growing phases of a black hole. This has largely resolved the question of weakly isolated horizon due to Ashtekar et al., except that the (allowable) scaling freedom in \(\xi\) has not been fixed.
X. TYPE-CHANGING HORIZONS

While the above properties indicate that a future outer trapping horizon serves as a practical definition of black hole, it should be noted that a trapping horizon may change its type under evolution. For instance, in gravitational collapse, it is common for an inner horizon to form simultaneously with an outer horizon. In fact they join smoothly and are really two parts of a single horizon in a space-time sense, manifesting as distinct horizons in a spatial slicing (Fig. 4). According to the basic laws, the transition occurs when the horizon is null, with the outer horizon achronal and the inner horizon causal.

Booth et al.\textsuperscript{74} constructed examples where an outer horizon becomes past-null, turning into a past-causal inner horizon, which can then turn into an outer horizon again, and so on. Also, numerical examples of Schnetter et al.\textsuperscript{71} show that a trapping horizon can be partly outer and partly inner on a given marginal surface, interpolating between regions where it is strictly inner or strictly outer.

One application of such ideas is to evaporating black holes. Ingoing Hawking radiation tends to have negative energy density, $T_{++} < 0$, violating NEC, and if it dominates over gravitational radiation in the sense that $T_{++} + \Theta_{++} < 0$, the basic laws reverse: the outer horizon becomes causal and shrinks, while the inner horizon becomes spatial and grows. Thus it is possible that they simply reunite smoothly, closing off the region of trapped surfaces (Fig. 4). The black hole has then evaporated. If no singularity ever formed, the space-time would have the global structure of Minkowski space-time and there would be no possibility of information loss and the associated information puzzle. Explicit examples of such space-times have been constructed\textsuperscript{75}.

Another application is to the coalescence of binary black holes, of great interest as generators of strong gravitational waves. Typically a common outer horizon forms around the original two outer horizons. As above, it joins smoothly to an inner horizon, though the latter is not always tracked in simulations. It can be conjectured that the inner horizon later joins with the original outer horizons. For instance, the inner horizon may pinch off and change topology from one sphere to two spheres, which close in on the original outer horizons and merge smoothly with them, leaving the entire region inside the common outer horizon composed of trapped surfaces (Fig. 5). This appears to be consistent with simulations\textsuperscript{71}, where the original outer horizons are slowly evolving and appear to be stable, while the inner horizon appears to be unstable and is rapidly evolving. More complex topologies are possible, and indeed there is numerical evidence that the original outer horizons can osculate and then intersect each other\textsuperscript{76}, allowing a self-intersecting inner horizon. It is also possible that singularities may intervene first, but this would be occurring in untrapped regions and so violate a version of strong cosmic censorship.

According to the above picture, the entire set of horizons actually form a single trapping horizon in space-time, despite manifesting temporally as different and apparently unrelated horizons. The whole horizon can be smooth, though the spatial sections will have at least one point of non-smoothness where the topology changes. The irreducible mass $M$, being given by the area $A$, may be evaluated throughout the entire process, satisfying the energy conservation law. The separate areas $A_1$ and $A_2$ add to the common area $A$ at the topology change,
\[ A = A_1 + A_2, \text{ so that, assuming NEC,} \]
\[ A \geq A_1 + A_2 \quad (46) \]

at any stage where either exists, where one follows the original outer horizons outwards in a space-time sense until the trapping horizon becomes a common outer horizon. The inequality holds also during inner stages, where one should note that the inner horizon is past-causal in an outward sense, so the area is still increasing outwards. This is an area-increase law for coalescing black holes, reminiscent of Hawking’s famous theorem for event horizons. It also implies
\[ M \geq \sqrt{M_1^2 + M_2^2}. \quad (47) \]

However, note that the total irreducible mass is actually reducible in such a process, indeed it necessarily decreases discontinuously at the change of topology, \( M < M_1 + M_2 \). The greatest fractional loss of total irreducible mass, \((M_1 + M_2 - M)/(M_1 + M_2)\), occurs in the equal-mass case and is \(1 - 1/\sqrt{2} \approx 29\%\), the same figure as obtained by the classical argument involving event horizons. In practice, the figure will be much less, since \( M \) decreases only at the change of topology and is otherwise increasing outwards, particularly during the violent type-changing phase.

Here one might prefer on physical grounds to use the effective energy \( E \) instead of \( M \) as a measure of available energy, which would require knowledge of angular momentum \( J \). One may also evaluate \( J \) throughout the whole process, but here there is an issue of whether it goes smoothly through the topology change, due to the more indirect construction and the numerically observed spin-flip phenomenon. This is of some interest by itself, as a potential way to relate initial and final angular momenta, and could be studied either analytically or by pushing simulations inside the region which is normally excised.

\section{XI. CONCLUDING REMARKS}

It seems appropriate to conclude with some important recent results and issues. Firstly, there is the uniqueness of trapping horizons. Ashtekar & Galloway have shown that the structure of a given dynamical horizon is unique, in that there is only one way to foliate it by marginal surfaces. They also showed that such horizons are not so numerous as to foliate a space-time region, so that they will tend to interweave one another.

Andersson, Mars & Simon\cite{73,84} defined a strictly stable marginally outer trapped surface by a condition similar to the outer condition and showed that, given such a surface in one of a foliation of spatial hypersurfaces, such surfaces exist locally in the foliation. This shows that trapping horizons are not unique in a given space-time, indeed one is locally generated from a given marginal surface however the hypersurface is evolved. In practice, this is a useful quality since it means that a trapping horizon is likely to be found in numerical simulations, provided the foliation is chosen reasonably well.

They also showed that, assuming NEC, the horizon is, on any one marginal surface, either spatial everywhere or null everywhere. Thus transitions between isolated and dynamical phases happen simultaneously with respect to the foliation.

Andersson & Metzger\cite{85} have shown that, given a spatial hypersurface with an outer trapped inner boundary and an outer untrapped outer boundary, there exists a stable marginally outer trapped surface between, which is smooth and unique as the outermost such surface. This strengthens an earlier result which assumed piecewise smoothness. Since this is the situation expected in numerical simulations of black holes, it guarantees the existence of an outermost trapping horizon with respect to the foliation.

Another issue is that, in a spherically symmetric space-time such as Vaidya, it is possible to find trapped or outer trapped surfaces which lie partly outside the spherically symmetric trapping horizon, as shown numerically by Schnetter & Krishnan\cite{86} and analytically by Ben-Dov\cite{87}. The surfaces are roughly spherical and inside the horizon except for a long tentacle which extends through it. These are rather strange-looking surfaces, so one interpretation is that, in order to locate a black hole in a physically acceptable way, the type of trapping horizon needs to be refined further. For instance, it seems reasonable to require marginal or trapped surfaces to have positive Gaussian curvature, which would tend to exclude such tentacles. Alternatively or additionally, one might require any surface sufficiently close to a marginal surface and inside it to be a (non-strictly) trapped surface.

This review has not addressed quantum issues directly, for which one may consult the review of Ashtekar & Krishnan\cite{88}. One remarkable feature of the framework for isolated horizons is a derivation of black-hole entropy proportional to area \( A \). Much less is known about dynamical situations. A very recent result is that, in spherical symmetry, a tunnelling method derives a local Hawking temperature \( k/2\pi \) precisely for future outer trapping horizons. Thus this framework for dynamical black holes seems to be well adapted to the issues raised by Hawking radiation.

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