Fischer decomposition by inframonogenic functions

Helmuth R. Malonek*,1, Dixan Peña Peña*,2 and Frank Sommen†,3

*Department of Mathematics, Aveiro University, 3810-193 Aveiro, Portugal
†Department of Mathematical Analysis, Ghent University, 9000 Gent, Belgium

1e-mail: hrmalon@ua.pt
2e-mail: dixanpena@ua.pt; dixanpena@gmail.com
3e-mail: fs@cage.ugent.be

Abstract

Let $\partial x$ denote the Dirac operator in $\mathbb{R}^m$. In this paper, we present a refinement of the biharmonic functions and at the same time an extension of the monogenic functions by considering the equation $\partial_x f \partial_x = 0$. The solutions of this “sandwich” equation, which we call inframonogenic functions, are used to obtain a new Fischer decomposition for homogeneous polynomials in $\mathbb{R}^m$.

Keywords: Inframonogenic functions; Fischer decomposition.

Mathematics Subject Classification: 30G35; 31B30; 35G05.

1 Introduction

Let $\mathbb{R}_{0,m}$ be the $2^m$-dimensional real Clifford algebra constructed over the orthonormal basis $(e_1, \ldots, e_m)$ of the Euclidean space $\mathbb{R}^m$ (see [6]). The multiplication in $\mathbb{R}_{0,m}$ is determined by the relations $e_j e_k + e_k e_j = -2\delta_{jk}$ and a general element of $\mathbb{R}_{0,m}$ is of the form $a = \sum_A a_A e_A$, $a_A \in \mathbb{R}$, where for

*accepted for publication in CUBO, A Mathematical Journal
\(A = \{j_1, \ldots, j_k\} \subset \{1, \ldots, m\}, j_1 < \cdots < j_k, e_A = e_{j_1} \cdots e_{j_k}\). For the empty set \(\emptyset\), we put \(e_{\emptyset} = 1\), the latter being the identity element.

Notice that any \(a \in \mathbb{R}_{0,m}\) may also be written as \(a = \sum_{k=0}^{m} [a]_k\) where \([a]_k\) is the projection of \(a\) on \(\mathbb{R}_{0,m}\). Here \(\mathbb{R}_{0,m}^{(k)}\) denotes the subspace of \(k\)-vectors defined by

\[
\mathbb{R}_{0,m}^{(k)} = \left\{ a \in \mathbb{R}_{0,m} : a = \sum_{|A|=k} a_A e_A, \quad a_A \in \mathbb{R} \right\}.
\]

In particular, \(\mathbb{R}_{0,m}^{(1)}\) and \(\mathbb{R}_{0,m}^{(0)} \oplus \mathbb{R}_{0,m}^{(1)}\) are called, respectively, the space of vectors and paravectors in \(\mathbb{R}_{0,m}\). Observe that \(\mathbb{R}^{m+1}\) may be naturally identified with \(\mathbb{R}_{0,m}^{(0)} \oplus \mathbb{R}_{0,m}^{(1)}\) by associating to any element \((x_0, x_1, \ldots, x_m) \in \mathbb{R}^{m+1}\) the paravector \(x = x_0 + x = x_0 + \sum_{j=1}^{m} x_j e_j\).

Conjugation in \(\mathbb{R}_{0,m}\) is given by

\[
\overline{a} = \sum_A a_A e_A, \quad a_A = (-1)^{|A|+1} e_A.
\]

One easily checks that \(\overline{ab} = \overline{b} \overline{a}\) for any \(a, b \in \mathbb{R}_{0,m}\). Moreover, by means of the conjugation a norm \(|a|\) may be defined for each \(a \in \mathbb{R}_{0,m}\) by putting

\[
|a|^2 = [a \overline{a}]_0 = \sum_A a_A^2.
\]

The \(\mathbb{R}_{0,m}\)-valued solutions \(f(x)\) of \(\partial_{\bar{x}} f(x) = 0\), with \(\partial_{\bar{x}} = \sum_{j=1}^{m} e_j \partial_{x_j}\) being the Dirac operator, are called left monogenic functions (see e.g. \([4, 8]\)). The same name is used for null-solutions of the operator \(\partial_x = \partial_{x_0} + \partial_{\bar{x}}\) which is also called generalized Cauchy-Riemann operator.

In view of the non-commutativity of \(\mathbb{R}_{0,m}\) a notion of right monogenicity may be defined in a similar way by letting act the Dirac operator or the generalized Cauchy-Riemann operator from the right. Functions that are both left and right monogenic are called two-sided monogenic.

One can also consider the null-solutions of \(\partial_{\bar{x}}^k\) and \(\partial_x^k\) \((k \in \mathbb{N})\) which gives rise to the so-called \(k\)-monogenic functions (see e.g. \([2, 3, 15]\)).

It is worth pointing out that \(\partial_{\bar{x}}\) and \(\partial_x\) factorize the Laplace operator in the sense that

\[
\Delta_{\bar{x}} = \sum_{j=1}^{m} \partial_{x_j}^2 = -\partial_{\bar{x}}^2, \quad \Delta_x = \partial_{x_0}^2 + \Delta_{\bar{x}} = \partial_{\bar{x}} \partial_x = \overline{\partial_x} \partial_x.
\]

Let us now introduce the main object of this paper.
Definition 1 Let $\Omega$ be an open set of $\mathbb{R}^m$ (resp. $\mathbb{R}^{m+1}$). An $\mathbb{R}_{0,m}$-valued function $f \in C^2(\Omega)$ will be called an inframonogenic function in $\Omega$ if and only if it fulfills in $\Omega$ the “sandwich” equation

$$\partial_x f \partial_x = 0 \quad (\text{resp. } \partial_x f \partial_x = 0).$$

Here we list some motivations for studying these functions.

1. If a function $f$ is inframonogenic in $\Omega \subset \mathbb{R}^m$ and takes values in $\mathbb{R}$, then $f$ is harmonic in $\Omega$.

2. The left and right monogenic functions are also inframonogenic.

3. If a function $f$ is inframonogenic in $\Omega \subset \mathbb{R}^m$, then it satisfies in $\Omega$ the overdetermined system $\partial_x^3 f = 0 = f \partial_x^3$. In other words, $f$ is a two-sided 3-monogenic function.

4. Every inframonogenic function $f \in C^4(\Omega)$ is biharmonic, i.e. it satisfies in $\Omega$ the equation $\Delta_x^2 f = 0$ (see e.g. [1, 11, 13, 16]).

The aim of this paper is to present some simple facts about the inframonogenic functions (Section 2) and establish a Fischer decomposition in this setting (Section 3).

2 Inframonogenic functions: simple facts

It is clear that the product of two inframonogenic functions is in general not inframonogenic, even if one of the factors is a constant.

Proposition 1 Assume that $f$ is an inframonogenic function in $\Omega \subset \mathbb{R}^m$ such that $e_j f$ (resp. $f e_j$) is also inframonogenic in $\Omega$ for each $j = 1, \ldots, m$. Then $f$ is of the form

$$f(x) = c x + M(x),$$

where $c$ is a constant and $M$ a right (resp. left) monogenic function in $\Omega$.

Proof. The proposition easily follows from the equalities

$$\partial_x(e_j f(x)) \partial_x = -2 \partial_x f(x) \partial_x e_j - e_j(\partial_x f(x) \partial_x),$$

$$\partial_x(f(x)e_j) \partial_x = -2 \partial_x f(x) e_j - (\partial_x f(x) \partial_x)e_j,$$

$$\partial_x f \partial_x = 0 \quad (\text{resp. } \partial_x f \partial_x = 0).$$

(1)
For a vector $x$ and a $k$-vector $Y_k$, the inner and outer product between $x$ and $Y_k$ are defined by (see [8])

\[
x \cdot Y_k = \begin{cases} [xY_k]_{k-1} & \text{for } k \geq 1 \\ 0 & \text{for } k = 0 \end{cases} \quad \text{and} \quad x \land Y_k = [xY_k]_{k+1}.
\]

In a similar way $Y_k \cdot x$ and $Y_k \land x$ are defined. We thus have that

\[
x Y_k = x \cdot Y_k + x \land Y_k,
Y_k x = Y_k \cdot x + Y_k \land x,
\]

where also

\[
x \cdot Y_k = (-1)^{k-1} Y_k \cdot x,
Y_k \land x = (-1)^k Y_k \land x.
\]

Let us now consider a $k$-vector valued function $F_k$ which is inframonogenic in the open set $\Omega \subset \mathbb{R}^m$. This is equivalent to say that $F_k$ satisfies in $\Omega$ the system

\[
\begin{cases}
\partial_\bar{z} \cdot (\partial_\bar{z} \cdot F_k) = 0 \\
\partial_\bar{z} \land (\partial_\bar{z} \cdot F_k) - \partial_\bar{z} \cdot (\partial_\bar{z} \land F_k) = 0 \\
\partial_\bar{z} \land (\partial_\bar{z} \land F_k) = 0.
\end{cases}
\]

In particular, for $m = 2$ and $k = 1$, a vector-valued function $f = f_1 e_1 + f_2 e_2$ is inframonogenic if and only if

\[
\begin{cases}
\partial_{x_1} f_1 - \partial_{x_2} f_2 - 2\partial_{x_1} f_2 = 0 \\
\partial_{x_1} f_2 - \partial_{x_2} f_1 + 2\partial_{x_1} f_1 = 0.
\end{cases}
\]

We now try to find particular solutions of the previous system of the form

\[
f_1(x_1, x_2) = \alpha(x_1) \cos(nx_2),
\]
\[
f_2(x_1, x_2) = \beta(x_1) \sin(nx_2).
\]

It easily follows that $\alpha$ and $\beta$ must fulfill the system

\[
\alpha'' + n^2 \alpha + 2n\beta' = 0
\]
\[
\beta'' + n^2 \beta + 2n\alpha' = 0.
\]

Solving this system, we get

\[
f_1(x_1, x_2) = (c_1 + c_2 x_1) \exp(nx_1) + (c_3 + c_4 x_1) \exp(-nx_1) \cos(nx_2), \quad (2)
f_2(x_1, x_2) = (c_3 + c_4 x_1) \exp(-nx_1) - (c_1 + c_2 x_1) \exp(nx_1) \sin(nx_2). \quad (3)
\]
Therefore, we can assert that the vector-valued function
\[ f(x_1, x_2) = \left( (c_1 + c_2 x_1) \exp(nx_1) + (c_3 + c_4 x_1) \exp(-nx_1) \right) \cos(nx_2)e_1 + \left( (c_3 + c_4 x_1) \exp(-nx_1) - (c_1 + c_2 x_1) \exp(nx_1) \right) \sin(nx_2)e_2, \quad c_j, n \in \mathbb{R}, \]
is inframonogenic in \( \mathbb{R}^2 \). Note that if \( c_1 = c_3 \) and \( c_2 = c_4 \), then
\[ f_1(x_1, x_2) = 2(c_1 + c_2 x_1) \cosh(nx_1) \cos(nx_2), \]
\[ f_2(x_1, x_2) = -2(c_1 + c_2 x_1) \sinh(nx_1) \sin(nx_2). \]
Since the functions (2) and (3) are harmonic in \( \mathbb{R}^2 \) if and only if \( c_2 = c_4 = 0 \), we can also claim that not every inframonogenic function is harmonic.

Here is a simple technique for constructing inframonogenic functions from two-sided monogenic functions.

**Proposition 2** Let \( f(\underline{x}) \) be a two-sided monogenic function in \( \Omega \subset \mathbb{R}^m \). Then \( \underline{x} f(\underline{x}) \) and \( f(\underline{x}) \underline{x} \) are inframonogenic functions in \( \Omega \).

**Proof.** It is easily seen that
\[ (\underline{x} f(\underline{x})) \partial_{\underline{x}} = \sum_{j=1}^{m} \partial_{x_j} (\underline{x} f(\underline{x})) e_j = \underline{x} (f(\underline{x}) \partial_{\underline{x}}) + \sum_{j=1}^{m} e_j f(\underline{x}) e_j = \sum_{j=1}^{m} e_j f(\underline{x}) e_j. \]
We thus get
\[ \partial_{\underline{x}} (\underline{x} f(\underline{x})) \partial_{\underline{x}} = -\sum_{j=1}^{m} e_j (\partial_{\underline{x}} f(\underline{x})) e_j - 2 f(\underline{x}) \partial_{\underline{x}} = 0. \]
In the same fashion we can prove that \( f(\underline{x}) \underline{x} \) is inframonogenic. \( \square \)

We must remark that the functions in the previous proposition are also harmonic. This may be proved using the following equalities
\[ \Delta_{\underline{x}} (\underline{x} f(\underline{x})) = 2 \partial_{\underline{x}} f(\underline{x}) + \underline{x} (\Delta_{\underline{x}} f(\underline{x})), \]
\[ \Delta_{\underline{x}} (f(\underline{x}) \underline{x}) = 2 f(\underline{x}) \partial_{\underline{x}} + (\Delta_{\underline{x}} f(\underline{x})) \underline{x}, \]
and the fact that every monogenic function is harmonic. At this point it is important to notice that an \( \mathbb{R}_{0,m} \)-valued harmonic function is in general not inframonogenic. Take for instance \( h(\underline{x}) e_j, h(\underline{x}) \) being an \( \mathbb{R} \)-valued harmonic function. If we assume that \( h(\underline{x}) e_j \) is also inframonogenic, then from (1) it
may be concluded that $\partial_x h(x)$ does not depend on $x_j$. Clearly, this condition is not fulfilled for every harmonic function.

We can easily characterize the functions that are both harmonic and inframonogenic. Indeed, suppose that $h(x)$ is a harmonic function in a star-like domain $\Omega \subset \mathbb{R}^m$. By the Almansi decomposition (see [12, 15]), we have that $h(x)$ admits a decomposition of the form

$$h(x) = f_1(x) + x f_2(x),$$

where $f_1(x)$ and $f_2(x)$ are left monogenic functions in $\Omega$. It is easy to check that

$$\partial_x h(x) = -m f_2(x) - 2 E_x f_2(x),$$

$E_x = \sum_{j=1}^m x_j \partial_{x_j}$ being the Euler operator. Thus $h(x)$ is also inframonogenic in $\Omega$ if and only if $m f_2(x) + 2 E_x f_2(x)$ is right monogenic in $\Omega$. In particular, if $h(x)$ is a harmonic and inframonogenic homogeneous polynomial of degree $k$, then $f_1(x)$ is a left monogenic homogeneous polynomial of degree $k$ while $f_2(x)$ is a two-sided monogenic homogeneous polynomial of degree $k - 1$.

The following proposition provides alternative characterizations for the case of $k$-vector valued functions.

**Proposition 3** Suppose that $F_k$ is a harmonic (resp. inframonogenic) $k$-vector valued function in $\Omega \subset \mathbb{R}^m$ such that $2k \neq m$. Then $F_k$ is also inframonogenic (resp. harmonic) if and only if one of the following assertions is satisfied:

(i) $F_k(x)x$ is left 3-monogenic in $\Omega$;

(ii) $x F_k(x)$ is right 3-monogenic in $\Omega$;

(iii) $x F_k(x)x$ is biharmonic in $\Omega$.

**Proof.** We first note that

$$e_j e_A e_j = \begin{cases} (-1)^{|A|} e_A & \text{for } j \in A, \\ (-1)^{|A|+1} e_A & \text{for } j \notin A, \end{cases}$$

which clearly yields $\sum_{j=1}^m e_j e_A e_j = (-1)^{|A|}(2|A| - m)e_A$. It thus follows that for every $k$-vector valued function $F_k$,

$$\sum_{j=1}^m e_j F_k e_j = (-1)^k (2k - m) F_k.$$
Using the previous equality together with (4) and (5), we obtain
\[
\partial_x \Delta_x (F_k(x)x) = 2 \partial_x F_k(x) + (\partial_x \Delta_x F_k(x))x + (-1)^k(2k - m) \Delta_x F_k,
\]
\[
\Delta_x (x F_k(x)) \partial_x = 2 \partial_x F_k(x) x + x (\Delta_x F_k(x) \partial_x) + (-1)^k(2k - m) \Delta_x F_k,
\]
\[
\Delta_x^2 (x F_k(x)x) = 4 (2 \partial_x F_k(x) \partial_x + (\partial_x \Delta_x F_k(x))x
\]
\[
\quad + x (\Delta_x F_k(x) \partial_x)) + x (\Delta_x^2 F_k(x))x.
\]

The proof now follows easily. □

Before ending the section, we would like to make two remarks. First, note that if \( m \) even, then a \( m/2 \)-vector valued function \( F_{m/2}(x) \) is inframonogenic if and only if \( F_{m/2}(x) \) and \( x F_{m/2}(x) \) are left 3-monogenic, or equivalently, \( F_{m/2}(x) \) and \( x F_{m/2}(x) \) are right 3-monogenic. Finally, for \( m \) odd the previous proposition remains valid for \( \mathbb{R}_{0,m} \)-valued functions.

3 Fischer decomposition

The classical Fischer decomposition provides a decomposition of arbitrary homogeneous polynomials in \( \mathbb{R}^m \) in terms of harmonic homogeneous polynomials. In this section we will derive a similar decomposition but in terms of inframonogenic homogeneous polynomials. For other generalizations of the Fischer decomposition we refer the reader to [5, 7, 8, 9, 10, 12, 14, 17, 18].

Let \( P(k) (k \in \mathbb{N}_0) \) denote the set of all \( \mathbb{R}_{0,m} \)-valued homogeneous polynomials of degree \( k \) in \( \mathbb{R}^m \). It contains the important subspace \( \mathfrak{l}(k) \) consisting of all inframonogenic homogeneous polynomials of degree \( k \).

An inner product may be defined in \( P(k) \) by setting
\[
\langle P_k(x), Q_k(x) \rangle_k = \left[ \overline{P_k(x)} \partial_x Q_k(x) \right]_0, \quad P_k(x), Q_k(x) \in P(k),
\]

\( \overline{P_k(\partial_x)} \) is the differential operator obtained by replacing in \( P_k(x) \) each variable \( x_j \) by \( \partial_{x_j} \) and taking conjugation.

From the obvious equalities
\[
[e_j a b]_0 = -[\overline{a e_j} b]_0,
\]
\[
[\overline{a e_j} b]_0 = -[\overline{e_j a b}]_0, \quad a, b \in \mathbb{R}_{0,m},
\]
we easily obtain
\[
\langle x P_{k-1}(x), Q_k(x) \rangle_k = - \langle P_{k-1}(x), \partial_x Q_k(x) \rangle_{k-1},
\]
\[
\langle P_{k-1}(x) x, Q_k(x) \rangle_k = - \langle P_{k-1}(x), Q_k(x) \partial_x \rangle_{k-1},
\]
with $P_{k-1}(x) \in P(k-1)$ and $Q_k(x) \in P(k)$. Hence for $P_{k-2}(x) \in P(k-2)$ and $Q_k(x) \in P(k)$, we deduce that

$$\langle xP_{k-2}(x), Q_k(x) \rangle_k = \langle P_{k-2}(x), \partial_x Q_k(x) \partial_x \rangle_{k-2}. \quad (6)$$

**Theorem 1 (Fischer decomposition)** For $k \geq 2$ the following decomposition holds:

$$P(k) = l(k) \oplus xP(k-2)x.$$

Moreover, the subspaces $l(k)$ and $xP(k-2)x$ are orthogonal w.r.t. the inner product $\langle , \rangle_k$.

**Proof.** The proof of this theorem will be carried out in a similar way to that given in [8] for the case of monogenic functions.

As $P(k) = xP(k-2)x \oplus (xP(k-2)x)^\perp$ it is sufficient to show that

$$l(k) = (xP(k-2)x)^\perp.$$

Take $P_k(x) \in (xP(k-2)x)^\perp$. Then for all $Q_{k-2}(x) \in P(k-2)$ it holds

$$\langle Q_{k-2}(x), \partial_x P_k(x) \partial_x \rangle_{k-2} = 0,$$

where we have used (6). In particular, for $Q_{k-2}(x) = \partial_x P_k(x) \partial_x$, we get that $\partial_x P_k(x) \partial_x = 0$ or $P_k(x) \in l(k)$. Therefore $(xP(k-2)x)^\perp \subset l(k)$.

Conversely, let $P_k(x) \in l(k)$. Then for each $Q_{k-2}(x) \in P(k-2)$,

$$\langle xQ_{k-2}(x)x, P_k(x) \rangle_k = \langle Q_{k-2}(x), \partial_x P_k(x) \partial_x \rangle_{k-2} = 0,$$

whence $P_k(x) \in (xP(k-2)x)^\perp$. \hfill \square

By recursive application of the previous theorem we get:

**Corollary 1 (Complete Fischer decomposition)** If $k \geq 2$, then

$$P(k) = \bigoplus_{s=0}^{[k/2]} x^s l(k-2s)x^s.$$

**Acknowledgments**

D. Peña Peña was supported by a Post-Doctoral Grant of Fundação para a Ciência e a Tecnologia, Portugal (grant number: SFRH/BPD/45260/2008).
References

[1] S. Bock and K. Gürlebeck, On a spatial generalization of the Kolosov-Muskhelishvili formulae, Math. Methods Appl. Sci. 32 (2009), no. 2, 223–240.

[2] F. Brackx, On (k)-monogenic functions of a quaternion variable, Funct. theor. Methods Differ. Equat. 22–44, Res. Notes in Math., no. 8, Pitman, London, 1976.

[3] F. Brackx, Non-(k)-monogenic points of functions of a quaternion variable, Funct. theor. Meth. part. Differ. Equat., Proc. int. Symp., Darmstadt 1976, Lect. Notes Math. 561, 138–149.

[4] F. Brackx, R. Delanghe and F. Sommen, Clifford analysis, Research Notes in Mathematics, 76, Pitman (Advanced Publishing Program), Boston, MA, 1982.

[5] P. Cerejeiras, F. Sommen and N. Vieira, Fischer decomposition and special solutions for the parabolic Dirac operator, Math. Methods Appl. Sci. 30 (2007), no. 9, 1057–1069.

[6] W. K. Clifford, Applications of Grassmann’s Extensive Algebra, Amer. J. Math. 1 (1878), no. 4, 350–358.

[7] H. De Bie and F. Sommen, Fischer decompositions in superspace, Function spaces in complex and Clifford analysis, 170–188, Natl. Univ. Publ. Hanoi, Hanoi, 2008.

[8] R. Delanghe, F. Sommen and V. Souček, Clifford algebra and spinor-valued functions, Mathematics and its Applications, 53, Kluwer Academic Publishers Group, Dordrecht, 1992.

[9] D. Eelbode, Stirling numbers and spin-Euler polynomials, Experiment. Math. 16 (2007), no. 1, 55–66.

[10] N. Faustino and U. Kähler, Fischer decomposition for difference Dirac operators, Adv. Appl. Clifford Algebr. 17 (2007), no. 1, 37–58.

[11] K. Gürlebeck and U. Kähler, On a boundary value problem of the biharmonic equation, Math. Methods Appl. Sci. 20 (1997), no. 10, 867–883.

[12] H. R. Malonek and G. Ren, Almansi-type theorems in Clifford analysis, Math. Methods Appl. Sci. 25 (2002), no. 16-18, 1541–1552.
[13] V. V. Meleshko, *Selected topics in the history of the two-dimensional biharmonic problem*, Appl. Mech. Rev. 56 (2003), no. 1, 33-85.

[14] G. Ren and H. R. Malonek, *Almansi decomposition for Dunkl-Helmholtz operators*, Wavelet analysis and applications, 35–42, Appl. Numer. Harmon. Anal., Birkhäuser, Basel, 2007.

[15] J. Ryan, *Basic Clifford analysis*, Cubo Mat. Educ. 2 (2000), 226–256.

[16] L. Sobrero, *Theorie der ebenen Elastizität unter Benutzung eines Systems hyperkomplexer Zahlen*, Hamburg. Math. Einzelschriften, Leipzig, 1934.

[17] F. Sommen, *Monogenic functions of higher spin*, Z. Anal. Anwendungen 15 (1996), no. 2, 279–282.

[18] F. Sommen and N. Van Acker, *Functions of two vector variables*, Adv. Appl. Clifford Algebr. 4 (1994), no. 1, 65–72.