On Haag Duality for Pure States of Quantum Spin Chain

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Abstract: In this note, we consider quantum spin chains and their translationally invariant pure states. We prove Haag duality for quasilocal observables localized in semi-infinite intervals \((\infty, -1]\) and \([0, \infty)\) when the von Neumann algebra generated by observables localized in \([0, \infty)\) is non type I.

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1 Introduction.

In local Quantum Field Theory, the Haag duality is a crucial notion in structure analysis. (See [9].) In this note, we consider the Haag duality for quantum spin systems on a one-dimensional lattice in an irreducible representation. By Haag duality we mean that the von Neumann algebra \( M_\Lambda \) generated by observables localized in an infinite subset \( \Lambda \) of \( \mathbb{Z} \) is the commutant of the von Neumann algebra \( M_{\Lambda^c} \) generated by observables localized in the complement \( \Lambda^c \) of \( \Lambda \). This duality plays a crucial role in analysis of entanglement property of states of infinite spin chain. See [10] and [11].

If these von Neumann algebra \( M_\Lambda \) is of type I, the duality is very easy to show. However, even if the representation of a whole quasi-local algebra is irreducible, the restriction to an infinite region may give rise to a non-type I von Neumann sub-algebra. For example, the restriction of the ground state of massless XY model to the semi-infinite interval \([0, \infty)\) gives rise to a type III von Neumann algebra and we believe that the same is true for the spin 1/2 massless antiferromagnetic XXZ chain. Though Haag duality is a basic concept, it seems that the proof of Haag duality is not obtained so far for the general case when both \( \Lambda \) and its complement \( \Lambda^c \) are infinite sets. We will see that the duality holds when the representation contains a vector state which is translationally invariant and \( \Lambda = [1, \infty) \).

To explain our results more precisely, we introduce our notation now. By \( \mathfrak{A} \), we denote the UHF \( C^* \)-algebra \( d^\infty \) (the infinite tensor product of \( d \) by \( d \) matrix algebras):

\[
\mathfrak{A} = \bigotimes_{\mathbb{Z}} M_d(C)^{C^*}.
\]

Each component of the tensor product above is specified with a lattice site \( j \in \mathbb{Z} \). By \( Q^{(j)} \) we denote the element of \( \mathfrak{A} \) with \( Q \) in the \( j \)th component of the tensor product and the identity in any other component. For a subset \( \Lambda \) of \( \mathbb{Z} \), \( \mathfrak{A}_\Lambda \) is defined as the \( C^* \)-subalgebra of \( \mathfrak{A} \) generated by elements \( Q^{(j)} \) with all \( j \) in \( \Lambda \). We set

\[
\mathfrak{A}_{\text{loc}} = \bigcup_{\Lambda \subset \mathbb{Z} : |\Lambda| < \infty} \mathfrak{A}_\Lambda
\]

where the cardinality of \( \Lambda \) is denoted by \( |\Lambda| \). We call an element of \( \mathfrak{A}_{\text{loc}} \) a local observable or a strictly local observable.

When \( \varphi \) is a state of \( \mathfrak{A} \), the restriction of \( \varphi \) to \( \mathfrak{A}_\Lambda \) will be denoted by \( \varphi_\Lambda : \)

\[
\varphi_\Lambda = \varphi|_{\mathfrak{A}_\Lambda}.
\]

We set

\[
\mathfrak{A}_R = \mathfrak{A}_{[1, \infty)}, \mathfrak{A}_L = \mathfrak{A}_{(-\infty, 0]}, \varphi_R = \varphi_{[1, \infty)}, \varphi_L = \varphi_{(-\infty, 0]}.
\]

By \( \tau_j \) we denote the automorphism of \( \mathfrak{A} \) determined by \( \tau_j(Q^{(k)}) = Q^{(j+k)} \) for any \( j \) and \( k \) in \( \mathbb{Z} \). \( \tau_j \) is referred to as the lattice translation of \( \mathfrak{A} \).
Given a representation \( \pi \) of \( \mathfrak{A} \) on a Hilbert space, the von Neumann algebra generated by \( \pi(A) \) is denoted by \( \mathfrak{M}_{\Lambda} \). We set

\[
\mathfrak{M}_R = \mathfrak{M}_{(1, \infty)} = \pi(\mathfrak{A}_R)^{\prime\prime}, \quad \mathfrak{M}_L = \mathfrak{M}_{(-\infty, 0]} = \pi(\mathfrak{A}_L)^{\prime\prime}.
\]

For a state \( \psi \) of a \( C^* \)-algebra \( \mathfrak{A} \) we denote the GNS triple by \( \{ \pi_{\psi}(\mathfrak{A}), \Omega_{\psi}, \mathfrak{H}_{\psi}\} \) where \( \pi_{\psi} \) is the GNS representation and \( \Omega_{\psi} \) is the GNS cyclic vector in the GNS Hilbert space \( \pi_{\psi} \).

**Theorem 1.1** Let \( \varphi \) be a translationally invariant pure state of the UHF algebra \( \mathfrak{A} \) and let \( \{ \pi_{\varphi}(\mathfrak{A}), \Omega_{\varphi}, \mathfrak{H}_{\varphi}\} \) be the GNS triple for \( \varphi \). Then, the Haag duality holds:

\[
\mathfrak{M}_R = \mathfrak{M}_L \tag{1.1}
\]

**Remark 1.2** We consider the situation that the state may not be faithful. In Proposition 4.2 [11], we have shown that \( \mathfrak{M}_R \) appearing in our context cannot be a type \( II_1 \) factor. Precise statement and its proof is included here in Lemma 2.2. We are not aware of any example of \( \mathfrak{M}_R \) which is of type \( II_\infty \). \( \mathfrak{M}_R \) is of type \( III \) in generic cases. For example, when the state \( \varphi_R \) is faithful, \( \mathfrak{M}_R \) is of type \( III_1 \) due to Theorem 4 of [13] by R.Longo. More precisely, An endomorphism \( \hat{\tau} \) of a factor is strongly asymptotically abelian if

\[
\lim_n \langle \hat{\tau}^n(Q), R \rangle = 0
\]

in strong operator topology for any \( Q \) and \( R \) in \( \mathfrak{M} \). If \( \varphi_R \) is faithful, the restriction of the (normal extension) shift \( \tau_1 \) of \( \mathfrak{M} \) to \( \mathfrak{M}_R \) is an strongly asymptotic abelian endomorphism of \( \mathfrak{M}_R \) and \( \mathfrak{M}_R \) is a type \( III_1 \) factor. See [13].

**Remark 1.3** In our proof of Haag duality, we consider a gauge invariant extension of the state \( \varphi \) to a state of the tensor product \( O_d \otimes O_d \) of Cuntz algebras and show the Haag duality at this level. We use ideas of [2] in our proof, though, our way of proof is different from [2]. In [2], O.Bratteli, P. Jorgensen, A.Kishimoto and R.Werner focus on dilation of Popescu systems to representations of Cuntz algebras and their pure states while our starting point is a pure state of (two-sided infinite) UHF algebras and go down to Popescu systems.

At first look, the section 7 of the paper [2] may give an impression that Proof of Theorem 7.1 of [2] implies Haag duality. (c.f. Lemma 7.7 and Lemma 7.8) However, for the KMS state of the standard \( U(1) \) gauge action of \( O_d \) the assumption of Theorem 7.1 of [2] are satisfied both Lemma 7.7 and Lemma 7.8 do not hold.

Let \( S_j \) be the Cuntz generator and consider the gauge action \( \gamma_z \) defined in Section 2. Then the \( \beta = \ln d \) KMS state \( \psi \) is unique, in particular it is faithful and the gauge invariant extension of the trace of \( \mathfrak{M}_R \). Then the assumption of Theorem 7.1 of [2] is satisfied for the GNS representation \( \{ \pi_{\psi}, \Omega_{\psi}, \mathfrak{H}_{\psi}\} \) of \( O_d \) associated with \( \psi \) if we set \( K = \Omega_{\psi} \cdot V_j = \pi_{\psi}(S_j) \). Then, \( \hat{V}_j = J\pi_{\psi}(S_j^*)J \) and

\[
\Omega_{\psi} = \Omega_0, \quad E|_K = P = I_K.
\]
Nevertheless, the state $\omega$ is not pure and the equivalence of conditions (i) and (iii) of Theorem 7.1 of [5] is valid. We prove that Lemma 7.6 of [5] is valid when the state of $A$ is pure, and for that purpose we introduce new ideas in Section 3. Our ideas are based on the observation that the translation $\tau_2$ is an inner automorphism of $O_d \otimes O_d$. We do not use Commuting Lifting Theorem of [5] for our proof of Haag duality.

2 Split Property

One key word in our analysis is split property or split inclusion.

Let $M_1$ and $M_2$ be factors acting on a Hilbert space $H$ satisfying $M_1 \subset M_2$. We say the inclusion is split if and only if there exists an intermediate type I factor $N$ such that $M_1 \subset N \subset M_2$.

The split inclusion is introduced for analysis of local QFT and of von Neumann algebras in 1980’s. (c.f. [8]) In [12] R.Longo used the notion for his solution to the factorial Stone-Weierstrass conjecture.

If mutually commuting factors $M_1$ and $M_2$ acting on a Hilbert space $H$ have a common cyclic and separating vector, say $\Omega$, the inclusion $M_1 \subset M_2$ is called standard. The standard split inclusion is a weak notion of independence of two quantum systems. Let $\varphi$ be the vector state associated with the common cyclic and separating vector $\Omega$ for $M_1$ and $M_2$.

A standard inclusion $M_1 \subset M_2$ is split if and only if $\varphi$ is quasi-equivalent to a product state $\psi_1 \otimes \psi_2$ where $\psi_1$ (resp. $\psi_2$) is a normal state of $M_1$ (resp. $M_2$) (c.f. [12]) . In our case, $M_1 = \pi(A_\Lambda)''$, $M_2 = \pi(A_{\Lambda^c})''$. We note that $M_1$ and $M_2$ may not have a common cyclic and separating vector in the GNS Hilbert space associated with a translationally invariant pure state and our inclusion may not be standard.

**Definition 2.1** A state $\varphi$ of the UHF algebra $\mathcal{A}$ for a one-dimensional quantum spin system has split property with respect to $\Lambda$ and $\Lambda^c$ if and only if $\varphi$ is quasi-equivalent to the product state $\varphi_\Lambda \otimes \varphi_{\Lambda^c}$.

It is easy to see that $\varphi$ has the split property if and only if $\varphi$ is quasi-equivalent to another product state $\psi_1 \otimes \psi_2$. When $\varphi$ is pure, $\varphi$ is unitarily equivalent to a product state $\psi_1 \otimes \psi_2$ where $\psi_1$ (resp. $\psi_2$) is a normal state of $M_1$ (resp. $M_2$) (c.f. [12]). In our case, $M_1 = \pi(A_\Lambda)''$, $M_2 = \pi(A_{\Lambda^c})''$. We note that $M_1$ and $M_2$ may not have a common cyclic and separating vector in the GNS Hilbert space associated with a translationally invariant pure state and our inclusion may not be standard.

**Lemma 2.2** Let $\varphi$ be a pure state of $\mathcal{A}$. If the von Neumann algebra $M_\Lambda$ generated $\pi_\varphi(A_\Lambda)$ is of type $I$, then $M_\Lambda = M_\Lambda'$. 

**Proof.** As the pure state $\varphi$ of $\mathcal{A}$ is split with respect to $\Lambda$ and $\Lambda^c$, $\varphi$ is unitarily equivalent to $\psi_1 \otimes \psi_2$ where $\psi_1$ (resp. $\psi_2$) is a state of $\mathcal{A}_\Lambda$ (resp. $\mathcal{A}_{\Lambda^c}$). The GNS Hilbert space $H_\varphi$ associated with $\varphi$ is unitarily equivalent to the tensor product $H_\psi_1 \otimes H_\psi_2$ and $M_\Lambda = B(H_\psi_1) \otimes 1_{H_\psi_2}$. $M_\Lambda^c = 1_{H_\psi_1} \otimes B(H_\psi_2) = M_{\Lambda^c}$. End of Proof.
As a consequence, in our proof of Haag duality, we concentrate on pure states \( \varphi \) which are not quasi-equivalent to \( \varphi_\Lambda \otimes \varphi_\Lambda^c \). Existence of a translationally invariant pure without the split property for \( \Lambda = [1, \infty) \), is highly non-trivial. In [13], we have shown that ground states of some spin 1/2 systems satisfy these requirement.

When \( \varphi \) is a translationally invariant factor state of \( A \), \( \varphi_R \) gives rise to a shift of the von Neumann algebra \( M_R \) in the following way. As there exists a unitary \( U \) implementing the shift \( \tau_1 \) specified with \( U \pi(Q) \Omega_\varphi = \pi(\tau_1(Q)) \Omega_\varphi \) for \( Q \in A \). \( \text{Ad}(U) \) gives rise to an endomorphism on the factor \( M_R \) generated by \( \pi_\varphi(A_R) \).

We denote this endomorphism of \( M_R \) by \( \hat{\tau}_1 \):
\[
\cap_{n=0}^{\infty} \hat{\tau}_1^n(M_R) = C1.
\]

**Lemma 2.3** Let \( \varphi \) be a translationally invariant pure state and let \( M_R \) be the von Neumann algebra generated by \( \pi_\varphi(A_R) \). \( M_R \) cannot be of type \( II_1 \).

**Proof.**
Suppose that \( M_R \) is of type \( II_1 \) and let \( tr \) be its unique normal tracial state. The shift endomorphism of \( A_R \) is a limit of cyclic permutations of \( (1, 2, \ldots, n) \) of lattice site which is implemented by unitary \( U_n, \tau_1(Q) = \lim U_nQU_n^* \). It turns out that the trace is invariant under \( \hat{\tau}_1 \) because
\[
tr(\hat{\tau}_1 \pi_\varphi(Q)) = tr(\pi_\varphi(\tau_1(Q))) = \lim_{n \to \infty} tr(\pi_\varphi(U_nQU_n^*)) = tr(\pi_\varphi(Q))
\]
Thus, as \( \varphi \) is the unique normal shift invariant state, \( \varphi_R = tr \). Then, the two sided translationally invariant extension of \( tr \) to \( A \) is a trace and this contradicts with our assumption that \( \varphi \) is pure. **End of Proof.**

If a translationally invariant pure state \( \varphi \) has the split property, the endomorphism \( \Theta_R \) of \( A_R \) defined as the restriction of \( \tau_1 \) to \( A_R \) is weakly inner on the GNS subspace associated with \( \varphi \). More precisely, let \( \Theta_R \) be an endomorphism of \( A \) determined by \( \Theta_R(Q) = \tau_1(Q) \) for \( Q \in A_R \) and \( \Theta_R(Q) = Q \) for \( Q \in A_L \).

If \( \varphi \) is a translationally invariant pure state of \( A \) with the split property, there exist isometries \( S_j \) \( (j = 1, 2, \ldots, d) \) acting on the GNS space associated with \( \varphi \) satisfying generating relations of the Cuntz algebra (c.f. the next section)
\[
S_j^* S_i = \delta_{ij} 1, \sum_{k=1}^d S_k S_k^* = 1
\]

As a consequence of weakly inner property of \( \Theta_R \), \( \varphi \) and \( \varphi \circ \Theta_R \) are mutually quasi-equivalent.

When \( \varphi \) is a state without split property \( \varphi \) and \( \varphi \circ \Theta_R \) may not be mutually quasi-equivalent. For example, the (unique) infinite volume ground state of the massless XY model with spin 1/2 (d=2) gives rise to such non-equivalence.
Proposition 2.4 Let $\varphi$ be the unique infinite volume ground state of the massless XY model with the following Hamiltonian $H$:

$$H = -\sum_{j\in\mathbb{Z}} \{\sigma^{(j)}_x \sigma^{(j+1)}_x + \sigma^{(j)}_y \sigma^{(j+1)}_y\}$$  \tag{2.2}$$

where $\sigma^{(j)}_x$ and $\sigma^{(j)}_y$ are Pauli spin matrices at the site $j$ in one-dimensional integer lattice $\mathbb{Z}$.

Then, $\Theta_R$ cannot be weakly inner in the sense specified in (2.1). In other words, the representations of $\mathfrak{A}$ associated with $\varphi$ and $\varphi \circ \Theta_R$ are disjoint.

Does non-split property of a translationally invariant pure state imply impossibility of obtaining a representation of the Cuntz algebra implementing $\Theta_R$ on $\mathfrak{A}$? At the moment we are not able to prove it. For the proof of Haag duality we do not need an answer to this question, though, we have to keep Proposition 2.4 in mind.

Sketch of Proof

The XY model is formally equivalent to the free Fermion on the one-dimensional lattice $\mathbb{Z}$. Our proof of Proposition 2.4 relies deeply on $\mathfrak{C}^*$-algebraic methods of [4] and results on quasifree states of CAR algebras. As these topics are not related to the proof of Haag duality we present here only a sketch of proof of Proposition 2.4.

Let $c_j$ and $c_j^*$ be the creation annihilation operators of Fermions on $\mathbb{Z}$ satisfying Canonical Anti-Commutation Relations (CAR), $\{c_j, c_k^*\} = \delta_{jk}$ etc. For $f = f(j)$ in $l^2(\mathbb{Z})$ we set $c^*(f) = \sum_{j\in\mathbb{Z}} c_j^* f_j$ and $c(f) = (c^*(f))^*$. By $\mathfrak{A}_{CAR}$ we denote the $\mathfrak{C}^*$-algebra generated by $c_j$ and $c_j^*$. We introduce the parity automorphism $\Theta_{parity}$ of $\mathfrak{A}_{CAR}$ and the spin algebra $\mathfrak{A}$ determined by $\Theta_{parity}(c_j) = -c_j$ and $\Theta_{parity}(\sigma^{(j)}_{x,y}) = -\sigma^{(j)}_{x,y}$. We set

$$\mathfrak{A}^\pm_{CAR} = \{Q \in \mathfrak{A}_{CAR}| \Theta_{parity}(Q) = \pm Q\}, \quad \mathfrak{A}^\pm = \{Q \in \mathfrak{A}| \Theta_{parity}(Q) = \pm Q\}.$$

A gauge invariant quasifree state $\psi$ of $\mathfrak{A}_{CAR}$ is determined by the covariance operator $A$ defined by $\psi(c^*(f)c(g)) = (g, Af)_{l^2(\mathbb{Z})}$ where the right-hand side is the inner product of $l^2(\mathbb{Z})$. Any bounded selfadjoint operator $A$ on $l^2(\mathbb{Z})$ satisfying $0 \leq A \leq 1$ gives rise to a quasifree state in this way, so by $\psi_A$ we denote the gauge invariant quasifree state of $\mathfrak{A}_{CAR}$ determined by $\psi_A(c^*(f)c(g)) = (g, Af)_{l^2(\mathbb{Z})}$.

Via Jordan-Wigner transformation and $\mathbb{Z}_2$ cross product, Pauli spin matrices (on $\mathbb{Z}$) are written in terms of $c_j$ and $c_j^*$ and $\mathfrak{A}^+_{CAR} = \mathfrak{A}^+$. The infinite volume ground state $\varphi$ of the XY model (2.2) is $\Theta_{parity}$ invariant and is determined by a quasifree state $\psi_p$ of $\mathfrak{A}_{CAR}$:

$$\varphi|_{\mathfrak{A}^+} = \psi_p.$$
In this formula, with help of Fourier series, \( l^2(\mathbb{Z}) \) is identified with \( L^2([−\pi, \pi]) \) and \( p \) is the multiplication operator of the characteristic function \( \chi_{[0,\pi]} \).

To show that \( \Theta_R \) is not weakly inner on the GNS space of the ground state \( \varphi \) of the XY model, it suffices to show that \( \varphi \) and \( \varphi \circ \Theta_R \) are not quasi-equivalent. To prove this claim, we focus our attention to the representation of \( \mathfrak{A}_{CAR}^+ \). The representation of \( \mathfrak{A}_{CAR}^+ \) on the GNS space associated with \( \varphi \) has decomposition into two components, both of which are irreducible.

Now look at
\[
\varphi \circ \Theta_R|_{\mathfrak{A}^+} = \psi_{u^*pu}|_{\mathfrak{A}^+}
\]

where \( u \) is an isometry on \( l^2(\mathbb{Z}) \). On \( L^2([−\pi, \pi]) \), \( u^*pu \) is an operator with a kernel function. If \( \varphi \circ \Theta_R \) and \( \varphi \) both restricted to \( \mathfrak{A}^+ \) are quasi-equivalent, the quasifree states \( \psi_p \) and \( \psi_{u^*pu} \) of the CAR \( \mathfrak{A}_{CAR} \) must be quasi-equivalent. (See the argument on the top of page 99 in [1]). So \( p - (u^*pu)^{1/2} \) and \( (1 - p) - (1 - u^*pu)^{1/2} \) are of Hilbert Schmidt class. These conditions imply that \( X = p - u^*pu \) is a Hilbert Schmidt operator. However, the kernel \( k(\theta_1, \theta_2) \) for the operator \( X \) has a singularity of order \( |\theta_1 - \theta_2|^{-2} \) at the diagonal part. Thus
\[
\text{Tr}(X^*X) = \text{Tr}((p - u^*pu)^2) = \infty. \tag{2.3}
\]

Thus (2.3) leads a contradiction if \( \varphi \) and \( \varphi \circ \Theta_R \) are quasi-equivalent.

End of Sketch of Proof 2.4.
3 \( O_d \otimes O_d \)

Our basic strategy to prove Theorem 1.1 is the following. We consider the gauge invariant extension \( \psi \) of the state \( \varphi \) to the Cuntz algebra \( O_d \otimes O_d \) and examine conditions of factoriality of \( \varphi \). Then, we consider a pure state \( \psi \) of \( O_d \otimes O_d \) which is a pure state extension of \( \varphi \) and prove Haag duality at the level of the Cuntz algebra.

Next we introduce our notation for the Cuntz algebra \( O_d \). The Cuntz algebra \( O_d \) is a simple \( C^\ast \)-algebra generated by isometries \( S_1, S_2 \cdots S_d \) satisfying \( S_k S_l = \delta_{kl} 1 \), \( \sum_{k=1}^d S_k S_k^\ast = 1 \). The gauge action \( \gamma_U \) of the group \( U(d) \) of \( d \) by \( d \) unitary matrices is defined via the following formula:

\[
\gamma_U(S_k) = \sum_{l=1}^d U_{kl} S_l.
\]

where \( U_{kl} \) is the \( k \) \( l \) matrix element for \( U \) in \( U(d) \). Consider the diagonal circle group \( U(1) = \{ z \in \mathbb{C} | |z| = 1 \} \) and \( \gamma_z \) on \( O_d \), \( \gamma_z(S_j) = z S_j, (j = 1, 2, \cdots d) \). The fixed point algebra \( O_d^{U(1)} \) for this action of \( U(1) \) is the UHF algebra \( d_\infty \) which we will identify with \( \mathfrak{A}_R = \mathfrak{A}_{[1, \infty)} \) as follows: Let \( I \) and \( J \) be \( m \)-tuples of ordered indices, \( I = (i_1, i_2, i_3, \cdots, i_m) , J = (j_1, j_2, j_3, \cdots, j_m) \) \((i_k, j_l \in \{1, 2, \cdots, d\})\) and set \( S_I = S_{i_1} S_{i_2} \cdots S_{i_m}, S_J = S_{j_1} S_{j_2} \cdots S_{j_m} \). Then, we identify the matrix unit of \( \mathfrak{A}_R \) and the \( U(1) \) gauge invariant part of \( O_d \) via the following equation:

\[
S_I S_J^* = e_{i_1 j_1}^{(1)} e_{i_2 j_2}^{(2)} \cdots e_{i_m j_m}^{(m)}
\]

where \( e_{ij} \) is the matrix element of the one-site matrix algebra. The canonical endomorphism \( \Theta \) of \( O_d \) is determined by

\[
\Theta(Q) = \sum_{k=1}^d S_k QS_k^* , \quad Q \in O_d.
\]

It is easy to see that the restriction of \( \Theta \) to \( \mathfrak{A}_R \) is the lattice translation \( \tau_1 \).

**Lemma 3.1** Let \( \varphi \) be a translationally invariant factor state of \( \mathfrak{A} \). Consider the restriction \( \varphi_R \) of \( \varphi \) to \( \mathfrak{A}_R \). Let \( \psi \) be the \( U(1) \) gauge invariant extension of \( \varphi_R \) to \( O_d \). Suppose further that \( \psi \) is not factor.

Then, there exists a positive \( k \) such that \( \tau_k \) acting on \( \mathfrak{A}_R \) is weakly inner on the GNS spaces associated with \( \varphi \) and \( \psi \). More precisely, there exists a representation \( \hat{\pi}(\cdot) \) of the Cuntz algebra \( O_{d \times k} \) on the GNS space \( \mathcal{H}_\psi \) such that

\[
\hat{\pi}(S_i) \in \pi_\psi(\mathfrak{A}_R)^\prime, \quad \sum_{i=1}^{dk} \hat{\pi}(S_i) \pi_\psi(Q) \hat{\pi}(S_i^*) = \pi_\psi(\tau_k(Q)). \tag{3.1}
\]

\( \hat{\pi}(S_i) \) implements the canonical endomorphism of \( O_{d \times k} \) as well.
Conversely, if there exist operators $T_j$ in $\pi_\psi(\mathfrak{A}_R)^{''}$ satisfying
\[ \sum_{j=1}^{d} T_j \pi_\psi(Q) T_j^* = \pi_\psi(\Theta(Q)) \]
for any $Q$ in $O_d$, $\psi$ is not a factor.

Proof. Let $\tilde{\psi}$ be the $U(1)$ gauge invariant extension of $\varphi$ to $O_d$ and $\{\pi(O_d), \Omega, \mathfrak{H}\}$ be the GNS representation associated with $\tilde{\psi}$. (\(\Omega\) is the GNS cyclic vector.) There exists a unitary representation $U_z$ of $U(1)$ satisfying
\[ U_z \Omega = \Omega, \quad U_z \pi(Q) U_z^* = \pi(\gamma_z(Q)) \quad \text{for } Q \in O_d. \tag{3.2} \]
We set $\mathcal{N} = \pi(O_d)^{''}$ and $\mathcal{C} = \mathcal{N} \cap \mathcal{N}'$. Using $U_z$ we have introduced the normal extension $\gamma_z$ of $U(1)$ action to the von Neumann algebra $\mathcal{N}$. (By abuse of notation we use the same symbol $\gamma_z$ for this action.) Let $Q$ be an element of $\mathcal{N} = \pi(O_d)^{''}$. and consider Fourier expansion of $Q$:
\[ Q = \sum_{k=-\infty}^{\infty} Q_k, \quad Q_k = \int dzz^{-k} U_z Q U_z^* \tag{3.3} \]
Let $\mathcal{N}_k$ be the subspace generated by operators $Q_k$:
\[ \mathcal{N}_k = \{ Q \in \mathcal{N} \mid \gamma_z(Q) = z^k Q \}, \quad \mathcal{N}_0 = \pi(\mathfrak{A}_R)^{''}. \tag{3.4} \]
Let
\[ C_k = \mathcal{N}_k \cap \mathcal{C} = \{ Q \in \mathcal{C} \mid \gamma_z(Q) = z^k Q \}. \]
As we assumed that $\mathcal{N}$ is not a factor, we can find a non-trivial self-adjoint element $c$ of the center $\mathcal{C}$. As $\mathcal{N}_0$ is a factor on $\mathcal{N}_0 \Omega$ and $\Omega$ is cyclic for $\mathcal{N}$, $c_0$ is a scalar multiple of the identity, i.e. $c_0 = c1$. As $c_k c_{-k}$ and $c_k c_k^*$ belong to $\mathcal{C}$, and since we assume that $\mathcal{C}$ is self-adjoint $c_k c_{-k} = c_k c_k^*$ is scalar. By the same reason, $c_{-k} c_k$ and $c_k^* c_k$ are scalar as well. Thus by rescaling we can assume that any non-vanishing $c_k$ is a unitary. Moreover if $C_k$ is not 0 it is one-dimensional. To see this take another central element $c^1$ and consider its Fourier component $c_k^1$. As $c_k^1 c_{-k}$ belongs to $\mathcal{C}_0$, it is a scalar.

Take the smallest positive $k$ such that $C_0$ is one-dimensional and for a multi-index $I$ with $|I| = k$, we set
\[ \tilde{\pi}(S_I) = \pi(S_I) c_k^*, \quad \tilde{\pi}(S_I^*) = \pi(S_I^*) c_k. \]
Both $\tilde{\pi}(S_I)$ and $\tilde{\pi}(S_I^*)$ are $\gamma_z$ invariant and their restriction to $\mathfrak{H}_{\varphi_R}$ satisfies (\ref{eq:3.1}).

Next let $T_j$ be a operators in $\pi_\psi(\mathfrak{A}_R)^{''}$ implementing the canonical endomorphism $\Theta$ of $O_d$. Then the operator $\pi_\psi(S_j^*) T_i$ commutes with any element of $\pi_\psi(O_d)^{''}$ because of
\[ \pi_\psi(S_j^*) \pi_\psi(\Theta(Q)) = \pi_\psi(Q) \pi_\psi(S_j^*). \]
Thus $\pi_\varphi(\mathcal{A})$ is not a factor. End of Proof.

The following lemma is known. (See, for example, Lemma 6.10 and 6.11 of [5].)

**Lemma 3.2** Let $\varphi$ be a translationally invariant factor state of $\mathcal{A}$. Suppose that for a positive $k$, the restriction $\tau_k$ to $\mathcal{A}_R$ is implemented by a representation $\tilde{\pi}(O_{d\times k})$ of the Cuntz algebra $O_{d\times k}$ on the GNS space $\mathcal{H}_\varphi$ and the gauge invariant part of $\tilde{\pi}(O_{d\times k})$ coincides with $\mathcal{A}_R$. More precisely,

$$\tilde{\pi}(SS^*) = \pi_{\varphi_R}(e^{i1})$$

Suppose that the gauge action $\gamma_z$ does not admit a normal extension to the von Neumann algebra $\tilde{\pi}(O_{d\times k})$ for any $z$. Then, $\tau_k$ is weakly inner in the sense of (3.4), namely $\tilde{\pi}(O_{d\times k})'' = \mathcal{A}_R''$.

**Proof:** By abuse of notation $\varphi_R$ is regarded as a state of the fixed point sub-algebra $(O_{d\times k})^{U(1)}$. Consider a vector state $\psi_0$ of $(O_{d\times k})^{U(1)}$ associated with the GNS vector for $\varphi_R$ and let $\psi$ be the $U(1)$ invariant extension of $\varphi_R$ to $(O_{d\times k})^{U(1)}$. Then, $\int \psi_0 \circ \gamma_z dz = \psi$ and at the level of the GNS representation,

$$\mathcal{H}_\psi = \int_0^\infty \mathcal{H}_{\psi_0} dz = \mathcal{H}_{\psi_0} \otimes L^2(S^1), \quad \pi_\psi = \int_0^\infty \pi_{\psi_0} \circ \gamma_z \, dz$$

Due to our assumption that $\gamma_z$ does not admit any normal extension to $\tilde{\pi}(O_{d\times k})''$ for any $z$, the von Neumann algebra $N = \pi_\psi(\mathcal{A}_R)''$ is isomorphic to $\mathfrak{M} \otimes L^\infty(S^1)$ where the gauge action acts as the rotation on $S^1$. $\pi_\psi(\mathcal{A}_R)''$ is the commutant of the unitaries implementing the rotation.

$$\pi_\psi(\mathcal{A}_R)'' = \mathfrak{M} \otimes 1 \tag{3.5}$$

By definition, $\pi_\psi(Q) = \pi_{\psi_0}(Q) \otimes 1$ for $Q$ in $\mathcal{A}_R$ and we have

$$\pi_\psi(\mathcal{A}_R)'' = \pi_{\varphi_R}(\mathcal{A}_R)'' \otimes 1 \tag{3.6}$$

Looking at each fiber of equations (3.5) and (3.6), we conclude that $\tilde{\pi}(O_{d\times k})'' = \mathcal{A}_R''$. End of Proof.

Next we consider a pair of Cuntz algebras denoted by $O_{d}^{(L)}$ and $O_{d}^{(R)}$ and we set $\mathcal{B} = O_{d}^{(L)} \otimes O_{d}^{(R)}$. The Cuntz generators are denoted by $S_{1}^{(L)}$ and $S_{1}^{(R)}$ etc. The algebra $\mathcal{B}$ is naturally equipped with the $U(1) \otimes U(1)$ gauge action $\gamma_{z_L,z_R} = \gamma_{z_L} \otimes \gamma_{z_R}$:

$$\gamma_{z_L,z_R}(S_{1}^{(L)}) = z_{L}^{1}|S_{1}^{(L)}|, \quad \gamma_{z_L,z_R}(S_{1}^{(R)}) = z_{R}^{1}|S_{1}^{(R)}| \quad (z_L,z_R \in U(1))$$

As $\mathcal{A} = \mathcal{A}_L \otimes \mathcal{A}_R$ we identify $\mathcal{A}$ with the $U(1) \otimes U(1)$ fixed point sub-algebra $\mathcal{B} = O_{d}^{L} \otimes O_{d}^{R}$. The canonical endomorphisms of $\mathcal{B}$ is defined via the following equation:

$$\Theta_{k,l} = \Theta_{L}^{k} \otimes \Theta_{R}^{l}$$
where $\Theta_L$ (resp. $O_d^R$) is the canonical endomorphism of $O_d^{(L)}$ (resp. $O_d^{(R)}$).

The lattice translation automorphism $\tau_1$ has an extension to $B$ as an inner automorphism. To see this, set

$$V = \sum_{j=1}^{d} (S_j^{(L)})^* S_j^{(R)}. \quad (3.7)$$

Then, $V$ satisfies

$$VV^* = V^*V = 1, \quad V e_{kl}^{(0)} V^* = V S_k^{(L)} (S_l^{(L)})^* V^* = S_k^{(R)} (S_l^{(R)})^* = \epsilon_{kl}^{(1)} \quad (3.8)$$

which shows that

$$\text{Ad}(V)(Q) = \tau_1(Q) \quad Q \in \mathfrak{A} \quad (3.9)$$

We extend $\tau_1$ to $B$ via the above equation (3.9).

Let $k$ be a positive integer and we regard $O_{d \times k}$ is a subalgebra of $O_d$ which is generated by $S_I$ and $S_J^*$ with $|I| = kn$, $|J| = km$ $(n, m = 1, 2, \cdots)$. Set

$$B^k = O_{d \times k}^L \otimes O_{d \times k}^R \subset B. \quad (3.10)$$

**Lemma 3.3** Let $\varphi$ be a pure state of $\mathfrak{A}$. Suppose that there exists a representation $\tilde{\pi}$ of $B^k$ on the GNS space $\mathcal{H}_\varphi$ associated with $\varphi$ such that

$$\tilde{\pi}(S_I^{(L)} (S_J^{(L)})^*) = \pi_\varphi (e_{ij}^{(0)} e_{i_2j_2}^{(-1)} \cdots e_{i_kj_k}^{(-k+1)}),$$
$$\tilde{\pi}(S_I^{(R)} (S_J^{(R)})^*) = \pi_\varphi (e_{ij}^{(1)} e_{i_2j_2}^{(2)} \cdots e_{i_kj_k}^{(k)}). \quad (3.10)$$

Then, $\tilde{\pi}(O_{d \times k}^{(L)}) \subset \pi_\varphi (\mathfrak{A}_L)'$, and $\tilde{\pi}(O_{d \times k}^{(R)}) \subset \pi_\varphi (\mathfrak{A}_R)'$.

**Proof:** Due to Lemma 3.2, we have only to show the gauge action $\gamma$ does not have a normal extension to the von Neumann algebra $\tilde{\pi}(B)'$. Any normal homomorphism of a type I factor is implemented by a unitary. As $\tilde{\pi}(B)$ is irreducible, we suppose there exists a unitary $W$ such that

$$W \tilde{\pi}(S_i^{(L)}) W^* = z_L S_i^{(L)}, \quad W \tilde{\pi}(S_i^{(R)}) W^* = z_R S_i^{(R)}. \quad (3.11)$$

Then, due to (3.10) $W$ commutes with the gauge invariant part $\mathfrak{A}_L$ and $\mathfrak{A}_R$. As $\varphi$ is pure, $W$ is a scalar multiple of the identity and $z_L = z_R = 1$.

**End of Proof**
Lemma 3.4 Let \( \varphi \) be a translationally invariant pure state of \( \mathcal{A} \) and let \( \overline{\psi} \) be the \( U(1) \times U(1) \) gauge invariant extension of \( \varphi \) to \( \mathcal{B} \).

\[
\overline{\psi}(Q) = \varphi \left( \int_{U(1) \times U(1)} \gamma_{zLzR}(Q) \, dzLdzR \right) \quad Q \in \mathcal{B}.
\] (3.12)

\( \overline{\psi} \) is not a pure state.

**Proof:** Let \( \{ \pi(\mathcal{B}), \Omega, \mathcal{F} \} \) be the GNS triple. As \( \overline{\psi} \) is \( \gamma_{zLzR} \) invariant, there exists a unitary \( U_{zLzR} \) satisfying

\[
U_{zLzR} \pi(Q) U_{zLzR}^* = \pi(\gamma_{zLzR}(Q)), \quad U_{zLzR} \Omega = \Omega
\]

We consider the restriction of \( \pi \) to \( \mathcal{A} \) and the Fourier decomposition of \( \mathcal{F} \) with respect to \( U_{zLzR} \).

\[
\mathcal{F} = \sum_{k,l} \mathcal{F}_{kl}.
\]

If \( \overline{\psi} \) is pure,

\[
\pi(\mathcal{A})'' = \pi(\mathcal{B})'' \cap C' = C'
\]

where \( C \) is the abelian von Neumann algebra generated by \( U_{zLzR} \). As \( \pi(\mathcal{A})'' \) is the commutant of \( C \), the center of \( \pi(\mathcal{A})'' \) is \( C \). Each irreducible representation \( \pi(\mathcal{A}) \) appearing in \( \pi(\mathcal{A})'' \) is of the form \( \pi(Q) \pi(P) \) where \( P \) is a central projection of \( \pi(\mathcal{A})'' \). Thus \( \pi(\mathcal{A})'' \) is decomposed into irreducible representations \( \pi_{kl} \) on \( \mathcal{F}_{kl} \). \( \pi_{kl} \) and \( \pi_{nm} \) are equivalent if and only if \( k = n \) and \( l = m \). \( \pi_{00} \) is equivalent to the GNS representation associated with \( \varphi \). However the operator \( \pi(V) \) gives rise to unitary equivalence between \( \pi_{00} \) and \( \pi_{1-1} \), which implies contradiction. Thus \( \overline{\psi} \) cannot be pure. *End of Proof.*

By the same line of argument in Lemma 3.1, we can show that the Fourier component \( C_{ij} \) of \( C \) in Lemma 3.4 is either one or zero dimensional. Furthermore \( C \) is generated by \( C_{k,-k} \) for some \( k \) when the canonical endomorphism is not weakly inner in \( \pi_{\varphi}(\mathcal{A})'' \). We show this claim rather implicitly in the next step.

We introduce the diagonal action \( \gamma^d_z \) of \( U(1) \) on \( \mathcal{B} \) via the equation: \( \gamma^d_z = \gamma_{z, z} \) and similarly the diagonal action \( \gamma^d_{z,k} \) of \( U(1) \) on \( \mathcal{B}^k \) Set

\[
\mathcal{D} = \{ Q \in \mathcal{B} \mid \gamma^d_z(Q) = Q \quad \text{for any } z \}.
\]

**Lemma 3.5**

(i) \( \mathcal{D} \) is generated by \( \mathcal{A} \) and \( V \), hence \( \mathcal{D} \) is isomorphic to the crossed product of \( \mathcal{A} \) by the action \( \tau_j \) of \( \mathbb{Z} \).

(ii) Let \( \varphi \) be a translationally invariant state of \( \mathcal{A} \). There exists a state \( \tilde{\varphi} \) of \( \mathcal{D} \) satisfying

\[
\tilde{\varphi}(V) = 1, \quad \tilde{\varphi}(Q) = \varphi(Q) \quad Q \in \mathcal{A}.
\] (3.13)

The state \( \tilde{\varphi} \) of \( \mathcal{D} \) satisfying (3.13) is unique. 

(iii) \( \tilde{\varphi} \) is pure if \( \varphi \) is factor.
Proof: (i) \( D \) is generated by \( S_i^{(L)}(S_j^{(R)})^*Q \) where multi-indices \( I \) and \( J \) satisfy \(|I| - |J| = 0\) and \( Q \) is an element of \( \mathfrak{A} \). By direct calculation, we have \( VS_i^{(L)}(S_j^{(R)})^* = S_i^{(R)}(S_j^{(R)})^* \). Thus
\[
S_i^{(L)}(S_j^{(R)})^*Q = V^*VS_i^{(L)}(S_j^{(R)})^*Q = V^*S_i^{(R)}(S_j^{(R)})^*Q
\]
which shows that \( S_i^{(L)}(S_j^{(R)})^*Q \) is written by a product of \( V \) and elements in \( \mathfrak{A} \).

(ii) Consider the GNS triple \( \{ \pi_\varphi(\mathfrak{A}), \Omega, \Delta_\varphi \} \) associated with \( \varphi \). As the state \( \varphi \) is translationally invariant we have a unitary \( W \) implementing \( \tau_1 \) and \( W\Omega = \Omega \). Then we set \( \pi_\varphi(V) = W \) the vector state \( \tilde{\varphi} \) of \( D \) associated with \( \Omega \) satisfies \( \tilde{\varphi} \). Conversely, if a state \( \tilde{\varphi} \) satisfies (3.13), the GNS cyclic vector \( \Omega_{\tilde{\varphi}} \) is invariant under \( \pi_{\tilde{\varphi}}(V) \) due to the identity:
\[
||[(\pi_{\tilde{\varphi}}(V) - 1)\Omega_{\tilde{\varphi}}]||^2 = 2 - \tilde{\varphi}(V) - \tilde{\varphi}(V^*) = 2 - 1 - 1 = 0.
\]

Thus \( W = \pi_{\tilde{\varphi}}(V) \).

(iii) As \( \varphi \) is factor, for \( Q \in \mathfrak{A} \)
\[
w - \lim_{k \to \infty} \pi_\varphi(\tau_k(Q)) = \varphi(Q)1.
\]

Suppose \( P \) commutes with \( \pi_{\tilde{\varphi}}(V) \) and \( \pi_\varphi(\mathfrak{A}) \). Then,
\[
(\Omega, \pi_{\tilde{\varphi}}(Q)P\Omega) = (\Omega, \pi_\varphi(Q)P\pi_{\tilde{\varphi}}(V^{-k})\Omega) = (\Omega, \pi_\varphi(Q)\pi_{\tilde{\varphi}}(V^{-k})P\Omega)
\]
\[
= (\Omega, \pi_\varphi(Q)\pi_{\tilde{\varphi}}(V^{k})P\Omega) = (\Omega, \pi_\varphi(\tau_k(Q))P\Omega)
\]
\[
= \lim_{k \to \infty} (\Omega, \pi_\varphi(\tau_k(Q))P\Omega) = \varphi(Q)(\Omega, P\Omega)
\]
(3.14)
which implies \( P\Omega = (\Omega, P\Omega)\Omega, \ P = (\Omega, P\Omega)1 \).

End of Proof.

Lemma 3.6 Let \( \varphi \) be a translationally invariant pure state of \( \mathfrak{A} \). Then, for a positive \( k \) there exists a pure state extension \( \psi \) of \( \varphi \) to \( B^k \) such that \( \psi \) is invariant under \( \tau_k \) and
\[
\sum_{|I| = k} \psi((S_i^{(L)})^*S_j^{(R)}) = 1.
\]

Furthermore, one of the following mutually exclusive conditions is valid.
(i) \( \psi \) is invariant under \( \gamma_z^{d,k} \).
(ii) \( \psi \circ \gamma_z^{d,k} \) is not equivalent to \( \psi \) for any \( z \).

When (ii) is valid, the assumptions of Lemma 3.3 are satisfied.

Proof: Consider the state \( \tilde{\varphi} \) of \( D \) satisfying (3.13). Let \( \tilde{\psi} \) be the \( \gamma_z \) invariant extension of \( \tilde{\varphi} \) to \( B \).

If \( \psi \) is pure, we set \( \tilde{\psi} = \psi \) and as \( \varphi \) is translationally invariant, there exists a unitary \( W \) on \( \Delta_\varphi = \Delta_0 \) satisfying
\[
W\pi_{\tilde{\varphi}}(Q)W^* = \pi(\tau_1(Q)), \quad W\Omega_{\tilde{\varphi}} = \Omega_{\tilde{\varphi}}.
\]
Then the operator $\pi \tilde{\varphi}(V)W^*$ acting on $\mathcal{H}_{\tilde{\varphi}}$ commutes with $\pi \varphi(A)$. This shows that $\pi \tilde{\varphi}(V)W^*$ is a scalar. After a gauge transformation of $O^{(L)}_d$ we have

$$\pi \tilde{\varphi}(V)\Omega_{\tilde{\varphi}} = \Omega_{\tilde{\varphi}}$$

which is equivalent to the equation $\gamma_{d,k}$. By definition the state $\psi$ is $\gamma_{d,k}$ invariant.

Next we consider the case that $\tilde{\psi}$ is not pure. Let $U(z)$ be the unitary on the GNS space $\mathcal{H}_{\tilde{\varphi}}$ associated with $\tilde{\psi}$ such that

$$\pi \tilde{\varphi}(\gamma_{d,k}(Q)) = U(z)\pi \varphi(Q)U(z)^*, \quad U(z)\Omega_{\tilde{\varphi}} = \Omega_{\tilde{\varphi}}.$$ 

The GNS representation $\pi \varphi$ restricted to $\mathcal{D}$ is a direct sum of $\pi_j(\mathcal{D})$ on $\mathcal{H}_j$:

$$\mathcal{H}_{\tilde{\varphi}} = \bigoplus_{j \in \mathbb{Z}} \mathcal{H}_j,$$

$$\pi \tilde{\varphi} = \bigoplus_{j \in \mathbb{Z}} \pi_j, \quad \pi_j = \pi \varphi|_{\mathcal{H}_j}$$

Note that the representations $\pi_j$ and $\pi_i$ are disjoint when $i \neq j$.

The Fourier component of the commutant $\mathcal{C}$ of $\pi \tilde{\varphi}(\mathcal{B})$ is denoted by $\mathcal{C}_j$. For $Q$ in $\mathcal{C} = \pi \tilde{\varphi}(\mathcal{B})$ we have

$$Q = \sum_{k=-\infty}^{\infty} Q_k, \quad Q_k = \int dz z^{-k} U_z Q U_z^*$$

Let $C_k$ be the subspace generated by operators $Q_k$:

$$C_k = \{ Q \in \mathcal{C} | \gamma_z(Q) = z^k Q \}$$

As the state $\varphi$ is pure $C_0$ is one dimensional, $C_0 = \mathbb{C}1$ because $C_0$ commutes with $\pi \varphi(\mathfrak{A})$. By the similar argument in proof of Lemma 3.1 it is possible to show the dimension of $C_k$ is zero or one and $\mathcal{C}$ is generated by a single unitary $U$ in $C_k$ for some $k$.

Now we introduce a representation $\pi(\mathcal{B}_k)$ of $\mathcal{B}_k$ on $\mathcal{H}_0 = \mathcal{H}_{\tilde{\varphi}}$ determined by

$$\pi(S^{(L)}_j) = e^{i\theta} \pi \varphi(S^{(L)}_j)|_{\mathcal{H}_0}, \quad \pi(S^{(R)}_j) = \pi \varphi(S^{(R)}_j)|_{\mathcal{H}_0}$$

for $|I| = |J| = k$ where the phase factor $e^{i\theta}$ is determined later. By definition $\pi(Q) = \pi \varphi(Q)$ for $Q$ in $\mathfrak{A}$ while on $\mathcal{H}_0$, $\pi \varphi(\mathfrak{A})$ acts irreducibly. Let $\psi$ be the vector state of $\mathcal{B}_k$ associated with $\Omega_{\varphi}$. As $\varphi$ is translationally invariant, there exists a unitary $W$ on $\mathcal{H}_{\varphi} = \mathcal{H}_0$ satisfying

$$W \pi \varphi(Q)W^* = \pi(\tau_k(Q)), \quad W\Omega_{\varphi} = \Omega_{\varphi}.$$

Set

$$V^{(k)} = \sum_{|I|=k} (S^{(L)}_I)^* S^{(R)}_I.$$
Then the operator $V^k W^*$ commutes with $\pi_\varphi(\mathcal{A})$. This shows that $\pi(V^{(k)}) W^*$ is a scalar. By suitably choosing the phase factor $e^{i\theta}$ we have

$$\pi(V^{(k)}) = W, \quad \pi(V^{(k)}) \Omega_\varphi = \Omega_\varphi.$$ 

$\psi$ is the state satisfying our requirement. \textit{End of Proof.}
4 Proof of Theorem 1.1

We consider Haag duality for the Cuntz algebras $O_d \otimes O_d$ first. The same duality (Proposition 4.2) is stated in [5]. However, due to the reason stated in the introduction of this paper, we present our proof here. To show the Haag duality for the Cuntz algebra we apply Tomita-Takesaki Theory. The state $\varphi_R$ or its extension to $O_d$ may not be faithful so we consider reduction of the von Neumann algebra generated by $O_d$ by support projection and apply the Tomita modular conjugation to obtain the (reduced) commutant. Then we apply the following lemma.

**Lemma 4.1** Let $M_1 \subset M_2$ be a pair of factor-subfactor on a separable Hilbert space $H$. Suppose that there exists a projection $P$ in $M_1$ such that $PM_1P = PM_2P$. Then, $M_1$ and $M_2$ coincide: $M_1 = M_2$.

**Proof.** Suppose that we have a matrix unit $e_{ij}$ ($i, j = 1, 2, \cdots$) in $M_1$ such that

$$e_{11} = P, \quad \sum_{j=1}^{\infty} e_{jj} = 1 \quad (4.1)$$

Let $Q$ be an element of $M_2$. Then $e_{ii}Qe_{jj}$ is an element of $M_1$ because

$$e_{ii}Qe_{jj} = e_{i1}e_{1j}e_{jj}, \quad e_{1j}Qe_{j1} \in M_1.$$ 

Thus if we have a matrix unit satisfying (4.1) any $Q$ in $M_2$ is an element of $M_1$. When $M_1$ has a tracial state $tr$ and $1/trP$ is not an integer, the matrix unit satisfying $\sum_{j=1}^{\infty} e_{jj} = 1$ does not exists. In such a case, we consider another projection $q$ in $M_1$ such that $q \leq P$ and $1/trq$ is a positive integer. Then we apply the above argument to $qM_1q = qM_2q$. End of Proof.

Without loss of generality, we assume that $k = 1$ in Lemma 3.6 for the proof of Haag duality. Let $\psi$ be a translationally invariant state. From now on, $\psi$ is the pure state extension of $\varphi$ to $B$ such that $\psi$ is invariant under $\tau_1$. Recall that due to the equation (3.15),

$$\pi_\psi(V)\Omega_\psi = \Omega_\psi.$$

Hence,

$$(S_I^{(R)})^*V = (S_I^{(L)})^*, \quad \pi_\psi(S_I^{(R)})^*\Omega_\psi = \pi_\psi(S_I^{(L)})^*\pi_\psi(V)\Omega_\psi = \pi_\psi(S_I^{(L)})^*\pi_\psi(V)\Omega_\psi.$$

As a consequence the Hilbert space $H_\psi$ is generated by the following vectors:

$$\pi_\psi(S_I^{(L)}\pi_\psi(S_R^{(L)}))\pi_\psi(Q)\pi_\psi(S_I^{(R)}), \quad Q \in \mathfrak{A}_R. \quad (4.3)$$
Proposition 4.2 Suppose $\psi$ is the pure state extension of $\varphi$ to $\mathcal{B}$ such that $\psi$ is invariant under $\tau_1$. Then,

$$\pi_\psi(O_d^{(L)})'' = \pi_\psi(O_d^{(R)})'.$$ (4.4)

The equation (4.2) is crucial in our proof of (4.4). We need some preparation for our proof of (4.4).

Let $E_R$ be the support projection of $\psi$ for $\pi_\psi(O_d^{(R)})''$. By $E_R'$ we denote the projection with range $[\pi_\psi(O_d^{(R)})'' \Omega_\psi]$ where $[\pi_\psi(O_d^{(R)})'' \Omega_\psi]$ is the closed subspace of $H_\psi$ generated by $\pi_\psi(O_d^{(R)}) \Omega_\psi$. Similarly, by $E_L'$ we denote the projection to $[\pi_\psi(O_d^{(L)})'' \Omega_\psi]$.

Set $P = E_R E_R'$ and $\mathfrak{N} = P\Omega_\psi$. The range of $P$ is $[E_R \pi_\psi(O_d^{(R)})'' \Omega_\psi]$.

Now we denote the von Neumann algebra $E_R \pi_\psi(O_d^{(R)})'' E_R$ by $\mathfrak{M}$. $\Omega_\psi$ is a cyclic and separating vector for $\mathfrak{M}$ acting on $\mathfrak{N}$. Let $\Delta$ and $J$ be the Tomita modular operator and the modular conjugation associated with $\Omega_\psi$ for $E_R \pi_\psi(O_d^{(R)})'' E_R$. Set $v_j = P \pi_\psi(S_j^{(R)}) P$. As

$$\langle \Omega_\psi, \left( (P \pi_\psi(S_j^{(R)}) P - P \pi_\psi(S_j^{(R)})) \left( P \pi_\psi(S_j^{(R)})^* P - \pi_\psi(S_j^{(R)})^* P \right) \Omega_\psi \right) = 0,$$

we have

$$P \pi_\psi(S_j^{(R)}) P = P \pi_\psi(S_j^{(R)})$$

and

$$\sum_{j=1}^d v_j v_j^* = 1.$$

Set $\tilde{v}_j = J \Delta^{1/2} v_j^* \Delta^{1/2}$ and $\tilde{v}_j^* = J \Delta^{1/2} v_j \Delta^{-1/2}$. The closure of $\tilde{v}_j$ and $\tilde{v}_j^*$ are bounded operators satisfying

$$\sum_{j=1}^d \tilde{v}_j \tilde{v}_j^* = 1$$

because

$$\sum_{j=1}^d ||\tilde{v}_j Q \Omega_\psi||^2 = \sum_{j=1}^d ||Q \tilde{v}_j \Omega_\psi||^2$$

$$= \sum_{j=1}^d ||Q \pi_\psi(S_j^{(R)})^* \Omega_\psi||^2 = \psi(\tau_1(Q^* Q)) = ||Q \Omega_\psi||^2$$

for $Q \in P \pi_\psi(O_d^{(R)}) P$. Moreover,

$$P \pi_\psi(S_j^{(L)}) P = \tilde{v}_j^*.$$ (4.5)
The above inclusion tells us.

By the symmetry of Lemma 4.5, we show

Proof: Let \( \pi \) be generated by the vectors \( \pi \) as in Proposition 4.2 and let \( \mathcal{N}_1 \) be the von Neumann algebra on \( \mathcal{N} \) generated by \( \tilde{v}_j \). Then, \( \mathcal{N}^\prime = \mathcal{N}_1 \).

Lemma 4.3: Let \( \mathcal{N} \) be the von Neumann algebra on \( \mathcal{N} \) generated by \( v_j \) as in Proposition 4.2, and let \( \mathcal{N}_1 \) be the von Neumann algebra on \( \mathcal{N} \) generated by \( \tilde{v}_j \).

Proof: \( \mathcal{N}^\prime \) is generated by \( Jv_J \) and \( \mathcal{N}_1 \subset \mathcal{N}^\prime \). The modular operator \( \Delta_1 \) and the conjugation \( J_1 \) of \( \mathcal{N}_1 \) acting on \( [\mathcal{N}_1, \Omega] \) are the restriction of those for \( \mathcal{N}^\prime \) on \( \mathcal{N} \). Then \( Jv_J = \Delta^{-1/2} \tilde{v}_j \Delta^{-1/2} \) is in \( \mathcal{N}_1 \). End of Proof.

Lemma 4.4

\[
[ E_R \pi_\psi(O_d^{(R)}) \Omega_\psi ] = [ E_L \pi_\psi(O_d^{(L)}) \Omega_\psi ]
\] (4.9)

Proof: By Lemma 4.3, the commutant of \( \mathcal{N} = E_R \pi_\psi(O_d^{(R)})^{\prime\prime} E_R \) acting on \( \mathcal{N} \) is \( P \pi_\psi(O_d^{(L)})^{\prime\prime} P \). Obviously \( P = E_R E_R^{\prime\prime} \leq E_R E_L \). Then,

\[
[ E_R \pi_\psi(O_d^{(R)})^{\prime\prime} \Omega_\psi ] = [ P \pi_\psi(O_d^{(L)})^{\prime\prime} \Omega_\psi ] \subset [ E_R E_L \pi_\psi(O_d^{(L)})^{\prime\prime} \Omega_\psi ] = [ E_L \pi_\psi(O_d^{(L)})^{\prime\prime} \Omega_\psi ]
\] (4.10)

The above inclusion tells us

\[
[ E_R \pi_\psi(O_d^{(R)}) \Omega_\psi ] \subset [ E_L \pi_\psi(O_d^{(L)}) \Omega_\psi ].
\]

By the symmetry of \( L \) and \( R \) we have reverse inclusion. End of Proof.

Lemma 4.5

\[
P = E_R E_L
\] (4.11)

Proof: We show \( P = E_R E_R^{\prime\prime} \geq E_R E_L \). Due to (4.2), the Hilbert space \( S \) is generated by the vectors \( \pi_\psi(S_i^{(L)}) \pi_\psi(S_j^{(R)}) \pi_\psi(S_j^{(R)})^{*} \Omega_\psi \). It suffices to show that the vector \( \xi = E_R E_L \pi_\psi(S_i^{(L)}) \pi_\psi(S_j^{(R)}) \pi_\psi(S_j^{(R)})^{*} \Omega_\psi \) is in \( \mathcal{R} \) (= the range of \( P \)). Due to the previous Lemma, \( \eta = E_L \pi_\psi(S_i^{(L)}) \Omega_\psi \) is in \( \mathcal{R} \). Thus,

\[
E_R E_L \pi_\psi(S_i^{(L)}) \pi_\psi(S_j^{(R)}) \pi_\psi(S_j^{(R)})^{*} \Omega_\psi
\]

\[
= E_R \pi_\psi(S_j^{(R)}) \pi_\psi(S_j^{(R)})^{*} E_L \pi_\psi(S_i^{(L)}) \Omega_\psi
\]

\[
= E_R \pi_\psi(S_j^{(R)}) \pi_\psi(S_j^{(R)})^{*} \eta
\]

\[
= E_R \pi_\psi(S_j^{(R)}) \pi_\psi(S_j^{(R)})^{*} E_R \eta \in \mathcal{R}
\] (4.12)
End of Proof.

Now we return to proof of Proposition 4.2. First we look at the commutant of $E_R \pi_\psi ( O^{(R)}_d )'' E_R$ on $E_R \mathcal{F}_\psi$. Obviously, $E_R \pi_\psi ( O^{(L)}_d )'' E_R \subset ( E_R \pi_\psi ( O^{(R)}_d ) E_R )'$ on $E_R \mathcal{F}_\psi$. By Lemma 4.3 and Lemma 4.5,

$$E_R E_L \pi_\psi ( O^{(L)}_d )'' E_L E_R = ( E_R E_L \pi_\psi ( O^{(R)}_d ) E_L E_R )'$$

Then, due to Lemma 4.1, $\pi_\psi ( O^{(L)}_d )'' = \pi_\psi ( O^{(R)}_d )'$ on $E_R \mathcal{F}_\psi$.

Next we consider the inclusion $\pi_\psi ( O^{(R)}_d )'' \subset \pi_\psi ( O^{(L)}_d )'$ on $\mathcal{F}_\psi$. As we already know that $E_R \pi_\psi ( O^{(R)}_d )'' E_R = E_R \pi_\psi ( O^{(L)}_d )' E_R$ we apply Lemma 4.1 again and conclude that $\pi_\psi ( O^{(R)}_d )'' = \pi_\psi ( O^{(L)}_d )'$. End of Proof of Proposition 4.2.

Proof of Theorem 1.1

Now recall Lemma 3.6. We have two cases (i) and (ii). In the case (ii), our previous analysis shows that the pair of Cuntz algebras $B$ is in the von Neumann algebra $\pi_\psi ( A )''$ and the duality follows from Proposition 4.2.

Hence we consider the case where the pure state $\psi$ of $B$ is invariant under $\gamma_d$. Let $U_z$ be the unitary implementing $\gamma_d$ and satisfying $U_z \Omega_\psi = \Omega_\psi$. We use the previous notation in our proof for Proposition 4.2. By the duality for Cuntz algebras (Proposition 4.2), $E_R = E'_L$, $E_L = E'_R$. $E_R$ commutes with $U_z$ due to $\gamma_d$ invariance of $\psi$. As a result, the support projection of $\varphi$ for $\pi_\psi ( \mathfrak{A}_R )''$ is the $E_R$ restricted to $\mathcal{F}_\varphi$. So we use the same notation $E_R$ (resp. $E_L$) for the support projection of $\varphi$ for $\pi_\psi ( \mathfrak{A}_R )''$ (resp. $\pi_\psi ( \mathfrak{A}_L )''$).

To show Haag duality we proceed as before. Taking into account of $P = E_R E_L$ and $\pi_\psi ( \mathfrak{A}_L )'' \subset \pi_\psi ( \mathfrak{A}_R )'$, it suffices to show

$$P \pi_\psi ( \mathfrak{A}_L )'' P = P \pi_\psi ( \mathfrak{A}_R )' P. \tag{4.13}$$

On $\mathfrak{R}_0 = P \mathcal{F}_\varphi$ we apply Tomita-Takesaki theorem. $P \pi_\psi ( \mathfrak{A}_R )' P$ is generated by $J_0 v_I v_K J_0$, where $I$ and $K$ are multi-indices satisfying $|I| = |K|$ and $J_0$ is the restriction of $J$ to $\mathfrak{R}_0$. By Haag duality for Cuntz algebras, $J v_I J$ and $J v_K^* J$ are approximated in strong operator topology by elements $w_\alpha$ and $x_\alpha$ of $P \pi_\psi ( O_d ) P$. Using Fourier decomposition (with help of $U_z$) we may assume that

$$U_z w_\alpha U_z^* = z^{|I|} w_\alpha, \quad U_z x_\alpha U_z^* = z^{|K|} x_\alpha. \tag{4.14}$$

As a consequence, $J v_I v_K^* J$ is approximated by elements of $P \pi_\psi ( O_d ) P \cap \{ U_z | z \in U(1) \}$. Thus $J v_I v_K^* J$ is contained in

$$P \pi_\psi ( O_d )'' P \cap \{ U_z | z \in U(1) \}' = P \pi_\psi ( \mathfrak{A}_L )'' P$$

on $\mathfrak{R}$. By taking restriction to $\mathfrak{R}_0$, we see (4.13). End of Proof.
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