On Accurate Domination in Graphs

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Abstract

A dominating set of a graph \( G \) is a subset \( D \subseteq V_G \) such that every vertex not in \( D \) is adjacent to at least one vertex in \( D \). The cardinality of a smallest dominating set of \( G \), denoted by \( \gamma(G) \), is the domination number of \( G \). The accurate domination number of \( G \), denoted by \( \gamma_a(G) \), is the cardinality of a smallest set \( D \) that is a dominating set of \( G \) and no \(|D|\)-element subset of \( V_G \setminus D \) is a dominating set of \( G \). We study graphs for which the accurate domination number is equal to the domination number. In particular, all trees \( G \) for which \( \gamma_a(G) = \gamma(G) \) are characterized. Furthermore, we compare the accurate domination number with the domination number of different coronas of a graph.

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1 Introduction and notation

We generally follow the notation and terminology of [1] and [6]. Let $G = (V_G, E_G)$ be a graph with vertex set $V_G$ of order $n(G) = |V_G|$ and edge set $E_G$ of size $m(G) = |E_G|$. If $v$ is a vertex of $G$, then the open neighborhood of $v$ is the set $N_G(v) = \{u \in V_G : uv \in E_G\}$, while the closed neighborhood of $v$ is the set $N_G[v] = N_G(v) \cup \{v\}$. For a subset $X$ of $V_G$ and a vertex $x$ in $X$, the set $p_{NG}(x, X) = \{v \in V_G | N_G[v] \cap X = \{x\}\}$ is called the $x$-private neighborhood of the vertex $x$, and it consists of those vertices of $N_G[x]$ which are not adjacent to any vertex in $X \setminus \{x\}$; that is, $p_{NG}(x, X) = N_G[x] \setminus N_G[X \setminus \{x\}]$. The degree $d_G(v)$ of a vertex $v$ in $G$ is the number of vertices in $N_G(v)$. A vertex of degree one is called a leaf and its neighbor is called a support vertex. The set of leaves of a graph $G$ is denoted by $L_G$, while the set of support vertices by $S_G$. For a set $S \subseteq V_G$, the subgraph induced by $S$ is denoted by $G[S]$, while the subgraph induced by $V_G \setminus S$ is denoted by $G - S$. Thus the graph $G - S$ is obtained from $G$ by deleting the vertices in $S$ and all edges incident with $S$. Let $\kappa(G)$ denote the number of components of $G$.

A dominating set of a graph $G$ is a subset $D$ of $V_G$ such that every vertex not in $D$ is adjacent to at least one vertex in $D$, that is, $N_G(x) \cap D \neq \emptyset$ for every $x \in V_G \setminus D$. The domination number of $G$, denoted by $\gamma(G)$, is the cardinality of a smallest dominating set of $G$. An accurate dominating set of $G$ is a dominating set $D$ of $G$ such that no $|D|$-element subset of $V_G \setminus D$ is a dominating set of $G$. The accurate domination number of $G$, denoted by $\gamma_a(G)$, is the cardinality of a smallest accurate dominating set of $G$. We call a dominating set of $G$ of cardinality $\gamma(G)$ a $\gamma$-set of $G$, and an accurate dominating set of $G$ of cardinality $\gamma_a(G)$ a $\gamma_a$-set of $G$. Since every accurate dominating set of $G$ is a dominating set of $G$, we note that $\gamma(G) \leq \gamma_a(G)$. The accurate domination in graphs was introduced by Kulli and Kattimani [7], and further studied in a number of papers. A comprehensive survey of concepts and results on domination in graphs can be found in [6].

We denote the path and cycle on $n$ vertices by $P_n$ and $C_n$, respectively. We denote by $K_n$ the complete graph on $n$ vertices, and by $K_{m,n}$ the complete bipartite graph with partite sets of size $m$ and $n$. The accurate domination numbers of some common graphs are given by the following formulas:

**Observation 1.** The following holds.

(a) For $n \geq 1$, $\gamma_a(K_n) = \left\lceil \frac{n}{2} \right\rceil + 1$ and $\gamma_a(K_{n,n}) = n + 1$.

(b) For $n > m \geq 1$, $\gamma_a(K_{m,n}) = m$.

(c) For $n \geq 3$, $\gamma_a(C_n) = \left\lfloor \frac{n}{2} \right\rfloor - \left\lceil \frac{3}{2} \right\rceil + 2$.

(d) For $n \geq 1$, $\gamma_a(P_n) = \left\lceil \frac{n}{2} \right\rceil$ unless $n \in \{2, 4\}$ when $\gamma_a(P_n) = \left\lceil \frac{n}{2} \right\rceil + 1$ (see Corollary 1).

In this paper we study graphs for which the accurate domination number is equal to the domination number. In particular, all trees $G$ for which $\gamma_a(G) = \gamma(G)$ are characterized. Furthermore, we compare the accurate domination number with the domination number of different coronas of a graph. Throughout the paper, we use the symbol $A_\gamma(G)$ (respectively, $A_{\gamma_a}(G)$) to denote the set of all minimum dominating sets (respectively, minimum accurate dominating sets) of $G$. 

2
2 Graphs with $\gamma_a$ equal to $\gamma$

We are interested in determining the structure of graphs for which the accurate domination number is equal to the domination number. The question about such graphs has been stated in [7]. We begin with the following general property of the graphs $G$ for which $\gamma_a(G) = \gamma(G)$.

**Lemma 1.** Let $G$ be a graph. Then $\gamma_a(G) = \gamma(G)$ if and only if there exists a set $D \in A_\gamma(G)$ such that $D \cap D' \neq \emptyset$ for every set $D' \in A_\gamma(G)$.

**Proof.** First assume that $\gamma_a(G) = \gamma(G)$, and let $D$ be a minimum accurate dominating set of $G$. Since $D$ is a dominating set of $G$ and $|D| = \gamma_a(G) = \gamma(G)$, we note that $D \in A_\gamma(G)$. Now let $D'$ be an arbitrary minimum dominating set of $G$. If $D \cap D' = \emptyset$, then $D' \subseteq V_G \setminus D$, implying that $D'$ would be a $|D|$-element dominating set of $G$, contradicting the fact that $D$ is an accurate dominating set of $G$. Hence, $D \cap D' \neq \emptyset$.

Now assume that there exists a set $D \in A_\gamma(G)$ such that $D \cap D' \neq \emptyset$ for every set $D' \in A_\gamma(G)$. Then, $D$ is an accurate dominating set of $G$, implying that $\gamma_a(G) \leq |D| = \gamma(G) \leq \gamma_a(G)$. Consequently, we must have equality throughout this inequality chain, and so $\gamma_a(G) = \gamma(G)$. \qed

It follows from Lemma 1 that if $G$ is a disconnected graph, then $\gamma_a(G) = \gamma(G)$ if and only if $\gamma_a(H) = \gamma(H)$ for at least one component $H$ of $G$. In particular, if $G$ has an isolated vertex, then $\gamma_a(G) = \gamma(G)$. It also follows from Lemma 1 that for a graph $G$, $\gamma_a(G) = \gamma(G)$ if $G$ has one of the following properties: (1) $G$ has a unique minimum dominating set (see, for example, [3] or [5] for some characterizations of such graphs); (2) $G$ has a vertex which belongs to every minimum dominating set of $G$ (see [5]); (3) $G$ has a vertex adjacent to at least two leaves. Consequently, there is no forbidden subgraph characterization for the class of graphs $G$ for which $\gamma_a(G) = \gamma(G)$, as for any graph $H$, we can add an isolated vertex (or two leaves to one vertex of $H$), and in this way form a graph $H'$ for which $\gamma_a(H') = \gamma(H')$.

The *corona* $F \circ K_1$ of a graph $F$ is the graph formed from $F$ by adding a new vertex $v'$ and edge $vv'$ for each vertex $v \in V(F)$. A graph $G$ is said to be a *corona graph* if $G = F \circ K_1$ for some connected graph $F$. We note that each vertex of a corona graph $G$ is a leaf or it is adjacent to exactly one leaf of $G$. Recall that we denote the set of all leaves in a graph $G$ by $L_G$, and set of support vertices in $G$ by $S_G$.

**Lemma 2.** If $G$ is a corona graph, then $\gamma_a(G) > \gamma(G)$.

**Proof.** Assume that $G$ is a corona graph. If $G = K_1 \circ K_1$, then $G = K_2$ and $\gamma_a(G) = 2$ and $\gamma(G) = 1$. Hence, we may assume that $G = F \circ K_1$ for some connected graph $F$ of order $n(F) \geq 2$. If $v \in V_{G} \setminus L_G$, then let $\overline{v}$ denote the unique leaf-neighbor of $v$ in $G$. Now let $D$ be an arbitrary minimum dominating set of $G$, and so $D \in A_\gamma(G)$. Then, $|D \cap \{v, \overline{v}\}| = 1$ for every $v \in V_{G} \setminus L_G$. Consequently, $D$ and its complement $V_G \setminus D$ are minimum dominating sets of $G$. Thus, $D$ is not an accurate dominating set of $G$. This is true for every minimum dominating set of $G$, implying that $\gamma_a(G) > \gamma(G)$. \qed
Lemma 3. If $T$ is a tree of order at least three, then there exists a set $D \in \mathcal{A}_\gamma(T)$ such that the following hold.

(a) $S_T \subseteq D$.
(b) $N_T(v) \subseteq V_T \setminus D$ or $|pn_T(v, D)| \geq 2$ for every $v \in D \setminus S_T$.

Proof. Let $T$ be a tree of order $n(T) \geq 3$. Among all minimum dominating sets of $T$, let $D \in \mathcal{A}_\gamma(T)$ be chosen that

1. $D$ contains as many support vertices as possible.
2. Subject to (1), the number of components $\kappa(T[D])$ is as large as possible.

If the set $D$ contains a leaf $v$ of $T$, then we can simply replace $v$ in $D$ with the support vertex adjacent to $v$ to produce a new minimum dominating set with more support vertices than $D$, a contradiction. Hence, the set $D$ contains no leaves, implying that $S_T \subseteq D$. Suppose, next, that there exists a vertex $v$ in $D$ that is not a support vertex of $T$ and such that $N_T(v) \not\subseteq V_T \setminus D$. Thus, $v$ has at least one neighbor in $D$; that is, $N_T(v) \cap D \neq \emptyset$. By the minimality of the set $D$, we therefore note that $pn_T(v, D) \neq \emptyset$. If $|pn_T(v, D)| = 1$, say $pn_T(v, D) = \{u\}$, then letting $D' = (D \setminus \{v\}) \cup \{u\}$, the set $D' \in \mathcal{A}_\gamma(T)$ and satisfies $S_T \subseteq D' \setminus \{v\} \subseteq D'$ and $\kappa(T[D']) > \kappa(T[D])$, which contradicts the choice of $D$. Hence, if $v \in D$ is not a support vertex of $T$ and $N_T(v) \not\subseteq V_T \setminus D$, then $|pn_T(v, D)| \geq 2$. \hfill \square

We are now in a position to present the following equivalent characterizations of trees for which the accurate domination number is equal to the domination number.

Theorem 1. If $T$ is a tree of order at least two, then the following statements are equivalent:

1. $T$ is not a corona graph.
2. There exists a set $D \in \mathcal{A}_\gamma(T)$ such that $\kappa(T - D) > |D|$.
3. $\gamma_a(T) = \gamma(T)$.
4. There exists a set $D \in \mathcal{A}_\gamma(T)$ such that $D \cap D' \neq \emptyset$ for every $D' \in \mathcal{A}_\gamma(T)$.

Proof. The statements (3) and (4) are equivalent by Lemma 1. The implication (3) $\Rightarrow$ (1) follows from Lemma 2. To prove the implication (2) $\Rightarrow$ (3), let us assume that $D \in \mathcal{A}_\gamma(T)$ and $\kappa(T - D) > |D|$. Thus, $\gamma(T - D) \geq \kappa(T - D) > |D| = \gamma(T)$. This proves that $D$ is an accurate dominating set of $T$, and therefore $\gamma_a(T) = \gamma(T)$.

Thus it suffices to prove that (1) implies (2). The proof is by induction on the order of a tree. The implication (1) $\Rightarrow$ (2) is obvious for trees of order two, three, and four. Thus assume that $T$ is a tree of order at least five and $T$ is not a corona graph. Let $D \in \mathcal{A}_\gamma(T)$ and assume that $S_T \subseteq D$. Since $T$ is not a corona graph, the tree $T$ has a vertex which is neither a leaf nor adjacent to exactly one leaf. We consider two cases, depending on whether $d_T(v) \geq 3$ for some vertex $v \in S_T$ or $d_T(v) = 2$ for every vertex $v \in S_T$.

Case 1. $d_T(v) \geq 3$ for some $v \in S_T$. Let $v'$ be a leaf of $T$ adjacent to $v$. Let $T'$ be a component of $T - \{v, v'\}$. Now let $T_1$ and $T_2$ be the subtrees of $T$ induced on the vertex sets $V_{T'} \cup \{v, v'\}$ and $V_T \setminus V_{T'}$, respectively. We note that both trees $T_1$ and $T_2$ have order strictly less than $n(T)$. Further, $V(T_1) \cap V(T_2) = \{v, v'\}$, $E(T_1) \cap E(T_2) = \{vv'\}$, and at
Consequently, \( \kappa \) components induced by leaves of \( w \) contradicting the assumption that \( n \) belong to \( D \) is a corona graph. By the induction hypothesis, there exists a set \( \kappa \) such that \( \kappa(T_2 - D_2) = |D_2| \). In both cases, there exists a set \( D_2 \subseteq A_n(T_2) \) such that \( \kappa(T_2 - D_2) \geq |D_2| \). We may assume that all support vertices of \( T_1 \) and \( T_2 \) are in \( D_1 \) and \( D_2 \), respectively. Thus, \( v \in D_1 \cap D_2 \), the union \( D_1 \cup D_2 \) is a \( \gamma \)-set of \( T \), and \( \kappa(T - (D_1 \cup D_2)) = \kappa(T_1 - D_1) + \kappa(T_2 - D_2) - 1 > |D_1| + |D_2| - 1 = |D_1 \cup D_2| \).

Case 2. \( d_T(v) = 2 \) for every \( v \in S_T \). We distinguish two subcases, depending on whether \( D \setminus S_T \neq \emptyset \) or \( D \setminus S_T = \emptyset \).

Case 2.1. \( D \setminus S_T \neq \emptyset \). Let \( v \) be an arbitrary vertex belonging to \( D \setminus S_T \). It follows from the second part of Lemma $3$ that there are two vertices \( v_1 \) and \( v_2 \) belonging to \( N_T(v) \setminus D \). Let \( R \) be the tree obtained from \( T \) by adding a new vertex \( v' \) and the edge \( vv' \). We note that \( D \) is a minimum dominating set of \( R \) and \( S_R \subseteq D \). Let \( R' \) be the component of \( R - \{v, v'\} \) containing \( v_1 \). Now let \( R_1 \) and \( R_2 \) be the subtrees of \( R \) induced by the vertex sets \( V_{R'} \cup \{v, v'\} \) and \( V_R \setminus V_{R'} \), respectively. We note that both trees \( R_1 \) and \( R_2 \) have order strictly less than \( n(T) \). Further, \( V(R_1) \cap V(R_2) = \{v, v'\} \), \( E(R_1) \cap E(R_2) = \{vv'\} \), and neither \( R_1 \) nor \( R_2 \) is a corona graph. By the induction hypothesis, there exists a set \( \kappa \in A_n(R_1) \) and a set \( D_2 \subseteq A_n(R_2) \) such that \( \kappa(R_1 - D_1) > |D_1| \) and \( \kappa(R_2 - D_2) > |D_2| \). We may assume that all support vertices of \( R_1 \) and \( R_2 \) are in \( D_1 \) and \( D_2 \), respectively. Thus, \( v \in D_1 \cap D_2 \), the union \( D_1 \cup D_2 \) is a \( \gamma \)-set of \( R \), and

\[
\kappa(T - (D_1 \cup D_2)) &= \kappa(R - (D_1 \cup D_2)) - 1 \\
&= (\kappa(R_1 - D_1) + \kappa(R_2 - D_2) - 1) - 1 \\
&= (\kappa(R_1 - D_1) - |D_1| + \kappa(R_2 - D_2) - |D_2|) - 2 + |D_1| + |D_2| \\
&\geq |D_1| + |D_2| \\
&\geq |D_1 \cup D_2| + 1 \\
&> |D_1 \cup D_2|.
\]

Case 2.2. \( D \setminus S_T = \emptyset \). In this case, we note that \( D = S_T \). Let \( v \) be an arbitrary vertex belonging to \( D \) and assume that \( N_T(v) = \{u, w\} \), where \( u \in L_T \). If \( w \in L_T \), then \( T = K_{1,2} \), contradicting the assumption that \( n(T) \geq 5 \). If \( w \in S_T \), then \( T = P_4 = K_2 \circ K_1 \), contradicting the assumption that \( T \) is not a corona graph (and the assumption that \( n(T) \geq 5 \)). Therefore, \( w \in V_T \setminus (L_T \cup S_T) \). Thus, \( V_T \setminus (L_T \cup S_T) \) is nonempty and \( T - D \) has \( |D| \) one-element components induced by leaves of \( T \) and at least one component induced by \( V_T \setminus (L_T \cup S_T) \). Consequently, \( \kappa(T - D) \geq |D| + 1 \geq |D| \). This completes the proof of Theorem $1$.

The equivalence of the statements (1) and (3) of Theorem $1$ shows that the trees \( T \) for which \( \gamma_a(T) = \gamma(T) \) are easy to recognize. From Theorem $1$ and from the well-known fact that \( \gamma(P_n) = \lceil n/3 \rceil \) for every positive integer \( n \), we also immediately have the following corollary which provides a slight improvement on Proposition 3 in $7$.

**Corollary 1.** For \( n \geq 1 \), \( \gamma_a(P_n) = \gamma(P_n) = \lceil n/3 \rceil \) if and only if \( n \in \mathbb{N} \setminus \{2, 4\} \).
3 Domination of general coronas of a graph

Let \( G \) be a graph, and let \( \mathcal{F} = \{ F_v : v \in V_G \} \) be a family of nonempty graphs indexed by the vertices of \( G \). By \( G \circ \mathcal{F} \) we denote the graph with vertex set

\[
V_{G \circ \mathcal{F}} = (V_G \times \{0\}) \cup \bigcup_{v \in V_G} (\{v\} \times V_{F_v})
\]

and edge set determined by open neighborhoods defined in such a way that

\[
N_{G \circ \mathcal{F}}((v, 0)) = (N_G(v) \times \{0\}) \cup (\{v\} \times V_{F_v})
\]

for every \( v \in V_G \), and

\[
N_{G \circ \mathcal{F}}((v, x)) = \{(v, 0)\} \cup (\{v\} \times N_{F_v}(x))
\]

if \( v \in V_G \) and \( x \in V_{F_v} \). The graph \( G \circ \mathcal{F} \) is said to be the \( \mathcal{F} \)-corona of \( G \). Informally, \( G \circ \mathcal{F} \) is the graph obtained by taking a disjoint copy of \( G \) and all the graphs of \( \mathcal{F} \) with additional edges joining each vertex \( v \) of \( G \) to every vertex in the copy of \( F_v \). If all the graphs of the family \( \mathcal{F} \) are isomorphic to one and the same graph \( \mathcal{F} \) (as it was defined by Frucht and Harary [4]), then we simply write \( G \circ F \) instead of \( G \circ \mathcal{F} \). Recall that a graph \( G \) is said to be a corona graph if \( G = \mathcal{F} \circ K_1 \) for some connected graph \( \mathcal{F} \).

The 2-subdivided graph \( S_2(G) \) of a graph \( G \) is the graph with vertex set

\[
V_{S_2(G)} = V_G \cup \bigcup_{vu \in E_G} \{(v, vu), (u, vu)\}
\]

and the adjacency is defined in such a way that

\[
N_{S_2(G)}(x) = \{(x, xy) : y \in N_G(x)\}
\]

if \( x \in V_G \subseteq V_{S_2(G)} \), while

\[
N_{S_2(G)}((x, xy)) = \{x\} \cup \{(y, xy)\}
\]

if \( (x, xy) \in \bigcup_{vu \in E_G} \{(v, vu), (u, vu)\} \subseteq V_{S_2(G)} \). Less formally, \( S_2(G) \) is the graph obtained from \( G \) by subdividing every edge with two new vertices; that is, by replacing edges \( vu \) of \( G \) with disjoint paths \( (v, (v, vu), (u, vu), u) \).

For a graph \( G \) and a family \( \mathcal{P} = \{P(v) : v \in V_G\} \), where \( P(v) \) is a partition of the neighborhood \( N_G(v) \) of the vertex \( v \), by \( G \circ \mathcal{P} \) we denote the graph with vertex set

\[
V_{G \circ \mathcal{P}} = (V_G \times \{1\}) \cup \bigcup_{v \in V_G} (\{v\} \times P(v))
\]

and edge set

\[
E_{G \circ \mathcal{P}} = \bigcup_{v \in V_G} \{(v, 1)(v, A) : A \in P(v)\} \cup \bigcup_{uv \in E_G} \{(v, A)(u, B) : (u \in A) \land (v \in B)\}.
\]
The graph $G \circ \mathcal{P}$ is called the \textit{$\mathcal{P}$-corona} of $G$ and was defined by Dettlaff et al. in [2]. It follows from this definition that if $\mathcal{P}(v) = \{N_G(v)\}$ for every $v \in V_G$, then $G \circ \mathcal{P}$ is isomorphic to the corona $G \circ K_1$. On the other hand, if $\mathcal{P}(v) = \{u : u \in N_G(v)\}$ for every $v \in V_G$, then $G \circ \mathcal{P}$ is isomorphic to the 2-subdivided graph $S_2(G)$ of $G$. Examples of $G \circ K_1$, $G \circ \mathcal{F}$, $G \circ \mathcal{P}$, and $S_2(G)$ are shown in Fig. 1. In this case $G$ is the graph $(K_2 \cup K_1) + K_1$ with vertex set $V_G = \{v, u, w, z\}$ and edge set $E_G = \{vu, vw, uw, wz\}$, where the family $\mathcal{F}$ consists of the graphs $F_v = F_w = K_1$, $F_z = K_2$, and $F_u = K_2 \cup K_1$, while $\mathcal{P} = \{\mathcal{P}(x) : x \in V_G\}$ is the family in which $\mathcal{P}(v) = \{\{u, w\}\}$, $\mathcal{P}(u) = \{\{v, \}\}, \mathcal{P}(w) = \{\{u, v, z\}\}$, and $\mathcal{P}(z) = \{\{w\}\}$.

![Figure 1: Coronas of $G = (K_2 \cup K_1) + K_1$.](image)

We now study relations between the domination number and the accurate domination number of different coronas of a graph. Our first theorem specifies when these two numbers are equal for the $\mathcal{F}$-corona $G \circ \mathcal{F}$ of a graph $G$ and a family $\mathcal{F}$ of nonempty graphs indexed by the vertices of $G$.

**Theorem 2.** If $G$ is a graph and $\mathcal{F} = \{F_v : v \in V_G\}$ is a family of nonempty graphs indexed by the vertices of $G$, then the following holds.

1. $\gamma(G \circ \mathcal{F}) = |V_G|$. 
2. $\gamma_a(G \circ \mathcal{F}) = \gamma(G \circ \mathcal{F})$ if and only if $\gamma(F_v) > 1$ for some vertex $v$ of $G$.
3. $|V_G| \leq \gamma_a(G \circ \mathcal{F}) \leq |V_G| + \min\{|V_{F_v}| : v \in V_G\}$.

**Proof.** (1) It is obvious that $V_G \times \{0\}$ is a minimum dominating set of $G \circ \mathcal{F}$ and therefore $\gamma(G \circ \mathcal{F}) = |V_G \times \{0\}| = |V_G|$.
(2) If \( \gamma(F_v) > 1 \) for some vertex \( v \) of \( G \), then
\[
\gamma(G \circ F - (V_G \times \{0\})) = \sum_{v \in V_G} \gamma((G \circ F)[\{v\} \times V_{F_v}]) = \sum_{v \in V_G} \gamma(F_v) > |V_G| = |V_G \times \{0\}|
\]
and this proves that no subset of \( V_{G \circ F - (V_G \times \{0\})} \) of cardinality \( |V_G \times \{0\}| \) is a dominating set of \( G \circ F \). Consequently \( V_G \times \{0\} \) is a minimum accurate dominating set of \( G \circ F \) and therefore \( \gamma_a(G \circ F) = \gamma(G \circ F) \).

Assume now that \( G \) and \( F \) are such that \( \gamma_a(G \circ F) = \gamma(G \circ F) \). We claim that \( \gamma(F_v) > 1 \) for some vertex \( v \) of \( G \). Suppose, contrary to our claim, that \( \gamma(F_v) = 1 \) for every vertex \( v \) of \( G \). Then the set \( U_v = \{ x \in V_{F_v} : N_{F_v}[x] = V_{F_v} \} \), the set of universal vertices of \( F_v \), is nonempty for every \( v \in V_G \). Now, let \( D \) be any minimum dominating set of \( G \circ F \). Then, \( |D| = \gamma(G \circ F) = |V_G \times \{0\}| = |V_G| \), \( |D \cap ((\{v\} \times \{0\}) \cup (\{v\} \times U_v))| = 1 \), and the set \( (\{v\} \times \{0\}) \cup (\{v\} \times U_v)) - D \) is nonempty for every \( v \in V_G \). Now, if \( \overline{D} \) is a system of representatives of the family \( \{(\{v\} \times \{0\}) \cup (\{v\} \times U_v)\} - D : v \in V_G \) \), then \( \overline{D} \) is a minimum dominating set of \( G \circ F \). Since \( \overline{D} \) and \( D \) are disjoint, \( D \) is not an accurate dominating set of \( G \circ F \). Consequently, no minimum dominating set of \( G \circ F \) is an accurate dominating set and therefore \( \gamma(G \circ F) < \gamma_a(G \circ F) \), a contradiction.

(3) The lower bound is obvious as \( |V_G| = \gamma(G \circ F) \leq \gamma_a(G \circ F) \). Since \( (V_G \times \{0\}) \cup (\{v\} \times V_{F_v}) \) is an accurate dominating set of \( G \circ F \) for every \( v \in V_G \), we also have the inequality \( \gamma_a(G \circ F) \leq |V_G| + \min\{|V_{F_v}| : v \in V_G \} \). This completes the proof of Theorem 2.

As a consequence of Theorem 2 we have the following result.

**Corollary 2.** If \( G \) is a graph, then \( \gamma_a(G \circ K_1) = \gamma(G \circ K_1) + 1 = |V_G| + 1 \).

**Proof.** Since \( \gamma(K_1) = 1 \), it follows from Theorem 2 that \( \gamma_a(G \circ K_1) \geq \gamma(G \circ K_1) + 1 = |V_G| + 1 \). On the other hand the set \( (V_G \times \{0\}) \cup (\{v,1\}) \) is an accurate dominating set of \( G \circ K_1 \) and therefore \( \gamma_a(G \circ K_1) \leq |(V_G \times \{0\}) \cup (\{v,1\})| = |V_G| + 1 \). Consequently, \( \gamma_a(G \circ K_1) = \gamma(G \circ K_1) = |V_G| + 1 \).

From Theorem 2 we know that \( \gamma_a(G \circ F) = \gamma(G \circ F) = |V_G| \) if and only if the family \( F \) is such that \( \gamma(F_v) > 1 \) for some \( F_v \in F \), but we do not know any general formula for \( \gamma_a(G \circ F) \) if \( \gamma(F_v) = 1 \) for every \( F_v \in F \). The following theorem shows a formula for the domination number and general bounds for the accurate domination number of a \( P \)-corona of a graph.

**Theorem 3.** If \( G \) is a graph and \( P = \{P(v) : v \in V_G\} \) is a family of partitions of the vertex neighborhoods of \( G \), then the following holds.

1. \( \gamma(G \circ P) = |V_G| \).
2. \( \gamma_a(G \circ P) \geq |V_G| \).
3. \( \gamma_a(G \circ P) \leq |V_G| + \min\{|P(v)| : v \in V_G\}, 1 + \min\{|A| : A \in \bigcup_{v \in V_G} P(v)\} \).

**Proof.** It follows from the definition of \( G \circ P \) that \( V_G \times \{1\} \) is a dominating set of \( G \circ P \), and therefore \( \gamma(G \circ P) \leq |V_G \times \{1\}| = |V_G| \). On the other hand, let \( D \in \mathcal{A}_\gamma(G \circ P) \). Then
For every \( v \in V_G \), and, since the sets \( N_{G \circ P}[(v,1)] \) form a partition of \( V_{G \circ P} \), we have

\[
\gamma(G \circ P) = |D| = \sum_{v \in V_G} |D \cap N_{G \circ P}[(v,1)]| \geq |V_G|.
\]

Consequently, we have \( |V_G| = \gamma(G \circ P) \leq \gamma_a(G \circ P) \), which proves (1) and (2).

From the definition of \( G \circ P \) it also follows that each of the sets \( (V_G \times \{1\}) \cup N_{G \circ P}[(v,1)] \) (for every \( v \in V_G \)) and \( (V_G \times \{1\}) \cup N_{G \circ P}[(v,A)] \) (for every \( v \in V_G \) and \( A \in P(v) \)) is an accurate dominating set of \( G \circ P \). Hence,

\[
|V_G| = \gamma_a(G \circ P) \leq \gamma_a(S_2(G)) \leq |V_G| + 2.
\]

This completes the proof of Theorem 3.

We do not know all the pairs \((G,P)\) achieving equality in the upper bound for the accurate domination number of a \( P \)-corona of a graph, but Theorem 4 and Corollaries 3 and 4 show that the bounds in Theorem 3 are best possible. The next theorem also shows that the domination number and the accurate domination number of a 2-subdivided graph are easy to compute.

**Theorem 4.** If \( G \) is a connected graph, then the following holds.

1. \( \gamma(S_2(G)) = |V_G| \).
2. \( |V_G| \leq \gamma_a(S_2(G)) \leq |V_G| + 2 \).
3. \( \gamma_a(S_2(G)) = \begin{cases} |V_G| + 2, & \text{if } G \text{ is a cycle}, \\ |V_G| + 1, & \text{if } G = K_2, \\ |V_G|, & \text{otherwise}. \end{cases} \)

**Proof.** The statement (1) follows from Theorem 3(1).

(2) The inequalities \( |V_G| \leq \gamma_a(S_2(G)) \leq |V_G| + 2 \) are obvious if \( G = K_1 \). Thus assume that \( G \) is a connected graph of order at least two. Let \( u \) and \( v \) be adjacent vertices of \( G \). Then,
Thus assume that neighborhoods of $S$ If Proof.

\[ \gamma \]

Now, since $2 + 1 = 3$ then $S$ For Proof.

\[ \gamma \]

Case 1. $|E_G| > |V_G|$. In this case $S_2(G) - V_G$ has $|E_G|$ components and therefore no $|V_G|$-element subset of $V_{S_2(G)} \setminus V_G$ dominates $S_2(G)$. Hence, $V_G$ is an accurate dominating set of $S_2(G)$ and $\gamma_a(S_2(G)) = |V_G|$.

Case 2. $|E_G| = |V_G|$. In this case, $G$ is a unicyclic graph. First, if $G$ is a cycle, say $G = C_n$, then $S_2(G) = C_3n$ and $\gamma_a(S_2(G)) = \gamma_a(C_3n) = n + 2 = |V_G| + 2$ (see Proposition 3 in [7]). Thus assume that $G$ is a unicyclic graph which is not a cycle. Then $G$ has a leaf, say $v$. Now, if $u$ is the only neighbor of $v$, then $(V_G \setminus \{v\}) \cup \{(v, vu)\}$ is a minimum dominating set of $S_2(G)$. Since $S_2(G) - ((V_G \setminus \{v\}) \cup \{(v, vu)\})$ has $|V_G| + 1$ components, $(V_G \setminus \{v\}) \cup \{(v, vu)\}$ is a minimum accurate dominating set of $S_2(G)$ and $\gamma_a(S_2(G)) = |(V_G \setminus \{v\}) \cup \{(v, vu)\}| = |V_G|$.

Case 3. $|E_G| = |V_G| - 1$. In this case, $G$ is a tree. Now, if $G = K_1$, then $S_2(G) = K_1$ and $\gamma_a(S_2(G)) = \gamma_a(K_1) = 1 = |V_G|$. If $G = K_2$, then $S_2(G) = P_4$ and $\gamma_a(S_2(G)) = \gamma_a(P_4) = 3 = 2 + 1 = |V_G| + 1$. Finally, if $G$ is a tree of order at least three, then the tree $S_2(G)$ is not a corona graph and by (1) and Theorem $C$ we have $\gamma_a(S_2(G)) = \gamma(S_2(G)) = |V_G|$.

As a consequence of Theorem $A$ we have the following results.

Corollary 3. If $T$ is a tree and $\mathcal{P} = \{P(v) : v \in V_T\}$ is a family of partitions of the vertex neighborhoods of $T$, then

\[
\gamma_a(T \circ \mathcal{P}) = \begin{cases} 
|V_T| + 1 & \text{if } |P(v)| = 1 \text{ for every } v \in V_T \\
|V_T| & \text{if } |P(v)| > 1 \text{ for some } v \in V_T.
\end{cases}
\]

Proof. If $|P(v)| = 1$ for every $v \in V_T$, then $T \circ \mathcal{P} = T \circ K_1$ and the result follows from Corollary $B$. If $|P(v)| > 1$ for some $v \in V_T$, then the tree $T \circ \mathcal{P}$ is not a corona and the result follows from Theorem $C$ and Theorem $C$ (1).

Corollary 4. For $n \geq 3$, if $\mathcal{P} = \{P(v) : v \in V_{C_n}\}$ is a family of partitions of the vertex neighborhoods of $C_n$, then

\[
\gamma_a(C_n \circ \mathcal{P}) = \begin{cases} 
n + 1 & \text{if } |P(v)| = 1 \text{ for every } v \in V_{C_n} \\
n + 2 & \text{if } |P(v)| = 2 \text{ for every } v \in V_{C_n} \\
n & \text{otherwise}.
\end{cases}
\]

Proof. If $|P(v)| = 1$ for every $v \in V_{C_n}$, then $C_n \circ \mathcal{P} = C_n \circ K_1$. Thus, by Theorem $B$ we have $\gamma_a(C_n \circ \mathcal{P}) = \gamma_a(C_n \circ K_1) = \gamma(C_n \circ K_1) = |V_{C_n}| + 1 = n + 1$.

If $|P(v)| > 1$ (and therefore $|P(v)| = 2$) for every $v \in V_{C_n}$, then $C_n \circ \mathcal{P} = S_2(C_n) = C_{3n}$. Now, since $\gamma_a(C_{3n}) = n + 2$ (as it was observed in [7]), we have $\gamma_a(C_n \circ \mathcal{P}) = \gamma_a(C_{3n}) = n + 2$. 

\[ 10 \]
Finally assume that there are vertices $u$ and $v$ in $C_n$ such that $|P(v)| = 1$ and $|P(u)| = 2$. Then the sets

$$V^1_{C_n} = \{x \in V_{C_n} : |P(x)| = 1\} \quad \text{and} \quad V^2_{C_n} = \{y \in V_{C_n} : |P(y)| = 2\}$$

form a partition of $V_{C_n}$. Without loss of generality we may assume that $x_1, x_2, \ldots, x_k$, $y_1, y_2, \ldots, y_\ell, \ldots, z_1, z_2, \ldots, z_p, t_1, t_2, \ldots, t_q$ are the consecutive vertices of $C_n$, where

$$x_1, x_2, \ldots, x_k \in V^1_{C_n}, y_1, y_2, \ldots, y_\ell \in V^2_{C_n}, \ldots, z_1, z_2, \ldots, z_p \in V^1_{C_n}, t_1, t_2, \ldots, t_q \in V^2_{C_n},$$

and $k + \ell + \ldots + p + q = n$. It is easy to observe that $D = \{(x_i, N_{C_n}(x_i)) : i = 1, \ldots, k\} \cup \{(y_j, 1) : j = 1, \ldots, \ell\} \cup \cdots \cup \{(z_i, N_{C_n}(z_i)) : i = 1, \ldots, p\} \cup \{(t_j, 1) : j = 1, \ldots, q\}$ is a dominating set of $C_n \circ \mathcal{P}$. Since the set $D$ is of cardinality $n = |V_{C_n}|$ and $n = \gamma(C_n \circ \mathcal{P})$ (by Theorem 3.1), $D$ is a minimum dominating set of $C_n \circ \mathcal{P}$. In addition, since $C_n \circ \mathcal{P} - D$ has $k + (2 + (\ell - 1)) + \ldots + p + (2 + (q - 1)) > k + \ell + \ldots + p + q = n$ components, that is, since $\kappa(C_n \circ \mathcal{P} - D) > n$, no $n$-element subset of $V_{C_n \circ \mathcal{P}} \setminus D$ is a dominating set of $C_n \circ \mathcal{P}$. Thus, $D$ is an accurate dominating set of $C_n \circ \mathcal{P}$ and therefore $\gamma(C_n \circ \mathcal{P}) = n$. \hfill \Box

4 Closing open problems

We close with the following list of open problems that we have yet to settle.

**Problem 1.** Find a formula for the accurate domination number $\gamma_a(G \circ \mathcal{F})$ of the $\mathcal{F}$-corona of a graph $G$ depending only on the family $\mathcal{F} = \{F_v : v \in V_G\}$ such that $\gamma(F_v) = 1$ for every $v \in V_G$.

**Problem 2.** Characterize the graphs $G$ and the families $\mathcal{P} = \{P(v) : v \in V_G\}$ for which $\gamma_a(G \circ \mathcal{P}) = |V_G| + \min\{\min\{|P(v)| : v \in V_G\}, 1 + \min\{|A| : A \in \bigcup_{v \in V_G} P(v)\}\}$.

**Problem 3.** It is a natural question to ask if there exists a nonnegative integer $k$ such that $\gamma_a(G \circ \mathcal{P}) \leq |V_G| + k$ for every graph $G$ and every family $\mathcal{P} = \{P(v) : v \in V_G\}$ of partitions of the vertex neighborhoods of $G$.

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