A SELF-DUAL MODEL FOR THE
SU(2) GAUGE THEORY

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Abstract

A model of SU(2) gauge theory in the space-time $R \times S^3$ is constructed in terms of local gauge-invariant variables. A metric tensor $g_{\mu\nu}$ is defined starting with the components of the strength tensor $F_{\mu\nu}^k$ and of its dual $\tilde{F}_{\mu\nu}^k$. It is shown that the components $g_{\mu\nu}$ determine the gauge field equations if some supplementary constraints are imposed. Two families of analitical solutions of the field equations are also obtained.
1 Introduction

The gauge theory are usually formulated in terms of non-gauge-invariant variables, like potentials $A^k_\mu(x)$. However, the physical observables are gauge invariant. The connection between potentials and observables is established by choosing a specific gauge. This rises many difficulties even in the classical theories. For example, some solutions of the field equations can be spherical symmetric in a chosen gauge, but they may nor have this symmetry in other gauges. In addition, the potentials can be singular in one and non-singular in others.

At the quantum level these difficulties are even more serious. For example, the quantization of one and the same theory in different gauges can give rise to non-equivalent theories.

However, the physicists prefer to use the non-gauge-invariant potentials because of the following advantages:

(i) the theory has a local character;
(ii) the relativistic invariance is manifest;
(iii) the Lagrangian formalism applies in the standard form

Recently, some models of gauge theories on Euclidean and Minkowskian 3-dimensional spaces have been developed in terms of gauge-invariant variables [2], [3]. The fundamental quantity is the gauge-invariant tensor $g_{ij} = -\frac{1}{2}Tr(*F_i^*F_j)$, where $*F_i = \frac{1}{2}\epsilon_{ijk}F^j_k$ is the dual of the gauge field tensor $F^ij$. It has been shown that this metric satisfies the Einstein equations with the right-hand side of a very simple form [3].

In a recent work [4] we generalized this theory to the case of a curved space-time. Namely, we developed a $SU(2)$ gauge theory on the three-dimensional sphere $S^3$. The manifold $S^3$ is a space with constant curvature, and the generalization of the theory to this case is not trivial. We used the advantage that the dimensions of the $SU(2)$ group and of the $S^3$ sphere are the same.

In this paper, we develop a model of $SU(2)$ gauge theory in terms of local gauge-invariant variables defined over a 4-dimensional space-time. We define a metric tensor $g_{\mu\nu}$, $\mu, \nu = 0, 1, 2, 3$, starting with the components $F^k_\mu$ and $\bar{F}^k_\mu$ of the tensor associated to the Yang-Mills field and, respectively, to its dual. The components $g_{\mu\nu}$ are interpreted as new local gauge variables and they are calculated for a particular gauge field defined over the $R \times S^3$ space-time. It is shown that these components determine the gauge field
equations when some supplementary conditions are imposed.

Our metric $g_{\mu \nu}$ do not make the field strength self-dual. In order to assure this property a convenient scale factor $\Delta$ is introduced in the expression of the metric $g_{\mu \nu}$. It is concluded that the self-dual variables are nor compatible with the geometric structure of the space-time $R \times S^3$.

Two families of analytical solutions for the field equations of the gauge fields are also given. It is proven that these solutions are not of solitonic type, but they can be used to determine the structure of vacuum state for the gauge fields considered in this paper.

2 Gauge Potentials

We consider the $SU(2)$ Yang-Mills theory in the space-time $R \times S^3$ endowed with the metric:

$$ds^2 = -dt^2 + d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\varphi^2)$$ (2.1)

where $t$ is the time coordinate on the real line $R$ and $\chi, \theta$ and $\varphi$ are the angular coordinates on the three-dimensional sphere $S^3$ \[5\]. Let $P(M, G, \pi)$ be the principal fibre bundle with $M = R \times S^3$ as the base manifold and $G = SU(2)$ as the structural group. The mapping $\pi : P \to M$ is the natural projection of $P$ onto $M$. The infinitesimal generators of the $SU(2)$ group are chosen in the form:

$$T_k = \frac{1}{2i} \sigma_k , k = 1, 2, 3$$ (2.2)

where $\sigma_k$ are the Pauli matrices.

The corresponding structure equations are:

$$[T_k, T_l] = \varepsilon_{klm} T_m$$ (2.3)

where $\varepsilon_{klm}$ is the antisymmetric tensor of rank 3 with $\varepsilon_{123} = 1$.

The gauge potentials $A_{\mu} = A_{\mu}^k T_k$, with values in the Lie algebra of the group $SU(2)$, determine a connection on the principal fibre bundle $P(M, G, \pi)$ \[6\]. We will consider the following ansatz \[7\] for the gauge potentials:

$$A_0 = -\Phi T_3, \quad A_1 = 0, \quad A_2 = W T_2, \quad A_4 = \cos \theta T_3 - W \sin \theta T_1$$ (2.4)

where $\Phi$ and $W$ are two unknown functions depending only on the variable $\chi$. Therefore, the components $A_{\mu}^k$ of these gauge potentials are the following:
The tensor of the gauge fields \( F^{\mu \nu} = F^k T_k \), with values in the Lie algebra of SU(2), is defined by the formula \( \Box \):

\[
F^{\mu \nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu} + [A^{\mu}, A^{\nu}]
\]

where \( \partial^{\mu} \) denotes the derivatives with respect to the variables \( x^\mu = (t, \chi, \theta, \phi) \) and \( \mu = 0, 1, 2, 3 \).

The non-null components of this tensor are:

\[
\begin{align*}
F^{12} &= W \Phi, & F^{13} &= -W' \sin \theta, \\
F^{23} &= W', & F^{21} &= \Phi W \sin \theta, \\
F^{30} &= \Phi', & F^{32} &= (W^2 - 1) \sin \theta
\end{align*}
\]

where \( \Phi' = \frac{d \Phi}{d \chi} \) and \( W' = \frac{d W}{d \chi} \) are the derivatives of the two functions \( \Phi \) and \( W \) with respect to the variable \( \chi \).

In the next section we will determine a metric tensor \( g^{\mu \nu} \) starting with the tensor \( F^{\mu \nu} \) and its dual \( \tilde{F}^{\mu \nu} \). The components of this metric tensor will be interpreted as local gauge-invariant variables for the SU(2) Yang-Mills theory on the space-time \( R \times S^3 \).

### 3 Local gauge-invariant variables

The gauge potentials \( A^{\mu}_k \) are not invariant under the gauge transformations. In order to obtain new variables, which are invariant, we define first the dual \( \tilde{F}^{\mu \nu} \) of the strength tensor field \( F^{\mu \nu} \) \( \Box \):

\[
F^{\mu \nu} = \frac{1}{2} \varepsilon^{\mu \nu \sigma \tau} \tilde{F}_{\sigma \tau}
\]

where \( \varepsilon^{\mu \nu \sigma \tau} \) is the antisymmetric tensor of rank 4, with \( \varepsilon_{0123} = 1 \). Then, using the above result (2.7), we obtain the following non-null components of the dual \( \tilde{F}^{\mu \nu} \):

\[
\begin{align*}
\tilde{F}^{02} &= W' \sin \theta, & \tilde{F}^{12} &= -W \Phi, \\
\tilde{F}^{13} &= \Phi W \sin \theta, & \tilde{F}^{23} &= W', \\
\tilde{F}^{30} &= (W^2 - 1) \sin \theta & \tilde{F}^{32} &= \Phi'
\end{align*}
\]

Now, we introduce new local gauge-invariant variables \( g^{\mu \nu} \), given by \( \Box \):
\[ g_{\mu\nu} = \frac{1}{3\Delta^{1/3}} \varepsilon^{klm} F_{\mu\alpha} \tilde{F}_{\alpha\beta}^{l} \tilde{F}_{\beta\nu}^{m} \quad (3.3) \]

and:

\[ g^{\mu\nu} = \frac{2}{3\Delta^{1/3}} \varepsilon^{klm} \tilde{F}_{\mu\alpha}^{l} F_{\alpha\beta}^{m} \tilde{F}_{\beta\nu}^{l} \quad (3.4) \]

Here, \( \Delta \) is a scale factor which will be chosen of a convenient form in what follows.

Introducing the expressions (2.7) and (3.2) of the tensor \( F_{\mu\nu} \) and respectively of its dual \( \tilde{F}_{\mu\nu} \), we obtain the following non-null components of \( g_{\mu\nu} \):

\[
\begin{align*}
    g_{00} &= \frac{2}{\Delta^{1/3}} W^2 \Phi^2 \Phi' \sin \theta, \\
    g_{11} &= \frac{2}{\Delta^{1/3}} W^2 \Phi' \sin \theta, \\
    g_{22} &= \frac{2}{\Delta^{1/3}} WW' \Phi (W^2 - 1) \sin \theta, \\
    g_{33} &= \frac{2}{\Delta^{1/3}} WW' \Phi (W^2 - 1) \sin^2 \theta,
\end{align*}
\]

(3.5)

Having these quantities determined, we introduce a new metric manifold, whose line element written in the variables \((t, \chi, \theta, \varphi)\) is:

\[
d\sigma^2 = g_{00} dt^2 + g_{11} d\chi^2 + g_{22} d\theta^2 + g_{33} d\varphi^2
\]

or:

\[
d\sigma^2 = \frac{2W^2 \Phi^2 \Phi' \sin \theta}{\Delta^{1/3}} \left[ -dt^2 - \frac{W^2}{W^2 \Phi} d\chi^2 + \frac{W'(1-W^2)}{W \Phi \Phi'} \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right) \right]
\]

(3.7)

If we chose now the scale factor \( \Delta \) in the form:

\[
\Delta^{1/3} = -2W^2 \Phi^2 \Phi' \sin \theta
\]

(3.8)

then (3.7) reduces to:

\[
d\sigma^2 = -dt^2 - \frac{W^2}{W^2 \Phi} d\chi^2 + \frac{W'(1-W^2)}{W \Phi \Phi'} \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right)
\]

(3.9)

The line element (3.9) coincides with that of the space-time \( R \times S^3 \) given in (2.1) if we impose the following supplementary conditions:

\[
W' = \frac{dW}{d\chi} = iW \Phi, \quad \Phi' = \frac{d\Phi}{d\chi} = i \frac{1 - W^2}{\sin^2 \chi}
\]

(3.10)

But the conditions (3.10) are nothing than the field Yang-Mills equations for the potentials \( A_k^\mu \). In fact, the Yang-Mills equations are differential
equations of second order; in the case of the ansatz (2.40), they have the following form:

\[
\frac{d}{d\chi} \left( \frac{d\Phi}{d\chi} \sin^2 \chi \right) = 2W^2\Phi \tag{3.11}
\]

\[
\frac{d}{d\chi} \left( \frac{dW}{d\chi} \right) = -W\Phi^2 - \frac{(1 - W^2)}{\sin^2 \chi} W \tag{3.12}
\]

It is easy to verify that the equations (3.11)-(3.12) are equivalent with the first order equations given in (3.10).

Therefore, we conclude that the scale factor \( \Delta \) chosen in (3.8) together with the field equations of the gauge potentials (2.5), reduce the metric \( g_{\mu\nu} \) to that of the space-time \( R \times S^3 \).

It is important to remark that choosing \( \Delta \) as in (3.8) we do not obtain a self-dual gauge theory, i.e. \( F_{\mu\nu} \) is not a self-dual tensor. This means that the following condition of self-duality is not satisfied \[10\]:

\[
\frac{1}{2\sqrt{g}} \varepsilon^{\alpha\beta\sigma\tau} g_{\mu\alpha} g_{\nu\beta} F_{\sigma\tau} = F_{\mu\nu} \tag{3.13}
\]

where \( g = \det(g_{\mu\nu}) \). Therefore, the metric (2.1) of the space-time is not compatible with a self-dual \( SU(2) \) gauge theory.

In order to satisfy the condition (3.13) we should choose the factor \( \Delta \) in the form:

\[
\Delta = 8\Phi^2W^2W'2(W^2 - 1)\sin^3 \theta \tag{3.14}
\]

In this case we obtain:

\[
g = \det(g_{\mu\nu}) = \frac{1}{4}\Delta^{2/3} \tag{3.15}
\]

However, the metric components \( g_{\mu\nu} \) in (3.5) do not correspond to the geometry of the space-time \( R \times S^3 \) when \( \Delta \) is given by (3.14). Therefore, the space-time \( R \times S^3 \) do not admit a self-dual \( SU(2) \) gauge theory.

4 Solutions of the field equations

We have obtained two families of analytic solutions of the field equations (3.10) \[9\]. First of them is:

\[
\Phi(\chi) = i\mu \cot(\mu \chi) - i \cot(\chi) \tag{4.1}
\]

\[
W(\chi) = \frac{\mu \sin \chi}{\sin(\mu \chi)} \tag{4.2}
\]
where $\mu$ is an arbitrary constant. The Chern index of this solution is:

$$c_2 = \int_0^\pi \left[ \Phi'(1 - W^2) - 2WW'\Phi \right] = \Phi(1 - W^2) |_0^\pi = 0$$  \hspace{1cm} (4.3)

This value of $c_2$ shows that the solutions (4.1)-(4.2) in first family are not of instanton type.

The second family of solutions for the field equations 93.10) is given by

$$\Phi(\chi) = i\nu \coth(\mu\chi) - i\cot(\chi) \hspace{1cm} (4.1)$$

$$W(\chi) = -\frac{\nu \sin \chi}{\sinh(\nu\chi)} \hspace{1cm} (4.2)$$

where $\nu$ is an arbitrary real constant. The Chern index has, in this case, the value $c_2 = \infty$. Therefore, the solutions in this family are not of instanton type too.

However, the solutions in the two families can be used for studying the vacuum structure of the fields under considerations.

5 Concluding remarks

In this paper we have developed a gauge theory in terms of local gauge-invariant variables defined over the space-time $\mathbb{R} \times S^3$. A metric tensor $g_{\mu\nu}$, $\mu, \nu = 0, 1, 2, 3$, has been constructed starting with the components of the strength tensor $F_{\mu\nu}$ and of its dual $\tilde{F}_{\mu\nu}$. The components $g_{\mu\nu}$ have been interpreted as new local gauge-invariant variables and they have been calculated for some particular gauge fields defined over the space-time $\mathbb{R} \times S^3$.

We proved that these components determine the gauge field equations if some supplementary constraints are imposed. On the other hand we showed that the metric tensor $g_{\mu\nu}$ do not make the field strength $F_{\mu\nu}$ self-dual. In order to assure this property, a convenient scale factor $\Delta$ has been introduced in the expression of $g_{\mu\nu}$. We concluded that the self-dual variables are not compatible with the geometric structure of the space-time $\mathbb{R} \times S^3$.

Finally, two families of analytical solutions of the gauge field equations have been obtained. We proved that these solutions are not of instanton type, but they could be used to determine the structure of the vacuum state for the gauge fields considered in this paper.
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