Comparison of Extended Irreversible Thermodynamics with thermodynamics based on a distribution containing the first-passage time.

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Abstract
An analogy is drawn between version of non-equilibrium thermodynamics a distribution-based containing an additional thermodynamic first-passage time parameter and Extended Irreversible Thermodynamics with flows as an additional thermodynamic parameter. Thermodynamics containing an additional thermodynamic first-passage time parameter maps to Extended Irreversible Thermodynamics. Various conditions for the dependence of the distribution parameters of the first-passage time on the random value of energy, the first thermodynamic parameter, are considered. It is shown that more meaningful results are recorded when the time parameter relaxation time τ of Extended Irreversible Thermodynamics is replaced by the average first-passage time. Expressions are obtained for the average first-passage time through the flows.

Extended Irreversible Thermodynamics (EIT) [1-4] goes beyond the local equilibrium hypothesis. The paradox of propagation of signals with infinite speed is eliminated. The laws of Fourier, Fick, Stokes and Newton are generalized. Considerations include: memory effects (for fast processes and polymers); non-local effects (micro- and nanodevices); non-linear effects (high powers). The main innovations of the theory are: raising dissipative flows to the status of state variables; the central role is given to the generalized entropy, assumed to be a given function of the entire set of variables, and the rate of production of which is always positive definite.

The EIT [1-4] uses an additional thermodynamic parameter, flows of thermodynamic quantities. For simplicity, we restrict ourselves to the case of heat flow. The change in entropy in this case is

\[ s(u, q) = s_{eq}(u) - \frac{1}{2 \rho \lambda \theta^2} q^2, \]  

(1)

where \( \rho \) is the total mass density, \( \lambda \) is the heat conductivity, depending generally on the temperature, \( \theta \) is the non-equilibrium temperature [1], \( q \) is the heat flux, \( T \) is the local equilibrium temperature, \( T(u) \), \( u \) is the specific internal energy, \( s \) is a local specific entropy.

The term containing the time \( \tau \) represents the heat flux relaxation. When the relaxation time \( \tau \) of the heat flux is negligible or when the time variation of the heat flux is slow, this equation reduces to Fourier’s law. For homogeneous solids, \( \tau \) describes molecular-scale energy transfer by either phonons or electrons, and it is very small, of the order of time between two successive collisions at the microscopic level. Therefore, in most practical heat transfer problems, infinite propagation is not relevant as those parts of the signals with infinite velocity are strongly damped at room temperature. However, when slow internal degrees of freedom are involved, as in polymers, superfluids, porous media, or organic tissues, \( \tau \) reflects the time required to transfer energy between different degrees of freedom and it may be relatively large, of the order or larger than 1 s [4]. The value \( \Delta \) (3) in EIT is proportional to the relaxation time \( \tau \). In this case, the time of the process is constant and equal to \( \tau \). Appendix A explains this behavior of \( \Delta \).
In [5-6], thermodynamics is developed, in which an additional thermodynamic parameter is the first-passage time (FPT). The entropy in this case is equal to the average logarithm of the distribution containing FPT \( [7-10] \). Note that FPT is widely used in various fields \([11–17]\). Appendix B briefly describes the main features of thermodynamics with FPT (TFPT). The introduction of additional state variables is a common feature of EIT and TFPT. At the same time, fluxes and FPT have a lot in common. Therefore, it is interesting to compare these areas of nonequilibrium thermodynamics. With EIT, the approach proposed in [5-6] is united by the fact that in both cases an additional thermodynamic parameter is introduced. In EIT it is flows, in thermodynamics with first-passage time (TFPT) it is FPT. A similar value is implicitly contained in the method of nonequilibrium statistical operator (NSO) \([18-19]\); in it, the quasi-equilibrium distribution is averaged over the distribution of the lifetime of the system \([20]\).

Equate (A1) and (B5) \( ds_{EIT} = \theta^{-1} d\beta u_{\beta} - \frac{\tau}{\rho \lambda T} q dq \) to \( ds_{\gamma} = \beta d\langle u \rangle + \gamma d\langle T \rangle \) from TFPT. Mean values \( u_{\beta} = \frac{1}{Z_{\beta}} \int e^{-\beta\omega(u) du} \), \( Z_{\beta} = \int e^{-\beta\omega(u) du} \), and \( \langle u \rangle = \frac{\partial \ln Z(\beta, \gamma)}{\partial \beta} \mid_{\gamma} \) are generally different \((A4)\). But for the independent variables \( u \) and \( T_{\gamma} \) (when the distribution of FPT \( T_{\gamma} \) does not depend on the random variable \( u \)) they coincide, \( u_{\gamma} = \langle u \rangle \):

\[
\frac{ds}{T} = \frac{d\theta^{-1}}{u} - a_{\beta} q dq = \beta d\bar{u} + \gamma d\bar{T}_{\gamma}.
\]

We equate the entropy changes \( \Delta \) in EIT \((1), (A1)\) and TFPT \((B8)\):

\[
\Delta = \Delta s_{EIT} = \frac{1}{2} \frac{\tau}{\rho \lambda T} q^{2} = s_{eq}(u) - s(u, q) = \Delta = \Delta s_{\gamma} = s_{\beta} - s = -[\gamma T_{\gamma} + \ln Z(\beta, \gamma) - \ln Z_{\beta} + \beta \bar{u} - \beta u_{\beta}],
\]

\[
s_{eq} = s_{\beta} = \beta u_{\beta} + \ln Z_{\beta}.
\]

The expressions \((C4), (C8), (C9)\) obtained from such an equation in Appendix C seem to be of little information and not entirely correct. And the dependence with a fixed value of the parameter \( \gamma \), taken in order for the Wald distribution \((C28), (C29)\) to satisfy condition \((C8)\), looks artificial. However, the main reason for replacing the time constant \( \tau \) with \( T_{\gamma} \) is that the constancy of the process time, with a constant parameter \( \tau \), is not actually fulfilled.

For these reasons, in expression \((1), (A1)\) to change the entropy, the parameter \( \tau \) is replaced by \( T_{\gamma} \). The entropy changes \( \Delta = \Delta s_{\gamma} = s_{\beta} - s \) in TFPT is equated to \( \Delta = \Delta s_{\gamma} = s_{\beta} - s = \frac{1}{2} \frac{T_{\gamma} q q}{\rho \lambda T^{2}} \) from EIT in the form \( \Delta = \Delta s_{\gamma} = s_{\beta} - s = \frac{1}{2} \frac{T_{\gamma} q q}{\rho \lambda T^{2}} = \Delta_{1} \), \( \Delta_{1} = \frac{q q}{2 \rho \lambda T^{2}} \). Then

\[
\Delta = \Delta s_{\gamma} = s_{\beta} - s = -[\gamma T_{\gamma} + \ln Z(\beta, \gamma) - \ln Z_{\beta} + \beta \bar{u} - \beta u_{\beta}] = \frac{1}{2} \frac{T_{\gamma} q q}{\rho \lambda T^{2}}.
\]

Wherein

\[
-d\Delta = \gamma dT_{\gamma} + \beta(d\bar{u} - du_{\beta}).
\]

A similar approach to entropy change using FPT was applied in \([21]\). Appendix D gives the results obtained from expressions \((4)-(5)\).

Possibly, in this case one should interpret the average FPT \( T_{\gamma} \) as the parameter of relaxation time \( \tau \) of the Cattaneo’s equation \((C32)\) for the processes indicated after expression \((1)\) in a small volume. In this case, the value \( \tau = T_{\gamma} \) is not a constant value, but depends on the parameter \( \gamma \) from
the expression (B1), but through the parameter $\gamma$ and on the change in the entropy $\Delta$ (C33) and flows $q$.

The possibility of considering the variables $u$ and $T_\gamma$ independent remains an open question. The independence of the distribution of FPT $T_\gamma$ on the random variable $u$ [7] seems rather artificial. This is an approximation connected with the replacement of the random variable $u$ by the mean value $<u>$. Conditions (C9) and (C8) immediately follow from it. It is also necessary to investigate the question of what physical conditions correspond to different models of the dependences of the FPT distribution on the random value of energy $u$.

Another important question is about the time constant $\tau$ in (1). Replacing the time constant $\tau$ by $\bar{T}_\gamma$ in EIT makes it possible to obtain meaningful thermodynamic relations relating the fluxes from EIT with the parameter $\gamma$ from TFPT conjugate to $T_\gamma$. And it's physically justified. The process time is not constant, but itself depends on the process. In the proposed approach, the mean FPT $\bar{T}_\gamma$ depends on $\Delta$, the change in entropy, which in turn depends on $\bar{T}_\gamma$. The resulting expressions are self-consistent: the change in entropy depends on the average value of FPT, which depends on the change in entropy. What limits are reached by FPT? In [21] it is written: «The observable $O$, the threshold $D$, and the time $\tau := \inf\{t \geq 0 : O(\omega t) \in D\}$, define the physical process and its duration. For concreteness, $\tau$ may be the minimum time to displace a mass, or to exchange a given amount of energy or particles with a reservoir. In general, it represents the time needed for a specific physical process to be carried out by the system». This quantity $\tau$ differs significantly from the relaxation time $\tau$. The deviation of the stationary value of fluxes that exceeds the standard deviation, as well as a number of other physical quantities, for example, the time to reach a new stationary or equilibrium state, can be considered as a threshold.

The proposed version of non-equilibrium thermodynamics assumes that the Laplace transform function of the probability density distribution FPT is known. However, in equilibrium thermodynamics, in order to apply it to specific systems, one must know the equation of state and other characteristics of the system. There are a lot of FPT distributions, as well as random processes themselves. Classes of this kind of distributions can be divided according to the stages of the evolution of the system [18-19].

Two examples are considered: the exponential distribution and the Wald distribution. The exponential distribution was used in the nonequilibrium statistical operator [18-20]. In [21], the exponential distribution for FPT was also used. This was justified by the rarity of events. The exponential distribution can be compared with the hydrodynamic stage of the evolution of the system (in the terminology of [18-19]), and the Wald distribution with the kinetic stage.

By itself, the value of the average FPT $\bar{T}_\gamma$ is informative and important in many areas of science and its applications [11-13]. And the expression of this quantity in terms of the entropy change parameters $\Delta$ (3), flows $q$ and $\gamma$ (B1) should be useful. The parameters $\Delta$ and $q$ are known and important. The parameter $\gamma$ is also important. Its meaning is not entirely clear. Based on the symmetry of the thermodynamic parameters $u$ and $T_\gamma$ from (B1), this parameter is similar to the parameter $\beta$ - the reciprocal temperature. The equation for the distribution of $T_\gamma$ is conjugate to the equation for the energy $u$ [14], the basic quantity of equilibrium thermodynamics. If $\gamma\sim\sigma^4$ is proportional to entropy production, then what is proportional to the inverse entropy production?

A detailed mesoscopic description can be obtained using explicit stochastic models of the system, such as diffusion type [11-16] or stochastic storage processes [22]. The latter can be considered as models of a system excited by generalized noise. They take into account the
interaction with the thermostat (reservoir), as well as the moments of degeneracy and loss of stationarity under certain conditions.

Appendix C shows dependencies derived from the assumption that the distribution of FPT \( f(T) \) depends not only on the mean values \( \langle u \rangle \), but also on the random variable \( u \). Appendix D shows the consequences of the assumption that the relaxation time \( \tau \) in (1), (A1) is replaced by the average FPT \( \bar{T} \). It can be seen that these assumptions significantly increase the possibilities of describing nonequilibrium systems.

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Appendix A: Fundamentals of the EIT. How does the parameter $\tau$ appear.

In the EIT the independent thermodynamic variables are the energy $u_\beta$ and flows $q$ [1-4]. For simplicity, we restrict ourselves to the case of the simple problem of heat conduction in rigid bodies, without considering other transport processes. In TFPT, the variables are mean energy $\langle u \rangle$ and FPT $T_\gamma$. The EIT gives an explicit expression for entropy outside (local) equilibrium up to second-order terms in $q$ [1-4]:

$$s(u_\beta, q) = s_{eq}(u_\beta) - \frac{1}{2} \frac{\tau}{\rho \lambda \theta^2} q^2,$$

(A1)

$$ds = \theta^{-1} du_\beta - \frac{\tau}{\rho \lambda \theta^2} dq d\theta,$$

(A2)

$$\sigma^s = \frac{1}{\lambda \theta^2} q q, \quad a_\beta = \frac{\tau}{\rho \lambda \theta^2},$$

(A3)

$$\bar{u}_\beta = u_\beta = \frac{\partial \ln Z_{\beta}}{\partial \beta} \bigg|_{\gamma} \quad Z_{\beta} = \int e^{-\rho u - \omega(u)} du \quad \bar{u} = \bar{u}(\beta, \gamma) = \frac{\partial \ln Z(\beta, \gamma)}{\partial \beta} \bigg|_{\gamma},$$

(A4)

where $\sigma^s$ is the entropy production, function $Z(\beta, \gamma)$ is defined in (B2), $Z_{\beta} = Z(\beta) = Z(\beta, \gamma)_{\gamma=0}$.

In EIT, the equation for the material time derivative of the entropy is written in the general form of the entropy balance equation

$$\rho \dot{s} = -\nabla \dot{J}^s + \sigma^s.$$  

(A5)

In this case, entropy flux $\dot{J}$ and the (positive) entropy production $\sigma^s$ given $J^s = T^{-1} q$ and
\[ \sigma' = q(\nabla T^{-1} - T^{-1} \alpha_i \dot{q}) \geq 0. \] (A6)

Relation (A6) has the structure of a bilinear form \( \sigma' = q X \) in the flux \( q \) and the force \( X \), identified as the quantity within parentheses in (A6); it differs from the classical thermodynamic force, which is simply \( \nabla T^{-1} \), by the presence of a term in the time derivative of the heat flux. The simplest way to obtain an evolution equation for \( q \) compatible with the positiveness of \( \sigma' \) is to assume that the force \( X \) is linear in \( q \), namely

\[ \nabla T^{-1} - T^{-1} \alpha_i \dot{q} = \mu(q), \] (A7)

where the phenomenological coefficient \( \mu_1 \) may depend on \( u \) but not on \( q \); third-order contributions in \( q \) are omitted. Introduction of (A7) into (A6) results in \( \sigma' = \mu_1 q q \geq 0 \), from which it is inferred that \( \mu_1 > 0 \). Expression (A7) contains two non-defined coefficients \( \alpha_i \) and \( \mu_1 \), which must be identified on physical grounds. Under steady state conditions, (A7) simplifies to \( q = -\nabla T/(\mu_1 T^2) \); \( \mu_1 = (\lambda T^2)^{-1} \); by comparing (A7) with Cattaneo’s equation (C32), one is led to \( \alpha_i = \tau_\lambda / \beta T \).

With the above identifications of \( \mu_1 \) and \( \alpha_i \), the generalized Gibbs’ equation \( ds = T^{-1} du \theta - T^{-1} v \alpha_1 q dq \) takes the form (\( v \) is the specific volume (the reciprocal of the mass density))

\[ ds = T^{-1} du \theta - (\tau / \rho \lambda T^2 ) q dq, \] (A8)

wherein it is important to observe that the coefficient of the new term in \( dq \) is completely identified in terms of physical quantities, namely the relaxation time \( \tau \) and the heat conductivity \( \lambda \). After integration of (A8), the explicit expression for the entropy outside (local) equilibrium up to second-order terms in \( q \) is (1), (A2). Note that in (A7) the simplest (but not the only) way of choosing the thermodynamic force \( X \) is chosen. Coefficient \( \alpha_i \) represents functions only from \( u_\theta \); her dependence on \( q^2 \) is assumed to be negligible.

### Appendix B: Fundamentals of the TFPT

In [7-10] a statistical distribution is used, which contains the time it takes for a random process to reach a certain level (first-passage time [7-9] or lifetime in terms of [10]).

In [7-10] the statistical distribution of the random variable of the system the first passage time (lifetime) \( T_\gamma \) was obtained. In the theory of random processes, a more general concept is used - the first passage time it takes for a random process to reach a given level [11-13]. The microscopic density of this distribution in the phase space of variables \( z \) (coordinates and momenta of all particles of the system) has the form

\[ \rho_{rel}(z; u, T_\gamma) = \frac{e^{-\beta u(z)-\gamma T_\gamma(z)}}{Z(\beta, \gamma)}, \quad \beta = 1/T, \quad p(u, T_\gamma) = \frac{e^{-\beta u - \gamma T_\gamma}}{Z(\beta, \gamma)}, \] (B1)

where \( u \) is the energy of the system (we assume this value to be a random process), \( \beta = 1/T \) is the inverse temperature of the reservoir (\( k_B = 1, k_B \) is Boltzmann constant),

\[ Z(\beta, \gamma) = \int e^{-\beta u - \gamma T_\gamma} dz = \int du d T_\gamma \omega(u, T_\gamma) e^{-\beta u - \gamma T_\gamma} \] (B2)

is the partition function. The right side of expression (B1) is obtained by changing variables, passing from variables \( z \) to variables \( u, T_\gamma \). The transition from the left side of expression (B1) to the right one corresponds to the transition from the microscopic description to the macroscopic one. Parameters \( \beta \) and \( \gamma \) are the Lagrange multipliers satisfying the following expressions for the averages:
\[ \langle u \rangle = -\frac{\partial \ln Z(\beta, \gamma)}{\partial \beta} , \quad \langle T \rangle = -\frac{\partial \ln Z(\beta, \gamma)}{\partial \gamma}. \]  

(B3)

The factor \( \omega(u) \) in the case of a distribution for \( u \) is replaced by \( \omega(u, T_\gamma) \), which is the volume of the hypersurface in the phase space containing fixed values of \( u \) and \( T_\gamma \). If \( \mu(u, T_\gamma) \) is the number of states in the phase space with parameter values less than \( u \) and \( T_\gamma \), then \( \omega(u, T_\gamma) = e^\beta \mu(u, T_\gamma) / du dT_\gamma \). Moreover, \( \int \omega(u, T_\gamma) dT_\gamma = \omega(u) \). The number of phase points with parameters in the interval between \( u, u + du; T_\gamma, T_\gamma + dT_\gamma \), is \( \omega(u, T_\gamma) dT_\gamma \).

The factor \( \omega(u, T_\gamma) \) is the joint probability for \( u \) and \( T_\gamma \), considered as the stationary probability of this process. We rewrite the value \( \omega(u, T_\gamma) \) in the form

\[ \omega(u, T_\gamma) = \omega(u) \omega_\lambda(u, T_\gamma) = \omega(u) \sum_{k=1}^{n} R_k f_k(T_\gamma, u). \]  

(B4)

In Eq. (B4) it is assumed that there are \( n \) classes of states in the system; \( R_k \) is the probability that the system is in the \( k \)-th class of states, \( f_k(T_\gamma, u) \) is the density of the distribution of \( FPT \ T_\gamma \) in this class of (ergodic) states (in the general case, \( f_k(T_\gamma, u) \) depends on \( u \)).

Assuming that \( FPT \) can be instrumentally measured, we introduce the local specific entropy \( s \) corresponding to the distribution (B1) \((u \) is specific internal energy) by the relation \([14, 15]\)

\[ s = -\langle \ln \rho_{ed}(z; u, T_\gamma) \rangle = \beta \langle u \rangle + \gamma \langle T_\gamma \rangle + \ln Z(\beta, \gamma); \quad ds = \beta d \langle u \rangle + \gamma d \langle T_\gamma \rangle. \]  

(B5)

At

\[ \beta F_{T_\gamma} = -[\ln Z(\beta, \gamma) + \gamma <T_\gamma>], \quad \beta d = \beta<d_u> - F_{T_\gamma}; \quad ds = \beta d <u> + \gamma d <T_\gamma>. \]  

(B6)

At a constant value of the average \( FPT \ <T_\gamma> \) from (B5)-(B6) we obtain thermodynamic relations that coincide with the equilibrium:

\[ \beta = 1 / T = \partial \delta \langle u \rangle / \partial <T_\gamma>; \quad s = -\partial F_{T_\gamma} / \partial T_{\langle T_\gamma \rangle}; \quad <u> = F_{T_\gamma} T \partial F_{T_\gamma} / \partial T_{\langle T_\gamma \rangle}; \quad c_v = \partial <u> / \partial T_{\langle T_\gamma \rangle} = T \partial \delta / \partial T_{\langle T_\gamma \rangle}, \]

where

\[ c_v = \beta^2 D / \Delta T; \quad D = \Delta u \Delta T - \Delta^2; \quad \Delta u = - \partial <u> / \partial \beta; \quad \Delta = - \partial <u> / \partial \gamma; \]

\[ \Delta_T = - \partial <T_\gamma> / \partial \gamma = \langle T_\gamma^2 \rangle - <T_\gamma>^2; \quad \Delta = - \partial <u> / \partial \gamma; \quad \Delta_T = \langle T_\gamma u > - <T_\gamma> <u> \],

(averaging is carried out over the distribution (B1)). \( c_v \) is an analog of the equilibrium heat capacity; it can also be \( c_p \), since the volume coordinate is not considered.

For spatially inhomogeneous systems, the quantities \( \beta \) and \( \gamma \) generally depend on the spatial coordinate. Distributions (B1) can be considered for a small volume element, in which the values of \( \beta \) and \( \gamma \) are replaced by mean values constant in this volume over this volume element. In nonequilibrium thermodynamics, the densities of extensive thermodynamic quantities (entropy, internal energy, mass fraction of a component) are considered. We follow this approach, including here the FPT. Entropy (B5) has two terms:

\[ s = s + s = s - \Delta = \gamma T + \beta \bar{u} + \ln Z = \beta u + \ln Z - \Delta, \quad -\Delta = s - s, \quad s = \beta \bar{u} + \ln Z. \]  

(B8)

In the case of independent variables

\[ -\Delta = s - \gamma T + \ln Z, \]  

(B9)

(s is the entropy density). We got a match with the expression \( s = -\langle \ln \rho(z; u, T_\gamma) \rangle \).

Let us make an assumption about the independence of the variables \( u \) and \( T_\gamma \). Let us define partition function (B2). In this case, we use expressions (B2)-(B3) and the approximation that the distribution of the first passage time \( f(T_\gamma) \) does not depend on the random value of energy, the
variables are separated, and equilibrium and nonequilibrium components are distinguished in the total average energy [7],

\[ \bar{u} = -\frac{\partial \ln Z(\beta, \gamma)}{\partial \beta}, \quad Z(\beta, \gamma) = Z_\beta Z_\gamma, \quad Z_\beta = \int e^{-\beta u} \omega(u) du, \quad Z_\gamma = \int_0^\infty e^{-\gamma T_\gamma f(T_\gamma) dT_\gamma}, \quad \beta = 1/T_1, \quad (B10) \]

\[ Z_\gamma = \int_0^\infty e^{-\gamma T_\gamma \sum_{j=0}^n P_j f(T_j, u) dT_\gamma}. \quad (B10) \]

where \( f(T_\gamma, u) = f(T_\gamma, \bar{u}) = f(T_\gamma) \) is the probability density of the distribution of the FPT. In (B10), we assume that the parameters included in the distribution of the first passage time depend on the average values of the energy, and not on their random values; \( n=1 \) from (B4). By expressions (B10)-(B11), the partition function factor describing the non-equilibrium behavior of the system \( Z_\gamma \) is expressed in terms of the Laplace transform of the first-passage time \( T_\gamma \) distribution density \( f(T_\gamma) \).

**Appendix C: Comparison of TFPT and EIT with entropy (1)**

Consider expressions (2)-(3).

\[ ds_{EIT} = \theta^{-1} d\bar{u}_\beta - \frac{\tau}{\beta \theta} q dz = ds_{TFPT} = \beta d\bar{u} + \gamma d\bar{T}_\gamma, \quad (C1) \]

\[ -d\Delta s = \beta (d\bar{u}_\beta - du_\beta) + \gamma d\bar{T}_\gamma, \quad (C2) \]

where \( \Delta s_{EIT} \) is chosen in the form (1), (3). If the variables \( u \) and \( T_\gamma \) are independent, then in (C2), \( d\bar{u} = du_\beta \), and

\[ -d\Delta s_{EIT} = \gamma d\bar{T}_\gamma. \quad (C3) \]

Let us substitute the expression (B5), (B8) into the left side of (C3),

\[ \Delta = \Delta s \equiv s_\beta - s = -[\gamma \bar{T}_\gamma + \ln Z(\beta, \gamma) - \ln Z_\beta + \beta \bar{u}_\beta - \beta u_\beta] \to -(\gamma \bar{T}_\gamma + \ln Z_\gamma), \]

where the arrow indicates the transition from the general case to the case of independent variables \( u \) and \( T_\gamma \).

Differentiating in (C3) with respect to \( \gamma \), we obtain an identity. Differentiation with respect to \( \beta \) leads to the relation

\[ \frac{\partial \ln Z_\gamma}{\partial \beta} = 0, \quad (C4) \]

which corresponds to the expression \( \bar{u}_\gamma = 0 \), where the value \( \bar{u}_\gamma = -\frac{\partial \ln Z_\gamma}{\partial \beta} \bigg|_{\beta=0} [7]. \]

Consider the case of exponential distribution for first-passage time; \( T_0 \) does not depend on \( u \):

\[ f(T_\gamma) = (T_0)^{-1} \exp(-T_\gamma/T_0). \quad (C5) \]

The exponential distribution (C5) is valid for large times, \( t \to \infty \). In this case

\[ Z(\gamma) = 1/(1 + \gamma T_0), \quad \bar{T}_\gamma = \frac{T_0}{1 + x}, \quad x = \gamma T_0, \quad \frac{\partial \bar{T}_\gamma}{\partial \gamma} = -(\bar{T}_\gamma)^2. \quad (C6) \]

Entropy (B5) is

\[ s = s_\beta + \frac{x}{1 + x} \ln(1 + x), \quad x = \gamma T_0. \quad (C7) \]

From (C4) we get
\[
\frac{\partial T_0}{\partial \beta} = 0, \quad \frac{\partial T_\gamma}{\partial \beta} = 0, \quad T_0 = T_{\beta=0}. \quad (C8)
\]

According to Maxwell’s relations
\[
\frac{\partial u_\beta}{\partial \gamma} = 0. \quad (C9)
\]

Expression (C9) is valid for all distributions for the first-passage time, provided that the variables \(u\) and \(T_\gamma\) are independent. You can start from the expression (C9), getting from it (C8).

Assuming the value of \(s_{eq}\) from (2) equal to the value of \(s_\beta\) (C7), and comparing (2) and (C7), we obtain the relation
\[
x = \frac{3(1 - e^m)}{(6 - 5e^m)} \pm \sqrt{\frac{3(1 - e^m)}{(6 - 5e^m)^2} - \frac{2(1 - e^m)}{(6 - 5e^m)}}. \quad (C10)
\]

Relation (C10) can be considered as a relation for the expression \(x(m)\), and for the expression \(m(x)\), \(q(x)\). In the case of small fluxes \(q\), the expansion into a series of relation (C10) up to \(e^m\) leads to the transition from the transcendental equation (C10) to the quadratic equation for \(x(m)\) with the solution
\[
x = \frac{3m \pm \sqrt{2m}}{(6 - 5e^m)}, \quad x_+ \approx \sqrt{2m} \frac{1 - 3m/2}{1 - 5m}, \quad x_- \approx -\sqrt{2m} \frac{1 + 3m/2}{1 - 5m}, \quad \sqrt{2m} = q\sqrt{a_\beta}. \quad (C12)
\]

If we choose the value \(x_+\), then \(T_\gamma < T_0\). For \(x\) values \(T_\gamma \geq T_0\).

The relation for \(q\) obtained from (C10) has the
\[
q = \left[\frac{2\rho \lambda \theta^2}{\tau} \left(\ln(1 + x) - \frac{x}{1 + x}\right)\right]^{1/2}. \quad (C13)
\]

Let us now consider the case of dependent variables \(u\) and \(T_\gamma\).

For the exponential distribution (C5), the parameter \(T_0\) is the mean value. But averaging is carried out over the distribution of the random variable \(T_\gamma\). No averaging is carried out over the parameter \(u\). Therefore, in the general case, the dependence \(T_0(u)\) is possible. It was assumed above that \(T_0\) may depend on the average values of the energy \(u\), and not on their random values. In cases of possible dependences of \(T_0\) on \(u\), random values of \(u\) were replaced by their average values.

Consider the case of a linear dependence
\[
T_0 = T_{00} + uT_{10}, \quad (C14)
\]

where \(T_{00}\) and \(T_{10}\) are some parameters. In this case, after integrating over \(T_\gamma\), we obtain
\[
Z(\beta, \gamma) = \int e^{-\beta u} \omega(u) \frac{du}{1 + \gamma(T_{00} + uT_{10})}. \quad (C15)
\]

For small \(\gamma\)
\[
Z(\beta, \gamma) \approx \int e^{-\beta u} \omega(u)[1 - \gamma(T_{00} + uT_{10})]du = Z_\beta[1 - \gamma(T_{00} + u_\beta T_{10})],
\]
\[
u_\beta = \frac{1}{Z_\beta} \int u e^{-\beta u} \omega(u)du, \quad Z_\beta = \int e^{-\beta u} \omega(u)du,
\]
\[
\ln Z(\beta, \gamma) \approx \ln Z_\beta + \ln[1 - \gamma(T_{00} + u_\beta T_{10})],
\]
(C16)
\[
\bar{u} = -\frac{\partial \ln Z(\beta, \gamma)}{\partial \beta} = u_\beta + \frac{\gamma(T_{10}\partial u_\beta / \partial \beta + u_\beta \partial T_{10} / \partial \beta + \partial T_{00} / \partial \beta)}{1 - \gamma(T_{00} + u_\beta T_{10})},
\]
(C17)
\[
\bar{\gamma} = -\frac{\partial \ln Z(\beta, \gamma)}{\partial \gamma} = \frac{T_{00} + u_\beta T_{10}}{1 - \gamma(T_{00} + u_\beta T_{10})}.
\]
(C18)

The expression (C4) is replaced by the expression
\[
\frac{\partial (\ln Z(\beta, \gamma) - \ln Z_\beta)}{\partial \beta} = 0,
\]
(C19)
which leads to the relation
\[
\frac{\partial (T_{00} + u_\beta T_{10})}{\partial \beta} = 0, \quad \frac{\partial (T_{00})}{\partial \beta} = -\frac{\partial (u_\beta T_{10})}{\partial \beta}.
\]
(C20)

Suppose that in expression (C15) the function \(\omega(u) \approx \omega(\bar{u})\). Then for relation (C15) we obtain
\[
Z(\beta, \gamma) \approx -\omega(\bar{u})e^{f(1+\gamma T_{00})/T_{10}} \frac{1}{\gamma T_{10}} E_i(-\frac{\beta(1+\gamma T_{00})}{\gamma T_{10}}),
\]
where \(E_i(z)\) is an integral exponential function [23]. In the case of small values of \(\gamma\), the function \(E_i(z) = -E_i(-z)\) has an asymptotic expansion \(E_i(z) = \frac{e^{-z}}{z^\frac{1}{2}}(1 - \frac{1}{2} + ...).\) Then
\[
Z(\beta, \gamma) = \omega(\bar{u}) \frac{1 - \gamma T_{10} / \beta(1 + \gamma T_{00})}{\beta(1 + \gamma T_{00})}.
\]
Since at \(\omega(u) \approx \omega(\bar{u})\), \(Z_\beta = \omega(\bar{u}) / \beta\), we can write some effective nonequilibrium reciprocal temperature of the form
\[
\beta_{\gamma/\beta}(\beta, \gamma) = \beta(1 + \gamma T_{00}) \approx \beta(1 + \gamma T_{00}) + \gamma T_{10}, \quad \beta_{\gamma/\beta}(\beta, \gamma = 0) = \beta.
\]
(C21)

For dependence \(T_{\delta(u)}\) of the form
\[
T_0 = \frac{T_{00}}{1 + u T_{10}}
\]
(C22)
integration over \(T_\gamma\) leads to the expression
\[
\frac{1}{1 + \gamma T_0} \approx \frac{1}{1 + \gamma T_{00} / (1 + u T_{10})} \approx 1 - \gamma T_{00} / (1 + u T_{10}).
\]
Then
\[
Z(\beta, \gamma) = Z_\beta - \gamma T_{00} \int e^{-\mu} \omega(u) \frac{du}{1 + u T_{10}}.
\]
(C23)

If the values of \(T_{10}\) are small, then \(Z(\beta, \gamma) \approx Z_\beta[1 - \gamma T_{00}(1 - u_\beta T_{10})]\). Expressions for \(\bar{u} = \bar{u}(\beta, \gamma)\) and \(\bar{\gamma} = \bar{\gamma}(\beta, \gamma)\) are equal
\[
\bar{u} = -\frac{\partial \ln Z(\beta, \gamma)}{\partial \beta} = u_\beta + \frac{\gamma((\partial T_{00} / \partial \beta)(1 - u_\beta T_{10}) - T_{00}(T_{10}\partial u_\beta / \partial \beta + u_\beta \partial T_{10} / \partial \beta))}{1 - \gamma T_{00}(1 - u_\beta T_{10})},
\]
(C24)
\[
\bar{\gamma} = -\frac{\partial \ln Z(\beta, \gamma)}{\partial \gamma} = \frac{T_{00}(1 - u_\beta T_{10})}{1 - \gamma T_{00}(1 - u_\beta T_{10})}.\]
(C25)
In the approximation $\omega(u) \approx \omega(\bar{u})$, the expression for $1/\beta$ in approximation (C22) is replaced by
\[
\frac{1}{\beta_{\gamma}(\beta, \gamma)} = \frac{1}{\beta} + \gamma e^{-\beta(1+\gamma T_{0})/T_{0}} E_{i}(\frac{\beta(1+\gamma T_{0})}{T_{10}}).
\] (C26)

Let's consider another example. The Langevin equation for drift-diffusion process has the form
\[
\frac{dX(t)}{dt} = v + \zeta(t),
\] (C27)
where $X(t)$ from (C27) is the coordinate, $v = F/\zeta$ is the drift velocity, $F$ is an external force, $\zeta$ is a friction coefficient, and $\zeta(t)$ is a Gaussian white noise with zero mean $\langle \zeta(t) \rangle = 0$ and with autocorrelation $\langle \zeta(t)\zeta(t') \rangle = D\delta(t-t')$, $D = k_{B}T/\zeta$ is the diffusion coefficient.

In diffusion processes of the form (C27) for FPT, the Wald distribution (for example, [24], [7]) $f(T_{\gamma})$ of the form
\[
F_{T_{\gamma}}(t) = \sqrt{|L|/4\pi D_{\gamma} t} \exp[-(L-vt)^{2}/4D_{\gamma} t], \quad \langle t \rangle = L/v, \quad a = v^{2}/D, \quad T_{0} = T_{\gamma=0} = \langle t \rangle.
\] (C28)
The Laplace transform of distribution (C28) is
\[
Z_{\gamma}(s) = \int_{0}^{\infty} e^{-st} F_{T_{\gamma}}(t) dt = \exp\left\{ \frac{aT_{0}}{2}(1-\sqrt{1+4\gamma/a}) \right\}.
\] (C29)
The expression obtained from relation (C29) for the average time to reach the level $L$ is
\[
\overline{T}_{\gamma} = -\frac{\partial \ln Z_{\gamma}}{\partial \gamma} = \frac{T_{0}}{\sqrt{1+4\gamma/a}}, \quad T_{0} = \langle t \rangle.
\] (C30)

From (C4) it follows that the value $\ln Z_{\gamma} = c(\gamma)$ does not depend on $\beta$. Expression (C4) follows from the assumption that the distributions for $u$ and $T_{\gamma}$ are independent. Expressions (C8)-(C9) are executed only for a fixed value of the parameter $\gamma$. They can also be performed with the assumption of the dependence
\[
v(\gamma) = \frac{v_{0}}{\sqrt{1+4\gamma/a}}, \quad v_{0} = v(\gamma = 0) \quad \text{when} \quad T_{0} = (L/v_{0})\sqrt{1+4\gamma/a}.
\]
But the quantity $T_{0} = \overline{T}_{\gamma=0}$, by its definition, should not depend on $\gamma$.

Expressions (1), (A1)-(A3) are set in EIT. TFPT uses dimensionless entropy, in units of $k_{B}$
\[
\Delta = \Delta s = s_{\beta} - s = -[\gamma T_{\gamma} + \ln Z(\beta, \gamma) - \ln Z_{\beta} + \beta\bar{u} - \beta u_{\beta}], \quad s_{\beta} = \beta u_{\beta} + \ln Z_{\beta}, \quad \bar{u} = -\frac{\partial \ln Z(\beta, \gamma)}{\partial \beta}.
\] (C31)

In EIT [1-4]
\[
\rho s = -\nabla(q/\theta) + q q/\lambda \theta^{2} = -\nabla q/\theta, \quad q = -\lambda \nabla \theta.
\] (C32)

If the parameters $\lambda, \tau$ weakly depend on the coordinate $r$, then
\[
\nabla q = -\lambda \nabla^{2}T - \tau \nabla q/\lambda, \quad \lambda \nabla T = (\partial T/\partial t)(\rho c_{J}/\lambda), \quad \rho s = (\rho c_{J}\partial t/\partial t + \tau \nabla q/\partial t)/\theta.
\]
If $\theta$ weakly depends on $t$ (this is a rough assumption), then
\[
-\Delta s = \int \rho s dt = \left[ \rho c_{J}(T-T_{0}) + \tau \nabla q \right]/\theta, \quad -\Delta q = \tau \nabla q/\lambda, \quad \rho s = \Delta s/\overline{T}_{\gamma}, \quad \Delta s = \tau p s = \Delta s/\overline{T}_{\gamma}, \quad \tau = \overline{T}_{\gamma}.
\]
Therefore, in $\Delta_{\text{EIT}}$ we replace $\tau$ by $\overline{T}_\gamma$, and we replace the relation
\[ \Delta = \Delta s = s_\beta - s = \tau qq / 2 \rho \lambda T^2 = -[\gamma \overline{T}_\gamma + \ln Z(\beta, \gamma) - \ln Z_\beta + \beta \overline{u} - \beta u_\beta] \]
on the ratio
\[ \Delta = \Delta s = s_\beta - s = \overline{T}_\gamma qq / 2 \rho \lambda T^2 = -[\gamma \overline{T}_\gamma + \ln Z(\beta, \gamma) - \ln Z_\beta + \beta \overline{u} - \beta u_\beta]. \quad (C33) \]

**Appendix D: Thermodynamics after replacing the EIT relaxation time $\tau$ with the mean FPT $\overline{T}_\gamma$.**

The comparison of TFPT with EIT, in which $\Delta = \Delta s = \frac{1}{2} \frac{\tau qq}{\rho \lambda T^2}$ [1-4], carried out above, in Appendix C, raises many questions. The resulting expressions (C4), (C8)-(C9) seem to be uninformative and doubtful. For the Wald distribution, relation (C8) holds only for a certain value of $\gamma$, which looks artificial. In order to get meaningful results, consider the thermodynamic relationships obtained from comparing TFPT with EIT, in which $\tau$ is replaced by $\overline{T}_\gamma$, as in (C33).

Wherein
\[ \Delta s = s_\beta - s = \frac{1}{2} \frac{\overline{T}_\gamma qq}{\rho \lambda T^2}. \quad (D1) \]

Relation (2) takes the form
\[ ds = \theta^{-1} d\overline{u}_\beta - \frac{\overline{T}_\gamma}{\rho \lambda \theta^2} q dq = \beta d\overline{u} + \gamma d\overline{T}_\gamma, \quad (D2) \]
\[ -d\Delta s = \beta(d\overline{u} - du_\beta) + \gamma d\overline{T}_\gamma, \quad (D3) \]
where $\Delta s$ is written as (D1). If the variables $u$ and $T_\gamma$ are independent, then in (D3) $d\overline{u} = du_\beta$, and
\[ -d\Delta s = \gamma d\overline{T}_\gamma. \quad (D4) \]

Consider an example of an exponential distribution (C5) in which $T_0$ does not depend on $u$. Substituting (D1) into (D4), we get
\[ (\gamma + \Delta_1) d\overline{T}_\gamma = -\overline{T}_\gamma d\Delta_1, \quad \Delta_1 = \frac{\Delta s}{\overline{T}_\gamma} = \frac{1}{2} \frac{qq}{\rho \lambda T^2}. \quad (D5) \]

If we differentiate in (D5) with respect to $\gamma$, then, taking into account (C6), we obtain a differential equation for $\Delta_1$ of the form
\[ \frac{\partial \Delta_1}{\partial \gamma} - \Delta_1 \frac{T_0}{1 + \gamma T_0} = \frac{\gamma T_0}{1 + \gamma T_0}, \quad (D6) \]
with a solution
\[ \Delta_1 = \frac{\Delta s}{\overline{T}_\gamma} = \frac{1}{2} \frac{qq}{\rho \lambda T^2} = \frac{1}{T_0} [(1 + \gamma T_0) \ln(1 + \gamma T_0) - \gamma T_0]. \quad (D7) \]

In contrast to (C8), we obtain non-zero expressions
\[ \frac{\partial T_0}{\partial \beta} = \frac{T_0^2}{\gamma T_0 - \ln(1 + \gamma T_0)} \frac{\partial \Delta_1}{\partial \beta}, \quad \frac{\partial \overline{T}_\gamma}{\partial \beta} = \frac{\partial T_0}{\partial \beta} / \frac{\partial \beta}{\partial \beta} = (1 + \gamma T_0)^2. \quad (D8) \]
Expression (D7) is the same as (C10) after replacing \( \tau \) to \( \overline{T}_y \). From (D7), the dependence \( q(\gamma) \) is determined in the form
\[
q = \pm \left( \frac{2\rho \lambda T^2}{T_0} \right)[(1+x)\ln(1+x)-x]^{1/2}, \quad x = \gamma T_0,
\]
(D9)
which coincides with (C13) at \( \tau \to T_0/(1+x) \). For an explicit expression for \( \gamma(q) \), from (D7) we obtain a transcendental equation.

For the Wald distribution (C28)-(C29) with independent variables \( u \) and \( T_\gamma \), from relation (D4) we obtain, as above, a differential equation for \( \Delta_l \) with the solution
\[
\Delta_l = \frac{\Delta s}{T_\gamma} = \frac{1}{2} \frac{qq}{\rho \lambda T^2} = \frac{q[(2\gamma/a) - 1]\sqrt{1 + 4\gamma/a + 1}}{6\sqrt{1 + 4\gamma/a}}.
\]
(D10)
From (D10) we get
\[
q = \pm \left( \frac{2\rho \lambda T^2}{T_0} \right) \frac{q[(2\gamma/a) - 1]\sqrt{1 + 4\gamma/a + 1}}{6\sqrt{1 + 4\gamma/a}}.
\]
(D11)
For the inverse dependence \( \gamma(q) \), the cubic equation is written
\[
\Delta_1 y = \frac{1}{6} \left( \frac{1}{2} y^2 - \frac{3}{2} + 1 \right), \quad y = \sqrt{1 + 4\gamma/a}, \quad \gamma = \frac{a}{4}(y^2 - 1).
\]
(D12)
Let us pass to the above cases (C14), (C22) of dependence \( T_0(u) \). We use equation (D3).

For the exponential distribution (C5) with the parameter \( T_0(u) \) linearly dependent on \( u \) (C14) we obtain, by substituting into (D3) the expressions obtained from (C17), (C18), as above, (D6), (D7), the linear differential equation for \( \Delta_l \) with solution
\[
\Delta_l = \frac{\Delta s}{T_\gamma} = \frac{1}{2} \frac{qq}{\rho \lambda T^2} = -\frac{1}{B} [(1 - \gamma B) \ln(1 - \gamma B) + \gamma B(1 + \frac{\beta \partial B}{B})], \quad B = T_0 + u \beta T_0.
\]
(D13)
For \( T_0(u) \) of the form (C14) and the Wald distribution (C28), similar expressions are written:
\[
\ln Z(\beta, \gamma) \approx \ln Z(\beta_{ef}) + (aT_{00}/2)(1 - \sqrt{1/4\gamma/a}), \quad \beta_{ef} = \beta - (aT_{00}/2)(1 - \sqrt{1/4\gamma/a}),
\]
(D14)
\[
\bar{u} = -\frac{\partial \ln Z(\beta, \gamma)}{\partial \beta} = \bar{u}(\beta_{ef}) \frac{\partial \beta_{ef}}{\partial \beta} - E_a \sqrt{1 + 4\gamma/a} \left[ \frac{aT_{00}}{2} (1 - \sqrt{1 + 4\gamma/a}) + \frac{\gamma T_{10}}{\sqrt{1 + 4\gamma/a}} \right],
\]
\[
\frac{\partial \beta_{ef}}{\partial \beta} = 1 - E_a \sqrt{1 + 4\gamma/a} \left[ \frac{aT_{00}}{2} (1 - \sqrt{1 + 4\gamma/a}) + \frac{\gamma T_{10}}{\sqrt{1 + 4\gamma/a}} \right], \quad \frac{\partial \gamma}{\partial \beta} = aE_a \sqrt{1 + 4\gamma/a},
\]
(D15)
\[
\overline{T}_\gamma = -\frac{\partial \ln Z(\beta, \gamma)}{\partial \gamma} = \frac{T_{10}}{\sqrt{1 + 4\gamma/a}} + \frac{T_{00}}{\sqrt{1 + 4\gamma/a}}.
\]
(D16)
Substituting the expressions (D14)-(D16) into (D3) gives the expression
\[
\left[ \frac{1}{2} \frac{qq}{\rho \lambda T^2} + \gamma \right][\bar{u}(\beta_{ef}) \frac{T_{10}}{\sqrt{1 + 4\gamma/a}} + \frac{T_{00}}{\sqrt{1 + 4\gamma/a}}] = -\left[ \frac{aT_{00}}{2} (1 - \sqrt{1 + 4\gamma/a}) + \beta \bar{u}(\beta_{ef}) \frac{\partial \beta_{ef}}{\partial \beta} - E_a \sqrt{1 + 4\gamma/a} \left( \frac{aT_{00}}{2} (1 - \sqrt{1 + 4\gamma/a}) + \frac{\gamma T_{10}}{\sqrt{1 + 4\gamma/a}} \right) \right].
\]
(D17)
For the Wald distribution from (D3) (the same relations are also written from (D19)) we obtain
\[
\frac{\partial \Delta_1}{\partial \gamma} \bar{T}_\gamma + \Delta_1 \frac{\partial \bar{T}_\gamma}{\partial \gamma} = -\gamma \frac{\partial \bar{T}_\gamma}{\partial \gamma} - \beta \frac{\partial \bar{u}}{\partial \gamma}. \quad \text{(D18)}
\]

Relations (C7)-(C8) are replaced by the expression

\[
\frac{\partial \bar{u}(\beta, \gamma)}{\partial \gamma} = \frac{\partial \bar{u}(\beta_\sigma)}{\partial \beta_\sigma} \frac{\partial \beta_\sigma}{\partial \gamma} + \bar{u}(\beta_\sigma) \frac{\partial^2 \beta_\sigma}{\partial \beta \partial \gamma} + E_a(\sqrt{1+4\gamma/a}) \left[ \frac{T_{00} - T_{10}}{\sqrt{1+4\gamma/a}} + \frac{2\gamma T_{10}}{a(1+4\gamma/a)\sqrt{1+4\gamma/a}} \right] - E_a \left[ \frac{T_{00}(1-\sqrt{1+4\gamma/a})}{\sqrt{1+4\gamma/a}} + \frac{2\gamma T_{10}}{a(1+4\gamma/a)} \right].
\]

If we consider (D18) as a differential equation for \( \Delta_1 \), then this equation can be solved, although the solution is cumbersome. We obtain the expression (D19).

Many different examples of this kind can be written down. All of them obey the general ratios. Let's repeat what was said above. The change in entropy \( \Delta = \Delta s = s_\beta - s \) is equated to

\[
\Delta = \Delta s = s_\beta - s = \frac{1}{2} \frac{T_{qq} - \bar{T}_{qq}}{\rho T^2} \text{ from EIT in the form}
\]

\[
\Delta = \Delta s = s_\beta - s = \frac{1}{2} \frac{T_{qq} - \bar{T}_{qq}}{\rho \lambda T^2} \bar{T}_{\Delta}, \quad \Delta = \Delta s = s_\beta - s = -\frac{1}{2} \frac{T_{qq} - \bar{T}_{qq}}{\rho \lambda T^2} \bar{T}_{\Delta}, \quad \Delta = \Delta s = s_\beta - s = -\frac{1}{2} \frac{T_{qq} - \bar{T}_{qq}}{\rho \lambda T^2} \bar{T}_{\Delta} = \Delta_1 = \frac{1}{2} \frac{T_{qq} - \bar{T}_{qq}}{\rho \lambda T^2}. \quad \text{(D19)}
\]

Wherein

\[
-d\Delta = \gamma d\bar{T}_\gamma + \beta (d\bar{u} - d\bar{u}_\beta).
\]

And the solution of differential equations (D6), (D18) leads to (D19).