Holographic geometries for condensed matter applications

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Holographic modeling of strongly correlated many-body systems motivates the study of novel spacetime geometries where the scaling behavior of quantum critical systems is encoded into spacetime symmetries. Einstein-Dilaton-Maxwell theory has planar black brane solutions that exhibit Lifshitz scaling and in some cases hyperscaling violation. Entanglement entropy and Wilson loops in the dual field theory are studied by inserting simple geometric probes involving minimal surfaces into the black brane geometry. Coupling to background matter fields leads to interesting low-energy behavior in holographic models, such as U(1) symmetry breaking and emergent Lifshitz scaling.

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1. Introduction

In recent years gauge theory - gravity duality, also referred to as holographic duality, has emerged as a versatile tool to investigate strongly coupled dynamics in a variety of physical systems, ranging from the hydrodynamics of quark-gluon plasma formed in heavy ion collisions to quantum critical phenomena in condensed matter physics and ultra-cold atomic systems (for reviews see e.g.[1–3]). The motivation comes from the AdS/CFT conjecture of duality between superstring theory and maximally supersymmetric Yang-Mills gauge theory.[4–6] The original conjecture and various refinements of it have passed many nontrivial tests and their validity is by now accepted by most high-energy theorists, even if it remains unproven. Building on this, considerable effort has been put into extending the duality to settings where there is no supersymmetry or even conformal symmetry to constrain the dynamics. This is clearly more speculative, but may bring us closer to the study of real world systems where these symmetries are known to be absent. The gravity dual then provides a phenomenological description of strongly coupled physics on the field theory side, and, as such, it can be useful even if the underlying dynamics is poorly understood. The success of this approach will in the end be judged by its ability to match and ultimately predict experimental results for strongly coupled systems.

The duality relates gravity in spacetime with a negative cosmological constant to field theoretic systems, which are a priori far removed from physics in curved spacetime. More conventional applications of gravitational theory in astrophysics and cosmology involve vanishing or positive cosmological constant and a spacetime geometry that is asymptotically flat or de Sitter. Interest in strongly correlated many-body systems thus motivates the study of novel gravitational solutions that would otherwise be of limited interest. This includes domain-wall like analogs of
black holes with a planar horizon and carrying various types of matter field ‘hair’. In the following we review several holographic constructions that have been developed to model interesting strong coupling physics. As befits a Marcel Grossman meeting, the focus will be on the geometries arising in these constructions and various geometric probes used to study their properties. The presentation will be brief and many important topics left out but we hope to give the reader an impression of this new and rapidly developing area of application of general relativity.

A lot of the work on condensed matter applications of gauge theory - gravity duality has been aimed at quantum critical systems and we begin our discussion there. In what is usually referred to as a bottom-up approach, we consider simple gravity models where the scaling behavior of quantum critical systems is encoded into spacetime symmetries and many-body interactions are modeled by coupling appropriate matter fields to gravity. The important issue of to what extent these models may be obtained in a top-down fashion as low-energy limits of a consistent background in string theory or supergravity will not be addressed here.

2. Quantum critical points

In a quantum phase transition the ground state of a system at zero temperature changes as a physical parameter (pressure, external magnetic field, etc.) is varied. The transition between superconducting and insulating behavior in certain thin metallic films as a function of their thickness is a classic example. Another example is provided by the transition to anti-ferromagnetic order as a function of doping in certain heavy fermion alloys and a quantum critical point may also play a role in explaining the behavior of high $T_c$ superconductors at low doping.

When a second order quantum critical point is approached, characteristic length scales of the system diverge and there is an emergent scaling symmetry under

$$t \rightarrow \lambda^z t, \quad \vec{x} \rightarrow \lambda \vec{x}.$$  \hspace{1cm} (1)

In a relativistic system energy and distance are inversely related. In this case the dynamical scaling exponent is $z = 1$ and the scaling symmetry is enhanced to a conformal symmetry. In non-relativistic systems, on the other hand, the scaling at a quantum critical point can be asymmetric between the temporal and spatial directions so that $z \neq 1$, commonly referred to as Lifshitz scaling in the literature.

Finite temperature introduces a length scale that breaks the scaling symmetry at a quantum critical point. In an otherwise scaling symmetric theory, the temperature dependence of a characteristic length scale is $\ell \sim T^{-1/z}$. If the corresponding zero temperature system is not exactly at the quantum critical point, the temperature dependence exhibits a more general scaling form,

$$\ell \sim T^{-1/z} \eta(T^{-1/z} \lambda_i),$$  \hspace{1cm} (2)

where $\lambda_i$ are a set of deformation parameters, defined so that they have dimensions of inverse length and $\lambda_i = 0$ corresponds to infinite $\ell$ at $T = 0$, and $\eta$ is some generic function of its arguments.
The systematic study of non-relativistic holographic systems that realize scale symmetry without conformal symmetry has progressed considerably in recent years and is now at a point where such models can be applied to condensed matter systems with a degree of confidence that approaches that of their conformally invariant (non-supersymmetric) counterparts. In the following we will use a relatively simple gravitational model to illustrate the holographic approach. The first step is to realize the asymmetric scaling symmetry (1) as an isometry of a higher dimensional space-time. This is achieved by introducing an extra radial dimension and considering the $d+2$ dimensional Lifshitz geometry\cite{12,13}

$$ds^2 = L^2 (- r^{2z} dt^2 + r^2 dx^2 + \frac{dr^2}{r^2}),$$

where $L$ is a characteristic length scale that we set to $L = 1$ from now on.

The Lifshitz metric is invariant under the transformation

$$t \to \lambda^z t, \quad \vec{x} \to \lambda \vec{x}, \quad r \to \frac{r}{\lambda},$$

which includes \cite{14} acting on the $d+1$ coordinates $t, \vec{x}$ of the dual field theory. The scaling acts inversely on the radial coordinate compared with the transverse spatial coordinates. Under the duality the asymptotic large $r$ region corresponds to short distance UV physics in the field theory while low-energy IR physics is encoded at small $r$. So far, this is in direct analogy to the usual AdS/CFT correspondence where the conformal symmetry of a relativistic field theory is realized as the isometry of $\text{AdS}_{d+2}$. Indeed, the Lifshitz metric \cite{3} reduces to that of AdS spacetime in Poincaré coordinates when $z=1$. The Lifshitz metric for $z \neq 1$ is singular at $r=0$. All curvature invariants remain finite but tidal forces diverge as $r \to 0$. This is a puzzling feature but it does not pose any immediate problem for condensed matter physics applications as real world systems always have a non-vanishing temperature and then the singularity is cloaked by the event horizon of a black brane.

To develop the analogy further, one looks for a gravitational model which has the Lifshitz metric for generic $z \geq 1$ as a solution of its field equations. There are a few different options available but the following Einstein-Dilaton-Maxwell (EDM) theory turns out to be a convenient choice\cite{15}

$$S_{\text{EDM}} = \int d^4 x \sqrt{-g} \left( R - 2\Lambda - \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \frac{1}{4} \sum_{i=1}^{2} e^\lambda_i \phi F_{\mu\nu}^{(i)} F^{(i)\mu\nu} \right).$$

Here we have set $d = 2$ to obtain a dual description of a 2+1 dimensional field theory. The formulas generalize to other values of $d$ in a straightforward way.

The Lifshitz metric \cite{3}, with $z$ related to the cosmological constant through $\Lambda = -(z+2)(z+1)/2$, is a solution of the model when the dilaton field and one of the $U(1)$ gauge fields have the following background values,

$$e^\phi = \left( \frac{r}{r_0} \right)^{2\sqrt{z-1}}, \quad F_{\mu\nu}^{(1)} = \sqrt{2(z-1)(z+2)} r_0^{-1} \left( \frac{r}{r_0} \right)^{z+1}.$$
Here $r_0$ is an arbitrary reference value of the radial variable. The role of the auxiliary gauge field, $F_{\mu \nu}^{(1)}$, is to modify the asymptotic behavior of the metric from that of AdS spacetime to Lifshitz via the gravitational back-reaction to its background value. In the limit $z \to 1$ the dilaton becomes independent of $r$ and the auxiliary gauge field vanishes.

The gauge coupling parameter is required to be $\lambda_1 = -2/\sqrt{z-1}$, which means that $F_{\mu \nu}^{(1)}$ is strongly coupled in the asymptotic $r \to \infty$ region. This strong coupling is not of immediate concern as long as $F_{\mu \nu}^{(1)}$ only couples to the gravitational sector and not to matter fields and we are only interested in classical solutions of the model. Furthermore, the logarithmic form of the dilaton field in (6) breaks the scale symmetry unless the scale transformation is generalized to include a shift in the dilaton. We minimize the effect of this breaking of scale invariance by only considering observables that do not directly couple to the dilaton.

In order to model finite charge density in the dual field theory the EDM theory contains a second $U(1)$ gauge field that couples to charged matter in the bulk gravitational spacetime. Under the duality, scalar and spinor fields, that are charged under the second $U(1)$ in the EDM theory, correspond to charged operators in the dual field theory.

### 3. Thermodynamics

Finite temperature is introduced by constructing static black hole solutions, or more precisely black branes with a planar event horizon, whose Hawking temperature corresponds to the temperature of the dual field theoretic system. For $\lambda_2 = \sqrt{z-1}$, the field equations obtained from (5) have the following one-parameter family of electrically charged black brane solutions:

\[
\begin{align*}
    ds^2 & = -r^2 f(r) dt^2 + r^2 dx^2 + \frac{dr^2}{r^2 f(r)}, \\
    f(r) & = 1 - \left(1 + \frac{\rho_2^2}{4z} \right) \left( \frac{r_0}{r} \right)^{z+2} + \frac{\rho_2^2}{4z} \left( \frac{r_0}{r} \right)^{2z+2}, \\
    F_{rt}^{(2)} & = \rho_2 r^{z-1} \left( \frac{r_0}{r} \right)^{z+1},
\end{align*}
\]

and the dilaton and $F_{\mu \nu}^{(1)}$ having the same background values as in (6). As expected, the metric and the physical gauge field reduce to those of a standard AdS-Reissner-Nordström black brane when $z = 1$.

The scalar potential of the physical gauge field is obtained by integrating the field strength in (9) with respect to $r$,

\[
    A_\nu^{(2)} = \mu - \frac{\rho_2 r_0^2}{4z} \left( \frac{r_0}{r} \right)^2.
\]

According to the AdS/CFT prescription for the generating functional in the dual field theory, the boundary value of $A_\mu^{(2)}$ acts as the source for the corresponding $U(1)$
current. The integration constant \( \mu \) therefore plays the role of a chemical potential in the field theory. In order for the gauge connection to be regular at the horizon the scalar potential must go to zero at \( r = r_0 \), and as a result the chemical potential and the charge density of a Lifshitz black brane are related, \( \mu = \rho_2 r_0^2 / z \).

The Hawking temperature is determined in the usual manner by considering the Euclidean version of the black brane metric and requiring smoothness at the horizon,

\[
T = \frac{r_0^{z+1}}{4\pi} f'(r_0) = \frac{r^z}{4\pi} \left( z + 2 - \frac{\rho_2^2}{4} \right). \tag{11}
\]

The metric (8) is invariant under a Lifshitz rescaling (4) combined with \( r_0 \rightarrow \lambda^{-1} r_0 \), but the value of the Hawking temperature scales under this transformation. Due to the underlying scale invariance of the system, the Hawking temperature by itself does not have physical meaning but only dimensionless combinations such as \( T/\mu \).

The free energy of the dual field theory is obtained from the gravitational theory by computing the on-shell Euclidean action, including the Gibbons-Hawking boundary term\(^{16}\) and boundary counterterms for holographic renormalization\(^{17-19}\).

For the charged Lifshitz black brane (8) we find a scaling form,

\[
F = V T^{(z+2)/2} g(T/\mu), \tag{12}
\]

that generalizes the \( F \propto V T^3 \) scaling found for \( z = 1 \) branes in 2+1 dimensions, where \( V \) is the volume in the field theory, or rather an area since we are considering a theory in 2+1 dimensions.

If the gravitational theory has more than one Euclidean solution then the one with lowest free energy dominates. We will see examples below where the system makes a phase transition from one type of solution to another when \( T/\mu \) crosses a critical value.

4. Geometric probes

Different types of minimal surfaces ending on the boundary of the spacetime provide a natural set of geometric observables that correspond to entanglement entropy and Wilson loops in the dual field theory. Entanglement entropy in a \( d + 1 \) dimensional field theory is obtained by dividing the \( d \) dimensional space, on which the theory is defined, into several parts and then taking a trace of the density matrix over the quantum mechanical Hilbert space of the local degrees of freedom in some of the parts. This leads to a reduced density matrix for the remaining system that generically has non-vanishing entropy \( S = -\text{Tr} \rho \log \rho \), which measures the degree of entanglement between different spatial parts of the system\(^{20,21}\). The entanglement entropy depends on the state of the system and can be computed both in pure and mixed states.

In the following we consider a special case where the \( d \) dimensional spatial boundary in the dual gravity theory is divided into two parts \( A \) and \( B \). The holographic entanglement entropy\(^{22}\) between \( A \) and \( B \) is then given by the area of the \( d \)
dimensional minimal surface that ends on the $d - 1$ dimensional perimeter between the two regions,

$$S_{\text{ent}} = \frac{1}{4G_N} \int d^d \sigma \sqrt{\det ab \left( g_{\mu \nu} \frac{\partial x^\mu}{\partial \sigma^a} \frac{\partial x^\nu}{\partial \sigma^b} \right)}.$$  (13)

From the geometric point of view the holographic entanglement entropy can be regarded as a generalization of the Bekenstein-Hawking entropy.\(^{23,24}\) As a concrete example, we consider the Lifshitz spacetime with $d = 2$ at zero temperature and take $A$ to be a disk of radius $a$ in the transverse $\vec{x}$ plane. The minimal surface inherits the rotational symmetry of the disk and can be parametrized by a single function $u(\rho)$, where $\rho$ is a polar coordinate on the spatial boundary, $u \equiv 1/r$, and $u(a) = 0$. The area functional becomes

$$S_{\text{ent}} = \frac{\pi}{2G_N} \int_0^a d\rho \sqrt{1 + \left( \frac{du}{d\rho} \right)^2},$$  (14)

which coincides with the corresponding area functional in AdS$_4$ since the spatial metric components of Lifshitz spacetime are the same as those of AdS in Poincare coordinates. The resulting Euler-Lagrange equation is solved by $u(\rho) = \sqrt{a^2 - \rho^2}$ and one finds that the dual field theory with Lifshitz scale-invariance has the same entanglement entropy for a disk of radius $a$ as a 2 + 1 dimensional CFT\(^{22}\)

$$S_{\text{ent}} = \frac{\pi}{2G_N} \left( \frac{a}{\epsilon} - 1 \right).$$  (15)

Here $\epsilon$ is a short distance cutoff in the dual field theory, related to a long distance cutoff in the radial direction in the gravitational theory. The first term is divergent in the limit as the cutoff is taken to zero and corresponds to the well known ‘area law’ of entanglement entropy.\(^{20}\) It is accompanied by a finite term, which is independent of $a$ and thus manifestly scale invariant.

At finite temperature the Lifshitz spacetime is replaced by the planar black brane.\(^7\) Let us again consider the holographic entanglement entropy of a disk of radius $a$ in the transverse plane. The area functional depends on the black brane metric but at low temperature, i.e. when $ar_0 \ll 1$, the minimal surface remains well separated from the event horizon of the brane and the holographic entanglement entropy depends only weakly on the temperature. At high temperature, when $ar_0 \gg 1$, it instead becomes advantageous for the minimal surface to be as close as possible to the brane horizon. It approaches a limiting form that consists of a cylinder, $|\vec{x}| = a$, extending from the spatial boundary to the brane horizon, and a disk, $|\vec{x}| < a$, at $r = r_0$. The holographic entanglement entropy becomes

$$S_{\text{ent}} = \frac{\pi a}{4G_N \epsilon} + \frac{A}{4G_N} + \ldots,$$  (16)

where $A = \pi a^2 r_0^2$ is the proper area covered by the disk at the horizon and we have neglected finite terms that are linear in $a$. Thus the entanglement entropy of a large region, or equivalently at high temperature, contains the same cutoff dependent
term as we saw at zero temperature and an additional term that equals the thermal entropy contained in that region.\footnote{This is physically reasonable from the dual field theory perspective. The divergent short distance entanglement across the perimeter of the region is not affected by the finite temperature but the remaining finite part of the entropy is no longer scale invariant and is given by the thermal entropy.} Next we consider Wilson loops in a gauge theory with a gravity dual. These are gauge invariant observables located on a loop in the spacetime of the field theory. If the loop has a rectangular shape extending in the time direction, the Wilson loop gives the potential energy \( V(l) \) between charges in the fundamental and the anti-fundamental representation of the gauge group through the identification

\[
\langle W \rangle \approx e^{-iTV(l)},
\]  

(17)

where \( l \) is the spatial length of the rectangle and \( T \) is its temporal length.

In holography, the expectation value of a Wilson loop is given by the worldvolume of a classical string ending on the corresponding loop at the spacetime boundary.\footnote{Strictly speaking, to justify the Wilson loop formula, one should consider duality where the gravitational theory is a full fledged string theory. Here we will continue on a more phenomenological path and assume the Wilson loop to be given by the on-shell value of the Nambu-Goto action, \[ S_{NG} = -\frac{1}{\alpha'} \int d^2 \sigma \sqrt{-\det \left( g_{\mu\nu} \frac{\partial X^\mu}{\partial \sigma^a} \frac{\partial X^\nu}{\partial \sigma^b} \right)} \]. (18) From the conserved "Hamiltonian" \( H = \frac{\delta L}{\delta (\partial_x r)} \partial_x r - L \), one obtains a first order differential equation for \( r(x) \) that can be integrated to give

\[
l = 2 \int_{r_*}^\infty \frac{dr}{r^2 \sqrt{\left( \frac{r}{r_*} \right)^2 + 1}} = 2 \int_{1}^\infty \frac{ds}{s^2 \sqrt{s^2 + 1}}, \]  

(20)

where \( r_* \) is the midpoint of the hanging string. Similarly the action of the string evaluated on the solution is given by

\[
S_{NG} = -\frac{2T}{\alpha'} r_*^2 \int_{1}^{1/(r_*)} \frac{ds s^{2z}}{\sqrt{s^{2z} + 1}}. \]  

(21)

The resulting energy of a charge-anti charge configuration in the dual field theory is

\[
V(l) = \frac{a}{\epsilon^2} - \frac{1}{\alpha' T^2}. \]  

(22)

One can also consider Wilson loops in other representations of the gauge group.
where \( a \) and \( c \) are (\( z \) dependent) constants.\(^{27}\) The first term is a divergent self-energy while the second term is the potential energy between the charges as a function of their separation, with a functional form consistent with scale invariance.

One can also compute the Wilson loop at finite temperature by considering a black brane background. When \( l \) is small, the string worldvolume is located at large \( r \) giving rise to a potential that is close to the vacuum result \(^{22}\). When \( l \) is increased above a critical value \( l_c \propto 1/\sqrt{T} \), the lowest energy string solution consists of a pair of vertical strings penetrating the black hole horizon. This results in screening at distances above a critical value\(^{28,29}\) analogous to the \( z = 1 \) case.

5. Hyperscaling violation

An interesting generalization of the Lifshitz geometry is provided by the so-called hyperscaling violating metrics\(^{30,31}\)

\[
ds^2 = r^{-2\theta/d} \left( -r^{2z} dt^2 + r^2 dx^2 + \frac{dr^2}{r^2} \right),
\]

in \( d+2 \) dimensions, which are covariant \( ds^2 \rightarrow \lambda^{2\theta/d} ds^2 \) under the scaling \( r \rightarrow \lambda^{-1} r \), \( t \rightarrow \lambda^z t \) and \( x \rightarrow \lambda x \), when \( \theta \) is non-vanishing. Such metrics can appear, for instance, in Einstein-Maxwell-Dilaton theories when a potential for the dilaton field is included\(^{31}\) and also in the context of holographic superfluids in the zero temperature limit when the scalar and gauge fields in the bulk gravitational theory have non-minimal couplings.\(^{32}\) We will have more to say about holographic superfluids in Section 6.2 below.

The hyperscaling violating metric \(^{23}\) has a naked singularity at \( r = 0 \) but at finite temperature it is cloaked by the event horizon of a black brane. We consider a specific generalization of the EMD theory \(^{5}\) with a potential \( U(\phi) = U_0 e^{\gamma \phi} \) for the dilaton.\(^{33}\) This theory has neutral black brane solutions given by

\[
ds^2 = r^{-2\theta/d} \left( -r^{2z} f(r) dt^2 + r^2 dx^2 + \frac{dr^2}{f(r)r^2} \right), \quad f(r) = 1 - \left( \frac{r_0}{r} \right)^{z+d-\theta}. \tag{24}
\]

The Hawking temperature is obtained as in \(^{11}\) and scales as \( T \propto r_0^{z}\). The entropy of the black brane is proportional to the horizon area \( S \propto V_d r_0^{d-\theta} \), where \( V_d \) is the spatial volume of the field theory, or when expressed in terms of the temperature, \( S \propto V_d T^{d-\theta}/r \). \( \tag{25}\)

This differs from the behavior of a scale invariant system, where the entropy scales as \( S \propto V_d T^{d/z} \). In particular, for \( \theta = d-1 \), the entropy scales with temperature as if the relevant low energy degrees of freedom lived in one dimension. This is precisely what one expects to find for the entropy in a system with a Fermi surface, where the low energy excitations are located near a \( d-1 \) dimensional surface in momentum space and only disperse along the direction orthogonal to that surface.\(^{31}\)

Another hint of a Fermi surface at \( \theta = d-1 \) comes from the holographic entanglement entropy. Consider a disk shaped region in \( d = 2 \), as in section \(^4\) but this
time at the spatial boundary of a 4-dimensional, zero temperature, hyperscaling violating spacetime [23]. The area functional (13) is given by

\[ S_{\text{ent}} = \frac{\pi}{2G_N} \int_0^a d\rho \rho u^{d-2} \sqrt{1 + \left(\frac{du}{d\rho}\right)^2}, \]  

(26)

with \( \rho \) and \( u(\rho) \) defined as in section 4. For the case of interest, \( \theta = d - 1 = 1 \), the holographic entanglement entropy exhibits an \( a \log (a/\epsilon) \) form, where \( a \) is the radius of the disk in the transverse \( \vec{x} \) plane and \( \epsilon \) is a short distance cutoff in the dual field theory. The \( a \log a \) behavior is characteristic of a system with a Fermi surface [31,34–36].

At \( \theta = 1 \) the Euler-Lagrange equation takes the form

\[ \frac{d^2 u}{d\rho^2} + \frac{1}{1 + \left(\frac{du}{d\rho}\right)^2} \frac{1}{\rho} \frac{du}{d\rho} + \frac{1}{u} = 0, \]  

(27)

and one looks for a solution \( u(\rho) \), such that \( u(a) = 0 \) and \( u(\rho) > 0 \) for \( 0 \leq \rho < a \). Unlike the \( \theta = 0 \) case, we are not able to find a closed form solution to the \( \theta = 1 \) equation, but it is straightforward to obtain a numerical solution starting from initial data \( u(0) = 1, \ u'(0) = 0 \). As expected, the numerical solution is concave and hits \( u = 0 \) at some finite \( \rho = \rho_0 \).

The leading behavior of the holographic entanglement entropy, including the UV divergent part, is determined by the small \( u \) asymptotics near the spatial boundary and these can be obtained analytically as follows, without having to rely on a numerical solution. Introducing a new dimensionless variable \( s \) through

\[ \frac{1}{s} = \frac{u}{a} \sqrt{1 + \left(\frac{du}{d\rho}\right)^2}, \]  

(28)

and viewing \( \rho \) and \( u \) as functions of \( s \), allows us to rewrite the Euler-Lagrange equation (27) as

\[ \rho \frac{d\rho}{ds} = s \left(\frac{du}{ds}\right)^2. \]  

(29)

The previous two equations can be re-expressed as

\[ \dot{\rho} = -\frac{\sqrt{1 - s^2 \dot{u}^2}}{\dot{u}} \frac{d\dot{u}}{ds}, \quad \frac{d\dot{\rho}}{ds} = -\frac{s \dot{u}}{\sqrt{1 - s^2 \dot{u}^2}} \frac{d\dot{u}}{ds}, \]  

(30)

where \( \dot{\rho} = \rho/a \) and \( \dot{u} = u/a \). At the spatial boundary \( \dot{\rho} \to 1 \) and \( \dot{u} = \alpha e^{-s} + \ldots \), with \( \alpha > 0 \) a constant. The limit is approached as \( s \to \infty \) and near the boundary the pair of equations (30) can be solved order by order in \( e^{-s} \). The solution takes
the form
\[ \hat{u}(s) = \alpha e^{-s} + \sum_{k=1}^{\infty} \alpha^{2k+1} e^{-(2k+1)s} \hat{u}_k(s), \]
and
\[ \hat{\rho}(s) = 1 - \sum_{k=1}^{\infty} \alpha^{2k} e^{-2ks} \rho_k(s). \] (31)

The \( \hat{u}_k(s) \) and \( \rho_k(s) \) are universal polynomials in \( s \), independent of \( a \) and \( \alpha \), that are determined recursively from (30),
\[ u_1(s) = \frac{1}{4} s^2 - \frac{1}{8}, \quad \rho_1(s) = \frac{1}{2} s + \frac{1}{4}, \quad \ldots. \] (32)

The area functional (26) can be expressed as
\[ S_{\text{ent}} = \pi a^2 G_N \int_{s_{\text{min}}}^{s_{\text{max}}} ds \left( \frac{1}{\hat{u}} \frac{d\hat{u}}{ds} \right)^2, \] (33)
where \( s_{\text{min}} \) and \( s_{\text{max}} \) are the values of \( s \) at \( \rho = 0 \) and \( u = \epsilon \) respectively, and we have introduced a short distance cutoff in the field theory, as in section 4. For \( \epsilon \ll a \) the upper limit of the \( s \) integration is at \( s_{\text{max}} = \log(a/\epsilon) + \log \alpha + \ldots \) and at large \( s \) the integrand reduces to 1 at leading order in \( e^{-s} \). The holographic entanglement entropy is thus given by
\[ S_{\text{ent}} = \frac{\pi}{2 G_N} (a \log a - a \log \epsilon + \ldots), \] (34)
as promised. This result is universal in the sense that the free parameter \( \alpha \) in the solution for \( \hat{u} \) and \( \hat{\rho} \) does not enter in the leading \( a \log a \) behavior of the finite part of the entropy but only in subleading terms, indicated by \( \ldots \) in (34).

6. Coupling to scalar matter

6.1. Probe fields

Correlation functions of local operators play a central role in gauge theory - gravity duality. Gauge invariant local operators involving a single trace over the gauge theory color indices are dual to local fields in the bulk gravitational theory.

As an example, consider a free scalar field governed by the action
\[ S = -\frac{1}{2} \int d^4x \sqrt{-g} \left( (\partial \phi)^2 + m^2 \phi^2 \right). \] (35)

Two point correlation functions of the corresponding dual operator are obtained by solving the classical equation of motion of the bulk field, \( (\Box - m^2)\phi = 0 \). The solutions have the asymptotic form
\[ \phi(r, \omega, k) = A(\omega, k) r^{-\Delta_-} + B(\omega, k) r^{-\Delta_+} + \ldots, \] (36)

In the string theory context this identification assumes that the string length \( \sqrt{\alpha'} \) is small compared to the characteristic length scale of the spacetime. In this case, states that correspond to string excitations can be approximated by local fields.
where
\[ \Delta_\pm = \frac{2 + z}{2} \pm \sqrt{\frac{(2 + z)^2}{4} + m^2}, \] (37)
and we have performed a Fourier transform \( \varphi(x, t) = \int \frac{d^2 k \omega}{(2\pi)^3} e^{-i\omega t + i k \cdot x} \phi(r, \omega, k) \).

The leading term \( A(\omega, k) \) is identified as a source for the operator dual to the field \( \phi \), while \( B(\omega, k) \) is identified as the expectation value of the dual operator \( \langle \mathcal{O} \rangle \). In linear response theory one identifies the retarded correlation function as the ratio of the expectation value and the source\(^{35-40}\)

\[ G_R(\omega, k) \propto \frac{\langle \mathcal{O}(\omega, k) \rangle}{A(\omega, k)} \propto \frac{B(\omega, k)}{A(\omega, k)}. \] (38)

The retarded correlator describes the response of the system to turning on a source. It is a causal quantity and vanishes in the past of the earlier operator inside the expectation value. One is thus lead to impose ingoing boundary conditions with vanishing boundary data on any past horizons that are present in the spacetime.

The boundary conditions fix \( B(\omega, k) \) in terms of \( A(\omega, k) \) and thereby determine the correlation function (38) up to an overall normalization. It is straightforward to generalize the above procedure to interacting scalar fields and fields other than scalars. The bulk field equations can be also solved in position space, leading directly to correlation functions in position space in the dual field theory.

In the limit of large scalar field mass, the path integral for the two point correlation function can be evaluated in a saddle point approximation\(^{41,42}\)

\[ G(x_2, x_1) = \langle \mathcal{O}(x_2) \mathcal{O}(x_1) \rangle \propto e^{iS_{cl}}, \] (39)

where \( S_{cl} \) is the classical action of a relativistic particle of mass \( m \) along a geodesic connecting the two boundary points \( x_1 \) and \( x_2 \),

\[ S_{cl} = m \int d\lambda \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}. \] (40)

The geodesic approximation provides a simple way to calculate position space correlators. To illustrate its use, we consider the equal time correlation function of scalar operators in the field theory dual to the Lifshitz spacetime \(^3\), where spacelike geodesics satisfy

\[ \frac{dt}{d\lambda} = \frac{E}{r^{2z}}, \quad \frac{dx}{d\lambda} = \frac{P}{r^2}, \quad \left( \frac{dr}{d\lambda} \right)^2 = r^2 - P^2 + E^2 r^{2-2z}. \] (41)

Here \( E \) and \( P \) are constants (for the equal time correlator we set \( E = 0 \)) and we have used translation and rotation invariance in the \( \tilde{x} \)-plane to align the geodesic along the \( x \)-axis. The geodesic equations are easily integrated and the resulting action is

\[ S = im(\lambda_2 - \lambda_1) = 2im \log \left( \frac{|x_2 - x_1|}{\epsilon} \right), \] (42)
where $\lambda_1$ and $\lambda_2$ are the values of the affine parameter at the endpoints of the geodesic and we have cut off the integral at $r = 1/\epsilon$. The action takes an imaginary value since the equal time geodesic is spacelike. This leads to the correlation function

$$\langle O(x_1)O(x_2) \rangle \propto \frac{1}{|x_2 - x_1|^{2\Delta}}, \quad (43)$$

with $\Delta = m$. The algebraic decay of the correlation is a sign of scale invariance at a quantum critical point. It goes beyond the geodesic approximation and (43) holds for a scalar field of any mass, with $\Delta$ given by (37).

Next we consider the equal time correlation function in a non-extremal Lifshitz black brane background (7). An equal time geodesic connecting two well separated boundary points $x_1$ and $x_2$, such that $|x_2 - x_1| r_0 \gg 1$, consists of two ‘vertical’ segments $r_0 < r < 1/\epsilon$ at $x = x_1$ and $x = x_2$, respectively, connected by a ‘horizontal’ segment along the horizon at $r = r_0$. The classical action (40) of such a geodesic is given by

$$S_{cl} = 2im \log \left( \frac{1}{\epsilon} \right) + im r_0 |x_2 - x_1| + \ldots, \quad (44)$$

where $\ldots$ are finite terms that do not depend on $|x_2 - x_1|$. This leads to an exponentially decaying correlation,

$$G(x_2, x_1) \propto e^{-mr_0|x_2 - x_1|}, \quad (45)$$

characteristic of a thermal system with a correlation length $\xi = 1/(mr_0)$. This result holds quite generally as long as the metric function $f(r)$ in (8) has a simple zero at $r = r_0$. The geodesic approximation breaks down for small scalar field masses but numerical calculations confirm that equal time correlation functions have a thermal character in this case as well.

Two-point correlation functions involving operators at timelike separated boundary points are more difficult to calculate as they arise from quantum tunneling. In particular, there are no real valued geodesics that connect the boundary points in this case. The geodesic approximation can still be applied, using Euclidean time methods and analytically continuing the answers to real time at the end, provided the Euclidean result is known in closed form. More generally, timelike correlators can be computed directly in real time by solving the scalar field equations following from (35) for a general diagonal and time independent metric,

$$\frac{1}{\sqrt{-g}} \partial_r (\sqrt{-g} g^{rr} \partial_r \phi) - (k^2 g^{xx} + \omega^2 g^{tt} + m^2) \phi = 0. \quad (46)$$

In the simple example of a BTZ black brane in 2+1 dimensions, with $f(r) = 1 - \frac{r^2}{L^2}$, the scalar field equation (46) has a closed form solution in terms of hypergeometric functions, from which it is straightforward to calculate the retarded correlation function using (38). The spectral function, which is the imaginary part of the retarded correlator, is given by

$$\text{Im}(G_R) \propto \sinh \left( \frac{\omega}{2T} \right) \Gamma \left( \frac{\Delta}{2} - \frac{i}{4\pi T}(\omega - k) \right) \Gamma \left( \frac{\Delta}{2} - \frac{i}{4\pi T}(\omega + k) \right)^2. \quad (47)$$
The spectral function only has poles in the lower half of the complex $\omega$ plane,

$$\omega = \pm k - 4\pi iT(\frac{\Delta}{2} + n), \quad n \in \{0, 1, 2, \ldots\},$$

so the correlation function decays exponentially in time. Small perturbations away from thermal equilibrium decay on a time scale $t = 1/(2\pi T\Delta)$, determined by the pole at which $\omega$ has the smallest imaginary part. The poles of the correlation function correspond to solutions, called quasinormal modes, for which $A(\omega, k)$ in (48) vanishes.

6.2. **Holographic superfluids**

As opposed to the neutral BTZ background considered above, quasinormal modes of a scalar field can migrate to the upper half complex $\omega$ plane in charged black brane backgrounds\(^{15}\) This occurs in many gravity models with a negative cosmological constant and indicates an instability of the black brane towards a configuration with non-vanishing background scalar field. In other words, black branes can grow scalar hair. To see how this may happen, consider a charged scalar field in the charged black brane background\(^{1}\),

$$\Box - (m^2 + e^2 g^{tt} A_t^2) \phi = 0,$$

where $e$ is the charge of $\phi$ and $A_t(r)$ is the brane gauge field\(^{10}\) The combination

$$m_{\text{eff}}^2 = m^2 + g^{tt} A_t^2,$$

(50)

can be interpreted as an effective mass squared for the scalar field and if the original scalar field mass squared is not too large, the gauge field term may dominate. Since $g^{tt}$ is negative outside the brane horizon, the effective mass is then tachyonic and if it violates the Breitenlohner-Freedman bound\(^{16}\) one expects an instability towards condensation of the scalar field outside the black brane. Near the spacetime boundary $r \to \infty$ the effective mass is dominated by the bare mass term $m^2$, so any condensate that forms near the brane horizon will fall off at large $r$.

There is another mechanism for a scalar field instability in a charged black brane background\(^{13}\) The near-horizon geometry of an extremal black brane in asymptotically Lifshitz spacetime is given by $\text{AdS}_2 \times R^2$. At low temperatures the brane is near-extremal and the near-horizon region approaches $\text{AdS}_2 \times R^2$. The Breitenlohner-Freedman bound is stronger in $\text{AdS}_2$ than in the asymptotic region,

$$m_{BF}^2 \bigg|_{\text{AdS}_2} = -\frac{1}{4} > -\frac{(z + d)^2}{4} = m_{BF}^2 \bigg|_{r \to \infty},$$

(51)

so even a neutral scalar field can be sufficiently tachyonic in the near-horizon region to cause condensation at low temperature if the mass squared is in the above range.

To illustrate the scalar condensation, we consider a large $e$ probe limit where the back reaction of the scalar field and the gauge field on the brane geometry can

\(^{c}\) The scalar field only couples to the physical gauge field $A^{(2)}_\mu$ so we drop the \(^{(2)}\) superscript.
be neglected. Further assuming that $d = 2$, $z = 1$, and $m^2 = -2$, the problem is reduced to considering

$$S = \frac{1}{e^2} \int d^4x \sqrt{-g} \left( -\frac{1}{4} F_{\mu\nu}^2 - |D_\mu \phi|^2 + 2|\phi|^2 \right),$$

in the background geometry of an AdS-Schwarzschild black brane. A time independent equilibrium configuration with translation and rotation symmetry in the transverse plane amounts to an ansatz of the form $\phi = \phi(r)$, $A_t = A_t(r)$. The $r$ component of Maxwell’s equations requires the phase of $\phi$ to be a constant, which may be chosen so that $\phi$ is real valued. The field equations obtained from (52) then reduce to

$$\phi'' + \left( \frac{f'}{f} + \frac{2}{r} \right) \phi' + \frac{A_t^2}{f^2} \phi + \frac{2}{f} \phi = 0,$$

$$A_t'' + 2r A_t' - 2 \frac{\phi^2}{f} A_t = 0,$$  \hspace{1cm} (53)  \hspace{1cm} (54)

where $f(r) = -g_{tt} = g^{rr}$ is the function that appears in the background metric. The boundary conditions for the fields at the horizon follow from regularity: For $A = A_t(r) dt$ to have finite norm we must have $A_t = 0$ at the horizon, and, requiring $\phi''$ to be finite, the equation of motion of $\phi$ implies $f' \phi' + 2 \phi = 0$ at the horizon. The behavior of the bulk fields near the AdS boundary provides input data for the dual field theory. The boundary value of $A_t$ gives the field theory chemical potential $\mu$ and we require that the scalar operator that is dual to $\phi$ has vanishing source. For the chosen value of $m$ these conditions amount to

$$\phi = \frac{\phi_0}{r^2} + O(r^{-3}), \quad A_t = \mu + \frac{\rho}{r} + O(r^{-2}).$$  \hspace{1cm} (55)

The field equations can be solved numerically, subject to these boundary conditions. The chemical potential introduces a reference scale and when the Hawking temperature of the black brane is smaller than a critical temperature $T_c \approx 0.059 \mu$ one finds solutions with a non-vanishing scalar field profile satisfying the above conditions. The critical value corresponds to the temperature at which the first scalar quasi-normal mode crosses to the upper half of the complex $\omega$ plane. Below the critical temperature the scalar field condenses and this spontaneously breaks the global $U(1)$ symmetry of the dual field theory. By studying fluctuations around the condensate background one finds that it is stable, unlike the normal phase with $\phi = 0$ and $A_t = \mu - \rho/r$.

The condensate solution is thermodynamically stable. The free energy in the field theory is identified with the Euclidean on shell action of the bulk system. Since the temperature and the chemical potential are the only scales in the problem, the free energy has a scaling form

$$\Delta F = F_{\text{cond}} - F_{\text{normal}} = VT^3 g(T/|\mu|),$$  \hspace{1cm} (56)

where $V$ is the transverse volume, $F_{\text{cond}}$ is the free energy of the condensate solution, and $F_{\text{normal}}$ is the free energy of the normal phase solution. Comparing the free
energies of the numerical solution for $\phi$ and $A_t$ in the condensed phase and the known analytic solution for the normal phase leads to the result shown on the left in Figure 1. The condensed phase, when it exists, is indeed found to have the lower free energy. The phase transition at the critical temperature is seen to be continuous and of second order.

As usual in AdS/CFT the asymptotic behavior of bulk fields near the AdS boundary provides information about physical quantities in the dual field theory. For instance the coefficients in (55) determine the one point functions of the $U(1)$ current and the scalar operator dual to $\phi$,

$$\langle O \rangle \propto \phi, \quad \langle J^0 \rangle = \rho.$$  

The expectation value of the scalar operator $O$ is shown on the right in Figure 1.

Fitting the numerical data near the critical temperature yields

$$\langle O \rangle \propto (T_c - T)^\beta,$$

with $\beta \approx 1/2$, consistent with mean field scaling at a second order phase transition.

The above scalar field instability is not restricted to the probe approximation and the main conclusions for the model with $d = 2$ and $z = 1$ remain unchanged when the back reaction of the scalar field and the gauge field is taken into account. Qualitatively similar results are found for other values of $d$ and more general $z$. The nature of the phase transition is sensitive to details of the gravitational model. Adding strong enough self interactions for the scalar field or a non-minimal gauge field coupling can, for instance, change the scaling exponent $\beta$ in (58) and the order of the phase transition can go from second order to first order. The AdS$_2 \times R^2$ near-horizon limit of an extremal charged brane without hair is unphysical in the sense that it corresponds to finite entropy density at zero temperature in the dual field theory. The condensate solution behaves rather differently at low temperature because of the strong back reaction due to the scalar field on the near-horizon geometry. The end result is model dependent and here we will restrict

Fig. 1. Left: Free energy difference between the condensate and the normal phase. Right: The expectation value of the scalar operator dual to $\phi$ in the condensed phase.
out attention to simple models of the form\(^5\)

\[
S = \frac{1}{2e^2} \int d^4x \sqrt{-g} \left( R - \frac{1}{4e^2} F_{\mu\nu}^2 - |D_\mu \phi|^2 - U(|\phi|^2) \right), \tag{59}
\]

where the potential \(U\) is assumed to be bounded from below.\(^4\) The spacetime will approach AdS\(_4\) in the UV \(r \to \infty\), and then a charged scalar condensate builds up towards IR at small values of \(r\). Rather than attempting to construct the full spacetime at zero temperature, we concentrate on the possible IR limits. As \(U\) is bounded from below, it seems reasonable to assume that \(\phi\) will settle to a constant value \(\phi_0\) at small \(r\). We also assume that the ground state preserves translation and rotation symmetry in the transverse \(\vec{x}\) plane. The geometry can then be viewed as a domain wall interpolating between the UV AdS\(_4\) vacuum and an IR vacuum with \(\phi = \phi_0\). The scalar field equation can be satisfied by a constant field provided that field value minimizes the effective potential,

\[
U_{\text{eff}} = U(|\phi_0|^2) + g^{\mu\nu}(r) A_\mu(r)^2 \phi_0^* \phi_0. \tag{60}
\]

This can happen in two ways. Either both terms in (60) are separately extremised, or the variations of the two terms cancel each other. In the first case \(A_\mu = 0\) and \(\phi_0\) extremises the potential \(U\) leading to an AdS\(_4\) spacetime in the IR with a cosmological constant that in general differs from the UV value. In the second case, where the variations of the two terms cancel, the \(r\) dependence of \(A_\mu\) has to be \(A_\mu \propto \sqrt{-g_{\mu\nu}} = g(r)\). The Maxwell equations then imply that \(rg'/g\) must be a constant. Denoting this constant by \(z\) gives \(A_\mu \propto r^z\). Einstein’s equations can then be easily solved to find

\[
ds^2 = -\left( \frac{r}{L_0} \right)^z dt^2 + \frac{L_0^2}{r^2} dr^2 + r^2 d\vec{x}^2, \quad \phi = \phi_0, \quad A_\mu = \sqrt{2 - \frac{2}{z} \left( \frac{r}{L_0} \right)^z}. \tag{61}
\]

For a given potential \(U\) and scalar field charge \(e\) one can determine \(z\) and \(\phi_0\) from Einstein’s equations, although real solutions for \(z\) and \(\phi_0\) might not always exist. A more detailed analysis\(^5\) indicates that for large enough charge \(e\), the IR spacetime is AdS\(_4\), while for small \(e\), the IR spacetime will be a Lifshitz spacetime. From the field theory perspective, the different IR limits correspond to whether the operator \(J^t\) is relevant or irrelevant in the renormalization group sense in the IR. When it is irrelevant, conformal symmetry including Lorentz invariance emerges in the IR. On the other hand, when \(J^t\) is a relevant operator, the dual field theory flows to a scale invariant fixed point with Lifshitz symmetry.

Again one can ask how much of this story depends on the particular action we are considering. More general models can certainly lead to other zero temperature geometries. In particular, one can obtain hyperscaling violating geometries as zero temperature IR limits if one allows non-minimal couplings between the gauge field and the scalar field\(^3\).\(^2\)

\(^{5}\)The case where \(U\) includes only a mass term and is not necessarily bounded from below has also been considered in detail.\(^3\)
7. Fermionic matter

Fermions are ubiquitous in condensed matter physics and by coupling fermions to the bulk gravitational theory the AdS/CFT prescription can be extended to correlation functions of fermion operators in the dual field theory. Analogous to the scalar case considered in section 6.1, the retarded two point correlation function of operators dual to a bulk fermion of mass $m$ and charge $q$ is obtained by solving the Dirac equation,

$$\left( \Gamma^\mu D_\mu + m \right) \Psi = 0,$$

(62)

with ingoing boundary conditions on $\Psi$ at the black brane horizon. We will not describe the calculation here but simply note some results of a detailed study at finite charge density and zero temperature involving $z = 1, \theta = 0$ extremal branes.

The correlation function develops a pole at $\omega = 0$ for a certain discrete value of the momentum $k = k_F \sim \mu$. The dual finite density system then has gapless states on a shell in momentum space, which is a signature of a Fermi surface. The physics near the Fermi surface depends on the scaling behavior of solutions to (62) near the brane horizon through the parameter $\nu = \sqrt{m^2 - q^2 + \frac{k_F^2}{\mu^2}}$. Suppressing spinor indices, the correlation function has the form

$$G_R(\omega, k) \approx \frac{Z(k)}{\omega - \omega_*(k) + i\Gamma(k)},$$

(63)

for $k \approx k_F$. For $\nu > 1/2$ we have $\omega_* \approx v_F(k - k_F)$ and $\Gamma \sim (k - k_F)^{2\nu} \ll \omega_*$ as $k \to k_F$, indicating the presence of stable quasi-particles. The value of the Fermi velocity $v_F \sim \mu$ is determined by the UV physics at $r \to \infty$. For general parameter values the width of the holographic quasi-particles differs from the $\Gamma \sim \omega_*^2$ found in Landau-Fermi liquid theory. Further departure from Fermi liquid behavior is seen for $\nu < 1/2$. In this case, the frequency and width remain comparable as $k \to k_F$ and the would be quasi-particles are unstable. For the special case $\nu = 1/2$, the correlation function is that of a so called marginal Fermi liquid, with the quasi-particle width suppressed compared to the frequency but only logarithmically while the quasi-particle residue $Z$ vanishes logarithmically as $k \to k_F$.

So far we have ignored interactions among the bulk fermions and any back reaction on the metric and the gauge field. Going beyond this fermion probe approximation rather quickly leads to computational complexity rivaling that of the original strongly coupled many body problem that the gravity dual is supposed to model. As a final topic, we will briefly describe an interesting set of geometries that arise when the bulk fermions are treated in a Thomas-Fermi approximation as a continuous charged fluid. The Compton wavelength of the bulk fermion is assumed to be small compared to the AdS length scale, which means that the scaling dimension of the corresponding fermion operator in the dual field theory is large, $\Delta \approx m \gg 1$, which is not the range of parameters of direct interest for condensed matter applications, but simplifies the bulk fermion problem dramatically.
We set $d = 2$, $z = 1$ and assume translation and rotation symmetry in the transverse plane. The fermions are described as a charged perfect fluid with energy momentum tensor $T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}$ and current density $J_\mu = \sigma u_\mu$, where $\rho, p, \sigma$, and $u_\mu$ are the local energy density, pressure, charge density and four velocity of the fluid, respectively. Using a free fermion equation of state for the fluid takes into account the fermionic character of the particles but ignores their mutual interactions. For static configurations, the coupled system of Einstein, Maxwell, and fluid equations reduce to a set of coupled ordinary differential equations in the radial variable $r$, in a holographic analogy to the standard Tolman-Oppenheimer-Volkov equations for stellar structure.

At high temperature, or more precisely high $T/\mu$, the only solution with given mass and charge density is an AdS-Reissner-Nordström charged black brane with vanishing fluid density everywhere. As the temperature is lowered, at fixed chemical potential, a second solution becomes possible where a ‘cloud’ of fermion fluid is suspended over the horizon of a black brane with sharply defined inner and outer edges at which the fluid density goes to zero. The cloud can be viewed as a fermion analog of the scalar hair on the black brane in a holographic superfluid. The sharp edges of the cloud are an artifact of treating the fermions as a continuous fluid and are smoothed out in more quantum mechanical treatments. Whenever the solution with a fluid cloud outside the brane is allowed it has lower free energy than the AdS-Reissner-Nordström solution at the same temperature. In the fluid approximation, the system undergoes a third order continuous phase transition from the black brane to the fermion cloud phase, but the phase transition is first order in a model based on a WKB approximation to the Dirac equation for the bulk fermions.

In the zero temperature limit the fermion cloud expands in the radial direction, the black brane horizon recedes, and the solution approaches that of an ‘electron star’ which interpolates between AdS$_4$ in the UV at $r \to \infty$ and four dimensional Lifshitz spacetime in the IR at $r \to 0$ with a dynamical critical exponent that depends on model parameters. This is analogous to the emergent IR Lifshitz behavior at zero temperature seen in certain models of holographic superfluids, as described in section above.

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