Home-range search provides advantage under high uncertainty

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Many search processes are conducted in the vicinity of a favored location, i.e., a home, which is visited repeatedly. Foraging animals return to their dens and nests to rest, scouts return to their bases to resupply, and drones return to their docking stations to recharge or refuel. And yet, despite its prevalence, very little is known about home-range search as its analysis is much more challenging than that of unconstrained, free-range, search. Some attempts to treat the home-range problem have been made, but simplifying assumptions cripple existing models and render them inadequate for the description of realistic scenarios. To this end, we develop a theoretical framework for home-range search. This does not make any assumptions on the underlying search process and is furthermore suited to treat generic return and home-stay strategies. We show that the solution to the home-range problem can then be given in terms of the solution to the corresponding free-range problem—which not only reduces overall complexity but also gives rise to a simple, and universal, phase-diagram for search. This reveals that home-range search outperforms free-range search in conditions of high uncertainty. And so, when living gets rough, a home will not only provide warmth and shelter but also allow one to locate food and other resources quickly and more efficiently than in its absence.

Consider a falcon that is roaming the sky in search of prey that is well hidden amongst the grass below. The falcon will wander around for a while, but if prey is not found it will eventually return to its nest empty-handed. Other animals—humans included—display similar behaviour while foraging and when engaged in search activities. Home-return capabilities are now routinely built into robots and drones to avoid running out of fuel or battery power. And yet, while the observation that most natural search processes are home-bound goes back to Darwin, it is still unclear if this situation merely reflects the prevalence of permanent dwellings, or rather is a result of evolutionary convergence to a superior search strategy. To start answering this question, we must first understand how being home-bound affects search and the time it takes to locate a target.

A free-range searcher will set off from a certain location and look for a target until it is found. In contrast, home-range search is a cyclic process which consists of three stages: search, return, and home (Fig. 1A). How much time does it take a home-range searcher to find its target? At face value, it seems that this question can be answered by taking advantage of the existing theory of search and first-passage process, and on recent advancements in our understanding of first-passage under restart. Indeed, home-range search can be seen as a first-passage process which is effectively restarted by home returns. However, basic models of first-passage under restart are a far-cry from reality as they assume that home-returns are instantaneous and that home-stays can be neglected. Slightly more sophisticated models lump together return and home times assuming that the search stage is followed by some delay. This is a step in the right direction: it takes time to get from one place to another, and time spent home to e.g., recover, recharge, or refuel, may not be negligible. However, the time it takes a searcher to return home will typically depend on its velocity, the distance home, and on other parameters which govern motion. All these are currently assumed irrelevant.

Existing formulations of home-range search are non-realistic as they completely ignore even the simplest spatio-temporal correlations that arise during everyday motion. For example, places that are further away take more time to be reached, but return times in models of home-range search are insensitive to the position of the searcher at the end of the search stage. This crippling situation is in many ways similar to that which hindered the acceptance of the continuous time random walk (CTRW) model before the development of space-time coupled CTRW. These introduced explicit correlations between time and distance traveled and cured many illnesses of the original CTRW. Here, we propose to do the same for home-range search. In what follows, we will do away with non-physical assumptions to provide a realistic description of this widely observed search process.

A theoretical framework for home-range search. Consider a searcher which starts at the origin (home) of a d-dimensional arena (can be infinite) at time zero. In the absence of home returns, the searcher will locate one of the existing targets in the arena following a random time T. This time is a property of the free-range problem, and we will henceforth refer to it as the free-range first-passage time (FPT). We will not make any assumptions on the arena, the search process, and target distribution that govern T. However, and in contrast to free-range search, here we will consider a situation where the searcher returns home if it fails to locate the target within a time R (can be random) which we will henceforth refer to as the restart time. Thus, if $T < R$ the searcher finds the target before it is required to return and the search process completes. Otherwise, the searcher will stop looking for the target and start its return back home (Fig. 1B - Search).

The time it takes the searcher to return home will typically depend on the searcher’s position at the end of the search stage (Fig. 1B - Return). For example, the searcher may return home by moving at a constant speed along the shortest possible path. The return time is then simply given by the distance home over the speed of travel. However, various constraints e.g., topographic, may force the searcher to follow a different route and may also affect its velocity. Such situations will result in more complicated relations between the position of the searcher...
and its return time. To capture this, we allow the return time \( \tau(\vec{x}) \) to be a general function of the searcher’s position \( \vec{x} \). After the searcher returns home it stays there for some generic time \( W \) which can also be random (Fig. 1B - Home). This, search-return-home, cycle repeats itself until a target is found.

The above description allows us to write a renewal equation for the home-range FPT, i.e., the time it takes the home-range searcher to find a target. Denoting this time by \( T_R \), we have

\[
T_R = \begin{cases} 
T & \text{if } T < R, \\
R + \tau(\vec{x}) + W & \text{if } R \leq T,
\end{cases}
\]

(1)

where \( T, R, \tau(\vec{x}), \) and \( W \) were defined above; and \( T_R \) is an independent and identically distributed copy of \( T \). Taking expectations in Eq. (1), we obtain

\[
\langle T_R \rangle = \frac{\min(T, R)}{\Pr(T < R)} + \frac{\langle (R \leq t) \phi(\vec{x}) \rangle}{\Pr(T < R)} + \frac{\Pr(R \leq T)(W)}{\Pr(T < R)}
\]

(2)

where \( I(R \leq T) \) is an indicator function which takes the value one if \( R \leq T \), i.e., with probability \( \Pr(R \leq T) \), and is zero otherwise; and different contributions to the sum were labeled according to their source.

The first term on the right-hand side of Eq. (2) gives the FPT of the searcher in an idealized scenario where return and home times can be neglected \( (\tau(\vec{x}) = 0, W = 0) \). The second term gets its contribution from the time it takes the searcher to return home and the third term comes from the time spent at home. Evaluating the first and third terms is straightforward given \( R, T, \) and \( W \) (SI). The second term is slightly more delicate because it depends on \( \vec{x} \)—the random position of the searcher at the end of the search stage. To evaluate this term, we let \( f_R(t) \) denote the probability density function of the restart time \( R \). We then observe that

\[
\langle I(R \leq T) \tau(\vec{x}) \rangle = \int_0^\infty dt f_R(t) \langle \tau(\vec{x}(t)) | I(R \leq T) = t \rangle
\]

(3)

where we have first conditioned on restart happening at time \( t \), and then on \( R \) being smaller or larger than this time. Note, that a non-zero contribution is obtained only for \( T \geq r \), i.e., only when the target is not found and a return actually takes place.

In order to proceed, we define the free-range propagator, \( G_0(\vec{x}, t) \), as the probability to find the searcher at position \( \vec{x} \) at time \( t \) given that it started at the origin. Note that this propagator is called free-range because it is defined in the presence of targets but in the absence of home-returns. Thus, the free-range survival probability is given by \( \Pr(T \geq t) = \int_D d\vec{x} G_0(\vec{x}, t) \), where \( D \) is the available search domain. The internal expectation in Eq. (3) can then be written as \( \langle \tau(\vec{x}(t)) | R = t, T \geq t \rangle = \frac{1}{\Pr(T \geq t)} \int_D d\vec{x} \tau(\vec{x}) G_0(\vec{x}, t) \). Substituting this expression into Eq. (2), we obtain

\[
\langle I(R \leq T) \tau(\vec{x}) \rangle = \int_0^\infty dt f_R(t) \int_D d\vec{x} \tau(\vec{x}) G_0(\vec{x}, t).
\]

(4)

Equation (4) asserts that the second term in Eq. (2) can be evaluated given the free-range propagator \( G_0(\vec{x}, t) \).

So far, we have made no assumptions on the distribution of the time \( R \) which governs restart. One may continue analyzing the problem in whole generality (SI), but we find that much insight can be gained by focusing on the case where \( R \) is exponentially distributed with rate \( r \). Letting \( \tilde{G}_0(\vec{x}, r) = \int_0^\infty dt e^{-rt} G_0(\vec{x}, t) \) and

\[
\tilde{T}(r) = \int_0^\infty dt e^{-rt} f(t) = 1 - r \int_D d\vec{x} \tilde{G}_0(\vec{x}, r)
\]

stand respectively for the Laplace transforms of \( G_0(\vec{x}, t) \) and \( f(t) \) at \( r \), we find

\[
\langle T_r \rangle = \frac{1 - \tilde{T}(r)}{r \tilde{T}(r)} + \frac{1 - \tilde{T}(r)}{\tilde{T}(r)} \langle \tau(\vec{x}) \rangle + \frac{1 - \tilde{T}(r)}{\tilde{T}(r)} \langle W \rangle,
\]

(5)

where \( \langle \tau(\vec{x}) \rangle_r \equiv \int_D d\vec{x} \tau(\vec{x}) \phi(\vec{x}) \) is the mean return time taken with respect to the probability measure \( \phi(\vec{x}) = \tilde{G}_0(\vec{x}, r) \int_D d\vec{x} \tilde{G}_0(\vec{x}, r) \). Starting from Eq. (1), and proceeding similarly to the above, the Laplace transform of the home-range FPT, \( T_r \), can also be obtained. This reads (SI)

\[
T_r(s) = \frac{\tilde{T}(s + r)}{1 - r W(s) \int_D d\vec{x} e^{s \tau(\vec{x})} G_0(\vec{x}, s + r)}.
\]

(6)
with \( \tilde{W}(s) = \langle e^{-sW} \rangle \) standing for the Laplace transform of \( W \). Equation (6) asserts that the distribution of the home-range FPT can always be written in terms of free-range propagator \( G_0(x',t) \), and the random variables \( T, R \) and \( W \).

**Diffusive home-range search.** To set ideas and illustrate how the framework developed above can be utilized in practice, we examine a paradigmatic case study. Consider a 1-d search process in which a particle that starts at the origin diffuses until it hits a stationary target; and let \( D \) and \( L \) denote respectively the diffusion constant and the initial distance from the target. To make this home-range search, we also assume that search is restarted at a constant rate \( \tau_b \) upon which the searcher returns home at a constant speed \( v_r \) (Fig. 2A). In what follows, the time spent home will be neglected as its stand-alone contribution is already well-understood.

To progress, we recall that the free-range propagator of this problem is given by

\[
G_0(x,t) = \frac{1}{\sqrt{4\pi Dt}} \left( e^{-\frac{x^2}{4Dt}} - e^{-\frac{(L-x)^2}{4Dt}} \right). \tag{7}
\]

To get the home-range mean FPT, we observe that the time penalty due to a ballistic home-return from position \( x \) is given by \( \tau(x) = |x|/v_r \). Plugging in the above into Eq. (5) gives (SI)

\[
\langle T_r \rangle = \frac{1}{2} \left( \tau_b \right)_{\text{search}} + \left( \frac{2 \sinh(\sqrt{\tau_b}D)}{\sqrt{\tau_b}D} \right)_{\text{return}} - 1, \tag{8}
\]

where

\[
\tau_d = \frac{L^2}{D}, \quad \tau_b = \frac{L}{v_r}, \tag{9}
\]

stand respectively for the diffusive and ballistic time scales in the problem.

In the limit \( \tau_b \to 0 \), Eq. (8) boils down to the classical result for the mean FPT of diffusion with resetting but we would now like to understand the effect of non-instantaneous, and space time coupled, home-returns. In Fig. 2B, we plot \( \langle T_r \rangle \) as a function of the restart rate for \( \tau_b = 1/2 \) and different values of \( \tau_b \). We then observe that diffusive home-range search is always superior to diffusive free-range search—regardless of how slow home returns are. This can also be seen directly from Eq. (8) by noting that \( \langle T_r \rangle \) is finite for \( r > 0 \), but diverges for \( r = 0 \) where the searcher does not return home.

Diving deeper, we observe that two things happen as we increase the ballistic (return) time scale: (i) it takes more time for the searcher to locate the target, i.e., \( \langle T_r \rangle \) becomes larger; and (ii) the optimal restart rate, \( r^* \), which minimizes \( \langle T_r \rangle \) becomes smaller. The first effect is easy to understand by inspection of the return term in Eq. (6). Quantitative analysis of the second effect reveals a non-trivial scaling relation.

When \( \tau_b = 0 \), the optimal restart rate \( r^*_0 \) can be determined by minimizing the first term in Eq. (6). One then finds\(^{24,39} \) \( r^*_0 = \frac{z}{d} / \tau_d \) with \( z = 1.593 \ldots \) standing for the solution to the following transcendental equation

\[ \tau_b = \frac{z}{d} / \tau_d, \tag{10} \]

Noting that \( z \) is uniquely determined by the ratio \( \tau_b/\tau_d \) on the right hand side of Eq. (10) (Fig. 2C, inset), we conclude that \( r^*/r^*_0 \approx \frac{1}{z} \) for \( \tau_b \ll \tau_d \).

In the limit \( \tau_b \gg \tau_d \), one has \( r^*/r^*_0 \approx 1 \) by definition. In the other extreme, \( \tau_b \gg \tau_d \) which in turn implies \( z' \to 0 \) (Fig. 2C, inset). Expanding \( \mathcal{F}(z) \) around \( z = 0 \), we find \( \mathcal{F}(z) = \frac{z^2}{z^2} + O(z^2) \) (SI). Equating this with \( \tau_b/\tau_d \) on the right side of Eq. (10), we conclude that (Fig. 2C)

\[ r^*/r^*_0 \approx \left\{ \begin{array}{ll} 1 & \text{for } \tau_b \ll \tau_d \\ \left( \frac{1}{z^2} \right)^{2/3} \left( \frac{\tau_b}{\tau_d} \right)^{2/3} & \text{for } \tau_b \gg \tau_d. \end{array} \right. \tag{11} \]

We thus see that the interplay between search and home-returns gives rise to a power law which governs the optimal restart rate in the case of diffusive home-range search. Consequently, by substituting Eq. (11) into Eq. (6), we find that the optimal mean FPT obeys (SI)

\[ \langle T_r \rangle \sim \left\{ \begin{array}{ll} \tau_b & \text{for } \tau_b \ll \tau_d \\ \tau_d & \text{for } \tau_b \gg \tau_d. \end{array} \right. \tag{12} \]

And so, while arbitrary restart rates may easily lead to a situation where \( \langle T_r \rangle \gg \max(\tau_b, \tau_d) \), the optimal mean FPT for home-range search asymptotically scales like \( \langle T_r \rangle \sim \max(\tau_b, \tau_d) \).

**A phase-diagram for search.** The above example illustrates a situation where home-range search offers significant performance advantage over free-range search. To generalize, one
only needs to observe that since the home-range mean FPT in Eq. (5) is finite for \( r > 0 \) (under mild regularity conditions: \( \langle W \rangle < \infty \), \( \int_0^\infty d\bar{\tau} \tau \phi_0(\bar{\tau}) < \infty \)—home-range search offers a huge performance advantage in conditions where the mean FPT of the underlying free-range process diverges (Fig. 3A left). This suggests that home-range search performs best when search conditions are at their worst, but how to quantify and further extend this statement to situations where the underlying free-range FPT has a finite mean is not immediately clear as either free or home-range search may perform better in this case (Fig. 3A right).

When does the introduction of home returns to a free-range search process lower the mean FPT to the target? To answer this question, one should take \( T_r \) in Eq. (5) and check when \( d(T_r)/dr|_{r=0} < 0 \), which we find happens when (SI)

\[
CV^2 > 1 + 2\left(\frac{\tau(\bar{\tau})}{T} + \frac{\langle W \rangle}{T}\right),
\]

(13)

where \( T \) and \( CV = \alpha(T)/\langle T \rangle \) are the mean and relative standard deviation (coefficient of variation) of the free-range FPT, \( \langle W \rangle \) is the mean home-stay time, and \( \tau(\bar{\tau}) = \int_0^\infty d\bar{\tau} \tau(\bar{\tau}) \phi_0(\bar{\tau}) = \frac{1}{\langle T \rangle} \int_0^\infty d\bar{\tau} \tau(\bar{\tau}) \) \( \phi_0(\bar{\tau}) \) standing for the mean return time in the limit \( r \to 0 \).

The condition in Eq. (13) relates three dimensionless quantities and reveals that home-range outperforms free-range search in conditions of high uncertainty. Indeed, on the left hand side of Eq. (13) stands the \( CV \) which quantifies the relative magnitude of fluctuations, or uncertainty, around the free-range mean FPT. These fluctuations need to be large in order for the introduction of home-returns to be beneficial. On the other side of the inequality stand the relative mean return time, \( \tau(\bar{\tau})/\langle T \rangle \), and the relative mean home time, \( \langle W \rangle/\langle T \rangle \), which act as penalties against home-range search and set the bar for the critical magnitude of fluctuations at which the transition between the free-range and home-range phases occurs. The resulting phase-diagram for search is graphically illustrated in panels B & C of Fig. 3.

To demonstrate how the universal result in Eq. (13) manifests itself in a concrete example, we consider a simple model for search in the presence of guidance cues. Namely, we consider the same diffusive home-range search scenario as in Fig. 2A above, but now assume that the particle also drifts at an average velocity \( v \). Note that when the particle drifts away from the target \( (v < 0) \) the free-range mean FPT diverges and home-range search is always preferable (see discussion above). We thus focus on the \( v > 0 \) case which could e.g., model search in the presence of an attractant (potential field) that biases the searcher’s motion in the direction of the target.

The free-range propagator of drift-diffusion in the presence of an absorbing boundary (target) is known to be given by

\[
G_0(x,t) = \frac{1}{\sqrt{4\pi D t}} \left[ e^{-v(x-vt)^2/4Dt} - e^{-\bar{v}(x-vt)^2/4Dt} \right].
\]

(14)

To build the search phase space, we rewrite the condition in Eq. (13) in terms of the natural parameters of drift-diffusion. Setting off from Eq. (14), a straightforward calculation gives \( T = L/v \) and \( CV^2 = 1/\bar{v}^2 \), where \( \bar{v} = Lv/2D \) is the Péclet number, i.e., the ratio between the rates of advective and diffusive transport. In addition, we find \( \tau(\bar{\tau}) = \frac{1}{\bar{v}} (1 - e^{-2v(L/\bar{v})^2} - e^{-2\bar{v}(L/v)²})/\bar{v}^2 \), with \( v \) standing once again for the home-return speed.

When \( \bar{v} \geq 1 \) drift rules over diffusion which alternatively means that guidance cues towards the target are strong. Uncertainty in the free-range FPT is then relatively small and the condition in Eq. (13) cannot be satisfied since \( CV^2 = 1/\bar{v}^2 \leq 1 \). On the other hand, when \( 0 < \bar{v} < 1 \), diffusion rules over drift which means that guidance cues towards the target are weak. Uncertainty in the free-range FPT is then larger and the condition in Eq. (13) is satisfied whenever

\[
v \geq v \cdot G(\bar{v})
\]

(15)

with \( G(\bar{v}) = \frac{\bar{v}e^{-2\bar{v}(L/v)²}}{\bar{v}²(1 - e^{-2v(L/\bar{v})²})} - 1 \). This means that the introduction of

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**Figure 3:** A. Home-range search wins over free-range search whenever the mean FPT of the latter diverges. However, when the free-range mean FPT is finite, e.g., when the search arena is finite or when motion is biased in the direction of the target, either free or home-range search can have a lower mean FPT. B & C. The phase-space determined by Eq. (15) is spanned by three dimensionless parameters: the coefficient of variation of the free-range FPT, \( CV = \alpha(T)/\langle T \rangle \), the relative mean return time, \( \tau(\bar{\tau})/\langle T \rangle \), and the relative mean home time \( \langle W \rangle/\langle T \rangle \). When system parameters belong to the home-range (free-range) phase the introduction of home-returns is asserted to decrease (increase) the mean FPT to the target.

**Figure 4:** A. The phase space of drift-diffusive search as determined by Eq. (15). The free and home-range phases are separated by \( G(\bar{v}) \) which is a simple function of the Péclet number (see main text). B. The mean FPT of drift-diffusive home-range search (SI) vs. the restart rate. Here, \( L = v = 1 \), and other parameters are set by position in phase-space (see numbers, panel A). When system parameters belong to the home-range phase, e.g., for curve number (2), the introduction of home-returns decreases the mean FPT to the target. The converse happens for curves (1) & (3) whose parameters belong to the free-range phase.
home returns will be beneficial whenever the return velocity $v_r$ is greater than a critical velocity $v_c = v \cdot G(Pe)$. Measured in units of the drift velocity $v$, the critical return velocity is uniquely determined by the Péclet number and hence by the relative uncertainty of the free-range FPT. When $Pe \ll 1$, $v_c \approx v$, but in the limit $Pe \to 1$, we have $v_c \sim v/(1 - Pe)$. Thus, as guidance cues (drift) towards the target become stronger the return velocity must also increase sharply in order for home-range search to remain beneficial (Fig. 1).

**Conclusions and outlook.** Home-range search is a process that is widely observed in nature, but its analysis has so far been extremely challenging. We developed a theoretical framework for home-range search and used it to show that solutions to the home-range problem can always be given in terms of solutions to the corresponding free-range problem. The latter are known for a plethora of cases as first-passage time problems have been studied for decades; but even when this is not the case, the framework developed herein can still be useful as it reduces a complicated problem to a much simpler one.

Our analysis of home-range search also reveals key benefits of central place foraging. Living organisms heavily rely on a steady supply of nutrients and other essential resources. This means that even when the time taken to locate a resource is, on average, short enough to support life—large fluctuations around the average are deleterious and may result in death. In face of this uncertainty home-range search offers two important advantages: (i) it reduces the average time taken to locate a resource; and (ii) it renders resource supply more regular by limiting fluctuations around the average.

When fluctuations in the free-range FPT are high such that the inequality in Eq. (13) holds, the introduction of home-range search, but only in conditions of low uncertainty which suggests that home-range search may have evolved as a bet-hedging strategy which performs best when search conditions are at their worst.

**Supplementary Information** is available in the online version of the paper.

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