Abstract

We use the fusion construction in the twisted quantum affine algebras to obtain a unified method to deform the wedge product for classical Lie algebras. As a byproduct we uniformly realize all non-spin fundamental modules for quantized enveloping algebras of classical types, and show that they admit natural crystal bases as modules for the (derived) twisted quantum affine algebra. These crystal bases are parametrized in terms of the $q$-wedge products.

0 Introduction

Let us start with recalling a basic fact in representation theory of finite-dimensional simple Lie algebras, which may be found in an elementary textbook of Lie algebras. Let $V$ be the vector representation of the simple Lie algebra $\mathfrak{sl}_n$. The $k$-fold exterior space $\bigwedge^k V$ ($1 \leq k < n$) turns out to be a new representation of $\mathfrak{sl}_n$, and

$$\bigwedge^k V \cong V(\omega_k)$$

as $\mathfrak{sl}_n$-modules.

Here $V(\omega_k)$ stands for the irreducible finite-dimensional representation with the $k$-th fundamental weight $\omega_k$ as the highest weight.

Let us now consider the quantized enveloping algebra $U_q(\mathfrak{sl}_n)$ in the sense of Drinfel’d and Jimbo. We also have the fundamental representation $V(\omega_k)$ for $U_q(\mathfrak{sl}_n)$. We would like to realize $V(\omega_k)$ as a $\textit{wedge}$-like representation. Since the coproduct of the Hopf algebra $U_q(\mathfrak{sl}_n)$ is non-cocommutative, the construction becomes a bit tricky. Let $V = \bigoplus_{i=1}^n Q(q)v_i$ be the vector representation of $U_q(\mathfrak{sl}_n)$. Our problem can be stated as follows.

Find a suitable vector subspace $W$ of $V \otimes V$ such that
(1) \( W \) is invariant under the action of \( U_q(\mathfrak{sl}_n) \),

(2) at \( q = 1 \), \( W \) is generated by the vectors \( v_i \otimes v_i \) and \( v_i \otimes v_j + v_j \otimes v_i \ (i \neq j) \),

(3) \( V^k / \sum_{i=0}^{k-2} V^i \otimes W \otimes V^{(k-2-i)} \simeq V(\omega_k) \) as \( U_q(\mathfrak{sl}_n) \)-modules.

In the case of \( U_q(\mathfrak{sl}_n) \), the answer is given by
\[
W = \text{span} \{ v_i \otimes v_i, v_i \otimes v_j + q v_j \otimes v_i \ (i > j) \}.
\]

An important remark here is that \( W \) is identified with the image of the so-called \( R \)-matrix \( \hat{R}(z) \) acting on \( V \otimes V \) at \( z = q^2 \).

\[
\hat{R}(z) = (1 - q^2 z) \sum_i E_{ii} \otimes E_{ii} + q(1 - z) \sum_{i \neq j} E_{ij} \otimes E_{ji} \\
+ (1 - q^2) \left( \sum_{i > j} + z \sum_{i < j} \right) E_{ii} \otimes E_{jj}.
\]

Here \( E_{ij} \) denotes the matrix unit, i.e. \( E_{ij} v_l = \delta_{jl} v_i \).

The purpose of this article is to quantize the wedge module \( \bigwedge^k V \) of the vector representation \( V \) for the simple Lie algebras of types \( B_n, C_n \) and \( D_n \) à la fusion construction (e.g. \[5\]). A weird thing here is that one needs to use the \( R \)-matrices for affine Lie algebras of twisted types \( A(2)_{2n} \) and \( A(2)_{2n-1} \). Furthermore, one has to use different \( R \)-matrices of \( A(2)_{2n-1} \) corresponding to different choices of the 0-vertex (See Figure 1). If we compliantly choose the \( R \)-matrices for \( B(1)_n, C(1)_n \) or \( D(1)_n \), we cannot obtain a \( q \)-analogue of the wedge modules, as seen in \[4\] or the \( C(1)_n \) case in this article. We have nevertheless included this \( C(1)_n \) case since it can be treated in a similar fashion. All the modules constructed here are not only \( U_q(\mathfrak{g}) \)-modules but also \( U'_q(\mathfrak{g}) \)-modules (See Table 1), and are shown to have a crystal base in the sense of Kashiwara \[6\]. Moreover, except for the case of \( C(1)_n \) they appear to be perfect of level one, which will be discussed elsewhere.

A byproduct of our construction is that we also have a unified way to realize all non-spin fundamental modules for quantum enveloping algebras of classical types.

1 Preliminaries

In this section we shall fix conventions for the quantized enveloping algebra \( U_q(\mathfrak{g}) \) and recall the expressions of \( R \)-matrices.
1.1 Dynkin datum

We follow the notations of the affine Lie algebra in [4]. Let $\Lambda_i (i = 0, \cdots, n)$ be the fundamental weights. Let $\alpha_i$ and $h_i = \alpha_i^\vee (i = 0, \cdots, n)$ be the simple roots and coroots. The generator of the null roots is denoted by $\delta = \sum_{i=0}^{n} a_i \alpha_i$ and the canonical central element by $c = \sum_{i=0}^{n} a_i^\vee h_i$. $d$ stands for the degree operator.

We consider the affine Lie algebras $g = A^{(2)}_{2n} (n \geq 2), A^{(2)}_{2n-1} (n \geq 3), C^{(1)}_n (n \geq 2)$. For the Dynkin diagram of $A^{(2)}_{2n-1}$, we distinguish two different labelings of the vertices by $A^{(2)}_{2n-1}$ and $A^{(2)\dagger}_{2n-1}$. The Dynkin diagrams for these affine Lie algebras are given in Figure 1. The marks $(a_i)_{0 \leq i \leq n}$ and comarks $(a_i^\vee)_{0 \leq i \leq n}$ are given by

- $(a_i) = (1, 2, \cdots, 2, 1), \quad (a_i^\vee) = (2, 2, \cdots, 2, 1)$ for $g = A^{(2)}_{2n}$,
- $(a_i) = (1, 1, 2, \cdots, 2, 1), \quad (a_i^\vee) = (2, 2, \cdots, 2, 1)$ for $g = A^{(2)}_{2n-1}$,
- $(a_i) = (1, 2, \cdots, 2, 1), \quad (a_i^\vee) = (1, 1, \cdots, 1)$ for $g = C^{(1)}_n$.

It is convenient to introduce an orthogonal basis $\{\epsilon_j\}_{1 \leq j \leq n}$ with which the simple roots are given as follows:

- $\alpha_0 = \delta - 2\epsilon_1$ for $g = A^{(2)}_{2n}, A^{(2)\dagger}_{2n-1}, C^{(1)}_n$.

![Figure 1: Dynkin diagrams](image-url)
The invariant bilinear form $\langle \cdot, \cdot \rangle$ in $\mathfrak{g}$ is normalized so that $\langle \theta | \theta \rangle = 2a_0^2$, where $\theta = \delta - \alpha_0$. (Note that $a_0 = 1$ in our cases.) With this normalization we have $\langle \epsilon_i | \epsilon_j \rangle = \frac{1}{2} \delta_{ij}$ for $\mathfrak{g} = C_n^{(1)}$, $\delta_{ij}$ for the other cases.

### 1.2 Quantized enveloping algebra

We recall the definition of the quantized enveloping algebra $U_q(\mathfrak{g})$. Set $P^* = \bigoplus_{i=0}^n \mathbb{Z} h_i \oplus \mathbb{Z} \mathfrak{g}$. $U_q(\mathfrak{g})$ is the associative algebra generated by the symbols $e_i, f_i (i = 0, \ldots, n)$ and $q^h$ ($h \in P^*$) satisfying the following relations:

\[
q^0 = 1, \quad q^h q^{h'} = q^{h + h'}, \\
q^he_iq^{-h} = q^{(h, \alpha_i)}e_i, \quad q^hf_iq^{-h} = q^{-(h, \alpha_i)}f_i, \\
[e_i, f_j] = \delta_{ij} \frac{t_i - t_i^{-1}}{q_i - q_i^{-1}}
\]

where $q_i = q^{(\alpha_i, \alpha_i)/2}, t_i = q^{(\alpha_i, \alpha_i)h_i/2}$,

\[
\sum_{k=0}^b (-1)^k e_i^{(k)} e_j^{(b-k)} = \sum_{k=0}^b (-1)^k f_i^{(k)} f_j^{(b-k)} = 0 \quad (i \neq j)
\]

where $b = 1 - (h_i, \alpha_j)$. Here we set $[k]_i = (q_i^k - q_i^{-k}) / (q_i - q_i^{-1}), [k]_i! = \prod_{m=1}^{m=k} [m]_i, e_i^{(k)} = e_i^k / [k]_i!, f_i^{(k)} = f_i^k / [k]_i!$.

There are several coproducts of $U_q(\mathfrak{g})$. We use the ‘lower’ one $\Delta = \Delta_-$ given by

\[
\Delta(e_i) = e_i \otimes t_i^{-1} + 1 \otimes e_i, \quad (1.1) \\
\Delta(f_i) = f_i \otimes 1 + t_i \otimes f_i, \\
\Delta(q^h) = q^h \otimes q^h.
\]

For the relations among different coproducts, see [8] for example. We also define $U_q(\mathfrak{g})$ as the subalgebra of $U_q(\mathfrak{g})$ generated by the elements $e_i, f_i, t_i (i = 0, \ldots, n)$, and $U_q(\mathfrak{g})$ as the subalgebra generated by $e_i, f_i, t_i (i = 1, \ldots, n)$. $U_q(\mathfrak{g})$ is canonically viewed as the quantized enveloping algebra associated to the simple Lie algebra $\mathfrak{g}$.
Table 1: Associated simple Lie algebras

\[
\mathfrak{g} = A_{2n}^{(2)} \quad A_{2n-1}^{(2)} \quad A_{2n-1}^{(2)\dagger} \quad C_n^{(1)}
\]

\[
\mathfrak{f} = B_n \quad C_n \quad D_n \quad C_n
\]

1.3 Representations and the R–matrix

Define an index set \( J \) by

\[
J = \{0, \pm 1, \ldots, \pm n\} \quad \text{for} \quad \mathfrak{g} = A_{2n}^{(2)},
\]

\[
J = \{\pm 1, \ldots, \pm n\} \quad \text{for} \quad \mathfrak{g} = A_{2n-1}^{(2)}, A_{2n-1}^{(2)\dagger}, C_n^{(1)}.
\]

We introduce a linear order \( \prec \) in \( J \) by

\[
1 \prec 2 \prec \cdots \prec n \quad (\prec 0) \prec -n \prec \cdots \prec -2 \prec -1.
\]

We now consider the ‘vector’ representation \((\pi, V)\) of \( U_q'(\mathfrak{g}) \). Let \( \{v_j \mid j \in J\} \) be the lower global crystal base \( [7] \). Denoting the matrix units by \( E_{ij} \) i.e. \( E_{ij}v_k = \delta_{jk}v_i \), the actions of the generators read as follows. \((\pi(f_i) = \pi(e_i)^t, \) unless otherwise stated.\)

\[
\pi(e_0) = E_{-1,1} \quad \text{for} \quad \mathfrak{g} = A_{2n}^{(2)}, A_{2n-1}^{(2)}, C_n^{(1)},
\]

\[
= E_{-1,2} + E_{-2,1} \quad \text{for} \quad \mathfrak{g} = A_{2n-1}^{(2)},
\]

\[
\pi(t_0) = \sum_{j \in J} q^{-2\delta_{j,1}+2\delta_{j,-1}} E_{jj} \quad \text{for} \quad \mathfrak{g} = A_{2n}^{(2)}, A_{2n-1}^{(2)}, C_n^{(1)},
\]

\[
= \sum_{j \in J} q^{-\delta_{j,1}-\delta_{j,2}+\delta_{j,-1}+\delta_{j,-2}} E_{jj} \quad \text{for} \quad \mathfrak{g} = A_{2n-1}^{(2)},
\]

\[
\pi(e_i) = E_{i,i+1} + E_{-i-1,-i} \quad (1 \leq i \leq n-1),
\]

\[
\pi(t_i) = \sum_{j \in J} q^{\delta_{j,i+1}+\delta_{j,-1}-\delta_{j,-i}} E_{jj} \quad (1 \leq i \leq n-1),
\]

\[
\pi(e_n) = [2]_q E_{n0} + E_{0,-n} \quad \text{for} \quad \mathfrak{g} = A_{2n}^{(2)},
\]

\[
= E_{n,-n} \quad \text{for} \quad \mathfrak{g} = A_{2n-1}^{(2)}, C_n^{(1)},
\]

\[
= E_{n-1,-n} + E_{-n,-n+1} \quad \text{for} \quad \mathfrak{g} = A_{2n-1}^{(2)},
\]

\[
\pi(f_n) = E_{0n} + [2]_q E_{-n,0} \quad \text{for} \quad \mathfrak{g} = A_{2n}^{(2)},
\]

\[
\pi(t_n) = \sum_{j \in J} q^{\delta_{j,n}-\delta_{j,-n}} E_{jj} \quad \text{for} \quad \mathfrak{g} = A_{2n}^{(2)},
\]

\[
= \sum_{j \in J} q^{2\delta_{j,n}-2\delta_{j,-n}} E_{jj} \quad \text{for} \quad \mathfrak{g} = A_{2n-1}^{(2)}, C_n^{(1)},
\]

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Here uniquely determined by for the other cases. Also Here \( \epsilon \)

The explicit expressions for our \( \mathcal{R} \) by setting \( V_z = V[z, z^{-1}], \pi_z(e_i) = z^{\delta_{i0}} \pi(e_i), \pi_z(f_i) = z^{-\delta_{i0}} \pi(f_i), \pi_z(t_i) = \pi(t_i) \). Let \( \mathcal{R} \) be the \( R \)-matrix, which satisfies

\[
\mathcal{R}(z_1/z_2)(\pi_{z_1} \otimes \pi_{z_2}) \Delta(x) = (\pi_{z_2} \otimes \pi_{z_1}) \Delta(x) \mathcal{R}(z_1/z_2) \quad \text{for all } x \in U'_q(g). \quad (1.2)
\]

The explicit expressions for our \( \mathcal{R} \) have been known in [3], though we need a slight modification due to a different choice of the basis and the coproduct, except for the \( A_{2n-1}^{(2)} \) case. Note that the expressions of \( \mathcal{R} \) for \( A_{2n-1}^{(2)} \) and \( A_{2n-1}^{(2)\dagger} \) should be different due to different choices of the 0-vertex. The expression of \( \mathcal{R} \) for \( A_{2n-1}^{(2)} \) seems to be new. Up to an overall function of \( z \), \( \mathcal{R} \) is uniquely determined by

\[
\mathcal{R}(z) = (1 - q^2 z)(1 - \xi z) \sum_{i \neq 0} E_{ii} \otimes E_{ii} \\
+ q(1 - z)(1 - \xi z) \sum_{i \neq \pm j} E_{ij} \otimes E_{ji} \\
+ (1 - q^2)(1 - \xi z) \left( \sum_{i > j, i \neq -j} + z \sum_{i < j, i \neq -j} \right) E_{ii} \otimes E_{jj} \\
+ \sum_{i, j} a_{ij}(z) E_{-i, j} \otimes E_{i, -j}.
\]

Here

\[
a_{ij}(z) = \begin{cases} 
(q^2 - \xi z)(1 - z) + \delta_{i0}(1 - q)(q + z)(1 - \xi z) & (i = j) \\
(1 - q^2)(\varepsilon_j \varepsilon_i^{-1} - q^{3\delta_{ij}})(z - 1) + \delta_{i, -j}(1 - \xi z)) & (i < j) \\
(1 - q^2)z(\varepsilon_j \varepsilon_i^{-1} \xi(-q)^{3\delta_{ij}})(z - 1) + \delta_{i, -j}(1 - \xi z)) & (i > j).
\end{cases} \quad (1.3)
\]

Here \( \varepsilon_j = 1 \ (j \in J) \) for \( g = A_{2n-1}^{(2)\dagger} \) and \( \varepsilon_j = 1 \ (j > 0), \ [2]_n \ (j = 0), -1 \ (j < 0) \) for the other cases. Also \( \overline{j} \) is defined by

\[
\overline{j} = \begin{cases} 
j & (j = 1, \ldots, n) \\
\overline{n} + 1 & (j = 0) \\
j + N & (j = -n, \ldots, -1),
\end{cases}
\]

and \( N, \xi \) are given below.

\[
g = A_{2n}^{(2)} \quad A_{2n-1}^{(2)} \quad A_{2n-1}^{(2)\dagger} \quad C_n^{(1)} \\
N = 2n + 1 \quad 2n + 2 \quad 2n \quad 2n \\
\xi = -q^{2n+1} \quad -q^{2n} \quad -q^{2n} \quad q^{2n+2}
\]

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2 $q$-wedge constructions

In this section we will use the fusion construction to deform the wedge product. The basis for the deformed wedge product relies on the $R$-matrix and its spectral decomposition.

2.1 $R$-matrices and their spectral decompositions

Let $\lambda$ be the classical part of a weight $\lambda$. Let $V(\lambda)$ be the highest weight $U_q(\mathfrak{g})$-module with highest weight $\lambda$. Recall $V(\lambda) = V(\Lambda_1)$ as a $U_q(\mathfrak{g})$-module) is the vector representation of the quantized enveloping algebra $U_q(\mathfrak{g})$. Direct computation gives the following proposition.

Proposition 2.1 As a $U_q(\mathfrak{g})$-module, the tensor product $V \otimes V$ decomposes itself as follows.

$$V(\Lambda_1) \otimes V(\Lambda_1) = V(2\Lambda_1) \oplus V(\Lambda_2) \oplus V(0),$$

and the highest weight vectors are respectively:

$$u_{2\Lambda_1} = v_1 \otimes v_1,$$

$$u_{\Lambda_2} = v_1 \otimes v_2 - qv_2 \otimes v_1,$$

$$u_0 = \sum_{i=1}^{n} (-q)^{i-1} v_i \otimes v_{-i} - \sum_{i=1}^{n} (-q)^{i-1} \xi' v_{-i} \otimes v_i - \frac{(-q)^{n-1}}{[2]^n} v_0 \otimes v_0,$$

where we take $v_0 = 0$ except for type $A_{2n}^{(2)}$ and $\xi' = \xi$ except $\xi' = q^{2n+2}$ for type $A_{2n-1}^{(2)}$.

Proposition 2.2 Let $P_{2\Lambda_1}$, $P_{\Lambda_2}$ and $P_0$ be the projections from $V(\Lambda_1) \otimes V(\Lambda_1)$ to the corresponding $U_q(\mathfrak{g})$-irreducible components in Prop. (2.1) respectively. Then we have the following spectral decomposition of $\tilde{R}(z)$.

$$\tilde{R}(z) = (1 - q^2 z)(1 - \xi z)P_{2\Lambda_1} \oplus (1 - \xi z)(z - q^2)P_{\Lambda_2} \oplus (1 - q^2 z)(z - \xi)P_0$$

for $\mathfrak{g} = A_{2n}^{(2)}$, $A_{2n-1}^{(2)}$, $C_n^{(1)}$,

and

$$\tilde{R}(z) = (1 - q^2 z)(1 - \xi z)P_{2\Lambda_1} \oplus (1 - \xi z)(z - q^2)P_{\Lambda_2} \oplus (z - q^2)(z - \xi)P_0$$

for $\mathfrak{g} = A_{2n-1}^{(2)}$. 

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Proof. Since $\hat{R}(z)$ commutes with the action of $U_q(\mathfrak{g})$, it is a constant on each irreducible component. From Proposition 2.1, the $R$-matrix $\hat{R}(z)$ can be written as:

$$\hat{R}(z) = c_{2\mathfrak{p}_1}P_{2\mathfrak{p}_1} + c_{\mathfrak{p}_2}P_{\mathfrak{p}_2} + c_0P_0.$$ 

Since the other cases are found in the literature (see [1] for $\mathfrak{g}$ being irreducible component. From Proposition 2.1 the

Proof. Proposition 2.3 is generated by the vectors $W$.

Let $W$ be the image of $\hat{R}(q^2)$ in $V^{\otimes 2}$, then from the spectral decomposition of the $R$-matrix (Proposition 2.2), it easily follows that

$$W = \text{Im} \hat{R}(q^2) = \text{Ker} \hat{R}(q^{-2}) \simeq V^{\otimes 2}/\text{Ker} \hat{R}(q^2).$$

Proposition 2.3 $W$ is generated by the vectors

$$v_i \otimes v_i \ (i \neq 0), \quad v_i \otimes v_j + qv_j \otimes v_i \ (i > j, i \neq \pm j),$$

$$v_{-i} \otimes v_i + q^2 v_i \otimes v_{-i} + q(v_{i+1} \otimes v_{-i-1} + v_{-i-1} \otimes v_{i+1}) \ (1 \leq i < n),$$

plus the following additional ones:

$$v_{-1} \otimes v_1 + v_1 \otimes v_{-1}, \quad v_{-n} \otimes v_n + q^2 v_n \otimes v_{-n} + q^2 v_0 \otimes v_0 \quad \text{for } \mathfrak{g} = A_{2n}^{(2)},$$

$$v_{-n} \otimes v_n + q^2 v_n \otimes v_{-n} \quad \text{for } \mathfrak{g} = A_{2n-1}^{(2)},$$

$$v_{-1} \otimes v_1 + v_1 \otimes v_{-1} \quad \text{for } \mathfrak{g} = A_{2n-1}^{(2)},$$

$$v_{-1} \otimes v_1 + v_1 \otimes v_{-1}, \quad v_{-n} \otimes v_n + q^2 v_n \otimes v_{-n} \quad \text{for } \mathfrak{g} = C_n^{(1)}.$$
Proof. The $R$ matrix $\hat{R}(z)$ preserves the weight, thus it is enough to show that the listed vectors generate the spanning set $\{\hat{R}(q^2)(v_i \otimes v_j) \mid i, j \in J\}$.

For $i \in J \setminus \{0\}$ we have

$$\hat{R}(z)(v_i \otimes v_i) = (1 - q^2 z)(1 - \xi z)v_i \otimes v_i,$$

which implies that $v_i \otimes v_i \in W$.

For $i \neq j$ we have

$$\hat{R}(q^2)(v_i \otimes v_j) = q(1 - q^2)(1 - \xi q^2)v_j \otimes v_i$$

$$+ (1 - \xi q^2)(1 - q^2) \times \begin{cases} v_i \otimes v_j & \text{if } i > j \\ q^2 v_i \otimes v_j & \text{if } i < j \end{cases}$$

$$= \begin{cases} (1 - q^2)(1 - \xi q^2)(qv_j \otimes v_i + v_i \otimes v_j) & \text{if } i > j \\ q(1 - q^2)(1 - \xi q^2)(v_j \otimes v_i + qv_i \otimes v_j) & \text{if } i < j, \end{cases}$$

thus $v_i \otimes v_j + qv_j \otimes v_i (i > j, i \neq -j) \in W$.

As for the weight 0 vectors $\hat{R}(q^2)(v_j \otimes v_{-j})$, we only show it for the case of $A_{2n-1}^{(2)}$ for the purpose of simplicity, since all other cases are similar.

Let $A_j = A_j(z) = \sum_{i \in J} a_{ij}(z)v_{-i} \otimes v_i = \hat{R}(z)(v_j \otimes v_{-j})$, where $a_{ij}(z)$ are defined in (1.3). Write $u_i = v_i \otimes v_{-i}$. We have for $1 \leq j \leq n - 1$

$$A_j + q^{-1} A_{j+1} = (z - 1)q(z \xi - 1)u_{-j-1} + (1 - z)(1 - \xi z)u_{-j}$$

$$+ z(1 - q^2)(1 - \xi z)u_j + q^{-1} z(1 - q^2)(1 - \xi z)u_{j+1}$$

$$= (1 - \xi z)\{(1 - z)(qu_{j-1} + u_{-j}) - z(1 - q^2)(u_j + q^{-1}u_{j+1})\}$$

At $z = q^2$ we see that $u_{-j} + q^2 u_j + q(u_{j+1} + u_{j-1}) \in W$. Similarly we have at $z = q^2$

$$A_{-j-1} + q^{-1} A_{-j} = (1 - q^2)(1 - q^2 \xi)\{q^{-1} u_{-j} + u_{-j-1} + u_{j+1} + qu_j\},$$

which does not produce any new vectors in $W$. Also we have

$$A_n - A_{-n} = (z - q^2)(1 - \xi z)(u_n - u_{-n})$$

which vanishes when $z = q^2$. These relations show that $W$ is generated by the vectors

$$v_i \otimes v_i, v_i \otimes v_j + qv_j \otimes v_i (i > j, i \neq j), u_{i+1} + u_{i-1} + (q^{-1}u_{i} + qu_i),$$

plus one of the vectors $A_j(q^2)$.

We claim that for positive $j \in J$

$$A_j(q^2) \equiv q^{-2(n-j)} A_{-j}(q^2)$$

$$\equiv (-q)^j q^2 (1 - q^4)(u_1 + u_{-1}) \mod u_{i+1} + u_{i-1} + (q^{-1}u_{i} + qu_i).$$
In fact we have
\[ A_1 = (q^2 - \xi z)(1 - z)u_{-1} + (1 - q^2)z (\xi (-q)^{2-2n}(1 - \xi z)) u_1 \]
\[ + \sum_{i=2}^{n} (1 - q^2)z \xi (z - 1) ((-q)^{1-i} u_{-1} + (-q)^{1-2n+i} u_i) \]
\[ \equiv (q^2 - \xi z)(1 - z)u_{-1} + (1 - q^2)z (\xi (-q)^{2-2n}(1 - \xi z)) u_1 \]
\[ + (1 - q^2)z \xi (z - 1) ((-q)^{-1} + (-q)^{-3} + \ldots + (-q)^{3-2n}) (-q^{-1} u_{-1} - q u_1) \]
\[ = z(1 - z^2)u_{-1} + z ((1 - q^2 z) + \xi (q^2 - z)) u_1, \]
which is equivalent to \(q^2(1 - q^4)(u_1 + u_{-1})\) at \(z = q^2\). \(\square\)

Define the action of \(x = e_i, f_i, t_i \in U_q'(g)\) on \(V^\otimes k\) by
\[ x(u_1 \otimes \cdots \otimes u_k) = (\pi_{e_i} \otimes \cdots \otimes \pi_{t_i}) \Delta^{(k)}(x)(u_1 \otimes \cdots \otimes u_k). \]
Here \(z_j = (-q)^{2-j} k\) and \(\Delta^{(k)} = (\Delta \otimes \id \otimes \cdots \otimes \id) \circ \cdots \circ (\Delta \otimes \id) \circ \Delta\). Explicitly, it reads as
\[ e_i(u_1 \otimes u_2 \otimes \cdots \otimes u_k) = \sum_{j=1}^{k} u_1 \otimes \cdots \otimes u_{j-1} \otimes z_j^{\delta} e_i u_j \otimes t_i^{-1} u_{j+1} \otimes \cdots \otimes t_k^{-1} u_k \]
\[ f_i(u_1 \otimes u_2 \otimes \cdots \otimes u_k) = \sum_{j=1}^{k} t_i u_1 \otimes \cdots \otimes t_i u_{j-1} \otimes z_j^{-\delta} f_i u_j \otimes u_{j+1} \otimes \cdots \otimes u_k \]
\[ t_i(u_1 \otimes u_2 \otimes \cdots \otimes u_k) = t_i u_1 \otimes \cdots \otimes t_i u_k. \]

Because of the intertwining property \(\{1, 2\}\), it is immediate to see that the subspace \(\sum_{i=1}^{k-1} V^\otimes (i-1) \otimes W \otimes V^\otimes (k-i-1)\) is a \(U_q'(g)\)-submodule of \(V^\otimes k\). Thus setting
\[ V^k = V^\otimes k / \sum_{i=1}^{k-1} V^\otimes (i-1) \otimes W \otimes V^\otimes (k-i-1), \]
\(V^k\) turns out to be a \(U_q'(g)\)-module. For simplicity we will write the image of \(u_1 \otimes u_2 \otimes \cdots \otimes u_k\) as \(u_1 \wedge u_2 \wedge \cdots \wedge u_k\).

At \(q = 1\) the module \(V^k\) degenerates into the wedge product for the simple Lie algebra except for the cases when the algebra is \(C_n^{(1)}\) or \(A_{2n-1}^{(2)\dagger}\) and \(k = n\). We will include some examples at the end of this section.

We set \(\omega_i = \epsilon_1 + \cdots + \epsilon_i\) and \(\overline{\omega}_n = \epsilon_1 + \cdots + \epsilon_{n-1} - \epsilon_n\). The relations between \(\omega_i\) and \(\Lambda_i\) are as follows.
\[ \omega_i = \begin{cases} 2\Lambda_n, & i = n; B_n, D_n, \\ \Lambda_{n-1} + \Lambda_n, & i = n - 1; D_n, \\ \Lambda_i, & \text{otherwise} \end{cases} \]
\[ \overline{\omega}_n = 2\Lambda_{n-1}, \quad D_n. \]
Theorem 2.4 For each $k$ the module $V^k$ is isomorphic to the highest weight $U_q(\mathfrak{g})$-module with the highest weight $\omega_k$ except for

$$V^n = V(\omega_n) \oplus V(\varpi_n), \quad \text{for } A^{(2)\dagger}_{2n-1},$$

$$V^k = V(\omega_k) \oplus V(\omega_{k-2}) \oplus \cdots \oplus V(\omega_{k \mod 2}), \quad \text{for } A^{(2)}_{2n-1}.$$

Proof. By the relations of $V^k$ we see that the space is generated by the set of vectors:

$$v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_k},$$

where $i_1 < i_2 < \cdots < i_k$ are $k$ indices from the set $J$. Therefore the dimension of the space is less than or equal to

$$\binom{|J|}{k}.$$

On the other hand the module $V^k$ contains the highest weight vector $v_1 \wedge v_2 \wedge \cdots \wedge v_k$ of weight

$$\epsilon_1 + \epsilon_2 + \cdots + \epsilon_k = \omega_k.$$

When $k = n$ in the case of $A^{(2)\dagger}_{2n-1}$, there is another highest weight vector $v_1 \wedge v_2 \wedge \cdots \wedge v_{n-1} \wedge v_n$, which is of the weight

$$\epsilon_1 + \epsilon_2 + \cdots + \epsilon_{n-1} - \epsilon_n = \varpi_n.$$

When $k \leq n$ in the case of $A^{(2)}_{2n-1}$, there are $\lfloor k/2 \rfloor + 1$ highest weight vectors. For a multi-index set $I = (i_1, i_2, \cdots, i_l)$ denote

$$|I| = i_1 + i_2 + \cdots + i_l.$$

We claim that in the case of $A^{(2)}_{2n-1}$, for each $l = 0, 1, \cdots, \lfloor k/2 \rfloor$ the following vector $w_{k-2l}$ is a highest weight vector of weight $\omega_{k-2l} = \epsilon_1 + \cdots + \epsilon_{k-2l}$:

$$w_{k-2l} = \sum_{i=1}^{\lfloor k/2 \rfloor} (-q)^{|I|} v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_{k-2l}} \wedge v_{i_{k-2l+1}} \wedge \cdots \wedge v_{i_l} \wedge v_{i_{l+1}} \wedge \cdots \wedge v_{i_{l+k-2l-1}}.$$

In fact we can check directly that the action of $e_i|_{V^k} = \sum 1 \otimes \cdots 1 \otimes e_i \otimes t_i^{-1} \otimes \cdots \otimes t_i^{-1} (i \neq 0)$ kills the vector. Since $wt(v_{\pm i}) = \pm \epsilon_i$, we have $wt(w_{k-2l}) = \omega_{k-2l}$.

First we consider the cases other than $C^{(1)}_n$. By the dimension formula for irreducible modules of simple Lie algebras [10] it follows that

$$\dim V^k \geq \dim V(\omega_k) = \binom{|J|}{k}, \quad \text{for } A^{(2)\dagger}_{2n-1}(k \neq n), A^{(2)}_{2n}.$$
First we make several observations about the \( q \)-wedge products.

We will use the formula
\[
\dim V^k \geq \dim V(\omega_n) + \dim V(\omega_{n-1}) = \binom{|J|}{k}, \quad \text{for } A_{2n-1}^{(2)}(k=n),
\]

\[
\dim V^k \geq \dim V(\omega_k) + \dim V(\omega_k-2) + \cdots + \dim V(\omega_{k \mod 2})
= \binom{|J|}{k}, \quad \text{for } A_{2n-1}^{(2)}.
\]

In the \( C_n^{(1)} \) case we need the formula
\[
\sum_{j=1}^{n} (-q)^j v_j \wedge v_{-j} = 0, \quad (2.4)
\]
which is easily shown from the relations in Proposition 2.3 (cf. Prop. 2.5).

Since the action of \( e_i \) (\( i \neq 0 \)) is the same as in the \( A_{2n-1}^{(2)} \) case, (2.3) give rise to highest weight vectors if they are not zero. Moreover, from the properties of weights, they are the only possible highest weight vectors. Thus it suffices to check \( w_{k-2l} = 0 \) if \( l > 0 \). Interchanging \( i_l \) and \( i_{l-1} \) and using relations, we have
\[
q^2 w_{k-2l} = \sum_{k-2l < i_1 < \cdots < i_{l-2} < i_{l-1} < i_l} (-q)^{|I|} v_{1,\ldots,k-2l,i_1,\ldots,i_{l-2},i_{l-1},i_l}.
\]
where \( v_{j_1,\ldots,j_k} \) denotes \( v_{j_1} \wedge \cdots \wedge v_{j_k} \). Similarly we have
\[
q^{2l-2j} w_{k-2l} = \sum_{k-2l < i_1 < \cdots < i_{l-1} < i_l < \cdots < i_j} (-q)^{|I|} v_{1,\ldots,k-2l,i_1,\ldots,i_{l-1},i_l,\ldots,i_j}.
\]

If \( i_l \) is equal to some other \( i_j \), we see \( v_{i_1,\ldots,i_l} = 0 \). Thus using (2.4) we have
\[
(1 + q^2 + \cdots + q^{2l-2}) w_{k-2l}
= \sum_{k-2l < i_1 < \cdots < i_{l-1}} \sum_{k-2l < i_l} (-q)^{|I|} v_{1,\ldots,k-2l,i_1,\ldots,i_{l-1},i_l}
= -\sum_{k-2l < i_1 < \cdots < i_{l-1} < i_l} (-q)^{|I|} v_{1,\ldots,k-2l,i_1,\ldots,i_{l-1},i_l},
\]
which is zero unless \( q^{2l} = 1 \).

Thus the theorem is proved. \( \square \)

### 2.3 Remarks and Examples

We will use the \( q \)-wedge product to calculate some analog of classical identities. First we make several observations about the \( q \)-wedge products.

In the cases of \( A_{2n-1}^{(2)}, A_{2n}^{(2)} \) and \( C_n^{(1)} \), the vector
\[
u_0 = \sum_{i=1}^{n} (-q)^{i-1} v_i \wedge v_{-i} - \sum_{i=1}^{n} (-q)^{2n-i+1} v_{-i} \wedge v_i - \frac{(-q)^{n-1}}{[2]_n} v_0 \wedge v_0 = 0.
\]
In the case of \( A^{(2)}_{2n-1} \), we have
\[
\omega_0 = \sum_{i=1}^{n} (-q)^{i-1}v_i \wedge v_{-i} - \sum_{i=1}^{n} (-q)^{-i-1}q^i v_{-i} \wedge v_i
\]
\[
= (1 + q^{2n+2}) \sum_{i=1}^{n} (-q)^{i-1}v_i \wedge v_{-i}. \quad (2.5)
\]

This is proved by an inductive calculation based on the wedge relations. Let \( v'_0 \) be the vector \( \sum_{i=1}^{n} (-1)^{i-1}q^{-i+1}v_{-i} \wedge v_i \).

\[
v'_0 = v_{-1} \wedge v_1 - q^{-1}v_{-2} \wedge v_2 + \cdots + (-q)^{n-1}v_{-n} \wedge v_n
\]
\[
= -q^2v_1 \wedge v_{-1} - q(v_2 \wedge v_{-2} + v_{-2} \wedge v_2) - q^{-2}v_2 \wedge v_2 + \cdots
\]
\[
= -q^2v_1 \wedge v_{-1} + q^2v_2 \wedge v_{-2} + q[2](v_{-3} \wedge v_3 + v_3 \wedge v_{-3}) + q^{-2}v_3 \wedge v_3 + \cdots
\]
\[
= -q^2v_1 \wedge v_{-1} + \cdots + (-1)^{i-1}q^i v_{i-1} \wedge v_{i-1} + (-1)^{i-1}q(i-1)[v_{i-1} \wedge v_i
\]
\[
+ v_i \wedge v_{i-1}] + (-q)^{-i+1}v_{-i} \wedge v_i + \cdots + (-q)^{n-1}v_{-n} \wedge v_n
\]

where we assume inductively at the \( i \)-step. Using the relations
\[
v_{-i} \wedge v_i = -q^2v_i \wedge v_{-i} - q(v_{i-1} \wedge v_{i+1} + v_{i+1} \wedge v_{i-1}),
\]
\[
[i] = q[i - 1] + q^{-i+1},
\]
we see by induction that
\[
v'_0 = -q^2v_1 \wedge v_{-1} + \cdots + (-1)^{n-1}q^n v_{n-1} \wedge v_{n-1} + (-1)^{n-1}q[n-1](v_{n-1} \wedge v_n
\]
\[
+ v_n \wedge v_{n-1}) + \cdots + (-q)^{n-1}v_{n-1} \wedge v_n
\]
\[
= -q^2v_1 \wedge v_{-1} + \cdots + (-1)^{n-1}q^n v_{n-1} \wedge v_{n-1}
\]
\[
+ (-1)^{n-1}(q[n-1] - q^2[n])v_n \wedge v_{n-1}
\]
\[
= -q^2v_1 \wedge v_{-1} + q^2v_2 \wedge v_{-2} + \cdots - (-q)^{n+1}v_n \wedge v_{n-1}
\]

which proves (2.5).

From the relations it is easy to obtain the following formula.
\[
v_{-i} \wedge v_i = -q^2v_i \wedge v_{-i} + (1 - q^2) \sum_{k=1}^{n-i} (-q)^k v_{i+k} \wedge v_{-(i+k)}
\]
for \( i = 1, \ldots, n-1 \).

Note that except for the case of \( C^{(1)}_n \), the theorem implies that the set of vectors \( v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_k} \), where \( i_1 < i_2 < \cdots < i_k \) are \( k \) indices from the set \( J \), forms a basis for the module \( V^k \). At \( q = 1 \) the module \( V^k \) specializes to the ordinary wedge representation of the subalgebra \( \mathfrak{g} \), except for \( C^{(1)}_n \). In fact,
it is well-known that the wedge representations for classical Lie algebras are as follows [10]:

\[ \bigwedge^k V(A_1) \simeq \begin{cases} 
V(\omega_k) & \text{for } A_n, B_n, D_n \text{ and } k \neq n \\
V(\omega_n) \oplus V(\bar{\omega}_n) & \text{for } D_n \text{ and } k = n \\
\bigoplus_{t=0}^{[k/2]} V(\omega_{k-2t}) & \text{for } C_n 
\end{cases} \]

*Example 1.* For \( g = \mathfrak{a}^{(2)\dagger}_{2n-1} = \mathfrak{a}^{(2)\dagger}_7 \),

\[ V^4 = V^{\otimes 4}/(V \otimes V \otimes V \otimes V) \simeq V(\omega_4) \oplus V(\bar{\omega}_4), \]

where the highest weight vectors are \( u_{\omega_4} = v_1 \wedge v_2 \wedge v_3 \wedge v_4 \) and \( u_{\bar{\omega}_4} = v_1 \wedge v_2 \wedge v_3 \wedge v_{-4} \).

\[ V^3 = V^{\otimes 3}/(V \otimes V \otimes V) \simeq V(\omega_3), \]

where the highest weight vector is \( v_1 \wedge v_2 \wedge v_3 \). Note that

\[ e_3(v_1 \wedge v_2 \wedge v_{-2} - q v_1 \wedge v_3 \wedge v_{-3} + q^2 v_1 \wedge v_3 \wedge v_{-4}) = -q(1 + q^2)v_1 \wedge v_3 \wedge v_4. \]

\[ V^2 = V^{\otimes 2}/W \simeq V(\omega_2), \]

where the highest weight vector is \( v_1 \wedge v_2 \) and we have

\[ v_1 \wedge v_{-1} - q v_2 \wedge v_{-2} + q^2 v_3 \wedge v_{-3} - q^3 v_4 \wedge v_{-4} = -q^6(v_{-1} \wedge v_1 - q^{-1} v_{-2} \wedge v_2 + q^{-2} v_{-3} \wedge v_3 - q^{-3} v_{-4} \wedge v_4) \]

*Example 2.* For \( g = \mathfrak{a}^{(2)}_{2n-1} = \mathfrak{a}^{(2)}_5 \),

\[ V^4 = V^{\otimes 4}/(V \otimes V \otimes W \otimes V) \simeq V(\omega_4) \oplus V(W \otimes V), \]

where the highest weight vectors are \( u_{\omega_2} = v_1 \wedge v_2 \) and \( u_0 = v_1 \wedge v_{-1} - q v_2 \wedge v_{-2} + q^2 v_3 \wedge v_{-3} \).

\[ V^3 = V^{\otimes 3}/(V \otimes W \otimes W) \simeq V(\omega_3) \oplus V(\omega_1), \]

where the highest weight vectors are \( u_{\omega_3} = v_1 \wedge v_2 \wedge v_3 \) and \( u_{\omega_1} = v_1 \wedge v_2 \wedge v_{-2} - q v_1 \wedge v_3 \wedge v_{-3} \).

*Example 3.* For \( g = \mathfrak{c}^{(1)}_n = \mathfrak{c}^{(1)}_4 \),

\[ V^2 = V^{\otimes 2}/W \simeq V(\omega_2) \]

where the highest weight vector is \( v_1 \wedge v_2 \). It is a good exercise to check directly from the relations of \( W \) that

\[ v_{-1} \wedge v_1 - q^{-1} v_{-2} \wedge v_2 + q^{-2} v_{-3} \wedge v_3 - q^{-3} v_{-4} \wedge v_4 = v_1 \wedge v_{-1} - q v_2 \wedge v_{-2} + q^2 v_3 \wedge v_{-3} - q^3 v_4 \wedge v_{-4} = 0. \]
In fact denote the two vectors in the first two lines by \( u'_0 \) and \( u_0 \) respectively. Let \( W_1 \) be the submodule generated by the common relations (2.1-2.2) plus \( v_{-1} \wedge v_1 = -v_1 \wedge v_{-1} \), and \( W_2 \) to be the submodule generated by the common relations (2.1-2.2) plus \( v_{-n} \wedge v_n = -q^2 v_n \wedge v_{-n} \).

It can be seen immediately that
\[
u'_0 \equiv -q^2 u_0 \mod W_2
\]
as in the case of \( A_{2n-1}^{(2)} \) (cf. 2.4). Swapping the role of 1 and \( n \) and straightening backward we have
\[
u'_0 \equiv -q^{-(2n-2)} u_0 \mod W_1.
\]
As long as \( q^{2n} \neq 1 \) we have
\[
u_0 = u'_0 = 0.
\]

Summarizing previous calculations we obtain the following result.

**Proposition 2.5** Let \( V \) be the 2\( n \)-dimensional vector space generated by basis vectors \( v_j, j \in J = \{1, 2, \cdots, n, -n, \cdots, -1\} \). Let \( W_1 \) be the subspace of \( V \otimes V \) generated by relations:
\[
v_i \otimes v_i, \quad v_i \otimes v_j + q v_j \otimes v_i \ (i > j, i \neq \pm j),
v_{-i} \otimes v_i + q^2 v_i \otimes v_{-i} + q(v_{i+1} \otimes v_{i-1} + v_{i-1} \otimes v_{i+1}) \ (1 \leq i < n),
v_1 \otimes v_1.
\]
Let \( W_2 \) be the subspace of \( V \otimes V \) with the same relations except that \( v_1 \otimes v_1 \) is replaced by \( v_n \otimes v_n + q^2 v_n \otimes v_{-n} \). Then we have
\[
\sum_{i=1}^{n} (-q)^{-(i-1)} v_{-i} \otimes v_i \equiv -q^2 \sum_{i=1}^{n} (-q)^{i-1} v_i \otimes v_{-i} \mod W_2
\]
\[
\equiv -q^{-(2n-2)} \left( \sum_{i=1}^{n} (-q)^{i-1} v_i \otimes v_{-i} \right) \mod W_1.
\]
Also we have
\[
v_{-i} \otimes v_i \equiv -v_i \otimes v_{-i} + (1 - q^2) \sum_{k=1}^{i-1} (-q)^{-k} v_{i-k} \otimes v_{-(i-k)} \mod W_1
\]
\[
\equiv -q^2 v_i \otimes v_{-i} + (1 - q^2) \sum_{k=1}^{n-k} (-q)^{k} v_{i+k} \otimes v_{-(i+k)} \mod W_2.
\]
3 Crystal structures

In this section we show that the $U'_q(g)$-modules $V^k$ constructed in the last section admit crystal base in the sense of Kashiwara. Since $V^k$ for $A_{2n-1}^{(2)}$ and $A_{2n-1}^{(2)\dagger}$ are the same as $U'_q(g)$-modules (apart from the labeling of the vertices of the Dynkin diagram), we consider the $A_{2n-1}^{(2)\dagger}$ case. The crystal structure of $V^k$ for $A_{2n-1}^{(2)}$ is rather difficult to describe. Before going into details we recall necessary facts about crystal base from [6].

3.1 Basics of crystal base

Let $\mathbf{P} = \sum_{j=1}^{n} \mathbf{Z} \Lambda_j$ be the weight lattice and $\mathbf{P}^+$ denote the set of dominant integral weights for $g$. Let $V$ be a finite dimensional $U'_q(g)$-module. Then $V = \bigoplus_{\mu \in \mathbf{P}} V^\mu$, where $V^\mu = \{ u \in V \mid t_i u = q_i^{\mu(h_i)} u \}$. We shall explain the crystal base for $V$. Kashiwara defined the operators $\tilde{e}_i$ and $\tilde{f}_i$ acting on $V$ by

\[ \tilde{e}_i f_i^{(k)} u = f_i^{(k-1)} u, \tilde{f}_i f_i^{(k)} u = f_i^{(k+1)} u \text{ for } u \in V_{\mu + k \alpha_i} \cap \text{Ker } e_i. \]

Let $A$ be the ring of rational functions regular at $q = 0$. The crystal base for $V$ is a pair $(L, B)$ satisfying the following properties.

- $L$ is a free $A$-submodule of $V$ such that $\mathbf{Q} \otimes_A L \simeq V$. \hfill (3.1)
- $B$ is a base of the $\mathbf{Q}$-vector space $L/qL$. \hfill (3.2)
- $\tilde{e}_i L \subseteq L$ and $\tilde{f}_i L \subseteq L$ for any $i$. \hfill (3.3)
  Hence $\tilde{e}_i$ and $\tilde{f}_i$ act on $L/qL$.
- $\tilde{e}_i B \subseteq B \cup \{ 0 \}$ and $\tilde{f}_i B \subseteq B \cup \{ 0 \}$. \hfill (3.4)
- $L = \bigoplus_{\lambda \in \mathbf{P}} L_\lambda$ and $B = \bigsqcup_{\lambda \in \mathbf{P}} B_\lambda$, \hfill (3.5)
  where $L_\lambda = L \cap V_\lambda$ and $B_\lambda = B \cap (L_\lambda/qL_\lambda)$.
- For $b, b' \in B$, $b' = \tilde{f}_i b$ if and only if $b = \tilde{e}_i b'$. \hfill (3.6)

$L$ is called the crystal lattice and $B$ the crystal for $V$. The set $B$ has a graph structure which is called the crystal graph for $V$.

Let $V(\lambda)$ be the irreducible $U'_q(g)$-module with highest weight $\lambda \in \mathbf{P}^+$ and highest weight vector $u_\lambda$, and set

\[ L(\lambda) = A(\tilde{f}_{i_1} \cdots \tilde{f}_{i_t} u_\lambda \mid t \geq 0, 0 \leq i_1, \ldots, i_t \leq n), \]
\[ B(\lambda) = L(\lambda)/qL(\lambda) \setminus \{ 0 \}. \]

Then up to a trivial isomorphism, $(L(\lambda), B(\lambda))$ is the unique crystal base of $V(\lambda)$. It is also known that if $V = V(\lambda_1) \oplus V(\lambda_2)$ ($\lambda_1, \lambda_2 \in \mathbf{P}^+$) and $(L(\lambda_1), B(\lambda_1))$
Lemma 3.1

is the crystal base for \( V(\lambda_i) \) (\( i = 1, 2 \)), then \( (L(\lambda_1) \oplus L(\lambda_2), B(\lambda_1) \sqcup B(\lambda_2)) \) is the crystal base for \( V \). For \( \lambda \in \mathfrak{P}^\tau \), the explicit realization of the crystal \( B(\lambda) \) for the \( U_q(\mathfrak{g}) \)-module \( V(\lambda) \) is given in \([3]\).

Our strategy to show that \( V^k \) has a crystal is as follows. Since we already know the structure of \( V^k \) as a \( U_q(\mathfrak{g}) \)-module, we take \( (L(\omega_1), B(\omega_1)) \) (or \( (L(\omega_n) \oplus L(\omega_n), B(\omega_n) \sqcup B(\omega_n)) \) when \( g = A_{2n-1}^2 \), \( k = n \)) as a candidate for crystal base. Then it is clear that the conditions \([3.3], [3.4] \text{ and } [3.6] \) are valid except \([3.3], [3.4] \text{ and } [3.6] \) for \( i = 0 \). Therefore, we are left to check those conditions for \( i = 0 \). We do it with the help of the explicit description of the crystal structure given in \([3]\).

### 3.2 Crystal bases

Let us set

\[
L^k = L(\omega_n) \oplus L(\omega_n), \quad B^k = B(\omega_n) \sqcup B(\omega_n) \quad \text{for } \mathfrak{g} = D_n, k = n, \\
L^k = L(\omega_k), \quad B^k = B(\omega_k) \quad \text{otherwise.}
\]

We shall show that \( \tilde{e}_0, \tilde{f}_0 \) preserve \( L^k, B^k \) \& \{0\}, and their actions on \( B^k \) is described explicitly. We recall the description of \( B^k \) in \([3]\).

\[
B^k = \left\{ b_{i_1, i_2, \ldots, i_k} \mid \begin{array}{|l}
\text{1. } \forall \nu \leq i \nu < k, \quad \text{if } \mathfrak{g} = C_n, \\
\text{2. } \nu < i, \nu < k, \quad \text{if } \mathfrak{g} = B_n, \\
\text{3. } \nu < i, \nu < k, \quad \text{if } \mathfrak{g} = D_n.
\end{array} \right\}
\]

Let \( v_{i_1, i_2, \ldots, i_k} \) denote \( v_{i_1} \land v_{i_2} \land \cdots \land v_{i_k} \) in \( V^k \) for \( i_1, i_2, \ldots, i_k \in J \). By the uniqueness theorem of crystal base, as a \( U_q(\mathfrak{g}) \)-crystal, we identify \( (L^k, B^k) \) with the crystal base of \( V^k \) by \( b_{i_1, i_2, \ldots, i_k} = v_{i_1, i_2, \ldots, i_k} \mod qL^k \) as \( \mathfrak{g} \) = \( D_n, k = n \), and \( b_{i_1, i_2, \ldots, i_k} = v_{i_1, i_2, \ldots, i_k} \mod qL^k \) otherwise.

By the explicit actions of \( \tilde{e}_i, \tilde{f}_i \) \( (1 \leq i \leq n) \) on \( B^k \) given in \([3]\) we have the following result.

**Lemma 3.1** We have

1. Except for \( g = A_{2n-1}^2 \) and \( k = n \), if \( b_{i_1, i_2, \ldots, i_k} = f_{a_1} \cdots f_{a_1}b_{i_2, \ldots, i_k} \), then \( 1 \notin \{a_1, \ldots, a_1\} \) and furthermore, \( f_{a_1} \cdots f_{a_1}b_{i_2, \ldots, i_k} = b_{i_2, \ldots, i_k} \).

2. For \( g = A_{2n-1}^2 \) and \( k = n \), if \( b_{i_1, i_2, \ldots, i_n} = f_{a_1} \cdots f_{a_1}b_{i_2, \ldots, i_n} \), then \( 1 \notin \{a_1, \ldots, a_1\} \) and furthermore, \( f_{a_1} \cdots f_{a_1}b_{i_2, \ldots, i_n} = b_{i_2, \ldots, i_n} \).

3. Except for \( g = A_{2n-1}^2 \) and \( k = n \), if \( b_{i_1, \ldots, i_k, -1} = e_{a_1} \cdots e_{a_1}b_{-1} \cdots, -2, -1 \), then \( 1 \notin \{a_1, \ldots, a_1\} \) and furthermore, \( e_{a_1} \cdots e_{a_1}b_{1, \ldots, -1} = b_{1, \ldots, -1} \).
Theorem 3.3

We have
\[
\text{for } a_1, \ldots, a_l \}
\text{and furthermore,}
\]
\[
\hat{e}_{a_1} \cdots \hat{e}_{a_l} b_{±,n,(n-1),\ldots,-2} = b_{,i,\ldots,i_1}.
\]

Define the following operators.
\[
F_k = f_1^2 \cdots f_{k-1} f_{k} \cdots f_{n-1} f_n f_{n-1} \cdots f_{k+1} f_k, \quad \text{for } g = C_n^{(1)},
\]
\[
= f_1^2 \cdots f_{k-1} f_{k} \cdots f_{n-1} f_n f_{n-1} \cdots f_{k+1} f_k, \quad \text{for } g = A_n^{(2)},
\]
\[
= f_1^2 \cdots f_{k-1} f_{k} \cdots f_{n-2} f_n f_{n-1} \cdots f_{k+1} f_k, \quad \text{for } g = A_n^{(2)}, k \neq n,
\]
\[
F_n^+ = f_1^2 \cdots f_{n-2} f_n \quad F_n^- = f_1^2 \cdots f_{n-2} f_n, \quad \text{for } g = A_n^{(2)}, k = n.
\]

The operators \( E_k \) or \( E_n^{±} \) are defined similarly by replacing \( f_i \) with \( \hat{e}_i \). Using the actions of \( f_i \) and \( e_i \) on \( V^k \) we have the following lemma.

Lemma 3.2 We have

1. Except for \( g = A_n^{(2)} \) and \( k = n \), we have \( F_k v_{1,2,\ldots,k} = v_{2,3,\ldots,k-1} \) and \( \hat{f}_0 v_{2,3,\ldots,k-1} = v_{2,3,\ldots,k} \).
2. For \( g = A_n^{(2)} \) and \( k = n \), we have \( F_n v_{1,2,\ldots,n-1,±n} = v_{2,\ldots,n-1,±n} \) and \( \hat{f}_0 v_{2,\ldots,n-1,±n} = v_{1,2,\ldots,n-1,±n} \).
3. Except for \( g = A_n^{(2)} \) and \( k = n \), we have \( E_k v_{-k,\ldots,-2,1} = v_{-k,\ldots,-3,2} \) and \( \hat{e}_0 v_{-k,\ldots,-3,2} = v_{-k,\ldots,-2,1} \).
4. For \( g = A_n^{(2)} \) and \( k = n \), we have
\[
E_n^{±} v_{±,n,(n-1),\ldots,-2,1} = v_{±,n,(n-1),\ldots,-2,1} \quad \text{and} \quad \hat{e}_0 v_{±,n,(n-1),\ldots,-2} = v_{±,n,(n-1),\ldots,-2,1}.
\]

Now we have the following theorem declaring the existence of a crystal base for \( V^k \).

Theorem 3.3 We have

1. \( \hat{e}_0 \) and \( \hat{f}_0 \) preserve \( L^k \).
2. \( \hat{f}_0 b_{i_1,i_2,\ldots,i_k} = b_{i_1,i_2,\ldots,i_k} \) if \( i_k = -1, = 0 \) otherwise.
3. \( \hat{e}_0 b_{i_1,i_2,\ldots,i_k} = b_{i_2,\ldots,i_k} \) if \( i_1 = 1, = 0 \) otherwise.
Proof. Note that (2) and (3) ensures (1).

We first prove (2) except for the case of $g = A_{2n-1}^{(2)}$, $k = n$. If $i_k \neq -1$, $\tilde{f}_0 b_{i_1, \ldots, i_k} = 0$ from the weight consideration. If $i_k = -1$, take $a_1, \ldots, a_l \in \{2, \ldots, n\}$ such that $b_{1, i_1, \ldots, i_{k-1}} = \tilde{f}_{a_1} \cdots \tilde{f}_{a_l} b_{1, 2, \ldots, k}$. Note that $f_0\tilde{f}_i = \tilde{f}_i f_0$ for $i \neq 0, 1$ and $b_{2, \ldots, k-1} = v_{2, \ldots, k-1} \mod qL^k$ from Lemma 3.2 (1). Using Lemma 3.1 (1) and Lemma 3.2 (1), we have

$$\tilde{f}_0 b_{i_1, \ldots, i_{k-1}, -1} = f_0 \tilde{f}_{a_1} \cdots \tilde{f}_{a_l} b_{2, \ldots, k-1} = f_0 \tilde{f}_{a_1} \cdots \tilde{f}_{a_l} v_{2, \ldots, k-1}$$

$$= \tilde{f}_{a_1} \cdots \tilde{f}_{a_l} f_0 v_{2, \ldots, k-1} = \tilde{f}_{a_1} \cdots \tilde{f}_{a_l} v_{1, 2, \ldots, k} = b_{1, i_1, \ldots, i_{k-1}} \mod qL^k.$$

For the case of $g = A_{2n-1}^{(2)}$, $k = n$, the proof is similar. The only difference is that we have to choose + or − of $b_{1, i_1, \ldots, i_{n-1}} = \tilde{f}_{a_1} \cdots \tilde{f}_{a_l} b_{1, 2, \ldots, n}$, since $B^n$ has two connected components.

For the proof of (3), observe that $b_{\pm n, -(n-1), \ldots, -2, -1} = v_{\pm n, -(n-1), \ldots, -2, -1} \mod qL^n$ for $g = A_{2n-1}^{(2)}$, $k = n$, and $b_{-k, \ldots, -2, -1} = v_{-k, \ldots, -2, -1} \mod qL^k$ otherwise. \qed

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