Connected correlations, fluctuations and current of magnetization in the steady state of boundary driven XXZ spin chains

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Abstract. We show how to exploit algebraic relations among the operators (or matrices) which constitute the non-equilibrium matrix product steady state of a boundary driven quantum spin chain to find partial differential equations determining all the \(m\)-point correlation functions in the continuum (or thermodynamic) limit. These partial differential equations, the order of which is determined by scaling of the non-equilibrium partition function, are readily solved if we also know the boundary conditions. In this way we can avoid resorting to representation theory of the matrix product algebra. We apply our methods to study the distributions, or moments, of the magnetization and the spin current observables in boundary driven open XXZ spin chains, as well as some connected correlation functions. Interestingly, we find that the transverse connected correlation functions are thermodynamically non-decaying and long range at the isotropic point \(\Delta = 1\).

Keywords: spin chains, ladders and planes (theory), correlation functions (theory), stationary states, current fluctuations
1. Introduction

The matrix product ansatz (MPA) for steady states has a long history in non-equilibrium physics. The MPA was first used to find the non-equilibrium steady states of a 1D asymmetric exclusion model analytically [1] and was also extended to other classical driven diffusive systems [2]. Later the MPA was also applied to steady states of open quantum spin systems, which are described as fixed points of Lindblad master equations [3]. In these models the spin systems are coupled to two Markovian baths, which drive the system out of equilibrium. These types of setup have attracted a lot of attention lately in the context of transport theory (see, e.g. [4–9, 11–15]), as their study has become accessible to experiments through various quantum simulation techniques, such as those using cold atoms [16–19]. They also find potential application in the context of, e.g. quantum computing [20–22].

When we are studying the statistical properties of such systems either the continuum or the thermodynamic limit is the most interesting. Due to the richness of phenomena and the highly non-trivial nature of out-of-equilibrium systems, interesting and useful information can be gained from studying not merely the expectation value of physical quantities, but their statistical properties (fluctuations) as well. For instance, one may be interested in, e.g. cumulants, correlators or connected correlation functions in various contexts.
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Calculation of the statistical properties can be greatly eased through the use of large deviation theory and the related method of full counting statistics, by means of which one can in principle calculate the full probability distributions of physical quantities of interest [23]. These two methods were only very recently applied in full to quantum systems [24] and even more recently to both non-interacting (see, e.g., [25–28]) and interacting [29–31] many-body systems. Using these methods open quantum systems can also be seen to exhibit interesting properties near phase transitions [32–35] and one may also extend them to study closed systems [36, 37].

However there are only a few known exact analytical results for the full counting statistics of non-interacting many-body quantum systems [26, 27, 31], and likewise, there is only a single analytical result for an interacting case of the XXZ spin chain [29]. The latter was achieved only perturbatively in system-bath coupling. In fact, analytically studying interacting many-body quantum systems under the open quantum system framework seemed like a formidable task, but became feasible after two recent results [7, 8] for the open XXZ spin chain, later understood through the underlying quantum integrability of the system [10, 38, 39] (for a review see [4]).

Throughout our work we shall call the matrices constituting the matrix product steady state auxiliary space operators (ASOs). In the aforementioned solutions the ASOs fulfil certain algebraic relations, which we shall call the matrix product algebra. To compute observables one must usually employ an appropriate representation of the algebra. We shall show however that what one requires, in principle, is only the defining relations of the algebra, which in the continuum limit (or thermodynamic limit) lead to partial differential equations (and of course the corresponding boundary conditions we need to solve these equations). What we study in this article is essentially a simple generalization of the procedure employed in [8] to calculate the magnetization profiles and two-point connected longitudinal spin correlation function (the same method was also later used in [40] to compute the profiles and currents—but not the correlations—for more general, so-called twisted boundary conditions).

Using this method we then find explicit expressions for several $m$-point connected correlation functions and spin current fluctuations for the boundary driven open XXZ spin chain [4, 8]. Interestingly, for the critical $\Delta = 1$ case the transverse connected correlation functions are non-decaying and thus exhibit genuine long-range order, similar to what has been previously observed numerically in a related case [41]. We also study the probability distribution of total magnetization and the moments of the spin current operator.

Note that we compute these correlators for the non-equilibrium steady state, which can be contrasted with other dynamical studies of both open quantum systems [42] and systems undergoing quantum quenches [43]. We should also note that Verstraete and Cirac [44] introduced continuous matrix product states (cMPSs) for quantum fields. Our approach is not related to this. Instead of constructing matrix product states for quantum fields, which are continuum limits of lattice theories, we shall take a discrete matrix product and study the continuum limit.

In this paper we discuss a general procedure for computing the continuum limit of a steady state (assumed to be given in the form of a matrix product state). A key step in taking this continuum limit is a perturbative expansion in lattice spacing, the validity of which is not known. The second result is the computation of connected correlators and fluctuations of current in the steady state for the open maximally driven XXZ spin
chain in the continuum limit. Using a known discrete solution for this model [8] we can check the validity of our method.

More specifically, in section 2 we review the properties of matrix product steady states. In section 3 we outline our method for computing the continuum limit of the steady state equation and steady state (under certain assumption discussed). Later, in section 4 using this method (and also aided by the known solution for the discrete steady state [8]) we compute the correlation and connected correlation functions for the steady state of the aforementioned open maximally driven XXZ spin chain, focusing mostly on the non-trivial isotropic XXX case. Afterwards, in sections 5 and 6 we study the fluctuations of spin current and total magnetization in the steady state of the open maximally driven XXZ spin chain, aided by our previous computation of the connected correlation functions.

2. Matrix product steady states

We shall be interested in non-equilibrium steady states (NESSs) $\rho_\infty$ of one-dimensional spin-1/2 systems (quantum spin chains). Let the system have $n$ sites described by a $2^n$-dimensional Hilbert space $\mathcal{H}$ on which act operators constructed from the Pauli matrices, $\sigma^+_j, \sigma^-_j, \sigma^z_j := 1$, where $j = 1, \ldots, n$ labels the site position. Let the dynamics of the system, described by the density matrix $\rho(t)$, be determined by a quantum Liouville equation,

$$\frac{d}{dt}\rho(t) = \hat{\mathcal{L}} \rho(t),$$

where $\hat{\mathcal{L}}$ can be understood as a superoperator acting on the space of operators $\mathcal{B}(\mathcal{H})$, spanned by the Pauli matrices. The space $\mathcal{B}(\mathcal{H})$ may also be considered as a Hilbert space itself if one defines an inner product in the Hilbert–Schmidt sense, i.e. $A, B \in \mathcal{B}(\mathcal{H}), (A, B) = \text{tr} A^\dagger B$. From equation (1) the defining equation for the NESS is

$$\hat{\mathcal{L}} \rho_\infty = 0.$$  

The key assumption we shall use is that the NESS is given in the form of a homogenous MPA,

$$\rho_\infty = \frac{S_n}{\text{tr} S_n}$$

where

$$S_n = \langle L | L^\otimes n | R \rangle,$$

such that

$$S_n = \langle L \left( \begin{array}{cc} O_1 & O_- \\ O_+ & O_2 \end{array} \right)^\otimes n | R \rangle,$$
and $O_j \in \text{End}(\mathfrak{S})$ are called the auxiliary space operators (ASOs)\(^1\) acting over the vector space $\mathfrak{S}$, which is also the space on which the representation of the symmetry algebra of the model acts. Importantly, these operators satisfy some algebraic relations, which are assumed to be known. The states $|\mathcal{R}, \mathcal{L}\rangle \in \mathfrak{S}$ are referred to as the boundary vectors. We shall also define four important operators, $O_0, O_z, O_x,$ and $O_y$,

\begin{align*}
O_0 & := O_1 + O_2, \\
O_z & := O_1 - O_2, \\
O_x & := O_+ + O_-, \\
O_y & := -i(O_+ - O_-).
\end{align*}

One usually assumes that the representation (used to calculate the steady state explicitly) of the algebra satisfied by the ASOs is known. We shall not do so here, but shall first illustrate the approach one takes if it is known. Observables can be calculated from the MPA in the following simple way, provided one knows the representation of the ASO algebra.

Define a general, not necessarily local, operator $\sigma_1 \cdots \sigma_n$, where $\sigma_j \in \{x, y, z, 0\}$ denote the components of the corresponding Pauli matrices [4, 46]. Its expectation value in the steady state is given by (from equation (3))

\begin{equation}
\langle B_{\alpha_1, \ldots, \alpha_n} \rangle = \text{tr}(\rho_{\text{st}} B_{\alpha_1, \ldots, \alpha_n}) = \frac{\text{tr}(S_n B_{\alpha_1, \ldots, \alpha_n})}{\text{tr} S_n}.
\end{equation}

An object which will be of central importance is the so-called non-equilibrium partition function, $\mathcal{Z}_n$,

\begin{equation}
\mathcal{Z}_n = \text{tr} S_n.
\end{equation}

It is related to currents flowing through the system in the NESS for a wide variety of exactly solvable 1D systems, including both classical processes [2] and various out-of-equilibrium quantum spin chains [4].

Using the MPA form in equation (5), equation (8) can be written as

\begin{equation}
\langle B_{\alpha_1, \ldots, \alpha_n} \rangle = \frac{\langle \mathcal{L} \left[ \text{tr} \left( \left( \begin{array}{cc}
O_1 & O_2 \\
O_+ & O_-
\end{array} \right)^{\otimes n} B_{\alpha_1, \ldots, \alpha_n} \right) \right] |\mathcal{R}\rangle \rangle \rangle \rangle}{\text{tr} S_n},
\end{equation}

where we have used the fact that the trace $\text{tr} := \text{tr}_p$ is taken only over the physical $2^n$-dimensional Hilbert space $\mathcal{H}$ and not over the auxiliary space $\mathfrak{S}$ by definition (equation (8)). Then it is merely a matter of simple matrix multiplication (in the physical space) and using repeatedly the property of the trace that $\text{tr}(A \otimes B) = \text{tr}(A)\text{tr}(B)$ (together with definitions (6) and (7)) to find that

\begin{equation}
\langle B_{\alpha_1, \ldots, \alpha_n} \rangle = \frac{\langle \mathcal{L} |O_{\alpha_1} O_{\alpha_2} \cdots O_{\alpha_n} |\mathcal{R}\rangle \rangle}{\langle \mathcal{L} |O_0^j |\mathcal{R}\rangle \rangle},
\end{equation}

where $\alpha_j \in \{x, y, z, 0\}$. Note that this also means that the non-equilibrium partition function (9) can be written as

\begin{equation}
\mathcal{Z}_n = \text{tr} S_n = \langle \mathcal{L} |O_0^j |\mathcal{R}\rangle \rangle.
\end{equation}

\(^1\) These operators were sometimes also called vertex operators in [4, 7, 8], but they have no relation to the more standard concept of vertex operators (see, e.g. [45]).
We can thus define a mapping from expectation values of observables to their corresponding auxiliary space operators, e.g.
\[
\langle \sigma_{j}^{\alpha} \sigma_{k}^{\alpha} \rangle = \frac{\langle L|O_{j}^{0}O_{k}^{0}O_{0}^{n-k}O_{0}^{j-1}O_{0}^{0}|R \rangle}{\langle L|O_{0}^{0}|R \rangle}.
\] (13)

In order to actually calculate the expectation values of operators in the NESS we also need know the representation of the ASOs $O_{j}$. For interacting systems these representation are generically infinite dimensional. Even though the representations are near diagonal in the integrable (in the sense of [4]) cases and thus allow for efficient computation, calculating the expectation values when the operators are not ultraslocal (acting only on a single site) or for very large systems is impossible. Using the approach discussed in the next section we show that we can bypass this difficulty (at least up to a multiplicative prefactor) by employing only the asymptotic ($n \to \infty$) form of the non-equilibrium partition function $\mathcal{Z}_{n}$, together with the algebra satisfied by the ASOs, to calculate all the $m$-point correlators (and equivalently the entire steady state of the system) in the continuum limit without resorting to an explicit representation of the ASOs.

3. Continuum limit of the NESS

Our procedure is similar to that used for one- and two-point functions in [8] and later in [40]. Motivated by these known examples we shall consider only ASOs which satisfy at most cubic algebraic relations of the form
\[
k_{3,1}^{\alpha_{1}}O_{i_{1}}O_{j_{1}}O_{0} + k_{3,2}^{\alpha_{2}}O_{0}O_{i_{1}}O_{0} + k_{3,3}^{\alpha_{3}}O_{0}O_{0}O_{0} + k_{2,1}^{\alpha_{1}}O_{i_{1}} + k_{2,2}^{\alpha_{2}}O_{0}O_{0} + k_{1,1}^{\alpha_{1}}O_{0} = 0,
\] (14)

where $\alpha \in \{x, y, z\}$ for some constants $k_{i,j}^{\alpha}$. In principle our approach can be generalized to other cases as well, though we shall not discuss this here. Let us now introduce a lattice spacing $a = 1/n$, such that the total length of the system is unity, and so the continuum limit $a \to 0$ corresponds to the thermodynamic limit $n \to \infty$.

We wish to find a set of partial differential equations for an $m$-point correlator, $C_{j_{1},j_{2},\ldots,j_{m}}^{\alpha_{1},\alpha_{2},\ldots,\alpha_{m}} = \langle \sigma_{j_{1}}^{\alpha_{1}}\sigma_{j_{2}}^{\alpha_{2}}\ldots\sigma_{j_{m}}^{\alpha_{m}} \rangle$, where $\alpha_{j} \in \{x, y, z\}$ for each of the $m$ operator coordinates.

We shall first find them for $\alpha_{1} (j_{1})$ by multiplying equation (14) for $\alpha = \alpha_{1}$ by $O_{j_{2}^{\ell}}^{i_{2}^{\ell}}\ldotsO_{j_{m}^{\ell}}^{i_{m}^{\ell}}O_{0}^{n-j_{1}^{\ell}-1}O_{0}^{0}O_{0}^{0}$ from the right and by $\langle L|O_{0}^{i}O_{0}^{0}|R \rangle$ from the left and dividing it by the non-equilibrium partition function $\mathcal{Z}_{n} = \langle L|O_{0}^{0}|R \rangle$. We then use equation (11) to find, similarly to [4, 40],
\[
k_{3,1}^{\alpha_{1}}C_{j_{1},j_{2},\ldots,j_{m}}^{\alpha_{1},\alpha_{2},\ldots,\alpha_{m};n+1} + k_{3,2}^{\alpha_{1}}C_{j_{1}+1,j_{2},\ldots,j_{m}+1}^{\alpha_{1},\alpha_{2},\ldots,\alpha_{m};n+1} + k_{3,3}^{\alpha_{1}}C_{j_{1}+2,j_{2}+3,\ldots,j_{m}+3}^{\alpha_{1},\alpha_{2},\ldots,\alpha_{m};n+1} + k_{2,1}^{\alpha_{1}}C_{j_{1}+1,j_{2}+2,\ldots,j_{m}+2}^{\alpha_{1},\alpha_{2},\ldots,\alpha_{m};n}
+ k_{2,2}^{\alpha_{1}}C_{j_{1}+1,j_{2}+2,\ldots,j_{m}+2}^{\alpha_{1},\alpha_{2},\ldots,\alpha_{m};n} \frac{\mathcal{Z}_{n-1}}{\mathcal{Z}_{n}} + k_{1,1}^{\alpha_{1}}C_{j_{1}+1,j_{2}+1,\ldots,j_{m}+1}^{\alpha_{1},\alpha_{2},\ldots,\alpha_{m};n-1} \frac{\mathcal{Z}_{n-2}}{\mathcal{Z}_{n}} = 0,
\] (15)

where the superscript $n$ over $C_{j_{1},j_{2},\ldots,j_{m}}^{\alpha_{1},\alpha_{2},\ldots,\alpha_{m}}$ denotes that this correlator is computed for system size $n$. Define $x_{k} = j_{k}/n$, $x_{k} + a := (j_{k} + 1)/n$, with inter-site (lattice) spacing...
denoted as \( a := 1/n \), and likewise \( C^{\alpha_1, \ldots, \alpha_n}(x_1 \ldots x_m) := C^{\alpha_1, \ldots, \alpha_n}_{\alpha_1, \ldots, \alpha_n} \). Note that as in [4, 8, 40] one may instead equivalently define \( x_k = (j_k - 1)/(n - 1) \) and \( a = 1/(n - 1) \) as we shall do in the next section. Assume that we can expand for large \( n \) as

\[
\frac{\mathcal{F}_{n-1}}{\mathcal{F}_n} = \sum_{m=0}^{\infty} z^{(m)} n^{-m},
\]

where the first few coefficients \( z^{(0)}, z^{(1)}, \) etc may be vanishing. As we shall see later, this type of expansion can be performed for the open maximally driven XXZ spin chain at \( \Delta \leq 1 \) and for some other models such as open \( SU(N) \)-symmetric quantum gases [39], XXZ spin chains with twisted boundary driving (also for \( \Delta \leq 1 \)) [10, 40], the open spin-1 Lai–Sutherland chain [38] and some other cases discussed in the review article [4]. We then take the continuum limit \( n \to \infty \) by expanding \(^2\) in \( 1/n \),

\[
C^{\alpha_1, \ldots, \alpha_n;n}(x_1, \ldots, x_m) = C^{(0)}(x_1, \ldots, x_m) + \frac{C^{(1)}(x_1, \ldots, x_m)}{n} + \mathcal{O}\left(\frac{1}{n^2}\right), \quad x_k = \frac{j_k}{n}
\]

and performing Taylor series expansion \(^3\) in \( a = 1/n \) around zero to find an infinite system of partial differential equations up to arbitrary order in \( 1/n \) (where, for brevity, \( k_i := k_{i,1}^{\alpha_1} + k_{i,2}^{\alpha_1} + k_{i,3}^{\alpha_1}, k_{i,2}^{\alpha_1} := k_{i,2,1}^{\alpha_1} + k_{i,2,2}^{\alpha_1} \) and \( C^{(k)}(x_1, \ldots, x_m) := C^{(k)}(\bar{x}) \)),

\[
(z^{(0)} k_2^{\alpha_1} + k_3^{\alpha_1}) C^{(0)}(\bar{x}) = 0,
\]

\[
z^{(1)} k_2^{\alpha_1} C^{(0)}(\bar{x}) + (z^{(0)} k_3^{\alpha_1} + k_3^{\alpha_1}) C^{(1)}(\bar{x}) + (k_3^{\alpha_1} - k_3^{\alpha_2} + [(k_2^{\alpha_1} x_1 - k_2^{\alpha_2} x_1)]) \partial_{x_1} C^{(0)}(\bar{x})
\]

\[
+ z^{(0)} \left( k_2^{\alpha_1} \sum_{j=1}^{\infty} (x_j - 1) \partial_{x_j} C^{(0)}(\bar{x}) \right) = 0,
\]

\[
k_2^{\alpha_1} (z^{(2)} C^{(0)}(\bar{x}) + z^{(1)} C^{(1)}(\bar{x})) + (k_3^{\alpha_1} + z^{(0)} k_3^{\alpha_1}) C^{(2)}(\bar{x}) + k_2^{\alpha_1} \sum_{j=1}^{\infty} \left( (x_j - 1) (z^{(0)} \partial_{x_j} C^{(1)}(\bar{x}) + z^{(1)} \partial_{x_j} C^{(0)}(\bar{x})) \right)
\]

\[
+ \frac{1}{2} k_2^{\alpha_1} (x_j - 1)^2 z^{(0)} \partial_{x_j}^2 C^{(0)}(\bar{x}) + (k_2^{\alpha_1} (x_j - 1) + k_2^{\alpha_1} x_1) z^{(0)} \partial_{x_j} C^{(0)}(\bar{x})
\]

\[
+ (k_3^{\alpha_1} - k_3^{\alpha_2} + (k_2^{\alpha_1} x - k_2^{\alpha_2} z) z^{(0)}) \partial_{x_1} C^{(1)}(\bar{x})
\]

\[
+ \sum_{j=1}^{\infty} (k_2^{\alpha_1} (x_j - 1) + k_2^{\alpha_1} x_1) (x_j - 1) z^{(0)} \partial_{x_j} C^{(0)}(\bar{x})
\]

\[
+ \frac{1}{2} [k_3^{\alpha_1} + k_3^{\alpha_2} + (k_2^{\alpha_1} (1 - 2 x_1) + k_2^{\alpha_1} x_1)^2 z^{(0)}] \partial_{x_1}^2 C^{(0)}(\bar{x}) = 0.
\]

\(^2\) Note that we have suppressed the superscript \( \alpha_1, \ldots, \alpha_n \) in \( C^{(k)}(x_1, \ldots, x_m) \); when we do this we refer to a general \( m \)-point correlator, i.e. \( C(x_1, \ldots, x_m) = C^{\alpha_1, \ldots, \alpha_n}(x_1, \ldots, x_m) \).

\(^3\) We assume that we can perform this expansion. The points when the indices coincide can introduce extra boundary conditions and even cause our expansion to fail when these indices are close to each other in the continuum limit. For the open maximally driven XXZ case we study later we have the added benefit of having a discrete solution for the NESS which was previously found to check our results [8].

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We then continue by deriving the equations for $j_2$ by first multiplying equation (14) by $(\Lambda \Omega_0^{k-1} \Omega_{\alpha} \Omega_0^{k-1})$ from the left etc and likewise for all $\alpha_j$. We are finally left with a set of coupled partial differential equations for every order in $1/n$. These are in general complicated for arbitrary orders, but if we only focus on the leading order $(1/n)^0$ they are generically quite simple. The leading order will be determined by the first non-zero equation in the set equation (18). For instance if $z^{(0)}k_2^0 + k_3^0 = 0$ the leading order is given by $C^{(0)}(x) = 0$.

Note that everything, except the boundary conditions, is fully determined by the algebraic relations equation (14) and the asymptotic scaling $n \to \infty$ of the non-equilibrium partition function $Z_n$ equation (9).

We did not consider the representation of the ASOs at all. One may object that the representation is relevant when one wishes to find the boundary conditions to solve these partial differential equations and that it comes into play via the boundary vectors $I(R)$ and $I(L)$, which we used when deriving equation (15). However, in the leading order at least this can be circumvented for a quite general set of Liouvillians equation (1), which define the non-equilibrium steady state equation (2),

$$\mathcal{L}_q \rho_\infty = (\mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_n) \rho_\infty = 0$$

where $\mathcal{L}_0$ acts in general in the bulk of the system and $\mathcal{L}_1$ and $\mathcal{L}_n$ act only ultralocally (on the boundary sites 1 and $n$, respectively). These are the types of Liouvillian one most often encounters when studying non-equilibrium matrix product steady states.

As mentioned previously, we work with one-dimensional spin-1/2 systems and thus we take $\mathcal{L}_0$ as being associated with a model with finite lattice spacing $a$. For instance, in the case of the Lindblad master equation (which we shall study in more detail in the next section) $\mathcal{L}_0 \rho = -i[H, \rho]$, where $H$ is the Hamiltonian of the system.

Assume that $\mathcal{L}_0$ can be written in terms of local two-site interaction operators, $\mathcal{L}_0 = \sum_j \sum_d \mathcal{J}_d^{(a)}$, where $j, j + 1$ denote the sites on which the operator acts. We then perform the continuum limit as before by first setting $a = 1/n$, $x = j/n$, $x + a = (j + 1)/n$. Namely,

$$\mathcal{J}_d^{(a)} j \to \mathcal{J}_d \left( x = \frac{j}{n} \right) \quad \mathcal{J}_d^{(a)} j+1 \to \mathcal{J}_d \left( x + a = \frac{j + 1}{n} \right).$$

Formally then, when taking the continuum limit, $a = 1/n \to 0$ we may expand for small $a$, $\mathcal{L}_0(a) = \mathcal{L}_0^{(0)} + \mathcal{O}(a)$, where the $a$ in $\mathcal{L}_0(a)$ denotes that we are now dealing with an operator which depends on lattice spacing $a$ after we have used equation (20).

Likewise we may formally take the continuum limit of the NESS by first writing out in the operator basis

$$\rho_\infty = N \left( 1 + \sum_k \sum_{\alpha} \langle \sigma_k^\alpha \rangle \sigma_k^\alpha + \sum_{k=m} \sum_{\alpha, \beta} \langle \sigma_k^\alpha \sigma_m^\beta \rangle \sigma_k^\alpha \sigma_m^\beta \cdots \right),$$

where $N$ is a normalization coefficient (such that $\text{tr} \rho_\infty = 1$), and then taking the same continuum limit $\rho_\infty(a) = \rho^{(0)} + \rho^{(1)}a + \mathcal{O}(a^2)$. We do this in the following manner. First rewrite equation (21) as discussed,
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\[
\rho_\infty = N \left( 1 + \sum_k \sum_\alpha \langle \sigma^\alpha(x = ka) \rangle \sigma^\alpha(x = ka) + \sum_{k=m} \sum_{\alpha,\beta} \langle \sigma^\alpha(x_1 = ka) \sigma^\beta(x_2 = ma) \rangle \sigma^\alpha(x_1 = ka) \sigma^\beta(x_2 = ma) \ldots \right) .
\]

(22)

We then formally expand the correlation functions in equation (22) for small \( a = 1/n \),

\[
\langle \sigma^\alpha(x = ka) \rangle = \langle \sigma^\alpha(x) \rangle^{(0)} + \langle \sigma^\alpha(x) \rangle^{(1)} a + \mathcal{O}(a^2), \langle \sigma^\alpha(x_1 = ka) \sigma^\beta(x_2 = ma) \rangle = \langle \sigma^\alpha(x_1) \sigma^\beta(x_2) \rangle^{(0)} + \langle \sigma^\alpha(x_1) \sigma^\beta(x_2) \rangle^{(1)} a + \mathcal{O}(a^2), \ldots
\]

Plugging this back into equation (22) and gathering terms in \( a \) we identify

\[
\rho^{(0)} = N \left( 1 + \lim_{a \to 0} \sum_k \sum_\alpha \langle \sigma^\alpha(x = ka) \rangle^{(0)} \sigma^\alpha(x = ka) + \lim_{a \to 0} \sum_{k=m} \sum_{\alpha,\beta} \langle \sigma^\alpha(x_1 = ka) \sigma^\beta(x_2 = ma) \rangle^{(0)} \sigma^\alpha(x_1 = ka) \sigma^\beta(x_2 = ma) \ldots \right)
\]

(23)

\[
\rho^{(1)} = N \left( \lim_{a \to 0} \sum_k \sum_\alpha \langle \sigma^\alpha(x = ka) \rangle^{(1)} \sigma^\alpha(x = ka) + \lim_{a \to 0} \sum_{k=m} \sum_{\alpha,\beta} \langle \sigma^\alpha(x_1 = ka) \sigma^\beta(x_2 = ma) \rangle^{(1)} \sigma^\alpha(x_1 = ka) \sigma^\beta(x_2 = ma) \ldots \right).
\]

(24)

Let us pause to make a few comments. When the difference between \( x_1 \) and \( x_2 \) etc is of the order of the lattice spacing \( a \) the expansion may be ill defined. In fact, this will turn out to be the case for the \( \Delta < 1 \) case studied later. When we know the discrete solution for the NESS, which will be the case when we shall later study the open XXZ spin chain, we can use this solution to check our results for the continuum limit. Otherwise, one may have to simply assume that the continuum limit can be taken. Furthermore, the normalization coefficient \( N \) depends on \( a = 1/n \). However, it can be seen to cancel in the equation for the NESS (19) and thus does not influence the physical results.

We also assume that \( \hat{L}_1 \) and \( \hat{L}_n \) do not depend on \( a \). We then have in the leading order \( a^0 \),

\[
(\hat{L}_1 + \hat{L}_n + \hat{L}_0^{(0)}) \rho^{(0)} = 0,
\]

where \( \rho^{(0)} \) is essentially almost equivalent to knowledge of all the correlators in the continuum limit.

Note that this continuum limit is the same as the one we took when calculating the differential equations for the correlator (18). In other words, if we already know the discrete solution for the NESS \( \rho_\infty \), taking the continuum limit for the correlators as we did when finding equation (18) also gives the solution perturbatively in lattice spacing \( a \). Since \( \hat{L}_1 \) and \( \hat{L}_n \) are ultralocal (assumed to be acting on one site each), solving equation (25) and thus obtaining the boundary conditions needed to solve the leading order of the set of partial differential equations (18) is simple.
It is important to note that, even though we may circumvent the issue of not knowing the boundary vectors $|R\rangle$ and $|L\rangle$ and the representation of the ASOs using the above discussed approach, it is not necessary to do so. In the case of the already solved problem of the steady state of the open non-equilibrium boundary driven XXZ spin chain [8] in terms of an MPA, the boundary vectors and the representation of the ASOs are known. One can then simply use this to find the appropriate boundary conditions when solving the partial differential equations (18) in a manner similar to what was done in [8, 40] for a less general set of correlators.

We shall use equation (25) in the next section to find the boundary conditions for the partial differential equations determining the NESS of a boundary driven open XXX spin chain.

We shall now turn to an example of a previously solved MPA steady state of an open non-equilibrium boundary driven XXZ spin chain and compute the $m$-point correlators in this case.

4. The $m$-point correlators of the maximally boundary driven XXZ spin chain

The Lindblad master equation is a useful tool for describing out-of-equilibrium physics [47]. It can represent both driving and decoherence by a set of infinite baths coupled to a system under the Born–Markov and rotating wave approximations [3]. It also has an important property of being the most general form of a time-local Markovian quantum master equation which is both completely positive and trace preserving. The Lindblad master equation is

$$
\frac{d}{dt} \rho(t) = \mathcal{L} \rho(t) = -i[H, \rho(t)] + \mathcal{D}(\rho(t)),
$$

$$
\mathcal{D}(\rho(t)) := \sum_k \Gamma_k \left( L_k \rho(t) L_k^\dagger - \frac{1}{2} \{ L_k^\dagger L_k, \rho(t) \} \right),
$$

(26)

where we shall take $H$ to be the XXZ spin chain Hamiltonian,

$$
H = \sum_{j=1}^n 2(\sigma_j^+ \sigma_{j+1}^- + \sigma_j^- \sigma_{j+1}^+) + \Delta \sigma_j^x \sigma_{j+1}^x,
$$

(27)

and Lindblad operators acting only on the boundary sites 1 and $n$,

$$
L_1 = \sqrt{\epsilon} \sigma_1^+, \quad L_n = \sqrt{\epsilon} \sigma_n^+.
$$

(28)

They represent maximum driving; that is, the left bath incoherently only injects magnetization into the system and the right one only takes it out. The equation for the NESS is exactly solvable (where we define the superoperator $(\text{ad} \ H) \rho \equiv [H, \rho]$),

$$
\mathcal{L} \rho_\infty = -i \text{ad} H \rho_\infty + \mathcal{D}(\rho_\infty) = 0,
$$

(29)

due to the underlying integrability structure of the XXZ spin chain. The solution was found in [8] (see [4] for a more comprehensive overview).
4.1. The isotropic point $\Delta = 1$

At the isotropic point $\Delta = 1$, it is known that the ASOs satisfy the following cubic algebraic relations [4, 40, 46]:

$$[O_0, [O_0, O_\alpha]] + 2\{O_0, O_\alpha\} - 8p^2O_\alpha = 0, \quad \alpha = x, y, z, \quad p = \frac{4i}{\varepsilon}. \quad (30)$$

It is also known that for $\varepsilon \gg 1/n$ the partition function scales as [4]

$$\frac{Z_{n-1}}{Z_n} = \frac{\pi^2}{4n^2} + O(n^{-3}). \quad (31)$$

Using the method discussed in the previous section we immediately arrive at a set of decoupled second order partial differential equations for the $m$-point correlators in the continuum limit$^4$, 

$$\frac{\partial^2}{\partial x_k^2} C^{(0)}(\vec{x}) = -\pi^2 C^{(0)}(\vec{x}) \quad k = 1, \ldots, m, \quad (32)$$

where we emphasize again, to avoid confusion, that $C^{(0)}(\vec{x})$ denotes the leading order in $1/n$ of a general $m$-point correlator $C(\vec{x})$,

$$C(\vec{x}) := C(\sigma_1^\alpha \sigma_2^\alpha \ldots \sigma_m^\alpha) := \langle \sigma_1^\alpha(x_1)\sigma_2^\alpha(x_2) \ldots \sigma_m^\alpha(x_m) \rangle, \quad x_k = \frac{j_k}{n}, \quad (33)$$

i.e.

$$C(\vec{x}) = \sum_{k=0}^{\infty} C^{(k)}(\vec{x}) \left(\frac{1}{n}\right)^k. \quad (34)$$

Now we shall show how one can obtain the boundary conditions as outlined briefly in section 3. First we take $x = j/n = j a$ and set $\sigma_j^\alpha \rightarrow \sigma^\alpha(x)$ and $\sigma_{j+1}^\alpha \rightarrow \sigma^\alpha(x + a)$ in the Hamiltonian (27) (for $\Delta = 1$), while keeping the length fixed, $n a = 1$, and then expand in lattice spacing $a$. To do this first look at a pair of local densities $\sigma_j^\alpha \rightarrow \sigma^\alpha(x)$ and $\sigma_{j+1}^\alpha \rightarrow \sigma^\alpha(x + a)$ in the discrete Hamiltonian (27) (noting that $H = \sum_{j=1}^{n-1} h_{j,j+1}$, which in the continuum limit go as

$$h_{j,j+1} + h_{j-1,j} \rightarrow h(x, x + a) + h(x - a, x) = \sum_{\alpha = x, y, z} \sigma^\alpha(x - a)\sigma^\alpha(x) + \sigma^\alpha(x)\sigma^\alpha(x + a). \quad (35)$$

Expanding in $a$, we find

$$h(x, x + a) + h(x - a, x) = \sum_{\alpha = x, y, z} 2\sigma^\alpha(x)\sigma^\alpha(x) + 2\sigma^\alpha(x) \frac{\partial^2 \sigma^\alpha(x)}{\partial x^2} a^2 + O(a^3), \quad (36)$$

i.e. the term of order $a$ containing the first derivative cancels and the first non-trivial term is the one with the second order derivative. It may be interesting to note that this cancellation of the terms with the first derivatives mimics the one we obtained when

$^4$ Note that it turns out due to the cubic algebra (30) of this problem that the leading order terms after performing the expansion via equation (18) do not depend on $\varepsilon$ or $p$. 

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The leading term here in equation (37) is divergent as \( a \to 0 \). However, this is not much of a problem as this term commutes with any operator in \( \text{ad} H \) in the steady state (29) and thus has no influence on the final result for the NESS. Likewise the term of order \( a^0 \) commutes with any operator and does not influence \( \text{ad} H \).

Therefore, the first non-trivial term in equation (37) is of order \( a \). We note for the interested reader that, after using integration by parts, this term corresponds to what is known as the quantum Landau–Lifshitz model (e.g. [48, 49]) or the \( SU(2) \) quantum continuous Heisenberg magnet [50].

We can also expand \( \rho_\infty \) using equation (21) for small \( a \). Then the leading order in \( a \) is given by (using equations (25), (23) and (29))

\[
\hat{\mathcal{D}}(\rho^{(0)}) = 0,
\]

where \( \hat{\mathcal{D}} \) is given by equation (26) and \( L_1 = \sqrt{\varepsilon} \sigma^+(0), L_n = \sqrt{\varepsilon} \sigma^-(1) \). Note that assuming that \( \varepsilon \neq 0 \) we can cancel it in equation (38). Splitting the dissipator into \( \hat{\mathcal{D}} = \hat{\mathcal{D}}_1 + \hat{\mathcal{D}}_n \), where \( \hat{\mathcal{D}}_{1,n} = \left( L_{1,n} \rho L_{1,n}^\dagger \frac{1}{2} \{ L_{1,n}^\dagger, L_{1,n} \}, \right) \), we observe the following action on the basis operators:

\[
\hat{\mathcal{D}}_1(1) = \varepsilon \sigma^z(0), \quad \hat{\mathcal{D}}_1(\sigma^y(0)) = -\varepsilon \sigma^y(0), \quad \hat{\mathcal{D}}_1(\sigma^{x,y}(0)) = -\frac{\varepsilon}{2} \sigma^{x,y}(0),
\]

\[
\hat{\mathcal{D}}_n(1) = -\varepsilon \sigma^z(1), \quad \hat{\mathcal{D}}_n(\sigma^y(1)) = \varepsilon \sigma^y(1), \quad \hat{\mathcal{D}}_n(\sigma^{x,y}(1)) = -\frac{\varepsilon}{2} \sigma^{x,y}(1).
\]

Using this equation (39) and requiring a solution to equation (38) for each of the operators in the basis (21) we arrive at the following boundary conditions:

\[
C^{(0)}(\sigma^x(x_0 = 0)) = \langle \sigma^{\alpha_1}(x_1) \ldots \sigma^{\alpha_{k-1}}(x_{k-1}) \sigma^x(x_k = 0) \sigma^{\alpha_{k+1}}(x_{k+1}) \ldots \sigma^{\alpha_m}(x_m) \rangle = \langle \sigma^{\alpha_1}(x_1) \ldots \sigma^{\alpha_{k-1}}(x_{k-1}) \sigma^x(x_k = 0) \sigma^{\alpha_{k+1}}(x_{k+1}) \ldots \sigma^{\alpha_m}(x_m) \rangle,
\]

\[
C^{(0)}(\sigma^y(x_{k-1}) = 1) = \langle \sigma^{\alpha_1}(x_1) \ldots \sigma^{\alpha_{k-1}}(x_{k-1}) \sigma^z(x_k = 1) \sigma^{\alpha_{k+1}}(x_{k+1}) \ldots \sigma^{\alpha_m}(x_m) \rangle = -\langle \sigma^{\alpha_1}(x_1) \ldots \sigma^{\alpha_{k-1}}(x_{k-1}) \sigma^z(x_k = 1) \sigma^{\alpha_{k+1}}(x_{k+1}) \ldots \sigma^{\alpha_m}(x_m) \rangle,
\]

\[
C^{(0)}(\sigma^x(x_k = 0, 1)) = \langle \sigma^{\alpha_1}(x_1) \ldots \sigma^{\alpha_{k-1}}(x_{k-1}) \sigma^x(x_k = 0, 1) \sigma^{\alpha_{k+1}}(x_{k+1}) \ldots \sigma^{\alpha_m}(x_m) \rangle = 0.
\]

The first order equation for the Liouvillian would be given by continuing the expansion of (29) in lattice spacing \( a = 1/n \), using equation (37) (for the Hamiltonian) and the explicit forms of the NESS expansion given by equations (23) and (24),

\[
-i \text{ad} H^{(1)} \rho^{(0)} + \hat{\mathcal{D}}_1 \rho^{(1)} = 0.
\]

However, we do not need this as we already know the algebraic relations of the ASOs defining the steady state (equation (30)), which fully determine the leading order \( \rho^{(0)} \)

\[
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\]

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via the partial differential equations (32) up to the boundary conditions for these equations. We have found these boundary conditions for the \( m \)-point correlators using the preceding discussion and they are given in equations (40)–(42), and thus we find

\[
C^{(0)}(x) = A \prod_{k=1}^{m} \sin \left( a_k \frac{\pi}{2} + \pi x_k \right),
\]

(44)

where \( a_k = 0 \) if the operator depending on the \( k \)th coordinate \( j_k \) is \( \sigma^x \) or \( \sigma^y \) (i.e. \( \alpha_k = x, y \)) and \( a_k = 1 \) if it is \( \sigma^z \) (or \( \alpha_k = z \)) and \( A \) is some yet to be determined constant.

Let us recap: we have used the expansion of the equation for the NESS (29) in lattice spacing \( a = 1/n \) and via equation (25) found the boundary conditions for the NESS of an open XXZ spin chain under boundary driving given by equation (28). It is important to note that this preceding discussion on the perturbation expansion in \( a \) (via equation (25)), which we used only to find the boundary conditions for the partial differential equations (32), is not necessary provided that we also know the action of the representation of the ASOs (which define the MPA for the NESS in equation (4)); i.e. we can use how the ASOs act on the boundary vectors \( |R\rangle|L\rangle \) to find the boundary conditions. This is in fact known for the maximally driven open XXZ spin chain [8], and some of the boundary conditions we found here in equations (40)–(42) were found in that article.

Furthermore, in order to find the scaling factor \( A \) we need to employ the boundary conditions given by the appropriate representation of the ASOs [8]. Doing this for fairly large system sizes we find the following behaviour. If we let \( p_x \) denote the number of \( \sigma^x \) and \( p_y \) the number of \( \sigma^y \) operators in the correlator, then \( A = 0 \) if \( p_x \) or \( p_y \) are odd; otherwise, \( A \) is given by the following recurrence relation (\( p := p_x + p_y \)) [51]:

\[
A(p, p_x - p_y) = \begin{cases} 
\frac{(2p)!}{(m!)^2} & \text{if } p_x - p_y = 0 \\
\frac{1}{(m!)^2} (4A(p - 1, p_x - p_y - 1) - A(p, p_x - p_y - 1)) & \text{if } p_x - p_y \neq 0
\end{cases}
\]

We have not been able to prove this in general; however, exact calculations for up to \( n = 20 \) support this conjecture. The initial terms \( A(p, 0) \) and \( A(p + 1, 0) \) can be calculated using the first line of the recurrence relation and, in turn, can be used to compute \( A(p + 1, 1), A(p + 1, 2), \ldots \) Also we take \( A(0, k) := 0 \).

In order to calculate with higher precision than \( \mathcal{O}(n^0) \) we shall again need to employ the boundary conditions given by the appropriate representation of the ASOs [8]. These corrections will be important when we want to look at the connected correlation functions. For higher precision in \( 1/n \) we shall likewise need a more precise scaling of the non-equilibrium partition function found in [4],

\[
\frac{Z_{n-1}^{(n)}}{Z_n^{(n)}} = \frac{\pi^2}{4(n - \alpha)^2} (1 + \mathcal{O}(n^{-2})).
\]

(45)

Interestingly, the constant \( \alpha \) can be found through self-consistency conditions imposed by the algebra (30) and the boundary conditions, as will be shown later. We find, using Taylor series expansion from equation (18) for equation (30), that the \( 1/n \) corrections
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$C^{(1)}$ are given by the following coupled partial differential equations (omitting $\Delta = 1$, setting $\beta = 4(1 - \alpha)$ and switching to a more compact PDE notation):

$$
C^{(1)}_{x_k x_k} + \pi^2 C^{(1)} = \sum_{p<k,j>k} \frac{\pi^2}{2} ((\beta - 4) C^{(0)}(x) + 2(1 - x) C^{(0)} - (1 - 2x_k) C^{(0)} + 2 C^{(0)}_{x_k x_k}).
$$

Likewise we may find the higher order corrections. The PDEs determining them become exceedingly complicated and we omit writing them explicitly.

Note that these correction terms actually depend on whether we expand the series in $1/n$ or $1/(n - 1)$ (i.e., the correction terms do not depend on whether we define $x_k$ as $x_k = j_k/n$ or $x_k = (j_k-1)/(n-1)$). The $k$th order corrections are then of either order in $1/n^k$ or $1/(n - 1)^k$. Naturally, the leading orders $1/n^0$ and $1/(n - 1)^0$ are equal, but the higher orders are not. This reflects the fact that the higher orders give increasingly precise corrections for large, but still finite, system size $n$ and it is merely a matter of convention in what we shall expand—they give equivalent and consistent information about the correlation functions.

We can now turn to computing the connected correlation functions. They are given by the standard generating function,

$$
\tilde{C}(\sigma_1^a \sigma_2^a \ldots \sigma_m^a) = \langle \sigma_1^a \sigma_2^a \ldots \sigma_m^a \rangle := \frac{\partial}{\partial z_1} \ldots \frac{\partial}{\partial z_m} \log \text{tr}_{\rho_{\infty}} \left( \exp \sum_j z_j \sigma_j^a \right)|_{z_j = 0}.
$$

We shall write out the two- and three-point connected correlators explicitly,

$$
\tilde{C}(\sigma_1^a \sigma_2^a \sigma_3^a) = \langle \sigma_1^a \sigma_2^a \sigma_3^a \rangle - \langle \sigma_1^a \sigma_2^a \rangle \langle \sigma_3^a \rangle - \langle \sigma_1^a \sigma_3^a \rangle \langle \sigma_2^a \rangle - \langle \sigma_2^a \sigma_3^a \rangle \langle \sigma_1^a \rangle + 2 \langle \sigma_1^a \rangle \langle \sigma_2^a \rangle \langle \sigma_3^a \rangle.
$$

The expressions for higher connected correlators are quite long so we omit writing them. Now using equation (47) and the previous results from equation (46) (as well as the higher order equations obtained from equation (18)) and the boundary conditions given by the matrix representation of the ASOs [4, 8] we can find arbitrary $m$-point connected correlators.

We shall show how to obtain some of the lower $m$-point connected correlators. We first note that, like the correlators (44), all the connected correlators which contain an odd number of $\sigma^x$ and $\sigma^y$ operators are zero.

First let us then start with the two-point connected correlators; $\tilde{C}(\sigma_j^x \sigma_j^y)$ was already found in [4, 8]. It was shown to scale inversely with the system size, $\sim 1/n$. Since the expectation values of the transverse operators $\langle \sigma_j^x \rangle = \langle \sigma_j^y \rangle = 0$ we immediately see that

$$
\tilde{C}^{(0)}(\sigma_j^x \sigma_j^y) = \frac{1}{2} \sin(\pi x_1) \sin(\pi x_2), \quad x_{1,2} = \frac{j_{1,2} - 1}{n - 1},
$$

where we have taken $x_{1,2} = \frac{j_{1,2} - 1}{n - 1}$ to conform with previous approaches [4, 8, 40]. One sees then that the transverse two-point connected correlators do not decay with system size.
size, making them truly long range in the sense of [41]. Since the basis of decoherence is determined by the dissipator equation (28) to be in the $z$-direction, this can be understood as a purely quantum effect.

Furthermore, when we find the $1/n$ correction we require also (as is given by the explicit representation of the ASOs [4, 8]) that the boundary conditions,

$$\frac{\partial \tilde{C}^{(1)}(\sigma^x_j \sigma^x_j)}{\partial x_1} \bigg|_{x_1=x_2=1} = 0,$$

are satisfied. This fixes uniquely that $\beta = 1$, or $\alpha = 3/4$ in equation (45). Thus,

$$\frac{8}{\pi} \tilde{C}^{(1)}(\sigma^x_j \sigma^x_j) = 2(1 - y_2) \cos(\pi y_2)[\pi y_1 \cos(\pi y_1) - \sin(\pi y_1)] - 2y_1 \cos(\pi y_1)$$

$$-\pi[y_1(y_1 - 1) + y_2(1 - y_2)] \sin(\pi y_1) \sin(\pi y_2),$$

where, as before, $x_{1,2} = \frac{j_{1,2} - 1}{n - 1}$ and $y_1 = \min(x_1, x_2)$, $y_2 = \max(x_1, x_2)$.

We find that the mixed transverse two-point correlator $\tilde{C}(\sigma^x_j \sigma^y_j) = C(\sigma^x_j \sigma^y_j)$ decays with system size as $1/n$ in leading order,

$$\tilde{C}^{(1)}(\sigma^x_j \sigma^y_j) = \frac{\pi}{\epsilon} \sin(\pi(x_1 - x_2)), \quad \tilde{C}^{(0)}(\sigma^x_j \sigma^y_j) = 0, \quad x_{1,2} = \frac{j_{1,2} - 1}{n - 1}. \quad (52)$$

The $m$-point connected correlators however become quickly more complicated for higher orders.

The simplest non-zero three-point connected correlation function is $\langle \sigma^x_j \sigma^x_j \sigma^x_{j'} \rangle$. It scales as $1/n$ and is given by

$$\tilde{C}^{(1)}(\sigma^x_j \sigma^x_j \sigma^x_{j'}) = \frac{1}{4} \pi^2(x_1 - 1) \cos(\pi y_2) \cos(\pi x_1) \cos(\pi y_2)$$

$$-\frac{1}{4} \pi(x_1 - 1) \cos(\pi y_2) [(\cos(\pi x_1) + \pi x_1 \sin(\pi x_1)) \sin(\pi y_2)]$$

$$-4 \cos(\pi y_2) [\pi^2 y_2 \cos(\pi x_1) - (2 + 3\pi^2)(-1 + y_2)] \sin(\pi x_1)]$$

$$+ \frac{1}{16\pi} \left[\{(8 - 5\pi - 2\pi^3)[(1 - y_2)x_1 + (1 - y_2)x_2 + (1 - y_2)x_3]\} \cos(\pi x_1)
+ 2\pi^2(1 - 2x_1) \sin(\pi y_2) \sin(\pi x_1) \sin(\pi y_2)\right] , \quad x_1 < y_2 < y_3, \quad (53)$$

$$\tilde{C}^{(1)}(\sigma^x_j \sigma^x_j \sigma^x_{j'})$$

$$= \frac{1}{16\pi} \left[2\pi \cos(\pi y_2)[-2\pi^2 x_1(-1 + y_3) \sin(\pi x_1) \sin(\pi y_2)
+ \cos(\pi x_1) \cos(\pi y_2)(2\pi^2 y_2(-1 + y_3) + (-16 + 3\pi - 2\pi y_3) \sin(\pi y_2)]
-4\pi^2 \cos(\pi y_2)(y_2 \cos(\pi x_1) + \pi x_1(-1 + y_2) \sin(\pi x_1))
-\{(8 - 5\pi + 2\pi^3)(-1 + x_1) + (1 - y_2)x_2 + (1 - y_2)x_3\} \cos(\pi x_1)
+ 2(1 + \pi^2(1 - 2x_1)) \sin(\pi x_1) \sin(\pi y_2) \sin(\pi y_3)\right], \quad y_2 < x_1 < y_3, \quad (54)$$

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\[ C^{(1)}(\sigma_{j_1}^{y} \sigma_{j_2}^{x} \sigma_{j_3}^{y}) = \frac{1}{4} \pi(1 - x_1) \cos(\pi y_2) [\pi x_1 \sin(\pi x_1)] \sin(\pi y_2) - \pi y_2 \cos(\pi x_1) \cos(\pi y_2) - (\cos(\pi x_1)) \]

\[ C^{(0)}(\sigma_{j_1}^{z} \sigma_{j_2}^{z} \sigma_{j_3}^{z}) = -\frac{3}{8} \sin(\pi x_1) \sin(\pi x_2) \sin(\pi x_3) \sin(\pi x_4), \quad x_k = j_k - \frac{1}{n-1}, \]

where \( x_k = \frac{j_k - 1}{n-1} \) and \( y_2 = \min(x_2, x_3), \quad y_3 = \max(x_2, x_3) \).

The simplest four-point function is \( C^{(1)}(\sigma_{j_1}^{y} \sigma_{j_2}^{x} \sigma_{j_3}^{y}) = C^{(0)}(\sigma_{j_1}^{z} \sigma_{j_2}^{z} \sigma_{j_3}^{z}) \) and is again long range in the sense that it does not decay with system size \( n \) in the leading order,

\[ C^{(l)}(\sigma_{j_1}^{x} \sigma_{j_2}^{y} \sigma_{j_3}^{x}) = \frac{1}{16\pi} [-4 \cos(\pi y_2) [\pi^2 y_2 \cos(\pi x_1)] - (2 + \pi^2 x_1(-1 + y_2)) \sin(\pi x_1)] \]

\[ - \{8 - 5\pi - 2\pi^3 [(1 + x_1)x_1 - (1 - y_2)y_2 + (1 - y_3)y_3] \} \cos(\pi x_1) \]

\[ -2\pi^2(1 + 2x_1) \sin(\pi x_1) \sin(\pi y_2) \sin(\pi y_3), \quad y_2 < y_3 < x_1, \tag{55} \]

The other correlation functions studied and plotted in figures 1 and 2 include

\[ \langle \sigma_{j_1}^{x} \sigma_{j_2}^{y} \sigma_{j_3}^{x} \rangle \propto 1/n^1, \quad \langle \sigma_{j_1}^{y} \sigma_{j_2}^{x} \sigma_{j_3}^{y} \rangle \propto 1/n^3 \quad \text{and} \quad \langle \sigma_{j_1}^{x} \sigma_{j_2}^{y} \sigma_{j_3}^{z} \rangle \propto 1/n. \]

Based on our results, although we were unable to prove this in general, we conjecture that, to leading order in \( 1/n \), \( m \)-point connected correlation functions containing only \( \sigma^z \) operators scale as \( 1/n^{m-1} \), and that all purely transverse \( m \)-point connected correlators, i.e. those containing only an even number of \( \sigma^x \) (\( \sigma^y \)), do not decay in the leading order, \( \sim (1/n)^{-1} \).

4.2. The easy-plane regime \( \Delta < 1 \)

For \( \Delta < 1 \) only the cubic relation for \( O_z \) is known [39, 46] and it is given by

\[ \kappa_0(\gamma, s)(O_0O_0O_z + O_zO_0O_0) + O_0O_zO_0 + \kappa_1(\gamma, s)O_zO_z + \kappa_2(\gamma, s)O_z = 0, \tag{57} \]

where

\[ \kappa_0(\gamma, s) = \frac{1}{2} - \cos(2\gamma), \]

\[ \kappa_1(\gamma, s) = 1 + \cos(2\gamma) + \cos(4\gamma) - 4 \cos(2\gamma s), \]

\[ \kappa_2(\gamma, s) = 12 \cos(2\gamma s) - 2 \cos(4\gamma) - 10 - 16 \cos(2\gamma) \sin^2(\gamma s) + (8 - 4 \cos(2\gamma s))s^2, \tag{58} \]

and \( \Delta = \cos(\gamma), \quad q = \exp(i\gamma), \quad [x]_q := (q^x - q^{-x})/(q - q^{-1}) \) and \( \varepsilon[s]_q = 4i \cos(\gamma s) \).

This leads to

\[ C^{(0)}(x)|_{\Delta < 1} = 0, \tag{59} \]

for correlators containing only \( \sigma^z \). This is incorrect for, e.g. two-point correlators when \( x_1 = x_2 \) since \( \sigma^z^2 = 1 \). This obviously means that there is a discontinuity and hence the assumption that we may expand in a Taylor series, which we made when deriving equation (18), is possibly not valid. In fact computing the correlators when the difference between \( x_1 \) and \( x_2 \) is of the order of the lattice spacing using the known discrete solution [8] shows that the trivial result (59) is not valid there. We cannot therefore make any claims near these points, but calculating these correlators for large but finite \( n \) shows that they are simply constant functions in \( x_k \). For finite distances among all the \( x_k \) and
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\[ n \langle \sigma(x_1)\sigma(x_2) \rangle_C \]

\[ n\langle \sigma(x_1)\sigma(x_{1+1})\sigma(x_2) \rangle_C \]

\[ n^2 \langle \sigma(x_1)\sigma(x_{2+1})\sigma(x_2) \rangle_C \]

\[ n^2\langle \sigma(x_1)\sigma(x_2)\sigma(x_{2+1}) \rangle_C \]

**Figure 1.** The two- and three-point connected correlation function for \( \Delta = 1 \) and \( \epsilon = 10 \).

\( x_k \) in the continuum limit (meaning that \( |j_k - j_{k'}| \rightarrow \infty \) as \( n \rightarrow \infty \)) the trivial result (59) holds. In other words, on the ‘large scale’ there is no non-smoothness. This is also the reason why the ‘large-scale’ results also hold for \( \Delta = 1 \) even though there \( \langle \sigma^0_i(\sigma^0_j) = 1, \) when \( i = j, \) as well.

Even though we could not find any similar cubic relations for \( O_k \) or \( O \), we find that correlators containing these operators behave like those with only \( \sigma^z; \) i.e. they are vanishing. In any case, the behaviour in this regime is quite trivial as all expectation values of non-local operators decay to zero exponentially fast, whereas the (ultra-)local expectation values are a constant function throughout the chain [4].

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5. Fluctuations of spin current and magnetization operators

Quantum mechanics is an intrinsically probabilistic theory; therefore, in order to achieve a deeper understanding of any physical system we are studying we need
to go beyond calculating merely the expectation values of observables we are interested in.

We shall study the probability distribution of the total magnetization in the maximally driven open XXZ spin chain, \( P(M := \sum_j \sigma_j^z) \).

We also study the fluctuations of the instantaneous spin current flowing through the system once the system has reached NESS. That is, avoiding certain ambiguities with defining current statistics \([36]\), we want to merely study simply the moments of the instantaneous spin current measured in the long time limit of the system (NESS), i.e. \( \langle j_k^m \rangle \).

The local spin current operator is defined via the continuity equation, \( \frac{\partial \sigma_k^z}{\partial t} = i[H, \sigma_k^z] = j_k - j_{k-1} \), and for the XXZ spin chain is
\[
j_k = 2i(\sigma_k^+\sigma_{k+1}^- - \sigma_k^-\sigma_{k+1}^+).
\]

However, we notice a problem immediately, namely \( j_k^m \propto j_k \) when \( m \) is odd and \( j_k^m \propto (1 - \sigma_k^z\sigma_{k+1}^z) \) when \( m \) is even, which makes studying this quite trivial.

A possible solution to this is that we may define a multiple site current operator averaged over \( K \) sites, similar to the one studied in \([27]\) and in \([52]\), \( J_k^{(K)} = \sum_{m=k}^{k+K-1} j_m / K \). For example, \( J_k^{(1)} = j_k \), \( J_k^{(2)} = (j_k + j_{k+1})/2 \), etc. The same type of ‘space-integrated’ current was previously studied in different settings in \([53, 54]\).

These operators physically correspond to the average flow of magnetization between site \( k \) and site \( K + k \), i.e.
\[
\frac{1}{K} \frac{\partial (\sigma_k^z + \sigma_{k+1}^z + \ldots + \sigma_{k+K}^z)}{\partial t} = J_k^{(K)} - J_{k+1}^{(K)}.
\]

Note that the expectation values \( \langle J_k^{(K)} \rangle = \langle J_k^{(K)} \rangle \) for all \( K_1, K_2 \), which follows from the continuity equation in the long time limit. However, the higher moments are not equal for different \( K_1 \) and \( K_2 \).

Also, these operators still have the property that with their \((K + 1)\)th power \( (J_k^{(K)})^{K+1} \propto J_k^{(K)} \), so we shall define an extensive (up to a prefactor of \( 1/(n - 1) \)) quantity \( J = J^{(n-1)} \), where \( n \) is the system size. That is,
\[
J = \sum_{k=1}^{n-1} \frac{j_k}{n - 1}.
\]

We also define a moment generating function for the total magnetization operator, \( M = \sum_{k=1}^n \sigma_k^z \) in the steady state,
\[
G_M(\chi) = \langle e^{i\chi M} \rangle = \text{tr}(e^{i\chi M} \rho_\infty).
\]

6. Fluctuations of spin current and magnetization in the maximally boundary driven XXZ spin chain

Using the results from section 4 we can study the fluctuations of the spin current and magnetization operators, as defined in section 5. Recall that we defined the averaged
spin current operator as $J = \sum_{k=1}^{n-1} \frac{j_k}{n-1}$. The ASO corresponding to the local spin current $W := i(O_x \sigma \cdot O_x)$ is proportional to $O_0 [8, 46]$,

$$W = -2i[s_j]_0 O_0,$$

where, as before, $\Delta = \cos(\gamma), q = \exp(i\gamma), [x_j] := (q^r - q^{-r})/(q - q^{-1})$ and $\varepsilon[s_j] = 4i \cos(\gamma s)$. The fact that $W \propto O_0$ guarantees the validity of the continuity equation for the magnetization $\partial \sigma^z_k / \partial t = i[H, \sigma^z_k] = j_k - j_{k-1}$, in the long time limit when the system has reached the NESS, i.e. $\partial(\sigma^z_k) / \partial t = 0 = \langle j_k \rangle - \langle j_{k-1} \rangle$ (in other words the expectation values of spin current $j_k$ are equal on all sites $k$).

We square (take the third power of) equation (62), and then using the definition equation (60) the properties of the Pauli matrices and the previously mentioned fact that the NESS expectation values $\langle j_k \rangle = \langle j_m \rangle$ for all $k, m$ we find (grouping equal terms together)

$$\langle J^2 \rangle = \frac{1}{(n-1)^2} \left( (n-2)(n-3)\langle j_{1,2} j_{3,4} \rangle + \frac{n-1}{2} - \sum_{k=1}^{n-1} \left( \langle \sigma^z_k \sigma^z_{k+1} \rangle + 2 \langle \sigma^z_k \sigma^z_{k+2} + \sigma^z_{k+1} \sigma^z_{k+3} \rangle \right) \right),$$

and

$$\langle J^3 \rangle = \frac{1}{(n-1)^3} \left\{ - (n-3)(n-4)(n-5)\langle j_{1,2} j_{3,4} j_{4,5} \rangle - 3(n-2)(n-3)\langle j_{1,2} \rangle 
+ i \sum_k (3 \langle \sigma^z_k \sigma^z_{k+3} - \sigma^z_{k+1} \sigma^z_{k+2} \rangle - \langle \sigma^z_k \sigma^z_{k+1} \sigma^z_{k+2} \rangle + \langle \sigma^z_k \sigma^z_{k+2} \rangle) + \sum_k (3(n-3)\langle j_{1,2} \rangle \langle \sigma^z_k \sigma^z_{k+1} \rangle - 12(n-4)\langle j_{1,2} \rangle \langle \sigma^z_k \sigma^z_{k+2} + \sigma^z_{k+1} \rangle) \right\}. $$

Let us first consider the isotropic case $\Delta = 1 (q \to 0)$. The second moment $\langle J^2 \rangle$ contains sums of expectation values of three different types of operator which are non-trivial in the thermodynamic limit $n \to \infty$: $\langle j_{1,2} j_{3,4} \rangle, \langle \sigma^z_k \sigma^z_{k+1} \rangle$ and $\langle \sigma^z_k \sigma^z_{k+2} + \sigma^z_{k+1} \sigma^z_{k+3} \rangle$. Using equation (64) and the asymptotic form of the non-equilibrium partition function equation (45), it is easy to show that $\langle j_{1,2} j_{3,4} \rangle \propto 1/n^2$ in the leading order.

Furthermore we note that the leading order of the other terms cancels exactly. In fact, using the next to leading order of $\langle \sigma^z_k \sigma^z_{k+1} \rangle$, which was shown in [4] to be (where, as before, $x_{1,2} = j_{1,2} - 1 / n$)

$$C^{(4)}(\sigma^z_j \sigma^z_{j+1}) = \frac{\pi}{4} \cos(\pi x_1) \left[ \pi \{ (-1 + x_1)x_1 + (-1 + x_2)x_2 \} \cos(\pi x_2) + (1 - 2x_2) \sin(\pi x_2) + \sin(\pi x_1) \{ (1 - 2x_1) \cos(\pi x_2) - 2\pi x_1(-1 + y_2) \sin(\pi x_2) \} \right],$$

together with equation (51), we see that the next-to-leading order $1/n$ cancels as well. However expanding around $x_2 = x_1 + 1/(n-1)$ for large $n$ we are left with a finite
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order of the continuum limit up to a scaling factor (for which a recursion relation was conjectured based on a known discrete solution [8]). We derived explicit expressions for certain (up to four-point) connected correlators and observed that the connected correlators of operators transversal to the basis of decoherence in the $z$-direction (i.e. tensor products of an even number of $\sigma^x$ and $\sigma^y$) exhibit long-range order—that is, they do not decay with system size in the leading order. Similar behaviour has been previously observed numerically for a related system [41].

Finally, we defined two statistical quantities of interest—fluctuations of the total spin current (studied previously in [53, 54] in different contexts and also related to a quantity studied in [27] and in [52]) and the magnetization operators. We computed the second and third moments of the spin current operator at the isotropic point $\Delta = 1$ and all the moments for $\Delta < 1$ when $\Delta$ can be expressed via rational multiples of $\pi$ as $\Delta = \cos(\pi n/m)$. At the isotropic point we found that for system size $n$ the second moment decays $\propto 1/n^3$ and the third moment $\propto 1/n^5$. For $\Delta < 1$ we find that none of the moments decay with the system size.

We need to stress that it is possible that the continuum limit is not well defined when the difference between the operators in the basis for the steady state is of the
order of the lattice spacing. In this article we have focused on the continuum limit of the previously mentioned open XXZ spin chain for which a discrete solution is known \[8\], and this allowed us to check the validity of the continuum limit.

Even though our treatment for the continuum limit is exact, it would be interesting to see whether one can find a solution for an open quantum non-equilibrium steady state directly for a continuum system (and thus avoiding knowing an exact solution for any finite, discrete system), either on the level of a quantum field theory or at least perturbatively in lattice spacing in the leading order (in the sense of equation \(25\)).

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