1. Introduction

Let $M$ be an oriented 4-dimensional Riemannian manifold. For each point of $M$, we can consider the complexification $T_p^\mathbb{C}M$ of the tangent space $T_pM$ at the point $p$. There are two natural extensions of the inner product on $T_p^\mathbb{C}M$ to the complexified tangent space $T_p^\mathbb{C}M$, namely, as a complex bilinear form $(\cdot, \cdot)$ or as a Hermitian inner product $\langle \cdot, \cdot \rangle$. These extensions are related by $\langle u, v \rangle = (u, \bar{v})$ for $u, v \in T_p^\mathbb{C}M$. The tensor curvature of $M$ induces the curvature operator $R : \Lambda^2 T_p^\mathbb{C}M \to \Lambda^2 T_p^\mathbb{C}M$ on the fiber bundle of 2-forms $\Lambda^2 M$. After complexification, it extends to complex vectors by linearity, and we can define the complex sectional curvature of a two-dimensional subspace $\pi$ of $T_p^\mathbb{C}M$ by

$$K(\pi) = \langle R(u \wedge v), \bar{v} \wedge u \rangle,$$

where $\{u, v\}$ is any unitary basis of $\pi$. A vector $u \in T_p^\mathbb{C}M$ is said isotropic if $(u, u) = 0$. While a subspace $E \subset T_p^\mathbb{C}M$ is said isotropic if every vector $u \in E$ is isotropic. Moreover, $M$ is said to have nonpositive isotropic curvature if $K(\pi) \leq 0$ for all isotropic two-dimensional subspace $\pi$ of $T_p^\mathbb{C}M$. The notions of nonnegative, negative and positive isotropic curvature can be defined in a similar way.

It is well known that for an oriented 4-dimensional Riemannian manifold $M^4$ the Hodge star operator splits the fiber bundle of 2-forms $\Lambda^2 M = \Lambda^+ \oplus \Lambda^-$, where $\Lambda^\pm = \{\omega \in \Lambda^2 M \mid \ast \omega = \pm \omega\}$ denote the $\pm$eigenspaces of that operator. At this context, the Weyl curvature tensor $W$ commutes with the Hodge star operator, and it is, therefore, an endomorphism of $\Lambda^2 M$, such that $W = W^+ \oplus W^-$. Using such a decomposition, we see that the non positivity of the isotropic curvature operator is equivalent to the non positivity of the endomorphisms $P^\pm = s I_{\Lambda^\pm} \pm W^\pm$, where $s$ stands for the scalar curvature of $M^4$.

In [6], Micallef and Wang proved that if $M$ is a compact Einstein 4-manifold with nonnegative isotropic curvature, then $M$ is locally symmetric. Recently, Brendle [1] proved that a compact Einstein manifold with nonnegative isotropic curvature must be a locally symmetric space of compact type. On the other hand, Seshadri [8] showed that any compact Riemannian 4-manifold $M$ admits a Riemannian metric

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with strictly negative isotropic curvature. Despite the result of Seshadri, we can propose the following questions:

**Question 1.1.** Is a compact Einstein 4-manifold with nonpositive isotropic curvature locally symmetric?

Since nonpositive curvature operator implies nonpositive isotropic curvature, we can propose the following weak version of Question 1.1.

**Question 1.2.** Is a compact Einstein 4-manifold with nonpositive curvature operator locally symmetric?

Very recently, Fine, Kasnov and Panov [5] have proposed a similar question. More precisely, they posed the following question.

**Question 1.3.** Do there exists compact Einstein 4−manifolds with scalar curvature $s < 0$, for which $\frac{1}{12}I + W^+$ is negative definite, besides hyperbolic and complex-hyperbolic 4−manifolds?

It is important to highlight that if an oriented 4−manifold has endomorphism $\frac{1}{12}I + W^+$ negative definite, then the endomorphism $P^+ = \frac{s}{6} - W^+$ is also negative definite.

There are some evidences in favor of a positive answers to Questions 1.1 and 1.2. More precisely, we have the following useful informations:

(a) The only known examples of compact Einstein 4−manifolds $M$ with negative Ricci curvature and nonpositive isotropic curvature are locally symmetric spaces whose universal covering of $M$ is isometric to either complex hyperbolic space $CH^2$, real hyperbolic space $H_4^c$ or a product of two hyperbolic spaces $H_2^c \times H_2^c$.

(b) 4−manifolds of nonpositive curvature operator or negative 1/4−pinched sectional curvature have nonpositive isotropic curvature.

(c) In [10], Ville proved that if a compact oriented Riemannian 4−manifold $M$ has negative 1/4−pinched sectional curvature, then the Euler characteristic $\chi$ and its signature $\tau$ satisfy $\chi \geq 3|\tau|$. Moreover, the equality occurs if and only if the universal covering of $M$ is isometric to $CH^2$.

(d) In [11], Zheng and Yau proved that if a compact Kähler-surface has negative 1/4−pinched sectional curvature, then the universal covering of $M$ is isometric to complex hyperbolic space $\mathbb{C}H^2$ with its standard metric.

(e) In [12], F. Zheng showed that if a compact Kähler-surface $M$ has nonpositive sectional curvature, then $M$ has signature $\tau \geq 0$. Moreover, $\tau = 0$ if and only if the universal covering of $M$ is isometric to $H_2^c \times H_2^c$.

Proceeding, it is well-known that a compact oriented Einstein 4−manifold satisfies

\begin{align*}
8\pi^2 \chi &= \int_M (|W|^2 + \frac{s^2}{24}) \, dV, \quad (1) \\
12\pi^2 \tau &= \int_M (|W^+|^2 - |W^-|^2) \, dV. \quad (2)
\end{align*}
These formulae tell us that a compact oriented Einstein 4-manifold must to satisfy the well-known Hitchin-Thorpe inequality:

\[ \chi \geq \frac{3|\tau|}{2}. \]

Our first result gives a similar obstruction to the existence of Einstein metrics with nonpositive isotropic curvature on 4-manifolds. More precisely, we have the following result.

**Theorem 1.1.** Let \( M \) be a compact oriented Einstein 4–manifold with negative Ricci curvature and nonpositive isotropic curvature. Then we have:

\[ (3) \quad \chi \geq \frac{15}{8} |\tau|. \]

In addition, if \( M \) is a Kähler-Einstein manifold, then

\[ (4) \quad \chi \geq 3|\tau|. \]

Moreover, the equality in (4) occurs if and only if the universal covering \( \tilde{M} \) of \( M \) is isometric to \( \mathbb{C}H^2 \).

**Theorem 1.2.** Let \( M \) be a compact oriented Einstein 4–manifold with negative Ricci curvature and volume \( V \).

1. If \( M \) has nonpositive curvature operator, then \( \chi \geq 3|\tau| \). Moreover, the equality occurs if and only if the universal covering \( \tilde{M} \) of \( M \) is isometric to \( \mathbb{C}H^2 \).

2. If \( M \) has nonpositive curvature operator, then \( \chi \leq \frac{\rho^2V}{8\pi^2} \). Moreover, the equality occurs if and only if the universal covering \( \tilde{M} \) of \( M \) is isometric to \( \mathbb{H}^2 \times \mathbb{H}^2 \).

3. If the sectional curvature \( K \) of \( M \) satisfies \( \sup K \leq \frac{\rho}{6} \), then \( \chi \leq \frac{\rho^2V}{12\pi^2} \). Furthermore, the equality occurs if and only if the universal covering \( \tilde{M} \) of \( M \) is isometric to \( \mathbb{C}H^2 \).

2. **Proof of Theorem 1.1**

The first statement of Theorem 1.1 is a straightforward consequence of the following lemma.

**Lemma 2.1.** Let \( M \) be a compact oriented Einstein 4–manifold with negative Ricci curvature. If \( M \) has nonpositive isotropic curvature, then

\[ 2\chi + 3|\tau| \leq \frac{3\rho^2V}{2\pi^2} \leq 18\chi - 27|\tau|. \]

**Proof:** We follow the ideas developed in [2]. Indeed, for each \( x \in M \) there exist an orthonormal basis of \( T_xM \) and a corresponding orthonormal basis of \( \Lambda^\pm \), such that \( \mathcal{W}^\pm \) has the respective eigenvalues

\[ \lambda_1 \pm \mu_1 - \rho/3 \leq \lambda_2 \pm \mu_2 - \rho/3 \leq \lambda_3 \pm \mu_3 - \rho/3, \]

where \( \lambda_1, \lambda_2, \) and \( \lambda_3 \) are the principal sectional curvatures of \( M \). Hence, \( \lambda_1 + \lambda_2 + \lambda_3 = \rho \) and \( \mu_1 + \mu_2 + \mu_3 = 0 \). It will be convenient to see \( \lambda = (\lambda_1, \lambda_2, \lambda_3) \) and \( \mu = (\mu_1, \mu_2, \mu_3) \) as vectors in \( \mathbb{R}^3 \) with canonical inner product \( \langle , \rangle \) and its canonical norm \( |\cdot| \).
Using these settings, (1) and (2) become
\[ 4\pi^2 \chi = \int_M (|\lambda|^2 + |\mu|^2) \, dV, \]
and
\[ 3\pi^2 \tau = \int_M \langle \lambda, \mu \rangle \, dV. \]

We point out that the non positivity of the isotropic curvature of \( M \):
\[ R_{1313} + R_{1414} + R_{2323} + R_{2424} + 2R_{1234} \leq 0 \]
is equivalent to
\[ \lambda_1 \pm \mu_1 - \rho \geq 0. \]
Then, by setting \( \alpha_\pm = \lambda_i \pm \mu_i - \rho \), we have
\[ 0 \leq \alpha_1^\pm \leq \alpha_2^\pm \leq \alpha_3^\pm. \]
Now, we notice that (7) implies
\[ 4\rho^2 = (\alpha_1^\pm + \alpha_2^\pm + \alpha_3^\pm)^2 \geq (\alpha_1^\pm)^2 + (\alpha_2^\pm)^2 + (\alpha_3^\pm)^2. \]
But
\[ (\alpha_1^\pm)^2 + (\alpha_2^\pm)^2 + (\alpha_3^\pm)^2 = |\lambda|^2 + |\mu|^2 \pm 2\langle \lambda, \mu \rangle + \rho^2. \]
From here it follows that
\[ |\lambda|^2 + |\mu|^2 \pm 2\langle \lambda, \mu \rangle \leq 3\rho^2. \]
From (5) and (6) we deduce
\[ 2\chi + 3|\tau| \leq \frac{3\rho^2 V}{2\pi^2}. \]
On the other hand, using once more (5) and (6) we arrive at
\[ 2\pi^2 (2\chi \pm 3\tau) = \int_M \left( |\lambda|^2 + |\mu|^2 \pm 2\langle \lambda, \mu \rangle \right) \, dV \]
\[ = \int_M |\lambda \pm \mu|^2 \, dV \geq \frac{\rho^2 V}{3}, \]
so that
\[ 2\pi^2 (2\chi - 3|\tau|) \geq \frac{\rho^2 V}{3}. \]
Finally, by combining (8) and (9) we get
\[ 2\chi + 3|\tau| \leq \frac{3\rho^2 V}{2\pi^2} \leq 18\chi - 27|\tau|. \]
This concludes the proof of lemma.

In order to prove the last statement of Theorem 1.1, we consider \( M \) to be a Kähler-Einstein manifold with negative Ricci curvature \( \rho \). In this case, the eigenvalues of \( W^+ \) are \( 2\rho/3, -\rho/3 \) and \( -\rho/3 \). Thus, we can use (1) and (2) to infer
\[ 8\pi^2 \chi = \int_M (|W^+|^2 + 4\rho^2/3) \, dV \]
and
\[ 12\pi^2 \tau = \int_M (2\rho^2/3 - |W^+|^2) \, dV. \]
These two above inequalities gives
\[8\pi^2(\chi - 3\tau) = 3 \int_M |W^-|^2 \, dV.\]
So, we conclude that \(\chi \geq 3\tau\).

On the other hand, since that \(M\) has nonpositive isotropic curvature, we deduce that the eigenvalues of \(W^-\) satisfy \(\gamma_3 \geq \gamma_2 \geq \gamma_1 \geq 2\rho/3\) and \(\gamma_1 + \gamma_2 + \gamma_3 = 0\).
From here it follows that \(0 \leq \gamma_3 \leq -4\rho/3\) and \(2\rho/3 \leq \gamma_1 \leq 0\). With this setting, a simple computation yields
\[|W^-|^2 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 \leq 8\rho^2/3.\]
Sum (10) and (11), and then we plug (12) to obtain \(\chi \geq -3\tau\). Hence, \(\chi \geq 3|\tau|\).

3. PROOF OF THEOREM 1.2

The first and second assertions follow directly from the following Lemma 3.1.

**Lemma 3.1.** Let \(M\) be a compact oriented Einstein 4–manifold with negative Ricci curvature. If \(M\) has nonpositive curvature operator, then
\[2\chi + 3|\tau| \leq \frac{\rho^2V}{2\pi^2} \leq 6\chi - 9|\tau|.\]

**Proof:** First of all, observe that the eigenvalues of the curvature operator of \(M\) are \(\lambda_i \pm \mu_i\), for \(i = 1, 2, 3\). Hence, the non positivity of the curvature operator implies that \(\lambda_i \pm \mu_i \leq 0\). Using this data, we have
\[
\rho^2 = \left(\sum_i (\lambda_i \pm \mu_i)^2\right)^2 = |\lambda \pm \mu|^2 + 2\sum_{i<j} (\lambda_i \pm \mu_i)(\lambda_j \pm \mu_j)
\geq |\lambda \pm \mu|^2 = |\lambda|^2 + |\mu|^2 \pm 2\langle \lambda, \mu \rangle,
\]
Thus
\[|\lambda|^2 + |\mu|^2 \pm 2\langle \lambda, \mu \rangle \leq \rho^2.\]
Upon integrating (13) we use (10) and (9) to arrive at
\[2\chi + 3|\tau| \leq \frac{\rho^2V}{2\pi^2}.\]

Clearly, the combination of (14) and (9) yields
\[2\chi + 3|\tau| \leq \frac{\rho^2V}{2\pi^2} \leq 6\chi - 9|\tau|.\]
This concludes the proof of Lemma 3.1.

From now on we shall characterize the equality in the first and second assertions of Theorem 1.2. Initially, we notice that if \(\chi = 3|\tau|\), then the universal covering of \(M\) is isometric to \(\mathbb{C}\mathbb{H}^2\). Next, if \(\chi = \frac{\rho^2V}{4\pi^2}\), then we have from (14) that \(\tau = 0\). This implies that \(\lambda_i \pm \mu_i = 0\), for \(i = 2, 3\). So, the eigenvalues of the curvature operator of \(M\) are constant and \(M\) is locally symmetric. Therefore, \(W^+ \neq 0\) and \(\tau = 0\), which tell us that \(\tilde{M}\) is isometric to \(\mathbb{H}^2 \times \mathbb{H}^2\).

In order to proof the third item of Theorem 1.2 we suppose that the sectional curvature \(K\) of \(M\) satisfies \(\sup K \leq \rho/6\). We then use the inequality (2.8) in [2] to
infer $\chi \leq \frac{e^{3/4}}{6\pi^2}$. Moreover, if $\chi = \frac{e^{3/4}}{6\pi^2}$, then is locally symmetric. Therefore, $\widetilde{M}$ is isometric to $\mathbb{C}P^2$, as we wanted to prove.

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