Pattern formation for colloidal systems

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Abstract
In this paper we study pattern formation for a local/nonlocal interaction functional where the local attractive term is given by the 1-perimeter and the nonlocal repulsive term is the Yukawa (or screened Coulomb) potential. This model is physically interesting since close to those used in simulations to model pattern formation in colloidal systems [2, 3, 7, 5]. Following a strategy introduced in [9] we prove that in a suitable regime minimizers are periodic stripes, in any space dimension. Doing so, we are able to prove pattern formation for a physical continuous model which is symmetric w.r.t. permutation of coordinates.

1 Introduction
The interaction between local attractive and nonlocal repulsive forces is often responsible for pattern formation.

The ability of block copolymers to spontaneously organize themselves in periodic patterns has for example important applications in the production of nanosized micelles for drug delivery in the human body or in the production of nanometer memory cells. The most famous and studied model for diblock copolymers is due to Ohta and Kawasaki [28]. For the sharp interface model and under the assumption of volume fraction 1/2, the formation of periodic stripes is conjectured, but the available results are still far from a full characterization of minimizers (see among others [1], [30], [25] and for numerical simulations [29, 8]).

Another important instance of spontaneous pattern formation at a mesoscopic scale is that showed by certain suspensions of charged colloids and polymers and also by protein solutions, when the attractive and repulsive forces compete at some strength ratio [31]. In particular, one can observe gathering of the particles in lamellas (stripes) or bubbles (clusters) according to the different regimes between the two mutual interactions. These self-assembly processes play a crucial role in applications such as therapeutic monoclonal antibodies, nanolithography or gelation processes.

For colloidal systems, the long-range repulsive forces have been shown on theoretical grounds to be represented by a Yukawa (or screened Coulomb) potential [10, 32] (the so-called DLVO Theory). The kernel of the Yukawa potential is the following

\[ K(\zeta) := \frac{e^{-\mu|\zeta|}}{|\zeta|^{d-2}}, \quad \mu > 0 \]  

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if $d \geq 3$, with the denominator substituted by minus a logarithm in dimension two.
The Yukawa potential was introduced in the 30s by Yukawa in particle physics \[33\]. Other than for electrolytes and colloids, it is used also in plasma physics, where it represents the potential of a charged particle in a weakly nonideal plasma, in solid state physics, where it describes the effects of a charged particle in a sea of conduction electrons, and in quantum mechanics.

Pattern formation in models for colloid particles involving the Yukawa potential as repulsive term has been numerically studied in several papers (see e.g. \[2, 3, 4, 5, 7, 11, 12, 19\]) and lamellar (striped) phases has been observed in suitable regimes.

In particular, some of these models (see e.g. \[4, 19\]) use as short range attractive term the Yukawa potential with opposite sign and parameter $\mu$ much larger than the one appearing in the repulsive Yukawa. Putting in front of the short range attracting Yukawa a suitable rescaling, such models become close to the following model: for $d \geq 1$, $L > 0$, $E \subset \mathbb{R}^d \times [0, L]^d$-periodic and $J > 0$, consider

$$\tilde{F}_{J,L}(E) := \frac{1}{L^d} \left( J \text{Per}_{1}(E, [0, L]^d) - \int_{[0,L]^d} \int_{\mathbb{R}^d} |\chi_E(x + \zeta) - \chi_E(x)| K(\zeta) \, d\zeta \, dx \right), \quad (1.2)$$

where

$$\text{Per}_{1}(E, [0, L]^d) := \int_{\partial E \cap [0,L]^d} \nu^E(x)|_1 \, d\mathcal{H}^{d-1}(x), \quad |z|_1 = \sum_{i=1}^d |z_i|,$$

with $\nu^E(x)$ exterior normal to $E$ in $x$, is the 1-perimeter of $E$, and $K$ is the Yukawa-type potential

$$K(\zeta) := e^{-|\zeta|_1} \frac{|\zeta|_1}{|\zeta|_1^{d-2}}, \quad (1.3)$$

namely the potential obtained substituting the Euclidean norm in \[1.1\] with the 1-norm. Notice that, since we assume periodicity of the sets w.r.t. $[0, L]^d$, the choice of the norm does not reduce the underlying symmetries of the problem, namely those w.r.t. permutation of coordinates.

A functional of this kind (namely with the perimeter as attractive term and the Yukawa as repulsive term), but in a different regime, appears also in a series of papers \[26, 20, 21\] in connection with the sharp interface limit of the Ohta-Kawasaki model for small volume fractions.

The potential \[1.3\] behaves, for short range interactions, like the Coulomb potential and, for long range interactions, like a strong decaying potential, analogously to the generalized anti-ferromagnetic potentials considered in \[9, 22, 18\].

Our aim in this paper is to characterize minimizers of \[1.2\] for a suitable range of $J$.

The main difficulty in showing pattern formation lies in the fact that the functional exhibits a larger group of symmetries than the expected minimizers.

For one-dimensional models, where the symmetry breaking does not occur, pattern formation has been proved among others in \[27, 16\], using either convexity methods or reflection positivity techniques.

In more space dimensions, in the discrete setting, pattern formation was shown in \[18\] for the kernel $\tilde{K}(\zeta) = \frac{1}{(|\zeta|_1 + 1)^p}$ and $p > 2d$.

In the continuous setting, pattern formation was shown for the first time for a functional which is invariant under permutation of coordinates in \[9\]. There the kernel $\tilde{K}$ can have any exponent $p \geq d + 2$. 

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In [9] we proved that, for a sufficiently small range of $J$ below the critical constant $J_c$ above which minimizers become trivial, minimizers of the local/nonlocal interaction functional are periodic unions of stripes. Recall that a union of stripes in the continuous setting is a $[0, L]^d$-periodic set which is, up to Lebesgue null sets, of the form $V_i^+ + \hat{E}e_i$ for some $i \in \{1, \ldots, d\}$, where $V_i^+$ is the $(d-1)$-dimensional subspace orthogonal to $e_i$ and $\hat{E} \subset \mathbb{R}$ with $\hat{E} \cap [0, L) = \bigcup_{k=0}^{N-1} (s_i, t_i)$. A union of stripes is periodic if $\exists h > 0, \nu \in \mathbb{R}$ s.t. $\hat{E} \cap [0, L) = \bigcup_{k=0}^{N} (2kh + \nu, (2k+1)h + \nu)$.

In this paper we prove an analogous result for the functional (1.2).

While for the power-like potential $\tilde{K}$ the physical exponents $p = d + 1$ (3D-micromagnetics) and $p = d - 2$ (Coulomb potential) remain excluded by the results in [9], here we are able to prove pattern formation for a physical model.

Let us now state our results precisely. First of all, there exists also in this case a critical constant $\tilde{J}_\infty$ such that if $J > \tilde{J}_\infty$ then the minimal among these values as

$$\tilde{J}_\infty := \int_{\mathbb{R}^d} |\zeta_1| K(\zeta) \, d\zeta.$$ 

What one expects is that for values of $J$ strictly below $\tilde{J}_\infty$ minimizers are periodic unions of stripes of optimal period.

Therefore one considers the functional

$$\tilde{\mathcal{F}}_{J_M, L}(E) = \frac{1}{L^d} \left( \int_{\mathbb{R}^{d-1}} \int_{-M}^{M} |\zeta_1| K(\zeta) \, d\zeta \Per (E; \{0, L\}^d) - \int_{\{0, L\}^d} \int_{\mathbb{R}^d} |\chi_E(x + \zeta) - \chi_E(x)| K(\zeta) \, d\zeta \, dx \right).$$

One can see that minimizing (1.4) in the class of periodic unions of stripes, for $1 \ll M \ll L$, those with optimal energy have width and distance of order $h_M \leq M, \frac{\hat{E}M}{M} \to 1$ as $M \to +\infty$ and energy of order $e_M^* \geq -e^{-\alpha_M M}$, with $\alpha_M \leq 1, \alpha_M \to 1$ as $M \to +\infty$.

Therefore it is natural to rescale the spacial variables and the functional so that the optimal width and distance for unions of stripes is $O(1)$ and the energy is $O(1)$.

Setting $M \zeta' = \zeta, M x' = x$, and $\tilde{\mathcal{F}}_{J_M, L}(E) = -e_M^* \tilde{\mathcal{F}}_{M, L/M}(E/M)$ one ends up considering the rescaled functional

$$\mathcal{F}_{M, L}(E) = \frac{M^2}{L^d} \left( J_M \Per (E; \{0, L\}^d) - \int_{\{0, L\}^d} \int_{\mathbb{R}^d} |\chi_E(x + \zeta) - \chi_E(x)| K_M(\zeta) \, d\zeta \, dx \right)$$

where

$$K_M(\zeta) = \frac{-1}{e_M^* |\zeta|^{d-2}} e^{-M|\zeta|}.$$

and

$$J_M = \int_{\mathbb{R}^{d-1}} \int_{-1}^{1} K_M(\zeta)|\zeta_1| \, d\zeta.$$

For fixed $M > 0$, consider first for all $L > 0$ the minimal value obtained by $\mathcal{F}_{M, L}$ on $[0, L]^d$-periodic unions of stripes and then the minimal among these values as $L$ varies in $(0, +\infty)$. We will denote this value by $e_M^*$.

By the reflection positivity technique (see Section [3]), this value is attained on periodic stripes of width and distance $h_M^* > 0$, which is unique provided $M$ is large enough (see Theorem 3.1).

Our main theorem is the following:
**Theorem 1.1.** There exists a constant $M_0$ such that for every $M > M_0$ and $L = 2kh_M^*$ for some $k \in \mathbb{N}$, then the minimizer of $F_{M,L}$ are optimal stripes of width and distance $h_M^*$.

In Theorem 1.1 notice that $M_0$ is independent of $L$.

Notice that the $[0, L)^d$-periodic boundary conditions were imposed in order to give sense to the functional which is otherwise not well-defined. If one is interested to show that optimal periodic stripes of width and distance $h_M^*$ are "optimal" if one varies also the periodicity, then it is not difficult to see that Theorem 1.3 is sufficient. This corresponds to the "thermodynamic limit" and is relevant in physics.

Reading the proof of Theorem 1.1 one can see that, as in [9], one can provide a characterization of minimizers of $F_{M,L}$ also for arbitrary $L$, but this time with $M$ larger than a constant depending on $L$. Namely, one has the following

**Theorem 1.2.** Let $L > 0$. Then there exists $\bar{M} > 0$ such that $\forall M \geq \bar{M}$ there exists $h_{M,L}$ such that the minimizers of $F_{M,L}$ are periodic stripes of width and distance $h_{M,L}$.

According to the next theorem, when $L$ is large then $h_{M,L}$ is close to $h_M^*$.

**Theorem 1.3.** There exists $C > 0$ and $\hat{M} > 0$ such that for every $M > \hat{M}$ the width and distance $h_{M,L}$ of minimizers of $F_{M,L}$ satisfies

$$|h_{M,L} - h_M^*| \leq \frac{C}{L}.$$ 

In this paper we will focus on the proof of Theorem 1.1, stronger w.r.t. the independence of $M_0$ on $L$, referring for Theorem 1.2 to the strategy adopted in Sections 3 and 6 of [9] and to the novelties presented in this paper in the proof of Theorem 1.1.

The general strategy of the proof is similar to that adopted in [9] and consists in the following steps: decomposition of the functional in terms which penalize deviations from being a stripe ([22, 9]); decomposition of $\mathbb{R}^d$ in "good" and "bad" regions according to how much in a region the set $E$ "resembles" a stripe ([18, 9]); rigidity estimate to prove that in the limit $M \to +\infty$ minimizers approach a striped structure ([22, 9]); stability estimates to prove that once close to a stripe the most convenient thing is to be flat ([9]); use of the reflection positivity technique.

However, the rigidity estimate (see Lemma 4.1) and the stability lemma (see Lemma 5.2) base on the specific properties of the Yukawa kernel and are therefore quite different.

**1.1 Structure of the paper**

This paper is organized as follows: in Section 2, after setting the notation, we estimate the energy and width of optimal stripes and we rescale the functional accordingly. Then, we identify as in [9] the suitable quantities that penalize deviations from being a union of stripes. In Section 3 we show that in the interesting regime the width of minimizers is uniquely determined and minimizers are periodic. Section 4 contains the main rigidity estimate and Section 5 the proof of Theorem 1.1.

**2 Setting and preliminary results**

In this section, we set the notation and we introduce some preliminary results in the spirit of those given in [22] and [9], which will be necessary to carry on our analysis.
2.1 Notation and preliminary definitions

In the following, we let \( \mathbb{N} = \{ 1, 2, \ldots \} \), \( d \geq 1 \). On \( \mathbb{R}^d \), we let \( \langle \cdot, \cdot \rangle \) be the Euclidean scalar product and \( |\cdot| \) be the Euclidean norm. We let \( (e_1, \ldots, e_d) \) be the canonical basis in \( \mathbb{R}^d \) and for \( y \in \mathbb{R}^d \) we let
\[
y_i = (y, e_i) e_i \quad \text{and} \quad y_i^\perp := y - y_i.
\]
For \( y \in \mathbb{R}^d \), let \( |y|_1 = \sum_{i=1}^d |y_i| \) be its 1-norm and \( |y|_\infty = \max_i |y_i| \) its \( \infty \)-norm.

Given a measurable set \( A \subset \mathbb{R}^d \), let us denote by \( \mathcal{H}^{d-1}(A) \) its \( (d-1) \)-dimensional Hausdorff measure and \( |A| \) its Lebesgue measure. Moreover, let \( \chi_A : \mathbb{R}^d \to \mathbb{R} \) the function defined by
\[
\chi_A(x) = \begin{cases} 
1 & \text{if } x \in A \\
0 & \text{if } x \in \mathbb{R}^d \setminus A
\end{cases}
\]
and by \( 1_A : \mathbb{R}^d \to \mathbb{R} \cup \{ +\infty \} \) the function
\[
1_A(x) = \begin{cases} 
+\infty & \text{if } x \in A \\
0 & \text{if } x \in \mathbb{R}^d \setminus A
\end{cases}
\]

A set \( E \subset \mathbb{R}^d \) is of (locally) finite perimeter if the distributional derivative of \( \chi_E \) is a (locally) finite measure. We let \( \partial E \) be the reduced boundary of \( E \). We call \( \nu_E \) the exterior normal to \( E \). (see [?]).

The first term of our functional is, up to a constant, the 1-perimeter of a set relative to \([0, L]^d\), namely
\[
\text{Per}_1(E, [0, L]^d) := \int_{\partial E \cap [0, L]^d} |\nu_E(x)|_1 \, d\mathcal{H}^{d-1}(x)
\]
and, for \( i \in \{ 1, \ldots, d \} \)
\[
\text{Per}_{1i}(E, [0, L]^d) = \int_{\partial E \cap [0, L]^d} |\nu_i^E(x)| \, d\mathcal{H}^{d-1}(x),
\]
thus \( \text{Per}_1(E, [0, L]^d) = \sum_{i=1}^d \text{Per}_{1i}(E, [0, L]^d) \). Notice that in the definition of \( \text{Per}_1 \) the norm applied to the exterior normal \( \nu_E \) is the 1-norm, instead of the Euclidean norm used to define the standard perimeter.

Because of periodicity, w.l.o.g. we always assume that \( |D\chi_E|(\partial [0, L]^d) = 0 \), being \( \chi_E \) the characteristic function of \( E \) and \( D\chi_E \) its distributional derivative.

When \( d = 1 \) one can define
\[
\text{Per}_1(E, [0, L]) = \text{Per}(E, [0, L]) = \#(\partial E \cap [0, L]),
\]
where \( \partial E \) is the reduced boundary of \( E \).

While writing slicing formulas, with a slight abuse of notation we will sometimes identify \( x_i \in [0, L]^d \) with its coordinate in \( \mathbb{R} \) w.r.t. \( e_i \) and \( \{ x_i^\perp : x \in [0, L]^d \} \) with \( [0, L]^{d-1} \subset \mathbb{R}^{d-1} \).

In Section [4] we will have to apply slicing on small cubes around a point. Therefore we need to introduce the following notation. For \( z \in [0, L]^d \) and \( r > 0 \), we define \( Q_r(z) = \{ x \in \mathbb{R}^d : |x - z|_\infty \leq r \} \). For \( r > 0 \) and \( x_i^\perp \) we let \( Q_r^+(x_i^\perp) = \{ z_i^\perp : |x_i^\perp - z_i^\perp|_\infty \leq r \} \) or we think of \( x_i^\perp \in [0, L]^{d-1} \) and \( Q_r^+(x_i^\perp) \) as a subset of \( \mathbb{R}^{d-1} \). Since the subscript \( i \) will be always present in the centre (namely \( x_i^\perp \)) of such \((d-1)\)-dimensional cube, the implicit dependence on \( i \) of \( Q_r^+(x_i^\perp) \) should be clear. We denote also by \( Q_r^+(t_i) \subset \mathbb{R} \) the interval of length \( r \) centred in \( t_i \).
Now, let us turn to the elements defining the nonlocal term in (1.2). The kernel is the so called Yukawa kernel
\[ K(\zeta) = \frac{e^{-|\zeta|_1}}{|\zeta|_{d-2}^d}, \quad \zeta \in \mathbb{R}^d, \] (2.4)
up to considering the 1-norm in the exponential instead of the Euclidean norm.

As commented in the Introduction, such a kernel is both physical and reflection positive, namely it satisfies the following property (see property (2.6) in [9]): the function
\[ \hat{K}(t) := \int_{\mathbb{R}^d-1} K(t, \zeta_2, \ldots, \zeta_d) d\zeta_2 \cdots d\zeta_d. \]
is the Laplace transform of a nonnegative function (see Lemma 3.2).

Notice that
\[ \hat{K}(t) = e^{-|t|}C(t), \] (2.5)
where \(0 < C(t) \leq \hat{C}\) and \(C(t) \to 0\) as \(|t| \to +\infty\).

As in [18, 22, 9], there exists a critical constant \(\tilde{J}_\infty\) such that if \(J > \tilde{J}_\infty\), the functional \(\tilde{F}\) is nonnegative and therefore has trivial minimizers. Such a constant is given by
\[ \tilde{J}_\infty := \int_{\mathbb{R}^d} |\zeta_1|K(\zeta) d\zeta. \]
A proof of this fact is analogous to [22][Proposition 3.5], so we omit it.

Letting, for \(M \geq 0\),
\[ \tilde{J}_M := \int_{\mathbb{R}^d-1} \int_{-M}^{M} |\zeta_1|K(\zeta) d\zeta = \int_{-M}^{M} |\zeta_1|\hat{K}(\zeta_1) d\zeta_1 \]
and using the fact that, for all \(J < \tilde{J}_\infty\), \(J = \tilde{J}_M\) for some \(M \geq 0\), we come to the functional
\[ \tilde{F}_{\tilde{J}_M, L}(E) = \frac{1}{L^d} \left( \text{Per}_1(E, [0, L]^d) \int_{\mathbb{R}^d-1} \int_{-M}^{M} |\zeta_1|K(\zeta) d\zeta - \int_{[0, L]^d} \int_{R^d} |\chi_E(x + \zeta) - \chi_E(x)|K(\zeta) d\zeta dx \right). \] (2.6)

### 2.2 Energy and width of optimal stripes

We are interested in showing pattern formation for \(J = \tilde{J}_M\), with \(M\) large but finite. Since we expect the width of the stripes to become larger and larger and the value of the functional to approach 0 as \(M\) tends to \(+\infty\), it is convenient to find the optimal energy and width among all the \([0, L]^d\)-periodic stripes so as to find suitable rescaling parameters.

Let
\[ E_h = \bigcup_{j=1}^{N} ((2j - 1)h, 2jh) \times \mathbb{R}^{d-1} \]
be a periodic union of stripes of width and distance \(h\) (in particular, \(L = Nh\)) and assume that
\[ 1 \ll M \ll L, \]
The energy of $E_h$, that we denote by $e_M(h)$, is given by the following formula:

\[ e_M(h) = \frac{2}{L^d} \left[ L^{d-1} \frac{L}{h} \int_0^M \zeta_1 \tilde{K}(\zeta_1) \, d\zeta_1 - L^{d-1} \frac{L}{h} \int_0^{+\infty} \min\{h, \zeta_1\} \tilde{K}(\zeta_1) \, d\zeta_1 \right]. \quad (2.7) \]

In particular, if $h < M$ then

\[ e_M(h) = \frac{1}{h} \int_h^M \zeta_1 \tilde{K}(\zeta_1) \, d\zeta_1 - \frac{1}{h} \int_h^{+\infty} h \tilde{K}(\zeta_1) \, d\zeta_1, \]

while if $h > M$

\[ e_M(h) = -\frac{1}{h} \int_M^{+\infty} \min\{h, \zeta_1\} \tilde{K}(\zeta_1) \, d\zeta_1. \]

It is not difficult to check that

\[ e_M(h) > e_M(M), \quad \forall h > M. \]

Moreover, let us fix $0 \leq C < 1$ and assume $h < CM$. Then, there exists $M_0 > 0$ such that if $M > M_0$ then $e_M(h) > 0$. Therefore, if $h_{M,L}$ is an optimal width for some $M$ and $L$, then

\[ \frac{h_{M,L}}{M} \leq 1, \quad \frac{h_{M,L}}{M} \to 1 \quad \text{as} \quad M \to +\infty. \]

As a consequence, the following holds.

\[ e_M(h) \sim -e^{-\alpha_M M} \quad (2.8) \]

for some $\alpha_M \leq 1$, $\alpha_M \to 1$ as $M \to +\infty$.

### 2.3 Rescaling

Let us denote by $h_{M,L}$ an optimal width and distance and by $e_M^*$ the minimal energy for the functional $\mathcal{F}_{\hat{j}_{M,L}}$.

As we already saw

\[ M/2 \leq h_{M,L} \leq M \quad \text{and} \quad e_M^* \geq -e^{-\alpha_M M} \]

where $\alpha_M \to 1$ and $h_{M,L}/M \to 1$ as $M \to +\infty$ for $1 << M << L$.

In this section we will rescale the spatial variables and the functional so that the optimal width and distance for unions of stripes is $O(1)$ and the energy is $O(1)$.

By the change of variables $M\zeta' = \zeta$, $Mx' = x$ we have that

\[ \text{Per}_1(E/M, [0, L/M]^d) = M^{1-d} \text{Per}_1(E, [0, L]^d), \]

\[ J_M := -\int_{\mathbb{R}^{d-1}} \int_1^{-1} \frac{\zeta_1'}{e_M^* |\zeta'|^{d-2}} \exp(-M |\zeta'|) \, d\zeta' = -e_M^*^{-1} M^{-3} \int_{\mathbb{R}^{d-1}} \int_{-M}^{M} |\zeta_1| K(\zeta) \, d\zeta = -e_M^*^{-1} M^{-3} \tilde{J}_M \]

and

\[ \int_{[0,L]^d} \int_{\mathbb{R}^d} |\chi_E(x + \zeta) - \chi_E(x)| K(\zeta) \, dx \, d\zeta = M^{d+2} \int_{[0,L/M]^d} \int_{\mathbb{R}^d} \left| \frac{\chi_E/M(x' + \zeta') - \chi_E/M(x')}{|\zeta'|^{d-2}} \right| e^{-M |\zeta'|} \, dx' \, d\zeta'. \]
Thus we have that
\[ \tilde{J}_M \text{Per}_1(E, [0, L]^d) = -M^{d+2} J_M e_M^* \text{Per}_1(E/M, [0, L/M]^d). \]
Finally defining
\[ K_M(\zeta) = \frac{-1}{e_M^*|\zeta|_1 d^{-2} e^{-M|\zeta|_1}} \]
and putting everything together we have that
\[ \tilde{F}_{\tilde{J}_M, L}(E) = -e_M^* F_{M, \tilde{L}}(\tilde{E}) \]
Then let us set
\[ \tilde{F}_{\tilde{J}_M, L}(E) = -e_M^* F_{M, \tilde{L}}(\tilde{E}) \]
where \( \tilde{L} = L/M \) and \( \tilde{E} = E/M \), and let us drop the tildes in the r.h.s. Hence
\[ F_{M, L}(E) = \frac{M^2}{L^d} \left( J_M \text{Per}_1(E, [0, L]^d) - \int_{[0,L/M]^d} \int_{\mathbb{R}^d} |\chi_E(x + \zeta) - \chi_E(x)| K_M(\zeta) \, dx \, d\zeta \right). \] (2.10)
Notice also that, due to (2.8), there exists a constant \( 1 > \gamma_M > 0 \) such that
\[ K_M(\zeta) \geq \frac{1}{|\zeta|_1 d^{-2} e^{-M(|\zeta|_1 - \gamma_M)}} \] (2.11)
with \( \gamma_M \to 1 \) as \( M \to +\infty \).
For simplicity of notation we define
\[ \hat{K}_M(t) := \int_{\mathbb{R}^{d-1}} K_M(t, \zeta_1, \ldots, \zeta_d) \, d\zeta_2 \ldots d\zeta_d. \] (2.12)

### 2.4 Splitting
As in [22] and in [9], after splitting the nonlocal term, one gets to the following lower bound:
\[
F_{M, L}(E) \geq \frac{M^2}{L^d} \sum_{i=1}^{d} \left[ \int_{[0, L)^d \cap \partial E} \int_{\mathbb{R}^{d-1}} \int_{-1}^{1} |\nu_i^E(x)||\zeta_i| K_M(\zeta) \, d\zeta \, dH^{d-1}(x) \right. \\
- \int_{[0, L)^d} \int_{\mathbb{R}^d} |\chi_E(x + \zeta) - \chi_E(x)| K_M(\zeta) \, d\zeta \, dx \\
+ \frac{2}{d} \frac{M^2}{L^d} \sum_{i=1}^{d} \left( \int_{[0, L)^d} \int_{\mathbb{R}^d} |\chi_E(x + \zeta) - \chi_E(x)||\chi_E(x + \zeta^\perp) - \chi_E(x)| K_M(\zeta) \, d\zeta \, dx \right) \\
= \frac{M^2}{L^d} \left( \sum_{i=1}^{d} G_{M, L}^i(E) + \sum_{i=1}^{d} I_{M, L}^i(E) \right), \tag{2.13}
\]
where

\[ G_{M,L}^i(E) := \int_{[0,L]^d} \int_{\mathbb{R}^{d-1}} |\nu^E_i(x)||\zeta_i|K_M(\zeta) \, d\zeta \, d\mathcal{H}^{d-1}(x) - \int_{[0,L]^d} |\chi_E(x+\zeta_i)-\chi_E(x)|K_M(\zeta) \, d\zeta \, dx \]

and

\[ I_{M,L}^i(E) := \frac{2}{d} \int_{[0,L]^d} \int_{\mathbb{R}^{d}} |\chi_E(x+\zeta_i)-\chi_E(x)||\chi_E(x+\zeta_i)-\chi_E(x)|K_M(\zeta) \, d\zeta \, dx. \]

We let moreover

\[ I_{M,L}(E) := \sum_{i=1}^d I_{M,L}^i(E). \]

As in [9] Section 7, one can further express \( G_{M,L}^i(E) \) as a sum of contributions obtained by first slicing and then considering interactions with neighbouring points on the slice lying on \( \partial E \), namely

\[ G_{M,L}^i(E) = \int_{[0,L)^d} \sum_{s \in \partial E_{t_i^+}} r_{i,M}(E, t_i^+, s) \, dt_i^+ \] (2.14)

where for \( s \in \partial E_{t_i^+} \)

\[ r_{i,M}(E, t_i^+, s) := \int_{-1}^1 |\zeta_i|\hat{K}_M(\zeta_i) \, d\zeta_i - \int_{s^-}^s \int_0^{+\infty} |\chi_{E_{t_i^+}}(u + \rho) - \chi_{E_{t_i^+}}(u)|\hat{K}_M(\rho) \, d\rho \, du \]

\[ - \int_{s}^{s^+} \int_{-\infty}^0 |\chi_{E_{t_i^+}}(u + \rho) - \chi_{E_{t_i^+}}(u)|\hat{K}_M(\rho) \, d\rho \, du \] (2.15)

and

\[ s^+ = \inf\{t' \in \partial E_{t_i^+}, \text{with } t' > s\} \]

\[ s^- = \sup\{t' \in \partial E_{t_i^+}, \text{with } t' < s\}. \] (2.16)

Using a notation similar to [9], we set

\[ f_E(t_i^+, t_i, t_i^+, t_i') := |\chi_E(t_i^+ + t_i + t_i') - \chi_E(t_i + t_i^+)||\chi_E(t_i^+ + t_i + t_i^+) - \chi_E(t_i + t_i^+)|. \] (2.17)

Then we can rewrite the last term in the r.h.s. of (2.13) as

\[ I_{M,L}^i(E) = \frac{2}{d} \int_{[0,L)^d} \int_{\mathbb{R}^{d}} f_E(t_i^+, t_i, t_i^+, t_i')K_M(\zeta) \, d\zeta \, dt_i^+ = \int_{[0,L)^d} \sum_{s \in \partial E_{t_i^+} \cap \partial E_t} v_{i,M}(E, t_i^+, s) \, dt_i^+ \]

\[ + \int_{[0,L)^d} w_{i,M}(E, t_i^+, t_i) \, dt \] (2.18)

where

\[ w_{i,M}(E, t_i^+, t_i) = \frac{1}{d} \int_{\mathbb{R}^{d}} f_E(t_i^+, t_i, t_i^+, t_i')K_M(\zeta) \, d\zeta. \] (2.19)

and

\[ v_{i,M}(E, t_i^+, s) = \frac{1}{2d} \int_{s^-}^{s^+} \int_{\mathbb{R}^{d}} f_E(t_i^+, u, t_i^+, t_i')K_M(\zeta) \, d\zeta \, du \] (2.20)
and $s^+, s^-$ as in (2.10).
As in [9], the role of the term $r_{i,M}$ is to penalize sets whose slices in direction $i$ have boundary points at distance smaller than some given constant (see Lemma 4.2). The term $v_{i,M}$ penalizes oscillations in direction $e_i$ whenever the neighbourhood of the point $(t^+_i + s e_i)$ is close in $L^1$ to a stripe oriented along $e_j$. This statement is made precise in Lemma 5.2.

2.5 Averaging

In this subsection we define the “local contribution” to the energy in a cube $Q_l(z)$, where $z \in [0, L)^d$ and $0 < l < L$ is a length which in the Section 4 will be fixed independently of $L$ ($l$ will depend only on the dimension).

We let

$$F_{i,M}(E, Q_l(z)) := \frac{1}{l^d} \left[ \int_{Q_l^i(z_i^+)} \sum_{s \in \partial E_{t_i^+}} (v_{i,M}(E, t_i^+, s) + r_{i,M}(E, t_i^+, s)) dt_i^+ + \int_{Q_l(z)} w_{i,M}(E, t_i^+, t_i) dt \right],$$

$$\bar{F}_M(E, Q_l(z)) := \sum_{i=1}^d F_{i,M}(E, Q_l(z)).$$

(2.21)

Thanks to Lemma 7.2 in [9], one has that the r.h.s. of (2.13) is equal to

$$\frac{M^2}{L^d} \int_{[0,L)^d} F_M(E, Q_l(z)) \, dz. \quad (2.22)$$

This implies that

$$\mathcal{F}_{M,L}(E) \geq \frac{M^2}{L^d} \int_{[0,L)^d} \bar{F}_M(E, Q_l(z)) \, dz. \quad (2.23)$$

Given that, in the above inequality, equality holds for stripes, if we show that the minimizers of (2.22) are periodic optimal stripes, then the same claim holds for $\mathcal{F}_{M,L}$.

2.6 A distance from unions of stripes

In the next definition we recall from [9] a quantity which measures the distance of a set from being a union of stripes.

**Definition 2.1.** For every $\eta > 0$ we denote by $A_{\eta}^i$ the family of all sets $F$ such that

(i) they are union of stripes oriented along the direction $e_i$,

(ii) their connected components of the boundary are distant at least $\eta$.

We denote by

$$D_{\eta}^i(E, Q) := \inf \left\{ \frac{1}{\text{vol}(Q)} \int_Q |\chi_E - \chi_F| : F \in A_{\eta}^i \right\} \quad \text{and} \quad D_{\eta}(E, Q) = \inf_i D_{\eta}^i(E, Q). \quad (2.24)$$

Finally, we let $A_\eta := \cup_i A_{\eta}^i$. 

We recall now some properties of the distance (2.24) (see Remark 7.4 in [9]).

Lemma 2.2.

(i) Let $E, F \subset \mathbb{R}^d$. Then, the map $z \mapsto D_{\eta}(E, Q_l(z))$ is Lipschitz. The Lipschitz constant can be shown to be $C_d/l$, where $C_d$ is a constant depending only on the dimension $d$.

(ii) For every $\varepsilon > 0$, there exists $\delta_0 = \delta_0(\varepsilon)$, such that for every $\delta \leq \delta_0$ and whenever $D_\eta^i(E, Q_l(z)) \leq \delta$ and $D_\eta^j(E, Q_l(z)) \leq \delta$ for $i \neq j$ and $\eta > 0$, the following hold

$$\min(\{|Q_l(z) \setminus E|, |E \cap Q_l(z)|\}) \leq \varepsilon. \quad (2.25)$$

3 The one-dimensional problem

We consider the following one-dimensional functional: on an $L$-periodic set $E \subset \mathbb{R}$ of locally finite perimeter

$$F_{M,L}^1(E) = \frac{M^2}{L} \left( \int_{-1}^{1} \hat{K}_M(\rho) \left( \text{Per}(E, [0,L])|\rho| - \int_0^L |\chi_E(s) - \chi_E(s + \rho)| \, ds \right) \, d\rho \right),$$

where $\hat{K}_M$ has been defined in (2.12).

The functional $F_{M,L}^1$ corresponds to $F_{M,L}(E)$ when the set $E$ is a union of stripes. Namely, given $E \subset \mathbb{R}^d$ and such that $E = \hat{E} \times \mathbb{R}^{d-1}$ where $E$ is $L$-periodic, then

$$F_{M,L}(E) = F_{M,L}(\hat{E}).$$

The purpose of this section is to show that the periodic sets are minimizers among the sets composed of stripes, whenever $M$ is large enough. For the above one-dimensional problem there are some standard techniques available in the literature. In particular, our proof will rely on the reflection positivity technique, introduced in the context of quantum field theory by Osterwalder and Schrader and then applied for the first time in statistical mechanics by Fröhlich, Simon and Spencer. For works where the reflection positivity is used in models with competing interactions, see e.g. [13], [14], [16], [17], [15], [22], [9]. As the technique is nowadays standard, we will only outline briefly the steps and show some of the differences with respect to the literature.

Before showing optimality of periodic stripes, we show that there exists a unique optimal period $h^*_M$, provided $M$ is large enough. For $h > 0$, let $E_h = \bigcup_{j \in \mathbb{Z}} [(2j)h, (2j+1)h]$ and define the energy

$$e_M(h) = F_{M,2h}^1(E_h) = \frac{M^2}{h} \left[ \int_h^{1} (\rho-h) \hat{K}_M(\rho) \, d\rho - \int_1^{+\infty} h\hat{K}_M(\rho) \, d\rho + \sum_{k \in \mathbb{N}} \int_0^{h} \int_{2kh}^{(2k+1)h} \hat{K}_M(u-v) \, dv \, du \right].$$

We prove the following

**Theorem 3.1.** There exists $\tilde{M} > 0$ such that $\forall M > \tilde{M}$, there exists a unique minimizer of $e_M(\cdot)$, $h^*_M$. 

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Proof. In Section 2.2 we showed that, as $M \to +\infty$, minimizers to $e_M$ tend to 1. So, in order to prove the theorem it is sufficient to show that there exist $\delta > 0$, $\tilde{M}$ such that $e''_M(1) \geq \delta$ for all $M > \tilde{M}$. Computing $e''_M(1)$, one finds

$$e''_M(1) = -2M^2 \left\{ \int_1^{+\infty} -\hat{K}_M(\rho) \, d\rho + \sum_{k \in \mathbb{N}} \left[ \int_{2k}^{2k+1} \hat{K}_M(1-\rho) \, d\rho \right. \right.$$

$$+ (2k + 1) \int_0^1 \hat{K}_M(\rho-(2k+1)) \, d\rho - 2k \int_0^1 \hat{K}_M(\rho-2k) \, d\rho \left. \right] \right.$$

$$+ 2M^2 \left[ \int_1^{+\infty} -\hat{K}_M(\rho) \, d\rho + \sum_{k \in \mathbb{N}} \int_0^1 \int_{2k}^{2k+1} \hat{K}_M(u-v) \, dv \, du \right. \right.$$

$$+ M^2 \left[ \hat{K}_M(1) + \sum_{k \in \mathbb{N}} \left[ \int_{2k}^{2k+1} \hat{K}'_M(1-\rho) \, d\rho \right. \right.$$

$$+ 2(2k + 1)\hat{K}_M(-2k) - 4k\hat{K}_M(1-2k)$$

$$- (2k + 1)^2 \int_0^1 \hat{K}_M(\rho-(2k+1)) \, d\rho + (2k)^2 \int_0^1 \hat{K}_M(\rho-2k) \, d\rho \left. \right] \right.$$ 

$$= M^2 \sum_{k \in \mathbb{N}} \left[ \hat{K}_M(-2k)(-1 + 2(2k+1) - (2k+1)^2) - (2k)^2 \right.$$ 

$$+ \hat{K}_M(1-2k)(1 - 4k + (2k)^2) + \hat{K}_M(-1-2k)(2k+1)^2 \right]$$ 

$$+ M^2 \hat{K}_M(1)$$ 

$$+ 2M^2 \sum_{k \in \mathbb{N}} \left[ \int_0^1 \int_{2k}^{2k+1} \hat{K}_M(u-v) \, du \, dv - \int_{2k}^{2k+1} \hat{K}_M(1-\rho) \, d\rho \right.$$ 

$$- (2k + 1) \int_0^1 \hat{K}_M(\rho-(2k+1)) \, d\rho - 2k \int_0^1 \hat{K}_M(\rho-2k) \, d\rho \left. \right] \right.$$. 

(3.1)

One can see that, since the kernel is positive and monotone decreasing, for $M$ large enough the terms in (3.1), (3.2) and (3.3) are nonnegative, larger than some $\delta > 0$ for $M$ large enough and then the theorem is proved. \hfill \Box

Let us now return to the issue of showing that optimal periodic stripes are optimal among all stripes. This fact follows from the reflection positivity technique and is well-known in literature. We refer the reader to the references given at the beginning of this section. Only for notational reasons we also refer to [9, 22]. The only part needed is the reflection positivity of the Yukawa kernel.

**Lemma 3.2.** The Yukawa kernel is reflection positive, namely there exists a positive Borel measure $\mu$ such that

$$\hat{K}(t) = \int_0^{+\infty} e^{-\alpha t} \, d\mu(\alpha).$$

**Proof.** Due to the Hausdorff-Bernstein-Widder theorem, one has that a function $\varphi$ is reflection positive if and only if it is completely monotone, namely $(-1)^n \frac{d^n}{dt^n} \varphi \geq 0$. 

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By using the complete monotone property one has that the set of functions which are reflection positive is an algebra. Indeed, given two functions \( \varphi, \psi \) which are completely monotone by the Leibniz rule one has that

\[
(-1)^n \frac{d^n}{d^m}(\varphi \psi) = \sum_{k=0}^{n} (-1)^k \frac{d^k \varphi}{d^k} \cdot (-1)^{n-k} \frac{d^{n-k} \psi}{d^{n-k}} \geq 0
\]

In order to conclude the proof, we need to show that the map

\[
t \mapsto e^{-t} \int_{\mathbb{R}^{d-1}} \frac{1}{(t + |\zeta_2| + \cdots + |\zeta_d|)^{d-2}} e^{-(|\zeta_2| + \cdots + |\zeta_d|)}
\]

is completely monotone. Given the complete monotone functions are an algebra, we need to check that the single terms in the product above are completely monotone. This can be done easily with explicit calculations.

Once reflection positivity of the kernel is shown the proof follows by standard means in the literature. We refer the reader to [22, 9] where further details are given and a similar notation is used.

4 A local rigidity estimate

The main purpose of this section is to prove the following proposition:

**Proposition 4.1 (Local Rigidity).** For every \( N > 1, l, \delta > 0 \), there exist \( \bar{M}, \bar{\eta} > 0 \) such that whenever \( M > \bar{M} \) and \( \bar{F}_M(E, Q_l(z)) < N \) for some \( z \in [0, L]^d \) and \( E \subset \mathbb{R}^d \) \( \bar{M} \)-periodic, with \( L > l \), then it holds \( D_\eta(E, Q_l(z)) \leq \delta \) for every \( \eta < \bar{\eta} \). Moreover \( \bar{\eta} \) can be chosen independent on \( \delta \). Notice that \( \bar{M} \) and \( \bar{\eta} \) are independent of \( L \).

In particular, such a rigidity estimate tells us that on small cubes minimizers of the functional are, for \( M \) large enough, close to stripes of a given minimal width.

In order to prove Proposition 4.1 we will need to analyze the behaviour of \( \bar{F}_M \) for large \( M \). First of all, we start with the following lemma, about the term \( r_{i,M} \). In particular this tells us that given a sequence of sets \( \{E_M\}_{M \geq 0} \subset \mathbb{R}^d \) of bounded local energy \( \bar{F}_M(E, Q_l(z)) \), if \( M \) is large enough their boundary points on the slices have distance at least 1 and then they converge to a set of locally finite perimeter \( E_0 \).

**Lemma 4.2.** There exists a function \( g : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) such that, for all \( E \subset \mathbb{R}^d \) of locally finite perimeter, \( t_i^+ \in [0, L)^{d-1}, s \in \partial E_{t_i^+} \)

\[
r_{i,M}(E, t_i^+, s) \geq g((\gamma_M - \min(|s - s^-|, |s - s^+|), M) \tag{4.1}
\]

with \( \gamma_M \) defined in (2.11). The function \( g \) satisfies the following: \( g(v, M) \geq g(v', M) \) whenever \( v \geq v' \), \( g(v, M) \geq e^{-cM} \) for some \( c > 0 \) and \( g(v, M) \to +\infty \) as \( M \to +\infty \) provided \( v > 0 \).

In particular, for every \( 0 < \eta_0 < 1 \) there exists \( M_0 \) such that, for all \( M > M_0 \) if \( \min\{|s - s^-|, |s - s^+|\} < \eta_0 \) then \( r_{i,M}(E, t_i^+, s) > 0 \).
Proof. The proof of this lemma uses the following inequality, introduced in [22] and used also in [9]: for every $E \subset \mathbb{R}^d$ of locally finite perimeter, $\forall t_i^+ \subset [0, L)^{d-1}$,

$$\forall \rho > 0, \int_{s^-}^{s^+} |\chi_{E_{t_i^+}}(u) - \chi_{E_{t_i^+}}(u + \rho)| \, du \leq \min(\rho, s - s^-)$$

$$\forall \rho < 0, \int_{s^-}^{s^+} |\chi_{E_{t_i^+}}(u) - \chi_{E_{t_i^+}}(u + \rho)| \, du \leq \min(-\rho, s^+ - s). \tag{4.2}$$

Indeed,

$$\int_0^1 \zeta_i \tilde{K}_M(\zeta_i) \, d\zeta_i - \int_{-\infty}^{s^-} \int_0^{s^+} |\chi_{E_{t_i^+}}(u + \rho) - \chi_{E_{t_i^+}}(u)| \tilde{K}_M(\rho) \, d\rho \, du \geq \int_0^1 \zeta_i \tilde{K}_M(\zeta_i) \, d\zeta_i - \int_{0}^{s^+} \min(|s - s^-|, \zeta_i) \tilde{K}_M(\zeta_i) \, d\zeta_i$$

$$\geq \int_0^1 (\zeta_i - |s - s^-|) \tilde{K}_M(\zeta_i) \, d\zeta_i - \int_{-\infty}^{s^-} |s - s^-| \tilde{K}_M(\zeta_i) \, d\zeta_i \tag{4.3}$$

and analogously

$$\int_0^1 |\zeta_i| \tilde{K}_M(\zeta_i) \, d\zeta_i - \int_{-\infty}^{s^-} \int_0^{s^+} |\chi_{E_{t_i^+}}(u + \rho) - \chi_{E_{t_i^+}}(u)| \tilde{K}_M(\rho) \, d\rho \, du$$

$$\geq \int_0^1 (|\zeta_i| - |s - s^-|) \tilde{K}_M(\zeta_i) \, d\zeta_i - \int_{-\infty}^{s^-} |s - s^-| \tilde{K}_M(\zeta_i) \, d\zeta_i. \tag{4.4}$$

Then, by definition of $r_{i,M}$ and using (2.11), since $\gamma_M < 1$ one gets (4.1) and the statement of the lemma. \qed

Remark 4.3. From Lemma 4.2 it follows as well that the function

$$r_{i,\infty}(E, t_i^+, s) := \liminf_{M \to +\infty} r_{i,M}(E, t_i^+, s)$$

satisfies

$$r_{i,\infty}(E, t_i^+, s) = +\infty \quad \text{whenever} \quad \min(|s - s^-|, |s - s^+|) < 1. \tag{4.5}$$

In particular, if $\{E_M\}_{M>0} \subset \mathbb{R}^d$ is a family of sets of locally finite perimeter with $\sup_M \tilde{F}_M(E_M) \leq N$, then for a.e. $t_i^+ \in Q_i^+(z_i^+)$ and for every $I \subset \mathbb{R}$ open interval,

$$\liminf_{M \to +\infty} \min\left\{|s_i^M - s_i^{M+1}| : \partial E_{M,t_i^+} \cap I = \{s_i^M\}_{i=1}^{m(M)}\right\} \geq 1 \tag{4.6}$$

In particular, $E_M$ converges in $L^1_{\text{loc}}$ to a set $E_\infty$ of locally finite perimeter such that

$$\min\{|s_i^\infty - s_j^\infty| : \partial E_{\infty,t_i^+} \cap I = \{s_k^\infty\}_{k=1}^{m(\infty)}\} \geq 1. \tag{4.7}$$

For details of how to deduce (4.7) form (4.6) see the proof of Lemma 7.5 in [9].
Remark 4.4. Let us notice the following: the family of kernels $K_M$ is monotone increasing in $M$ as $M \to +\infty$. Let $K_\infty$ be defined by

$$K_\infty(\zeta) := \liminf_{M \to +\infty} K_M(\zeta).$$

From (2.11) we get that

$$K_\infty \geq 0 \quad \text{and} \quad K_\infty(\zeta) = +\infty \text{ whenever } |\zeta| < 1.$$  \hfill (4.9)

Let us now proceed to the proof of Proposition 4.1. The main steps can be summarized as follows.

1. Let us now proceed to the proof of Proposition 4.1. The main steps can be summarized as follows.

(i) one has that $M_k \to +\infty$, $L_k > l$, $\eta_k \downarrow 0$, $z_k \in [0, L_k]^d$;

(ii) the family of sets $E_{\eta_k}$ is $[0, L_k]^d$-periodic

(iii) one has that $D_{\eta_k}(E_{\eta_k}, Q_l(z_k)) > \delta$ and $\tilde{F}_{\eta_k}(E_{\eta_k}, Q_l(z_k)) < N$.

Given that $\eta \mapsto D_\eta(E, Q_l(z))$ is monotone increasing, we can fix $\tilde{\eta}$ sufficiently small instead of $\eta_k$ with $D_\tilde{\eta}(E_{\tilde{\eta}}, Q_l(z_k)) > \delta$. In particular, $\tilde{\eta}$ will be chosen at the end of the proof depending only on $N, l$.

W.l.o.g. (taking e.g. $E_{\eta_k} - z_k$ instead of $E_{\eta_k}$) we can assume there exists $z \in \mathbb{R}^d$ such that $z_k = z$ for all $k \in \mathbb{N}$.

Because of Remark 4.3 one has that $\sup_k \text{Per}_1(E_{\eta_k}, Q_l(z)) < +\infty$. Thus up to subsequences there exists $E_\infty$ such that $E_{\eta_k} \to E_\infty$ in $L^1(Q_l(z))$ with

$$D_{\tilde{\eta}}(E_\infty, Q_l(z)) > \delta.$$  \hfill (4.10)

In order to keep the notation simpler, we will write $M \to +\infty$ instead of $M_k \to +\infty$ and $E_M \to E_\infty$ instead of $E_{\eta_k} \to E_\infty$.

Define $J_i := (z_i - l/2, z_i + l/2)$.

By Lebesgue’s theorem, there exists a subsequence of $M$ such that for almost every $t_i^+ \in Q_l(z_i^+)$ one has that $E_{M_{i_{k_i}}^\perp} \cap J_i$ converges to $E_{\infty_{i_{k_i}}}^\perp \cap J_i$ in $L^1(Q_l(z))$.

By using (2.21) and the fact that $v_{i,M} \geq 0$, we have that

$$N \geq \tilde{F}_M(E_M, Q_l(z)) \geq \frac{1}{l^d} \sum_{i=1}^d \int_{Q_l^+(z_i^+)} \sum_{s \in \partial E_M^\perp} r_{i,M}(E_M, t_i^+, s) \, dt_i^+ + \int_{Q_l(z)} w_{i,M}(E_M, t_i^+, t_i) \, dt_i^+ \, dt_i.$$  \hfill (4.11)
By the Fatou lemma, we have that

\[ l^d M \geq \liminf_{M \to +\infty} \sum_{i=1}^{d} \int_{Q_i^+} \sum_{s \in \partial E_{M,t_i^+}} r_{i,M}(E_M, t_i^+, s) \, dt_i^+ \geq \sum_{i=1}^{d} \int_{Q_i^+} \liminf_{M \to +\infty} \sum_{s \in \partial E_{M,t_i^+}} r_{i,M}(E_M, t_i^+, s) \, dt_i^+ \]

where

\[ B_i = \left\{ t_i^+ \in Q_i^+ (z_i^+): \min \{|s_i^\infty - s_j^\infty|: \partial E_{E_{\infty}, t_i^+} = \{s_k^\infty \}_{k=1}^{m(\infty, t_i^+)} < 1 \right\}, \]

and in the last inequality we have used Remark 4.3.

For the last term in (4.11), namely

\[
\liminf_{M \to +\infty} \int_{Q_i(z)} w_{i,M}(E_M, t_i^+, t_i) \, dt_i^+ \, dt_i \\
\geq \liminf_{M \to +\infty} \frac{1}{d} \int_{Q_i(z)} \int_{Q_i(z)} f_{E_M}(t_i^+, t_i, t_i^+, t_i) K_M(t) \, dt \, dt' \\
= \liminf_{M \to +\infty} \frac{1}{d} \int_{Q_i(z)} \int_{Q_i(z)} f_{E_M}(t_i^+, t_i, t_i^+, t_i^+ - t_i^+ - t_i^+ - t_i) K_M(t - t') \, dt \, dt' \\
\geq \frac{1}{d} \int_{Q_i(z)} \int_{Q_i(z)} f_{E_{\infty}}(t_i^+, t_i, t_i^+, t_i^+ - t_i^+ - t_i^+ - t_i) K_{\infty}(t - t') \, dt \, dt',
\]

where in the third line we have used a change of variables.

Indeed, in order to prove (4.12) we fix \( M' > 0 \) and by using initially \( E_M \to E_{\infty} \) in \( L^1(Q_i(z)) \) and afterwards the monotonicity of \( M \mapsto K_M(\zeta) \) we have that

\[
\liminf_{M \to +\infty} \int_{Q_i(z)} w_{i,M}(E_M, t_i^+, t_i) \, dt_i^+ \, dt_i \geq \sup_{M' \to +\infty} \int_{Q_i(z)} w_{i,M'}(E_M, t_i^+, t_i) \, dt_i^+ \, dt_i \\
\geq \sup_{M'} \frac{1}{d} \int_{Q_i(z)} \int_{Q_i(z)} f_{E_{\infty}}(t_i^+, t_i, t_i^+, t_i^+ - t_i^+ - t_i^+ - t_i) K_{M'}(t - t') \, dt \, dt' \\
\geq \frac{1}{d} \int_{Q_i(z)} \int_{Q_i(z)} f_{E_{\infty}}(t_i^+, t_i, t_i^+, t_i^+ - t_i^+ - t_i^+ - t_i) K_{\infty}(t - t') \, dt \, dt'.
\]

Thus, we have shown that

\[
\sum_{i=1}^{d} \frac{1}{d} \int_{Q_i(z)} \int_{Q_i(z)} f_{E_{\infty}}(t_i^+, t_i, t_i^+ - t_i^+ - t_i^+ - t_i) K_{\infty}(t - t') \, dt \, dt' \\
+ \sum_{i=1}^{d} \int_{Q_i^+ (z_i^+)} 1_{B_i}(t_i^+) \, dt_i^+ \lesssim l^d N. \quad (4.13)
\]
Defining
\[ \text{Int}(t_i^+, t_i^-) := \int_{Q_i^+(z_i)} \int_{Q_i^-(z_i)} f_{E_\infty}(t_i^+, t_i, t_i^+ - t_i, t_i^-) K_\infty(t - t') \, dt \, dt', \]  
(4.14)

one has
\[ \int_{Q_i^+(z_i)} \int_{Q_i^-(z_i)} \text{Int}(t_i^+, t_i^-) \, dt \, dt' \lesssim \lambda^d N < +\infty \]  
(4.15)

Given \( \lambda \in (0, \frac{1}{2}) \), \( u \in (z_i - \lambda, z_i + \lambda) \) and \( t_i^+ \in Q_i^+(z_i^+) \), we denote by
\[ r_i^+(u, t_i^+) := \min \left\{ \inf \{ |u - s| : s \in \partial E_{\infty, t_i^+} \text{ and } s \in (z_i - \lambda, z_i + \lambda) \}, |u - z_i - \lambda|, |z_i + \lambda - u| \right\} \]
\[ r_i^-(t_i^-) := \inf_{s \in \partial E_{\infty, t_i^+} \cap Q_i^+(z_i)} \min(s^+ - s, s - s^-), \]  
(4.16)

where \( s^+, s^- \) are defined in (2.10). Notice that, since \( \int_{Q_i^+(z_i)} \mathbb{1}_{B_i}(t_i^+) \, dt_i < +\infty \), for a.e. \( t_i^+ \) \( r_i^+(t_i^+) \geq 1 \).

As noticed in [9], the map \( r_i^+(\cdot, t_i^+) \) is well-defined for almost every \( t_i^+ \) and measurable. The role of \( \lambda > 0 \) is to deal with the boundary, since \( E_\infty \) is not \([0, l]^d\)-periodic.

Suppose that, for every \( u \), one has that \( r_i^+(u, \cdot) \) is constant almost everywhere: if this holds for every \( i \), then it is not difficult to see that \( E_\infty \) is (up to null sets) either a union of stripes or a checkerboards, where by checkerboards we mean any set whose boundary is the union of affine subspace orthogonal to coordinate axes, and there are at least two of these directions.

The checkerboards can be ruled out easily via an energetic argument (see the comment at the end of this section).

In order to obtain that \( r_i^+(u, \cdot) \) is constant almost everywhere we proceed in the following way.

In the next lemma we give a lower bound for the interaction term.

**Lemma 4.5.** Let \( \lambda \in (0, L/2) \) and let \( t_i^+, t_i^- \in Q_i^+(z_i^+) \), \( t_i^+ \neq t_i^- \) be such that \( \min(r_i^+(t_i^+), r_i^-(t_i^-)) > |t_i^+ - t_i^-| \) and \( |t_i^+ - t_i^-| \leq \min(\lambda, 1/2) \). Then for every \( u \in (z_i - \lambda, z_i + \lambda) \) it holds
\[ \text{Int}(t_i^+, t_i^-) \geq \mathbb{1}_{\{ (t_i^+, t_i^-) : r_i^+(u, t_i^+), r_i^-(u, t_i^-) \}}(t_i^+, t_i^-). \]
(4.17)

**Proof.** The idea of the proof is similar to the proof of Lemma 3.5 in [9], so some steps are analogous and we report them here for completeness. However, the different kernel \( K_\infty \) gives a different estimate.

W.l.o.g., let us assume that \( r_i^+(u, t_i^+) < r_i^-(u, t_i^-) \). In particular this implies that \( r_i^+(u, t_i^+) < \min(|u - z_i + \lambda|, |z_i + \lambda - u|) \), and hence there exists a point \( s_o \in (z_i - l + \lambda, z_i + l - \lambda) \) such that
\[ |u - s_o| = \inf \{ |u - s| : s \in \partial E_{\infty, t_i^+}, s \in (z_i - l + \lambda, z_i + l - \lambda) \}. \]

Let us denote by \( \delta = |t_i^+ - t_i^-| \) and by \( r = |r_i^+(u, t_i^+) - r_i^-(u, t_i^-)| \). Given that \( r_o(t_i^+) > \delta \), the following holds
\[ (s_o - \delta, s_o + \delta) \cap E_{\infty, t_i^+} = (s_o, s_o) \quad \text{or} \quad (s_o - \delta, s_o + \delta) \cap E_{\infty, t_i^+} = (s_o - \delta, s_o). \]
Notice that since $\lambda \geq \delta$, we have that $(s_0 - \delta, s_0 + \delta) \subset Q^i_t(z_i)$. In the following, we will assume that

$$(s_0 - \delta, s_0 + \delta) \cap E_{\infty, t_i^+} = (s_0, s_0 + \delta)$$

(4.18)

The other case is analogous.
We will distinguish two subcases:

(i) Suppose $r > \delta/2$. From the definition of $\delta$ and $r$, for every slice in $t_i^{1/2}$ it holds

$$(s_0 - \delta/2, s_0 + \delta/2) \cap E_{\infty, t_i^+} = (s_0 - \delta/2, s_0 + \delta/2) \quad \text{or} \quad (s_0 - \delta/2, s_0 + \delta/2) \cap E_{\infty, t_i^+} = \emptyset.$$ 

Indeed on the slice $E_{\infty, t_i^+}$, the closest jump point to $s_0$ is at least $r$ distant and $r > \delta/2$. We will assume the first of the alternatives above. The other case is analogous.

For every $a \in (s_0 - \delta/2, s_0)$ and $a' \in (s_0, s_0 + \delta/2)$, one has that

$$f_{E_{\infty}}(t_i^{1/2}, a, t_i^{1/2} - t_i^{1/2}, a' - a) = 1.$$ 

Given that $r_\lambda(u, t_i^{1/2}) \leq L$, one has that

$$\text{Int}(t_i^{1/2}, t_i^{1/2}) = \int_{Q_i^r(z_i)} \int_{Q_i^r(z_i)} f_{E_{\infty}}(t_i^{1/2}, t_i, t_i^{1/2} - t_i^{1/2}, t_i - t_i) K\infty(t' - t) dt_i dt'_i$$

$$\geq \int_{s_0-r/2}^{s_0} \int_{s_0}^{s_0+\delta/2} f_{E_{\infty}}(t_i^{1/2}, t_i, t_i^{1/2} - t_i^{1/2}, t_i - t_i) K\infty(t' - t) dt_i dt'_i$$

$$\geq \int_{s_0-r/2}^{s_0} \int_{s_0}^{s_0+\delta/2} K\infty(t' - t) dt_i dt'_i \geq \left\{ (t_i^{1/2}, a, a' - a) = 1 \right\}$$

since $|t - t'| < 1$ if $\max(|t_i^{1/2} - t_i^{1/2}|, |t - t'|) \leq 1/2$.

(ii) Let us assume now that $r \leq \delta/2$. Given that $r_\lambda(t_i^{1/2}), r_\lambda(t_i^{1/2}) > \delta$, one has that either

$$(s_0 - r, s_0 + \delta/2) \cap E_{\infty, t_i^+} = (s_0 - r, s_0 + \delta/2) \quad \text{or} \quad (s_0 - r, s_0 + \delta/2) \cap E_{\infty, t_i^+} = \emptyset$$

or

$$(s_0 - \delta/2, s_0 + r) \cap E_{\infty, t_i^+} = (s_0 - \delta/2, s_0 + r) \quad \text{or} \quad (s_0 - \delta/2, s_0 + r) \cap E_{\infty, t_i^+} = \emptyset.$$ 

Indeed if none of the above were true we would have that $\#(\partial E_{\infty, t_i^+} \cap (s_0 - \delta/2, s_0 + \delta/2)) \geq 2,$

which contradicts $r_\lambda(t_i^{1/2}) > \delta$. W.l.o.g. we will assume

$$(s_0 - r, s_0 + \delta/2) \cap E_{\infty, t_i^+} = (s_0 - r, s_0 + \delta/2).$$

The other cases are similar.

Then for every $a \in (s_0 - r, s_0)$ and $a' \in (s_0, s_0 + \delta/2)$, one has that $f_{E_{\infty}}(t_i^{1/2}, a, t_i^{1/2} - t_i^{1/2}, a' - a) = 1$.

Thus

$$\text{Int}(t_i^{1/2}, t_i^{1/2}) = \int_{Q_i^r(z_i)} \int_{Q_i^r(z_i)} f_{E_{\infty}}(t_i^{1/2}, t_i, t_i^{1/2} - t_i^{1/2}, t_i - t_i) K\infty(t' - t) dt_i dt'_i$$

$$\geq \int_{s_0-r}^{s_0} \int_{s_0}^{s_0+\delta/2} f_{E_{\infty}}(t_i^{1/2}, t_i, t_i^{1/2} - t_i^{1/2}, t_i - t_i) K\infty(t' - t) dt_i dt'_i$$

$$\geq \int_{s_0-r}^{s_0} \int_{s_0}^{s_0+\delta/2} K\infty(t' - t) dt_i dt'_i \geq \left\{ (t_i^{1/2}, a, a' - a) = 1 \right\}$$

following at the end the same argument in case (i).
Thus, since this holds for every \( t_i^+ \in Q_{t_i}^i(z_i^+) \), we may assume that around this edge there are at least two of these directions. Recall that by a checkerboard we mean any set whose boundary is the union of affine hyperplanes orthogonal to coordinate axes. Let \( B \) be the set defined by

\[
B := \left\{ (t_i^+, t_i^-) \in [0, L)^{d-1} \times [0, L)^{d-1} : |t_i^+ - t_i^-| \leq \min(\lambda, 1/2) \right\}.
\]

Then, by (4.15) and Lemma 4.5,

\[
\int_{Q_i(z)} \int_{Q_i(z)} 1_B(t_i^+, t_i^-) \leq I_{\infty,L}(E_\infty) < +\infty.
\]

Hence, \( r_\lambda^i(u, t_i^+) = r_\lambda^i(u, t_i^-) \) whenever \( |t_i^+ - t_i^-| \leq \min(\lambda, 1/2) \) and therefore the statement of the lemma follows.

From Lemma 4.6, one has that \( r_\lambda^i(u, \cdot) \) is constant almost everywhere for every \( u \) and for every \( i \). Fix \( u \) and \( \lambda \) sufficiently small such that \( r_\lambda^i(u, \cdot) \neq \min(|u - z_i + l - \lambda|, |z_i + l - \lambda - u|) \). If this is not possible, then \( E_\infty \cap Q_i(z) \) is either \( Q_i(z) \) or \( \emptyset \), which contradicts (4.10). Therefore for every \( t_i^+ \), the minimizers of \( |s - u| \) for \( s \in \partial E_\infty, t_i^+ \) are either \( u + r_\lambda^i(u, t_i^+) \) or \( u - r_\lambda^i(u, t_i^+) \). The fact that \( r_\lambda^i(u', t_i^+) \) is also constant almost everywhere for \( u' \in (u - \varepsilon, u + \varepsilon) \) implies that one of the following three cases holds

(a) \( u + r_\lambda^i(u, t_i^+) \in \partial E_\infty, t_i^+ \) for all \( t_i^+ \in Q_{t_i}^i(z_i^+) \)

(b) \( u - r_\lambda^i(u, t_i^+) \in \partial E_\infty, t_i^- \) for all \( t_i^- \in Q_{t_i}^i(z_i^-) \)

(c) \( u + r_\lambda^i(u, t_i^+) \in \partial E_\infty, t_i^+ \) for all \( t_i^+ \in Q_{t_i}^i(z_i^+) \) and \( u - r_\lambda^i(u, t_i^+) \in \partial E_\infty, t_i^- \) for all \( t_i^- \in Q_{t_i}^i(z_i^-) \).

Thus, since this holds for every \( i \), we have that \( E_\infty \) must be a checkerboard or a union of stripes. We recall that by a checkerboard we mean any set whose boundary is the union of affine hyperplanes orthogonal to coordinate axes, and there are at least two of these directions.

However, the checkerboard can be ruled out immediately. To see this we consider the contribution to the energy given in a neighbourhood of an edge. W.l.o.g. we may assume that around this edge the set \( E_\infty \) is of the following form \( -\varepsilon \leq x_1 \leq 0 \) and \( -\varepsilon \leq x_2 \leq 0 \) and \( x_i \in (-\varepsilon, \varepsilon) \) for \( i \neq 1, 2 \). Notice that for every \( \zeta \) such that \( \zeta_1 + x_1 > 0, \zeta_2 + x_2 > 0 \) and \( \zeta_i \in (-\varepsilon, \varepsilon) \) for \( i \neq 1, 2 \), the integrand in \( |x_1 + \zeta_1| \) is equal to \( +\infty \). Therefore also the first term in (4.13) must be \( +\infty \), which contradicts our assumptions.

Moreover, since the second term in the l.h.s. of (4.13) explodes for stripes with minimal width tending to zero, one has that there exists \( \eta = \eta(N, l) \geq 1 \) such that \( D_{\eta}(E_\infty, Q_i(z)) = 0 \). This contradicts that \( D_{\eta}(E_\infty, Q_i(z)) > \delta \), which was assumed at the beginning of the proof. 

\[ \square \]
5 Proof of Theorem 1.1

The main purpose of this section is to prove Theorem 1.1. Let us first give a brief idea of the proof, referring for a more detailed account on the strategy to Section 7 in [9].

As in Section 2.5, instead of the functional $\mathcal{F}_{M,L}$ it is convenient to consider the r.h.s. of (2.23) and show that its minimizers are stripes, for $l$ and $M$ sufficiently large and any $L > l$ of the form $2kh_M^*$ with $k \in \mathbb{N}$. Recall that $h_M^*$ is the width of the periodic stripes which minimize the energy density $\mathcal{F}_{M,L}$ among all periodic stripes as $L$ varies.

The aim is first to prove that minimizers are close according to the distance defined in (2.24) to stripes in some fixed direction on the whole cube $[0,L]^d$. Then, using a stability argument (see Lemma 5.3) one can show that, once close to stripes in one direction it is not convenient anymore to deviate from being exactly stripes in that direction. Moreover, by Theorem 3.1 and Lemma 3.2 such stripes have to be periodic of period $h_M^*$.

Let us focus on the first point. Let $E_M$ be a minimizer for $\bar{F}_M(E_M,Q_l(z))$. Then, $[0,L]^d = A_{-1} \cup A_0 \cup \ldots \cup A_l$ where

- $A_i$ with $i > 0$ are the set of points $z$ such that there is only one direction $e_i$, such that $E_M \cap Q_l(z)$ is close to stripes oriented in direction $e_i$.
- $A_{-1}$ is the set of points $z$ such that there exist directions $e_i$ and $e_j$ ($i \neq j$) and stripes $S_i$ (oriented in direction $e_i$) and $S_j$ (stripes oriented in direction $e_j$) such that $E_M \cap Q_l(z)$ is close to both $S_i \cap Q_l(z)$ and $S_j \cap Q_l(z)$. In particular, this implies that either $|E_M \cap Q_l(z)| \ll l^d$ or $|E_M^c \cap Q_l(z)| \ll l^d$ (see Remark 2.2 (ii)).
- $A_0$ is the set of points $z$ where none of the above points is true.

The aim is then to show that $A_0 \cup A_{-1} = \emptyset$ and that there exists only one $A_i$ with $i > 0$.

As in [9], as soon as there are $A_i \neq \emptyset$ and $A_j \neq \emptyset$ for $i \neq j$, then $A_0 \cup A_{-1} \neq \emptyset$ and it separates the different $A_i$, namely every continuous curve $\gamma : [0,T] \to [0,L]^d$ intersecting $A_i$ and $A_j$ has necessarily to intersect $A_0 \cup A_{-1}$. Moreover, $A_0$ and $A_{-1}$ are “thick”, namely the possess a neighbourhood with similar properties.

What happens is now the following:

(i) for any $z \in A_i$ with $i > 0$, when slicing in directions orthogonal to $e_i$, alternation between regions in $E_M$ and regions in $E_M^c$ should increase the energy. Thus one expects the contribution of $A_i$ to be bigger than

$$e_M^* |A_i|/L^d - C_l g(\partial A_i)/L^d$$

where $g$ is some positive real function, $C_l$ is a constant depending on $l$ and $e_M^*$ is the energy of periodic stripes of width $h_M^*$.

(ii) for any $z \in A_0$ or $z \in A_{-1}$ we expect contributions larger than $e_M^*$. Indeed, the functional $\bar{F}_M(E_M,Q_l(z))$ contains by construction a term of the form

$$\frac{1}{l^d} \sum_{i=1}^d \int_{Q_l(z)} w_{i,M}(E,t_1,t_2^i) \, dt.$$  (5.1)
As proved in the local rigidity Proposition 4.1, such term is larger than some given $N$ if $D_0(E, Q_l(z)) > \delta$, for $M > M(\ell, N, \delta)$ as in $A_0$. Moreover, as shown in Lemma 5.4, on $A_{-1}$ the contribution becomes also close to zero if $l$ is large enough depending only on the dimension. Thus having $A_0 \cup A_{-1}$ is not energetically convenient. Since $A_0 \cup A_{-1}$ separates the different $A_i$, and is “thick”, one has that $|A_0 \cup A_{-1}|$ acts like a boundary term and compensates the boundary term $g(\partial A_i)$ in (i) provided $M$ is large enough (independently of $L$).

5.1 Preliminary Lemmas

In order to simplify notation, we will use $A \lesssim B$, whenever there exists a constant $C_d$ depending on the dimension $d$ such that $A \leq C_d B$.

For notational reasons it is convenient to introduce the one-dimensional analogue of (2.15). Namely, let $E \subset \mathbb{R}$ be a set of locally finite perimeter and let $s^-, s, s^+ \in \partial E$. We define

$$r_M(E, s) := -1 + \int_{\mathbb{R}} |n| \hat{K}_M(n) \, dn - \int_{s^-}^{s^+} \int_{0}^{+\infty} |\chi_E(n + u) - \chi_E(n)| \hat{K}_M(n) \, dn \, du$$

$$- \int_{s^-}^{s^+} \int_{-\infty}^{0} |\chi_E(n + u) - \chi_E(n)| \hat{K}_M(n) \, dn \, du. \quad (5.2)$$

The quantities defined in (2.15) and (5.2) are related via $r_{i,M}(E, t_i^+, s) = r_M(E_t^+, s)$. The following is a technical lemma needed in the proof of Lemma 5.3, analogous to Lemma 7.7 in [9]. It says that given a set $E \subset \mathbb{R}$, and $I \subset \mathbb{R}$ an interval, then the one-dimensional contribution to the energy, namely $\sum_{s \in \partial E \cap I} r_M(E, s)$, is comparable to the periodic case up to a constant $C_0$ depending only on the dimension.

**Lemma 5.1.** There exists $C_0 > 0$ such that the following holds. Let $E \subset \mathbb{R}$ be a set of locally finite perimeter and $I \subset \mathbb{R}$ be an open interval. Let $s^-, s$ and $s^+$ be three consecutive points on the boundary of $E$ and $r_M(E, s)$ defined as in (5.2). Then there exists $M_0 > 0$ such that for all $M > M_0$ it holds

$$\sum_{s \in \partial E \cap I} r_M(E, s) \geq e_M |I| - C_0. \quad (5.3)$$

The proof is analogous to that of Lemma 7.7 in [9] and therefore we omit it.

The next lemma is the so called local stability Lemma. Informally, it shows that if we are in a cube where the set $E \subset \mathbb{R}^d$ is close to a set $E'$ which is a union of stripes in direction $e_i$ (according to Definition 2.1), then it is not convenient to oscillate in direction $e_j$ with $j \neq i$ (namely, on the slices in direc

**Lemma 5.2** (Local Stability). Let $(t_i^+ + se_i) \in (\partial E) \cap [0,1]^d$ and $0 < \eta_0 < 1$ and $M_0$ as Lemma 4.2. Then, for every $\varepsilon < \eta_0$ there exists $M = \tilde{M}(\varepsilon) > M_0$ such that for every $M > \tilde{M}$ the following holds: assume that

(a) $\min(|s - l|, |s|) > \eta_0$

(b) $D_0^j(E, [0,1]^d) \leq \frac{d}{16\varepsilon^2}$ for some $\eta > 0$ and with $j \neq i$ (this condition expresses that $E \cap [0,1]^d$ is close to stripes oriented along a direction orthogonal to $e_i$)

Then $r_{i,M}(E, t_i^+, s) + v_{i,M}(E, t_i^+, s) \geq 0$.  

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Proof. Let \( s^-, s, s^+ \) be three consecutive points for \( \partial E_{t_i^+} \). By Lemma 4.2 for all \( 0 < \eta_0 < 1 \), there exists \( M_0 > 0 \) such that if \( M > M_0 \)

\[
\min(|s - s^-|, |s^+ - s|) < \eta_0 \quad \text{then} \quad r_{i,M}(E, t_i^+, s) > 0.
\]

Thus without loss of generality we may assume that \( \min(|s - s^-|, |s^+ - s|) \geq \eta_0 \).

Thus, given that, for every \( s \), \( r_{i,M}(E, t_i^+, s) > -e^{-cM} \) for some \( c > 0 \) (see Lemma 4.2), one has that

\[
r_{i,M}(E, t_i^+, s) + v_{i,M}(E, t_i^+, s) \geq -e^{-cM} + \frac{1}{2d} \int_{s^-}^{s^+} \int_{\mathbb{R}^d} f_E(t_i^+, u, \zeta_i, \zeta_i) K_M(\zeta) \, d\zeta \, du \quad (5.4)
\]

Let now \( 0 < \varepsilon < \eta_0 \). By assumption, for some \( t_i \in \partial E_{t_i^+} \) one of the following holds:

(i) \( (t_i - \varepsilon, t_i) \subset E_{t_i^+} \) and \( (t_i, t_i + \varepsilon) \subset E_{t_i^+}^c \)

(ii) \( (t_i - \varepsilon, t_i) \subset E_{t_i^+}^c \) and \( (t_i, t_i + \varepsilon) \subset E_{t_i^+} \).

W.l.o.g., we may assume that (i) above holds and that \( i = d \).

As shown in [9] Lemma 6.1, Lemma 7.8, hypothesis (b) implies that

\[
\max \left( \frac{|Q_{\varepsilon}^+(t_d^+ \times (t_d - \varepsilon, t_d) \cap E^c|}{|Q_{\varepsilon}^+(t_d^+ \times (t_d - \varepsilon, t_d) \cap E^c|} \right) \geq \frac{7}{16}. \quad (5.5)
\]

Thus, we can further assume that

\[
(t_d - \varepsilon, t_d) \subset E_{t_d^+} \quad \text{and} \quad \frac{|Q_{\varepsilon}^+(t_d^+ \times (t_d - \varepsilon, t_d) \cap E^c|}{|Q_{\varepsilon}^+(t_d^+ \times (t_d - \varepsilon, t_d) \cap E^c|} \geq \frac{7}{16}. \quad (5.6)
\]

For every \( s \in (t_d - \varepsilon, t_d) \), \( (\zeta_d^+, s) \not\in E \) and \( \zeta_d + s \in (t_d, t_d + \varepsilon) \) we have that \( f_E(t_d^+, s, \zeta_d^+, \zeta_d) = 1 \).

Thus by integrating initially in \( \zeta_d \) and using (2.11), we have that

\[
\int_{t_d - \varepsilon}^{t_d + \varepsilon - s} \int_{Q_{\varepsilon}^+(t_d^+)} f_E(t_d^+, s, \zeta_d^+, \zeta_d) K_M(\zeta) \, d\zeta_d \, ds \geq \frac{e^{M(\gamma_M - \varepsilon)}}{e^{d - 2}} \varepsilon \int_{Q_{\varepsilon}^+(t_d^+)} \int_{t_d - \varepsilon}^{t_d} |E_{t_d^+}^d(s) - E_{t_d^+ + \varepsilon_d^+}^d(s)| \, ds \, d\zeta_d^+
\]

\[
\geq \frac{e^{M(\gamma_M - \varepsilon)}}{e^{d - 2}} \varepsilon \int_{Q_{\varepsilon}^+(t_d^+)} \int_{t_d - \varepsilon}^{t_d} |1 - \chi_{E_{t_d^+}^d + \varepsilon_d^+}^d(s)| \, ds \, d\zeta_d^+
\]

\[
\geq \frac{e^{M(\gamma_M - \varepsilon)}}{e^{d - 2}} \varepsilon |Q_{\varepsilon}^+(t_d^+ \times (t_d - \varepsilon, t_d) \cap E^c| \geq \frac{7e^{M(\gamma_M - \varepsilon)} \varepsilon^{d + 1}}{16e^{d - 2}},
\]

which tends to \(+ \infty\) as \( M \to + \infty \).

Therefore, for \( M \) sufficiently large depending on \( \varepsilon \) the r.h.s. of (5.4) is positive. Up to a permutation of coordinates, this naturally holds also for \( i = 2, \ldots, d - 1 \). Therefore the lemma is proved.

The following Lemma, analogue of Lemma 7.9 in [9], gives an estimate from below to the contribution of the energy on a segment of a slice in direction \( e_i \).
Lemma 5.3. Let \( 0 < \eta_0 < 1, \tilde{M} \) as in Lemma 5.2. Let \( \delta = \varepsilon^d/(16l^d) \) with \( 0 < \varepsilon \leq \eta_0 \), \( M > \tilde{M} \) and \( l > C_0/(-e_M^*) \), where \( C_0 \) is the constant appearing in Lemma 5.1. Let \( t_i^+ \in [0,L)^{d-1} \) and \( \eta > 0 \). The following statements hold: there exists \( C_1 \) constant independent of \( l \) (but depending on the dimension) such that

(i) Given \( J \subset \mathbb{R} \) such that for every \( s \in J \) it holds \( D_{\eta}^j(E,Q_l(t_i^+ + se_i)) \leq \delta \) with \( j \neq i \), then

\[
\int_J \bar{F}_{i,M}(E,Q_l(t_i^+ + se_i)) \, ds \geq -\frac{C_1}{l}.
\]

Moreover, if \( J = [0,L) \), then

\[
\int_J \bar{F}_{i,M}(E,Q_l(t_i^+ + se_i)) \, ds \geq 0.
\]

(ii) Given \( J = (a,b) \subset \mathbb{R} \). If for \( s = a \) and \( s = b \) it holds \( D_{\eta}^j(E,Q_l(t_i^+ + se_i)) \leq \delta \) with \( j \neq i \), then

\[
\int_J \bar{F}_{i,M}(E,Q_l(t_i^+ + se_i)) \, ds \geq |J|e_M^* - \frac{C_1}{l},
\]

otherwise

\[
\int_J \bar{F}_{i,M}(E,Q_l(t_i^+ + se_i)) \, ds \geq |J|e_M^* - C_1l.
\]

Moreover, if \( J = [0,L) \), then

\[
\int_J \bar{F}_{i,M}(E,Q_l(t_i^+ + se_i)) \, ds \geq |J|e_M^*.
\]

Proof. For the proof we refer to Lemma 7.9 in [9]. \( C_1 \) correspond to \( M_0 \) there, \( M \) to \( \tau, \eta_0 \) to \( \varepsilon \) and \( e_M^* \) to \( C_\tau^* \). In the proof one uses Lemma 4.2 and Lemma 5.2. Here, the estimate \( r_{i,\tau}(E,t_i^+,s) \geq -1 \) is replaced by \( r_{i,M}(E,t_i^+,s) \geq -e^{-cM} \) (see Lemma 4.2). \( \Box \)

The purpose of the next lemma is to give a lower bound on the energy in the case that almost all the volume of \( Q_l(z) \) is filled by \( E \) or \( E^c \) (as on the set \( A_{-1} \) defined in (5.24)).

Lemma 5.4. Let \( E \) be a set of locally finite perimeter such that \( \min(|Q_l(z) \setminus E|, |E \cap Q_l(z)|) \leq \nu l^d \), for some \( \nu > 0 \). Then

\[
\bar{F}_M(E,Q_l(z)) \geq -\frac{e^{-cM}\nu d}{\eta_0},
\]

where \( \eta_0 < 1 \), provided \( M \geq M_0(\eta_0) \) as in Lemma 4.2.

For the proof we refer to Lemma 7.11 in [9], substituting the lower bound \( r_{i,\tau}(E,t_i^+,s) \geq -1 \) with \( r_{i,M}(E,t_i^+,s) \geq -e^{-cM} \) and \( \delta \) with \( \nu \).
5.2 Proof of Theorem 1.1

The strategy is analogous to that of [9][Theorem 1.4]. The sets defined in the proof and the main estimates will depend on a set of parameters \(l, \delta, \rho, N, \eta\) and \(M\). The validity of the theorem relies on a suitable choice of such parameters. We fix them now, making clear how they depend on each other. The reason for such choice may seem now obscure but it will be clarified during the proof.

Let \(0 < \sigma < -\varepsilon^* / 2\), where \(\varepsilon^* = \lim_{M \to +\infty} \varepsilon_M^*\). Notice that \(\varepsilon^* < 0\).

- Initially we fix \(l > 0\) such that
  \[
  l > \max \left\{ \frac{dC_d}{-\varepsilon^* - \sigma}, \frac{C_0}{-\varepsilon^* - \sigma} \right\},
  \tag{5.12}
  \]
  where \(C_d\) is a constant (depending only on the dimension \(d\)) that appears in (5.25), and \(C_0\) is the constant which appears in the statement of Lemma 5.1.

- Let \(0 < \eta_0 < 1\), \(M_0\) as in Lemma 4.2 and \(0 < \varepsilon < \eta_0\). Then from Lemma 5.2, have the parameter \(\tilde{M} = \tilde{M}(\varepsilon, M_0)\).

- We then fix \(M > \tilde{M}\) as in Lemma 5.3. Thus we obtain \(\delta\) defined by \(\delta = \frac{\varepsilon^d}{16}\). Moreover, by choosing \(\varepsilon\) sufficiently small we can additionally assume that if for some \(\eta > 0\)
  \[
  D_\eta^i(E, Q_l(z)) \leq \delta \quad \text{and} \quad D_\eta^j(E, Q_l(z)) \leq \delta, \ i \neq j \quad \Rightarrow \quad \min\{|E \cap Q_l(z)|, |E^c \cap Q_l(z)|\} \leq \varepsilon^{d-1}.
  \tag{5.13}
  \]
  This follows from Remark 2.2 (ii).

- Thanks to Remark 2.2 (i), we then fix
  \[
  \rho \sim \delta l,
  \tag{5.14}
  \]
  in such a way that for any \(\eta\) the following holds
  \[
  \forall z, z' \text{ s.t. } D_\eta(E, Q_l(z)) \geq \delta, |z - z'|_{\infty} \leq \rho \quad \Rightarrow \quad D_\eta(E, Q_l(z')) \geq \delta/2.
  \tag{5.15}
  \]

- Then, we fix \(N\) such that
  \[
  \frac{N \rho}{2d} > C_1 l.
  \tag{5.16}
  \]

- From Lemma 4.1 we obtain \(\bar{\eta} = \bar{\eta}(N, l)\) and \(\tilde{M} = \tilde{M}(N, l, \delta/2)\). Hence we fix
  \[
  0 < \eta < \bar{\eta}, \quad \bar{\eta} = \bar{\eta}(N, l)
  \tag{5.17}
  \]

- Finally, we choose \(M > 0\) s.t.
  \[
  M > M_0 \quad \text{as in Lemma 4.2}
  \tag{5.18}
  \]
  \[
  M > \tilde{M}, \quad \tilde{M} \text{ as in Lemma 5.2 and Lemma 5.3}
  \tag{5.19}
  \]
  \[
  M > \bar{M}, \quad \bar{M} \text{ as in Lemma 4.1 depending on } N, l, \delta/2 \text{ and } \eta
  \tag{5.20}
  \]
  and
  \[
  M \text{ s.t. } e_M^* < e^* + \sigma.
  \tag{5.21}
  \]
Notice that, by the local rigidity Proposition 4.1, \( \exists \hat{M} \) such that if \( M > \hat{M} \), then (5.24) holds. In particular, (5.12) is satisfied with \(-e_M^* \) instead of \(-e^* - \sigma\).

Given such parameters, let us prove the theorem for any \( L > l \) of the form \( L = 2kh_M^* \), with \( k \in \mathbb{N} \).

Let \( E \) be a minimizer of \( F_{M,L} \). Since \( E \) is \( L \)-periodic, we can consider \( E \subset \mathbb{T}_L^d \), where \( \mathbb{T}_L^d \) is the \( d \)-dimensional torus of size \( L \). With a slight abuse of notation, we will denote by \([0,L]^d \) the cube of size \( L \) with the usual identification of the boundary.

We define

\[
\tilde{A}_0 := \left\{ z \in [0,L)^d : D_\eta(E,Q_l(z)) \geq \delta \right\}.
\]

Hence, by the choice of \( \eta \) and \( M \) made in (5.17) and (5.20) and by Lemma 4.1, for every \( z \in \tilde{A}_0 \) one has that \( \tilde{F}_M(E,Q_l(z)) > N \).

We now introduce the set \( \tilde{A}_{-1} \)

\[
\tilde{A}_{-1} := \left\{ z \in [0,L)^d : \exists i,j \text{ s.t. } D_\eta(E,Q_l(z)) \leq \delta, D^i_\eta(E,Q_l(z)) \leq \delta \right\}.
\]

It is not difficult to see that \( \tilde{A}_0 \) and \( \tilde{A}_{-1} \) are closed.

Given (5.14), one has that (5.15) holds, namely for every \( z \in \tilde{A}_0 \) and \( |z - z'|_\infty \leq \rho \) one has that \( D_\eta(E,Q_l(z')) > \delta/2 \).

Moreover, since \( \delta \) satisfies (5.13), whenever \( z \in \tilde{A}_{-1} \), one has that \( \min(|E \cap Q_l(z)|, |Q_l(z) \setminus E|) \leq l^d - 1 \).

Thus, using Lemma 5.4 with \( \nu = 1/l \), one has that

\[
\tilde{F}_M(E,Q_l(z)) \geq \frac{1}{l}.
\]

Moreover, let now \( z' \) such that \( |z - z'|_\infty \leq 1 \) with \( z \in \tilde{A}_{-1} \). It is not difficult to see that if \( |Q_l(z) \setminus E| \leq l^d - 1 \) then \( |Q_l(z') \setminus E| \leq l^d - 1 \). Thus from Lemma 5.4 one has that

\[
\tilde{F}_M(E,Q_l(z')) \geq -\frac{C_d}{l}.
\]

Therefore we define

\[
A_0 := \left\{ z' \in [0,L)^d : \exists z \in \tilde{A}_0 \text{ with } |z - z'|_\infty \leq \rho \right\},
\]

\[
A_{-1} := \left\{ z' \in [0,L)^d : \exists z \in \tilde{A}_{-1} \text{ with } |z - z'|_\infty \leq 1 \right\},
\]

By the choice of the parameters and the observations above, for every \( z \in A_0 \) one has that \( \tilde{F}_M(E,Q_l(z)) > N \) and for every \( z \in A_{-1} \), \( \tilde{F}_M(E,Q_l(z)) \geq -\tilde{C}_d/l \).

Let us denote by \( A := A_0 \cup A_{-1} \). The advantage of replacing \( \tilde{A}_0 \), \( \tilde{A}_{-1} \) with \( A_0 \), \( A_{-1} \) is that now there exists \( \rho \) (independent of \( L, M \)) such that if \( z \in A_0 \), then \( \exists z' \text{ s.t. } Q_\rho(z') \subset A_0 \) and \( z \in Q_\rho(z') \), while if \( z \in A_{-1} \) then \( \exists z' \text{ s.t. } Q_\rho(z') \subset A_{-1} \) and \( z \in Q_\rho(z') \).

Let us consider the set \( [0,L)^d \setminus A \): for every \( z \in [0,L)^d \setminus A \), there exists \( i \in \{1, \ldots, d\} \) such that \( D^i_\eta(E,Q_l(z)) \leq \delta \) and for every \( k \neq i \) one has that \( D^k_\eta(E,Q_l(z)) > \delta \).

Since the set \( A \) is closed, we can consider the connected components \( C_1, \ldots, C_n \) of \([0,L)^d \setminus A\). Such components are path-wise connected and for each of them, \( C_j \), there exists \( i \) such that \( D^i_\eta(E,Q_l(z)) \leq \delta \) for every \( z \in C_j \) and for every \( k \neq i \) one has that \( D^k_\eta(E,Q_l(z)) > \delta \). We then say that \( C_j \) is oriented in direction \( e_i \).
We define $A_i$ as the union of the connected components $C_j$ such that $C_j$ is oriented along the direction $e_i$.

In this way, the sets $A = A_{-1} \cup A_0, A_1, A_2, \ldots, A_d$ form a partition of $[0, L]^d$ and one can show, analogously to $[9]$, that for every $z \in A_i$ and $z' \in A_j$ one has that there exists a point $\tilde{z}$ in the segment connecting $z$ to $z'$ lying in $A_0 \cup A_{-1}$.

Let us denote by $B$ the set

Our main goal is to prove the following lower bound: for every $i$, $M$ as in (5.18)-(5.20), the following holds

\[ \frac{1}{L^d} \int_B \tilde{F}_{i,M}(E, Q_l(z)) \, dz + \frac{1}{dL^d} \int_A \tilde{F}_M(E, Q_l(z)) \, dz \geq \frac{e_M^* |A_i|}{L^d} - C_d |A| \]

(5.25)

for some constant $C_d$ depending on the dimension $d$. Indeed, from (5.25), summing over $i$ we obtain

\[ \mathcal{F}_{M,L}(E) \geq \sum_{i=1}^d \frac{1}{L^d} \int_{[0,L]^d} \tilde{F}_{i,M}(E, Q_l(z)) \, dz \geq \frac{e_M^*}{L^d} \sum_{i=1}^d |A_i| - \frac{dC_d |A|}{L^d} \]

(5.26)

\[ \geq e_M^* - e_M^* |A| - \frac{dC_d |A|}{L^d} \]

where in the above $e_M^*$ is the energy density of optimal stripes of width $h_M^*$ and we have used that $e_M^* < 0$ and that $|A| + \sum_{i=1}^d |A_i| = |[0, L]^d| = L^d$.

Notice that, in (5.26), equality holds only if $|A| = 0$ and therefore since this happens only if there is just one $A_i$, $i > 0$ with $|A_i| > 0$, it has been proved that there exists $i > 0$ with $A_i = [0, L)^d$. To conclude, let us consider

\[ \frac{1}{L^d} \int_{[0,L]^d} \tilde{F}_M(E, Q_l(z)) \, dz = \frac{1}{L^d} \int_{[0,L]^d} \tilde{F}_{i,M}(E, Q_l(z)) \, dz \]

(5.27)

\[ + \frac{1}{L^d} \int_{j \neq i} \int_{[0,L]^d} \tilde{F}_{j,M}(E, Q_l(z)) \, dz \]

(5.28)

Let us now apply Lemma 5.3 with $j = i$ and slice the cube $[0, L)^d$ in direction $e_i$. From (5.28), one has that (5.27) is nonnegative and strictly positive unless the set $E$ is a union of stripes in direction $e_i$. On the other hand, from (5.11), one has the r.h.s. of (5.21) is minimized by a periodic union of stripes in direction $e_i$ and with width $h_M^*$ and the theorem is proved.

Now notice that (5.25) follows from the analogous statement on the slices, namely that for every $t_i^+ \in [0, L)^{d-1}$, it holds

\[ \frac{1}{L^d} \int_{B_{t_i^+}} \tilde{F}_{i,M}(E, Q_l(t_i^+ + se_i)) \, ds + \frac{1}{dL^d} \int_{A_{t_i^+}} \tilde{F}_M(E, Q_l(t_i^+ + se_i)) \, ds \geq \frac{e_M^* |A_i|}{L^d} - C_d |A_{t_i^+}| \]

(5.29)

Indeed by integrating (5.29) over $t_i^+$ we obtain (5.25). Therefore, the rest of the proof will be devoted to proving (5.29).

In order to prove (5.29) one proceeds using Lemma 5.3 and the properties of the sets $A_i$, $A$ as in $[9]$. We include the proof for the reader’s convenience.
Let \( \{I_1, \ldots, I_n\} \) such that \( \bigcup_{j=1}^n I_j = B_{i^+} \) with \( I_j \cap I_k = \emptyset \) whenever \( j \neq k \). We can further assume that \( I_i \leq I_{i+1} \), namely that for every \( s \in I_i \) and \( s' \in I_{i+1} \) it holds \( s \leq s' \). By construction there exists \( J_j \subset A_{i^+} \) such that \( I_j \leq J_j \leq I_{j+1} \).

Thus we have that

\[
\frac{1}{L^d} \int_{B_{i^+}} \bar{F}_{i,M}(E, Q_l(t_i^+ + se_i)) \, ds + \frac{1}{dL^d} \int_{A_{i^+}} \bar{F}_{M}(E, Q_l(t_i^+ + se_i)) \, ds \\
\geq \frac{1}{L^d} \sum_{j=1}^n \int_{I_j} \bar{F}_{i,M}(E, Q_l(t_i^+ + se_i)) \, ds + \frac{1}{dL^d} \sum_{j=1}^n \int_{J_j} \bar{F}_{M}(E, Q_l(t_i^+ + se_i)) \, ds \\
\geq \frac{1}{L^d} \sum_{j=1}^n \left( \int_{I_j} \bar{F}_{i,M}(E, Q_l(t_i^+ + se_i)) \, ds + \frac{1}{2d} \int_{J_j \setminus J_{j-1}} \bar{F}_{M}(E, Q_l(t_i^+ + se_i)) \, ds \right).
\]

Let first \( I_j \subset A_{i^+} \). By construction, we have that \( \partial I_j \subset A_{i^+} \).

If \( \partial I_j \subset A_{-1,t_i^+} \), thanks to (5.12) and (5.21), we can apply (5.9) in Lemma 5.3 with \( J = I_i \) and obtain

\[
\frac{1}{L^d} \int_{I_j} \bar{F}_{i,M}(E, Q_l(t_i^+ + se_i)) \, ds \geq \frac{1}{L^d} \left( I_j|e_M^* - C_1 \right). \tag{5.30}
\]

If \( \partial I_j \cap A_{0,t_i^+} \neq \emptyset \), thanks to (5.12) and (5.21), we can apply (5.10) in Lemma 5.3 with \( J = I_i \) and obtain

\[
\frac{1}{L^d} \int_{I_j} \bar{F}_{i,M}(E, Q_l(t_i^+ + se_i)) \, ds \geq \frac{1}{L^d} \left( J_j|e_M^* - C_1 l \right). \tag{5.31}
\]

On the other hand, if \( \partial I_j \cap A_{0,t_i^+} \neq \emptyset \), we have that either \( J_j \cap A_{0,t_i^+} \neq \emptyset \) or \( J_j-1 \cap A_{0,t_i^+} \neq \emptyset \). Remembering that whenever \( J_j \cap A_{0,t_i^+} \neq \emptyset \), we have that \( |J_j| > \rho \), that for every \( z \in A_0 \) one has that \( \bar{F}_{M}(E, Q_l(z)) > N \) and that, by (5.22) for every \( z \in A_{-1} \) \( \bar{F}_{M}(E, Q_l(z)) \geq -\bar{C}_d/|l| \), then

\[
\frac{1}{2dL^d} \int_{J_{j-1}} \bar{F}_{M}(E, Q_l(t_i^+ + se_i)) \, ds + \frac{1}{2dL^d} \int_{I_j} \bar{F}_{M}(E, Q_l(t_i^+ + se_i)) \, ds \\
\geq \frac{N \rho}{2dL^d} \frac{|J_{j-1} \cap A_{-1,t_i^+}| \bar{C}_d}{2dL^d} - \frac{|J_j \cap A_{-1,t_i^+}| \bar{C}_d}{2dL^d}. \tag{5.32}
\]

Since \( N \) satisfies (5.16), in both cases \( \partial I_j \subset A_{-1,t_i^+} \) or \( \partial I_j \cap A_{0,t_i^+} \neq \emptyset \), summing (5.32) to either (5.30) or (5.31) we have that

\[
\frac{1}{L^d} \int_{I_j} \bar{F}_{i,M}(E, Q_l(t_i^+ + se_i)) \, ds + \frac{1}{2dL^d} \int_{J_{j-1}} \bar{F}_{M}(E, Q_l(t_i^+ + se_i)) \, ds + \frac{1}{2dL^d} \int_{J_j} \bar{F}_{M}(E, Q_l(t_i^+ + se_i)) \, ds \\
\geq \frac{e_M^* |I_j|}{L^d} - \frac{|J_{j-1} \cap A_{-1,t_i^+}| \bar{C}_d}{2dL^d} - \frac{|J_j \cap A_{-1,t_i^+}| \bar{C}_d}{2dL^d}.
\]

If \( I_j \subset A_{k,t_i^+} \) with \( k \neq i \) from Lemma 5.3 Point (i) it holds

\[
\frac{1}{L^d} \int_{I_j} \bar{F}_{i,M}(E, Q_l(t_i^+ + se_i)) \, ds \geq \frac{C_1}{L^d}.
\]

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In general for every $J_j$ we have that

$$\frac{1}{2dL^d} \int_{J_j} \tilde{F}_M(E, Q_l(t_i^+ + se_i)) \, ds \geq \frac{|J_j \cap A_{0,t_i^+}| |N|}{2dL^d} - \frac{\tilde{C}_d}{2dL^d} |J_j \cap A_{-1,t_i^+}|.$$

For $I_j \subset A_{k,t_i^+}$ such that $(I_j \cup J_{j-1}) \cap A_{0,t_i^+} \neq \emptyset$ with $k \neq i$, remembering that whenever $J_j \cap A_{0,t_i^+} \neq \emptyset$ one has $|J_j| > \rho$, we have that

$$\frac{1}{L^d} \int_{I_j} \tilde{F}_i,M(E, Q_l(t_i^+ + se_i)) \, ds + \frac{1}{2dL^d} \int_{J_{j-1}} \tilde{F}_M(E, Q_l(t_i^+ + se_i)) \, ds + \frac{1}{2dL^d} \int_{J_j} \tilde{F}_M(E, Q_l(t_i^+ + se_i)) \, ds$$

$$\geq -\frac{C_1}{2dL^d} + \frac{N\rho}{2dL^d} \frac{|J_{j-1} \cap A_{-1,t_i^+}| \tilde{C}_d}{2dL^d} - \frac{|J_j \cap A_{-1,t_i^+}| \tilde{C}_d}{2dL^d}.$$

where the last inequality is true due to (5.10).

For $I_j \subset A_{k,t_i^+}$ such that $(I_j \cup J_{j-1}) \subset A_{-1,t_i^+}$ with $k \neq i$, we have that

$$\frac{1}{L^d} \int_{I_j} \tilde{F}_i,M(E, Q_l(t_i^+ + se_i)) \, ds + \frac{1}{2dL^d} \int_{J_{j-1}} \tilde{F}_M(E, Q_l(t_i^+ + se_i)) \, ds + \frac{1}{2dL^d} \int_{J_j} \tilde{F}_M(E, Q_l(t_i^+ + se_i)) \, ds$$

$$\geq -\frac{C_1}{2dL^d} \frac{|J_{j-1} \cap A_{-1,t_i^+}| \tilde{C}_d}{2dL^d} - \frac{|J_j \cap A_{-1,t_i^+}| \tilde{C}_d}{2dL^d}.$$

where in the last inequality we have used that $|J_j \cap A_{-1,t_i^+}| \geq 1$, $|J_{j-1} \cap A_{-1,t_i^+}| \geq 1$, since by the “thickness” of $A_{-1}$ whenever $J_j \cap A_{-1,t_i^+} \neq \emptyset$ then $|J_i| > 1$.

Summing over $j$, and taking $C_d = \max \left( C_1, \frac{\tilde{C}_d}{d} \right)$, one obtains (5.29) as desired.

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