Equipartition of energy and the first law of thermodynamics at the apparent horizon

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We apply the holographic principle and the equipartition law of energy to the apparent horizon of a Friedmann-Robertson-Walker universe and derive the Friedmann equation describing the dynamics of the universe. We also show that the equipartition law of energy can be interpreted as the first law of thermodynamics at the apparent horizon. The consistency check shows that our derivation is correct for $-1 < w < -\frac{1}{3}$, a value matches the recent cosmological observations.

Keywords: Holography, Friedmann Equations, Equipartition Law, Apparent Horizon Thermodynamics.

1. Introduction

The entropy of a black hole is 1/4 of the area of the event horizon of the black hole measured in Planck units. This area law of entropy was generalized to more general systems, and later was taken as a general principle: the so called holographic principle. The strongest evidence of the holographic principle was provided by the AdS/CFT correspondence which states that all information about a gravitational system in a spatial region is encoded in its boundary. On the other hand, the thermodynamic laws of black holes and de-Sitter space-time suggest a deep connection between gravitation and thermodynamics. By using the area law of entropy for all local Rindler horizons, Einstein field equation was derived from the first law of thermodynamics.

Recently, the thermodynamic relation $S = E/2T$ between the entropy $S$, active gravitational energy $E$ and temperature $T$ was reinterpreted as the equipartition law of energy $E = NT/2$, where $N$ is the number of bits on the horizon. Combining the equipartition law of energy and the holographic principle, Einstein field equation was derived. Furthermore, gravity was explained as an entropic force caused by changes in the information associated with the positions of material bodies, and
this idea was extensively discussed in the literature.

Motivated by Bekenstein's original thought experiment about black holes, Verlinde considered a small piece of a spherical holographic screen, and a particle with mass $m$ approaching it from the side, with the change of entropy near the screen assumed to be

$$\Delta S = 2\pi \frac{mc}{\hbar} \Delta x. \quad (1)$$

The effective force acting on the particle due to the change of entropy is

$$F \Delta x = T \Delta S. \quad (2)$$

Because an observer in an accelerated frame has the Unruh temperature

$$T = \frac{1}{2\pi} \hbar a, \quad (3)$$

so we get Newton's second law $F = ma$. Although it seems that we derived Newton's second law for any force, actually the derivation is satisfied for gravitational force only. Suppose the closed holographic screen with radius $R$ can be divided into $N = A/(c_1 L_P^2)$ microscopic cells, where $L_P \equiv \sqrt{G\hbar}$ (we use units with $c = k_B = 1$ throughout this paper) is the Planck length and $c_1$ is a numerical factor whose value will be determined later. If each cell has $c_2$ microscopic configurations, then the entropy of the screen is

$$S = N \ln c_2 = \frac{4 \ln c_2}{c_1} \frac{A}{4 L_P^2}. \quad (4)$$

Once $4 \ln c_2 = c_1$ is chosen, the standard area law, $S = (A/4L_P^2)$ recovers. Recently it was shown that the entropy of a holographic screen is 1/4 of its area for static holographic screens if gravity is interpreted as an entropic force. If one assumes that each cell of area $c_1 L_P^2$ contributes an energy $\frac{1}{2}T$, according to the equipartition law, we get the total energy

$$\mathcal{E} = \frac{1}{2} N T = \frac{1}{2} \frac{A}{c_1 L_P^2} T = M. \quad (5)$$

so the Unruh temperature reads

$$T = \frac{2M c_1 L_P^2}{4\pi R^2} = \frac{1}{2\pi} \hbar a, \quad (6)$$

and the acceleration is

$$a = \frac{c_1 GM}{R^2}. \quad (7)$$

If $c_1 = 1$, then we get the Newton's law of gravitation. So we choose $c_1 = 1$.

By generalizing this approach to the relativistic case, Einstein equation can be derived. As we discussed above, Einstein field equation can also be derived from the first law of thermodynamics, and the Friedmann equation for several gravity theories was derived from the first law of thermodynamics at the apparent horizon
(AH) a natural question raised is whether there is any connection between
the first law of thermodynamics and the equipartition law of energy. In this paper,
we address this problem by applying this approach to the apparent horizon in cos-
mos. We first show that the first law of thermodynamics at the apparent horizon
is equivalent to the equipartition law of energy at the apparent horizon, then we
show that Friedmann equation can be derived from the equipartition law of energy
and the holographic principle at the apparent horizon. Finally, we make a consis-
tency check of our derivation. It tells us that our derivation is correct only when the
state parameter satisfies $-1 < w < -\frac{1}{3}$, a value matches the recent cosmological
observations. This constraint opens a door for applying Verlinde’s scheme to the
study of dark energy.

2. Derivation of the Friedmann equations

Generally speaking, except the above two hypotheses, i.e., the equipartition law and
the holographic principle at the apparent horizon, our derivation of the Friedmann
equations needs one more assumption: there is energy flux through the apparent
horizon. The basic idea is that by assuming an energy flux through the apparent
horizon with area $A$, the energy $\mathcal{E}$ which is enclosed by the apparent horizon in-
creases with the time. According to the equipartition law, this leads to the changes
of temperature and number of bits of the screen, explicitly:

$$\Delta \mathcal{E} = \frac{1}{2} \Delta (NT) = \frac{1}{2} T_A \Delta N + \frac{1}{2} N \Delta T_A,$$

where $T_A$ and $N$, respectively, represent the temperature and the number of bits
of the holographic screen. By identifying $\Delta N$ with the changes of the area of the
apparent horizon $\Delta A$ through the holographic principle, we derive the Friedmann
equation. This is roughly sketched in figure (1). This counter-intuition picture can
be well understood as following: we take the apparent horizon as a holographic
screen, increase of the energy of the region enclosed by a horizon inevitably leads
to the increase of the entropy of that region. Holographic principle tells us that this
corresponds to the increase of the horizon area. On the other hand, the flux of fluid
through the horizon implies our uncertainty about the region outside the horizon
increases. This is equivalent to the increase of the entropy of the horizon, and again
leads to the increase of the horizon area.

To address our derivation more explicitly, let us focus on a (3+1)-dimensional

\footnote{Note that this does not mean that we are using two holographic screens with different temperatures. Instead, we consider the same screen at an infinitesimal later time with number of bits $(N + \Delta N)$ and temperature $(T_A + \Delta T_A)$. Eq. (8) is the first-order approximation, i.e.,

$$\Delta \mathcal{E} = \frac{1}{2} (N + \Delta N)(T_A + \Delta T_A) - \frac{1}{2} N T_A = \frac{1}{2} N \Delta T_A + \frac{1}{2} T_A \Delta N + \text{higher orders}.$$}
Fig. 1. Fluid flow out through the apparent horizon (AH) leads to the increase of the radius of the AH from $R$ to $R + dR$.

Friedmann-Robertson-Walker (FRW) universe with the metric

$$ds^2 = h_{ab}dx^a dx^b + \tilde{r}^2 d\Omega_2^2,$$

where $\tilde{r} = a(t)r$ and the 2-dimensional metric $h_{ab} = \text{diag}(-1, a^2/(1 - kr^2))$ with $k = 0, -1, 1$ corresponding to a flat, open and closed universe respectively. The apparent horizon, which is defined by the relation $h_{ab} \partial_a \tilde{r} \partial_b \tilde{r} = 0$, turns out to be

$$\tilde{r}_A = \frac{1}{\sqrt{H^2 + k/a^2}},$$

where $H \equiv \dot{a}/a$ denotes the Hubble parameter. The apparent horizon with area $A = 4\pi \tilde{r}_A^2$ carries $N = 4\pi \tilde{r}_A^2/L_p^2$ bits of information. Suppose during the infinitesimal time interval $dt$, the radius of the apparent horizon evolves from $\tilde{r}_A$ to $\tilde{r}_A + d\tilde{r}_A$, then the change of the area of the AH is $dA = 8\pi \tilde{r}_A d\tilde{r}_A$. So the number of bit is increased by

$$dN = \frac{8\pi \tilde{r}_A}{L_p^2} d\tilde{r}_A.$$

On the other hand, the change of the Hawking temperature $T_A = \hbar/(2\pi \tilde{r}_A)$ is

$$dT_A = -\frac{\hbar}{2\pi \tilde{r}_A} d\tilde{r}_A.$$

From Eq. (5), we get the changes of the total energy (similar to Eq. (8))

$$dE = \frac{1}{2}dT_A + \frac{1}{2}T_A dN = \frac{d\tilde{r}_A}{G}.$$

\(^b\text{Although the definition of temperature for a dynamic horizon like apparent horizon, as compared to a stationary horizon, has not yet fully understood, it was shown that there exists a Hawking radiation with temperature } T_A = \hbar/2\pi \tilde{r}_A \text{ for locally defined AH}\)
Note that the entropy of the apparent horizon is\[ S_A = \frac{\pi \tilde{r}_A^2}{L_P^2}, \]so\[ T_A dS_A = d\tilde{r}_A/G. \]This shows the equivalence between the equipartition law of energy and the first law of thermodynamics,\[ d\mathcal{E} = T_A dS_A. \]From the definition of the apparent horizon\[ (10), \]we get\[ d\tilde{r}_A = -H\tilde{r}_A^3 \left( \dot{H} - \frac{k}{a^2} \right) dt. \] (14)

Now we discuss the energy flow through the apparent horizon within a time interval\[ dt. \]Because the energy-momentum tensor of the matter in the universe is a perfect fluid,\[ T_{\mu\nu} = (\rho + p)U_\mu U_\nu + pg_{\mu\nu}, \]where\[ \rho \]and\[ p \]are the energy density and pressure, respectively, the energy flow out through the horizon is\[ -dE = d\mathcal{E} = 4\pi \tilde{r}_A^2 T_{\mu\nu} k^\mu k^\nu dt = 4\pi \tilde{r}_A^3 (\rho + p) H dt, \] (15)

where the (approximate) Killing vector or the (approximate) generator of the horizon, the future directed ingoing null vector field\[ k^\mu = (1, -Hr, 0, 0). \]Combining Eqs. (13), (14) and (15), we get the following equation\[ \dot{H} - \frac{k}{a^2} = -4\pi G (\rho + p). \] (16)

Using the energy conservation equation\[ \dot{\rho} + 3H (\rho + p) = 0, \] (17)
we get the Friedmann equation\[ H^2 + \frac{k}{a^2} = \frac{8\pi G}{3} \rho. \] (18)

In the above formulae, an integration constant, which can be regarded as a cosmological constant, has been dropped out.

### 3. Consistency check of the derivation

During our derivation of the Friedmann equations, we have, as mentioned in the last section, made an assumption that the energy is flowed out through the apparent horizon. This is the only robust of our derivation, and hence it deserves further investigation. To prove or disprove this assumption one can make a consistency check of the derivation\(^\text{e}\). In other words, starting from the correct Friedmann equations obtained by standard way (for example by solving the Einstein’ field equations), we consider if it is possible to have an energy flux out through the apparent horizon.

For simplicity, we only consider\[ k = 0, \]cases with\[ k \neq 0 \]can be done in the same way. Then from the definition of\[ \dot{r}_A \]we obtain the expansion rate of the apparent horizon\[ \dot{r}_A = -\frac{\dot{H}}{H^2}. \]

\(^e\)We would like to thank Professor Yong-Shi Wu for his suggestion on this consistency check and for his valuable and stimulated discussions.
After combining the Friedmann equations (16) and (18), this can be written as
\[ \dot{\tilde{r}}_A = \frac{3}{2}(1 + w), \] (20)
where we have introduced a parameter \( w \) which is associated with the equation of state \( P = w \rho \).

To examine if the fluids will flow out through the horizon, we only need to consider the fluid very close to the horizon. Recalling to the definition of the apparent horizon which can be defined as the boundary above which the fluid has velocities more than 1—the velocity of light, we then obtain the velocity of the fluid (inside the horizon) very close to the horizon (denoted by \( \dot{\tilde{r}}_f \))
\[ \dot{\tilde{r}}_f = 1 - \varepsilon, \] (21)
where \( 0 < \varepsilon \ll 1 \). Comparing \( \dot{\tilde{r}}_A \) with \( \dot{\tilde{r}}_f \) we are able to learn if the fluid flows out through the horizon, i.e.,
\[ \dot{\tilde{r}}_A - \dot{\tilde{r}}_f = \frac{1}{2} + \frac{3}{2} w + \varepsilon \simeq \frac{1}{2} + \frac{3}{2} w. \] (22)

According to our derivation of Friedmann equations, two conditions should be fulfilled: the flux of energy flows out through the apparent horizon and the area of the horizon increases. This is equivalent to require that \( \dot{\tilde{r}}_A - \dot{\tilde{r}}_f < 0 \) and \( \dot{\tilde{r}}_A > 0 \). From Eqs. (20) and (22) this can be satisfied when \(-1 < w < -\frac{1}{3}\). Therefore, our derivation of the Friedmann equations is self-consistent if we impose a constraint on the state parameter by \(-1 < w < -\frac{1}{3}\), a value matches recent cosmological observations\( \text{d} \).

4. Conclusions

Motivated by the work on the origin of gravity\( \text{8} \), we derived the Friedmann equation from the equipartition law of energy and the holographic principle. We also show that the equipartition law of energy can be interpreted as the first law of thermodynamics at the apparent horizon. This suggests that the equipartition law of energy plays a fundamental role. Although there are many unresolved issues on Verlinde’s proposal, it has, at least in some extent, transmitted a message that it is possible for the gravity to have a thermodynamic origin, and our universe in this context is driven by the so-called entropic force.

The consistency check shows that the derivation in our way is not always effective—it will be invalid if the state parameter is out of the region \(-1 < w < -\frac{1}{3}\). This defect does not affect the correctness of our derivation, on the contrary, the “defect” may provide us with a natural requirement of dark energy\( \text{13} \).

\( \text{d} \)Actually, this constraint simply implies that the universe is accelerated, since from (19)–(22), we see that
\[ \dot{\tilde{r}}_A - \dot{\tilde{r}}_f \simeq -\frac{\dot{H}}{H^2} - 1 = -\frac{\ddot{a}}{a^2} < 0 \]
implies that \( \ddot{a} > 0 \).
It also should be noted that our derivation of Friedmann equations is different from Jacobson’s previous work\footnote{5} in that: (i) Our derivation is not restricted to the local Rindler causal horizon, and a uniformly accelerated frame is not needed; (ii) Instead of using the first law of thermodynamics, we are using the equipartition law of energy.

**Note added.** After the completion of our paper, there are two relative works\footnote{14,15} on the derivation, with different methods, of Friedmann equations appeared.

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