Time-Dependent Open String Solutions in 2+1 Dimensional Gravity

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Abstract

We find general, time-dependent solutions produced by open string sources carrying no momentum flow in 2+1 dimensional gravity. The local Poincaré group elements associated with these solutions and the coordinate transformations that transform these solutions into Minkowski metric are obtained. We also find the relation between these solutions and the planar wall solutions in 3+1 dimensions.
1. Introduction

There has recently been much interest in $2 + 1$ dimensional gravity [1]. Three dimensional Einstein gravity has the important feature that spacetime is locally flat away from sources. Nevertheless solutions can be globally non-trivial. It also admits a gauge formulation with the action given by the Chern-Simons action for the $2 + 1$ dimensional Poincaré group $ISO(2, 1)$ [2]. In this work we study general, time-dependent solutions produced by open string sources (which are translational invariant along the direction of the string) carrying no momentum flow. Static string solutions in $2 + 1$ dimensions were previously studied by Deser and Jackiw [3]. Stationary string solutions were also studied [4].

The remainder of this paper is organized as follows. In Sec.II we obtain vacuum solutions away from the string, and study their gauge formulation. In Sec.III we match these vacuum solutions across the string using junction conditions. We then obtain the relation between these solutions and the $3 + 1$ dimensional planar wall solutions studied by Vilenkin, Ipser and Sikivie [5] [6]. Brief conclusions are given in Sec.IV. We study the geodesic motion of test particles in a typical background metric in the appendix.

2. Vacuum Solutions and Their Gauge Formulation

In this section we shall find vacuum solutions for the following “line symmetric” metric

$$ds^2 = -A^2(t, z)dt^2 + C^2(t, z)dz^2 + B^2(t, z)dx^2$$  \hspace{1cm} (2.1)

This is the most general form one can have if one requires the metric be invariant under $x$ translation $x \rightarrow x + a$ and inversion $x \rightarrow -x$, the $x$-axis being where the string resides. One can further make $A = C$ by appropriate coordinate transformations. (It is well known in elementary differential geometry that any two-dimensional metric can be brought into this “isothermal” form.) The position of the string, $z = 0$, will in general be transformed into $z = h(t)$. One can then
transform it back to $z = 0$, while retaining the $A = C$ form of the metric, by a transformation of the form

$$t + z \rightarrow f(t + z)$$
$$t - z \rightarrow g(t - z)$$ (2.2)

where $f$ and $g$ are subject to the constraint

$$f'(t + z)g'(t - z) > 0$$ (2.3)

and a prime denotes differentiation of a function with respect to its argument. Therefore to find general solutions, we only need to find solutions with $A = C$. However, we would like to leave $A$ and $C$ arbitrary as far as we can, so that results are directly applicable to solutions not in the $A = C$ form. It is also easier to obtain some particular solutions when $A$ and $C$ are arbitrary. We will illustrate this later.

To proceed, we choose the dreibein $e_a \equiv e_{a\mu} dx^\mu$ to be

$$e_0 = -A dt, \quad e_1 = C dz, \quad e_2 = B dx$$ (2.4)

Latin indices are raised and lowered by $\eta_{ab} = \text{diag}(-1, 1, 1)$ and Greek indices by $g_{\mu\nu}$. It follows that the connection 1-forms $\omega_{ab} \equiv \omega_{\mu ab} dx^\mu$ and the curvature two-forms $R_{ab} \equiv R_{\mu\nu ab} dx^\mu \wedge dx^\nu$ are given by

$$\omega_{01} = -\frac{\partial_z A}{C} dt - \frac{\partial_t C}{A} dz$$
$$\omega_{02} = -\frac{\partial_t B}{A} dx$$
$$\omega_{12} = -\frac{\partial_z B}{C} dx$$ (2.5)

$$R_{01} = \left[ \partial_z \left( \frac{\partial_z A}{C} \right) - \partial_t \left( \frac{\partial_t C}{A} \right) \right] dt \wedge dz$$
$$R_{02} = \left[ \partial_z \left( \frac{\partial_z A}{C} \right) - \partial_t \left( \frac{\partial_t B}{A} \right) \right] dt \wedge dx + \left[ \frac{\partial_t C}{A} \frac{\partial_z B}{C} - \partial_z \left( \frac{\partial_t B}{A} \right) \right] dz \wedge dx$$ (2.6)
$$R_{12} = \left[ \partial_z A \frac{\partial_z B}{C} - \partial_t \left( \frac{\partial_t A}{C} \right) \right] dt \wedge dx + \left[ \frac{\partial_t C}{A} \frac{\partial_z B}{C} - \partial_z \left( \frac{\partial_t B}{C} \right) \right] dz \wedge dx$$

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In $2+1$ dimensions, the curvature tensor is proportional to the Einstein tensor

$$R_{\beta\nu}^{\alpha\mu} = \epsilon^{\alpha\mu\lambda} \epsilon_{\beta\nu\delta} C_{\lambda}^{\delta}$$  \hspace{1cm} (2.7)$$

Therefore vacuum Einstein equations can be solved by setting the curvature tensor $R_{\mu\nu\alpha\beta}$ to zero. In the case of static solutions, one has

$$\partial_z \left( \frac{\partial_z A}{C} \right) = 0, \quad \partial_z \left( \frac{\partial_z B}{C} \right) = 0, \quad (\partial_z A)(\partial_z B) = 0$$  \hspace{1cm} (2.8)$$

To find time-dependent solutions, we define

$$F \equiv \frac{\partial_z A}{C}, \quad G \equiv \frac{\partial_t C}{A}, \quad P \equiv \frac{\partial_t B}{A}, \quad Q \equiv \frac{\partial_z B}{C}$$  \hspace{1cm} (2.9)$$

We then obtain the following five equations. Only four of them are independent, but all are needed to make the whole set of equations symmetric.

$$\partial_z F - \partial_t G = 0$$
$$\partial_t P = FQ, \quad \partial_t Q = FP$$
$$\partial_z P = GQ, \quad \partial_z Q = GP$$  \hspace{1cm} (2.10)$$

The last four equations tell us that $P^2 - Q^2$ is a $(t, z)$ independent constant. Thus solutions fall into two classes:

**Type-I solutions**

The solutions are characterized by $P^2 - Q^2 = \pm \beta^2 \neq 0$. With the $+$ sign, the solutions are given by

$$\frac{\partial_z A}{C} = \partial_t S, \quad \frac{\partial_t C}{A} = \partial_z S$$
$$\frac{\partial_t B}{A} = \beta \cosh S, \quad \frac{\partial_z B}{C} = \beta \sinh S$$  \hspace{1cm} (2.11)$$

where $\beta$ is an arbitrary constant and $S = S(t, z)$ an arbitrary function. Interchanging $\cosh S$ and $\sinh S$ in the above also gives solutions. This corresponds to choosing the $-$ sign. We will refer to these solutions as type-I($\pm$) solutions. For simplicity, the distinction ($\pm$) will sometimes not be made.
**Type-II solutions**

This case corresponds to $P^2 - Q^2 = 0$. The solutions are

\[
\frac{\partial_z A}{C} = \pm \frac{\partial_t P}{P}, \quad \frac{\partial_t C}{A} = \pm \frac{\partial_z P}{P}, \quad \frac{\partial_t B}{A} = P, \quad \frac{\partial_z B}{C} = \pm P
\]  

(2.12)

where $P = P(t, z)$ is an arbitrary function. We choose either all upper signs or all lower signs. Eq.(2.11) with $\beta$ set to zero also gives type-II solutions.

According to Eq.(2.7), the vacuum spacetime is maximally symmetric with zero curvature. Therefore despite its appearance, the spacetime is actually homogeneous and isotropic about all points. The metric can always be transformed into the Minkowski one

\[
ds^2 = -d\tilde{t}^2 + d\tilde{z}^2 + d\tilde{x}^2
\]

(2.13)

We assume that this coordinate transformation is described by

\[
\partial_\mu q^a = \frac{\partial q^a}{\partial x^\mu} = \Lambda^a_b e^b_\mu
\]

(2.14)

where $x^\mu \equiv (t, z, x)^T$ and $q^a \equiv (\tilde{t}, \tilde{z}, \tilde{x})^T$. This defines the (spacetime dependent) matrix $\Lambda$. That $\Lambda$ is a Lorentz matrix follows from the fact that $e^a_\mu$ is the dreibein. To see this, consider a vector $V^\mu$ and its transformation into Minkowski coordinates $\tilde{V}^a$. We have

\[
(\partial_\mu q^a) V^\mu = \tilde{V}^a
\]

(2.15)

or

\[
V^\mu = e_{b\mu}(\Lambda^{-1})_a^b \tilde{V}^a
\]

(2.16)

Comparing this with

\[
V^\mu = (\partial_\mu q^a) \tilde{V}_a = \Lambda^a_b e^b_\mu \tilde{V}_a
\]

(2.17)
one obtains in matrix notation

\[ \Lambda^{-1} = \eta \Lambda^T \eta \] \hspace{1cm} (2.18)

There are six Killing vectors. In Minkowski coordinates \( q^a \), they are given by

\[ l^a = c^a \] \hspace{1cm} (2.19)

and

\[ l^a = J^a_b q^b \] \hspace{1cm} (2.20)

where \( c^a \) are constant vectors and \( J \) the Lorentz generators. If we define the Lie bracket of two vectors \( l^a \) and \( m^a \) to be

\[ [l^a, m^a] \equiv l^a \partial_a m^b - m^a \partial_a l^b \equiv n^b \] \hspace{1cm} (2.21)

then the Lie bracketing of these Killing vectors reproduces the commutators of the Poincaré group [7]. Transforming these vectors into \( x^\mu \) coordinates we obtain the vectors

\[ \chi_\mu = \Lambda^a_b e^b_\mu l_a \] \hspace{1cm} (2.22)

which must satisfy the Killing equation

\[ \nabla_\mu \chi_\nu + \nabla_\nu \chi_\mu = 0 \] \hspace{1cm} (2.23)

Substituting Eq.(2.22) into this we obtain, with \( l_a \) “stripped off”,

\[ 0 = \nabla_\mu (\Lambda^a_b e^b_\nu) + \nabla_\nu (\Lambda^a_b e^b_\mu) \]
\[ = (\partial_\mu \Lambda^a_b) e^b_\nu + (\partial_\nu \Lambda^a_b) e^b_\mu + \Lambda^a_b (\nabla_\mu e^b_\nu + \nabla_\nu e^b_\mu) \] \hspace{1cm} (2.24)

(The extra contributions obtained when the differential operators act on the \( l_a \) given by Eq.(2.20) cancel due to Eq.(2.14) and the antisymmetry of the Lorentz
generators \( J^a_{b,} \). Since
\[
\nabla_\mu e^b_\nu = -\omega^b_\mu e^a_\nu
\] (2.25)
we obtain
\[
(\Lambda^{-1}\partial_\mu \Lambda)^a_{b,} e^b_\nu + (\Lambda^{-1}\partial_\nu \Lambda)^a_{b,} e^b_\mu = \omega^a_\mu b^b_\nu + \omega^a_\nu b^b_\mu
\] (2.26)
Note that, as we will show later, \( (\Lambda^{-1}\partial_\mu \Lambda)^a_{b,} \) is anti-symmetric in \( a \) and \( b \) for any Lorentz matrix \( \Lambda \). Eq.(2.26) determines
\[
(\Lambda^{-1}\partial_\mu \Lambda)^a_{b,} = \omega^a_\mu b
\] (2.27)
Eqs.(2.14) and (2.27) reproduce in an instructive way the familiar equations in the Poincaré gauge formulation of the Einstein gravity. These can also be viewed as a general proof of Gerbert’s claim that the vector \( q^a \) satisfying Eq.(2.14) is the local coordinate transformation that transforms the metric \( g_{\mu\nu} \) into \( \eta_{ab} \), which he made based on the study of a class of point source solutions [8].

In terms of the gauge formulation, since the gauge fields vanish (The connection 1-forms Eq.(2.5) already satisfy the torsion free condition, while Eq.(2.6) sets the Riemann curvature to zero.), the gauge potentials associated with these vacuum solutions must be pure gauges
\[
A_\mu \equiv e^a_\mu P_a + \omega^a_\mu J_a = U\partial_\mu U^{-1}
\] (2.28)
Here \( U \) is an element of the three dimensional Poincaré group \( ISO(2,1) \), \( \omega^a_\mu = -\frac{1}{2}\epsilon^{abc}\omega_{\mu bc}, \) and \( J_a, P_a \) are group generators obeying
\[
[J^a, J^b] = -\epsilon^{abc} J^c, \quad [J^a, P^b] = -\epsilon^{abc} P^c, \quad [P^a, P^b] = 0
\] (2.29)
with \( \epsilon^{012} = 1 \). We wish to present the Poincaré group elements \( U \) corresponding to the solutions Eqs.(2.11) and (2.12). In the meantime, we will find the coordinate transformations that transform these solutions into the Minkowski metric. We use
a $4 \times 4$ representation for $ISO(2,1)$ where an arbitrary group element $V$ has the form [9]

$$V = \begin{pmatrix} \Lambda^a_b & q^a \\ 0 & 1 \end{pmatrix}$$  \hspace{1cm} (2.30)

where $q^a$ is a three-vector and $\Lambda^a_b$ is a $3 \times 3$ Lorentz matrix. If we denote $V$ by $(\Lambda, q)$, then Eq.(2.28) reads with $U^{-1} \equiv (\Lambda, q)$

$$A_\mu = U\partial_\mu U^{-1} \equiv \begin{pmatrix} \Lambda^{-1} \partial_\mu \Lambda & \Lambda^{-1} \partial_\mu q \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \omega^a_{\mu} J_a & \epsilon^b_{\mu} P_b \\ 0 & 0 \end{pmatrix}$$  \hspace{1cm} (2.31)

where

$$(J_a)^b_c = \epsilon^b_{a c}, \quad (P_a)^b = \delta^b_a$$  \hspace{1cm} (2.32)

are the group generators. More explicitly

$$J_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_1 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$  \hspace{1cm} (2.33)

Since the right-hand side of Eq.(2.31) is known, we can solve the equation to obtain $(\Lambda, q)$. To this end, we introduce

$$J_+ \equiv \frac{1}{2}(J_0 + J_1), \quad J_- \equiv \frac{1}{2}(J_0 - J_1)$$  \hspace{1cm} (2.34)

Thus

$$[J_2, J_+] = J_+, \quad [J_2, J_-] = -J_-, \quad [J_+, J_-] = -\frac{1}{2}J_2$$  \hspace{1cm} (2.35)

We consider an arbitrary Lorentz group element $\Lambda$, parametrized by$^1$

$$\Lambda = e^{uJ_+ - vJ_-} e^{wJ_2}$$  \hspace{1cm} (2.36)

$^1$ The fact that any rotation group element can be brought into this form was used to construct representations for the three dimensional Euclidean group, see [10].
Using the commutators Eq.(2.35) and

\[ \partial_\alpha e^\Theta = \int_0^1 ds \ e^{\Theta} \partial_\alpha \Theta e^{(1-s)\Theta} \]  

(2.37)

we compute [11]

\[
\Lambda^{-1} \partial_\alpha \Lambda = \partial_\alpha w J_2 + e^{-w} \partial_\alpha u J_+ - e^w \partial_\alpha v J_-
\]

\[
+ (u \partial_\alpha v - v \partial_\alpha u) \left[ \frac{1 - \cosh \sqrt{uv}}{uv} J_2 + \sqrt{uv} - \sinh \sqrt{uv} (ue^{-w} J_+ + ve^w J_-) \right]
\]

(2.38)

Comparing this with Eqs.(2.31), (2.5) and (2.11), we get for type-I(+) solutions

\[ \Lambda = e^{-\beta x} h e^{S J_2} \]

(2.39)

or, more explicitly,

\[
\Lambda = \begin{pmatrix}
\cosh \beta x \cosh S & \cosh \beta x \sinh S & \sinh \beta x \\
\sinh S & \cosh S & 0 \\
\sinh \beta x \cosh S & \sinh \beta x \sinh S & \cosh \beta x
\end{pmatrix}
\]

(2.40)

To find out the desired coordinate transformations, we only need to integrate \( \partial_\mu q = \Lambda e^b_\mu P_b \). When this is done, we get \( q = (\tilde{t}, \tilde{z}, \tilde{x})^T \) with

\[
\tilde{t} = \frac{1}{\beta} B \cosh \beta x
\]

\[
\tilde{x} = \frac{1}{\beta} B \sinh \beta x
\]

(2.41)

\[
\tilde{z} = \int dt \ A \sinh S = \int dz \ C \cosh S
\]

where \( S \) is given in terms of the metric via Eq.(2.11). Similarly, for type-I(−)
solutions we have

\[ \Lambda = e^{\beta x J_0} e^{SJ_2} \]  \hspace{1cm} (2.42)

and the transformations are

\[ \tilde{t} = \int dt \ A \cosh S = \int dz \ C \sinh S \]

\[ \tilde{z} = \frac{1}{\beta} B \cos \beta x \]  \hspace{1cm} (2.43)

\[ \tilde{x} = \frac{1}{\beta} B \sin \beta x \]

For type-II(+) solutions, the \( \Lambda \) matrix is given by

\[ \Lambda = e^{x(J_0-J_1)} e^{(\ln P)J_2} \]  \hspace{1cm} (2.44)

That is,

\[ \Lambda = \frac{1}{2} \begin{pmatrix}
(1 + x^2)P + P^{-1} & (1 + x^2)P - P^{-1} & 2x \\
(1 - x^2)P - P^{-1} & (1 - x^2)P + P^{-1} & -2x \\
2xP & 2xP & 2
\end{pmatrix} \]  \hspace{1cm} (2.45)

The general transformations are

\[ \tilde{t} = \frac{1}{2} B x^2 + \frac{1}{2} \int dt \ A \left( P + \frac{1}{P} \right) = \frac{1}{2} B x^2 + \frac{1}{2} \int dz \ C \left( P - \frac{1}{P} \right) \]

\[ \tilde{z} = -\frac{1}{2} B x^2 + \frac{1}{2} \int dt \ A \left( P - \frac{1}{P} \right) = -\frac{1}{2} B x^2 + \frac{1}{2} \int dz \ C \left( P + \frac{1}{P} \right) \]  \hspace{1cm} (2.46)

\[ \tilde{x} = B x \]

where \( P \) is given in terms of the metric via Eq.(2.12). Type-II(−) solutions are characterized by

\[ \Lambda = e^{-x(J_0+J_1)} e^{-(\ln P)J_2} \]  \hspace{1cm} (2.47)

One can easily obtain the transformation laws.
Next we specialize to solutions with $A = C$. We only consider "+" type solutions. The "−" type solutions can be similarly studied. For type-II solutions, the integrability condition for the first two equations in Eq.(2.12) tells us that $\ln P$ satisfies the wave equation

$$
(\partial_t^2 - \partial_z^2) \ln P = 0 \quad (2.48)
$$

Thus the general solution for $P$ may be written as

$$
P = \frac{f'^{1/2}(t + z)}{g'^{1/2}(t - z)} \quad (2.49)
$$

where a prime denotes differentiation of a function with respect to its argument. The choice of this form for $P$ is for future convenience. It follows that the metric is given by

$$
ds^2 = f'(t + z)g'(t - z)(-dt^2 + dz^2) + f^2(t + z)dx^2 \quad (2.50)
$$

where $f$ and $g$ are arbitrary functions subject to the constraint Eq.(2.3). Under a coordinate transformation

$$
\tilde{t} + \tilde{z} = f(t + z) \\
\tilde{t} - \tilde{z} = g(t - z) + f(t + z)x^2 \\
\tilde{x} = f(t + z)x
$$

the above metric becomes Minkowski. Similarly, type-I, $A = C$ solutions are given by

$$
e^S = \frac{f'^{1/2}(t + z)}{g'^{1/2}(t - z)} \quad (2.52)
$$

and

$$
ds^2 = f'(t + z)g'(t - z)(-dt^2 + dz^2) + \frac{1}{4} \beta^2 [f(t + z) + g(t - z)]^2 dx^2 \quad (2.53)
$$

where $\beta$ is a constant and $f, g$ are arbitrary functions satisfying Eq.(2.3). This can
be transformed into Minkowski metric by the transformations

\[
\tilde{t} = \frac{1}{2} [f(t + z) + g(t - z)] \cosh \beta x \\
\tilde{x} = \frac{1}{2} [f(t + z) + g(t - z)] \sinh \beta x \quad (2.54) \\
\tilde{z} = \frac{1}{2} [f(t + z) - g(t - z)]
\]

Although the transformations Eqs.(2.51) and (2.54) may be directly obtained without using the gauge formulation (They can certainly be verified without reference to the gauge formulation.), the \(x\) dependences in these transformations are better understood within the gauge formalism.

Note that, the spacetime Eq.(2.53) covers only (a portion of) the region \(|\tilde{t}| > |\tilde{x}|\), while there is no such restriction for Eq.(2.50). (There could be other restrictions, of course.) Spacetimes associated with Eq.(2.43) are periodic in \(x\); other types of solutions do not share this property.

We need to clarify what we mean by “time-dependent solutions”. Normally, when we say a spacetime is time-dependent, we mean that it is locally time-dependent, that is, it does not have a time-like Killing vector. These vacuum spacetimes are locally Minkowski and certainly not time-dependent. Once we put a string at \(z = 0\), the spacetime will be globally time-dependent if, in the Minkowski coordinates, the world sheet of the string does not lie in a \(\tilde{t} = \) constant surface (modulo a Lorentz transform). Practically, since all static solutions are obtained in [3], solutions that are not simply related to those must be time-dependent ones.

This completes our study of the vacuum solutions.
3. Open String Solutions and the Relation to Planar Walls

We will look for reflection symmetric solutions of the form

\[ ds^2 = -A^2(t, |z|)dt^2 + C^2(t, |z|)dz^2 + B^2(t, |z|)dx^2 \]  

(3.1)

The energy-momentum tensor of an open string, positioned at \( z = 0 \), is given by

\[ T^{\mu\nu}(t, z) = S^{\mu\nu}(t)\delta(z) \]  

(3.2)

We consider sources for which

\[ S^{\mu\nu}(t) = \tilde{\sigma}(t)u^\mu u^\nu - \tilde{\xi}(t)(h^{\mu\nu} + u^\mu u^\nu) \]  

(3.3)

where \( \tilde{\sigma} \) and \( \tilde{\xi} \) correspond to the energy density and tension of the string, respectively. For the metric Eq.(3.1), the three velocity of the string is \( u^\mu = (1/A, 0, 0) \) and the normal to the \( z = 0 \) hypersurface is \( n^\mu = (0, 1/C, 0) \). Thus the induced two dimensional metric is

\[ h^{\mu\nu} = g^{\mu\nu} - n^\mu n^\nu = \text{diag}\left(-\frac{1}{A^2}, 0, \frac{1}{B^2}\right) \]  

(3.4)

Conservation of the energy-momentum tensor is given by \( \nabla_\mu T^{\mu\nu} = 0 \). The \( x \) component of this equation is trivially satisfied. The \( z \) component is also identically satisfied for reflection symmetric solutions, due to the following prescription [12]

\[ \Gamma^z_{\alpha\alpha}(z = 0) = \lim_{\epsilon \to 0} \frac{1}{2} \left[ \Gamma^z_{\alpha\alpha}(z = +\epsilon) + \Gamma^z_{\alpha\alpha}(z = -\epsilon) \right] = 0 \]  

(3.5)

Finally, the \( t \) component gives us the conservation law

\[ \partial_t (BC\tilde{\sigma}) = \tilde{\xi}C \partial_t B \]  

(3.6)

In the regions \( z > 0 \) or \( z < 0 \), the solutions are given by the vacuum solutions
found in the last section. We need to match these solutions using junction conditions. Junction conditions can be obtained by integrating the Einstein equations

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 2\pi G T_{\mu\nu} \]  

(3.7)

across the string. In an orthonormal basis, we have

\[ R_{01} = R_{0212} = 0 \]
\[ R_{22} = -R_{0202} + R_{1212} = 2\pi G \tilde{\sigma} \delta(z) \]  

(3.8)
\[ R_{00} = R_{0101} + R_{0202} = -2\pi G \tilde{\xi} \delta(z) \]
\[ R_{11} = -R_{0101} + R_{1212} = 2\pi G (\tilde{\sigma} + \tilde{\xi}) \delta(z) \]

If we apply \( \int_{-\epsilon}^{+\epsilon} dz \) to these equations, only terms with a \( \delta(z) \) singularity survive. On the left-hand side, only terms containing a double \( z \) derivative \( \partial_z^2 \) survive. Using the explicit expressions for the curvature tensor, Eq.(2.6), we obtain, for reflection symmetric solutions,

\[ \pi G \tilde{\sigma} = - \frac{1}{BC} \frac{\partial_z B}{C} \bigg|_{z=\pm\epsilon} \]
\[ \pi G \tilde{\xi} = - \frac{1}{AC} \frac{\partial_z A}{C} \bigg|_{z=\pm\epsilon} \]  

(3.9)

For any solutions satisfying Einstein equations and the junction conditions, the energy-momentum conservation is automatically satisfied.

One may compare these with the junction conditions obtained using Gaussian normal coordinates \((\tau, \eta, x)\), in terms of which the metric is given by (For a discussion of obtaining junction conditions with Gaussian normal coordinates, see [12] and [13].)

\[ ds^2 = g_{\tau\tau}(\tau, \eta)d\tau^2 + d\eta^2 + \tilde{B}^2(\tau, \eta)dx^2 \]  

(3.10)

where \( g_{\tau\tau}(\tau, 0) = -1, \tilde{B}(\tau, \eta) = B(t(\tau, \eta), z(\tau, \eta)) \) and \( \eta(t, z = 0) = 0, z(\tau, \eta =
0) = 0. The energy-momentum tensor is $T^{\mu\nu}(\tau, \eta) = S^{\mu\nu}(\tau)\delta(\eta)$ with

$$S^{\mu\nu}(\tau) = \sigma(\tau)u^\mu u^\nu - \xi(\tau)(h^{\mu\nu} + u^\mu u^\nu) \tag{3.11}$$

Now $u^\mu = (1, 0, 0)$ and the two-dimensional induced metric is

$$ds^2 = -d\tau^2 + \tilde{B}^2(\tau, 0)dx^2 \tag{3.12}$$

The conservation law now reads

$$\partial_\tau(\tilde{B}\sigma) = \xi\partial_\tau\tilde{B} \tag{3.13}$$

or equivalently, $\partial_t(B\sigma) = \xi\partial_tB$. The junction conditions are

$$K^i_j \bigg|_{\eta=+\epsilon} = -\pi G(S^i_j - \delta^i_j \text{Tr}S) \tag{3.14}$$

for reflection symmetric solutions. The non-zero components of the extrinsic curvature are

$$K_{xx} = \frac{1}{2} n^{\mu}\partial_\mu B^2 = \frac{1}{C}B\partial_z B$$

$$K_{\tau\tau} = -n_\mu u^\nu \nabla_\nu u^\mu = -C\Gamma_{ij}^z u^i u^j = -\frac{\partial_z A}{AC} \tag{3.15}$$

So we get

$$\pi G\sigma = -\frac{1}{C} \frac{\partial_z B}{B} \bigg|_{z=+\epsilon}$$

$$\pi G\xi = -\frac{1}{C} \frac{\partial_z A}{A} \bigg|_{z=+\epsilon} \tag{3.16}$$

Eqs.(3.9) and (3.16) look different, but they can be reconciled by noting that $\delta(z) = C\delta(\eta)$ and hence that $\sigma = C\tilde{\sigma}$ and $\xi = C\tilde{\xi}$.

We present some special solutions as examples. These are obtained by choosing $S$ and $P$ in Eqs.(2.11) and (2.12) to be simple functions. We use Eq.(3.11) for the
energy-momentum tensor, and thus Eq.(3.16) for the junction conditions. For type-I solutions we obtain

\[ ds^2 = -dt^2 + dz^2 + (1 - \alpha|z| + \beta t)^2 dx^2 \]

\[ \pi G\sigma = \frac{\alpha}{1 + \beta t}, \quad \pi G\xi = 0 \]  

(3.17)

\[ ds^2 = -(1 - \alpha|z|)^2 dt^2 + dz^2 + (1 - \alpha|z|)^2 \sinh^2 \alpha t dx^2 \]

\[ \pi G\sigma = \alpha, \quad \pi G\xi = \alpha \]  

(3.18)

and

\[ ds^2 = -dt^2 + (1 + \alpha t)^2 dz^2 + (1 + \alpha t)^2 \cosh^2 \alpha z dx^2 \]

\[ \pi G\sigma = \pi G\xi = 0 \]  

(3.19)

where \( \alpha, \beta \) are arbitrary constants. They correspond to choosing \( S = \text{constant} \), \(-\alpha t\) and \( \alpha z\), respectively. Eq.(3.17) is a generalization of the static dust string solutions [3]. Note that there are no static reflection symmetric solutions with \( \sigma \) and \( \xi \) both non-vanishing. This can be easily seen from Eq.(2.8) and the junction conditions Eq.(3.16). Eq.(3.19) describes a pure vacuum solution. Type-II solutions can be similarly constructed. We have

\[ ds^2 = -(1 - \alpha|z|)^2 dt^2 + dz^2 + (1 - \alpha|z|)^2 e^{2\alpha t} dx^2 \]

\[ \pi G\sigma = \alpha, \quad \pi G\xi = \alpha \]  

(3.20)

\[ ds^2 = -dt^2 + (1 - \alpha t)^2 dz^2 + (1 - \alpha t)^2 e^{-2\alpha|z|} dx^2 \]

\[ \pi G\sigma = \frac{\alpha}{1 + \alpha t}, \quad \pi G\xi = 0 \]  

(3.21)

where \( \alpha \) is a constant. They correspond to \( P = \alpha e^{\alpha t}, \alpha e^{-\alpha z} \), respectively.

One can transform these solutions into the \( A = C \) form. For example, from Eqs.(3.18) and (3.20) we get

\[ ds^2 = e^{-2\alpha|z|}(-dt^2 + dz^2) + e^{-2\alpha|z|} \sinh^2 \alpha t dx^2 \]

\[ \pi G\sigma = \pi G\xi = \alpha \]  

(3.22)
and
\[ ds^2 = e^{-2\alpha|z|}(-dt^2 + dz^2) + e^{-2\alpha|z|}e^{2\alpha t}dx^2 \]
\[ \pi G\sigma = \pi G\xi = \alpha \] respectively. The coordinate singularities at \( z = \pm 1/\alpha \) in the original solutions are now transformed to \( z = \pm \infty \).

Solutions with \( \pi G\sigma = \pi G\xi = \alpha = \text{constant} \) are sometimes called vacuum string solutions. They correspond to domain wall solutions in \( 3 + 1 \) dimensions. General vacuum string solutions with \( A = C \) can be obtained by solving the junction conditions Eq.(3.16). Type-I solutions are found to be given by Eq.(2.53) with
\[ g^{1/2} = -\frac{f^{1/2}}{\alpha f}, \quad g = -\frac{1}{\alpha^2 f} \] where \( f \) is still an arbitrary function. Similarly, type-II solutions are given by Eq.(2.50) with \( g \) given by Eq.(3.24). One can transform these into Minkowski metric using Eqs.(2.51) and (2.54). For both types of solutions, the world sheet of the string is transformed into (a portion of) the hyperboloid
\[ -\tilde{t}^2 + \tilde{z}^2 + \tilde{x}^2 = \frac{1}{\alpha^2} \] (3.25)

As point sources solutions in \( 2 + 1 \) dimensions are related to cosmic string solutions in \( 3 + 1 \) dimensions, so should open string solutions be related to planar wall solutions. Indeed, the solution of Eq.(3.20) exhibits a remarkable resemblance to Vilenkin’s domain wall solution [5], which takes the form
\[ ds^2 = -(1 - \alpha|z|)^2dt^2 + dz^2 + (1 - \alpha|z|)^2e^{2\alpha t}(dx^2 + dy^2) \]
\[ \pi G\sigma = 2\alpha, \quad \pi G\xi = 2\alpha \] (3.26)
in terms of our convention. On the other hand, there is no four-dimensional solution corresponding to Eq.(3.18); and one can verify that
\[ ds^2 = -(1 - \alpha z)^2dt^2 + dz^2 + (1 - \alpha z)^2 \sinh^2 \alpha t (dx^2 + dy^2) \] (3.27)
is not a vacuum solution in \( 3 + 1 \) dimensions.
To establish the general relations, we use the reduction formulae

\[ ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta + N^2(dx^3)^2 \]

\[ K_{\alpha\beta} = \frac{1}{2N} \partial_3 g_{\alpha\beta} \]

\[ \bar{R}_{\alpha\beta\mu\nu} = \frac{3}{N} (D_\nu K_{\beta\mu} - D_\nu K_{\beta\mu}) \]

\[ R_{\beta\mu\nu} = \frac{1}{N} (D_\nu K_{\beta\mu} - D_\nu K_{\beta\mu}) \]

\[ R_{\beta\nu} = \frac{1}{N} \partial_3 K_{\beta\nu} - \frac{1}{N} D_\nu D_\beta N + K_{\alpha\beta} K_{\nu} \]

where \( \alpha, \beta, \mu, \nu = 0, 1, 2 \) or \( t, z, x \) and \( D_\alpha \) is the covariant derivative associated with the three dimensional metric \( g_{\alpha\beta} \)

\[ D_\beta T_\alpha \equiv \partial_\beta T_\alpha - \Gamma^\lambda_{\alpha\beta} T_\lambda \] (3.29)

We see that given a \( 2 + 1 \) dimensional vacuum solution Eq.(2.1), the metric

\[ ds^2 = -A^2(t, z)dt^2 + C^2(t, z)dz^2 + B^2(t, z)(dx^2 + dy^2) \] (3.30)

will be a flat spacetime solution in \( 3 + 1 \) dimensions if and only if

\[ D_\alpha D_\beta B = \partial_\alpha \partial_\beta B - \Gamma^\lambda_{\alpha\beta} \partial_\lambda B = 0 \] (3.31)

Performing a dimensional reduction once more to \( 1 + 1 \) dimensions, we see that only possibly non-vanishing terms are \( D_\alpha D_x B \). Using the explicit expressions for \( \Gamma^\lambda_{\alpha\beta} \), we see that the only possible non-zero term is

\[ -D_x D_x B = \Gamma^t_{xx} \partial_t B + \Gamma^z_{xx} \partial_z B = B \left[ \left( \frac{\partial_t B}{A} \right)^2 - \left( \frac{\partial_z B}{C} \right)^2 \right] \] (3.32)

Therefore a three-dimensional vacuum solution corresponds to a four-dimensional flat spacetime solution if it is a type-II solution. Type-I solutions don’t even correspond to vacuum solutions in four dimensions. On the other hand, given any flat spacetime solution in four dimensions of the form Eq.(3.30), Eq.(2.1) is automatically a type-II vacuum solution in three dimensions.
Next we consider junction conditions. If we normalize $G$ as in Eq.(3.7), the junction conditions in 3 + 1 dimensions (with energy-momentum tensor given by Eq.(3.11)) are

$$\frac{1}{2} \pi G \sigma = - \frac{1}{C} \frac{\partial_z B}{B} \bigg|_{z=+\epsilon}$$

$$\frac{1}{2} \pi G (-\sigma + 2\xi) = - \frac{1}{C} \frac{\partial_z A}{A} \bigg|_{z=+\epsilon}$$  \hspace{1cm} (3.33)

We see that a domain wall solution in 3 + 1 dimensions corresponds to a vacuum string solution in 2 + 1 dimensions (Both terms mean $\sigma = \xi = \text{constant}$). All class-I domain wall solutions of Ipser and Sikivie are flat [6]; they correspond to the type-II vacuum string solutions in three dimensions. Their class-II solutions are unphysical because of the curvature singularity. Our type-I vacuum string solutions do not have any curvature singularities, since the curvature identically vanishes away from sources in three dimensions. Thus there are more solutions in 2 + 1 dimensions.

Ipser and Sikivie have noted the “gravitational repulsiveness” of a planar wall [6]. Consider a test particle, initially at rest, in the vicinity of a planar wall. From the geodesic equation we obtain its initial acceleration in the $z$ direction:

$$\left. \frac{d^2 z}{d\lambda^2} \right|_{z=+\epsilon} = -\Gamma^z_{tt} \left( \frac{dt}{d\lambda} \right)^2 = - \frac{A \partial_z A}{C^2} \left( \frac{dt}{d\lambda} \right)^2$$  \hspace{1cm} (3.34)

From the junction conditions we see that this is positive if $-\sigma + 2\xi$ is positive. The wall is then said to be repulsive. Essentially the same calculation applied to the 2 + 1 dimensional theory tells us that a string is gravitationally repulsive if the string tension $\xi$ is positive.
4. Conclusions

In conclusion, we have found general, line symmetric vacuum solutions and we have matched them across the string using the junction conditions. For any metric, we have given the corresponding string energy density and tension. We have studied the gauge formulation of these solutions and found their local Poincaré group elements. The relation of these solutions to the four dimensional planar walls is also obtained.

Recently, Cangemi et al. have studied the gauge formulation of the 2 + 1 dimensional black hole in anti-de Sitter space [14]. It would be interesting to study string solutions and their gauge formulation when a cosmological constant is present.

Acknowledgements:

I thank Hsien-Chung Kao, Professor Kimyeong Lee and Professor Erick Weinberg for helpful discussions. I also thank Professor Erick Weinberg for valuable comments on the manuscript, Professor Roman Jackiw for a critical reading of the paper.

APPENDIX

In this appendix, we study geodesics in the background metric

\[ ds^2 = -z^2 dt^2 + dz^2 + z^2 e^{2t} dx^2, \quad -1 < z < \infty \]  

(A.1)

which is related to the solution Eq.(3.20) with \( \alpha = 1 \) (in the \( z > 0 \) region) by a coordinate transformation. The position of the string is now at \( z = -1 \) and the coordinate singularity at \( z = 0 \). Free motion of a test particle can be determined from the Lagrangian

\[ L = -z^2 \left( \frac{dt}{d\lambda} \right)^2 + \left( \frac{dz}{d\lambda} \right)^2 + z^2 e^{2t} \left( \frac{dx}{d\lambda} \right)^2 \]  

(A.2)

Since there is only one cyclic coordinate, it is difficult to solve the differential equations. Thus we will use the Hamilton-Jacobi equation, which turns out to be
separable
\[-\frac{1}{z^2} \left( \frac{\partial W}{\partial t} \right)^2 + \left( \frac{\partial W}{\partial z} \right)^2 + \frac{1}{z^2} e^{-2t} \left( \frac{\partial W}{\partial x} \right)^2 = E \] (A.3)

Letting \( W(t, z, x) = xp_x + W_1(t) + W_2(z) \), we get
\[ \left( \frac{dW_1}{dt} \right)^2 - e^{-2t} p_x^2 = \eta, \quad \left( \frac{dW_2}{dz} \right)^2 - \frac{\eta}{z^2} = E \] (A.4)

Hence

\[ W(t, z, x) = xp_x + \int \sqrt{E + \frac{\eta}{z^2}} dz + \int \sqrt{\eta + e^{-2t} p_x^2} dt \] (A.5)

where \( p_x, E, \eta \) are arbitrary constants. The motion is described by

\[ x_0 = \frac{\partial W}{\partial p_x} = x + \int \frac{e^{-2t} p_x}{\sqrt{\eta + e^{-2t} p_x^2}} dt \]
\[ \zeta = 2 \frac{\partial W}{\partial \eta} = \int \frac{dz}{z^2 \sqrt{E + z^{-2} \eta}} + \int \frac{dt}{\sqrt{\eta + e^{-2t} p_x^2}} \] (A.6)
\[ \lambda - \lambda_0 = \frac{\partial W}{\partial E} = \frac{1}{2} \int \frac{dz}{\sqrt{E + z^{-2} \eta}} \]

where \( p_x, \eta, E, x_0, \zeta, \lambda_0 \) are constants. The integrals can all be carried out. As a simple example, we consider the motion with \( \eta = 0 \). The geodesic is

\[ t = \ln \left| \frac{1 - \alpha \lambda}{\lambda} \right|, \quad z = \lambda, \quad x = \frac{\lambda}{1 - \alpha \lambda} \] (A.7)

where \( \alpha \) is a constant. The metric Eq.(A.1) can be flattened by the transformation

\[ \tilde{t} + \tilde{z} = z e^t, \quad \tilde{t} - \tilde{z} = -z e^{-t} + z e^t x^2, \quad \tilde{x} = z e^t x \] (A.8)

In terms of these new coordinates, the above geodesic becomes manifestly a straight line

\[ \tilde{t} + \tilde{z} = 1 - \alpha \lambda, \quad \tilde{t} - \tilde{z} = 0, \quad \tilde{x} = \lambda \] (A.9)

On the other hand, an arbitrary straight line in \((\tilde{t}, \tilde{z}, \tilde{x})\) coordinates is not necessarily a geodesic, since it may be out of the region that the spacetime of Eq.(A.1) covers.
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