SEMI-INFINITE COMBINATORICS IN REPRESENTATION THEORY

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ABSTRACT. In this work we discuss some appearances of semi-infinite combinatorics in representation theory. We propose a semi-infinite moment graph theory and we motivate it by considering the (not yet rigorously defined) geometric side of the story. We show that it is possible to compute stalks of the local intersection cohomology of the semi-infinite flag variety, and hence of spaces of quasi maps, by performing an algorithm due to Braden and MacPherson.

1. Introduction

Semi-infinite combinatorics occurs - or it is expected to occur - in representation theory of quantum groups at a root of unity, of Lie algebras and algebraic groups in positive characteristics, of complex affine Kac-Moody algebras. This paper does not aim at furnishing an exhaustive list of such occurrences, but rather at presenting the ones the author has personally been working on.

While looking for evidences for his modular conjecture, Lusztig introduced the periodic module, a certain $\mathbb{Z}[v^\pm 1]$-module equipped with an action of the affine Hecke algebra [Lu80]. In loc. cit. he studied a periodic analogue of Kazhdan-Lusztig basis elements and in his 1990 ICM paper [Lu90] he related them to the geometry of the periodic Schubert varieties: Lusztig’s generic polynomials were declared to play in this context the role played by Kazhdan-Lusztig polynomials in the usual Schubert variety setting. Such a fact was stated in [Lu80] without a proof. A more precise statement (and its proof) involving Drinfeld’s spaces of quasi maps was given in [FFKM].

In the same year, independently, Feigin and Frenkel also considered these varieties, using the denomination semi-infinite, which seems to be now the preferred name in the literature. Their motivation to introduce the semi-infinite flag variety was a geometric construction of a class of modules attached to any affine Kac-Moody algebra, the Wakimoto modules, which had been previously considered in the $\hat{sl}_2$-case by Wakimoto [Wak]. Wakimoto modules are realised in [FF] as zero extensions of constant sheaves on semi-infinite Schubert cells. On the other hand, under the Beilinson-Bernstein and Riemann-Hilbert correspondences, (dual) Verma modules can be obtained as zero-extensions of constant sheaves on Schubert cells and we are hence allowed to think of Wakimoto modules as the semi-infinite analogue of (dual) Verma modules.
We have already mentioned that Lusztig investigated in [Lu80] a periodic analogue of Kazhdan-Lusztig basis elements. In fact, he studied two families of polynomials appearing in this periodic context and both of them could be considered as a periodic analogue of Kazhdan-Lusztig polynomials: Lusztig’s generic polynomials, which have already appeared in this introduction, and Lusztig’s periodic polynomials. We will recall their definition in Section 6. In the same way as Kazhdan-Lusztig polynomials control Jordan-Hölder multiplicities of Verma modules, also periodic and generic polynomials (are expected to) govern Jordan-Hölder multiplicities of certain standard objects in various representation categories (see §7).

The idea of using moment graph techniques for solving multiplicity formula problems is due to Fiebig and it is motivated by Soergel’s approach to the Kazhdan-Lusztig conjecture on the characters of irreducible modules for finite dimensional complex Lie algebras. The main point of such a strategy is to combinatorially describe intersection cohomology groups of Schubert varieties. This can be done by either using the theory of Soergel bimodules or of Braden-MacPherson sheaves on Bruhat graphs.

Here we show that the theory of Braden-MacPherson sheaves on semi-infinite graphs can be applied to calculate local intersection cohomology of semi-infinite Schubert varieties (once made sense of them). This is the only new result of this paper and its proof consists of combining results of [FFKM] and [La15]. We hope to be able to obtain a new proof, independent of [FFKM], in a forthcoming paper.

All in all, the aim of this paper is to convince the reader that it is extremely natural to consider semi-infinite structures in representation theory and that a theory of sheaves on semi-infinite moment graphs will have applications in different branches of representation theory. This is further motivated by results in [BFGM], [ABBGM], where the geometry of the spaces of quasi maps and of the semi-infinite flag manifold are used to study Lie algebras in positive characteristic and quantum groups at a root of unity.

2. BRUHAT ORDER VS SEMI-INFINITE ORDER

Let \((W, S)\) be a Coxeter system. Every element \(w\) of \(W\) can be thought of as a (non-unique) word in the alphabet \(S\) and its length \(\ell(w)\) is the minimum number of letters necessary for writing such a word. Recall that the set of reflections of \(W\) is \(T = \{wsw^{-1} \mid w \in W, s \in S\}\) and the Bruhat order \(\leq\) on \(W\) is the partial order generated by the relations \(w \leq tw\) if \(\ell(w) < \ell(tw)\) for a \(t \in T\).

Since in this paper we are mainly interested in the representation theoretic side of the story, we will focus on the case of \(W\) being the Weyl group of an affine Kac-Moody algebra. More precisely, let \(g\) be a simple finite dimensional complex Lie algebra, then we consider its affinisation \(\hat{g}\). As a vector space,

\[\hat{g} \simeq g \otimes \mathbb{C}[t^\pm] \oplus CK \oplus CD\]
We are not going to recall the Lie algebra structure on \( g \), which can be found, for example, in [Kac], but we limit ourselves to mention that \( K \) is the central element and \( D \) the derivation operator.

Recall that the Weyl group of \( \hat{g} \) together with the set of reflections indexed by simple affine reflections is a Coxeter system, so that we can consider \( W \) endowed with the structure of a poset with respect to the Bruhat order.

There is a further partial order we want to equip \( W \) with, but in order to introduce it, we need to recall the alcove picture. Let \( h \) be a Cartan subalgebra of \( g \), let \( R \subset h^* \), resp. \( R^\vee \subset h \), be the root, resp. coroot, system of \( g \) and let \( \Lambda = ZR \subset h^* \), resp. \( \Lambda^\vee = ZR^\vee \subset h \), be its root, resp. coroot, lattice. Consider the Euclidean vector space \( V := \Lambda \otimes \mathbb{R} \). The Weyl group of \( \hat{g} \) can be hence realised as the group of affine transformations of \( V \) generated by the reflections across the affine hyperplanes

\[
H_{\alpha,n} := \{ v \in V \mid \langle v, \alpha^\vee \rangle = n \}, \quad (\alpha \in R^+, \ n \in \mathbb{Z})
\]

where \( \langle \cdot, \cdot \rangle : V \times V^* \to \mathbb{R} \) denotes the natural pairing, \( R^+ \) the set positive roots, and \( \alpha^\vee \in R^\vee \) is the coroot corresponding to \( \alpha \) (by abuse of notation, we denote by \( \alpha \), resp. \( \alpha^\vee \), also its image in \( \Lambda \otimes \mathbb{R} = V \), resp. in \( \Lambda^\vee \otimes \mathbb{R} = V^* \)). Thus the (left) action of \( W \) on \( V \) is given by

\[
s_{\alpha,n}(v) = v - (\langle v, \alpha^\vee \rangle - n)\alpha.
\]

The connected components of \( V \setminus \bigcup_{\alpha,n} H_{\alpha,n} \) are called alcoves and, since \( W \) acts on the set of alcoves \( \mathcal{A} \) freely and transitively, \( W \) and \( \mathcal{A} \) are in bijection as sets. In order to make such a bijection into an identification, we need to choose an alcove and look at its orbit under the \( W \)-action. Let \( A^-_0 \) be the only alcove which contains the origin in its closure and such that \( \langle v, \alpha^\vee \rangle < 0 \) for any \( v \in A^-_0 \) and all \( \alpha \in R^+ \). We identify \( W \) with \( \mathcal{A} \) via \( w \mapsto w(A^-_0) \). Let \( \mathcal{S} \) be the set of reflections across the hyperplanes containing the walls of \( A^-_0 \). Thus \( (W, \mathcal{S}) \) is a Coxeter system.

The semi-infinite (Bruhat) order, or Lusztig’s generic order [Lu80], on \( \mathcal{A} \) is the partial order generated by the relations \( A \preceq \succeq_{\alpha,n} A \) if \( \langle v, \alpha^\vee \rangle > n \) for any \( v \in A \) (where \( \alpha \in R^+ \) and \( n \in \mathbb{Z} \)).

**Remark 2.1.** Observe that, while the poset \( (W, \leq) \) has a minimal element (the identity), but not a maximal one, the poset \( (\mathcal{A}, \preceq_{\alpha}) \) has neither a minimal nor a maximal element and it is hence unbounded in both directions. It is moreover clear that the semi-infinite order is stable under root translations, that is \( A \preceq_{\alpha} B \) if and only if \( A + \gamma \preceq_{\alpha} B + \gamma \) for any \( \gamma \in \Lambda \).

We conclude this section by recalling that the semi-infinite order can be thought of as a limit at \( -\infty \) of the Bruhat order. The following lemma makes this statement precise.
Lemma 2.2 (cf. [Soe, Claim 4.4]). Let $\gamma \in \Lambda$ be such that $\langle \gamma, \alpha^\vee \rangle < 0$ for any $\alpha \in R^+$ and let $A, B \in \mathcal{A}$. There exists a non-negative integer $n_0 = n_0(A, B, \gamma)$ such that the following are equivalent:

1. $A \leq_B B$
2. $A + n\gamma \leq B + n\gamma$ for all $n \geq n_0$.

Remark 2.3. The reader should be aware that we defined the semi-infinite order using a convention which is opposite to the one in [Lu80] and [Soe], but which agrees with the one in [A].

3. Inclusions of orbit closures

Recall that the Bruhat order on Weyl groups comes from geometry. We briefly remind such a fact in the case of an affine Weyl group (see, for example, [IM]).

Let $G$ be the connected, simply connected, complex algebraic group with root system $R$. For any subgroup $M$ of $G$, denote by $M((t))$, resp. $M[[t]]$, the group of $\mathbb{C}((t))$-, resp. $\mathbb{C}[[t]]$-, points of $M$. For example, $G((t))$ is the loop group associated with $G$. The Iwahori subgroup $I$ of $G((t))$ is the preimage of the Borel subgroup $B \subseteq G$ (corresponding to $R^+$) under the map $G[[t]] \to G$ given by $t \mapsto 0$. The affine flag variety $\mathcal{F}l = G((t))/I$ is naturally equipped with a left action of $I$. The $I$-orbits give the Iwahori decomposition

$$\mathcal{F}l = G((t))/I = \bigsqcup_{w \in W} X_w,$$

where $X_w = IwI/I \simeq \mathbb{C}^{i(w)}$ is called Bruhat cell and its closure $\overline{X_w}$ is a Schubert variety, which turns out to be the union of the Schubert cells indexed by the elements which are less or equal than $w$ in the Bruhat order:

$$\overline{X_w} = \bigsqcup_{y \leq w} X_y.$$

Therefore the affine flag variety is an honest ind-scheme, stratified by (finite dimensional) $I$-orbits, whose ind-structure is given by the Schubert varieties, which are in fact schemes of finite type.

In order to discuss the semi-infinite setting we first need to introduce some notation. Let $N$ be the unipotent radical of $B$ and $T$ the maximal torus of $G$ such that $B = NT$. The semi-infinite flag variety, as introduced in [Lu90] and [FF], is

$$\mathcal{F}l^\infty = G((t))/N((t))T[[t]].$$

Again, the Iwahori acts naturally on $\mathcal{F}l^\infty$ and its orbits (the semi-infinite Schubert cells) are in bijection with the affine Weyl group. Unluckily, the space above is not a good algebro geometric object, since it cannot be realised as an ind-scheme, as in the case of the affine flag variety: one could hope to play the same game as before and get the ind-structure on the semi-infinite flag variety from closures of $I$-orbits on it, but the $I$-orbits have now infinite dimension and
infinite codimension in $\mathcal{F}l^{\hat{T}}$. Therefore it is not even clear how to rigorously define $I$-orbit closures in this setting and hence semi-infinite Schubert varieties $X_w^{\geq}$. Anyway, if the closure $X_w^{\geq}$ made sense, it should be equal to the union of the $X_y^{\geq}$ with $y \leq_w w$. In fact, in [FFKM, Section 5] it is proven that the adjacency order on the set of Schubert strata in the quasi map spaces (see [FFKM] for the definition) is equivalent to the semi-infinite order. We conclude that the semi-infinite order should come from the inclusion relation of semi-infinite Schubert varieties (once made sense of them), in the same way as the Bruhat order coincides with the order relation given by the inclusion of closures of $I$-orbits on the affine flag variety.

4. Torus actions and moment graphs

Let $Y$ be a lattice, i.e. a free abelian group, of finite rank. A moment graph on $Y$ is a graph whose edges are labelled by non-zero elements of $Y$ and whose set of vertices is equipped with the structure of a poset such that two vertices are comparable if they are connected. We assume moreover that there are no loops.

Let us keep the same notation as in the previous sections. The extended torus $\hat{T} := T \times \mathbb{C}^\times$ acts on the affine flag variety $\mathcal{F}l: T$ by left multiplication on $G$ and $\mathbb{C}^\times$ by “rotating the loop” (i.e., by rescaling the variable $t$) and we can consider the 1-skeleton of such an action (cf. [Ku, Chapter 7]). The fixed point set consists of isolated points and it is in bijection with the affine Weyl group $W$ (each Schubert cell contains exactly one fixed point), so that from now on we will identify $\mathcal{F}l^{\hat{T}}$ with $W$. Moreover, the closure of any 1-dimensional orbit is smooth and contains exactly two fixed points. Therefore the set of 0- and 1-dimensional obits of the $\hat{T}$-action gives us a graph. The closure of each 1-dimensional orbit $O \simeq \mathbb{C}^\times$ is a one-dimensional representation of the extended torus and we can hence label the edge $O$ by the corresponding character.

Let us consider the finite dimensional Lie algebra $\mathfrak{g} := \text{Lie}(G)$ and its Cartan subalgebra $\mathfrak{h} := \text{Lie}(T)$, and denote, as in Section 2, by $\hat{\mathfrak{g}}$ its affinisation. Let us write $\hat{\mathfrak{h}}^*$ for the dual of the affinisation of $\mathfrak{h}$. Recall that there exists an element $\delta \in \hat{\mathfrak{h}}^*$ such that the set of real roots of the affine Kac-Moody group associated to $G$ (which is a central extension of $G((t))$) is equal to $\{\alpha + n\delta \mid \alpha \in R, n \in \mathbb{Z}\}$. By [Ku, Chapter 7], two fixed points $x \neq y$ are in the closure of the same 1-dimensional orbit $O$ if and only if there exist $\alpha \in \mathbb{R}^+$ and $n \in \mathbb{Z}$ such that $x = s_{\alpha,n}y$ and $\hat{T}$ acts on $\overline{O}$ via $\pm(\alpha + n\delta)$. Notice that the label of an edge is well-defined up to a sign. Anyway, once the sign is fixed, nothing will depend on this choice. Observe that if two vertices of the graph are connected, then they are comparable with respect to the Bruhat order on the affine Weyl group. The affine Bruhat graph $\mathcal{G}$ is the graph on $\mathbb{Z}\hat{R} := \mathbb{Z}R \oplus \mathbb{Z}\delta$ given by the previous data, together with the Bruhat order on its set of vertices. Each full subgraph $\mathcal{G}_w$ of $\mathcal{G}$ having as a set of vertices the elements which are less or equal than a given
$w \in W$ coincides with the moment graph associated by Braden and MacPherson in \cite{BMP} to the Schubert variety $\overline{X_w}$, as a stratified variety with a nice enough $\hat{T}$-action.

The extended torus $\hat{T}$ acts on the semi-infinite flag variety too. The set of fixed points is once again in 1-1 correspondence with the set of $I$-orbits and, hence, we will identify it with $W$. Even if the geometric side is not yet rigorously defined or understood, we will pretend to be able to associate with the $\hat{T}$-action on $\mathcal{F}l^\infty_2$ a moment graph on $\mathbb{Z}\hat{R}$. We hope to be able to make this construction more natural, by using Drinfeld’s spaces of quasi maps (see, for example, \cite{B}), in a forthcoming paper. We hence define the semi-infinite moment graph $G^\infty_2$ as follows: it has same set of vertices, edges and label function as $G_2$, but the structure of poset on its set of vertices is given by the semi-infinite order.

5. Structure algebras and cohomology of equivariantly formal spaces

Let $G$ be a moment graph on a lattice $Y$, $k$ a field and $S$ the symmetric algebra of the $k$-vector space $Y \otimes k$. We will write $\mathcal{V}$, respectively $\mathcal{E}$, for the set of vertices, respectively edges, of $G$. For an edge $E \in \mathcal{E}$ we will denote by $l(E) \in Y$ its label.

The structure algebra of $G$ is

$$Z := \left\{ (f_x) \in \prod_v S \bigg| f_x - f_y \in l(E)S \quad \text{if} \ E = (x, y) \in \mathcal{E} \right\}. $$

Notice that $S$ is diagonally embedded in $Z$ and that $Z$ is equipped with a structure of $S$-algebra, given by componentwise addition and multiplication.

Observe moreover that the definition of $G$ is independent of the partial order on the set of vertices of $G$. This has in fact a geometric reason. In the previous section we saw an example of a moment graph arising as the 1-skeleton of the action of an algebraic torus on a stratified variety. The partial order on the set of vertices was coming from the stratification, so forgetting about the partial order is somehow equivalent to forgetting about the stratification. In fact, if the moment graph $G$ coincides with the one skeleton of the action of an algebraic torus $T$ on a complex equivariantly formal variety $X$ (no stratification needed!), Goresky, Kottwitz and MacPherson \cite{GKM} showed that (for $\text{char } k = 0$) $H^*_T(X, k) \simeq Z$ as $\mathbb{Z}$-graded $S$-modules, where the $\mathbb{Z}$-grading on $Z$ is induced by the $\mathbb{Z}$-grading on $S$ given by $\deg Y := 2$. The structure of $S$-module on $H^*_T(X, k)$ is given by the classical identification of $S$ with $H^*_T(pt)$. Goresky, Kottwitz and MacPherson showed also that it is possible to recover the usual cohomology just by change of base:

$$H^*(X, k) \simeq Z \otimes_S k.$$

A $T$-space is equivariantly formal if its $T$-equivariant cohomology is free as an $S$-module. For example, a $T$-variety whose odd cohomology groups all vanish
is equivariantly formal. So Schubert varieties are an example of equivariantly formal spaces.

Let \( w \in W \) be an element of the affine Weyl group and consider the graph \( G_w \) of the previous section. If we denote by \( Z_w \) its structure algebra, then \( Z_w \cong H^*_T(\overline{X}_w) \) and \( Z_w \otimes k \cong H^*(\overline{X}_w) \). Taking inductive limits we also get \( Z \cong H^*_T(\mathcal{F}) \), where now \( Z \) denotes the structure algebra of the affine Bruhat graph.

Recall that, once forgotten the partial order on the set of vertices, \( G \) and \( G^\infty \) coincide, so that their structure algebras also coincide. We have not tried to make sense of the \( \check{T} \)-equivariant cohomology of \( \mathcal{F}^\infty \) yet and do not know whether the equality of the structure algebras of \( G \) and \( G^\infty \) has a rigorous geometric interpretation, but it is certainly compatible with [FF, Proposition 1].

6. Hecke Modules

In this section we recall the definition of the Hecke algebra and of certain modules for the action of the affine Hecke algebra, whose connection with representation theory will be discussed in Section 7. We use Soergel’s notation and normalisation [Soe].

Denote by \( L \) the ring of Laurent polynomials in one variable with integer coefficients \( \mathbb{Z}[v^\pm] \). The Hecke algebra \( H \) associated with the Coxeter system \((W, S)\) is the free \( L \)-module with basis \( \{H_y\} \) indexed by \( W \) and whose structure of associative \( L \)-algebra (and hence of right \( H \)-module over itself) is uniquely determined by

\[
H_y(H_s + v) = \begin{cases} H_{ys} + vH_y & \text{if } ys > y, \\ H_{ys} + v^{-1}H_y & \text{if } ys < y. \end{cases}
\]

It follows that, for any simple reflection \( s \in S \), \( H_s^2 = (v^{-1} - v)H_s + H_e \) and hence \( H_s^{-1} = H_s - (v^{-1} - v) \). Moreover, if \( y = s_{i_1} \cdots s_{i_r} \), where \( s_{i_1}, \ldots, s_{i_r} \in S \) is a reduced expression (that is, \( r = \ell(y) \)), then \( H_y = H_{s_{i_1}} \cdots H_{s_{i_r}} \) and hence \( H_y^{-1} = H_{s_{1}} \cdots H_{s_{r}} \in H \) for any \( y \in W \). Thus we can define the bar involution \( \overline{} : H \to H \), which is the \( \mathbb{Z} \)-linear involutive automorphism of the affine Hecke algebra (as a \( \mathbb{Z} \)-algebra) given by: \( v^{\pm 1} \mapsto v^{\mp 1} \) and \( H_y \mapsto H_{y^{-1}} \). The following is a classical and well-known result by Kazhdan and Lusztig. The formulation we give here is not the original one, but can be found, for example, in [Soe].

**Theorem 6.1** ([KL79], [Soe, Theorem 2.1]). For any \( w \in W \) there is a unique element \( \overline{H}_w \in H \) such that \( \overline{H}_w = H_w \) and \( H_w \in H_w + \sum_{y \in W \setminus \{w\}} v \mathbb{Z}[v]H_y \).

The coefficients of the change of basis matrix \((h_{y,w})\) from \( \{H_y\} \) to \( \{H_w\} \) are, by the above result, polynomials in \( v \) and are called Kazhdan-Lusztig polynomials. Notice that \( H_s + v \) is self dual, so that \( \overline{H}_s = H_s + v \) and hence \( \overline{H} \) is in fact describing the right action of \( H \) on itself in terms of multiplication by \( H_s \).

As in the previous sections, we now want to focus on the case of \( W \) being an affine Weyl group. Let \( S \) denote the set of Coxeter generators for \( W \) given in
Section 2 and let $\mathcal{H}$ be the corresponding Hecke algebra. We will briefly recall from [Lu80] Lusztig’s construction of three $\mathcal{H}$-modules.

The periodic Hecke module $\mathcal{P}$ is the free $\mathcal{L}$-module with basis $\mathcal{A}$ and with a structure of right $\mathcal{H}$-module given by

$$A \cdot H_s = \begin{cases} As + vA & \text{if } As > \frac{1}{2} A, \\ As + v^{-1}A & \text{if } As < \frac{1}{2} A. \end{cases}$$

Notice that, once replaced the Bruhat order by the semi-infinite order, (1) and (2) coincide.

In order to introduce the second $\mathcal{H}$-module we want to deal with, some more notation is needed. First, recall that in Section 2 we have denoted by $A_0^-$ the anti-fundamental alcove, that is the only alcove which contains the origin in its closure and such that $\langle v, \alpha^\vee \rangle < 0$ for any $v \in A_0^-$ and all $\alpha \in R^+$. Let $W_0 \subset W$ be the finite Weyl group, which we identify with the stabiliser in $W$ of $0 \in V$. Let $Q \subset V$ be the set of integral weights, that is $Q = \{ v \in V | \langle v, \alpha^\vee \rangle \in \mathbb{Z} \text{ for any } \alpha \in R \}$. For any $\lambda \in Q$ we set

$$E_\lambda := \sum_{x \in W_0} v^{\ell(x)} x(A_0^-) + \lambda$$

and we denote by $\mathcal{P}^0$ the $\mathcal{H}$-submodule of $\mathcal{P}$ generated by the set $\{ E_\lambda \mid \lambda \in Q \}$.

A map $\psi : \mathcal{M} \to \mathcal{N}$ of $\mathcal{H}$-modules is called $\mathcal{H}$-skew linear if $f(m \cdot h) = f(m) \cdot h$ for any $m \in \mathcal{M}$ and $h \in \mathcal{H}$. In [Lu80] it is shown that there exists a unique $\mathcal{H}$-skew linear involution $\cdot : \mathcal{P}^0 \to \mathcal{P}^0$ such that $E_\lambda = E_{\lambda}$ for any $\lambda \in Q$.

In this setting the analogue of Kazhdan-Lusztig’s Theorem is the following result.

**Theorem 6.2 ([Lu80]).** For any $A \in \mathcal{A}$ there is a unique element $P_A \in \mathcal{P}^0$ such that $P_A = P_A$ and $P_A \in A + \sum_{B \in \mathcal{A} \setminus \{ A \}} v\mathbb{Z}[v]B$.

Moreover, the set $\{ P_A \mid A \in \mathcal{A} \}$ is an $\mathcal{L}$-basis of $\mathcal{P}^0$ and the polynomials $p_{B,A} \in v\mathcal{L}$ defined by $P_A = \sum_{B \in \mathcal{A}} p_{B,A} B$ are called periodic polynomials.

The definition of the third $\mathcal{H}$-module we want to deal with needs a sort of completion of $\mathcal{P}$. We say that a map $f : \mathcal{A} \to \mathcal{L}$ is bounded from above if there exists an alcove $C \in \mathcal{A}$ such that whenever $f(A) \not\equiv 0$, then $A \leq C \leq \infty$. We consider the space of all bounded from above maps:

$$\hat{\mathcal{P}} := \{ f : \mathcal{A} \to \mathcal{L} \mid f \text{ is bounded from above} \}$$

and we identify it with a set of formal $\mathcal{L}$-linear combinations of alcoves via $f \mapsto \sum f(A)A$. At this point it is clear that it is possible to extend the right action of $\mathcal{H}$ and the $\mathcal{H}$-skew-linear involution on $\mathcal{P}$ to $\hat{\mathcal{P}}$. The second semi-infinite variation of Kazhdan-Lusztig’s Theorem is hence the following:

**Theorem 6.3 ([Kat]).** For any $A \in \mathcal{A}$ there is a unique element $\hat{P}_A \in \hat{\mathcal{P}}$ such that $\hat{P}_A = \hat{P}_A$ and $\hat{P}_A \in A + \sum_{B \in \mathcal{A}} v^{-1}\mathbb{Z}[v^{-1}]B \in \hat{\mathcal{P}}$. 


We are now ready to define the last family of polynomials we are interested in in this paper: the generic polynomials $q_{B,A} \in \mathbb{Z}[v]$ are such that $P_A = \sum_{B \in \mathcal{A} \setminus \{A\}} q_{B,A}B$, where the latter is allowed to be (and indeed it is) an infinite sum.

7. Multiplicity formulae

7.1. Affine Kac-Moody algebras. Let $\hat{g}$ denote the affinisation of the simple Lie algebra $g$ as in Section 2 and recall that $K$ is a fixed generator of the central line and $D$ the derivation operator. Let $\mathfrak{h} \subset \mathfrak{b}$ be a Cartan and a Borel subalgebra of $g$ (corresponding to $R^+$) and let $\hat{\mathfrak{h}}$ and $\hat{\mathfrak{b}}$ be the corresponding affine Cartan and Borel subalgebras of $\hat{g}$ (so, in particular, $\hat{\mathfrak{h}} \simeq \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}D$). The affine BGG category $O$ is the full subcategory of $\hat{g}$-modules on which $\hat{\mathfrak{h}}$ acts semi-simply and $\hat{\mathfrak{b}}$ locally finitely. Then the affine BGG category $O$ decomposes in levels $O_\kappa$, $\kappa \in \mathbb{C}$, according to the action of the central element $K$. Let us fix an element $\rho$ in the dual of the Cartan $\hat{\mathfrak{h}}^*$ having the property that $\rho(\gamma^\vee) = 1$ for any simple affine root $\gamma$. Observe that $\rho$ is uniquely determined only up to a multiple of the smallest positive imaginary root $\delta$. This is in fact not a problem, since nothing is going to depend on this choice. The critical level is the value $\text{crit} := -\rho(K)$. For any weight $\lambda \in \hat{\mathfrak{h}}^*$, we denote by $\Delta(\lambda)$ and $L(\lambda)$ the Verma module and the irreducible representation of highest weight $\lambda$, respectively.

7.1.1. Negative level. Let us consider a regular weight $\lambda$ with $\lambda(\gamma^\vee) \in \mathbb{Z}_{<1}$ for any simple affine root $\gamma$. Then we have the following affine version of the Kazhdan-Lusztig conjecture:

$$[\Delta(\gamma \cdot \lambda) : L(w \cdot \lambda)] = \bar{h}_{y,w}(1)$$

where, for any $x \in W$, the $\rho$-shifted action of $W$ on $\hat{\mathfrak{h}}^*$ is defined as $x \cdot \lambda := x(\lambda + \rho) - \rho$, and $\bar{h}_{y,w}$ denotes the inverse Kazhdan-Lusztig polynomial, that is the polynomial such that $\sum_y (-1)^{\ell(u) - \ell(y)}h_{u,y}\bar{h}_{y,w} = \delta_{u,w}$. The above multiplicity statement was proven by Kashiwara and Tanisaki using the theory of $D$-modules on the affine flag variety $[KT95]$. An alternative, algebraic proof can be deduced from [EW], where Elias and Williamson prove a more general statement about indecomposable Soergel bimodules, known as Soergel’s conjecture. We will briefly discuss the moment graph formulation of Soergel’s conjecture later in Section 11.

7.1.2. Positive level. If a regular weight $\lambda$ is such that $\lambda(\gamma^\vee) \in \mathbb{Z}_{>1}$ for any simple affine root $\gamma$, then it is enough to substitute $\bar{h}_{y,w}$ in the multiplicity statement (3) by the Kazhdan-Lusztig polynomial $h_{y,w}$. Such a result was proven by Kashiwara and Tanisaki $[KT98]$ using once again the theory of $D$-modules, but in this case, they had to deal with sheaves of differential operators on Kashiwara’s thick flag variety, whose definition we do not want to recall. It is sufficient for us to mention that such a variety, as $\mathcal{F}l$ and $\mathcal{F}\mathcal{T}$, is an appropriate quotient of a loop group, on which the Iwahori acts with orbits of infinite dimension (but
finite codimension). As for the negative level case, an algebraic proof of the affine Kazhdan-Lusztig conjecture at a positive level can be deduced from \[EW\].

7.1.3. Critical level. Finally, we want to look at the case of a dominant regular \(\lambda \in \widehat{\mathfrak{h}}^*\) such that \(\lambda(K) = \text{crit}\). (Moreover, \(\lambda\) should satisfy some technical conditions that we do not want to list and that can be found in \[AF, \S 1.4\]). We will state a (still in general conjectural) further multiplicity formula involving restricted Verma modules. Tensoring by the one-dimensional representation \(L(\delta)\) is an autoequivalence of \(\mathcal{O}\) and a block is stabilised by such an equivalence if and only if it is of critical level, so that tensoring by \(L(\delta)\) is also an auto-equivalence of any critical block. In order to define restricted Verma modules the notion of graded centre of a critical block is needed. Its degree \(n\) part consists of the space of natural transformations from the functor \(\cdot \otimes L(n\delta)\) to the identity functor on the block (satisfying certain extra conditions, which are described, for example, in \[AF, \S 1.4\]). Then the restricted Verma module \(\Delta(\lambda)\) is obtained by quotienting \(\Delta(\lambda)\) by the ideal generated by the homogeneous components of degree \(\neq 0\) of the centre. The following formula has been conjectured by Feigin, Frenkel and Lusztig, independently:

\[
\mathfrak{Z}(y \cdot \lambda) : L(w \cdot \lambda) = p_{y(A_{n}),w(A_{n})}(1).
\]

7.1.4. Wakimoto modules. We leave to \[AL\] the discussion of a multiplicity formula for Wakimoto modules at a non critical level involving Lusztig’s generic polynomials evaluated at one.

7.2. Quantum groups at a root of unity. Let \(\mathfrak{g}\) be again a simple complex finite dimensional Lie algebra and consider (Lusztig’s version of) its quantum group at a \(p\)-th root of unity \[Lu90b\], where \(p\) is an odd integer (prime to 3 in the G\(_2\)-case). Denote by \(u_p(\mathfrak{g})\) the (finite dimensional) small quantum group. Then \(u_p(\mathfrak{g})\) admits a triangular decomposition and the standard objects \(\{Z(\mu)\}\) in this setting have the same realisation as Verma modules for \(\mathfrak{g}\): they are obtained by inflating a linear form of the 0-part to the positive part and then inducing it to the whole \(u_p(\mathfrak{g})\). The module \(Z(\mu)\) has a unique simple quotient, that we denote \(L(\mu)\). Let \(p\) be greater than the Coxeter number of \(\mathfrak{g}\), and \(\lambda \in \mathfrak{h}^*\) such that \(\langle \lambda,\alpha^\vee \rangle < -1\) for any simple root \(\alpha\) of \(\mathfrak{g}\) and \(\langle \lambda,\varphi^\vee \rangle > p - 1\), where \(\varphi\) is the highest root of \(\mathfrak{g}\). Denote by \(\rho_0\) half the sum of all positive roots. We consider the \(p\)-dilated \(\rho_0\)-shifted action \(\cdot_p\) of the affine Weyl group on \(\mathfrak{h}^*\): this means that we shift by \(-\rho_0\) the action of the affine Weyl group \(W\) which has been rescaled in such a way that \(s_{\alpha,n}(v) = v - (\langle v, \alpha^\vee \rangle - pn)\alpha\). Then the following multiplicity formula is a restatement of Lusztig’s conjecture on type 1 finite dimensional modules for quantum groups at a root of unity (cf. \[Lu90b\]):

\[
[Z(y \cdot_p \lambda) : L(w \cdot_p \lambda)] = p_{y(A_{n}),w(A_{n})}(1),
\]

where \(y \cdot \lambda\) is dominant and \(w\) is a minimal element in \(w\text{Stab}_{W \cdot_p}(\lambda)\).
7.3. $G_1 T$-modules. Let $k$ be a field of characteristic $p$, $G$ a semisimple simply connected algebraic group over $k$ and $G_1 \subset G$ be the kernel of the Frobenius. Fix once and for all a Borel subgroup $B$, a maximal torus $T \subseteq B$, and denote by $\mathfrak{b}$ and $\mathfrak{h}$ the corresponding Lie algebras. The Lie algebra $\mathfrak{g}$ of $G$ is a $p$-algebra and we denote by $U^{\text{res}}(\mathfrak{g})$ its restricted Lie algebra, which is a finite dimensional quotient of the enveloping algebra $U(\mathfrak{g})$ (see, for example [AJS, Introduction]). Moreover, let $G_1 T \subset G$ be the group scheme generated by $T$ and $G_1$. The representation category we are interested in has as objects the finite dimensional $G_1 T$-representations, which can be identified with $U^{\text{res}}(\mathfrak{g})$-modules graded by the group of characters $X$ of $T$. Let $\mu \in X$, and differentiate it to get $\mu^\ast \in \mathfrak{h}^\ast$. The Baby Verma module $Z(\mu)$ is defined as the induction to the whole $U^{\text{res}}(\mathfrak{g})$ of the $X$-graded one-dimensional $U(\mathfrak{b})$-representation concentrated in degree $\mu$ (which is obtained, as usual, by inflating to $U(\mathfrak{b})$ the one-dimensional $\mathfrak{h}$-module $k^\mu$). Denote by $L(\mu)$ the unique simple quotient of $Z(\mu)$. As in §7.2, let $W$ be the affinisation of the Weyl group of $G$ and, as in §7.2, denote by $\cdot$ the $\rho_0$-shifted action of it on $\mathfrak{h}^\ast$. Then the $G_1 T$-version of Lusztig’s conjecture on the characters of modular representations (of regular restricted highest weight $s$) is the following (cf. [Fie10, Conjecture 3.4])

\begin{equation}
[Z(y \cdot 0) : L(w \cdot 0)] = p_{y(A^\alpha_0),w(A^\alpha_0)}(1),
\end{equation}

for $w, y \in W$ and $w$ such that $-p < \langle w \cdot 0, \alpha^\vee \rangle \leq 0$. For $p \gg 0$, in [AJS] the above multiplicity formula is derived from the quantum group at a root of unity analogue, and an explicit huge bound on $p$, depending on the root system of $G$, for the statement to be true was found by Fiebig in [Fie12]. Till June 2013, when Williamson announced the first counterexample [W], Lusztig’s conjecture was expected to hold for $p$ greater than the Coxeter number $h$ of $G$ (this was Kato’s hope [Kat], even more optimistic than Lusztig’s suggestion of $p \geq 2h - 3$). A modified conjecture is not available yet, so that new tools are now needed.

8. Sheaves on moment graphs

Let $G$ be a moment graph on $Y$ and $S$ the symmetric algebra of $Y \otimes k$ for a given field $k$, $\mathbb{Z}$-graded as in Section 5. Denote by $S - \text{mods}^\mathbb{Z}$ the category of $\mathbb{Z}$-graded $S$-modules. A sheaf $\mathcal{F}$ on $G$ is given by two collections of $\mathbb{Z}$-graded $S$-modules

$$(\mathcal{F}^x \in S - \text{mods}^\mathbb{Z})_{x \in V}, \quad (\mathcal{F}^E \in S - \text{mods}^\mathbb{Z} \mid l(E)\mathcal{F}^E = (0))_{E \in \mathcal{E}}$$

and a collection of maps of $\mathbb{Z}$-graded $S$-modules

$$(\rho^{x,E} : \mathcal{F}^x \to \mathcal{F}^E)_{E \in \mathcal{E}, x \text{ is a vertex of } E}.$$

For a vertex $x$, we will often refer to the module $\mathcal{F}^x$ as the stalk in $x$ of $\mathcal{F}$. A morphism $f$ between two sheaves $\mathcal{F}_1$ and $\mathcal{F}_2$ on $G$ consists of two collections of
morphism of \( \mathbb{Z} \)-graded \( S \)-modules

\[
(f^x : F^x_1 \rightarrow F^x_2)_{x \in \mathcal{V}}, \quad (f^E : F^E_1 \rightarrow F^E_2)_{E \in \mathcal{E}}
\]
compatible with the \( \rho \)-maps, that is \( f^E \circ \rho^{x,E}_1 = \rho^{x,E}_2 \circ f^x \) for any edge \( E \) and any vertex \( x \) of \( \mathcal{E} \).

The set of sections of \( \mathcal{F} \) over \( \mathcal{I} \subseteq \mathcal{V} \), is

\[
\Gamma(\mathcal{I}, \mathcal{F}) := \left\{(f_x) \in \prod_{x \in \mathcal{I}} F^x \mid \rho^{x,E}(f_x) = \rho^{y,E}(f_y) \text{ if } E = (x, y) \in \mathcal{E} \right\}.
\]

Notice that \( \Gamma(\mathcal{I}, \mathcal{F}) \) has naturally a structure of \( \mathbb{Z} \)-graded \( S \)- and \( \mathcal{Z} \)-module, on which \( S \) acts diagonally and \( \mathcal{Z} \) componentwise.

Observe that the structure algebra can be realised as the set of global sections (i.e, sections over the whole \( \mathcal{V} \)) of the sheaf \( \mathcal{F} \) given by

\[
(\mathcal{F}^x_S = S)_{x \in \mathcal{V}}, \quad (\mathcal{F}^E_S = S/l(E)S)_{E \in \mathcal{E}}
\]
and the \( \rho \)-maps \( \rho^{x,E} : S \rightarrow S/l(E)S \) (for \( x \) vertex of \( E \)) are the canonical quotient maps.

### 8.1. Braden-MacPherson sheaves.

We recall here the definition of a class of sheaves on a moment graph introduced by Braden and MacPherson [BMP] in order to compute the intersection cohomology of varieties acted upon by an algebraic torus \( T \), which are equivariantly formal and equipped with a \( T \)-stable stratification (see [BMP] §1.1 for the exact assumptions on the varieties). Their construction makes sense also for moment graphs not coming from geometry.

Recall that, by definition, the vertex set of any moment graph \( \mathcal{G} \) is equipped with a partial order \( \leq \). For a vertex \( x \in \mathcal{V} \), denote by \( \{> x\} \) the set of elements which are strictly greater than \( x \). Let \( \mathcal{F} \) be a sheaf and call \( \rho^{\delta x} \) the following composition of maps:

\[
\Gamma(\{> x\}, \mathcal{F}) \hookrightarrow \bigoplus_{y>x} \mathcal{F}^y \rightarrow \bigoplus_{y>x \text{ s.t. } E=(x,y)\in \mathcal{E}} \mathcal{F}^y \oplus \rho^{y,E} \rightarrow \bigoplus_{E=(x,y)\in \mathcal{E} \text{ s.t. } y>x} \mathcal{F}^E.
\]

We denote by \( \mathcal{F}^{\delta x} \) the \( \mathbb{Z} \)-graded \( S \)-module \( \rho^{\delta x}(\Gamma(\{> x\}, \mathcal{F})) \).

For any \( w \in \mathcal{V} \), the indecomposable Braden-MacPherson sheaf \( \mathcal{B}(w) \) is inductively defined as follows. We start by setting:

\[
\mathcal{B}(w)^y = (0) \text{ if } y \not\leq w, \quad \mathcal{B}(w)^w \simeq S,
\]

\[
\mathcal{B}(w)^{(y,z)} = (0) \text{ and } \rho^{y,(y,z)} = \rho^{z,(y,z)} = 0 \text{ if } y \not\leq w \text{ or } z \not\leq w
\]
and we then assume that \( \mathcal{B}(w)^y \) and \( \mathcal{B}(w)^E \) have been constructed for any vertex \( y > x \) and \( E = (y, z) \in \mathcal{E} \) with \( y, z > x \). We define

\[
\mathcal{B}(w)^E = \mathcal{B}(w)^y / l(E) \mathcal{B}(w)^y \text{ if } E = (x, y) \text{ and } y > x
\]
and, for such an \( E \), the morphism \( \rho^{y,E} : \mathcal{B}(w)^y \rightarrow \mathcal{B}(w)^y / l(E) \mathcal{B}(w)^y \) is the canonical quotient map. Now it is possible to consider the module \( \mathcal{B}(w)^{\delta x} \) and
to take its projective cover (which always exists in the category of $\mathbb{Z}$-graded $S$-modules):

$$p_x : \mathcal{B}(w)^x \to \mathcal{B}(w)_{\delta x}$$

is a projective cover.

Obviously, for $(x, z) \in E$ and $z > x$, one obtains the map $p^{x,(x,z)}$ as the composition of the following morphisms

$$\mathcal{B}(w)^x \xrightarrow{p_x} \mathcal{B}(w)_{\delta x} \hookrightarrow \bigoplus_{E = (x,y) \in E \text{ s.t. } y > x} \mathcal{B}E \twoheadrightarrow \mathcal{B}(x,z).$$

9. **Braden-MacPherson sheaves and intersection cohomology**

As already mentioned, Braden-MacPherson sheaves were introduced with the aim of providing a combinatorial algorithm to compute equivariant intersection cohomology, with coefficients in a field $k$ of characteristic zero, of a sufficiently nice complex algebraic variety $X$ equipped with the action of an algebraic torus $T$. More precisely, with any such a variety one can associate a moment graph $G$ on the character lattice $Y$ of $T$, whose set of vertices is given by the $T$-fixed points $X^T$ (see [BMP, §1.2]). The order on the set of vertices is induced by a fixed stratification on $X$ having the property that any stratum contains exactly one fixed point and hence there is a unique maximal vertex $\bar{v}$. Let $S$ be the symmetric algebra of the vector space $Y \otimes k$. If $X$ satisfies all the assumptions in [BMP, §1.1], then there are canonical identifications: (cf [BMP, Theorem 1.5 and Theorem 1.6]):

$$IH^*_T(X) \simeq \Gamma(V, \mathcal{B}(\bar{v}))$$

as $\mathbb{Z}$-modules, and hence also as $S$-modules, and, for each point $x \in X$,

$$IH^*_T(x) \simeq \mathcal{B}(\bar{v})^y$$

as $S$-modules,

where $y \in X^T$ is the fixed point contained in the same stratum as $x$. As the previous modules are all free over $S$, non-equivariant global and local intersection cohomology are obtained by base change:

$$IH^*(X) \simeq \Gamma(V, \mathcal{B}(\bar{v})) \otimes_S k, \quad IH^*_T(x) \simeq \mathcal{B}(\bar{v})^y \otimes_S k$$

as $k$-vector spaces.

For example, any affine Schubert variety $\overline{X_w}$ satisfies the assumptions in [BMP, §1.1] and we can consider the moment graph $G_w$ as in Section 4. It has a unique maximal vertex, $w$, and hence

$$IH^*_T(\overline{X_w}) \simeq \Gamma(V, \mathcal{B}(w)), \quad \text{and} \quad IH^*_T(\overline{X_w})_y \simeq \mathcal{B}(w)^y \text{ for any } y \leq w$$

In positive characteristic (under some technical assumption on $k$), Braden-MacPherson sheaves compute hypercohomology of parity complexes on $X$ [FieW]. Parity complexes are not perverse in general and this fact is related to the presence
of torsion in the intersection cohomology groups of Schubert varieties. The discovery of these torsion phenomena allowed Williamson to produce many counterexamples to Lusztig’s modular conjecture [W].

10. Local intersection cohomology of $\mathcal{F}l^{\mathbb{Z}}$

Let $M$ be a finitely generated, free $\mathbb{Z}$-graded $S$-module, then there are integers $j_1, \ldots, j_r$, uniquely determined up to reordering, such that $M = \bigoplus_{i=1}^r S[j_i]$ (where the shift in the grading is such that $M[j_i]_n = M_{j_i+n}$). With such an $M$ we can associate its graded rank, that is a Laurent polynomial in $v$ which keeps track of the shifts: $\text{rk} M := \sum_{i=1}^r v^{-j_i} \in \mathbb{Z}[v^\pm 1]$. Let $\mathcal{G}$ be a moment graph. Observe that for any pair of vertices $w, y \in V$ the stalk $\mathcal{B}(w)^y$ is by construction finitely generated and free as a $\mathbb{Z}$-graded $S$-module, so that it makes sense to consider its graded rank.

Let $\text{char } k = 0$. In [La15] we investigated the stable moment graph $\mathcal{G}^{\text{stab}}$, which is the full subgraph of $\mathcal{G}^{\mathbb{Z}}$ having as set of vertices the set $A^-$ of alcoves such that if $v \in A \in A^-$ then $\langle v, \alpha^\vee \rangle < 0$ for any simple finite root $\alpha$. (To be precise, in $\mathcal{G}^{\mathbb{Z}}$ we were dealing with the upside down setting, since we had defined the semi-infinite order by giving the affine hyperplanes an orientation which is opposite to the one considered here). For any alcove $A \in A^-$, denote by $\mathcal{B}^{\text{stab}}(A)$ the corresponding indecomposable Braden-MacPherson sheaf, where the “stab” is there to remind us that we are considering sheaves on $\mathcal{G}^{\text{stab}}$.

Let $\delta(A, B)$ be Lusztig’s semi-infinite length function (see [Lu80]). The main result of [La15] is the following:

**Theorem 10.1.** Assume that $A, B \in A^-$ are deep enough in $A^-$, then

$$\text{rk} \mathcal{B}^{\text{stab}}(A)^B = v^{\delta(A, B)} q_{B,A}.$$  

Next, consider the semi-infinite graph $\mathcal{G}^{\mathbb{Z}}$ and the Braden-MacPherson sheaves on it. Since for any $A \in A$ the set of elements less than $A$ with respect to $<_{\mathbb{Z}}$ is infinite, the algorithm described in the previous section will never end. Nevertheless, all intervals $[B, A] := \{C \in W \mid B \leq_{\mathbb{Z}} C \leq_{\mathbb{Z}} A\}$ have finite cardinality and hence it is, in principle, possible to compute $\mathcal{B}^{\mathbb{Z}}(A)^B$, where, again, $\mathcal{G}^{\mathbb{Z}}$ reminds us that we are dealing with sheaves on $\mathcal{G}^{\mathbb{Z}}$. Taking their graded rank returns again Lusztig’s generic polynomials.

**Lemma 10.2.** $\text{rk} \mathcal{B}^{\mathbb{Z}}(A)^B = v^{\delta(A, B)} q_{B,A}.$

**Proof.** If $A, B \in A^-$ and they are both deep enough in $A^-$, then the statement coincides with the statement of Theorem 10.1. Otherwise, there exists an antidominant $\gamma$ such that $[B + \gamma, A + \gamma] \subset A^-$ and $A, B$ are deep enough in $A^-$. For any two alcoves $C, D \in A$, denote by $\mathcal{G}^{\mathbb{Z}}_{[C,D]}$ the full subgraph of $\mathcal{G}^{\mathbb{Z}}$ having as set of vertices the interval $[C, D]$. There is an induced morphism of moment graphs $T_\gamma : \mathcal{G}^{\mathbb{Z}}_{[A,B]} \to \mathcal{G}^{\mathbb{Z}}_{[A+\gamma, B+\gamma]}$ which is in fact an isomorphism in the sense of
By [La12, Lemma 5.1], $B^\infty_\infty(A)^B \cong B^\infty_\infty(A + \gamma)^{B+\gamma}$ and now the lemma follows from Theorem 10.1.

We want to relate all of this to the geometry of the semi-infinite flag variety. In [FFKM] the singularities of $\mathcal{F}l^\infty$ are investigated via the study of Drinfeld’s spaces of quasi maps. For any $\gamma \in \mathbb{Z}R^\vee$, let us denote by $Q^\gamma$ the space of quasi maps of degree $\gamma$ from a curve $C$ of genus zero to the flag variety $G/B$ (where $R^\vee$ is the set of coroots of $G$). We are not going to recall the definition of these objects which can be found in, for example, [FFKM]. Let $w, y \in W = \mathbb{Z}R^\vee \rtimes W_0$, then $w = t^\lambda u$ and $y = t^\mu v$, for $\lambda, \mu \in \mathbb{Z}R^\vee$ and $u, v \in W_0$, then we set

$$IH^\bullet \left( X^\infty_w \right)_y := IH^\bullet \left( Q^\lambda_u \right)_{-\mu,v},$$

where $IH^\bullet \left( Q^\lambda_u \right)_{-\mu,v}$ denotes the stalk of $IH \left( Q^\lambda_u \right)$ at the generic point of $Q^\mu_v$ (see [FFKM, 6.4.4.] for the definition of the closed variety $Q^\gamma$). Our choice of orientation of the affine hyperplanes, opposite to the one in [FFKM], requires the “-” sign in front of $\mu$ and $\lambda$.

The following result had been anticipated in [La90, §11] and proved in [FFKM]:

**Theorem 10.3.** For any pair $y, w \in W$

$$\sum_i \dim IH^{2i} \left( X^\infty_w \right)_y = \nu^b(y(A_0^-),w(A_0^-)) q_y(A_0^-),w(A_0^-).$$

Moreover, by the considerations in [FFKM, §3.2] the equivariant cohomology $IH_T^\bullet \left( Q^\lambda_u \right)_{-\mu,v}$ is isomorphic, as an $S$-module, to $S \otimes IH^\bullet \left( Q^\lambda_u \right)_{-\mu,v}$, so that we have

$$\text{rk} IH_T^\bullet \left( Q^\lambda_u \right)_{-\mu,v} = \nu^b(y(A_0^-),w(A_0^-)) q_y(A_0^-),w(A_0^-).$$

By combining the results of this section, we get:

**Corollary 10.4.** The Braden-MacPherson algorithm computes the local cohomology of the semi-infinite flag variety, that is for any pair $y, w \in W$

$$IH_T^\bullet \left( X^\infty_w \right)_y \cong \mathcal{B}(w(A_0^-))^\mu(A_0^-),$$

$$IH^\bullet \left( X^\infty_w \right)_y \cong \mathcal{B}(w(A_0^-))^\mu(A_0^-) \otimes k.$$
11. STALKS OF BRADEN-MACPherson SHEAVES AND MULTIPLICITY FORMULAE

Assume $G$ is the affine Bruhat moment graph of Section 4 and let $k = \mathbb{C}$. The vertices of $G$ are indexed by elements of an affine Weyl group $W$ and for any pair $y, w \in W$

$$\text{rk} \mathcal{B}(w)^y = v^{l(y) - l(w)} h_{y,w}. \tag{7}$$

The above equality follows from [BMP] and the fact that affine Kazhdan-Lusztig polynomials are (up to a shift in our normalisation) Poincaré polynomials for the stalks of local intersection cohomology of the affine flag variety with coefficients in a field of characteristic zero [KL80]. It is in fact possible to associate with any reflection faithful representation of a Coxeter system a moment graph and to study indecomposable Braden-MacPherson sheaves on it. The moment graph formulation of Soergel’s conjecture states that (7) holds in this more general setting too and has been proven, as we have already mentioned, by Elias and Williamson [EW].

In the affine Bruhat graph case (and, more in general, for a moment graph associated with a symmetrisable Kac-Moody algebra), Fiebig [Fie08] proved a moment graph localisation theorem for modules in (a deformed version of) category $\mathcal{O}$ admitting a Verma flag, that is a finite filtration with subquotients isomorphic to Verma modules. The fundamental application of his result is that one can use Braden-MacPherson sheaves on $G$ to compute multiplicities of simples in Verma modules at a negative level without passing through geometry:

$$\text{rk} \mathcal{B}(w)^y(1) = [\Delta(y \cdot \lambda) : L(w \cdot \lambda)]. \tag{8}$$

Once we know (8), it is clear that from (7) follows (3), the affine version of the Kazhdan-Lusztig conjecture at a negative level for the affinisation of a simple complex Lie algebra $g$.

Next, we want to focus on the semi-infinite graph $G^\infty$. In [AL] we prove a (semi-infinite) moment graph localisation theorem for modules in (a deformed and truncated version of) the affine BGG category $\mathcal{O}$ admitting a Wakimoto flag, that is a finite filtration with subquotients isomorphic to Wakimoto modules and we expect to be able to use Theorem 10.1 to interpret multiplicities of simple quotients of Wakimoto modules in terms of generic polynomials.

We conclude this section by briefly mentioning the main result of [FieLa15a]. In [FieLa15b] we introduce the notion of group actions on a moment graph and in [FieLa15a] we consider the special case of the root lattice acting on $G^\infty$. This allows us to define a sort of pushforward functor from the category of sheaves on $G^\infty$ to a certain category of modules over the structure algebra of the quotient graph, equipped with a particularly nice filtration (see [FieLa15a]). Under this pushforward functor, the indecomposable Braden-MacPherson sheaves $\mathcal{B}^\infty(A)$ decompose and looking at the indecomposable summands enables us to define a
certain object $P(A)$ for any $A \in \mathcal{A}$. For any $B, A \in \mathcal{A}$, it is possible to consider a finitely generated, free $\mathbb{Z}$-graded $S$-module $P(A)_{[B]}$ and we have:

**Theorem 11.1.** $\text{rk } P(A)_{[B]} = p_{B,A}$.

The above theorem suggests that our construction should have applications in the questions discussed in Sections 7.3 and 7.1.3 where Lusztig’s periodic polynomials appear.

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