A Decision Procedure for a Theory of Finite Sets with Finite Integer Intervals

MAXIMILIANO CRISTIÁ, Universidad Nacional de Rosario and CIFASIS, Argentina
GIANFRANCO ROSSI, Università di Parma, Italy

In this paper we extend a decision procedure for the Boolean algebra of finite sets with cardinality constraints ($L_{|·|}$) to a decision procedure for $L_{|[·]}$ extended with set terms denoting finite integer intervals ($L_{|[·]}$). In $L_{|[·]}$ interval limits can be integer linear terms including unbounded variables. These intervals are a useful extension because they allow to express non-trivial set operators such as the minimum and maximum of a set, still in a quantifier-free logic. Hence, by providing a decision procedure for $L_{|[·]}$ it is possible to automatically reason about a new class of quantifier-free formulas. The decision procedure is implemented as part of the $\{\log\}$ (‘setlog’) tool. The paper includes a case study based on the elevator algorithm showing that $\{\log\}$ can automatically discharge all its invariance lemmas, some of which involve intervals.

CCS Concepts: • Theory of computation → Automated reasoning; Constraint and logic programming; Logic and verification;

Additional Key Words and Phrases: $\{\log\}$, set theory, integer intervals, decision procedure, constraint logic programming

ACM Reference format:
Maximiliano Cristiá and Gianfranco Rossi. 2023. A Decision Procedure for a Theory of Finite Sets with Finite Integer Intervals. ACM Trans. Comput. Logic 25, 1, Article 3 (November 2023), 34 pages. https://doi.org/10.1145/3625230

1 INTRODUCTION

In the context of formal verification and analysis, it is necessary to discharge a number of verification conditions or proof obligations. Tools capable of automating such proofs are essential to render the development process cost-effective. At the base of proof automation is the concept of decision procedure. Methods and tools based on set theory, such as B [1], ProB [2] and Atelier-B [3], provide a proven vehicle for formal modeling, specification, analysis and verification of software systems. These methods would benefit from new decision procedures for fragments of set theory.

On the basis of this observation, in this paper we provide a decision procedure for the Boolean algebra of finite sets extended with cardinality constraints and finite integer intervals or ranges.1

1From now on, we will say integer intervals or just intervals meaning finite integer intervals.
To the best of our knowledge this is the first time this fragment of set theory is proven to be decidable. The Boolean algebra of finite sets extended with cardinality constraints, denoted $L_{\mathbb{1}}$ (read ‘l-card’), is known to be decidable since quite some time and solvers supporting it already exist [4–7]. The addition of integer intervals (i.e., $[k, m] = \{p \in \mathbb{Z} \mid k \leq p \leq m\}$) allows to reason about operations such as the minimum of a set, the $i$-th smallest element of a set, partitioning of a set in the elements below and above a given number, and so on. Besides, integer intervals are important in the verification of programs with arrays [8] and static analysis [9]. Although the other approaches just mentioned above could potentially reason about these set operations and solve problems in this field, the most challenging and original aspect of our approach is to obtain these results by successive extensions of a base theory of sets.

Indeed, the base theory of sets is the one supported by a Constraint Logic Programming (CLP) language known as CLP($\mathbb{S}\mathbb{T}$) [10] providing a decision procedure for the Boolean algebra of hereditarily finite sets, i.e., finitely nested sets that are finite at each level of nesting. CLP($\mathbb{S}\mathbb{T}$) has been extended in several ways [11, 12]. In particular, $L_{\mathbb{1}}$ extends CLP($\mathbb{S}\mathbb{T}$) with cardinality and linear integer constraints [7]. In this paper we show how a further extension, called $L_{\mathbb{1}}$ (read ‘l-int’), can deal with integer intervals to provide a decision procedure.

CLP($\mathbb{S}\mathbb{T}$) and all of its extensions have been implemented in a tool called $\{\log\}$ (read ‘setlog’), itself implemented in Prolog [13, 14]. $\{\log\}$ can be used as a CLP language and as a satisfiability solver. This duality is reflected in the fact that $\{\log\}$ code behaves as both a formula and a program. Then, $\{\log\}$ users can write programs and prove their properties using the same and only code and tool, all within set theory. As a satisfiability solver, $\{\log\}$ has proved to solve non-trivial problems [15, 16]. After presenting the implementation of $L_{\mathbb{1}}$ in $\{\log\}$ we provide empirical evidence that it is useful in practice through a preliminary empirical evaluation and a case study based on the elevator algorithm. The case study shows the duality formula-program that can be exploited in $\{\log\}$.

Structure of the paper. The paper is structured as follows. In Section 2 we show an example where intervals combined with sets are needed to motivate this work. Section 3 presents a detailed account of the syntax and semantics of $L_{\mathbb{1}}$. The algorithm to decide the satisfiability of $L_{\mathbb{1}}$ formulas, called $\mathbb{S}\mathbb{A}\mathbb{T}_{\mathbb{1}}$ (read ‘sat-int’), is described in Section 4. $\mathbb{S}\mathbb{A}\mathbb{T}_{\mathbb{1}}$ is proved to be a decision procedure for $L_{\mathbb{1}}$ in Section 5. In Section 6 we provide several examples of non-trivial operations definable as $L_{\mathbb{1}}$ formulas as well as the kind of automated reasoning $\mathbb{S}\mathbb{A}\mathbb{T}_{\mathbb{1}}$ is capable of. The implementation of $\mathbb{S}\mathbb{A}\mathbb{T}_{\mathbb{1}}$ as part of the $\{\log\}$ tool is discussed in Section 7, including an initial empirical evaluation in Section 7.3. In order to provide further empirical evidence that $\{\log\}$ can reason about intervals combined with sets we present a case study based on the elevator problem in Section 8. In Section 9 we put our work in the context of some results about interval reasoning. Section 10 presents our conclusions. Three appendices include technical details which are duly referenced throughout the paper.

2 MOTIVATION

Informally, $L_{\mathbb{1}}$ is a language including the classic operators of set theory plus the cardinality operator. Set operators are provided as constraints. For example, $\text{un}(A, B, C)$ is interpreted as $C = A \cup B$; $A \mid \parallel B$ is interpreted as $A \cap B = \emptyset$; and $\text{size}(A, j)$ as $|A| = j$. $L_{\mathbb{1}}$ includes so-called negative constraints implementing the negation of constraints. For example, $x \notin A$ corresponds to $\neg x \in A$, and $\text{nun}(A, B, C)$ corresponds to $\neg C = A \cup B$. In $L_{\mathbb{1}}$ sets are finite and are built from the empty set ($\emptyset$) and a set constructor, called extensional set, of the form $\{x \cup A\}$ whose interpretation is $\{x\} \cup A$. In $\{x \cup A\}$, both $x$ and $A$ can be variables. Set elements can be integer numbers, some other
ur-elements or sets.\textsuperscript{2} Besides, $L_{\lbrack 1 \rbrack}$ includes linear integer terms and constraints. Variables can denote integer numbers, ur-elements or sets. Formulas are conjunctions and disjunctions of constraints. $L_{\lbrack 1 \rbrack}$ is a decidable language whose solver is called $SAT_{\lbrack 1 \rbrack}$ which has been implemented as part of the $\{\log\}$ tool \cite{7}.

$L_{\lbrack 1 \rbrack}$ can express, for example, that a set is the disjoint union of two sets of equal cardinality:

\[
un(A, B, C) \land A \parallel B \land size(A, j) \land size(B, j)
\] (1)

However it is unclear how $L_{\lbrack 1 \rbrack}$ can express that $C$ is an integer interval. This is a limitation if $C$ is the collection of items numbered from $k$ to $m$ that are processed by agents $A$ and $B$ who should process an equal number of them. Clearly, the post-condition of such a system is:

\[
un(A, B, [k, m]) \land A \parallel B \land size(A, j) \land size(B, j)
\] (2)

where $[k, m]$ denotes the set $\{p \in \mathbb{Z} \mid k \leq p \leq m\}$, with $k$ and $m$ variables. Hence, $L_{\lbrack 1 \rbrack}$ and $SAT_{\lbrack 1 \rbrack}$ are not enough to describe such a system nor to automatically prove properties of it.

In this paper we show how $L_{\lbrack 1 \rbrack}$ and $SAT_{\lbrack 1 \rbrack}$ can be extended with set terms of the form $[k, m]$ where either of the limits can be integer linear terms including unbounded variables—making $[k, m]$ a finite set. The extensions are noted $L_{\lbrack 1 \rbrack}$ and $SAT_{\lbrack 1 \rbrack}$, and $SAT_{\lbrack 1 \rbrack}$ is proved to be a decision procedure for $L_{\lbrack 1 \rbrack}$ formulas. One of the two key ideas behind this extension is the application of the following identity true of any set $A$ and any non-empty integer interval\textsuperscript{3} $[k, m]$ (see the proof in Appendix B):

\[
k \leq m \Rightarrow (A = [k, m] \equiv A \subseteq [k, m] \land |A| = m - k + 1)
\] (3)

By means of this identity, (2) can be rewritten as follows:\textsuperscript{4}

\[
un(A, B, N) \land A \parallel B \land size(A, j) \land size(B, j) \land N \subseteq [k, m] \land size(N, m - k + 1)
\] (4)

where $N$ is a fresh variable. In this way (4) can be divided into a $L_{\lbrack 1 \rbrack}$ formula

\[
un(A, B, N) \land A \parallel B \land size(A, j) \land size(B, j) \land size(N, m - k + 1)
\] (5)

plus the constraint $N \subseteq [k, m]$.

Observe that in $un(A, B, [k, m])$ in (2), $A$ and $B$ can be variables and extensional sets built with the set constructor $\{ \cdot \cup \cdot \}$. In general, $L_{\lbrack 1 \rbrack}$ allows to express and reason about formulas where set variables, extensional sets and integer intervals can be freely combined. Integer intervals are a particular kind of set. Even operations such as $[k, m] \setminus [i, j]$, whose result is not necessarily an interval, are dealt with correctly. Furthermore, $L_{\lbrack 1 \rbrack}$ can also deal with formulas where intervals are set elements, e.g., $[k, m] \in \{(\{1, x, y, 3\}, \{-3, 2, z\}\}$, thanks to set unification \cite{17}. One of the keys for this result is to encode integer intervals in terms of the cardinality and integer constraints already provided by $L_{\lbrack 1 \rbrack}$. In this sense, $L_{\lbrack 1 \rbrack}$ takes a different direction than logics dealing only with intervals, e.g., \cite{18}.

In general, any $L_{\lbrack 1 \rbrack}$-formula $\Phi$ can be rewritten by means of (3) into a conjunction of the form $\Phi_{\subseteq \lbrack 1 \rbrack} \land \Phi_{\leq \lbrack 1 \rbrack}$ where $\Phi_{\lbrack 1 \rbrack}$ is a $L_{\lbrack 1 \rbrack}$ formula and $\Phi_{\subseteq \lbrack 1 \rbrack}$ is a conjunction of constraints of the form $X \subseteq [p, q]$ where $X$ is a variable and either $p$ or $q$ are variables. We will refer to such intervals as variable-intervals. In this way, $SAT_{\lbrack 1 \rbrack}$ relies on $SAT_{\lbrack 1 \rbrack}$ as follows. If $SAT_{\lbrack 1 \rbrack}$ finds $\Phi_{\lbrack 1 \rbrack}$ unsatisfiable then $\Phi$ is unsatisfiable. However, if $SAT_{\lbrack 1 \rbrack}$ finds $\Phi_{\leq \lbrack 1 \rbrack}$ satisfiable we still need to check if $\Phi_{\subseteq \lbrack 1 \rbrack}$

\textsuperscript{2}Ur-elements (also known as atoms or individuals) are objects which contain no elements but are distinct from the empty set.

\textsuperscript{3}Note that $[k, m]$ is not empty only when $k \leq m$.

\textsuperscript{4}The case $m < k$ is not considered as it adds nothing to the understanding of the problem.

ACM Transactions on Computational Logic, Vol. 25, No. 1, Article 3. Publication date: November 2023.
does not compromise the satisfiability of $\Phi_{[\cdot]}$. At this point the second key idea of our method comes into play. First, $\mathcal{SAT}_1$ asks $\mathcal{SAT}_1$ to compute a minimal solution of $\Phi_{[\cdot]}$—roughly, a solution of $\Phi_{[\cdot]}$ where sets have the minimum number of elements [7]. Second, if the computed minimal solution of $\Phi_{[\cdot]}$ is a solution of $\Phi$, then $\Phi$ is clearly satisfiable. Third, we have proved that if any minimal solution of $\Phi_{[\cdot]}$ is not a solution of $\Phi$, then $\Phi$ is unsatisfiable. That is, if any minimal solution of $\Phi_{[\cdot]}$ is not a solution of $\Phi$, then any larger solution will not be a solution of $\Phi$.

In Section 6, we further study the power of $\mathcal{L}_1$ and $\mathcal{SAT}_1$ by showing several problems they can describe and solve and in Section 8 we present a case study based on the elevator algorithm where we show how $\{\log\}$ can automatically discharge all the invariance lemmas.

3 \ $\mathcal{L}_1$: A LANGUAGE FOR SETS AND INTEGER INTERVALS

In this section we describe the syntax and semantics of our set-based language $\mathcal{L}_1$. $\mathcal{L}_1$ is an extension of $\mathcal{L}_{\cdot}$ [1]. Although $\mathcal{L}_{\cdot}$ has been thoroughly presented elsewhere [7] here we reproduce that presentation with the extensions to integer intervals. Hence, $\mathcal{L}_1$ is a multi-sorted first-order predicate language with three distinct sorts: the sort Set of all the terms which denote sets, the sort Int of terms denoting integer numbers, and the sort Ur of all the other terms. Terms of these sorts are allowed to enter in the formation of set terms (in this sense, the designated sets are hybrid), nesting restrictions being enforced (in particular, membership chains of any finite length can be modeled). A handful of reserved predicate symbols endowed with a pre-designated set-theoretic meaning is available. The usual linear integer arithmetic operators are available as well. Formulas are built in the usual way by using conjunction, disjunction and negation of atomic predicates. A few more complex operators (in the form of predicates) are defined as $\mathcal{L}_1$ formulas, thus making it simpler for the user to write complex formulas.

3.1 Syntax

The syntax of the language is defined primarily by giving the signature upon which terms and formulas are built.

**Definition 1 (Signature).** The signature $\Sigma_{[\cdot]}$ of $\mathcal{L}_{[\cdot]}$ is a triple $\langle \mathcal{F}, \Pi, \mathcal{V} \rangle$ where:

- $\mathcal{F}$ is the set of function symbols along with their sorts, partitioned as $\mathcal{F} \equiv \mathcal{F}_S \cup \mathcal{F}_Z \cup \mathcal{F}_U$, where $\mathcal{F}_S \equiv \{\emptyset, \cdot \cup \cdot, \cdot, [\cdot, \cdot]\}$, $\mathcal{F}_Z = \{0, -1, 1, -2, 2, \ldots\} \cup \{+, -, \ast\}$ and $\mathcal{F}_U$ is a set of uninterpreted constant and function symbols.
- $\Pi$ is the set of predicate symbols along with their sorts, partitioned as $\Pi \equiv \Pi_{\ast} \cup \Pi_S \cup \Pi_{\text{size}} \cup \Pi_Z$, where $\Pi_{\ast} \equiv \{=, \neq\}$, $\Pi_S \equiv \{\in, \notin, \text{un}, \|\}$, $\Pi_{\text{size}} \equiv \{\text{size}\}$, and $\Pi_Z \equiv \{\leq\}$.
- $\mathcal{V}$ is a denumerable set of variables partitioned as $\mathcal{V} \equiv \mathcal{V}_S \cup \mathcal{V}_Z \cup \mathcal{V}_U$.

Intuitively, $\emptyset$ represents the empty set; $\{x \cup A\}$ represents the set $\{x\} \cup A$; $[m, n]$ represents the set $\{p \in \mathbb{Z} \mid m \leq p \leq n\}$; and $\mathcal{V}_S$, $\mathcal{V}_Z$ and $\mathcal{V}_U$ represent sets of variables ranging over sets, integers and ur-elements, respectively.

Sorts of function and predicate symbols are specified as follows: if $f$ (resp., $\pi$) is a function (resp., a predicate) symbol of arity $n$, then its sort is an $n + 1$-tuple $\langle s_1, \ldots, s_{n+1} \rangle$ (resp., an $n$-tuple $\langle s_1, \ldots, s_n \rangle$) of non-empty subsets of the set of sorts $\{\text{Set}, \text{Int}, \text{Ur}\}$. This notion is denoted by $f : \langle s_1, \ldots, s_{n+1} \rangle$ (resp., by $\pi : \langle s_1, \ldots, s_n \rangle$). Specifically, the sorts of the elements of $\mathcal{F}$ and $\mathcal{V}$ are the following:

\[5\] In this context, a larger solution is a solution where at least one of the sets involved in the minimal solution has at least one more element w.r.t. the cardinality of the minimal solution.
Definition 2 (Sorts of Function Symbols and Variables). The sorts of the symbols in $\mathcal{F}$ are as follows:

- $\emptyset : \langle \{\text{Set}\} \rangle$
- $\cdot : \langle \{\text{Set}, \text{Int}, \text{Ur}\}, \{\text{Set}\} \rangle$
- $[\cdot : \cdot] : \langle \{\text{Int}, \{\text{Int}\} \rangle$
- $c : \langle \{\text{Int}\} \rangle$, for any $c \in \{0, 1, -1, 1, -2, 2, \ldots\}$
- $\cdot +, \cdot -, \cdot \cdot \cdot : \langle \{\text{Int}, \{\text{Int}\} \rangle$
- $f : \langle \{\text{Ur}, \ldots, \{\text{Ur}\}\} \rangle \in \langle \{\text{Ur}\} \rangle^{n+1}$, if $f \in \mathcal{F}_{\text{Ur}}$ is of arity $n \geq 0$.

The sorts of variables are as follows:

- $v : \langle \{\text{Set}\} \rangle$, if $v \in \mathcal{V}_{\text{S}}$
- $v : \langle \{\text{Int}\} \rangle$, if $v \in \mathcal{V}_{\text{Z}}$
- $v : \langle \{\text{Ur}\} \rangle$, if $v \in \mathcal{V}_{\text{U}}$

Definition 3 (Sorts of Predicate Symbols). The sorts of the predicate symbols in $\Pi$ are as follows (symbols un and size are prefix; all other symbols in $\Pi$ are infix):

- $=, \neq : \langle \{\text{Set}, \text{Int}, \text{Ur}\}, \{\text{Set}, \text{Int}, \text{Ur}\} \rangle$
- $\in, \notin : \langle \{\text{Set}, \text{Int}, \text{Ur}\}, \{\text{Set}\} \rangle$
- $\text{un} : \langle \{\text{Set}\}, \{\text{Set}\}, \{\text{Set}\} \rangle$
- $\| : \langle \{\text{Set}\}, \{\text{Set}\} \rangle$
- $\leq : \langle \{\text{Int}\}, \{\text{Int}\} \rangle$

Note that arguments of $=$ and $\neq$ can be of any of the three considered sorts. We do not have distinct symbols for different sorts, but the interpretation of $=$ and $\neq$ (see Section 3.2) depends on the sorts of their arguments.

The set of admissible (i.e., well-sorted) $L_{\{\}}$-terms is defined as follows:

Definition 4 ([]-terms). The set of $\{\}$-terms, denoted by $\mathcal{F}_{\{\}}$, is the minimal subset of the set of $\Sigma_{\{\}}$-terms generated by the following grammar complying with the sorts as given in Definition 2:

- $C ::= 0 | -1 | 1 | -2 | 2 | \ldots$
- $\mathcal{T}_{\text{Z}} ::= C | \mathcal{V}_{\text{Z}} | C \cdot \mathcal{V}_{\text{Z}} | \mathcal{V}_{\text{Z}} \cdot C | \mathcal{T}_{\text{Z}} + \mathcal{T}_{\text{Z}} | \mathcal{T}_{\text{Z}} - \mathcal{T}_{\text{Z}}$
- $\mathcal{T}_{\{\}} ::= \mathcal{T}_{\text{Z}} | \mathcal{T}_{\text{Ur}} | \mathcal{V}_{\text{U}} | \text{Set}$
- $\text{Set} ::= \emptyset | \mathcal{V}_{\text{S}} | \{ [\mathcal{T}_{\{\}} \cdot \cdot \cdot \mathcal{V}_{\text{Z}} \cdot \cdot \cdot \mathcal{T}_{\text{Z}} \cdot \cdot \cdot \mathcal{T}_{\text{Ur}} \cdot \cdot \cdot \} | [ \mathcal{T}_{\text{Z}} \cdot \cdot \cdot \mathcal{T}_{\text{Z}} \cdot \cdot \cdot ]$

where $\mathcal{T}_{\text{Z}}$ (resp., $\mathcal{T}_{\{\}}$) represents any non-variable $\mathcal{T}_{\text{Z}}$-term (resp., $\mathcal{T}_{\{\}}$-term).

As can be seen, the grammar allows only integer linear terms.

If $t$ is a term $f(t_1, \ldots, t_n), f \in \mathcal{F}$, $n \geq 0$, and $\langle s_1, \ldots, s_{n+1} \rangle$ is the sort of $f$, then we say that $t$ is of sort $\langle s_{n+1} \rangle$. The sort of any $\{\}$-term $t$ is always $\langle \{\text{Set}\} \rangle$ or $\langle \{\text{Int}\} \rangle$ or $\langle \{\text{Ur}\} \rangle$. For the sake of simplicity, we say that $t$ is of sort Set or Int or Ur, respectively. In particular, we say that a $\{\}$-term of sort Set is a set term, that set terms of the form $\{ \cdot \cdot \cdot \}$ are extensional set terms, and set terms of the form $\{ [\cdot \cdot \cdot \cdot \} are integer intervals or just intervals. The first parameter of an extensional set term is called element part and the second is called set part. Observe that one can write terms representing sets which are nested at any level. The parameters of intervals are called left and
right limits, respectively. It is important to remark that interval limits can be integer linear terms including variables.

Hereafter, we will use the following notation for extensional set terms: \( \{ t_1, t_2, \ldots, t_n \cup t \} \) is a shorthand for \( \{ t_1 \cup \{ t_2 \cup \cdots \{ t_n \cup t \} \cdots \} \} \), while \( \{ t_1, t_2, \ldots, t_n \} \) is a shorthand for \( \{ t_1, t_2, \ldots, t_n \cup \emptyset \} \).

Moreover, we will use the following name conventions: \( A, B, C, D \) for terms of sort \( \text{Set} \) excluding integer intervals; \( i, j, k, m, p, q \) for terms of sort \( \text{Int} \); \( a, b, c, d \) for terms of sort \( \text{Ur} \); and \( x, y, z \) for terms of any of the three sorts.

**Example 1 (Set Terms).** The following \( \Sigma \)-terms are set terms:

\[
\emptyset
\{ x \cup A \}
\{ 4 + k, f(a, b) \}, \text{ i.e., } \{ 4 + k \cup \{ f(a, b) \cup \emptyset \} \}, \text{ where } f \text{ is a (uninterpreted)}
\text{symbol in } \mathcal{F}_U
\[-3, 2 \ast m]
\]

On the opposite, \( \{ x \cup 17 \} \) is not a set term.

The sets of well-sorted \( \mathcal{L} \)-constraints and formulas are defined as follows:

**Definition 5 (\([\,]\)-constraints).** If \( \pi \in \Pi \) is a predicate symbol of sort \( \langle s_1, \ldots, s_n \rangle \), and for each \( i = 1, \ldots, n, t_i \) is a \( [\,] \)-term of sort \( \langle s'_i \rangle \) with \( s'_i \subseteq s_i \), then \( \pi(t_1, \ldots, t_n) \) is a \( [\,] \)-constraint. The set of \( \Sigma \)-constraints is denoted by \( C \).

**Example 2.** If \( k, m \in \mathcal{V}_2 \) and \( A \in \mathcal{V}_5 \), then \( [k, m] = 3 \cup A \) is a \( [\,] \)-constraint but \( [k, \emptyset] = 3 \cup A \) is not, as \( [k, \emptyset] \) is not a \( [\,] \)-term because \( \emptyset \) does not belong to \( \mathcal{F}_S \).

\( [\,] \)-constraints whose arguments are of sort \( \text{Set} \) (including size-constraints) will be called set constraints; \( [\,] \)-constraints whose arguments are of sort \( \text{Int} \) will be called integer constraints.

**Definition 6 (\([\,]\)-formulas).** The set of \( [\,] \)-formulas, denoted by \( \Phi \), is given by the following grammar:

\[
\Phi : = \text{true} \mid \text{false} \mid C \mid \Phi \land \Phi \mid \Phi \lor \Phi
\]

where \( C \) represents any element belonging to the set of \( \Sigma \)-constraints.

**Example 3 ([\,]-formulas).** The following are \( [\,] \)-formulas:

\[
a \in [k, i] \land a \notin B \land un([m, k + 1], B, C) \land C = \{ x \cup D \}
un(A, B, [k, j]) \land n + k > 5 \land size([k, j], n) \land B \neq \emptyset
\]

On the contrary, \( un(A, B, 23) \) is not a \( [\,] \)-formula because \( un(A, B, 23) \) is not a \( [\,] \)-constraint (23 is not of sort \( \text{Set} \) as required by the sort of \( un \)).

**Remark 1.** \( \mathcal{L}_\Sigma \) coincides with \( \mathcal{L}_\Sigma \) without integer interval terms. That is, if the function symbol \( [\,] \) is removed from \( \mathcal{F}_S \) (and consequently from the other definitions of the elements of the language), we get \( \mathcal{L}_\Sigma \).

### 3.2 Semantics

Sorts and symbols in \( \Sigma \) are interpreted according to the interpretation structure \( \mathcal{R} = \langle D, (\cdot)^\mathcal{R} \rangle \), where \( D \) and \( (\cdot)^\mathcal{R} \) are defined as follows:

**Definition 7 (Interpretation Domain).** The interpretation domain \( D \) is partitioned as \( D = D_{\text{Set}} \cup D_{\text{Int}} \cup D_{\text{Ur}} \) where:

ACM Transactions on Computational Logic, Vol. 25, No. 1, Article 3. Publication date: November 2023.
• $D_{\text{Set}}$ is the set of all hereditarily finite hybrid sets built from elements in $D$. Hereditarily finite sets are those sets that admit (hereditarily finite) sets as their elements, that is sets of sets.

• $D_{\text{int}}$ is the set of integer numbers, $\mathbb{Z}$.

• $D_{\text{Ur}}$ is a collection of other objects.

**Definition 8 (Interpretation Function).** The interpretation function $(\cdot)^R$ is defined as follows:

- Each sort $X \in \{\text{Set}, \text{Int}, \text{Ur}\}$ is mapped to the domain $D_X$.

- The constant and function symbols in $F_S$ are interpreted as follows:
  - $\emptyset$ is interpreted as the empty set
  - $\{x \cup A\}$ is interpreted as the set $\{x^R \cup A^R\}$
  - $[k, m]$ is interpreted as the set $\{p \in \mathbb{Z} \mid k^R \leq p \leq m^R\}$

- The constant and function symbols in $F_Z$ are interpreted as follows:
  - Each element in $\{0,-1,1,-2,2,\ldots\}$ is interpreted as the corresponding integer number
  - $i + j$ is interpreted as $i^R + j^R$
  - $i - j$ is interpreted as $i^R - j^R$
  - $i * j$ is interpreted as $i^R * j^R$

- The predicate symbols in $\Pi$ are interpreted as follows:
  - $x = y$, where $x$ and $y$ have the same sort $X, X \in \{\text{Set}, \text{Int}, \text{Ur}\}$, is interpreted as the identity between $x^R$ and $y^R$ in $D_X$; conversely, if $x$ and $y$ have different sorts, $x = y$ is interpreted in such a way as its truth value will be always false
  - $x \in A$ is interpreted as $x^R \in A^R$
  - $\text{un}(A, B, C)$ is interpreted as $C^R = A^R \cup B^R$
  - $A \parallel B$ is interpreted as $A^R \cap B^R = \emptyset$
  - $\text{size}(A, k)$ is interpreted as $|A^R| = k^R$
  - $i \leq j$ is interpreted as $i^R \leq j^R$
  - $x \neq y$ and $x \notin A$ are interpreted as $\neg x = y$ and $\neg x \in A$, respectively.

Note that integer intervals in $L_{[\cdot]}$ denote always finite sets given that their limits can assume only integer values—in other words, integer limits cannot be $\pm \infty$.

The interpretation structure $\mathcal{R}$ is used to evaluate each $[\cdot]$-formula $\Phi$ into a truth value $\Phi^R = \{\text{true, false}\}$ in the following way: set constraints (resp., integer constraints) are evaluated by $(\cdot)^R$ according to the meaning of the corresponding predicates in set theory (resp., in number theory) as defined above; $[\cdot]$-formulas are evaluated by $(\cdot)^R$ according to the rules of propositional logic. In particular, observe that equality between two set terms is interpreted as the equality in $D_{\text{Set}}$; that is, as set equality between hereditarily finite hybrid sets. Such equality is regulated by the standard extensionality axiom, which has been proved to be equivalent, for hereditarily finite sets, to the following equational axioms [10]:

\[
\begin{align*}
\{x, x \cup A\} &= \{x \cup A\} & (Ab) \\
\{x, y \cup A\} &= \{y, x \cup A\} & (Cl)
\end{align*}
\]

Axiom (Ab) states that duplicates in a set term do not matter (Absorption property). Axiom (Cl) states that the order of elements in a set term is irrelevant (Commutativity on the left). These two properties capture the intuitive idea that, for instance, the set terms $\{1, 2\}, \{2, 1\}$, and $\{1, 2, 1\}$ all denote the same set.

A valuation $\sigma$ of a formula $\Phi$ is an assignment of values from $D$ to the free variables of $\Phi$ which respects the sorts of the variables. $\sigma$ can be extended to terms in a straightforward manner. In the case of formulas, we write $\Phi[\sigma]$ to denote the application of a valuation to a formula $\Phi$. $\sigma$ is a successful valuation (or, simply, a solution) if $\Phi[\sigma]$ is true in $\mathcal{R}$.
3.3 Derived Constraints

$L_{[1]}$ can be extended to support other set and integer operators definable by means of suitable $L_{[1]}$ formulas.

Dovier et al. [10] proved that the collection of predicate symbols in $Π ∧ Π_S$ is sufficient to define constraints implementing the set operators $∩$, $⊆$ and $\setminus$. For example, $A ⊆ B$ can be defined by the $L_{[1]}$ formula $un(A, B, B)$. In a similar fashion, $\{=,\neq\} ∪ Π_S$ is sufficient to define $<$, $>$ and $≥$. With a little abuse of terminology, we say that the set and integer predicates that are given as $[\cdot]$-formulas are derived constraints. In Section 6, we introduce more complex derived constraints that can be written only when integer intervals are available.

Whenever a formula contains a derived constraint, the constraint is replaced by its definition turning the given formula into a $L_{[1]}$ formula. Precisely, if formula $φ$ is the definition of constraint $c$, then $c$ is replaced by $φ$ and the solver checks satisfiability of $φ$ to determine satisfiability of $c$. Thus, we can completely ignore the presence of derived constraints in the subsequent discussion about constraint solving and formal properties of our solver.

The negated versions of set and integer operators can be introduced as derived constraints, as well. The derived constraint for $\neg ∪$ and $\neg ∥$ (called $nun$ and $\|$, respectively) are shown in [10]. For example, $\neg (A ∪ B = C)$ is introduced as:

$$nun(A, B, C) ≡ (n ∈ C ∧ n ∉ A ∧ n ∉ B) ∨ (n ∈ A ∧ n ∉ C) ∨ (n ∈ B ∧ n ∉ C)$$

(6)

With a little abuse of terminology, we will refer to these predicates as negative constraints.

Thanks to the availability of negative constraints, (general) logical negation is not strictly necessary in $L_{[1]}$.

Now that we have derived and negative constraints it is easy to see that $L_{[1]}$ expresses the Boolean algebra of sets with cardinality extended with integer intervals. That is, one can write Boolean formulas where arguments are extensional sets and integer intervals.

Remark 2. From now on, we will use $A ⊆ B$ as a synonym of the $L_{[1]}$ constraint $un(A, B, B)$. In particular we will write $X ⊆ [k, m]$ in place of $un(X, [k, m], [k, m])$.

Remark 3 (A Stack of Constraint Languages and Solvers). As we have said, $L_{[1]}$ and $SAT_{[1]}$ are built on top of $L_{[\cdot]}$ and $SAT_{[\cdot]}$. In turn, $L_{[\cdot]}$ and $SAT_{[\cdot]}$ are built on top of CLP(SET) [10]. CLP(SET) is based on a constraint language including $Π_S$ and $Π_S$; formulas in CLP(SET) are built as in $L_{[\cdot]}$. One of the main concepts behind CLP(SET) is set unification [17]. $L_{[\cdot]}$ effectively extends CLP(SET) by introducing size and integer constraints; and $L_{[1]}$ effectively extends $L_{[\cdot]}$ by admitting integer intervals. Set unification goes all the way up to $SAT_{[\cdot]}$; it is also pervasive in other CLP schemas developed by the authors [11, 12]. $\{\log\}$ implements $L_{[\cdot]}$ and $SAT_{[\cdot]}$, while the extension of $\{\log\}$ to implement $L_{[1]}$ is presented in this paper.

4 $SAT_{[1]}$: A CONSTRAINT SOLVING PROCEDURE FOR $L_{[1]}

In this section, we show how $SAT_{[\cdot]}$ can be further extended to support set constraints whose arguments can be integer intervals. The resulting constraint solving procedure, i.e., $SAT_{[\cdot]}$, is a decision procedure for $L_{[1]}$ formulas. Furthermore, it produces a finite representation of all the possible solutions of any satisfiable $L_{[1]}$ formula (see Section 5).

4.1 The Solver

The overall organization of $SAT_{[\cdot]}$ is shown in Algorithm 1. Basically, $SAT_{[\cdot]}$ uses four routines: gen_size_leq, STEPs[1] (called from step_loop⁶), remove_neq and solve_size.

⁶As step_loop merely loops calling STEPs[1], we will talk about the latter rather than the former. STEPs[1] is the key procedure in Algorithm 1.

ACM Transactions on Computational Logic, Vol. 25, No. 1, Article 3. Publication date: November 2023.
ALGORITHM 1: The solver \( \mathcal{SAT} \). \( \Phi \) is the input formula.

\[
\begin{align*}
\Phi & \leftarrow \text{gen\_size\_leq}(\Phi); \\
\text{repeat} & \quad \Phi' \leftarrow \Phi; \\
\text{until} & \quad \Phi = \Phi'; \quad \text{[end of main loop]} \\
\text{let} & \quad \Phi \text{ be } \Phi_{\mid \cdot} \land \Phi_{\subseteq [\cdot];} \\
\text{let} & \quad \Phi_{\mid \cdot} \text{ be } \Phi_1 \land \Phi_2; \\
\text{if} & \quad \Phi_{\subseteq [\cdot]} \neq \text{true} \quad \text{then} \\
\text{return} & \quad \text{step\_loop}(\text{solve\_size\_minsol}(\Phi_1) \land \Phi_2 \land \Phi_{\subseteq [\cdot]}); \\
\text{else} & \quad \text{return} \quad \text{solve\_size}(\Phi_1) \land \Phi_2 \\
\text{end if} \\
\text{procedure} & \quad \text{step\_loop}(\Phi) \\
\text{repeat} & \quad \Phi' \leftarrow \Phi; \\
\text{until} & \quad \Phi = \Phi'; \\
\text{return} & \quad \Phi \\
\text{end procedure}
\end{align*}
\]

\( \text{gen\_size\_leq} \) simply adds integer constraints to the input formula \( \Phi \) to force the second argument of \( \text{size}\)-constraints in \( \Phi \) to be non-negative integers. \( \text{STEP}_{\mid \cdot} \) includes the constraint solving procedures for the \( \mathcal{L}_{\mid \cdot} \) fragment as well as the new constraint solving procedures for set constraints whose arguments are intervals. \( \text{STEP}_{\mid \cdot} \) applies specialized rewriting procedures to the current formula \( \Phi \) and returns either \( \text{false} \) or the modified formula. Each rewriting procedure applies a few non-deterministic rewrite rules which reduce the syntactic complexity of \( [\cdot] \)-constraints of one kind. \( \text{remove\_neq} \) deals with the elimination of \( \neq \)-constraints involving set variables. Its motivation and definition will be made evident later in Section 4.9. \( \text{solve\_size} \) is the adaptation of the decision procedure proposed by C. Zarba for cardinality constraints \([4]\) to our CLP framework. Both \( \text{STEP}_{\mid \cdot} \) and \( \text{solve\_size} \) use the SWI-Prolog CLP(Q) library to solve linear integer arithmetic problems \([19]\). These problems may be part of \( \Phi \) or they are generated during set processing. CLP(Q) provides the library predicate \( \text{bb\_inf} \) to implement a decision procedure for linear integer arithmetic. Besides, \( \text{solve\_size} \) uses a SAT solver implemented in Prolog by Howe and King \([20]\) to help in the implementation of Zarba’s algorithm. \( \text{solve\_size\_minsol}(\Phi_1) \) is the call to \( \text{solve\_size} \) in \( \text{minimal solution mode} \), i.e., asking \( \text{solve\_size} \) to compute a minimal solution of formula \( \Phi_1 \)—which in turn is implemented with a suitable call to \( \text{bb\_inf} \). See \([7]\) for further details. We will formally analyze the use of \( \text{solve\_size\_minsol} \) within \( \mathcal{SAT}_{\mid \cdot} \) in the proof of Theorem 3.

The execution of \( \text{STEP}_{\mid \cdot} \) and \( \text{remove\_neq} \) is iterated until a fixpoint is reached, i.e., the formula is irreducible. These routines return \( \text{false} \) whenever (at least) one of the procedures in it rewrites \( \Phi \) to \( \text{false} \). In this case, a fixpoint is immediately detected.

As we will show in Section 5, when all the non-deterministic computations of \( \mathcal{SAT}_{\mid \cdot}(\Phi) \) return \( \text{false} \), then we can conclude that \( \Phi \) is unsatisfiable; otherwise, when at least one of them does not return \( \text{false} \), then we can conclude that \( \Phi \) is satisfiable and each solution of the formulas returned by \( \mathcal{SAT}_{\mid \cdot} \) is a solution of \( \Phi \), and vice versa.

---

\( \text{bb\_inf}(\text{Vars}, \text{Expr}, \text{Min}, \text{Vert}) \) finds the vertex (\( \text{Vert} \)) of the minimum (\( \text{Min} \)) of the expression (\( \text{Expr} \)) subjected to the integer constraints present in the constraint store and assuming all the variables in \( \text{Vars} \) take integer values.

---

ACM Transactions on Computational Logic, Vol. 25, No. 1, Article 3. Publication date: November 2023.
\( (x \cup A) = (y \cup B) \quad \rightarrow \)
\[
(x = y \land A = B) \lor (x = y \land (x \cup A) = B) \lor (x = y \land A = (y \cup B)) \lor (A = (y \cup N) \land (x \cup N) = B)
\]

\(un((x \cup C), A, B) \quad \rightarrow \)
\[
\{x \in C\} = \{x \cup N_1\} \land x \not\in N_1 \land B = \{x \cup N\}
\]
\[
\land (x \not\in A \land un(N_1, A, N) \lor A = \{x \cup N_2\} \land x \not\in N_2 \land un(N_1, N_2, N))
\]
\[
size(\{x \cup A\}, m) \quad \rightarrow \quad (x \not\in A \land m = 1 + n \land size(A, n) \land 0 \leq n) \lor (A = \{x \cup N\} \land x \not\in N \land size(N, m))
\]

Fig. 1. Some key rewrite rules inherited from CLP(SET) and \(S\mathcal{AT}_{\mid\mid}\).

Apart from the new rewrite rules, \(S\mathcal{AT}_{\mid}\) works exactly as \(S\mathcal{AT}_{\mid\mid}\) up until the end of the main loop. After the main loop, \(S\mathcal{AT}_{\mid}\) differs from \(S\mathcal{AT}_{\mid\mid}\) in: (a) dividing \(\Phi\) into \(\Phi_{\mid}\) and \(\Phi_{\parallel}\); and (b) adding the then branch. \(\Phi_{\mid}\) is a \(\mathcal{L}_{\mid}\) formula; and \(\Phi_{\parallel}\) is a conjunction of constraints of the form \(X \subseteq [p, q]\) where \([p, q]\) is a variable-interval. In turn, \(\Phi_{\mid}\) is divided into \(\Phi_1\) and \(\Phi_2\) as in \(S\mathcal{AT}_{\mid\mid}\): \(\Phi_1\) contains all the integer constraints and all the \(un, \parallel\) and size constraints, and \(\Phi_2\) is the rest of \(\Phi_{\mid}\) (i.e., \(!\equiv\) constraints, and \(=\) and \(!\) constraints not involving integer terms). If \(\Phi_{\parallel}\) is true, \(S\mathcal{AT}_{\mid}\) executes the else branch which corresponds to the implementation of \(S\mathcal{AT}_{\mid\mid}\). This means that when in \(\Phi\) there are no integer intervals, \(S\mathcal{AT}_{\mid}\) reduces to \(S\mathcal{AT}_{\mid\mid}\). The then branch is exclusive of \(S\mathcal{AT}_{\mid}\) and is entered only if at this point a constraint of the form \(X \subseteq [p, q]\) is still present in \(\Phi\). However, \(\Phi_{\parallel}\) is not passed in to solve\_size because Zarba’s algorithm cannot deal with integer intervals.

**4.2 Rewrite Rules of CLP(SET) and \(S\mathcal{AT}_{\mid}\)**

The rewrite rules used by \(S\mathcal{AT}_{\mid}\) are defined as follows.

**Definition 9 (Rewrite Rules).** If \(\pi\) is a symbol in \(\Pi\) and \(\phi\) is a \(\mathcal{I}\)-constraint based on \(\pi\), then a rewrite rule for \(\pi\)-constraints is a rule of the form \(\xi \rightarrow \Phi_1 \lor \cdots \lor \Phi_n\), where \(\Phi_i, 1 \leq i \leq n\), are \(\mathcal{I}\)-formulas. Each \(\Sigma_{\mathcal{I}}\)-predicate matching \(\phi\) is non-deterministically rewritten to one of the \(\Phi_i\).

Variables appearing in the right-hand side but not in the left-hand side are assumed to be fresh variables, implicitly existentially quantified over each \(\Phi_i\).

A rewriting procedure for \(\pi\)-constraints consists of the collection of all the rewrite rules for \(\pi\)-constraints. The first rule whose left-hand side matches the input \(\pi\)-constraint is used to rewrite it. Constraints that are rewritten by no rule are called irreducible. Irreducible constraints are part of the final answer of \(\text{STEP}_S\mathcal{I}\) (see Definition 10).

The following conventions are used throughout the rules. \(\dot{x}\), for any name \(x\), is a shorthand for \(x \in \mathcal{V}\), i.e., \(\dot{x}\) represents a variable. In particular, variable names \(n, \dot{n}, N, \dot{N}\) denote fresh variables of sort \(\mathbb{N}\) and \(\mathbb{S}\), respectively.

Before introducing the new rewrite rules added to \(S\mathcal{AT}_{\mid}\) we show some of the key rewrite rules inherited from CLP(SET) and \(S\mathcal{AT}_{\mid\mid}\) (see Figure 1). Without the new rewrite rules, \(S\mathcal{AT}_{\mid}\) can deal with \(\mathcal{L}_{\mathcal{I}}\) formulas as long as integer intervals are not present.

Rule (7) is the main rule of set unification. It states when two non-empty, non-variable sets are equal by non-deterministically and recursively computing four cases. These cases implement the (\(Ab\)) and (\(Cf\)) axioms shown in Section 3.2. As an example, by applying rule (7) to \([1] = \{1, 1\}\) we get: \([1 = 1 \land \emptyset = \{1\}] \lor (1 = 1 \land \{1\} = \{1\}) \lor (1 = 1 \land \emptyset = \{1, 1\}) \lor (\emptyset = \{1 \cup N\} \land \{1 \cup \dot{N}\} = \{1\})\), which turns out to be true (due to the second disjunct).

In turn, rule (8) is one of the main rules for \(un\)-constraints. It deals with \(un\)-constraints where the first argument is an extensional set and the last one a variable. Observe that this rule is based
Fig. 2. Rewrite rules for =-constraints involving intervals.

\[
[k, m] = \emptyset \rightarrow m < k \tag{10}
\]

\[
[k, m] = \{y \cup B\} \rightarrow \{y \cup B\} \subseteq [k, m] \land \text{size}(\{y \cup B\}, m - k + 1) \tag{11}
\]

\[
[k, m] = [i, j] \rightarrow (k \leq m \land i \leq j \land k = i \land m = j) \lor (m < k \land j < i) \tag{12}
\]

\[
[k, m] \neq \emptyset \rightarrow k \leq m \tag{13}
\]

\[
[k, m] \neq \{y \cup B\} \rightarrow (\hat{n} \in [k, m] \land \hat{n} \not\in \{y \cup B\}) \lor (\hat{n} \not\in [k, m] \land \hat{n} \in \{y \cup B\}) \tag{14}
\]

\[
[k, m] \neq [i, j] \rightarrow (k \leq m \land (m \neq j \lor j < i \lor k \neq i)) \lor (i \leq j \land (m \neq j \lor m < k \lor k \neq i)) \tag{15}
\]

on set unification (i.e., on rule (7)). It computes two cases: \(x\) does not belong to \(A\), and \(x\) belongs to \(A\) (in which case \(A\) is of the form \(\{x \cup \hat{N}_2\}\) for some set \(\hat{N}_2\)). In the latter, \(x \not\in \hat{N}_2\) prevents Algorithm 1 from generating infinite terms denoting the same set. The rest of the rewrite rules of CLP(SET) can be found in [10] and online [21].

One of the rewrite rules concerning size-constraints implemented in \(\mathcal{SAT}_1\) is rule (9). It computes the size of any extensional set by counting the elements that belong to it while taking care of avoiding duplicates. This means that, for instance, the first non-deterministic choice for a formula such as size(\(\{1, 2, 3, 1, 4\}, m\)) will be:

\[1 \not\in \{2, 3, 1, 4\} \land m = 1 + \hat{n} \land \text{size}(\{2, 3, 1, 4\}, \hat{n}) \land 0 \leq \hat{n}\]

which will eventually lead to a failure due to the presence of \(1 \not\in \{2, 3, 1, 4\}\). This implies that 1 will be counted in its second occurrence. Besides, the second choice becomes size(\(\{2, 3, 1, 4\}, m\)) which is correct given that \(|\{1, 2, 3, 1, 4\}| = |\{2, 3, 1, 4\}|\).

Part of the work of extending \(\mathcal{SAT}_1\) to integer intervals is to make rewrite rules such as those shown in Figure 1 to correctly deal with the same constraints but when at least one of their arguments is an integer interval. This is shown in Sections 4.3–4.7; and in Section 4.8 we briefly discuss the new rewrite rules.

### 4.3 Rules for =-constraints

The main rewrite rules for =-constraints are listed in Figure 2. Rule (10) is straightforward. Rule (11) is based on the identity (3). Hence, the rule decides the satisfiability of \([k, m] = \{y \cup B\}\) by deciding the satisfiability of \(\{y \cup B\} \subseteq [k, m] \land \text{size}(\{y \cup B\}, m - k + 1)\). This might seem odd but, due to (3), we know that \(\{y \cup B\}\) is actually the interval \([k, m]\). Observe that (3) is correctly applied as \(k \leq m\) is implicit because \(\{y \cup B\}\) is a non-empty set. In fact, if \(m < k\) then \(\{y \cup B\} \subseteq [k, m]\) will fail (see Section 4.6). We will discuss the intuition behind identity (3) in Section 4.8.

Rule (12) takes care of whether the intervals denote the empty set or not; if not, their corresponding limits must be equal. Rules (13) and (15) are the negations of rules (10) and (12), respectively—rule (15) includes some Boolean simplification. Instead of negating rule (11), rule (14) uses set extensionality to find out whether or not both sets are different; this is so for efficiency reasons.

Rules for =-constraints where the interval is at the right-hand side have not been included in the figure since they can be trivially obtained from those shown in Figure 2.

Besides, note that an equality of the form \(\hat{A} = [k, m]\) is not rewritten as a solution for variable \(A\) has been found.
4.4 Rules for $\in$-constraints

The rewrite rules for $\in$-constraints are listed in Figure 3. Note that all $\in$-constraints involving integer intervals are rewritten into integer constraints.

4.5 Rules for $\parallel$-constraints

The main rewrite rules for $\parallel$-constraints are listed in Figure 4. Rule (19) uses the identity (3); note that in this case a new variable is introduced. Then, this rule decides the satisfiability of $\hat{N} \parallel A$, instead of $[k, m] \parallel A$, because we know that $\hat{N} = [k, m]$.

Rule (20) considers all the possible cases when two intervals can be disjoint. For example, the first case corresponds to the left-hand interval being the empty set; while the third considers the two cases when both are non-empty intervals but they are one after the other in the $\mathbb{Z}$ line.

Rule (21) uses a general criterion to decide the satisfiability of $\not\parallel$-constraints by asking for a new variable ($\hat{n}$) to be an element of both sets. In this way, the $\not\parallel$-constraint is rewritten into two $\in$-constraints where the first one is dealt with by the first rule of Figure 3 and the second one by rules introduced elsewhere [10].

Rules for $\parallel$-constraints where the interval is at the right-hand side have not been included in the figure.

4.6 Rules for un-constraints

The main rewrite rules for un-constraints are listed in Figure 5 and more rules can be found in Appendix A. In these figures, $A$ and $B$ represent either variables or extensional set terms; $C$ represents any set term (including intervals).

Rules (22)–(24) are special cases as they, actually, implement the subset relation (recall that we are using $A \subseteq B$ just as a synonym of the $L_{[1]}$ constraint $un(A, B, B)$). The first of these rules is trivial. The second one states that a constraint of the form $\hat{A} \subseteq [k, m]$ is left unchanged. This is so because these constraints are always satisfiable by substituting $\hat{A}$ by the empty set. The importance of this property will be evident in Section 5. The third rule walks over all the elements of an extensional set until the empty set or a variable is found. In each step two integer constraints are generated. Note that when the recursion arrives at the end of the set, rules (22) or (23) are applied and a fixpoint is reached.
Fig. 5. Rewrite rules for \textit{un}-constraints involving intervals.

\[ \begin{align*}
\text{Rule (25)} & \text{ is based on the identity (3). This rule is crucial as it permits to reconstruct an integer interval from two sets. For instance, this rule covers constraints such as } \text{un}(\{x \cup \hat{A}, B, \hat{[k, m]}\}). \text{ Note that the interval disappears from the } \text{un}-\text{constraint. Rule (26) is based on (3) as well. In this case we apply the identity twice to transform the intervals in the } \text{un}-\text{constraint into extensional sets. Although in rule (27) it would be possible to use (3), it is more efficient to rely on the fact that the union of two integer intervals is equal to an interval if some linear integer arithmetic conditions hold. As far as we understand, the same approach cannot be used in the other rules. A particular case that is true only in } \mathbb{Z} \text{ is considered in the rows labeled } \text{in } \mathbb{Z}. \text{ Indeed, for instance, } i \leq m + 1 \text{ takes care of a case such as } \text{un}(\{2, m\}, \{m + 1, 10\}, \{2, 10\}) \text{ which does not hold outside of } \mathbb{Z} \text{ as there are infinite numbers between } m \text{ and } m + 1. \text{ Rules for } \text{un}-\text{constraints where } A \text{ or } B \text{ are the empty set have not been included in Figure 5 nor in Appendix A.}
\end{align*} \]

\subsection*{4.7 Rule for size-constraints}

The rewrite rules for \textit{size}-constraints are listed in Figure 6. Note that all \textit{size}-constraints involving integer intervals are rewritten into integer constraints.

\textit{Remark 4.} The rewrite rules shown in Sections 4.3–4.7 can be used both for variable-intervals and for non-variable-intervals. However, at implementation level, it is convenient to rewrite some non-variable-intervals into extensional sets as soon as possible.
4.8 Discussion
As can be seen by inspecting the rewrite rules, after step_loop terminates, integer intervals remain only in constraints of the form $X \subseteq [p, q]$ where $X$ is a variable and $[p, q]$ is a variable-interval (i.e., either $p$ or $q$ are variables). Besides, these constraints remain irreducible (i.e., there are no rewrite rules dealing with them). This makes the formula returned by the main loop of Algorithm 1 very similar to formulas returned at the same point by $\mathcal{SAT} \downarrow$ [7]. Precisely, the only difference is the presence of constraints of the form $X \subseteq [p, q]$.

Note that the new rewrite rules added to $\text{STEP}_3$ rewrite set constraints into integer constraints, whenever possible. We do so because, in general, linear integer arithmetic formulas can be solved much more efficiently than set formulas. When that is impossible we use (3) to somewhat trick the solver making it to solve what is, essentially, a $\mathcal{L} \downarrow$ problem.

Intuitively, (3) forces an interval to become an extensional set. Let us see this by applying rule (25) to an example.

Example 4. Consider the following constraint:

$$\text{un}(\{3, x, 1\}, \{y, 5\}, [k, m])$$

which is rewritten by rule (25) into:

$$N \subseteq [k, m] \land \text{size}(N, m - k + 1) \land \text{un}(\{3, x, 1\}, \{y, 5\}, N)$$

where $N$ is intended to be equal to $[k, m]$. At this point rule (8) rewrites the un-constraint yielding:

$$N = \{3, x, 1, y, 5\}.$$ See that $N$ is now an extensional set instead of an interval. Then, $N$ is substituted by $\{3, x, 1, y, 5\}$ in the rest of the formula:

$$\{3, x, 1, y, 5\} \subseteq [k, m] \land \text{size}(\{3, x, 1, y, 5\}, m - k + 1)$$

Now, rule (24) is applied several times yielding:

$$k \leq 3 \leq m \land k \leq x \leq m \land k \leq 1 \leq m \land k \leq y \leq m \land k \leq 5 \leq m \land \text{size}(\{3, x, 1, y, 5\}, m - k + 1)$$

Rule (9) is applied to the size-constraint opening several non-deterministic choices as $\{\{3, x, 1, y, 5\}\} \in \{3, 4, 5\}$ depending on the values of $x$ and $y$. In this case, all these choices are encoded as integer problems. For instance, when rule (9) considers the alternative where $\{\{3, x, 1, y, 5\}\} = 5$ the formula to solve becomes:

$$k \leq 3 \leq m \land k \leq x \leq m \land k \leq 1 \leq m \land k \leq y \leq m \land k \leq 5 \leq m \land x \neq y \land x \neq 1 \land x \neq 3 \land x \neq 5 \land y \neq 1 \land y \neq 3 \land y \neq 5 \land m - k + 1 = 5$$

Then, $k = 1$, $m = 5$ and $x$ can be 2 and $y$ can be 4 or vice versa.

However, when rule (9) takes $y = 5 \land x \notin \{3, 1, 5\}$, then $\{3, x, 1, y, 5\}$ becomes $\{3, x, 1, 5\}$, which cannot be an interval regardless of the value of $x$ as there are two holes in it (i.e., 2 and 4). In this case $\mathcal{SAT} \uparrow$ returns false.

4.9 Inequality Elimination (remove_neq)
The $\uparrow$-formula returned by Algorithm 1 when $\text{STEP}_3$ reaches a fixpoint is not necessarily satisfiable.

Example 5 (Unsatisfiable Formula Returned by $\text{STEP}_3$). The $\uparrow$-formula:

$$\text{un}(A, B, C) \land \text{un}(A, B, D) \land C \neq D$$

cannot be further rewritten by any of the rewrite rules of $\text{STEP}_3$. Nevertheless, it is clearly unsatisfiable.
In order to guarantee that $\text{SAT}_{\text{[\_\_\_]}}$ returns either $false$ or satisfiable formulas (see Theorem 3), we still need to remove all inequalities of the form $\dot{A} \neq t$, where $\dot{A}$ is of sort $\text{Set}$, occurring as an argument of $\text{[\_]$\pi$}$-constraints based on $\text{un}$ or $\text{size}$. This is performed (see Algorithm 1) by executing the routine $\text{remove}_\text{neq}$, which applies the rewrite rule described by the generic rule scheme of Figure 7. Basically, this rule exploits set extensionality to state that non-equal sets can be distinguished by asserting that a fresh element ($\dot{n}$) belongs to one but not to the other. Notice that the rule (\#$_{\text{Un}_l}$) is necessary when $u$ is a non-set term. In this case by just using rule (\#$_{\text{Set}}$) we would miss the solution $\dot{A} = \emptyset$.

Example 6 (Elimination of $\neq$-constraints). The $\text{[\_\_\_]$\pi$}$-formula of Example 5 is rewritten by rule (\#$_{\text{Set}}$) to:

$$
\text{un}(A, B, C) \land \text{un}(A, B, D) \land C \neq D 
\longrightarrow
\left( \text{un}(A, B, C) \land \text{un}(A, B, D) \land \dot{n} \in C \land \dot{n} \notin D \right) \lor 
\left( \text{un}(A, B, C) \land \text{un}(A, B, D) \land \dot{n} \notin C \land \dot{n} \in D \right)
$$

Then, the $\varepsilon$-constraint in the first disjunct is rewritten into a $\equiv$-constraint (namely, $C = \{\dot{n} \cup \dot{N}\}$), which in turn is substituted into the first $\text{un}$-constraint. This constraint is further rewritten by rules such as those shown in Figure 1 and [10], binding either $A$ or $B$ (or both) to a set containing $\dot{n}$, which in turn forces $D$ to contain $\dot{n}$. This process will eventually return $false$, at which point the second disjunct is processed in a similar way.

### 4.10 Irreducible Constraints

When no rewrite rule applies to the current $\text{[\_\_\_]$\pi$}$-formula $\Phi$ and $\Phi$ is not $false$, the main loop of $\text{SAT}_{\text{[\_\_\_]}}$ terminates returning $\Phi$ as its result. This formula can be seen, without loss of generality, as $\Phi_S \land \Phi_Z$, where $\Phi_Z$ contains all (and only) integer constraints and $\Phi_S$ contains all other constraints occurring in $\Phi$.

The following definition precisely characterizes the form of atomic constraints in $\Phi_S$.

**Definition 10 (Irreducible Formula).** Let $\Phi$ be a $\text{[\_\_\_]$\pi$}$-formula, $A$ and $A_i$ $\text{[\_]$\pi$}$-terms of sort $\text{Set}$, $t$ and $X$ $\text{[\_]$\pi$}$-terms of sort $\langle\text{Set, Ur}\rangle$, $x$ a $\text{[\_]$\pi$}$-term of any sort, $c$ a variable or a constant integer number, and $k$ and $m$ are terms of sort $\text{Int}$. A $\text{[\_]$\pi$}$-constraint $\phi$ occurring in $\Phi$ is irreducible if it has one of the following forms:

(i) $X = t$, and neither $t$ nor $\Phi \setminus \{\phi\}$ contains $X$;
(ii) $X \neq t$, and $X$ does not occur either in $t$ or as an argument of any constraint $\pi(\ldots)$, $\pi \in \{\text{un, size}\}$, in $\Phi$;
(iii) $x \notin \dot{A}$, and $\dot{A}$ does not occur in $x$;
(iv) $\text{un}(\dot{A}_1, \dot{A}_2, \dot{A}_3)$, where $\dot{A}_1$ and $\dot{A}_2$ are distinct variables;
(v) $\dot{A}_1 \parallel \dot{A}_2$, where $\dot{A}_1$ and $\dot{A}_2$ are distinct variables;
(vi) $\text{size}(\dot{A}, c), c \neq 0$;
(vii) $\dot{A} \subseteq [k, m]$, where $k$ or $m$ are variables.
A [ ]-formula $\Phi$ is irreducible if it is true or if all its [ ]-constraints are irreducible.

$\Phi_S$, as returned by $\mathcal{SAT}_[\cdot]$ once it finishes its main loop, is an irreducible formula. This fact can be checked by inspecting the rewrite rules presented in [7, 10] and those given in this section. This inspection is straightforward as there are no rewrite rules dealing with irreducible constraints and all non-irreducible form constraints are dealt with by some rule.

Putting constraints of the form $X \subseteq [k, m]$ aside, $\Phi_S$ is basically the formula returned by $\mathcal{SAT}_[\cdot]$. Cristiá and Rossi [7, Theorem 2] show that such a formula is always satisfiable, unless the result is false.

It is important to observe that the atomic constraints occurring in $\Phi_S$ are indeed quite simple. In particular:

(a) all extensional set terms occurring in the input formula have been removed, except those occurring as right-hand sides of $=$ and $\neq$ constraints; and
(b) all integer interval terms occurring in the input formula have been removed, except those occurring at the right-hand side of $\subseteq$-constraints. Thus, all (possibly complex) equalities and inequalities between set terms have been solved. Furthermore, all arguments of $\neq$ and $\parallel$ constraints are necessarily simple variables or variable-intervals—and only in constraints of the form $X \subseteq [k, m]$.

5 $\mathcal{SAT}_[\cdot]$ IS A DECISION PROCEDURE FOR $L_\mathcal{I}$

In this section we analyze the soundness, completeness and termination properties of $\mathcal{SAT}_[\cdot]$. The complete proofs of some theorems can be found in Appendix B.

The following theorem ensures that each rewriting rule used by $\mathcal{SAT}_[\cdot]$ preserves the set of solutions of the input formula.

**Theorem 1 (Equisatisfiability).** Let $\phi$ be a [ ]-constraint based on symbol $\pi \in \Pi \setminus \Pi_\mathcal{I}$, and $\phi \rightarrow \Phi_1 \lor \cdots \lor \Phi_n$ a rewrite rule for $\pi$-constraints. Then, each solution $\sigma$ of $\phi^8$ is a solution of $\Phi_1 \lor \cdots \lor \Phi_n$, and vice versa, i.e., $R \models \phi[\sigma] \iff R \models (\Phi_1 \lor \cdots \lor \Phi_n)[\sigma]$.

**Proof.** The proof is based on showing that for each rewrite rule the set of solutions of left and right-hand sides is the same. For those rules dealing with set terms different from integer intervals the proofs can be found in [11] and [7].

The proof of equisatisfiability for the rules dealing with integer intervals is as follows. The equisatisfiability of rules (10), (12)–(18), (20)–(24) is trivial as these rules implement basic results of set theory and integer intervals. The equisatisfiability of rules (11), (19), (25) and (26) depends on basic facts of set theory and integer intervals (e.g., the first branch of rule (25)), and on the application of the identity (3). It is easy to check that (3) has been consistently applied on each rule. Note that the same argument can be applied to the rules included in Appendix A. The proof of equisatisfiability of rule (27) can be found in Appendix B.

The next theorem ensures that, after termination, the whole rewriting process implemented by $\mathcal{SAT}_[\cdot]$ is correct and complete.

**Theorem 2 (Soundness and Completeness).** Let $\Phi$ be a [ ]-formula and $\Phi^1, \Phi^2, \ldots, \Phi^n$ be the collection of [ ]-formulas returned by $\mathcal{SAT}_[\cdot](\Phi)$. Then, every possible solution of $\Phi$ is a solution of one of the $\Phi^i$ and, vice versa, every solution of one of these formulas is a solution for $\Phi$.

**Proof.** According to Definition 4.10, each formula $\Phi_i$ returned at the end of $\mathcal{SAT}_[\cdot]$’s main loop is of the form $\Phi_S^i \land \Phi_Z^i$, where $\Phi_S^i$ is a [ ]-formula in irreducible form and $\Phi_Z^i$ contains all integer constraints encountered during the processing of the input formula.

8More precisely, each solution of $\phi$ expanded to the variables occurring in $\Phi^i$ but not in $\phi$, so as to account for the possible fresh variables introduced into $\Phi^i$. 

ACM Transactions on Computational Logic, Vol. 25, No. 1, Article 3. Publication date: November 2023.
As concerns $\Phi_i$, no rewriting is actually performed on the constraints occurring in it. Thus, the set of solutions is trivially preserved.

Considering also the calls to $\text{solve\_size}$ and $\text{gen\_size\_leq}$, we observe that the first is just a check which either returns $false$ or has no influence on its input formula, while the second simply adds constraints entailed by the definition of set cardinality.

As concerns constraints in $\Phi_i$, we observe that $SAT_1$ is just the repeated execution of the rewriting rules described in the previous section, for which we have individually proved equisatisfiability (see Theorem 1). No other action of Algorithm 1 can add or remove solutions from the input formula.

Thus, the whole $SAT_1$ process preserves the set of solutions of the input formula. \hfill \Box

**Theorem 3 (Satisfiability of the Output Formula).** Any $[\cdot]$-formula different from false returned by $SAT_1$ is satisfiable w.r.t. the underlying interpretation structure $R$.

**Proof [Sketch].** Given an input formula $\Phi$, containing at least one variable-interval, at the end of the main loop of Algorithm 1 we have $\Phi \equiv \Phi_1 \land \Phi_\subseteq$, where $\Phi_1$ is a $[\cdot]$-formula and $\Phi_\subseteq$ is a conjunction of constraints of the form $X \subseteq [k, m]$ with $k$ or $m$ variables. As can be seen in Algorithm 1, $\text{solve\_size}$ is called on $\Phi_1$ (actually, a sub-formula of it). If $SAT_1$ finds $\Phi_1$ unsatisfiable then $\Phi$ is unsatisfiable. However, if $SAT_1$ finds $\Phi_1$ satisfiable we still need to check if $\Phi_\subseteq$ does not compromise the satisfiability of $\Phi_1$. To this end, $SAT_1$ asks $SAT_1$ to compute a minimal solution of $\Phi_1$. If the minimal solution of $\Phi_1$ is a solution of $\Phi$, then $\Phi$ is clearly satisfiable. Otherwise (i.e., the minimal solution of $\Phi_1$ is not a solution of $\Phi$), we show that $\Phi$ is unsatisfiable. That is, if the minimal solution of $\Phi_1$ is not a solution of $\Phi$, then any larger solution (w.r.t. the minimal solution) will not be a solution of $\Phi$. \hfill \Box

Now, we can state the termination property for $SAT_1$.

**Theorem 4 (Termination).** The $SAT_1$ procedure can be implemented in such a way that it terminates for every input $L_1$ formula.

**Proof [Sketch].** Termination of $SAT_1$ is a consequence of: (a) termination of $SAT_1$; (b) the individual termination of each new rewrite rule added to $STEP_S$; and (c) the collective termination of all the rewrite rules of $STEP_S$.

Assuming (b) and (c), the same arguments used in [7, Theorem 3] can be applied to Algorithm 1. That is, Algorithm 1 uses $STEP_S$ instead of the $STEP_S$ procedure used by $SAT_1$ and adds the then branch after the main loop. $STEP_S$ differs from $STEP_S$ in the new rewrite rules introduced in Section 4. Therefore, it is enough to prove that $STEP_S$ terminates as $STEP_S$ does. In turn, this entails to prove (b) and (c)—as done when the termination of CLP(SET) and $SAT_1$ were proved. \hfill \Box

**Theorem 5 (Decidability).** Given a $[\cdot]$-formula $\Phi$, then $\Phi$ is satisfiable with respect to the intended interpretation structure $R$ if and only if there is a non-deterministic choice in $SAT_1(\Phi)$ that returns a $[\cdot]$-formula different from false. Conversely, if all the non-deterministic computations of $SAT_1(\Phi)$ terminate with false, then $\Phi$ is surely unsatisfiable. Hence, $SAT_1$ is a decision procedure for $L_1$.

**Proof.** Thanks to Theorem 2 we know that, if $SAT_1$ terminates, the initial input formula $\Phi$ is equisatisfiable to the disjunction of formulas $\Phi^1, \Phi^2, \ldots, \Phi^n$ non-deterministically returned by $SAT_1$. Thanks to Theorem 3, we know that any $[\cdot]$-formula different from false returned by $SAT_1$ is surely satisfiable w.r.t. the underlying interpretation structure $R$. Then, if $SAT_1$ terminates, the initial input formula $\Phi$ is satisfiable iff the formula $\Phi^1 \lor \cdots \lor \Phi^n$ is satisfiable,
that is, at least one of the $\Phi^i$ is different from $false$. Thanks to Theorem 4, we know that $SAT$ terminates for all admissible $[]$-formulas. Hence, $SAT$ is always able to decide whether the input formula $\Phi$ is satisfiable or not.

In Section 6, we show several formulas that $SAT$ is able to detect to be unsatisfiable.

Note that many of the rewriting procedures given in the previous section will stop even when returning relatively complex formulas.

**Example 7.** Assuming all the arguments are variables, $SAT$ called on the formula:

$\{x \sqcup A\} = [k, m]$

will return the following two formulas:

$k \leq x \land x \leq m \land A \subseteq [k, m] \land x \notin A \land size(A, N_2) \land 1 \leq N_1 \land N_2 = N_1 - 1 \land N_1 = m - k + 1$

$A = \{x \cup N_1\} \land k \leq x \land x \leq m \land N_1 \subseteq [k, m] \land x \notin N_1 \land size(N_1, N_3) \land 1 \leq N_2 \land N_3 = N_2 - 1 \land N_2 = m - k + 1$

This is so because there is no rewrite rule for constraints such as $size(A, N_2)$ when both arguments are variables. However, Theorem 3 ensures that both formulas are satisfiable. For example, the first one is satisfiable with $N_1 = 1, N_2 = 0, m = k = x, A = \emptyset$.

**5.1 Complexity of $SAT$**

$SAT$ strongly relies on set unification. In fact, most rewrite rules dealing with integer intervals rely on the identity (3) which, roughly speaking, forces an interval to become an extensional set and then, ultimately, to be managed through set unification.

Hence, complexity of our decision procedure strongly depends on complexity of set unification. As observed in [17], the decision problem for set unification is NP-complete. A simple proof of the NP-hardness of this problem has been given in [13]. The proof is based on representing 3-SAT as a set unification problem; thus, solving the latter in polynomial time could also be exploited for solving 3-SAT in polynomial time. Concerning NP-completeness, the algorithm presented here clearly does not belong to NP since it applies syntactic substitutions. Nevertheless, it would be possible to encode this algorithm using well-known techniques that avoid explicit substitutions, maintaining a polynomial time complexity along each non-deterministic branch of the computation.

Moreover, the implementation of the solve_size procedure requires to perform, among others, the following actions [7]: compute the set of solutions of a Boolean formula derived from the irreducible form; and solve an integer linear programming problem for each subset of the Boolean solutions, which entails to compute the powerset of the Boolean solutions. Both these problems are inherently exponential in the worst case.

Finally, observe that, $SAT$ deals not only with the decision problem for set unification but also with the associated function problem (i.e., it can compute solutions for the problem at hand). Since solving the function problem clearly implies solving the related decision problem, the complexity of $SAT$ can be no better than the complexity of the decision problem for set unification.

**6 EXPRESSIVENESS OF $L[1]$, POWER OF $SAT$**

The presence of integer intervals in $L[1]$ is a sensible extension as it can express many operators and problems that (at least) are hard to express in $L[|\cdot|]$. It is important to observe that all these operators are introduced as $[]$-formulas, i.e., as quantifier-free formulas. In this section we explore the expressiveness of $L[1]$ by means of several examples while we show examples of what kind of automated reasoning $SAT$ is capable of. More examples can be found in Appendix C.
6.1 Minimum and Maximum of a Set

$L_1$ can express the minimum and maximum of a set as a quantifier-free formula:

\[ s_{\text{min}}(S, m) \equiv m \in S \land S \subseteq [m, \_] \]

\[ s_{\text{max}}(S, m) \equiv m \in S \land S \subseteq [\_, m] \]

where “\_” stands for an anonymous variable as in Prolog. That is, if \( m \) is the minimum of \( S \) then \( m \in S \) and every other element in \( S \) must be greater than \( m \). This second condition is achieved by stating \( S \subseteq [m, \_] \) because there is no \( x \) \( \in S \) such that \( x < m \) and \( x \notin [m, \_] \) given that \( m \) is the minimum of \( [m, \_] \).

Concerning the automated reasoning that \( S\mathcal{AT}_{\_} \) can perform, it can easily prove, for instance, the following propositions by proving that their negations are unsatisfiable.

\[ s_{\text{min}}(S, m) \Rightarrow \forall x \in S : m \leq x \]  

(33)

\[ s_{\text{min}}(S, m) \land s_{\text{max}}(S, n) \Rightarrow m \leq n \]  

(34)

For example, the negation of (33) is the following \( L_{\_} \) formula:

\[ s_{\text{min}}(S, m) \land x \in S \land x < m \]  

(35)

where \( x \) is implicitly existentially quantified.

6.2 The \( i \)-th Smallest Element of a Set

The definition of minimum of a set can be generalized to a formula computing the \( i \)-th smallest element of a set:

\[ s_{\text{nth}}(S, i, e) \equiv \text{un}(S_{\text{min}}, S_{\text{max}}, S) \land S_{\text{min}} \parallel S_{\text{max}} \]

\[ \wedge m \in S_{\text{min}} \land e \in S_{\text{min}} \]

\[ \wedge \text{size}(S_{\text{min}}, i) \land S_{\text{min}} \subseteq [m, e] \]

\[ \wedge S_{\text{max}} \subseteq [e + 1, \_] \]  

(36)

The formula partitions \( S \) into two disjoint sets \( S_{\text{min}} \) and \( S_{\text{max}} \). Intuitively, \( S_{\text{min}} \) contains the \( i \)-th smallest elements of \( S \) while \( S_{\text{max}} \) contains the rest of \( S \). Then \( m \) is intended to be the minimum of \( S \) which actually belongs to \( S_{\text{min}} \). Then \( S_{\text{min}} \) is forced to hold \( i \) elements including \( e \) and to be a subset of \( [m, e] \). In this way we know that all the elements of \( S_{\text{min}} \) are between \( m \) (the minimum of \( S \)) and \( e \) (the \( i \)-th smallest element of \( S \)). Finally, \( S_{\text{max}} \) is forced to be a subset of \( [e + 1, \_] \) because otherwise some \( x \in S \cap [m, e] \) could be put in \( S_{\text{max}} \) and we do not want that. Note, however, that we do not require \( e + 1 \in S_{\text{max}} \).

**Example 8.** If \( S\mathcal{AT}_{\_} \) is called as follows it binds \( e \) as indicated in each case.

\[ s_{\text{nth}}([7, 8, 2, 14], 1, e) \rightarrow e = 2 \]

\[ s_{\text{nth}}([7, 8, 2, 14], 2, e) \rightarrow e = 7 \]

\[ s_{\text{nth}}([7, 8, 2, 14], 3, e) \rightarrow e = 8 \]

\[ s_{\text{nth}}([7, 8, 2, 14], 4, e) \rightarrow e = 14 \]

As the above example shows, \( s_{\text{nth}} \) provides a logic iterator for sets whose elements belong to a total order. Without \( s_{\text{nth}} \), peeking the “first” element of a set becomes totally non-deterministic as any element of the set can be the first one. On the contrary, \( s_{\text{nth}} \) provides a deterministic iterator for sets as the \( i \)-th smallest element of a set is unique—if its elements belong to a total order. Furthermore, as \( S\mathcal{AT}_{\_} \) is based on constraint programming, \( s_{\text{nth}} \) allows to compute the index of a given element.
Example 9. If \( i \) is a variable, then \( \text{SAT} \) will bind \( i \) to 3 if the following query is run: \( \text{snth}([7, 8, 2, 14], i, 8) \). Furthermore, if \( x \) is a variable, then \( \text{SAT} \) will yield conditions for \( x \) that make \( \text{snth}([7, 8, x, 14], 3, 8) \) true—specifically, \( x < 7 \).

Concerning the automated reasoning that \( \text{SAT} \) can perform, it can prove, for instance, the following propositions by proving that their negations are unsatisfiable.

\[
\text{snth}(S, i_1, e_1) \land \text{snth}(S, i_2, e_2) \land i_1 < i_2 \Rightarrow e_1 < e_2 \tag{37}
\]

\[
\text{snth}(S, i, e_1) \land \text{snth}(S, i + 1, e_2) \Rightarrow \neg \exists x \in S : e_1 < x < e_2 \tag{38}
\]

Observe that (38) is basically a proof of correctness for \( \text{snth} \).

The \( i \)-th greatest element of a set can be defined likewise.

6.3 Partitioning of a Set w.r.t. a Number

Consider a set of integer numbers \( S \) and any integer \( i \notin S \). The operator called \( \text{mxlb}_\text{mnub} \) partitions \( S \) into the elements strictly below \( i \) (\( L \)) and those strictly above \( i \) (\( U \)). Besides, it computes the maximum of \( L \) (\( \text{max} \)) and the minimum of \( U \) (\( \text{min} \)), if they exist—either \( L \) or \( U \) can be the empty set in some border cases. Hence, \( \text{mxlb}_\text{mnub} \) computes the maximum (minimum) of the ‘lower’ (‘upper’) elements of \( S \) w.r.t. \( i \).

\[
\text{mxlb}_\text{mnub}(S, i, L, \text{max}, U, \text{min}) \equiv \text{un}(L, U, S) \land L \parallel U \land (\text{max} < i \land \text{smax}(L, \text{max}) \lor L = \emptyset) \land (i < \text{min} \land \text{smin}(U, \text{min}) \lor U = \emptyset) \tag{39}
\]

It would be possible to remove from the interface of \( \text{mxlb}_\text{mnub} \) the arguments \( \text{max} \) and \( \text{min} \), and compute them from \( L \) and \( U \) by calling \( \text{smax} \) and \( \text{smin} \). However, since \( \text{max} \) and \( \text{min} \) have to be computed inside \( \text{mxlb}_\text{mnub} \) to compute \( L \) and \( U \) it makes sense to include \( \text{max} \) and \( \text{min} \) as arguments to avoid a double computation. Besides, note that \( \text{mxlb}_\text{mnub} \) fails if \( i \in S \).

Concerning the automated reasoning that \( \text{SAT} \) can perform, it can prove, for instance, the following propositions by proving that their negations are unsatisfiable.

\[
\text{mxlb}_\text{mnub}(S, i, L, \text{max}, U, \text{min}) \land \text{smin}(S, k) \land i < k \Rightarrow L = \emptyset \tag{40}
\]

\[
S \subseteq T \land \text{mxlb}_\text{mnub}(S, i, L_{\text{s}}, \text{maxs}, U_{\text{s}}, \text{mins}) \land \text{mxlb}_\text{mnub}(T, i, L_{\text{t}}, \text{maxt}, U_{\text{t}}, \text{mint}) \Rightarrow U_{\text{s}} \subseteq U_{\text{t}} \tag{41}
\]

\( \text{mxlb}_\text{mnub} \) is a key operator used in the case study presented in Section 8 because it allows to compute the next floor to be served by the elevator either when moving up or down.

6.4 Proper Maximal Intervals of a Set

Consider a set of integer numbers \( S \). It might be useful to find out the maximal proper intervals contained in \( S \). That is, we look for intervals \([k, m] \subseteq S\) with \( k < m \) such that there is no other interval in \( S \) including \([k, m]\). Such intervals may represent, for instance, the longest continuous paths in a list or graph.

\[
\text{max}_\text{int}(S, k, m) \equiv \text{un}([k, m], R, S) \land [k, m] \parallel R \land k < m \land k - 1 \notin R \land m + 1 \notin R \tag{42}
\]

Example 10. If \( \text{SAT} \) is called on \( \text{max}_\text{int}([5, 3, 8, 2, 4, 7, 1], k, m) \) it first binds \( k \) to 1 and \( m \) to 5 and then to 7 and 8. It can also be called on \( \text{max}_\text{int}(S, 1, 5) \) in which case it returns \( S = \{1, 2, 3, 4, 5 \parallel N\} \) plus constraints forcing \([1, 5]\) to be the maximal subinterval in \( S \)—specifically \( 0 \notin N \land 6 \notin N \).
Concerning the automated reasoning that SAT can perform, it can prove, for instance, the following propositions by proving that their negations are unsatisfiable.

\[
\begin{align*}
\text{max_int}(S, k, m) \land [a, m] \subseteq S & \Rightarrow k \leq a \\
\end{align*}
\]

(43)

\[
\begin{align*}
a < b \land b + 2 < c \land \text{un}([a, b], [b + 2, c], S) \land \text{max_int}(S, k, m) & \Rightarrow (k = a \land m = b \lor k = b + 2 \land m = c)
\end{align*}
\]

(44)

Note that if in (44) the left limit of the second interval is \(b + 1\) then the maximal interval is \([a, c]\).

If in (42) \(k < m\) is removed, then \text{max_int} would return solutions for the empty interval and for singleton intervals in some cases. Clearly, \text{max_int} can be generalized to compute only intervals of a minimum cardinality \(c\) by stating \(c \leq m - k + 1\) instead of \(k < m\).

7 \{log\}'s Implementation of SAT

\(L_{[1]}\) is implemented by extending the solver provided by the publicly available tool \{log\} (pronounced 'setlog') [14]. \{log\} is a Prolog program that can be used as a constraint solver, as a satisfiability solver and as a constraint logic programming language. It also provides some programming facilities not described in this paper.

The main syntactic differences between the abstract syntax used in previous sections and the concrete syntax used in \{log\} is made evident by the following examples.

Example 11. The formula \text{max_int} given in Section 6.4 is written in \{log\} as follows:

\[
\begin{align*}
\text{max_int}(S, K, M) :& - \\
\text{un}(\text{int}(K, M), R, S) & \land \\
\text{disj}(\text{int}(K, M), R) & \land \\
K & < M & \land \\
K1 & \text{is} \ K - 1 & \text{and} \ K1 \ \text{nin} \ R & \land \\
M1 & \text{is} \ M + 1 & \text{and} \ M1 \ \text{nin} \ R.
\end{align*}
\]

where names beginning with a capital letter represent variables, and all others represent constants and function symbols. As can be seen, \text{int}(K, M) corresponds to the integer interval \([K, M]\); \& to \land; \text{disj}(\text{int}(K, M), R) to \([k, m] \parallel R\); and \(K1\) is \(K - 1\) and \(K1\ \text{nin} \ R\) to \(k - 1 \notin R\).

In \{log\} interval limits and the cardinality of \text{size}-constraints can only be variables or constants. Besides, the extensional set constructor \{·\⊔·\} is encoded as \{·/\_\}. All this is shown in the following example.

Example 12. A formula such as:

\[
\begin{align*}
\text{un}([X \sqcup A], B, [k + 1, m]) \land \text{size}(A, p) \land \text{size}(B, p - 3)
\end{align*}
\]

is encoded in \{log\} as follows:

\[
\begin{align*}
\text{un}([X/A], B, \text{int}(K1, M)) & \land K1 \ \text{is} \ K + 1 & \text{and} \ \text{size}(A, P) \land \text{size}(B, P3) \land P3 \ \text{is} \ P - 3.
\end{align*}
\]

In other words, constraints such as \text{un}(A, B, \text{int}(K + 1, M)) or \text{size}(A, P - 3) make \{log\} to output an error message.

More examples on how to use \{log\} are given in Section 8.

7.1 Rewrite Rules for Subset, Intersection and Difference

As we have said in Section 3.3, subset, intersection (\textslash\text{inters}) and difference (\text{diff}) are definable in terms of union and disjoint. This means that when a formula including subset, intersection or difference is processed it is first transformed into a \(L_{[1]}\) formula by substituting these operators
by union and disjoint. This works well from the theoretical perspective but in practice it leads to performance penalties.

Therefore, we extend the implementation of \( \mathcal{SAT} \) in \( \{\log\} \) by including rewrite rules for subset, intersection and difference—this follows the implementation of CLP(SET) and \( \mathcal{SAT} \). As with the primitive constraints, the rewrite rules for subset, intersection and difference are based either on simple mathematical results (e.g., \( \text{inters}([k, m], \emptyset, A) \rightarrow A = \emptyset \)); on the application of the identity (3); or on integer arithmetic constraints—such as rule (27) for \( \text{un} \)-constraints. As an example, Figure 8 shows rule 45 for \( \text{diff} \)-constraints where we rely on integer arithmetic constraints as much as possible until the last case where rule (26) is called—which in turn is based on (3). This last case can be graphically represented over the \( \mathbb{Z} \) line as follows:

\[
\begin{align*}
\text{Observ...}
\end{align*}
\]

Observe that, in spite that rule (45) calls rule (26), it does not cause termination problems as the rules for \( \text{un} \)-constraints do not call rules for \( \text{diff} \)-constraints.

7.2 A Memoizing Schema

\( \{\log\} \) processes formulas by rewriting one constraint at a time. As we have seen, some rewrite rules apply the identity (3) to substitute an integer interval by a new variable plus some constraints. In this way, if a given integer interval appears in two or more constraints which are rewritten by rules that apply (3), that interval will be substituted by different variables. The following example illustrates this.

\[\text{Example 13. When the last alternative of rule (26) is applied to the following formula:}\]

\[\text{un}(\text{int}(K,M),\text{int}(I,J),A) \& \text{un}(\text{int}(I,J),\text{int}(K,M),B) \& A \neq B\]

the result is a formula such as:

\[
\begin{align*}
\text{un}(W,X,A) & \& \text{un}(Y,Z,B) & \& A \neq B & \\
K & =< M & I & => J & \\
\text{subset}(W,\text{int}(K,M)) & \& \text{size}(W,P1) & \& P1 \text{ is } M - K + 1 & \\
\text{subset}(X,\text{int}(I,J)) & \& \text{size}(X,P2) & \& P2 \text{ is } J - I + 1 & \\
\text{subset}(Y,\text{int}(I,J)) & \& \text{size}(Y,P3) & \& P3 \text{ is } J - I + 1 & \\
\end{align*}
\]
subset(Z,int(K,M)) & size(Z,P4) & P4 is M - K + 1

Note how, for instance, int(K,M) has been substituted by W and Z. Clearly, this formula implies W = Z but \{log\} will deduce this after many rewriting steps.

This rewriting schema makes some formulas unnecessarily complex and, in general, degrades \{log\} efficiency when dealing with integer intervals. In order to avoid this problem we have implemented a memoizing schema that keeps track of what variable has been used to substitute a given integer interval. Then, when an integer interval is about to be substituted, \{log\} looks up if it has already been substituted and in that case it reuses the variable used in the first substitution. In this way, any given integer interval is always substituted by the same variable.

**Example 14.** With the memoizing schema, the formula of Example 13 is rewritten as follows:

\[
\text{un}(W,X,A) & \text{un}(X,W,B) & A \neq B & \text{subset}(W,int(K,M)) & \text{size}(W,P1) & P1 is M - K + 1 & \\
\text{subset}(X,int(I,J)) & \text{size}(X,P2) & P2 is J - I + 1
\]

where it is evident that the two \text{un}-constraints share the same variables W and X.

This memoizing schema makes a linear search over a list every time (3) is applied. Since set solving (especially cardinality solving) can be exponential in time, the memoizing schema produces a sensible gain in efficiency. It may degrade the efficiency only in very specific cases which are nonetheless solved quickly. \{log\} with the memoizing schema solves the formula of Example 13 ten times faster than without it.

### 7.3 An Initial Empirical Evaluation

Several in-depth empirical evaluations provide evidence that \{log\} is able to solve non-trivial problems [11, 12, 22, 23]; in particular as an automated verifier of security properties [15, 16].

As far as we know there are no benchmarks for a language like L_1. There are a couple of benchmarks for languages performing only interval reasoning, i.e., intervals cannot be mixed with sets and not all set operators are supported. These languages are meant to solve specific verification problems—for instance, model-checking of interval temporal logic [24].

Then, besides the case study presented in Section 8, we have gathered 60 L_1 formulas stating properties of the operators defined in Section 6—including all the properties stated in that section. \{log\} solves\(^9\) all the problems in 62.13 seconds thus averaging 1.03 seconds per problem. Only 10 problems take more than 1 second of which 2 take more than 5 seconds. In particular, \{log\} needs more than 19 seconds to solve formula (38).

The benchmark can be found here [http://people.dmi.unipr.it/gianfranco.rossi/SETLOG/setlog-intervals.zip](http://people.dmi.unipr.it/gianfranco.rossi/SETLOG/setlog-intervals.zip), along with instructions to reproduce our results.

### 8 CASE STUDY

In this section we present a case study using the implementation of L_1 and SAT [ in \{log\}. The intention of the case study is to show that \{log\} is useful in practice when it comes to solving problems involving integer intervals, especially concerning automatically discharging proof obligations. Here we present a simplified version to make the presentation more amenable. The \{log\} program can be found in the file lift.pl located in the same URL indicated above.

\(^9\)These problems and the case study of Section 8 were solved on a Latitude E7470 with a 4 core Intel(R) Core™ i7-6600U CPU at 2.60GHz with 8 Gb of main memory, running Linux Ubuntu 18.04.5, SWI-Prolog 7.6.4 and \{log\} 4.9.8-9g.
The case study is based on the elevator problem. That is, there is an elevator receiving service requests from the floors and from inside it. The control software should move the elevator up and down according to the requests it must serve. The key requirement is that the elevator shall move in one direction as long as there are requests that can be served in that direction. In particular, the elevator shall serve first the nearest request in the direction of movement. We have included in the case study requirements about stopping the elevator, opening and closing the door, and so on.

8.1 A \(\{\log\}\) Program

The resulting program consists of a 180 LOC \(\{\log\}\) program implementing seven operations of the elevator control software (add a request, serve next request, close the door, start the elevator, pass by a floor, stop the elevator and open the door). The number of LOC might look too small but this is due, in part, to the fact that many complex operations can be written very compactly by using set theory. The \(\{\log\}\) code shown below corresponds to one of the main operations of the program, namely \texttt{nextRequestUp}, which computes the next request to be served when the elevator is moving up.

\begin{verbatim}
nextRequestUp(Lift,Lift_):- 
    Lift = [F,Nf,D,C,M,R] & 
    M = up & 
    diff(R,[F],R1) & 
    mxlb_mnub(R1,F,_,_,Ub,Nf_) & 
    (Ub neq {} & M_ = M 
    or 
    Ub = {} & Nf_ = Nf & M_ = none 
    ) & 
    Lift_ = [F,Nf_,D,C,M_,R].
\end{verbatim}

\texttt{Lift} and \texttt{Lift}\_ represent the before and after states of the elevator, respectively. That is, \texttt{Lift}\_ plays the same role as \(Lift'\) in notations such as B and Z. As can be seen, \texttt{Lift} is a 6-tuple where each variable holds part of the state of the system: \(F\) represents the floor which the elevator is currently passing by; \(Nf\) is the next floor to be served; \(D\) represents the elevator’s door (open or closed); \(C\) specifies whether the elevator is moving or halted; \(M\) is the direction of movement (up, down or none); and \(R\) is the set of requests to be served. The next state is updated in the last line by unifying \texttt{Lift}\_ with \([F,Nf_,D,C,M_,R]\) where some of the variables are different from those used in the initial tuple. The same naming convention is used: \(M\ (M_\) is the current (next) direction of movement.

\texttt{nextRequestUp} computes the next floor to be served (\(Nf_\)) by calling \texttt{mxlb_mnub} (see Section 6.3) but only paying attention to the upper bounds (\(Ub\)) of the requests to be served w.r.t. the current floor. Clearly, if the elevator is moving up then it should keep that direction unless there are no more requests in that direction. Then, \texttt{nextRequestUp} distinguishes two cases: \(Ub\) is not empty and so it takes the minimum of \(Ub\) as the next floor to be served by putting \(Nf_\) as the second component of \texttt{Lift}\_; or \(Ub\) is empty and so the direction of movement is changed to none. In this last case, the software can change the direction to down if \(R\) is not empty (this is done by an operation called \texttt{nextRequestNone} which becomes enabled when \(M =\) none). The three alternatives to compute the next floor are assembled in one operation:

\begin{verbatim}
nextRequest(Lift,Lift_):- 
    nextRequestNone(Lift,Lift_) or nextRequestUp(Lift,Lift_) 
    or nextRequestDown(Lift,Lift_).
\end{verbatim}
8.2 Simulations

With this code we can run simulations to evaluate how the system works by setting the current state and calling some operations. Simulations are encoded as \(\text{\{log\}}\) formulas. For example:

\[
\text{Lift} = [3,3,\text{closed},\text{halted},\text{up},\{2,5,8,1,0\}] \& \text{nextRequest(Lift, Lift_)}.
\]

returns the next state:

\[
\text{Lift}_\text{=} = [3,5,\text{closed},\text{halted},\text{up},\{2,5,8,1,0\}]
\]

We can see that the next floor to be served is 5 because it is the nearest requested floor going up. Instead, if the elevator is moving down we get 2 as the next floor to be served:

\[
\text{Lift} = [3,3,\text{closed},\text{halted},\text{down},\{2,5,8,1,0\}] \& \text{nextRequest(Lift, Lift_)}
\]

\[
\text{Lift}_\text{=} = [3,2,\text{closed},\text{halted},\text{down},\{2,5,8,1,0\}]
\]

It is also possible to call more than one operation:\(^{10}\)

\[
\text{Lift}_1 = [3,3,\text{closed},\text{halted},\text{up},\{2,5,8,1,0\}] \& \text{addRequest(0,20,Lift1,4,Lift2)} \& \text{nextRequest(Lift2, Lift3)}
\]

\[
\begin{align*}
\text{Lift}_2 &= [3,3,\text{closed},\text{halted},\text{up},\{2,5,8,1,0,4\}], \\
\text{Lift}_3 &= [3,4,\text{closed},\text{halted},\text{up},\{2,5,8,1,0,4\}]
\end{align*}
\]

Note that \(\text{\{log\}}\) produces the state trace. Since a request to the 4th floor has been added, the next floor to be served is the 4th. If addRequest adds the 12th floor, the next floor to be served would be the 5th.

8.3 Automated Proofs

Being able to run simulations on the code is good to have a first idea on how the system works, but this cannot guarantee the program is correct. If we need stronger evidence on the correctness of the program we should try to prove some properties true of it. In this section we show that we can use the same representation of the control software and the same tool that we have used to run simulations (i.e., \(\text{\{log\}}\)), also to automatically prove properties of it.

In this context, one of the canonical class of properties to be proved are state invariants. Therefore, we have stated 7 state invariants and we have used \(\text{\{log\}}\) to prove that all the state operations preserve all of them. This amounts to automatically discharge 49 invariance lemmas—plus 7 proving that the initial state satisfies all the state invariants. \(\text{\{log\}}\) discharges all these proof obligations in about half of a second—on a standard laptop computer. The state invariants and the lemmas are encoded with 900 LOC of \(\text{\{log\}}\) code.

As with simulations, state invariants and invariance lemmas are encoded as \(\text{\{log\}}\) formulas. Just to give an idea of what is this all about, in Figure 9 we reproduce one state invariant and in Figure 10 one invariance lemma. Both figures present a mathematical encoding and the corresponding \(\text{\{log\}}\) encoding. As we have explained, logical implication has to be encoded as disjunction. In Figure 10, the \(\text{\{log\}}\) encoding corresponds to the negation of the mathematical encoding, given that \(\text{\{log\}}\) proves a lemma by proving that its negation is unsatisfiable. Then, \(n_{\text{liftInv3}}\) is the Boolean negation of liftInv3.

It is important to observe that if a formula that is supposed to be a lemma is run on \(\text{\{log\}}\) but it happens not to be valid, then \(\text{\{log\}}\) will return a counterexample.

\(^{10}\)addRequest(0,20,Lift1,4,Lift2) states that the elevator runs in a building with floors numbered from 0 to 20 and a request to the 4th floor is added.
Example 15. If in liftInv3 the inequality \( F =< Nf \) is changed to \( F < Nf \), then operation passFloor will not preserve that invariant because this operation can increment \( F \) in one. Then, the formula to be run is:

\[
\text{Lift} = [F,Nf,D,C,Di,R] \& (Di \neq \text{up} \text{ or } C \neq \text{moving} \text{ or } F < Nf) \& \\
\text{passFloor}(\text{MinF,MaxF,Lift,Lift}) \& \\
\text{Lift}_\text{=} = [F_,Nf_,D_,C_,Di_,R_] \& \\
\text{Di}_\text{=} = \text{up} \text{ and } C_\text{=} = \text{moving} \text{ and } F_\text{=} >= Nf_\text{=}. 
\]

In this case \{log\} returns a counterexample stating that:

\( F_\text{=} = Nf, \ Nf_\text{=} = Nf, \ Nf \text{ is } F + 1 \)

That is, initially, \( F < Nf \) but passFloor increments \( F \) in one and ‘assigns’ this value to \( F_\text{=} \) while leaving \( Nf \) unchanged. Then, after passFloor has executed the current floor can be equal to the next floor to be served.

More concrete counterexamples can be obtained by changing the default integer solver to CLP(FD) (\{log\} command int_solver(clpfd)), although in that case the user has to give values for MinF and MaxF and has to state that \( F \) ranges between those limits, i.e., \( F \text{ in int(MinF,MaxF)} \).

The case study provides evidence that \{log\} can deal with verification problems involving integer intervals by providing simulation and proving capabilities over the same representation of the system.

9 RELATED WORK

Tools such as Atelier B [25] and ProB [2] are very good in performing automated reasoning and a variety of analysis over B specifications. B specifications are based on a set theory including \( \mathcal{L}_[] \). We are not aware of these tools implementing a decision procedure for that fragment of set theory. Integrating \{log\} into these tools would constitute a promising line of work.

There are a number of works dealing with constraints admitting integer intervals but where their limits are constants (e.g., [26, 27]). For this reason, in these approaches full automated reasoning is not possible. Some of these approaches accept non-linear integer constraints. In general, they aim at a different class of problems, notably constraint programming to solve hard combinatorial problems.

Allen [24] defines an interval-based temporal logic which later on has been widely studied (e.g., [28, 29]). In Allen’s logic, intervals are defined over the real line. This logic can be expressed as a relation algebra [30]. The algebra has been proved to be decidable. Answer Set Programming (ASP) is closely related to CLP. Janhunen and Sioutis [31] use ASP to solve problems expressed in Allen’s interval algebra over the rational line.
A possible abstract representation of an array is as a function over the integer interval \([1, n]\) where \(n\) is the array length;\(^{11}\) then, \([1, n]\) becomes the array domain. The point here is that research on the theory of arrays sometimes needs to solve problems about integer intervals. For instance, Bradley et al. \([8]\) define a decision procedure for a fragment of the theory of arrays that allows to reason about properties holding for the array components with indexes in \([k, m]\). \{log\} might help in that context. For example, if we have proved that the components of array \(A\) with indexes in two sets, \(I\) and \(J\), verify some property we may want to prove that the property holds for the whole array by proving that un\((I, J, [1, n])\). It will be interesting to investigate whether or not the solving capabilities of \{log\} concerning partial functions \([11]\) combined with the results of this paper could deal with the decidable fragment found by Bradley et al.

Also motivated by research on formal verification of programs with arrays, Eriksson and Parsa \([18]\) define a domain specific language for integer interval reasoning. They pay particular attention to partition diagrams that divide an array domain into several (disjoint) integer intervals. Different properties of an array are true of each interval in the diagram. The DSL has been prototyped in the Why3 platform.

Integer interval reasoning is also used in static program analysis—sometimes in connection with arrays. Su and Wagner \([9]\) present a polynomial time algorithm for a general class of integer interval constraints. In this work, the authors define a lattice of intervals and interval constraints are defined over the partial order of the lattice. The language allows for infinite intervals where limits can be \(\pm \infty\).

After reviewing these works we can draw two conclusions. First, some have addressed the problem of reasoning about languages where the only available sets are intervals \([9, 18, 24]\). Without other kinds of sets it is not possible to define and reason about the operators discussed in Section 6. Instead, \(L_{[1]}\) can, for instance, express all the interval relations defined by Allen, over the integer line. The second conclusion is that no approach to interval reasoning seems to be rooted in set theory. On the contrary, the extension of \{log\} to integer intervals seamlessly integrate interval reasoning with set reasoning allowing to freely combine sets with intervals. This makes our approach more general and coherent, perhaps paying the price of a reduced efficiency when it comes to specific problems.

10 CONCLUSIONS AND FUTURE WORK

We have presented a language and a decision procedure for the algebra of finite sets extended with cardinality constraints and finite integer intervals. As far as we know, this is the first time that a language with such features is proved to be decidable. The implementation of the decision procedure as part of the \{log\} tool has also been presented. Initial empirical evidence showing that \{log\} could be used in practice is available.

\{log\} supports complex relational constraints such as composition, domain and domain restriction \([11]\). When integer intervals are combined with relational and cardinality constraints it is possible to model (finite) arrays:

\[
array(A, n) \equiv 0 < n \land size(A, n) \land dom(A, D) \land D \subseteq [1, n] \land size(D, n)
\]

where \(n\) is the length of array \(A\). That is, \(A\) is a function (set of ordered pairs) whose domain is the integer interval \([1, n]\)—note that the cardinality of \(A\) and its domain \(D\) is the same and \(D = [1, n]\) due to (3). Therefore, our next step is to investigate what are the decidable fragments concerning arrays. This would yield a powerful tool to work on the automated verification of programs with arrays.

\(^{11}\)The interval can also be \([0, n - 1]\) depending on the convention used for array indexes.
APPENDICES

A  MORE REWRITE RULES FOR \textit{un}-CONSTRAINTS

Below some more rewrite rules included in \textit{STEP}_{\text{S1}} for \textit{un}-constraints are listed. \textit{STEP}_{\text{S1}} also includes rules symmetric to rules (46) and (47) when the interval \([k, m]\) is the second argument and not the first.

\begin{align*}
\text{un}([k, m], A, B) &\rightarrow (m < k \land A = B) \lor (k \leq m \land \tilde{N} \subseteq [k, m] \land \text{size}(\tilde{N}, m - k + 1) \land \text{un}(\tilde{N}, A, B)) \\
\text{un}([k, m], A, [i, j]) &\rightarrow (j < i \land [k, m] = A = \emptyset) \lor (i \leq j \land m < k \land A = [i, j]) \lor (k \leq m \land i \leq j \land \tilde{N}_1 \subseteq [k, m] \land \text{size}(\tilde{N}_1, m - k + 1) \land \tilde{N}_2 \subseteq [i, j] \land \text{size}(\tilde{N}_2, j - i + 1) \land \text{un}(\tilde{N}_1, A, \tilde{N}_2))
\end{align*}

(46)

(47)

B  PROOFS

This section contains the proofs of some results referred to in the main document.

The proof of the fundamental identity (3).

\textsc{Lemma 1.} If \(A\) is any finite set, then:

\(\forall k, m \in \mathbb{Z} : k \leq m \Rightarrow (A = [k, m] \iff A \subseteq [k, m] \land |A| = m - k + 1)\)

\textit{Proof.} Assuming \(k \leq m\) the proof of:

\(A = [k, m] \Rightarrow A \subseteq [k, m] \land |A| = m - k + 1\)

is trivial. Note that without assuming \(k \leq m\) the result is not true because if \(m < k\) then \(|[k, m]| = 0\) while \(m - k + 1\) is not necessarily 0.

Now, assuming \(k \leq m\) the proof of:

\(A \subseteq [k, m] \land |A| = m - k + 1 \Rightarrow A = [k, m]\)

is also simple because knowing that \(|[k, m]| = m - k + 1\) then \(A\) and \([k, m]\) are two sets of the same cardinality with one of them being a subset of the other. This implies the two sets are indeed the same. Then, \(A = [k, m]\).

\(\square\)

\textit{Proof of equisatisfiability of rule (27).}

\textsc{Lemma 2.}

\(\text{un}([k, m], [i, j], [p, q]) \iff m < k \land [i, j] = [p, q]\)

\(\lor j < i \land [k, m] = [p, q]\)

\(\lor k \leq m \land i \leq j \land k \leq i \land i \leq m + 1 \land m \leq j \land p = k \land q = j\)

\(\lor k \leq m \land i \leq j \land k \leq i \land i \leq m + 1 \land j < m \land p = k \land q = m\)

\(\lor k \leq m \land i \leq j \land i < k \land k \leq j + 1 \land m \leq j \land p = i \land q = j\)

\(\lor k \leq m \land i \leq j \land i < k \land k \leq j + 1 \land j < m \land p = i \land q = m\)

\[\text{[1st]}\]

\[\text{[2nd]}\]

\[\text{[3rd]}\]

\[\text{[4th]}\]

\[\text{[5th]}\]

\[\text{[6th]}\]
Proof. The proof of the first two branches is trivial.

The proof of the 3rd and 4th branches is symmetric to the proof of the 5th and 6th branches. This symmetry comes from considering whether \([k, m]\) is at the left of \([i, j]\) or vice versa. In turn, this is expressed by stating \(k \leq i\) in the 3rd and 4th branches and \(i < k\) in the 5th and 6th.

Then, we will only prove the 3rd and 4th branches with the help of a geometric argument over the \(\mathbb{Z}\) line. Indeed if \([k, m]\) is at the left of \([i, j]\) we have the following cases.

The first case is depicted as follows.

In this case the union of \([k, m]\) and \([i, j]\) cannot yield an integer interval because there is a hole (\(x\)) in between them. This case is avoided by stating \(i \leq m + 1\) in both branches.

The second case is depicted as follows.

In this case the union of \([k, m]\) and \([i, j]\) is equal to \([k, j]\); that is, \(p = k\) and \(q = j\). This case is covered in the 3rd branch when \(i = m + 1\). Note that this case is only valid on the \(\mathbb{Z}\) line because there are no integer numbers between \(m\) and \(m + 1\) (i.e., \(i\)).

The third case is depicted as follows.

In this case the union of \([k, m]\) and \([i, j]\) is again equal to \([k, j]\); that is, \(p = k\) and \(q = j\). This case is covered also in the 3rd branch when \(i < m\).

The fourth and last case is depicted as follows.

In this case the union of \([k, m]\) and \([i, j]\) is equal to \([k, m]\); that is, \(p = k\) and \(q = m\). This case is covered in the 4th branch by stating \(j < m\). □

The next is the proof of Theorem 3.

Proof. As we have analyzed at the end of Section 4.8, when there are no integer intervals in the input formula \(\Phi\), \(\mathcal{SAT}_{\{\}}\) behaves exactly as \(\mathcal{SAT}_{\{\}\}^\{|\}\). Hence, for such formulas the theorem is proved elsewhere [7].

Now, consider an input formula \(\Phi\) containing at least one variable-interval. Hence, at the end of the main loop of Algorithm 1 we have \(\Phi \equiv \Phi_{\{\}^{|\}} \land \Phi_{\{\}^{|\}}\), where \(\Phi_{\{\}^{|\}}\) is a \(|\cdot|\)-formula and \(\Phi_{\{\}^{|\}}\) is a conjunction of constraints of the form \(X \subseteq [k, m]\) with \(k\) or \(m\) variables. As can be seen in Algorithm 1, \(\Phi_{\{\}^{|\}}\) is divided into \(\Phi_1\) and \(\Phi_2\). Now solve_size is called on \(\Phi_1\) in minimal solution mode. In this way, all sets of the size-constraints in the formula returned by solve_size are bound to bounded sets of the minimal possible cardinality as to satisfy the input formula. Given that \(\Phi_1\) is a \(|\cdot|\)-formula the answer returned by solve_size is correct.

Therefore, if solve_size(\(\Phi_1\)) returns false it is because \(\Phi_1\) is unsatisfiable. If \(\Phi_1\) is unsatisfiable, \(\Phi\) is unsatisfiable just because \(\Phi \equiv \Phi_1 \land \Phi_2 \land \Phi_{\{\}^{|\}}\).
On the other hand, if solve_size($\Phi_1$) does not return $false$ it is because it returns $\Phi_1 \land \bigwedge_{i=1}^{n} s_i = v_i$ where $s_i$ are the second arguments of the size-constraints in $\Phi_1$ and $v_i$ the corresponding minimal values. Now $STEP_{S_1}$ is called on $\Phi_1 \land (\bigwedge_{i=1}^{n} s_i = v_i) \land \Phi_2 \land \Phi_{\subseteq[1]}$. We can think that in a first iteration, $STEP_{S_1}$ substitutes $s_i$ by $v_i$ in the rest of the formula:

$$ (\Phi_1 \land \Phi_2 \land \Phi_{\subseteq[1]})[\forall i \in 1 \ldots n : s_i \leftarrow v_i] $$

This means that all size-constraints become $size(\hat{A}_i, v_i)$, with $v_i$ constant. In a second iteration $STEP_{S_1}$ rewrites these constraints as follows:

$$ size(\hat{A}_i, v_i) \rightarrow \hat{A}_i = \{y_1^i, \ldots, y_{v_i}^i\} \land \bigwedge_{p=1}^{v_i} \bigwedge_{q=1}^{v_i} y_p^i \neq y_q^i \quad \text{[if } p \neq q \text{]}$$

where $y_1^i, \ldots, y_{v_i}^i$ are all new variables. Hence, all size-constraints are eliminated from the formula. Furthermore, if any $\hat{A}_i$ is at the left-hand side of a constraint in $\Phi_{\subseteq[1]}$, $STEP_{S_1}$ will perform the following substitution:

$$ \Phi_{\subseteq[1]}[\forall i \in 1 \ldots n : \hat{A}_i \leftarrow \{y_1^i, \ldots, y_{v_i}^i\}] $$

Now, $STEP_{S_1}$ applies rule (24) to each constraint in $\Phi_{\subseteq[1]}$ where the substitution has been performed. Then, all those constraints where (24) is applied will be rewritten into a conjunction of integer constraints:

$$ \{y_1^i, \ldots, y_{v_i}^i\} \subseteq [k^i, m^i] \rightarrow \bigwedge_{p=1}^{v_i} k^i \leq y_p^i \leq m^i $$

This implies that all the remaining constraints in $\Phi_{\subseteq[1]}$ are of the form $\hat{A} \subseteq [k, m]$ and there is no size-constraint with $\hat{A}$ as first argument.

At this point, the resulting formula is either $false$ or a conjunction of: (i) a CLP(SET) formula in solved form; (ii) a linear integer formula and; (iii) a conjunction of constraints of the form $\hat{A} \subseteq [k, m]$, with $k$ or $m$ variables, and where there is no size-constraint with $\hat{A}$ as first argument. The case when the resulting formula is $false$ will be analyzed below. Then, let us consider the case when the resulting formula is a conjunction of i-iii. The linear integer formula is satisfiable because $STEP_{S_1}$ calls CLP(Q) at each iteration. The CLP(SET) formula in solved form is satisfiable due to results presented elsewhere [10]. Furthermore, a solution can be obtained by substituting all set variables by the empty set. Finally, the third conjunct is also satisfiable by substituting all set variables by the empty set. That is, a constraint of the form $\hat{A} \subseteq [k, m]$ has as a solution $\hat{A} = \emptyset$. This solution does not conflict with other occurrences of $\hat{A}$ in the CLP(SET) formula because, there, it is substituted by the empty set, therefore, we can conclude that the input formula, $\Phi$, is satisfiable. Basically, we have proved that a minimal solution of $\Phi_1$ is a solution of $\Phi_1 \land \Phi_2 \land \Phi_{\subseteq[1]}$.

Now, we will analyze the case when $STEP_{S_1}$ returns $false$ after the second call to step_loop. As we have said, in that call $STEP_{S_1}$ is called on $\Phi_1 \land (\bigwedge_{i=1}^{n} s_i = v_i) \land \Phi_2 \land \Phi_{\subseteq[1]}$, where $\bigwedge_{i=1}^{n} s_i = v_i$ encodes the minimal solution computed by solve_size for $\Phi_1$. Clearly, if this call answers $false$ it might be due to the minimal solution of $\Phi_1$ not being a solution for $\Phi_1 \land \Phi_2 \land \Phi_{\subseteq[1]}$. If $\Phi_1 \land \Phi_2 \land \Phi_{\subseteq[1]}$ is satisfiable there should be other solutions. We will prove that if the minimal solution of $\Phi_1$ is not a solution of $\Phi_1 \land \Phi_2 \land \Phi_{\subseteq[1]}$, then this formula is indeed unsatisfiable and so it is $\Phi$. We will prove this by proving that any larger solution (w.r.t. the minimal solution) is not a solution of the formula being analyzed.

Then, we assume that the following is $false$:

$$ \Phi_1 \land \left(\bigwedge_{i=1}^{n} s_i = v_i\right) \land \Phi_2 \land \Phi_{\subseteq[1]} $$

(49)
On the other hand:

\[ \Phi_1 \land \left( \bigwedge_{i=1}^{n} s_i = v_i \right) \land \Phi_2 \]  

(50)

is satisfiable due to the properties of \( \mathcal{SAT} \_1 \)—that is, if \( \bigwedge_{i=1}^{n} s_i = v_i \) is a solution of \( \Phi_1 \) is also a solution of \( \Phi_1 \land \Phi_2 \). Then, (49) becomes unsatisfiable due to interactions with \( \Phi_{\subseteq} \).

As we have seen in the first part of the proof, when the minimal solution is propagated in \( \Phi_{\subseteq} \) rule (24) is applied, at least, to some of its \( \subseteq \)-constraints. Hence, a \( \subseteq \)-constraint in \( \Phi_{\subseteq} \) either remains unchanged or is rewritten as in (48). If the \( \subseteq \)-constraint remains unchanged, then a larger solution than the minimal one will not make this \( \subseteq \)-constraint to be rewritten, and so a larger solution will not change the satisfiability of (49). Now, if the \( \subseteq \)-constraint is rewritten as in (48) we know that:

1. All the \( y^d_j \) are of sort \( \text{Int} \).
2. All the \( y^d_j \) are such that \( k^i \leq y^d_j \leq m^i \).
3. There are at least \( v_i \) integer numbers in \([k^i, m^i]\)—because all the \( y^d_p \) are different from each other.

Therefore, if (49) is unsatisfiable when (50) is satisfiable it must be because \( \Phi_1 \land \Phi_2 \) implies at least one of the following (for some \( x, p \) and \( i \) such that \( A_i \subseteq [k^i, m^i] \):

1. \( x \in \hat{A}_i \land (x < k^i \lor m^i < x) \), when \( \hat{A}_i \subseteq [k^i, m^i] \) making the whole formula trivially unsatisfiable.
2. size(\( \hat{A}_i, q_i \)) \land m^i - k^i + 1 < q_i
   
   \[ m^i - k^i + 1 < q_i, \text{ size}(\hat{A}_i, q_i) \land \hat{A}_i \subseteq [k^i, m^i] \] is trivially unsatisfiable.
3. \( 0 \leq p \land (\bigwedge_{j=1}^{p+1} x_j \notin \hat{A}_i \land k^i \leq x_j \leq m^i) \land \text{size}(\hat{A}_i, m^i - k^i + 1 - p) \)
   
   (a) \( \bigwedge_{j=1}^{p+1} k^i \leq x_j \leq m^i \) implies that there are \( p + 1 \) elements in \([k^i, m^i] \).
   
   (b) size(\( \hat{A}_i, m^i - k^i + 1 - p \)) \land \hat{A}_i \subseteq [k^i, m^i] \] implies that there are \( m^i - k^i + 1 - p \) elements in \([k^i, m^i] \).
   
   (c) \( \bigwedge_{j=1}^{p+1} x_j \notin \hat{A}_i \) implies that the \( p + 1 \) elements of 3a are disjoint from the \( m^i - k^i + 1 - p \) elements of 3b.
   
   (d) Then, there are \( p + 1 + m^i - k^i + 1 - p = m^i - k^i + 2 \) elements in \([k^i, m^i] \) when its cardinality is \( m^i - k^i + 1 \).

If any of the above makes (49) unsatisfiable, any solution larger than the minimal will not change that because (1)–(3) are implied by \( \Phi_1 \land \Phi_2 \) and so any of its solutions will imply the same.

The next is the proof of Theorem 4.

**Proof.** Termination of \( \mathcal{SAT} \_1 \) is a consequence of: (a) termination of \( \mathcal{SAT} \_1 \) [7, Theorem 3]; (b) the individual termination of each new rewrite rule added to \( \text{STEP}_{\mathcal{S}[1]} \); and (c) the collective termination of all the rewrite rules of \( \text{STEP}_{\mathcal{S}[1]} \).

Assuming (b) and (c), the same arguments used in [7, Theorem 3] can be applied to Algorithm 1. That is, Algorithm 1 uses \( \text{STEP}_{\mathcal{S}[1]} \) instead of the \( \text{STEP}_3 \) procedure used by \( \mathcal{SAT} \_1 \) and adds the then branch after the main loop. \( \text{STEP}_{\mathcal{S}[1]} \) differs from \( \text{STEP}_3 \) in the new rewrite rules introduced in Section 4. Therefore, it is enough to prove that \( \text{STEP}_{\mathcal{S}[1]} \) terminates as \( \text{STEP}_3 \) does. In turn, this entails to prove (b) and (c)—as done when the termination of \( \text{CLP}(\mathcal{SET}) \) and \( \mathcal{SAT} \_1 \) were proved.

Concerning (b) (i.e., individual termination), observe that in Figures 2–5 and A the only self recursive rule is (24). However, termination of (24) is guaranteed as, at some point, it arrives at the
set part of the extensional set. At this point (24) becomes disabled and (22) or (23) are executed. Note that (24) is not mutually recursive.

Concerning (c) (i.e., collective termination), we can observe the following:

1. Rewrite rules of $SAT_{\cdot\cdot}$ do not call the new rewrite rules as $L_{\cdot\cdot}$ does not admit integer intervals. Then, if one of the new rewrite rules generates a constraint dealt by the old rewrite rules, the latter will not call the former. Then, termination is proved in what concerns the interaction between new and old rewrite rules.

2. Rules (10), (12), (13), (15), (16), (17), (18), (20), (22), (27), (28) and (29), produce only integer constraints or true. All these constraints are processed by CLP(Q) which only generates integer constraints. Then, termination is proved.

3. Rules (16) and (17) trivially terminate. This is important because other rules in Figures 2–5 produce constraints of the form $\cdot \in [\cdot, \cdot]$ or $\cdot \not\in [\cdot, \cdot]$. So when that happens, termination is proved.

4. Rules (22)–(24) are either non-recursive or self recursive, but they never make a recursion on another rule. This is important because these are the rules called when the identity (3) is applied. Then, when (22)–(24) are called, termination is proved.

5. Rule (11) generates a size-constraint in which case 1 applies; and a constraint of the form $\cdot \subseteq [\cdot, \cdot]$ in which case 4 applies. Then, termination is proved.

6. Rule (14) calls rules (16) and (17), in which case 3 applies. Then, termination is proved.

7. Rule (19) generates a size and a $\parallel$ constraints in which case 1 applies; and a constraint of the form $\cdot \subseteq [\cdot, \cdot]$ in which case 4 applies. Then, termination is proved.

8. Rule (21) generates a constraint of the form $\cdot \in [\cdot, \cdot]$ in which case 3 applies. Then, termination is proved.

9. Rules (25) and (26) generate equality, size and un constraints in which case 1 applies. They also generate constraints of the form $[\cdot, \cdot] = \cdot$ in which case 5 applies; and constraints of the form $\cdot \subseteq [\cdot, \cdot]$ in which case 4 applies. Then, termination is proved. □

C TWO OPERATORS DEFINABLE AS $L_{\cdot\cdot}$ FORMULAS

In this section we show two more operators definable as $L_{\cdot\cdot}$ formulas.

\[ lb_{\_ub}(S, Lb, Ub) \equiv un(Lb, Ub, S) \land Lb \parallel Ub \land Lb \subseteq [\_ m] \land size(Lb, k) \land m < n \land Ub \subseteq [n, \_] \land size(Ub, k) \]

\[ \text{ssucc}(A, x, y) \equiv \]

\[ x < y \land A = \{x, y \cup B\} \land x \not\in B \land y \not\in B \land un(\text{Inf}, \text{Sup}, B) \land \text{Inf} \parallel \text{Sup} \land \text{Inf} \subseteq [\_ x - 1] \land \text{Sup} \subseteq [y + 1, \_] \]

REFERENCES

[1] J.-R. Abrial. 1996. The B-book: Assigning Programs to Meanings. Cambridge University Press, New York, NY, USA.

[2] M. Leuschel and M. Butler. 2003. ProB: A model checker for B. In FME, Vol. 2805 of Lecture Notes in Computer Science. A. Keijiro, S. Gnesi, D. Mandrioli (Eds.), Springer-Verlag, 855–874.

[3] Clearsy, Atelier B home page, http://www.atelierb.eu/

[4] C. G. Zarba. 2002. Combining sets with integers. In Frontiers of Combining Systems, 4th International Workshop, FroCoS 2002, Santa Margherita Ligure, Italy, April 8–10, 2002, Proceedings, A. Armando (Ed.), Vol. 2309 of Lecture Notes in Computer Science. Springer, 103–116. DOI: https://doi.org/10.1007/3-540-45988-X_9
A Decision Procedure for a Theory of Finite Sets with Finite Integer Intervals

[23] M. Cristiá and G. Rossi. 2020. Solving quantifier-free first-order constraints over finite sets and binary relations. J. Autom. Reason. 56, 4 (2021), 463–478. DOI: https://doi.org/10.1007/s10817-020-09577-6

[24] J. Eriksson and M. Parsa. 2020. A DSL for integer range reasoning: Partition, interval and mapping diagrams, In Practical Aspect of Declarative Languages-22nd International Symposium, PADL 2020, New Orleans, LA, USA, January 20–21, 2020, E. Komendantskaya, Y. A. Liu (Eds.), Proceedings, Vol. 12007 of Lecture Notes in Computer Science, Springer, 196–212. DOI: https://doi.org/10.1007/978-3-030-39197-3_13

[25] M. Cristiá, G. Rossi, and C. S. Frydman. 2013. \{log\} as a test case generator for the Test Template Framework. In SEFM, R. M. Hierons, M. G. Merayo, M. Bravetti (Eds.), Proceedings, Vol. 11194 of Lecture Notes in Computer Science, Springer, 8137 of Lecture Notes in Computer Science, Springer, 229–243.

[26] J. F. Allen. 1983. Maintaining knowledge about temporal intervals. Commun. ACM 26, 11 (1983), 832–843. DOI: https://doi.org/10.1145/82358434

[27] M. Cristiá and G. Rossi. 2018. A set solver for finite set relation algebra. In Relational and Algebraic Methods in Computer Science - 17th International Conference, RAMiCS 2018, Groningen, The Netherlands, October 29 - November 1, 2018, J. Desharnais, W. Guttman, S. Joosten (Eds.), Proceedings, Vol. 11194 of Lecture Notes in Computer Science, Springer, 333–349. DOI: https://doi.org/10.1007/978-3-030-02149-8_20

[28] M. Cristiá, G. Rossi, and C. S. Frydman. 2013. \{log\} as a test case generator for the Test Template Framework. In SEFM, R. M. Hierons, M. G. Merayo, M. Bravetti (Eds.), Proceedings, Vol. 8137 of Lecture Notes in Computer Science, Springer, 229–243.

[29] A. A. Krokhin, P. Jeavons, and P. Jonsson. 2003. Reasoning about temporal relations: The tractable subalgebras of Allen’s interval algebra. J. ACM 50, 5 (2003), 591–640. DOI: https://doi.org/10.1145/876638.876639

ACM Transactions on Computational Logic, Vol. 25, No. 1, Article 3. Publication date: November 2023.
[30] P. B. Ladkin. 1987. The completeness of a natural system for reasoning with time intervals. In Proceedings of the 10th International Joint Conference on Artificial Intelligence. Milan, Italy, August 23-28, 1987, Morgan Kaufmann, J. P. McDermott (Ed.), 462–465. DOI: http://ijcai.org/Proceedings/87-1/Papers/091.pdf

[31] T. Janhunen and M. Sioutis. 2019. Allen’s interval algebra makes the difference. In Declarative Programming and Knowledge Management - Conference on Declarative Programming, DECLARE 2019, Unifying INAP, WLP, and WFLP, Cottbus, Germany, September 9-12, 2019, P. Hofstedt, S. Abreu, U. John, H. Kuchen, D. Seipel (Eds.), Revised Selected Papers, Vol. 12057 of Lecture Notes in Computer Science, Springer, 89–98. DOI: https://doi.org/10.1007/978-3-030-46714-2_6

Received 3 November 2022; accepted 29 August 2023