Arrangements of Pseudocircles: On Circularizability

Stefan Felsner1 · Manfred Scheucher1

Received: 3 September 2018 / Revised: 28 January 2019 / Accepted: 4 March 2019 / Published online: 3 April 2019
© Springer Science+Business Media, LLC, part of Springer Nature 2019

Abstract
An arrangement of pseudocircles is a collection of simple closed curves on the sphere or in the plane such that any two of the curves are either disjoint or intersect in exactly two crossing points. We call an arrangement intersecting if every pair of pseudocircles intersects twice. An arrangement is circularizable if there is a combinatorially equivalent arrangement of circles. In this paper we present the results of the first thorough study of circularizability. We show that there are exactly four non-circularizable arrangements of 5 pseudocircles (one of them was known before). In the set of 2131 digon-free intersecting arrangements of 6 pseudocircles we identify the three non-circularizable examples. We also show non-circularizability of eight additional arrangements of 6 pseudocircles which have a group of symmetries of size at least 4. Most of our non-circularizability proofs depend on incidence theorems like Miquel’s. In other cases we contradict circularizability by considering a continuous deformation where the circles of an assumed circle representation grow or shrink in a controlled way. The claims that we have all non-circularizable arrangements with the given properties are based on a program that generated all arrangements up to a certain size. Given the complete lists of arrangements, we used heuristics to find circle representations. Examples where the heuristics failed were examined by hand.

Dedicated to the memory of Ricky Pollack.

Editor in Charge: Kenneth Clarkson

Partially supported by the DFG Grants FE 340/11-1 and FE 340/12-1. Manfred Scheucher was partially supported by the ERC Advanced Research Grant No. 267165 (DISCONV). We gratefully acknowledge the computing time granted by TBK Automatisierung und Messtechnik GmbH and by the Institute of Software Technology, Graz University of Technology. We also thank the anonymous reviewers for valuable comments.

Stefan Felsner
felsner@math.tu-berlin.de

Manfred Scheucher
scheucher@math.tu-berlin.de

1 Institut für Mathematik, Technische Universität Berlin, 10623 Berlin, Germany
1 Introduction

Arrangements of pseudocircles generalize arrangements of circles in the same vein as arrangements of pseudolines generalize arrangements of lines. The study of arrangements of pseudolines was initiated by Levi [18] in 1918. Since then arrangements of pseudolines were intensively studied. The handbook article on the topic [8] lists more than 100 references. To the best of our knowledge the study of arrangements of pseudocircles was initiated by Grünbaum [14] in the 1970s.

A pseudocircle is a simple closed curve in the plane or on the sphere. An arrangement of pseudocircles is a collection of pseudocircles with the property that the intersection of any two of the pseudocircles is either empty or consists of two points where the curves cross. Other authors also allow touching pseudocircles, e.g. [1]. The (primal) graph of an arrangement $A$ of pseudocircles has the intersection points of pseudocircles as vertices, the vertices split each of the pseudocircles into arcs, these are the edges of the graph. Note that this graph may have multiple edges and loop edges without vertices. The graph of an arrangement of pseudocircles comes with a plane embedding, the faces of this embedding are the cells of the arrangement. A cell of the arrangement with $k$ crossings on its boundary is a $k$-cell. A 2-cell is also called a digon (some authors call it a lens), and a 3-cell is also called a triangle. An arrangement $A$ of pseudocircles is

- **simple** if no three pseudocircles of $A$ intersect in a common point;
- **connected** if the graph of the arrangement is connected;
- **intersecting** if any two pseudocircles of $A$ intersect;
- **cylindrical** if there are two cells of the arrangement $A$ which are separated by each of the pseudocircles.

Note that every intersecting arrangement is connected. In this paper we assume that arrangements are simple and connected. The few cases where these assumptions do not hold will be clearly marked.

Two arrangements $A$ and $B$ are isomorphic if they induce homeomorphic cell decompositions of the compactified plane, i.e., on the sphere. Stereographic projections can be used to map between arrangements of pseudocircles in the plane and arrangements of pseudocircles on the sphere. Such projections are also considered isomorphisms. In particular, the isomorphism class of an arrangement of pseudocircles in the plane is closed under changes of the unbounded cell.

Figure 1 shows the three connected arrangements of three pseudocircles and Fig. 2 shows the 21 connected arrangements of four pseudocircles. We call the unique digon-free intersecting arrangement of three (pseudo)circles the Krupp. The second intersecting arrangement is the NonKrupp; this arrangement has digons. The non-intersecting arrangement is the 3-Chain.

Every triple of great-circles on the sphere induces a Krupp arrangement, hence, we call an arrangement of pseudocircles an arrangement of great-pseudocircles if every subarrangement induced by three pseudocircles is a Krupp.

---

1 This name refers to the logo of the Krupp AG, a German steel company. Krupp was the largest company in Europe at the beginning of the 20th century.
Some authors think of arrangements of great-pseudocircles when they speak about arrangements of pseudocircles, this is e.g. common practice in the theory of oriented matroids. In fact, arrangements of great-pseudocircles serve to represent rank 3 oriented matroids, cf. [3]. Planar partial cubes can be characterized as the duals of so-called ‘non-separating’ arrangements of pseudocircles, these are certain arrangements such that no triple forms a NonKrupp, see [2].

**Definition 1.1** An arrangement of pseudocircles is *circularizable* if there is an isomorphic arrangement of circles.

Preceding our work there have been only few results about circularizability of arrangements of pseudocircles. Edelsbrunner and Ramos [7] presented an intersecting arrangement of 6 pseudocircles (with digons) which has no realization with circles, i.e., it is not circularizable (see Fig. 21a). Linhart and Ortner [19] found a non-circularizable non-intersecting arrangement of 5 pseudocircles with digons (see Fig. 3b). They also proved that every intersecting arrangement of at most 4 pseudocircles is circularizable. Kang and Müller [15] extended the result by showing that all arrangements with at most 4 pseudocircles are circularizable. They also proved that deciding circularizability for connected arrangements is NP-hard.

**2 Overview**

In Sect. 3 we present some background on arrangements of pseudocircles and provide tools that will be useful for non-circularizability proofs.

In Sect. 4 we study arrangements of great-pseudocircles—this class of arrangements of pseudocircles is in bijection with projective arrangements of pseudolines. Our main theorem in this section is the Great-Circle Theorem which allows to transfer knowledge regarding arrangements of pseudolines to arrangements of pseudocircles.

**Theorem 2.1** (Great-Circle Theorem) An arrangement of great-pseudocircles is circularizable (i.e., has a circle representation) if and only if it has a great-circle representation.

Subsequent to the theorem, we present several direct consequences such as the \( \exists \mathbb{R} \)-completeness of circularizability. The complexity class \( \exists \mathbb{R} \) consists of problems, that can be reduced in polynomial time to solvability of a system of polynomial inequalities in several variables over the reals, and lies between \( \text{NP} \) and \( \text{PSPACE} \). Further background on \( \exists \mathbb{R} \) can be found in [21,25].
Fig. 2 The 21 connected arrangements of \( n = 4 \) pseudocircles. The 8 first arrangements (a)–(h) are intersecting. The arrangements (a), (b), and (m) are digon-free. The arrangement (s) is the unique non-cylindrical one.

In Sects. 5 and 6, we present a full classification of circularizable and non-circularizable arrangements among all connected arrangements of 5 pseudocircles and all digon-free intersecting arrangements of 6 pseudocircles. With the aid of computers
we generated the complete lists of connected arrangements of \( n \leq 6 \) pseudocircles and of intersecting arrangements of \( n \leq 7 \) pseudocircles. The respective numbers are shown in Table 1. Given the complete lists of arrangements, we used automatized heuristics to find circle representations. Examples where the heuristics failed had to be examined by hand.

Computational issues and algorithmic ideas are deferred until Sect. 8. There we also sketch the heuristics that we have used to produce circle representations for most of the arrangements. The encoded lists of arrangements of up to \( n = 6 \) pseudocircles and circle representations are available on our webpage [9]. Section 8 also contains asymptotic results on the number of arrangements of \( n \) pseudocircles as well as results on their flip-graph.

The list of circle representations at [9] together with the non-circularizability proofs given in Sect. 5 yields the following theorem.

Theorem 2.2 The four isomorphism classes of arrangements \( \mathcal{N}_5^1, \mathcal{N}_5^2, \mathcal{N}_5^3, \) and \( \mathcal{N}_5^4 \) (shown in Fig. 3) are the only non-circularizable ones among the 984 isomorphism classes of connected arrangements of \( n = 5 \) pseudocircles.

Corollary 2.3 The isomorphism class of arrangement \( \mathcal{N}_5^1 \) is the unique non-circularizable one among the 278 isomorphism classes of intersecting arrangements of \( n = 5 \) pseudocircles.

We remark that the arrangements \( \mathcal{N}_5^1, \mathcal{N}_5^2, \mathcal{N}_5^3, \) and \( \mathcal{N}_5^4 \) have symmetry groups of order 4, 8, 2, and 4, respectively. Also, note that none of the four examples is digon-free. Non-circularizability of \( \mathcal{N}_5^2 \) was previously shown by Linhart and Ortner [19]. We give an alternative proof which also shows the non-circularizability of \( \mathcal{N}_5^3 \). Jonathan Wild and Christopher Jones, contributed sequences A250001 and A288567 to the On-Line Encyclopedia of Integer Sequences (OEIS) [26]. These sequences count certain classes of arrangements of circles and pseudocircles. Wild and Jones also looked at circularizability and independently found Theorem 2.2 (personal communication).

Concerning arrangements of 6 pseudocircles, we were able to fully classify digon-free intersecting arrangements.
Fig. 3 The four non-circularizable arrangements on \( n = 5 \) pseudocircles: (a) \( N^1_5 \), (b) \( N^2_5 \), (c) \( N^3_5 \), and (d) \( N^4_5 \).

Fig. 4 The three non-circularizable digon-free intersecting arrangements for \( n = 6 \): (a) \( N^\Delta_6 \), (b) \( N^2_6 \), and (c) \( N^3_6 \). Inner triangles are colored gray. Note that in (b) and (c) the outer face is a triangle.

**Theorem 2.4** The three isomorphism classes of arrangements \( N^\Delta_6 \), \( N^2_6 \), and \( N^3_6 \) (shown in Fig. 4) are the only non-circularizable ones among the 2131 isomorphism classes of digon-free intersecting arrangements of \( n = 6 \) pseudocircles.

In Sect. 6, we give non-circularizability proofs for \( N^\Delta_6 \), \( N^2_6 \), and \( N^3_6 \). In fact, for the non-circularizability of \( N^\Delta_6 \) and \( N^2_6 \), respectively, we have two proofs of different flavors: One proof (see Sect. 6) uses continuous deformations like the proof of the Great-Circle Theorem (Theorem 2.1) and the other proof is based on an incidence theorem. The incidence theorem used for \( N^\Delta_6 \) may be of independent interest:

**Theorem 2.5** Let \( a, b, c, d, w, x, y, z \) be 8 points in \( \mathbb{R}^3 \) such that \( a, b, c, d \) are affinely independent and each of the following 5 subsets of 4 points is coplanar:

\[
\{a, b, w, x\}, \{a, c, w, y\}, \{a, d, w, z\}, \{b, c, x, y\}, \text{ and } \{b, d, x, z\}.
\]

Then \( \{c, d, y, z\} \) is also coplanar.

We did not find a source for this incidence theorem. The proof we give is based on determinant cancelation, a technique that we learned from Richter-Gebert, cf. [24].

An instance of Theorem 2.5 is obtained by assigning the eight letters appropriately to the corners of a cube. Each of the six involved sets then corresponds to the four corners covered by two opposite edges of the cube. The appropriate assignment of the letters can be derived from Fig. 16.

We remark that the arrangements \( N^\Delta_6 \), \( N^2_6 \), and \( N^3_6 \) have symmetry groups of order 24, 3, and 6, respectively. Particularly interesting is the arrangement \( N^\Delta_6 \) (Fig. 4a, see
This is the unique intersecting digon-free arrangement of 6 pseudocircles which attains the minimum 8 for the number of triangles (see [10]).

Even though we could not complete the classification for intersecting arrangements of 6 pseudocircles, we provide some further nice incidence theorems and non-circularizability proofs in Sect. 7. One of them (Fig. 21a) is the example of Edelsbrunner and Ramos [7]. Details on the current status are deferred to Sect. 9, where we also present some further results and discuss some open problems.

It may be worth mentioning that, by enumerating and realizing all arrangements of \( n \leq 4 \) pseudocircles, we have an alternative proof of the Kang and Müller result, that all arrangements of \( n \leq 4 \) pseudocircles are circularizable [15].

3 Preliminaries: Basic Properties and Tools

Stereographic projections map circles to circles (if we consider a line to be a circle containing the point at infinity), therefore, circularizability on the sphere and in the plane is the same concept. Arrangements of circles can be mapped to isomorphic arrangements of circles via Möbius transformations. In this context, the sphere is identified with the extended complex plane \( \mathbb{C} \cup \{ \infty \} \). Note that, for \( n \geq 3 \), the isomorphism class of an arrangement of \( n \) circles is not covered by Möbius transformations. Indeed, if \( \mathcal{C} \) is a simple arrangement of circles, then \( \varepsilon \)-perturbations of the circles in size and position will result in an isomorphic arrangement when \( \varepsilon \) is chosen small enough.

Let \( \mathcal{C} \) be an arrangement of circles represented on the sphere. Each circle of \( \mathcal{C} \) spans a plane in 3-space, hence, we obtain an arrangement \( \mathcal{E}(\mathcal{C}) \) of planes in \( \mathbb{R}^3 \). In fact, with a sphere \( S \) we get a bijection between (not necessarily connected) circle arrangements on \( S \) and arrangements of planes with the property that each plane of the arrangement intersects \( S \).

Consider two circles \( C_1, C_2 \) of a circle arrangement \( \mathcal{C} \) on \( S \) and the corresponding planes \( E_1, E_2 \) of \( \mathcal{E}(\mathcal{C}) \). The intersection of \( E_1 \) and \( E_2 \) is either empty (i.e., \( E_1 \) and \( E_2 \) are parallel) or a line \( \ell \). The line \( \ell \) intersects \( S \) if and only if \( C_1 \) and \( C_2 \) intersect, in fact, \( \ell \cap S = C_1 \cap C_2 \).

With three pairwise intersecting circles \( C_1, C_2, C_3 \) we obtain three planes \( E_1, E_2, E_3 \) intersecting in a vertex \( v \) of \( \mathcal{E}(\mathcal{C}) \). It is notable that \( v \) is in the interior of \( S \) if and only if the three circles form a Krupp in \( \mathcal{C} \). We save this observation for further reference.

**Fact 1** Let \( \mathcal{C} \) be an arrangement of circles represented on the sphere. Three circles \( C_1, C_2, C_3 \) of \( \mathcal{C} \) form a Krupp if and only if the three corresponding planes \( E_1, E_2, E_3 \) intersect in a single point in the interior of \( S \).

Digons of \( \mathcal{C} \) are also nicely characterized in terms of \( \mathcal{E}(\mathcal{C}) \) and \( S \).

**Fact 2** Let \( \mathcal{C} \) be an arrangement of circles represented on the sphere \( S \). A pair of intersecting circles \( C_1, C_2 \) in \( \mathcal{C} \) forms a digon of \( \mathcal{C} \) if and only if the line \( E_1 \cap E_2 \) has no intersection with any other plane \( E_3 \) corresponding to a circle \( C_3 \not\in \{ C_1, C_2 \} \) inside of \( S \).

As a consequence from the previous two paragraphs, we obtain that a connected digon-free arrangement of circles contains a Krupp. This is also true for connected
digon-free arrangements of pseudocircles, for the proof look at the circular sequence of intersections on a fixed circle.

3.1 Incidence Theorems

The smallest non-stretchable arrangements of pseudolines are closely related to the incidence theorems of Pappos and Desargues. A construction already described by Levi [18] is depicted in Fig. 5a. Pappos’s Theorem states that, in a configuration of 8 lines as shown in the figure in black, the 3 white points are collinear, i.e., a line containing two of them also contains the third one. Therefore, the arrangement including the red pseudoline has no corresponding arrangement of straight lines, i.e., it is not stretchable.

Miquel’s Theorem asserts that, in a configuration of 5 circles as shown in Fig. 5b in black, the 4 white points are cocircular, i.e., a circle containing three of them also contains the fourth one. Therefore, the arrangement including the red pseudocircle cannot be circularized.

Next we state two incidence theorems which will be used in later proofs of non-circularizability. In the course of the paper we will meet further incidence theorems such as Lemma 5.1, Lemma 5.2, Theorem 6.3, Lemma 6.6, and again Miquel’s Theorem (Theorem 7.1).

Lemma 3.1 (First Four-Circles Incidence Lemma) Let \( C \) be an arrangement of four circles \( C_1, C_2, C_3, C_4 \) such that none of them is contained in the interior of another one, and such that \( (C_1, C_2), (C_2, C_3), (C_3, C_4) \), and \( (C_4, C_1) \) are touching. Then there is a circle \( C^* \) passing through these four touching points in the given cyclic order.

We point the interested reader to the website “Cut-the-Knot.org” [4], where this lemma is stated (except for the cyclic order). The website also provides an interactive GeoGebra applet, which nicely illustrates the incidences.

Proof Apply a Möbius transformation \( \Gamma \) that maps the touching point of \( C_1 \) and \( C_2 \) to the point \( \infty \) of the extended complex plane. This maps \( C_1 \) and \( C_2 \) to a pair \( L_1, L_2 \) of parallel lines. The discs of \( C_1 \) and \( C_2 \) are mapped to disjoint halfplanes. We may assume that \( L_1 \) and \( L_2 \) are horizontal and that \( L_1 \) is above \( L_2 \). Circles \( C_3 \) and \( C_4 \) are...
mapped to touching circles $C'_3$ and $C'_4$. Moreover, $C'_3$ is touching $L_2$ from above and $C'_4$ is touching $L_1$ from below. Figure 6 shows a sketch of the situation.

Let $\ell'$ be the line tangent to $C'_3$ and $C'_4$ at their touching point $p$. Consider the two segments from $p$ to $C'_3 \cap L_2$ and from $p$ to $C'_4 \cap L_1$. Elementary considerations show the following equalities of angles: $\alpha_3 = \alpha_4$, $\beta_3 = \gamma_3$, $\beta_4 = \gamma_4$, and $\gamma_3 = \gamma_4$ (cf. Fig. 6). Hence, there is a line $\ell$ containing the images of the four touching points. Consequently, the circle $C^* = \Gamma^{-1}(\ell)$ contains the four touching points of $\mathcal{C}$, i.e., they are cocircular.

The following theorem (illustrated in Fig. 7) is mentioned by Richter-Gebert [24, p. 26] as a relative of Pappos’s and Miquel’s Theorems.

**Theorem 3.2** ([24]) Let $C_1, C_2, C_3$ be three circles in the plane such that each pair of them intersects in two points, and let $\ell_i$ be the line spanned by the two points of intersection of $C_j$ and $C_k$, for $\{i, j, k\} = \{1, 2, 3\}$. Then $\ell_1, \ell_2,$ and $\ell_3$ meet in a common point.

**Proof** Use a stereographic projection $\phi$ to map the three circles to circles $C'_1, C'_2, C'_3$ on a sphere $S$. Consider the planes $E'_1, E'_2, E'_3$ spanned by $C'_1, C'_2, C'_3$. Let $\ell'_i$ be the line $E'_i \cap E'_k$, for $\{i, j, k\} = \{1, 2, 3\}$. Since the arrangement is simple and intersecting, the lines $\ell'_1, \ell'_2, \ell'_3$ are distinct and the intersection $E'_1 \cap E'_2 \cap E'_3$ is a single projective point $p$, which is contained in each of $\ell'_1, \ell'_2, \ell'_3$. The inverse of $\phi$ can be interpreted as a central projection from 3-space to the plane. In this interpretation of $\phi^{-1}$, the lines $\ell'_1, \ell'_2, \ell'_3$ are mapped to $\ell_1, \ell_2, \ell_3$ and $p$ is mapped to a projective point, i.e., either $p$ is a point or the lines are parallel.

\[ \square \]
3.2 Flips and Deformations of Pseudocircles

Let $C$ be an arrangement of circles. Imagine that the circles of $C$ start moving independently, i.e., the position of their centers and their radii depend on a time parameter $t$ in a continuous way. This yields a family $C(t)$ of arrangements with $C(0) = C$. Let us assume that the set $T$ of all $t$ for which $C(t)$ is not simple or contains touching circles is discrete and for each $t \in T$ the arrangement $C(t)$ contains either a single point where 3 circles intersect or a single touching. If $t_1 < t_2$ are consecutive in $T$, then all arrangements $C(t)$ with $t \in (t_1, t_2)$ are isomorphic. Selecting one representative from each such class, we get a list $C_0, C_1, \ldots$ of simple arrangements such that two consecutive (non-isomorphic) arrangements $C_i, C_{i+1}$ are either related by a triangle flip or by a digon flip, see Fig. 8.

We will make use of controlled changes in circle arrangements, in particular, we grow or shrink specified circles of an arrangement to produce touchings or points where 3 circles intersect. The following lemma will be of use frequently.

**Lemma 3.3** (Digon Collapse Lemma) Let $C$ be an arrangement of circles in the plane and let $C$ be one of the circles of $C$, which intersects at least two other circles from $C$ and does not fully contain any other circle from $C$ in its interior. If $C$ has no incident triangle in its interior, then we can shrink $C$ into its interior such that the combinatorics of the arrangement remains the same except that two digons collapse to touchings. Moreover, the two corresponding circles touch $C$ from the outside.

**Proof** As illustrated on the left hand side of Fig. 9, we shrink the radius of $C$ until the first event occurs. Since $C$ already intersects all other circles of $C$, no new digon can be created. Moreover, since $C$ has no incident triangles in its interior, an interior digon collapses. We obtain a point where $C$ touches another circle that lies outside of $C$. (Note that several digons might collapse at the same time.)

If $C$ has only one touching point $p$, we shrink the radius and simultaneously move the center towards $p$ (cf. the right hand side of Fig. 9) so that $p$ stays a touching until
a second digon becomes a touching. Again the touching point is with a circle that lies outside of \( C \).

In the following we will sometimes use the dual version of the lemma, whose statement is obtained from the Digon Collapse Lemma by changing interior to exterior and outside to inside. The validity of the dual lemma is seen by applying a Möbius transformation which exchanges interior and exterior of \( C \).

Triangle flips and digon flips are also central to the work of Snoeyink and Hershberger [27]. They have shown that an arrangement \( C \) of pseudocircles can be swept with a sweepfront \( \gamma \) starting at any pseudocircle \( C \in \mathcal{C} \), i.e., \( \gamma_0 = C \). The sweep consists of two stages, one for sweeping the interior of \( C \), the other for sweeping the exterior. At any fixed time \( t \) the sweepfront \( \gamma_t \) is a closed curve such that \( C \cup \gamma_t \) is an arrangement of pseudocircles. Moreover, this arrangement is simple except for a discrete set \( T \) of times where sweep events happen. The sweep events are triangle flips or digon flips involving \( \gamma_t \).

4 Arrangements of Great-Pseudocircles

Central projections map between arrangements of great-circles on a sphere \( S \) and arrangements of lines on a plane. Changes of the plane to which we project preserve the isomorphism class of the projective arrangement of lines. In fact, arrangements of lines in the projective plane are in one-to-one correspondence to arrangements of great-circles.

In this section we generalize this concept to arrangements of pseudolines and show that there is a one-to-one correspondence to arrangements of great-pseudocircles. As already mentioned, this correspondence is not new (see e.g. [3]).

A pseudoline is a simple closed non-contractible curve in the projective plane. A (projective) arrangement of pseudolines is a collection of pseudolines such that any two intersect in exactly one point where they cross. We can also consider arrangements of pseudolines in the Euclidean plane by fixing a “line at infinity” in the projective plane—we call this a projection.

A Euclidean arrangement of \( n \) pseudolines can be represented by \( x \)-monotone pseudolines, a special representation of this kind is the wiring diagram, see e.g. [8]. As illustrated in Fig. 10, an \( x \)-monotone representation can be glued with a mirrored
copy of itself to form an arrangement of \( n \) pseudocircles. The resulting arrangement is intersecting and has no NonKrupp subarrangement, hence, it is an arrangement of great-pseudocircles.

For a pseudocircle \( C \) of an arrangement of \( n \) great-pseudocircles the cyclic order of crossings on \( C \) is antipodal, i.e., the infinite sequence corresponding to the cyclic order crossings of \( C \) with the other pseudocircles is periodic of order \( n - 1 \). If we consider projections of projective arrangements of \( n \) pseudolines, then this order does not depend on the choice of the projection. In fact, projective arrangements of \( n \) pseudolines are in bijection with arrangements of \( n \) great-pseudocircles.

### 4.1 The Great-Circle Theorem and Its Applications

Let \( \mathcal{A} \) be an arrangement of great-pseudocircles and let \( \mathcal{L} \) be the corresponding projective arrangement of pseudolines. Central projections show that, if \( \mathcal{L} \) is realizable with straight lines, then \( \mathcal{A} \) is realizable with great-circles, and conversely.

In fact, due to Theorem 2.1, it is sufficient that \( \mathcal{A} \) be circularizable to conclude that \( \mathcal{A} \) is realizable with great-circles and \( \mathcal{L} \) is realizable with straight lines.

**Theorem 2.1** (Great-Circle Theorem) An arrangement of great-pseudocircles is circularizable (i.e., has a circle representation) if and only if it has a great-circle representation.

**Proof of Theorem 2.1** Consider an arrangement of circles \( \mathcal{C} \) on the unit sphere \( \mathbb{S} \) that realizes an arrangement of great-pseudocircles. Let \( \mathcal{E}(\mathcal{C}) \) be the arrangement of planes spanned by the circles of \( \mathcal{C} \). Since \( \mathcal{C} \) realizes an arrangement of great-pseudocircles, every triple of circles forms a Krupp, hence, the point of intersection of any three planes of \( \mathcal{E}(\mathcal{C}) \) is in the interior of \( \mathbb{S} \).

Imagine the planes of \( \mathcal{E}(\mathcal{C}) \) moving towards the origin. To be precise, for time \( t \geq 1 \) let \( \mathcal{E}_t := \{ \frac{1}{t} \cdot E : E \in \mathcal{E}(\mathcal{C}) \} \). Since all intersection points of the initial arrangement \( \mathcal{E}_1 = \mathcal{E}(\mathcal{C}) \) are in the interior of the unit sphere \( \mathbb{S} \), the circle arrangement obtained by intersecting the moving planes \( \mathcal{E}_t \) with \( \mathbb{S} \) remains the same (isomorphic). Moreover, every circle in this arrangement converges to a great-circle as \( t \to +\infty \), and the statement follows.

The Great-Circle Theorem has several interesting consequences. The following corollary allows the transfer of results from the world of pseudolines into the world of (great-)pseudocircles.

**Corollary 4.1** An arrangement of pseudolines is stretchable if and only if the corresponding arrangement of great-pseudocircles is circularizable.

Since deciding stretchability of arrangements of pseudolines is known to be \( \exists \mathbb{R} \)-complete (see e.g. [21,22,25]), the hardness of stretchability directly carries over to hardness of circularizability. To show containment in \( \exists \mathbb{R} \), the circularizability problem has to be modeled with polynomial inequalities. This can be done by taking the centers and radii of the circles as variables and using polynomial inequalities to prescribe the order of the intersections along the circles.
Corollary 4.2  Deciding circularizability is \(\exists R\)-complete, even when the input is restricted to arrangements of great-pseudocircles.

It is known that all (not necessarily simple) arrangements of \(n \leq 8\) pseudolines are stretchable and that the simple non-Pappos arrangement is the unique non-stretchable simple projective arrangement of 9 pseudolines, see e.g. [8]. This again carries over to arrangements of great-pseudocircles.

Corollary 4.3  All arrangements of up to 8 great-pseudocircles are circularizable and the arrangement corresponding to the simple non-Pappos arrangement of pseudolines is the unique non-circularizable arrangement of 9 great-pseudocircles.

Note that the statement of the corollary also holds for the non-simple case. A non-simple arrangement of great-pseudocircles is an arrangement where three pseudocircles either form a Krupp or the intersection of the three pseudocircles consists of two points. Grünbaum [14] denoted such arrangements as “symmetric”.

Bokowski and Sturmfels [6] have shown that infinite families of minimal non-stretchable arrangements of pseudolines, i.e., non-stretchable arrangements where every proper subarrangement is stretchable, exist. Again, this carries over to arrangement of pseudocircles.

Corollary 4.4  There are infinite families of minimal non-circularizable arrangements of (great-)pseudocircles.

Mnëv’s Universality Theorem [22], see also [23], has strong implications for pseudoline arrangements and stretchability. Besides the hardness of stretchability, it also shows the existence of arrangements of pseudolines with a disconnected realization space, that is, there are isomorphic arrangements of lines such that there is no continuous transformation which transforms one arrangement into the other within the isomorphism class. Suvorov [31] gave an explicit example of two such arrangements on \(n = 13\) lines.

Corollary 4.5  There are circularizable arrangements of (great-)pseudocircles with a disconnected realization space.

5 Arrangements of Five Pseudocircles

In this section we prove Theorem 2.2.

Theorem 2.2  The four isomorphism classes of arrangements \(N_5^1, N_5^2, N_5^3,\) and \(N_5^4\) (shown in Fig. 3) are the only non-circularizable ones among the 984 isomorphism classes of connected arrangements of \(n = 5\) pseudocircles.

On the webpage [9] we have the data for circle realizations of 980 out of the 984 connected arrangements of 5 pseudocircles. The remaining four arrangements in this class are the four arrangements of Theorem 2.2. Since all arrangements with \(n \leq 4\) pseudocircles have circle representations, there are no disconnected non-circularizable
examples with \( n \leq 5 \). Hence, the four arrangements \( \mathcal{N}_5^1, \mathcal{N}_5^2, \mathcal{N}_5^3 \), and \( \mathcal{N}_5^4 \) are the only non-circularizable arrangements with \( n \leq 5 \). Since \( \mathcal{N}_5^2, \mathcal{N}_5^3, \) and \( \mathcal{N}_5^4 \) are not intersecting, \( \mathcal{N}_5^1 \) is the unique non-circularizable intersecting arrangement of 5 pseudocircles, this is Corollary 2.3.

5.1 Non-circularizability of \( \mathcal{N}_5^1 \)

The arrangement \( \mathcal{N}_5^1 \) is depicted in Figs. 3a and 12. Since Fig. 3a is meant to illustrate the symmetry of \( \mathcal{N}_5^1 \) while Fig. 12 illustrates our non-circularizability proof, the two drawings of \( \mathcal{N}_5^1 \) differ in the Euclidean plane. However, we have marked one of the cells of \( \mathcal{N}_5^1 \) by black cross in both figures to highlight the isomorphism.

For the non-circularizability proof of \( \mathcal{N}_5^1 \) we will make use of the following additional incidence lemma.

**Lemma 5.1** (Second Four-Circles Incidence Lemma) Let \( \mathcal{C} \) be an arrangement of four circles \( C_1, C_2, C_3, C_4 \) such that every pair of them is touching or forms a digon in \( \mathcal{C} \), and every circle is involved in at least two touchings. Then there is a circle \( C^* \) passing through the digon or touching point of each of the following pairs of circles \( (C_1, C_2), (C_2, C_3), (C_3, C_4), \) and \( (C_4, C_1) \) in this cyclic order.

**Proof** We first deal with the case where \( C_1 \) and \( C_3 \) form a digon. The assumptions imply that there is at most one further digon which might then be formed by \( C_2 \) and \( C_4 \). In particular, the four pairs mentioned in the statement of the lemma form touchings and, as illustrated in the first row of Fig. 11, we will find a circle \( C^* \) that is incident to those four touching points. In the following let \( p_{i,j} \) denote the touching point of \( C_i \) and \( C_j \).

Think of the circles as being in the extended complex plane. Apply a Möbius transformation \( \Gamma \) that maps one of the points of intersection of \( C_1 \) and \( C_3 \) to the point \( \infty \). This maps \( C_1 \) and \( C_3 \) to a pair of lines. The images of \( C_2 \) and \( C_4 \) are circles which touch the two lines corresponding to \( C_1 \) and \( C_3 \) and mutually either touch or form a digon. The first row of Fig. 11 gives an illustration. Since the centers of \( C_2 \) and \( C_4 \) lie on
the bisector $\ell$ of the lines $\Gamma(C_1)$ and $\Gamma(C_3)$, the touchings of $C_2$ and $C_4$ are symmetric with respect to $\ell$. Therefore, there is a circle $C$ with center on $\ell$ that contains the images of the four points $p_{1,2}$, $p_{2,3}$, $p_{3,4}$, and $p_{4,1}$. The circle $C^* = \Gamma^{-1}(C)$ contains the four points, i.e., they are cocircular.

Now we consider the case where $C_1$ and $C_3$ touch. Again we apply a Möbius transformation $\Gamma$ that sends $p_{1,3}$ to $\infty$. This maps $C_1$ and $C_3$ to parallel lines, each touched by one of $C_2$ and $C_4$. The second row of Fig. 11 shows that there is a circle $C$ such that $C^* = \Gamma^{-1}(C)$ has the claimed property. $\square$

**Proof (Non-circularizability of $N^1_5$)** Suppose for a contradiction that there is an isomorphic arrangement $C$ of circles. We label the circles as illustrated in Fig. 12. We apply the Digon Collapse Lemma (Lemma 3.3) to shrink $C_2$, $C_3$, and $C_4$ into their respective interiors. We also use the dual of the Digon Collapse Lemma for $C_1$. In the resulting subarrangement $C'$ formed by these four transformed circles $C'_1$, $C'_2$, $C'_3$, $C'_4$, each of the four circles is involved in at least two touchings. By applying Lemma 5.1 to $C'$ we obtain a circle $C^*$ which passes through the four points $p_{1,2}$, $p_{2,3}$, $p_{3,4}$, and $p_{4,1}$ (in this order) which respectively are touching points or points from the digons of $(C'_1, C'_2)$, $(C'_2, C'_3)$, $(C'_3, C'_4)$, and $(C'_4, C'_1)$.

Moreover, since the intersection of $C'_i$ and $C'_j$ in $C'$ is contained in the intersection of $C_i$ and $C_j$ in $C$, each of the four points $p_{12}$, $p_{23}$, $p_{34}$, and $p_{41}$ lies in the original digon of $C$. It follows that the circle $C_5$ has $p_{12}$ and $p_{34}$ in its interior but $p_{23}$ and $p_{41}$ in its exterior (cf. Fig. 12). By applying Lemma 5.1 to $C'$ we obtain a circle $C^*$ which passes through the points $p_{12}$, $p_{23}$, $p_{34}$, and $p_{41}$ (in this order). Now the two circles $C_5$ and $C^*$ intersect in four points. This is impossible, and hence $N^1_5$ is not circularizable. $\square$

### 5.2 Non-circularizability of the Connected Arrangements $N^2_5$, $N^3_5$, and $N^4_5$

The non-circularizability of $N^2_5$ has been shown by Linhart and Ortner [19]. We give an alternative proof which also shows the non-circularizability of $N^3_5$. The two arrangements $N^2_5$ and $N^3_5$ are depicted in Fig. 3b and c, and also in Fig. 13a and b.

**Proof (Non-circularizability of $N^2_5$ and $N^3_5$)** Suppose for a contradiction that there is an isomorphic arrangement $C$ of circles. We label the circles as illustrated in Fig. 13.
Since the respective interiors of $C_1$ and $C_3$ are disjoint, we can apply the Digon Collapse Lemma (Lemma 3.3) to $C_1$ and $C_3$. This yields an arrangement $C'$ with four touching points $p_{12}$, $p_{23}$, $p_{34}$, $p_{41}$, where $p_{ij}$ is the touching point of $C'_i$ and $C'_j$.

From Lemma 3.1 it follows that there is a circle $C^*$ which passes through the points $p_{12}$, $p_{23}$, $p_{34}$, and $p_{41}$ in this cyclic order. Since the point $p_{ij}$ lies inside the digon formed by $C_i$ and $C_j$ in the arrangement $C$, it follows that the circle $C_5$ has $p_{12}$, $p_{34}$ in its interior and $p_{23}$, $p_{41}$ in its exterior. Therefore, the two circles $C_5$ and $C^*$ intersect in four points. This is impossible and, therefore, $N^2_5$ and $N^3_5$ are not circularizable. □

It remains to prove that $N^4_5$ (shown in Figs. 3d and 15) is not circularizable. In the proof we make use of the following incidence lemma.

**Lemma 5.2** (Third Four-Circles Incidence Lemma) Let $C$ be an arrangement of four circles $C_1$, $C_2$, $C_3$, $C_4$ such that $(C_1, C_2)$, $(C_2, C_3)$, $(C_4, C_1)$, and $(C_3, C_4)$ are touching, moreover, $C_4$ is in the interior of $C_1$ and the exterior of $C_3$, and $C_2$ is in the interior of $C_3$ and the exterior of $C_1$, see Fig. 14. Then there is a circle $C^*$ passing through the four touching points in the given cyclic order.

**Proof** Since $C_1$ is touching $C_2$ and $C_4$ which are respectively inside and outside $C_3$, the two circles $C_1$ and $C_3$ intersect. Apply a Möbius transformation $\Gamma$ that maps a crossing point of $C_1$ and $C_3$ to the point $\infty$ of the extended complex plane. This maps $C_1$ and $C_3$ to a pair $L_1$, $L_3$ of lines. The images $C'_2$, $C'_4$ of $C_2$ and $C_4$ are separated.
by the lines \(L_1, L_3\) and each of them is touching both lines. Figure 14 illustrates the situation. The figure also shows that a circle \(C’\) through the four touching points exists. The circle \(C^* = \Gamma^{-1}(C’)\) has the claimed properties.

\(\blacksquare\)

**Proof** (Non-circularizability of \(N^4_5\)) Suppose for a contradiction that there is an isomorphic arrangement \(\mathcal{C}\) of circles. We label the circles as illustrated in Fig. 15. We shrink the circles \(C_2\) and \(C_4\) so that each of the pairs \((C_1, C_2), (C_2, C_3), (C_3, C_4),\) and \((C_4, C_1)\) touch. (Note that each of these pairs forms a digon in \(\mathcal{C}\).) With these touchings the four circles \(C_1, C_2, C_3, C_4\) form the configuration of Lemma 5.2. Hence there is a circle \(C^*\) containing the four touching points in the given cyclic order. Now the two circles \(C^*\) and \(C_5\) intersect in four points. This is impossible and, therefore, \(N^4_5\) is not circularizable.

(\(\blacksquare\))

### 6 Intersecting Digon-Free Arrangements of Six Pseudocircles

In this section we prove Theorem 2.4.

**Theorem 2.4** The three isomorphism classes of arrangements \(N^{\Delta}_6, N^2_6, \) and \(N^3_6\) (shown in Fig. 4) are the only non-circularizable ones among the 2131 isomorphism classes of digon-free intersecting arrangements of \(n = 6\) pseudocircles.

We remark that all three arrangements do not have the intersecting arrangement \(N^1_5\) as a subarrangement—otherwise non-circularizability would follow directly. In fact, \(N^1_5\) has no extension to an intersecting digon-free arrangement of six pseudocircles.

On the webpage [9] we have the data of circle realizations of all 2131 intersecting digon-free arrangements of 6 pseudocircles except for the three arrangements mentioned in Theorem 2.4. In the following, we present two non-circularizability proofs for \(N^{\Delta}_6\) and \(N^2_6\), respectively, and a non-circularizability proof for \(N^3_6\).

#### 6.1 Non-circularizability of \(N^\Delta_6\)

The arrangement \(N^\Delta_6\) (shown in Fig. 4a) is an intersecting digon-free arrangement. Our interest in \(N^\Delta_6\) was originally motivated by our study [10] of arrangements of pseudocircles with few triangles. From a computer search we know that \(N^\Delta_6\) occurs as

![Fig. 15 Illustration of the non-circularizability proof of the arrangement \(N^4_5\). The circle \(C^*\) is drawn dashed](image-url)
a subarrangement of every digon-free arrangement for \( n = 7, 8, 9 \) with \( p_3 < 2n - 4 \) triangles. Since \( \mathcal{N}_6^\Delta \) is not circularizable, neither are these arrangements. It thus seems plausible that for every arrangement of \( n \) circles \( p_3 \geq 2n - 4 \). This is the Weak Grünbaum Conjecture stated in [10].

Our first proof is an immediate consequence of the following theorem, whose proof resembles the proof of the Great-Circle Theorem (Theorem 2.1).

**Theorem 6.1** Let \( \mathcal{A} \) be a connected digon-free arrangement of pseudocircles. If every triple of pseudocircles which forms a triangle is NonKrupp, then \( \mathcal{A} \) is not circularizable.

**Proof** Assume for a contradiction that there exists an isomorphic arrangement of circles \( \mathcal{C} \) on the unit sphere \( \mathbb{S} \). Let \( \mathcal{E}(\mathcal{C}) \) be the arrangements of planes spanned by the circles of \( \mathcal{C} \).

Imagine the planes of \( \mathcal{E}(\mathcal{C}) \) moving away from the origin. To be precise, for time \( t \geq 1 \) let \( \mathcal{E}_t := \{ t \cdot E : E \in \mathcal{E}(\mathcal{C}) \} \). Consider the arrangement induced by intersecting the moving planes \( \mathcal{E}_t \) with the unit sphere \( \mathbb{S} \). Since \( \mathcal{C} \) has NonKrupp triangles, it is not a great-circle arrangement and some planes of \( \mathcal{E}(\mathcal{C}) \) do not contain the origin. All planes from \( \mathcal{E}(\mathcal{C}) \), which do not contain the origin, will eventually lose the intersection with \( \mathbb{S} \), hence some event has to happen.

When the isomorphism class of the intersection of \( \mathcal{E}_t \) with \( \mathbb{S} \) changes, we see a triangle flip, or a digon flip, or some isolated circle disappears. Since initially there is no digon and no isolated circle, the first event is a triangle flip. By assumption, triangles of \( \mathcal{C} \) correspond to NonKrupp subarrangements, hence, the intersection point of their planes is outside of \( \mathbb{S} \) (Fact 1). This shows that a triangle flip event is also impossible. This contradiction implies that \( \mathcal{A} \) is non-circularizable. \( \square \)

**Proof** (First proof of non-circularizability of \( \mathcal{N}_6^\Delta \)) The arrangement \( \mathcal{N}_6^\Delta \) is intersecting, digon-free, and each of the eight triangles of \( \mathcal{N}_6^\Delta \) is formed by three circles which are a NonKrupp configuration. Hence, Theorem 6.1 implies that \( \mathcal{N}_6^\Delta \) is not circularizable. \( \square \)

All arrangements known to us whose non-circularizability can be shown with Theorem 6.1 contain \( \mathcal{N}_6^\Delta \) as a subarrangement—which already shows non-circularizability. Based on this data we venture the following conjecture:

**Conjecture 6.2** Every connected digon-free arrangement \( \mathcal{A} \) of pseudocircles with the property, that every triple of pseudocircles which forms a triangle in \( \mathcal{A} \) is NonKrupp, contains \( \mathcal{N}_6^\Delta \) as a subarrangement.

Our second proof of non-circularizability of \( \mathcal{N}_6^\Delta \) is based on an incidence theorem for circles (Theorem 6.3) which is a consequence of an incidence theorem for points and planes in 3-space (Theorem 2.5). Before going into details, let us describe the geometry of the arrangement \( \mathcal{N}_6^\Delta \): Consider the non-simple arrangement \( \mathcal{A}^\bullet \) obtained from \( \mathcal{N}_6^\Delta \) by contracting each of the eight triangles into a single point of triple intersection. The arrangement \( \mathcal{A}^\bullet \) is circularizable. A realization is obtained by taking a cube inscribed in the sphere \( S \) such that each of the eight corners is touching the sphere \( S \). The arrangement \( \mathcal{A}^\bullet \) is the intersection of \( S \) with the six planes which are spanned by pairs of diagonally opposite edges of the cube.
Theorem 2.5 Let \( a, b, c, d, w, x, y, z \) be 8 points in \( \mathbb{R}^3 \) such that \( a, b, c, d \) are affinely independent and each of the following 5 subsets of 4 points is coplanar:

\[
\{a, b, w, x\}, \{a, c, w, y\}, \{a, d, w, z\}, \{b, c, x, y\}, \text{ and } \{b, d, x, z\}.
\]

Then \( \{c, d, y, z\} \) is also coplanar.

Proof With an affine transformation we can make \( a, b, c, d \) the corners of a unit tetrahedron. We then embed the point configuration (in \( \mathbb{R}^3 \)) into \( \mathbb{R}^4 \) so that the four points become the elements of the standard basis, namely, \( a = e_1, b = e_2, c = e_3, \) and \( d = e_4. \) Now coplanarity of 4 points can be tested by evaluating the determinant. Coplanarity of \( \{a, b, w, x\} \) yields \( \det(abwx) = \det(e_1, e_2, w, x) = 0. \) On the basis of the 5 collinear sets, we get the following determinants and equations:

\[
\begin{align*}
\det(abwx) &= 0 \quad \det(cayw) = 0 \quad \det(adwz) = 0 \quad \det(chyx) = 0 \quad \det(bdxz) = 0 \\
w_3x_4 &= w_4x_3 \quad w_4y_2 = w_2y_4 \quad w_2z_3 = w_3z_2 \quad x_1y_4 = x_4y_1 \quad x_3z_1 = x_1z_3
\end{align*}
\]

Take the product of the left sides of the six equations and the product of the right sides. These products are the same. Canceling as much as possible from the resulting equations yields \( y_2z_1 = y_1z_2. \) This implies that \( \det(e_3, e_4, y, z) = 0, \) i.e., the coplanarity of \( \{c, d, y, z\}. \)

The theorem implies the following incidence theorem for circles.

Theorem 6.3 Let \( C_1, C_2, C_3, C_4 \) be four circles and let \( a, b, c, d, w, x, y, z \) be eight distinct points in \( \mathbb{R}^2 \) such that \( C_1 \cap C_2 = \{a, w\}, C_3 \cap C_4 = \{b, x\}, C_1 \cap C_3 = \{c, y\}, \) and \( C_2 \cap C_4 = \{d, z\}. \) If there is a circle \( C \) containing \( a, b, w, x, \) then there is a circle \( C' \) containing \( c, d, y, z. \) Moreover, if one of the triples of \( C_1, C_2, C_3, C_4 \) forms a Krupp, then \( c, d, y, z \) represents the circular order on \( C'. \)

Proof Consider the arrangement of circles on the sphere. The idea is to apply Theorem 2.5. The coplanarity of the 5 sets follows because the respective 4 points belong to \( C, C_1, C_2, C_3, C_4 \) in this order. The points \( a, b, c, d, z \) are affinely independent because those five points would otherwise lie on a common plane, whence \( C_2 = C_4 \) (a contradiction to \( C_2 \cap C_4 = \{d, z\} \)). Since \( d \) and \( z \) can be exchanged, we can assume without loss of generality that \( a, b, c, d \) are affinely independent. This shows that Theorem 2.5 can be applied. Regarding the circular order on \( C' \), suppose that \( C_1, C_2, C_3 \) is a Krupp. This implies that \( C_2 \) separates \( c \) and \( y. \) Since \( C' \cap C_2 = \{d, z\} \) the points of \( \{c, y\} \) and \( \{d, z\} \) alternate on \( C' \), this is the claim.

It is worth mentioning that the second part of the theorem can be strengthened: If one of the triples of \( C_1, C_2, C_3, C_4 \) forms a Krupp, then the arrangement together with \( C \) and \( C' \) is isomorphic to the simplicial arrangement \( \mathcal{A}^* \) obtained from \( N_6^A \) by contracting the eight triangles into triple intersections. A simplicial arrangement is a non-simple arrangement where all cells are triangles. The arrangement \( \mathcal{A}^* \) can be extended to larger simplicial arrangements by adding any subset of the three circles \( C_1^*, C_2^*, C_3^* \) which are defined as follows: \( C_1^* \) is the circle through the four points...
\((C_1 \cap C_4) \cup (C_2 \cap C_3); C^*_2\) is the circle through the four points \((C_1 \cap C_4) \cup (C \cap C')\); \(C^*_3\) is the circle through the four points \((C_2 \cap C_3) \cup (C \cap C')\). In each case the cocircularity of the four points defining \(C^*_i\) is a consequence of the theorem.

The following lemma is similar to the Digon Collapse Lemma (Lemma 3.3). By changing interior to exterior and outside to inside (by applying a Möbius transformation), we obtain a dual version also for this lemma.

**Lemma 6.4** (Triangle Collapse Lemma) Let \(\mathcal{C}\) be an arrangement of circles in the plane and let \(C\) be one of the circles of \(\mathcal{C}\), which intersects at least three other circles from \(\mathcal{C}\) and does not fully contain any other circle from \(\mathcal{C}\) in its interior. If \(C\) has no incident digon in its interior, then we can continuously transform \(C\) so that the combinatorics of the arrangement remains except that two triangles collapse to points of triple intersection. Moreover, it is possible to prevent a fixed triangle \(T\) incident to \(C\) from being the first one to collapse.

**Proof** Shrink the radius of \(C\) until the first flip occurs, this must be a triangle flip, i.e., a triangle is reduced to a point of triple intersection. If \(C\) has a point \(p\) of triple intersection, shrink \(C\) towards \(p\), i.e., shrink the radius and simultaneously move the center towards \(p\) so that \(p\) stays incident to \(C\). With the next flip a second triangle collapses.

For the extension let \(q \in T \cap C\) be a point. Start the shrinking process by shrinking \(C\) towards \(q\). This prevents \(T\) from collapsing. \(\square\)

**Proof** (Second proof of non-circularizability of \(\mathcal{N}_{6}^\Delta\)) Suppose for a contradiction that \(\mathcal{N}_{6}^\Delta\) has a realization \(\mathcal{C}\). Each circle of \(\mathcal{C}\) has exactly two incident triangles in the inside and exactly two on the outside. Apply Lemma 6.4 to \(C_5\) and to \(C_3\) (we refer to the circles with the colors and labels used in Fig. 16). This collapses the triangles labeled \(a, b, x, w\), i.e., all the triangles incident to \(C_5\). Now the green circle \(C_1\), the magenta circle \(C_2\), the black circle \(C_3\), the blue circle \(C_4\), and the red circle \(C_5\) are precisely the configuration of Theorem 6.3 with \(C_5\) in the role of \(C\). The theorem implies that there is a circle \(C'\) containing the green-black crossing at \(c\), the blue-magenta crossing at \(d\), the green-black crossing at \(y\), and the blue-magenta crossing at \(z\) in this order. Each consecutive pair of these crossings is on different sides of the yellow circle \(C_6\), hence, there are at least four crossings between \(C'\) and \(C_6\). This is impossible for circles, whence, there is no circle arrangement \(\mathcal{C}\) realizing \(\mathcal{N}_{6}^\Delta\). \(\square\)

**Fig. 16** The arrangement \(\mathcal{N}_{6}^\Delta\) with a labeling of its eight triangles
6.2 Non-circularizability of $\mathcal{N}_6^2$

The arrangement $\mathcal{N}_6^2$ is shown in Figs. 4b and 18a. We give two proofs for the non-circularizability of $\mathcal{N}_6^2$. The first one is an immediate consequence of the following theorem, which—in the same flavor as Theorem 6.1—can be obtained similarly as the proof of the Great-Circle Theorem (Theorem 2.1).

**Theorem 6.5** Let $A$ be an intersecting arrangement of pseudocircles which is not an arrangement of great-pseudocircles. If every triple of pseudocircles which forms a triangle is Krupp, then $A$ is not circularizable.

We outline the proof: Suppose a realization of $A$ exists on the sphere. Continuously move the planes spanned by the circles towards the origin. The induced arrangement will eventually become isomorphic to an arrangement of great-circles. Now consider the first event that occurs. As the planes move towards the origin, there is no digon collapse. Since $A$ is intersecting, no digon is created, and, since all triangles are Krupp, the corresponding intersection points of their planes is already inside $S$. Therefore, no event can occur—a contradiction.

**Proof** (First proof of non-circularizability of $\mathcal{N}_6^2$) The arrangement $\mathcal{N}_6^2$ is intersecting but not an arrangement of great-pseudocircles ($\mathcal{N}_6^2$ contains digons while arrangements of $n \geq 3$ great-pseudocircles are digon-free) and each triangle in $\mathcal{N}_6^2$ is Krupp. Hence, Theorem 6.5 implies that $\mathcal{N}_6^2$ is not circularizable.

Besides $\mathcal{N}_6^2$, there is exactly one other arrangement of 6 pseudocircles (with digons, see Fig. 17) where Theorem 6.5 implies non-circularizability. For $n = 7$ there are eight arrangements where the theorem applies; but each of them has one of the two $n = 6$ arrangements as a subarrangement.

Our second proof of non-circularizability of $\mathcal{N}_6^2$ is based on Theorem 3.2.

**Proof** (Second proof of non-circularizability of $\mathcal{N}_6^2$) Suppose that $\mathcal{N}_6^2$ has a representation as a circle arrangement $C$. We refer to circles and intersection points via the labels of the corresponding objects in Fig. 18b.

Let $\ell_i$ be the line spanned by $p_i$ and $p_i'$ for $i = 1, 2, 3$. The directed line $\ell_1$ intersects $C_4$ and $C_5$ in the points $p_1$, $p_1'$ and has its second intersection with the yellow circle.
C_2 between these points. After p'_1, the line has to cross C_3 (magenta), C_1 (black), and C_6 (red) in this order, i.e., the line behaves as shown in the figure. Similarly \ell_2 and \ell_3 behave as shown. Let T be the triangle spanned by the intersection points of the three lines \ell_1, \ell_2, \ell_3. Observe that the gray interior triangle T' of \mathcal{C} is fully contained in T. By applying Theorem 3.2 to the circles C_4, C_5, C_6, we obtain that \ell_1, \ell_2, \ell_3 meet in a common point, and therefore, T and T' are degenerated. This contradicts the assumption that \mathcal{C} is a realization of \mathcal{N}_6^2, whence this arrangement is non-circularizable.

\[\square\]

6.3 Non-circularizability of \(\mathcal{N}_6^3\)

The arrangement \(\mathcal{N}_6^3\) is shown in Figs. 4c and 19b. To prove its non-circularizability, we again use an incidence lemma. The following lemma is mentioned by Richter-Gebert as a relative of Pappos’s Theorem, cf. [24, p. 26]. Fig. 19a gives an illustration.

**Lemma 6.6** Let \(\ell_1, \ell_2, \ell_3\) be lines, \(C'_1, C'_2, C'_3\) be circles, and \(p_1, p_2, p_3, q_1, q_2, q_3\) be points, such that for \(\{i, j, k\} = \{1, 2, 3\}\) point \(p_i\) is incident to line \(\ell_i\), circle \(C'_j\), and circle \(C'_k\), while point \(q_i\) is incident to circle \(C'_i\), line \(\ell_j\), and line \(\ell_k\). Then \(C'_1, C'_2, \) and \(C'_3\) have a common point of intersection.

![Fig. 18](image1.png) (a) The non-circularizable arrangement \(\mathcal{N}_6^2\). (b) An illustration for the proof

![Fig. 19](image2.png) (a) An illustration for Lemma 6.6. (b) The non-circularizable arrangement \(\mathcal{N}_6^2\) with 3 dashed pseudolines illustrating the proof
Proof (Non-circularizability of $N_6^3$) Suppose that $N_6^3$ has a representation $C$ as a circle arrangement in the plane. We refer to circles and intersection points via the label of the corresponding object in Fig. 19b. As in the figure, we assume without loss of generality that the triangular cell spanned by $C_4$, $C_5$, and $C_6$ is the outer cell of the arrangement.

Consider the region $R := R_{24} \cup R_{35}$ where $R_{ij}$ denotes the intersection of the respective interiors of $C_i$ and $C_j$. The two straight-line segments $p_1p_1'$ and $p_3p_3'$ are fully contained in $R_{35}$ and $R_{24}$, respectively, and have alternating end points along the boundary of $R$, hence they cross inside the region $R_{24} \cap R_{35}$.

From rotational symmetry we obtain that the three straight-line segments $p_1p_1'$, $p_2p_2'$, and $p_3p_3'$ intersect pairwise.

For $i = 1, 2, 3$, let $\ell_i$ denote the line spanned by $p_i$ and $p_i'$, let $q_i$ denote the intersection point of $\ell_i+1$ and $\ell_i+2$, and let $C'_i$ denote the circle spanned by $q_i$, $p_{i+1}$, $p_{i+2}$ (indices modulo 3). Note that $\ell_i$ contains $p_i, q_{i+1}, q_{i+2}$. These are precisely the conditions for the incidences of points, lines, and circles in Lemma 6.6. Hence, the three circles $C'_1, C'_2$, and $C'_3$ intersect in a common point (cf. Fig. 19a).

Let $T$ be the triangle with corners $p_1, p_2, p_3$. Since $p_2$ and $p_3$ are on $C_1$, and $q_1$ lies inside of $C_1$, we find that the intersection of the interior of $C'_1$ with $T$ is a subset of the intersection of the interior of $C_1$ with $T$. The respective containments also hold for $C'_2$ and $C_2$ and for $C'_3$ and $C_3$. Moreover, since $C'_1, C'_2$, and $C'_3$ intersect in a common point, the union of the interiors of $C'_1, C'_2$, and $C'_3$ contains $T$. Hence, the union of interiors of the $C_1, C_2$, and $C_3$ also contains $T$. This shows that in $C$ there is no face corresponding to the gray triangle; see Fig. 19b. This contradicts the assumption that $C$ is a realization of $N_6^3$, whence the arrangement is non-circularizable.

7 Additional Arrangements with $n = 6$

In the previous two sections we have exhibited all non-circularizable arrangements with $n \leq 5$ and all non-circularizable intersecting digon-free arrangements with $n = 6$. With automatized procedures we managed to find circle representations of 98% of the connected digon-free arrangements and of 90% of the intersecting arrangements of 6 pseudocircles. Unfortunately, the numbers of remaining candidates for non-circularizability are too large to complete the classification by hand. In this section we show non-circularizability of a few of the remaining examples which we consider to be interesting. As a criterion for being interesting we used the order of the symmetry group of the arrangement. The symmetry groups have been determined as the automorphism groups of the primal-dual graphs using SageMath [28,30].

In Sect. 7.1 we show non-circularizability of the three intersecting arrangements of $n = 6$ pseudocircles (with digons) depicted in Fig. 20. The symmetry group of these three arrangements is of order 6. All the remaining examples of intersecting arrangements with $n = 6$, where we do not know about circularizability, have a symmetry group of order at most 3.

In Sect. 7.2 we show non-circularizability of the three connected digon-free arrangements of 6 pseudocircles depicted in Fig. 24. The symmetry group of these three arrangements is of order 24 or 8. In Sect. 7.2.3 we show non-circularizability of two
additional connected digon-free arrangements of 6 pseudocircles. The examples are shown in Fig. 27, the symmetry group of these two arrangements is of order 4. All the remaining examples of connected digon-free arrangements with \( n = 6 \), where we do not know about circularizability, have a symmetry group of order 2 or 1.

7.1 Non-circularizability of Three Intersecting Arrangements with \( n = 6 \)

In this subsection we prove non-circularizability of the three arrangements \( N_{6}^{ER} \), \( N_{6}^{i6:2} \), and \( N_{6}^{i6:3} \) shown in Fig. 20. The non-circularizability of \( N_{6}^{ER} \) was already shown by Edelsbrunner and Ramos [7], the name of the arrangement reflects this fact. The other names are built, so that the subscript of the \( N \) is the number of pseudocircles, the first part of the superscript indicates that the arrangement is intersecting with a symmetry group of order 6, and the number after the colon is the counter. Accordingly, the arrangement \( N_{6}^{ER} \) can also be denoted as \( N_{6}^{i6:1} \).

7.1.1 Non-circularizability of the Edelsbrunner–Ramos Example \( N_{6}^{ER} \)

The arrangement \( N_{6}^{ER} \) is shown in Fig. 20a. As in the original proof [7] the argument is based on considerations involving angles.

Figure 21a shows a representation of the arrangement \( N_{6}^{ER} \) consisting of a sub-arrangement \( A_{O} \) formed by the three outer pseudocircles \( C_{1}, C_{2}, C_{3} \) and a second subarrangement \( A_{I} \) formed by the three inner circles \( C_{4}, C_{5}, C_{6} \).

Suppose that there is a circle representation \( \mathcal{C} \) of \( N_{6}^{ER} \). Let \( \mathcal{C}_{O} \) and \( \mathcal{C}_{I} \) be the subarrangements of \( \mathcal{C} \) which represent \( A_{O} \) and \( A_{I} \), respectively. For each outer circle \( C_{i} \) from \( \mathcal{C}_{O} \) consider a straight-line segment \( s_{i} \) that connects two points from the two digons which are formed by \( C_{i} \) with inner circles. The segment \( s_{i} \) is fully contained in \( C_{i} \). Let \( \ell_{i} \) be the line supporting \( s_{i} \) and let \( T \) be the triangle bounded by \( \ell_{1}, \ell_{2}, \) and \( \ell_{3} \).

We claim that \( T \) contains the inner triangle of \( \mathcal{C}_{O} \). Indeed, if three circles form a NonKrupp where the outer face is a triangle and with each circle we have a line which intersects the two digons incident to the circle, then the three lines form a triangle containing the inner triangular cell of the NonKrupp arrangement.

The inner triangle of \( \mathcal{C}_{O} \) contains the four inner triangles of \( \mathcal{C}_{I} \). Let \( a, b, c \) be the three crossing points on the outer face of the subarrangement \( \mathcal{C}_{I} \). Comparing the inner
angle at $a$, a crossing of $C_4$ and $C_5$, and the corresponding angle of $T$, i.e., the angle formed by $\ell_1$ and $\ell_2$, we claim that the inner angle at $a$ is smaller. To see this let us assume that the common tangent $h$ of $C_4$ and $C_5$ on the side of $a$ is horizontal. Line $\ell_1$ has both crossings with $C_5$ above $a$ and also intersects with $C_6$. This implies that the slope of $\ell_1$ is positive but smaller than the slope of the tangent at $C_5$ in $a$. Alike the slope of $\ell_2$ is negative but larger than the slope of the tangent at $C_4$ in $a$. This is the claim, see Fig. 21b.

The respective statements hold for the inner angles at $b$ and $c$, and the corresponding angles of $T$. Since the sum of angles of $T$ is $\pi$, we conclude that the sum of the inner angles at $a$, $b$, and $c$ is less than $\pi$.

The sum of inner angles at $a$, $b$, $c$ equals the sum of inner angles at $a'$, $b'$, $c'$, see Fig. 21c. This sum, however, clearly exceeds $\pi$. The contradiction shows that $\mathcal{N}^{ER}_6$ is not circularizable.

### 7.1.2 Non-circularizability of $\mathcal{N}^{16:2}_6$

The arrangement $\mathcal{N}^{16:2}_6$ is shown in Fig. 20b and again in Fig. 22a. This figure also shows some shaded triangles, three of them are gray and three are pink.

Suppose that $\mathcal{N}^{16:2}_6$ has a circle representation $\mathcal{C}$. Each of $C_1$, $C_2$, and $C_3$ has two triangles and no digon on its interior boundary. One of the two triangles is gray, the
other pink. Lemma 6.4 allows to shrink the three circles $C_1$, $C_2$, $C_3$ of $C$ into their respective interiors so that in each case the shrinking makes a pink triangle collapse. Let $p_i$ be the point of triple intersection of $C_i$, for $i = 1, 2, 3$. Further shrinking $C_i$ towards $p_i$ makes another triangle collapse. At this second collapse two triangles disappear, one of them a gray one, and $C_i$ gets incident to $p_{i-1}$ (with the understanding that $1 - 1 = 3$). Having done this for each of the three circles yields a circle representation for the (non-simple) arrangement shown in Fig. 22b.

To see that this arrangement has no circle representation apply a Möbius transformation that maps the point $p_1$ to the point $\infty$ of the extended complex plane. This transforms the four circles $C_1, C_2, C_4, C_5$, which are incident to $p_1$, into lines. The two remaining circles $C_3$ and $C_6$ intersect in $p_2$ and $p_3$. The lines of $C_2$ and $C_5$ both have their second intersections with $C_3$ and $C_6$ separated by $p_2$, hence, they both avoid the lens formed by $C_3$ and $C_6$. The line of $C_1$ has its intersections with $C_2$ and $C_5$ in the two components of the gray double-wedge of $C_2$ and $C_5$, see Fig. 22c. Therefore, the slope of $C_1$ belongs to the slopes of the double-wedge. However, the line of $C_1$ has its second intersections with $C_3$ and $C_6$ on the same side of $p_3$ and, therefore, it has a slope between the tangents of $C_3$ and $C_6$ at $p_3$. These slopes do not belong to the slopes of the gray double-wedge. This contradiction shows that a circle representation of $\mathcal{N}_6^{16:2}$ does not exist.

7.1.3 Non-circularizability of $\mathcal{N}_6^{16:3}$

The arrangement $\mathcal{N}_6^{16:3}$ is shown in Fig. 20c and again in Fig. 23a. This figure also shows some shaded triangles, three of them are gray and three are pink.

Suppose that $\mathcal{N}_6^{16:2}$ has a circle representation $\mathcal{C}$. Each of $C_1$, $C_2$ and $C_3$ has two triangles and no digon on its exterior boundary. One of the two triangles is gray, the other is pink (we disregard the exterior triangle because it will not appear in a first flip when expanding a circle). The dual form of Lemma 6.4 allows to expand the three circles $C_1, C_2, C_3$ of $\mathcal{C}$ into their respective exteriors so that in each case the expansion makes a pink triangle collapse.

Figure 23b shows a pseudocircle representation of the arrangement after this first phase of collapses. In the second phase, we modify the circles $C_i$, for $i = 4, 5, 6$. We explain what happens to $C_5$, the two other circles are treated alike with respect to the rotational symmetry. Consider the circle $C_5'$, which contains $p_1$ and shares $p_3$ and

![Fig. 23](image-url) (a) The arrangement $\mathcal{N}_6^{16:3}$ with some triangle faces emphasized. (b) After collapsing the pink shaded triangles. (c) A detail of the arrangement after the second phase of collapses.
the red point with $C_5$. This circle is obtained by shrinking $C_5$ on one side of the line containing $p_3$ and the red point, and by expanding $C_5$ on the other side of the line. It is easily verified that the collapse of the triangle at $p_1$ is the first event in this process.

Figure 23c shows the inner triangle formed by $C_1$, $C_2$, and $C_3$ together with parts of $C'_4$, $C'_5$, and $C'_6$. At each of the three points, the highlighted red angle is smaller than the highlighted gray angle. However, the red angle at $p_i$ is formed by the same two circles as the gray angle at $p_{i+1}$, whence, the two angles are equal. This yields a contradictory cyclic chain of inequalities. The contradiction shows that a circle representation $\mathcal{C}$ of $\mathcal{N}_6^{6:3}$ does not exist.

7.2 Non-circularizability of Three Connected Digon-Free Arrangements with $n = 6$

In this subsection we prove non-circularizability of the three arrangements $\mathcal{N}_6^{c:24}$, $\mathcal{N}_6^{c:8:1}$, and $\mathcal{N}_6^{c:8:2}$ shown in Fig. 24.

7.2.1 Non-circularizability of $\mathcal{N}_6^{c:24}$ and $\mathcal{N}_6^{c:8:1}$

The proof of non-circularizability of the two arrangements is based on Miquel’s Theorem. For proofs of the theorem we refer to [24].

**Theorem 7.1** (Miquel’s Theorem) Let $C_1$, $C_2$, $C_3$, $C_4$ be four circles and let $C_1 \cap C_2 = \{a, w\}$, $C_2 \cap C_3 = \{b, x\}$, $C_3 \cap C_4 = \{c, y\}$, and $C_4 \cap C_1 = \{d, z\}$. If there is a circle $C$ containing $a, b, c, d$, then there is a circle $C'$ containing $w, x, y, z$.

The arrangement $\mathcal{N}_6^{c:24}$ is shown in Fig. 24a and again in Fig. 25a. Suppose that $\mathcal{N}_6^{c:24}$ has a circle representation $\mathcal{C}$. Circle $C_5$ has six triangles in its exterior. These triangles are all incident to either the crossing of $C_1$ and $C_4$ or the crossing of $C_2$ and $C_3$. Hence, by Lemma 6.4 we can grow $C_5$ into its exterior to get two triple intersection points $p_1 = C_1 \cap C_4 \cap C_5$ and $p_2 = C_2 \cap C_3 \cap C_5$. The situation in the interior of $C_6$ is identical to the situation in the exterior of $C_5$. Hence, by shrinking $C_6$ we get two additional triple intersection points $p_3 = C_1 \cap C_2 \cap C_6$ and $p_4 = C_3 \cap C_4 \cap C_6$. This yields the (non-simple) arrangement shown in Fig. 25b. Now grow the circles $C_1$, $C_2$, $C_3$, $C_4$ to the outside while keeping each of them incident to its two points $p_i$, this makes them shrink into their inside at the ‘short arc’. Upon this growth process, the gray crossings

![Fig. 24](https://example.com/fig24.png)

**Fig. 24** A digon-free connected arrangement of $n = 6$ pseudocircles with symmetry group of order 24, and two with a symmetry group of order 8: (a) $\mathcal{N}_6^{c:24}$, (b) $\mathcal{N}_6^{c:8:1}$, (c) $\mathcal{N}_6^{c:8:2}$
\[p_{13}, p_{23}, p_{34}, \text{ and } p_{14}\] move away from the blue circle \(C_6\). Hence the process can be continued until the upper and the lower triangles collapse, i.e., until \(p_{12}\) and \(p_{34}\) both are incident to \(C_5\). Note that we do not care about \(p_{23}\) and \(p_{14}\), they may have passed to the other side of \(C_5\). The collapse of the upper and the lower triangle yields two additional triple intersection points \(C_1 \cap C_2 \cap C_5\) and \(C_3 \cap C_4 \cap C_5\). The circles \(C_1, C_2, C_3, C_4\) together with \(C_5\) in the role of \(C\) form an instance of Miquel’s Theorem (Theorem 7.1). Hence, there is a circle \(C'\) traversing the four points \(p_3, p_{14}, p_4, p_{23}\). The points \(p_3, p_4\) partition \(C'\) into two arcs, each containing one of the gray points \(p_{14}, p_{23}\). Now \(C'\) shares the points \(p_3\) and \(p_4\) with \(C_6\) while the gray points \(p_{14}, p_{23}\) are outside of \(C_6\). This is impossible, whence, there is no circle representation of \(N_6^{c24}\).

The arrangement \(N_6^{c8:1}\) is shown in Fig. 24b and again in Fig. 25c. The proof of non-circularizability of this arrangement is exactly as the previous proof, just replace \(N_6^{c24}\) by \(N_6^{c8:1}\) and think of an analog of Fig. 25b.

Originally, we were aiming at deriving the non-circularizability of \(N_6^{c24}\) as a corollary to the following theorem. Turning things around we now prove it as a corollary to the non-circularizability of \(N_6^{c24}\). We say that a polytope \(P\) has the combinatorics of the cube if \(P\) and the cube have isomorphic face lattices. The graph of the cube is bipartite, hence, we can speak of the white and black vertices of a polytope with the combinatorics of the cube.

**Theorem 7.2** Let \(S\) be a sphere. There is no polytope \(P\) with the combinatorics of the cube such that the black vertices of \(P\) are inside \(S\) and the white vertices of \(P\) are outside \(S\).

**Proof** Suppose that there is such a polytope \(P\). Let \(E\) be the arrangement of planes spanned by the six faces of \(P\) and let \(C\) be the arrangement of circles obtained from the intersection of \(E\) and \(S\). This arrangement is isomorphic to \(N_6^{c24}\). To see this, consider the eight triangles of \(N_6^{c24}\) corresponding to the black and gray points of Fig. 25b. Triangles corresponding to black points are Krupp and triangles corresponding to gray points are NonKrupp. By Fact 1 this translates to corners of \(P\) being outside and inside \(S\), respectively.

\[\square\]

### 7.2.2 Non-circularizability of \(N_6^{c8:2}\)

The arrangement \(N_6^{c8:2}\) is shown in Fig. 24c and again in Fig. 26a.

---

![Fig. 25](https://example.com/fig25.png)

**(a)** The arrangement \(N_6^{c24}\). **(b)** \(N_6^{c24}\) after collapsing four triangles. **(c)** The arrangement \(N_6^{c8:1}\).
Fig. 26 (a) The arrangement $\mathcal{N}_{6}^{8:2}$ with four gray triangles. (b) $\mathcal{N}_{6}^{8:2}$ after collapsing the gray triangles. (c) After moving the point $p$ to infinity

Fig. 27 Two non-circularizable arrangements of $n = 6$ pseudocircles with a symmetry group of order 4. The arrangements are denoted as (a) $\mathcal{N}_{6}^{c:4:1}$ and (b) $\mathcal{N}_{6}^{c:4:2}$

Suppose that $\mathcal{N}_{6}^{c:8:2}$ has a circle representation $\mathcal{C}$. Circle $C_1$ only has two triangles in the exterior, in the figure they are gray. Circle $C_6$ only has two triangles in the interior. With Lemma 6.4 these four triangles can be collapsed into points of triple intersection. This results in a (non-simple) arrangement as shown in Fig. 26b. Note that we do not care, whether the circles $C_1$ and $C_6$ cross or not.

Apply a Möbius transformation that maps the point $p = C_4 \cap C_5 \cap C_6$ to the point $\infty$ of the extended complex plane. This maps $C_4$, $C_5$, and $C_6$ to lines, while $C_1$, $C_2$, and $C_3$ are mapped to circles. From the order of crossings, it follows that the situation is essentially as shown in Fig. 26c. This figure also shows the line $\ell_1$ through the two intersection points of $C_1$ and $C_2$, the line $\ell_2$ through the two intersection points of $C_2$ and $C_3$, and the line $\ell_3$ through the two intersection points of $C_3$ and $C_1$. The intersection points of $\ell_3$ with the other two are separated by the two defining points of $\ell_3$. According to Theorem 3.2, however, the three lines should share a common point. The contradiction shows that there is no circle representation of $\mathcal{N}_{6}^{c:8:2}$.

7.2.3 Non-circularizability of $\mathcal{N}_{6}^{c:4:1}$ and $\mathcal{N}_{6}^{c:4:2}$

The arrangements $\mathcal{N}_{6}^{c:4:1}$ and $\mathcal{N}_{6}^{c:4:2}$ are shown in Fig. 27 and again in Fig. 28. These are the only two connected digon-free arrangements of 6 pseudocircles with a symmetry group of order 4 which are not circularizable. The two proofs of non-circularizability are very similar.

Suppose that $\mathcal{N}_{6}^{c:4:1}$ has a circle representation $\mathcal{C}$. In Fig. 28a the pseudocircles $C_5$ and $C_6$ each have two gray triangles on the outside and these are the only triangles on
the outside of the two pseudocircles. With Lemma 6.4 the respective circles in $C$ can be grown until the gray triangles collapse into points of triple intersection or until a digon flip occurs. In the case of a digon flip $C_5$ and $C_6$ become intersecting, and no further triangles incident to $C_5$ and $C_6$ are created. Therefore, it is possible to continue the growing process until the four triangles collapse. In the following, we do not care whether the digon flip occurred during the growth process, i.e., whether $C_5$ and $C_6$ intersect. The four points of triple intersection are the points $p_1, p_2, p_3,$ and $p_4$ in Fig. 29a.

There is a circle $C'_3$ which shares the points $p_1$ and $p_3$ with $C_3$ and also contains the point $q_1$, which is defined as the intersection point of $C_1$ and $C_5$ inside $C_3$. Similarly there is a circle $C'_4$ which shares the points $p_2$ and $p_4$ with $C_4$ and also contains the point $q_2$, which is defined as is the intersection point of $C_2$ and $C_5$ inside $C_4$. By construction $C_5$ is incident to one intersection point of each of the pairs $C_1, C'_3,$ and $C'_2, C_5, C_2, C_4, C'_3, C_4, C_1$. Miquel’s Theorem (Theorem 7.1) implies that there is a circle $C^*$ through the second intersection points of these pairs. It can be argued that on $C^*$ the two points $p_3$ and $p_4$ separate $q_3$ and $q_4$. The circle $C_6$ shares the points $p_3$ and $p_4$ with $C^*$ and contains the crossing of the pairs $C_1, C_3$ and $C_2, C_4$ which are ‘close to’ $q_3$ and $q_4$ in its interior (the two points are emphasized by the arrows in the figure). Hence $p_3$ and $p_4$ are separated by $q_3$ and $q_4$. This is impossible, whence, $N_6^{c4:1}$ is not circularizable.
Suppose that $\mathcal{N}^{c:4:2}_6$ has a circle representation $C$. In Fig. 28b the pseudocircles $C_3$ and $C_4$ each have two gray triangles on the outside and these are the only triangles on the outside of the two pseudocircles. By Lemma 6.4 the respective circles in $C$ can be grown to make the gray triangles collapse into points of triple intersection (we do not care whether during the growth process $C_3$ and $C_4$ become intersecting). The four points of triple intersection are the points $p_1$, $p_2$, $p_3$, and $p_4$ in Fig. 29b.

There is a circle $C'_3$ which shares the points $p_1$ and $p_3$ with $C_3$ and also contains the point $q_1$, this point $q_1$ is the intersection point of $C_2$ and $C_5$ inside $C_3$. Similarly there is a circle $C'_4$ which shares the points $p_2$ and $p_4$ with $C_4$ and also contains the point $q_2$, this point $q_2$ is the intersection point of $C_1$ and $C_5$ inside $C_4$. By construction $C_5$ contains one intersection point of the pairs $C_1$, $C'_3$, and $C'_3$, $C_2$, and $C_2$, $C'_4$ and $C'_4$, $C_1$. Miquel’s Theorem (Theorem 7.1) implies that there is a circle $C^*$ through the second intersection points of these pairs. It can be argued that on $C^*$ the points $q_3$ and $q_4$ belong to the same of the two arcs defined by the pair $p_3$, $p_4$. The circle $C_6$ shares the points $p_1$ and $p_4$ with $C^*$ and has the crossing of the pair $C_2$, $C_3$ which is outside $C^*$ in its inside and the crossing of the pair $C_1$, $C_4$ which is inside $C^*$ in its outside (the two points are emphasized by the arrows in the figure). This is impossible, whence, $\mathcal{N}^{c:4:2}_6$ is not circularizable.

8 Enumeration and Asymptotics

Recall from Sect. 1, that the primal graph of a connected arrangement of $n \geq 2$ pseudocircles is the plane graph whose vertices are the crossings of the arrangement and edges are pseudoarcs, i.e., pieces of pseudocircles between consecutive crossings. The primal graph of an arrangement is a simple graph if and only if the arrangement is digon-free.

The dual graph of a connected arrangement of $n \geq 2$ pseudocircles is the dual of the primal graph, i.e., vertices correspond to the faces of the arrangement and edges correspond to pairs of faces which are adjacent along a pseudoarc. The dual graph of an arrangement is simple if and only if the arrangement remains connected after the removal of any pseudocircle. In particular, dual graphs of intersecting arrangements are simple.

The primal-dual graph of a connected arrangement of $n \geq 2$ pseudocircles has three types of vertices; the vertices correspond to crossings, pseudoarcs, and faces of the arrangement. Edges represent incident pairs of a pseudoarc and a crossing, and of a pseudoarc and a face. Figure 30 shows the primal graph, the dual graph, and the primal-dual graph of the NonKrupp arrangement. We now show that crossing free embeddings of dual graphs and primal-dual graphs on the sphere are unique. This will allow us to disregard the embedding and work with abstract graphs.

If $G$ is a subdivision of a 3-connected graph $H$, then we call $G$ almost 3-connected. If $H$ is planar and, hence, has a unique embedding on the sphere, then the same is true for $G$.

In the dual and the primal-dual graph of an arrangement of pseudocircles the only possible 2-separators are the two neighboring vertices of a vertex corresponding to a digon. It follows that these graphs are almost 3-connected. We conclude the following.
Proposition 8.1  The dual graph of a simple intersecting arrangement of $n \geq 2$ pseudocircles has a unique embedding on the sphere.

Proposition 8.2  The primal-dual graph of a simple connected arrangement of $n \geq 2$ pseudocircles has a unique embedding on the sphere.

Note that the statement of Proposition 8.1 clearly holds for $n = 2$, where the dual graph is the 4-cycle.

8.1 Enumeration of Arrangements

The database of all intersecting arrangements of up to $n = 7$ pseudocircles was generated with a recursive procedure. Arrangements of pseudocircles were represented by their dual graphs. The recursion was initiated with the unique arrangement of two intersecting pseudocircles. Given the dual of an arrangement we used a procedure which generates all possible extensions by one additional pseudocircle. The procedure is based on the observation that a pseudocircle in an arrangement of $n$ pseudocircles corresponds to a cycle of length $2n - 2$ in the dual graph. A problem is that an arrangement of $n$ pseudocircles is generated up to $n$ times. Since the embedding of the dual graph is unique, we could use the canonical labeling provided by the Graph-package of SageMath [28] to check whether an arrangement was found before.\footnote{We recommend the Sage Reference Manual on Graph Theory [30] and its collection of excellent examples.}

Another way for obtaining a database of all intersecting arrangements of $n$ pseudocircles for a fixed value of $n$, is to start with an arbitrary intersecting arrangement of $n$ pseudocircles and then perform a recursive search in the flip-graph using the triangle flip operation (cf. Sect. 8.4).

Recall that the dual graph of a connected arrangement contains multiple edges if the removal of one of the pseudocircles disconnects the arrangement. Hence, to avoid problems with non-unique embeddings, we modeled connected arrangements with their primal-dual graphs. To generate the database of all connected arrangements for $n \leq 6$, we used the fact that the flip-graph is connected, when both triangle flips and digon flips are used (cf. Sect. 3.2). The arrangements were created with a recursive search on the flip-graph.
8.2 Generating Circle Representations

Having generated the database of arrangements of pseudocircles, we were then interested in identifying the circularizable and the non-circularizable ones.

Our first approach was to generate arrangements of circles $C_1, \ldots, C_n$ with centers $(x_i, y_i)$ and radii $r_i$ by choosing triples $x_i, y_i, r_i$ at random from $\{1, \ldots, K\}$ for a fixed constant $K \in \mathbb{N}$. In the database the entries corresponding to the generated arrangements were marked circularizable. Later we used known circle representations to find new ones by perturbing values. In particular, whenever a new circle arrangement was found, we tried to locally modify the parameters to obtain further new ones. With these quantitative approaches we managed to break down the list for $n = 5$ to few “hard” examples, which were then treated “by hand”.

For later computations on $n = 6$ (and $n = 7$), we also used the information from the flip-graph on all arrangements of pseudocircles. In particular, to find realizations for a “not-yet-realized” arrangement, we used neighboring arrangements which had already been realized for perturbations. This approach significantly improved the speed of realization.

Another technique to speed up our computations was to use floating point operations and, whenever a solution suggested that an additional arrangement is circularizable, we verified the solution using exact arithmetics. Note that the intersection points of circles, described by integer coordinates and integer radii, have algebraic coordinates, and can therefore be represented by minimal polynomials. All computations were done using the computer algebra system SageMath [28].

As some numbers got quite large during the computations, we took efforts to reduce the “size” of the circle representations, i.e., the maximum over all parameters $|x_i|, |y_i|, r_i$. It turned out to be effective to scale circle arrangements by a large constant, perturb the parameters, and divide all values by the greatest common divisor. This procedure allowed to reduce the number of bits significantly when storing the circle $(x - a)^2 + (y - b)^2 = r^2$ with $a, b, r \in \mathbb{Z}$.

8.3 Counting Arrangements

Projective arrangements of pseudolines are also known as projective abstract order types or oriented matroids of rank 3. The precise numbers of such arrangements are known for $n \leq 11$, see [16,17]. Hence the numbers of great-pseudocircle arrangements given in Table 1 are not new. Moreover, it is well known that there are $2^{\Theta(n^2)}$ arrangements of pseudolines and only $2^{\Theta(n \log n)}$ arrangements of lines [8,13]. Those bounds directly translate to arrangements of great-(pseudo)circles. In this subsection we show that the number of arrangements of pseudocircles and circles are also $2^{\Theta(n^2)}$ and $2^{\Theta(n \log n)}$, respectively.

Proposition 8.3 There are $2^{\Theta(n^2)}$ arrangements on $n$ pseudocircles.

For more details, we refer to the Sage Reference Manual on Algebraic Numbers and Number Fields [29].
Proof The primal-dual graph of a connected arrangement of \(n\) pseudocircles is a plane quadrangulation on \(O(n^2)\) vertices. A quadrangulation can be extended to a triangulation by inserting a diagonal edge in every quadrangular face. It is well known that the number of triangulations on \(s\) vertices is \(2^{\Theta(s)}\) \cite{32}. Hence, the number of connected arrangements of \(n\) pseudocircles is bounded by \(2^{O(n^2)}\).

Since every not necessarily connected arrangements \(A\) on \(n\) pseudocircles can by extended by \(n\) further pseudocircles to a connected arrangement \(A\), and since \(O((2n)^2) = O(n^2)\), the bound \(2^{O(n^2)}\) also applies to the number of (not necessarily connected) arrangements on \(n\) pseudocircles. \(\blacksquare\)

Proposition 8.4 There are \(2^{\Theta(n \log n)}\) arrangements on \(n\) circles.

The proof relies on a bound for the number of cells in an arrangement of zero sets of polynomials (the underlying theorem is associated with the names Oleinik-Petrovsky, Milnor, Thom, and Warren). The argument is similar to the one given by Goodman and Pollack \cite{12} to bound the number of arrangements of lines, see also Matoušek \cite[Chap. 6.2]{20}.

Proof An arrangement \(C\) of \(n\) circles on the unit sphere \(S\) is induced by the intersection of \(n\) planes \(E = \{E_1, \ldots, E_n\}\) in 3-space with \(S\). Plane \(E_i\) can be described by the linear equation \(a_i x + b_i y + c_i z + d_i = 0\) for some reals \(a_i, b_i, c_i, d_i\); we call them the parameters of \(E_i\). Below we define a polynomial \(P_{ijk}\) of degree 6 in the parameters of the planes, such that \(P_{ijk} = 0\) iff the three circles \(E_i \cap S, E_j \cap S,\) and \(E_k \cap S\) have a common point of intersection. We also define a polynomial \(Q_{ij}\) of degree 8 in the parameters of the planes, such that \(Q_{ij} = 0\) iff the circles \(E_i \cap S\) and \(E_j \cap S\) touch.

Transforming an arrangement \(C\) into an arrangement \(C'\) in a continuous way corresponds to a curve \(\gamma\) in \(\mathbb{R}^{4n}\) from the parameter vector of \(E\) to the parameter vector of \(E'\). If a triangle flip or a digon flip occurs when transforming \(C\) to \(C'\), then \(\gamma\) intersects the zero set of a polynomial \(P_{ijk}\) or \(Q_{ij}\). Hence, all the points in a fixed cell of the arrangement defined by the zero set of the polynomials \(P_{ijk}\) or \(Q_{ij}\) with \(1 \leq i < j < k \leq n\) are parameter vectors of isomorphic arrangements of circles.

The number of cells in \(\mathbb{R}^d\) induced by the zero sets of \(m\) polynomials of degree at most \(D\) is upper bounded by \((50Dm/d)^d\) (Theorem 6.2.1 in \cite{20}). Consequently the number of non-isomorphic arrangements of \(n\) circles, is bounded by \((50 \cdot 8 \cdot 2^{(n^3)/4n})^{4n}\) which is \(n^{O(n)}\).

For the definition of the polynomial \(P_{ijk}\) we first note (see e.g. \cite[Sect. 12.3]{24}) that the homogeneous coordinates of the point \(I_{ijk}\) of intersection of the three planes \(E_i, E_j, E_k\) are given by

\[
\bigotimes_4 \left( (a_i, b_i, c_i, d_i), (a_j, b_j, c_j, d_j), (a_k, b_k, c_k, d_k) \right),
\]

If the three planes \(E_i, E_j, E_k\) intersect in a common line, we still take the expression as a definition for \(I_{ijk}\), i.e., the homogeneous coordinates are all zero.

\(\odot\) Springer
where $\bigotimes_n$ denotes the $(n - 1)$-ary analogue of the cross product in $\mathbb{R}^n$,

$$\bigotimes_n(v_1, \ldots, v_{n-1}) := \begin{vmatrix} v_1^{(1)} & \ldots & v_{n-1}^{(1)} & e_1 \\ \vdots & \ddots & \vdots & \vdots \\ v_1^{(n)} & \ldots & v_{n-1}^{(n)} & e_n \end{vmatrix}.$$ 

Each component of $I_{ijk}$ is a cubic polynomial in the parameters of the three planes. Since a homogeneous point $(x, y, w, \lambda)$ lies on the unit sphere $S$ if and only if $x^2 + y^2 + z^2 - \lambda^2 = 0$, we get a polynomial $P_{ijk}$ of degree 6 in the parameters of the planes such that $P_{ijk} = 0$ iff $I_{ijk} \in S$.

To define the polynomial $Q_{ij}$ we need some geometric considerations. Note that the two circles $E_i \cap S$ and $E_j \cap S$ touch if and only if the line $L_{ij} = E_i \cap E_j$ is tangential to $S$. Let $E^*_{ij}$ be the plane normal to $L_{ij}$ which contains the origin. The point $I^*_{ij}$ of intersection of the three planes $E_i, E_j, E^*_{ij}$ is on $S$ if and only if $E_i \cap S$ and $E_j \cap S$ touch.

A vector $N_{ij}$ which is parallel to $L_{ij}$ can be obtained as $\bigotimes_3((a_i, b_i, c_i), (a_j, b_j, c_j))$. The components of $N_{ij}$ are polynomials of degree 2 in the parameters of the planes in $E$. The components of $N_{ij}$ are the first three parameters of $E^*_{ij}$; the fourth parameter is zero. The homogeneous components of $I^*_{ij}$ are obtained by a $\bigotimes_4$ from the parameters of the three planes $E_i, E_j, E^*_{ij}$. Since the parameters of $E^*_{ij}$ are polynomials of degree 2, the components of $I^*_{ij}$ are polynomials of degree 4. Finally we have to test whether $I^*_{ij} \in S$, this makes $Q_{ij}$ a polynomial of degree 8 in the parameters of the planes.

8.4 Connectivity of the Flip-Graph

Given two arrangements of $n$ circles, one can continuously transform one arrangement into the other. During this transformation (the combinatorics of) the arrangement changes whenever a triangle flip or a digon flip occurs.

Snoeyink and Hershberger [27] showed an analog for arrangements of pseudocircles: Given two arrangements on $n$ pseudocircles, a sequence of digon flips and triangles flips can be applied to transform one arrangement into the other. In other words, they have proven connectivity of the flip-graph of arrangements of $n$ pseudocircles, which has all non-isomorphic arrangements as vertices and edges between arrangements that can be transformed into each other using single flips.

For arrangements of (pseudo)lines, it is well known that the triangle flip-graph is connected (see e.g. [11]). A triangle flip in an arrangement of (pseudo)lines corresponds to an operation in the corresponding arrangement of great-(pseudo)circles where two “opposite” triangles are fliped simultaneously.

With an idea as in the proof of the Great-Circle Theorem (Theorem 2.1) we can show that the flip-graph for arrangements of circles is connected.

Theorem 8.5 The triangle flip-graph on the set of all intersecting digon-free arrangements of $n$ circles is connected for every $n \in \mathbb{N}$.

\[\Box\] Springer
**Proof** Consider an intersecting arrangement of circles \( C \) on the unit sphere \( S \). Imagine the planes of \( \mathcal{E}(C) \) moving towards the origin. To be precise, for time \( t \geq 1 \) let \( \mathcal{E}_t := \{1/t \cdot E : E \in \mathcal{E}(C)\} \). During this process only triangle flips occur as the arrangement is already intersecting and eventually the point of intersection of any three planes of \( \mathcal{E}_t \) is in the interior of the unit sphere \( S \). Thus, in the circle arrangement obtained by intersecting the moving planes \( \mathcal{E}_t \) with \( S \) every triple of circles forms a Krupp, that is, the arrangement becomes a great-circle arrangement. Since the triangle flip-graph of line arrangements is connected, we can use triangle flips to get to any other great-circle arrangement. Due to Fact 2, no digon occurs in the arrangement during the whole process.

Consequently, any two arrangements of circles \( C \) and \( C' \) can be flipped to the same great-circle arrangement without digons to occur, and the statement follows. 

Based on the computational evidence for \( n \leq 7 \), we conjecture that the following is true.

**Conjecture 8.6** The triangle flip-graph on the set of all intersecting digon-free arrangements of \( n \) pseudocircles is connected for every \( n \in \mathbb{N} \).

### 9 Further Results and Discussion

In course of this paper, we generated circle representations or proved non-circularizability for all connected arrangements of \( n \leq 5 \) pseudocircles (cf. Sect. 5) and for all digon-free intersecting arrangements of \( n \leq 6 \) pseudocircles (cf. Sect. 5). Besides that, we also investigated the next larger classes and found

- about 4,400 connected digon-free arrangements of 6 circles (which is about 98%),
- about 130,000 intersecting arrangements of 6 circles (which is about 90%), and
- about 2 millions intersecting digon-free arrangements of 7 circles (which is about 66%).

For our computations (especially the last two additional items), we had up to 24 CPUs running over some months with the quantitative realization approaches described in Sect. 8.2.

We further investigated arrangements that were not realized by our computer program and have high symmetry or other interesting properties. Non-circularizability proofs for some of these candidates were presented in Sect. 7. Since we have no automated procedure for proving non-circularizability, these proofs had to be done by hand.

**Problem 9.1** Find an algorithm for deciding circularizability which is practical.

In the case of stretchability of arrangements of pseudolines, the method based on final polynomials, i.e., on the Graßmann–Plücker relations, would qualify as being practical (cf. [5]).
References

1. Agarwal, P.K., Nevo, E., Pach, J., Pinchasi, R., Sharir, M., Smorodinsky, S.: Lenses in arrangements of pseudo-circles and their applications. J. ACM 51, 139–186 (2004)

2. Albenque, M., Knauer, K.: Convexity in partial cubes: the hull number. Discrete Math. 339, 866–876 (2016)

3. Björner, A., Las Vergnas, M., White, N., Sturmfels, B., Ziegler, G.M.: Oriented Matroids, Encyclopedia of Mathematics and Its Applications, vol. 46, 2nd edn. Cambridge University Press, Cambridge (1999)

4. Bogomolny, A.: Cut-the-knot: four touching circles. http://www.cut-the-knot.org/Curriculum/Geometry/FourTouchingCircles.shtml#Explanation

5. Bokowski, J., Richter, J.: On the finding of final polynomials. Eur. J. Comb. 11, 21–34 (1990)

6. Bokowski, J., Sturmfels, B.: An infinite family of minor-minimal nonrealizable 3-chirotopes. Mathematische Zeitschrift 200, 583–589 (1989)

7. Edelsbrunner, H., Ramos, E.A.: Inclusion-exclusion complexes for pseudodisk collections. Discrete Comput. Geom. 17, 287–306 (1997)

8. Felsner, S., Goodman, J.E.: Pseudoline arrangements. In: Toth, O’Rourke, Goodman, (eds.) Handbook of Discrete and Computational Geometry, 3rd edn. CRC Press, Boca Raton (2018)

9. Felsner, S., Scheucher, M.: Webpage: Homepage of Pseudocircles. http://www3.math.tu-berlin.de/pseudocircles

10. Felsner, S., Scheucher, M.: Arrangements of Pseudocircles: Triangles and Drawings (2017). arXiv:1708.06449

11. Felsner, S., Weil, H.: A theorem on higher Bruhat orders. Discrete Comput. Geom. 23, 121–127 (2000)

12. Goodman, J.E., Pollack, R.: Upper bounds for configurations and polytopes in $\mathbb{R}^d$. Discrete Comput. Geom. 1, 219–227 (1986)

13. Goodman, J.E., Pollack, R.: Allowable sequences and order types in discrete and computational geometry. In: Pach, J. (ed.) New Trends in Discrete and Computational Geometry, pp. 103–134. Springer, New York (1993)

14. Grünbaum, B.: Arrangements and Spreads, CBMS Regional Conference Series in Mathematics, AMS, vol. 10 (1972) (reprinted 1980)

15. Kang, R.J., Müller, T.: Arrangements of pseudocircles and circles. Discrete Comput. Geom. 51, 896–925 (2014)

16. Knuth, D.E.: Axioms and Hulls, LNCS 606. Springer, New York (1992)

17. Krasser, H.: Order types of point sets in the plane, PhD thesis, Graz University of Technology, Austria (2003)

18. Levi, F.: Die Teilung der projektiven Ebene durch Gerade oder Pseudogerade. Berichte über die Verhandlungen der Sächsischen Akademie der Wissenschaften zu Leipzig, Mathematisch-Physische Klasse 78, 256–267 (1926)

19. Linhart, J., Ortner, R.: An arrangement of Pseudocircles Not Realizable with Circles. Beiträge zur Algebra und Geometrie 46, 351–356 (2005)

20. Matoušek, J.: Lectures on Discrete Geometry. Springer, New York (2002)

21. Matoušek, J.: Intersection graphs of segments and $\mathbb{R}$ (2014). arXiv:1406.2636

22. Mnëv, N.E.: The universality theorems on the classification problem of configuration varieties and convex polytopes varieties. In: Topology and Geometry—Rohlin Seminar, LNM 1346, pp. 527–543. Springer (1988)

23. Richter-Gebert, J.: Mnëv’s Universality Theorem Revisited. Séminaire Lotharingien de Combinatoire 34 (1995)

24. Richter-Gebert, J.: Perspectives on Projective Geometry—A Guided Tour through Real and Complex Geometry. Springer, New York (2011)

25. Schaefer, M., Štefankovič, D.: Fixed points, nash equilibria, and the existential theory of the reals. Theory Comput. Syst. 60, 172–193 (2017)

26. Sloane, N.J.A.: The On-Line Encyclopedia of Integer Sequences, sequences A250001 and A288567. http://oeis.org

27. Snoeyink, J., Hershberger, J.: Sweeping arrangements of curves. In: Goodman, J.E., Pollack, R.D., Steiger, W.L. (eds.) Discrete & Computational Geometry, DIMACS DMTCS Series, vol. 6, pp. 309–349. AMS (1991)

28. Stein, W.A., et al.: Sage Mathematics Software (Version 8.0), The Sage Development Team (2017). http://www.sagemath.org
29. Stein, W.A., et al.: Sage Reference Manual: Algebraic Numbers and Number Fields (Release 8.0) (2017). http://doc.sagemath.org/pdf/en/reference/number_fields/number_fields.pdf
30. Stein, W.A., et al.: Sage Reference Manual: Graph Theory (Release 8.0). (2017). http://doc.sagemath.org/pdf/en/reference/number_fields/number_fields.pdf
31. Suvorov, P.: Isotopic but not rigidly isotopic plane systems of straight lines. In: Topology and Geometry – Rohlin Seminar, LNM 1346, pp. 545–556 Springer (1988)
32. Tutte, W.T.: A census of planar triangulations. Can. J. Math. 14, 21–38 (1962)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.