THE STOCHASTIC 3D GLOBALLY MODIFIED NAVIER-STOKES EQUATIONS: EXISTENCE, UNIQUENESS AND ASYMPTOTIC BEHAVIOR

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(Communicated by Martino Bardi)

1. Introduction. It is well known that the three-dimensional (3D) Navier-Stokes equations describe the time evolution of an incompressible fluid and are given by
\[
\begin{align*}
\frac{\partial u}{\partial t} &= \nu \Delta u - (u, \nabla) u + \nabla p + f(t), \\
\text{div } u &= 0,
\end{align*}
\]
where \(u\) represents the velocity field, \(\nu\) is the viscosity constant, \(p(t, x)\) denotes the pressure and \(f\) is an external force field acting on the fluid. While the 2D Navier-Stokes equations have been studied extensively in the literature, there exist serious obstacles to tackle the 3D Navier-Stokes equations. One of them is the lack of uniqueness.

There exist many modified versions of the 3D Navier-Stokes equations due to Leray and others with mollification (and/or cut off) of the nonlinear term as a way to approximate the original problem, see for instance the review paper of Constantin [9]. We also mention the paper [15] by Flandoli and Maslowski with a global cut off function used for the 2D Navier-Stokes equations. In 2006, Caraballo, Kloeden and Real [6] proposed a 3-dimensional model where the nonlinear term included a cut off factor \(F_N(\|u\|)\) based on the norm of the gradient of the solution in the whole domain. Namely, for \(N \in (0, +\infty)\) the function \(F_N : [0, +\infty) \to (0, 1]\) is defined by
\[
F_N(r) := \min\{1, \frac{N}{r}\}.
\]
They called the resulting system
\[
\begin{align*}
\frac{\partial u}{\partial t} - \nu \Delta u + F_N(\|u\|) [(u, \nabla) u] + \nabla p &= f(t), \\
\text{div } u &= 0,
\end{align*}
\]
the globally modified Navier-Stokes equations (GMNSE). They established the well-posedness of the model, in particular the absence of blow-up of solutions, as well

2000 Mathematics Subject Classification. 35R60,35Q35,60H15,76M35,86A05.
Key words and phrases. Globally modified, Navier-Stokes, stochastic, strong solutions, stability.

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as the existence of global $H^1$-attractors. The GMNSE are useful in obtaining new results about the 3D Navier-Stokes equations. Indeed, they were used in [6] to establish the existence of bounded entire weak solutions for the 3D Navier-Stokes equations. Also in [21], the model was used to show that the attainability set of the weak solutions of the 3D Navier-Stokes equations satisfying an energy inequality are weakly compact and weakly connected. For convergence results of solutions of GMNSE to solutions of the 3D Navier-Stokes equations, see [6, 26]. See [22, 12, 7, 27, 28, 23, 29] for other studies and applications of the GMNSE as well as the review paper [7].

Stochastic Partial Differential Equations (SPDEs) are powerful tool for understanding and investigating mathematically hydrodynamic and turbulence theory. To model turbulent fluids, mathematicians often use stochastic equations obtained from adding a noise term in the dynamical equations of the fluids. This approach is basically motivated by Reynolds work which stipulates that turbulent flows are composed of slow (deterministic) and fast (stochastic) components. Recently by following the statistical approach of turbulence theory, Flandoli et al [16], Kupiainen [24] confirm the importance of studying the stochastic version of fluids dynamics. Indeed, the authors of [16] pointed out that some rigorous information on questions of turbulence theory might be obtained from these stochastic versions. It is worth emphasizing that the presence of the stochastic term (noise) in these models often leads to qualitatively new types of behavior for the processes. Since the pioneering work of Bensoussan and Temam [35], there has been an extensive literature on stochastic Navier-Stokes equations with Wiener noise and related equations, we refer to [1, 2, 4, 5, 8, 13, 30].

In the present paper, we shall study the stochastic 3D globally modified Navier-Stokes equations. Let us now describe our model equation. Let $\mathcal{M} \subset \mathbb{R}^3$ be an open bounded set with regular boundary $\Gamma$ and $T > 0$ be a final time. We consider the following stochastic 3D system of globally modified Navier-Stokes equations

$$
\begin{align*}
\left\{ 
\begin{array}{l}
\frac{du(t)}{dt} = [\nu \Delta u - F_N(\|u\|)][(u, \nabla)u] + \nabla p - F(u) \] dt \\
+ \sum_{k=1}^{\infty} \sigma_k(u(t))dW_k(t), \\
\nabla u = 0 \text{ in } (0, T) \times \mathcal{M}, \\
u = 0, \text{ on } (0, T) \times \Gamma, \\
u(0, x) = \nu_0(x), \ x \in \mathcal{M},
\end{array}
\right.
\end{align*}
$$

(1)

where $\{W^k_t, \ t \geq 0, k = 1, 2, \ldots\}$ is a sequence of independent one dimensional standard Brownian motions on some complete filtration probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in (0, T)})$. If $(e_k)_{k \geq 1}$ is an orthonormal basis of $l^2$, we may formally define $W$ by taking $W = \sum_k W_k e_k$. As such $W$ is a cylindrical Brownian motion evolving over $l^2$. We recall that $l^2$ is the Hilbert space consisting of all sequences of square summable real numbers. We define the auxiliary space $\mathcal{U}_0 \subset l^2$ via $\mathcal{U}_0 = \{v = \sum_{k=1}^{\infty} \alpha_k e_k : \sum_{k=1}^{\infty} \alpha_k^2 k^{-2} < \infty\}$ endowed with the norm $|v|_{\mathcal{U}_0}^2 := \sum_{k=1}^{\infty} \alpha_k^2$ for $v = \sum_{k=1}^{\infty} \alpha_k e_k$. Note that the embedding of $l^2 \subset \mathcal{U}_0$ is Hilbert-Schmidt. Moreover, using standard martingale arguments with the fact that each $W_k$ is almost surely continuous (see [10]), we have that for almost every $\omega \in \Omega$, $W(\omega) \in C([0, T]; \mathcal{U}_0)$. See Section 3 for the precise assumptions on the coefficients $F$ and $\sigma_k$ for $k = 1, \ldots, \infty$.

In this paper, we shall prove the existence of a unique strong solution to our stochastic 3D GMNSE (1) under some assumptions on $F$ and $\sigma_k$ for $k = 1, \ldots, \infty$. 

Here the word “strong” means “strong” both in the sense of the theory of stochastic differential equations and the theory of partial differential equations. The proof combines the Galerkin approximation, the strong monotonicity of the coefficients and a Gronwall lemma for stochastic processes (see Lemma 5.5). To obtain the strong solution in the sense of partial differential equations, we prove that the solution of the Galerkin scheme is a Cauchy sequence in probability in $L^\infty([0,T]; H^1)$. Moreover, as in the deterministic case, we study the convergence of the strong solution of the stochastic 3D GMNSE when $N \to \infty$. This enables us to prove the existence of a martingale solution for the stochastic 3D Navier-Stokes equations. One of the main difficulties here is the passage to the limit on the nonlinear term containing $F_N$.

The layout of the present paper is as follows. In Section 2, we recall some spaces useful for the abstract framework and some properties and estimates related to the operators involved in the model. Section 3 is concerned with the existence and uniqueness of strong solution for the stochastic 3D GMNSE. The convergence of the strong solution of the model when $N \to \infty$, is treated in Section 4. Finally in the Appendix for the reader’s convenience, we recall two compacts embedding theorems, a convergence theorem for the stochastic integral and a stochastic Gronwall lemma.

2. Preliminaries. To set our problem in the abstract framework, we consider the following usual abstract spaces (see Lions [25] and Temam [35, 36]):

$$ V = \{ u \in C^\infty_0(\mathcal{M})^3 : \text{div} u = 0 \}, $$

$H$ is the closure of $V$ in $(L^2(\mathcal{M}))^3$ with inner product $(.,.)$ and associate norm $|.|$, where for $u, v \in (L^2(\mathcal{M}))^3$,

$$ (u, v) = \sum_{j=1}^3 \int_\mathcal{M} u_j(x)v_j(x)dx. $$

$V$ is the closure of $V$ in $(H^1_0(\mathcal{M}))^3$ with the scalar product $((.,.))$ and associate norm $\|\|$, where for $u, v \in (H^1_0(\mathcal{M}))^3$,

$$ ((u, v)) = (u, v) + (\nabla u, \nabla v). $$

It follows that $V \subset H \equiv H' \subset V'$, where the injections are dense and compact. Finally, we will use $\|\|_*$ for the norm in $V'$ and $\langle.,.\rangle$ for the duality pairing between $V$ and $V'$.

Now we consider the trilinear form $b$ on $V \times V \times V$ by

$$ b(u, v, w) = \sum_{i,j=1}^3 \int_\mathcal{M} u_i \frac{\partial v_j}{\partial x_i} w_j dx, \forall u, v, w \in V, $$

and we denote

$$ b_N(u, v, w) = F_N(\|v\|)b(u, v, w), \forall u, v, w \in V. $$

The form $b_N$ is linear in $u$ and $w$, but it is nonlinear in $v$. We have $b_N(u, v, v) = 0$ for all $u, v \in V$. Moreover from the properties of $b$ (see [35] or [32]), and the definition of $F_N$, one easily obtains the existence of a constant $c_1 > 0$ only dependent on $\Omega$ such that

$$ |b_N(u, v, w)| \leq c_1 |u|^\frac{2}{3} \|v\|^\frac{2}{3} \|w\|^\frac{2}{3}, \forall u, v, w \in V, $$

$$ |b_N(u, v, w)| \leq Nc_1 \|u\| \|w\|, \forall u, v, w \in V. $$

If we denote

$$ \langle B_N(u, v) , w \rangle = b_N(u, v, w), \forall u, v, w \in V, $$
we have
\[ \|B_N(u, v)\|_* \leq c_1 |u|^\frac{1}{2} |v|^\frac{3}{2}, \forall u, v \in V, \]
\[ \|B_N(u, v)\|_* \leq N c_1 \|u\|, \forall u, v \in V. \]
If \( u = v \), we write \( B_N(u) = B_N(u, u) \).

We also consider \( A : V \to V' \) defined by \( \langle Au, v \rangle = ((u, v)) \). Denoting \( D(A) = (H^2(M))^3 \cap V \), then \( Au = -\Delta u, \forall u \in D(A) \) is the Stokes operator \( (P \) is the orthogonal projection from \( (L^2(\Omega))^3 \) onto \( H \))

We recall (see [36]) that there exists a constant \( c_2 > 0 \) depending only on \( \Omega \) such that
\[ \|u\|_{(L^\infty(M))^3} \leq c_2 |Au|, \forall u \in D(A), \]
\[ |b(u, v, w)| \leq c_2 |u|^\frac{1}{2} |Au|^\frac{3}{2} \|w\|, \forall u \in D(A), v \in V, w \in H, \]
\[ |b(u, v, w)| \leq c_2 |u|^\frac{1}{2} |Au|^\frac{3}{2} \|w\|, \forall u \in D(A), v \in V, w \in H. \]
\[ |b(u, v, w)| \leq c_2 \|u\|^\frac{1}{2} \|v\|^\frac{1}{2} \|w\|, \forall u, v, w \in V. \]

The following lemmas give some important properties of the map \( F_N \) (see [6, 33] for the proof)

**Lemma 2.1.**
\[ |F_N(p) - F_N(r)| \leq \frac{|p - r|}{r} \]
for all \( p, r \in \mathbb{R}^+ \).

**Lemma 2.2.** For any \( u, v \in V \), and each \( N > 0 \),
\[ 1) \ 0 \leq \|u\| F_N(\|u\|) \leq N, \]
\[ 2) \ |F_N(\|u\|) - F_N(\|v\|)| \leq \frac{1}{N} F_N(\|u\|) F_N(\|v\|) \|u - v\|. \]

The next lemma shows that \( B_N \) is locally Lipschitz.

**Lemma 2.3.** The map \( B_N : V \to V' \) is locally Lipschitz continuous i.e. for every \( r > 0 \), there exists a constant \( L_r \) such that
\[ \|B_N(u) - B_N(v)\|_{V'} \leq L_r \|u - v\| \]
for \( u, v \in V \) with \( \|u\|, \|v\| \leq r \).

**Proof.** For \( u, v \) and \( w \in V \), we have
\[ \langle B_N(u) - B_N(v), w \rangle \]
\[ = b_N(u, u, w) - b_N(v, v, w) \]
\[ = F_N(\|v\|) b(u - v, v, w) + (F_N(\|v\|) - F_N(\|u\|)) b(u, v, w) + F_N(\|u\|) b(u, u - v, w) \]
\[ \leq c_1 \|u - v\| \|v\| \|w\| \]
\[ + c_1 \|u\| \|v\| \|w\| + c_1 \|u\| \|u - v\| \|w\|, \]
where we have used the estimate of Lemma 2.1. From this, we deduce that the map \( B_N \) is locally Lipschitz. \( \square \)

The next result will be useful in our study of the stochastic 3D globally modified Navier-Stokes equations. See [33] for the proof.
Lemma 2.4. Consider the operator $\mathcal{N}\mathcal{L} : V \times V \rightarrow V'$ given by

$$\langle \mathcal{N}\mathcal{L}(u,v), w \rangle = F_N(u) b(u,u,w) - F_N(v) b(v,v,w)$$

for all $u, v, w \in V$. There exists a constant $c_3 > 0$ which depends on $c_2$ and $\nu$ such that

$$\langle \mathcal{N}\mathcal{L}(u,v), u - v \rangle \leq \frac{\nu}{2} ||u - v||^2 + c_3 N^4 |(u-v)(t)|^2.$$  \hspace{1cm} (9)

3. Existence and uniqueness. Let $(W_k(t), k \geq 1)$ be a sequence of independent $\mathcal{F}_t$-Brownian motions defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. Consider the stochastic 3D globally modified Navier-Stokes equations

$$\begin{cases}
    du(t) = -\nu A u dt - B_N(u) dt - F(u) dt + \sum_{k=1}^{\infty} \sigma_k(u(t)) dW_k(t), \\
    u(0) = u_0 \in V.
\end{cases}$$ \hspace{1cm} (10)

Here $F$ is a mapping from $V$ (resp. $H$) into $V$ (resp. $H$). $\sigma_k(\cdot), k \geq 1$ is a sequence of mapping from $V$ (resp. $H$) into $V$ (resp. $H$). Consider the following hypothesis.

1. $|F(u) - F(v)|^2 \leq c|u - v|^2$,  \hspace{1cm} (11)
2. $\|F(u) - F(v)\|^2 \leq c\|u - v\|^2$,  \hspace{1cm} (12)
3. $\sum_{k=1}^{\infty} |\sigma_k(u)|^2 < \infty$ for $u \in H$,  \hspace{1cm} (13)
4. $\sum_{k=1}^{\infty} \|\sigma_k(u)\|^2 < \infty$ for $u \in V$,  \hspace{1cm} (14)
5. $\sum_{k=1}^{\infty} |\sigma_k(u) - \sigma_k(v)|^2 \leq c|u - v|^2$,  \hspace{1cm} (15)
6. $\sum_{k=1}^{\infty} \|\sigma_k(u) - \sigma_k(v)\|^2 \leq c\|u - v\|^2$.  \hspace{1cm} (16)

Remark 1. The hypothesis (13)-(14) imply that for every $u \in V$ (resp. $H$) the map $\sigma(u) := (\sigma_k(u))_{k \in \mathbb{N}} : \ell^2 \rightarrow V$ (resp. $H$) defined by

$$\sigma(u) h := \sum_{k=1}^{\infty} \sigma_k(u) h_k, \quad h = (h_k)_{k \in \mathbb{N}} \in \ell^2,$$

is in $L_2(\ell^2, V)$ (resp. $H$). $\sigma(u) : \ell^2 \rightarrow V$ (resp. $H$) and (15)-(16) imply that $u \mapsto \sigma(u)$ is Lipschitz.

We write $G(u) := -\nu A u - B_N(u) - F(u)$.

The next lemma shows some strong monotonicity of the operator $G$.

Lemma 3.1. For $u_1, u_2 \in D(A) \subset V$, we have

$$\langle G(u_1) - G(u_2), u_1 - u_2 \rangle \geq -\frac{\nu}{2} \|u_1 - u_2\|^2 + (\nu + c_3 N^4 + c) \|u_1 - u_2\|^2,$$ \hspace{1cm} (17)

where the constant $c_3$ is given by Lemma 2.4.

Proof. We have

$$\begin{align*}
    \langle G(u_1) - G(u_2), u_1 - u_2 \rangle_H &= -\nu \langle A(u_1) - A(u_2), u_1 - u_2 \rangle_H \\
    &\quad - \langle B_N(u_1) - B_N(u_2), u_1 - u_2 \rangle_H - \langle F(u_1) - F(u_2), u_1 - u_2 \rangle_H \\
    &\leq -\nu \|u_1 - u_2\|^2 + \nu \|u_1 - u_2\|^2 + \frac{\nu}{2} \|u_1 - u_2\|^2 + c_3 N^4 \|u_1 - u_2\|^2 + c \|u_1 - u_2\|^2.
\end{align*}$$
Theorem 3.2. Assume that the hypotheses (11)-(16) hold and \( u_0 \in L^2(\Omega, \mathcal{F}_0; V) \). Then there exists a unique solution to the stochastic 3D system of globally modified Navier-Stokes equations (10) that satisfies the following energy inequality

\[
\mathbb{E} \left( \sup_{t \in [0,T]} \|u(t)\|^2 \right) + \int_0^T \mathbb{E} \left[ \|u(t)\|_{H^2}^2 \right] dt < \infty.
\]

Proof. I) Uniqueness. Let \( X \) and \( \tilde{X} \) two solutions of Problem 1 starting from the same initial value \( u_0 \). For any \( r > 0 \) and \( R > 0 \), define the stopping time

\[
\tau_R := \inf \{ t \in [0,T] : \|X(t)\| \vee \|\tilde{X}(t)\| \geq R \}.
\]

Set \( U(t) = X(t) - \tilde{X}(t) \). Then by Itô’s formula, we have

\[
|U(t)|^2 = -2\nu \int_0^t \langle AU(s), U(s) \rangle ds - 2 \int_0^t (B_N(X(s)) - B_N(\tilde{X}(s)), U(s)) ds
- \int_0^t (F(X(s)) - F(\tilde{X}(s)), U(s)) ds + \sum_{k=1}^{\infty} \int_0^t |\sigma_k(X(s)) - \sigma_k(\tilde{X}(s))|^2 ds
+ 2 \sum_{k=1}^{\infty} \int_0^t (\sigma_k(X(s)) - \sigma_k(\tilde{X}(s)), U(s)) dW_k(s).
\] (19)

We are going to estimate each term in (19). We have

\[
- \langle AU(s), U(s) \rangle_{H} = -\|U(s)\|^2 + |U(s)|^2.
\] (20)

From Lemma 2.4, there exists a constant \( c_3 > 0 \) such that

\[
\langle B_N(X(s)) - B_N(\tilde{X}(s)), U(s) \rangle_H \leq \frac{\nu}{3} \|U(s)\|^2 + c_3 N^4 |U(s)|^2.
\] (21)

Combining the estimates (20)-(21) with (19), we get

\[
\mathbb{E}|U(t \wedge \tau_R)|^2 \leq C_{R,T} \int_0^t \mathbb{E}|U(s \wedge \tau_R)|^2 ds.
\]

By Gronwall’s inequality, we get for any \( t \in [0,T] \)

\[
\mathbb{E}|U(t \wedge \tau_R)|^2 = 0.
\]

And the uniqueness follows by letting \( R \to \infty \) and Fatou’s lemma.

II) Existence. We will use the Galerkin approximation combined with the strong monotonicity of the stochastic 3D globally modified Navier-Stokes equations. We shall do this in two steps:

Step 1: Assume \( u_0 \in L^6(\Omega, \mathcal{F}_0; V) \).

Let \( \{e_i : i \geq 1\} \subset D(A) \) be a fixed orthonormal basis of \( H \) consisting of eigenvectors of \( \Delta \), so that it is also orthogonal in \( V \). Denote \( \pi_n \) the orthogonal projection from \( H \) onto the finite dimensional space \( H_n := \text{span} \{e_1, e_2, ..., e_n\} \):

\[
\pi_n u := \sum_{i=1}^{n} \langle u, e_i \rangle e_i.
\]

Thus \( \pi_n \) is also the orthogonal projection onto \( H_n \) in \( V \).
Consider the following finite dimensional stochastic differential equations in $H_n$:

$$
\begin{cases}
    du_n(t) = [\pi_n G(u_n(t))] dt + \sum_{k=1}^{\infty} \pi_n \sigma_k(u_n(t)) dW_k(t), \\
    u_n(0) = \pi_n u_0.
\end{cases}
$$

(22)

We have for $u \in H_n$,

$$
\langle u, \pi_n G(u) \rangle \leq C_N (1 + |u|^2_{H_n}),
$$

$$
\sum_{k=1}^{\infty} \|\pi_n \sigma_k(u)\|_{H_n}^2 \leq c(1 + |u|^2_{H_n}).
$$

Moreover by Lemma 2.3, (11) and (15), the maps

$$
u \in H_n \mapsto \pi_n G(u) \in H_n$$

$$
u \in H_n \mapsto \pi_n \sigma$$

are respectively locally Lipschitz continuous and Lipschitz continuous. Then by the theory of stochastic differential equations (see [19, 17]), there exists a unique continuous ($\mathcal{F}_t$)-adapted process $u_n(t)$ satisfying

$$
u_n(t) = u_n(0) + \int_0^t \pi_n G(u_n(s)) ds + \sum_{k=1}^{\infty} \int_0^t \pi_n \sigma_k(u_n(s)) dW^k_s$$

and for any $n \geq i$,

$$
\langle \nu_n(t), e_i \rangle = \langle u_0, e_i \rangle + \int_0^t \langle \pi_n G(u_n(s)), e_i \rangle ds + \sum_{k=1}^{\infty} \int_0^t \langle \pi_n \sigma_k(u_n(s)), e_i \rangle dW^k_s.
$$

We now prove some a priori estimates of the approximated solution.

\textbf{Lemma 3.3.} There exists a constant $c$ such that

1) $\sup_{t \in [0, T]} \mathbb{E} (\|u_n(t)\|^2) + \int_0^T \mathbb{E} (\|u_n(s)\|_{H(A)}^2) ds \leq c (1 + \mathbb{E} \|u_0\|^2),

(23)

2) $\mathbb{E} \left( \sup_{t \in [0, T]} \|u_n(t)\|^2 \right) \leq c (1 + \mathbb{E} \|u_0\|^2),

(24)

3) $\sup_n \int_0^T \mathbb{E} \|u_n(t)\|^6 dt \leq c (1 + \mathbb{E} \|u_0\|^6),

(25)

4) $\sup_n \int_0^T \mathbb{E} \|\pi_n G(u_n(t))\|^2 dt < \infty.

(26)

\textbf{Proof.} 1) By Itô’s formula, we have

$$
\|u_n(t)\|^2 = \|u_n(0)\|^2 - 2 \int_0^t \langle Au_n(s), u_n(s) \rangle ds - 2 \int_0^t \langle B_N(u_n(s)), u_n(s) \rangle ds
$$

$$
- \int_0^t \langle F(u_n(s)), u_n(s) \rangle ds + \sum_{k=1}^{\infty} \int_0^t \langle \sigma_k(u_n(s)), u_n(s) \rangle dW^k(s)
$$

$$
+ \sum_{k=1}^{\infty} \int_0^t \|\pi_n \sigma_k(u_n(s))\|^2 ds.
$$

(27)
For $u \in D(A)$, we have
\[
- \langle Au, u \rangle_V = -|u|^2_{D(A)} + |\nabla u|^2 + |u|^2 = -|u|^2_{D(A)} + \|u\|^2. \tag{28}
\]
From the estimate (4) and the properties of $F$, we get
\[
- (B_N(u), u)_V = -(B_N(u), (I - \Delta)u)_H = -b_N(u, u, (I - \Delta)u) 
\leq c_2 N \|u\|^\frac{1}{2} |Au| \frac{1}{2} \leq \frac{\nu'}{2} \|Au\|^2 + c |u|^2 
= \frac{\nu'}{2} |u|^2_{D(A)} + c \|u\|^2. \tag{29}
\]
The properties of $F$ give
\[
- (F(u), u)_V \leq c(1 + \|u\|^2). \tag{30}
\]
The estimates (28)-(30) in (27) yield
\[
\|u_n(t)\|^2 \leq \|u_0\|^2 - \nu \int_0^t \|u_n(s)\|^2_{D(A)} ds + c_N \int_0^t (1 + \|u_n(s)\|^2) ds 
+ 2 \sum_{k=1}^\infty \int_0^t \|\sigma_k(u_n(s)), u_n(s)\| dW_k(s). \tag{31}
\]
Taking expectation, we get
\[
\mathbb{E}\left[\|u_n(t)\|^2\right] \leq \mathbb{E}\|u_0\|^2 - \nu \int_0^t \mathbb{E}\|u_n(s)\|^2_{D(A)} ds + c_N \int_0^t (1 + \mathbb{E}\|u_n(s)\|^2) ds. \tag{32}
\]
Hence by Gronwall’s inequality, we have for any $T > 0$,
\[
\sup_{t \in [0, T]} \mathbb{E}\|u_n(t)\|^2 + \nu \int_0^T \mathbb{E}\|u_n(s)\|^2_{D(A)} ds \leq C_N (1 + \mathbb{E}\|u_0\|^2). \tag{33}
\]
This proves 1).

2) Applying the Burkholder’s inequality to the martingale
\[
M_t = 2 \sum_{k=1}^\infty \int_0^t \langle \sigma_k(u_n(s)), u_n(s) \rangle_V dW_k(s),
\]
we have
\[
\mathbb{E}\left(\sup_{t \in [0, T]} \left| \sum_{k=1}^\infty \int_0^t \langle \sigma_k(u_n(s)), u_n(s) \rangle_V dW_k(s) \right| \right) 
\leq c \mathbb{E}\left(\int_0^T \sum_{k=1}^\infty \|\sigma_k(u_n(s)), u_n(s)\|_V^2 ds \right)^\frac{1}{2} 
\leq c \mathbb{E}\left(\sup_{s \in [0, T]} \|u_n(s)\|^2\right)^\frac{1}{2} \left(\int_0^T (1 + \|u_n(s)\|^2) ds \right)^\frac{1}{2}
\leq \epsilon \mathbb{E}\left(\sup_{s \in [0, T]} \|u_n(s)\|^2\right) + C \epsilon \int_0^T (1 + \|u_n(s)\|^2) ds \tag{34}
\]
Combining (31), (33) and (34), we get
\[
\mathbb{E}\left(\sup_{t \in [0, T]} \|u_n(t)\|^2\right) \leq C_N \epsilon (1 + \mathbb{E}\|u_0\|^2) + \epsilon \mathbb{E}\left(\sup_{t \in [0, T]} \|u_n(t)\|^2\right)
\]
Choosing $c$ small enough, we obtain
\[
E \left( \sup_{t \in [0,T]} \|u_n(t)\|^2 \right) \leq C_{T,N} \left( 1 + E\|u_0\|^2 \right).
\]
This ends the proof of 2).

3) We apply Itô’s formula the function $f(x) = x^3$ and the real-valued process $Y(t) = \|u_n(t)\|^2$ we get
\[
\|u_n(t)\|^6 \leq \|u_n(0)\|^6 + 6 \int_0^t \|u_n(s)\|^4((G(u_n(s), u_n(s)))ds
+ 3 \sum_{k=1}^{\infty} \int_0^t \|u_n(s)\|^4\|\pi_n \sigma_k(u_n(s))\|^2 ds
+ 12 \sum_{k=1}^{\infty} \int_0^t \|u_n(s)\|^2((\pi_n \sigma_k(u_n(s)), u_n(s)))^2 ds
+ 6 \sum_{k=1}^{\infty} \int_0^t \|u_n(s)\|^4((\pi_n \sigma_k(u_n(s)), u_n(s)))dW_k(s),
\]
or
\[
\|u_n(t)\|^6 \leq \|u_n(0)\|^6 - 6\nu \int_0^t \|u_n(s)\|^4\|u_n(s)\|^2_{\mathcal{D}(A)} ds + 6\nu \int_0^t \|u_n(s)\|^6 ds
- 3\nu \int_0^t \|u_n(s)\|^4\|u_n(s)\|^2_{\mathcal{D}(A)} ds + 6\nu \int_0^t \|u_n(s)\|^6 ds
+ 6c \int_0^t \|u_n(s)\|^4 ds + 6c \int_0^t \|u_n(s)\|^6 ds
+ 3c \int_0^t \|u_n(s)\|^4 ds + 3c \int_0^t \|u_n(s)\|^6 ds
+ 12c \int_0^t \|u_n(s)\|^4(1 + \|u_n(s)\|^2) ds
+ 6 \sum_{k=1}^{\infty} \int_0^t \|u_n(s)\|^4((\pi_n \sigma_k(u_n(s)), u_n(s)))dW_k(s). \tag{35}
\]
Using the inequality $|x|^{2p-2} \leq 1 + |x|^{2p}$ for $p \geq 1$, we get
\[
\|u_n(t)\|^6 + 9\nu \int_0^t \|u_n(s)\|^4\|u_n(s)\|^2_{\mathcal{D}(A)} ds
\leq \|u_0\|^6 + c \int_0^t (1 + \|u_n(s)\|^6) ds + 6 \sum_{k=1}^{\infty} \int_0^t \|u_n(s)\|^4((\pi_n \sigma_k(u_n(s)), u_n(s)))dW_k(s).
\]
Taking the supremum over $[0,t]$, we obtain
\[
\sup_{s \in [0,t]} \|u_n(s)\|^6 + 9\nu \int_0^t \|u_n(s)\|^4\|u_n(s)\|^2_{\mathcal{D}(A)} ds \leq \|u_0\|^6 + c \int_0^t (1 + \|u_n(s)\|^6) ds
+ 6 \sup_{s \in [0,t]} \left( \sum_{k=1}^{\infty} \int_0^s \|u_n(s)\|^4((\pi_n \sigma_k(u_n(s)), u_n(s)))dW_k(s) \right). \tag{36}
\]
By Burkholder’s inequality, we have
\[
\mathbb{E} \sup_{s' \in [0,t]} \left( \sum_{k=1}^{\infty} \int_0^{s'} \|u_n(s)\|^4 (\pi_n \sigma_k(u_n(s)), u_n(s)) dW_k(s) \right)
\]
\[
\leq c \mathbb{E} \left( \int_0^t \sum_{k=1}^{\infty} \|u_n(s)\|^8 (\pi_n \sigma_k(u_n(s)), u_n(s))^2 ds \right)^{\frac{1}{4}}
\]
\[
\leq c \mathbb{E} \left( \int_0^t \sum_{k=1}^{\infty} \|u_n(s)\|^8 \|\pi_n \sigma_k(u_n(s))\|^2 \|u_n(s)\|^2 ds \right)^{\frac{1}{4}}
\]
\[
\leq \mathbb{E} \left( \int_0^t \|u_n(s)\|^{10} (1 + \|u_n(s)\|^2) ds \right)^{\frac{1}{2}}
\]
\[
\leq c \mathbb{E} \left( \sup_{s \in [0,t]} \|u_n(s)\|^6 \right)^{\frac{1}{2}} \left( \mathbb{E} \int_0^t \|u_n(s)\|^4 (1 + \|u_n(s)\|^2) ds \right)^{\frac{1}{4}}
\]
\[
\leq c \mathbb{E} \left( \sup_{s \in [0,t]} \|u_n(s)\|^6 \right) + \frac{c}{2} \mathbb{E} \int_0^t \|u_n(s)\|^4 ds + \frac{c}{2} \mathbb{E} \int_0^t \|u_n(s)\|^6 ds
\]
\[
\leq c \mathbb{E} \left( \sup_{s \in [0,t]} \|u_n(s)\|^6 \right) + \frac{c}{2} T + \frac{c}{2} \mathbb{E} \int_0^t \|u_n(s)\|^6 ds. \quad (37)
\]

Taking the expectation in (36) and using (37), we obtain
\[
\mathbb{E} \sup_{s \in [0,t]} \|u_n(s)\|^6 + 9 \nu \mathbb{E} \int_0^t \|u_n(s)\|^4 \|u_n(s)\|^2_{D(A)} ds
\]
\[
\leq \mathbb{E} \|u_0\|^6 + CT + \int_0^t \mathbb{E} \sup_{s' \in [0,s]} \|u_n(s')\|^6 ds
\]
\[
+ \frac{c}{2} T + \frac{c}{2} \mathbb{E} \sup_{s \in [0,t]} \|u_n(s)\|^6 + \frac{c}{2} \mathbb{E} \int_0^t \sup_{s' \in [0,s]} \|u_n(s')\|^6 ds.
\]

Taking \(\epsilon\) sufficiently small, we get
\[
\mathbb{E} \sup_{s \in [0,t]} \|u_n(s)\|^6 + 9 \nu \mathbb{E} \int_0^t \|u_n(s)\|^4 \|u_n(s)\|^2_{D(A)} ds
\]
\[
\leq \mathbb{E} \|u_0\|^6 + (CT + \frac{c}{2} T) + \left(1 + \frac{c}{2} \epsilon\right) \int_0^t \mathbb{E} \sup_{s' \in [0,s]} \|u_n(s')\|^6 ds. \quad (38)
\]

Applying Gronwall’s inequality, we obtain
\[
\mathbb{E} \sup_{s \in [0,t]} \|u_n(s)\|^6 + 9 \nu \mathbb{E} \int_0^t \|u_n(s)\|^6 ds \leq c \left( \mathbb{E} \|u_0\|^6 + 1 \right). \quad (39)
\]

This ends the proof of 3).

4) We have
\[
|\pi_n G(u_n(t))|^2 \leq c \left( |A u_n|^2 + |B_N(u_n)|^2 + |F(u_n)|^2 \right) \quad (40)
\]

Using the estimate (4), we get
\[
\langle B_N(u_n), v \rangle \leq c \|u_n\| \|A u_n\| \|v\| \leq c \|u_n\| \|A u_n\| \|v\|.\]
This implies that
\[ |B_N(u_n)|^2 \leq c\|u_n\|^3|A_{u_n}| \leq \frac{c\epsilon}{2}\|u_n\|^6 + \frac{c}{2\epsilon}|u_n|^2_{D(A)}. \] (41)
We also have
\[ |F(u_n)|^2 \leq c(1 + |u_n|^2) \leq c(1 + \|u_n\|^2). \] (42)
Using (41) and (42) in (40), we get
\[ |\pi_n G(u_n(t))|^2 \leq c(|u_n|^2_{D(A)} + \|u_n\|^6 + 1 + \|u_n\|^2) \]
and
\[ \mathbb{E} \int_0^T |\pi_n G(u_n(t))|^2 < \infty. \]
This ends the proof of the lemma.

Let \( \Omega_T = \Omega \times [0, T] \). Using the energy estimates (23)-(26) along with the Banach-Alaoglu theorem, we can extract a subsequence of \{u_n\} still denoted by \{u_n\}_n and processes \( \tilde{u} \in L^2(\Omega_T; H^0) \cap L^2(\Omega; L^\infty([0, T]; V)) \), \( F \in L^2(\Omega_T; H^0) \) and \( \tilde{\sigma} := (\tilde{\sigma}_k)_{k \in \mathbb{N}} \in L_2(l_2; V) \) for which the following hold:

i) \( u_n \rightharpoonup \tilde{u} \) weakly in \( L^2(\Omega_T; D(A)) \), hence weakly in \( L^2(\Omega_T; H^1) \),

ii) \( u_n \rightharpoonup \tilde{u} \) in \( L^2(\Omega, L^\infty([0, T]; V)) \) with respect to the weak star topology,

iii) \( \pi_n G(u_n) \rightharpoonup G \) weakly in \( L^2(\Omega_T; H^0) \),

iv) \( \pi_n \sigma(u_n) \rightharpoonup \tilde{\sigma} \) weakly in \( L^2(\Omega_T; l_2(V)) \),

v) \( u_n \rightarrow \tilde{u} \) weakly also in \( L^6(\Omega_T; V) \).

For \( 0 \leq t \leq T \), define
\[ u(t) = u_0 + \int_0^t G(s) ds + \sum_{k=1}^{\infty} \int_0^t \tilde{\sigma}_k(s) dW_k(s). \]

It follows from [31] that \( u = \tilde{u} \) \( dt \times \mathbb{P} \)-a.e. and \( u \) has continuous paths in \( V \). To complete the proof of the theorem, we need to show that
\[ F(s) = F(\tilde{u}(s)) \) and \( \tilde{\sigma}(s) = \sigma_k(\tilde{u}(s)) - a.e. on \Omega_T. \]
The proof follows the same steps as in [34]. Fix an integer \( K \). Take \( v \in L^2(\Omega_T, H_K) \) where \( H_K \) is the linear span of \( e_1, e_2, ..., e_K \). By Itô’s formula, writing \( u = u - v + v \), we have
\[
\mathbb{E} \left[ |u(t)|^2 e^{-r(t)} \right] - \mathbb{E} \left[ |u_0|^2 \right] = \mathbb{E} \left[ \int_0^t 2e^{-r(s)} \langle G(s), u(s) \rangle_{H^0} ds \right] \\
+ \mathbb{E} \left[ \int_0^t e^{-r(s)} \sum_{k=1}^{\infty} |\tilde{\sigma}_k(s)|^2 ds \right] - \mathbb{E} \left[ \int_0^t e^{-r(s)} r'(s) |u(s)|^2 ds \right] \\
= \mathbb{E} \left[ \int_0^t 2e^{-r(s)} \langle F(s), u(s) \rangle_{H^0} ds \right] + \mathbb{E} \left[ \int_0^t e^{-r(s)} \sum_{k=1}^{\infty} |\tilde{\sigma}_k(s)|^2 ds \right] \\
- \mathbb{E} \left[ \int_0^t e^{-r(s)} r'(s) \left( |u(s) - v(s)|^2 + 2\langle u(s) - v(s), v(s) \rangle_{H^0} + |v(s)|^2 ds \right) ds \right], \] (43)
where \( r(t) \) is a non-negative stochastic process which is absolutely continuous and to be chosen later. A similar expression also holds for \( \mathbb{E} \left[ |u_n(t)|^2 e^{-r(t)} \right] - \mathbb{E} \left[ |u_0|^2 \right] \), that is
\[
\mathbb{E} \left[ |u_n(t)|^2 e^{-r(t)} \right] - \mathbb{E} \left[ |u_0|^2 \right] = \mathbb{E} \left[ \int_0^t 2e^{-r(s)} \langle G(u_n(s)), u_n(s) \rangle_{H^0} ds \right].
\]
For any non-negative \( \psi \in L^\infty([0, T], \mathbb{R}) \), the weak convergence implies that
\[
\int_0^T \psi(t)dt \mathbb{E} \left[ |u(t)|^2 e^{-\tau(t)} \right] - \mathbb{E} \left[ |u_0|^2 \right] = \int_0^T \psi(t)dt \mathbb{E} \left[ |\tilde{u}(t)|^2 e^{-\tau(t)} \right] - \mathbb{E} \left[ |u_0|^2 \right].
\]
(45)

By substituting the corresponding expressions, (45) becomes
\[
\int_0^T \psi(t)dt \left\{ \mathbb{E} \left[ \int_0^t 2e^{-\tau(s)} \langle G(u(s)), u(s) \rangle_{H^0} ds \right] + \mathbb{E} \left[ \int_0^t e^{-\tau(s)} \sum_{k=1}^{\infty} |\tilde{\sigma}_k(u(s))|^2 ds \right] \right\}
- \int_0^T \psi(t)dt \left\{ \mathbb{E} \left[ \int_0^t e^{-\tau(s)} \rho'(s) \{ |u(s) - v(s)|^2 + 2(u(s) - v(s), v(s))_{H^0} \} ds \right] \right\}
\leq \liminf_{n \to \infty} \int_0^T \psi(t)dt \left\{ \mathbb{E} \left[ \int_0^t 2e^{-\tau(s)} \langle G(u_n(s)), u_n(s) \rangle_{H^0} ds \right] + \liminf_{n \to \infty} \int_0^T \psi(t)dt \mathbb{E} \left[ \int_0^t e^{-\tau(s)} \sum_{k=1}^{\infty} |\tilde{\pi}_n \sigma_k(u_n(s))|^2 ds \right] 
- \liminf_{n \to \infty} \int_0^T \psi(t)dt \mathbb{E} \left[ \int_0^t e^{-\tau(s)} \rho'(s) \{ |u_n(s) - v(s)|^2 + 2(u_n(s) - v(s), v(s))_{H^0} \} ds \right] \right\}
= \liminf_{n \to \infty} Z_n,
\]
(46)

where \( Z_n = Z_n^1 + Z_n^2 + Z_n^3 \) with
\[
Z_n^1 = \int_0^T \psi(t)dt \mathbb{E} \int_0^t e^{-\tau(s)} \{ -\rho'(s) |u_n(s) - v(s)|^2 + 2(G(u_n(s)) - G(v(s)), u_n(s) - v(s))_{H^0} \} ds
+ \int_0^T \psi(t)dt \mathbb{E} \int_0^t \sum_{k=1}^{\infty} |\tilde{\pi}_n \sigma_k(u_n(s)) - \pi_n \sigma_k(v(s))|^2 ds,
\]
(47)

\[
Z_n^2 = \int_0^T \psi(t)dt \mathbb{E} \int_0^t e^{-\tau(s)} \{ -2\rho'(s) (u_n(s) - v(s), v(s))_{H^0} + 2(G(u_n(s)), v(s))_{H^0} \} ds
+ \int_0^T \psi(t)dt \mathbb{E} \int_0^t \sum_{k=1}^{\infty} \langle \tilde{\pi}_n \sigma_k(u_n(s)), \sigma_k(v(s)) \rangle_{H^0} ds,
\]
(48)

\[
Z_n^3 = \int_0^T \psi(t)dt \mathbb{E} \int_0^t e^{-\tau(s)} \left\{ 2 \sum_{k=1}^{\infty} \langle \tilde{\pi}_n \sigma_k(u_n(s)), \pi_n \sigma_k(v(s)) - \sigma_k(v(s)) \rangle_{H^0} \right\} ds
\]
Therefore by the dominated convergence theorem, we get
\[
- \int_0^T \psi(t) dt \mathbb{E} \int_0^t e^{-\tau(s)} \sum_{k=1}^\infty |\pi_n \sigma_k(v(s))|^2 ds.
\] (49)
Set \( r'(s) = c + 2(\nu + c_3 N^4 + c) \). In view of (17) and (16) we see that \( Z_n^1 \leq 0 \). By the weak convergence, it is clear that \( Z_n^2 \to Z^2 \), where
\[
Z^2 = \int_0^T \psi(t) dt \mathbb{E} \int_0^t e^{-\tau(s)} \{ -2r'(s)(u(s) - v(s)) + 2\langle G(s), v(s) \rangle_{H^0} \} ds
+ 2 \int_0^T \psi(t) dt \mathbb{E} \int_0^t e^{-\tau(s)} \{ \langle G(v(s)), u(s) \rangle_{H^0} - 2\langle G(v(s)), v(s) \rangle_{H^0} \} ds,
\]
\[
+ 2 \int_0^T \psi(t) dt \mathbb{E} \int_0^t e^{-\tau(s)} \sum_{k=1}^\infty (\bar{\sigma}_k(s), \sigma_k(v(s)))_{H^0} ds.
\] (50)
Also
\[
Z_n^3 \to Z^3 := - \int_0^T \psi(t) dt \sum_{k=1}^\infty \mathbb{E} \left[ \int_0^t |\sigma_k(v(s))|^2 ds \right].
\] (51)
Combining (46)-(51) after cancellations it turns out that
\[
\int_0^T \psi(t) dt \mathbb{E} \int_0^t e^{-\tau(s)} \{ -r'(s)|u(s) - v(s)|^2 + 2\langle G(s) - G(v(s)), u(s) - v(s) \rangle_{H^0} \} ds
\]
\[
+ \int_0^T \psi(t) dt \mathbb{E} \int_0^t e^{-\tau(s)} \sum_{k=1}^\infty |\bar{\sigma}_k(s) - \sigma_k(v(s))|^2 ds \leq 0.
\] (52)
As \( K \) is arbitrary, by approximation it is seen that (52) holds true for every \( v \in L^2(\Omega_T, H^2) \). In particular, take \( v(s) = u(s) \) in (52) to obtain \( \bar{\sigma}_k(s) = \sigma_k(u(s)) \) for every \( k \geq 1 \). For \( \lambda \in [-1, 1], \tilde{v} \in L^\infty(\Omega_T, H^2) \), set \( v_\lambda(s) = u(s) - \lambda \tilde{v}(s) \). Replace \( v \) by \( v_\lambda \) in (52) to get
\[
\mathbb{E} \left[ \int_0^T e^{-r_\lambda(s)} \{ -\lambda^2 r_\lambda'(s) |\tilde{v}(s)|^2 + 2\lambda \langle G(s) - G(v_\lambda(s)), \tilde{v}(s) \rangle_{H^0} \} ds \right] \leq 0,
\] (53)
where \( r_\lambda(s) \) is defined as \( r(s) \) with \( v \) replaced by \( v_\lambda \). Dividing (53) by \( \lambda \) we obtain
\[
\mathbb{E} \left[ \int_0^T e^{-r_\lambda(s)} \{ -\lambda r_\lambda'(s) |\tilde{v}(s)|^2 + 2\lambda \langle G(s) - G(v_\lambda(s)), \tilde{v}(s) \rangle_{H^0} \} ds \right] \leq 0,
\] (54)
for \( \lambda > 0 \), and
\[
\mathbb{E} \left[ \int_0^T e^{-r_\lambda(s)} \{ -\lambda r_\lambda'(s) |\tilde{v}(s)|^2 + 2\lambda \langle G(s) - G(v_\lambda(s)), \tilde{v}(s) \rangle_{H^0} \} ds \right] \geq 0
\] (55)
for \( \lambda < 0 \).
By (17), we have
\[
|\langle G(u(s)) - G(v_\lambda(s)), \tilde{v}(s) \rangle_{H^0}| \leq \frac{|\lambda|}{2} \| \tilde{v}(s) \|^2 + (\nu + c_3 N^4 + c)|\lambda| \| \tilde{v}(s) \|^2.
\] (56)
Therefore by the dominated convergence theorem, we get
\[
\lim_{\lambda \to 0} \mathbb{E} \left[ \int_0^T e^{-r_\lambda(s)} \langle G(s) - G(v_\lambda(s)), \tilde{v}(s) \rangle_{H^0} ds \right] = \mathbb{E} \left[ \int_0^T e^{-r_0(s)} \langle G(s) - G(u(s)), \tilde{v}(s) \rangle_{H^0} ds \right].
\] (57)
Letting $\lambda \to 0^+$ in (54) and $\lambda \to 0^-$ in (55), we obtain
\[
E \left[ \int_0^T e^{-\tau_0(s)} \langle G(s) - G(u(s)), \tilde{v}(s) \rangle_{H^0} ds \right] = 0.
\]

As $\tilde{v}$ is arbitrary, we conclude that $G(s) = G(u(s))$ a.e. on $\Omega_T$. Then
\[
u(t) = u_0 - \int_0^t \nu A u(s) ds - \int_0^t B_N(u(s)) ds - \int_0^t F(u(s)) ds + \sum_{k=1}^\infty \int_0^t \sigma_k(u(s)) dW_k(s).
\]

**Step 2:** General case: $E \left( \|u_0\|^2 \right) < \infty$.

Let $X_n(0) \in L^6(\Omega, F_0, V)$ such that $E\|X_n(0) - u_0\|^2_{D(A)} \to 0$.

Let $X_n(t), t \geq 0$ be the solution of the following equation
\[
\begin{cases}
dX_n(t) = -\nu AX_n(t) dt - B_N(X_n(t)) dt - F(X_n(t)) dt + \sum_{k=1}^\infty \sigma_k(X_n(t)) dW_k(t) \\
X_n(0) = X_n(0) \in V.
\end{cases}
\]

The existence of $X_n$ is guaranteed by Step 1. As in the proof of (23), we can show that
\[
\sup_n \left\{ E \sup_{t \in [0,T]} \|X_n(t)\|^2 + \int_0^T E \|X_n(t)\|^2_{D(A)} dt \right\} \leq c \sup_n E\|X_n(0)\|^2 < \infty.
\]

This implies that there exist a subsequence of $X_n$ still denoted by the same symbol and a process $X \in L^2(\Omega; L^\infty([0,T]; V)) \cap L^2(\Omega_T; D(A))$ such that
i) $X_n \to X$ weakly in $L^2(\Omega_T; D(A))$,
ii) $X_n \to X$ in $L^2(\Omega; L^\infty([0,T]; V))$ equipped with the weak star topology.

Next, we show that $X_n$ also converges to $X$ in probability in $L^\infty([0,T]; H^1)$.

For $R > 0$, define the stopping time
\[
\tau^n_R := \inf \{ t \in [0,\infty) : \|X_n(t)\| > R \} \quad \text{or} \quad \int_0^T \|X_n(s)\|^2_{D(A)} ds \geq R
\]

$\tau^n_R$ is a stopping time since $X_n$ is continuous in $V$. It follows from (62) that there exists a constant $M$, independent of $n, R$ so that
\[
P(\tau^n_R < T) \leq P \left( \sup_{t \in [0,T]} \|X_n(t)\| > R \right) + P \left( \int_0^T \|X_n(t)\|^2_{D(A)} dt > R \right) \leq \frac{M}{R^2} + \frac{M}{R}.
\]

We are going to prove that
\[
E \left( \sup_{0 \leq t \leq \tau^n_R} \|X_n(t) - X_m(t)\|^2 \right) \leq C_{R,T} \|X_n(0) - X_m(0)\|^2.
\]

Let $X_{n,m}(t) = X_n(t) - X_m(t)$ and $\tau^n_{R,m} = \tau^n_R \wedge \tau^m_R$. By Itô’s formula, we have
\[
d\|X_{n,m}(t)\|^2 = 2\nu \langle AX_{n,m}(t), X_{n,m}(t) \rangle_V dt + 2\langle B_N(X_{n,m}(t) - B_N(X_m(t)), X_{n,m}(t) \rangle_V dt + \langle F(X_n(t)) - F(X_m(t)), X_{n,m}(t) \rangle_V dt \\
= 2 \sum_{k=1}^{\infty} \langle \sigma_k(X_n(t)) - \sigma_k(X_m(t)), X_{n,m}(t) \rangle_V dW_k(t)
\]
+ \sum_{k=1}^{\infty} \|\sigma_k(X_n(t)) - \sigma_k(X_m(t))\|^2 dt \\

\text{We now estimate each term of } (62).

i) \langle AX_n, n(t), X_n, m(t)\rangle_V = -\|X_n, m(t)\|_{D(A)}^2 + \|X_n, m(t)\|^2.

ii) \\

\langle B_N(X_n - B_N(X_m), X_n - X_m)\rangle_V \\

= \langle B_N(X_n - B_N(X_m), (I - \Delta)(X_n, m)) \rangle_{H} \\

= B_N(X_n, X_n, (I - \Delta)X_n, m) - B_N(X_m, X_m, (I - \Delta)X_n, m) \\

= F_N(\|X_n\|) b(X_n - X_m, X_n, (I - \Delta)X_n, m) \\

+ F_N(\|X_m\|) b(X_m, X_n, (I - \Delta)X_n, m) \\

+ F_N(\|X_m\|) b(X_m, X_n - X_m, (I - \Delta)X_n, m) \\

We then estimate each term of this equality as follows. From the properties of 

\( F_N \) and (4), we have 

\[ F_N(\|X_n\|) b(X_n - X_m, X_n, (I - \Delta)X_n, m) \leq c_2 \frac{N}{\|X_n\|} \|X_n\| (I - \Delta)X_n, m \|^2 \]

\[ \leq \frac{\nu'}{8} (I - \Delta)X_n, m \|^2 + C' \|X_n\|^2 \]

\[ \leq \frac{\nu'}{8} \|X_n, m\|^2_{D(A)} + C' \|X_n\|^2. \quad (63) \]

For the second term, we get 

\[ |F_N(\|X_n\|) - F_N(\|X_m\|)| b(X_m, X_n, (I - \Delta)X_n, m) \]

\[ \leq \frac{\nu'}{4} (I - \Delta)X_n, m \|^2 + C'' |AX_m|^2 \|X_n, m\|^2 \]

\[ \leq \frac{\nu'}{4} \|X_n, m\|^2_{D(A)} + C'' |AX_m|^2 \|X_n, m\|^2. \quad (64) \]

For the third term, we have 

\[ F_N(\|X_m\|) b(X_m, X_n, (I - \Delta)X_n, m) \]

\[ \leq |AX_m| \|X_n, m\| (I - \Delta)X_n, m \]

\[ \leq \frac{\nu'}{8} (I - \Delta)X_n, m \|^2 + C''' |AX_m|^2 \|X_n, m\|^2 \]

\[ \leq \frac{\nu'}{8} \|X_n, m\|^2_{D(A)} + C''' \|X_m\|^2_{D(A)} \|X_n, m\|^2. \quad (65) \]

Combining (63)-(65), we get 

\[ |(B_N(X_n) - B_N(X_m), X_n - X_m)\rangle_V \]

\[ \leq \frac{\nu'}{2} \|X_n, m\|^2_{D(A)} + C \left( 1 + \|X_n\|^2_{D(A)} \right) \|X_n, m\|^2. \quad (66) \]

iii) Finally \[ |F(X_n) - F(X_m), X_n - X_m)\rangle_V \leq C \|X_n, m\|^2. \]

By (62), for any pair of stopping times \( 0 \leq \sigma_a \leq \sigma_b \leq \tau_R^\nu \wedge \tau_R^{\nu_1} \), we have 

\[ \mathbb{E} \left( \sup_{t \in [\sigma_a, \sigma_b]} \|X_n, m(t)\|^2 + \nu \int_{\sigma_a}^{\sigma_b} \|X_n, m(t)\|^2_{D(A)} dt \right) \]
\[
\begin{align*}
&\leq c \mathbb{E} \left( \| X_{n,m}(\tau_0) \|^2 + \int_{\tau_0}^{\tau_b} (1 + \| X_m(s) \|_{D(A)}^2) \| X_{n,m}(s) \|^2 ds \right) \\
&+ \mathbb{E} \left( \sup_{t \in [\tau_a, \tau_b]} \left| 2 \sum_{k=1}^{\infty} \int_{\tau_a}^{t} \langle \sigma_k(X_n) - \sigma_k(X_m), X_n - X_m \rangle_V dW_k \right| \right) \\
&\quad \leq c \mathbb{E} \left( \sum_{k=1}^{\infty} \int_{\tau_a}^{\tau_b} \langle \sigma_k(X_n) - \sigma_k(X_m), X_n - X_m \rangle_V^2 dt \right)^{\frac{1}{2}} \\
&\quad \leq \frac{1}{2} \mathbb{E} \sup_{t \in [\tau_a, \tau_b]} \| X_{n,m}(t) \|^2 + C \mathbb{E} \int_{\tau_a}^{\tau_b} \| X_{n,m}(s) \|^2 ds
\end{align*}
\]

Equation (67)

For the last term in (67), the Burkholder-Davis-Gundy inequality implies

\[
\mathbb{E} \left( \sup_{t \in [\tau_a, \tau_b]} \left| 2 \sum_{k=1}^{\infty} \int_{\tau_a}^{t} \langle \sigma_k(X_n) - \sigma_k(X_m), X_n - X_m \rangle_V dW_k \right| \right) \\
\leq \frac{1}{2} \mathbb{E} \sup_{t \in [\tau_a, \tau_b]} \| X_{n,m}(t) \|^2 + C \mathbb{E} \int_{\tau_a}^{\tau_b} \| X_{n,m}(s) \|^2 ds
\]

Combining (67) and (68), we get

\[
\begin{align*}
\mathbb{E} \left( \sup_{t \in [\tau_a, \tau_b]} \| X_{n,m}(t) \|^2 + \nu \int_{\tau_a}^{\tau_b} \| X_{n,m}(t) \|_{D(A)}^2 dt \right) &\leq c \mathbb{E} \left( \| X_{n,m}(\tau_0) \|^2 + \int_{\tau_a}^{\tau_b} (1 + \| X_m(s) \|_{D(A)}^2) \| X_{n,m}(s) \|^2 ds \right),
\end{align*}
\]

where \( c \) is a constant independent of the choice of \( \tau_0, \tau_b \).

By definition of \( \tau_R \), we have

\[
\int_{0}^{\tau_R^{n} \wedge \tau_R} (1 + \| X_m \|_{D(A)}^2) \leq (R + 1) \mathbb{P} - a.s..
\]

Then by application of the Gronwall lemma for stochastic processes (see Lemma 5.5), we obtain

\[
\mathbb{E} \left( \sup_{0 \leq t \leq \tau_R^{n} \wedge \tau_R} \| X_n(t) - X_m(t) \|^2 \right) \leq C_{R,T} \mathbb{E} \| X_n(0) - X_m(0) \|^2.
\]

and this proves (61).

For \( \eta > 0 \) and any \( R > 0 \), we get

\[
\mathbb{P} \left( \sup_{t \in [0,T]} \| X_n(t) - X_m(t) \| > \eta \right) \\
\leq \mathbb{P}(\tau_R^{n} \leq T) + \mathbb{P}(\tau_R^{m} \leq T) + \mathbb{P} \left( \sup_{t \in [0,\tau_R^{n} \wedge \tau_R]} \| X_n(t) - X_m(t) \| > \eta \right).
\]

Equation (70)

Given an arbitrary small constant \( \delta > 0 \), in view of (60), one can choose \( R \) such that \( \mathbb{P}(\tau_R^{n} \leq T) \leq \frac{\delta}{4} \) and \( \mathbb{P}(\tau_R^{m} \leq T) \leq \frac{\delta}{4} \). For such \( R \), by (61) there exists \( N_0 \) such that for \( m, n \geq N_0 \),

\[
\mathbb{P} \left( \sup_{t \in [0,\tau_R^{n} \wedge \tau_R]} \| X_n(t) - X_m(t) \| > \eta \right) \leq \frac{\delta}{4}.
\]

Therefore

\[
\mathbb{P} \left( \sup_{t \in [0,T]} \| X_n(t) - X_m(t) \| > \eta \right) \leq \delta.
\]
and
\[ \lim_{n,m \to \infty} \mathbb{P} \left( \sup_{t \in [0,T]} \|X_n(t) - X_m(t)\| > \eta \right) = 0 \]

This proves that \( X_n \) converges to \( X \) in probability in \( L^\infty([0,T]; H^1) \). Finally we want to show that \( X \) solves (10). To this end, it suffices to prove that for \( v \in V \),
\[
(X(t), v) = (u_0, v) - \int_0^T \langle AX(s), v \rangle ds - \int_0^T \langle B_N(X(s)), v \rangle ds \\
- \int_0^T (F(X(s)), v) ds + \sum_{k=1}^\infty \int_0^T (\sigma_k(X(s)), v) dW_k(s). \tag{71}
\]

But for every \( n \geq 1 \), we know that
\[
(X_n(t), v) = (u_0, v) - \int_0^T \langle AX_n(s), v \rangle ds - \int_0^T \langle B_N(X_n(s)), v \rangle ds \\
- \int_0^T (F(X_n(s)), v) ds + \sum_{k=1}^\infty \int_0^T (\sigma_k(X_n(s)), v) dW_k(s). \tag{72}
\]

Since \( X_n \) converges to \( X \) in probability in \( L^\infty([0,T]; H^1) \), there exists a subsequence of \( X_n \) (still denoted by the same symbol) such that \( X_n \) converges to \( X \) in \( H^1 \) for almost all \( t \in [0,T] \), that is
\[ X_n \to X \text{ in } H^1 \text{ p.p. } t \in [0,T]. \tag{73} \]

Since
\[
\langle B_N(X_N(s)), v \rangle = F_N(\|X_N(s)\|)b(X_n(s), X_n(s), v)
= -F_N(\|X_N(s)\|)b(X_n(s), v(s), X_n(s)),
\]
we also have
\[
-\int_0^T F_N(\|X_n(s)\|)b(X_n(s), v(s), X_n(s)) ds \to -\int_0^T F_N(\|X(s)\|)b(X, v, X)
= \int_0^T F_N(\|X(s)\|)b(X, v, X).
\]

Therefore
\[
\langle B_n(X_n(t)), v \rangle \to \langle B_N(X(t)), v \rangle \text{ p.p.t.}
\]

and
\[
\lim_{n \to \infty} \int_0^T F_N(\|X_n(s)\|)b(X_n(s), X_n(s), v) ds = \int_0^T F_N(\|X(s)\|)b(X(s), X(s), v) ds.
\]

We also have
\[
\mathbb{E} \left( \int_0^T F_N(\|X_n(s)\|)b(X_n(s), X_n(s), v) ds \right)^2 \\
\leq c_2 \mathbb{E} \left( \int_0^T \|X_n(s)\|^{\frac{3}{2}} |AX_n(s)|^{\frac{1}{2}} \|X_n\| |v| ds \right)^2 \\
\leq c \mathbb{E} \left( \int_0^T |AX_n(s)| \|X_n(s)\| |v| ds \right)^2 \\
\leq c|v|^2 \mathbb{E} \left( \int_0^T |AX_n|^2 ds \right) \mathbb{E} \left( \int_0^T \|X_n(s)\|^2 ds \right) < \infty.
\]
From Vitali’s theorem, we conclude that
\[
\int_0^t F_N(\|X_n(s)\|)b(X_n(s), X_n(s), v)ds \to \int_0^t F_N(\|X(s)\|)b(X(s), X(s), v)ds
\]
in $L^2(\Omega)$.
We also have
\[
\int_0^t (F(X_n(s)), v)ds \to \int_0^t (F(X(s)), v)ds,
\]
\[
\sum_{k=1}^{\infty} \int_0^t (\sigma_k(X_n(s)), v)dW_k(s) \to \sum_{k=1}^{\infty} \int_0^t (\sigma_k(X(s)), v)dW_k(s)
\]
in $L^2(\Omega)$.
By the weak convergence, we have
\[
\int_0^T \langle AX_n(s), v \rangle ds \to \int_0^T \langle AX(s), v \rangle ds
\]
in $L^2(\Omega)$.
Collecting all these convergences, $X$ satisfies (71) and this ends the proof of the existence.

4. Convergence to martingale solutions of the stochastic 3D Navier-Stokes equations. Let $\mu_0$ be a probability measure on $V$ such that \( \int_V \|U\|^6 d\mu_0(U) < \infty \). Let $u_0$ be an $F_0$-random variable in $V$ with distribution $\mu_0$. Let $u^N$ be the unique strong solution of the stochastic 3D Globally Modified Navier-Stokes equations. In this section, we are going to study the asymptotic behavior of $u^N$ when $N \to \infty$.

4.1. Some a priori estimates.

**Lemma 4.1.** We have the following a priori estimates on $u^N$

\[
\mathbb{E} \sup_{s \in [0, T]} |u^N(s)|^2 \leq c_1,
\]
(74)
\[
\mathbb{E} \int_0^T \|u^N(s)\|^2 ds \leq c_2,
\]
(75)
\[
\mathbb{E} \sup_{s \in [0, T]} |u^N(s)|^4 \leq c_3,
\]
(76)
\[
\mathbb{E} \left( \int_0^T \|u^N(s)\|^2 ds \right)^2 \leq c_4,
\]
(77)
where the constants $c_1, c_2, c_3$ and $c_4$ are independent of $N$.

**Proof.** By Itô’s formula, we get
\[
|u^N(t)|^2 = |u_0|^2 - 2\nu \int_0^t \langle Au^N, u^N \rangle_V ds - 2 \int_0^t \langle B_N(u^N), u^N \rangle_{V', V} ds
\]
\[
- \int_0^t (F(u^N(s)), u^N(s))ds + 2 \sum_{k=1}^{\infty} \int_0^t (\sigma_k(u^N), u^N) dW_k(s)
\]
\[
+ \sum_{k=1}^{\infty} \int_0^t |\sigma_k(u^N(s))|^2 ds.
\]
(78)
For \( u \in V \), we have

\[
\begin{align*}
  a) & \quad -\nu \langle Au, u \rangle_{V, V'} = -\nu \|u\|^2, \\
  b) & \quad (B_N(u), u)_{V, V'} = 0, \\
  c) & \quad (F(u), u) \leq c(1 + |u|^2), \\
  d) & \quad \sum_{k=1}^{\infty} |\sigma_k(u)|^2 \leq c(1 + |u|^2).
\end{align*}
\]  

(79-82)

Using the estimates (79)-(82) in (78), we arrive at

\[
\begin{align*}
  |u^N|^2 + 2\nu \int_0^t \|u^N(s)\|^2 ds \\
  \leq |u_0|^2 + c \int_0^t (1 + |u^N|^2) ds + 2 \sum_{k=1}^{\infty} \int_0^t (\sigma_k(u^N), u^N) dW_k(s). \\
\end{align*}
\]

(83)

Taking the supremum over \([0, T]\), we get

\[
\begin{align*}
  \sup_{s \in [0, T]} |u^N(s)|^2 + 2\nu \int_0^T \|u^N(s)\|^2 ds \\
  \leq |u_0|^2 + c \int_0^T (1 + |u^N|^2) ds + 2 \sup_{t \in [0, T]} \left( \sum_{k=1}^{\infty} \int_0^t (\sigma_k(u^N), u^N) dW_k(s) \right). \\
\end{align*}
\]

(84)

Raising both sides to the power \( \frac{p}{2} \) for \( p \geq 2 \), then taking expectations, we obtain with the Minkowski inequality and Fubini’s theorem

\[
\begin{align*}
  \mathbb{E} \sup_{s \in [0, T]} |u^N(s)|^p \leq & \mathbb{E}|u_0|^p + c \int_0^T (1 + |u^N|^p) ds \\
  + & 2^{\frac{p}{2}} \mathbb{E} \sup_{t \in [0, T]} \left( \sum_{k=1}^{\infty} \int_0^t (\sigma_k(u^N), u^N) dW_k(s) \right)^{\frac{p}{2}}. \\
\end{align*}
\]

(85)

For the stochastic term, we use the Burkholder-Davis-Gundy inequality

\[
\begin{align*}
  \mathbb{E} \sup_{t \in [0, T]} \left( \sum_{k=1}^{\infty} \int_0^t (\sigma_k(u^N), u^N) dW_k(s) \right)^{\frac{p}{2}} \\
  \leq c \mathbb{E} \left( \sum_{k=1}^{\infty} \int_0^T |u^N|^2 |\sigma_k(u^N)|^2 ds \right)^{\frac{p}{4}} \\
  \leq c \mathbb{E} \left( \sup_{s \in [0, T]} |u^N|^2 \int_0^T (1 + |u^N|^2) dt \right)^{\frac{p}{4}} \\
  \leq \frac{1}{2} \mathbb{E} \sup_{s \in [0, T]} |u^N|^p + c' \mathbb{E} \int_0^T |u^N|^p dt + c'. \\
\end{align*}
\]

(86)

Applying the above estimate to (85), we obtain

\[
\frac{1}{2} \mathbb{E} \sup_{t \in [0, T]} |u^N(s)|^p \leq \mathbb{E}|u_0|^p + c' \mathbb{E} \int_0^T |u^N(s)|^p ds + c'.
\]

(87)
Since
\[ \mathbb{E} \int_0^T |u^N(t)|^p \, dt \leq \int_0^T \mathbb{E} \sup_{s \in [0,t]} |u^N(s)|^p \, dt, \]
the deterministic Gronwall lemma implies that
\[ \mathbb{E} \sup_{t \in [0,T]} |u^N(t)|^p \leq \mathbb{E}|u_0|^p + c'. \] (88)

Letting \( p = 4 \) and \( p = 2 \), we obtain the estimates (74) and (76).

The estimate (83) implies
\[ 2 \nu \int_0^t \|u^N(s)\|^2 \, ds \leq |u_0|^2 + c \int_0^t (1 + |u^N|^2) \, ds + 2 \sum_{k=1}^{\infty} \int_0^t (\sigma_k(u^N), u^N) \, dW_k(s). \] (89)

Taking the supremum over \([0, T]\), raising both sides to the power 2 then taking expectation, we obtain with Minkowski’s inequality and Fubini’s theorem
\[ \mathbb{E} \left( \int_0^T \|u^N(s)\|^2 \, ds \right)^2 \leq c \mathbb{E}|u_0|^4 + c \mathbb{E} \int_0^T |u^N(s)|^4 \, ds \]
\[ \quad + 4 \mathbb{E} \sup_{t \in [0,T]} \left( \sum_{k=1}^{\infty} \int_0^t (\sigma_k(u^N), u^N) \, dW_k(s) \right)^2. \] (90)

For the stochastic term, we have
\[ \mathbb{E} \sup_{t \in [0,T]} \left( \sum_{k=1}^{\infty} \int_0^t (\sigma_k(u^N), u^N) \, dW_k(s) \right)^2 \]
\[ \leq c \mathbb{E} \sum_{k=1}^{\infty} \int_0^T |u^N|^2 |\sigma_k(u^N)|^2 \, ds \]
\[ \quad + \frac{1}{2} \mathbb{E} \sup_{t \in [0,T]} |u^N(t)|^4 + c' \int_0^T |u^N(s)|^4 \, ds + c'. \]

This together with (90) implies the estimate (77). The proof of Lemma 4.1 is complete. \( \square \)

4.2. \textbf{Estimates in fractional Sobolev spaces.} We will apply the compactness result based on fractional Sobolev spaces in Lemma 5.2 (of the Appendix) with
\[ \mathcal{Y} = L^2(0, T; V) \cap W^{\alpha, 2}(0, T; D(A^{-1})), \quad 0 < \alpha < \frac{1}{2}. \] (91)

For this purpose, we will need the following estimates on fractional derivatives of \( u^N \).

\textbf{Lemma 4.2.}

\[ \mathbb{E}|u^N|_{\mathcal{Y}} \leq k_1, \]
\[ \mathbb{E} \left| u^N - \int_0^t \sigma(u^N) \, dW(s) \right|_{H^1(0, T; V')}^2 \leq k_2, \]
\[ \mathbb{E} \left| \int_0^t \sigma(u^N) \, dW(s) \right|_{W^{\alpha, 6}(0, T; H)}^2 \leq k_3, \quad \forall \alpha < \frac{1}{2}, \]

where the constants \( k_1, k_2 \) and \( k_3 \) are independent of \( N \).
Proof. \( u^N \) can be written as

\[
u^N(t) = u_0 - \nu \int_0^t A u^N(s) ds - \int_0^t B_N(u^N) ds - \int_0^t F(u^N) ds + \sum_{k=1}^{\infty} \int_0^t \sigma_k(u^N(s))dW_k(s) =: J_1 + J_2 + J_3 + J_4 + J_5.
\]

(95)

For \( J_2 \), we have

\[|Au^N|_{V'} \leq c \|u^N\|.\]

(96)

With (75), we obtain

\[E|J_2|^2_{W^{1,2}(0,T;V')}\]

is bounded independently of \( N \).

For \( J_3 \), we observe that for \( v \in D(A) \)

\[\langle B_N(u^N), v \rangle = b_N(u^N, u^N, v) = F_N(||u^N||)b(u^N, u^N, v) = c ||u^N||v||_{L^\infty(\Omega)}^3 \leq c ||u^N||||Av||,\]

(98)

where we have used the relation (2) in the last inequality. This implies that

\[E|B_N(u^N)|_{L^2(0,T;D(A^{-1}))}^2 \leq c E \left( \sup_{s \in [0,T]} |u^N(s)|^2 \int_0^T ||u^N(s)||^2 ds \right) \]

\[\leq c \left( E \sup_{s \in [0,T]} |u^N(s)|^4 \right)^{\frac{1}{2}} \left( E \left( \int_0^T ||u^N(s)||^2 ds \right)^2 \right)^{\frac{1}{2}}.
\]

(99)

This along with (76) and (77) conclude that

\[E|J_3|^2_{W^{1,2}(0,T;D(A^{-1}))}\]

is bounded independently of \( N \).

For \( J_4 \), using the estimate (74), we have

\[E|J_4|^2_{W^{1,2}(0,T;V')}\]

is also bounded independently of \( N \).

For the term \( J_5 \), Lemma 5.2 implies that, \( \forall \alpha < \frac{1}{2} \)

\[E|J_5|_{W^{\alpha,6}(0,T;H)}^6 = \left[ \sum_{k=1}^{\infty} \int_0^T \sigma_k(u^N(s))dW_k(s) \right]_{W^{\alpha,2}(0,T;H)}^2 \leq C(\alpha) E \int_0^T \sum_{k=1}^{\infty} |\sigma_k(u^N(s))|^6 ds \leq C(\alpha) E \int_0^T (1 + ||u^N||^6) ds.
\]

This together with (88) imply that

\[E|J_5|^2_{W^{\alpha,6}(0,T;H)}\]

is bounded independently of \( N \), \( \forall \alpha < \frac{1}{2} \). Hence we obtain (94). Collecting the estimates (97)-(102), we obtain

\[E\|u^N\|_{W^{\alpha,2}(0,T;D(A^{-1}))}^2\]

(103)
is bounded independently of $N$. By (75), we deduce
\[ \mathbb{E}[u^N|_{L^2(0,T;V)}] \] is bounded independently of $N$. From (103) and (104), we obtain (92).

We observe from (95) that $u^N(t) - \int_0^t \sigma(u^N) dW(s) = J_1 + J_2 + J_3 + J_4$ combined with the estimates (97)-(101), we obtain (93) as desired.

### 4.3. Compactness arguments for $\{(u^N, W)\}_N$.

With the estimates independent of $N$, we can establish the compactness of the family $(u^N, W)$. For this purpose, we consider the following phase spaces:
\[
X_u = L^2(0,T;H) \cap C([0,T];D(A^{-1})) \quad \text{and} \quad X_W = C([0,T];U_0), \quad X = X_u \times X_W.
\] We then define the probability laws of $u^N$ and $W$ respectively in the corresponding phase spaces:
\[
\mu^N_u(.) = \mathbb{P}(u^N \in .), \quad \mu^N_w(.) = \mathbb{P}(W \in .).
\]

This defines a family of probability measures $\mu^N = \mu^N_u \times \mu^N_w$ on the phase space $X$. We now prove that this family is tight in $N$. More precisely:

**Lemma 4.3.** Consider the measures $\mu^N$ on $X$ defined according to (106) and (107). Then the family $\{\mu^N\}_N$ is tight and therefore weakly compact over the phase space $X$.

**Proof.** We can use the same technic as in the proof of Lemma 4.1 in [11]. The main idea is to apply Lemma 5.2 (of the Appendix) and Chebychev’s inequality to (92)-(94).

Strong convergence as $N \to \infty$. Since the family of measures $\{\mu^N\}$ associated with the family $(u^N, W)$ is weakly compact on $X$, we deduce that $\mu^N$ converges weakly to a probability $\mu$ on $X$ up to a subsequence. We can apply the Skorokhod embedding theorem (see Theorem 2.4 in [10], also [20]) to deduce the strong convergence of a further subsequence, that is:

**Proposition 1.** There exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, and a subsequence $N_k$ of random vectors $(\tilde{u}^{N_k}, \tilde{W}^{N_k})$ with values in $X$ such that

(i) $(\tilde{u}^{N_k}, \tilde{W}^{N_k})$ have the same probability distributions as $(u^{N_k}, W^{N_k})$.

(ii) $(\tilde{u}^{N_k}, \tilde{W}^{N_k})$ converges almost surely as $N_k \to \infty$, in the topology of $X$, to an element $(\tilde{u}, \tilde{W}) \in X$, i.e.
\[
\tilde{u}^{N_k} \to \tilde{u} \text{ strongly in } L^2(0,T;H) \cap C([0,T];D(A^{-1})) \text{ a.s.,}
\]
\[
\tilde{W}^{N_k} \to \tilde{W} \text{ strongly in } C([0,T];U_0) \text{ a.s.,}
\]

where $(\tilde{u}, \tilde{W})$ has distribution $\mu$.

(iii) $\tilde{W}^{N_k}$ is a cylindrical Wiener process, relative to the filtration $\tilde{\mathcal{F}}^{N_k}$, given by the completion of the $\sigma$-algebra generated by $\{(\tilde{u}^{N_k}(s), \tilde{W}^{N_k}(s)); s \leq t\}$.

(iv) All the statistical estimates on $u^{N_k}$ are valid for $\tilde{u}^{N_k}$, in particular, the estimates (74)-(77) hold.
Each pair \((\tilde{u}^{N_k}, \tilde{W}^{N_k})\) satisfies (10) as an equation in \(L^2(M)\), that is
\[
\begin{cases}
    d\tilde{u}^{N_k}(t) = -\nu \tilde{u}^{N_k} \, dt - B_N(\tilde{u}^{N_k}) \, dt - F(\tilde{u}^{N_k}) \, dt + \sum_{i=1}^{\infty} \sigma_i(\tilde{u}^{N_k}(t)) \, d\tilde{W}_i^{N_k}(t), \\
    \tilde{u}^{N_k}(0) = \tilde{u}_0^{N_k}.
\end{cases}
\]

(110)

The following lemma proves that \(\tilde{u}^{N_k}\) is weakly continuous with value in \(H\).

**Lemma 4.4.** The stochastic processes \(\tilde{u}^{N_k}\) and \(\tilde{u} \in C([0,T];H_u) \overset{\mathbb{P}}{\longrightarrow} a.s..\)

**Proof.** The proof follows from the fact that \(\tilde{u}^{N_k} \in L^\infty(0,T;H) \cap C([0,T], D(A^{-1}))\) a.s., hence \(\tilde{u}^{N_k}\) is weakly continuous with values in \(H\) a.s.. \(\square\)

### 4.4. Passage to the limit.

With the strong convergence in (108), we can pass to the limit in (110). Thanks to (76) and (75), we deduce the existence of an element
\[\tilde{u} \in L^4(\hat{\Omega}; L^\infty(0,T; L^2(M))) \cap L^2(\hat{\Omega}; L^2(0,T; H^1_0(M)),\]
and a subsequence still denoted as \(N_k\) such that
\[\tilde{u}^{N_k} \rightharpoonup \tilde{u}\] weak star in \(L^4(\hat{\Omega}; L^\infty(0,T; L^2(M)))\),

(111)

and
\[\tilde{u}^{N_k} \rightarrow \tilde{u}\] weakly in \(L^2(\hat{\Omega}; L^2(0,T; H^1_0(M))\).

(112)

Combining the strong convergence (108), the estimate (76) and the Vitali convergence theorem, we get
\[\tilde{u}^{N_k} \rightarrow \tilde{u}\] strongly in \(L^2(\hat{\Omega}; L^2(0,T; H))\),

(113)

and, possibly extracting a new subsequence denoted in the same way to save notation, one has also
\[\tilde{u}^{N_k} \rightarrow \tilde{u}\] for almost all \(\omega, t\) with respect to the measure \(d\hat{\mathbb{P}} \otimes dt\).

(114)

Fix \(w \in D(A)\). Using the weak convergence (112), we can pass to the limit in the linear term.

We are going to prove that
\[\int_0^T F_{N_k}(\|\tilde{u}^{N_k}(s)\|)b(\tilde{u}^{N_k}(s), \tilde{u}^{N_k}(s), w) \, ds \rightarrow \int_0^T b(\tilde{u}(s), \tilde{u}(s), w) \, ds\]
in \(L^1(\hat{\Omega} \times (0,T))\).

(115)

The following lemma will be crucial for the proof of (115).

**Lemma 4.5.** We have
\[F_N(\|\tilde{u}^N(s)\|) \rightarrow 1 \text{ in } L^p(\hat{\Omega}; L^p(0,T; \mathbb{R})) \text{ as } N \rightarrow \infty \text{ and } p > 1.\]

(116)

**Proof.** From the estimate (75), we have
\[\tilde{E} \int_0^T \|\tilde{u}^N(s)\|^2 \, ds \leq k_1.\]

Let
\[O_N = \{s \in (0,T), \|\tilde{u}^N(s)\| \geq N \text{ a.s.}\}\]
and \(|O_N|\) the Lebesgue measure of \(O_N\). Then
\[N^2 \tilde{E}|O_N| \leq \tilde{E} \int_0^T \tilde{u}^N(s) \, ds \leq k_1,\]
and so
\[\tilde{E}|O_N| \leq \frac{k}{N^2} \rightarrow 0 \text{ as } N \rightarrow \infty.\]
Observing that
\[ T - |O_N| = \int_{[0,T]-O_N} F_N(\|u^N(s)\|) ds, \]
we deduce that
\[ T - \tilde{E} |O_N| \leq \tilde{E} \int_0^T F_N(\|\tilde{u}^N(s)\|) ds \leq T. \]
These inequalities show that
\[ \tilde{E} \int_0^T F_N(\|\tilde{u}^N(s)\|) ds \to \int_0^T 1 ds \text{ as } N \to \infty. \]

But as \( 0 \leq F_N(\|\tilde{u}^N(s)\|) \leq 1 \), we get
\[ \tilde{E} \int_0^T |1 - F_N(\|u^N(s)\|)| ds = \tilde{E} \int_0^T (1 - F_N(\|u^N(s)\|)) ds \to 0 \text{ as } N \to \infty. \]

Finally, since \(|1 - F_N(\|\tilde{u}^N(s)\|)| \leq 1\), we arrive at
\[ \tilde{E} \int_0^T |1 - F_N(\|u^N(s)\|)|^{p-1} ds \leq \tilde{E} \int_0^T |1 - F_N(\|u^N(s)\|)| ds \to 0 \text{ as } N \to \infty. \]

This ends the proof of the lemma.

For the proof of (115), we introduce the abbreviations as in [6],
\[ F^{N_k}(s) = F_{N_k}(\|\tilde{u}^{N_k}(s)\|), \]
\[ b^{N_k}(s) = b(\tilde{u}^{N_k}(s), \tilde{u}^{N_k}(s), w), \]
\[ b(s) = b(\tilde{u}(s), \tilde{u}(s), w). \]

To prove (115), we write
\[
\tilde{E} \int_0^T \left( \int_0^t (F^{N_k}(s)b^{N_k}(s) - b(s)) ds \right) dt \\
= \tilde{E} \int_0^T \left( \int_0^t (F^{N_k}(s) - 1)b^{N_k}(s) ds \right) dt + \tilde{E} \int_0^T \left( \int_0^t b^{N_k}(s) - b(s) ds \right) dt. \tag{117}
\]

Reasoning as in the proof of the existence of martingale solutions of the 3D Navier-Stokes equations, the second term of this equality tends to 0, that is
\[
\int_0^t b^{N_k}(s) ds \to \int_0^t b(s) ds \text{ in } L^1(\tilde{\Omega} \times (0,T)). \tag{118}
\]

For the first term, we get
\[
\tilde{E} \int_0^T \left( \int_0^t (F^{N_k}(s) - 1)b^{N_k}(s) ds \right) dt \\
\leq \left( \tilde{E} \int_0^T \int_0^t |F^{N_k}(s) - 1|^2 ds dt \right)^{\frac{1}{2}} \left( \tilde{E} \int_0^T \int_0^t |b^{N_k}(s)|^2 ds dt \right)^{\frac{1}{2}} \\
\leq T \left( \tilde{E} \int_0^T |F^{N_k}(s) - 1|^2 ds \right)^{\frac{1}{2}} \left( \tilde{E} \int_0^T |b^{N_k}(s)|^2 ds \right)^{\frac{1}{2}},
\]
and

\[ \int_0^t (F^{N_k}(s) - 1) b^{N_k}(s) ds \to 0 \text{ in } L^1(\tilde{\Omega} \times (0, T)), \]

since

\[ \tilde{E} \int_0^T |b^{N_k}(s)|^2 ds \leq c_2 |Aw|^2 \tilde{E} \int_0^T |\tilde{u}^{N_k}(s)|^2 \|\tilde{u}^{N_k}(s)\|^2 ds \]

\[ \leq c_2 |Aw|^2 \left( \tilde{E} \sup_{s \in [0, T]} |\tilde{u}^{N_k}(s)|^4 \right) \left( \tilde{E} \left( \int_0^T \|\tilde{u}^{N_k}(s)\|^2 ds \right)^2 \right)^{\frac{1}{2}} \]

\[ < \infty, \]

and Lemma 4.5 shows that

\[ \tilde{E} \int_0^T |F^{N_k}(s) - 1|^2 ds \to 0. \]

This proves (115).

The convergence

\[ \int_0^t (F(\tilde{u}^{N_k}(s), w) ds \to \int_0^t (F(u), w) ds \text{ in } L^1(\tilde{\Omega} \times (0, T)) \] (119)

follows from estimate (76), the Lipschitz condition on \( F \) and the Vitali convergence theorem.

For the stochastic term, by (114), we obtain

\[ |\tilde{u}^{N_k} - \tilde{u}|^2 \to 0, \text{ for a.e. } (\omega, t) \in \tilde{\Omega} \times (0, T). \]

Thus, along with Lipschitz condition on \( \sigma \), we deduce

\[ |\sigma(\tilde{u}^{N_k}) - \sigma(\tilde{u})|_{L_2^2} \to 0 \text{ for a.e. } (\omega, t) \in \tilde{\Omega} \times (0, T). \]

On the other hand

\[ \sup_{N_k} \tilde{E} \left( \int_0^T |\sigma(\tilde{u}^{N_k})|^4_{L_2^2} ds \right) \leq \sup_{N_k} \tilde{E} \left( \int_0^T (1 + |\tilde{u}^{N_k}(s)|^4) ds \right). \]

We therefore infer from (76) that \( |\sigma(\tilde{u}^{N_k})|_{L_2^2} \) is uniformly integrable for \( N_k \) in

\[ L^q(\tilde{\Omega} \times (0, T)) \]

for any \( q \in [1, 4] \).

With the Vitali convergence theorem, we deduce that for all such \( q \in [1, 4] \),

\[ \sigma(\tilde{u}^{N_k}) \to \sigma(\tilde{u}) \text{ in } L^q(\tilde{\Omega}, L^q(0, T; L_2^2(H))). \] (120)

In particular, we get the convergence in probability of \( \sigma(\tilde{u}^{N_k}) \) in \( L^2(0, T; L_2^2(H)) \).

Thus along with the convergence (109), we apply Lemma 5.4 (of the Appendix) and deduce that

\[ \int_0^t \sigma(\tilde{u}^{N_k}) d\tilde{W}^{N_k} \to \int_0^t \sigma(\tilde{u}) d\tilde{W} \] (121)

in probability in \( L^2(0, T; L^2(M)) \).

By (121) and Vitali convergence theorem, we infer a stronger convergence result:

\[ \int_0^t \sigma(\tilde{u}^{N_k}) d\tilde{W}^{N_k} \to \int_0^t \sigma(\tilde{u}) d\tilde{W} \text{ in } L^2(\tilde{\Omega}; L^2(0, T; L^2(M))). \] (122)

Collecting all the convergence results, we obtain

\[ (\tilde{u}(t), w) + \nu \int_0^t (\nabla \tilde{u}(s), \nabla w) ds + \int_0^t b(\tilde{u}(s), \tilde{u}(s), w) ds + \int_0^t (F(\tilde{u}(s)), w) ds \]
\[ (\tilde{u}(0), w) + \int_0^t (\sigma(\tilde{u}(s), w) \, d\tilde{W}(s), \quad (123) \]
for all \( w \in D(A) \) and for a.e. \( \omega \in \tilde{\Omega}, t \in (0, T) \). The equality (123) is also valid for \( w \in V \) by density argument.

We have then proved the following result.

**Proposition 2.** The pair \( (\tilde{S}, \tilde{u}) \) where \( \tilde{S} = (\tilde{\Omega}, \tilde{\mathcal{F}}, \{ \tilde{\mathcal{F}}_t \}, \tilde{\mathbb{P}}, \tilde{W}) \) is a martingale solution of the stochastic 3D Navier-Stokes equations.

We now summarize the result obtained in the following theorem which says that, up to a subsequence the solution \( u^N \) of the stochastic 3D GMNSE converges in law to a martingale solution of the original 3D stochastic Navier-Stokes equations when \( N \) tends to infinity.

**Theorem 4.6 (Convergence of the 3D stochastic GMNSE).** There exists a martingale solution \( (\tilde{\Omega}, \tilde{\mathcal{F}}, \{ \tilde{\mathcal{F}}_t \}, \tilde{\mathbb{P}}, \tilde{W}, \tilde{u}) \) of the stochastic 3D Navier-Stokes equations and a sequence \( (\tilde{u}^N) \) of random processes defined on \( \tilde{\Omega} \), with the same law as \( u^N \), so that up to a subsequence, the following convergence holds:

\[ \tilde{u}^N \to \tilde{u} \text{ in } L^2(\tilde{\Omega} \times [0, T] \times \mathcal{M}). \]

5. **Appendix.** In section 5.1 we recall some results of deterministic nature. In Section 5.2 we present a result of probabilistic nature.

5.1. **Compact embedding theorems.** We recall the theorems from [13] and [14] (see also [11] for Lemma 5.2)

**Definition 5.1.** (The Fractional Derivative Space) We assume that \( H \) is a separable Hilbert space. Given \( p \geq 2, \alpha \in (0, 1) \), \( W^{\alpha,p}(0, T; H) \) denotes the Sobolev space of all \( h \in L^p(0, T; H) \) such that

\[ \int_0^T \int_0^T \frac{|h(t) - h(s)|^p_H}{|t-s|^{1+\alpha p}} \, dt \, ds \]

which is endowed with the Banach norm

\[ |h|_{W^{\alpha,p}(0, T; H)} = \left( \int_0^T |h(t)|^p_H \, dt + \int_0^T \int_0^T \frac{|h(t) - h(s)|^p_H}{|t-s|^{1+\alpha p}} \, dt \, ds \right)^{\frac{1}{p}} < \infty. \]

**Lemma 5.2.** (i) Let \( \mathcal{E}_0 \subset \mathcal{E} \subset \mathcal{E}_1 \) be Banach spaces, \( \mathcal{E}_0 \) and \( \mathcal{E}_1 \) reflexive, with continuous injections and a compact embedding of \( \mathcal{E}_0 \) in \( \mathcal{E} \). Let \( \mathcal{Y} \) be the space

\[ \mathcal{Y} = L^p(0, T; \mathcal{E}_0) \cap W^{\alpha,p}(0, T; \mathcal{E}_1), \]

endowed with the natural norm. Then the embedding of \( \mathcal{Y} \) in \( L^p(0, T; \mathcal{E}) \) is compact.

(ii) If \( \mathcal{E} \subset \hat{\mathcal{E}} \) are two Banach spaces with \( \mathcal{E} \) compactly in \( \hat{\mathcal{E}}, 1 < p < \infty \) and \( \alpha \in (0, 1) \) satisfy

\[ \alpha p > 1, \]

then the space \( W^{\alpha,p}(0, T; \mathcal{E}) \) is compactly embedded into \( \mathcal{C}([0, T]; \hat{\mathcal{E}}) \).

The following lemma is based on the Burkholder-Davis-Gundy inequality and the notion of fractional derivatives (see [13] for the proof).
Lemma 5.3. Let $q \geq 2, \alpha < \frac{1}{q}$ be given so that $q\alpha > 1$ Then for any predictable process $h \in L^2(\Omega \times (0,T); L^2(U, H))$, we have
\[
\int_0^t h(s)dW(s) \in L^q(\Omega, W^{\alpha,q}(0,T; H)),
\]
and there exists a constant $c' = c'(q, \alpha) \geq 0$ independent of $h$ such that
\[
\mathbb{E}\left[ \int_0^t h(s)dW(s) \right]^{q}_{W^{\alpha,q}(0,T; H)} \leq c'(q, \alpha) \mathbb{E}\int_0^t |h(s)|^q_{L^2(U,H)} ds.
\]

5.2. Convergence theorem for the noise term. The following convergence theorem for the stochastic integral is used to facilitate the passage to the limit. The statements and proofs can be found in [2, 11].

Lemma 5.4. Let $(\Omega, F, \mathbb{P})$ be a fixed probability space, and $X$ a separable Hilbert space. Consider a sequence of stochastic bases $S_n := (\Omega, F, \{F^n_t\}_{t \geq 0}, \mathbb{P}, W^n)$, such that each $W^n$ is a cylindrical Brownian motion (over $U$) with respect to $\{F^n_t\}_{t \geq 0}$. We suppose that the $\{G^n\}_{n \geq 1}$ are a sequence of $X$-valued $F^n_t$-predictable processes so that $G^n \in L^2((0,T); L^2(U,X)) \ a.s.$ Finally consider $S := (\Omega, F, \{F_t\}_{t \geq 0}, \mathbb{P}, W)$ and a function $G \in L^2((0,T); L^2(U,X))$, which is $F_t$-predictable. If
\[
W^n \to W \text{ in probability in } C([0,T]; U_0),
\]
\[
G^n \to G \text{ in probability in } L^2((0,T); L^2(U,X)),
\]
then
\[
\int_0^t G^n dW^n \to \int_0^t G dW \text{ in probability in } L^2((0,T); X).
\]

5.3. A stochastic Gronwall lemma. The following Gronwall lemma for stochastic processes is useful to prove the existence of strong solution for the stochastic 3D globally modified Navier-Stokes equations. See [18] for the proof.

Lemma 5.5. Fix $T > 0$. Assume that $X, Y, Z : [0,T] \times \Omega \to \mathbb{R}$ are real-valued, non-negative stochastic processes. let $\tau < T$ be a stopping time so that
\[
\mathbb{E}\int_0^\tau (RX + Z) ds < \infty.
\]
Assume, moreover that for some fixed constant $k$,
\[
\int_0^\tau R ds < k, a.s.
\]
Suppose that for all stopping times $0 \leq \tau_a < \tau_b \leq \tau$
\[
\mathbb{E}\left( \sup_{t \in [\tau_a, \tau_b]} X + \int_{\tau_a}^{\tau_b} Y ds \right) \leq c_0 \mathbb{E}\left( X(\tau_a) + \int_{\tau_a}^{\tau_b} (RX + Z) ds \right),
\]
where $c_0$ is a constant independent of the choice of $\tau_a, \tau_b$. Then
\[
\mathbb{E}\left( \sup_{t \in [0,\tau]} X + \int_0^\tau Y ds \right) \leq c \mathbb{E}\left( X(0) + \int_0^\tau Z ds \right),
\]
where $c = c(c_0, T, k)$. 

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Received October 2016; revised July 2017.

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