Area of a product torus of a link

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Abstract

A 2-component link $C_1 \cup C_2$ gives a torus $C_1 \times C_2$ in the set of the pairs of points in $S^3$. We study the area of the torus with respect to a natural pseudo-Riemannian structure of set of the pairs of points in $S^3$ which is compatible with Möbius transformations. We show that a link attains the minimum area 0 if and only if it is the “best” Hopf link.

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1 Introduction

Geometric knot theory is the study of measuring the geometric complexity of curves by “energy functionals”. One of its origins is [O1], where the first example of energy functionals for knots, the energy $E$, was introduced. It is the renormalization of modified electrostatic energy of charged knots. Later, Freedman, He, and Wang showed that this energy is invariant under Möbius transformations. On the other hand, in his attempt to the Willmore conjecture, Langevin has studied integral geometry on the space of sub-spheres of $S^3$ from a viewpoint of conformal geometry.

In this paper we introduce an energy functional for 2-component links by studying the space of oriented 0-spheres in $S^3$, $S(0, 3)$. It has symplectic structure and pseudo-Riemannian structure both of which are conformally invariant, i.e. compatible with Möbius transformations. They appear quite naturally in the study of conformal geometry. The former is obtained by identifying $S(0, 3)$ with the total space of the cotangent bundle of $S^3$. The latter is obtained by identifying $S(0, 3)$ with a Grassmannian manifold of oriented 2-planes through the origin of the 5-dimensional Minkowski space that intersect the light cone transversely.

A 2-component link $C_1 \cup C_2$ produces a torus $C_1 \times C_2$ in $S(0, 3)$ since a pair of points $(x, y)$, where $x \in C_1$ and $y \in C_2$, is a point in $S(0, 3)$. We define its “(imaginary) signed area” and the area with respect to the pseudo-Riemannian
structure of $S(0,3)$. Our main theorem (Theorem 3.3) claims that the former vanishes for any link and that the latter attains the minimum value 0 if and only if the link is the “best” Hopf link.

The key of the proof is the infinitesimal cross ratio, which was introduced in the joint paper with Langevin [LO]. It is the cross ratio of $x, x + dx, y, y + dy$, where these four points are considered complex numbers by identifying a sphere through them with the Riemann sphere $\mathbb{C} \cup \{\infty\}$. It can be considered a complex valued 2-form on $C_1 \times C_2$. It is conformally invariant. Moreover, it is the unique conformal invariant of a pair of 1-jets of a given curve up to multiplication by a constant. We showed that the energy $E$ can be expressed as the integration of the difference of the absolute value and the real part of the infinitesimal cross ratio ([LO]).

Our theorem can be proved as follows: The first part can be proved by showing that the (imaginary) signed area element of the torus $C_1 \times C_2$ is equal to the real part of the infinitesimal cross ratio. It turns out to coincide with the pull-back of the canonical symplectic form of the cotangent bundle of $S^3$. Therefore the (imaginary) signed area element is exact. The second part follows from geometric argument on the “conformal angle” which can express the infinitesimal cross ratio.

Throughout the paper, a link means a smooth (or at least of class $C^1$) 2-component link.

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2 Space $S(0,3)$ of oriented 0-spheres in $S^3$

Let $S(0,3)$ be the set of oriented 0-spheres in $S^3$. Topologically speaking, it can be identified with $S^3 \times S^3 \setminus \Delta$. We show that it admits natural pseudo-Riemannian structure and symplectic structure which are conformally invariant.

2.1 Symplectic structure

Let us fix a diffeomorphism between $S^3 \times S^3 \setminus \Delta$ and the total space of the cotangent bundle of $S^3$.

Assume $S^3$ is the unit sphere in $\mathbb{R}^4$. Let $x$ be a point in $S^3$, $\Pi_x$ the orthogonal complement of $\text{Span}(x)$, and $p_x : S^3 \setminus \{x\} \to \Pi_x$ a stereographic projection. We identify $T_xS^3$ with $\Pi_x$. Then we can identify $S^3 \setminus \{x\}$ with $T_x^*S^3$ through a bijection given by

$$\varphi_x : \{x\} \ni y \mapsto (T_xS^3 \ni v \mapsto p_x(y) \cdot v \in \mathbb{R}) \in T_x^*S^3,$$

where $\cdot$ denotes the standard inner product in $\mathbb{R}^4$ (Figure 1). It induces a canonical bijection between $S^3 \times S^3 \setminus \Delta$ and $T^*S^3$

$$\varphi : S^3 \times S^3 \setminus \Delta \ni (x, y) \mapsto (x, \varphi_x(y)) \in T^*S^3. \quad (1)$$
As a cotangent bundle of a manifold has the canonical symplectic form, the space $S(0, 3)$ admits a natural symplectic structure. The canonical symplectic form is conformally invariant, namely:

**Lemma 2.1** ([LO]) Let $\omega_{S^3}$ be the canonical symplectic form of the cotangent bundle $T^*S^3$, and $\varphi$ the bijection from $S^3 \times S^3 \setminus \Delta$ to $T^*S^3$ given by (1). Then the pull-back $\varphi^*\omega_{S^3}$ is invariant under the diagonal action of a Möbius transformation $T: (T \times T)^*\varphi^*\omega_{S^3} = \varphi^*\omega_{S^3}$.

### 2.2 Pseudo-Riemannian structure

We show that $S(0, 3)$ is a Grassmannian manifold of oriented 2-planes through the origin of the 5-dimensional Minkowski space that intersect the light cone transversely. The pseudo-Riemannian structure of the Minkowski space naturally induces that of $S(0, 3)$.

#### 2.2.1 Minkowski spaces

The *Minkowski space* $\mathbb{R}_1^5$ is $\mathbb{R}^5$ with indefinite inner product

$$\langle x, y \rangle = -x_0y_0 + x_1y_1 + \cdots + x_4y_4.$$ 

A vector $v$ in $\mathbb{R}_1^5$ is called spacelike if $\langle v, v \rangle > 0$, lightlike if $\langle v, v \rangle = 0$ and $v \neq 0$, and timelike if $\langle v, v \rangle < 0$. The set of lightlike vectors and the origin $L = \{v \in \mathbb{R}_1^5 \mid \langle v, v \rangle = 0\}$ is called the light cone. The “pseudosphere” $\Lambda = \{v \in \mathbb{R}_1^5 \mid \langle v, v \rangle = 1\}$ is called the de Sitter space. Let $\{e_0, e_1, \cdots, e_4\}$ be the standard pseudo-orthonormal basis of $\mathbb{R}_1^5$ with $e_0$ being timelike.

Let $\Pi$ be a 2-dimensional vector subspace of $\mathbb{R}_1^5$. There are three cases which are mutually exclusive. Let $\langle , \rangle|_\Pi$ denote the restriction of $\langle , \rangle$ to $\Pi$.

1. The case when $\langle , \rangle|_\Pi$ is non-degenerate. This case can be divided into two cases.

   1-a) The case when $\langle , \rangle|_\Pi$ is indefinite. It happens if and only if $\Pi$ intersects the light cone transversely. In this case $\Pi$ is said to be *timelike*. 

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Figure 1: The bijection between $S^3 \times S^3 \setminus \Delta$ and $T^*S^3$
(1-b) The case when $\langle \cdot, \cdot \rangle |_{\Pi}$ is positive definite. It happens if and only if $\Pi$ intersects the light cone only at the origin. In this case $\Pi$ is said to be spacelike.

(2) The case when $\langle \cdot, \cdot \rangle |_{\Pi}$ is degenerate. It happens if and only if $\Pi$ is tangent to the light cone. In this case $\Pi$ is said to be isotropic.

The 3-sphere $S^3$ can be realized in $\mathbb{R}^5_1$ as the set of lines through the origin in the light cone. We will denote it by $S^3(\infty)$. It can be identified with the intersection of the light cone and the hyperplane given by $\{x_0 = 1\}$. We will denote it by $S^3(1)$.

2.2.2 The set of 0-spheres as a Grassmann manifold

An oriented 0-sphere in $S^3$ can be realized in $\mathbb{R}^5_1$ as the transversal intersection of $S^3(\infty)$ and an oriented 2-dimensional timelike vector subspace. Therefore the set $S(0, 3)$ can be identified with the Grassmannian manifold $\overline{\text{Gr}}(2; \mathbb{R}^5_1)$ of oriented 2-dimensional timelike subspaces of $\mathbb{R}^5_1$. It follows that the set $S(0, 3)$ is a homogeneous space $SO(4, 1)/SO(3) \times SO(1, 1)$. We give its pseudo-Riemannian structure explicitly in what follows.

2.2.3 Plücker coordinates

Let us recall Plücker coordinates of Grassmannian manifolds. Let $\Pi$ be an oriented 2-dimensional vector subspace in $\mathbb{R}^5_1$ with an ordered basis $\{x, y\}$ which gives the orientation. Define $p_{i_1i_2}$ ($0 \leq i_1, i_2 \leq 4$) by

$$p_{i_1i_2} = \begin{vmatrix} x_{i_1} & x_{i_2} \\ y_{i_1} & y_{i_2} \end{vmatrix}.$$  (2)
They satisfy the Plücker relations:

\[ \sum_{k=1}^{3} (-1)^k p_{i_1 j_2 p(\{j_1 j_2 j_3\})} = 0. \]  

(3)

There are five non-trivial Plücker relations and exactly three of them are independent.

As we are concerned with the orientation of the subspaces, we use the Euclidean spaces for the Plücker coordinates in this article instead of the projective spaces which are used in most cases. The exterior product of \( x \) and \( y \) in \( \mathbb{R}^5_1 \) is given by

\[ x \wedge y = (\cdots, p_{i_1 i_2}, \cdots) \in \mathbb{R}^{10} \quad (i_1 < i_2) \]

through the identification \( \wedge \mathbb{R}^5_1 \cong \mathbb{R}^{10} \).

### 2.2.4 Pseudo-Riemannian structure of \( \wedge \mathbb{R}^5_1 \)

The indefinite inner product of the Minkowski space \( \mathbb{R}^5_1 \) naturally induces that of \( \wedge \mathbb{R}^5_1 \cong \mathbb{R}^{10} \) by

\[ \langle x \wedge y, x' \wedge y' \rangle = -\left| \begin{array}{cc} \langle x, x' \rangle & \langle x, y' \rangle \\ \langle y, x' \rangle & \langle y, y' \rangle \end{array} \right| \]  

(4)

for \( x, y, x', \) and \( y' \) in \( \mathbb{R}^5_1 \).

The above formula is a natural generalization of the one for the exterior products of four vectors which we obtained in [LO], where we studied the set of oriented spheres in \( S^3 \).

It follows that \( \wedge \mathbb{R}^5_1 \) can be identified with \( \mathbb{R}^{10} \) with the pseudo-Riemannian structure with index 6, which we denote by \( \mathbb{R}^{10}_6 \), with \( \{e_{i_1} \wedge e_{i_2}\}_{i_1 < i_2} \) being a pseudo-orthonormal basis with

\[ \langle e_{i_1} \wedge e_{i_2}, e_{i_1} \wedge e_{i_2} \rangle = \begin{cases} -1 & \text{if } i_1 \geq 1, \\
+1 & \text{if } i_1 = 0. \end{cases} \]  

(5)

The norm \( |v| \) of a vector \( v \) in \( \wedge \mathbb{R}^5_1 \cong \mathbb{R}^{10}_6 \) is given by \( \sqrt{|\langle v, v \rangle|} \).

### 2.2.5 \( S(0, 3) \) as a pseudo-Riemannian submanifold of \( \wedge \mathbb{R}^5_1 \)

Let us realize \( S(0, 3) \) as a pseudo-Riemannian submanifold of \( \wedge \mathbb{R}^5_1 \cong \mathbb{R}^{10}_6 \).

**Lemma 2.2** Let \( \Pi \) be a 2-dimensional vector subspace in \( \mathbb{R}^5_1 \) spanned by \( x \) and \( y \). Then \( \Pi \) is timelike if and only if

\[ \langle x \wedge y, x \wedge y \rangle > 0. \]
Proof. Let us use the classification given in subsubsection 2.2.1. 

Case (1): Suppose \( \Pi \) is not isotropic. We may assume without loss of generality that \( \{ x, y \} \) is a pseudo-orthonormal basis of \( \Pi \). If \( \Pi \) is timelike, one of \( x \) and \( y \) is timelike, and therefore \( \langle x \wedge y, x \wedge y \rangle = 1 \) by (4). If \( \Pi \) is spacelike, then \( \langle x \wedge y, x \wedge y \rangle = -1 \).

Case (2): Suppose \( \Pi \) is isotropic. Then \( \Pi \) is tangent to the light cone at a lightlike line \( l \). Now we may assume without loss of generality that \( \{ x, y \} \) is a pseudo-orthogonal basis of \( \Pi \) and that \( x \) belongs to \( l \). Then we have \( \langle x \wedge y, x \wedge y \rangle = 0 \).

This Lemma and Lemma 2.7 in the next subsubsection imply

Proposition 2.3 Let \( \Theta(0, 3) \) be a subspace of \( \bigwedge^2 \mathbb{R}_1^5 \cong \mathbb{R}_6^{10} \) given by

\[
\Theta(0, 3) = \left\{ \cdots, p_{i_1 i_2}, \cdots \in \mathbb{R}_6^{10} \left| \begin{array}{c}
- \sum_{i_1 \geq 1} p_{i_1 i_2}^2 + \sum_{i_2 \geq 1} p_{0 i_2}^2 - 1 = 0 \\
\sum_{k=1}^3 (-1)^k p_{i_1 j_k} p_{(j_1 j_2 j_3) \setminus \{j_k\}} = 0
\end{array} \right. \right\} \quad (6)
\]

It is a 6-dimensional pseudo-Riemannian submanifold of \( \mathbb{R}_6^{10} \) with index 3. The set \( S(0, 3) \) of oriented 0-spheres in \( S^3 \) can be identified with \( \Theta(0, 3) \) through a bijection \( \psi \) given by

\[
\psi : S(0, 3) \xrightarrow{\cong} \mathbb{G}_{-}(2; \mathbb{R}_1^5) \xrightarrow{\cong} \Theta(0, 3) \subset \bigwedge^2 \mathbb{R}_1^5 \cong \mathbb{R}_6^{10}
\]

\[
\Pi \cap S^3(\infty) \mapsto \Pi = \text{Span}(x, y) \mapsto \frac{x \wedge y}{|x \wedge y|}.
\]

2.2.6 Conformal invariance of the pseudo-Riemannian structure

We show that the pseudo-Riemannian structure of \( S(0, 3) \) is conformally invariant, namely, a Möbius transformation of \( S^3 \) induces a pseudo-orthogonal transformation of \( \Theta(0, 3) \subset \bigwedge^2 \mathbb{R}_1^5 \). Let \( O(4, 6) \) denote the pseudo-orthogonal group of \( \mathbb{R}_6^{10} \).

Definition 2.4 Define a map \( \Psi : M_5(\mathbb{R}) \to M_{10}(\mathbb{R}) \) by

\[
\Psi : M_5(\mathbb{R}) \ni A = (a_{ij}) \mapsto \Psi(A) = (\tilde{a}_{IJ}) \in M_{10}(\mathbb{R}),
\]

where \( I = (i_1 i_2) \) and \( J = (j_1 j_2) \) are multi-indices, and \( \tilde{a}_{IJ} \) is given by

\[
\tilde{a}_{IJ} = \begin{vmatrix}
a_{i_1 j_1} & a_{i_1 j_2} \\
a_{i_2 j_1} & a_{i_2 j_2}
\end{vmatrix}.
\]
Lemma 2.5  

(1) We have

\[(Ax) \land (Ay) = \Psi(A)(x \land y) \quad (\forall x, y \in \mathbb{R}^5)\]  \hspace{1cm} (8)

for \( A \in M_5(\mathbb{R}) \).

(2) If \( A \in O(4, 1) \) then \( \Psi(A) \in O(4, 6) \).

(3) The restriction of \( \Psi \) to \( O(4, 1) \) is a homomorphism.

We can say more, although we do not give a proof here: The matrix \( \Psi(A) \) can be characterized by the equation (8). The reverse statement of (2) also holds.

The restriction of \( \Psi \) to \( Gl(5, \mathbb{R}) \) is a homomorphism whose kernel consists of \( \{ \pm I \} \).

Proof.  

(1) The definition of \( \tilde{a}_{IJ} \) implies \( \Psi(A)(e_{i_1} \land e_{i_2}) = (Ae_{i_1}) \land (Ae_{i_2}) \).

(2) If \( A \in O(4, 1) \) then (4) and (8) imply

\[\langle \Psi(A)(e_{i_1} \land e_{i_2}), \Psi(A)(e_{j_1} \land e_{j_2}) \rangle = \langle e_{i_1} \land e_{i_2}, e_{j_1} \land e_{j_2} \rangle,\]

which implies \( \Psi(A) \in O(4, 6) \).

(3) Routine calculation in linear algebra implies \( \Psi(AB)_{IJ} = \sum_K \tilde{a}_{IK} \tilde{b}_{KJ} \).

Then the formula (7) implies

Corollary 2.6  

We have

\[\psi(A \cdot \Sigma) = \Psi(A)\psi(\Sigma)\]

for \( \Sigma \in S(0, 3) \) and \( A \in O(4, 1) \), where \( \psi \) is the bijection from \( S(0, 3) \) to \( \Theta(0, 3) \subset \mathbb{R}^{10}_6 \) given by (7) and \( \Psi \) the homomorphism from \( O(4, 1) \) to \( O(4, 6) \) given in Definition 2.4.

Lemma 2.7  

The restriction of the indefinite inner product of \( \mathbb{R}^{10}_6 \) to each tangent space of \( \Theta(0, 3) \) induces an non-degenerate indefinite inner product of index 3.

Proof.  

The conformal invariance of the pseudo-Riemannian structure allows us to assume that a 0-sphere \( \Sigma \) is given by a pair of antipodal points, say \((\pm 1, 0, 0, 0)\). The index can be calculated in several ways.

(i) It corresponds to \( H = \text{Span}(e_0, e_1) \) in the Grassmannian \( \widetilde{Gr}^{-}(2; \mathbb{R}^5) \). The tangent space \( T_H \widetilde{Gr}^{-}(2; \mathbb{R}^5) \) is isomorphic to \( \text{Hom}(H, H^\perp) \), which is isomorphic to \( M_{2,3}(\mathbb{R}) \). We can construct six vectors which form a pseudo-orthonormal basis of the tangent space explicitly. It turns out that three of them are timelike and the other three are spacelike.

(ii) The tangent space \( T_\Sigma \Theta(0, 3) \) can be identified with the pseudo-orthogonal complements of the vector subspace spanned by gradients of the defining functions of \( \Theta(0, 3) \) which appear in (6). The gradients of the Plücker relations, three
of which are independent, span timelike subspace, and that of \( -\sum_{i_2 \geq 1} p_{i_1 i_2}^2 + \sum_{i_2 \geq 1} p_{i_2}^2 - 1 \) spacelike. Hence the index can be given by \( 6 - 3 = 3 \). □

The index can also be implied by Proposition 3.2.6 of [Ko-Yo] if we use the fact that \( \mathcal{S}(0, 3) \) is a homogeneous space \( SO(4, 1)/SO(3) \times SO(1, 1) \).

3 Area of a product torus in \( \mathcal{S}(0, 3) \) of a link

A pair of ordered points can be considered an oriented 0-sphere. Therefore the set of pairs of points \( \{(x, y) \mid x \in C_1, y \in C_2\} \) forms a torus in \( \mathcal{S}(0, 3) \). Let us call it the product torus of a link \( L = C_1 \cup C_2 \). We identify \( C_1 \times C_2 \) with the torus in \( \mathcal{S}(0, 3) \) in what follows.

Let \( \sigma \) be the composition of maps:

\[
\sigma : C_1 \times C_2 \xrightarrow{\iota} S^3 \times S^3 \setminus \Delta \xrightarrow{\cong} \mathcal{S}(0, 3) \xrightarrow{\psi} \Theta(0, 3),
\]

where \( \psi \) is the bijection given by (7). We identify \( \sigma(C_1 \times C_2) \) with \( C_1 \times C_2 \). The area element \( dv \) associated with the pseudo-Riemannian structure of \( \Theta(0, 3) \) is given by

\[
\sigma^* dv = \sqrt{\left| \begin{array}{cc}
\langle \sigma_x, \sigma_x \rangle & \langle \sigma_x, \sigma_y \rangle \\
\langle \sigma_y, \sigma_x \rangle & \langle \sigma_y, \sigma_y \rangle
\end{array} \right|} \, dx \wedge dy,
\]

where \( \sigma_x \) and \( \sigma_y \) denote \( \frac{\partial \sigma}{\partial x}(x, y) \) and \( \frac{\partial \sigma}{\partial y}(x, y) \) in \( T_{\sigma(x,y)} \Theta(0, 3) \).

Lemma 3.1 Both \( \sigma_x \) and \( \sigma_x \) are null vectors. Namely we have \( \langle \sigma_x, \sigma_x \rangle = \langle \sigma_y, \sigma_y \rangle = 0 \).

We give a computational proof here, as a geometric proof need some preparation.

Proof. Suppose points in \( C_1 \) and \( C_2 \) are parametrized as \( x = x(s) \) and \( y = y(t) \). Let \( \bar{x} \) and \( \bar{y} \) be maps to \( S^3(1) \) in \( \mathbb{R}^5 \) given by \( \bar{x}(s) = (1, x(s)) \) and \( \bar{y}(t) = (1, y(t)) \). Put

\[
p(s, t) = \bar{x}(s) \wedge \bar{y}(t) \in \bigwedge^2 \mathbb{R}^5_1 \cong \mathbb{R}^1_0.
\]

If we put \( \bar{\sigma}(s, t) = \sigma(x(s), y(t)) \) then it is given by

\[
\bar{\sigma}(s, t) = \frac{p(s, t)}{\langle p(s, t), p(s, t) \rangle^{\frac{1}{2}}}.
\]

Since \( \bar{x} \) and \( \bar{y} \) are lightlike vectors, (4) implies

\[
\langle p, p \rangle = -\left| \begin{array}{cc}
\langle \bar{x}, \bar{x} \rangle & \langle \bar{x}, \bar{y} \rangle \\
\langle \bar{y}, \bar{x} \rangle & \langle \bar{y}, \bar{y} \rangle
\end{array} \right| = \langle \bar{x}, \bar{y} \rangle^2
\]

\[
\langle p, p_s \rangle = \langle \bar{x}, \bar{y} \rangle \langle \bar{x}_s, \bar{y} \rangle
\]

\[
\langle p_s, p_s \rangle = -\left| \begin{array}{cc}
\langle \bar{x}_s, \bar{x}_s \rangle & \langle \bar{x}_s, \bar{y} \rangle \\
\langle \bar{y}, \bar{x}_s \rangle & \langle \bar{y}, \bar{y} \rangle
\end{array} \right| = \langle \bar{x}_s, \bar{y} \rangle^2.
\]
Therefore,
\[
\langle \tilde{s}_x, \tilde{s}_s \rangle = \frac{(p,p)(p_s,p_s) - (p,p_s)^2}{(p,p)^2} = 0
\]

It follows that the area element \( dv \) is given by
\[
\sigma^* dv = \sqrt{-\langle \sigma_x, \sigma_y \rangle^2} \, dx \wedge dy.
\]

**Definition 3.2** (1) Define the signed area element \( \alpha \) of \( \sigma(C_1 \times C_2) \) by
\[
\sigma^* \alpha = \langle \sigma_x, \sigma_y \rangle \, dx \wedge dy,
\]
and the signed area of the surface \( \sigma(C_1 \times C_2) \) by
\[
\text{SA}(C_1 \times C_2) = \int_{\sigma(C_1 \times C_2)} \alpha = \int_{x \in C_1} \int_{y \in C_2} \langle \sigma_x, \sigma_y \rangle \, dx \wedge dy.
\]

(2) Define the area of the surface \( \sigma(C_1 \times C_2) \) by
\[
\text{Area}(C_1 \times C_2) = \int_{\sigma(C_1 \times C_2)} |\alpha| = \int_{x \in C_1} \int_{y \in C_2} |\langle \sigma_x, \sigma_y \rangle| \, dx \wedge dy.
\]

**Theorem 3.3** (1) The signed area of a product torus \( \sigma(C_1 \times C_2) \) of any link \( L = C_1 \cup C_2 \) is equal to 0.

(2) The area of a product torus \( \sigma(C_1 \times C_2) \) of a link \( L = C_1 \cup C_2 \) takes its minimum value 0 if and only if \( L \) is an image of the standard Hopf link
\[
\{(z, w) \in \mathbb{C}^2 : |z| = 1, w = 0\} \cup \{(z, w) \in \mathbb{C}^2 : z = 0, |w| = 1\} \subset S^3 \tag{9}
\]
by a Möbius transformation.

### 4 Infinitesimal cross ratio

Let us explain the infinitesimal cross ratio of a link \( L = C_1 \cup C_2 \), which serves as a bridge between two structures of \( S(0,3) \). We assume that both components \( C_1 \) and \( C_2 \) are oriented.

#### 4.1 Geometric definition of the infinitesimal cross ratio

Suppose \( x \in C_1 \) and \( y \in C_2 \). Let \( \Sigma_L(x, y) \) denote a bitangent sphere at \( x \) and \( y \), i.e. the 2-sphere which is tangent to \( C_1 \) at \( x \) and to \( C_2 \) at \( y \). It can be considered the 2-sphere that passes through four points \( x, x+dx, y, \) and \( y+dy \) (Figure 3). It is generically determined uniquely unless these four points are concircular, which is a codimension 2 phenomenon. Identify \( \Sigma_L(x, y) \) with the Riemann sphere \( \mathbb{C} \cup \{\infty\} \) through a stereographic projection \( p \) (Figure 4). Then the four points \( x, x+dx, y, \) and \( y+dy \) can be considered four complex numbers
\[ \Sigma_L(x, y) \]

\[ \Omega_L(x, y) = \frac{\tilde{x} + \tilde{d}x - \tilde{x}}{\tilde{y} - \tilde{d}y} \quad : \quad \frac{\tilde{y} - \tilde{x}}{\tilde{y} - \tilde{d}y} \sim \frac{\tilde{d}x\tilde{d}y}{(\tilde{x} - \tilde{y})^2}. \] (10)

**Remark:**

1. To be precise, we need an orientation of \( \Sigma_L(x, y) \) to avoid the ambiguity of complex conjugacy (the reader is referred to [O2] for the details). We omit the argument of orientation in what follows as we only need the real part of the infinitesimal cross ratio in this paper.

2. The infinitesimal cross ratio does not depend on the stereographic projection by the following reason. Suppose we use another stereographic projection. Then we get another quadruplet of complex numbers, which can be obtained from the former by a linear fractional transformation. Since a linear fractional transformation does not change the cross ratio, \( \Omega_L(x, y) \) is determined uniquely. (We have omitted the argument of orientation here.)

**Definition 4.1** ([LO]) We call \( \Omega_L(x, y) \) the infinitesimal cross ratio of a link \( C_1 \cup C_2 \).

**4.2 Infinitesimal cross ratio as a complex valued 2-form**

We show that the infinitesimal cross ratio can be considered a complex valued 2-form on \( C_1 \times C_2 \).

**Definition 4.2** (Doyle and Schramm) Let \( \Gamma(x, y) \) be an oriented circle which is tangent to \( C_1 \) at \( x \) that passes through \( y \), and \( \Gamma(y, y, x) \) an oriented circle tangent to \( C_2 \) at \( y \) through \( x \). Suppose \( \Gamma(x, x, y) \) and \( \Gamma(y, y, x) \) are oriented by the tangent vectors to \( C_1 \) at \( x \) and to \( C_2 \) at \( C \) respectively. Let \( \theta (0 \leq \theta \leq \pi) \) be the angle between \( \Gamma(y, y, x) \) and \( \Gamma(x, x, y) \). We call it the conformal angle between \( x \) and \( y \) and denote it by \( \theta_L(x, y) \).
Generically, \( \Gamma(x, x, y) \) and \( \Gamma(y, y, x) \) are different. If so, the bitangent sphere \( \Sigma_L(x, y) \) is the unique sphere that contains both \( \Gamma(x, x, y) \) and \( \Gamma(y, y, x) \). The conformal angle is conformally invariant, i.e. we have \( \theta_{T(L)}(T(x), T(y)) = \theta_L(x, y) \) for a Möbius transformation \( T \).

**Proposition 4.3 ([LO])**

1. The infinitesimal cross ratio \( \Omega_L(x, y) \) has the absolute value \( \frac{dxdy}{|x - y|^2} \) and the argument \( \theta_L(x, y) \). Therefore it can be considered a complex valued 2-form on \( C_1 \times C_2 \) given by
   \[
   \Omega_L(x, y) = e^{i\theta_L(x, y)} \frac{dxdy}{|x - y|^2}.
   \]

2. The infinitesimal cross ratio is conformally invariant. Namely, if \( T \) is a Möbius transformation then
   \[
   (T \times T)^* \Omega_{T(L)} = \Omega_L,
   \]
   where \( T \times T \) denotes the diagonal action.

The conformal angle \( \theta : C_1 \times C_2 \to [0, \pi] \) may not be a smooth function of \( x \) and \( y \) where it takes 0 or \( \pi \). It behaves like an absolute value of a smooth function. Therefore, the imaginary part of the infinitesimal cross ratio may be singular where the conformal angle vanishes.

**4.3 The real part of the infinitesimal cross ratio**

We show that the real part of the infinitesimal cross ratio can be interpreted in two ways according to the two structures of \( S(0, 3) \) explained in section 2, namely, as the canonical symplectic form and as the signed area element.
Lemma 4.4 ([LO]) The real part of the infinitesimal cross ratio is equal to the minus one half of pull-back of the canonical symplectic form $\omega_{S^3}$ of the cotangent bundle $T^*S^3$;
\[
\Re \Omega_L = -\frac{1}{2} t^* \varphi^* \omega_{S^3},
\]
where $\varphi$ is the bijection from $S^3 \times S^3 \setminus \Delta$ to $T^*S^3$ given by (1) and $t$ is an inclusion map from $C_1 \times C_2$ to $S^3 \times S^3 \setminus \Delta$.

Lemma 4.5 The real part of the infinitesimal cross ratio is equal to the half of the pull-back of the signed area element of $\sigma(C_1 \times C_2)$:
\[
\Re \Omega_L = \frac{1}{2} \sigma^* \alpha = \frac{1}{2} \langle \sigma_x, \sigma_y \rangle dx \wedge dy.
\]

We give a computational proof here. (We can also give a geometric proof using pseudo-orthogonal basis of $S(0,3)$ which can be obtained by pencils.)

Proof. Let $x(s), y(t), \bar{x}(s), \bar{y}(t), p(s,t)$, and $\tilde{\sigma}(s,t)$ be as in the proof of Lemma 3.1.

Let $\theta_L(x, y)$ denote the conformal angle. The pull-back of the real part of the infinitesimal cross ratio is given by
\[
((x \times y)^* \Re \Omega_L)(s,t) = \frac{\cos \theta_L(x(s), y(t))}{|x(s) - y(t)|^2} |x'(s)||y'(t)| ds \wedge dt.
\]
On the other hand, the pull-back of the signed area element is given by
\[
((x \times y)^* (\langle \sigma_x, \sigma_y \rangle dx \wedge dy))(s,t) = \langle \tilde{\sigma}_s, \tilde{\sigma}_t \rangle(s,t) ds \wedge dt.
\]
Therefore, we have only to show
\[
\langle \tilde{\sigma}_s, \tilde{\sigma}_t \rangle(s,t) = 2 \frac{|x'(s)||y'(t)|}{|x(s) - y(t)|^2} \cos \theta_L(x(s), y(t))
\]
for any fixed $(s_0, t_0)$.

The conformal invariance of the both sides allows us to assume that $x(s_0)$ and $y(t_0)$ are antipodal, say, $x(s_0) = (0,0,1,0)$ and $y(t_0) = (0,0,1,0)$.

Since
\[
\langle \tilde{\sigma}_s, \tilde{\sigma}_t \rangle = \frac{\langle p, p_s \rangle \langle p, p_t \rangle - \langle p, p_s \rangle \langle p, p_t \rangle}{\langle p, p \rangle^2}
\]

the formula (4) implies that the following hold at \((s_0, t_0)\),
\[
\langle p, p \rangle(s_0, t_0) = \langle \bar{x}(s_0) \wedge \bar{y}(t_0), \bar{x}(s_0) \wedge \bar{y}(t_0) \rangle \\
= - \begin{vmatrix} \langle \bar{x}(s_0), \bar{x}(s_0) \rangle & \langle \bar{x}(s_0), \bar{y}(t_0) \rangle \\ \langle \bar{y}(t_0), \bar{x}(s_0) \rangle & \langle \bar{y}(t_0), \bar{y}(t_0) \rangle \end{vmatrix} = - \begin{vmatrix} 0 & -2 \\ -2 & 0 \end{vmatrix} \\
= 4,
\]
\[
\langle p, p_s \rangle(s_0, t_0) = \langle \bar{x}(s_0) \wedge \bar{y}(t_0), \bar{x}'(s_0) \wedge \bar{y}(t_0) \rangle \\
= - \begin{vmatrix} \langle \bar{x}(s_0), \bar{x}'(s_0) \rangle & \langle \bar{x}(s_0), \bar{y}(t_0) \rangle \\ \langle \bar{y}(t_0), \bar{x}(s_0) \rangle & \langle \bar{y}(t_0), \bar{y}(t_0) \rangle \end{vmatrix} = - \begin{vmatrix} 0 & -2 \\ -2 & 0 \end{vmatrix} \\
= 0,
\]
\[
\langle p_s, p_t \rangle(s_0, t_0) = \langle \bar{x}'(s_0) \wedge \bar{y}(t_0), \bar{x}(s_0) \wedge \bar{y}'(t_0) \rangle \\
= - \begin{vmatrix} \langle \bar{x}'(s_0), \bar{x}(s_0) \rangle & \langle \bar{x}'(s_0), \bar{y}(t_0) \rangle \\ \langle \bar{y}(t_0), \bar{x}(s_0) \rangle & \langle \bar{y}(t_0), \bar{y}'(t_0) \rangle \end{vmatrix} = - \begin{vmatrix} 0 & x'(s_0) \cdot y'(t_0) \\ -2 & 0 \end{vmatrix} \\
= -2x'(s_0) \cdot y'(t_0).
\]

Therefore,
\[
\langle \bar{s}_s, \bar{s}_t \rangle(s_0, t_0) = -\frac{1}{2} x'(s_0) \cdot y'(t_0).
\]

Since \(x_0 = x(s_0)\) and \(y_0 = y(t_0)\) are antipodal we have (Figure 7)
\[
\theta_L(x(s_0), y(t_0)) = \pi - \angle x'(s_0) \cdot y'(t_0).
\]

It follows that
\[
\langle \bar{s}_s, \bar{s}_t \rangle(s_0, t_0) = -\frac{1}{2} x'(s_0) \cdot y'(t_0) = 2 \frac{|x'(s_0)||y'(t_0)|}{|x(s_0) - y(t_0)|^2} \cos \theta_L(x(s_0), y(t_0)),
\]

which completes the proof. \(\square\)

The above proof implies that the surface \(\sigma(C_1 \times C_2)\) is generically of mixed type. Namely,
Corollary 4.6 The restriction of the pseudo-Riemannian structure of $S(0, 3)$ to a tangent space of the torus $\sigma(C_1 \times C_2)$ induces a non-degenerate indefinite quadratic form with index 1 if the conformal angle $\theta(x, y)$ between $x$ and $y$ (Definition 4.2) is not equal to $\frac{\pi}{2}$; otherwise it induces a degenerate one.

5 Proof of the main theorem

Proof of Theorem 3.3 (1) Lemma 4.5 and Lemma 4.4 imply that the signed area element $\alpha$ of the torus $\sigma(C_1 \times C_2)$ is equal to the minus of the pull-back of the canonical symplectic form $\omega_{S^3}$ of $T^*S^3$:

$$\sigma^* \alpha = -\iota^* \varphi^* \omega_{S^3}.$$  (14)

Since a canonical symplectic form of a cotangent bundle is an exact form, the signed area element is an exact form.

(2) Lemma 4.5 and Proposition 4.3 imply that the area of of the surface $\sigma(C_1 \times C_2)$ is given by

$$\text{Area}(C_1 \times C_2) = 2 \int_{C_1 \times C_2} |\Re \Omega_L| = 2 \int_{C_1 \times C_2} \frac{\cos \theta_L(x, y)}{|x-y|^2} \, dx \, dy.$$  (15)

It follows that $\text{Area}(C_1 \times C_2)$ is equal to 0 if and only if the conformal angle $\theta(x, y)$ is equal to $\frac{\pi}{2}$ for any $x \in C_1$ and $y \in C_2$.

Suppose $\text{Area}(C_1 \times C_2) = 0$. Let $x$ be a point in $C_1$. Let $C_x$ be the set of the circles which are tangent to $C_1$ at $x$. Then $C_2$ can intersect circles in $C_x$ only in the right angle (Figure 8). Consider a stereographic projection $\pi$ from $S^3 \setminus \{x\}$ to $\mathbb{R}^3$. It maps $C_x$ to the set of parallel lines. Since $\pi(C_2)$ can intersect lines of $\pi(C_x)$ only in the right angle, $\pi(C_2)$ is contained in a 2-plane which is orthogonal to the lines in $\pi(C_2)$ (Figure 9). Therefore, $C_2$ is contained in a sphere $\Sigma_x$ which intersects $C_1$ in the right angle at $x$ (Figure 10).

Let $x'$ be a point of $C_1$ close to $x$. As $C_1$ intersects $\Sigma_x$ orthogonally at $x$, we can take $x'$ outside $\Sigma_x$. Therefore, $\Sigma_x \neq \Sigma_{x'}$. Since $C_2$ is contained in the intersection $\Sigma_x \cap \Sigma_{x'}$, $C_2$ must be a circle (Figure 11). The same argument shows that $C_1$ is also a circle.
Consider the stereographic projection $\pi$ again. Since $C_1$ is a circle, $\pi(C_1)$ is a line. Then $\pi(C_2)$ is the intersection of two spheres which intersect the line $\pi(C_1)$ in the right angle (Figure 12). Therefore, $\pi(C_2)$ is symmetric in the line $\pi(C_1)$. It follows that $\pi(C_1) \cup \pi(C_2)$ is an image of the standard Hopf link (Figure 12).

**Remark:** The formula (14) does not hold for surfaces in $\Theta(0, 3)$ which cannot be obtained as “products” of curves in $S^3$ in general.

Let $[L]$ denote an isotopy class of a link $L$. Let $\text{Area}([L])$ denote the infimum of the areas of the product tori of the links which belong to $[L]$:

$$\text{Area}([L]) = \inf_{C_1' \cup C_2' \in [L]} \text{Area}(C_1' \times C_2').$$

**Corollary 5.1** If $L$ is a separable link or a satellite link of a Hopf link then $\text{Area}([L]) = 0$.

**Proof.** Suppose $L = C_1 \cup C_2$ is a separable link in $\mathbb{R}^3$. We can make $|x - y|$ $(x \in C_1, y \in C_2)$ as big as we like. Now the conclusion follows from the formula (15).

Suppose $L = C_1 \cup C_2$ is a satellite link of a Hopf link. Then it can be captured in a very thin tubular neighbourhood of the standard Hopf link given by (9). Furthermore, for any positive constants $\delta_1$ and $\delta_2$ the link can be placed so that outside a small region of $C_1 \times C_2$ whose measure is $\delta_1 \cdot \text{Length}(C_1) \cdot \text{Length}(C_2)$ the conformal angle satisfies $|\theta_L - \frac{\pi}{2}| \leq \delta_2$. Since $|x - y|$ $(x \in C_1, y \in C_2)$ is bounded below, the formula (15) implies that it completes the proof. \qed

We like to end this article with a conjecture.
Conjecture 5.2 We conjecture that $\text{Area}([L])$ does not always vanish. For example, if $L = C_1 \cup C_2$ is a hyperbolic link each component of which is a non-trivial knot, then there is no solid torus $H_1$ so that $C_1$ is contained in $H_1$ and $C_2$ in $\mathbb{R}^2 \setminus H_1$. We conjecture that $\text{Area}([L])$ is positive for such a link type.

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