Harmonic Analysis of Stochastic Equations and Backward Stochastic Differential Equations

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Abstract

The BMO martingale theory is extensively used to study nonlinear multi-dimensional stochastic equations (SEs) in $\mathcal{R}^p$ ($p \in [1, \infty)$) and backward stochastic differential equations (BSDEs) in $\mathcal{R}^p \times \mathcal{H}^p$ ($p \in (1, \infty)$) and in $\mathcal{R}^\infty \times \mathcal{H}^\infty_{BMO}$, with the coefficients being allowed to be unbounded. In particular, the probabilistic version of Fefferman’s inequality plays a crucial role in the development of our theory, which seems to be new. Several new results are consequently obtained. The particular multi-dimensional linear case for SDEs and BSDEs are separately investigated, and the existence and uniqueness of a solution is connected to the property that the elementary solutions-matrix for the associated homogeneous SDE satisfies the reverse Hölder inequality for some suitable exponent $p \geq 1$. Finally, we establish some relations between Kazamaki’s quadratic critical exponent $b(M)$ of a BMO martingale $M$ and the spectral radius of the solution operator for the $M$-driven SDE, which lead to a characterization of Kazamaki’s quadratic critical exponent of BMO martingales being infinite.

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1 Preliminaries

Let $T > 0$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ be a complete filtered probability space on which a one-dimensional standard Brownian motion $\{W(t)\}_{t \geq 0}$ is defined such that $\{\mathcal{F}_t\}_{t \geq 0}$ is the natural filtration generated by $W(\cdot)$, augmented by all the $P$-null sets in $\mathcal{F}$. Let $H$ be a Banach space. We denote by $\mathcal{L}^p_{\mathcal{F}}(0,T;H)$ ($p \geq 1$) the Banach space consisting of all $H$-valued $\{\mathcal{F}_t\}_{t \geq 0}$-optional processes $X(\cdot)$ such that $E(\|X(\cdot)\|_{L^p(0,T;H)}^2) < \infty$, with the canonical norm; by $\mathcal{L}^\infty_{\mathcal{F}}(0,T;H)$ the Banach space consisting of all $H$-valued $\{\mathcal{F}_t\}_{t \geq 0}$-optional bounded processes; and by $\mathcal{L}^2_{\mathcal{F}}(\Omega;C([0,T];H))$ the Banach space consisting of all $H$-valued $\{\mathcal{F}_t\}_{t \geq 0}$-adapted continuous processes $X$ such that $E(\|X\|_{L^2(C([0,T];H))}^2) < \infty$, with the canonical norm.

**Definition 1.1.** Let $p \in [1,\infty)$. The space $\mathcal{R}^p$ is the space of all continuous adapted processes $Y$ such that

$$\|Y\|_{\mathcal{R}^p} := \|Y_T\|_{L^p} \quad \text{with} \quad Y_T^* := \max_{0 \leq t \leq T} |Y_t|$$

is finite. $\mathcal{H}^p$ is the Banach space of continuous $\{\mathcal{F}_t, 0 \leq t \leq T\}$-adapted local martingales such that

$$\|Y\|_{\mathcal{H}^p} := \|\langle Y \rangle_T^{1/2}\|_{L^p}$$

is finite. $\langle Y \rangle$ denotes the quadratic variation process of a semi-martingale, and $\langle X, Y \rangle$ denotes the covariance process between the two semi-martingales $X$ and $Y$.

Let $M$ be a continuous martingale. Define

$$a(M) := \sup\{a \geq 0 : \sup_\tau E[\exp(a|M_\infty - M_\tau|) | \mathcal{F}_\tau]\}_\infty < \infty \quad (1.3)$$

and

$$b(M) := \sup\left\{b \geq 0 : \sup_\tau \left\| E \left[ \exp \left( \frac{1}{2} b^2 \left( \langle M \rangle_\infty - \langle M \rangle_\tau \right) \right) \middle| \mathcal{F}_\tau \right\|_\infty < \infty \right\}. \quad (1.4)$$

In both expressions, $\tau$ is an arbitrary stopping time.

**Definition 1.2.** Let $Y = (Y_t)_{0 \leq t \leq T}$ be a uniformly integrable martingale. Then $Y$ is said to belong to BMO if there is a constant $C > 0$ such that for every stopping time $\tau$

$$E[\|Y_T - Y_\tau\|_{\mathcal{F}_\tau}] \leq C \quad \text{P-a.s.} \quad (1.5)$$

This definition is independent of $p$. Usually we define $\|Y\|_{\text{BMO}}$ as the smallest constant $c$ such that for all stopping time $\tau$,

$$E[\|Y_T - Y_\tau\|_{\mathcal{F}_\tau}^2] \leq c^2 \quad \text{P-a.s.} \quad (1.6)$$

**Definition 1.3.** The nonzero-valued process $L$ is said to satisfy the reverse Hölder inequality under $P$, denoted by $R_p(P)$, where $p \in [1, +\infty]$, if there is a constant $C > 0$ such that for every stopping time $\tau$, we have

$$E\left[ \left\| \frac{L_T}{L_\tau} \right\|_{\mathcal{F}_\tau}^p \right] \leq C. \quad (1.7)$$
For $p = +\infty$, we require that $\frac{L_T}{L_T}$ is essentially bounded by $C$ (see Kazamaki [32, Definition 3.1]).

**Lemma 1.1.** (Kunita-Watanabe inequality) Let $X$ and $Y$ be two semi-martingales, and let $H$ and $K$ be two measurable processes. Then, we have almost surely

$$
\int_0^\infty |H_s||K_s||d[X,Y]_s| \leq \left( \int_0^\infty H_s^2 d[X,X]_s \right)^{1/2} \left( \int_0^\infty K_s^2 d[Y,Y]_s \right)^{1/2}.
$$

(1.8)

More generally, for $p \in [1, \infty)$, we have

$$
\int_0^\infty |H_s||K_s||d[X,Y]_s| \leq \left( \int_0^\infty H_s^p d[X,X]_s \right)^{1/p} \left( \int_0^\infty K_s^q d[Y,Y]_s \right)^{1/q}
$$

with $\frac{1}{p} + \frac{1}{q} = 1$.

**Lemma 1.2.** (Fefferman’s inequality) If $X \in H^1$ and $Y \in \text{BMO}$, then

$$
E \left[ \int_0^T |d\langle X, Y \rangle_s| \right] \leq \sqrt{2} \|X\|_{H^1} \|Y\|_{\text{BMO}}.
$$

(1.10)

About the expression of the duality between $H^1$ and BMO space, we have (see Kazamaki [32, Theorem 2.7, page 38]):

**Lemma 1.3.** Let $X$ be a continuous local martingale. Then, we have

$$
\|X\|_{H^1} \leq \sup \{E[\langle X, Y \rangle_\infty] : \|Y\|_{\text{BMO}} \leq 1\},
$$

$$
\|X\|_{\text{BMO}} \leq \sup \{E[\langle X, Y \rangle_\infty] : \|Y\|_{H^1} \leq 1\}.
$$

(1.11)

From Fefferman’s inequality, we can show the following lemma.

**Lemma 1.4.** Let $p \in [1, \infty)$. Assume that $X \in \mathcal{R}^p$ and $M \in \text{BMO}$. Then, $X \circ M \in \mathcal{H}^p$. Moreover, we have the following estimate

$$
\|X \circ M\|_{\mathcal{H}^p} \leq \sqrt{2} \|X\|_{\mathcal{R}^p} \|M\|_{\text{BMO}}.
$$

(1.12)

for $p \in (1, \infty)$ and

$$
\|X \circ M\|_{\mathcal{H}^1} \leq \|X\|_{\mathcal{R}^1} \|M\|_{\text{BMO}}
$$

(1.13)

(corresponding to the case of $p = 1$).

**Proof of Lemma 1.4.** (i) The case $p \in (1, \infty)$. Take any $N \in \mathcal{H}^q$. We have

$$
E[\langle X \circ M, N \rangle_\infty] \leq E[\langle X \circ N, M \rangle_\infty]
\leq \sqrt{2} \|X \circ N\|_{\mathcal{H}^1} \|M\|_{\text{BMO}} \quad \text{(using Fefferman’s inequality)}
\leq \sqrt{2} \|X\|_{\mathcal{R}^p} \|N\|_{\mathcal{H}^q} \|M\|_{\text{BMO}} \quad \text{(using Hölder’s inequality)}
$$

(1.14)
(ii) The case $p = 1$. We have

$$\int_0^\infty X_s^2 d\langle M \rangle_s \leq X_*^\infty \int_0^\infty |X_s| d\langle M \rangle_s \leq X_*^\infty \int_0^\infty X_s^\prime d\langle M \rangle_s \leq X_*^\infty \left( X_*^\infty \langle M \rangle_\infty - \int_0^\infty \langle M \rangle_s dX_s^* \right) \leq X_*^\infty \left( \langle M \rangle_\infty - \langle M \rangle_s dX_s^* \right).$$  \hspace{1cm} (1.15)

Therefore,

$$E \left[ \left( \int_0^\infty X_s^2 d\langle M \rangle_s \right)^{1/2} \right] \leq E \left[ \left( X_*^\infty \int_0^\infty (\langle M \rangle_\infty - \langle M \rangle_s) dX_s^* \right)^{1/2} \right] \leq \left\{ E [X_*^\infty]^1 \right\}^{1/2} \left\{ E \left[ \int_0^\infty (\langle M \rangle_\infty - \langle M \rangle_s) dX_s^* \right] \right\}^{1/2} \leq \|X\|_{R_*^1}^{1/2} \|M\|_{BMO} \{E [X_*^\infty]\}^{1/2} \leq \|X\|_{R_*^1} \|M\|_{BMO}. \hspace{1cm} (1.16)$$

The proof is complete. \hfill \Box

For the case of $X \in \mathcal{H}^p (\subset \mathcal{R}^p)$, the first assertion in Lemma 1.4 is included in Bañuelos and Bennett [1, Theorem 1.1 (i), page 1227]. The following lemma is obvious from the definition of $BMO$ norm, see Bañuelos and Bennett [1, Theorem 1.1 (ii), page 1227].

**Lemma 1.5.** If $X \in \mathcal{R}^\infty$ and $M \in BMO$, then $X \circ M \in BMO$ and $\|X \circ M\|_{BMO} \leq \|X\|_{R_*^\infty} \|M\|_{BMO}$.

**Lemma 1.6.** Let $p \in [1, \infty)$. Assume that $X \in \mathcal{H}^p$ and $M \in BMO$. Then, $\langle X, M \rangle_\infty \in L^p$. Moreover, we have the following estimate

$$\|\langle X, M \rangle_\infty\|_{L^p} \leq \sqrt{2p}\|X\|_{\mathcal{H}^p} \|M\|_{BMO}. \hspace{1cm} (1.17)$$

The first assertion in Lemma 1.6 can be found in Bañuelos and Bennett [1, Theorem 1.1 (iii), page 1227]. For convenience of the reader, we give a full proof.

**Proof of Lemma 1.6.** For the case $p = 1$, noting that

$$|\langle X, M \rangle_\infty| \leq \int_0^\infty |d\langle X, M \rangle|,$$ \hspace{1cm} (1.18)

it is immediate from Fefferman’s inequality to get the desired results. In what follows, we consider the case $p \in (1, \infty)$. Then, $q \in (1, \infty)$. Take any $\xi \in L^q$. Write $Y_t := E[\xi | \mathcal{F}_t]$ for
\( t \in [0, \infty] \). We have \( Y_\infty = \xi \) and
\[
E [\langle X, M \rangle_\infty \xi] = E \left[ \int_0^\infty Y_s d\langle X, M \rangle_s \right] = E \left[ \int_0^\infty d\langle X, Y \circ M \rangle_s \right]
\]
\[
\leq \|X\|_{\mathcal{H}^p} \|Y \circ M\|_{\mathcal{H}^q}
\]
(\text{using both Kunita-Watanabe inequality and Hölder’s inequality})
\[
\leq \sqrt{2} \|X\|_{\mathcal{H}^p} \|M\|_{\text{BMO}} \|Y\|_{\mathcal{H}^q} \quad (\text{using Lemma 1.4})
\]
\[
\leq \sqrt{2p} \|X\|_{\mathcal{H}^p} \|\xi\|_{L^p} \|M\|_{\text{BMO}}. \quad (\text{using Doob’s inequality})
\]

\[ \square \]

\textbf{Definition 1.4.} An integrable random variable \( \xi \) is said to be in \( \text{BMO} \) if the local martingale 
\( \{E[\xi|\mathcal{F}_t], t \in [0, T]\} \in \text{BMO} \).

\textbf{Lemma 1.7.} Let \( X \in \text{BMO} \) and \( M \in \text{BMO} \). Then, \( \langle X, M \rangle_\infty \in \text{BMO} \). Moreover, 
\[ \|\langle X, M \rangle_\infty\|_{\text{BMO}} \leq \sqrt{2} \|X\|_{\text{BMO}} \|M\|_{\text{BMO}}. \]

\textbf{Proof.} Take \( Y \in \mathcal{H}^1 \). We have
\[
|E[Y \langle X, M \rangle]| = \left| E \left[ \int_0^\infty Y_s d\langle X, M \rangle_s \right] \right|
\]
\[
= |E[(Y \circ X, M)]| \leq \sqrt{2} \|Y \circ X\|_{\mathcal{H}^q} \|M\|_{\text{BMO}} \quad (\text{Fefferman’s inequality})
\]
\[
\leq \sqrt{2} \|Y\|_{\mathcal{H}^q} \|X\|_{\text{BMO}} \|M\|_{\text{BMO}}. \quad (\text{Lemma 1.4})
\]

Using Lemma 1.3, we have the desired results.

\[ \square \]

The following fundamental Burkóhder-Davis-Gundy (abbreviated as BDG) inequality will be frequently used in our paper: for any \( p \in (0, \infty) \), there are two universal positive constants \( c_p \) and \( C_p \) such that for any local continuous martingale \( M \) with \( M_0 = 0 \), we have
\[
C_p^{-p} E \left[ \langle M \rangle_T^{p/2} \right] \leq E \left[ (\langle M_T \rangle^p)^{\frac{p}{2}} \right] \leq c_p^{-p} E \left[ \langle M \rangle_T^{p/2} \right],
\]
\[ (1.21) \]
or in a different form,
\[
C_p^{-1} \|M\|_{\mathcal{H}^p} \leq \|M\|_{\mathcal{R}^p} \leq c_p^{-1} \|M\|_{\mathcal{R}^p}.
\]
\[ (1.22) \]
See Yor [50, page 100].

The following definition is based on that of Emery [17, 18] (see also Protter [44, page 248]).

\textbf{Definition 1.5.} Let \( M \in \text{BMO} \) and \( \varepsilon > 0 \). A finite sequence of stopping times \( 0 = T_0 \leq T_1 \leq \cdots \leq T_k \) is said to \( \varepsilon \)-slice \( M \) if \( M = M_{T_k} \) and \( |(M - M_{T_i})_{T_{i+1}}|_{\text{BMO}} \leq \varepsilon \), for \( i = 0, 1, \cdots, k-1 \). If such a sequence of stopping times exists, we say that \( M \) is \( \varepsilon \)-sliceable in \( \text{BMO} \).

\textbf{Definition 1.6.} \( M \) is called sliceable in \( \text{BMO} \) if for \( \forall \varepsilon > 0 \), \( M \) is \( \varepsilon \)-sliceable in \( \text{BMO} \), i.e., there are a positive integer \( N \) and a finite increasing sequence of stopping times \( \{T_i, i = 1, 2, \ldots, N\} \) with \( T_0 = 0 \) and \( T_{N+1} = \infty \) such that \( T_i M_{T_{i+1}} := M_{T_{i+1}} - M_{T_i} \) satisfies
\[
\|T_i M_{T_{i+1}}\|_{\text{BMO}} \leq \varepsilon.
\]
\[ (1.23) \]

This is equivalent to \( M \in \mathcal{H}^\infty_{\text{BMO}} \) by Schachermayer’s result [45].
For more knowledge on local martingales and semi-martingales, the reader is referred to, among others, the following books: Dellacherie and Meyer [9], He, Wang, and Yan [25], Kazamaki [32], and Protter [44].

Throughout the rest of the paper, $N_1, N_2,$ and $M$ are supposed to be continuous local martingales on the time interval $[0, T]$, being equal to zero at time $t = 0$.

Since Itô’s initial works [29, 30, 31], stochastic differential equations (abbreviated hereafter as SDEs) driven by general semimartingales, instead of just Brownian motion, have been studied by Doléans-Dade [12], Doléans-Dade and Meyer [13], Protter [43, 44], and Emery [18, 18] among others. The theory of existence and uniqueness on SDEs driven by general semi-martingales is already quite general. However, the rather general result presented in the literature is concerned with existence and uniqueness in a very large space like $\cup_{p \geq 1} H^p$. In this subsection, we present some new sufficient conditions on existence and uniqueness of solutions in $H^p$ for some fixed $p \in [1, \infty)$. These conditions are more general than those presented in Protter [44], allowing the coefficients to be unbounded. We make best use of the deep property of Fefferman’s inequality on BMO martingales, which seems to be new in the study of SDEs.

Similar situations also exist for the research into BSDEs. Since Bismut’s initial works [3, 4, 5] and Pardoux and Peng’s seminal paper [42], BSDEs driven by general local martingales in the space $\mathcal{R}^p \times H^p$ for general $p \in (1, \infty)$ instead of just $p = 2$, have been studied by Buckdahn [8] (with the restriction that $p \in [2, \infty)$) and El Karoui, Peng and Quenez [15] (the underlying driving martingale is assumed to be a Brownian motion) among others. In El Karoui, Peng and Quenez [15], the coefficients of BSDEs are restricted to be uniformly Lipschitz in the unknown variables. The existence results in the space $\mathcal{R}^p \times H^p$ for some $p \in [2, \infty)$ existing in Buckdahn [8] requires—though the coefficients of BSDEs are allowed to be unbounded—that the data $(\xi, J)$ (see BSDE (2.2) below) lie in a space $\mathcal{R}^{p+\epsilon}$ for some $\epsilon > 0$, a stronger integrability. Roughly speaking, the integrability of the adapted solution of BSDEs is less than that of the data in Buckdahn [8]. Note that BSDEs with unbounded coefficients have also been studied by El Karoui and Huang [14], but requiring that both the solution and the data lie in the square integrable space which is weighted in relevance to the coefficients. In this paper, the BMO martingale theory, in particular Fefferman’s inequality on BMO martingales, is applied to study BSDEs with unbounded coefficients. New existence results are proved where the adapted solutions of BSDEs—even though the coefficients are unbounded—have the same integrability index $p$ to the underlying data $(\xi, J)$ for $p \in (1, \infty)$. The critical case of $p = +\infty$ is also discussed, and some interesting results are obtained.

It seems to be necessary to mention some applications of BMO martingales in the study of BSDEs. Bismut [5] has already used some properties of BMO martingales when he discussed the existence and uniqueness of adapted solutions of backward stochastic Riccati equation in some particular case. He chose the BMO space for the second unknown variable. In the work of Delbaen et al. [10, 11] on hedging contingent claims in mathematical finance, BMO
martingales are connected to some closedness in some suitable Banach spaces of the set of attainable claims for the agent’s wealth equation, which is essentially a problem of existence and uniqueness of a linear BSDE, but with unbounded coefficients. In the conference on mathematical finance, held in Konstanz in the year of 2000, the role of BMO martingales received a special emphasis in the study of backward stochastic Riccati equation and related linear quadratic stochastic optimal control problems. See Kohlmann and Tang [35, 36, 37]. In particular in Kohlmann and Tang [37], the second component of the adapted solution pair for a general backward stochastic Riccati equation—which is a multi-dimensional BSDE with the generator being a quadratic form of the second unknown variable—is shown to be a BMO martingale. Later, such kind of results are widely obtained and used, among others, by Hu, Imkeller, and Müller [26], Hu and Zhou [28], Barrieu and El Karoui [2], Briand and Hu [6, 7], and Hu et al. [27].

The rest of the paper consists of three sections, and is organized as follows.

Section 2 consists of three subsections. In Subsection 2.1, a rather general nonlinear multi-dimensional SE (2.1) driven by semimartingales with unbounded coefficients is discussed, and a new existence result in \((\mathcal{R}^p)^n (p \in [1, \infty))\) is proved under some suitable sliceability in the BMO space of the coefficients, which is stated in Theorem 2.1. In Subsection 2.2, a rather general nonlinear multi-dimensional BSDE (2.19) driven by a continuous local martingale with unbounded coefficients is discussed, and a new existence result in \((\mathcal{R}^p)^n \times (\mathcal{H}^p)^{(2n)} (p \in (1, \infty))\) is proved under some suitable sliceability in the BMO space of the coefficients, which is stated in Theorem 2.2. For the critical case of \(p = \infty\), a new existence result in \(\cap_{p>1} (\mathcal{R}^p)^n \times (\text{BMO})^{(2n)}\) is also obtained, but for a less general BSDE (2.42), and it is stated in Theorem 2.3. In Subsection 2.3, we give a sufficient condition on the suitable sliceability in the BMO space of the coefficients required in Theorems 2.1, 2.2, and 2.3. They are stated in Theorems 2.4, 2.5, and 2.6, respectively. Moreover, when the data \((\xi, J) \in (L^\infty(F_T))^n \times (\mathcal{R}^\infty)^n\), a new existence result in \((\mathcal{R}^\infty)^n \times (\mathcal{H}^\infty)^{(2n)}\) is proved for the rather general nonlinear multi-dimensional BSDE (2.19) with a nice application of Fefferman’s inequality, the John-Nirenberg inequality, and the Garnett-Jones’s Theorem, and it is stated in Theorem 2.7.

Section 3 is concerned with the linear BSDEs and SDEs with unbounded coefficients. The existence and uniqueness of the solution is connected to some reverse Hölder inequality property. It consists of two subsections. Subsection 3.1 is concerned with linear BSDEs with unbounded coefficients, while Subsection 3.2 is concerned with linear SDEs with unbounded coefficients.

Finally, in Section 4, the solution operator \(\phi\) from \(\mathcal{H}^p\) to \(\mathcal{H}^p\) of the one-dimensional SDE driven by a BMO martingale \(M\) receives a special consideration, whose spectral radius is estimated in terms of the Kazamaki’s quadratic critical exponent \(b(M)\) for the underlying BMO martingale \(M\). This estimation leads to a characterization of \(b(M) = \infty\).
2 The nonlinear multi-dimensional case

2.1 Unbounded SEs

Let $\mathcal{D}$ denote the space of $\{\mathcal{F}_t, 0 \leq t \leq T\}$-adapted càdlàg processes, and $\mathcal{D}^n$ the space of $n$-dimensional vector processes whose components are in $\mathcal{D}$.

Consider the following nonlinear SEs:

$$X_t = J(t) + \int_0^t f(s, X) \, d\langle N_1, N_2 \rangle_s + \int_0^t g(s, X) \, dM_s, \quad t \in [0, T]. \quad (2.1)$$

Here, $J \in (\mathbb{R}^p)^n$, $f$ and $g$ denote $\mathbb{R}^n$-valued functionals defined on $\Omega \times [0, T] \times \mathcal{D}^n$.

**Theorem 2.1.** Let $p \in [1, \infty)$. Assume that

(i) There are two $\{\mathcal{F}_t, 0 \leq t \leq T\}$-adapted processes $\alpha(\cdot)$ and $\beta(\cdot)$ such that

$$f(t, 0) = 0; \quad |f(t, x_1) - f(t, x_2)| \leq \alpha(t)(x_1 - x_2)^*(t), \quad x_1, x_2 \in \mathcal{D}^n \quad (2.2)$$

and

$$g(t, 0) = 0; \quad |g(t, x_1) - g(t, x_2)| \leq \beta(t)(x_1 - x_2)^*(t), \quad x_1, x_2 \in \mathcal{D}^n. \quad (2.3)$$

(ii) The martingale $\alpha \circ N_1 \in \text{BMO}$. The martingale $N_2 \in \text{BMO}$ is $\varepsilon_1$-sliceable in the space $\text{BMO}$ and the martingale $\beta \circ M \in \text{BMO}$ is $\varepsilon_2$-sliceable in the space $\text{BMO}$. Let

$$\rho_1 := 2p\varepsilon_1 |\alpha \circ N_1|_{\text{BMO}} + \sqrt{2}\varepsilon_2 C_p < 1. \quad (2.4)$$

Then for any $J \in (\mathbb{R}^p)^n$, there is unique solution $X \in (\mathbb{R}^p)^n$ to equation (2.1). Furthermore, there is a constant $K_p$, which is independent of $J$, such that

$$\|X\|_{\mathbb{R}^p} \leq K_p\|J\|_{\mathbb{R}^p}. \quad (2.5)$$

If $J \in (\mathbb{R}^p)^n$ is a semi-martingale, then so is the solution.

**Proof.** We shall use the contraction mapping principle to look for a fix-point. For this purpose, consider the following map $I$ in the Banach space $(\mathbb{R}^p)^n$:

$$I(X)_t := J(t) + \int_0^t f(s, X) \, d\langle N_1, N_2 \rangle_s + \int_0^t g(s, X) \, dM_s, \quad t \in [0, T]. \quad (2.6)$$

We have

$$E\left[ \max_{0 \leq t \leq T} \left| \int_0^t f(s, X) \, d\langle N_1, N_2 \rangle_s \right|^p \right] \leq E\left[ \int_0^T |f(s, X)| \, |d\langle N_1, N_2 \rangle_s|^p \right] \leq E\left[ \int_0^T \alpha_s X^*_s |d\langle N_1, N_2 \rangle_s|^p \right] \leq (\sqrt{2}p)^p \|\alpha \circ N_1\|_{\text{BMO}}^p \|X^* \circ N_2\|_{(\mathbb{R}^p)^n}^p \quad (\text{using Lemma 1.6})$$

$$\leq (2p)^p \|\alpha \circ N_1\|_{\text{BMO}}^p \|X\|_{(\mathbb{R}^p)^n}^p \|N_2\|_{\text{BMO}}^p \quad (\text{using Lemma 1.4})$$

$$\leq (2p)^p \|\alpha \circ N_1\|_{\text{BMO}}^p \|X\|_{(\mathbb{R}^p)^n}^p \|N_2\|_{\text{BMO}}^p$$
and
\[
E \left[ \max_{0 \leq t \leq T} \left| \int_0^t g(s, X_s) \, dM_s \right|^p \right]
\leq C_p^2 E \left[ \int_0^T |g(s, X_s)|^2 \, d\langle M \rangle_s \right]^{p/2}
\leq C_p^2 e \left[ \int_0^T \beta_s^2(X_s^*)^2 \, d\langle M \rangle_s \right]^{p/2}
\leq C_p^2 \|\beta \circ M \|^p_{BMO} \|X\|_{(R^p)^n}^p.
\] (from the Lipschitz assumption on \( g \))

Therefore, \( I(X) \in (R^p)^n \) for \( X \in (R^p)^n \).

For \( X^1, X^2 \in (R^p)^n \), proceeding similarly to the above arguments, we have
\[
E \left[ \max_{0 \leq t \leq T} \left| \int_0^t \left[ f(s, X^1_s) - f(s, X^2_s) \right] \, d\langle N_1, N_2 \rangle_s \right|^p \right]
\leq (2p)^2 \|\alpha \circ N_1\|^p_{BMO} \|X^1 - X^2\|^p_{(R^p)^n} \|N_2\|^p_{BMO}
\] (2.9)

and
\[
E \left[ \max_{0 \leq t \leq T} \left| \int_0^t \left[ g(s, X^1_s) - g(s, X^2_s) \right] \, dM_s \right|^p \right]
\leq (\sqrt{2C_p})^p \|\beta \circ M\|^p_{BMO} \|X^1 - X^2\|^p_{(R^p)^n}.
\] (2.10)

Therefore, we have
\[
\|I(X^1) - I(X^2)\|_{(R^p)^n}
\leq \left[ (\sqrt{2C_p})^p \|\beta \circ M\|^p_{BMO} + (2p)^p \|\alpha \circ N_1\|^p_{BMO} \|N_2\|^p_{BMO} \right]^{1/p} \|X^1 - X^2\|_{(R^p)^n}
\] (2.11)

Since the martingale \( N_2 \in BMO \) is \( \varepsilon_1 \)-sliceable and \( \beta \circ M \in BMO \) is \( \varepsilon_2 \)-sliceable, there is a finite sequence of stopping times \( \{T_i, i = 1, 2, \ldots, \tilde{T}\} \) such that the following are satisfied:
(i) \( 0 = T_0 \leq T_1 \leq T_2 \leq \cdots \leq T_{\tilde{T}} \leq T_{\tilde{T} + 1} = \tilde{T} \);
(ii) \( |N_2|_{BMO} \leq \varepsilon_1 \) and \( |\beta \circ M|_{BMO} \leq \varepsilon_2 \) where \( N_{2i} := N_{2i}^{T_{i+1}} - N_{2i}^{T_i} \) and \( M_i := M^{T_{i+1}} - M^{T_i} \) are defined on \( [T_i, T_{i+1}] \). Since \( |\alpha \circ N_{1i}|_{BMO} \leq |\alpha \circ N_1|_{BMO} \), we have
\[
\rho_{1i} := 2p\varepsilon_1 |\alpha \circ N_{1i}|_{BMO} + \sqrt{2}\varepsilon_2 C_p \leq \rho_1
\] (2.12)

with \( N_{1i} := N_{1i}^{T_{i+1}} - N_{1i}^{T_i} \) for \( i = 0, 1, 2, \ldots, \tilde{T} \). Set \( R_i^p := R^p[T_i, T_{i+1}] \). Set \( X_{-1} := 0 \). Consider the map \( I_i : (R_i^p)^n \rightarrow (R_i^p)^n \), defined by
\[
I_i(X)_t := J_i(t) + \int_0^t f(s, X_s) \, d\langle N_{1i}, N_{2i} \rangle_s + \int_0^t g(s, X_s) \, dM_{is}, \quad t \in [T_i, T_{i+1}].
\] (2.13)
where \( J_i(\cdot) := J_{T_{i+1}} - J_{T_i} + X_{i-1}(T_i) \) is defined on \([T_i, T_{i+1}].\)

Similar to the derivation of inequality (2.11), we have

\[
\begin{align*}
\|I_i(X^1) - I_i(X^2)\|_{(R^p)^n} &\leq \left[ \sqrt{2}C_p\|\beta \circ M_1\|_{BMO} + 2p\|\alpha \circ N_{1i}\|_{BMO}\|N_{2i}\|_{BMO} \right]\|X^1 - X^2\|_{(R^p)^n} \\
&\leq \left[ \sqrt{2}C_p\|\beta \|_{C^2} + 2p\|\alpha \circ N_{1i}\|_{BMO} \right]\|X^1 - X^2\|_{(R^p)^n} \\
&= \rho_{1i}\|X^1 - X^2\|_{(R^p)^n} \leq \rho_1\|X^1 - X^2\|_{(R^p)^n}
\end{align*}
\]

for any \( X^1, X^2 \in (R^p)^n \). In view of the second assumption of the theorem, we see that the map \( I_i \) is a contraction map, and satisfies the following estimate:

\[
\|I(X)\|_{(R^p)^n} \leq \rho_1\|X\|_{(R^p)^n} + \|J\|_{(R^p)^n}
\]

for any \( X \in (R^p)^n \). Therefore, in an inductive way, we show that the following stochastic equation

\[
X_t = J_i(t) + \int_{T_i}^t f(s, X) d\langle N_{1i}, N_{2i} \rangle_s + \int_{T_i}^t g(s, X) dM_s, \quad t \in [T_i, T_{i+1}]
\]

has a unique solution \( X_i(\cdot) \) in \((R^p)^n\) for \( i = 0, 1, \cdots, \tilde{I} \). Moreover, we have

\[
\|X_i\|_{(R^p)^n} \leq (1 - \rho_1)^{-1}\|J_i\|_{(R^p)^n}.
\]

Then, the process

\[
X(t) := \sum_{i=0}^{\tilde{I}} X_i(t) X_{[T_i, T_{i+1})}(t), \quad t \in [0, T]
\]

lies in \((R^p)^n\) and is the unique solution to equation (2.1). The desired a priori estimate (2.5) is immediate from the assumption (2.4) and the inequality (2.17). The last assertion of the theorem is obvious.

### 2.2 Unbounded BSDEs

Consider the following nonlinear BSDEs:

\[
\begin{align*}
Y_t &= \xi + J_T - J_t + \int_t^T f(s, Y_s) d\langle N_1, N_2 \rangle_s + \int_t^T g(s, Y_s, Z_s) d\langle M \rangle_s \\
&\quad - \int_t^T Z_s dM_s - \int_t^T dM^\perp_s, \quad t \in [0, T]; \quad \langle M, M^\perp \rangle = 0.
\end{align*}
\]

Here, \( \xi \) is an \( R^n \)-valued \( \mathcal{F}_T \)-measurable random variable, \( J \) is an \( R^n \)-valued optional continuous process, and the \( R^n \)-valued random fields \( f \) and \( g \) are defined on \( \Omega \times [0, T] \times R^n \) and \( \Omega \times [0, T] \times R^n \times R^n \), respectively.
Theorem 2.2. Let $p \in (1, \infty)$ and $q$ be the conjugate number. Assume that

(i) There are three $\{F_t, 0 \leq t \leq T\}$-adapted processes $\alpha(\cdot), \beta(\cdot)$ and $\gamma(\cdot)$ such that

\[ f(\cdot, 0) = 0; \quad |f(t, y_1) - f(t, y_2)| \leq \alpha(t)|y_1 - y_2| \]  \hspace{1cm} (2.20)

for $y_1, y_2 \in \mathbb{R}^n$ and

\[ g(\cdot, 0, 0) = 0; \quad |g(t, y_1, z_1) - g(t, y_2, z_2)| \leq \beta(t)|y_1 - y_2| + \gamma(t)|z_1 - z_2| \]  \hspace{1cm} (2.21)

for $y_1, y_2, z_1, z_2 \in \mathbb{R}^n$.

(ii) The martingale $\alpha \circ N_1 \in \text{BMO}$. The martingale $N_2 \in \text{BMO}$ is $\varepsilon_1$-sliceable in the space $\text{BMO}$, the martingale $\sqrt{\beta} \circ M \in \text{BMO}$ is $\varepsilon_2$-sliceable in the space $\text{BMO}$ and the martingale $\gamma \circ M \in \text{BMO}$ is $\varepsilon_3$-sliceable in the space $\text{BMO}$. Set $\overline{\rho}_2 := q(1 + C_p) + C_p$. Let

\[ \rho_2 := \overline{\rho}_2 \max \left\{ \sqrt{2} \rho_3, \ 2p \| \alpha \circ N_1 \|_{\text{BMO}} \varepsilon_1 + 2p \varepsilon_2^2 \right\} < 1. \]  \hspace{1cm} (2.22)

Then for any $(\xi, J) \in (L^p(\mathcal{F}_T))^n \times (\mathcal{H}^p)^n$, the BSDE (2.19) has a unique solution $(Y, Z \circ M, M^\perp) \in (\mathcal{R}^p)^n \times (\mathcal{H}^p)^{2n}$. Moreover, there is a universal constant $K_p$, which is independent of $(\xi, J)$, such that

\[ \|Y\|_{(\mathcal{R}^p)^n} + \|(M, M^\perp)\|_{(\mathcal{H}^p)^{2n}} \leq K_p \left[ \|\xi\|_{(L^p)^n} + \|J\|_{(\mathcal{R}^p)^n} \right]. \]  \hspace{1cm} (2.23)

Proof of Theorem 2.2. We shall still use the contraction mapping principle and look for a fix-point. Consider the following map $I$ in the Banach space $(\mathcal{R}^p)^n \times (\mathcal{H}^p)^n$: for $(y, z \circ M) \in (\mathcal{R}^p)^n \times (\mathcal{H}^p)^n$, define $I(y, z \circ M)$ to be components $(Y, Z \circ M)$ of the unique adapted solution $(Y, Z \circ M, M^\perp)$ of the following BSDE:

\[ \begin{aligned}
Y_t &= \xi + J_T - J_t + \int_t^T f(s, y_s) \, d\langle N_1, N_2 \rangle_s + \int_t^T g(s, y_s, z_s) \, d\langle M \rangle_s \\
&\quad - \int_t^T Z_s \, dM_s - \int_t^T dM^\perp_s, \quad t \in [0, T]; \quad \langle M, M^\perp \rangle = 0.
\end{aligned} \]  \hspace{1cm} (2.24)

We have

\[ \begin{split}
Y_t &= E \left[ \xi + (J_T - J_t) + \int_t^T f(s, y_s) \, d\langle N_1, N_2 \rangle_s + \int_t^T g(s, y_s, z_s) \, d\langle M \rangle_s \middle| \mathcal{F}_t \right] \\
&= -J_t + E \left[ \xi + J_T \middle| \mathcal{F}_t \right] + E \left[ \int_t^T f(s, y_s) \, d\langle N_1, N_2 \rangle_s \middle| \mathcal{F}_t \right] \\
&\quad + E \left[ \int_t^T g(s, y_s, z_s) \, d\langle M \rangle_s \middle| \mathcal{F}_t \right]. \hspace{1cm} (2.25)
\]
In view of Doob’s inequality, we have

\[
\|Y\|_{(R^p)^n} \leq \|J\|_{(R^p)^n} + \left\{ E \left[ \max_{0 \leq t \leq T} \left| E \left[ |\xi + J_T | \mathcal{F}_t | \right|^p \right] \right] \right\}^{1/p} \\
+ \left\{ E \left[ \max_{0 \leq t \leq T} \left| E \left[ \int_0^T f(s, y_s) d\langle N_1, N_2 \rangle_s | \mathcal{F}_t | \right|^p \right] \right] \right\}^{1/p} \\
+ \left\{ E \left[ \max_{0 \leq t \leq T} \left| E \left[ \int_0^T g(s, y_s, z_s) d\langle M \rangle_s | \mathcal{F}_t | \right|^p \right] \right] \right\}^{1/p} \\
\leq \|J\|_{(R^p)^n} + q \|\xi + J_T\|_{(L^p)^n} \\
+ \left\{ E \left[ \max_{0 \leq t \leq T} \left( E \left[ \int_0^T |f(s, y_s)| d\langle N_1, N_2 \rangle_s | \mathcal{F}_t | \right]^p \right] \right] \right\}^{1/p} \\
+ \left\{ E \left[ \max_{0 \leq t \leq T} \left( E \left[ \int_0^T |g(s, y_s, z_s)| d\langle M \rangle_s | \mathcal{F}_t | \right]^p \right] \right] \right\}^{1/p} \\
\leq \|J\|_{(R^p)^n} + q \|\xi + J_T\|_{(L^p)^n} + q \left\{ E \left[ \left( \int_0^T |f(s, y_s)| d\langle N_1, N_2 \rangle_s \right)^p \right] \right\}^{1/p} \\
+ q \left\{ E \left[ \left( \int_0^T |g(s, y_s, z_s)| d\langle M \rangle_s \right)^p \right] \right\}^{1/p}.
\]

(2.26)

Proceeding identically as in the derivation of inequality (2.7) in the proof of Theorem 2.1, we have

\[
E \left[ \left( \int_0^T |f(s, y_s)| d\langle N_1, N_2 \rangle_s \right)^p \right] \leq \left( \sqrt{2} p \right)^p \|\alpha \circ N_1\|_{BMO}^p y \circ N_2 \|_{(R^p)^n}^p.
\]

(2.27)

Proceeding similarly as in the derivation of inequality (2.8), using the Lipschitz assumption on \( g \), we have

\[
\left\{ E \left[ \left( \int_0^T |g(s, y_s, z_s)| d\langle M \rangle_s \right)^p \right] \right\}^{1/p} \leq \left\{ E \left[ \left( \int_0^T (\beta_s |y_s| + \gamma_s |z_s|) d\langle M \rangle_s \right)^p \right] \right\}^{1/p} \\
= \left\| \langle \sqrt{\beta} \circ M, \sqrt{\beta} |y| \circ M \rangle_T + \langle \gamma \circ M, |z| \circ M \rangle_T \right\|_{L^p}.
\]

(2.28)
Therefore, we have

\[
\|Y\|_{(R^p)^n} \leq \|J\|_{(R^p)^n} + q \|\xi + J_T\|_{(L^p)^n} + \sqrt{2}pq \|\alpha \circ N_1\|_{BMO} \|y \circ N_2\|_{(H^p)^n} \\
+ q \left\| \langle \sqrt{\beta} \circ M, \sqrt{\beta} |y| \circ M \rangle_T + \langle \gamma \circ M, |z| \circ M \rangle_T \right\|_{L^p} \\
\leq \|J\|_{(R^p)^n} + q \|\xi + J_T\|_{(L^p)^n} + \sqrt{2}pq \|\alpha \circ N_1\|_{BMO} \|y \circ N_2\|_{(H^p)^n} \\
+ \sqrt{2}pq \left\| \sqrt{\beta} \circ M \right\|_{BMO} \left\| \sqrt{\beta} |y| \circ M \right\|_{H^p} \\
+ \sqrt{2}pq \|\gamma \circ M\|_{BMO} \|z \circ M\|_{(H^p)^n}
\]

(using Lemma 1.6)

\[
\leq \|J\|_{(R^p)^n} + q \|\xi + J_T\|_{(L^p)^n} + 2pq \|\alpha \circ N_1\|_{BMO} \|N_2\|_{BMO} \|y\|_{(R^p)^n} \\
+ 2pq \left\| \sqrt{\beta} \circ M \right\|_{BMO}^2 \|y\|_{(R^p)^n} + \sqrt{2}pq \|\gamma \circ M\|_{BMO} \|z \circ M\|_{(H^p)^n}
\]

(using Lemma 1.4)

\[
\leq \|J\|_{(R^p)^n} + q \|\xi + J_T\|_{(L^p)^n} + \sqrt{2}pq \|\gamma \circ M\|_{BMO} \|z \circ M\|_{(H^p)^n} \\
+ 2pq \left( \|\alpha \circ N_1\|_{BMO} \|N_2\|_{BMO} + \left\| \sqrt{\beta} \circ M \right\|_{BMO}^2 \right) \|y\|_{(R^p)^n}.
\]

Further, we have

\[
\int_t^T Z_s \, dM_s + \int_t^T dM_s^\perp \\
= \xi + J_T - J_t - Y_t + \int_t^T f(s, y_s) \, d\langle N_1, N_2 \rangle_s \\
= + \int_t^T g(s, y_s, z_s) \, d\langle M \rangle_s, \quad t \in [0, T]; \quad \langle M, M^\perp \rangle = 0.
\]
From the BDG inequality and using the similar arguments to the above, we have

\[ \| z \circ M \|_{(H^p)^n} \leq C_p \| z \circ M + M^+ \|_{(R^p)^n} \]

\[ \leq C_p \| \xi + J_T \|_{(L^p)^n} + C_p \| J \|_{(R^p)^n} + C_p \| Y \|_{(R^p)^n} \]

\[ + C_p \left\{ E \left[ \max_{0 \leq t \leq T} \left| \int_t^T f(s, y_s) \, d\langle N_1, N_2 \rangle_s \right|^p \right] \right\}^{1/p} \]

\[ + C_p \left\{ E \left[ \max_{0 \leq t \leq T} \left| \int_t^T g(s, y_s, z_s) \, d\langle M \rangle_s \right|^p \right] \right\}^{1/p} \]

\[ \leq C_p \| \xi + J_T \|_{(L^p)^n} + C_p \| J \|_{(R^p)^n} + C_p \| Y \|_{(R^p)^n} \]

\[ + C_p \left\{ E \left[ \left( \int_0^T |f(s, y_s)| \, |d\langle N_1, N_2 \rangle_s| \right)^p \right] \right\}^{1/p} \]

\[ + C_p \left\{ E \left[ \left( \int_0^T |g(s, y_s, z_s)| \, |d\langle M \rangle_s| \right)^p \right] \right\}^{1/p} \quad (2.31) \]

Concluding the above, we have

\[ \| Y \|_{(R^p)^n} + \| z \circ M \|_{(H^p)^n} \]

\[ \leq (1 + C_p) \| Y \|_{(R^p)^n} + C_p \| \xi + J_T \|_{(L^p)^n} + C_p \| J \|_{(R^p)^n} \]

\[ + 2pC_p \| \alpha \circ N_1 \|_{BMO} \| N_2 \|_{(H^p)^n} \]

\[ + 2pC_p \| \gamma \circ M \|_{BMO} \| z \circ M \|_{H^p} \]

\[ \leq \| \xi + J_T \|_{(L^p)^n} + (1 + 2C_p) \| J \|_{(R^p)^n} \]

\[ + 2pC_p \| \alpha \circ N_1 \|_{BMO} \| N_2 \|_{BMO} \| y \|_{(R^p)^n} \]

\[ + 2pC_p \| \gamma \circ M \|_{BMO} \| z \circ M \|_{H^p} \quad (2.32) \]

Let \((y^i, z^i \circ M) \in (R^p)^n \times (H^p)^n\) with \(i = 1, 2\). Denote by \((Y^i, Z^i \circ M)\) the image
Consider the map $T_N$ and its sliceable, there is a finite sequence of stopping times $\{T_i, i = 1, 2, \cdots, \tilde{T}\}$ such that the following are satisfied:

(i) $0 = T_0 \leq T_1 \leq T_2 \leq \cdots \leq T_{\tilde{T} + 1} = T$;

(ii) $\|N_2\|_{BMO} \leq \varepsilon_1$, $\|\sqrt{\beta} \circ M_i\|_{BMO} \leq \varepsilon_2$ and $\|\beta \circ M_i\|_{BMO} \leq \varepsilon_3$ where $N_{2i} := N_{2i+1} - N_{2i}$ and $M_i := M_{T_{i+1}} - M_{T_i}$ are defined on $[T_i, T_{i+1}]$.

Since $\|\alpha \circ N_1\|_{BMO} \leq \|\alpha \circ N_1\|_{BMO}$, we have

$$
\rho_{2i} := C_p \max \left\{ \sqrt{2} \rho \varepsilon_3, \ 2p \|\alpha \circ N_1\|_{BMO} \varepsilon_1 + 2p \varepsilon_2^2 \right\} \leq \rho_2.
$$

with $N_{1i} := N_{1i+1} - N_{1i}$ for $i = 0, 1, 2, \cdots, \tilde{T}$.

Set $\mathcal{R}_p^i := \mathcal{R}_p[T_i, T_{i+1}]$ and $\mathcal{H}_p^i := \mathcal{H}_p(T_i, T_{i+1})$ for $i = 0, 1, \cdots, \tilde{T}$. Set $Y^T(T) = \xi$. Consider the map $I_i$ in the Banach space $(\mathcal{R}_p^n) \times (\mathcal{H}_p^n)$: for $(y, z \circ M) \in (\mathcal{R}_p^n) \times (\mathcal{H}_p^n)$, define $I_i(y, z \circ M)$ to be components $(Y, Z \circ M)$ of the unique adapted solution $(Y, Z \circ M, M^\perp)$ of the following BSDE:

\[
\begin{cases}
Y_t = Y^i_{Ti+1} + (J_{Ti+1} - J_t) + \int_t^{Ti+1} f(s, y_s) d\langle N_{1i}, N_{2i} \rangle_s + \int_t^{Ti+1} g(s, y_s, z_s) d\langle M_i \rangle_s \\
- \int_t^{Ti+1} z_s dM_{is} - \int_t^{Ti+1} dM^\perp_s, & t \in [T_i, T_{i+1}]; \quad \langle M, M^\perp \rangle = 0.
\end{cases}
\]
Similar to the derivation of inequality (2.33), we have
\[
\|I_i(y^1, z^1) - I_i(y^2, z^2)\|_{(\mathcal{R}_p^i)^n \times (\mathcal{H}_p^i)^n} \\
\leq \max \left\{ \sqrt{2p} \|\gamma \circ M_i\|_{BMO}, 2p \|\alpha \circ N_i\|_{BMO} \|N_{ti}\|_{BMO} + 2p \left\| \sqrt{\beta} \circ M_i \right\|^2_{BMO} \right\} \\
\times \mathcal{C}_p \left[ \left\|y^1 - y^2\right\|_{(\mathcal{R}_p^i)^n} + \left\| (z^1 - z^2) \circ M \right\|_{(\mathcal{H}_p^i)^n} \right]
\leq \rho_{2i} \left[ \left\|y^1 - y^2\right\|_{(\mathcal{R}_p^i)^n} + \left\| (z^1 - z^2) \circ M \right\|_{(\mathcal{H}_p^i)^n} \right]
\leq \rho_2 \left[ \left\|y^1 - y^2\right\|_{(\mathcal{R}_p^i)^n} + \left\| (z^1 - z^2) \circ M \right\|_{(\mathcal{H}_p^i)^n} \right]
\]
(2.36)
for any \((y^1, z^1), (y^2, z^2) \in (\mathcal{R}_p^i)^n \times (\mathcal{H}_p^i)^n\). In view of inequality (2.22) in the second assumption of the theorem, we see that for each \(i = 0, 1, \ldots, \tilde{I}, I_i\) is a contraction map on \((\mathcal{R}_p^i)^n \times (\mathcal{H}_p^i)^n\). More precisely, first, since \(I_{\tilde{I}}\) is a contraction, the following BSDE:
\[
\begin{cases} 
Y_t = \xi + (J_T - J_t) + \int_t^T f(s, Y_s) d\langle N_{1 \tilde{I}}, N_{2 \tilde{I}} \rangle_s + \int_t^T g(s, Y_s, Z_s) d\langle M_{\tilde{I}} \rangle_s \\
- \int_t^T Z_s dM_{\tilde{I}} - \int_t^T dM_s^\perp, \quad t \in [T_{\tilde{I}}, T]; \quad \langle M, M^\perp \rangle = 0
\end{cases}
\]
(2.37)
has a unique solution \((Y^{\tilde{I}}, Z^{\tilde{I}} \circ M_{\tilde{I}}, M^{\tilde{I}}) \in (\mathcal{R}_p^i)^n \times (\mathcal{H}_p^i)^{2n}\). Second, consider the following BSDE:
\[
\begin{cases} 
Y_t = Y^i_{T_{i+1}} + (J_{T_{i+1}} - J_t) + \int_t^{T_{i+1}} f(s, Y_s) d\langle N_{1 \tilde{I}}, N_{2 \tilde{I}} \rangle_s + \int_t^{T_{i+1}} g(s, Y_s, Z_s) d\langle M_{i+1} \rangle_s \\
- \int_t^{T_{i+1}} Z_s dM_{i+1} - \int_t^{T_{i+1}} dM_s^\perp, \quad t \in [T_i, T_{i+1}); \quad \langle M, M^\perp \rangle = 0
\end{cases}
\]
(2.38)
Since the map \(I_{T_{i+1}}\) is a contraction in \((\mathcal{R}_p^i)^n \times (\mathcal{H}_p^i)^n\), it has a unique solution \((Y^{i+1}, Z^{i+1} \circ M^{i+1}) \in (\mathcal{R}_p^i)^n \times (\mathcal{H}_p^i)^{2n}\). Inductively in a backward way, we can show that the following BSDE:
\[
\begin{cases} 
Y_t = Y^{i+1}_{T_{i+1}} + (J_{T_{i+1}} - J_t) + \int_t^{T_{i+1}} f(s, Y_s) d\langle N_{1 \tilde{I}}, N_{2 \tilde{I}} \rangle_s + \int_t^{T_{i+1}} g(s, Y_s, Z_s) d\langle M_{i+1} \rangle_s \\
- \int_t^{T_{i+1}} Z_s dM_{i+1} - \int_t^{T_{i+1}} dM_s^\perp, \quad t \in [T_i, T_{i+1}); \quad \langle M, M^\perp \rangle = 0
\end{cases}
\]
(2.39)
has a unique solution \((Y^i, Z^i, M^{i, \perp}) \in (\mathcal{R}_p^i)^n \times (\mathcal{H}_p^i)^{2n}\) for \(i = 0, 1, \ldots, \tilde{I}\). Moreover, we have
\[
(1 - \rho_2) \left( \|Y^i\|_{(\mathcal{R}_p^i)^n} + \|Z^i\|_{(\mathcal{H}_p^i)^n} \right) \\
\leq \mathcal{C}_p \left\| Y_{T_{i+1}}^{i+1} + J_{T_{i+1}} \right\|_{L^p(F^{T_{i+1}})} + (2C_p + 1) \|J\|_{(\mathcal{R}_p^i)^n}
\]
(2.40)
for \(i = 0, 1, \ldots, \tilde{I}\).
Then, the triple of processes \((Y, Z \circ M, M^\perp)\) given by

\[
X(t) := \sum_{i=0}^{\bar{t}} Y^i_t \chi_{[T_i, T_{i+1})} (t), \quad t \in [0, T],
\]

\[
Z(t) := \sum_{i=0}^{\bar{t}} Z^i_t \chi_{[T_i, T_{i+1})} (t), \quad t \in [0, T], \tag{2.41}
\]

\[
M^\perp(t) := M^0_t \chi_{[0, T_1)} (t) + \sum_{i=1}^{\bar{t}} \left[ M^{i, \perp}_t + M^{i-1, \perp}_T \right] \chi_{[T_i, T_{i+1})} (t), \quad t \in [0, T]
\]

lies in \((\mathcal{R}^p)^n \times (\mathcal{H}^p)^{2n}\) and is the unique adapted solution to BSDE (2.19). The estimate (2.23) is a consequence of the inequalities (2.40).

Consider BSDE (2.19) for the case of \(f = 0, J = 0\) and \(g\) being independent of \(y\). That is, consider the following nonlinear BSDEs:

\[
\begin{cases}
Y_t = \xi + \int_t^T g(s, Z_s) \, d\langle M \rangle_s \\
\quad - \int_t^T Z_s \, dM_s - \int_t^T dM^\perp_s, \quad t \in [0, T]; \quad \langle M, M^\perp \rangle = 0.
\end{cases} \tag{2.42}
\]

For the extremal case of \(p = \infty\), we have the following result.

**Theorem 2.3.** Assume that

(i) There is an \({\mathcal{F}_t, 0 \leq t \leq T}\)-adapted processes \(\gamma(\cdot)\) such that

\[
g(\cdot, 0) = 0; \quad |g(t, z_1) - g(t, z_2)| \leq \gamma(t)|z_1 - z_2| \tag{2.43}
\]

for \(z_1, z_2 \in \mathbb{R}^n\).

(ii) The martingale \(\gamma \circ M \in \text{BMO}\) is \(\varepsilon\)-sliceable in the space \(\text{BMO}\) such that

\[
\sqrt{2\varepsilon} < 1. \tag{2.44}
\]

Then for \(\xi \in (\text{BMO})^n\), the BSDE (2.42) has a unique solution \((Y, Z \circ M, M^\perp)\) such that \((Z \circ M, M^\perp) \in (\text{BMO})^{2n}\). Moreover, there is a universal constant \(K\), which is independent of \(\xi\), such that

\[
\| (Z \circ M, M^\perp) \|_{(\text{BMO})^{2n}} \leq K \| \xi \|_{(\text{BMO})^n}. \tag{2.45}
\]

**Proof of Theorem 2.3.** We shall still use the contraction mapping principle and look for a fix-point. Consider the following map \(I\) in the Banach space \((\text{BMO})^n\): for \(z \circ M \in (\text{BMO})^n\), define \(I(z \circ M)\) to be component \(Z \circ M\) of the unique adapted solution \((Y, Z \circ M, M^\perp)\) of the following BSDE:

\[
\begin{cases}
Y_t = \xi + \int_t^T g(s, z_s) \, d\langle M \rangle_s \\
\quad - \int_t^T Z_s \, dM_s - \int_t^T dM^\perp_s, \quad t \in [0, T]; \quad \langle M, M^\perp \rangle = 0.
\end{cases} \tag{2.46}
\]
The following shows that $I(z \circ M)$ is in the BMO space for any $z \circ M \in (BMO)^n$:

$$
\|Z \circ M\|_{(BMO)^n} \leq \|Z \circ M + M\|_{(BMO)^n} 
= \left\| \xi + \int_0^T g(s, z_s) \, d\langle M \rangle_s \right\|_{(BMO)^n} 
\leq \|\xi\|_{(BMO)^n} + \left\| \int_0^T g(s, z_s) \, d\langle M \rangle_s \right\|_{(BMO)^n} 
\leq \|\xi\|_{(BMO)^n} + \int_0^T \gamma_s z_s \, d\langle M \rangle_s \right\|_{(BMO)^n} 
= \|\xi\|_{(BMO)^n} + \|\langle \gamma \circ M, z \circ M \rangle_T\|_{(BMO)^n} 
\leq \|\xi\|_{(BMO)^n} + \sqrt{2}\|\gamma \circ M\|_{BMO} \|z \circ M\|_{(BMO)^n} 
$$

(using Lemma 1.7).

Let $z^i \circ M \in (BMO)^n$ with $i = 1, 2$. Denote by $Z^i \circ M$ the image $I(z^i \circ M)$ for $i = 1, 2$. Similar to the above arguments, we can show that

$$
\|Z^1 \circ M - Z^2 \circ M\|_{(BMO)^n} \leq \sqrt{2} \|\gamma \circ M\|_{BMO} \|z^1 \circ M - z^2 \circ M\|_{(BMO)^n}. 
$$

(2.48)

The rest of the proof is identical to that of Theorem 2.2. \qed

**Remark 2.1.** Theorem 2.3 is not implied by Theorem 2.2 due to the fact that the assumption (ii) of the latter involves $p$. In fact, the proof of the former appeals to Lemma 1.7, while the proof of the latter appeals to Lemma 1.6. Lemma 1.7 is not implied by Lemma 1.6.

### 2.3 Comments on the slice-ability assumption in the space $BMO$

on the martingales $N_2, \gamma \circ M$, and $\beta \circ M$ in Theorems 2.1, 2.2, and 2.3

Schachermayer [45] shows that any martingale in $\mathcal{H}_{\infty}^{BMO}$ is sliceable in the space $BMO$. Therefore, the suitable slice-ability assumption in the space $BMO$ in the preceding subsection on the martingales $N_2, \gamma \circ M$, and $\beta \circ M \in BMO$ is automatically true when they are in the space $\mathcal{H}_{\infty}^{BMO}$. Therefore, we have the following

**Theorem 2.4.** Let $p \in [1, \infty)$. Assume that

(i) There are two $\{\mathcal{F}_t, 0 \leq t \leq T\}$-adapted processes $\alpha(\cdot)$ and $\beta(\cdot)$ such that

$$
f(t, 0) = 0, \quad |f(t, x_1) - f(t, x_2)| \leq \alpha(t) \max_{0 \leq s \leq t} |x_1(s) - x_2(s)|, \quad x_1, x_2 \in \mathcal{D}^n
$$

(2.49)

and

$$
g(t, 0) = 0, \quad |g(t, x_1) - g(t, x_2)| \leq \beta(t) \max_{0 \leq s \leq t} |x_1(s) - x_2(s)|, \quad x_1, x_2 \in \mathcal{D}^n.
$$

(2.50)
(ii) The martingale $\alpha \circ N_1 \in BMO$. Both martingales $N_2$ and $\beta \circ M$ are in the space $\mathcal{H}^{\infty\text{BMO}}$.

Then for any $J \in (\mathbb{R}^p)^n$, there is unique solution in $(\mathbb{R}^p)^n$ to equation (2.1). Furthermore, there is a constant $K_p$, which is independent of $J$, such that

$$\|X\|_{\mathbb{R}^p} \leq K_p \|J\|_{\mathbb{R}^p}.$$  \hfill (2.51)

If $J \in (\mathbb{R}^p)^n$ is a semi-martingale, then so is the solution.

**Remark 2.2.** Theorem 2.4 more or less generalizes Protter [43, Lemma 2, page 252].

**Corollary 2.1.** There are three real valued nonnegative $\{F_t, 0 \leq t \leq T\}$-adapted processes $\alpha(\cdot), \beta(\cdot)$ and $\gamma(\cdot)$ such that the adapted $\mathbb{R}^{n \times n}$-valued processes $A, B,$ and $D$ are bounded respectively by $\alpha, \beta,$ and $\gamma$. Assume that the martingale $\alpha \circ N_1 \in BMO$, and the martingales $N_2, \sqrt{\beta \circ M},$ and $\gamma \circ M$ are all in the space $\mathcal{H}^{\infty\text{BMO}}$. Let $S(\cdot)$ be the fundamental solution matrix process to the following SDE:

$$\begin{align*}
    dS(t) &= [A_t d\langle N_1, N_2 \rangle_t + B_t d\langle M \rangle_t + D_t dM_t] S(t), \quad t \in [0, T]; \\
    S(0) &= I.
\end{align*}$$  \hfill (2.52)

Then, for any $p \in [1, \infty)$, there is a universal constant $K_p$ such that for any stopping time $\tau$, we have

$$E \left[ \max_{\tau \leq t \leq T} |S^{-1}(\tau)S(t)|^p \right] \leq K_p^n.$$  \hfill (2.53)

The last inequality implies that $S(\cdot)$ satisfies the reverse Hölder property $(R_p)$ for any $p \in [1, \infty)$.

**Proof of Corollary 2.1.** The assumptions of Theorem 2.4 are all satisfied except that the two continuous local martingales $N_1, N_2,$ and the real nonnegative process $\alpha$ in Theorem 2.4 correspond to two two-dimensional vector-valued continuous local martingales $(N_1, M), (N_2, \sqrt{\beta \circ M}),$ and the two-dimensional vector-valued processes $(\alpha, \sqrt{\beta})$ in this corollary.

Consider any stopping time $\tau$. Take any $G \in F_\tau$. For $J = S(\tau)\chi_G$, it is easy to see that $X := S\chi_G$ is the unique solution to the SDE (2.52) with the initial condition being replaced with $X(\tau) = S(\tau)\chi_G$. The assertions of Theorem 2.4 are still true for $X$. In view of the estimate (2.51) of Theorem 2.4, we have

$$E \left[ \max_{\tau \leq t \leq T} |S(t)\chi_G|^p \right] \leq K_p^n E \left[ |S(\tau)\chi_G|^p \right].$$  \hfill (2.54)

Therefore, we have

$$E \left[ \max_{\tau \leq t \leq T} |S(t)|^p \chi_G \right] \leq K_p^n E \left[ |S(\tau)|^p \chi_G \right].$$  \hfill (2.55)

This implies the inequality (2.53). The proof is complete.
Theorem 2.5. Let $p \in (1, \infty)$. Assume that

(i) There are three $\{\mathcal{F}_t, 0 \leq t \leq T\}$-adapted processes $\alpha(\cdot), \beta(\cdot)$ and $\gamma(\cdot)$ such that

$$f(\cdot, 0) = 0; \quad |f(t, y_1) - f(t, y_2)| \leq \alpha(t)|y_1 - y_2|$$

(2.56)

for $y_1, y_2 \in \mathbb{R}^n$ and

$$g(\cdot, 0, 0) = 0; \quad |g(t, y_1, z_1) - g(t, y_1, z_2)| \leq \beta(t)|y_1 - y_2| + \gamma(t)|z_1 - z_2|$$

(2.57)

for $y_1, y_2, z_1, z_2 \in \mathbb{R}^n$.

(ii) The martingale $\alpha \circ N_1 \in BMO$. The martingales $N_2, \sqrt{\beta} \circ M$, and $\gamma \circ M$ are all in the space $\overline{H}_\infty^{BMO}$.

Then for $(\xi, J) \in (L^p(\mathcal{F}_T))^n \times (\mathcal{R}^p)^n$, BSDE (2.19) has a unique solution $(Y, Z \circ M, M^\perp) \in (\mathcal{R}^p)^n \times (\mathcal{H}^p)^{2n}$. Moreover, there is a universal constant $K_p$, which is independent of $J$, such that

$$\|Y\|_{(\mathcal{R}^p)^n} + \|(M, M^\perp)\|_{(\mathcal{H}^p)^{2n}} \leq K_p \left[\|\xi\|_{(L^p)^n} + \|J\|_{(\mathcal{R}^p)^n}\right].$$

(2.58)

Note that the existence and uniqueness of Föllmer-Schweizer decomposition (see Föllmer and Schweizer [20]) is exactly the existence and uniqueness of a one-dimensional linear BSDE, but possibly and typically with unbounded coefficients. Theorem 2.5 includes as particular cases the existence and uniqueness results on linear BSDEs not only for bounded coefficients by Bismut [5], but also for unbounded coefficients by Monat and Stricker [39, 40, 41] and by Schweizer [46, 47]—where $\gamma \circ M$ is assumed to be in $\mathcal{H}_\infty$—in the case of no jumps in $\gamma \circ M$, and by Delbaen et al [10, 11] in the case of $\gamma \circ M \in \overline{H}_\infty^{BMO}$. Note that $\lambda \circ M \in \overline{H}_\infty^{BMO}$ when the process $\lambda$ is a uniformly bounded adapted process and the local martingale $M$ is a Brownian motion stopped at a finite deterministic time $T$. Therefore, Theorem 2.5 also includes as particular cases the existence and uniqueness results on nonlinear BSDEs of Pardoux and Peng [42] (for $L^2$ solutions), and El Karoui, Peng, and Quenez [15, Theorem 5.1, page 54] (for $L^p$ solutions ($p > 1$)).

Theorem 2.6. Assume that

(i) There is an $\{\mathcal{F}_t, 0 \leq t \leq T\}$-adapted processes $\gamma(\cdot)$ such that

$$\int_0^T |g(t, 0)||d(M)_s| \in BMO; \quad |g(t, z_1) - g(t, z_2)| \leq \gamma(t)|z_1 - z_2|$$

(2.59)

for $z_1, z_2 \in \mathbb{R}^n$.

(ii) The martingale $\gamma \circ M \in \overline{H}_\infty^{BMO}$.

Then for $\xi \in (BMO)^n$, BSDE (2.42) has a unique solution $(Y, Z \circ M, M^\perp)$ such that $(Z \circ M, M^\perp) \in (BMO)^{2n}$. Moreover, there is a universal constant $K$, which is independent of $\xi$, such that

$$\|(Z \circ M, M^\perp)\|_{(BMO)^{2n}} \leq K\|\xi\|_{(BMO)^n}.$$
When the data (i.e., the terminal state $\xi$ and the “zero” term $J$) is essentially bounded, instead of just being in the BMO space, the unique adapted solution $(Y, Z \circ M, M^\perp)$ to BSDE (2.19) can be further proved to lie in the better space: $(\mathcal{R}^\infty)^n \times \left( \mathcal{H}^\infty_{BM O} \right)^n$.

**Theorem 2.7.** Assume that

(i) There are three \{${\mathcal{F}}_t, 0 \leq t \leq T$\}-adapted processes $\alpha(\cdot), \beta(\cdot)$ and $\gamma(\cdot)$ such that

$$f(\cdot, 0) = 0; \quad |f(t, y_1) - f(t, y_2)| \leq \alpha(t)|y_1 - y_2|$$

for $y_1, y_2 \in \mathbb{R}^n$ and

$$g(\cdot, 0, 0) = 0; \quad |g(t, y_1, z_1) - g(t, y_1, z_2)| \leq \beta(t)|y_1 - y_2| + \gamma(t)|z_1 - z_2|$$

for $y_1, y_2, z_1, z_2 \in \mathbb{R}^n$.

(ii) The martingale $\alpha \circ N_1 \in BM O$. The martingales $N_2, \sqrt{\beta} \circ M$, and $\gamma \circ M$ are all in the space $\mathcal{H}^\infty_{BM O}$.

For any $\xi \in L^\infty(\mathcal{F}_T)$ and $J \in \mathcal{R}^\infty$, there is unique adapted solution $(Y, Z \circ M, M^\perp)$ to BSDE (2.19), with $Y \in (\mathcal{L}^\infty)^n$ and $Z \circ M + M^\perp \in \left( \mathcal{L}^\infty_{BM O} \right)^n$.

**Proof of Theorem 2.7.** From Theorem 2.5, we know that there is a unique adapted solution $(Y, Z \circ M, M^\perp) \in (\mathcal{R}^p)^n \times (\mathcal{H}^p)^{2n}$ to BSDE (2.19) for any $p \in (1, \infty)$. The proof is divided into the following three steps.

Step 1. We show that $Y \in (\mathcal{L}^\infty)^n$. In fact, BSDE (2.19) can be written into the following linear form:

$$Y_t = \xi + J_T - J_t + \int_t^T A_r^s Y_s d\langle N_1, N_2 \rangle_s + \int_t^T (D_r^s Z_s + B_r^s Y_s) d\langle M \rangle_s - \int_t^T Z_s dM_s - \int_t^T dM^\perp_s, \quad t \in [0, T]; \quad \langle M, M^\perp \rangle = 0,$$

with the adapted matrix-valued processes $A, B,$ and $D$ being bounded respectively by $\alpha, \beta,$ and $\gamma$. Let $S(\cdot)$ be the fundamental solution matrix process to the SDE (2.52). Then, we have

$$Y_t = E \left[ S^\tau(t)^{-1} S^\tau(T)(\xi + J_T) - \int_t^T S^\tau(t)^{-1} S^\tau(s)[A^s J_s d\langle N_1, N_2 \rangle_s + B^s J_s d\langle M \rangle_s] \bigg| \mathcal{F}_t \right] - J_t.$$

In view of Corollary 2.1, $S(\cdot)$ satisfies the reverse Hölder property $(R_p)$ for any $p \in [1, \infty)$,
and the inequality (2.53) hold. Therefore, we have
\[
|Y_t| \leq |J_t| + E \left[ |S^\tau(t)^{-1}S^\tau(T)| \cdot |\xi + J_T| |\mathcal{F}_t\right] \\
+ J_{(R^\infty)^n} E \left[ \left( \sup_{s \in [t,T]} |S^\tau(t)^{-1}S^\tau(s)| \right) \int_t^T \left[ \|A_s\| |d\langle N_1, N_2 \rangle_s| + |B_s| d\langle M_s \rangle \left| \mathcal{F}_t \right| \right] \right] \\
\leq |J_t| + E \left[ |S^\tau(t)^{-1}S^\tau(T)| \cdot |\xi + J_T| |\mathcal{F}_t\right] \\
+ J_{(R^\infty)^n} E \left[ \left( \sup_{s \in [t,T]} |S^\tau(t)^{-1}S^\tau(s)| \right) \int_t^T \left[ \alpha_s |d\langle N_1, N_2 \rangle_s| + \beta_s d\langle M_s \rangle \left| \mathcal{F}_t \right| \right] \right] \\
\leq \|J\|_{(R^\infty)^n} + K_1 \left( \|\xi\|_{(L^\infty)^n} + \|J\|_{(R^\infty)^n} \right) \\
+ K_2 \|J\|_{(R^\infty)^n} \left\{ E \left[ \int_t^T \left( |d\langle \alpha \circ N_1, N_2 \rangle_s| + d\langle \sqrt{\beta} \circ M \rangle_s \right)^2 \left| \mathcal{F}_t \right| \right] \right\}^{1/2} \\
\leq \|J\|_{(R^\infty)^n} + K_1 \left( \|\xi\|_{(L^\infty)^n} + \|J\|_{(R^\infty)^n} \right) \\
+ K_2 \|J\|_{(R^\infty)^n} \left\{ E \left[ 2 \left| \langle \alpha \circ N_1, N_2 \rangle_t \right|^2 + 2 \left| \langle \sqrt{\beta} \circ M \rangle_t \right|^2 \left| \mathcal{F}_t \right| \right] \right\}^{1/2} .
\] (2.65)

Here, $K_1$ and $K_2$ are introduced in Corollary (2.1). In view of the assumption (ii) of the theorem, using Kazamaki [32, Lemma 2.6, page 48] and the John-Nirenber inequality (see Kazamaki [32, Theorem 2.2, page 29]), we have $b(N_2) = b(\sqrt{\beta} \circ M) = \infty$ and
\[
E \left[ \exp \left( \epsilon \langle \alpha \circ N_1 \rangle_t^T \right) \left| \mathcal{F}_t \right| \right] \in \mathcal{L}^\infty
\] (2.66)
for $\epsilon < \|\alpha \circ N_1\|_{(BMO)^n}^{-2}$. Then, the following process
\[
E \left[ 2 \left| \langle \alpha \circ N_1, N_2 \rangle_t^T \right|^2 + 2 \left| \langle \sqrt{\beta} \circ M \rangle_t^T \right|^2 \left| \mathcal{F}_t \right| \right] \\
\leq E \left[ 2\langle \alpha \circ N_1 \rangle_t^T \langle N_2 \rangle_t^T + 2 \left| \langle \sqrt{\beta} \circ M \rangle_t^T \right|^2 \left| \mathcal{F}_t \right| \right] \\
\leq E \left[ \epsilon \langle \alpha \circ N_1 \rangle_t^T \left| \mathcal{F}_t \right| + \epsilon^2 E \left[ \left| \langle N_2 \rangle_t^T \right|^2 \left| \mathcal{F}_t \right| \right] + 2 \left| \langle \sqrt{\beta} \circ M \rangle_t^T \right|^2 \left| \mathcal{F}_t \right| \right] \\
\leq E \left[ \exp \left( \epsilon \langle \alpha \circ N_1 \rangle_t^T \right) \left| \mathcal{F}_t \right| \right] + \epsilon^2 E \left[ \left| \langle N_2 \rangle_t^T \right|^2 \left| \mathcal{F}_t \right| \right] + 2 \left| \langle \sqrt{\beta} \circ M \rangle_t^T \right|^2 \left| \mathcal{F}_t \right| \\
+ 2 \left| \exp \left( \langle \sqrt{\beta} \circ M \rangle_t^T \right) \left| \mathcal{F}_t \right| \right| \in \mathcal{L}^\infty
\] (2.67)
for $\epsilon \in (0, \|\alpha \circ N_1\|_{(BMO)^n}^{-2})$. Consequently, we have $Y \in (\mathcal{L}^\infty)^n$.

Step 2. We show that $Z \circ M + M \perp \in BMO$. To simplify the exposition, set
\[
C_{YJ} := \|Y\|_{(R^\infty)^n} + \|J\|_{(R^\infty)^n} .
\] (2.68)
In view of BSDE (2.63), using Itô’s formula and standard arguments, we can obtain the following estimate for any stopping time $\sigma$:

$$
E \left[ (Z \circ M + M^{\perp})^T_{\sigma} | \mathcal{F}_\sigma \right] \\
\leq E \left[ |\xi + J|^2 + 2 \int_{\sigma}^T |Y_s + J_s| \left[ \alpha_s d\langle N_1, N_2 \rangle_s + (\beta_s |Y_s| + \gamma_s |Z_s|) \right] d\langle M \rangle_s \right] \mathcal{F}_\sigma \\
\leq 2\|\xi\|^2_{L^\infty} + 2\|J\|^2_{R^n} + 2C_{Y,J}E \left[ \int_{\sigma}^T \alpha_s d\langle N_1, N_2 \rangle_s + \int_{\sigma}^T (\beta_s |Y_s| + \gamma_s |Z_s|) d\langle M \rangle_s \right] \mathcal{F}_\sigma \\
\leq 2\|\xi\|^2_{L^\infty} + 2\|J\|^2_{R^n} + 2C_{Y,J}E \left[ \int_{\sigma}^T d\langle \alpha \circ N_1, N_2 \rangle_s \right] \mathcal{F}_\sigma \\
+ 2C_{Y,J}E \left[ \langle \sqrt{\beta} \circ M \rangle_{\sigma}^{T} \right] \mathcal{F}_\sigma + 2C_{Y,J}E \left[ \langle \gamma \circ M, |Z| \circ M \rangle_{\sigma}^{T} \right] \mathcal{F}_\sigma \\
\leq 2\|\xi\|^2_{L^\infty} + 2\|J\|^2_{R^n} + 2C_{Y,J}E \left[ \langle \alpha \circ N_1 \rangle_{\sigma}^{T} \right]^{1/2} \left( \langle N_2 \rangle_{\sigma}^{T} \right)^{1/2} \mathcal{F}_\sigma \\
+ 2C_{Y,J}E \left[ \langle \sqrt{\beta} \circ M \rangle_{\sigma}^{T} \right] \mathcal{F}_\sigma \\
+ 2C_{Y,J}E \left[ \langle \gamma \circ M \rangle_{\sigma}^{T} \right]^{1/2} \left( \langle |Z| \circ M \rangle_{\sigma}^{T} \right)^{1/2} \mathcal{F}_\sigma \\
\leq 2\|\xi\|^2_{L^\infty} + 2\|J\|^2_{R^n} + 2C_{Y,J}\|\alpha \circ N_1\|_{BMO} \|N_2\|_{BMO} \\
+ 2C_{Y,J}\|\sqrt{\beta} \circ M\|_{BMO}^2 \\
+ 2C_{Y,J}\|\gamma \circ M\|_{BMO}^2 + \frac{1}{2} E \left[ \langle |Z| \circ M \rangle_{\sigma}^{T} \right] \mathcal{F}_\sigma \\
\right) .

(2.69)

Using the elementary Cauchy inequality, we have

$$
E \left[ (Z \circ M + M^{\perp})^T_{\sigma} \mathcal{F}_\sigma \right] \\
\leq 2\|\xi\|^2_{L^\infty} + 2\|J\|^2_{R^n} + 2C_{Y,J}\|\alpha \circ N_1\|_{BMO} \|N_2\|_{BMO} \\
+ 2C_{Y,J}\|\sqrt{\beta} \circ M\|_{BMO}^2 \\
+ 2C_{Y,J}\|\gamma \circ M\|_{BMO}^2 + \frac{1}{2} E \left[ \langle |Z| \circ M \rangle_{\sigma}^{T} \right] \mathcal{F}_\sigma .

(2.70)

The last inequality yields the following

$$
\frac{1}{2} E \left[ (Z \circ M + M^{\perp})^T_{\sigma} \right] \mathcal{F}_\sigma \\
\leq 2\|\xi\|^2_{L^\infty} + 2\|J\|^2_{R^n} + 2C_{Y,J}\|\alpha \circ N_1\|_{BMO} \|N_2\|_{BMO} \\
+ 2C_{Y,J}\|\sqrt{\beta} \circ M\|_{BMO}^2 + 2C_{Y,J}\|\gamma \circ M\|_{BMO}^2 .

(2.71)

Let $K$ denote the right hand side of the last inequality. We then have $Z \circ M + M^{\perp} \in BMO$ with $\|Z \circ M + M^{\perp}\|_{BMO} \leq \sqrt{2K}$. 

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Step 3. It remains to prove that $Z \circ M + M^\perp \in \mathcal{L}^{BM\O}_{\infty}$. In view of the probabilistic version of the Garnett and Jones theorem [21] (due to Varopoulos [49] and Emery [19], see Kazamaki [32, Theorem 2.8, page 39]), it is sufficient to show that for any $\lambda > 0$,

$$\sup_\sigma \left\| E \left[ \exp \left( \lambda |(Z \circ M)_T + (M^\perp)_T - (Z \circ M)_\sigma - (M^\perp)_\sigma| \right) | \mathcal{F}_\sigma \right] \right\|_{L^\infty} < \infty. \quad (2.72)$$

Since

$$Y_\sigma = \xi + (J_T - J_\sigma) + \int_\sigma^T f(s, Y_s) d\langle N_1, N_2 \rangle_s$$

$$+ \int_\sigma^T g(s, Y_s, Z_s) d\langle M \rangle_s - \int_\sigma^T Z_s dM_s - \int_\sigma^T dM^\perp_s$$

and the random variable $Y_\sigma + J_\sigma - \xi - J_T \in (L^\infty(\mathcal{F}_T))^n$, it is sufficient to prove the following

$$\sup_\sigma \left\| E \left[ \exp \left( \lambda | - \int_\sigma^T f(s, Y_s) d\langle N_1, N_2 \rangle_s - \int_\sigma^T g(s, Y_s, Z_s) d\langle M \rangle_s \right) \right] | \mathcal{F}_\sigma \right\|_{L^\infty} < \infty. \quad (2.74)$$

the left hand side of inequality (2.74) is equal to the following

$$\sup_\sigma \left\| E \left[ \exp \left( \lambda \left( \int_\sigma^T |f(s, Y_s)| |d\langle N_1, N_2 \rangle_s| + \int_\sigma^T |g(s, Y_s, Z_s)| d\langle M \rangle_s \right) \right) \right\|_{L^\infty} \leq \sup_\sigma \left\| E \left[ \exp \left( \lambda \int_\sigma^T \alpha_s |d\langle N_1, N_2 \rangle_s| + \lambda \int_\sigma^T (\beta_s |Y_s| + \gamma_s |Z_s|) d\langle M \rangle_s \right) \right] | \mathcal{F}_\sigma \right\|_{L^\infty}. \quad (2.75)$$

While for any $\epsilon > 0$

$$\lambda \int_\sigma^T \alpha_s |d\langle N_1, N_2 \rangle_s| + \lambda \int_\sigma^T (\beta_s |Y_s| + \gamma_s |Z_s|) d\langle M \rangle_s$$

$$\leq 2\epsilon (\alpha \circ N_1)_T + 2\epsilon^{-1} \lambda^2 (N_2)_T + \lambda \|Y\|_{(\mathcal{R}^\infty)^n} (\sqrt{\beta} \circ M)_T^T + 2\epsilon^{-1} \lambda^2 (\gamma \circ M)_T^T + 2\epsilon (Z \circ M)_T^T,$$

in view of the facts that $b(N_2) = b(\sqrt{\beta} \circ M) = b(\gamma \circ M) = \infty$ (due to the assumption (ii) of the theorem), it is sufficient to prove the following for some $\epsilon > 0$

$$\sup_\sigma \left\| E \left[ \exp \left( 4\epsilon (\alpha \circ N_1)_T + 4\epsilon (Z \circ M)_T^T \right) \right] | \mathcal{F}_\sigma \right\|_{L^\infty} < \infty. \quad (2.76)$$

Since $\alpha \circ M, Z \circ M \in BM\O$, in view of the John-Nirenberg inequality (see Kazamaki [32, Theorem 2.2, page 29]), we have

$$\sup_\sigma \left\| E \left[ \exp \left( 8\epsilon (\alpha \circ N_1)_T \right) \right] | \mathcal{F}_\sigma \right\|_{L^\infty} \leq \frac{1}{1 - 8\epsilon \|\alpha \circ N_1\|_{BM\O}^2} < \infty \quad (2.78)$$

and

$$\sup_\sigma \left\| E \left[ \exp \left( 8\epsilon (Z \circ M)_T^T \right) \right] | \mathcal{F}_\sigma \right\|_{L^\infty} \leq \frac{1}{1 - 8\epsilon \|Z \circ M\|_{BM\O}^2} < \infty \quad (2.79)$$
for sufficiently small \( \varepsilon > 0 \). Therefore, the inequality (2.77) hold when \( \varepsilon \) is sufficiently small. The proof is then complete.

When the generator of a BSDE is not Lipschitz in the second unknown variable, we should not expect that \( Z \circ M \in L^{\infty}_{BM} \) as in the last theorem. From Kazamaki [32, Theorem 2.14, page 48], we have \( H^{\infty}_{BM} \subseteq L^{\infty}_{BM} \). Therefore, we should not expect that \( Z \circ M \in H^{\infty}_{BM} \), neither. We have the following negative result.

**Theorem 2.8.** Let \( M \) be a one-dimensional standard Brownian motion, and \( \{ \mathcal{F}_t, 0 \leq t \leq 1 \} \) be the completed natural filtration. Assume that \((Y, Z)\) solves the following BSDE:

\[
\begin{align*}
    dY_t &= Z_t \, dM_t + aZ^2_t \, dt, \quad t \in [0, 1]; \\
    Y_1 &= \xi \in L^{\infty}(\mathcal{F}_1).
\end{align*}
\]

Then, \( Y \in L^{\infty} \) and \( Z \circ M \in BM, \) but it is not always true that \( Z \circ M \in L^{\infty}_{BM} \).

**Proof.** Without loss of generality, we assume \( a = \frac{1}{2} \). From Kobylansky [33, 34] and Briand and Hu [6], we see that \( Y \in L^{\infty} \) and \( Z \circ M \in BM \).

Consider the following process \( X \):

\[
X_t := \int_0^t \frac{1}{\sqrt{1-s}} \, dM_s, \quad t \in [0, 1).
\]

Define the following stopping time \( \tau \):

\[
\tau := \inf \{ t \in [0, 1) : |X_t|^2 > 1 \}.
\]

It is easy to see that \( \tau \) is a.s. well-defined and \( \tau < 1 \). Set

\[
\begin{align*}
    \xi &= -\log(X_\tau + 2); \\
    Y_t &= -\chi_{[0, \tau]}(t) \log(X_t + 2) + \xi \chi_{(\tau, 1]}(t), \\
    Z_t &= -\frac{\chi_{[0, \tau]}(t)}{(X_t + 2)\sqrt{1-t}}, \quad t \in [0, 1].
\end{align*}
\]

Then, we can verify that \((Y, Z)\) is the unique adapted solution of BSDE (2.80). Further, in view of the fact that \( X_t + 2 \in [1, 3] \), we have

\[
E \left[ \exp \left( \lambda \int_0^1 Z_s^2 \, ds \right) \right] \geq E \left[ \exp \left( \lambda \int_0^\tau \frac{1}{9(1-s)} \, ds \right) \right] = E \left[ \exp \left( \frac{\lambda}{9} \langle X \rangle_t^\tau \right) \right].
\]

It is known (see Kazamaki [32, Lemma 1.3, pages 11–12] for a similar result) that

\[
E \left[ \exp \left( \frac{\lambda}{9} \langle X \rangle_0^\tau \right) \right] = \infty
\]

for \( \lambda \geq \frac{9}{8} \pi^2 \). Consequently, we have

\[
E \left[ \exp \left( \lambda \int_0^1 Z_s^2 \, ds \right) \right] = \infty
\]
for $\lambda \geq \frac{9}{8}\pi^2$. In view of Kazamaki [32, Lemma 2.6, page 48] and BSDE (2.80), we have $Z \circ M \notin \mathcal{H}^{\infty}_{BMO}$, and for $\lambda \geq \frac{9}{8}\pi^2$,

$$E\left[\exp\left(\lambda|Z \circ W|\right)\right] = \infty$$

(2.87)
due to both facts that $Y \in \mathcal{L}^\infty$ and $\xi \in L^\infty(\mathcal{F}_1)$.

Again, using the probabilistic version of the Garnett and Jones theorem [21] (see also Kazamaki [32, Theorem 2.8, page 39]), we conclude the proof. \[\square\]

3 The linear case

The study of linear BSDEs goes back to J. M. Bismut’s Ph. D. Thesis, which presented a rather extensive study on stochastic control, optimal stopping, and stochastic differential games. Also there, the concept of BSDEs was introduced and the theory of linear BSDEs was initiated, though only for the case of uniformly bounded coefficients and $L^2$-integrable adapted solutions.

3.1 BSDEs

Assume that $A : \Omega \times [0,T] \to \mathbb{R}^{n \times n}$ is $\{\mathcal{F}_t, 0 \leq t \leq T\}$-optional. Let $M$ be a continuous local martingale such that $A \circ M \in BMO$. Consider the following linear SDE:

$$dX_t = A_t X_t \, dM_t + dV_t, \quad x_0 = 0.$$ (3.1)

**Definition 3.1.** Consider the homogeneous linear SDE:

$$dX_t = A_t X_t \, dM_t, \quad X_0 = I_{n \times n}.$$ (3.2)

Its unique strong solution is denoted by $S(\cdot)$. It is said that $S(\cdot)$ satisfies the reversed Hölder inequality $(R_p)$ for some $p \in [1, \infty)$ if for any stopping time $\sigma$ and any matrix norm $| \cdot |$, we have

$$E[|S(T)|^p | \mathcal{F}_\sigma] \leq C|S(\sigma)|^p.$$ (3.3)

**Remark 3.1.** Note that $S = \mathcal{E}(A \circ M)$ if $n = 1$. In this case, it is known that $S(\cdot)$ satisfies the reverse Hölder inequality $(R_p)$ for all $p \in [1, \infty)$ if $A \circ M \in L^\infty_{BMO}$. See Kazamaki [32, Theorem 3.8, page 66] for details. Since $\langle B^T \rangle_T = T$ and thus $B^T \in \mathcal{H}^\infty \subset \mathcal{L}^\infty_{BMO}$, an immediate consequence is the obvious fact that the stochastic exponential $\mathcal{E}(B^T)$ of a one-dimensional Brownian motion, stopped at a deterministic time $T$, satisfies the reverse Hölder inequality $(R_p)$ for all $p \in [1, \infty)$, which can be verified by some straightforward explicit computations.

**Remark 3.2.** Assume that $A \circ M \in \overline{\mathcal{H}^\infty_{BMO}}$. From Theorem 2.1, we see that $S(\cdot)$ is uniformly integrable.
Similar to the proof of Kazamaki [32, Corollary 3.2, page 60], we can prove (by taking $U = |S(T)|^p$) the following result.

**Theorem 3.1.** Assume that $S(\cdot)$ is a uniformly integrable matrix martingale, and let $p \in (1, \infty)$. If $S(\cdot)$ satisfies the reverse Hölder inequality $(R_p)$, then it satisfies $(R_{p'})$ for some $p' > p$.

We have the following
\[ dS(t)^{-1} = -S(t)^{-1}[A_t dM_t - A_t^2 d\langle M \rangle_t]. \] (3.4)

**Theorem 3.2.** Let $A \circ M \in BMO$ and $S(\cdot)$ be an adapted continuous process that satisfies the reverse Hölder inequality $(R_{p'})$ for $p' > 1$. Let $q$ be the conjugate of $p \in (1, p')$, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. Then, for $\xi \in L^q(\mathcal{F}_T)$ and $f \in L^q(0, T)$, the following BSDE
\[ dY_t = -[A_t' Z_t d\langle M \rangle_t + f_t dt] + Z_t dM_t + M_t^\perp, \quad \langle M, M^\perp \rangle = 0, \] (3.5)

has a unique adapted solution $(Y, Z \circ M, M^\perp) \in (\mathcal{R}^q)^n \times (\mathcal{H}^q)^{2n}$. Moreover, we have some universal constant $K_q$ such that
\[ \|Y\|_{(\mathcal{R}^q)^n} + \|\langle Y \rangle_T^{1/2}\|_{L^q} \leq K_q \left[ \|\xi\|_{(L^q)^n} + \|f\|_{(L^q)^n} \right]. \] (3.6)

**Remark 3.3.** In view of Theorem 3.1, we can take $p = p'$ in Theorem 3.2 if furthermore $S(\cdot)$ is assumed to be uniformly integrable.

**Proof of Theorem 3.2.** First for $s \geq t$, set
\[ \tilde{Y}_t := E \left[ S^\perp(t)^{-1} S^\perp(T) \xi + \int_t^T S^\perp(t)^{-1} S^\perp(s) f_s ds \bigg| \mathcal{F}_t \right]. \] (3.7)

We have
\[ \tilde{Y}_T = \xi \] (3.8)

and
\[ \tilde{Y}_t = E \left[ S^\perp(t)^{-1} S^\perp(T) \xi + \int_t^T S^\perp(t)^{-1} S^\perp(T) f_s ds \bigg| \mathcal{F}_t \right]. \] (3.9)

Since $S(\cdot)$ satisfies the reverse Hölder inequality $(R_{p'})$, letting $q'$ be the conjugate of $p'$, we see that
\begin{align*}
\|	ilde{Y}_t\| &\leq E \left[ |\xi|^{q'} \bigg| \mathcal{F}_t \right]^{1/q'} E \left[ |S^\perp(t)^{-1} S^\perp(T)|^{p'} \bigg| \mathcal{F}_t \right]^{1/p'} \\
&\quad + E \left[ \int_t^T |f_s|^{q'} ds \bigg| \mathcal{F}_t \right]^{1/q'} E \left[ |S^\perp(t)^{-1} S^\perp(T)|^{p'} \bigg| \mathcal{F}_t \right]^{1/p'} \\
&\leq C \left( E \left[ |\xi|^{q'} \bigg| \mathcal{F}_t \right]^{1/q'} + E \left[ \int_t^T |f_s|^{q'} ds \bigg| \mathcal{F}_t \right]^{1/q'} \right).
\end{align*} (3.10)
Therefore, and using Doob’s inequality, we have
\[
\left( E \left| \tilde{Y}_t^q \right|^\frac{1}{q} \right) \leq C \left( E \left| \xi \right|^\frac{1}{q} + E \left[ \int_t^T |s|^q \, ds \right]^\frac{1}{q} \right).
\] (3.11)

Now it is clear that \( \tilde{Y} \in (\mathcal{R}^q)^n \).

We have
\[
dS^\tau(t) = S^\tau(t)A_t^\tau \, dM_t,
\]
dS^\tau(t)^{-1} = - [A_t^\tau \, dM_t - (A_t^\tau)^2 \, d\langle M \rangle_t] \, S^\tau(t)^{-1}
\] (3.12)
and
\[
S^\tau(t)\tilde{Y}_t := E \left[ S^\tau(T)\xi + \int_0^T S^\tau(s)f_s \, ds \mid \mathcal{F}_t \right] - \int_0^t S^\tau(s)f_s \, ds.
\] (3.13)

From the martingale decomposition theorem, there is an \( \{ \mathcal{F}_t, 0 \leq t \leq T \} \)-adapted process \( z \) and a martingale \( m^\perp \) such that
\[
S^\tau(T)\xi + \int_0^T S^\tau(s)f_s \, ds = E \left[ S^\tau(T)\xi + \int_0^T S^\tau(s)f_s \, ds \right] + \int_0^T z_s \, dM_s + \int_0^T d\langle m^\perp \rangle_t, \quad \langle M, m^\perp \rangle = 0.
\] (3.14)

Then, we have
\[
S^\tau(t)\tilde{Y}_t = E \left[ S^\tau(T)\xi + \int_0^T S^\tau(s)f_s \, ds \right] + \int_0^t z_s \, dM_s + \int_0^t dm^\perp_t - \int_0^t S^\tau(s)f_s \, ds.
\] (3.15)

Denote by \( X_t \) the right hand side of the last equality. Then, we have
\[
\tilde{Y}_t = S^\tau(t)^{-1}X_t, \quad dX_t = -S^\tau(t)f_t \, dt + z_t \, dM_t + dm^\perp_t;
\] (3.16)
and from Itô’s formula, we further have
\[
d\tilde{Y}_t = -A_t^\tau Z_t \, d\langle M \rangle_t - f_t \, dt + Z_t \, dM_t - dm^\perp_t
\] (3.17)
where
\[
Z_t := A_t^\tau \tilde{Y}_t - S^\tau(t)^{-1}z_t, \quad M^\perp_t := \int_0^t S^\tau(s)^{-1} \, dm^\perp_s.
\] (3.18)

Noting that \( \int_0^t A_t^\tau Z_s \, d\langle M \rangle_s \) is the quadratic variation of \( \tilde{Y} \) and the BMO martingale \( A \circ M \), and then applying the a priori estimate of Yor [50, Proposition 2, page 116], we have
\[
E \left[ \langle \tilde{Y} \rangle^q/2 \right] \leq C_p \left( 1 + \| A \circ M \|^q_{BMO} \right) E \left( |\tilde{Y}^q|_T \right).
\] (3.19)

The last inequality, together with inequality (3.11), shows that
\[
Z \circ M, M^\perp \in (\mathcal{H}^q)^n,
\] (3.20)
and the desired estimate (3.6). The proof for the existence is complete.

The uniqueness follows immediately from the a priori estimate (3.6).

For the special case of \( p = 1 \) (i.e., the conjugate number \( q = \infty \)) and \( n = 1 \), we have the following deeper result.
Theorem 3.3. Let $A \circ M \in BMO$ and $S(\cdot) := \mathcal{E}(A \circ M)$ be its stochastic exponent. Then, for $\xi \in BMO(P)$ and $\int_0^T |f_s| \, ds \in BMO(P)$, BSDE (3.5) has a unique adapted solution $(Y, Z \circ M, M^\perp) \in (\mathbb{R}^q)^n \times (\mathcal{H}^p)^{2n}$ for any $q > 1$. Moreover, we have $Z \circ M + M^\perp \in BMO(P)$, and the following estimate:

$$
\|Z \circ M + M^\perp\|_{BMO} \leq C \left( \|\xi\|_{BMO} + \left\| \int_0^T |f_s| \, ds \right\|_{BMO} \right) \tag{3.21}
$$

for some universal constant $C$ which depends on the BMO norm of $A \circ M$.

Proof of Theorem 3.3. The first assertion follows immediately from Theorem 3.2 and the fact that $S(\cdot)$ satisfies the reverse Hölder inequality $(R_p)$ for $p \in [1, p')$ for some $p' > 1$. It remains to show the second assertion. Without loss of generality, assume $f \equiv 0$.

First it is well known (see Kazamaki [32]) that $\xi^Q : \xi - \langle A \circ M, \xi \rangle \in BMO(Q)$ and $M^Q := M - \langle A \circ M, M \rangle \in BMO(Q)$ due to the fact that $\xi, M \in BMO(P)$. In fact, for some $p > 1$, $\mathcal{E}(A \circ M)$ satisfies the reverse Hölder inequality $(R_p)$. See Kazamaki [32] for this assertion. Therefore, we have

$$
E_Q \left[ \langle \xi^Q \rangle_t^T \bigg| \mathcal{F}_t \right] \leq E_Q \left[ \langle \xi \rangle_t^T \bigg| \mathcal{F}_t \right]
$$

$$
= E \left[ \frac{\mathcal{E}(A \circ M)_t^T \langle \xi \rangle_t^T}{\mathcal{E}(A \circ M)_t} \bigg| \mathcal{F}_t \right]
$$

$$
\leq E \left[ \left( \frac{\mathcal{E}(A \circ M)_t^T}{\mathcal{E}(A \circ M)_t} \right)^{p} \bigg| \mathcal{F}_t \right]^{1/p} E \left[ \left| \langle \xi \rangle_t^T \right|^q \bigg| \mathcal{F}_t \right]^{1/q}
$$

$$
\leq C \|\xi\|_{BMO},
$$

and the same is true for $M^Q$.

From BSDE (3.5), we have $\xi^Q = Z \circ M^Q + M^\perp \in BMO(Q)$. Therefore, we have $Z \circ M + M^\perp \in BMO(P)$. Moreover, we have the following estimate

$$
\|Z \circ M + M^\perp\|_{BMO} \leq C_1\|\xi^Q\|_{BMO(Q)} \leq C_2\|\xi\|_{BMO}. \tag{3.23}
$$

The proof is then complete.

3.2 SDEs

For the multidimensional linear case, we have

Theorem 3.4. Let $A \circ M \in BMO$ such that $S(\cdot)$ satisfies the reverse Hölder inequality $(R_{p'})$ for some $p' \in (1, \infty)$, and $p \in (1, p')$. Then for $V \in (\mathcal{H}^p)^n$, the process

$$
X_t = S(t) \int_0^t S(s)^{-1} dV^Q_s, \quad 0 \leq t \leq T \tag{3.24}
$$

solves SDE (3.4) and lies in $\mathcal{H}^p$. Here, $V^Q := V - \langle A \circ M, V \rangle$.
Remark 3.4. In view of Theorem 3.1, we can take $p = p'$ in Theorem 3.4 if furthermore $S(\cdot)$ is assumed to be uniformly integrable.

The proof of Theorem 3.4 is based on a duality argument, and will appeal to Theorem 3.2.

**Proof of Theorem 3.4.** Set

$$\tilde{X}_t := S(t) \int_0^t S(s)^{-1}dV_s^Q, \quad t \in [0, T].$$

Using Itô’s formula, we can show that

$$d\tilde{X}_t = A_t \tilde{X}_tdM_t + dV_t.$$  \hspace{1cm} (3.26)

For any $\xi \in (L^q(F_T))^n$ and $f \equiv 0$, BSDE 3.5 has a unique adapted solution $(Y, Z, M)$. From Itô’s formula, we have

$$d(\tilde{X}_\tau^\tau Y_t) = Y_\tau^\tau dV_t + \tilde{X}_\tau^\tau (Z_t dM_t + dM_\tau^\tau) + d\langle Y, V \rangle_t, 0 \leq t \leq T.$$  \hspace{1cm} (3.27)

Therefore, applying the inequality (1.9), we have

$$E\langle \xi^\tau, \tilde{X} \rangle = E\langle \xi^\tau \tilde{X}_T \rangle = E\langle Y, V \rangle_T \leq \|\langle Y \rangle_T^{1/2}\|_{L^q(F_T)} \|V\|_{(H^p)^n}. \hspace{1cm} (3.28)$$

In view of the a priori estimate (3.6) of Theorem 3.2, this shows $\tilde{X} \in (H^p)^n$. \hfill \Box

As in Delbaen et al. [10, 11], we have the following theorem.

**Theorem 3.5.** Let $p \in [1, \infty)$ and $A \circ M \in BMO$. Suppose that the solution operator for SDE (3.4):

$$V \mapsto X = S(\cdot) \int_0^\cdot S(s)^{-1}dV_s^Q \in (H^p)^n$$

is continuous from $(H^p)^n \to (H^p)^n$. Here $V^Q := V - \langle A \circ M, V \rangle = \langle V, \cdots, V_n \rangle^\tau - \langle A \circ M, V_1, \cdots, A \circ M, V_n \rangle^\tau$ for $V := \langle V_1, \cdots, V_n \rangle^\tau$. Then $S(\cdot)$ satisfies $(R_p(P))$. Moreover, if $S(\cdot)$ is a uniformly integrable matrix martingale, then the above solution operator remains to be continuous from $H^{p'} \to H^{p'}$ for some $p' > p$.

**Proof of Theorem 3.5.** In view of Theorem 3.1, the second assertion is an immediate consequence of the first one. Therefore, it is sufficient to prove the first assertion.

For any stopping time $\sigma$, we are to show that

$$E\left[|S(T)S(\sigma)^{-1}|^p |\mathcal{F}_\sigma\right] \leq C \hspace{1cm} (3.30)$$

for some constant $C$. For any $B \in \mathcal{F}_\sigma$, take

$$V_i = \chi_{[\sigma,T]} \chi_B A \circ M, \quad i = 1, 2, \cdots, n. \hspace{1cm} (3.31)$$
We have
\[ \|V\|_{(H^p)^n}^p = E \left( (A \circ M)_{T} - (A \circ M)_{\sigma}\right)^{p/2} \chi_B \]
\[ = E \left( \chi_B E \left( (A \circ M)_{T} - (A \circ M)_{\sigma}\right)^{p/2} \left| \mathcal{F}_{\sigma}\right) \right) \]
\[ \leq C \|A \circ M\|_{BMO}^p P(B). \] (3.32)

While
\[ V^Q = \chi_{[\sigma, T]} \chi_B A \circ M - \chi_{[\sigma, T]} \chi_B (A \circ M)^{(1, \cdots, 1)^T}, \]
\[ X_T = \int_0^T \chi_{[\sigma, T]}(s) S(s)^{-1} d(A \circ M)^Q \]
\[ = -S(T) \int_0^T \chi_{[\sigma, T]}(s) dS(s)^{-1} \]
\[ = -\chi_B S(T) \int_{\sigma}^T dS(s)^{-1} \]
\[ = S(T) [S(T)^{-1} - S(\sigma)^{-1}] \chi_B \]
\[ = \chi_B [S(T) S(\sigma)^{-1} - I]. \] (3.33)

From the assumption of the underlying theorem, we have
\[ \|\chi_B [S(T) S(\sigma)^{-1} - I]\|_{(H^p)^{n \times n}} = \|X_T\|_{(H^p)^{n \times n}} \leq C \|V\|_{(H^p)^n} \leq C \|A \circ M\|_{BMO}^p P(B) \] (3.34)
for some constant \( C > 0 \). Therefore, in view of the BDG inequality, the quantity
\[ \|S(T) S(\sigma)^{-1} \chi_B\|_{(L^p)^{n \times n}} \]
is bounded by \( P(B) \). This implies that \( S(\cdot) \) satisfies the reverse Hölder inequality \( (R_p) \).

**Remark 3.5.** From Theorems 2.4 and 3.5, we see that if \( A \circ M \in \mathcal{H}^{\infty BMO} \), then \( S(\cdot) \) satisfies the reverse Hölder inequality \( (R_p) \) for all \( p \in [1, \infty) \).

For the special case of \( p = 1 \) and \( n = 1 \), we have the following better result.

**Theorem 3.6.** Let \( n = 1 \). Assume that \( A \circ M \in BMO \). Then for \( V \in \mathcal{H}^1 \), the local martingale
\[ X_t = S(t) \int_0^t S(s)^{-1} dV^Q_s, \quad t \in [0, T] \] (3.35)
solves SDE (3.4) and lies in \( \mathcal{H}^1 \). Here, \( V^Q := V - (A \circ M, V) \).

**Proof of Theorem 3.6.** Take any \( \xi \in BMO \) and \( f \equiv 0 \). Let \((Y, Z \circ M, M^\perp)\) be the unique solution of BSDE (3.5) for the data \((\xi, f)\). As shown in the proof of Theorem 3.4, we have
\[ E(\xi^T, \tilde{X}) = E(\xi^T \tilde{X}_T) = E(\xi^T Y)_T = E(Z \circ M + M^\perp, V)_T. \] (3.36)
Applying Fefferman’s inequality, we have
\[ E\langle \xi^\tau, \tilde{X}\rangle \leq \sqrt{2}\|Z \circ M + M^\perp\|_{BMO}\|V\|_{(\mathcal{H}^1)^n}. \] (3.37)

In view of the a priori estimate (3.21) of Theorem 3.3, we have
\[ E\langle \xi^\tau, \tilde{X}\rangle \leq C\|\xi\|_{BMO}\|V\|_{(\mathcal{H}^1)^n}, \quad \forall \xi \in BMO \] (3.38)

for some positive constant C. In view of Lemma 1.3, the last inequality implies that \( \tilde{X} \in (\mathcal{H}^1)^n \). The proof is then complete.

4 One-dimensional linear case: the characterization of Kazamaki’s critical quadratic exponent being infinite.

In Section 2, we have applied Fefferman’s inequality to prove new results for SEs and BSDEs. In what follows, we present an operator approach to Kazamaki’s critical quadratic exponent on BMO martingales. We establish some relations between Kazamaki’s critical quadratic exponent \( b(M) \) of a BMO martingale \( M \) and the solution operator for the associated \( M \)-driven SDE. Throughout this section, all processes will be considered in \([0, \infty)\).

Let \( M \in BMO \) be real and \( p \in [1, \infty) \). Consider the operator \( \hat{\phi} : \phi(X) = X \circ M \) for \( X \in \mathcal{H}^p \). Define the complex version \( \tilde{\phi} : \mathcal{H}^p(\mathbb{C}) \to \mathcal{H}^p(\mathbb{C}) \) as follows:
\[ \tilde{\phi}(U + iV) := U \circ M + iV \circ M. \] (4.1)

Since
\[ \|\tilde{\phi}(U + iV)\|_p \leq \|\tilde{\phi}(U)\|_{\mathcal{H}^p} + \|\tilde{\phi}(V)\|_{\mathcal{H}^p} \leq \|\phi\|\|U\|_{\mathcal{H}^p} + \|\phi\|\|V\|_{\mathcal{H}^p} \leq 2\|\phi\|\|U + iV\|_{\mathcal{H}^p}, \] (4.2)

we have
\[ \|\tilde{\phi}\| \leq 2\|\phi\|, \quad \|\tilde{\phi}^n\| \leq 2\|\phi^n\| \quad \text{(since } M \text{ is real!).} \] (4.3)

Their spectral radii are equal, denoted by \( r_p \):
\[ \lim_{n \to \infty} \|\tilde{\phi}^n\|^{1/n} = \lim_{n \to \infty} \|\phi^n\|^{1/n} = r_p. \] (4.4)

For \( \lambda \in \mathbb{C} \), define
\[ M^\lambda : \lambda M - \lambda^2\langle M \rangle \] (4.5)

and
\[ \mathcal{E}(\lambda M)_t := \exp\left( \lambda M_t - \frac{1}{2}\lambda^2\langle M \rangle_t \right) \] (4.6)

which is a complex local martingale.

Using the same procedure as in the real case, we have
Proposition 4.1. Suppose that $\lambda \in \mathbb{C}$ and that $(Id - \lambda \tilde{\phi})$ has an inverse on $\mathcal{H}^p(\mathbb{C})$ for some $p \in [1, \infty)$. Then $\mathcal{E}(\lambda M)$ satisfies the following stronger property than $(R_p)$: there is a positive constant $K$ such that

$$E \left[ \left| \frac{\mathcal{E}(\lambda M)_\sigma}{\mathcal{E}(\lambda M)_\tau} \right|^p \right| \mathcal{F}_\tau] \leq K$$

for any stopping times $\tau$ and $\sigma$ such that $0 \leq \tau \leq \sigma \leq \infty$.

Remark 4.1. Even for $p = 1$, Proposition 4.1 yields information in the complex case.

Proof of Proposition 4.1. Use stopping to make all integrals bounded. For $A \in \mathcal{F}_\tau$, define the process $g$ as follows:

$$g(t) = \chi_A \chi_{(t, \infty)}(t), \quad t \in [0, \infty).$$

Then, we have

$$\| g \circ M \|_{\mathcal{H}^p} \leq C[P(A)]^{\frac{1}{2}}.$$  

Indeed, we have

$$E \left[ \langle g \circ M \rangle_\infty^{p/2} \right] = E \left[ \chi_A \left( \int_\tau^\infty \langle M \rangle_t^{p/2} \right) \right]
= E \left[ \chi_A (\langle M \rangle_\infty)^{p/2} \right]
\leq CP(A).$$  

On the other hand, we have

$$\phi^\lambda(g \circ M)_\sigma := \mathcal{E}(\lambda M)_\sigma \int_\tau^\sigma \mathcal{E}(\lambda M)^{-1}_s g(s) dM^\lambda_s
= \mathcal{E}(\lambda M)_\sigma \int_\tau^\sigma \mathcal{E}(\lambda M)^{-1}_s \chi_A dM^\lambda_s
= \mathcal{E}(\lambda M)_\sigma \left( \mathcal{E}(-\lambda M^\lambda)_\tau - \mathcal{E}(-\lambda M^\lambda)_\sigma \right) \chi_A$$

(4.11)

Since the map $\phi^\lambda : g \circ M \rightarrow \phi^\lambda(g \circ M)$ is the operator $(Id - \lambda \phi)$, we get by hypothesis that there is a constant $K$ (changing from line to line) such that

$$E \left[ \chi_A \left| \frac{\mathcal{E}(\lambda M)_\sigma}{\mathcal{E}(\lambda M)_\tau} - 1 \right|^p \right] \leq K \| g \circ M \|_{\mathcal{H}^p} \leq KP(A).$$

(4.12)

Therefore,

$$E \left[ \left| \frac{\mathcal{E}(\lambda M)_\sigma}{\mathcal{E}(\lambda M)_\tau} - 1 \right|^p \right| \mathcal{F}_\tau] \leq K,$$

(4.13)

which implies the following

$$E \left[ \left| \frac{\mathcal{E}(\lambda M)_\sigma}{\mathcal{E}(\lambda M)_\tau} \right|^p \right| \mathcal{F}_\tau] \leq K.$$  

(4.14)
Proposition 4.2. Suppose that $\mathcal{E}(\lambda M)$ satisfies $(R_p)$ for some $\lambda \in \mathbb{C}$ and some $p \in (1, \infty)$, that is, there is a positive constant $K$ such that

$$E \left[ \left| \mathcal{E}(\lambda M)^\sigma \right|^{\frac{p}{2}} \right| \mathcal{F}_\tau \right] \leq K \tag{4.15}$$

for any stopping times $\tau$ and $\sigma$ such that $0 \leq \tau \leq \sigma \leq \infty$. Then $(\text{Id} - \lambda \tilde{\phi})$ has an inverse on $\mathcal{H}^p(\mathbb{C})$.

Kazamaki [32, Lemma 2.6, page 48] states that

$$\frac{1}{\sqrt{2d_2(M, \mathcal{H}^\infty)}} \leq b(M). \tag{4.16}$$

Schachermayer [45] has shown that the reverse is not true in the following sense:

$$b(M) = +\infty \not\Rightarrow M \in \mathcal{H}^{BMO}. \tag{4.17}$$

There seems to be no hope to establish a relation between $\text{dist}(M, \mathcal{H}^\infty)$ and $b(M)$.

Proposition 4.3. If $\lambda \in \mathbb{C}$ satisfies

$$|\lambda| < \frac{b(M)}{\sqrt{2p(2p - 1)}}, \tag{4.18}$$

then $(\text{Id} - \lambda \tilde{\phi})$ has an inverse on $\mathcal{H}^p(\mathbb{C})$.

Proof of Proposition 4.3. In view of Proposition 4.2, it is sufficient to show that $\mathcal{E}(\lambda M)$ satisfies $(R_p)$, i.e., there is a positive constant $C$ such that

$$E \left[ |\mathcal{E}(\lambda N)|^p \right| \mathcal{F}_\tau \right] \leq C \tag{4.19}$$

with $N := M - M^\tau$.

Denote $\lambda := u + iv$ with $u$ and $v$ being real numbers. We have

$$\left| \exp \left( p(u + iv)N_\infty - \frac{1}{2}(u + iv)^2\langle N \rangle_\infty \right) \right| = \exp \left( puN_\infty - \frac{1}{2}p(u^2 - v^2)\langle N \rangle_\infty \right) \tag{4.20}$$

$$= \exp \left( puN_\infty - p^2u^2\langle N \rangle_\infty \right) \exp \left( p^2u^2\langle N \rangle_\infty - \frac{1}{2}pu^2\langle N \rangle_\infty + \frac{1}{2}pv^2\langle N \rangle_\infty \right).$$

Taking the conditional expectation and using the Cauchy-Schwarz inequality, we have

$$E \left[ |\mathcal{E}(\lambda N)_\infty|^p \right| \mathcal{F}_\tau \right] \leq E \left[ \exp \left( 2puN_\infty - 2p^2u^2\langle N \rangle_\infty \right) \right| \mathcal{F}_\tau \right] \frac{1}{2} E \left[ \exp \left( \langle N \rangle_\infty \left( 2p^2u^2 - pu^2 + pv^2 \right) \right) \right] \frac{1}{2}. \tag{4.21}$$
Since $M$ is in BMO and so is $2puM$, $\mathcal{E}(2puM)$ is uniformly integrable. Hence, we have
\[
E\left[\exp\left(2puN_\infty - 2p^2u^2(N)_\infty\right)\big|\mathcal{F}_\tau\right] = 1. \tag{4.22}
\]
Concluding the above, we have
\[
E\left[|\mathcal{E}(\lambda N)_\infty|^p \big|\mathcal{F}_\tau\right] \leq E\left[\exp\left((N)_\infty (2p^2u^2 - pu^2 + pv^2)\right)\big|\mathcal{F}_\tau\right]^{\frac{1}{2}}. \tag{4.23}
\]
In view of the fact that
\[
2p^2u^2 - pu^2 + pv^2 \leq p(2p - 1)(u^2 + v^2) = p(2p - 1)|\lambda|^2 < \frac{1}{2}b^2(M), \tag{4.24}
\]
we obtain the desired inequality (4.19).

The spectral radius $r_p$ of $\tilde{\phi}: \mathcal{H}^p(\mathbb{C}) \to \mathcal{H}^p(\mathbb{C})$ is estimated by $b(M)$.

**Corollary 4.1.** We have
\[
r_p \leq \frac{\sqrt{2p(2p - 1)}}{b(M)}. \tag{4.25}
\]

**Proof.** Since
\[
(1 - \lambda \tilde{\phi})^{-1} \text{ exists } \iff (\lambda^{-1} - \tilde{\phi})^{-1} \text{ exists }, \tag{4.26}
\]
we have from Proposition 4.3 that
\[
|\lambda^{-1}| > r_p \tag{4.27}
\]
for all $\lambda \in \mathbb{C}$ such that
\[
|\lambda| < \frac{b(M)}{\sqrt{2p(2p - 1)}}. \tag{4.28}
\]
This implies immediately the desired inequality.

**Proposition 4.4.** We have
\[
r_p \geq \frac{\sqrt{p}}{b(M)}. \tag{4.29}
\]

**Proof of Proposition 4.4.** Take $\lambda \in R$ such that $\lambda < r_p^{-1}$. Then $(Id - i\lambda \tilde{\phi})$ has inverse. From Proposition 4.1, we see that there is a positive constant $C$ such that
\[
E\left[\left|\mathcal{E}(\lambda M)_\infty\right|^p \big|\mathcal{F}_\tau\right] \leq C. \tag{4.30}
\]

Set $N := M - M^\tau$. We have
\[
E\left[\exp\left(\frac{1}{2}p\lambda^2 N\right)\big|\mathcal{F}_\tau\right] \leq C, \tag{4.31}
\]
which is equivalent to the following
\[
E\left[\exp\left(\frac{1}{2}p\lambda^2 \langle N \rangle\right)\big|\mathcal{F}_\tau\right] \leq C. \tag{4.32}
\]
Therefore, by the definition of \( b(M) \), we have

\[ p\lambda^2 \leq b^2(M) \]  

(4.33)

for all \( \lambda \in \mathbb{R} \) such that \( \lambda < r_p^{-1} \). The desired inequality then follows immediately.

We combine the above two propositions into the following theorem.

**Theorem 4.1.** We have

\[ \frac{\sqrt{p}}{b(M)} \leq r_p \leq \frac{\sqrt{2p(2p-1)}}{b(M)}. \]  

(4.34)

We have the following equivalent conditions.

**Theorem 4.2.** The following statements are equivalent.

(i) \( \forall \lambda \in \mathbb{C}, \forall p \in [1, \infty), \text{ the map } \text{Id} - \lambda \varphi : \mathcal{H}^p \rightarrow \mathcal{H}^p \text{ is an isomorphism} \).

(ii) For some \( p \in [1, \infty) \), the map \( \text{Id} - \lambda \varphi : \mathcal{H}^p \rightarrow \mathcal{H}^p \text{ is an isomorphism for } \forall \lambda \in \mathbb{C} \).

(iii) \( \forall \lambda \in \mathbb{C}, \forall p \in [1, \infty), \text{ there is } K > 0 \text{ such that} \)

\[ E \left[ \left| \frac{\mathcal{E}(\lambda M)_{\sigma}}{\mathcal{E}(\lambda M)_\tau} \right|^p \right] \leq K \]  

(4.35)

for any stopping times \( \tau \) and \( \sigma \) such that \( 0 \leq \tau \leq \sigma \leq \infty \).

(iv) For some \( p \in [1, \infty) \) and \( \forall \lambda \in \mathbb{C} \), there is \( K > 0 \) such that

\[ E \left[ \left| \frac{\mathcal{E}(\lambda M)_{\sigma}}{\mathcal{E}(\lambda M)_\tau} \right|^p \right] \leq K \]  

(4.36)

for any stopping times \( \tau \) and \( \sigma \) such that \( 0 \leq \tau \leq \sigma \leq \infty \).

(v) \( \tilde{\phi} : \mathcal{H}^p(\mathbb{C}) \rightarrow \mathcal{H}^p(\mathbb{C}) \) is quasinilpotent (i.e. \( r_p = 0 \)) for \( \forall p \in [1, \infty) \).

(vi) \( \tilde{\phi} : \mathcal{H}^p(\mathbb{C}) \rightarrow \mathcal{H}^p(\mathbb{C}) \) is quasinilpotent (i.e. \( r_p = 0 \)) for some \( p \in [1, \infty) \).

(vii) \( b(M) = +\infty \).

(viii) \( \lim_{n \to \infty} ||\phi^n||^{1/n} = \lim_{n \to \infty} ||\phi^n||^{1/n} = 0. \)

**Proof.** We show that (vii) \( \implies \) (iii). For \( \forall \lambda = \lambda_1 + i\lambda_2 \in \mathbb{C} \), we have

\[ \left| \frac{\mathcal{E}(\lambda M)^\infty}{\mathcal{E}(\lambda M)_\tau} \right| = \exp \left[ \frac{1}{2} (\langle M \rangle^\infty - \langle M \rangle_\tau) \right] \]

\[ \leq \exp \left[ |\lambda_1||M^\infty - M_\tau| \right] \exp \left[ \frac{1}{2} \lambda_2^2 (\langle M \rangle^\infty - \langle M \rangle_\tau) \right]. \]  

(4.37)

Hence,

\[ \left| \frac{\mathcal{E}(\lambda M)^\infty}{\mathcal{E}(\lambda M)_\tau} \right|^p \leq \exp \left[ |\lambda_1| p |M^\infty - M_\tau| \right] \exp \left[ \frac{1}{2} \lambda_2^2 p (\langle M \rangle^\infty - \langle M \rangle_\tau) \right]. \]  

(4.38)

On the other hand, we have for \( \forall \lambda = \lambda_1 + i\lambda_2 \in \mathbb{C} \),

\[ \exp \left[ \frac{1}{2} (\lambda_2^2 - 2\lambda_1^2) (\langle M \rangle^\infty_\tau) \right] = \exp \left( \frac{1}{2} \lambda_2^2 p (\langle M \rangle^\infty_\tau) \right) \cdot \mathcal{E}(\lambda M^\infty_\tau) \cdot \mathcal{E}(\lambda_1 M^\infty_\tau), \]

(4.39)

from which we can derive that (iii) \( \implies \) (vii).
References

[1] R. Bañuelos and A. G. Bennett, *Paraproducts and Commutators of martingale transforms*, Proc. American Mathematical Society, 103 (1988), 1226–1234.

[2] P. Barrieu and N. El Karoui, *Optimal derivatives design under dynamic risk measures*, In: Mathematics of Finance, 13–25, Contemp. Math., 351, Amer. Math. Soc., Providence, RI, 2004.

[3] J. M. Bismut, *Conjugate convex functions in optimal stochastic control*, J. Math. Anal. Appl., 44 (1973), 384–404.

[4] J. M. Bismut, *Linear quadratic optimal stochastic control with random coefficients*, SIAM J. Control, 14 (1976), 419–444.

[5] J. M. Bismut, *Contrôle des systèmes linéaires quadratiques: applications de l’intégrale stochastique*, Séminaire de Probabilités XII (eds.: C. Dellacherie, P. A. Meyer, and M. Weil), Lecture Notes in Mathematics 649, 180–264, Springer-Verlag, Berlin/Heidelberg, 1978.

[6] P. Briand and Y. Hu, *Ying BSDE with quadratic growth and unbounded terminal value*, Probab. Theory Related Fields, 136 (2006), no. 4, 604–618.

[7] P. Briand and Y. Hu, *Quadratic BSDEs with convex generators and unbounded terminal conditions*, to appear in Probab. Theory Related Fields, 2007.

[8] R. Buckdahn, *Backward stochastic differential equations driven by a martingale*, preprint, 1993.

[9] C. Dellacherie and P. A. Meyer, *Probabilités et Potentiels. Théorie des Martingales*. Hermann, 1980.

[10] F. Delbaen, P. Monat, W. Schachermayer, M. Schweizer, C. Stricker, *Inégalités de normes avec poids et fermeture d’un espace d’intégrales stochastiques*, Comptes Rendus Acad. Sci. Paris 319 (1994), Série I, 1079-1081.

[11] F. Delbaen, P. Monat, W. Schachermayer, M. Schweizer, C. Stricker, *Weighted norm inequalities and hedging in incomplete markets*, Finance and Stochastics, 1 (1997), 181–227.

[12] C. Doléans-Dade, *On the existence and unicity of solutions of stochastic differential equations*, Z. Wahrscheinlichkeitstheorie verw. Gebiete, 36 (1976), 93–101.

[13] C. Doléans-Dade and P. A. Meyer, *Équations différentielles stochastiques*. Séminaire de Probabilités XI, Lecture Notes in Mathematics 581, 376–382, Springer-velag, Berlin, 1977.
[14] N. El Karoui and S. Huang, *A general result of existence and uniqueness of backward stochastic differential equations*, in: (eds. N. El Karoui and Mazliak), Backward Stochastic Differential Equations, Pitman research Notes in Math. Series, 364, 27–36 (1997).

[15] N. El Karoui, S. Peng, and M. C. Quenez, *Backward stochastic differential equations in finance*, Mathematical Finance, 7 (1997), 1-71.

[16] M. Emery, *Sur l’exponentielle d’une martingale de BMO*, in: Séminaire de Probabilités XVIII (eds.: J. Azéméa and M. Yor), Lecture Notes in Mathematics 1059, page 500, Springer-Verlag, Berlin/Heidelberg, 1984.

[17] M. Emery, *Stabilité des solutions des équations différentielles stochastiques; applications aux intégrales multiplicatives stochastiques*. Z. Wahrscheinlichkeitstheorie verw. Gebiete, 41 (1978), 241–262.

[18] M. Emery, *Équations différentielles stochastiques lipschitziennes: étude de la stabilité*. Séminaire de Probabilités XIII, Lecture Notes in Mathematics 721, 281–293, Springer-velag, Berlin, 1979.

[19] M. Emery, *Le théorème de Garnett-Jones d’après Varopoulos*, In: Séminaire de Probabilités XV, (eds.: J. Azéma and M. Yor), Université de Strasbourg, Lecture Notes in Mathematics 721, pages 278–284, Berlin Heidelberg New York, Springer, 1985.

[20] H. Föllmer and M. Schweizer, *Hedging of contingent claims under incomplete information*, in: Applied Stochastic Analysis, Stochastics Monograph, (eds.: M. H. A. Davis and R. J. Elliott), 389–414, Gordon and Breach, 1991, London.

[21] J. Garnett and P. Jones, *The distance in BMO to $L^\infty$*, Ann. Math., 108 (1978), 373–393.

[22] P. Grandits, *On a conjecture of Kazamaki*, Séminaire de Probabilités, XXX, 357–360, Lecture Notes in Mathematics, 1626, Springer, Berlin, 1996.

[23] P. Grandits and L. Krawczyk, *Closedness of some spaces of stochastic integrals*, Séminaire de Probabilités, XXXIII, 73–85, Lecture Notes in Mathematics, 1686, Springer, Berlin, 1998.

[24] P. Grandits, *Some remarks on $L^\infty$, $H^\infty$ and BMO*. Séminaire de Probabilités, XXXIII, 342–348, Lecture Notes in Mathematics, 1709, Springer, Berlin, 1999.

[25] S. He, J. Wang, and J. Yan, *Semimartingale Theory and Stochastic Calculus*, Science Press and CRC Press Inc, Beijing/New York, 1992.

[26] Y. Hu, P. Imkeller, and M. Müller, *Utility maximization in incomplete markets*, Ann. Appl. Probab., 15 (2005), 1691–1712.
[27] Y. Hu, J. Ma, S. Peng, and S. Yao, *Representation theorems for quadratic $\mathcal{F}$-consistent nonlinear expectations*, Prépublication 07-26, April 2007; see also arXiv:0704.1796v1 [math.PR], April 13, 2007.

[28] Y. Hu and X. Zhou, *Constrained stochastic LQ control with random coefficients, and application to portfolio selection.*, SIAM J. Control Optim., 44 (2005), 444–466.

[29] K. Itô, *Differential equations determining Markov processes (in Japanese)*, Zenkoku Shijō Sūgaku Danwakai, 1077 (1942), 1352–1400.

[30] K. Itô, *On a stochastic integral equation*, Proc. Imp. Acad. Tokyo, 22 (1946), 32–35.

[31] K. Itô, *On stochastic differential equations*, Mem. Amer. Math. Soc., 4 (1951), 1–51.

[32] N. Kazamaki, *Continuous Exponential Martingales and BMO*, Lecture Notes in Mathematics 1579, Berlin, Heidelberg: Springer 1994.

[33] M. Kobylansky, *Existence and uniqueness results for backward stochastic differential equations when the generator has a quadratic growth*, C. R. Acad. Sci., Ser. I-Math., 324 (1997), 81–86.

[34] M. Kobylansky, *Backward stochastic differential equations and partial differential equations with quadratic growth*, The Annals of Probability, 28 (2000), 558–602.

[35] M. Kohlmann and S. Tang, *New developments in backward stochastic Riccati equations and their applications*, Mathematical Finance (Konstanz, 2000), 194–214, Trends Math., Birkhäuser, Basel, 2001.

[36] M. Kohlmann and S. Tang, *Global adapted solution of one-dimensional backward stochastic Riccati equations, with application to the mean-variance hedging*, Stochastic Process. Appl. 97 (2002), no. 2, 255–288.

[37] M. Kohlmann and S. Tang, *Minimization of risk and linear quadratic optimal control theory*, SIAM J. Control Optim. 42 (2003), no. 3, 1118–1142.

[38] J. Ma, J. Yong, *Forward-Backward Differential Equations and Their Applications*, Lecture Notes in Mathematics, 1702, Berlin, Springer, 1999.

[39] P. Monat and C. Stricker, *Décomposition de Föllmer-Schweizer et fermeture de $G_T(\Theta)$*, C. R. Acad. Sci. Sér. I, 318 (1994), 573–576.

[40] P. Monat and C. Stricker, *Föllmer-Schweizer decomposition and mean-variance hedging for general claims*, The Annals of Probability, 23 (1995), 605–628.

[41] P. Monat and C. Stricker, *Fermeture de $G_T(\Theta)$ et de $L^2(\mathcal{F}_0) + G_T(\Theta)$*, —Séminaire de Probabilités XXVIII, Lecture Notes in Mathematics 1583, 189–194, Springer, 1994, New York.
[42] E. Pardoux and S. Peng, *Adapted solution of a backward stochastic differential equation*, Systems Control Lett., 14 (1990), 55–61.

[43] P. Protter, *On the existence, uniqueness, convergence, and explosions of solutions of systems of stochastic differential equations*, Ann. Probab., 5 (1977), 243–261.

[44] P. Protter, *Stochastic Integration and Differential Equations*, Second Edition, Springer-Verlag, New York, 2004.

[45] W. Schachermayer, *A characterisation of the closure of $H^\infty$ in BMO*, Séminaire de Probabilités, XXX, 344–356, Lecture Notes in Mathematics, 1626, Springer, Berlin, 1996.

[46] M. Schweizer, *Approximating random variables by stochastic integrals*, The Annals of Probability, 22 (1994), 1536–1575.

[47] M. Schweizer, *A projection result for semimartingales*, Stochastics and Stochastic Reports, 50 (1994), 175–183.

[48] S. Tang, *General linear quadratic optimal stochastic control problems with random coefficients: linear stochastic Hamilton systems and backward stochastic Riccati equations*, SIAM J. Control Optim. 42 (2003), no. 1, 53–75.

[49] N. Th. Varopoulos, *A probabilistic proof of the Garnett-Jones theorem on BMO*, Proc. J. Math., 90 (1980), 201–221.

[50] M. Yor, *Inégalités de martingales continues arrêtées à un temps quelconque, I: théorèmes généraux*, In: Grossissements de filtrations: exemples et applications (eds.: Th. Jeulin and M. Yor), 110-146, Lecture Notes in Mathematics 1118, Springer-Verlag, Berlin Heidelberg, 1985.