Super-biderivations of the contact Lie superalgebra \( K(m, n; \mathfrak{t}) \)

Xiaodong Zhao\(^1,2\), Yuan Chang\(^3\), Xin Zhou\(^1,2\), Liangyun Chen\(^1\)

\(^1\) School of Mathematics and Statistics, Northeast Normal University, Changchun, 130024, CHINA
\(^2\) School of Mathematics and Statistics, Yili Normal University, Yining, 835000, CHINA
\(^3\) School of Mathematics, Dongbei University of Finance and Economics, Dalian, 116025, CHINA

Abstract

Let \( K \) denote the contact Lie superalgebra \( K(m, n; \mathfrak{t}) \) over a field of characteristic \( p > 3 \), where \( m, n \in \mathbb{N} + 1 \) and \( \mathfrak{t} = (t_1, t_2, \ldots, t_m) \) is an \( m \)-tuple of positive integers, and \( K \) has a finite \( \mathbb{Z} \)-graded structure. Let \( T_K \) be the canonical torus of \( K \), which is an abelian subalgebra of \( K_0 \) and semisimple on \( K_{-1} \). Utilizing the weight space decomposition of \( K \) with respect to \( T_K \), we prove in this paper that each skew-symmetric super-biderivation of \( K \) is inner.

Key words: Torus; Weight space decomposition; Super-biderivation.
Mathematics Subject Classification(2010): 17B05; 17B40; 17B50

1 Introduction

Let \( L \) be a Lie algebra over an arbitrary field \( \mathbb{F} \). An \( \mathbb{F} \)-linear map \( D : L \to L \) is a derivation satisfying

\[
D([x, y]) = [D(x), y] + [x, D(y)],
\]

for all \( x, y \in L \). A bilinear map \( \psi : L \times L \to L \) is called a biderivation if it is a derivation with respect to both components, meaning that

\[
\psi(x, [y, z]) = [\psi(x, y), z] + [y, \psi(x, z)],
\]

(1.1)
\[ \psi([x, y], z) = [\psi(x, z), y] + [x, \psi(y, z)], \]  
(1.2)

for all \( x, y, z \in L \). A biderivation \( \psi \) is called skew-symmetric if \( \psi(x, y) = -\psi(y, x) \) for all \( x, y \in L \). Obviously, if a biderivation \( \psi \) is skew-symmetric, we can omit one of the equations (1.1) and (1.2). Meanwhile, we can view \( \psi(x, \cdot) \) or \( \psi(\cdot, x) \) as a derivation of \( L \).

The study of biderivations traces back to the research on the commuting map in the associative ring \([1]\), where the author showed that all biderivations on associative prime rings are inner. The notation of biderivations of Lie algebras was introduced in \([2]\). In recent years, there exist a lot of interests in studying biderivations and commuting maps on Lie algebras \([3\,\text{to}\,9]\). Moreover, the authors gave the notion of the skew-symmetric super-biderivation in \([1]\). So the results about the skew-symmetric super-biderivation of Lie superalgebras arise in \([11]\,\text{to}\,[13]\).

The Cartan modular Lie superalgebra is an important branch of the modular Lie superalgebra, which is a Lie superalgebra over an algebraically closed field of characteristic \( p > 0 \). And the contact Lie superalgebra \( K(m, n; \mathcal{L}) \) is an important class of Cartan modular Lie superalgebras. There are many research results about the contact Lie superalgebra \( K(m, n; \mathcal{L}) \), such as, derivation superalgebras \([14]\,\text{to}\,[16]\), noncontractible filtrations \([17]\), nondegenerate associative bilinear forms \([18]\).

In this paper, we prove that each skew-symmetric super-biderivation of \( K(m, n; \mathcal{L}) \) is inner. The paper is organized as follows. In Section 2, we recall the basic notation. In Section 3, we use the weight space decomposition of \( K(m, n; \mathcal{L}) \) with respect to the canonical torus \( T_K \) to prove that all skew-symmetric super-biderivation of \( K \) is inner (Theorem 3.14).

## 2 Preliminaries

Let \( F \) denote the prime field of the characteristic \( p > 2 \) and \( \mathbb{Z}_2 = \{0, 1\} \) the additive group of two elements. For a vector superspace \( V = V_0 \oplus V_1 \), we use \( p(x) \) for the parity of \( x \in V_\alpha, \alpha \in \mathbb{Z}_2 \). If \( V = \bigoplus_{i \in \mathbb{Z}} V_i \) is a \( \mathbb{Z} \)-graded vector space and \( x \in V \) is a \( \mathbb{Z} \)-homogeneous element, write \( |x| \) for the \( \mathbb{Z} \)-degree of \( x \). Once the symbol \( p(x) \) or \( |x| \) appears in this paper, it implies that \( x \) is a \( \mathbb{Z}_2 \)-homogeneous element or that \( x \) is a \( \mathbb{Z} \)-homogeneous element. Throughout this paper all vector spaces or algebras are over \( F \).

### 2.1 Skew-symmetric super-biderivations of a Lie superalgebra

Let us recall some facts related to the superderivation and skew-symmetric super-biderivation of Lie superalgebras. A Lie superalgebra is a vector superspace \( L = L_0 \oplus L_\mathcal{T} \) with an even bilinear mapping \([,\,] : L \times L \to L\) satisfying the following axioms:

\[
[x, y] = -(-1)^{p(x)p(y)}[y, x],
\]

\[
[x, [y, z]] = [[x, y], z] + (-1)^{p(x)p(y)}[y, [x, z]],
\]

\[
[\psi([x, y], z) = [\psi(x, z), y] + [x, \psi(y, z)],
\]  
(1.2)
for all \( x, y, z \in L \). We call a linear mapping \( D : L \times L \to L \) a superderivation of \( L \) if it satisfies the following axiom:

\[
D([x, y]) = [D(x), y] + (-1)^{p(D)p(x)}[x, D(y)],
\]

for all \( x, y \in L \), where \( p(D) \) denotes the \( \mathbb{Z}_2 \)-degree of \( D \). Write \( \text{Der}_\mathbb{Z}(L) \) (resp. \( \text{Der}_\mathbb{T}(L) \)) for the set of all superderivations of \( \mathbb{Z}_2 \)-degree \( \mathbb{Z} \) (resp. \( \mathbb{T} \)) of \( L \).

We call a bilinear mapping \( \phi : L \times L \to L \) a skew-symmetric super-biderivation of \( L \) if it satisfies the following axioms:

\[
\begin{align*}
\text{skew - symmetry} : & \quad \phi(x, y) = -(-1)^{p(x)p(y)}\phi(y, x), \\
\phi([x, y], z) = & \quad (-1)^{p(\phi)p(x)}\phi([x, y], z) + (-1)^{p(y)p(x)}\phi(x, [y, z]), \\
\phi(x, [y, z]) = & \quad [\phi(x, y), z] + (-1)^{p(\phi)+p(x)p(y)}[y, \phi(x, z)],
\end{align*}
\]

for all \( \mathbb{Z}_2 \)-homogeneous elements \( x, y, z \in L \). A super-biderivation \( \phi \) of \( \mathbb{Z}_2 \)-degree \( \gamma \) of \( L \) is a super-biderivation such that \( \phi(L_\alpha, L_\beta) \subseteq L_{\alpha+\beta+\gamma} \) for any \( \alpha, \beta \in \mathbb{Z}_2 \). Denote by \( \text{BDer}_\gamma(L) \) the set of all skew-symmetric super-biderivations of \( \mathbb{Z}_2 \)-degree \( \gamma \). Obviously,

\[
\text{BDer}(L) = \text{BDer}_\mathbb{Z}(L) \oplus \text{BDer}_\mathbb{T}(L).
\]

Specially, if the bilinear map \( \phi_\lambda : L \times L \to L \) is defined by \( \phi_\lambda(x, y) = \lambda [x, y] \) for all \( x, y \in L \), where \( \lambda \in \mathbb{F} \), then it is easy to check that \( \phi_\lambda \) is a super-biderivation of \( L \). This class of super-biderivations is called inner. Denote by \( \text{IBDer}(L) \) the set of all inner super-biderivations.

### 2.2 Contact Lie superalgebras \( K(m, n; \mathbb{F}) \)

We propose to construct a \( \mathbb{Z}_2 \)-gradation tensor algebra via a divided power algebra and a exterior superalgebra. In the follow, we introduce the divided power algebra \( \mathcal{O}(m) \) and the exterior superalgebra \( \Lambda(n) \). Fix two positive integers \( m > 1 \) and \( n > 1 \). For \( \alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m \), where \( \mathbb{N} \) denote the set of natural numbers, put \( |\alpha| = \sum_{i=1}^{m} \alpha_i \).

For two \( m \)-tuples \( \alpha = (\alpha_1, \ldots, \alpha_m) \) and \( \beta = (\beta_1, \ldots, \beta_m) \in \mathbb{N}^m \), we write \( \binom{n}{\beta} = \prod_{i=1}^{m} \binom{\alpha_i}{\beta_i} \) and define \( \beta \preceq \alpha \iff \beta_i \leq \alpha_i, 1 \leq i \leq m \). Let \( \mathcal{O}(m) \) denote the \( \mathbb{F} \)-algebra of divided power series in the variable \( x_1, \ldots, x_m \), which is called a divided power algebra. For convenience, we replace \( x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_m^{\alpha_m} \) by \( x^{(\alpha)} = (\alpha_1, \alpha_2, \ldots, \alpha_m) \). Obviously, \( \mathcal{O}(m) \) has an \( \mathbb{F} \)-basis \( \{x^{(\alpha)}|\alpha \in \mathbb{N}^m\} \) and satisfies the formula:

\[
x^{(\alpha)} x^{(\beta)} = \binom{\alpha + \beta}{\alpha} x^{(\alpha + \beta)}, \quad \forall \ \alpha, \beta \in \mathbb{N}^m.
\]  

Let \( \Lambda(n) \) denote the exterior superalgebra over \( \mathbb{F} \) with \( n \) variables \( x_{m+1}, \ldots, x_s \), where \( s = m + n \). The tensor product \( \mathcal{O}(m, n) = \mathcal{O}(m) \otimes_\mathbb{F} \Lambda(n) \) is an associative superalgebra with a \( \mathbb{Z}_2 \)-gradation induced by the trivial \( \mathbb{Z}_2 \)-gradation of \( \mathcal{O}(m) \) and the natural \( \mathbb{Z}_2 \)-gradation of \( \Lambda(n) \). Obviously, \( \mathcal{O}(m, n) \) is super-commutative. For \( g \in \mathcal{O}(m), f \in \Lambda(n) \), it
is customary to write \( gf \) instead of \( g \otimes f \). Including the formula (2.1), the following formulas also hold in \( \mathcal{O}(m, n) \):

\[
x_k x_l = -x_l x_k, \ \forall \ k, l \in \{m + 1, \ldots, s\}; \\
x^{(\alpha)} x_k = x_k x^{(\alpha)}, \ \forall \ \alpha \in \mathbb{N}^m, k \in \{m + 1, \ldots, s\}.
\]

For \( k = 1, \ldots, n \), set

\[
\mathbb{B}_k := \{(i_1, i_2, \ldots, i_k) \mid m + 1 \leq i_1 < i_2 < \ldots < i_k \leq s\}
\]

and \( \mathbb{B} := \cup_{k=0}^n \mathbb{B}_k \), where \( \mathbb{B}_0 = \emptyset \). For \( u = (i_1, i_2, \ldots, i_k) \in \mathbb{B}_k \), set \( |u| := k \), \( x^u = x_{i_1} \cdots x_{i_k} \).

Specially, we define \( \{0\} = 0 \), \( x^0 = 1 \), \( |\omega| = n \) and \( x^\omega = x_{m+1} \cdots x_{m+n} \). Clearly, the set \( \{x^{(\alpha)} x^u \mid \alpha \in \mathbb{N}^m, u \in \mathbb{B}\} \) constitutes an \( \mathbb{F} \)-basis of \( \mathcal{O}(m, n) \).

Let \( I_0 := \{1, \ldots, m\} \), \( I_1 := \{m+1, \ldots, m+n\} \) and \( I := I_0 \cup I_1 \). Let \( \partial_1, \partial_2, \ldots, \partial_s \) be the linear transformations of \( \mathcal{O}(m, n) \) such that \( \partial_i(x^{(\alpha)}) = x^{(\alpha-\epsilon_i)} \) for \( i \in I_0 \), and \( \partial_i(x_k) = \delta_{ik} \), \( k \in I_1 \), for \( i \in I_1 \), where \( \delta_{ij} \) is the Kronecker symbol. Obviously, \( p(\partial_i) = \mathbb{B} \) if \( i \in I_0 \) and \( p(\partial_i) = \mathbb{T} \) if \( i \in I_1 \). Then \( \partial_1, \partial_2, \ldots, \partial_s \) are superderivations of the superalgebra \( \mathcal{O}(m, n) \). Let

\[
W(m, n) := \left\{ \sum f_r \partial_r \mid f_r \in \mathcal{O}(m, n), r \in I \right\}.
\]

Then \( W(m, n) \) is an infinite-dimensional Lie superalgebra contained in \( \text{Der}(\mathcal{O}(m, n)) \).

One can verify that

\[
[f \partial_i, g \partial_j] = f \partial_i(g) \partial_j - (-1)^{p(f)p(g)} g \partial_j(f) \partial_i, 
\]

(2.2)

for all \( f, g \in \mathcal{O}(m, n) \) and \( i, j \in I \).

Fix two \( m \)-tuples of positive integers \( \underline{t} = (t_1, t_2, \ldots, t_m) \) and \( \pi = (\pi_1, \pi_2, \ldots, \pi_m) \), where \( \pi_i = p^{t_i} - 1 \) for all \( i \in I_0 \) and \( p \) is denoted the characteristic of the basic field \( \mathbb{F} \). For two \( m \)-tuples \( \alpha = (\alpha_1, \ldots, \alpha_m) \) and \( \beta = (\beta_1, \ldots, \beta_m) \in \mathbb{N}^m \), we have \( (\alpha+\beta) = 0 \) if there is some \( i \in \{1, \ldots, m\} \) satisfying \( \alpha_i + \beta_i \geq p^{t_i} \). Hence the set

\[
\mathcal{O}(m, n; \underline{t}) = \{x^{(\alpha)} x^u \mid 0 \leq \alpha \leq \pi, u \in \mathbb{B}\}
\]

is a subalgebra of \( \mathcal{O}(m, n) \) and the set

\[
W(m, n; \underline{t}) = \text{span}_\mathbb{F}\{x^{(\alpha)} x^u \partial_r \mid 0 \leq \alpha \leq \pi, u \in \mathbb{B}, r \in I\}
\]

is a finite-dimensional simple subalgebra of \( W(m, n) \), which is called the generalized Witt Lie superalgebra. \( W(m, n; \underline{t}) \) possesses a \( \mathbb{Z} \)-graded structure:

\[
W(m, n; \underline{t}) = \bigoplus_{r=1}^{\xi-1} W(m, n; \underline{t})_r,
\]

where \( W(m, n; \underline{t})_r := \text{span}_\mathbb{F}\{x^{(\alpha)} x^u \partial_r \mid |\alpha| + |u| = r + 1, j \in I\} \) and \( \xi := |\pi| + n \). For \( i \in I_0 \), we abbreviate \( x^{(\epsilon_i)} \) to \( x_i \), where \( \epsilon_i \) is denoted the \( m \)-tuple with 1 as the \( i \)-th entry and 0 elsewhere.
Hereafter, suppose \( m = 2r + 1 \) is odd and \( n = 2t \) is even. Let \( J = I \setminus \{ m \} \) and \( J_0 = I_0 \setminus \{ m \} \). For \( i \in J \), put

\[
i' := \begin{cases} 
i + r, & 1 \leq i \leq r, \\
i - r, & r < i \leq 2r, \\
i, & i = m, \\
i + t, & m < i \leq m + t, \\
i - t, & m + t < i \leq s; \end{cases} \quad \sigma(i) := \begin{cases} 1, & 1 \leq i \leq r \\
-1, & r < i \leq 2r \\
1, & 2r < i \leq s. \end{cases}
\]

Define a linear mapping \( D_K : \mathcal{O}(m, n) \to \mathcal{W}(m, n) \) by means of

\[
D_K(f) = \sum_{i \in J} (-1)^{p(i)+p(f)} (x_i \partial_m(f) + \sigma(i') \partial_v(f)) \partial_i + (2f - \sum_{i \in J} x_i \partial_m(f)) \partial_m.
\]

The restricted linear mapping of \( D_K \) on \( \mathcal{O}(m, n; t) \) still is denoted by \( D_K \), that is

\[
D_K : \mathcal{O}(m, n; t) \to \mathcal{K}(m, n; t).
\]

Let \( \tilde{K}(m, n; t) \) denote the image of \( \mathcal{O}(m, n; t) \) under \( D_K \). Consider the derived algebra of \( \tilde{K}(m, n; t) \):

\[
K(m, n; t) = [\tilde{K}(m, n; t), \tilde{K}(m, n; t)].
\]

The derived algebra \( K(m, n; t) \) is a finite dimensional simple Lie superalgebra, which is called the contact Lie superalgebra. We define a Lie bracket \( \langle \cdot, \cdot \rangle \) on the tensor superalgebra \( \mathcal{O}(m, n; t) \) by

\[
\langle f, g \rangle := D_K(f)(g) - 2\partial_m(f)(g),
\]

for all \( f, g \in \mathcal{O}(m, n; t) \). Since \( D_K \) is injective and \( D_K(\langle f, g \rangle) = [D_K(f), D_K(g)] \), there exists an isomorphism, that is,

\[
(K(m, n; t), [\cdot, \cdot]) \cong (\mathcal{O}(m, n; t), \langle \cdot, \cdot \rangle).
\]

For convenience, we use \( W \) and \( W_r \) denote \( W(m, n; t) \) and its \( \mathbb{Z} \)-graded subspace \( W(m, n; t)_r \), respectively, \( K(m, n; t) \) is denoted by \( K \).

## 3 Skew-symmetry Super-biderivation of \( K(m, n; t) \)

**Lemma 3.1.** [10] Let \( L \) be a Lie superalgebra. Suppose that \( \phi \) is a skew-symmetric super-biderivation on \( L \), then

\[
[\phi(x, y), [u, v]] = (-1)^{p(x)[p(y)]} [[x, y], \phi(u, v)]
\]

for any homogenous element \( x, y, u, v \in L \).
Lemma 3.2. \([10]\) Let \(L\) be a Lie superalgebra. Suppose that \(\phi\) is a skew-symmetric super-biderivation on \(L\). If \(p(x) + p(y) = 0\), then
\[
[\phi(x, y), [x, y]] = 0
\]
for any homogenous element \(x, y \in L\).

Lemma 3.3. \([10]\) Let \(L\) be a Lie superalgebra. Suppose that \(\phi\) is a skew-symmetric super-biderivation on \(L\). If \([x, y] = 0\), then \(\phi(x, y) \in C_L([L, L])\), where \(C_L([L, L])\) is the centralizer of \([L, L]\).

Lemma 3.4. Let \(K\) denote the contact Lie superalgebra. Suppose \(\phi\) is a skew-symmetric super-biderivation on \(K\). If \([x, y] = 0\) for \(x, y \in K\), then \(\phi(x, y) = 0\).

Proof. Since \(K\) is a simple Lie superalgebra, it is obvious that \(K = \langle K, K \rangle \) and \(C(K) = 0\). If \([x, y] = 0\) for \(x, y \in K\), by Lemma 3.3, we obtain \(\phi(x, y) \in C'_K(\langle K, K \rangle ) = C(K) = 0\).

Set \(T_K = \text{span}_F \{x_i x_{i'} \mid i \in J\}\). Obviously, \(T_K \subseteq K(m, n; L) \cap K(m, n; J)\). \(T_K\) is an abelian subalgebra of \(K\). For any \(x^{(a)} x^{u} \in K\), we have
\[
\langle x_i x_{i'}, x^{(a)} x^{u} \rangle = (\alpha_{i'} - \alpha_i + \delta_{i' \in u} - \delta_{i \in u}) x^{(a)} x^{u}, \tag{3.1}
\]
where \(\delta_{\{P\}} = 1\) if the proposition \(P\) is true, \(= 0\) if the proposition \(P\) is false. Fixed an \(m\)-tuple \(\alpha\), where \(\alpha \in \mathbb{N}^m\), \(0 \leq \alpha \leq \pi\) and \(u \in \mathbb{B}\), we define a linear function \((\alpha + (u)) : T \to F\) such that
\[
(\alpha + (u))(x_i x_{i'}) = \alpha_{i'} - \alpha_i + \delta_{i' \in u} - \delta_{i \in u}.
\]
Further, \(K\) has a weight space decomposition with respect to \(T_K\):
\[
K = \bigoplus_{(\alpha + (u))} K_{(\alpha + (u))}.
\]

Lemma 3.5. Suppose that \(\phi\) is a \(\mathbb{Z}_2\)-homogeneous skew-symmetric super-biderivation on \(K\). Let \(x^{(a)} x^{u} \in K\) such that
\[
\phi(x_i x_{i'}, x^{(a)} x^{u}) \in K_{(\alpha + (u))},
\]
for any \(x_i x_{i'} \in T_K\).

Proof. The equation by Lemma 3.4 it follows that \(\phi(x_i x_{i'}, x_j x_{j'}) = 0\) for any \(i, j \in J\) from \([x_i x_{i'}, x_j x_{j'}] = 0\). Note that \(p(x l x_{l'}) = \overline{0}\) for all \(l \in J\), then all \(x^{(a)} x^{u} \in K\), it is clear that
\[
\langle x_i x_{i'}, \phi(x_i x_{i'}, x^{(a)} x^{u}) \rangle
\]
\[= (-1)^{p(\phi) + p(x l x_{l'})} \phi(x_i x_{i'}, x^{(a)} x^{u}) - \langle \phi(x_i x_{i'}, x_i x_{i'}), x^{(a)} x^{u} \rangle
\]
\[= (\alpha_{i'} - \alpha_i + \delta_{i' \in u} - \delta_{i \in u}) \phi(x_i x_{i'}, x^{(a)} x^{u}).
\]
The proof is completed.

6
Remark 3.6. Due to Lemma 3.5, we can find that any $\mathbb{Z}_2$-homogeneous skew-symmetric super-biderivation on $K$ is an even bilinear map. Since $\phi(x, x^u, x^{(\alpha)}x^u)$ and $x^{(\alpha)}x^u$ have the same $\mathbb{Z}_2$-degree. Then the $\mathbb{Z}_2$-degree of $\phi$ is even.

Lemma 3.7. [17] Let $M = \{ x^{(\kappa_i\varepsilon_i)} | 0 \leq \kappa_i \leq \pi_i, i \in I_0 \}$ and $N = \{ x_i | i \in I_1 \}$. Then $K$ is generated by $M \cup N$.

Lemma 3.8. Let $i \in J_0$, $j \in I_1$ and $q_i \in \mathbb{N}$, $1 \leq q_i \leq \pi_i$. Then the following statements hold:

(1) $K(m, n; l)_0 = \sum_{0 \leq \alpha \leq \pi, \bar{u} \in \mathbb{B}} F(\prod_{l \in J_0} x^{(\alpha_l \varepsilon_l)}x^{(\alpha_m \varepsilon_m)}x^{\bar{a}})$

(2) $K(m, n; l)_{(q_i \varepsilon_i)} = \sum_{0 \leq \alpha \leq \pi, \bar{u} \in \mathbb{B}} F(\prod_{l \in J_0 \setminus \{l'\}} x^{(\alpha_l \varepsilon_l)}x^{(\alpha_m \varepsilon_m)}x^{\bar{a}})$

(3) $K(m, n; l)_{(j)} = \sum_{0 \leq \alpha \leq \pi, \bar{u} \in \mathbb{B}} F(\prod_{l \in J_0} x^{(\alpha_l \varepsilon_l)}x^{(\alpha_m \varepsilon_m)}x^{\bar{a}})$

where $i$ and $i'$ are both in $\bar{u}$ for $i \in J$, and $\alpha_i^q$ is denoted some integer and $\alpha_i^q \equiv q \pmod{p}$.

Proof. (1) We first discuss the vector of the same weight with 1 in $K$ with respect to $T_K$. Since we have the equation

$$\langle x_ix^u, 1 \rangle = D_K(x_ix^u)(1) - 2\partial_m(x_ix^u)(1) = 0.$$

For any $l \in J$, in contrast with equation (3.1), we get that

$$\alpha_l - \alpha_i + \delta_{(i \in u)} - \delta_{(l \in u)} = 0.$$

Then if $l \in J_0$, it is obvious that is $\alpha_l - \alpha_i \equiv 0 \pmod{p}$. If $l \in I_1$, it is obvious that $l$ and $l'$ are both in $\bar{u}$. It proves that

$$K(m, n; l)_0 = \sum_{0 \leq \alpha \leq \pi, \bar{u} \in \mathbb{B}} F(\prod_{l \in J_0} x^{(\alpha_l \varepsilon_l)}x^{(\alpha_m \varepsilon_m)}x^{\bar{a}}).$$

(2) Without loss of generality, we choose a fixed element $i \in J$. For any $l \in J$, we have the equation

$$\langle x_ix^u, x^{(\alpha_i \varepsilon_i)} \rangle = D_K(x_ix^u)(x_i) - 2\partial_m(x_ix^u)(x_i) = -q_i x_i \delta_{(i)},$$

For any $l \in J$, by equation (3.1) we have that

$$\alpha_l - \alpha_i + \delta_{(i \in u)} - \delta_{(l \in u)} = -q_i \delta_{(i)}.$$
Then we try to discuss the choice of \( l \in J \). If \( l \in J_0 \setminus \{i, i'\} \), it is obvious that \( \alpha_{l'} - \alpha_l \equiv 0 \pmod{p} \). If \( l = i \), it is obvious that \( \alpha_{l'} - \alpha_l \equiv -q_i \pmod{p} \). If \( l \in I_1 \), we have that \( l \) and \( l' \) are both in \( \bar{u} \). So we prove that

\[
K(m, n; l)_{(q, \varepsilon_i)} = \sum_{0 \leq \alpha \leq \pi, \, \bar{u} \in \mathbb{B}} \mathbb{F}( \prod_{l \in J_0 \setminus \{i, i'\}} x^{(\alpha \varepsilon_i)} ) \prod_{l \in J_0} x^{(\alpha_{l'} - \alpha_l) + \delta_{l' u} - \delta_{l u}}. \]

(3) Without loss of generality, we choose a fixed element \( i \in J \). For any \( l \in J \), we have the equation

\[
\langle x_l x_{l'}, x_j \rangle = D_K(x_l x_{l'})(x_j) - 2\partial_m(x_l x_{l'})(x_j) = -x_l \delta_{i j}.
\]

By equation (3.1), for any \( l \in J \), we have that

\[
\alpha_{l'} - \alpha_l + \delta_{l' u} - \delta_{l u} = -\delta_{i j}.
\]

Then we try to discuss the choice of \( l \in J \). If \( l \in J_0 \), it is obvious that \( \alpha_{l'} - \alpha_l \equiv 0 \pmod{p} \). If \( l \in I_1 \), we have that \( l \) and \( l' \) are both in \( \bar{u} \). It proves that

\[
K(m, n; l)_{(l)} = \sum_{0 \leq \alpha \leq \pi, \, \bar{u} \in \mathbb{B}} \mathbb{F}( \prod_{l \in J_0} x^{(\alpha \varepsilon_i)} ) \prod_{l \in J_0} x^{(\alpha_{l'} - \alpha_l) + \delta_{l' u} - \delta_{l u}}.
\]

Lemma 3.9. Suppose that \( \phi \) is a \( \mathbb{Z}_2 \)-homogeneous skew-symmetric super-biderivation on \( K \). For any \( x_l x_{l'} \in T_K \) and \( x^{(q_m \varepsilon_m)} \in M \), where \( 0 \leq q_m \leq \pi_m \), we have

\[
\phi(x_l x_{l'}, x^{(q_m \varepsilon_m)}) = 0.
\]

\textbf{Proof.} When \( q_m = 0 \), by Lemma 3.4 it is obvious that \( \phi(x_l x_{l'}, 1) = 0 \) for \( l \in I \) from the equation \( \langle x_l x_{l'}, 1 \rangle = 0 \). When \( q_m \neq 0 \), we have that

\[
\langle x_l x_{l'}, x^{(q_m \varepsilon_m)} \rangle = D_K(x_l x_{l'})(x^{(q_m \varepsilon_m)}) - 2\partial_m(x_l x_{l'})(x^{(q_m \varepsilon_m)})
\]

\[
=(2(x_l x_{l'}) - \sum_{l \in J} x_l \partial_l(x_l x_{l'})) \partial_m(x^{q_m \varepsilon_m})
\]

\[
=(2(x_l x_{l'}) - 2(x_l x_{l'})) \partial_m(x^{q_m \varepsilon_m})
\]

\[
=0.
\]

Hence, we have that

\[
\phi(x_l x_{l'}, x^{(q_m \varepsilon_m)}) = 0.
\]

The proof is completed. \( \Box \)
Lemma 3.10. Suppose that \( \phi \) is a \( \mathbb{Z}_2 \)-homogeneous skew-symmetric super-biderivation on \( K \). For any \( i \in J \) and \( x_i x_{i'} \in T_K \), there is an element \( \lambda_i \in F \) such that

\[
\phi(x_i x_{i'}, x_i) = \lambda_i \langle x_i x_{i'}, x_i \rangle,
\]

where \( \lambda_i \) is dependent on the second component.

Proof. Without loss of generality, we choose a fixed element \( i \in J \). By Lemma 3.4, it is obvious that \( \phi(x_i x_{i'}, x_i) = 0 \) for \( l \in J \setminus \{i, i'\} \) from \( \langle x_i x_{i'}, x_i \rangle = 0 \). So we only need to discuss the case with the condition \( l = i \).

When \( q_i = 1 \), by Lemma 3.4 (2), we can suppose that

\[
\phi(x_i x_{i'}, x_i) = \sum_{0 \leq \alpha \leq \pi, \bar{u} \in B} a(\alpha, \bar{u})(\prod_{l \in J_0 \setminus \{i, i'\}} x^{(\alpha_l \varepsilon_l)} x^{(\bar{\alpha}_l - \bar{\alpha}_l \varepsilon_l)} x^{(\alpha_m \varepsilon_m)} x^{\bar{u}}).
\]

It is obvious that

\[
0 = \phi(x_i x_{i'}, \langle 1, x_i \rangle) - \langle \phi(x_i x_{i'}, 1), x_i \rangle
= (-1)^{p(\phi) + p(x_i x_{i'})} \langle 1, \phi(x_i x_{i'}, x_i) \rangle
= D_K(1) \langle \phi(x_i x_{i'}, x_i) \rangle - 2 \partial_m(1) \langle \phi(x_i x_{i'}, x_i) \rangle
= \partial_m \langle \phi(x_i x_{i'}, x_i) \rangle
= \partial_m \left( \sum_{0 \leq \alpha \leq \pi, \bar{u} \in B} a(\alpha, \bar{u})(\prod_{l \in J_0 \setminus \{i, i'\}} x^{(\alpha_l \varepsilon_l)} x^{(\bar{\alpha}_l - \bar{\alpha}_l \varepsilon_l)} x^{(\alpha_m \varepsilon_m)} x^{\bar{u}}) \right).
\]

By computing the equation, we find that \( a(\alpha, \bar{u}) = 0 \) if \( \alpha_m > 0 \). Putting \( l \in J \setminus \{i, i'\} \), we have that

\[
0 = (-1)^{p(\phi) + p(x_i x_{i'})} p(x_i) \langle \phi(x_i x_{i'}, \langle x_l, x_i \rangle) - \langle \phi(x_i x_{i'}, x_l), x_i \rangle \rangle
= \langle x_i, \phi(x_i x_{i'}, x_i) \rangle
= D_K(x_i) \langle \phi(x_i x_{i'}, x_i) \rangle - 2 \partial_m(x_i) \langle \phi(x_i x_{i'}, x_i) \rangle
= \partial \langle \phi(x_i x_{i'}, x_i) \rangle
= \partial \left( \sum_{0 \leq \alpha \leq \pi, \bar{u} \in B} a(\alpha, i)(\prod_{l \in J_0 \setminus \{i, i'\}} x^{(\alpha_l \varepsilon_l)} x^{\bar{\alpha}_l \varepsilon_l} x^{\bar{u}}) \right).
\]

By computing the equation, we find that \( a(\alpha, \bar{u}) = 0 \) if \( \alpha_l > 0 \) for \( l \in J_0 \setminus \{i, i'\} \) or \( |\bar{u}| > 0 \). Then we can suppose that

\[
\phi(x_i x_{i'}, x_i) = \sum_{0 \leq \alpha \leq \pi} a(\alpha) x^{(\bar{\alpha}_l - \bar{\alpha}_l \varepsilon_l)} x^{\bar{\alpha}_l \varepsilon_l}.
\]

Since \( p(x_i x_{i'}) + p(x_i) = 0 \) for any \( i \in I_0 \), by Lemma 3.2, we have

\[
0 = \langle \phi(x_i x_{i'}, x_i), \langle x_i x_{i'}, x_i \rangle \rangle
\]
\[\begin{align*}
&= \langle \phi(x_i x_{i'}', x_i), -x_i \rangle \\
&= D_K(x_i)(\phi(x_i x_{i'}, x_i)) - 2\partial_m(x_i)(\phi(x_i x_{i'}, x_i)) \\
&= \partial_r(\phi(x_i x_{i'}, x_i)) \\
&= \partial_r(\sum_{0 \leq \alpha \leq \pi} a(\alpha)x^{(\alpha_{i-1})_{x_{i'}}}\lambda_{(\alpha_{i'})}x_{(\alpha_{i})}).
\end{align*}\]

By computing the equation, we find that \(a(\alpha) = 0\) if \(\alpha_i - 1 > 0\). Hence we get that \(\alpha_i = 1\). Let \(\lambda_i = a(\varepsilon_i)\). From what has been discussed above, for any \(i \in J_0\) we have that
\[\phi(x_i x_{i'}, x_i) = -x_i = \lambda_i\langle x_i x_{i'}, x_i \rangle,\]
where \(\lambda_i\) is dependent on the second component.

Similarly, we choose a fixed element \(j \in I_1\). By Lemma 6.8 (3), we can suppose that
\[\phi(x_j x_{j'}, x_j) = \sum_{0 \leq \alpha \leq \pi, \bar{u} \in B} a(\alpha, \bar{u}, j)(\prod_{l \in J_0} x^{(\alpha_{l \varepsilon_i})}\lambda_{(\alpha_{l})}e_{(\alpha_{m})}x_{(\bar{u})}x_j),\]
where \(a(\alpha, \bar{u}, j) \in \mathbb{F}\). By the definition of the skew-symmetric super-biderivation, we have
\[0 = \langle \phi(x_j x_{j'}, \langle 1, x_j \rangle) - \langle \phi(x_j x_{j'}, 1), x_j \rangle \rangle = (-1)^{(p(\phi)+p(x_j x_{j'}))p(1)} \langle 1, \phi(x_j x_{j'}, x_j) \rangle \]
\[= D_K(1)(\phi(x_j x_{j'}, x_j)) - 2\partial_m(1)(\phi(x_j x_{j'}, x_j)) = 2\partial_m(\phi(x_j x_{j'}, x_j)) \]
\[= \partial_r(\sum_{0 \leq \alpha \leq \pi, \bar{u} \in B} a(\alpha, \bar{u}, j)(\prod_{l \in J_0} x^{(\alpha_{l \varepsilon_i})}\lambda_{(\alpha_{l})}e_{(\alpha_{m})}x_{(\bar{u})}x_j),\]
where \(a(\alpha, \bar{u}, j) \in \mathbb{F}\). By the definition of the skew-symmetric super-biderivation, we have
\[0 = \langle \phi(x_j x_{j'}, \langle 1, x_j \rangle) - \langle \phi(x_j x_{j'}, 1), x_j \rangle \rangle = (-1)^{(p(\phi)+p(x_j x_{j'}))p(1)} \langle 1, \phi(x_j x_{j'}, x_j) \rangle \]
\[= D_K(1)(\phi(x_j x_{j'}, x_j)) - 2\partial_m(1)(\phi(x_j x_{j'}, x_j)) = 2\partial_m(\phi(x_j x_{j'}, x_j)) \]
\[= \partial_r(\sum_{0 \leq \alpha \leq \pi, \bar{u} \in B} a(\alpha, \bar{u}, j)(\prod_{l \in J_0} x^{(\alpha_{l \varepsilon_i})}\lambda_{(\alpha_{l})}e_{(\alpha_{m})}x_{(\bar{u})}x_j),\]
By computing the equation, we find that \(a(\alpha, \bar{u}, j) = 0\) if \(\alpha_m > 0\). For \(k \in J \setminus \{j, j'\}\), it is obvious that
\[0 = (-1)^{(p(\phi)+p(x_k x_{k'}))p(\phi(x_k x_{k'})(x_k, x_j))} \langle x_k, \phi(x_k x_{k'}, x_j) \rangle = \langle x_k, \sum_{0 \leq \alpha \leq \pi, \bar{u} \in B} a(\alpha, \bar{u}, j)(\prod_{l \in J_0} x^{(\alpha_{l \varepsilon_i})}\lambda_{(\alpha_{l})}e_{(\alpha_{m})}x_{(\bar{u})}x_j) \rangle.
\]
Putting \(k \in I_1\), we can deduce \(a(\alpha, \bar{u}, j) = 0\) if \(|\bar{u}| > 0\). Putting \(k \in J_0\), we have that \(a(\alpha, \bar{u}, j) = 0\) if \(\alpha_k > 0\). Let \(a(0, 0, j) = \lambda_j\). Hence, for any \(j \in I_1\), we have that
\[\phi(x_k x_{j'}, x_j) = -a(0, 0, j)x_j = \lambda_j\langle x_k x_{j'}, x_j \rangle.
\]
The proof is completed. \(\square\)
Lemma 3.11. Suppose that $\phi$ is a $\mathbb{Z}_2$-homogenous skew-symmetric super-biderivation on $K$. For any $x^{(q;e_i)} \in M$, where $1 \leq q_i \leq \pi_i$, $i \in J_0$, there is an element $\lambda_i \in \mathbb{F}$ such that
\[
\phi(x_i x_{i'}, x^{(q;e_i)}) = \lambda_i \langle x_i x_{i'}, x^{(q;e_i)} \rangle.
\]

Proof. Without loss of generality, we choose a fixed element $i \in I_0$. By Lemma 3.4, it is obvious that $\phi(x_i x_{i'}, x^{(q;e_i)}) = 0$ for $i \in J \setminus \{i, i'\}$ from $\langle x_i x_{i'}, x^{(q;e_i)} \rangle = 0$. So we only need to consider the condition that $i = i$. It is clear that $\langle x_i x_{i'}, x^{(q;e_i)} \rangle = -q_i x^{(q;e_i)}$. If $q_i > 1$, by Lemma 3.11 and 3.10 we have
\[
0 = \langle \phi(x_k x_k', x_k), x^{(q;e_i)} \rangle - (-1)^p(\phi(x_k x_k', x_k)) \langle x_k x_k', x_k, x^{(q;e_i)} \rangle
\]
\[
= \langle x_k, x^{(q;e_i)} \rangle - (-x_k, \phi(x_i x_{i'}, x^{(q;e_i)}))
\]
\[
= \langle x_k, -\lambda_k q_i x^{(q;e_i)} + \phi(x_i x_{i'}, x^{(q;e_i)}) \rangle.
\]
(3.2)

Because of $C_{K-1}(K) = \{ f \in K | f(x_i) = 0, \forall i \in I \} = K_{-2} = \mathbb{F}1$, the equation (3.2) implies that
\[
\phi(x_i x_{i'}, x^{(q;e_i)}) = \lambda_i \langle x_i x_{i'}, x^{(q;e_i)} \rangle + b,
\]
where $\lambda_i$ is denoted in Lemma 3.10 and $b \in \mathbb{F}$. Since $\phi(x_i x_{i'}, x^{(q;e_i)}) \in K(m,n;\mathbb{Z}(q;e_i))$ by Lemma 3.5. It is easily seen from Lemma 3.8 (2) that $K_{-2} \cap K(q;e_i) = \emptyset$ for $q_i > 1$. So $b = 0$ and
\[
\phi(x_i x_{i'}, x^{(q;e_i)}) = \lambda_i \langle x_i x_{i'}, x^{(q;e_i)} \rangle,
\]
where $\lambda_i$ is dependent on the second component. \hfill $\square$

Lemma 3.12. Suppose that $\phi$ is a $\mathbb{Z}_2$-homogenous skew-symmetric super-biderivation on $K$. For any $x^{(q_m;e_m)} \in M$, where $0 \leq q_m \leq \pi_m$, there is an element $\lambda \in \mathbb{F}$ such that
\[
\phi(1, x^{(q_m;e_m)}) = \lambda(1, x^{(q_m;e_m)}).
\]

Proof. When $q_m = 1$, we suppose that
\[
\phi(1, x_m) = \sum_{0 \leq \alpha \leq \pi, \; u \in \mathbb{B}} c_{(\alpha, u)} x^{(\alpha)} x^u,
\]
where $c_{(\alpha, u)} \in \mathbb{F}$. For $k \in J_0$, by the definition of the skew-symmetric super-biderivation, we have the equation
\[
0 = (-1)^{(p(\phi)+p(1))p(1)}(\phi(1, \langle 1, x_m \rangle) - \langle \phi(1, 1), x_m \rangle)
\]
\[
= 1, \sum_{0 \leq \alpha \leq \pi, \; u \in \mathbb{B}} c_{\alpha} x^{(\alpha)} x^u
\]
\[
= 2 \sum_{0 < \alpha \leq \pi, \; u \in \mathbb{B}} c_{\alpha} x^{(\alpha - e_m)} x^u.
\]
By computing the equation, we find that \( c_{(\alpha,u)} = 0 \) if \( \alpha_m - \varepsilon_m \geq 0 \). We suppose that

\[
\phi(1, x_m) = \sum_{0 \leq \alpha_m \leq \tau_m, \ u \in \mathbb{B}} c_{\alpha(\hat{m},u)} x^{(\alpha_m)} x^u,
\]

where \( \hat{m} \) represents an m-tuple with 0 as the m-th entry. For \( k \in J \), by the definition of the skew-symmetric biderivation, we have the equation

\[
0 = (-1)^{(p(\phi) + p(1))p(x_m - x_m)} \langle \phi(x_k, (1, x_m)) - \langle \phi(x_k, 1), x_m \rangle \rangle
= \langle x_k, \phi(1, x_m) \rangle + \sum_{0 \leq \alpha_m \leq \tau_m, \ u \in \mathbb{B}} c_{\alpha(\hat{m},u)} x^{(\alpha_m)} x^u
= \sum_{0 \leq \alpha_m, \ u \in \mathbb{B} \leq \tau_m} c_{\alpha(\hat{m},u)} x^{(\alpha_m - \varepsilon_{m'})} x^u.
\]

By computing the equation, we find that \( c_{\alpha(\hat{m},u)} = 0 \) if \( \alpha_m - \varepsilon_{m'} > 0 \) or \( |u| > 0 \). Then we can suppose that

\[
\phi(1, x_m) = c_0.
\]

Set \( \lambda = \frac{c_0}{2} \), then we can get that

\[
\phi(1, x_m) = \lambda \langle 1, x_m \rangle.
\]

When \( q_m \geq 2 \), we suppose that

\[
\phi(1, x^{(q_m \varepsilon_m)}) = \sum_{0 \leq \alpha \leq \tau, \ u \in \mathbb{B}} c_{(\alpha,u)} x^{(\alpha)} x^u.
\]

By Lemma 3.1 and the conclusion of the case \( q_m = 1 \), we have the equation

\[
0 = \langle \phi(1, x_m), (1, x^{(q_m \varepsilon_m)}) \rangle - \langle (1, x_m), \phi(1, x^{(q_m \varepsilon_m)}) \rangle
= \lambda \langle 1, x^{(q_m \varepsilon_m)} \rangle - \langle 1, x_m, \sum_{0 \leq \alpha \leq \tau, \ u \in \mathbb{B}} c_{\alpha} x^{(\alpha)} x^u \rangle
= 2 \lambda \langle 1, x^{((q_m - 1) \varepsilon_m)} \rangle - \sum_{0 \leq \alpha \leq \tau, \ u \in \mathbb{B}} c_{(\alpha,u)} x^{(\alpha)} x^u
= 2 \lambda x^{((q_m - 1) \varepsilon_m)} - \sum_{0 \leq \alpha \leq \tau, \ u \in \mathbb{B}} c_{(\alpha,u)} x^{(\alpha - \varepsilon_m)} x^u.
\]

By computing the equation, we find that \( c_{(\alpha,u)} = 0 \) if \( \alpha - (q_m - 1) \varepsilon_m \neq 0 \) or \( |u| > 0 \). And \( c_{(q_m - 1)} = 2 \lambda \). We suppose that

\[
\phi(1, x^{(q_m \varepsilon_m)}) = \sum_{0 \leq \alpha_m \leq \tau_m} c_{\alpha_m} x^{(\alpha_m)} + 2 \lambda x^{((q_m - 1) \varepsilon_m)}.
\]

12
For any $i \in J_0$, by Lemma 3.4, we have the equation

$$0 = \langle \phi(1, x^{(qm \varepsilon_m)}), \langle x_m, x_i \rangle \rangle - \langle \langle x^{(qm \varepsilon_m)}, \phi(x_m, x_i) \rangle \rangle$$

$$= \langle \sum_{0 \leq \alpha \leq \tau_m} c_{\alpha \alpha} x^{(\alpha \alpha)} + 2 \lambda x^{((q - 1) \varepsilon_m)} - x_i \rangle$$

$$= \langle x_i, \sum_{0 \leq \alpha \leq \tau_m} c_{\alpha \alpha} x^{(\alpha \alpha)} + 2 \lambda x^{((q - 1) \varepsilon_m)} - 2 \lambda x^{((q - 1) \varepsilon_m)} \rangle.$$ 

Since $C_{K-1}(K) = K-2$, we have that

$$\sum_{0 \leq \alpha \leq \tau_m} c_{\alpha \alpha} x^{(\alpha \alpha)} + 2 \lambda x^{((q - 1) \varepsilon_m)} - 2 \lambda x^{((q - 1) \varepsilon_m)} \in F.$$ 

Then we have $c_{\alpha \alpha} = 0$ for $\alpha > 0$ and $\lambda = \lambda_i$ for $i \in J_0$. Then we can get that

$$\phi(1, x^{(qm \varepsilon_m)}) = c_0 + 2 \lambda x^{((q - 1) \varepsilon_m)}.$$ 

Utilizing the definition of the skew-symmetry biderivation, by Lemma 3.4, we have that

$$0 = \langle \phi(1, x^{(qm \varepsilon_m)}), \langle 1, x^{(qm \varepsilon_m)} \rangle \rangle$$

$$= \langle c_0 + 2 \lambda x^{((q - 1) \varepsilon_m)} \rangle$$

$$= 2 c_0 x^{((q - 2) \varepsilon_m)}.$$ 

It is obvious that $c_0 = 0$ for $p > 2$. So we can get that

$$\phi(1, x^{(qm \varepsilon_m)}) = \lambda(1, x^{(qm \varepsilon_m)}).$$ 

The proof is complete. □

Remark 3.13. We claim that $\lambda_1 = \cdots = \lambda_m = \cdots = \lambda_{m+n}$. Choose two mutually different elements $i, j \in J$. Since the characteristic $p > 3$, there are two positive integers $q_i$ and $q_m$, which are greater than 1 and are neither congruent to 0 modulo $p$, such that we have

$$0 = \langle \phi(x_i x_i', x_i), \langle 1, x^{(qm \varepsilon_m)} \rangle \rangle - \langle \langle x_i x_i', x_i \rangle, \phi(1, x^{(qm \varepsilon_m)}) \rangle$$

$$= \langle \lambda_i \langle x_i x_i', x_i \rangle, 2 \lambda x^{((q - 1) \varepsilon_m)} \rangle - \langle -x_i, \lambda(1, x^{(q \varepsilon_m)}) \rangle$$

$$= \langle -2 \lambda_i x^{((q - 1) \varepsilon_m)} + 2 \lambda x^{((q - 1) \varepsilon_m)} \rangle$$

$$= 2 \lambda x^{((q - 1) \varepsilon_m)}.$$ 

By direct calculation, it is easily seen that $\lambda_i = \lambda$ for any $i \in I$. Set $\lambda := \lambda_1 = \cdots = \lambda_m = \lambda_{m+1} = \cdots = \lambda_{m+n}$. Then we can conclude that for any $x^{(q \varepsilon_i)} \in M$, $1 \leq q_i \leq \pi_i$ and $x_i x_i' \in T$, there is an element $\lambda \in F$ such that

$$\phi(x_i x_i', x^{(q \varepsilon_i)}) = \lambda(x_i x_i', x^{(q \varepsilon_i)}).$$ 

where $\lambda$ depends on neither $x^{(q \varepsilon_i)}$ nor $x_i x_i'$. 

13
Theorem 3.14. Let $K$ be the contact Lie superalgebra $K(m, n; \mathfrak{t})$ over the prime field $\mathbb{F}$ of the characteristic $p > 3$, where $m, n \in \mathbb{N} + 1$ and $\mathfrak{t} = (t_1, t_2, \ldots, t_m)$ is an $m$-tuple of positive integers. Then

$$\text{BDer}(K) = \text{IBDer}(K).$$

Proof. Suppose that $\phi$ is a skew-symmetric super-biderivation on $K$. By Lemmas 3.9 and 3.10 there is an element $\lambda \in \mathbb{F}$ such that $\phi(x_ix_j, x_i) = \lambda \langle x_i x_j, x_i \rangle$ for all $i \in J$. For any $x^{(\alpha)} x^u, x^{(\beta)} x^v \in K$ and $x_i x_j \in T_k$, by Lemma 3.11 and Remark 3.6, we have the equation

$$0 = \langle \phi(x_i x_j, x_i), \langle x^{(\alpha)} x^u, x^{(\beta)} x^v \rangle \rangle - \langle \langle x_i x_j, x_i \rangle, \phi(x^{(\alpha)} x^u, x^{(\beta)} x^v) \rangle$$

$$= \langle \langle x_i x_j, x_i \rangle, \lambda \langle x^{(\alpha)} x^u, x^{(\beta)} x^v \rangle \rangle - \langle \langle x_i x_j, x_i \rangle, \phi(x^{(\alpha)} x^u, x^{(\beta)} x^v) \rangle$$

$$= \langle x_i, \phi(x^{(\alpha)} x^u, x^{(\beta)} x^v) - \lambda \langle x^{(\alpha)} x^u, x^{(\beta)} x^v \rangle \rangle.$$

Since $C_{K-1}(K) = K_{-2}$, we have that

$$\phi(x^{(\alpha)} x^u, x^{(\beta)} x^v) = \lambda \langle x^{(\alpha)} x^u, x^{(\beta)} x^v \rangle + b,$$

where $\lambda$ is denoted in Remark 3.13 and $b \in \mathbb{F}$. By Lemma 3.11 and Remark 3.6, we have

$$0 = \langle \phi(x^{(\alpha)} x^u, x^{(\beta)} x^v), (1, x^{(2m)}) \rangle - \langle \langle x^{(\alpha)} x^u, x^{(\beta)} x^v \rangle, \phi(1, x^{(2m)}) \rangle$$

$$= \langle \lambda \langle x^{(\alpha)} x^u, x^{(\beta)} x^v \rangle + b, 2x_m \rangle - \langle \langle x^{(\alpha)} x^u, x^{(\beta)} x^v \rangle, \lambda \langle 1, x^{(2m)} \rangle \rangle$$

$$= \langle \lambda \langle x^{(\alpha)} x^u, x^{(\beta)} x^v \rangle + b, 2x_m \rangle - \langle \langle x^{(\alpha)} x^u, x^{(\beta)} x^v \rangle, \lambda 2x_m \rangle$$

$$= \langle b, 2x_m \rangle$$

$$= 4b.$$

Then $b = 0$. Hence, $\phi(x^{(\alpha)} x^u, x^{(\beta)} x^v) = \lambda \langle x^{(\alpha)} x^u, x^{(\beta)} x^v \rangle$ for any $x^{(\alpha)} x^u, x^{(\beta)} x^v \in K$ and $\phi$ is an inner super-biderivation.

Acknowledgements The authors would like to thank the referee for valuable comments and suggestions on this article.

References

[1] M. Brešar, Commuting maps: a survey, Taiwanese J. Math, 8 (2004), 361–397.

[2] D. Wang, X. Yu, Z. Chen, Biderivations of the parabolic subalgebras of simple Lie algebras, Comm. Algebra, 39 (2011), 4097–4104.

[3] Z. Chen, Biderivations and linear commuting maps on simple generalized Witt algebras over a field, Electron. J. Linear Algebra, 31 (2016), 1–12.

[4] X. Han, D. Wang, C. Xia, Linear commuting maps and biderivations on the Lie algebras $\mathcal{W}(a, b)$, J. Lie Theory, 26 (2016), 777–786.
[5] X. Tang, Biderivations of finite-dimensional complex simple Lie algebras, Linear Multilinear Algebra, 66 (2018), 250–259.

[6] D. Wang, X. Yu, Biderivations and linear commuting maps on the Schrödinger-Virasoro Lie algebra, Comm. Algebra, 41 (2013), 2166–2173.

[7] M. Brešar, K. Zhao, Biderivations and commuting linear maps on Lie algebras, J. Lie Theory, 28 (2018), 885–900.

[8] Y. Chang, L. Chen, Biderivations and linear commuting maps on the restricted Cartan-type Lie algebras $W(n;1)$ and $S(n;1)$, Linear Multilinear Algebra, DOI:10.1080/03081087.2018.1465525.

[9] Y. Chang, L. Chen, X. Zhou, Biderivations and linear commuting maps on the restricted Cartan-type Lie algebras $H(n;1)$, Comm. Algebra 47 (2019), 1311-1326.

[10] G. Fan, X. Dai, Super-biderivations of Lie superalgebras, Linear Multilinear Algebra, 65 (2017), 58–66.

[11] C. Xia, D. Wang, X. Han, Linear super-commuting maps and super-biderivations on the super-Virasoro algebras, Comm. Algebra, 44 (2016), 5342–5350.

[12] J. Yuan, X. Tang, Super-biderivations of classical simple Lie superalgebras, Aequationes Math., 92 (2018), 91–109.

[13] Y. Chang, L. Chen, Y. Cao, Super-biderivations of the generalized Witt Lie superalgebra $W(m,n;1)$, Linear Multilinear Algebra, DOI:10.1080/03081087.2019.1593312.

[14] F. Ma, Q. Zhang, Derivation algebras for K-type modular Lie superalgebras, J. Math. (Wuhan), 20 (2000), 431–435.

[15] B. Guan, W. Liu, Derivations of the even part into the odd part for modular contact superalgebra, J. Math. (Wuhan) 32 (2012), 402–414.

[16] B. Guan, L. Chen, Derivations of the even part of contact Lie superalgebra, J. Pure Appl. Algebra 216 (2012), 1454–1466.

[17] Y. Zhang, W. Liu, Modular Lie superalgebras, Science Press, Beijing, 2005.

[18] Y. Wang, Y. Zhang, The associative forms of the graded Cartan type Lie superalgebras. Adv. Math. (in Chinese), 29 (2000), 65–70.