The surface code on the rhombic dodecahedron

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I present a jaunty little [[14, 3, 3]] non-CSS surface code that can be described using a rhombic dodecahedron. Do with it what you will.

The surface. Figure 1 depicts the rhombic dodecahedron in wireframe, azimuthal, and net projections. Enlarge, cut out, fold, and tape the net diagram together to create your own handy three-dimensional model. Better yet, print one using a 3D printer for maximum enjoyment.[14]

(a) 3D wireframe.  
(b) Azimuthal projection.

![3D wireframe and azimuthal projection of the rhombic dodecahedron](image)

(c) 2D net.

FIG. 1: (Color online.) Depictions of the rhombic dodecahedron.

The rhombic dodecahedron has 14 vertices, 24 edges, and 12 faces. Eight vertices have degree three and six vertices have degree four. Each of its dozen faces is a rhombus of the same size and shape, with the long diagonal of each rhombus having a length that is \( \sqrt{2} \) times the length of the short diagonal. (The Bilinski rhombic dodecahedron has a slightly different geometry, for golden ratio aficionados [1, 2].) The faces can be colored red, green, and blue so that no two adjacent faces share the same color.

The code. Locate a qubit on each vertex and a check on each face of the rhombic dodecahedron; \( X \), \( Y \), and \( Z \) Pauli checks are on the the red, green, and blue faces respectively. Because the product of all face checks is the identity, there are only 11 independent checks, meaning that the code encodes three logical qubits. Another way to arrive at this conclusion is to interpret the degree-three vertices as disclination twists [15], where each twist after the first pair contributes a quantum dimension of \( \sqrt{2} \) to the logical Hilbert space, following the quasiparticle-based reasoning discussed in Ref. [3].

Subject to the vertex labeling and face coloring in the figures, the algebraic representations of the checks are the \( S_j \) operators below, with the operator \( S_4^Z \) written in parentheses because it is not an independent check—it is the product of the preceding ones.

\[
S_1^X = XXX I I X I I I I I I I I
S_2^X = I I X X I I X I I I I I I I I
S_3^X = I I I I I I X I I X X I
S_4^X = X I I I I I I I I X X
S_1^Y = I Y Y Y I I I I I I I I
S_2^Y = I I Y I Y Y I I I I I I I I
S_3^Y = I I I Y I I Y Y I I Y Y
S_1^Z = I I I I I I I I Y Y Y Y
S_2^Z = I I I I I I I I I I I I I I
S_3^Z = I I I I I I I I I I I I I I
S_4^Z = Z Z I I Z I I I I Z I I I (S_4^Z) = Z I I I I Z I I I I Z I Z I I Z.
\]

It may seem puzzling at first that a surface code on an object that is topologically equivalent to a sphere can hold any logical qubits because the surface code is a homological code, meaning that the logical operators have a homological character, yet the homology of a sphere is trivial. The resolution to this puzzle is the presence of the disclination twist defects—they provide locations on which string-like logical operators can begin and end.

We can construct a basis for the code’s logical space in the following way. Each collection of faces of the same color forms an “equator” on the rhombic dodecahedron. Along each of the three colored equators, construct a weight-four Pauli operator that commutes with all of the checks and other equator operators but which cannot be expressed as a product of them—these are representatives for the logical \( \overline{X} \) operators for the code. Algebraically, the operators are

\[
\overline{X}_r = Z I Y I I I I Z I I I I Y I
\overline{X}_g = I I X I Z I I I I I I Z I X I
\overline{X}_b = X I I I Y I I X I I Y I I I.
\]
To each logical $X$ operator, associate a logical $Z$ operator in the following way. For each “equator,” locate a collection of three qubits that forms a “V” shape that points along the equator; the “V” is supported on a string of qubits that connects twist defects on either side of the equator. Choose these three “V” sets so that they correspond to disjoint sets of qubits. On each of these “V” sets, form a Pauli operator that anticommutes with the corresponding logical $X$ operator on the equator but commutes with all other checks and logical operators. Algebraically, one choice for the logical $Z$ operators is

$$
Z_r = I I X Z I I Z I I I I I I I I
$$

$$
Z_g = I I I I Y I I I X I I I I
$$

$$
Z_b = Z I I I I Y I I I I I I I Y
.$$  \hfill (3)

Figure 2 depicts the logical $X$ and $Z$ operators for the code using the azimuthal projection of the rhombic dodecahedron.

![Logical Operators](image)

(a) Logical $X$ operators.  \hspace{5cm}  (b) Logical $Z$ operators.

FIG. 2: (Color online.) A basis of logical operators for the code.

Observe, from Eq. (3), that the distance of the surface code on the rhombic dodecahedron is at most three. Using the symmetry of the rhombic dodecahedron, it is a straightforward task to verify that all weight-two Pauli operators anticommute with some check. This proves that three is, in fact, the minimum distance of the code.

Although standard surface codes [4–7] are CSS codes [8, 9], the presence of disclination twist defects allows the presence of $Y$ checks [10], making it a non-CSS code. The consequence of all of these considerations is that the surface code on the rhombic dodecahedron is a $[14, 3, 3]$ non-CSS code.

**Ruminations.** A natural follow-on question to ask is, “Does the surface code on the rhombic dodecahedron generalize to larger polyhedral codes?” Perhaps it does, but as far as I can tell, there is no unique systematic method of generalization; there are an infinite number of larger face-three-colorable polyhedra. However, turning the other direction, the surface code on the cube is an obvious progenitor of the surface code on the rhombic dodecahedron. (In fact, one can glue a pyramid to each face of the cube to generate the rhombic dodecahedron, as depicted in Fig. 3.) This “cubic surface code” has eight qubits and eight disclination twist defects, so three logical qubits total. The distance turns out to be two, so it is an $[8, 3, 2]$ code, just like the (distinct!) “smallest interesting colour code” [11]. Perhaps you will be the one to discover interesting generalizations of the rhombic dodecahedron surface code. Be sure to enjoy your time doing so.

![Rhombic Dodecahedron](image)

FIG. 3: (Color online.) The rhombic dodecahedron can be formed by gluing pyramids to each face of the cube.

**Acknowledgments.** Thanks to my many colleagues patient enough to listen to me muse about this code while I whirled a 3D model of it around in my fingers, including (in alphabetical order) Jonas Andersson, Dave Bacon, Steve Brierley, Ben Brown, Bob Carr, Dan Browne, Stephen DiPippo, Anand Ganti, Tomas Joeyhm-O’Connor, Cody Jones, Markus Kesselring, Isaac Kim, Vadym Kliuchnikov, Alex Kubica, Setso Metodi, Naomi Nickerson, Tom O’Brien, John Preskill, Robert Raussendorf, Ciarán Ryan-Anderson, John Sirola, Jaimie Stephens, Barbara Terhal, Dave Wecker, Nathan Wiebe, Wayne Witzel, and Ted Yoder. A special thanks to Jaimie Stephens for generating the 3D colored figures in this paper.

This work was performed, in part, at the Aspen Center for Physics, which is supported by National Science Foundation Grant PHY-1607611.

Sandia National Laboratories is a multimission laboratory managed and operated by National Technology and Engineering Solutions of Sandia, LLC., a wholly owned subsidiary of Honeywell International, Inc., for the U.S. Department of Energy’s National Nuclear Security Administration under contract DE-NA-0003525.

This paper describes objective technical results and analysis. Any subjective views or opinions that might be expressed in the paper do not necessarily represent the views of the U.S. Department of Energy or the United States Government.

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[14] Thanks to Matt Curry and Jaimie Stephens for 3D printing a rhombic dodecahedron for me.

[15] The word “twist” is used ambiguously in the literature on lattice defects in topological codes [12] to mean either a lateral dislocation [13] or a rotational disclination [10]. I use the word “disclination” to clarify which I mean.