Determination of the CKM matrix elements $|V_{cb}|$

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We review the current status of $|V_{cb}|$, $|V_{ub}|$ and $|V_{tb}|$, the absolute values of the matrix elements in the CKM third column.

1. Introduction

In the Standard Model (SM), the strength of weak charged current interactions of quarks is codified inside the Cabibbo-Kobayashi-Maskawa (CKM) matrix, but the actual values of the CKM matrix elements are not predictable within the SM. It is important to ascertain such values precisely, since the departure of the CKM matrix from the unit matrix is at the origin of flavour and CP violating processes in the SM. We briefly review recent progress in the determination of $|V_{cb}|$, $|V_{ub}|$ and $|V_{tb}|$, that is the absolute values of the matrix elements in the CKM third column (see Fig. (1)).

\[
\begin{pmatrix}
V_{ud} & V_{us} & V_{ub} \\
V_{cd} & V_{cs} & V_{cb} \\
V_{td} & V_{ts} & V_{tb}
\end{pmatrix}
\]

Fig. 1. The CKM matrix

These matrix elements only contribute to weak transitions involving the heavy $b$-quark and it is a theoretical advantage the possibility to exploit, in most cases, the setting of the heavy flavour effective theory. Another advantage is that they can be determined directly by analyzing tree-level decays only. The exchange of a new physics (NP) particle is strongly constrained at tree level. A clean determination of CKM parameters from tree level processes is therefore a valuable input for other NP more sensitive estimates. Conversely, relations between $|V_{cb}|$, $|V_{ub}|$ and other parameters, such as, e.g. the $\epsilon_K$ dependency on $|V_{cb}|^4$, can be exploited to estimate their values, within or beyond the SM.
$|V_{cb}|$ and $|V_{ub}|$ play a considerable role in the analysis of the unitarity triangle. The so-called unitarity clock, the circle around the origin in the $\bar{\rho} - \bar{\eta}$ plane is proportional to the ratio $|V_{ub}|/|V_{cb}|$, and $|V_{cb}|$ normalizes the whole unitarity triangle. Most precise determinations of $|V_{cb}|$ and $|V_{ub}|$ are currently inferred from semi-leptonic decays, exclusive and inclusive ones. In Sect. 2 we outline the theoretical approaches and set the notations for semi-leptonic decays. Sections 3, 4, 5 report in detail on the determinations of $|V_{cb}|$, $|V_{ub}|$ and $|V_{tb}|$, respectively.

2. Semi-leptonic decays - Outline

At the moment, the most precise values of $|V_{cb}|$ and $|V_{ub}|$ are inferred from semi-leptonic decays, exclusive and inclusive ones. Data is provided by electron-positron machines, as LEP and CLEO, but above all by the dedicated Beauty ($B$-) Factories, BaBar and Belle, which have greatly reduced the errors on previous branching ratio determinations, allowing to attain an unprecedented high level of precision. In the $B$-factories, a $\bar{B} - B$ pair is produced nearly at rest in the $\Upsilon(4S)$ frame and the $\bar{B} - B$ production accounts for approximately $1/4$ of the $e^+ e^- \rightarrow$ hadrons cross-section. The decay products of the $B$ mesons overlap, and the neutrino from the semileptonic $B$ decay goes undetected. As a result, in order to unambiguously associate hadrons with a semileptonic $B$ decay, the second $B$ meson in the event need to be fully reconstructed.

Semileptonic exclusive $B$ decays are also studied at hadron colliders. However, the measurements of $|V_{cb}|$ and $|V_{ub}|$ imply the reconstruction, in the $b$-hadron rest frame, of observables difficult to measure at hadron colliders, such as the squared invariant mass of the lepton pair $q^2$. At LHCb, it is possible to improve the $q^2$ resolution by exploiting the separation between primary and secondary vertices, determining the $B$ flight direction vector and measuring the neutrino momentum with a two-fold ambiguity. With about 1.2 million $B^0 \rightarrow D^{*+} \mu \nu$ decays reconstructed in 1 fb$^{-1}$, it is worthwhile to explore the LHCb potential for the $|V_{cb}|$ determination.

The inclusive and exclusive determinations of $|V_{cb}|$ and $|V_{ub}|$ rely on different theoretical calculations, each with different (independent) uncertainties, and on different experimental techniques which have, to a large extent, uncorrelated statistical and systematic uncertainties. This independence makes the comparison of $|V_{cb}|$ and $|V_{ub}|$ determination from inclusive and exclusive decays a powerful test of our physical understanding. In Sects. 2.1 and 2.2 we summarize the main points of the theoretical approach and set notations.

2.1. Exclusive decays

Let us consider a generic semi-leptonic decay $H \rightarrow Pl\nu$, where $H$ and $P$ denote a heavy and a light pseudoscalar meson, respectively. The transition $H \rightarrow P$ is

*For recent, dedicated reviews to semi-leptonic decays see also Refs. 4, 5, 6.*
mediated by the vector current $V^\mu$ and the hadronic matrix element between
the initial and final state can be decomposed in a Lorentz covariant form built from
the independent four-momenta of the decay
\begin{equation}
\langle P(p_F)|V^\mu|H(p_H)\rangle = f_+(q^2) \left( p_H^\mu + p_F^\mu - \frac{m_H^2 - m_P^2}{q^2} q^\mu \right) + f_0(q^2) \frac{m_H^2 - m_P^2}{q^2} q^\mu \tag{1}
\end{equation}
The two form factors $f_+(q^2)$ and $f_0(q^2)$ depend only on $q^\mu = p_H^\mu - p_F^\mu$, the
momentum transferred to the lepton pair.

If the hadronic final state is a vector meson $V$, both the vector and axial currents
contribute to the semileptonic decay $H \to Vl\nu$
\begin{align}
\langle V(p_V)|V^\mu|H(p_H)\rangle &= V(q^2) \varepsilon_{\mu\nu\rho\sigma}^* \varepsilon^\nu \varepsilon^\rho \varepsilon^\sigma \frac{2p_H^\nu p_V^\rho}{m_H + m_V} \\
\langle V(p_V)|A^\mu|H(p_H)\rangle &= i\varepsilon^\nu \left[ A_0(q^2) \frac{2m_V q^\nu q^\nu}{q^2} + A_1(q^2) (m_H + m_V) \eta^\mu\nu - A_2(q^2) \frac{(p_H + p_V)_\sigma q^\nu}{m_H + m_V} \eta^\mu\sigma \right] \tag{2}
\end{align}
where $\varepsilon_{\mu\nu\rho\sigma}$ is the usual totally antisymmetric tensor, $\epsilon^\mu$ is the $D^*$ polarization
vector and $\eta^\mu\nu \equiv g^\mu\nu - q^\mu q^\nu/q^2$. Here the momentum transferred to the lepton pair
is $q^\mu = p_H^\mu - p_V^\mu$, while $m_H$ and $m_V$ are the meson $H$ and $V$ masses, respectively.

A great advantage of $B$ decays is that the mass $m_b$ of the $b$-quark is large
compared to the QCD scale and therefore approximations and techniques of the
heavy quark effective theory (HQET) can be used. In $B \to D^{(*)}$ semi-leptonic
decays also the mass $m_c$ of the $c$-quark can be considered large compared to the
QCD scale, allowing further approximations. In the heavy flavour limit, $m_b, m_c \to \infty$ ($m_b/m_c$ fixed),
when the weak current changes the flavour $b \to c$, the light
degrees of freedom inside the meson become aware of the change in the heavy
quark velocities, $v_B \to v_{D^{(*)}}$ ($v_B = p_B/m_B$, $v_{D^{(*)}} = p_{D^{(*)}}/m_{D^{(*)}}$), rather than
of the change in momenta. The form factors depend on $\omega = v_B \cdot v_{D^{(*)}}$, the only
scalar formed from the velocities ($v_B^2 = v_{D^{(*)}}^2 = 1$ by definition). The scalar $\omega$
is related to $q^2$, the momentum transferred to the lepton pair, according to the relation
$\omega = (m_B^2 + m_{D^{(*)}}^2 - q^2)/(2m_B m_{D^{(*)}})$. Approximate heavy-quark symmetries impose
constraints on the form factors that become more transparent with a basis of form
factors different from the one given in Eqs. (1) and (2), that is
\begin{align}
\langle D|V^\mu|B\rangle &= h_+(\omega)(v_B + v_D)^\mu + h_-(\omega)(v_B - v_D)^\mu \\
\langle D^*|V^\mu|B\rangle &= h_V(\omega)\epsilon^\mu\nu\epsilon_{\nu\rho\sigma} v_{D\rho} v_{D\sigma}^* \\
\langle D^*|A^\mu|B\rangle &= i\hbar A_1(\omega)(1 + \omega)\epsilon^\mu\nu - i [h_A_2(\omega)v_B^\mu + h_A_3(\omega)v_D^\mu] \epsilon^\nu \cdot v_B \tag{3}
\end{align}
For the polarization vector, $\sum_{\alpha=1}^3 \epsilon^\mu_{\alpha} \epsilon^*_{\alpha} = -g^{\mu\nu} + v_B^\mu v_B^\nu$, holds. The factor
$1/\sqrt{m_B m_{D^{(*)}}}$ changes the conventional, relativistic normalization of the meson.
respectively. In Eq. (7),
\[ \chi \text{ in terms of a single form factor } G \]
where \( r = m_D / m_B \). Similar relations hold for the other form factors. Let us also define the ratios
\[ R_1(\omega) = \frac{h_V(\omega)}{h_A(\omega)} \quad R_2(\omega) = \frac{h_A_1(\omega) + rh_A_2(\omega)}{h_A(\omega)} \]
that appear in the description of \( B \to D^{(*)} l\nu \) decays. In the heavy flavour limit, there is only one form factor, the Isgur-Wise function \( \xi(\omega) \) \[\text{I}\] in that limit, the form factors become
\[ h_+(\omega) = h_V(\omega) = h_A_1(\omega) = h_A_3(\omega) = \xi(\omega) \quad h_-(\omega) = h_A_2(\omega) = 0 \]

For negligible lepton masses \( l = e, \mu \), the differential ratios for the semi-leptonic decays \( B \to D^{(*)} l\nu \) can be written as
\[ \frac{d\Gamma}{d\omega}(B \to D l\nu) = \frac{G_F^2}{48\pi^3} (m_B + m_D)^2 m_D^3 (\omega^2 - 1)^{3/2} |V_{cb}|^2 G^2(\omega) \]
\[ \frac{d\Gamma}{d\omega}(B \to D^* l\nu) = \frac{G_F^2}{48\pi^3} (m_B - m_D^*)^2 m_D^3 \chi(\omega)(\omega^2 - 1)^{3/2} |V_{cb}|^2 F^2(\omega) \]
in terms of a single form factor \( G(\omega) \) and \( F(\omega) \), for \( B \to D l\nu \) and \( B \to D^* l\nu \), respectively. In Eq. \[\text{I}\], \( \chi(\omega) \) is a phase space factor which reads
\[ \chi(\omega) = (w + 1)^2 \left( 1 + \frac{4\omega}{\omega + 1} \frac{m_B^2 - 2\omega m_B m_D^* + m_D^*}{(m_B - m_D^*)^2} \right) \]
The form factor \( G(\omega) \) is a combination of \( h_+(\omega) \) and \( h_-(\omega) \)
\[ G(\omega) = h_+(\omega) - \frac{m_B - m_D}{m_B + m_D} h_-(\omega) \]
Similarly, the form factor \( F(\omega) \) can be written as a function of \( h_A_1(\omega) \), \( R_1(\omega) \) and \( R_2(\omega) \).

The Isgur-Wise function is normalized to unity at the zero recoil point \( \omega = 1 \), when \( D^{(*)} \) is at rest with respect to \( B \) \[\text{II}\] Indeed, at that point, the light constituents of the initial and final hadrons are not affected by the transition \( b \to c \), and there is a complete overlap between the initial and final hadronic quantum states. It follows
\[ G(1) = F(1) = 1 \]

Aside from short distance QCD and EW corrections, for finite values of the quark masses, the unity value of the form factors \( G \) and \( F \) is altered by inverse powers of the masses, to be calculated nonperturbatively. Let us observe that for \( F \) nonperturbative linear corrections are absent at zero recoil and the leading terms are quadratic in \( 1/m_{b,c} \). We can write, schematically
\[ F(1) = \eta_{EW} \eta_A (1 + \delta_{1/m^2} + \ldots) \]
where $\delta_{1/m^2}$ are power corrections which are suppressed by a factor of at least $\Lambda_{QCD}^2/m_c^2 \sim 3\%$. $\eta_{EW}$ is the enhancement factor 1.007, due to the electroweak corrections to the four-fermion operator mediating the semileptonic decay and $\eta_A(\alpha_s)$ is a short distance QCD coefficient known at order $\alpha_s^2$. A similar relation holds for $G(1)$, with the addition of linear corrections $\delta_{1/m}$, since in this case they, although kinematically suppressed, are not zero.

For heavy to light transitions like $B \to \pi l\nu$, $B \to \rho l\nu$, etc., the impact of heavy quark symmetry is less significant and is mostly reduced to flavour symmetry relations among $B$ and $D$ decay semileptonic form factors. The form factors are generally parameterized according to Eq. (1). In the approximation where the leptons are massless, only the form factor $f_+(q^2)$ enters the partial rate. In that case, the differential rate for, e. g., $B \to \pi l\nu$ decay reads

$$\frac{d\Gamma(B \to \pi l\nu)}{dq^2} = \frac{G_F^2 |p_\pi|^3}{24\pi^3} |V_{ub}|^2 |f_+(q^2)|^2$$

where $p_\pi$ is the momentum of the pion in the $B$ meson rest frame and $0 < q^2 < (m_B - m_\pi)^2 \simeq 26.4$ GeV. Non perturbative theoretical predictions for form factors are usually confined to particular regions of $q^2$. Complementary regions are spanned by Light Cone Sum Rules (LCSR) (low $q^2$) and lattice QCD (high $q^2$).

Lattice high-statistics calculations have been performed in the kinematic region where the outgoing light hadron carries little energy ($q^2 \geq 16$ GeV$^2$). At low $q^2$, with light hadrons carrying large momentum of order 2 GeV, direct simulations require a very fine lattice which is not yet accessible in calculations with dynamical fermions. The Compton wavelength of charm and bottom quarks $\sim 1/m_{c,b}$ may be similar to or even smaller than the lattice spacing, introducing large discretization errors. Recurring to HQET is one possible solution. By formulating the theory such that the large energy scale is explicitly separated from the low energy degrees of freedom, one can treat only the low energy part (which concerns the non-perturbative dynamics) in the lattice simulation; the high energy part can be reliably treated in perturbation theory.

The dependence of the form factor from $q^2$ is parameterized according several models, the most used ones being the Becirevic Kaidalov (BK) parameterization and the so-called z-expansion.

LCSR combines the standard sum rule framework with elements of the theory of hard exclusive processes. In the sum rule approach, the $B \to \pi$ matrix element is obtained from the correlation function of quark currents, such that, at large space-like external momenta, the operator-product expansion (OPE) near the light-cone is applicable. Within OPE, the correlation function is factorized in a series of hard-scattering amplitudes convoluted with the pion light-cone distribution amplitudes of growing twist. The contributions corresponding to higher twist and/or higher multiplicity pion distribution amplitudes are suppressed by inverse powers of the $b$-quark virtuality, allowing one to truncate the expansion after a few low twist contributions.
2.2. Inclusive decays

Let us consider the inclusive $\bar{B} \to X_q l \bar{\nu}$ decays, where the final state $X_q$ is an hadronic state originated by the quark $q$. In inclusive decays, $X_q$ refers to the sum of all possible final states, no matter if single-particle or multi-particle states. In the limit of large $b$-quark mass, the wavelengths associated with the $b$-quark decay are considered short enough to not interfere with the hadronization process. Inclusive heavy flavour decays are regarded as occurring in two separate steps: the heavy quark decay and the final hadron composition. The second step is not expected to determine gross characteristic like total rates, etc. and quark-hadron duality is generally assumed. Long distance dynamics of the meson can be factorized by using an OPE approach, which, combined with HQET, gives to inclusive transition rates the form of a heavy quark expansion, schematically written as

$$\Gamma(B \to X_q l \nu) = \frac{G_F^2 m_B^5}{192\pi^3} |V_{qb}|^2 \left[ c_3 \langle O_3 \rangle + c_5 \frac{\langle O_5 \rangle}{m_B^b} + c_6 \frac{\langle O_6 \rangle}{m_B^b} + O \left( \frac{1}{m_B^b} \right) \right]$$

(13)

Duality violation effects are hard to classify; in practice they would appear as unnaturally large coefficients of higher order terms in the expansion. In Eq.(13), $c_d (d = 3, 5, 6 \ldots)$ are short distance coefficients, which depend on the parton level characteristics of the hadronic final state and are calculable in perturbation theory as a series in the strong coupling $\alpha_s$. $O_d$ denote local operators of (scale) dimension $d$, whose hadronic expectation values $\langle O_d \rangle$ encode the nonperturbative corrections. In the definition

$$\langle O_d \rangle \equiv \frac{\langle B | O_d | B \rangle}{2m_B}$$

(14)

the $B$-meson mass, $m_B$, is included for the relativistic normalization of the state $|B \rangle$ and for dimensional counting. These matrix elements can be systematically expanded in powers of $1/m_B$. A remarkable feature of the total decay width in Eq.(13) is the absence of a contribution of order $1/m_B$, due to the absence of an independent gauge invariant operator of dimension four once the equation of motion is imposed. The leading operator is $O_3 = \bar{b}b$, whose hadronic expectation value $\langle \bar{b}b \rangle = 1 + O(1/m_B^2)$ incorporates the parton model result which dominates asymptotically, i.e. for $m_B \to \infty$. The fact that nonperturbative, bound state effects in inclusive decays are strongly suppressed (at least two powers of the heavy quark mass) explains a posteriori the success of the description in terms of the parton model.

Basically the same cast of operators in Eq.(13), albeit with different weights, appears in semi-leptonic, radiative and non-leptonic rates as well as distributions. While we can identify these operators and their dimensions, in general we cannot compute their hadronic expectation values from first principles, and we have to rely on a number of HQET parameters, which increase with powers of $1/m_B$. A certain degree of universality is attained by the fact that the HQET parameters not depending on the final state appear in different inclusive $B$ meson observables.
and can be measured in experiments. By measuring spectra plus as many moments as possible, one can perform what is generally indicated as a global fit, that is a simultaneous fit to HQET parameters, quark masses and absolute values of CKM matrix elements.

In a global fit, it is important to understand which kind of quark mass is to be employed, since for confined quarks there exists no a priori natural choice. Given a specific scheme, masses and HQET parameters depend on it. Using the pole masses is computationally most convenient, but require overcoming problems due to misbehaved perturbative series. QCD is not Borel summable and the presence of (infrared) renormalons, representing poles in the Borel plane, leads to an additive mass renormalization generating an uncertainty in the size of the pole mass. Renormalons cancel when the inclusive decay rate is written in terms of the minimal subtraction (MS) mass, rather than the pole mass. However, the MS scheme sets the scale of order of the $b$-quark mass, which is considered unnaturally high, due to the presence of typical scales significantly below, down to the order of 1 GeV. It reflects in large corrections at lower orders in the perturbation series. Alternative definitions of the $b$-quark mass have been introduced, in order to give a better convergence of the first (few) orders of the perturbative series and consequently reduce the theoretical errors.

Global fits to extract $|V_{cb}|$ are currently employed in two implementations, based on either the kinetic scheme or the 1S scheme. They belong to the so-called low subtracted (or threshold) mass schemes, where non perturbative contribution to the heavy quark pole mass are subtracted by making contact to some physical observable. Care must be taken in converting from one mass scheme to another due to the presence of truncated perturbative expressions.

The OPE-based, fixed order, framework just described is not applicable in the whole phase space. The phase space region includes a region of singularity, also called endpoint or threshold region, corresponding to a kinematic region near the limits of both the lepton energy $E_l$ and $q^2$ phase space, where the rate is dominated by the production of low mass final hadronic states. This region is plagued by the presence of large double (Sudakov-like) perturbative logarithms at all orders in the strong coupling. Corrections can be large and need to be resummed at all orders. This can be intuitively understood, since by integrating out the heavy flavor masses in HQET, the only remaining scales in the hadronic subprocess are $m_X$ and a hard scale $E_X$, where $m_X$ is the invariant mass of the hadronic system $X_q$ and $E_X$ its energy. Infrared logarithms occur in the ratio $E_X/m_X$ and become large in the threshold region, where $m_X^2 \sim O(E_X \Lambda_{QCD})$. A resummation formalism analogous to the one used to factorize Sudakov threshold effects for parton distribution functions in usual hard processes, such as deep inelastic or Drell-Yan scattering, can be applied. The relative low energy threshold region is also sensitive to non perturbative effects.

$^b$for theoretical aspects of threshold resummation in $B$ decays see Refs. 24, 25, 26, 27, 28, 29, and references therein.
One is the so-called Fermi motion, which classically can be described as a small vibration of the heavy quark inside the B meson due to the momentum exchange with the valence quark. This effect is important in the end-point region, because it produces some smearing of the partonic spectra. Let us illustrate it with the simplest example, occurring in the inclusive radiative decay $B \to X_s \gamma$. To leading order in $1/m_b$, the $B \to X_s \gamma$ decay is described by the quark level transition $b \to s \gamma$ and no events can be generated beyond the quark level kinematical boundary, i.e. with $E_\gamma > m_b/2$. On the other hand, the true kinematical boundary is set by the higher hadron mass $m_B/2$. It is intuitively clear the physical solution: the $b$-quark is not at rest inside the B meson and its Fermi motion spreads the photon line out over a region of order $m_B - m_b$.

For $b \to c$ semileptonic decays, the effect of the small region of singularity is not very important; in addition, corrections are not expected as singular as in the $b \to u$ case, being cutoff by the charm mass. However, in $B \to X_u l \nu$ decays, the copious background from the $B \to X_c l \nu$ process, which has a rate about 50 times higher, stands in the way, and data taken is pushed towards restricted regions of phase space, where such background is highly suppressed by kinematics. The OPE framework can reliably predict the inclusive decay rate as long as it is integrated over a large region of the phase space. The experimental cuts required to suppress the background violate this requirement. The weight of the threshold region within the phase space region increases and the above mentioned theoretical issues need to be addressed. Several theoretical approaches have been proposed, that will be discussed in Sect. 4.3.

Finally, let us observe that, very recently, the branching fractions of $B_s \to X l \nu$ decays have been measured at BaBar\cite{32} and Belle\cite{33} in datasets obtained from energy scans above the $\Upsilon(4S)$, with uncertainty going down as much as 5-6\%.\cite{33} In this section, we have always implicitly alluded to $B$ decays, but semileptonic $B_s$ decays can also probe CKM matrix elements, within the approach just outlined. The presence of the heavier spectator strange quark is bound to introduce some amount of SU(3) symmetry breaking.

3. $|V_{cb}|$

In 1983, Mark II and MAC collaborations presented the first measurements of the average lifetime of $b$-quark, obtained at the $e^+e^-$ storage ring PEP located at SLAC, and gave the first estimate of $|V_{cb}|$ (setting $|V_{ub}|$ to zero)\cite{34,35}. Currently, $|V_{cb}|$ is inferred from semi-leptonic decays; about 25% of all B mesons decay semileptonically via the tree-level $b \to c$ quark transition.

In Fig. 2 we plot recent $|V_{cb}|$ determinations, exclusive and inclusive, discussed in detail in Sects. 3.1 and 3.2. To facilitate the comparison, the errors shown are the squared average of theoretical and experimental errors. We can observe a certain tension between the inclusive and the exclusive unquenched lattice results, around $2\sigma$ with the most recent lattice results by FNAL/MILC\cite{36} having a considerable
reduced theoretical error. New lattice results are in progress and expected soon. We also compare with $|V_{cb}| = (42.07 \pm 0.64) \times 10^{-3}$ by the UTfit collaboration\textsuperscript{37} and $|V_{cb}| = (40.77^{+0.13}_{-0.48}) \times 10^{-3}$ by the CKMfitter collaboration (at 1σ).\textsuperscript{38} Indirect fits prefer a value for $|V_{cb}|$ that is closer to the (higher) inclusive determination.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2.png}
\caption{Comparison among exclusive, inclusive and indirect determinations of $|V_{cb}|$ ($10^{-3}$). For the exclusive decays the legend refers to the theoretical determination of the form factor: for $B \to D^*l\nu$, unquenched lattice FNAL/MILC\textsuperscript{36} and ZRSR;\textsuperscript{39,40} for $B \to Dl\nu$, unquenched lattice FNAL/MILC,\textsuperscript{41} quenched lattice Rome/Tor Vergata (RM/TV),\textsuperscript{42,43} HQ/BPS expansion.\textsuperscript{44} For inclusive decays, HFAG\textsuperscript{45} results are reported in the kinetic scheme with the $m_c$ constraint, and in the 1S scheme with the $B \to X_s\gamma$ constraint. Indirect fits are from Utfit\textsuperscript{37} and CKMfitter collaborations.\textsuperscript{38}}
\end{figure}

### 3.1. Exclusive determination

Let us consider the tree level driven $\bar{B} \to D(\ast)l\nu$ weak decays, where $l$ is an electron or a muon. Neglecting the charged lepton and neutrino masses, their differential ratio can be parameterized in terms of the form factors $G(\omega)$ and $F(\omega)$, according to Eq. (7).

The extraction of $|V_{cb}|$ can be reckoned as divided into two steps, the first one being the experimental fit of the products $|G(\omega)V_{cb}|$ and $|F(\omega)V_{cb}|$. Due to the kinematic suppression factors, $(\omega^2 - 1)^{3/2}$ and $(\omega^2 - 1)^{1/2}$, data are taken at $\omega \neq 1$. Step two is the theoretical evaluation of the form factors. It is generally performed at zero-recoil point, where the non perturbative evaluation of the operator matrix elements is simplified by heavy flavour approximations and symmetries. Since the zero recoil point is not accessible experimentally, the $|V_{cb}|$ estimates rely on the extrapolation from $\omega \neq 0$ to the zero recoil point. The dynamics of the decay is contained in the form factors $F(\omega)$ and $G(\omega)$, which can be parameterized in model dependent ways based on the HQET framework.\textsuperscript{46,47,48} The experimental fit of step one may also extend to parameters characterizing the dependence on $\omega$ of the form...
factors.

For the determination of $|V_{cb}|$, the decay $\bar{B} \to D^* l \bar{\nu}$ is preferred over $\bar{B} \to D l \bar{\nu}$ for both theoretical and experimental reasons. At present, the $\bar{B} \to D^* l \bar{\nu}$ decay, with an higher rate, is measured with a better accuracy than the $\bar{B} \to D l \bar{\nu}$ decay. On the theory side, nonperturbative linear corrections to the form factor $F$ are absent at zero recoil. The most recent Heavy Flavor Averaging Group (HFAG) experimental fit\textsuperscript{45} gives

$$|V_{cb}| F(1) = (35.90 \pm 0.45) \times 10^{-3}$$

(15)

The currently most precise measurement\textsuperscript{49} uses 711 fb\textsuperscript{−1} of data collected by the Belle experiment and fits four kinematic variables fully characterizing the form factor in the framework of Ref. 46, that is $F(1)|V_{cb}|, R_1(1), R_2(1)$ and a parameter $\rho^2$. Their results agree well with the ones from BaBar.\textsuperscript{50}

Nonperturbative corrections to the unit limit of $F(1)$ can be theoretically computed by means of lattice QCD. The first lattice calculation for $F(1)$ has been accomplished by FNAL\textsuperscript{51} in the quenched approximation, in which the effect of vacuum polarization of quark loops is neglected. Quenched lattice results are also available at finite momentum transfer ($\omega = 1.075$),\textsuperscript{52} and combined with 2008 BaBar data\textsuperscript{50} give

$$|V_{cb}| = (37.4 \pm 0.5_{\exp} \pm 0.8_{\th}) \times 10^{-3}$$

(16)

with a rather small nominal error.

Unquenched calculations take into account vacuum polarization effects, i.e., include up, down and strange sea quarks on the gauge configurations’ generation. The up and down quarks are usually taken to be degenerate, so those simulations are referred to as $n_f = 2 + 1$. The only unquenched calculations available for $F(1)$ have been performed by FNAL/MILC.\textsuperscript{53,54} Their latest update\textsuperscript{36} reduces the total uncertainty on $F(1)$ from about 2.6% to 1.8% and gives

$$F(1) = 0.908 \pm 0.017$$

(17)

by using lattice with the Fermilab action for $b$- and $c$-quarks, the asqtad staggered action for light valence quarks, and the MILC ensembles for gluons and light quarks. It includes the enhancement factor 1.007, due to the electroweak (EW) corrections to the four-fermion operator mediating the semileptonic decay.\textsuperscript{12} By combining Eq. (17) with the HFAG results in Eq. (15), one estimates

$$|V_{cb}| = (39.54 \pm 0.50_{\exp} \pm 0.74_{\th}) \times 10^{-3}$$

(18)

Let us observe that a further update by FNAL/MILC collaboration has been announced,\textsuperscript{54} claiming a reduction of discretization effects and of the error on $|V_{cb}|$ down to 1.6%, but no new value for $|V_{cb}|$ has been published until now.

The lattice calculations have to be compared with non-lattice ones. By using zero recoil sum rules, the value

$$F(1) = 0.86 \pm 0.02$$

(19)
has been recently reported. Let us remark that full $\alpha_s$ and up to $1/m^2$ corrections are included in this result; moreover, in order to compare with the lattice value in Eq. (17), one has to remove the EW factor $1.007$ from the latter. The related estimate, given the HFAG average in Eq. (15), yields to

$$|V_{cb}| = (41.6 \pm 0.6_{\text{exp}} \pm 1.9_{\text{th}}) \times 10^{-3} \quad (20)$$

The theoretical error is more than twice the error in the lattice determination given in Eq. (18). However, let us observe that the budget error from lattice has been recently questioned. The claim is that existing differences between the power-suppressed deviations from the heavy flavour symmetry in the lattice theory with heavy quarks and in continuum QCD may be compensated by a matching between the two theories that has been performed, at the best, only at lower levels.

Let us compare the previous determinations with the $|V_{cb}|$ value extracted from $\bar{B} \to D l \bar{\nu}$ decays. The HFAG average includes older Aleph, CLEO and Belle measurements, as well as the new 2008-2009 BaBar data, and adopts the parametrization in Ref. [46] where the form factor $G(\omega)$ is described by only two parameters: the normalization $G(1)|V_{cb}|$ and the slope $\rho^2$. The resulting global two-dimensional fit gives

$$|V_{cb}||G(1)| = (42.64 \pm 1.53) \times 10^{-3} \quad (21)$$

Unquenched calculations of the form factor $G(1)$ have been performed by the FNAL/MILC collaboration in 2005 with an update the year after giving

$$G(1) = 1.074 \pm 0.024 \quad (22)$$

after correcting by the usual EW factor of $1.007$. Let us observe that studies of this form factor at non-zero recoil are in progress and the FNAL/MILC collaboration has already reported some preliminary results. By combining Eqs. (21) and (22), the resulting estimate is

$$|V_{cb}| = (39.70 \pm 1.42_{\text{exp}} \pm 0.89_{\text{th}}) \times 10^{-3} \quad (23)$$

in good agreement with the lattice determination from $\bar{B} \to D^* l \bar{\nu}$, Eq. (18), although the experimental error is more than twice larger. At non-zero recoil, a lattice determination is already available, but only in the quenched approximation. By using 2009 BaBar data a slightly higher value is found

$$|V_{cb}| = (41.6 \pm 1.8 \pm 1.4 \pm 0.7_{FF}) \times 10^{-3} \quad (24)$$

The errors are statistical, systematic and due to the theoretical uncertainty in the form factor $G$, respectively.

The most recent non lattice calculation dates 2004 and combines the heavy quark expansion with expanding around the point where the kinetic energy is equal to the chromomagnetic moment $\mu^2 = \mu^2_G$ ("BPS" limit). A number of relations, connected to the form factor, receive corrections only to the second order expanding...
around this limit to any order in \(1/m_{c,b}\). Under this approximations, the form factor reads

\[
G(1) = 1.04 \pm 0.02
\]  
(25)

With such estimate the PDG finds

\[
|V_{cb}| = (40.7 \pm 1.5_{\text{exp}} \pm 0.8_{t1b}) \times 10^{-3}
\]  
(26)

in agreement, within the errors, with both lattice determinations \(^{[23]}\) and \(^{[24]}\).

Semileptonic B decays to orbitally-excited P-wave charm mesons (\(D^{**}\)) contribute as a source of systematic error in the \(|V_{cb}|\) measurements at the B factories (and previously at LEP), as a background to the direct decay \(B^0 \to D^*l\nu\) (see BaBar preliminary results in Ref. \(^{[59]}\). The knowledge on these semileptonic decays is not complete yet: one example for all, the so called "1/2 versus 3/2" puzzle\(^{[c]}\). Very recently, first dynamical lattice computation of the \(\bar{B} \to D^{**}l\nu\) form factors have been attempted, although still preliminary and needing extrapolation to the continuum \(^{[60]}\).

Semileptonic decays of B mesons to the \(\tau\) lepton are experimentally challenging to study because their rate is suppressed, due to the large \(\tau\) mass, and the final state contains not just one, but two or three neutrinos as a result of the \(\tau\) decay. Theoretically, an additional form factor is needed for both \(\bar{B} \to D\tau^-\bar{\nu}_\tau\) and \(\bar{B} \to D^\tau\tau^-\bar{\nu}_\tau\), since the \(\tau\) mass cannot be neglected. The first exclusive observation of \(B^0 \to D^{*-}\tau^+\nu_\tau\) decays was presented by Belle in 2007\(^{[61]}\). Since then, both BaBar and Belle have published improved measurements, and have found evidence for \(\bar{B} \to D\tau^\tau^-\bar{\nu}_\tau\) decays. The branching ratio measured values have consistently exceeded the SM expectations and the experimental precision starts to be enough to constrain NP. The most recent data from BaBar are not compatible with a charged Higgs boson in the type II two-Higgs-doublet model and with large portions of the more general type III two-Higgs-doublet model\(^{[62]}\). At present, semileptonic \(b \to \tau\) decays do not contribute to the determination of \(|V_{cb}|\), but are studied because of their NP sensitivity. The same is true for exclusive \(B_s\) decays, that are attracting a lot of attention, due to the avalanche of recent data and to the expectation of new ones\(^{[d]}\).

### 3.2. Inclusive decays

In Sect.\(^{[2.2]}\) the standard setting for the \(|V_{cb}|\) extraction using data on inclusive semileptonic decays \(B \to X_c l\nu\) has been outlined. The expansion \(^{[13]}\) is valid only for sufficiently inclusive measurements and away from perturbative singularities, therefore the relevant quantities to be measured are global shape parameters (the first few moments of various kinematic distributions) and the total rate. As already discussed in Sect.\(^{[2.2]}\) masses and HQET parameters need to be defined in a given

\(^{[c]}\) see, e.g., Refs. \(^{[3]}\) \(^{[63]}\) and references therein.
mass scheme: global fits to extract $|V_{cb}|$, currently employed by HFAG\cite{HFAG} have been performed in two schemes, the kinetic and the $1S$ schemes. Care must be exercised while passing from one scheme to another, due to the presence of different definitions and approximations. The reliability of the inclusive method depends also on the ability to control the higher order contributions in the expansion (13), a double series in $\alpha_s$ and $\Lambda_{QCD}/m_b$. The calculation of higher order effects permits to ascertain unwanted behavior of the double series and to reduce the theoretical uncertainty due to the truncation.

The leading term is the parton model, which is known completely to order $\alpha_s$ and $\alpha_s^2$, for the width and moments of the lepton energy and hadronic mass distributions (see Refs. \textsuperscript{64, 65, 66, 67} and references therein).

The coefficients of even the first non-perturbative corrections are not completely known to order $\alpha_s$. In the expansion, the leading power corrections arise from two dimension five operators: the kinetic operator $O_{\text{kin}}$ and the chromomagnetic operator $O_{\text{mag}}$. Different schemes are used to define these parameters, but to leading order and leading power they are given by

$$\langle O_{\text{kin}} \rangle \equiv \frac{1}{2m_B} \langle \bar{B}(p_B)|\bar{b}_v(iD)^2b_v|B(p_B)\rangle \equiv -\mu^2,$$

$$\langle O_{\text{mag}} \rangle \equiv \frac{1}{2m_B} \langle \bar{B}(p_B)|\bar{b}_v\frac{g}{2}\sigma_{\mu\nu}G_{\mu\nu}b_v|B(p_B)\rangle \equiv \mu_G^2 \tag{27}$$

where $b_v$ is the quark field in the HQET and $D$ is the covariant derivative with respect to the background gluon field. Sometimes in the definitions (27) the spatial component $D_{\mu} = (g^{\mu\nu} - v^{\mu}v^\nu)D_{\nu}$ is used instead; these expressions differ by higher-order terms in the expansion $1/m_b$. The HQET parameters $\mu^2$ and $\mu_G^2$ are also denoted by $-\lambda_1$ and $3\lambda_2$, respectively. For the total rate, the kinetic corrections have the same coefficient as the leading order. For other observables, such as partial rates and moments, the corrections to the coefficient of the kinetic matrix element have been evaluated at $O(\alpha_s)$ order. They lead to numerically modest modifications of the width and moments. Corrections at order $O(\alpha_s)$ to the coefficient of the matrix element of the chromomagnetic operator are not yet available, although a study is in progress. In the simpler case of inclusive radiative decay, these corrections have increased the coefficient by almost 20% in the rate.\textsuperscript{74,75,76}

At order $1/m_b^2$ the expansion (13) receives contributions from local dimension-six operators. There are also other sources of $1/m_b^3$ corrections. The matrix elements of Eq. (27) have an implicit dependence on $m_b$. At order $1/m_b^2$, this dependence can be neglected, but at higher orders this mass dependence has to be taken into account explicitly. Neglecting perturbative corrections, i.e. working at tree level, contributions to various observables have been computed to order $1/m_b^4$.\textsuperscript{26,27} At order $1/m_b^3$, terms with a sensitivity to the charm mass $m_c$ start to acquire relevance.\textsuperscript{24,25} Roughly speaking, since $m_c^2 \sim O(m_b\Lambda_{QCD})$ and $\alpha_s(m_c) \sim O(\Lambda_{QCD})$, contributions of order $1/m_b^3 m_c^2$ and $\alpha_s(m_c)1/m_b^3 m_c$ are expected comparable in size to the contributions of order $1/m_b^4$. Contributions $O(1/m_b^4)$ have been esti-
mated in the ground state saturation approximation resulting in a small 0.4% increase of $|V_{cb}|$. The usefulness of the ground state saturation has been recently questioned on the basis that the non-factorizable contributions can in general be comparable to the factorizable ones.

In order to perform a global fit to $|V_{cb}|$, the $b$-quark mass and the hadronic parameters, HFAG employs as experimental inputs the (truncated) moments of the lepton energy $E_l$ (in the $B$ rest frame) and the $m_X^2$ spectra in $B \to X_c l \nu$. A total of about 70 measurements is available, 80% of which performed at the $B$-factories. Let us underline that, since new non-perturbative parameters appear at each order in $1/m_b$ (e.g. as many as nine new expectation values at $O(1/m_b^2)$), only the parameters associated with $O(1/m_b^{2,3})$ corrections are routinely fitted from experiment.

The moments in $B \to X_c l \nu$ are sufficient for determining $|V_{cb}|$, but measure the $b$-quark mass only to about 50 MeV precision. To get higher precision, additional constraints are introduced: the photon energy moments in $B \to X_s \gamma$, or a precise constraint on the $c$-quark mass. Using the former constraint, in the kinetic scheme, the global fit yields

$$|V_{cb}| = (41.88 \pm 0.73) \times 10^{-3}$$

with the value $m_c^{\text{MS}}(3\text{GeV}) = (0.998 \pm 0.029) \text{ GeV}$, obtained using low-energy sum rules. In the 1S scheme, the $c$-quark mass constraint cannot be applied as the 1S expressions do not depend on this parameter. The result, using the $B \to X_s \gamma$ constraints, is

$$|V_{cb}| = (41.96 \pm 0.45) \times 10^{-3}$$

The central values are in excellent agreement in the two schemes. The precision is higher than in the exclusive determinations, being about 1.7% in the kinetic scheme and 1.1% in the 1S scheme.

High statistic $B$-factories have greatly contributed to the increase in measurement precision with respect to previous experiments. BaBar and Belle have collected about 1.5 ab$^{-1}$ high-quality data, and Belle II at SuperKEKB is expected to collect about 30 times more data by 2023, pushing the error on $|V_{cb}|$ down to 1%.

4. $|V_{ub}|$

It is likely the most studied CKM matrix element, on both theoretical and experimental aspects, but it is, comparatively, the less known. The order of magnitude of the ratio $|V_{ub}/V_{cb}| \sim 0.1$ has been known since the ’90s. The error on $|V_{ub}|$, that at the time was around 30%, is now reduced of about 1/3. The increased precision has made manifest a tension between the values of $|V_{ub}|$ extracted from the exclusive and inclusive semileptonic decays. In Fig. we show some recent exclusive and inclusive determinations, that will be commented in Sects. 4.1, 4.2 and 4.3.

The leptonic decay $B \to \tau \nu$, first observed by Belle in 2006, can also provide information on $|V_{ub}|$, which we do not display in Fig. Indeed, previous data have
Fig. 3. Comparison among exclusive, inclusive and indirect determinations of $|V_{ub}|$ ($10^{-3}$). For inclusive decays, we refer to the HFAG estimates\cite{HFAG} following ADFR\cite{ADFR, DGE, GGOU, BLNP} determinations. For the exclusive decays we report, from the most recent estimates\cite{BLNP} the full $q^2$ range fit to data and FNAL/MILC LQCD results\cite{LQCD_fit} and the QCDSR based determination below $q^2 = 16$ GeV.\cite{LCSR} Indirect fits are from UTfit\cite{UTfit} and CKMfitter collaborations.\cite{CKMfitter}

The largest among theoretical and experimental errors have been depicted.

shown a disagreement of the measured branching ratio with the SM prediction, which has softened significantly with the new data from Belle collaboration.\cite{Belle}

In Fig. 3 we also compare with indirect fits, $|V_{ub}| = (3.65 \pm 0.13) \times 10^{-3}$ by UTfit\cite{UTfit} and $|V_{ub}| = (3.49^{+0.21}_{-0.10}) \times 10^{-3}$ at 1$\sigma$ by CKMfitter.\cite{CKMfitter} At variance with the $|V_{cb}|$ case, the results of the global fit prefer a value for $|V_{ub}|$ that is closer to the (lower) exclusive determination.

4.1. Exclusive semileptonic decays

Among all charmless $B$ meson semileptonic channel presently observed, $B \to \pi l \nu$ decays benefit of more precise branching fraction measurements and are currently the channels of election to determine $|V_{ub}|$ exclusively. The $B \to \pi l \nu$ decays are affected by a single form factor $f_+(q^2)$, in the limit of zero leptonic masses (see the differential ratio in Eq. 12). The first lattice determinations of $f_+(q^2)$ based on unquenched simulations have been obtained by the Fermilab/MILC collaboration\cite{LQCD_fit} and the HPQCD collaboration\cite{HPQCD} and they are in substantial agreement. In Ref.\cite{HPQCD} the $b$-quark is simulated by using the so-called Fermilab heavy-quark method, while the dependence of the form factor from $q^2$ is parameterized according to the $z$-expansion.\cite{HPQCD, z_expansion} In Ref.\cite{LQCD_fit} the $b$-quark is simulated by using nonrelativistic QCD and the BK parameterization\cite{BK} is extensively used for the $q^2$ dependence. Recent results are also available on a fine lattice (lattice spacing $a \sim 0.04$ fm) in the quenched approximations by the QCDSF collaboration.\cite{QCDSF}

Latest data on $B \to \pi l \nu$ decays coming from Belle and BaBar\cite{Belle, BaBar} are not yet included in the HFAG averages.\cite{HFAG} The measured partial branching fractions can be
fit at low and high $q^2$ according to LCSR and lattice approaches, respectively, the latter providing generally better fits. Both BaBar and Belle collaborations determine the magnitude of the CKM matrix element $|V_{ub}|$ using two different methods. In one case, the $|V_{ub}|$ value is extracted in a limited range of $q^2$ from the measured partial branching fraction using the relation $|V_{ub}| = \sqrt{\Delta B/(\tau \Delta \zeta)}$, where $\tau$ is the $B$ lifetime and $\Delta \zeta = \Gamma/|V_{ub}|^2$ is the normalized partial decay width derived in different theoretical approaches. In the other, a simultaneous fit to lattice results and experimental data is performed, to exploit all the available information on the form factors from the data (shape) and theory (shape and normalization). The simultaneous fit to the data over the full $q^2$ range and the FNAL/MILC lattice QCD result has given the following average value

$$|V_{ub}| = (3.43 \pm 0.33) \times 10^{-3}$$  \hspace{1cm} (30)

More recently, the analogous fit by BaBar has yielded

$$|V_{ub}| = (3.25 \pm 0.31) \times 10^{-3}$$  \hspace{1cm} (31)

in the full $q^2$ range. The $|V_{ub}|$ determinations inferred by using lattice QCD direct calculations, in the kinematic region $q^2 > 16$ GeV$^2$, with no extrapolations, are in agreement with the previous values.\cite{91,90}

In the complementary kinematic region, at large recoil, with an upper limit for $q^2$ varying between 6 and 16 GeV$^2$, several direct calculations of the semileptonic form factor are available, based on LCSR. There has been recent progress in pion distribution amplitudes, NLO and LO higher order twists (see e.g. \cite{92,98,99,100,101,102,103,104} and references therein). The latest determination, from BaBar collaboration\cite{90} using LCSR results below $q^2 = 16$ GeV\cite{92} gives

$$|V_{ub}| = (3.46 \pm 0.06 \pm 0.08\pm0.37) \times 10^{-3}$$  \hspace{1cm} (32)

where the three uncertainties are statistical, systematic and theoretical, respectively. The Belle collaboration\cite{97} has used the LCSR form factor determination in Ref.\cite{99} with the same $q^2 = 16$ GeV as upper limit, and found a consistent value

$$|V_{ub}| = (3.64 \pm 0.11\pm0.60) \times 10^{-3}$$  \hspace{1cm} (33)

Recently, BaBar and Belle collaborations have reported significantly improved branching ratios of other heavy-to-light semileptonic decays, that reflects on increased precision for $|V_{ub}|$ values inferred by these decays. $|V_{ub}|$ has been extracted from $B^+ \rightarrow \omega l^+\nu$, with the LCSR form factor determination\cite{105} at $q^2 < 20.2$ GeV, yielding\cite{90}

$$|V_{ub}| = (3.20 \pm 0.21 \pm 0.12\pm0.45) \times 10^{-3}$$  \hspace{1cm} (34)

where the three uncertainties are statistical, systematic and theoretical, respectively. By comparing the measured distribution in $q^2$, with an upper limit at $q^2 = 16$ GeV, for $B \rightarrow \rho l\nu$ decays, with LCSR predictions for the form factors\cite{105} the $|V_{ub}|$ value read\cite{105}

$$|V_{ub}| = (2.75 \pm 0.24) \times 10^{-3}$$  \hspace{1cm} (35)
Other interesting channels are $B \rightarrow \eta(l)l\nu$ but a value of $|V_{ub}|$ has not been extracted because the theoretical partial decay rate is not sufficiently precise yet. There has also been recent progress on the form factor evaluation of the $|V_{ub}|$ sensitive $\Lambda_b \rightarrow p\nu$ decay in the LCSR framework and from lattice with static $b$-quarks.

4.2. **Purely leptonic decays**

The decay $B^- \rightarrow \tau^{-}\bar{\nu}_\tau$ has been the first purely leptonic B decay to be observed. In the absence of new physics, $B \rightarrow l\nu$ decays ($l=e,\mu,\tau$) are simple tree-level decays, where the two quarks in the initial state, $b$ and $\bar{u}$, annihilate to a $W^-$ boson. They are particularly sensitive to physics beyond the SM, since a new particle, for example, a charged Higgs boson in supersymmetry or a generic two-Higgs doublet model, may lead the decay taking the place of the $W^-$ boson. In the SM, the $B^- \rightarrow \tau^{-}\bar{\nu}_\tau$ branching ratio is

$$B(B^- \rightarrow \tau^{-}\bar{\nu}_\tau) = \frac{G_F^2 m_B^2 m_{\tau}^2}{8\pi} \left(1 - \frac{m_{\tau}^2}{m_B^2}\right)^2 f_B^2 |V_{ub}|^2 \tau_B$$

and its measurement provides a direct experimental determination of the product $f_B |V_{ub}|$. Experimentally, it is challenging to identify the $B^- \rightarrow \tau^{-}\bar{\nu}_\tau$ decay because it involves more than one neutrino in the final state and therefore cannot be kinematically constrained. At $B$ factories, one can reconstruct one of the $B$ mesons in the $e^+e^- \rightarrow \Upsilon(4S) \rightarrow \bar{B}B$ chain, either in hadronic decays or in semileptonic decays. One then compares properties of the remaining particle(s) to those expected for signal and background. In contrast with previous data, the new Belle result, is in substantial agreement with the SM predictions. The amount of the agreement varies if we compare with predictions based on specific or averaged $|V_{ub}|$ exclusive and inclusive determinations, indirect $|V_{ub}|$ fits, or estimates where the dependency from $|V_{ub}|$ is eliminated, e.g. by using the unitarity conditions of the CKM matrix.

4.3. **Inclusive $|V_{ub}|$**

The extraction of $|V_{ub}|$ from inclusive decays requires to address theoretical issues absent in the inclusive $|V_{cb}|$ determination, as outlined in Sect. 2.2. On the experimental side, efforts have been made to enlarge the experimental range, so as to reduce, on the whole, the weight of the endpoint region. Latest results by Belle access $\sim 90\%$ of the $B \rightarrow X_u l\bar{\nu}_l$ phase space, claiming an overall uncertainty of $7\%$ on $|V_{ub}|$. A similar portion of the phase space is covered also by the most recent BaBar analysis. From the theoretical side, several available theoretical schemes are available. All of them are tailored to analyze data in the threshold region, but
differ significantly in their treatment of perturbative corrections and the parameterization of non-perturbative effects.

The average values for $|V_{ub}|$ have been extracted by HFAG from the partial branching fractions, adopting a specific theoretical framework and taking into account correlations among the various measurements and theoretical uncertainties.

The latest experimental analysis, Ref. 113, and the HFAG averages in Ref. 45 rely on at least four different QCD calculations of the partial decay rate: BLNP by Bosch, Lange, Neubert, and Paz; DGE, the dressed gluon exponentiation, by Andersen and Gardi; ADFR by Aglietti, Di Lodovico, Ferrara, and Ricciardi; and GGOU by Gambino, Giordano, Ossola and Uraltsev. These QCD theoretical calculations are the ones taking into account the whole set of experimental results, or most of it, starting from 2002 CLEO data. They can be roughly divided into approaches based on the estimation of the shape function (BLN, GGOUP) and on resummed perturbative QCD (DGE, ADFR). Other theoretical schemes have been described in Refs. 115, 116, 117.

The shape function approach is based on the introduction of a nonperturbative distribution function (shape function) that at leading order is universal. The shape function takes care of singular terms in the theoretical spectrum; it has the role of a momentum distribution function of the $b$-quark in the $B$ meson. However, the OPE does not predict the shape function and an ansatz is needed for its functional form. The subleading shape functions are difficult to constrain and are not process independent.

Predictions based on resummed perturbative QCD use resummed perturbation theory in moment space to provide a perturbative calculation of the on-shell decay spectrum in the entire phase space. They extend the standard Sudakov resummation framework by adding non-perturbative corrections in the form of power corrections, whose structure is determined by renormalon resummation or by an effective QCD coupling. The shape of the spectrum in the kinematic region, where the final state is jet-like, is largely determined by a calculation, and less by parametrization. In principle, there is no preclusion to why an effective coupling inserted in the perturbative resummation formula cannot adequately describe the non-perturbative Fermi motion as well as a fitting function (see e.g. Ref. 118). In ADFR, the physical picture implied is that $B$ fragmentation into the $b$-quark and the spectator quark can be described as a radiation process off the $b$-quark with a proper coupling. This effective determination of $|V_{ub}|$ from semileptonic $B$ decays is universal in the sense that describes radiative decay processes as well as $B$ fragmentation processes; once it is fixed, for instance on the basis of minimal analyticity arguments, there are no free parameters to be fitted in the model.

Although conceptually quite different, all the above approaches generally lead to roughly consistent results when the same inputs are used and the theoretical errors are taken into account. Recent HFAG estimates are reported in Table 1 and plotted in Fig. 3. They give values in the range $\sim (3.9 - 4.6) \times 10^{-3}$; let us observe that the theoretical uncertainty among determinations can reach 10%.
Table 1. Inclusive $|V_{ub}|$ averages

| Theory | $|V_{ub}| \times 10^4$ |
|--------|------------------|
| BLNP   | 4.40 ± 0.15$^{+0.19}_{-0.21}$ |
| DGE    | 4.45 ± 0.15$^{+0.15}_{-0.16}$  |
| ADFR   | 4.03 ± 0.13$^{+0.18}_{-0.12}$  |
| GGOU   | 4.39 ± 0.15$^{+0.12}_{-0.20}$  |

5. $|V_{tb}|$

The $t$-quark decays before it can hadronize, since its lifetime $\tau \simeq (1.5 \text{ GeV})^{-1}$ is much less than the QCD scale $\simeq (200 \text{ MeV})^{-1}$; there are no top mesons or baryons. Information on the value of $|V_{tb}|$ has been traditionally obtained indirectly, analyzing loop-dominated observables sensitive to $|V_{tb}|$, e.g., $B_d$ and $B_s$ mixing or the radiative decay $b \rightarrow s\gamma$. The term indirect emphasizes that in order to extract the desired information one has to consider loop processes and/or make some usage of SM properties, such as CKM unitarity. In the SM, unitarity constrains $|V_{tb}|$ to be very close to one,$^{38}$

$$|V_{tb}| = 0.999142^{+0.000043}_{-0.000025} \quad (38)$$

An indirect measurement of $|V_{tb}|$ that does not require the assumption of unitarity has been performed on electroweak data from LEP, SLC, the Tevatron, and neutrino experiments. The result mostly comes from two-loop contributions to $\Gamma(Z \rightarrow b\bar{b})$ and yields,$^{119}$

$$|V_{tb}| = 0.77^{+0.18}_{-0.24} \quad (39)$$

At hadron machines, the top quark is mainly produced in top-antitop pairs via strong interactions. However, the pure electroweak production of a single top (or anti-top) quark has a remarkably competitive cross-section. A test of $|V_{tb}|$ can be made from the measurement of $R = B(t \rightarrow Wb)/B(t \rightarrow Wq)$ where $B(t \rightarrow Wq)$ is the branching fraction of the top quark to a W boson and a quark ($q = b$, $s$, $d$). This quantity has been measured at the Tevatron. The latest results, from the DØ collaboration,$^{120}$ are $R = 0.90 \pm 0.04$ (stat.+syst.), which agrees within approximately 2.5 standard deviations with the SM prediction of $R$ close to one. A simultaneous measurement of $R = 0.94 \pm 0.09$ and $\sigma_{\bar{t}t}$ has been recently performed by CDF; they found,$^{121}$

$$|V_{tb}| = 0.97 \pm 0.05 \quad (40)$$

The single top quark production cross section is directly proportional to the square of $|V_{tb}|$, allowing a direct measurement of $|V_{tb}|$ without assuming unitarity of the CKM matrix or three fermion generations. The top quark was discovered at Tevatron in 1995, but, due to the very low signal-over-background, it was possible to see single top at the Tevatron only in 2009,$^{122,123}$ 14 years later. Instead, ATLAS and CMS, thanks to the much larger cross sections and better signal-over-background
available at the LHC, observed single top already in 2011 and measured its cross section the year after. The current single top cross section measurements, (including the most recent one by CMS at 8 TeV) have uncertainties at the level of 10%, too large to challenge the SM. The only exception is the CMS 7 TeV measurement with uncertainty of $\sim 5\%$, which yields

$$|V_{tb}| = 1.02 \pm 0.05 \pm 0.02$$

(41)

Due to the large LHC statistics, these measurements are (mostly) systematics limited. Dedicated strategies need to be developed to increase precision and usefully employ this process in the NP search. A deviation from the SM prediction in Eq. 58 could arise from NP contributions that violate unitarity. Possibly the simplest way to violate unitarity is enlarging the fermion sector, by including a fourth quark generation or vector-like quarks, that appear in many models, Randall-Sundrum or E6 GUTs amongst others (see, e.g.126,127,128,129,130).

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