Self-similarity of dynamical structure factors for fractional quantum Hall states with long-range interactions

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Dynamical structure factors are central quantities in the analysis of quantum many-body states, since they describe the response of many-body systems to external perturbations and hence directly correspond to observables in experiments. In this paper, we evaluate a momentum-averaged dynamical density structure factor for the fermionic $\nu = 1/3$ fractional quantum Hall state on a torus, using the continued fraction method to compute the dynamical correlation function. We highlight the discrepancy between the structure factors corresponding to the short-range $V_1$ and long-range Coulomb interactions, despite both models yielding ground states in the same universality class. Motivated by this, we establish a scaling relation for the screened Coulomb structure factor with respect to interaction range, and expose an inherent self-similarity of structure factors for long-range interactions in the frequency domain. Finally, we demonstrate that Haldane pseudopotentials may be used to approximate the structure factor for long-range interactions on the torus, provided that the interaction is sufficiently screened. These results highlight the self-similar properties of structure factors for fractional quantum Hall states with long-range interactions and show how they can be efficiently approximated in numerical models.

I. INTRODUCTION

One of the key observables yielding insights into interacting quantum systems is the dynamical structure factor $S(q, \omega)$, which captures the complete momentum- and energy-resolved spectrum of particle excitations. Apart from its central role in the dynamics of quantum many-body systems, the structure factor has a number of appealing properties that continue to stimulate a broad range of research. We focus on its applications to fractional quantum Hall (FQH) systems, which have been known for several decades to host a rich spectrum of collective modes,$^{1-4}$ and have in recent years been extended to both lattice models$^{5,6}$ and effective field theories.$^{7,8}$ Since the structure factor is directly related to the correlation function, it can be computed in a variety of ways, such as via Feynmann diagram resummation$^3$ or continued fractions.$^9$ From an experimental standpoint, the structure factor can be directly probed in a two-dimensional electron gas. For example, surface acoustic waves have been used to measure the linear response of FQH systems in order to analyze the propagation of collective excitations, such as rotons$^{10}$ and magnons.$^{11}$ More recently, Raman scattering has also been proposed as a way to reveal additional spin properties of the response function.$^{12}$ Despite its rich structure and experimental applicability, however, numerical studies that systematically investigate the structure factor of FQH states are comparatively rare.

In this paper, we study a type of dynamical density structure factor$^{13}$ for the $\nu = 1/3$ fermionic Laughlin state on a torus, using the continued fraction method to compute the dynamical correlation function. In particular, two aspects of the structure factor are investigated: (i) the effect of interaction range and (ii) self-similarity. We highlight the discrepancy between the structure factors for the short-range $V_1$ and long-range Coulomb interactions. Building on this, we tune between these interactions to reveal a scaling relation with respect to screening. Motivated by the fractality of continued fraction Green’s functions,$^{14,15}$ we then study the self-similarity of the structure factors for long-range interactions in the frequency domain, across several orders of magnitude. Finally, we evaluate the effectiveness of Haldane pseudopotentials on the torus. In all cases, we present systematic exact diagonalization computations, which we scale with system size. Our results emphasize that interaction range can have a significant impact on the structure factor, even for Hamiltonians with ground states in the same universality class. We reveal that FQH dynamical structure factors are self-similar with respect to interaction range, and in the frequency domain for long-range interactions. We also point out that, under certain conditions, Haldane pseudopotentials may be used to approximate the Coulomb structure factor on the torus. Apart from providing a deeper insight into the spectral properties of FQH systems, these results may be exploited to compute response functions more efficiently.

The structure of the paper is as follows. In Sec. II we define the FQH system under examination and in Sec. III we describe the method for computing the structure factors and particle interactions. Subsequently, in Sec. IV we present our exact diagonalization results. In Sec. IV A, we compute the structure factors for the $V_1$ interaction, as well as the Coulomb interaction, defined explicitly through its Fourier transform. In Sec. IV B, we...
tune between these two interactions and study the effect of screening. In Sec. IV C, we examine the self-similarity of the Coulomb structure factor as the frequency domain is rescaled, and in Sec. IV D we compute the structure factor using Haldane pseudopotentials. Finally, in Sec. V we discuss the relevance of our results with respect to future numerical investigations, as well as recent experiments.

II. MODEL

We consider a two-dimensional system of $N$ spin-polarized fermions of mass $m$ and charge $q$ in a perpendicular magnetic field $B$ on the $xy$-plane with periodic boundary conditions. Building on earlier work, the torus geometry has recently experienced a revival of interest, particularly regarding numerical advances and model wavefunctions, and effective theories which motivate our choice. We consider the Landau gauge such that the momentum $k_\parallel$ is a good quantum number. The energy spectrum of this FQH set-up is split into Landau levels, the lowest of which we fill up to a filling factor $\nu = N/N_\phi$, where $N_\phi$ is the number of flux quanta in the system. Moreover, we focus on the regime where the interaction is weak compared to the Landau level spacing (given by the cyclotron frequency $\omega_c = qB/m$). Hence, to a good approximation, we may project the interaction Hamiltonian to the lowest Landau level (LLL), such that

$$H = H_{\text{kin}} + \sum_{i<j}^N P_{\text{LLL}}V(|r_i - r_j|)P_{\text{LLL}}, \quad (1)$$

where $H_{\text{kin}}$ is the kinetic part of the Hamiltonian, $P_{\text{LLL}}$ is the LLL projection operator, $V$ is the interaction potential, and $r_i$ is the displacement of particle $i$. The relevant length scale in the problem is the magnetic length $l_B = 1/\sqrt{qB}$.

In this paper, we start by considering the Coulomb $V^C(r) \sim r^{-1}$ and Yukawa $V^Y(r) \sim r^{-1}e^{-\lambda r}$ interactions explicitly by diagonalizing the Hamiltonian directly in Fourier space. However, since it is not always possible or desirable to directly account for a long-range interaction in this way, we complement our analysis by using a Haldane pseudopotential formalism. Haldane showed that for systems with a translation and rotation invariant two-body interaction, the interaction Hamiltonian may be written as

$$H_{\text{int}} = \sum_{i<j}^N \sum_L V_L P_{ij}^L, \quad (2)$$

where $V_L$ are the Haldane pseudopotentials, $L$ is the relative angular momentum quantum number between particles $i$ and $j$, and $P_{ij}^L$ is the corresponding projection operator. This simplifies a certain class of long-range interactions into a simple sum of projectors, which has found a diverse set of applications from accelerating early numerical computations on the sphere through to modeling realistic semiconductor heterojunctions. The derivation of the Haldane pseudopotentials for the Coulomb and Yukawa interactions is discussed in Appendix A and the convergence of the Haldane pseudopotentials on the torus is presented in Appendix B.

Throughout our study, we focus on the primary Laughlin state defined at the filling factor $\nu = 1/3$. Laughlin famously proposed a wavefunction ansatz for the ground state of a FQH system with particles interacting via the Coulomb potential in a $1/m$-filled LLL, where $m$ is an odd integer. Although the Laughlin ansatz is a successful description of the problem since it is in the correct universality class, it is not the exact ground state for the Coulomb interaction. Rather, it was later shown to be the unique, highest-density, zero energy state for the $V_1$ Haldane pseudopotential. In this paper, we investigate the $\nu = 1/3$ state in both limits. When we discuss the "Laughlin state", we refer to the general ground-state solution to a FQH system with a $1/m$-filled LLL and not the Laughlin ansatz wavefunction in particular.

III. METHOD

In order to efficiently find the eigenspectrum of the many-body Hamiltonian in Eq. 1, we employ the Lanczos algorithm. This method works by using an orthogonal Krylov basis in which the original Hamiltonian $H$ is transcribed to a tridiagonal form $\tilde{H}$, defined such that

$$H \left| \Psi_i \right\rangle = E_i \left| \Psi_i \right\rangle \text{ with } i = 0, \ldots, D - 1, \quad (3)$$

$$\tilde{H} \left| \tilde{\Psi}_j \right\rangle = \tilde{E}_j \left| \tilde{\Psi}_j \right\rangle \text{ with } j = 0, \ldots, L - 1. \quad (4)$$

Convergence in the Krylov space is rapid, since many degrees of freedom are simultaneously used in the optimization, and memory efficient, since only two vectors of length $N$ need to be stored. The algorithm is consequently a widely-used approach in large-scale exact diagonalization computations and has been optimized to diagonalize sparse matrices as large as $\text{dim}(H) \sim 10^9$. The Lanczos algorithm was later extended by Haydock et al. and applied to compute observables in physical systems with a large number of particles. In particular, Haydock showed that the resolvent of the Hamiltonian can be efficiently computed using a continued fraction expansion, which is useful for calculating local quantities, such as the single-particle density matrix and the density of states. Crucially, when the sparse Hamiltonian $H$ is written as a tridiagonal Hamiltonian $\tilde{H}$ in the Krylov basis, the problem is effectively reduced to a chain of length $L$, which accelerates the computation.

For our system, we work in momentum space and consider the zero-temperature dynamical correlation function for the operator $O_\mathbf{q}$ in the Lehmann representation,
which using the Krylov basis may be approximated as
\[
\tilde{G}_O(q, \omega + i\epsilon) = \sum_{j=0}^{L-1} \frac{|\langle \Psi_0|O_q^\dagger \Psi_j \rangle|^2}{E_0 + \omega + i\epsilon - E_j}
\]
\[= \langle \Psi_0|O_q^\dagger \frac{1}{E_0 + \omega + i\epsilon - \hat{H}} O_q |\Psi_0 \rangle, \tag{5}
\]
where \(q = (q_x, q_y)\) are the Fourier components of the \(O\) operator, \(\omega\) is the frequency, and \(\epsilon\) is a small parameter used to avoid poles in the expansion. From this formula, it is straightforward to show that for the symmetric tridiagonal Hamiltonian
\[
\hat{H} = \begin{pmatrix}
a_0 & b_1 & 0 & 0 & \cdots \\
b_1 & a_1 & b_2 & 0 & \cdots \\
0 & b_2 & a_2 & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & b_{L-1} & a_{L-1}
\end{pmatrix}, \tag{7}
\]
the correlation function may be written as a continued fraction expansion
\[
\tilde{G}_O(q, z) = \frac{\langle \Psi_0|O_q^\dagger O_q |\Psi_0 \rangle}{z - a_0 - \frac{b_1^2}{z - a_1 - \frac{b_2^2}{\ldots}}}, \tag{8}
\]
which terminates at \(-b_{L-1}^2/(z - a_{L-1})\), where \(z = E_0 + \omega + i\epsilon\). This form of the correlation function converges rapidly to machine precision.34

Specifically, we are interested in studying the density-density correlation functions arising from the density operator
\[
\rho_q = \int dr \, e^{iqr} c^\dagger(r)c(r), \tag{9}
\]
where \(r \equiv (x, y)\) is the position operator conjugate to \(q\). Given our choice of Landau gauge with definite momentum \(k_y\), we are particularly interested in resolving the \(q_y\) Fourier components of the density operator. We therefore choose to integrate out the \(q_x\) modes on the torus to avoid an additional free variable and consider the \(q_y\)-momentum-averaged density operator, setting
\[
O_{q_y} \equiv \bar{\rho}_{q_y} = \sum_{m=0}^{N_y-1} \rho_{q_y= \frac{2\pi m}{L_y}} q_y, \tag{10}
\]
where \(L_x \times L_y\) are the system dimensions. We have separately verified, by evaluating at specific \(q_x\) values, that the density operator is only weakly dependent on \(q_x\). The full derivation of the momentum-averaged density operator is presented in Appendix C. Finally, we may use this operator to compute the corresponding dynamical density structure factor
\[
\tilde{I}_{\bar{\rho}}(q_y, \omega) = -\frac{1}{\pi} \lim_{\epsilon \to 0} \text{Im} \tilde{G}_{\bar{\rho}}(q_y, \omega + i\epsilon). \tag{11}
\]

The crucial property of the continued fraction expansion is that the structure factor in the Krylov representation \(\tilde{I}\) accurately reproduces the moments of the structure factor in the Hilbert representation \(I\), and so we now drop the check marks.9

In this paper, we compute the momentum-averaged dynamical density structure factor \(I_{\rho}\) for the \(\nu = 1/3\) Laughlin state stabilized by a variety of interactions. We start by computing the structure factor for the Laughlin state stabilized by a \(V_1\) Haldane pseudopotential and by an explicit \(V^C(y)\) Coulomb interaction, as well as the (loosely written) linear combination of the two: \((1 - \alpha)V_1 + \alpha V^C\), with a tuning parameter \(\alpha \in [0, 1]\). We then proceed to investigate the effect of interaction range by considering the generalized linear combination \((1 - \alpha)V_1 + \alpha V^C\), with a tuning parameter \(\beta > 0\) an odd integer. The tuning parameter \(\alpha\) allows us to interpolate between two common ground-state solutions in the same universality class; the truncation parameter \(\beta\) allows us to comment on the influence of interactions between particles with increasing relative angular momentum; and the Yukawa mass \(\lambda\) enables us to vary the interaction range and recover the \(V_1\) and Coulomb limits.

### IV. RESULTS

In this section we present our exact diagonalization results. In Sec. IV A, we highlight the discrepancy between the momentum-averaged dynamical density structure factors for the Laughlin state due to the \(V_1\) and Coulomb interactions, and, motivated by this, in Sec. IV B we tune from the \(V_1\) to the screened Coulomb interaction. In Sec. IV C, we expose a self-similarity of structure factors for long-range interactions in the frequency domain, and in Sec. IV D, we study the convergence of the structure factor using truncated Haldane pseudopotentials.

#### A. Structure factors for the \(V_1\) and Coulomb interactions

To begin our investigation, we compute the structure factors corresponding to the two most common approaches for stabilizing the \(\nu = 1/3\) Laughlin state, via the \(V_1\) and Coulomb interactions, shown in Fig. 1. We present our initial results for a 6-particle system in all \(q_y\) momentum sectors. Here, we choose a small system size to simply illustrate the discrepancy and interpolation between the two interaction Hamiltonians.

In Fig. 1(a), we show the single-peak form of the structure factors corresponding to the \(V_1\) interaction. We omit any numerical artifacts that appear in the limit of short-range interactions, as discussed in Appendix D. The height and width of the peak scales with our small
FIG. 1. Momentum-averaged dynamical density structure factors. Structure factors $I_p$ as a function of angular frequency $\omega$ for the $\nu = 1/3$ FQH state on a torus, with $N = 6$ particles and $N_{\Phi} = 18$ flux quanta, stabilized by the (a) $V_1$ pseudopotential and (b) Coulomb interaction in the LLL. The spectra are additionally resolved with respect to their $q_y$ momentum sector. The computations were performed with a resolution of $\Delta \omega = 10^{-5}$, $\Delta I = 10^{-5}$, and $\epsilon = 10^{-4}$.

parameter $\epsilon$ such that the area under the curve is constant. Hence, in the limit of $\epsilon \to 0$ the structure factor corresponds to the Dirac delta function, which holds for all $q_y$. When an electron is removed from the system, the final states generated are spanned by all states of three quasiholes, and for the $V_1$ Hamiltonian these states are degenerate, since the quasiholes do not interact. Consequently, there are no spectral peaks at finite $\omega$ or gaps in the single-particle spectral function. Surprisingly, we see this phenomenon also with the momentum-averaged structure factor for the $V_1$ Hamiltonian. As a caveat, we note that in the context of chiral gravitons in the FQHE, it was previously observed that a single peak could gain structure as the system size is increased. However, we see a single peak for all of the system sizes that we considered ($N < 10$).

In Fig. 1(b) we show the form of the structure factor corresponding to the Coulomb interaction. In contrast to the $V_1$ example, the Coulomb interaction shown here is computed by exactly diagonalizing the Hamiltonian (Eq. 1) in Fourier space and not via a Haldane pseudopotential formalism (Eq. 2). Again, the many-body energy spectrum has a three-fold degenerate ground state, which corresponds to the $\nu = 1/3$ FQH state on a torus. However, on this occasion, the inclusion of the Coulomb interaction (or indeed any higher pseudopotentials) breaks the degeneracy of quasihole states that applies only to the $V_1$ Hamiltonian. Consistently with this, we observe a broad spectrum in the momentum-averaged structure factor. Numerically, there may be a slight variation of the heights of the peaks, up to $\sim 1\%$ per peak, due to rounding error but the shape of the envelope holds for all runs and for all $q_y$. The results for the $V_1$ and Coulomb interactions not only allow us to benchmark our algorithm but they also emphasize the stark contrast in the structure factors for two $\nu = 1/3$ FQH ground states that are in the same universality class.

B. Tuning from short- to long-range interactions

Motivated by the discrepancy between the structure factors for the $V_1$ and Coulomb interactions, we progress by studying a family of systems interpolating between
these cases such that \( H_{\text{int}} = (1 - \alpha) H_{V_1} + \alpha H_{V_{\infty}} \), where \( H_{V_1} \) and \( H_{V_{\infty}} \) are the interaction Hamiltonians corresponding to the \( V_1 \) and \( V_{\infty} \) interactions, respectively, and \( \alpha \in [0, 1] \). In Fig. 2, we present the evolution of the structure factor as a function of the tuning parameter \( \alpha \).37 For this analysis, we stress that the “self-similarity” with respect to \( \alpha \) is a trivial scaling of the Coulomb interaction, and the transition point between a single- and multi-peak spectrum is simply an artifact of our simulation resolution. We investigate this scaling primarily to introduce our methodology and benchmark the effect of numerical noise. In Fig. 2(a) we identify the point at which the single-peak structure is decomposed into multiple peaks for the simulation parameters in Fig. 1, which occurs at \( 2 < \alpha / 10^{-4} < 3 \). Moreover, in Fig. 2(b) we plot the complete evolution from \( \alpha = 10^{-4} \) to the pure Coulomb interaction at \( \alpha = 1 \), which highlights the reduction in height and increase in spread of the distribution. By increasing the \( \omega \) and \( \epsilon \) resolution by two orders of magnitude, we reveal the underlying “self-similar” scaling of the structure factor in Fig. 2(c). This plot shows that the structure of the response function is similar across at least four orders of magnitude for \( \alpha \). To calibrate our numerics, we quantify this scaling by computing the self-similarity parameter \( \kappa = \log M_y / \log M_z \) in Fig. 2(d), where \( M_i \) is the magnification factor of the \( i \) axis. Specifically, we use the range of the set of angular frequencies that coincide with maxima in the structure factor, \( \Omega \), as a quantifier of spread. The line of best fit has an \( R^2 \approx 1 \), which shows that the scaling is self-similar, and the self-similarity parameter is \( \kappa \approx 1 \). Further details on the analysis of self-similar data series are given in Appendix E. Despite this conclusive consistency check, it can be seen from the data in Fig. 2(c) that the distributions are not self-identical. To quantify the effect of numerical imprecision, we plot the number, and average value, of peaks \( I_{\text{max}} \) as a function of \( \alpha \) in Figs. 2(e) and 2(f), respectively. This analysis shows that there is a significant influence of numerical noise, particularly at small \( \alpha < 10^{-3} \). Over the entire \(-4 < \log \alpha < 0\) range, there is a 12% standard deviation in number of peaks and a 13% standard deviation in their mean values. As previously mentioned, the “self-similar” scaling is to be expected since we are effectively multiplying the Coulomb interaction by a scale factor modulo a change to the \( V_1 \) term. Nevertheless, this analysis illustrates the connection between the disparate structure factors in Fig. 1 and exposes the effect of numerical noise.

Although tuning between the \( V_1 \) and Coulomb interactions succinctly links the structure factors from two common types of interaction used to stabilize the Laughlin state, it cannot be used to directly comment on the effect of interaction range. This is because even a small component of \( H_{V_{\infty}} \) added to the interaction Hamiltonian is an inherently long-range contribution. Consequently, in order to isolate the effect of interaction range, we proceed by tuning from the \( V_1 \) to the generalized screened Coulomb potential, known as the Yukawa interaction, such that \( H_{\text{int}} = (1 - \alpha) H_{V_1} + \alpha H_{V_\infty} \). As before, the Yukawa interaction is incorporated explicitly by diagonalizing \( H_{V_\infty} \) in Fourier space. In this case, however, the Yukawa mass \( \lambda \) may be varied to tune the interaction range. In particular, the limit of \( \lambda \rightarrow \infty \) corresponds to the \( V_1 \) pseudopotential and the limit of \( \lambda = 0 \) reproduces the Coulomb interaction, for all \( \alpha \neq 0 \). The scaling of the structure factor from the \( V_1 \) to the Yukawa interaction is presented in Fig. 3. From Fig. 3(a), we can immediately see that the scaling is trivially self-similar as a function of \( \alpha \), for all \( \lambda \), with respect to quantifiers for both the center and spread of the distribution.38 We note that \( \alpha < 10^{-3} \) values for \( \lambda = 10 \) and \( \alpha < 10^{-2} \) values for \( \lambda = 100 \) are omitted, since the computational requirements exceeded standard machine precision. From the figure, we observe that the average of the distribution is unaffected by increases in \( \lambda \) up to \( \lambda \sim 10^{-1} \), on the scale of the plot, after which the center of the distribution is seen to exponentially approach zero. This is to be expected since in the limit of \( \lambda \rightarrow \infty \) we tend to the Dirac delta function in Fig. 1(a). Analogously, we observe that the spread of the distribution responds to \( \lambda \) in the same way, which accords with a narrowing of the spectrum as we approach the \( V_1 \) limit. We note also that the curve for small \( \lambda \approx 10^{-4} \) in the top panel of Fig. 3(a) is indistinguishable from the Coulomb curve in Fig. 2(d), on the scale of the plot. Comparing top and bottom panels, we find that range(\( \Omega \)) is more stable to changes in \( \alpha \).
than \( \tilde{\Omega} \), owing to the fact that \( \tilde{\Omega} \) involves more points in the sample. Moreover, numerical noise at smaller values of \( \alpha \) and larger values of \( \lambda \) is more likely due to the increased rounding error for the extremely small numbers involved. Crucially, the data in Fig. 3(b) explicitly show that the continuous connection between \( \alpha \) and \( \lambda \) translates to a non-trivial self-similar scaling with interaction range, albeit with two different regimes. As identified in Fig. 3(a): for \( \lambda < 10^{-1} \), the structure factor is unaffected by changes in \( \lambda \); for \( \lambda > 0 \), the mean and spread of the structure factor exponentially converge to zero; and for \( 10^{-1} \leq \lambda \leq 0 \), there is a smooth transition region. A finite-size scaling of the \( \alpha = 1 \) curve from Fig. 3(b) is shown in Fig. 3(c). Here we can verify the convergence of the distribution center with system size and identify the transition point to be at \( \lambda \approx 1 \).

C. Self-similarity of the structure factor

In the previous section, we demonstrated that the structure factor is trivially self-similar with respect to a linear interpolation between the \( V_t \) and Coulomb interactions, as well as non-trivially self-similar with respect to interaction range. In both cases, this self-similarity is the result of tuning parameters; namely, \( \alpha \) and \( \lambda \). In this section, we investigate a form of self-similarity with respect to the frequency domain, which cannot be explicitly linked to a tuning parameter.

In Fig. 4, we investigate the distribution of the peaks in the Coulomb structure factor from Fig. 1(b) as we scale the frequency domain. We start with an initial interval of \( \omega_0 \in [0, 5] \), which we symmetrically shrink relative to the mid-point \( \omega_{\text{mid}} = 2.5 \). Note that the choice of initial domain and the way in which we shrink the interval is arbitrary. In order to keep the scaling numerically consistent, we correspondingly scale the frequency resolution, \( \Delta \omega \), and the \( \epsilon \) value in our simulations. In Fig. 4(a), we plot the first two moments of the distribution against the domain scale factor to compute the self-similarity parameters,\(^{39,40}\) as introduced in the previous section. We additionally scale this with system size up to \( N = 9 \) particles. We find that there is a contiguous linear region, which grows with system size, and has a correlation coefficient of \( R^2 > 0.99 \) for both \( \mu_{\text{max}} \) and \( \sigma_{\text{max}} \). This shows that there is a self-similarity of the Coulomb structure factor with respect to frequency domain scaling. In fact, the self-similarity parameters for the mean and standard deviation are \( m_\mu = -1.08 \pm 0.0096 \) and \( m_\sigma = -1.07 \pm 0.00917 \), respectively. As mentioned in Sec. IV A, since a reduction of \( \epsilon \) increases the heights of the peaks, it is consistent that the self-similarity parameters \( m \sim -1 \) and that the \( I\epsilon \) spectrum is approximately self-identical. After halving the frequency domain more than four times, the self-similarity relation breaks down for particle numbers \( N \leq 8 \), which we proceed to examine in the following figures. In Fig. 4(b), we compare the magnitude of frequency interval reduction, \( |\Delta \tilde{\omega}_i - \Delta \tilde{\omega}_{i-1}| \), with the average spacing between the peaks, \( \langle \Delta \Omega \rangle \). Since the frequency domain scaling analysis above relies on the assumption that we exclude a finite section of the structure factor on each iteration, examining the relative spacing between the peaks provides one way to quantify the validity of our procedure. We can see that, in the self-similar region, the average spacing between the peaks is consistently smaller than the size of the frequency interval being removed, as expected. For certain system sizes, however, the average spacing between the peaks increases, particularly for small frequency windows. After halving the frequency domain

![Fig. 4](image-url)
more than four times, the average spacing between the peaks for \( N = 8 \), for example, is of the order of the frequency interval reduction, which could potentially lead to a breakdown of the rescaling procedure. In Fig. 4(c), we examine the number of peaks in the structure factor with frequency domain rescaling. Since the moments of a distribution are sensitive to the sample size, analyzing the number of \( I_{\text{max}} \) points provides a further method to quantify the validity of our self-similarity relation. We find that the number of peaks in the structure factor decreases exponentially with frequency domain reduction. In fact, for \( N = 8 \) the number of peaks in the spectrum is \( n(I_{\text{max}}) \sim 1 \) below its breakdown point, which explains the transition.\(^{41} \) For the \( N = 6, 7 \) system sizes, the numerous peaks in the structure factor coupled with their small average spacing, results in a non-physical discontinuity in the linear scaling across the breakdown in Fig. 4(a). Finally, in Fig. 4(d), we analyze the average value of \( I_{\text{max}} \) as we shrink the frequency domain. Since the height of the peaks increases with decreasing \( \epsilon \), one may naively expect that \( \mu_I_{\text{max}}/\epsilon \) is constant in the linear region. Although this statement approximately holds here with respect to range(\( I_{\text{max}}/\epsilon \)) \sim 0.1, the trend deviates in proportion to the domain reduction to peak spacing ratio in Fig. 4(b), as well as the number of peaks in Fig. 4(c). We note that the trend holds most closely for the largest system size of \( N = 9 \).

Following on from the self-similarity of the structure factor for the Coulomb interaction with respect to \( \omega \), it is natural to ask how this is affected by interaction range. As in Sec. IV B, we employ the Yukawa interaction and tune the Yukawa mass \( \lambda \) to probe the role of the interaction range in our system. In Fig. 5, we present the self-similarity of the structure factor with respect to \( \omega \) for the Yukawa interaction with \( N = 6, 7, 8 \), analogously to Fig. 3. We study both the center and spread of the distribution in our analysis, which in this case are quantified by the mean and standard deviation. For the small values of \( \lambda \), we extend the curves in the breakdown region with dashed lines, whereas for larger values of \( \lambda \) we truncate the data before they diverge, for readability. In Fig. 5(a), we see that the self-similarity breaks down for all frequency intervals \( \log_2(\text{range}(\omega)/\text{range}(\omega_0)) < -4 \) for \( \lambda = 10^{-4}, 10^{-3}, 10^{-2} \), similarly to the Coulomb interaction in Fig. 4(a). At larger values of \( \lambda \), the self-similarity breaks down sooner, for example at \( \log_2(\text{range}(\omega)/\text{range}(\omega_0)) = -3 \) for \( \lambda = 10^{-1} \). We notice this behavior in both the center and spread of the distribution, although the spread is more sensitive to changes in frequency interval. In Fig. 5(b), we replot the data on different axes so that we can clearly illustrate the role of \( \lambda \). From the figure, we find that the small \( \lambda \) region of the curves is approximately constant, which shows that there is little deviation from the self-similar scaling, whereas as \( \lambda \) is increased, the curves deviate significantly from the straight line. For reference, we present the finite-size scaling of the \( \text{range}(\omega) = \text{range}(\omega_0) \) curve in Fig. 5(c). This gives an indication of the magnitude of finite-size effects in the data. Note that the dashed lines representing the breakdown region in Fig. 5(a) are also dashed in Fig. 5(b), however the colors between the subfigures do not correspond. In general, the larger the value of \( \lambda \) and hence shorter the interaction range, the smaller the region of self-similarity. Moreover, the self-similarity holds more precisely for longer-range interactions with smaller values of \( \lambda \).

**D. Tuning from lower- to higher-order Haldane pseudopotentials**

To complement our analysis of the Coulomb and Yukawa interactions using an explicit diagonalization in Fourier space, we proceed by studying the interactions using a Haldane pseudopotential description. The Haldane pseudopotential formalism is useful because most of the physics of interacting electrons in the LLL can be captured by its first few values, which greatly simplifies both analytical and computational complexity. Although originally applied to short-range interactions on the sphere, it has since been generalized to accommodate more diverse
interactions on a range of geometries. Nevertheless, it is important to note that, unlike for the sphere, pseudopotential algebra is not exact on the torus and so a convergence of the energy spectrum will only be reached for the first few energy levels with short-range interactions / large system sizes. The convergence of Haldane pseudopotentials on the torus is further discussed in Appendix B.

In order to quantify the effectiveness of the first few Haldane pseudopotentials in capturing the physics of the Laughlin state on the torus, we present the scal- [Fig. 6](image). Box plots showing the spread of the structure factor $I_\nu(0, \omega)$ for the $\nu = 1/3$ FQH state on a torus, with $N = 6$ particles and $N_\Phi = 18$ flux quanta, stabilized by the LLL Haldane pseudopotentials corresponding to the Yukawa interaction $\{V_{\lambda,L}^Y\} = \{V_{\lambda,0}^Y, V_{\lambda,1}^Y, ..., V_{\lambda,\beta}^Y\}$, as a function of truncation parameter $\beta \in [1,3,\ldots,99]$ at a variety of $\lambda$. The median is labeled with an orange line, the interquartile range (IQR) is drawn with a box and the whiskers extend to 1.5 times the IQR. All data points outside of this range are plotted as outliers. In (a) we additionally plot the asymptotic behavior of $1/V_{\lambda,1}^{C(0)}$ in green; in (d) we overlay the IQR and median for the Coulomb interaction from Fig. 1(b) in red; and in (f) we overlay the IQR and median for the $V_1$ interaction from Fig. 1(a) in blue. The computations were performed with a resolution of $\Delta \omega = 10^{-3}$, $\Delta I = 10^{-3}$, and $\epsilon = 10^{-3}$.

cal to consider only $\beta \ll N_\Phi/2$, since larger values of $\beta$ may reflect eigenstate mixing and boundary conditions, we show an extended range of $\beta \in [1,3,\ldots,99]$ to additionally comment on numerical effects. In Fig. 6(a), we show the structure factor scaling in the Coulomb limit $\lambda \to 0$. From the plot, it can be clearly seen that the Haldane pseudopotentials do not accurately reproduce the true Coulomb structure factor (Fig. 1(b)) for any value of $\beta$. Instead, the structure factor is inversely proportional to the exact values of the Haldane pseudopotentials in the asymptotic limit, plotted in green. This steady increase across all $\beta$ indicates that the structure factor does not converge for the long-range Coulomb interaction with higher-order pseudopotentials and that lower-order pseudopotentials are not sufficient to accurately capture the Coulomb interaction at this system size. As we increase the value of $\lambda$, and hence decrease the interaction range, we eventually reach a physical convergence at $\lambda = 1$, which corresponds to the Coulomb potential modulated by an exponential decay. In this case, the interaction is sufficiently short-range and, to a good approximation, reproduces the structure factor for the Coulomb interaction, overlaid in red. Further increasing the value of $\lambda$ and decreasing the interaction range, we take the $V_1$ limit: $\lambda \to \infty$. In this regime, the spread of the structure factor diminishes and its median approaches zero. In the large $\lambda$ limit, the inter-quartile range tends to that for the $V_1$ interaction, overlaid in blue in Fig. 6(f). These results stress that Haldane pseudopotentials can work well on the torus provided that the interactions are sufficiently short-range.

Having shown that Haldane pseudopotentials are unsuitable for modeling the Coulomb interaction on the torus at accessible system sizes, we now focus on the approximation obtained at $\lambda = 1$. In Fig. 7, we present a finite-size scaling of the structure factor from Fig. 6(d) for the particle numbers $N = 6, 7, 8$ and the physical values of the truncation parameter $\beta < (N_\Phi - 1)/2$. From Fig. 7(a), we find a positively-skewed distribution akin to Fig. 1(b), albeit with fewer peaks and a reduced spread. The smaller number of peaks is due to the truncated Haldane pseudopotential description and can be seen to increase with $\beta$, whereas the reduced spread is due to the modulation of the interaction potential by an exponential factor, and continues to diminish as $\lambda$ is increased, as shown in Fig. 6. These points notwithstanding, the structure factors presented here are a fair approximation of that for the long-range Coulomb potential in Fig. 1(b) and come at a significantly reduced numerical cost. In Fig. 7(b), we note that convergence is reached at $\beta = 3$, where the median is almost exactly reproduced. In fact, for the $N_\Phi = 18$ system size, the energy levels only converge to within a 1% error up to $\beta = 3$ for the two-particle system, and so it is reasonable to expect that this is the optimal value to reproduce the structure factor (as discussed in Appendix B). Similar results are found for the $N = 7$ and 8 system sizes in Figs. 7(c-f), which show that the close approximation to the Coulomb structure
In this paper, we studied numerically the momentum-averaged dynamical density structure factors \( I_\beta \) for the \( \nu = 1/3 \) Laughlin state on the torus, using the continued fraction method. In Sec. IV A, we emphasized the stark contrast between the structure factors corresponding to the \( V_1 \) and Coulomb interactions, despite the fact that they both yield ground states in the same universality class. In Sec. IV B, we verified the expansion of the structure factor as we tuned between the \( V_1 \) and Coulomb interactions, which exposed the effect of numerical noise in our simulations. Building on this, we demonstrated that the trivial self-similar expansion with respect to our tuning parameter translates to a non-trivial self-similar expansion with respect to interaction range. In Sec. IV C, we then exposed our central result, demonstrating an inherent self-similarity that cannot be explicitly linked to a tuning parameter: the self-similarity of the structure factor in the frequency domain. We found that this holds most precisely, and for the largest range of frequency values, for long-range interactions. Finally, in Sec. IV D, we analyzed the effectiveness of Haldane pseudopotentials in recovering the structure factors. We showed that, for the system sizes accessible to us, the Haldane pseudopotentials are generally not suitable for modeling the long-range Coulomb interaction on the torus. However, a reasonable approximation can be made to the Coulomb structure factor with an exponential modulation of the interaction potential.

Our results highlight the effect of interaction range on, and the self-similarity of, dynamical structure factors \( I_\beta \) for FQH systems. In terms of the Laughlin ansatz, which is the exact zero-energy ground state wavefunction for the \( V_1 \) interaction, it may be deduced analytically that the single-particle spectral function for the Laughlin Hamiltonian takes the form of a single peak. In contrast, the Coulomb interaction does not yield a degeneracy of quasi-hole states and so it is natural that the corresponding FQH Hamiltonian hosts multiple peaks in its dynamical structure factor. Nevertheless, our results emphasize that the interaction range can have a more general impact on the response functions of FQH systems. Furthermore, there have been a wealth of studies on the self-similarity, fractality, and chaos of continued fractions in a mathematical context and so it is reasonable to expect this to be reflected in observables derived from the Green’s function. In this paper, we have explicitly demonstrated the fine structure of the structure factor in the frequency domain, which stems from its continued fraction representation. This leaves open questions regarding further manifestations of self-similarity due to this recurrence relation, as well as the implications of other continued fractions in physics, such as the Haldane hierarchy and the Lanczos algorithm in general. Aside from these physical insights, our results also offer two numerical methods to efficiently approximate the Coulomb structure factor. The first method is to compute the structure factor...
by diagonalizing the Yukawa interaction Hamiltonian in Fourier space. At large $\lambda$, this is a short-range interaction that is efficient to implement, and, provided the simulation resolution is high enough, this result can be smoothly tuned to the bare Coulomb limit. The second method is to use the Haldane pseudopotentials for the Coulomb interaction modulated by an exponential decay. This modulation yields a sufficiently short-range interaction relative to the system size that allows the first few pseudopotentials to reproduce the Coulomb structure factor to a reasonable approximation.

There are several ways in which this work could be extended in the future. First and foremost, in this paper we have demonstrated scaling relations and self-similarity for Hamiltonians with ground states in the same universality class at $\nu = 1/3$ filling. It would be natural to extend this to FQH ground states for other filling factors and, in particular, states that do not share a universality class and inherently require a long-range interaction to be stabilized.\textsuperscript{43,44} Second, it would be useful to analyze the trade-off between interaction range / frequency window and simulation resolution using this approach to find the optimal efficiency benefit for a series of FQH states. Finally, there is strong motivation to apply this method to compute the full density-density response function to the optimal efficiency benefit for a series of FQH states.

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\textbf{Appendix A: Derivation of the Haldane pseudopotentials}

The derivation in this section is performed on a plane and in atomic units, where the Coulomb constant $k_e = (4\pi\epsilon)^{-1}$, the electronic charge $e = 1$, and the magnetic length $l_B = (2\pi N_p)^{-1/2} = 1$.

The Haldane pseudopotentials\textsuperscript{35} in the $n$-th Landau level, $V_{m}^{(n)} \equiv \langle n, m | V | n, m \rangle$, may be written in momentum space as
\begin{equation}
V_{m}^{(n)} = \int \frac{dq}{2\pi} V(q) \langle n, m | e^{iq \cdot r} | n, m \rangle,
\end{equation}
where $m$ is the pseudopotential index and $\mathbf{r}$, $\mathbf{q}$ are the position and momentum vectors, respectively.

Using the fact that the FQHE in the $n$-th Landau level with interaction potential $V(q)$ is equivalent to the FQHE in the 0-th Landau level with interaction potential $[L_n(q^2/2)]^2 V(q)$, coupled with the result $\langle m | e^{iq \cdot r} | m \rangle = e^{-q^2} L_m(q^2)$, allows us to write
\begin{equation}
V_{m}^{(n)} = \int \frac{dq}{2\pi} V(q) [L_n(q^2/2)]^2 L_m(q^2) e^{-q^2},
\end{equation}
where $L_n$ is the $n$-th Laguerre polynomial.\textsuperscript{45,46}

Finally, converting the momentum integral to polar coordinates yields
\begin{equation}
V_{m}^{(n)} = \int_0^\infty q dq V(q) [L_n(q^2/2)]^2 L_m(q^2) e^{-q^2}.
\end{equation}

\textbf{1. Coulomb interaction}

The Coulomb interaction, $V^C(r) = r^{-1}$, may be written in momentum space as
\begin{equation}
V^C(q) = \int \frac{dr}{2\pi} V(r) e^{-iq \cdot r}.
\end{equation}

Converting the momentum integral to polar coordinates, coupled with the fact that $e^{-iq \cdot r} = \cos(q \cdot r)$ since the Coulomb interaction is a real and even function of $r$, allows us to write
\begin{equation}
V^C(q) = \int_0^\infty rdr V(r) \int_0^{2\pi} \frac{d\theta}{2\pi} \cos(qr \cos \theta).
\end{equation}

Since the angular integral is a standard Bessel integral, this expression reduces to
\begin{equation}
V^C(q) = \int_0^\infty rdr V(r) J_0(qr) = \frac{1}{q},
\end{equation}
where $J_n$ is the $n$-th Bessel function of the first kind.

Finally, inserting this result into the expression for the Haldane pseudopotentials, Eq. A3, yields
\begin{equation}
V_{m}^{C,(n)} = \int_0^\infty dq [L_n(q^2/2)]^2 L_m(q^2) e^{-q^2}.
\end{equation}

In the LLL ($n = 0$), this integral can be evaluated analytically and written in closed form, such that
\begin{equation}
V_{m}^{C,(0)} = \sqrt{\frac{\pi}{2}} 2F_1(1/2, -m; 1; 1) = \sqrt{\frac{\pi}{2}} \frac{(2m-1)!}{2^{-m} m!},
\end{equation}
where $2F_1$ is the Gauss hypergeometric function. The asymptotic scaling as $m \rightarrow \infty$ is given as
\begin{equation}
V_{m}^{C,(0)} \sim 2^{-5/4 + \cos(2m\pi)/4} \frac{\pi^2 m^{2}(m\pi/2)^{2}}{\sqrt{m}}.
\end{equation}
2. Yukawa interaction

Similarly, the Yukawa interaction, $V_Y^\lambda(r) = r^{-1}e^{-\lambda r}$, may be written in momentum space as

$$V_Y^\lambda(q) = \int_0^\infty r dr V_\lambda(r) J_0(qr) = \frac{1}{\sqrt{\lambda^2 + q^2}},$$

(A10)

where $\lambda$ is the Yukawa scaling constant.

Inserting this result into the expression for the Haldane pseudopotentials, Eq. A3, yields

$$V_{Y,\lambda,m}^{(n)} = \int_0^\infty dq q \frac{1}{\sqrt{\lambda^2 + q^2}} [L_n(q^2/2)]^2 L_m(q^2)e^{-q^2}.$$

(A11)

In the LLL, this expression simplifies to

$$V_{Y,\lambda,m}^{(0)} = \frac{\lambda}{2\sqrt{\pi}} \Gamma(m + 1/2) U(m + 1, 3/2, \lambda^2),$$

(A12)

where $\Gamma$ is the gamma function and $U$ is the confluent hypergeometric function of the second kind. The asymptotic scaling as $m \to \infty$ is given as

$$V_{Y,\lambda,m}^{(0)} \sim \frac{\lambda}{\sqrt{2}} e^{-m/2} U(m + 1, 3/2, \lambda^2).$$

(A13)

Plots of the Haldane pseudopotentials for both the Coulomb and Yukawa interactions are shown in Fig. 8.

Appendix B: Convergence of two-particle energy spectra

One of the fundamental properties of Haldane pseudopotentials is their correspondence with the two-particle energy spectrum. On a sphere, it is straightforward to verify that the energy levels are completely equivalent to the Haldane pseudopotential coefficients. On a torus, however, the pseudopotential algebra is not exact and so different eigenstates can mix, as well as potentially be affected by boundary conditions. In light of this, we examine the convergence of the energy levels in a two-particle system relative to the pseudopotential coefficients.

We consider a system on a torus with two particles in the LLL, with a size defined by the number of flux quanta $l$. Subsequently, we compute the energy spectrum of this system for a particular Haldane pseudopotential $V_i$, for the two-particle energy spectrum on a torus in the LLL, as a function of the number of flux quanta $l$. The mean $\bar{E}_i$ and standard deviation $\sigma_i$ of $\{E_{i,j}\}$ are plotted and the $\pm 0.01 V_i$ region is shaded blue. The first value of $l$ for which $\bar{E}_i \pm \sigma_i$ is within 1% of $V_i(l_{\text{crit}})$ is marked with a red dashed line. (j) Scaling of $l_{\text{crit}}$ as a function of Haldane pseudopotential coefficient $i$.

From Fig. 9, we can gain several insights into the effectiveness of the Haldane pseudopotential representation on a torus. First and foremost, we can see that the higher-order Haldane pseudopotential coefficients are more difficult to reproduce than for lower orders, with the required system size $l_{\text{crit}}$ scaling linearly with the order $i$. For example, although the first pseudopotential $V_1$ is
already reproduced at \( t_{\text{crit}} = 12 \), the third pseudopotential \( V_5 \) requires double this with \( t_{\text{crit}} = 24 \), and so on. Second, we can conclude that for a given system with \( N_B \) flux quanta, the largest Haldane pseudopotential coefficient that should be employed in the interaction is \( \beta < N_B/2 \). Physically, any coefficients larger than this correspond to wrapping around the system and so are severely effected by the boundary conditions. Finally, we can see that at inadequate \( \ell \), not all of the energies are resolved and so the quasi-degenerate set of \( \{ E_{i,j} \} \) contains at least one zero, which pulls the whole down unpredictable. At larger system sizes, we observe the expected symmetric and monotonic convergence of the mean and standard deviation towards the Haldane pseudopotential coefficient in all cases.

Building on this, we consider the case of Haldane pseudopotentials modeling the Yukawa interaction. From Fig. 8, we can see that the form of the Yukawa pseudopotential drastically changes in the region \( 10^{-2} < \lambda < 1 \). At \( \lambda = 10^{-2} \), we observe a pseudopotential indistinguishable from the Coulomb pseudopotential on the scale of the plot, whereas at \( \lambda = 1 \) we observe a sharp cut-off, where only the first few pseudopotentials take significant non-zero values. Due to this sharp transition, it is no longer critical to consider higher-order pseudopotentials in order to accurately model the Yukawa interaction at large \( \lambda \). Since the analysis in Fig. 9 is performed on a pseudopotential at a time, the results are simply scaled for the Yukawa interaction. Hence, the relative error thresholds are unaffected and the scaling in Fig. 9(j) still holds. However, since the higher-order pseudopotentials are significantly smaller for the Yukawa interaction with large \( \lambda \), the absolute error threshold \( |E_i - V_i| < 1\% \) is satisfied at drastically smaller system sizes.

In summary, although the Haldane pseudopotentials have a complete equivalence with their two-particle energy spectra on a sphere, this does not hold for the torus. On a torus geometry with \( N_B \) flux quanta, the Haldane pseudopotentials can only be used to model short-range interactions described by \( \{ V_1, V_3, \ldots, V_\beta \} \), where \( \beta < N_B/2 \).

**Appendix C: Derivation of the momentum-averaged density operator on a torus**

Consider a free electron in the \( xy \)-plane in the presence of a perpendicular magnetic field \( \mathbf{B} = B\mathbf{z} \). The electron motion is typically parameterized using its center-of-mass coordinates \( (X,Y) \) and characterized by the magnetic length \( l_B \). In Landau gauge with a conserved \( y \)-momentum, \( k_y \), it can be shown that the single-particle ground states may be written as

\[
\phi(x, y) \sim e^{ik_yy} \exp \left( -\frac{(X-x)^2}{2l_B^2} \right),
\]

where \( X = -k_y l_B^2 \). This shows that the wavefunctions are localized in the \( x \)-direction but extended in \( y \).

When the electron is confined to a rectangular sample of sides \( L_x \) by \( L_y \), then the degeneracy of the LLL is determined by the number of allowed \( k_y \) such that \( 0 \leq X < L_x \). Applying periodic boundary conditions to Eq. (C1), we obtain \( k_y = X_j / l_B^2 = 2\pi j / L_y \), where \( j = 0, \ldots, N_B - 1 \) is an integer bounded by the degeneracy of the LLL, \( N_B = l_x l_y / 2\pi l_B^2 \). There are consequently \( N_B \) LLL eigenstates, which may be written as

\[
\phi_j(x, y) \sim \sum_{m=-\infty}^{\infty} e^{i(x_j + mL_x)y/l_B^2} \exp \left( -(X_j + mL_x - x)^2 / 2l_B^2 \right),
\]

where \( 0 \leq j < N_B \). The degeneracy is equivalent to the total number of flux quanta, such that \( N_B \Phi = \Phi_0 \), where \( \Phi \) is the magnetic flux and \( \Phi_0 \) is the flux quantum. From now on, we set the magnetic length to one \( l_B = 1 \).

In order to compute the density operator on the torus we expand in the basis of LLL orbitals, such that

\[
\rho_\mathbf{q} = \sum_{j,j'} \hat{\rho}_{j,j'}(\mathbf{q}) c_j^{(1)} c_{j'}^*,
\]

where \( c_j^{(1)} \) are the annihilation (creation) operators for the Landau level orbitals \( \phi_j \) (Eq. C2) and \( \hat{\rho}_{j,j'}(\mathbf{q}) \) is the Fourier transform of the normalized particle density coefficients, or form factor, defined as

\[
\hat{\rho}_{j,j'}(\mathbf{q}) = \int d\mathbf{r} e^{i\mathbf{r} \cdot \mathbf{q}} \hat{\phi}_j(\mathbf{r}) \hat{\phi}_{j'}(\mathbf{r}),
\]

\( \mathbf{r} \equiv (x, y) \) and \( \mathbf{q} \equiv (q_x, q_y) \) are position and momentum conjugate variables, and the hats denote normalization.

Starting with the form factor, we compute the Fourier transform of

\[
\hat{\rho}_{j,j'}(x, y) = \frac{\phi_j(x, y) \phi_{j'}^*(x, y)}{N},
\]

where \( N \) is a normalization constant, corresponding to the total particle number.

Computing the denominator of Eq. C5, the total particle number

\[
N = \int_0^{L_x} \int_0^{L_y} dx dy \phi_j^*(x, y) \phi_j(x, y),
\]

yields, in terms of a number repetitions of the simulation cell in the \( x \)- and \( y \)-directions, \( N_x, N_y \),

\[
N = \frac{1}{N_x N_y} \int_0^{L_x} \int_0^{L_y} dx dy \phi_j^*(x, y) \phi_j(x, y).
\]

Working with large \( N_x, N_y \) allows us to extend the integration range. Performing the \( y \) integral yields

\[
N = \frac{2\pi L_y}{N_x L_x} \sum_{m=-\infty}^{\infty} \int dx \exp \left( -\frac{1}{2} x^2 - \frac{1}{2} (x - mL_x)^2 \right),
\]
where we have invoked the momentum periodicity in the \( y \)-direction. Subsequently, performing the \( x \) integral yields
\[
N = 2\pi \sqrt{\pi L_y} \sum_{m=-\infty}^{\infty} \exp \left( -m^2 L_x^2 / 4 \right).
\] (C9)

In a similar fashion, we can compute the Fourier transform of the numerator of Eq. C5, the unnormalized particle density coefficients
\[
\rho_{j,j'}(q) = \frac{1}{N_x N_y} \int \exp i q \cdot r \, \phi_j(r) \phi_{j'}(r).
\] (C10)

As before, working with large \( N_x, N_y \) allows to extend the integration range, and we can ultimately take \( N_t \to \infty \) to obtain complete Gaussian integrals along \( x \). Performing the \( y \) integral yields
\[
\rho_{j,j'}(q) = \frac{2\pi L_y}{N_x} \delta(X_j - X_{j'} + q_y) \sum_{m=-\infty}^{\infty} \int dx e^{iq_x x} \exp \left( -\frac{1}{2} (x - X_j)^2 - \frac{1}{2} (x - X_{j'} - mL_x)^2 \right),
\] (C11)

where we have again employed the momentum periodicity in the \( y \)-direction. Finally, evaluating the integral along \( x \) yields
\[
\rho_{j,j'}(q) = 2\pi \sqrt{\pi L_y} \delta(X_j - X_{j'} + q_y) \sum_{m=-\infty}^{\infty} \exp \chi_{j,j',m}(q_x),
\] (C12)

where
\[
\chi_{j,j',m}(q_x) \equiv \frac{(X_{j'} + X_j + mL_x + iq_x)^2}{2} - \frac{(X_j + mL_x)^2}{2} - \frac{X_{j'}^2}{2},
\] (C13)

and hence the form factor is given as
\[
\hat{\rho}_{j,j'}(q) = \delta(X_j - X_{j'} + q_y) \sum_{m=-\infty}^{\infty} \exp \chi_{j,j',m}(q_x) - \sum_{m=-\infty}^{\infty} \exp (m^2 L_x^2 / 4).
\] (C14)

By invoking the \( q_x \) periodicity of the form factor, we can integrate out the \( q_x \) dependence by summing over momentum modes on the torus. This leaves us with an expression for the density operator that is only dependent on the \( q_y \) modes, which has analogous properties to the conserved \( k_y \) momentum in Landau gauge. Hence, we arrive at our final expression for the \( q_x \)-momentum-averaged density operator used in the main text
\[
\bar{\rho}_{q_y} = \sum_{m=0}^{N_y-1} \rho_{q_y + \frac{mN_x}{L_x}},
\] (C15)

\[\textbf{FIG. 10.} \textbf{Satellite peaks.} \text{Structure factor} I_{\alpha}(0, \omega) \text{for the} \mu = 1/3 \text{FQH state on a torus, with} N = 6 \text{ particles and} N_\phi = 18 \text{ flux quanta, stabilized by} V = V_1 + \alpha V_{17}. \text{The evolution of} I_{\alpha}(0, \omega) \text{with} \alpha \text{is shown as a (a) 2D and (b) 3D plot. The computations were performed with a resolution of} \Delta \omega = 10^{-5}, \Delta I = 10^{-5}, \text{ and} \epsilon = 10^{-4}, \text{corresponding to Fig. 1(a).}
\]

\textbf{Appendix D: Satellite peaks}

When computing the structure factor for the \( V_1 \) interaction, or other interactions in the short-range limit, it is common to observe two “satellite peaks” symmetrically distributed either side of the central peak, as shown in Fig. 10. These peaks are a numerical artifact that arise due to the way that the structure factor is computed, outlined in Sec. III. As we add long-range components to the interaction, these satellite peaks symmetrically converge to the physical peak and rescale its magnitude. This is demonstrated for a \( V_{17} \) component gradually added to the \( V_1 \) interaction, such that \( V = V_1 + \alpha V_{17} \), in Fig. 10. We can clearly see two satellite peaks for \( \alpha = 10^{-4} \), at negative and positive frequencies. As the value of \( \alpha \), and hence the \( V_{17} \) component, is increased, these two peaks exponentially converge to the center and eventually merge with the physical peak at \( \alpha = 1 \). We note that we chose this simple \( V_1 + \alpha V_{17} \) example to clearly illustrate the convergence of the satellite peaks. In realistic scenarios, where there are multiple long-range components, these satellite peaks merge almost immediately, even when the magnitude of the long-range components is much smaller than that of the leading short-range component. Since these peaks in the structure factor are of numerical origin and are only present in the extreme limit of short-range interactions, we omit them from the discussion in the main text.

\textbf{Appendix E: Fractals and self-similarity}

A fractal is defined by two properties: self-similarity and fractional dimension. Self-similarity implies that a subsection of a structure resembles the whole, ad infinitum, but does not imply a fractional dimension. In order to better understand this distinction, we study these con-
cepts in two different contexts: geometry and analysis.

From a geometrical point of view, dimensionality is typically defined as $n = s^{-D}$, where $n$ is the number of units in the whole object, $s$ is the scale factor, and $D$ is the dimension. For example, a square can be divided into 4 equal sections, each with a scale factor of 1/2, and so its dimension is 2. Since each subsection of a square is identical to the whole, it is a self-similar object. However, since it has an integer dimension, it does not fulfill the requirements of a fractal. On the other hand, a Koch curve replicates into 4 equal sections on each iteration, each with a scale factor of 1/3, and so its dimension is $D \approx 1.26$. Again, each subsection is identical to the whole and so it is a self-similar object. Moreover, the Koch curve has a fractional dimension and so it is also a fractal.

In the context of functional analysis, especially of real-world data, these definitions are slightly relaxed. Although, mathematically, self-similarity should hold on all scales, in practice, there will be lower and upper bounds on where this behavior applies. Moreover, although geometrical self-similarity implies that a subsection of an object is identical to the whole, in functional analysis self-similarity is defined when a subsection of a function statistically resembles the entire function, and hence can be quantified by a parameter. Specifically, self-similarity is defined for a function of one variable $y(x)$ as

$$y(x) \equiv s^n y\left(\frac{x}{s}\right), \quad (E1)$$

where $s$ is a scale factor and $\kappa$ is the self-similarity parameter. In Eq. E1, “=” implies that the distributions on both sides of the equation are statistically identical. However, in practice, this is typically approximated by examining the first and second moments (mean and variance). In contrast to geometry, a general function of one variable is in a two-dimensional space where each axis represents different physical quantities. Consequently, two magnification factors are required to quantify self-similarity, such that

$$\kappa = \frac{\log M_y}{\log M_x}, \quad (E2)$$

where $M_i$ is the magnification factor of the $i$-axis. Note that this takes an analogous form to the definition of fractal dimension discussed above, albeit with a different interpretation. The fractional dimension of a function $y(x)$ is often difficult to quantify. However, since the demonstration of self-similarity for any non-trivial curve indicates detail across many orders of magnitude, which precludes an integer dimension, this is typically taken as sufficient evidence to show that a curve is a fractal with respect to the axes on which the magnification occurs.

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