On a maximal inequality and its application to SDEs with singular drift

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Abstract

In this paper we present a Doob type maximal inequality for stochastic processes satisfying the conditional increment control condition. If we assume, in addition, that the margins of the process have uniform exponential tail decay, we prove that the supremum of the process decays exponentially in the same manner. Then we apply this result to the construction of the almost everywhere stochastic flow to stochastic differential equations with singular time dependent divergence-free drift.

Key words: Doob’s maximal inequality, Kolmogorov’s criteria, divergence-free, Aronson estimate

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1 Introduction

Let \( \Omega, \{ \mathcal{F}_t \}_{t \geq 0}, \mathcal{F}, \mathbb{P} \) be a filtered probability space satisfying the usual conditions. Let \( \{ X_t \}_{t \in [0,T]} \) be an \( \{ \mathcal{F}_t \} \)-adapted stochastic process. There has been abundant research on the distribution of the supremum \( \sup_{t \in [0,T]} |X_t| \) since Doob’s martingale maximal inequality. See, for example, [2, 11, 14, 15]. Here we consider continuous processes \( \{ X_t \}_{t \in [0,T]} \) satisfying the conditional increment control condition as follows.

Definition 1.1. Let \( \{ X_t \}_{t \in [0,T]} \) be a continuous \( \{ \mathcal{F}_t \} \)-adapted stochastic process, and let \( p > 1, 0 < h \leq 1, \; ph > 1 \). \( \{ X_t \}_{t \in [0,T]} \) is said to satisfy the conditional increment control with parameter \( (p, h) \) if \( X_t \in L^p(\Omega, \mathbb{R}^d) \) for all \( t \in [0, T] \) and there exists a constant \( A_{p,h} \geq 0 \) independent of \( s \) and \( t \) such that

\[
\mathbb{E} \left[ \| \mathbb{E}(X_t | \mathcal{F}_s) - X_s \|^p \right] \leq A_{p,h} |t-s|^{ph},
\]

for all \( 0 \leq s < t \leq T \). (1.1)

For processes satisfying condition (1.1), we prove a Doob type maximal inequality as follows.

Theorem 1.2. Suppose \( \{ X_t \}_{t \in [0,T]} \) is a continuous \( \{ \mathcal{F}_t \} \)-adapted process satisfying condition (1.1). Let \( 0 \leq s_0 < t_0 \leq T \), and \( X^* = \sup_{u \in [s_0,t_0]} |X_u| \). Then for any \( 1 < q \leq p \),

\[
\| X^* \|_{L^q} \leq \frac{q}{q-1} \left[ c_{p,h}^{1/p} A_{p,h}^{1/p} |t_0-s_0|^h + \| X_{t_0} \|_{L^q} \right]
\]

for some constant \( C_{p,h} > 0 \).

Under condition (1.1), we further study the tail decay of \( \sup_{t \in [0,T]} |X_t| \) when the margins of \( \{ X_t \}_{t \in [0,T]} \) have uniform \( \alpha \)-exponential decay for some \( \alpha > 0 \), i.e. there exist \( C_1, C_2 > 0 \) such that

\[
\mathbb{P} \left( |X_t| \geq \lambda \right) \leq C_2 \exp \left( -C_1 \lambda^\alpha \right), \quad \text{for all } \lambda > 0 \text{ and all } t \in [0,T].
\]

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In Theorem 2.4, we prove that if a continuous process \( \{X_t\}_{t \in [0,T]} \) has uniform \( \alpha \)-exponential marginal decay and satisfies the conditional increment control for \( (p,h) \) with \( p > 1, 0 < h \leq 1 \) and \( ph > 1 \), then its supremum \( \sup_{t \in [0,T]} |X_t| \) decays in the same manner as its margins, i.e.

\[
P \left( \sup_{t \in [0,T]} |X_t| \geq \lambda \right) \leq C \exp(-C\lambda^\alpha).
\]

Our results here are closely related to the celebrated theorems of Kolmogorov and Doob. The conditional increment control condition can be easily deduced from Kolmogorov’s continuity and tightness criteria, which means our result can be applied to a large class of diffusion processes as well as Gaussian processes like fractional Brownian motion. Moreover, the conditional increment control condition is also satisfied by martingales, which makes Theorem 1.2 a generalization of Doob’s maximal inequality. In addition to being mathematically interesting, our results also have practical significance since we have no structural assumption on the processes. We only assume the conditional increment control and exponential marginal decay, which can be directly verified using empirical data. In this article, we show an application of our results to the study of stochastic differential equations

\[
dX_t = b(t, X_t)dt + dB_t, \quad (1.2)
\]

where \( b \) is a time dependent divergence-free vector field and \( B_t \) is the standard Brownian motion on \( \mathbb{R}^d \). Our main result of this application is Theorem 3.8, which states that there is unique almost everywhere stochastic flow \( X(\omega, x) : \Omega \times \mathbb{R}_d \) to SDE (1.2) if \( b \in L^1(0, T; W^{1,p}(\mathbb{R}^d)) \cap L^1(0, T; L^{q}(\mathbb{R}^d)) \) with \( d \geq 3, p \geq 1, \frac{2}{q} + \frac{d}{2} \in [1, 2] \). Here \( X(\omega, x) \) is defined almost every \( (\omega, x) \in \Omega \times \mathbb{R}^d \) under \( \mathbb{P} \times m \), where \( m \) is the Lebesgue measure on \( \mathbb{R}^d \).

For SDEs with singular drift, Aronson [1] proved that there is a unique weak solution to (1.2) when \( b \in L^1(0, T; L^{q}(\mathbb{R}^d)) \) with \( \frac{2}{q} + \frac{d}{2} < 1 \). Moreover, it was proved that the transition probability of \( \{X_t\}_{t \in [0,T]} \) satisfies the Aronson estimate

\[
\frac{1}{C_1(t-\tau)^{d/2}} \exp \left( -C_2 \left( \frac{|x-\xi|^2}{t-\tau} \right) \right) \leq \Gamma(t, x; \tau, \xi) \leq \frac{C}{(t-\tau)^{d/2}} \exp \left( -\frac{C}{C} \left( \frac{|x-\xi|^2}{t-\tau} \right) \right).
\]

Actually the Aronson estimate is true in more general cases when the diffusion coefficient is uniformly elliptic and \( b \in L^1(0, T; L^{q}(\mathbb{R}^d)) \) with \( \frac{2}{q} + \frac{d}{2} \leq 1, q > d \). With the same condition on \( b \) as in Aronson [1], Krylov and Röckner [9] proved that there exists a unique strong solution to (1.2). Regularity results about the strong solution are obtained in Fedrizzi and Flandoli [6, 7]. If we assume boundedness of \( \text{div } b \), since the introduction of the renormalized solutions by DiPerna and Lions [4], there has been lots of work on ODEs \( dX_t = b(t, X_t)dt \). In particular, Crippa and De Lellis [3] developed new estimate on ODEs with Sobolev coefficient \( b \) and gave a new approach to construct the Diperna-Lions flow. This idea is extended to solve SDEs in [5, 16, 17]. In Zhang [16], in addition to boundedness of \( \text{div } b \), it assumes that \( \nabla b \in L \log L(\mathbb{R}^d) \) to control \( X_t \) locally and that \( b/(1+|x|) \in L^\infty(\mathbb{R}^d) \) to control \( X_t \) from explosion. Together with Sobolev condition on the diffusion coefficient, Zhang [16] proved existence of a unique almost everywhere stochastic flow to SDEs. Since it is harder to control the growth of solutions to SDEs than ODEs, the linear growth condition on \( b \) is needed in Zhang [16], while it is not necessary in Crippa and De Lellis [3]. Fang, Luo and Thalmaier [5] extend it to SDEs in Gaussian space with Sobolev diffusion and drift coefficients. In Zhang [17], it relaxes the boundedness of \( \text{div } b \) to only the negative part of \( \text{div } b \), and proved large deviation principle for the corresponding SDEs.

In Section 3, we prove the existence of the almost everywhere stochastic flow to (1.2) through approximation. Take smooth approximation sequence \( b_n \to b \), such that \( \{b_n\}_{n \in \mathbb{N}} \) is uniformly bounded in \( L^1(0, T; W^{1,p}(\mathbb{R}^d)) \) and \( L^1(0, T; L^{q}(\mathbb{R}^d)) \) and divergence-free. Using the Aronson type upper bound estimate of the transition probability proved in Qian and Xi [13, Corollary 9], we can show that
\{X_t(n)\}_{n \in \mathbb{N}} satisfies uniform conditional increment control and exponential marginal decay. Hence Theorem 2.4 implies that \{\sup_{t \in [0,T]} |X_t(n)|\}_{n \in \mathbb{N}} can be controlled uniformly and allows us to remove the linear growth condition on \(b\) used in Zhang [16]. Then following the idea of Zhang [16], in Theorem 3.8 we prove that the sequence \{X_t(n)\}_{n \in \mathbb{N}} converges to a unique limit \(X_t\), which is the unique almost everywhere stochastic flow to SDE (1.2). It worth noting that the proof of Theorem 3.8 only uses the moment estimate of the supremum \(\sup_{t \in [0,T]} |X_t|\) proved in Proposition 3.4. The moment estimate actually can be obtained from the Aronson estimate using Kolmogorov’s continuity theorem as well, while the exponential decay of the supremum proved in Proposition 3.4 is new. In a special case, Theorem 3.8 is true when divergence-free \(b \in L^2(0,T;H^1(\mathbb{R}^3))\), which is of particular interest since the Leray-Hopf weak solutions to the 3-dimensional Navier-Stokes equations are in this space. However, the existence of a unique almost everywhere stochastic flow does not imply the uniqueness of the weak solutions to the corresponding parabolic equations. Since the stochastic flow is defined for almost everywhere initial data \(x \in \mathbb{R}^d\), it actually disguises the “bad” points in the measure zero set. For more discussion on the non-uniqueness of parabolic equations, we refer to Modena and Székelyhidi [12]. For simplicity, in this article we only discuss the case when the drift is divergence-free and the diffusion is the standard Brownian motion. But actually, using the idea in [13, Corollary 9], we can obtain the Aronson type estimate and hence extend the result in Theorem 3.8 for \(\text{div} \ b \in L^1(0,T;L^d(\mathbb{R}^d))\) with \(\frac{2}{p} + \frac{d}{q} < 2\). Moreover, the diffusion coefficient also can be extended to be Sobolev as in Zhang [16].

2 Maximal Inequality

In this section, we always assume that the process \(\{X_t\}_{t \in [0,T]}\) satisfies the conditional increment control (1.1) with \(p > 1, 0 < h \leq 1\) and \(ph > 1\). Under this condition, we prove the Doob type maximal inequality in Theorem 1.2 and the exponential tail decay for the supremum \(\sup_{t \in [0,T]} |X_t|\) in Theorem 2.4. Before starting the proof, we give below some examples of processes which satisfy the conditional increment control.

- Any continuous martingales. The conditional increment control is satisfied for any \((p,h)\) with \(A_{p,h} = 0\).
- Fractional Brownian motions with Hurst parameter \(h \in (0,1)\).
- Let \(\{X_t\}_{t \in [0,T]}\) be a continuous stochastic process satisfying

\[
\mathbb{E}(|X_t - X_s|^p) \leq A_{p,h}|t - s|^ph, \quad \text{for all } 0 \leq s < t \leq T. \tag{2.1}
\]

Using Jensen’s inequality, it is easy to see that the conditional increment control is satisfied with the same parameter \((p,h)\) and the same constant \(A_{p,h}\). Processes satisfying (2.1) are archetypal examples considered in the rough paths theory for which canonical constructions of associated geometric rough paths are available and well-studied (see [8, 10]). This type of processes also arises as solutions to SDEs.

2.1 A Doob Type Maximal Inequality

Before proving the maximal inequality, we will need two lemmas. Firstly, we prove the following estimate for the supremum of the conditional increment.

**Lemma 2.1.** Suppose \(\{X_t\}_{t \in [0,T]}\) is a continuous \(\mathcal{F}_t\)-adapted process satisfying conditional increment control with parameter \((p,h)\). For any \(0 \leq s_0 < t_0 \leq T\), there holds that

\[
\mathbb{E} \left( \sup_{s_0 \leq s < t < t_0} |\mathbb{E}(X_t | \mathcal{F}_s) - X_s|^p \right) \leq C_{p,h}A_{p,h}|t_0 - s_0|^{ph},
\]
where
\[ C_{p,h} = [2ζ(θ)]^{p-1} \left( \frac{p}{p-1} \right)^p \left( \frac{4}{ph-1} \right)^{θ(p-1)+1} Γ[θ(p-1) + 1]. \]

Here θ > 1 is an arbitrary constant, \( ζ(θ) := \sum_{m=1}^{∞} m^{-θ} < ∞ \) for all θ > 1 and Γ(θ) is the Gamma function.

Proof. Let \( s, t \in [s_0, t_0] \), \( s < t \) be fixed temporarily. Denote the dyadic intervals
\[ I^m_t = [t^m_{l-1}, t^m_l] = s_0 + (t_0 - s_0) \ast \left[ \frac{l-1}{2m}, \frac{l}{2m} \right]. \]

Then we will construct a sequence of intervals \( \{J_k\} \subset \{I^m_t : 1 < l < 2^m, m ≥ 0\} \) which gives a partition to \([s, t]\) such that

(i) \( J_k, k = 1, 2, \cdots \), have mutually disjoint interior;

(ii) For any \( m ≥ 1 \), there are at most two elements of \( \{J_k\} \) with length \((t_0 - s_0)2^{-m}\);

(iii) \((s, t) \subset \cup_{k=0}^{m} J_k \subset [s, t]\).

Suppose \( m_0 = \min\{m ∈ \mathbb{N} : 3l ≤ l < 2^m \text{ such that } I^m_t \subset [s, t] \} \). For \( m_0 \), either there is only one \( 1 ≤ l_0 < 2^m_0 \) such that \( I^m_{l_0} \subset [s, t] \) and \([s, t] = [s, t^m_{l_0-1}] \cup [t^m_{l_0}, t^m_t] \), or there are two consecutive \( 1 ≤ l_0 < l_0 + 1 < 2^m_0 \) such that \( I^m_{l_0} \cup I^m_{l_0+1} \subset [s, t] \) and \([s, t] = [s, t^m_{l_0-1}] \cup I^m_{l_0} \cup [t^m_{l_0+1}, t^m_t] \). Here we only deal with the first case and the second case follows the same argument. Notice that \([s - t^m_{l_0-1}] \) and \([t^m_t - t] \) are smaller than \((t_0 - s_0)2^{-m_0}\). For \([t^m_{l_0}, t] \), we set \( m_{k+1} = \min\{m > m_k : 3l ≤ l ≤ 2^m \text{ such that } I^m_{l_k} \subset [t^m_{l_k}, t] \} \).

There is at most one \( 1 ≤ l_{k+1} < 2^{m_{k+1}} \) such that \( I^m_{l_{k+1}} \subset [t^m_{l_k}, t] \) since \([t^m_{l_k} - t] < (t_0 - s_0)2^{-m_k}\). Then \( \{I^m_{l_k}\} \) forms a dyadic partition to \([t^m_{l_k}, t] \), together with \( I^m_{l_k} \) and a dyadic partition to \([t^m_{l_k-1}, l]\) following similar argument, we obtained the collection of intervals \( \{J_k\} \) satisfying (i)-(iii).

Suppose \( J_k = [u_{k-1}, u_k] \). Then
\[
|E(X_t | \mathcal{F}_s) - X_s| = \sum_{k=1}^{∞} E(\Delta X_{J_k} | \mathcal{F}_s)
\leq \sum_{k=1}^{∞} E \left[ |E(\Delta X_{J_k} | \mathcal{F}_{u_{k-1}})| | \mathcal{F}_s \right]
= \sum_{m=1}^{∞} \sum_{\{J_k : J_k = [l_0 - s_0]2^{-m}\}} E \left[ |E(\Delta X_{J_k} | \mathcal{F}_{u_{k-1}})| | \mathcal{F}_s \right],
\]

where \( \Delta X_{J_k} = X_{u_k} - X_{u_{k-1}} \). Let \( ζ^m = E(\Delta X_t | \mathcal{F}_{u_{m+1}}) \) for \( 1 ≤ l ≤ 2^m, m = 1, 2, \cdots \). For any θ > 1, take θ = \( \sum_{m=1}^{∞} m^{-θ} \) to be the Riemann zeta function. By Jensen’s inequality, we have
\[
|E(X_t | \mathcal{F}_s) - X_s|^p ≤ \left( \sum_{m=1}^{∞} \frac{1}{ζ(θ)m^θ} \sum_{\{J_k : J_k = [l_0 - s_0]2^{-m}\}} E \left[ |E(\Delta X_{J_k} | \mathcal{F}_{u_{k-1}})| | \mathcal{F}_s \right] \right)^p
\leq \sum_{m=1}^{∞} \frac{1}{ζ(θ)m^θ} \left( \sum_{\{J_k : J_k = [l_0 - s_0]2^{-m}\}} E \left[ |E(\Delta X_{J_k} | \mathcal{F}_{u_{k-1}})| | \mathcal{F}_s \right] \right)^p
= ζ(θ)^{p-1} \sum_{m=1}^{∞} m^{θ(p-1)} \left( \sum_{\{J_k : J_k = [l_0 - s_0]2^{-m}\}} E \left[ |E(\Delta X_{J_k} | \mathcal{F}_{u_{k-1}})| | \mathcal{F}_s \right] \right)^p
\leq (2ζ(θ))^p \sum_{m=1}^{∞} m^{θ(p-1)} \sum_{\{J_k : J_k = [l_0 - s_0]2^{-m}\}} \left( E \left[ |E(\Delta X_{J_k} | \mathcal{F}_{u_{k-1}})| | \mathcal{F}_s \right] \right)^p.\]
where the inequality in the fourth line is due to property (ii) of \( \{ J_k \} \). Notice that for \( s \leq t_2 \leq t_1 \), by Jensen’s inequality we have
\[
E \left[ |E(X_n|s,F_s) - E(X_n|t,F_s)|^p \right] 
= E \left[ E \left( |E(X_n|s,F_s) - E(X_n|t,F_s)|^p \right| F_s \right] \leq E \left( E \left( |E(X_n|s,F_s) - E(X_n|t,F_s)|^p \right| s \right) 
= E \left| E(X_n|F_s) - E(X_n|F_t) \right|^p 
\leq A_{p,h}|t_1 - t_2|^{ph}.
\]

By Kolmogorov’s continuity theorem, for any fixed \( s \in [s_0,t_0] \), \( E(X_n|s,F_s) \) is a continuous process with respect to \( t \). Recall that the filtration satisfies the usual conditions. \( E(X_n|s,F_s) \) can be regarded as a process of \( s \) for fixed \( t \) and it has a càdlàg modification by Doob’s regularization theorem. Hence, \( \sup_{s_0 \leq s \leq t \leq t_0} |E(X_n|s,F_s) - X_n|^p \) is measurable with respect to \( F_t \) and we have
\[
\sup_{s_0 \leq s \leq t \leq t_0} |E(X_n|s,F_s) - X_n|^p \leq [2 \zeta(\theta)]^{p-1} \sum_{m=1}^{\infty} m^{\theta(p-1)} \sum_{l=1}^{2m} E \left( \sup_{r \in [s_0,t_0]} \| \zeta_r \| \right)^p.
\]

By Doob’s maximal inequality for martingales,
\[
E \left( \sup_{s_0 \leq s \leq t \leq t_0} |E(X_n|s,F_s) - X_n|^p \right) 
\leq [2 \zeta(\theta)]^{p-1} \sum_{m=1}^{\infty} m^{\theta(p-1)} \sum_{l=1}^{2m} E \left( \sup_{r \in [s_0,t_0]} \| \zeta_r \| \right)^p 
\leq [2 \zeta(\theta)]^{p-1} \left( \frac{p}{p-1} \right)^p \sum_{m=1}^{\infty} m^{\theta(p-1)} \sum_{l=1}^{2m} E \left( \| \zeta_l \| \right)^p 
\leq A_{p,h} [2 \zeta(\theta)]^{p-1} \left( \frac{p}{p-1} \right)^p \sum_{m=1}^{\infty} m^{\theta(p-1)} \left( \frac{|t_0 - s_0|}{2m} \right)^{ph} 
= C_{p,h} A_{p,h} |t_0 - s_0|^{ph},
\]
where
\[
C_{p,h} = [2 \zeta(\theta)]^{p-1} \left( \frac{p}{p-1} \right)^p \sum_{m=1}^{\infty} m^{\theta(p-1)} 2^{-m(ph-1)}.
\]

Notice that
\[
\sum_{m=1}^{\infty} m^{\theta(p-1)} 2^{-m(ph-1)} \leq \sum_{m=1}^{\infty} \left( e^{\theta(p-1)} \int_{m-1}^{m} r^{\theta(p-1)} dr \right) 2^{-m(ph-1)} 
\leq \sum_{m=1}^{\infty} e^{\theta(p-1)} \int_{m-1}^{m} r^{\theta(p-1)} e^{-r(ph-1) \ln 2} dr 
\leq \frac{e}{(ph-1) \ln 2} \int_{0}^{\infty} r^{\theta(p-1)} e^{-r} dr 
\leq \frac{4}{(ph-1)} \Gamma[\theta(p-1)+1].
\]

Now the proof is complete. \( \square \)

Using Lemma 2.1, we show a Doob type inequality for processes satisfying condition (1.1). To this end, we shall need the following elementary result.
Lemma 2.2. Let \( \{Y_t\}_{t \in [0,T]} \) be a continuous stochastic process such that \( \mathbb{E}(|Y_t|) < \infty \) for all \( t \in [0,T] \). Let \( 0 \leq s_0 < t_0 \leq T \). Then

(1) For any stopping time \( \tau \) with \( s_0 \leq \tau \leq t_0 \), we have

\[
|\mathbb{E}(Y_{\tau}|\mathcal{F}_\tau) - Y_\tau| \leq \mathbb{E}\left[ \sup_{u \in [s_0,t_0]} |\mathbb{E}(Y_{\tau}|\mathcal{F}_u) - Y_u| \right] .
\]

(2) For any \( \lambda > 0 \), we have

\[
\mathbb{P}\left( \sup_{u \in [s_0,t_0]} |Y_u| \geq \lambda \right) \leq \frac{1}{\lambda} \int \left[ \sup_{u \in [s_0,t_0]} |\mathbb{E}(Y_{\tau}|\mathcal{F}_u) - Y_u| + |Y_\tau| \right] d\mathbb{P} .
\]

Proof. (1) By the right continuity of \( Y_t \) and \( \mathbb{E}(Y_{\tau}|\mathcal{F}_\tau) \), we may assume that \( \tau \) takes only countably many values \( \{u_k : k = 1,2, \cdots \} \subset [s_0,t_0] \). Then

\[
|\mathbb{E}(Y_{\tau}|\mathcal{F}_\tau) - Y_\tau| = \sum_{k=1}^{\infty} |\mathbb{E}(Y_{\tau}|\mathcal{F}_\tau) - Y_\tau| \mathbb{1}_{\{\tau = u_k\}}
\]

\[
= \sum_{k=1}^{\infty} |\mathbb{E}\left( (Y_{\tau} - Y_\tau) \mathbb{1}_{\{\tau = u_k\}} \mid \mathcal{F}_{u_k} \cap \{\tau = u_k\} \right)|
\]

\[
= \sum_{k=1}^{\infty} |\mathbb{E}\left( (Y_{\tau} - Y_\tau) \mathbb{1}_{\{\tau = u_k\}} \mid \sigma(\mathcal{F}_{u_k} \cap \{\tau = u_k\}) \right)|
\]

\[
\leq \sum_{k=1}^{\infty} \mathbb{E}\left[ \left( \sup_{u \in [s_0,t_0]} |\mathbb{E}(Y_{\tau}|\mathcal{F}_u) - Y_u| \right) \mathbb{1}_{\{\tau = u_k\}} \mid \sigma(\mathcal{F}_{u_k} \cap \{\tau = u_k\}) \right]
\]

\[
= \sum_{k=1}^{\infty} \mathbb{E}\left[ \sup_{u \in [s_0,t_0]} |\mathbb{E}(Y_{\tau}|\mathcal{F}_u) - Y_u| \mid \mathcal{F}_{u_k} \right] \mathbb{1}_{\{\tau = u_k\}}
\]

\[
= \mathbb{E}\left[ \sup_{u \in [s_0,t_0]} |\mathbb{E}(Y_{\tau}|\mathcal{F}_u) - Y_u| \mid \mathcal{F}_\tau \right] .
\]

(2) Let \( \tau = \inf\{u \in [s_0,t_0] : |Y_u| \geq \lambda \} \wedge T \). Then \( \sup_{u \in [s_0,t_0]} |Y_u| \geq \lambda \} = \{\tau < T\} \cup \{\tau = t_0, |Y_{t_0}| \geq \lambda \} \in \mathcal{F}_\tau \). Therefore, by (2.2) we have

\[
\int_{\sup_{u \in [s_0,t_0]} |Y_u| \geq \lambda} |Y_\tau| d\mathbb{P} \leq \int_{\sup_{u \in [s_0,t_0]} |Y_u| \geq \lambda} |\mathbb{E}(Y_{\tau}|\mathcal{F}_\tau) - Y_\tau| d\mathbb{P} + \int_{\sup_{u \in [s_0,t_0]} |Y_u| \geq \lambda} |\mathbb{E}(Y_{\tau}|\mathcal{F}_\tau)| d\mathbb{P}
\]

\[
\leq \int_{\sup_{u \in [s_0,t_0]} |Y_u| \geq \lambda} \mathbb{E}\left[ \sup_{u \in [s_0,t_0]} |\mathbb{E}(Y_{\tau}|\mathcal{F}_u) - Y_u| \mid \mathcal{F}_\tau \right] + \mathbb{E}\left( |Y_\tau| \mid \mathcal{F}_\tau \right) d\mathbb{P}
\]

\[
\leq \int_{\sup_{u \in [s_0,t_0]} |Y_u| \geq \lambda} \left[ \sup_{u \in [s_0,t_0]} |\mathbb{E}(Y_{\tau}|\mathcal{F}_u) - Y_u| + |Y_\tau| \right] d\mathbb{P} .
\]

Finally, using

\[
\mathbb{P}\left( \sup_{u \in [s_0,t_0]} |Y_u| \geq \lambda \right) \leq \frac{1}{\lambda} \int_{\sup_{u \in [s_0,t_0]} |Y_u| \geq \lambda} |Y_\tau| d\mathbb{P},
\]

and we complete the proof. □

Now we are ready to prove Theorem 1.2.
Proof of Theorem 1.2. Denote $Z = \sup_{a \in [0, t]} |\mathbb{E}(X_0 \mid \mathcal{F}_a) - X_a| + |X_0|$ and fix a $q \in (1, p]$. By (2.3) and Lemma 2.1

\[
\|X^*\|_{L^q}^q = q \int_0^\infty \lambda^{q-1} \mathbb{P}(X^* \geq \lambda) d\lambda \\
\leq q \int_0^\infty \lambda^{q-2} \int_{\{X^\ast \geq \lambda\}} Z d\lambda d\lambda \\
= q \int_\Omega \left( \int_0^{\infty} \lambda^{q-2} d\lambda \right) Z d\lambda \\
= \frac{q}{q-1} \int_\Omega |X^*|^{q-1} Z d\lambda \\
\leq \frac{q}{q-1} \|X^\ast\|_{L^q} \|Z\|_{L^\infty}.
\]

Therefore,

\[
\|X^*\|_{L^q} \leq \frac{q}{q-1} \|Z\|_{L^\infty} \\
\leq \frac{q}{q-1} \left[ \sup_{a \in [0, t]} \left\| \mathbb{E}(X_0 \mid \mathcal{F}_a) - X_a \right\|_{L^q} + \|X_0\|_{L^q} \right] \\
\leq \frac{q}{q-1} \left[ \mathbb{C}_p A_{p,h}^1 \lambda^{1/p} |t_0 - s_0| h + \|X_0\|_{L^q} \right].
\]

\]

2.2 Tail decay for the supremum

Here, we shall show that the distribution of the supremum $\sup_{t \in [0, T]} |X_t|$ has $\alpha$-exponential decay if the margins of $\{X_t\}_{t \in [0, T]}$ have uniform $\alpha$-exponential decay as in the following definition.

Definition 2.3. Given $\alpha > 0$, a continuous stochastic process $\{X_t\}_{t \in [0, T]}$ is said to have uniform $\alpha$-exponential marginal decay if there exists constants $C_1, C_2 > 0$ such that

\[
\mathbb{P}(|X_t| \geq \lambda) \leq C_2 \exp(-C_1 \lambda^\alpha), \quad \text{for all } \lambda > 0 \text{ and all } t \in [0, T].
\]

Remark. Suppose

\[
\mathbb{P}(|X_t| \geq \lambda) \leq C_2 \exp(-C_1 \lambda^\alpha), \quad \text{for all } \lambda > M
\]

with $M$ being a large enough constant. Since we always have $\mathbb{P}(|X_t| \geq \lambda) \leq 1$, actually (2.5) implies (2.4) for another pair of constants $(C_1, C_2)$. In the following, we will always use (2.4).

Notice that if $\{X_t\}_{t \in [0, T]}$ satisfies (2.4), one has

\[
\mathbb{E}(|X_t|^q) \leq C_2 C_1^{-q/\alpha} \Gamma\left(\frac{q}{\alpha} + 1\right)
\]

(2.6) for any $q > 0$. Now we state our theorem.

Theorem 2.4. Suppose stochastic process $\{X_t\}_{t \in [0, T]}$ satisfies condition (1.1) with parameter $(p, h)$ and has uniform $\alpha$-exponential marginal decay. Then

\[
\mathbb{P}\left(\sup_{t \in [0, T]} |X_t| \geq \lambda\right) \leq C_2 \lambda^{-1/h} \exp\left[-\frac{C_1}{2^{\alpha+2}} \left(1 - \frac{1}{ph}\right) \lambda^\alpha\right]
\]

for large enough $\lambda$, where $C$ depends on $(C_1, C_2, \alpha, p, h, A_{p,h})$. 

Proof. For $N \in \mathbb{N}_+$, let $I_n = [t_{n-1}, t_n] = [(n - 1)T/N, nT/N]$, $1 \leq n \leq N$. Then

$$
\left\{ \sup_{t \in [0, T]} |X_t| \geq 2 \lambda \right\} \subset \bigcup_{n=1}^{N} \left( \left\{ \sup_{t \in I_n} |E(X_t | \mathcal{F}_t) - X_t| \geq \lambda \right\} \bigcup \left\{ \sup_{t \in I_n} |E(X_t | \mathcal{F}_t)| \geq \lambda \right\} \right).
$$

Therefore

$$
P \left( \sup_{t \in [0, T]} |X_t| \geq 2 \lambda \right) \leq \sum_{n=1}^{N} P \left( \sup_{t \in I_n} |E(X_t | \mathcal{F}_t) - X_t| \geq \lambda \right) + \sum_{n=1}^{N} P \left( \sup_{t \in I_n} |E(X_t | \mathcal{F}_t)| \geq \lambda \right). \tag{2.7}
$$

By Lemma 2.1,

$$
P \left( \sup_{t \in I_n} |E(X_t | \mathcal{F}_t)| \geq \lambda \right) \leq C_{p,h} A_{p,h} \lambda^{-p} \left( \frac{T}{N} \right)^{ph}. \tag{2.8}
$$

Next we need to estimate $P \left( \sup_{t \in I_n} |E(X_t | \mathcal{F}_t)| \geq \lambda \right)$. Notice that

$$
E \left[ \exp \left( \frac{C_1}{4} \sup_{t \in I_n} |E(X_t | \mathcal{F}_t)|^\alpha \right) \right] \leq \sum_{q=0}^{\infty} \frac{(C_1/4)^q}{q!} E \left( \sup_{t \in I_n} |E(X_t | \mathcal{F}_t)|^{\alpha q} \right).
$$

Here we fix an arbitrary constant $\beta > 1$. When $\alpha q \leq \beta$, by Doob's maximal inequality and (2.6), we have that

$$
E \left( \sup_{t \in I_n} |E(X_t | \mathcal{F}_t)|^{\alpha q} \right) \leq \left( \frac{\beta}{\beta - 1} \right)^{\frac{\alpha q}{\beta}} E \left( |X_n|^{\alpha q} \right) \leq \left( \frac{\beta}{\beta - 1} \right)^{\frac{\alpha q}{\beta}} \left( C_2 C_1^{-\beta/\alpha} \Gamma \left( \frac{\beta}{\alpha} + 1 \right) \right)^{\frac{\alpha q}{\beta}}.
$$

When $\alpha q > \beta$, again by Doob's maximal inequality and (2.6), we have that

$$
E \left( \sup_{t \in I_n} |E(X_t | \mathcal{F}_t)|^{\alpha q} \right) \leq \left( \frac{\beta}{\beta - 1} \right)^{\frac{\beta}{\beta}} E \left( |X_n|^{\alpha q} \right) \leq \left( \frac{\beta}{\beta - 1} \right)^{\frac{\beta}{\beta}} C_2 C_1^{-\beta/\alpha} \Gamma (q + 1).
$$

Therefore

$$
E \left[ \exp \left( \frac{C_1}{4} \sup_{t \in I_n} |E(X_t | \mathcal{F}_t)|^\alpha \right) \right] \leq \sum_{q=0}^{\left( \frac{\beta}{\beta - 1} \right)^{\frac{\alpha q}{\beta}}} \frac{(C_1/4)^q}{q!} \left( \frac{\beta}{\beta - 1} \right)^{\frac{\alpha q}{\beta}} \left( C_2 C_1^{-\beta/\alpha} \Gamma \left( \frac{\beta}{\alpha} + 1 \right) \right)^{\frac{\alpha q}{\beta}}
$$

$$
+ \sum_{q=\left( \frac{\beta}{\beta - 1} \right)^{\frac{\alpha q}{\beta} + 1}}^{\infty} \frac{(C_1/4)^q}{q!} \left( \frac{\beta}{\beta - 1} \right)^{\frac{\beta}{\beta}} C_2 C_1^{-\beta/\alpha} \Gamma (q + 1)
$$

$$
\leq C + \sum_{q=\left( \frac{\beta}{\beta - 1} \right)^{\frac{\alpha q}{\beta} + 1}}^{\infty} 4^{-q} \left( \frac{\beta}{\beta - 1} \right)^{\frac{\beta}{\beta}} C_2
$$

$$
\leq C,
$$

8
where the constant $C$ depends on $(\alpha, \beta, C_1, C_2)$. By Chebyshev’s inequality,

$$
\mathbb{P} \left( \sup_{t \in \mathbb{I}} |\mathbb{E}(X_t) - \mathfrak{U}_t| \geq \lambda \right) \leq C \exp \left( -\frac{C_1}{4} \lambda^2 \right) .
$$

(2.9)

Hence, for any $N \in \mathbb{N}_+$, by (2.8) and (2.9), we have that

$$
\mathbb{P} \left( \sup_{t \in [0,T]} |X_t| \geq 2\lambda \right) \leq NC_{p,h}A_{p,h}^{\gamma}(\frac{T}{N})^{\gamma} + NC \exp \left( -\frac{C_1}{4} \lambda^2 \right)
$$

$$= EN^{1-p} + FN,$n

where $E = C_{p,h}A_{p,h}T^{\gamma} \lambda^{-p}$ and $F = C \exp \left( -\frac{C_1}{4} \lambda^2 \right)$. When $\lambda$ is large enough, notice that $E \gg F$ and $E/F$ is large enough. Then we can set $N$ to be the greatest integer less than or equal to $(\frac{C}{p})^{\frac{1}{\gamma}}$ to obtain that

$$
\mathbb{P} \left( \sup_{t \in [0,T]} |X_t| \geq 2\lambda \right) \leq CE^{1-p}F^{1-\frac{1}{\gamma}}
$$

$$= CE^{1-p}A_{p,h}^\gamma T^{\gamma} \lambda^{-p} \exp \left( -\frac{C_1}{4} \left( 1 - \frac{1}{p} \right) \lambda^2 \right),
$$

where $C$ depends on $C_1, C_2, \gamma, h$ and $\alpha$. Finally, replace $2\lambda$ by $\lambda$ and the proof is complete. \(\square\)

3 SDEs with singular drift

In this section, we apply the results in the previous section to solve SDEs

$$
dX_t = b(t, X_t)dt + dB_t,
$$

(3.1)

where divergence-free vector field $b \in L^1(0, T; L^d(\mathbb{R}^d)) \cap L^1(0, T; W^{1,p}(\mathbb{R}^d))$ with $\frac{2}{p} + \frac{d}{q} = \gamma \in [1, 2)$, $d \geq 3$ and $p \geq 1$. We use the idea of approximation and \textit{a priori} estimates to show that the approximation sequence converges in probability. Together with the $L^k(\Omega \times B_r; C([0, T]))$ bound of the approximation sequence, we have that the sequence converges in $L^k(\Omega \times B_r; C([0, T]))$ for any $k \geq 1$. Here $B_r$ is the ball in $\mathbb{R}^d$ of radius $r$ and center at the origin.

3.1 Supremum of solutions to SDEs

To control the supremum $\sup_{t \in [0,T]} |X_t|$, we will need the following Aronson type estimate of the transition probability of $X_t$ proved in [13, Corollary 9].

Theorem 3.1. Suppose $b$ is divergence-free and $b \in L^1(0, T; L^q(\mathbb{R}^d))$ for some $d \geq 3$, $l > 1$ and $q > \frac{d}{\gamma}$ such that $\frac{d}{p} + \frac{2}{q} = \gamma \in [1, 2)$. In addition, we assume that $b$ is smooth with bounded derivatives. If $\mu := \frac{2}{2 - \gamma + \frac{2}{\gamma}} > 1$, the transition probability has upper bound

$$
\Gamma(t, x; \tau, \xi) \leq \begin{cases} 
\frac{C_1}{(t - \tau)^{p/2}} \exp \left( -\frac{1}{C_2} \left( \frac{|x - \xi|^2}{(t - \tau)^{p/2}} \right) \right) & \frac{|x - \xi|^2}{(t - \tau)^{p/2}} < 1 \\
\frac{C_1}{(t - \tau)^{p/2}} \exp \left( -\frac{1}{C_2} \left( \frac{|x - \xi|^2}{(t - \tau)^{p/2}} \right)^{1/\gamma} \right) & \frac{|x - \xi|^2}{(t - \tau)^{p/2}} \geq 1,
\end{cases}
$$

where $\nu = \frac{2 - \gamma}{2 - \gamma + \frac{2}{\gamma}}$, $\Lambda = \|b\|_{L^1(0, T; L^q(\mathbb{R}^d))}$, $C_1 = C_1(l, q, d)$, $C_2 = C_2(l, q, d, \Lambda)$. If $\mu = 1$, which implies $q = \infty$, we have

$$
\Gamma(t, x; \tau, \xi) \leq \frac{C_1}{(t - \tau)^{d/2}} \exp \left( -\frac{(C_1 \Lambda(t - \tau)^{\nu} - |x - \xi|^2)^2}{4C_1(t - \tau)} \right).
$$

(9)
Remark 3.2. This theorem is actually true for diffusion processes corresponding to parabolic equations
\[ \partial_t u(t, x) - \sum_{i,j=1}^{d} \partial_j(a_{ij}(t, x)\partial_i u(t, x)) + \sum_{i=1}^{d} b_i(t, x)\partial_i u(t, x) = 0 \]
for uniformly elliptic \{a_{ij}\}, \( b \in L^1(0, T; L^d(\mathbb{R}^d)) \) with \( \frac{2}{d} + \frac{d}{q} = \gamma \in [1, 2) \) and \( \text{div}b \in L^q(0, T; L^d(\mathbb{R}^d)) \)
with \( \frac{2}{d} + \frac{d}{q} = \gamma' \in [1, 2) \). The proof follows the idea in [13] with small modification.

This upper bound estimate of the transition probability implies the following.

**Proposition 3.3.** Suppose \( b \) is a smooth divergence-free vector field with bounded derivatives and \( b \in L^1(0, T; L^d(\mathbb{R}^d)) \) for some \( d \geq 3, \lambda > 1 \) and \( q > \frac{d}{2} \) such that \( 1 \leq \gamma < 2 \). Then the solution \( \{X_t\}_{t \geq 0} \) to (3.1) satisfies that
\[ E|X_t - X_s|^p \leq C (t - s)^{\frac{2}{(2-\gamma)(d-1)} - \frac{1}{2}} \]
for \( 0 \leq s < t \leq T \). Moreover, there exists a constant \( \alpha > 1 \) depending on \( (l, q, d) \) such that
\[ P(|X_t - X_s| > \lambda) \leq C \exp(-C \lambda^\alpha) \]
for large enough \( \lambda \) and \( 0 \leq s < t \leq T \). The constant \( C \) depends on \( (l, q, d, \Lambda) \).

**Proof.** Without loss of generality, we may take \( s = 0 \) and \( X_0 = 0 \). When \( \mu > 1 \), we have
\[ E|X_t - X_0|^p = \int_{\mathbb{R}^d} |x|^p \Gamma(t, x; 0, 0) dx \]
\[ \leq \int_{\mathbb{R}^d} |x|^p C_1 \frac{C_1^r}{t^{d/2}} \exp \left( -\frac{1}{C_2} \left( \frac{|x|^2}{t} \right) \right) dx + \int_{\mathbb{R}^d} |x|^p C_1 \frac{C_1^r}{t^{d/2}} \exp \left( -\frac{1}{C_2} \left( \frac{|x|^\mu}{t^{\frac{\gamma}{\mu}}} \right) \right) dx \]
\[ \leq C \left( t^{\frac{d^2}{2}} + t^{\frac{2-\gamma(d-1)}{2}} \right) \]
\[ \leq Ct^{\frac{2-\gamma(d-1)}{2}}. \]

When \( \mu = 1 \), following similar argument, we have that
\[ E|X_t - X_0|^p = \int_{\mathbb{R}^d} |x|^p \Gamma(t, x; 0, 0) dx \]
\[ \leq \int_{|x| < C_1} |x|^p \Gamma(t, x; 0, 0) dx + \int_{|x| > C_1} |x|^p \Gamma(t, x; 0, 0) dx \]
\[ \leq \int_{\mathbb{R}^d} |x|^p C_1 \frac{C_1^r}{t^{d/2}} \exp \left( -\frac{C|x|^2}{4C_1 t} \right) dx + \int_{|x| < C_1} |x|^p C_1 \frac{C_1^r}{t^{d/2}} dx \]
\[ \leq C \left( t^{d^2/2} + t^{2-\gamma(d-1)/2} \right) \]
\[ \leq Ct^{2-\gamma(d-1)/2}. \]

Now we prove the uniform \( \alpha \)-exponential marginal decay for \( X_t \). When \( \mu > 1 \), we have that
\[ P(|X_t| > \lambda) \leq \int_{|x| > \lambda} C_1 \frac{C_1^r}{t^{d/2}} \exp \left( -\frac{1}{C_2} \left( \frac{|x|^2}{t} \right) \right) dx + \int_{|x| > \lambda} C_1 \frac{C_1^r}{t^{d/2}} \exp \left( -\frac{1}{C_2} \left( \frac{|x|^\mu}{t^{\frac{\gamma}{\mu}}} \right) \right) dx \]
\[ = C_1 \int_{|x| > \lambda} \exp \left( -\frac{|x|^2}{C_2} \right) dx + \frac{C_1}{t^{d(\gamma-1)/2}} \int_{|x| > \lambda} \exp \left( -\frac{|x|^\mu}{C_2} \right) dx \]
\[ \leq C \exp(-C(\lambda t^{-\frac{1}{2}})^2) + \frac{C}{t^{d(\gamma-1)/2}} \exp \left( -C(\lambda t^{-\frac{2-\gamma}{\mu}})^{\frac{\mu}{\gamma-1}} \right) \]
\[ \leq C \exp(-C\lambda^2) + C \exp(-C\lambda^{\frac{d}{2}}) \]

for \(0 < t \leq T\) and large enough \(\lambda\). Similarly, when \(\mu = 1\), we have

\[ \mathbb{P}(|X_t| > \lambda) \leq \int_{|x| > \lambda} \frac{C_1}{t^{d/2}} \exp \left( -\frac{(C_1 \lambda t^\gamma - |x|)^2}{4C_1 t} \right) \, dx \]
\[ \leq C_1 \int_{|x| > \lambda t^{-\frac{1}{2}}} \exp \left( -C(\lambda t^{-\frac{1}{2}} - |x|)^2 \right) \, dx. \]

Recall that \(\mu = 1\) implies \(q = \infty\) and hence \(v = \frac{2-\gamma}{2} \in (0, \frac{1}{2}]\). For large enough \(\lambda\), we have \(|x| > \lambda t^{-\frac{1}{2}} \gg C t^{-\frac{1}{2}}\) and

\[ \mathbb{P}(|X_t| > \lambda) \leq C_1 \int_{|x| > \lambda t^{-\frac{1}{2}}} \exp \left( -C|x|^2 \right) \]
\[ \leq C \exp(-C\lambda^2). \]

Now we can apply Theorem 2.4 to obtain the following result.

**Proposition 3.4.** Suppose \(b\) is a smooth divergence-free vector field with bounded derivatives and \(b \in L^1(0, T; L^q(\mathbb{R}^d))\) for some \(d \geq 3, l > 1\) and \(q > \frac{d}{4}\) such that \(1 \leq q < 2\). Then the solution \(\{X_t\}_{t \in [0,T]}\) to (3.1) satisfies that

\[ \mathbb{P} \left( \sup_{t \in [0,T]} |X_t - X_0| > \lambda \right) \leq C \exp(-C\lambda^\alpha) \]

(3.4)

with the same \(\alpha\) as in Proposition 3.3. Moreover, for any \(p \geq 1\) we have that

\[ \mathbb{E} \left[ \sup_{t \in [0,T]} |X_t - X_0|^p \right] < C, \]

(3.5)

where \(C\) depends on \(l, q, p, d\) and \(\|b\|_{L^1(0, T; L^q(\mathbb{R}^d))}\).

**Remark.** Using Kolmogorov’s continuity theorem, for any large enough \(p\) such that \(\frac{(2-\gamma)(p+d)-d}{2} > 1\), inequality (3.2) implies that there exists \(\alpha > 0\) such that

\[ |X_t - X_s| \leq K|t - s|^\alpha \quad \text{for all } 0 \leq s < t \leq T. \]

Here \(K\) is a random variable satisfying \(\mathbb{E}[K^p] < C\). Since this is true for any large enough \(p\), in this way we can also prove the moment estimate (3.5) for any \(p \geq 1\), although it can not prove the exponential decay (3.4). But actually Theorem 3.8 below uses only the moment estimate (3.5). This provides an alternative proof to Theorem 3.8.

### 3.2 Existence and uniqueness of strong solutions

Now we can construct a unique almost everywhere stochastic flow to (3.1) using approximation. The argument essentially follows Zhang [16]. Firstly, we define almost everywhere stochastic flow as follows.

**Definition 3.5.** Suppose \(\{X_t\}_{t \in [0,T]}\) is a \(\mathbb{R}^d\)-valued stochastic process defined on \(\Omega \times \mathbb{R}^d \times [0,T]\). We say that \(\{X_t\}_{t \in [0,T]}\) is an almost everywhere stochastic flow to (3.1) if

1. for \(\mathbb{P} \times m\)-almost all \((\omega, x) \in \Omega \times \mathbb{R}^d, t \rightarrow X_t(\omega, x)\) is a \(\mathbb{R}^d\)-continuous function on \([0, T];\)
Proof. Consider for any \( x \in \Omega \), under mapping \( x \rightarrow X_t(\omega, x) \), the push-forward of the Lebesgue measure \( m \) restricted to any Borel set \( A \subset \mathbb{R}^d \) has density, i.e. \( (m1_A) \circ X_t^{-1}(\omega) = \rho_t(\omega, A, x)dx \), where the density satisfies that \( \rho_t(\omega, A, \cdot) \leq 1 \) for all \( x \in \mathbb{R}^d \) and \( \int_{\mathbb{R}^d} \rho_t(\omega, A, x)dx = m(A) \);

(3) for any \( t \in [0, T] \), we have

\[
X_t(x) = x + \int_0^t b(s, X_s(x))ds + \int_0^t dB_s
\]

for \( \mathbb{P} \times m \)-almost all \((\omega, x) \in \Omega \times \mathbb{R}^d \).

When the vector field \( b \) is smooth and divergence-free, the strong solution \( X_t \) to (3.1) preserves the Lebesgue measure in the sense that

\[
\mathbb{P} [\omega \in \Omega : m(X_t(\omega, A)) = m(A)] = 1,
\]

where \( X_t(\omega, A) \) is the image of any Borel set \( A \in \mathbb{R}^d \) under mapping \( x \rightarrow X_t(\omega, x) \). Clearly \( X_t \) satisfies Definition 3.5.

We first recall the following lemma in Crippa and De Lellis [3, Lemma A.3].

**Lemma 3.6.** Let \( M_Rf \) be the local maximal function of locally integrable function \( f \) defined as

\[
M_Rf(x) = \sup_{0 < r < R} \frac{1}{|B_r|} \int_{B_r(x)} f(y)dy.
\]

Suppose \( f \in BV_{loc}(\mathbb{R}^d) \), then

\[
|f(x) - f(y)| \leq C|x - y| [M_R|\nabla f|(x) - M_R|\nabla f|(y)]
\]

for \( x, y \in \mathbb{R}^d \setminus N \), where \( N \) is a negligible set in \( \mathbb{R}^d \), \( R = |x - y| \) is the distance between \( x \) and \( y \), and constant \( C \) depends only on the dimension \( d \).

We denote by \( Mf \) the maximal function

\[
Mf(x) = \sup_{0 < r < \infty} \frac{1}{|B_r|} \int_{B_r(x)} f(y)dy
\]

and clearly inequality (3.7) is also true if we replace \( M_R|\nabla f| \) with \( M|\nabla f| \).

**Lemma 3.7.** Suppose \( X_t(x) \) and \( \tilde{X}_t(x) \) are almost everywhere stochastic flows to SDE (3.1) driven by the same Brownian motion, with initial data \( x \) and drifts \( b \) and \( \tilde{b} \) in \( L^1(0, T; W^{1,p}(\mathbb{R}^d)) \), \( p \geq 1 \) respectively. Then for any \( r > 0 \) and \( \theta > 0 \),

\[
\mathbb{E} \left[ \int_{\mathbb{R}^d} \log \left( \frac{\sup_{0 \leq t \leq T} |X_t(x) - \tilde{X}_t(x)|^2}{\theta^2} + 1 \right) dx \right] \leq C \left( \|\nabla b\|_{L^1(0, T; L^p(\mathbb{R}^d))} + \frac{1}{\theta} \|b - \tilde{b}\|_{L^1(0, T; L^p(\mathbb{R}^d))} \right),
\]

where the constant \( C \) depends on \((r, p, d)\).

**Proof.** Consider

\[
\frac{d}{dt} \log \left( \frac{|X_t(x) - \tilde{X}_t(x)|^2}{\theta^2} + 1 \right) \leq \frac{|X_t(x) - \tilde{X}_t(x)||b(t, X_t(x)) - \tilde{b}(t, \tilde{X}_t(x))|}{|X_t(x) - \tilde{X}_t(x)|^2 + \theta^2}
\]

\[
\leq \frac{|b(t, X_t(x)) - b(t, \tilde{X}_t(x))|}{\sqrt{|X_t(x) - \tilde{X}_t(x)|^2 + \theta^2}} + \frac{|\tilde{b}(t, \tilde{X}_t(x)) - \tilde{b}(t, \tilde{X}_t(x))|}{\sqrt{|X_t(x) - \tilde{X}_t(x)|^2 + \theta^2}}
\]

\[
= g_1(x) + g_2(x).
\]
Integrate both sides on $B_r$ and take expectation, then by Lemma 3.6 we have that

$$
\mathbb{E} \left[ \int_{B_r} g_1(x) \, dx \right] \leq \mathbb{E} \left[ \int_{B_r} \frac{C |X_t(x) - \bar{X}_t(x)| |M| |\nabla b|(t, X_t(x)) + M |\nabla b|(t, \bar{X}_t(x)) \, dx}{\sqrt{|X_t(x) - \bar{X}_t(x)|^2 + \theta^2}} \right]
$$

$$
\leq C \int_{\Omega} \left( \int_{B_r} M |\nabla b|(t, X_t(x)) \, dx + \int_{B_r} M |\nabla b|(t, \bar{X}_t(x)) \, dx \right) \, d\mathbb{P}(\omega).
$$

$$
= C \int_{\Omega} \left( \int_{\mathbb{R}^d} M |\nabla b|(t, x) \rho_t(\omega, B_r, x) \, dx + \int_{\mathbb{R}^d} M |\nabla b|(t, x) \rho_t(\omega, B_r, x) \, dx \right) \, d\mathbb{P}(\omega).
$$

Here $|M| |\nabla b| \leq C |\nabla b|_{L^p(\mathbb{R}^d)}$ and $L^p(\mathbb{R}^d) \subset L^1(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$. For any $f \in L^p(\mathbb{R}^d)$, we have $f = f_1 + f_2$, where $f_1 = f 1_{\{|f| \leq \|f\|_{L^p}\}}$ and $f_2 = f 1_{\{|f| > \|f\|_{L^p}\}}$. It is easy to verify that $\|f_1\|_{L^p(\mathbb{R}^d)} \leq \|f\|_{L^p(\mathbb{R}^d)}$ and $\|f_2\|_{L^1(\mathbb{R}^d)} \leq \|f\|_{L^p(\mathbb{R}^d)}$. By (2) in Definition 3.5, we have that \( \|\rho_t(\omega, B_r, \cdot)\|_{L^p(\mathbb{R}^d)} \leq 1, \|\rho_t(\omega, B_r, \cdot)\|_{L^1(\mathbb{R}^d)} \leq |B_r| \) and the same is true for $\tilde{p}_t(\omega, B_r)$. Hence

$$
\mathbb{E} \left[ \int_{B_r} g_1(x) \, dx \right] \leq 2C \int_{\Omega} (1 + |B_r|) \|\nabla b\|_{L^p(\mathbb{R}^d)} \, d\mathbb{P}(\omega)
$$

$$
= 2C(1 + |B_r|) \|\nabla b\|_{L^p(\mathbb{R}^d)}.
$$

Similarly, we have

$$
\mathbb{E} \left[ \int_{B_r} g_2(x) \, dx \right] \leq \frac{1}{\theta^2} \mathbb{E} \left[ \int_{B_r} |b(t, X_t(x)) - \tilde{b}(t, \bar{X}_t(x))| \, dx \right]
$$

$$
\leq \frac{1}{\theta^2} \int_{\Omega} \int_{\mathbb{R}^d} |b - \tilde{b}|(t, x) \rho_t(\omega, B_r, x) \, dx \, d\mathbb{P}(\omega)
$$

$$
\leq \frac{1}{\theta^2} (1 + |B_r|) \|b - \tilde{b}\|_{L^p(\mathbb{R}^d)}.
$$

Finally, we integrate in $t$, and take supremum over time $t$ for $\log \left( \frac{|X_t(x) - \bar{X}_t(x)|^2 + 1}{\theta^2} \right)$ and the proof is complete.

Now we are ready to prove the main result in this section.

**Theorem 3.8.** Given a divergence-free vector field $b \in L^1(0, T; W^{1,p}(\mathbb{R}^d)) \cap L^1(0, T; L^q(\mathbb{R}^d))$ with $d \geq 3$, $p \geq 1$, $\frac{2}{p} + \frac{2}{q} \in [1, 2)$, there is a unique almost everywhere stochastic flow $X(\omega, x) : \Omega \times \mathbb{R}^d \rightarrow C([0, T], \mathbb{R}^d)$ to

$$
dX_t(\omega, x) = b(t, X_t(\omega, x)) \, dt + dB_t(\omega), \quad X_0(\omega, x) = x
$$

in space $L^k(\Omega \times B_r; C([0, T], \mathbb{R}^d))$ for any $k \geq 1$ and $r > 0$.

**Proof.** Step 1: We prove the existence of solution $X_t$ using approximation. By cut-off and mollification, we can find a sequence of divergence-free $b^{(n)}(\omega) \in C([0, T], C_0^\infty(\mathbb{R}^d))$ such that $b^{(n)} \rightarrow b$ in $L^1(0, T; W^{1,p}(\mathbb{R}^d)) \cap L^1(0, T; L^q(\mathbb{R}^d))$ and denote by $X^{(n)}_t$ the corresponding solution. We first prove that $X^{(n)}_t$ is a Cauchy sequence in space $L^k(\Omega \times B_r; C([0, T]))$ for any $k \geq 1$ and $r > 0$. Denote

$$
\Omega_{n,m}(\omega) = \left\{ x \in \mathbb{R}^d : \sup_{0 \leq t \leq T} |X^{(n)}_t(\omega, x)| < R, \sup_{0 \leq t \leq T} |X^{(m)}_t(\omega, x)| < R \right\}
$$

and by Proposition 3.4 we have that for any fixed $r > 0$,

$$
\lim_{R \rightarrow \infty} \sup_{n, m} \sup \mathbb{P}(x \notin \Omega_{n,m}(\omega)) = 0.
$$

(3.8)
Set $S_T^{(n,m)}(\omega,x) = \sup_{0 \leq t \leq T} |X_t^{(n)}(\omega,x) - X_t^{(m)}(\omega,x)|^2$, then for any fixed $\delta > 0$ we have

$$
P \left( \omega : \int_{B_r} S_T^{(n,m)}(\omega,x) dx \geq 2\delta \right) \leq P \left( \omega : \int_{B_r \cap O_R^\theta(\omega)} S_T^{(n,m)}(\omega,x) dx \geq \delta \right) + \int_{B_r \cap O_R^\theta(\omega)} S_T^{(n,m)}(\omega,x) dx \geq \delta \right)
$$

To show convergence in probability of $X_t^{(n)}$, for any $\epsilon > 0$, we find $R$ and large enough $n,m$ such that $I_i^{(n,m)} \leq \epsilon$, $i = 1, 2$. We first estimate the second term $I_2^{(n,m)}$

$$
I_2^{(n,m)} \leq \frac{1}{\delta} \mathbb{E} \left[ \int_{B_r \cap O_R^\theta(\omega)} \sup_{0 \leq t \leq T} |X_t^{(n)}(\omega,x) - X_t^{(m)}(\omega,x)|^2 dx \right]
$$

$$
\leq \frac{1}{\delta} \int_{B_r} \int_{\Omega} \sup_{0 \leq t \leq T} |X_t^{(n)}(\omega,x) - X_t^{(m)}(\omega,x)|^2 1_{\{\omega \notin O_R^\theta(\omega)\}} dP(\omega) dx
$$

$$
\leq \frac{1}{\delta} \int_{B_r} 4 \mathbb{E} \left[ \sup_{0 \leq t \leq T} |X_t^{(n)}(\omega,x) - x|^4 + \sup_{0 \leq x \leq t} |X_t^{(m)}(\omega,x) - x|^4 \right] \mathbb{P}(\omega \notin O_R^\theta(\omega))^{\delta/2} dx
$$

$$
\leq \epsilon
$$

for large enough $R > M$ by (3.8) and Proposition 3.4. Here the choice of $M$ is independent of $(n,m)$. To obtain the estimate that $I_2^{(n,m)} \leq \epsilon$, we fixed an $R$. With the same $R$, next we estimate $I_1^{(n,m)}$. For any $\omega \in \Omega$, if

$$
\int_{B_r} \log \left( \frac{S_T^{(n,m)}(\omega,x)}{\theta^2} + 1 \right) dx \leq L,
$$

we have that $||S_T^{(n,m)}(\omega,x) \geq \theta^2(e^{L^2} - 1)|| \leq \frac{1}{e^2}$, which implies

$$
\int_{B_r \cap O_R^\theta(\omega)} S_T^{(n,m)}(\omega,x) dx = \int_{B_r \cap O_R^\theta(\omega)} S_T^{(n,m)}(\omega,x) 1_{\{S_T^{(n,m)}(\omega,x) \geq \theta^2(e^{L^2} - 1)\}} dx
$$

$$
+ \int_{B_r \cap O_R^\theta(\omega)} S_T^{(n,m)}(\omega,x) 1_{\{S_T^{(n,m)}(\omega,x) < \theta^2(e^{L^2} - 1)\}} dx
$$

$$
\leq \theta^2(e^{L^2} - 1)|B_r| + 4R^2 \frac{1}{L}.
$$

Now we set $\theta^{(n,m)} = \|b^{(n)} - b^{(m)}\|_{L^2}$ to obtain

$$
\sup_{n,m} \mathbb{E} \left[ \int_{B_r} \log \left( \frac{S_T^{(n,m)}(\omega,x)}{\theta^{(n,m)}} + 1 \right) dx \right] \leq C
$$

by Lemma 3.7, which implies that

$$
\sup_{n,m} \mathbb{P} \left( \int_{B_r} \log \left( \frac{S_T^{(n,m)}(\omega,x)}{\theta^{(n,m)}} + 1 \right) dx \geq L \right) \leq \frac{C}{L}
$$

For fixed $\delta > 0$ and the fixed $R$ obtained from the estimate of $I_2^{(n,m)}$, we can first choose $L$ large enough and then choose $(n,m)$ large enough, which means $\theta^{(n,m)}$ is small enough, such that

$$
(\theta^{(n,m)})^2(e^{L^2} - 1)|B_r| + 4R^2 \frac{1}{L} < \delta \quad \text{and} \quad \frac{C}{L} \leq \epsilon.
$$
\[
\mathbb{P} \left( \omega : \int_{B_r \cap O_{\varepsilon,m}(\omega)} S_T^{m,n}(\omega,x)dx \geq \delta, \int_{B_r} \log \left( \frac{S_T^{m,n}(\omega,x)}{(\theta(n,m))^2} \right) + 1 \right) dx \leq L = 0,
\]

which implies that
\[
I_1^{(n,m)} = \mathbb{P} \left( \omega : \int_{B_r \cap O_{\varepsilon,m}(\omega)} S_T^{m,n}(\omega,x)dx \geq \delta, \int_{B_r} \log \left( \frac{S_T^{m,n}(\omega,x)}{(\theta(n,m))^2} \right) + 1 \right) dx > L
\]
\[
\leq \mathbb{P} \left( \omega : \int_{B_r} \log \left( \frac{S_T^{m,n}(\omega,x)}{(\theta(n,m))^2} \right) + 1 \right) dx > L
\]
\[
\leq \varepsilon.
\]

Now we have that for any \( \varepsilon > 0 \), there is \((n,m)\) large enough such that
\[
\mathbb{P} \left( \omega : \int_{B_r} S_T^{m,n}(\omega,x)dx \geq 2\delta \right) \leq 2\varepsilon.
\]

Hence
\[
\lim_{n,m \to \infty} \mathbb{P} \left( \omega : \int_{B_r} S_T^{m,n}(\omega,x)dx \geq 2\delta \right) = 0
\]

for any fixed \( \delta \). This implies that for any fixed \( r > 0 \), \( \{X_{t}^{(n)}\} \) converges in probability under the finite measure \( \mathbb{P} \times m 1_{B_r} \) as functions \( X^{(n)} : \Omega \times B_r \to C([0,T], \mathbb{R}^d) \). Recall that for any \( k \geq 1 \) we have
\[
\sup_{n, t \in B_r} \mathbb{E} \left[ \sup_{0 \leq s \leq t} |X_{s}^{(n)}(x)|^k \right] < \infty
\]

by Proposition 3.4, which means that for any fixed \( k \geq 1 \), \( \sup_{0 \leq t \leq T} |X_{t}^{(n)}(x)|^k \) are uniformly integrable. This implies that for any fixed \( r > 0 \), \( \{X_{t}^{(n)}\}_n \) is a Cauchy sequence in \( L^1(\Omega \times B_r; C([0,T])) \) and we denote the limit as \( X_{t} \). Since each \( X_{t}^{(n)} \) satisfies Definition 3.5, it is easy to verify that their limit \( X_{t} \) also satisfies (1) and (2) of Definition 3.5.

Step 2: Now we verify that the limit \( X_{t} \) satisfies (3) of Definition 3.5, i.e. if we define
\[
Y_{t}(x) = x + \int_{0}^{t} b(s, X_{s}(x))ds + \int_{0}^{t} dB_{s},
\]

then \( X_{t} = Y_{t} \) in space \( L^1(\Omega \times B_r; C([0,T])) \). Consider
\[
\sup_{0 \leq t \leq T} |X_{t}^{n}(x) - Y_{t}(x)| \leq \int_{0}^{T} |b^n(t, X_{t}^{(n)}(x)) - b(t, X_{t}(x))|dt.
\]

Then integrate both sides on \( B_r \) and take expectation, we have that
\[
\mathbb{E} \left[ \int_{B_r} \sup_{0 \leq t \leq T} |X_{t}^{n}(x) - Y_{t}(x)|dx \right] \leq \mathbb{E} \left[ \int_{B_r} \int_{0}^{T} |b^n(t, X_{t}^{(n)}(x)) - b(t, X_{t}(x))|dt dx \right]
\]
\[
+ \mathbb{E} \left[ \int_{B_r} \int_{0}^{T} |b(t, X_{t}^{(n)}(x)) - b(t, X_{t}(x))|dt dx \right] = I_1 + I_2.
\]
Again by (2) in Definition 3.5, we have that\( I_1 \leq \langle 1 + |B_x| \rangle \| b - b^{(n)} \|_{L^p} \). For the second term, we will find another smooth \( b_\varepsilon \) such that \( \| b_\varepsilon - b \|_{L^p} \leq \varepsilon \) and then separate \( I_2 \) into three parts

\[
I_2 \leq \mathbb{E} \left[ \int_{B_\varepsilon} \int_0^T |b_\varepsilon(t,X^{(n)}(x)) - b_\varepsilon(t,X_t(x))|dt\,dx \right] + \mathbb{E} \left[ \int_{B_\varepsilon} \int_0^T |b_\varepsilon(t,X^{(n)}(x)) - b(t,X^{(n)}_t(x))|dt\,dx \right] + \mathbb{E} \left[ \int_{B_\varepsilon} \int_0^T |b_\varepsilon(t,X_t(x)) - b(t,X_t(x))|dt\,dx \right].
\]

For the second and the third terms, we control them just as \( I_1 \). The first term converges to 0 as \( n \rightarrow 0 \) since \( X^{(n)}_t \rightarrow X_t \) in \( L^k(\Omega \times B_t;C([0,T])) \) and now we can conclude that \( X_t = Y_t \) \( \mathbb{P} \times m \)-almost everywhere.

Step 3: Finally we prove that the solution is unique. Suppose that we have two almost everywhere stochastic flows \( X_t \) and \( \tilde{X}_t \) corresponding to the same \( b \in L^1(0,T;W^{1,p}(\mathbb{R}^d)) \cap L^q(0,T;L^q(\mathbb{R}^d)) \) and we apply Lemma 3.7 to them to deduce that

\[
\mathbb{E} \left[ \int_{B_\varepsilon} \log \left( \frac{\sup_{0 \leq t \leq T} |X_t(x) - \tilde{X}_t(x)|^2}{\theta^2} + 1 \right) dx \right] \leq C \| \nabla b \|_{L^p}^2,
\]

which is uniform for all \( \theta > 0 \). Hence we can take \( \theta \rightarrow 0 \) and now the proof is complete.

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