Sample, Computation vs Storage Tradeoffs for Classification Using Tensor Subspace Models

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Abstract—In this paper, we exhibit the tradeoffs between (training) sample, computation and storage complexity for the problem of supervised classification using signal subspace estimation. Our main tool is the use of tensor subspaces, i.e. subspaces with a Kronecker structure, for embedding the data into lower dimensions. Among the subspaces with a Kronecker structure, we show that using subspaces with a hierarchical structure for representing data leads to improved tradeoffs. One of the main reasons for the improvement is that embedding data into these hierarchical Kronecker structured subspaces prevents overfitting at higher latent dimensions.

I. INTRODUCTION

The principle of dimensionality reduction is important for many machine learning and statistical signal processing tasks. The simplest of these approaches, embeds the data into a low-dimensional linear subspace or a locally linear subspace. In this paper we exploit a sub-class of subspaces that have a tensor or Kronecker structure, namely that they are constructed out of tensor product of other low-dimensional subspaces. Among these we study the Tucker subspace [1], the Hierarchical-Tucker (HT) subspace [2], and the Tensor-Train (TT) [3] subspace models, though further generalizations are possible[1]. It is to be noted that finding the optimal tensor subspace representation in general is a computationally hard problem, although there exist several efficient algorithms [4]. In this paper we consider the approximations obtained by variations of the higher order and hierarchical singular value decomposition algorithms for finding tensor subspace representations [4, 5]. The main objective of the paper is to numerically study the tradeoffs between the sample complexity, computational cost of projection, storage (of the subspace representation) and error tradeoffs when using tensor subspaces for supervised classification [6–9]. While implicit, to the best of our knowledge, these tradeoffs are brought to attention for the first time via a direct numerical study.

Main results: For a fixed classification error, in this paper we note the following points regarding the tradeoffs.

- The storage complexity of the HT subspace is much higher compared to the Tucker subspace. This is not surprising since specification of HT subspace requires more parameters[2]. On the other hand the total cost of the projection onto the tensor subspace is lower for the HT compared to the Tucker model.
- The sample complexity, i.e. the number of data examples required to learn the subspace, is lower for HT compared to the Tucker. This is due to the fact that the overall dimension of the manifold of fixed rank HT subspace (polynomial in tensor order) is much smaller compared to the overall dimension of the manifold of fixed rank Tucker subspace (exponential in tensor order).

The rest of the paper is organized as follows. In section II we provide necessary notation and technical background. In section III we outline a simple algorithm for finding the hierarchical Tucker subspace fitting the data. In section IV we show numerical results for the problem of supervised classification and highlight the variety of tradeoffs between classification error, storage and computational complexity that can be achieved using these subspaces on a variety of image databases.

II. NOTATION AND BACKGROUND

Throughout this paper, we denote the set of integers \{1, \ldots, N\} via \([N]\). We denote the size of the set \(S\) via \(|S|\). We denote vectors with small bold-face letters like \(b\), matrices by capital bold-face letters like \(B\). The \(i^{th}\) column of a matrix \(B\) is denoted by \((B)_i\) and the \((i,j)\) element is denoted by \(B_{ij}\). Matrix fibers are extracted by using the colon notation. For example, columns of \(i\) to \(j\) are denoted by \(B(:,i:j)\). We depict trees with \(\mathbb{T}\) symbols. Tensors are denoted by bold-face calligraphic letters like \(\mathcal{X} \in \mathbb{R}^{I_1 \times \cdots \times I_n}\), where \(I_i\) is the size along the \(i^{th}\) dimension/direction of the tensor and \(n\) denotes the order of the tensor.

Tensor Subspaces: Most of the development has been distilled from [11]. All subspaces in this paper are subspaces of \(\mathbb{R}^d\) of appropriate dimension \(d\). The dimension \(d\) will be clear from the context. Recall that a single subspace \(\mathcal{S}\) of dimension \(r\) can be expressed as,

\[
\mathcal{S} = \text{col} - \text{span}\{\mathbf{u}\}
\]

\(\text{2}\)Technically speaking one can further reduce the cost of storage of the HT Subspace representations by considering the overall dimension of the subspace obtained by quotienting out equivalent representations [10]. However this does not reduce the orders and using this optimally compressed representation dramatically increases the projection costs by requiring a representation to be computed explicitly.
for some rank-$r$ matrix $U$. Given two subspaces $S_1 = \text{col} - \text{span}\{U_1 \in \mathbb{R}^{1 \times r_1}\}$, $S_2 = \text{col} - \text{span}\{U_2 \in \mathbb{R}^{2 \times r_2}\}$, a tensor subspace denoted $S_1 \otimes S_2$ is defined via,

\[ S_1 \otimes S_2 = \text{col} - \text{span}(U_1 \otimes U_2), \]

This construction can also be naturally extended to a collection of subspaces $S_1, ..., S_n$ yielding $\otimes_{i=1}^n S_i = \text{col} - \text{span}(\otimes_{i=1}^n U_i)$, where the notation $\otimes_{i=1}^n U_i$ is a short-hand for tensor/Kronecker products. Note that a single element say $x$ from this tensor product of subspaces can be expressed as,

\[ x_i = (U_1 \otimes \cdots \otimes U_n)c \]  

where $b$ is a vector of size $r_1 r_2 \cdots r_n$. One can reshape this vector into a core tensor $C$ of size $r_1 \times \cdots \times r_n$, thereby obtaining what is referred to as the Tucker decomposition $\text{span}$ of $x$ or of a tensor $X$ obtained by reshaping $x_i$.

**Hierarchical tensor subspaces** - Instead of a single level construction, one can construct tensor subspaces in a hierarchical manner. These subspaces are referred to as Hierarchical Tensor Subspaces or Hierarchical Tucker (HT) subspaces [5], and are essentially subspaces of $\otimes_{i=1}^n U_i$. Note that any subspace of $\otimes_{i=1}^n U_i$ of rank $r$ can be written as $(\otimes_{i=1}^n U_i)B$ where $B$ is a rank $r$ matrix. This matrix is also referred to as the transfer matrix. The Hierarchical Tucker subspace construction endows the matrix $B$ with an additional Kronecker structure, which in turn corresponds to a dimension tree.

From this expression it can be seen that the Hierarchical Tucker subspace defined by $\text{col} - \text{span}(U_{1,2,3,4})$ corresponds to and is defined with respect to the balanced dimension tree as shown in figure (1a).

The same approach could be Applied to another dimension tree as shown in figure (1b), referred to as the Tensor Train tree [3]. In this construction each higher-subspace comes from two lower-subspaces: $U_i$ where $i \in T = \{1,\ldots,n\}$ and the second subspace $U_j$ where $j \in T' = \{1,\ldots,n\}\setminus i$. Backslash means that $i$ and $j$ are mutually exclusive. Corresponding to this tree we note that,

\[
U_{1,2,3,4} = U_{1,2,3} \otimes U_4 = ((U_{1,2} \otimes U_3)(B_{1,2,3} \otimes I) \otimes U_4) \\
= (((U_1 \otimes U_2)B_{1,2}) \otimes U_3)(B_{1,2,3} \otimes I) \otimes U_4 \\
= (U_1 \otimes U_2 \otimes U_3 \otimes U_4)(B_{1,2} \otimes I)(B_{1,2,3} \otimes I)
\]

In general a dimension tree is defined as follows,

**Definition 2.1:** Dimension Tree [5]: A (binary) dimension tree $\mathcal{T}$ is a tree whose nodes are represented by a set $S$, that is a set of subsets of $[n]$ with root corresponding to the set $[n]$ and $n$ leaf nodes $\{i\}$ corresponding to the base subspaces $S_i$, $i = 1, 2, ..., n$. Every node $s \in S$ that is not a leaf has two sons $s_1, s_2$ that form an ordered partition of $S$, i.e., $s_1 \cup s_2 = S$, $s_1 \cap s_2 = \emptyset$. For example, the balanced HT tree for an order-4 tensor can be defined as,

\[ S = \{(1, 2, 3, 4), (1, 2), (3, 4), (1, 3, 4), (2, 3), (1, 4), (2, 4), (3, 4), (1), (2), (3), (4)\}, \]

Here the node $(3, 4)$ is the parent of 3 and 4, $(1, 2, 3, 4)$ is the parent of $(1, 2)$ and $(3, 4)$ and so on. We will use subscript to denote the transfer matrix corresponding to the nodes in a tree. For example, in the balanced tree example, $B_{2,3}$ denotes the transfer matrix for node $(2, 3)$.

### III. Estimating Hierarchical Subspaces

We are given $N$ data points $X_i \in \mathbb{R}^{I_1 \times \cdots \times I_n}$, $i \in [N]$ and a dimension tree $\mathcal{T}$ and the ranks corresponding to the nodes in the tree. The problem is to find the best HT subspace, i.e. estimate $U_j \in \mathbb{R}^{I_j \times r_j}$, $j \in [n]$ and the transfer matrices $B_s$ for all $s \in S$, that fits the data well in terms of least squares error in projection.

We note that estimating both the dimension tree and the subspaces is a hard problem as there are combinatorially many dimension trees possible. In this paper we restrict ourselves to the balanced trees, which with slight abuse of notation will be referred to as the Hierarchical Tucker (HT), and the tree corresponding to Tensor Train (TT). We now present an algorithm for estimating the Hierarchical Subspace. Further finding the best HT subspace approximation is also computationally hard and in this paper we consider a suboptimal algorithm.

**Hierarchical Subspace Learning Algorithm:** Before we describe the algorithm we need one more definition.

**Definition 3.1:** (see [5], [10], [12]) Unfolding: Let $s_1$ and $s_2$ be a partition of $[n]$. For an order-$n$ tensor $X \in \mathbb{R}^{I_1 \times \cdots \times I_n}$, 

\[ X_{s_1} = \mathcal{U} \times_s \mathcal{V} \times_s \mathcal{W}, \]

where $\mathcal{U} \in \mathbb{R}^{I_{s_1,1} \times \cdots \times I_{s_1,n}}$, $\mathcal{V} \in \mathbb{R}^{I_{s_2,1} \times \cdots \times I_{s_2,n}}$, and $\mathcal{W}$ is the tensor of size $I_{s_1} \times \cdots \times I_{s_2}$. The unfoldings of three tensors are defined by $X_{s_1 \cap s_2}$ to be the matrix obtained by the multiplication of $X$ with $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$, respectively.
$\mathbb{R}^{I_1 \times \cdots \times I_n}$, unfolding along $s_1$, denoted by $Unfold(\mathbf{X}) = \mathbf{X}^{(s_1)}$ is a matrix of size $I_{s_1} \times I_{s_2}$ where,

$$I_{s_1} = \prod_{i \in s_1} I_i \quad I_{s_2} = \prod_{j \in s_2} I_j,$$

(4)

obtained by lexicographically combining the indices belonging to $s_1$ into row indices, and those belonging to $s_2$ into column indices.

Algorithm 1 is a simple variant of the Hierarchical SVD [13] computing H-Tucker representation of a single datum. The algorithm takes $N$ tensors $\mathbf{X}_1, \ldots, \mathbf{X}_n \in \mathbb{R}^{I_1 \times \cdots \times I_n}$, which belong to one class of data as input. Using the specified dimension tree, the algorithm computes the hierarchical subspace, returned as the column span of the matrix $\mathbf{H}$, using a Depth First Traverse (leaves to root) on the dimension tree.

The subspaces corresponding to each node are computed using a truncated SVD on the node unfolding and the transfer tensors are computed using the projections on the tensor product of subspace of the node’s children (except the root).

Algorithm 1 Hierarchical Tensor subspace learning

INPUT: $\mathbf{X}_1, \ldots, \mathbf{X}_n \in \mathbb{R}^{I_1 \times \cdots \times I_n}$, $T$ the tree, $r$ the rank of each node of the tree except the root.

$S \leftarrow$ Depth-First- Traverse($T$)

Stack all $\mathbf{X}_i$ to make $\mathbf{X} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_n \times N}$ for $i$ in reverse($S$)

if $|S| = 1$ then

$\mathbf{X}^{(s)} \leftarrow Unfold(\mathbf{X})$ along $s$ (see Definition 3.1)

$\mathbf{X}^{(s)} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T \%$ SVD

$\mathbf{U}_s \leftarrow \mathbf{U}(;1:r_s)$,

save $\mathbf{U}_s$

else if $s$ not root then

Split $s$ into its children; $s_1$ and $s_2$

\[ \mathbf{U} \leftarrow \mathbf{U}_{s_2} \otimes \mathbf{U}_{s_1} \]

$\mathbf{X}^{(s)} \leftarrow Unfold(\mathbf{X})$ along $s$

$\mathbf{X} \leftarrow \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$

$\mathbf{X} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$

$\mathbf{U}_s \leftarrow \mathbf{U}(;1:r_s) \%$ Top $r_s$ singular vectors

$\mathbf{B}_s \leftarrow \mathbf{U}^\top\mathbf{U}_s$

store $\mathbf{U}_s$ and $\mathbf{B}_s$

else if $s$ is root then

split root into its children $s_1$ and $s_2$

$\mathbf{H} = \mathbf{U}_{s_1} \otimes \mathbf{U}_{s_2}$

end if

end for

Return $\mathbf{H}$

IV. TRADEOFFS BETWEEN ERROR, STORAGE AND PROJECTION COST USING TENSOR SUBSPACES

We now investigate the storage and projection tradeoffs for various tensor subspace representations on some real data sets. In the following experiments, we use the PIE [14] and Weizmann [15] face data sets.

Preprocessing: We centered all of the images by subtracting the mean of the samples. For PIE database each picture is of size 486 x 640 and for Weizmann database each image is of size 512 x 288. We reshaped these images to $\mathbf{X} \in \mathbb{R}^{18 \times 27 \times 32 \times 20}$ (PIE data) and $\mathbf{X} \in \mathbb{R}^{16 \times 32 \times 16 \times 18}$ (Weizmann data). In short, we reshape each 2-D matrix to a 4-D tensor. We set aside 50 percent of the samples of each subject for testing and 50 percent for training.

Procedure: For training, we select 4 subjects (classes) out of a collection of 18 subjects randomly for each experiment and estimate the Hierarchical Tensor subspace for each subject. For testing and classifying a data point, we project it on the Hierarchical Subspaces computed for each subject and choose the one with the maximal projection energy.

For Hierarchical Tucker, one can take two different approaches for computing the projection $\mathbf{X}_{pr}$ from a test data point, say, $\mathbf{X}_0$, all of which have different projection and storage costs.

1) $\text{Hier approach 1: } \mathbf{X}_{pr} = \mathbf{U}_{1,2}^T \mathbf{X}_{0}^{(1,2)} (\mathbf{U}_{3,4})^T$. Recall that $\mathbf{X}_0^{(1,2)}$ is an unfolding of $\mathbf{X}_0$ - c.f. Definition 3.1.

We only have to store $\mathbf{U}_{1,2}$ and $\mathbf{U}_{3,4}$. In a case where $\mathbf{U}_{1,2}, \mathbf{U}_{3,4} \in \mathbb{R}^{n_s \times r'}$, the cost of projection is $n_s r' + n_s^2 r' + r'^2 + r'^2$ and the cost of storage is $2n_s^2 r'$.

2) $\text{Hier approach 2: } \mathbf{X}_{pr} = ((\mathbf{U}_1 \otimes \mathbf{U}_2) \mathbf{B}_{1,2})^T (\mathbf{X}_0^{(1,2)} (\mathbf{U}_3 \otimes \mathbf{U}_4) \mathbf{B}_{3,4})^T$.

In this method, we store the leaf subspaces and the transfer matrices and then build $\mathbf{U}_{1,2}$ and $\mathbf{U}_{3,4}$. In a case where all of the leaf subspaces are in $\mathbb{R}^{n_s \times r}$ and the transfer matrices are in $\mathbb{R}^{r_s \times r'}$, the cost of projection is $n_s r' + n_s^2 r^2 + r'^2 + 2n_s^2 r + 2n_s^2 r^2 + 2n_s^2 r'^2$ and the cost of storage is $4nr + 2nr^2$.

For the Tensor Train, we use multiplication in the following format $\mathbf{X}_{pr} = \mathbf{U}_{1,2,3}^{(1,2,3)} (\mathbf{U}_4^T)$, where we only have to store $\mathbf{U}_{1,2,3}$ and $\mathbf{U}_4$. In a case where $\mathbf{U}_{1,2,3} \in \mathbb{R}^{n_s \times r}$ and $\mathbf{U}_4 \in \mathbb{R}^{r_s \times r'}$, the cost of projection is $n_s r + n_s^2 r r'$ and the cost of storage is $n_s r^3 + r s^3$. For Tucker, we only store the subspaces. For example, in a case where all of the subspaces are in $\mathbb{R}^{n_s \times r}$, the cost of projection is $n_s^2 r + n_s^2 r^2 + n_s^2 r^3 + n_s^2 r^3 + n_s^2 r^4 + r^4$ and the cost of storage is $4nr$.

After computing the projection of each test point onto the subspaces corresponding to each class, we label the test tensor with the class that has the largest Frobenius norm of the projection. We use,

$$\text{error} = \frac{\# \text{misclassified points}}{\# \text{total test points}}$$

as the measure of misclassification. We repeat each experiment (selecting the subject randomly from the set of 18 subjects) 10 times and average over the results.

Cost Comparison - In the following experiments, we varied $r_i, i \in 1, 2, 3, 4$ from 10% to 100% of the full rank. For Hierarchical and Tensor Train (TT), we set the leaf level subspaces to 70% of the full rank and varied $r' = r_{1,2} = r_{3,4}$. As seen in the figures (2a, 2b, 3a and 3b), we plotted the error rates versus the cost of normalized: 1. Storage 2. Projection. We normalized the costs by dividing them by the number of the dimensions of the vectorized tensors. We can see that different limitations in storage or projection play role in choosing which algorithm performs better.
In figures (2a) and (3a), we observe that the cost of storage of the Tucker subspace representation is smaller than Hierarchical Tucker and Tensor Train, however the error rates of Hierarchical Tucker and Tensor Train are much smaller.

We can also observe that using the Tucker subspace leads to a very strong overfitting for higher ranks. This is due to using large ranks which brings about extra complexity. The bad performance of using the Tucker subspace representation at higher ranks demonstrates that the Tucker method is sensitive to noise, however Hierarchical Tucker is much more robust and Tensor Train demonstrates no overfitting at all.

In figures (2b) and (3b), we observe that the cost of projection for HT and TT are almost the same as the Tucker representation. We also observe that the classification error of Tensor Train is smaller than the Tucker method at any given computation cost.

Sample complexity vs error: We further evaluate the methods for sample complexity, i.e. the number of samples required to achieve a given classification error. As demonstrated in figures (4a) and (4b), Hierarchical methods (TT and HT) tend to perform better in a sense that they need fewer points in order to achieve the same classification error. Among Hierarchical Subspace models, the Tensor Train performs better compared to Hierarchical Tucker (corresp. to the balanced tree). This is particularly interesting, since it shows how the choice of the tree can affect the performance.
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