A generalized Trudinger-Moser inequality on a compact Riemannian surface with conical singularities

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Abstract In this paper, using the method of blow-up analysis, we establish a generalized Trudinger-Moser inequality on a compact Riemannian surface with conical singularities. Precisely, let $(\Sigma, D)$ be such a surface with divisor $D = \sum_{i=1}^{m} p_i$, where $p_i > -1$ and $p_i \in \Sigma$ for $i = 1, \ldots, m$, and $g$ be a metric representing $D$. Denote $b_0 = 4\pi (1 + \min_{1 \leq i \leq m} \beta_i)$. Suppose $\psi : \Sigma \to \mathbb{R}$ is a continuous function with $\int_{\Sigma} \psi \, dv_g \neq 0$ and define

$$
\lambda_1^+(\Sigma, g) = \inf_{u \in H^1(\Sigma, g), \int_{\Sigma} \psi \, dv_g = 0, \int_{\Sigma} u^2 \, dv_g = 1} \int_{\Sigma} |\nabla_g u|^2 \, dv_g.
$$

Then for any $\alpha \in [0, \lambda_1^+(\Sigma, g))$, we have

$$
\sup_{u \in H^1(\Sigma, g), \int_{\Sigma} \psi \, dv_g = 0, \int_{\Sigma} u^2 \, dv_g = 1} \int_{\Sigma} e^{bu^2} \, dv_g < +\infty.
$$

When $b > b_0$, the integrals $\int_{\Sigma} e^{bu^2} \, dv_g$ are still finite, but the supremum is infinity. Moreover, we prove that the extremal function for the above inequality exists.

Keywords Trudinger-Moser inequality, the extremal function, blow-up analysis, conical singularity

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1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain, and $H^1_0(\Omega)$ be the usual Sobolev space. The classical Trudinger-Moser inequality (see [20–22,27,34]) says that

$$
\sup_{u \in H^1_0(\Omega), \int_{\Omega} |\nabla u|^2 \, dx \leq 1} \int_{\Omega} e^{4\pi u^2} \, dx < +\infty.
$$

When $a > 4\pi$, the integrals $\int_{\Omega} e^{au^2} \, dx$ are still finite, but the supremum is infinity. A natural question is whether the supremum in (1.1) can be attained by some extremal function. This was answered by Carleson and Chang [4], Struwe [24], Flucher [10] and Lin [17].
Let \( \{u_k\} \subset H^1_0(\Omega) \) with \( \int_\Omega |\nabla u_k|^2 dx = 1 \) such that \( u_k \rightharpoonup u_0 \) weakly in \( H^1_0(\Omega) \) as \( k \to \infty \). It was proved by Lions [18] that for any \( p < 1/(1 - \int_\Omega |\nabla u_0|^2 dx) \),

\[
\limsup_{k \to \infty} \int_\Omega e^{4\pi u_k^2} dx < +\infty. \quad (1.2)
\]

When \( u_0 \not\equiv 0 \), (1.2) gives more information than (1.1). While if \( u_0 \equiv 0 \), (1.2) gives nothing more than (1.1). Later, Adimurthi and Druet [1] defined

\[
C_\alpha(\Omega) = \sup_{u \in H^1_0(\Omega), f_\alpha |\nabla u|^2 dx = 1} \int_\Omega e^{4\pi u^2(1 + f_\alpha u^2 dx)} dx.
\]

Let \( \lambda_1(\Omega) \) be the first eigenvalue of the Laplacian with Dirichlet boundary condition in \( \Omega \). They proved the following: (i) For any \( \alpha < \lambda_1(\Omega) \), \( C_\alpha(\Omega) < +\infty \); (ii) for any \( \alpha \geq \lambda_1(\Omega) \), \( C_\alpha(\Omega) = +\infty \). This result was generalized by Yang [28] to the higher dimensional case, and by Lu and Yang [19] to a version involving \( L^p \)-norm. For any \( \alpha < \lambda_1(\Omega) \), Tintarev [25] proved that

\[
\sup_{u \in H^1_0(\Omega), f_\alpha |\nabla u|^2 dx - \alpha f_\alpha u^2 dx \leq 1} \int_\Omega e^{4\pi u^2} dx < +\infty, \quad (1.3)
\]

while existence of the extremal function for (1.3) was obtained by Yang [30].

It is also of significance to consider the singular Trudinger-Moser inequality. For solving singular PDEs like

\[-\Delta u = \frac{ue^{bu^2}}{|x|^{2\beta}}, \quad u \in H^1_0(\Omega), \quad u \geq 0 \quad \text{in} \quad \Omega,\]

Adimurthi and Sandeep [2] studied the functional \( \int_\Omega \frac{e^{b u^2}}{|x|^{2\beta}} dx \) and proved that for any \( \beta \in (0, 1) \), it holds that

\[
\sup_{u \in H^1_0(\Omega), f_\alpha |\nabla u|^2 dx \leq 1} \int_\Omega \frac{e^{4\pi(1-\beta)u^2}}{|x|^{2\beta}} dx < +\infty. \quad (1.4)
\]

The extremal function for (1.4) was obtained recently by Csató and Roy [8]. Yang and Zhu [32] extended (1.4) to the following: For any \( \beta \in (0, 1) \) and \( \alpha \in [0, \lambda_1(\Omega)) \), we have

\[
\sup_{u \in H^1_0(\Omega), f_\alpha |\nabla u|^2 dx = 1} \int_\Omega \frac{e^{4\pi(1-\beta)u^2}}{|x|^{2\beta}} dx < +\infty. \quad (1.5)
\]

Moreover, the extremal function for (1.5) was also obtained in [32]. In [3], Adimurthi and Yang generalized (1.4) to the whole plane in order to solve a type of nonhomogeneous singular PDEs like

\[-\Delta u + V(x)u = \frac{f(x, u)}{|x|^{2\beta}} + ch(x), \quad x \in \mathbb{R}^2,\]

where \( V \in C^0(\mathbb{R}^2) \) has a positive lower bound, \( f(x, s) \in C^0(\mathbb{R}^2 \times \mathbb{R}) \) behaves like \( e^{\beta |s|^2} \) as \( |s| \to \infty \), and \( h \) belongs to the dual space of \( H^1(\mathbb{R}^2) \). Recently, Li and Yang [13] proved existence of the extremal function for the singular Trudinger-Moser inequality in [3].

A generalization of (1.1) to a Riemannian surface is important in analysis and geometry. Let \( (\Sigma, g) \) be a compact Riemannian surface and \( H^1(\Sigma, g) \) be the usual Sobolev space. It was proved by Moser [20] and Fontana [11] that

\[
\sup_{u \in H^1(\Sigma, g), \int_\Sigma u dv_g = 0, f_\alpha |\nabla u|^2 dv_g \leq 1} \int_\Sigma e^{4\pi u^2} dv_g < +\infty. \quad (1.6)
\]
When $a > 4\pi$, the integrals $\int_{\Sigma} e^{au^2} dv_g$ are still finite, but the supremum is infinity. Existence of the extremal function for (1.6) was due to Li [14, 15]. The Trudinger-Moser inequality of Adimurthi-Druet type on a Riemannian surface was obtained by Yang [29]. Precisely, denoting

$$
\lambda_1(\Sigma, g) = \inf_{u \in H^1(\Sigma, g), \int_{\Sigma} u dv_g = 0, \int_{\Sigma} u^2 dv_g = 1} \int_{\Sigma} |\nabla_g u|^2 dv_g,
$$

he proved that for any $\alpha < \lambda_1(\Sigma, g)$,

$$
\sup_{u \in H^1(\Sigma, g), \int_{\Sigma} u dv_g = 0, \int_{\Sigma} u^2 dv_g = 1} \int_{\Sigma} e^{4\pi u^2 (1 + \alpha)} dv_g < +\infty,
$$

and the supremum is infinity for any $\alpha \geq \lambda_1(\Sigma, g)$. Furthermore, the extremal function for (1.7) exists when $\alpha$ is sufficiently small. In [30], Yang obtained the extremal function for the inequality

$$
\sup_{u \in H^1(\Sigma, g), \int_{\Sigma} u dv_g = 0, \int_{\Sigma} u^2 dv_g = 1} \int_{\Sigma} e^{4\pi u^2} dv_g < +\infty
$$

for all $\alpha < \lambda_1(\Sigma, g)$. Besides, Yang [31] considered a Trudinger-Moser inequality on a Riemannian surface involving the Gaussian curvature. Let $K_g$ be the Gaussian curvature of $(\Sigma, g)$. Suppose $\int_{\Sigma} K_g dv_g \neq 0$ and denote

$$
\lambda_1^*(\Sigma, g) = \inf_{u \in H^1(\Sigma, g), \int_{\Sigma} K_g u dv_g = 0, \int_{\Sigma} u^2 dv_g = 1} \int_{\Sigma} |\nabla_g u|^2 dv_g.
$$

He proved that for any $\alpha < \lambda_1^*(\Sigma, g)$,

$$
\sup_{u \in H^1(\Sigma, g), \int_{\Sigma} K_g u dv_g = 0, \int_{\Sigma} u^2 dv_g = 1} \int_{\Sigma} e^{4\pi u^2} dv_g < +\infty;
$$

the extremal function for (1.8) was also obtained.

When a Riemannian surface has conical singularities, the situation becomes more subtle. To study the problem of prescribing Gaussian curvature on compact surfaces with conical singularities, Troyanov [26] derived a Trudinger-Moser inequality on such surfaces. Precisely, suppose $(\Sigma, D)$ has conical singularities of divisor $D = \sum_{i=1}^{m} \beta_i p_i$, where $\beta_i \in (-1, +\infty)$ and $p_i \in \Sigma$ for $i = 1, \ldots, m$, and $g$ represents $D$. Denote $b_0 = 4\pi(1 + \min_{1 \leq i \leq m} \beta_i)$. He proved that for any $b \in (0, b_0)$,

$$
\sup_{u \in H^1(\Sigma, g), \int_{\Sigma} u dv_g = 0, \int_{\Sigma} |\nabla_g u|^2 dv_g = 1} \int_{\Sigma} e^{bu^2} dv_g < +\infty.
$$

With an isoperimetric inequality, Chen [5] proved that (1.9) still holds when $b = b_0$. Moreover, Chen showed by constructing a sequence of Moser’s functions that (1.9) fails to hold when $b > b_0$. Recently, Iula and Mancini [12] proved existence of the extremal function for (1.9) with $b = b_0$.

In this paper, we shall establish a generalized Trudinger-Moser inequality on a Riemannian surface with conical singularities by using the method of blow-up analysis (see Section 2 below).

The rest of this paper is organized as follows: In Section 2, we give some notation about a Riemannian surface with conical singularities and present our main results. We devote Section 3 to the proof of Theorem 2.1. In Section 4, we show existence of the extremal function and prove Theorem 2.3. Throughout this paper, we use $C$ to denote a constant whose value may change from line to line, and we do not distinguish a sequence and its subsequences.

## 2 Notation and main results

In this section, we shall review some knowledge about a Riemannian surface with conical singularities and present our main results.
Troyanov [26] defined a metric representing a divisor as follows: Let \( \Sigma \) be a compact Riemannian surface, and \( D = \sum_{i=1}^{m} \beta_i p_i \) be a divisor, where \( \beta_i > -1 \) and \( p_i \in \Sigma \) for \( i = 1, \ldots, m \). Denote the support of \( D \) by \( \text{supp} \, D = \{ p_1, \ldots, p_m \} \). A metric \( g \) on \( \Sigma \) is said to represent the divisor \( D \) if \( g \) is a \( C^2 \) Riemannian metric on \( \Sigma \setminus \text{supp} \, D \) such that if \( z_i \) is a coordinate defined in a neighborhood \( U_i \) of \( p_i \), then there exists a continuous function \( u : U_i \to \mathbb{R} \), which is of class \( C^2 \) on \( U_i \setminus \{ p_i \} \), and such that in \( U_i \)

\[
g = e^{2u}|z_i - z_i(p_i)|^{2\beta_i}|dz_i|^2.
\]

By the definition we know, for any smooth metric \( g_0 \) on \( \Sigma \), there exists a function \( \rho \) on \( \Sigma \) such that \( g = \rho g_0 \). In [26], Troyanov proved the following:

**Fact 1.** \( H^1(\Sigma, g) = H^1(\Sigma, g_0) \).

**Fact 2.** The embedding \( H^1(\Sigma, g) \subset L^p(\Sigma, g) \) is compact for all \( p \in [1, +\infty) \).

**Fact 3.** Let \( \psi \) be a continuous function on \( \Sigma \) such that \( \int_\Sigma \psi \, dv_g \neq 0 \). Then there exists a positive constant \( C \) such that for any \( u \in H^1(\Sigma, g) \) with \( \int_\Sigma \psi u \, dv_g = 0 \) it holds that

\[
\int_\Sigma u^2 \, dv_g \leq C \int_\Sigma |\nabla_g u|^2 \, dv_g.
\]

Our first result reads as follows.

**Theorem 2.1.** Let \( (\Sigma, D) \) be a compact Riemannian surface with conical singularities of divisor \( D = \sum_{i=1}^{m} \beta_i p_i \), where \( \beta_i > -1 \) and \( p_i \in \Sigma \) for \( i = 1, \ldots, m \), and \( g \) be a metric representing \( D \). Denote \( b_0 = 4\pi(1 + \min_{1 \leq i \leq m} \beta_i) \). Suppose \( \psi : \Sigma \to \mathbb{R} \) is a continuous function with \( \int_\Sigma \psi \, dv_g \neq 0 \) and denote

\[
\lambda_1^*(\Sigma, g) = \inf_{u \in H^1(\Sigma, g), \int_\Sigma \psi u \, dv_g = 0, \int_\Sigma |\nabla_g u|^2 \, dv_g = 1} \int_\Sigma |\nabla_g u|^2 \, dv_g.
\]

Then for any \( \alpha \in [0, \lambda_1^*(\Sigma, g)) \), we have

\[
\sup_{u \in H^1(\Sigma, g), \int_\Sigma \psi u \, dv_g = 0, \int_\Sigma |\nabla_g u|^2 \, dv_g = 1} \int_\Sigma e^{b_0 u^2} \, dv_g < +\infty.
\]

When \( b > b_0 \), the integrals \( \int_\Sigma e^{b_0 u^2} \, dv_g \) are still finite, but the supremum is infinity.

**Remark 2.2.** When \( \psi \equiv 1 \) and \( \alpha = 0 \), Theorem 2.1 is just Chen’s main result in [5].

Concerning the extremal function, we have the following theorem.

**Theorem 2.3.** Under the same assumptions as in Theorem 2.1, for any \( \alpha \in [0, \lambda_1^*(\Sigma, g)) \), the supremum in (2.1) can be attained by some \( u_0 \in H^1(\Sigma, g) \) with \( \int_\Sigma \psi u_0 \, dv_g = 0 \) and \( \int_\Sigma |\nabla_g u|^2 \, dv_g = 1 \).

As an easy consequence of Theorem 2.1, the following corollary holds.

**Corollary 2.4.** Assume \( \psi : \Sigma \to \mathbb{R} \) is a continuous function with \( \int_\Sigma \psi \, dv_g = 0 \). Then for any \( \alpha \in [0, \lambda_1^*(\Sigma, g)) \), there exists a constant \( C \) such that for all \( u \in H^1(\Sigma, g) \) with \( \int_\Sigma \psi u \, dv_g = 0 \),

\[
\int_\Sigma e^{u} \, dv_g \leq Ce^{b_0} \int_\Sigma |\nabla_g u|^2 \, dv_g - \alpha \int_\Sigma u^2 \, dv_g.
\]

**Remark 2.5.** When \( g \) is smooth and \( \psi \equiv 1 \), Corollary 2.4 was proved by Ding et al. [9] for \( \alpha = 0 \) and by Yang and Zhu [33] for \( \alpha \in (0, \lambda_1^*(\Sigma, g)) \). When \( g \) is smooth, we derived this corollary in [35].

### 3 Proof of Theorem 2.1

In this section, we give the proof of Theorem 2.1. We shall divide the proof into four subsections. In the rest of this paper, we always assume \( \int_\Sigma \psi \, dv_g \neq 0 \).
3.1 Extremal functions for subcritical inequalities

In this subsection, we prove that extremal functions for subcritical inequalities exist. Precisely, we prove the following lemma.

Lemma 3.1. For any $\epsilon \in (0, 1)$, there exists a $u_\epsilon \in H^1(\Sigma, g) \cap C^1(\Sigma \setminus \text{supp } D) \cap C^0(\Sigma)$ with $\int_{\Sigma} \nabla_g u_\epsilon \cdot dv_g = 0$ and $\int_{\Sigma} |\nabla_g u_\epsilon|^2 dv_g = 1$ such that

$$\int_{\Sigma} e^{b_\epsilon(1-\epsilon)u_\epsilon^2} dv_g = \sup_{u \in H^1(\Sigma, g), \int_{\Sigma} \nabla_g u \cdot dv_g = 0, \int_{\Sigma} |\nabla_g u|^2 dv_g \leq 1} \int_{\Sigma} e^{b_\epsilon(1-\epsilon)u^2} dv_g.$$

Moreover, the equation of $u_\epsilon$ is

$$\begin{cases}
\Delta_g u_\epsilon + \alpha u_\epsilon = -\lambda_\epsilon^{-1} u_\epsilon e^{b_\epsilon(1-\epsilon)u_\epsilon^2} + \frac{\psi}{\int_{\Sigma} \psi} \mu_\epsilon, \\
\lambda_\epsilon = \int_{\Sigma} u_\epsilon^2 e^{b_\epsilon(1-\epsilon)u_\epsilon^2} dv_g, \\
\mu_\epsilon = \lambda_\epsilon^{-1} \int_{\Sigma} u_\epsilon e^{b_\epsilon(1-\epsilon)u_\epsilon^2} dv_g + \alpha \int_{\Sigma} u_\epsilon dv_g.
\end{cases}$$

Proof. For any $\epsilon \in (0, 1)$, we choose $u_j \in H^1(\Sigma, g)$ with $\int_{\Sigma} \nabla_g u_j \cdot dv_g = 0$ and $\int_{\Sigma} |\nabla_g u_j|^2 dv_g \leq 1$ such that

$$\int_{\Sigma} e^{b_\epsilon(1-\epsilon)u_j^2} dv_g \to \sup_{u \in H^1(\Sigma, g), \int_{\Sigma} \nabla_g u \cdot dv_g = 0, \int_{\Sigma} |\nabla_g u|^2 dv_g \leq 1} \int_{\Sigma} e^{b_\epsilon(1-\epsilon)u^2} dv_g$$

as $j \to \infty$. By the Poincaré inequality (see Fact 3), $u_j$ is bounded in $H^1(\Sigma, g)$. Thus in view of Sobolev embedding (see Fact 2), we can assume up to a subsequence that

$$\begin{cases}
u_j \to u_\epsilon \text{ weakly in } H^1(\Sigma, g), \\
u_j \to u_\epsilon \text{ strongly in } L^p(\Sigma, g), \ \forall p \geq 1, \\
u_j \to u_\epsilon \text{ almost everywhere in } \Sigma.
\end{cases}$$

Then we have

$$\int_{\Sigma} |\nabla_g u_j|^2 dv_g \leq \liminf_{j \to \infty} \int_{\Sigma} |\nabla_g u_j|^2 dv_g.$$  

By (3.3) and (3.4) one obtains

$$\int_{\Sigma} |\nabla_g u_j|^2 dv_g - \alpha \int_{\Sigma} u_j^2 dv_g \leq 1.$$

Thus,

$$\int_{\Sigma} |\nabla_g (u_j - u_\epsilon)|^2 dv_g = \int_{\Sigma} |\nabla_g u_j|^2 dv_g - \int_{\Sigma} |\nabla_g u_\epsilon|^2 dv_g + o_j(1)$$

$$\leq 1 - \left( \int_{\Sigma} |\nabla_g u_\epsilon|^2 dv_g - \alpha \int_{\Sigma} u_\epsilon^2 dv_g \right) + o_j(1),$$

where $o_j(1) \to 0$ as $j \to \infty$. It follows from (3.3) that $\int_{\Sigma} (u_j - u_\epsilon) \cdot dv_g \to 0$ as $j \to +\infty$. This together with (3.5) and Troyanov’s inequality (1.9) tells us that $e^{b_\epsilon(1-\epsilon)(u_j - u_\epsilon)^2}$ is bounded in $L^{p_1}(\Sigma, g)$ for some $p_1 > 1$. Then by Cauchy’s and Hölder’s inequalities, we have

$$e^{b_\epsilon(1-\epsilon)u_j^2} \text{ is bounded in } L^{p_2}(\Sigma, g) \text{ for some } p_2 > 1.$$  

Hence, by the mean value theorem and Hölder’s inequality we know

$$\lim_{j \to \infty} \int_{\Sigma} e^{b_\epsilon(1-\epsilon)u_j^2} dv_g = \int_{\Sigma} e^{b_\epsilon(1-\epsilon)u^2} dv_g.$$
Combining (3.2) and (3.7), we have
\[ \int_{\Sigma} e^{b_0(1-\epsilon)u^2} dv_g = \sup_{u \in H^1(\Sigma, g), \int_{\Sigma} \psi u dv_g = 0, \int_{\Sigma} |\nabla u|^2 dv_g - \alpha \int_{\Sigma} u^2 dv_g \leq 1} \int_{\Sigma} e^{b_0(1-\epsilon)u^2} dv_g. \]

It is obvious that \( u_\epsilon \in H^1(\Sigma, g) \) satisfying \( \int_{\Sigma} \psi u_\epsilon dv_g = 0 \) and \( \int_{\Sigma} |\nabla u_\epsilon|^2 dv_g - \alpha \int_{\Sigma} u_\epsilon^2 dv_g = 1 \). By a direct calculation, one obtains (3.1). The regularity of \( u_\epsilon \) follows from standard elliptic estimates.

**Lemma 3.2.** It holds that
\[ \lim_{\epsilon \to 0} \int_{\Sigma} e^{b_0(1-\epsilon)u^2} dv_g = \sup_{u \in H^1(\Sigma, g), \int_{\Sigma} \psi u dv_g = 0, \int_{\Sigma} |\nabla u|^2 dv_g - \alpha \int_{\Sigma} u^2 dv_g \leq 1} \int_{\Sigma} e^{b_0(1-\epsilon)u^2} dv_g. \]  

**Proof.** By Lemma 3.1, we know \( \int_{\Sigma} e^{b_0(1-\epsilon)u^2} dv_g \) increases when \( \epsilon \) tends to 0. Thus, the limit on the right-hand side of (3.8) is meaningful. For any \( u \in H^1(\Sigma, g) \) with \( \int_{\Sigma} \psi u dv_g = 0 \) and \( \int_{\Sigma} |\nabla u|^2 dv_g - \alpha \int_{\Sigma} u^2 dv_g \leq 1 \), one has
\[ \int_{\Sigma} e^{b_0u^2} dv_g = \lim_{\epsilon \to 0} \int_{\Sigma} e^{b_0(1-\epsilon)u^2} dv_g \]
\[ \leq \sup_{u \in H^1(\Sigma, g), \int_{\Sigma} \psi u dv_g = 0, \int_{\Sigma} |\nabla u|^2 dv_g - \alpha \int_{\Sigma} u^2 dv_g \leq 1} \int_{\Sigma} e^{b_0(1-\epsilon)u^2} dv_g \]
\[ = \lim_{\epsilon \to 0} \int_{\Sigma} e^{b_0(1-\epsilon)u^2} dv_g. \]

Thus, we have
\[ \sup_{u \in H^1(\Sigma, g), \int_{\Sigma} \psi u dv_g = 0, \int_{\Sigma} |\nabla u|^2 dv_g - \alpha \int_{\Sigma} u^2 dv_g \leq 1} \int_{\Sigma} e^{b_0u^2} dv_g \leq \lim_{\epsilon \to 0} \int_{\Sigma} e^{b_0(1-\epsilon)u^2} dv_g. \]  

On the other hand, since \( u_\epsilon \in H^1(\Sigma, g) \) satisfying \( \int_{\Sigma} \psi u_\epsilon dv_g = 0 \) and \( \int_{\Sigma} |\nabla u_\epsilon|^2 dv_g - \alpha \int_{\Sigma} u_\epsilon^2 dv_g = 1 \), we have
\[ \lim_{\epsilon \to 0} \int_{\Sigma} e^{b_0(1-\epsilon)u_\epsilon^2} dv_g \leq \sup_{u \in H^1(\Sigma, g), \int_{\Sigma} \psi u dv_g = 0, \int_{\Sigma} |\nabla u|^2 dv_g - \alpha \int_{\Sigma} u^2 dv_g \leq 1} \int_{\Sigma} e^{b_0u^2} dv_g. \]  

Combining (3.9) and (3.10) we obtain (3.8). This completes the proof.

### 3.2 Blow-up analysis

Since \( u_\epsilon \in H^1(\Sigma, g) \) satisfying \( \int_{\Sigma} \psi u_\epsilon dv_g = 0 \) and \( \int_{\Sigma} |\nabla u_\epsilon|^2 dv_g - \alpha \int_{\Sigma} u_\epsilon^2 dv_g = 1 \), we have \( u_\epsilon \) is bounded in \( H^1(\Sigma, g) \) by the Poincaré inequality (see Fact 3). In view of Sobolev embedding (see Fact 2), one could assume without loss of generality that
\[ \begin{cases} u_\epsilon \to u_0 & \text{weakly in } H^1(\Sigma, g), \\ u_\epsilon \to u_0 & \text{strongly in } L^p(\Sigma, g), \quad \forall p \geq 1, \\ u_\epsilon \to u_0 & \text{almost everywhere in } \Sigma. \end{cases} \]  

Denote \( c_\epsilon = u_\epsilon(x_\epsilon) = \max_{\Sigma} |u_\epsilon| \). In the rest of this subsection, we always assume \( c_\epsilon \) is not bounded. Without loss of generality, one could assume that \( c_\epsilon = u_\epsilon(x_\epsilon) \to +\infty \) and \( x_\epsilon \to x_0 \) as \( \epsilon \to 0 \).

**Lemma 3.3.** We have
\[ \liminf_{\epsilon \to 0} \lambda_\epsilon > 0 \quad \text{and} \quad \limsup_{\epsilon \to 0} |\mu_\epsilon| < +\infty. \]
Proof. By the elementary inequality $e^t \leq 1 + te^t$ \((t \geq 0)\), we have
\[
\lambda_\varepsilon = \int_{\Sigma} u_\varepsilon^2 e^{b_\varepsilon(1-\varepsilon)u_\varepsilon^2} \, dv_g \geq \frac{1}{b_\varepsilon(1-\varepsilon)} \int_{\Sigma} (e^{b_\varepsilon(1-\varepsilon)u_\varepsilon^2} - 1) \, dv_g.
\]
This together with Lemma 3.2 tells us that $\lim \inf_{\varepsilon \to 0} \lambda_\varepsilon > 0$. Since $u_\varepsilon \in H^1(\Sigma, g)$ satisfying $\int_{\Sigma} \psi u_\varepsilon \, dv_g = 0$ and $\int_{\Sigma} |\nabla u_\varepsilon|^2 \, dv_g - \alpha \int_{\Sigma} u_\varepsilon^2 \, dv_g = 1$, by Hölder’s inequality and the Poincaré inequality (see Fact 3), there exists a constant $C$ which does not depend on $\varepsilon$ such that
\[
\int_{\Sigma} |u_\varepsilon| \, dv_g \leq C.
\]
Therefore,
\[
|\mu_\varepsilon| \leq \lambda_\varepsilon^{-1} \int_{\{x \in \Sigma; |u_\varepsilon(x)| > 1\}} |u_\varepsilon| e^{b_\varepsilon(1-\varepsilon)u_\varepsilon^2} \, dv_g + \lambda_\varepsilon^{-1} \int_{\{x \in \Sigma; |u_\varepsilon(x)| \leq 1\}} |u_\varepsilon| e^{b_\varepsilon(1-\varepsilon)u_\varepsilon^2} \, dv_g + \alpha \int_{\Sigma} |u_\varepsilon| \, dv_g
\]
\[
\leq 1 + \lambda_\varepsilon^{-1} e^{b_\varepsilon(1-\varepsilon) \vol_g(\Sigma)} + C.
\]
This together with $\lim \inf_{\varepsilon \to 0} \lambda_\varepsilon > 0$ tells us that $\lim \sup_{\varepsilon \to 0} |\mu_\varepsilon| < +\infty$. This completes the proof of Lemma 3.3.

With the similar arguments to those in [31, Lemma 11], one can obtain the following lemma.

Lemma 3.4. We have $u_0 \equiv 0$, and $|\nabla_g u_\varepsilon|^2 \, dv_g \rightarrow \delta_{x_0}$ weakly in the sense of measure, where $\delta_{x_0}$ is the usual Dirac measure centered at $x_0$.

Proof. By (3.11), one has $\int_{\Sigma} |\nabla_g u_\varepsilon|^2 \, dv_g - \alpha \int_{\Sigma} u_\varepsilon^2 \, dv_g = 1$ and $u_\varepsilon \rightarrow u_0$ strongly in $L^2(\Sigma, g)$ as $\varepsilon \rightarrow 0$. Hence,
\[
\int_{\Sigma} |\nabla_g u_\varepsilon|^2 \, dv_g = 1 + \alpha \int_{\Sigma} u_\varepsilon^2 \, dv_g + o_\varepsilon(1) \tag{3.12}
\]
and
\[
\int_{\Sigma} |\nabla_g (u_\varepsilon - u_0)|^2 \, dv_g = 1 - \left( \int_{\Sigma} |\nabla_g u_0|^2 \, dv_g - \alpha \int_{\Sigma} u_0^2 \, dv_g \right) + o_\varepsilon(1). \tag{3.13}
\]
If $u_0 \neq 0$, from (3.13) we have
\[
\int_{\Sigma} |\nabla_g (u_\varepsilon - u_0)|^2 \, dv_g < 1
\]
for $\varepsilon > 0$ sufficiently small. Using a similar argument to that in (3.6), one has
\[
e^{b_\varepsilon(1-\varepsilon)u_\varepsilon^2} \text{ is bounded in } L^q(\Sigma, g) \text{ for some } q > 1. \tag{3.14}
\]
In view of (3.11) and (3.14), one has $u_\varepsilon$ is uniformly bounded on $\Sigma$ by elliptic estimates, which contradicts the assumption that $c_\varepsilon \rightarrow +\infty$ as $\varepsilon \rightarrow 0$. Therefore, $u_0 \equiv 0$.

Recall that $g = \rho g_0$ and $g_0$ is a smooth metric on $\Sigma$. By (3.12) and $u_0 \equiv 0$ we have
\[
\int_{\Sigma} |\nabla_g u_\varepsilon|^2 \, dv_{g_\varepsilon} = \int_{\Sigma} |\nabla_g u_\varepsilon|^2 \, dv_g = 1 + o_\varepsilon(1). \tag{3.15}
\]
Assume $|\nabla_g u_\varepsilon|^2 \, dv_{g_\varepsilon} \rightarrow \nu$ as $\varepsilon \rightarrow 0$ in the sense of measure. We claim $\nu = \delta_{x_0}$. If the assertion would not hold, by (3.15) one could choose a sufficiently small $r_0 > 0$ and a cut-off function $\eta \in C^1_0(B_{r_0}(x_0))$ such that $0 \leq \eta \leq 1$, $\eta \equiv 1$ on $B_{r_0/2}(x_0)$, $|\nabla_g \eta| \leq C/r_0$ and
\[
\lim \sup_{\varepsilon \rightarrow 0} \int_{\Sigma} |\nabla_g (\eta u_\varepsilon)|^2 \, dv_g = \lim \sup_{\varepsilon \rightarrow 0} \int_{B_{r_0}(x_0)} |\nabla_g (\eta u_\varepsilon)|^2 \, dv_{g_\varepsilon} < 1. \tag{3.16}
\]
Here and in the sequel, we use $B_\epsilon(x)$ to denote the geodesic ball in $\Sigma$ with respect to the smooth metric $g_0$. Since $u_0 \equiv 0$, it follows from (3.11) that
\[ \int_\Sigma \eta u_\epsilon dv_g \to 0 \quad \text{as} \quad \epsilon \to 0. \] (3.17)

Then Troyanov’s inequality (1.9) together with (3.16) and (3.17) tells us that $e^{b_0(1-\epsilon)(\eta u_\epsilon)^2}$ is bounded in $L^s(\Sigma, g)$ for some $s > 1$. Since $\eta \equiv 1$ on $B_{r_0/2}(x_0)$, we have by using elliptic estimates to (3.1) that $u_\epsilon$ is uniformly bounded in $B_{r_0/4}(x_0)$. This contradicts $c_\epsilon \to +\infty$ as $\epsilon \to 0$. Then we have $|\nabla_g u_\epsilon|^2 dv_g = |\nabla_g u_\epsilon|^2 dv_{g_0} \to \delta_{x_0}$ as $\epsilon \to 0$ in the sense of measure. This completes the proof of the lemma.

Set
\[ r_\epsilon = \lambda_\epsilon^{1/2} c_\epsilon^{-1} e^{-b_0(1-\epsilon)c_\epsilon^2/2}. \]

We have the following lemma.

**Lemma 3.5.** For any $b \in [0, b_0)$, it holds that $r_\epsilon^2 e^{bc_\epsilon^2} \to 0$ as $\epsilon \to 0$.

**Proof.** By Hölder’s inequality we have
\[
\lambda_\epsilon = \int_\Sigma u_\epsilon^2 e^{b_0(1-\epsilon)u_\epsilon^2} dv_g \leq e^{\frac{b_0-b}{2}c_\epsilon^2} \int_\Sigma u_\epsilon^6 e^{b_0(1-\epsilon)u_\epsilon^2} dv_g \leq e^{\frac{b_0-b}{2}c_\epsilon^2} \left( \int_\Sigma u_\epsilon^6 dv_g \right)^{1/3} \left( \int_\Sigma e^{\frac{3}{2}(b_0+b)u_\epsilon^2} dv_g \right)^{2/3}.
\]

This together with (3.11) and Troyanov’s inequality (1.9) tells us that
\[
\lambda_\epsilon \leq C e^{\frac{b_0-b}{2}c_\epsilon^2}.
\]

Thus we have
\[
r_\epsilon^2 e^{bc_\epsilon^2} \leq C e^{-2c_\epsilon^2} \to 0 \quad \text{as} \quad \epsilon \to 0.
\]

This completes the proof. □

Choose an isothermal coordinate system $(U, \phi)$ around $x_0$ such that the metric $g_0$ can be represented by $g_0 = e^f (dx_1^2 + dx_2^2)$, where $f \in C^1(\Omega, \mathcal{R})$, $\Omega = \phi(U) \subset \mathbb{R}^2$ and $f(0) = 0$. Denote $\tilde{F} = F \circ \phi^{-1}$ for a function $F$ on $U$. Define
\[
\beta(x) = \begin{cases} 
\beta_i, & \text{if } x = p_i \text{ for some } i \in \{1, \ldots, m\}, \\
0, & \text{otherwise}.
\end{cases}
\]

By the definition of conical singularities, we can assume
\[
\rho(x) = h(x) |\phi(x)|^{2\beta(x_0)}, \quad x \in \phi^{-1}(\mathbb{B}_{\delta_0}(0)) \subset (U, \phi),
\]

for a sufficiently small $\delta_0 > 0$, where $0 < h \in C^0(\phi^{-1}(\mathbb{B}_{\delta_0}(0)))$.

Denote $y_\epsilon = \phi(x_\epsilon)$ and $\beta_0 = \min\{\beta_1, \ldots, \beta_m\}$. We consider two cases to discuss the blow-up phenomenon of $u_\epsilon$.

**Case 1.** The case $\beta(x_0) \geq 0$.

Define on $\Omega_{0, \epsilon} = \{y \in \mathbb{R}^2 : y_\epsilon + r_\epsilon y \in \mathbb{B}_{\delta_0}(0)\}$,
\[
\psi_{0, \epsilon}(y) = c_\epsilon^{-1} \bar{u}_\epsilon(y_\epsilon + r_\epsilon y), \quad \varphi_{0, \epsilon}(y) = c_\epsilon (\bar{u}_\epsilon(y_\epsilon + r_\epsilon y) - c_\epsilon).
\]

By (3.1) and a direct calculation, one has on $\Omega_{0, \epsilon}$ that
\[
\Delta_{g_\epsilon^2} \psi_{0, \epsilon}(y) = -\alpha e^{f(y_\epsilon + r_\epsilon y)} \bar{h}(y_\epsilon + r_\epsilon y) |y_\epsilon + r_\epsilon y|^{2\beta(x_0)} r_\epsilon^2 \psi_{0, \epsilon}(y)
\]
It then follows from the classification theorem of Chen and Li [6, Theorem 1] that

\[
- \epsilon f(y_t + r, y) \overline{h}(y_t + r, y)|y_t + r, y|^2 \overline{\psi}(y) e^{b_0(1-\epsilon)c^2}\overline{\psi}(y) - 1
+ \epsilon f(y_t + r, y) \overline{h}(y_t + r, y)|y_t + r, y|^2 e^{-1} c^2 \frac{\overline{\psi}(y) + r, y}{\int_{\Sigma} \psi dv_y}.
\]

In view of Lemmas 3.3 and 3.5, since \(|\psi_{0, \epsilon}| \leq 1\), we have by elliptic estimates that \(\psi_{0, \epsilon} \rightarrow \psi_0\) in \(C^{4}_{\text{loc}}(\mathbb{R}^2)\) as \(\epsilon \rightarrow 0\), where \(\psi_0\) satisfies

\[
\begin{cases}
\Delta \psi_0 = 0 & \text{in } \mathbb{R}^2, \\
\psi_0(y) \leq \psi_0(0) = 1, & \forall y \in \mathbb{R}^2.
\end{cases}
\]

Then by the Liouville theorem one knows \(\psi_0 \equiv 1\). Still we have on \(\Omega_{0, \epsilon}\) that

\[
\Delta \varphi_0(y) = -\alpha \epsilon f(y_t + r, y) \overline{h}(y_t + r, y)|y_t + r, y|^2 e^{2b_0(\overline{\psi}(y) - 1)} \psi_0(y)
+ \epsilon f(y_t + r, y) \overline{h}(y_t + r, y)|y_t + r, y|^2 e^{-1} c^2 \frac{\overline{\psi}(y) + r, y}{\int_{\Sigma} \psi dv_y}.
\]

We claim that \(\beta(x_0) = 0\). Otherwise, we have \(\beta(x_0) > 0\). In view of Lemmas 3.3 and 3.5, since \(\psi_{0, \epsilon} \rightarrow 1\) in \(C^{4}_{\text{loc}}(\mathbb{R}^2)\) as \(\epsilon \rightarrow 0\), we have by elliptic estimates that \(\varphi_{0, \epsilon} \rightarrow \varphi_0^1\) in \(C^{4}_{\text{loc}}(\mathbb{R}^2)\) as \(\epsilon \rightarrow 0\), where \(\varphi_0^1\) satisfies

\[
\begin{cases}
\Delta \varphi_0^1(y) = 0 & \text{in } \mathbb{R}^2, \\
\varphi_0^1(y) \leq \varphi_0^1(0) = 0, & \forall y \in \mathbb{R}^2.
\end{cases}
\]

Then by the Liouville theorem one knows \(\varphi_0^1 \equiv 0\). For any fixed \(R > 0\), we have

\[
\int_{B_R(0)} h(x)dy \leq \limsup_{\epsilon \rightarrow 0} \int_{B_R(0)} h(x) e^{2b_0(\overline{\psi}(y) - 1)} dy
= \limsup_{\epsilon \rightarrow 0} \int_{B_{R, \epsilon}(y_t)} h(x) e^{b_0(1-\epsilon)(\overline{\psi}(y_t + r) - 1)} dy
\leq \limsup_{\epsilon \rightarrow 0} \lambda_{\epsilon}^{-1} \int_{\phi^{-1}(B_{R, \epsilon}(y_t))} \varphi_0^1(x) e^{b_0(1-\epsilon) \varphi_0^1(x)} \rho(x) dx
\leq 1.
\]

So one has \(\int_{\mathbb{R}^2} h(x)dy \leq 1\) and then \(h(x) = 0\), which leads to a contradiction. Then we have \(\beta(x_0) = 0\). In view of Lemmas 3.3 and 3.5, since \(\psi_{0, \epsilon} \rightarrow 1\) as \(\epsilon \rightarrow 0\), we have by elliptic estimates that \(\varphi_{0, \epsilon} \rightarrow \varphi_0\) in \(C^{4}_{\text{loc}}(\mathbb{R}^2)\) as \(\epsilon \rightarrow 0\), where \(\varphi_0\) satisfies

\[
\begin{cases}
\Delta \varphi_0(y) = -h(x) e^{2b_0(\overline{\psi}(y) - 1)} & \text{in } \mathbb{R}^2, \\
\varphi_0(y) \leq \varphi_0(0) = 0, & \forall y \in \mathbb{R}^2, \\
\int_{\mathbb{R}^2} h(x) e^{2b_0(\overline{\psi}(y) - 1)} dy \leq 1.
\end{cases}
\]

It then follows from the classification theorem of Chen and Li [6, Theorem 1] that

\[
\varphi_0(y) = -\frac{1}{4\pi} \log(1 + \pi h(x)) \left| y \right|^2
\]

and

\[
\int_{\mathbb{R}^2} h(x) e^{8\pi \varphi_0(y)} dy = 1. \tag{3.18}
\]

**Case II.** The case \(-1 < \beta(x_0) < 0\).
(a) $r\epsilon^{-1}|y_e|^{1+\beta(x_0)} \to +\infty$ as $\epsilon \to 0$. Since $r\epsilon|y_e|^{-\beta(x_0)} \to 0$ as $\epsilon \to 0$, we use $t_\epsilon := r\epsilon|y_e|^{-\beta(x_0)}$ as the blow-up rate in this situation. Define on $\Omega_{1,\epsilon} = \{ y \in \mathbb{R}^2 : y_e + t_\epsilon y \in B_{\delta_0}(0) \}$,

$$
\psi_{1,\epsilon}(y) = c_\epsilon^{-1} \tilde{u}_\epsilon (y_e + t_\epsilon y), \quad \varphi_{1,\epsilon}(y) = c_\epsilon (\tilde{u}_\epsilon (y_e + t_\epsilon y) - c_\epsilon).
$$

With the similar arguments to those in Case I above, one can prove that $\psi_{1,\epsilon} \to 1$ in $C^1_{\text{loc}}(\mathbb{R}^2)$ as $\epsilon \to 0$. In addition, $\varphi_{1,\epsilon} \to \varphi_1$ in $C^1_{\text{loc}}(\mathbb{R}^2)$ as $\epsilon \to 0$, where $\varphi_1$ satisfies

$$
\begin{cases}
\Delta_{\mathbb{R}^2} \varphi_1(y) = -h(x_0) e^{2\beta_{0}\varphi_1(y)} & \text{in } \mathbb{R}^2, \\
\varphi_1(y) \leq \varphi_1(0) = 0, & \forall y \in \mathbb{R}^2, \\
\int_{\mathbb{R}^2} h(x_0) e^{2\beta_{0}\varphi_1(y)} dy \leq 1.
\end{cases}
$$

It then follows from Chen-Li's classification theorem (see [6, Theorem 1]) that

$$
\int_{\mathbb{R}^2} h(x_0) e^{2\beta_{0}\varphi_1(y)} dy \geq \frac{4\pi}{b_0} > 1.
$$

The contradiction tells us that (a) never happens.

(b) $r\epsilon^{-1}|y_e|^{1+\beta(x_0)} \leq C$ as $\epsilon \to 0$. We shall use $s_\epsilon := r\epsilon^{1/(1+\beta(x_0))}$ as the blow-up rate in this situation. Define on $\Omega_{2,\epsilon} = \{ y \in \mathbb{R}^2 : y_e + s_\epsilon y \in B_{\delta_0}(0) \}$,

$$
\psi_{2,\epsilon}(y) = c_\epsilon^{-1} \tilde{u}_\epsilon (y_e + s_\epsilon y), \quad \varphi_{2,\epsilon}(y) = c_\epsilon (\tilde{u}_\epsilon (y_e + s_\epsilon y) - c_\epsilon).
$$

With the similar arguments to those in Case I above, one can prove that

$$
\psi_{2,\epsilon} \to 1 \quad \text{in } C^1_{\text{loc}}(\mathbb{R}^2 \setminus \{-y^*\}) \cap C^0_{\text{loc}}(\mathbb{R}^2) \cap W^{2,s}_{\text{loc}}(\mathbb{R}^2)
$$

and

$$
\varphi_{2,\epsilon} \to \varphi_2 \quad \text{in } C^1_{\text{loc}}(\mathbb{R}^2 \setminus \{-y^*\}) \cap C^0_{\text{loc}}(\mathbb{R}^2) \cap W^{2,s}_{\text{loc}}(\mathbb{R}^2)
$$

as $\epsilon \to 0$, where $s \in [1, -1/\beta(x_0))$ is arbitrary and $\varphi_2$ is a distributional solution of the equation

$$
\begin{cases}
\Delta_{\mathbb{R}^2} \varphi_2(y) = -h(x_0) |y^* + y|^{2\beta(x_0)} e^{2\beta_{0}\varphi_2(y)} & \text{in } \mathbb{R}^2, \\
\varphi_2(y) \leq \varphi_2(0) = 0, & \forall y \in \mathbb{R}^2, \\
\int_{\mathbb{R}^2} h(x_0) |y^* + y|^{2\beta(x_0)} e^{2\beta_{0}\varphi_2(y)} dy \leq 1.
\end{cases}
$$

It then follows from the classification theorem of Chen and Li [7, Theorem 3.1] (one can also see [16, 23]) that

$$
\beta(x_0) = \beta_0, \quad \varphi_2(y) = -\frac{1}{4\pi(1+\beta_0)} \log \left( 1 + \frac{\pi h(x_0)}{1+\beta_0} |y^* + y|^{2(1+\beta_0)} \right).
$$

Since $\varphi_2(0) = 0$, we have $y^* = 0$ and

$$
\varphi_2(y) = -\frac{1}{4\pi(1+\beta_0)} \log \left( 1 + \frac{\pi h(x_0)}{1+\beta_0} |y|^{2(1+\beta_0)} \right).
$$

Then we have

$$
\int_{\mathbb{R}^2} h(x_0) |y|^{2\beta_{0}} e^{2\beta_{0}\varphi_2(y)} dy = 1. \quad (3.19)
$$

Therefore, we have the following key lemma, which states that the blow-up (if it happens) must happen at the point with the minimal singularities.
Lemma 3.6. Let $s_c = r_c^{1/(1+\beta_0)}$. Define on $\Omega_c = \{y \in \mathbb{R}^2; \ y_c + s_c y \in B_{\delta_0}(0)\}$,
\[
\psi_c(y) = c_e^{-1} \tilde{u}_c(y_c + s_c y), \quad \varphi_c(y) = c_e (\tilde{u}_c(y_c + s_c y) - c_e).
\]
We have $\psi_c \to 1$ and $\varphi_c \to \varphi_0$ as $\epsilon \to 0$ in $C^1_{\text{loc}}(\mathbb{R}^2)$ when $\beta_0 = 0$; in $C^1_{\text{loc}}(\mathbb{R}^2 \setminus \{0\}) \cap C^0_{\text{loc}}(\mathbb{R}^2) \cap W^{2,s}_{\text{loc}}(\mathbb{R}^2)$ when $-1 < \beta_0 < 0$, where $\epsilon \in [1, -1/\beta_0]$ is arbitrary and
\[
\varphi_0(y) = -\frac{1}{4\pi(1 + \beta_0)} \log \left(1 + \frac{\pi h(x_0)}{1 + \beta_0}|y|^{2(1 + \beta_0)}\right).
\]
Like in [1, 14], we define
\[
u_{\epsilon, \gamma} = \min\{\nu_{\epsilon}, \gamma c_\epsilon\}, \quad \forall \gamma \in (0, 1).
\]
With the similar arguments to those in [32, Lemma 7], we have

Lemma 3.7. For any $\gamma \in (0, 1)$, it holds that
\[
\lim_{\epsilon \to 0} \int_{\Sigma} |\nabla_{\epsilon} u_{\epsilon, \gamma}|^2 dv_{\epsilon} = \gamma.
\]

Proof. By (3.1) and (3.11), one knows that
\[
\int_{\Sigma} |\nabla_{\epsilon} u_{\epsilon, \gamma}|^2 dv_{\epsilon} = \int_{\Sigma} \nabla_{\epsilon} u_{\epsilon, \gamma} \cdot \nabla_{\epsilon} u_{\epsilon} dv_{\epsilon}
\]
\[
= \alpha \int_{\Sigma} u_{\epsilon, \gamma} u_{\epsilon} dv_{\epsilon} + \lambda_{\epsilon}^{-1} \int_{\Sigma} u_{\epsilon, \gamma} u_{\epsilon} e^{b_\epsilon(1-\epsilon)}u^2 dv_{\epsilon} - \mu_{\epsilon} \int_{\Sigma} \frac{1}{\psi_{\epsilon}} \psi_{\epsilon} dv_{\epsilon}
\]
\[
\geq \lambda_{\epsilon}^{-1} \int_{\phi^{-1}(B_{R_{\epsilon}}(y_c))} u_{\epsilon, \gamma} u_{\epsilon} e^{b_\epsilon(1-\epsilon)}u^2 p dv_{\epsilon} + o_\epsilon(1)
\]
\[
\geq \lambda_{\epsilon}^{-1} \int_{B_{R}(0)} \gamma(1 + o_\epsilon(1)) c_\epsilon^2 e^{b_\epsilon(1-\epsilon)} \tilde{u}_\epsilon^2 (y_c + s_c y) h(y_c + s_c y) |y_c + s_c y|^{2b_\epsilon} e^{\epsilon(y_c + s_c y)} s_c^2 dy
\]
\[
+ o_\epsilon(1)
\]
\[
\gamma(1 + o_\epsilon(1)) \int_{B_{R}(0)} h(x_0) |y|^{2b_\epsilon} e^{b_\epsilon(1-\epsilon)} dy + o_\epsilon(1).
\]
Therefore, by letting $\epsilon \to 0$ first and then $R \to +\infty$ in (3.20), we have
\[
\lim_{\epsilon \to 0} \int_{\Sigma} |\nabla_{\epsilon} u_{\epsilon, \gamma}|^2 dv_{\epsilon} \geq \gamma.
\]
Notice that
\[
|\nabla_{\epsilon} (u_{\epsilon} - \gamma c_\epsilon)|^2 = |\nabla_{\epsilon} u_{\epsilon} - \gamma c_\epsilon|^2 \nabla_{\epsilon} u_{\epsilon} \quad \text{on} \quad \Sigma,
\]
\[
(u_{\epsilon} - \gamma c_\epsilon)^2 = (1 + o_\epsilon(1))(1 - \gamma)c_\epsilon \quad \text{on} \quad \phi^{-1}(B_{R_{\epsilon}}(y_c)).
\]
By the same proof as (3.21), we have
\[
\lim_{\epsilon \to 0} \int_{\Sigma} |\nabla_{\epsilon} (u_{\epsilon, \gamma} - \gamma c_\epsilon)|^2 dv_{\epsilon} \geq 1 - \gamma.
\]
Because of
\[
|\nabla u_{\epsilon}|^2 = |\nabla_{\epsilon} u_{\epsilon, \gamma}|^2 + |\nabla_{\epsilon} (u_{\epsilon, \gamma} - \gamma c_\epsilon)|^2 \quad \text{almost everywhere on} \quad \Sigma,
\]
we have
\[
\int_{\Sigma} |\nabla_{\epsilon} u_{\epsilon, \gamma}|^2 dv_{\epsilon} + \int_{\Sigma} |\nabla_{\epsilon} (u_{\epsilon, \gamma} - \gamma c_\epsilon)|^2 dv_{\epsilon} = \int_{\Sigma} |\nabla u_{\epsilon}|^2 dv_{\epsilon} = 1 + o_\epsilon(1).
\]
Rearranging the above equality and letting $\epsilon \to 0$, we have by (3.22) that
\[
\limsup_{\epsilon \to 0} \int_{\Sigma} |\nabla g u_{\epsilon, \gamma}|^2 dv_g = 1 - \liminf_{\epsilon \to 0} \int_{\Sigma} |\nabla g (u_{\epsilon, \gamma} - \gamma c)|^2 dv_g \leq \gamma.
\]
This together with (3.21) tells us that
\[
\lim_{\epsilon \to 0} \int_{\Sigma} |\nabla g u_{\epsilon, \gamma}|^2 dv_g = 1 - \limsup_{\epsilon \to 0} \int_{\Sigma} |\nabla g (u_{\epsilon, \gamma} - \gamma c)|^2 dv_g + \frac{\lambda\epsilon}{\gamma^2 c^2}.
\]
In view of Lemma 3.7, using a similar argument to that in (3.6) one has $e^{b_0(1-\epsilon)u^2} v_g$ is bounded in $L^q(\Sigma, g)$ for some $q > 1$. By (3.11) we know $u_{\epsilon, \gamma} \to 0$ on $\Sigma$ almost everywhere as $\epsilon \to 0$. Therefore, we have
\[
\lim_{\epsilon \to 0} \int_{\Sigma} e^{b_0(1-\epsilon)u^2} dv_g = \text{vol}_g(\Sigma).
\]
By letting $\epsilon \to 0$ in (3.23) first and then $\gamma \to 1$ one has
\[
\lim_{\epsilon \to 0} \int_{\Sigma} e^{b_0(1-\epsilon)u^2} dv_g \leq \text{vol}_g(\Sigma) + \limsup_{\epsilon \to 0} \frac{\lambda\epsilon}{\gamma^2 c^2}.
\]
This completes the proof of the lemma. \(\square\)

As a direct consequence of Lemma 3.8, one has the following corollary.

**Corollary 3.9.** For any $\theta < 2$, it holds that
\[
\limsup_{\epsilon \to 0} \frac{\lambda\epsilon}{c^2} = +\infty.
\]

**Proof.** If the assertion would not hold, one has $\lambda\epsilon/c^2 \to 0$ as $\epsilon \to 0$. Then for any $v \in H^1(\Sigma, g)$ with $\int_{\Sigma} \psi dv_g = 0$ and $\int_{\Sigma} |\nabla g v|^2 dv_g - \alpha \int_{\Sigma} v^2 dv_g = 1$, we have by Lemmas 3.2 and 3.8 that
\[
\int_{\Sigma} e^{b_0u^2} dv_g \leq \sup_{u \in H^1(\Sigma, g), \int_{\Sigma} \psi u dv_g = 0, \int_{\Sigma} |\nabla g u|^2 dv_g - \alpha \int_{\Sigma} u^2 dv_g \leq 1} \int_{\Sigma} e^{b_0(1-\epsilon)u^2} dv_g \leq \text{vol}_g(\Sigma).
\]
This is impossible since $v \not\equiv 0$. Then we finish the proof of this corollary. \(\square\)

Still, we have the following lemma.
Lemma 3.10. As $\epsilon \to 0$, we have $c_{\epsilon} u_{\epsilon} \to G$ weakly in $W^{1,q}(\Sigma, g_0)$ for any $q \in (1, 2)$ and $c_{\epsilon} u_{\epsilon} \to G$ in $C^1(\Sigma \setminus \text{supp } D) \cap C^0(\Sigma \setminus \{x_0\}) \cap W^{2,s}_{\text{loc}}(\Sigma \setminus \{x_0\}, g_0)$, where $s \in [1, -1/\beta_0]$ is arbitrary and $G$ is the Green function satisfying

$$
\begin{align*}
\Delta_{g_0} G + \alpha G &= \frac{1 + \alpha}{f_0} \int_{\Sigma} G \rho \delta_{x_0} \psi \rho - \delta_{x_0} \text{ on } \Sigma, \\
\int_{\Sigma} \psi G \rho \delta_{x_0} &= 0.
\end{align*}
$$

Moreover, in a small neighborhood of $x_0$, one has

$$
G(x) = -\frac{1}{2\pi} \log r + A_{x_0} + \psi_\alpha,
$$

where $r = r(x)$ is the distance from $x$ to $x_0$ on $(\Sigma, g_0)$ and $\psi_\alpha$ belongs to $C^1$ in a small neighborhood of $x_0$.

Proof. With a similar proof to that in [32, Lemma 9, (44)] one can prove that: For any $\eta \in C^0(\Sigma)$, it holds that

$$
\lim_{\epsilon \to 0} \lambda_\epsilon^{-1} \int_{\Sigma} c_{\epsilon} u_{\epsilon} e^{b_0(1-\epsilon)u^2} \eta dv_g = \eta(x_0).
$$

In fact, it is clear that

$$
\lambda_\epsilon^{-1} \int_{\Sigma} c_{\epsilon} u_{\epsilon} e^{b_0(1-\epsilon)u^2} \eta dv_g = \lambda_\epsilon^{-1} \int_{\{x \in \Sigma: u_{\epsilon}(x) \leq \gamma c_{\epsilon}\}} c_{\epsilon} u_{\epsilon} e^{b_0(1-\epsilon)u^2} \eta dv_g + \lambda_\epsilon^{-1} \int_{\{x \in \Sigma: u_{\epsilon}(x) > \gamma c_{\epsilon}\}} c_{\epsilon} u_{\epsilon} e^{b_0(1-\epsilon)u^2} \eta dv_g.
$$

(3.27)

We now turn to estimate $I_1$, $I_2$ and $I_3$ in (3.27), respectively. It follows from Lemma 3.7 and Corollary 3.9 that

$$
I_1 = o_{\epsilon}(1).
$$

(3.28)

Then, we estimate $I_2$, i.e.,

$$
|I_2| \leq \max_{\Sigma} |\eta| \cdot \lambda_\epsilon^{-1} \int_{\{x \in \Sigma: u_{\epsilon}(x) > \gamma c_{\epsilon}\}} c_{\epsilon} u_{\epsilon} e^{b_0(1-\epsilon)u^2} dv_g
$$

$$
\leq \max_{\Sigma} |\eta| \cdot \lambda_\epsilon^{-1} \int_{\Sigma} e^{b_0(1-\epsilon)u^2} \eta dv_g
$$

$$
= \frac{\max_{\Sigma} |\eta|}{\gamma} \left(1 - \lambda_\epsilon^{-1} \int_{\phi^{-1}(B_{R_{\epsilon}}(y_\epsilon))} u_2 e^{b_0(1-\epsilon)u^2} dv_g\right)
$$

$$
= \frac{\max_{\Sigma} |\eta|}{\gamma} \left(1 - \int_{\phi^{-1}(B_{R_{\epsilon}}(y_\epsilon))} h(x_0) \eta^2 e^{b_0(1-\epsilon)u} dy + o_{\epsilon}(1)\right)
$$

$$
= o_{\epsilon}(1) + o_R(1),
$$

(3.29)

where in the last equality we have used (3.18) and (3.19). Here and in the sequel, when $o_{\epsilon}(1)$ and $o_R(1)$ appear in one sentence, it always means that we let $\epsilon \to 0$ first and then $R \to +\infty$.

For sufficiently small $\epsilon > 0$, since $\phi^{-1}(B_{R_{\epsilon}}(y_\epsilon)) \subset \{x \in \Sigma: u_{\epsilon}(x) > \gamma c_{\epsilon}\}$, one has by (3.18) and (3.19) that

$$
I_3 = (\eta(x_0) + o_{\epsilon}(1)) \lambda_\epsilon^{-1} \int_{\phi^{-1}(B_{R_{\epsilon}}(y_\epsilon))} c_{\epsilon} u_{\epsilon} e^{b_0(1-\epsilon)u^2} dv_g
$$

$$
= (\eta(x_0) + o_{\epsilon}(1)) \lambda_\epsilon^{-1} \int_{\phi^{-1}(B_{R_{\epsilon}}(y_\epsilon))} c_{\epsilon} u_{\epsilon} e^{b_0(1-\epsilon)u^2} dv_g.
$$
\[
\int_{B(y_0)} \frac{\psi}{x(y_0)} \int_{\Sigma} c_\varepsilon u_\varepsilon \, dv_g \leq \alpha \int_{\Sigma} |\xi_\varepsilon - \xi_0|^s \rho \, dv_{g_0} = \left( \frac{1}{1/p} \right)^{1/p} \left( \int_{\Sigma} \rho^s \, dv_{g_0} \right)^{1/p} \to 0 \quad \text{as} \quad \varepsilon \to 0.
\]

Thus \( \|\xi_\varepsilon\|_{L^s(\Sigma, g_0)} = \lim_{\varepsilon \to 0} \|\xi_\varepsilon\|_{L^s(\Sigma, g_0)} = 1 \). This contradicts \( \xi_0 = 0 \).

In view of (3.31) we know \( c_\varepsilon u_\varepsilon \) is a distributional solution to

\[
\Delta_{g_0}(c_\varepsilon u_\varepsilon) = -\alpha c_\varepsilon u_\varepsilon \rho + \frac{\psi}{x(y_0)} \int_{\Sigma} c_\varepsilon u_\varepsilon \rho \, dv_{g_0} + \rho u_\varepsilon \quad \text{on} \quad (\Sigma, g_0).
\]

Since \(-\alpha c_\varepsilon u_\varepsilon \rho + \frac{\psi}{x(y_0)} \int_{\Sigma} c_\varepsilon u_\varepsilon \rho \, dv_{g_0} + \rho u_\varepsilon\) is bounded in \( L^1(\Sigma, g_0) \), we have by Green's representation and elliptic estimates that \( c_\varepsilon u_\varepsilon \) is bounded in \( W^{1,q}(\Sigma, g_0) \) for any \( q \in (1, 2) \). Thus, there exists some \( G \in \bigcap_{q \in (1, 2)} W^{1,q}(\Sigma, g_0) \) such that \( \varepsilon \to 0 \), we have \( c_\varepsilon u_\varepsilon \to G \) weakly in \( W^{1,q}(\Sigma, g_0) \) for any \( q \in (1, 2) \), and that \( c_\varepsilon u_\varepsilon \to G \) strongly in \( L^p(\Sigma, g_0) \) for any \( p > 1 \). Since \( |g_\varepsilon \rho - \delta \xi_0| \to 0 \) as \( \varepsilon \to 0 \) in the sense of measure on \( (\Sigma, g_0) \), we have that \( G \) is a distributional solution to (3.24). Then the proof of this lemma is ended by elliptic estimates.

\[
\int_{\Sigma} c_\varepsilon e^{b_0(1-\varepsilon)u_\varepsilon^2} \, dv_g
\]

It is obvious that

\[
\int_{\Sigma} |\nabla g_\varepsilon u_\varepsilon|^2 \, dv_{g_0} = \int_{\Sigma} |\nabla g_\varepsilon u_\varepsilon|^2 \, dv_{g_0} = \int_{\Sigma} |\nabla g_\varepsilon u_\varepsilon|^2 \, dv_{g_0} + \int_{\Sigma \setminus B(g(y_0))} |\nabla g_\varepsilon u_\varepsilon|^2 \, dv_{g_0}.
\]

Using the similar arguments to what we have used in [32, Subsection 2.3], we define

\[
\mathcal{W}_\varepsilon = \{ u \in W^{1,2}(B(\delta(y_\varepsilon)) \setminus B_{R_\varepsilon}(y_\varepsilon)) : u |_{\partial B(\delta(y_\varepsilon))} = a_\varepsilon, u |_{\partial B_{R_\varepsilon}(y_\varepsilon)} = b_\varepsilon \}.
\]
It is known to all that
\[
\inf_{u \in W} \int_{\mathcal{B}_{\delta}(y_{e}) \setminus \mathcal{B}_{R_{\delta}}(y_{e})} |\nabla_{\mathbb{R}^{2}} u|^{2} \, dy
\]
is uniquely attained by \(\Phi_{0}\) satisfying
\[
\begin{cases}
\Delta_{g_{0}} \Phi_{0} = 0 & \text{in } \mathcal{B}_{\delta}(y_{e}) \setminus \mathcal{B}_{R_{\delta}}(y_{e}), \\
\Phi_{0} = a_{e} & \text{on } \partial \mathcal{B}_{\delta}(y_{e}), \\
\Phi_{0} = b_{e} & \text{on } \partial \mathcal{B}_{R_{\delta}}(y_{e})
\end{cases}
\]
and
\[
\Phi_{0}(y) = a_{e} \left( \log |y - y_{e}| - \log(R_{\delta}) \right) + b_{e} \left( \log \delta - \log |y - y_{e}| \right) \left/ \log \delta - \log(R_{\delta}) \right.
\]
so
\[
\int_{\mathcal{B}_{\delta}(y_{e}) \setminus \mathcal{B}_{R_{\delta}}(y_{e})} |\nabla_{\mathbb{R}^{2}} \Phi_{0}(y)|^{2} \, dy = \frac{2\pi(a_{e} - b_{e})^{2}}{\log \delta - \log(R_{\delta})} \tag{3.32}
\]
Let
\[
a_{e} = \sup_{\partial \mathcal{B}_{\delta}(y_{e})} \Phi_{0}, \quad b_{e} = \inf_{\partial \mathcal{B}_{R_{\delta}}(y_{e})} \Phi_{0}, \quad \bar{u}_{e,a_{e},b_{e}} = \max\{b_{e}, \min\{a_{e}, \bar{u}_{e}\}\}.
\]
Then \(\bar{u}_{e,a_{e},b_{e}} \in W_{\delta}(y_{e})\) and \(|\nabla_{\mathbb{R}^{2}} \bar{u}_{e,a_{e},b_{e}}| \leq |\nabla_{\mathbb{R}^{2}} \bar{u}_{e}|\) almost everywhere in \(\mathcal{B}_{\delta}(y_{e}) \setminus \mathcal{B}_{R_{\delta}}(y_{e})\) when \(\epsilon\) is sufficiently small. Therefore, we have
\[
\int_{\mathcal{B}_{\delta}(y_{e}) \setminus \mathcal{B}_{R_{\delta}}(y_{e})} |\nabla_{\mathbb{R}^{2}} \Phi_{0}(y)|^{2} \, dy \leq \int_{\mathcal{B}_{\delta}(y_{e}) \setminus \mathcal{B}_{R_{\delta}}(y_{e})} |\nabla_{\mathbb{R}^{2}} \bar{u}_{e,a_{e},b_{e}}(y)| \, dy
\]
\[
\leq \int_{\mathcal{B}_{\delta}(y_{e}) \setminus \mathcal{B}_{R_{\delta}}(y_{e})} |\nabla_{\mathbb{R}^{2}} \bar{u}_{e}(y)|^{2} \, dy
\]
\[
= \int_{\phi^{-1}(\mathcal{B}(y_{e})) \setminus \phi^{-1}(\mathcal{B}_{R_{\delta}}(y_{e}))} |\nabla_{g_{0}} \bar{u}_{e}|^{2} \, dv_{g_{0}}
\]
\[
= 1 + \alpha \int_{\Sigma} u_{e}^{2} \rho \, dv_{g_{0}} - \int_{\Sigma \setminus \phi^{-1}(\mathcal{B}_{\delta}(y_{e}))} |\nabla_{g_{0}} u_{e}|^{2} \, dv_{g_{0}}
\]
\[
- \int_{\phi^{-1}(\mathcal{B}_{R_{\delta}}(y_{e}))} |\nabla_{g_{0}} u_{e}|^{2} \, dv_{g_{0}}. \tag{3.33}
\]
Now we calculate the integrals \(\int_{\Sigma \setminus \phi^{-1}(\mathcal{B}(y_{e}))} |\nabla_{g_{0}} u_{e}|^{2} \, dv_{g_{0}}\) and \(\int_{\phi^{-1}(\mathcal{B}_{R_{\delta}}(y_{e}))} |\nabla_{g_{0}} u_{e}|^{2} \, dv_{g_{0}}\) in (3.33), respectively.

By (3.25), we have
\[
\int_{\Sigma \setminus \phi^{-1}(\mathcal{B}(y_{e}))} |\nabla_{g_{0}} u_{e}|^{2} \, dv_{g_{0}}
\]
\[
= \frac{1}{c_{e}^{2}} \left( \int_{\Sigma \setminus \phi^{-1}(\mathcal{B}(y_{e}))} |\nabla_{g_{0}} G|^{2} \, dv_{g_{0}} + o_{\epsilon}(1) \right)
\]
\[
= \frac{1}{c_{e}^{2}} \left( \frac{1}{2\pi} \log \frac{1}{\delta} + A_{x_{0}} + \alpha \int_{\Sigma} G^{2} \rho \, dv_{g_{0}} + o_{\epsilon}(1) + o_{\epsilon}(1) \right). \tag{3.34}
\]
Since \(\varphi_{e} \to \varphi_{0}\) in \(W^{2,s}_{\text{loc}}(\mathbb{R}^{2})\) as \(\epsilon \to 0\) for some \(s > 1\), in particular in \(W^{1,2}_{\text{loc}}(\mathbb{R}^{2})\), one has by Lemma 3.6 that
\[
\int_{\phi^{-1}(\mathcal{B}_{R_{\delta}}(y_{e}))} |\nabla_{g_{0}} u_{e}|^{2} \, dv_{g_{0}}
\]
\[
= \frac{1}{c_{e}^{2}} \left( \int_{\mathcal{B}(0)} |\nabla_{\mathbb{R}^{2}} \varphi_{0}(y)|^{2} \, dy + o_{\epsilon}(1) \right)
\]
\[
= \frac{1}{c_{e}^{2}} \left( \int_{\mathcal{B}(0)} |\nabla_{\mathbb{R}^{2}} \varphi_{0}(y)|^{2} \, dy + o_{\epsilon}(1) \right)
\]
Then we have (2.1) by combining (3.39) and (3.40).

Hence, we have

\[
2\pi(a_\epsilon - b_\epsilon)^2 = 2\pi c_\epsilon^2 - \frac{1}{1 + \beta_0} \log \left( 1 + \frac{\pi h(x_0)}{1 + \beta_0} R^{2(1 + \beta_0)} \right) + 2\log \delta - 4\pi A_{x_0} + o(1).
\]  

(3.36)

Recalling the definition of \(s_\epsilon\) one has

\[
\log \delta - \log (R s_\epsilon) = \log \delta - \log R - \frac{1}{2(1 + \beta_0)} \log \frac{\lambda}{c_\epsilon^2} + 2\pi(1 - \epsilon)c_\epsilon^2.
\]  

(3.37)

Substituting (3.36) and (3.37) into (3.32), and then inserting (3.34) and (3.35) into (3.33), we have

\[
\frac{2\pi c_\epsilon^2 - \frac{1}{1 + \beta_0} \log \left( 1 + \frac{\pi h(x_0)}{1 + \beta_0} R^{2(1 + \beta_0)} \right) + 2\log \delta - 4\pi A_{x_0} + o(1)}{\log \delta - \log R - \frac{1}{2(1 + \beta_0)} \log \frac{\lambda}{c_\epsilon^2} + 2\pi(1 - \epsilon)c_\epsilon^2} \\
\leq 1 + \alpha \int_{\Sigma} u^2 \rho dv_{g_0} - \frac{1}{c_\epsilon^2} \left( \frac{1}{2} \log \frac{\delta}{\delta} + A_{x_0} + \alpha \int_{\Sigma} G^2 \rho dv_{g_0} + o(1) + o(1) \right) \\
- \frac{1}{c_\epsilon^2} \left( \frac{1}{2\pi} \log R + \frac{1}{4\pi(1 + \beta_0)} \log \frac{\pi h(x_0)}{1 + \beta_0} - \frac{1}{4\pi(1 + \beta_0)} + o(1) + o(R(1)) \right).
\]

Rearranging the above inequality and letting \(\epsilon \to 0\), \(\delta \to 0\) and \(R \to +\infty\), we have

\[
\limsup_{\epsilon \to 0} \frac{\lambda}{c_\epsilon^2} \leq \frac{\pi h(x_0)}{1 + \beta_0} e^{1+b_0 A_{x_0}}.
\]

Then by Lemma 3.8 one has

\[
\lim_{\epsilon \to 0} \int_{\Sigma} e^{b_0(1-\epsilon)u^2} dv_g \leq \text{vol}_g(\Sigma) + \frac{\pi h(x_0)}{1 + \beta_0} e^{1+b_0 A_{x_0}}.
\]  

(3.38)

### 3.4 Completion of the proof of Theorem 2.1

Let \(u_0\) be the weak limit of \(u_\epsilon\) in (3.11). If \(c_\epsilon\) is bounded, then by the Lebesgue dominated convergence theorem, we have for any \(u \in H^1(\Sigma, g)\) with \(\int_{\Sigma} \psi u dv_g = 0\) and \(\int_{\Sigma} |\nabla_g u|^2 dv_g = \alpha \int_{\Sigma} u^2 dv_g \leq 1\) that

\[
\int_{\Sigma} e^{b_0 u^2} dv_g = \lim_{\epsilon \to 0} \int_{\Sigma} e^{b_0(1-\epsilon)u^2} dv_g \leq \lim_{\epsilon \to 0} \int_{\Sigma} e^{b_0(1-\epsilon)u^2} dv_g = \int_{\Sigma} e^{b_0 u^2} dv_g.
\]

Hence

\[
\sup_{u \in H^1(\Sigma, g), \int_{\Sigma} \psi u dv_g = 0, \int_{\Sigma} |\nabla_g u|^2 dv_g = \alpha \int_{\Sigma} u^2 dv_g \leq 1} \int_{\Sigma} e^{b_0 u^2} dv_g = \int_{\Sigma} e^{b_0 u^2} dv_g < +\infty.
\]  

(3.39)

If \(c_\epsilon\) is not bounded, we have by (3.8) and (3.38) that

\[
\sup_{u \in H^1(\Sigma, g), \int_{\Sigma} \psi u dv_g = 0, \int_{\Sigma} |\nabla_g u|^2 dv_g = \alpha \int_{\Sigma} u^2 dv_g \leq 1} \int_{\Sigma} e^{b_0 u^2} dv_g < +\infty.
\]  

(3.40)

Then we have (2.1) by combining (3.39) and (3.40).
When \( b > b_0 \), for any \( u \in H^1(\Sigma, g) \) with \( \int_\Sigma \psi udv_g = 0 \) and \( \int_\Sigma |\nabla g u|^2 dv_g - \alpha \int_\Sigma u^2 dv_g \leq 1 \), it follows from Cauchy’s inequality and Troyanov’s inequality (1.9) that the integral \( \int_\Sigma e^{bu^2} dv_g \) is still finite. But the supremum

\[
\sup_{u \in H^1(\Sigma, g), \int_\Sigma \psi udv_g=0, \int_\Sigma |\nabla g u|^2 dv_g - \alpha \int_\Sigma u^2 dv_g \leq 1} \int_\Sigma e^{bu^2} dv_g \\
\geq \sup_{u \in H^1(\Sigma, g), \int_\Sigma \psi udv_g=0, \int_\Sigma |\nabla g u|^2 dv_g \leq 1} \int_\Sigma e^{bu^2} dv_g = +\infty.
\]  

(3.41)

In fact, for any \( u \in H^1(\Sigma, g) \) with \( \int_\Sigma \psi udv_g = 0 \) and \( \int_\Sigma |\nabla g u|^2 dv_g \leq 1 \), we have by the Poincaré inequality (see Fact 3) that \( |\int_\Sigma udv_g| \leq C \). We have by Young’s inequality that

\[
\int_\Sigma e^{bu^2} dv_g \geq C \int_\Sigma e^{\frac{b_0}{b} ((u - \psi g_c(\phi)) \psi g_c)} dv_g.
\]

Then (3.41) follows from Chen’s result for the supercritical inequality in [5]. This completes the proof of Theorem 2.1.

4 Existence of extremal functions

In this section, we shall construct a sequence of functions \( \{\phi_\epsilon\}_{\epsilon > 0} \) such that

\[
\int_\Sigma |\nabla g \phi_\epsilon|^2 dv_g - \alpha \int_\Sigma (\phi_\epsilon - (\phi_\epsilon)^\alpha)^2 dv_g = 1
\]

(4.1)

and

\[
\int_\Sigma e^{b_0(\phi_\epsilon - (\phi_\epsilon)^\alpha)^2} dv_g > \text{vol}_g(\Sigma) + \frac{\pi h(x_0)}{1 + \beta_0} e^{1 + b_0 A_{x_0}}
\]

(4.2)

for sufficiently small \( \epsilon > 0 \), where

\[
(\phi_\epsilon)^\alpha = \frac{1}{\int_\Sigma \psi dv_g} \int_\Sigma \psi \phi_\epsilon dv_g.
\]

Since (4.2) contradicts (3.38), one has that \( c_\epsilon \) is bounded and ends the proof of Theorem 2.3.

Set

\[
\phi_\epsilon(x) = \begin{cases} 
  c + \frac{1}{c} \left( - \frac{1}{4\pi(1 + \beta_0)} \log \left( 1 + \frac{\pi}{1 + \beta_0} \left( \frac{r}{\epsilon} \right)^{2(1 + \beta_0)} \right) + B \right) & \text{for } r \leq R\epsilon, \\
  \frac{G - \eta \psi_c}{c} & \text{for } R\epsilon < r \leq 2R\epsilon, \\
  \frac{\Delta g_\epsilon}{c} & \text{for } r \geq 2R\epsilon,
\end{cases}
\]

where \( \psi_\alpha \) is given in (3.25), \( r = r(x) \) denotes the geodesic from \( x \) to \( x_0 \) with respect to the smooth metric \( g_\epsilon \), and \( R_\epsilon \) depends on \( \epsilon \) and \( R \to +\infty, R \to 0 \) as \( \epsilon \to 0 \). \( \eta \in C^1_0(B_{2R\epsilon}(x_0)) \) is a cut-off function satisfying \( 0 \leq \eta \leq 1 \) on \( B_{2R\epsilon}(x_0) \), \( \eta \equiv 1 \) on \( B_{R\epsilon}(x_0) \) and \( |\nabla g_\epsilon \eta| = O(\frac{1}{R\epsilon}) \), and \( c \) and \( B \) are two constants depending only on \( \epsilon \) to be determined later.

We need \( \phi_\epsilon \in H^1(\Sigma, g) \), which can be ensured by letting

\[
c^2 = -\frac{1}{2\pi} \log(R\epsilon) + A_{x_0} + \frac{1}{4\pi(1 + \beta_0)} \log \left( 1 + \frac{\pi}{1 + \beta_0} R^{2(1 + \beta_0)} \right) - B.
\]

(4.3)

Since \( \int_\Sigma \psi G dv_{g_\epsilon} = 0 \), we have

\[
\int_{\Sigma \setminus B_{R\epsilon}(x_0)} \nabla g_\epsilon G \nabla G dv_{g_\epsilon} = - \int_{\Sigma \setminus B_{R\epsilon}(x_0)} G \Delta g_\epsilon G dv_{g_\epsilon} + \int_{\partial B_{R\epsilon}(x_0)} G \frac{\partial G}{\partial n} dv_{g_\epsilon}
\]
It follows from (4.4)–(4.6) that
\[ = \alpha \int_{\Sigma \setminus B_{R_\varepsilon}(x_0)} G^2 \rho dv_{g_0} - \frac{1}{c^2} \int_{\Sigma} G \rho dv_{g_0} \int_{\Sigma \setminus B_{R_\varepsilon}(x_0)} \psi G \rho dv_{g_0} \]
\[ - \int_{\partial B_{R_\varepsilon}(x_0)} \frac{\partial G}{\partial n} ds_{g_0} \]
\[ = - \frac{1}{2\pi} \log(Re) + \alpha \int_{\Sigma} G^2 \rho dv_{g_0} + A_{x_0} + O(Re \log(Re)). \] (4.4)

Noting that \( \psi_\alpha \) is \( C^1 \) in a small neighborhood of \( x_0 \) and \( \psi_\alpha(x_0) = 0 \), one has \( |\nabla_{g_0}(\eta \psi_\alpha)| = O(1) \) as \( \varepsilon \to 0 \). So we have
\[ \int_{B_{2R_\varepsilon}(x_0) \setminus B_{R_\varepsilon}(x_0)} |\nabla_{g_0}(\eta \psi_\alpha)|^2 dv_{g_0} = O((Re)^2) \] (4.5)

and
\[ \int_{B_{2R_\varepsilon}(x_0) \setminus B_{R_\varepsilon}(x_0)} \nabla_{g_0} G \nabla_{g_0}(\eta \psi_\alpha) dv_{g_0} = O(Re). \] (4.6)

It follows from (4.4)–(4.6) that
\[ \int_{\Sigma \setminus B_{R_\varepsilon}(x_0)} |\nabla_{g_0} \phi_\varepsilon|^2 dv_{g_0} = \frac{1}{c^2} \int_{\Sigma \setminus B_{R_\varepsilon}(x_0)} |\nabla_{g_0} G|^2 dv_{g_0} + \frac{1}{c^2} \int_{B_{2R_\varepsilon}(x_0) \setminus B_{R_\varepsilon}(x_0)} |\nabla_{g_0}(\eta \psi_\alpha)|^2 dv_{g_0} \]
\[ - \frac{2}{c^2} \int_{B_{2R_\varepsilon}(x_0) \setminus B_{R_\varepsilon}(x_0)} \nabla_{g_0} G \nabla_{g_0}(\eta \psi_\alpha) dv_{g_0} \]
\[ = \frac{1}{c^2} \left( - \frac{1}{2\pi} \log(Re) + \alpha \int_{\Sigma} G^2 \rho dv_{g_0} + A_{x_0} + O(Re) \right). \] (4.7)

A direct calculation tells us that
\[ \int_{B_{R_\varepsilon}(x_0)} |\nabla_{g_0} \phi_\varepsilon|^2 dv_{g_0} = \frac{1}{4\pi(1 + \beta_0)c^2} \left( \log \left( 1 + \frac{\pi}{1 + \beta_0} R^{2(1 + \beta_0)} \right) + \frac{1}{1 + \frac{\pi}{1 + \beta_0} R^{2(1 + \beta_0)}} - 1 \right). \] (4.8)

By (4.7) and (4.8) we obtain
\[ \int_{\Sigma} |\nabla_{g_0} \phi_\varepsilon|^2 dv_{g_0} \]
\[ = \frac{1}{c^2} \left( - \frac{\log \varepsilon}{2\pi} + \alpha \int_{\Sigma} G^2 \rho dv_{g_0} + A_{x_0} + \frac{\log \frac{\pi}{1 + \beta_0}}{4\pi(1 + \beta_0)} - \frac{1}{4\pi(1 + \beta_0)} + O\left( \frac{1}{R^{2(1 + \beta_0)}} \right) \right). \] (4.9)

By a direct calculation, we have
\[ \int_{\Sigma} \psi \phi_\varepsilon \rho dv_{g_0} = \frac{1}{c} O((Re)^{2(1 + \beta_0)} \log(Re)). \]

Hence
\[ (\phi_\varepsilon)^a_\psi = \frac{1}{c} O((Re)^{2(1 + \beta_0)} \log(Re)). \] (4.10)

By (4.10), one has
\[ \int_{\Sigma} (\phi_\varepsilon - (\phi_\varepsilon)^a_\psi)^2 dv_g = \int_{\Sigma} \phi_\varepsilon^2 \rho dv_{g_0} + ((\phi_\varepsilon)^a_\psi)^2 vol_g(\Sigma) - 2(\phi_\varepsilon)^a_\psi \int_{\Sigma} \phi_\varepsilon \rho dv_{g_0} \]
\[ = \frac{1}{c^2} \left( \int_{\Sigma} G^2 \rho dv_{g_0} + O((Re)^{2(1 + \beta_0)} \log(Re)) \right). \] (4.11)

Combining (4.9) and (4.11), we obtain
\[ \int_{\Sigma} |\nabla_{g_0} \phi_\varepsilon|^2 dv_{g_0} - \alpha \int_{\Sigma} (\phi_\varepsilon - (\phi_\varepsilon)^a_\psi)^2 dv_g \]
\[
\begin{align*}
&= \frac{1}{c^2} \left( -\log\epsilon + A x_0 + \frac{\log \pi}{4\pi(1+\beta_0)} - \frac{1}{4\pi(1+\beta_0)} + O\left(\frac{1}{R^{2(1+\beta_0)}}\right) + O((R\epsilon)^{2(1+\beta_0)} \log(R\epsilon)) \right) \\
&= \frac{1}{c^2} \left( c^2 + B - \frac{1}{4\pi(1+\beta_0)} + O\left(\frac{1}{R^{2(1+\beta_0)}}\right) + O((R\epsilon)^{2(1+\beta_0)} \log(R\epsilon)) \right).
\end{align*}
\]

Letting \( \phi_\epsilon \) satisfy (4.1), we have
\[
\frac{1}{c^2} \left( c^2 + B - \frac{1}{4\pi(1+\beta_0)} + O\left(\frac{1}{R^{2(1+\beta_0)}}\right) + O((R\epsilon)^{2(1+\beta_0)} \log(R\epsilon)) \right) = 1. \tag{4.12}
\]

Then by (4.3) and (4.12) we have
\[
B = \frac{1}{4\pi(1+\beta_0)} + O\left(\frac{1}{R^{2(1+\beta_0)}}\right) + O((R\epsilon)^{2(1+\beta_0)} \log(R\epsilon)). \tag{4.13}
\]

It is clear that on \( B_{R\epsilon}(x_0) \) one has
\[
b_0(\phi_\epsilon - (\phi_\epsilon)_0^\beta)^2 \geq 4\pi(1+\beta_0)c^2 - 2\log \left( 1 + \frac{\pi}{1+\beta_0} \left( \frac{r}{\epsilon} \right)^{2(1+\beta_0)} \right) \\
+ 4\pi(1+\beta_0)B + O((R\epsilon)^{2(1+\beta_0)} \log(R\epsilon)).
\]

This together with (4.12) and (4.13) gives
\[
\int_{B_{R\epsilon}(x_0)} e^{b_0(\phi_\epsilon - (\phi_\epsilon)_0^\beta)^2} \rho dv_{g_0} \geq \frac{\pi h(x_0)}{1+\beta_0} e^{1+b_0 A x_0} + O\left(\frac{1}{R^{2(1+\beta_0)}}\right) + O((R\epsilon)^{2(1+\beta_0)} \log(R\epsilon)). \tag{4.14}
\]

Still, we have
\[
\int_{\Sigma \setminus B_{R\epsilon}(x_0)} e^{b_0(\phi_\epsilon - (\phi_\epsilon)_0^\beta)^2} \rho dv_{g_0} \geq \int_{\Sigma \setminus B_{R\epsilon}(x_0)} (1 + b_0(\phi_\epsilon - (\phi_\epsilon)_0^\beta)^2) \rho dv_{g_0} \\
\geq \text{vol}_g(\Sigma) + \frac{b_0}{c^2} \int_{\Sigma} G^2 \rho dv_{g_0} + o\left(\frac{1}{c^2}\right). \tag{4.15}
\]

Choosing \( R = (-\log\epsilon)^{1/(1+\beta_0)} \), one has \( O(\frac{1}{R^{2(1+\beta_0)}}) = o\left(\frac{1}{c^2}\right) \) and \( O((R\epsilon)^{2(1+\beta_0)} \log(R\epsilon)) = o\left(\frac{1}{c^2}\right) \). Then by (4.14) and (4.15) we have
\[
\int_{\Sigma} e^{b_0(\phi_\epsilon - (\phi_\epsilon)_0^\beta)^2} dv_g \geq \text{vol}_g(\Sigma) + \frac{\pi h(x_0)}{1+\beta_0} e^{1+b_0 A x_0} + \frac{b_0}{c^2} \int_{\Sigma} G^2 dv_g + o\left(\frac{1}{c^2}\right).
\]

Then we obtain (4.2) for sufficiently small \( \epsilon > 0 \). This completes the proof of Theorem 2.3.

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