Analytic Formulation of 21 cm Signal from Cosmic Dawn: Ly\(\alpha\) Fluctuations

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Abstract

We present an analytic formalism to compute the fluctuating component of the H\(\alpha\) signal and extend it to take into account the effects of partial Ly\(\alpha\) coupling during the era of cosmic dawn. We use excursion set formalism to calculate the size distribution of randomly distributed self-ionized regions. These ionization bubbles are surrounded by partially heated and Ly\(\alpha\) coupled regions, which create spin temperature \(T_S\) fluctuations. We use the ratio of number of Ly\(\alpha\) to ionizing photons (\(f_L\)) and number of X-ray photons emitted per stellar baryon (\(N_{\text{heat}}\)) as modeling parameters. Using our formalism, we compute the global H\(\alpha\) signal, its autocorrelation, and its power spectrum in the redshift range \(10 \leq z \leq 30\) for the \(\Lambda\)CDM model. We check the validity of this formalism for various limits and simplified cases. Our results agree reasonably well with existing results from \(N\)-body simulations, in spite of following a different approach and requiring orders of magnitude less computation power and time. We further apply our formalism to study the fluctuating component corresponding to the recent observation by the Experiment to Detect the Global Epoch of reionization Signature (EDGES) that shows an unexpectedly deep absorption trough in the global H\(\alpha\) signal in the redshift range \(15 < z < 19\). We show that, generically, the EDGES observation predicts a larger signal in this redshift range but a smaller signal at higher redshifts. We also explore the possibility of negative real-space autocorrelation of spin temperature and show that it can be achieved for partial Ly\(\alpha\) coupling in many cases corresponding to simplified models and a complete model without density perturbations.

Key words: cosmology: theory – dark ages, reionization, first stars

1. Introduction

Our current understanding of the history of the universe suggests that the dark ages of the universe ended around redshift \(z \approx 35\) with the formation of the first large-scale structures (epoch of cosmic dawn). These collapsed structures emitted radiation that heated and ionized their surrounding medium (epoch of reionization, or EoR) by \(z \approx 8\) (Barkana & Loeb 2001; Morales & Wyithe 2010; Pritchard & Loeb 2012; Natarajan & Yoshida 2014). The physics of first stars and galaxies is only partially understood theoretically and is poorly constrained by observations. The current observational bound on the cumulative history of reionization is provided by the detection of the Gunn–Peterson effect, which indicates that the universe was fully ionized by \(z \approx 6\). The cosmic microwave background (CMB) temperature and polarization anisotropy detections by WMAP and Planck give the redshift of reionization, \(z_{\text{reion}} = 7.75 \pm 0.73\) (Fan et al. 2000; Becker et al. 2001; Hinshaw et al. 2013; Planck Collaboration et al. 2014, 2016, 2018).

The cleanest probe of the physics of EoR is through the detection of the redshifted hyperfine 21 cm line of neutral hydrogen (H\(\alpha\)). This signal carries crucial information about the first sources of radiation in the universe and their spectrum in three frequency bands: ultraviolet (UV) radiation (which ionizes the surrounding medium), Ly\(\alpha\) radiation (which determines the relative population of neutral hydrogen atoms in hyperfine states), and X-ray photons (which heat and partially ionize the medium). In addition, the sources that emitted soft radio photons would also affect the observable H\(\alpha\) signal considerably (e.g., Ewall-Wice et al. 2018; Feng & Holder 2018). Along with primordial density perturbations given by the \(\Lambda\)CDM model, the inhomogeneities of these radiation fields establish the length scales of the fluctuating component of the signal.

The epoch of cosmic dawn and EoR has been studied in detail using numerical, semianalytic, and analytic methods (e.g., Pritchard & Furlanetto 2007; Mesinger et al. 2011, 2013; Visbal et al. 2012; Tashiro & Sugiyama 2013; Fialkov et al. 2014, 2017; Pacucci et al. 2014; Ghara et al. 2015).

The recent detection of a broad global absorption trough of strength 500 mK by the Experiment to Detect the Global Epoch of reionization Signature (EDGES) (Bowman et al. 2018) in a frequency range of 50–150 MHz, corresponding to a redshift range \(25 > z > 8\) (e.g., Madau et al. 1997; Tozzi et al. 2000; Gnedin & Shaver 2004; Sethi 2005). The fluctuating component of the signal is expected to be an order of magnitude smaller on scales in the range 3–100 Mpc, which implies angular sizes \(\approx 1–30\) arcmin (e.g., Furlanetto et al. 2004a, 2004b; Zaldarriaga et al. 2004; Pritchard & Furlanetto 2007; for comprehensive reviews see e.g., Morales & Wyithe 2010; Pritchard & Loeb 2012; Natarajan & Yoshida 2014). Many of the ongoing and upcoming experiments have the capability of detecting this signal in hundreds of hours of integration (e.g., Mesinger et al. 2014; Ahn et al. 2015). Upper limits on the fluctuating component of the H\(\alpha\) signal have been obtained by many ongoing experiments, such as GMRT, MWA, PAPER, and LOFAR (Paciga et al. 2013; Ali et al. 2015; Beardsley et al. 2016; Patil et al. 2017), with the best upper limits of \(\approx 50\) mK\(^2\) for \(k \approx 0.1\) Mpc\(^{-1}\).
detected by EDGES will also open avenues to investigate exotic physics (e.g., Barkana 2018; Fraser et al. 2018; Lambise & Mohanty 2018; Muñoz & Loeb 2018).

Numerical simulations can provide us insight into the morphology and evolution of the sources in the early universe. However, given the uncertainty in the astrophysics of this epoch, it is useful to develop fast analytic methods that can analyze the signal for a large range of scales for different combinations of physics inputs and modeling parameters. In our previous work (Raste & Sethi 2018, RS18 from now on), we developed a formalism to analytically compute the autocorrelation and power spectrum of the H I signal in the early phase of cosmic dawn and EoR, when the medium is partially heated and ionized. For simplicity, we had assumed a complete coupling of hydrogen spin temperature \( T_S \) with matter kinetic temperature \( T_K \). In this paper, we expand the formalism to include the effect of inhomogeneous Ly\( \alpha \) coupling on the H I signal. We also apply our method to study the fluctuating signal that would correspond to the global H I signal observed by the EDGES group.

In the next section, we review the H I signal from the EoR and the expected photoionization and heating of the intergalactic medium (IGM) due to UV and X-ray. In Section 2.3, we model fluctuations in the signal due to inhomogeneous Ly\( \alpha \) coupling. In Section 3, we briefly present the formalism for computing the two-point correlation function of the H I signal and discuss a few approximations. In Section 4, we present our results and explore their dependence on our modeling parameters. We summarize the derivation of our formalism in Appendix D. We present our conclusions in Section 5. Throughout this paper, we assume the spatially flat \( \Lambda \)CDM model with the following parameters: \( \Omega_m = 0.310, \Omega_b = 0.049, h = 0.677 \), and \( n_s = 0.967 \), with the overall normalization corresponding to \( \sigma_8 = 0.808 \) (Planck Collaboration et al. 2018).

### 2. Cosmic Dawn and EoR

The hyperfine splitting of neutral hydrogen in its ground state causes an energy difference corresponding to wavelength \( \lambda = 21.1 \) cm. The spin (or excitation) temperature of this line, \( T_S \), is a function of the ratio of atoms in two hyperfine states. This ratio is determined by three processes in the early universe: absorption and stimulated emission of CMB photons at temperature \( T_{\text{CMB}} \), collisions with atoms and charged particles, and the mixing of hyperfine levels owing to Ly\( \alpha \) photons (Wouthuysen–Field effect). We can express \( T_S \) in terms of the color temperature of Ly\( \alpha \) photons, \( T_{\alpha} \), gas kinetic temperature \( T_K \), and \( T_{\text{CMB}} \) (Field 1958; Pritchard & Loeb 2012):

\[
T_S = \frac{T_{\text{CMB}} + \chi_\alpha T_{\alpha} + \chi_I T_K}{1 + \chi_\alpha + \chi_I},
\]

Here, \( \chi_\alpha \approx n_\text{H} \), \( n_e \) (number density of neutral hydrogen atom or electrons; neutral atoms dominate for a small ionized fraction) and \( \chi_I \approx n_\text{H}\alpha \) (number density of Ly\( \alpha \) photons). During the dark ages, \( 1000 < z < 100 \), \( T_S \) relaxes to \( T_{\text{CMB}} \). After matter thermally decouples from CMB, for \( 100 < z < 30 \), collisions couple \( T_S \) to the matter temperature \( T_K \). When the first sources of radiation form during cosmic dawn, the production of Ly\( \alpha \) photons couples the spin temperature to the color temperature of Ly\( \alpha \) \( T_{\alpha} \), which is relaxed to \( T_K \) by multiple scattering of Ly\( \alpha \) photons with H I (e.g., Field 1959; Rybicki & dell’Antonio 1994; Chen & Miralda-Escudé 2004). If, at any redshift, \( \chi_\alpha + \chi_I \gtrsim T_{\text{CMB}}/T_K \), then \( T_S \) is strongly coupled to \( T_K \). Otherwise, in the absence of these coupling mechanisms, it relaxes to \( T_{\text{CMB}} \).

The H I is observable in emission or absorption depending on whether its spin temperature \( T_S \) is greater than or less than \( T_{\text{CMB}} \). The CMB spectral distortion caused by this effect is observable and can be expressed as follows (e.g., Madau et al. 1997; Shaver et al. 1999; Gnedin & Shaver 2004; Sethi 2005; Pritchard & Loeb 2012):

\[
\Delta T_b \approx 26.25 n(1 + \delta)\left(1 - \frac{T_{\text{CMB}}}{T_S}\right) \times \left(\frac{1 + z}{10} \Omega_b h^2\right)^2 \Omega_m h^2 0.022 \text{ mK}, \tag{2}
\]

Here the redshift-space distortion is ignored. We have expressed H I number density as \( n_H = n_{\text{H}} n(1 + \delta) \), and the mean density \( n_H \) has been absorbed in the prefactor of Equation (2). Here, \( \delta \) is overdensity of the gas. We have assumed that a small volume at any point is either completely neutral or completely ionized, so a variable \( n \) is defined that is unity if the medium is neutral and zero otherwise. We further define dimensionless temperature fluctuation as (Zaldarriaga et al. 2004)

\[
\psi = n(1 + \delta)(1 - s), \tag{3}
\]

which captures the density \( \delta \), ionization \( n \), and spin temperature, \( T_S \), inhomogeneities. Here, \( s = T_{\text{CMB}}/T_K \). These quantities are functions of position, and thus they contribute to the spatial fluctuation of the signal. We suppress this dependence for notational clarity.

At the end of the dark ages, high-density regions of the universe collapse and form structures of a range of masses. The radiation emitted by them change the properties of their surrounding medium. In our work, we assume that the smallest mass that can collapse corresponds to a H I-cooled halo (e.g., Barkana & Loeb 2001; Dayal & Ferrara 2018):

\[
M_{\text{min}} = 3.915 \times 10^8 \frac{\Omega_m^{1/2}}{h (1 + z)^{3/2}} M_\odot. \tag{4}
\]

We consider fluctuations of hydrogen-ionizing, Ly\( \alpha \), and X-ray radiation fields emitted by the sources on the brightness temperature inhomogeneities.

#### 2.1. Photoionization

The hydrogen-ionizing (ultraviolet—UV) photons emitted from the star within the collapsed structures are absorbed in the immediate vicinity of the sources and carve out H II regions around them in the IGM. We use excursion set formalism to compute a size distribution of the ionization bubbles by defining self-ionized regions (Furlanetto et al. 2004b). Such regions have enough sources to ionize all gas within them. Their sizes are determined by the ionization efficiency factor, defined as \( \zeta = f_{\text{coll}} = M_{\text{coll}}/M_{\text{coll}} \). Here, \( f_{\text{coll}} \) is the fraction of collapsed mass inside the self-ionized region, and \( \zeta \) is a
function of the properties of the sources as well as the surrounding halo (Furlanetto et al. 2004b):
\[
\zeta = f_\star f_{\text{esc}} N_{\text{ion}} N_{\text{rec}}^{-1}.
\]

Here, \(f_\star\) is the fraction of collapsed baryons that are converted into stars, \(f_{\text{esc}}\) is the fraction of ionizing photons that escape the source halo, \(N_{\text{ion}}\) is the number of UV photons created per stellar baryon, and \(N_{\text{rec}}\) is the number of recombinations. We assume \(\zeta\) to be independent of redshift in this paper, even though it can evolve with time owing to the evolution of quantities used to define it. For higher values of \(\zeta\), the reionization is completed at higher redshift (e.g., RS18). These self-ionized regions are larger than the H II regions of a single source because they are created by highly clustered multiple sources in the early universe (for the LCDM model). For our work, we assume the region to be spherical.

### 2.2. X-Ray Heating

Photons of energy \(E \gg 13.6\) eV (X-rays) escape the H II region into the surrounding medium. These photons ionize the neutral gas up to 10% and can impart up to 20% of their energy into heating the gas through photoionization and secondary collisional ionization and excitation (e.g., Shull & van Steenberg 1985; Venkatesan et al. 2001). In our study, we have assumed the medium outside the ionized region to be completely neutral \((n = 1)\) because the fraction of ionization due to this process is generally small.

The photoionization cross section of X-ray photons falls as \(\sigma (\nu) = \sigma_0 (\nu/\nu_i)^{-3}\), with \(\nu_i\) being the ionization threshold frequency of species \(i\). In this work, we only consider two species: neutral hydrogen and neutral helium, with their relative fractions \(x_i = 12/13\) and \(1/13\) of the baryon number, respectively. This is a valid assumption because the metallicity in the early universe is very low. Because the low-energy X-ray photons are absorbed with higher probability, they contribute to heating the medium immediately surrounding the H II region, whereas the higher energy photons free-stream through the medium and might get absorbed far away from any source. These photons uniformly heat up the whole IGM to some background temperature \(T_{bg}\).

We assume the X-ray photon source luminosity to be given by a power law (e.g., Mesinger et al. 2011 and references therein), \(N_\nu = N_\nu (\nu/\nu_{\text{min}})^{-\alpha}\), where \(\nu_{\text{min}}\) is the lowest frequency of X-ray photons escaping from source halos, and \(N_{\text{heat}}\) is the number of X-ray photons emitted per stellar baryon. We assume that \(f_{HI} = 0.15\) is the fraction of energy of emitted photelectrons that goes into heating the medium (Shull & van Steenberg 1985; Venkatesan et al. 2001). Other than the adiabatic expansion of the universe, we neglect all other cooling processes.

We divide the neutral hydrogen volume into two zones. In the near zone, the heating is dominated by X-ray photons from an individual self-ionized region. In the far zone, the contribution from all of the faraway background sources is taken into account. For more details and explanations, see RS18, where the increase in temperature due to a self-ionized region of radius \(R_x\) at a distance \(R_0\) from the center of the ionized region was calculated at redshift \(z_1\):
\[
\Delta T' = \frac{h f_{HI} \alpha N_{\text{heat}} f_\star N_\nu \nu_{\text{min}} \nu^3}{3k_B} \frac{R_0^3}{(1 + z_1)^2} \int_{R_{\text{min}}}^{R_{\text{max}}} \frac{dt' f_{\text{coll}}(t')}{f_{\text{coll}}(t)} \int_{\nu_{\text{min}}}^{\nu_{\text{max}}} d\nu' \nu'^{-1} e^{-\tau(R_0, \nu')} \sum_i (\nu' - \nu_i) \sigma_{i0} \nu_i^3.
\]

The primed quantities are calculated at the receiving point (point \(P\)), unprimed quantities are at the source (point \(S\)), and quantities with 0 subscript are comoving quantities. Here, \(\alpha\) is the X-ray spectra power-law index, and \(f_i\) are global ionization fractions.

### 2.3. Lyα Radiation

For EoR studies, all of the radiation between Lyα and the Lyman limit is referred to as Lyα radiation emitted from the source, and we shall follow this convention. The Lyα contribution at any point arises from two main factors: Lyα emitted from the sources and Lyα created by X-ray photoelectrons (Venkatesan et al. 2001). The latter is generally small, and we neglect it in this paper.

Lyα photons emitted from the sources escape the surrounding H II region and redshift until their frequency nearly equals the resonant frequency of one of the Lyman series lines. When a photon redshifts to the frequency corresponding to one of the Lyman series lines, it gets scattered by the neutral hydrogen and eventually cascades to a frequency corresponding to Lyα. Given the complicated frequency structure of Lyman series lines, these photons are absorbed at varying distances from the source. Thus, the coupling depends on two factors: the region of influence of the Lyα radiation and the coupling coefficient \(\gamma_\alpha\). Photons between Lyα and Lyβ frequencies can be absorbed by \(H_2\) molecules, but we ignore this effect as the density of \(H_2\) is very low in the IGM.

We define the Ly-n influence region as the distance traveled by the Ly-(n + 1) photons to redshift to Ly-n frequency. If these photons were emitted at \(z = z_e\) and absorbed at \(z = z_a\) with \(\nu_e = \nu_{n+1}\) and \(\nu_a = \nu_n\), then the comoving distance traveled by the photon before it is absorbed in an expanding universe is \((n \geq 2)\):
\[
R_{\text{max}}(n) \approx \frac{16040}{(1 + z_n)^{1/2} \left[\left(\frac{n^3(n+2)}{(n+1)^3(n-1)}\right)^{1/2} - 1\right]}.\]

These influence regions become smaller with increasing \(n\). We note that \(R_{\text{max}}(2)\) (Lyα influence region) is much larger than the mean distance between ionization bubbles at any redshift. For \(\zeta = 7.5\), the values of mean comoving distance between bubbles for redshifts 25, 20, and 15 are 7.85 Mpc, 2.29 Mpc, and 0.96 Mpc, respectively. Therefore, Lyα regions are very large and merge very early. However, this would create homogeneous coupling to H I atoms only if the Lyα coupling coefficient, \(\gamma_\alpha\), is high enough (Equation (1)); \(\gamma_\alpha\) is a function of Lyα photon (physical) number density \(n_\alpha^0\) (Field 1958;...
Chen & Miralda-Escudé (2004):

\[
\chi_0 = \frac{\sqrt{2} \hbar c^4}{36\pi^2 \nu_0^2 K} \frac{A_{\alpha} \nu_{10} n'_{\alpha}}{2kT R_n \nu_0} \tag{8}
\]

To calculate the number density of Ly\(\alpha\) photons \(n'_{\alpha}\) received at any point from ionizing sources, we use the method used to calculate X-ray heating in RS18. Assuming a flat spectrum between Ly\(\alpha\) and the Lyman limit, the number density of Ly\(\alpha\) photons at a comoving distance \(R_0\) from the source is

\[
n_{\alpha,0} = \frac{\mathcal{N}_{\alpha}}{4\pi c R_0^2 \nu_0 (1 + z')^3/2}. \tag{9}
\]

Here, \(\Delta \nu_0 = 2kT_0/m_e c^2 \nu_0\) is the Doppler line width. This factor arises because at the source the photons are emitted with frequencies between \(\nu_\beta\) and \(\nu_\alpha\), but the only frequencies that are absorbed at redshift \(z'\) are in the range of \(\Delta \nu_0\) around \(\nu_\alpha\).

The Ly\(\alpha\) luminosity, \(\mathcal{N}_{\alpha}\), can be expressed in terms of the size of the ionization halo, assuming that the Ly\(\alpha\) luminosity scales with ionizing luminosity with a factor \(f_L\). Using the balance between ionization and recombination in the ionizing region, we have

\[
\mathcal{N}_{\alpha} = f_L \frac{4}{3} n_0^2 \alpha_B C R_l^3 (1 + z)^3.
\]

It would be reasonable to assume that \(f_L > 1\) because Ly\(\alpha\) photons escape the halo more easily than ionizing photons. However, for the sake of completeness, we take \(0.1 < f_L < 1000\) in this paper. Combining all these, we get

\[
\chi_{\alpha,1,0} T_k \approx \frac{S_{\alpha}}{54\pi \nu_0^2 \alpha_B R_0 n_0^2 A_{\alpha} (1 + z')^3 (1 + z)^2}. \tag{10}
\]

Here, \(S_{\alpha}\) is a correction factor of order unity as defined in Chen & Miralda-Escudé (2004), which depends on the photon spectrum around the Ly\(\alpha\) line (Hirata 2006; Pritchard & Loeb 2012). This expression has been derived in a somewhat different manner in Pritchard & Loeb (2012).

We find contributions to \(\chi_{\alpha}\) from higher-order Lyman transitions (e.g., from Ly\(\gamma\) to Ly\(\beta\)) in a similar way, by counting the number of Ly\(\alpha\) influence regions the point falls within. The effect of higher-order transitions is expected to be subdominant because, for a continuum source, the total number of photons between Ly\(\beta\) to the Lyman limit is smaller than in the frequency range Ly\(\beta\) and Ly\(\alpha\). However, these photons can have a substantial impact near an ionizing source because they are absorbed closer to the source given their smaller influence regions. If the distance of a point from the source \(R_0\) is such that \(R_{\max} (n + 1) < R_0 < R_{\max} (n)\), then the point is in question will have \(n\) Ly\(\alpha\) photons in its vicinity rather than one photon if only the transition between Ly\(\alpha\) and Ly\(\beta\) is considered. This means that the Ly\(\alpha\) flux from the source center generally falls more rapidly than \(1/r^2\) when this effect is taken into account.

In this paper, we only consider Lyman series lines that have influence regions larger than the ionization bubble radius.\(^4\)

### 2.4. Collisional Coupling

The collisional coupling of spin temperature \(T_s\) and matter kinetic temperature \(T_K\) due to scattering of neutral hydrogen and electrons (Equation (1)) can play an important role at lower redshift \((z \lesssim 30)\) too. The coupling coefficient is proportional to the number density of colliding particles:

\[
\chi_c = \frac{(n_H kT_0^H + n_e kT_0^e)}{A_{\alpha}} \frac{T_s}{K}. \tag{11}
\]

For collision rate coefficients, we use the following fits (Zygelman 2005; Pritchard & Loeb 2012):

\[
k_{10} = \begin{cases} \exp \left(\frac{-9.607 + 0.5 \log(T_k) \exp \left(\frac{-(\log(T_k))^4.5}{1800}\right)}{10^4 K}\right) \quad & T_K \leq 10^4 K \\ \exp \left(\frac{6.035}{T_k}\right) \quad & T_K > 10^4 K \end{cases}
\]

\[
k_{10}^H = \begin{cases} 3.6 \times 10^{-16} T_k^{3.640} \exp \left(\frac{6.035}{T_k}\right) \quad & T_K \lesssim 10 K \\ 3.1 \times 10^{-11} T_k^{3.375} \exp \left(-\frac{32}{T_k}\right) \quad & T_K > 10 K. \end{cases} \tag{12}
\]

These rates increase with temperature, which means that there is stronger collisional coupling for hotter gas than for cool gas. This effect is important if the gas was colder during the cosmic dawn, due to unknown physics: the preionization absorption trough might be shallower instead of steeper, in spite of having a larger contrast of matter temperature from the CMB temperature. During EoR, the electron scattering is more effective near the sources where there is partial ionization and high temperature. However, since we assume the ionization fraction to be just the residual fraction outside of ionization bubbles, this effect is negligible in our work.

### 3. Autocorrelation of Dimensionless Brightness Temperature \(\psi\)

In our work, we wish to find the autocorrelation of \(\psi\) (Equation (3)), which can be defined as

\[
\mu = \langle \psi_1 \psi_2 \rangle - \langle \psi \rangle^2 = \langle n_1 (1 + \delta_1) (1 - s_1) n_2 (1 + \delta_2) (1 - s_2) \rangle - \langle n_1 (1 + \delta_1) (1 - s_1) \rangle^2 \tag{13}
\]

Here, \(n_1, \delta_1, s_1\) and \(n_2, \delta_2, s_2\) are values of ionization, overdensity, and heating \((T_{\mathrm{CMB}}/T_k)\) at point 1 \((\rho_1)\) and at point 2 \((\rho_2)\), respectively. Since the process of reionization is statistically

\(^4\)Photons with influence regions smaller than the ionization bubble will redshift to lower and lower Lyman series lines until they cross the ionization region boundary. However, their effect would be very small and very close to the boundary of the ionization region. It is not useful to model them in more detail because several other assumptions would break down so close to the boundary (e.g., sphericity of bubbles, sharp boundary of ionization regions).
homogeneous and isotropic, the autocorrelation function $\mu$ is a function of $r = |\mathbf{r}_1 - \mathbf{r}_2|$. To calculate $\mu$, we need to find all of the pairs of points that are separated by a distance $r$, and take their weighted average over the entire space. To calculate the probability of a pair with certain values, we use geometric arguments as described in RS18 and briefly summarized in Appendix B.

In this paper, we assume that the density has no correlation with ionization or heating ($\langle n \delta \rangle = \langle \delta \rangle = 0$). This gives us

$$\langle \psi \rangle = (1 + \delta) \langle n (1 - s) \rangle = f_n - \langle ns \rangle$$

where $\langle \delta \rangle = 0$ and $\xi = \langle \delta (n) \delta (n) \rangle$ is the autocorrelation function of the H1 density perturbation. We compute $\xi$ using the $\Lambda$CDM model power spectrum and assume the relative bias between the dark matter and the H1, $b = 1$. Here, $f_n = \langle n \rangle$ is defined as the average neutral volume fraction at that redshift. We also define $\phi = n (1 - s)$ to explore simplifying cases where we temporarily ignore the effect of density correlation $\xi$.

In such case, the two-point correlation function is

$$\mu = (1 + \xi) \langle n (1 - s) \rangle - \langle n (1 - s) \rangle^2$$

$$= (1 + \xi) \langle \phi \rangle^2 - \langle \phi \rangle^2. \quad (15)$$

The correlation functions we calculate in RS18 and this paper are analytically derived. However, a function $\mu(r)$ is a valid correlation function only if it follows certain properties: (1) it should be a finite positive value at $r = 0$, $\mu(0)$; (2) at any $r > 0$, the value of the correlation function $\mu(r) < \mu(0)$; (3) the correlation should go to zero at very large distance; (4) the Fourier transform of the correlation function should be a positive definite function (power spectrum); (5) between $r = 0$ and $r \rightarrow \infty$, the correlation function can be positive or negative, with the condition that its integration over all space must be zero. We do not satisfy this condition. However, we check for consistency of our formalism with the rest of the conditions.

We first discuss a few limiting cases (for details see RS18). For scales greater than the largest bubbles, the fluctuations due to the ionization, heating, and Ly$\alpha$ coupling inhomogeneities vanish, and the H1 correlation function is determined by only density perturbations. In this limit we get

$$\mu = \xi (f_n - \langle s \rangle)^2. \quad (16)$$

Here the correlation function scales as the density correlation function $\xi$. We also note that the correlation function at large scale vanishes when $f_n = \langle s \rangle$ (close to the global heating transition). If the heating and Ly$\alpha$ coupling are uniform, the neutral gas of the IGM is at the same spin temperature $T_{bg}$. The correlation function simplifies in this case:

$$\mu = (1 - s_b)^2 ((1 + \xi) \langle n (n_2) \rangle - f_n^2). \quad (17)$$

Here, $s_b = T_{CMB}/T_{bg}$. If the ionization fraction is very small, $\langle n (n_2) \rangle \approx f_n^2 \approx 1$. This gives us $\mu = \xi (1 - s_b)^2$. Here the density fluctuations are enhanced by the temperature contrast between the IGM and the CMB. At late times, $T_{bg} \gg T_{CMB}$ owing to X-ray heating, which drives $s_b$ to zero. This reduces the correlation function to the one dominated by density and ionization inhomogeneities:

$$\mu = -f_n^2 + (1 + \xi) \langle n (n_2) \rangle \quad (18)$$

$$\langle n (n_2) \rangle = f_n - f_n \sum_{R_i} N (R_i) \frac{4\pi}{3} R_i^3 C (r, 0, R_i, R_b). \quad (19)$$

This result is derived and explored in Zaldarriaga et al. (2004) and RS18.

3.1. Modeling and Notations

Our aim in this paper is to analytically model the correlation function of H1 brightness temperature fluctuations from the early phase of EoR owing to scales that emerge from ionization, heating, and Ly$\alpha$ coupling inhomogeneities. These inhomogeneities are caused by bubbles of a given size distribution, which evolves with time. These bubbles determine the scales of correlation. The details of our modeling and main assumptions are discussed extensively in RS18. We briefly summarize them here for the sake of completeness, as only a subset of the details are given in this paper. We have assumed a topology where there are isolated, spherical self-ionized bubbles, surrounded by isotropic, smooth $T_S$ profiles that might overlap with one another and smoothly merge with the background. Given the statistical isotropy (we neglect redshift-space distortion) and homogeneity of the process of reionization and heating, our assumption of spherical bubbles and isotropic spin temperature profiles holds even though the individual bubbles might not be spherical.

We compute ionization and spin temperature autocorrelation and ionization–spin temperature cross-correlation. We neglect, as noted above, the cross-correlation of density with ionization and spin temperature inhomogeneities because these contribute negligibly on the scales of interest. We neglect the clustering of self-ionized regions; a self-ionized region already accounts for clustering of ionizing sources at smaller scales. In RS18 we present cases that account for the clustering of self-ionized regions. This does not substantially alter our main results, but it might introduce new scales in the problem that correspond to the correlation scales of bubble centers. We note that this assumption is better for higher redshifts as the mean bubble separation is larger.

At any redshift, using excursion set formalism (Section 2.1) and the matter power spectrum given by the $\Lambda$CDM model, we generate a size distribution of self-ionized bubbles. Using Equations (1), (6), (9), and (10), we calculate the spin temperature in shells around these bubbles. The ionization volume fraction and volume fraction due to these shells is, respectively,

$$f_i = \sum_{R_i} \frac{4\pi}{3} N (R_i) R_i^3 \quad (20)$$

$$f_{hh} = \sum_{R_i} \frac{4\pi}{3} N (R_i) (R_{hh}^3 - R_i^3), \quad (21)$$

where $R_i$ are radii of ionization bubbles, $N (R_i)$ is the number density of bubbles with ionization radius $R_i$, and $R_{hh}$ corresponds to the outer radius of the spin temperature profile for a given $R_i$ (Figure 1). During the initial phase, the ionization bubbles and surrounding temperature profiles are

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5 These cross-correlations can be computed using excursion set formalism, e.g., Furlanetto et al. (2004b), for density-ionization cross-correlation. These terms are generally subdominant to other terms we retain (for discussion, see, e.g., RS18 and references therein).

6 As discussed in RS18, to compute ionization inhomogeneities, we assume that the probability of finding an ionized region outside an ionization bubble is the global ionization fraction $f_i$. While this is an excellent assumption for computing the correlation function on scales of interest to us, it violates the integral constraint on the correlation function.
nonoverlapping; we discuss the case of overlapping heating and Lyα coupled regions later in this section.

For any bubble of size $R_x$, we take shells of thickness $\Delta R_x(R_x, s)$ between radii $R_x$ and $R_y$, having temperature $s = T_{\text{CMB}}/T_3$. A detailed description of notations followed in this paper is given in Table 1. To compute correlations, we assume two random points separated by a distance $r$, as shown in Figure 1. The formalism used for the computation of correlation functions is described in Appendix D.

3.2. Overlap

At low redshift, spin temperature profiles around ionization bubbles are very large and overlap significantly. Thus, in such cases, $f_{hb}$ can be much larger than 1. In our previous paper (RS18), we had discussed a method to consistently take into account the overlap of heated regions. In this paper, our principal aim is to model the fluctuation of Lyα radiation. Unlike the heated regions that merge after the heating transition, the mean free path of lower Lyman series photons is larger than the mean interbubble distance at all times for $z < 30$, the starting redshift of our study. However, when the overlapped volume is very large ($f_{hb} \gg 1$), our formalism fails to generate a valid correlation function, as its Fourier transform (Appendix B) could fluctuate and yield negative values at large $k$. To avoid such unphysical results, we have taken a different approach to modeling overlaps in this paper.

We calculate kinetic temperature and Lyα profiles up to very large distances and start shedding outermost shells until $f_{hb}$ approaches 1. The energy and Lyα photons from excess shells are uniformly distributed in the neutral universe (background as well as remaining shells). The bubbles are still likely to overlap because of randomness in their positions. To account for that, we use (Appendix C)

$$f_h \simeq \left(1 + \frac{f}{f_{hb}}\right)(1 - e^{-f_h}) - f$$

$$f_{hb} = 1 - f - f_h,$$

where $f_h$ corresponds to the actual volume fraction occupied by heating bubbles (single counting of overlapped region), and $f_b$ is the background fraction. Note that $f_h$ approaches $f_{hb}$ when the $T_3$ profile volume fraction and ionization fraction are small. However, $f_h$ remains less than unity even if the value of $f_{hb}$ becomes much larger than unity.

3.3. Modeling Parameters

In our analysis, we explore two parameters to model X-ray heating and Lyα coupling:

1. $N_{\text{heat}}$: Number of X-ray photons emitted per stellar baryon. For our study, we assume $N_{\text{heat}}$ in the range 0.1–10.0. For larger values of $N_{\text{heat}}$, the heating is stronger, with higher values of $T_K$.
2. $f_L$: Ratio of source luminosity of photons between Lyα and the Lyman limit to the luminosity of UV photons. We take $0.1 < f_L < 1000$ in this paper. As the value of $f_L$ increases, the coupling between $T_S$ and $T_K$ becomes stronger. This increases the absolute value of $\Delta T_B$ in both emission and absorption.

In this paper, we have not considered scenarios in which the modeling parameters evolve with time.

4. Results

In the early phase of EoR, the brightness temperature fluctuations are determined by perturbations on the scales of
heated and Ly\textalpha
coupled regions. In the complete model, these regions have a range of
sizes and have diffuse pro-
file
ions.
region outside the second shell (background region) is neither
heated nor coupled (Figure 3). This gives three values of
\phi = n(1 - T_{CMB}/T_h) when ignoring density fluctuations:
\phi_1 = 0 in the ionized regions, \phi_1 = \phi > 0 in the first shell,
\phi_2 = -\phi < 0 in the second shell, and \phi_b = 0 in the
background region. For simplicity, we have assumed that the
volume occupied by the two shells is the same, \phi_1 = \phi_2 = \phi_0.

From Figure 3, we can see that the correlation will be
positive if both points are either in shell 1 or both in shell 2. If
one point is in shell 1 and another in shell 2, then the
correlation is negative. The two-point correlation function
without density perturbation is

\begin{align}
\mu &= \phi_1^2 P(\phi_1 \cap \phi_1) + \phi_2^2 P(\phi_2 \cap \phi_2) + 2\phi_1\phi_2 P(\phi_1 \cap \phi_2) \\
&= \phi_1^2 P(\phi_1) - P(\phi_1 \cap \phi_1) - P(\phi_1 \cap \phi_b) - P(\phi_1 \cap \phi_2) \\
&+ P(\phi_2) - P(\phi_2 \cap \phi_1) - P(\phi_2 \cap \phi_b) \\
&- P(\phi_2 \cap \phi_2) - 2P(\phi_1 \cap \phi_2),
\end{align}

which can be expanded using the same logic used in RS18 and
Appendix D:

\begin{align}
\frac{\mu}{\phi_1^2} &= \phi_1(1 - C(r, 0, R_c, R_h)) - \phi_1(1 - f_1) \\
&\times [C(r, 0, R_c, R_b) - C(r, 0, R_c, R_b)] - (1 - f_1)\frac{f_1}{2}
\end{align}

4.2. A Simple Model: One Bubble Size, Two Shells, \langle \psi \rangle = 0

In the case where Ly\textalpha
coupling is inhomogeneous, we can
construct another simple scenario where there is only one
ionized bubble size and the profile around the sources has two
shells. The first shell is heated and coupled through Ly\textalpha
radiation. The second shell is nonheated but still coupled. The
region outside the second shell (background region) is neither
heated nor coupled (Figure 3). This gives three values of
\phi = n(1 - T_{CMB}/T_h) when ignoring density fluctuations:
\phi_1 = 0 in the ionized regions, \phi_1 = \phi > 0 in the first shell,
\phi_2 = -\phi < 0 in the second shell, and \phi_b = 0 in the
background region. For simplicity, we have assumed that the
volume occupied by the two shells is the same, \phi_1 = \phi_2 = \phi_0.

From Figure 3, we can see that the correlation will be
positive if both points are either in shell 1 or both in shell 2. If
one point is in shell 1 and another in shell 2, then the
correlation is negative. The two-point correlation function
without density perturbation is

\begin{align}
\mu &= \phi_1^2 P(\phi_1 \cap \phi_1) + \phi_2^2 P(\phi_2 \cap \phi_2) + 2\phi_1\phi_2 P(\phi_1 \cap \phi_2) \\
&= \phi_1^2 P(\phi_1) - P(\phi_1 \cap \phi_1) - P(\phi_1 \cap \phi_b) - P(\phi_1 \cap \phi_2) \\
&+ P(\phi_2) - P(\phi_2 \cap \phi_1) - P(\phi_2 \cap \phi_b) \\
&- P(\phi_2 \cap \phi_2) - 2P(\phi_1 \cap \phi_2),
\end{align}

which can be expanded using the same logic used in RS18 and
Appendix D:

\begin{align}
\frac{\mu}{\phi_1^2} &= \phi_1(1 - C(r, 0, R_c, R_h)) - \phi_1(1 - f_1) \\
&\times [C(r, 0, R_c, R_b) - C(r, 0, R_c, R_b)] - (1 - f_1)\frac{f_1}{2}
\end{align}
This result can also be obtained by simplifying the complete model (Equation (41)) for approximations used in this section. At large scales, all of the functions $C(\ldots)$ tend to unity. In this case, Equation (25) approaches zero. This is expected because, at large scale ($r > 2R_b$), the two-point correlation should vanish if we ignore density correlation. When $r = 0$, $\mu = f_0\phi^2$. This is also as expected because there is no probability of two points being in different shells at $r = 0$. This expression takes a negative value for a range of values of $r$, where the two points are more likely to be in different shells than in the same shell.

In Figure 4, we have shown a case for this simple model where the autocorrelation of dimensionless brightness temperature is negative at certain scales. For scales $R_x + R_y < r < R_x + R_b$, depending on the values of various radii, two randomly chosen points can have a higher probability of being in different shells than in the same shell, driving the correlation to a negative value. For larger scales, $R_x + R_y < r < 2R_b$, both points have a finite probability of being in the outer shell and a zero probability of being in different shells of the same bubble, which leads the overall correlation at these scales to be positive. On scales $r > 2R_b$, the correlation function is zero. We note that the possibility of negative correlation at any scale depends on values of $R_x$, $R_y$, and $R_b$ and their differences. It is entirely possible to have models where the correlation function remains positive at all scales of interest.

For the complete model (discussed in detail in the next section and Appendix D), identifying the scales is more difficult because the ionization bubbles have a range of sizes, and the spin temperature profiles have a number of shells. However, if we ignore $\xi$, which is positively correlated at all scales of interest, we can still identify cases where the autocorrelation of $\phi$ goes negative. We show one such case in Figure 5. This constitutes the first important result of our analysis, which underlines the importance of correlation analysis in real space to extract the relevant physics. If the correlation function is negative at certain scales, it will point to a very specific geometry of the heated and Ly$\alpha$ coupled regions surrounding the self-ionized bubbles.

### 4.3. Complete Model

In this paper, we have explored the correlation function and power spectrum evolution due to inhomogeneity of spin temperature using two modeling parameters: number of X-ray photons per stellar baryon $N_{\text{heat}}$ and the ratio of luminosity of Ly$\alpha$ photons to ionizing photons of the sources $f_L$. We have taken $\zeta \sim 7.5$ (Equation (5)), which is in agreement with the reionization optical depth ($\tau_{\text{reion}} \approx 0.055$) given by Planck (Planck Collaboration et al. 2018). We first present results for the $\Lambda$CDM model in the redshift range 10–30, without any additional cooling mechanism needed to explain the recent EDGES results (Bowman et al. 2018). The results below redshift $z \approx 12$ are not entirely reliable because several approximations used in our formalism (including excursion set formalism) become less valid and eventually break down when the ionization volume fraction is large ($f_i > 0.1$; Furlanetto & Oh 2016; Giri et al. 2018). As noted above, to compute the brightness temperature mean and fluctuations, we have taken a size distribution of ionization bubbles, given by excursion set formalism and calculated spin temperature profiles surrounding these bubbles with a number of shells.

In Figure 6, we show the evolution of global H$\text{I}$ brightness temperature as a function of redshift for various combinations of modeling parameters. At $z \approx 30$, the global signal starts slightly negative as weak collisional coupling drives the spin temperature toward the matter temperature, which is below the CMB temperature (Equations (1) and (2)). At smaller values of $z$, the behavior is completely determined by the modeling...
The redshift of the heating transition strength of the signal is larger at all redshifts before complete coupling is achieved. The redshift of the heating transition strength of the signal is larger at all redshifts before complete coupling is achieved.

For higher values of $f_L$, the coupling starts earlier as well as stronger, and the overall strength of the signal is larger at all redshifts before complete coupling is achieved. The redshift of the heating transition (when the average gas temperature $T_K = T_{CMB}$ and the signal goes from absorption to emission) is only dependent on the heating ($N_{heat}$) and is independent of Ly$\alpha$ coupling ($f_L$). It happens sooner for higher values of $N_{heat}$.

In Figure 7, we show the evolution of the correlation function at scales $r = 0.5$–8 Mpc for different values of $N_{heat}$ and $f_L$, using Equations (2), (6), (9), and (41). The correlation functions are large at small scales and decrease as the distance between two points increases. At very large scale, they approach Equation (16), where the density inhomogeneity is enhanced by the average of $(1 - T_{CMB}/T_S)$. On intermediate scales, the structure of the correlation function is determined by the size distribution of bubbles and their surrounding $T_S$ profiles.

As the main aim of this paper is to model brightness temperature fluctuations owing to incomplete Ly$\alpha$ coupling, we first discuss it qualitatively. As in the case of incomplete heating, the Ly$\alpha$ coupling can also be separated into near and far zones. In the near zone, the emission from a nearby source dominates the coupling. The strength of this coupling is determined by Equation (9), which shows that there always is a distance from the source at which complete Ly$\alpha$ coupling ($\gamma_a, T_s > T_{CMB}$) can be established. Two conditions have to be met to ensure this creates a new length scale in the correlation function. First, this distance must exceed the size of the ionization region. Because the coupling strength scales as $R_{in}^3$, larger ionization bubbles meet this requirement more easily.

Second, the Ly$\alpha$ flux at this point from all of the background sources must be smaller than the flux from the nearby source, or we are in the limit of homogeneous coupling. The background intensity at any point is proportional to $R_{max}(n)$ (Equation (7)) multiplied by the number density of sources at any redshift. In the initial phase, the background intensity is not high enough to cause complete Ly$\alpha$ coupling. During this phase, there are regions around individual sources that attain complete coupling, and these regions are surrounded by a background in which only partial coupling has been attained. This creates inhomogeneities in Ly$\alpha$ coupling on the scales of these regions. As the intensity builds in the background owing to the birth of new self-ionized regions and it reaches levels sufficient to cause complete coupling, these inhomogeneities disappear. The nature of these inhomogeneities is determined by $f_L$, the atomic structure of neutral hydrogen, and the excursion set formalism.

At $z \approx 30$, only the collisional coupling is effective. Because it is weak and the Ly$\alpha$ intensity is small, all of the curves shown in the figure start with small values of the correlation function with similar strength. The fluctuations are dominated by density perturbations because the number density of ionizing sources is small and the ionization, heating, and Ly$\alpha$ coupled fractions are tiny. For small $f_L$, the correlation declines with time as the collisional coupling weakens, owing to the fall of the number density of particles with the expansion of the universe. This situation is only reversed when Ly$\alpha$ coupling becomes efficient. For higher values of $f_L$, this coupling occurs sooner, leading to an increase in fluctuations with time until complete coupling is achieved. After this period, the fluctuations are determined by heating and ionization inhomogeneities. The position of the first peak in

Figure 7. Evolution of autocorrelation of H$\text{I}$ brightness temperature for $r = 0.5$ Mpc (top left panel), $r = 2.0$ Mpc (top right panel), $r = 4$ Mpc (bottom left panel), and $r = 8$ Mpc (bottom right panel) for $z = 7.5$, a range of $f_L$ varying from 0.1 to 1000 and three values of $N_{heat}$: 10 (solid lines), 1.0 (long dashed lines), and 0.1 (short dashed lines). The dotted-dashed lines represent a fiducial model that matches EDGES observations.

parameters. For higher values of $N_{heat}$, the heating starts earlier and the absorption troughs are shallower. For higher values of $f_L$, the coupling starts earlier as well as stronger, and the overall strength of the signal is larger at all redshifts before complete coupling is achieved.
The correlation function approaches zero near the heating transition. At large scales, the effect of inhomogeneous $T_S$ decreases, and the main source of fluctuations is ionization inhomogeneities. Their effect is suppressed if the heating or coupling is not saturated (very small values of $N_{\text{heat}}$ and $f_L$). In general, larger $N_{\text{heat}}$ creates larger heating profiles and causes correlation at larger scales. However, these profiles also merge sooner and wipe out heating fluctuations at those scales.

The H I signal from the epoch of cosmic dawn and reionization has been extensively studied in the literature using semianalytic methods and large-scale simulations (e.g., Pritchard & Furlanetto 2007; Santos et al. 2008, 2010; Baek et al. 2010; Visbal et al. 2012; Mesinger et al. 2013, 2016; Tashiro & Sugiyama 2013; Pacucci et al. 2014; Fialkov et al. 2015, 2017; Ghara et al. 2015; Ross et al. 2017). Our formalism has been built on geometric arguments, which are intuitively easier to visualize in real space. The entire correlation structure can be written in terms of a single polynomial function: $C(\ldots \ldots \ldots )$ (Appendix B). Taking a Fourier transform with respect to the first argument $r$ of this function yields the power spectrum. In Figure 8, we show the evolution of the power spectrum $\Delta^2 = k^3P(k)/2\pi^2$ for a range of $k$. These figures show an evolutionary trend similar to the correlation functions (Figure 7). While comparing our results with simulations, we have focused on three features: (1) the number of peaks in the power spectrum, (2) the amplitude of $\Delta^2(k)$ for a range of scales $k \sim 0.1-0.5$ Mpc$^{-1}$, and (3) the difference between the redshift of the heating transition and the redshift of the power spectrum minimum.

Existing results in the literature show that, for $k \simeq 0.1-0.5$ Mpc$^{-1}$, there are generally two or three peaks of a power spectrum as a function of redshift (Santos et al. 2008; Baek et al. 2010; Mesinger et al. 2013, 2016; Ghara et al. 2015; Fialkov et al. 2017). At high redshifts, when the Ly$\alpha$ coupling and X-ray heating commence, they create fluctuations of $T_S$ in the medium. If fluctuations in these two fields dominate at widely different times, there will be two distinct peaks at high redshift: one due to coupling inhomogeneities and the other (generally at lower redshift than the former) due to heating inhomogeneities (Pritchard & Furlanetto 2007; Chen & Miralda-Escudé 2008; Ahn et al. 2015). After the heating transition, there is a third, smaller peak at low redshifts, when the power spectrum is dominated by ionization inhomogeneities (e.g., Pritchard & Furlanetto 2007; Fialkov et al. 2014; Ghara et al. 2015). We studied the two peaks that are due to heating and ionization inhomogeneities in RS18.

The third peak owing to the inhomogeneous Ly$\alpha$ coupling is expected at early times for the following reason: as large self-ionizing regions are born, these fluctuations should initially build and then diminish as the contrast between Ly$\alpha$ coupling...
in the near and far zone reduces, finally disappearing when complete Lyα coupling is established. We see this additional peak in Figure 9 in which the effect of density perturbations is neglected. However, in the complete model, we find a weak peak owing to this effect only at small scales in the power spectrum ($k = 2 \text{ Mpc}^{-1}$ in Figure 8). Generally this possible additional peak is masked by density perturbations. The strength of this peak can be understood in terms of the evolution of number density of large self-ionized regions at early times and the influence region of Lyα photons. For $z < 30$, the number density of the self-ionizing bubble builds exponentially in the excursion set formalism. While this creates inhomogeneities owing to the geometry seen in Figure 3, it also causes a rapid build-up of the background Lyα photons, rapidly destroying the contrast between the near and far zones. At any point, the background flux gets a nearly equal contribution from sources within the (comoving) radius $R_{\text{max}}(n)$ (Equation (7)). This radius is close to 600 comoving Mpc for $n = 2$ at $z \approx 25$. This large influence region contributes to wiping out the contrast in a short span. We note that $R_{\text{max}}(n)$ (for small principal quantum numbers $n > 2$) determines the length scale at which the physics needs to be captured to study the Lyα generated inhomogeneities. It might be difficult to achieve it using an $N$-body simulation as the box size is generally smaller than this length scale. In our results for large $N_{\text{heat}}$ and small $f_{\text{L}}$, we get three peaks at large $k$ (top left panel of Figure 8) because the heating inhomogeneities, and by extension collisional coupling inhomogeneities, dominate before the Lyα inhomogeneities commence.

While $\Delta^2(k)$ agrees with the results of simulations for smaller scales, for $k \approx 0.1 \text{ Mpc}^{-1}$, our results give less power as compared to simulations (bottom right panel of Figure 8). Given the small sizes of ionization bubbles at high redshifts, the only contribution to fluctuations at $k \sim 0.1 \text{ Mpc}^{-1}$ is due to density and spin temperature fluctuations. For the models of heating and Lyα coupling used in this paper, the contribution due to heating and coupling is dominated by faraway sources, which diminishes fluctuations at large scales. Therefore, at these scales, the spin temperature inhomogeneities are negligible, and only the density fluctuations are enhanced by the average contrast of H I spin temperature with CMB temperature. While it is conceivable that the higher power at large scales in the simulation is due to the finite box size (for a discussion, see Zahn et al. 2011; Ghara et al. 2015), which might not allow one to take into account the contribution of faraway sources whose effect tends to homogenize the fluctuations of $T_s$ at large scales, a more detailed comparison with simulations is hard as the parameter range used is generally not the same.

When the average spin temperature equals the CMB temperature during the heating transition, the power spectrum at small $k$ reaches a minimum value (Figure 8). However, for larger $k$ (small $r$), even during the heating transition there are significant fluctuations due to inhomogeneities of spin temperature and ionization, which delay the minima of the power spectrum. Figure 8 shows that, for $k = 2 \text{ Mpc}^{-1}$, the minimum of the power spectrum depends on the value of $f_\alpha$ even though the heating transition is independent of it. In general, the minimum of the power spectrum occurs during or after the global heating transition, depending on the scales and modeling parameters. This is in agreement with simulations that explore the dependence of these inhomogeneities on modeling parameters (Mesinger et al. 2013, 2016; Ghara et al. 2015; Fialkov et al. 2017).

4.4. Implication of EDGES Detection

Recent EDGES observations (Bowman et al. 2018) reported a sky-averaged absorption feature of strength $\Delta T_s \approx -300 \text{ mK}$ in the frequency range 70–90 MHz, corresponding to a redshift range 15–19 for the redshifted H I line. It can be shown that for a standard recombination and thermal history, the minimum temperature of the gas at $z \approx 17$ is $T_K \approx 6 \text{ K}$. It follows from Equations (1) and (2) that the absorption trough should not have been deeper than $-180 \text{ mK}$.

One possible explanation of the EDGES result is that there is additional radio background in the redshift range 15 < $z$ < 19; in this case, we can replace $T_{\text{CMB}}$ with the $T_{\text{CMB}} + T_{\text{radio}}$ in Equation (1) in the relevant redshift range (Ewall-Wice et al. 2018; Feng & Holder 2018; Sharma 2018). With this replacement and a suitable choice of $T_{\text{radio}}$, we can rederive all our results in this paper for compatibility with the EDGES result.

Another plausible explanation invokes the additional cooling of baryons that is due to interaction between dark matter and baryons. In this case, we can explain the EDGES detection using Equations (1) and (2) if (1) Lyα photons globally couple the spin temperature to matter temperature, that is, $T_K Y_{\alpha} \gg T_{\text{CMB}}$, such that $T_s = T_K$ at $z \approx 19$; and (2) $T_K \approx 2.5 \text{ K}$. We consider this theoretical extension to explain the EDGES result in this paper.

We consider a fiducial model that roughly fits the EDGES results. We assume dark matter–baryon interaction of the form

\[ f_L \approx 0, 1, 10, 100, 1000 \text{ EDGES} \]

\[ f_L = 0.7, 1, 10 \text{ (solid lines), 1.0 (long dashed curves), and 0.1 (short dashed lines). The dotted–dashed line represents a fiducial model that matches EDGES observations.} \]
described in Barkana (2018), with a cross section $\sigma_1 = 5 \times 10^{-24} \text{cm}^2$ and the ratio of dark matter to proton mass $m_{\text{DM}}/m_p = 0.001$. Such an interaction helps cool the baryon gas temperature sufficiently to explain the EDGES result. For $f_L = 2$ and $N_{\text{heat}} = 0.08$, we get an absorption trough in the global signal similar to the one observed by EDGES data (Figure 6). In Figures 7 and 8, we show the correlation functions and power spectrum for a range of scales, taking into account our fiducial model to replicate the EDGES results.

The main effect of EDGES observations on the expected correlations is to boost the signal by nearly an order of magnitude in the redshift range $15 < z < 19$, even as compared to the most optimistic models (low $N_{\text{heat}}$ and high $f_L$) in the usual case. This is entirely owing to a decrease in $T_S$ that boosts $s$ in Equation (3). The correlation function scales roughly as $(1 - s)^3$. Even though this result has been arrived at within the framework of a model involving baryon cooling, this result holds when excess radio background is responsible for the deep absorption feature because $s$ increases by a similar factor in that case too.

This result is also robust to change in other modeling parameters $f_L$ and $N_{\text{heat}}$. Therefore, an important prediction of the EDGES detection is that the deeper absorption trough in the global signal gives a corresponding increase in the fluctuating component too. Even when the spin temperature field is very uniform, with $L_y$-coupled and unheated gas, the low spin temperature would enhance the underlying density inhomogeneities (Equation (17)). As we have shown in Section 4.2, the spin temperature field can be negatively correlated at some scales. In such scenarios, the fluctuating component of the signal could be less than given by Equation (17). We tried to produce such models for parameters needed to explain the EDGES data and found it very difficult to anticorrelate the spin temperature field. Hence we infer that the minimum value of the correlation function that corresponds to the EDGES signal is given by Equation (17), using the temperature derived from the trough in the global signal. If the global signal has a trough of $\sim 500 \text{ mK}$ at $z \simeq 17$, then assuming complete coupling and no heating (no fluctuations due to these two fields), the autocorrelation function at $r = 2 \text{ Mpc}$ and $r = 4 \text{ Mpc}$ should be $\sim 5100 \text{ (mK)}^2$ and $\sim 2500 \text{ (mK)}^2$, respectively.

Many currently operational (e.g., LOFAR, MWA, PAPER, GMRT) and upcoming radio interferometers (HERA, SKA) have the capability to detect the fluctuating component of the H I signal in the redshift range $8 < z < 25$ (for details, see, e.g., Mesinger et al. 2014; Ahn et al. 2015; Koopmans et al. 2015). We discuss the detectability of this signal especially in light of the recent EDGES results. These radio interferometers directly measure visibilities and their correlations, which can be related to the power spectrum of the H I signal (e.g., Bharadwaj & Sethi 2001; Zaldarriaga et al. 2004). This data analysis can readily be extended to the image plane (which is a byproduct of the analysis pipeline; e.g., Patil et al. 2017 for LOFAR). This means real-space correlation functions can also be used for computation of the signal (e.g., Sethi & Haiman 2008). As an approximate rule, one could use $r \simeq \pi/k$ to shift from Fourier to real space.

SKA1-LOW is expected to detect the H I signal at $z \simeq 16$ with a signal-to-noise ratio varying from 100 to 10 for $0.1 < k < 0.6 \text{ Mpc}^{-1}$ for a signal strength $\Delta^2(k) \simeq 10^2 \text{ (mK)}^2$ (for details, see, e.g., Koopmans et al. 2015). EDGES results predict a signal strength nearly a factor of five larger, which means the signal-to-noise ratio would be significantly higher (Figure 8). The ongoing experiment LOFAR’s best upper limit corresponds to $(80 \text{ mK})^2 (k \simeq 0.05 \text{ Mpc}^{-1})$ in the redshift range $9.6 < z < 10.6$ in 10 hr of integration. LOFAR has the frequency range to probe the redshift range of EDGES detection. If the noise properties at smaller frequencies $(\sim 80 \text{ MHz})$ behave roughly as the one observed at higher frequencies $(\sim 110 \text{ MHz})$, LOFAR might be able to detect the signal in a few hundred hours of integration.

SKA1-LOW (and SKA2) have the capability to detect the EoR signal in the redshift range $20 < z < 25$. A signal of $(100 \text{ (mK)}^2)$ at $z \simeq 25$ is detectable by a deep SKA1-LOW survey with signal-to-noise ratio of five for $k < 0.1 \text{ Mpc}^{-1}$ (Koopmans et al. 2015). However, we notice that the EDGES result places a significant constraint on the signal in this era. This, as noted above, is because the EDGES results imply a smaller value of $f_L$, which results in a smaller signal for $z > 19$ as compared to models with a larger $f_L$, which give a significantly higher signal (Figure 8).

5. Summary and Conclusions

The main aim of this paper is to extend the analytic formalism presented in our previous work (Raste & Sethi 2018) to explore the epoch of partial $L_y$ coupling. Following RS18, we generate the size distribution of self-ionized regions using excursion set formalism for the $\Lambda$CDM model. Around these bubbles, we create spin temperature profiles due to X-ray heating and $L_y$ coupling, which merge smoothly with a uniform background. We model these spin temperature $T_S$ profiles using two parameters: $N_{\text{heat}}$, the number of X-ray photons per stellar baryon, and $f_L$, the ratio of $L_y$ to ionizing photons. Our analytic formulation allows us to explore relevant physical processes and study both their individual and combined effect on the H I signal.

We study the evolution of correlation properties of H I in the redshift range $10 < z < 30$ for many possible scenarios, with a greater focus on higher redshifts at which partial $L_y$ coupling plays an important role (Figures 4, 9, 7, 8). We find a reasonable agreement with existing semianalytic and $N$-body simulation results. As we compute correlation functions in both real and Fourier space, we find a possible case where the correlation function in real space is negative owing to partial heating and $L_y$ coupling (Figure 4).

We also analyze the implications of the recent EDGES detection of the global H I signal in the redshift range $15 < z < 19$. Generically, EDGES detection results in a higher correlation signal in the redshift range of the detection but a lower signal at higher redshifts, as compared to the most

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10 We have not incorporated the enhancement in the signal due to the inhomogeneous velocity-dependent cooling of gas within this model (Fialkov et al. 2018; Muñoz et al. 2018).

11 We notice a decrease in the signal in this case at $z \simeq 30$. This model-dependent decrement is due to cooler baryons causing a decrease in the efficiency of coupling from collisions. 

12 The angular scale above which the H I signal can be reliably measured for most ongoing and upcoming radio interferometers is a few arcminutes; 1" corresponds to nearly 3 Mpc (comoving) at $z \simeq 15$, or these telescopes are sensitive to linear scales larger than 5–10 Mpc (comoving). However, these telescopes have frequency resolutions that correspond to much smaller linear scales; for example, MWA’s frequency resolution of 40 kHz corresponds to nearly 1 Mpc (comoving) along the line of sight. Or the 3D H I signal is probed with a different resolution on the sky plane as compared to the line of sight.
optimistic models, which do not take into account this detection (Figures 7 and 8).

In this paper, we study the implications of the standard ΛCDM model. The resultant size distribution of self-ionized regions and hence the fluctuation scales of brightness temperature would be different for an extension of this model, which we hope to explore in future work (e.g., Sethi & Subramanian 2009; Sarkar et al. 2016).

Understanding theoretically the H1 signal from EoR/cosmic dawn remains a challenge. Given the large amount of uncertainty in the physics of ionizing sources, IGM, feedback mechanisms, and so on, it is important to explore a wide set of modeling paradigms. While N-body simulations are important to understand and image the H1 field, analytic methods, like the one presented in this paper, are suited to predicting the statistical quantities like correlation function and power spectrum. Since our formalism is not limited by the size of the simulation box, we can easily incorporate a variety of physical processes at very small or very large scale, for example, the influence region of Lyα photons. Also, this formalism is computationally cheaper, which means we can explore a large set of modeling parameters and their degeneracies at a fraction of the computation resources taken by N-body simulation. Since N-body simulations, semi-analytic, and analytic formalisms each have their own set of assumptions, strengths, and weaknesses, it is beneficial to apply all these methods to unravel the complex physics of reionization.

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### Appendix A Probability

\[
P(A | B) = \frac{P(A \cap B)}{P(B)} \quad (26)
\]

\[
P((A \cap B) | C) = \frac{P(A | (B \cap C))}{P(B | C)} \quad (27)
\]

### Appendix B Geometry

If point 1 is located between distance \( P \) and \( Q \) \((P < Q)\) from the center of a sphere, then \( C(x, P, Q, R) \) is the probability that its neighbor point 2 at distance \( x \) from point 1 is located outside the concentric sphere of radius \( R \):

\[
P(A \cap B) = P(A) - P(A \cap \bar{B}). \quad (28)
\]

### Appendix C Overlap

Three-dimensional Fourier transform of \( 1 - C(x, P, Q, R) \) is useful while calculating the power spectrum:

\[
\text{FT}(1 - C(x, P, Q, R)) = \frac{12\pi}{k^3(Q^3 - P^3)} \\
(kQ \cos(kQ) - \sin(kQ)) - (kP \cos(kP) - \sin(kP)) \\
\times [kR \cos(kR) - \sin(kR)].
\]
with the total volume fraction occupied by the sphere $g_i = g_i + g_r$. Here we are only interested in the volume fraction occupied by the outer shells. Thus, we call this fraction (for $N = 1$) $g_1 = g_r$. For $N = 2$, we need to randomly place the second sphere after the first sphere is already placed in the box. Across multiple experiments, where the second sphere is placed randomly, the average total volume fraction that is occupied by the outer shell of these two spheres is

$$g_2 = g_1 + g_1(1 - g_1) = g_1 + g_1(1 - g_1 + g_2),$$

where the first term $g_1$ is due to the first sphere, and the second term is due to the second sphere, which has $(1 - g_1)$ probability of overlap with the first sphere. Placing a third sphere in this scenario ($N = 3$), we have

$$g_3 = g_2 + g_1(1 - (2g_1 + g_2)), $$

where the first term $g_2$ is due to the first two spheres, and the second term is due to the third sphere, which can overlap with these two spheres. This can be expressed recursively for $N$ spheres as

$$g_N = g_{N-1} + g_1(1 - ((N - 1)g_1 + g_{N-1}))$$

$$= g_{N-1} - (1 - g_1)g_1 + g_1(1 - (N - 1)g_1)$$

$$= \sum_{k=0}^{N-1} g_1(1 - g_1)^k(1 - (N - 1 - k)g_1)$$

$$= \left(1 + \frac{g_1}{g_1}\right)(1 - (1 - g_1)^N) - Ng_i. \quad (30)$$

Here we can define $f_i = Ng_i$, the total volume fraction occupied by the inner shells of all spheres (since they do not overlap, it is simple adition), and $f_{ob} = Ng_{ob}$, the total sum of the fractions occupied by the outer shells independently, including the multiple counting of overlapped parts. The actual volume fraction occupied by the outer shells is $f_o = g_N$. In the limit where $g_1 \ll 1$ and $N \gg 1$, we have

$$1 - (1 - g_1)^N \approx 1 - \exp(-Ng_1).$$

Thus, Equation (30) can be written as

$$f_o = \left(1 + \frac{f_i}{f_{ob}}\right)(1 - \exp(-f_{ob})) - f_i. \quad (31)$$

The results match very well with the simulation where a large number of spheres are randomly arranged within a box with the above conditions.

### Appendix D

**Complete Model ($\mu = \langle \psi_1^* \psi_2 \rangle - \langle \psi \rangle^2$)**

In this section, we develop a formalism to calculate the correlation of dimensionless brightness temperature $\psi$ for epochs at which the ionization volume fraction is small. This ensures that ionization bubbles are separate and nonoverlapping. However, as described in Section 3.2 and Appendix C, our formulation allows us to deal with overlap of heating bubbles.

Since we have already assumed that the cross-correlation of density with ionization or spin temperature is negligible, we try to find the autocorrelation of $\phi = n(1 - s)$ (henceforth referred to as “temperature” in this section; Equation (15)). Therefore, to calculate the correlation at scale $r$, we need to find pairs of points such that $\phi_1 \neq 0$ and $\phi_2 \neq 0$, where $\phi_1$ and $\phi_2$ are temperatures of points 1 and 2 that are separated by distance $r$:

$$\langle \phi_1 \phi_2 \rangle = \phi_i^2 \mathbb{P}(\{\phi_1 = \phi_b\} \cap \{\phi_2 = \phi_b\})$$

$$+ 2\phi_i \sum_{r_i} \sum_{\phi_b} \phi_b \mathbb{P}(\{\phi_1 = \phi_b\} \cap \{\phi_2 = \phi\})$$

$$+ \sum_{r_i} \sum_{\phi_b} \sum_{\phi_b} \mathbb{P}(\{\phi_1 = \phi_b\} \cap \{\phi_2 = \phi_b\}). \quad (32)$$

Here, $\phi_1$ and $\phi_2$ are the temperatures of the ionized region and background regions, respectively; $\mathbb{P}(\{\phi_1 = \phi_b\} \cap \{\phi_2 = \phi_b\})$ is the joint probability of point 1 having temperature $\phi_b$ and point 2 having temperature $\phi_b$. In the above equation, the first, second, and third terms correspond respectively to (1) both points being in the background region, (2) one point being in the background and another in a heated bubble, and (3) both points being in some heated bubble. If one or both points are in the ionized region, the term corresponding to that pair will be zero. When both points are in the background, we use Equation (28):

$$P(\{\phi_1 = \phi_b\} \cap \{\phi_2 = \phi_b\})$$

$$= P(\phi_1 = \phi_b) - P(\{\phi_1 = \phi_b\} \cap \{\phi_2 \neq \phi_b\})$$

$$= P(\phi_1 = \phi_b) - P(\{\phi_1 = \phi_b\} \cap \{\phi_2 = \phi_b\})$$

$$- \sum_{r_i} \sum_{\phi_b} \mathbb{P}(\{\phi_1 = \phi_b\} \cap \{\phi_2 = \phi\}). \quad (33)$$

In the case where both points are partially heated, these points can be within the same bubble or different bubbles. Within the same bubble, they can be in the same shell ($\phi_1 = \phi_2$) or in different shells. This gives

$$\sum_{r_i} \sum_{\phi_b} \sum_{\phi_b} \mathbb{P}(\{\phi_1 = \phi_b\} \cap \{\phi_2 = \phi_b\})$$

$$= \sum_{r_i} \sum_{\phi_b} \mathbb{P}(\{\phi_1 = \phi_b\} \cap \{\phi_2 = \phi_b\} \cap \{\phi_2 \neq \phi_b\})$$

$$+ \sum_{r_i} \sum_{\phi_b} \sum_{\phi_b} \mathbb{P}(\{\phi_1 = \phi_b\} \cap \{\phi_2 = \phi_b\} \cap \{\phi_2 \neq \phi_b\})$$

$$+ \sum_{r_i} \sum_{\phi_b} \sum_{\phi_b} \mathbb{P}(\{\phi_1 = \phi_b\} \cap \{\phi_2 = \phi_b\} \cap \{\phi_2 = \phi\} \cap \{\phi_2 \neq \phi\}) \quad (34)$$

Here, $P(\{\phi_1 = \phi_b\} \cap \{\phi_2 = \phi_b\} \cap \{\phi_2 = \phi_b\})$ is the probability that both points are in the same bubble with the same temperature $\phi_b$. This is straightforward to calculate. However, because we allow overlap of heated profiles, the simplest derivation is not necessarily correct. Instead, we expand it
further:

\[ \sum_{R_s, \phi_p(R_s)} \phi^2_p P(\{\phi_1 = \phi_p\} \cap \{\phi_2 = \phi_p\} \cap \{\text{same}\}) \]
\[ = \sum_{R_s, \phi_p(R_s)} \phi^2_p P(\{\phi_1 = \phi_p\}) - P(\{\phi_1 = \phi_p\} \cap \{\phi_2 = \phi_p\}) \]
\[ = \sum_{R_s, \phi_p(R_s)} \phi^2_p P(\{\phi_1 = \phi_p\}) - P(\{\phi_1 = \phi_p\} \cap \{\phi_2 = \phi_p\}) \]
\[ - \sum_{R_s, \phi_p(R_s)} \sum_{\phi_p(R_s) \neq \phi_p} \phi^2_p P(\{\phi_1 = \phi_p\} \cap \{\phi_2 = \phi_p\} \cap \{\text{same}\}) \]
\[ - \sum_{R_s, \phi_p(R_s)} \sum_{R_s', \phi_p(R_s')} \phi^2_p P(\{\phi_1 = \phi_p\} \cap \{\phi_2 = \phi_p\} \cap \{\text{diff}\}). \]

(35)

Putting Equations (33)–(35) in (32), we get

\[ \langle \phi_1 \phi_2 \rangle = \phi^2_b P(\{\phi_1 = \phi_b\}) - \phi^2_p P(\{\phi_1 = \phi_p\} \cap \{\phi_2 = \phi_p\}) \]
\[ + \sum_{R_s, \phi_p(R_s)} \phi^2_p P(\{\phi_1 = \phi_p\}) - \sum_{R_s, \phi_p(R_s)} \phi^2_p \]
\[ \times P(\{\phi_1 = \phi_p\} \cap \{\phi_2 = \phi_p\}) \]
\[ - \sum_{R_s, \phi_p(R_s)} (\phi_2 - \phi_1)^2 P(\{\phi_1 = \phi_2\} \cap \{\phi_2 = \phi_1\}) \]
\[ + \sum_{R_s, \phi_p(R_s)} \sum_{\phi_p(R_s) \neq \phi_p} \phi_1^2 (\phi_q - \phi_1) P(\{\phi_1 = \phi_p\} \cap \{\phi_2 = \phi_q\}) \]
\[ \times (\{\phi_2 = \phi_q\} \cap \{\text{same}\}) \]
\[ + \sum_{R_s, \phi_p(R_s)} \sum_{R_s', \phi_p(R_s')} \phi_1^2 (\phi_q - \phi_1) P(\{\phi_1 = \phi_p\} \cap \{\phi_2 = \phi_q\}) \]
\[ \times (\{\phi_2 = \phi_q\} \cap \{\text{diff}\}). \]

(36)

Now we calculate each term separately. Here, \( P(\phi_1 = \phi_b) = f_b \) is the probability of a point being in the background region, and \( P(\phi_1 = \phi_p) = 4\pi/3 N(R_s) f_s (R_s + \Delta R_s)^3 - R_s^3 \) is the probability of a point having temperature \( \phi_p \) in a bubble of ionization radius \( R_s \).

The probability of one point being in the background and a second point being ionized can be calculated by assuming that point 1 is in an ionized region of radius \( R_s \), and calculating the probability that its neighbor point 2 at distance \( r \) is outside the temperature profile of that bubble \((C(r, 0, R_s, R_b))\). Given this condition, the probability of point 2 being in the background region is equal to the average background fraction \( f_b \). Thus, we have

\[ P(\{\phi_1 = \phi_b\} \cap \{\phi_2 = \phi_b\}) \]
\[ = f_b \sum_{R_s, \phi_p(R_s)} N(R_s) \frac{4\pi}{3} R_s^3 C(r, 0, R_s, R_b). \]

When point 1 has temperature \( \phi_p \) and point 2 is ionized, they both can be in the same bubble or in different bubbles, which respectively give the first and second terms on the right-hand side of the following equation:

\[ \sum_{R_s, \phi_p(R_s)} \phi^2_p P(\{\phi_1 = \phi_p\} \cap \{\phi_2 = \phi_p\}) \]
\[ = \sum_{R_s, \phi_p(R_s)} \phi^2_p \frac{4\pi}{3} N(R_s) \left( R_s^3 (1 - f_b) \right) \]
\[ \times[C(r, 0, R_s, R_p) - C(r, 0, R_s, R_s + \Delta R_s)] \]
\[ + f_b (R_s + \Delta R_s)^3 - R_s^3 \]
\[ \times \sum_{R_s, \phi_p(R_s)} \frac{4\pi}{3} N(R_s') R_s'^3 C(r, 0, R_s', R_b') \].

(37)

When point 1 is in the background and point 2 has temperature \( \phi \), we can expand it as

\[ P(\{\phi_1 = \phi_b\} \cap \{\phi_2 = \phi\}) \]
\[ = P(\phi_1(\text{out other})|\phi_1(\text{out same}) \cap \{\phi_2 = \phi\})) \]
\[ \times P(\phi_1(\text{out same}) \cap \{\phi_2 = \phi\}), \]

where the first term on the right-hand side, \( P(\phi_1(\text{out other})|\phi_1(\text{out same}) \cap \{\phi_2 = \phi\})) \), is the probability that point 1 is in the background region given that point 2 is partially heated with temperature \( \phi \) and point 1 is not inside the same bubble in which point 2 is. This probability equals the fraction of the universe heated to background temperature, \( f_b \). The second term on the right-hand side, \( P(\phi_1(\text{out same}) \cap \{\phi_2 = \phi\}) \), is the probability that point 1 is out of the bubble in which point 2 is, and point 2 is partially heated with temperature \( \phi \). As point 2 can be in a bubble with any ionization radius \( R_s \), \( P(\phi_1(\text{out same})|\{\phi_2 = \phi\}(R_s)) \) is the probability that point 1 is outside the bubble that has ionization radius \( R_s \) and that contains point 2 with temperature \( \phi \). This equals the probability that point 1 is outside of the bubble with outer radius \( R_b \) in which point 2 is located between radius \( R_s \) and \( R_s + \Delta R_s \).

\[ \sum_{R_s, \phi_p(R_s)} P(\{\phi_1 = \phi_b\} \cap \{\phi_2 = \phi\}) \]
\[ = f_b \sum_{R_s} N(R_s) \frac{4\pi}{3} \sum_{\phi_p(R_s)} f_s (R_s + \Delta R_s)^3 - R_s^3 \]
\[ \times (r, R_s, R_s + \Delta R_s, R_b). \]

(38)

Note that \( P(\{\phi_1 = \phi_p\} \cap \{\phi_2 = \phi_q\} \cap \{\text{diff}\}) \) gives the probability that point 1 has \( \phi = \phi_p \), point 2 has \( \phi = \phi_q \), and they both belong to different bubbles. We take a simple assumption that if point 2 is outside the bubble in which point 1 is, then its probability of having \( \phi = \phi_q \) is equal to the global probability of \( \phi_q \) temperature shell. Here, either \( p \) or \( q \) could
have been point 1. Therefore,
\[
\sum_{R_i} \sum_{\phi_i(R_i)} \sum_{\phi_j(R_j)} \phi_p \phi_q - \phi_p \phi_p P(\{\phi_1 = \phi_p\} \cap \{\phi_2 = \phi_q\} \cap \{\text{same}\})
\]
\[
= \sum_{R_i} N(R_i) \frac{4\pi}{3} \int_{\phi(R_i)} f_p ((R_{i+} + \Delta R_{i+})^3 - R_{i+}^3)
\]
\[
\sum_{R_i} N(R_i) \frac{4\pi}{3} \int_{\phi(R_i)} f_p ((R_{i+} + \Delta R_{i+})^3 - R_{i+}^3) \phi_p (\phi_q - \phi_p)
\]
\[
C(r, R_{i+}, R_{i+} + \Delta R_{i+}, R_{i+}) + C(r, R_{i+}', R_{i+}' + \Delta R_{i+}', R_{i+}')
\]
\[
= \frac{2}{3} (39)
\]

Here, \( P(\{\phi_1 = \phi_p\} \cap \{\phi_2 = \phi_q\} \cap \{\text{same}\}) \) gives the probability that point 1 has \( \phi = \phi_p \), point 2 has \( \phi = \phi_q \), and they both belong to the same bubble. If point 1 is located at a distance between \( R_{i+} \) and \( R_{i+} + \Delta R_{i+} \) from the center of the sphere, then the fraction of its neighbors at distance \( r \) that are outside the sphere of radius \( R_{i+} \) and inside the sphere of radius \( R_{i+} + \Delta R_{i+} \) can be computed. However, because bubbles can overlap, point 2 can be neutral or ionized, which leads to

\[
\sum_{R_i} \sum_{\phi_i(R_i)} \sum_{\phi_j(R_j)} \phi_p \phi_q P(\{\phi_1 = \phi_p\} \cap \{\phi_2 = \phi_q\} \cap \{\text{same}\})
\]
\[
= \sum_{R_i} N(R_i) \frac{4\pi}{3} \int_{\phi(R_i)} f_p ((R_{i+} + \Delta R_{i+})^3 - R_{i+}^3)
\]
\[
\sum_{R_i} \phi_p \phi_q (1 - f_i) (C(r, R_{i+}, R_{i+} + \Delta R_{i+}, R_{i+})
\]
\[
- C(r, R_{i+}', R_{i+}' + \Delta R_{i+}', R_{i+}')
\]
\[
= (40)
\]

Putting all equations together, including the influence of \( \xi \) and simplifying in terms of \( F(x, P, Q, R) = 1 - C(x, P, Q, R) \), we get
\[
\mu = \xi (f_b \phi_b + \langle \phi_b \rangle)^2 + (1 + \xi)
\]
\[
\times \left( \phi_p^2 f_b + \langle \phi_p \rangle^2 \right) \frac{4\pi}{3} N(R_i) R_i^2 F(r, 0, R_i, R_b)
\]
\[
+ f_b \sum_{R_i} \frac{4\pi}{3} N(R_i) \sum_{\phi(R_i)} (\phi_p - \phi)^2 f_b
\]
\[
\times ((R_i + \Delta R_i)^3 - R_i^3) F(r, R_i, R_i + \Delta R_i, R_{i+})
\]
\[
+ \frac{4\pi}{3} N(R_i) R_i^3 \sum_{\phi(R_i)} \phi^2 (1 - f_i)
\]
\[
\times \left[ F(r, 0, R_i, R_i) - F(r, 0, R_i, R_i + \Delta R_i) \right]
\]
\[
- \sum_{R_i} \frac{4\pi}{3} N(R_i) \sum_{\phi(R_i)} f_b ((R_{i+} + \Delta R_{i+})^3 - R_{i+}^3)
\]
\[
\times \sum_{\phi_j(R_i)} \phi_p (\phi_q - \phi_p) (1 - f_i)
\]
\[
= (41)
\]

where
\[
\langle \phi_0 \rangle = \sum_{R_i} \frac{4\pi}{3} N(R_i) \sum_{\phi(R_i)} \phi f_b ((R_i + \Delta R_i)^3 - R_i^3)
\]
\[
\langle \phi_0^2 \rangle = \sum_{R_i} \frac{4\pi}{3} N(R_i) \sum_{\phi(R_i)} \phi^2 f_b ((R_i + \Delta R_i)^3 - R_i^3).
\]

We also calculate the correlation at the same point using
\[
\langle \phi_0 \phi_0 \rangle = \langle \phi_0^2 \rangle - (\langle \phi_0 \rangle)^2,
\]
where \( \xi \) is the correlation function.

Equation (41) matches the expression derived in RS18 for most of the cases. However, this expression is more robust as it gives expected results in all simplifying cases and boundary conditions.

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