THE STATISTICS OF THE TRAJECTORY OF A CERTAIN BILLIARD IN A FLAT TWO-TORUS

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Abstract. We consider a billiard in the punctured torus obtained by removing a small disk of radius \( \varepsilon > 0 \) from the flat torus \( T^2 \), with trajectory starting from the center of the puncture. In this case the phase space is given by the range of the velocity \( \omega \) only. Let \( \tilde{\tau}_\varepsilon(\omega) \), and respectively \( \tilde{R}_\varepsilon(\omega) \), denote the first exit time (length of the trajectory), and respectively the number of collisions with the side cushions when \( T^2 \) is being identified with \( [0, 1)^2 \). We prove that the probability measures on \( [0, \infty) \) associated with the random variables \( \varepsilon \tilde{\tau}_\varepsilon \) and \( \varepsilon \tilde{R}_\varepsilon \) are weakly convergent as \( \varepsilon \to 0^+ \) and explicitly compute the densities of the limits.

1. Introduction and main results

Various ergodic and statistical properties of the periodic Lorentz gas were studied during the last decades (see [1], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [17], [20], [22], [23] for a non-exhaustive list of references). In the case of periodically distributed circular scatterers of radius \( \varepsilon \in (0, \frac{1}{2}) \) in \( \mathbb{R}^2 \), one considers the region

\[ Z_\varepsilon = \{ x \in \mathbb{R}^2 ; \text{dist}(x, \mathbb{Z}^2) \geq \varepsilon \} \]

and the first exit time (called free path length by some authors) defined as

\[ \tau_\varepsilon(x, \omega) = \inf \{ \tau > 0 ; x + \tau \omega \in \partial Z_\varepsilon \} , \]

where \( x \in Y_\varepsilon = Z_\varepsilon / \mathbb{Z}^2 \) and \( \omega \) belongs to the unit circle \( T \), which will be steadily identified with \( [0, 2\pi) \) throughout the paper. Equivalently, one can consider the free motion of a point-like particle in the billiard table \( Y_\varepsilon \) obtained by removing pockets of the form of quarters of a circle of radius \( \varepsilon \). If we identify \( T^2 = \mathbb{R}^2 / \mathbb{Z}^2 \) with \( [0, 1)^2 \), then \( Y_\varepsilon \) can be regarded as a punctured two-torus. The reflection in the side cushions of the table is specular and the trajectory between two such reflections is rectilinear. Assume that the particle has constant speed, say equal to 1, and leaves the table when it reaches one of the four pockets. In this setting \( \tau_\varepsilon(x, \omega) \) denotes the exit time from the table (or equivalently the length of the trajectory).

In this paper we will study the case where the trajectory starts at the origin \( O = (0, 0) \) with initial velocity \( \omega \). In this case the first exit time is \( \tilde{\tau}_\varepsilon(\omega) := \tau(O, \omega) \) and one averages over \( \omega \) only. We shall give very precise estimates on the average of \( \tilde{\tau}_\varepsilon \) and of the number \( \tilde{R}_\varepsilon(\omega) \) of collisions of the

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Research partially supported by ANSTI grant C6189/2000.
particle with the side cushions as \( \varepsilon \to 0^+ \). The related problem of estimating the moments

\[
   c_r = \frac{1}{2\pi} \int_0^{2\pi} \tilde{\tau}_\varepsilon'(\omega) \, d\omega, \quad r > 0,
\]

was raised by Ya.G. Sinai in 1981. An answer was given in [3], where it was proved for any interval \( I \subseteq [0, \frac{\pi}{4}] \) and any (small) \( \delta > 0 \) that

\[
   (1.1) \quad \varepsilon^r \int_I \tilde{\tau}_\varepsilon'(\omega) \, d\omega = c_r |I| + O_{r,\delta}(\varepsilon^{\frac{1}{4} - \delta}) \quad \text{as} \quad \varepsilon \to 0^+,
\]

where

\[
   c_r = \frac{2}{\zeta(2)} \int_0^{1/2} \left( x(2^{r-1} + (1-x)^r-1) \right) dx.
\]

Since all the probability measures \( \tilde{\mu}_\varepsilon \) defined by

\[
   \tilde{\mu}_\varepsilon(f) = \frac{1}{2\pi} \int_0^{2\pi} f \left( \varepsilon \tilde{\tau}_\varepsilon(\omega) \right) \, d\omega, \quad f \in C_c([0, \infty)),
\]

have the support included into a common compact (see Lemma 3.1 in [3] or Lemma 2.1 in this paper), the equality (1.1) implies that \( \tilde{\mu}_\varepsilon \to \tilde{\mu} \) weakly as \( \varepsilon \to 0^+ \), and that the moments of \( \tilde{\mu} \) are given by

\[
   \frac{1}{2\pi} \int_0^{2\pi} \omega^r d\mu(\omega) = c_r.
\]

However, this does not directly provide an explicit formula for the density of \( \tilde{\mu} \). The primary aim of this paper is to give a direct proof for this convergence, computing in the meantime the density of \( \tilde{\mu} \) in closed form. For
obvious symmetry reasons we may only consider the interval \([0, \pi/4]\) instead of \([0, 2\pi]\). To state the main result we consider the repartition function

\[
H_\varepsilon(t) = \frac{4}{\pi} \left\{ \omega \in \left[0, \frac{\pi}{4}\right] ; \varepsilon \tilde{\tau}_\varepsilon(\omega) > t \right\}
\]

of \(\tilde{\mu}_\varepsilon\) and the function

\[
(1.2) \quad \psi(x) = \frac{1 - x}{x} \left(1 + \ln \frac{x}{1 - x}\right), \quad x \in \left[\frac{1}{2}, 1\right).
\]

It is seen that \(\psi\) is non-decreasing and satisfies \(\psi(\frac{1}{2}) = 1, \psi(1^-) = 0\), and especially

\[
(1.3) \quad \int_{1/2}^{1} \psi(x) \, dx = \frac{\zeta(2) - 1}{2} = \frac{\pi^2}{12} - \frac{1}{2}.
\]

We can now state the main result.

**Theorem 1.1.** For each \(t > 0\), there exists \(H(t) = \lim_{\varepsilon \to 0^+} H_\varepsilon(t)\). Moreover, one has

\[
H_\varepsilon(t) = H(t) + O_\delta(\varepsilon^{\frac{1}{3}\delta})
\]

for any (small) \(\delta > 0\), where

\[
H(t) = \begin{cases} 
1 - \frac{2t}{\zeta(2)} & \text{if } 0 < t < \frac{1}{2}; \\
\frac{2}{\zeta(2)} \int_{1/2}^{t} \psi(x) \, dx & \text{if } \frac{1}{2} < t < 1; \\
0 & \text{if } t > 1.
\end{cases}
\]

In particular \(\tilde{\mu}_\varepsilon \to \tilde{\mu}\) weakly as \(\varepsilon \to 0^+\), where \(d\tilde{\mu}(t) = h(t) \, dt\) and

\[
h(t) = -H'(t) = \frac{2}{\zeta(2)} \begin{cases} 
1 & \text{if } 0 < t \leq \frac{1}{2}; \\
\psi(t) & \text{if } \frac{1}{2} < t \leq 1; \\
0 & \text{if } t \geq 1.
\end{cases}
\]
This result deserves some comments. Firstly, since \( \tilde{\mu} \) is a probability measure on \([0, 1]\), one must have that 
\[
\frac{1}{2} + \int_{1/2}^{1} \psi(t) \, dt = \frac{\zeta(2)}{2},
\]
that is to formula (1.3), and we got a first proof of that formula.

However, one can prove (1.3) in other ways. For instance, writing \( \psi(x) = \frac{1}{x} \ln \frac{1}{1-x} + \ln(1 - x) + \frac{1}{2} \ln x - 1 - \ln x \), integrating each term from \( t \) to 1 and using the Taylor expansion for \( \frac{1}{x} \ln \frac{1}{1-x} \), we gather

\[
(1.4) \quad \int_{t}^{1} \psi(x) \, dx = \text{Li}_2(1) - \text{Li}_2(t) - \frac{\ln^2 t}{2} + (1 - t) \left( \ln \frac{1 - t}{t} - 1 \right),
\]

where \( \text{Li}_2(t) = \sum_{n=1}^{\infty} \frac{t^n}{n^2} \) denotes the dilogarithm function. Plugging now \( t = \frac{1}{2} \) in (1.4) and using the formula (cf. relation (1.6) in [21])

\[
(1.5) \quad \text{Li}_2 \left( \frac{1}{2} \right) = \frac{\pi^2}{12} - \frac{\ln^2 2}{2} = \frac{\zeta(2) - \ln^2 2}{2}
\]

we arrive again at relation (1.3). Thus, the fact that \( \tilde{\mu} \) has mass one is equivalent to the (non-trivial) equality (1.5) concerning the dilogarithm.

Secondly, Theorem 1.1 should be compared with the most recent results of the first two authors ([4]), who proved the existence of the limit repartition

\[
\frac{4}{\pi} \left\{ (x, \omega) \in [0, 1)^2 \times \left[ 0, \frac{\pi}{4} \right]; \varepsilon \tau_{\varepsilon}(x, \omega) > t \right\} \quad \text{as} \quad \varepsilon \to 0^+.
\]

This repartition function is no longer compactly supported; it was proved in [4] to be of the form

\[
4\zeta(2) \sum_{n=1}^{\infty} \frac{2^n - 1}{n^2(n + 1)^2(n + 2)^n} t^n + O_{\delta}(\varepsilon^{\frac{1}{2} - \delta}), \quad \forall t \geq 2.
\]

Interestingly, the density \( h \) coincides\(^1\) for \( t \in [0, 1] \) with the density of the limit measure of the geometric free path, proved to exist in the small-scatterer limit in [4], and computed independently in [13] and [4]. Another interesting feature of the free path lengths in the small-scatterer limit is that their limit repartition functions are closely related with dilogarithms in the homogeneous case treated in this paper, and with dilogarithms and possibly trilogarithms in the nonhomogeneous case treated in [4], where one averages over a phase space defined by both velocity and position, and where the limit measure has a tail at \( +\infty \).

We finally estimate the repartition function

\[
F_{\varepsilon}(t) = \frac{4}{\pi} \left\{ \omega \in \left( 0, \frac{\pi}{4} \right]; \varepsilon \tilde{R}_{\varepsilon}(\omega) > t \right\}
\]

\(^1\)after scaling \( t \) by 2 since the factor \( \varepsilon \) is replaced by \( 2\varepsilon \) in [4].
of the random variable $\varepsilon \tilde{R}_\varepsilon$ in the small-scatterer limit, and prove

**Theorem 1.2.** For each $t > 0$, there exists $F(t) = \lim_{\varepsilon \to 0^+} F_\varepsilon(t)$. Moreover, one has

$$F_\varepsilon(t) = F(t) + O_\delta(\varepsilon^{\frac{1}{2} - \delta})$$

for any (small) $\delta > 0$, where

$$F(t) = \frac{4}{\pi} \int_0^{\pi/4} H \left( \frac{t}{\cos \omega + \sin \omega} \right) d\omega = \frac{4}{\pi} \int_{\pi/4}^{\pi/2} H \left( \frac{t}{\sqrt{2} \sin \omega} \right) d\omega.$$

In particular this implies that the probability measures $\tilde{\nu}_\varepsilon$ defined by

$$\tilde{\nu}_\varepsilon(f) = \frac{1}{2\pi} \int_0^{2\pi} f(\varepsilon \tilde{R}_\varepsilon(\omega)) d\omega, \quad f \in C_c([0, \infty)),$$

converge weakly to the probability measure $\tilde{\nu}$ with repartition function $F$. It is clear that the support of $\tilde{\nu}$ coincides with the interval $[0, \sqrt{2}]$. Moreover, the function $F$ is linear on $[0, \frac{\pi}{4}]$ and on this interval one has

$$F(t) = \frac{4}{\pi} \int_{\pi/4}^{\pi/2} \left( 1 - \frac{t\sqrt{2}}{\zeta(2) \sin \omega} \right) d\omega = 1 + \frac{4t\sqrt{2}}{\pi\zeta(2)} \cdot \ln \tan \frac{\pi}{8}$$

$$= 1 - \frac{4\sqrt{2} \ln(1 + \sqrt{2}) t}{\pi\zeta(2)}.$$

One may replace the range $[0, \frac{\pi}{4}]$ of $\omega$ by an arbitrary interval $I \subseteq [0, \frac{\pi}{4}]$, proving the existence of the weak limit of the probability measures associated with the random variables $\varepsilon \tilde{\tau}_\varepsilon$ and $\varepsilon \tilde{R}_\varepsilon$ when $\omega \in I$, and also computing their limits. This can be done only with minimal modifications in Section 4.

## 2. Formulas for sectors ending at Farey points

For each integer $Q \geq 1$, let $F_Q$ denote the set of Farey fractions of order $Q$, that is the set of irreducible rational numbers in $[0, 1]$ with denominators less or equal than $Q$. It is well-known that if $\frac{a}{q} < \frac{a'}{q'}$ are consecutive elements in $F_Q$, then one has

$$a'q - aq' = 1 \quad \text{and} \quad q + q' > Q.$$  

Conversely, if $q, q' \in \{1, \ldots, Q\}$ and $q + q' > Q$, then there exist $a \in \{1, \ldots, q - 1\}$ and $a' \in \{1, \ldots, q' - 1\}$ such that $\frac{a}{q} < \frac{a'}{q'}$ are consecutive in $F_Q$.

For each interval $I \subseteq [0,1]$, we consider the set

$$F_Q(I) = I \cap F_Q$$
of Farey fractions of order $Q$ from $I$ of cardinality

$$N_{Q,I} = \# \mathcal{F}_Q(I) = \frac{Q^2 |I|}{2\zeta(2)} + O(Q \ln Q).$$

For each $h > 0$ we consider the vertical scatterers of height $2h$, 

$$\tilde{V}_{q,a,h} = \{q\} \times [a-h, a+h], \quad (q,a) \in \mathbb{Z}^2,$$

and the set

$$\tilde{Z}_h = \bigcup_{(q,a) \in \mathbb{Z}^2} \tilde{V}_{q,a,h}.$$

The free path length in this model is then given by

$$l_h(\omega) = \inf \{\tau > 0 ; (\tau \cos \omega, \tau \sin \omega) \in \tilde{Z}_h \}.$$ 

We denote by $R_h(\omega)$ the number of reflections in the side cushions in the billiard model in the case of vertical scatterers considered above.

We also define for each $\varepsilon > 0$, each interval $I \subseteq [0,1]$ and each integer $Q \geq 1$ the quantities

$$\tilde{H}_{\varepsilon,I,Q}(t) = \left| \left\{ \omega ; \tan \omega \in I, \ l_{\frac{1}{Q}}(\omega) > \frac{t}{\varepsilon} \right\} \right|$$

and

$$\tilde{F}_{\varepsilon,I,Q}(t) = \left| \left\{ \omega ; \tan \omega \in I, \ R_{\frac{1}{Q}}(\omega) > \frac{t}{\varepsilon} \right\} \right|.$$ 

The repartition of Farey fractions in $\mathcal{F}_Q(I)$ will play a central role in the next section while proving an asymptotic formula for $\tilde{H}_{\varepsilon,I,Q}(t)$ as $\varepsilon \to 0^+$, $|I| \to 0$ and $Q \to \infty$ in a suitable way.

In the remainder of this section $Q \geq 1$ will be a fixed integer. A key remark is that every line of slope between 0 and 1 through the origin $O$ will necessarily intersect the set

$$\mathcal{V}_Q = \bigcup_{a/q \in \mathcal{F}_Q} \tilde{V}_{a/q, \frac{1}{Q}}$$

consisting in $N_Q = N_{Q,[0,1]}$ vertical scatterers of height $\frac{2}{Q}$ centered at the integer points $(q,a)$ with $\frac{2}{Q}$ irreducible fraction in $\mathcal{F}_Q$. This is contained in the next statement which is essentially Lemma 3.1 in [3].

**Lemma 2.1.** For any $\omega \in [0, \frac{\pi}{4}]$ one has

$$\{(\tau \cos \omega, \tau \sin \omega) ; \tau > 0\} \cap \mathcal{V}_Q \neq \emptyset.$$

**Proof.** Let $t_P$ denote the slope of the line $OP$. We use the inequalities $q + q' \geq Q + 1 > \max(q,q')$ to infer

$$t_A = \frac{a}{q} \leq t_S = \frac{a' - \frac{1}{Q} q}{q'} < t_N = \frac{a + \frac{1}{Q} q}{q} \leq t_A' = \frac{a'}{q'},$$

where we set $A = (q,a)$, $A' = (q',a')$, $N = (q,a + \frac{1}{Q})$, $N' = (q',a' + \frac{1}{Q})$, $S = (q,a - \frac{1}{Q})$, $S' = (q',a' - \frac{1}{Q})$, $W = (q - \frac{1}{Q},a)$, $W' = (q' - \frac{1}{Q},a')$. This
clearly shows that for any $\omega \in [0, \pi/4]$, the line of slope $\omega$ through the origin will necessarily intersect $V_Q$. \hfill \square

We need a few more elementary things concerning Farey fractions. Suppose next that $a'/q' < a/q < a''/q''$ are three consecutive fractions in $\mathcal{F}_Q$. Then the relation $aq' - a'q = 1$ yields that $q' = \bar{a} \pmod{q}$, where $\bar{a}$ denotes the multiplicative inverse of $a \pmod{q}$. Since $q' \in (Q - q, Q]$, then $q' - \bar{a}$ is the unique multiple of $q$ in the interval $(Q - q - \bar{a}, Q - \bar{a}]$. Hence $q' - \bar{a} = q\lfloor \frac{Q - \bar{a}}{q} \rfloor$, and so we get

\begin{equation}
q' = q \left\lfloor \frac{Q - \bar{a}}{q} \right\rfloor + \bar{a}.
\end{equation}

Making use of $a''q - aq'' = 1$ and of $q'' \in (Q - q, Q]$, we arrive in a similar way at

\begin{equation}
q'' = q \left\lceil \frac{Q + \bar{a}}{q} \right\rceil - \bar{a}.
\end{equation}

We consider the partition $I_q^{(1)} \cup I_q^{(2)} \cup I_q^{(3)} \cup I_q^{(4)}$ of $[0, q]$, where

- $I_q^{(1)} = [\max(2q - Q, 0), \min(Q - q, q)];$
- $I_q^{(2)} = [\max(2q - Q, Q - q), q];$
- $I_q^{(3)} = [0, \min(2q - Q - 1, Q - q)];$
- $I_q^{(4)} = [Q - q, 2q - Q].$

It is clear that $I_q^{(1)} = \emptyset$ unless $q \leq \frac{2Q}{3}$, $I_q^{(2)} = I_q^{(3)} = \emptyset$ unless $q \geq \frac{Q}{2}$, and $I_q^{(4)} = \emptyset$ unless $q \geq \frac{2Q}{3}$. Taking into account (2.1) and (2.2) we see that $q' > q \iff \bar{a} \leq Q - q$

and $q'' > q \iff \bar{a} \geq 2q - Q$.

Since $\bar{a}$ cannot actually take the values 0 and $q$, one has
Lemma 2.2. (i) \( \min(q', q'') > q \iff q \leq \frac{2Q}{3} \) and \( \bar{a} \in I_q^{(1)} \).

(ii) \( q' < q < q'' \iff q \geq \frac{Q}{2} \) and \( \bar{a} \in I_q^{(2)} \).

(iii) \( q'' < q < q' \iff q \geq \frac{Q}{2} \) and \( \bar{a} \in I_q^{(3)} \).

(iv) \( q > \max(q', q'') \iff q \geq \frac{2Q}{3} \) and \( \bar{a} \in I_q^{(4)} \).

For each \( \omega \) we put \( e_{\omega,Q} = (q, a) \) if the half-line \( \mathbb{R}_{+} \omega \) first intersects the scatterer \( (q, a) + V_{1/Q} \) among the \( NQ \) components of \( V_Q \). We denote by \( \omega_{q,a} \) the angle determined by the trajectories which end near the lattice point \( (q, a) \), that is

\[
\omega_{q,a} = \left\{ \omega \in \left[0, \frac{\pi}{4}\right) : e_{\omega,Q} = (q, a) \right\}.
\]

Figure 4. The cases \( q < \min(q', q'') \) and \( q' < q < q'' \)

Figure 5. The cases \( q'' < q < q' \) and \( q > \max(q', q'') \)

Explicit formulas for \( \omega_{q,a} \) can be given using Lemma 2.2 and

\[
\arctan(x + h) - \arctan x = \frac{h}{1 + x^2} + O(h^2)
\]

(2.3)

\[
= \frac{h}{1 + (x + h)^2} + O(h^2), \quad x \in [0, 1], \ h > 0 \text{ small},
\]

as follows
Lemma 2.3. (i) If \( q \leq \frac{Q}{2} \), then \( I_q^{(1)} = [0, q] \), \( I_q^{(2)} = I_q^{(3)} = I_q^{(4)} = \emptyset \), and
\[
\omega_{q,a} = \frac{2}{Qq(1 + \gamma^2)} + O\left(\frac{1}{Q^2q}\right).
\]

(ii) If \( \frac{Q}{2} < q \leq \frac{2Q}{3} \), then \( I_q^{(4)} = \emptyset \) and
\[
\omega_{q,a} = \begin{cases} \\
\frac{2}{Qq(1 + \gamma^2)} + O\left(\frac{1}{Q^2q}\right) & \text{if } \bar{a} \in I_q^{(1)} = [2q - Q, Q - q]; \\
\frac{Q - q + \bar{a}}{Qq(1 + \gamma^2)} + O\left(\frac{1}{Q^2q}\right) & \text{if } \bar{a} \in I_q^{(2)} = [Q - q, q]; \\
\frac{Q - \bar{a}}{Qq(q - \bar{a})(1 + \gamma^2)} + O\left(\frac{1}{Q^2q}\right) & \text{if } \bar{a} \in I_q^{(3)} = [0, 2q - Q].
\end{cases}
\]

(iii) If \( q > \frac{2Q}{3} \), then \( I_q^{(1)} = \emptyset \) and
\[
\omega_{q,a} = \begin{cases} \\
\frac{Q - q + \bar{a}}{Qq(1 + \gamma^2)} + O\left(\frac{1}{Q^2q}\right) & \text{if } \bar{a} \in I_q^{(2)} = [2q - Q, q]; \\
\frac{Q - \bar{a}}{Qq(q - \bar{a})(1 + \gamma^2)} + O\left(\frac{1}{Q^2q'} + \frac{1}{Q^2q''}\right) & \text{if } \bar{a} \in I_q^{(3)} = [0, Q - q]; \\
\frac{Q - \bar{a}}{Qq(1 + \gamma^2) + O\left(\frac{1}{Q^2q} + \frac{1}{Q^2q''}\right)} & \text{if } \bar{a} \in I_q^{(4)} = [Q - q, 2q - Q].
\end{cases}
\]

Proof. If \( q > \frac{Q}{2} \), then one has, according to Lemma 2.2, that \( q < \min(q', q'') \). In this case we infer from (2.3) that
\[
\omega_{q,a} = \arctan \frac{a + \gamma}{q} - \arctan \frac{a - \gamma}{q} = \frac{2}{Qq(1 + \frac{(a - \gamma)^2}{q^2})} + O\left(\frac{1}{Q^2q^2}\right).
\]

This implies, in connection with
\[
\frac{1}{1 + \frac{(a - \gamma)^2}{q^2}} - \frac{1}{1 + \frac{a^2}{q^2}} \ll \frac{a^2}{q^2} - \frac{(a - \gamma)^2}{q^2} \ll \frac{2a}{Qq} \ll \frac{1}{Q},
\]
the desired formula for \( \omega_{q,a} \).

In (ii) and (iii) we split the discussion in the following three cases:
1) \( \bar{a} \in I_q^{(2)} \), that is \( q' < q < q'' \).
In this case \( \frac{a - \frac{1}{q}}{q} < \frac{a' + \frac{1}{q'}}{q'} \), we may write
\[
\omega_{q,a} = \arctan \left( \frac{a + \frac{1}{q}}{q} \right) - \arctan \left( \frac{a' + \frac{1}{q'}}{q'} \right) = \frac{1 - \frac{q - q'}{Q q'q''}}{qq'q''(1 + (a + \frac{1}{q})^2)q''} + O \left( \frac{1}{q^2q''^2} \right)
\]
\[
= \frac{Q - q + q'}{Q q'q''(1 + \frac{a^2}{q^2})} + O \left( \frac{1}{Q q^2q''} + \frac{(Q - q + q')^2}{Q^2q^2q''^2} \right)
\]
\[
= \frac{Q - q + q'}{Q q'q''(1 + \gamma^2)} + O \left( \frac{Q - q + q'}{Q q''q''} + \frac{1}{Q q^2q''} + \frac{(Q - q + q')^2}{Q^2q^2q''^2} \right).
\]

The desired formula for \( \omega_{q,a} \) follows in this case from the formula above, the fact that \( q' = \bar{a} \) as a result of \( \frac{Q - a}{q} = 0 \), from \( q \geq \frac{Q}{q} \), and from \( \frac{Q - q + q'}{qq'} \left( \frac{a}{q} - \frac{a'}{q'} \right) = \frac{Q - q + q'}{q^2q'^2} < \frac{2q'}{q'^2q'^2} = \frac{2}{q'q''} \ll \frac{1}{q'^2q''} \), \( \frac{1}{Q q^2q''} \ll \frac{1}{Q q'^2} \), and from \( \frac{(Q - q + q')^2}{q^2q'^2} \ll \frac{1}{q^2q'^2} \ll \frac{1}{Q q'^2} \).

2) \( \bar{a} \in I_q^{(3)} \), that is \( q'' < q < q' \).

In this case \( q'' = q - \bar{a}, q > \frac{Q}{q} \), and \( \frac{a - \frac{1}{q}}{q} < \frac{a' + \frac{1}{q'}}{q'} < \frac{a}{q} < \frac{a'' - \frac{1}{q''}}{q''} < \frac{a + \frac{1}{q}}{q} \). Proceeding as in the case \( \bar{a} \in I_q^{(2)} \) we find
\[
\omega_{q,a} = \arctan \frac{a''}{q''} - \arctan \frac{a}{q} = \frac{1 - \frac{q - q''}{Q q''}}{q q''(1 + (a - \frac{1}{q})^2)} + O \left( \frac{1}{Q^2q''^2} \right)
\]
\[
= \frac{Q - \bar{a}}{Q q(q - \bar{a})(1 + \gamma^2)} + O \left( \frac{1}{Q^2q''} \right),
\]
as required.

3) \( \bar{a} \in I_q^{(4)} \), that is \( q > \max(q', q'') \).

In this case \( \frac{a - \frac{1}{q}}{q} < \frac{a' + \frac{1}{q'}}{q'} < \frac{a'}{q'} < \frac{a'' - \frac{1}{q''}}{q''} < \frac{a + \frac{1}{q}}{q} \), and we write
\[
\omega_{q,a} = \arctan \frac{a'' - \frac{1}{q''}}{q''} - \arctan \frac{a}{q} = \omega_{q,a}^{(1)} + \omega_{q,a}^{(2)},
\]
with
\[
\omega_{q,a}^{(1)} = \arctan \frac{a'' - \frac{1}{q''}}{q''} - \arctan \frac{a}{q} = \frac{Q - q}{Q q''(1 + \gamma^2)} + O \left( \frac{1}{Q^2q''^2} \right),
\]
\[
\omega_{q,a}^{(2)} = \arctan \frac{a}{q} - \arctan \frac{a'}{q'} = \frac{Q - q}{Q q'q''(1 + \gamma^2)} + O \left( \frac{1}{Q^2q'^2} \right).
\]
Since in this case \( q'' = q - \bar{a} \) and \( q' = \bar{a} \) (thus \( q' + q'' = q \)) we arrive at

\[
\omega_{q,a} = \frac{Q - q}{Q\bar{a}(q - \bar{a})(1 + \gamma^2)} + O\left(\frac{1}{Q^2q'^2} + \frac{1}{Q^2q''^2}\right),
\]

as required. \( \square \)

3. The case of vertical scatterers

In this section we estimate \( \tilde{H}_{\varepsilon,I,Q}(t) \) as \( \varepsilon \to 0^+ \) in the hypothesis that \( |I| \to 0 \) and \( Q \to \infty \) in a controlled way. More precisely we prove

**Proposition 3.1.** Let \( \theta, \theta_1 \in (0,1) \) and suppose that \( I = [\tan \omega_0, \tan \omega_1] \) is a subinterval of \([0,1]\) of size \( |I| \approx \varepsilon^\theta \), and \( Q \) is an integer such that \( Q = \frac{\cos \omega_0}{\varepsilon} + O(\varepsilon^{-\theta}) \). Then one has

\[
\tilde{H}_{\varepsilon,I,Q}(t) = c_I H(t) + O(\mathcal{E}_{\theta,\theta_1,\delta}(\varepsilon)) \quad \text{as } \varepsilon \to 0^+,
\]

uniformly for \( t \) in a compact subset of \((0,\infty) \setminus \{1,2\} \), where \( H \) has been explicitly defined in Theorem 1.1, and we denote

\[
c_I = \int_I \frac{dt}{1 + t^2} = \omega_1 - \omega_0, \quad \mathcal{E}_{\theta,\theta_1,\delta}(\varepsilon) = \varepsilon^{\min(2\theta, \frac{1}{2} - 2\theta_1 - \theta, \theta_1 + \delta)}.
\]

It is convenient to write

\[
(3.1) \quad \tilde{H}_{\varepsilon,I,Q}(t) = \left| \left\{ \omega ; \tan \omega \in I, \ l_{tQ}(\omega) \cos \omega > \frac{t \cos \omega}{\varepsilon} \right\} \right|.
\]

We also define for each interval \( I \subseteq [0,1] \) and each integer \( Q \geq 1 \) the quantity

\[
\tilde{H}_{I,Q}(t) = \left| \left\{ \omega ; \tan \omega \in I, \ l_{tQ}(\omega) \cos \omega > tQ \right\} \right|.
\]

Let \( Q^- \) and \( Q^+ \) be two integers such that

\[
Q^- \leq \frac{\cos \omega_1}{\varepsilon} \leq \frac{\cos \omega_0}{\varepsilon} \leq Q^+.
\]

This also gives for any \( Q \) as in the statement of Proposition 3.1

\[
(3.2) \quad \frac{Q^\pm}{Q} - 1 = O(\varepsilon^\theta).
\]

Since \( \varepsilon \mapsto \tilde{H}_{\varepsilon,I,Q}(t) \) is monotonically increasing, one has

\[
(3.3) \quad \tilde{H}_{I,Q}\left(\frac{tQ^+}{Q}\right) \leq \tilde{H}_{\varepsilon,I,Q}(t) \leq \tilde{H}_{I,Q}\left(\frac{tQ^-}{Q}\right).
\]

We now estimate \( \tilde{H}_{I,Q}(t) \) in
Proposition 3.2. Let $\theta, \theta_1 \in (0, 1)$ and suppose that $I = [\tan \omega_0, \tan \omega_1]$ is a subinterval of $[0, 1]$ of size $|I| \asymp Q^{-\theta}$. Then one has
\[
\tilde{H}_{I,Q}(t) = c_I H(t) + O\left(E_{\theta, \theta_1, \delta}(Q)\right) \quad \text{as } Q \to \infty,
\]
where we denote
\[
E_{\theta, \theta_1, \delta}(Q) = Q^{\max\left(2\theta_1 - \frac{1}{2} + \delta, -\theta - \theta_1 + \delta\right)}.
\]

The key technical tools we shall employ to estimate $\tilde{H}_{I,Q}(t)$ are the following two lemmas from [3] and [2].

Lemma 3.3. ([3, Lemma 2.2]) Suppose that $q \geq 1$ is an integer, $I$ and $J$ are intervals of length lesser than $q$, and that $f$ is a $C^1$ function on $I \times J$. Then one has
\[
\sum_{a \in I, b \in J} f(a, b) = \frac{\varphi(q)}{q^2} \iint_{I \times J} f(x, y) \, dx \, dy
\]
\[
+ O\left(T^2 q^{\frac{1}{2} + \delta} \|f\|_\infty + T q^{\frac{1}{2} + \delta} \|Df\|_\infty + \frac{|I||J|\|Df\|_\infty}{T}\right)
\]
for any integer $T \geq 1$ and any $\delta > 0$, where $\| \cdot \|$ denotes the $L^\infty$ norm on $I \times J$, $Df = |\frac{\partial f}{\partial x}| + |\frac{\partial f}{\partial y}|$, and $\varphi$ is Euler’s totient function.

Lemma 3.4. ([2, Lemma 2.3]) Suppose that $0 < a < b$ and that $f$ is a $C^1$ function on $[a, b]$. Then one has
\[
\sum_{a < k \leq b} \frac{\varphi(k)}{k} f(k) = \frac{1}{\zeta(2)} \int_a^b f(x) \, dx + O\left(\ln b \left(\|f\|_\infty + \int_a^b |f'|\right)\right).
\]

We shall need the next two corollaries of these two lemmas.

Corollary 3.5. (i) For each $0 < t_1 < t_2 \leq 1$ one has
\[
\sum_{t_1 Q < q \leq t_2 Q} \sum_{a \in I, a \in [0, q], \bar{a} = 1 \mod q} \frac{1}{Qq(1 + \gamma^2)} = \frac{c_I(t_2 - t_1)}{\zeta(2)} + O\left(E_{\theta, \theta_1, \delta}(Q)\right).
\]

(ii) For each $\frac{1}{2} < t_1 < t_2 \leq 1$ one has
\[
\sum_{t_1 Q < q \leq t_2 Q} \sum_{a \in I, a \in [0, Q-q], \bar{a} = 1 \mod q} \frac{1}{Qq(1 + \gamma^2)} = \frac{c_I}{\zeta(2)} \int_{t_1}^{t_2} \frac{1 - x}{x} \, dx + O\left(E_{\theta, \theta_1, \delta}(Q)\right);
\]
\[
\sum_{t_1 Q < q \leq t_2 Q} \sum_{a \in I, a \in [2Q-Q], \bar{a} = 1 \mod q} \frac{1}{Qq(1 + \gamma^2)} = \frac{c_I}{\zeta(2)} \int_{t_1}^{t_2} \frac{1 - x}{x} \, dx + O\left(E_{\theta, \theta_1, \delta}(Q)\right).
\]
(iii) For each \( t \in \left( \frac{1}{2}, \frac{3}{2} \right) \) one has

\[
\sum_{tQ < q \leq \frac{3q}{2}} \sum_{\bar{a} \in [2q-Q,Q-q]} \frac{1}{Q(q(1 + \gamma^2))} = c_I \int_t^{2/3} \frac{2 - 3x}{x} \, dx + O(E_{\theta,\theta_1,\delta}(Q)).
\]

Corollary 3.6. For each \( \frac{1}{2} < t_1 < t_2 \leq 1 \), both sums

\[
\sum_{t_1Q < q \leq t_2Q} \sum_{a \in qI, \bar{a} \in [Q-q,q]} \frac{Q-q}{Qq\bar{a}(1 + \gamma^2)}
\]

and

\[
\sum_{t_1Q < q \leq t_2Q} \sum_{a \in qI, \bar{a} \in [0,2q-Q]} \frac{Q-q}{Qq(q-a)(1 + \gamma^2)}
\]

are of the form

\[
\frac{c_I}{\zeta(2)} \int_{t_1}^{t_2} \frac{1-x}{x} \ln \frac{x}{1-x} \, dx + O(E_{\theta,\theta_1,\delta}(Q)).
\]

We only give the proof of Corollary 3.6. The proof of Corollary 3.5 is easier and we leave it as an exercise to the reader.

Proof of Corollary 3.6. We only prove the first equality. The second one is proved in a similar way by changing \( \bar{a} \) to \( q - \bar{a} \). To estimate the inner sum we consider for each \( q \in (t_1Q,t_2Q) \subset (Q/2,Q] \) the function

\[
f(a,\bar{a}) = f_q(a,\bar{a}) = \frac{Q-q}{Qq\bar{a}(1 + \gamma^2)} = \frac{q(Q-q)}{Q(q^2 + a^2)\bar{a}}, \quad a \in qI, \bar{a} \in [Q-q+1,q].
\]

Since \( Q-q+1 \leq a \), one has

\[
\|f\|_{\infty} \leq \frac{1}{Qq}, \quad \left\| \frac{\partial f}{\partial a} \right\|_{\infty} \ll \frac{1}{Qq^2} \quad \text{and} \quad \left\| \frac{\partial f}{\partial \bar{a}} \right\|_{\infty} \ll \frac{1}{Qq(Q-q+1)}.
\]

Applying Lemma 3.3 to \( f \) with \( T = [Q^{\theta_1}], \, I = [Q-q+1,q] \) and \( J = qI \), we gather

\[
(3.4) \sum_{a \in qI, \bar{a} \in [Q-q,q]} \frac{Q-q}{Qq\bar{a}(1 + \gamma^2)} = \frac{\varphi(q)}{q} \cdot W_Q(q) + O(F_Q(q)),
\]

where

\[
W_Q(q) = \frac{Q-q}{Qq^2} \int_{qI} \frac{da}{1 + \frac{a^2}{Q}} \int_{Q-q+1}^q \frac{d\bar{a}}{\bar{a}} = \frac{c_I(Q-q)}{Qq} \ln \frac{q}{Q-q+1},
\]
and the error terms

\[ F_Q(q) = Q^{2\theta_1 - 1} q^{-\frac{1}{2} + \delta} + Q^{\theta_1 - 1} \frac{q^{\frac{1}{2} - \delta}}{Q - q + 1} + \frac{q Q^{-\theta}(2q - Q) Q^\frac{1}{2} Q_{Q - q + 1}}{Q^\theta}, \]

sum up over \( q \) to (we may take \( \delta < \theta_1 \))

\[ \sum_{\frac{Q}{2} < q \leq Q} F_Q(q) \ll Q^{2\theta_1 - \frac{3}{2} + \delta} + Q^{\theta_1 - \frac{1}{2} + \delta} \ln Q + Q^{-\theta - \theta_1} \ln Q \ll F_{\theta,\theta_1,\delta}(Q). \]

Using the inequality \( \ln(1 + x) \leq x \), we see immediately that

\[ \int_{t_1}^{t_2 Q} W_Q(q) \, dq = c_I \int_{t_1}^{t_2 Q} \frac{Q - q}{Q q} \ln \frac{q}{Q - q} \, dq + O\left(\frac{1}{Q}\right) \]

\[ = c_I \int_{t_1}^{t_2} \frac{1 - x}{x} \ln \frac{x}{1 - x} \, dx + O(Q^{-1}). \]

Finally, the inequalities \( \|W_Q\|_{\infty,[t_1 Q,t_2 Q]} \ll Q^{\delta - 1} \) and \( \|W_Q'\|_{\infty,[t_1 Q,t_2 Q]} \ll q^{\delta - 2} \) show that Lemma 3.3 applied to the function \( W_Q \) yields

\[ \sum_{t_1 Q < q \leq t_2 Q} \frac{\varphi(q)}{q} W_Q(q) = \frac{1}{\zeta(2)} \int_{t_1 Q}^{t_2 Q} W_Q(q) \, dq + O(Q^{2\delta - 1}). \]

Finally we put together \[ \boxed{3.4}, \boxed{3.5}, \boxed{3.6} \] and \[ \boxed{3.7} \] to get the desired estimate in the first formula of the statement. \( \square \)

**Proof of Proposition 3.2.** We use the explicit formulas for \( \omega_{q,a} \) found in Section 2. The crude estimate

\[ \max \left( \sum_{\frac{q}{2} \in F_Q(I)} \frac{1}{Q^2 q}, \sum_{\frac{q}{2} \in F_Q(I)} \frac{1}{Q^2 q'}, \sum_{\frac{q}{2} \in F_Q(I)} \frac{1}{Q^2 q''} \right) \leq \frac{1}{Q^2} \sum_{q=1}^{Q} \#(q I \cap \mathbb{Z}) + 1 \]

\[ \ll \frac{1}{Q^2} \sum_{q=1}^{Q} \frac{q |I| + 1}{q} \ll \ln Q + \frac{|I|}{Q} \ll Q^{-\theta - 1} \]

and Lemma 2.3 show that

\[ \tilde{H}_{I,Q}(t) = S_{I,Q}(t) + O(Q^{-\theta - 1}), \]

where

\[ S_{I,Q}(t) = \sum_{\frac{q}{2} \in F_Q(I)} \tilde{\omega}_{q,a}, \]
with

\[
\tilde{\omega}_{q,a} = \begin{cases} 
\frac{2}{Qq(1 + \gamma^2)} & \text{if } \bar{a} \in I_q^{(1)}; \\
\frac{Q - q + \bar{a}}{Qq(1 + \gamma^2)} & \text{if } \bar{a} \in I_q^{(2)}; \\
\frac{Q - \bar{a}}{Q(q - \bar{a})(1 + \gamma^2)} & \text{if } \bar{a} \in I_q^{(3)}; \\
\frac{Q - q}{Q\bar{a}(q - \bar{a})(1 + \gamma^2)} & \text{if } \bar{a} \in I_q^{(4)}. 
\end{cases}
\]  

(3.8)

It is clear that

\[
\sum_{\bar{a} \in F_q(I)} \tilde{\omega}_{q,a} = \int_{\omega_0}^{\omega_1} d\omega + O\left(\frac{1}{Q}\right) = c_I + O(Q^{-1}).
\]  

(3.9)

Suppose now that \(0 < t < \frac{1}{2}\). In this case we may write in view of Lemma 2.3, (3.8), (3.9) and Corollary 3.5 (i)

\[
S_{I,Q}(t) = \sum_{\frac{tQ}{2} < q \leq Q} \sum_{\bar{a} \in F_q(I)} \tilde{\omega}_{q,a} - \sum_{\frac{tQ}{2} < q \leq Q} \sum_{\bar{a} \in F_q(I)} \tilde{\omega}_{q,a}
\]

\[= c_I - \sum_{0 < q \leq tQ} \sum_{\frac{a}{q} \in qI, \bar{a} \in \{0, q\}} \frac{2}{Qq(1 + \gamma^2)} + O(Q^{-1})
\]

\[= c_I - \frac{2c_I t}{\zeta(2)} + O\left(E_{q,a}, \delta(Q)\right).
\]

If \(t \geq \frac{2}{3}\), then Lemma 2.3 and 3.8 yield

\[
S_{I,Q}(t) = \sum_{tQ < q \leq Q} \left( \sum_{\bar{a} \in F_q(I)} \tilde{\omega}_{q,a} + \sum_{\bar{a} \in qI, \bar{a} \in I_q^{(2)}} \tilde{\omega}_{q,a} + \sum_{\bar{a} \in qI, \bar{a} \in I_q^{(3)}} \tilde{\omega}_{q,a} \right)
\]

\[= \sum_{tQ < q \leq Q} \sum_{\bar{a} \in qI, \bar{a} \in \{2q - Q, q\}} \frac{Q - q + \bar{a}}{Qq(1 + \gamma^2)}
\]

\[+ \sum_{tQ < q \leq Q} \sum_{\bar{a} \in qI, \bar{a} \in [0, Q - q]} \frac{Q - \bar{a}}{Q(q - \bar{a})(1 + \gamma^2)}
\]

\[\quad + \sum_{tQ < q \leq Q} \sum_{\bar{a} \in qI, \bar{a} \in (Q - q, 2q - Q)} \frac{Q - q}{Q\bar{a}(q - \bar{a})(1 + \gamma^2)},
\]
which can also be written as

\[
\sum_{tQ < q \leq Q} \sum_{a \in qI, \bar{a} \in (Q - q), a \bar{a} \equiv 1 (\text{mod } q)} \frac{Q - q}{Qq\bar{a}(1 + \gamma^2)}
\]

\[
+ \sum_{tQ < q \leq Q} \sum_{a \in qI, \bar{a} \in [0, 2q - Q), a \bar{a} \equiv 1 (\text{mod } q)} \frac{Q - q}{Qq(q - \bar{a})(1 + \gamma^2)}
\]

\[
+ \sum_{tQ < q \leq Q} \sum_{\bar{a} \in [0, Q - q), Q, Q \bar{a} \equiv 1 (\text{mod } q)} \frac{1}{Qq(1 + \gamma^2)}.
\]

Applying Corollaries 3.6 and 3.5 (ii) we find that

\[
S_{I, Q}(t) = \frac{2c_I}{\zeta(2)} \int_{t}^{1} \left( 1 + \ln \frac{x}{1 - x} \right) dx + O\left(E_{\theta, \theta_1, \delta}(Q)\right)
\]

\[
(3.10)
\]

\[
= \frac{2c_I}{\zeta(2)} \int_{t}^{1} \psi(x) dx + O\left(E_{\theta, \theta_1, \delta}(Q)\right).
\]

Finally, in the case where \(\frac{1}{2} < t < \frac{2}{3}\), Lemma 2.3 yields

\[
S_{I, Q}(t) = \sum_{tQ < q \leq Q} \sum_{\frac{2}{3}} \left( \sum_{a \in qI, \bar{a} \equiv 1 (\text{mod } q)} \tilde{\omega}_{q, a} + \sum_{a \in qI, \bar{a} \equiv 1 (\text{mod } q)} \tilde{\omega}_{q, a} + \sum_{a \in qI, \bar{a} \equiv 1 (\text{mod } q)} \tilde{\omega}_{q, a} \right).
\]

where

\[
T_{I, Q}(t) = \sum_{tQ < q \leq Q} \left( \sum_{a \in qI, \bar{a} \equiv 1 (\text{mod } q)} \tilde{\omega}_{q, a} + \sum_{a \in qI, \bar{a} \equiv 1 (\text{mod } q)} \tilde{\omega}_{q, a} + \sum_{a \in qI, \bar{a} \equiv 1 (\text{mod } q)} \tilde{\omega}_{q, a} \right).
\]

Using also 3.8 the sum \(T_{I, Q}(t)\) can be successively written as

\[
\sum_{tQ < q \leq \frac{2Q}{3}} \left( \sum_{a \in [2q - Q, Q - q], a \bar{a} = 1 (\text{mod } q)} \frac{2}{Qq(1 + \gamma^2)} + \sum_{a \in qI, \bar{a} \equiv 1 (\text{mod } q)} \frac{Q - q + \bar{a}}{Qq\bar{a}(1 + \gamma^2)} + \sum_{a \in [0, 2q - Q], a \bar{a} = 1 (\text{mod } q)} \frac{Q - \bar{a}}{Qq(q - \bar{a})(1 + \gamma^2)} \right)
\]

\[
= \sum_{tQ < q \leq \frac{2Q}{3}} \sum_{a \in [2q - Q, Q - q], a \bar{a} = 1 (\text{mod } q)} \frac{Q - q}{Qq(1 + \gamma^2)} + \sum_{tQ < q \leq \frac{2Q}{3}} \sum_{a \in [0, 2q - Q), a \bar{a} = 1 (\text{mod } q)} \frac{Q - q}{Qq(q - \bar{a})(1 + \gamma^2)}
\]

\[
+ \sum_{tQ < q \leq \frac{2Q}{3}} \sum_{a \in [0, q], a \bar{a} = 1 (\text{mod } q)} \frac{1}{Qq(1 + \gamma^2)} + \sum_{tQ < q \leq \frac{2Q}{3}} \sum_{a \in [2q - Q, Q - q], a \bar{a} = 1 (\text{mod } q)} \frac{1}{Qq(1 + \gamma^2)}.
\]
Applying Corollaries 3.5 and 3.6 we find that

\[
T_{I,Q}(t) = 2c_I \int_t^{2/3} \frac{1 - x}{x} \ln \frac{x}{1 - x} + c_I \int_t^{2/3} \left( 1 + \frac{2 - 3x}{x} \right) dx + O(E_{\theta, \theta_1, \delta}(Q))
\]

\[
= 2c_I \int_t^{2/3} \psi(x) dx + O(E_{\theta, \theta_1, \delta}(Q)).
\]

The desired estimate now follows from (3.10), (3.11) and the equality above.

\[\square\]

**Proof of Proposition 3.7.** By Proposition 3.2, the mean value theorem and (3.2) we infer that

\[
\tilde{H}_{I,Q}(t Q) = c_I H(t Q) + O(E_{\theta, \theta_1, \delta}(Q))
\]

\[
= c_I (H(t) + O(\varepsilon^\theta)) + O(E_{\theta, \theta_1, \delta}(\theta))
\]

\[
= c_I H(t) + O(Q^{-\min(2\theta, 1/2, 2\theta_1 - \delta, \theta_1 + \delta)}).
\]

The statement in Proposition 3.1 now follows from this inequality and (3.3).

\[\square\]

**Proposition 3.7.** Let \( \theta, \theta_1 \in (0, 1) \) and suppose that \( I = [\tan \omega_0, \tan \omega_1] \) is a subinterval of \([0, 1]\) of size \(|I| \asymp \varepsilon^\theta\), and \( Q \) is an integer such that \( Q = \frac{\cos \omega_0}{\varepsilon} + O(\varepsilon^{-\theta}) \). Then one has

\[
\tilde{F}_{\varepsilon,I,Q}(t) = c_I H\left( \frac{t Q^{\pm}}{Q} \right) + O(E_{\theta, \theta_1, \delta}(\varepsilon))
\]

uniformly for \( t \) in a compact subset of \((0, 1) \cup (1, 2) \cup (2, \infty)\), with \( E_{\theta, \theta_1, \delta}(\varepsilon) \) as in Proposition 3.1.

**Proof.** We write \( I = [\tan \omega_0, \tan \omega_1] \). Since \( |R_{1/Q}(\omega) - (q + a)| \leq 1 \) whenever \( \varepsilon_{\omega,Q} = (q, a) \), it follows that

\[
|R_h(\omega) - l_h(\omega)(\cos \omega + \sin \omega)| \leq 1, \quad \forall h, \forall \omega.
\]

This manifestly implies for all \( \omega \in [\omega_0, \omega_1] \)

\[
\tilde{H}_{\varepsilon,I,Q}\left( \frac{t + \varepsilon}{\cos \omega_1 + \sin \omega_1} \right) \leq \tilde{H}_{\varepsilon,I,Q}\left( \frac{t + \varepsilon}{\cos \omega + \sin \omega} \right)
\]

\[
= \left\{ \omega ; \tan \omega \in I, l_{\frac{1}{Q}}(\omega)(\cos \omega + \sin \omega) > \frac{t}{\varepsilon} + 1 \right\}
\]

\[
\leq \tilde{F}_{\varepsilon,I,Q}(t)
\]

\[
\leq \left\{ \omega ; \tan \omega \in I, l_{\frac{1}{Q}}(\omega)(\cos \omega + \sin \omega) > \frac{t}{\varepsilon} - 1 \right\}
\]

\[
= \tilde{H}_{\varepsilon,I,Q}\left( \frac{t - \varepsilon}{\cos \omega_0 + \sin \omega_0} \right) \leq \tilde{H}_{\varepsilon,I,Q}\left( \frac{t - \varepsilon}{\cos \omega_0 + \sin \omega_0} \right).
\]
By Proposition 3.1, the mean value theorem and
\[ \frac{t + \varepsilon}{\cos \omega_1 + \sin \omega_1} - \frac{t}{\cos \omega_0 + \sin \omega_0} = O(\varepsilon^\theta), \]
we get
\[
\tilde{H}_{\varepsilon, I, Q} \left( \frac{t + \varepsilon}{\cos \omega_1 + \sin \omega_1} \right) = c_I H \left( \frac{t + \varepsilon}{\cos \omega_1 + \sin \omega_1} \right) + O(\varepsilon^\theta) + O(\|H\|_\infty \varepsilon^\theta),
\]
and
\[
\tilde{H}_{\varepsilon, I, Q} \left( \frac{t - \varepsilon}{\cos \omega_0 + \sin \omega_0} \right) = c_I H \left( \frac{t}{\cos \omega_0 + \sin \omega_0} \right) + O(\varepsilon^\theta) + O(\|H\|_\infty \varepsilon^\theta). \tag{3.13}
\]
In a similar way one gets
\[
\tilde{H}_{\varepsilon, I, Q} \left( \frac{t - \varepsilon}{\cos \omega_0 + \sin \omega_0} \right) = c_I H \left( \frac{t}{\cos \omega_0 + \sin \omega_0} \right) + O(\varepsilon^\theta) + O(\|H\|_\infty \varepsilon^\theta). \tag{3.14}
\]
The statement follows from (3.12), (3.13) and (3.14). \( \square \)

4. Proof of Theorems 1.1 and 1.2

We focus now on the case of circular scatterers of radius \( \varepsilon \). For each integer lattice point \((q, a) \in \mathbb{Z}^2\), let \((q, a \pm \varepsilon_\pm)\) be the intersection of the line \( x = q \) with the tangents from \( O \) to the circle of center \((q, a)\) and radius \( \varepsilon \) (see Figure 6).

The quantities \( \varepsilon_\pm \) are computed from
\[
\varepsilon = \frac{|a - \frac{a \pm \varepsilon_\pm}{q} \cdot q|}{\sqrt{1 + (\frac{a \pm \varepsilon_\pm}{q})^2}} = \frac{\varepsilon_\pm q}{\sqrt{q^2 + (a \pm \varepsilon_\pm)^2}},
\]

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure6.png}
\caption{A circular scatterer}
\end{figure}
which eventually gives
\[
\varepsilon_\pm(q, a) = \frac{\varepsilon}{q}, \quad q = \frac{a \pm a \varepsilon^2}{q^2 - \varepsilon^2} = \frac{\varepsilon}{\cos \arctan \frac{a}{q} + O \left( \varepsilon^2 \right)}.
\]

Let \( I = [\tan \omega_0, \tan \omega_1] \subseteq [0, 1] \) be an interval of size \(|I| \propto \varepsilon^\theta\) with \(0 < \theta < \frac{1}{2}\). It follows from (4.1) that there exists \( \varepsilon_0 = \varepsilon_0(\theta) > 0 \) such that for all \( \varepsilon < \varepsilon_0 \) and all \( \frac{a}{q} \in \mathcal{F}_Q(I) \) one has
\[
\frac{\varepsilon - \varepsilon^{3/2}}{\cos \omega_0} \leq \varepsilon_\pm(q, a) \leq \frac{\varepsilon + \varepsilon^{3/2}}{\cos \omega_1}.
\]

Since \( \theta < \frac{1}{2} \) it also follows that
\[
\frac{\cos \omega_0}{\varepsilon - \varepsilon^{3/2}} - \frac{\cos \omega_1}{\varepsilon + \varepsilon^{3/2}} \ll \varepsilon^{\theta - 1}.
\]

Thus we find integers \( Q^- \) and \( Q^+ \) such that
\[
(4.3) \quad Q^- \leq \frac{\cos \omega_1}{\varepsilon + \varepsilon^{3/2}} \leq \frac{\cos \omega_0}{\varepsilon - \varepsilon^{3/2}} \leq Q^+ \quad \text{and} \quad Q^+ - Q^- \ll \varepsilon^{\theta - 1}.
\]

By (4.2) and (4.3) it follows for \( \varepsilon < \varepsilon_0 \) and \( \frac{a}{q} \in \mathcal{F}_Q(I) \) that
\[
\frac{1}{Q^+} \leq \varepsilon_\pm(q, a) \leq \frac{1}{Q^-}
\]
and
\[
(4.4) \quad Q^\pm = \frac{\cos \omega_0}{\varepsilon} + O(\varepsilon^{\theta - 1}).
\]

Obvious monotonicity properties of
\[
H_{\varepsilon, I}(t) = \left\{ \omega; \tan \omega \in I, \ t_{\varepsilon}(\omega) > \frac{t}{\varepsilon} \right\}
\]
and of \( \tilde{H}_{\varepsilon, I, Q}(t) \) imply that
\[
(4.5) \quad H_{\varepsilon, I}(t) \leq \left\{ \omega; \tan \omega \in I, \ l_{\varepsilon, I, Q}(\omega) > \frac{t}{\varepsilon} - 2\varepsilon \right\} = \tilde{H}_{\varepsilon, I, Q^+}(t - 2\varepsilon^2)
\]
and
\[
(4.6) \quad H_{\varepsilon, I}(t) = \tilde{H}_{\varepsilon, I, Q^-}(t + 2\varepsilon^2).
\]

Due to (4.4) we may apply Proposition 8.1 to get
\[
(4.7) \quad \tilde{H}_{\varepsilon, I, Q^+}(t + 2\varepsilon) = c_I H(t + 2\varepsilon) + O(\varepsilon^{\min(2\theta, 1 - 2\theta_1 - \delta, \theta + \theta_1 - \delta)}).
\]

In view of (4.5), (4.6) and (4.7) we gather for any \( t \neq 1, 2 \)
\[
(4.8) \quad H_{\varepsilon, I}(t) = c_I H(t) + O(\varepsilon^{\min(2\theta, 1 - 2\theta_1 - \delta, \theta + \theta_1 - \delta)}).
\]

Finally we choose a partition \([0, 1] = \bigcup_{j=1}^N I_j\) with intervals \( I_j \) of equal size \(|I_j| = \frac{1}{N} \propto \varepsilon^\theta\). Summing in (4.8) over \( I \in \{I_1, \ldots, I_N\} \) we arrive at
\[
H_{\varepsilon}(t) = \frac{4H(t)}{\pi} \sum_{j=1}^N c_j + O(\varepsilon^{\min(\theta, 1 - \theta_1 - \delta, \theta + \theta_1 - \delta)}).
\]
The proof of Theorem 1.1 is complete once we choose \( \theta = \theta_1 = \frac{1}{8} \).

**Proof of Theorem 1.2.** Arguing as above we get from Proposition 3.7 taking \( \theta = \theta_1 = \frac{1}{8} \):

\[
F_{\varepsilon,J}(t) = \tilde{F}_{\varepsilon,J,Q}^\pm (t \mp 2\varepsilon^2) = c_J H \left( \frac{t}{\cos \omega_0 + \sin \omega_0} \right) + O(\varepsilon^{\frac{1}{8}-\delta})
\]

for any interval \( J = [\tan \omega_0, \tan \omega_1] \subseteq [0,1] \). Thus if the intervals \( I_j = [\tan \omega_j, \tan \omega_{j+1}], 1 \leq j \leq N = \lceil \varepsilon^{-\frac{1}{8}} \rceil \), are such that \( |I_j| = \frac{1}{N} \times \varepsilon^{\frac{1}{8}} \), then

\[
F_{\varepsilon}(t) = \frac{4}{\pi} \sum_{j=1}^{N} F_{\varepsilon,I_j}(t) = \frac{4}{\pi} \sum_{j=1}^{N} (\omega_{j+1} - \omega_j) H \left( \frac{t}{\cos \omega + \sin \omega} \right) + O(\varepsilon^{\frac{1}{8}-\delta}).
\]

It only remains to compare the main terms in the relation above and in Theorem 1.2. This is achieved by merely applying the mean value theorem:

\[
\frac{4}{\pi} \int_0^{\pi/4} H \left( \frac{t}{\cos \omega + \sin \omega} \right) d\omega = \frac{4}{\pi} \sum_{j=1}^{N} \int_{\omega_j}^{\omega_{j+1}} H \left( \frac{t}{\cos \omega + \sin \omega} \right) d\omega
\]

\[
= \frac{4}{\pi} \sum_{j=1}^{N} (\omega_{j+1} - \omega_j) \left( H \left( \frac{t}{\cos \omega_j + \sin \omega_j} \right) + O(\varepsilon^{\frac{1}{8}}) \right)
\]

\[
= \frac{4}{\pi} \sum_{j=1}^{N} (\omega_{j+1} - \omega_j) H \left( \frac{t}{\cos \omega_j + \sin \omega_j} \right) + O(\varepsilon^{\frac{1}{8}}),
\]

which concludes the proof of Theorem 1.2.

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