Almost local generation of EPR entanglement

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The generation of entanglement is studied in a minimal model consisting of two independent Gaussian parties embedded in a common heat bath. We consider the case of weak reservoir-induced interactions which themselves are insufficient to generate entanglement. Local driving by an external classical field, however, can promote this weak interaction to a source of entanglement. Presence or absence of the effect depends on the specific pulse shape of the external control, which we determine through optimal control techniques.

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Introduction.- Entanglement is one of the most fascinating manifestations of quantum non-locality in systems with more than one degree of freedom. It has thus been investigated as a key resource for quantum information in case of both discrete and continuous variables. Both fundamental properties of bipartite Gaussian entanglement and dynamical process producing entangled states have been investigated extensively, with varying roles assigned to a dissipative environment, and to external control fields.

The non-local interaction required for the generation of entanglement is sometimes postulated as an explicit feature of the system. Alternatively, entanglement can be generated through forcing the system state towards the pointer states of a strongly dissipative reservoir, or by relying on the effective interaction potential arising from the exchange of virtual reservoir excitations. The latter type of interaction may be strong even in the case of weak dissipation; its main features can be revealed through the adiabatic elimination of high-frequency reservoir excitations. The cases of strong dissipation or strong environment-induced coupling may be undesirable or infeasible in intended applications. The entangled states thus produced are useful only if the strong interactions that produced them can reliably be quenched, i.e., if the system-reservoir coupling can be controlled at will.

In this Letter, we therefore consider a more subtle dissipative mechanism which will not generate entanglement by itself. However, when leveraged by suitably chosen classical driving fields, the combined effect of driving and dissipation can result in entanglement. We do not assume environments which require specific engineering, but rather are known as standard models for decoherence in solid state devices. Remarkably, even local parametric driving is sufficient to promote the reservoir from an entanglement-degrading to an entanglement-promoting feature. This effect is quite sensitive to the exact time dependence of the driving fields, which we determine through optimal control techniques.

We find that a weakly damped bipartite harmonic system in a Gaussian state (Fig. 1) can be driven into an entangled state by local driving. The immediate effect of local driving on modes A and B is single-mode squeezing. Dissipation in the reduced dynamics of the bipartite system transforms this single-mode squeezing into two-mode squeezing (when viewing symmetric and anti-symmetric modes) or entanglement (when viewing local modes A and B). This effect is found over a wide temperature range. Suitable pulse shapes yielding significant entanglement can be found even for the case of driving only mode A. These findings are of potential relevance for current experiments with superconducting circuits. They also open new possibilities for teleportation without an explicit state transfer.

Open system dynamics.- We consider a compound system with Hamiltonian $H = H_S(t) + H_I + H_R$, where two harmonic oscillators of equal mass $M$ and frequency $\Omega$ form a distinguished system with

$$H_S(t) \equiv H_A(t) + H_B(t) = \sum_{j=A,B} \frac{p_j^2}{2M} + \frac{M\Omega^2}{2} q_j^2 + \frac{u_j(t)}{2} q_j^2.$$ (1)

The last term in Eq. (1) represents local parametric driving of strength $u_j(t)$. The oscillators interact with a common reservoir which has the conventional form of a thermal reservoir used in a quantum Brownian motion.

FIG. 1. Two independent harmonic modes A and B are embedded in a common reservoir (green), which equilibrates the modes to a non-entangled stationary state. Time-dependent control fields applied locally to the A and B change the reservoir into a source of entanglement.
context [12,21,24], i.e.,

\begin{equation}
H_R = \sum_k \frac{p_k^2}{2m_k} + \frac{m_k\omega_k^2}{2} x_k^2
\end{equation}

\begin{equation}
H_I = (q_A + q_B) \sum_k c_k x_k + (q_A + q_B)^2 \sum_k \frac{c_k^2}{2m_k\omega_k^2}
\end{equation}

The reduced dynamics depends on the properties of the reservoir only through the inverse thermal energy \( \beta \) and through the spectral density \( J(\omega) = \pi \sum_k \frac{c_k^2}{2m_k\omega_k^2} \delta(\omega - \omega_k) \), formed from the parameters in Eqs. (2) and (3). The last part in \( H_I \) (known as “counterterm” in the context of the Caldeira-Leggett model) should not be misread as an interaction term for the modes; it ensures a vanishing net effect of the reservoir on the dynamics if adiabatic elimination is performed. If the reservoir is traced out from the exact, full dynamics, only reservoir fluctuations and velocity-dependent memory friction affect the system dynamics. In the present two-mode model, the memory friction includes “mutual drag” between the oscillators, which is the only feature of the model that qualifies as a source of quantum non-locality [12].

While the complete density operator of system and reservoir obeys the standard Liouville-von Neumann equation, the treatment of the dynamics of the relevant reduced density is a formidable challenge. This is particularly true in case of low temperatures and \textit{a priori} unknown control signals where conventional perturbative expansions like Born-Markov master equations fail. In this situation stochastic Liouville-von-Neumann equations (SLN) have been proven as formally exact and numerically powerful tools to capture on the same footing the non-Markovianity of the reduced time evolution and arbitrary external driving fields also for nonlinear systems. They are based on a stochastic representation of the Feynman-Vernon influence functional such that the physical reduced density is obtained by averaging the time evolution according to the SLN over proper noise realizations [22].

For Gaussian modes, the reduced density matrix is fully determined by the first and second cumulants of the elements \( x_j, j = 1, \ldots 4 \) of the vector \( x = (q_A, p_A, q_B, p_B) \) of phase space operators. Since first moments can always be adjusted by local unitary operations, they cannot affect entanglement properties. The central quantity is thus the covariance matrix \( \sigma \) with \( \sigma_{ij} = \frac{1}{2} \langle x_i x_j + x_j x_i \rangle - \langle x_i \rangle \langle x_j \rangle \). In the \( 2 \times 2 \) block structure [3]

\begin{equation}
\sigma = \begin{pmatrix} \alpha & \gamma \\ \gamma^T & \beta \end{pmatrix}
\end{equation}

\( \alpha, \beta \) correspond to the covariance matrices of the respective sub-units A and B, while \( \gamma \) (transpose \( \gamma^T \)) carries the mixed cumulants and thus non-local information. Now, a well-defined measure for entanglement in bipartite Gaussian systems is given by the logarithmic negativity [8, 26]

\begin{equation}
E_N = \max\{0, -\ln(\tilde{\nu}_-)\}
\end{equation}

with \( \tilde{\nu}_- \) being the smallest symplectic eigenvalue of the partially transposed density matrix. Entanglement only exists if \( E_N > 0 \) while \( E_N = 0 \) corresponds to purely classical and/or local quantum correlations. Note that \( \det \gamma < 0 \) is necessary for \( E_N \) to be positive [27].

Without external driving, any initially separable two-mode state will remain separable indefinitely in the scenario considered here. This applies in particular to the steady state, which can be calculated non-perturbatively exactly [21]. The resulting expression \( \tilde{\nu}_2 = \langle (q_A + q_B)^2 \rangle_\beta \langle (p_A + p_B)^2 \rangle_\beta / (\hbar/2)^2 \) contains an uncertainty product of the thermal variances always exceeding the ground state limit (cf. Fig. 2) so that \( \det \gamma > 0 \). It is to be noted, however, that this does not preclude discord as a further type of non-classical correlations [28, 29].

**Optimal control of entanglement.**- Although we have stated that interaction with a common reservoir alone is not sufficient to generate entangled states, it nevertheless plays a key role in the generation of entanglement through an external control signal: Since the system Hamiltonian [11] provides no two-body interactions (neither static nor controlled), it is needed as an additional ingredient beyond local controls. In the sequel, we not only show that entangled states can be created by this combination of factors, but also perform a numerical search for the maximal entanglement that can be achieved in finite time. For this purpose, the recently developed optimal control formalism for open quantum systems [23] is exploited to maximize the entanglement at a given final time \( t_f \), i.e., we seek a maximum of the functional

\begin{equation}
F[u_A(t), u_B(t); \sigma(t)] = E_N(\sigma(t_f))
\end{equation}

of the control fields \( u_A(t), u_B(t) \).

We consider an initial preparation with both oscillators in the ground state, and without initial correlations between system and reservoir. This is followed by propagation of the system under the influence of control fields \( u_A(t), u_B(t) \) and interaction with an Ohmic bath, 

\( J(\omega) = \frac{1}{1+(\omega/\omega_0)^2} \), parameterized by a coupling constant \( \eta \), high frequency cut-off \( \omega_0 \), and at inverse temperature \( \beta = 1/k_B T \). We use dimensionless units with frequencies scaled with \( \Omega \) and lengths scaled with \( \sqrt{\hbar / M} \).

Fig. 2 displays bar graphs of optimized covariance matrices (right) compared to ground and stationary states (left). Apparently, substantial off-diagonal correlations are built up, positive and negative valued, so that one indeed has \( \det \gamma < 0 \). Notably, the same is true for single-site control \( u_B(t) \equiv 0 \).

Further insight is gained through a transformation to normal modes \( Q_\pm = (q_A \pm q_B) / \sqrt{2}, P_\pm = (p_A \pm p_B) / \sqrt{2} \) which transforms the system Hamiltonian [11] into \( H_S = H_R + H_I \).
FIG. 2. Covariance matrices for reservoir parameters $\beta = 1$, $\eta = 0.1$, and $\omega_c = 50$. Left: autonomous dynamics; ground state as initial state (top), stationary final state (bottom). Right: final states of parametrically driven dynamics with $u_A = u_B$ (top) and $u_A \neq 0$, $u_B = 0$ (bottom) at $t_f = 6\pi$. Dimensionless units are used, see text for details.

FIG. 3. Reduced Wigner functions for the anti-symmetric $Q_-, P_-$ (top) and symmetric $Q_+, P_+$ (bottom) normal modes at $t_f = 6\pi$ for two-site (left) and single-site (right) control. The green line indicates the ground state width; parameters are as in Fig. 2.

$$\sum_{n=\pm} \frac{P_n^2}{2} + \frac{1}{2}(1 + u_A + u_B)Q_n^2 + \frac{1}{2}(u_A - u_B)Q_+ Q_-.$$ Then, for $u_A(t) \equiv u_B(t)$, entanglement ($E_N > 0$) requires

$$4 \det \gamma = \Delta_P \Delta_Q - (|Q_+ P_+| - |Q_- P_-|)^2 \quad (7)$$

to be negative, where $\Delta_X = \langle X_f^2 \rangle - \langle X_0^2 \rangle$, $X = P, Q$. Apparently, this is only possible if the evolution of the two normal modes is not degenerate. Lifting this degeneracy is the decisive role the heat bath plays in this setting. However, external driving is needed as well in order to render the normal mode states dissimilar enough to result in entangled A and B states.

Entanglement generation is thus a cooperative effect of local driving and global dissipation in the present setting—either factor by itself is neutral or detrimental to entanglement. A tailored control pulse changes the non-local effects of the heat bath from a destructive influence on quantum resources to an asset promoting entanglement.

The correlators in Eq. (7) can be obtained from the Wigner functions plotted in Fig. 3 for the final state under an optimized driving protocol. For symmetric control (left), $u_A \equiv u_B$, a strongly squeezed antisymmetric mode results, while the symmetric mode is close to a thermal state. This leads to opposite signs of the terms $\Delta_P$ and $\Delta_Q$, therefore the r.h.s. of Eq. (7) is negative, and the local modes $q_A$ and $q_B$ are entangled. For a more quantitative analysis [22], it is convenient to represent the correlation matrix elements $\langle Q_+^2 \rangle$, $\langle P_+^2 \rangle$ and $\langle Q_\pm P_\pm \rangle$ by a different parameter set: squeezing parameters $r_\pm$, squeezing angles $\varphi_\pm$, and width parameters $a_\pm$.

Using these parameters, one finds that $\det \gamma < 0$ if the difference $|r_- - r_+|$ is sufficiently large, with a threshold that depends on $\varphi_- - \varphi_+$ and on the ratio $a_-/a_+$. This observation shows a similarity between the states considered here and two-mode EPR states: EPR states can be seen as factorized pure states of squeezed symmetric and antisymmetric modes, with $r_- = -r_+$ and identical squeezing angles [30]. The states produced in our numerical simulations show strong squeezing in the antisymmetric mode, but not in the symmetric mode, which is close to a thermal state. They might therefore be labeled ‘semi-EPR’ states.

Even using a more rudimentary single-site control ($u_A \neq 0, u_B \equiv 0$), we find that the cooperative effect between driving and dissipation persists, although the similarity to EPR states is diminished, see Fig. 3 (right). Moreover, additional correlations between the symmetric and antisymmetric modes are needed to fully characterize the quantum state. The gradual (non-monotonous) build-up of entanglement over time is shown for the cases of symmetric and single-site control in Fig. 4 (top), with a rapid final approach to values of $E_N(t_f) = 2.33$ (single-site control) and $E_N(t_f) = 4.37$ (symmetric control). Simulations at lower temperatures show further improvements.
locality to be quite robust, particularly for
protocol. Qualitatively, one may expect quantum non-
fundamental solutions \(\phi^+ [1 + \phi] \equiv \phi^+ + [1 + u(t)]\phi = 0\).

Our analysis in the accompanying supplemental ma-
terial reveals that the final state of the \(Q_+\) mode
differs very little from a thermal state for the para-
ters covered by our numerical simulations. The condition
det \(\gamma < 0\) can thus be re-stated to a good approxima-
tion in the simple form

\[
cosh 2r_\gamma > \coth \beta. \tag{8}
\]

This indicates that raising temperature does not preclude
entanglement per se. However, it gives a lower bound on
the squeezing required to obtain entanglement at a given
temperature, which becomes more and more stringent as
temperature is raised. Given an absolute upper bound
of the achievable squeezing \(r_\gamma\), Eq. \(8\) indicates the
highest temperature at which the condition det \(\gamma < 0\)
can be fulfilled. This result is largely independent of the
dissipative coupling strength; it remains valid as long as
the oscillators are underdamped.

Interestingly, the generated entanglement in the
present scenario even persists once the external control
is switched off, as illustrated in Fig. \(5\) for \(\beta = 0.1\). This
feature may be attractive for potential applications.

Discussion.- We have shown that dynamical symme-
try breaking for two-mode Gaussian parties due to the
combined impact of common dissipation and local para-
tmetric control efficiently generates entanglement. This
proves that quantum non-locality can be induced in a
minimal model where it is absent in the thermodynamic
state and only appears in non-equilibrium. Local op-
timal control allows to achieve substantial logarithmic
negativity in finite time and even at high temperatures
and in a simple, experimentally realizable system. The
protocol may thus be applicable to various bipartite sys-
tems in condensed phase devices. Particular examples
include two Cooper pair boxes spatially well separated
in a "bad" cavity or two impurity fermions embedded
in a Bose-Einstein condensate. Potentially, NV centers
in diamonds provide a test-bed to dynamically induce
entanglement at high temperatures. Beyond these
direct realizations, there are also consequences for quan-
tum teleportation to be explored. Instead of transferring

FIG. 4. Top: Logarithmic negativity \(E_N(t)\) (solid) for two-
site (red) and single-site (blue) control. Negative values
for \(-\ln(\tilde{\nu})\) corresponding to \(E_N(t) = 0\) are also depicted
(dashed). Parameters are as in Fig. 2. Bottom: \(-\ln(\tilde{\nu})\) for
various temperatures (two-site control) with other para-
eters held fixed. The inset displays the final time range for the
control with \(E_N(t_f) > 0\) for \(\gamma \geq 0.0005\).

FIG. 5. Entanglement dynamics for \(\beta = 0.1\) for a protocol
where the control is switched off at \(t_f = 6\pi\). Other parameters
are as in Fig. 2.

The numerical values of the entanglement measure
may be related to the number of states involved, allow-
ing a rough comparison to qubit-based entanglement. For
two qubits, \(E_N = 1\) corresponds to a Bell state. Fur-
thermore, for two-mode squeezed vacuum states with
squeezing parameter \(r\) and negativity \(E_N(r) = 2r\), the num-
ber of excited states in each mode dominantly contributing to
the entanglement can be estimated as \(m \approx \exp[|E_N(r)|]/2\).

While, as discussed above, the situation here is different
due to the dissipative \(Q_+\) mode, one may at least esti-
mate that symmetric (single-site) control involves about
\(m \approx 40\) (\(m \approx 5\)) states in each of the oscillators.

Let us now discuss the temperature dependence of the
protocol. Qualitatively, one may expect quantum non-
locality to be quite robust, particularly for \(u_A = u_B\).
This is indeed seen in Fig. 4 (bottom) where the logarithmic
negativity is of order 1 even for inverse temperatures
around \(\beta = 0.01\). Further understanding can be gained
by observing that the fundamental solutions of the clas-
sical equation of motion play a prominent role even in
the fluctuations of a harmonic oscillator \(21\). General-
izing results for the parametrically driven oscillator \(21\),
a semi-analytical theory for the case of symmetric driving
\(u(t) = u_A(t) = u_B(t)\) can be found based on the
fundamental solutions \(\phi_{1,2}(t)\) of the equation of motion
\(\dot{\phi} + [1 + u(t)]\phi = 0\).

Our analysis in the accompanying supplemental ma-
terial reveals that the final state of the \(Q_+\) mode

\[
\phi^+ (t) = \exp \left[ \int_0^t \frac{\gamma}{\beta} \, d\tau \right] \phi(0) + \int_0^t \exp \left[ \int_\tau^t \frac{\gamma}{\beta} \, d\tau' \right] u(\tau) \, d\tau \phi(0)
\]

determines the logarithmic negativity
\(E_N(t)\) at time \(t\). The first term \(\phi^+ (t)\) represents
the contribution from the initial state \(\phi(0)\) and the
second term \(\int_0^t \exp \left[ \int_\tau^t \frac{\gamma}{\beta} \, d\tau' \right] u(\tau) \, d\tau \phi(0)\)
represents the contribution from the control.
one half of an entangled pair from A to B, it may be possible to create a spatially separated entangled pair in place through a dynamical process involving a (possibly “dirty”) shared medium or reservoir.

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