Power sums of Coxeter exponents

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Abstract

Consider an irreducible finite Coxeter system. We show that for any nonnegative integer \( n \) the sum of the \( n \)th powers of the Coxeter exponents can be written uniformly as a polynomial in four parameters: \( h \) (the Coxeter number), \( r \) (the rank), \( \alpha \), \( \beta \) (two further parameters).

1 Introduction

Let \((W,S)\) be an irreducible finite Coxeter system of rank \( r \) with \( S = \{s_1, \ldots, s_r\} \) its set of simple reflections. The Coxeter transformation \( c := s_1 \ldots s_r \in W \) has order \(|c| = h\) known as the Coxeter number, and the eigenvalues of \( c \) in the reflection representation of \( W \) are of the form \( e^{2\pi m_1/h}, \ldots, e^{2\pi m_r/h} \) with \( 1 = m_1 \leq m_2 \leq \cdots \leq m_r = h - 1 \) the exponents of \((W,S)\). Furthermore, for any permutation \( \sigma \) of \( \{1, \ldots, r\} \) the elements \( c \) and \( s_{\sigma(1)} \ldots s_{\sigma(r)} \) are conjugate in \( W \). Hence the exponents do not depend on the enumeration of the simple reflections. Recall that the symmetry \( m_i + m_{r+1-i} = h \) follows from the facts that \( c \) has no eigenvalue 1 and that the reflection representation is defined over the reals.

In this note we will derive uniform expressions for the power sums \( \sum_{i=1}^{r} m_i^n \) for any \( n \in \mathbb{Z}_{\geq 0} \). Of course, for \( n = 0 \) the sum is \( r \), and for \( n = 1 \) the symmetry \( m_i + m_{r+1-i} = h \).
shows that the sum is $\frac{1}{2}rh$. We shall see that
\[
\sum_{i=1}^{r} m_i^n = n! r Td_n(\gamma_1, \ldots, \gamma_n)
\]
where $Td_n(\gamma_1, \ldots, \gamma_n)$ denotes the $n$th Todd polynomial evaluated at $\gamma_1, \ldots, \gamma_n$ (for $n$ odd $Td_n(\gamma_1, \ldots, \gamma_n)$ does not depend on $\gamma_n$, as follows from Proposition 3.1). The $\gamma_i$'s can be chosen to be polynomials in four parameters (details below) with integer coefficients. This answers Panyushev’s question in [6].

2 Some history and preliminaries

For type $A_r$ the exponents are just $1, 2, \ldots, r$ and one has Bernoulli’s formula
\[
\sum_{i=1}^{r} i^n = \frac{1}{n+1} \left( B_{n+1}(r+1) - B_{n+1}(1) \right)
\]
where $B_{n+1}(x)$ is the $(n+1)$st Bernoulli polynomial, defined by the expansion
\[
\sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} = \frac{te^{tx}}{e^t - 1}.
\]

For general types uniform formulae for the power sums up to third power are listed in the epilogue of [8]. Besides the Coxeter number $h$ and the rank $r$ they depend (for the squares and the cubes) on a further parameter $\gamma$ which is defined for the crystallographic types with crystallographic root system $\Phi$ (= $\Phi_+ \cup \Phi_-$ a decomposition into the sets of positive and negative roots) by the formula (see [2, Ch. VI, § 1, no. 12])
\[
\sum_{\varphi \in \Phi} \langle \lambda | \varphi \rangle \langle \mu | \varphi \rangle = \gamma \langle \lambda | \mu \rangle \quad (\lambda, \mu \in \text{span}_\mathbb{R} \Phi)
\]

where $\langle | \rangle$ denotes the Killing form on $\text{span}_\mathbb{R} \Phi$, which is the $W$-invariant (symmetric) bilinear form characterized by
\[
\langle \lambda | \mu \rangle = \sum_{\varphi \in \Phi} \langle \lambda | \varphi \rangle \langle \mu | \varphi \rangle \quad (\lambda, \mu \in \text{span}_\mathbb{R} \Phi).
\]

It turns out that $\gamma = kgg'$ where $k = \langle \theta | \theta \rangle / \langle \theta_s | \theta_s \rangle \in \{1, 2, 3\}$ with $\theta, \theta_s \in \Phi_+$ the highest resp. highest short roots, and $g = 1/\langle \theta | \theta \rangle \in \mathbb{Z}_{>0}$ is the dual Coxeter number of $\Phi$ whereas $g'$ is the dual Coxeter number of the dual root system $\Phi'$. So $\gamma = h^2$ if $\Phi$ is simply-laced. For the noncrystallographic types $\gamma = 2m^2 - 5m + 6$ for $I_2(m)$ (the formula is also valid for the crystallographic types, where $m = 3, 4, 6$); $\gamma = 124$ for type $H_3$; and $\gamma = 1116$ for type $H_4$.

The formulae from [8] read as follows:
\[
\sum_{i=1}^{r} m_i^n = \begin{cases} 
  r & \text{if } n = 0, \\
  \frac{1}{2}rh & \text{if } n = 1, \\
  \frac{1}{6}r(h^2 + \gamma - h) & \text{if } n = 2, \\
  \frac{1}{4}rh(\gamma - h) & \text{if } n = 3.
\end{cases}
\]
Remark 2.1 The power sum for the fourth powers is not of the form \( r \) times a function depending only on \( h \) and \( \gamma \), as a computation for the types \( A_{h-1} \) and \( D_{(h+2)/2} \) shows.

Panyushev recently gave the universal formula [6, Proposition 3.1]

\[
\sum_{\varphi \in \Phi_+} \text{ht}(\varphi)^2 = \frac{1}{12} r(h+1)\gamma
\]  

(2.4)

for the sum of the heights squares of all positive roots. He then suspects [6, Remark 3.4] that for the sum of the heights of all positive roots there is no similar formula in the general case; however, for simply-laced root systems he mentions

\[
\sum_{\varphi \in \Phi_+} \text{ht}(\varphi) = \frac{1}{6} r(h^2 + h)
\]  

(2.5)

and asks for which values of \( n \) there is a nice closed expression for \( \sum_{\varphi \in \Phi_+} \text{ht}(\varphi)^n \). Our result shows that there are universal formulae for all \( n \in \mathbb{Z} \geq 0 \). In fact, let \( (k_1, \ldots, k_{h-1}) \) be the partition dual to \( (m_r, \ldots, m_1) \); then it is well-known (see, e.g., [4, Section 3.20]) that there are exactly \( k_j \) roots of height \( j \) in \( \Phi_+ \). Hence

\[
\sum_{\varphi \in \Phi_+} \text{ht}(\varphi)^n = \sum_{i=1}^{r} (1^n + 2^n + \cdots + m_i^n).
\]  

(2.6)

In particular, using (2.3) we recover (2.4) and have

\[
\sum_{\varphi \in \Phi_+} \text{ht}(\varphi) = \sum_{i=1}^{r} m_i^2 + m_i = \frac{1}{12} r(h^2 + \gamma + 2h)
\]  

(2.7)

which generalizes (2.5) to all types.

Alternatively, using the symmetry \( m_i + m_{r+1-i} = h \) we can write as in [3, Proposition 2.1]

\[
h^2 \sum_{i=1}^{r} m_i - 3h \sum_{i=1}^{r} m_i^2 + 2 \sum_{i=1}^{r} m_i^3 = 0.
\]  

(2.8)

Hence

\[
\sum_{\varphi \in \Phi_+} \text{ht}(\varphi)^2 \overset{2.6}{=} \sum_{i=1}^{r} \frac{m_i(m_i + 1)(2m_i + 1)}{6} = \sum_{i=1}^{r} \frac{m_i^3}{3} + \sum_{i=1}^{r} \frac{m_i^2}{2} + \sum_{i=1}^{r} \frac{m_i}{6}
\]

\[
\overset{2.3}{=} -h^2 \sum_{i=1}^{r} \frac{m_i}{6} + h \sum_{i=1}^{r} \frac{m_i^2}{2} + \sum_{i=1}^{r} \frac{m_i^2}{2} + \sum_{i=1}^{r} \frac{m_i}{6}
\]

\[
= (h+1) \sum_{i=1}^{r} \frac{m_i(m_i + 1)}{2} - \left( \frac{h+1}{2} + \frac{h^2 - 1}{6} \right) \sum_{i=1}^{r} m_i
\]

\[
\overset{2.7}{=} (h+1) \sum_{\varphi \in \Phi_+} \text{ht}(\varphi) - (h+1) \frac{rh(h+2)}{12} = \frac{rh}{2}
\]

so that (2.7) is recovered from (2.4).

We shall stick to the exponents rather than the heights in order not to restrict our considerations to the crystallographic types.
3 Power sums and Todd polynomials

The observation that (2.3) can be written as

\[ \sum_{i=1}^{r} m_i^n = \begin{cases} 
0! r Td_0 & \text{if } n = 0, \\
1/2 r h & \text{if } n = 1, \\
1/6 (r^2 + \gamma - h) & \text{if } n = 2, \\
1/4 r h(\gamma - h) & \text{if } n = 3,
\end{cases} \]  

(3.1)

where \( Td_0 = 1, Td_1(c_1) = \frac{1}{2} c_1, Td_2(c_1, c_2) = \frac{1}{12} (c_1^2 + c_2), \) and \( Td_3(c_1, c_2, c_3) = \frac{1}{24} c_1 c_2 \) are Todd polynomials (the general definition will be recalled in the proof of Theorem 3.3), suggests the ansatz

\[ \sum_{i=1}^{r} m_i^n = n! r Td_n(\gamma_1, \ldots, \gamma_n). \]  

(3.2)

From (3.1) and (3.2) we get

\[ \gamma_1 = h \text{ and } \gamma_2 = \gamma - h \]  

(3.3)

and are looking for solutions \( \gamma_3, \gamma_4, \ldots. \) Note that the symmetry \( m_i + m_{r+1-i} = h \) implies the identities (for \( a, b \in \mathbb{Z}_{\geq 0} \))

\[ \sum_{j=0}^{a} (-1)^{a-j} \binom{a}{j} h^j \sum_{i=1}^{r} m_i^{a+b-j} = \sum_{j=0}^{b} (-1)^{b-j} \binom{b}{j} h^j \sum_{i=1}^{r} m_i^{a+b-j} \]  

(3.4)

that generalize (2.8), which is (3.4) for \( \{a, b\} = \{1, 2\} \).

**Proposition 3.1** For \( a, b \in \mathbb{Z}_{\geq 0} \) one has the identity

\[ \sum_{j=0}^{a} (-1)^{a-j} \binom{a}{j} c_1^j (a + b - j)! Td_{a+b-j}(c_1, \ldots, c_{a+b-j}) = \sum_{j=0}^{b} (-1)^{b-j} \binom{b}{j} c_1^j (a + b - j)! Td_{a+b-j}(c_1, \ldots, c_{a+b-j}). \]  

(3.5)

**Proof.** For instance, one verifies the formula (3.5) for \( a = 0 \) and all \( b \in \mathbb{Z}_{\geq 0} \) by using a generating series and then proceeds by induction on \( a \). \( \square \)

Strictly speaking we don’t need Proposition 3.1. But it is worth noting that it indicates that we seem to be on the right track when using the ansatz (3.2).

**Lemma 3.2** Let \( m_1 \leq \cdots \leq m_r \in \mathbb{Z}_{\geq 0} \) be such that there are multisets \( V_+ \) and \( V_- \) of positive integers satisfying

\[ \sum_{i=1}^{r} q^{m_i} = \frac{q \prod_{v \in V_+} (1 - q^v)}{\prod_{v \in V_-} (1 - q^v)}. \]  

(3.6)

Then

\[ \prod_{v \in V_+} v = r \prod_{v \in V_-} v \]  

(3.7)

\[ |V_+| = |V_-|. \]  

(3.8)
Proof. The equality (3.7) is clear from the $q \to 1$ limit in (3.6); (3.8) follows since $1 - q^v$ has exactly one factor $1 - q$ and the polynomial on the left hand side in (3.6) has neither a zero nor a pole at $q = 1$. Note also that $m_1 = 1$ and $m_2 > 1$ if $r \geq 2$. \[ \square \]

**Theorem 3.3** Let $m_1 \leq \cdots \leq m_r \in \mathbb{Z}_{>0}$ be such that there are multisets $V_+$ and $V_-$ of positive integers satisfying

\[
\sum_{i=1}^{r} q^{m_i} = \frac{q \prod_{v \in V_+} (1 - q^v)}{\prod_{v \in V_-} (1 - q^v)}. \tag{3.6}
\]

We fix (for simplicity) a positive integer $p$ and define $\gamma_0(= 1), \gamma_1, \gamma_2, \gamma_3, \ldots$ by the generating series

\[
\sum_{n=0}^{\infty} \gamma_n t^n = \prod_{v \in V_-} (1 - vt) \prod_{v \in V_+} \frac{1 + pt}{1 - pt} \prod_{v \in V_-} (1 - q^v). \tag{3.9}
\]

Then for $n \in \mathbb{Z}_{\geq 0}$

\[
\sum_{i=1}^{r} m_i^n = n! \cdot r \cdot Td_n(\gamma_1, \ldots, \gamma_n). \tag{3.10}
\]

Proof. We consider the exponential generating series (with $q := e^t$) of both sides in (3.10)

\[
\sum_{n=0}^{\infty} \left( \sum_{i=1}^{r} m_i^n \right) \frac{t^n}{n!} = \sum_{i=1}^{r} e^{m_i n} = \sum_{i=1}^{r} q^{m_i} = q \prod_{v \in V_+} (1 - q^v) \prod_{v \in V_-} (1 - q^v) \tag{3.11}
\]

\[
\sum_{n=0}^{\infty} \left( n! \cdot r \cdot Td_n(\gamma_1, \ldots, \gamma_n) \right) \frac{t^n}{n!} = r \sum_{n=0}^{\infty} Td_n(\gamma_1, \ldots, \gamma_n) t^n = r \prod_{j=1}^{\infty} \frac{x_j t}{1 - e^{-x_j t}} \tag{3.12}
\]

where the last equality incorporates the definition of the Todd polynomials by means of their generating series in $t$ with coefficients in the elementary symmetric functions in $x_1, x_2, \ldots$ so that

\[
\prod_{j=1}^{\infty} (1 + x_j t) = \sum_{n=0}^{\infty} \gamma_n t^n
\]

and hence by (3.9)

\[
\frac{(1 - pt) \prod_{v \in V_+} (1 - vt)^p}{(1 + pt) \prod_{v \in V_-} (1 - vt)^p} \prod_{j=1}^{\infty} (1 + x_j t)^p = 1
\]

(that is, the supersymmetric elementary symmetric functions in $-p, -v$ ($p$ times, for every $v \in V_+$), $x_1$ ($p$ times), $x_2$ ($p$ times), \ldots; $-p, v$ ($p$ times, for every $v \in V_-$) all vanish) so that the formal expansion

\[
\left( \frac{-pt}{1 - e^{pt}} \right) \prod_{v \in V_+} \left( \frac{1 - vt}{1 - e^{vt}} \right) \prod_{v \in V_-} \left( \frac{1 - e^{vt}}{1 - vt} \right) = 1
\]

or taking $p$th roots (look at $t = 0$ to choose the correct branch)

\[
\prod_{j=1}^{\infty} \frac{x_j t}{1 - e^{-x_j t}} = e^t \prod_{v \in V_+} \left( \frac{1 - e^{vt}}{-vt} \right) \prod_{v \in V_-} \left( \frac{-vt}{1 - e^{vt}} \right).
\]
Therefore we can write the right hand side in (3.12) as (recall $q = e^t$)

$$r \prod_{j=1}^{\infty} \frac{x_j t}{1 - e^{-x_j t}} = \frac{r \prod_{v \in V_+} v}{\prod_{v \in V_+} (1 - q^v)} \cdot \frac{q \prod_{v \in V_+} (1 - q^v)}{\prod_{v \in V_-} (1 - q^v)}$$

where we have used (3.8) $|V_+| = |V_-|$ to cancel factors $t$ and then (3.7) to simplify the product. Thus the right hand side of (3.12) is identical to the right hand side of (3.11), which proves (3.10).

\[ \square \]

**Remark 3.4** Instead of the definition (3.9) for $\gamma_0, \gamma_1, \gamma_2, \gamma_3, \ldots$ one can define more generally

$$\sum_{n=0}^{\infty} \gamma_n t^n = \prod_{v \in V_-} (1 - vt) \prod_{k=1}^{K} \frac{1 + \pi_k t}{1 - \pi_k t}^{\mu_k}$$

with $\pi_1, \ldots, \pi_K \in \mathbb{R}$ and $\mu_1, \ldots, \mu_K \in \mathbb{Q}$ satisfying $\sum_{k=1}^{K} \pi_k \mu_k = 1$ (and for general $m_1$ (with $q^{m_1}$ instead of $q$ as first factor in the right hand side of (3.6)) just require that $\sum_{k=1}^{K} \pi_k \mu_k = m_1$).

### 4 Root system considerations

To apply Theorem 3.3 in the context of root systems we need the following proposition.

**Proposition 4.1** Let $m_1 \leq \cdots \leq m_r$ be the exponents of an irreducible (crystallographic (and reduced) or noncrystallographic) finite root system (of rank $r$). Then there are multisets $V_+$ and $V_-$ of positive integers such that

$$\sum_{i=1}^{r} q^{m_i} = \frac{q \prod_{v \in V_+} (1 - q^v)}{\prod_{v \in V_-} (1 - q^v)}. \quad (3.6)$$

Furthermore, $|V_\pm| \leq 2$ if $V_+ \cap V_- = \emptyset$.

**Proof.** According to the first note added in proof in [7] I. G. Macdonald was acquainted with the fact that (3.6) holds for all irreducible finite Coxeter groups.

The classification shows that the following three cases exhaust all possible types.

1. For the types $A_r$, $C_r/B_r$, and types of rank $\leq 3$ the sequence of exponents forms an arithmetic progression $1, m_2, \ldots, 1 + (r - 1)(m_2 - 1)$ (or just 1 if $r = 1$). Hence

$$\sum_{i=1}^{r} q^{m_i} = \begin{cases} q & \text{if } r = 1 \\ q \frac{(1 - q^{r(m_2 - 1)})}{1 - q^{m_2 - 1}} & \text{if } r \geq 2 \end{cases}$$

so that we can take $V_+ = V_- = \emptyset$ if $r = 1$ and $V_+ = \{r(m_2 - 1)\}$ and $V_- = \{m_2 - 1\}$ if $r \geq 2$. 


(2) For the types of rank 4 we have
\[ \sum_{i=1}^{4} q^{m_i} = q + q^{m_2} + q^{h-m_2} + q^{h-1} = \frac{q(1 - q^{2(m_2-1)}(1 - q^{2(h-m_2-1)})}{(1 - q^{m_2-1})(1 - q^{h-m_2-1})} \]
so that we can take \( V_+ = \{2(m_2-1), 2(h-m_2-1)\} \) and \( V_- = \{m_2-1, h-m_2-1\} \).

(3) For the simply-laced types (ADE) the root system is the Weyl group orbit of the highest root: \( \Phi = W\theta \). The stabilizer of \( \theta \) is \( W_\perp \), the reflection group generated by those simple reflections in \( W \) that fix \( \theta \). The root system is thus isomorphic as a \( W \)-set to \( W/W_\perp \). We need the usual length function \( \ell : W \to \mathbb{Z}_{\geq 0} \) defined as \( \ell(w) = k \) if \( w \) can be written as a product of \( k \) but not less than \( k \) simple reflections. If \( \varphi = w\theta \) is any positive root with \( w \) chosen such that \( \ell(w) \) is minimal, then \( \text{ht}(\varphi) = \text{ht}(\theta) - \ell(w) = h - 1 - \ell(w) \). Since the reflection along a simple root \( \psi \) maps \( \psi \) (of height 1) to \( -\psi \) (of height \(-1\)), we have similarly the equality \( \text{ht}(\varphi) = \ell(w) - 1 = h - 2 - \ell(w) \) if \( \varphi = w\theta \) is any negative root with \( w \) chosen such that \( \ell(w) \) is minimal. So we have
\[ \sum_{w \in W_\perp \theta \in W/W_\perp} q^{\ell(w)} = \sum_{\varphi \in \Phi_+} (q^{h-1-\text{ht}(\varphi)} + q^{h-2+\text{ht}(\varphi)}) \]
and since \( 1, \ldots, m_1, 1, \ldots, m_2, \ldots, 1, \ldots, m_r \) enumerates \( \text{ht}(\varphi) \) as \( \varphi \) runs over \( \Phi_+ \), we can continue
\[ = \sum_{i=1}^{r} \sum_{j=1}^{m_i} (q^{h-1-j} + q^{h-2+j}) \]
and using the symmetry \( m_i + m_{r+1-i} = h \) we obtain
\[ = \sum_{i=1}^{r} \sum_{j=0}^{h-1} q^{m_{i-1}+j} = \left( \sum_{i=1}^{r} q^{m_{i-1}} \right) \frac{1 - q^h}{1 - q}. \]

On the other hand by the Chevalley-Solomon identity for the Poincaré series of finite Coxeter groups (see, e. g., [4, Section 3.15]) we have
\[ \sum_{w \in W_\perp \theta \in W/W_\perp} q^{\ell(w)} = \left( \prod_{i=1}^{r} \frac{1 - q^{m_{i-1}+1}}{1 - q} \right) \left( \prod_{i=1}^{s} \frac{1 - q}{1 - q^{m_{i-1}+1}} \right) \]
where \( \tilde{m}_1, \ldots, \tilde{m}_s \) lists the exponents of all the irreducible components of \( W_\perp \). Since \( m_r + 1 = h \) we finally get
\[ \sum_{i=1}^{r} q^{m_i} = \frac{q}{(1 - q)^{r-s-1}} \prod_{i=1}^{r-1} (1 - q^{m_{i-1}+1}) \prod_{i=1}^{s} (1 - q^{m_{i-1}+1}) \]
and the following table finishes the proof. (We have left out the types \( A_r \) which were already dealt with in case (1).)
Multisets are needed for type $D_4$. Note that for $r \geq 2$ (3.6) implies that $m_2 - 1 \in V_-$. Furthermore, for all the crystallographic types except $A_1$ and $G_2$, $m_2 - 1 = d$ is the largest coefficient of the highest root (when written as a linear combination of the simple roots). This observation extends to the noncrystallographic types $H_3$ and $H_4$ if we define $d = 4$ and $d = 10$, respectively, as suggested by the following folding procedure, $D_6 \rightsquigarrow H_3$ and $E_8 \rightsquigarrow H_4$.

For $l_2(m)$ we have $m_2 - 1 = m - 2$, but the folding procedure gives $d = \lfloor \frac{m}{2} \rfloor$. In fact, for $m = 2k + 1$ odd $A_{2k} \rightsquigarrow l_2(2k + 1)$ with $d = k$ (and the other coefficient is $k$, too). For $m = 2k$ even $D_{k+1} \rightsquigarrow l_2(2k)$ with $d = k$ (and the other coefficient is $k - 1$); alternatively we can fold $A_{2k-1} \rightsquigarrow l_2(2k)$ and also $E_6 \rightsquigarrow l_2(12)$, $E_7 \rightsquigarrow l_2(18)$, and $E_8 \rightsquigarrow l_2(30)$.

For $A_r$, $C_r$, $B_r$, $l_2(m)$, and $H_3$ one can append the same element(s) to both $V_+$ and $V_-$ to make all the above multisets $V_+$ and $V_-$ have cardinality 2. The following proposition gives a uniform description of multisets $V_+ = \{A, B\}$ and $V_- = \{\alpha, \beta\}$ satisfying (3.6) in terms of three parameters: the Coxeter number $h$, the coefficient $d$, and $\nu : =$ the number of times $d$ occurs among the marks in the extended Dynkin diagram minus 1, and extended to the noncrystallographic types as displayed in the following table. The table also shows the values of $\gamma$ (see (2.2) and the text afterwards). Some parameters $\beta$ (and for type $A_1$ also $\alpha$) are irrelevant and are left unspecified. Clearly, one can interchange $A \leftrightarrow B$ and also $\alpha \leftrightarrow \beta$.

| type   | $r$ | $h$ | $\gamma$ | $d$ | $A, B$ | $\alpha, \beta$ | $\nu$ |
|--------|-----|-----|----------|-----|--------|-----------------|-------|
| $A_1$  | 1   | 2   | 4        | 1   | $\alpha, \beta$ | $\alpha, \beta$ | 1     |
| $A_r$  | $r \geq 2$ | $r + 1$ | $(r + 1)^2$ | 1 | $r, \beta$ | 1 | $r$ |
| $C_r/B_r$ | $r \geq 2$ | $2r$ | $4r^2 + 2r - 2$ | 2 | $2r, \beta$ | 2 | $r - 2$ |
| $D_r$  | $r \geq 4$ | $2r - 2$ | $(2r - 2)^2$ | 2 | $r, 2(r - 2)$ | 2 | $r - 2$ |
| $E_6$  | 6   | 12  | 144      | 3   | 8, 9   | 3   | 4     |
| $E_7$  | 7   | 18  | 324      | 4   | 12, 14 | 4   | 6     |
| $E_8$  | 8   | 30  | 900      | 6   | 20, 24 | 6   | 10    |
| $F_4$  | 4   | 12  | 162      | 4   | 8, 12  | 4   | 6     |
| $G_2 = l_2(6)$ | 2   | 6   | 48       | 3   | 8, $\beta$ | 4 | $\beta$ |
| $H_2 = l_2(5)$ | 2   | 5   | 31       | 2   | 6, $\beta$ | 3 | $\beta$ |
| $H_3$  | 3   | 10  | 124      | 4   | 12, $\beta$ | 4 | $\beta$ |
| $H_4$  | 4   | 30  | 1116     | 10  | 20, 36 | 10 | 18    |
| $l_2(2k + 1)$ | $k \geq 3$ | 2 | $2k + 1$ | $8k^2 - 2k + 3$ | $k$ | 4$k - 2, \beta$ | 2$k - 1, \beta$ | 1 |
| $l_2(2k)$ | $k \geq 4$ | 2 | $2k$ | $8k^2 - 10k + 6$ | $k$ | 4$k - 4, \beta$ | 2$k - 2, \beta$ | 0 |
redefined parameters $d$ and $\nu$ for $l_2(2k+1)$ ($k \geq 2$)

| type   | $r$ | $h$ | $\gamma$ | $d$ | $A, B$ | $\alpha, \beta$ | $\nu$ |
|--------|-----|-----|----------|-----|--------|-----------------|------|
| $l_2(m)$ ($m \geq 4$) | 2    | $m$ | $2m^2 - 5m + 6$ | $\frac{m}{2}$ | $2m - 4, \beta$ | $m - 2, \beta$ | 0    |

The table shows that in the cases where $\beta$ has a well-defined value (and $\alpha = m_2 - 1$), this value is $m_3 - 1$ except for $D_r$ ($r \geq 7$), where $\beta = m_{l(r+1)/2} - 1$. With the redefinition of $d$ and $\nu$ for the types $l_2(2k+1)$ ($k \geq 2$) the formula $h = \frac{d}{2}(r + 2 + \nu)$ is true in general, and it is also true for $H_2 = l_2(5)$ with the original parameters $d = 2$ and $\nu = 1$.

**Proposition 4.2** The equality (3.6) in Proposition 4.1 holds if the multisets $V_{\pm}$ are given as

$$V_- = \{d, 2d - 2 + \nu\} \text{ and } V_+ = \{4d - 4 + dv, h - d - (d - 1)\nu\}$$

with $d = \frac{m}{2}$ and $\nu = 0$ for $l_2(m)$ ($m \geq 4$); and for $H_2 = l_2(5)$ the original values $d = 2$ and $\nu = 1$ also work.

The choice in Proposition 4.2 of the irrelevant parameters is thus $\alpha = \beta = 1$ for type $A_1$ and as shown in the following table.

| type   | $A_r$ | $C_r/B_r$ | $G_2$ | $H$ with $d = 2, \nu = 1$ | $H_3$ | $l_2(m)$ with $d = \frac{m}{2}, \nu = 0$ |
|--------|-------|----------|-------|---------------------------|-------|-------------------------------|
| $\beta$ | $r$   | $r$      | 3     | 2                         | 6     | $\frac{m}{2}$                 |

**Proof.** Let us first look at those exceptional types for which $d \mid h$ (including $l_2(m)$ ($m \geq 5$)). Here we have $\nu = 0$ and the (multi)set of exponents is

$$\{m_1, \ldots, m_r\} = \left\{1 + jd \mid 0 \leq j \leq \frac{h}{d} - 2\right\} \cup \left\{2d - 1 + jd \mid 0 \leq j \leq \frac{h}{d} - 2\right\}$$

(see [3] Theorem 3.2 (i)) adding $H_4$ and $l_2(m)$) so that

$$\sum_{i=1}^{r} q^{m_i} = \sum_{j=0}^{\frac{h}{d} - 2} (q^{1+jd} + q^{2d-1+jd}) = q(1 + q^{2d-2}) \sum_{j=0}^{\frac{h}{d} - 2} q^{jd} = \frac{q(1 - q^{4d-4})(1 - q^{h-d})}{(1 - q^d)(1 - q^{2d-2})}$$

in agreement with the expressions for $V_{\pm}$ (with $\nu = 0$).

For the remaining types we use the following table.

| type   | $h$ | $d$ | $\nu$ | $4d - 4 + dv, h - d - (d - 1)\nu$ | $d, 2d - 2 + \nu$ |
|--------|-----|-----|-------|---------------------------------|-------------------|
| $A_r$   ($r \geq 1$) | $r+1$ | 1    | $r$   | $r, r$                        | 1, $r$            |
| $C_r/B_r$ ($r \geq 2$) | 2$^r$ | 2    | $r - 2$ | $2r, r$                     | 2, $r$           |
| $D_r$ ($r \geq 4$) | $2r - 2$ | 2    | $r - 4$ | $2r - 4, r$ | 2, $r - 2$ |
| $E_7$   | 18   | 4    | 0     | $12, 14$                       | 4, 6             |
| $H_2$   | 5    | 2    | 1     | $6, 2$                         | 2, 3             |
| $H_3$   | 10   | 4    | 0     | $12, 6$                        | 4, 6             |

This is in agreement with the table before Proposition 4.2. □
Remark 4.3 For the DE types one has $V_- = \{ a, b \}$ and $V_+ = \{ b, r_2 \}$, where the parameters $a$ and $b$ are as in Kostant’s article [5]. Note also that for those types $\frac{b}{2} = d$ and $\frac{b}{2} = h + 2 - d$. We can already look ahead and use (5.4) to obtain $h = dr - 4d + 6$; from (5.5) and $h^2 = \gamma$ (still for the DE types) and using the equality $h = dr - 4d + 6$ we get $d(h - 2r - 6d + 26) = 24$.

5 Synthesis and further computations

Proposition 4.1 shows that Theorem 3.3 can be applied in the context of root systems with $V_+ = \{ A, B \}$ and $V_- = \{ a, b \}$ as in the table before Proposition 4.2.

Define $\gamma_0, \gamma_1, \gamma_2, \gamma_3, \ldots$ (depending on a parameter $p$) by the series expansion

$$\sum_{n=0}^{\infty} \gamma_n t^n = \frac{(1 - \alpha t)(1 - \beta t)}{(1 - At)(1 - Bt)} \sqrt{1 + pt} \frac{1 + pt}{1 - pt}. \quad (5.1)$$

The series expansions

$$\frac{(1 - \alpha t)(1 - \beta t)}{(1 - At)(1 - Bt)} = (1 - (\alpha + \beta)t + \alpha \beta t^2) \sum_{n=0}^{\infty} \left( \sum_{j=0}^{n} A^j B^{n-j} \right) t^n \quad (5.2)$$

and

$$p \sqrt{\frac{1 + pt}{1 - pt}} = \left( \sum_{j=0}^{\infty} \left( \frac{1}{p} \right) (pt)^j \right) \left( \sum_{k=0}^{\infty} \left( \frac{-1}{k} \right) (-pt)^k \right) =: \sum_{n=0}^{\infty} p_n t^n \quad (5.3)$$

specializing for $p = 1$ and $p = 2$

$$\frac{1 + t}{1 - t} = 1 + 2 \sum_{n=1}^{\infty} t^n$$

$$\sqrt{\frac{1 + 2t}{1 - 2t}} = \sum_{n=0}^{\infty} \binom{2n}{n} (1 + 2t)^{2n} = 1 + 2t + 2t^2 + 4t^3 + 6t^4 + 12t^5 + \ldots$$

can be used to write down an explicit formula for $\gamma_n$ defined in (5.1).

Note that the series expansion of $\left( (1 + pt)/(1 - pt) \right)^{1/p}$ has integer coefficients if $p = 2^k$ with $k \in \mathbb{Z}_{\geq 0}$. In fact, for $f(t) = 1 + \sum_{n=1}^{\infty} a_n t^n$ we let

$$Tf(t) := \sqrt{f(2t)} = 1 + \sum_{n=1}^{\infty} b_n t^n.$$

A comparison of coefficients shows that

$$b_n = 2^{n-1} a_n - \frac{1}{2} \sum_{j=1}^{n-1} b_j b_{n-j},$$
and hence if \( a_1 \) is even and all \( a_n \) are integers, then all \( b_n \) are even. Starting with 
\[
f(t) := \frac{1+t}{1-t} = 1+2\sum_{n=1}^{\infty} t^n,
\]
we get 
\[
((1+2^k t)/(1-2^k t))^{1/2^k} = T^k f(t) \in 1+2t\mathbb{Z}[t].
\]
(Note also that in the limit \( p \to 0 \) we get the power series expansion of \( e^{2t} \), which is a fixed point of the transformation \( T \)).

**Remark 5.1** The transformation \( T \) on (generating series of) integer sequences starting with 1 and having an even integer as next term may be investigated. Here is a tiny list of examples:

| \( a_0, a_1, a_2, \ldots \) | \( T \) | \( b_0, b_1, b_2, \ldots \) |
|---|---|---|
| \( a_n = n+1 \) | \( T \) | \( b_n = 2^n \) |
| \( a_n = 2^n \) | \( T \) | \( b_n = \binom{2n}{n} \) |
| \( a_n = C_{n+1} = \frac{1}{n+2} \binom{2n+2}{n+1} \) | \( T \) | \( b_n = 2^n C_n \) |

More generally, one may fix a positive integer \( \ell \) and look at the transformation 
\[
f(t) \mapsto \ell \sqrt{f(\ell t)}
\]
for \( f(t) = 1 + \sum_{n=1}^{\infty} a_n t^n \) with \( \ell \mid a_1 \) and \( a_n \in \mathbb{Z} \).

**Lemma 5.2** The elementary symmetric polynomials in \( A \) and \( B \) can be written as follows.

\[
A + B = h - 2 + \alpha + \beta \tag{5.4}
\]
\[
AB = h^2 - \gamma + (h-2)(\alpha + \beta - 1) + \alpha \beta \tag{5.5}
\]

Furthermore,
\[
X_n := \sum_{j=0}^{n} A^j B^{n-j}
\]
\[
= \frac{1}{2} \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \binom{n-j}{j} (h - 2 + \alpha + \beta)^{n-2j} (h^2 - \gamma + (h-2)(\alpha + \beta - 1) + \alpha \beta)^j. \tag{5.6}
\]

**Proof.** From (5.1) we get using (3.3)
\[
\gamma_1 = 2 + (A + B) - (\alpha + \beta) = h
\]
\[
\gamma_2 = 2 + 2(A + B) + (A^2 + AB + B^2) - 2(\alpha + \beta) - (\alpha + \beta)(A + B) + \alpha \beta = \gamma - h
\]
and solving for the elementary symmetric polynomials in \( A \) and \( B \) we get (5.4) and (5.5). Note that for \( n \geq 2 \)
\[
X_n = (A + B) \sum_{j=0}^{n-1} A^j B^{n-1-j} - AB \sum_{j=0}^{n-2} A^j B^{n-2-j}
\]
\[
= (h - 2 + \alpha + \beta) X_{n-1} - (h^2 - \gamma + (h-2)(\alpha + \beta - 1) + \alpha \beta) X_{n-2}
\]
with \( X_0 = 1 \) and \( X_1 = h - 2 + \alpha + \beta \). By solving the recursion we have (5.6). \qed
Proposition 5.3 The invariant $\gamma$ is expressible as a polynomial in $h, r, \alpha, \beta$, namely,

$$\gamma = h^2 + (h - 2)(\alpha + \beta - 1) - (r - 1)\alpha \beta. \quad (5.7)$$

Proof. From (5.7) we have $AB = r\alpha \beta$. The formula (5.7) follows by combining with (5.5). \qed

Remark 5.4 The formula $h = \frac{d}{2}(r + 2 + \nu)$ follows by inserting into $AB = r\alpha \beta$ the expressions in Proposition 4.2 for $V_+ = \{A, B\}$ and $V_- = \{\alpha, \beta\}$ and using the fact that the product $(d(\nu - r) + 4(d - 1))(d - 2)\nu$ vanishes.

To summarize we state the following theorem.

Theorem 5.5 Let $m_1 \leq \cdots \leq m_r$ be the exponents of an irreducible (crystallographic (and reduced) or noncrystallographic) finite root system (of rank $r$) with Coxeter number $h$ and parameters $\gamma$ and $d$ as in the table before Proposition 4.2. Put

$$\alpha := \begin{cases} \text{arbitrary} & \text{if } r = 1, \\ m_2 - 1 & \text{if } r \geq 2, \end{cases}$$

and define

$$\beta := \begin{cases} \text{arbitrary} & \text{if } h = (r - 1)\alpha + 2, \\ \frac{h^2 - \gamma + (h - 2)(\alpha - 1)}{2 + (r - 1)\alpha - h} & \text{if } h \neq (r - 1)\alpha + 2. \quad (5.8) \end{cases}$$

Let

$$\sum_{n=0}^{\infty} \gamma_n t^n = (1 - (\alpha + \beta)t + \alpha \beta t^2) \left( \sum_{n=0}^{\infty} X_n t^n \right) \left( \sum_{n=0}^{\infty} p_n t^n \right)$$

with $X_n$ as in (5.6) and $p_n$ as in (5.3). (So $\gamma_n$ a polynomial in $h, \gamma, \alpha, \beta$ (or, by (5.7), alternatively in $h, r, \alpha, \beta$) (symmetric in $\alpha, \beta$) and depends on an additional parameter $p$ which can be chosen arbitrarily.) Then

$$\sum_{i=1}^{r} m_i^n = n! \cdot Td_n(\gamma_1, \ldots, \gamma_n). \quad (5.9)$$

Proof. As already mentioned, this is an application of Theorem 3.3 in the context of root systems, that is, using Proposition 4.1 with $V_+ = \{A, B\}$ and $V_- = \{\alpha, \beta\}$ and inserting (5.2), (5.3), and (5.6) into the series expansion (5.1). The expression for $\beta$ follows from Proposition 5.3.

For $r = 1$ there is nothing more to say. So let’s assume $r \geq 2$. Here is the reason why we can take $\alpha = d$ instead of $\alpha = m_2 - 1$: $d = m_2 - 1$ in all cases except possibly for types $I_2(m)$, but then we get $\beta = m_2 - 1 = m - 2$. Or still slightly more generally: for the types $A_r$ ($r \geq 2$), $C_r/B_r$, $I_2(m)$, and $H_3$ we could choose $\alpha \neq m_2 - 1$ and automatically get $\beta = m_2 - 1$ from (5.8). \qed
Let’s continue by writing down $\gamma_3$ and $\gamma_4$ in terms of $h, \gamma, \alpha, \beta$ (and $p$)

\[
\gamma_3 = -h^3 + 2h\gamma - 2\gamma + \frac{1}{3}(2p^2 + 4) - (h^2 - \gamma - h + 2)(\alpha + \beta) - (h - 2)\alpha\beta \\
\gamma_4 = -h^4 + h^2\gamma + \gamma^2 + 3h^3 - 6h\gamma - h^2 + 2\gamma + \frac{2}{3}h(p^2 + 5) - 2 \\
- (h^2 - \gamma - h + 2)((2h - 2 + \alpha + \beta)(\alpha + \beta) - \alpha\beta) \\
- (h - 2)(2h - 2 + \alpha + \beta)\alpha\beta.
\] (5.10)

By inserting (3.3), (5.10), and (5.11) into (5.9) using the formulae $T_d(c_1, c_2, c_3, c_4) = \frac{1}{720}(-c_4 + 4c_1^2c_2 + c_1c_3 + 3c_2^2 - c_4)$ and $T_d(c_1, c_2, c_3, c_4, c_5) = \frac{1}{1440}(-c_3^2 + 3c_1^2c_2 - c_1c_3 + 3c_2^2 - c_4)$ we get

\[
\sum_{i=1}^{r} m_i^4 = \frac{r}{30}(-h^4 + 5h^2\gamma + 2\gamma^2 - 7h^3 - 2h\gamma + 4h^2 - 2\gamma - 2h + 2 + R_{45}) \\
\sum_{i=1}^{r} m_i^5 = \frac{r}{12}h(2\gamma^2 - 2h^3 - 2h\gamma + 4h^2 - 2\gamma - 2h + 2 + R_{45})
\] (5.12)

where

\[
R_{45} = (h^2 - \gamma - h + 2)((h - 2 + \alpha + \beta)(\alpha + \beta) - \alpha\beta) + (h - 2)(h - 2 + \alpha + \beta)\alpha\beta.
\] (5.14)

Surely, one could continue and give explicit formulae for higher power sums. Let’s stop here and display formulae for the sum of the heights cubes and fourth powers.

**Proposition 5.6** With $R_{45}$ as in (5.14) above we have

\[
\sum_{\varphi \in \Phi_+} \text{ht}(\varphi)^3 = \frac{r}{120}(-h^4 + 5h^2\gamma + 2\gamma^2 - 7h^3 + 13h\gamma - 6h^2 + 3\gamma - 7h + 2 + R_{45}) \\
\sum_{\varphi \in \Phi_+} \text{ht}(\varphi)^4 = \frac{r}{60}(h + 1)(2\gamma^2 - 3h^3 + 3h\gamma - 2\gamma - 3h + 2 + R_{45}).
\]

**Proof.** Insert (2.3), (5.12), and (5.13) into (2.6). $\square$

**Remark 5.7** Using the power series expansions for (5.1) one computes the following explicit expressions for the quantities $\gamma_n$. For the types $A_r$ one gets for $n \geq 1$

\[
\gamma_n|_{p=1} = r^n + r^{n-1}
\]

and has

\[
\sum_{i=1}^{r} i^n = n!r T_d(r + 1, r^2 + r, \ldots, r^n + r^{n-1})
\]

as an alternative to Bernoulli’s formula (2.1).
For the types $C_r \ (r \geq 2)$ one gets

$$\gamma_n|_{p=1} = (2r)^n - 2 \sum_{j=0}^{n-2} (2r)^j$$

but it looks somewhat more natural to specialize to $p = 2$

$$\gamma_n|_{p=2} = -2 \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} C_{j-1} (2r)^{n-2j}$$

where $C_k = \frac{1}{k+1} \binom{2k}{k}$ is the $k$th Catalan number (for $k \geq 0$) and employing the $(-1)$st Catalan number $C_{-1} = -\frac{1}{2}$.

One may ask whether as an alternative to our considerations using generating series a more geometric/combinatorial approach via toric geometry/counting lattice points in polytopes can be found (see also [1, Section 2.4], where the Bernoulli polynomials are recognized as lattice point enumerators of certain pyramids).

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