Classification of Material and Type of Ellipsoid based on the First Order Polarization Tensor

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Abstract. The terminology polarization tensor (PT) has many useful practical applications especially in electrics and electromagnetics. In this case, it must be firstly determined by some appropriate methods. Besides, understanding some properties of the PT might also be very useful in order to apply it. In this study, we investigate the first order PT for ellipsoids to determine its material and type based on the PT. First of all, we show that the first order PT for ellipsoid is a nonzero matrix. After that, we prove that, instead of classifying the first order PT for ellipsoid by the material of the ellipsoid, we can also classify the conductivity and the material of the ellipsoid by determining whether the first order PT for ellipsoid is either positive or negative definite matrix. After that, by further investigating the first order PT for the ellipsoid, we can classify the semi principal axes of the ellipsoid according to the elements of its first order PT.

1. Introduction
Polarization Tensor (PT) has being applied in electrical imaging for industrial purposes or metal detection. Classically, PT is introduced to study fluid mechanics and potential theory \cite{1, 2, 3}. After several developments on the researches (see \cite{4, 5, 6}), Ammari and Kang \cite{6} have proposed a method to improve electrical imaging by using PT. Moreover, in electrical imaging of electrosensing fish \cite{7, 8, 9} and metal detection \cite{10, 11, 12, 13, 14, 15, 16} for airport security screening or landmine clearance, PT is specifically used to describe and characterize objects rather than reconstructing the image of the objects. In this case, PT offers lower computational cost than full image reconstruction especially since PT can be computed explicitly based on mathematical formula.

Generally, PT for electrosensing fish is different with the PT for metal detection but both PTs can be related with some specific conditions \cite{14, 15}. At the moment, PT for electrosensing fish or electrical imaging has stronger and established mathematical foundations while there were a few ongoing studies to further investigate the PT for metal detection. Here, PT for electrosensing fish represents the perturbation in electric field due to the presence of conducting object \cite{7, 8, 9} whereas PT for metal detection represents the perturbation in electromagnetic field due to the presence of conducting and magnetic object \cite{14, 15, 16}. Both PTs can be determined by solving a specific set of integral equations if the geometry and material of the object are known so, the
PT can be used to describe and characterize that object. When the object is an ellipsoid, PT for electrosensing fish or electrical imaging can be further simplified [6] where, this simplified formula has been used for investigating the efficiency of numerical methods when approximating the PT using the general formula [17, 18]. According to the simplified version of the PT for ellipsoid, one can also show that the first order PT for ellipsoid can be expressed in terms of depolarization factors, a classical terminology in mathematics and physics [19, 20, 21]. It is not yet discovered so far an alternative or simpler formula of the PT for ellipsoid in metal detection.

In this paper, we present some results regarding the first order PT for ellipsoid for PT in electrical imaging or electrosensing fish specifically for characterization of the ellipsoids based on the first order PT. We study ellipsoid as previous studies have shown that the first order PT for some objects can be related to the first order PT for ellipsoid [9, 22]. Generally, this study is an extension of our previous studies for the special case spheroid in Ahmad Khairuddin et al [23]. Therefore, understanding the first order PT for ellipsoid might help us to understand the first order PT for object related to it. We also believe that the results for the first order PT for ellipsoid can be further generalized for any object and applied to the PT in metal detection as well.

The paper now proceeds as follows. We will discuss the mathematical formula of the first order PT for ellipsoid in the next section. Since the first order PT for ellipsoid is related to depolarization factors for ellipsoid, some properties regarding the depolarization factors will also be revised. After that, we will present our results about characterization of the first order PT for ellipsoid. The paper ends with a brief discussion and conclusions.

2. Mathematical Formulation of the First Order Polarization Tensor for Ellipsoid

Let $B$ be a small object presented in the space $\mathbb{R}^3$. The conductivity $\sigma(x)$ is then defined such that for any point $x \in \mathbb{R}^3$,

$$
\sigma(x) = \begin{cases} 
  k, & \text{if } x \in B \\
  1, & \text{if } x \in \mathbb{R}^3
\end{cases}
\quad (1)
$$

where $k$ is a constant depending on the material of $B$. Equation (1) suggests that there exists a conductivity contrast between $\mathbb{R}^3$ (conductivity equal to 1) and $B$ (conductivity equal to $k$). According to Ammari and Kang [6], if there is an electrical field in $\mathbb{R}^3$ with the presence of $B$, the perturbation caused by the conductivity contrast can be mathematical presented by an asymptotic formula, where, the dominant term in the formula is called as the Generalized Polarization Tensor (GPT). GPT can be determined by solving system of integral equations if $B$ and $k$ are known so, GPT can be used to describe $B$ and its material. In this study, we are interested with the simplest form of GPT called as the first order GPT (or simply the first order PT) and is denoted by $M$. Here, $M$ for $B$ at conductivity $k$ denoted by $M(k, B)$ where $0 < k \neq 1 < +\infty$ is a real $3 \times 3$ matrix and it is proven in [6] that $M$ is symmetric. Moreover, [6] have also shown that $M$ is positive definite if $k > 1$ whereas, it is negative definite if $0 < k < 1$.

In addition, by adapting Ammari and Kang [6], Mohamad Yunus and Ahmad Khairuddin [20, 21] have proposed a slightly different explicit formula of the first order PT when $B$ is an ellipsoid. If $B$ is an ellipsoid with semi principal axes $a$, $b$, and $c$, that can be represented by $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$ in the three dimensional Cartesian coordinate system, $M(k, B)$ is given by [20] as

$$
M(k, B) = (k - 1)|B| \begin{bmatrix} m_1 & 0 & 0 \\
0 & m_2 & 0 \\
0 & 0 & m_3 \end{bmatrix},
\quad (2)
$$

where $|B|$ is the volume of $B$ and for $i = 1, 2$ and $3$, $m_i = \frac{1}{(1 - d_1) + kd_1}$. $d_i$ are constants called
as the depolarization or demagnetizing factors for the ellipsoid \( B \) \([4, 24, 25]\) given by \( d_i = \frac{abc}{2} I_i \) and

\[
I_1 = \int_0^\infty \frac{1}{(a^2 + y)^{3/2} \sqrt{b^2 + y^2}} dy, \tag{3}
\]

\[
I_2 = \int_0^\infty \frac{1}{(b^2 + y)^{3/2} \sqrt{a^2 + y^2}} dy, \tag{4}
\]

\[
I_3 = \int_0^\infty \frac{1}{(c^2 + y)^{3/2} \sqrt{a^2 + y^2}} dy. \tag{5}
\]

Previously, depolarization factors were classically appeared in the study of composites \([4]\) and also had been used by \([24, 25, 26, 27]\) to study electromagnetism. In this study, equation (2) together with (3), (4) and (5) will be used to investigate the first order PT for ellipsoid.

Now, in the next section, we shall recall some established results regarding the depolarization factors for ellipsoid that will be used in this particular study.

### 3. Some Properties of the Depolarization Factors for Ellipsoid

The terminology depolarization factors originates from the study of electromagnetism for examples by Thomson and Tait \([26]\) or Maxwell \([27]\) (The second volume of the updated book by Maxwell is listed in the references, not the original version). After that, some investigations about the depolarization factors for general ellipsoids were done classically by Osborn \([24]\) and Stoner \([25]\). We refer to these four authors and also the more recent one by Milton \([4]\) to determine some properties of the depolarization factors for ellipsoid which are essential in this study.

Now, we will refer to the equations (3), (4) and (5) in order to revise some properties of the depolarization factors for ellipsoid. The following property is mentioned in Milton \([4]\). However, the proof in details of it can only be found in \([28]\).

**Proposition 1** The depolarization factors \( d_i \) are nonnegative numbers.

Proposition 1 is actually a direct consequence of the semi principal axes \( a, b, c > 0 \) by definition. Furthermore, Stoner \([25]\) and Milton \([4]\) have also suggested and shown that \( d_1 + d_2 + d_3 = 1 \). Based on this relation and Proposition 1, the following property is further proven by \([28]\).

**Proposition 2** The depolarization factors \( d_i \) for \( i = 1, 2, 3 \) each satisfies \( 0 < d_i < 1 \).

In this study, Proposition 2 will be used to classify the materials of ellipsoids based on their first order PT. Finally, we will restate one more property of the depolarization factors for ellipsoids taken also from \([28]\). This property categorizes the depolarization factors for ellipsoid and its semi-principal axes.

**Proposition 3** The semi principal axes of an ellipsoid satisfy \( a \leq b \leq c \) if and only if the depolarization factors for the ellipsoid satisfy \( d_1 \geq d_2 \geq d_3 \).

### 4. Characterization of Ellipsoid based on its First Order Polarization Tensor

In this section, based on the previous properties of the depolarization factors, we will present some mathematical theories regarding characterization of ellipsoid based on its first order PT. At this moment, our characterization is basically the classification of material and type for a conducting ellipsoid. When a conducting ellipsoid is presented in an electric field, the ellipsoid can always be represented by a nonzero first order PT. This result is stated here in Theorem 4. The theorem is due to \( k > 0 \) and Proposition 1 as given in the proof.
Theorem 4
The first order PT for an ellipsoid when $0 < k \neq 1 < +\infty$ is always a nonzero matrix.

Proof:
Suppose $0 < k \neq 1 < +\infty$. We must show that the first order PT for an ellipsoid is a nonzero matrix. Let $M(k, B)$ be the first order PT for ellipsoid, following (2), we have

$$M = (k - 1)|B| \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix},$$

where $m_i = \frac{1}{(1 - d_i) + kd_i}$ for $i = 1, 2$ and 3. Since $0 < k \neq 1 < +\infty$ then $(k - 1) \neq 0$. Similarly, $|B| \neq 0$ as $a, b, c > 0$. By referring to Proposition 2 where $0 < d_i < 1$ and again the assumption $0 < k \neq 1 < +\infty$, we also have $m_i \neq 0$. Therefore, it is proven that the first order PT for an ellipsoid when $0 < k \neq 1 < +\infty$ is always a nonzero matrix.

Besides, Ammari and Kang [6] have shown that the first order PT for ellipsoid can be categorized as positive definite matrix or negative definite matrix based on the value of $k$. Reversely, we can then classify the value of $k$ representing the material of an ellipsoid by identifying whether its first order PT is positive or negative matrix. We state this result in Theorem 5. In order to prove the theorem, we will recall the definition of positive and negative definite matrix from [29]. According to [29], a matrix is a positive definite if all of its eigenvalues are nonnegative while it is negative definite if all of its eigenvalues are negative.

Theorem 5
Let $M(k, B)$ be the first order PT for ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ at a specified $k$ where $0 < k \neq 1 < +\infty$.

i. $k > 1$ if and only if $M(k, B)$ is a positive definite matrix.

ii. $k < 1$ if and only if $M(k, B)$ is a negative definite matrix.

Proof:
We will prove part (i) first. Assume that $k > 1$. We need to show $M(k, B)$ is a positive definite matrix. Let $M(k, B)$ be the first order PT for ellipsoid. We have

$$M(k, B) = (k - 1)|B| \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}.$$
Next, we will prove part (ii). First, assume that $k < 1$. We need to show $M(k, B)$ is a negative definite matrix. Similarly,

$$
M(k, B) = (k - 1)|B| \begin{bmatrix}
  m_1 & 0 & 0 \\
  0 & m_2 & 0 \\
  0 & 0 & m_3
\end{bmatrix}.
$$

Let $\alpha_i = (k - 1)|B|m_i$, for $i = 1, 2$ and $3$, where $\alpha_i$ are the eigenvalues of $M(k, B)$. Obviously $|B| > 0$. Moreover, since $0 < k \neq 1 < +\infty$ and $0 < d_i < 1$ according to Proposition 2, $m_i > 0$. However, since $k < 1$ then $(k - 1) < 0$. Therefore, $a_i < 0$ which implies that $M(k, B)$ is a negative definite matrix.

Now, assume that $M(k, B)$ is a negative definite matrix. We must show that $k < 1$. We have

$$
M(k, B) = (k - 1)|B| \begin{bmatrix}
  m_1 & 0 & 0 \\
  0 & m_2 & 0 \\
  0 & 0 & m_3
\end{bmatrix}.
$$

Let $\alpha_i = (k - 1)|B|m_i$, for $i = 1, 2$ and $3$, where $\alpha_i$ are the eigenvalues of $M(k, B)$. Since $M(k, B)$ is a negative definite matrix, then $a_i < 0$. Here, $|B| > 0$ and $m_i > 0$ due to $0 < d_i < 1$ based on Proposition 2 and also $0 < k \neq 1 < +\infty$. In order for $a_i < 0$, we must have $(k - 1) < 0$ which also means that $k < 1$. 

As we can see, Proposition 2 is specifically used to prove Theorem 5. Moreover, Theorem 5 is a generalization of the results given in Ahmad Khairuddin et al [23] for spheroids with principal axes $a > b = c$ and $a < b = c$\footnote{In Ahmad Khairuddin et al [23], the semi principal axes in Proposition 1, Proposition 2, Theorem 1 and Theorem 2 should be $a > b = c$ and $a < b = c$ not $a \geq b = c$ and $a \leq b = c$.}. Next, for simplification, we rewrite (2) as

$$
M(k, B) = \begin{bmatrix}
  M_1 & 0 & 0 \\
  0 & M_2 & 0 \\
  0 & 0 & M_3
\end{bmatrix}, \quad (6)
$$

where $M_i = (k - 1)|B|m_i$. Finally, we now propose the next theorem and prove it by using Proposition 3 to help us determine the relationship between the semi principal axes of an ellipsoid and its first order PT at a fixed $k$. This theorem extends the results in [30] for spheroid and helps us to identify the type of any ellipsoid based on its first order PT.

**Theorem 6**

Let $M(k, B)$ be the first order PT for ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ at a specified $k$ where $0 < k \neq 1 < +\infty$.

i. $a \leq b \leq c$ if and only if $M_1 \leq M_2 \leq M_3$.

ii. $c \leq b \leq a$ if and only if $M_3 \leq M_2 \leq M_1$.

**Proof:**

We will prove part (i) as similar steps can be used to prove part (ii).

Assume that $a \leq b$ and we must show $M_1 \geq M_2$. According to Proposition 3, $a \leq b$ implies $d_1 \geq d_2$ where $d_1, d_2 > 0$ based on Proposition 1. Thus, we will have

$$
d_1(k - 1) \geq d_2(k - 1).
$$

In Ahmad Khairuddin et al [23], the semi principal axes in Proposition 1, Proposition 2, Theorem 1 and Theorem 2 should be $a > b = c$ and $a < b = c$ not $a \geq b = c$ and $a \leq b = c$. 


We will have two cases here. If \( k > 1 \) then
\[
\begin{align*}
  d_1(k - 1) & \geq d_2(k - 1), \\
  kd_1 - d_1 & \geq kd_2 - d_2, \\
  (1 - d_1) + kd_1 & \geq (1 - d_2) + kd_2, \\
  \frac{1}{(1 - d_1) + kd_1} & \leq \frac{1}{(1 - d_2) + kd_2}, \\
  m_1 & \leq m_2, \\
  (k - 1)|B|m_1 & \leq (k - 1)|B|m_2, \\
  M_1 & \leq M_2.
\end{align*}
\]

For the second case, where \( k < 1 \), we have
\[
\begin{align*}
  d_1(k - 1) & \leq d_2(k - 1), \\
  kd_1 - d_1 & \leq kd_2 - d_2, \\
  (1 - d_1) + kd_1 & \leq (1 - d_2) + kd_2, \\
  \frac{1}{(1 - d_1) + kd_1} & \geq \frac{1}{(1 - d_2) + kd_2}, \\
  m_1 & \geq m_2, \\
  (k - 1)|B|m_1 & \leq (k - 1)|B|m_2, \\
  M_1 & \leq M_2.
\end{align*}
\]

Therefore, for both cases, we have shown that \( M_1 \leq M_2 \).

Next, suppose \( b \leq c \). We want to prove \( M_2 \leq M_3 \). According to Proposition 3, \( b \leq c \) implies \( d_2 \geq d_3 \) where \( d_2, d_3 > 0 \) based on Proposition 1. Thus, we will have
\[
d_2(k - 1) \geq d_3(k - 1).
\]

Again, we have two cases here. If \( k > 1 \) then
\[
\begin{align*}
  d_2(k - 1) & \geq d_3(k - 1), \\
  kd_2 - d_2 & \geq kd_3 - d_3, \\
  (1 - d_2) + kd_2 & \geq (1 - d_3) + kd_3, \\
  \frac{1}{(1 - d_2) + kd_2} & \leq \frac{1}{(1 - d_3) + kd_3}, \\
  m_2 & \leq m_3, \\
  (k - 1)|B|m_2 & \leq (k - 1)|B|m_3, \\
  M_2 & \leq M_3.
\end{align*}
\]

For the second case, where \( k < 1 \), we have
\[
\begin{align*}
  d_2(k - 1) & \leq d_3(k - 1), \\
  kd_2 - d_2 & \leq kd_3 - d_3, \\
  (1 - d_2) + kd_2 & \leq (1 - d_3) + kd_3, \\
  \frac{1}{(1 - d_2) + kd_2} & \geq \frac{1}{(1 - d_3) + kd_3}, \\
  m_2 & \geq m_3, \\
  (k - 1)|B|m_2 & \leq (k - 1)|B|m_3, \\
  M_2 & \leq M_3.
\end{align*}
\]
Therefore, for both cases, it is proven that $M_2 \leq M_3$.

By combining the results, we have shown that $M_1 \leq M_2 \leq M_3$ if $a \leq b \leq c$.

Now, assume that $M_1 \leq M_2$ and we must show $a \leq b$. Since $M_1 \leq M_2$, we will have

\[(k - 1)|B|m_1 \leq (k - 1)|B|m_2,\]
\[m_1 \leq m_2,\]
\[\frac{1}{(1 - d_1) + kd_1} \leq \frac{1}{(1 - d_2) + kd_2},\]
\[(1 - d_1) + kd_1 \geq (1 - d_2) + kd_2,\]
\[kd_1 - d_1 \geq kd_2 - d_2,\]
\[d_1(k - 1) \geq d_2(k - 1),\]
\[d_1 \geq d_2.\]

Since $d_1 \geq d_2$, according to Proposition 3, we will have $a \leq b$.

Now, suppose $M_2 \leq M_3$. We must show $b \leq c$. Since $M_2 \leq M_3$, then

\[(k - 1)|B|m_2 \leq (k - 1)|B|m_3,\]
\[m_2 \leq m_3,\]
\[\frac{1}{(1 - d_2) + kd_2} \leq \frac{1}{(1 - d_3) + kd_3},\]
\[(1 - d_2) + kd_2 \geq (1 - d_3) + kd_3,\]
\[kd_2 - d_2 \geq kd_3 - d_3,\]
\[d_2(k - 1) \geq d_3(k - 1),\]
\[d_2 \geq d_3.\]

Therefore, $d_2 \geq d_3$ gives $b \leq c$ based on Proposition 3.

Thus, by combining all results, we have shown that $a \leq b \leq c$ if $M_1 \leq M_2 \leq M_3$.

\[\square\]

5. Discussion and Conclusion

In this study, by using depolarization factors for general ellipsoids in the explicit formula of the first order PT for ellipsoids, we have investigated some properties of the first order PT for ellipsoids. After showing that it is a nonzero matrix, the first order PT for ellipsoid is then categorized based on the conductivity and also the semi principal axes of the of the ellipsoid. Consequently, the results can then be also used to describe an ellipsoid in an electrical field according to its first order PT for any related applications.

Acknowledgment

The authors would like to thank Research Management Centre-UTM for the financial support through the vote number 02K98 and also to Ministry of Higher Education Malaysia (MOHE) for scholarship MyBrainSc.

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