Isometric copies of $\ell^n_\infty$ and $\ell^n_1$ in transportation cost spaces on finite metric spaces

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Abstract. Main results: (a) If a metric space contains $2n$ elements, the transportation cost space on it contains a 1-complemented isometric copy of $\ell^n_1$. (b) An example of a finite metric space whose transportation cost space contains an isometric copy of $\ell^4_\infty$. Transportation cost spaces are also known as Arens-Eells, Lipschitz-free, or Wasserstein 1 spaces.

Keywords: Arens-Eells space, Banach space, duality in linear programming, earth mover distance, Edmonds matching algorithm, Kantorovich-Rubinstein distance, Lipschitz-free space, perfect matching, transportation cost, Wasserstein distance

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1 Introduction

The introduced below notions go back at least to Kantorovich and Gavurin [KG49]. We use the terminology and notation of [OO19+]. History of the notions introduced below as well as related terminology (Arens-Eells space, earth mover distance, Edmonds matching algorithm, Kantorovich-Rubinstein distance, Lipschitz-free space, perfect matching, transportation cost, Wasserstein distance) is discussed in [OO19+, Section 1.6] and references therein.

Definition 1.1. Let $(M, d)$ be a metric space. Consider a real-valued finitely supported function $f$ on $M$ with a zero sum, that is,

$$\sum_{v \in M} f(v) = 0. \tag{1}$$

A natural and important interpretation of such a function is the following:

- $f(v) > 0$ means that $f(v)$ units of a certain product are produced or stored at point $v$;
- $f(v) < 0$ means that $-f(v)$ units of the same product are needed at $v$.

The number of units can be any real number. With this in mind, $f$ may be regarded as a transportation problem. For this reason, we denote the vector space of all real-valued functions finitely supported on $M$ with a zero sum by $\text{TP}(M)$, where TP stands for transportation problems.

One of the standard norms on the vector space $\text{TP}(M)$ is related to the transportation cost and is defined in the following way:

A transportation plan is a plan of the following type: we intend to deliver
\begin{itemize}
  \item $a_1$ units of the product from $x_1$ to $y_1$,
  \item $a_2$ units of the product from $x_2$ to $y_2$,
  \item \ldots
  \item $a_n$ units of the product from $x_n$ to $y_n$,
\end{itemize}

where $a_1, \ldots, a_n$ are nonnegative real numbers, and $x_1, \ldots, x_n, y_1, \ldots, y_n$ are elements of $M$, which do not have to be distinct.

This transportation plan is said to \textit{solve the transportation problem} $f$ if
\[
  f = a_1(1_{x_1} - 1_{y_1}) + a_2(1_{x_2} - 1_{y_2}) + \cdots + a_n(1_{x_n} - 1_{y_n}),
\]
where $1_u(x)$ for $u \in M$ is the \textit{indicator function} defined as:
\[
  1_u(x) = \begin{cases}
    1 & \text{if } x = u, \\
    0 & \text{if } x \neq u.
  \end{cases}
\]

The \textit{cost} of transportation plan \eqref{eq:transportation-plan} is defined as $\sum_{i=1}^n a_i d(x_i, y_i)$. We introduce the \textit{transportation cost norm} (or just \textit{transportation cost}) $\|f\|_{TC}$ of a transportation problem $f$ as the minimal cost of transportation plans solving $f$. It is easy to see that the transportation plan of the minimum cost exists. We introduce the \textit{transportation cost space} $\text{TC}(M)$ on $M$ as the completion of $\text{TP}(M)$ with respect to the norm $\| \cdot \|_{TC}$.

It is worth mentioning that the norm of an element in $\text{TC}(M)$ can be computed using Linear Programming, see \cite{MG07} and \cite{Sch86}, see also related historical comments in \cite[pp. 221–223]{Sch86}.

Arens and Eells \cite{AE56} observed that if we pick a \textit{base point} $O$ in the space $M$, then the \textit{canonical embedding} of $M$ into $\text{(TP}(M), \| \cdot \|_{TC})$ given by the formula:
\[
  v \mapsto 1_v - 1_O
\]
is an isometric embedding. This observation can be easily derived from the following characterization of optimal transportation plans.

Let $0 \leq C < \infty$. A real-valued function $l$ on a metric space $(M, d)$ is called \textit{C-Lipschitz} if
\[
  \forall x, y \in M \quad |l(x) - l(y)| \leq Cd(x, y).
\]
The \textit{Lipschitz constant} of a function $l$ on a metric space containing at least two points is defined as
\[
  \text{Lip}(l) = \max_{x, y \in M, \ x \neq y} \frac{|l(x) - l(y)|}{d(x, y)}.
\]
Theorem 1.2 ([KG49]). A plan \( f = a_1(1_{x_1} - 1_{y_1}) + a_2(1_{x_2} - 1_{y_2}) + \cdots + a_n(1_{x_n} - 1_{y_n}) \) (4)
is optimal if and only if there exist a 1-Lipschitz real-valued function \( l \) on \( M \) such that
\[ l(x_i) - l(y_i) = d(x_i, y_i) \] (5)
for all pairs \( x_i, y_i \) for which \( a_i > 0 \).

The mentioned above observation of Arens and Eells makes transportation cost spaces an important object in the theory of metric embeddings, see [Ost13, Chapter 10] and [OO19+, Section 1.4]. This theory makes it very important to study the conditions of isometric embeddability of spaces \( \ell^n_\infty \) into \( \text{TC}(M) \).

Problems on isometric embeddability of spaces \( \ell^n_1 \) and \( \ell^n_\infty \) into \( \text{TC}(M) \) are also motivated by the following definitions, the first of which goes back to Kantorovich and Gavurin [KG49].

Definition 1.3. Let \( f_1, \ldots, f_n \) be nonzero transportation problems in \( \text{TP}(M) \) and \( x_1, \ldots, x_n \) be their normalizations, that is, \( x_i = f_i/\|f_i\|_{\text{TC}} \).

We say that transportation problems \( f_1, \ldots, f_n \) are completely unrelated, if
\[ \left\| \sum_{i=1}^n a_i x_i \right\|_{\text{TC}} = \sum_{i=1}^n |a_i| \]
for every collection \( \{a_i\}_{i=1}^n \) of real numbers.

We say that transportation problems \( f_1, \ldots, f_n \) are completely intertwined, if
\[ \left\| \sum_{i=1}^n a_i x_i \right\|_{\text{TC}} = \max_{1 \leq i \leq n} |a_i| \]
for every collection \( \{a_i\}_{i=1}^n \) of real numbers.

Remark 1.4. The notion of completely unrelated problems has a natural meaning in applications: we cannot decrease the total cost by combining the transportation plans for a set of completely unrelated transportation problems.

The notion of completely intertwined problems describes the very unusual situation: we have several transportation problems \( \{x_i\}_{i=1}^n \) such that each of them has cost 1 and the sum \( \sum_{i=1}^n \theta_i x_i \) (of \( n \) summands with cost 1 each) has cost 1 for every collection \( \theta_i = \pm 1 \).

It is clear that problems are completely unrelated if and only if their normalizations are isometrically equivalent to the unit vector basis of \( \ell^n_1 \) and problems are completely intertwined if and only if their normalizations are isometrically equivalent to the unit vector basis of \( \ell^n_\infty \).

The main goal of this paper is to study embeddability of \( \ell^n_1 \) and \( \ell^n_\infty \) into \( \text{TC}(M) \) for finite metric spaces \( M \). The following theorem is our main result.
Theorem 1.5. If a metric space $M$ contains $2n$ elements, then $TC(M)$ contains a 1-complemented subspace isometric to $\ell_1^n$. If the space $M$ is such that triangle inequalities for all distinct triples in $M$ are strict, then $TC(M)$ does not contain a subspace isometric to $\ell_1^{n+1}$.

Remark 1.6. It can be easily seen from the proof that in the case where a finite metric space $M$ contains more than $2n$ elements, the space $TC(M)$ also contains a 1-complemented subspace isometric to $\ell_1^n$. This is not completely obvious only if $|M|$ is odd. In this case we add to $M$ one point in an arbitrary way, apply Theorem 1.5 and then observe that all elements of standard basis of the constructed space, except one, are contained in $TC(M)$.

Theorem 1.5 solves [DKO18+] Problem 3.3 by strengthening [DKO18+] Theorem 3.1 which states that for $M$ with $2n$ elements the space $TC(M)$ contains a 2-complemented subspace 2-isomorphic to $\ell_1^n$.

Problems of isometric embeddability of $\ell_1^n$ into $TC(M)$ on infinite metric spaces $M$ were considered in [CJ17, OO19+].

The existing knowledge on embeddability of $\ell_1^n$ is very limited. The most important sources in this direction are [Bou86] and [GK03]. In Section 3 we present a special case of one of the results of [GK03] in the form which, in our opinion, helps to understand the phenomenon. Bourgain [Bou86] proved (see also a presentation in [Ost13 Section 10.4]) that $TC(\ell_1)$ contains almost isometric copies of $\ell_\infty^n$ for all $n$.

Our contribution to the case of $\ell_\infty^n$ (Section 3) consists in examples of relatively small finite metric spaces $M_3$ and $M_4$ such that $TC(M_3)$ and $TC(M_4)$, respectively, contain $\ell_\infty^n$ and $\ell_\infty^4$ isometrically. The reason for our interest to $M_3$ is that it is smaller than $M_4$. We do not know whether such finite metric spaces can be constructed for $\ell_\infty^n$ with $n \geq 5$.

In this connection it is natural to recall the well-known fact that the spaces $\ell_1^2$ and $\ell_\infty^2$ are isometric. It is easy to see that the standard proof of this can be stated as:

Observation 1.7. The transportation problems $f_1$ and $f_2$ are completely unrelated if and only if the transportation problems $g_1 = \frac{1}{2}(f_1 + f_2)$ and $g_2 = \frac{1}{2}(f_1 - f_2)$ are completely intertwined.

2 Proof of Theorem 1.5

We use terminology of [Die17]. Consider the metric space $M$ as a weighted complete graph with $2n$ elements, we denote it also $G = (V(G), E(G))$, the weight of an edge is the distance between its ends. We consider matchings containing $n$ edges in this graph, such matchings are called perfect matchings or 1-factors. We pick among all perfect matchings a matching of minimum weight (the weight of a matching is
defined as the sum of weights of its edges). Let \( e_1 = u_1v_1, \ldots, e_n = u_nv_n \) be a perfect matching of minimum weight. We claim that the transportation problems \( f_1 = 1_{u_1} - 1_{v_1}, \ldots, f_n = 1_{u_n} - 1_{v_n} \) are completely unrelated.

We need to show that for any set \( \{a_i\}_{i=1}^n \) of real numbers we have

\[
\left| \sum_{i=1}^n a_i (1_{u_i} - 1_{v_i}) \right| = \sum_{i=1}^n |a_i|d(u_i, v_i).
\]

Assume for simplicity that all \( a_i \) are positive (all other cases can be done similarly, we can just interchange \( u_i \) and \( v_i \) for those \( i \) for which \( a_i < 0 \)).

The inequality

\[
\left| \sum_{i=1}^n a_i (1_{u_i} - 1_{v_i}) \right| \leq \sum_{i=1}^n |a_i|d(u_i, v_i)
\]

is obvious. To prove the inverse inequality, assume the contrary, that is,

\[
\left| \sum_{i=1}^n a_i (1_{u_i} - 1_{v_i}) \right| < \sum_{i=1}^n |a_i|d(u_i, v_i).
\]

In such a case there exists transportation plans with lower cost that the straightforward plan (by the straightforward plan we mean the plan in which everything is moved from \( u_i \) to \( v_i \)). We call each such plan a better plan, and show that lowering (possibly non-strictly) the cost of any better plan we get the straightforward transportation plan, thus getting a contradiction.

In fact, it is easy to see that among all plans which can be obtained by not-necessarily-strict improvement of any better plan there is a plan with the maximal intersection with the straightforward plan. Here by intersection we mean a sum of the form \( \sum_{i=1}^n s_i (1_{u_i} - 1_{v_i}) \) satisfying \( s_i \leq a_i \) which is present as a part of the better plan. Such intersection is called maximal if there are no intersections \( \sum_{i=1}^n t_i (1_{u_i} - 1_{v_i}) \) with \( t_i \geq s_i \) for all \( i \) and a strict inequality for at least one value of \( i \). If the plan with the maximal intersection coincides with the straightforward plan, we get a contradiction and the proof is completed.

If the better plan with the maximal intersection with the straightforward plan does not coincide with the straightforward plan, it means that for some \( i \), which we denote by \( n(0) \), some amount is moved from \( u_{n(0)} \) to \( v_{n(1)} \) with \( n(1) \neq n(0) \). This implies, that some amount out of \( u_{n(1)} \) has to be moved to some \( v_{n(2)} \) with \( n(2) \neq n(1) \). This implies, that some amount has to be moved out of \( u_{n(2)} \) to some \( v_{n(3)} \) with \( n(3) \neq n(2) \). So on, since we consider a finite set, at some point \( n(k) = n(0) \). Let \( \alpha > 0 \) be a minimum out of all these amounts. Now, we replace the part of the plan dealing with moving these \( \alpha \) units by the following: we move that amount \( \alpha \) from \( u_{n(1)} \) to \( v_{n(1)} \), from \( u_{n(2)} \) to \( v_{n(2)} \), \ldots, from \( u_{n(k)} \) to \( v_{n(k)} \). It is easy to see that the outcome will be a valid transportation plan for our transportation problem, whose cost does not exceed the cost of the better transportation plan which we consider, but the intersection with the straightforward plan becomes larger. This is
a contradiction with the assumption that the plan with the maximal intersection can be different from the straightforward plan. This contradiction proves the existence in $\text{TC}(M)$ of the subspace isometric to $\ell^1_n$.

Now, assume that $M$ is such that all triangle inequalities in $M$ are strict. Let $f_1, \ldots, f_k$ be completely unrelated transportation problems on $M$.

**Lemma 2.1.** The functions $f_i$ have disjoint supports.

This lemma is essentially known [OO19+, Lemma 3.3], for convenience of the reader we provide a proof.

**Proof.** Assume the contrary, let $v \in M$ be in the supports of both $f_i$ and $f_j$, $i \neq j$. Without loss of generality we assume that $f_i(v) > 0$ and $f_j(v) < 0$, changing signs of $f_i$ and $f_j$ if needed (the change of signs does not affect complete unrelatedness).

To get a contradiction it suffices to show that $\|f_i + f_j\|_{\text{TC}} < \|f_i\|_{\text{TC}} + \|f_j\|_{\text{TC}}$. This can be done in the following way. In an optimal plan for $f_i$ some amount of units, denote it $\alpha > 0$, is moved from $v$ to some $u \in M$. In an optimal plan for $f_j$ some amount of units, denote it $\beta > 0$, is moved to $v$ from some $w \in M$ ($w$ can be the same as $u$).

Let $\gamma = \min\{\alpha, \beta\}$. Now we combine the optimal plans for $f_i$ and $f_j$ with the following exception: we move $\gamma$ units of the product directly from $w$ to $u$. Since, by our assumption, $d(w, u) < d(w, v) + d(v, u)$, the cost of the obtained plan is $< \|f_i\|_{\text{TC}} + \|f_j\|_{\text{TC}}$. \qed

Finally, since support of each function $f_i$ contains at least two points, we get that $k \leq n$. This proves the last statement of Theorem 1.5.

It remains to show that there is a projection of norm 1 onto the subspace spanned by $\{f_i\}_{i=1}^n$. We show that a linear operator $P$ is a norm-1 projection onto the subspace spanned by $\{1_{u_i} - 1_{v_i}\}_{i=1}^n$ if and only if it can be represented in the form

$$P(f) = \sum_{i=1}^n l_i(f) \frac{f_i}{\|f_i\|_{\text{TC}}}, \quad (6)$$

where:

- $f_i = 1_{u_i} - 1_{v_i}$
- $l_i$ are Lipschitz functions, and $l_i(f_j) = \delta_{i,j}\|f_j\|_{\text{TC}} = \delta_{i,j}d(u_j, v_j)$ ($\delta_{i,j}$ is the Kronecker delta).
- $\|Pf\|_{\text{TC}} \leq \|f\|_{\text{TC}}$ for every $f \in \text{TC}(M)$ of the form $f = 1_w - 1_z$ for $w, z \in M$.

Since $\{f_i\}_{i=1}^n$ are linearly independent and the dual of $\text{TC}(M)$ is the space of the Lipschitz functions on $M$, which take value 0 at the base point (see [Ost13, Theorem 10.2]), any projection onto the subspace spanned by $\{f_i\}_{i=1}^n$ is of the form (6) for some Lipschitz functions $\{l_i\}$ satisfying $l_i(f_j) = \delta_{i,j}\|f_j\|_{\text{TC}} = \delta_{i,j}d(u_j, v_j)$. 6
It remains to show the condition \(\|P f\|_{TC} \leq \|f\|_{TC}\) for \(f \in TC(M)\) of the form \(f = 1_w - 1_z\) implies that \(\|P\| \leq 1\). This follows from our definitions and observations made above: Any \(g \in TC(M)\) can be written as a sum of functions \(f_i\) of the form \(f_i = 1_{w_i} - 1_{z_i}\) in such a way that \(\|g\|_{TC} = \sum_{i=1}^{n} \|f_i\|_{TC}\). Therefore we get

\[
\|P g\|_{TC} = \left\| \left(\sum_{i=1}^{n} f_i\right)\right\|_{TC} \leq \sum_{i=1}^{n} \|P f_i\|_{TC} \leq \sum_{i=1}^{n} \|f_i\|_{TC} = \|g\|_{TC}.
\]

Our approach to the construction of suitable functions \(l_i\) is based on the Duality Theorem of Linear Programming and the Edmonds [Edm65] algorithm for the minimum weight perfect matching problem. We use the description of the algorithm in the form given in [LP09, Theorem 9.2.1], where it is shown that the minimum weight perfect matching problem on a complete graph \(G\) with even number of vertices and weight \(w : E(G) \to \mathbb{R}, w \geq 0\), can be reduced to the following linear program. (An odd cut in \(G\) is the set of edges joining a subset of \(V(G)\) of odd cardinality with its complement, a trivial odd cut is a set of edges joining one vertex with its complement. If \(x\) is a real-valued function on \(E(G)\) and \(A\) is a set of edges, we write \(x(A) = \sum_{e \in A} x(e)\).)

- **(LP1)** minimize \(w^\top \cdot x\) (where \(x : E(G) \to \mathbb{R}\))

  - subject to
    1. \(x(e) \geq 0\) for each \(e \in E(G)\)
    2. \(x(C) = 1\) for each trivial odd cut \(C\)
    3. \(x(C) \geq 1\) for each non-trivial odd cut \(C\).

We introduce a variable \(y_C\) for each odd cut \(C\).

The dual program of the program (LP1) is:

- **(LP2)** maximize \(\sum_C y_C\)

  - subject to
    1. \(y_C \geq 0\) for each non-trivial odd cut \(C\)
    2. \(\sum_C y_C \leq w(e)\) for every \(e \in E(G)\).

The Duality in Linear Programming [Sch86, Section 7.4] (see also a summary in [LP09, Chapter 7]) states that the optima (LP1) and (LP2) are equal. (In the general case we need to require the existence of vectors satisfying the constraints and finiteness of one of the optima.)

This means that the total length of the minimum weight perfect matching coincides with the sum of entries of the optimal solution of the dual program.

We complete our proof of the existence of norm-1 projection \(P\) of the desired form by proving the following two lemmas.
Lemma 2.2. Suppose that there is an optimal dual solution satisfying $y_C \geq 0$ for all odd cuts $C$ including trivial ones. Then there exist functions $l_i$ for which $P$ defined by (6) is a norm-1 projection.

Lemma 2.3. If the weight function $w : E(G) \to \mathbb{R}$ corresponds to a metric on $V(G)$ (this means that $w(uv) = d(u, v)$ for some metric $d$ on $V(G)$), then there is an optimal dual solution satisfying $y_C \geq 0$ for all odd cuts, including trivial ones.

Proof of Lemma 2.2. Let $M$ be the minimum weight perfect matching, then $e \in M$ is of the form $u_i v_i$. We introduce the function $l_i : V(G) \to \mathbb{R}$ by

$$l_i(w) = \begin{cases} 0 & \text{if } w = u_i \\ \sum_C \text{contains } u_i v_i \text{ and separates } u_i \text{ and } w y_C & \text{if } w \neq u_i. \end{cases}$$

(7)

We claim that the function $l_i$ has the following desired properties:

1. $l_i$ is 1-Lipschitz.
2. $l_i(v_i) - l_i(u_i) = d(v_i, u_i)$.
3. $l_i(v_j) - l_i(u_j) = 0$ if $j \neq i$.
4. $\sum_{i=1}^n |l_i(w) - l_i(z)| \leq d(w, z)$ for every $w, z \in M = V(G)$.

The discussion following (6) implies that these conditions imply that the obtained $P$ is a norm-1 projection.

Proofs of 1–4:

1. $|l_i(w) - l_i(z)| \leq \sum_C \text{separates } w \text{ and } z y_C \leq w(wz) = d(w, z)$, where in the first inequality we used the definition of $l_i$, in the second we used (D2). Observe also that item 1 follows from the stronger inequality in item 4, which we prove below.

2. $l_i(v_i) - l_i(u_i) = d(v_i, u_i)$.

The corresponding argument is shown in [LP09, p. 371]. We reproduce it. We have

$$w(M) = \sum_{e \in M} w(e) \geq \sum_{e \in M} \sum_{C \text{ containing } e} y_C = \sum_{C} |M \cap C| y_C \geq \sum_{C} y_C,$$  

(8)

where in the first inequality we used (D2) and in the second inequality we used $|M \cap C| \geq 1$ for each odd cut.

If $y_C$ is an optimal dual solution, we get that the leftmost and the rightmost sides in (8) coincide, and therefore

$$w(e) = \sum_{C \text{ containing } e} y_C$$

(9)
for each $e \in M$ and

$$|M \cap C| = 1 \text{ for each non-trivial odd cut } C \text{ satisfying } y_C > 0 \quad (10)$$

Equality (9) implies $l_i(v_i) - l_i(u_i) = \sum_{C \text{ containing } u_i v_i} y_C - 0 = w(u_i v_i) = d(u_i, v_i)$.

3. $l_i(v_j) - l_i(u_j) = 0$ if $j \neq i$.

This equality follows from (10). In fact, equality (10) implies that none of the cuts with $y_C > 0$ containing $u_i v_i$ can contain $u_j v_j$ for $j \neq i$, and thus $l_i(v_j) = l_i(u_j)$ for all $j \neq i$.

4. $\sum_{i=1}^{n} |l_i(w) - l_i(z)| \leq d(w, z)$ for every $w, z \in M$.

To prove this inequality we observe that $|l_i(w) - l_i(z)| \leq \sum_{C \in S_i(w, z)} y_C$, where $S_i(w, z)$ is the set of cuts $C$ with $y_C > 0$ which simultaneously separate $u_i$ from $v_i$ and $w$ from $z$. It is important to observe that (10) implies that the sets $\{S_i(w, z)\}_{i=1}^{n}$ are disjoint. Therefore, by (D2), $\sum_{i=1}^{n} |l_i(w) - l_i(z)| \leq d(w, z)$.

\[ \square \]

Proof of Lemma 2.3. We follow the presentation in [LP09, Section 9.2] of the Edmonds algorithm for construction of an optimal dual solution. To prove the lemma it suffices to show that the assumption that $w$ corresponds to a metric implies that when we run the algorithm we maintain $y_C \geq 0$ in each step, even for trivial odd cuts.

We decided not to copy the whole Section 9.2 at a price that we expect readers (who do not remember the algorithm) to have [LP09, Section 9.2] handy.

The beginning of the algorithm can be described as follows: we assign the number $y_C = \frac{1}{2 \min_{u,v} d(u, v)}$ to all trivial cuts $C$ and set $y_C = 0$ for all nontrivial cuts $C$. This function on the set of all odd cuts satisfies the conditions (D1) and (D2). Such functions are called dual solutions. For a dual solution $y$ we form a graph $G_y$ whose vertex set is $V(G)$ and edge set is defined by

$$E_y = \left\{ e \in E(G) : \sum_{C \text{ containing } e} y_C = w(e) \right\}.$$ 

It is clear that with $y_C$ defined as above we get a graph $G_y$ which can contain any number of edges between 1 and $\frac{n(n-1)}{2}$.

In each step of the Edmonds algorithm we construct not only the function $y_C$, but also a set $\mathcal{H}$ of odd cardinality subsets of $V(G)$ satisfying four conditions listed in [LP09, (P-1)-(P-4), page 372]. We list only the first two conditions, because the contents of the last two conditions does not affect our modification of the argument in [LP09, Section 9.2].
(P-1) $\mathcal{H}$ is nested, that is, if $S, T \in \mathcal{H}$, then either $S \subset T$ or $T \subset S$ or $S \cap T = \emptyset$.

(P-2) $\mathcal{H}$ contains all singletons of $V(G)$.

At the end of the first step described above the set $\mathcal{H}$ is let to be the set of singletons (and all of the desired conditions are satisfied).

After that the following step is repeated and the function $y_C$ is modified till the graph $G'_y$ (described below) becomes a graph having perfect matching.

Let $S_1, \ldots, S_k$ be the (inclusionwise) maximal members of $\mathcal{H}$. It follows from (P-1) that $S_1, \ldots, S_k$ are mutually disjoint and from (P-2) that they form a partition of $V(G)$. Let $G'_y$ denote the graph obtained from $G_y$ by contracting each $S_i$ to a single vertex $s_i$. Since $|V(G)|$ is even, but $S_j$ is odd, it follows that $k := |V(G'_y)|$ is even.

Suppose that $G'_y$ does not have a perfect matching. Let $A(G'_y)$, $C(G'_y)$, and $D(G'_y)$ be the sets of the Gallai-Edmonds decomposition for $G'_y$ (see [LP09, Section 3.2]).

We use the notation $A(G'_y) = \{s_1, \ldots, s_m\}$ and denote the components of the subgraph of $G'_y$ induced by $D(G'_y)$ by $H_1, \ldots, H_{m+d}$, where $d$ is the number of vertices which are not matched in a maximum matching in $G'_y$. Let

$$T_i = \bigcup_{s_j \in V(H_i)} S_j.$$

Now we modify the dual solution $y$ as follows (by $\nabla(S)$ we denote the set of edges connecting a vertex set $S$ with its complement):

$$y^L_{\nabla(S_j)} = y_{\nabla(S_j)} - t \quad (1 \leq j \leq m),$$

$$y^L_{\nabla(T_i)} = y_{\nabla(S_i)} + t \quad (1 \leq i \leq m + d),$$

$$y^C = y_C, \text{ otherwise.}$$

In this formula $t$ is chosen as the minimum of three numbers, $t_1, t_2, t_3$, defined as:

$$t_1 = \min\{y_{\nabla(S_j)} : 1 \leq j \leq m, |S_j| > 1\},$$

$$t_2 = \min\{w(e) - \sum_{e \in C} y_C : e \in \nabla(T_1) \cup \cdots \cup \nabla(T_{m+d}) \setminus (\nabla(S_1) \cup \cdots \cup \nabla(S_m))\},$$

$$t_3 = \frac{1}{2} \min\{w(e) - \sum_{e \in C} y_C : e \in (\nabla(T_i) \cap \nabla(T_j)), 1 \leq i < j \leq m + d\}.$$
Because of the positive surplus condition in [LP09, Theorem 3.2.1 (c)], the vertex \( v \) is connected in \( G_y \) with at least two of the sets \( \{ T_i \}_{i=1}^{m+d} \), suppose that these are sets \( T_{i_1} \) and \( T_{i_2} \). Let \( u \in T_{i_1} \) and \( w \in T_{i_2} \) be adjacent to \( v \) in \( G_y \). Let \( \{ U_p \}_{p=1}^\tau \) be the elements of \( H \) containing \( u \) and let \( \{ W_q \}_{q=1}^\sigma \) be the elements of \( H \) containing \( w \).

Since the edges \( uv \) and \( wv \) are in \( G_y \), we have

\[
\begin{align*}
    w(uv) &= y_{\nabla(v)} + \sum_{p=1}^\tau y_{\nabla U_p} \\
    w(wv) &= y_{\nabla(v)} + \sum_{q=1}^\sigma y_{\nabla W_q}
\end{align*}
\]

On the other hand, the definition of \( t_3 \) and our choice of \( S_1, \ldots, S_k \) imply that

\[
\begin{align*}
t_3 \leq \frac{1}{2} \left( w(uw) - \sum_{p=1}^\tau y_{\nabla U_p} - \sum_{q=1}^\sigma y_{\nabla W_q} \right) \\
&\leq \frac{1}{2} \left( \left( w(uv) - \sum_{p=1}^\tau y_{\nabla U_p} \right) + \left( w(vw) - \sum_{q=1}^\sigma y_{\nabla W_q} \right) \right) \\
&= y_{\nabla(v)},
\end{align*}
\]

where in the second inequality we use the triangle inequality for the distance corresponding to weight \( w \), and in the last equality we use (11) and (12).

\[ \square \]

3 Isometric copies of \( \ell_n^\infty \) in \( \text{TC}(M) \)

As is well-known the spaces \( \{ \ell_n^\infty \} \) admit low-distortion and even isometric embeddings into some transportation cost spaces. This follows from the basic property of \( \text{TC}(M) \): it contains an isometric copy of \( M \) (see (3)).

Another related fact is the following immediate consequence of the Bourgain discretization theorem (see [Bou87], [GNS12], [Ost13, Section 9.2]): for sufficiently large \( m \) the transportation cost space on the set of integer points in \( \ell_n^\infty \) with absolute values of coordinates \( \leq n \) contains an almost-isometric copy of \( \ell_n^\infty \).

In the next example we need the following well-known fact (see [Wea18, Section 3.3], [OO19+, Section 1.6]): If \( (M, d) \) is a complete metric space, then \( \text{TC}(M) \) contains the vector space of differences between finite positive compactly supported measures \( \mu \) and \( \nu \) on \( M \) with the same total masses and \( \| \mu - \nu \|_{\text{TC}} \) is equal to the quantity \( \mathcal{F}_1(\mu, \nu) \) defined in the following way.

A \textit{coupling} of a pair of finite positive Borel measures \( (\mu, \nu) \) with the same total mass on \( M \) is a Borel measure \( \pi \) on \( M \times M \) such that \( \mu(A) = \pi(A \times M) \) and \( \nu(A) = \pi(M \times A) \) for every Borel measurable \( A \subset M \). The set of couplings of \( (\mu, \nu) \)
is denoted $\Pi(\mu, \nu)$. We define

$$\mathcal{T}_1(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \left( \int \int_{M \times M} d(x, y) \, d\pi(x, y) \right).$$

The result of Godefroy and Kalton [GK03, Theorem 3.1] has the following special case:

**Example 3.1.** Let us consider the following (non-discrete) transportation problems on the unit cube $[0, 1]^n$ with its $\ell_\infty$-distance:

$P_i$: “available” is the Lebesgue measure on the face $x_i = 0$, “needed” is the Lebesgue measure on the face $x_i = 1$.

It is clear that $P_i$ has cost 1, and actually any measure-preserving transportation from bottom to top does the job. The easiest transportation plan is to move each point from the the face $x_i = 0$ to the point with the same coordinates, changing only $x_i$ from 0 to 1.

It is not that easy to see that $\sum_{i=1}^n \theta_i P_i$ has cost 1. This can be done as follows. By symmetry it suffices to consider the case where all $\theta_i = 1$. In this case we move each point from the surface with “availability” to the surface with “need” in the direction of the diagonal $(1, \ldots, 1)$. It is easy to see that it will be a bijection between points of “availability” and “need”. The cost can be computed as the following integral:

$$n \int_0^1 t(-d(1-t)^{n-1}) = n(n-1) \int_0^1 t(1-t)^{n-2} dt$$

$$= n(n-1) \int_0^1 ((1-t)^{n-2} - (1-t)^{n-1}) dt$$

$$= n(n-1) \left( \frac{(1-t)^n}{n} - \frac{(1-t)^{n-1}}{n-1} \right) \bigg|_0^1 = 1.$$

We are interested in constructing finite metric spaces $M$ for which $\text{TC}(M)$ contains $\ell_\infty^n$ isometrically. So far we succeeded to do this only for $n = 3$ and $n = 4$ (the case $n = 2$ is easy, see Observation [1.7]).

**Example 3.2 (Finite $M$ with $\text{TC}(M)$ containing $\ell_\infty^n$ isometrically).** The set $M$ which we consider is a subset of the surface of the cube $[0, 1]^3$ endowed with its $\ell_\infty$ distance. Transportation problem $P_i$ is described in the following way: “available” is $\frac{1}{6}$ at each midpoint of the edge in the face $x_i = 0$ and $\frac{1}{3}$ at the center of the face; “needed” is at the similar points with $x_i = 1$.

The transportation cost for $P_i$ is 1 - just shift from $x_i = 0$ to $x_i = 1$. Again by symmetry it suffices to show that the cost of $P_1 + P_2 + P_3$ is 1.

Consider faces with $x_i = 0$ as colored “red” and faces with $x_i = 1$ as colored “blue”. It is clear that availability and need on two-dimensional faces which are on
the boundary between blue and red cancel each other. There will be 6 points of availability left. Three of them are on edges, and three are centers of faces. The value is $\frac{1}{3}$ at each. So to achieve cost 1 it suffices to match red and blue vertices in such a way that the distance between any two matched vertices is $\frac{1}{2}$.

This is possible. To achieve this we match red points which are centers of edges with blue vertices which are centers of faces and red points which are centers of faces with blue vertices which are centers of edges.

Our example in dimension 4 is even more symmetric.

**Example 3.3** (Finite $M$ with $\text{TC}(M)$ containing $\ell^4_\infty$ isometrically). The set $M$ which we consider is a subset of the surface of the cube $[0,1]^4$ endowed with its $\ell_\infty$ distance. Transportation problem $P_i$ is described in the following way: “available” is $\frac{1}{6}$ at the center of each of each 2-dimensional face of the face $x_i = 0$; “needed” is at the similar points with $x_i = 1$.

The transportation cost for $P_i$ is 1 - just shift from $x_i = 0$ to $x_i = 1$. Again by symmetry it suffices to show that the cost of $P_1 + P_2 + P_3 + P_4$ is 1.

As in the above discussion with blue and red we see that half of the availability and need will cancel each other.

The remaining availability of value $\frac{1}{3}$ will be concentrated at 6 centers of 2-dimensional faces of 3-dimensional faces. Each of these centers will have coordinates $\frac{1}{2}, \frac{1}{2}, 1, 1$ in some order. The need of value $\frac{1}{3}$ will concentrated at 6 points with coordinates $\frac{1}{2}, \frac{1}{2}, 0, 0$. Cancellation will occur at points with coordinates $\frac{1}{2}, \frac{1}{2}, 0, 1$.

To get the transportation plan of cost 1 we need to find a matching between points with coordinates $\frac{1}{2}, \frac{1}{2}, 1, 1$ and points with coordinates $\frac{1}{2}, \frac{1}{2}, 0, 0$, such that the distance between each pair of matched vertices is $\frac{1}{2}$. Such matching is obvious.

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