Quantization of $n$ coupled scalar field theory

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Abstract

We study a model of $n$ coupled scalar fields in Minkowski spacetime where all masses degenerate, which is considered as a toy model of polycritical gravity on AdS spacetime. We quantize this model within the Becchi-Rouet-Stora-Tyutin (BRST) scheme by introducing $n$ Faddeev-Popov (FP) ghost fields. Extending a BRST quartet generated by two scalars and two FP ghosts to $n$ scalars and $n$ FP ghosts, there remains a physical subspace with positive norm for odd $n$, but there exists only the vacuum for even $n$. This clearly shows a non-triviality of odd-higher order derivative scalar field theories. This is helpful to understand the truncation mechanism which is used to obtain a unitary conformal field theory dual to linearized polycritical gravity. It turns out that the truncation mechanism is nothing but a general quartet mechanism appeared when introducing the FP ghost action.

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I. INTRODUCTION

The quantization of the system with first-class constraints [1] has been performed using the BRST symmetry [2–4]. The system with second-class constraints could be quantized by converting these to a first-class theory in an extended phase space [5, 6].

Even though we are seeking to find a consistent quantum gravity [7, 8], we focus on the quantization of the scalar theory but not the gauge and gravity theories because of its simplicity. To this end, a chiral boson is a well-known example of the second-class theory in two dimensions. After the BRST quantization of a chiral boson, the quartet mechanism forces all states to have a zero norm, leaving the vacuum [9]. Importantly, it was argued that all higher derivative scalar theories are trivial because these have the BRST symmetry when introducing FP ghosts, and have only the vacuum when imposing the BRST quartet [10]. However, for the $2n$-order Klein-Gordon theory with different masses [8], the odd $n$ and the even $n$ cases feature qualitative differences. For odd $n$, one has $(n-1)/2$ ghost (physical) fields and $(n+1)/2$ physical (ghost) fields according to the overall negative (positive) sign of the free part of the Lagrangian [11]. Here ghost (physical) fields represent their scalar propagators with negative (positive) norm states. For even $n$, one finds $n/2$ fields of each type. This distinguishes the odd $n$ case from the even $n$ case. However, the degenerate cases are ruled out in that approach because the higher-order Green’s function (equivalent second-order Lagrangian) blows up after performing the partial fraction.

In this work, we investigate a model of $n$ coupled scalar fields in Minkowski spacetime where all masses degenerate, which leads to a $2n$-order single scalar theory when eliminating $n-1$ auxiliary scalar fields [10]. In particular, for $n = 2$, the model was considered as a toy model of the fourth-order critical gravity on AdS$_3$ spacetime which appeared in the pursuit of quantum gravity. Both of $n$ coupled scalar field and polycritical gravity theories may have the same rank-$n$ logarithmic conformal field theory (LCFT) as their duals [12–14], which still suffers from the non-unitarity. A truncation mechanism has been introduced to cure the non-unitarity [15, 16]. However, up to now, there is no consistent truncation mechanism to provide a unitary CFT. Furthermore, it was pointed out that these linearized approaches of polycritical gravities have pathologies when considering the non-linear level [17]. This implies that calculations on the linearized level seemed to lend support to the possibility of truncating the theory. In this sense, we have to regard our model of the $n$-coupled scalar
field theory as a toy model of (linearized) polycritical gravities.

We wish to quantize the $n$ coupled scalar field theory within the BRST quantization scheme by constructing the FP ghost action composed of $n$ FP ghost fields. Extending a BRST quartet generated by two scalars and FP ghosts to $n$ scalars and FP ghosts, there remains a physical subspace with positive norm for odd $n$, while there exists only the vacuum for even $n$. This shows the non-triviality of odd-higher order derivative scalar field theories clearly, which might provide a hint to resolve the non-unitarity issue appeared in developing the higher-derivative quantum gravity.

II. $n$ COUPLED SCALAR FIELD THEORY

Let us start with the $n$ coupled scalar field model with degenerate masses [12]

$$ S_0 = -\frac{1}{2} \int d^4x \sum_{i,j=1}^{n} (X_{ij} \partial_\mu \phi_i \partial^\mu \phi_j + Y_{ij} \phi_i \phi_j), \tag{1} $$

where we adopt the Minkowskian convention of $\eta_{\mu\nu} = \text{diag.}(-+++)$ and $x^\mu = (t, \vec{x})$. The ($n \times n$)-matrices $X_{ij}$ and $Y_{ij}$ for the scalar fields are given by

$$ X_{ij} = \begin{pmatrix}
0 & \cdots & 0 & 1 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 1 & 0 \\
0 & \cdots & 0 & 1
\end{pmatrix}, \quad Y_{ij} = \begin{pmatrix}
0 & \cdots & 0 & m^2 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 1 & 0 \\
m^2 & \cdots & 0 & 0
\end{pmatrix} \tag{2} $$

for $i, j = 1, \ldots, n$ ($n \geq 2$). In this work we will not consider the non-degenerate case with different masses for $Y_{ij}$, because it could not be considered as a toy model of the polycritical gravity.

The equations of motion for the $n$ coupled scalar fields are

$$ (\Box - m^2) \phi_1 = 0, \tag{3} $$

$$ (\Box - m^2) \phi_i = \phi_{i-1}, \quad i = 2, \ldots, n, \tag{4} $$

which lead to

$$ (\Box - m^2)^n \phi_n = 0 \tag{5} $$
for a scalar field $\phi_n$. In the classical aspect of the theory, the other fields $\phi_l$ with $l = 1, \cdots, n - 1$ are considered as auxiliary fields used to lower the number of derivatives in the single scalar $\phi_n$ action

$$S_n = -\frac{1}{2} \int d^4x (\Box - m^2)^{\frac{n}{2}} \phi_n (\Box - m^2)^{\frac{n}{2}} \phi_n. \quad (6)$$

However, in the quantum aspect of the theory, $\{\phi_n, \phi_l\}$ will be treated equally as scalar fields.

In order to obtain the BRST invariant action, we have to construct the corresponding FP ghost action. Usually, the BRST symmetry was found in gauge theories as a symmetry of the gauge-fixed action [2–4]. Its purpose is definitely to remove unphysical fields (negative norm states) associated with gauge invariance. On the other hand, physical fields are defined as those which have zero ghost number and are invariant under BRST transformations.

The BRST symmetry in this work is not due to gauge symmetry after a gauge-fixing. Surely, it takes into account a feature of giving the higher-order derivative structure starting from the second-order action (1) via the scalar coupling. At this stage, we emphasize that the model (1) [or (6)] inherently possesses ghost states. In order to eliminate the ghosts arising from the higher derivative action, we need to construct the corresponding FP ghost action. Here, we use the same FP terminology which was used in the gauge and gravity theories to distinguish between FP ghosts and ghost fields with negative norm state (poltergeist [8]). In the U(1) gauge theory, the FP ghosts are introduced to remove the unphysical fields of scalar and longitudinal photons by imposing the quartet, leaving two transverse photons [4]. More precisely, the ‘gauge’ FP ghosts are used for the quantization of gauge and gravity theories, while the ‘higher-derivative’ FP ghosts are introduced to take into account the higher-derivative nature of the $n$ coupled scalar field theory (1) [or (6)].

Recently, we have studied a sixth order derivative ($n = 3$) scalar field model in Minkowski spacetime in a BRST invariant manner [18] as a toy model of critical gravity theories. There, the ‘higher-derivative’ FP ghost action was included to require that the resultant action is invariant under the BRST transformation. By extending the $n = 3$ analysis to the $n$ coupled scalar field theory, we find the ghost action composed of $n$ FP ghost fields $c_i$ as

$$S_g = -\frac{1}{2} \int d^4x \sum_{i,j=1}^{n} (Z_{ij} \partial_{\mu} c_i \partial^{\mu} c_j + W_{ij} c_i c_j), \quad (7)$$
whose even $Z_{ij}$ and $W_{ij}$ are given by

$$Z_{ij} = \begin{pmatrix}
0 & \cdots & 0 & 0 & 1 \\
0 & \cdots & 0 & 1 & 0 \\
0 & \cdots & 0 & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & -1 & 0 \\
-1 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0
\end{pmatrix}, \quad W_{ij} = \begin{pmatrix}
0 & \cdots & 0 & 0 & m^2 \\
0 & \cdots & 0 & m^2 & 1 \\
0 & \cdots & 0 & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 1 & 0 \\
0 & \cdots & 0 & -m^2 & \cdots \\
-1 & 0 & \cdots & 0 & 0 \\
-1 & 0 & \cdots & 0 & 0
\end{pmatrix}, \quad (8)$$

for $i, j = 1, \ldots, 2k (= n)$. On the other hand, odd $Z_{ij}$ and $W_{ij}$ matrices take the forms

$$Z_{ij} = \begin{pmatrix}
0 & \cdots & 0 & 0 & 1 \\
0 & \cdots & 0 & 1 & 0 \\
0 & \cdots & 0 & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & -1 & 0 \\
-1 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0
\end{pmatrix}, \quad W_{ij} = \begin{pmatrix}
0 & \cdots & 0 & 0 & m^2 & 0 \\
0 & \cdots & 0 & m^2 & 1 & 0 \\
0 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 1 & 0 & 0 \\
0 & \cdots & 0 & -m^2 & \cdots & 0 \\
-1 & 0 & \cdots & 0 & 0 & 0 \\
-1 & 0 & \cdots & 0 & 0 & 0
\end{pmatrix}, \quad (9)$$

for $i, j = 1, \ldots, 2k + 1 (= n)$. We note that in the odd case (9), the last-null row and column are added to the even case (8). For non-degenerate case with different masses for $W_{ij}$ in (8) and (9), we could not find the BRST invariant action because the BRST symmetry is not nilpotent. We explain how the ghost action (7) is nontrivially constructed, depending on $n$ because the $n$ coupled scalar action (11) has already known. For $n = 2, 3$, two FP ghosts is enough to have the BRST invariant action for a dipole ghost field (singleton). This was a known case. We need to introduce more FP ghost fields to construct the BRST invariant action as $n$ increases. For example, we have 4 FP ghosts for $n = 4, 5$, 6 FP ghosts for $n = 6, 7$, and so on. This indicates clearly nontrivial terms for $n > 3$ when comparing the known cases of $n \leq 3$. 

5
Now, we show that the total action

\[ S_t = S_0 + S_g, \]  

is invariant under the BRST transformation

\[ \begin{align*}
\delta \phi_1 &= 0, \quad \cdots, \quad \delta \phi_k = 0, \quad \delta \phi_{k+1} = c_k, \quad \cdots, \quad \delta \phi_{2k} = c_1, \\
\delta c_1 &= 0, \quad \cdots, \quad \delta c_k = 0, \quad \delta c_{k+1} = \phi_k, \quad \cdots, \quad \delta c_{2k} = \phi_1,
\end{align*} \]  

for the even \((n = 2k)\) case, while

\[ \begin{align*}
\delta \phi_1 &= 0, \quad \cdots, \quad \delta \phi_k = 0, \quad \delta \phi_{k+1} = 0, \quad \delta \phi_{k+2} = c_k, \quad \cdots, \quad \delta \phi_{2k+1} = c_1, \\
\delta c_1 &= 0, \quad \cdots, \quad \delta c_k = 0, \quad \delta c_{k+1} = \phi_k, \quad \cdots, \quad \delta c_{2k+1} = \phi_1,
\end{align*} \]  

for the odd \((n = 2k+1)\) case. Here we wish to point out that for the odd case of (12), there exists an additional BRST-invariant field like \(\phi_{k+1}\) when comparing it with the even case of (11). In the odd \(n\) coupled scalar theory, only \(\phi_{k+1}\) is a physical field, whereas all remaining fields belong to unphysical fields. This might explain an origin of existing a physical state with positive norm state.

Finally, we derive \(n\) coupled equations for \(n\) ghost fields

\[ \begin{align*}
(\Box - m^2)c_{i-1} &= c_i, \quad i = 2, ..., k, \\
(\Box - m^2)c_k &= 0, \\
(\Box - m^2)c_{i-1} &= c_i, \quad i = k + 2, ..., 2k, \\
(\Box - m^2)c_{2k} &= 0,
\end{align*} \]

where \(k = \lfloor \frac{n}{2} \rfloor \) \((n \geq 2)\) is the greatest integer which is less than \(\frac{n}{2}\). Evidently, these are different from the FP ghost equations of the gauge theory. Note that by making successive eliminations of the smaller indices, these equations reduce to two FP ghosts equations

\[ \begin{align*}
(\Box - m^2)^k c_1 &= 0, \\
(\Box - m^2)^k c_{k+1} &= 0,
\end{align*} \]

which are just two FP ghost equations for a single field \(\phi_n\) \([6]\) \([10]\).
III. BRST TRANSFORMATIONS OF MODES

In this section, instead of scalar and FP ghost fields \( \{\phi_i(x), c_i(x)\} \), we find the BRST transformations of corresponding modes from solutions to Eqs. (3)-(4) and (13)-(16). First of all, making use of an ansatz

\[
\phi_1(x) = \int \frac{d^3 k}{(2\pi)^{3/2}\sqrt{2\omega}} \phi_1(\vec{k}, t)e^{i\vec{k} \cdot \vec{x}},
\]

(19)

Eq. (3) becomes one dimensional equation for \( \phi_1(\vec{k}, t) \) as

\[
\left( \frac{d^2}{dt^2} + \omega^2 \right) \phi_1(\vec{k}, t) = 0
\]

(20)

with \( \omega^2 = \vec{k}^2 + m^2 \). This is solved to give a solution

\[
\phi_1(\vec{k}, t) = iN_1 \left( a_1(\vec{k})e^{-i\omega t} - a_1^\dagger(\vec{k})e^{i\omega t} \right)
\]

(21)

with two Fourier modes \( a_1(\vec{k}) \) and \( a_1^\dagger(\vec{k}) \). Here, we have introduced a coefficient \( N_1 \) which may be taken to be 1. On the other hand, choosing \( N_1 = -\sqrt{m} \) reproduces the particle theory’s result appeared in Ref. [10]. Using the ansatz for Eq. (4)

\[
\phi_i(x) = \int \frac{d^3 k}{(2\pi)^{3/2}\sqrt{2\omega}} \phi_i(\vec{k}, t)e^{i\vec{k} \cdot \vec{x}},
\]

(22)

we obtain the time-dependent equations as

\[
\left( \frac{d^2}{dt^2} + \omega^2 \right) \phi_i(\vec{k}, t) = -\phi_{i-1}(\vec{k}, t), \quad i = 2, ..., n.
\]

(23)

Eq. (23) can be further separated into two first-order differential equations as

\[
\left( \frac{d}{dt} + i\omega \right) \psi_i(\vec{k}, t) = -\phi_{i-1}(\vec{k}, t), \quad (24)
\]

\[
\left( \frac{d}{dt} - i\omega \right) \phi_i(\vec{k}, t) = \psi_i(\vec{k}, t), \quad (25)
\]

whose solutions are given by

\[
\psi_i(\vec{k}, t) = -e^{-i\omega t} \int dt' e^{i\omega t'} \phi_{(i-1)}(\vec{k}, t'),
\]

(26)

\[
\phi_i(\vec{k}, t) = e^{i\omega t} \int dt' e^{-i\omega t'} \psi_i(\vec{k}, t'),
\]

(27)

respectively.
As a result, we find the first-five iterative solutions as
\begin{equation}
\phi_1(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega}} \left( iN_1 \right) \left( a_1(\vec{k}) e^{-i\omega t + ik \cdot \vec{x}} - c.c. \right),
\end{equation}
\begin{equation}
\phi_2(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega}} \left( -\frac{N_1}{2\omega^2} \right) \left[ \left( a_2(\vec{k}) - a_1(\vec{k})\omega t \right) e^{-i\omega t + ik \cdot \vec{x}} + c.c. \right],
\end{equation}
\begin{equation}
\phi_3(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega}} \left( -\frac{iN_1}{4\omega^4} \right) \left[ \left( a_3(\vec{k}) - \left( \frac{i}{2} a_1(\vec{k})\omega t + a_2(\vec{k}) \right) \omega t + \frac{1}{2} a_1(\vec{k})\omega^2 t^2 \right) e^{-i\omega t + ik \cdot \vec{x}} - c.c. \right],
\end{equation}
\begin{equation}
\phi_4(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega}} \left( \frac{N_1}{8\omega^6} \right) \left[ \left( a_4(\vec{k}) + \left( \frac{1}{2} a_1(\vec{k}) - \frac{i}{2} a_2(\vec{k}) - a_3(\vec{k}) \right) \omega t + \left( \frac{i}{2} a_1(\vec{k}) + \frac{1}{2} a_2(\vec{k}) \right) \omega^2 t^2 - \frac{1}{6} a_1(\vec{k})\omega^3 t^3 \right) e^{-i\omega t + ik \cdot \vec{x}} + c.c. \right],
\end{equation}
\begin{equation}
\phi_5(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega}} \left( \frac{iN_1}{16\omega^8} \right) \left[ \left( a_5(\vec{k}) + \left( \frac{5i}{8} a_1(\vec{k}) + \frac{1}{2} a_2(\vec{k}) - \frac{i}{2} a_3(\vec{k}) - a_4(\vec{k}) \right) \omega t + \left( -\frac{5}{8} a_1(\vec{k}) + \frac{i}{2} a_2(\vec{k}) + \frac{1}{2} a_3(\vec{k}) \right) \omega^2 t^2 - \left( \frac{i}{4} a_1(\vec{k}) + \frac{1}{6} a_2(\vec{k}) \right) \omega^3 t^3 \right] + \frac{1}{24} a_1(\vec{k})\omega^4 t^4 \right) e^{-i\omega t + ik \cdot \vec{x}} + c.c. \right]
\end{equation}
with five sets of Fourier modes \{a_i, a_i^\dagger\}. Here, we observe that \( \phi_i \sim t^{i-1} \) reflects the classical solution to a higher-order degenerate equation of \( (\Box - m^2)^i \phi_i = 0 \) and an inclusion of all previous modes in \( \phi_i \) represents the coupled nature of the second-order equation \[ (\Box - m^2)\phi_i = -\phi_{i-1} \] in Eq. \( \langle 1 \rangle \).

At this stage, it is appropriate to comment that we have iteratively found the solutions to \( \langle 1 \rangle \) up to \( n = 7 \) through the steps of \( \langle 22 \rangle \text{-} \langle 27 \rangle \). However, we encounter a difficulty to write them down in a compact way.

Similarly, introducing the ansatz for the \( n \) ghost fields
\begin{equation}
c_i(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega}} c_i(\vec{k}, t) e^{i\vec{k} \cdot \vec{x}},
\end{equation}
Eqs. \( \langle 13 \rangle \text{-} \langle 16 \rangle \) are reduced to one dimensional equations
\begin{equation}
\left( \frac{d^2}{dt^2} + \omega^2 \right) c_{i-1}(\vec{k}, t) = -c_i(\vec{k}, t), \quad i = 2, \ldots, k,
\end{equation}
\begin{equation}
\left( \frac{d^2}{dt^2} + \omega^2 \right) c_k(\vec{k}, t) = 0,
\end{equation}
\begin{equation}
\left( \frac{d^2}{dt^2} + \omega^2 \right) c_{i-1}(\vec{k}, t) = -c_i(\vec{k}, t), \quad i = k + 2, \ldots, 2k,
\end{equation}
\begin{equation}
\left( \frac{d^2}{dt^2} + \omega^2 \right) c_{2k}(\vec{k}, t) = 0.
\end{equation}
Corresponding to the solutions of the $n = 5$ coupled scalar field theory, we write down the first-four ghost solutions.

For $n = 2$ case,

\[
c_1(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega}} \left( -\frac{N_1}{8\omega^2} \right) \left( c_1(\mathbf{k})e^{-i\omega t + i\mathbf{k} \cdot \mathbf{x}} + c.c. \right) ,
\]
\[
c_2(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega}} (iN_1) \left( c_2(\mathbf{k})e^{-i\omega t + i\mathbf{k} \cdot \mathbf{x}} - c.c. \right) .
\]

For $n = 3$ case,

\[
c_1(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega}} \left( -\frac{iN_1}{4\omega^4} \right) \left( c_1(\mathbf{k})e^{-i\omega t + i\mathbf{k} \cdot \mathbf{x}} - c.c. \right) ,
\]
\[
c_2(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega}} (iN_1) \left( c_2(\mathbf{k})e^{-i\omega t + i\mathbf{k} \cdot \mathbf{x}} - c.c. \right) .
\]

For $n = 4$ case,

\[
c_1(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega}} \left( \frac{N_1}{8\omega^6} \right) \left[ (c_1(\mathbf{k}) - c_2(\mathbf{k})\omega t) e^{-i\omega t + i\mathbf{k} \cdot \mathbf{x}} + c.c. \right] ,
\]
\[
c_2(t) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega}} \left( -\frac{iN_1}{4\omega^4} \right) \left( c_2(\mathbf{k})e^{-i\omega t + i\mathbf{k} \cdot \mathbf{x}} - c.c. \right) ,
\]
\[
c_3(t) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega}} \left( -\frac{N_1}{2\omega^2} \right) \left[ (c_3(\mathbf{k}) - c_4(\mathbf{k})\omega t) e^{-i\omega t + i\mathbf{k} \cdot \mathbf{x}} + c.c. \right] ,
\]
\[
c_4(t) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega}} (iN_1) \left( c_4(\mathbf{k})e^{-i\omega t + i\mathbf{k} \cdot \mathbf{x}} - c.c. \right) .
\]

Finally, for $n = 5$ case,

\[
c_1(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega}} \left( \frac{iN_1}{16\omega^8} \right) \left[ (c_1(\mathbf{k}) - c_2(\mathbf{k})\omega t) e^{-i\omega t + i\mathbf{k} \cdot \mathbf{x}} - c.c. \right] ,
\]
\[
c_2(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega}} \left( -\frac{N_1}{8\omega^6} \right) \left( c_2(\mathbf{k})e^{-i\omega t + i\mathbf{k} \cdot \mathbf{x}} + c.c. \right) ,
\]
\[
c_3(t) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega}} \left( -\frac{N_1}{2\omega^2} \right) \left[ (c_3(\mathbf{k}) - c_4(\mathbf{k})\omega t) e^{-i\omega t + i\mathbf{k} \cdot \mathbf{x}} + c.c. \right] ,
\]
\[
c_4(t) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega}} (iN_1) \left( c_4(\mathbf{k})e^{-i\omega t + i\mathbf{k} \cdot \mathbf{x}} - c.c. \right) .
\]

Here we distinguish number of ghost modes between 2 for $n = 2, 3$ and 4 for $n = 4, 5$. With these, we obtain the BRST transformation for all modes $a_i(\mathbf{k})$ and $c_i(\mathbf{k})$ as

\[
\delta a_1(\mathbf{k}) = 0, \quad \cdots, \quad \delta a_k(\mathbf{k}) = 0, \quad \delta a_{k+1}(\mathbf{k}) = c_k(\mathbf{k}), \quad \cdots, \quad \delta a_{2k}(\mathbf{k}) = c_1(\mathbf{k}),
\]
\[
\delta c_1(\mathbf{k}) = 0, \quad \cdots, \quad \delta c_k(\mathbf{k}) = 0, \quad \delta c_{k+1}(\mathbf{k}) = a_k(\mathbf{k}), \quad \cdots, \quad \delta c_{2k}(\mathbf{k}) = a_1(\mathbf{k}),
\]
for the even \((n = 2k)\) case, while
\[
\begin{align*}
\delta a_1(\vec{k}) &= 0, \cdots, \delta a_k(\vec{k}) = 0, \delta a_{k+1}(\vec{k}) = 0, \delta a_{k+2}(\vec{k}) = c_k(\vec{k}), \cdots, \\
\delta c_1(\vec{k}) &= 0, \cdots, \delta c_k(\vec{k}) = 0, \delta c_{k+1}(\vec{k}) = a_k(\vec{k}), \cdots,
\end{align*}
\]
(51)
for the odd \((n = 2k + 1)\) case. Note from Eq. (51) that the mode \(a_{k+1}(\vec{k})\) is invariant under the BRST transformation. In order to see the role of remaining modes, we have to compute all commutators between the modes.

**IV. GENERAL QUARTET MECHANISM**

After a tedious computation, we derive the first-four commutation relations between \(A_a\) and \(A_b^\dagger\) where \(A_a\ (A_b^\dagger)\) denotes the set of the modes \(\{a_i(\vec{k}), c_j(\vec{k})\}\) \((\{a_i^\dagger(\vec{k}), c_j^\dagger(\vec{k})\}\) with \(i = 1, \cdots, n, j = 1, \cdots, n\) for even \(n\) and \(i = 1, \cdots, n, j = 1, \cdots, n - 1\) for odd \(n\).

For \(n = 2\), one has
\[
\begin{aligned}
[A_a, A_b^\dagger]_x &= \frac{2\omega^2}{N_1^2} \begin{pmatrix}
\begin{array}{cccc}
a_2^\dagger & a_1^\dagger & c_1^\dagger & c_2^\dagger \\
0 & i & 0 & 0 \\
0 & -i & -1 & 0 \\
0 & 0 & 0 & -i \\
0 & 0 & -i & 0
\end{array}
\end{pmatrix} \delta^3(\vec{k} - \vec{k}'),
\end{aligned}
\]
(52)
which form a quartet to give the zero norm state. This was designed for a dipole ghost pair for the singleton \([4, 10, 19]\). Note that the subscripts \(-\) \((+)\) denote the commutator (anti-commutator) for the bosonic (fermionic) fields. On the other hand, commutators between bosonic and fermionic fields vanish.

For \(n = 3\) in Ref. [18], the commutators take the forms
\[
\begin{aligned}
[A_a, A_b^\dagger]_x &= \frac{4\omega^4}{N_1^2} \begin{pmatrix}
\begin{array}{cccc}
a_2^\dagger & a_1^\dagger & a_3^\dagger & c_1^\dagger & c_2^\dagger \\
1 & 0 & -i & 0 & 0 \\
0 & 0 & -1 & 0 & 0 \\
i & -1 & \frac{3}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & 0
\end{array}
\end{pmatrix} \delta^3(\vec{k} - \vec{k}'),
\end{aligned}
\]
(53)
which shows that \([a_2(\vec{k}), a_3^\dagger(\vec{k}')]_− = \frac{4a^-}{N_1} \delta^3(\vec{k} − \vec{k}')\) defines a physical commutator, while the remaining four modes form the quartet to give the zero norm state. The factor of \(3/2\) in \([a_3(\vec{k}), a_3^\dagger(\vec{k}')]_−\) represents the higher-derivative nature of \((\Box − m^2)^3\phi_3 = 0\). This corresponds to the first case for having a physical subspace without the non-unitarity.

For \(n = 4\), we have

\[
[A_a, A_b^\dagger]_+= \frac{8\omega^6}{N_1^2} \begin{pmatrix}
0 & 0 & 0 & -i & 0 & 0 & 0 & 0 \\
0 & 0 & i & 1 & 0 & 0 & 0 & 0 \\
0 & -i & -1 & \frac{3i}{2} & 0 & 0 & 0 & 0 \\
i & 1 & -\frac{3i}{2} & -\frac{5}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & i \\
0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \\
0 & 0 & 0 & 0 & -1 & -i & 0 & 0 \\
0 & 0 & 0 & i & 0 & 0 & 0 & 0
\end{pmatrix} \delta^3(\vec{k} − \vec{k}'),
\]

which form an octet to give the zero norm state, leaving the vacuum. The factor of \(-5/2\) in \([a_4(\vec{k}), a_4^\dagger(\vec{k}')]_−\) denotes the higher-derivative nature of \((\Box − m^2)^4\phi_4 = 0\).

For \(n = 5\), we have

\[
[A_a, A_b^\dagger]_+= \frac{16\omega^8}{N_1^2} \begin{pmatrix}
1 & 0 & 0 & -i & \frac{3i}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
i & 0 & -1 & \frac{3}{2} & -\frac{5}{2} & 0 & 0 & 0 & 0 \\
-3i & 1 & -\frac{5}{2} & \frac{35}{8} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0
\end{pmatrix} \delta^3(\vec{k} − \vec{k}'),
\]

which indicates that \([a_3(\vec{k}), a_3^\dagger(\vec{k})]_-= \frac{16\omega^8}{N_1} \delta^3(\vec{k} − \vec{k}')\) defines a physical commutation relation, whereas the remaining eight modes form an octet to give the zero norm state. Here we observe that for odd \(n\), there exists only one physical commutator of \([a_2(\vec{k}), a_2^\dagger(\vec{k}')]_−\) for \(n = 3\) and \([a_3(\vec{k}), a_3^\dagger(\vec{k})]_−\) for \(n = 5\) which show that they describe the states with positive
norm, while for even $n$, all commutators belong to a quartet for $n = 2$ and an octet for $n = 4$ which indicate unphysical modes. A factor $35/8$ in $[a_5(k), a_5^\dagger(k')]_-$ reflects the higher-derivative nature of $(\Box - m^2)^5\phi_5 = 0$.

Inductively, we insist that there remains a physical subspace with positive norm for odd $n$, but there exists the vacuum only for even $n$.

Finally, Ref. [10] has stated that all higher derivative scalar theories are trivial after performing the BRST quantization. However, looking into his model equation of (1) closely, it describes even power of higher derivative operators only and thus, his model hits our even $n$ case, leaving the odd $n$ untouched.

V. GENERAL QUARTET MECHANISM AND TRUNCATION MECHANISM

In this section we compare the general quartet mechanism in Minkowski space with the truncation mechanism in the AdS/CFT correspondence.

Before we proceed, the authors [17] have discussed the specific case of non-linear critical gravity of rank-3 in AdS$_3$ and AdS$_4$ spacetimes with the result that truncations that appear to be unitary at the linearized level may be inconsistent at the non-linear level. The argument given there seems to extend to the general case independently of how the linearized theory is completed. This suggests that the unitary subsector might exist only in the linearized approximation.

Here we use the same bilinear action (1), but the difference is the background spacetime: the $n$ coupled scalar theory in Minkowski and the $n$ coupled scalar theory (a toy model of the polycritical gravity) on AdS$_3$ spacetime.

The truncation mechanism without FP ghosts was used to resolve the non-unitarity in the LCFT, dual to the fourth-order critical gravity on AdS$_3$ spacetime [15, 16]. A rank $n$ of the LCFT refers to the dimensionality of the Jordan cell on the boundary, while it represents the $2n$-order polycritical gravity on the bulk side. Explicitly, the two-point correlation functions
in the rank-\(n\) LCFT are given by

\[
\langle \mathcal{O}^i \mathcal{O}^j \rangle \sim \begin{pmatrix}
0 & \cdots & 0 & 0 & \text{CFT} \\
0 & \cdots & 0 & 0 & \text{CFT} & L \\
0 & \cdots & 0 & \text{CFT} & L & L^2 \\
\vdots & \ddots & L & L^2 & L^3 \\
\vdots & \ddots & \ddots & \ddots & \ddots & L^4 \\
\vdots & & & & & \\
\end{pmatrix},
\tag{56}
\]

where \(i, j = \text{KG}, \log, \log^2, \cdots\). KG represents the Klein-Gordon correlation function, CFT denotes the CFT correlation function, L represents log-correlation function, and \(L^2\) is log\(^2\)-correlation function, etc. For example, the LCFT dual to a fourth-order critical gravity has a rank-2 Jordan cell and thus, an operator has a log-mode as a logarithmic partner. For a \(2 \times 2\) submatrix of the top-right \((56)\), one may truncate out L by imposing the AdS boundary conditions to avoid the non-unitarity. After truncation of the rank-2 LCFT, there remains nothing (0) for the unitary CFT. This is also the case for all even rank-\(n\) LCFTs. On the other hand, the LCFT dual to a sixth-order tricritical gravity has a rank-3 Jordan cell and an operator has two logarithmic partners. For a \(3 \times 3\) submatrix of the top-right \((56)\), we throw away all correlation functions which generate the third column and row of this matrix. Hence the non-zero correlation functions is proportional to the unitary CFT correlation function. Actually, a truncation may allow an odd rank-\(n\) LCFT to be a unitary CFT, while all remaining correlators of an even rank-\(n\) LCFT vanish and the theory contains null states after the truncation.

For the \(n\) coupled scalar field theory without the FP ghost modes in Minkowski spacetime,
the commutation relations can be recast into the following matrix form:

\[
\begin{pmatrix}
 a_1^\dagger & a_2^\dagger & \cdots & a_{n-3}^\dagger & a_{n-2}^\dagger & a_{n-1}^\dagger & a_n^\dagger \\
 a_1 & 0 & 0 & 0 & 0 & 0 & -1 \\
a_2 & 0 & 0 & 0 & 0 & 1 & -i \\
 \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
 \vdots & \vdots & \ddots & 0 & -1 & i & \frac{3}{2} \\
 \vdots & \vdots & \ddots & \ddots & \ddots & \frac{35}{8} & -\frac{63}{8}i
\end{pmatrix}
\]

where \( \alpha_n = \frac{(iN)^n}{(2\omega^2)^{n-1}} \) is the coefficient of Eqs. (28)-(32), etc. Roughly, the correlators in (56) are replaced by commutators in (57). Also, we observe that (56) and (57) have the same lower triangular matrix. For a \( 2 \times 2 \) submatrix of the top-right (57), one may truncate out the second column and row by hand to avoid the non-unitarity. After truncation, there remains nothing (0) for a unitary scalar theory. This is also the case for all even \( n \) coupled scalar theories. On the other hand, for a \( 3 \times 3 \) submatrix of the top-right (57), we throw away all modes which generate the third column and row of this matrix. Hence the only non-zero commutator is \([a_2, a_2^\dagger]_- = 1\) for \( \alpha_1 = 1/\alpha_3 \). Actually, a truncation allows an odd \( n \) coupled scalar theory to be a unitary scalar theory with positive norm states, while all commutators of an even \( n \) coupled scalar field theory vanish and the theory contains null states after truncation.

Hence, it is evident from the above that without the FP ghosts, there is no consistent way to remove the ghost states which arise from the higher derivative theories of \( n \) coupled scalar and polycritical gravity theories. Noting that our action (1) without the FP ghosts on AdS_3 is dual to the rank-\( n \) LCFT [12, 13], it is not enough to find a unitary CFT consistently. We need the \( n \) coupled scalar action (1) as well as its FP action (7) to confine all unphysical fields to the zero norm state, arriving at the unitary scalar theory with positive norm states. Finally, we insist that the truncation mechanism is nothing but a general quartet mechanism when including the FP ghost action. As was pointed out previously, the truncation mechanism is valid for the linearized theory [17].

\[
\delta^3(\mathbf{k} - \mathbf{k}'), \quad (57)
\]
VI. SUMMERY AND DISCUSSIONS

We first summarize our main results.

- We have considered the degenerate \( n \) coupled scalar field theory (1). For non-degenerate case with different masses for \( W_{ij} \) in (8) and (9), we could not find the BRST invariant action because the BRST symmetry is not nilpotent. This implies that a higher-derivative Lee-Wick model [20, 21] is not suitable for a consistent quantized scalar model, even though it shows clearly which one has positive (negative) norm states.

- The \( n = 2 \) corresponds to a dipole ghost field for the singleton [19, 22, 23]. They form a quartet to give the zero norm state when including the FP ghost action, leaving the vacuum only.

- The \( n = 3 \) case is enough to have a physical subspace with positive norm states. This implies that the six-order derivative theory [18] provides a physical scalar field. No higher than \( n = 3 \) coupled scalar theory is necessary to give a unitary scalar theory.

- Without the FP ghost action, we could not obtain the consistent truncation mechanism. This is why we have constructed the FP ghost action (7) which has non-trivial terms for \( n > 3 \) when comparing the known cases of \( n \leq 3 \).

- The truncation mechanism becomes the general quartet mechanism when introducing FP ghost action. The truncation mechanism works for the boundary CFT theory via the AdS/CFT correspondence, while the general quartet mechanism works for the bulk theory of the \( n \) coupled scalar theory in Minkowski spacetime. In this sense, the general quartet mechanism is dual to the truncation mechanism.

- The physical field is given by the \( \phi_{k+1} \) for the odd \( n \) coupled scalar theory. This implies that even though \( \phi_n \) satisfies \( (\Box - m^2)^n \phi_n = 0 \) and \( \phi_l \) are regarded as auxiliary fields in the classical aspect, \( \{\phi_n, \phi_l\} \) are treated equally as scalar fields in the quantum aspect. A centered field \( \phi_{k+1} \) between \( \phi_1 \) and \( \phi_n \) is considered as a physical field with positive norm state in the odd \( n \) coupled scalar field theory.

- We need to introduce the higher-order FP ghosts when quantizing the higher-order derivative gauge and gravity theory in addition to the gauge FP ghosts.

Finally, we wish to mention our implications to quantum gravity. Even though our model is a non-interacting scalar field model in Minkowski spacetime, the similar statements could be made for interacting spin-2 models. In the three-dimensional AdS gravity theory, the
most general Einstein-Hilbert action is given by

\[ I_{3D\text{AdS}} = \frac{1}{16\pi G} \int d^3x \sqrt{-g} \left[ \sigma R - 2\Lambda_0 + \alpha R^2 + \beta R^\mu{}^\nu R_{\mu\nu} + \mathcal{L}_{\Box R} \right], \tag{58} \]

with a sixth-derivative combination

\[ \mathcal{L}_{\Box R} = b_1 \nabla_\mu R \nabla^\mu R + b_2 \nabla_\rho R_{\mu\nu} \nabla^\rho R^{\mu\nu}. \tag{59} \]

Imposing the condition of avoiding scalar gravitons,

\[ b_1 = -\frac{3}{8} b_2, \quad \alpha = \frac{\Lambda}{8} b_2 - \frac{3}{8} \beta, \quad \Lambda = -\frac{1}{\ell^2} \tag{60} \]

we find the parity-even tricritical (PET) gravity which is proposed as a promising model of quantum gravity. At the tricritical point of \( \beta = -4\sigma/\Lambda \) and \( b_2 = -2\sigma/\Lambda^2 \), its linearized equation takes the form

\[ \mathcal{G}_{\mu\nu}(\mathcal{G}(h)) = 0, \tag{61} \]

where the gauge-fixed linearized Einstein tensor is given by

\[ \mathcal{G}_{\mu\nu} = -\frac{1}{2}(\Box - 2\Lambda) h_{\mu\nu}. \tag{62} \]

We note that tensor equation (61) is similar to the \( n = 3 \) scalar equation of \((\Box - m^2)^3 \phi_3 = 0\) when replacing \( 2\Lambda \) by \( m^2 \). It shows that the \( n = 3 \) coupled scalar theory is a toy model of (58). Accordingly, it is possible to reformulate the PET gravity as a two-derivative tensor theory upon introducing two auxiliary fields \( f_{\mu\nu} \) and \( \lambda_{\mu\nu} \). This model is promising because it will be a ghost-free theory if one introduces gauge FP ghosts for the metric perturbation \( h_{\mu\nu} \) and two higher-order FP ghosts for two auxiliary perturbations \( k_{2\mu\nu} \) and \( k_{1\mu\nu} \) to lower higher-derivative terms in the bilinear action of (58). Considering the connection between \( (\phi_1, \phi_2, \phi_3) \) in the \( n = 3 \) coupled scalar theory and \( (k_{2\mu\nu}, k_{1\mu\nu}, h_{\mu\nu}) \) in the PET gravity, it conjectures that a physical tensor would be \( k_{2\mu\nu} \) with positive norm states. However, we need more time to find a quantum gravity model in Minkowski spacetime because the tricritical gravity is still unknown in Minkowski spacetime.

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