On quantum interferometric measurements of temperature

Marcin Jarzyna∗ and Marcin Zwierz†
Faculty of Physics, University of Warsaw, Pasteura 5, 02-093 Warsaw, Poland
(Dated: 19th December 2014)

We provide a detailed description of the interferometric thermometer, which is a device that estimates the temperature of a sample from measurements of the optical phase. For the first time, we rigorously analyze the operation of such a device by studying the interaction of the optical probe system with a heated sample. We find that this approach to thermometry is capable of measuring the temperature of a sample in the nK regime. Furthermore, we compare the theoretical precision of interferometric thermometers with the precision offered by the idealized pyrometers, which infer the temperature from a measurement of the total thermal radiation emitted by the sample.

PACS numbers: 07.20.Dt, 06.20.-f, 42.50.St, 42.50.Lc

I. INTRODUCTION

Temperature is one of the fundamental and arguably one of the most frequently measured physical quantities. Apart from the central role played by temperature in the fields of thermodynamics and statistical physics, precise temperature measurements are important for all branches of modern science and technology. Indeed, precise knowledge of temperature of a sample proved indispensable for many advancements in physics, biology, chemistry, atmospheric sciences, but also in material science and microelectronic industry. In this paper, we study the classical and quantum limitations that constrain the precision with which we can measure temperature. These limitations are imposed by the combination of the laws of statistics, statistical physics, and quantum mechanics.

The theoretical limit on the precision of classical thermometers is known as the standard quantum limit or the shot-noise limit \( \Delta T \sim \frac{c_{\text{SQL}}}{\sqrt{\bar{N}}} \), where \( c_{\text{SQL}} \) is a constant depending on the properties of the thermometer and the sample, and \( \bar{N} \) is the number of resources, which for the classical thermometers, typically, reduces to the (mean) number of uncorrelated particles that make up the thermometer or the time it takes to make a measurement [1]. The standard quantum limit predicts that in order to measure the temperature of a sample with high precision we need \( \bar{N} \) to be large, which translates to enlarging our classical thermometer. A typical measurement consists of bringing the thermometer into a physical contact with the sample and then letting the two systems to thermalize. However, if we wish to use a large, and thus a highly precise thermometer, then this approach is not optimal as it will, in most cases, significantly disturb the temperature of the sample we are probing. Fortunately, there is a way of avoiding this problem. Namely, we could measure the temperature without putting our thermometer and the sample in a direct contact. In such a case the sample of interest is observed remotely. There are many different techniques of the so-called non-invasive or non-contact thermometry that are used across many branches of science and industry. Most noninvasive techniques infer temperature from the electromagnetic spectrum. One of the most common optical techniques that is used for temperature measurements is the measurement of the thermal infrared radiation naturally emitted by all heated samples [2, 3]. This is exactly the principle used in the commercially available pyrometers.

On the other hand, recent years have witnessed a growing interest in applying various ideas taking advantage of quantum mechanical features of nature to the problem of temperature measurements [1, 4–9]. Specifically, in Ref. [1], it was shown that it is possible to map the problem of measuring the temperature onto the problem of estimating an unknown phase. Moreover, it was further shown that using the interferometric techniques of phase estimation theory the classical standard-quantum-limited precision in temperature estimation may be improved to the so-called Heisenberg limit \( \Delta T \sim \frac{c_{\text{HL}}}{\bar{N}} \), where \( c_{\text{HL}} \) is of the order of unity and \( \bar{N} \) is the number of resources, which now usually denotes the (mean) number of correlated particles that make up the thermometer. Unfortunately, this improvement occurs only for measurements in the absence of decoherence; typically, the presence of noise reduces the Heisenberg-limited precision back to the classical shot-noise-limited scaling \( \Delta T \sim \frac{c}{\sqrt{\bar{N}}} \) with a possible advantage constrained to the scaling constant \( c \),...
where \( c < c_{\text{SQL}} \). Nevertheless, the prospect of using interferometric tools of quantum-enhanced phase estimation for temperature measurements is rather intriguing, and we believe, has not been fully explored.

The main advantages of interferometric thermometry are very fast response time of interferometric devices, which means that rapid variations in temperature can be measured, the ability to measure a wide range of temperatures and the high spacial resolution allowing for temperature measurements of micrometer and even nanometer size spots. When combined in a single measurement setup, these advantages open up a possibility of preparing temperature maps with very high spacial and temporal resolutions that could find a wide range of important applications in the microelectronic industry, and in material and life sciences. Surprisingly, in spite of those most immediate benefits interferometric thermometry met with limited attention in the past. The interferometric thermometry was used mostly in the microelectronic industry for temperature measurements of the semiconductor electronic devices; in industry, this approach to thermometry is used not only for local temperature measurements but also as a kind of a scanning device for measuring the thermal expansion coefficient. In academia, apart from the theoretical proposal discussed above, the idea of interferometric thermometry realized with atomic quantum dots was most recently employed to study temperature measurements of the Bose-Einstein condensates.

To the best of our knowledge, this is the first work that rigorously studies the interferometric approach to thermometry. We determine the fundamental precision of optical interferometric thermometers. To this end, we use the single-mode Gaussian states of light to probe the temperature variations in a general medium modeled as a thermal dissipative reservoir. We then compare our precision limits against the theoretical limit on the precision offered by commonly used classical noninvasive temperature sensors the so-called pyrometers. This comparison is provided for one of the most important classes of materials used in many quantum technologies, namely, the non-linear crystals. A precise sensing of tiny temperature variations followed by an active temperature stabilization is required in many experimental setups based on the non-linear crystals such as the PPKTP crystal. The non-linear crystals are used as a source of indistinguishable photons generated via the spontaneous parametric down-conversion process and the quality of those photons depends strongly on the temperature of the crystal. Hence, new practical methods for ultra-precise temperature sensing and stabilization, which could be easily incorporated into the existing experimental setups are of the paramount importance for the development of various optical quantum technologies that rely heavily on the bright sources of a truly indistinguishable photons. The interferometric approach to thermometry that we study seems ideal for such applications.

The paper is organized as follows. In Sec. II we introduce the basic concepts of the classical and quantum estimation theory. In Sec. III we derive the fundamental precision limits for the idealized classical noninvasive pyrometers that we use as a benchmark to judge the usefulness of the interferometric thermometry. Sec. IV introduces in detail the idea of interferometric thermometry. In Sec. V we find the precision of temperature estimation obtained via the interferometric phase measurement in the presence of a thermal dissipative reservoir focusing on temperature measurements in non-linear crystals. In Sec. VI we conclude with final remarks.

II. CLASSICAL AND QUANTUM ESTIMATION THEORY

The task of temperature measurement can be translated using the language of estimation theory to the problem of parameter estimation. The generic scheme for such a problem is depicted in Fig. 1. In order to estimate the value of parameter \( \theta \) we send a probe system prepared in an initial quantum state \( \rho_0 \) through a sample, which depends on \( \theta \). The interaction between our probe system and the sample can then be modeled as an evolution under a quantum channel \( \Lambda_\theta \). Hence, following the interaction the
probe system is left in the output quantum state \( \rho_\theta = \Lambda_\theta [\rho_0] \). Next, we subject the probe system to a general quantum measurement, described by a POVM \( \{ \Pi_x \} \), which outputs measurement results \( x \). Finally, we use a special function \( \hat{\theta}(x) \) called an estimator to calculate the estimated value of \( \theta \). This scheme describes not only the quantum estimation tasks but also the classical ones.

The procedure described above naturally never returns the true value of \( \theta \). We quantify the discrepancy between the estimated value and the true value of \( \theta \) with the root-mean-square error \( \Delta \theta = \sqrt{\langle (\hat{\theta}(x) - \theta)^2 \rangle} \), where the average is taken with respect to the probability distribution \( p(x|\theta) \) of the measurement outcomes \( x \). In the most common case when we have a specific POVM measurement \( \{ \Pi_L \} \), corresponding to an observable \( A \), and we wish to estimate the value of \( \theta \) from the mean value of \( A \), the precision is given by a standard error propagation formula

\[
\Delta \theta = \frac{\Delta A}{|d(A)/d \theta|},
\]

where \( \Delta A \) is the standard deviation of the observable \( A \) calculated for the output state \( \rho_\theta \). We remind the reader that \( A \) may as well be a classical observable.

The above equation is valid for any observable but it explicitly assumes an estimation from the mean value of \( A \). It may happen that this is not the optimal approach and we could improve the precision by choosing another estimator. Moreover, it may also happen that the measurement we choose is not optimal either. In order to avoid such problems we need to optimize Eq. (1) over all possible measurements and estimators, which is not an easy task. Fortunately, the solution to this optimization problem is provided by the classical and quantum Cramér-Rao inequalities [16][18], which state that for any unbiased estimator

\[
\Delta \theta \geq \frac{1}{\sqrt{k F_\theta}} \geq \frac{1}{\sqrt{k Q_\theta}},
\]

where \( k \) is the number of independent experimental repetitions and \( F_\theta \) is the classical Fisher information defined as

\[ F_\theta = \sum_x \frac{1}{p(x)} \left[ \frac{\partial p(x)}{\partial \theta} \right]^2. \]

The classical Fisher information gives a lower bound on the precision optimized over all unbiased estimators for a specific POVM measurement \( \{ \Pi_L \} \). Further optimization over all conceivable measurements gives a lower bound on the precision expressed via the quantum Fisher information (QFI) \( Q_\theta \), which can be computed from \( Q_\theta = \text{Tr}[\rho_\theta L_\theta^2] \), where \( L_\theta \) is a Hermitian operator called the symmetric logarithmic derivative (SLD) that is implicitly defined via

\[ \frac{d \rho_\theta}{d \theta} = \rho_\theta \circ L_\theta, \]

where we used the notation of the symmetric product for operators, \( A \circ B = \frac{1}{2}(AB + BA) \). In a situation, where the parameter is encoded by a unitary transformation, i.e., \( \rho_\theta = \Lambda_\theta [\rho_0] = U_\theta \Lambda [\rho_0] U_\theta^\dagger \), where \( U_\theta \) is unitary and \( \Lambda \) is a \( \theta \)-independent quantum channel describing possible decoherence processes, then \( L_\theta = U_\theta L_0 U_\theta^\dagger \), where \( L_0 \) is the SLD for the state \( \Lambda [\rho_0] \). In such a case the QFI does not depend on \( \theta \), that is, \( Q_\theta = Q \).

The quantum Cramér-Rao bound (QCRB) and its classical counterpart are known to be saturable in the limit of large number of repetitions \( k \to \infty \). The optimal measurement consists of a set of projectors on the eigenbasis of the SLD and outputs the measurement results, which are then processed with the maximum likelihood estimator. Such projective measurements are typically very hard to implement reliably in the laboratory, however, very often it is possible to saturate the quantum Cramér-Rao bound with a more natural set of measurements, for example, with the optical homodyne or the parity detection [19][21]. Since the QCRB is already optimized over all possible POVMs and unbiased estimators, we are usually only left with the optimization over all possible input quantum states \( \rho_0 \).

### III. NONINVASIVE CLASSICAL THERMOMETERS

As we have already mentioned, one of the most widely used optical noninvasive temperature sensors relies on the measurement of thermal infrared radiation [2][3]. This measurement principle is used in the commercially available pyrometers. To be more specific, these devices measure the flux of thermal radiation coming from the sample and using the well-known Stefan-Boltzmann law infer the temperature [2][3]. What is the fundamental precision of pyrometers? Assuming for simplicity that such devices are sensitive to all frequencies of electromagnetic radiation [22] and excluding all sources of loss in the detection process, the error propagation formula in Eq. (1) dictates the following theoretical limit on the precision in temperature estimation (for the deriva-
The classical Fisher information for thermal state allows us to estimate the photon number probability distribution measurement. This is a measurement that allows us to estimate the temperature of a sample using the classical Fisher information for a photon-number resolved device. Therefore, it is always optimal to measure the mean number of photons at the output for each frequency as we have assumed in the derivation of Eq. (10) or simply measure the total flux of thermal radiation.

\[
\Delta T = \sqrt{\frac{k_B}{4\sigma S\delta tT}} \quad \text{with} \quad \sigma = \frac{\pi^2 k_B^4}{60\hbar^2 c^2},
\]

where \(\sigma\) is the Stefan-Boltzmann constant, \(S\) is the surface area of the sample we are probing, which we assume is optimized to be equal to the size of the detector area used in our pyrometer, and \(\delta t\) is the response time of the device, that is, the time it takes to make a measurement. This result has a standard-quantum-limited scaling with respect to the response time \(\delta t\), which here corresponds to the amount of resources \(N\) \cite{23}; therefore, the longer we probe the sample the more photons the detector registers. Modern pyrometers operate in the ms regime, hence, we assume \(\delta t = 10\,\text{ms}\). Now further assuming that we measure the temperature of a surface with \(S = 1\,\text{cm}^2\), we find that the local precision of such a measurement near \(T = 298\,\text{K}\) is \(\Delta T \approx 0.452\,\mu\text{K}\). Therefore, at the fundamental level pyrometers are highly precise devices. Naturally, a real commercially available pyrometer estimates temperature with a much lower precision \cite{24}.

This, however, is not the only way we could use thermal light to infer the temperature of a sample. We could, for example, imagine a more sophisticated device that estimates temperature based on the detailed knowledge of the spectral photon number distribution of thermal radiation. Such a device would measure the photon number probability distribution for each frequency. This kind of measurement would provide more information about the character of the thermal light, and therefore, should allow for improved temperature estimation. In order to calculate this enhanced precision we need to find the classical Fisher information for a photon-number resolving measurement. This is a measurement that allows us to estimate the photon number probability distribution \(p(n)\), which enters into the definition of the thermal state

\[
\rho_N = \sum_{n=0}^{\infty} p(n)\ket{n}\bra{n} \quad \text{with} \quad p(n) = \frac{N^n}{(1 + N)^{n+1}},
\]

where \(N\) is the mean number of photons present in the thermal state distributed according to the Bose-Einstein statistics \cite{25}, as

\[
N = \left[ \exp \left( \frac{\hbar\omega}{k_B T} \right) - 1 \right]^{-1}.
\]

The classical Fisher information for \(p(n)\) for a single frequency \(\omega\) is

\[
F_T(\omega) = \left( \frac{\hbar\omega}{k_B T^2} \right)^2 N(N + 1).
\]

Surprisingly, when we calculate the total Fisher information per unit area for all frequencies

\[
F_T = \int_0^\infty F_T(\omega)\varphi(\omega)d\omega,
\]

where \(\varphi(\omega) = \omega^2/(4\pi^2 c^2)\) is the density of modes per unit interval in \(\omega\) in a unit area, we obtain via the classical Cramér-Rao inequality Eq. (2) the very same expression as in Eq. (5), that is,

\[
\Delta T \geq \frac{1}{\sqrt{MSF_T}} = \sqrt{\frac{k_B}{4\sigma S\delta tT}},
\]

This result means that inferring temperature from the total flux of thermal radiation is already optimal. Any additional knowledge of the photon number distribution does not contribute to the information about temperature. Furthermore, as can be easily checked, the calculation of the QFI for \(\rho_N\) returns the same result because thermal state is diagonal in the Fock basis and so is the corresponding SLD. Therefore, it is always optimal to measure the mean number of photons at the output for each frequency as we have assumed in the derivation of Eq. (10) or simply measure the total flux of thermal radiation.

IV. QUANTUM INTERFEROMETRIC THERMOMETERS

Having explained how the noninvasive classical thermometers work, we now describe in detail the operation of interferometric thermometers. It may be somewhat confusing to consider the interferometric thermometry as noninvasive or non-contacting since as it will become apparent soon the photons in the probe beam are clearly interacting with the sample. However, as long as this interaction does not cause a significant change in sample’s temperature we can consider this method as truly non-contacting \cite{3}. The scope of this work is to investigate a possible advantage offered by such interferometric devices that typically rely on the use of non-classical states of light. The basic scheme for a quantum interferometric thermometer is depicted in Fig. (2). A single-mode Gaussian state of light prepared in an initial state \(\rho_0\) propagates through a sample with temperature \(T\) and transmissivity \(\eta\), resulting in a mixed output state \(\rho_\varphi\). Following the propagation stage, a general quantum measurement described by a POVM

---

\[\text{Appendix A}\]
A single-mode Gaussian state of light $\rho$ is sent through a sample with temperature $T$ and transmissivity $\eta$, resulting in a mixed output state $\rho_x$, which is then measured using a general POVM $\{\Pi_x\}$ measurement. Assuming that the phase shift $\varphi$ is temperature dependent, the measurement outcomes are then used to find an estimated value of $T$ via the estimator function $\tilde{T}(x)$. The bright yellow beam depicts a classical reference, which allows us to define the phase shift in a meaningful way. The propagation of light through a heated sample with transmissivity $\eta$ can be decomposed into two distinct processes: first the single-mode Gaussian beam undergoes the phase shift $\varphi$ relative to the reference beam and then it undergoes a photon loss process, which is modeled with a help of a beam splitter with transmissivity $\eta$, where the second port is filled with light in the thermal state $\rho_N$.

The mean number of photons $\bar{N}$ in such an input Gaussian state is given by

$$\bar{N} = \frac{1}{2} \left[ \left( N_0 + \frac{1}{2} \right) \cosh 2r_0 + x_0^2 + p_0^2 - 1 \right].$$

In the next section we describe how the propagation through a sample with temperature $T$ and transmissivity $\eta$ affects the above single-mode Gaussian state.

### B. Evolution and the output state

The evolution of an arbitrary state $\rho$ in the presence of a thermal Gaussian dissipative reservoir is described by the following master equation

$$\frac{d\rho}{dt} = \mathcal{G}(\omega, N, \Gamma)\rho$$

with the superoperator $\mathcal{G}(\omega, N, \Gamma)$ defined as

$$\mathcal{G}(\omega, N, \Gamma) = -i\omega H + \frac{\Gamma}{2} (NL[a] + (N+1)L[a^\dagger]).$$
where \( L[\rho] = 2a^\dagger a \rho - \rho a^\dagger a - \rho a a^\dagger \) and \( H \rho = [a^\dagger a, \rho] \). The first term in the superoperator \( \mathcal{G} \) describes a free unitary evolution of a single bosonic mode \( a \) with frequency \( \omega \). [Naturally, the frequency appearing in the definition of \( N \) in Eq. (7) is equal to the frequency \( \omega \) of the incident Gaussian light.] The second term accounts for a coupling of the bosonic mode to a thermal reservoir (with mean photon number \( N \)) with strength \( \Gamma \) [29].

According to the above master equation, after a time \( t \), an initial state \( \rho_0 \) evolves to \( \rho_\varphi = \exp[\mathcal{G}(\omega, N, \Gamma) t] \rho_0 = S(\varphi, N, \gamma) \rho_0 \), where \( S(\varphi, N, \gamma) = \exp[\mathcal{G}(\varphi, N, \gamma)] \) and here \( \gamma = \Gamma t \) is the effective coupling to a thermal reservoir (which is related to the photon loss coefficient \( \eta \) via the relation \( \eta = e^{-\gamma} \) and \( \varphi = \omega t \) is the optical phase that we are interested in estimating.

Now, when the initial state \( \rho_0 \) corresponds to our initial single-mode Gaussian state then following the above evolution, the output state \( \rho_\varphi \) is still a single-mode Gaussian state but with the changed first and second moments. By transforming the master equation in (15) into a partial differential equation, the so-called Fokker-Planck-type equation, for the Wigner quasi-probability distribution \( W(\alpha, \alpha^*, t) \) [25]

\[
\frac{\partial W}{\partial t} = \left( \frac{\gamma}{2} + i \varphi \right) \frac{\partial}{\partial \alpha} [\alpha W] + \left( \frac{\gamma}{2} - i \varphi \right) \frac{\partial}{\partial \alpha^*} [\alpha^* W] + \frac{\gamma}{2} \left( N + \frac{1}{2} \right) \frac{\partial^2 W}{\partial \alpha \partial \alpha^*}
\]

(17)

and then solving this equation for the complex variables \( \alpha \) and \( \alpha^* \) [25] we can show that the first moments of the output state \( \rho_\varphi \) are given by

\[
\bar{x} = \sqrt{\eta} (\cos \varphi \bar{x}_0 + \sin \varphi \bar{p}_0),
\]

(18)

\[
\bar{p} = \sqrt{\eta} (-\sin \varphi \bar{x}_0 + \cos \varphi \bar{p}_0).
\]

(19)

The second moments of the output state are given by the covariance matrix

\[
\Sigma = \Sigma_\eta (\Sigma_\varphi - \Sigma_N) \Sigma_\eta + \Sigma_N,
\]

(20)

where \( \Sigma_\eta = \sqrt{\eta} I \), \( \Sigma_N = (N + \frac{1}{2}) I \) and \( \Sigma_\varphi = R(\varphi) \Sigma_0 R^\dagger(\varphi) \) is the covariance matrix \( \Sigma_0 \) of the input state \( \rho_0 \) rotated by an angle \( \varphi \) [28]. Because in our setup the optical phase \( \varphi \) is encoded by a unitary transformation (as \( H \) commutes with \( L[a] \) and \( L[a^\dagger] \)) the QFI for \( \varphi \) will not depend on the actual value of \( \varphi \). Therefore, in the remaining sections we always neglect the rotation about \( \varphi \) by setting \( \varphi = 0 \).

The above analysis shows that the propagation of light through the sample may be modeled as a Gaussian channel, which may be effectively decomposed into two distinct processes: first, the single-mode Gaussian beam acquires the phase shift \( \varphi \) relative to the reference beam and then it undergoes a photon loss process, which is modeled with a help of a beam splitter with transmissivity \( \eta \), where the second port is filled with thermal light \( \rho_N \) with temperature \( T \). Physically, (the temperature-dependent part of) the acquired phase shift is caused by the thermal expansion of a sample and small temperature-dependent changes in the refractive index \( n \). Assuming a sample with a length \( L \) and the refractive index \( n \) both known for a specific temperature \( T \), the thermoptic coefficient \( n' = dn/dT \) and the thermal expansion coefficient \( \alpha_T \), which is probed with light with frequency \( \omega \), the acquired phase shift is usually described by a simple relation \( \varphi = n(\delta T) \omega L(\delta T)/c \), where \( n(\delta T) = n + n' \delta T \) and \( L(\delta T) = L(1 + \alpha_T \delta T) \). The temperature-dependant part of the acquired phase shift given to the first order in \( \delta T \) can be written as

\[
\varphi = \frac{\omega L}{c} (n\alpha_T + n') \delta T = \alpha_T \delta T,
\]

(21)

where for the sake of simplicity we replaced \( \delta T \) with \( T \). However, we emphasize that in this work, we always estimate or probe tiny deviations of temperature \( \delta T \) from a known value.

In the following section, we determine the best possible precision with which we can estimate the temperature of a non-linear PPKTP crystal with the interferometric phase measurement.

V. TEMPERATURE ESTIMATION WITH APPLICATION TO NON-LINEAR CRYSTALS

We now wish to calculate the QFI for temperature. To this end, we need to first calculate the QFI for the optical phase \( \varphi \). In general, any calculation involving a single-mode Gaussian beam interacting with a thermal Gaussian reservoir is very complicated to the point, where in many cases only numerical results can be obtained. The calculation of the QFI for the optical phase shift acquired in such a setup is no exception. Fortunately, in Ref. [30] the authors developed a very powerful technique for the derivation of the SLD and the QFI for arbitrary Gaussian probe states propagating through general dissipative Gaussian reservoirs. Here, we adopt this technique to find the SLD and the QFI for the optical phase. [We should mention here two related results presented in Refs. [31] and [32] that can also be used to find the SLD and the QFI for the optical phase but these results were obtained using rather
different methods.] The details of the derivation of the SLD for the optical phase given in terms of the first and second moments of the output state $\rho_x$ are presented in Appendix A. Here we only present the resulting QFI

$$Q_\phi = \frac{4(\Sigma_{22} - \Sigma_{11})^2}{1 + 4\Sigma_{11}\Sigma_{22}} + \frac{\bar{x}^2}{\Sigma_{22}} + \frac{\bar{p}^2}{\Sigma_{11}},$$

(22)

where $\Sigma_{11}$ and $\Sigma_{22}$ are the diagonal elements of the covariance matrix $\Sigma$ of the output state $\rho_x$, and $\bar{x}$ and $\bar{p}$ are the mean displacements in the respective canonical position and momentum quadratures all of which are calculated with $\phi = 0$. Based on the above formula for $Q_\phi$ and the error propagation formula we can easily find the following QFI for temperature

$$Q_T = \left(\frac{d\phi}{dT}\right)^2 Q_\phi.$$  

(23)

We now focus on finding the maximum value of $Q_\phi$ (and by implication the maximum value of $Q_T$) and the optimal state that asymptotically attains this value. To this end, we need to optimize Eq. (22) over the input state parameters, that is, optimize over the mean displacements $\bar{x}_0$ and $\bar{p}_0$, the mean number of thermal photons $N_0$ and the amount of squeezing $r_0$. In the limit of a large number of input photons $N \gg 1$, the numerical optimization of Eq. (22) predicts that the optimal input state is a squeezed-vacuum state, which implies $\bar{x}_0 = \bar{p}_0 = 0$ and $N_0 = 0$. Hence, for the squeezed-vacuum input state the asymptotic error of temperature estimation is lower bounded via the Cramér-Rao inequality and reads

$$\Delta T \geq \frac{1}{\alpha} \sqrt{\frac{(1 - \eta)(1 + 2N)}{4\eta N}},$$

(24)

This bound scales as $c_{SQL}/\sqrt{N}$. The readers familiar with the problem of phase estimation in the presence of dissipative reservoirs will certainly notice that the above bound, neglecting for a moment the prefactor $1/\alpha$, resembles a bound that is typically obtained for phase estimation in the presence of photon loss [10][11]. However, our bound has an additional coefficient $\sqrt{1 + 2N}$ because we consider here a dissipative reservoir prepared in a thermal state $\rho_N$, whereas in the lossy phase estimation it is more common to assume a reservoir prepared in the vacuum state [33].

It is also instructive to determine how well we can estimate temperature if we use coherent states (implying $N_0 = 0$ and $r_0 = 0$) as the input states to our setup instead of the optimal squeezed-vacuum states. In this case, the Cramér-Rao inequality predicts that the error is lower bounded by

$$\Delta T \geq \frac{1}{\alpha} \sqrt{\frac{1 + 2(1 - \eta)N}{4\eta N}},$$

(25)

which for a zero-temperature reservoir, that is, $N = 0$, and neglecting the prefactor $1/\alpha$, is equivalent to the well-known bound in lossy phase estimation [21].

A. Noninvasiveness of interferometric thermometers

From the above bounds we could draw a conclusion that it is best to send as much light into the sample as possible because with increasing $N$ the error in temperature estimation becomes smaller. However, if we were to use very strong squeezed-vacuum or coherent states then because of the absorption of light this would at some point disturb sample’s temperature. We take this effect into account by adding the magnitude of this disturbance to our precision bound. If the sample absorbs on average $N_{abs} = (1 - \eta)N$ photons then its temperature is at worst disturbed by $\delta T = h\omega N_{abs}/(MC_s)$, where $M$ and $C_s$ are the mass and the specific heat of the sample, respectively. Hence, as long as $\delta T \ll \Delta T$ then our interferometric thermometer is noninvasive.

The effect of disturbance can be included into our asymptotic error bounds by adding $\delta T$ on the right-hand side of Eqs. (24) and (25). We note that by doing so we are combining in a single formula two different types of errors: the statistical error and the systematic error associated with the heating of the sample; however, this allows us to depict both of these errors neatly on the same figure. The resulting formulas for the asymptotic error of temperature estimation are given by

$$\Delta T \geq \frac{1}{\alpha} \sqrt{\frac{(1 - \eta)(1 + 2N)}{4\eta N} + \frac{(1 - \eta)h\omega N}{MC_s}},$$

(26)

$$\Delta T \geq \frac{1}{\alpha} \sqrt{\frac{1 + 2(1 - \eta)N}{4\eta N} + \frac{(1 - \eta)h\omega N}{MC_s}},$$

(27)

for the squeezed-vacuum and coherent states, respectively. From the above formulas we see that there is a maximum value $\hat{N}_{max}$ of $N$ for which the error is minimal and increasing $N$ from that point would only decrease the precision because of the heating of the sample. This behavior is clearly visible in Fig. 3 (which was prepared for the PPKTP crystal; all necessary parameters for this material
Figure 3. The precision of the interferometric thermometer near $T = 298$K for a non-linear PPKTP crystal with a length of 1cm plotted as a function of the mean number of photons $\bar{N}$ with $\lambda = 1064$nm ($\omega = 1.77 \times 10^{14}$Hz). In our model for the PPKTP crystal, we set $M = 3g$, $C_s = 688J/(kgK)$, $n = 1.74$, $\alpha_T = 1.1 \times 10^{-5}K^{-1}$, $n' = 0.6 \times 10^{-5}K^{-1}$ and $\eta = \exp[-L\alpha_{abs}] = 0.9998$ with $\alpha_{abs} = 0.0002cm^{-1}$ [34]. The red curve represents the exact lower bound obtained via the QCRB with $k = 1$ and $Q_T$ given in Eq. (23) with the tail correcting for a possible heating up of the crystal. The gray areas depict the precision regions lying below the asymptotic bound for the single-mode squeezed-vacuum state given in Eq. (26) and lying above the asymptotic bound for the coherent state given in Eq. (25). The black dashed line represents the fundamental precision for the idealized pyrometer given in Eq. (6) with $S = 1cm^2$ and $\delta t = 10ms$.

are given in figure’s caption), where for moderately small $\bar{N}$ the error decreases but eventually it starts to increase when growing $\bar{N}$ disturbs sample’s temperature. Fortunately, this maximum value $\bar{N}_{\text{max}}$ is of order of $10^{13}$--$10^{14}$ photons, and thus the interferometric thermometers are noninvasive for most of the realistic probe states.

B. Comparison with noninvasive classical thermometers

We compare the precision offered by our interferometric thermometer with the theoretical limit imposed on the idealized pyrometer that we found in Sec. III. We perform this comparison for the PPKTP crystal which is commonly used in many quantum optics experiments and its basic physical properties can be easily found [34]. The results of our calculations for the PPKTP crystal near $T = 298$K are presented in Fig. (3). As can be easily checked the best precision achieved by our interferometric approach is $\Delta T \approx 1.4nK$ for light with wavelength $\lambda = 1064$nm and $\bar{N} = 3.2 \times 10^{13}$ photons prepared in the squeezed-vacuum state. This is much higher than the precision of the commercially available pyrometers and even higher than the theoretical limit for the idealized pyrometer of $\Delta T = 452nK$. Surprisingly, even if we use coherent input states, the precision given by the interferometric thermometer is still excellent with $\Delta T \approx 24.2nK$ obtained with $\bar{N} = 5.6 \times 10^{14}$ photons. Therefore, we find that the interferometric thermometry does not need to use specially designed quantum states of light to surpass the classical optical thermometers.

One could argue that our analysis is not comprehensive because we have not fully exploited all possible sources of information about the temperature in our setup. For example, we could have additionally tried to measure the mean number of thermal photons $N$ being radiated by the sample at frequency $\omega$, which clearly is temperature dependent. Furthermore, it very much depends on the type of the sample, but very often the transmissivity, or in other words, the magnitude of photon loss $\eta$ also depends on the temperature. Therefore, we could have tried to estimate the temperature from $\eta$ as well. The QFIs for estimation of $N$ and $\eta$ were already calculated in Ref. [29] and as we found the estimation of these additional parameters provides negligible information about temperature when compared with the information obtained from the estimation of the optical phase. Moreover, if, in spite of that, we still wish to estimate the mean number of thermal photons $N$, then it is optimal to send a vacuum state through the sample [29], which is equivalent to measuring the thermal radiation emitted by the sample at a single frequency $\omega$, and interestingly provides information that we have already found in Eq. (8) in the section devoted to the idealized pyrometers.

VI. DISCUSSION AND FUTURE DIRECTIONS

In summary, we provided a detailed description of the interferometric thermometer by analyzing the interaction between the single-mode Gaussian states of light and a heated sample modeled as a dissipative thermal reservoir. We found that the single-mode squeezed-vacuum state is optimal for temperature measurement offering precision in the nK range; although, coherent input states provide an excellent performance as well. Moreover, we also found a very elegant formula Eq. (6) that lower bounds the fundamental precision of the idealized pyrometer, which
infers the temperature from a measurement of the total thermal radiation emitted by a heated sample. The interferometric thermometer provides a superior performance in temperature sensing even when compared with this idealized pyrometer. One of the main advantages of interferometric thermometry is its noninvasiveness as highly precise temperature measurements can be obtained without disturbing sample’s temperature. We believe this feature to be crucial and hope that interferometric thermometers could be used, for example, for active temperature stabilization of non-linear crystals such as the PP-KTP crystal.

Apart from non-linear crystals, we predict that interferometric thermometers should prove very useful for temperature measurements of atomic vapors. At the moment, the most popular optical technique for temperature measurements in gaseous mediums such as atomic vapors is the absorption spectroscopy in which a laser light is shined on an atomic-vapor cell producing the absorption spectrum. In order to estimate the temperature of atomic vapor, it is then necessary to fit the observed spectrum to a theoretical model described by the so-called Voigt profile, which accurately models the intensity profile of absorption spectral lines by including the contributions of natural linewidth and Doppler broadening. This fitting procedure normally involves a prior knowledge of atomic-vapor parameters such as the density of the vapor, which has a complicated dependence on the temperature. All this makes it very hard to say with any certainty how precise the absorption spectroscopy is for temperature measurements of atomic vapors. However, in practice the precision of this method is limited to at most ±1 K, which is very poor. Furthermore, when we try to estimate the temperature of atomic vapor by scanning the whole absorption spectrum, then we will very likely disturb its state when we probe it around resonant transitions. This is very problematic if we wish to use our atomic-vapor cell as a quantum memory. Therefore, it is clear that an alternative simpler method of temperature measurements for atomic vapors is required. We strongly believe that our noninvasive and non-contact interferometric approach is ideally suited for this kind of applications and it would be of great interest to determine the fundamental precision offered by the interferometric thermometer for temperature measurements in realistic atomic vapors.

Finally, we note that interferometric thermometers operating at the telecom wavelengths of 1550nm can also be readily used for temperature measurements of silicon semiconductor electronic devices as silicon is almost transparent at these longer wavelengths.

VII. ACKNOWLEDGMENTS

We would like to thank Rafal Demkowicz-Dobrzański and Radosław Chrapkiewicz for many helpful discussions and comments. This work was supported by the SIQS project and the European Union Seventh Framework Programme (FP7/2007-2013) under grant agreement n° 316244.
Appendix A: Fundamental precision of the idealized pyrometer

Pyrometers estimate temperature by measuring the flux of thermal radiation emitted by an object modeled as a black body in thermal equilibrium. Typically, those devices take into account the non-unit emissivity $\varepsilon$ of most physical objects, however, for the purpose of this derivation we assume that we are dealing with a perfect black body. The flux of thermal radiation $\Phi$, that is, the total energy radiated per unit surface area of a black body across all wavelengths per unit time, is quantified by the Stefan-Boltzmann formula

$$\Phi = \sigma T^4 \quad \text{with} \quad \sigma = \frac{\pi^2 k_B^4}{60 h^3 c^2},$$

where $\sigma$ is the Stefan-Boltzmann constant and $T$ is the temperature of the object. In order to calculate the precision of temperature estimation based on the Stefan-Boltzmann law we use the error propagation formula given in Eq. (1) in the main text with $\langle A \rangle = \Phi$. Using the definition of the variance in the thermal photon number $(\Delta N)^2 = N(N + 1)$ with $N$ being the mean number of thermal photons distributed according to the Bose-Einstein statistics [25], as

$$N = \left[ \exp \left( \frac{\hbar \omega}{k_B T} \right) - 1 \right]^{-1},$$

we can calculate

$$\Delta \Phi = \sqrt{\int_0^\infty \langle \hbar \omega \Delta N \rangle^2 \varphi(\omega) d\omega} = \sqrt{4 k_B \sigma T^5},$$

where $\varphi(\omega) = \omega^2/(4 \pi^2 c^2)$ is the density of modes per unit interval in $\omega$ in a unit area. This allows us to find the fundamental precision in temperature estimation for the idealized pyrometer

$$\Delta T = \sqrt{\frac{k_B}{4 \sigma T}}.$$ 

The above error represents the square root of the inverse of the information collected per unit surface area per unit measurement time. In order to include the total information collected by the pyrometer we need to divide Eq. (A4) by $\sqrt{S \delta t}$, where $S$ and $\delta t$ are the surface area of the sample we probe or equivalently the area of the detector, and the duration of the measurement, respectively. This leads to Eq. (5) given in the main text.
Appendix B: Derivation of the QFI for the optical phase

In order to calculate the QFI for the optical phase we adopt a technique presented in Ref. [39]. We recall that according to the master equation given in Eq. (15), after a time \( t \), an initial state \( \rho_0 \) evolves to \( \rho_\varphi = \exp[G(\omega, N, \Gamma)t]\rho_0 = S(\varphi, N, \gamma)\rho_0 \), where \( S(\varphi, N, \gamma) = \exp[G(\varphi, N, \gamma)] \) and here \( \gamma = \Gamma t \) is the effective coupling to a thermal reservoir (which is related to the photon loss coefficient \( \eta \) via the relation \( \eta = e^{-\gamma} \)) and \( \varphi = \omega t \) is the optical phase we wish to estimate. In the following derivation, we will drop the dependence of the superoperators \( G \) and \( S \) on the mean number of photons \( N \) and the photon loss \( \gamma \) because we are only concerned with finding the SLD for phase.

We begin by finding the partial derivative of \( \rho_\varphi \) with respect to \( \varphi \), which can be neatly expressed as

\[
\partial_\varphi \rho_\varphi = \partial_\varphi S(\varphi)\rho_0 = \partial_\varphi \exp[G(\varphi)]\rho_0. \tag{B1}
\]

Using the following relation [30]

\[
\partial_\varphi \exp[G(\varphi)] = \int_0^1 e^{uG(\varphi)} \partial_\varphi G(\varphi) e^{(1-u)G(\varphi)} du
\]

\[
= \int_0^1 e^{uG(\varphi)} \partial_\varphi G(\varphi) e^{-uG(\varphi)} du S(\varphi) \tag{B2}
\]

we obtain

\[
\partial_\varphi \rho_\varphi = D_\varphi S(\varphi)\rho_0 = D_\varphi \rho_\varphi, \tag{B3}
\]

where the superoperator \( D_\varphi \) is given by

\[
D_\varphi = \int_0^1 e^{uG(\varphi)} \partial_\varphi G(\varphi) e^{-uG(\varphi)} du, \tag{B4}
\]

which can be easily calculated using the Baker-Campbell-Hausdorff formula resulting in \( D_\varphi = -iH \).

Recalling the definition of the SLD \( \partial_\varphi \rho_\varphi = L_\varphi^\dagger \rho_\varphi \), where we have introduced the symmetric product for operators \( A \odot B = \frac{1}{2}(AB + BA) \), we can write

\[
L_\varphi \rho_\varphi = \beta_{ij} [\hat{\chi}_i \hat{\chi}_j, \rho_\varphi], \tag{B5}
\]

where we have introduced \( \chi = (a, a^\dagger) \) and

\[
\beta = \left( \begin{array}{cc} 0 & -i/2 \\ -i/2 & 0 \end{array} \right) \tag{B6}
\]

and Einstein summation convention is assumed. Eq. (B5) is an example of the Sylvester equation \( Y = Z \odot X \) which has the following formal solution

\[
L_\varphi = 2 \int_0^\infty e^{-v\rho_\varphi} \beta_{ij} [\hat{\chi}_i \hat{\chi}_j, \rho_\varphi] e^{-v\rho_\varphi} dv. \tag{B7}
\]

In the next step we introduce \( \tilde{x}_i = \chi_i - \langle \chi_i \rangle \) and calculate

\[
\int_0^\infty e^{-v\rho_\varphi} \tilde{x}_i \tilde{x}_j \rho_\varphi e^{-v\rho_\varphi} dv = \int_0^\infty e^{-v\rho_\varphi} \tilde{x}_i e^{v\rho_\varphi} \tilde{x}_j e^{-v\rho_\varphi} \rho_\varphi e^{-v\rho_\varphi} dv
\]

\[
= \sum_{m,n} \frac{(-1)^{m+n}}{m!n!} \{(F - 1)^m \langle F \rangle^n \} \tilde{x}_i (F - 1)^m \langle F \rangle^n \tilde{x}_j \rho_\varphi
\]

\[
= \sum_{m,n} \frac{(-1)^{m+n}}{2m+n+1} \binom{m+n}{n} \{(F - 1)^m \langle F \rangle^n \} \tilde{x}_i (F - 1)^m \langle F \rangle^n \tilde{x}_j \rho_\varphi
\]

\[
= \sum_{m,n} \frac{(-1)^{m+n}}{2m+n+1} \binom{m+n}{n} \{(F - 1)^m \langle F \rangle^n \} \tilde{x}_i (F - 1)^m \langle F \rangle^n \tilde{x}_j \rho_\varphi
\]

\[
= \sum_{m,n} \frac{(-1)^{m+n}}{2m+n+1} \binom{m+n}{n} \{(F - 1)^m \langle F \rangle^n \} \tilde{x}_i (F - 1)^m \langle F \rangle^n \tilde{x}_j \rho_\varphi
\]

\[
= \sum_{q=0}^\infty \frac{(-1)^q}{2q+1} \sum_{n=0}^q \binom{q}{n} \{(F - 1)^q \langle F \rangle^n \} \tilde{x}_i (F - 1)^q \langle F \rangle^n \tilde{x}_j
\]

\[
= \sum_{q=0}^\infty \frac{(-1)^q}{2q+1} \{(F - 1)^q \langle F \rangle + 1 \} \tilde{x}_i (F - 1)^q \langle F \rangle \tilde{x}_j
\]

\[
= \{(F \odot F + 1 \odot (F - 1))^{-1}\} \tilde{x}_i \tilde{x}_j,
\]

\[
= \{(F \odot F + 1 \odot (F - 1))^{-1}\} \tilde{x}_i \tilde{x}_j,
\]

\[
= \{(F \odot F + 1 \odot (F - 1))^{-1}\} \tilde{x}_i \tilde{x}_j,
\]

\[
= \{(F \odot F + 1 \odot (F - 1))^{-1}\} \tilde{x}_i \tilde{x}_j,
\]

\[
= \{(F \odot F + 1 \odot (F - 1))^{-1}\} \tilde{x}_i \tilde{x}_j,
\]
where $F = H^\dagger f(\tilde{\Sigma})H$ with $H$ being the Hadamard matrix

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix},$$  \hspace{1cm} (B9)

and $f(x) = \frac{x - i/2}{x+i/2}$ and $\tilde{\Sigma} = \Sigma \Omega$, where $\Sigma$ is the covariance matrix of the output state $\rho_\varphi$ with $\Omega$ begin the symplectic matrix

$$\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$  \hspace{1cm} (B10)

In the derivation of Eq. \eqref{B8} we used a number identities, which can be all found in Appendix C of Ref. \cite{39}: (i) in the second line we used

$$e^{-\nu \varphi_c} \tilde{\chi}^i e^{\nu \varphi_c} = \sum_m \frac{(-1)^m}{m!} v^m \left[ (F - 1)^m \right] \tilde{\chi}^k \varphi^m;$$  \hspace{1cm} (B11)

(ii) in the fourth line we used $\rho_\varphi^m \tilde{\chi}^i \rho_\varphi^m = |\tilde{\chi}^i|^2$;

(iii) in the sixth line we changed the variables $m + n = q$ and replaced $\sum_{m,n=0}$ with $\sum_{q=0}^\infty \sum_{n=0}$. Similarly, we can show that

$$\int_0^\infty e^{-\nu \varphi_c} \rho_\varphi \tilde{\chi}^i \tilde{\chi}^j e^{-\nu \varphi_c} dv$$  \hspace{1cm} (B12)

Replacing now $\chi^i$ with $\tilde{\chi}^i - \langle \tilde{\chi}^i \rangle$ in Eq. \eqref{B7} and using Eqs. \eqref{B8} and \eqref{B12} we obtain an expression for the SLD for the optical phase $\varphi$

$$L_\varphi = 2 \tilde{\beta}_{ij} [F \otimes \mathbb{1} \otimes \mathbb{1} - \mathbb{1} \otimes F \otimes \mathbb{1}] \tilde{\chi}^k \tilde{\chi}^l$$

$$+ \left[ (\mathbb{1} + F)^{-1} - (1 + F^{-1})^{-1} \right] \tilde{\chi}^k \langle \tilde{\chi}^l \rangle$$

$$+ \left[ (\mathbb{1} + F)^{-1} - (1 + F^{-1})^{-1} \right] \tilde{\chi}^l \langle \tilde{\chi}^k \rangle,$$  \hspace{1cm} (B13)

which after a very tedious but rather straightforward algebra can be further simplified to

$$L_\varphi = i \tilde{\beta}_{ij} \left[ |D^{-1}(\tilde{\Sigma} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \tilde{\Sigma})| \right] \tilde{\chi}^k \tilde{\chi}^l (\tilde{R}^k \tilde{R}^l + i\Omega_{kl}/2) + \langle \tilde{\Sigma}^{-1} \rangle \tilde{\chi}^k \tilde{\chi}^l + \langle \tilde{\Sigma}^{-1} \rangle \tilde{\chi}^k \langle \tilde{\chi}^l \rangle \tilde{R}^k + \langle \tilde{\Sigma}^{-1} \rangle \tilde{\chi}^l \langle \tilde{\chi}^k \rangle \tilde{R}^k,$$  \hspace{1cm} (B14)

where we have defined

$$D = \tilde{\Sigma} \otimes \mathbb{1} - \mathbb{1} \otimes \mathbb{1} \otimes \tilde{\Sigma} = \frac{1}{4} \mathbb{1} \otimes \mathbb{1}$$  \hspace{1cm} (B15)

and $\tilde{\beta}_{ij} = \beta_{j} (H^\dagger \otimes H^\dagger)^{ij}$, which is explicitly given by

$$\tilde{\beta} = \begin{pmatrix} -i/2 & 0 \\ 0 & -i/2 \end{pmatrix}.$$  \hspace{1cm} (B16)

and $R^k = H^\dagger \tilde{\chi}^k$ and similarly $\tilde{R}^k = H^\dagger \tilde{\chi}^l$. The last transformation means that $R = (x, p) \tilde{R} = (\tilde{x}, \tilde{p})$, where $x$ and $p$ are the canonical position and momentum operators $[\tilde{x}$ and $\tilde{p}$ are the displaced canonical position and momentum operators].

In the final step we rewrite the SLD for the optical phase as

$$L_\varphi = \frac{4(\Sigma_{11} - \Sigma_{12})}{1 + 4\Sigma_{11} \Sigma_{22}} \tilde{x} \circ \tilde{p} + \tilde{p} \frac{\Sigma_{11}}{\Sigma_{11}} \tilde{x} - \tilde{x} \frac{\Sigma_{22}}{\Sigma_{22}} \tilde{p},$$  \hspace{1cm} (B17)

where $\Sigma_{11}$ and $\Sigma_{22}$ are the diagonal elements of the covariance matrix $\Sigma$ of the output state $\rho_\varphi$ given in Eq. \eqref{20}, and $\tilde{x}$ and $\tilde{p}$ are the mean displacements in the respective canonical position and momentum quadratures given in Eqs. \eqref{18} and \eqref{19}, all of which are calculated with $\varphi = 0$. At this point we need to make two comments: (i) the SLD for the optical phase has zero expectation, that is, $\text{Tr}[\rho_\varphi L_\varphi] = 0$, as any legitimate SLD should, and (ii) it is expressed in terms of the first and the second moments of the output Gaussian state $\rho_\varphi$ with $\varphi = 0$ because in our setup the QFI for the optical phase $\varphi$ is independent of the actual value of $\varphi$, hence, we choose to present the SLD in its simplest form.

Given the above SLD we can calculate the corresponding QFI for the optical phase via

$$Q_\varphi = \text{Tr}[\rho_\varphi L_\varphi^2],$$  \hspace{1cm} (B18)

which is given in Eq. \eqref{22} in the main text.