Schrödinger equation solution for q-deformed Scarf II potential plus Pöschl-Teller potential and trigonometric Scarf potential

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Abstract. The Schrödinger equation for q-deformed Scarf II potential plus Pöschl-Teller potential and trigonometric Scarf potential was solved using Asymptotic Iteration Method (AIM). The combination of three potentials was substituted into the Schrödinger equation, then we separated the three-dimensional Schrodinger equation into three one-dimensional differential equation, consist of the radial part, polar part and azimuth part. The radial, polar and azimuth part differential equations were solved by reducing them into the hypergeometric intermediaries equation, to further we resolved it using AIM. By using AIM, the energy equation and the variable separation constant equations, $\lambda_1$ for the polar part and $\lambda_2$ for the azimuth part were obtained, where both are interrelated between the quantum numbers. The value of energies were calculated numerically using Matlab software, where the increasing in the radial quantum number $n_r$ causes increase and decrease in the energy. Radial, polar and azimuth wave functions were defined as hypergeometric functions and were visualized using Matlab software. The results show that the disturbance of Poschl-Teller potential and trigonometric Scarf potential give effect in probability

1. Introduction
Schrodinger equation is a wave equation as a representation of electrons or microscopic particles in the non-relativistic case. The solution of the Schrodinger equation was obtained wave function equation and energy equation which will be used to find out how the behavior of electrons. The various potentials have been investigated using Schrodinger equation to explain the behavior of the particle. Several types of potentials, especially the exact settlement type of the Schrodinger equation are possible only for angular momentum $= 0$. But when the angular momentum is $\neq 0$ the Schrodinger equation can only be solved by an appropriate method approach such as Asymptotic Iteration method, $1/N$ expansion method, the Nikiforov-Uvarov method [1-2], and the Supersymmetry method [3-4] in quantum mechanic. In quantum mechanics, different approaches are used in determining the quantities associated with particle motion. Completion of the wave function of moving particles can be obtained using the Schrodinger equation. [5] Approximate solution of Schrodinger equation for modified Poschl-Teller plus trigonometric Rosen-Morse non-central potentials in terms of finite Romanovski polynomial has been investigated by Suparmi, et al. in [6]. Deta, et al. in 2014 solved the solution of Schrodinger equation in D-dimensions for Scarf Hyperbolic plus non-central Poschl-Teller potential using Nikiforov-Uvarov method [7]. The solution of three-dimensional Schrodinger equation for Eckart and Manning-Rosen non-central potential using asymptotic iteration method has been
investigated by Farizky, et al. [8]. From the previous research, it gave information that the certainty method can be used to solve the Schrodinger equation which refers to the potential.

The method to solve the Schrodinger equation for a charged particle at a central and noncentral potential has been developed using Asymptotic Iteration Method [8]. Asymptotic iteration method is one of a method to solve the second order differential equation by using the asymptotic approach [9]. AIM has been used in various solutions, not only for Schrodinger equation but also for any solution in the Dirac equation [10-12] and Klein-Gordon equation[13-14].

In this study, we used the q-deformed Scarf II potential (V_{SFII}(r)), Pöschl-Teller potential (V_{PT}(\theta)) and trigonometric Scarf potential (V_{SF}(\varphi)) in Schrodinger equation to find the solution using AIM. The presence of trigonometric Scarf potential in azimuth part gives influence in the value of energy, which is still rare in solution the three-dimensional Schrodinger equation. The three-dimensional Schrodinger equation was written in section 2, which was followed by the brief review of AIM method. The results and discussion can be found in section 3, and for the conclusion in section 4.

2. Basic Theory

2.1. Schrodinger equations in 3 dimensions

The Schrodinger equation independent of time is given as [5]

\[
\frac{-\hbar^2}{2m} \nabla^2 \psi(r, \theta, \varphi) + V\psi(r, \theta, \varphi) = E \psi(r, \theta, \varphi)
\]  

(1)

where the non-central potential is written as:

\[
V(r, \theta, \varphi) = V_{SFII}(r) + \frac{V_{PT}(\theta)}{r^2} + \frac{V_{SF}(\varphi)}{r^2 \sin^2 \theta}
\]  

(2)

In this study, the potential of the radial part is the q-deformed Scarf II potential, is given as

\[
V_{SFII}(r) = \frac{\hbar^2}{2m} \alpha^2 \frac{(b^2 + a(a+1))}{\sin^2 \alpha r^2} - \frac{2b(a+\frac{1}{2}) \cosh \alpha r}{\sinh \alpha r} \]

(3)

where \(b, a\) are potential constants, \(q\) is deformation constant and \(\alpha\) is range of q-deformed Scarf II potential in system. while, for non-central potential consist of Posch-Teller potential and Trigonometric Scarf potential, are written as, respectively

\[
V_{PT}(\theta) = \frac{\hbar^2}{2m} \left( \frac{k(k-1)}{\sin^2 \theta} + \frac{A(\lambda-1)}{\cos^2 \theta} \right)
\]  

(4)

\[
V_{SF}(\varphi) = \frac{\hbar^2}{2m} \left( \frac{b^2 + A(A-1)}{\cos^2 \varphi} + \frac{2B(\lambda-\frac{1}{2}) \sin \varphi}{\cos^2 \varphi} \right)
\]  

(5)

where \(\lambda, k, B, A\) are potential constants. The Pöschl–Teller potential is used to explain spectrum vibration and interaction of atomic systems [15], while the Scarf potential is applied to explain the atomic or molecular force [16].

By substituting equations (3-5) into equation (1), and by setting the wave function \(\psi(r, \theta, \varphi) = R(r)\Theta(\theta)\Phi(\varphi)\) for the spherical coordinates, and using the separation variable, we obtained the three one-dimensional differential equation, consist of radial part, polar part and azimuth part, respectively are given as,

\[
\frac{1}{R(r)} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} R(r) \right) - \frac{2m}{\hbar^2} r^2 \left( \frac{b^2 + a(a+1)}{\sin^2 \alpha r^2} - \frac{2b(a+\frac{1}{2}) \cosh \alpha r}{\sinh \alpha r} \right) - E = -\lambda_1 = 0
\]  

(6)

\[
\sin \theta \frac{1}{\Theta(\theta)} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \Theta(\theta) \right) - \sin^2 \theta \left( \frac{k(k-1)}{\sin^2 \theta} + \frac{A(\lambda-1)}{\cos^2 \theta} \right) + \lambda_1 \sin^2 \theta - \lambda_2 = 0
\]  

(7)

\[
\frac{1}{\Phi(\varphi)} \frac{\partial^2}{\partial \varphi^2} \Phi(\varphi) - \left( \frac{b^2 + A(A-1)}{\cos^2 \varphi} + \frac{2B(\lambda-\frac{1}{2}) \sin \varphi}{\cos^2 \varphi} \right) + \lambda_2 = 0
\]  

(8)

where \(\lambda_1, \lambda_2\) are variable separation constants. Equation (6-8) are the basis equation which will be solved using Asymptotic Iteration Method (AIM) to obtain the energy equation and wavefunction equation.
2.2. Asymptotic Iteration Method (AIM)

The Asymptotic Iteration Method (AIM) has the second order differential equation in term,

\[ y''_n = \lambda_0(x)y_n + s_0(x)y_n \]  \hspace{1cm} (9)

If equation (9) is decreased to \( x \) by \( n \) times, then equation (9) become,

\[ y''_n(x) = \lambda_i-2(x)y_n(x) - s_i-2(x)y_n(x) = 0 \]  \hspace{1cm} (10)

for \( i = 1, 2, 3, \ldots \), where

\[ \lambda_i(x) = \lambda_{i-1} + \lambda_i \Lambda_0 + s_i \]

\[ s_i(x) = s_{i-1} + s_i \lambda_i-1 \]

By using the asymptotic aspect of the iteration method for a sufficiently large value of \( i \), we had

\[ \frac{s_i}{\lambda_i} = \frac{s_{i-1}}{\lambda_{i-1}} = \alpha \]  \hspace{1cm} (13)

The eigen value of equation (9) can be obtained using the termination conditions, is given as

\[ \lambda_i(x)s_{i-1}(x) - \lambda_{i-1}(x)s_i(x) = 0 = \Delta_i, i = 1, 2, 3 \]  \hspace{1cm} (14)

To obtain the wave function of the Schrodinger equation, we can use the second order differential equation as follows,

\[ y''(x) = 2 \left( \frac{tx^{N+1}}{1-bx^{N-2}} - \frac{c+1}{x} \right) y'(x) - \frac{Wx^N}{1-bx^{N/2}} \]  \hspace{1cm} (15)

Equation (15) is the other term of AIM-type differential equation in equation (9) to obtain the wave function simpler. Refer to equation (15), the wave function solution can be obtained using,

\[ y_n(x) = (-1)^n c^{-N/2} p_n(x) F_1(-n, p + n, \sigma, bx^{N/2}) \]  \hspace{1cm} (16)

where

\[ (\sigma)_n = e^{p(n+1)/\sigma} \] \hspace{1cm} (13), \hspace{1cm} \sigma = \frac{2c + N + 3}{N + 2} \] \hspace{1cm} (14), \hspace{1cm} p = \frac{(2c + 1)b + 2t}{(N + 2)b} \] \hspace{1cm} (17)

So by reducing the Schrodinger equation to AIM-type differential equation similar to equations (9,15), we can obtain the solution by using AIM.

3. Solution and Discussion

3.1. The solution of radial part

By using equation (6), and by setting \( R(r) = \frac{\psi(r)}{r} \), we obtained

\[ U''(r) - \frac{2m}{r^2} U(r) \left( \frac{h^2}{2m} \alpha^2 \left( \frac{b^2 + a(a+1)}{\sin h^2 \alpha r} - \frac{2b(a+1)\cos h\alpha r}{\sin h^2 \alpha r} \right) - E \right) = - \frac{\lambda}{r^2} U(r) = 0 \]  \hspace{1cm} (18)

Equation (18) can be solved by using the approximation \( \frac{1}{r^2} \approx -\frac{a^2}{\sin h^2 \alpha r} \), and by setting the suitable variable \( \cos h\alpha r = (1 - 2z)\sqrt{q} \), yields

\[ z(1 - z) \frac{\partial^2}{\partial z^2} U(z) + \frac{1}{2}(1 - 2z) \frac{\partial}{\partial z} U(z) + \left[ \frac{b+2ab}{4q} - \frac{(b^2 + a(a+1) + \lambda)}{4z} \right] + \frac{2mE}{h^2 a^2} \frac{\partial}{\partial z} U(z) = 0 \]  \hspace{1cm} (19)

Equation (19) must be transformed to AIM-type equation by setting the new wave function,

\[ U(z) = z^\beta (1 - z)^\beta f(z) \]  \hspace{1cm} (20)

so, equation (19) become

\[ f''(z) + \left[ \frac{2 \rho + 2}{z(1-z)} \right] f'(z) + \left[ \frac{2mE}{h^2 a^2} (\rho + \beta)^2 \right] f(z) = 0 \]  \hspace{1cm} (21)

where

\[ \rho = \frac{1}{4} \pm \frac{1}{2} \sqrt{\frac{-(b+2ab)}{q} + (b^2 + a(a+1) + \lambda)} + \frac{1}{4} \]  \hspace{1cm} (22)
Equation (21) is similar to the AIM-type equation in equation (9), where
\[ \lambda_0(z) = \frac{(2\rho + 2\beta + 1)z - (2\rho + 1)}{x(1-x)} \]
\[ s_0(z) = \frac{(\rho + \beta)^2 + \frac{2mE}{\hbar^2}}{x(1-x)} \]
By using equations (11,14) corresponding to equations (21,24-25), we obtained the eigenvalue of equation (21), is given as in energy equation,
\[ E = -\alpha^2 \left( \frac{1}{2} - \frac{1}{2} \sqrt{-(b+2ab)\sqrt{q} + (b^2 + a(a+1) + \lambda_1}} + \frac{1}{4} - \frac{1}{2} \frac{(b+2ab)\sqrt{q} + (b^2 + a(a+1) + \lambda_1}}{q} \right)^2 \]
where \( \lambda_1, \lambda_2 \) are variable separation constants, which were obtained from the polar part solution in equation (36) and azimuth part solution in equation (45) Equation (26) is energy equation, which was influenced by quantum number \( n_r, n_\theta, n_\phi \). From the numerically results, we visualized the energy value in radial quantum number \( n_r \) function, was shown in Figure 1.

Figure 1 gives information about the energy value for variation radial quantum number. It shows that in the increasing radial quantum number cause the increasing and decreasing value of energy. The variation of q-deformed Scarf II potential constant \( a \) give influence in the value of energy.

For the radial part wave function, we used equation (16-17) and by using equation (20), we obtained the radial wave function, is given as
\[ U(r) = \left( \frac{1}{2} \right)^{\rho+\beta} \left( 1 - \frac{\cosh a r}{\sqrt{q}} \right)^{\rho} \left( 1 + \frac{\cosh a r}{\sqrt{q}} \right)^{\beta} (-1)^{n_r} C'(1)^{n_r} (2\rho + \frac{1}{2})_{\frac{n_r}{2}} F_1(1) (-n_r, 2\rho + 2\beta + n_r, 2\rho + \frac{1}{2} - \frac{1}{2} \frac{\cosh a r}{\sqrt{q}}) \]
where \( C' \) is the radial normalization constant and \( _2F_1 \) is a hypergeometric function. Equation (27) is the radial wave function equation depending on the distance of the particles as far as \( r \) on the potential existence. The visualization of radial wave function from equation (27) was shown in Figure 2. Figure 2 shows that for the increasing value of potential constant \( a \) causes the increasing in amplitude of wave function. It shows that for the increasing value of potential constant causes the particle moves away from the nucleus, so can be said that the increasing potential constant give effect in lowering the binding energy of particle.
3.2. Solution of polar part

For the angular part, we used equation (7) by setting $\Theta(\theta) = \frac{T(\theta)}{\sin^{1/2}\theta}$, so equation (7) becomes

$$T''(\theta) + \left(\frac{1}{4}\frac{\lambda - k(k-1)}{\sin^{3/2}\theta} - \frac{\lambda(k-1)}{\cos^{3/2}\theta} + \lambda_1 + \frac{1}{4}\right) T(\theta) = 0 \quad (28)$$

By setting $\sin^2\theta = z$ in equation (28), yield

$$z(1 - z) \frac{\partial^2 T(z)}{\partial z^2} + \frac{1}{2} (1 - 2z) \frac{\partial T(z)}{\partial z} + \left(\frac{1}{4}\frac{\lambda - k(k-1)}{4z} - \frac{\lambda(k-1)}{4(1-z)} + \frac{\lambda_1 + \frac{1}{4}}{4}\right) T(z) = 0 \quad (29)$$

Equation (29) must be reduced to AIM-type differential equation by substituting the new wave function,

$$T(z) = z^\gamma (1 - z)^\omega f(z) \quad (30)$$

in equation (29), so we got

$$f''(z) + \left[\frac{(2y+1) - (2y+2)\omega + 1} {z(1-z)}\right] f'(z) + \left[\frac{\lambda_1 + \frac{1}{4} - (\omega + 1)^2} {z(1-z)}\right] f(z) = 0 \quad (31)$$

where,

$$\gamma = \frac{1}{4} \pm \frac{1}{2} \sqrt{\lambda_2 + k(k-1)} \quad (32)$$

$$\omega = \frac{1}{4} \pm \left(\frac{1}{2} \lambda - \frac{1}{4}\right) \quad (33)$$

Equation (31) is AIM-type differential equation, with

$$\lambda_0(z) = \frac{(2y+2)\omega + 1} {z(1-z)} \quad (34)$$

$$s_0(z) = \frac{z(1-z)} {z(1-z)} \quad (35)$$

By using equations (11,14) corresponding to equations (31,34-35), we obtained the eigenvalue of equation (31), is given as the polar part variable separation constant,

$$\lambda_1 = -\frac{1}{4} + 4\left(\frac{1}{2} \pm \frac{1}{2} \sqrt{\lambda_2 + k(k-1) \pm \left(\frac{1}{2} \lambda - \frac{1}{4}\right) + n_\theta}\right)^2 \quad (36)$$
And then for polar wave function, by using equations (15-17), equations (30-31), we obtained the polar wave function is given as,

\[ T(\theta) = (\sin^2\theta)^\gamma (\cos^2\theta)^\omega (-1)^n_\theta C''(1)^n_\theta (2\gamma + \frac{1}{2}) n_\theta \, _2F_1\left( -n_\theta, 2\gamma + 2\omega + n_\theta, 2\gamma + \frac{1}{2}, \sin^2\theta \right) \]  

(37)

where \( C'' \) is the normalization constant of polar part and \( _2F_1 \) is hypergeometric function. By using equation (37) to obtain the polar wave function, where \( \Theta(\theta) = \frac{T(\theta)}{\sin^{1/2} \theta} \), we visualized in Figure 3 for polar wave function in \( n_\theta = 2 \) (left side) and for \( n_\theta = 3 \) (right side).

![Figure 3. The visual of polar wave function in 3-dimensions (up side) and 2-dimension (bottom side) for \( n_\theta = 2 \) (left side) and for \( n_\theta = 3 \) (right side)](image)

Figure 3 (up side) are three-dimensional wave function in spherical coordinates, it shows that for the increasing polar quantum number \( n_\theta \) causes the increasing amplitude of the wave function. It means that the probability to find particle in \( n_\theta = 3 \) is wider than in \( n_\theta = 2 \). The amplitude of polar wave function can be seen more clear from Figure 3 (bottom side) in visual 2-dimension.

3.3. Solution of azimuth part

In this part, we used equation (8) to obtain the azimuth quantum number and wave function. By setting \( \sin \varphi = 1 - 2z \) in equation (8), yield
\[ z(1-z) \frac{\partial^2 \phi(z)}{\partial z^2} + \left( \frac{1}{2} - z \right) \frac{\partial \phi(z)}{\partial z} + \left( \lambda - \frac{2B(A-1)}{2z} + B^2 + A(A-1) \right) \phi(z) = 0 \quad (38) \]

Equation (38) must be reduced to AIM type differential equation by substituting the new wave function, is given as

\[ \phi(z) = z^{\epsilon}(1-z)^\mu f(z) \quad (39) \]

so equation (38) becomes

\[ f''(z) + \left[ \frac{2z^2 + (2\epsilon + \mu + 1)z}{z(1-z)} \right] f'(z) + \left[ \frac{\lambda z - (\epsilon + \mu)^2}{z(1-z)} \right] f(z) = 0 \quad (40) \]

where

\[ \epsilon = \frac{1}{4} \pm \frac{1}{2} \sqrt{(A + B)^2 - (A + B) + \frac{1}{4}} \quad (41) \]

\[ \mu = \frac{1}{4} \pm \frac{1}{2} \sqrt{(A - B)^2 - (A - B) + \frac{1}{4}} \quad (42) \]

Equation (40) is AIM-type differential equation, where

\[ \lambda_0(z) = \frac{(2\epsilon + \mu + 1)z - (2\epsilon + 1)}{z(1-z)} \quad (43) \]

\[ s_0(z) = \frac{-\lambda z - (\epsilon + \mu)^2}{z(1-z)} \quad (44) \]

By using AIM method to solve equation (40), we obtained the eigenvalue by using equations (11-12,14), which was expressed,

\[ \lambda_2 = \left( \frac{1}{2} \pm \frac{1}{2} \sqrt{(A + B)^2 - (A + B) + \frac{1}{4} \pm \frac{1}{2} \sqrt{(A - B)^2 - (A - B) + \frac{1}{4} + n_\varphi}} \right)^2 \quad (45) \]

Equation (45) is the variable separation constant of azimuth part which influences the value of energy. And then for the azimuth wave function, we used equations (15-17) and equations (39-42) to obtain the general equation of azimuth wave function, is given as

\[ \phi(\varphi) = \left( \frac{1-\sin \varphi}{2} \right)^{\epsilon} \left( \frac{1+\sin \varphi}{2} \right)^{\mu} (-1)^{n_\varphi} C^{(2\epsilon + 1)}_{n_\varphi, 2} F_1 \left( -n_\varphi, 2\epsilon + 2\mu + n_\varphi, 2\epsilon + \frac{1}{2} + \frac{1-\sin \varphi}{2} \right) \quad (46) \]

Here \( C^{(m)}_{n} \) is the normalization constant of azimuth part and \( F_1 \) is hypergeometric function. By using equation (46), the visual of azimuth wave function can be obtained which was shown in Figure 4.

Figure 4. The visual of azimuth wave function in 3-dimensions (left side) and 2-dimension (right side) for \( n_\varphi = 2 \)

Figure 4 (left side) for 3-dimension visual wave function give information that the particle was bounded in certainty area refer to that figure. For accurately, we can see in Figure 4 (right side) in
visualization 2-dimensions, it shows that for azimuth wave function for $n_{\varphi} = 2$, the most probability to find the particle is approximately on the range value of $\varphi = 180 - 210$ and of $\varphi = 300 - 360$, while on the others area weren’t found the existence particle.

4. Conclusion
In this research, we solved the Schrodinger equation for potential combination, consist of q-deformed Scarf II potential, Poschl-Teller potential and Trigonometric Scarf potential using asymptotic iteration method. The energies value were calculated numerically using Matlab software, and it was found that the q-deformed Scarf II potential constant $a$ influenced the value of energies. The wave functions were obtained in terms of hypergeometric function and were visualized using Matlab software. The wave function gives information about the probability to find particle in the system.

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