ON THE EXTENSION OF HOLOMORPHIC SECTIONS FROM REDUCED UNIONS OF STRATA OF DIVISORS

CHEN-YU CHI

Abstract. In this paper we study the problem of extension of holomorphic sections of line bundles/vector bundles from reduced unions of strata of divisors. An extension theorem of Ohsawa–Takegoshi type is proved. As consequences we deduce several qualitative results on extension from snc divisors and generic global generation of vector bundles.

1. Introduction

In the current work we study the possibility of extending holomorphic sections of an adjoint or a pluricanonical bundle on a complex manifold simultaneously from several strata of an effective divisor. The situation of simple normal crossing will be our main concern. Before describing the question to study and stating the main results we first fix terminology and notation which will be in force throughout the whole paper. In the following $X$ will be a connected complex manifold and $S: (S_j, \sigma_j, h_j) \ (j = 1, \ldots, q)$ be a family of data which consist of a holomorphic line bundle $S_j$ on $X$, a nonzero holomorphic section $\sigma_j$ of $S_j$, and a smooth hermitian metric $h_j$ on $S_j$ for every $j$. The Cartier divisor $\text{div}(\sigma_j)$ will also be denoted by $S_j$. Note that we allow data with different indices to coincide.

Definition 1.1. (1) For any $J \subseteq \{1, \ldots, q\}$ we let $S_J = \cap_{j \in J} S_j$ (as an analytic subspace of $X$) and $(S^J, \sigma^J, h^J) = \bigotimes_{j \notin J} (S_j, \sigma_j, h_j)$. We denote $(S^\emptyset, \sigma^\emptyset, h^\emptyset)$ by $(S, \sigma_S, h_S)$. For any $\alpha_\bullet = (\alpha_1, \ldots, \alpha_q) \in \mathbb{R}^q$, we let

$$|\sigma_S|_{h_S}^{2\alpha_\bullet} = \prod_{j=1}^q |\sigma_j|_{h_j}^{2\alpha_j} \quad \text{and} \quad |\sigma^J|_{h^J}^{2\alpha_\bullet} = \prod_{j \notin J} |\sigma_j|_{h_j}^{2\alpha_j} \ (J \subseteq \{1, \ldots, q\}).$$

(2) (Transversal strata) A nonempty analytic set $W$ in $X$ is called an $(S)$-transversal stratum associated to some $J \subseteq \{1, \ldots, q\}$ if $W$ is an irreducible

This work was partially supported by the Ministry of Science and Technology project (grant No. 107-2115-M-002-013 from August 2018) and by the European Research Council project Algebraic and Kähler Geometry (ERC-ALKAGE, grant No. 670846 from September 2015).
component of $S_J$ such that there exists a nowhere dense analytic subset $W_1$ of $W$ satisfying the following conditions: along $W \setminus W_1$ the divisors $S_j (j \in J)$ are smooth and intersect transversally, and $(W \setminus W_1) \cap \bigcup_{j \notin J} S_j = \emptyset$. If this is the case, we denote $J$ by $J_{W_1}$.

(3) **(Adjunction maps)** For a transversal stratum $W$ associated to $J$, we let $S^W$ denote the line bundle $S^J|_{W_{\text{reg}}}$. For any integer $m > 0$ and any holomorphic vector bundle $F$, we have the adjunction map

$$
\Gamma(X, (K_X \otimes S)^{\otimes m} \otimes F) \xrightarrow{(\cdot)|_{W}} \Gamma(W_{\text{reg}}, (K_{W_{\text{reg}}} \otimes S^W)^{\otimes m} \otimes F|_{W_{\text{reg}}}),
$$

which depends at most on the choice of the defining sections $\sigma_j$ and on the order in which $S_j$ are listed.

(4) **(Admissible families)** A family $\mathcal{W}$ of transversal strata is called $(S)$-admissible if there is no inclusion relation between the members of $\mathcal{W}$. For such a family we let

$$J(\mathcal{W}) := \bigcup_{W \in \mathcal{W}} J_W$$

and

$$J_0(\mathcal{W}) := \{ J_W \mid W \in \mathcal{W} \}.$$

(5) **(Underlying spaces of admissible families)** For an admissible family $\mathcal{W}$ we let

$$\mathcal{W} := \left( \bigcup_{W \in \mathcal{W}} W \right)_{\text{red}}.$$

**Question** (Simultaneous extension from admissible family of starta). Given an admissible family $\mathcal{W}$ and a section $u \in \Gamma(\mathcal{W}, (K_X \otimes S)^{\otimes m} \otimes F|_{\mathcal{W}})$, can one find a section $U \in \Gamma(X, (K_X \otimes S)^{\otimes m} \otimes F)$ such that $U|_{\mathcal{W}} = u$? Suppose that $m = 1$ and $u$ is $L^2$ with respect to some metric, can $U$ be chosen so that its $L^2$-norm (with respect to certain corresponding metric) is controlled by that of $u$ in some manner?

Our main technical result is the following quantitative extension theorem, the statement of which is tedious. The author may want to first go to its qualitative consequences (by starting from Definition 1.3, Theorems 2, 4, and 5 and

---

1It is clear that if $j \in J_W$ for some admissible $W$ then $S_j \neq S_{j'}$ (as divisors) for every $j' \neq j$.\]
their corollaries, and come back for the precise conditions and estimates when necessary.

**Theorem 1.** Let \((X, \mathcal{S}, (F, h_F), \mathcal{W}, \gamma_\bullet, \delta_\bullet, \varepsilon, A, H)\) consist of

(i) a complex manifold \(X\) admitting a projective morphism to a Stein space,
(ii) a family of data \(\mathcal{S} : (S_j, \sigma_j, h_j) (j = 1, \ldots, q)\) which consist of a holomorphic line bundle \(S_j\) on \(X\), a nonzero holomorphic section \(\sigma_j\) of \(S_j\), and a smooth hermitian metric \(h_j\) on \(S_j\) for every \(j\),
(iii) a holomorphic vector bundle \(F\) on \(X\) with a measurable hermitian metric \(h_F\),
(iv) an \(\mathcal{S}\)-admissible family \(\mathcal{W}\),
(v) numbers \(\gamma_j \geq 0, \delta > 0, 0 \leq \delta_j < 1,\) and \(\varepsilon > 0\) such that
\[
\gamma_j > 0 \quad \text{and} \quad \delta_j = \delta \quad \text{for every} \quad j \in J(\mathcal{W}),
\]
(vi) \(A \in (-\infty, \infty]\) such that \(|\sigma_S|^{2\gamma_\bullet}_{h_S} < e^A\),
(vii) a function \(H\) on \((-A, \infty)\) which is strictly positive, nonincreasing, and integrable with \(\lim_{y \to -A} H(y) \in (0, \infty)\), which we denote by \(H(-A^+)\).

Assume that either \(h_F\) is smooth or \((F, h_F)\) is a line bundle with a singular metric. If on \(X \setminus \bigcup_{j=1}^q S_j\) we have

\[
\Theta_{h_F} - \sum_{j \in J(\mathcal{W})} \left( \delta(t_j - 1) + \varepsilon H(-A^+)s\gamma_j \right) \Theta_{h_j} \otimes \text{id}_F
\]
\[
- \sum_{j \notin J(\mathcal{W})} \left( \varepsilon H(-A^+)s\gamma_j - \delta_j \right) \Theta_{h_j} \otimes \text{id}_F \geq_{\text{Nak}} 0
\]

\[
(1.1)
\]

for \(s \in [0, 1]\) and \((t_j) \in \prod_{j \in J(\mathcal{W})} [0, T_j] \quad \left(T_j := \sum_{W : j \in J_W} |J_W|\right).

---

2A measurable hermitian metric \(h\) on a holomorphic line bundle \(F\) on \(X\) is called a singular metric if the local weights of \(h\) are locally integrable quasi-psh functions. For any locally closed complex submanifold \(Y\) of \(X\), the restriction \(h|_Y\) may not be a singular metric.

**Convention.** For any measurable section \(u\) of \(K_Y \otimes F|_Y\), the condition \(\int_Y \langle u \rangle^2_h < \infty\) will always mean that \(\int_{Y'} \langle u \rangle^2_h < \infty\) and \(u|_{Y'} = 0\) where \(Y'\) is the union of components of \(Y\) along which \(h\) takes the value \(\infty\) almost everywhere. We define the multiplier ideal sheaf \(\mathcal{I}(h|_Y)\) to be zero if \(h|_Y\) takes the value \(\infty\).
and if
\begin{equation}
(1.2) \quad H\left(-\log |\sigma_S|_{h_S}^{2\delta^*}\right) \prod_{J \in \mathcal{J}(\mathcal{W})} \left(\sum_{j \in J} |\sigma_j|_{h_j}^2\right)^{\delta/|J|} \ \text{is locally bounded from below},
\end{equation}
then for any collection of sections
\[ u_W \in \Gamma\left(W_{\text{reg}}, K_{W_{\text{reg}}} \otimes S^W \otimes F|_W\right) \quad (W \in \mathcal{W}) \]
with\footnote{Let $h$ be a measurable hermitian metric on a complex vector bundle $E$ over a complex manifold $M$. For any measurable section $u$ of $K_M \otimes E$, we first define a nonnegative $(n, n)$-form $\langle u \rangle_h^2$ as follows: on a chart $(V, \{z_k = x_k + iy_k\})$ we let $\langle u \rangle_h^2 \mid_V = |f|^2_{\bar{h}}\, dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n$ if $u \mid_V = dz_1 \wedge \cdots \wedge dz_n \otimes f$. The $L^2$-norm of $u$ with respect to $h$ is defined to be $\int_M \langle u \rangle_h^2 \in [0, \infty)$.}
\begin{equation}
(1.3) \quad \int_{W_{\text{reg}}} \prod_{J' \in \mathcal{J}(\mathcal{W}) \setminus \{J\}} \left(\sum_{j \in J'} |\sigma_j|_{h_j}^2\right)^{-\delta/|J'|} \left<\frac{u_W}{\sigma_j}\right>_{h_{J'}}^2 < \infty \quad (W \in \mathcal{W}),
\end{equation}
there exists $U \in \Gamma(X, K_X \otimes S \otimes F)$ such that $U_W = u_W$ for every $W \in \mathcal{W}$ and
\begin{equation}
(1.4) \quad \int_X H\left(-\log |\sigma_S|_{h_S}^{2\delta^*}\right) \prod_{J \in \mathcal{J}(\mathcal{W})} \left(\sum_{j \in J} |\sigma_j|_{h_j}^2\right)^{\delta/|J|} \left<\frac{U}{\sigma_j}\right>_{h_{J'}}^2 \leq \left(\frac{1}{\varepsilon} + \int_{-A^+}^{\infty} H(t) dt\right) \sum_{W \in \mathcal{W}} \sum_{j \in W} \gamma_j \int_{W_{\text{reg}}} \prod_{J' \in \mathcal{J}(\mathcal{W}) \setminus \{J\}} \left(\sum_{j \in J'} |\sigma_j|_{h_j}^2\right)^{-\delta/|J'|} \left<\frac{u_W}{\sigma_j}\right>_{h_{J'}}^2
\end{equation}
where
\begin{equation}
B_{k-1}(\delta) = \pi^k \int_{\Delta_{k-1}} \frac{d\sigma_1 \cdots d\sigma_{k-1}}{(\sigma_1 \cdot \cdots \cdot \sigma_{k-1} (1 - \sigma_1 - \cdots - \sigma_{k-1}))^{(1-\delta)}} \quad (B_0(\delta) := 1)
\end{equation}
with
\[ \Delta_{k-1} = \{(\sigma_1, \ldots, \sigma_{k-1}) \in [0, \infty)^k \mid \sigma_1 + \cdots + \sigma_{k-1} \leq 1\}. \]
Roughly speaking, given a holomorphic section $u$ on a “center” $Z$, the first part of the argument modifies a rather arbitrary extension $U_0$, the $L^2$-norm of which one does not have estimate of, to obtain an extension $F_t$ with $L^2$-norm bounded by a quantity $I_t$ with a parameter $t$. $I_t$ usually appears as the $L^2$-norm of $U_0$ with respect to some “density” $\chi_t$ supported in a neighborhood $V_t$ of $Z$ on which $u$ is defined; when $t \to -\infty$, the “mass” associated to the $\chi_t$ concentrates to the center. The second part of the argument rests on showing that $\liminf_{t \to -\infty} I_t$ is bounded by the $L^2$-norm of $u$ over $Z$ with respect to a suitable measure. In Section 2 we formalize the first part of argument into a metatheorem (Theorem 2.3), the proof of which follows very closely that of Theorem 2.1 in [20]. A difference between our treatment and that of [20] is that we introduce not only one but two “weight functions” $\psi$ and $\phi$. They usually appear as the same function in previous work of other authors (e.g., $\Psi$ in [20]), but allowing them to be different makes the curvature conditions in our extension theorem more flexible. As for the second part of argument, an upper bound of $\liminf_{t \to -\infty} I_t$ can be given by the $L^2$-norm of $u$ with respect to an abstractly defined measure due to Ohsawa [29] (see also [20] and [14]). However, even when $Z$ is a simple normal crossing divisor, Ohsawa’s measure for a general singular metric does not seem to have explicit expressions, which is often necessary in practical applications. When the centers of extension bear sufficient regularity one may obtain a more explicit upper bound. We give in Section 3 an explicit upper bound of $\liminf_{t \to -\infty} I_t$ by direct computations. Roughly speaking, the function $\psi$ defines the center $Z$, and $\phi$ determines the way the aforementioned neighborhood $V_t$ shrinks. In previous studies where explicit upper bounds are obtained, one mainly considers the situation $\phi = \psi = \log |\sigma_S|^2_{\bar{h}_S}$ with $\sigma_S$ a holomorphic section of some vector bundle $E$ such that $\wedge^{rk} E d\sigma_S$ is essentially nonzero along its zero set $Z = Z(\sigma_S)$; if this is the case, $V_t$ shrinks to $Z$ as a usual tubular neighborhood shrinks when its radius tends to zero, and the computation is rather simple. In our proof of Lemma 3.3 both $\phi$ and $\psi$ are less regular and hence complicate the calculation,

\footnote{However, the author was informed by Dano Kim that more explicit understanding of the Ohsawa measure does exist according to his unpublished work.}
which was achieved by using suitably chosen local coordinate systems related to blow-ups.

**Remark 1.2.** We emphasize that in Theorem 1 the numbers $\gamma_j$ and $\delta_j$ with $j \notin J(W)$ are allowed to zero. When they are set to be zero, the curvatures $\Theta_j$ ($j \notin J(W)$) do not appear in (1.1), and this is the reason in the statements of many results below these curvatures play no role. It could be possible that in some situations taking some of the above numbers to be nonzero may help validating (1.1); if this is the case, it is not hard to modify suitably the curvature conditions in the statements of corresponding results.

The condition (1.1) suggests the following definition.

**Definition 1.3.** Given two real $(1,1)$-currents $T_1$ and $T_2$ on $X$, we say that $T_1$ locally dominates sufficiently small multiples of $T_2$ (Notation: $T_1 \gtrapprox \pm T_2$) if for every point $p \in X$ there exist a neighborhood $V$ of $p$ and a number $c_0 > 0$ such that $(T_1 - cT_2)|_V$ is a positive $(1,1)$-current for every $c \in (-c_0, c_0)$. (Note that $T_1$ is not only required to dominate positive but also negative multiples of $T_1$. This always holds when, e.g., $T_1$ is a Kähler current.) There is obviously a natural extension of this definition to endomorphism-valued $(1,1)$-currents in the sense of Nakano.

In the following we assume that $X$, $(S_j, \sigma_j, h_j)$ ($j = 1, \ldots, q$), and $W$ are as in the statement of Theorem 1.

**Definition 1.4.** (1) **(SNC families)** We say that an admissible family $W$ is $(S_\text{snc})$ if $S_1 + \cdots + S_q$ has simple normal crossing along $W$ for every $W \in \mathcal{W}$. (Note that all admissible families are snc when the divisor $S$ is reduced and of simple normal crossing.)

(2) **(Minimal transversal strata)** An transversal stratum is called a minimal stratum if $W \cap S_j = \emptyset$ for every $j \notin J_W$.

(3) **(Derived families)** For an $S$-snc family $\mathcal{W}$ we let $\mathcal{W}'$ be the family consisting
of maximal members (with respect to inclusion) of
\[
\left\{ W' \left| W' \text{ is a stratum of the snc divisor } W \cap S^j W \text{ on } W \text{ for some } W \in \mathcal{W} \right. \right\}.
\]
\( \mathcal{W}' \) is also an \( S \)-snc family. More generally, we let \( \mathcal{W}'^{(k)} = (\mathcal{W}'^{(k-1)})' \) for all \( k \in \mathbb{N} \) (\( \mathcal{W}'^{(0)} := \mathcal{W} \)).

**Theorem 2.** Suppose that \( \mathcal{W} \) is an \( S \)-snc family. if \((F, h_F)\) is either
- a holomorphic vector bundle with a smooth hermitian metric, or
- a holomorphic line bundle with a singular hermitian metric such that
  \[ \mathcal{I}(h_F|_W) = \mathcal{O}_W \text{ for every } W \in \bigcup_{k=0}^{\dim X} \mathcal{W}^{(k)}, \]
  and if \( \sqrt{-1} \Theta_{h_F} \geq \pm \sqrt{-1} \Theta_{h,j} \otimes \text{id}_F \) for every \( j \in J(\mathcal{W}) \), then the restriction map
  \[
  (1.5) \quad \Gamma(X, K_X \otimes S \otimes F) \longrightarrow \Gamma(\mathcal{W}, (K_X \otimes S \otimes F)|_W)
  \]
is surjective.

We remark that in the case that \( S = S_1 + \cdots + S_q \) itself be an snc divisor and \( F \) be an ample line bundle, if \( \mathcal{W} \) is the (scheme-theoretic) complete intersection
\[ S_1 \cap \cdots \cap S_k \text{ for some } k \leq q, \]
then the surjectivity of (1.5) can be obtained as follows: we consider the Koszul resolution
\[
0 \longrightarrow M_N \longrightarrow \cdots \longrightarrow M_1 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_W \longrightarrow 0
\]
where
\[ M_l = \bigoplus_{J \subseteq \{1, \ldots, k\}, |J| = l} \mathcal{O}_X(-S_J) \]
and the morphisms are induced by the sections \( \sigma_j \). Applying the Kawamata–Viehweg vanishing theorem, we have for \( l > 0 \) that
\[ H^l(X, M_l \otimes \mathcal{O}_X(K_X \otimes S \otimes F)) \cong \bigoplus_{J \subseteq \{1, \ldots, k\}, |J| = l} H^l(X, \mathcal{O}_X(K_X \otimes S^J \otimes F)) = 0, \]
and simple diagram chasing yields the desired surjectivity. However, the above argument cannot be applied for much more general \( \mathcal{W} \) as in Theorem 2 even if \( F \) is assumed ample. On the other hand, Cao, Demailly, and Matsumura \cite{4} have obtained a very general qualitative extension theorem for an adjoint
line bundle $K_X \otimes E$ equipped with a singular metric $h$ together with some quasi-psh function $\psi$ with analytic singularities satisfying certain curvature conditions. However, the singular metric case of Theorem 2 is not an immediate consequence of their result, in the sense that [4] extends sections from the analytic subspace $V(I(h e^{-\psi}))$, which does not seem to have an explicit description when $h$ is a less explicit singular metric, even if $V(I(\psi))$ defines a simple normal crossing divisor; in our case the center for extension is rather explicit, namely, the underlying space associated to an admissible family. The proof of Theorem 2 will be given in Section 4 based on a strategy of successive extension-correction suggested by Demailly, the idea of which appeared already in [14]. Validity of the strong openness conjecture [21] (especially the slightly stronger forms in [22] and [25]) is used in the proof of the singular metric case of Theorem 2 (cf. Lemma 4.1), and this is the place the triviality of multiplier sheaves plays a role. It is tempting to believe that applying the strong openness will enable removing the triviality condition on multiplier sheaves, but the situation is actually subtler. It is thus interesting to ask the following question:

**Question 1.** Suppose that $\mathcal{W}$ is an $S$-snc family. Let $(F, h_F)$ be a holomorphic line bundle on $X$ with a singular hermitian metric such that

$$\sqrt{-1} \Theta_{h_F} \geq \pm \sqrt{-1} \Theta_{h_j} \otimes \text{id}_F \quad (j \in J(\mathcal{W})).$$

Suppose that $h_F|_{W \cap S_j}$ is well-defined for every $W$ and for every $J$. Define $\Gamma(S, I(h_F|_S)(K_X \otimes S \otimes F)|_S)$ to consist of all elements $u = (u_W)$ such that $u_W|_{W \cap S_j}$ is locally $L^2$ with respect to $h_F|_{W \cap S_j}$ for every $W \in \mathcal{W}$ and $J \subseteq \{1, \ldots, q\}$. Is it true that $\Gamma(S, I(h_F|_S)(K_X \otimes S \otimes F)|_S)$ lies in the image of the restriction map (1.5)?

Note that if we drop the well-definedness requirement then Ohsawa [31] has provided a counterexample to the statement considered in the above question.

The following is an immediate consequence of Theorem 2.

**Corollary 3.** Suppose that $F$ is a holomorphic vector bundle such that $\mathcal{O}_{\mathbb{P}(F)}(1)$ admits a smooth hermitian metric with strictly positive curvature. For an snc
family \( \mathcal{W} \), the restriction map
\[
\Gamma(X, K_X \otimes S \otimes \det F \otimes \text{Sym}^k F) \rightarrow \Gamma(\mathcal{W}, K_X \otimes S \otimes \det F \otimes \text{Sym}^k F|_\mathcal{W})
\]
is surjective.

**Proof.** Let \( r = \text{rk} F \) and let \( \mathbf{P}(F) \rightarrow X \) be the natural projection. We let
\[
\tilde{S} : (\tilde{S}_j, \tilde{\sigma}_j, \tilde{h}_j) := p^*(S_j, \sigma_j, h_j)
\]
and let \( \tilde{W} \) be the family consisting of \( p^{-1}W \) (\( W \in \mathcal{W} \)). We have the following commutative diagram
\[
\begin{array}{ccc}
\Gamma(\mathbf{P}(F), K_{\mathbf{P}(F)} \otimes \tilde{S} \otimes \mathcal{O}_{\mathbf{P}(F)}(r + k)) & \rightarrow & \Gamma(\tilde{\mathcal{W}}, K_{\mathbf{P}(F)} \otimes \tilde{S} \otimes \mathcal{O}_{\mathbf{P}(F)}(r + k)|_{\tilde{\mathcal{W}}}) \\
\approx & \downarrow & \approx \\
\Gamma(X, K_X \otimes S \otimes \det F \otimes \text{Sym}^k (F)) & \rightarrow & \Gamma(\tilde{\mathcal{W}}, K_X \otimes S \otimes \det F \otimes \text{Sym}^k (F)|_{\tilde{\mathcal{W}}})
\end{array}
\]
The proof is then completed by applying Theorem 2 to \( (\mathbf{P}(F), \tilde{S}) \) with \( \tilde{W} \) and \( F = \mathcal{O}_{\mathbf{P}(F)}(r + k) \). \( \square \)

The following analogue of invariance of plurigenera can be proved by incorporating Theorem 1 with essentially the same argument of [6], a variant of that of [32] and [35]. For completeness we provide a proof of it in Section 5.

**Theorem 4.** Suppose that \( \mathcal{W} \) is an \( \mathcal{S} \)-snc family of minimal strata, and that \( (L_i, h_{L_i}) \) \( (i = 1, \ldots, m) \) are holomorphic line bundles on \( X \) with singular hermitian metrics such that
\[
(1.6) \quad \sqrt{-1} \Theta_{h_{L_i}} \succ \pm \sqrt{-1} \Theta_{h_j} \quad (i = 1, \ldots, m \text{ and } j \in J(\mathcal{W})).
\]
Then for any collection of sections
\[
uW \in \Gamma\left(W, \mathcal{I}(h_{L_1}|_W) \cdot \mathcal{I}(h_{L_m}|_W) \cdot \otimes_{i=1}^m K_{W} \otimes S^W \otimes L_i|_W\right) \quad (W \in \mathcal{W})
\]
there exists a section
\[
U \in \Gamma\left(X, \mathcal{I}(\sqrt{(h_{L_1} \otimes \cdots \otimes h_{L_m})^{\frac{1}{m}}}) \cdot \otimes_{i=1}^m K_X \otimes S \otimes L_i\right)
\]
such that \( U_W = \nu_W \) for every \( W \in \mathcal{W} \).
It is natural to ask whether or not we may drop the condition that $\mathcal{W}$ consists of only minimal strata, when, for example, all $h_{L_i}$ are smooth. It is tempting to attack the general case with the aforementioned successive extension-correction process, as exploited in the proof of Theorem 2, but there is at least one essential obstacle in establishing a pluricanonical analogue of Lemma 4.1(2). Roughly speaking, in applying Theorem 1 to extend a section $u = (u_W)$, one has to verify the finiteness of the integrals

$$\int_W \prod_{J' \in J_0(W) \setminus \{J_W\}} \left( \sum_{j \in J'} |\sigma_j|_{h_j}^2 \right) \frac{\langle u_W \rangle^2}{\sigma^{J_W}_W} \left( \sum_{j \in J'} \sigma_j \right) \delta_{|J'|} \bigg( \sum_{j \in J'} |\sigma_j|_{h_j}^2 \bigg) \frac{\langle u_W \rangle^2}{\sigma^{J_W}_W} \left( \sum_{j \in J'} \sigma_j \right) \delta_{|J'|} (W \in \mathcal{W}).$$

In the case of Lemma 4.1(2) the problematic vanishing of the factors in the denominator of the integrand is taken care of by the vanishing assumption $u \in \ker \beta$ together with the triviality of $\mathcal{I}(h_F|_{W \cap S_j})$ (and validity of the strong openness conjecture). In the pluricanonical case an obstacle appears when one proceed Păun’s one-tower argument as the proof of Lemma 5.2.1. More precisely, at that stage one uses the singular metric $h_{k-1}$ defined by some collection $U_{k-1}$ of extensions of $u$ twisted by some auxiliary objects. Intuitively speaking, since $h_{k-1}$ runs out the vanishing of $u$, nothing can take care of the denominator of the integrand. On the other hand, in situations with slightly more positivity and better integrability one may overcome the above obstacle. For example, we have the following extension theorem:

**Theorem 5.** Suppose that

1. $L_i (i = 1, \ldots, m - 1)$ are holomorphic line bundles on $X$ with singular hermitian metrics $h_{L_i}$ such that $\sqrt{-1} \Theta_{h_{L_i}} \geq 0$ for every $i$,
2. $L$ is a $\mathbb{Q}$-line bundle with $L^{\otimes m} = \otimes_{i=1}^{m-1} L_i$ and $h_L$ is a singular metric on $L$ such that $\sqrt{-1} \Theta_{h_L} \geq \pm \sqrt{-1} \Theta_{h_j} \big( j \in J(W) \big)$ and $\mathcal{I}(h_L) = \mathcal{O}_X$,
3. $h_T$ is a singular metric on $(K_X \otimes S \otimes L)^{\otimes m}$ with $\sqrt{-1} \Theta_{h_T} \geq 0$.

Assume that for every $W \in \bigcup_{k=0}^{\dim X} W^{(k)}$ and $J \subseteq \{1, \ldots, q\}$ we have

$$\mathcal{I}(\langle |\sigma_S|_{h_S}^{-2} \sigma_{h_{L_i}} \rangle^2) = \mathcal{O}_W \quad (1.7)$$

---

6 Note that we do not require $h_L^{m} = h_1 \otimes \cdots \otimes h_{m-1}$. 10
and there exists a number $\varepsilon_0 > 0$ such that\footnote{11}

\begin{equation}
I\left(\left|\sigma_{J^W}^{\delta + \varepsilon}h_{L_i}^{1-\delta}h_T^{\varepsilon_0}\right|_W\right) = \mathcal{O}_W \text{ for all } \delta > 0 \text{ and } \varepsilon > 0.
\end{equation}

Then the restriction map

$\Gamma(X, (K_X \otimes S \otimes L)^{\otimes_m}) \xrightarrow{\alpha} \Gamma(W, (K_X \otimes S \otimes L)^{\otimes_m}|_W)$

is surjective.

Theorem 5 will be proved in Section 6 by the successive extension-correction process together with Lemma 6.1, a pluricanonical analogue of Lemma 4.1. The proof of Lemma 6.1 incorporates the strong openness conjecture with the approach of Păun \[33\], \[1\], and \[3\]. The metrics considered in this approach do not run out the vanishing coming from a section to be extended, and hence help overcome the aforementioned obstacle. A typical situation fulfilling the conditions (1), (2), and (1.7) in Theorem 5 is the following. Let $A$ be an ample $\mathbb{Q}$-divisor and $\Delta$ an effective $\mathbb{Q}$-divisor. Suppose that $(X, S + \Delta + A)$ is a log-smooth dlt pair with $S + \Delta + A_{\downarrow} = S$ such that $m(\Delta + A)$ is Cartier for some $m \in \mathbb{N}$. Then we may write $m(\Delta + A)$ as $\Delta_1 + \cdots + \Delta_{m-1}$ (cf. \[16\]) where $\Delta_i$ are all reduced simple normal crossing divisors. We may simply take $L_i$ (resp. $h_{L_i}$) to be $\Delta_i$ (resp. the metric associated to a defining section of $\Delta_i$), and take $h_L$ to be $h_A \cdot h_{\Delta}$ where $h_A$ is a smooth metric with strictly positive curvature and $h_{\Delta}$ is the metric associated to a defining section of $\Delta$.

Applying Theorem 4 with $W = \{\{x\}\}$ and $J(W) = J(x)$, we obtain the following funny nonvanishing result.

**Corollary 6.** If $(L_i, h_{L_i}) (i = 1, \ldots, m)$ are holomorphic line bundles on $X$ with singular hermitian metrics and if $x$ is a 0-dimensional transversal stratum of $S$ at which all $h_{L_i}$ take finite values, and if

$$\sqrt{-1}\Theta_{h_{L_i}} \geq \pm \sqrt{-1}\Theta_{h_j} \quad (i = 1, \ldots, m \text{ and } j \in J\{x\})$$

then $I\left((h_{L_1} \otimes \cdots \otimes h_{L_m})^{\frac{1}{m}}\right)\mathcal{O}_X(\otimes_{i=1}^m K_X \otimes S \otimes L_i)$ admits a global section which is nonvanishing at $x$.\footnote{Note that (1.8) implies that $h_{L_i}|_W$ and $h_T|_W$ are well-defined singular metrics on $W$ according to our convention.}
The following result is obtained by taking $W = \{\{x\}\}$ in the proof of Corollary 3 and replace the bundle $K_X \otimes S \otimes \det F$ by its $m$-th power.

**Corollary 7.** If $F$ is a holomorphic vector bundle such that $\mathcal{O}_{\mathbb{P}(F)}(1)$ admits a smooth hermitian metric with strictly positive curvature, and if $x$ is a 0-dimensional transversal stratum of $\mathcal{S}$, then

$$\mathcal{O}_X((K_X \otimes S \otimes \det F)^{\otimes m} \otimes \text{Sym}^k F)$$

is generated by globally sections at $x$, and hence generically globally generated, for every $k,m \geq 0$.

**Remark 1.5.** As parallel to the vanishing theorems of Griffiths [19] and of Demailly [9], the same extension/generic generation statements as in Corollaries 3 and 7 hold with $(K_X \otimes S \otimes \det F)^{\otimes m} \otimes \text{Sym}^k F$ replaced by the bundle $V$ in the following situations:

1. For holomorphic vector bundles $F_i (i = 1, \ldots, h)$ such that $\mathcal{O}_{\mathbb{P}(F_i)}(1)$ admit smooth hermitian metrics with strictly positive curvature, we may take

$$V = (K_X \otimes S)^{\otimes m} \otimes \bigotimes_{i=1}^{h}(\det F_i)^{\otimes m} \otimes \text{Sym}^{k_i} F_i$$

since $\bigotimes_{i=1}^{h}\text{Sym}^{k_i} F_i$ is a direct summand of $\text{Sym}^k (F_1 \oplus \cdots \oplus F_h)$ ($k = k_1 + \cdots + k_h$).

2. For a holomorphic vector bundle $F$ such that $\mathcal{O}_{\mathbb{P}(F)}(1)$ admit smooth hermitian metrics with strictly positive curvature, we may take

$$V = (K_X \otimes S)^{\otimes m} \otimes (\det F)^{\otimes m} \otimes \Gamma^\lambda F.$$  

This can be seen by Manivel’s idea that $\Gamma^\lambda F$ is a direct summand of $\bigotimes_{i=1}^{h(\lambda)} \text{Sym}^{\lambda_i} F$ (cf. [17] 8.3, Corollary 2).

Finally we would like to talk about the original motivation of the current paper and propose a question. In their seminal work [15], Demailly, Hacon, and Păun considers the following statements.

---

8For every $r$-dimensional complex vector space $E$ the (equivalent classes of) irreducible representations $E^\lambda$ of $GL(E)$ (cf. [17] 8.2) can be listed according to parameters $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{Z}_{\geq 0}$ such that $\lambda_1 \geq \cdots \geq \lambda_r$. We let $h(\lambda) := |\{k \mid \lambda_k > 0\}|$. For any vector bundle $F$ we let $\Gamma^\lambda F$ denote the vector bundle associated to the $\lambda$-th irreducible representation.
Existence of good minimal models (GMM\(_n\)). Every \(n\)-dimensional klt pseudoeffective pair \((X, \Delta)\) admits a good minimal model.

**Nonvanishing (NV\(_n\)).** If \((X, \Delta)\) is an \(n\)-dimensional klt pair and \(K_X + \Delta\) is pseudoeffective, then \(\kappa(K_X + \Delta) \geq 0\).

**DLT Extension (DLT\(_n\)).** If \((X, S + B)\) is an \(n\)-dimensional dlt pair such that 
\(\triangleleft S + B \triangleleft = S\), \(K_X + S + B\) is nef, and \(K_X + S + B \sim Q D\) for some effective \(D\) such that \(S \subseteq \text{supp} \, D\), then the restriction map
\[
H^0(X, m(K_X + S + B)) \longrightarrow H^0(S, m(K_X + S + B)|_S)
\]
is surjective for sufficiently divisible \(m > 0\).

We let (NV\(_n^{\text{slc}}\)) stand for the statement of (NV\(_n\)) with klt replaced by semi-log canonical. It is shown ([15] Theorems 1.4 and 1.8) that (GMM\(_n\)) holds if both (NV\(_n^{\text{slc}}\)) and (DLT\(_n\)) hold; (DLT\(_n\)) holds if \((X, S+B)\) is assumed to be a plt pair, i.e., if \(S\) is a prime divisor. Along this line the problem of extension from simple normal crossing divisors becomes crucial. In essentially all known extension results with \(L^2\)-estimates, when the center is not smooth, only sections which vanish along suitable analytic subsets can be extended. In view of this, the author started to examine the possibility of proving a variant of (DLT\(_n\)), which aims at extending sections from minimal strata instead of from the whole snc divisor, and this was the original motivation of the current work. On the other hand, if one insists to apply extension theorems to attack the snc extension problem, it seems that not many strategies of dealing with the vanishing issue, if any, have come out other than the aforementioned successive extension-correction. However, even with the method successive extension-correction, one has to start from the bottom level, namely, extension from minimal strata. Unfortunately, the results established in this paper, which require stronger curvature assumption, is not sufficient for establishing the expected variant of (DLT\(_n\)). Nevertheless, we ask

\[9\] There appeared a different attempt towards proving (DLT\(_n\)), due to Gongyo and Matsumura [18], by using theorems of injectivity instead of theorems of Ohsawa–Takegoshi type. An advantage of their approach is that the condition \(S \subseteq D\) in (DLT\(_n\)) is naturally incorporated in the framework of theorem of injectivity. However, the task is then turned to finding singular metrics with specific behaviour, the existence of which is not clear at the moment.
the following question, an affirmative answer to which will be very helpful in proving that \((GMM_n)\) holds if \((NV_n)\) does, as indicated in the survey\(^5\).

**Question 2.** Is the following statement true?

Let \((X, S, W, c_1, \ldots, c_q)\) consist of

- a projective manifold \(X\),
- a family of data \(S : (S_j, \sigma_j, h_j) (j = 1, \ldots, q)\) which consist of a holomorphic line bundle \(S_j\) on \(X\), a nonzero holomorphic section \(\sigma_j\) of \(S_j\), and a smooth hermitian metric \(h_j\) on \(S_j\) for every \(j\) such that (the divisor) \(S_1 + \cdots + S_q\) is of simple normal crossing,
- a minimal stratum \(W\) of \(S\), and
- a collection of positive numbers \(c_1, \ldots, c_q\).

There exists a constant \(C\) depending only on \((X, S, W, c_1, \ldots, c_q)\) such that, for any holomorphic line bundle \(F\) on \(X\) with a singular hermitian metric \(h_F\) such that
\[
\sqrt{-1} \Theta_{h_F} \geq 0 \quad \text{and} \quad \sqrt{-1}(c_1 \Theta_{h_1} - \cdots - c_q \Theta_q) \geq 0,
\]
if \(u\) is a holomorphic section of \(K_W \otimes S^W \otimes F\) on \(W\), then there exists a holomorphic section \(U\) of \(K_X \otimes S \otimes F\) with
\[
\int_X \langle U \rangle^2_{h_S \otimes h_F} \leq C \int_W \langle u \rangle^2_{h_{hW} \otimes h_F}.
\]

**Acknowledgement.** The main part of this paper is written during the period I visited Institut Fourier in 2018-2019. I am most grateful to Professor Demailly for his warm hospitality and generosity in sharing many of his interesting and challenging ideas and in sparing plenty of time for discussions. The strategy of proving Theorem 2 was essentially suggested by him, and some of the crucial settings of Theorem 3.1 are inspired by discussions with him. I am indebted to Professors Qi’an Guan and Xiangyu Zhou for patiently answering my several questions about their work [20]. I also like to thank Mario Chan, Jiun-Cheng Chen, Shin-Yao Jow, Yih Sung, and Sz-Sheng Wang for many stimulating discussions, and thank Professors Dano Kim, and Takeo Ohsawa for giving comments. Finally, I would like to thank Professors Shing-Tung Yau, Yng-Ing Lee, Hui-Wen Lin and Chin-Lung Wang for their constant supports and encouragements.
2. A general scheme for extension of holomorphic sections

In this section we provide the first part of the proof of Theorem 1 in a form of a metatheorem, Theorem 2.3. Although our argument models on that for Theorem 2.1 in [20], we include a proof here for completeness since some adjustments are made as explained in the introduction.

The following setting will be in force throughout the current section.

**Setting.** Let \((Y, (E, h), Z, Z_0, \psi, \phi, A, G)\) consist of

(i) a Kähler manifold \(Y\),

(ii) an analytic subspace \(Z = V(\mathcal{I}_Z)\) of \(Y\),

(iii) an analytic set \(Z_0\) of \(Y\),

(iv) a holomorphic vector bundle \(E\) on \(Y\) with a smooth hermitian metric \(h\),

(v) two usc functions \(\psi : Y \rightarrow [-\infty, \infty)\) and \(\phi : Y \rightarrow [-\infty, A)\) with \(A \in (-\infty, \infty]\),

and

(vi) a strictly positive function \(G \in C^\infty((-A, \infty))\)

such that

\[(2.1) \quad \phi|_{Y \setminus Z_0} \text{ is smooth and } \phi^{-1}(-\infty) = Z_0,\]

\[(2.2) \quad \psi|_{Y \setminus Z_0} \text{ is smooth and } \mathcal{I}(\psi) \subseteq \mathcal{I}_Z,\]

\[(2.3) \quad \sqrt{-1} \left(\Theta_h + \overline{\partial} \partial \psi \otimes \text{id}_E\right) \geq 0 \text{ in the sense of Nakano on } Y \setminus Z_0,\]

and

\[(2.4) \quad \sqrt{-1} \left(\Theta_h + \left(\overline{\partial} \partial \psi + \frac{G'(-\phi)}{G(-\phi)} \overline{\partial} \partial \phi\right) \otimes \text{id}_E\right) \geq_{\text{Nak}} 0 \text{ on } Y \setminus Z_0.\]

Before giving the precise statement of the theorem, we first introduce two families of functions, one of which will be used to construct weights for \(L^2\)-norms, and another will be used to construct truncations when dealing with singularity.
Definition 2.1.\footnote{$G_A$ here essentially consists of all functions $G$ with $G''(\cdot) = c_A(\cdot)e^{-\cdot}$ fulfilling the conditions in Theorem 2.1 of \cite{20}.} For $A \in (-\infty, \infty]$, we define $G_A$ to be the set of functions $G \in C^\infty((-A, \infty))$ such that

$$G^{(j)}(-A^+) := \lim_{y \to -A^+} G^{(j)}(y) \ (j = 0, 1, 2)$$

all exist and are strictly positive,

$$G''(\cdot) > 0, \ G'^2 - GG'' > 0, \text{ and}$$

$$G'(\infty) := \lim_{y \to \infty} G'(y) \text{ exists and is strictly positive.} \quad (2.5)$$

It is clear that $G > 0$ and $G' > 0$ for every $G \in G_A$. See Section 4 for a practical construction of such functions $G$.

Definition 2.2. For every $0 < a < b$ we let

$$\mathcal{X}_{a,b} := \left\{ \chi \in C^\infty(\mathbb{R}) \left| \chi \geq 0, \ \text{supp} \chi \subseteq (a, b), \text{ and } \int_{\mathbb{R}} \chi = 1 \right. \right\}.$$ 

For any $\chi \in \mathcal{X}_{a,b}$ we define $\tau_\chi \in C^\infty(\mathbb{R})$ by integrating $\chi$ twice with initial values $\tau'_\chi(b) = 1$ and $\tau_\chi(b) = b$.

We remark that $\tau_\chi$ is convex and remains constant on $(-\infty, a]$, and $\tau_\chi(y) = y$ for every $y \in [b, \infty)$; in particular, $\tau_\chi \geq a$ and $0 \leq \tau'_\chi \leq 1$.

We state the following metatheorem.

Theorem 2.3. Given $(u, \Omega, U_0, G)$ consisting of

(i) a section $u \in \Gamma(Z, \mathcal{O}_Y(K_Y \otimes E)|_Z)$,

(ii) a relatively compact open subset $\Omega$ of $Y$ admitting a complete Kähler metric,

(iii) a section $U_0 \in \Gamma(Y, \mathcal{O}_Y(K_Y \otimes E))$ such that $U_0|_{Z \cap \Omega} = u|_{Z \cap \Omega}$, and

(iv) $G \in G_A$,

if $\chi \in \mathcal{X}_{a,b}$ for some $a < b$ and

$$\liminf_{t \to -\infty} \int_{\Omega} \chi(\phi - t) e^{-\psi}(U_0)^2_h < \infty,$$

then there exists $U_\Omega \in \Gamma(\Omega, \mathcal{O}_Y(K_Y \otimes E))$ such that $U_\Omega|_{Z \cap \Omega} = u|_{Z \cap \Omega}$ and

$$\int_{\Omega} G''(-\phi) e^{-\psi}(U_\Omega)^2_h \leq G'(\infty) \liminf_{t \to -\infty} \int_{\Omega} \chi(\phi - t) e^{-\psi}(U_0)^2_h. \quad (2.6)$$
The proof of this theorem will be reduced to the following result.

**Lemma 2.4.** Let \((u, G, \Omega, U_0)\) be as in Theorem 2.3 and \(a < b\) be real numbers such that \((G/G')(-b) \geq b - a\). For every \(\chi \in X_{a,b}\) there exists a section \(U_\chi \in \Gamma(\Omega, \mathcal{O}_Y(K_Y \otimes E))\) such that \(U_\chi|_{\Omega \cap Z} = u|_{\Omega \cap Z}\) and

\[
\int _\Omega G''(-\tau_\chi(\phi)) e^{-\psi} \langle U_\chi - (1 - \tau'_\chi(\phi))U_0 \rangle_h^2 \leq G'(\infty) \int _\Omega \chi(\phi) e^{-\psi} \langle U_0 \rangle_h^2. 
\]

**Proof of Theorem 2.3 assuming Lemma 2.4.** Select a sequence \(t_j \to -\infty\) as \(j \to \infty\) such that

\[
\lim_{j \to \infty} \int _\Omega \chi(\phi - t_j) e^{-\psi} \langle U_0 \rangle_h^2 = \liminf_{t \to -\infty} \int _\Omega \chi(\phi - t) e^{-\psi} \langle U_0 \rangle_h^2.
\]

Let \(a_j := a + t_j\) and \(b_j := b + t_j\). We have \((G/G')(-b_j) \geq b_j - a_j = b - a\) for \(j\) sufficiently large. Applying Lemma 2.4 with the functions \(\chi_j(\cdot) := \chi(\cdot - t_j) \in X_{a_j, b_j}\) we obtain sections

\[U_{\chi_j} \in \Gamma(\Omega, \mathcal{O}_Y(K_Y \otimes E))\]

such that

\[
\int _\Omega G''(-\tau_{\chi_j}(\phi)) e^{-\psi} \langle U_{\chi_j} - (1 - \tau'_{\chi_j}(\phi))U_0 \rangle_h^2 \leq G'(\infty) \int _\Omega \chi_j(\phi) e^{-\psi} \langle U_0 \rangle_h^2.
\]

Fix a sequence of relatively compact open sets

\[O_1 \subseteq O_2 \subseteq \cdots \subseteq O_m \subseteq O_{m+1} \subseteq \cdots\]

of \(Y\) such that \(Y \setminus Z = \bigcup_{m=1}^{\infty} O_m\). There exists a strictly increasing sequence \(j_m\) such that \(b + t_{j_m} \leq \min_{O_m} \phi\) for every \(m \in \mathbb{N}\). Then \(\tau_{\chi_{j_m}}(\phi)|_{O_m} = \phi|_{O_m}\) and \(\tau'_{\chi_{j_m}}(\phi)|_{O_m} = 1\), and hence

\[
\int _{O_m} G''(-\tau_{\chi_{j_m}}(\phi)) e^{-\psi} \langle U_{\chi_{j_m}} \rangle_h^2 \leq G'(\infty) \int _\Omega \chi_{j_m}(\phi) e^{-\psi} \langle U_0 \rangle_h^2 \quad (m \in \mathbb{N}).
\]

To get the desired section, we seek to apply Lemma A.2.2 with

\[(M, U_m, w_m) = \left(\Omega, U_{\chi_{j_m}}, G''(-\tau_{\chi_{j_m}}(\phi))\right).
\]

---

11This holds when \(b \to -\infty\) with \(b - a\) bounded by a fixed constant. By (2.3) we have \(\lim_{y \to \infty} \frac{G(y)}{y} = \lim_{y \to \infty} \frac{G'(y)}{y} = G'(\infty) > 0\), and hence \((G'/G)(y) \to 0\) as \(y \to \infty\).
Note that $w_m$ converges to $w = G''(-\phi)$ almost everywhere as $m \to \infty$, and it is direct to check that the conditions of Lemma A.2.2 are fulfilled to yield a section $U_\Omega \in \Gamma(\Omega, \mathcal{O}_Y(K_Y \otimes E))$ such that $U_\Omega|_{\partial \Omega} = u|_{\partial \Omega}$ and
\[
\int_\Omega G''(-\phi) e^{-\psi} \langle U_\Omega \rangle_h^2 \leq G'(\infty) \liminf_{m \to \infty} \int_\Omega \chi_{j_m}(\phi) e^{-\psi} \langle U_0 \rangle_h^2.
\]
\[\square\]

Now it remains to prove Lemma 2.4, and the rest of this section constitutes a proof.

2.1 (\(\overline{\partial}\)-equations with truncated data). For simplicity, in the following discussions we denote $\tau_\chi$ by $\tau$ for $\chi \in \mathcal{X}_{a,b}$. We manually extend the domains of $\tau$, $\tau'$, and $\tau'' = \chi$ to include $-\infty$ by defining
\[
(2.7) \quad \tau^{(j)}(-\infty) := \lim_{y \to -\infty} \tau^{(j)}(y) = \tau^{(j)}(a) \quad (j = 0, 1, 2).
\]
Then $\tau(\phi)$, $\tau'(\phi)$, and $\tau''(\phi)$ are all smooth functions on $Y$ whose restrictions to the open neighborhood $\phi^{-1}([-\infty, a])$ of $Z_0$ are constant functions. In particular,
\[
(2.8) \quad \eta_\chi := \overline{\partial} \left( (1 - \tau'(\phi)) U_0 \right),
\]
can be viewed as a smooth $E$-valued $(n, 1)$-form on $Y$. We also note that $\tau'(\phi)$, $\tau''(\phi)$, and $\eta_\chi$ all vanish on $\phi^{-1}([-\infty, a])$. We will first obtain a solution $\gamma = \gamma_\chi$ to the \(\overline{\partial}\)-equation
\[
(2.9) \quad \overline{\partial} \gamma = \eta
\]
on $\Omega \setminus Z$ with $L^2$-estimate and define
\[
U_\chi := (1 - \tau'(\phi)) U_0 - \gamma_\chi.
\]
Then we shall obtain by a limiting process a holomorphic extension $U_\Omega$ of $u|_{\partial \Omega}$ on $\Omega$ with $L^2$-estimate.

2.2 (Choosing auxiliary weight functions by solving ODEs). We first explain the presence of part of the conditions in Definition 2.1, following the ODE approach in [20]. To exploit Theorem A.1.1 to solve the aforementioned
\(\overline{\partial}\)-equation with \(L^2\)-estimate, for any \(\chi \in \mathcal{X}_{a,b}\) we consider functions \(\lambda, \mu, \text{ and } \nu\) of the form (with \(\tau_\chi\) denoted by \(\tau\))

\[
\lambda = s(-\tau \circ \phi), \quad \mu = t(-\tau \circ \phi), \quad \text{and } \quad \nu = w(-\tau \circ \phi)
\]

where \(s > 0, t > 0, \text{ and } w\) are smooth functions on \((-A, \infty)\), and define

\[
(2.10) \quad \hat{h}(\cdot, \cdot) := h(\cdot, \cdot)e^{-\psi} \quad \text{and} \quad \tilde{h}(\cdot, \cdot) = h(\cdot, \cdot)e^{-\psi}e^{-\nu},
\]

whose restriction on \(Y \setminus Z_0\) is then smooth. To check the validity of (A.1), we proceed a direct calculation and obtain\(^{12}\)

\[
\lambda \Theta_{\hat{h}} - \overline{\partial} \overline{\partial} \lambda - \frac{1}{\mu} \partial \lambda \wedge \overline{\partial} \lambda = \lambda \Theta_{\hat{h}} + \lambda \partial \overline{\partial} \nu - \overline{\partial} \partial \lambda - \frac{1}{\mu} \partial \lambda \wedge \overline{\partial} \lambda
\]

\[
= s(-\tau \circ \phi) \Theta_{\hat{h}} + (s' - sw') \circ (-\tau \circ \phi) \cdot (\tau' \circ \phi) \partial \overline{\partial} \phi + (\tau'' \circ \phi) \partial \phi \wedge \overline{\partial} \phi
\]

\[
+ \left( sw'' - s'' - \frac{s'^2}{t} \right) \circ (-\tau \circ \phi) \cdot \partial (\tau \circ \phi) \wedge \overline{\partial} (\tau \circ \phi)
\]

\[
= (s' - sw') \circ (-\tau \circ \phi) \cdot (\tau'' \circ \phi) \partial \phi \wedge \overline{\partial} \phi
\]

\[
+ s(-\tau \circ \phi) \Theta_{\hat{h}} + (s' - sw') \circ (-\tau \circ \phi) \cdot (\tau' \circ \phi) \partial \overline{\partial} \phi
\]

\[
+ \left( sw'' - s'' - \frac{s'^2}{t} \right) \circ (-\tau \circ \phi) \cdot \partial (\tau \circ \phi) \wedge \overline{\partial} (\tau \circ \phi).
\]

As in [20], we impose the following conditions on \(s, t, \text{ and } w\):

\[
(2.11) \quad s > 0, t > 0, s' - sw' = 1 \quad \text{and} \quad sw'' - s'' - \frac{s'^2}{t} = 0.
\]

Then \((e^{-w}s)' = e^{-w}\), and hence \(s\) is of the form \(e^w \int e^{-w}\). By setting \(G = \int e^{-w}\), we have \(G' = e^{-w} > 0\),

\[
(2.12) \quad s = \frac{G}{G'}, \quad \text{and} \quad w = -\log G'.
\]

The second condition in (2.11) implies that \(s'' - sw'' - s'w' = (s' - sw')' = 0\), and hence

\[
(2.13) \quad t = \frac{-s'^2}{s'' - sw''} = \frac{s'}{w'}.
\]

A direct computation yields that

\[
(s'/s)w' = \left( \frac{G'}{G} - \frac{G''}{G'} \right) \left( -\frac{G''}{G'} \right) = -\frac{(G'^2 - GG'')G''}{GG'^2}
\]

\(^{12}\)Hereinafter we omit \(\otimes \text{id}_E\) from all appearances of (1, 1)-forms for simplicity.
and
\[ s + t = s \left( 1 - \frac{s' / s}{w'} \right) = s \left( \frac{w' - (s' / s)}{w'} \right) = s \left( \frac{-1 / s}{w'} \right) = -\frac{1}{w'} = \frac{G'}{G''}. \]

Therefore, that \( G = \int e^{-w} \) and the strictly positivity of \( s \) and \( t \) imply that
\[ G > 0, \quad G' > 0, \quad G'' > 0, \quad \text{and} \quad G'^2 - GG'' > 0, \]
(2.14)

It is easy to see that, conversely, if \( s, t, \) and \( w \) are determined by (2.12) and (2.13) according to a smooth function \( G \) on \((-\infty, \infty)\) satisfying (2.14), then (2.11) holds.

Note that every \( G \in G_A \) fulfills (2.14). We note for later use that
\[ e^{-w} = G' \quad \text{and} \quad \frac{e^{-w}}{s + t} = G''. \]

2.3 (Solving the \( \overline{\partial} \)-equations). Now we assume that \( G \in G_A \) and that the functions \( t \) and \( u \) are constructed as in (2.12) and (2.13).

Lemma 2.3.1. If \( b - a \leq s(-b) \), then
\[ \sqrt{-1} \left( \lambda \Theta_h - \partial \overline{\partial} \lambda - \frac{1}{\mu} \partial \lambda \wedge \overline{\partial} \lambda \right) \geq \text{Nak} \sqrt{-1} \chi(\phi) \partial \phi \wedge \overline{\partial} \phi \]
for every \( \chi \in \mathcal{X}_{a,b} \).

Proof. We have
\[ \sqrt{-1} \left( \lambda \Theta_h - \partial \overline{\partial} \lambda - \frac{1}{\mu} \partial \lambda \wedge \overline{\partial} \lambda \right) \]
\[ = \sqrt{-1} \left( (\tau'' \circ \phi) \partial \phi \wedge \overline{\partial} \phi + s(-\tau \circ \phi) \Theta_h + (\tau' \circ \phi) \overline{\partial} \phi \right) \]
\[ = \sqrt{-1} \left( (\chi \circ \phi) \partial \phi \wedge \overline{\partial} \phi + \left[ s(-\tau \circ \phi) (\Theta_h + \partial \overline{\partial} \psi) + (\tau' \circ \phi) \partial \overline{\partial} \phi \right] \right). \]

It suffices to show that the last term is nonnegative if \( b - a \leq s(-b) \). Note that \( s = \frac{G}{G'} \). By applying Lemma 2.3.2 below with \( S = s \), we see that
\[ \sqrt{-1} \left( s(-\tau \circ \phi) (\Theta_h + \partial \overline{\partial} \psi) + (\tau' \circ \phi) \partial \overline{\partial} \phi \right) \]
\[ \geq \sqrt{-1} (\tau' \circ \phi) \left( s(-\phi)(\Theta_h + \partial \overline{\partial} \psi) + \partial \overline{\partial} \phi \right) \geq 0 \]
by our curvature assumption (2.3) and (2.4). \( \Box \)

\[ \text{This conclusion corresponds to (5.7) in [20], which was stated there without a proof. It can be verified via Lemma 2.3.2 here, which I learned via private communication with Guan and Zhou. The proof given here follows essentially their argument.} \]
Lemma 2.3.2. For any $\chi \in \mathcal{X}_{a, b}$ and any $C^1$ function $S$, if $S' < 1$ and $b - a \leq S(-b)$, then $S(-\tau_\chi(y)) \geq S(-y) \tau_\chi(y)$.

Proof. We only need to consider those $y$ lying in $(a, b)$ since $\tau'(y) = 0$ if $y \leq a$, and $\tau(y) = y$ if $y \geq b$. For $y \in (a, b)$, then
\[
S(-\tau(y)) - S(-y) \tau'(y)
= S(-\tau(y)) - S(-y) + S(-y) - S(-y) \tau'(y)
\geq -\tau(y) + y + S(-y)(1 - \tau'(y)) \quad \text{by the mean value theorem and } S' < 1
= \int_y^b (\tau'(z) - 1)dz + S(-y)(1 - \tau'(y))
\geq (b - y)(\tau'(y) - 1) + S(-y)(1 - \tau'(y)) \quad \text{since } (\tau' - 1)' = \tau'' \geq 0
\geq (1 - \tau'(y))(S(-b) - b + a) \geq 0.
\]

Recall the right hand side of (2.9):
\[
\eta_x = \overline{\partial}((1 - \tau'(\phi))U_0) = -\chi(\phi) \overline{\partial} \phi \wedge U_0
\]
To apply Theorem A.1.1, first note that $\Omega \setminus Z_0$ admits a complete Kähler metric by Lemma A.2.1. Besides, since $\lambda$ and $\mu$ are smooth on $Y$, they are bounded on the relatively compact set $\Omega \setminus Z_0$. As for condition (A.1), for every $v \in \mathcal{D}^{n,1}(\Omega \setminus Z_0, E|_{\Omega \setminus Z_0})$ and (a fixed Kähler metric $g$) we have pointwise that
\[
(\eta_x, v)_{g, \overline{h}} = -\left(\chi(\phi) \overline{\partial} \phi \wedge U_0, v\right)_{g, \overline{h}} = -\left(\sqrt{\chi(\phi)} U_0, \sqrt{\chi(\phi)} \overline{\partial} \phi, v\right)_{g, \overline{h}},
\]
and hence by Lemma 2.3.1,
\[
\left|\left(\eta_x, v\right)_{g, \overline{h}}\right|^2 = \left|\int_{\Omega \setminus Z_0} (\sqrt{\chi(\phi)} U_0, \sqrt{\chi(\phi)} \overline{\partial} \phi, v)_{g, \overline{h}} \, dV_g\right|^2
\leq \left(\int_{\Omega \setminus Z_0} \chi(\phi) |U_0|_{g, \overline{h}}^2 \, dV_g\right) \left(\int_{\Omega \setminus Z_0} \sqrt{-1} \chi(\phi) (\overline{\partial} \phi \wedge \overline{\partial} \phi) [v, v]_{g, \overline{h}} \, dV_g\right)
\leq C' \left(\int_{\Omega \setminus Z_0} \sqrt{-1} (\lambda \Theta_{\overline{h}} - \overline{\partial} \partial \lambda) [v, v]_{g, \overline{h}} \, dV_g\right).
\]
where
\[
C' = \int_{\Omega \setminus Z_0} \chi(\phi) |U_0|_{g, \overline{h}}^2 \, dV_g = \int_{\Omega \setminus Z_0} C'(\tau_\chi(\phi)) e^{-\psi} \chi(\phi) (U_0)^2_{\overline{h}}.
\]
Since \( w' = \frac{s' - 1}{s} < 0 \) and \( e^{-w} = G' \), we have \( C' \leq C''(\infty) I_\chi \) with

\[
I_\chi := \int_{\Omega \setminus Z_0} \chi(\phi) e^{-\psi} (U_0)^2_h < \infty.
\]

By Theorem A.1.1 there exists a measurable \( E \)-valued \((n, 0)\)-form \( \gamma_\chi \) such that

\[
(2.16) \quad \overline{\partial'} \gamma_\chi = \overline{\partial'} \left( (1 - \tau'(\phi)) U_0 \right) \text{ on } \Omega \setminus Z_0 \text{ in the sense of current}
\]

and

(2.17) \[
\int_{\Omega \setminus Z_0} G''(\tau(\phi)) e^{-\psi} |\gamma_\chi|^2_{g, h} dV_g \leq G'(\infty) I_\chi.
\]

We let

\[
U_\chi := (1 - \tau'(\phi)) U_0 - \gamma_\chi.
\]

By (2.17), \( U_\chi \) is a holomorphic section of \( K_Y \otimes E \) on \( \Omega \setminus Z_0 \). Since \( G''(\tau(\phi))|\Omega \) has strictly positive lower bounds, (2.18) implies that

(2.18) \[
\int_{\Omega \setminus Z_0} (1 - \tau'(\phi)) U_0)^2_h < \infty,
\]

and hence

\[
\int_{\Omega \setminus Z_0} U_\chi)^2_h < \infty.
\]

Since \( h \) is smooth, \( U_\chi \) extends to a holomorphic section of \( K_Y \otimes E \) on \( \Omega \). As noted right after (2.8), \( \tau'(\phi) \) vanishes on an open neighborhood of \( Z_0 \), say, \( V \). Thus, the measurable section \( \gamma_\chi|_{V \cap \Omega} \), which coincides with the holomorphic section \( U_\chi|_{V \cap \Omega} - U_0|_{V \cap \Omega} \), is holomorphic. Now the last inequality in (2.19) implies that the local expression of \( \gamma_\chi|_{V \cap \Omega} \) lies in the ideal sheaf \( \mathcal{I}(\psi) \subseteq \mathcal{I}_Z \). This shows that \( U_\chi|_{\Omega \cap Z} = U_0|_{\Omega \cap Z} = u|_{\Omega \cap Z} \). Finally, the required \( L^2 \)-estimate is simply (2.18). This completes the proof of Lemma 2.4.

3. Extension of twisted canonical sections from an admissible family of strata with \( L^2 \)-estimates

In this section we prove an extension theorem which is slightly more general than Theorem 1. This will be done by applying Theorem 2.3 with a particular
choice of the functions \( \psi \) and \( \varphi \). The estimate (1.4) will be obtained by establishing an explicit upper bound of the right hand side of (2.6). See Lemma 3.3 for the detail.

**Theorem 3.1.** Suppose that \((X, S, (F, h_F), W, \delta, \delta\bullet, \gamma\bullet, A)\) and

\[
u_W \in \Gamma(W_{\text{reg}}, K_{W\text{reg}} \otimes S^W \otimes F|_W) \quad (W \in \mathcal{W}),
\]

satisfy (i)-(vii) and (1.3) in the statement of Theorem 1, and that \( G \in \mathcal{G}_A \).

(I) Assume that \( h_F \) is smooth. If on \( X \setminus \bigcup_{j=1}^q S_j \) we have

\[
\tilde{\Theta} := \sqrt{-1} \left( \Theta_{h_F} + \left( \sum_{j=1}^q \delta_j \Theta_{h_j} + \delta \sum_{W \in \mathcal{W}} |J_W| \partial \bar{\partial} \log \sum_{j \in J_W} |\sigma_j|_{h_j}^2 \right) \otimes \text{id}_F \right) \geq 0
\]

and

\[
\sqrt{-1} \left( \tilde{\Theta} - \frac{G'(\log |\sigma_S|_{h_S}^{2\gamma\bullet})}{G(\log |\sigma_S|_{h_S}^{2\gamma\bullet})} \sum_{j=1}^q \gamma_j \Theta_{h_j} \otimes \text{id}_F \right) \geq 0
\]

in the sense of Nakano, and if

\[
G''(\log |\sigma_S|_{h_S}^{2\gamma\bullet}) \frac{|\sigma_S|_{h_S}^{2\delta\bullet-2}}{\prod_{J \in J_0(W)} \left( \sum_{j \in J} |\sigma_j|_{h_j}^2 \right)^{\delta_j|J|}} \]

is locally bounded from below,

then for any collection of sections

\[
u_W \in \Gamma(W_{\text{reg}}, K_{W\text{reg}} \otimes S^W \otimes F|_W) \quad (W \in \mathcal{W})
\]

such that

\[
\int_{W_{\text{reg}}} \prod_{J' \in J_0(W) \setminus \{J_W\}} \left( \sum_{j \in J'} |\sigma_j|_{h_j}^2 \right)^{\delta_{|J'|}} \left( \frac{\nu_W}{\sigma^{J_W}} \right)^{2} |J_W| < \infty \quad (W \in \mathcal{W}),
\]
there exists $U \in \Gamma(X, K_X \otimes S \otimes F)$ such that $U_W = u_W$ for every $W \in \mathcal{W}$ and

$$
\int_X G''(\log |s_S|^{2\gamma \bullet}_{h_S}) \prod_{J \in J_0(\mathcal{W})} \left( \sum_{j \in J} |s_j|^2_{h_j} \right)^{\delta_{|J|}} \left< \frac{U}{s_S} \right>_{h_F}^2
\leq G'(\infty) \sum_{W \in \mathcal{W}} \sum_{j \in J_W} B_{|J_W| - 1}(\delta) \int_{\mathcal{W}_{\text{reg}}} \prod_{J' \in J_0(\mathcal{W}) \setminus \{J_W\}} \left( \sum_{j \in J'} |s_j|^2_{h_j} \right)^{\delta_{|J'|}} \left< \frac{u_W}{\sigma_{J_W}} \right>_{h_F}^2
$$

with

$$B_{k-1}(\delta) = \pi^k \int_{\Delta_{k-1}} \frac{d\sigma_1 \cdots d\sigma_{k-1}}{(\sigma_1 \cdots \sigma_{k-1}(1 - \sigma_1 - \cdots - \sigma_{k-1}))^{(1-\delta)}} \quad (B_0(\delta) := 1)$$

and

$$\Delta_{k-1} = \{(\sigma_1, \ldots, \sigma_{k-1}) \in [0, \infty)^k \mid \sigma_1 + \cdots + \sigma_{k-1} \leq 1\}.$$

(II) Suppose that $(F, h_F)$ is a line bundle with a hermitian metric. The statement in (I) still holds if we drop the smoothness assumption on $h_F$ and replace (3.1) and (3.2) by the following family of conditions:

$$\Theta_{h_F} - \sum_{j \in J(\mathcal{W})} \left( \frac{\delta(t_j - 1)}{G'(-A^+)} s_{\gamma_j} \right) \Theta_{h_j} - \sum_{j \notin J(\mathcal{W})} \left( \frac{G'(-A^+)}{G(-A^+)} s_{\gamma_j} \right) \Theta_{h_j} \geq 0$$

for $s \in [0, 1]$ and $(t_j) \in \prod_{j \in J(\mathcal{W})} [0, T_j]$ with $T_j := \sum_{W: j \in J_W} |J_W|$.\[3.6\]

Proof. We may restrict the above data to $X \setminus H$ for any analytic hypersurface $H$ which contains no $W \in \mathcal{W}$ and, by (1.2) and Riemann’s extension theorem, it suffices to prove the theorem on $X \setminus H$. In particular, we may assume that $X$ is Stein, all vector bundles involved are trivial, and $S$ is a reduced snc divisor. Thus we have the adjunction maps

$$\Gamma(X, K_X \otimes S \otimes F) \xrightarrow{(s_J)_{s_j}} \Gamma(S_J, K_{S_J} \otimes (S^J \otimes F)|_{S_J}).$$

In the situation of (II), we may apply the standard regularization process (e.g., 4.1 of [15]) and reduce to the (I) with $X$ a Stein manifold and $F$ a line bundle. Note that (3.6) is preserved by the regularization process since $\hat{\Theta}_{s,t,\bullet}$ is a linear combination of $\Theta_{h_F}$ and $\Theta_j$ ($j = 1, \ldots, q$) with constant coefficients.
Besides, when \( h_F \) is smooth, (3.6) implies both (3.1) and (3.2). This can be seen by Lemma A.3.1 and the fact that \( G'/G \) is strictly decreasing. Therefore, it remains to prove (I) assuming that \( X \) is Stein.

We may assume that, for every \( J \subseteq \{1, \ldots, q\} \), the family \( \mathcal{W} \) contains either all or none of the irreducible components of \( S_J \). More precisely, for every \( J \subseteq \{1, \ldots, q\} \) let \( Q_J \) be the union of all irreducible components of \( S_J \) which are not a member of \( \mathcal{W} \). It suffices to remove from \( X \) a hypersurface which contains \( \bigcup_J Q_J \) but contains no \( W \in \mathcal{W} \).

Let \( J_0 = J_0(W)( = \{ J_W \mid W \in \mathcal{W}\}) \). For every \( J \in J_0 \), let

\[ u_J \in \Gamma(S_J, K_{S_J} \otimes S^J \otimes F|_{S_J}) \]

be the section determined by \( u_J|_W = u_W \) for every \( W \) associated to \( J \), and fix an “initial extension” \( U_0 \in \Gamma(X, K_X \otimes S \otimes F) \) such that \( (U_0)_{S_J} = u_J \) \((J \in J_0)\). We fix an exhaustion \( \Omega_1 \Subset \Omega_2 \Subset \cdots \) of \( X \) with strongly pseudoconvex open subsets.

It is clear that the proof will be completed by applying Lemma 3.2 below with \( \Omega = \Omega_k \) for every \( k \) together with a normal family argument. \( \square \)

**Lemma 3.2.** If \( \Omega \) is a relatively compact open subset of \( X \) admitting a complete Kähler metric, then there exists \( U_\Omega \in \Gamma(\Omega, \mathcal{O}_X(K_X \otimes S \otimes F)) \) such that \( (U_\Omega)_{S_J \cap \Omega} = u_J|_{S_J \cap \Omega} \) for every \( J \in J_0 \) and

\[
\int_\Omega G''(\log |\sigma_S|_{h_S}^{2\gamma}) \prod_{J_0} \left( \sum_{j \in J} |\sigma_j|_{h_j}^{2\gamma} \right)^{\delta[J]} \left\langle \frac{U}{\sigma_S} \right\rangle_{h_F}^2 \leq G'(\infty) \sum_{J_0} \sum_{j \in J_0} B_{|J|}(\delta) \int_{S_J} \left( \sum_{J' \subseteq J_0 \setminus \{J\}} |\sigma_{J'}|_{h_{J'}}^{2\gamma} \right)^{\delta[J]} \left\langle \frac{u_J}{\sigma_{J'}} \right\rangle_{h_F}^2.
\]

**Proof.** Applying Theorem 2.3 with

\[
Y = X \setminus \left( \bigcup_{j \notin J_0} S_J \cup \left( \bigcup_{J \in J_0} S_J \right)_{\text{sing}} \right),
\]

\[h = h_S \otimes h_F, \ Z = Y \cap \bigcup_{j \in J_0} S_J, \ Z_0 = Y \cap \bigcup_{J \in J_0} S_J,\]


\[ \psi = \log \left( |\sigma_S|_{h_S}^{2(1-\delta^*)} \prod_{J \in J_0} \left( \sum_{j \in J} |\sigma_j|_{h_j}^2 \right)^{\delta|J|} \right), \text{ and } \phi = \log |\sigma_S|_{h_S}^{2\gamma^*}. \]

we see that the proof will be completed by Lemma 3.3 below. \[\square\]

**Lemma 3.3.**

\[ \limsup_{t \to -\infty} \int_\Omega \chi(\phi - t) e^{-\psi} \langle U_0 \rangle_{h_S}^2 = \int_\Omega \frac{\chi(e^\phi / \varepsilon) \langle U_0 \rangle_{h_S}^2}{\prod_{J \in J_0} \left( \sum_{j \in J} |\sigma_j|_{h_j}^2 \right)^{\delta|J|}}. \]

**Proof.** It will be convenient to set \( t = \log \varepsilon \) and define \( \tilde{\chi}(s) := \chi(\log s) \). We now analyse the integral

\[ I(\varepsilon) = \int_\Omega \chi(\phi - \log \varepsilon) e^{-\psi} \langle U_0 \rangle_{h_S}^2 = \int_\Omega \frac{\tilde{\chi}(e^\phi / \varepsilon) \langle U_0 \rangle_{h_S}^2}{\prod_{J \in J_0} \left( \sum_{j \in J} |\sigma_j|_{h_j}^2 \right)^{\delta|J|}}. \]

We choose for every \( p \in \Omega \) a relatively compact open neighborhood \( V_p \) satisfying the following conditions: if \( \mathcal{W} = \emptyset \), then \( V_p \cap \mathcal{W} = \emptyset \); if \( p \in \mathcal{W} \), then \( V_p \cap \mathcal{W} = V_p \cap S_{J_p} \). We let

\[ A_p(x) := |\sigma_{J_p}(x)|_{h_{J_p}}^{2(1-\delta^*)} \prod_{J \in J_0 \setminus (J_p)} \left( \sum_{j \in J} |\sigma_j(x)|_{h_j}^2 \right)^{\delta|J|} \quad (x \in X) \]

and

\[ \tilde{A}_p(x) := \left( |\sigma_{J_p}(x)|_{h_{J_p}}^{2\gamma^*} \right)^{\frac{1}{\sum_{j \in J} \gamma_j}} = \left( \prod_{j \notin J_p} |\sigma_j(x)|_{h_j}^{2\gamma_j} \right)^{\frac{1}{\sum_{j \notin J_p} \gamma_j}} \quad (x \in X). \]

If \( J_p \) consists of \( j_1 < \cdots < j_k \), there exist an open neighborhood \( V_p \) of \( p \), a coordinate system \((z_1, \ldots, z_n)\) on \( V_p \), and a frame \( e_j \) of \( S_j|V_p \) for every \( j \in \{1, \ldots, q\} \) such that \( \sigma_j|_{V_p} = z_i e_j \) (\( i = 1, \ldots, k = |J_p| \)). We let \((x_i, y_i) := (\text{Re } z_i, \text{Im } z_i)\), and let \( \varphi_j \) \((j = 1, \ldots, q)\) and \( \varphi^j_p \) denote the local weights of \( h_j \) \((j = 1, \ldots, q)\) and \( h^j_p \) with respect to the associated local frames, respectively. Note that both \( A_p \) and \( \tilde{A}_p \) are strictly positive on \( V_p \). We let \((\xi_i, \eta_i) = (\text{Re } \zeta_i, \text{Im } \zeta_i)\) with

\[ \zeta_i := z_i e^{-\frac{1}{2} \varphi_i \tilde{A}_p^2} \quad (i = 1, \ldots, |J_p|). \]
We will write $|J_p|$ as $k = k(p)$ from now on for simplicity. By shrinking $V_p$ suitably we may assume that the following statements hold.

$$(\xi_1, \eta_1, \ldots, \xi_k, \eta_k, x_{k+1}, y_{k+1}, \ldots, x_n, y_n)$$
is a real coordinate system on $V_p$ with respect to which the open set $V_p$ is represented by the Cartesian product $D_p \times V_p'$ with

$$D_p = \left\{ (\xi, \eta) \in \mathbb{R}^{2k} \left| \sum_{i=1}^k |\xi_i|^2 + |\eta_i|^2 \leq R^2 \right. \right\}$$

for some $R = R_p < 1$ and $V_p' \subseteq \mathbb{R}^{2n-2k} \cong \mathbb{C}^{n-k}$ an open set.

The closure $\overline{V}_p$ is compact and is disjoint from $D_j$ if $j \notin J_p$; in particular, $A_p$ and $\tilde{A}_p$ both have strictly positive lower bounds on $\overline{V}_p$.

The coordinate systems $(z_1, \ldots, z_n)$ and $(\xi_1, \eta_1, \ldots, \xi_k, \eta_k, x_{k+1}, y_{k+1}, \ldots, x_n, y_n)$ and the frame sections $e_1, \ldots, e_q$ are all defined on a neighborhood of $\overline{V}_p$.

With respect to such a coordinate system we have

$$e^\psi = \begin{cases} 
|\xi_1^2 \cdots \xi_k^2|^{1-\delta} (|\xi_1|^2 + \cdots + |\xi_k|^2)^{\delta k} A_p \tilde{A}_p^{-k} & \text{if } p \in \overline{\Omega} \cap \bigcup_{J \in J_0} S_J; \\
|\xi_1^2 \cdots \xi_k^2|^{1-\delta} A_p \tilde{A}_p^{-(1-\delta)k} & \text{if } p \in \overline{\Omega} \setminus \bigcup_{J \in J_0} S_J
\end{cases}$$

and

$$e^\phi = |\xi_1|^{2\alpha_1} \cdots |\xi_k|^{2\alpha_k} \quad \text{with} \quad (\alpha_1, \ldots, \alpha_k) := (\gamma_{j_1}, \ldots, \gamma_{j_k}).$$

Choose a partition of unity $\{\varrho_l\}_{l \in \mathbb{N}}$ subordinate to $\{V_p\}_{p \in X}$. By the local finiteness of the partition of unity, we may assume that there is a finite set $P \subseteq \overline{\Omega}$ and $l(p) \in \mathbb{N}$ such that $(\sum_{p \in P} \varrho_{l(p)})|_{\overline{\Omega}} = 1$ and $\varrho_{l(p)} \subseteq V_p$ for every $p \in P$.

Note that $I(\varepsilon) \leq \sum_{p \in P} I_p(\varepsilon)$ where

$$I_p(\varepsilon) := \int_{V_p} \varrho_{l(p)} \tilde{\chi}(e^\phi /\varepsilon) e^{-\psi} \langle U_0 \rangle_{h_s \otimes h_F}^2.$$
We adopt the following abbreviations
\[(x, y) = (x_1, y_1, \ldots, x_n, y_n), \quad dx \, dy = dx_1 \, dy_1 \cdots dx_n \, dy_n,\]
\[(\xi, \eta) = (\xi_1, \eta_1, \ldots, \xi_k, \eta_k), \quad d\xi \, d\eta = d\xi_1 \, d\eta_1 \cdots d\xi_k \, d\eta_k,\]
\[(x', y') = (x_{k+1}, y_{k+1}, \ldots, x_n, y_n), \quad dx' \, dy' = dx_{k+1} \, dy_{k+1} \cdots dx_n \, dy_n\]
and write
\[U_0|_{V_p} = U_p \otimes e_1 \otimes \cdots \otimes e_q \otimes (dz_1 \wedge \cdots \wedge dz_n)\]
with \(U_p\) a holomorphic section of \(F\) on \(V_p\). If in particular \(p \in \bigcup_{J \in J_0} S_J\), we also write
\[u_J|_{S_J \cap V_p} = u_p \otimes e_J^p \otimes (dz_{k+1} \wedge \cdots \wedge dz_n),\]
with \(u_p\) a holomorphic section of \(F|_{S_J}\) on \(S_J \cap V_p\); if this is the case, since \((U_0)_{\mathcal{J}} = u_J\) for every \(J \in J_0\), we have on \(S_J \cap V_p\)
\[(3.10) \quad U_p(0, \ldots, 0, z_{k+1}, \ldots, z_n) = u_p(z_{k+1}, \ldots, z_n).\]

We let
\[(3.11) \quad \Xi(p)(\xi, \eta, x', y') := \varrho_{l(p)}(\xi, \eta, x', y') \cdot |U_p(z)|^2_{h^p} e^{-\varphi_{l(p)}} A_{p}^{-1}.\]
It is direct to see that there exist bounded functions \(\tau_i, \upsilon_i, \tilde{\tau}_i, \text{ and } \tilde{\upsilon}_i (i = 1, \ldots, k)\) on \(\mathcal{V}_p\) such that
\[(3.12) \quad e^{-\sum_{j \in J_p} \varphi_j} \, dx \, dy = (\tilde{A}_p)^{-k} \left(1 + \sum_{i=1}^{k} (\xi_i \tilde{\tau}_i + \eta_i \upsilon_i)\right) \, d\xi \, d\eta \, dx' \, dy'\]
and
\[(3.13) \quad \Xi(p)(\xi, \eta, x', y') - \Xi(p)(0, 0, x', y') = \sum_{i=1}^{k} (\xi_i \tilde{\tau}_i + \eta_i \tilde{\upsilon}_i).\]

Now we are ready to analyse \(I_p(\varepsilon)\).

**Claim 1.** \(I_p(\varepsilon) \to 0\) as \(\varepsilon \to 0\) if \(p \in P \setminus \bigcup_{J \in J_0} S_J\).

Consider the polar coordinate systems in the \(\xi_i \eta_i\)-directions:
\[\xi_i + \sqrt{-1} \eta_i = \zeta_i = r_i \, e^{\sqrt{-1} \theta_i} \quad (i = 1, \ldots, k).\]
By (3.8), (3.9), (3.12), (3.13), and that both $A_p$ and $\tilde{A}_p$ have strictly positive lower bounds on $V_p$, there exists a constant $C_p > 0$ independent of $\varepsilon$ such that

$$I_p(\varepsilon) = \int_{V_p} \tilde{\chi}(e^{\phi}/\varepsilon) \Xi^{(p)}(\xi, \eta, x', y') \left(1 + \sum_{i=1}^{k} (\xi_i \tau_i + \eta_i \upsilon_i)\right) d\xi d\eta dx' dy' \leq C_p \int_{\Pi[0, R^{1/\alpha_i}]} \chi \left(\log \frac{r_1^{2\alpha_1} \cdots r_k^{2\alpha_k}}{\varepsilon}\right) (r_1 \cdots r_k)^{2\varepsilon} \frac{dr_1}{r_1} \cdots \frac{dr_k}{r_k},$$

which converges to 0 as $\varepsilon \to 0$ by the dominated convergence theorem, since $\chi$ is compactly supported.

**Claim 2.** If $p \in P \cap \bigcup_{j \in J_0} S_j$, then

$$\limsup_{\varepsilon \to 0} I_p(\varepsilon) \leq \frac{B_{|J_p|-1}(\delta)}{\sum_{j \in J_p} \gamma_j} \int_{V'_p} \Xi^{(p)}(0, 0, x', y') \, dx' \, dy'.$$

To see this, note that $\chi, \tau_j, \upsilon_j, \tilde{\tau}_j, \tilde{\upsilon}_j$, and $\Xi^{(p)}$ are all bounded on $V_p$, and hence there exists a constant $C_p > 0$ independent of $\varepsilon$ such that

$$I_p(\varepsilon) = \int_{V_p} \tilde{\chi}(e^{\phi}/\varepsilon) \Xi^{(p)}(\xi, \eta, x', y') \left(1 + \sum_{i=1}^{k} (\xi_i \tau_i + \eta_i \upsilon_i)\right) d\xi d\eta dx' dy' \leq I_p(0) \int_{V'_p} \Xi^{(p)}(0, 0, x', y') \, dx' \, dy' + 2C_p I_p(1)(\varepsilon) \int_{V'_p} \, dx' \, dy'$$

where

$$I_p^{(m)}(\varepsilon) := \int_{BR(0)} \frac{\tilde{\chi}(e^{\phi}/\varepsilon)(|\xi_1|^2 + \cdots + |\xi_k|^2)^{\frac{m}{2}}} {(|\xi_1^2 \cdots \xi_k^2|)^{1-\delta} (|\xi_1|^2 + \cdots + |\xi_k|^2)^{\delta_k}} d\xi d\eta \quad (m = 0, 1).$$

We consider the following “spherical coordinate system” in the total $\xi \eta$-direction:

$$\xi_i + \sqrt{-1} \eta_i = \zeta_i = \sigma_i^\frac{1}{2} r \, e^{\sqrt{-1} \theta_i} \quad (i = 1, \ldots, k; \ \sigma_k := 1 - \sigma_1 - \cdots - \sigma_{k-1}),$$

with

$$r \in [0, 1), (\sigma_1, \ldots, \sigma_{k-1}) \in \Delta_{k-1}, \text{ and } (\theta_1, \ldots, \theta_k) \in [0, 2\pi]^k.$$
where

\[ \Delta_{k-1} = \{ (\sigma_1, \ldots, \sigma_{k-1}) \in [0, \infty)^k \mid \sigma_1 + \cdots + \sigma_{k-1} \leq 1 \}. \]

A direct commutation shows that

\[ |z_1^2 \cdots z_k^2|^{1-\delta} (|z_1|^2 + \cdots + |z_k|^2)^{\delta_k} = (\sigma_1 \cdots \sigma_{k-1} \sigma_k)^{1-\delta} r^{2k}, \]

and

\[ e^\phi = \sigma_1^{\alpha_1} \cdots \sigma_{k-1}^{\alpha_{k-1}} \sigma_k^{\alpha_k} r^{2(\alpha_1 + \cdots + \alpha_k)}, \]

and

\[ d\xi d\eta = \frac{1}{2^{k-1}} r^{2k} \frac{dr}{r} d\sigma_1 \cdots d\sigma_{k-1} d\theta_1 \cdots d\theta_k. \]

Thus

\[ I_p^{(m)}(\varepsilon) = 2\pi^k \int_{\Delta_{k-1}} \left( \int_0^{R} \chi \left( \log \left( \prod_i \frac{\sigma_i^{\alpha_i} r^{2 \sum_i \alpha_i}}{\varepsilon} \right) \right) r^m \frac{dr}{r} \right) \frac{d\sigma_1 \cdots d\sigma_{k-1}}{(\sigma_1 \cdots \sigma_{k-1} \sigma_k)^{1-\delta}}. \]

Note that

\[ \int_{\Delta_{k-1}} \frac{d\sigma_1 \cdots d\sigma_{k-1}}{(\sigma_1 \cdots \sigma_{k-1} \sigma_k)^{1-\delta}} < \infty \text{ for every } \delta \in (0, 1]. \]

Since \( \chi \) is compactly supported,

\[ \chi \left( \log \left( \prod_i \frac{\sigma_i^{\alpha_i} r^{2 \sum_i \alpha_i}}{\varepsilon} \right) \right) \to 0 \text{ as } \varepsilon \to 0 \]

for almost every \((\sigma_1, \ldots, \sigma_{k-1}, r) \in \Delta_{k-1} \times [0, R], \) and hence \( I_p^{(1)}(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \) by the dominated convergence theorem. Finally,

\[ I_p^{(0)}(\varepsilon) = 2\pi^k \int_{\Delta_{k-1}} \left( \int_0^{R} \chi \left( \log \left( \prod_i \frac{\sigma_i^{\alpha_i} r^{2 \sum_i \alpha_i}}{\varepsilon} \right) \right) \frac{dr}{r} \right) \frac{d\sigma_1 \cdots d\sigma_{k-1}}{(\sigma_1 \cdots \sigma_{k-1} \sigma_k)^{1-\delta}} \]

\[ = \frac{\pi^k}{\sum_i \alpha_i} \int_{\Delta_{k-1}} \left( \int_0^{\left( \prod_i \sigma_i^{\alpha_i} r^{2 \sum_i \alpha_i} \right) \frac{dv}{\varepsilon}} \chi(\log v) \frac{dv}{v} \right) \frac{d\sigma_1 \cdots d\sigma_{k-1}}{(\sigma_1 \cdots \sigma_{k-1} \sigma_k)^{1-\delta}} \]

\[ \to \frac{\pi^k}{\sum_i \alpha_i} \int_{\Delta_{k-1}} \frac{d\sigma_1 \cdots d\sigma_{k-1}}{(\sigma_1 \cdots \sigma_{k-1} \sigma_k)^{1-\delta}} \text{ as } \varepsilon \to 0 \]

by the monotone convergence theorem ( \( \int_0^\infty \chi(\log v) \frac{dv}{v} = \int_\mathbb{R} \chi = 1 \)). This completes the proof of Claim 2.
In summary, \[
\limsup_{\varepsilon \to 0} I(\varepsilon) \leq \sum_{p \in P \cap \bigcup_{J \in J_0} S_J} \frac{B_{|J| - 1}(\delta)}{\sum_{j \in J} \gamma_j} \int_{V_p'} \Xi^{(p)}(0, 0, x', y') \, dx' \, dy'.
\]

By (3.7), (3.10), and (3.11),
\[
\int_{V_p'} \Xi^{(p)}(0, 0, x', y') \, dx' \, dy' = \int_{V_p'} \varrho_l(p)(0, 0, x', y') \frac{|u_p(z_{k(p)} + 1, \ldots, z_n)|^2 h_F}{A_p(0, 0, x', y')} e^{-\varphi_{J_p}(0, 0, x', y')} \, dx' \, dy'
\]
\[
= \int_{S_J \cap V_p} \varrho_l(p) \frac{\langle u_{J_p} \rangle^2 h_F}{|\sigma_{J_p}|^2 h_p} \prod_{J \in J_0 \setminus \{J_p\}} \left( \sum_{j \in J} |\sigma_j|_{h_j}^2 \right)^{\delta[J]} \delta[J].
\]

Therefore,
\[
\limsup_{\varepsilon \to 0} I(\varepsilon) \leq \sum_{J \in J_0} \frac{B_{|J| - 1}(\delta)}{\sum_{j \in J} \gamma_j} \int_{S_J} \prod_{J' \in J_0 \setminus \{J\}} \left( \sum_{j \in J'} |\sigma_j|_{h_j}^2 \right)^{\delta[J']} \frac{\langle u_J \rangle^2}{\sigma^J_{h_F}}.
\]

\[
4. \text{ Quantitative and qualitative consequences of Theorem 1}
\]

4.1 (Proof of Theorem 1). Theorem 1 is a specialization of Theorem 3.1 with a particular type of function \(G\). As pointed out in [20] (in a slightly different manner), a function \(G \in C^\infty(-A, \infty)\) lies in \(G_A\) if
- \(G^{(j)}(-A^+)\) \((j = 0, 1, 2)\) all exist and are strictly positive,
- \(G'(-A^+)^2 - G(-A^+)G''(-A^+) \geq 0\),
- \(G''\) is strictly positive and nonincreasing, and
- \(G'(\infty) := \lim_{y \to \infty} G'(y)\) exists and is strictly positive.

For example, given a number \(\varepsilon > 0\) and a smooth function \(H\) on \((-A, \infty)\) which is strictly positive, nonincreasing, and integrable with \(H(-A^+) \in (0, \infty)\), one
can construct a function $G_{\varepsilon,H} \in \mathcal{G}_A$ as follows. Let

$$G_1(y) := \frac{1}{\varepsilon} + \int_{-A^+}^{y} H(t)\,dt$$

and

$$G_{\varepsilon,H}(y) := \frac{1}{\varepsilon^2} H(-A^+) + \int_{-A^+}^{y} G_1(t)\,dt.$$ 

It is direct to see the four conditions above hold for $G_{\varepsilon,H}$, and hence $G_{\varepsilon,H} \in \mathcal{G}_A$. Compare the statements of Theorem 1 and Theorem 3.1. We have

$$G'_{\varepsilon,H}(\infty) = \frac{1}{\varepsilon} + \int_{-A^+}^{\infty} H(t)\,dt$$

and

$$\frac{G'_{\varepsilon,H}(-\log |\sigma_S|^{2\gamma}_{h_S})}{G_{\varepsilon,H}(-\log |\sigma_S|^{2\gamma}_{h_S})} \leq \frac{G'_{\varepsilon,H}(A^+)}{G_{\varepsilon,H}(-A^+)} = \varepsilon H(-A^+).$$

Now we examine conditions (3.1), (3.2), and (3.6) with $G = G_{\varepsilon,H}$. When $F$ is a line bundle (3.6) is equivalent to (1.1). When $h_F$ is smooth, by Lemma A.3.1, we see that (3.1) and (3.2) are implied by taking $s = 0$ and $s = 1$ in (1.1). The rest of the conditions are easily translated.

4.2 (Proof of Theorem 2). In the following we assume that $\mathcal{W}$ is an $S$-snc family. We first prove a lemma which enables extension of sections satisfying certain vanishing conditions.

Consider the restriction maps:

$$
\begin{array}{ccc}
\Gamma(X, K_X \otimes S \otimes F) & \xrightarrow{\alpha} & \Gamma(\mathcal{W}, (K_X \otimes S \otimes F)|_{\mathcal{W}}) \\
\downarrow & & \downarrow \\
\Gamma(\mathcal{W}', (K_X \otimes S \otimes F)|_{\mathcal{W}'}) & \xrightarrow{\beta} & \\
\end{array}
$$

Lemma 4.1. (1) $\ker \beta \subseteq \text{im} \alpha$ if $(F, h_F)$ is a holomorphic vector bundle with a smooth hermitian metric such that $\sqrt{-1} \Theta_{h_F} \geq \pm \sqrt{-1} \Theta_{h_j} \otimes \text{id}_F$ for $j \in J(\mathcal{W})$. (2) Suppose that $(F, h_F)$ is a holomorphic line bundle with a singular hermitian metric such that $\sqrt{-1} \Theta_{h_F} \geq \pm \sqrt{-1} \Theta_{h_j}$ for $j \in J(\mathcal{W})$. Given $u \in \ker \beta$, if $\mathcal{I}(h_F|_W) = \mathcal{O}_W$ for every $W \in \mathcal{W}$ along which $u$ is not identically 0, then $u \in \text{im} \alpha$. 

32
Proof. Suppose we have any \( u = (u_W)_{W \in \mathcal{W}} \in \ker \beta \). In order to apply Theorem 1 it suffices to take \( H(y) = y^{-2} \) and show that the integrability condition (1.3)

\[
\int_W \prod_{J' \in J_0(W) \setminus \{J_W\}} \left( \sum_{j \in J'} |\sigma_j|^2_{h_j} \right)^{\delta |J'|} \left( \frac{\langle u_W \rangle}{\sigma^{J_W} h_F} \right)^2 < \infty \quad (W \in \mathcal{W})
\]

holds for some \( 0 < \delta < 1 \). We are allowed to replace \( X \) with a suitable relatively compact open subset, as indicated in Remark 5.1 in Section 5, and hence the verification of (1.3) is purely local. For a point \( p \in W \) suppose that

\[ \{j \in J_W \mid p \in S_j \} = \{1, \ldots, k\} \]

and \((V_p, \{z_j\})\) a coordinate patch with center \( p \) on which \( F \) and all \( S_j \) are trivial and \( \sigma_j \) is represented by \( z_j \) for \( j = 1, \ldots, k \). Express \( u_W \) as a holomorphic vector-valued function \((u_1, \ldots, u_r)\) and \( h_F \) is a measurable hermitian matrix \( h_{ab} \) of order \( r \). Ignoring nonvanishing factors in the integrand, it suffices to show that, for a fixed set of positive integers

\[ m_{j_1 \cdots j_m} \quad (1 \leq j_1 \leq \cdots \leq j_l \leq k \quad \text{and} \quad 1 \leq l \leq k) \]

we have

\[
\int_{V_p} \frac{\sum_{a,b} h_{ab} u_a \overline{u_b}}{\prod_{l=1}^k \prod_{1 \leq j_1 \leq \cdots \leq j_l \leq k} \left( \sum_{m=1}^l |z_{j_m}|^2 \right)^{\delta l m_{j_1 \cdots j_l}} |z_1 \cdots z_k|^2} < \infty
\]

for sufficiently small \( \delta > 0 \). Since \( u \in \ker \beta, u_1, \ldots, u_r \) all vanish along the zero set of the function \( z_1 \cdots z_k \), i.e., there exist holomorphic functions \( v_1, \ldots, v_r \) on \( V \) such that \( u_a = z_1 \cdots z_k v_a (a = 1, \ldots, r) \), and hence the integral becomes

\[
\int_{V_p} \frac{\sum_{a,b} h_{ab} v_a \overline{v_b}}{\prod_{l=1}^k \prod_{1 \leq j_1 \leq \cdots \leq j_l \leq k} \left( \sum_{m=1}^l |z_{j_m}|^2 \right)^{\delta l m_{j_1 \cdots j_l}} |z_1 \cdots z_k|^2}.
\]

For (1) of Theorem 2 since \( h_F \) is smooth, the integral is clearly finite for sufficiently small \( \delta > 0 \). As for (2), the integral is finite for sufficiently small \( \delta > 0 \) by the affirmative answer to the strong openness conjecture. □

\[ ^{14} \text{Since } \mathcal{W} \text{ is reduced and } \overline{\mathcal{W}} \text{ is an snc family, a section } u \in \Gamma(\overline{\mathcal{W}}, (K_X \otimes S \otimes F)|_{\mathcal{W}}) \text{ can always be represented uniquely by its "restrictions" (via adjunction) } u_W \in \Gamma(W, K_W \otimes S^W \otimes F|_W) (W \in \mathcal{W}). \]
Now we can proceed the proof of Theorem 2, which is a slight modification from a strategy suggested by Demailly. Suppose that $W^{(m)} \supseteq (W^{(m+1)}) = \emptyset$. Consider a section $u \in \Gamma(W, (K_X \otimes S \otimes F)|_W)$ and its restrictions:

$$\Gamma(X, K_X \otimes S \otimes F) \xrightarrow{u} \Gamma(W, (K_X \otimes S \otimes F)|_W) \xrightarrow{u} \Gamma(W^{(1)}, (K_X \otimes S \otimes F)|_{W^{(1)}}) \xrightarrow{\cdots} \Gamma(W^{(m)}, (K_X \otimes S \otimes F)|_{W^{(m)}})$$

By Lemma 4.1, $u_m = U_m|_{W^{(m)}}$ for some $U_m \in \Gamma(X, K_X \otimes S \otimes F)$. In particular, $u_{m-1} - U_m|_{S_{W^{(m)}}} = 0$. Suppose we have created $U_m, \ldots, U_{m-k+1} \in \Gamma(X, K_X \otimes S \otimes F)$ such that

$$u_{m-k} - \left( U_m + \cdots + U_{m-k+1} \right)|_{W^{(m-k+1)}} = 0.$$ 

Lemma 4.1 again yields some $U_{m-k} \in \Gamma(X, K_X \otimes S \otimes F)$ such that

$$u_{m-k} - \left( U_m + \cdots + U_{m-k+1} \right)|_{W^{(m-k)}} = U_{m-k}|_{W^{(m-k)}}.$$ 

Repeating this procedure we will reach the stage $k = m$ and obtain the desired extension.

5. Extension of pluricanonical forms from strata of a snc divisor

In this section we prove Theorem 4. In the proof we will assume $X$ to be projective for simplicity, as is justified by the following remark.

**Remark 5.1.** Let $X \xrightarrow{\pi} T$ be a projective morphism to a Stein space given by assumption. Consider the statement of Theorem 4. Let $Z = W$, which is a (not necessarily connected) closed submanifold of $X$, and let $\mathcal{I}_{W,i}$ denote $\mathcal{I}(h_{L_i}|_W)$ and let $\mathcal{I}_{Z,i}$ denote the ideal sheaf of $\mathcal{O}_Z$ such that $\mathcal{I}_{Z,i}|_W$ is $\mathcal{I}_{W,i}$ for every $W \in \mathcal{W}$.
The canonical maps

\[ \Gamma(X, \otimes_{i=1}^{m} K_X \otimes S \otimes L_i) \xrightarrow{\rho(T)} \prod_{W \in W} \Gamma(W, \otimes_{i=1}^{m} K_W \otimes S^W \otimes L_i|_W) \]

are obtained by taking global sections of the following canonical morphisms of \( \mathcal{O}_T \)-modules:

\[ \pi_* \mathcal{O}_X(\otimes_{i=1}^{m} K_X \otimes S \otimes L_i) \xrightarrow{\rho} \pi_* \mathcal{O}_X(\otimes_{i=1}^{m} K_X \otimes S \otimes L_i) \otimes_{\mathcal{O}_X} \mathcal{O}_X / \mathcal{I}_Z \]

where \( \mathcal{I}' \) denotes the preimage of the ideal product \( \mathcal{I}_{1,Z} \cdots \mathcal{I}_{m,Z} \) of \( \mathcal{O}_Z \) under the natural map \( \mathcal{O}_X \to \mathcal{O}_Z \). Proving Theorem 4 amounts to showing that \( \text{im} \, \iota(T) \subseteq \text{im} \, \rho(T) \); since \( T \) is Stein, this is further equivalent to show that \( \text{im} \, \iota \subseteq \text{im} \, \rho \) or, equivalently, that \( \text{im} \, \iota(T') \subseteq \text{im} \, \rho(T') \) for every relatively compact open subset \( T' \) of \( T \). We will see soon that, in the proof of Theorem 4, we need to tensor the relevant sheaves by some auxiliary line bundles to achieve global generation of some coherent sheaves and surjectivity of some maps between coherent sheaves. When replacing \( T \) by any of such \( T' \), Serre’s Theorems A and B about tensoring with coherent \( \pi \)-ample bundles both hold, and the aforementioned global generation and surjectivity holds as in the case \( T \) is a point.

We will use the following special form of Theorem 1:

**Lemma 5.2.** Suppose that \( \mathcal{W} \) consists of minimal strata and \((F, h_F)\) is a holomorphic line bundle on \( X \) with a singular metric such that

\[ (5.1) \quad \sqrt{-1} \Theta_{h_F} \succcurlyeq \pm \sqrt{-1} \Theta_{h_j} \quad (j \in J(\mathcal{W})). \]

Then for every \( \delta > 0 \) sufficiently small, there exists a constant \( C = C(S, \delta) > 0 \) such that, for every collection of sections \( v_W \in \Gamma(W, K_W \otimes S^W \otimes F|_W) \) \( (W \in \mathcal{W}), \)
if
(5.2) \[ \int_W \langle v_W \rangle^2_{h^jW \otimes h_F} < \infty, \]
then there exists \( V \in \Gamma(X, K_X \otimes S \otimes F) \) such that \( V_W = v_W \) for every \( W \in \mathcal{W} \) and
(5.3) \[ \int_X \langle V \rangle^2_{hS \otimes h_F} \leq C \int_W \langle v_W \rangle^2_{h^jW \otimes h_F}. \]

This special form can be obtained from Theorem 1 with \( H(y) = e^{-\frac{1}{2}y}, \varepsilon = \delta, \delta_j = 0 \) for \( j \notin J(W) \), and \( \gamma_j = 1 \) (resp. \( 0 \)) for \( j \in J(W) \) (resp. \( j \notin J(W) \)). Since \( \mathcal{W} \) consists of minimal strata, \( \sigma^{jW} \) and \( \sum_{j \in J'} |\sigma_j|^2_{h_j} \) (\( J' \in J_0(\mathcal{W}) \setminus \{ J_W \} \)) are all nonvanishing along every \( W \in \mathcal{W} \). This takes care of all factors of in the denominator of the integrand (1.3).

We first fix some auxiliary objects to be used later. We choose an ample line bundle \( A \) on \( X \) and a smooth metric \( h_A \) on \( A \) such that the following conditions hold:

(A0) Every holomorphic section of \( \otimes_{i=1}^m (K_X \otimes S \otimes L_i \otimes A)|_W \) on \( \mathcal{W} \) extends to \( X \) holomorphically.

(A1) For each \( r = 0, 1, \ldots, m - 1 \), the line bundle \( (m - r)A \) is generated by global sections \( \{ t_{p}^{(r)} \}_{1 \leq p \leq N} \).

(A2) The coherent sheaves \( \mathcal{I}_{W,i} \cdot (K_W \otimes S^W \otimes L_i|_W + A|_W) \) is generated by global sections \( \{ s_{i,l} \}_{1 \leq l \leq N} \) for \( 1 \leq i \leq m \) and \( W \in \mathcal{W} \).

(A3) For every \( W \in \mathcal{W} \) the natural multiplicative map
\[ \bigotimes_{i=1}^m \Gamma(W, \mathcal{I}_{W,i} \cdot (K_W \otimes S^W \otimes L_i|_W + A|_W)) \]
\[ \longrightarrow \Gamma(W, \mathcal{I}_{1,W} \cdots \mathcal{I}_{m,W} \cdot \otimes_{i=1}^m (K_W \otimes S^W \otimes L_i|_W + A|_W)) \]
is surjective (cf. [6] Appendix 2).

5.1 (Reduction to the construction of a metric \( h_\infty \) on \( \otimes_{i=1}^m (K_X \otimes S \otimes L_i) \)).

To obtain the desired section \( U \) on \( X \), we will apply Lemma 5.2 with some singular metric \( h_F \) on
\[ F := (K_X \otimes S)^{(m-1)} \otimes \otimes_{i=1}^m L_i \]
to the collection

\[ v_W = d\sigma^{(m-1)}_W \wedge u_W \quad (W \in \mathcal{W}) \]

(\(v_W\) viewed as sections of \(K_W \otimes S^W \otimes F|_W\)). More precisely, we will define

\[ h_F := h_{\infty}^{\frac{m-1}{m}} (h_{L_1} \otimes \cdots \otimes h_{L_m})^{-\frac{1}{m}} \]

where \(h_{\infty}\) is a singular metric on \(\otimes_{i=1}^{m} K_X \otimes S \otimes L_i\) such that

\[ \sqrt{-1} \Theta_{h_{\infty}} \geq 0 \quad (5.4) \]

and

\[ |d\sigma^m_W \wedge u_W|_{h_{\infty}} \leq 1 \quad \text{for every } W \in \mathcal{W} \text{ such that } u_W \text{ is not identically 0.} \quad (5.5) \]

For \(h_F\) so constructed, (5.1) holds for sufficiently small \(\delta > 0\) in view of (1.6) and (5.4). For (5.2), we only need to consider those \(W\) along which \(u_W\) is not identically zero. On these components, by (5.5), we have

\[ \langle d\sigma^{(m-1)}_W \wedge u_W \rangle_{h_{J_W} \otimes h_F}^2 \leq \langle u_W \rangle_{(h_{J_W} \otimes h_{L_1}) \otimes \cdots \otimes (h_{J_W} \otimes h_{L_m})}^2. \]

By \((A_3)\), for \(p = 1, \ldots, N\) there exist

\[ t_{W;i;p,k} \in \Gamma(W, \mathcal{I}_{W;i} \cdot (K_W \otimes S^W \otimes L_i|_W \otimes A|_W)) \quad (i = 1, \ldots, m \text{ and } k = 1, \ldots, n_p) \]

such that

\[ u_W \otimes t_p^{(0)} = \sum_{k=1}^{n_p} t_{W;1;p,k} \otimes \cdots \otimes t_{W;m;p,k}. \]

Then

\[ \left( \sum_{p=1}^{N} \left| t_p^{(0)} \right|_{h_{A}^m}^2 \right)^{\frac{1}{2}} \langle u_W \rangle_{(h_{J_W} \otimes h_{L_1}) \otimes \cdots \otimes (h_{J_W} \otimes h_{L_m})}^{\frac{1}{m}} \leq \sum_{p=1}^{N} \sum_{k=1}^{n_p} \langle t_{W;1;p,k} \otimes \cdots \otimes t_{W;m;p,k} \rangle_{(h_{J_W} \otimes h_{L_1}) \otimes \cdots \otimes (h_{J_W} \otimes h_{L_m})}^{\frac{1}{m}}. \]

\[ \text{Here } d\sigma^m_W \text{ is defined to be } d\sigma_{j_1} \wedge \cdots \wedge d\sigma_{j_k} \text{ as a section of } \text{det} N^*_W \otimes S^W \text{ if } J_W \text{ consists of } j_1 < \cdots < j_k. \]

\[ \text{Let } h \text{ be a measurable hermitian metric on a complex vector bundle } E \text{ over a complex manifold } M. \text{ For any measurable section } u \text{ of } K^m \otimes E, \text{ we first define a nonnegative } (n, n)-\text{form } \langle u \rangle_{h}^{\frac{2}{m}} \text{ as follows:} \]

\[ \text{on a chart } (V, \{z_k = x_k + iy_k\}) \text{ we let } \langle u \rangle_{h}^{\frac{2}{m}} |_V = f_{h}^{\frac{2}{m}} dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n \text{ if } |u|_V = dz_1 \wedge \cdots \wedge dz_n \otimes f. \]

When \(m = 1\), the \(L^2\)-norm of \(u\) with respect to \(h\) is defined to be \(\int_M (u)^2_{h} \in [0, \infty]\).
Let
\[ M_W := \min_W \sum_p |h_p^{(0)}|^{\frac{m}{h_A}} > 0. \]

By (A1) and Hölder’s inequality,
\[ \int_W \langle d\sigma_W^{(m-1)} \wedge u_W \rangle^2_{h_W \otimes h_p} \leq \frac{1}{M_W} \sum_p \sum_m \prod_{i=1}^{m} \left( \int_W \langle t_{W_i} ; p, k \rangle^2_{h_W \otimes h_{L_i} \otimes h_A} \right)^{\frac{1}{m}} < \infty. \]

Now it remains to construct the metric \( h_\infty \) satisfying (5.4) and (5.5).

5.2 (Păun’s simplification of Siu’s iteration). For every positive integer \( k \) we write \( k = \lfloor \frac{k}{m} \rfloor m + r \). We let \( L^{(k)} := (L_1 \otimes \cdots \otimes L_m)^{\otimes \lfloor \frac{k}{m} \rfloor} \otimes L_1 \otimes \cdots \otimes L_r \)

and let
\[ B_k := (K_X \otimes S)^{\otimes k} \otimes L^{(k)} \otimes A^{\otimes m} \]

where \( A \) is the ample line bundle chosen in [5.1]. We will use several specific sets of indices. We let
\[ \Lambda_r := \{1, \ldots, N\}^r \quad (r = 1, \ldots, m - 1) \]

and define
\[ s^{(r)}_I := s_{i_1} \otimes \cdots \otimes s_{i_r} \quad (I = (i_1, \ldots, i_r) \in \Lambda_r) \]

with the convention that \( \Lambda_0 = \{0\} \) and \( s^{(0)}_0 := 1 \) for \( r = 0 \). We let
\[ \hat{\Lambda}_m := \{1, \ldots, N\}^m \]

and define
\[ \hat{s}^{(m)}_I = s_{i_1} \otimes \cdots \otimes s_{i_r} \quad (I = (i_1, \ldots, i_r) \in \Lambda_r). \]

We consider for each \( k \geq m \) the following statement:

\( (E_k) \): There exists a family of sections
\[ U^{(k)}_{I,p} \in \Gamma(X, B_k) \quad (I \in \Lambda_r, 1 \leq p \leq N) \]

such that
\[ (5.6) \quad U^{(k)}_{I,p} |_W = d\sigma_W^{\otimes k} \wedge u_W^{\otimes [k/m]} \otimes s^{(r)}_I \otimes t^{(r)}_p \]

for every \( I \in \Lambda_r \) and \( p = 1, \ldots, N \).

We let \( h_k \) denote the singular metric on \( B_k \) defined by the family \( U_k \) of sections.
Lemma 5.2.1. \((E_k)\) holds for all \(k \geq m\). Moreover, there exists a constant \(C_0 > 0\) which only depends on \(S\), \(\delta\), \(u_W\), \(\{t_p^{(r)}\}\), and \(\{s_l,i\}\) such that

\[
\int_X \sum_{I \in \Lambda_r} \langle U_{I,p}^{(k)} \rangle^2_{h_S \otimes h_{k-1} \otimes h_{r^*}} \leq C_0
\]

for all \(k > m\) where \(r = k - \lfloor k/m \rfloor m\) and

\[
r^* := \begin{cases} r & \text{if } r \neq 0, \\ m & \text{if } r = 0. \end{cases}
\]

Proof. \((E_m)\) holds by \((A_0)\) in 5.1. Now assume the validity of \((E)_{k-1}\) for some \(k > m\). Note that \(B_k = K_X \otimes S \otimes B_{k-1} \otimes L_{r^*}\) and hence

\[
B_k|_W = K_W \otimes \det N_{W/X}^* \otimes S_W \otimes (B_{k-1} \otimes L_{r^*})|_W.
\]

We will apply Lemma 5.2 with

\[
(F, h_F) = (B_{k-1} \otimes L_{r^*}, h_{k-1} \otimes h_{r^*}).
\]

In the following we will show that

\[
\int_W \langle t_W^{[k/m]} \otimes s_I^{(r)} \otimes t_p^{(r)} \wedge d\sigma_W^{(k-1)} \rangle^2_{h_{k-1} \otimes h_{r^*}} \leq C' \quad (W \in \mathcal{W})
\]

for some positive number \(C'\) which only depends on the data \(S\) and the choices of \(\{t_p^{(r)}\}\) and \(\{s_l,i\}\) in \((A_2)\) and \((A_3)\) in 5.1. Were this done, \((5.8)\) implies \((E_k)\) by Lemma 5.2. The verification of \((5.8)\) is separated into two cases:

Case 1. \(r \neq 0\), i.e., \([k/m] = [(k-1)/m]\).

We fix smooth auxiliary metrics \(h'\) and \(h^{(r-1)}\) on

\[
(K_W \otimes L^{(m)}|_W)^{[k/m]} \otimes (\det N_{W/X}^* \otimes S_W)^{(k-1)} \quad \text{and} \quad K_W^{(r-1)} \otimes L^{(r-1)}|_W,
\]

\[
\int_X \sum_{I \in \Lambda_r} \langle U_{I,p}^{(k)} \rangle^2_{h_S \otimes h_{k-1} \otimes h_{r^*}} \leq C_0
\]

for all \(k > m\) where \(r = k - \lfloor k/m \rfloor m\) and

\[
r^* := \begin{cases} r & \text{if } r \neq 0, \\ m & \text{if } r = 0. \end{cases}
\]
respectively. We let \( h := h' \otimes h^{(r-1)} \otimes h_A^{\otimes m} \) on \( B_{k-1}|_W \). Writing \( I = (I', i) \) with \( I' \in \Lambda_{r-1} \), we have

\[
\langle u_W \otimes [k/m] \otimes s_I^{(r)} \otimes t^{(r)}_p \wedge d\sigma_W^{(k-1)} \rangle_{h_{k-1} \otimes h_{r, i}}^2 = \sum_{I' \in \Lambda_{r-1}} \frac{\max_{p'} \left| t^{(r)}_{p'} \right|^2_{h_A^{(m-r)}} \langle s_{r, i} \rangle_{h_A^{(m-r)}}^2 \langle s_{r, i} \rangle_{h_A^{(m-r)}}^2}{\sum_{p' = 1}^N \left| t^{(r-1)}_{p'} \right|^2_{h_A^{(m-r+1)}} \langle s_{r, i} \rangle_{h_A^{(m-r+1)}}^2} < \infty.
\]

By (A_1) and the choices of \( s_{r, i} \),

\[
C_{1, W} := \max_{i, r} \int_W \sum_{p' = 1}^N \left| t^{(r-1)}_{p'} \right|^2_{h_A^{(m-r+1)}} \langle s_{r, i} \rangle_{h_A^{(m-r+1)}}^2 < \infty.
\]

**Case 2.** \( r = 0 \), i.e., \([k/m] = [(k-1)/m] + 1\) and \( k - 1 = \lfloor k/m \rfloor m + m - 1\).

We fix smooth metrics \( h^{(m-1)}, \tilde{h}, \text{and } h' \) on

\[
K_W^{(m-1)} \otimes L^{(m-1)}|_W, \quad K_W \otimes (L_m \otimes A)|_W,
\]

and

\[
(K_W^{\otimes m} \otimes L^{(m)}|_W)^{\otimes [(k-1)/m]} \otimes (\det N^*_W X \otimes S^W)^{\otimes (k-1)},
\]

respectively, and let \( h := h' \otimes h^{(m-1)} \otimes h_A^{\otimes m} \) on \( B_{k-1}|_W \). We have

\[
\langle u_W \otimes [k/m] \otimes s_0^{(0)} \otimes t^{(0)}_p \wedge d\sigma_W^{(k-1)} \rangle_{h_{k-1} \otimes h_{L_m}}^2 = \sum_{I' \in \Lambda_{m-1}} \frac{\max_{p'} \left| t^{(0)}_{p'} \right|^2_{h^{(m-1)} \otimes h_{A}^{\otimes m} \otimes h_{L_m}} \langle s_{r, i} \rangle_{h_{A}^{(m-r)}}^2 \langle s_{r, i} \rangle_{h_{A}^{(m-r)}}^2}{\sum_{p' = 1}^N \left| t^{(m-1)}_{p'} \right|^2_{h_{A}^{(m-r+1)}} \langle s_{r, i} \rangle_{h_{A}^{(m-r+1)}}^2} < \infty.
\]
By multiplying both the numerator and the denominator by the same positive factor $\sum_{i=1}^{N} |s_{m,i}|^2_h$, the expression becomes

$$\left| \frac{u_W \otimes t_p^{(m-1)} |^{2}_{h(m-1) \otimes \tilde{h} \otimes h_A} \sum_{I \in \Lambda} |s_{I}^{(m)}|^2_{h(m-1) \otimes \tilde{h} \otimes h_A} \right|^2_{h_A \otimes h_{Lm}} \sum_{i=1}^{N} \left\langle s_{m,i} \right|^{2}_{h_A \otimes h_{Lm}} \sum_{i=1}^{N} \left\langle t_p^{(m-1)} \right|^{2}_{h_A} \sum_{p' = 1}^{N} \left| t_p^{(m-1)} \right|^{2}_{h_A}$$

By $(A_1)$ and the choice of $s_{m,i}$,

$$C_{2,W} := \sum_{i=1}^{N} \int W \frac{\left\langle s_{m,i} \right|^{2}_{h_A \otimes h_{Lm}}}{\sum_{p' = 1}^{N} \left| t_p^{(m-1)} \right|^{2}_{h_A}} < \infty.$$

By $(A_2)$ and $(A_3)$, $u_W \otimes t_p^{(0)}$ is a linear combination of $\{s_{I}^{(m)}\}_{I \in \Lambda_m}$. The Cauchy–Schwartz inequality implies that

$$C_{3,W} := \max_{p=1, \ldots, N} \sup_{W} \left| \frac{u_W \otimes t_p^{(0)} |^{2}_{h(m-1) \otimes \tilde{h} \otimes h_A} \sum_{I \in \Lambda} |s_{I}^{(m)}|^2_{h(m-1) \otimes \tilde{h} \otimes h_A} \right|^2_{h_{Lm}} < \infty.$$

In this case we have

$$\int_{W} \langle u_{W}^{(k)} \otimes s_{I}^{(r)} \otimes t_p^{(r)} \otimes d\sigma_{W}^{(k-1)} \rangle^{2}_{h_{k-1} \otimes h_{Lm}} \leq C_{2,W} C_{3,W}.$$

In summary, it suffices to take

$$C' := \max_{W \in W} \{C_{1,W}, C_{2,W} C_{3,W}\}.$$

By Lemma 5.2, there exists a family of sections

$$U_{k} : \quad U_{I,p}^{(k)} \quad (I \in \Lambda_k, 1 \leq p \leq N)$$

of $B_k$ over $X$ such that

$$U_{I,p}^{(k)} |_{W} = u_{W}^{(k)} \otimes s_{I}^{(r)} \otimes t_p^{(r)} \otimes d\sigma_{W}^{(k)}$$

and

$$\int_{X} \sum_{I \in \Lambda_r} \langle U_{I,p}^{(k)} \rangle^{2}_{h_{k-1} \otimes h_{r}} \leq C_0 := CC'.$$

This completes the proof.
5.3 (Siu’s construction of the metric $h_\infty$). We first make some general setting which will be in force till the end of 5.3. For any $w_0 = (w_0^1, \ldots, w_0^n) \in \mathbb{C}^n$ and any $r > 0$, we let $D_r(w_0)$ denote $\{(w^1, \ldots, w^n) : |w^\nu - w_0^\nu| < r, 1 \leq \nu \leq n\}$, the polydisk in $\mathbb{C}^n$ centered at $w_0$ with polyradii $(r, \ldots, r)$. Choose a finite open cover $\{V'_\alpha\}_{\alpha \in A}$ of $X$ such that each $V'_\alpha$ is biholomorphic to $D_1(0)$ and $V_\alpha$ ($\alpha \in A$) also cover $X$ where $V_\alpha \subseteq V'_\alpha$ corresponds to $D_{1/3}(0)$. We let $V''_\alpha$ be the open set corresponding to $D_{1/2}(0)$. We also require that $L_i|_{V'_\alpha}$ ($i = 1, \ldots, m$), $D_j|_{V'_\alpha}$ ($j = 1, \ldots, q$), and $A|_{V'_\alpha}$ (and hence $B_k|_{V'_\alpha}$, $k \geq m$) are trivial for all $\alpha$. Finally, suppose that $U'_{\alpha; I, p}$ given by Lemma 5.2.1 are represented by holomorphic functions $F^{(k)}_{\alpha; I, p}$ on $V'_\alpha$.

**Lemma 5.3.1.** There exists $C'_0 > 0$ such that

$$\frac{1}{l} \log \sum_{p=1}^{N} |F^{(lm)}_{\alpha; 0, p}(x)|^2 \leq C'_0$$

for all $x \in V_\alpha$ ($\alpha \in A$) and $l \in \mathbb{N}$.

The essential part of this result is the uniformity of $C'_0$ with respect to $l \in \mathbb{N}$.

**Proof.** For each $x \in V''_\alpha$ with coordinate $w_x = (w_x^1, \ldots, w_x^n)$, we let $D_x$ be the subset of $V'_\alpha$ corresponding to $D_{1/3}(w_x)$. Since $\bigcup_{x \in V_\alpha} D_x \subseteq V'_\alpha$, there exists $M > 0$ such that on all $D_x$ we have (according to the notation in 5.2)

$$\sum_{I, p} |F^{(k+1)}_{\alpha; I, p}|^2 \leq M \sum_{I, p} (U^{(k+1)}_{I, p})^2 \langle U^{(k+1)}_{I, p} \rangle_{h_D \otimes h_{k-1} \otimes h_r}$$

(5.9)

where $dV = du^1 \wedge dv^1 \wedge \cdots \wedge du^n \wedge dv^n$ and $(w^\nu, v^\nu) = (\text{Re } w^\nu, \text{Im } w^\nu)$. For each $m \leq k \leq lm - 1$, by Jensen’s inequality, (5.9), and Lemma 5.2.1.
\[
\frac{1}{\text{Vol}(D_{1/3}(w_x))} \int_{D_{1/3}(w_x)} \log \sum_{I,p} |F_{\alpha;I,p}^{(k+1)}|^2 \, dV
\]
\[
- \frac{1}{\text{Vol}(D_{1/3}(w_x))} \int_{D_{1/3}(w_x)} \log \sum_{I',p'} |F_{\alpha;I',p'}^{(k)}|^2 \, dV
\]
\[
\leq \log \left( \frac{1}{\text{Vol}(D_{1/3}(w_x))} \int_{D_{1/3}(w_x)} \sum_{I,p} |F_{\alpha;I,p}^{(k+1)}|^2 \, dV \right)
\]
\[
\leq \log \left( \frac{9}{\pi} \int_{D_x} \sum_{I,p} \langle U_{I,p}^{(k+1)} \rangle^2_{\mathcal{H}_D \otimes \mathbb{H}_{k-1} \otimes \mathbb{H}_r} \right) \leq \log \frac{9MC_0}{\pi}.
\]

Summing telescopically the above computation from \( k = m \) to \( k = lm - 1 \), by the sub-mean-value inequality, we obtain

\[
\frac{1}{l} \log \sum_{p=1}^{N} |F_{\alpha;0,p}^{(lm)}(x)|^2
\]
\[
\leq \frac{1}{l\text{Vol}(D_{1/3}(w_x))} \int_{D_{1/3}(w_x)} \log \sum_{p=1}^{N} |\bar{f}_{\alpha;0,p}^{(lm)}|^2 \, dV
\]
\[
\leq \frac{9^n}{\pi^n} \int_{D_{1/3}(w_x)} \log \sum_{p=1}^{N} |F_{\alpha;0,p}^{(m)}|^2 \, dV + \frac{(l-1)m}{l} \log \frac{9^nMC_0}{\pi^n}.\]

Note that \( \int_{D_{1/3}(w_x)} \log \sum_{p=1}^{N} |F_{\alpha;0,p}^{(m)}|^2 \, dV \), when viewed as a function in \( x \) on \( V_{\alpha}'' \) is a psh function. Therefore it is bounded from above on \( V_{\alpha} \). Since we have only finitely many \( \alpha \), it suffices to take

\[
C_0' = \frac{9^n}{l\pi^n} \max_{\alpha} \sup_{x \in V_{\alpha}} \int_{D_{1/3}(w_x)} \log \sum_{p=1}^{N} |F_{\alpha;0,p}^{(m)}|^2 \, dV + \frac{(l-1)m}{l} \log \frac{9^nMC_0}{\pi^n}(< \infty).
\]

The last step is a limiting process to obtain the metric \( h_{\infty} \).

**Lemma 5.3.2.** There exists a singular metric \( h_{\infty} \) on \((K_X \otimes S)^{\otimes m} \otimes L^{(m)}\) satisfying (5.4) and (5.5).
Proof. On each $\alpha$ we let
\[
F^{(\infty)}_{\alpha} := \lim_{k \to \infty} \left( \sup_{l \geq k} \frac{1}{l} \log \sum_{p=1}^{N} \left| F^{(lm)}_{\alpha;0,p} \right|^2 \right)^*,
\]
where ( )* stands for the operation of upper semicontinuous regularization. By Lemma 5.3.1,
\[
\left\{ \left( \sup_{l \geq k} \frac{1}{l} \log \sum_{p=1}^{N} \left| F^{(lm)}_{\alpha;0,p} \right|^2 \right)^* \right\}_{k \in \mathbb{N}}
\]
is a decreasing sequence of plurisubharmonic functions on $V_\alpha'$ which are bounded from above by $C'_0$ on $V_\alpha$, and hence $F^{(\infty)}_{\alpha}$ is also plurisubharmonic and bounded from above by $C'_0$ on $V_\alpha$.

Let $g_{\alpha\beta}$ and $a_{\alpha\beta} \in \mathcal{O}_X'(V_\alpha \cap W_\beta)$, $\alpha, \beta \in I$ be the transition functions of $(K_X \otimes S)^{\otimes m} \otimes L^{(m)}$ and $A$ respectively. By the definition of $F^{(lm)}_{\alpha;0,p}$, we have
\[
F^{(lm)}_{\alpha;0,p} = (g_{\alpha\beta})^l a_{\alpha\beta} F^{(lm)}_{\beta;0,p}
\]
and hence
\[
\frac{1}{l} \log \sum_{p=1}^{N} \left| F^{(lm)}_{\alpha;0,p} \right|^2 = \log |g_{\alpha\beta}|^2 + \frac{m}{l} \log |a_{\alpha\beta}|^2 + \frac{1}{l} \log \sum_{p=1}^{N} \left| F^{(lm)}_{\beta;0,p} \right|^2.
\]
Taking $\lim_{k \to \infty} (\sup_{l \geq k})$* to both sides, we get rid of the term involving $a_{\alpha\beta}$ and obtain
\[
e^{-F^{(\infty)}_{\beta}} = |g_{\alpha\beta}|^2 e^{-F^{(\infty)}_{\alpha}}.
\]
This shows that the set of local data $e^{-F^{(\infty)}_{\alpha}} (\alpha \in \mathcal{A})$ define a singular metric on $(K_X \otimes S)^{\otimes m} \otimes L^{(m)}$, which we denote by $h_\infty$, with $\sqrt{-1} \Theta_{h_\infty} \geq 0$.

It remains to show (5.5). Suppose that $u_W$ is not identically zero along $W$. By Lemma 5.2.1
\[
U^{(lm)}_{0,p} |_W = d\sigma^{(lm)}_W \wedge u^{(0)}_W \otimes t^{(0)}_p
\]
for $p = 1, \ldots, N$. Suppose that $d\sigma^{(m)}_W \wedge u_W$ and $t^{(0)}_p$ are represented by functions $u_\alpha$ and $t^{(0)}_{\alpha;p}$ on $V_\alpha \cap W$, respectively. Then we have $F^{(lm)}_{\alpha;0,p} = u^{l} t^{(0)}_{\alpha;p}$ for each $p$, and hence
\[
\frac{1}{l} \log \sum_{p=1}^{N} \left| F^{(lm)}_{\alpha;0,p} \right|^2 |_{W \cap V_\alpha} = \log |u_\alpha|^2 + \frac{1}{l} \log \sum_{p=1}^{N} \left| t^{(0)}_{\alpha;p} \right|^2.
\]
Since
\[
\left( \sup_{l \geq k} \frac{1}{l} \log \sum_{p=1}^{N} \left| F^{(lm)}_{\alpha;0,p} \right|^2 \right)^* \left| W \cap V_{\alpha} \right| \geq \sup_{l \geq k} \frac{1}{l} \log \sum_{p=1}^{N} \left| F^{(lm)}_{\alpha;0,p} \right|^2 \left| W \cap V_{\alpha} \right|,
\]
we obtain that
\[
F^{(\infty)}_{\alpha}|_{W \cap V_{\alpha}} = \lim_{k \to \infty} \left( \sup_{l \geq k} \frac{1}{l} \log \sum_{p=1}^{N} \left| F^{(lm)}_{\alpha;0,p} \right|^2 \right)^* \left| W \cap V_{\alpha} \right| \geq \log |u_{\alpha}|^2.
\]
This shows that \(e^{-F^{(\infty)}_{\alpha}} |u_{\alpha}|^2 \leq 1\) for each \(\alpha\) and hence completes the proof. \(\square\)

6. Extension of pluricanonical forms from strata of a snc divisor 2

In this section we prove a pluricanonical analogue (Lemma 6.1) of Lemma 4.1, from which Theorem 5 can be derived by the successive extension-correction process. We adopt the following setting which will be in force throughout the whole section:

**Setting.** Let \((X, \mathcal{S}, (L_\bullet, h_\bullet), (L, h_L), h_T, \mathcal{W})\) consist of

(i) a complex manifold \(X\) admitting a projective morphism to a Stein space,
(ii) a family of data \(\mathcal{S} : (S_j, \sigma_j, h_j) (j = 1, \ldots, q)\) which consist of a holomorphic line bundle \(S_j\) on \(X\), a nonzero holomorphic section \(\sigma_j\) of \(S_j\), and a smooth hermitian metric \(h_j\) on \(S_j\) for every \(j\),
(iii) holomorphic line bundles \(L_i (i = 1, \ldots, m - 1)\) on \(X\) with singular hermitian metrics \(h_L\) such that \(\sqrt{-1} \Theta_{h_L} \geq 0\) for every \(i\),
(iv) a \(\mathbb{Q}\)-line bundle \(L\) with \(L^m = \bigotimes_{i=1}^{m-1} L_i\) and a singular metric \(h_L\) such that \(\sqrt{-1} \Theta_{h_L} \geq \pm \sqrt{-1} \Theta_{h_j} (j \in J(\mathcal{W}))\) and \(\mathcal{I}(h_L) = \mathcal{O}_X\),
(v) a singular metric \(h_T\) on \((K_X \otimes S \otimes L)^{\otimes m}\) with \(\sqrt{-1} \Theta_{h_T} \geq 0\), and
(vi) an \(\mathcal{S}\)-admissible family \(\mathcal{W}\).

Assume that for every \(W \in \mathcal{W}\) we have
\[
\mathcal{I}(|\sigma_S|_{h_S}^{-2+\frac{2}{m} h_L}|_W) = \mathcal{O}_W
\]
and there exists a number \(0 < \varepsilon_0 < 1\) such that
\[
\mathcal{I}(|\sigma^{J_\mathcal{W}}|_{h^{1-\delta} h_T^{\varepsilon_0}|_W} = \mathcal{O}_W \text{ for all } \delta > 0 \text{ and } \varepsilon > 0.
\]
The proof of the following lemma incorporates the validity of the strong openness conjecture with the argument initiated by Păun [33] (see also [1], [3]), which will occupy the rest of this section.

**Lemma 6.1.** \( \ker \beta \subseteq \text{im} \alpha \) for the restriction maps:

\[
\begin{array}{c}
\Gamma(X, (K_X \otimes S \otimes L)^{\otimes m}) \xrightarrow{\alpha} \Gamma(W, (K_X \otimes S \otimes L)^{\otimes m}|_W) \\
\downarrow \beta \\
\Gamma(W', (K_X \otimes S \otimes L)^{\otimes m}|_{W'}). \\
\end{array}
\]

In view of Remark 5.1, we will assume that \( X \) is projective from now on. We will use the following special form of Theorem 1 repeatedly, which is obtained in exactly the same way we derived Lemma 5.2:

**Lemma 6.2.** Suppose that \((F, h_F)\) is a holomorphic line bundle on \( X \) with a singular metric such that

\[
\sqrt{-1} \Theta_{h_F} \gtrsim \pm \sqrt{-1} \Theta_{h_j} \quad (j \in J(W)).
\]

Then for every \( \delta_0 > 0 \) sufficiently small, there exists a constant \( C_0 = C_0(S, \delta_0) > 0 \) such that, for every collection of sections

\[
v_W \in \Gamma(W, K_X \otimes S^W \otimes F|_W) \quad (W \in \mathcal{W}),
\]

if

\[
\int_W \frac{\langle v_W \rangle_{h_F}^2}{|\sigma_{J_W}|_{h_W}^2 \prod_{j' \in J_0(W) \setminus \{J_W\}} \left( \sum_{j \in J'} |\sigma_{j'_W h_j}^2| \right)^{\delta_0 |J'|}} < \infty \quad (W \in \mathcal{W}),
\]

then there exists \( V \in \Gamma(X, K_X \otimes S \otimes F) \) such that \( V_W = v_W \) for every \( W \in \mathcal{W} \) and

\[
\int_X \langle V \rangle_{h_S \otimes h_F}^2 \leq C_0 \int_W \frac{\langle v_W \rangle_{h_F}^2}{|\sigma_{J_W}|_{h_W}^2 \prod_{j' \in J_0(W) \setminus \{J_W\}} \left( \sum_{j \in J'} |\sigma_{j'_W h_j}^2| \right)^{\delta_0 |J'|}}.
\]

Now let \( u = (u_W) \in \ker \beta \). We will show that there exists a section

\[
U \in \Gamma(X, (K_X \otimes S \otimes L)^{\otimes m})
\]
such that $U_W = u_W$ for every $W \in \mathcal{W}$. Many parts of the following discussions will be parallel to those in Section 5.

We first fix some auxiliary objects to be used later. Let

$$Q^{(r)} := \bigotimes_{i=1}^{r} K_X \otimes S \otimes L_i \quad (0 \leq r \leq m - 1).$$

We fix a smooth metric $\tilde{h}_{Q^{(r)}}$ on $Q^{(r)}$ for every $r$ and choose an ample line bundle $A$ on $X$ with a smooth metric $h_A$ on $A$ such that the following conditions hold:

(A'$_0$) Every holomorphic section of $(K_X \otimes S \otimes L)^{\otimes m} \otimes A|_W$ on $W$ extends to $X$ holomorphically.

(A'$_1$) For each $r = 0, 1, \ldots, m - 1$, the line bundle $Q^{(r)} \otimes A$ is generated by global sections $\{t_p^{(r)}\}_{1 \leq p \leq N}$.

(A'$_3$) $\sqrt{-1}(\Theta_{h_A} + \Theta_{h^{(r)}})$ is strictly positive for $0 \leq r \leq m - 1$.

6.1 (Reduction to the construction of a metric $h_\infty$ on $(K_X \otimes S \otimes L)^{\otimes m}$).

Similarly, it suffices to create a singular metric $h_\infty$ on $(K_X \otimes S \otimes L)^{\otimes m}$ such that

$$\sqrt{-1} \Theta_{h_\infty} \geq 0$$

and

$$|d\sigma^m_W \wedge u_W|_{h_\infty} \leq 1 \quad \text{for every } W \in \mathcal{W} \text{ such that } u_W \text{ is not identically 0.}$$

We will apply Lemma 6.2 to the collection

$$v_W = d\sigma^{\otimes(m-1)}_W \wedge u_W \quad (W \in \mathcal{W})$$

($v_W$ viewed as sections of $K_W \otimes S^W \otimes F|_W$) with the singular metric

$$h_F = h^{\frac{m-1}{m}}_\infty \cdot h_L$$

on

$$F = (K_X \otimes S)^{\otimes(m-1)} \otimes L^{\otimes m}.$$ 

The curvature condition (6.3) holds by (6.6) since $\sqrt{-1} \Theta_{h_L} \geq \pm \sqrt{-1} \Theta_j$ for every $j \in J(\mathcal{W})$ by assumption. For (6.4), we only need to consider those $W$ along which $u_W$ is not identically zero. On these components, by (6.7), we have

$$\langle d\sigma^{\otimes(m-1)}_W \wedge u_W \rangle_{h^iW \otimes h_F}^2 \leq \langle u_W \rangle_{(h^iW \cdot h_L)^m}^2,$$
and hence the integral in (6.4) is dominated by

\[
\int_W \frac{\langle u_W \rangle^{\frac{2}{m}}_{(h^J_{W} \cdot h_L)^m}}{|\sigma^{J_{W}}_{h_{J_{W}}} \Pi_{J' \in J_0 \setminus \{J_{W}\}} (\sum_{j \in J'} |\sigma_j|_{h_j}^2)^{\delta_0 |J'|}}.
\]

Since \( u \in \ker \beta \), it vanishes along the divisor \( W \cap S^{J_{W}} \) on \( W \). The condition (6.1) implies that

\[
I(|\sigma^{J_{W}}_{h_{J_{W}}} - 2 + \frac{2}{m} h_L|_W) = O_W (i = 1, \ldots, m - 1),
\]

and hence

\[
\int_W \frac{\langle u_W \rangle^{\frac{2}{m}}_{(h^J_{W} \cdot h_L)^m}}{|\sigma^{J_{W}}_{h_{J_{W}}}^2}|_W < \infty.
\]

Thus (6.8) is finite if we choose \( \delta_0 > 0 \) to be sufficiently small, by the strong openness conjecture. Then we may apply Lemma 6.2 to obtain the desired section \( U \).

6.2 (Păun’s modified iteration). Write every integer \( k = mq + r \) with integers \( q \geq 0 \) and \( 0 \leq r \leq m - 1 \).

**Lemma 6.2.1.** For every \( k \geq m \), there exists a family of sections

\[
U_{k} : \quad U_{p}^{(k)} \in \Gamma(X, (K_X \otimes S \otimes L)^{\otimes mq} \otimes Q^{(r)} \otimes A) \quad (1 \leq p \leq N)
\]

such that

\[
U_{p}^{(k)}|_W = (d\sigma^{\otimes m}_W \land u_W)^q \otimes t_{p}^{(r)} \quad (p = 1, \ldots, N).
\]

**Remark 6.3.** For future references, we note that the proof of Lemma 6.2.1 does not use any property of \((L, h_L)\) but that \( L^m = \otimes_{i=1}^{m-1} L_i \).

**Proof.** The statement holds for the \( m \)-th stage by \((A_0')\). For \( k \geq m \), when the statement for the \( k \)-th stage is proved, we let \( h_k \) denote the singular metric on \((K_X \otimes S \otimes L)^{\otimes mq} \otimes Q^{(r)} \otimes A\) defined by the family \( U_{k} \) of sections. Fix smooth metrics \( h_{L_i} \) on \( L_i \) \((i = 1, \ldots, m - 1)\).

Now suppose that the statement holds before the \((k + 1)\)-th stage.

**Case 1.** \( r < m - 1 \).
Note that \( k + 1 = mq + (r + 1) \) and \((K_X \otimes S \otimes L)^{\otimes mq} \otimes Q^{(r+1)} \otimes A = K_X \otimes S \otimes L^{r+1} \otimes (K_X \otimes S \otimes L)^{\otimes mq} \otimes Q^{(r)} \otimes A\).

We consider metrics on \( L^{r+1} \otimes (K_X \otimes S \otimes L)^{\otimes mq} \otimes Q^{(r)} \otimes A \)
of the form

\[
h_{(k+1)}^{(k+1)} = (h_{L^{r+1}}^{1-\delta} \cdot \tilde{h}_{L^{r+1}}^{\delta}) \otimes (h_{k-\delta}^{1-\varepsilon} \cdot (h_T^{\otimes q} \otimes \tilde{h}_{Q^{(r)}}^{(r)} \otimes h_A)^\varepsilon)
\]

\(0 < \delta < 1, 0 < \varepsilon < 1\), the curvatures of which are

\[
\Theta_{(k+1)}^{(k+1)} = (1 - \delta)\Theta_{h_{L^{r+1}}} + [(1 - \delta)\Theta_{h_{L^{r+1}}} + (1 - \varepsilon)\Theta_{h_{k-\delta}} + \varepsilon(q\Theta_T + \Theta_{\tilde{h}_{Q^{(r)}}}) + \Theta_A)].
\]

Note that the current in the last parenthesis is strictly positive by \((A_3)\) and that \(\sqrt{-1} \Theta_T \geq 0\). Therefore, once \(\varepsilon > 0\) is chosen (no matter how small it is), if \(\delta > 0\) is taken to be sufficiently small, \(\sqrt{-1}[\cdots]\) must dominate a Kähler form.

Since \(\sqrt{-1} \Theta_{L_{r+1}}\) by assumption, we have

\[
\sqrt{-1} \Theta_{(k+1)}^{(k+1)} \succ \pm \sqrt{-1} \Theta_j \quad (j \in J(W)),
\]

and this verifies \((6.3)\). Now we analyse the singularity of the integrand in \((6.4)\):

\[
\langle (d\sigma^m_W \wedge u_W)^q \otimes t_p^{(r+1)} \rangle_{h^{J_W} \otimes h_{k-\delta}^{(k+1)}}^{2}\frac{\sigma^{J_W}_{|h^{J_W}_{|J_W} \otimes \prod_{J' \in J_W \setminus \{J_W\}} (\sum_{j \in J'} |\sigma_{J'}_{|h_{J'}}|^2)^{\delta_0} |J'|}}{|\sigma^{J_W}_{|h^{J_W}_{|J_W}}|^2}
\]

\(u_W\) in the numerator and the factor \(h_{k-\delta}^{1-\varepsilon}\) in the metric will contribute together a factor \(|u|^{2q\varepsilon}\) by the construction of \(h_{k-\delta}^{1-\varepsilon}\). Since \(u = (u_W) \in \ker \beta\), this vanishing factor together with \(|\sigma^{J_W}_{|h^{J_W}_{|J_W}}|^2\) in the denominator form a factor which is not more singular than \(|\sigma^{J_W}_{|h^{J_W}_{|J_W}}|^{-2+2q\varepsilon}\). On the other hand, the rest source of singularity in the numerator comes from \(h_{L_{r+1}}^{1-\delta} h_T^{q\varepsilon}\). By choosing \(\varepsilon > 0\) so small that \(q\varepsilon \leq \varepsilon_0\), we have

\[
\mathcal{I}(\sigma^{J_W}_{|h^{J_W}_{|J_W}}^{-2+2q\varepsilon} h_{L_{r+1}}^{1-\delta} h_T^{q\varepsilon} W) = \mathcal{O}_W,
\]

and hence

\[
\int_W \frac{\langle (d\sigma^m_W \wedge u_W)^q \otimes t_p^{(r+1)} \rangle_{h^{J_W} \otimes h_{k-\delta}^{(k+1)}}^{2}}{|\sigma^{J_W}_{|h^{J_W}_{|J_W}}|^2} < \infty \quad (W \in \mathcal{W}).
\]
Finally, we may take $\delta_0 > 0$ sufficiently small so that (6.4) holds by the strong openness conjecture. Applying Lemma 6.2 then proves the statement of the $(k + 1)$-th stage in this case.

Case 2. $r = m - 1$.

We have $k + 1 = m(q + 1) + 0$ and

$$(K_X \otimes S \otimes L)^\otimes m(q+1) \otimes A = K_X \otimes S \otimes (K_X \otimes S \otimes L)^m Q^{(m-1)} \otimes A.$$ 

As $Q$-line bundles we have

$$(K_X \otimes S \otimes L)^{mq} \otimes Q^{(m-1)} \otimes A = (K_X \otimes S \otimes L)^{mq+m-1} \otimes L \otimes A.$$ 

Therefore it has two particular metrics

$$h_k = h_{mq+m-1}$$ and $$h_T^{mq+m-1} \cdot (h_{L_1} \otimes \cdots \otimes h_{L_{m-1}}) h_A,$$

and we consider their convex combinations

$$h^{(k+1)}_\varepsilon = h_k^{1-\varepsilon} \cdot (h_T^{mq+m-1} \cdot (h_{L_1} \otimes \cdots \otimes h_{L_{m-1}}) h_A) \varepsilon \quad (0 < \varepsilon < 1),$$

the curvatures of which are

$$\Theta^{(k+1)}_\varepsilon = (1 - \varepsilon)\Theta_{h_k} + \varepsilon ((q + 1 - m^{-1})\Theta_T + m^{-1}\sum_{i=1}^{m-1} \Theta_{h_{L_i}} + \Theta_A).$$

All the curvature forms involved are semipositive and $\sqrt{-1}\Theta_A$ is a Kähler form, and hence (6.3) holds. We may choose $\varepsilon > 0$ so that $\varepsilon(q + 1 - m^{-1}) \leq \varepsilon_0$. Verification of (6.4) in this case is similar to that in Case 1 and is slightly simpler: the vanishing of $|\sigma^J W| h_{sW}^2$ is completely cancelled out by that of $u_W$, and the integrability can then be obtained by (6.2) and Hölder’s inequality. Applying Lemma 6.2 then proves the statement of the $(k + 1)$-th stage completely. □

6.3 (The construction of the metric $h_\infty$). By ($A'_1$) and Lemma 6.2.1, we know in particular that for every section $t$ (fixed in this section) of $A$, the sets

$$E_q := \{ F \in \Gamma(X, (K_X \otimes S \otimes L)^mq \otimes A) \mid F|_{\mathcal{W}} = u^{\otimes q} \otimes t \} \quad (q \in \mathbb{N})$$

are nonempty. We define for every

$$F \in \Gamma(X, (K_X \otimes S \otimes L)^mq \otimes A)$$
that
\[ \|F\|_{\frac{2}{mq}} := \int_X \langle F \rangle_{(h_S, h_L)^{mq} \otimes A}^2, \]
which is finite by the initial assumption (iv) in the initial setting. Then for every \( t \) there exists a section \( F_{\text{min}}(t) = F_{\text{min}} \in E_q \) such that \( \|F_{\text{min}}\|_{\frac{2}{mq}} \leq \|F\|_{\frac{2}{mq}} \) for every \( F \in E_q \). More precisely, for a fixed \( q \) we pick a sequence \( F_n \in E_q \) such that \( \|F_n\|_{\frac{2}{mq}} \) converges to \( \inf_{F \in E_q} \|F\|_{\frac{2}{mq}} \) as \( n \to \infty \). By the sub-mean-value inequality \( F_n \) admits a subsequence which converges uniformly to some \( F_{\infty} \). By Fatou’s lemma we see that \( \|F_{\infty}\| = \inf_{F \in E_q} \|F\|_{\frac{2}{mq}} \), and it suffices to take \( F_{\text{min}} = F_{\infty} \).

**Lemma 6.3.1.** There exists a constant \( C > 0 \) such that
\[ \|F_{\text{min}}\|_{\frac{2}{mq}} \leq C \quad \text{for all} \quad q \in \mathbb{N}. \]

*Proof.* We will apply Lemma 6.2 to extend the section \( u^\otimes q \otimes t \) from \( \mathcal{W} \) to a section \( F' \) on \( X \) with finite \( L^2 \)-norm. Write
\[ (K_X \otimes S \otimes L)^{mq} \otimes A = K_X \otimes S \otimes (K_X \otimes S)^{mq-1} \otimes (L^m)^{\otimes q} \otimes A \]
and equip the underlined line bundle (which is
\[ \left((K_X \otimes S \otimes L)^{mq} \otimes A\right)^{\frac{mq-1}{mq}} \otimes L \otimes A^{\frac{1}{mq}} \]
as a \( \mathbb{Q} \)-divisor) with the metric \( h^\frac{mq-1}{mq} \cdot h_L \cdot h^\frac{1}{mq}_A \). The curvature condition (6.3) holds obviously by the presence of the factor \( h^{\frac{1}{mq}}_A \). As for (6.4), the integrand is
\[ \langle d\sigma_W^{\otimes (mq-1)} \wedge u_W^\otimes q \otimes t \rangle^2_{h^{mq-1} \otimes h_{F_{\text{min}}}^{mq-1} \cdot h_L \cdot h^{\frac{1}{mq}}_A} \]
\[ |\sigma_{JW}|^2_{h^{JW}} \prod_{J' \in J_0(\mathcal{W}) \setminus \{J_W\}} \left( \sum_{j \in J'} |\bar{\sigma}_j h_j|^2 \right)^{\delta_0 |J'|}. \]
We may write the numerator as follows:

$$
\langle d\sigma^\otimes (mq-1) \wedge u^\otimes q \otimes t \rangle^2_{h^J W \otimes h^q F_{\min} \cdot h_L \cdot h_A}^{mq-1} \cdot \frac{1}{h^J W} = \frac{t^2_{h_A} \cdot \langle u W \rangle_{h^J W \cdot h_L}^m}{|\sigma^J W|_{h^J W}^2}.
$$

Then

$$
\frac{\langle d\sigma^\otimes (mq-1) \wedge u^\otimes q \otimes t \rangle^2_{h^J W \otimes h^q F_{\min} \cdot h_L \cdot h_A}^{mq-1}}{|\sigma^J W|^2_{h^J W}} = \frac{t^2_{h_A} \cdot \langle u W \rangle_{(\sigma^J W) \otimes m}^m}{h^m L}.
$$

Since \( u \in \ker \beta \), the singularity of \( \langle u W \rangle_{(\sigma^J W) \otimes m}^m \) is not worse than \( |\sigma^J W|^2_{h^J W} \cdot \frac{2}{h^m L} \).

By the initial assumption (6.1) we have

$$
\mathcal{I}(|\sigma S|_{h^J S}^{-2+\frac{2}{h^m L}}) = \mathcal{O}_W \quad (W \in \mathcal{W}).
$$

For sufficiently small \( \delta_0 > 0 \) we have, by the strong openness conjecture,

$$
\int_W \frac{\langle d\sigma^\otimes (mq-1) \wedge u^\otimes q \otimes t \rangle^2_{h^J W \otimes h^q F_{\min} \cdot h_L \cdot h_A}^{mq-1}}{|\sigma^J W|^2_{h^J W} \cdot \prod_{J' \in J_0(W) \setminus \{J_W\}} \left( \sum_{j \in J'} |\sigma_j|_{h^J j}^2 \right) \delta_0 |J'|} \lesssim \int_W \prod_{J' \in J_0(W) \setminus \{J_W\}} \left( \sum_{j \in J'} |\sigma_j|_{h^J j}^2 \right) \delta_0 |J'| \frac{t^2_{h_A} \cdot \langle u W \rangle_{(\sigma^J W) \otimes m}^m}{h^m L} < \infty.
$$

52
Then we apply Lemma 6.2 to obtain an extension $F'$ such that
\[
\int_X \left\langle F' \right\rangle^2 \frac{m_{q-1}}{h_S \otimes h_{F_{\min}}^{m_{q-1}} \cdot h_L \cdot h_A^{m_{q-2}}} =: C_0(S, \delta_0) \int_W \prod_{J' \in J_0} \left( \sum_{j \in J'} |\sigma_j|^2 \right)^{\delta_0} \left( \frac{u_W}{(\sigma_J^{h_L})^{m_{q-1}}} \right)^{\frac{2}{m}} = C_1.
\]

Now we write the integrand on the left hand side more explicitly:
\[
\left\langle F' \right\rangle^2 \frac{m_{q-1}}{h_S \otimes h_{F_{\min}}^{m_{q-1}} \cdot h_L \cdot h_A^{m_{q-2}}} = \left\langle F' \otimes m_q \right\rangle^2 \frac{2}{h_{F_{\min}}^{m_{q-1}} \otimes (h_S \cdot h_L)^{m_q \cdot h_A}} = \left\langle F' \otimes m_q \right\rangle^2 \frac{2}{h_{F_{\min}}^{m_{q-1}} \otimes (h_S \cdot h_L)^{m_q \cdot h_A}}.
\]

Thus,
\[
\int_X \left\langle \frac{F' \otimes m_q}{F_{\min}^{\otimes (m_q-1)}} \right\rangle^2 \frac{2}{(h_S \cdot h_L)^{m_q \cdot h_A}} \leq C_1 \quad (C_1 \text{ being independent of } q \in \mathbb{N}).
\]

The proof of lemma will be completed if we can show that
\[
\left\| F_{\min} \right\|_{\frac{2}{m}}^2 \leq \int_X \left\langle \frac{F' \otimes m_q}{F_{\min}^{\otimes (m_q-1)}} \right\rangle^2 \frac{2}{(h_S \cdot h_L)^{m_q \cdot h_A}} = \left\| F' \otimes m_q \right\|_{\frac{2}{m}}^2.
\]

Were this false, by Hölder’s inequality, we have
\[
\left\| F' \right\|_{\frac{2}{m}}^2 = \left\| F' \otimes m_q \right\|_{\left( \frac{2}{m_q} \right)^2} = \left\| \frac{F' \otimes m_q}{F_{\min}^{\otimes (m_q-1)}} \otimes F_{\min}^{\otimes (m_q-1)} \right\|_{\left( \frac{2}{m_q} \right)^2}^2 \leq \left\| \frac{F' \otimes m_q}{F_{\min}^{\otimes (m_q-1)}} \right\|_{\left( \frac{2}{m_q} \right)^2} \left\| \frac{F_{\min}^{\otimes (m_q-1)}}{F_{\min}^{\otimes (m_q-1)}} \right\|_{\left( \frac{2}{m_q} \right)^2}^2 = \left\| F_{\min} \right\|_{\frac{2}{m}}^2 \cdot \left\| F_{\min}^{\otimes (m_q-1)} \right\|_{\left( \frac{2}{m_q} \right)^2} \frac{m_q(m_q-1)}{2} \left( \frac{2}{m_q} \right)^2,
\]

a contradiction to the minimality of $F_{\min}$.

Now we apply the above discussion to the sections $t^{(0)}_p$ ($p = 1, \ldots, N$) to obtain corresponding minimal extensions
\[
F_{q,p,\min} \in \Gamma\left( X, (K_X \otimes S \otimes L)^{\otimes m_q} \otimes A \right)
\]
of $u^\otimes q \otimes t_p^{(0)} \ (p = 1, \ldots, N)$. Let $F_{\alpha,q,p}$ be the functions locally representing $F_{q,p,\min}$ on trivializing charts. For every $q \in \mathbb{N}$, we have a singular metric on the $\mathbb{Q}$-line bundle $(K_X \otimes S \otimes L)^{\otimes m} \otimes A^{1/2}$ with local weights

$$\varphi_{q,\alpha} = \frac{1}{q} \log \sum_{p=1}^{N} |F_{q,p,\alpha}|^2.$$  

As in [5.3], the construction will be completed if the regularized upper limit $\varphi_{\alpha}$ of $\varphi_{q,\alpha}$ exists as $q \to \infty$ and is a locally integrable psh function on those $W \in \mathcal{W}$ along which $u$ is not identically zero for every $\alpha$. It suffices to show that $\varphi_{q,\alpha} \ (q \in \mathbb{N})$ are uniformly bounded from above on every compact set. This can be seen by applying Jensen’s inequality and Lemma [6.3.1].

**APPENDIX A. Preliminaries for solving variants of the $\overline{\partial}$-equation**

Let $X$ be a complex manifold of complex dimension $n$.

**A.1 (Solving twisted $\overline{\partial}$-equations).** The following theorem serves as a standard framework of solving twisted $\overline{\partial}$-equations with $L^2$-estimate, which can be obtained following the proof of (6.1) Theorem in [7].

**Theorem A.1.1.** Let $(X, g)$ be an $n$-dimensional Kähler manifold admitting a complete Kähler metric (which is not necessarily $g$), $(E, h)$ a holomorphic vector bundle on $X$ with a smooth hermitian metric, and $\lambda$ and $\mu$ two strictly positive bounded smooth functions. Let

$$R = R_{h,\lambda,\mu} := \sqrt{-1} \left( \lambda \Theta_h - \left( \partial \overline{\partial} \lambda - \frac{1}{\mu} \partial \lambda \wedge \overline{\partial} \lambda \right) \otimes \text{id}_E \right).$$

For any $\eta \in L^2_{g,h}(X, \wedge^{n,q} T^* X \otimes E)$ (with $q \geq 1$) and any nonnegative constant $C$, if $CR$ is semipositive, $\overline{\partial} \eta = 0$, and

$$|(\eta, v)_{L^2_{g,h}}|^2 \leq C \int_X R[v, v]_{g,h} dV_g$$

for every $v \in D^{n,q}(X, E)$, then there exist a locally Lebesgue integrable $E$-valued $(n, q-1)$-form $\gamma$ and a locally Lebesgue integrable $E$-valued $(n, q)$-form $\beta$ such that

$$\overline{\partial} \gamma = \eta \quad \text{in the sense of current}$$
and
\[ \int_X \frac{1}{\lambda + \mu} |\gamma|_{g,h}^2 dV_g \leq C. \]

**A.2 (Auxiliary facts when dealing with complements of analytic subsets).** The following lemma is crucial in obtaining estimates in $L^2$-norms with weight functions singular along analytic subsets from which holomorphic sections are to be extended. In our discussions it will mainly be applied to fulfil the completeness requirement in Lemma A.1.1.

**Lemma A.2.1** ([8] Lemma 1.5). Given an analytic set $Z$ and a relatively compact open set $\Omega$ in a Kähler manifold $X$, if $\Omega$ admits a complete Kähler metric, so does $\Omega \setminus Z$.

For sequences of holomorphic sections of a holomorphic vector bundle with a smooth hermitian metric, convergence locally in $L^2$ coincides with uniform convergence on every compact subset. This fact has the following generalization, which is a slight modification of Lemma 4.6 of [20]:

**Lemma A.2.2.** Let $Z$ be an analytic subset of a hermitian manifold $(M, g)$. Suppose that there are

1. open subsets $O_1 \subseteq O_2 \subseteq \cdots \subseteq O_m$ of $M$ such that $M \setminus Z = \bigcup_m O_m$,
2. a sequence $U_m$ of Lebesgue measurable sections of $K_M \otimes E$, $E$ being a hermitian holomorphic vector bundle equipped with a smooth hermitian metric $h$, for every point $p \in M \setminus (Z)_{\text{sing}}$ there exists an open neighborhood $V_p$ of $p$ and an index $m_p$ such that $U_m|_{V_p}$ (for $m \geq m_p$) are all holomorphic, and
3. a sequence $w_m$ of positive Lebesgue measurable functions on $M$ such that for every compact subset $K$ of $M \setminus Z$ there exists an index $m_K$ such that the family of functions $w_m$ (for $m \geq m_K$) are uniformly bounded away both from 0 and from $\infty$, and

If $w_m$ converges to a function $w$ almost everywhere, and if $\liminf_{m \to \infty} \int_{O_m} w_m(U_m)^2 h$ exists as a real number, then $U_m$ admits a subsequence which converges uniformly
on every compact subset of $M$ to a section $U \in \Gamma \left( M, O_M \otimes K_M \otimes E \right)$ such that

$$\int_M w(U)^2_h \leq \liminf_{m \to \infty} \int_{O_m} w_m(U_m)^2_h.$$

**Proof.** By passing to a subsequence we may assume that $\int_{O_m} w_m(U_m)^2_h$ actually converges as $m \to \infty$. We may also assume $Z$ to be a submanifold by considering $(M \setminus Z_{\text{sing}}, Z \setminus Z_{\text{sing}})$ instead of $(M, Z)$ and then extending the obtained section on $M \setminus Z_{\text{sing}}$ to $M$ by the second Riemann extension theorem. Besides, it suffices to show, as we will do in the next paragraph, that every point $p \in M$ admits an open neighborhood $N_p$ on which $U_m$ with $m$ sufficiently large form a normal family. More precisely, were this done, then $M$ is covered by countably many such open sets $N_{p_1}, N_{p_2}, \ldots$, and an application of the diagonal method yields a subsequence $U_{m_k}$ which converges uniformly on every compact subset of $M$ to a limit $U$, which is clearly a holomorphic section of $K_M \otimes E$ on $M$ by the condition (2). Then for every $j$ we have

$$\int_{O_j} w(U)^2_h \leq \lim_{k \to \infty} \int_{O_{m_k}} w_{m_k}(U_{m_k})^2_h$$

by Fatou’s lemma, and hence

$$\int_M w(U)^2_h = \int_{M \setminus Z} w(U)^2_h = \lim_{j \to \infty} \int_{O_j} w(U)^2_h \leq \lim_{m \to \infty} \int_{O_m} w_m(U_m)^2_h$$

By the monotone convergence theorem. Therefore it remains to find the desired neighborhood $N_p$ for every $p \in M$.

Since the statement is purely local, we may assume that $M$ is an open subset of $\mathbb{C}^n$. It suffices to find a neighborhood $N_p$ for every $p \in M$ such that $U_m$ for $m$ sufficiently are uniformly bounded with respect to the euclidean $L^2$ norms on $N_p$. For $p \in M \setminus Z$ such $N_p$ exists obviously since both $w_m$ and the metric $h$ have strictly positive uniform lower and upper bounds. For $p \in Z$, we may further assume that

$$(M, Z, p) = \left( D_2(0)^n, \{0\} \times D_2(0)^{n-1}, (0, \ldots, 0) \right)$$

where $D_R(0) := \{ z \in \mathbb{C} \mid |z| < R \}$, and that $E$ is the trivial bundle. Without any control on $w_m$, we need to apply the following elementary fact (cf. Lemma
4.4 of [20]): for fixed $0 < r < 1$ and for every holomorphic function $F$ on $D_2(0)^n$, we have
\[
\int_{D_1(0)^n} |U|^2 d\lambda \leq \frac{1}{1 - r^2} \int_{(D_1(0) \setminus D_r(0)) \times D_1(0)^{n-1}} |U|^2 d\lambda.
\]
This inequality can be directly verified by expanding $F$ as a power series centered at $(0, \ldots, 0)$ and using the fact that different monomial functions are mutually orthogonal with respect to the $L^2$-norms on both sides. For $m$ sufficiently large $(D_1(0) \setminus D_r(0)) \times D_1(0)^{n-1}$ is covered by $O_m$, and hence
\[
\int_{(D_1(0) \setminus D_r(0)) \times D_1(0)^{n-1}} |U|^2 d\lambda
\]
is bounded by a fixed multiple of $\int_{O_m} w_m(U_m)_h$. This completes the proof. \(\square\)

A.3 (Comparison of curvatures).

Lemma A.3.1. Let $v_1, \ldots, v_q$ be strictly positive smooth functions on a complex manifold $X$. For an integer $1 \leq l \leq q$ we have
\[
\sqrt{-1} \partial \bar{\partial} \log \sum_{i=1}^{k} v_i \geq \sum_{i=1}^{k} \frac{v_i}{\sum_j v_j} \sqrt{-1} \partial \bar{\partial} \log v_i.
\]

Proof. We let $\lambda_i := \frac{v_i}{\sum_j v_j}$.
\[
\sqrt{-1} \partial \bar{\partial} \log \sum_{i=1}^{k} v_i = \sqrt{-1} \left( \sum_{i} \frac{\partial \bar{\partial} v_i}{\sum_j v_j} - \sum_{r} \frac{\partial v_r}{\sum_j v_j} \wedge \sum_{s} \frac{\partial v_s}{\sum_j v_j} \right)
\]
\[
= \sqrt{-1} \left( \sum_{i} \frac{\partial \bar{\partial} v_i}{v_i} - \left( \sum_{r} \frac{\partial v_r}{v_r} \wedge \sum_{s} \frac{\partial v_s}{v_s} \right) \right)
\]
\[
= \sqrt{-1} \left( \sum_{i} \lambda_i \frac{\partial \bar{\partial} v_i}{v_i} - \left( \sum_{r} \lambda_r \frac{\partial v_r}{v_r} \wedge \sum_{s} \lambda_s \frac{\partial v_s}{v_s} \right) \right)
\]
\[
= \sum_{i} \lambda_i \sqrt{-1} \left( \frac{\partial \bar{\partial} v_i}{v_i} - \frac{\partial v_i}{v_i} \wedge \frac{\partial v_i}{v_i} \right) + \sqrt{-1} \sum_{r,s} \left( \lambda_r \delta_{rs} - \lambda_r \lambda_s \right) \frac{\partial v_r}{v_r} \wedge \frac{\partial v_s}{v_s}
\]
\[
= \sum_{i} \lambda_i \sqrt{-1} \partial \bar{\partial} \log v_i + \sqrt{-1} \sum_{r,s} \left( \lambda_r \delta_{rs} - \lambda_r \lambda_s \right) \frac{\partial v_r}{v_r} \wedge \frac{\partial v_s}{v_s}.
\]
The second sum in the last line can be seen to be a semipositive \((1,1)\)-form. More precisely, for any \((a_1, \ldots, a_m) \in \mathbb{C}^k\) we have
\[
\sum_{r,s} (\lambda_r \delta_{rs} - \lambda_r \lambda_s) a_r \overline{a_s} = \left( \sum_i \lambda_i |a_i|^2 \right) \left( \sum_i \lambda_i \right) - \left| \sum_r \lambda_r a_r \right|^2 \geq 0
\]
by the Cauchy-Schwarz inequality.

References

[1] C. Birkar and M. Păun, Minimal models, flips and finite generation: a tribute to V. V. Shokurov and Y.-T. Siu, Classification of algebraic varieties, 77–113, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2011.
[2] B. Berndtsson, The openness conjecture for plurisubharmonic functions, arXiv:1305.5781.
[3] B. Berndtsson and M. Păun, Quantitative extensions of pluricanonical forms and closed positive currents, Nagoya Math. J. 205 (2012), 25–65.
[4] J. Cao, J.-P. Demailly, and S. Matsumura, A general extension theorem for cohomology classes on non reduced analytic subspaces, Sci. China Math. 60 (2017), no. 6, 949–962.
[5] C.-Y. Chi, Y. Sung, and S.-S. Wang, Extensions of pluri-canonical sections from strata of a divisor of simple normal crossing II, to appear.
[6] C.-Y. Chi, C.-L. Wang, and S.-S. Wang, Extensions of multiply twisted pluri-canonical forms, Pure Appl. Math. Quarterly 7 (2011), no. 4, 1129–1164, special issue dedicated to Eckart Viehweg.
[7] J.-P. Demailly, Complex analytic and differential geometry, online resource.
[8] J.-P. Demailly, Estimations \(L^2\) pour l’opérateur \(\overline{\partial}\) d’un fibré vectoriel holomorphe semi-positif au-dessus d’une variété kählerienne complète, Ann. Sci. École Norm. Sup. (4) 15 (1982), no. 3, 457–511.
[9] J.-P. Demailly, Vanishing theorems for tensor powers of an ample vector bundle, Invent. Math. 91 (1988), no. 1, 203–220.
[10] J.-P. Demailly, \(L^2\) estimates for the \(\overline{\partial}\) operator on complex manifolds, Notes de cours, Ecole d’été de Mathématiques (Analyse Complex), Institut Fourier, Grenoble, Juin 1996.
[11] J.-P. Demailly, On the Ohsawa-Takegoshi-Manivel \(L^2\) extension theorem, Complex analysis and geometry (Paris, 1997), 47–82, Progr. Math., 188, Birkhäuser, Basel, 2000.
[12] J.-P. Demailly, Analytic methods in algebraic geometry, Surveys of Modern Mathematics, 1 Int. Press, Somerville, MA; Higher Education Press, Beijing, 2012.
[13] J.-P. Demailly, On the cohomology of pseudoeffective line bundles, Complex geometry and dynamics, 51–99, Abel Symp., 10, Springer, Cham, 2015.
[14] J.-P. Demailly, Extension of holomorphic functions defined on non reduced analytic subvarieties. The legacy of Bernhard Riemann after one hundred and fifty years. Vol. I, 191–222, Adv. Lect. Math., 35.1, Int. Press, Somerville, MA, 2016.
[15] J.-P. Demailly, C.D. Hacon, and M. Păun, Extension theorems, non-vanishing and the existence of good minimal models, Acta Math. 210 (2013), no. 2, 203–259.
[16] L. Ein and M. Popa, Extension of sections via adjoint ideals, Math. Ann. 352 (2012), no. 2, 373–408.
[17] W. Fulton, Young tableaux, with applications to representation theory and geometry, London Mathematical Society Student Texts, 35. Cambridge University Press, Cambridge, 1997.
[18] Y. Gongyo and S. Matsumura, Versions of injectivity and extension theorems, Ann. Sci. Éc. Norm. Supér. (4) 50 (2017), no. 2, 479–502.
[19] P. Griffiths, Hermitian differential geometry, Chern classes, and positive vector bundles, Global Analysis (Papers in Honor of K. Kodaira) 185–251 Univ. Tokyo Press, Tokyo, 1969.
[20] Q. Guan and X. Zhou, A solution of an \(L^2\) extension problem with an optimal estimate and applications, Ann. of Math. (2) 181 (2015), no. 3, 1139–1208.
21] Q. Guan and X. Zhou, *A proof of Demailly’s strong openness conjecture*, Ann. of Math. (2) 182 (2015), no. 2, 605–616
22] P. H. Hiep, *The weighted log canonical threshold*, C. R. Math. Acad. Sci. Paris 352 (2014), no. 4, 283–288.
23] L. Hörmander, *\(L^2\) estimates and existence theorems for the \(\overline{\partial}\) operator*, Acta. Math. 113 (1965), 89–152.
24] R. Lazarsfeld, *Positivity in Algebraic Geometry II*, Erg. Math. 49, Springer, 2004.
25] L. Lempert, *Modules of square integrable holomorphic germs*, Analysis meets geometry, 311–333, Trends Math., Birkhäuser/Springer, Cham, 2017.
26] J. McNeal and D. Varolin, *Analytic inversion of adjunction: \(L^2\) extension theorems with gain*, Ann. Inst. Fourier (Grenoble) 57 (2007), no. 3, 703–718.
27] L. Manivel, *Un théorème de prolongement \(L^2\) de sections holomorphes d’un fibré vectoriel*, Math. D. 212 (1993), 107–122.
28] T. Ohsawa, *On the extension of \(L^2\) holomorphic functions. III. Negligible weights*, Math. Z. 219 (1995), no. 2, 215–225.
29] T. Ohsawa, *On the extension of \(L^2\) holomorphic functions. V. Effects of generalization*, Nagoya Math. J. 161 (2001), 1–21.
30] T. Ohsawa and K. Takegoshi, *On the extension of \(L^2\) holomorphic functions*, Math. D. 195 (1987), 197–204.
31] T. Ohsawa, *On a curvature condition that implies a cohomology injectivity theorem of Kollár-Skoda type*, Publ. Res. Inst. Math. Sci. 41 (2005), no. 3, 565–577.
32] M. Păun, *Siu’s invariance of plurigenera: a one-tower proof*, J. Diff. Geom. 76 (2007), 485–493.
33] M. Păun, *Relative critical exponents, non-vanishing and metrics with minimal singularities*, Invent. Math. 187 (2012), no. 1, 195–258.
34] Y.-T. Siu, *Every Stein subvariety admits a Stein neighborhood*, Invent. Math. 38 (1976), 89–100.
35] Y.-T. Siu, *Extension of twisted pluricanonical sections with plurisubharmonic weight and invariance of semipositively twisted plurigenera for manifolds not necessarily of general type*, Complex Geometry, Göttingen, 2000, Springer, Berlin, (2002), 223–277.
36] D. Varolin, *A Takayama-type extension theorem*, Compositio Math. 144 no.2 (2008), 522–540.

CHEN-YU CHI: DEPARTMENT OF MATHEMATICS, NATIONAL TAIWAN UNIVERSITY, TAIPEI 10617, TAIWAN
E-mail address: chi@math.ntu.edu.tw