Higher Dimensional Hypercategories

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Abstract

We introduce higher dimensional hypergraphs, which is a generalization of Baez-Dolans’s opetopic sets and Hermida-Makkai-Power’s multitopes. This is based on a simple combinatorial structure called shells and the formal composites of pasting diagrams based on the closure of open shells. We give two types of graphical representation of higher dimensional cells which show effectively the relationship of cells of different dimensions. Using the hypergraphs, we define strict hypercategories and illustrate its use by taking Lafont’s interaction combinator \[5\] as an example. We also give a definition of weak \(\omega\)-hypercategories and show that usual category is identified with a special kind of weak hypercategory as a sample of arguments in our framework.

Introduction

The purpose of this paper is to give an elementary reformulation of Baez-Dolan’s definition \[6\] of weak higher dimensional categories from the following points of views:

(H1) In view of the importance of the higher dimensional category theory in wide area of mathematical science, it should be formulated elementary without using advanced category theory.

(H2) The shapes of cells and pasting diagrams, prepared by the framework, should be as general as possible, in order not to restrict the potential application.

(H3) Graphical representations of cells should be devised which describe succinctly the interrelationship among cells of various dimensions.

The point (H1) may not seem realistic in view of the necessity of the usage of advanced category theory in various formulations \[2, 10\] but its feasibility is seen clearly in the Baez-Dolan’s formulation which radically reconstructs category theory in the following points: (i) The composites of a pasting diagram of \(n\)-cells, if they existed at all, should be explicitly described \(n+1\)-cells which they call universal. (ii) The result of composition of \(n\)-cells depends on the universal \(n+1\)-cells, but the results are as it were isomorphic. (iii) However, “isomorphism” is a primitive notion, not defined one as in the usual category theory. In fact, in their formulation, the composition and isomorphisms are defined mutually recursively. (iv) The existence of cells which play the role of identities are derived from the axiom on universal cells. See \[6\] for another radical viewpoint of category theory.

The point (H2) has importance in view of the fact that the “computad”, a sort of 2-graph, whose 2-cells have general polyhedron shapes \(8, 9\) is indispensable in 2-category theory. However usually they are considered merely as a useful informal graphical apparatus, which is theoretically unnecessary. For example, the pasting diagram shown in the upper Figure \[6\] with 2-cells of general form is considered as an informal representation of the sequences of globular 2-cells drawn in the lower, which is not unique but equivalent to any other such sequence by virtue of the interchange law. The upper diagram seems to be usually more helpful in actual reasoning compared with the lower one.

Most of the current proposals of the formulation of higher dimensional category restrict the shape of cells, e.g. to globular forms \(2\) or to tree forms \(1, 3, 7\). The

\[1\]Note that the usual definition of inverse makes no sense since the composition is not unique by (ii) and the very notion of identity does not exist by (iv).
definition of “weak n-category” seems to be considerably simplified by removing such
restriction. For example, we will see that by using the cells of general form, we
can in a sense unify the notion of universality and balancedness in the Baez-Dolan’s
formulation.

Another obstacle of higher dimensional category theory seems to be the lack of
methods of clearly drawing higher dimensional cells which can conceal unnecessary
lower dimensional details appropriately. Note that the direct generalization of usual 2-
pasting diagrams is not feasible for dimensions higher than 2 and for higher dimensional
multicategories.

**Outline of the definition** We proceed as follows. (i) We introduction of a simple
combinatorial structure called shell, which describes universal shapes of cells and
pasting diagrams. (ii) We define the notion of labeling of shells, which substantiate
shells, and certain classes of intermediate substantiation are called cells, frames and
pasting diagrams. We define the closure operation and formal composite of pasting
diagrams, which is the most important combinatorial apparatus of this paper. (iii)
A hypergraph is defined inductively by substantiating some of the frames of previous
dimension. (iv) We show that there is a monad on the category of n-hypergraphs.
Strict hypercategories is defined to be the algebra over this monad or its submonads.
(v) To show the usefulness of our framework, we describe the process of graph rewrit-
ing in the Lafont’s interaction network as certain 2-dimensional cells in a free strict
2-hypercategory. This framework seems to possess a nice connection with graphical
language. This give also semantics for a programming language. (vi) A hypercat-
egory is an ω-hypergraph endowed with special cells called universal such that (iii-a)
pasting diagram of dimension n can be completed to an n + 1-cell, giving composition
like operations which however may not have unique result, (iii-b) universal cells are
closed under the “composition”, which play the role both of composition and of equiv-
alence, (iii-c)universal cells has inverses with conjugate frames. (vii) As an example
of arguments of our formulation, we prove that a 1-hypercategory of certain type is
nothing but a usual category. The points of proof are the construction of the identity
maps and the proof of the associativity.
1 Preliminaries

Trees A tree is a directed graph with a node called the root to which there is a unique path from every node other than the root. Nodes different from the root are called general. When there is an edge from \( c \) to \( p \), we call \( p \) the parent of \( c \) and \( c \) a child of \( p \). When there is a path from a node \( x \) to a node \( y \), \( x \) is called a descendant of \( y \) and \( y \) an antecedent of \( x \).

Let \( T \) be a tree. The set of the children of a node \( p \) is denoted by \( \text{Child}_T(p) \) or often simply by \( \text{Child}(p) \). A node \( x \) and its descendants form a tree denoted by \( T_x \).

The length of the unique path from a node \( x \) to the root is called the depth of \( x \). We denote by \( T[k] \) the set of all the nodes of depth \( k \), so that \( T[1] = \text{Child}(o_T) \) and \( T[i+1] = \bigsqcup_{j \in T[i]} \text{Child}(j) \). Note that if the node \( x \) is of depth \( k \) then \( T_x[k+i] \subseteq T[k+i] \).

The sets of nodes of depth \( \geq i \) is denoted by \( T[i] \), namely, \( T[i] := \bigsqcup_{j \geq i} T[j] \). A tree is called of height \( \leq n \) if the depth of its nodes are \( \leq n \) and of height \( n \) if it is of height \( \leq n \) and there is at least one node of depth \( n \).

\( \mathcal{C} \)-words Let \( \mathcal{C} \) be a category. A \( \mathcal{C} \)-word is a family of objects indexed by a finite set. A \( \mathcal{C} \)-word is denoted as \( w = (w_i)_{i \in |w|} \). An isomorphism \( f : w \rightarrow u \) consists of a bijection \( |f| : |w| \rightarrow |u| \) and a family of iso’s \( f_i : w_i \rightarrow u_i |f| (i) \) for \( i \in |w| \). A \( \mathcal{C} \)-word can be described in various ways.

For example, a \( \mathcal{C} \)-word \( w \) with \( |w| = \{2, 3, 5, a, b\} \) and \( w_2 = A \), \( w_3 = B \), \( w_5 = A \), \( w_a = C \), \( w_b = A \) can be drawn as follows.

\( \mathcal{C} \)-trees Let \( \mathcal{C} \) be a category. A \( \mathcal{C} \)-tree, or a tree of \( \mathcal{C} \)-objects, is a family of \( \mathcal{C} \)-objects indexed by the nodes of a tree. We write a \( \mathcal{C} \)-tree as \( w = (w_p)_{p \in T_w} \), where \( T_w \) is the underlying tree of \( w \). An isomorphism \( f : w \rightarrow u \) consists of a tree isomorphism \( |f| : T_w \rightarrow T_u \) and isomorphisms \( f_p : w_p \rightarrow u_{|f| (p)} \) for each nonnegative integer \( n \), called the depth \( n \) layer of \( w \).
A finite groupoid is called a link type if it has no nontrivial endoarrows and its nontrivial isomorphism called an involution, is composable only with its own inverse and the identities. More explicitly, a finite groupoid $L$ is a link type if

- $L(a, a) = \{ \text{id} \}$,
- if $a \neq b$, then $|L(a, b)| \leq 1$,
- if $L(a, b) \cap L(b, c) \neq \emptyset$ with $a \neq b \neq c$, then $a = c$.

The following is a typical example of a link type,

where the identity arrows are omitted.

An object of a link type is called internal if it is the domain of an involution and external otherwise. External objects form a discrete full subcategory denoted by $L_{\text{ext}}$, which is just a finite set considered as a discrete category. A link type without external objects is called closed. A link type which is not closed is called open.

A $C$-link $\varphi$ is a functor from a link type $|\varphi|$ to $C$, which we describe explicitly as

$$\varphi = ((\varphi_i)_{i \in |\varphi|}, (\varphi_m : \varphi_i \to \varphi_j)_{m : i \to j}).$$

We specify a $C$-link $\varphi$ by a $C$-word $(\varphi_i)_{i \in |\varphi|}$ with a set of isomorphisms each of which is composable only with its own inverses in the set. We call those isomorphisms also as the involutions of the $C$-link $\varphi$.

We can define the groupoid of $C$-links, whose isomorphism $\kappa : \varphi \to \psi$ between $C$-links is a pair $(|\kappa|, \{ \kappa_i : \varphi_i \to \psi_i \}_{i \in |\varphi|})$ of an isomorphic functor $|\kappa| : |\varphi| \to |\psi|$ and isomorphisms

$$\kappa_i : \varphi_i \to \psi_{\kappa(\varphi_i)}$$

which makes the following commutative for $m : i \to j$ in $|\varphi|$:
\( \mathcal{C} \)-links \( \varphi_i \) have coproduct \( \coprod_{i \in I} \varphi_i \) whose link type is the coproduct of the link types of \( \varphi_i \) and the functor \( \coprod_{i \in I} \to \mathcal{C} \) is the direct sum of the functors \( \varphi_i \).

There is a forgetful functor \( U \) from the groupoid of \( \mathcal{C} \)-links to that of \( \mathcal{C} \)-words and isomorphisms defined by throwing away the arrow parts: \( \varphi \mapsto U \varphi = (\varphi_i)_{i \in |\varphi|} \). A link structure on a \( \mathcal{C} \)-word \( w \) is a \( \mathcal{C} \)-link \( \varphi \) with \( U \varphi = w \).

2 Shells

2.1 Motivating example

First we explain an example which motivate the definition of shells. The following is the process of blowing up the tetrahedron, first along edges and then at the vertices.

The final result is can be described as in Fig. 2.

Figure 2: The tree representation of the 3-shell representing the blow-up structure of the tetrahedron. The two nodes are considered different components even they have the same label. The vertex 0 in the tetrahedron appears as 6 0-components in \( S[3] \) which are linked one another by certain succession of isomorphisms \( \sigma^* \).

2.2 Definition

We formalize this combinatorial object as follows.
For nonnegative integers $n$, the notions of $n$-shells, their isomorphisms and closedness are defined by induction on $n$ as follows.

For $n = 0$, an $n$-shell is simply a singleton set, regarded as the tree consisting solely of the root. The unique map between singleton sets are the isomorphisms. Every 0-shell is closed.

For $n = 1$, an $n$-shell has the underlying tree $T_S$ of depth $\leq 1$. Its nodes of depth 1 is regarded as 0-shells. An isomorphism of 1-shells is simply a tree iso. Every 1-shell is closed.

For $n \geq 2$, an $n$-shell consists of the following three data.

(Shell–1) Its underlying tree $T_S$ of depth $n$,

(Shell–2) For each node $x \in T_S[1]$, a closed $n-1$-shell $S^x$ with the underlying tree $T^x_S$.

Note that not only the $n-1$-components but also every $i$-component $x$ with $i < n-1$ of an $n$-shell accompany a closed $n-i$-shell $S^x$ and hence a link structure $\sigma^i$ on the word $(S^y)_{y \in S^x[2]}$. From this it follows that for each $i > 0$, there is an involution

$$\sigma_{n-i-2} : T[[i+2]] \to T[[i+2]]$$

and a partial involution

$$\sigma_{n-2} : T[[2]] \to T[[2]]$$

which commutes whenever composition is possible.

(Shell–3) A link structure $\sigma_S$ on the word $(S^x)_{x \in T_S[2]}$ of $n-2$-shells.

An isomorphism $f : S \to S'$ between $n$-shells consists of the following three data.

(Iso–1) A tree isomorphism $|f| : T_S \to T_{S'}$.

(Iso–2) A shell isomorphism $f^x : S^x \to S'^{f^x}$ for each $x \in T_S[1]$.

Note that this induces $f^x : S^x \to S'^{f^x}$ for all $x \in T_S[2]$.

(Iso–3) A link isomorphism $\kappa : \sigma_S \to \sigma_{S'}$ such that

$$|\kappa| := |f|_{T_S[2]} : |\sigma_S| = T_S[2] \to |\sigma_{S'}| = T_{S'}[2]$$

and, for each $x \in T_S[2]$,

$$\kappa_x := f^x : \sigma_S(x) = S^x \to \sigma_{S'}((f^x)(x)) = T_{S'}[2].$$

An $n$-shell $S$ is closed if the link structure $\sigma_S$ is closed, which concludes the inductive definition.

An element of $T_S[i]$ of an $n$-shell $S$ is called of dimension $n-i$ and of codimension $i$. We call an element of dimension $k$ simply as a $k$-component.

A shell is called open if it is not closed.

Two components are called linked if the one is mapped to the other by some involution $\sigma_i$. 

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2.3 Graphical representations

2.3.1 Tree representation

A complete representation up to isomorphisms of the structure of an \( n \)-shell is given by tree representation: We draw the underlying tree and the tree iso’s \( \sigma^* \) by linking \( x \) and \( \sigma^*x \) for \( x \) in the domain of \( \sigma^* \).

2.3.2 Link representations

Link representation allows us to focus our attention to crucial aspects of shells by suitably neglecting inessential details of lower dimensional components. For example, the closed 3-shell represented by the tree diagram for the tetrahedron can be represented by the middle one in Figure 3 ignoring the 0-components. If necessary we can draw the information on the 0-components as in the upper Figure 3, which for 3-shells has complete information. The lower Figure is another representation of link diagram. For dimension greater than 4, it seems necessary to have an appropriate method of concealing lower dimensional components.

Figure 4 draws an open 3-shell in two ways.

2.4 Closure of open shells

From each open \( n \)-shell \( S \) we construct a closed \( n \)-shell \( \overline{S} \), called its closure.

First define a new tree \( \overline{T} \) by adding new nodes to \( \overline{T} := T_S \) as follows. First we add a node \( c_S \) of depth 1 so that

\[
\overline{T}[1] := T[1] \bigcup \{ c_S \}.
\]

Let \( T_{ext}[2] \) denotes the external indices of the link structure \( \sigma_S \) and denote by \( T_{ext}[k] \) the set of nodes of depth \( k \) which have antecedents in \( T_{ext}[2] \). Define for \( k \geq 2 \)

\[
\overline{T}[k] := T[k] \bigcup T_{ext}[k].
\]

Note that \( t \in T_{ext}[k] \) appears twice in the right hand side, so we use \( t \) for its occurrence in the first summand and \( t^* \) for the second. We say the nodes \( c_s \) and \( t^* \)'s are new whereas the nodes \( t \)'s old.

Now we define a shell structure on \( \overline{T} \).

First we need to determine the \( n-1 \)-shell \( S^cS \). The only missing information is the closed link structure on the word of \( n-3 \)-shells indexed by the set of external nodes \( T_{ext}[3] \subseteq T[3] \). Let \( \tau \) be the graph of \( [S-d] \) with the vertex set \( T[3] \). The external nodes in \( T[3] \) are nothing but the nodes of degree one and hence from each external node there is a unique maximal chain with another external node as a terminal. Hence we obtained an involution \( \mu \) on \( T_{ext}[3] \). A chain connecting \( t \) and \( \mu t \) are accompanied by tree iso’s in the obvious way.

The Fig. 4 illustrates this construction for the open 3-shell in Fig. 4.
Figure 3: Link representations of the 3-shell of the tetrahedron. The upper one keeps all the information whereas the middle one, used frequently, neglects information on 0-components. The lower one is another link diagram which gives two-dimensional forms to 2-components.
Figure 4: Descriptions of an open 3-shell. In the left diagram, $T_w[1] = \{ 1, 2, 3 \}$ and the 2-shells $S^i$ are the 2-shell of the blow-up of a triangle. The involution $\sigma_3$ is represented by thin curves connecting the leaves and the involution $\sigma_2$ by thick curves. Note that, for example, the thick link $\sigma$ accompany an isomorphism between the 1-shell $T$ and $S$. The right diagram also describes the same pasting diagram, which seems to be more appealing intuitively.

Figure 5: The construction of a link on the external nodes of depth 3 for the open 3-shell in Fig.4. The six nodes in circles are the external ones. The first figure describes two link types on $T_w[3]$, the dotted one from $\lambda$ and the solid one from $\kappa$. The second one describes the graph $\Gamma$. The third one exhibits the path joining $x, y \in L$. The bottom one is the link structure for $S^{c,s}$.
For the root $\overline{r} := o_r$, the link $\sigma^r$ is defined by extending the involutions of $\sigma^o$, and defining for the external $t \in T[[2]]$ as the tree isomorphism $T^t \simeq T^t^r$ which maps $u$ to $u^r$ for external $u \in T^t$.

For a new node $t^r$ other than $c_S$, we define $\sigma^{t^r}$ as the composition of the tree iso’s:

\[
T^u \simeq T^v \overset{\sigma^t}{\simeq} T^v \simeq T^u^r
\]

for $u, v \in T^t[[2]]$.

It is obvious that $\overline{S}$ is a closed shell. This completes the construction of the closure $\overline{S}$ of the shell $S$.

3 Substantiation of shells

3.1 Labeling of shells

Let $\Sigma$ be a set endowed with an involution $x \leftrightarrow x^*$ called the conjugation and a conjugation invariant grading $\Sigma \rightarrow \mathbb{Z}_+ := \{ 0, 1, 2, 3, \cdots \}$. The set of labels of grade $i$ is denoted by $\Sigma_i$. We call such a set with a conjugation a labeling set. We fix a labeling set $\Sigma$ in this section.

Let $S$ be an $n$-shell. A partial map $\lambda : T_S \rightarrow \Sigma$ with the domain $|\lambda| \subseteq T_S$ is called a labeling of the shell $S$ with the label set $\Sigma$ or simply a labeling on $S$ if

- it is graded in the sense that the grade of $\lambda(x)$ is $k$ if $x$ is $k$-dimensional,
- the domain $|\lambda|$ is descendant closed in the sense that if $\lambda$ is defined on $x$ then it is defined on the descendants of $x$,
- it is compatible with the link structures, namely, two components have conjugate labels whenever they are linked.

The conjugate labeling $\lambda^*$ of a labeling $\lambda$ is defined by

\[
\lambda^*(s) := \lambda(s)^*.
\]

A pair $(S, \lambda)$ of an $n$-shell and a labeling $\lambda$ is called a partial cell. A partial cell $(S, \lambda)$ extends a partial cell $(S, \mu)$ if $|\mu| \subset |\lambda|$ and $\mu$ is the restriction of $\lambda$.

A labeling on an $n$-shell whose domain is the set of all the nodes of depth $\geq n - k$ is called a $k$-dimensional labeling on $S$ over the label set $\Sigma$, or simply a $k$-labeling on $S$ over $\Sigma$. If the domain is the whole tree $T_S$, the pair $(S, \lambda)$ is called an $n$-cell over $\Sigma$.

Note that a partial labeling $\lambda$ restricts to labeling on subsets of its domain. In particular, if a component $s$ satisfies $S^s \subseteq |\lambda|$, then $\lambda$ defines a total labeling on $S^s$. When $s$ is $i$-dimensional, the cell $(S^s, \lambda|S^s)$ is called an $i$-face of the partial cell $(S, \lambda)$. For example, a $k$-labeling $\lambda$ defines boundary $i$-cells $(S^s, \lambda|S^s)$ for $i$-components $s$ with $i \leq k$.  

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3.2 Cells and pasting diagrams

We give the following names to pairs $(S, \lambda)$ of $n$-shells $S$ and $k$-labeling $\lambda$.

| $S$      | $k = n$         | $k = n - 1$         |
|----------|-----------------|---------------------|
| closed   | $n$-cell        | $n$-frame           |
| open     | $(n - 1)$-pasting diagram |

An $n$-cell $(S, \lambda)$ defines the $n$-frame $(S, |S[1]|)$, called its boundary frame or simply the boundary. The $n$-cell $(S, \lambda)$ is said to extend the $n$-frame $(S, |S[1]|)$.

We note also that every $n$-cell $(S, \lambda)$ defines an $n$-pasting diagram $(\tilde{S}, \lambda)$ where $\tilde{S}$ is an $n + 1$-shell defined by $\tilde{S}[i] = S[i - 1]$ for $i \geq 0$, which we call the $n$-pasting diagram with the only one $n$-component $(S, \lambda)$.

3.3 Examples

The following shows examples of 1-pasting diagram, 2-frame and 2-cell

\[
\begin{array}{ccc}
\begin{array}{c}
\quad f \\
\quad c \\
\quad h \\
\end{array} & \begin{array}{c}
\quad f \\
\quad g \\
\quad m \\
\quad h \\
\end{array} & \begin{array}{c}
\quad f \\
\quad h \\
\quad n \\
\end{array}
\end{array}
\]

1-pasting diagram 2-frame 2-cell

in the usual way of describing cells of 2-categories.

The Fig. 6 describes a 1-pasting diagram $w$ using the representation method explained in §2.3.

3.4 The formal composites of pasting diagrams

An $n$-pasting diagram describes an arbitrary combinatorially possible way of composing $n$-cells. The following theorem shows that an $n$-pasting diagram uniquely defines a closed $n$-shell, called its formal composite, which will be the underlying shell of the actual composite.

**Theorem 3.1** Let $(S, \lambda)$ be an $n$-pasting diagram and $\overline{S}$ be the closure of the open $n$-shell $S$. Then the labeling $\lambda$ extends uniquely to a labeling on $\overline{S}[2] \cup S[1]$. In particular, it induces an $n - 1$ labeling $\overline{\lambda}$ on the formal composite $c(S)$ of $S$. The $n$-frame $(c(S), \overline{\lambda}|c(S))$ is called the formal composite of the pasting diagram $(S, \lambda)$ and is denoted by $c(S, \lambda)$.

**Proof.** Since every new component in $\overline{S}$ are linked to some component in $S$, there is at most one extension of the labeling. The existence of the extension is obvious for components of dimension less than $n - 3$. The compatibility of the labeling of new $n - 3$-components follows from the definition of $\overline{S}_{n-3}$.
Figure 6: A 1-pasting diagram. The symbols $f, g, h$ stand for 1-hyperoperators, $A, B, C, D, E, F$ for 0-hyperoperators with $A^* = F$, $B^* = D$, $C^* = E$, and $w, w_p, w_q, w_r, 1, 2, 3, \cdots$ for the component of the underlying shell.
An \(n + 1\)-cell \((\overline{S}, \overline{\lambda})\) is called a composer of the pasting diagram \((S, \lambda)\) if \(S \res \overline{\lambda} = \lambda\). The \(n\)-cell \((c(S), \lambda|c(S))\) is called the composite of the pasting diagram \((S, \lambda)\) by the \(n + 1\) cell \((\overline{S}, \overline{\lambda})\) or simply a composite.

The lower is the formal composite of the upper 1-pasting diagram, drawn in two ways.

4 Hypergraphs

4.1 Definition

For \(n \geq 0\) we define the notions of \(n\)-hypergraphs, the boundary operator and the coherence of labeling of \(n + 1\)-shells over \(\Sigma\) by induction on \(n\). The set of coherent \(n\)-frames of an \(n - 1\)-hypergraph \(\Sigma = \bigsqcup_{0 \leq i \leq n-1} \Sigma_i\) is denoted by \(\text{frame}_n \Sigma\). Element of \(\Sigma_i\) is called a hyperoperator of dimension \(i\) or simply an \(i\)-hyperoperator.

1. For \(n = 0\), every labeling set with \(\Sigma = \Sigma_0\) is a 0-hypergraph. Every labeling on a 1-shell is defined to be coherent.

2. A 1-hypergraph is a labeling set \(\Sigma = \Sigma_0 \bigsqcup \Sigma_1\) with the boundary operator \(\delta : \Sigma_1 \to \text{frame}_1(\Sigma_0)\) which commutes with the conjugation operators, i.e., \(\delta(c^*) = \delta(c)^*\). A labeling \(\lambda\) on a 2-shell \(S\) over \(\Sigma\) is called coherent if for every \(t \in T_S[1] \cap |\lambda|\), the 1-frame \((T_S^0, \lambda|T_S^0[1])\) coincides with the boundary 1-frame \(\delta(\lambda(t))\) of the 1-hyperoperator \(\lambda(t) \in \Sigma_1\).

3. For \(n \geq 2\), an \(n\)-hypergraph is a labeling set

\[\Sigma = \prod_{0 \leq i \leq n} \Sigma_i\]

with boundary operators \(\delta_i : \Sigma_i \to \text{frame}_i(\Sigma)\) for \(1 \leq i \leq n\) such that

\[(\Sigma \setminus \Sigma_n, \delta_1, \ldots, \delta_{n-1})\]

is an \(n - 1\)-hypergraph. A labeling \(\lambda\) on an \(n + 1\)-shell \(S\) is called coherent if for every \(i\)-component \(t\) of \(S\), the \(i\)-frame \((S^t, \lambda|S^t[1])\) coincides with the boundary of the \(i\)-hyperoperator \(\lambda(t)\).
We denote an $n$-hypergraph as
$$\big(\coprod_{0 \leq i \leq n} \Sigma_i, \delta_1, \ldots, \delta_n\big)$$
or simply as $(\Sigma, \delta: \Sigma \to \text{frame}(\Sigma))$. An $\omega$-hypergraph is a labeling set $\Sigma$ with the boundary operator $\Sigma_m \to \text{frame}_m(\Sigma)$ for all natural number $m$.

Hereafter all labelings over hypergraphs will be assumed to be coherent.

4.2 Category of hypergraphs

Let $H, H'$ be $n$-hypergraphs. A graded map $f: H \to H'$ commuting with the conjugation is called a hypergraph map if it commutes with the boundary map, in the sense that for every $i$-edge $t$ of $H$,
$$\delta(f(t)) = f_i \delta(t),$$
where $f_i: \text{frame}_i(H) \to \text{frame}_i(H')$ is defined by composing $f$ to the labeling map. We denote by $\mathcal{HG}_{\text{raph}}^n$ the category of $n$-hypergraphs and hypergraph maps.

5 The monad structure

Let $(H_0, \cdot \cdot \cdot, H_n, \delta_1, \cdot \cdot \cdot, \delta_n)$ be an $n$-hypergraph. Denote by $L_n H$ the set of $n$-pasting diagrams over $H$. Since the formal composite of a coherent pasting diagram is a coherent frame, we have
$$C: L_n H \to \text{frame}_n(H) = \text{frame}_n(L(H)),$$
where $L$ is a labeling set with $(L(H))_i = H_i$ for $i < n$ and $(L(H))_n = L_n H$, with the conjugation on $L_n H$ given by the conjugation of the labeling.

5.1 Multiplication on $L$

This induces an endofunctor $L$ of $\mathcal{HG}_{\text{raph}}^n$ defined by
$$LH = (H_0, \cdot \cdot \cdot, H_{n-1}, L_n H, \delta_1, \cdot \cdot \cdot, \delta_{n-1}, C: L_n H \to \text{frame}_n(L(H))).$$
This endofunctor has a monad structure whose multiplication
$$\mu: L LH \to LH$$
is defined as follows. Since $(L LH)_i = H_i$ for $i < n$, we define $\mu_i$ to be the identity map. So we need only to define
$$\mu_n: (L LH)_n = L_n(LH) \to L_n H.$$

Let $(S, \lambda)$ be an $n$-pasting diagram over $L H$, where $S$ is an open $n + 1$-shell. For $t \in TS[1]$, the labeling $\lambda(t) \in L_n H$ is an $n$-pasting diagram $(U_t, \lambda_t)$ over $H$ whose boundary is the $n$-frame $(S^t, \lambda[S^t[[1]])$, i.e., $S^t$ is the external components of $T_{U_t}$. 

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Define now an $n + 1$-shell $V$. Its underlying tree is defined by

$$T_V[i] = \coprod_{t \in T_S[1]} T_{U_t}[i]$$

for $i \geq 1$. For each $i \in T_{U_t}[i]$ for $i > 0$, we define $i$-shell $V^i := U_t'$. The link structure for the root of $T_V$ is defined by joining those for the roots of $T_{U_t}$'s and that for the root of $T_S$ transferred to the link structure for the words of shells indexed by the external nodes of $T_{U_t}$'s. Fig. 7 illustrates this construction.

\begin{figure}
\centering
\begin{tikzpicture}
  \node (S) at (0,0) {$S$};
  \node (T1) at (-2,-2) {$T_1$};
  \node (T2) at (0,-2) {$T_2$};
  \node (T3) at (2,-2) {$T_3$};

  \draw (S) -- (T1);
  \draw (S) -- (T2);
  \draw (S) -- (T3);

  \draw (T1) -- (T11);
  \draw (T1) -- (T12);
  \draw (T1) -- (T13);

  \draw (T2) -- (T21);
  \draw (T2) -- (T22);
  \draw (T2) -- (T23);

  \draw (T3) -- (T31);
  \draw (T3) -- (T32);
  \draw (T3) -- (T33);

\end{tikzpicture}
\caption{The lower one is the composition of the upper pasting diagram of the pasting diagram.}
\end{figure}

The unit natural transformation $\eta : H \to \mathcal{L}H$ is given by defining $\eta_n(s)$ to be the $n$-pasting diagram consisting solely of $s$.

It is straightforward to see the following.

**Theorem 5.1** The triple $(\mathcal{L}, \mu, \eta)$ is a monad on the category of hypergraphs.

### 5.2 Submonads of $(\mathcal{L}, \mu, \alpha)$

By restricting the shapes of the pasting diagrams, we obtain a few submonads of $\mathcal{L}$ which are equally important.

An $n$-pasting diagram $\varphi = (S, \lambda)$ over $\mathcal{H}$ defines a graph $\Gamma_\varphi$ with the vertex set $T_S[2]$. An element $\{i, j\}$ is its edge if and only if there is an $n$-component
$s \in T_S[1]$ whose link structure $\sigma^*$ on $T_S[2] \subseteq T_S[3]$ connects a child of $i$ with that of $j$. For example, the 3-cell of Fig. 4 is as follows:

A pasting diagram $\varphi$ is called acircuit if the graph $\Gamma_\varphi$ has no circuits. It is called connected if the graph $\Gamma_\varphi$ is connected. Denote by $\mathcal{T}\mathcal{H}$ the collection of connected acircuit pasting diagrams of $\mathcal{H}$.

Suppose now that every $n-1$-hyperoperator $x$ of an $n$-hypergraph $\mathcal{H}$ has signature $\text{sgn}(x) \in \{-1,1\}$ with $\text{sgn}(x^*) = -\text{sgn}(x)$. Let $\varphi = (S, \lambda)$ be an $n$-pasting diagram. We give the direction to the graph $\Gamma_\varphi$ by $i \rightarrow j$ if there are linked $p \in \text{Child}(i)$ and $q \in \text{Child}(j)$ with $\lambda p$ positive and $\lambda q$ negative. We say the pasting diagram $\varphi$ is acyclic if this graph has no cycles. Let $\mathcal{M}\mathcal{A}$ be the collection of connected acircuit pasting diagrams of $\mathcal{A}$.

We have

**Proposition 5.2** Both $\mathcal{T}$ and $\mathcal{M}$ are submonad of $\mathcal{L}$.

Thus we obtain two monads $(\mathcal{T}, \mu, \eta)$ and $(\mathcal{M}, \mu, \eta)$.

## 6 Strict Hypercategories

### 6.1 Definition

Let $n \geq 1$. A **strict $n$-hypercategory** is a $\mathcal{L}$-algebra $(\mathcal{H}, \alpha : \mathcal{L}\mathcal{H} \rightarrow \mathcal{H})$ where $\mathcal{H}$ is an $n$-hypergraph.

Each submonad of $\mathcal{L}$, defines a variant of strict $n$-hypercategory. We call a $\mathcal{T}$-algebra an acircuit hypercategory. Similarly we define an acyclic hypercategory when the hyperoperators have parities.

The monad multiplication $\mu$ defines the free $n$-hypercategory $\mathcal{L}\mathcal{H}$ generated by an $n$-hypergraph $\mathcal{H}$.

We first give examples of 1-hypercategories and then illustrate the usefulness of 2-hypercategories by explicitly representing the rewriting process of the Lafont’s interaction combinator as a 2-pasting diagram of a strict 2-hypergraph.

### 6.2 Examples of 1-hypercategories

#### 6.2.1 Classical simple logic

Let $\mathcal{H}_0$ be a set of propositional variables together with their negations. The conjugation is the negation. Fix a truth assignment of the variables. Write $\vdash P_1, \ldots, P_m$ when at least one of $P_1, \ldots, P_m$ is true. If we define $\mathcal{H}_1$ the set of finite sets $\{P_1, \ldots, P_m\}$ of
propositional variables with \( \vdash P_1, \cdots, P_m \), then we have a strict acircuit 1-hypercategory with the multiplication given by the following cut rule:

\[
\vdash P, P_1, \cdots, P_n \quad \vdash P^*, Q_1, \cdots, Q_m
\]
\[
\vdash P_1, \cdots, P_n, Q_1, \cdots, Q_m
\]

6.2.2 Categories as acircuit hypercategories

Let \( C \) be a category. We define simply

\[
A^* = \overline{A} \quad \overline{A} = A.
\]

An arrow \( f : A \to B \) defines an arrow \( f \) with

\[ \delta f = (A, B). \]

An acircuit pasting diagram is simply a linear diagram of composable sequence of arrows.

**Proposition 6.1** There is a bijection between categories with objects \( C_0 \) and acircuit hypercategories over \( C_0 \) \( \bigcup C_0 \) whose hyperedges have boundaries of the form \( (A, \overline{B}) \).

6.2.3 Multicategories

Let \( M \) be a multicategory in the sense of Lambek. For each object \( A \) we prepare its conjugate \( \overline{A} \) and define \( H_0 \) to be the collection of objects and their conjugates.

An arrow \( \Gamma : A_1, \cdots, A_n \to B \) corresponds to an arrow \( \varphi \Gamma \) with the boundary \( (A_1, \cdots, A_n, \overline{C}) \). Then by the associativity of the cut rule, we obtain a strict acircuit 1-hypercategory. In fact it can be seen easily that there is a bijection between multicategories with objects \( H_0 \) and acircuit hypercategories over \( (C_0 \bigcup C_0) \).

6.3 Lafont’s interaction net

Let \( H_0 \) be the singleton set \( \{ a \} \). Let \( H_1 \) consist of 1-hyperoperators \( \{ 0, \epsilon, s, +, \times, \delta \} \), whose boundaries are as follows:

\[
\begin{array}{cccccc}
0 & \epsilon & s & + & \times & \delta \\
 a & a & a & a & a & a
\end{array}
\]

We denote these as

\[
0 \quad \epsilon \quad s \quad + \quad \times \quad \delta
\]

Let \( H_2 \) be the set of 2-hyperoperators whose boundaries are described in Fig. 8. For example, the \( s+ \) 2-cell is described by tree form as:
Figure 8: The set of 2-hyperoperators

The following is an example of pasting diagram of the lafont’s 2-hypergraph.
The Figure is the pasting diagram which corresponds to the sequence of actions of interaction net which calculates $2 \times 2 = 4$.

7 Hypercategory

We sketch a formulation of weak $\omega$-hypercategory based on hypergraphs.

7.1 Definition

We assume that the underlying $\omega$-hypergraph $\mathcal{H}$ satisfy the following conditions. (i) Every hyperoperator $f$ has the parity $\sgn(f) \in \{ 1, -1 \}$ with $\sgn(f^*) = -\sgn(f)$. A partial $m + 1$-cell $(S, \lambda)$ over $\mathcal{H}$ is called pure if the labels of $m - 1$-components are of the same sign. (ii) The boundary of hyperoperators are pure.

A hypercategory is an $\omega$-hypergraph $\mathcal{H} = ((\mathcal{H}_i)_{i=0,1,...,\delta})$ with certain elements called universal are singled out and satisfy the following conditions. (H$_1$) Every pure $n$-pasting diagram $C$ has a universal composer $U$. The composite of $C$ with respect to the composer $U$ is called universally composed $C$. Similarly every pure $n + 1$-frame $F$ has a universal $n + 1$-cell $U$ with $\partial U = F$.

(H$_2$) Every universal $m$-cell $U$ has a universal cell $U^\dagger$ of the same sign with the frame conjugate to $\partial U$, called a transpose.

(H$_3$) If a pure $n$-pasting diagram is universal in the sense that the labeling of $n$-components are universal, then its composites universally composed are universal.

The condition (H$_2$) replaces the role of that involving balancedness in Baez-Dolan’s definition. In fact we can easily show the following.\footnote{This should not be confused with $U^*$ which has opposite parity and exists for all $U$. Moreover there are usually more than one transposes.}
Figure 9: The pasting diagram of the interactions calculating $2 \times 2 = 4$. 
Proposition 7.1 Let \( \mathcal{H} \) be a hypercategory and suppose an \( n \)-pasting diagram \((S, \lambda)\) has two composites \( C, C' \). (i) If \( C \) is universally composed, then there is an \( n + 1 \)-cell \( M \) whose frame has the underlying \( n + 1 \)-shell, denoted by \( F_{n+1}(C^*, C') \), whose components of dimension \( \leq n \) belongs either to \( C \) or to \( C' \) and the involution on \( n - 1 \)-components is given by the identity of the \( n \)-shell \( S \).

(ii) If \( C' \) and \( C \) are both universal, then then there is a universal \( n + 1 \)-cell \( C \) whose boundary is \( F_{n+1}(C^*, C') \).

Proof. Just take the following \( n + 1 \)-pasting diagram. Its \( n + 1 \)-components are a transpose of the universal composer of \((S, \lambda)\) composing \( C \) and the \( n + 1 \)-cell giving the composite \( C' \). The \( n - 1 \) involution the identity on \( S[[1]] \). Then its composite gives the asserted \( n + 1 \) cell.

A hypercategory \( \mathcal{H} \) is called of dimension \( n \) or simply an \( n \)-hypercategory if every cell of dimension greater than \( n \) is universal. It is called \( m \)-weak if every pure pasting diagram of dimension greater than \( m \) has a unique composite.

7.2 Hypercategories over an \( n \)-hypergraph

Baez-Dolan give a method of restricting class of \( n \)-categories by restricting the type of shells. Some of their procedure can be described by using prototype \( n \)-hypergraph.

Let \( \Sigma \) be an \( n \)-hypergraph. An \( \omega \)-hypergraph \( \mathcal{H} \) with a hypergraph map \( \varphi : \mathcal{H} \rightarrow \Sigma \) is called an \( \omega \)-hypergraph over \( \Sigma \). A hypercategory with the underlying hypergraph over \( \Sigma \) is said to be of type \( \Sigma \).

7.3 0-weak 1-hypercategory

To show some aspects of arguments in our formulation, we show that usual category is obtained from 0-weak 1-hypercategory. Let \( \Sigma_0 = \{ a, a^* \}, \Sigma_1 = \{ b, b^* \} \) and \( \delta b = (a^*, a) \). A 0-weak 1-hypercategory over \( \Sigma \) corresponds to a usual category in the following way.

First of all, pure 1-pasting diagrams are nothing but composable sequences of arrows and by the 0-weakness, they have the unique composite, called the composition which is universally composed.

Lemma 7.2 Let \( f, g : A \rightarrow B \). If there is a universal 2-cell \( u : f \rightarrow g \), then \( f = g \).

Proof. Note that \( u \) is a universal composer of \( f \) regarded as a 1-pasting diagram. Since \( u \) is universal, there is another universal \( \overset{1}{u} : g \rightarrow f \) which together gives a universal composite \( f \rightarrow f \), which is another universal composer of the \( f \) regarded as a 1-pasting diagram and hence must coincide with \( u \). In particular \( f = g \).

Proposition 7.3 The composition is associative.
Proof. For simplicity, let us prove $f \circ (g \circ h) = (f \circ g) \circ h$. The pasting diagram gives a universal 2-cell $f \circ (g \circ h) \to (f \circ g) \circ h$, whence the associativity follows from the above lemma.

Each object $A \in \mathcal{H}_0$, regarded as a 0-pasting diagram has universal composer which we call quasi-identities temporarily.

**Proposition 7.4** (i) Let $u : A \to A$ be a quasi-identity. If $f : A \to B$, then $f \circ u = f$. Similarly if $g : B \to A$, then $u \circ g = g$. (ii) For each $A$, there is only one quasi-identity.

**Proof.** Let $0_u$ be the universal composer of the 1-pasting diagram $u$. Then the 2-pasting diagram gives a composite $f \circ u \to f$ universally composed, whence by the lemma $f \circ u = f$. If $u, v$ are quasi-identities, then $u = u \circ v = v$.

Similarly multicategory can be identified with a class of 0-weak 1-hypercategories.

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