Metric Spaces with Linear Extensions Preserving Lipschitz Condition

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Abstract
We study a new bi-Lipschitz invariant λ(M) of a metric space M; its finiteness means that Lipschitz functions on an arbitrary subset of M can be linearly extended to functions on M whose Lipschitz constants are expanded by a factor controlled by λ(M). We prove that λ(M) is finite for several important classes of metric spaces. These include metric trees of arbitrary cardinality, groups of polynomial growth, Gromov-hyperbolic groups, certain classes of Riemannian manifolds of bounded geometry and the finite direct sums of arbitrary combinations of these objects. On the other hand we construct an example of a two-dimensional Riemannian manifold M of bounded geometry for which λ(M) = ∞.

Contents
1 Introduction 2
2 Formulation of the Main Results 6
3 Examples 13
4 Open Problems 17

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1 Introduction

A. The concept of a metric space is arguably one of the oldest and important in mathematics, whereas analysis on metric spaces has been developed only within last few decades. The important part of this development is devoted to the selection and study of classes of metric spaces ”supporting” certain basic analytic facts and theories known for $\mathbb{R}^n$. Some results and problems appearing in this area are presented, in particular, in the surveys [CW], [Gr1] Appendix B (by Semmes) and [HK]. The main topic of our paper also belongs to this direction of research and is devoted to study of classes of metric spaces possessing the following property.

Definition 1.1 A metric space $(M,d)$ has the Lipschitz condition preserving linear extension property (abbreviated $\mathcal{LE}$), if for each of its subspaces $S$ there is a linear continuous extension operator acting from $\text{Lip}(S)$ into $\text{Lip}(M)$.

Here $\text{Lip}(S)$ is the space of real-valued functions on $S$ equipped with the seminorm

$$|f|_{\text{Lip}(S)} := \sup_{m' \neq m''} \frac{|f(m') - f(m'')|}{d(m', m'')} ;$$

(1.1)

hence the linear operator $E : \text{Lip}(S) \to \text{Lip}(M)$ of this definition meets the following conditions.

(a) The restriction of $Ef$ to $S$ satisfies

$$Ef|_S = f , \quad f \in \text{Lip}(S) .$$

(1.2)

(b) The norm of $E$ given by

$$||E|| := \sup \{ |Ef|_{\text{Lip}(M)} : |f|_{\text{Lip}(S)} \leq 1 \}$$

(1.3)

is finite.
In the sequel the linear space of these operators will be denoted by \( \text{Ext}(S, M) \), that is to say,

\[
\text{Ext}(S, M) := \{ E \in \mathcal{L}(\text{Lip}(S), \text{Lip}(M)) : E \text{ satisfies (a) and (b)} \} .
\]  

The adjective “linear” in Definition 1.1 drastically changes the situation compared to that for nonlinear extensions of Lipschitz functions. In fact, McShane [Mc] gave two simple nonlinear formulas for the extension of functions \( f \in \text{Lip}(S) \) preserving their Lipschitz constants. At the same year Kirszbraun [Ki] proved existence of a Lipschitz constant preserving (nonlinear) extension for maps between two Euclidean spaces; then Valentine [V] remarked that this result remains true for the case of general Hilbert spaces. Another generalization of Kirszbraun’s theorem was due to Lang and Schroeder [LSch] who proved such a result for Lipschitz maps between metric path spaces with upper and lower curvature bounds. Most of these results either fail to be true or are unknown for the linear extension case (even for scalar functions). For instance, Theorem 2.16 below presents an example of a Riemannian two-dimensional manifold \( \Sigma \) of bounded geometry and a (metric) subspace \( S \) such that \( \text{Ext}(S, \Sigma) = \emptyset \). (In the forthcoming paper [BB] we prove a similar result for infinite-dimensional Hilbert spaces.)

In the present paper we will study the following quantitative characteristic of spaces with \( \mathcal{L}\mathcal{E} \).

**Definition 1.2** Given a metric space \((M, d)\) one defines its Lipschitz condition preserving linear extension constant \( \lambda(M) \) by

\[
\lambda(M) := \sup_{S \subset M} \inf \{ ||E|| : E \in \text{Ext}(S, M) \} .
\]  

It is proved for a wide class of metric spaces, see Theorem 2.6 below, that finiteness of (1.5) is equivalent to the \( \mathcal{L}\mathcal{E} \) of \( M \). In particular, \( \mathbb{R}^n \) equipped with an arbitrary norm and the hyperbolic space \( \mathbb{H}^n \) with the inner path (geodesic) metric belong to this class. In the former case, the following estimate can be derived from the classical Whitney extension theorem [W1]

\[
\lambda(l^n_1) \leq c^n
\]  

where \( c \) is some absolute (numerical) constant. The proof of (1.6) is based on the Whitney covering lemma that is not true for a relatively simple metric spaces, e.g., for \( \mathbb{H}^n \). Using a new approach based on a quasi-isometric representation of \( \mathbb{H}^n \) as a space of balls in \( \mathbb{R}^{n-1} \) with a corresponding path metric, it was proved in [BSh2, Proposition 5.33] that

\[
\lambda(\mathbb{H}^n) \leq c^n
\]  

with an absolute constant \( c \).

In the present paper, we essentially enlarge the class of metric spaces with finite \( \lambda(M) \) and, in some cases, give even relatively sharp estimates of this constant. For example, we show that for some \( 0 < c < 1 \)

\[
c\sqrt{n} \leq \lambda(\mathbb{Z}^n) \leq 24n ,
\]
for $\mathbb{Z}^n$ regarded as an abelian group with the word metric, and that the same upper estimate holds for an arbitrary Carnot group of homogeneous dimension $n$.

One of the main tools of our approach is the finiteness property of the characteristic (1.5), see Theorem 2.1 and Corollaries 2.2 and 2.3 below, asserting, in particular, that

$$\lambda(M) = \sup_S \lambda(S)$$

where $S$ runs through all finite point subspaces of $M$ (with the induced metric).

As the first consequence of this fact we prove, see Theorem 2.4 below, that for the direct sum of arbitrary nontrivial metric trees $T_i$ with path metrics $d_{T_i}$, $1 \leq i \leq n$,

$$C_1 \sqrt{n} \leq \lambda(\oplus_{i=1}^n T_i) \leq C_2 n$$

with absolute constants $C_i > 0$. This implies a similar estimate for the Cayley graph of the direct product of free groups of arbitrary cardinality.

The next important result, Theorem 2.9, gives sufficient conditions for finiteness of $\lambda(M)$ in the case of locally doubling metric spaces $M$ (see corresponding definitions of this and other notions used here in the next section). In particular, this implies the corresponding results for a metric space of bounded geometry and for one framed by a group of its isometries acting freely, properly and cocompactly, see Corollaries 2.11 and 2.15.

The problem of nontrivial lower estimates of $\lambda(M)$ is unsolved even for the “relatively simple” case of the $n$-dimensional Euclidean space. However, we do prove such a result for the space $l_p^n$ with $p \neq 2$ based on a sharp in order estimate of $\lambda_{\text{conv}}(l_p^n)$, see Theorem 2.17 below. Here $\lambda_{\text{conv}}$ is defined for metric spaces $M$ with convex structure, e.g., for normed spaces, by the same formula (1.5) but with $S$ running through convex subsets of $M$.

Finally, Theorem 2.21 gives sufficient conditions for finiteness of $\lambda(M)$ for a wide class of metric spaces that includes, in particular, fractals, Carnot groups, groups of polynomial growth, Gromov-hyperbolic groups and certain Riemannian manifolds with curvature bounds, see section 3 below.

B. The linear extension problem for spaces of continuous functions was first studied by Borsuk [Bor] in 1933. Important results in this area were obtained by Kakutani, Dugundji, Lindenstrauss, Pelczynski and many other mathematicians, see [P] and [BL, Chapters 2 and 3] and references therein. For the case of uniformly continuous functions a negative result was proved by Pelczynski [P, Remarks to §2]. His argument, going back to the paper [L] by Lindenstrauss, can be modified (cf. the proof of Proposition 9.1 below) to establish that there is no linear bounded extension operator from $\text{Lip}(Y)$ into $\text{Lip}(X)$, if $Y$ is a reflexive subspace of a Banach space $X$. It was noted in the cited book [P] that “... our knowledge of existence of linear extension operators for uniformly continuous or Lipschitz functions is rather unsatisfactory”. Not much research has, however, been done in this area. Several important linear extension results were however proved (but linearity there was not

\[ \text{In the forthcoming paper [BB] we show that in this case } \lambda(M) \geq c \sqrt[n]{n} \text{ for a numerical constant } c > 0. \]
formulated explicitly) in another area of research on Lipschitz extensions. The main point there is to estimate (nonlinear) Lipschitz extension constants for mappings from finite metric spaces into Banach spaces, see [MP], [JL], [JLS]. In particular, in the paper [JLS] by Johnson, Lindenstrauss and Schechtman the proposed extension operator is linear and therefore their results give for an $n$-dimensional Banach space $B$ the estimate $\lambda(B) \leq Cn$, and for a finite metric space $M$ the estimate $\lambda(M) \leq C \log(\text{card}M)$, where $C$ is an absolute constant. Another important result was proved by Matoušek [Ma]; for scalar valued functions this gives an estimate of $\lambda(T)$ for an arbitrary metric tree $T$ by some universal constant.

For differentiable functions on $\mathbb{R}^n$ a method of linear extension was discovered by Whitney [W1] in 1934. It has been then used in variety of problems of Analysis. To discuss the few results in this field we recall that $C^k_b(\mathbb{R}^n)$ and $C^k_u(\mathbb{R}^n)$ are spaces of $k$-times continuously differentiable functions on $\mathbb{R}^n$ whose higher derivatives are, respectively, bounded or uniformly continuous. We also introduce the space $C^{k,\omega}(\mathbb{R}^n) \subset C^k(\mathbb{R}^n)$ defined by the seminorm

$$|f|_{C^{k,\omega}} := \max_{|\alpha|=k} \sup_{x,y \in \mathbb{R}^n} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{\omega(|x-y|)}.$$  

Here $\omega: \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing, equal to 0 at 0 and concave; we will write $C^{k,s}(\mathbb{R}^n)$ for $\omega(t) := t^s$, $0 < s \leq 1$.

Finally, $\Lambda^\omega(\mathbb{R}^n)$ stands for the Zygmund space defined by the seminorm

$$|f|_{\Lambda^\omega} := \sup_{x\neq y} \frac{|f(x) - 2f\left(\frac{x+y}{2}\right) + f(y)|}{\omega(|x-y|)};$$

(1.9)

here $\omega: \mathbb{R}_+ \to \mathbb{R}_+$ is as in (1.8), but we assume now that $\omega(\sqrt{t})$ is concave.

Let now $S \subset \mathbb{R}^n$ be an arbitrary closed subset and $X$ be one of the above introduced function spaces. Then $X|_S$ denotes the linear space of traces of functions from $X$ to $S$ endowed with the seminorm

$$|f|_X^S := \inf\{|g|_X : g|_S = f\}.$$  

(1.10)

Then the linear extension problem can be formulated as follows.

Does there exist a linear continuous extension operator from $X|_S$ into $X$?

One can also consider the restricted linear extension problem with $S$ belonging to a fixed class of closed subspaces of $\mathbb{R}^n$.

Whitney’s paper [W2] is devoted to a criterium for a function $f \in C(S)$ with $S \subset \mathbb{R}$ to belong to the trace space $C^k_b(\mathbb{R})|_S$ and gives, in fact, a positive solution to the linear extension problem for $C^k_b(\mathbb{R})$. It was noted in [BSh2] that Whitney’s method gives the same result for the spaces $C^{k,\omega}(\mathbb{R})$ and $C^k_u(\mathbb{R})$.

The situation for the multidimensional case is much more complicated. The restricted problem, for the class of compact subsets of $\mathbb{R}^n$ was solved positively by G. Glaeser [Gl] for the space $C^1_b(\mathbb{R}^n)$ using a special construction of the geometry of subsets in $\mathbb{R}^n$. However, for the space $C^1_b(\mathbb{R}^n)$, $n \geq 2$, the linear extension problem fails to be true, see [BSh2, Theorem 2.5]. In [BSh2] (see also [BSh1]) the linear extension problem was solved positively for the spaces $C^{1,\omega}(\mathbb{R}^n)$ and $\Lambda^\omega(\mathbb{R}^n)$. A recent
breakthrough due to Ch. Fefferman [F1] in the problem of a constructive characterization of the trace space $C^{k,1}(\mathbb{R}^n)|_S$, allowed him to solve the linear extension problem for the space $C^{k,\omega}(\mathbb{R}^n)$, see [F2], [F3] and [F4].

C. The paper is organized as follows.

Section 2 introduces basic classes of metric spaces involved in our considerations and formulates the main results, Theorems 2.1, 2.4, 2.6, 2.16, 2.17 and their corollaries.

The next section presents some important examples of metric spaces possessing $\mathcal{LE}$, while in section 4 we discuss several open problems.

All the remaining sections are devoted to proofs of the aforementioned main theorems and corollaries.

Finally, the Appendix presents an alternative proof of a Kantorovich-Rubinshtein duality type theorem used in the proof of Theorem 2.1.

Notations. Throughout the paper we often suppress the symbol $d$ in the notation $(M,d)$ and simply refer to $M$ as a metric space. The same simplification will be used for all notation related to $M$, e.g., we will write $\text{Lip}(M)$, $\lambda(M)$, see (1.1) and (1.5), and use similar notations $\text{Lip}(S)$ and $\lambda(S)$ for $S \subset M$ regarded as a metric subspace of $M$ (with the induced metric). Points of $M$ are denoted by $m,m',m''$ etc, and $B_r(m)$ stands for the open ball of $M$ centered at $m$ and of radius $r$. We will write $M \in \mathcal{LE}$ for $M$ satisfying Definition 1.1. Recall also that $\text{Ext}(S,M)$ has already introduced by (1.4). Finally, we define the Lipschitz constant of a map $\phi : (M,d) \to (M_1,d_1)$ by

$$|\phi|_{\text{Lip}(M,M_1)} := \sup_{m' \neq m''} \frac{d_1(\phi(m'),\phi(m''))}{d(m',m'')}.$$  (1.11)

Let us recall that $\phi$ is a quasi-isometry (or bi-Lipschitz equivalence), if $\phi$ is a bijection and Lipschitz constants for $\phi$ and $\phi^{-1}$ are finite. If, in addition,

$$\max\{|\phi|_{\text{Lip}(M,M_1)} , |\phi^{-1}|_{\text{Lip}(M_1,M)}\} \leq C$$

for some $C > 0$, then $\phi$ is called a $C$-isometry (and isometry, if $C = 1$).

2 Formulation of the Main Results

To formulate our first result we require the notion of a dilation. This is a quasi-isometrty $\delta$ of $M$ such that the operator $\Delta : \text{Lip}(M) \to \text{Lip}(M)$ given by

$$(\Delta f)(m) := f(\delta(m)) , \quad m \in M ,$$

satisfies

$$||\Delta|| \cdot ||\Delta^{-1}|| = 1 .$$  (2.1)

Theorem 2.1 Assume that $S$ is a subspace of $M$ such that for some dilation $\delta : M \to M$ we have
(a) \( S \subset \delta(S) \);
(b) \( \cup_{j=0}^\infty \delta^j(S) \) is dense in \( M \).

Then

\[ \lambda(M) = \sup_{F \subset S} \lambda(F) \]

where \( F \) runs through all finite point subspaces of \( S \).

Choosing \( S = M \) and \( \delta \) equal to the identity map we get from this

**Corollary 2.2**

\[ \lambda(M) = \sup_{F} \lambda(F) \] \quad (2.2)

where \( F \) runs through all finite point subspaces of \( M \).

Together with Theorem 2.1 this immediately implies

**Corollary 2.3** Let \( S \subset M \) satisfy the assumptions of Theorem 2.1. Then

\[ \lambda(M) = \lambda(S) \, . \]

The results presented above will be used in almost all subsequent proofs. As the first application we give a rather sharp estimate of \( \lambda(M) \) for \( M \) being the direct sum of metric trees. To formulate the result let us recall the corresponding notions.

A tree \( T \) is a connected graph with no cycles, see, e.g. [R, Ch.9] for more details. We turn \( T \) into a path metric space by identifying each edge \( e \) with a bounded interval of \( \mathbb{R} \) of length \( l(e) \) and then determining the distance between two points of the 1-dimensional CW-complex formed by these edges to be the infimum of the lengths of the paths joining them. Since every two vertices of a tree can be joined by a unique path, the metric space \((T, d_T)\) obtained in this way is, in fact, a geodesic space, see e.g. [BH, pp. 8-9].

Let now \( (M_i, d_i), 1 \leq i \leq n \), be metric spaces. Their direct \( p \)-sum \( \oplus_p \{(M_i, d_i)\}_{1 \leq i \leq n} \) is a metric space with the underlying set \( \prod_{i=1}^n M_i \) and a metric \( d \) given by

\[ d(m, m') := \left( \sum_{i=1}^n d_i(m_i, m'_i)^p \right)^{1/p} ; \]

here \( m = (m_1, \ldots, m_n), m' = (m'_1, \ldots, m'_n) \).

**Theorem 2.4** Let \( T_i \) be a nontrivial metric tree, \( 1 \leq i \leq n \). Then for \( p = 1, \infty \)

\[ c_0 \sqrt{n} \leq \lambda(\oplus_p \{T_i\}_{1 \leq i \leq n}) \leq cn \]

where \( c_0, c \) are absolute constants.
The basic fact of independent interest used along with Corollary 2.2 in the proof of this theorem asserts that every infinite metric tree with uniformly bounded vertex degrees admits a quasi-isometric embedding into the hyperbolic plane with distortion\(^2\) bounded by a numerical constant. It seems to be strange to use here the hyperbolic plane instead of a Euclidean space of some dimension. Strikingly, by a result of Bourgain [Bou] this cannot be done even if we use an infinite dimensional Hilbert space.

For \(n = 1\) the above result was proved by Matoušek [Ma] by another method. It is worth noting that an important class of spaces, Gromov-hyperbolic spaces of bounded geometry, have metric structure close to that of metric trees. This implies the corresponding Lipschitz extension result for spaces of this class, see Corollary 2.13 below.

Our next result relates the \(LE\) of \(M\) to the finiteness of \(\lambda(M)\). For its formulation we introduce the following two classes of metric spaces.

**Definition 2.5**  
(a) A metric space \(M\) is said to be proper (or boundedly compact), if every closed ball in \(M\) is compact.  
(b) A metric space \(M\) has the weak transition property (WTP), if for some \(C \geq 1\) and every finite set \(F\) and open ball \(B\) in \(M\) there is a \(C\)-isometry \(\sigma : M \to M\) such that

\[ B \cap \sigma(F) = \emptyset. \]

**Theorem 2.6** Assume that \(M\) is either proper or has the WTP. Then the \(LE\) of \(M\) is equivalent to the finiteness of \(\lambda(M)\).

We now shall discuss some general conditions under which a metric space possesses the required extension property. For this purpose we use a modification of the well-known doubling condition.

**Definition 2.7** A metric space \(M\) is locally doubling, if for some \(R > 0\) and integer \(N\) each ball of radius \(r < R\) in \(M\) can be covered by at most \(N\) balls of radius \(r/2\).

The class of such spaces will be denoted by \(\mathcal{D}(R, N)\). The class of doubling metric spaces is then \(\bigcup_N (\bigcap_{R > 0} \mathcal{D}(R, N))\). We will write \(M \in \mathcal{D}(N)\), if the assumption of Definition 2.7 holds for all \(r < \infty\).

The second notion that will be used is introduced by

**Definition 2.8** A set \(\Gamma \subset M\) is said to be an \(R\)-lattice, if the family of open balls \(\{B_{R/2}(\gamma) : \gamma \in \Gamma\}\) forms a cover of \(M\), while the balls \(B_{cR}(\gamma), \gamma \in \Gamma\), are pairwise disjoint for some \(c = c_\Gamma \in (0, 1/4]\).

The existence of \(R\)-lattices follows easily from Zorn’s lemma.

\(^2\)recall that distortion of a quasi-isometry \(\phi : M_1 \to M_2\) of metric spaces is defined by 

\[ |\phi|_{\text{Lip}(M_1, M_2)} \cdot |\phi^{-1}|_{\text{Lip}(M_2, M_1)}.\]
**Theorem 2.9** Assume that a metric space $M \in \mathcal{D}(R,N)$ and $\Gamma \subset M$ is an R-lattice. Assume also that the constants $\lambda(\Gamma)$ and

$$\lambda_R := \sup\{\lambda(B_R(m)) : m \in M\}$$  \hfill (2.4)

are finite.

Then $\lambda(M)$ is bounded by a constant depending only on $\lambda(\Gamma)$, $\lambda_R$, $c_\Gamma$, $R$ and $N$.

In order to formulate a corollary of this result we introduce a subclass of the class $\cup_{N,R} \mathcal{D}(R,N)$ consisting of metric spaces of bounded geometry, cf. the corresponding definition in [CG] for the case of Riemannian manifolds.

**Definition 2.10** A metric space $M$ is of bounded geometry with parameters $n \in \mathbb{N}$, $R, C > 0$ (written $M \in \mathcal{G}_n(R,C)$), if each open ball of radius $R$ in $M$ is $C$-isometric to a subset of $\mathbb{R}^n$.

Let us note that if $B_R(m)$ is $C$-isometric to a subset $S$ of $\mathbb{R}^n$, then

$$C^{-2} \cdot \lambda(S) \leq \lambda(B_R(m)) \leq C^2 \cdot \lambda(S),$$

and by the classical Whitney extension theorem, see, e.g., [St, Ch.6], $\lambda(S) \leq \lambda(\mathbb{R}^n) < \infty$. So the previous theorem leads to

**Corollary 2.11** Let $M \in \mathcal{G}_n(R,C)$. Then $\lambda(M)$ is finite if and only if for some $R$-lattice $\Gamma$ we have

$$\lambda(\Gamma) < \infty.$$

To formulate the second corollary we recall the definition of Gromov hyperbolicity [Gr3]. We choose a definition attributed (by Gromov) to Rips (see equivalent formulations in [BH] Chapter 3).

Let $(M,d)$ be a geodesic metric space; this means that every two points $m,n \in M$ can be joined by a geodesic segment, the image of a map $\gamma : [0,a] \to M$ such that $\gamma(0) = m$, $\gamma(a) = n$ and $d(\gamma(t),\gamma(s)) = |t-s|$ for all $t, s$ in $[0,a]$.

**Definition 2.12** A geodesic metric space $M$ is said to be $\delta$-hyperbolic, if every geodesic triangle in $M$ is $\delta$-slim, meaning that each of its sides is contained in the $\delta$-neighbourhood of the union of the remaining sides.

**Corollary 2.13** Let $M$ be the finite direct $p$-sum of hyperbolic metric spaces of bounded geometry, $1 \leq p \leq \infty$. Then $\lambda(M)$ is finite.

The next consequence concerns a path metric space\(^3\) with a group action. For its formulation we need

**Definition 2.14** (see, e.g., [BH, p.131]). A subgroup $G$ of the group of isometries of a metric space $M$ acts properly, freely and cocompactly on $M$, if

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\(^3\)i.e., the distance between every pair of points equals the infimum of the lengths of curves joining the points.
(a) for every compact set \( K \subset M \) the set \( \{ g \in G : g(K) \cap K \neq \emptyset \} \) is finite;
(b) for every point \( m \in M \) the identity \( g(m) = m \) implies that \( g = 1 \);
(c) there is a compact set \( K_0 \subset M \) such that
\[
M = G(K_0) .
\] (2.5)

By the Švarc-Milnor lemma, see, e.g., [BH, p.140], the group \( G \) of this definition is finitely generated whenever \( M \) is a path space. If \( A \) is a (finite) generating set for \( G \), then \( d_A \) stands for the word metric on \( G \) determined by \( A \), see, e.g., [Gr1,p.89]. Replacing \( A \) by another (finite) generating set one obtains a corresponding word metric bi-Lipschitz equivalent to \( d_A \). Therefore the \( \mathcal{LE} \) of \( G \) regarded as a metric space in this way does not depend on the choice of \( A \).

**Corollary 2.15** Let \( M \) be a path space framed by a group \( G \) acting on \( M \) by isometries. Assume that

(a) \( M \) is a metric space of bounded geometry;
(b) \( G \) acts on \( M \) properly, freely and cocompactly.

Then \( \lambda(M) \) is finite if and only if \( \lambda(G, d_A) \) is.

In view of Corollary 2.11 it would be natural to conjecture that \( \lambda(M) \) is finite for every \( M \) of bounded geometry. The following counterexample disproves this assertion.

**Theorem 2.16** There exists a connected two-dimensional metric space \( M_0 \) of bounded geometry such that
\[
\text{Ext}(S, M_0) = \emptyset
\]
for some subset \( S \subset M_0 \).

The basic step used in our construction of \( M_0 \) is the following result of independent interest. For its formulation we introduce the functional
\[
\lambda_{\text{conv}}(M) := \sup_C \lambda(C, M)
\]
where \( C \) runs through all convex subsets of a normed linear space \( M \). Here we set
\[
\lambda(S, M) := \inf \{ ||E|| : E \in \text{Ext}(S, M) \} .
\]

**Theorem 2.17** There exists an absolute constant \( c_0 > 0 \) such that for all \( n \) and \( 1 \leq p \leq \infty \)
\[
c_0 \leq n^{-\frac{1}{p} - \frac{1}{2}} \cdot \lambda_{\text{conv}}(p^n) \leq 1 .
\]

Let us now return to metric spaces of bounded geometry. Corollary 2.11 tells us that the existence of the desired extension property is reduced to that for lattices. The example of Theorem 2.16 makes the following conjecture to be rather plausible.
**Conjecture 2.18** A lattice $\Gamma \subset M$ has the LE, if it is uniform.

The latter means that for some increasing function $\phi_\Gamma : \mathbb{R}_+ \to \mathbb{R}_+$ and constant $0 < c \leq 1$ the number of points of $\Gamma \cap B_R(m)$ for every $R > 0$ and $m \in \Gamma$ satisfies

$$c\phi_\Gamma(R) \leq |\Gamma \cap B_R(m)| \leq \phi_\Gamma(R). \quad (2.6)$$

We confirm this conjecture for lattices of polynomial growth, i.e., for $\phi_\Gamma(R) = aR^n$ for some $a, n \geq 0$ and for some other lattices including even those of exponential growth. These will follow from an extension result presented below.

In its introduction we use the notion of a measure doubling at a point $m$. This is a nonnegative Borel measure $\mu$ on $M$ such that every open ball centered at $m$ is of finite strictly positive $\mu$-measure and the doubling constant

$$D_m(\mu) := \sup_{R > 0} \frac{\mu(B_{2R}(m))}{\mu(B_R(m))}$$

is finite. If, in addition,

$$D(\mu) := \sup_{m \in M} D_m(\mu) < \infty,$$

the $\mu$ is said to be a doubling measure.

A metric space endowed with a doubling measure is said to be of homogeneous type [CW].

Our basic class of metric spaces is presented by

**Definition 2.19** A metric space $(M, d)$ is said to be of pointwise homogeneous type if there is a fixed family $\{\mu_m\}_{m \in M}$ of Borel measures on $M$ satisfying the following properties.

(i) Uniform doubling condition:

$\mu_m$ is doubling at $m$ and

$$D := \sup_{m \in M} D_m(\mu_m) < \infty. \quad (2.7)$$

(ii) Consistency with the metric:

For some constant $C > 0$ and all $m_1, m_2 \in M$ and $R > 0$

$$|\mu_{m_1} - \mu_{m_2}|(B_R(m)) \leq \frac{C\mu_m(B_R(m))}{R}d(m_1, m_2) \quad (2.8)$$

where $m = m_1$ or $m_2$.

**Remark 2.20** The conditions (2.7), (2.8) hold trivially for $M$ equipped with a doubling measure $\mu$ (i.e., in this case $\mu_m = \mu$ for all $m$). So metric spaces of homogeneous type belong to the class introduced by this definition.

**Theorem 2.21** If $M$ is of pointwise homogeneous type with the optimal constants $C$ and $D$, then the following inequality

$$\lambda(M) \leq k_0(C + 1)(\log_2 D + 1) \quad (2.9)$$

holds with some numerical constant $k_0$. 

11
In particular, for a metric space $M$ of homogeneous type (when $C = 0$) $D$ is greater than, say, $\sqrt{2}$, see, e.g., [CW], and (2.9) gives the inequality

$$\lambda(M) \leq 2k_0 \log_2 D.$$ 

**Remark 2.22** The last inequality can be easily derived from Theorem 1.4 of the paper [LN] by Lee and Naor on “absolute” Lipschitz extendability of doubling metric spaces.\(^4\) Their proof is based on a probabilistic argument. Using a modification of the proof of Theorem 2.21 one can give a constructive proof of the Lee-Naor result.

We formulate several consequences of Theorem 2.21.

**Definition 2.23** A metric space $(M, d)$ is said to be of pointwise $(a, n)$-homogeneous type, $n \geq 0$, $a \geq 1$, with respect to a family of Borel measures $\{\mu_m\}_{m \in M}$ on $M$, if it satisfies condition (2.8) and the condition

$$\frac{\mu_m(B_{lR}(m))}{\mu_m(B_R(m))} \leq al^n$$

for arbitrary $l \geq 1$, $m \in M$ and $R > 0$.

**Corollary 2.24** If $M$ is of pointwise $(a, n)$-homogeneous type, then

$$\lambda(M) \leq K_0 (C + 1)a^2(n + 1)$$

where $K_0$ is a numerical constant ($< 225$) and $C$ is the constant in (2.8).

Let us note that for a metric space of homogeneous type condition (2.8) trivially holds, and we can take $C = 0$ in (2.11).

We now single out a special case of the above result with a better estimate of $\lambda(M)$. Specifically, suppose now that for all balls in $M$

$$\mu_m(B_R(m)) = \gamma R^n, \quad \gamma, n > 0.$$  \hspace{1cm} (2.12)

Under this assumption the following holds.

**Corollary 2.25**

$$\lambda(M) \leq 24(n + C).$$

Finally, we establish the finiteness of $\lambda(M)$ for a metric space being the direct sum of spaces of pointwise homogeneous type. Unlike the situation for spaces of homogeneous type this case requires an additional restriction on the related families of Borel measures introduced as follows.

**Definition 2.26** A family of measures $\{\mu_m\}$ on a metric space $M$ is said to be $K$-uniform ($K \geq 1$), if for all $m_1, m_2$ and $R > 0$

$$\mu_{m_1}(B_R(m_1)) \leq K\mu_{m_2}(B_R(m_2))$$

---

\(^4\)A metric space is doubling, if for every $R > 0$ there is an integer $N > 1$ such that any closed ball of radius $2R$ can be covered by $N$ balls of radius $R$. 

12
Let $M_i$ be of pointwise homogeneous type with respect to a $K_i$-uniform family of Borel measures $\{\mu^i_m\}_{m \in M_i}$ satisfying conditions of Definition 2.19 with the optimal constants $D_i, C_i, 1 \leq i \leq N$. Then the following inequality

$$\lambda(\oplus_p \{(M_i, d_i)\}_{1 \leq i \leq N}) \leq c_0(\bar{C}_p + 1)(\log_2 D + 1)$$

(2.13)

is true with

$$D := \prod_{i=1}^N D_i, \quad \bar{C}_p := \left( \sum_{i=1}^N C_i^q \right)^{1/q} \prod_{i=1}^N K_i.$$

Here $c_0$ is a numerical constant and $q$ is the exponent conjugate to $p$, i.e., $\frac{1}{p} + \frac{1}{q} = 1$.

If, in particular, (2.12) holds for $M_i$ with $n = n_i$ and $\gamma = \gamma_i, 1 \leq i \leq N$, then

$$\lambda(\oplus_{\infty} \{(M_i, d_i)\}_{1 \leq i \leq N}) \leq 24 \sum_{i=1}^N (n_i + C_i).$$

(2.14)

The extension results of this section are true for Banach-valued Lipschitz functions, if the Banach space is complemented in its second dual space (e.g., dual Banach spaces possess this property [Di]). This can be derived straightforwardly from the scalar results. However, the Banach-valued version of Theorem 2.21 is true without any restriction. It can be established by an appropriate modification of the proof presented here. This and other results in that direction will be presented in a forthcoming paper.

3 Examples

3.1. Groups with a metric space structure.

A. Carnot Groups (see [FS] and [He] for basic facts).

A Carnot group (also known as a homogeneous group) is a connected and simply connected real Lie group $G$ whose Lie algebra $g$ admits a stratification

$$g = \bigoplus_{i=1}^m V_i \quad \text{with} \quad [V_i, V_i] = V_{i+1};$$

(3.1)

here $V_{m+1} = \{0\}$ and $V_m \neq \{0\}$.

Being nilpotent, $G$ is diffeomorphic to $\mathbb{R}^n$ with $n := \dim G$. Together with the topological dimension $n$ an important role is played by the homogeneous dimension of $G$ given by

$$\dim_h G := \sum_{j=1}^m j \dim V_j.$$

(3.2)

The group $G$ can be equipped with a left-invariant (Carnot-Carathéodory) metric $d$ for which the ball $B_r(x) := \{y \in G : d(x, y) < r\}$ satisfies

$$|B_r(x)| = r^Q, \quad x \in G, \ r > 0.$$
Here $|\cdot|$ is the (normed) left-invariant Haar measure on $G$ and $Q := \dim_h G$. Therefore Corollary 2.25 immediately implies that

$$\lambda(G,d) \leq 24 \dim_h G .$$

(3.3)

The simplest example of a Carnot group is $\mathbb{R}^n$. In this case, $\dim_h G = \dim G = n$, and an arbitrary Banach norm on $\mathbb{R}^n$ defines a Carnot-Caratheodory metric. This gives the aforementioned extension result of [JLS] with a better constant. In particular, (3.3) and Theorem 2.17 yield

$$c_0 n^{\frac{1}{p} - \frac{1}{2}} \leq \lambda(l_p^n) \leq 24 n$$

(3.4)

with $c_0 > 0$ independent of $n$ and $p$.

Another interesting example of a Carnot group is the Heisenberg group $H_n(\mathbb{R})$ that, as a set, is equal to $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$. The group operation is defined by

$$(x, y, t) \cdot (x', y', t') = (x + x', y + y', t + t' + <x', y>)$$

(3.5)

where the first two coordinates are vectors in $\mathbb{R}^n$ and $<\cdot, \cdot>$ is the standard scalar product. The topological dimension of $H_n(\mathbb{R})$ is clearly $2n + 1$ while its homogeneous dimension equals $2n + 2$. Finally, a Carnot-Caratheodory metric $d$ is given by

$$d((x, y, t), (x', y', t')) := |(x, y, t)^{-1} \cdot (x', y', t')|$$

(3.6)

where $|(x, y, t)| := (x^2 + y^2 + t^2)^{1/4}$.

For the metric space $(H_n(\mathbb{R}), d)$ inequality (3.3) gives an upper bound $\lambda(H_n(\mathbb{R})) \leq 48(n + 1)$. It is interesting to note that the Whitney extension method does not work even to prove that $\lambda(H_n(\mathbb{R})) < \infty$. In fact, its basic geometric ingredient, Whitney’s covering lemma cannot be proved in this setting in a form allowing the required Lipschitz partition of unity.

Finally, consider the discrete subgroup $H_n(\mathbb{Z})$ of $H_n(\mathbb{R})$ consisting of elements of the set $\mathbb{Z}^n \times \mathbb{Z}^n \times \mathbb{Z}$. It is easily seen that the map $\delta : (x, y, t) \mapsto \frac{1}{2}(x, y, t)$ is a dilation in the sense used in Theorem 2.1. Since $\delta(H_n(\mathbb{Z})) \supset H_n(\mathbb{Z})$ and $\delta^j H_n(\mathbb{Z})$ is dense in $H_n(\mathbb{R})$, Theorem 2.1 implies that

$$\lambda(H_n(\mathbb{Z})) = \lambda(H_n(\mathbb{R})).$$

(3.7)

B. Groups of Polynomial Growth.

Let $G$ be a finitely generated group with the word metric $d_A$ associated with a set of generators $A$. This group is said to be of polynomial growth, if for every $R > 0$ the number of elements in a ball of radius $R$ is bounded by $cR^n$ with fixed constants $c, n > 0$. By the Gromov result [Gr2] such a group is virtually nilpotent and therefore by [B] for every ball $B_R(g_0)$ the inequality

$$c_1 R^Q \leq |B_R(g_0)| \leq c_2 R^Q$$

(3.8)

is true. Here $|\cdot|$ is the counting measure, $Q$ is the homogeneous degree (3.2) of the Zariski closure of the maximal torsion free nilpotent subgroup of $G$ and $c_1, c_2 > 0$ depend only on $G$. Then Theorem 2.21 implies for this group the inequality

$$\lambda(G) \leq c$$

(3.9)
with $c$ depending on $G$. In case $G$ being torsion free nilpotent, the constants in (3.8) depend only on $Q$, see [B], and $c$ in (3.9) does, as well.

For the special case of the abelian group $\mathbb{Z}^n$ this result can be sharpened. In this case we use another representation of the metric space $(G, d_A)$ related to the Cayley graph $\mathcal{C}_A(G)$. The latter is a metric graph whose vertices are in a one-to-one correspondence with elements of $G$ and which has edges $e_a$ of length one joining each $g \in G$ with $ga$, $a \in A \cup A^{-1}$. The metric subspace $G \subset \mathcal{C}_A(G)$ is then isometric to $(G, d_A)$, see, e.g. [BH, p.8]. For $\mathbb{Z}^n$ with the set of generators $A := \{a_1, \ldots, a_n\}$ being the standard basis of $\mathbb{R}^n$ the Cayley graph $\mathcal{C}_A(\mathbb{Z}^n)$ is the 1-dimensional CW-complex determined by $\mathbb{Z}^n$ with the metric induced from $l_1^n$. Therefore $(\mathbb{Z}^n, d_A)$ coincides with $\mathbb{Z}^n_1 \subset l_1^n$ and the application of Theorem 2.1 with the dilation $\delta : x \mapsto \frac{1}{2}x$, $x \in \mathbb{R}^n$, yields
\[
\lambda(\mathbb{Z}^n, d_A) = \lambda(l_1^n) .
\]
This and Corollary 2.25 immediately imply that
\[
c\sqrt{n} \leq \lambda(\mathbb{Z}^n, d_A) \leq 24n
\] (3.10)
with $c$ independent of $n$.

C. Gromov-Hyperbolic Groups.

A finitely generated group $G$ is called Gromov-hyperbolic if its Cayley graph with respect to some finite generating set is a Gromov hyperbolic metric space. Every such group is finitely presented, and conversely, in a certain statistical sense, almost every finitely presented group is hyperbolic [Gr3]. It is also known that any infinite, non-virtually cyclic hyperbolic group is of exponential growth. On the other hand, Corollary 2.13 asserts that $\lambda(G)$ is finite for these groups $G$. We now present several examples of hyperbolic groups.

(1) Any finite group is hyperbolic.
(2) Any free group of finite rank is 0-hyperbolic.
(3) The fundamental group of a compact Riemann surface is hyperbolic.
(4) A discrete cocompact group $G$ of isometries of the hyperbolic $n$-space $\mathbb{H}^n$ is hyperbolic, see [Bo] and [GrP] for existence of such arithmetic and nonarithmetic groups $G$.

D. Free Groups.

Let $\mathcal{F}(A)$ be a free group with the set of generators $A$ of arbitrary cardinality. It is easily seen that the Cayley graph of $\mathcal{F}(A)$ is a metric tree rooted at the unit of $\mathcal{F}(A)$ (the empty word). Hence $\mathcal{F}(A)$ equipped with the word metric $d_A$ is an infinite rooted metric tree with all edges of length one. In turn, the direct product $\prod_{i=1}^n \mathcal{F}(A_i)$ with the word metric generating by the generating set $\prod_{i=1}^n A_i$ is isometric to $\oplus_1\{\mathcal{F}(A_i), d_{A_i}\}_{1 \leq i \leq n}$. Therefore Theorem 2.4 immediately implies that
\[
c_1\sqrt{n} \leq \lambda(\prod_{i=1}^n \mathcal{F}(A_i)) \leq c_2n
\]
with $0 < c_1 < c_2$ independent of $n$.

We conjecture that this quantity is actually equivalent to $n$ as $n \to \infty$.

3.2. Riemannian Manifolds.

A. Nilpotent Lie Groups.
Let $G$ be a simply connected real nilpotent Lie group of dimension $N$ equipped with a left-invariant Riemannian metric. Unlike Carnot groups the unit ball of $G$ is far from being a Euclidean ellipsoid, see Figure 1 in [K] plotting the ball of a big radius for the Heisenberg group $H^3(\mathbb{R})$ equipped with a Riemannian metric. Nevertheless, it was shown in that paper that the volume of the ball $B_R(g)$ of $G$ (with respect to the geodesic metric) satisfies the inequality

$$a R^Q \leq v(B_R(g)) \leq b R^Q,$$

where $a, b > 0$ are independent of $R$ and $g$, and $Q$ is the homogeneous dimension of $G$, see (3.2). A similar inequality with $N$ instead of $Q$ holds for $0 \leq R \leq 1$ (this follows from the definition of the Riemannian structure on $G$). Hence the dilation function for the measure $v$ is bounded by $(b/a)^Q$ for $l > 1$, and the metric space $G$ is of homogeneous type with respect to the measure $v$. Then the conditions of Corollary 2.24 hold with $C = 0$ and $n = Q$. This implies the inequality

$$\lambda(G) \leq K_0 (b/a)^2 (Q + 1)$$

with $K_0 < 225$.

**B. Regular Riemannian Coverings.**

Let $p : M \to C$ be a regular Riemannian covering of a compact Riemannian manifold $C$. By definition, the deck transformation group of this covering $G_p$ acts on $M$ by isometries properly, freely and cocompactly. Hence by Corollary 2.15 $\lambda(M)$ is finite, if $\lambda(G_p)$ is. For instance, finiteness of the latter is true for $G_p$ being one of the discrete groups presented in section 3.1. In particular, if $C$ is a compact $n$-dimensional Riemannian manifold whose mean curvature tensor is everywhere positive definite the group $G_p$ is of a polynomial growth [M]. Hence $\lambda(M) < \infty$ in this case. Suppose now that $C$ is a compact Riemannian manifold whose sectional curvature is bounded by a negative number. Then its fundamental group $\pi_1(C)$ is Gromov-hyperbolic, see, e.g., [BH, p.448]. Moreover, this group acts properly, freely and cocompactly on the universal covering $M$ of $C$. Hence $\lambda(M) < \infty$ by Corollary 2.15.

**C. Riemannian Manifolds of Nonnegative Ricci Curvature.**

Let $M_n$ be a complete noncompact $n$-dimensional Riemannian manifold regarded as a metric space with the geodesic metric. Assume that the Ricci curvature of $M_n$ is nonnegative. Then by the Laplacian comparison theorem, see, e.g., [Gr, p.283], the volume of its balls satisfies

$$\frac{v(B_{R_2}(m))}{v(B_{R_1}(m))} \leq \left(\frac{R_2}{R_1}\right)^n, \quad 0 < R_1 \leq R_2.$$

Hence $M_n$ equipped with the measure $v$ is of homogeneous type and the dilation function $D(l)$ of $v$ is bounded by $l^n$. Applying Corollary 2.24 with $C = 0$ and $a = 1$, see (2.11), we get

$$\lambda(M_n) \leq K_0 (n + 1)$$

with $K_0 < 225$.

**D. Riemannian Manifolds of Pinched Negative Sectional Curvature.**
Let $M$ be a complete, simply-connected Riemannian manifold whose sectional curvature $\kappa$ satisfies $-b^2 < \kappa < -a^2 < 0$ for some $a, b \in \mathbb{R}$. Then $M$ is a Gromov-hyperbolic metric space, see, e.g., [BH]. Rauch’s comparison theorem implies also that $M$ is of bounded geometry, see, e.g., [CE]. Now, application of Corollary 2.13 immediately yields the finiteness of $\lambda(M)$.

E. Other Riemannian Manifolds.

Let $H^{n+1}_\rho$ be a complete Riemannian manifold with the underlying set $\mathbb{R}^{n+1}_+ := \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : t > 0\}$ and the Riemannian metric

$$ds^2 := \rho(t)^{-2}(dx_1^2 + \ldots + dx_n^2 + dt^2).$$

We assume that $\rho$ is continuous and nondecreasing and $\rho(0) = 0$, while $\rho(t)/t^2$ is nonincreasing. We endow $H^{n+1}_\rho$ with the inner (geodesic) metric and show that the metric space obtained possesses the $LE$. For this goal we use the collection of measures $\{\mu_m : m \in H^{n+1}_\rho\}$ introduced in [BSh2]. They are given by the formula

$$\mu_m(U) := \int_U \chi(t-s) \hat{\rho}(s) \prod_{i=1}^n \hat{\rho}(t + |y_i - x_i|) \ dy_1 \cdots dy_n \ ds$$

provided that $m = (x, t) \in \mathbb{R}^{n+1}_+$ and $U \subset \mathbb{R}^{n+1}_+$. Here $\hat{\rho} := 1/\rho$ and $\chi$ is the Heaviside function, indicator of $[0, \infty)$. It was proved in [BSh2, pp.537-540] that these measures satisfy the conditions of Definition 2.19. Hence the application of Theorem 2.21 yields the inequality

$$\lambda(H^{n+1}_\rho) < \infty.$$

4 Open Problems

(a) Does there exist a metric space $M \in LE$ but with $\lambda(M) = \infty$?

(b) It follows from Theorem 2.17 that

$$\lambda(l_p) = \infty, \quad \text{if} \quad p \neq 2.$$

Is the same true for infinite dimensional Hilbert spaces?\(^5\)

(c) It was established within the proof of Theorem 2.17, that there is a surjection $\phi$ of $l^n_p$ onto its arbitrary convex subset $C$ such that

$$|\phi|_{Lip(l^n_p, C)} \leq n|\frac{1}{p} - \frac{1}{2}|, \quad 1 \leq p \leq \infty. \quad (4.1)$$

Is it true that the metric projection of $l^n_p$ onto $C$ satisfies a similar inequality with a constant $c(p)$ depending only on $p$? The result is known for $p = 2$ with $c(p) = 1$, but in general $c(p)$ should be more than 1. E.g., the sharp Lipschitz constant for the metric projection in $l^2_\infty$ is 2, not $\sqrt{2}$ (V. Dol’nikov, a personal communication).

(c) Is it true that $\lambda(\Gamma)$ is finite for any uniform lattice $\Gamma$ of a metric space $M$?

\(^5\)The answer is positive. In the forthcoming paper [BB] we show that $\lambda(X) = \infty$ for an arbitrary infinite dimensional Banach space $X$. 

17
(Conjecture 2.18)?
(d) Is it true that for a complete simply connected length space \( M \) of a non-positive curvature in the A. D. Alexandrov sense \( \lambda(M) < \infty \)?
(e) Assume that \( (M_i, d_i) \) satisfy \( \lambda(M_i) < \infty, i = 1, 2 \). Is the same true for \( M_1 \times M_2 \) endowed with the metric \( \max(d_1, d_2) \)?

5 Proof of Theorem 2.1: the Case of \( S = M \)

5.1. By a technical reason we prefer to deal with a punctured metric space \((M, d, m^*)\) with a designed point \( m^* \in M \). A subspace \( S \) of this space has to contain \( m^* \) while \( \text{Lip}_0(S) \) stands for a closed subspace of \( \text{Lip}(S) \) given by the condition

\[
f(m^*) = 0.
\]

Since \( f \mapsto f - f(m^*) \) is a projection of norm 1 from \( \text{Lip}(S) \) onto \( \text{Lip}_0(S) \), the constant \( \lambda(S) \) is unchanged after replacing \( \text{Lip}(S) \) by \( \text{Lip}_0(S) \).

In the sequel we will exploit the following fact.

**Theorem A** There is a Banach space \( K(S) \) predual to \( \text{Lip}_0(S) \) and such that all evaluation functionals \( \delta_m : f \mapsto f(m), m \in S, \text{belong to } K(S) \).

For compact metric spaces \( S \) this can be derived from the Kantorovich-Rubinshtein theorem [KR] and for separable \( S \) from the result of Dudley [Du] who generalized the Kantorovich-Rubinshtein construction to this case. For bounded metric spaces a predual space with the required in Theorem A property can be found in [We], see also [GK]. For the convenience of the reader we present below a simple alternative proof of Theorem A, see Appendix.

5.2. We begin with the next result constituting the first part of the proof (for \( S = M \), i.e., for Corollary 2.2).

**Proposition 5.1** Let \( F \) be a given finite point subset of \((M, m^*)\). Assume that for every finite \( G \supset F \) there is an extension operator \( E_G \in \text{Ext}(F,G) \) from \( F \) to \( G \) and

\[
A := \sup_G ||E_G|| < \infty.
\]

Then there is \( E \in \text{Ext}(F,M) \) such that

\[
||E|| \leq A.
\]

**Proof.** Introduce a map \( J : M \to K(M) \) by

\[
J(m) := \delta_m.
\]

**Lemma 5.2** (a) \( J \) is an isometric embedding;

(b) \( K(M) \) is the minimal closed subspace containing \( J(M) \).
**Proof.** Let $f \in K(M)^* = Lip_0(M)$. Then for $m', m'' \in M$

$$| < f, \delta_{m'} - \delta_{m''} > | := |f(m') - f(m'')| \leq ||f||_{Lip(M)} d(m', m'').$$

Taking supremum over $f$ from the unit ball of $Lip_0(M)$ to have

$$|| J(m') - J(m'') ||_{K(M)} \leq d(m', m'').$$

To prove the converse, one defines a function $g \in Lip_0(M)$ by

$$g(m) := d(m, m') - d(m^*, m').$$

Then

$$| g(m') - g(m'') | = d(m', m'') \quad \text{and} \quad | g(m_1) - g(m_2) | \leq d(m_1, m_2), \quad m_1, m_2 \in M.$$

In particular,

$$|| g ||_{Lip(M)} = 1 \quad \text{and} \quad || J(m') - J(m'') ||_{K(M)} \geq | g(m') - g(m'') | = d(m', m''),$$

and the first assertion is done.

Let now $X := \overline{linJ(M)} \neq K(M)$. Then there is a nonzero linear functional $f \in K(M)^*(= Lip_0(M))$ which is zero on $X$. By the definition of $X$

$$f(m) = < f, \delta_m - \delta_m^* > = < f, J(m) > = 0$$

for every $m \in M$ and so $f = 0$, a contradiction. \ \box

Let now

$$\kappa_S : K(S) \to K(S)^{**} \quad \text{(5.5)}$$

be the canonical isometric embedding. Since $K(F)$ is finite-dimensional, $\kappa_F$ is an isomorphism onto $K(F)^{**}$.

For a finite $G \supset F$ one introduces a linear operator $\rho_G : K(G) \to K(F)^{**}$ by

$$\rho_G := E_G^* \kappa_G \quad \text{(5.6)}$$

where the star designates a conjugate operator. Then define a vector-valued function $\phi_G : G \to K(F)^{**}$ by

$$\phi_G(m) := \rho_G J(m), \quad m \in G. \quad \text{(5.7)}$$

**Lemma 5.3** (a) $\phi_G \in Lip(G, K(F)^{**})$ and its norm satisfies

$$|| \phi_G || \leq A. \quad \text{(5.8)}$$

(b) For $m \in F$

$$\phi_G(m) = \kappa_G J(m) \quad \text{(5.9)}$$

In particular, $\phi_G(m^*) = 0$. 19
Proof. (a) Let $m \in G$ and $h \in Lip_0(F) (= K(F)^*)$. By (5.6) and (5.7)

$$ < \phi_G(m), h >= < \kappa_G J(m), E_G h >= < E_G h, J(m) >= (E_G h)(m) . $$

This immediately implies that

$$ | < \phi_G(m'), h > - < \phi_G(m''), h > | \leq \| E_G \| d(m', m'') \| h \|_{Lip_0(F)} , \quad m', m'' \in G . $$

This, in turn, gives (5.8).

(b) Since $(E_G h)(m) = h(m), m \in F$, and $h(m^*) = 0$, the previous identity implies (5.9). $\square$

The family of the functions $\{ \phi_G \}$ is indexed by the elements $G \supset F$ forming a net. We now introduce a topology on the set $\Phi$ of functions $\psi : M \to K(F)^{**}$ satisfying the inequality

$$ \| \psi(m) \|_{K(F)^{**}} \leq Ad(m,m^*) , \quad m \in M , \quad (5.10) $$

that allows to find a limit point of the family $\{ \phi_G \}$. Denote by $B_m, m \in M$, the closed ball in $K(F)^{**}$ centered at 0 and of radius $Ad(m,m^*)$. Then (5.10) means that for every $\psi \in \Phi$

$$ \psi(m) \in B_m , \quad m \in M . $$

Let $Y := \prod_{m \in M} B_m$ equipped with the product topology. Since $K(F)^{**}$ is finite-dimensional, $B_m$ is compact and therefore $Y$ is compact, as well. Let $\tau : \Phi \to Y$ be the natural bijection given by

$$ \psi \mapsto (\psi(m))_{m \in M} . $$

Identifying $\Phi$ with $Y$ one equips $\Phi$ with the topology of $Y$. Then $\Phi$ is compact.

Let now $\hat{\phi}_G : M \to K(F)^{**}$ be the extension of $\phi_G$ from $G$ by zero. By Lemma 5.3 $\hat{\phi}_G$ meets condition (5.10), i.e., $\{ \hat{\phi}_G \} \subset \Phi$. By compactness of $\Phi$ there is a subnet $N = \{ \hat{\phi}_{G\alpha} \}$ of the net $\{ \hat{\phi}_G : G \supset F \}$ such that

$$ \lim_{\alpha} \hat{\phi}_{G\alpha} = \phi \quad (5.11) $$

for some $\phi \in \Phi$, see e.g. [Ke, Chapter 5, Theorem 2]. By the definition of the product topology one also has

$$ \lim_{\alpha} \hat{\phi}_{G\alpha}(m) = \phi(m) , \quad m \in M , \quad (5.12) $$

(convergence in $K(F)^{**}$).

Show that $\phi$ is Lipschitz. Let $m', m'' \in M$ be given, and $\tilde{N}$ be a subnet of $N$ containing those of $\hat{\phi}_{G\alpha}$ for which $m', m'' \subset G\alpha$. Then by (5.6), (5.7) and (5.12) one has for $h \in Lip_0(F)$

$$ < \phi(m') - \phi(m''), h >= \lim_{\tilde{N}} < \phi_{G\alpha}(m') - \phi_{G\alpha}(m''), h >= $$

$$ \lim_{\tilde{N}} < E_{G\alpha} h, J(m') - J(m'') >= \lim_{\tilde{N}} [(E_{G\alpha} h)(m') - (E_{G\alpha} h)(m'')] . $$

20
Together with (5.2) this leads to the inequality
\[ | < \phi(m') - \phi(m''), h > | \leq A ||h||_{Lip_0(F)} d(m', m'') , \]
that is to say,
\[ ||\phi||_{Lip(M,K(F)^* *)} \leq A . \]  (5.13)
Using (5.9) we also similarly prove that for \( m \in F \)
\[ \phi(m) = \kappa_F J(m) , \quad \text{and} \quad \phi(m^*) = 0 . \]  (5.14)

Utilizing the function \( \phi \) we, at last, define the required extension operator \( E : Lip_0(F) \rightarrow Lip_0(M) \) as follows. Let
\[ \bar{\kappa}_F : K(F)^* \rightarrow K(F) *** \]  (5.15)
be the canonical embedding (an isometry in this case). For \( h \in Lip_0(F) = K(F)^* \) we define \( Eh \) by
\[ (Eh)(m) := < \bar{\kappa}_F h, \phi(m) > , \quad m \in M . \]  (5.16)
Then by (5.13)
\[ |(Eh)(m') - (Eh)(m'')| = | < \phi(m') - \phi(m''), h > | \leq A ||h||_{Lip_0(F)} d(m', m'') , \]
and (5.3) is proved.

Now by (5.14) we have for \( m \in F \)
\[ (Eh)(m) = < \phi(m), h > = < \kappa_F J(m), h > = h(m) ; \]
in particular, \((Eh)(m^*) = 0.\)

The proof of the proposition is done. \( \square \)

We are now ready to prove Theorem 2.1 for the case of \( S = M.\)

Since the inequality
\[ A := \sup_F \lambda(F) \leq \lambda(M) \]  (5.17)
with \( F \) running through finite subspaces of \((M,m^*)\) is trivial, we have to establish the converse. In other words, we have to prove that for every \( S \subset (M,m^*) \) and \( \epsilon > 0 \) there is \( E \in Ext(S,M) \) with
\[ ||E|| \leq A + \epsilon . \]  (5.18)
By the definition of \( A, \) for each pair \( F \subset G \) of finite subspaces of \( S \) there is \( E_G \in Ext(F,G) \) with \( ||E_G|| \leq A + \epsilon.\) Applying to family \( \{E_G\} \) Proposition 5.1 we find for every finite \( F \) an operator \( \bar{E}_F \in Ext(F,M) \) with
\[ ||\bar{E}_F|| \leq A + \epsilon . \]  (5.19)
To proceed with the proof we need the following fact.

**Lemma 5.4** There is a linear isometric embedding \( I_S : K(S) \rightarrow \kappa_M(K(M)) .\)
**Proof.** Let \( R_S : f \mapsto f|_S \) be the restriction to \( S \subset (M,m^*) \). Clearly, \( R_S \) is a linear mapping from \( \text{Lip}_0(M) \) onto \( \text{Lip}_0(S) \) with norm bounded by 1. Since each \( f \in \text{Lip}_0(S) \) has a preserving norm extension to \( \text{Lip}_0(M) \), see e.g. [Mc],

\[
||R_S|| = 1 . \tag{5.20}
\]

Introduce now the required linear operator \( I_S \) by

\[
I_S := R_S^* \kappa_S . \tag{5.21}
\]

Since \( R_S^* \) maps the space \( \text{Lip}_0(S)^* = K(S)^{**} \) into \( \text{Lip}_0(M)^* = K(M)^{**} \) and \( \kappa_S : K(S) \to K(S)^{**} \) is the canonical embedding, the operator \( I_S \) maps \( K(S) \) into \( K(M)^{**} \). Show that it, in fact, sends \( K(S) \) into \( \kappa_M(K(M)) \). Let \( m \in S \) and \( g \in \text{Lip}_0(M) \). Then by (5.21) and the definition of \( R_S \) we get

\[
< I_S J(m), g > = < \kappa_S J(m), g |_S > = < g, J(m) > = < \kappa_M J(m), g >
\]

which implies the embedding

\[
I_S(J(S)) \subset \kappa_M(J(M)) .
\]

According to Lemma 5.2 this, in turn, implies the required embedding of \( I_S(K(S)) \) into \( \kappa_M(K(M)) \). At last, by (5.20) and (5.21)

\[
||I_S|| = ||R_S^*|| = ||R_S|| = 1 ,
\]

and the result is done. \( \square \)

Let now \( \tilde{E}_F \) be the operator from (5.19). Let us define an operator \( P_F : K(M) \to K(F)^{**} \) by

\[
P_F := \tilde{E}_F^* \kappa_M . \tag{5.22}
\]

Using the isometric embedding \( I_F : K(F) \to \kappa_S(K(S)) \) of Lemma 5.4 we then introduce an operator \( Q_F : K(M) \to \kappa_S(K(S)) \subset K(S)^{**} \) by

\[
Q_F := I_F(\kappa_F)^{-1} P_F . \tag{5.23}
\]

Since \( \dim K(F) < \infty \), this is well-defined. Introduce, at last, a vector-valued function \( \phi_F : M \to K(S)^{**} \) by

\[
\phi_F(m) := Q_F J(m) , \quad m \in M . \tag{5.24}
\]

Arguing as in Lemma 5.3 and using the estimate \( ||Q_F|| \leq ||P_F|| \leq ||\tilde{E}_F|| \leq A + \epsilon \), see (5.22), (5.23) and (5.19), we obtain the inequality

\[
||\phi_F||_{\text{Lip}(M,K(S)^{**})} \leq A + \epsilon . \tag{5.25}
\]

Moreover, for each \( m \in F \) and \( h \in \text{Lip}_0(M) \)

\[
< \phi_F(m), h > =< R_F^* \kappa_F(\kappa_F)^{-1} P_F J(m), h |_F > =< h |_F, P_F J(m) >=
\]

\[
< E_F h, J(m) >= h(m) ,
\]

22
Moreover, by \((5.28)\) we have for \(m\)
\[
\phi_F(m) = \kappa_M J(m);
\]
(5.26)
in particular, \(\phi_F(m^*) = 0\).

From here and \((5.25)\) we derive that the set \(\{\phi_F(m)\}\) with \(F\) running through the net of all finite subspaces of \(S\) is a subset of the closed ball \(B_m \subset K(S)^{**}\) centered at 0 and of radius \((A + \epsilon)d(m, m^*)\). In the weak* topology \(B_m\) is compact. From this point our proof repeats word for word that of Proposition 5.1. Namely, consider the set \(\Phi\) of functions \(\psi : M \rightarrow K(S)^{**}\) satisfying
\[
\|\psi(m)\|_{K(S)^{**}} \leq (A + \epsilon)d(m, m^*), \quad m \in M.
\]
Equip \(B_m\) with the weak* topology and introduce the set \(Y := \prod_{m \in M} B_m\) equipped with the product topology. Then \(Y\) is compact and, so, \(\Phi\) is too in the topology induced by the bijection \(\Phi : \psi \mapsto (\psi(m))_{m \in M} \in Y\). Then there is a subnet \(N\) of the net \(\{\phi_F : (F, m^*) \subset (S, m^*)\}, \#F < \infty\) such that
\[
\lim_{N} \phi_F = \phi
\]
for some \(\phi \in \Phi\). By the definition of the product topology
\[
\lim_{N} \phi_F(m) = \phi(m), \quad m \in M
\]
(convergence in the weak* topology of \(K(S)^{**}\)). Arguing as in the proof of Proposition 5.1, see \((5.13)\), we derive from \((5.25)\) that
\[
\|\phi\|_{\text{Lip}(M, K(S)^{**})} \leq A + \epsilon
\]
(5.27)
and, moreover, for \(m \in S\)
\[
\phi(m) = \lim_{N'} \phi_F(m) = \kappa_M J(m).
\]
(5.28)
Here \(N' := \{\phi_F \in N : m \in F \subset S\}\) is a subnet of \(N\), and we have used \((5.26)\).

Using now the canonical embedding \(\tilde{k}_S : K(S)^* = \text{Lip}_0(S) \rightarrow K(S)^{***}\) we introduce the required extension operator \(\tilde{E} \in \text{Ext}(S, M)\) by
\[
(\tilde{E}h)(m) := < \tilde{k}_S h, \phi(m) >, \quad m \in M, \ h \in \text{Lip}_0(S).
\]
Since \(\phi(m) \in K(S)^{**}\), this is well-defined. Then for \(m', m'' \in M\) we get from \((5.27)\)
\[
\|(\tilde{E}h)(m') - (\tilde{E}h)(m'')\| \leq \|h\|_{\text{Lip}_0(S)}\|\phi(m') - \phi(m'')\|_{K(S)^{**}} \leq (A + \epsilon)\|h\|_{\text{Lip}_0(S)}d(m', m'').
\]
Moreover, by \((5.28)\) we have for \(m \in S\)
\[
(\tilde{E}h)(m) = < \tilde{k}_S h, \kappa_M J(m) > = < h, J(m) > = h(m).
\]
Hence \(E \in \text{Ext}(S, M)\) and \(\|E\| \leq A + \epsilon\). This implies the converse to \((5.17)\) inequality
\[
\lambda(M) \leq \sup_{F} \lambda(F)(= A).
\]

Proof of Theorem 2.1 for \(S = M\) is complete. \(\Box\)

23
6 Proof of Theorem 2.1: the Final Part

We begin with a slight modification of Proposition 5.1 using an increasing sequence of subspaces \( \{S_j\}_{j \geq 0} \) of the space \((M, m^*)\) instead of the net \( \{G\} \) of its finite point subspaces containing a given \( F \). Repeating line-to-line the proof of Proposition 5.1 for this setting we obtain as a result an extension operator \( E \in \text{Ext}(F, S_\infty) \) where \( S_\infty := \cup_j S_j \) with the corresponding bound for \( ||E|| \). If, in addition, \( S_\infty \) is dense in \( M \), then there is a canonical isometry \( \text{Lip}_0(S) \leftrightarrow \text{Lip}_0(M) \) generated by continuous extensions of functions from \( \text{Lip}_0(S_\infty) \). This leads to the following assertion.

**Proposition 6.1** Assume that \( \{S_j\}_{j \geq 0} \) is an increasing sequence of subspaces of \((M, m^*)\) whose union \( S_\infty \) is dense in \( M \), and \( F \) is a finite point subspace in \( \cap_{j \geq 0} S_j \). Suppose that for every \( j \) there exists \( E_j \in \text{Ext}(F, S_j) \) and

\[
A := \sup_j ||E_j|| < \infty . \tag{6.1}
\]

Then there is an operator \( E \in \text{Ext}(F, M) \) with \( ||E|| \leq A \). \( \Box \)

Let now \( \{m_k^j\}_{j \in \mathbb{N}} \) be a sequence in \( M \) convergent to \( m_k \), \( 1 \leq k \leq n \). Set \( F := \{m^*, m_1, \ldots, m_n\} \) and \( F_j := \{m^*, m_1^j, \ldots, m_n^j\} \).

**Proposition 6.2** Assume that for each \( j \) there is \( E_j \in \text{Ext}(F_j, M) \) and

\[
A := \sup_j ||E_j|| < \infty . \tag{6.2}
\]

Then there exists \( E \in \text{Ext}(F, M) \) with

\[
||E|| \leq A . \tag{6.3}
\]

**Proof.** Let a linear operator \( L_j : \text{Lip}_0(F) \to \text{Lip}_0(F_j) \) be given by

\[
(L_j f)(m_l^j) := f(m_l) , \quad 1 \leq l \leq n , \quad \text{and} \quad (L_j f)(m^*) = 0 .
\]

Then its norm is bounded by \( \sup_{l' \neq l''} \{d(m_{l'}, m_{l''})/d(m_l^j, m_l'^j)\} \) and therefore

\[
\limsup_{j \to \infty} ||L_j|| = 1 \quad \text{and} \quad ||L_j|| \leq 2 , \quad j \geq j_0 , \tag{6.4}
\]

for some \( j_0 \).

Introduce now a linear operator \( \widetilde{E}_j : \text{Lip}_0(F) \to \text{Lip}_0(M) \) by

\[
\widetilde{E}_j := E_j L_j .
\]

Applying (6.4) to have

\[
\limsup_{j \to \infty} ||\widetilde{E}_j|| \leq A \quad \text{and} \quad ||\widetilde{E}_j|| \leq 2A , \quad j \geq j_0 . \tag{6.5}
\]

Let \( R_j : K(M) \to K(F)^{**} \) be given by

\[
R_j := \widetilde{E}_j^* \kappa_M ,
\]

24
and a vector-valued function $G_j : M \to K(F)^{**}$ be defined by

$$G_j(m) := R_j(J(m)) , \ m \in M .$$

Arguing as in Lemma 5.3 we then establish that $G_j$ is Lipschitz and

$$||G_j||_{Lip(M,K(F)^{**})} \leq 2A , \ j \geq j_0 , \ (6.6)$$

and, moreover,

$$G_j(m) = \kappa_M(J(m)) , \ m \in F . \ (6.7)$$

In particular, $G_j(m^*) = 0$. These allow to assert that there is a subnet $\{G_{j_k}\}$ of the net $\{G_j\}$ such that $G(m) := \lim_{k} G_{j_k}(m)$ exists for each $m \in M$ in the topology of $K(F)^{**}$ (see the argument of Proposition 5.1 after Lemma 5.3). It remains to define $(Eh)(m)$ for $h \in Lip_0(F)$ and $m \in M$ by

$$(Eh)(m) := <\kappa_F h, G(m) > ,$$

cf. (5.16). The argument at the end of the proof of Proposition 5.1 with $E_j$ instead of $E_G$ and (6.5) instead of (5.2) leads to existence of $E \in Ext(F,M)$ with

$$||E|| \leq \limsup_{j \to \infty} ||E_j|| \leq A . \quad \Box$$

Now we will finalize the proof of Theorem 2.1. Since the inequality

$$\sup_{F \in S} \lambda(F) = \lambda(S) \leq \lambda(M) \quad (6.8)$$

is clear, we have to prove the following assertion.

Given a finite $F \subset (M,m^*)$ and $\epsilon > 0$, there is $E \in Ext(F,M)$ such that

$$||E|| \leq \lambda(S) + \epsilon . \quad (6.9)$$

Since $\lambda(M) = \sup_F \lambda(F)$, this will prove the converse to (6.8).

To establish this assertion one considers first the case of $F$ containing in $S_\infty := \cup_{j \geq 0} S_j$ where $S_j := \delta^j(S), \ j = 0, 1, \ldots$. Since $S_j$ is increasing, see assumption (a) of Theorem 2.1, there is $j = j(F)$ such that $F \subset S_j$ and so

$$F_j := \delta^{-j}(F) \subset S .$$

Then there is $E_j \in Ext(F_j,S)$ such that

$$||E_j|| \leq \lambda(S) + \epsilon .$$

Introduce now linear operators $D_j : Lip(F) \to Lip(F_j)$ and $H_j : Lip(S) \to Lip(S_j)$ by

$$(D_j f)(m) := f(\delta^j(m)) , \ f \in Lip(F) , \ m \in F_j ,$$

$$(H_j f)(m) := f(\delta^{-j}(m)) , \ f \in Lip(S) , \ m \in S_j .$$

25
Then the operator
\[ \tilde{E}_j := H_j E_j D_j \] (6.10)
clearly belongs to \( Ext(F, S_j) \). Then we have
\[ D_j R_F = R_{F^j} \Delta^j \quad \text{and} \quad H_j R_S = R_{S^j} \Delta^{-j}, \]
where \( R_K : f \mapsto f|_K \) stands for the restriction operator to \( K \subset M \), and \( \Delta \) is given by (2.1).

Since each \( f \in Lip_0(K) \) has a preserving norm extension to \( Lip_0(M) \), the above identities imply that
\[ ||D_j|| = ||D_j R_F|| = ||R_{F^j} \Delta^j|| \leq ||\Delta||^j \quad \text{and} \]
\[ ||H_j|| = ||H_j R_S|| = ||R_{S^j} \Delta^{-j}|| \leq ||\Delta^{-1}||^j. \]

This, in turn, leads to the estimate
\[ ||\tilde{E}_j|| \leq ||H_j|| \cdot ||E_j|| \cdot ||D_j|| \leq ||E_j||(||\Delta|| \cdot ||\Delta^{-1}||)^j \leq \lambda(S) + \epsilon. \] (6.11)

So we have constructed a sequence \( \{\tilde{E}_j\} \) of operators whose norms are bounded by (6.11). Moreover, \( \{S_j\} \) is increasing and its union is dense in \( M \), see assumption (b) of Theorem 2.1. Hence we are under the conditions of Proposition 6.1 that guarantees existence of an operator \( E \in Ext(F, M) \) with \( ||E|| \leq \lambda(S) + \epsilon \). Therefore the inequality (6.9) is done for such \( F \).

Let now \( F := \{m^*, m_1, \ldots, m_n\} \) be arbitrary. Since \( S_\infty \) is dense in \( M \), one can find a sequence \( F_j := \{m^*, m^j_1, \ldots, m^j_n\}, j \in \mathbb{N}, \) of subsets of \( S_\infty \) such that
\[ \lim_{j \to \infty} m^j_l = m_l, \quad 1 \leq l \leq n. \]

It had just proved that for each \( j \) there is \( E_j \in Ext(F_j, M) \) such that
\[ ||E_j|| \leq \lambda(S) + \epsilon, \quad j \in \mathbb{N}. \] (6.12)

Applying to this setting Proposition 6.2 we conclude that there is \( E \in Ext(F, M) \) satisfying (6.9). Together with (6.8) this completes the proof of Theorem 2.1. \( \square \)

**Remark 6.3**  
(a) Using the compactness argument of the proof of Proposition 5.1 one can also prove that for every \( S \subset M \) there is an extension operator \( E_{min} \in Ext(S, M) \) such that
\[ ||E_{min}|| = \inf \{ ||E|| : E \in Ext(S, M) \} \] (6.13)
(b) The same argument combining with some additional consideration allow to establish the following fact.

The set function \( S \mapsto \lambda(S) \) defined on closed subsets of \( M \) is lower semicontinuous in the Hausdorff metric.
7 Proof of Theorem 2.4

Trees with all edges of length one. In accordance with Corollary 2.2 one has to find for every pair \( S < S' \) of finite sets in \( M_p := \oplus_p \{ T_i \}_{1 \leq i \leq n} \), an operator \( E \in \text{Ext}(S, S') \) whose norm is bounded by

\[
||E|| \leq cn \tag{7.1}
\]

with \( c \) independent of \( S \) and \( S' \); recall that \( p = 1 \) or \( \infty \).

To accomplish this we first find a subset \( S'' = \prod_{i=1}^{n} S_i, S_i \subset T_i \), such that \( S' \subset S'' \). Further, every finite subset of a tree can be isometrically embedded into an infinite rooted tree \( T_k \) with vertices of degree \( k + 2 \) for some \( k \in \mathbb{N} \) (and all edges of length one in our case). Note that the \textit{degree of vertex } \( v \), written \( \text{deg } v \), is the number of its children plus 1. Taking some \( k \) such that every set \( S_i \) is an isometric part of \( T_k \), we therefore can derive (7.1) from a similar inequality with \( S' \) substituted for \( \oplus_p \{ T_k \}_{1 \leq i \leq n} (= \text{the direct } p\text{-sum of } n \text{ copies of } T_k) \). Hence we have reduced the required result to the following assertion:

\[
\lambda(\oplus_p \{ T_k \}_{1 \leq i \leq n}) \leq cn \tag{7.2}
\]

with \( c \) independent of \( n \) and \( k \).

The proof of this inequality is divided into two parts. We first prove that each \( T_k \) can be quasi-isometrically embedded into the hyperbolic plane \( \mathbb{H}^2 \), see Proposition 7.1 below. From here we derive that

\[
\lambda(\oplus_p \{ T_k \}_{1 \leq i \leq n}) \leq 256 \lambda(\oplus_p \{ \mathbb{H}^2 \}_{1 \leq i \leq n}). \tag{7.3}
\]

Then we estimate the right-hand side applying inequalities (2.13) and (2.14) of Theorem 2.27 for the case \( M_i = \mathbb{H}^2 \) for all \( 1 \leq i \leq n \). Combining these we prove the right-hand side inequality of Theorem 2.4 for trees with edges of length one.

We begin with establishing the desired quasi-isometric embedding of \( T_k \) into \( \mathbb{H}^2 \). In the formulation of this result, \( d \) and \( \rho \) are, respectively, the path metrics on \( T_k \) and \( \mathbb{H}^2 = \{ x \in \mathbb{R}^2 : x_2 > 0 \} \).

**Proposition 7.1** For every \( k \geq 2 \) there is an embedding \( I : T_k \to \mathbb{H}^2 \) such that for all \( m_1, m_2 \in T_k \)

\[
A d(m_1, m_2) \leq \rho(I(m_1), I(m_2)) \leq B d(m_1, m_2) \tag{7.4}
\]

with constants \( 0 < A < B \) independent of \( m_1, m_2 \) and satisfying

\[
BA^{-1} \leq 256. \tag{7.5}
\]

**Proof.** It will be more appropriate to work with another metric on \( \mathbb{H}^2 \) given by

\[
\rho_0(x, y) := \max_{i=1,2} \log \left( 1 + \frac{|x_i - y_i|}{\min(x_2, y_2)} \right). \tag{7.6}
\]

The following result establishes an equivalence of this to the hyperbolic metric \( \rho \) for pairs of points far enough from each other.
Lemma 7.2  (a) $\rho \leq 4\rho_0$;

(b) If $|x - y| \geq \frac{1}{2} \min(x_2, y_2)$, then

$$\rho(x, y) \geq \frac{1}{8} \rho_0(x, y).$$

Here and below $|x|$ is the Euclidean norm of $x \in \mathbb{R}^2$.

Proof. For definiteness assume that

$$\min(x_2, y_2) = y_2. \tag{7.7}$$

Use for a while the complex form of $\mathbb{H}^2$ with the underlying set $\{z \in \mathbb{C} : \text{Im} z > 0\}$. Then the metric $\rho$ is given by

$$\rho(z_1, z_2) = \log \frac{1 + \frac{|z_1 - z_2|}{|z_1 - z_2|}}{1 - \frac{|z_1 - z_2|}{|z_1 - z_2|}}.$$  \hspace{1cm} \tag{7.8}

Identifying $z = x_1 + ix_2$, with $(x_1, x_2) \in \mathbb{H}^2$ we rewrite this as

$$\rho(x, y) = \log \frac{(|x - y^+| + |x - y|)^2}{|x - y^+|^2 - |x - y|}$$

where $y^+ := (y_1, -y_2)$ is the reflexion of $y$ in the $x_1$-axis. Since the denominator in (7.8) equals $4x_2y_2$ and

$$|x - y^+| + |x - y| \leq 2|x - y| + |y^+ - y| = 2(|x - y| + y_2)$$

we derive from (7.8) and (7.7) the inequality

$$\rho(x, y) \leq 2 \log \frac{y_2 + |x - y|}{y_2} \leq 4 \log \left(1 + \max_{i=1,2} \frac{|x_i - y_i|}{y_2}\right).$$

By (7.6) and (7.7) this implies the required result formulated in (a).

In case (b) we use an equivalent formula for $\rho$ given by

$$\cosh \rho(x, y) = 1 + \frac{|x - y|}{2x_2y_2}.$$  \hspace{1cm} \tag{7.9}

Since $\cosh t \leq e^t$ for $t \geq 0$, this yields

$$\rho(x, y) \geq \log \left(1 + \frac{|x - y|^2}{2x_2y_2}\right). \tag{7.9}$$

Consider two possible cases

$$y_2 \leq x_2 \leq 2y_2; \tag{7.10}$$

$$2y_2 < x_2. \tag{7.11}$$
In the first case we use (7.7) and the assumption $\frac{2|x-y|}{y^2} \geq 1$ to derive from (7.9)

$$\rho(x, y) \geq \log \left(1 + \left(\frac{|x-y|}{2y^2}\right)^2\right) \geq \frac{1}{8} \log \left(1 + \frac{|x-y|}{y^2}\right) \geq \frac{1}{8} \log \left(1 + \frac{\max_{i=1,2} |x_i - y_i|}{y^2}\right) := \frac{1}{8} \rho_0(x, y).$$

In the second case we have from (7.11) $|x-y| \geq |x_2 - y_2| \geq \frac{1}{2} x_2$. Inserting this in (7.9) we obtain the required result

$$\rho(x, y) \geq \log \left(1 + \frac{|x-y|}{4y^2}\right) \geq \frac{1}{4} \log \left(1 + \frac{\max_{i=1,2} |x_i - y_i|}{y^2}\right) := \frac{1}{4} \rho_0(x, y). \quad \Box$$

We now begin to construct the required embedding $I : \mathcal{T}_k \to \mathbb{H}^2$. To this end we first introduce coordinates for the set of vertices $\mathcal{V}_k$ of the $\mathcal{T}_k = (R, \mathcal{V}_k, \mathcal{E}_k)$ where $R$ stands for the root. Actually, we assign to $v \in \mathcal{V}_k$ the pair of integers $(j_v, l_v)$ determined as follows. The number $l_v$ is the level of $v$, the length of the unique path from the root to $v$. To define $j_v$ we visualize $\mathcal{T}_k$ using the natural isometric embedding of $\mathcal{T}_k$ into $\mathbb{R}^2$. Then $j_v$ is the number of $v$ in the ordering of the vertices of the $l_v$-th level from the left to the right. We use in this ordering the set of numbers $0, 1, \ldots$ and therefore

$$0 \leq j_v < (k+1)^{l_v},$$

since the number of children of each vertex equals $k + 1$. We also assign $(0, 0)$ to the root $R$ of $\mathcal{T}_k$.

Using this we relate the coordinates of $v$ and its parent $v^+$. To this end one uses the $(k + 1)$-ary digital system to present $j_v$ as

$$j_v = \sum_{s=1}^{l_v} \delta_s(v)(k+1)^{s-1} \quad (7.12)$$

where $\delta_s(v) \in \{0, 1, \ldots, k\}$ are the digits. Then the coordinates of $v$ and $v^+$ are related by

$$l_{v^+} = l_v - 1; \quad (7.13)$$

$$\delta_{s}(v^+) = \delta_{s+1}(v) + 1, \quad s = 1, \ldots, l_{v^+}. \quad (7.14)$$

To express the distance between $v, w \in \mathcal{V}_k$ in their coordinates, we first introduce the notion of the common ancestor $a(v, w)$ of these vertices. This is the vertex of the biggest level in the intersection of the paths joining the root with $v$ and $w$, respectively. Hence there are sequences $v := v_1, v_2, \ldots, v_p := a(v, w)$ and $w := w_1, w_2, \ldots, w_q := a(v, w)$ such that $v_{i+1} = v^+_i$ (the parent of $v_i$), $w_{i+1} = w^+_i$ and the children $v_{p-1}$ and $w_{q-1}$ of $a(v, w)$ are distinct. The distance between $v$ and $w$ in the $\mathcal{T}_k$ is therefore given by

$$d(v, w) = l_v + l_w - 2l_{a(v, w)} \quad (7.15)$$
We now define the required embedding $I : \mathcal{T}_k \to \mathbb{H}^2$ on the set of vertices $\mathcal{V}_k \subset \mathcal{T}_k$. To this end we assign to each $v \in \mathcal{V}_k$ a square $Q(v)$ in $\mathbb{R}^2_+$ in the following fashion. For the root $R$ we define $Q(R)$ to be the square in $\mathbb{R}^2_+$ whose center $c(R)$ and lengthside $\mu(R)$ are given by

$$c(R) = (c_1(R), c_2(R)) = (0, 1), \quad \mu(R) = \frac{2(n - 1)}{n + 1} \tag{7.16}$$

where here and below

$$n := k^2 + 1. \tag{7.17}$$

Define now $Q(v)$ for a child $v$ of $R$. In this case $l_v = 1$ and $j_v = \delta_1(v)$, see (7.12), and we introduce $Q(v)$ to be a square in $\mathbb{R}^2_+$ whose center $c(v)$ and lengthside $\mu(v)$ are given by

$$c_1(v) := \frac{1}{2} \mu(v) (2\delta_1(v)k - k^2), \quad c_2(v) := \frac{1}{n}, \quad \mu(v) = \frac{2(n - 1)}{n + 1} \cdot \frac{1}{n}. \tag{7.18}$$

Note that these squares are introduced by the following geometric construction. Divide the bottom side of $Q(R)$ into $n$ equal intervals and construct outside $Q(R)$ $n$ squares with these intervals as their sides. Number those from the left to the right. Then the squares numbered by $1, k + 1, 2k + 1, \ldots, k \cdot k + 1 := n$ form the set $\{Q(v) : l_v = 1\}$. Apply now this construction to each $Q(v)$ with $l_v = 1$ to obtain $(k + 1)^2$ squares corresponding to vertices of the level $l_v = 2$ and so on. Straightforward evaluation leads to the following formulas related to the coordinates of $c(v)$, the center of $Q(v)$, and its lengthside $\mu(v)$ to those for the square $Q(v^+)$ associated with the parent $v^+$ of $v$

$$c_1(v) = c_1(v^+) + \frac{1}{2} \mu(v) (2\delta_1(v)k - k^2). \tag{7.19}$$

In particular, for the second coordinate of $c(v)$ and for $\mu(v)$ we get

$$c_2(v) = \frac{1}{n^{l_v}}, \quad \mu(v) = \frac{2(n - 1)}{n + 1} \cdot \frac{1}{n^{l_v}}. \tag{7.19}$$

Compare now the metrics $\rho$ and $\rho_0$ on the set of the centers $c(v)$, $v \in \mathcal{V}_k$. Let $v, w$ be distinct vertices of $\mathcal{V}_k$. Without loss of generality we assume that

$$l_v \geq l_w, \tag{7.20}$$

so that

$$\min(c_2(v), c_2(w)) = n^{-l_v}. \tag{7.21}$$

Moreover, if $l_v = l_w$ then

$$|c_1(v) - c_1(w)| \geq |c_1(v) - c_1(\hat{v})|$$
The Lemma 7.3

Proof. Show that inequality, Lemma 7.3 and (7.15) we then have (7.22) it suffices to work with the metric space \((\mathbb{R}^n, \rho)\) of the desired estimates.

Let \(\hat{v}\) has coordinates satisfying \(l_\hat{v} = l_v\) and \(|j_{\hat{v}} - j_v| = 1\). If now \(l_v > l_w\), then \(|c_2(v) - c_2(w)| \geq |c_2(v) - c_2(v')|\) provided that \(v'\) satisfies \(l_v - l_{v'} = 1\). By (7.18) and (7.19) the right-hand side of the inequality for \(\hat{v}\) is equal to \(\frac{2(n-1)k}{n^v(n+1)}\), while the right-hand side of the inequality for \(v'\) is equal to \(\frac{n-1}{n^v}\). Together with (7.21) these inequalities yield the estimate

\[
\frac{|c(v) - c(w)|}{\min(c_2(v), c_2(w))} \geq \frac{n - 1}{n} \geq \frac{1}{2}.
\]

Hence the assumption of Lemma 7.2 holds for \(x := c(v)\) and \(y := c(w)\) and we have

\[
\frac{1}{8} \rho_0(c(v), c(w)) \leq \rho(c(v), c(w)) \leq 4 \rho_0(c(v), c(w)). \tag{7.22}
\]

We now introduce the required embedding \(I : T_k \to \mathbb{H}^2\) beginning with its definition on the subset \(\mathcal{V}_k \subset T_k\) of vertices; namely, we let

\[
I(v) := c(v), \quad v \in \mathcal{V}_k.
\]

Show that \(I|_{\mathcal{V}_k}\) is a bi-Lipschitz equivalence with the constants satisfying (7.5). By (7.22) it suffices to work with the metric space \((\mathbb{R}^2_+, \rho_0)\). So we have to compare \(\rho_0(c(v), c(w))\) with the distance \(d(v, w)\) in the tree \(T_k\).

**Lemma 7.3** The \(\rho_0\)-distance between \(c(v)\) and \(c(v^+)\) equals \(\log n\).

**Proof.** By (7.19)

\[
\log \left(1 + \frac{|c_2(v) - c_2(v^+)|}{\min(c_2(v), c_2(v^+))}\right) = \log(1 + n - 1) = \log n.
\]

On the other hand, the similar expression with \(c_2\) replaced by the first coordinates in the numerator equals \(\log \left(1 + \frac{n}{n+1} |2\delta_1(v) k - k^2|\right)\), see (7.18). Since \(0 \leq \delta_1(v) \leq k\), this is at most \(\log n\). Hence

\[
\rho_0(c(v), c(v^+)) := \max_{i=1,2} \log \left(1 + \frac{|c_i(v) - c_i(v^+)|}{\min(c_2(v), c_2(v^+))}\right) = \log n. \quad \square
\]

Thus, the length of each edge in \(I(\mathcal{V}_k)\) equals \(\log n\). Using this we prove the first of the desired estimates.

Let \(a(v, w)\) be the common ancestor of \(v\) and \(w\) and \(v := v_1, v_2, \ldots, v_p := a(v, w)\) and \(w := w_1, w_2, \ldots, w_q := a(v, w)\) are the corresponding connecting paths. So \(v_{i+1} = v_i^+, w_{i+1} = w_i^+\) and \(p := l_v - l_{a(v, w)} + 1, q = l_w - l_{a(v, w)} + 1\). By the triangle inequality, Lemma 7.3 and (7.15) we then have

\[
\rho_0(c(v), c(w)) \leq \sum_{i=1}^{p-1} d(v_i, v_{i+1}) + \sum_{i=1}^{q-1} d(w_i, w_{i+1}) \leq
\]

\[
(l_v - l_{a(v, w)}) \log n + (l_w - l_{a(v, w)}) \log n = d(v, w) \log n.
\]

31
So we get
\[ \rho_0(I(v), I(w)) \leq \log n \cdot d(v, w) \tag{7.23} \]
and it remains to establish the inverse inequality. To this end we consider two cases. First, suppose that \( w = a(v, w) \). Then by (7.19), (7.21) and (7.15) we have
\[ \rho_0(c(v), c(w)) \geq \log \left(1 + \frac{|c_2(v) - c_2(w)|}{\min(c_2(v), c_2(w))}\right) = \log n^{l_v - l_w} = d(v, w) \log n . \tag{7.24} \]
Suppose now that \( w \neq a(v, w) \), then we use the inequality
\[ \rho_0(c(v), c(w)) \geq \log \left(1 + \frac{|c_1(v) - c_1(w)|}{\min(c_2(v), c_2(w))}\right) \geq l_v \log n + \log |c_1(v) - c_1(w)| , \tag{7.25} \]
see (7.21). To estimate the second summand one notes that, by our geometric construction, the orthogonal projection of \( Q(v) \) onto the bottom side of \( Q(v^+) \) lies inside this side. Applying this consequently to the vertices of the chains \( \{v_i\}_{1 \leq i \leq p} \) and \( \{w_i\}_{1 \leq i \leq q} \) joining \( v \) and \( w \) with \( a(v, w) \), see the proof of (7.23), and taking into account that \( v_{i+1} = v_i^+ \) and \( w_{i+1} = w_i^+ \) we conclude that the orthogonal projections of \( c(v) := c(v_i) \) and \( c(w) := c(w_i) \) onto the bottom side of \( Q(a(v, w)) = Q(v_{p-1}^+) = Q(w_{q-1}^+) \) lie, respectively, inside the top sides of the squares \( Q(v_{p-1}) \) and \( Q(w_{q-1}) \) adjoint to \( Q(a(v, w)) \). Hence
\[ |c_1(v) - c_1(w)| \geq \text{dist}(Q(v_{p-1}), Q(w_{q-1})) \]
and this distance is at least \( \frac{\kappa_\mu(a(v, w))}{n} \) by the definition of the squares involved. By the equality \( n := k^2 + 1 \), \( k \geq 2 \), and formula (7.19) we derive from here that
\[ \log |c_1(v) - c_1(w)| \geq \log \left(\frac{2(n - 1)^{3/2}}{n(n + 1)} n^{-l_a}\right) \geq -(l_a + 1/2) \log n , \quad a := a(v, w) . \]
Together with (7.25) this yields
\[ \rho_0(c(v), c(w)) \geq (l_v - l_a - 1/2) \log n . \]
Now note that (7.20) and the inequality \( l_v - l_a \geq 1 \) imply that
\[ l_v - l_a - 1/2 \geq \frac{1}{8} (l_v + l_w - 2l_a) = \frac{1}{8} d(v, w) , \]
see (7.15). Hence
\[ \rho_0(c(v), c(w)) \geq \frac{1}{8} \log n \cdot d(v, w) . \]
Together with (7.23) and (7.24) this yields the required bi-Lipschitz equivalence
\[ \frac{1}{8} \log n \cdot d(v, w) \leq \rho_0(I(v), I(w)) \leq \log n \cdot d(v, w) , \quad v, w \in V_k . \tag{7.26} \]
We now extend $I$ to the whole $T_k$ by defining it on each edge $[v, v^+] \subset T_k$; recall that $[v, v^+]$ is identified with the unit interval of $\mathbb{R}$ and therefore there is a curve $\gamma_v : [0, 1] \to [v, v^+]$ so that

$$|\gamma_v(t) - \gamma_v(t')| = |t - t'|, \quad \gamma_v(1) = 1.$$ 

To define the extension we join $c(v)$ and $c(v^+)$ by the geodesic segment in $\mathbb{H}^2$ (the subarc of a Euclidean circle or a straight line intersecting the $x_1$-axis orthogonally). Denote this by $[c(v), c(v^+)]$. By our geometric construction and properties of the geodesics of the hyperbolic plane the interiors of any two such segments do not intersect. Therefore the union of all $[c(v), c(v^+)]$, $v \in V_k \setminus \{R\}$, forms a metric tree whose edges $[c(v), c(v^+)]$ are isometric to the closed intervals of $\mathbb{R}$ of lengths $\rho_v := \rho(c(v), c(v^+))$. Let now $\tilde{\gamma}_v : [0, \rho_v] \to [c(v), c(v^+)]$ be the canonical parameterization of the geodesic $[c(v), c(v^+)]$ so that

$$\rho(\tilde{\gamma}_v(t'), \tilde{\gamma}_v(t'')) = |t' - t''|, \quad \tilde{\gamma}_v(\rho_v) = \rho(c(v), c(v^+))(:= \rho_v).$$

Let us extend the map $I$ to a point $m = \gamma_v(t) \in [v, v^+] \subset T_k$, $0 \leq t \leq 1$, by

$$I(m) := \tilde{\gamma}_v(\rho_v t).$$

Then for $m, m' \in [v, v^+]$ we get

$$\rho(I(m), I(m')) = \rho_v |t - t'| = \rho_v d(m, m').$$

Using now (7.22) and Lemma 7.3 we obtain the estimate

$$\frac{1}{8} \log n \leq \rho_v \leq 4 \log n.$$

Together with the previous equality this yields

$$\frac{1}{8} \log n \cdot d(m, m') \leq \rho(I(m), I(m')) \leq 4 \log n \cdot d(m, m'). \tag{7.27}$$

Let now $m \in [v, v^+]$, $m' \in [w, w^+]$ and $v \neq w$. Since $\rho$ and $d$ are path metrics, the latter inequality together with (7.26) and (7.22) imply that

$$\frac{1}{64} \log n \cdot d(m, m') \leq \rho(I(m), I(m')) \leq 4 \log n \cdot d(m, m').$$

Hence $I$ is a bi-Lipschitz embedding of $T_k$ into $\mathbb{H}^2$ and the inequality (7.4) of Proposition 7.1 holds with $A := \frac{1}{64} \log n$ and $B := 4 \log n$.

This proves Proposition 7.1. \qed

Using now the established embedding (7.4) we derive the required inequality (7.3). To complete the proof of the right-hand side inequality of Theorem 2.4 for this case it remains to derive inequality (7.2) from Theorem 2.27. To this end we use the following particular case of the result from [BSh2, Proposition 5.33] with $n = 1$.

There is a distance $\rho_0$ on $\mathbb{H}^2$ and a family of measures $\{\mu_x\}_{x \in \mathbb{H}^2}$ such that
(a) \((\mathbb{H}^2, \rho_0)\) is of pointwise homogeneous type with respect to this family;
(b) \(\rho_0\) is equivalent to the hyperbolic metric \(\rho\);
(c) For every ball \(B_R^0(x) := \{ y \in \mathbb{H}^2 : \rho_0(x, y) \leq R \}\) we have
\[
\mu_x(B_R^0(x)) = 2R^2.
\]

Apply now Theorem 2.27 for \(M_i = (\mathbb{H}^2, \rho_0), 1 \leq i \leq n\). The above formulated statement shows that the conditions of this theorem are true in this case and therefore inequalities (2.13) and (2.14) yield
\[
\lambda(\oplus_p\{(\mathbb{H}^2, \rho_0)\}_{1 \leq i \leq n}) \leq c_1n
\]
for some numerical constant \(c_1\) and \(p = 1, \infty\). It remains to note that \(\rho_0 \sim \rho\) and therefore the metrics of \(\oplus_p\{(\mathbb{H}^2)\}_{1 \leq i \leq n}\) and \(\oplus_p\{(\mathbb{H}^2, \rho_0)\}_{1 \leq i \leq n}\) are equivalent with constants independent of \(n\).

So inequality (7.2) has proved, and the right-hand side inequality of Theorem 2.4 is established for this case.

**The general case.** Let now \(T_i\) be an arbitrary metric tree with edges \(e\) of lengths \(l_i(e) > 0, 1 \leq i \leq n\). The argument of the previous subsection shows that in order to prove the right-hand side inequality of the theorem it suffices to derive the inequality
\[
\lambda(\oplus_p\{T_i\}_{1 \leq i \leq n}) \leq 256\lambda(\oplus_p\{\mathbb{H}^2\}_{1 \leq i \leq n})
\]
for arbitrary finite rooted metric trees \(T_i\).

We first establish this for finite \(T_i\) with edges of lengths being rational numbers. Let \(N\) be the least common denominator of all these numbers for all \(i\). Introduce a new rooted metric tree \(T_i^N, 1 \leq i \leq n\), whose sets of vertices \(V_i^N\) and edges \(E_i^N\) are defined as follows. Let \(e\) be an edge of \(T_i\) and \(l_i(e) := \frac{M_i(e)}{N}\) where \(M_i(e)\) is a natural number. Insert in this edge \(M_i(e) - 1\) equally distributed new vertices; recall that \(e\) is regarded as the closed interval of \(\mathbb{R}\) of length \(l_i(e)\). In this way we obtain the new rooted tree \(T_i^N\), a triangulation of \(T_i\), that we endow by the path metric \(D_i^N\) induced by the metric \(Nd_i\) (here \(d_i\) is the path metric on \(T_i\)). Note that every \(T_i^N\) has all edges of length 1 and therefore it can be embedded into the infinite tree \(T_k\) with a suitable \(k\) (the same for all \(i\)). Hence the inequality (7.3) yields
\[
\lambda(\oplus_p\{T_i^N\}_{1 \leq i \leq n}) \leq 256\lambda(\oplus_p\{\mathbb{H}^2\}_{1 \leq i \leq n}) .
\]
On the other hand, \((T_i, Nd_i)\) is a metric subspace of \(T_i^N, 1 \leq i \leq n\), and therefore
\[
\lambda(\oplus_p\{T_i\}_{1 \leq i \leq n}) = \lambda(\oplus_p\{(T_i, Nd_i)\}_{1 \leq i \leq n}) \leq \lambda(\oplus_p\{T_i^N\}_{1 \leq i \leq n}) .
\]
Together with (7.29) this implies the required result (7.28).

Consider now the general situation of finite rooted metric trees \(T_i\) with arbitrary lengths of edges. Given \(\epsilon > 0\) one replaces the metric of \(T_i, 1 \leq i \leq n\), by a path
metric $d_{i,\epsilon}$ which remains to be linear on edges but such that lengths of edges $l_{i,\epsilon}(e)$ in this metric are rational numbers satisfying

$$l_i(e) \leq l_{i,\epsilon}(e) \leq (1 + \epsilon)l_i(e).$$

Let $\mathcal{T}_{i,\epsilon}$ be a rooted metric tree with the underlying set $\mathcal{T}_i$ and the path metric $d_{i,\epsilon}$. It is already proved that

$$\lambda(\oplus_p\{\mathcal{T}_{i,\epsilon}\}_{1 \leq i \leq n}) \leq 256 \lambda(\oplus_p\{\mathbb{H}^2\}_{1 \leq i \leq n}).$$

On the other hand, the identity map $\mathcal{T}_i \to \mathcal{T}_{i,\epsilon}$ is a bi-Lipschitz equivalence with the constant of equivalence equals $1 + \epsilon$. Therefore

$$\lambda(\oplus_p\{\mathcal{T}_{i}\}_{1 \leq i \leq n}) \leq (1 + \epsilon)^2 \lambda(\oplus_p\{\mathcal{T}_{i,\epsilon}\}_{1 \leq i \leq n}).$$

Since $\epsilon$ is arbitrary, the last two inequalities prove (7.28) in the general case.

Thus we have proved the right-hand side inequality of Theorem 2.4.

To complete the proof of the theorem it remains to prove the lower estimate

$$\lambda(\oplus_p\{\mathcal{T}_i\}_{1 \leq i \leq n}) \geq c_0\sqrt{n}$$

for $p = 1, \infty$ and $c_0 > 0$ independent of $n$.

To this end choose in every $\mathcal{T}_i$ a path $P_i$ incident to the root of $\mathcal{T}_i$ of length $l_i > 0$.

Then the interval $[0, l_i]$ is isometrically embedded into $\mathcal{T}_i$ and the parallelepiped

$$\Pi := \prod_{i=1}^n[0, l_i]$$

equipped with the $l_p^n$-metric is an isometric part of $\oplus_p\{\mathcal{T}_i\}_{1 \leq i \leq n}$. By Corollary 2.3 with $S := \Pi$ and $\delta(x) := 2(x - c)$, $x \in \mathbb{R}^n$, where $c$ is the center of $\Pi$, one has

$$\lambda(\Pi) = \lambda(l_p^n)$$

and the latter is greater than $\lambda_{conv}(l_p^n)$. This, in turn, is at least $c_0\sqrt{n}$ for $p = 1, \infty$ with $c_0 > 0$ independent of $n$ (see Theorem 2.17).

This completes the proof of the required lower bound (and the theorem). $\square$

### 8 Proof of Theorem 2.6

We begin with the case of $M$ possessing the WTP. Assume that $M$ has the Lipschitz preserving linear extension property, but $\lambda(M) = \infty$. The latter implies existence of a sequence of finite sets $F_j$ with $\lambda(F_j) \geq j, j \in \mathbb{N}$; see Corollary 2.2. This, in turn, leads to the inequalities

$$\inf\{\|E\| : E \in Ext(F_j, M)\} \geq j, j \in \mathbb{N}. \quad (8.1)$$

Using the WTP of $M$ to choose an appropriate sequence of $C$-isometries $\sigma_j$ such that for $G_j := \sigma_j(F_j)$ the following holds

$$\text{dist}(G_j, \cup_{i \neq j}G_i) \geq C \text{ diam } F_j, j \in \mathbb{N}. \quad (8.2)$$
For every \( j \in \mathbb{N} \), fix a point \( m_j^* \in G_j \). From (8.2) we derive that an operator \( N_j \) given for every \( f \in \text{Lip}(G_j) \) by
\[
(N_j f)(m) := \begin{cases} f(m), & m \in G_j \\ f(m_j^*), & m \in \cup_{i \neq j} G_i \end{cases} \tag{8.3}
\]
belongs to \( \text{Ext}(G_j, G_\infty) \) where \( G_\infty := \cup_{i \in \mathbb{N}} G_i \), and, besides,
\[
\|N_j\| = 1 . \tag{8.4}
\]
In fact, if \( f \in \text{Lip}(G_j) \) and \( m' \in G_j, m'' \in G_\infty \setminus G_j = \cup_{i \neq j} G_i \), then
\[
|N_j f(m') - N_j f(m'')| = |f(m') - f(m_j^*)| \leq \|f\|_{\text{Lip}(G_j)}\|\|diam G_j \leq (C \text{diam } F_j)\|f\|_{\text{Lip}(G_j)} .
\]
Together with (8.2) this leads to
\[
|N_j f(m') - N_j f(m'')| \leq \|f\|_{\text{Lip}(G_j)}d(m', m'') .
\]
Since this holds trivially for all other choices of \( m', m'' \), the equality (8.4) is done.

Now by the \( \mathcal{LE} \) of \( M \) there is an operator \( E \in \text{Ext}(G_\infty, M) \) with \( \|E\| \leq A \) for some \( A > 0 \). By (8.4) the operator \( E_j := EN_j \in \text{Ext}(G_j, M) \) and \( \|E_j\| \leq A \). Then an operator \( \tilde{E}_j \) given by the formula
\[
(\tilde{E}_j f)(m) := (E_j (f \circ \sigma_j^{-1}))(\sigma_j(m)) , \quad m \in M , \quad f \in \text{Lip}(F_j) ,
\]
with the above introduced \( C \)-isometries \( \sigma_j \) belongs to \( \text{Ext}(F_j, M) \) and its norm is bounded by \( C^2 A \). Comparing with (8.1) to get for each \( j \)
\[
C^2 A \geq j ,
\]
a contradiction.

Let now \( M \) be proper. In order to prove that \( \lambda(M) < \infty \) we need

**Lemma 8.1** For every \( m \in M \) there is an open ball \( B_m \) centered at \( m \) such that \( \lambda(B_m) < \infty \).

**Proof.** Assume that this assertion does not hold for some \( m \). Then there is a sequence of balls \( B_i := B_{r_i}(m) \), \( i \in \mathbb{N} \), centered at \( m \) of radii \( r_i \) such that \( \lim_{i \to \infty} r_i = 0 \) and \( \lim_{i \to \infty} \lambda(B_i) = \infty \). According to Corollary 2.2 this implies existence of finite subsets \( F_i \subset B_i, i \in \mathbb{N} \), such that
\[
\inf\{\|E\| : E \in \text{Ext}(F_i, B_i)\} \to \infty , \quad \text{as } i \to \infty . \tag{8.5}
\]
We may and will assume that \( m \in F_j, j \in \mathbb{N} \). Otherwise we replace \( F_j \) by \( G_j := F_j \cup \{m\} \) and show that (8.5) remains true for \( G_i \) and \( B_i, i \in \mathbb{N} \). In fact, let \( L_i \) be an operator given for every \( f \in \text{Lip}(F_i) \) by
\[
(L_i f)(m') := \begin{cases} f(m_i), & m' = m \\ f(m'), & m' \in F_i \end{cases}
\]

where \( m_i \) is the closest to \( m \) point from \( F_i \). Then \( L_i \in Ext(F_i, G_i) \) and \( ||L_i|| \leq \) 2, since
\[
|(L_i f)(m)-(L_i f)(m')| = |f(m_i)-f(m')| \leq ||f||_{Llip(F_i)}d(m_i, m') \leq 2||f||_{Llip(F_i)}d(m, m') .
\]

If now (8.5) does not hold for \( \{G_i\} \) substituted for \( \{F_i\} \), then there is a sequence \( E_i \in Ext(G_i, B_i) \) such that \( \sup_i ||E_i|| < \infty \). But then the same will be true for the norms of \( \tilde{E}_i := E_iL_i \in Ext(F_i, B_i), i \in \mathbb{N} \), in contradiction with (8.5).

The proof will be now finished by the argument of section 4.1. Actually, choose a subsequence \( F_{i_k} \subset B_{i_k} := B_{r_{i_k}}(m), k \in \mathbb{N} \), such that
\[
r_{i_k+1} < \min\{r_{i_k}, \text{dist}(F_{i_{k+1}} \setminus \{m\}, \cup_{s<k+1}F_s \setminus \{m\})\} .
\]

Without loss of generality we assume that the sequence \( \{F_i\} \) already satisfies this condition, i.e.,
\[
r_{i+1} < \min\{r_i, \text{dist}(F_{i+1} \setminus \{m\}, \cup_{s<i+1}F_s \setminus \{m\})\} \quad (8.6)
\]

Set \( F_\infty := \cup_{s \in \mathbb{N}} F_s \) and show that
\[
Ext(F_\infty, M) = \emptyset \quad (8.7)
\]
which gives the required contradiction to the \( \mathcal{L}E \) of \( M \). To prove this we choose the point \( m \) as a marked point of \( M \). Then all \( F_i \) are subspaces of \( (M, m) \) and \( f(m) = 0 \) if \( f \in Lip_0(F_i), i \in \mathbb{N} \). Define now an operator \( N_i \) by
\[
(N_i f)(m') := \begin{cases} f(m'), & m' \in F_i \\ 0, & m' \in F_\infty \setminus F_i \end{cases} .
\]
Then for \( f \in Lip_0(F_i) \) and \( m' \in F_i \setminus \{m\} \) and \( m'' \in F_\infty \setminus F_i \) we have
\[
|(N_i f)(m') - (N_i f)(m'')| = |f(m') - f(m)| \leq ||f||_{Lip_0(F_i)}d(m', m) .
\]
Moreover, \( m'' \in B_j \) for some \( j \neq i \). Assume, first, that \( j > i \). Then by (8.6)
\[
d(m', m) \leq d(m', m'') + d(m'', m) \leq d(m', m'') + r_j \leq d(m', m'') + \text{dist}(F_j \setminus \{m\}, F_i \setminus \{m\}) \leq 2d(m', m'') .
\]
If now \( j < i \), then by (8.6) we have
\[
d(m', m) \leq r_i < \text{dist}(F_i \setminus \{m\}, F_j \setminus \{m\}) \leq d(m', m'') .
\]
Combining these we prove that \( N_i \in Ext(F_i, F_\infty) \) and \( ||N_i|| \leq 2 \).

If now (8.7) is not true, then there is an operator \( E \in Ext(F_\infty, M) \), and so every operator \( \tilde{E}_i := EN_i \) belongs to \( Ext(F_i, M) \) and \( ||\tilde{E}_i|| \leq 2||E||, i \in \mathbb{N} \), a contradiction to (8.5). Hence (8.7) holds and the proof is complete. \( \square \)

**Remark 8.2** In this proof properness of \( M \) is not used.
To prove the next important fact on \( \lambda \) we need

**Lemma 8.3** Let \( \mathcal{U} \) be a finite cover of a compact set \( C \subset M \) by open sets. Then there is a partition of unity \( \{ \rho_U \}_{U \in \mathcal{U}} \) on \( C \) subordinate to \( \mathcal{U} \) such that every \( \rho_U \) is Lipschitz with a constant depending only on the cover.

Let us recall its proof.

Define \( d_U : M \to \mathbb{R}_+ \) by

\[
d_U(m) := \text{dist}(m, M \setminus U), \quad m \in M.
\]

This is supported on \( U \) and is Lipschitz with constant 1. Moreover, \( \sum_{U \in \mathcal{U}} d_U > 0 \) on \( C \), as \( \mathcal{U} \) is a cover of \( C \). Putting now

\[
\rho_U(m) := \frac{d_U(m)}{\sum_{U \in \mathcal{U}} d_U(m)}, \quad m \in C \cap U, \ U \in \mathcal{U},
\]

we get the required partition. \( \square \)

The next result implies finiteness of \( \lambda(M) \) for compact \( M \).

**Lemma 8.4** For every compact set \( C \subset M \) the constant \( \lambda(C) \) is finite.

**Proof.** We have to show that for every \( S \subset C \) there is an operator \( E \in \text{Ext}(S, C) \) such that

\[
\sup_S ||E|| < \infty . \tag{8.8}
\]

We may and will assume that \( S \) and \( C \) are subspaces of \((M, m^*)\) so that \( f(m^*) = 0 \) for \( f \) belonging to \( \text{Lip}_0(S) \) or \( \text{Lip}_0(C) \). By compactness of \( C \) and Lemma 8.1 there is a finite cover \( \{ U_i \}_{1 \leq i \leq n} \) of \( C \) by open balls such that for some constant \( A > 0 \) depending only on \( C \) we have

\[
\lambda(U_i) < A , \quad 1 \leq i \leq n .
\]

By the definition of \( \lambda \) this implies existence of \( E_i \in \text{Ext}(S \cap U_i, U_i) \) with

\[
||E_i|| \leq A , \quad 1 \leq i \leq n . \tag{8.9}
\]

For \( f \in \text{Lip}_0(S) \) one sets

\[
f_i := \begin{cases} f|_{S \cap U_i} , & \text{if } S \cap U_i \neq \emptyset \\ 0 , & \text{if } S \cap U_i = \emptyset \end{cases} \tag{8.10}
\]

and introduces a function \( f_{ij} \) given on \( U_i \cap U_j \) by

\[
f_{ij} := \begin{cases} E_i f_i - E_j f_j , & \text{if } U_i \cap U_j \neq \emptyset \\ 0 , & \text{if } U_i \cap U_j = \emptyset \end{cases} \tag{8.11}
\]

here \( E_i f_i := 0 \), if \( f_i = 0 \). Then (8.9) implies that

\[
||f_{ij}||_{\text{Lip}(U_i \cap U_j)} \leq 2A ; \tag{8.12}
\]

38
besides, we get
\[ f_{ij} = 0 \quad \text{on} \quad S \cap U_i \cap U_j . \tag{8.13} \]

At last, introduce the function \( g_i \) on \( C \cap U_i \) by
\[
g_i(m) := \sum_{1 \leq j \leq n} \rho_j(m) f_{ij}(m) , \quad m \in C \cap U_i , \tag{8.14}\]
where \( \rho_j := \rho_{U_j} \), \( 1 \leq j \leq n \), is the partition of unity of Lemma 8.3.

A straightforward computation leads to the equalities:
\[
g_i - g_j = f_{ij} \quad \text{on} \quad U_i \cap U_j \cap C \tag{8.15}\]
and, moreover,
\[
g_i \mid_{S \cap U_i} = 0 . \tag{8.16}\]

Introduce now an operator \( E \) on \( f \in Lip_0(S) \) by the formula
\[
(Ef)(m) := (E_i f_i - g_i)(m) , \quad \text{if} \quad m \in U_i \cap C , \tag{8.17}\]
and show that \( E \) is an extension operator. In fact, if \( m \in S \), then \( m \in S \cap U_i \) for some \( 1 \leq i \leq n \) and by (8.16) and (8.10) we get
\[
(Ef)(m) = (E_i f_i)(m) = f(m) .
\]

Show now that \( E \in Ext(S,C) \) and \( ||E|| \) is bounded by a constant depending only on \( C \). To this end we denote by \( \delta = \delta(C) > 0 \) the Lebesgue number of the cover \( U \), see, e.g., [Ke]. So every subset of \( \bigcup_{i=1}^n U_i \) of diameter at most \( \delta \) lies in one of \( U_i \). Using this we first establish the corresponding Lipschitz estimate for \( m', m'' \in (\bigcup_{i=1}^n U_i) \cap C \) with
\[
d(m', m'') \leq \delta . \tag{8.18}\]

In this case both \( m', m'' \in U_{i_0} \) for some \( i_0 \). Further, (8.14)-(8.17) imply that for \( m \in U_{i_0} \cap C \)
\[
(Ef)(m) = \sum_{U_i \cap U_{i_0} \neq \emptyset} (\rho_i E_i f_i)(m) . \tag{8.19}\]

In this sum each \( \rho_i \) is Lipschitz with a constant \( L(C) \) depending only on \( C \) and \( 0 \leq \rho_i \leq 1 \). In turn, \( E_i f_i \) is Lipschitz on \( U_i \cap U_{i_0} \) with constant \( A ||f||_{Lip_0(S)} \), see (8.9) (recall that \( E_i f_i = 0 \) if \( S \cap U_i = \emptyset \)). If now \( m \in U_i \) with \( S \cap U_i \neq \emptyset \), then for arbitrary \( m_i \in S \cap U_i \)
\[
|E_i f_i(m)| \leq |E_i f_i(m) - (E_i f_i)(m_i)| + |(E_i f_i)(m_i)| \leq A ||f||_{Lip_0(S)} d(m, m_i) + |f(m_i) - f(m^*)| \leq A ||f||_{Lip_0(S)} (d(m, m_i) + d(m_i, m^*)) .
\]

This implies for all \( m \in C \cap U_i \) the inequality
\[
|(E_i f_i)(m)| \leq 2A \text{diam } C \ ||f||_{Lip_0(S)} . \tag{8.20}\]
Together with (8.19) this leads to the estimate
\[ |(Ef)(m') - (Ef)(m'')| \leq An(2L(C) \text{diam } C + 1) \| f \|_{Lip(S)} d(m', m'') , \quad (8.21) \]
provided \( m', m'' \in U_{i_0} \cap C \). To prove a similar estimate for \( d(m', m'') > \delta, m', m'' \in C \), we note that the left-hand side in (8.21) is bounded by
\[
2 \sup_{m \in C} |(Ef)(m)| \leq 4A \text{diam } C \| f \|_{Lip(S)} , \]
see (8.20). In turn, the right-hand side of the last inequality is \( \leq 4\delta^{-1} A \text{diam } C \| f \|_{Lip(S)} d(m', m'') \).
Together this implies that \( E \) belongs to \( Ext(S, C) \) and its norm is bounded by a constant depending only on \( C \). \( \Box \)

Our last basic step is a lemma which formulation uses the notation
\[
\lambda(S, M) := \inf \{ \| E \| : E \in Ext(S, M) \} ; \quad (8.22)
\]
here we set \( \lambda(S, M) := 0 \), if \( S = \emptyset \).

**Lemma 8.5** Assume that for a sequence of finite subsets \( F_i \subset (M, m^*) \), \( i \in \mathbb{N} \),
\[
\sup_i \lambda(F_i, M) = \infty . \quad (8.23)
\]
Then for every closed ball \( \overline{B} \) centered at \( m^* \)
\[
\sup_i \lambda(F_i \setminus \overline{B}, M) = \infty . \quad (8.24)
\]

**Proof.** Arguing as in the proof of Lemma 8.1, we can assume that \( m^* \in F_i, i \in \mathbb{N} \).
If now (8.24) is not true, then for some \( A_1 > 0 \) there is a sequence of operators \( E^1_i, i \in \mathbb{N} \), such that
\[
E^1_i \in Ext(F_i \setminus \overline{B}, M) \quad \text{and} \quad \| E^1_i \| \leq A_1 ; \quad (8.25)
\]
here we set \( E^1_i := 0 \), if \( F_i \setminus \overline{B} = \emptyset \). Let \( 2B \) be the open ball centered at \( m^* \) and of twice more radius than that of \( B \). Introduce an open cover of \( M \) by
\[
U_1 := M \setminus \overline{B} \quad \text{and} \quad U_2 := 2B , \quad (8.26)
\]
and let \( \{\rho_1, \rho_2\} \) be the corresponding Lipschitz partition of unity, cf. Lemma 8.3, given by
\[
\rho_j(m) := \frac{d_{U_j}(m)}{d_{U_1}(m) + d_{U_2}(m)} , \quad m \in M , \quad j = 1, 2 .
\]
By this definition
\[
|\rho_j(m_1) - \rho_j(m_2)| \leq \frac{3d(m_1, m_2)}{\max_{k=1,2} \{d_{U_1}(m_k) + d_{U_2}(m_k)\}} , \quad m_1, m_2 \in M . \quad (8.27)
\]
Set now \( H_i := F_i \cap 2B, i \in \mathbb{N} \). Since these are subsets of the compact set \( 2\overline{B} \), Lemma 8.4 gives
\[
\sup_i \lambda(H_i, 2B) \leq \lambda(2\overline{B}) < \infty .
\]
This, in turn, implies existence of operators $E_i^2$, $i \in \mathbb{N}$, such that
\[ E_i^2 \in Ext(H_i, 2B) \quad \text{and} \quad ||E_i^2|| \leq A_2 \tag{8.28} \]
with $A_2$ independent of $i$. We now follow the proof of Lemma 8.4 where the set $S$ and the compact set $C \supset S$ are replaced by $F_i$ and the (noncompact) space $M$, respectively, and the cover (8.26) is used. Since
\[ H_i = F_i \cap U_2 \quad \text{and} \quad F_i \setminus \overline{B} = F_i \cap U_1, \]
we can use in our derivation the operators $E_i^j$, $j = 1, 2$, instead of those in (8.9). By (8.25) and (8.28) inequalities similar to (8.9) hold for these operators. Then we set $f_j := f|_{F_i \cap U_j}$, and define for $f \in Lip_0(F_i)$ functions $f_{12} := -f_{21}$ on $U_1 \cap U_2$, $g_1$ on $U_1$ and $g_2$ on $U_2$ by
\[ f_{12} := E_i^1 f_1 - E_i^2 f_2, \quad g_1 := \rho_2 f_{12}, \quad g_2 := \rho_1 f_{21}. \]

At last, we introduce the required operator $E_i$ on $Lip_0(F_i)$ by
\[ (E_i f)(m) := (E_i^1 f_j)(m) - g_j(m), \quad m \in U_j, \quad j = 1, 2. \]

As in Lemma 8.4 $E_i$ is an operator extending functions from $F_i$ to the whole $M$. To estimate the Lipschitz constant of $E_i f$ we extend each $E_i^j f_j$ outside $U_j$ so that the extensions $\tilde{E}_j$ satisfy
\[ ||\tilde{E}_j||_{Lip(M)} = ||E_i^j f_j||_{Lip(U_j)}. \]

Now, the definition of $E_i$ implies that
\[ E_i f := \rho_1 \tilde{E}_1 + \rho_2 \tilde{E}_2. \]

Assume without loss of generality that $F_i \cap U_1 \neq \emptyset$, and choose a point $m' \in F_i \cap U_1$. Then as in the proof of (8.20) for arbitrary $m \in M$ we obtain
\[ |\tilde{E}_j(m)| \leq \begin{cases} A_1 ||f||_{Lip_0(F_i)}(d(m, m') + d(m', m^*)), & \text{if } j = 1 \\ A_2 ||f||_{Lip_0(F_i)}d(m, m^*), & \text{if } j = 2 \end{cases} \]

This implies for all $m$ the inequality
\[ |\tilde{E}_j(m)| \leq A(d(m, m^*) + d(m', m^*))||f||_{Lip_0(F_i)} \tag{8.29} \]
with $A := 2\max(A_1, A_2)$.

Together with (8.27) this leads to the estimate
\[ |(E_i f)(m_1) - (E_i f)(m_2)| \leq \left( \frac{3(d(m_1, m^*) + d(m', m^*))}{\max_{k=1,2}\{d_{U_1}(m_k) + d_{U_2}(m_k)\}} + 1 \right) 2A ||f||_{Lip_0(F_i)}d(m_1, m_2). \tag{8.30} \]
Since \( \max_{k=1,2}\{d_{U_1}(m_k) + d_{U_2}(m_k)\} \geq R \), the radius of \( B \), and
\[
\lim_{d(m_1, m^*) \to \infty} \frac{d(m_1, m^*) + d(m', m^*)}{d_{U_1}(m_1) + d_{U_2}(m_1)} = 1 ,
\]
(8.30) implies that \( E_i \in Ext(F_i, M) \) and its norm is bounded by a constant independent of \( i \). By definition (8.22) this gets
\[
\sup_i \lambda(F_i, M) < \infty
\]
in a contradiction with (8.23).  \( \square \)

Now we will complete the proof of Theorem 2.6. Recall that it has already proved for compact \( M \), see Lemma 8.4. So it remains to consider the case of a proper \( M \) with
\[
diam M = \infty . \quad (8.31)
\]
In this case we will show that
\[
\sup_F \lambda(F, M) < \infty , \quad (8.32)
\]
where \( F \) is running through all finite subsets \( F \subset (M, m^*) \). Since \( \sup_F \lambda(F) \) is bounded by the supremum in (8.32), this leads to finiteness of \( \lambda(M) \), see Corollary 2.2, and proves the result.

Let, to the contrary, (8.32) does not hold. Then there is a sequence of finite subsets \( F_i \subset (M, m^*) \), \( i \in \mathbb{N} \), that satisfies the assumption of Lemma 8.5, see (8.23).
We use this to construct a sequence \( G_i, i \in \mathbb{N} \), such that
\[
\lambda(G_i, M) \geq i - 1 \quad \text{and} \quad dist(G_{i+1}, G_i) \geq dist(m^*, G_i) , \quad i \in \mathbb{N} . \quad (8.33)
\]
As soon as it is done we set \( G_{\infty} := \cup_{i \in \mathbb{N}} G_i \), and use with minimal changes the argument of Lemma 8.1 to show that
\[
Ext(G_{\infty}, M) = \infty .
\]
Since this contradicts to the \( LE \) of \( M \), the result will be done.

To construct the required \( \{G_i\} \), set \( G_1 := F_1 \) and assume that the first \( j \) terms of this sequence have already defined. Choose the closed ball \( \overline{B} \) such that
\[
dist(G_j, M \setminus \overline{B}) \geq dist(m^*, G_j) ;
\]
it exists because of (8.31). Then apply Lemma 8.5 to find \( F_{i(j)} \) such that
\[
\lambda(F_{i(j)} \setminus \overline{B}, M) \geq j - 1 .
\]
Setting \( G_{j+1} := F_{i(j)} \setminus \overline{B} \) we obtain the next term satisfying the condition (8.33).

The proof is complete.  \( \square \)
9 Proof of Theorem 2.17

We first prove that

\[ \lambda_{\text{conv}}(l^n_2) = 1. \]  
(9.1)

Since the lower bound 1 is evident, we have to prove that

\[ \lambda_{\text{conv}}(l^n_2) \leq 1. \]  
(9.2)

Let \( C \subset (l^n_2, 0) \) be a closed convex set containing 0, and \( p_C(x) \) be the (unique) closest to \( x \) point from \( C \). Then, see, e.g., [BL, Sect.3.2], the metric projection \( p_C \) is Lipschitz and

\[ \|p_C(x) - p_C(y)\|_2 \leq \|x - y\|_2. \]  
(9.3)

Using this we introduce a linear operator \( E \) given on \( \text{Lip}_0(C) \) by

\[ (Ef)(x) := (f \circ p_C)(x), \quad x \in l^n_2. \]  
(9.4)

Since \( p_C \) is identity on \( C \) and \( p_C(0) = 0 \) as \( 0 \in C \), this operator belongs to \( \text{Ext}(C, l^n_2) \).

Moreover, by (9.3)

\[ \|(Ef)(x) - (Ef)(y)\|_2 \leq \|f\|_{\text{Lip}_0(C)} \|x - y\|_2, \]

i.e., \( \|E\| \leq 1 \), and (9.2) is established.

Using now the inequalities

\[ \|x\|_p \leq \|x\|_2 \leq n^{\frac{1}{2} - \frac{1}{p}} \|x\|_p \quad \text{for} \quad 2 \leq p \leq \infty, \]  
(9.5)

and

\[ n^{\frac{1}{2} - \frac{1}{p}} \|x\|_p \leq \|x\|_2 \leq \|x\|_p \quad \text{for} \quad 1 \leq p \leq 2 \]  
(9.6)

we derive from (9.2) the required upper bound:

\[ \lambda(l^n_2) \leq n^{\frac{1}{2} - \frac{1}{p}}. \]  
(9.7)

In order to prove the lower estimate we need the following result where Banach spaces are regarded as punctured metric spaces with \( m^* = 0 \).

**Proposition 9.1** Let \( Y \) be a linear subspace of a finite dimensional Banach space \( X \), and an operator \( E \) belongs to \( \text{Ext}(Y, X) \). Then there is a linear projection \( P \) from \( X \) onto \( Y \) such that

\[ \|P\| \leq \|E\|. \]  
(9.8)

**Proof.** We use an argument similar to that in [P, Remarks to §2]. First, we introduce an operator \( S : \text{Lip}_0(Y) \to \text{Lip}_0(X) \) given at \( z \in X \) by

\[ (Sf)(z) := \int_X \left\{ \int_Y [(Ef)(x + y + z) - (Ef)(x + y)]dy \right\} dx. \]  
(9.9)

Here \( f_A \ldots da \) is a translation invariant mean on the space \( l_\infty(A) \) of all bounded functions on an abelian group \( A \), see, e.g., [HR]. Since the function within [ ] is
bounded for every fixed $z$ (recall that $Ef \in Lip(X)$), this operator is well-defined. Moreover, as $\int_A da = 1$ we get
\[
\|Sf\|_{Lip_0(X)} \leq \|E\| \cdot \|f\|_{Lip_0(Y)}.
\] (9.10)

By translation invariance of $dx$ we then derive from (9.9) that
\[
(Sf)(z_1 + z_2) = (Sf)(z_1) + (Sf)(z_2), \quad z_1, z_2 \in X.
\]
Together with (9.10) and the equality
\[
\|f\|_{Lip_0(X)} = \|f\|_{X^*}, \quad f \in X^*,
\] (9.11)
this shows that $Sf$ belongs to $X^*$ and therefore $S$ maps $Lip_0(Y)$ linearly and continuously in $X^*$. Further, $Y^*$ is a linear subset of $Lip_0(Y)$ whose norm coincides with that induced from $Lip_0(Y)$. Therefore the restriction
\[
T := S|_{Y^*}
\]
is a linear bounded operator from $Y^*$ to $X^*$. Show that $T$ satisfies
\[
(Tf)(z) = f(z), \quad z \in Y,
\] (9.12)
i.e., $T$ is an extension from $Y^*$. To this end write
\[
(Tf)(z) = \int_X \left\{ \int_Y [(Ef)(x + y + z) - (Ef)(y + z)] dy \right\} dx + \int_X \left\{ \int_Y [(Ef)(y + z) - (Ef)(x + y)] dy \right\} dx.
\]
Since $z \in Y$ and $dy$ is translation invariant with respect to translations by elements of $Y$, we can omit $z$ in the first summand. Moreover, $(Ef)(y) = f(y)$ for $f \in Y^* \subset Lip_0(Y)$. Thus, the right-hand side is equal to
\[
\int_X \left\{ \int_Y [(Ef)(x + y) - f(y) + (Ef)(y + z) - (Ef)(x + y)] dy \right\} dx.
\]
Since $(Ef)(y + z) = f(y + z) = f(y) + f(z)$, this integral equals
\[
f(z) \int_X dx \int_Y dy = f(z),
\]
and (9.12) is done.

Consider now the conjugate to $T$ operator $T^*$ acting from $X^{**} = X$ to $Y^{**} = Y$. Since $T$ is a linear extension operator from $Y^*$, its conjugate is a projection onto $Y$. At last, (9.10) and (9.11) give for the norm of this projection the required estimate
\[
\|T^*\| = \|T\| \leq \|E\|.
\]

The proposition is proved. \qed
Proposition 9.2 Let $X$ be either $l^n_1$ or $l^n_\infty$. Then there is a subspace $Y \subset X$ such that $\dim Y = \lfloor n/2 \rfloor$ and its projection constant\(^6\) satisfies
\[
\pi(X, Y) \geq c_0 \sqrt{n} \tag{9.13}
\]
with $c_0$ independent of $n$.

Proof. The inequality follows from Theorem 1.2 of the paper [S] by Sobczyk with the optimal $c_0$ greater than $1/4$. \(\square\)

We now complete the proof of Theorem 2.17. Applying Propositions 9.1 and 9.2 we get for an arbitrary $E \in \text{Ext}(Y, l^n_1)$ the inequality
\[
||E|| \geq c_0 \sqrt{n} \tag{9.14}
\]
with $c_0 > 0$ independent of $n$. A similar estimate is valid for $E \in \text{Ext}(Y^\perp, l^n_\infty)$, as well. Hence for $p = 1, \infty$
\[
\lambda_{\text{conv}}(l^n_p) \geq c_0 \sqrt{n}. \tag{9.15}
\]
Using this estimate for $p = 1$ and applying a similar to (9.6) inequality comparing $||x||_1$ and $||x||_p$ we get for $1 \leq p \leq 2$ the estimate
\[
\lambda(l^n_p) \geq c_0 n^{\frac{1}{p} - \frac{1}{2}}.
\]
Then using (9.15) for $p = \infty$ and a similar to (9.5) inequality comparing $||x||_p$ and $||x||_\infty$ we get for $2 \leq p \leq \infty$ the inequality
\[
\lambda(l^n_p) \geq c_0 n^{\frac{1}{p} - \frac{1}{2}}.
\]

The proof of the theorem is complete. \(\square\)

10 Proof of Theorem 2.16

A metric graph without $\mathcal{LE}$. A construction presented here may be of independent interest. To formulate the result we recall several notions of Graph Theory, see, e.g., [R].

Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a graph with the sets of vertices $\mathcal{V}$ and edges $\mathcal{E}$. We consider below only simple (i.e., without loops and double edges) and connected graphs. The latter means that for every two vertices $v', v''$ there is a path\(^7\) whose head and tail are $v'$ and $v''$, respectively. The distance $d_\Gamma(v', v'')$ between two vertices $v', v''$ is the length (number of edges) of a shortest path between them. To introduce the metric graph $\mathcal{M}_\Gamma$ associated with $\Gamma$ we regard every $e \in \mathcal{E}$ as the unit interval of $\mathbb{R}$ and equip the 1-dimensional CW complex obtained in this way with the path (length) metric generated by $d_\Gamma$. Thus the restriction of this metric to $\mathcal{V}$ coincides with $d_\Gamma$, and every edge is isometric to $[0, 1] \subset \mathbb{R}$, see, e.g., [BH] for further details.

\[^6\] $\pi(Y, X) := \inf ||P||$ where $P$ runs through all linear projections from $X$ onto $Y$.

\[^7\] i.e., an alternative sequence $\{v_0, e_1, v_1, \ldots, e_n, v_n\}$ with pairwise distinct edges $e_i$ such that $e_i$ joins $v_{i-1}$ and $v_i$. Vertices $v_0$ and $v_n$ are the head and the tail of this path.
Proposition 10.1 There exists a graph $\Gamma = (V, E)$ and a subset $S \subset V$ such that

(a) the degree of every vertex $v \in V$ is at most 3;

(b) $\text{Ext}(S, M_\Gamma) = \emptyset$.

Remark 10.2 In fact, we will prove that

$$\text{Ext}(S, V) = \emptyset. \quad (10.1)$$

Here and above $S$ and $V$ are regarded as metric subspaces of $M_\Gamma$. Since $\lambda(S, M_\Gamma) \geq \lambda(S, V)$, equality (10.1) implies statement (b).

Proof. Our argument is based on the following result.

Let $\mathbb{Z}_p^n$ denote $\mathbb{Z}^n$ regarding as a metric subspace of $l_p^n$.

Lemma 10.3 There is an absolute constant $c_0 > 0$ such that for every $n$

$$\lambda(\mathbb{Z}_1^n) \geq c_0 \sqrt{n}. \quad (10.2)$$

Proof. We apply Theorem 2.1 with $M := l_1^n$, $S := \mathbb{Z}_1^n$ and the dilation $\delta : x \mapsto \frac{1}{2}x$. Then $\delta(S) = \frac{1}{2} \mathbb{Z}^n \supset \mathbb{Z}^n$ and the Lipschitz constants of $\delta$ and $\delta^{-1}$ equal $\frac{1}{2}$ and 2, respectively. Consequently, the assumptions of Theorem 2.1 hold for this case and therefore

$$\lambda(l_1^n) = \lambda(\mathbb{Z}_1^n). \quad (10.3)$$

Since $\lambda(l_1^n) \geq \lambda_{\text{conv}}(l_1^n)$, it remains to apply to (10.3) the lower estimate of Theorem 2.17 with $p = 1$. \qed

Remark 10.4 The same argument gives

$$\lambda(l_p^n) = \lambda(\mathbb{Z}_p^n) \geq c_0 n^{\frac{1}{2} - \frac{1}{p}}. \quad (10.4)$$

Let now $\mathbb{Z}_1^n(l)$ denote the discrete cube of the lengthside $l \in \mathbb{N}$, i.e.

$$\mathbb{Z}_1^n(l) := \mathbb{Z}_1^n \cap [-l, l]^n.$$ 

Lemma 10.5 For every $n \in \mathbb{N}$ there is an integer $l = l(n)$ and a subset $S_n \subset \mathbb{Z}_1^n(l)$ such that

$$\lambda(S_n, \mathbb{Z}_1^n(l)) \geq c_1 \sqrt{n} \quad (10.5)$$

with $c_1 > 0$ independent of $n$. 

---

8i.e., the number of edges incident to $v$. This is denoted by $\text{deg } v$
**Proof.** By Corollary 2.2

\[ \lambda(\mathbb{Z}^n_1) = \sup_F \lambda(F) \]

where \( F \) runs through all finite subsets \( F \subset \mathbb{Z}^n \). On the other hand

\[ \lambda(\mathbb{Z}^n_1) \geq \sup_{l \in \mathbb{N}} \lambda(\mathbb{Z}^n_1(l)) . \]

At last, Corollary 2.2 gives

\[ \lambda(\mathbb{Z}^n_1(l)) = \sup_{F \subset \mathbb{Z}^n_1(l)} \lambda(F) . \tag{10.6} \]

These three relations imply that

\[ \lambda(\mathbb{Z}^n_1) = \sup_{l \in \mathbb{N}} \lambda(\mathbb{Z}^n_1(l)) . \]

Together with (10.2) this gives for some \( l = l(n) \)

\[ \lambda(\mathbb{Z}^n_1(l(n))) > \frac{c_0}{2} \sqrt{n} . \]

Applying now (10.6) with \( l := l(n) \) we then find \( S_n \subset \mathbb{Z}^n_1(l(n)) \) such that for \( l = l(n) \)

\[ \lambda(S_n, \mathbb{Z}^n_1(l)) := \inf\{ ||E|| : E \in Ext(S_n, \mathbb{Z}^n_1(l)) \} \geq \frac{c_0}{2} \sqrt{n} . \]

The result is done. \( \square \)

Let now \( G_n := (\mathbb{Z}^n, \mathcal{E}^n) \) be a graph whose set of edges is given by

\[ \mathcal{E}^n := \{ (x, y) : x, y \in \mathbb{Z}^n, \ ||x - y||_{l^1} = 1 \} . \]

Let \( \Gamma_n := (V_n, E_n) \) be a subgraph of \( G_n \) whose set of vertices is

\[ V_n := \mathbb{Z}^n \cap [-l(n), l(n)]^n \]

where \( l(n) \) is defined in Lemma 10.5. So the set \( S_n \) from (10.5) contains in \( V_n \). The metric graph \( \mathcal{M}_{\Gamma_n} \) is then a (metric) subspace of the space \( l^1_\mathbb{N} \), but it also can and will be regarded below as a subspace of \( l^2_\mathbb{N} \) with the path metric induced by the Euclidean metric.

**Lemma 10.6** There is a finite connected graph \( \hat{\Gamma}_n := (\hat{V}_n, \hat{E}_n) \) and a subset \( \hat{S}_n \subset \hat{V}_n \) such that

(a) for every vertex \( v \in \hat{V}_n \)

\[ \deg v \leq 3 ; \tag{10.7} \]

(b) the underlying set of the metric graph \( \mathcal{M}_{\hat{\Gamma}_n} \) is a subset of the \( n \)-dimensional Euclidean space and its metric is the path metric generated by the Euclidean one;

47
there is an absolute constant $c > 0$ such that for every $n$

$$\lambda(\hat{S}_n, \hat{V}_n) \geq c\sqrt{n} ;$$

(10.8)

here $\hat{S}_n$ and $\hat{V}_n$ are regarded as subspaces of $\mathcal{M}_{\Gamma_n}$.

**Proof.** Let $\epsilon := \frac{1}{q\sqrt{2}}$ for some natural $q \geq 2$. For a vertex $v \in V_n \subset l^2_0$ of $\Gamma_n$, let $S(v)$ stand for the $(n - 1)$-dimensional Euclidean sphere centered at $v$ and of radius $\frac{\epsilon}{1+2\epsilon}$. Then $S(v)$ intersects $N(v)$ ($n \leq N(v) \leq 2n$) edges of $\mathcal{M}_{\Gamma_n}$ at some points denoted by $p_i(v)$, $i = 0, \ldots, N(v) - 1$. The ordering is choosing in such a way that any interval $\text{conv}\{p_i(v), p_{i+1}(v)\}$ does not belong to $\mathcal{M}_{\Gamma_n}$ (here and below $p_{N(v)}(v)$ is identified with $p_0(v)$). Let us introduce a new graph with the set of vertices \{\(p_i(v) : i = 0, \ldots, N(v) - 1, \ v \in V_n\}\} and the set of edges defined as follows. This set contains edges determined by all pairs \((p_i(v), p_{i+1}(v))\) with $0 \leq i \leq N(v) - 1$ and $v \in V_n$ and, moreover, all edges formed by all pairs \((p_i(v'), p_j(v''))\) where $v', v''$ are the head and the tail of an edge $e \in E_n$, and $i \neq j$ satisfy the condition

$$\text{conv}\{p_i(v'), p_j(v'')\} \subset e ;$$

(10.9)

here $e$ is regarded as a subset (interval) of $\mathcal{M}_{\Gamma_n}$. In this way we obtain a new graph (and an associated metric space with the path metric induced by the Euclidean one) whose vertices have degree at most 3.

The lengths of edges $(p_i(v), p_{i+1}(v))$ of this graph are $\frac{1}{q(1+2\epsilon)}$ while the lengths of edges $(p_i(v'), p_j(v''))$ satisfying (10.9) are $\frac{1}{1+2\epsilon}$. Then we add new vertices (and edges) by inserting into every edge satisfying (10.9) the $(q - 1)$ equally distributed new vertices. (Note that every new vertex obtained in this way has degree 2.) At last, by dilation (with respect to $0 \in l^2_0$) with factor $q(1+2\epsilon)$ we obtain a new graph $\tilde{\Gamma}_n := (\tilde{E}_n, \tilde{V}_n)$ whose edges are of length one, and such that

$$\text{deg} v \leq 3, \quad v \in \tilde{V}_n, \quad \text{and} \quad \exists v_n \in \tilde{V}_n : \text{deg} v_n = 2 .$$

Moreover, the metric graph $\mathcal{M}_{\tilde{\Gamma}_n}$ is a (metric) subspace of $l^m$ equipped with the path metric induced by that of $l^m$.

It remains to introduce the required subset $\hat{S}_n \subset \hat{V}_n$. At this end we define a map $i : V_n \rightarrow \hat{V}_n$ by

$$i(v) := q(1 + 2\epsilon) \cdot p_0(v) , \quad v \in V_n .$$

Recall that $p_0(v)$ is a point of the sphere $S(v) \subset l^2_0$. Since our construction depends on $\epsilon$ continuously, and $\Gamma_n$ is a finite graph, we clearly have for a sufficiently small $\epsilon$

$$(q/2)(1+2\epsilon) \cdot d(v', v'') \leq \tilde{d}(i(v'), i(v'')) \leq 2q(1+2\epsilon)(d(v', v''), \quad v', v'' \in V_n .$$

(10.10)

Here $d, \tilde{d}$ are the metrics of $\mathcal{M}_{\Gamma_n}$ and $\mathcal{M}_{\tilde{\Gamma}_n}$, respectively. Note now that the constant $\lambda(S_n, V_n)$ does not change if we replace the metric $d$ by $q(1+2\epsilon) \cdot d$. Therefore (10.10) and (10.5) imply the estimate

$$\lambda(i(S_n), i(V_n)) \geq \frac{1}{4} \lambda(S_n, V_n) \geq \frac{c_1}{4} \sqrt{n} .$$

48
We set \( \hat{S}_n := i(S_n) \cup \{v_n\} \) where \( v_n \in \hat{V}_n \) satisfies \( \text{deg} v_n = 2 \). Noting that \( i(V_n) \subset \hat{V}_n \) and there is an \( L \in \text{Ext}(i(S_n), \hat{S}_n) \) such that \( ||L|| \leq 2 \) (cf. the proof of Lemma 8.1), we get from here

\[
\lambda(\hat{S}_n, \hat{V}_n) \geq \frac{1}{2} \lambda(i(S_n), i(V_n)) \geq \frac{c_1}{8} \sqrt{n} .
\]

This proves (10.8) and the lemma. □

Let now \( \hat{\Gamma}_n = (\hat{V}_n, \hat{E}_n) \) and \( \hat{S}_n \subset \hat{V}_n \) be as in Lemma 10.6. Then \( \mathcal{M}_{\hat{\Gamma}_n} \) is a subset of the space \( l_2^n \) equipped with the path metric generated by the Euclidean metric. We now identify \( l_2^n \) with its isometric copy \( P_n \), an \( n \)-dimensional plane of the Hilbert space \( l_2(\mathbb{N}) \) orthogonal to the line \( \{x \in l_2(\mathbb{N}) : x_i = 0 \text{ for } i > 1\} \) and intersecting this line at the point \( v_n := (n, 0, \ldots) \). Then \( \mathcal{M}_{\hat{\Gamma}_n} \) is a subset of \( P_n \subset l_2(\mathbb{N}) \) equipped with the path metric generated by the metric of \( l_2(\mathbb{N}) \). Using an appropriate translation we also may and will assume that

\[
v_n \subset \hat{S}_n \text{ and } \text{deg} v_n = 2 .
\]

Note that

\[
dist(P_n, P_{n+1}) = ||v_n - v_{n+1}|| = 1
\]

and therefore the sets \( \mathcal{M}_{\hat{\Gamma}_n} \) are pairwise disjoint.

Introduce now the set of vertices and edges of the required graph \( \Gamma = (\mathcal{V}, \mathcal{E}) \) by letting

\[
\mathcal{V} := \bigcup_{n \in \mathbb{N}} \hat{V}_n
\]

and, moreover,

\[
\mathcal{E} := \bigcup_{n \in \mathbb{N}} (\hat{E}_n \cup e_n)
\]

where \( e_n \) denotes the new edge joining \( v_n \) with \( v_{n+1} \).

This definition and Lemma 10.6 imply that

\[
\text{deg} v \leq 3, \quad v \in \mathcal{V} ,
\]

and so assertion (a) of Proposition 10.1 holds.

Set now

\[
S := \bigcup_{n \in \mathbb{N}} \hat{S}_n .
\]

We claim that this \( S \) satisfies (10.1) and assertion (b) of the proposition. If, to the contrary, there is an operator \( E \in \text{Ext}(S, \mathcal{V}) \), we choose \( n \) so that

\[
c\sqrt{n} > ||E||
\]

with the constant \( c \) from (10.8). Introduce for this \( n \) an operator \( T_n \) given on \( f \in \text{Lip}(\hat{S}_n) \) by

\[
(T_n f)(v) := \begin{cases} 
  f(v), & \text{if } v \in \hat{S}_n \\
  f(v_n), & \text{if } v \in S \setminus \hat{S}_n 
\end{cases}
\]

49
Show that $T_n$ maps $\text{Lip}(\hat{S}_n)$ into $\text{Lip}(S)$ and its norm is 1. To accomplish this we have to show that for $v' \in \hat{S}_n$ and $v'' \in S \setminus \hat{S}_n$

$$|(T_n f)(v') - (T_n f)(v'')| \leq ||f||_{\text{Lip}(\hat{S}_n)} d(v',v'').$$

But the left-hand side here is

$$|f(v') - f(v_n)| \leq ||f||_{\text{Lip}(\hat{S}_n)} d(v',v_n)$$

and $d(v',v_n) \leq d(v',v'')$ by the definition of $S$ and the metric $d$ of $M_\Gamma$. This implies the required statement for $T_n$.

Finally, introduce the restriction operator $R_n : \text{Lip}(\mathcal{V}) \to \text{Lip}(\hat{V}_n)$ by

$$R_n f = f|_{\hat{V}_n}$$

and set $E_n := R_n E T_n$. Then $E_n \in \text{Ext}(\hat{S}_n, \hat{V}_n)$ and its norm is bounded by $||E||$. This immediately implies that

$$\lambda(\hat{S}_n, \hat{V}_n) \leq ||E||$$

in contradiction with (10.8) and our choice of $n$, see (10.11).

So we establish (10.1) and complete the proof of the proposition. □

**Two-dimensional metric space without $\mathcal{L}\mathcal{E}$.** In order to complete the proof of Theorem 2.16 we have to construct a connected two-dimensional metric space $M$ of bounded geometry so that

$$\text{Ext}(S,M) = \emptyset$$

for some its subspace $S$. In fact, $M$ will be a Riemannian manifold (with the geodesic (inner) metric). This will be done by sewing surfaces of three types along the metric graph $M_\Gamma$ of the previous part. At the first stage we introduce an open cover of $M_\Gamma$ by balls and a related coordinate system and partition of unity. To simplify evaluation we replace the metric of $M_\Gamma$ by $\bar{d}_\Gamma := 4d_\Gamma$. So every edge $e \subset M_\Gamma$ is a closed interval of length 4. (Note that the abstract graph $\Gamma = (\mathcal{V}, \mathcal{E})$ remains unchanged.) The required cover $\{ B(v) \}_{v \in \mathcal{V}}$ is given by

$$B(v) := \{ m \in M_\Gamma : \bar{d}(m,v) < 3 \} .$$

(10.12)

So $B(v)$ is the union of at most three intervals of length 3 each of which has a form $e \cap B(v)$ where every $e$ belongs to the set of edges $\mathcal{E}(v)$ incident to $v$. We numerate these intervals by numbers from the set $\omega \subset \{ 1,2,3 \}$ where $\omega = \{ 1 \}$, \{1,2\} or \{1,2,3\}, if $\deg v = 1$, 2 or 3, respectively. This set of indices will be denoted by $\omega(v)$ and $i(e,v)$ (briefly, $i(e)$) will stand for the number of $e \cap B(v)$ in this numeration.

We then introduce a coordinate system $\psi_v : B(v) \to \mathbb{R}^3$, $v \in \mathcal{V}$, of $M_\Gamma$ as follows. Let $\{ b_1, b_2, b_3 \}$ be the standard basis in $\mathbb{R}^3(= \bar{b}_3^2)$. We define $\psi_v$ as the isometry sending $v$ to 0 and each interval $e \cap B(v)$, $e \in \mathcal{E}(v)$, to the interval $\{ t b_i : 0 \leq t < 3 \}$ of the $x_i$-axis with $i := i(e)$. 

50
Now we introduce the desired partition of unity \( \{ \rho_v \}_{v \in \mathcal{V}} \) subordinate to the cover \( \{ B(v) \}_{v \in \mathcal{V}} \). To this end one first considers a function \( \tilde{\rho}_v : \psi_v(B(v)) \to [0, 1] \) with support strictly inside of its domain such that \( \tilde{\rho}_v = 1 \) in the neighbourhood \( \cup_{e \in E(v)} \{ b_{i(e)} : 0 \leq t \leq 1 \} \) of 0 and is \( C^\infty \)-smooth outside 0. This function gives rise to a function \( \rho_v : \mathcal{M}_T \to [0, 1] \) equals \( \tilde{\rho}_v \circ \psi_v \) on \( B(v) \) and 0 outside. It is important to note that there exist only three types of the functions \( \tilde{\rho}_v \) corresponding to the types of the balls \( B(v) \). Finally we determine the required partition of unity by setting

\[
\rho_v := \frac{\hat{\rho}_v}{\sum_v \hat{\rho}_v} , \quad v \in \mathcal{V} .
\] (10.13)

At the second stage we introduce the building blocks of our construction, \( C^\infty \)-smooth surfaces \( \Sigma_{\{1\}} \), \( \Sigma_{\{1,2\}} \) and \( \Sigma_{\{1,2,3\}} \) embedded in \( \mathbb{R}^3 \). We begin with a \( C^\infty \)-function \( f : [-1,3) \to [0,1] \) given by

\[
f(t) := \begin{cases} 
\sqrt{1-t^2}, & \text{if } -1 \leq t \leq 1 - \epsilon := \frac{3}{4} \\
\sqrt{1-10t^2}, & \text{if } 1 \leq t < 3
\end{cases}
\]

In the remaining interval \([1-\epsilon,1](:=[3/4,1]) \) \( f \) is an arbitrary decreasing function smoothly joining the given endpoint values. Then we introduce \( \Sigma_{\{1\}} \) as a surface of revolution

\[
\Sigma_{\{1\}} := \{(t, f(t) \cos \theta, f(t) \sin \theta) \in \mathbb{R}^3 : -1 \leq t < 3, \ 0 \leq \theta < 2\pi \} ,
\] (10.14)

the result of rotating the graph of \( f \) about the \( x_1 \)-axis. By the definition of \( f \) this surface is the union of the unit sphere \( S^2 \subset \mathbb{R}^3 \) with the spherical hole \( S(b_1) \) centered at \( b_1 \) and of the curvilinear (near the bottom) cylinder \( T(b_1) \) attached to the circle \( \partial S(b_1) \) (of radius \( \sqrt{1-(1-\epsilon)^2} \)). In turn, \( T(b_1) \) is the union of the curvilinear cylinder and that of circular. The latter, denoted by \( \hat{T}(b_1) \), is of height 2.

Similarly \( \Sigma_{\{1,2\}} \) and \( \Sigma_{\{1,2,3\}} \) are the unions of the unit sphere \( S^2 \) with the holes \( S(b_i) \) and of the cylinders \( T(b_i) \) attached to \( \partial S(b_i) \); here \( i = 1,2 \) or \( i = 1,2,3 \), respectively. Note that each \( T(b_i) \) with \( i \neq 1 \) is obtained from \( T(b_1) \) by a fixed turn around the \( x_j \)-axis, \( j \neq 1, i \). This determines the isometry

\[
J_i : \hat{T}(b_i) \to \hat{T}(b_1) , \quad i = 1,2,3
\] (10.15)

where \( J_1 \) stands for the identity map.

Using these blocks and the previous notations for \( B(v) \) we now assign to every \( v \in \mathcal{V} \) the smooth surface

\[
\Sigma(v) := \Sigma_{\omega(v)} \subset \mathbb{R}^3
\] (10.16)

We denote by \( S(v) \subset \Sigma_{\omega(v)} \) the corresponding sphere \( S^2 \) with holes \( \{ S(b_i) \}, i \in \omega(v) \), and by \( T(e), e \in E(v) \), the corresponding curvilinear cylinder \( (= T(b_{i(e)}) \)\). The circular part of the latter is denoted by \( \hat{T}(e) \) and the corresponding isometry of \( \hat{T}(e) \) onto \( \hat{T}(b_1) \) is denoted by \( J_e (= J_{i(e)})^9 \). Finally, we equip \( \Sigma(v) \) with the

\(^9\)Since \( e \) belongs to two different sets, say, \( E(v) \) and \( E(v') \), we will also write \( T(e, v), \hat{T}(e, v) \) and \( J_{e,v} \) to distinguish them from the corresponding objects determined by \( e \) as an element of \( E(v') \).
Riemannian metric induced by the canonical Riemannian structure of $\mathbb{R}^3$, and denote the corresponding geodesic metric by $d_v$.

According to our construction there exists for every $\Sigma(v)$ a continuous surjection $p_v : \Sigma(v) \to \psi_v(B(v))$ such that the restriction of $p_v$ to every cylinder $\hat{T}(e, v)$, $e \in \mathcal{E}(v)$, is the orthogonal projection onto its axis $I_e := \{tb_i(v) : 1 < t < 3\}$.

Using this and the polar coordinate $\theta$ from (10.14) we then equip each $x \in \hat{T}(e, v)$ with the cylindrical coordinates:

$$r(x) := \psi_v^{-1}(p_v(x)) \ , \ \theta(x) := \theta(J_{e,v}(x)) \ .$$

Now we define the required smooth surface $M$ as the quotient of the disjoint union $\sqcup_{v \in V} \Sigma(v)$ by the equivalence relation:

$$x \sim y \ , \ \text{if} \ x \in \hat{T}(e, v_0) \ , \ y \in \hat{T}(e, v_1) \ \text{for some} \ e \in \mathcal{E}(v_0) \cap \mathcal{E}(v_1) \ \text{and}$$

$$(r(x), \theta(x)) = (r(y), \theta(y)) \ .$$

Let $\pi : \sqcup_{v \in V} \Sigma(v) \to M$ be the quotient projection. Then $\{\pi(\Sigma(v))\}_{v \in V}$ is an open cover of $M$. Using the partition of unity (10.13) we now introduce a partition of unity subordinate to this cover as follows. Define a function $\hat{\phi}_v : \Sigma(v) \to [0, 1]$ as a pullback of the function $\rho_v : B(v) \to [0, 1]$ given by

$$\hat{\phi}_v := \rho_v(\psi_v^{-1}(p_v(x))) \ , \ x \in \Sigma(v). \quad (10.17)$$

By the definitions of all functions used here, the function $\hat{\phi}_v$ is $C^\infty$-smooth in every $\hat{T}(e, v)$ and is equal to 1 outside $\sqcup_{e \in \mathcal{E}(v)} \hat{T}(e, v)$. In particular, $\hat{\phi}_v$ is $C^\infty$-smooth and its support is strictly inside $\Sigma(v)$. Since $\pi|_{\Sigma(v)}$ is a smooth embedding, the function $\phi_v : M \to [0, 1]$ equals $\hat{\phi}_v \circ \pi$ on $\pi(\Sigma(v))$ and 0 outside is $C^\infty$-smooth. By (10.17) the family $\{\phi_v\}_{v \in V}$ forms the required partition of unity subordinate to the cover $\{\pi(\Sigma(v))\}_{v \in V}$.

Using this we now determine a Riemannian metric tensor $R$ of $M$ by

$$R := \sum_{v \in V} \phi_v \cdot (\pi^{-1})^*(R_v)$$

where $R_v$ stands for the metric tensor of $\Sigma(v)$. If now $d$ is the geodesic (inner) metric of $M$ determined by $R$, then the metric space $(M, d)$ is clearly of bounded geometry, because in this construction we used the objects of only three different types.

It remains to find a subspace $\bar{S}$ of $(M, d)$ such that

$$Ext(M, \bar{S}) = \emptyset \ . \quad (10.18)$$

To this end we first consider two the hole spheres $\pi(S(v_i)) \subset M$, $i = 1, 2$, such that $v_1$ and $v_2$ are joined by an edge. Let $m_i \in \pi(S(v_i))$ be arbitrary points, $i = 1, 2$. Then by the definition of the metric $d$ and by a compactness argument

$$0 < c \leq d(m_1, m_2) \leq C \quad (10.19)$$

52
where \( c, C \) are independent of \( m_i \) and \( v_i \). On the other hand in the space \( \mathcal{M}_\Gamma \)

\[
d_\Gamma(v_1, v_2) = 1
\]

(10.20)

for this choice of \( v_i \).

Let now \( m_a \in \pi(S(v_a)) \subset M, a \in \{A, B\} \), be arbitrary points and \( v_A, v_B \) are distinct and may not necessarily be joined by an edge. Let \( \{v_i\}_{i=1}^n \) be a path in the graph \( \Gamma \) joining \( v_A \) and \( v_B \) (here \( v_1 := v_A \) and \( v_n := v_B \)) such that

\[
d_\Gamma(v_A, v_B) = \sum_{i=1}^{n-1} d_\Gamma(v_i, v_{i+1}).
\]

Together with (10.19) and (10.20) this implies that

\[
ccd_\Gamma(v_A, v_B) \leq \sum_{i=1}^{n-1} d(m_i, m_{i+1}) \leq Cd_\Gamma(v_A, v_B).
\]

(10.21)

On the other hand, the definitions of \( M \) and \( d \) get

\[
\tilde{c} \cdot \sum_{i=1}^{n-1} d(m_i, m_{i+1}) \leq d(m_A, m_B) \leq \sum_{i=1}^{n-1} d(m_i, m_{i+1})
\]

(10.22)

with some \( \tilde{c} > 0 \) independent of \( m_i \)'s.

Introduce now a map \( T : \mathcal{V} \to M \) sending a point \( v \in \mathcal{V} \) to an arbitrary point \( T(v) \in \pi(S(v)) \). Because of (10.21) and (10.22) \( T \) is a quasi-isometric embedding of \( \mathcal{V} \subset \mathcal{M}_\Gamma \) into \( M \). We then define the required subset \( \tilde{S} \) as the image under \( T \) of the set \( S \subset \mathcal{V} \) for which \( \text{Ext}(S, \mathcal{V}) = \emptyset \), see (10.1). Then we have for \( \tilde{S} := T(S) \)

\[
\text{Ext}(\tilde{S}, M) = \emptyset.
\]

The proof of Theorem 2.16 is complete. \( \square \)

**Remark 10.7** Using the Nash embedding theorem one can realize the Riemannian manifold \( M \) as a \( C^\infty \)-surface in an open ball of \( \mathbb{R}^3 \) (with the Riemannian quadratic form induced from the canonical Riemannian structure of \( \mathbb{R}^3 \)).

**11 Proofs of Theorem 2.9 and Corollaries 2.13 and 2.15**

**Proof of Theorem 2.9.** Let \( \Gamma \) be the \( R \)-lattice and \( \mathcal{B} := \{B_R(\gamma)\}_{\gamma \in \Gamma} \). By the definition of an \( R \)-lattice, \( \mathcal{B} \) and \( \frac{1}{2}\mathcal{B} := \{B_{R/2}(\gamma)\}_{\gamma \in \Gamma} \) are covers of \( M \).

**Lemma 11.1** Multiplicity of \( \mathcal{B} \) is bounded by a constant \( \mu \) depending only on \( c_\Gamma \) and \( N = N_M \).
For convenience of the reader we outline the proof of this well-known fact.

Let $B_R(\gamma_i), 1 \leq i \leq k$, contain a point $m$. Then all $\gamma_i$ are in the ball $B_R(m)$. Since $d(\gamma_i, \gamma_j) > 2cR$, $i \neq j$, see Definition 2.7, any cover of $B_R(m)$ by balls of radius $cR$ separates $\gamma_i$, i.e., distinct $\gamma_i$ lie in the distinct balls. Hence cardinality of such a cover is at least $k$. On the other hand the doubling condition implies that there is a cover of $B_R(m)$ by balls of radius $cR$ and cardinality $N^s$ where $s := \lfloor \log_2 \frac{1}{cR} \rfloor + 1$. So multiplicity of $\mathcal{B}$ is bounded by $N^s$. □

Lemma 11.2 There is a partition of unity $\{\rho_\gamma\}_{\gamma \in \Gamma}$ subordinate to $\mathcal{B}$ such that

$$K := \sup_{\gamma} \|\rho_\gamma\|_{Lip(M)} < \infty$$

(11.1)

where $K$ depends only on $c_T$, $N = N_M$ and $R = R_M$.

Proof. Set

$$B_\gamma := B_R(\gamma) \quad \text{and} \quad ^cB_\gamma := M \setminus B_\gamma$$

and define

$$d_\gamma(m) := dist(m, ^cB_\gamma), \quad m \in M.$$  

It is clear that

$$\text{supp } d_\gamma \subset B_\gamma \quad \text{and} \quad \|d_\gamma\|_{Lip(M)} \leq 1.$$  

(11.2)

Let now $\phi : \mathbb{R}_+ \to [0, 1]$ be continuous, equal one on $[0, R/2]$, zero on $[R, \infty)$ and linear on $[R/2, R]$. Introduce the function

$$s := \sum_\gamma \phi \circ d_\gamma.$$  

(11.3)

By Lemma 11.1 only at most $\mu$ terms here are nonzero at every point. Therefore

$$\|s\|_{Lip(M)} \leq 2\mu\|\phi\|_{Lip(\mathbb{R}_+)} \sup_\gamma \|d_\gamma\|_{Lip(M)}$$

and by (11.2) and the definition of $\phi$ we get

$$\|s\|_{Lip(M)} \leq 4\mu/R.$$  

(11.4)

On the other hand, every $m \in M$ is contained in some ball $B_{R/2}(\gamma)$ of the cover $\frac{1}{2}\mathcal{B}$. For this $\gamma$

$$(\phi \circ d_\gamma)(m) \geq \phi(R/2) = 1$$

and therefore

$$s \geq 1.$$  

(11.5)

Introduce now the required partition by

$$\rho_\gamma := \frac{\phi \circ d_\gamma}{s}, \quad \gamma \in \Gamma.$$  

Then $\{\rho_\gamma\}$ is clearly a partition of unity subordinate to $\mathcal{B}$. Moreover, we have

$$|\rho_\gamma(m) - \rho_\gamma(m')| \leq \frac{\phi(d_\gamma(m)) - \phi(d_\gamma(m'))}{s(m)} + \frac{\phi(d_\gamma(m'))}{s(m) \cdot s(m')} \cdot |s(m) - s(m')|$$
and application of (11.5), (11.4) and (11.2) leads to the desired inequality

\[ \|\rho_\gamma\|_{\text{Lip}(M)} \leq \frac{2}{R}(2\mu + 1). \]

\[ \blacksquare \]

**Lemma 11.3**

\[ \text{Ext}(\Gamma, M) \neq \emptyset. \]

**Proof.** By the assumption (2.4) of the theorem, for every \( \gamma \in \Gamma \) there is a linear operator \( E_\gamma \in \text{Ext}(\Gamma \cap B_\gamma, B_\gamma) \) such that

\[ \|E_\gamma\| \leq \lambda R, \quad \gamma \in \Gamma. \quad (11.6) \]

Using this we introduce the required linear operator by

\[ Ef := \sum_{\gamma \in \Gamma} (E_\gamma f_\gamma) \rho_\gamma, \quad f \in \text{Lip}(\Gamma), \quad (11.7) \]

where \( \{\rho_\gamma\} \) is the partition of unity from Lemma 11.2 and \( f_\gamma := f|_{\Gamma \cap B_\gamma} \); here we assume that \( E_\gamma f_\gamma \) is zero outside of \( B_\gamma \). We have to show that

\[ Ef|_{\Gamma} = f|_{\Gamma} \quad (11.8) \]

and estimate \( \|Ef\|_{\text{Lip}(M)} \).

Given \( \hat{\gamma} \in \Gamma \) we can write

\[ (Ef)(\hat{\gamma}) = \sum_{B_\gamma \ni \hat{\gamma}} (E_\gamma f_\gamma)(\hat{\gamma}) \rho_\gamma(\hat{\gamma}). \]

Since \( E_\gamma \) is an extension from \( B_\gamma \cap \Gamma \) we get

\[ (E_\gamma f_\gamma)(\hat{\gamma}) = f_\gamma(\hat{\gamma}) = f(\hat{\gamma}). \]

Moreover, \( \sum_{B_\gamma \ni \hat{\gamma}} \rho_\gamma(\hat{\gamma}) = 1 \), and (11.8) is done.

To estimate the Lipschitz constant of \( Ef \), we extend \( E_\gamma f_\gamma \) outside of \( B_\gamma \) so that the (non-linear) extension \( F_\gamma \) satisfies

\[ \|F_\gamma\|_{\text{Lip}(M)} = \|E_\gamma f_\gamma\|_{\text{Lip}(B_\gamma)}. \quad (11.9) \]

Since \( \rho_\gamma F_\gamma = \rho_\gamma E_\gamma f_\gamma \), we have

\[ Ef = \sum_\gamma F_\gamma \rho_\gamma. \quad (11.10) \]

Given \( \hat{\gamma} \in \Gamma \) introduce a function \( G_{\hat{\gamma}} \) by

\[ G_{\hat{\gamma}} := \sum_\gamma (F_\gamma - F_{\hat{\gamma}}) \rho_\gamma := \sum_\gamma F_{\gamma \hat{\gamma}} \rho_\gamma. \quad (11.11) \]

Then we can write for every \( \hat{\gamma} \)

\[ Ef = F_{\hat{\gamma}} + G_{\hat{\gamma}}. \quad (11.12) \]
It follows from (11.6) and (11.9) that

$$||F_\gamma||_{Lip(M)} \leq \lambda_R ||f||_{Lip(\Gamma)}.$$  \hspace{1cm} (11.13)

Prove now that

$$|F_{\gamma\hat{\gamma}}(m)| \leq 4R\lambda_R||f||_{Lip(\Gamma)}, \quad m \in B_\gamma \cap B_{\hat{\gamma}}.$$  \hspace{1cm} (11.14)

In fact, we have for these \(m\)

$$|F_{\gamma\hat{\gamma}}(m)| = |(E_\gamma f_\gamma - E_{\hat{\gamma}} f_{\hat{\gamma}})(m)| \leq |f(\gamma) - f(\hat{\gamma})| + |(E_\gamma f_\gamma)(m) - (E_{\hat{\gamma}} f_{\hat{\gamma}})(\gamma)| + |(E_{\hat{\gamma}} f_{\hat{\gamma}})(m) - (E_{\hat{\gamma}} f_{\hat{\gamma}})(\hat{\gamma})|.$$  

Using now (11.6) to estimate the right-hand side we get

$$|F_{\gamma\hat{\gamma}}(m)| \leq \lambda_R||f||_{Lip(\Gamma)}(d(\gamma, \hat{\gamma}) + d(m, \gamma) + d(m, \hat{\gamma})) \leq 4R\lambda_R||f||_{Lip(\Gamma)}.$$  

We apply this to estimate

$$\Delta G_{\hat{\gamma}} := G_{\hat{\gamma}}(m) - G_{\hat{\gamma}}(m')$$  

provided that \(m, m' \in B_{\hat{\gamma}}\). We get

$$|\Delta G_{\hat{\gamma}}| \leq \sum_{B_{\gamma} \cap B_{\hat{\gamma}} \ni m} |\Delta \rho_\gamma| \cdot |F_{\gamma\hat{\gamma}}(m)| + \sum_{B_{\gamma} \cap B_{\hat{\gamma}} \ni m'} \rho_\gamma(m') \cdot |\Delta F_{\gamma\hat{\gamma}}|$$

(here \(\Delta \rho_\gamma\) and \(\Delta F_{\gamma\hat{\gamma}}\) are defined similarly to \(\Delta G_{\hat{\gamma}}\)). The first sum is estimated by (11.14), (11.1) and Lemma 11.1, while the second sum is at most \(2\lambda_R||f||_{Lip(\Gamma)}d(m, m')\) by (11.9) and (11.6). This leads to the estimate

$$|\Delta G_{\hat{\gamma}}| \leq (8RK\mu + 2) \cdot \lambda_R \cdot ||f||_{Lip(\Gamma)}d(m, m'), \quad m, m' \in B_{\hat{\gamma}}.$$  

Together with (11.13) this gives for these \(m, m'\):

$$|(Ef)(m) - (Ef)(m')| \leq C||f||_{Lip(\Gamma)}d(m, m').$$  \hspace{1cm} (11.15)

Here and below \(C\) denotes a constant depending only on the basic parameters, that may change from line to line.

It remains to prove (11.15) for \(m, m'\) belonging to distinct balls \(B_\gamma\). Let \(m \in B_\gamma\) and \(m'\) be a point of some \(B_{R/2}(\hat{\gamma})\) from the cover \(\frac{1}{2}B\). Then \(m \in B_\gamma \setminus B_{\hat{\gamma}}\) and so

$$d(m, m') \geq R/2.$$  \hspace{1cm} (11.16)

Using now (11.12) we have

$$|(Ef)(m) - (Ef)(m')| \leq |F_{\gamma}(m) - F_{\hat{\gamma}}(m')| + |G_{\gamma}(m) - G_{\hat{\gamma}}(m')| := I_1 + I_2.$$  

By the definition of \(F_{\gamma}\), we then get

$$I_1 \leq |f(\gamma) - f(\hat{\gamma})| + |(E_\gamma f_\gamma)(m) - (E_{\hat{\gamma}} f_{\hat{\gamma}})(\gamma)| + |(E_{\hat{\gamma}} f_{\hat{\gamma}})(m') - (E_{\hat{\gamma}} f_{\hat{\gamma}})(\hat{\gamma})|.$$
Together with (11.6) this leads to the estimate
\[ I_1 \leq \lambda R \| f \|_{Lip(\Gamma)} (d(\gamma, \hat{\gamma}) + d(m, \gamma) + d(m', \hat{\gamma})) \leq 2\lambda R \| f \|_{Lip(\Gamma)} (d(m, m') + d(m, \gamma) + d(m', \hat{\gamma})). \]

Since \( d(m, \gamma) + d(m', \hat{\gamma}) \leq 2R \leq 4d(m, m') \) by (11.16), we therefore have
\[ I_1 \leq C \| f \|_{Lip(\Gamma)} d(m, m'). \] (11.17)

To estimate \( I_2 \), note that for \( m \in B_\gamma \)
\[ G_\gamma(m) = \sum_{B_{\gamma'} \cap B_\gamma \ni m} (\rho_{\gamma'} F_{\gamma'\gamma})(m). \]

In combination with (11.14) and (11.16) this gives
\[ |G_\gamma(m)| \leq 4\lambda R \| f \|_{Lip(\Gamma)} \leq C \| f \|_{Lip(\Gamma)} d(m, m'). \]

The same argument estimates \( G_{\hat{\gamma}}(m') \) for \( m' \in B_{\hat{\gamma}} \). Hence
\[ I_2 \leq |G_\gamma(m)| + |G_{\hat{\gamma}}(m')| \leq C \| f \|_{Lip(\Gamma)} d(m, m'). \]

Together with (11.17) and (11.15) this leads to the inequality
\[ \| Ef \|_{Lip(M)} \leq C \| f \|_{Lip(\Gamma)}. \]

Hence \( E \) is an operator from \( Ext(\Gamma, M) \).

We now in a position to complete the proof of Theorem 2.9. According to Theorem 2.1 we have to show that
\[ \sup_F \lambda(F) < \infty \] (11.18)

where \( F \) runs through all finite point subspaces of \( M \). To this end consider a “\( \Gamma \)-envelope” of such \( F \) given by
\[ \hat{F} := \{ \gamma \in \Gamma : B_\gamma \cap F \neq \emptyset \}. \]

Then \( \{ B_\gamma : \gamma \in \hat{F} \} \subset B \) is an open cover of \( F \). By assumption (2.4) of the theorem for every \( \gamma \in \hat{F} \) there is an operator \( E_\gamma \in Ext(F \cap B_\gamma, B_\gamma) \) such that
\[ \| E_\gamma \| \leq \lambda R. \]

Introduce now a linear operator \( T \) given on \( f \in Lip(F) \) by
\[ (Tf)(\gamma) := (E_\gamma f_\gamma)(\gamma), \quad \gamma \in \hat{F} \]

where \( f_\gamma := f|_{B_\gamma \cap F} \). Show that
\[ T : Lip(F) \to Lip(\hat{F}) \quad \text{and} \quad \| T \| \leq \lambda R(2/c_\Gamma + 1). \] (11.19)

57
Actually, let $\gamma_i \in \hat{F}$ and $m_i \in B_{\gamma_i} \cap F$, $i = 1, 2$. Then $(E_{\gamma_i}f_{\gamma_i})(m_i) = f(m_i)$ and 

$$|(Tf)(\gamma_1) - (Tf)(\gamma_2)| \leq \sum_{i=1,2} |(E_{\gamma_i}f_{\gamma_i})(\gamma_i) - (E_{\gamma_i}f_{\gamma_i})(m_i)| + |f(m_1) - f(m_2)| \leq \lambda_R \|f\|_{Lip(F)}(d(\gamma_1, m_1) + d(\gamma_2, m_2) + d(m_1, m_2)) .$$

The sum in the brackets does not exceed $2R + d(m_1, m_2) \leq 4R + d(\gamma_1, \gamma_2)$. Moreover, by the definition of an $R$-lattice, $d(\gamma_1, \gamma_2) \geq 2c_\Gamma R$. Combining these estimates to have 

$$|(Tf)(\gamma_1) - (Tf)(\gamma_2)| \leq \lambda_R \|f\|_{Lip(F)}(2/c_\Gamma + 1)d(\gamma_1, \gamma_2) .$$

This establishes (11.19).

Now the assumption (2.4) of the theorem implies that there is an operator $L$ from $Ext(\hat{F}, \Gamma)$ whose norm is bounded by $\lambda_\Gamma$. Composing $T$ and $L$ with the operator $E \in Ext(\Gamma, M)$ of Lemma 11.3 we obtain the operator

$$\tilde{E} := ELT : Lip(F) \to Lip(M) \quad (11.20)$$

whose norm is bounded by a constant depending only on $\lambda(\Gamma), \lambda_R, c_\Gamma, R$ and $N$. This definition also implies that

$$(\tilde{E}f)(\gamma) := (E_{\gamma}f_\gamma)(\gamma), \quad \gamma \in \hat{F} . \quad (11.21)$$

Unfortunately, $\tilde{E}$ is not extension from $F$ and we modify it to obtain the required extension operator. To accomplish this we, first, introduce an operator $\hat{T}$ given on $f \in Lip(F)$ by

$$(\hat{T}f)(m) := \begin{cases} (\tilde{E}f)(m), & \text{if } m \in \hat{F} \\ f(m), & \text{if } m \in F \setminus \hat{F} \end{cases} \quad (11.22)$$

**Lemma 11.4** $\hat{T} : Lip(F) \to Lip(F \cup \hat{F})$ and $||\hat{T}|| \leq C$.

**Proof.** It clearly suffices to estimate

$$I := |(\hat{T}f)(m_1) - (\hat{T}f)(m_2)|$$

for $m_1 \in \hat{F}$ and $m_2 \in F \setminus \hat{F}$. Let, first, these points belong to a ball $B_\gamma$ (hence $m_1 = \gamma$). Then (11.21) and the implication $E_\gamma \in Ext(F \cap B_\gamma, B_\gamma)$ imply

$$I = |(E_\gamma f_\gamma)(\gamma) - f(m_2)| = |(E_\gamma f_\gamma)(\gamma) - (E_\gamma f_\gamma)(m_2)| \leq \lambda_R \|f\|_{Lip(F)}d(\gamma, m_2) := \lambda_R \|f\|_{Lip(F)}d(m_1, m_2) .$$

In the remaining case $m_2 \in B_{\hat{\gamma}} \setminus B_\gamma$ for some $\hat{\gamma} \in \hat{F}$ and therefore

$$d(m_1, m_2) = d(\gamma, m_2) \geq R . \quad (11.23)$$

Similarly to the previous estimate we now get

$$I \leq |(\tilde{E}f)(\gamma) - (\tilde{E}f)(\hat{\gamma})| + |(E_{\hat{\gamma}} f_{\hat{\gamma}})(\hat{\gamma}) - (E_{\gamma} f_{\gamma})(m_2)| \leq 

\left(\|\tilde{E}\|d(\gamma, \hat{\gamma}) + \lambda_R d(\hat{\gamma}, m_2)\right)\|f\|_{Lip(F)} .$$

58
Moreover, (11.23) implies
\[ d(\gamma, \hat{\gamma}) + d(\hat{\gamma}, m_2) \leq d(\gamma, m_2) + 2d(\hat{\gamma}, m_2) \leq d(\gamma, m_2) + 2R \leq 3d(\gamma, m_2) = 3d(m_1, m_2). \]
Together with the previous inequalities this gives the required estimate of $I$. \qed

The next operator that will be used in our construction is defined on $f \in \text{Lip}(F)$ by
\[ (\hat{S}f)(m) := (\hat{T}f)(m) - (\hat{E}f)(m), \quad m \in F \cup \hat{F}. \] (11.24)

**Lemma 11.5**

\[ ||\hat{S}f||_{l_\infty(F \cup \hat{F})} \leq C ||f||_{\text{Lip}(F)} \text{ and, moreover,} \]
\[ \hat{S} : \text{Lip}(F) \to \text{Lip}(F \cup \hat{F}) \quad \text{and} \quad ||\hat{S}|| \leq C. \]

**Proof.** The second statement follows straightforwardly from (11.24). If, now, $m \in B_\gamma \cap (F \cup \hat{F})$, then by the same definition
\[ (\hat{S}f)(m) = [(\hat{T}f)(m) - (\hat{T}f)(\gamma)] + [(\hat{E}f)(\gamma) - (\hat{E}f)(m)] \]
which implies that
\[ |(\hat{S}f)(m)| \leq C ||f||_{\text{Lip}(F)} d(m, \gamma) \leq CR ||f||_{\text{Lip}(F)} . \] \qed

Finally we introduce an operator $\hat{K}$ given on $g \in (\text{Lip} \cap l_\infty)(F \cup \hat{F})$ by
\[ \hat{K}g := \sum_\gamma (E_\gamma g_\gamma) \rho_\gamma ; \] (11.25)

here $\gamma$ runs through the set $\{ \gamma \in \Gamma : (F \cup \hat{F}) \cap B_\gamma \neq \emptyset \}$, and $\{ \rho_\gamma \}$ is the partition of unity of Lemma 11.2. Besides, $E_\gamma$ is an operator from $\text{Ext}((F \cup \hat{F}) \cap B_\gamma, B_\gamma)$ with
\[ ||E_\gamma|| \leq \lambda_R, \] (11.26)
and $g_\gamma := g|_{(F \cup \hat{F}) \cap B_\gamma}$.

**Lemma 11.6**

\[ ||\hat{K}g||_{\text{Lip}(M)} \leq C ||g||_{(\text{Lip} \cap l_\infty)(F \cup \hat{F})} . \]

**Proof.** As in the proof of Lemma 11.3 it is convenient to extend every $E_\gamma g_\gamma$ outside $B_\gamma$ so that the extension $F_\gamma$ satisfies
\[ ||F_\gamma||_{\text{Lip}(M)} = ||E_\gamma g_\gamma||_{\text{Lip}(B_\gamma)} . \] (11.27)

Then we clearly have
\[ \hat{K}g = \sum_\gamma F_\gamma \rho_\gamma . \] (11.28)
Now, according to (11.25) we get for $m \in F \cup \hat{F}$
\[ (\hat{K}g)(m) = \sum_\gamma \rho_\gamma(m)g(m) = g(m) . \]
So it remains to estimate the right-hand side of the inequality

\[ I := \| (\tilde{K} g)(m_1) - (\tilde{K} g)(m_2) \| \leq \sum_{\gamma} |\rho_{\gamma}(m_1) - \rho_{\gamma}(m_2)| \cdot |F_{\gamma}(m_1)| + \sum_{\gamma} \rho_{\gamma}(m_2) |F_{\gamma}(m_1) - F_{\gamma}(m_2)| . \]

By (11.27) the second sum is at most \( \left( \sup_{\gamma} \| E_{\gamma} \| \right) \| g \|_{Lip(F \cup \hat{F})} d(m_1, m_2) \) and together with (11.26) this leads to an appropriate bound. In turn, the first sum is at most

\[ 2 \mu \cdot K \cdot \max_{\gamma} \| (E_{\gamma} g_{\gamma})(m_1) \| \cdot d(m_1, m_2) \]

where \( \mu \) (multiplicity) and \( K \) are defined in Lemmata 11.1 and 11.2. To estimate the maximum, one notes that \( (E_{\gamma} g_{\gamma})(\gamma) = g(\gamma) \) and therefore

\[ \| (E_{\gamma} g_{\gamma})(m_1) \| \leq \| (E_{\gamma} g_{\gamma})(m_1) - (E_{\gamma} g_{\gamma})(\gamma) \| + |g(\gamma)| \leq \lambda_R d(m_1, \gamma) \| g \|_{Lip(F \cup \hat{F})} + \| g \|_{Lip(F \cup \hat{F})} \leq (R \lambda_R + 1) \| g \|_{Lip(F \cup \hat{F})} . \]

Together with the estimate of the second sum this proves the lemma. \( \square \)

We are now ready to define the required operator \( \hat{E} \) from \( Ext(F, M) \). Actually, we use the above introduced operators \( \hat{E}, \tilde{K} \) and \( \hat{S} \) and set

\[ \hat{E} f := \tilde{E} f + \tilde{K} \left( \hat{S} f \right)_{F \cup \hat{F}} . \]

If \( m \in F \), we then have

\[ (\hat{E} f)(m) = (\tilde{E} f)(m) + (\hat{S} f)(m) = (\tilde{E} f)(m) + (\tilde{T} f)(m) - (\tilde{E} f)(m) = f(m) , \]

i.e., \( \hat{E} \) is an extension from \( F \). To obtain the necessary estimate of \( \| \hat{E} f \|_{Lip(M)} \) it suffices by (11.20) to estimate \( \| \tilde{K} \left( \hat{S} f \right)_{F \cup \hat{F}} \|_{Lip(M)} \). The latter by Lemmata 11.6, 11.5, 11.4 and (11.20) is bounded by

\[ C \| \hat{S} f \|_{Lip(F \cup \hat{F})} \leq C \| f \|_{Lip(F)} . \]

Hence \( \hat{E} \in Ext(F, M) \) and its norm is bounded as required.

The proof of the theorem is complete. \( \square \)

**Proof of Corollary 2.13.** Our initial proof derives this corollary from Theorem 2.9 and an important result by Bonk and Schramm [BoSch]; it will be outlined below. However, a recently established embedding theorem, see [NPSS], allows to prove the desired result as an immediate consequence of Theorem 2.4. The embedding theorem is formulated as follows.

*Let \( M \) be a \( \delta \)-hyperbolic space of bounded geometry. Then there exist constants \( N \in \mathbb{N} \) and \( C > 0 \) (depending on \( M \)) such that \( M \) is \( C \)-isometric to a subset of the direct sum of \( N \) metric trees.*

This and Theorem 2.4 immediately imply Corollary 2.13. \( \square \)
Remark 11.7 We outline another proof of Corollary 2.13 based (as the above formulated embedding theorem) on the main result of [BoSch]. This result asserts:

Let $M$ be a $\delta$-hyperbolic space of bounded geometry. Then there exists an integer $n$ such that $M$ is roughly similar to a convex subset of hyperbolic $n$-space $\mathbb{H}^n$.

Recall that a map $f : (M, d) \to (M', d')$ is a $(K, L)$-rough similarity, if for all $m, n$ from $M$ it is true that

$$K^{-1}d(m, n) - L \leq d'(f(m), f(n)) \leq Kd(m, n) + L.$$ 

As a consequence of this result we obtain that the finite direct $p$-sum $M = \oplus_p \{M_i\}_{1 \leq i \leq k}$ of Gromov-hyperbolic metric spaces $M_i$ is roughly similar to a subset of the direct $p$-sum $\oplus_p \{\mathbb{H}^{n_i}\}_{1 \leq i \leq k}$ for some natural $n_i$. Now, the required corollary can be easily derived from the above result and Theorem 2.9 if we observe the following.

1. Restriction $f|_{\Gamma}$ of a rough similar map $f$ to an $R$-lattice $\Gamma$ (see Definition 2.9) with $R$ big enough is a $C$-isometric embedding into $M'$ for an appropriate $C$. If, in addition, $\lambda(M') < \infty$, then $\lambda(\Gamma)$ is finite.

2. If $M_i$ is a geodesic metric space of $R_0$-bounded geometry, then every its ball of radius $R$ can be covered by at most $k = k(R, R_0)$ balls of radius $R_0$. Therefore the same is true for the finite direct $p$-sum $M = \oplus_p \{M_i\}_{1 \leq i \leq k}$ of such spaces. From here one easily deduce that for the space $M$ the constants $\lambda_R$ defined by (2.4) are finite for any $R$.

3. According to [BSh2, Proposition 5.33] every $\mathbb{H}^{n_i}$ satisfies the assumptions of Corollary 2.27. Therefore $\lambda(\oplus_p \{\mathbb{H}^{n_i}\}_{1 \leq i \leq k}) < \infty$.

Proof of Corollary 2.15. We first prove that the condition

$$\lambda(G, d_A) < \infty \quad (11.29)$$

of Corollary 2.15 is necessary for finiteness of $\lambda(M)$. Let the latter be true. Then for a $G$-orbit $G(m) := \{g(m) : g \in G\}$ we have

$$\lambda(G(m)) \leq \lambda(M) < \infty \quad (11.30)$$

Then the Švarc-Milnor lemma, see, e.g., [BH, p.140], states that under the hypothesis (b) of Corollary 2.15 there is a constant $C \geq 1$ independent of $m$ so that

$$C^{-1}d_A(g, h) \leq d(g(m), h(m)) \leq Cd_A(g, h) \quad (11.31)$$

for all $g, h \in G$. This, in particular, means that the metric subspace $G(m)$ is quasi-isometric to the metric space $(G, d_A)$. Hence (11.30) implies the required inequality (11.29).

To prove sufficiency of the condition (11.29) for finiteness of $\lambda(M)$, we choose a point $m_0$ of the generating compact set $K_0$ from Definition 2.14, see (2.5), and show that the $G$-orbit $G(m_0)$ is an $R$-lattice for some $R > 0$. Let $B_{R_0}(m_0)$ be a ball containing $K_0$. Then we have by (2.5) for $\Gamma := G(m_0)$

$$\bigcup_{m \in \Gamma} B_{R_0}(m) = G(B_{R_0}(m_0)) \supset G(K_0) = M.$$
Hence the family of balls \( \{ B_{R_0}(m) : m \in \Gamma \} \) covers \( M \). Moreover, (11.31) implies that for \( m := g(m_0), m' := h(m_0) \) with \( g \neq h \)

\[
d(m, m') \geq C^{-1}d_A(g, h) \geq C^{-1},
\]

that is to say, the family \( \{ B_{cR_0}(m) : m \in \Gamma \} \) with \( c = c_\Gamma := \frac{1}{2d(K_0)} \) consists of pairwise disjoint balls. So \( \Gamma \) is an \( R \)-lattice, \( R := 2R_0 \), satisfying, by (11.29) and (11.31), the condition

\[
\lambda(\Gamma) < \infty.
\]

We now apply Theorem 2.9 with that \( R \)-lattice \( \Gamma \) to derive finiteness of \( \lambda(M) \). To this end we have to establish validity of the assumptions of the theorem with this \( R \).

First we prove that \( M \) belongs to the class of doubling metric spaces \( \mathcal{D}(R, N) \) for some \( N = N(R, M) \). In other words, we show that every ball \( B_r(m) \) with \( r < R \) can be covered by at most \( N \) balls of radius \( r/2 \). Indeed, by the hypothesis (a) of the corollary, \( M \in \mathcal{G}_n(\tilde{R}, \tilde{C}) \) for certain \( \tilde{R}, \tilde{C} \) and \( n \). This implies that \( M \in \mathcal{D}(\tilde{R}/2, N) \) for some \( N = N(\tilde{C}, n) \) and shows that the required statement is true for \( R \leq \tilde{R}/2 \).

Suppose now that

\[
\tilde{R}/2 \leq r < R.
\]  

(11.32)

Note that it suffices to consider balls with \( m \in K_0 \). In fact, \( G(K_0) = M \) and therefore \( g_0(m) \in K_0 \) for some isometry \( g_0 \in G \). Hence \( g_0(B_r(m)) = B_r(g_0(m)) \) and we can work with \( B_r(m) \) for \( m \in K_0 \). Let us fix a point \( m_0 \in K_0 \) and set \( R' := R + \text{diam } K_0 \). Then

\[ B_r(m) \subset B_{R'}(m_0), \quad m \in K_0, \]  

(11.33)

and it remains to show that \( B_{R'}(m_0) \) can be covered by a finite number, say \( N \), of (open) balls of radius \( r/2 \) with \( N \) independent of \( r \). We use the following

**Lemma 11.8** Suppose that \( G \) acts properly, freely and cocompactly on a path space \( M \) by isometries. Then every bounded closed set \( S \subset M \) is compact.

**Proof.** For every \( m \in S \) there is a finite number of isometries \( g_{im} \in G, i = 1, \ldots, k_m \), such that \( g_{im}(m) \in K_0 \). Here \( K_0 \) is the generating compact of Definition 2.14 (c). Let \( H := \{ g_{im}^{-1} \in G : 1 \leq i \leq k_m, \ m \in S \} \). Then \( S \subset H(K_0) \), and, by that definition, \( \text{diam } H(K_0) < \infty \). For a fixed \( m_0 \in K_0 \) let us consider the orbit \( H(m_0) \). Show that \( H(m_0) \) consists of a finite number of points. Otherwise there is a sequence of points \( m_i := h_i(m_0) \in H(m_0) \) such that \( d_A(h_i, 1) \to \infty \) as \( i \to \infty \). This and inequality (11.31) imply \( d(m_i, m_0) \to \infty \) in \( M \) as \( i \to \infty \) and this contradicts the condition \( \text{diam } H(K_0) < \infty \). From finiteness of \( H(m_0) \) we also obtain that \( H \) is finite. Thus \( S \) is covered by a finite number of compact sets, and, since \( S \) is closed, it is compact. \( \square \)

According to this lemma the closure \( \overline{B_{R'}(m_0)} \) is compact. Thus \( B_{R'}(m_0) \) can be covered by a finite number \( N \) of open balls of radius \( \tilde{R}/4 \). This, (11.32) and (11.33) show that \( B_r(m) \) can be covered by \( N \) open balls of radius \( r/2 \) as it is required.
To establish the second condition of Theorem 2.9, finiteness of
\[
\lambda_R := \sup \{ \lambda(B_R(m)) : m \in M \},
\]
we first use the previous argument and (11.33) which immediately get
\[
\lambda_R \leq \lambda(B'_R(m_0)).
\]
Show that the right-hand side is bounded. Since \( M \in \mathcal{G}_n(\tilde{R}, \tilde{C}) \), for every \( m \in M \),
\[
\lambda(B_R'(m)) < \infty.
\]
This, compactness of \( B_R'(m_0) \) and the argument used in the proof of Lemma 8.4 lead to the required inequality
\[
\lambda(B_R'(m_0)) < \infty.
\]

The proof of Corollary 2.15 is completed. \( \square \)

12 Proofs of Theorem 2.21 and its Corollaries

Proof of Theorem 2.21; Part I. Given a metric space \((M, d)\) of pointwise homogeneous type of Definition 2.19 and a subspace \( S \subset M \) we will construct an operator \( E \in \text{Ext}(S,M) \) whose norm is bounded by a constant depending only on the constants \( C \) of (2.8), and
\[
D(l) := \sup_{m \in M} \sup_{R > 0} \frac{\mu_m(B_{lR}(m))}{\mu_m(B_R(m))}
\]
with \( l > 1 \) that will be specified later. By the uniform doubling condition (2.7), this is finite and depends only on the constant in (2.7). Our construction is similar to that of [BSh2, pp.535-540]; unfortunately, the latter used a Borel measurable selection of the multivalued function \( m \mapsto \{ m' \in M : d(m, S) \leq d(m, m') \leq 2d(m, S) \} \) for some specific spaces \((M, d)\) (including e.g. the \( n \)-dimensional hyperbolic space). Generally speaking, such a selection may not exist even in the case of the Euclidean plane\(^{10}\). Fortunately, Theorem 2.1 allows to restrict our consideration to the case of finite point subspaces \( S \) in which case the corresponding measurable selection trivially exists.

So we consider a finite point metric subspace \( S \subset M \) and construct in this case an operator \( E \in \text{Ext}(S, M) \) with
\[
\|E\| \leq K = K(D(l), C) < \infty.
\]
By Theorem 2.1 \( \lambda(M) \) will be then bounded by the same constant \( K \).

\(^{10}\text{see, in particular, the corresponding counterexample in [N].}\)
For this purpose let us arrange $S$ in a sequence $s_1, \ldots, s_l$ and introduce functions $d : M \to \mathbb{R}_+$ and $p : M \to S$ by the conditions

$$d(m) := \min\{d(m, m') : m' \in S\} \quad (12.3)$$

and

$$p(m) := s_i \quad (12.4)$$

where $i$ is the minimal number for which $s_i \in \{m' \in S : d(m) = d(m', m)\}$.

**Lemma 12.1** (a) For every $m_1, m_2 \in M$

$$|d(m_1) - d(m_2)| \leq d(m_1, m_2). \quad (12.5)$$

(b) If $f : S \to \mathbb{R}$ is an arbitrary function, then $f \circ p$ is Borel measurable.

**Proof.** (a) follows directly from (12.3). To check (b) note that for each $1 \leq i \leq l$ the set $p^{-1}(\{s_1, \ldots, s_i\}) \subset M$ is closed. This implies the required result. \(\square\)

To introduce the required extension operator $E$ we also use the average with respect to the Borel measure $\mu_m$ of Definition 2.19, letting for a Borel measurable function $g : M \to \mathbb{R}$

$$I(g; m, R) := \frac{1}{\mu_m(B_R(m))} \cdot \int_{B_R(m)} g \, d\mu_m, \quad m \in M, R > 0. \quad (12.6)$$

Finally, we define $E$ on functions $f \in \text{Lip}(S)$ by

$$(Ef)(m) := \begin{cases} f(m), & \text{if } m \in S \\ I(f \circ p; m, d(m)), & \text{if } m \in M \setminus S \end{cases} \quad (12.7)$$

We now have to show that for every $m_1, m_2$

$$|(Ef)(m_1) - (Ef)(m_2)| \leq K||f||_{\text{Lip}(S)}d(m_1, m_2) \quad (12.8)$$

where $K = K(D(l), C)$ will be specified later.

It suffices to consider only two cases:

(a) $m_1 \in S$ and $m_2 \notin S$;

(b) $m_1, m_2 \notin S$.

We assume without loss of generality that

$$||f||_{\text{Lip}(S)} = 1 \quad (12.9)$$

and simplify computations by introducing the following notations:

$$R_i := d(m_i), \mu_i := \mu_{m_i}, B_{ij} := B_{R_j}(m_i), v_{ij} := \mu_i(B_{ij}), \quad 1 \leq i, j \leq 2. \quad (12.10)$$

We assume also for definiteness that

$$0 < R_1 \leq R_2. \quad (12.11)$$
By Lemma 12.1 we then have
\[ 0 \leq R_2 - R_1 \leq d(m_1, m_2) . \] (12.12)

In what follows we will prove the required result under the next additional assumption on the family \( \{ \mu_m \} \):

There is a constant \( A > 0 \) such that for all \( 0 < R_1 \leq R_2 \) and \( m \in M \)
\[ \mu_m(B_{R_2}(m)) - \mu_m(B_{R_1}(m)) \leq \frac{A \mu_m(B_{R_2}(m))}{R_2} (R_2 - R_1) . \] (12.13)

This restriction will be removed at the final stage of the proof.

Under the notations and the assumptions introduced the following is true
\[ v_{i_2} - v_{i_1} \leq \frac{A v_{i_2}}{R_2} (R_2 - R_1) , \] (12.14)
\[ |\mu_1 - \mu_2|(B_{ij}) \leq \frac{C v_{ij}}{R_j} d(m_1, m_2) , \] (12.15)
see (12.13) and (2.8).

To estimate the difference in (12.8) for \( m_1 \in S \) we need Lemma 12.2

It is true that
\[ \max\{|\tilde{f}(m)| : m \in B_{i_2}\} \leq 4R_2 + (i - 1)d(m_1, m_2) ; \] (12.16)
here \( i = 1, 2 \) and
\[ \tilde{f}(m) := (f \circ p)(m) - (f \circ p)(m_1) . \] (12.17)

**Proof.** Let first \( i = 1 \) and \( m \in B_{i_2} \). By (12.17), (12.9), and the triangle inequality
\[ |\tilde{f}(m)| \leq d(p(m), p(m_1)) \leq d(m) + d(m, m_1) + d(m_1) . \]
But \( d(m, m_1) \leq R_2 \), since \( m \in B_{i_2} \). Besides, \( d(m) \leq d(m, m_1) + d(m_1) \leq 2R_2 \) by Lemma 12.1. Taking these together to get
\[ |\tilde{f}(m)| \leq 4R_2 , \ m \in B_{i_2} . \]

Let now \( i = 2 \) and \( m \in B_{2_2} \). As before, the triangle inequality gives
\[ |\tilde{f}(m)| \leq d(m, m_1) + d(m, m_2) + d(m_1) + d(m_2) . \]
Since \( d(m, m_2) \leq R_2 \) and \( d(m, m_1) \leq d(m, m_2) + d(m_1, m_2) \), we therefore have
\[ |\tilde{f}(m)| \leq d(m_1, m_2) + 4R_2 , \ m \in B_{2_2} . \square \]

Prove now (12.8) for \( m_1 \in S \) and \( m_2 \not\in S \). We clearly have under the notation (12.10)
\[ |(Ef)(m_2) - (Ef)(m_1)| = \frac{1}{v_{22}} \left| \int_{B_{2_2}} \tilde{f}(m)d\mu_2 \right| \leq \max_{B_{2_2}} |\tilde{f}| . \]
Applying (12.16) with \(i = 2\) we then bound this difference by \(4R_2 + d(m_1, m_2)\). But \(m_1 \in S\) and so
\[ R_2 = d(m_2) \leq d(m_1, m_2) , \]
and therefore (12.8) holds in this case with \(K = 5\).

The remaining case \(m_1, m_2 \notin S\) requires some additional auxiliary results. To their formulations we first write
\[(Ef)(m_1) - (Ef)(m_2) := D_1 + D_2 \tag{12.18} \]
where
\[ D_1 := I(\tilde{f}; m_1, R_1) - I(\tilde{f}; m_1, R_2) \]
\[ D_2 := I(\tilde{f}; m_1, R_2) - I(\tilde{f}; m_2, R_2) , \tag{12.19} \]
see (12.7) and (12.17).

**Lemma 12.3** It is true that
\[ |D_1| \leq 8Ad(m_1, m_2) . \]

Recall that \(A\) is the constant in (12.13).

**Proof.** By (12.19), (12.17) and (12.10),
\[ D_1 = \frac{1}{v_11} \int_{B_11} \tilde{f}d\mu_1 - \frac{1}{v_12} \int_{B_12} \tilde{f}d\mu_1 = \left( \frac{1}{v_11} - \frac{1}{v_12} \right) \int_{B_11} \tilde{f}d\mu_1 - \frac{1}{v_12} \int_{B_12 \setminus B_11} \tilde{f}d\mu_1 . \]
This immediately implies that
\[ |D_1| \leq 2 \cdot \frac{v_11 - v_12}{v_12} \cdot \max_{B_12} |\tilde{f}| . \]
Applying now (12.14) and (12.12), and then Lemma 12.2 with \(i = 1\) we get the desired estimate. \(\Box\)

To obtain a similar estimate for \(D_2\) we will use the following two facts.

**Lemma 12.4** Assume that for a given \(l > 1\)
\[ d(m_1, m_2) \leq (l - 1)R_2 . \tag{12.20} \]

Let for definiteness
\[ v_{22} \leq v_{12} . \tag{12.21} \]

Then it is true that
\[ \mu_2(B_{12} \Delta B_{22}) \leq 2(A + C)D(l)\frac{v_{12}}{R_2}d(m_1, m_2) \tag{12.22} \]
(\(\text{here } \Delta \text{ denotes symmetric difference of sets}).
Proof. Set
\[ R := R_2 + d(m_1, m_2). \]

Then \( B_{12} \cup B_{22} \subset B_R(m_1) \cap B_R(m_2) \) and
\[ \mu_2(B_{12} \Delta B_{22}) \leq (\mu_2(B_R(m_1)) - \mu_2(B_{12})) + (\mu_2(B_R(m_2)) - \mu_2(B_{22})). \tag{12.23} \]
The first summand on the right-hand side is at most
\[ |\mu - \mu_1|(B_R(m_1)) + |\mu_2 - \mu_1|(B_{R_2}(m_1)) + (\mu_1(B_R(m_1)) - \mu_1(B_{R_2}(m_1))). \]
Estimating the first two summands by (2.8) and the third by (12.13) we bound this sum by
\[ C \left( \frac{\mu_1(B_R(m_1))}{R} + \frac{\mu_1(B_{R_2}(m_1))}{R_2} \right) + A \frac{\mu_1(B_R(m_1))}{R} (R - R_2). \]

Besides, \( R_2 \leq R \leq lR_2 \) and \( R - R_2 := d(m_1, m_2) \), see (12.20); taking into account (12.1) and the notations (12.10) we therefore have
\[ \mu_2(B_R(m_1)) - \mu_2(B_{12}) \leq [C(D(l) + 1) + AD(l)] \frac{v_{12}}{R_2} d(m_1, m_2). \]
Similarly, by (12.13) and (12.21)
\[ \mu_2(B_R(m_2)) - \mu_2(B_{22}) \leq A \frac{\mu_2(B_R(m_2))}{R} (R - R_2) \leq AD(l) \frac{v_{22}}{R_2} d(m_1, m_2) \leq AD(l) \frac{v_{12}}{R_2} d(m_1, m_2). \]

Combining the last two estimates with (12.23) we get the result. \( \square \)

Lemma 12.5 Under the assumptions of the previous lemma it is true that
\[ v_{12} - v_{22} \leq 3(A + C)D(l) \frac{v_{12}}{R_2} d(m_1, m_2). \tag{12.24} \]

Proof. By (12.10) the left-hand side is bounded by
\[ |\mu_1(B_{12}) - \mu_2(B_{12})| + \mu_2(B_{12} \Delta B_{22}). \]
Estimating these summands by (12.15) and (12.22) we get the result. \( \square \)

We now estimate \( D_2 \) from (12.19) beginning with

Lemma 12.6 Under the conditions of Lemma 12.4 it is true that
\[ |D_2| \leq K(l)d(m_1, m_2) \]
where
\[ K(l) := 6(A + C)D(l)(l + 3). \tag{12.25} \]
Proof. By the definition of \( D_2 \) and our notations, see (12.19), (12.16) and (12.10),

\[
|D_2| := \left| \frac{1}{v_{12}} \int_{B_{12}} \tilde{f} \, d\mu_1 - \frac{1}{v_{22}} \int_{B_{22}} \tilde{f} \, d\mu_2 \right| \leq \frac{1}{v_{12}} \int_{B_{12}} |\tilde{f}| \, d|\mu_1 - \mu_2| + \frac{1}{v_{12}} \int_{B_{12} \Delta B_{22}} |\tilde{f}| \, d\mu_2 + \frac{1}{v_{22}} \int_{B_{22}} |\tilde{f}| \, d\mu_2 := J_1 + J_2 + J_3.
\]

By (12.15), (12.16) with \( i = 1 \) and (12.20)

\[
J_1 \leq \frac{1}{v_{12}} |\mu_1 - \mu_2| (B_{12}) \sup_{B_{12}} |\tilde{f}| \leq \frac{C}{R_2} d(m_1, m_2)(d(m_1, m_2) + 4R_2) \leq C(l+3)d(m_1, m_2).
\]

In turn, by (12.22), (12.11) and (12.16)

\[
J_2 \leq \frac{1}{v_{12}} \mu_2 (B_{12} \Delta B_{22}) \sup_{B_{12} \Delta B_{22}} |\tilde{f}| \leq \frac{2(A + C)D(l)}{R_2} d(m_1, m_2)(d(m_1, m_2) + 4R_2) \leq 2(A + C)D(l)(l + 3)d(m_1, m_2).
\]

Finally, (12.24), (12.16) and (12.20) yield

\[
J_3 \leq 3(A + C)D(l)(l + 3)d(m_1, m_2).
\]

Combining we get the required estimate. \( \square \)

It remains to consider the case of \( m_1, m_2 \in M \) satisfying the inequality

\[
d(m_1, m_2) > (l - 1)R_2
\]

converse to (12.20). Now the definition (12.19) of \( D_2 \) and (12.16) imply that

\[
|D_2| \leq 2 \sup_{B_{12} \cup B_{22}} |\tilde{f}| \leq 2(4R_2 + d(m_1, m_2)) \leq 2 \left( \frac{4}{l-1} + 1 \right) d(m_1, m_2).
\]

Combining this with the inequalities of Lemmas 12.3 and 12.6 and equality (12.18) we obtain the required estimate of the Lipschitz norm of the extension operator \( E \) defined by (12.7). Actually, we have proved that

\[
||E|| \leq 8A + \max \left( \frac{2(l + 3)}{l - 1}, K(l) \right)
\]

where \( K(l) \) is the constant in (12.25). Hence Theorem 2.21 has proved under the additional assumption (12.13) with \( \lambda(M) \) estimated by (12.26).

Remark 12.7 (a) Let \( M \) be a metric space of homogeneous type with respect to a doubling measure \( \mu \). Taking \( \mu_m := \mu \) for all \( m \in M \) and noting that (12.15) is now trivially held we improve the estimate (12.26) as follows:

\[
\lambda(M) \leq 8A + \max \left( \frac{2(l + 3)}{l - 1}, K_\mu(l) \right)
\]
where $K_{\mu}(l) := 2A(l + 3)D_{\mu}(l)$. Here $D_{\mu}(l)$ is the dilation function for $\mu$, see (12.1).

In fact, (12.23) is now bounded by $\frac{2AD(l)}{R_2}d(m_1, m_2)$ and this constant appears in (12.26). Besides, $J_1 = J_3 = 0$ in this case.

(b) If, on the other hand, for some $a, n > 0$ and all $B_R(m)$

$$\mu_m(B_R(m)) = aR^n,$$  \hspace{1cm} (12.28)

the estimate (12.27) can be sharpen. Note that in this case condition (12.13) clearly holds with $A = n$. Hence Theorem 2.21 had already proved in this case. Besides, in the proof of Lemma 12.6, $J_1$ is now bounded by $C(l + 3)d(m_1, m_2)$, and $J_2$ and $J_3$ by $n(l + 3)l^{n-1}d(m_1, m_2)$ and $nl^n(l + 3)d(m_1, m_2)$, respectively. Collecting these we get in this case

$$\lambda(M) \leq 8n + \max\left(\frac{2(l + 3)}{l - 1}, K_n(l)\right)$$  \hspace{1cm} (12.29)

where

$$K_n(l) := (l + 3)(C + 2nl^{n-1}).$$

(c) Finally, for the case of the doubling measure $\mu$ of part (a) satisfying condition (12.28) the constant $C$ in (12.29) disappears and we get the estimate (12.29) with

$$K_n(l) = 2n(l + 3)l^{n-1}.$$  

Let us recall that $l > 1$ is arbitrary and we may and will optimize all these estimates with respect to $l$.

**Proof of Theorem 2.21; Part II.** We apply now the result of the previous part to a metric space $(\hat{M}, \hat{d})$ of pointwise homogeneous type satisfying the following two conditions

(a) *The original metric space $(M, d)$ embeds isometrically to $(\hat{M}, \hat{d})$.*

This immediately implies the inequality

$$\lambda(M) \leq \lambda(\hat{M}).$$  \hspace{1cm} (12.30)

(b) *Condition (12.13) holds for $(\hat{M}, \hat{d})$.*

This implies validity of estimate (12.26) for the extension operator $E \in Ext(S, \hat{M})$. Of course, this estimate includes now the dilation function and the consistency constant for $\hat{M}$ which must be evaluated via the corresponding amounts for $M$, see (12.1) and (2.8). This goal will be achieved by two auxiliary results, Lemmas 12.8 and 12.9 presented below.

To their formulation, introduce a metric space $(M_N, d_N)$ by

$$M_N := M \times l_1^N;$$

where $l_1^N$ is the $N$-dimensional vector space defined by the metric

$$\delta_1^N(x, y) := \|x - y\|_1 = \sum_{i=1}^{N}|x_i - y_i|, \hspace{0.5cm} x, y \in \mathbb{R}^N.$$
In turn, $d_N$ is given by

$$d_N(\tilde{m}, \tilde{m}') := d(m, m') + \delta_N(x, x')$$

where here and below

$$\tilde{m} := (m, x) \text{ with } m \in M \text{ and } x \in l_1^N.$$  

At last, we equip $M_N$ with a family $\tilde{F} := \{\mu_{\tilde{m}}\}_{\tilde{m} \in M_N}$ of positive Borel measures on $M_N$ introduced by

$$\mu_{\tilde{m}} := \mu_m \otimes \lambda_N$$

where $\lambda_N$ is the Lebesgue measure of $\mathbb{R}^N$ and $F := \{\mu_m\}_{m \in M}$ is the family of doubling measures from the definition of $M$.

Now we estimate the dilation function of family $\tilde{F}$ using the corresponding doubling inequality for family $F$. Recall that this inequality, see (2.7), implies that

$$\mu_m(B_{2R}(m)) \leq D \mu_m(B_R(m))$$

for all $m \in M$ and $R > 0$.

Thus, we consider now the function

$$D_N(l) := \sup \left\{ \frac{\mu_{\tilde{m}}(B_{IR}(\tilde{m}))}{\mu_{\tilde{m}}(B_R(\tilde{m}))} \right\}$$

where the supremum is taken over all $\tilde{m} \in M_N$ and $R > 0$.

**Lemma 12.8** Assume that $N$ is related to the doubling constant $D$ of (12.31) by

$$N \geq \lceil \log_2 D \rceil + 5 .$$

Then it is true that

$$D_N(1 + 1/N) \leq \frac{6}{5}e^4 .$$

**Proof.** Note that open ball $B_R(\tilde{m})$ of $M_N$ is the set

$$\{(m', y) \in M \times l_1^N : d(m', m) + ||x - y||_1 < R\}$$

(recall that $\tilde{m} = (m, x)$). Therefore application of Fubini’s theorem yields

$$\mu_{\tilde{m}}(B_R(\tilde{m})) = \gamma_N \int_{B_R(m)} (R - d(m, m'))^Nd\mu_m(m') ;$$

here $B_R(m)$ is a ball of $M$ and $\gamma_N$ is the volume of the unit $l_1^N$-ball.

Estimate this measure with $M$ and $\gamma_N$ is the volume of the unit $l_1^N$-ball.

Estimate this measure with $R$ replaced by

$$R_N := (1 + 1/N)R .$$

Split the integral in (12.34) into those over $B_{3R/4}(m)$ and over the remaining part $B_{R_N}(m) \setminus B_{3R/4}(m)$. Denote these integrals by $I_1$ and $I_2$. For $I_2$ we get from (12.34)

$$I_2 \leq \gamma_N(R_N - 3R/4)^Nd\mu_m(m') = \gamma_N \left( \frac{1}{4} + \frac{1}{N} \right)^Nd\mu_m(m') .$$

70
Using the doubling constant for $\mathcal{F} = \{\mu_m\}$, see (12.31), we further have

$$\mu_m(B_{R_N}(m)) \leq D \mu_m(B_{R_N/2}(m)).$$

Moreover, by (12.33), $D < 2^{\log_2 D} + 1 \leq \frac{1}{16} 2^N$. Combining all these inequalities we obtain

$$I_2 \leq \gamma_N \frac{1}{16} 2^{-N} \left(1 + \frac{4}{N}\right)^N R^N \mu_m(B_{R_N/2}(m)). \quad (12.35)$$

To estimate $I_1$ we present its integrand (which equals to that in (12.34) with $R$ replaced by $R_N$) in the following way:

$$\left(1 + \frac{1}{N}\right)^N (R - d(m, m'))^N \left(1 + \frac{d(m, m')}{(N+1)(R - d(m, m'))}\right)^N.$$

Since $d(m, m') \leq 3R/4$ for $m' \in B_{3R/4}(m)$, the last factor is at most $\left(1 + \frac{3}{N+1}\right)^N$. Hence, we have

$$I_1 \leq \gamma_N \left(1 + \frac{1}{N}\right)^N \left(1 + \frac{3}{N+1}\right)^N \int_{B_{3R/4}(m)} (R - d(m, m'))^N d\mu_m(m').$$

Using then (12.34) we, finally, obtain

$$I_1 \leq e^4 \mu_{\tilde{m}}(B_R(\tilde{m})). \quad (12.36)$$

To estimate $D_N(l)$ with $l = 1 + 1/N$ it remains to bound fractions

$$\tilde{I}_k := \frac{I_k}{\mu_{\tilde{m}}(B_R(\tilde{m}))}, \quad k = 1, 2.$$

For $k = 2$ estimate the denominator from below as follows. Since $R_N < 2R$, we bound $\mu_{\tilde{m}}(B_R(\tilde{m}))$ from below by

$$\gamma_N \int_{B_{R_N/2}(m)} (R - d(m, m'))^N d\mu_m(m') \geq \gamma_N 2^{-N} \left(1 - \frac{1}{N}\right)^N R^N \int_{B_{R_N/2}(m)} d\mu_m(m') = \gamma_N 2^{-N} \left(1 - \frac{1}{N}\right)^N R^N \mu_m(B_{R_N/2}(m)).$$

Combining this with (12.35) to get

$$\tilde{I}_2 \leq \frac{1}{16} \left(1 - \frac{1}{N}\right)^{-N} \left(1 + \frac{4}{N}\right)^N.$$

Since $\left(1 - \frac{1}{N}\right)^{-N} \leq \left(1 - \frac{1}{N}\right)^{-5}$ as $N \geq 5$, we finally obtain

$$\tilde{I}_2 \leq \frac{1}{5} e^4.$$

71
For $\tilde{I}_1$ using (12.36) one immediately has

$$\tilde{I}_1 \leq e^4.$$  

Hence, by the definition of $D_N$, see (12.32), we have

$$D_N(1 + 1/N) \leq \sup_{\tilde{m}, R}(\tilde{I}_1 + \tilde{I}_2) < \frac{6}{3} e^4. \quad \square$$

Our next auxiliary result evaluate the consistency constant $C_N$ for family $\tilde{F} = \{\mu_{\tilde{m}}\}$ using that for $F := \{\mu_m\}$. Recall that the latter constant stands in the inequality, see (2.8),

$$|\mu_{m_1} - \mu_{m_2}|(B_R(m_i)) \leq \frac{C \mu_{m_i}(B_R(m_i))}{R}d(m_1, m_2) \quad (12.37)$$

where $m_1, m_2$ are arbitrary points of $M$, $R > 0$ and $i = 1, 2$.

**Lemma 12.9**

$$C_N \leq \left(1 + \frac{4e}{3}\right) NC.$$  

**Proof.** Using the Fubini theorem, rewrite (12.34) in the form

$$\mu_{\tilde{m}}(B_R(\tilde{m})) = \beta_N \int_0^R \mu_m(B_s(m))(R - s)^{N-1}ds \quad (12.38)$$

where $\beta_N$ is the volume of the unit sphere in $l_1^N$. Then for $i = 1, 2$ we have

$$|\mu_{m_1} - \mu_{m_2}|(B_R(\tilde{m}_i)) \leq \beta_N \int_0^R |\mu_{m_1} - \mu_{m_2}|(B_s(m_i)) \cdot (R - s)^{N-1}ds .$$

Divide here the interval of integration into subintervals $[0, R/N]$ and $[R/N, R]$ and denote the corresponding integrals over these intervals by $I_1$ and $I_2$. It suffices to find appropriate upper bounds for $I_k$. Replacing $B_s(m_i)$ in $I_1$ by bigger ball $B_{s+R/N}(m_i)$ and applying (12.37) we obtain

$$I_1 \leq C \left(\beta_N \int_0^{R/N} \frac{\mu_{m_i}(B_{s+R/N}(m_i))}{s + R/N} (R - s)^{N-1}ds \right) d(m_1, m_2) .$$

Changing $s$ by $t = s + R/N$ we bound the expression in the brackets by

$$\left(\beta_N \int_{R/N}^{2R/N} \mu_{m_i}(B_t(m_i))(R - t)^{N-1}dt \right) \max_{R/N \leq t \leq 2R/N} \frac{(R + R/N - t)^{N-1}}{t(R - t)^{N-1}} .$$

Since $[R/N, 2R/N] \subset [0, R]$ and the maximum $< \frac{N}{R} \left(1 + \frac{1}{N-2}\right)^{N-1} < \frac{4e}{3} \frac{N}{R}$, as $N \geq 5$, this and (12.38) yield

$$I_1 \leq \frac{4e}{3} CN \frac{\mu_{m_i}(B_R(\tilde{m}_i))}{R}d(m_1, m_2) .$$
For the second term we get from (12.37)
\[ I_2 \leq C \left( \beta_N \int_{R/N}^{R} \frac{\mu_{m_i}(B_s(m_i))}{s} (R - s)^{N-1} ds \right) d(m_1, m_2) \]
and by (12.38) the factor in the brackets is at most \( \mu_{\tilde{m}}(B_R(\tilde{m})) \cdot \frac{N}{R} \). Hence, we have
\[ I_2 \leq CN \frac{\mu_{\tilde{m}}(B_R(\tilde{m}))}{R} d(m_1, m_2) . \]
Further note that \( d(m_1, m_2) \leq d_N(\tilde{m}_1, \tilde{m}_2) \). Hence, we obtain finally the inequality
\[ |\mu_{\tilde{m}_1} - \mu_{\tilde{m}_2}|(B_R(\tilde{m})) \leq \left( 1 + \frac{4e}{3} \right) NC \frac{\mu_{\tilde{m}}(B_R(\tilde{m}))}{R} d(\tilde{m}_1, \tilde{m}_2) \]
whence \( C_N \leq \left( 1 + \frac{4e}{3} \right) NC \). \( \square \)

At the last step we introduce the desired metric space \((\hat{M}, \hat{d})\) simply setting
\[ \hat{M} := M_N \times \mathbb{R} \quad (= M \times l_1^{N+1}) \quad \text{and} \quad \hat{d} := d_{N+1} \]
with \( N := \lceil \log_2 D \rceil + 5 \). Moreover, we introduce the family of measures \( \hat{F} := \{ \mu_{\hat{m}} \}_{\hat{m} \in \hat{M}} \) by
\[ \mu_{\hat{m}} := \mu_{\hat{m}} \otimes \lambda ; \]
here and below \( \hat{m} := (\tilde{m}, x) \) where \( \tilde{m} \in M_N \) and \( x \in \mathbb{R} \), and \( \lambda \) is the Lebesgue measure on \( \mathbb{R} \).

By Lemma 12.8 the dilation function \( \hat{D} = D_{N+1} \) of family \( \hat{F} \) at point \( 1 + \frac{1}{N+1} \) is estimated as
\[ \hat{D} \left( 1 + \frac{1}{N+1} \right) \leq \frac{6}{5} e^4 . \]  \( (12.39) \)
Moreover, by Lemma 12.9 the consistency constant \( \hat{C} = C_{N+1} \) of \( \hat{F} \) satisfies
\[ \hat{C} \leq \left( 1 + \frac{4e}{3} \right) (N + 1)C . \]  \( (12.40) \)

Show now that family \( \hat{F} = \{ \mu_{\hat{m}} \} \) satisfies condition (12.13) with constant \( \hat{A} \) satisfying
\[ \hat{A} \leq \frac{6}{5} e^4 (N + 1) . \]  \( (12.41) \)
In fact, \( \mu_{\hat{m}} = \mu_{\tilde{m}} \otimes \lambda \) and by the Fubini theorem we have for \( 0 < R_1 < R_2 \)
\[ \mu_{\hat{m}}(B_{R_2}(\tilde{m})) - \mu_{\hat{m}}(B_{R_1}(\tilde{m})) = 2 \int_{R_1}^{R_2} \mu_{\tilde{m}}(B_s(\tilde{m})) ds \leq \frac{2R_2 \mu_{\tilde{m}}(B_{R_2}(\tilde{m}))}{R_2} (R_2 - R_1) . \]
Prove that for arbitrary \( l > 1 \) and \( R > 0 \)
\[ R \mu_{\hat{m}}(B_R(\tilde{m})) \leq \frac{1D_N(l)}{2(l-1)} \mu_{\hat{m}}(B_R(\tilde{m})) . \]  \( (12.42) \)

73
Together with the previous inequality this will yield
\[ \mu_{\hat{m}}(BR_2(\hat{m})) - \mu_{\hat{m}}(BR_1(\hat{m})) \leq \frac{lD_N(l)}{l-1} \cdot \frac{\mu_{\hat{m}}(BR_2(\hat{m}))}{R_2}(R_2 - R_1), \]
that is, inequality (12.13) for family \( \{\mu_{\hat{m}}\} \) will be proved with
\[ \hat{A} \leq \frac{lD_N(l)}{l-1}. \tag{12.43} \]

Finally choose here \( l = 1 + \frac{1}{N} \) and use Lemma 12.8. This gives the required inequality (12.41).

Hence, it remains to establish (12.42). By the definition of \( D_N(l) \), see (12.1), we have for \( l > 1 \)
\[ \mu_{\hat{m}}(B_{lR}(\hat{m})) = 2l \int_0^R \mu_{\hat{m}}(B_{ls}(\hat{m})) ds \leq lD_N(l)\mu_{\hat{m}}(BR(\hat{m})). \]

On the other hand, replacing \([0, R]\) by \([l^{-1}R, R]\) we also have
\[ \mu_{\hat{m}}(B_{lR}(\hat{m})) \geq 2l \mu_{\hat{m}}(BR(\hat{m}))(R - l^{-1}R) = 2(l - 1)R\mu_{\hat{m}}(BR(\hat{m})). \]

Combining the last two inequalities to get (12.42).

**Remark 12.10** For the proofs of corollaries it is useful to single out the next two inequalities
\[ \tilde{D}(l) \leq lD_N(l) \quad \text{and} \quad \tilde{C} \leq \frac{C_N}{l-1}D_N(l). \tag{12.44} \]

The first of them follows from the inequality next to (12.43). To prove the second one, write for \( i = 1, 2 \)
\[ |\mu_{\hat{m}_1} - \mu_{\hat{m}_2}|(BR(\hat{m}_i)) \leq 2 \int_0^R |\mu_{\hat{m}_1} - \mu_{\hat{m}_2}|(B_s(\hat{m}_i)) ds \leq 2C_N\mu_{\hat{m}_i}(BR(\hat{m}_i))d(\hat{m}_1, \hat{m}_2). \]

Combining this with inequality (12.42) we obtain the second inequality in (12.44).

We will use inequalities (12.43) and (12.44) for \( N = 0 \), i.e., for \( C_N \) equals the consistency constant \( C \) for \((M, d)\) and \( D_N(l) = D(l) \).

Now use the main result of Part I for the case of \( S \subset M \subset \hat{M} \). We conclude from here that there exists an extension operator \( \hat{E} \in Ext(S, \hat{M}) \) with norm satisfying the inequality
\[ ||\hat{E}|| \leq 8\hat{A} + \max\left(\frac{2(l + 3)}{l-1}, K(l)\right) \]
where
\[ K(l) = 6(\hat{A} + \tilde{C})\tilde{D}(l)(l + 3), \]
see (12.25) and (12.26).

Choose here \( l := 1 + \frac{1}{N+1} \) and apply inequalities (12.39)- (12.41) with \( N = [\log_2 D] + 5 \). This yields
\[ ||\hat{E}|| \leq a_0(C + a_1)(\log_2 D + 6) \tag{12.45} \]
with some $a_0 \ (< 7575)$ and $a_1 \ (< 15)$. Then the restriction of $\hat{E}f$ to $M$ gives
the required extension operator from $Ext(S,M)$ with the norm bounded by the
right-hand side of (12.45).

The proof of Theorem 2.21 is complete. 

**Proof of Corollary 2.24.** According to (2.10) the dilation function for \{\mu_m\} satisfies
\[ D(l) \leq a l^n, \quad 1 \leq l < \infty, \] (12.46)
with $a \geq 1$ and $n \geq 0$. To derive the require estimate of $\lambda(M)$ we first use inequality
(12.26) for the space $(\hat{M}, \hat{d})$ where $\hat{M} := M \times \mathbb{R}$ and $\hat{m}$, $\hat{d}$ and \{\mu_{\hat{m}}\} are defined
as in the above proof, i.e., $\hat{m} := (m, x)$ with $m \in M$ and $x \in \mathbb{R}$, $\hat{d} = d_1$ and
$\mu_{\hat{m}} := \mu_\hat{m} \otimes \lambda_1$. Then
\[ \lambda(M) \leq 8\hat{A} + \max \left( \frac{2(l + 3)}{l - 1}, \hat{K}(l) \right) \] (12.47)
where
\[ \hat{K}(l) = 6(\hat{A} + \hat{C})(l + 3)\hat{D}(l) \]
and the quantities with the hat are estimated in (12.43) and (12.44) with $N = 0$
(see Remark 12.10). In particular, one has $\hat{D}(l) \leq a l^{n+1}$, $\hat{A} \leq \frac{a l^{n+1}}{l - 1}$ and $\hat{C} \leq \frac{a l^{n+1}}{l - 1} \hat{C}$.
Taking in the last two inequalities $l = 1 + \frac{1}{n+1}$ we have
\[ \hat{A} \leq ea(n + 1), \quad \hat{C} \leq ea(n + 1)C. \]
Inserting this in (12.47) we get
\[ \lambda(M) \leq 8ea(n + 1) + \max \left( \frac{2(l + 3)}{l - 1}, \hat{K}(l) \right) \]
where now
\[ \hat{K}(l) \leq 6ea(n + 1)(C + 1)(l + 3)\hat{D}(l), \]
and $\hat{D}(l) \leq a l^{n+1}$. Since $l := 1 + \frac{1}{n+1}$, we straightforwardly obtain the inequality
\[ \lambda(M) \leq 225 a^2 (C + 1)(n + 1). \]

**Proof of Corollary 2.25.** We follow the same argument using now the inequality
(12.29) and then choosing for $n \geq 1$ the value $l = 1 + \frac{2}{3n}$. Then a straightforward
computation yields for this choice of $l$
\[ \lambda(M) \leq 8n + \max \left( \frac{2(l + 3)}{l - 1}, (l + 3)(C + 2nl^{n-1}) \right) \leq 24(n + C). \]
For $n < 1$ we simply choose $l = 2$. 

**Proof of Theorem 2.27.** (a) Let, first, $p = \infty$. Since the metric in $M := \oplus_{\infty} \{(M_i, d_i)\}_{1 \leq i \leq N}$ is given by $d(m, m') := \max_{1 \leq i \leq N} d_i(m_i, m'_i)$, the ball $B_R(m)$ of

75
$M$ is the product of balls $B_R(m_i)$ of $M_i$, $1 \leq i \leq N$. Therefore for a family of doubling measures $\{\mu_m\}_{m \in M}$ given by the tensor product

$$\mu_m := \bigotimes_{i=1}^{N} \mu_{m_i}^{i}, \quad m = (m_1, \ldots, m_N),$$

we get

$$\mu_m(B_R(m)) = \prod_{i=1}^{N} \mu_{m_i}^{i}(B_R(m_i)).$$

Hence for the dilation function (12.1) of the family $\{\mu_m\}_{m \in M}$ we get

$$D(l) = \prod_{i=1}^{N} D_i(l)$$

where $D_i$ is the dilation function of $\{\mu_{m_i}^{j}\}_{m \in M_i}$. In particular, $\{\mu_m\}_{m \in M}$ satisfies the uniform doubling condition (2.7) with $D := D_1 \cdots D_N$.

Check that the condition (2.8) holds for this family with the constant

$$\tilde{C}_\infty := \left(\prod_{i=1}^{N} K_i\right) \sum_{i=1}^{N} C_i.$$

In fact, the identity

$$\mu_m - \mu_{\tilde{m}} = \sum_{i=1}^{N} \left(\bigotimes_{j \neq i}^{i-1} \mu_{m_j}^{j} \right) \otimes \left(\mu_{m_i}^{i} - \mu_{\tilde{m}_i}^{i}\right) \otimes \left(\bigotimes_{j=i+1}^{N} \mu_{m_j}^{j}\right)$$

(12.52)

together with (12.49), and (2.8) and $K_j$-uniformity of $\{\mu_{m_i}^{j}\}_{m \in M_j}$ implies that for $\tilde{m} = m$ or $\tilde{m}$

$$|\mu_m - \mu_{\tilde{m}}|(B_R(\tilde{m})) \leq \sum_{i=1}^{N} \left(\prod_{j \neq i}^{i-1} K_j\right) C_i \frac{\mu_m(B_R(m))}{R} d_i(m, \tilde{m}_i) \leq \tilde{C}_\infty \frac{\mu_m(B_R(m))}{R} d(m, \tilde{m}).$$

Thus $\oplus_\infty \{(M_i, d_i)\}_{1 \leq i \leq N}$ is of pointwise homogeneous type with respect to the family (12.48) with the optimal constants bounded by $D$ and $\tilde{C}_\infty$ (and so we have the required estimate for $\lambda(M)$ in this case).

Let now $\mu_{m_i}^{i}(B_R(m)) = \gamma_i R^{n_i}$ for some $\gamma_i, n_i > 0$ and all $m \in M_i$ and $R > 0$, $1 \leq i \leq N$. In this case $\{\mu_{m_i}^{i}\}_{m \in M_i}$ is clearly $K_i$-uniform with $K_i = 1$. Moreover, by (12.49)

$$\mu_m(B_R(m)) = \gamma R^n, \quad n := \sum_{i=1}^{N} n_i.$$

Hence $M$ equipped with the family (12.48) satisfies the conditions of Corollary 2.25 with this $n$ and $C = \sum_{i=1}^{N} C_i$, see (12.51). Applying this corollary we get

$$\lambda(\oplus_\infty \{(M_i)\}_{1 \leq i \leq N}) \leq 24 \sum_{i=1}^{N} (n_i + C_i).$$
(b) Let now \(1 \leq p < \infty\). In this case we cannot estimate the optimal constants \(C\) and \(D\) for the space

\[
(M, d) := \bigoplus_p \{ (M_i, d_i) \}_{1 \leq i \leq N}
\]

(12.53)
directly. To overcome this difficulty we use the argument of Theorem 2.21 and isometrically embed this space into the space

\[
(\hat{M}, \hat{d}) := (M, d) \oplus_1 l^a
\]

with a suitable \(a\). Hence, a point \(\hat{m} \in \hat{M}\) is an \((N + a)\)-tuple

\[
\hat{m} := (m, x) := (m_1, \ldots, m_N, x_1, \ldots, x_a)
\]

with \(m \in \prod_{i=1}^N M_i\) and \(x \in \mathbb{R}^a\). Moreover, the metric \(\hat{d}\) is given by

\[
\hat{d}(\hat{m}, \hat{m}') := \left( \sum_{i=1}^N d_i(m_i, m_i')^p \right)^{1/p} + \sum_{i=1}^a |x_i - x_i'|.
\]

Endow \(\hat{M}\) with a family of measures given by the tensor product

\[
\mu_{\hat{m}} := \mu_m \otimes \Lambda_a, \quad \hat{m} \in \hat{M},
\]

where \(\Lambda_a\) is the Lebesgue measure on \(\mathbb{R}^a\) and \(\mu_m := \otimes_{i=1}^N \mu_{m_i}\).

We will show that \(\lambda(\hat{M})\) is bounded as required in Theorem 2.27. This immediately gets the desired estimate for \(\lambda(M)\) and completes the proof of the theorem.

To accomplish this we need

**Lemma 12.11** The optimal uniform doubling constant \(D\) of the family \(\{\mu_m\}_{m \in M}\) satisfies

\[
D \leq \prod_{i=1}^N D_i.
\]

Recall that \(D_i\) is the optimal uniform doubling constant of \(\{\mu_{m_i}^i\}_{m_i \in M_i}\).

**Proof** (induction on \(N\)). For the \(\mu_m\)-measure of the ball

\[
B_{2R}(m) := \{ m' \in M : \sum_{i=1}^N d_i(m_i, m_i')^p \leq (2R)^p \}
\]

we get by the Fubini theorem:

\[
\mu_m(B_{2R}(m)) = \int_{d^1 < (2R)^p} d\mu_1^1(m') \int_{d_1 < (2R)^p} \mu_1(m_1') d\mu_1(m_1').
\]

Here we set for simplicity:

\[
d_a := \sum_{i=2}^N d_i(m_i, m_i')^p, \quad d_1 := d(m_1, m_1')^p, \quad \mu_1 := \otimes_{i=2}^N \mu_{m_i}^i, \quad \mu_1 := \mu_{m_1}^1.
\]
The second integral is the $\mu_1$-measure of the ball $B_{2\rho}(m_1)$ where $\rho := \sqrt[2-p]{d_1}$ which is bounded by $D_1 \mu_1(B_{\rho}(m_1))$. This and the Fubini theorem imply

$$
\mu_m(B_{2R}(m)) \leq D_1 \int_{d_1 < R^{\rho - 2p} \delta_1} d\mu_1(m') \int_{d_1 < R^{\rho - 2p} \delta_1} d\mu_1(m_1) =
D_1 \int_{d_1 < R^{\rho - 2p} \delta_1} d\mu_1(m_1) \int_{d_1 < (2R)^{\rho - 2p} \delta_1} d\mu_1(m').
$$

By the induction hypothesis the inner integral in the right-hand side is bounded by

$$
\left( \prod_{i=2}^{N} D_i \right) \mu_1(B_{\rho^{1}}(m_2, \ldots, m_N)) = \prod_{i=2}^{N} D_i \int_{d_1 < R^{\rho - d_1}} d\mu_1(m').
$$

Combining this with the previous inequality to get the required result:

$$
\mu_m(B_{2R}(m)) \leq \left( \prod_{i=1}^{N} D_i \right) \mu_m(B_{R}(m)). \quad \Box
$$

Using Lemma 12.11 we estimate now the dilation function $D_a(s)$ of the family $\{\mu_{\tilde{m}}\}$. Recall that for $s > 1$

$$
D_a(s) := \sup_{\tilde{m} \in \tilde{M}} \left\{ \frac{\mu_{\tilde{m}}(B_{sR}(\tilde{m}))}{\mu_{\tilde{m}}(B_{R}(\tilde{m}))} \right\} \quad (12.54)
$$

To this end we simply apply to this setting Lemma 12.8 with $D$ replaced by $\prod_{i=1}^{N} D_i$ and $N$ by $a$. This gets

**Lemma 12.12** If $a \geq \lfloor \log_2 \prod_{i=1}^{N} D_i \rfloor + 5$, then

$$
D_a(1 + 1/a) \leq \frac{6}{5} e^4. \quad \Box
$$

Now we estimate the consistency constant for the family $\{\mu_{\tilde{m}}\}_{\tilde{m} \in \tilde{M}}$, see Definition 2.19. To this goal we use (12.52) for $\mu_{\tilde{m}} - \mu_{\tilde{m}'}$ and then apply the Fubini theorem to have for $\tilde{m}'' := \tilde{m}$ or $\tilde{m}'$

$$
|\mu_{\tilde{m}} - \mu_{\tilde{m}'}|(B_{R}(\tilde{m}'')) \leq
\sum_{i=1}^{N} \int_{\delta_a < R} d\Lambda_a \int_{d_1 < (R - \delta_a)^p} d\mu_i d\mu_i \int_{d_1 < (R - \delta_a)^p - d^i} d\mu_{m_i}^i - \mu_{m_i}^i. \quad (12.55)
$$

Here we use the notations:

$$
\delta_a := \sum_{j=1}^{a} |x_j - x_j'|, \quad d^i := \sum_{j \neq i} d_j(m''_j, m_j)^p, \quad d_i := d(m''_i, m_i)^p,
$$

$$
\mu_i' := \bigotimes_{j<i} \mu_{m_j}' \quad \mu_i := \bigotimes_{j>i} \mu_{m_j}.
$$

78
Recall that $\hat{m} = (m, x) \in M \times \mathbb{R}^a$.

The inner integral in the $i$-th term of the right-hand side of (12.55) equals $|\mu_{m_i}^i - \mu_{m_i}^{i'}(B_\rho(m_i^{i'}))|$ where $\rho := \sqrt{(R - \delta_a)^p - d^i}$. Replacing here $\rho$ by $\rho_a := \sqrt{(R_a - \delta_a)^p - d^i}$ with $R_a := (1 + \frac{1}{a})R$ and applying the consistency inequality for $(M_i, d_i)$ we then bound this inner integral by

$$\frac{C_i \mu_{m_i}^{i'}(B_{\rho_a}(m_i^{i'}))}{\rho_a} d_i(m_i, m_i').$$

Since $d^i \leq (R - \delta_a)^p$, the denominator here is at least $R_a - R = \frac{1}{a}R$. Therefore the inner integral is bounded by

$$\frac{a C_i d_i(m_i, m_i')}{R} \int_{d_i < (R_a - \delta_a)^p - d^i} d\mu_{m_i}^{i'},$$

Inserting this in (12.55) and replacing there $R$ by $R_a$ we get

$$|\mu_\hat{m} - \mu_\hat{m'}|(B_R(\hat{m}'')) \leq \frac{a}{R} \sum_{i=1}^N C_i d_i(m_i, m_i') \int_{B_{R_a}(\hat{m}'')} d\Lambda_a d\mu_i d\mu_i'.
$$

To replace in this inequality each $\mu_{m_i}^j$ (or $\mu_{m_j}^i$) by $\mu_{m_i}^{j'}$ we now use $K_j$-uniformity of the family $\{\mu_{m_j}^j\}_{m_j \in M_j}$, see Definition 2.26. Applying this to the right-hand side of the previous inequality and recalling definition (12.54) we estimate the $i$-th integral there by

$$\left(\prod_{i=1}^N K_i\right) \int_{B_{R_a}(\hat{m}'')} d\Lambda_a d\mu_{m_i}^{j'} = \left(\prod_{i=1}^N K_i\right) \mu_{\hat{m}''}(B_{R_a}(\hat{m}'')).$$

Combining with the previous inequality we get for $\hat{m}'' = \hat{m}$ or $\hat{m}'$

$$|\mu_\hat{m} - \mu_\hat{m'}|(B_R(\hat{m}'')) \leq \frac{a D_a(1 + 1/a)}{R} \left(\prod_{i=1}^N K_i\right) \left(\sum_{i=1}^N C_i d_i(m_i, m_i')\right) \mu_{\hat{m}''}(B_R(\hat{m}'')).
$$

By the Hölder inequality the sum in the brackets is at most

$$\left(\sum_{i=1}^N C_i^q\right)^{1/q} \left(\sum_{i=1}^N \left(d_i(m_i, m_i')\right)^p\right)^{1/p} =: \left(\sum_{i=1}^N C_i^q\right)^{1/q} d(m, m');$$

here $\frac{1}{p} + \frac{1}{q} = 1$. Hence the consistency constant $\hat{C}$ of the family $\{\mu_{\hat{m}}\}_{\hat{m} \in \hat{M}}$ satisfies

$$\hat{C} \leq a D_a(1 + 1/a) \left(\prod_{i=1}^N K_i\right) \left(\sum_{i=1}^N C_i^q\right)^{1/q}.
$$

(12.56)
Choose now \( a := [\log_2 \prod_{i=1}^{N} D_i] + 5 \) and use (12.43) for the space \((\hat{M}, \hat{d})\) equipped with the family \( \{\mu_m\}_{\hat{m} \in \hat{M}} \). Since \( \hat{A} \) in (12.43) is bounded by \( \frac{D_0(s)}{s-1} \), with \( s = 1 + 1/a \), we therefore get from Lemma 12.12
\[
\hat{A} \leq \frac{6}{5} e^4 \left( \log_2 \left( \prod_{i=1}^{N} D_i \right) + 6 \right).
\]
Combining Lemma 12.12 with (12.56) and the above inequality we finally obtain the required result (see (12.26))
\[
\lambda(\hat{M}) \leq c_0(\tilde{C}_p + 1) \left( \log_2 \left( \prod_{i=1}^{N} D_i \right) + 1 \right)
\]
with \( \tilde{C}_p := \left( \sum_{i=1}^{N} C_i^q \right)^{1/q} \left( \prod_{i=1}^{N} K_i \right) \) and \( \frac{1}{p} + \frac{1}{q} = 1 \). \( \Box \)

### 13 Appendix: A Duality Theorem

Our goal is to prove Theorem A of section 5, that is, we have to find a Banach space \( V \) such that its dual
\[
V^* = \text{Lip}_0(M)
\]
and all evaluations \( \delta_m : \phi \to \phi(m), \phi \in \text{Lip}_0(M), m \in M, \) belong to \( V \).

We introduce this as a (closed) subspace of the Banach space \( l_\infty(B) \) where \( B \) is the closed unit ball of \( \text{Lip}_0(M) \). To this end, define a map
\[
\Phi : M \to l_\infty(B)
\]
given for \( m \in M \) by
\[
\Phi(m)(b) := b(m), \quad b \in B.
\]
As all functions of \( \text{Lip}_0(M) \) vanish at a prescribed point \( m^* \), we have for \( b \in B \)
\[
|b(m) - b(m^*)| \leq d(m, m^*),
\]
and \( \Phi(M) \) is, actually, a subset of \( l_\infty(B) \).

We now define the desired Banach space by
\[
V := \text{span} \ \Phi(M),
\]
the closure in \( l_\infty(B) \) of the linear span of \( \Phi(M) \) endowed by the norm induced from \( l_\infty(B) \).

Then introduce the required isometry \( I \) of the dual \( V^* \) to \( V \) onto \( \text{Lip}_0(M) \) as the pullback of the map \( \Phi : M \to V \); that is to say, we let for \( l \in V^* \)
\[
I(l)(m) := l(\Phi(m)), \quad m \in M.
\]

**Assertion 1.** The linear operator \( I \) is an injection.
In fact, if \( I(l) = 0 \) for some \( l \in V^* \), then \( l|_{\Phi(M)} = 0 \) and, by (13.3), \( l = 0 \).

**Assertion 2. It is true that**

\[
\text{Lip}_0(M) \subset I(V^*) , \quad \text{and} \quad ||I|| := \sup\{||I(l)||_{\text{Lip}(M)} : ||l|| \leq 1\} \geq 1 .
\]  

(13.5)

Actually, \( \Phi(m^*) = 0 \) and therefore each function \( I(l) \) vanishes at \( m^* \). Let now \( b \in B \) and \( \pi_b : l_\infty(B) \to \mathbb{R} \) be the canonical projection given by

\[
\pi_b(x) := x(b) , \quad x \in l_\infty(B) .
\]  

(13.7)

Then by (13.4) and (13.2)

\[
I(\pi_b|_V) (m) = \pi_b(\Phi(m)) = \Phi(m)(b) = b(m)
\]  

(13.8)

for all \( m \in M \). Since \( \pi_b|_V \in V^* \) and \( b \) is an arbitrary element of the unit ball in \( \text{Lip}_0(M) \), the embedding (13.5) holds. Besides,

\[
||I|| \geq \sup_{b \in B} ||I(\pi_b|_V)||_{\text{Lip}_0(M)} = 1 ,
\]  

(13.6)

and (13.6) is also true.

**Assertion 3. It is true that**

\[
I(V^*) \subset \text{Lip}_0(M) , \quad \text{and} \quad ||I|| \leq 1 .
\]  

(13.9)

(13.10)

Let \( l \in V^* \) and \( m_1 \neq m_2 \in M \). We have to show that

\[
|I(l)(m_1) - I(l)(m_2)| \leq d(m_1,m_2) ||l|| ;
\]  

(13.11)

as, in addition, \( I(l)(m^*) = 0 \), this will prove the assertion.

To establish (13.11), extend \( l \) by the Hahn-Banach theorem to \( \widehat{l} \in l_\infty(B)^* \). Hence,

\[
\widehat{l}|_V = l \quad \text{and} \quad ||\widehat{l}|| = ||l|| .
\]  

(13.12)

Now, using the Gelfand transform we identify \( l_\infty(B) \) with the space \( C(\beta B) \) of continuous functions on the space \( \beta B \) of maximal ideals of the Banach algebra \( l_\infty(B) \). In fact, \( \beta B \) is the Stone-Čech compactification of \( B \) regarded as a topological space endowed by the discrete topology, see, e.g., [Lo]. By the F. Riesz theorem there exists a bounded (regular) Borel measure \( \mu_\widehat{l} \) on \( \beta B \) such that

\[
\widehat{l}(x) := \int_{\beta B} g(x)d\mu_\widehat{l} , \quad x \in l_\infty(B) ;
\]  

(13.13)

here \( g(x) \in C(\beta B) \) is the Gelfand transform of \( x \); recall that in this case \( g(x) \) is the continuous extension of \( x \) from \( B \) to \( \beta B \).
Let now $x \in V$, and $\{U_i\}$ be a finite open cover of $\beta B$ such that the oscillation of $g(x)$ on each $U_i$ is at most $\epsilon$. Since $B$ is dense in $\beta B$, every $U_i$ contains a point $b_i \in B$, and therefore

$$|g(x)(\omega) - g(x)(b_i)| = |g(x)(\omega) - x(b_i)| < \epsilon$$

for every $\omega \in U_i$. Hence for such an $x$

$$\left| \int_{\beta B} g(x)d\mu_l - \sum x(b_i)\mu_l(U_i) \right| < \epsilon \cdot \text{Var}\mu_l = \epsilon ||l||,$$

see (13.12). The sum here can be written as $l_\epsilon(x)$ where $l_\epsilon \in V^*$ is given by

$$l_\epsilon := \sum (\pi_{b_i}|_V)\mu_l(U_i),$$

see (13.7) and (13.8). So, together with (13.13) and (13.12) this leads to the estimate

$$|l(x) - l_\epsilon(x)| < \epsilon ||l||.$$

Choose here $x := \Phi(m_1) - \Phi(m_2)$ and use (13.4). This implies that

$$|(I(l)(m_1) - I(l)(m_2)) - (I(l_\epsilon)(m_1) - I(l_\epsilon)(m_2))| < \epsilon ||l||.$$

Besides, by the definition of $l_\epsilon$

$$I(l_\epsilon)(m) = \sum b_i(m)\mu_l(U_i), \quad m \in M,$$

and so $I(l_\epsilon) \in Lip_0(M)$ and

$$||I(l_\epsilon)||_{Lip_0(M)} \leq \text{Var}\mu_l = ||l||,$$

see (13.12) and (13.13). Together with the previous inequality this yields

$$|I(l)(m_1) - I(l)(m_2)| \leq d(m_1, m_2)||l|| + \epsilon ||l||.$$

Letting $\epsilon$ to 0, we conclude that $I(l) \in Lip_0(M)$ and $||I(l)||_{Lip_0(M)} \leq ||l||$. This proves (13.9) and (13.10).

Now the Assertions 1-3 prove that $I$ is an isometry of $V^*$ onto $Lip_0(M)$. Besides, the evaluation functionals $\delta_m : \phi \to \phi(m), \phi \in Lip_0(M)$, can be presented as $\delta_m = \Phi(m)$, see (13.2), and therefore belong to $V$.

The proof is complete. $\square$

**Remark 13.1** Let $U \subset V$ be the closure of the convex hull of the set

$$\{v_{x,y} := (x - y)/d(x,y) : x, y \in \Phi(M), x \neq y\}.$$ It is easily seen that $U$ is the closed unit ball of $V$. 

82
References

[Ba] H. Bass, The degree of polynomial growth of finitely generated groups, Proc. London Math. Soc., 25 (1972), 603-614.

[BB] A. Brudnyi and Yu. Brudnyi, Linear and nonlinear extensions of Lipschitz functions from subsets of metric spaces, Preprint (2005), 9 pp.

[BH] M. R. Bridson and A. Haefliger, Metric spaces of non-positive curvature, Springer, 1999.

[BL] Y. Benyamini and J. Lindenstrauss, Geometric nonlinear functional analysis, Vol 1, American Mathematical Society Colloquium Publications, 48, Providence, RI, 2000.

[Bo] A. Borel, Compact Clifford-Klein forms of symmetric spaces, Topology, 2, No. 2 (1963), 111-122.

[Bor] K. Borsuk, Über Isomorphic der Funktionalräume, Bull. Int. Acad. Pol. Sci. (1933), 1-10.

[BoSch] M. Bonk and O. Schramm, Embeddings of Gromov hyperbolic spaces, Geom. Func. Anal., Vol. 10 (2000), 266-306.

[BSh1] Yu. Brudnyi and P. Shvartsman, A linear extension operator for a space of smooth functions defined on closed subsets of \(\mathbb{R}^n\), Dokl. Acad. Nauk SSSR, 280 (1985), 268-272. English transl. in Soviet Math. Dokl., 31 (1985), 48-51.

[BSh2] Yu. Brudnyi and P. Shvartsman, The Whitney problem of existence of a linear extension operator, J. Geom. Analysis, 7, No. 4 (1997), 515-574.

[Bou] J. Bourgain, The metrical interpretation of superreflexivity in Banach spaces, Israel J. Math., 56, No. 2 (1986), 222-230.

[CE] J. Cheeger and D. Ebin, Comparison theorems in Riemannian Geometry, North Holland, 1975.

[CG] J. Cheeger and M. Gromov, Bounds of the von Neumann dimension of \(L^2\)-cohomology and the Gauss-Bonnet theorem for open manifolds, J. Diff. Geometry, 21 (1985), 1-34.

[CW] R. R. Coifman and G. Weiss, Extensions of Hardy spaces and their use in analysis, Bull. Amer. Math. Soc., 83 (1977), 569-645.

[Du] R. M. Dudley, Real Analysis and Probability, Chapman and Hall, 1989.

[F1] Ch. Fefferman, A sharp form of Whitney’s extension theorem, Annals of Math. (to appear), 76 pps.
[F2] Ch. Fefferman, Interpolation and extrapolation of smooth functions by linear operators, Revista Math. Iberoamericana (to appear), 43 pps.

[F3] Ch. Fefferman, Whitney’s extension problems for certain function spaces, Revista Math. Iberoamericana (to appear), 78 pps.

[F4] Ch. Fefferman, Extension of $C^{k,\omega}(\mathbb{R}^n)$-functions by linear operators, Preprint, Princeton (2004), 59 pps.

[FS] G.B. Folland and E.M. Stein, Hardy Spaces on Homogeneous Groups, Math. Notes, 28, Princeton Univ. Press, 1982.

[GK] G. Godefroy and N. J. Kalton, Lipschitz-free spaces, Studia Math., 159 (2003), 121-141.

[Gl] G. Glaeser, Étude de quelques algèbres tayloriennes, J. d’Analyse Math., 6 (1958), 1-124.

[Gr1] M. Gromov (with Appendix by M. Katz, P. Pansu and S. Semmes), Metric structures for Riemannian and non-Riemannian spaces, Progress in Mathematics, 152, Birkhäuser, 1999.

[Gr2] M. Gromov, Groups of polynomial growth and expanding maps, Publ. Math. IHES, 53 (1981), 53-78.

[Gr3] M. Gromov, Hyperbolic groups, Essays in Group Theory (S. M. Gersten, ed.), MSRI Publ., 8 (1987), 75-263.

[GrP] M. Gromov and I. Piatetski-Shapiro, Nonarithmetic groups in Lobachevsky spaces, Publ. Math. IHES, 66 (1988), 93-103.

[He] J. Heinonen, Calculus on Carnot Groups, Ber. Univ. Jyväskylä Math. Inst., 68 (1995), 1-31.

[HK] P. Hajlasz and P. Koskela, Sobolev met Poincaré, Memoirs Amer. Math. Soc., 145, No. 688 (2000), 1-101.

[HR] E. Hewitt and K.A. Ross, Abstract Harmonic Analysis, Springer, 1963.

[JL] W. B. Johnson and J. Lindenstrauss, Extensions of Lipschitz mappings into a Hilbert space, Contemp. Math., 26, Amer. Math. Soc. Providence, RI, 1984, 189-206.

[JLS] W. B. Johnson, J. Lindenstrauss and G. Schechtman, Extension of Lipschitz maps into Banach spaces, Israel J. of Math., 54, No. 2 (1986), 128-138.

[K] R. Karidi, Geometry of balls in nilpotent Lie groups, Duke Math. J., 74, No. 2 (1994), 301-317.

[Ke] J. L. Kelley, General Topology, D. Van Nostrand Company, 1955.
[Ki] M. D. Kirszbraun, Über die zusammenziehende und Lipschitsche Transformationen, Fund. Math., 22 (1934), 77-108.

[KR] L. V. Kantorovich and G. Sh. Rubinshtein, On the space of completely additive functions, Vestnik LGU, Ser. Mat., Mekh. i Astron., 7 (1958), 52-59.

[L] J. Lindenstrauss, On non-linear projections in Banach spaces, Michigan Math. J., 11 (1964), 268-287.

[LN] J. R. Lee and A. Naor, Extending Lipschitz functions via random metric partition, Invent. Math., 160 (2005), 50-95.

[Lo] I. H. Loomis, An Introduction to Abstract Harmonic Analysis, Van Nostrand, Princeton, 1953.

[LSch] U. Lang and V. Schoeder, Kirszbraunn’s theorem and metric spaces of bounded curvature, GAFA, vol. 7 (1987), 535-560.

[Ma] J. Matoušek, Extension of Lipschitz mappings on metric trees, Comment. Math. Univ. Carolinae 31, no.1 (1990), 99-104.

[Mc] E. McShane, Extension of range of functions, Bulletin AMS, 40, No. 12 (1934), 837-842.

[M] J. Milnor, A note on curvature and fundamental groups, J. Diff. Geometry, 2 (1968), 1-7.

[MP] M. B. Markus and G. Pisier, Characterization of almost surely continuous p-stable random Fourier series and strongly stationary processes, Acta Math., 152, No. 3-4 (1984), 245-301.

[N] P. S. Novikov, Sur les fonctions implicites measurables B, Fund. Math., vol. XVII (1931), 8-25.

[NPSS] A. Naor, Y. Peres, O. Schramm and S. Sheffield, Markov chains in smooth Banach spaces and Gromov hyperbolic metric spaces, Preprint (2004), available at http://arxiv.org/abs/math.FA/0410422.

[P] A. Pełczyński, Linear extensions, linear averagings, and their applications to linear topological classification of spaces of continuous functions, Dissertationes Math. LVIII, PAN, Warszawa, 1968.

[R] K. H. Rosen, Handbook of Discrete and Combinatorial Mathematics, CRC Press, 2000.

[S] A. Sobczyk, Projections in Minkowski and Banach spaces, Duke Math. J., 8 (1941), 78-106.

[St] E. M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton Univ. Press, 1970.
[V] F. A. Valentine, A Lipschitz condition preserving extension for a vector function, Amer. J. Math., 67 (1945), 89-93.

[W1] H. Whitney, Analytic extensions of differentiable functions defined on closed sets, Trans. Amer. Math. Soc., 36 (1934), 63-89.

[W2] H. Whitney, Differentiable functions defined on closed sets I, Trans. Amer. Math. Soc., 36 (1934), 369-387.

[We] N. Weaver, Lipschitz algebras, World Scientific, Singapore, 1999.