Higher-Order Asymptotic Properties of Kernel Density Estimator with Global Plug-In and Its Accompanying Pilot Bandwidth

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Abstract

This study investigates the effect of bandwidth selection via a plug-in method on the asymptotic structure of the nonparametric kernel density estimator. We generalise the result of Hall and Kang (2001) and find that the plug-in method has no effect on the asymptotic structure of the estimator up to the order of \( O\left(\frac{nh_0}{n} - \frac{1}{2} + h_0^{L_0}\right) = O\left(\frac{1}{n} - \frac{L}{2L+1}\right) \) for a bandwidth \( h_0 \) and any kernel order \( L \) when the kernel order for pilot estimation \( L_p \) is high enough. We also provide the valid Edgeworth expansion up to the order of \( O\left(\frac{nh_0}{n} - \frac{1}{2} + h_0^{2L}L\right) = O\left(\frac{1}{n} - \frac{L+1}{2L+1}\right) \) and find that, as long as the \( L_p \) is high enough, the plug-in method has an effect from on the term whose convergence rate is \( O\left(\frac{nh_0}{n} - \frac{1}{2} + h_0^{L_0}\right) = O\left(\frac{1}{n} - \frac{L+1}{2L+1}\right) \). In other words, we derive the exact achievable convergence rate of the deviation between the distribution functions of the estimator with a deterministic bandwidth and with the plug-in bandwidth. In addition, we weaken the conditions on kernel order \( L_p \) for pilot estimation by considering the effect of pilot bandwidth associated with the plug-in bandwidth. We also show that the bandwidth selection via the global plug-in method possibly has an effect on the asymptotic structure even up to the order of \( O\left(\frac{nh_0}{n} - \frac{1}{2} + h_0^{L_0}\right) \). Finally, Monte Carlo experiments are conducted to see whether our approximation improves previous results.

Keywords: nonparametric statistics, kernel density estimator, plug-in bandwidth, Edgeworth expansion, coverage probability

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1 Introduction

In nonparametric statistics, the target of statistical inference is a function or an infinite dimensional vector \( f \) that is not specifically modelled itself (See Wasserman (2006) for introductory overviews, Giné and Nickl (2016) for mathematically unified understanding and Ichimura and Todd (2007) and Chen (2007) for overviews especially in the context of economic literature). One of the important components of the function \( f \) is the density function because, in statistics and its related fields, there are cases where we are interested in the distribution as a wage distribution (See e.g. DiNardo et al. (1996)) or where a target of statistical inference depends on the density function as a conditional expectation function. Although there are different methods for estimating a density function, we focus on the estimator based on the kernel method, namely kernel density estimator (KDE), also called Rosenblatt estimator or Rosenblatt-Parzen estimator after their pioneering works (Rosenblatt (1956) and Parzen (1962)).

The first-order asymptotic properties of KDE have been studied over a long period and it has been proven that, under certain conditions, KDE has pointwise consistency and asymptotic normality (see e.g. Parzen (1962), and the monograph by (Li and Racine, 2007, pp.28-30)). As we will review in Section 2, the rate of convergence of KDE is slower than the parametric rate, and furthermore, becomes slower as the dimension increases. This property is called the curse of dimensionality. We can understand this as being the cost of using local data to avoid misspecification. Hall (1991) has clarified the higher-order asymptotic properties of the estimator in both non-Studentised and Studentised cases. The asymptotic expansion of KDE is no longer a series of \( n^{-1/2} \) as parametric estimators, but a series of \( (nh)\) \(-1/2\), even in the non-Studentised case; it is a more complicated series in the Studentised case, where \( n \) and \( h \) are the sample size and bandwidth, respectively.

Bandwidth \( h \) specifies the flexibility of statistic models and is adjusted between the bias and variance trade-offs in the sense that creating flexible models and consequently decreasing the bias results in increasing variance while creating non-flexible models and decreases the variance results in increasing bias. It is well known that the performance of the kernel-based estimators depends greatly on the bandwidth, not so much on the kernel function. By defining a loss function, one can compute the theoretically optimal bandwidth \( h_0 \) that minimises loss. For example, mean integrated squared error (MISE) is the most commonly used global loss measure. However, in practice, such a bandwidth is typically infeasible because it depends on the unknown density. Therefore, one has to choose the bandwidth in a data-driven manner. Among the many bandwidth selection methods, two famous ones are cross-validation and plug-in method. In this paper, we focus on the latter.

It is natural to ask whether the choice of bandwidth affects the asymptotic structure of the estimator. Ichimura (2000) and Li and Li (2010) have considered the asymptotic distribution of kernel-based non/semiparametric estimators with data-driven bandwidth. They argue that, under certain conditions, the bandwidth selection has no effect on the first-order asymptotic structure of the estimators. Hall and Kang (2001) showed that the bandwidth selection by the global plug-in method also has no effect on the asymptotic structure of KDE up to the order of \( O(n^{-2/5}) \) for \( L = 2 \) and \( L_p = 6 \), where \( L \) and \( L_p \) are kernel orders for the density estimation and estimation of an unknown part of the optimal bandwidth, respectively.

Our contributions are fivefold. First, we provide the Edgeworth expansion of KDE with global plug-in bandwidth up to the order of \( O((nh_0)^{-1/2} + h_0^{L_p}) = O(n^{-1/2}) \) and show that the bandwidth selection by the plug-in method begins to affect the term whose convergence rate is \( O((nh_0)^{-1/2} h_0 + h_0^{L_p+1}) = O(n^{-1/2}) \) under the condition that \( L_p \) is large enough. Second, we generalise Theorem 3.2 of Hall and Kang (2001), which states that bandwidth selection via the global plug-in method has no effect on the asymptotic structure of KDE up to the order of \( O((nh_0)^{-1/2} + h_0^{L_p}) = O(n^{-2/5}) \). Their results limit the order of kernel functions \( K(u) \) and \( H(u) \) to \( L = 2, L_p = 6 \), respectively, but we show that they are valid for general orders \( L \) as well under the condition that \( L_p \) is large enough. Third, we explore Edgeworth expansion of KDE with deterministic bandwidth in more detail than Hall (1991). We show that Edgeworth expansion of Standardised KDE with deterministic bandwidth has the term of order \( O((nh_0)^{-1/2} + h_0^{L_p}) = O(n^{-2/5}) \) right after the term \( \Phi(z) \) with a gap between them. After that however, the terms decrease at the rate of \( O(h_0) = O(n^{-1/2}) \). However, the result of Hall and Kang (2001) and our results above need the kernel order \( L_p \) for the estimation of unknown parts of the optimal bandwidth to be high enough. We have two motivations to avoid imposing this condition on \( L_p \). One is that although the higher-order kernel is theoretically justified, in terms of implementation using a computer, it has undesirable properties. The other is that the condition forces pilot bandwidth to be relatively large but the range is restrictive especially in multidimensional settings. For details of the latter motivation, see the seminal works of Cattaneo et al. (2010, 2013, 2014a,b) and Cattaneo and Jansson (2018). Then, as a fourth contribution, we weaken this condition on \( L_p \) assumed by Hall and Kang (2001) and our Theorem 3.1 and provide the Edgeworth...
expansion including the effect of pilot bandwidth up to the order of $O((nh_0)^{-1} + h_0^{2L})$. In this situation, the bandwidth selection via the global plug-in method possibly has an effect on the asymptotic structure of KDE even up to the order of $O((nh_0)^{-1/2} + h_0^{L})$ (for example, when $L = 2$ and $L_p = 2$). Finally, we consider the intersectional effect of the bandwidth selection via the global plug-in method, its accompanying pilot bandwidth, and Studentisation. The proof of our main theorem owes much to Nishiyama and Robinson (2000). They have established the valid Edgeworth expansion for the semiparametric density-weighted averaged derivatives estimator of the single index model, which has an exact second-order $U$-statistic form. Although the higher-order asymptotic structure of $U$-statistics had been studied before Nishiyama and Robinson (2000) (See e.g. Callaert et al. (1980)), the estimator is different from standard $U$-statistics in that it is $U$-statistics whose kernel depends on the sample size $n$ through the bandwidth. Since KDE with plug-in bandwidth can also be approximated by a sum of first- and second-order $U$-statistics whose kernel depends on the sample size $n$ through the bandwidth, we can benefit from their proof.

The remainder of this paper is organised as follows. In the next section, we introduce KDE and review its known properties. Section 3 provides the main results, namely the Edgeworth expansion of the estimator with the global plug-in bandwidth. In section 4, we employ Monte Carlo studies to compare our results with those of previous works. Section 5 concludes and discusses future research directions.

2 Review of the Estimator’s Properties

2.1 Estimator and Its First Order Properties

Assumption 1. Let $\{X_i\}_{i=1}^n$ be a random sample with an absolutely continuous distribution with Lebesgue density $f$.

First, we introduce nonparametric KDE $\hat{f}$ for unknown density $f$. Estimator $\hat{f}$ at a point $x$ with a bandwidth $h$ is defined as follows:

$$\hat{f}_h(x) = \frac{1}{nh} \sum_{i=1}^n K\left(\frac{X_i - x}{h}\right) = \frac{1}{nh} \sum_{i=1}^n K_{i,h}(x),$$

where $K$ is a kernel function, and we say that $K$ is a $L$-th order kernel, for a positive integer $L$, if

$$\int x^l K(u) du = \begin{cases} 1 & (l = 0) \\ 0 & (1 \leq l \leq L - 1) \\ C \neq 0, < \infty & (l = L). \end{cases}$$

Assumption 2. In a neighbourhood of $x$, $f$ is $L$ times continuously differentiable and its first $L$ derivatives are bounded.

Assumption 3. Kernel function $K$ is a bounded, even function with a compact support, of order $L \geq 2$ and $\int K(u) du = 1$.

Assumption 4. $x$ is an interior point in the support of $X$.

Assumption 5. $h \to 0$, $nh \to \infty$ as $n \to \infty$

KDE has pointwise consistency and asymptotic normality for an interior point in the support of $X$. Although it also converges uniformly for an interior point in the support of $X$, we only review pointwise properties because we investigate the pointwise higher-order asymptotics of KDE with global plug-in bandwidth. Under Assumption 1–3, we can expand mean squared error (MSE) of $\hat{f}_h(x)$ as follows:

$$MSE[\hat{f}_h(x)] \equiv \mathbb{E}[\{\hat{f}_h(x) - f(x)\}^2] = \left(C_L f^{(L)}(x) h^L\right)^2 + \frac{R(K)f(x)}{nh} + o(h^{2L} + (nh)^{-1}),$$

where $R(K) = \int K(u)^2 du, C_L = \frac{1}{L!} \int u^L K(u) du$. Therefore, Markov’s inequality, Assumptions 1–5, and (2.1) imply pointwise consistency $\hat{f}_h(x) \overset{p}{\to} f(x)$. Moreover, we can show that KDE has asymptotic normality by applying Lindberg-Feller’s central limit theorem:

$$\sqrt{nh} \left(\hat{f}_h(x) - \mathbb{E}\hat{f}_h(x)\right) \overset{d}{\to} N\left(0, R(K)f(x)\right).$$
2.2 Plug-In Method

Bandwidth \( h \) is a parameter that analysts need to choose in advance. One of the criteria for bandwidth selection is the mean integrated squared error (MISE):

\[
MISE(h) = \int \mathbb{E}[(\hat{f}_h(x) - f(x))^2] dx.
\]

The theoretically optimal bandwidth is the one that minimises MISE and, from the MISE expansion, this bandwidth is defined as follows:

\[
h_0 = \left( \frac{R(K)}{2LC^2_d} \right)^{\frac{1}{2+\alpha}} n^{-\frac{1}{2+\alpha}},
\]

where \( I_L = \int f^{(L)}(x)^2 dx \). Although \( h_0 \) would perform the best, it is infeasible because \( I_L \) is unknown, so one has to select the bandwidth from the available data. We examine the effect of a certain plug-in method on the distribution of the estimator.

Several plug-in methods have been proposed so far (see e.g. Hall et al. (1991), Sheather and Jones (1991)). In this paper, we adopt as Hall and Kang (2001), a simple plug-in method that estimates \( I_L \) directly and nonparametrically using the estimator proposed by Hall and Marron (1987). Their estimator, \( \hat{I}_L \) for \( I_L \), is given as follows:

\[
\hat{I}_L = \left( \frac{n}{2} \right)^{-1} \sum_{i=1}^{n} \sum_{j=1}^{n} b^{-2L+1} H^{(2L)} \left( \frac{X_i - X_j}{b} \right) = \left( \frac{n}{2} \right)^{-1} \sum_{i=1}^{n} \sum_{j=1}^{n} \hat{I}_{ij},
\]

where \( b \) (called pilot bandwidth) is a bandwidth for estimation of \( I_L \), different from \( h \), and \( H \) is a kernel function of order \( L_p \).

Another estimator for \( I_L \) proposed by Hall and Marron (1987) is

\[
\int \left\{ \hat{f}^{(L)}(x) \right\}^2 dx = \frac{1}{nb^{2L+1}} \widehat{H}^{(L)}(0) + \frac{1}{n^2 b^{2L+1}} \sum_{i=1}^{n} \sum_{j \neq i} \hat{H}^{(L)} \left( \frac{X_i - X_j}{b} \right) + O(n^{-2})
\]

where \( \hat{f}^{(L)}(x) = \frac{1}{n b^{2L+1}} \sum_{i=1}^{n} K^{(L)} \left( \frac{X_i - x}{b} \right) \) and \( \hat{H}^{(L)}(v) = \int H^{(L)}(u) H^{(L)}(v - u) du \). Hall and Marron (1987) state that 'the first term does not make use of the data, and hence may be thought of as adding a type of bias in the estimator. This motivates the estimator'.

\[
\hat{I}_{L, \text{conv}} = \frac{1}{n(n-1) b^{2L+1}} \sum_{i=1}^{n} \sum_{j \neq i} \hat{H}^{(L)} \left( \frac{X_i - X_j}{b} \right).
\]

Remark 2. \( \hat{I}_L \) and \( \hat{I}_{L, \text{conv}} \) can be negative in small samples. Although they are asymptotically justified, it can cause problems in empirical applications. Hall and Kang (2001) avoid this problem by using \( \hat{I}_L \) instead of \( \hat{I}_L \). Another way is to use \( \int \left\{ \hat{f}^{(L)}(x) \right\}^2 dx \) instead of \( \hat{I}_{L, \text{conv}} \) in Section 4, we employ the Monte Carlo Study in these two ways.

Assumption 6. \( b = cn^{-2/(4L+2L_p+1)} \)

Proposition 2.1 provides the expansion of the plug-in bandwidth (defined as \( \hat{h} \)) and plays an essential role in the derivation of the asymptotic expansion of KDE with the plug-in bandwidth. We assume additional conditions for Proposition 2.1:
Assumption 7. In a neighbourhood of $x$, $f$ is $(2L + L_p)$-times continuously differentiable and its first $(2L + L_p)$ derivatives are bounded.

Assumption 8. Kernel function $H$ is a bounded, even function with compact support, of order $L_p \geq 2$, $(2L)$-times continuously differentiable and for all integers $k$ such that $1 \leq k \leq 2L - 1$, lim$_{u \to \pm \infty} |H^{(k)}(u)| \to 0$.

Assumption 7 gives regularity conditions on the smoothness of the estimand, which implies Assumption 2. Assumption 8 is on the kernel function $H$ for the estimation of $I_L$, and the condition at the infinity of $u$ is necessary for integration by parts in the expanding process of $(\hat{h} - h_0)/h_0$. These assumptions can be interpreted as a generalisation of assumption ($A_{g_p}$) of Hall and Kang (2001) to $K$ of order $L$ and $H$ of order $L_p$.

Proposition 2.1 (Expansion of Plug-In Bandwidth). Under Assumptions 1, 3, 4, 6, 7 and 8, and additionally 13 for Theorem 2.4, 15 for Theorem 3.1, and 16 and 17 for Theorem 3.5 and 3.6, we can expand $(\hat{h} - h_0)/h_0$ as follows:

$$\frac{\hat{h} - h_0}{h_0} = \frac{C_{pl}}{n} \sum_{i=1}^{n} V_i - \frac{C_{pl}}{2} \left( \frac{n}{2} \right) \left( \sum_{j \neq i}^{n} W_{ij} + O_p \left( (nh_0)^{-1} + h_0^{2L} \right) \right)$$

(2.4)

where

$$C_{pl} = \frac{2}{2L + 1} I_{L_p}^{-1},$$

$$V_i = \left\{ f^{(2L)}(X_i) - \mathbb{E} f^{(2L)}(X_i) \right\} + \frac{\int u^{2L} H(u) du}{(L_p)!} h^{-L_p} \left\{ f^{(2L+L_p)}(X_i) - \mathbb{E} f^{(2L+L_p)}(X_i) \right\} + O_p \left( n^{-1/2} b^{L_p} \right),$$

$$W_{ij} = \left\{ I_{L_p} - \mathbb{E} [I_{L_p}|X_i] - \mathbb{E} [I_{L_p}|X_j] + \mathbb{E} [I_{L_p}] \right\}.$$

The proof is in A.1.

Remark 3. The first term on the right-hand side of (2.4) reflects the projection term of the Hoeffding-decomposition of $\hat{I}_L$, whose convergence rate is $O_p(n^{-1/2})$. The second term reflects the quadratic term of the decomposed $\hat{I}_L$, whose convergence rate is $O_p(n^{-1} b^{-(4L+1)/2})$.

Remark 4. Since the MSE optimal rate of $b$ is $O_p \left( n^{\frac{L_p}{4L+2L_p+1}} \right)$ from Hall and Marron (1987), for example, when one chooses the pilot bandwidth via rule of thumb (see Silverman (1986)) or second-stage plug-in method, the convergence rate of the second term in (2.4) is

$O_p \left( n^{-1/2} b^{L_p} \right) = O_p \left( n^{\frac{L_p}{4L+2L_p+1}} \right)$. We can make the second term in $V_i$ as small as we like up to the order of $O(n^{-3/2})$ by letting kernel order $L_p$ be large enough. This is not an unrealistic statement; for example, when one uses a second order kernel function $K$, adopting a second order kernel function is sufficient to make the effect of the second order term negligible in the sense that they do not affect on the asymptotic structure of KDE up to the order of $O\left( (nh_0)^{-1} + h_0^{2L} \right)$.

Remark 5. Since the MSE optimal rate of $b$ is $O_p \left( n^{\frac{L_p}{4L+2L_p+1}} \right)$ from Hall and Marron (1987), for example, when one chooses the pilot bandwidth via rule of thumb (see Silverman (1986)), the convergence rate of the second term in (2.4) is

$O_p \left( n^{-1} b^{-(4L+1)/2} \right) = O_p \left( n^{\frac{2L_p}{4L+2L_p+1}} \right)$. This implies that we can also make the third term of (2.4) as small as we like up to the order of $O(n^{-1})$ by letting kernel order $L_p$ be large enough. Although we cannot immediately identify how large $L_p$ needs to be to make the effect of pilot bandwidth negligible without deriving the Edgeworth expansion with pilot bandwidth, as we will see later, one has to adopt a considerably large $L_p$.

Remark 6. Since the convergence rate of the second term is $O_p \left( n^{-1} b^{-(4L+1)/2} \right) = O_p \left( n^{\frac{2L_p}{4L+2L_p+1}} \right)$, if not $L_p > (4L + 1)/2$, the convergence rate of the second term is slower than that of the first term. In order to ignore the effect of the second term, Hall and Kang (2001) provide the expansion under the condition that $L = 2$ and $L_p = 6$. The generalised version of this assumption is provided as Assumption 13. In addition, we weaken the condition by considering the effect of pilot bandwidth. We provide such results as Theorem 3.5 and 3.6.
2.3 Review of Previous Studies

Theorem 2.1 of Hall (1991) established the Edgeworth expansion for KDE with a deterministic bandwidth, which we replicate in Proposition 2.2. Let \( S_h(x) \) be the Standardised version of KDE with a bandwidth \( h \):

\[
S_h(x) = \frac{\sqrt{nh} \{ \hat{f}_h(x) - E \hat{f}_h(x) \}}{\mu_{20}(h)^{1/2}},
\]

where \( K_{i,h}(x) = K \left( \frac{x_i - x}{h} \right) \) and

\[
\mu_{ld}(h) = h^{-1} \mathbb{E} \left[ \{ K_{i,h}(x) - \mathbb{E}[K_{i,h}(x)] \}^k \{ K_{i,h}(x)^2 - \mathbb{E}[K_{i,h}(x)]^2 \} \right].
\]

Assumption 9. \( h \to 0, \ nh/\log n \to \infty \) as \( n \to \infty \)

Assumption 10 (Cramér Condition). For a sufficiently small \( h \):

\[
\sup_{z \in \mathbb{R}} \int_{-\infty}^{\infty} \exp \{ itK(u) \} f(x - uh) du < 1.
\]

Remark 7. Assumption 10 is a high-level condition. Lemma 4.1 in Hall (1991) shows that primitive condition (2.1) in Hall (1991) implies Assumption 10. Moreover, Assumption 10 is weaker than the Cramér condition in Lemma 4.1 of Hall (1991). This is because Theorem 3.1 only deal with the Standardised case, while Hall (1991) also deals with the Studentised case. Our Theorem 3.6 needs the same Cramér condition as Hall (1991).

Remark 8. Assumption 10 rules out the uniform kernel, but many kernels which are practically used will satisfy this condition. However, as stated in Hall (1991), one can also derive the Edgeworth expansion in the case of the uniform kernel by routine methods for lattice-valued random variables.

Proposition 2.2 (Hall (1991), Expansion with a Deterministic Bandwidth). Under Assumptions 1, 3, 4, 9, and 10, the following expansions are valid:

\[
\sup_{z \in \mathbb{R}} \left| P(S_h(x) \leq z) - \Phi(z) - \phi(z) \left( (nh)^{-1/2} p_1(z) \right) \right| = o \left( (nh)^{-1/2} \right)
\]

\[
\sup_{z \in \mathbb{R}} \left| P(S_h(x) \leq z) - \Phi(z) - \phi(z) \left( (nh)^{-1/2} p_1(z) + (nh)^{-1} p_2(z) \right) \right| = o \left( (nh)^{-1} \right),
\]

where \( \Phi(z) \) and \( \phi(z) \) are the distribution and density functions at \( z \) of a standard normal random variable, respectively, and:

\[
p_1(z) = -\frac{1}{6} \mu_{20}(h)^{-3/2} \mu_{30}(h)(z^2 - 1),
\]

\[
p_2(z) = -\frac{1}{24} \mu_{20}(h)^{-2} \mu_{40}(h)(z^3 - 3z) - \frac{1}{72} \mu_{20}(h)^{-3} \mu_{50}(h)(z^5 - 10z^3 + 15z).
\]

See Hall (1991) for the proof.

These results are the Edgeworth expansion of KDE up to the order of \( O \left( (nh)^{-1/2} \right) \) and \( O \left( (nh)^{-1} \right) \), respectively. However, bandwidth in his results is still deterministic. In this paper, we study KDE with data-driven bandwidth \( \hat{f}_h \) at a point \( x \). The next proposition decomposes the \( \hat{f}_h \) into terms that include the effect of bandwidth selection and ones that do not.

Assumption 11. Kernel function \( K \) is twice continuously differentiable.

Proposition 2.3 (Expansion of KDE with Data-Driven Bandwidth). Under Assumptions 1, 4, 5, 7, 8, 9, and 11, expanding \( \hat{f}_h(x) \) around \( h = h_0 \) yields:

\[
\hat{f}_h(x) = \frac{1}{nh} \sum_{i=1}^{n} K_{i,h}(x)
\]
\[ \hat{f}_{h_0}(x) = \left( \frac{\hat{h} - h_0}{h_0} \right) \Gamma_{KDE_1} + \frac{1}{2} \left( \frac{\hat{h} - h_0}{h_0} \right)^2 \Gamma_{KDE_2} + o_p \left( \left( \frac{\hat{h} - h_0}{h_0} \right)^2 \Gamma_{KDE_2} \right), \tag{2.6} \]

where letting \( u_{i,h}(x) = \left( \frac{x - \bar{X}_i}{h} \right) \), \( \Gamma_{KDE_1} \) and \( \Gamma_{KDE_2} \) are defined as follows.

\[
\begin{align*}
\Gamma_{KDE_1} & = \frac{1}{n h_0} \sum_{i=1}^{n} \{ K'_{1,h_0}(x) u_{i,h_0}(x) + K_{1,h_0}(x) \}, \\
\Gamma_{KDE_2} & = \frac{1}{n h_0} \sum_{i=1}^{n} \{ 2K_{1,h_0}(x) + 4K'_{1,h_0}(x) u_{i,h_0}(x) + K''_{1,h_0}(x) u_{i,h_0}(x)^2 \}.
\end{align*}
\]

Let \( S_{pf}(x) \) be the Standardised version of KDE with global plug-in bandwidth and define \( \mu_{kl} = \mu_{kl}(h_0) \). Noting that expanding \( \hat{h}^{1/2} \) around \( \hat{h} = h_0 \) yields \( \hat{h}^{1/2} = h_0^{1/2} + \frac{1}{2} h_0^{-1/2} (\hat{h} - h_0) + o_p \{ (\hat{h} - h_0)^2 h_0^{-3/2} \} \), we have,

\[
S_{pf}(x) = \sqrt{n h \{ \hat{f}_h(x) - E \hat{f}_h(x) \}} - S_{h_0}(x) - \frac{\sqrt{n h_0} \left( \frac{\hat{h} - h_0}{h_0} \right) \Gamma_{KDE_1} - \frac{\sqrt{n h_0} \left( \frac{\hat{h} - h_0}{h_0} \right)^2 \Gamma_{KDE_2}}{1 + \frac{1}{2} \frac{S_{h_0}(x) \left( \frac{\hat{h} - h_0}{h_0} \right)}{\mu_{20}} + \frac{1}{6} \frac{S_{h_0}(x) \left( \frac{\hat{h} - h_0}{h_0} \right)^2}{\mu_{20}} + s.o. \tag{2.7}
\]

**Assumption 12.** \( \lim_{n \to \infty} |K(u)|u| \to 0 \)

The following theorem generalises the kernel orders of Theorem 3.2 in Hall and Kang (2001). Their theorem specifically sets the order of the kernels to be \( L = 2 \) and \( L_p = 6 \), and we prove that it holds for general kernel orders \( L \) and \( L_p \).

**Assumption 13.** \( L_p > (4L + 1)/2 \).

**Remark 9.** As stated in Remark 6, this assumption is also interpreted as the generalisation of \((A_{gpi})\) in Hall and Kang (2001).

**Theorem 2.4** (Second Order Equivalence). **Under Assumptions 1, 4, 7, 8, 10, 11, 12 and 13** the following expansion is valid:

\[
\sup_{z \in \mathbb{R}} \left| \mathbb{P}(S_{pf}(x) \leq z) - \Phi(z) - \Phi(z) \left( (n h_0)^{-1/2} p_1(z) \right) \right| = o \left( (n h_0)^{-1/2} + h_0^L \right). \tag{2.8}
\]

See A.2 for the proof. We note that \((n h_0)^{-1/2} \) and \( h_0^L \) have the same order of \( O(n^{-L/(2L+1)}) \), but we write the right hand side in this manner to clarify the effect of the variance and bias. Comparing this result with the first half of Proposition 2.2, we see that the bandwidth selection via the global plug-in method has no effect on the asymptotic structure of KDE up to the order of \( O(\{(n h_0)^{-1/2} + h_0^L\}) \) as long as the kernel order of the kernel function for pilot estimation is large enough to satisfy Assumption 13.

**Remark 10.** When Assumption 13 breaks, Theorem 2.4 does not hold. In other words, the bandwidth selection has an effect on the asymptotic structure of KDE up to the order of \( O(\{(n h_0)^{-1/2} + h_0^L\}) \). Theorem 3.5 deals with this issue.

### 3 Main Results

As stated in Theorem 3.2 in Hall and Kang (2001) or our Theorem 2.4, bandwidth selection via the global plug-in method has no effect on the asymptotic properties of KDE up to the order of \( O(\{(n h_0)^{-1/2} + h_0^L\}) = O(n^{-L/(2L+1)}) \) when one uses a sufficiently high order kernel for the estimation of \( \hat{L} \). Section 3.1 provides a valid Edgeworth expansion for KDE with plug-in bandwidth up to the order of \( O(\{(n h_0)^{-1/2} + h_0^L\}) = O(n^{-2L/(2L+1)}) \) in Theorem 3.1. This expansion
possesses a form comparable with that in Hall (1991). In Section 3.2, we rewrite the expansions in Proposition 2.2 and Theorem 3.1 to derive the expansions only in terms of $n$ and the $n$-independent coefficient functions without $h_0$ in Corollary 3.2 and 3.3. Using these results, we scrutinise the higher-order difference between the theoretical and plug-in bandwidths in Section 3.3. We realise that the global plug-in bandwidth selection starts to have an impact from on the order of $O\{(nh_0)^{-1/2}h_0 + h_0^{L+1}\} = O(h_0^{L+1})$, which is stated in Theorem 3.4. Section 3.4 provides a comprehensive example by considering the special case of $L = 2$.

In section 3.5, we develop the Edgeworth expansion for any $L_p$. This expansion implies that, when $L_p$ is small, bandwidth selection by the global plug-in method has an effect on the asymptotic structure up to the order of $O_p\{(nh_0)^{-1/2}+h_0^L\}$. In addition, we study the intersectional effect of the plug-in method, pilot bandwidth, and the Studentisation.

### 3.1 Edgeworth Expansion for KDE with Global Plug-In Bandwidth up to the order of $O\{(nh_0)^{-1}\}$

We introduce the following assumption:

**Assumption 14.** For $1 \leq k \leq L - 1$, $\lim_{u \to \pm \infty} |K(u)u^k| \to 0$ and $\lim_{u \to \pm \infty} |K'(u)u^2| \to 0$.

**Assumption 15.** $L_p > 8L - \frac{1}{2}$

**Remark 11.** Assumption 15 guarantees that the pilot bandwidth has no effect on the asymptotic structures of KDE up to the order of $O_p\{(nh_0)^{-1}+h_0^L\}$. We obtain this assumption from Edgeworth expansion including the pilot bandwidth (Theorem 3.5). Although this assumption may be unrealistic for empirical analysis (for example, when one uses $L = 2$, $L_p \geq 16$ is necessary), it can be considered of theoretical value in the sense that this theorem clarifies the inevitable effect of bandwidth selection via the global plug-in method (i.e. estimating $\int f''(x)^2 \, dx$). For empirical application, Theorem 3.6, which considers the simultaneous effect of the bandwidth selection via the global plug-in method, its accompanying pilot bandwidth, and Studentisation, is more valuable.

We have the following theorem which is proved in A.3.

**Theorem 3.1 (The Effect of Estimation of $\int f^{(L)}(x)^2 \, dx$).** Under Assumptions 1, 4, 6, 7, 8, 10, 11, 14 and 15, the following expansion is valid:

$$
\sup_{z \in \mathbb{R}} \left| P(S_{pl}(x) \leq z) - \Phi(z) \right|
- \phi(z) \left[ (nh_0)^{-1/2} p_1(z) + \sum_{l=0}^{L-1} h_0^{L+l+1} p_{3,l}(z) + n^{-1/2} h_0^{1/2} p_4(z) + (nh_0)^{-1} p_2(z) \right]
= o\{(nh_0)^{-1} + h_0^{2L}\},
$$

(3.1)

where

$$
p_{3,l}(z) = -C_{pl}C_{l}(x)p_{11}\mu_2^{-1} z,
$$

$$
p_4(z) = -C_{pl}p_{11}\xi_{11}\mu_2^{-3/2}(z^2 - 1) + \frac{1}{2}C_{pl}p_{11}\mu_2^{-1/2} z^2,
$$

$$
C_{l}(x) = -\left( \int x^{l+1} K(u) du \right) f^{(L+l)}(x) \bigg/ (L+l-1)!,
$$

$$
\xi_{kl} = h_0^{-1}(k \geq 1) \cup (l \geq 1) \mathbb{E} \left[ \left\{ K_{i,h_0}(x) - \mathbb{E}[K_{i,h_0}(x)] \right\}^k \times \left\{ K_{i,h_0}^l(x)u_{i,h_0}(x) + K_{i,h_0}(x) - \mathbb{E}[K_{i,h_0}^l(x)u_{i,h_0}(x) + K_{i,h_0}(x)] \right\}^l \right],
$$

$$
\rho_{kl} = h_0^{-1}(k \geq 1) \mathbb{E} \left[ \left\{ K_{i,h_0}(x) - \mathbb{E}[K_{i,h_0}] \right\}^k \left\{ f^{(2l)}(X_i) - \mathbb{E}[f^{(2l)}(X_i)] \right\}^l \right].
$$
Hall (1991) does not specify the bandwidth order, but we consider the use of plug-in bandwidth with a fixed convergence rate. Note that $h_0$ satisfies Assumption 9. Comparing (2.5) with $h = h_0$ and (3.1), we see that $h_0^{1+1/2} p_{3,j}(z)$ and $n^{-1/2} h_0^{1/2} p_4(z)$ reflect the effect of bandwidth selection via global plug-in methods. However, the results in Proposition 2.2, Theorem 2.4, and Theorem 3.1 are still insufficient for identifying the exact difference because the head term of the projection term of decomposed $I_{1L}^{cov}$, the result of this theorem does not change, even if the estimator is changed from $I_{1L}$ to $I_{1L}^{cov}$.

### 3.2 Edgeworth Expansions in Powers of $n^{-1/(2L+1)}$

Hall (1991) does not specify the bandwidth order, but we consider the use of plug-in bandwidth with a fixed convergence rate. Note that $h_0$ satisfies Assumption 9. Comparing (2.5) with $h = h_0$ and (3.1), we see that $h_0^{1+1/2} p_{3,j}(z)$ and $n^{-1/2} h_0^{1/2} p_4(z)$ reflect the effect of bandwidth selection via global plug-in methods. However, the results in Proposition 2.2, Theorem 2.4, and Theorem 3.1 are still insufficient for identifying the exact difference because $\mu_{kl}, \rho_{kl}, \xi_{kl}$. Accordingly $p_1(z), p_2(z), p_{3,j}(z),$ and $p_4(z)$ in the expansions depend on $h_0$ and, consequently, the relationship between the terms in the expansions is unclear.

For $S_{h_0}(x)$, we have to expand $p_1(z)$ and $p_2(z)$ in terms of only $n$, without $h_0$. $p_2(z)$ is easy to handle because only its leading term affects the Edgeworth expansion up to the order of $O\{(nh_0)^{-1}\}$. For $p_1(z)$, recalling that $p_1(z) = -\frac{1}{6} \mu_{30}^{-3/2} \mu_{30}(z^2 - 1)$, we expand $\mu_{30}^{-3/2}$ up to the term whose convergence rate is $O(h_0^2)$. Letting $\kappa_{kl} \equiv \int u^k K(u) du$ and, from straightforward computation, we can expand $\mu_{20}, \mu_{30}$ as follows:

\[\mu_{20} = \kappa_{02} f(x) - f(x)^2 h_0 + \sum_{i=2}^{L} \kappa_{2i} \frac{f^{(i)}(x)}{i!} h_0^i + o(h_0^i), \quad (3.2)\]

\[\mu_{30} = \kappa_{03} f(x) - 3 \kappa_{02} f(x)^2 h_0 + \left\{ \kappa_{23} \frac{f^{(2)}(x)}{2!} + 2 f(x)^3 \right\} h_0^2 \]

\[+ \sum_{i=3}^{L} \left\{ \kappa_{3i} \frac{f^{(i)}(x)}{i!} - 3 \kappa_{i-1,2} \frac{f^{(i-1)}(x)}{(i-1)!} f(x) \right\} h_0^i + o(h_0^i). \quad (3.3)\]

For notational simplicity, we rewrite $\mu_{20}, \mu_{30}$ as a series of $h_0$:

\[\mu_{20} = \sum_{i=0}^{L} m_{2,i}(x) h_0^i + o(h_0^i), \quad \mu_{30} = \sum_{i=0}^{L} m_{3,i}(x) h_0^i + o(h_0^i). \]

Then, expanding $\mu_{30}^{-3/2} \mu_{20}^{3/2}$ yields:

\[\mu_{30}^{-3/2} \mu_{20}^{3/2} = \left\{ \sum_{i=0}^{L} m_{3,i}(x) h_0^i + o(h_0^i) \right\} \left\{ \sum_{j=0}^{L} m_{2,j}(x) h_0^j + o(h_0^j) \right\}^{-3/2} \]

\[= \left\{ \sum_{i=0}^{L} m_{3,i}(x) h_0^i + o(h_0^i) \right\} \times \left[ \sum_{k=0}^{L} \frac{(-1)^k (2k + 1)!!}{2^k k!} \left\{ m_{2,0}(x) \right\}^{-(2k+3)/2} \left( \sum_{j=0}^{L} m_{2,j}(x) h_0^j \right)^k + o(h_0^k) \right] \]

\[= \sum_{l=0}^{L} \sum_{k=0}^{L} \frac{(-1)^k (2k + 1)!!}{2^k k!} \left\{ m_{2,0}(x) \right\}^{-(2k+3)/2} \sum_{l_1=0}^{L} \cdots \sum_{l_k=0}^{L} \sum_{i_l=1}^{L} \cdots \sum_{i_l=1}^{L} m_{3,i_1}(x) m_{2,i_1}(x) \cdots m_{2,i_k}(x) h_0^{l_1+\cdots+l_k} + o(h_0^k). \]

We define $\gamma_{1,0}(x), \gamma_{1,1}(x), \gamma_{1,0}(x)$ and $\gamma_{2,0}(x)$ as follows:

\[\gamma_{1,0}(x) = -\frac{1}{6} \kappa_{02}^{-3/2} \kappa_{03} f(x), \quad (3.4)\]

\[\gamma_{1,1}(x) = \frac{1}{2} \left( \kappa_{02}^{-1/2} f(x)^{1/2} - \kappa_{02}^{-5/2} \kappa_{03} f(x)^{1/2} \right), \quad (3.5)\]
\[
\begin{align*}
\gamma_{2,1,0}(x) &\equiv -\frac{1}{24} \lambda_{02}^{-2} \lambda_{04} f(x)^{-1}, \\
\gamma_{2,2,0}(x) &\equiv -\frac{1}{72} \lambda_{02}^{-3} \lambda_{03} f(x)^{-1}.
\end{align*}
\]

From the above results, we obtain the following corollary.

**Corollary 3.2** (Expansion of Hall (1991) in powers of \( n^{-(2L+1)} \)). Under Assumptions 1, 3, 4, and 10:

\[
\sup_{z \in \mathbb{R}} \left| \mathbb{P}(S_{h_0}(x) \leq z) - \Phi(z) - \phi(z) \sum_{j=0}^L a_j(z,x) n^{-\frac{(2j+1)}{2}} \right| = o\left( (nh_0)^{-1} + h_0^{2L} \right),
\]

where the definitions of \( a_j(z,x) \) are given as follows for \( 2 \leq q \leq L - 1 \):

\[
\begin{align*}
a_0(z,x) &= \gamma_{1,0}(x)(z^2 - 1), \\
a_1(z,x) &= \gamma_{1,1}(x)(z^2 - 1), \\
a_q(z,x) &= \sum_{l=0}^{L} \sum_{k=0}^{l} \frac{(-1)^k(2k+1)!!}{2^k k!} \{ m_{2,0}(x) \}^{-(2k+3)} \times \sum_{i_1+\cdots+i_k+l=q} m_{3,1}(x) m_{2,i_1}(x) \cdots m_{2,i_k}(x) h_0^{i_1+\cdots+i_k+l}(z^2 - 1), \\
a_L(z,x) &= \gamma_{2,0,1}(x)(z^3 - 3z) + \gamma_{2,0,2}(x)(z^5 - 10z + 15) + \sum_{l=0}^{L} \sum_{k=0}^{l} \frac{(-1)^k(2k+1)!!}{2^k k!} \{ m_{2,0}(x) \}^{-(2k+3)} \times \sum_{i_1+\cdots+i_k+l=L} m_{3,1}(x) m_{2,i_1}(x) \cdots m_{2,i_k}(x) h_0^{i_1+\cdots+i_k+l}(z^2 - 1).
\end{align*}
\]

**Remark 13.** Note that \( a_0 \) and \( a_1 \) are special cases of \( a_q \), but we explicitly write these terms for comparison of this result with the next corollary.

**Remark 14.** From this corollary, we identify that the Edgeworth expansion of the Standardised KDE with deterministic bandwidth has the term of order \( O\{(nh_0)^{-1/2}\} = O(n^{1/2}) \) right after the term \( \Phi(z) \), with a gap between them, but the subsequent terms decrease at the rate of \( O(h_0) = O(n^{1/4}) \), which is not clear in Hall (1991).

Next, for (3.1), we also need to expand \( p_{3,1}(z) \) and \( p_{3,2}(z) \). Although we do not provide the details here, one can use a similar process for \( p_1(z) \). We define:

\[
\begin{align*}
\tau_1 &\equiv \int u \{ K(u) K'(u) u + K(u)^2 \} du, \\
\mathcal{L}'(x) &\equiv f^{(2L)}(x) - \mathbb{E}[f^{(2L)}(x)], \\
\gamma_{1,0,0}(x) &\equiv -C_{pL} C_{1,0}(x) K_{02}^{-1} \mathcal{L}'(x), \\
\gamma_{1,1,1}(x) &\equiv C_{pL} C_{1,0}(x) K_{02}^{-2} \mathcal{L}'(x) f(x) \\
\gamma_{1,0,1}(x) &\equiv -C_{pL} K_{02}^{-3/2} \mathcal{L}'(x) f^{1/2}(x), \\
\gamma_{1,1,1}(x) &\equiv -\frac{3}{2} C_{pL} K_{02}^{-5/2} \mathcal{L}'(x) f(x)^{3/2}, \\
\gamma_{2,0,0}(x) &\equiv \frac{1}{2} C_{pL} K_{02}^{-1/2} \mathcal{L}'(x) f(x)^{1/2}, \\
\gamma_{2,1,1}(x) &\equiv -\frac{1}{4} C_{pL} K_{02}^{-3/2} \mathcal{L}'(x) f(x)^{3/2}.
\end{align*}
\]

Then, we have the next corollary.

**Corollary 3.3** (Main Theorem in powers of \( n^{-(2L+1)} \)). Under the same assumptions as in Theorem 3.1:

\[
\sup_{z \in \mathbb{R}} \left| \mathbb{P}(S_{pL}(x) \leq z) - \Phi(z) - \phi(z) \sum_{j=0}^L b_j(z,x) n^{-\frac{(2j+1)}{2}} \right| = o\left( (nh_0)^{-1} + h_0^{2L} \right),
\]

where

\[
\begin{align*}
b_0(z,x) &= a_0(z,x), \\
b_1(z,x) &= a_1(z,x) + \gamma_{1,0,0}(x) z + \gamma_{1,0,1}(x)(z^2 - 1) + \gamma_{2,0,0}(x) z^2, \\
b_2(z,x) &= a_2(z,x) + \gamma_{1,1,1}(x) z + \gamma_{1,1,1}(x)(z^2 - 1) + \gamma_{2,1,1}(x) z^2.
\end{align*}
\]
Here, we do not provide the definitions of \( b_j(z, x) \), \( j = 3, \ldots \) because they are too lengthy and tedious, but they can be obtained in a straightforward manner.

Hall and Kang (2001) and Theorem 2.4 state that the global plug-in method has no effect on the terms up to whose convergence rates are \( O\{ (nh_0)^{-1/2} \} \); in other words, \( b_0(z, x) \) does not include the effect of bandwidth selection in view of Corollary 3.3. Comparing \( a_1(z, x) \) and \( b_1(z, x) \), the bandwidth selection via the global plug-in method starts to have an effect on the term with the order of \( O\{ (nh_0)^{-1/2}h_0 \} = O(n^{-\frac{(L+1)}{2L+1}}) \). The deviation between \( b_0(z, x) \) (the smallest term not affected by bandwidth selection) and \( b_1(z, x) \) (the largest term affected by bandwidth selection) is only of order \( O(h_0) = O(n^{-1/(2L+1)}). \)

**Remark 15.** Although we omit \( b_3(z, x) \) and the subsequent terms, we can show that these terms are also affected by bandwidth selection via the global plug-in method in the same way as the process of deriving Corollary 3.2. However, the most important point is that the influence of the bandwidth selection via the global plug-in method starts to appear at \( b_1(z, x) \).

### 3.3 Difference between \( S_{h_0}(x) \) and \( S_{PI}(x) \)

From Corollaries 3.2 and 3.3, we can easily deduce the following theorem, which states the exact order of the difference between \( S_{h_0}(x) \) and \( S_{PI}(x) \). See Appendix A.4 for the proof.

**Theorem 3.4 (Exact Evaluation of the Deviation).** Under the same assumptions as in Theorem 3.1:

\[
\sup_{z \in \mathbb{R}} \left| P(S_{h_0}(x) \leq z) - P(S_{PI}(x) \leq z) \right| - \phi(z) \left( \gamma_{1,0,0}(z^2 - 1) + \gamma_{4,0}(z^2) \right) n^{-\frac{(L+1)}{2L+1}} = O(n^{-\frac{(L+1)}{2L+1}}),
\]

and the order is exact.

This theorem implies that:

\[
\sup_{z \in \mathbb{R}} \left| P(S_{h_0}(x) \leq z) - P(S_{PI}(x) \leq z) \right| = O(n^{-\frac{(L+1)}{2L+1}}).
\]

We can only claim that this deviation is \( o\{ (nh_0)^{-1/2} \} = o(n^{-L/(2L+1)}) \) from Theorem 3.2 in Hall and Kang (2001) and our Theorem 2.4, whereas Theorem 3.4 gives a stronger result, stating that the convergence rate is exactly \( O\{ (nh_0)^{-1/2}h_0 \} = O(n^{-L/(2L+1)}) \).

**Remark 16.** The larger the kernel order \( L \) we use, the slower the convergence rate of the approximation in Theorems 2.4, 3.1, and 3.4 will be. This is because we centralise at \( \mathbb{E} f_{h_0}(x) \). However, as stated in Section 5, one of the final goals would be to examine the effect of bandwidth selection and 'debias' simultaneously (we are in the process of working on it), and it is unclear if the second-order kernel \( L = 2 \) is optimal.

### 3.4 Special Case

Since the previous results are difficult to interpret because of their generality, we consider a special case of \( L = 2 \). Here, we also provide the details of the expansions of \( p_{3,1}(z) \) and \( p_4(z) \) as well as that of \( p_1(z) \).

First, we have to expand \( p_1(z) \) and \( p_2(z) \). From (3.2) and (3.3), we can expand \( p_1(z) \) as follows (see C):

\[
p_1(z) = -\frac{1}{6} \mu_{30} \mu_{20}^{-3/2} (z^2 - 1) = -\frac{1}{6} \left( \left( \frac{f(x)}{\mu_{12}/\mu_{12}} \right)^{1/2} \right) h_0 + \left( \frac{-3}{4} \right) \left( \frac{v_{02} f(x)}{\mu_{12} v_{02}} \right)^{-5/2} \left( \frac{v_{02} f(x)}{\mu_{20} v_{02}} \right)^{-5/2} \left( \frac{v_{02} f(x)}{\mu_{20} v_{02}} \right)^{4} + \frac{15}{8} \left( \frac{v_{02} f(x)}{\mu_{20} v_{02}} \right)^{-7/2} \left( \frac{v_{02} f(x)}{\mu_{20} v_{02}} \right)^{5} \right) h_0 \]
and since for \( p_2(z) \) we need only the leading term; a straightforward computation yields:

\[
p_2(z) = -\frac{1}{24} \kappa_{20}^2 \kappa_{04} f(x)^{-1} (z^3 - 3z) - \frac{1}{72} \kappa_{20}^3 \kappa_{03} f(x)^{-1} (z^5 - 10z^3 + 15) + o(1).
\]

From the above results, in the special case of \( L = 2 \), expansion (2.5) is as follows:

\[
\sup_{z \in \mathbb{R}} \left| P(S(x) \leq z) - \Phi(z) - \phi(z) \left[ a_0(z,x)n^{-2/5} + a_1(z,x)n^{-3/5} + a_2(z,x)n^{-4/5} \right] \right| = o\{(nh_0)^{-1} + h_0^4\},
\]

where

\[
a_0(z,x) = \gamma_{0,0}(x)(z^2 - 1), \\
a_1(z,x) = \gamma_{1,1}(x)(z^2 - 1), \\
a_2(z,x) = \gamma_{2,0}(x)(z^2 - 1) + \gamma_{2,1,0}(x)(z^3 - 3z) + \gamma_{2,2,0}(x)(z^5 - 10z^3 + 15).
\]

Next, we expand \( p_{3,0}(z), p_{3,1}(z) \) and \( p_4(z) \). From a straightforward computation, noting \( \tau_1 = 0 \) from the properties of the odd function, we can expand \( p_{11} \) and \( \xi_{11} \) as follows:

\[
p_{11} = \mathcal{L}(x)f(x) + O(h_0^4), \\
\xi_{11} = \tau_0 f(x) + \tau_1 f^{(1)}(x)h_0 + o(h_0) = \tau_0 + o(h_0).
\]

These imply:

\[
p_{3,0}(z) = -C_{\Pi}^2\kappa_{0}\mu_{11} z
\]

\[
= -C_{\Pi}^2\kappa_{0}\mu_{11} \mathcal{L}(x)f(x)z + C_{\Pi}^2\kappa_{0}\mu_{11} \mathcal{L}(x)f(x)z h_0 + o(h_0).
\]

See C for the second equality. Noting that \( C_{\Gamma,1}(x) = 0 \) from the properties of the odd function:

\[
p_{3,1}(z) = -C_{\Pi}^2\kappa_{1,1}(x)\mu_{11} z = 0,
\]

and, as shown in C:

\[
p_4(z) = -C_{\Pi}^2\rho_{11}\xi_{11} \mu_{20}^{3/2} (z^2 - 1) + \frac{1}{2} C_{\Pi}^2 \rho_{11} \mu_{20}^{-1/2} z
\]

\[
= -C_{\Pi}^2 \kappa_{02}^{-3/2} \phi_0 \mathcal{L}(x)f(x)^{1/2} (z^2 - 1) + \frac{3}{2} C_{\Pi}^2 \kappa_{02}^{-5/2} \phi_0 \mathcal{L}(x)f(x)^{3/2} (z^2 - 1)h_0 + o(h_0)
\]

\[
+ \frac{1}{2} C_{\Pi}^2 \kappa_{02}^{-1/2} \mathcal{L}(x)f(x)^{1/2} z^2 - \frac{1}{4} C_{\Pi}^2 \kappa_{02}^{-3/2} \mathcal{L}(x)f(x)^{3/2} z^2 h_0 + o(h_0).
\]

From the above results, in the special case of \( L = 2 \), the expansion (3.1) is as follows.

\[
\sup_{z \in \mathbb{R}} \left| P(S_P(x) \leq z) - \Phi(z) - \phi(z) \left[ b_0(z,x)n^{-2/5} + b_1(z,x)n^{-3/5} + b_2(z,x)n^{-4/5} \right] \right| = o\{(nh_0)^{-1} + h_0^4\},
\]

where the definitions of \( b_0(z,x) \), \( b_1(z,x) \) are given as follow.

\[
b_0(z,x) = a_0(z,x) \\
b_1(z,x) = a_1(z,x) + \gamma_{3,1,0}(x)z + \gamma_{4,1,0}(x)(z^2 - 1) + \gamma_{4,2,0}(x)z^2 \\
b_2(z,x) = a_2(z,x) + \gamma_{3,1,1}(x)z + \gamma_{4,1,1}(x)(z^2 - 1) + \gamma_{4,2,1}(x)z^2
\]
3.5 Edgeworth Expansion Including Pilot Bandwidth and Studentisation

In this section, we provide two more expansions. One is the Edgeworth Expansion of Standardised KDE with global plug-in bandwidth and its accompanying pilot bandwidth. Here, we allow $L_p$ to be small so that $b$ affects the expansion (note that Theorems 2.4 and 3.1 set $L_p$ sufficiently large such that $b$ does not appear in the expansion). The other is the Edgeworth Expansion of Studentised KDE with the global plug-in bandwidth and its accompanying pilot bandwidth.

Let $H^{(2L)}\left(\frac{X_i-X_j}{b}\right) = H_{ij,b}^{(2L)}$ and define

\[
\omega_{111} = h_0^{-1} b^{-1} E \left\{ \left\{ K_{i,h_0}(x) - E[K_{i,h_0}(x)] \right\} \left\{ K_{j,h_0}(x) - E[K_{j,h_0}(x)] \right\} \right\} 
\times \left\{ H_{ij,b}^{(2L)} - E[H_{ij,b}^{(2L)}] X_i - E[H_{ij,b}^{(2L)} | X_j] + E[H_{ij,b}^{(2L)}] \right\}.
\]

\[
\psi_{111} = \frac{h_0^{-1} b^{-1}}{\sqrt{2\pi}} \right. \left. \left\{ K_{i,h_0}(x) - E[K_{i,h_0}(x)] \right\} \left\{ K_{j,h_0}(x) + K'_{i,h_0}(x) u_i,h_0(x) - E[K_{i,h_0}(x) + K'_{i,h_0}(x) u_i,h_0(x)] \right\} \right\} 
\times \left\{ H_{ij,b}^{(2L)} - E[H_{ij,b}^{(2L)}] X_i - E[H_{ij,b}^{(2L)} | X_j] + E[H_{ij,b}^{(2L)}] \right\}.
\]

**Assumption 16.** $L_p > \frac{2L}{3} + \frac{1}{10}$

**Assumption 17.** $L_p > \frac{2L}{3} + 1$

**Remark 17.** Assumption 16 is for the following expansion (3.4) and Assumption 17 is for the following expansion (3.5). Owing to these assumptions, we can assume $\left(\frac{h-h_0}{h_0}\right)^2$ in (2.7) is negligible. Without this assumption, when $L$ is large enough, Edgeworth expansion (3.4) and (3.5) have the term associated with $\left(\frac{h-h_0}{h_0}\right)^2, \left(\frac{h-h_0}{h_0}\right)^4, \ldots$. However, when $L = 2, L_p = 4$ is sufficient for Assumption 17 and any $L_p$ satisfies Assumption 16, so these assumptions are not unrealistic unlike Assumption 15.

**Remark 18.** We do not provide the mathematically rigorous proof for the following expansions. However, one can prove their validity of them in the same way as our Theorem 3.1.

**Theorem 3.5** (Edgeworth Expansion Including Pilot Bandwidth). Under Assumptions 1, 4, 6, 7, 8, 11, 14 and 16

\[
\mathbb{P}(S_{PL}(x) \leq z) = \Phi(z) + \phi(z) \left\{ p_1(z)(nh_0)^{-1/2} + \sum_{l=0}^{L-1} p_{1,l}(z) h_0^{(2L+2l+1)/2} b^{-2L} \right. 
+ p_2(z) h_0^{-1/2} b^{-2L} + o\left( (nh_0)^{-1/2} + h_0^2 \right) \right\} (3.4)
\]

\[
\mathbb{P}(S_{PL}(x) \leq z) = \Phi(z) + \phi(z) \left\{ p_1(z)(nh_0)^{-1/2} + p_2(z)(nh_0)^{-1} + \sum_{l=0}^{L-1} p_{3,l}(z) h_0^{L+l+1} \right. 
+ \sum_{l=0}^{L-1} p_{1,l}(z) h_0^{-1/2} b^{-2L} \right. 
+ \sum_{l=0}^{L-1} p_{2,l}(z) h_0^{L+l+1} b^{-2L} + o\left( (nh_0)^{-1} + h_0^{2L} \right) \right\} (3.5)
\]

where the definitions of $p_{1,l}(z)$ and $p_2(z)$ are

\[p_{1,l}(z) = -C_{1,1} C_{T,1}(x) \frac{1}{2} \mu_2 (-3/2) \omega_{111} (z^2 - 1)\]

\[p_2(z) = -C_{1,1} \left( \frac{1}{2} \mu_2 \xi_1 \omega_{111} (z^3 - 3z) + \mu_2^{-2} \psi_{111} z - \frac{1}{4} \mu_2^{-2} \omega_{111} (z^3 - z) \right)\]

**Remark 19.** $p_{1,l}(z)$ and $p_2(z)$ reflect the effect of pilot bandwidth.
Remark 20. When one uses $\hat{I}_{n,L}^{\text{convo}}$ instead of $\hat{I}_L$, the definitions of $\eta_k, \omega_{111}$ and $\psi_{111}$ are changed as follows.

$$
\omega_{111}^{\text{convo}} \equiv h_0^{-1}b^{-1}E \left\{ K_{i,h_0}(x) - E[K_{i,h_0}(x)] \right\} \left\{ K_{i,h_0}(x) - E[K_{i,h_0}(x)] \right\}
\times \left[ \hat{A}_{i,j,b}^{(L)} - E\left[ \hat{A}_{i,j,b}^{(L)} | X \right] - E \left[ \hat{A}_{i,j,b}^{(L)} | X \right] + E \left[ \hat{A}_{i,j,b}^{(L)} \right] \right]
$$

$$
\psi_{111}^{\text{convo}} \equiv h_0^{-1}b^{-1}E \left\{ K_{i,h_0}(x) - E[K_{i,h_0}(x)] \right\} \left\{ K_{i,h_0}(x) + K_{i,h_0}'(x)u_{i,h_0}(x) - E[K_{i,h_0}(x) + K_{i,h_0}'(x)u_{i,h_0}(x)] \right\}
\times \left[ \hat{H}_{i,j,b}^{(L)} - E\left[ \hat{H}_{i,j,b}^{(L)} | X \right] - E \left[ \hat{H}_{i,j,b}^{(L)} | X \right] + E \left[ \hat{H}_{i,j,b}^{(L)} \right] \right]
$$

where $\hat{H}^{(L)}(u) \equiv \int H^{(L)}(v)H^{(L)}(u-v)dv$.

The following Theorem 3.6 is a formal expansion of the Studentised KDE with global plug-in bandwidth. Although we have options for variance estimation, we adopt the following natural estimator as Hall (1991, 1992a) and Hall and Kang (2001).

$$
\mu_20(h) \equiv h^{-1} \left\{ \frac{1}{n} \sum_{i=1}^{n} K_{i,h}(x)^2 - \left( \frac{1}{n} \sum_{i=1}^{n} K_{i,h}(x) \right)^2 \right\}.
$$

In addition, as Hall and Kang (2001), let bandwidth used for the estimation of $\mu_{20}$ be $\hat{h}$. Consequently, Studentised KDE with global plug-in bandwidth is given by

$$
T_{Pr}(x) = \frac{\sqrt{n\hat{h}(\hat{f}_h(x) - E\hat{f}_{h_0}(x))}}{\mu_{20}(\hat{h})^{1/2}}
$$

Define

$$
\delta = h_0^{-1} \left\{ E \left[ K_{i,h_0}(x)K_{i,h_0}'(x)u_{i,h_0}(x) \right] - E \left[ K_{i,h_0}(x) \right] K_{i,h_0}'(x)u_{i,h_0}(x) \right\}
$$

Theorem 3.6 (Edgeworth Expansion Including Pilot Bandwidth and the effect of Studentisation), under Assumptions 1, 4, 6, 7, 8, 11, 14 and 17

$$
P(T_{Pr}(x) \leq z) = \Phi(z) + \phi(z) \left\{ q_1(z)(nh_0)^{-1/2} + \left\{ q_2(z) + p_4(z) + q_1(z) \right\} n^{-1/2}h_0^{1/2} + q_3(z)(nh_0)^{-1}
\right.
\left. + \sum_{l=0}^{L-1} p_{1,l}(z)h_0^{l+1} + \left. \sum_{l=0}^{L-1} p_{1,l}(z)n^{-1/2}h_0^{(2L+2l+1)/2}b^{-2L} + \left\{ p_2(z) + q_2(z) \right\} n^{-1}b^{-2L} \right\}
\right. + a_r \left\{ (nh_0)^{-1} + h_0^2 \right\}
$$

where the definitions of $q_1(z), q_2(z), q_3(z), q_1(z), q_2(z)$ and $q_3(z)$ are

$$
q_1(z) \equiv \frac{1}{2} \mu_{20}^{-3/2} \mu_{11} - \frac{1}{6} \mu_{20}^{-3/2} (\mu_{30} - 3 \mu_{11})(z^2 - 1)
$$

$$
q_2(z) \equiv -f(x)\mu_{20}^{-1}z^2
$$

$$
q_3(z) \equiv -\mu_{20}^{-3/2} \mu_{30}z - \left( \frac{2}{3} \mu_{20}^{-3/2} \mu_{30}^2 - \frac{1}{12} \mu_{20}^{-3/2} \mu_{40} \right)(z^3 - 3z) - \frac{1}{18} \mu_{20}^{-3} \mu_{30}^2 (z^5 - 10z^3 + 15z)
$$

$$
q_1(z) \equiv \frac{C_{pr}}{2} \left\{ 1 + \delta \mu_{20}^{-1} \right\} \mu_{20}^{-1/2} \rho_{11} \xi^2
$$

$$
q_2(z) \equiv \frac{C_{pr}}{4} \mu_{20}^{-1} \omega_{111} \left\{ 1 + \delta \mu_{20}^{-1} \right\} (z^3 - 2z)
$$
Remark 21. $q_1(z), q_2(z)$ and $q_3(z)$ reflect the effect of Studentisation and $q_1(z)$ and $q_2(z)$ reflect the simultaneous effect of Studentisation and bandwidth selection.

Remark 22. Although, all expansions in our paper are for the KDE centralised at $\hat{f}_{b_0}(x)$ as Hall and Kang (2001), centring at $f(x)$, as Hall (1992b) and Calonico et al. (2018), is more desirable from an empirical point of view. Additionally, one of the final goals of the theoretical analysis for KDE with data-driven bandwidth is to simultaneously clarify the effect of bandwidth selection, Studentisation and debiasing. However, Hall and Kang (2001) and this paper retain some value in the sense that they extract the pure effect of bandwidth selection and the simultaneous effect of bandwidth selection and Studentisation.

4 Simulation Study

4.1 Simulation Settings and Confidence Interval estimation

In order to examine the higher order improvements by the Edgeworth expansions, we compare the coverage accuracies of the normal approximation, the Cornish-Fisher expansion with optimal bandwidth (Hall (1991)), and the Cornish-Fisher expansions with plug-in bandwidth (Theorem 3.1 and 3.5). Following Marron and Wand (1992), the underlying distributions are chosen to be a standard normal distribution $N(0, 1)$ and a skewed unimodal density constructed as a mixture of $N(0, 1)$, $N(1/2, (2/3)^2)$ and $N(13/12, (5/9)^2)$ in the proportions of 1 : 1 : 3. We use the following kernel functions as Hall and Kang (2001):

$$K(u) = \frac{1}{\sqrt{2\pi}}e^{-u^2/2}, \quad H(u) = \frac{1}{8\sqrt{2\pi}}(u^4 - 10u^2 + 15)e^{-u^2/2},$$

namely $L = 2$ and $L_p = 6$.

Let $z_{\alpha, 2}$, $w_{\alpha, 2}^{PL}$, and $w_{\alpha, 2}^{Pilot}$ be the $100\alpha$%-quantile point of normal distribution, Cornish-Fisher expansion of KDE with optimal bandwidth (Hall (1991)), Cornish-Fisher expansion of the KDE with plug-in bandwidth and Cornish-Fisher expansion with plug-in bandwidth, respectively (Theorem 3.5). In this experiment, we set $\alpha = 0.05$. We construct the following confidence intervals and count the number of intervals that include $\hat{f}_{b_0}(x)$ out of 2000 iterations. We divide it by 2000 to compute the empirical coverage probability and evaluate the performance of each approximation by its closeness to the nominal coverage probability of 0.9500:

$$I_N = \left[ \frac{\hat{f}_{b_0}}{\sqrt{nh_{b_0}}}, \frac{\hat{f}_{b_0}}{\sqrt{nh_{b_0}}} \right], \quad I_H = \left[ \frac{\hat{f}_{b_0}}{\sqrt{nh_{b_0}}}, \frac{\hat{f}_{b_0}}{\sqrt{nh_{b_0}}} \right],$$

$$I_{PL} = \left[ \frac{\hat{f}_{b_0}}{\sqrt{nh_{b_0}}}, \frac{\hat{f}_{b_0}}{\sqrt{nh_{b_0}}} \right], \quad I_{Pilot} = \left[ \frac{\hat{f}_{b_0}}{\sqrt{nh_{b_0}}}, \frac{\hat{f}_{b_0}}{\sqrt{nh_{b_0}}} \right].$$

The experiment is conducted with MSE-optimal pilot bandwidth for sample sizes $n = 50, 100, 400, 1000$. The MSE-optimal pilot bandwidth is defined as follows (See Lemma 3.1 of Hall and Marron (1987) for the proof):

$$b_0 = (4L + 1) \left( \frac{\int f(x)^2 dx}{L_p(L_p + 1)^{-2} \left( \int u^{L_p} H(u) du \right)^2 \left( \int f(x) f^{(L_p)}(x) dx \right)^2} \right)^{1/(4L + 2L_p + 1)} n^{-2/(4L + 2L_p + 1)},$$

$$b_0^{Conv} = (4L + 1) \left( \frac{\int f(x)^2 dx}{L_p(L_p + 1)^{-2} \left( \int u^{L_p} (H * H)(u) du \right)^2 \left( \int f(x) f^{(L_p)}(x) dx \right)^2} \right)^{1/(4L + 2L_p + 1)} n^{-2/(4L + 2L_p + 1)},$$

where $*$ denotes the convolution.

4.2 Simulation Results with $\hat{I}_L$

Tables 1-5 in Section 4.2.1 report the nominal coverage probabilities for five evaluation points $x = 0, 0.5, 1.0, 1.5$ and 2.0 in the case of $N(0, 1)$ observations. In each table, results for the sample sizes of $n = \{50, 100, 400, 1000\}$ are shown.
when we approximate the distribution of $S_{N}(x)$ by $N(0,1)$, Hall (1991)'s Edgeworth expansion, Theorem 3.1, and 3.5. In each row, ** and * indicate the closest and second-closest value to the nominal coverage probability of 0.9500. Similarly, Tables 6-14 in Section 4.2.2 present the results for $x = -2, -1.5, -1.0, -0.5, 0, 0.5, 1.0, 1.5$ and 2.0 with skewed unimodal normal mixture.

We also conducted a simulation when we adopted $\hat{I}_{conv}$ to estimate $I_{L}$ but we suppressed the results because they are qualitatively similar. We provide them in the supplemental material (F).

### 4.2.1 Standard Normal

We adopt #1 : $N(0, 1)$ in Marron and Wand (1992). For sample size $n = (50, 100, 400, 1000)$, MSE optimal pilot bandwidths are $b_0 = (0.8448, 0.7908, 0.6930, 0.6351)$. We evaluate the accuracy at the point of $x = 0, 0.5, 1, 1.5$, and $x = 2$ in Tables 1-5 respectively.

From Tables 1-5, we observe that approximation by Theorems 3.1 and 3.5 outperform the $N(0,1)$ or Hall’s approximations with some exceptions with mainly small $n$ (see Tables 3 and 4). We also see, in Table 3, that $N(0,1)$ and Hall’s expansion provide the closest coverage rate to 0.9500, but the differences in the coverage probability with Theorems 3.1 and 3.5 are only marginal. It is not clear which performs better Theorem 3.1 or 3.5 depending on the evaluation points and sample size. We conclude that Edgeworth expansions obtained mostly improve the confidence interval estimation in this case.

We point out that coverage ratios in Tables 3 and 4 are satisfactory in the level, that is, are close to the nominal probability of 0.9500, while Tables 1, 2, and 5 provide dismal performance independent of the approximation methods. We further find, in Tables 1, 4, and 5, that increase in sample size does not improve the confidence interval estimation. We discuss this issue at the end of this section.

Naturally, in any situation, the average length of intervals gets shorter as the sample size increases. Moreover, in most cases, the confidence intervals created by Theorem 3.1 and 3.5 are longer than those by $N(0,1)$ and Hall (1991)'s expansion. Except for the case of $n = 50$ in Table 4, the coverage probabilities by the approximation of $N(0,1)$ are much less than 0.9500, and Theorem 3.1 and 3.5 correct the approximation error by providing relatively long confidence intervals.

#### Table 1: $x = 0, b = $ MSE optimal, scaled second derivative= 0.4122

|       | $n = 50$ |       | $n = 100$ |       | $n = 400$ |       | $n = 1000$ |
|-------|---------|-------|-----------|-------|-----------|-------|------------|
| CP    | Ave.Length | CP    | Ave.Length | CP    | Ave.Length | CP    | Ave.Length |
| N(0,1)| 0.5200   | 0.1264| 0.5065     | 0.1007| 0.5660     | 0.0670| 0.5730     | 0.0492     |
| Hall (1991)| 0.5150 | 0.1265| 0.5005     | 0.1008| 0.5615     | 0.0670| 0.5660     | 0.0492     |
| Theorem 3.1| 0.7070∗ | 0.1762| 0.6005*    | 0.1267| 0.6270**   | 0.0744| 0.6110**   | 0.0523     |
| Theorem 3.5| 0.7180**| 0.2115| 0.6020**   | 0.1438| 0.6240*    | 0.0785| 0.6085*    | 0.0538     |

#### Table 2: $x = 0.5, b = $ MSE optimal, scaled second derivative= 0.2728

|       | $n = 50$ |       | $n = 100$ |       | $n = 400$ |       | $n = 1000$ |
|-------|---------|-------|-----------|-------|-----------|-------|------------|
| CP    | Ave.Length | CP    | Ave.Length | CP    | Ave.Length | CP    | Ave.Length |
| N(0,1)| 0.6780   | 0.1266| 0.6070     | 0.0999| 0.7500     | 0.0654| 0.7605     | 0.0477     |
| Hall (1991)| 0.6700 | 0.1267| 0.6010     | 0.1000| 0.7460     | 0.0654| 0.7500     | 0.0477     |
| Theorem 3.1| 0.7195∗ | 0.1434| 0.6310*    | 0.1081| 0.7585*    | 0.0675| 0.7665*    | 0.0485     |
| Theorem 3.5| 0.7685**| 0.1729| 0.6550**   | 0.1222| 0.7690**   | 0.0708| 0.7700**   | 0.0497     |

#### Table 3: $x = 1, b = $ MSE optimal, scaled second derivative= 0

|       | $n = 50$ |       | $n = 100$ |       | $n = 400$ |       | $n = 1000$ |
|-------|---------|-------|-----------|-------|-----------|-------|------------|
| CP    | Ave.Length | CP    | Ave.Length | CP    | Ave.Length | CP    | Ave.Length |
| N(0,1)| 0.9265∗ | 0.1209| 0.9130*    | 0.0934| 0.9655     | 0.0592| 0.9650**   | 0.0424     |
| Hall (1991)| 0.9200 | 0.1209| 0.9060     | 0.0934| 0.9640**   | 0.0592| 0.9650**   | 0.0424     |
| Theorem 3.1| 0.9240 | 0.1209| 0.9065     | 0.0934| 0.9640**   | 0.0592| 0.9655     | 0.0424     |
| Theorem 3.5| 0.9540**| 0.1375| 0.9340**   | 0.1011| 0.9680     | 0.0609| 0.9670     | 0.0430     |
4.2.2 Skewed Unimodal

We adopt #2: $\frac{1}{2}N(0,1) + \frac{1}{2}N(\frac{1}{3}, (\frac{2}{3})^2) + \frac{3}{4}N(\frac{1}{4}, (\frac{3}{4})^2)$ in Marron and Wand (1992).

For sample size $n = (50, 100, 400, 1000)$, MSE optimal pilot bandwidths are $b_0 = (0.5227, 0.4893, 0.4287, 0.3929)$.

We evaluate the accuracy at the point of $x = -2, -1.5, -1, -0.5, 0, 0.5, 1, 1.5, \text{ and } x = 2$.

Similar to the case of standard normal observations in the previous section, we find that Theorems 3.1 and 3.5 outperform $N(0, 1)$ and Hall’s approximations in general.

We observe in Tables 12 and 14 that the general coverage probability level significantly differs from the nominal value of whichever approximation we adopt and further the results look to contradict the asymptotic theory. We discuss this in the next subsection.

As with the case of standard normal, as the sample size increases, the intervals also get shorter. Moreover, Theorem 3.1 and 3.5 provide the longer confidence intervals and thereby achieve coverage probabilities closer to 0.95 than $N(0, 1)$ and Hall (1991)’s approximation, at the point with poor coverage.

| Table 4: $x = 1.5, b = \text{MSE optimal, scaled second derivative}= 0.1673$ |
| --- |
| $n = 50$ | $n = 100$ | $n = 400$ | $n = 1000$ |
| CP | Ave.Length | CP | Ave.Length | CP | Ave.Length | CP | Ave.Length |
| $N(0, 1)$ | 0.9495** | 0.1022 | 0.9140 | 0.0773 | 0.8645 | 0.0476 | 0.8330 | 0.0333 |
| Hall (1991) | 0.9550* | 0.1021 | 0.9305 | 0.0772 | 0.8755 | 0.0472 | 0.8405 | 0.0333 |
| Theorem 3.1 | 0.9760 | 0.1131 | 0.9520** | 0.0830 | 0.8880* | 0.0489 | 0.8520** | 0.0339 |
| Theorem 3.5 | 0.9790 | 0.1188 | 0.9530* | 0.0856 | 0.8885** | 0.0495 | 0.8520** | 0.0341 |

| Table 5: $x = 2, b = \text{MSE optimal, scaled second derivative}= 0.1673$ |
| --- |
| $n = 50$ | $n = 100$ | $n = 400$ | $n = 1000$ |
| CP | Ave.Length | CP | Ave.Length | CP | Ave.Length | CP | Ave.Length |
| $N(0, 1)$ | 0.6865 | 0.0746 | 0.5995 | 0.0552 | 0.6410 | 0.0327 | 0.6240 | 0.0227 |
| Hall (1991) | 0.7295 | 0.0742 | 0.6350 | 0.0550 | 0.6655 | 0.0327 | 0.6445 | 0.0227 |
| Theorem 3.1 | 0.7865** | 0.0823 | 0.6625** | 0.0588 | 0.6850** | 0.0335 | 0.6570** | 0.0230 |
| Theorem 3.5 | 0.7780* | 0.0833 | 0.6580* | 0.0593 | 0.6845* | 0.0336 | 0.6550* | 0.0230 |

| Table 6: $x = -2, b = \text{MSE optimal, scaled second derivative}= 0.0173$ |
| --- |
| $n = 50$ | $n = 100$ | $n = 400$ | $n = 1000$ |
| CP | Ave.Length | CP | Ave.Length | CP | Ave.Length | CP | Ave.Length |
| $N(0, 1)$ | 0.8675 | 0.0424 | 0.8150 | 0.0299 | 0.9190 | 0.0179 | 0.9270 | 0.0125 |
| Hall (1991) | 0.9155 | 0.0396 | 0.8615* | 0.0288 | 0.9395** | 0.0176 | 0.9440** | 0.0124 |
| Theorem 3.1 | 0.9160* | 0.0397 | 0.8615* | 0.0288 | 0.9395** | 0.0176 | 0.9440** | 0.0124 |
| Theorem 3.5 | 0.9165** | 0.0398 | 0.8630** | 0.0288 | 0.9395** | 0.0177 | 0.9440** | 0.0124 |

| Table 7: $x = -1.5, b = \text{MSE optimal, scaled second derivative}= 0.0278$ |
| --- |
| $n = 50$ | $n = 100$ | $n = 400$ | $n = 1000$ |
| CP | Ave.Length | CP | Ave.Length | CP | Ave.Length | CP | Ave.Length |
| $N(0, 1)$ | 0.8635 | 0.0654 | 0.8070 | 0.0464 | 0.9210 | 0.0279 | 0.9430 | 0.0196 |
| Hall (1991) | 0.8925* | 0.0637 | 0.8335 | 0.0457 | 0.9355** | 0.0277 | 0.9525** | 0.0195 |
| Theorem 3.1 | 0.8925* | 0.0639 | 0.8340* | 0.0457 | 0.9355** | 0.0278 | 0.9525** | 0.0196 |
| Theorem 3.5 | 0.8955** | 0.0643 | 0.8350** | 0.0460 | 0.9355** | 0.0278 | 0.9525** | 0.0196 |
### Table 8: $x = -1, b = \text{MSE optimal}$, scaled second derivative $= 0.0503$

|       | $n = 50$ |         | $n = 100$ |         | $n = 400$ |         | $n = 1000$ |         |
|-------|---------|---------|-----------|---------|-----------|---------|------------|---------|
| CP    | Ave.Length | CP    | Ave.Length | CP    | Ave.Length | CP    | Ave.Length |
| N(0,1)| 0.8490  | 0.0933 | 0.7550 | 0.0664 | 0.8870 | 0.0402 | 0.9375 | 0.0283 |
| Hall (1991)| 0.8780 | 0.0923 | 0.7870 | 0.0660 | 0.9070 | 0.0401 | 0.9450 | 0.0283 |
| Theorem 3.1 | 0.8795* | 0.0927 | 0.7880* | 0.0662 | 0.9075* | 0.0402 | 0.9455** | 0.0283 |
| Theorem 3.5 | 0.8855** | 0.0941 | 0.7910** | 0.0669 | 0.9085** | 0.0403 | 0.9455** | 0.0284 |

### Table 9: $x = -0.5, b = \text{MSE optimal}$, scaled second derivative $= 0.1112$

|       | $n = 50$ |         | $n = 100$ |         | $n = 400$ |         | $n = 1000$ |         |
|-------|---------|---------|-----------|---------|-----------|---------|------------|---------|
| CP    | Ave.Length | CP    | Ave.Length | CP    | Ave.Length | CP    | Ave.Length |
| N(0,1)| 0.8685  | 0.1265 | 0.7410 | 0.0907 | 0.8540 | 0.0555 | 0.8950 | 0.0393 |
| Hall (1991)| 0.8945 | 0.1260 | 0.7675 | 0.0905 | 0.8720 | 0.0554 | 0.9030* | 0.0393 |
| Theorem 3.1 | 0.9055* | 0.1286 | 0.7765* | 0.0916 | 0.8725* | 0.0556 | 0.9030* | 0.0393 |
| Theorem 3.5 | 0.9120** | 0.1328 | 0.7830** | 0.0936 | 0.8735** | 0.0562 | 0.9055** | 0.0395 |

### Table 10: $x = 0, b = \text{MSE optimal}$, scaled second derivative $= 0.2029$

|       | $n = 50$ |         | $n = 100$ |         | $n = 400$ |         | $n = 1000$ |         |
|-------|---------|---------|-----------|---------|-----------|---------|------------|---------|
| CP    | Ave.Length | CP    | Ave.Length | CP    | Ave.Length | CP    | Ave.Length |
| N(0,1)| 0.9630** | 0.1626 | 0.9510** | 0.1187 | 0.9145 | 0.0744 | 0.9205 | 0.0533 |
| Hall (1991)| 0.9670* | 0.1625 | 0.9590* | 0.1186 | 0.9225 | 0.0744 | 0.9225 | 0.0533 |
| Theorem 3.1 | 0.9795 | 0.1777 | 0.9710 | 0.1259 | 0.9350* | 0.0763 | 0.9290* | 0.0540 |
| Theorem 3.5 | 0.9900 | 0.1929 | 0.9780 | 0.1328 | 0.9375** | 0.0780 | 0.9305** | 0.0547 |

### Table 11: $x = 0.5, b = \text{MSE optimal}$, scaled second derivative $= 0.1170$

|       | $n = 50$ |         | $n = 100$ |         | $n = 400$ |         | $n = 1000$ |         |
|-------|---------|---------|-----------|---------|-----------|---------|------------|---------|
| CP    | Ave.Length | CP    | Ave.Length | CP    | Ave.Length | CP    | Ave.Length |
| N(0,1)| 0.8595* | 0.1867 | 0.7375* | 0.1399 | 0.9185* | 0.0915 | 0.9545 | 0.0669 |
| Hall (1991)| 0.8520 | 0.1867 | 0.7290 | 0.1400 | 0.9105 | 0.0915 | 0.9500** | 0.0670 |
| Theorem 3.1 | 0.8510 | 0.1841 | 0.7260 | 0.1380 | 0.9065 | 0.0907 | 0.9475* | 0.0665 |
| Theorem 3.5 | 0.9120 | 0.2312 | 0.7905 | 0.1597 | 0.9265 | 0.0960 | 0.9560 | 0.0685 |

### Table 12: $x = 1, b = \text{MSE optimal}$, scaled second derivative $= 0.7019$

|       | $n = 50$ |         | $n = 100$ |         | $n = 400$ |         | $n = 1000$ |         |
|-------|---------|---------|-----------|---------|-----------|---------|------------|---------|
| CP    | Ave.Length | CP    | Ave.Length | CP    | Ave.Length | CP    | Ave.Length |
| N(0,1)| 0.6345 | 0.1923 | 0.3620 | 0.1457 | 0.4835 | 0.0973 | 0.5265 | 0.0720 |
| Hall (1991)| 0.6240 | 0.1925 | 0.3545 | 0.1458 | 0.4810 | 0.0973 | 0.5225 | 0.0720 |
| Theorem 3.1 | 0.8370* | 0.2930 | 0.5015* | 0.1977 | 0.5630** | 0.1130 | 0.5845** | 0.0788 |
| Theorem 3.5 | 0.8405** | 0.3556 | 0.5035** | 0.2279 | 0.5595* | 0.1209 | 0.5785* | 0.0819 |

### Table 13: $x = 1.5, b = \text{MSE optimal}$, scaled second derivative $= 0.1559$

|       | $n = 50$ |         | $n = 100$ |         | $n = 400$ |         | $n = 1000$ |         |
|-------|---------|---------|-----------|---------|-----------|---------|------------|---------|
| CP    | Ave.Length | CP    | Ave.Length | CP    | Ave.Length | CP    | Ave.Length |
| N(0,1)| 0.8530 | 0.1849 | 0.7390 | 0.1373 | 0.8480 | 0.0886 | 0.8895 | 0.0644 |
| Hall (1991)| 0.8485 | 0.1850 | 0.7290 | 0.1373 | 0.8430 | 0.0886 | 0.8865 | 0.0644 |
| Theorem 3.1 | 0.8525* | 0.1893 | 0.7305* | 0.1387 | 0.8430* | 0.0886 | 0.8865* | 0.0643 |
| Theorem 3.5 | 0.8985** | 0.2259 | 0.7765** | 0.1556 | 0.8625** | 0.0928 | 0.8905** | 0.0659 |
4.3 Difficulties in Confidence Interval Estimation with High Curvature

We provide the tables of simulation results in the above subsections. Note that at some points, the results contradict the asymptotic theory. Such phenomena seem to occur at points of high curvature of the density function (e.g. Table 1, 5, 12, 14, 15, 19, 26 and 28.) Nevertheless, tough Table 2, 6, 7, 8, 9, 11, 12, 13, 22, 23, 25 and 27 also contradict the asymptotic theory in cases of a small sample size. However, their accuracy is recovered in cases of a large sample size, indicating that the asymptotic theory would work in situations where the sample size is literally \( \leq \infty \). Some studies (i.e. Brockmann et al. (1993) and Fan et al. (1996)) have already found similar phenomenon where curve estimation with global bandwidth tends to be oversmoothing and displays have poor performance at points of large curvature. Hastie et al. (2009) introduce this phenomenon as 'trimming the hills' and 'filling the valleys' in the literature of local linear regression. However, since our Standardised statistics are centred at \( \mathbb{E}[f_{h_0}(x)] \), this phenomenon cannot occur. Additional simulation results show that, using optimal bandwidth \( h_0 \), KDEs are distributed around \( \mathbb{E}[\hat{f}_{h_0}(x)] \) (See Figure 5,6,7,8,9 and 10), so oversmoothing at the high curvature point in our simulation studies comes from the bandwidth selection. This is despite the fact that \( \hat{h}_L \)'s are distributed around \( h_L \) in a good manner for large sample sizes (See Figure 1,2,3, and 4). Hastie et al. (2009) state that one can avoid oversmoothing from 'trimming the hills' and 'filling the valleys' by using local polynomial regressions higher than second-order (for density estimation, one has to use local polynomial density Cattaneo et al. (2020) higher than third-order). However, one cannot not avoid oversmoothing from bandwidth selection in the way. The scope of this paper is to develop higher-order approximation of KDE with global plug-in bandwidth and solving the puzzle on the curvature is out of scope. Strategies for dealing with this difficulty are discussed in Section 5. However, for almost all points, our expansions provide more precise approximation than the normal approximations and the Edgeworth expansion with deterministic bandwidth.

5 Discussion and Conclusions

This study investigated the higher-order asymptotic properties of KDE with global plug-in bandwidth. The first contribution is that we provide the Edgeworth expansion of KDE with global plug-in bandwidth up to the order of \( O\{(nh_0)^{-1} + h_0^{2L}\} = O(n^{\frac{L}{2L+1}}) \) and show that the bandwidth selection by the plug-in method starts to have an effect from the term whose convergence rate is \( O\{(nh_0)^{-1/2} + h_0^{L+1}\} = O(n^{\frac{-L}{2L+1}}) \) under the condition that \( L_p \) is large enough. Second, we generalise Theorem 3.2 of Hall and Kang (2001), which states that bandwidth selection via the global plug-in method has no effect on the asymptotic structure of KDE up to the order of \( O\{(nh_0)^{-1/2} + h_0^L\} = O(n^{\frac{L}{2L+1}}) \). Their results limit the order of kernel functions \( K(u) \) and \( H(u) \) to \( L = 2, L_p = 6 \) respectively, but we show that they are valid for general orders \( L \) as well under the condition that \( L_p \) is large enough. Third, we explore Edgeworth expansion of KDE with deterministic bandwidth in more detail than Hall (1991). We show that Edgeworth expansion of Standardized KDE with deterministic bandwidth has the term of order \( O\{(nh_0)^{-1/2} + h_0^L\} = O(n^{\frac{L}{2L+1}}) \) right after the term \( \Phi(z) \) with a gap between them. After that however, the terms decrease at the rate of \( O(h_0) = O(n^{\frac{1}{2L+1}}) \). Fourth, we weaken this condition on \( L_p \) assumed by Hall and Kang (2001) and our Theorem 3.1 and provide the Edgeworth expansion including the effect of pilot bandwidth up to the order of \( O\{(nh_0)^{-1} + h_0^{2L}\} \). In this situation, the bandwidth selection via the global plug-in method possibly has an effect on the asymptotic structure of KDE even up to the order of \( O\{(nh_0)^{-1/2} + h_0^L\} \) (for example, when \( L = 2 \) and \( L_p = 2 \)). Finally, we consider the intersectional effect of the bandwidth selection via the global plug-in method, its accompanying pilot bandwidth, and Studentisation.

Simulation studies show that our higher-order approximation is more precise at the point where coverage probability of normal approximation is away from 0.9500 while less precise at the point where normal approximation is nearly 0.9500.
Another implication of simulation studies is that the estimation at the points of large curvature is difficult. One possible method to avoid this problem is to use locally adaptive bandwidth. However, locally adaptive bandwidth also has disadvantages. First, selecting bandwidth at each $x$ is computationally expensive, especially in multivariate case. Second, Hall and Kang (2001) have shown that nonparametric bootstrap procedures for KDE with locally adaptive bandwidth lack the asymptotic refinement, while those with global bandwidth do not. Finally, and most importantly, some authors state that locally adaptive procedures are not suited for the construction of confidence intervals; we quote (Wasserman, 2006, p. 212) [...]do adaptive methods work or not? If one needs accurate function estimates and the noise level is low, then the answer is that adaptive function estimators are very effective. But if we are facing a standard nonparametric regression problem and we are interested in confidence sets, then adaptive methods do not perform significantly better than other methods such as fixed bandwidth local regression. Another strategy employs partially adaptive bandwidth as Hall et al. (1995) and in domains where global bandwidths are used, our approximations might be useful.

As stated in Remark 1, centring at $E\hat{f}_h(x)$ leaves asymptotic bias under standard conditions. Two standard methods to deal with asymptotic bias (debias) are ‘undersmoothing’ and ‘explicit bias reduction’. The former refers to choosing the bandwidth satisfying $\sqrt{n}h \rightarrow 0$ and the latter directly estimates and removes the bias term. Hall (1992b) examined the effect of undersmoothing and explicit bias reduction on the asymptotic structure via the Edgeworth expansion up to the order of $O\{nh^{-1}\}$, and stated that undersmoothing provides better coverage than explicit bias correction. After that, Calonico et al. (2018) have proposed alternative bias correction methods and show that their method is comparable with undersmoothing by Edgeworth expansion up to the order of $O\{nh^{-1}\}$. However, the bandwidth in their expansion is still deterministic. We can interpret that Hall and Kang (2001), our study, and Hall (1992b), Calonico et al. (2018) studied these effects separately, that is, the pure effect of bandwidth selection and the pure effect of debias respectively. A goal for future research will be investigating the effect of bandwidth selection and debias simultaneously, on which we are working at the moment.

Among the recent topics in which the density estimator plays an important role is the manipulation test of regression discontinuity designs (RDD). Cattaneo et al. (2020) proposed a local polynomial density estimator for adaptability at or near the boundary points. We expect that the asymptotic structure of their estimator with the corresponding plug-in bandwidth has a similar structure to that of the KDE provided in this paper.

One of the other possible extensions of this work is, which we are in the process of working on, is investigating the effects of cross-validation methods on the asymptotic structure.

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Supplemental Materials for 'Higher-Order Asymptotic Properties of Kernel Density Estimator with Global Plug-In and Its Accompanying Pilot Bandwidth' (not for publication)

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A Proofs of Results

A.1 Proof of Proposition 2.1

Proof. Recall that the unknown part \( I_L \) of the theoretically optimal bandwidth which minimize MISE is estimated by

\[
I_L = \left( \frac{n}{2} \right)^{-1} n \sum_{i=1}^{n} \sum_{j=1}^{n} b^{-(2L+1)}H^{(2L)} \left( \frac{X_i - X_j}{b} \right) = \left( \frac{n}{2} \right)^{-1} n \sum_{i=1}^{n} \sum_{j=1}^{n} \hat{I}_{ij}.
\]

Since \( \hat{I}_L \) has a U-statistic form, we can use Hoeffding-Decomposition,

\[
\hat{I}_L = E[\hat{I}_{ij} | X_i] + 2 \sum_{i=1}^{n} \left( \hat{I}_{li} - E[\hat{I}_{ij}] \right) + \left( \frac{n}{2} \right)^{-1} n \sum_{i=1}^{n} \sum_{j=1}^{n} \{ \hat{I}_{ij} - \hat{I}_{li} - \hat{I}_{lj} + E[\hat{I}_{ij}] \}, \tag{A.1}
\]

where \( \hat{I}_{li} = E[\hat{I}_{ij} | X_i] \). In order to examine \( \hat{I}_L \), we have to compute \( E[\hat{I}_{ij}] \) and \( \hat{I}_{li} \).

\[
\hat{I}_{li} = E[\hat{I}_{ij} | X_i] = \int \frac{1}{b^{2L+1}} H^{(2L)} \left( \frac{X_i - x}{b} \right) f(x) dx
\]

\[
= \int \frac{1}{b^{2L}} H^{(2L)}(u) f(X_i + ub) du
\]

\[
= \int H(u) f^{(2L)}(X_i + ub) du
\]

\[
= \int H(u) \left\{ f^{(2L)}(X_i) + \frac{f^{2L+1}p}{Lp!} (ub)^{p} + o(b^{p}) \right\} du
\]

\[
= f^{(2L)}(X_i) + \frac{b^{p}}{(Lp)!} \left( \int u^{p} H(u) du \right) f^{(2L+1)p)}(X_i) + o_p(b^{p}), \tag{A.2}
\]

where the third equality follows from integration by part and the fourth equality follows from the expansion of \( f^{(2L)}(X_i + ub) \) around \( X_i \). This implies

\[
E[\hat{I}_{ij}] = E[f^{(2L)}(X_i)] + \frac{\int u^{p} H(u) du}{(Lp)!} E[f^{(2L+1)p)}(X_i)] b^{p} + o_p(b^{p}). \tag{A.4}
\]

From integration by parts the first term of the right-hand side of (A.4) is

\[
E[f^{(2L)}(X_i)] = \int f^{(2L)}(x) f(x) dx = \int f^{(L)}(x)^2 dx = I_L, \tag{A.5}
\]

Inserting (A.3), (A.4) and (A.5) into (A.1), we have

\[
\hat{I}_L = I_L + 2 \sum_{i=1}^{n} \left\{ f^{(2L)}(X_i) - E[f^{(2L)}(X_i)] \right\}
\]

\[
+ 2 \left( \int u^{p} H(u) du \right) \frac{b^{p}}{(Lp)!} \sum_{i=1}^{n} \left\{ f^{(2L+1)p)}(X_i) - E[f^{(2L+1)p)}(X_i)] \right\}
\]

\[
+ \left( \frac{n}{2} \right)^{-1} \sum_{i=1}^{n} \sum_{j=1}^{n} \{ \hat{I}_{ij} - \hat{I}_{li} - \hat{I}_{lj} + E[\hat{I}_{ij}] \} + o_p(n^{-1/2}b^{p}). \tag{A.6}
\]

Recall that Plug-In bandwidth is defined as follows,

\[
\hat{h} = \left( \frac{R(K)}{2LC_L^2 I_L} \right)^{\frac{1}{2L+1}} n^{-\frac{1}{2L+1}}. \tag{A.7}
\]
We evaluate the difference between \( \hat{h} \) and \( h_0 \) using (A.6).

\[
\hat{I}^{\frac{-1}{p+1}}_L - I^{\frac{-1}{p+1}}_L = -\frac{1}{2L+1} I^{\frac{-1}{p+1}}_L \left[ 2 \sum_{i=1}^n \{ f^{(2L)}(X_i) - \mathbb{E} f^{(2L)}(X_i) \} + \frac{2}{n} \left( \int u^{p+1} H(u) du \right) \frac{b^{p+1}}{(L_p)!} \sum_{i=1}^n \{ f^{(2L+L_p)}(X_i) - \mathbb{E} f^{(2L+L_p)}(X_i) \} + \left( \frac{n}{2} \right) \sum_{i=1}^n \sum_{j=1}^{n-1} \{ \hat{I}_{Lj} - \hat{I}_{Li} - \hat{I}_{Lj} + \mathbb{E} \hat{I}_{Lj} \} \right] + o_p(n^{-1/2} b^{p+1}).
\]

Inserting this expansion into (A.7) yields

\[
\hat{h} = h_0 - \frac{h_0}{2L+1} I^{\frac{-1}{p+1}}_L \left[ 2 \sum_{i=1}^n \{ f^{(2L)}(X_i) - \mathbb{E} f^{(2L)}(X_i) \} + \frac{2}{n} \left( \int u^{p+1} H(u) du \right) \frac{b^{p+1}}{(L_p)!} \sum_{i=1}^n \{ f^{(2L+L_p)}(X_i) - \mathbb{E} f^{(2L+L_p)}(X_i) \} + \left( \frac{n}{2} \right) \sum_{i=1}^n \sum_{j=1}^{n-1} \{ \hat{I}_{Lj} - \hat{I}_{Li} - \hat{I}_{Lj} + \mathbb{E} \hat{I}_{Lj} \} \right] + o_p(n^{-1/2} b^{p+1})
\]

This implies

\[
\frac{\hat{h} - h_0}{h_0} = -\frac{1}{2L+1} I^{\frac{-1}{p+1}}_L \left[ 2 \sum_{i=1}^n \{ f^{(2L)}(X_i) - \mathbb{E} f^{(2L)}(X_i) \} + \frac{2}{n} \left( \int u^{p+1} H(u) du \right) \frac{b^{p+1}}{(L_p)!} \sum_{i=1}^n \{ f^{(2L+L_p)}(X_i) - \mathbb{E} f^{(2L+L_p)}(X_i) \} + \left( \frac{n}{2} \right) \sum_{i=1}^n \sum_{j=1}^{n-1} \{ \hat{I}_{Lj} - \hat{I}_{Li} - \hat{I}_{Lj} + \mathbb{E} \hat{I}_{Lj} \} \right] + o_p(n^{-1/2} b^{p+1})
\]

\[
\implies \frac{\hat{h} - h_0}{h_0} = -\frac{C_p}{n} \sum_{i=1}^n \left( \{ f^{(2L)}(X_i) - \mathbb{E} f^{(2L)}(X_i) \} + \frac{1}{(L_p)!} \frac{b^{p+1}}{(L_p)!} \sum_{i=1}^n \{ f^{(2L+L_p)}(X_i) - \mathbb{E} f^{(2L+L_p)}(X_i) \} \right) - \frac{C_p}{2} \left( \frac{n}{2} \right) \sum_{i=1}^n \sum_{j=1}^{n-1} \{ \hat{I}_{Lj} - \hat{I}_{Li} - \hat{I}_{Lj} + \mathbb{E} \hat{I}_{Lj} \} + o_p\{ (nh_0)^{-1} \}.
\]

\[\square\]

A.2 Proof of Theorem 2.4

\textit{Proof.} In view of (2.7), if the following evaluation is correct,

\[
\mathbb{E} \left| \sqrt{nh} \left( \frac{\hat{h} - h_0}{h_0} \right) \Gamma_{KDE} \right| = o\{ (nh_0)^{-1/2} \}, \quad \mathbb{E} \left| S_{h_0}(x) \left( \frac{\hat{h} - h_0}{h_0} \right) \right| = o\{ (nh_0)^{-1/2} \}
\]

then bandwidth selection has no effect on the asymptotic structure up to the order of \( O\{ (nh_0)^{-1/2} \} \). From Cauchy-Schwarz inequality

\[
\mathbb{E} \left| \sqrt{nh_0} \left( \frac{\hat{h} - h_0}{h_0} \right) \Gamma_{KDE} \right| \leq \sqrt{nh_0} \left\{ \mathbb{E} \left| \frac{\hat{h} - h_0}{h_0} \right|^2 \mathbb{E} \Gamma_{KDE}^2 \right\}^{1/2}
\]

Since under the Assumption 13, \( \hat{h} - h_0/h_0 \) has the asymptotic linear form, straightforward calculation gives

\[
\mathbb{E} \left| \frac{\hat{h} - h_0}{h_0} \right|^2 = O(n^{-1}).
\]
Next, we evaluate $\mathbb{E}[\Gamma_{\text{KDE}}]|^2$.

$$
\mathbb{E}[\Gamma_{\text{KDE}}]|^2 = \frac{1}{(nh_0)^2} \mathbb{E} \left[ \sum_{i,j} \sum_{j \neq j, i} \left\{ K_{i,j}(x)u_{i,j}(x) + K_{i,j}(x) \right\} \left\{ K'_{i,j}(x)u_{i,j}(x) + K_{i,j}(x) \right\} \right]
+ \frac{1}{(nh_0)^2} \mathbb{E} \left[ \sum_{i=1}^n \left\{ K'_{i,j}(x)u_{i,j}(x) + K_{i,j}(x) \right\} \right]^2
= \frac{1}{h_0} \mathbb{E} \left[ \sum_{i,j} \left\{ K_{i,j}(x)u_{i,j}(x) + K_{i,j}(x) \right\} \left\{ K'_{i,j}(x)u_{i,j}(x) + K_{i,j}(x) \right\} \right] + O\{\mathbb{E}[\Gamma_{\text{KDE}}]|^2\}
+ \frac{1}{h_0} \left( \int \left\{ K' \left( \frac{z_1-x}{h_0} \right) \left( \frac{z_1-x}{h_0} \right) + K \left( \frac{z_1-x}{h_0} \right) \right\} f(z_1)dz_1 \right)^2 + O\{(nh_0)^{-1}\}
= \left( \int K(u)f(x+uh_0)du + \int K(u)f(x+uh_0)du \right)^2 + O\{(nh_0)^{-1}\}
= O(h_0^2L) + O\{(nh_0)^{-1}\}.
$$

The fifth equality follows from integration by part of the first term and Assumption 12, and the final equality follows from the expansion of $f'(x+uh_0)$ around $h_0 = 0$ and Assumption 7,11. Therefore form Cauchy-Schwarz inequality,

$$
\mathbb{E} \left[ \sqrt{nh_0} \left( \frac{\hat{h}-h_0}{h_0} \right) \Gamma_{\text{KDE}} \right] \leq \sqrt{nh_0} \left\{ \mathbb{E} \left[ \left( \frac{\hat{h}-h_0}{h_0} \right)^2 \right] \mathbb{E}[\Gamma_{\text{KDE}}]|^2 \right\}^{1/2}
= O(n^{1/2}h_0^{1/2}) \left( O(n^{-1})O(h_0^2 + (nh_0)^{-1}) \right)^{1/2} = O(h_0^{L+\frac{1}{2}} + n^{-1/2}) = o\{(nh_0)^{-1/2}\}.
$$

Similar to above evaluation, Cauchy-Schwarz inequality gives $\mathbb{E} \left| S(x) \left( \frac{\hat{h}-h_0}{h_0} \right) \right| = O(n^{-1/2}) = o\{(nh_0)^{-1/2}\}$. Therefore bandwidth selection via Plug-In Method has no effect on the asymptotic structure up to the order of $O\{(nh_0)^{-1/2}\}$.

**A.3 Proof of Theorem 3.1**

**Proof.** From Proposition 2.3 and Lemma 3, we have,

$$
\sqrt{n}\left( \hat{f}_h(x) - \mathbb{E}\hat{f}(x) \right) = \sqrt{nh_0} \left( \hat{f}_h(x) - \mathbb{E}\hat{f}_h(x) \right) - \sqrt{nh_0} \left( \frac{\hat{h}-h_0}{h_0} \right) \Gamma_{\text{KDE}} + \frac{1}{2} S_h(x) \left( \frac{\hat{h}-h_0}{h_0} \right) + o_P\{(nh_0)^{-1}\}.
$$

Noting that we provide Theorem 3.1 under Assumption 15 and this assumption guarantees that the quadratic term of $(\hat{h} - h_0)/h_0$ is negligible, Proposition 2.1 provides the expansion of plug-in bandwidth as follows.

$$
\frac{\hat{h}-h_0}{h_0} = -\frac{Cp}{n} \sum_{i=1}^n \left\{ f^{(2)}(X_i) - \mathbb{E}f^{(2)}(X_i) \right\} + o_P\{(nh_0)^{-1}\}.
$$

Define

$$
S_i = \mu_{20}^{-1/2} \left\{ K \left( \frac{X_i-x}{h_0} \right) - \mathbb{E}K \left( \frac{X_i-x}{h_0} \right) \right\},
$$

$$
\Gamma_i = K' \left( \frac{X_i-x}{h_0} \right) \left( \frac{X_i-x}{h_0} \right) + K \left( \frac{X_i-x}{h_0} \right) - \mathbb{E} \left\{ K' \left( \frac{X_i-x}{h_0} \right) \left( \frac{X_i-x}{h_0} \right) + K \left( \frac{X_i-x}{h_0} \right) \right\},
$$

$$
\mathbb{L}_i = f^{(2)}(X_i) - \mathbb{E}f^{(2)}(X_i).
$$

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Recalling that $S_{PI}(x)$ is defined as (2.7), we have from Lemma 1,

$$S_{PI}(x) = \frac{\sqrt{n} \left( \hat{f}_h(x) - \mathbb{E}\hat{f}_{h_0}(x) \right)}{\mu_{20}^{1/2}} = \frac{\sqrt{n}h_0 \left( \hat{f}_h(x) - \mathbb{E}\hat{f}_{h_0}(x) \right)}{\mu_{20}^{1/2}} \frac{\left( h - h_0 \right)}{h_0} \mathbb{E}_{KDE} + o_p\left( (nh_0)^{-1} \right)$$

$$= \frac{1}{\sqrt{n}h_0} \sum_{i=1}^{n} S_i + \frac{\left( 1 / \sqrt{n}h_0 \right) \sum_{i=1}^{\Gamma} \left( \frac{C_{PI} \left( \Gamma_i \right)}{n^{1/2}} \right) L_i - \frac{1}{2} \left( \frac{1}{\sqrt{n}h_0} \sum_{i=1}^{n} C_{PI} \left( \Gamma_i \right) L_i \right) + o_p\left( (nh_0)^{-1} \right)}{\sum_{i=1}^{n} C_{PI} \left( \Gamma_i \right) L_i}$$

Define $F_{PI}(z)$ and $\tilde{F}_{PI}(z)$ as follows,

$$F_{PI}(z) = \mathbb{P}\left( S_{PI}(x) \leq z \right),$$

$$\tilde{F}_{PI}(z) = \Phi(z) + \phi(z) \left( \left( (nh_0)^{-1/2} p_1(z) + (nh_0)^{-1/2} p_2(z) + \sum_{i=0}^{L-1} h_0^{L_i+1} p_1(z) + n^{-1/2} h_0^{L_1} p_4(z) \right) \right).$$

To show the Edgeworth expansion is valid, we have to confirm $\sup_{z \in \mathbb{R}} |F_{PI}(z) - \tilde{F}_{PI}(z)| = o\left( (nh_0)^{-1} \right)$. First, we evaluate the remainder term.

$$\sup_{z \in \mathbb{R}} |F_{PI}(z) - \tilde{F}_{PI}(z)| \leq \sup_{z \in \mathbb{R}} \left| \mathbb{P}\left( \left( S(x) + \Lambda_1(x) + \Lambda_2(x) + \Lambda_3(x) + \Lambda_4(x) + \Lambda_5(x) \right) \leq z \right) - \tilde{F}_{PI}(z) \right| + \mathbb{P}\left( \left| S_{PI}(x) - \left( S(x) + \Lambda_1(x) + \Lambda_2(x) + \Lambda_3(x) + \Lambda_4(x) + \Lambda_5(x) \right) \right| \geq a_n \right) + O\left( a_n^{-1} \right)$$

where $a_n = nh_0(\log n)$. Since

$$\left| S_{PI}(x) - \left( S(x) + \Lambda_1(x) + \Lambda_2(x) + \Lambda_3(x) + \Lambda_4(x) + \Lambda_5(x) \right) \right| = o_p\left( (nh_0)^{-1} \right),$$

we have

$$\mathbb{P}\left( \left| S_{PI}(x) - \left( S(x) + \Lambda_1(x) + \Lambda_2(x) + \Lambda_3(x) + \Lambda_4(x) + \Lambda_5(x) \right) \right| \geq a_n \right) = O\left( (nh_0)^{-1} a_n^{-1} \right) = o\left( (nh_0)^{-1} \right).$$

Obviously, $O(a_n^{-1}) = o\left( (nh_0)^{-1} \right)$. Then, we only need to evaluate

$$\sup_{z \in \mathbb{R}} \left| \mathbb{P}\left( \left( S(x) + \Lambda_1(x) + \Lambda_2(x) + \Lambda_3(x) + \Lambda_4(x) + \Lambda_5(x) \right) \leq z \right) - \tilde{F}_{PI}(z) \right|.$$

Define $\chi_{PI}(t)$ and $\tilde{\chi}_{PI}(t)$ as follows,

$$\chi_{PI}(t) \equiv \mathbb{E} \left\{ \exp \left( it \left( S(x) + \Lambda_1(x) + \Lambda_2(x) + \Lambda_3(x) + \Lambda_4(x) + \Lambda_5(x) \right) \right) \right\}.$$
\[ \tilde{\chi}_p(t) \equiv \exp \left( -\frac{t^2}{2} \right) \left\{ 1 + \frac{\mu_{20}}{6n^{1/2}h_0^{1/2}} (it)^2 \right\} + C_p \left( \sum_{j=0}^{t-1} C_i(t)h_0^{L+1} \right) (it)^3 + \frac{\mu_{20}}{72n h_0} (it)^5 \right\}.

From Esséen (1945) smoothing lemma,

\[
\sup_{z \in \mathbb{R}} \left| \mathbb{P}(S(x) + \Lambda_1(x) + \Lambda_2(x) + \Lambda_3(x) + \Lambda_4(x) + \Lambda_5(x) \leq z) - \tilde{F}_T(z) \right| \\
\leq \int_{-n^{2/3} \log n}^{n^{2/3} \log n} \left| \frac{\chi_p(t) - \tilde{\chi}_p(t)}{t} \right| dt + O \left( \frac{1}{n^{2/3} \log n} \right) \\
\leq \int_{-p}^{p} \left| \frac{\chi_p(t) - \tilde{\chi}_p(t)}{t} \right| dt + \int_{-p}^{p} \left| \frac{\tilde{\chi}_p(t)}{t} \right| dt + \int_{-p}^{p} \left| \frac{\tilde{\chi}_p(t)}{t} \right| dt + o\{\log n\} \\
\equiv (A) + (B) + (C) + o\{\log n\} \quad \text{(A.9)}
\]

where \( p = \min \left\{ \frac{n^{2/3} \log n}{\mu_{20}}, \frac{2}{\mu_{20}} \right\} \). To prove the validity of the Edgeworth expansion, we show that each term of (A.9) has the convergence rate \( o\{\log n\} \).

In order to evaluate (A), we represent \( \chi_p(t) \) as \( \tilde{\chi}_p(t) \) plus a remainder. From Lemmas 8, 9, 11, 12, and 13,
\[
\chi_p(t) = \mathbb{E} \left[ e^{\theta S(x)} \right] = \mathbb{E} \left[ e^{\theta S(x)} \right] \left\{ 1 + i t \Lambda_1(x) \right\} \left\{ 1 + i t \Lambda_2(x) \right\} \left\{ 1 + i t \Lambda_3(x) \right\} \left\{ 1 + i t \Lambda_4(x) \right\} \left\{ 1 + i t \Lambda_5(x) \right\} \\
+ O(t^2 \mathbb{E} |\Lambda_1(x)|^2) + O(t^2 \mathbb{E} |\Lambda_2(x)|^2) + O(|t| \mathbb{E} |\Lambda_3(x)|) + O(t^2 \mathbb{E} |\Lambda_4(x)|^2) + O(t^2 \mathbb{E} |\Lambda_5(x)|^2) \\
= \mathbb{E} \left[ e^{\theta S(x)} \right] \left\{ 1 + i t \Lambda_1(x) + i t \Lambda_2(x) + i t \Lambda_3(x) + i t \Lambda_4(x) + i t \Lambda_5(x) \right\} \\
+ O(t^2 \mathbb{E} |\Lambda_1(x)|^2) + O(t^2 \mathbb{E} |\Lambda_2(x)|^2) + O(|t| \mathbb{E} |\Lambda_3(x)|) + O(t^2 \mathbb{E} |\Lambda_4(x)|^2) + O(t^2 \mathbb{E} |\Lambda_5(x)|^2) \\
+ O(t^2 \mathbb{E} |\Lambda_1(x)\Lambda_2(x)|) + O(t^2 \mathbb{E} |\Lambda_1(x)\Lambda_4(x)|) + O(t^2 \mathbb{E} |\Lambda_1(x)\Lambda_5(x)|) \\
+ O(t^2 \mathbb{E} |\Lambda_2(x)\Lambda_3(x)|) + O(t^2 \mathbb{E} |\Lambda_2(x)\Lambda_5(x)|) + O(t^2 \mathbb{E} |\Lambda_4(x)\Lambda_5(x)|) \\
\equiv (I) + (II) + (III) + (IV) + (V) \\
+ O(t^2 \mathbb{E} |\Lambda_1(x)|^2) + O(t^2 \mathbb{E} |\Lambda_2(x)|^2) + O(|t| \mathbb{E} |\Lambda_3(x)|) + O(t^2 \mathbb{E} |\Lambda_4(x)|^2) + O(t^2 \mathbb{E} |\Lambda_5(x)|^2) \\
+ O(t^2 \mathbb{E} |\Lambda_1(x)\Lambda_2(x)|) + O(t^2 \mathbb{E} |\Lambda_1(x)\Lambda_4(x)|) + O(t^2 \mathbb{E} |\Lambda_1(x)\Lambda_5(x)|) \\
+ O(t^2 \mathbb{E} |\Lambda_2(x)\Lambda_3(x)|) + O(t^2 \mathbb{E} |\Lambda_2(x)\Lambda_5(x)|) + O(t^2 \mathbb{E} |\Lambda_4(x)\Lambda_5(x)|) \\
\equiv (I) + (II) + (III) + (IV) + (V) \\
+ O(t^2 \mathbb{E} |\Lambda_1(x)|^2) + O(t^2 \mathbb{E} |\Lambda_2(x)|^2) + O(|t| \mathbb{E} |\Lambda_3(x)|) + O(t^2 \mathbb{E} |\Lambda_4(x)|^2) + O(t^2 \mathbb{E} |\Lambda_5(x)|^2) \\
+ O(t^2 \mathbb{E} |\Lambda_1(x)\Lambda_2(x)|) + O(t^2 \mathbb{E} |\Lambda_1(x)\Lambda_4(x)|) + O(t^2 \mathbb{E} |\Lambda_1(x)\Lambda_5(x)|) \\
+ O(t^2 \mathbb{E} |\Lambda_2(x)\Lambda_3(x)|) + O(t^2 \mathbb{E} |\Lambda_2(x)\Lambda_5(x)|) + O(t^2 \mathbb{E} |\Lambda_4(x)\Lambda_5(x)|) \\
\equiv (I) + (II) + (III) + (IV) + (V) \\
+ O(t^2 \mathbb{E} |\Lambda_1(x)|^2) + O(t^2 \mathbb{E} |\Lambda_2(x)|^2) + O(|t| \mathbb{E} |\Lambda_3(x)|) + O(t^2 \mathbb{E} |\Lambda_4(x)|^2) + O(t^2 \mathbb{E} |\Lambda_5(x)|^2) \\
+ O(t^2 \mathbb{E} |\Lambda_1(x)\Lambda_2(x)|) + O(t^2 \mathbb{E} |\Lambda_1(x)\Lambda_4(x)|) + O(t^2 \mathbb{E} |\Lambda_1(x)\Lambda_5(x)|) \\
+ O(t^2 \mathbb{E} |\Lambda_2(x)\Lambda_3(x)|) + O(t^2 \mathbb{E} |\Lambda_2(x)\Lambda_5(x)|) + O(t^2 \mathbb{E} |\Lambda_4(x)\Lambda_5(x)|) \\
= (I) + (II) + (III) + (IV) + (V) + O(t^2 n^{-1}) + O(|t| n^{-1/2} h_0^{-1/2} + n^{-1}) + O(t^2 n^{-1} h_0) + O(t^2 n^{-1} h_0)
\]

where the fourth equality follows from Lemmas 10, 14, 15, 16, 17, and 18 and the final equality uses \( h_0 = O(n^{-1/(2L+1)}) \).

Define \( \gamma(t) = \mathbb{E} \left[ e^{\frac{\theta}{\sqrt{n}} S_1} \right] \). We have
\[
(I) = \mathbb{E} \left[ e^{\frac{\theta}{\sqrt{n}} S_1} \right] = \mathbb{E} \left[ e^{\frac{\theta}{\sqrt{n}} S_1} \right]^n = \gamma(t)^n, \quad \text{(A.11)}
\]
from Lemma 5,

\begin{align*}
(II) &= \mathbb{E} \left[ e^{itS(x)} (it \Lambda_1(x)) \right] \\
&= \gamma(t)^{-1} \frac{C_p h_0}{n^{1/2} \mu_0^{1/2}} \mathbb{E} \left[ e^{\frac{it}{\sqrt{n} \mu_0}} S_1 \mathcal{L}_1 \right] \left( \sum_{i=0}^{L-1} C_{\Gamma,i}(x) h_0^{i} \right) (it) \\
&= \gamma(t)^{-1} \frac{C_p h_0}{n^{1/2} \mu_0^{1/2}} \mathbb{E} \left[ \left\{ 1 + \frac{it}{(nh_0)^{1/2}} S_1 + \frac{(it)^2}{2(nh_0)^2} S_1^2 \right\} \mathcal{L}_1 \right] \left( \sum_{i=0}^{L-1} C_{\Gamma,i}(x) h_0^{i} \right) (it) + o\{ (nh)^{-1} \} \\
&= \gamma(t)^{-1} \frac{C_p h_0}{n^{1/2} \mu_0^{1/2}} \mathbb{E} \left[ S_1 \mathcal{L}_1 \right] \left( \sum_{i=0}^{L-1} C_{\Gamma,i}(x) h_0^{i} \right) (it)^2 + O(n^{-1/2} h_0^{2L+1/2}) \\
&= \gamma(t)^{-1} \frac{C_p h_0}{n^{1/2} \mu_0^{1/2}} \mathbb{E} \left[ S_1 \mathcal{L}_1 \right] \left( \sum_{i=0}^{L-1} C_{\Gamma,i}(x) h_0^{i} \right) (it)^2 + O(n^{-1/2} h_0^{2L+1/2}). \quad (A.12)
\end{align*}

from Lemma 5 and 6,

\begin{align*}
(III) &= \mathbb{E} \left[ e^{itS(x)} (it \Lambda_2(x)) \right] \\
&= \gamma(t)^{-2} C_p h_0 \frac{n(n-1)}{n^{3/2} \mu_0^{3/2}} \mathbb{E} \left[ e^{\frac{it}{\sqrt{n} \mu_0}} (S_1 + S_2) \mathcal{L}_2 \right] (it) \\
&= \gamma(t)^{-2} \frac{C_p h_0}{n^{3/2} \mu_0^{3/2}} \mathbb{E} \left[ \left\{ 1 + \frac{it}{(nh_0)^{1/2}} (S_1 + S_2) + \frac{(it)^2}{2(nh_0)^2} (S_1 + S_2)^2 + \frac{(it)^3}{6(nh_0)^{3/2}} (S_1 + S_2)^3 \right\} \mathcal{L}_2 \right] (it) + o\{ (nh)^{-1} \} \\
&= \gamma(t)^{-2} \frac{C_p h_0}{n^{3/2} \mu_0^{3/2}} \mathbb{E} \left[ S_1 \Gamma_1 \mathbb{E} [S_2 \mathcal{L}_2] (it)^3 + O(n^{-2} h_0^{2} O(h_0) O(h_0) \right. \\
&= \gamma(t)^{-2} \frac{C_p h_0}{n^{3/2} \mu_0^{3/2}} \mathbb{E} \left[ S_2 \mathcal{L}_2 \right] (it)^3 + O(n^{-2} h_0^{2} O(h_0) O(h_0)), \quad (A.13)
\end{align*}

from Lemma 5,

\begin{align*}
(IV) &= \mathbb{E} \left[ e^{itS(x)} (it \Lambda_4(x)) \right] \\
&= \gamma(t)^{-2} \frac{C_p h_0}{n^{3/2} \mu_0^{3/2}} \mathbb{E} \left[ \left\{ 1 + \frac{it}{(nh_0)^{1/2}} (S_1 + S_2) \right\} \mathcal{L}_2 \right] (it) \\
&= \gamma(t)^{-2} \frac{C_p h_0}{n^{3/2} \mu_0^{3/2}} \mathbb{E} \left[ S_2 \mathcal{L}_2 \right] (it)^3 + O(n^{-1/2} h_0^{-1/2}) O(h_0) O(h_0) \right. \\
&= \gamma(t)^{-2} \frac{C_p h_0}{n^{3/2} \mu_0^{3/2}} \mathbb{E} \left[ S_2 \mathcal{L}_2 \right] (it)^3 + O(n^{-1/2} h_0^{-1/2}) O(h_0) O(h_0)), \quad (A.14)
\end{align*}

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where the final equality uses \( \mathbb{E}[S_1^2] = h_0 \), and from Lemma 5,

\[
(V) = \mathbb{E} \left[ e^{itS(x)(it\Lambda_S(x))} \right]
= -\frac{C_{Pt}}{2n^{1/2}h_0^{1/2}} \gamma(t)r^{-1} \mathbb{E} \left[ e^{\frac{r}{\sqrt{h_0}}S_1} \right] (it)
= -\frac{C_{Pt}}{2n^{1/2}h_0^{1/2}} \gamma(t)r^{-1} \mathbb{E} \left[ \left\{ 1 + \frac{it}{\sqrt{h_0}}S_1 \right\} S_1 \right] (it) + o\{(nh)^{-1}\}
= -\frac{C_{Pt}}{2n^{1/2}h_0^{1/2}} \gamma(t)r^{-1} \mathbb{E}[S_1 \mathcal{L}_1](it) + O(t^2n^{-1})
\]

then

\[
\chi_{pr}(t) = (I) + (II) + (III) + (IV) + (V) + O(t^2n^{-1}) + O(|t|n^{-1})
= \gamma(t)^n + \gamma(t)^r \frac{C_{Pt}}{\mu_{20} \gamma(t)} \mathbb{E}[S_1 \mathcal{L}_1] \left( \sum_{l=0}^{L-1} C_l \gamma(t)^r \right) (it)^2
+ \gamma(t)^r - \frac{C_{Pt}}{\mu_{20} \gamma(t)} \frac{1}{n^{1/2}h_0^{3/2}} \mathbb{E}[S_1 \mathcal{L}_1 \mathcal{L}_2](it)^3
- \left\{ \gamma(t)^r (it)^3 + \gamma(t)^r (it) \right\} \frac{1}{n^{1/2}h_0^{1/2}} \mathbb{E}[S_1 \mathcal{L}_1]
+ O\left( (|t| + r^2 + |t|^3) n^{-1} \right).
\]

For \( m = 0, 1, 2 \), by (Feller, 1971, p535-536),

\[
\gamma(t)^{m-3} = \exp \left( \frac{-t^2}{2} \right) \left\{ 1 + \frac{\mu_{20} \mu_{20}^{3/2}}{6n^{1/2}h_0^{1/2}} (it)^3 + \frac{\mu_{20} \mu_{20}^{3/2}}{24nh_0} (it)^4 + \frac{\mu_{20} \mu_{20}^{3/2}}{72nh_0} (it)^6 \right\} + o\left( (nh_0)^{-1} (r^4 + |t|^6) e^{-t^2/4} \right).
\]

By (A.11), (A.12), (A.13), (A.14) and (A.15), noting \( \mathbb{E}[S_1 \mathcal{L}_1] = h_0 \mu_{20}^{-1/2} \rho_{11}, \mathbb{E}[S_1 \mathcal{L}_1 \mathcal{L}_2] = h_0 \mu_{20}^{-1/2} \xi_{11} \), and \( \mathbb{E}[S_2 \mathcal{L}_2] = h_0 \mu_{20}^{-1/2} \rho_{11} \),

\[
\chi_{pr}(t) = \exp \left( \frac{-t^2}{2} \right) \left\{ 1 + \frac{\mu_{20} \mu_{20}^{3/2}}{6n^{1/2}h_0^{1/2}} (it)^3 + \frac{\mu_{20} \mu_{20}^{3/2}}{24nh_0} (it)^4 + \frac{\mu_{20} \mu_{20}^{3/2}}{72nh_0} (it)^6 \right\}
+ C_{pr} \frac{\mu_{20} \mu_{20}^{3/2}}{6n^{1/2}h_0^{1/2}} \sum_{i=0}^{L-1} C_{r_{11}}(x) \frac{\mu_{20} \mu_{20}^{3/2}}{24nh_0} (it)^3 + C_{pr} \frac{\mu_{20} \mu_{20}^{3/2}}{6n^{1/2}h_0^{1/2}} \frac{\mu_{20} \mu_{20}^{3/2}}{24nh_0} \frac{\mu_{20} \mu_{20}^{3/2}}{72nh_0} (it)^6
+ O\left( (|t| + r^2 + |t|^3) n^{-1} \right) + o\left( (nh_0)^{-1} (r^4 + |t|^6) e^{-t^2/4} \right)
= \chi_{pr}(t) + O\left( (|t| + r^2 + |t|^3 + r^4) n^{-1} \right) + o\left( (nh_0)^{-1} (r^4 + |t|^6) e^{-t^2/4} \right).
\]

This implies

\[
\langle A \rangle = \int_{-p}^{p} \left| \frac{\chi_{pr}(t) - \tilde{\chi}_{pr}(t)}{t} \right| dt = o\{(nh_0)^{-1}\}
\]

Next, we confirm \( B = o\{(nh_0)^{-1}\} \), for \( p \leq |t| \leq n^{1/21} \log n \). Define

\[
S(x; m) = \frac{1}{n^{1/2}h_0^{1/2}} \sum_{i=1}^{m} S_i,
\]

\[
\Lambda_1(x; m) = \frac{C_{pr} \mu_{20}^{3/2}}{n^{1/2}h_0^{1/2}} \sum_{i=1}^{m} \mathcal{L}_1 \left( \sum_{l=0}^{L-1} C_{r_{11}} h_0 \right),
\]

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\[ \Lambda_2(x;m) \equiv \frac{C_{Pl}}{n^{3/2}h_0^{1/2}M_{20}^{1/2}} \sum_{i=1}^{m} \sum_{j \neq i} \Gamma_i \mathcal{L}_j, \]

\[ \Lambda_4(x;m) \equiv -\frac{C_{Pl}}{2n^{3/2}h_0^{1/2}M_{20}^{1/2}} \sum_{i=1}^{m} S_i \mathcal{L}_j \]

\[ \Lambda_5(x;m) \equiv -\frac{C_{Pl}}{2n^{3/2}h_0^{1/2}M_{20}^{1/2}} \sum_{i=1}^{m} S_i \mathcal{L}_j \]

then

\[
|\mathcal{X}_P(t)| = |e^{it|S(x)|} + \Lambda_2(x) + \Lambda_3(x) + \Lambda_4(x) + \Lambda_5(x)|
\]

\[
< |e^{it|S(x)|} + \Lambda_2(x) + \Lambda_3(x) + \Lambda_4(x) + \Lambda_5(x)| + O(|t|/|E\Lambda_3(x)|)
\]

\[
< |e^{it|S(x)|} + (\Lambda_2(x) - \Lambda_3(x)) + (\Lambda_2(x) - \Lambda_2(x)) + (\Lambda_2(x) - \Lambda_4(x)) + (\Lambda_2(x) - \Lambda_5(x)) + O(t^2/|E\Lambda_3(x)|)
\]

\[
\times \left\{ 1 + itA_1(x;m) \right\} \left\{ 1 + itA_2(x;m) \right\} \left\{ 1 + itA_4(x;m) \right\} \left\{ 1 + itA_5(x;m) \right\}
\]

\[
+ O(t^2 \{ E\Lambda_1(x;m) + E\Lambda_2(x;m) + E\Lambda_4(x;m) + E\Lambda_5(x;m) \}) + O(|t|/|E\Lambda_3(x)|)
\]

\[
< \left| e^{it|S(x)|} + (\Lambda_2(x) - \Lambda_3(x)) + (\Lambda_2(x) - \Lambda_2(x)) + (\Lambda_2(x) - \Lambda_4(x)) + (\Lambda_2(x) - \Lambda_5(x)) + O(t^2/|E\Lambda_3(x)|) \right|
\]

\[
\times \left\{ 1 + itA_1(x;m) + \Lambda_2(x;m) + \Lambda_4(x;m) + \Lambda_5(x;m) \right\}
\]

\[
+ O(t^2 \{ E\Lambda_1(x;m) + E\Lambda_2(x;m) + E\Lambda_4(x;m) + E\Lambda_5(x;m) \}) + O(|t|/|E\Lambda_3(x)|)
\]

\[ (A.16) \]

The first term of (A.16) is bounded as below:

\[
\left| e^{it|S(x)|} - e^{it|S(x) - S(x)|} + (\Lambda_2(x) - \Lambda_3(x)) + (\Lambda_2(x) - \Lambda_2(x)) + (\Lambda_2(x) - \Lambda_4(x)) + (\Lambda_2(x) - \Lambda_5(x)) \right|
\]

\[ (A.17) \]

Similarly, the second term of (A.16) divided by |t| is bounded by

\[ |E \{ e^{it|S(x)|} \Lambda_2(x;m) \} | + |E \{ e^{it|S(x)|} \Lambda_4(x;m) \} | + |E \{ e^{it|S(x)|} \Lambda_5(x;m) \} |, \]

where each term is bounded as follows. Let C(x) be some positive and bounded generic function.

\[ |E \{ e^{it|S(x)|} \Lambda_2(x;m) \} | = \gamma(t)^{-m-1} \frac{C_{pl}C_{\Gamma,0}(x)}{n^{1/2}h_0^{1/2}M_{20}^{1/2}} \frac{m^2}{e^{\delta_{m,m}'|S_1(x)|}} \mathcal{L}_1' + s.o. \]

\[ \leq |\gamma(t)|^{-m-1} \frac{m_{h_0}^{2/1}}{n^{1/2}h_0^{1/2}M_{20}^{1/2}} \frac{2^{2/1}}{C_{pl}C_{\Gamma,0}(x)} \left| E \left[ \frac{d_{\gamma_{mm}'}^1}{|S_1(x)|} \mathcal{L}_1' \right] \right| + s.o. \]

\[ \leq |\gamma(t)|^{-m-1} \frac{m_{h_0}^{2/1}}{n^{1/2}h_0^{1/2}M_{20}^{1/2}} \left| C_{pl}C_{\Gamma,0}(x) \right| \mathcal{L}_1'' + s.o. \]

\[ \leq |\gamma(t)|^{-m-1} \frac{m_{h_0}^{2/1}}{n^{1/2}h_0^{1/2}M_{20}^{1/2}} \left| C_{pl}C_{\Gamma,0}(x) \right| \left| E \left[ \frac{d_{\gamma_{mm}''}}{|S_1(x)|} \mathcal{L}_1' \right] \right| + s.o. \]

\[ \leq |\gamma(t)|^{-m-1} \frac{m_{h_0}^{2/1}}{n^{1/2}h_0^{1/2}M_{20}^{1/2}} \left[ C_{pl}C_{\Gamma,0}(x) \right] \left| E \left| \mathcal{L}_1' \right| \right| + s.o. \]
where the final inequality uses Lemma 7.

\[
\left| \mathbb{E} \left\{ e^{\mu S(x,m)} \Lambda_2(x;m) \right\} \right| = \left| \gamma(t)^{m-2} \frac{C_{pl}}{n^{3/2}h_0^{1/2} \mu_2^{1/2}} m(m-1) \mathbb{E} \left[ \frac{n}{e^{\sqrt{\Delta_0}}(S_1 + S_2)} \Gamma_1 \mathcal{L}_2 \right] \right| \\
\leq \left| \gamma(t)^{m-2} \frac{m(m-1)}{n^{3/2}h_0^{1/2} \mu_2^{1/2}} C_{pl} \mathbb{E} \left[ \frac{n}{e^{\sqrt{\Delta_0}}(S_1 + S_2)} \right] \right| \left| \mathbb{E} [\Gamma_1 \mathcal{L}_2] \right| \\
\leq \left| \gamma(t)^{m-2} \frac{m(m-1)}{n^{3/2}h_0^{1/2} \mu_2^{1/2}} C_{pl} \mathbb{E} \left[ \Gamma_1 \mathcal{L}_2 \right] \right| \\
\leq C(x) \left| \gamma(t)^{m-2} \frac{m(m-1)}{n^{3/2}} \right| h_0^{1/2},
\]

where the final inequality uses Lemma 7.

\[
\left| \mathbb{E} \left\{ e^{\mu S(x,m)} \Lambda_4(x;m) \right\} \right| = \left| \gamma(t)^{m-2} \frac{C_{pl} m(m-1)}{2n^{3/2}h_0^{1/2}} \gamma(t)^{m-2} \mathbb{E} \left[ \frac{n}{e^{\sqrt{\Delta_0}}(S_1 + S_2)} S_1 \mathcal{L}_2 \right] \right| \\
\leq \left| \gamma(t)^{m-2} \frac{m(m-1)}{2n^{3/2}h_0^{1/2}} C_{pl} \mathbb{E} \left[ \frac{n}{e^{\sqrt{\Delta_0}}(S_1 + S_2)} S_1 \mathcal{L}_2 \right] \right| \\
\leq \left| \gamma(t)^{m-2} \frac{m(m-1)}{2n^{3/2}h_0^{1/2}} C_{pl} \mathbb{E} \left[ \Gamma_1 \mathcal{L}_2 \right] \right| \\
\leq C(x) \left| \gamma(t)^{m-2} \frac{m(m-1)}{2n^{3/2}} \right| h_0^{1/2},
\]

where the final inequality uses Lemma 5 and 7.

\[
\left| \mathbb{E} \left\{ e^{\mu S(x,m)} \Lambda_5(x;m) \right\} \right| = \left| \gamma(t)^{m-1} \frac{C_{pl} m}{2n^{3/2}h_0^{1/2}} \gamma(t)^{m-1} \mathbb{E} \left[ \frac{n}{e^{\sqrt{\Delta_0}} S_1} S_1 \mathcal{L}_2 \right] \right| \\
\leq \left| \gamma(t)^{m-1} \frac{C_{pl} m}{2n^{3/2}h_0^{1/2}} C_{pl} \mathbb{E} \left[ \frac{n}{e^{\sqrt{\Delta_0}} S_1} S_1 \mathcal{L}_1 \right] \right| \\
\leq \left| \gamma(t)^{m-1} \frac{C_{pl} m}{2n^{3/2}h_0^{1/2}} C_{pl} \mathbb{E} \left[ \Gamma_1 \mathcal{L}_1 \right] \right| \\
\leq C(x) \left| \gamma(t)^{m-1} \frac{m}{2n^{3/2}} \right| h_0^{1/2}
\]

where the final inequality uses Lemma 5. (A.18), (A.19), (A.20) and (A.21) imply

\[
\left| t \right| \mathbb{E} e^{\mu (S(t) + (\Lambda_1(x) - \Lambda_1(x,m)) + (\Lambda_2(x) - \Lambda_2(x,m)) + (\Lambda_4(x) - \Lambda_4(x,m)) + (\Lambda_5(x) - \Lambda_5(x,m)))} \\
\times \left\{ \Lambda_1(x;m) + \Lambda_2(x;m) + \Lambda_4(x;m) + \Lambda_5(x;m) \right\} \\
\leq C(x) \left\{ \left| \gamma(t)^{m-1} \frac{m}{n^{1/2}} \right| \frac{m}{n^{1/2}} \left| \gamma(t)^{m-2} \frac{2^{m+1}}{n^{1/2}} \right| \frac{m}{n^{1/2}} \left| \gamma(t)^{m-1} \frac{m}{n^{1/2}} \right| \right\} \left| t \right|.
\]

(A.22)
Then, (A.16), (A.17), and (A.22) yield

\[ |\chi p(t)| \leq |\gamma(t)|^m + C(x) \left\{ |\gamma(t)|^{m-1} \frac{m^{2+1}}{n^{1/2}} + |\gamma(t)|^{m-2} \frac{m^2 h_0^{1/2}}{n^{3/2}} + |\gamma(t)|^{m-1} \frac{m h_0^{1/2}}{n^{3/2}} \right\} |t| + O(t^2 \{ E \Lambda_1(x;m)^2 + E \Lambda_2(x;m)^2 + E \Lambda_4(x;m)^2 + E \Lambda_5(x;m)^2 + E \Lambda_1(x;m) \Lambda_2(x;m) | + E \Lambda_1(x;m) \Lambda_4(x;m) | + E \Lambda_1(x;m) \Lambda_5(x;m) | + E \Lambda_2(x;m) \Lambda_4(x;m) | + E \Lambda_2(x;m) \Lambda_5(x;m) | + E \Lambda_4(x;m) \Lambda_5(x;m) | \}) + O(|t||E \Lambda_3(x;m)|) \leq C(x)|\gamma(t)|^{m-2} \left[ 1 + \left( \frac{m^{2+1}}{n^{1/2}} + \frac{m^2 h_0^{1/2}}{n^{3/2}} + \frac{m h_0^{1/2}}{n^{3/2}} \right) |t| \right] + O \left( t^2 \left\{ \frac{m^{2L+1} h_0^{1/2}}{n} + \frac{m^2}{n^3} + \frac{m^{3/2} h_0^{(2L+1)/2}}{n^2} \right\} \right) + O \left( |t| \left\{ n^{-1/2} h_0^{(2L+1)/2} + n^{-1} \right\} \right) \]

where second inequality uses $|\gamma(t)| \leq 1$ and Lemma 11,19,20,21,22 and 23.

We evaluate (B), partitioning its range of integration into two parts, $p \leq |t| \leq \frac{n^{1/2} h_0^{3/2}}{\mu_{20} \mu_{30}}$ and $\frac{n^{1/2} h_0^{3/2}}{\mu_{20} \mu_{30}} \leq |t| \leq \frac{n^{1/2} h_0^{3/2}}{\mu_{20} \mu_{30}} \log n$.

(i) For $p \leq |t| \leq \frac{n^{1/2} h_0^{3/2}}{\mu_{20} \mu_{30}}$,

Applying Taylor expansion to $e^{\frac{t}{\mu_0} S_1(x)}$ with respect to $t$, we have

\[ |\gamma(t) - 1 - \frac{t^2}{2n} | \leq \frac{|t|^3 \mu_{20}^{-3/2} \mu_{30}}{6n^3 h_0^{3/2}}, \]

then for $|t| \leq \frac{n^{1/2} h_0^{3/2}}{\mu_{20} \mu_{30}}$,

\[ |\gamma(t)| \leq 1 - \frac{t^2}{2n} + \frac{|t|^3 \mu_{20}^{-3/2} \mu_{30}}{6n^3 h_0^{3/2}} \leq 1 - \frac{t^2}{2n} + \frac{t^2}{6n} = 1 - \frac{t^2}{3n} \leq \exp \left( -\frac{t^2}{3n} \right), \]

then

\[ |\chi p(t)| \leq C(x)|\gamma(t)|^{m-2} \left[ 1 + \left( \frac{m^{2+1}}{n^{1/2}} + \frac{m^2 h_0^{1/2}}{n^{3/2}} + \frac{m h_0^{1/2}}{n^{3/2}} \right) |t| \right] + O \left( t^2 \left\{ \frac{m^{2L+1} h_0^{1/2}}{n} + \frac{m^2}{n^3} + \frac{m^{3/2} h_0^{(2L+1)/2}}{n^2} \right\} \right) + O \left( |t| \left\{ n^{-1/2} h_0^{(2L+1)/2} + n^{-1} \right\} \right) \leq C(x) \exp \left( -\frac{(m-2)t^2}{3n} \right) \left[ 1 + \left( \frac{m^{2+1}}{n^{1/2}} + \frac{m^2 h_0^{1/2}}{n^{3/2}} + \frac{m h_0^{1/2}}{n^{3/2}} \right) |t| \right] + O \left( t^2 \left\{ \frac{m^2}{n^3} + \frac{m^{3/2}}{n^5} \right\} \right) + O \left( |t|n^{-1} \right) \].

Using (A.21) in Nishiyama and Robinson (2000), we can take $m = \lfloor 9n \log n/t^2 \rfloor$ since $1 \leq m \leq n - 1$ holds for $p \leq |t| \leq \frac{n^{1/2} h_0^{3/2}}{\mu_{20} \mu_{30}}$ and sufficiently large $n$.

Because $m \geq (9n \log n)/t^2 - 1$, for $|t| \leq \frac{n^{1/2} h_0^{3/2}}{\mu_{20} \mu_{30}}$

\[ \exp \left( -\frac{(m-2)t^2}{3n} \right) = \exp \left( -\frac{(m+1)t^2}{3n} \right) \exp \left( \frac{3t^2}{3n} \right) \leq C \exp(-3 \log n) \leq C \frac{1}{n^3}, \]

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and this implies, using $m \leq (9n \log n)/\gamma^2$,

$$|\chi_{Pf}(t)| \leq \frac{C(x)}{n^3} \left[ 1 + n^{1/2}(\log n)h_0 \frac{2^{x+1}}{\gamma^2} \frac{1}{|t|} + n^{1/2}(\log n)^2 h_0 \frac{1}{|t|^3} + n^{-1/2}(\log n)h_0 \frac{1}{|t|} \right]$$

$$+ O \left( n^{-1}(\log n) + n^{-1}(\log n)^2 \frac{1}{t^2} + n^{-1}(\log n)^3 \frac{1}{|t|} \right) + O(|t|n^{-1})$$

Therefore, dropping the integral range $p \leq |t| \leq \frac{n^{1/2}h_0}{\mu_0^{1/2}/\rho_0}$ on the right-hand side,

$$\int_{p \leq |t| \leq \frac{n^{1/2}h_0}{\mu_0^{1/2}/\rho_0}} \left| \frac{\chi_{Pf}(t)}{t} \right| dt$$

$$\leq C(x) \left[ \left\{ n^{-3} + n^{-1}(\log n) \right\} \int \frac{dt}{|t|} + \left\{ n^{-5/2}(\log n)h_0^{1/2} + n^{-7/2}(\log n)h_0^{1/2} \right. \right.$$  

$$+ n^{-1}(\log n)^{3/2} \left\{ \frac{dt}{t} + n^{-1}(\log n)^2 \int \frac{dt}{|t|^3} + n^{-5/2}(\log n)^2 h_0^{1/2} \int \frac{dt}{|t|^4} \right\} + O(n^{-1})$$

$$= o\left((nh_0)^{-1}\right)$$

(ii) For $\frac{n^{1/2}h_0}{\mu_0^{1/2}/\rho_0} \leq |t| \leq n^2 \log n$, there exist $\eta \in (0, 1)$, such that $|\gamma(t)| \leq 1 - \eta$ from Assumption 10. We can take $m = \lfloor -3 \log n/\log(1 - \eta) \rfloor$ since $1 \leq m \leq n - 1$ for sufficiently large $n$. Then $\chi_{Pf}(t)$ is bounded as follow.

$$|\chi_{Pf}(t)|$$

$$\leq C(1 - \eta)^{-3\log n/\log(1 - \eta)}$$

$$\times \left[ 1 + \left( \frac{h_0^{1/2}}{n^{1/2}} + \frac{h_0^{1/2}}{n^{3/2}} \right) \frac{|t|}{\log(1 - \eta)} + \frac{h_0^{1/2}}{n^{3/2}} \frac{|t|}{\log(1 - \eta)} \right]$$

$$+ O \left( t^2 \left\{ n^{-2} \left( \frac{-3 \log n}{\log(1 - \eta)} \right) + n^{-3} \left( \frac{-3 \log n}{\log(1 - \eta)} \right)^2 + n^{-5/2} \left( \frac{-3 \log n}{\log(1 - \eta)} \right)^{3/2} \right\} \right)$$

Noting that

$$(1 - \eta)^{-3 \log n/\log(1 - \eta)} = (1 - \eta)^{n^{-3} \log(1 - \eta)} = (1 - \eta)^{\log(1 - \eta)e^{-3}} = n^{-3},$$

$$\int_{\frac{n^{1/2}h_0}{\mu_0^{1/2}/\rho_0} \leq |t| \leq n^2 \log n} \left| \frac{\chi_{Pf}(t)}{t} \right| dt$$

$$= O \left( \frac{\log(n)^2}{n^3} \frac{\log(n)^2 h_0^{1/2}}{n^3/2} \frac{\log(n)^2 h_0^{1/2}}{n^{3/2}} \left( n^2 \log n \right) \right)$$

$$+ O \left( n^2 \log n \left\{ n^{-2} \log(n) + n^{-2} \log(n)^2 + n^{-3/2} \log(n)^{3/2} \right\} \right)$$

$$= o\left((nh_0)^{-1}\right)$$

Finally, we evaluate (C). For some constant $C$,

$$(C) = \int_{p \leq |t| \leq \frac{m^{1/2}h_0}{\mu_0^{1/2}/\rho_0}} \left[ 1 + \frac{\mu_0^{1/2}h_0^{1/2}}{6n^{1/2}h_0^{1/2}} (it)^3 + \frac{\mu_0^{1/2}h_0^{1/2}}{24nh_0} (it)^4 + \frac{\mu_0^{1/2}h_0^{1/2}}{72nh_0} (it)^6 \right]$$

$$+ C_P \rho_1 \frac{\mu_0}{n^{1/2}} \left( \sum_{l=0}^{L-1} C_{\Gamma, l} (x) h_0^{l+1} \right) (it)^2 + C_P \rho_1 \frac{\mu_0^{1/2}h_0^{1/2}}{n^{1/2}} (it)^3 - C_P \rho_1 \frac{\mu_0^{1/2}h_0^{1/2}}{2n^{1/2}} (\log(n)^3 + (it)) dt$$

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Proof. Since\[ p(e^n + 1) = \varepsilon_n + 1. \] It follows that\[ E_\infty \equiv \int_0^\infty \frac{1}{e^{\varepsilon p} t^2} dt \]
and\[ \int_0^\infty \frac{1}{e^{\varepsilon p} t} dt. \]
Since \( p = \min\left\{ \frac{n^{1/2} \log n}{\mu_2}, \log n \right\} \), the first integral is smaller than \( p^{-2} \int_0^\infty \frac{1}{e^{\varepsilon p} t^2} dt = p^{-2} e^{-\varepsilon p^2/2} = o(n^{-1}) \), the second and fifth integrals are smaller than \( p^{-1} \int_0^\infty t e^{-\varepsilon p^2/2} dt = p^{-1} e^{-\varepsilon p^2/2} (p^2 + 2) = o(n^{-1}) \), the third integral is \( \int_0^\infty (t^3 + t^5) e^{-\varepsilon p^2/2} dt = e^{-\varepsilon p^2/2} (p^4 + 5p^2 + 10) = o(n^{-1}) \), the fourth integral is \( \int_0^\infty \frac{e^{-\varepsilon p^2/2}}{t} dt = e^{-\varepsilon p^2/2} = o(n^{-1}) \), and the final integral is \( p^{-1} e^{-\varepsilon p^2/2} (p^2 + 3) = o(n^{-1}) \). It follows that \( (C) = o\left( (nh_0)^{-1} \right) \). Thus the expansion is valid. \( \square \)

A.4 Proof of Theorem 3.4

Proof. Define \( \varepsilon \) and \( \varepsilon_{pl} \) as follows,

\[
\varepsilon \equiv \mathbb{P}(S(x) \leq z) - \Phi(z) - \phi(z) \left[ (nh_0)^{-1/2} p_1(z) + (nh_0)^{-1} p_2(z) \right],
\]

\[
\varepsilon_{pl} \equiv \mathbb{P}(S_{pl}(x) \leq z) - \phi(z)
\]

\[
\phi(z) \left[ (nh_0)^{-1/2} p_1(z) + h_0^{L+1} p_{3,0}(z) + n^{-1/2} h_0^{1/2} p_4(z) + (nh_0)^{-1} p_2(z) \right].
\]

Then we have,

\[
\sup_{z \in \mathbb{R}} \left| \mathbb{P}(S(x) \leq z) - \mathbb{P}(S_{pl}(x) \leq z) - \phi(z) \left[ h_0^{L+1} p_{3,0}(z) + n^{-1/2} h_0^{1/2} p_4(z) \right] \right| = \sup_{z \in \mathbb{R}} |\varepsilon - \varepsilon_{pl}| = o(h_0^{L+1} + n^{-1/2} h_0^{1/2}).
\]

\( \square \)

B Lemmas

Lemma 1. Under Assumptions 1, 4, 5, 7, 11 and 12

\[
\mathbb{E} \Gamma_{KDE_1} = \sum_{l=0}^{L-1} C_{\Gamma,l}(x) h_0^{L+l} + o(h_0^{2L-1}), \quad \text{where} \quad C_{\Gamma,l}(x) \equiv - \left( \int u^{L+l} K(u) du \right) \frac{f^{(L+l)}(x)}{(L+l-1)!}
\]

Proof.

\[
\begin{align*}
\mathbb{E} \Gamma_{KDE_1} &= \mathbb{E} \left[ \frac{1}{n h_0} \sum_{i=1}^{n} K' \left( \frac{X_i - x}{h_0} \right) \frac{1}{h_0} \sum_{i=1}^{n} K \left( \frac{X_i - x}{h_0} \right) \right] + \mathbb{E} \left[ \frac{1}{n h_0} \sum_{i=1}^{n} K \left( \frac{X_i - x}{h_0} \right) \right] \\
&= \frac{1}{h_0} \int K' \left( \frac{z - x}{h_0} \right) \left( \frac{z - x}{h_0} \right) f(z) dz + \frac{1}{h_0} \int K \left( \frac{z - x}{h_0} \right) f(z) dz \\
&= \int K'(u) u f(x + uh_0) du + \int K(u) f(x + uh_0) du \\
&= - \int K(u) f(x + uh_0) du - \int K(u) u f'(x + uh_0) h_0 du + \int K(u) f(x + uh_0) du \\
&= - \int K(u) u f'(x + uh_0) h_0 du \\
&= - \int K(u) \left\{ f^{(1)}(x) + \ldots + \frac{f^{(L)}(x)}{(L-1)!} (uh_0)^{L-1} + \ldots + \frac{f^{(2L)}(x)}{(2L-1)!} (uh_0)^{2L-1} \right\} h_0 du + o(h_0^{2L-1}) \\
&= - \sum_{l=0}^{L-1} \left( \int u^{L+l} K(u) du \right) \frac{f^{(L+l)}(x)}{(L+l-1)!} h_0^{L+l} + o(h_0^{2L-1})
\end{align*}
\]
\[
\sum_{l=0}^{L-1} C_{\Gamma_l} h_0^{L+l} + o(h_0^{2L-1})
\]

The fourth equality follows from integration by part of the first term and Assumption 2,12, the seventh equality follows from the expansion of \(f'(x + uh_0)\) around \(h_0 = 0\) and Assumption 2 and the eighth equality follows from Assumption 11.

**Lemma 2.** Under Assumptions 1, 2, 4, 5, 11, and 14,

\[
\mathbb{E} \Gamma_{KDE_2} = O(h_0^L)
\]

**Proof.**

\[
\begin{align*}
\mathbb{E} \Gamma_{KDE_2} &= \mathbb{E} \left[ \frac{1}{nh_0} \sum_{i=1}^{n} K'' \left( \frac{X_i - x}{h_0} \right) \left( \frac{X_i - x}{h_0} \right)^2 + \frac{4}{nh_0} \sum_{i=1}^{n} K' \left( \frac{X_i - x}{h_0} \right) \left( \frac{X_i - x}{h_0} \right) + \frac{2}{nh_0} \sum_{i=1}^{n} K \left( \frac{X_i - x}{h_0} \right) \right] \\
&= \mathbb{E} \left[ \frac{1}{h_0} \int K'' \left( \frac{z - x}{h_0} \right) \left( \frac{z - x}{h_0} \right)^2 f(z)dz + \mathbb{E} \left[ \frac{4}{h_0} \int K' \left( \frac{z - x}{h_0} \right) \left( \frac{z - x}{h_0} \right) f(z)dz \right] + \frac{2}{h_0} \int K \left( \frac{z - x}{h_0} \right) f(z)dz \right] \\
&= 2 \int K(u)f(x + uh_0)du \\
& \quad + 4 \left\{ - \int K(u)f(x + uh_0)du - \int K(u)uf'(x + uh_0)h_0du \right\} \\
& \quad + \left\{ -2 \int K'(u)uf(x + uh_0) - \int K'(u)u^2f'(x + uh_0)h_0du \right\} \\
&= 2 \int K(u)f(x + uh_0)du \\
& \quad + 4 \left\{ - \int K(u)f(x + uh_0)du - \int K(u)uf'(x + uh_0)h_0du \right\} \\
& \quad + \left\{ -2 \left[ - \int K(u)f(x + uh_0)du - \int K(u)uf'(x + uh_0)h_0du \right] \right\} \\
& \quad + \left\{ -2 \int K(u)f'(x + uh_0)h_0du - \int K(u)u^2f''(x + uh_0)h_0^2du \right\} \\
&= \int K(u)u^2f''(x + uh_0)h_0^2du = O(h_0^L)
\end{align*}
\]

The third equality follows from integration by parts of the first and second terms and Assumption 14, the fourth equality follows from integration by parts of the second term and Assumption 14 and the final equality follows from the expansion of \(f''(x + uh_0)\) around \(h_0 = 0\) and Assumptions 2, 11.

**Lemma 3.** Under Assumptions 1, 2, 4, 5, 11, 13 and 14,

\[
\mathbb{E} \left| \sqrt{nh_0} \left( \frac{\hat{h} - h_0}{h_0} \right) \Gamma_{KDE_2} \right| = o((nh_0)^{-1})
\]

**Proof.** Similar to the proof of Theorem 1.

**Lemma 4.** Under Assumptions 1, 2, 4, 7, 11 and 14,

\[
\mathbb{E} \left[ \Gamma_1 \mathcal{L}_1 \right] = O(h_0^{L+1})
\]

**Proof.** Letting \(g(x) = f^{(L)}(x)f(x)\)

\[
\mathbb{E} \left[ \Gamma_1 \mathcal{L}_1 \right] = \mathbb{E} \left[ \left\{ K' \left( \frac{X_1 - x}{h_0} \right) \left( \frac{X_1 - x}{h_0} \right) \right\} f^{(L)}(X_1) \right] \]

\[
= \mathbb{E} \left[ \left\{ K' \left( \frac{X_1 - x}{h_0} \right) \left( \frac{X_1 - x}{h_0} \right) \right\} f^{(L)}(X_1) \right] - \mathbb{E} \left[ \left\{ K' \left( \frac{X_1 - x}{h_0} \right) \left( \frac{X_1 - x}{h_0} \right) \right\} f^{(L)}(X_1) \right] \]
We can compute the first term as follow.

$$\mathbb{E}\left[ \left\{ K' \left( \frac{X_1-x}{h_0} \right) \left( \frac{X_1-x}{h_0} \right) + K \left( \frac{X_1-x}{h_0} \right) \right\} f^{(L)}(X_1) \right]$$

$$= \int \left\{ K' \left( \frac{z-x}{h_0} \right) \left( \frac{z-x}{h_0} \right) + K \left( \frac{z-x}{h_0} \right) \right\} f^{(L)}(z) f(z) \, dz$$

$$= h_0 \int \left\{ K'(u) u + K(u) \right\} g(x + u h_0) \, du$$

$$= h_0 \int \left\{ K'(u) u + K(u) \right\} \left\{ g(x) + \cdots + \frac{g^{(L)}(x)}{L!} (u h_0)^L + o(h_0^L) \right\} \, du$$

$$= h_0 \int \left\{ K'(u) u + K(u) \right\} g(x) \, du + h_0 \int \left\{ \frac{K'(u) u}{L!} \right\} (u h_0)^L \, du + o(h_0^L)$$

The fourth equality follows from the expansion of $l(x + u h_0)$ around $h_0 = 0$ and Assumption 7, and the fifth equality follows from integration by parts of the products of $K'(u) u$ and $l^{(k)}(x) u^k$, $0 \leq k \leq L - 1$ and Assumption 7 and 14. Next, we can compute the second term similarly to the first term.

$$\mathbb{E}\left[ \left\{ K' \left( \frac{X_1-x}{h_0} \right) \left( \frac{X_1-x}{h_0} \right) + K \left( \frac{X_1-x}{h_0} \right) \right\} \right] \mathbb{E}[f^{(L)}(X_1)]$$

$$= \left( -(L+1) \frac{g^{(L)}(x)}{L!} h_0^{L+1} \right) \int K(u) u^L \, du + \frac{f^{(L)}(x)}{L!} h_0^{L+1} \int K(u) u^L \, du \right) \mathbb{E}[f^{(L)}(X_1)]$$

$$= O(h_0^{L+1})$$

These imply the lemma holds.

**Lemma 5.** For any positive integer $k$ and any non-negative integer $l$,

$$\mathbb{E}[S^k_i \mathcal{L}^l] = O(h_0)$$

**Proof.** Straightforward.

**Lemma 6.** For any positive integer $k, l$,

$$\mathbb{E}[S^k_i \Gamma^l_1] = O(h_0)$$

**Proof.** Straightforward.

**Lemma 7.** For any positive integer $k, l \geq 2$,

$$\mathbb{E}[\Gamma^l_1]^k = O(h_0^k), \quad \mathbb{E}[\mathcal{L}^l_1]^k = O(1)$$

**Proof.** Straightforward.

**Lemma 8.** For any positive integer $r$,

$$\mathbb{E}[\Lambda_1(x)]^r = O(h_0^{\frac{(2L+1)}{2}})$$

**Proof.** From Lemma 7, for any positive integer $k$, and some positive bounded function $C(x)$,

$$\mathbb{E}[\Lambda(x)]^{2k} = \mathbb{E}[\Lambda(x)]^{2k} \leq \frac{h_0^{(2L+1)}}{n^k} \mathbb{E} \left[ \sum_{j=1}^n \mathcal{L}_j^2 \right]^{2k} + s.o.$$

$$= \frac{h_0^{(2L+1)}}{n^k} \mathbb{E} \left[ \mathcal{L}_j^2 \right]^{2k} + s.o. = O(h_0^{(2L+1)})$$

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From Holder’s inequality, for $0 < r < s$, $E|X|^r \leq \{E|X|^s\}^{r/s}$, thus for any positive integer $k$,
\[ E|\Lambda_1(x)|^{2k-1} \leq \{E|\Lambda_1(x)|^{2k}\}^{\frac{2k-1}{2k}} = O(h_0^{-\frac{(2k-1)(2k+1)}{2}}) \]
This implies the lemma holds.

**Lemma 9.** For any positive integer $r$,
\[ E|\Lambda_2(x)|^r = O(n^{-r/2}) \]

**Proof.** From Lemma 7, for any positive integer $k$,
\[ E|\Lambda_2(x)|^{2k} \leq \frac{1}{n^k h_0^{2k} \mu_2} E\left[ \left( \sum_{i=1}^{n} \sum_{j \neq i} \Gamma_i \mathcal{L}_j \right)^{2k} \right] \]
\[ = \frac{n^k(n-1)^k}{n^k h_0^{2k} \mu_2} E\left[ \Gamma_1^2 \right]^k E\left[ \mathcal{L}_2^2 \right]^k + s.o. = O(n^{-k}) \]
Then, similarly to the evaluation of $E|\Lambda_1(x)|^r$, the lemma holds.

**Lemma 10.**
\[ E|\Lambda_1(x)\Lambda_2(x)| = O(n^{-1/2}h_0^{2k+1}) \]

**Proof.** Lemma 8, 9 and Holder inequality implies
\[ E|\Lambda_1(x)\Lambda_2(x)| \leq E|\Lambda_1(x)||E|\Lambda_2(x)| = O(h_0^{-\frac{2k+1}{2}})O(n^{-1/2}) \]

**Lemma 11.**
\[ E|\Lambda_3(x)| = O(n^{-1/2}h_0^{3k+1} + n^{-1}) \]

**Proof.** From Lemma 4,
\[ E\Lambda_3(x)^2 \leq \frac{1}{n^k h_0^{2k} \mu_2} E\left[ \sum_{l=1}^{n} \sum_{j \neq l} \Gamma_l \mathcal{L}_j \Gamma_j \mathcal{L}_l \right] \]
\[ = \frac{1}{n^k h_0^{2k} \mu_2} E\left[ \sum_{l=1}^{n} \sum_{j \neq l} \Gamma_l \mathcal{L}_j \Gamma_j \mathcal{L}_l + \sum_{l=1}^{n} \Gamma_l^2 \mathcal{L}_l^2 \right] \]
\[ = \frac{n(n-1)}{n^k h_0^{2k} \mu_2} E[\Gamma_1^2 \mathcal{L}_1^2] + \frac{1}{n^k h_0^{2k} \mu_2} E[\Gamma_l^2 \mathcal{L}_l^2] \]
\[ = O(n^{-1}h_0^{-1})O(h_0^{2k+1}) + O(n^{-2}h_0^{-1})O(h_0) \]
Similarly to the evaluation of $E|\Lambda_1(x)|^r$, the lemma holds.

**Lemma 12.**
\[ E|\Lambda_4(x)|^2 = O(n^{-1}h_0) \]

**Proof.** The proof is similar to Lemma 9.

**Lemma 13.**
\[ E|\Lambda_5(x)|^2 = O(n^{-1}h_0) \]

**Proof.** The proof is similar to Lemma 11.

**Lemma 14.**
\[ E|\Lambda_1(x)\Lambda_4(x)| = O(n^{-1/2}h_0^{L+1}) \]
Proof. From Cauchy-Schwarz inequality and Lemma 8 and 12, this lemma holds.

Lemma 15.

\[ \mathbb{E}[\Lambda_1(x) \Lambda_5(x)] = O(n^{-1/2} h_0^{L+1}) \]

Proof. From Cauchy-Schwarz inequality and Lemma 8 and 13, this lemma holds.

Lemma 16.

\[ \mathbb{E}[\Lambda_2(x) \Lambda_4(x)] = O(n^{-1} h_0^{1/2}) \]

Proof. From Cauchy-Schwarz inequality and Lemma 9 and 12, this lemma holds.

Lemma 17.

\[ \mathbb{E}[\Lambda_2(x) \Lambda_5(x)] = O(n^{-1} h_0^{1/2}) \]

Proof. From Cauchy-Schwarz inequality and Lemma 9 and 13, this lemma holds.

Lemma 18.

\[ \mathbb{E}[\Lambda_4(x) \Lambda_5(x)] = O(n^{-1} h_0) \]

Proof. From Cauchy-Schwarz inequality and Lemma 12 and 13, this lemma holds.

Lemma 19.

\[ \mathbb{E}[\Lambda_1(x; m)]^2 = O\left( \frac{m h_0^{2L+1}}{n} \right) \]

Proof. From Lemma 7,

\[ \mathbb{E}[\Lambda_1(x; m)]^2 \lesssim \frac{h_0^{2L+1}}{n} \mathbb{E} \left[ \sum_{i=1}^{m} \sum_{j=1}^{m} \mathcal{L}_i \mathcal{L}_j \right] = \frac{h_0^{2L+1}}{n} \mathbb{E} \left[ \sum_{i=1}^{m} \sum_{j \neq i} \mathcal{L}_i \mathcal{L}_j + \sum_{i=1}^{m} \mathcal{L}_i^2 \right] = O\left( \frac{m h_0^{2L+1}}{n} \right) \]

Lemma 20.

\[ \mathbb{E}[\Lambda_2(x; m)]^2 = O\left( \frac{m^2}{n^3} \right) \]

Proof.

\[ \mathbb{E}[\Lambda_2(x; m)]^2 \lesssim \frac{1}{n^3 h_0} \mathbb{E} \left[ \sum_{i=1}^{m} \sum_{j \neq k \neq l} \sum_{i \neq k \neq l} \Gamma_i \Gamma_j \Gamma_k \Gamma_l \right] = \frac{m(m-1)}{n^3 h_0} \mathbb{E}[\Gamma_1^2] \mathbb{E}[\mathcal{L}_2^2] = O\left( \frac{m^2}{n^3} \right) \]

Lemma 21.

\[ \mathbb{E}[\Lambda_4(x; m)]^2 = O\left( \frac{m^2}{n^3} \right) \]

Proof. Proof is similar to Lemma 20

Lemma 22.

\[ \mathbb{E}[\Lambda_5(x; m)]^2 = O\left( \frac{m}{n^3} \right) \]
Proof.

\[\mathbb{E}[\Lambda_5(x;m)]^2 \lesssim \frac{1}{n^3 h_0} \mathbb{E} \left[ \sum_{i=1}^{m} \sum_{j=1}^{m} S_i \mathcal{L}_i S_j \mathcal{L}_j \right] = \frac{1}{n^3 h_0} \mathbb{E} \left[ \sum_{i=1}^{m} \sum_{j \neq i}^{m} S_i \mathcal{L}_i S_j \mathcal{L}_j + \sum_{i=1}^{m} S_i^2 \mathcal{L}_i^2 \right] = \frac{m}{n^3 h_0} \mathbb{E}[S_1^2 \mathcal{L}_1^2] = O \left( \frac{m}{n^3} \right)\]

□

Lemma 23.

\[
\begin{align*}
\mathbb{E}[\Lambda_1(x;m)\Lambda_2(x;m)] &= O \left( \frac{m^3 h_0^{2L+1}}{n^4} \right) = O \left( \frac{m^3 h_0^{2L+1}}{n^2} \right), \\
\mathbb{E}[\Lambda_1(x;m)\Lambda_4(x;m)] &= O \left( \frac{m^3 h_0^{2L+1}}{n^4} \right) = O \left( \frac{m^3 h_0^{2L+1}}{n^2} \right), \\
\mathbb{E}[\Lambda_1(x;m)\Lambda_5(x;m)] &= O \left( \frac{m^4}{n^6} \right) = O \left( \frac{m^4}{n^3} \right), \\
\mathbb{E}[\Lambda_2(x;m)\Lambda_4(x;m)] &= O \left( \frac{m^3}{n^6} \right) = O \left( \frac{m^3}{n^3} \right), \\
\mathbb{E}[\Lambda_2(x;m)\Lambda_5(x;m)] &= O \left( \frac{m^3}{n^6} \right) = O \left( \frac{m^3}{n^3} \right), \\
\mathbb{E}[\Lambda_4(x;m)\Lambda_5(x;m)] &= O \left( \frac{m^3}{n^6} \right) = O \left( \frac{m^3}{n^3} \right).
\end{align*}
\]

Proof. From Cauchy-Schwarz inequality and Lemma 19, 20, 21 and 22, this lemma holds. □

C Derivation of Expression for \( p_1(z), p_3(z) \) and \( p_4(z) \)

For \( p_1(z) \), we have,

\[
p_1(z) = -\frac{1}{6} \mu_{30} \mu_{20}^{-3/2} (z^2 - 1)
\]

\[
= -\frac{1}{6} \left[ \kappa_{03} f(x) - 3 \kappa_{02} f(x)^2 h_0 + \kappa_{22} f^{(2)}(x) h_0^2/2 + o(h_0^2) \right]
\]

\[
- \frac{3}{2} \left[ \kappa_{02} f(x) \right]^{-5/2} f(x)^2 h_0 \left\{ \frac{\kappa_{33} f^{(2)}(x)}{2} + 2 f(x)^3 \right\} h_0^2/2
\]

\[
+ \frac{15}{8} \left[ \kappa_{02} f(x) \right]^{-7/2} f(x)^4 h_0^2 (z^2 - 1) + o(h_0^2)
\]

\[
= -\frac{1}{6} \left[ \kappa_{02}^{-3/2} \kappa_{03} f(x) - 3 \left\{ f(x)^{1/2} \kappa_{02}^{-1/2} - \kappa_{03} f(x)^{1/2} / 2 \kappa_{02}^{5/2} \right\} h_0 \right.
\]

\[
+ \left\{ -\frac{3}{4} \left[ \kappa_{02} f(x) \right]^{-5/2} \kappa_{03} \kappa_{23} f^{(2)}(x) f(x) - 3 \left[ \kappa_{02} f(x) \right]^{-5/2} \kappa_{03} f(x)^4 \right\}
\]

\[
+ \frac{15}{8} \left[ \kappa_{02} f(x) \right]^{-7/2} \kappa_{03} f(x)^5 + \frac{9}{2} \kappa_{02}^{-3/2} f(x)^{3/2} \right\} h_0 \right)
\]

\[
(z^2 - 1) + o(h_0^2)
\]

\[37\]
\[ \gamma_0(x)(z^2 - 1) + \gamma_1(x)(z^2 - 1)h_0 + \gamma_2(x)(z^2 - 1)h_0^2 + o(h_0^2). \]

For \( p_{3,0}(z) \), we have

\[
p_{3,0}(z) = -C_p C_{p,0}(x)\rho_{11} \mu_{20}^{-1} z
\]
\[ = -C_p C_{p,0}(x) \frac{L(x)}{\mu_{20}^{-2} h_0 + o(h_0)} \]
\[ = -C_p C_{p,0}(x) \frac{L(x)}{h_0 + o(h_0)} \]
\[ \times \left( \sum \kappa_0 f(x)^{-1} \kappa_0 \frac{L(x)}{h_0 + o(h_0)} \right) z + o(h_0) \]
\[ = -C_p C_{p,0}(x) \frac{L(x)}{h_0 + o(h_0)} \]
\[ \equiv \gamma_{1,0}(x)z + \gamma_{1,1}(x)zh_0 + o(h_0), \]

while for \( p_4(z) \),

\[
p_4(z) = -C_p \xi_{11} \mu_{20}^{-3/2} (z^2 - 1) + \frac{1}{2} C_p \xi_{11} \mu_{20}^{-1/2}z^2
\]
\[ = -C_p \frac{L(x)}{h_0 + o(h_0)} \]
\[ \times \left( \sum \kappa_0 f(x)^{-3/2} \kappa_0 \frac{L(x)}{h_0 + o(h_0)} \right) z^2 \]
\[ = -C_p \xi_{11} \mu_{20}^{-3/2} \tau_0 \frac{L(x)}{h_0 + o(h_0)} + \frac{3}{2} C_p \xi_{11} \mu_{20}^{-5/2} \tau_0 \frac{L(x)}{h_0 + o(h_0)} \]
\[ \equiv \gamma_{4,0}(x)z + \gamma_{4,1}(x)zh_0 + o(h_0). \]

\[ \square \]

D Formal Derivation of Theorem 3.5

In this section, we derive Theorem 3.5 formally. There is no guarantee for the mathematical rigor. However, one can validate Theorem 3.5 in the same way as the proof of 3.1.

\[
S_{\text{plot}}(x) = \sqrt{n} h \left( \tilde{f}(x) - E \hat{f}_{h_0}(x) \right)
\]
\[ = \frac{1}{\sqrt{n} h_0} \sum_{i=1}^{n} S_i
\]
\[ + \frac{C_p h_0^{2L-1}}{n^{2L-1} h_0^{2L} \mu_{20}^{1/2}} \sum_{i=1}^{n} V_i \left( \sum_{l=0}^{L-1} C_{l,i}(x) h_l \right)
\]
\[ + \frac{C_p h_0^{2L-1}}{2n^{2L} (n-1) h_0^{2L} \mu_{20}^{1/2}} \sum_{i=1}^{n} \sum_{j \neq i} W_{ij} \left( \sum_{l=0}^{L-1} C_{l,j}(x) h_l \right)
\]
\[ - \frac{C_p}{n^{3L} h_0^{L-1} \mu_{20}^{1/2}} \left[ \sum_{i=1}^{n} n \sum_{j \neq i} S_{ij} + \sum_{i=1}^{n} S_{i} \right]
\]
\[ - \frac{C_p}{4n^{3L} (n-1) h_0^{L-1} \mu_{20}^{1/2}} \left[ \sum_{i=1}^{n} n \sum_{j \neq i} W_{ij} + \sum_{i=1}^{n} \sum_{j \neq i} n (W_{ij} + W_{ij}) \right].
\]
\[ + o_p\{(nh_0)^{-1}\} \]
\[ \equiv S(x) + \sum_{k=1}^{10} \lambda_k(x) + o_p\{(nh_0)^{-1}\} \]

In the following expansion of characteristic function of \(S_{\text{pilot}}(x)\), we use
\[
\begin{align*}
\mathbb{E}[S_1 \gamma_1] &= h_0 \mu_{20}^{-1/2} \xi_{11}, \\
\mathbb{E}[S_1 \gamma_1] &= h_0 \mu_{20}^{-1/2} \rho_{11}, \\
\mathbb{E}[S_1 S_2 W_{12}] &= b^{-2} h_0 \mu_{20}^{-1} \omega_{111}, \\
\mathbb{E}[S_1 \gamma_2 W_{12}] &= b^{-2} h_0 \mu_{20}^{-1/2} \psi_{111}.
\end{align*}
\]
where the reason why \(\mathbb{E}[S_1 V_1] = \mathbb{E}[S_1 \mathcal{Z}_1]\) holds is provided as Remark 4. Define the characteristic function of \(S_{\text{pilot}}(x)\) as \(\chi_{\text{pilot}}(x)\).

\[
\chi_{\text{pilot}}(t) = \mathbb{E} \left[ \exp \left( it \{ S(x) + \sum_{k=1}^{10} \lambda_k(x) + o_p\{(nh_0)^{-1}\} \} \right) \right]
\]
\[ = \mathbb{E} \left[ e^{it S(x)} \left\{ 1 + \sum_{k=1}^{10} it \lambda_k(x) \right\} + o_p\{(nh_0)^{-1}\} \right]
\]
\[ = (I) + (II) + (III) + (IV) + (V) + (VI) + (VII) + (VIII) + (IX) + (X) + (XI) + o_p\{(nh_0)^{-1}\} \]

In the following subsections, we expand each component of ch.f.

### D.1 Expansion of The Second Component

\[
(II) = \mathbb{E} \left[ e^{it S(x)} \right]
\]
\[ = \frac{C_p h_0^{2L+1/2}}{n^{1/2} \mu_{20}^{1/2}} \mathbb{E} \left[ e^{it S(x)} \sum_{i=1}^{n} V_i \left( \sum_{l=0}^{L-1} C_{I,J}(x) h_l \right) \right] (it)
\]
\[ = \frac{C_p h_0^{2L+1}}{n^{1/2} \mu_{20}^{1/2}} \mathbb{E} \left[ (1 + it \sqrt{nh_0}) S_1 V_1 \right] (it) + o\{(nh_0)^{-1}\}
\]
\[ = C_p h_0^{2L+1} (it)^2 \mathbb{E} [S_1 V_1] \left[ \sum_{l=0}^{L-1} C_{I,J}(x) h_l \right] (it)^2 + o\{(nh_0)^{-1}\}
\]
\[ = C_p h_0^{2L+1} (it)^2 \mu_{20}^{-1} \sum_{l=0}^{L-1} C_{I,J}(x) h_l (it)^2 + o\{(nh_0)^{-1}\} \]

### D.2 The Third Component

\[
(III) = \mathbb{E} \left[ e^{it S(x)} \lambda_2(x) \right]
\]
\[ = \frac{C_p h_0^{2L+1}}{2n^{1/2} (n-1) \mu_{20}^{1/2}} \mathbb{E} \left[ e^{it S(x)} \sum_{i=1}^{n} \sum_{j \neq i} W_{ij} \left( \sum_{l=0}^{L-1} C_{I,J}(x) h_l \right) \right] (it)
\]
\[ = \frac{C_p h_0^{2L+1}}{2n^{1/2} (n-1) \mu_{20}^{1/2}} \mathbb{E} \left[ \sum_{i=1}^{n} \sum_{j \neq i} (S_1 + S_2) \left( \frac{it}{\sqrt{nh_0}} (S_1 + S_2)^2 \right) W_{12} \right] \left[ \sum_{l=0}^{L-1} C_{I,J}(x) h_l \right] (it) + o\{(nh_0)^{-1}\}
\]
\[ = \frac{C_p h_0^{2L+1}}{2n^{1/2}} \mathbb{E} [S_1 S_2 W_{12}] \left[ \sum_{l=0}^{L-1} C_{I,J}(x) h_l \right] (it)^3 + o\{(nh_0)^{-1}\} \]
\[
C_{pl} \frac{n^{3/2}}{2^n h_0^{1/2} \mu_2^{1/2}} \gamma(t)^{n-2} \mu_2^{-2} \omega_{111} \left( \sum_{i=0}^{n-1} C_{\Gamma_j(x)} h_0^i \right) (it)^3 + o\{ (nh_0)^{-1} \}
\]

**D.3 The Fourth Component**

\[
(IV) = \mathbb{E} \left[ e^{i S(x)} (it \Lambda_3(x)) \right] = \frac{C_{pl}}{n^{3/2} h_0^{1/2} \mu_2^{1/2}} \mathbb{E} \left[ e^{i S(x)} \sum_{i=1}^{n} \sum_{j \neq i} V_i \Gamma_j \right] (it)
\]

\[
= \frac{C_{pl}}{n^{3/2} h_0^{1/2} \mu_2^{1/2}} n(n-1) \gamma(t)^{n-2} \mathbb{E} \left[ \left\{ 1 + \frac{it}{\sqrt{nh_0}} (S_1 + S_2) + \frac{(it)^2}{nh_0} (S_1 + S_2)^2 \right\} V_1 \Gamma_2 \right] (it) + o\{ (nh_0)^{-1} \}
\]

\[
= \frac{C_{pl} \rho_{111}^{2}}{n^{1/2} h_0^{1/2} \mu_2^{1/2}} \gamma(t)^{n-2} \mu_2^{-2} \rho_{111} \xi_{11} (it)^3 + o\{ (nh_0)^{-1} \}
\]

**D.4 The Fifth Component**

\[
(V) = \frac{C_{pl}}{n^{3/2} h_0^{1/2} \mu_2^{1/2}} \mathbb{E} \left[ e^{i S(x)} \sum_{i=1}^{n} V_i \Gamma_i \right] (it)
\]

\[
= \frac{C_{pl}}{n^{1/2} h_0^{1/2} \mu_2^{1/2}} \mathbb{E} [V_1 \Gamma_1] (it) + o\{ (nh_0)^{-1} \}.
\]

Since, from Lemma 4, \( \mathbb{E} [V_1 \Gamma_1] = O(h_0^{L+1}) \), \( (V) = o\{ (nh_0)^{-1} \} \)

**D.5 The Sixth Component**

\[
(VI) = \frac{C_{pl}}{2^n h_0^{1/2} \mu_2^{1/2}} \mathbb{E} \left[ e^{i S(x)} \sum_{i=1}^{n-1} \sum_{j \neq i} W_i \Gamma_j \right] (it)
\]

\[
= \frac{C_{pl}}{2^n h_0^{1/2} \mu_2^{1/2}} n(n-1) \gamma(t)^{n-3} \mathbb{E} \left[ \frac{e^{-it}}{\sqrt{\mu_0}} (S_1 + S_2 + S_3) W_1 \Gamma_3 \right] (it)
\]

\[
= \frac{C_{pl} \rho_{12}^{2} \mu_2^{1/2}}{2nh_0^{1/2}} \gamma(t)^{n-3} \mathbb{E} [S_1 S_2 W_1] \mathbb{E} [S_3 \Gamma_3] (it)^4 + o\{ (nh_0)^{-1} \}
\]

\[
= \frac{C_{pl} \rho_{12}^{2} \mu_2^{1/2}}{2nh_0^{1/2}} \gamma(t)^{n-3} \mu_2^{-2} \xi_{11} \omega_{111} (it)^4 + o\{ (nh_0)^{-1} \}
\]

**D.6 The Seventh Component**

\[
(VII) = \frac{C_{pl}}{2^n h_0^{1/2} \mu_2^{1/2}} \mathbb{E} \left[ e^{i S(x)} \sum_{i=1}^{n-1} \{ W_i \Gamma_i + W_i \Gamma_j \} \right] (it)
\]

\[
= \frac{C_{pl}}{2^n h_0^{1/2} \mu_2^{1/2}} n(n-1) \gamma(t)^{n-2} \mathbb{E} \left[ e^{-it/\mu_0} (S_1 + S_2) \{ W_1 \Gamma_1 + W_1 \Gamma_2 \} \right] (it)
\]

\[
= \frac{C_{pl} \rho_{12}^{2} \mu_2^{1/2}}{nh_0} \gamma(t)^{n-2} \mathbb{E} [W_1 \Gamma_1 S_2] (it)^2 + o\{ (nh_0)^{-1} \}
\]

\[
= \frac{C_{pl} \rho_{12}^{2} \mu_2^{1/2}}{nh_0} \gamma(t)^{n-2} \mathbb{E} \psi_{111} (it)^2 + o\{ (nh_0)^{-1} \}
\]
D.7 The Eighth Component

\[
(VIII) = \frac{-C_p I}{2n^{3/2}h_0^{1/2}} \mathbb{E} \left[ e^{i \beta x(x)} \sum_{i=1}^{n} \sum_{j \neq i} S_i V_j \right] (it) \\
= \frac{-C_p I}{2n^{3/2}h_0^{1/2}} n(n-1) \gamma(t)^{n-2} \mathbb{E} \left[ \frac{2}{\nu_0} (S_1 + S_2) S_i V_j \right] (it) \\
= \frac{-C_p I}{2n^{1/2}h_0^{1/2}} \gamma(t)^{n-2} \mathbb{E} [S_i^2] \mathbb{E} [S_2 V_2] (it)^3 + o((nh_0)^{-1}) \\
= \frac{-C_p I h_0^{1/2}}{2n^{1/2}} \gamma(t)^{n-2} \mu_1^{-1/2} \rho_{11} (it) + o((nh_0)^{-1}) 
\]

D.8 The Ninth Component

\[
(IX) = \frac{-C_p I}{2n^{3/2}h_0^{1/2}} \mathbb{E} \left[ e^{i \beta x(x)} \sum_{i=1}^{n} S_i V_i \right] (it) \\
= \frac{-C_p I}{2n^{3/2}h_0^{1/2}} n \gamma(t)^{n-1} \mathbb{E} \left[ \frac{2}{\nu_0} S_1 V_i \right] (it) \\
= \frac{-C_p I}{2n^{1/2}h_0^{1/2}} \gamma(t)^{n-1} \mathbb{E} [S_1 V_i] (it) + o((nh)^{-1}) \\
= \frac{-C_p I h_0^{1/2}}{2n^{1/2}} \gamma(t)^{n-1} \mu_1^{-1/2} \rho_{11} (it) + o((nh_0)^{-1}) 
\]

D.9 The Tenth Component

\[
(X) = \frac{-C_p I}{4n^{3/2}(n-1)h_0^{1/2}} \mathbb{E} \left[ e^{i \beta x(x)} \sum_{i=1}^{n} \sum_{j \neq i} W_{ij} S_k \right] (it) \\
= \frac{-C_p I}{4n^{3/2}(n-1)h_0^{1/2}} n(n-1)(n-2) \gamma(t)^{n-3} \mathbb{E} \left[ \frac{2}{\nu_0} (S_1 + S_2 + S_3) W_{123} \right] (it) \\
= \frac{C_p I(n-2)}{4n^{1/2}h_0^{1/2}} \gamma(t)^{n-3} \mathbb{E} \left[ \left\{ 1 + \cdots + \left( \frac{it}{(nh_0)3^{1/2}} \right)^3 W_{123} \right\} \right] (it) \\
= \frac{-C_p I}{4nh_0^2} \mathbb{E} (t)^{n-3} \mathbb{E} [S_1 S_2 W_{12}] \mathbb{E} [S_3^2] (it)^4 + o((nh)^{-1}) \\
= \frac{-C_p I}{4nh_0^2} \mathbb{E} (t)^{n-3} \mu_1^{-1} \omega_{111} (it)^4 + o((nh_0)^{-1}) 
\]

D.10 The Eleventh Component

\[
(XI) = \frac{-C_p I}{4n^{3/2}(n-1)h_0^{1/2}} \mathbb{E} \left[ e^{i \beta x(x)} \sum_{i=1}^{n} \sum_{j \neq i} \{ W_{ij} S_i + W_{ij} S_j \} \right] (it) \\
= \frac{-C_p I}{4n^{3/2}(n-1)h_0^{1/2}} n(n-1) \gamma(t)^{n-2} \mathbb{E} \left[ \frac{2}{\nu_0} (S_1 + S_2) \{ W_{12} S_1 + W_{12} S_2 \} \right] \\
= \frac{-C_p I}{2nh_0} \gamma(t)^{n-2} \mathbb{E} [S_1 S_2 W_{12}] (it)^2 + o((nh_0)^{-1}) \\
= \frac{-C_p I}{2nh_0^2} \gamma(t)^{n-2} \mu_1^{-1} \omega_{111} (it)^2 + o((nh_0)^{-1}) 
\]
Recalling
\[ \chi_{\text{pilot}}(t) = (I) + (II) + (III) + (IV) + (V) + (VI) + (VII) + (VIII) + (IX) + (XI) + o_p\{(nh_0)^{-1}\}, \]
we have
\[ \chi_{\text{pilot}}(t) = \exp\left(-\frac{t^2}{2}\right) \left[ 1 + \frac{\mu_0\mu_2^{3/2}}{6n^{1/2}h_0^{1/2}}(it)^3 + \frac{\mu_0\mu_2^2}{24nh_0}(it)^4 + \frac{\mu_0^2\mu_2^3}{72nh_0}(it)^6 \right] \]
\[ + C_{\text{P}} \mu_2^{-1} \rho_{11} \left( \sum_{i=0}^{L-1} C_\Gamma(x) h_0^{i+1} \right) (it)^2 \]
\[ + \frac{C_{\text{P}}}{2} n^{-1/2} h_0^{-1/2} b^{-2L} \mu_2^{-3/2} \omega_{11} \left( \sum_{i=0}^{L-1} C_\Gamma(x) h_0^{i+1} \right) (it)^3 \]
\[ + C_{\text{P}} n^{-1/2} h_0^{-1/2} \rho_{11} \left( \frac{\mu_2^{-3/2} \omega_{11}}{2} (it)^3 - \frac{\mu_2^{-1/2}}{2} \{ (it)^3 + (it) \} \right) \]
\[ + C_{\text{P}} n^{-1} b^{-2L} \left( \frac{1}{2} \mu_2^{-2} \omega_{11} (it)^4 + \mu_2^{-1} \psi_{11} (it)^2 - \frac{1}{4} \mu_2^4 \omega_{11} \{ (it)^4 + 2(it)^2 \} \right) + o\{(nh_0)^{-1}\} \]
Inverting \( \chi_{\text{pilot}}(t) \), we have Theorem 3.5.

### E  Formal Derivation of Theorem 3.6

In this section, we derive Theorem 3.6 formally. There is no guarantee for the mathematical rigor. However, one can validate Theorem 3.6 in the same way as the proof 3.1.

#### E.1  Expansion of \( \hat{\mu}_{20}(\hat{h}) \)

Let \( \hat{\mu}_{20} \) be the natural estimator for \( \mu_{20} \),

\[ \hat{\mu}_{20}(h) \equiv h^{-1} \left\{ \frac{1}{n} \sum_{i=1}^{n} K_{i,h}(x)^2 - \left( \frac{1}{n} \sum_{i=1}^{n} K_{i,h}(x) \right)^2 \right\}, \]
then studentized KDE with data-driven bandwidth \( \hat{h} \), the standard deviation is \( \hat{\mu}_{20}(\hat{h}) \). Under Assumption 17, expanding \( \hat{\mu}_{20}(\hat{h}) \) around \( \hat{h} = h_0 \) yields

\[ \hat{\mu}_{20}(h) = \hat{\mu}_{20}(h_0) + \hat{\mu}_{20,\hat{h}}(h_0)(\hat{h} - h_0) + o_p\{(nh_0)^{-1}\}. \]  \( \text{(E.1)} \)

where the definition of \( \hat{\mu}_{20,\hat{h}}(h_0) \) is

\[ \hat{\mu}_{20,\hat{h}}(h_0) = -h_0^{-2} \left\{ \frac{1}{n} \sum_{i=1}^{n} K_{i,h_0}(x)^2 - \left( \frac{1}{n} \sum_{i=1}^{n} K_{i,h_0}(x) \right)^2 \right\} \]
\[ - h_0^{-2} \left\{ \frac{2}{n} \sum_{i=1}^{n} K_{i,h_0}(x) K'_{i,h_0}(x) u_i,h_0(x) - \frac{2}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} K_{i,h_0}(x) K'_{j,h_0} u_{i,j,h_0} \right\}. \]  \( \text{(E.2)} \)
E.1.1 Transformation of $\hat{\mu}_{20}(h_0)$

We first transform $\hat{\mu}_{20}(h_0)$. (Hall (1991) and Hall (1992a) has already done this transformation, see (Hall, 1992a, p.212-213).) Define

$$\Delta_f(h) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ K_{i,h}(x)^i - \mathbb{E} K_{i,h}(x)^i \right\}.$$ 

Then, $\hat{\mu}_{20}(h_0)$ is

$$\hat{\mu}_{20}(h_0) = h_0^{-1} \left\{ \frac{1}{n} \sum_{i=1}^{n} K_{i,h_0}(x)^2 - \left( \frac{1}{n} \sum_{i=1}^{n} K_{i,h_0}(x) \right)^2 \right\}$$

$$= h_0^{-1} \left\{ \mathbb{E} [K_{i,h_0}(x)^2] + \frac{1}{n} \sum_{i=1}^{n} K_{i,h_0}(x)^2 - \mathbb{E} [K_{i,h_0}(x)]^2 - \left( \frac{1}{n} \sum_{i=1}^{n} K_{i,h_0}(x) \right)^2 \right\}$$

$$= h_0^{-1} \left\{ \mathbb{E} [K_{i,h_0}(x)^2] - \mathbb{E} [K_{i,h_0}(x)]^2 \right\}$$

$$+ h_0^{-1} \left\{ \frac{1}{n} \sum_{i=1}^{n} K_{i,h_0}(x)^2 - \mathbb{E} [K_{i,h_0}(x)]^2 \right\} - h_0^{-1} \left\{ \left( \frac{1}{n} \sum_{i=1}^{n} K_{i,h_0}(x) \right)^2 - \mathbb{E} [K_{i,h_0}(x)]^2 \right\}$$

$$= \mu_{20}(h_0) + (nh_0)^{-1/2} \Delta_2(h_0) - 2 \frac{h_0 f(x)}{\sqrt{nh_0}} \Delta_1(h_0) + O_p(n^{-1}). \quad (E.4)$$

where the third term in the final quality follows from the following equation.

$$\Delta_1(h_0)^2 = nh_0^{-1} \left\{ \left( \frac{1}{n} \sum_{i=1}^{n} K_{i,h_0}(x) \right)^2 - \mathbb{E} [K_{i,h_0}(x)]^2 \right\} - 2h_0^{-1} \mathbb{E} [K_{i,h_0}(x)] \Delta_1(h_0)$$

$$\Rightarrow h_0^{-1} \left\{ \left( \frac{1}{n} \sum_{i=1}^{n} K_{i,h_0}(x) \right)^2 - \mathbb{E} [K_{i,h_0}(x)]^2 \right\} = n^{-1} \Delta_1(h_0)^2 + 2 \frac{\mathbb{E} [K_{i,h_0}(x)]}{\sqrt{nh_0}} \Delta_1(h_0)$$

E.1.2 $\hat{\mu}_{20,\delta}(h_0)$

Next, we have to transform $\hat{\mu}_{20,\delta}(h_0)$. Define

$$\Psi^{(1)}(h) = \frac{1}{\sqrt{nh}} \sum_{i=1}^{n} \left\{ K_{i,h}(x) K'_{i,h}(x) u_{i,h}(x) - \mathbb{E} [K_{i,h}(x) K'_{i,h}(x) u_{i,h}(x)] \right\}$$

$$\Delta^{(1)}_1(h) = \frac{1}{\sqrt{nh}} \sum_{i=1}^{n} \left\{ K'_{i,h}(x) u_{i,h}(x) - \mathbb{E} [K'_{i,h}(x) u_{i,h}(x)] \right\}$$

$$\delta(h) = h^{-1} \left\{ \mathbb{E} [K_{i,h}(x) K'_{i,h}(x) u_{i,h}(x)] - \mathbb{E} [K_{i,h}(x) K'_{i,h}(x) u_{i,h}(x)] \right\}$$

Since $\hat{h} - h_0 = O_p \{ n^{-1/2} h_0 \lor n^{-1} b^{-(4L+1)/2} h_0 \} = O_p \{ n^{-1/2} h_0 \lor n^{-1} b^{-(4L+1)/2} h_0 \}$, we can ignore the terms whose convergence rates are faster than $O_p \{ n^{-1/2} h_0^2 \lor b^{-(4L+1)/2} h_0^{-2} \}$ for Edgeworth expansion up to the order of $O\{ (nh_0)^{-1} \}$.

$$\hat{\mu}_{20,\delta}(h_0) = -h_0^{-2} \left\{ \frac{1}{n} \sum_{i=1}^{n} K_{i,h_0}(x)^2 - \left( \frac{1}{n} \sum_{i=1}^{n} K_{i,h_0}(x) \right)^2 \right\}$$

$$- h_0^{-2} \left\{ \frac{2}{n} \sum_{i=1}^{n} K_{i,h_0}(x) K'_{i,h_0}(x) u_{i,h_0}(x) - \frac{2}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} K_{i,h_0}(x) K'_{j,h_0}(x) u_{j,h_0}(x) \right\}$$

$$= -h_0^{-1} \hat{\mu}_{20}(h_0)$$

$$- 2h_0^{-2} \left\{ \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} K_{i,h_0}(x) K'_{i,h_0}(x) u_{i,h_0}(x) \right] - \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} K_{i,h_0}(x) K'_{j,h_0}(x) u_{j,h_0}(x) \right] \right\}$$
\[-2h_0^{-2} \left\{ \frac{1}{n} \sum_{i=1}^{n} K_{i,j_0}(x) K'_{i,j_0}(x) u_{i,j_0}(x) - \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} K_{i,j_0}(x) K'_{i,j_0}(x) u_{i,j_0}(x) \right] \right\} \]

\[+ 2h_0^{-2} \left\{ \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i}^{n} K_{i,j_0}(x) K'_{j,j_0}(x) u_{j,j_0}(x) - \mathbb{E} \left[ \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i}^{n} K_{i,j_0}(x) K'_{j,j_0}(x) u_{j,j_0}(x) \right] \right\} \]

\[= -h_0^{-1} \mu_{20}(h_0) \]

\[-2h_0^{-2} \left\{ \left( 1 - \frac{1}{n} \right) \mathbb{E} \left[ K_{i,j_0}(x) K'_{i,j_0}(x) u_{i,j_0}(x) \right] - \left( 1 - \frac{1}{n} \right) \mathbb{E} \left[ K_{i,j_0}(x) K'_{j,j_0}(x) u_{j,j_0}(x) \right] \right\} \]

\[-2h_0^{-2} \left\{ \frac{1}{n} \sum_{i=1}^{n} K_{i,j_0}(x) K'_{i,j_0}(x) u_{i,j_0}(x) - \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} K_{i,j_0}(x) K'_{i,j_0}(x) u_{i,j_0}(x) \right] \right\} \]

\[+ 2h_0^{-2} \left\{ \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i}^{n} K_{i,j_0}(x) K'_{j,j_0}(x) u_{j,j_0}(x) - \mathbb{E} \left[ \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i}^{n} K_{i,j_0}(x) K'_{j,j_0}(x) u_{j,j_0}(x) \right] \right\} \]

\[= -h_0^{-1} \left\{ \mu_{20}(h_0) + 2 \delta(h_0) + 2(nh_0)^{-1/2} \Psi(1)(h_0) \right\} - 2(nh_0)^{-1/2} f(x) \Delta_1^{(1)}(h_0) + 2(nh_0)^{-1/2} f(x) \Delta_1(h_0) \]

\[+ o_p \left\{ n^{-1/2} h_0^{-2} \land b^{(4L+1)/2} h_0^{-2} \right\} \]

\[= -h_0^{-1} \mu_{20}(h_0) - 2h_0^{-1} \delta(h_0) + o_p \left\{ n^{-1/2} h_0^{-2} \land b^{(4L+1)/2} h_0^{-2} \right\} \]

\[= -h_0^{-1} \mu_{20}(h_0) - 2h_0^{-1} \delta(h_0) + o_p \left\{ n^{-1/2} h_0^{-2} \land b^{(4L+1)/2} h_0^{-2} \right\} \]

Note that (E.8) is $o_p \left\{ n^{-1/2} h_0^{-2} \land b^{(4L+1)/2} h_0^{-2} \right\}$. The second and third terms in (E.9) are another expression of (E.5) and (E.6) respectively. The last two terms of (E.9) follows from the following transformation of (E.7).

\[(nh_0)^{-1} \Delta_1(h_0) \Delta_1^{(1)}(h_0) = \frac{1}{nh_0^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left\{ K_{i,j_0}(x) - \mathbb{E} \left[ K_{i,j_0}(x) \right] \right\} \left\{ K'_{j,j_0}(x) u_{j,j_0}(x) - \mathbb{E} \left[ K'_{j,j_0}(x) u_{j,j_0}(x) \right] \right\} \]

\[= \frac{1}{nh_0^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left\{ K_{i,j_0}(x) K'_{j,j_0}(x) u_{j,j_0}(x) - \mathbb{E} \left[ K_{i,j_0}(x) K'_{j,j_0}(x) u_{j,j_0}(x) \right] \right\} \]

\[- \frac{1}{nh_0} \mathbb{E} \left[ K_{i,j_0}(x) \right] \sum_{i=1}^{n} \left\{ K'_{j,j_0}(x) u_{j,j_0}(x) - \mathbb{E} \left[ K'_{j,j_0}(x) u_{j,j_0}(x) \right] \right\} \]

\[- \frac{1}{nh_0} \mathbb{E} \left[ K'_{j,j_0}(x) u_{j,j_0}(x) \right] \sum_{i=1}^{n} \left\{ K_{i,j_0}(x) - \mathbb{E} \left[ K_{i,j_0}(x) \right] \right\} \]

\[= \frac{1}{nh_0^2} \sum_{i=1}^{n} \sum_{j \neq i}^{n} \left\{ K_{i,j_0}(x) K'_{j,j_0}(x) u_{j,j_0}(x) - \mathbb{E} \left[ K_{i,j_0}(x) K'_{j,j_0}(x) u_{j,j_0}(x) \right] \right\} \]

\[+ \frac{1}{nh_0^2} \sum_{i=1}^{n} \left\{ K_{i,j_0}(x) K'_{i,j_0}(x) u_{i,j_0}(x) - \mathbb{E} \left[ K_{i,j_0}(x) K'_{i,j_0}(x) u_{i,j_0}(x) \right] \right\} \]

\[- \frac{1}{nh_0} \mathbb{E} \left[ K_{i,j_0}(x) \right] \sum_{i=1}^{n} \left\{ K'_{i,j_0}(x) u_{i,j_0}(x) - \mathbb{E} \left[ K'_{i,j_0}(x) u_{i,j_0}(x) \right] \right\} \]

\[- \frac{1}{nh_0} \mathbb{E} \left[ K'_{i,j_0}(x) u_{i,j_0}(x) \right] \sum_{i=1}^{n} \left\{ K_{i,j_0}(x) - \mathbb{E} \left[ K_{i,j_0}(x) \right] \right\} \]

\[= \frac{1}{nh_0^2} \sum_{i=1}^{n} \sum_{j \neq i}^{n} \left\{ K_{i,j_0}(x) K'_{j,j_0}(x) u_{j,j_0}(x) - \mathbb{E} \left[ K_{i,j_0}(x) K'_{j,j_0}(x) u_{j,j_0}(x) \right] \right\} \]

\[+ o_p \left\{ (nh_0)^{-3/2} - n^{-1/2} h_0^{-3/2} \mathbb{E} \left[ K_{i,j_0}(x) \right] \Delta_1^{(1)}(h_0) - 2(nh_0)^{-1/2} h_0^{-3/2} \mathbb{E} \left[ K'_{i,j_0}(x) u_{i,j_0}(x) \right] \Delta_1(h_0) \right\} \]

\[= \frac{1}{nh_0^2} \sum_{i=1}^{n} \sum_{j \neq i}^{n} \left\{ K_{i,j_0}(x) K'_{j,j_0}(x) u_{j,j_0}(x) - \mathbb{E} \left[ K_{i,j_0}(x) K'_{j,j_0}(x) u_{j,j_0}(x) \right] \right\} \]

\[= (nh_0)^{-1} \Delta_1(h_0) \Delta_1^{(1)}(h_0) + (nh_0)^{-1/2} f(x) \Delta_1^{(1)}(h_0) \]

\[- (nh_0)^{-1/2} f(x) \Delta_1(h_0) + o_p \left\{ (nh_0)^{-3/2} \right\} \]
\[ = (nh_0)^{-1/2} f(x) \Delta_1^{(1)}(h_0) - (nh_0)^{-1/2} f(x) \Delta_1(h_0) + s.o. \]

E.2 \( T_{Pf}(x) \)

We write the studentized KDE with plug-in bandwidth as \( T_{Pf} \). Then, from (E.4) and (E.9),

\[
T_{Pf}(x) = \frac{\sqrt{n h_0} \{ \hat{f}_h(x) - E \hat{f}_h(x) \}}{\mu_20(h_0)^{1/2}} - \frac{\sqrt{n h_0} \{ \hat{f}_h(x) - E \hat{f}_h(x) \}}{2 \mu_20(h_0)^{3/2}} \left\{ \hat{\mu}_{20, \hat{h}}(h_0) - o_p \{(nh_0)^{-1}\} \right\} + o_p \{(nh_0)^{-1}\}
\]

\[
= \frac{\sqrt{n h_0} \{ \hat{f}_h(x) - E \hat{f}_h(x) \}}{\mu_20(h_0)^{1/2}} \left\{ \hat{\mu}_{20, \hat{h}}(h_0) - o_p \{(nh_0)^{-1}\} \right\} + o_p \{(nh_0)^{-1}\}
\]

Since

\[
\sqrt{n h_0} \{ \hat{f}_h(x) - E \hat{f}_h(x) \}
\]

\[
= \sqrt{n h_0} \{ \hat{f}_h(x) - E \hat{f}_h(x) \} - \sqrt{n h_0} \left( \frac{\hat{h} - h_0}{h_0} \right) \Gamma_{KDE_1} + \frac{\sqrt{n h_0} \{ \hat{f}_h(x) - E \hat{f}_h(x) \}}{2} \left( \frac{\hat{h} - h_0}{h_0} \right) + o_p \{(nh_0)^{-1}\},
\]

\( T_{Pf}(x) \) is

\[
T_{Pf}(x) = \frac{\sqrt{n h_0} \{ \hat{f}_h(x) - E \hat{f}_h(x) \}}{\mu_20(h_0)^{1/2}} - \frac{\sqrt{n h_0} \{ \hat{f}_h(x) - E \hat{f}_h(x) \}}{2 \mu_20(h_0)^{3/2}} \left( \frac{\hat{h} - h_0}{h_0} \right) \Gamma_{KDE_1} + \frac{\sqrt{n h_0} \{ \hat{f}_h(x) - E \hat{f}_h(x) \}}{2 \mu_20(h_0)^{1/2}} \left( \frac{\hat{h} - h_0}{h_0} \right)
\]

\[
- \frac{\sqrt{n h_0} \{ \hat{f}_h(x) - E \hat{f}_h(x) \}}{2 \mu_20(h_0)^{3/2}} \left\{ \mu_20(h_0) + 2 \delta(h_0) \right\} + o_p \{(nh_0)^{-1}\}
\]

\[
= \frac{\Delta_1(h_0)}{\mu_20(h_0)^{1/2}} - \frac{\sqrt{n h_0} \{ \hat{f}_h(x) - E \hat{f}_h(x) \}}{\mu_20(h_0)^{3/2}} \left( \frac{\hat{h} - h_0}{h_0} \right) \left\{ \left( \frac{\hat{\mu}_{20, \hat{h}}(h_0)}{\mu_20(h_0)} + o_p \{(nh_0)^{-1}\} \right) \right\} + \frac{\Delta_1(h_0)}{\mu_20(h_0)^{1/2}} \left( \frac{\hat{h} - h_0}{h_0} \right)
\]

\[
- \frac{\Delta_1(h_0)}{2 \mu_20(h_0)^{3/2}} \left\{ \mu_20(h_0) + 2 \delta(h_0) \right\} + o_p \{(nh_0)^{-1}\}
\]

\[
= S(x) + A_1(x) + A_2(x) + A_3(x) + A_4(x) + A_5(x) + A_6(x) + A_7(x) + A_8(x) + o_p \{(nh_0)^{-1}\}
\]

\* \( S(x) \) is standardized KDE.

\* \( A_1(x), A_2(x) \) and \( A_3(x) \) include the effect of global plug-in bandwidth.

\* \( A_4(x), A_5(x) \) and \( A_6(x) \) include the effect of studentization.

\* \( A_7(x) \) and \( A_8(x) \) interaction of studentization and global plug-in bandwidth.
E.3 Review of the Definitions of $S(x)$ and $\Lambda(x)$s.

Define $T_i$ as follows.

\[
T_i = K_{i,h_0}(x)^2 - \mathbb{E}K_{i,h_0}(x)^2.
\]

Then, the definitions of $S(x)$ and $\Lambda(x)$s are

\[
S(x) = \Delta_1\mu_2^{-1/2} = \frac{1}{(nh_0)^{1/2}} \sum_{i=1}^{n} S_i
\]

\[
\Lambda_1(x) = \sqrt{nh_0}\mu_2^{-1/2} \left( \frac{\hat{h} - h_0}{h_0} \right) \mathbb{E} \Gamma_{KDE_1} = \frac{C_{pl}h_0^{3/2}}{n^{1/2}\mu_2^{1/2}} \sum_{i=1}^{n} V_i \left( \sum_{j=0}^{l-1} C_{\Gamma,j}(x)h^j \right) + \frac{C_{pl}h_0^{3/2}}{2n^{1/2}(n-1)\mu_2^{1/2}} \sum_{i=1}^{n} \sum_{j \neq i} W_{ij} \left( \sum_{j=0}^{l-1} C_{\Gamma,j}(x)h^j \right)
\]

\[
\Lambda_2(x) = \sqrt{nh_0}\mu_2^{-1/2} \left( \frac{\hat{h} - h_0}{h_0} \right) \{ \Gamma_{KDE_1} - \mathbb{E} \Gamma_{KDE_1} \}
\]

\[
\Lambda_3(x) = \frac{1}{2} \mu_2^{-1/2} \Delta_1 \left( \frac{\hat{h} - h_0}{h_0} \right)
\]

\[
\Lambda_4(x) = -\frac{1}{2} (nh_0)^{-1/2} \mu_2^{-3/2} \Delta_1 \Delta_2
\]

\[
\Lambda_5(x) = n^{-1/2}h_0^{1/2} \mu_2^{-3/2} f(x) \Delta_1^2
\]

\[
\Lambda_6(x) = \frac{3}{8} (nh_0)^{-1/2} \mu_2^{-5/2} \Delta_1 \Delta_2^2
\]

\[
\Lambda_7(x) = \frac{1}{2} \mu_2^{-1/2} \Delta_1 \left( \frac{\hat{h} - h_0}{h_0} \right)
\]

\[
\Lambda_8(x) = \mu_2^{-3/2} \delta \Delta_1 \left( \frac{\hat{h} - h_0}{h_0} \right)
\]

\[
\Lambda_9(x) = \mu_2^{-3/2} \delta \Delta_1 \left( \frac{\hat{h} - h_0}{h_0} \right)
\]

\[
\Lambda_4(x) = \frac{1}{4} (nh_0)^{-3} \mu_2^{-2} \sum_{i=1}^{n} \sum_{j \neq i} \sum_{k \neq i} \sum_{l \neq i} S_i T_j S_k T_l
\]
\[
= \frac{1}{4}(nh_0)^{-3/2} \mu_{20}^{-1} \left[ \sum_{j=1}^{n} \sum_{j \neq i} \left( S_i T_j + S_j T_i \right) + \sum_{j=1}^{n} \sum_{i \neq j} \left( 4S_i T_j S_j T_i + \frac{S_i^2 T_j^2 T_k + S_j S_i T_i^2}{n} \right) \right] \\
+ \sum_{i=1}^{n} \sum_{j \neq i} \left( 2S_j T_j + 2S_i^2 S_j + S_j^2 T_j^2 \right) \sum_{i=1}^{n} S_j^2 T_i^2 
\]

E.4 Expansion of Characteristic Function

In the following expansion of characteristic function of \( S_{\text{power}}(x) \), we use

\[
\mathbb{E}[S_1 T_1] = h_0 \mu_{20}^{-1/2} \xi_{11}, \\
\mathbb{E}[S_1 V_1] = \mathbb{E}[S_1 \mathcal{X}_t] = h_0 \mu_{20}^{-1/2} \rho_{11}, \\
\mathbb{E}[S_1 S_2 W_{12}] = b^{-2} h_0 \mu_{20}^{-1} \omega_{111}, \\
\mathbb{E}[S_1 \Gamma_2 W_{12}] = b^{-2} h_0 \mu_{20}^{-1/2} \psi_{111}.
\]

Define the characteristic function of \( T_{\text{pr}} \) as \( \chi_{T_{\text{pr}}}(t) \) as follows.

\[
\chi_{T_{\text{pr}}}(t) = \mathbb{E} \left[ \exp \left\{ it \left( S(x) + \sum_{k=1}^{8} \Lambda_k(x) \right) \right\} \right] \\
= \mathbb{E} \left[ e^{itS(x)} \left( 1 + \sum_{k=1}^{8} \frac{it \Lambda_k(x)}{2} + \frac{1}{2} \left( \frac{it \Lambda_4(x)}{2} \right)^2 \right) \right] + o\{(nh_0)^{-1}\} \\
= (I) + (II) + (III) + (IV) + (V) + (VI) + (VII) + (VIII) + (IX) + (X) + o\{(nh_0)^{-1}\}
\]

Define \( \gamma(t) = \frac{\mu}{\sqrt{nh_0}} S_i \). Note that for \( m = 0, 1, 2, 3 \)

\[
\gamma(t)^{n-m} = \exp \left( \frac{-t^2}{2} \right) \left[ 1 + \frac{\mu_{30} \mu_{20}^{-3/2}}{6 \nu_1^2 h_0^{1/2}} (it)^3 + \frac{\mu_{40} \mu_{20}^{-2}}{24 \nu h_0} (it)^4 + \frac{\mu_{50} \mu_{20}^{-3}}{72 \nu h_0} (it)^6 \right] + o\{(nh_0)^{-1}\}.
\]

\[
(I) = \gamma(t)^n = \exp \left( \frac{-t^2}{2} \right) \left[ 1 + \frac{\mu_{30} \mu_{20}^{-3/2}}{6 \nu_1^2 h_0^{1/2}} (it)^3 + \frac{\mu_{40} \mu_{20}^{-2}}{24 \nu h_0} (it)^4 + \frac{\mu_{50} \mu_{20}^{-3}}{72 \nu h_0} (it)^6 \right] + o\{(nh_0)^{-1}\}
\]

Since we have already driven (II), (III) and (IV) in the proof of Theorem 3.5, we quote them.

Next, we expand (IV) and (V), which is including the effect of studentisation.

\[
(V) = \mathbb{E} \left[ e^{itS(x)} \right] \\
= -\frac{1}{2} (nh_0)^{-3/2} \mu_{20}^{-1/2} \mathbb{E} \left[ e^{itS(x)} \left( \sum_{i=1}^{n} \sum_{j \neq i} S_i T_i + \sum_{i=1}^{n} S_i T_i \right) \right] (it) \\
= -\frac{1}{2} n^{1/2} h_0^{3/2} \mu_{20}^{-1} \gamma(t)^{n-2} \mathbb{E} \left[ \left( 1 + \frac{it}{\sqrt{nh_0}} (S_1 + S_2) + \frac{(it)^2}{2 \nu h_0} (S_1 + S_2)^2 + \frac{(it)^3}{6 \sqrt{nh_0}} (S_1 + S_2)^3 \right) S_1 T_1 \right] (it) \\
- \frac{1}{2} n^{1/2} h_0^{-3/2} \mu_{20}^{-1} \gamma(t)^{n-1} \mathbb{E} \left[ \left( 1 + \frac{it}{\sqrt{nh_0}} S_1 \right) S_1 T_1 \right] (it) + o\{(nh)^{-1}\} \\
= -\frac{1}{2} n^{1/2} h_0^{-5/2} \gamma(t)^{n-2} \mu_{20}^{-1} \mathbb{E}[S_1^2] \mathbb{E}[S_1 T_2] (it) - \frac{1}{4} h_0^{-3} \gamma(t)^{n-2} \mu_{20}^{-1} \mathbb{E}[S_1] \mathbb{E}[S_1 T_2] + \mathbb{E}[S_1^2] \mathbb{E}[S_1 T_2] \mathbb{E}[S_1 T_1] (it)^2 + o\{(nh)^{-1}\} \\
- \frac{1}{2} n^{1/2} h_0^{-3/2} \gamma(t)^{n-1} \mu_{20}^{-1} \mathbb{E}[S_1 T_1] (it) - \frac{1}{2} n^{1/2} h_0^{-2} \gamma(t)^{n-1} \mu_{20}^{-1} \mathbb{E}[S_1^2] T_1 (it)^2 + o\{(nh)^{-1}\} \\
- \frac{1}{2} n^{1/2} h_0^{-1/2} \gamma(t)^{n-1} \mu_{20}^{-1} T_1 (it) - \frac{1}{2} n^{1/2} h_0^{-1/2} \gamma(t)^{n-1} \mu_{20}^{-1} T_2 (it) + o\{(nh)^{-1}\} + o\{(nh)^{-1}\}
\]
\[ \begin{align*}
&= \exp \left( -\frac{r^2}{2} \right) \left[ -\frac{1}{2} (nh_0)^{-1/2} \mu_{20}^{-3/2} \mu_{11} \{ (it)^3 + (it) \} - \frac{1}{12} (nh_0)^{-1} \mu_{20}^{-3} \mu_{30} \mu_{11} \{ (it)^6 + (it)^4 \} \\
&\quad - \frac{1}{4} (nh_0)^{-1} \mu_{20}^{-3} \mu_{30} \mu_{11} (it)^4 - \frac{1}{4} (nh_0)^{-1} \mu_{20}^{-2} \mu_{21} \{ (it)^4 + 2(it)^2 \} \right] + o\{(nh_0)^{-1}\} \\
&= \exp \left( -\frac{r^2}{2} \right) \left[ -\frac{1}{2} (nh_0)^{-1/2} \mu_{20}^{-3/2} \mu_{11} \{ (it)^3 + (it) \} - \frac{1}{12} (nh_0)^{-1} \mu_{20}^{-3} \mu_{30} \mu_{11} \{ (it)^6 + 4(it)^4 \} \\
&\quad - \frac{1}{4} (nh_0)^{-1} \mu_{20}^{-2} \mu_{21} \{ (it)^4 + 2(it)^2 \} \right] + o\{(nh_0)^{-1}\} \\
(VI) &= \mathbb{E} \left[ e^{i\mathbf{S}(s)t} \Lambda_5(x) \right] \\
&= n^{-3/2} h_0^{-1/2} \mu_{20}^{-1/2} f(x) \mathbb{E} \left[ e^{i\mathbf{S}(x)} \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} S_i S_j + \sum_{i=1}^{n} S_i^2 \right\} \right] (it) \\
&= n^{-1/2} h_0^{-1/2} \mu_{20}^{-1/2} f(x) \mathbb{E} \left[ \left\{ 1 + \frac{it}{\sqrt{nh_0}} (S_1 + S_2) + \frac{(it)^2}{2nh_0} (S_1 + S_2)^2 \right\} S_1 S_2 \right] (it) \\
&\quad + n^{-1/2} h_0^{-1/2} \mu_{20}^{-1/2} f(x) \gamma(t)^{n-1} \mathbb{E} \left[ S_1^2 \right] (it) + o\{(nh_0)^{-1}\} \\
&= n^{-1/2} h_0^{-3/2} \mathbb{E} \left[ \gamma(t)^{n-2} \mu_{20}^{-1/2} \left\{ (S_1 + S_2 + S_3)^2 \right\} S_1 T_2 S_3 \right] \gamma(t)^{n-1} \mu_{20}^{-1/2} \mathbb{E} \left[ S_1 T_2^2 \right] (it) + o\{(nh_0)^{-1}\} \\
&= \exp \left( -\frac{r^2}{2} \right) \left[ n^{-1/2} h_0^{-3/2} f(x) \mu_{20}^{-1/2} \{ (it)^3 + (it) \} \right] + o\{(nh_0)^{-1}\} \\
(VII) &= \mathbb{E} \left[ e^{i\mathbf{S}(s)t} \Lambda_6(x) \right] \\
&= \frac{3}{8} (nh_0)^{-5/2} \mu_{20}^{-2} \mathbb{E} \left[ e^{i\mathbf{S}(x)} \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k \neq i, j} S_i T_j T_k + \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k \neq i, j} (2S_i T_j T_k + S_i T_j^2) + \sum_{i=1}^{n} S_i T_i^2 \right\} \right] (it) \\
&= \frac{3}{8} n^{-1/2} h_0^{-5/2} \mathbb{E} \left[ \gamma(t)^{n-3} \left\{ 1 + \cdots + \frac{(it)^3}{(nh_0)^{3/2}} (S_1 + S_2 + S_3)^3 \right\} S_1 T_2 S_3 \right] (it) \\
&\quad + \frac{3}{8} n^{-1/2} h_0^{-5/2} \mathbb{E} \left[ \gamma(t)^{n-2} \mu_{20}^{-2} \left\{ 1 + \frac{it}{\sqrt{nh_0}} (S_1 + S_2) \right\} (S_1 T_2 + S_1 T_2^2) \right] (it) + o\{(nh_0)^{-1}\} \\
&= \frac{3}{8} n^{-1} h_0^{-4} \mathbb{E} \left[ \gamma(t)^{n-3} \mu_{20}^{-2} \mathbb{E} \left[ S_1^2 \right] \mathbb{E} \left[ S_1 T_2^2 \right] \right] \gamma(t)^{n-1} \mu_{20}^{-2} \mathbb{E} \left[ (2S_1 T_2 + S_1 T_2^2) \right] (it) + o\{(nh_0)^{-1}\} \\
&= \frac{3}{8} (nh_0)^{-1} \gamma(t)^{n-3} \mu_{20}^{-3} \mu_{11}^2 \mu_{20}^{-1} (it)^4 + \frac{3}{8} (nh_0)^{-1} \gamma(t)^{n-2} \mu_{20}^{-2} \left\{ 2\mu_{20}^{-1} \mu_{11}^2 + \mu_{12} \right\} (it)^2 \\
&= \exp \left( -\frac{r^2}{2} \right) \left[ \frac{3}{8} (nh_0)^{-1} \mu_{20}^{-3} \mu_{11}^2 \{ (it)^4 + 2(it)^2 \} + \mu_{20}^{-3} \mu_{12} \mu_{20}^{-1} \mathbb{E} \left[ S_1 T_2^2 \right] \right] (it)^2 + o\{(nh_0)^{-1}\} \\
(X) &= \mathbb{E} \left[ e^{i\mathbf{S}(x)} \frac{1}{2} (it) \Lambda_4(x) \right] \\
&= \frac{1}{8} (nh_0)^{-3} \mu_{20}^{-2} \mathbb{E} \left[ e^{i\mathbf{S}(x)} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k \neq i, j} S_i T_j S_k T_i + \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k \neq i, j} \{ 4S_i T_j S_k T_i + S_i^2 T_j T_k + S_j T_i S_k T_i \} \\
&\quad + \sum_{i=1}^{n} \sum_{j=1}^{n} \{ 2S_i^2 T_j T_k + 2S_i T_j^2 S_j + S_i^2 T_j^2 + 2S_j T_i S_k T_j \} + \sum_{i=1}^{n} S_i T_i^2 \right] (it) \\
&= \frac{1}{8} (nh_0)^{-1} \gamma(t)^{n-4} \mu_{20}^{-3} \mu_{11}^2 (it)^6 
\end{align*}\]
\begin{align*}
&\text{Finally, we expand (VIII) and (IX), which are including the simultaneous effect of studentisation and bandwidth selection.} \\
&\text{(VIII) } = E\left[ e^{\delta(S)_{ii}}A_{S}(x) \right] \\
&= -\frac{C_p}{2n^{3/2}h^{1/2}} E\left[ e^{\delta(S)_{ii}} \left\{ \sum_{i=1}^{n} \sum_{j \neq i}^{n} S_{ij} + \sum_{i=1}^{n} S_{ii} \right\} (it) \right] \\
&\quad - \frac{C_p}{4n^{3/2}h^{1/2}} E\left[ e^{\delta(S)_{ii}} \left\{ \sum_{i=1}^{n} \sum_{j \neq i}^{n} W_{ij}S_k + \sum_{i=1}^{n} \sum_{j \neq i}^{n} \left\{ W_{ij}S_i + W_{ij}S_j \right\} \right\} (it) \right] \\
&\quad - \frac{C_p(n-1)}{2n^{1/2}h^{1/2}} \gamma(t)^{n-2} E\left[ \left\{ 1 + \cdots + \frac{(it)^2}{2\Delta} (S_1 + S_2)^2 \right\} S_{1V2} \right] (it) - \frac{C_p}{2n^{1/2}h^{1/2}} \gamma(t)^{n-1} E\left[ S_{1V1} \right] (it) \\
&\quad - \frac{C_p(n-2)}{4n^{1/2}h^{1/2}} \gamma(t)^{n-3} E\left[ \left\{ 1 + \cdots + \frac{(it)^3}{3\Delta} (S_1 + S_2 + S_3)^3 \right\} W_{12S3} \right] (it) \\
&\quad - \frac{C_p}{2n^{1/2}h^{1/2}} \gamma(t)^{n-2} E\left[ \left\{ 1 + \frac{it}{\Delta} (S_1 + S_2) \right\} W_{12S1} \right] (it) + o\{nh_0^{-1}\} \\
&\text{(IX) } = E\left[ e^{\delta(S)_{ii}}A_{S}(x) \right] \\
&= -\frac{C_p}{n^{1/2}h^{1/2}} E\left[ e^{\delta(S)_{ii}} \left\{ \sum_{i=1}^{n} \sum_{j \neq i}^{n} S_{ij} + \sum_{i=1}^{n} S_{ii} \right\} (it) \right] \\
&\quad - \frac{C_p}{4n^{1/2}h^{1/2}} E\left[ e^{\delta(S)_{ii}} \left\{ \sum_{i=1}^{n} \sum_{j \neq i}^{n} W_{ij}S_k + \sum_{i=1}^{n} \sum_{j \neq i}^{n} \left\{ W_{ij}S_i + W_{ij}S_j \right\} \right\} \right] (it) \\
&\quad - \frac{C_p(n-1)}{2n^{1/2}h^{1/2}} \gamma(t)^{n-2} E\left[ S_{1V2} \right] (it) - \frac{C_p}{2n^{1/2}h^{1/2}} \gamma(t)^{n-1} E\left[ S_{1V1} \right] (it) \\
&\quad - \frac{C_p(n-2)}{4n^{1/2}h^{1/2}} \gamma(t)^{n-3} E\left[ W_{12S3} \right] (it) - \frac{C_p}{2n^{1/2}h^{1/2}} \gamma(t)^{n-2} E\left[ W_{12S1} \right] (it) + o\{nh_0^{-1}\} \\
&\quad - \frac{C_p}{2n^{1/2}h^{1/2}} \gamma(t)^{n-2} E\left[ S_{1V2} \right] (it)^3 - \frac{C_p}{n^{1/2}h^{1/2}} \gamma(t)^{n-1} E\left[ S_{1V1} \right] (it) \\
&\quad - \frac{C_p}{nh_0^{1/2}} \gamma(t)^{n-2} E\left[ S_{1V2} \right] (it)^4 - \frac{C_p}{nh_0^{1/2}} \gamma(t)^{n-1} E\left[ S_{1V1} \right] (it) \\
&\quad - \frac{C_p}{nh_0^{1/2}} \gamma(t)^{n-1} E\left[ \{it\} (it)^4 + 2(it)^2 \right] + o\{nh_0^{-1}\} \\
&\quad - \frac{C_p}{nh_0^{1/2}} \gamma(t)^{n-1} E\left[ \{it\} (it)^4 + 2(it)^2 \right] + o\{nh_0^{-1}\} \\
\end{align*}
As stated in (Hall, 1992a, p.214), if the non-negative integer $i$, $j$, $k$, $l$ satisfy $i + 2j = k + 2l$ then $\mu_{ij} - \mu_{kl} = O(h) \Rightarrow \mu_{ij} = \mu_{kl} + O(h)$. Using this statements, $\mu_{02} = \mu_{40} + O(h_0), \mu_{21} = \mu_{40} + O(h_0)$ and $\mu_{11} = \mu_{30} + O(h_0)$.

\[
(I) + (V) + (VI) + (VII) + (X)
\]

\[
= \exp \left( -\frac{t^2}{2} \right) \left[ \frac{\mu_{30} - \mu_{20}}{6n^{1/2}h_0^{1/2}} (it)^3 + \frac{\mu_{40} - \mu_{30}}{24n^{1/2}h_0^{1/2}} (it)^4 + \frac{\mu_{50} - \mu_{40}}{72n^{1/2}h_0^{1/2}} (it)^6 \right]
\]

\[
- \frac{1}{2}(nh_0)^{-1/2} \mu_{30}^{-3/2} \mu_{11} \{ (it)^3 + (it) \} - \frac{1}{12} (nh_0)^{-1} \left( \mu_{30}^2 \mu_{30} \mu_{11} \{ (it)^6 + 4(it)^4 \} \right) - \frac{1}{4} (nh_0)^{-1} \mu_{30}^{-2} \mu_{21} \{ (it)^4 + 2(it)^2 \}
\]

\[
+ n^{-1/2} h_0^{-1/2} f(x) \mu_{20}^{-1/2} \{ (it)^3 + (it) \} + \frac{3}{8} (nh_0)^{-1} \left( \mu_{30}^3 \mu_{11} \{ (it)^4 + 2(it)^2 \} + \mu_{20}^{-2} \mu_{02} \{ (it)^4 + (it)^2 \} \right) + o\{(nh_0)^{-1}\}
\]

\[
= \exp \left( -\frac{t^2}{2} \right) \left[ (nh_0)^{-1/2} \left\{ \frac{1}{2} \mu_{20}^{-3/2} \mu_{11} \{ it \} + \mu_{40}^{-3/2} (\mu_{30} - 3\mu_{11}) (it)^3 \right\} + n^{-1/2} h_0^{-1/2} f(x) \mu_{20}^{-1/2} \{ (it)^3 + (it) \} \right]
\]

\[
+ (nh_0)^{-1} \left\{ \left( \frac{1}{2} \mu_{20}^{-2} \mu_{21} + \frac{1}{6} \mu_{40}^{-2} (\mu_{30} - 3\mu_{11}) (it)^3 \right) + n^{-1/2} h_0^{-1/2} f(x) \mu_{20}^{-1/2} \{ (it)^3 + (it) \} \right\}
\]

\[
+ (nh_0)^{-1} \left\{ \left( \frac{1}{24} \mu_{40}^{-2} - \frac{1}{3} \mu_{30}^{-2} \mu_{30} \mu_{11} \mu_{11} - \frac{1}{4} \mu_{20}^{-1} \mu_{21} + \frac{1}{2} \mu_{20}^{-1} \mu_{11} \mu_{11} \right) (it)^4 \right. \]

\[
+ \left. \left( \frac{1}{72} \mu_{20}^{-2} \mu_{20}^{-3} - \frac{1}{12} \mu_{30}^{-2} \mu_{30} \mu_{11} \mu_{11} + \frac{1}{8} \mu_{20}^{-2} \mu_{20}^{-2} \mu_{11} \right) (it)^6 \right\} + o\{(nh_0)^{-1}\}
\]

As stated in (Hall, 1992a, p.214), if the non-negative integer $i$, $j$, $k$, $l$ satisfy $i + 2j = k + 2l$ then $\mu_{ij} - \mu_{kl} = O(h) \Rightarrow \mu_{ij} = \mu_{kl} + O(h)$. Using this statements, $\mu_{02} = \mu_{40} + O(h_0), \mu_{21} = \mu_{40} + O(h_0)$ and $\mu_{11} = \mu_{30} + O(h_0)$.

\[
(I) + (V) + (VI) + (VII) + (X)
\]

\[
= \exp \left( -\frac{t^2}{2} \right) \left[ \frac{\mu_{30} - \mu_{20}}{6n^{1/2}h_0^{1/2}} (it)^3 + \frac{\mu_{40} - \mu_{30}}{24n^{1/2}h_0^{1/2}} (it)^4 + \frac{\mu_{50} - \mu_{40}}{72n^{1/2}h_0^{1/2}} (it)^6 \right]
\]

\[
- \frac{1}{2}(nh_0)^{-1/2} \mu_{30}^{-3/2} \mu_{11} \{ (it)^3 + (it) \} - \frac{1}{12} (nh_0)^{-1} \left( \mu_{30}^3 \mu_{30} \mu_{11} \{ (it)^6 + 4(it)^4 \} \right) - \frac{1}{4} (nh_0)^{-1} \mu_{30}^{-2} \mu_{21} \{ (it)^4 + 2(it)^2 \}
\]

\[
+ n^{-1/2} h_0^{-1/2} f(x) \mu_{20}^{-1/2} \{ (it)^3 + (it) \} + \frac{3}{8} (nh_0)^{-1} \left( \mu_{30}^3 \mu_{11} \{ (it)^4 + 2(it)^2 \} + \mu_{20}^{-2} \mu_{02} \{ (it)^4 + (it)^2 \} \right) + o\{(nh_0)^{-1}\}
\]

\[
= \exp \left( -\frac{t^2}{2} \right) \left[ (nh_0)^{-1/2} \left\{ \frac{1}{2} \mu_{20}^{-3/2} \mu_{11} \{ it \} + \mu_{40}^{-3/2} (\mu_{30} - 3\mu_{11}) (it)^3 \right\} + n^{-1/2} h_0^{-1/2} f(x) \mu_{20}^{-1/2} \{ (it)^3 + (it) \} \right]
\]

\[
+ (nh_0)^{-1} \left\{ \left( \frac{1}{2} \mu_{20}^{-2} \mu_{21} + \frac{1}{6} \mu_{40}^{-2} (\mu_{30} - 3\mu_{11}) (it)^3 \right) + n^{-1/2} h_0^{-1/2} f(x) \mu_{20}^{-1/2} \{ (it)^3 + (it) \} \right\}
\]

\[
+ (nh_0)^{-1} \left\{ \left( \frac{1}{24} \mu_{40}^{-2} - \frac{1}{3} \mu_{30}^{-2} \mu_{30} \mu_{11} \mu_{11} - \frac{1}{4} \mu_{20}^{-1} \mu_{21} + \frac{1}{2} \mu_{20}^{-1} \mu_{11} \mu_{11} \right) (it)^4 \right. \]

\[
+ \left. \left( \frac{1}{72} \mu_{20}^{-2} \mu_{20}^{-3} - \frac{1}{12} \mu_{30}^{-2} \mu_{30} \mu_{11} \mu_{11} + \frac{1}{8} \mu_{20}^{-2} \mu_{20}^{-2} \mu_{11} \right) (it)^6 \right\} + o\{(nh_0)^{-1}\}
\]

This is consistent with the expansion in Hall (1991, 1992a).

E.4.2 Simplifying Terms in the Expansion which Includes the Effect of Global Plug-In Method

\[
(II) + (III) + (IV)
\]

\[
= \exp \left( -\frac{t^2}{2} \right) \left[ C_{\rho_1} \mu_{20}^{-1} \rho_{11} \left( \sum_{l=0}^{L-1} C_{\Gamma,l}(x) h_0^{l+1} \right) (it)^3 \right]
\]

\[
+ C_{\rho_1} \frac{n^{-1/2} h_0}{h_0^{1/2}} \left( \frac{b}{\omega_{11}} \sum_{l=0}^{L-1} C_{\Gamma,l}(x) h_0^{l} \right) (it)^3
\]
\[
+ C_{pn}^{-1/2} h_0^{1/2} \left( \frac{1}{2} \mu_{20}^{-3/2} \rho_{11} \xi_{11}(i) \right) \left( \frac{1}{2} \mu_{20}^{-1/2} \rho_{11} \xi_{11}(i) \right) \\
+ C_{pn}^{-1} b^{-2L} \left( \frac{1}{2} \mu_{20}^{-2} \xi_{11}(i)^4 + \mu_{20}^{-1} \psi_{11}(i)^2 - \frac{1}{4} \mu_{20}^{-1} \omega_{111} \{ (i)^4 + 2(i)^2 \} \right) + o\{ (nh_0)^{-1} \} \\
= \exp \left( -\frac{t^2}{2} \right) \left[ \sum_{i=0}^{L-1} C_{\Gamma,1}(x) h_0^L x^{-1} \right] (i)^3 \\
+ C_{pn}^{-1/2} h_0^{-1/2} \mu_{20}^{-1/2} \rho_{11} \xi_{11}(i)^3 - \frac{C_{pn}}{4n b^2 \pi} \mu_{20}^{-1} \omega_{111} \{ (i)^4 + 2(i)^2 \} \\
+ C_{pn}^{-1/2} h_0^{-1/2} \rho_{11} \xi_{11}(i)^3 - \frac{C_{pn}}{4n b^2 \pi} \mu_{20}^{-2} \omega_{111} \{ (i)^4 + 2(i)^2 \} + o\{ (nh_0)^{-1} \} \\
= \exp \left( -\frac{t^2}{2} \right) \left[ \sum_{i=0}^{L-1} C_{\Gamma,1}(x) h_0^L x^{-1} \right] (i)^3 \\
+ C_{pn}^{-1/2} h_0^{-1/2} \mu_{20}^{-1/2} \rho_{11} \xi_{11}(i)^3 - \frac{C_{pn}}{4n b^2 \pi} \mu_{20}^{-1} \omega_{111} \{ (i)^4 + 2(i)^2 \} \\
+ C_{pn}^{-1/2} h_0^{-1/2} \rho_{11} \xi_{11}(i)^3 - \frac{C_{pn}}{4n b^2 \pi} \mu_{20}^{-2} \omega_{111} \{ (i)^4 + 2(i)^2 \} + o\{ (nh_0)^{-1} \} \\
\]

E.4.3 Simplifying Terms in the Expansion which Includes the Simultaneous Effect of Plug-In Method and Studentisation

(VIII) + (IX)

Then inverting \( \chi_{\Gamma,1}(t) \), we have Theorem 3.6.

F Results of Additional Monte Carlo Studies

F.1 Simulation Results with \( \hat{r}^{\text{conv}}_L \)

F.1.1 Standard Normal

We adopt \#1 : \( N(0,1) \) in Marron and Wand (1992). For sample size \( n = (50, 100, 400, 1000) \), MSE optimal pilot bandwidths are \( b_0 = (0.8596, 0.8047, 0.7052, 0.6462) \). We evaluate the accuracy at the point of \( x = 0, 0.5, 1, 1.5 \) and \( x = 2 \). Simulation results are as follows.

|          | \( n = 50 \) | \( n = 100 \) | \( n = 400 \) | \( n = 1000 \) |
|----------|--------------|--------------|--------------|--------------|
|          | CP | Ave.Length | CP | Ave.Length | CP | Ave.Length | CP | Ave.Length |
| N(0,1)   | 0.7730 | 0.1339 | 0.6930 | 0.1067 | 0.5950 | 0.0672 | 0.5575 | 0.0490 |
| Hall (1991) | 0.7670 | 0.1341 | 0.6885 | 0.1068 | 0.5525 | 0.0672 | 0.5525 | 0.0490 |
| Theorem 3.1 | 0.9140** | 0.1868 | 0.8130** | 0.1343 | 0.6630** | 0.0746 | 0.6020** | 0.0521 |
| Theorem 3.5 | 0.9090* | 0.2035 | 0.8055* | 0.1421 | 0.6575* | 0.0763 | 0.5990* | 0.0526 |
F.1.2 Skewed Unimodal

We adopt #2 : \( \frac{1}{\sqrt{2\pi}} N(0,1) + \frac{1}{\sqrt{2\pi}} N\left(\frac{1}{\sqrt{3}}, \frac{2}{3}\right) + \frac{1}{\sqrt{2\pi}} N\left(\frac{1}{\sqrt{6}}, \frac{2}{3}\right) \) in Marron and Wand (1992). For sample size \( n = (50, 100, 400, 1000) \), MSE optimal pilot bandwidths are \( b_0 = (0.5318, 0.4978, 0.4363, 0.3998) \). We evaluate the accuracy at the point of \( x = -2, -1.5, -1, -0.5, 0, 0.5, 1, 1.5 \) and \( x = 2 \). Simulation results are as follows.

Table 20: \( x = -2, b = \text{MSE optimal}, \) scaled second derivative = 0.0173

|     | \( n = 50 \) | \( n = 100 \) | \( n = 400 \) | \( n = 1000 \) |
|-----|-------------|-------------|--------------|--------------|
| CP  | Ave.Length  | CP          | Ave.Length   | CP           | Ave.Length   |
| N(0.1) | 0.9140       | 0.0446      | 0.9220       | 0.0328       | 0.9300       | 0.0182       | 0.9300       | 0.0125       |
| Hall (1991) | 0.9510**     | 0.0417      | 0.9455**     | 0.0315       | 0.9470**     | 0.0180       | 0.9440**     | 0.0124       |
| Theorem 3.1 | 0.9520*      | 0.0418      | 0.9455**     | 0.0315       | 0.9470**     | 0.0180       | 0.9440**     | 0.0124       |
| Theorem 3.5 | 0.9525       | 0.0418      | 0.9455**     | 0.0315       | 0.9470**     | 0.0180       | 0.9440**     | 0.0124       |
### Table 21: $x = -1.5, b = \text{MSE optimal}$, scaled second derivative$= 0.0278$

|         | $n = 50$ | $n = 100$ | $n = 400$ | $n = 1000$ |
|---------|----------|-----------|-----------|------------|
| CP      | Ave.Length | CP | Ave.Length | CP | Ave.Length | CP | Ave.Length |
| $N(0, 1)$ | 0.9200 | 0.0688 | 0.9180 | 0.0508 | 0.9390 | 0.0284 | 0.9425 | 0.0196 |
| Hall (1991) | 0.9280 | 0.0670 | 0.9330 | 0.0500 | 0.9460 | 0.0283 | 0.9540** | 0.0195 |
| Theorem 3.1 | 0.9285* | 0.0672 | 0.9345* | 0.0283 | 0.9465** | 0.0283 | 0.9545* | 0.0196 |
| Theorem 3.5 | 0.9310** | 0.0674 | 0.9355** | 0.0502 | 0.9465** | 0.0283 | 0.9545* | 0.0196 |

### Table 22: $x = -1, b = \text{MSE optimal}$, scaled second derivative$= 0.0503$

|         | $n = 50$ | $n = 100$ | $n = 400$ | $n = 1000$ |
|---------|----------|-----------|-----------|------------|
| CP      | Ave.Length | CP | Ave.Length | CP | Ave.Length | CP | Ave.Length |
| $N(0, 1)$ | 0.9050 | 0.0727 | 0.8965 | 0.0727 | 0.9170 | 0.0410 | 0.9385 | 0.0283 |
| Hall (1991) | 0.9300 | 0.0970 | 0.9185 | 0.0723 | 0.9285** | 0.0409 | 0.9470* | 0.0283 |
| Theorem 3.1 | 0.9345* | 0.0975 | 0.9195* | 0.0725 | 0.9285** | 0.0409 | 0.9470* | 0.0283 |
| Theorem 3.5 | 0.9360** | 0.0982 | 0.9205** | 0.0728 | 0.9285** | 0.0410 | 0.9475** | 0.0284 |

### Table 23: $x = -0.5, b = \text{MSE optimal}$, scaled second derivative$= 0.1112$

|         | $n = 50$ | $n = 100$ | $n = 400$ | $n = 1000$ |
|---------|----------|-----------|-----------|------------|
| CP      | Ave.Length | CP | Ave.Length | CP | Ave.Length | CP | Ave.Length |
| $N(0, 1)$ | 0.9090 | 0.1330 | 0.8850 | 0.0993 | 0.8900 | 0.0566 | 0.8995 | 0.0393 |
| Hall (1991) | 0.9205 | 0.1325 | 0.9020 | 0.0565 | 0.9030 | 0.0565 | 0.9065 | 0.0393 |
| Theorem 3.1 | 0.9280* | 0.1353 | 0.9075* | 0.1002 | 0.9050** | 0.0567 | 0.9070** | 0.0393 |
| Theorem 3.5 | 0.9335** | 0.1375 | 0.9095** | 0.1013 | 0.9050** | 0.0570 | 0.9070** | 0.0394 |

### Table 24: $x = 0, b = \text{MSE optimal}$, scaled second derivative$= 0.2029$

|         | $n = 50$ | $n = 100$ | $n = 400$ | $n = 1000$ |
|---------|----------|-----------|-----------|------------|
| CP      | Ave.Length | CP | Ave.Length | CP | Ave.Length | CP | Ave.Length |
| $N(0, 1)$ | 0.9620** | 0.1709 | 0.9580** | 0.1299 | 0.9315 | 0.0759 | 0.9205 | 0.0533 |
| Hall (1991) | 0.9660* | 0.1709 | 0.9635* | 0.1298 | 0.9365 | 0.0758 | 0.9205 | 0.0533 |
| Theorem 3.1 | 0.9820 | 0.1868 | 0.9735 | 0.1378 | 0.9435* | 0.0777 | 0.9330** | 0.0540 |
| Theorem 3.5 | 0.9865 | 0.1955 | 0.9760 | 0.1416 | 0.9440** | 0.0785 | 0.9330** | 0.0543 |

### Table 25: $x = 0.5, b = \text{MSE optimal}$, scaled second derivative$= 0.1170$

|         | $n = 50$ | $n = 100$ | $n = 400$ | $n = 1000$ |
|---------|----------|-----------|-----------|------------|
| CP      | Ave.Length | CP | Ave.Length | CP | Ave.Length | CP | Ave.Length |
| $N(0, 1)$ | 0.9405** | 0.1963 | 0.9290* | 0.1532 | 0.9445** | 0.0933 | 0.9555 | 0.0669 |
| Hall (1991) | 0.9370 | 0.1964 | 0.9230 | 0.1532 | 0.9395 | 0.0933 | 0.9500** | 0.0669 |
| Theorem 3.1 | 0.9350 | 0.1936 | 0.9220 | 0.1511 | 0.9380 | 0.0924 | 0.9485 | 0.0665 |
| Theorem 3.5 | 0.9605* | 0.2176 | 0.9380** | 0.1621 | 0.9435* | 0.0947 | 0.9505* | 0.0673 |

### Table 26: $x = 1, b = \text{MSE optimal}$, scaled second derivative$= 0.7019$

|         | $n = 50$ | $n = 100$ | $n = 400$ | $n = 1000$ |
|---------|----------|-----------|-----------|------------|
| CP      | Ave.Length | CP | Ave.Length | CP | Ave.Length | CP | Ave.Length |
| $N(0, 1)$ | 0.7605 | 0.2022 | 0.6760 | 0.1595 | 0.5520 | 0.0992 | 0.5275 | 0.0720 |
| Hall (1991) | 0.7550 | 0.2024 | 0.6680 | 0.1596 | 0.5455 | 0.0992 | 0.5220 | 0.0720 |
| Theorem 3.1 | 0.9390** | 0.3080 | 0.8370** | 0.2164 | 0.6440** | 0.1152 | 0.5860** | 0.0788 |
| Theorem 3.5 | 0.9355* | 0.3378 | 0.8270* | 0.2308 | 0.6395* | 0.1184 | 0.5810* | 0.0799 |
Table 27: $x = 1.5, b = $MSE optimal $ $, scaled second derivative $= 0.1559$

|       | $n = 50$   |       | $n = 100$  |       | $n = 400$  |       | $n = 1000$ |       |
|-------|-----------|--|-----------|---|-----------|---|-----------|---|
|       | CP Ave.Length | CP Ave.Length | CP Ave.Length | CP Ave.Length | CP Ave.Length | CP Ave.Length | CP Ave.Length |
| $N(0,1)$ | 0.8965* | 0.1944 | 0.8855* | 0.1503 | 0.8810** | 0.0903 | 0.8930** | 0.0644 |
| Hall (1991) | 0.8875 | 0.1945 | 0.8805 | 0.1503 | 0.8735 | 0.0903 | 0.8880 | 0.0644 |
| Theorem 3.1 | 0.8965* | 0.1990 | 0.8830 | 0.1518 | 0.8735 | 0.0903 | 0.8875 | 0.0643 |
| Theorem 3.5 | 0.9245** | 0.2170 | 0.8960** | 0.1601 | 0.8795* | 0.0920 | 0.8895* | 0.0649 |

Table 28: $x = 2, b = $MSE optimal $ $, scaled second derivative $= 0.3591$

|       | $n = 50$   |       | $n = 100$  |       | $n = 400$  |       | $n = 1000$ |       |
|-------|-----------|--|-----------|---|-----------|---|-----------|---|
|       | CP Ave.Length | CP Ave.Length | CP Ave.Length | CP Ave.Length | CP Ave.Length | CP Ave.Length | CP Ave.Length |
| $N(0,1)$ | 0.8600 | 0.1419 | 0.7995 | 0.1060 | 0.6765 | 0.0603 | 0.6145 | 0.0418 |
| Hall (1991) | 0.8730 | 0.1415 | 0.8195 | 0.1058 | 0.6965 | 0.0602 | 0.6275 | 0.0418 |
| Theorem 3.1 | 0.9190** | 0.1681 | 0.8670** | 0.1202 | 0.7365** | 0.0641 | 0.6505** | 0.0433 |
| Theorem 3.5 | 0.9130* | 0.1710 | 0.8630* | 0.1214 | 0.7335* | 0.0643 | 0.6505** | 0.0434 |

F.2 Distributions of $\hat{I}_L$

In this section, we provide the distributions of $\hat{I}_L$ estimated with $\hat{h}$ and $h_{conv}$ respectively for sample size $n = (50, 100, 400, 1000, 5000)$. Whereas for the simulation to see coverage probabilities, simulation with $n = 5000$ is not carried out, we show the result of estimation of $I_L$ at the expense of reducing the iteration from 2000 to 300, in order to confirm the consistency of $\hat{I}_L$ to $I_L$. On the following figures, blue histograms represent the distribution of $\hat{I}_L$ and red lines $I_L$.

F.2.1 Standard Normal

Figure 1: Distributions of $\hat{I}_L$ estimated with $\hat{h}$

Figure 2: Distributions of $\hat{I}_L$ estimated with $h_{conv}$
F.2.2 Skewed Unimodal

Figure 3: Distributions of $\hat{I}_L$ estimated with $\hat{h}$

Figure 4: Distributions of $\hat{I}_L$ estimated with $h_{\text{convo}}$

F.3 Distributions of KDEs at relatively high curvature points

On the following figures, the blue histograms represent the distributions of KDEs with plug-in bandwidth ($\hat{f}_h(x)$ or $\hat{f}_{h_{\text{convo}}}(x)$), the orange histograms KDE with optimal bandwidth $\hat{f}_{h_0}(x)$, the red lines $E[\hat{f}_{h_0}(x)]$ and the black lines $f(x)$.

F.3.1 Standard Normal at $x = 0$

Figure 5: Distributions of $\hat{f}_h$ and $\hat{f}_{h_0}$

Figure 6: Distributions of $\hat{f}_{h_{\text{convo}}}$ and $\hat{f}_{h_0}$
F.3.2 Standard Normal at $x = 1.5$

Figure 7: Distributions of $\hat{f}_h$ and $\hat{f}_{h_0}$

F.3.3 Skewed Unimodal at $x = 1$

Figure 8: Distributions of $\hat{f}_{h_{\text{conv}}}$ and $\hat{f}_{h_0}$

Figure 9: Distributions of $\hat{f}_h$ and $\hat{f}_{h_0}$

Figure 10: Distributions of $\hat{f}_{h_{\text{conv}}}$ and $\hat{f}_{h_0}$
References

Armstrong, T. B. and Kolesár, M. (2018). Optimal inference in a class of regression models. *Econometrica*, 86:655–683.

Brockmann, M., Gasser, T., and Herrmann, E. (1993). Locally adaptive bandwidth choice for kernel regression estimators. *Journal of American Statistical Association*, 88:1302–1309.

Callaert, H., Janssen, P., and d Veraverbeke, N. (1980). An edgeworth expansion for u-statistics. *Annals of Statistics*, 8:299–312.

Calonico, S., Cattaneo, M. D., and Farrell, M. H. (2018). On the effect of bias estimation on coverage accuracy in nonparametric inference. *Journal of American Statistical Association*, 113:767–779.

Calonico, S., Cattaneo, M. D., and Farrell, M. H. (2020). Optimal bandwidth choice for robust bias-corrected inference in regression discontinuity designs. *Econometrics Journal*, 23:192–210.

Calonico, S., Cattaneo, M. D., and Farrell, M. H. (2022). Coverage error optimal confidence intervals for local polynomial regression. *Bernoulli*, forthcoming.

Calonico, S., Cattaneo, M. D., and Titunik, R. (2014). Robust nonparametric confidence intervals for regression-discontinuity designs. *Econometrica*, 82:2295–2326.

Cattaneo, M. D., Crump, R., and Jansson, M. (2010). Robust data-driven inference for density-weighted average derivatives. *Journal of the American Statistical Association*, 105:1070–1083.

Cattaneo, M. D., Crump, R., and Jansson, M. (2013). Generalized jackknife estimators of weighted average derivatives. *Journal of the American Statistical Association*, 108:1243–1268.

Cattaneo, M. D., Crump, R., and Jansson, M. (2014a). Bootstrapping density-weighted average derivatives. *Econometric Theory*, 30:1135–1164.

Cattaneo, M. D., Crump, R., and Jansson, M. (2014b). Small bandwidth asymptotics for density-weighted average derivatives. *Econometric Theory*, 30:176–200.

Cattaneo, M. D. and Jansson, M. (2018). Kernel-based semiparametric estimators: Small bandwidth asymptotics and bootstrap consistency. *Econometrica*, 86:955–995.

Cattaneo, M. D., Jansson, M., and Ma, X. (2020). Simple local polynomial density estimators. *Journal of American Statistical Association*, 115:1449–1455.

Chen, X. (2007). Large sample sieve estimation of semi-nonparametric models. *Handbook of Econometrics*, vol. 6B(Chapter 76):5549–5632.

DiNardo, J., Fortin, N., and Lemieux, T. (1996). Labor market institutions and the distribution of wages, 1973–1992: A semiparametric approach. *Econometrica*, 64:1001–1044.

Esséen, C. G. (1945). Fourier analysis of distribution functions : A mathematical study of the laplace-gaussian law. *Acta Mathematica*, 77:1–125.

Fan, J., Hall, P., and Patil, P. (1996). On local smoothing of nonparametric curve estimators. *Journal of American Statistical Association*, 91:258–266.

Feller, W. (1971). *An Introduction to the Probability Theory and Its Applications*, volume II. John Wiley & Sons.

Giné, E. and Nickl, R. (2016). *Mathematical Foundation of Infinite-Dimensional Statistical Model*. Cambridge University Press.

Hall, P. (1991). Edgeworth expansions for nonparametric density estimators, with applications. *Statistics*, 22:215–232.

Hall, P. (1992a). *The bootstrap and Edgeworth Expansion*. Springer Verlag.

Hall, P. (1992b). Effect of bias estimation on coverage accuracy of bootstrap confidence intervals for a probability density. *Annals of Statistics*, 20:675–694.
Hall, P. and Horowitz, J. L. (2013). A simple bootstrap method for constructing nonparametric confidence bands for function. *Annals of Statistics*, 41:1892–1921.

Hall, P. and Kang, K. H. (2001). Bootstrapping nonparametric density estimators with empirically chosen bandwidth. *Annals of Statistics*, 29:1443–1468.

Hall, P., Marron, J., and Titterington, D. (1995). On partial local smoothing rules for curve estimation. *Biometrika*, 82:575–587.

Hall, P. and Marron, J. S. (1987). Estimation of integrated squared density derivatives. *Statistics and Probability Letters*, 6:109–115.

Hall, P., Sheather, S. J., Jones, M. C., and Marron, J. (1991). On optimal data-based bandwidth selection in kernel density estimation. *Biometrika*, 78:263–270.

Hastie, T., Tibshirani, R., and Friedman, J. (2009). *Elements of Statistical Learning Data Mining, Inference, and Prediction Second Edition*. Springer New York, NY.

Ichimura, H. (2000). Asymptotic distribution of nonparametric and semiparametric estimators with data dependent smoothing parameters. *Unpublished Manuscript*.

Ichimura, H. and Todd, P. E. (2007). Implementing nonparametric and semiparametric estimators. *Handbook of Econometrics*, vol. 6B(Chapter 74):5369–5468.

Li, D. and Li, Q. (2010). Nonparametric/semiparametric estimation and testing of econometric models with data dependent smoothing parameters. *Journal of Econometrics*, 157:179–190.

Li, Q. and Racine, J. (2007). *Nonparametric Econometrics: Theory and Practice*. Princeton University Press.

Marron, J. S. and Wand, M. P. (1992). Exact mean integrated squared error. *Annals of Statistics*, 20:712–736.

Nishiyama, Y. and Robinson, P. M. (2000). Edgeworth expansions for semiparametric averaged derivatives. *Econometrica*, 68:931–979.

Parzen, E. (1962). On estimation of a probability density function and mode. *Annals of Mathematical Statistics*, 33:1065–1076.

Rosenblatt, M. (1956). Remarks on some nonparametric estimates of a density function. *Annals of Mathematical Statistics*, 27:832–837.

Schennach, S. M. (2020). A bias bound approach to nonparametric inference. *The Review of Economic Studies*, 87:2439–2472.

Sheather, S. J. and Jones, M. C. (1991). A reliable data-based bandwidth selection method for kernel density estimation. *Journal of Royal Statistical Society Series B*, 53:683–690.

Silverman, B. W. (1986). *Density Estimation for Statistics and Data Analysis*. CRC press.

Wasserman, L. (2006). *All of nonparametric statistics*. Springer Science & Business Media.