AN EISENSTEIN SERIES OF RATIONAL WEIGHT

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Abstract. We study an Eisenstein series of weight $20/9$, level 11 whose multiplier system is the same as $\eta^{44/9}(\tau)\eta^{-4/9}(11\tau)$. We prove that it is a modular form and obtain its Fourier expansion.

1. Introduction

In this note we consider, among more general functions, the following Eisenstein series

$$E_{20/9}(\tau) = 1 + \epsilon \left( \frac{5}{9} \right) \sum_{11|c>0, \; d \in \mathbb{Z} \; \gcd(c,d)=1} \epsilon \left( -\frac{22}{9} s(-d,c) + \frac{2}{9} s(-d,c/11) \right) (c\tau + d)^{-20/9},$$

where $\epsilon(z) = \exp 2\pi i z$ and $s(h,k)$ is the Dedekind sum defined by the formula

$$s(h,k) = \sum_{r=0}^{k-1} h r \left( \frac{k r}{k} \right) - \frac{1}{2}, \quad k \in \mathbb{Z}_{\geq 1}, \; h \in \mathbb{Z}, \; \gcd(h,k) = 1.$$

We shall prove that, this is a modular form of weight $20/9$, on the group $\Gamma_0(11)$, and its multiplier system is the same as the eta quotient $\eta^{44/9}(\tau)\eta^{-4/9}(11\tau)$, where $\Gamma_0(N)$ is the congruence subgroup of $\text{SL}_2(\mathbb{Z})$ consisting of matrices $\left( \begin{array}{cc} a & b \\ c & d \end{array} \right)$ with $N | c$, and $\eta(\tau)$ is the Dedekind eta function defined by

$$\eta(\tau) = q^{1/24} \prod_{n \in \mathbb{Z}_{\geq 1}} (1 - q^n), \quad q = \epsilon(\tau).$$

See Theorem 5.1 for details. The meaning of all rational powers occurring, such as $(c\tau + d)^{-20/9}$ and $\eta^{44/9}(\tau)$, will be explained in next sections. The variable $\tau$ is always assumed to take values from the upper half plane $\mathfrak{H} = \{ z \in \mathbb{C} : \Im(z) > 0 \}$.

After showing that it is a modular form, we calculate its Fourier expansion. The result (Theorem 5.2) is

$$E_{20/9}(\tau) = 1 + \frac{(2\pi)^{20/9}}{\Gamma(20/9)\zeta(20/9)(11^{20/9} - 1)} \sum_{n \in \mathbb{Z}_{\geq 1}} \sum_{c \in \mathbb{Z}_{\geq 1}} n^{11/9} c^{-20/9} g(c, n) q^n,$$

where $g(c, n)$ is a certain function.

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where $\Gamma(z)$ and $\zeta(z)$ are Euler Gamma function and Riemann zeta function respectively, and

$$g(c, n) = \sum_{\substack{d = 0 \\ \gcd(11, d) = 1}}^{11c - 1} \epsilon \left( \frac{nd}{11c} \right) \epsilon \left( -\frac{2}{9} (11s(-d, 11c) - s(-d, c)) \right).$$

In the process of discussing above results, we present and prove some basic facts concerned with modular forms, eta-quotients and general Eisenstein series of rational weights.

2. Modular forms of rational weights

As one may use the double cover of the modular group when dealing with modular forms of half-integral weights, it is necessary to use a multiple cover of the modular group for rational weights. Let $D$ be a positive integer. Then the $D$-cover of $SL_2(\mathbb{R})$ is defined as

$$SL_2(\mathbb{R})^D = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right), \varepsilon (c\tau + d)^\frac{k}{2} : \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL_2(\mathbb{R}), \varepsilon^D = 1 \right\},$$

where the fractional power is the principal branch, that is, $\varepsilon^\frac{m}{n} = \exp(\frac{m}{n} \log z)$ with $-\pi < \Im(\log z) \leq \pi$. The notation $\left( \begin{array}{cc} a & b \\ c & d \end{array} \right), \varepsilon$ stands for $\left( \begin{array}{cc} a & b \\ c & d \end{array} \right), \varepsilon (c\tau + d)^\frac{k}{2}$ for simplicity. The group composition of $SL_2(\mathbb{R})^D$ is given by the following formula:

$$\left( \begin{array}{cc} a_1 & b_1 \\ c_1 & d_1 \end{array} \right), \varepsilon_1 \left( \begin{array}{cc} a_2 & b_2 \\ c_2 & d_2 \end{array} \right), \varepsilon_2 = \left( \begin{array}{cc} a_1a_2 & a_1b_2 + b_1a_2 \\ c_1d_2 & c_1d_2 + d_1c_2 \end{array} \right), \varepsilon_1\varepsilon_2\delta,$$

where $\delta$ is a $D$-th root of unity determined by

$$\delta = \sqrt[2D]{c_1(a_2\tau + b_2)/(c_2\tau + d_2) + d_1\sqrt[c_1(a_2\tau + b_2) + d_1(c_2\tau + d_2)]{2D}}.$$

For a subgroup $G$, the notation $\bar{G}$ means the preimage of $G$ under the natural projection $SL_2(\mathbb{R})^D \rightarrow SL_2(\mathbb{R})$.

Fix a positive integer $D$. Let $k \in \frac{1}{D}\mathbb{Z}$, $G$ be a finite index subgroup of $SL_2(\mathbb{Z})$, and $\chi : \bar{G} \rightarrow \mathbb{C}^\times$ be a finite index character. The two prefixes ”finite index” mean that $[SL_2(\mathbb{Z}) : G] < +\infty$ and $[\bar{G} : \ker\chi] < +\infty$. We define the slash operator of weight $k$ as

$$f|_k \left( \begin{array}{cc} a & b \\ c & d \end{array} \right), \varepsilon \left( \tau \right) = \varepsilon^{-Dk}(c\tau + d)^{-k}f \left( \frac{a\tau + b}{c\tau + d} \right)$$

for a continuous function $f : \mathcal{H} \rightarrow \mathbb{C}$ and $\left( \begin{array}{cc} a & b \\ c & d \end{array} \right), \varepsilon \in SL_2(\mathbb{R})^D$. It is a right group action of $SL_2(\mathbb{R})^D$ on continuous functions (or holomorphic, or meromorphic functions). By a modular form of weight $k$, on the group $\bar{G}$, and of character $\chi$, we mean a holomorphic function $f : \mathcal{H} \rightarrow \mathbb{C}$ satisfying the following properties:

\begin{enumerate}
  \item $f|_k \gamma = \chi(\gamma)f$ for any $\gamma \in \bar{G}$,
  \item When $c = 0$ and $d = -1$, one notation may be confused with the other which must be treated with caution.
  \item If we use the group $SL_2(\mathbb{R})$, instead of its $D$-cover, then $\bar{G}$ without the factor $\varepsilon^{-Dk}$ does not give a group action. This is one advantage of considering $D$-covers of modular groups. Another advantage is that the multiplier system is really a group character.
\end{enumerate}
(2) \( \lim_{s \to +\infty} f|_{k}\gamma \) exists for any \( \gamma \in \SL_2(\mathbb{Z})^D \).

The character \( \chi \) is called the multiplier system of \( f \). The complex vector space of all modular forms is denoted by \( M_k(\Gamma^D; \chi) \). The reader may compare our definition with Ibukiyama’s ([Ib00] and [Ib02]). Ibukiyama constructs modular forms of rational weights using quotients of certain theta series and fractional power of the eta function. However, we think it is necessary and extremely convenient to use \( D \)-covers of modular groups when dealing with formal Eisenstein series of rational weights, as we shall do next.

### 3. Eisenstein series of rational weights

First some notations. If a group \( G \) acts from the left (right resp.) on the set \( X \), then we use \( G \backslash X \) (\( X/G \) resp.) to denote the set of orbits. If the subgroup \( H \) acts on \( G \), then the action is always understood as translations. For an element \( g \in G \), \( \langle g \rangle \) denotes the subgroup generated by \( g \). The group considered in this section is \( \Gamma_0(N)^D \) with \( N, D \) positive integers. The symbol \( \mathbb{P}^1(\mathbb{Q}) \) denotes \( \mathbb{Q} \) together with the infinity point \( i\infty \) as a subset of the Riemann sphere. Notice \( a/0 = \infty \) for nonzero integer \( a \). The group \( \Gamma_0(N) \) and so \( \Gamma_0(N)^D \) act (from the left) on \( \mathbb{P}^1(\mathbb{Q}) \) as \( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) s = \frac{as + b}{cs + d} \). For \( s \in \mathbb{P}^1(\mathbb{Q}) \), \( \overline{s} \) denotes its image in \( \mathbb{P}^1(\mathbb{Q}) \backslash \mathbb{P}^1(\mathbb{Q}) \), and is called a cusp. The width of \( \overline{s} \), denoted by \( w_s \), is defined as \( |\PSL_2(\mathbb{Z}) : \Gamma_0(N)\overline{s}| \), where the notation \( G_s \) means the isotropy group, and \( \Gamma_0(N) \) is the quotient of \( \Gamma_0(N) \) by \( \{ \pm I \} \). If \( \gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \SL_2(\mathbb{R}) \), then the notation \( \overline{\gamma} \) denotes \( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \) in \( \SL_2(\mathbb{R})^D \).

Put \( T = \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right), S = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \) and \( I = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \).

**Definition 3.1.** Let \( D, N \) be positive integers, \( \chi : \Gamma_0(N)^D \to \mathbb{C}^\times \) be a finite index character such that \( \chi(-I) = \epsilon(-k/2) \). Let \( k \in \frac{1}{D} \mathbb{Z} \) which is greater than 2. Let \( s \in \mathbb{P}^1(\mathbb{Q}) \) and \( \gamma_s \in \SL_2(\mathbb{Z}) \) such that \( \gamma_s(i\infty) = s \). If \( \chi(\gamma_s T^u w_s \overline{\gamma_s}^{-1}) = 1 \), then we define the Eisenstein series \( E_{\gamma_s, k} \) on the group \( \Gamma_0(N)^D \), of weight \( k \), with character \( \chi \) and at cusp \( \gamma_s \) as

\[
E_{\gamma_s, k}(\tau) = \sum_{\gamma \in \gamma_s(\mathbb{P}^1) \gamma_s^{-1} \backslash \Gamma_0(N)^D} \chi(\gamma)^{-1} \cdot 1|k\gamma_s^{-1} \gamma.
\]

The condition \( \chi(\gamma_s T^u w_s \overline{\gamma_s}^{-1}) = 1 \) makes the formal series well-defined, that is, the terms are independent of the choice of representatives of \( \gamma_s(\mathbb{P}^1) \gamma_s^{-1} \backslash \Gamma_0(N)^D \). The condition \( k > 2 \) and \( \chi \) is of finite index ensure that the series converges normally and hence defines a holomorphic function on \( \mathbb{H} \).

**Lemma 3.2.** *Assumptions as in Definition 3.1.* Put \( \gamma_s = \left( \begin{array}{cc} a_s & b_s \\ c_s & d_s \end{array} \right) \). Let \( \gamma_1 = \left( \begin{array}{cc} a_1 & b_1 \\ c_1 & d_1 \end{array} \right) \in \SL_2(\mathbb{Z}) \) be arbitrary. Then we have

\[
E_{\gamma_s, k}|_{k}\overline{\gamma_1}(\tau) = D \cdot \sum_{c,d} \chi \left( \left( \begin{array}{cc} a_s & b_s \\ c_s & d_s \end{array} \right) \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \left( \begin{array}{cc} a_1 & b_1 \\ c_1 & d_1 \end{array} \right)^{-1} \right)^{-1} (ct + d)^{-k},
\]

where the summation is over pairs of integers \( (c, d) \) satisfying gcd\((c, d) = 1 \) and \( N | csad_1 + dscd_1 - csbc_1 - d_ascd_1 \) for some \( a, b \) with \( ad - bc = 1 \).
Proof. By a change of variables, we may write

$$E_{\gamma,k}(\tau) = \sum_{\gamma \in (\tilde{T}^\infty)^{-1} \cap \Gamma_0(N)^D \gamma_1} \chi(\gamma_s \gamma \gamma_1^{-1})^{-1} \cdot 1|k\gamma.$$  

It can be verified directly that the following set is a complete system of representatives of \((\tilde{T}^\infty)^{-1} \cap \Gamma_0(N)^D \gamma_1\):

\[\tilde{R} = \{ \tilde{T}^r \cdot \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right), \varepsilon : 0 \leq r \leq u-1, c, d \in \mathbb{Z}, \gcd(c,d) = 1, \varepsilon D = 1 \},\]

where for each pair \((c, d)\) in elements of \(\tilde{R}\), we shall fix a pair of integers \((a, b)\) with \(ad - bc = 1\). Therefore, the set

\[\tilde{R}_1 = \{ \gamma \in \tilde{R} : \gamma \gamma_1^{-1} \in \Gamma_0(N)^D \} = \tilde{R} \cap \gamma_1^{-1} \Gamma_0(N)^D \gamma_1\]

forms a complete system of representatives of \((\tilde{T}^\infty)^{-1} \cap \Gamma_0(N)^D \gamma_1\). Write out elements explicitly, we find that \(\tilde{R}_1\) consists of exactly those \(\tilde{T}^r \cdot \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right), \varepsilon\) with \(0 \leq r \leq u-1\), \(c, d \in \mathbb{Z}, \gcd(c,d) = 1\), \(N | c_s(a + cr)d_1 + d_sc_1 - c_s(b + dr)c_1 - d_sc_1\) and \(\varepsilon D = 1\). It can be verified that if both \(\tilde{T}^r \cdot \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right), \varepsilon \) and \(\tilde{T}^r \cdot \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right), \varepsilon \) are in \(\tilde{R}_1\), then \(r_1 = r_2\). Inserting this into (3), and taking into account the fact \(\sum_{\varepsilon D = 1} \chi(I, \varepsilon)^{-1} \cdot \varepsilon^{Dk} = D\) (since \(\chi(-I) = \varepsilon(-k/2)\) and \(\tilde{T}^2 = (I, \varepsilon(1/D))\)), the desired formula follows.

**Theorem 3.3.** Assumptions as in Definition 3.1. Let \(\gamma_1 \in \text{SL}_2(\mathbb{Z})\) be arbitrary. Then we have

$$\lim_{\tau: \tau \rightarrow \infty} E_{\gamma_s,k}(\tau) = \begin{cases} 0, & \text{if } \gamma_s(\tau) \neq \gamma_1(\tau) \\ 2D \cdot \chi(g)^{-1} \cdot 1|k\gamma_1^{-1} g_1, & \text{else,} \end{cases}$$

where \(g\) is any matrix in \(\Gamma_0(N)\) satisfying \(\gamma_s(\tau) = g\gamma_1(\tau)\). As a consequence, we have \(E_{\gamma_s,k} \in M_k((\Gamma_0(N)^D \cap \text{SL}_2(\mathbb{Z}))^D)\). Moreover, the set

\[\{ E_{\gamma_s,k} : \gamma(\tilde{T}^\infty) \gamma_1^{-1} \in \Gamma_0(N)^D \cap \text{SL}_2(\mathbb{Z})^D, \chi(\gamma_1^{-1} g_1) = 1 \}\]

is \(\mathbb{C}\)-linearly independent.

**Proof.** Since the series (3) converges uniformly on any compact subset of \(\Omega\), we can change the order of the limit and the summation. Hence we obtain

$$\lim_{\tau: \tau \rightarrow \infty} E_{\gamma_s,k}(\tau) = \sum_{\gamma} \chi(\gamma_s \gamma \gamma_1^{-1})^{-1} \cdot 1|k\gamma,$$

where \(\gamma\) ranges over \((\tilde{T}^\infty)^{-1} \cap \Gamma_0(N)^D \gamma_1 \cap \text{SL}_2(\mathbb{Z})^D\). If \(\gamma_s(\tau) \neq \gamma_1(\tau)\), the summation range is empty, so the limit is zero. Otherwise, there is some \(g \in \Gamma_0(N)\) such that \(\gamma_s^{-1} g_1 \in \text{SL}_2(\mathbb{Z})^D\). Then the set \(\gamma_s^{-1} \Gamma_0(N)^D \gamma_1 \cap \text{SL}_2(\mathbb{Z})^D\) is a disjoint union of the the following \(2D\) cosets:

\[(\tilde{T}^\infty) \cdot (\tilde{T}^t)^{-1} \gamma_1^{-1} g_1, \quad t = 0, 1, \ldots, 2D - 1.\]

It follows that \(\lim_{\tau: \tau \rightarrow \infty} E_{\gamma_s,k}(\tau) = 2D \cdot \chi(g)^{-1} \cdot 1|k\gamma_1^{-1} g_1,\) since \(\sum_{t=0}^{2D-1} \chi((-I)^{-t})^{-1} 1|k(-I)^{t} = 2D\). The assertion on linear independence follows immediately. It remains to prove \(E_{\gamma_s,k} \in M_k((\Gamma_0(N)^D \cap \text{SL}_2(\mathbb{Z}))^D)\). The second condition of modular forms has been proved.
To prove the first, let $\gamma_0 \in \widetilde{\Gamma_0(N)}^D$ be arbitrary. We have, applying a change of variables,

$$E_{\gamma_0,k} = \chi(\gamma_0) \cdot \sum_{\gamma \in \widetilde{\Gamma_0(N)}^D \setminus \Gamma_0(N)^D} \chi(\gamma \gamma_0)^{-1} \cdot 1_{k \gamma_0^{-1} \gamma \gamma_0} = \chi(\gamma_0) E_{\gamma_0,k}.$$  

This concludes the proof. \hfill $\square$

4. ETA QUOTIENTS OF RATIONAL WEIGHTS

The simplest way to construct concrete examples of modular forms of rational weights, and to construct concrete characters of $D$-covers of modular groups is to use fractional powers of eta-quotients. For $r \in \mathbb{Q}$, we define

$$\log \eta(t) = \log \eta(i) + \int_{t}^{\infty} \frac{\eta'(z)}{\eta(z)} \, dz,$$

$$\eta^r(t) = \exp \left( r \cdot \log \eta(t) \right).$$

Note that $\eta(i)$ is a real number, and $\log \eta(i)$ is the real logarithm. The function $\log \eta$ is a single one, and should not be understood as the composition of log function and $\eta$ function. The propositions we will prove in the remaining section all rely on the classical formula of Dedekind:

$$\log \eta \left( \frac{at + b}{c \tau + d} \right) - \log \eta(\tau) = 2\pi i \left[ \frac{a + d}{24c} + \frac{1}{2} s(-d,c) - \frac{1}{8} \right] + \frac{1}{2} \log(ct + d),$$

where $\left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \text{SL}_2(\mathbb{Z})$ with $c > 0$, and $\log(ct + d)$ is the principal branch. For a proof, see [Apo90, Equation (12), §3.4].

Let $N, D$ be positive integers. By an eta-quotient on $\widetilde{\Gamma_0(N)^{2D}}$, we mean a product $\prod_{n \mid N} \eta(n \tau)^{r_n}$ with $r_n \in \frac{1}{D} \mathbb{Z}$.

**Lemma 4.1.** Put $f(\tau) = \prod_{n \mid N} \eta(n \tau)^{r_n}$, and $k = \frac{1}{2} \sum_{n \mid N} r_n$. For any $\gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right), \varepsilon \in \widetilde{\Gamma_0(N)^{2D}}$, we have $f|_k \gamma = \chi(\gamma)f$, where $\chi: \widetilde{\Gamma_0(N)^{2D}} \rightarrow \mathbb{C}^\times$ is a group character defined by $\chi(\gamma) = \varepsilon^{-2Dk} v(a, b, c, d)$, where

$$v(a, b, c, d) = \begin{cases} \frac{a + d}{24c} \sum_{n \mid N} nr_n + \frac{1}{2} \sum_{n \mid N} r_n s(-d, \frac{c}{n}) - \frac{k}{4} & \text{if } c > 0 \\ \frac{a + d}{24c} \sum_{n \mid N} nr_n + \frac{1}{2} \sum_{n \mid N} r_n s(d, \frac{c}{n}) + \frac{k}{4} & \text{if } c < 0 \\ \frac{k}{2D} \sum_{n \mid N} nr_n - \frac{k}{2} & \text{if } c = 0, a > 0 \\ \frac{k}{2D} \sum_{n \mid N} nr_n - \frac{k}{2} & \text{if } c = 0, a < 0. \end{cases}$$

**Proof.** Note that $f(\tau) = \exp \sum_{n \mid N} r_n \cdot \log \eta(n \tau)$. Hence the assertion on the value of $\chi(\gamma)$ follows from the definition of slash operators, Dedekind’s functional equation of $\log \eta$, the fact $\eta^n(n \tau)|_{r_n/2} = \varepsilon(n \tau/24 \eta^n(n \tau)$, and the formula

$$\left( \frac{-1}{0}, 1 \cdot (-1)^{\frac{k}{2D}} \right) \cdot \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right), \varepsilon \left( ct + d \right)^{-\frac{k}{2D}} = \left( \frac{-a}{-c}, 1 \cdot (-c^2 - d)^{\frac{k}{2D}} \right), \varepsilon \left( -ct - d \right)^{\frac{k}{2D}} \text{ if } c < 0.$$

The assertion that $\chi$ is a group character follows from the fact that slash operators give right group actions. \hfill $\square$
If a function satisfies the first condition of the definition of modular forms, we say it transforms like a modular form. Hence an eta-quotient (of rational powers) transforms like a modular form. For a non-zero holomorphic function \( f \) on \( \mathfrak{H} \) that transforms like a modular form and any \( \gamma \in \text{SL}_2(\mathbb{Z}) \), \( f|_k \gamma \) can be expressed as a function series of the form \( \sum_{n \in \mathbb{Z}} a_n q^n \) where \( m \) is a fixed positive integer. The least \( n \) such that \( a_n \neq 0 \) is called the order of \( f \) at cusp \( \gamma(\infty) \) (The order may be \( -\infty \)). We shall need the following formula concerning the order of an eta-quotient:

**Lemma 4.2.** Notations and Assumptions as in Lemma 4.1. Let \( a, c \) be coprime integers with \( c > 0 \). Then the order of \( f \) at the cusp \( a/c \in \Gamma_0(N) \backslash \mathbb{P}^1(\mathbb{Q}) \) is 
\[
\frac{1}{2} \sum_{n \mid N} \frac{r_n \cdot \gcd(n, c)^2}{n}.
\]

**Proof.** If all \( r_n \)'s are integers, the desired formula is equivalent to [Ono04, Theorem 1.65] (Caution: the meaning of “order” at cusps here is different from that in [Ono04]). Otherwise, apply the case just proven to \( f^D \), and use the fact
\[
(f|_k ((a \ b \ c \ d), \varepsilon (c \tau + d) \frac{1}{D}))^D_0 = f^D_0 | D_0 (a \ b \ c \ d), \varepsilon D_0 (c \tau + d) \frac{1}{D_0 D_0}
\]
for \( 0 < D_0 | 2D \). \( \square \)

We can construct modular forms of rational weights by eta-quotients of rational powers at order all cusps are non-negative.

We now turn to simplest characters, namely, those \( \chi \colon \Gamma_0(N) \backslash \mathbb{P}^1(\mathbb{Q}) \rightarrow \mathbb{C}^\times \) such that \( \chi(-I) = \varepsilon (-k/2) \) (this is a necessary condition for \( M_k(\Gamma_0(N) \backslash \mathbb{P}^1(\mathbb{Q}); \chi) \neq 0 \) and \( \chi(\gamma) = 1 \) for as many \( \gamma \) as possible. As a typical example, we consider the case \( N = 11 \). From a set of generators of \( \Gamma_0(11) \) (can be obtained by a SageMath [Sag21] command Gamma0(11).generators() which calculates generators using Farey symbols) we obtain a set of generators of \( \Gamma_0(11) \backslash \mathbb{P}^1(\mathbb{Q}) \):

\[
\bar{T}, \quad \bar{T}S^{-1}\bar{T}^3\bar{S}^{-1}\bar{T}^4\bar{S}, \quad \bar{T}\bar{S}^{-1}\bar{T}^4\bar{S}^{-1}\bar{T}^3\bar{S}, \quad \bar{S}^2 = \bar{I}.
\]

**Lemma 4.3.** Let \( r_1 \) and \( r_{11} \) be rational numbers, and \( D \) be a positive integer such that \( D \cdot r_1 \) and \( D \cdot r_{11} \) are both integers. Then the multiplier system \( \chi \colon \Gamma_0(11) \backslash \mathbb{P}^1(\mathbb{Q}) \rightarrow \mathbb{C}^\times \) of the eta-quotient \( \eta^{r_1}(\tau)\eta^{r_{11}}(11\tau) \) is determined by the following values:

\[
\bar{T} \mapsto \varepsilon \left( \frac{r_1 + 11 r_{11}}{24} \right) \quad \bar{T}\bar{S}^{-1}\bar{T}^3\bar{S}^{-1}\bar{T}^4\bar{S} \mapsto \varepsilon \left( \frac{11 r_1 + 13 r_{11}}{24} \right) \quad \bar{S}^2 \mapsto \varepsilon \left( \frac{-6 r_1 - 6 r_{11}}{24} \right) \quad \bar{T}\bar{S}^{-1}\bar{T}^4\bar{S}^{-1}\bar{T}^3\bar{S} \mapsto \varepsilon \left( \frac{11 r_1 + 13 r_{11}}{24} \right) .
\]

**Proof.** Apply Lemma 4.1 and take into account the fact
\[
\bar{T}\bar{S}^{-1}\bar{T}^3\bar{S}^{-1}\bar{T}^4\bar{S} = \left( \begin{array}{cc} 7 & -2 \\ 11 & -3 \end{array} \right), \varepsilon \left( \frac{1}{2D} (11\tau - 3) \right), \quad \bar{T}\bar{S}^{-1}\bar{T}^4\bar{S}^{-1}\bar{T}^3\bar{S} = \left( \begin{array}{cc} 8 & -3 \\ 11 & -4 \end{array} \right), \varepsilon \left( \frac{1}{2D} (11\tau - 4) \right).
\]

\( \square \)

\(^3\)It can be proved that, as in the case of integral weight modular forms, this series converges normally for \( \tau \in \mathfrak{H} \).
Lemma 4.4. Among the functions \( \eta^{r_1}(\tau)\eta^{r_{11}}(11\tau) \) with \( r_1 \) and \( r_{11} \) rationals, the following ones are exactly those modular forms on \( \Gamma_0(11)^{2D} \) for some positive integer \( D \) such that the multiplier system is trivial on the subgroup generated by \( \tilde{T}, \tilde{T}\tilde{S}^{-1}\tilde{T}^3\tilde{S}^{-1}\tilde{T}\tilde{S}^{-1}\tilde{T}^4\tilde{S}, \tilde{T}\tilde{S}^{-1}\tilde{T}^4\tilde{S}^{-1}\tilde{T}^3\tilde{S} \):

\[
\eta(\tau)^{-26m_1-12m_2} \cdot \eta(11\tau)^{22m_1+12m_2},
\]

\( m_1 \in \frac{1}{9} \mathbb{Z}, m_2 \in \mathbb{Z}, \frac{9}{5}m_1 \leq m_2 \leq -\frac{11}{5}m_1. \)

Moreover, one can choose \( 2D = 18 \).

Proof. A consequence of Lemma 4.2 and Lemma 4.3. \( \square \)

5. PROOF OF ASSERTIONS IN INTRODUCTION

Theorem 5.1. The function \( E_{20/9}(\tau) \) presented in Introduction belongs to the space \( M_{20/9}(\Gamma_0(11)^{18}; \chi) \), where \( \chi \) is the character on \( \Gamma_0(11)^{18} \) determined by following values on generators:

\[
\begin{align*}
\tilde{T} & \mapsto 1 \\
\tilde{S}^2 & \mapsto \epsilon(-1/9)
\end{align*}
\]

Proof. The given character is just the multiplier system of \( \eta^{44/9}(\tau)\eta^{-4/9}(11\tau) \) by Lemma 4.3. So \( E_{1,20/9} \) in Definition 3.1 is well-defined (with \( N = 11, D = 18, \sigma = i\infty \), and \( \chi \) the above one). Now applying Lemma 3.2 (with \( \gamma_1 = I \)) we find that

\[
E_{1,20/9}(\tau) = 18 \cdot \sum_{\gcd(c,d)=1} \chi \left( \frac{a}{c} \right) \left( \frac{b}{d} \right) - (c\tau + d)^{-20/9}.
\]

Inserting Lemma 4.1 into this expression we obtain \( E_{1,20/9} = 36 \cdot E_{20/9} \). The desired assertion follows from this relation and Theorem 5.1. \( \square \)

Theorem 5.2. The Fourier expansion of \( E_{20/9} \) is given by the expression (1).

Proof. Let \( \mu \) denote the Möbius function. Extend the definition of Dedekind sum by setting \( s(h,k) = s(h/d,k/d) \) where \( d = \gcd(h,k) \). We start from the fact

\[
\sum_{\tau \geq 1 \atop \gcd(11,\tau)=1} \mu(t) \left( \sum_{c \geq 0 \atop \gcd(c,d)=1} \frac{1}{\tau^{20/9}} \right) \sum_{c \geq 0 \atop \gcd(c,d)=1} \frac{1}{\tau^{20/9}} \sum_{c \geq 0 \atop \gcd(c,d)=1} \frac{1}{\tau^{20/9}}
\]

\[
\sum_{t \geq 1 \atop \gcd(11,t)=1} \frac{1}{\tau^{20/9}} \sum_{c \geq 0 \atop \gcd(c,d)=1} \frac{1}{\tau^{20/9}} \sum_{c \geq 0 \atop \gcd(c,d)=1} \frac{1}{\tau^{20/9}}
\]

\[
\sum_{c \geq 0 \atop \gcd(c,d)=1} \frac{1}{\tau^{20/9}} \sum_{c \geq 0 \atop \gcd(c,d)=1} \frac{1}{\tau^{20/9}} \sum_{c \geq 0 \atop \gcd(c,d)=1} \frac{1}{\tau^{20/9}}
\]

We shall use the Lipschitz summation formula

\[
\frac{1}{\tau + n} = e^{-\pi i n / (2\pi)} \frac{1}{\Gamma(s)} \sum_{n \geq 1} n^{-s-1} e^{2\pi i n \tau}, \quad \tau \in \mathbb{C}, \mathfrak{R}(s) > 1.
\]
From this and the fact \( s(h, k) = s(h + k, k) \) we obtain that

\[
\sum_{d \in \mathbb{Z}} \epsilon \left( -\frac{22}{9} s(-d, c) + \frac{2}{9} s(-11d, c) \right) (ct + d)^{-20/9} = c^{-20/9} \epsilon \left( -\frac{k}{4} \right) \left( \frac{2\pi}{\Gamma(20/9)} \right) \sum_{n \geq 1} n^{11/9} \sum_{d=0}^{c-1} \epsilon \left( -\frac{22}{9} s(-d, c) + \frac{2}{9} s(-11d, c) \right) \epsilon \left( \frac{nd}{c} \right) q^n.
\]

Similarly, for \( 0 < c | 11 \), we have

\[
\sum_{d \in \mathbb{Z}} \epsilon \left( -\frac{22}{9} s(-d, c) + \frac{2}{9} s(-d, c/11) \right) (ct + d)^{-20/9} = c^{-20/9} \epsilon \left( -\frac{k}{4} \right) \left( \frac{2\pi}{\Gamma(20/9)} \right) \sum_{n \geq 1} n^{11/9} \sum_{d=0}^{c-1} \epsilon \left( -\frac{22}{9} s(-d, c) + \frac{2}{9} s(-d, c/11) \right) \epsilon \left( \frac{nd}{c} \right) q^n.
\]

Inserting these two identities into (4), and using the identity \( \sum_{\gcd(p,t)=1} \mu(t) \frac{t^{p-1}}{p^\alpha} = \zeta(k)^{-1} \left( 1 - \frac{1}{p^\alpha} \right)^{-1} \) where \( p \) is a prime and \( k > 1 \), we obtain the desired formula. \( \square \)

**References**

[Apo90] Tom M. Apostol, *Modular functions and Dirichlet series in number theory*, second ed., Graduate Texts in Mathematics, vol. 41, Springer-Verlag, New York, 1990. MR 1027834

[Ibu00] T. Ibukiyama, *Modular forms of rational weights and modular varieties*, Abh. Math. Sem. Univ. Hamburg 70 (2000), 315–339. MR 1809555

[Ibu20] Tomoyoshi Ibukiyama, *Graded rings of modular forms of rational weights*, Res. Number Theory 6 (2020), no. 1, Paper No. 8, 13. MR 4047212

[Ono04] Ken Ono, *The web of modularity: arithmetic of the coefficients of modular forms and q-series*, CBMS Regional Conference Series in Mathematics, vol. 102, Published for the Conference Board of the Mathematical Sciences, Washington, DC, by the American Mathematical Society, Providence, RI, 2004. MR 2020489

[Sag21] SageMath, *The sage mathematics software system (version 9.3)*, The Sage Developers (2021).