Spectral Radius of Biased Random Walks 
on Regular Trees 

Song He 
School of Mathematics and Statistics, Huaiyin Normal University 
Huaian City, 223300, P. R. China 
songhe@hytc.edu.cn 

Abstract 
We consider biased random walk on regular tree and we obtain the spectral radius, first return probability and n-step transition probability. 

Keywords: Biased random walk, spectral radius, regular tree. 

1 Introduction 
Let $G = (V(G), E(G))$ be a locally finite, connected infinite graph, where $V(G)$ is the set of its vertices and $E(G)$ is the set of its edges. Fix a vertex $o$ of $G$ as the root, we assume that $o$ has at least one neighbor. For any vertex $x$ of $G$ let $|x|$ denote the graph distance between $x$ and $o$. Let $\mathbb{N} := \{1, 2, \ldots\}$ and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. For any $n \in \mathbb{Z}_+$:

$$B_G(n) = \{x \in V(G) : |x| \leq n\}, \quad \partial B_G(n) = \{x \in V(G) : |x| = n\}.$$

Let $d \in \mathbb{N}$, $d \geq 2$, $\mathbb{T}_d$ denotes $d$-regular trees. Fix a vertex $o$ of $\mathbb{T}_d$ as the root. For $\lambda > 0$, if an edge $e = \{x, y\}$ is at distance $n$ from $o$, its conductance is defined as $\lambda^{-n}$. Denote by $\text{RW}_\lambda$ the nearest-neighbour random walk $(X_n)_{n=0}^\infty$ among such conductances and call it the $\lambda$-biased random walk. In other words, $\text{RW}_\lambda$ on $\mathbb{T}_d$ has the following transition probabilities:
for \( v \sim u \) (i.e., if \( u \) and \( v \) are adjacent on \( \mathbb{T}_d \)),

\[
p(v, u) := p^\lambda_\mathbb{T}_d = \begin{cases} 
\frac{\lambda}{d} & \text{if } v = o, \\
\frac{\lambda}{d+\lambda} & \text{if } u \in \partial B_\mathbb{T}_d(|v| - 1) \text{ and } v \neq o, \\
\frac{1}{d+\lambda} & \text{otherwise.}
\end{cases} \tag{1.1}
\]

Notice \( RW_1 \) is just the simple random walk on \( \mathbb{T}_d \). By Rayleigh’s monotonicity principle (see [1], p. 35), there is a critical parameter \( \lambda_c(G) \in (0, \infty) \) such that \( RW_\lambda \) is transient for \( \lambda < \lambda_c(G) \) and is recurrent for \( \lambda > \lambda_c(G) \). In the following we will introduce some basic notations. Write

\[
p^{(n)}(x, y) := p^{(n)}_\lambda(x, y) = \mathbb{P}_x(X_n = y),
\]

where \( \mathbb{P}_x := \mathbb{P}^G_x \) is the law of \( RW_\lambda \) starting at \( x \). The Green function is given by

\[
\mathbb{G}(x, y | z) := \mathbb{G}_\lambda(x, y | z) = \sum_{n=0}^{\infty} p^{(n)}(x, y) z^n, \quad x, y \in V(G), \quad z \in \mathbb{C}, \quad |z| < R_G,
\]

where \( R_G = R_G(\lambda) = R_G(\lambda, x, y) \) is its convergence radius. Recall [1] Exercise 1.2,

\[
R_G = R_G(\lambda) = \frac{1}{\limsup_{n \to \infty} \sqrt[n]{p^{(n)}(x, y)}}
\]

is independent of \( x, y \) when \( RW_\lambda \) is irreducible, i.e., \( \lambda > 0 \). Call

\[
\rho_\lambda = \rho(\lambda) = \frac{1}{R_G} = \limsup_{n \to \infty} p^{(n)}(x, x)^{1/n} = \limsup_{n \to \infty} p^{(n)}(o, o)^{1/n}
\]

the spectral radius of \( RW_\lambda \). Let \( M_n = |\partial B_G(n)| \) be the cardinality of \( \partial B_G(n) \) for any \( n \in \mathbb{Z}_+ \). Define the growth rate of \( G \) as

\[
gr(G) = \liminf_{n \to \infty} \sqrt[n]{M_n}.
\]

Since the sequence \( \{M_n\}_{n=0}^\infty \) is submultiplicative, the limit \( \lim_{n \to \infty} \sqrt[n]{M_n} \) exists indeed.
The motivation for introducing RW\textsubscript{λ} was to design a Monte-Carlo algorithm for self-avoiding walks by Berretti and Sokal \[2\]. See [3–5] for refinements of this idea. Due to interesting phenomenology and similarities to concrete physical systems (\[6–10\]), biased random walks and biased diffusions in disordered media have attracted much attention in mathematical and physics communities since the 1980s.

In the 1990s, Lyons (\[11–13\]), and Lyons, Pemantle and Peres (\[14,15\]) made series of achievements in the study of RW\textsubscript{λ}'s. RW\textsubscript{λ} has also received attention recently, see \[16–19\] and the references therein. Ben Arous and Fribergh publish a survey on biased random walks on random graphs see \[20\]. For spectral radius, R. Lyons \[12\] showed that the critical parameter for RW\textsubscript{λ} on a general tree is exactly the exponential of the Hausdorff dimension of the tree boundary. And R. Lyons \[13\] proved that for Cayley graphs and degree bounded transitive graphs, the growth rate is exactly the critical parameter of the RW\textsubscript{λ}. This paper focuses on a specific properties of spectral radius of RW\textsubscript{λ}'s on non-random infinite graphs.

We are ready to state our main results. The proofs will be presented in Section 2.

**Theorem 1.1** For the \(d\)-regular tree \(T_d\), the following holds:

\[
\rho_{T_d}(\lambda) = \frac{2\sqrt{(d-1)\lambda}}{d-1 + \lambda}, \quad \lambda \in (0, \lambda_c(T_d)] = (0, d-1],
\]

and for \(\lambda \in (0, \infty)\) and \(n \to \infty\),

\[
f^{(2n)}_\lambda(o, o) \sim \frac{1}{\sqrt{\pi}} \left( \frac{2\sqrt{(d-1)\lambda}}{d-1 + \lambda} \right)^{2n} n^{-3/2}. \tag{1.2}
\]

Moreover,

\[
p^{(2n)}_\lambda(o, o) \sim \begin{cases} 
\frac{(d-1-\lambda)^2}{16(\pi \lambda)^{1/2}(d-1)^{1/2}} \rho_{T_d}(\lambda)^{2n} n^{-3/2} & \text{if } \lambda \in (0, d-1), \\
\frac{1}{\sqrt{\pi n}} & \text{if } \lambda = d-1.
\end{cases} \tag{1.3}
\]

**Remark 1.2** Since for the case \(\lambda > \lambda_c(T_d) = d-1\), RW\textsubscript{λ} is recurrence, it means that \(\rho_\lambda = 1\). Hence, the spectral radius \(\rho_\lambda\) is continuous in \(\lambda \in (0, \infty)\), and \(\rho(\lambda_c(T_d)) = 1\).
Remark 1.3 For $\lambda \in (0, d-1)$, the derivative of $\rho_{\tau_d}(\lambda)$:

$$\rho'_{\tau_d}(\lambda) = \frac{\sqrt{(d-1)/\lambda(d-1+\lambda)}}{(d-1-\lambda)^2} > 0.$$ 

It means that $\rho_{\tau_d}(\lambda)$ is strictly increasing for $\lambda \in (0, d-1)$.

2 Proof of Theorem 1.1

Proof of Theorem 1.1 Assume $\lambda > 0$. Notice that $\text{RW}_\lambda (X_n)_{n=0}^\infty$ must return to $o$ in even steps, and that $\{|X_n|\}_{n=0}^\infty$ with $|X_0| = 0$ is a Markov chain on $\mathbb{Z}_+$ with transition probabilities given by

$$p(x, y) = \begin{cases} 
1 & \text{if } x = 0, y = 1 \\
\frac{\lambda}{d-1+\lambda} & \text{if } y = x - 1 \text{ and } x \neq 0, \\
\frac{d-1}{d-1+\lambda} & \text{otherwise}.
\end{cases}$$

Recall for any $n \in \mathbb{N}$ and $k \in \mathbb{Z}_+$,

$$f_{\lambda}^{(2n)}(o, o) = \mathbb{P}_o (\tau_o^+ = 2n), \quad f_{\lambda}^{(2n-1)}(o, o) = 0, \quad \lambda \in (0, \infty),$$

and the $k$th Catalan number given by $c_k = \frac{1}{k+1} \binom{2k}{k}$, with the associated related generating function

$$C(x) := \sum_{k=0}^\infty c_k x^k = \frac{1 - \sqrt{1 - 4x}}{2x}, \quad x \in \left[\frac{1}{4}, \frac{1}{4}\right]. \quad (2.1)$$

Note the number of all $2n$-length nearest-neighbour paths $\gamma = w_0 w_1 \cdots w_{2n}$ on $\mathbb{Z}_+$ such that

$$w_0 = w_{2n} = 0, \quad w_j \geq 1, \quad 1 \leq j \leq 2n - 1$$

is precisely $c_{n-1}$. Hence for any $\lambda > 0$, 

$$f_{\lambda}^{(2n)}(o, o) = c_{n-1} \left( \frac{d-1}{d-1+\lambda} \right)^{n-1} \left( \frac{\lambda}{d-1+\lambda} \right)^n, \quad n \in \mathbb{N},$$
which readily yields (1.2) by means of Stirling’s formula.

By definition, for $\lambda > 0$,

$$U_{\lambda}(o, o | z) = \sum_{n=1}^{\infty} f^{(2n)}_{\lambda}(o, o) z^{2n} = \sum_{n=1}^{\infty} c_{n-1} \left( \frac{d - 1}{d - 1 + \lambda} \right)^{n-1} \left( \frac{\lambda}{d - 1 + \lambda} \right)^n z^{2n}$$

$$= \frac{\lambda}{d - 1 + \lambda} z^2 C \left( \frac{\lambda(d - 1)z^2}{(d - 1 + \lambda)^2} \right),$$

which, in view of (2.1), implies that for $|z| \leq \frac{d - 1 + \lambda}{2\sqrt{\lambda(d - 1)}}$,

$$U_{\lambda}(o, o | z) = \frac{(d - 1 + \lambda) - \sqrt{(d - 1 + \lambda)^2 - 4\lambda(d - 1)z^2}}{2(d - 1)}. \quad (2.2)$$

Taking $z = 1$ gives that

$$P_o (\tau^+_o < \infty) = U_{\lambda}(o, o | 1) = \frac{\lambda \wedge (d - 1)}{d - 1}. \quad (2.3)$$

Notice from (2.2) that when $0 < \lambda \leq d - 1$,

$$U_{\lambda} \left( o, o \left| \frac{d - 1 + \lambda}{2\sqrt{\lambda(d - 1)}} \right. \right) = \frac{d - 1 + \lambda}{2(d - 1)} \leq 1.$$

Hence, for $|z| < \frac{d - 1 + \lambda}{2\sqrt{\lambda(d - 1)}}$ and $0 < \lambda \leq d - 1$,

$$G_{\lambda}(o, o | z) = \frac{1}{1 - U_{\lambda}(o, o | z)} = \frac{2(d - 1)}{2(d - 1) - (d - 1 + \lambda) + \sqrt{(d - 1 + \lambda)^2 - 4\lambda(d - 1)z^2}}. \quad (2.3)$$

This implies that the convergence radius for $G_{\lambda}(o, o | z)$ is $\frac{d - 1 + \lambda}{2\sqrt{\lambda(d - 1)}}$. In other words,

$$\rho(\lambda) := \rho_{\tau_d}(\lambda) = \frac{2\sqrt{\lambda(d - 1)}}{d - 1 + \lambda}, \quad 0 < \lambda \leq d - 1.$$

It remains to show (1.3) for $\lambda \in (0, d - 1)$. Write $a(\lambda) = \frac{2(d - 1)}{d - 1 + \lambda}$ and $b(\lambda) = \frac{d - 1 - \lambda}{d - 1 + \lambda}$. Then
for any \(|z| \leq R_G(\lambda) = \frac{1}{\rho(\lambda)}\),

\[
G_\lambda(o, o | z) = \frac{2(d-1)}{d-1 + \lambda \frac{d-1-\lambda}{d-1+\lambda} + \sqrt{1 - \rho(\lambda)^2 z^2}} = \frac{a(\lambda)}{b(\lambda) + \sqrt{1 - \rho(\lambda)^2 z^2}}.
\]

Let

\[
\Phi(t) := \Phi(\lambda) = \frac{-a(\lambda)b(\lambda) + \sqrt{a(\lambda)^2 + \rho(\lambda)^2 (1 - b(\lambda)^2) t^2}}{1 - b(\lambda)^2}, \quad t \in \mathbb{R}.
\]

Then for any \(|z| \leq R_G(\lambda)\),

\[
G_\lambda(o, o | z) = \Phi(z G_\lambda(o, o | z)).
\]

Define

\[
\Psi(u, v) := \Phi(uv) - v, \quad u, v \in \mathbb{R}.
\]

Then

\[
\frac{\partial \Psi(u, v)}{\partial v} \bigg|_{(u, v) = (\frac{1}{\rho(\lambda)}, G_\lambda(o, o | \frac{1}{\rho(\lambda)})}) = 0,
\]

\[
c_1(\lambda) := \frac{\partial^2 \Psi(u, v)}{\partial v^2} \bigg|_{(u, v) = (\frac{1}{\rho(\lambda)}, G_\lambda(o, o | \frac{1}{\rho(\lambda)})}) = \frac{(d-1-\lambda)^3}{2(d-1)(d-1+\lambda)^2} \neq 0,
\]

\[
c_2(\lambda) := \frac{\partial \Psi(u, v)}{\partial u} \bigg|_{(u, v) = (\frac{1}{\rho(\lambda)}, G_\lambda(o, o | \frac{1}{\rho(\lambda)})}) = \frac{2\rho(\lambda)(d-1)}{d-1-\lambda} \neq 0.
\]

Applying the method of Darboux (see [21] Theorem 5), we obtain that

\[
p^{(2n)}(\lambda) \sim \left(\frac{c_1(\lambda)}{2\pi \rho(\lambda)c_2(\lambda)}\right)^{1/2} \rho(\lambda)^2 n^{3/2} \left(\frac{(d-1-\lambda)^2}{16(\pi \lambda)^{1/2}(d-1)^{3/2}\rho(\lambda)^2 n^{-3/2}}.
\]

The idea of using the method of Darboux to establish the asymptotics for \(p^{(2n)}(\lambda, o, o)\) is not new. For example, in Woess [22] Chapter III Section 17 pp. 181–189, examples of random walk on groups are given such that \(p^{(n)}(\lambda, o, o) \sim c\rho^n n^{-3/2}\) for some constant \(c > 0\). The exact
value of $c$ is not known in general.

For $z \in (-1, 1)$, $G_{d-1}(o, o \mid z) = \frac{1}{\sqrt{1-z^2}} = \sum_{n=0}^{\infty} \frac{(2n)!}{2^n (n!)^2} z^{2n}$. Thus

$$p^{(2n)}_{d-1}(o, o) = \frac{(2n)!}{2^{2n} (n!)^2} \sim \frac{1}{\sqrt{\pi n}}$$

**Acknowledgements.** The authors would like to thank an anonymous referee and Zhan Shi, Kainan Xiang, Longmin Wang for valuable comments and suggestions to improve the quality of the paper. HS’s research is supported by Jiangsu University Natural Science Research Project 21KJB110003 and by Xiang Yu Ying Cai 31SH002.

**References**

[1] R. Lyons and Y. Peres. Probability on trees and networks: Plate section. 10.1017/9781316672815, 2016.

[2] Alberto Berretti and Alan D. Sokal. New monte carlo method for the self-avoiding walk. *Journal of Statistical Physics*, 40(3-4):483–531, 1985.

[3] G. F. Lawler and A. D. Sokal. Bounds on the l^2 spectrum for markov chains and markov processes: A generalization of cheeger’s inequality. *Transactions of the American Mathematical Society*, 309(2):557–580, 1988.

[4] Sinclair Mark Jerrum. Approximate counting, uniform generation and rapidly mixing markov chains. *Information and Computation*, 1989.

[5] D. Randall. Counting in lattices: Combinatorial problems from statistical mechanics. *University of California at Berkeley*, 1998.

[6] M. Barma and D. Dhar. Directed diffusion in a percolation network. *Journal of Physics C Solid State Physics*, 16(8):1451, 2000.
[7] D. Dhar. Diffusion and drift on percolation networks in an external field. *Journal of Physics A General Physics*, 17(5):L257, 1984.

[8] Stauffer D. Dhar D. Drift and trapping in biased diffusion on disordered lattices. *International Journal Of Modern Physics C*, 9(2):349–355, 1998.

[9] Shlomo Havlin and Daniel Ben-Avraham. Diffusion in disordered media. *Chemometrics Intelligent Laboratory Systems*, 10(1–2):117–122, 1988.

[10] H. Song L. Wang K. N. Xiang Z. Shi, V. Sidoravicius. Uniform spanning forests associated with biased random walks on euclidean lattices. *ANN I H POINCARE-PR*, 57(3):1569–1582, 2021.

[11] R. Lyons. Random walks and percolation on trees. *The Annals of Probability*, 18(3):931–958, 1990.

[12] Lyons and Russell. Random walks, capacity and percolation on trees. *Annals of Probability*, 20(4):2043–2088, 1992.

[13] Lyons and Russell. Random walks and the growth of groups. *Comptes Rendus de l’Académie des Sciences - Series I - Mathematics*, 320:1361–1366, 1995.

[14] Russell Lyons, Robin Pemantle, and Yuval Peres. Random walks on the lamplighter group. *Annals of Probability*, 24(4):1993–2006, 1996.

[15] Russell Lyons, Robin Pemantle, and Yuval Peres. Biased random walks on galton-watson trees. *Probab Theory Rel*, 106(2):249–264, 1996.

[16] G. B. Arous, Y. Hu, S. Olla, and O. Zeitouni. Einstein relation for biased random walk on galton–watson trees. *ANN I H POINCARE-PR*, 2013,49(3)(-):698–721, 2013.

[17] Aidekon and E. Speed of the biased random walk on a galton-watson tree. *Probab Theory Rel*, 2014.
[18] Gérard Ben Arous, Alexander Fribergh, and Vladas Sidoravicius. Lyons-pemantle-peres monotonicity problem for high biases. *Communications on Pure Applied Mathematics*, 67(4):519–530, 2014.

[19] YY and Shi. The most visited sites of biased random walks on trees. *ELECTRON J PROBAB*, 2015.

[20] G. B. Arous and A. Fribergh. Biased random walks on random graphs. *Lecture Notes of the Institute for Computer Sciences Social Informatics Telecommunications Engineering*, 3(4):95–106, 2014.

[21] E. A. Bender. Asymptotic methods in enumeration. *Siam Review*, 16(4):485–515, 1974.

[22] W. Woess. *Random walks on infinite graphs and groups*. 2000.