Topical Review

Conformal field theory, tensor categories and operator algebras

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Abstract
This is a set of lecture notes on the operator algebraic approach to 2-dimensional conformal field theory. Representation theoretic aspects and connections to vertex operator algebras are emphasized. No knowledge on operator algebras or quantum field theory is assumed.

Keywords: conformal field theory, subfactor, tensor category, vertex operator algebra, operator algebra

1. Introduction

Quantum field theory is a vast theory in physics which has many deep connections to various fields of mathematics. Here we are interested in analytic aspects of quantum field theory typically represented by Wightman axioms.

In classical field theory, a field is some kind of function on a spacetime. In quantum mechanics, numbers are replaced with operators, so we consider operator-valued functions on a spacetime, but it turns out that we have to deal with something like a δ-function, so we consider operator-valued distributions on a spacetime. Such an operator-valued distribution on a spacetime is called a quantum field. From this viewpoint, one quantum field theory consists of a spacetime, its symmetry group and a family of operator-valued distributions on the spacetime. The Wightman axioms [135] give a mathematical axiomatization of such objects.

We have another approach to quantum field theory based on operator algebras, which is called algebraic quantum field theory [75]. We now explain this idea. The operator-valued distributions are technically difficult to handle, since distributions are more difficult than functions and we have to deal with unbounded operators. In algebraic quantum field theory,
we deal with only bounded linear operators. These operators appear as ‘observables’, so observables are more emphasized than states in this approach.

Consider a (bounded) spacetime region $O$ and an operator-valued distribution $\Phi$. Take a test function $f$ supported in $O$. Then $\langle \Phi, f \rangle$, the application of $\Phi$ to the test function $f$, is a (possibly unbounded) operator and it represents an ‘observable’ in the spacetime region $O$ if it is self-adjoint. In one quantum field theory, we have many operator-valued distributions, and we have many test functions, so we have many observable on $O$ arising in this way. We consider an operator algebra $A(O)$ of bounded linear operators generated by these operators. (From an unbounded self-adjoint operator, we obtain a bounded operator through exponentiation.) In this way, we have a family $\{A(O)\}$ of operator algebras parameterized by spacetime regions $O$. We then impose mathematical axioms on this which are ‘natural’ from a physical viewpoint. We explain three of basic axioms here.

First, for a larger spacetime region, we have more test functions from a mathematical viewpoint and more observables from a physical viewpoint, so we have a larger operator algebra. Second, if we have two spacelike separated regions, we have no interactions between the two even at the speed of light, so an operator representing an observable in one commutes with that in the other. This property is called locality, or Einstein causality. Finally, we have a projective unitary representation of the spacetime symmetry group and a certain covariance property of the family of operator algebras with respect to it.

A family of operator algebras satisfying these axioms is our mathematical object in algebraic quantum field theory. We try to construct examples of such families, classify them and study relations among their various properties.

A framework where many researchers have worked is a Minkowski space with the Poincaré group. However, from an axiomatic viewpoint, we have been unable to construct an example different from free fields, so we do not have much progress recently. We have seen much progress in the last 30 years on the (1+1)-dimensional Minkowski space with a higher symmetry, conformal symmetry, and this is conformal field theory we would like to explain in this text.

Our spacetime to start with is the (1+1)-dimensional Minkowski space, where the space coordinate is $x$ and the time coordinate is $t$. We have a certain decomposition machinery and can restrict the theory to two light rays $\{x = \pm t\}$. Then each light rays plays the role of a ‘spacetime’, where the space and the time are mixed into one dimension. We further compactify each line by adding the point at $\infty$ so that we can deal with higher symmetries including those moving $\infty$. We thus reach the ‘spacetime’ $S^1$, the one-dimensional circle. Our spacetime region is now an arc contained in $S^1$. We also have to specify the spacetime symmetry group, and we choose the orientation preserving diffeomorphism group $Diff(S^1)$. This is an infinite dimensional Fréchet Lie group and represents a large symmetry. With these choices, our quantum field theory is called chiral conformal field theory [8]. We say ‘chiral’ because we have only one of the two light rays.

We also have another mathematical theory to deal with chiral conformal field theory, that is, theory of vertex operator algebras. This approach is based on algebraic axiomatization of Fourier expansions of a family of operator-distributions on the circle $S^1$. It is not a purpose here to develop its full theory, but we would like to compare this approach to that based on operator algebras. It is expected that the operator algebraic approach to chiral conformal field theory is mathematically equivalent to the approach using vertex operator algebras, at least in nice situations, so we focus on the relations between the two approaches.

We also consider conformal field theory on the entire (1 + 1)-dimensional Minkowski space. Such a theory is called full conformal field theory. It is also interesting to consider the theory on the half Minkowski space $\{x > 0\}$. This is called a boundary conformal field theory. We present outlines of these theories at the end.
Many important results are scattered in literature, and sometimes different names are used for the same notion, and sometimes the same name mean different notions, so it is not easy for a beginner to grasp the structure of the theory. Our aim here is to present a clear outline of the entire theory.

For those who are interested in the operator algebraic approach to conformal field theory from other topics such as vertex operator algebras or quantum groups, it is often hard to understand the argument due to technical difficulty. Here we present rather ideas than rigorous proofs or technical details so that the entire structure is comprehensible without knowing technical details of operator algebras.

This text is based on the lectures given by the author in the fall of 2014 at the University of Tokyo. This text is also partly based on [92].

We refer the reader to [39] for general conformal field theory, [69] for background materials, [60, 61, 64–66], for another approach to full conformal field theory and tensor categories, and [130] for a recent review on the operator algebraic approach to conformal field theory.

2. Basics of operator algebras

We prepare basic facts about operator algebras which are necessary for studying conformal field theory. We do not include proofs. As standard references, we list the textbooks [136–138].

2.1. $C^*$-algebras and von Neumann algebras

We provide some minimal basics on theory of operator algebras. For simplicity, we assume all Hilbert spaces appearing in the text are separable. Our convention for notations is as follows. We use $A, B, \ldots, M, N, \ldots$ for operator algebras, $a, b, \ldots, u, v, \ldots, x, y, z$ for operators, $H, K, \ldots$ for Hilbert spaces and $\xi, \eta, \ldots$ for vectors in Hilbert spaces.

Let $H$ be a complex Hilbert space and $B(H)$ be the set of all bounded linear operators on $H$. We have a natural $\ast$-operation $x \mapsto x^\ast$ on $B(H)$. It is common to use seven topologies on $B(H)$ as in [136, section II.2], but here we need only two of them as follows.

Definition 2.1.

(1) The norm topology on $B(H)$ is induced by the operator norm $\|x\| = \sup_{\|\xi\| \leq 1} \|x\xi\|$.
(2) We define convergence $x_i \to x$ in the strong operator topology when we have $x_i\xi \to x\xi$ for all $\xi \in H$.

Note that the strong operator topology is weaker than the norm topology. This is because we have another topology called the weak operator topology, which is weaker than the strong operator topology. The norm convergence is uniform convergence on the unit ball of the Hilbert space and the strong operator convergence is pointwise convergence on the Hilbert space.

Definition 2.2.

(1) Let $M$ be a subalgebra of $B(H)$ which is closed in the $\ast$-operation and contains the identity operator $I$. We say $M$ is a von Neumann algebra if $M$ is closed in the strong operator topology.
Let \( A \) be a subalgebra of \( B(H) \) which is closed in the \( * \)-operation. We say \( A \) is a \( C^* \)-algebra if \( A \) is closed in the norm topology.

By this definition, a von Neumann algebra is automatically a \( C^* \)-algebra, but a von Neumann algebra is quite different from ‘ordinary’ \( C^* \)-algebras, so we often think that operator algebras have two classes, von Neumann algebras and \( C^* \)-algebras.

We have a natural notion of isomorphisms for \( C^* \)-algebras and von Neumann algebras. An isomorphism of a \( C^* \)-algebra onto another has norm continuity automatically. (See [136, corollary I.5.4].) An isomorphism of a von Neumann algebra onto another has appropriate continuity automatically. (See [136, corollary III.3.10].)

A commutative \( C^* \)-algebra containing the multiplicative unit is isomorphic to \( C(X) \), where \( X \) is a compact Hausdorff space and \( C(X) \) means the algebra of all complex-valued continuous functions. A commutative \( C^* \)-algebra without a multiplicative unit is isomorphic to \( C_0(X) \), the algebra of all complex-valued continuous functions on a locally compact Hausdorff space \( X \) vanishing at infinity. (See [136, theorem I.4.4].) A commutative von Neumann algebra is isomorphic to \( L^\infty(X, \mu) \), where \( (X, \mu) \) is a measure space. (See [138, proposition XIII.1.2].)

Easy examples are as follows.

**Example 2.3.** Let \( H \) be \( L^2([0, 1]) \). The polynomial algebra \( \mathbb{C}[x] \) acts on \( H \) by left multiplication. The image of this representation is a \( * \)-subalgebra of \( B(H) \). Its norm closure is isomorphic to \( C([0, 1]) \) and its closure in the strong operator topology is isomorphic to \( L^\infty([0, 1]) \).

If a \( C^* \)-algebra is finite dimensional, then it is also a von Neumann algebra, and it is isomorphic to \( \bigoplus_{j=1}^d M_n(C) \), where \( M_n(C) \) is the \( n \times n \)-matrix algebra. (See [136, theorem I.11.9].)

**Definition 2.4.** For \( X \subseteq B(H) \), we set

\[
X' = \{ y \in B(H) \mid xy = yx \text{ for all } x \in X \}.
\]

We call \( X' \) the commutant of \( X \).

We have the following proposition for von Neumann algebras. (See [136, proposition II.3.9].)

**Proposition 2.5.** Let \( M \) be a subalgebra of \( B(H) \) closed under the \( * \)-operation and containing \( I \). Then the double commutant \( M'' \) is equal to the closure of \( M \) in the strong operator topology.

Note that taking the commutant is a purely algebraic operation, but the above proposition says it contains information on the topology.

For von Neumann algebras \( M \subseteq B(H) \) and \( N \subseteq B(K) \), we have natural operations of the direct sum \( M \oplus N \subseteq B(H \oplus K) \) and the tensor product \( M \otimes N \subseteq B(H \otimes K) \).

We have the following proposition. (See [136, proposition II.3.12].)
Proposition 2.6. The following conditions are equivalent for a von Neumann algebra $M$.

1. The von Neumann algebra $M$ is not isomorphic to the direct sum of two von Neumann algebras.
2. The center $\mathcal{M} \cap M'$ of $M$ is $\mathbb{C}I$.
3. Any two-sided ideal of $M$ closed in the strong operator topology is equal to 0 or $M$.

A natural name for such a von Neumann algebra would be a simple von Neumann algebra, but for a historic reason, this name is not used and such a von Neumann algebra is called a factor instead.

2.2. Factors of types I, II and III

The matrix algebra $M_n(\mathbb{C})$ is a factor and the algebra $B(H)$ is also a factor. The former is called a factor of type I$_n$, and the latter is called a factor of type I$_\infty$ if $H$ is in finite dimensional. We introduce another example of a factor.

Example 2.7. For $x \in M_2(\mathbb{C}) \otimes \cdots \otimes M_2(\mathbb{C})$, we consider the embedding $x \mapsto x \otimes I_2 \in M_2(\mathbb{C}) \otimes \cdots \otimes M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$, where $I_2$ is the identity matrix in $M_2(\mathbb{C})$. We identify $M_2(\mathbb{C}) \otimes \cdots \otimes M_2(\mathbb{C})$ with $M_k(\mathbb{C})$, where $k$ is the number of the factorial components $M_2(\mathbb{C})$ so that the above embedding is compatible with this identification. Let $\mathrm{tr}$ be the usual trace $\mathrm{Tr}$ on $M_k(\mathbb{C})$ divided by $2^k$. Then this $\mathrm{tr}$ is compatible with the embedding $M_k(\mathbb{C})$ into $M_{k+1}(\mathbb{C})$. Let $A$ be the increasing union of $M_k(\mathbb{C})$ with respect to this embedding. This is a $\ast$-algebra and the linear functional $\mathrm{tr}$ is well-defined on $A$.

Setting $(x,y) = \mathrm{tr}(y^\ast x)$ for $x,y \in A$, we make $A$ a pre-Hilbert space. Let $H$ be its completion. For $x \in A$, let $\pi(x)$ be the multiplication operator $y \mapsto xy$ on $A$. This is extended to a bounded linear operator on $H$ and we still denote the extension by $\pi(x)$. Then $\pi$ is a $\ast$-homomorphism from $A$ into $B(H)$. The norm closure of $\pi(A)$ is a $C^\ast$-algebra called the type II$\infty$-uniformly hyperfinite (UHF) algebra or the canonical anticommutation relations (CAR) algebra. The closure $M$ of $\pi(A)$ in the strong operator topology is a factor and it is called the hyperfinite type II$_1$ factor. (Here the name ‘hyperfinite’ means that we have an increasing union of finite dimensional von Neumann algebras which is dense in the strong operator topology. A hyperfinite type II$_1$ factor is unique up to isomorphism [138, theorem XIV.2.4]. Sometimes, the terminology AFD, standing for ‘approximately finite dimensional’, is used instead of ‘hyperfinite’.)

The linear functional $\mathrm{tr}$ is extended to $M$ and satisfies the following properties.

1. We have $\mathrm{tr}(xy) = \mathrm{tr}(yx)$ for $x,y \in M$.
2. We have $\mathrm{tr}(x^\ast x) \geq 0$ for $x \in M$ and if $\mathrm{tr}(x^\ast x) = 0$, then we have $x = 0$.
3. We have $\mathrm{tr}(I) = 1$.

(See [138, section XIV.2].)

If an infinite dimensional von Neumann algebra has a linear functional $\mathrm{tr}$ satisfying the above three conditions, then it is called a type II$_1$ factor. Such a linear functional is unique on each type II$_1$ factor and called a trace. (See [136, section V.2].) There are many type II$_1$ factors which are not hyperfinite.

A type II$_\infty$ factor is a tensor product of a type II$_1$ factor and $B(H)$ for an infinite dimensional Hilbert space $H$. 


Definition 2.8. Two projections $p, q$ in a von Neumann algebra are said to be equivalent if we have $u$ in the von Neumann algebra satisfying $pu = uu^*$ and $qu = uu^*$. If $uu^*$ is a projection, then $uu^*$ is also automatically a projection, and such $u$ is called a partial isometry.

Definition 2.9. A factor is said to be of type III if any two non-zero projections in it are equivalent and it is not isomorphic to $\mathbb{C}$.

This definition is different from the usual definition of a type III factor, but means the same condition since we consider only separable Hilbert spaces. (See [136, definition V.1.17] and [136, proposition V.1.39].) Two equivalent projections are analogous to two sets having the same cardinality in set theory. Then the property analogous to the above in set theory would be that any two non-empty subsets have the same cardinality for a set which is not a singleton. Such a condition is clearly impossible in set theory. Still, based on this analogy, we interpret that the above property for a type III factor manifests a very high level of infiniteness. Because of this analogy, a type III factor is also called purely infinite.

The following is an example of a type III factor.

Example 2.10. Fix $\lambda$ with $0 < \lambda < 1$ and set $\phi_\lambda: M_2(\mathbb{C}) \to \mathbb{C}$ by

$$\phi_\lambda\left(\begin{array}{cc}a & b \\ c & d \end{array}\right) = \frac{a}{1 + \lambda} + \frac{\lambda}{1 + \lambda}.$$ 

Let $A$ be the same as in example 2.7. The linear functionals $\phi_\lambda \otimes \cdots \otimes \phi_\lambda$ on $M_2(\mathbb{C}) \otimes \cdots \otimes M_2(\mathbb{C})$ are compatible with the embedding, so $\phi^A = \otimes \phi_\lambda$ is well-defined on $A$. We set the inner product on $A$ by $(x, y) = \phi^A(y^*x)$ and set $H$ be its completion. Here we use a convention that an inner product is linear in the first variable. Let $\pi(x)$ be the left multiplication of $x$ on $A$, then it is extended to a bounded linear operator on $H$ again. The extension is still denoted by $\pi(x)$. The norm closure of $\pi(A)$ is isomorphic to the $2^\infty$ UHF algebra in example 2.7. The closure $M$ of $\pi(A)$ in the strong operator topology is a type III factor, and we have non-isomorphic von Neumann algebras for different values of $\lambda$. They are called the powers factors. (See [138, section XVIII.1].)

It is non-trivial that powers factors are of type III. Here we give a rough idea why this should be the case. On the one hand, two equivalent projections are regarded as ‘having the same size’. On the other hand, now the functional $\phi^A$ is also involved in measuring the size of projections. The two projections

$$\left(\begin{array}{cc}1 & 0 \\ 0 & 0 \end{array}\right), \left(\begin{array}{cc}0 & 0 \\ 0 & 1 \end{array}\right)$$

are equivalent, but have different ‘sizes’ according to $\phi^A$. Because of this incompatibility, we do not have a consistent way of measuring sizes of projections, and it ends up that all non-zero projections are of ‘the same size’ in the sense of equivalence.

Connes has refined the class of type III factors into those of type $\text{III}_1$ factors with $0 \leq \lambda \leq 1$. The powers factors as above are of type $\text{III}_\lambda$ with $0 < \lambda < 1$. If $M$ and $N$ are the powers factors of type $\text{III}_\lambda$ and $\text{III}_\mu$, respectively, and $\log \lambda \log \mu$ is irrational, then $M \otimes N$ is a factor of type $\text{III}_1$. The isomorphism class of $M \otimes N$ does not depend on $\lambda$ and $\mu$ as long as
log λ/log µ is irrational, and this factor is called the Araki–Woods factor of type III₁. The factor which appears in conformal field theory is this one. (See [137, chapter XIII].) The Powers and Araki–Woods factors are hyperfinite. There are many type III factors which are not hyperfinite, but they do not appear in conformal field theory.

For a von Neumann algebra \( M \subset B(H) \) and a unit vector \( \xi \in H \) with \( M\xi = M\xi = H \), we have the modular operator \( \Delta_H \), a positive and possibly unbounded operator, and the modular conjugation \( J_\xi \), an antunitary involution, on \( H \). For \( x \in M \) and \( t \in \mathbb{R} \), we have \( \sigma_t(x) = \text{Ad}(\Delta_H^{it})(x) \in M \). The one parameter automorphism group \( \sigma_t \) is called the modular automorphism group of \( M \) with respect to \( \xi \). This is the Tomita–Takesaki theory and classification of type III factors into type III₁, 0 ≤ λ ≤ 1, is based on this. (See [137, chapter VI] for more details in a more general setting.)

2.3. Dimensions and modules

First consider a trivial example of \( M_2(\mathbb{C}) \). We would like to find the ‘most natural’ Hilbert space on which \( M_2(\mathbb{C}) \) acts. One might think it is clearly \( \mathbb{C}^2 \), but from our viewpoint of infinite dimensional operator algebras, it is not the right answer. Instead, let \( M_2(\mathbb{C}) \) act on itself by the left multiplication and put a Hilbert space structure on \( M_2(\mathbb{C}) \) so that a natural system \( \{ e_{ij} \} \) of matrix units gives an orthonormal basis. Then the commutant of the left multiplication of \( M_2(\mathbb{C}) \) is exactly the right multiplication of \( M_2(\mathbb{C}) \) and thus the left and right multiplications are now symmetric. This is the ‘natural’ representation from our viewpoint, and we would like to consider its infinite dimensional analogue.

Let \( M \) be a type II₁ factor with \( \text{tr} \). Put an inner product on \( M \) by \( \langle x, y \rangle = \text{tr}(y^*x) \) and denote its completion by \( L^2(M) \). The left and right multiplications by an element of \( M \) on \( M \) extend to bounded linear operators on \( L^2(M) \). We say that \( L^2(M) \) is a left \( M \)-module and also a right \( M \)-module. (As long as we consider separable Hilbert spaces, any module of a factor of type II or III gives a representation which has appropriate continuity automatically. See [136, theorem 5.1].)

Let \( p \) be a projection in \( M \). Then \( L^2(M)p \) is naturally a left \( M \)-module. For projections \( p_n \in M \), we define \( \dim_M \bigoplus_n L^2(M)p_n = \sum_n \text{tr}(p_n) \). Then it turns out that any left \( M \)-module \( H \) is unitarily equivalent to this form and this number \( \dim_M H \in [0, \infty] \) is well-defined. It is called the dimension of a left \( M \)-module \( H \), and is a complete invariant up to unitary equivalence. Note that we have \( \dim_M L^2(M) = 1 \). (See [136, section V.3], where the dimension is called the coupling constant under a more general setting.)

For a type III factor \( M \), any two non-zero left \( M \)-modules are unitarily equivalent. (See [136, corollary V.3.2].)

In this sense, representation theory of a type II₁ factor is dictated by a single number, the dimension, and that of a type III factor is trivial. Note that a left module of a type II or III factor is never irreducible.

2.4. Subfactors

Let \( M \) be a type II₁ factor with \( \text{tr} \). Suppose \( N \) is a von Neumann subalgebra of \( M \) and \( N \) is also a factor of type II₁. We say \( N \subset M \) is a subfactor. (The unit of \( N \) is assumed to be the same as that of \( M \).) The Hilbert space \( L^2(M) \) is a left \( M \)-module, but it is also a left \( N \)-module and we have \( \dim_N L^2(M) \). This number is called the index of the subfactor and denoted by \( [M:N] \). The index value is in \([1, \infty]\). The celebrated theorem of Jones [86] is as follows. (Also see [51, theorem 9.16].)
Theorem 2.11. The set of the index values of subfactors is equal to
\[
\left\{ 4 \cos^2 \frac{n}{4} \right\}_{n=3, 4, 5, \ldots} \cup [4, \infty].
\]

There have been many results on the case $M$ is hyperfinite, when $N$ is automatically hyperfinite. It is often assumed that the index value is finite. A subfactor $N \subset M$ is said to be irreducible if we have $N' \cap M = \mathbb{C}$. Irreducibility of a subfactor is also often assumed.

A subfactor is an analogue of an inclusion $L^\infty(X, B_1, \mu) \subset L^\infty(X, B_2, \mu)$ of commutative von Neumann algebras where $B_1$ is a $\sigma$-subalgebra of $B_2$ on the space $X$ and $\mu$ is a probability measure. That is, a smaller commutative von Neumann algebra means that we have less measurable sets. For $f \in L^\infty(X, B_2, \mu)$, we regard it as an element in $L^\infty(X, B_1, \mu)$ and apply the orthogonal projection $P$ onto $L^\infty(X, B_1, \mu)$. Then $Pf$ is in $L^\infty(X, B_1, \mu)$, and this map from $L^\infty(X, B_2, \mu)$ onto $L^\infty(X, B_1, \mu)$ is called a conditional expectation. For a subfactor $N \subset M$ of type II$_1$, we have a similar map $E: M \rightarrow N$ satisfying the following properties.

1. $E(x^*x) \geq 0$ for all $x \in M$.
2. $E(x) = x$ for all $x \in N$.
3. $\text{tr}(xy) = \text{tr}(E(xy))$ for all $x \in M$, $y \in N$.
4. $E(axb) = aE(x)b$ for all $x \in M$, $a, b \in N$.
5. $E(x^*) = \overline{E(x)}$ for all $x \in M$.
6. $\|E(x)\| \leq \|x\|$ for all $x \in M$.

This map $E$ is also called the conditional expectation from $M$ onto $N$. Actually, properties 1, 4 and 5 follow from 2 and 6. (See [136, theorem III.3.4].)

Pimsner and Popa [123] proved that we have $E(x) \geq \frac{1}{[M : N]}x$ for all positive $x \in M$ and the coefficient $\frac{1}{[M : N]}$ is the best possible for this inequality under the convention $1/\infty = 0$. (See [51, theorem 9.48].) A general linear map $E$ from a von Neumann algebra $M$ onto a von Neumann subalgebra $N$ satisfying the above properties 2 and 5 is also called a conditional expectation. A conditional expectation $E$ is said to be faithful if $E(x) = 0$ for a positive $x \in M$ implies $x = 0$. A conditional expectation is said to be normal if it satisfies appropriate continuity. In this text, we simply say a conditional expectation for a normal faithful conditional expectation.

Kosaki [103] extended the definition of the index of a subfactor to the index of a conditional expectation $E: M \rightarrow N$ for a subfactor of type III. If such a conditional expectation does not exist, we interpret that the index of $N$ in $M$ is $\infty$. If the subfactor $N' \cap M$ is irreducible and we have a conditional expectation from $M$ to $N$, then such a conditional expectation is unique, so we call its index the index of the subfactor, $[M : N]$. Many results on indices of type III factors are parallel to those of type II factors. If one conditional expectation $E: M \rightarrow N$ has a finite index, the all other conditional expectations from $M$ onto $N$ have finite indices, and we have the unique conditional expectations achieving the minimum value of the indices. We define $[M : N]$ to be the index of this conditional expectation. We have the following results. (See [76] for details.)

Proposition 2.12.

(1) For two subfactors $N \subset M$ and $P \subset Q$, we have $[M \otimes Q : N \otimes P] = [M : N][Q : P]$.
(2) For a subfactor $N \subset M$, we have $[M : N'] = [N' : M']$. 

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2.5. Bimodules and relative tensor products

Let $M$ be a type II factor with tr. The Hilbert space $L^2(M)$ is a left $M$-module and a right $M$-module. Furthermore, the left action of $M$ and the right action of $M$ commute, so this is an $M$-$M$ bimodule. We consider a general $M$-$N$ bimodule $\mathcal{M} \mathcal{H}_N$ for type $\Pi_1$ factors $M$ and $N$. For a bimodule $\mathcal{M} \mathcal{H}_N$, we have $\dim \mathcal{H}_N$ defined in a similar way to the definition of $\dim_{\mathcal{M}}$. If we have $\dim_{\mathcal{M}} \dim \mathcal{H}_N < \infty$, we say that the bimodule is of finite type. We consider only bimodules of finite type.

Let $M, N, P$ be type $\Pi_1$ factors and consider a general $M$-$N$ bimodule $\mathcal{M} \mathcal{H}_N$ and an $N$-$P$ bimodule $\mathcal{M} \mathcal{K}_P$. Then we can define a relative tensor product $\mathcal{M} \mathcal{H} \otimes_{\mathcal{N}} \mathcal{K}_P$, which is an $M$-$P$ bimodule. This is again of finite type. We have

$$\mathcal{M} \otimes_{\mathcal{M}} \mathcal{H}_N \cong \mathcal{M} \otimes_{\mathcal{N}} \mathcal{H} \otimes_{\mathcal{N}} \mathcal{N} \otimes_{\mathcal{M}} \mathcal{H}_N.$$ 

(See [51, section 9.7].)

For an $M$-$N$ bimodule $\mathcal{M} \mathcal{H}_N$, we have the contragredient (or conjugate) bimodule $\mathcal{M} \mathcal{R}_{\mathcal{M}}$. As a Hilbert space, it consists of the vectors of the form $\xi$ with $\xi \in H$ and has operations $\xi + \bar{\eta} = \bar{\xi} + \bar{\eta}$ and $\alpha \xi = \bar{\alpha} \bar{\xi}$. The bimodule operation is given by $x \cdot \xi \cdot y = y^* \cdot \xi \cdot x^*$, where $x \in N$ and $y \in M$. This is again of finite type.

For $M$-$N$ bimodules $\mathcal{M} \mathcal{H}_N$ and $\mathcal{M} \mathcal{K}_N$, we say that a bounded linear map $T: H \to K$ is an intertwiner when we have $T(x \xi y) = x T(\xi) y$ for all $x \in M$, $y \in N$, $\xi \in H$. We denote the set of all the intertwiners from $H$ to $K$ by $\mathrm{Hom}(\mathcal{M} \mathcal{H}_N, \mathcal{M} \mathcal{K}_N)$. We say that $\mathcal{M} \mathcal{H}_N$ is irreducible if $\mathrm{Hom}(\mathcal{M} \mathcal{H}_N, \mathcal{M} \mathcal{H}_N) = \{0\}$. We have a natural notion of a direct sum $\mathcal{M} \mathcal{H}_N \oplus_{\mathcal{M}} \mathcal{K}_N$.

A bimodule $\mathcal{M} \mathcal{H}_N$ decomposes into a finite direct sum of irreducible bimodules, because we assume $\mathcal{M} \mathcal{H}_N$ is of finite type. (See [51, proposition 9.68].)

Start with a subfactor $N \subset M$ of type $\Pi_1$ with $[M: N] < \infty$. Then the $N$-$M$ bimodule $\mathcal{N} \mathcal{L}^2(M)_M$ is of finite type. The finite relative tensor products of $\mathcal{N} \mathcal{L}^2(M)_M$ and $\mathcal{M} \mathcal{L}^2(M)_M$ and their irreducible decompositions produce four kinds of bimodule, $N$-$N$, $N$-$M$, $M$-$N$ and $M$-$M$. They are all of finite type. We have only finitely many irreducible bimodules for one of them up to isomorphisms only if we have only finitely many irreducible bimodules for all four kinds. When this finiteness condition holds, we say the subfactor $N \subset M$ is of finite depth. If the index is less than 4, the subfactor is automatically of finite depth. (See [51, section 9] for more details.)

Consider a type $\Pi_1$ subfactor $N \subset M$ of finite depth and pick a representative from each of finitely many isomorphism classes of the $N$-$N$ bimodules arising in the above way. For each such $\mathcal{N} \mathcal{X}_N$, we have $\dim_{\mathcal{N}} \mathcal{X} = \dim \mathcal{X}_N$. For such $\mathcal{N} \mathcal{X}_N$ and $\mathcal{N} \mathcal{Y}_N$, the relative tensor product $\mathcal{X} \mathcal{N} \mathcal{Y} \mathcal{N}$ is isomorphic to $\bigoplus_{j=1}^k \mathcal{N} \mathcal{Z}_{ij} \mathcal{N}$, where $\{\mathcal{N} \mathcal{Z}_{ij} \mathcal{N}\}$ is the set of the representatives. This gives fusion rules and the bimodule $\mathcal{N} \mathcal{L}^2(M)_N$ plays the role of the identity for the relative tensor product. (Note that the name ‘fusion’ sometimes means the relative tensor product operation is commutative, but we do not assume this here.) For each $\mathcal{N} \mathcal{Z}_{ij} \mathcal{N}$, we have $k$ with $\mathcal{N} \mathcal{Z}_{ij} \mathcal{N} \cong \mathcal{Z}_{ij} \mathcal{N}$. We also have the Frobenius reciprocity, $\dim_{\mathcal{N}} \mathrm{Hom}(\mathcal{N} \mathcal{X} \mathcal{N} \mathcal{Y} \mathcal{N}, \mathcal{N} \mathcal{Z} \mathcal{N}) = \dim_{\mathcal{N}} \mathrm{Hom}(\mathcal{N} \mathcal{X} \mathcal{N} \mathcal{Y} \mathcal{N}, \mathcal{Z} \mathcal{N} \mathcal{Y} \mathcal{N})$. (See [51, section 9.8].) The $N$-$N$ bimodules isomorphic to finite direct sums of these representative $N$-$N$ bimodule make a unitary fusion category, which is an abstract axiomatization of this system of bimodules and is some kind of a tensor category. A basic model of unitary fusion category is that of finite dimensional unitary representations of a finite group. We recall the definitions for unitary fusion categories as follows. (See [3, 24].)
Definition 2.13. A category $\mathcal{C}$ is called an abelian category over $\mathcal{C}$ if we have the following.

1. All $\text{Hom}(U, V)$ are $\mathcal{C}$-vector spaces and the compositions

$$\text{Hom}(V, W) \times \text{Hom}(U, V) \to \text{Hom}(U, W), \quad (\phi, \psi) \mapsto \phi \circ \psi$$

are $\mathcal{C}$-bilinear, where $U, V, W$ are objects in $\mathcal{C}$.

2. We have a zero object $0$ in $\mathcal{C}$ with $\text{Hom}(0, V) \cong \text{Hom}(V, 0) = 0$ for all objects $V$ in $\mathcal{C}$.

3. We have finite direct sums in $\mathcal{C}$.

4. Every morphism $\phi \in \text{Mor}(\mathcal{C})$ has a kernel $\ker \phi \in \text{Mor}(\mathcal{C})$ and a cokernel $\text{coker} \phi \in \text{Mor}(\mathcal{C})$.

5. Every morphism is the composition of an epimorphism followed by a monomorphism.

6. If $\ker \phi = 0$, then we have $\ker(\text{coker} \phi) = 0$ and if $\text{coker} \phi = 0$, then we have $\phi = \text{coker}(\ker \phi)$.

Definition 2.14. An object $U$ in an abelian category $\mathcal{C}$ is called simple if any injection $V \hookrightarrow U$ is either 0 or an isomorphism.

An abelian category $\mathcal{C}$ is called semisimple if any object $V$ is isomorphic to a direct sum of simple ones, $V \cong \bigoplus n_i V_i$, where $V_i$ are simple objects, $n_i$ are multiplicities and only finitely many $n_i$ are non-zero.

Definition 2.15. An abelian category $\mathcal{C}$ is called a monoidal category if we have the following.

1. A bifunctor $\otimes: \mathcal{C} \times \mathcal{C} \to \mathcal{C}$.

2. A functorial isomorphism $\alpha_{U/VW}: UVW \to U \otimes (V \otimes W)$.

3. A unit object $1$ in $\mathcal{C}$ and functorial isomorphisms $\lambda_U: 1 \otimes V \cong V$ and $\rho_U: V \otimes 1 \cong V$.

4. If $X_1$ and $X_2$ are two expressions obtained from $V \otimes V \otimes \cdots \otimes V$ by inserting $1$’s and brackets. Then all isomorphisms composed of $\alpha$’s, $\lambda$’s, $\rho$’s and their inverses are equal.

5. The functor $\otimes$ is bilinear on the space of morphisms.

6. The object $1$ is simple and $\text{End}(1) = \mathbb{C}$.

Definition 2.16. Let $C$ be a monoidal category and $V$ be an object in $C$. A right dual to $V$ is an object $V^*$ with two morphisms $e_V: V^* \otimes V \to 1$ and $i_V: 1 \to V \otimes V^*$ such that we have $(id_V \otimes e_V)(i_V \otimes id_{V^*}) = id_V$ and $(e_V \otimes id_{V^*})(id_V \otimes i_V) = id_{V^*}$.

Similarly, we define a left dual of $V$ to be $^*V$ with morphisms $e_V: V \otimes V \to 1$ and $i_V: 1 \to V \otimes V$ satisfying similar axioms.

Definition 2.17. A monoidal category is called rigid if every object has right and left duals.

A tensor category is a rigid abelian monoidal category.

A fusion category is a semisimple tensor category with finitely many simple objects and finite dimensional spaces of morphisms.

Definition 2.18. A fusion category $\mathcal{C}$ over $\mathbb{C}$ is said to be unitary if we have the following conditions.

1. We have a Hilbert space structure on each $\text{Hom}$ space.

2. We have a contravariant endofunctor $^*$ on $\mathcal{C}$ which is the identity on objects.

3. We have $\|\phi \psi\| \leq \|\phi\| \|\psi\|$ and $\|\phi^* \phi\| = \|\phi\|^2$ for each morphism $\phi, \psi$ where $\phi$ and $\psi$ are composable.
4. We have $(\phi \otimes \psi)^n = \phi^n \otimes \psi^n$ for each morphism $\phi$, $\psi$.

5. All structure isomorphisms for simple objects are unitary.

For any such $N$-$N$ bimodule $\mathcal{N}_N$, we automatically have $\dim \mathcal{N}_N = \dim \mathcal{N}$. The index value of a subfactor of finite depth is automatically a cyclotomic integer by [49, theorem 8.51].

Conversely, we have the following theorem.

**Theorem 2.19.** Any abstract unitary fusion category is realized as that of $N$-$N$ bimodules arising from some (not necessarily irreducible) subfactor $N \subset M$ with finite index and finite depth, where $N$ and $M$ are hyperfinite type $\text{II}_1$ factors.

This is a slight generalization of [51, theorem 12.19] since such bimodules produce quantum $6j$-symbols in the sense of [51, section 12.2].

### 2.6. Classification of subfactors with small indices

Popa’s celebrated classification theorem [124] says that if a hyperfinite type $\text{II}_1$ subfactor $N \subset M$ has a finite depth, then the finitely many irreducible bimodules of the four kinds and the intertwiners between their tensor products contain complete information on the subfactor and recover $N \subset M$.

Classification of subfactors up to index 4 was announced in [117] as follows. (See also [51, theorem 11.24].)

**Theorem 2.20.** The hyperfinite $\text{II}_1$ subfactors with index less than 4 are labelled with the Dynkin diagrams $A_n$, $D_{2n}$, $E_6$ and $E_8$. Subfactors corresponding to $A_n$ and $D_{2n}$ are unique and those corresponding to $E_6$ and $E_8$ have two isomorphism classes each.

Classification of subfactors with index equal to 4 has been achieved in [124].

Recently, we have classification of subfactors with finite depth up to index 5. See [89] for details. Many of them are related to conformal field theory and quantum groups, but we see some exotic examples which have been so far unrelated to them. Up to index 5, we have three such exotic subfactors, the Haagerup subfactor [1], the Asaeda–Haagerup subfactor [1] and the extended Haagerup subfactor [9].

This is a very important active topic of the current research, but we refrain from going into details here.

### 2.7. Bimodules and endomorphisms

In this section, $M$ is a type III factor. We present another formulation of the bimodule theory which is more useful in conformal field theory.

We can also define $L^2(M)$ as a completion of $M$ with respect to some inner product arising from some positive linear functional on $M$. Then the left action of $M$ is defined usually, and we can also define the right action of $M$ on $L^2(M)$ using the modular conjugation in the Tomita–Takesaki theory. We then have an $M$-$M$ bimodule $\mathcal{N}, L^2(M)_M$, and the commutant of the left action of $M$ is exactly the right action of $M$. (See [137, section IX.1].)

Consider an $M$-$M$ bimodule $H$. The left actions of $M$ on $H$ and $L^2(M)$ are unitarily equivalent since $M$ is a type III factor. So by changing $H$ within the equivalence class of left $M$-modules, we may and do assume that $H = L^2(M)$ and the left actions of $M$ on $H$ and
\(L^2(M)\) are the same. Now consider the right action of \(M\) on \(H = L^2(M)\). It must commute with the left action of \(M\), but this commutant is exactly the right action of \(M\) on \(L^2(M)\), so this means that a general right action of \(M\) on \(H\) is given by a homomorphism of \(M\) into \(M\), that is, an endomorphism of \(M\). (We consider only unital homomorphisms and endomorphisms in this text.) Conversely, if we have an endomorphism \(\lambda\) of \(M\), then we can define an \(M-M\) bimodule \(L^2(M)\) with the standard left action and the right action given by \(x \cdot \xi \cdot y = x\xi\lambda(y)\). In this way, considering bimodules and considering endomorphisms are the same. We now see the corresponding notions of various ones in the setting of bimodules. We write \(\text{End}(M)\) for the set of all endomorphisms of \(M\).

Two endomorphisms \(\lambda_1\) and \(\lambda_2\) of \(M\) are said to be unitarily equivalent if we have a unitary \(u\) with \(\text{Ad}(u) \cdot \lambda_1 = \lambda_2\). The unitary equivalence of endomorphisms corresponds to the isomorphism of bimodules. A unitary equivalence class of endomorphisms is called a sector. This name comes from superselection sectors which appear later in this text. We write \([\lambda]\) for the sector of \(\lambda\).

For two endomorphisms \(\lambda_1\) and \(\lambda_2\) of \(M\), we define the direct sum \(\lambda_1 \oplus \lambda_2\) as follows. Since \(M\) is a factor of type III, we have isometries \(V_1, V_2 \in M\) with \(V_1^* V_1 + V_2^* V_2 = I\). Then we set \((\lambda_1 \oplus \lambda_2)(x) = V_1 \lambda_1(x) V_1^* + V_2 \lambda_2(x) V_2^*\). The unitary equivalence class of \(\lambda_1 \oplus \lambda_2\) is well-defined, and this direct sum of endomorphisms corresponds to the direct sum of bimodules.

An intertwiner in the setting of endomorphisms is given by

\[
\text{Hom}(\lambda_1, \lambda_2) = \{ T \in M | T \lambda_1(x) = \lambda_2(x) T \text{ for all } x \in M \}.
\]

For two endomorphisms \(\lambda_1, \lambda_2\), we set \(\langle \lambda_1, \lambda_2 \rangle = \dim \text{Hom}(\lambda_1, \lambda_2)\).

The relative tensor product of bimodules corresponds to composition of endomorphisms. The contragredient bimodule corresponds to the conjugate endomorphism. The conjugate endomorphism of \(\lambda\) is denoted by \(\bar{\lambda}\) and it is well-defined only up to unitary equivalence. The conjugate endomorphism is also given using the canonical endomorphism in \([107, \text{section 2}]\) arising from the modular conjugation in the Tomita–Takesaki theory. The canonical endomorphism for a subfactor \(N \subset M\) corresponds to the bimodule \(M_L M_N \otimes N \to L^2(M)\). The dual canonical endomorphism for a subfactor \(N \subset M\) is an endomorphism of \(N\) corresponding to the bimodule \(M_N\).

An endomorphism \(\lambda\) of \(M\) is said to be irreducible if \(\lambda(M) \cap M = \mathbb{C}\). This corresponds to irreducibility if bimodules. The index of \(\lambda\) is the index \([M : \lambda(M)]\). We set the dimension of \(\lambda\) to be \([M : \lambda(M)]^2\) and write \(d(\lambda)\) or \(d_\lambda\). Note that an endomorphism with dimension 1 is an automorphism. We have \(d(\lambda_1 \oplus \lambda_2) = d(\lambda_1) + d(\lambda_2)\) and \(d(\lambda_1 \lambda_2) = d(\lambda_1) d(\lambda_2)\). (See \([106, \text{theorem 5.5}]\) and \([108, \text{theorem 2.1}]\) for details.)

As a counterpart of \(M-N\) bimodule, we consider \(M-N\) morphisms. Let \(M, N\) be type III factors. (We do not assume \(N \subset M\).) As a general bimodule \(M_NH_N\) is isomorphic to \(M_L L^2(M)\) as a left \(M\)-module, so the right action of \(N\) on \(H\) gives a homomorphism from \(N\) to \(M\). Based on this observation, we say a homomorphism from \(N\) to \(M\) is an \(M-N\) morphism and denote the set of \(M-N\) morphisms by \(\text{Mor}(N, M)\). (Be careful about the order of \(M\) and \(N\).) Two \(M-N\) morphisms \(\lambda_1\) and \(\lambda_2\) are said to be unitarily equivalent when we have a unitary \(u \in N\) with \(\text{Ad}(u) \cdot \lambda_1 = \lambda_2\). A unitary equivalence class of \(M-N\) morphisms is called an \(M-N\) sector. We have \(M-N\) morphism \(\lambda_1, N-P\) morphism \(\lambda_2,\) and \(M-P\) morphism \(\lambda_3\), we also have the Frobenius reciprocity \(\text{Hom}(\lambda_1, \lambda_2, \lambda_3) \cong \text{Hom}(\lambda_1, \lambda_3, \lambda_2)\). For a subfactor \(N \subset M\) of type III, let \(\iota\) be the inclusion map \(N \to M\), which is an \(M-N\) morphism. Then we have that \(\iota\) is the dual canonical endomorphism as \(N-N\) morphisms. (See \([81]\) for details.)

Suppose we have a finite set \(\{\lambda_i | i = 0, 1, ..., n\}\) of endomorphisms of finite dimensions of \(M\) with \(\lambda_0\) being the identity automorphism. Suppose we have the following conditions.
1. Different $\lambda_i$ and $\lambda_j$ are not unitarily equivalent.
2. The composition $\lambda_i\lambda_j$ is unitarily equivalent to $\bigoplus_{k=1} m_k \lambda_k$, where $m_k$ is the multiplicity of $\lambda_k$.
3. For each $\lambda_i$, its conjugate $\bar{\lambda_i}$ is unitarily equivalent to some $\lambda_j$.

Then the set of endomorphisms of $M$ unitarily equivalent to finite direct sums of $\{\lambda_i\}$ gives a unitary fusion category. This is a counterpart of the unitary fusion category of bimodules. Conversely, any abstract unitary fusion category is realized as that of endomorphisms of the type III$_1$ Araki–Woods factor. This is a direct consequence of theorem 2.19.

### 2.8. A $Q$-system and an extension of a factor

We deal with abstract characterization of a subfactor $N \subset M$ in terms of tensor categories.

Suppose we have a type III$_1$ subfactor $N \subset M$ with finite index. Then the multiplication map $S_{N \otimes_N L^2(M)} L^2(M) \otimes_N L^2(M) \to N L^2(M)$ extending $x \otimes y \mapsto xy$ for $x, y \in M$ exists. It is a bimodule intertwiner and satisfies the associativity $S(id \otimes S) = S(S \otimes id)$. Conversely, if we have an intertwiner $S : N L^2(M) \otimes_N L^2(M) \to N L^2(M)$ with associativity, it essentially recovers $M$ with the product structure. Longo’s $Q$-system in [109] gives a precise formulation of this and this bimodule version was later given in [115]. Since what we use in conformal field theory is the original version based on endomorphisms, we introduce the definition in [109, theorem 6.1] as follows.

**Definition 2.21.** A $Q$-system $(\lambda, v, w)$ is a triple of an endomorphism of $M$ and isometries $v \in \text{Hom}(\text{id}, \lambda), w \in \text{Hom}(\lambda, \lambda^2)$ satisfying the following identities:

$$v^* w = \lambda(v^*) w \in \mathbb{R}_+,$$

$$\lambda(w) w = w^2.$$

If $N \subset M$ is a subfactor with finite index, the associated canonical endomorphism $\lambda$ gives a $Q$-system for appropriate $v \in \text{Hom}(\text{id}, \lambda), w \in \text{Hom}(\lambda, \lambda^2)$. Conversely, any $Q$-system determines a subfactor $N$ of $M$ such that $\lambda$ is the canonical endomorphism for $N \subset M$ and $N$ is given by $N = \{x \in M | x w = \lambda(x) w\}$. We then have $M = N v$. The $Q$-system also determines a larger factor $M_1 \supset M$ such that the dual canonical endomorphism for the subfactor $M \subset M_1$ is $\lambda$. Note that the intertwiner $w$ corresponds to the intertwiner $S$ in the bimodule setting, and the second condition on $w$ represents associativity. If $\lambda$ is an endomorphism in some unitary fusion category of endomorphisms of $M$, then the intertwiners are also in the category, and the conditions make sense within the fusion category.

A $Q$-system in the abstract language of tensor categories is the same as a $C^*$-Frobenius algebra, a special version of a special symmetric Frobenius algebra [38]. (Also see [60, 61, 64, 65, 101] for a special symmetric Frobenius algebra.)

Theory of bimodules over II$_1$ factors and that of endomorphisms of type III factors are parallel, but for some purpose, one is conceptually easier than the other, so it is convenient to have basic understanding of both. It is the latter which we use in conformal field theory.

### 3. Local conformal nets

We now present a precise formulation of chiral conformal field theory in the operator algebraic framework.
After introducing basic definitions, we present elementary properties, representation theory, the machinery of $\alpha$-induction, examples and classification theory.

3.1. Definition

We now introduce the axioms for a local conformal net. We say $I \subset S^1$ is an interval when it is a non-empty, connected, non-dense and open subset of $S^1$.

**Definition 3.1.** We say that a family of von Neumann algebras $\{A(I)\}$ parameterized by intervals $I \subset S^1$ acting on the same Hilbert space $H$ is a local conformal net when it satisfies the following conditions.

1. (Isotony) for two intervals $I_1 \subset I_2$, we have $A(I_1) \subset A(I_2)$.
2. (Locality) when two intervals $I_1, I_2$ satisfy $I_1 \cap I_2 = \emptyset$, we have $[A(I_1), A(I_2)] = 0$.
3. (Möbius covariance) we have a unitary representation $U$ of $PSL(2, \mathbb{R})$ on $H$ such that we have $U(g)A(I)U(g)^* = A(gI)$ for all $g \in PSL(2, \mathbb{R})$, where $g$ acts on $S^1$ as a fractional linear transformation on $\mathbb{R} \cup \{\infty\}$ and $S^1 \{−1\}$ is identified with $\mathbb{R}$ through the Cayley transform $C(z) = −i(z - 1)/(z + 1)$.
4. (Conformal covariance) we have a projective unitary representation, still denoted by $U$, of $Diff(S^1)$ extending the unitary representation $U$ of $PSL(2, \mathbb{R})$ such that

$$U(g)A(I)U(g)^* = A(gI), \quad g \in Diff(S^1),$$

$$U(g)xU(g)^* = x, \quad x \in A(I), \quad g \in Diff(I'),$$

where $I'$ is the interior of the complement of $I$ and $Diff(I')$ is the set of diffeomorphisms of $S^1$ which are the identity map on $I$.
5. (Positive energy condition) the generator of the restriction of $U$ to the rotation subgroup of $S^1$, the conformal Hamiltonian, is positive.
6. (Existence of the vacuum vector) We have a unit vector $\Omega \in H$, called the vacuum vector, such that $\Omega$ is fixed by the representation $U$ of $PSL(2, \mathbb{R})$ and $(\bigvee_{I \subset S^1} A(I))\Omega$ is dense in $H$, where $\bigvee_{I \subset S^1} A(I)$ is the von Neumann algebra generated by $A(I)$'s.
7. (Irreducibility) the von Neumann algebra $\bigvee_{I \subset S^1} A(I)$ is $B(H)$.

The convergence in $Diff(S^1)$ is defined by uniform convergence of all the derivatives. We say $\{A(I)\}$ is a local Möbius covariant net when we drop the conformal covariance axiom.

The name ‘net’ originally meant that the set of spacetime regions are directed with respect to inclusions, but now the set of intervals in $S^1$ is not directed, so this name is not appropriate, but has been widely used. Another name ‘pre-cosheaf’ has been used in some literatures.

If the Hilbert space is 1-dimensional and all $A(I)$ are just $\mathbb{C}$, all the axioms are clearly satisfied, but this example is of no interest, so we exclude this from a class of local conformal nets.

Locality comes from the fact that we have no interactions between two spacelike separated regions in the $(1 + 1)$-dimensional Minkowski space. Now because of the restriction procedure to two light rays, the notion of spacelike separation takes this simple form of disjointness.

The projective unitary representation of $Diff(S^1)$ in conformal covariance extending the unitary representation of $PSL(2, \mathbb{R})$ is unique if it exists, by [32].
The positive energy condition is our counterpart to what is called the spectrum condition in quantum field theory on the higher dimensional Minkowski space.

Irreducibility condition is equivalent to the uniqueness of the $PSL(2, \mathbb{R})$-invariant vector up to scalar, and is also equivalent to factoriality of each algebra $A(I)$. (See [74, proposition 1.2] for a proof.)

Note that a subfactor $N \subset M$ produces the Jones tower/tunnel

$$ \cdots \subset N_2 \subset N_1 \subset N \subset M \subset M_1 \subset M_2 \subset \cdots $$

as in [51, definition 9.24, definition 9.43]. If we set the interval $I_t, t \in (0, \pi)$, to be the arc between $(1, 0)$ and $(\cos t, \sin t)$ on the unit circle on the $xy$-plane, then the family $\{A(I_t)\}_t$ of the von Neumann algebras is a continuous analogue of the Jones tower.

It would be better to have some easy examples here, but unfortunately, there are no easy examples one can present immediately without preparations, so we postpone examples to a later section.

We have the following consequences from the axioms.

**Theorem 3.2.** (The Reeh–Schlieder theorem) for each interval $I \subset S^1$, both $A(I)\Omega$ and $A(I)'\Omega$ are dense in $H$, where $A(I)'$ is the commutant of $A(I)$.

The positive energy condition is used essentially for a proof of this theorem through analytic continuation. See [67, corollary 2.8] or [7, theorem 6.2.3] for a proof.

Let $S^1$ be the unit circle in the complex plane and $I$ the upper open half circle. Let $C: S^1 \to \mathbb{R} \cup \{\infty\}$ be the Cayley transform $C(z) = -i(z - 1)(z + 1)^{-1}$. We define a one-parameter diffeomorphism group $A_k(s)$ by $CA_k(s)C^{-1}x = e^s x$. Define $\tau_0$ by $\tau_0(z) = \bar{z}$ for $z \in S^1$. For a general interval $I \subset S^1$, choose $g \in PSL(2, \mathbb{R})$ so that $I = gI_1$ and we define $A_I = gA_k g^{-1}$, $\tau_I = g\tau_0 g^{-1}$. (They are independent of the choice of $g$.) The action of $\tau_0$ on $PSL(2, \mathbb{R})$ defined a semi-direct product $PSL(2, \mathbb{R}) \rtimes \mathbb{Z}_2$. Let $\Delta_I, J_I$ be the modular operator and the modular conjugation of $(A(I), \Omega)$. Then we have an extension of $U$ appearing in the definition of Möbius covariance such that $U(g)$ is a unitary or an antiunitary depending on whether $g$ preserves or reverses the orientation. We still write $U$ for this extension. Then we have the following Bisognano–Wichmann property.

**Theorem 3.3.** We have $U(\Lambda_I(2\pi t)) = \Delta_I^{it}, U(\tau_I) = J_I$ for this $U$.

See [21, theorems 2.3, 2.5] or [67, theorem 2.19] for details of this property. This immediately implies the following important result.

**Theorem 3.4.** (The Haag duality) we have $A(I)' = A(I')$.

The next result has been proved in [53, page 545].

**Theorem 3.5.** (Additivity) if a family $\{I_i\}$ of intervals and an interval $I$ satisfy $I \subset \bigcup_i I_i$, then $A(I)$ is contained in the von Neumann algebra generated by $\{A(I_i)\}$.

One has that the fixed point algebra of $A(I)$ of the modular automorphism group with respect to $\Omega$ is $C$, then this implies the following.
Theorem 3.6. Each $\mathcal{A}(I)$ is a factor of type $\text{III}_1$.

See [37, corollary 2.6] or [7, theorem 6.2.5] for details.

We now introduce extra properties of local conformal nets.

Definition 3.7. Removing one point from an interval $I$, we obtain a disjoint union of two intervals $I_1$ and $I_2$. We say that the local conformal net has strong additivity if the von Neumann algebra $\mathcal{A}(I)$ is always generated by $\mathcal{A}(I_1)$ and $\mathcal{A}(I_2)$.

Many important examples satisfy strong additivity, but there are also examples without it. In this text, we consider only local conformal nets with strong additivity.

Definition 3.8. Consider two intervals $I_1, I_2$ with $I_1 \cap I_2 = \emptyset$ and the map $x \otimes y \mapsto xy$ from the algebraic tensor product of $\mathcal{A}(I_1)$ and $\mathcal{A}(I_2)$ to the von Neumann algebra generated by $\mathcal{A}(I_1)$ and $\mathcal{A}(I_2)$, where $x \in \mathcal{A}(I_1)$ and $y \in \mathcal{A}(I_2)$. We say that the local conformal net has the split property if this map always extends to an isomorphism from $\mathcal{A}(I_1 \otimes I_2)$.

It has been proved in [37, theorem 3.2] that if we have $\text{Tr}(e^{-tL_0}) < \infty$ for all $t > 0$, where $L_0$ is the conformal Hamiltonian, then the local conformal net has the split property. Practically all known examples of local conformal nets satisfy the split property, and we consider only those with this property in this text. If we have the split property for $\{\mathcal{A}(I)\}$, then each $\mathcal{A}(I)$ is a hyperfinite type $\text{III}_1$ factor, which is isomorphic to the Araki–Woods factor of type $\text{III}_1$. (See [37, proposition 3.1] and [138, theorem XVIII.4.16].)

This means that in our setting the isomorphism class of each von Neumann algebra $\mathcal{A}(I)$ is unique for any interval and any local conformal net. So each $\mathcal{A}(I)$ has no information on conformal field theory, and it is the relative positions of the algebras $\mathcal{A}(I)$ that contain information of conformal field theory.

We remark that conformal field theory is supposed to be a certain scaling limit of statistical mechanical models. There have been many works on the operator algebraic approach to quantum statistical mechanics, so we expect that it is possible to realize this passage from quantum statistical mechanics to conformal field theory in the rigorous framework using operator algebras, but there has been no such work so far.

3.2. Superselection sectors and braiding

An important tool to study local conformal nets is their representation theory.

Each $\mathcal{A}(I)$ of a local conformal net acts on the Hilbert space $H$ from the beginning by definition, but consider representations of a family $\{\mathcal{A}(I)\}$ of factors on the common Hilbert space $H$. That is, we consider a family $\rho$ of representations $\rho_I: \mathcal{A}(I) \to B(H)$ such that the restriction of $\rho_I$ to $\mathcal{A}(I_1)$ is equal to $\rho_{I_1}$ for $I_1 \subset I_2$. Note that $H$ does not have a vacuum vector in general. The original identity representation on $H$ is called the vacuum representation.

For this notion of a representation, it is easy to define an irreducible representation, the direct sum of two representations and unitary equivalence of two representations. A unitary equivalence class of representations is called a superselection sector or a DHR (Doplicher–Haag–Roberts) sector.

We also introduce a notion of a covariant representation as follows.
Definition 3.9. A representation $\pi$ is said to be Möbius [resp. conformal] covariant if we have projective unitary positive energy representation $U_g$ of the universal cover $\text{PSL}(2, \mathbb{R})$ [resp. Diff$(S^1)$] such that $\pi(g) U(x) U(g)^* = U(g) \pi(x) U(g)^*$ for all $x \in A(I)$. Here the positive energy condition means that the generator of the one-parameter unitary group arising from the rotation subgroup of Diff$(S^1)$ is positive.

Any irreducible representation of a local conformal net is automatically conformal covariant by [26, proposition 2.2]. (Also see [36].)

We would like to define a notion of a tensor product of two representations. This is a nontrivial task, and an answer has been given in the Doplicher–Haag–Roberts theory [46, 47], which was originally developed for quantum field theory on the 4-dimensional Minkowski space. The Doplicher–Haag–Roberts theory adapted to conformal field theory is given as follows. (See [54, 55]. Also see [97, appendix B] for relations to the representation theory of nets on $\mathbb{R}$.)

Take a representation $\pi = \{\pi_I\}$ of a local conformal net $\{A(I)\}$. Fix an arbitrary interval $I_0 \subset S^1$ and consider the representation $\pi_{I_0}$ of $A(I_0)$. Since $A(I_0)$ is a type III factor, the identity representation $A(I_0) \hookrightarrow B(H)$ and $\pi_{I_0}$ are unitarily equivalent. By changing the representation $\pi$ within the unitary equivalence class if necessary, we may and do assume that $H = H_{I_0}$ and $\pi_{I_0}$ is the identity representation.

Take an interval $I_1$ with $I_1 \subset I_0$, and then take an interval $I_2$ containing both $I_1$ and $I_0$. For $x \in A(I_1)$, we have $\pi_{I_0}(xy) = \pi_{I_0}(yx)$ for any $y \in A(I_0)$. This implies $\pi_{I_1}(xy) = yx\pi_{I_1}(x)$, and thus the image of $A(I_1)$ by $\pi_{I_0}$ is contained in $A(I_0)^{\prime} = A(I_0)$. Additivity now implies that the image of $A(I_0)$ by $\pi_{I_0}$ is contained in $A(I_0)$, that is, $\pi_{I_0}$ gives an endomorphism of $A(I_0)$. We say that this representation $\pi$ is localized in $I_0$.

We can construct a universal $C^*$-algebra $C^*(A)$ from the local conformal net $\{A(I)\}$ with the property that $C^*(A)$ contains each $A(I)$ and for any representation $\{\pi_I\}$ of $\{A(I)\}$, we have a representation $\pi$ of $C^*(A)$ with $\pi_{A(I)} = \pi_I$. We identify a representation of $\{A(I)\}$ with the corresponding representation of $C^*(A)$. (See [52, page 382] for details.) Let $\pi_0$ be the representation of $C^*(A)$ corresponding to the vacuum representation of $\{A(I)\}$. For a representation $\pi = \{\pi_I\}$ of $\{A(I)\}$, we construct an endomorphism $\lambda$ of $C^*(A)$ so that $\pi$ is unitarily equivalent to $\pi_0 \circ \lambda$. The endomorphism $\lambda$ is given by a family $\{\lambda_I\}$ of homomorphisms $\lambda_I: A(I) \to C^*(A)$ with $\lambda_{I_0} = \lambda_{I_1}$ for $I_0 \subset I_1$.

Start with a representation $\pi$ of $\{A(I)\}$ localized in an interval $I_0$. We construct the corresponding family $\{\lambda_I\}$ of homomorphisms $\lambda_I: A(I) \to C^*(A)$. If $I \cup I_0$ is not dense in $S^1$, we choose an interval $I_1 \supset I \cup I_0$ and set $\lambda_I = \pi_{I_0} |_{A(I)}$. This is independent of $I$. If $I \cup I_0$ is dense in $S^1$, choose an interval $I_1 \subset I_0 \cap I^\prime$ and let $\pi^\prime$ be a representation of $\{A(I)\}$ which is localized in $I_1$ and unitarily equivalent to $\pi$. The unitary equivalence gives a unitary $V \in B(H)$ with $\text{Ad}(V) \circ \pi(x) = \pi^\prime(x)$ for all $x \in C^*(A)$, and the Haag duality implies $V \in \pi_0(A(I_0))$. Then we have a unitary $U \in A(I_0)$ with $\pi_0(U) = V$. We then set $\lambda_I = \text{Ad}(U)_A(I_0)$. This is independent of $I_0$, $\pi^\prime$, $V$. In this way, we obtain a family $\lambda = \{\lambda_I\}$ of homomorphisms $\lambda_I: A(I) \to C^*(A)$. This also gives an endomorphism, still denoted by $\lambda$, of $C^*(A)$. (See [52, page 383] for details.)

For the family $\lambda = \{\lambda_I\}$ of homomorphisms, we say $\lambda$ is localized in $I_0$ if we have $\lambda_{I_0} = \text{id}$ on $A(I_0)$. We say that $\lambda$ is transportable if for all intervals $I_1, I$ with $I \supset I_0 \cup I_1$, we have a unitary $U \in A(I)$ so that $\text{Ad}(U) \circ \lambda$ is localized in $I_1$. Such a unitary $U$ is called a transporter. We say that an endomorphism of $C^*(A)$ is a DHR endomorphism if it is of the
form $\text{Ad}(U) \cdot \lambda'$ where $U$ is a unitary in $\mathcal{C}^*(\mathcal{A})$ and $\lambda'$ is localized transportable endomorphism of $\mathcal{C}^*(\mathcal{A})$. Considering a representation of $\{\mathcal{A}(I)\}$ and considering a DHR endomorphism of $\mathcal{C}^*(\mathcal{A})$ are equivalent. Two representations of $\{\mathcal{A}(I)\}$ are unitarily equivalent if and only if one of the corresponding DHR endomorphisms of $\mathcal{C}^*(\mathcal{A})$ is an inner perturbation of the other.

We see that a composition of two DHR endomorphisms is again a DHR endomorphism. This defines a tensor product operation of two representations. This also gives us a tensor product of superselection sectors.

We define the dimension of a DHR endomorphism $\lambda$ to be the square root of the index $I_{\mathcal{A} \{()\} \mathcal{F}} [() : (() )]_{\mathcal{A} \{()\} \mathcal{F}}$ when $\lambda$ is localized in $I_{\mathcal{A} \{()\} \mathcal{F}}$. This is independent of $I_{\mathcal{A} \{()\} \mathcal{F}}$ by [74, proposition 2.1]. So the dimension is additive and multiplicative with respect to the direct sum and the tensor product of representations.

For any DHR endomorphism $\lambda$ of $\mathcal{C}^*(\mathcal{A})$, we localize it in the fixed interval $I_{\mathcal{A} \{()\} \mathcal{F}}$ and have an endomorphism $I_{\mathcal{A} \{()\} \mathcal{F}} \lambda$ of $\mathcal{C}^*(\mathcal{A})$. By [74, theorem 2.3], considering these endomorphisms with intertwiners in $\mathcal{C}^*(\mathcal{A})$ is equivalent to considering the representations of $\mathcal{A} \{()\} \mathcal{F}$ with intertwiners. This gives a notion of the contragredient representation of a local conformal net. So we have a unitary tensor category consisting of endomorphisms $I_{\mathcal{A} \{()\} \mathcal{F}} \lambda$ and this gives the representation category of the local conformal net $\mathcal{A} \{()\} \mathcal{F}$. We write $\text{Rep}(\mathcal{F})$ for this. When we have an endomorphism of $I_{\mathcal{A} \{()\} \mathcal{F}} \lambda$ arising from a DHR endomorphism of $\mathcal{C}^*(\mathcal{A})$ localized in $I_{\mathcal{A} \{()\} \mathcal{F}} \subset I_{\mathcal{A} \{()\} \mathcal{F}}$, we say it is localized in $I_{\mathcal{A} \{()\} \mathcal{F}}$.

We next introduce the braiding structure on this unitary tensor category. We first have the following lemma as in [13, lemma 2.1]. This is clear with strong additivity, but actually holds without it.

**Lemma 3.10.** Let $\lambda_1, \lambda_2$ be the endomorphisms of $I_{\mathcal{A} \{()\} \mathcal{F}} \lambda$ localized in $I_{\mathcal{A} \{()\} \mathcal{F}}$, respectively, with $I_{\mathcal{A} \{()\} \mathcal{F}} \subset I_{\mathcal{A} \{()\} \mathcal{F}}$. Then we have $\lambda_1 \cdot \lambda_2 = \lambda_2 \cdot \lambda_1$.

Suppose we have $I_{\mathcal{A} \{()\} \mathcal{F}}$, $I_{\mathcal{A} \{()\} \mathcal{F}}$ with $I_{\mathcal{A} \{()\} \mathcal{F}} \subset I_{\mathcal{A} \{()\} \mathcal{F}}$ and also suppose $\lambda, \mu$ are localized in $I_{\mathcal{A} \{()\} \mathcal{F}} \subset I_{\mathcal{A} \{()\} \mathcal{F}}$. Choose unitaries $U_1, U_2$ in $\mathcal{C}^*(\mathcal{A})$ so that $\text{Ad}(U_1) \cdot \lambda$ and $\text{Ad}(U_2) \cdot \mu$ are localized in $I_{\mathcal{A} \{()\} \mathcal{F}}$, respectively. Because of the above lemma, it is now clear that $\lambda$ and $\mu$ commute up to unitary equivalence. Set $e(\lambda, \mu) = (U_2^* U_1^* U_1 U_2) \lambda$. As in [13, lemma 2.2], $e(\lambda, \mu)$ depends only on the order of $I_{\mathcal{A} \{()\} \mathcal{F}}$ and $I_{\mathcal{A} \{()\} \mathcal{F}}$. When $I_{\mathcal{A} \{()\} \mathcal{F}}$ is forward of $I_{\mathcal{A} \{()\} \mathcal{F}}$ in the counterclockwise order, we write $e^+(\lambda, \mu)$, and for the other case, we write $e^-(\lambda, \mu)$. These are called statistic operators. We then have the following results as in [13, lemma 2.3, proposition 2.5].

**Theorem 3.11.** Let $\lambda, \mu, \nu$ be the localized endomorphisms of $I_{\mathcal{A} \{()\} \mathcal{F}}$. We have the following relations.

$$\text{Ad}(e^\pm(\lambda, \mu)) \cdot \lambda \cdot \mu = \mu \cdot \lambda,$$

$$e^\pm(\lambda, \mu) \in \mathcal{A} \{()\},$$

$$e^\pm(\lambda, \mu) = e^{\mp}(\mu, \lambda)^*,$$

$$e^\pm(\lambda, \mu) = e^{\mp}(\lambda, \nu)\lambda(e^{\pm}(\mu, \nu)),$$

$$e^\pm(\lambda, \mu) = (e^{\pm}(\lambda, \mu))e^{\pm}(\lambda, \mu),$$

$$\nu(t)e^\pm(\lambda, \nu) = e^{\pm}(\mu, \nu)t, \quad t \in \text{Hom}(\lambda, \mu),$$

$$te^{\pm}(\nu, \lambda) = e^{\pm}(\nu, \mu)e^{\pm}(\nu, \lambda), \quad t \in \text{Hom}(\lambda, \mu).$$

The last two identities imply the following braiding fusion equations (BFE’s).
Corollary 3.12. Let $\lambda, \mu, \nu, \rho$ be the localized endomorphisms of $A(I_0)$. We have the following identities for $s \in \text{Hom}(\lambda \cdot \mu, \nu)$.

$$
\rho(s)e^\pm(\lambda, \rho)\lambda\left(e^\pm(\mu, \rho)\right) = e^\pm(\nu, \rho)s,
$$

$$
\lambda\left(e^\pm(\rho, \mu)\right)e^\pm(\rho, \lambda) = e^\pm(\rho, \nu)s.
$$

In this way, DHR endomorphisms localized in $I_0$ gives a unitary braided tensor category of endomorphisms of $A(I_0)$ in the following sense. (See [3, chapter 1], [140, section I.1.2] for a general theory on braided tensor categories.)

Definition 3.13. Let $C$ be a monoidal category with functorial isomorphisms $\sigma_{VW}: V \otimes W \rightarrow W \otimes V$ for all objects $V, W$ in $C$.

For given objects $V_1, V_2, \ldots, V_n$ in $C$, we consider all expressions of the form

$$
\left(\left(V_i \otimes V_j\right) \otimes \left(1 \otimes V_k\right)\right) \otimes \ldots \otimes V_n
$$

obtained from $V_i \otimes V_j \otimes \ldots \otimes V_n$ by inserting some $\text{mathbf{1}}$’s and brackets. where $(i_1, i_2, \ldots, i_n)$ is a permutation of $\{1, 2, \ldots, n\}$. To any composition of $\alpha's, \lambda's, \rho's, \sigma's$ and their inverses acting on the element in the above tensor product, we assign an element of the braid group $B_n$ with the standard generators $b_i$ in $(1, 2, \ldots, 1)$ satisfying $b_i b_j = b_j b_i$ for $|i - j| > 1$ and $b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}$ as follows. To $\alpha, \lambda$ and $\rho$, we assign 1 and to $\sigma_{V_i}$, the generator $b_i$.

The category $C$ is called a braided tensor category if for any two expressions $X_1, X_2$ of the above form and any isomorphism $\phi: X_1 \rightarrow X_2$ obtained by composing $\alpha's, \lambda's, \rho's, \sigma's$ and their inverses, $\phi$ depends only on its image in the braid group $B_n$.

It is sometimes written as DHR $(A)$.

3.3. Graphical intertwiner calculus

We now explain a method to describe these relations involving braiding graphically as in [17, section 3]. We start a general setting of endomorphisms of a type III factor $M$ which is an abstract version of the set of irreducible DHR endomorphisms.

Definition 3.14. We call a finite subset $\Delta \subset \text{End}(M)$ a system of endomorphisms if it satisfies the following properties.

1. Each $\lambda \in \Delta$ is irreducible and has finite dimension.
2. Different elements in $\Delta$ are unitarily inequivalent.
3. We have $\text{id}_M \in \Delta$.
4. For any $\lambda \in \Delta$, we have an endomorphism $\lambda' \in \Delta$ giving the conjugate sector of $\lambda$.
5. The system $\Delta$ is closed under composition and subsequent irreducible decomposition.

The rules for irreducible decompositions of the compositions are called fusion rules.

Definition 3.15. We say that a system $\Delta$ of endomorphisms is braided if for any pair $\lambda, \mu \in \Delta$ there is a unitary operator $\epsilon(\lambda, \mu) \in \text{Hom}(\lambda \mu, \mu \lambda)$ satisfying the identities

$$
\epsilon(\text{id}_M, \mu) = \epsilon(\lambda, \text{id}_M) = 1,
$$

(1)
and the following BFE’s
\[
\begin{align*}
\rho(t) &\in \lambda \mapsto \xi(\lambda, \rho) = \xi(\mu, \rho)\mu(e(\nu, \rho))t,
\end{align*}
\]
\[
\text{for any } \lambda, \mu, \nu \in \Delta \text{ and } t \in \text{Hom}(\lambda, \mu).}
\]

We fix and consider a braided system of endomorphisms \( \Delta \subseteq \text{End}(M) \). We represent intertwiners by ‘wire diagrams’ where the (oriented) wires represent endomorphisms \( \lambda \in \Delta \). This works as follows. For an intertwiner \( x \in \text{Hom}(\lambda_1 \lambda_2 \cdots \lambda_n, \mu_1 \mu_2 \cdots \mu_m) \), we draw a (dashed) box with \( n \) (downward) incoming wires labelled by \( \lambda_1, \ldots, \lambda_n \) and \( m \) (downward) outgoing wires \( \mu_1, \ldots, \mu_m \) as in figure 1, \( \lambda_i, \mu_j \in \Delta \). This means the diagrammatic representation of \( x \) does not only specify it as an operator, it also specifies the intertwiner space to which it belongs. If a morphism \( \rho \in \Delta \) is applied to \( x \), then \( \rho x \in \text{Hom}(\rho \lambda_1 \lambda_2 \cdots \lambda_n, \rho \mu_1 \mu_2 \cdots \mu_m) \) is represented graphically by adding a straight wire on the left as in figure 2. Since we can consider \( x \) also as an intertwiner in \( \text{Hom}(\lambda_1 \lambda_2 \cdots \lambda_n, \rho \mu_1 \mu_2 \cdots \mu_m) \), we can always add (or remove) a straight wire on the right as in figure 3 without changing the intertwiner as an operator. We say that intertwiners \( x \in \text{Hom}(\lambda_1 \lambda_2 \cdots \lambda_n, \mu_1 \mu_2 \cdots \mu_m) \) and \( y \in \text{Hom}(\nu_1 \nu_2 \cdots \nu_k, \rho_1 \rho_2 \cdots \rho_l) \), \( \rho_i \in \Delta \), are diagrammatically composable if \( m = k \) and \( \mu_i = \nu_i \) for all \( i = 1, 2, \ldots, m \). Then the composed intertwiner \( xy \in \text{Hom}(\lambda_1 \lambda_2 \cdots \lambda_n, \rho_1 \rho_2 \cdots \rho_l) \) is represented graphically by putting the wire diagram for \( x \) on top of that for \( y \) as in figure 4. (Note that some authors use an opposite convention where we compose intertwiners from the bottom to the top.) Now let also \( x' \in \text{Hom}(\lambda_1' \lambda_2' \cdots \lambda_n', \mu_1' \mu_2' \cdots \mu_m') \) with \( \lambda_i', \mu_j' \in \Delta \). The
intertwining property of \( x \) give the identity
\[
\mu_1 \mu_2 \cdots \mu_n 
\]
and this is diagrammatically given in figure 5. We thus have some freedom in translating intertwiner boxes vertically without changing the represented intertwiner.

The intertwiners we consider are (sums over) compositions of elementary intertwiners arising from the unitary braiding operators \( \varepsilon(\lambda, \mu) \in \text{Hom}(\lambda \mu, \lambda \mu) \) and isometries \( t \in \text{Hom}(\lambda, \mu) \). The wire diagrams and boxes we deal with are compositions of ‘elementary boxes’ representing the elementary intertwiners. We now have to introduce some normalization convention. First, the identity intertwiner \( 1 = 1_M \) is naturally given by the ‘trivial box’ with only straight wires of any labels. The next elementary intertwiner is \( \rho_1 \rho_2 \cdots \rho_k(\varepsilon(\lambda, \mu)) \) for which we have a diagram as in figure 6 where the arbitrary labels \( \nu_1, \ldots, \nu_m \) are irrelevant.

Similarly, the box of figure 7 represents the elementary intertwiner \( d_\mu^{1/4} d_\nu^{1/4} d_\lambda^{-1/4} \rho_1 \rho_2 \cdots \rho_k(t) \), where \( t \in \text{Hom}(\lambda, \mu \nu) \) is an isometry. We label the trivalent vertex in the box with an isometry \( t \) to label elements in \( \text{Hom}(\lambda, \mu \nu) \). Finally, the elementary intertwiners \( \varepsilon(\lambda, \mu)^* = \varepsilon^-(\mu, \lambda) \) and \( d_\mu^{1/4} d_\nu^{1/4} d_\lambda^{-1/4} \rho_1 \rho_2 \cdots \rho_k(t)^* \) are similarly represented as the vertical
reflections. Note that $\epsilon \equiv \epsilon^+$ represents overcrossing and $\epsilon^-$ undercrossing of wires. We consider intertwiners which are products of diagrammatically composable elementary intertwiners. Note that if a wire diagram represents some intertwiner $x$, then $x^*$ is represented by the diagram obtained by vertical reflection and reversing all the arrows. Note that our resulting wire diagrams are then composed only from straight lines, over- and undercrossings and trivalent vertices.

So far, we have considered only wires with downward orientation. We now introduce also the reversed orientation in terms of conjugation as follows. Reversing the orientation of an arrow on a wire changes its label $\lambda$ to $\bar{\lambda}$. Also we usually omit drawing a wire labelled by $\text{id} \equiv \text{id}_M$. For each $\lambda \in \Delta$, we fix (the common phase of) isometries $r_\lambda \in \text{Hom} (\text{id}, \bar{\lambda} \lambda)$ and $\bar{r}_\lambda \in \text{Hom} (\text{id}, \lambda \bar{\lambda})$ arising from the Frobenius reciprocity $\text{Hom}(\lambda, \lambda) \equiv \text{Hom} (\text{id}, \lambda \bar{\lambda}) \equiv \text{Hom} (\text{id}, \lambda \bar{\lambda})$ so that we have $\lambda (r_\lambda)^\rho \bar{r}_\lambda = \bar{\lambda} (\bar{r}_\lambda)^\rho r_\lambda = \Delta^{-1}$ and in turn for $\sqrt{\Delta} r_\lambda$ we draw one of the equivalent diagrams in figure 8. So the normalized isometries and their adjoints appear in wire diagrams as ‘caps’ and ‘cups’, respectively. The

![Figure 8. Wire diagrams for $\sqrt{\Delta} r_\lambda$.](image)

![Figure 9. A topological invariance for intertwiners represented by wire diagrams.](image)

![Figure 10. Wire diagrams for the statistical dimension $d_\lambda$.](image)

![Figure 11. The first braiding fusion equation.](image)
point is that with our normalization convention, the relation $\tilde{\lambda}(r_\lambda)\tilde{r}_\lambda = d_{\lambda}^{-1}1$ (and its adjoint) gives a topological invariance for intertwiners represented by wire diagrams, displayed as in figure 9. Note that then the wire diagrams in figure 10 represent the scalar $d_\lambda$ (i.e., the intertwiner $d_\lambda 1 \in \text{Hom}(\text{id}, \text{id})$).

The BFE’s give another topological invariance, as in figure 11 for the first equation.

The second, third and fourth equations are obtained similarly. Up to conjugation they can also be obtained by changing over- to undercrossings in figure 11.

The topological invariance gives us the freedom to write down the intertwiner algebraically from a given wire diagram. We can deform the wire diagram by finite sequences of the above moves and then split it into elementary wire diagrams—in whatever way we may decompose the wire diagrams into horizontal slices of elementary intertwiners, we always obtain the same intertwiner due to our topological invariance identities.

Next, the statistics phase $\omega_\lambda$ is defined by the wire diagram on the left-hand side of figure 12, which is equal to the intertwiner $d_\lambda r_\lambda^* \tilde{\lambda}(\epsilon(\lambda, \lambda)) r_\lambda$. The diagram on the right-hand side expresses that $\omega_\lambda$ can also be obtained as $d_\lambda \tilde{\lambda}(r_\lambda)^* \epsilon(\tilde{\lambda}, \lambda) (r_\lambda)$. We also define a matrix element $Y_{\lambda,\mu} = Y_{\mu,\lambda}$ of Rehren’s Y-matrix [127, section 5] as $d_\lambda d_\mu \phi(\epsilon(\lambda, \mu) \epsilon(\mu, \lambda)) \tilde{r}_\mu^* \tilde{r}_\mu = d_\lambda d_\mu r_\mu^* \tilde{\mu}(\epsilon(\tilde{\mu}, \mu) \epsilon(\mu, \tilde{\lambda})) r_\mu$.

We have drawn the circle $\mu$ symmetrically relative to the straight wire $\lambda$ because it does not make a difference whether we put the ‘caps’ and ‘cups’ for the isometry $r_\mu$ and its conjugate $r_\mu^*$ on the left or on the right due to the braiding fusion relations. Since it is a scalar, we can write $Y_{\lambda,\mu} = r_\mu^* Y_{\mu,\lambda} r_\mu$ and therefore its expression $d_\lambda d_\mu r_\mu^* r_\mu^* \tilde{\lambda}(\epsilon(\mu, \lambda) \epsilon(\mu, \lambda)) r_\mu r_\mu$ exactly gives the Hopf link as the wire diagram for the matrix element $Y_{\lambda,\mu}$ given by the left-hand side of figure 13. The equality to the right-hand side is just the relation $Y_{\lambda,\mu} = Y_{\mu,\lambda}$. We put $S_{\lambda,\mu} = Y_{\lambda,\mu} \sqrt{w}$ and $T_{\lambda,\mu} = (\sigma | \sigma \rangle)^{\frac{1}{2}} \delta_{\lambda,\mu} \omega_{\lambda}$, where $w = \sum_{j \in A} d_j^2$ and $\sigma = \sum_{j \in A} d_j^2 \omega_j^{-1}$. We say that $\lambda$ is degenerate if we have $Y_{\lambda,\mu} = d_\lambda d_\mu$ for all $\mu \in \Delta$. This is equivalent to the condition that we have $\epsilon^*(\lambda, \mu) = \epsilon^*(\lambda, \mu)$ for all $\mu \in \Delta$ by [127, lemma on page 352]. We have the following proposition as in [127, page 351].
Proposition 3.16. The following are equivalent.

1. The identity morphism is the only degenerate morphism in $\Delta$.
2. The matrix $Y(\lambda \mu, \lambda' \mu')$ is invertible.

If the above two conditions hold, we can define a unitary representation $\pi$ of $SL(2, \mathbb{Z})$ by setting

$$
\pi \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = S, \quad \pi \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = T.
$$

(See [127, p 351].) In this case, we say that the tensor category consisting of endomorphisms of $M$ unitarily equivalent to a finite direct sum of endomorphisms in $\Delta$ is a unitary modular tensor category in the following abstract sense. (See [3, chapter 3], [140, section II.1.4] for more details on modular tensor categories. See [131, 140, chapter IV], [99] for a $(2 + 1)$-dimensional topological quantum field theory arising from a modular tensor category.) We also say that the braiding is non-degenerate in this case.

Definition 3.17. A ribbon category is a rigid braided tensor category $\mathcal{C}$ with functorial isomorphisms $\delta_V : V \to V^{**}$ satisfying $\delta_V \otimes w = \delta_V \otimes \delta_W$, $\delta_1 = \text{id}$, and $\delta_{V^*} = (\delta_V)^{-1}$ for all objects $V, W$ in $\mathcal{C}$.

Definition 3.18. A modular tensor category is a semisimple ribbon category with the following properties.

1. We have only finitely many isomorphism classes of simple objects.
2. The matrix $(Y_{\lambda \mu})$, defined as above for the representatives $\lambda, \mu$ for the isomorphism classes of simple objects, is invertible.

In this case, we have the celebrated Verlinde formula,

$$
N^V_{\lambda \mu} = \sum_{\sigma} \frac{S_{\lambda \sigma} S_{\mu \sigma} S_{v^* \sigma}}{S_{0, \sigma}},
$$

(3)

where the non-negative integer $N^V_{\lambda \mu}$ is determined by the fusion rules $\lambda \cdot \mu = \sum \lambda' N^V_{\lambda' \mu'}$.

Wire diagrams can also be used for intertwiners of morphisms between different factors. Let $M, N, P$ infinite factors, $\rho \in \text{Mor}(M, N)$, $\sigma \in \text{Mor}(P, N)$, $\tau \in \text{Mor}(M, P)$ irreducible morphisms and $t \in \text{Hom}(\rho, \sigma)$ an isometry. Then figure 14 represents the intertwiner $d_{1/4} d_{1/4} d_{-1/4} t$. Similarly we can draw a picture using a co-isometry. Along the lines of the previous paragraphs, we can similarly build up larger wire diagrams out of trivalent vertices involving different factors. So far we do not have a meaningful way to cross wires with differently labelled regions left and right, but all the arguments listed above which do not involve braiding can be used for intertwiners of morphisms between different factors exactly as proceeded above. Moreover, the diagrams may also involve wires where left and right
regions are labelled by the same factor, i.e., these wires correspond to endomorphisms of some factor which may well form a braided system, and then one may have crossings for those wires.

Let $MNP$, $PN\text{Mor}(\ ,\ )\rho\in MN\text{Mor}(\ ,\ )\tau\in MP\text{Mor}(\ ,\ )\sigma\in$ morphisms with finite dimensions. For $t\in \text{Hom}(\tau,\rho\sigma)$, we have the identity as in figure 15 from the Frobenius reciprocity. Here the trivalent vertex in the right diagram represents the co-isometry given by the Frobenius reciprocity.

If we have a link diagram whose components are labelled with elements of $\Delta$, the entire diagram denotes an intertwiner from $\text{id}$ to $\text{id}$, that is, a scalar. This is a $\mathbb{C}$-valued regular isotopy invariant of a colored link, since this number is invariant under the Reidemeister moves of types II and III [23, definition 1.13]. (Labeling a wire is called coloring in this context.) In general, this is not a topological invariant since the twist $\omega_{ij}$ is not necessarily 1 and the number is not invariant under the Reidemeister move of type I. This is essentially a generalization of the Jones polynomial at specific values. (See [87] for the original Jones polynomial.)

3.4. Complete rationality and modular tensor categories

A unitary braided tensor category of representations of a local conformal net is similar to that of those of a quantum group at a root of unity. In such a representation theory, it is important to consider a case where we have only finitely many irreducible representations up to unitary equivalence. Such finiteness is often called rationality. This name comes from the fact that such finiteness of representation theory gives rationality of various parameters in conformal field theory. Based on this, we introduce the following notion.

Definition 3.19. Let $\{A(I)\}$ be a local conformal net with the split property. Split the circle $S^1$ into four intervals and label them $I_1$, $I_2$, $I_3$, $I_4$ in the clockwise order. If the subfactor $\mathcal{A}(I_1) \vee \mathcal{A}(I_3) \subset (\mathcal{A}(I_2) \vee \mathcal{A}(I_4))'$ has a finite index, we say that the local conformal net $\{A(I)\}$ is completely rational.

The reason we call this complete rationality comes from the following theorem [97, theorem 33, corollary 37].

**Theorem 3.20.** When a local conformal net is completely rational, then it has only finitely many irreducible representations up to unitary equivalence, and all of them have finite dimensions. When this holds, the unitary braided tensor category of finite dimensional representations of $\{A(I)\}$ is a unitary modular tensor category and the index of the above subfactor $\mathcal{A}(I_1) \vee \mathcal{A}(I_3) \subset (\mathcal{A}(I_2) \vee \mathcal{A}(I_4))'$ is equal to the square sum of the dimensions of the irreducible representations.
We have finite dimensionality of the irreducible representations and this is why we have added the word 'completely'. Note that it is difficult in general to know all the irreducible representations, but the above theorem gives information on representations from a subfactor defined in the vacuum representation.

We call the index of the above subfactor $\mathcal{A}(I_{1}) \vee \mathcal{A}(I_{2}) \subset (\mathcal{A}(I_{2}) \vee \mathcal{A}(I_{1}))'$ the $\mu$-index of $[\mathcal{A}(I)]$. This index is independent of the choice of $I_{1}, I_{2}, I_{3}, I_{4}$. (See [97, proposition 51].) It has been proved in [114, theorem 5.3] that a completely rational local conformal net is strongly additive. (In this text, we consider only local conformal nets, but one sometimes deals with local Möbius covariant nets. Then one has to include strong additivity as one of the requirements.)

The above theorem implies that for a local conformal net with the split property, all of its irreducible representations are unitarily equivalent to the vacuum representation if and only if the local conformal net has $\mu$-index 1, since the vacuum representation has dimension 1. We call such a local conformal net holomorphic. This name comes from holomorphicity of the partition function of a full conformal field theory.

An outline of the proof of the first half of theorem 3.20 in [97] is as follows. Let $\rho$ and $\sigma$ be finite dimensional irreducible localized endomorphisms of $\{\mathcal{A}(I)\}$ localized in $I_{1}$ and $I_{2}$, respectively. Then $\rho \sigma |_{\mathcal{A}(I_{1}) \vee \mathcal{A}(I_{2})}$ gives an endomorphism of $\mathcal{A}(I_{1}) \vee \mathcal{A}(I_{2})$. Let $\lambda$ be the dual canonical endomorphism for the subfactor $\mathcal{A}(I_{1}) \vee \mathcal{A}(I_{2}) \subset (\mathcal{A}(I_{2}) \vee \mathcal{A}(I_{1}))'$. The Frobenius reciprocity applied to $\iota: \mathcal{A}(I_{1}) \vee \mathcal{A}(I_{2}) \hookrightarrow (\mathcal{A}(I_{2}) \vee \mathcal{A}(I_{1}))'$ gives

$$\text{Hom}(\lambda, \rho \sigma |_{\mathcal{A}(I_{1}) \vee \mathcal{A}(I_{2})}) \cong \text{Hom}(\iota, \iota \rho \sigma |_{\mathcal{A}(I_{1}) \vee \mathcal{A}(I_{2})})$$

since $\lambda = \iota_{I}$ and then we have an isometry $\nu \in (\mathcal{A}(I_{2}) \vee \mathcal{A}(I_{1}))'$ satisfying $\nu \sigma = \rho \sigma(\iota x) \nu$ for all $x \in \mathcal{A}(I_{2}) \vee \mathcal{A}(I_{1})$ if and only if $\rho \sigma |_{\mathcal{A}(I_{1}) \vee \mathcal{A}(I_{2})}$ appears in the irreducible decomposition of $\lambda$. If $\nu$ satisfies the above identity, the strong additivity implies the same identity for all $x \in \mathcal{A}(I)$ for all $I$. This gives $\sigma = \tilde{\rho}$ as representations of $\{\mathcal{A}(I)\}$ again by the Frobenius reciprocity. Conversely, if we have $\sigma = \tilde{\rho}$, then we have an isometry $\nu \in \mathcal{A}(I)$ satisfying $\nu \sigma = \rho \sigma(\iota x) \nu$ for all $x \in \mathcal{A}(I)$ where $\tilde{I}$ is given by $\tilde{I} = I_{1} \cup I_{2} \cup I_{3}$. Since $\sigma$ and $\rho$ act trivially on $\mathcal{A}(I_{2})$, we have $\nu \in \mathcal{A}(I_{2})' \cap \mathcal{A}(\tilde{I})$, and we also have $\mathcal{A}(I_{2})' \cap \mathcal{A}(\tilde{I}) = (\mathcal{A}(I_{2}) \vee \mathcal{A}(I_{1}))'$. This means $\lambda$ contains $\rho \sigma |_{\mathcal{A}(I_{1}) \vee \mathcal{A}(I_{2})}$ as a direct summand with multiplicity 1.

This now implies that for all the representations $\{\rho_{i}\}$ of $\{\mathcal{A}(I)\}$, the direct sum $\bigoplus_{i} \rho_{i} \tilde{\rho}_{i} |_{\mathcal{A}(I_{1}) \vee \mathcal{A}(I_{2})}$ is a direct summand in the irreducible decomposition of $\lambda$, where $\rho_{i}$ is localized in $I_{1}$ and $\tilde{\rho}_{i}$ is localized in $I_{2}$. Considering the representation theory of a new type of 'net' which assigns $\mathcal{A}(I) \bigotimes \mathcal{A}(j)$ to each interval $I$ where $j$ is the complex conjugation map on the unit circle $\mathbb{S}^{1}$ in the complex plane, we find that an irreducible direct summand of $\lambda$ must be of the form $\rho \sigma |_{\mathcal{A}(I_{1}) \vee \mathcal{A}(I_{2})}$, so this gives that $\bigoplus_{i} \rho_{i} \tilde{\rho}_{i} |_{\mathcal{A}(I_{1}) \vee \mathcal{A}(I_{2})}$ is actually equal to $\lambda$.

We also show that we do not have any infinite dimensional irreducible representation of $\{\mathcal{A}(I)\}$ by [97, corollary 39]. This completes the proof of the first half of theorem 3.20, because $d_{\lambda} = \mu$. An outline of the proof of the second half is given after introduction of $\mu$-induction.

A conformal subnet $\{\mathcal{B}(I)\}$ of $\{\mathcal{A}(I)\}$ is an isotonic map assigning $\mathcal{B}(I) \subset \mathcal{A}(I)$ to each interval $I$ satisfying $U(g) \mathcal{B}(I) U(g)^{*} = \mathcal{B}(gI)$ for all $g \in \text{Diff}(\mathbb{S}^{1})$. Each $\mathcal{B}(I)$ is automatically a factor since the modular automorphism group $\sigma_{\mathcal{B}}$ is ergodic on $\mathcal{B}(I)$. The Reeh–Schlieder theorem adapted to this setting implies that the Hilbert space $\mathcal{H}_{0} = \overline{\mathcal{B}(I^{+})}$ is independent of $I$. The restriction of $\{\mathcal{B}(I)\}$ to $\mathcal{H}_{0}$ is then a local conformal net and we write $\{\mathcal{B}_{0}(I)\}$ for this. We say that $\{\mathcal{B}_{0}(I)\}$ is a subnet of $\{\mathcal{A}(I)\}$ and we often identify $\{\mathcal{B}(I)\}$
with \( \{ B_0(I) \} \). We also say that the local conformal net \( \{ A(I) \} \) is an extension of \( \{ B(I) \} \).
When we have \( B(I) \cap A(I) = \emptyset \), we say that the extension is irreducible. The index
\( [A(I): B(I)] \) is independent of \( I \). We have the following theorem.

**Theorem 3.21.** Let \( \{ B(I) \subset A(I) \} \) be an inclusion of local conformal nets with the split property
having the index \( [A(I): B(I)] \). We write \( \mu_A \) and \( \mu_B \), for the \( \mu \)-indices of \( \{ A(I) \} \) and
\( \{ B(I) \} \), respectively. Then we have \( \mu_B = \mu_A \mu[A(I): B(I)]^2 \). In particular, if one of the two
local conformal nets \( \{ A(I) \} \), \( \{ B(I) \} \) is completely rational, then so is the other.

An outline of the proof of theorem 3.21 in [97] is as follows. On the Hilbert space \( H \) of
\( \{ A(I) \} \), the index of the subfactor \( A(I_1) \vee A(I_2) \subset (A(I_2) \vee A(I_1))' \) is \( \mu_A \). The indices of
the subfactor \( B(I_1) \vee B(I_2) \subset A(I_1) \vee A(I_2) \) and \( B(I_2) \vee B(I_1) \subset A(I_2) \vee A(I_1) \) are both
equal to \( [A(I): B(I)]^2 \) by the split property and proposition 2.12 (1). The index of the
subfactor \( B(I_1) \vee B(I_2) \subset (B(I_2) \vee B(I_1))' \). The Hilbert space \( H \) for \( \{ A(I) \} \) is larger than the one \( H_0 \) for
\( \{ B(I) \} \), so the index of the subfactor is multiplied by the square of the relative size of \( \{ A(I) \} \)
with respect to \( \{ B(I) \} \). See the proof of [97, proposition 24]. These relations give
\( \mu_A = \mu_B [A(I): B(I)]^2 \) with proposition 2.12 (2). The conclusion on the split property follows
from [110, lemma 22].

Suppose we have a local conformal net \( \{ A(I) \} \) and its irreducible extension \( \{ B(I) \} \).
Regarding the vacuum representation of \( \{ B(I) \} \) as a representation \( \{ A(I) \} \), we have an
irreducible decomposition \( \bigoplus n_j \lambda_j \), where \( n_j \) is an integer representing the multiplicity. The irreducibility implies that the multiplicity of the vacuum representation of \( \{ A(I) \} \) is 1. This gives a \( Q \)-system where the endomorphism is given as a direct sum of irreducible representations of \( \{ A(I) \} \). Conversely, if an endomorphism \( \bigoplus n_j \lambda_j \), where \( n_0 = 1 \) and \( \lambda_j \)'s are irreducible representations of a local conformal net \( \{ A(I) \} \), gives a \( Q \)-system with locality, then it gives an extension of \( \{ A(I) \} \). We first construct an extension \( B(I) \) for one interval \( I \), then we construct the family \( \{ B(I) \} \) using transporters. (See [111, theorem 4.9] for details. Conformal covariance of the extension does not follow from [111, theorem 4.9], but the arguments in the proof of [26, proposition 3.7 (b)] gives conformal covariance of the extension.) The index of the extension is equal to \( \sum n_j d_j \). (Here locality of the \( Q \)-system corresponds to locality of the extension.)

As we have seen, the finite dimensional representations of a local conformal net give a unitary modular tensor category. Conversely, consider the following realization problem.

**Problem 3.22.** For a given unitary modular tensor category \( C \), find a completely rational
local conformal net whose finite dimensional representations gives \( C \).

We believe this problem always has a positive answer, but this is still an open problem. The reason to expect a positive answer is that as long as we have appropriate amenability on operator algebras, similar realization problems for subfactors, group actions and so on, have always had positive answers. Typical examples where amenability controls algebraic structures uniformly are [33] and [124]. A completely rational local conformal net is ‘amenable’ in any aspect, since each factor for each interval is injective and the representation category has finitely many simple objects. (The latter should be regarded as an amenable case, just like finite groups are amenable.)

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As a particular case of the above problem, consider unitary modular tensor categories arising from the ‘quantum doubles’ of unitary fusion categories. The quantum double construction of Drinfel’d [48] has a subfactor counterpart and it was first considered by Ocneanu [118] under the name of the ‘asymptotic inclusion’. (See [51, chapter 12] for more details.) As we have seen in theorem 2.19, any unitary fusion category is realized as that of $M$-$M$ bimodules of a hyperfinite subfactor $N \subset M$ of type $\text{II}_1$ with finite index and finite depth. Then the $M_\infty$-$M_\infty$ bimodules of the asymptotic inclusion $M \vee (M' \cap M_\infty) \subset M_\infty$ give a unitary modular tensor category, which is the quantum double of the original unitary fusion category, where the $\text{II}_1$ factor $M_\infty$ is given as the limit of the Jones basic construction as in [51, chapter 12]. This construction has been formulated in the setting of type III subfactors using the $Q$-system in [111, proposition 4.10] and studied in detail in [82, 83]. The $Q$-system is described as follows.

Let $M$ be a factor of type III and suppose we have a system $\Delta = \{ \lambda_i \}_{i=0}^n$ of irreducible endomorphisms of $M$. Let $M^{opp}$ the opposite algebra of $M$, where the order of the multiplication is reversed, and $j$ be the natural map sending $x \in M$ to $x^* \in M^{opp}$. We set $\lambda_i = \bigotimes_{i=0}^n \lambda_i \otimes (j \cdot \lambda_i \cdot j^{-1})$. We can then find $v, w$ satisfying the axioms for the $Q$-system. The index of the subfactor arising from this $Q$-system is equal to $\sum d_i^2$.

In this context, the construction of a subfactor is often called the Longo–Rehren subfactor and the $Q$-system is called the Longo–Rehren $Q$-system. A generalization of the asymptotic inclusion to another direction has been given in [125].

A particular example of the quantum double construction is given as follows. Let $G$ be a finite group. The group semiring consisting of the elements of the form $\sum_{g \in G} n_g g$ where $n_g \in \{0, 1, 2, \ldots\}$ gives the objects of a unitary fusion category where we have $\text{Hom}(g, h) = \delta_{g,h} \mathbb{C}$. The quantum double of this unitary fusion category is the quantum double of $G$ in the usual sense. This finite group $G$ gives another unitary braided tensor category of finite dimensional unitary representations of $G$. The braiding $\epsilon^+ (\lambda, \mu) = \epsilon^- (\lambda, \mu)$ is given by the canonical switching of the two tensor components. The quantum double of this (degenerate) unitary braided tensor category is also the same.

We then consider the following problem.

**Problem 3.23.** For a given unitary fusion category, find a completely rational local conformal net whose finite dimensional representations give the quantum double of the given unitary fusion category.

If we find such a local conformal net, it is easy to see that it has an extension, using the dual of the Longo–Rehren $Q$-system given in [83, section 6], and then the $\mu$-index of the extension is 1 by theorem 3.21 because the original $\mu$-index and the square of the index of the Longo–Rehren $Q$-system are equal. So we expect to be able to construct an answer to the above problem as a subnet of a holomorphic local conformal net. This should be an ‘orbifold construction for a paragroup action’, but we do not know yet such a construction. (See example 3.35 for the orbifold construction.) The work [50] gives a first step to this direction for a particular example of the Haagerup subfactor.

### 3.5. $\alpha$-Induction and modular invariants

Consider an inclusion $\{ A(I) \subset B(I) \}$ of a local conformal net and its subnet. Suppose the index $[B(I) : A(I)]$ is finite. We are interested in the case the smaller local conformal net $\{ A(I) \}$ is completely rational.
In classical representation theory, if we have a representation of a group $H$ with a larger group $G \supset H$, then we have an induced representation of $G$. We would like to have its analogue for local conformal nets. Fix an interval $I \subset \mathcal{S}$. A representation of a local conformal net $\{\mathcal{A}(I)\}$ is realized as a localized endomorphism of $\mathcal{A}(I)$, so denote this endomorphism by $\lambda$. We would like to extend $\lambda$ to an endomorphism of $\mathcal{B}(I)$, using the braiding between $\lambda$ and the dual canonical endomorphism of $\mathcal{A}(I) \subset \mathcal{B}(I)$. This construction is called the $\alpha$-induction. This was first defined in [11], and examples and applications were first given in [14] and further developed in [13–15]. This has been unified with Ocneanu’s graphical construction [119] on the Dynkin diagrams and generalized in [17, 18].

The construction of the extension of the endomorphism is as follows. This works on a more abstract setting where we have only a unitary braided tensor category of endomorphisms, so we present the basics in this abstract setting.

Let $N \subset M$ be a type III subfactor with finite index. Suppose we have a braided system $\Delta$ of endomorphisms of $N$ and the dual canonical endomorphism for $N \subset M$ has an irreducible decomposition within $\Delta$. We write $\iota: N \leftrightarrow M$ for the inclusion map and $\theta = \iota \bar{\iota}$ for the dual canonical endomorphism of $N \subset M$. We recall that we have an isometry $\nu \in \mathcal{M}$ satisfying $\mathcal{M} \nu = \nu$ and $\nu x = \theta(x) \nu$ for all $x \in \mathcal{M}$.

**Definition 3.24.** For $\lambda$ unitarily equivalent to a finite direct sum of endomorphisms in $\Delta$, we set

$$\alpha^\pm = \iota^{-1} \cdot \text{Ad}(e^\pm(\lambda, \theta)) \cdot \lambda \cdot \iota,$$

and call it the $\alpha^\pm$-induction of $\lambda$.

It is easy to see that $\alpha^\pm(x) = \lambda(x)$ for any $x \in N$. For $\nu \in \mathcal{M}$, we have $\alpha^\pm(\nu) = e^\pm(\lambda, \theta)\nu$ as in [13, lemma 3.1], so the map $\alpha^\pm$ is an endomorphism of $\mathcal{M}$ and regarded as an $M$-$M$ sector. It is easier to see the meaning of this construction in the context of bimodules as follows. Suppose we have a braided system of $N$-$N$ bimodules for a type $\text{II}_1$ subfactor $N \subset M$ so that the bimodule $\mathcal{M}_N$ has an irreducible decomposition within the system. Then for an $N$-$N$ bimodule $\mathcal{M}_N$ in the system, we have an isomorphism $\gamma^\pm M_M \equiv N \otimes N X \otimes N M_N$ through the braiding. The left hand side expression has a right action of $M$ and the right hand side expression has a left action of $M$. We see that these two actions commute and we have an $M$-$M$ bimodule. This construction of an $M$-$M$ bimodule from an $N$-$N$ bimodule is the bimodule version of the $\alpha$-induction.

We also introduce the following property.

**Definition 3.25.** When we have $\epsilon^+(\theta, \theta)\nu = \nu$, we say that the $Q$-system for $N \subset M$ is local.

The Longo–Rehren $Q$-system arising from a braided system $\Delta$ is local [11, proposition 4.10]. If $N \subset M$ arises from an inclusion of local conformal nets, this locality holds as in [11, theorem 4.9] from the locality of the larger conformal net.

We have the following result as in [13, theorem 3.9].

**Theorem 3.26.** If we have locality, then we have

$$\langle \alpha^\pm, \alpha^\pm \rangle = \langle \lambda \cdot \theta, \mu \rangle.$$

An outline of a proof is as follows. We have
We can actually show that the locality implies an equality in the first inequality.

We now assume that the braiding on $\Delta$ is non-degenerate. Consider the two generators $S_{01}, T_{11}$ of $\text{SL}(2, \mathbb{Z})$. Since we have a unitary modular tensor category arising from $\Delta$, we have a finite dimensional unitary representation of $\text{SL}(2, \mathbb{Z})$ as before, whose dimension is equal to the number of the endomorphisms in $\Delta$. We drop the symbol for a representation and simply write $S_{T}$, for the images of the matrices $S_{T}, T$. Take $\lambda, \mu \in \Delta$ and set $Z_{\lambda, \mu} = \dim \text{Hom}(\alpha_{\lambda}, \alpha_{\mu})$. Then we have the following theorem ([17, corollary 5.8]).

**Theorem 3.27.** The above matrix $(Z_{\lambda, \mu})_{\lambda, \mu}$ has the following properties.

1. $Z_{\lambda, \mu} \in \mathbb{N} = \{0, 1, 2, ...\}$.
2. We have $Z_{\lambda, 0} = 1$, where the label 0 denotes the vacuum representation.
3. We have $Z_{S} = Z_{T}$.
4. We have $Z_{T} = Z_{T}$.

A proof of theorem 3.27 is based on graphical expression of $Z_{\lambda, \mu}$ as in figure 16, where $w = \sum_{\lambda \in \Delta} d_{\lambda}^2$ and $b, c$ are $N$-$M$ sectors. We also follow the graphical convention of [17, figures 33 and 41] here. (We actually do not need the non-degeneracy of the braiding here.)

A matrix $Z$ satisfying the above four conditions is called a **modular invariant**. It is easy to see that we have only finitely many modular invariants for each unitary modular tensor category. Modular invariants arising from $SU(2)_k$ and $SU(3)_k$ have been classified in [25, Table 1], [68], respectively.

The following terminology comes from [119].

**Definition 3.28.** When an $M$-$M$ sector appears in both of the irreducible decompositions of the images of $\alpha^z$-inductions, we say the $M$-$M$ sector is ambichiral.

The following is a combination of proposition 3.1 and theorem 4.2 in [18], because the dual canonical endomorphism $\theta$ for the local case is given by $\theta = \oplus_{\lambda} Z_{0,\lambda}$. Then we have the following theorem 3.26.
Theorem 3.29. Suppose the Q-system for $N \subset M$ is local. Let $d_1 = \sum_{\lambda \in \Delta} d_{\lambda}^2$ and $d_2$ is the square sum of the dimensions of the irreducible M-M sectors. Then we have $d_2[M; N] = d_1$.

The proof in [18] is based on graphical manipulations of diagrams. We do need the non-degeneracy of the braiding here.

In the application of the above construction we are interested in, we have an inclusion $\{A(I) \subset B(I)\}$ of a completely rational local conformal net and its extension with finite index. We fix an interval $I$ and set $N = A(I), M = B(I)$ and let $\Delta$ be the system of endomorphisms of $N$ arising from the irreducible DHR endomorphisms of $\{A(I)\}$ localized in $I$. The embedding $A(I) \hookrightarrow B(I)$ on the Hilbert space of $\{B(I)\}$ gives a representation of $\{A(I)\}$, so it decomposes within $\Delta$, and it also gives the dual canonical endomorphisms for $N \subset M$. Then the $\alpha$-induction $\alpha_\Delta^+$ of $\lambda \in \Delta$ gives an M-M sector.

We now present an outline of the proof of the second half of theorem 3.20 in [97].

Consider the unitary braided tensor category of finite dimensional representations of a completely rational local conformal net $\{A(I)\}$ and regard it as the one arising from a braided system $\Delta$ of endomorphisms of a type III factor $N$. Let $\{\mathcal{A}^{\text{opp}}(I)\}$ be the local conformal net which assigns $A(I)^{\text{opp}}$ to each interval $I$.

Consider the Longo–Rehren Q-system $(\xi, \nu, w)$ on $N \otimes N^{\text{opp}}$ corresponding to $\Delta$ of the irreducible endomorphisms of $N = A(I)$ arising from irreducible representations of $\{A(I)\}$. Since it is local, it gives an extension $A(I) \otimes \mathcal{A}^{\text{opp}}(I) \subset B(I)$ of a local conformal net. We apply the $\alpha$-induction for the subfactor $A(I) \otimes \mathcal{A}^{\text{opp}}(I) \subset B(I)$. If we have a degenerate braiding on $\Delta$, then we have a non-trivial degenerate endomorphism $\rho$ in $\Delta$, but we then have $\alpha_\rho = \alpha_\rho^{\otimes 4} = \alpha_\rho^{\otimes 4}$ by degeneracy of the braiding and this gives a representation of $\{B(I)\}$ by [111, proposition 3.9], and this is different from the identity endomorphism by theorem 3.26. This is a contradiction to the identity $\mu_\lambda = 1$ which follows from theorem 3.21, since the index of the Longo–Rehren Q-system is equal to the $\mu$-index of $\{A(I)\}$ and the $\mu$-index is multiplicative with respect to a tensor product. We have thus obtained the non-degeneracy of the braiding on the unitary braided tensor category of finite dimensional representations of the local conformal net $\{A(I)\}$.

The following is [13, theorem 3.21], which is proved directly from the definition of the $\alpha$-induction.

Theorem 3.30. Let $\{A(I)\}$ be a completely rational local conformal net and $\{B(I)\}$ be its extension with finite index. Let $N, M, I, \Delta, \lambda$ be as above and consider the $\alpha^2$-induction. We regard a representation $\sigma$ of $\{B(I)\}$ as an endomorphism of $M$. Then we have $\langle \alpha^2_\lambda, \sigma \rangle = (\lambda, i \cdot \sigma \cdot i)$.

We now have the following important consequence.

Theorem 3.31. Let $\{A(I)\}$ be a completely rational local conformal net and $\{B(I)\}$ be its extension with finite index. Let $I, N, M, \Delta, i$ be as above and consider the $\alpha^2$-induction. Then the ambichiral M-M sectors are identified with the localized endomorphisms of $\{B(I)\}$ in $I$.

An outline of the proof of this theorem is as follows. By theorem 3.30, we know that the irreducible decomposition of any representation of $\{B(I)\}$ occurs among the ambichiral M-M sectors. Then theorems 3.21 and 3.29 give the conclusion.
Note that the above theorem implies that the unitary modular tensor category of finite dimensional representations of \( \{ B(I) \} \) is ‘smaller’ than that of \( \{ A(I) \} \), and this is opposite to the case of classical representations. In this sense, \( \alpha \)-induction is similar to restriction of representations rather than induction to some extent.

Theorem 3.27 says that any irreducible extension \( \{ B(I) \} \) of a completely rational local conformal net \( \{ A(I) \} \) produces a modular invariant. (The index \( \{ B(I): A(I) \} \) is automatically finite in this situation. See [84, page 39] and [93, proposition 2.3].) Since we have only a small number of modular invariants usually, this gives a severe restriction on an extension \( \{ B(I) \} \) for a given \( \{ A(I) \} \).

We also have computations of the quantum doubles related to \( \alpha \)-induction in [19]. The same mathematical structure as \( \alpha \)-induction has been studied in the context of anyon condensation [2]. See table 1 on page 440 in [102].

### 3.6. Examples and construction methods

We now discuss how to construct local conformal nets. One way to construct a local conformal net is from a Kac–Moody Lie algebra, but from our viewpoint, it is easier to use a loop group for a compact Lie group. Consider a connected and simply connected Lie group, say, \( SU(N) \). Let \( L(SU(N)) \) be the set of all the \( C^\infty \)-maps from \( S^1 \) to \( SU(N) \). We fix a positive integer \( k \) called a level. Then we have finitely many irreducible projective unitary representations of \( L(SU(N)) \) called positive energy representations at level \( k \). (See [126] for details.) We have one distinguished representation, called the vacuum representation, among them. For each interval \( I \subset S^1 \), we denote the set of \( C^\infty \)-maps from \( S^1 \) to \( SU(N) \) such that the image outside the interval \( I \) is always the identity matrix by \( L_I(SU(N)) \). Then setting \( A(I) \) to be the von Neumann algebra generated by the image of \( L_I(SU(N)) \) by the vacuum representation, we have a local conformal net \( \{ A(I) \} \), which is labelled as \( SU(N)_k \). (See [67, 141] for details.) A similar construction for other Lie groups has been done in [139]. These examples correspond to the so-called Wess–Zumino–Witten models, and this name is also often attached to these local conformal nets. The local conformal nets corresponding to the Wess–Zumino–Witten model \( SU(N)_k \) are completely rational by [143]. For the case of \( SU(2)_k \), the irreducible representations are labelled as \( \{ \lambda_0, \lambda_1, ..., \lambda_k \} \) and the fusion rules are given as

\[
\lambda_j \cdot \lambda_m = \lambda_{\lfloor -m \rfloor} \oplus \lambda_{\lfloor -m \rfloor + 2} \oplus \lambda_{\lfloor -m \rfloor + 4} \oplus \cdots \oplus \lambda_{\lfloor m + 2k - l - m \rfloor}
\]

by [141], where \( \lambda_0 \) is the vacuum sector. The dimensions are given as

\[
d_{\lambda_0} = \frac{\sin((m + 1)\pi/(k + 2))}{\sin(\pi(k + 2))}.
\]

The statistical phase of \( \lambda_I \) is given as

\[
\exp(i\pi(l + 2)/(2k + 4)),
\]

which follows from the above fusion rules and the spin-statistics theorem in [74, theorem 3.13]. We also have

\[
S_{\lambda_I, \lambda_0} = \sqrt{\frac{2}{k + 2}} \sin \left( \frac{(l + 1)(m + 1)\pi}{k + 2} \right).
\]

Another construction of a local conformal net is from a lattice \( \Lambda \) in the Euclidean space \( \mathbb{R}^n \), that is, an additive subgroup of \( \mathbb{R}^n \) which is isomorphic to \( \mathbb{Z}^n \) and spans \( \mathbb{R}^n \) linearly. A lattice is called even when we have \( (x, y) \in \mathbb{Z} \) and \( (x, x) \in 2\mathbb{Z} \) for the inner products of \( x, y \in \Lambda \). One obtains a local conformal net from an even lattice \( \Lambda \). This is like a loop group construction for \( \mathbb{R}^n/\Lambda \). (See [45, 96] for details.) The local conformal nets arising from even lattices are also completely rational by [45]. Let \( A^n = \{ x \in \mathbb{R}^n | (x, y) \in \mathbb{Z} \text{ for all } y \in \Lambda \} \), the dual lattice of \( \Lambda \). Then the irreducible representations of the local conformal net arising from
Λ are labelled with the elements of Λ*/Λ and all have dimension 1. It is holomorphic if and only if we have Λ* = Λ.

The above complete rationality results imply complete rationality of many examples, but still complete rationality of many examples arising from representations of loop groups as in [139] is open, so we have the following problem.

**Problem 3.32.** Prove complete rationality of local conformal nets arising from positive energy representations of loop groups corresponding to Wess–Zumino–Witten models.

Another construction of a local conformal net is from a vertex operator algebra. We see this construction in the next chapter.

We next show methods to obtain new local conformal nets from known ones.

**Example 3.33.** For two local conformal nets \{A(I)\} and \{B(I)\}, we construct a new one \{A(I) ⊗ B(I)\}. This is called the tensor product of local conformal nets. Both the Hilbert space and the vacuum vector of the tensor product of local conformal nets are those of the tensor products. Each irreducible representation of the tensor product local conformal net is a tensor product of two irreducible representations of the two local conformal nets, up to unitary equivalence. That is, each finite dimensional representation of \{A(I) ⊗ B(I)\} is of the form \(λ \otimes μ\), where \(λ\) and \(μ\) are finite dimensional representations of \{A(I)\} and \{B(I)\}, respectively. We also have \(\text{Hom}(λ_1 \otimes μ_1, λ_2 \otimes μ_2) = \text{Hom}(λ_1, λ_2) \otimes \text{Hom}(μ_1, μ_2)\). This representation category of \{A(I) ⊗ B(I)\} is written as \(\text{Rep}(A) ⊠ \text{Rep}(B)\) and called the Deligne product of \(\text{Rep}(A)\) and \(\text{Rep}(B)\), which was introduced in [38]. (Also see [3, definition 1.1.15].)

**Example 3.34.** The next construction is called the simple current extension. This is an extension of a local conformal net \{A(I)\} with something similar to a semi-direct product with a group. (See example 4.24 for the initial appearance of this type of construction.) Suppose some irreducible representations of \{A(I)\} have dimension 1 and they are closed under the conjugation and the tensor product. If they further have all statistical phases 1, then they make a group of DHR automorphisms and a local Q-system by [128, lemma 4.4] and [14, corollary 3.7]. An automorphism used in this construction is called a simple current in physics literatures and this is the source of the name of the construction. This method also gives a realization of local conformal nets arising from even lattices as follows. There is an important conformal field theory called free bosons in physics literatures, and we have the corresponding local conformal net \{A(I)\}. Its representation theory with tensor product structure is given by \(R\) with the additive structure. The representation theory of the nth tensor power of \{A(I)\} is identified with \(R^n\). Its simple current extension with \(Λ \subset R^n\) gives the local conformal net corresponding to the even lattice \(Λ\).

**Example 3.35.** The next one is called the orbifold construction. An automorphism of a local conformal net \{A(I)\} on \(H\) is a unitary operator \(U\) on \(H\) satisfying \(UA(I)U^* = A(I)\) for all intervals \(I\) and \(UΩ = Ω\). (In this case, \(U\) automatically commutes with the action of Diff(S¹). See [32].) We then consider a group \(G\) of automorphisms of a local conformal net \{A(I)\} and define a subnet by \(B(I) = \{x ∈ A(I) | UxU^* = x, U ∈ G\}\). Replacing \(H\) with the closure of \(B(I)Ω\), which is independent of \(I\), we obtain a new local conformal net \{B(I)\}. This construction is called the orbifold construction. (See example 4.25 for the
Another construction is called the coset construction. Suppose we have two local conformal nets \([A(I)], [B(I)]\) where the latter is a subnet of the former. Then the family of von Neumann algebras \(A(I)' \cap \mathcal{B}(I)\) on the Hilbert space \((A(I)' \cap \mathcal{B}(I))\mathcal{H}\) gives a new local conformal net. (This Hilbert space is again independent of \(I\).) This construction is called the coset construction. (See [144] for details.)

There was a conjecture or an expectation that all completely rational local conformal nets would be given by combinations of the above construction methods, but this is unlikely to be true as we see in the later sections.

Theorem 3.21 implies that if \([B(I)]\) is completely rational and \([A(I)]\) is its orbifold theory by a finite group \(G\), then the index \([B(I): A(I)]\) is the order of \(G\) and this implies complete rationality of \([A(I)]\). (This was first proved in [145, proposition 4.2]. Now this is a consequence of general results in [114, theorem 5.3] and [97, theorem 33].) The corresponding statement in theory of vertex operator algebras is still open.

Also when we perform the coset construction for \([A(I) \subset B(I)]\) with completely rationality of \([B(I)]\) and finiteness of the index \([B(I): A(I) \vee (A(I)' \cap \mathcal{B}(I))]\), we have complete rationality of \([A(I)' \cap \mathcal{B}(I)]\).

We also have a construction based on an extension by a \(Q\)-system. We show this construction in the next section.

3.7. Classification of local extensions of \(SU(2)_k\)

In the next section, we deal with classification theory of local conformal nets, and it is important to classify all irreducible local extensions of certain completely rational local conformal nets. As an easier example, we classify all local extensions of the local conformal nets \(SU(2)_k\) first here. We first note that we have only finitely many irreducible extensions for a given completely local rational net. This is because any such extension is given by the corresponding local \(Q\)-system, and for any endomorphism \(\otimes_{i=1}^{k} n_i \lambda_i\) appearing in the \(Q\)-system, we have a bound for each \(n_i\) by [84, page 39] and we have only finitely many equivalence classes of \(Q\)-system for each such endomorphism by [85]. We list this as follows.

**Theorem 3.36.** For any local conformal net, we have only finitely many irreducible extensions.

We now consider the case of \(SU(2)_k\). Recall that the irreducible superselection sectors of the local conformal net \(SU(2)_k\) are labelled as \(\{\lambda_0, \lambda_1, \ldots, \lambda_k\}\) with the fusion rules

\[
\lambda_j \cdot \lambda_m = \lambda_{|l-m|} \bigoplus \lambda_{|l-m|+2} \bigoplus \lambda_{|l-m|+4} \bigoplus \cdots \bigoplus \lambda_{\min(l+m,2k-l-m)}.
\]

For the \(S\)- and \(T\)-matrices given before, the modular invariant matrices have been completely classified in [25]. Theorems 3.26 and 3.27 give that we have \(\theta = \otimes_{\lambda} Z_0, \lambda \) for the dual canonical endomorphism giving an extension. We then have only the following possibilities for \(\theta\).
\[ \theta = \text{id}, \quad \text{for the type } A_{k+1} \text{ modular invariant at level } k, \]
\[ \theta = \lambda_0 \oplus \lambda_{4n-4}, \quad \text{for the type } D_{2n} \text{ modular invariant at level } k = 4n - 4, \]
\[ \theta = \lambda_0 \oplus \lambda_k, \quad \text{for the type } E_k \text{ modular invariant at level } k = 12, \]
\[ \theta = \lambda_0 \oplus \lambda_{10} \oplus \lambda_{18} \oplus \lambda_{28}, \quad \text{for the type } E_8 \text{ modular invariant at level } k = 28. \]

As a sample, we list the modular invariant matrix labelled with \( E_6 \) for \( SU(2)_{10} \) as follows.

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
\]

The case \( \theta = \text{id} \) is trivial. For the case \( \theta = \lambda_0 \oplus \lambda_{4n-4} \), we have a unique realization as a simple current extension by [128, lemma 4.4] and [85]. For the remaining two cases, we have local \( Q \)-systems on the unitary modular tensor categories arising from \( SU(2)_{10} \) and \( SU(2)_{28} \), and they correspond to the so-called ‘conformal embeddings’ \( SU(2)_{10} \subset SO(5)_1 \) and \( SU(2)_{28} \subset (G_2)_1 \). (The former is an inclusion of a local conformal net labelled with \( SU(2)_{10} \) into another labelled with \( SO(5)_1 \).) The uniqueness of each local \( Q \)-system follows from [94, theorem 5.3]. This completes the classification of irreducible local extensions of the local conformal nets \( SU(2)_k \) and they are labelled as in the above list of \( \theta \).

Another way to realize the last two cases of the \( Q \)-systems is given purely combinatorially as in [18]. The locality follows from [16, proposition 3.2] and the uniqueness again follows from [94, theorem 5.3].

3.8. Classification of local conformal nets with \( c < 1 \)

We next discuss classification theory of local conformal nets. The most desirable result would be a complete listing of all completely rational local conformal nets, but this would be impossible, since any finite group produces an orbifold local conformal net from a tensor power of a holomorphic local conformal net which almost remembers the original finite group. On the one hand, the history of classification theory of von Neumann algebras with amenability suggests that we have some kind of algebraic complete invariant for completely rational local conformal nets. On the other hand, the above examples arising from holomorphic local conformal nets suggest that simple representation theoretic invariants would not work. Furthermore, there are many different examples of holomorphic local conformal nets, so the unitary modular tensor category of the finite dimensional representations is far from a complete invariant. (A holomorphic vertex operator algebra is known to have a central charge equal to a multiple of 8 as in [149]. The corresponding statement for holomorphic local conformal nets has not been known.) Some different holomorphic local conformal nets are distinguished by the central charge or the vacuum character, \( \text{Tr}(\varphi^{L_0-c/24}) \). In theory of vertex operator algebras, there are known examples of different vertex operator algebras with the same central charge and the same vacuum character. We see the difference from a part of the binary operations \( v_{i,w} \) as in [105]. We have the corresponding local conformal nets in
[100, but it is not known whether they are really different as local conformal nets or not, though clearly they should be different.

It may be that unitary vertex operator algebra with $C_2$-cofiniteness could be a complete invariant of completely rational local conformal nets, as we see in the next chapter. The planar algebra [88] of Jones gives one formulation of a complete invariant for type $\text{II}_1$ hyperfinite subfactors with finite index and finite depth, and the invariant consists of countably many finite dimensional vector spaces with countably many multi-linear operations with countably many compatibility conditions, so this has formal similarity to vertex operator algebras we see later. In this section, we explain classification results we have today.

The Virasoro algebra is an infinite dimensional Lie algebra generated by a central element $c$ and countably many generators $L_n, n \in \mathbb{Z}$ subject to the following relations.

$$[L_m, L_n] = (m - n) L_{m+n} + \frac{(m^3 - m) \delta_{m,-n}}{12} c,$$

where $\delta$ is Kronecker’s $\delta$. This is a unique central extension of the complexification of the Lie algebra corresponding to the infinite dimensional Lie group Diff($S^1$). The central element $c$ is called the central charge.

Suppose we have a local conformal net $\{ \mathcal{A}(I) \}$. It has a projective unitary representation of Diff($S^1$), which implies that we have a unitary representation of the Virasoro algebra, where unitarity means we have $L_n^* = L_{-n}$. It is known by [59, 71] that if we have an irreducible unitary representation of the Virasoro algebra, then the central charge $c$ is mapped to a positive real number and the set of all possible such values is

$$\left\{ 1 - \frac{6}{m(m+1)} | m = 3, 4, 5, \ldots \right\} \cup [1, \infty).$$

The value of $c$ in the image is also called the central charge. (See [32, section 2.2], [26, theorem A.1] for example. Also see [31, theorem 3.4].) We note that since early days of subfactor theory, this restriction of the values of the central charge is similar to that of the index value. The unitary representation of the Virasoro algebra arising from a local conformal net is not irreducible in general, still we can show that the image of $c$ is a scalar as in [93, proposition 3.5]. We call the value the central charge of the local conformal net and simply write $c$ for this value. This is a numerical invariant of a local conformal net.

For a local conformal net $\{ \mathcal{A}(I) \}$, consider the projective unitary representation $U$ of Diff($S^1$) associated with it. For an interval $I \subset S^1$, set $B(I)$ to be the von Neumann algebra generated by the image $U(\text{Diff}(I))$. By restricting the Hilbert space to $B(I)\Omega$, we see $\{ B(I) \}$ is a local conformal net and this depends only on the value of $c$. We call this the Virasoro net with the central charge $c$ and write $\{ \text{Vir}_c(I) \}$ for this. (See [93, proposition 3.5].)

Another way to construct this Virasoro net is as follows. We simply write $L_n$ for the image of $L_n$ under the unitary representation of the Virasoro algebra. We see that they are closable operators, so we denote their closures again by $L_n$. The Fourier expansion $\sum_{n \in \mathbb{Z}} L_n z^{-n-2}$ makes sense as an operator-valued distribution on $S^1$ and called the stress-energy tensor. (It is a matter of convention that the exponent of $z$ is $-n - 2$.) Constructing ‘observables’ with test functions supported in $I \subset S^1$, we obtain the Virasoro net $\{ \text{Vir}_c(I) \}$. (See [22].)

For the case $c = 1 - 6/m(m+1)$, the coset construction of [71] adapted to the operator algebraic setting shows that the Virasoro net $\{ \text{Vir}_c(I) \}$ is given from the coset construction for the embedding $SU(2)_{m-1} \subset SU(2)_{m-2} \times SU(2)_{1}$ as in [93, corollary 3.3]. Complete
rationality of $SU(2)_c$, which was shown in [143], gives that of the Virasoro nets with $c < 1$ as in [144, section 4.3] and [110, section 3.5.1].

For the central charge $c = 1 - 6/(m(m + 1))$, ($m = 2, 3, 4, ...$), we have $m(m - 1)/2$ irreducible superselection sectors $\lambda(p, q)$ of the Virasoro net $\{\text{Vir}_c(I)\}$ with $(p, q)$, $1 \leq p \leq m - 1$, $1 \leq q \leq m$ with the identification $\lambda(p, q) = \lambda(m - p, m + 1 - q)$. They have fusion rules as follows.

$$\lambda(p, q)\lambda(p', q') = \bigoplus_{r=s(q-q') +1, i} \bigoplus_{s=q-q' \text{ odd}} \lambda(r, s).$$

The $\mu$-index of the Virasoro net $\{\text{Vir}_c(I)\}$ is

$$\frac{m(m + 1)}{8 \sin^2 \frac{\pi}{m} \sin^2 \frac{\pi}{m+1}}$$

by [146, lemma 3.6]. The statistical phase of the superselection sector $\lambda(p, q)$ is

$$\exp 2\pi i \left(\frac{(m + 1)p^2 - mq^2 - 1 + m(m + 1)(p - q)^2}{4m(m + 1)}\right).$$

If $c < 1$ for a general local conformal net $\{A(I)\}$, it is an irreducible extension of the Virasoro net $\{\text{Vir}_c(I)\}$. Recall that such an extension produces a modular invariant of the unitary modular tensor category arising from $\{\text{Vir}_c(I)\}$. The unitary modular tensor categories arising from the finite dimensional representations of $\{\text{Vir}_c(I)\}$ have been studied well in other contexts and their modular invariants have been completely classified in [25, tables 2 and 3]. We can verify the unitary representations of $SU(2, \mathbb{Z})$ studied in [25] are the same as those arising from the unitary modular tensor categories of the finite dimensional representations of $\{\text{Vir}_c(I)\}$ by [74, theorem 3.13] and the fusion rules in [93, section 3], so we have some modular invariant listed in [25, tables 2 and 3]. Furthermore, the modular invariants in [25] are labelled as type I and II, but it is easy to see that we have only type I modular invariants by theorem 3.30.

The modular invariants in [25, tables 2 and 3] are labelled with pairs of the $A$-$D$-$E$ Dynkin diagrams whose Coxeter numbers differ by 1, and the type I modular invariants are labelled with those of the $A_{n-2}$. Such a modular invariant $Z$ arises from a local extension of $\{\text{Vir}_c(I)\}$, then its dual canonical endomorphism must be of the form $\bigoplus Z_{\lambda, \lambda} \lambda$ by theorem 3.26. As in the previous section, the case of simple current extensions are easy to handle, and we also have local $Q$-systems on the unitary modular tensor categories arising from $SU(2)_h$ and $SU(2)_h$. By copying these $Q$-systems, we see that such a modular invariant $Z$ arising from the type I modular invariants of $\{\text{Vir}_c(I)\}$ does have a corresponding $Q$-system. For example, when $c = 21/22$, the unitary fusion category arising from the irreducible sectors $\{\lambda_1, \lambda_3, \lambda_5, \ldots, \lambda_{11}\}$ is isomorphic to that arising from the irreducible sectors $\{\lambda_0, \lambda_2, \lambda_4, \ldots, \lambda_6\}$ of $SU(2)_h$, so the $Q$-system arising from $\lambda_0 \bigoplus \lambda_6$ on the former can be copied to the one arising from $\lambda_1 \bigoplus \lambda_7$ on the latter. Then the locality criterion in [16, proposition 3.2] gives the locality of the $Q$-system, hence the locality of the extension of $\{\text{Vir}_c(I)\}$. Furthermore, the 2-cohomology vanishing result [94, theorem 5.3] implies that such a $Q$-system is unique for each $\bigoplus Z_{\lambda, \lambda}$. (See [93, theorem 4.1] and [93, remark 2.5] based on [94] for more details. See [85] for a general theory of 2-cohomology of subfactors.) We thus have the following theorem.

**Theorem 3.37.** The complete list of local conformal nets with $c < 1$ is as follows.

1. The Virasoro nets $\text{Vir}_c$ ($c < 1$).
2. The simple current extensions of the Virasoro nets with \( c < 1 \) by \( \mathbb{Z}/2\mathbb{Z} \).

3. Four exceptionals at \( c = 21/22, 25/26, 144/145, 154/155 \).

The first class in the list is not exciting at all, and neither is the second class. Two of the four exceptionals in the third class had been conjectured to arise from the coset construction, and it has been proved in [93, section 6.2] that this is indeed the case. The case \( c = 21/22 \) has been proved to arise from a more complicated coset construction in [104]. The final remaining case \( c = 144/145 \) arises as an ‘extension by a \( Q \)-system’ because we directly construct a local \( Q \)-system with the dual canonical endomorphism being in the unitary modular tensor category of the finite dimensional representations of \( \text{Vir}_c(I) \) and this has not been constructed by any other method so far. This construction of the exceptionals has been generalized to some infinite series in [147] under the name of the mirror extensions. Note that the above classification has some formal similarity to the classification of subfactors with index less than 4, theorem 2.20.

Here we have used the classification result of modular invariants. Originally modular invariants arise in the setting of full conformal field theory as the coefficients of modular invariant partition functions. Note that the above use of the modular invariants is quite different from this and this is why we have only type I modular invariants.

### 3.9. Miscellaneous remarks

We discuss some related problems here.

Conformal field theory on Riemann surfaces as in [134] has been well-studied and conformal blocks play a prominent role there. It is not clear at all how to formulate this in our approach, so we have the following open problem.

**Problem 3.38.** Formulate conformal field theory on Riemann surfaces with operator algebraic methods.

See [60, 61, 64, 65] for some related results in the Euclidean field theory approach.

We have operator algebraic versions of \( N = 1 \) and \( N = 2 \) superconformal field theories as in [29] and [28], and there we have superconformal nets rather than local conformal nets. We also have connections to non-commutative geometry in [27, 28]. (See [62, 63] for an early work on connections between superconformal field theory and non-commutative geometry.) Also see [4–6] for another approach to local conformal nets.

### 4. Vertex operator algebras

We have a different axiomatization of chiral conformal field theory from a local conformal net and it is a theory of **vertex operator algebras**. It is a direct axiomatization of Wightman fields on the circle \( S^1 \). In physics literatures, certain operator-valued distributions are called vertex operators and this is the origin of the name ‘vertex operator algebra’. We explain this theory in comparison to that of local conformal nets. We emphasize relations to local conformal nets rather than a general theory of vertex operator algebras.

A certain amount of the theory has been devoted to a single example called the **moonshine vertex operator algebra**, so we explain the background of the moonshine conjecture for which it was constructed. (See [69] for more details.) We start with the following very general problem.
**Problem 4.1.** Suppose a finite group $G$ is given. Realize it as the automorphism group of some ‘interesting’ algebraic structure.

This is not a precisely formulated mathematical problem since the word ‘interesting’ is ambiguous, of course. One precise formulation is as follows.

**Problem 4.2.** Suppose a finite group $G$ is given. Realize it as the Galois group of an extension over $\mathbb{Q}$.

This is called the inverse Galois problem and still open today. Note that $\mathbb{Q}$ is the smallest field of characteristic 0. The hyperfinite $\text{II}_1$ factor is the ‘smallest’ infinite dimensional factor in some appropriate sense, and we can define a Galois group for its extension naturally. Then we consider a direct analogue of the above problem for this von Neumann algebraic version, but it turns out that a solution for all finite groups is easily given, and this is not a particularly exciting problem. We look for more interesting analogues of the above Galois theory problem.

Our ‘interesting’ algebraic structure now is that of a vertex operator algebra, which is infinite dimensional. (We have a certain Galois type theory for vertex operator algebras [43].) It looks that a natural vector space for a finite group is finite dimensional, but somehow in the moonshine conjecture and related topics, infinite dimensional vector spaces naturally appear in connection to finite groups.

Among finite groups, finite simple groups are clearly fundamental. Today we have a complete list of finite simple groups as follows. (See [57, 69] and texts cited there for more details.)

1. Cyclic groups of prime order.
2. Alternating groups of degree 5 or higher.
3. 16 series of groups of Lie type over finite fields.
4. 26 sporadic finite simple groups.

The third class consists of matrix groups such as $\text{PSL}(n, \mathbb{F}_q)$. The last class consists of exceptional structures, and the first ones were found by Mathieu in the 19th century. The largest group among the 26 groups in terms of the order is called the Monster group, and its order is

$$2^{46} \times 3^{20} \times 5^9 \times 7^6 \times 11^2 \times 13^3 \times 17 \times 19 \times 23 \times 29 \times 31 \times 41 \times 47 \times 59 \times 71,$$

which is approximately $8 \times 10^{53}$. This group was first constructed in [73] as the automorphism group of some commutative, but non-associative algebra of 196884 dimensions. From the beginning, it has been known that the smallest dimension of a non-trivial irreducible representation of the Monster group is 196883.

Now we turn to a different topic of the classical $j$-function. This is a function of a complex number $\tau$ with $\text{Im } \tau > 0$ given as follows.

$$j(\tau) = \frac{1 + 240 \sum_{n>0} \sigma_3(n) q^n}{q \prod_{n>0} (1 - q^n)^{24}} = q^{-1} + 744 + 196884q + 21493760q^2 + 864299970q^3 + \ldots,$$

where $\sigma_3(n)$ is a sum of the cubes of the divisors of $n$ and we set $q = \exp(2\pi i \tau)$.
This function has modular invariance property

\[ j(\tau) = j \left( \frac{a \tau + b}{c \tau + d} \right) \]

for

\[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}), \]

and this property and the condition that the Laurent series of \( q \) start with \( q^{-1} \) determine the \( j \)-function uniquely except for the constant term. The constant term 744 has been chosen for a historic reason and this has no significance, so we set \( J(\tau) = j(\tau) - 744 \) and use this from now on.

McKay noticed that the first non-trivial coefficient of the Laurent expansion of the \( J \)-function satisfies the equality \( 196884 = 196883 + 1 \), where 1 the dimension of the trivial representation of the Monster group and 196883 is the smallest dimension of its non-trivial representation. People suspected it is simply a coincidence, but it has turned out that all the coefficients of the Laurent expansion of the \( J \)-function with small exponents are linear combinations of the dimensions of irreducible representations of the Monster group with 'small' positive integer coefficients. (We have 1 as the dimension of the trivial representation, so it is trivial that any positive integer is a sum of the dimensions of irreducible representations of the Monster group, but it is highly non-trivial that we have small multiplicities.)

Based on this, Conway–Norton \cite{34} formulated what is called the moonshine conjecture today, which has been proved by Borcherds \cite{20}.

**Conjecture 4.3.**

1. We have some graded infinite dimensional \( \mathbb{C} \)-vector space \( V = \bigoplus_{n=0}^{\infty} V_n \) (dim\( V_n < \infty \)) with some natural algebraic structure and its automorphism group is the Monster group.
2. Each element \( g \) of the Monster group acts on each \( V_n \) linearly. The Laurent series

\[ \sum_{n=0}^{\infty} \left( \text{Tr} g \mid \nu \right) q^{n-1} \]

arising from the trace value of the \( g \)-action on \( V_n \) is a classical function called a Hauptmodul corresponding to a genus 0 subgroup of \( SL(2, \mathbb{R}) \). (The case \( g \) is the identity element is the \( J \)-function.)

The above Laurent series is called the McKay–Thompson series. The first statement is vague since it does not specify the 'natural algebraic structure', but Frenkel–Lepowsky–Meurman \cite{57} introduced the axioms for vertex operator algebras and constructed an example \( V \), called the moonshine vertex operator algebra, corresponding to the first statement of the above Conjecture. This was the starting point of the entire theory.

**4.1. Basic definitions**

There are various, slightly different versions of the definition of vertex operator algebras, so we fix our definition here. We follow \cite{31}. (See also \cite{57, 90}.)

Let \( V \) be a \( \mathbb{C} \)-vector space. We say that a formal series \( a(z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1} \) with coefficients \( a(n) \in \text{End}(V) \) is a field on \( V \), if for any \( b \in V \), we have \( a(n) b = 0 \) for all sufficiently large \( n \).
Definition 4.4. A C-vector space $V$ is a **vertex algebra** if we have the following properties.

1. (State-field correspondence) For each $a \in V$, we have a field $Y(a, z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1}$ on $V$.
2. (Translation covariance) We have a linear map $T \in \text{End}(V)$ such that we have $[T, Y(a, z)] = \frac{d}{dz} Y(a, z)$ for all $a \in V$.
3. (Existence of the vacuum vector) We have a vector $\Omega \in V$ with $T\Omega = 0, Y(\Omega, z) = \text{id}_V, a_{(-1),}\Omega = a$.
4. (Locality) For all $a, b \in V$, we have $(z - w)^N [Y(a, z), Y(b, w)] = 0$ for a sufficiently large integer $N$.

We then call $Y(a, z)$ a **vertex operator**.

A vertex operator is an algebraic version of the Fourier expansion of an operator-valued distribution on the circle. The state-field correspondence means that any vector in $V$ gives an operator-valued distribution. The locality axiom is one representation of the idea that $Y(a, z)$ and $Y(b, w)$ commute for $z \neq w$. (Recall that a distribution $T$ on $\mathbb{R}$ has supp $T \subset \{0\}$ if and only if there exists a positive integer $N$ with $x^N T = 0$.)

The following **Borcherds identity** is a consequence of the above axioms, where $a, b, c \in V$ and $m, n, k \in \mathbb{Z}$.

$$\sum_{j=0}^{\infty} \binom{m}{j} (a_{(n+j)} b)_{(m-k-j)} c = \sum_{j=0}^{\infty} (-1)^j \binom{n}{j} a_{(m+n-j)} b_{(k+j)} c - \sum_{j=0}^{\infty} (-1)^{j+n} \binom{n}{j} b_{(n+k-j)} a_{(m+j)} c.$$

Definition 4.5. We say a linear subspace $W \subset V$ is a **vertex subalgebra** if we have $\Omega \in W$ and $a_{(a)} b \in W$ for all $a, b \in W$ and $n \in \mathbb{Z}$. (In this case, $W$ is automatically $T$-invariant.) We say a linear subspace $J \subset V$ is an **ideal** if it is $T$-invariant and we have $a_{(a)} b \in J$ for all $a \in V, b \in J$ and $n \in \mathbb{Z}$. A vertex algebra is said to be **simple** if any ideal in $V$ is either 0 or $V$. A **(anti)linear homomorphism** from a vertex algebra $V$ to a vertex algebra $W$ is an (anti)linear map $\phi$ satisfying $\phi(a_{(a)} b) = \phi(a)_{(a)} \phi(b)$ for all $a, b \in V$ and $n \in \mathbb{Z}$. We similarly define an **automorphism**.

If $J$ is an ideal of $V$, we also have $a_{(a)} b \in J$ for all $a \in J, b \in V$ and $n \in \mathbb{Z}$.

We next introduce conformal symmetry in this context.

Definition 4.6. Let $V$ be a C-vector space and $L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$ be a field on $V$. If the endomorphisms $L_n$ satisfy the Virasoro algebra relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{(m^3 - m) \delta_{m+n,0}}{12} c,$$

with central charge $c \in \mathbb{C}$, then we say $L(z)$ is a **Virasoro field**. If $V$ is a vertex algebra and $Y(a, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$ is a Virasoro field, then we say $\omega \in V$ is a **Virasoro vector**. A Virasoro vector $\omega$ is called a **conformal vector** if $L_{-1} = T$ and $L_0$ is diagonalizable on $V$. (The latter means that $V$ is an algebraic direct sum of the eigenspaces of $L_0$.) Then the corresponding vertex operator $Y(\omega, z)$ is called the **energy-momentum field** and $L_0$ the
A vertex algebra with a conformal vector is called a conformal vertex algebra. We then say $V$ has central charge $c \in \mathbb{C}$.

**Definition 4.7.** A non-zero element $a$ of a conformal vertex algebra in $\text{Ker}(L_0 - \alpha)$ is said to be a homogeneous element of conformal weight $d_a = \alpha$. We then set $a_n = a(\alpha + d_n)$ for $n \in \mathbb{Z} - d_a$. For a sum $a$ of homogeneous elements, we extend $a_n$ by linearity.

**Definition 4.8.** A homogeneous element $a$ in a conformal vertex algebra $V$ and the corresponding field $Y(a, z)$ are called quasi-primary if $L_1 a = 0$ and primary if $L_n a = 0$ for all $n > 0$.

**Definition 4.9.** We say that a conformal vertex algebra $V$ is of CFT type if we have $\text{Ker}(L_0 - \alpha) \neq 0$ only for $\alpha \in \{0, 1, 2, 3, \ldots\}$ and $V_0 = \mathbb{C} \Omega$.

**Definition 4.10.** We say that a conformal vertex algebra $V$ is a vertex operator algebra if we have the following.

1. We have $V = \bigoplus_{n \in \mathbb{Z}} V_n$, where $V_n = \text{Ker}(L_0 - n)$.
2. We have $V_n = 0$ for all sufficiently small $n$.
3. We have $\dim(V_n) < \infty$ for $n \in \mathbb{Z}$.

**Definition 4.11.** An invariant bilinear form on a vertex operator algebra $V$ is a bilinear form $(\cdot, \cdot)$ on $V$ satisfying

$$
(Y(a, z)b, c) = \left\{ b, Y\left(e^{z-\frac{1}{2}}(-z^{-2})^{L_n} a, z^{-1}\right)c \right\}
$$

for all $a, b, c \in V$.

**Definition 4.12.** For a vertex operator algebra $V$ with a conformal vector $\omega$, an automorphism $g$ as a vertex algebra is called a VOA automorphism if we have $g(\omega) = \omega$.

**Definition 4.13.** Let $V$ be a vertex operator algebra and suppose we have a positive definite inner product $(\cdot | \cdot)$, where this is supposed to be antilinear in the first variable. We say the inner product is normalized if we have $(\Omega | \Omega) = 1$. We say that the inner product is invariant if there exists a VOA antilinear automorphism $\theta$ of $V$ such that $(\theta \cdot | \cdot)$ is an invariant bilinear form on $V$. We say that $\theta$ is a PCT operator associated with the inner product.

If we have an invariant inner product, we automatically have $(L_n a | b) = (a | L_n b)$ for $a, b \in V$ and also $V_n = 0$ for $n < 0$. The PCT operator $\theta$ is unique and we have $\theta^2 = 1$ and $(\theta a | \theta b) = (b | a)$ for all $a, b \in V$. (See [31, section 5.1] for details.)

**Definition 4.14.** A unitary vertex operator algebra is a pair of a vertex operator algebra and a normalized invariant inner product.

See [42] for details of unitarity. A unitary vertex operator algebra is simple if and only if we have $V_0 = \mathbb{C} \Omega$. (See [31, proposition 5.3] for details.)

For a unitary vertex operator algebra $V$, we write $\text{Aut}(V)$ for the automorphism group fixing the inner product.
Definition 4.15. A unitary subalgebra $W$ of a unitary vertex operator algebra $(V, \cdot \mid \cdot )$ is a vertex subalgebra with $\partial W \subseteq W$ and $L_1 W \subseteq W$.

4.2. Modules and modular tensor categories

We introduce a notion of a module of a vertex operator algebra, which corresponds to a representation of a local conformal net, as follows. (Also see [149, definition 1.2.3].)

Definition 4.16. Let $M$ be a $\mathbb{C}$-vector space and suppose we have a field $Y_{\mathbb{C}}(a, z) \in \mathcal{M}$ for any $a \in V$, where the map $a \mapsto Y_{\mathbb{C}}(a, z)$ is linear and $V$ is a vertex algebra. We say $M$ is a module over $V$ if we have $Y_{\mathbb{C}}(z, 0) = \text{id}_M$ and the following Borcherds identity for $a, b, c \in M, m, n, k \in \mathbb{Z}$.

$$\sum_{j=0}^{\infty} \binom{m}{j} (a_{(n+j)}b)_{(m+k-j)}^M c = \sum_{j=0}^{\infty} (-1)^j \binom{n}{j} (a_{(m+n-j)}b)_{(k+j)}^M c - \sum_{j=0}^{\infty} (-1)^j (n+j) \binom{n}{j} (a_{(m+j)}b)_{(m+n)}^M c.$$

In this section, we consider only simple vertex operator algebras of CFT type.

The fusion rules on modules have been introduced in [56] and they give the tensor product operations on modules.

We have a natural notion of irreducible module and it is of the form $M_0 \oplus \cdots$, where every $M_n$ is finite dimensional and $L_0$ acts on $M_n$ as a scalar $n + h$ for some constant $h$. The vertex operator algebra $V$ itself is a module of $V$ with $h = 0$, and if this is the only irreducible module, then we say $V$ is holomorphic.

We define the formal power series, the character of $M$, by $\text{ch}(M) = \sum_{n=0}^{\infty} \dim(M_n) q^n + h c/24$, where $c$ is the central charge. We introduce the following important notion.

Definition 4.17. If the quotient space $V/\langle v \mid w \rangle \cap V$ is finite dimensional, we say that the vertex operator algebra $V$ is $C_2$-cofinite.

It has been proved in [149] that under the $C_2$-cofiniteness condition and small other conditions which automatically hold in the unitary case, we have only finitely many irreducible modules $M_0, M_2, \ldots, M_r$ up to isomorphism, their characters are absolutely convergent for $|q| < 1$, and the linear span of $\text{ch}(M_1), \text{ch}(M_2), \ldots, \text{ch}(M_r)$ is closed under the action of $SL(2, \mathbb{Z})$ on the upper half plane through the fractional linear transformation on $\tau$ with $q = \exp(2\pi i \tau)$.

Under the $C_2$-cofiniteness assumption and small other assumptions which again automatically hold in the unitary case, Huang [78, 79] further showed that the $S$-matrix defined by the transformation $\tau \mapsto -1/\tau$ on the characters satisfy the Verlinde formula (3) with respect to the fusion rules and the tensor category of the modules is modular.

Note the characters and their modular transformations make sense in the setting of operator algebras since $L_0$ acts on each representation space, so we have the following conjecture.

Conjecture 4.18. For a completely rational local conformal net, we have convergent characters for all irreducible representations and they are closed under modular
transformations of $SL(2, \mathbb{Z})$. Furthermore, the $S$-matrix defined with braiding gives transformation rules of the characters under the transformation $\tau \mapsto -1/\tau$.

This conjecture was made in [67, page 625], and a completely rational local conformal net satisfying this is called modular in [95].

For an inclusion $\{A(I) \subset B(I)\}$ of local conformal nets, the $\alpha$-induction produces $B(I)$- $B(I)$ morphisms which do not arise from a representation of $\{B(I)\}$. (That is, the non-ambichiral $B(I)$-$B(I)$ morphisms.) To some extent, they are similar to twisted modules of a vertex operator algebra, but they look more general, so we have the following problem. (See [120, section 5], [121] for an abstract treatment in the setting of tensor categories.)

**Problem 4.19.** Develop the theory of $\alpha$-induction for vertex operator algebras.

### 4.3. Examples and construction methods

The construction methods of local conformal nets we have explained had been known in the context of vertex operator algebras earlier except for an extension by a $Q$-system. We explain other constructions for vertex operator algebras. The last construction based on a $Q$-system has now been also introduced in the context of vertex operator algebras in [80].

**Example 4.20.** We have a simple unitary vertex operator algebra $L(c, 0)$ with central charge $c$ arising from a unitary representation of the Virasoro algebra with central charge $c$ and the lowest conformal energy 0. See [42, section 4.1] for details.

**Example 4.21.** Let $\mathfrak{g}$ be a simple complex Lie algebra and $V_{k\mathfrak{g}}$ be the conformal vertex algebra generated by the unitary representation of the affine Lie algebra associated with $\mathfrak{g}$ having level $k$ and lowest conformal energy 0. Then $V_{k\mathfrak{g}}$ is a simple unitary vertex operator algebra. (See [58]. Also see [42, section 4.2] for unitarity.)

**Example 4.22.** Let $\Lambda \subset \mathbb{R}^n$ be an even lattice. That is, it is isomorphic to $\mathbb{Z}^n$ as an abelian group and linearly spans the entire $\mathbb{R}^n$, and we have $(x, y) \in 2\mathbb{Z}$ for all $x, y \in \Lambda$, where $(\cdot, \cdot)$ is the standard Euclidean inner product. There is a general construction of a unitary vertex operator algebra from such a lattice and we obtain $V_{\Lambda}$ from $\Lambda$. The central charge is the rank of $\Lambda$. (See [57]. Also see [42, section 4.4] for unitarity.)

**Example 4.23.** If we have two unitary vertex operator algebras $(V, (\cdot | \cdot))$ and $(W, (\cdot | \cdot))$, then the tensor product $V \otimes W$ has a natural inner product with which we have a unitary vertex operator algebra.

**Example 4.24.** For a unitary vertex operator algebra with certain modules called simple currents satisfying some nice compatibility condition, we can extend the unitary vertex operator algebra. This is called a simple current extension. This was first studied in [133]. Also see [77, section 4], [66, 80].

**Example 4.25.** For a unitary vertex operator algebra $(V, (\cdot | \cdot))$ and $G \subset \text{Aut}_{(\cdot)}(V)$, the fixed point subalgebra $V^G$ is a unitary vertex operator algebra. This is called an orbifold subalgebra. This was first studied in [40].
Example 4.26. Let \((V, \cdot | \cdot)\) be a unitary vertex operator algebra and \(W\) its subalgebra. Then

\[ W^c = \{ b \in V | [Y(a, z), Y(b, w)] = 0 \text{ for all } a \in W \} \]

is a vertex subalgebra of \(V\). This is called a coset subalgebra. This is also called the commutant of \(W\) in \(V\) and was introduced in [58]. If \(W\) is unitary, then \(W^c\) is also unitary.

4.4. Moonshine vertex operator algebras

A rough outline of the construction of the moonshine vertex operator algebra is as follows. We have an exceptional even lattice in dimension 24 called the Leech lattice. It is the unique 24-dimensional even lattice \(\Lambda\) with \(\Lambda = \Lambda^\dagger\) and having no vectors \(x \in \Lambda\) with \((x, x) = 2\). (See [35, [57, page 304] for details.) We have a corresponding unitary vertex operator algebra \(V_\Lambda\).

The involution \(x \mapsto -x\) on \(\Lambda\) induces an automorphism of \(V_\Lambda\) of order 2. Its fixed point vertex operator subalgebra has a non-trivial simple current extension of order 2. Taking this extension is called the twisted orbifold construction and we obtain \(V_\Lambda^\natural\) with this. This is the moonshine vertex operator algebra.

Miyamoto [116] has given a new construction of \(V_\Lambda^\natural\) as follows. The most fundamental vertex operator algebra among the Virasoro vertex operator algebras has central charge 1/2 and is denoted by \(L(1/2, 0)\). It has been known that the moonshine vertex operator algebra contains the 48th tensor power of \(L(1/2, 0)\) in [44]. Conversely, we start with the 48th tensor power of \(L(1/2, 0)\) and construct the moonshine vertex operator algebra as its extension. Based on analogy to lattice theory, a finite tensor power of \(L(1/2, 0)\) contained in another vertex operator algebra is called a Virasoro frame. An extension of a Virasoro frame is called a framed vertex operator algebra [41]. The moonshine vertex operator algebra is constructed as a framed vertex operator algebra. It has been conjectured that this is the unique vertex operator algebra \(V\) having only trivial irreducible modules with \(c = 24\) and \(V_1 = 0\).

The local conformal net corresponding to the moonshine vertex operator algebra has been constructed in [96] and its automorphism group in the operator algebraic sense is the Monster group. This local conformal net is an extension of the 48th tensor power of the Virasoro net with \(c = 1/2\) and this extension is a simple current extension of a simple current extension of the tensor power of the Virasoro net with \(c = 1/2\).

The following is [148, conjecture 3.4].

Conjecture 4.27. A holomorphic local conformal net with \(c = 24\) and the eigenspace of \(L_0\) with eigenvalue 1 being 0 is unique up to isomorphism.

The following is a problem corresponding to [148, conjecture 3.5] which arose from [132, where Schellekens gave 71 possible Lie algebra structures as an invariant for classifying holomorphic vertex operator algebras with \(c = 24\).

Problem 4.28. Classify all holomorphic local conformal nets with \(c = 24\).

4.5. Local conformal nets and vertex operator algebras

We now consider relations between vertex operator algebras and local conformal nets. Both are supposed to be mathematical axiomatizations of the same physical theory, so we might expect the two sets of axioms are equivalent in the sense that we have a canonical bijective correspondence between the mathematical objects satisfying one set of axioms and those for
the other. However, both axiomatizations are broad and may contain some weird examples, so it is expected that we have to impose some more conditions in order to obtain such a nice bijective correspondence.

In principle, when we have some idea, example or construction on local conformal nets or vertex operator algebras, one can often ‘translate’ it to the other side. For example, the local conformal net corresponding to the moonshine vertex operator algebra has been constructed in [96] and its automorphism group in the operator algebraic sense is the Monster group. Also, a construction of holomorphic framed vertex operator algebras in [105] has been translated to the setting of local conformal nets in [100]. For the converse direction, with the results in [80, theorem 3.37 on local conformal nets implies the corresponding classification of vertex operator algebras with \( c < 1 \) as explained in [91, page 351]. Such a translation has been done on a case-by-case basis. It is sometimes easy, sometimes difficult, and sometimes still unknown.

Here we deal with a construction of a local conformal net from a unitary vertex operator algebra with some extra nice properties.

**Definition 4.29.** Let \( (V, \langle \cdot | \cdot \rangle) \) be a unitary vertex operator algebra. We say that \( a \in V \) (or \( Y(a, z) \)) satisfies energy-bounds if we have positive integers \( s, k \) and a constant \( M > 0 \) such that we have

\[
\| a_n b \| \leq M \| (|n| + 1) (L_0 + 1)^k b \|
\]

for all \( b \in V \) and \( n \in \mathbb{Z} \). If every \( a \in V \) satisfies energy-bounds, we say \( V \) is energy-bounded.

We have the following, which is [31, proposition 6.1].

**Proposition 4.30.** If \( V \) is generated by a family of homogeneous elements satisfying energy-bounds, then \( V \) is energy-bounded.

Roughly speaking, we need norm estimates for \( (a_n b)_m c \) from those for \( a_n (b_n c) \) and \( h_0 (a_n c) \). This is essentially done with the Borcherds identity.

We also have the following, which is [31, proposition 6.3].

**Proposition 4.31.** If \( V \) is a simple unitary vertex operator algebra generated by \( V_1 \) and \( F \subset V_2 \) where \( F \) is a family of quasi-primary \( \theta \)-invariant Virasoro vectors, then \( V \) is energy-bounded.

We have certain commutation relations for elements in \( V_1 \) and \( F \), and this implies energy-bounds for them. Then the above proposition follows from the previous one.

For a unitary vertex operator algebra \( (V, \langle \cdot | \cdot \rangle) \), define a Hilbert space \( H \) by the completion of \( V \) with respect to the inner product \( \langle \cdot | \cdot \rangle \). For any \( a \in V \) and \( n \in \mathbb{Z} \), we regard \( a_{(n)} \) as a densely defined operator on \( H \). By the invariance of the scalar product, the operator \( a_{(n)} \) has a densely defined adjoint, so it is closable. Suppose \( V \) is energy-bounded and let \( f(z) \) be a smooth function on \( S^1 = \{ z \in \mathbb{C} ||z| = 1 \} \) with Fourier coefficients

\[
\hat{f}_n = \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-int\theta} \frac{d\theta}{2\pi}
\]
for $n \in \mathbb{Z}$. For every $a \in V$, we define the operator $Y_0(a, f)$ with domain $V$ by

$$Y_0(a, f)b = \sum_{n \in \mathbb{Z}} \hat{f}_n a_nb$$

for $b \in V$. The convergence follows from the energy-bounds and $Y_0(a, f)$ is a densely defined operator. This is again closable. We denote by $Y(a, f)$ the closure of $Y_0(a, f)$ and call it a smeared vertex operator.

We define $A_{V(\mathbb{R}^1)}(I)$ to be the von Neumann algebra generated by the (possibly unbounded) operators $Y(a, f)$ with $a \in V$, $f \in C^\infty(S^1)$ and suppf $\subset I$. (For a family of closed operators $\{T_i\}$, we apply the polar decomposition to each $T_i$ and consider the von Neumann algebra generated by the partial isometry part of $T_i$ and the spectral projections of the self-adjoint part of $T_i$.) The family $\{A_{V(\mathbb{R}^1)}(I)\}$ clearly satisfies isotony. We can verify that $(\sqrt{I} A_{V(\mathbb{R}^1)}(I))\Omega$ is dense in $H$. A proof of conformal covariance is non-trivial, but can be done as in [72] and [139] by studying the representations of the Virasoro algebra and $\text{Diff}(S^1)$. We also have the vacuum vector $\Omega$ and the positive energy condition. However, locality is not clear at all from our construction, so we make the following definition.

**Definition 4.32.** We say that a unitary vertex operator algebra $(V, (\cdot, \cdot))$ is strongly local if it is energy-bounded and we have $A_{V(\mathbb{R}^1)}(I) \subset A_{V(\mathbb{R}^1)}(I)'$ for all intervals $I \subset S^1$.

Difficulty in having strong locality is seen as follows. It is well-known that if $A$ and $B$ are unbounded self-adjoint operators, having $AB = BA$ on a common core does not imply commutativity of the spectral projections of $A$ and $B$. That is, having commutativity of spectral projections from certain algebraic commutativity relations is a non-trivial task. A strongly local unitary vertex operator algebra produces a local conformal net through the above procedure by definition. The following is [31, theorem 6.9].

**Theorem 4.33.** Let $V$ be a strongly local unitary vertex operator algebra and $\{A_{V(\mathbb{R}^1)}(I)\}$ the corresponding local conformal net. Then we have $\text{Aut}(A_{V(\mathbb{R}^1)}) = \text{Aut}(V)$. If $\text{Aut}(V)$ is finite, then we have $\text{Aut}(A_{V(\mathbb{R}^1)}) = \text{Aut}(V)$.

We now have the following theorem for a criterion of strong locality [31, theorem 8.1].

**Theorem 4.34.** Let $V$ be a simple unitary energy-bounded vertex operator algebra and $F \subset V$. Suppose $F$ contains only quasi-primary elements, $F$ generates $V$ and $A_{(F(\mathbb{R}^1))}(I) \subset A_{(F(\mathbb{R}^1))}(I)'$ for some interval $I$, where $A_{(F(\mathbb{R}^1))}(I)$ is defined similarly to $A_{V(\mathbb{R}^1)}(I)$. We then have $A_{(F(\mathbb{R}^1))}(I) = A_{V(\mathbb{R}^1)}(I)$ for all intervals $I$, which implies strong locality of $A_{V(\mathbb{R}^1)}(I)$.

From this, we can prove the following result, [31, theorem 8.3].

**Theorem 4.35.** Let $V$ be a simple unitary vertex operator algebra generated by $V_1 \cup F$ where $F \subset V_2$ is a family of quasi-primary $\theta$-invariant Virasoro vectors, then $V$ is strongly local.

We also have the following result, [31, theorem 7.1]

**Theorem 4.36.** Let $V$ be a simple unitary strongly local vertex operator algebra and $W$ its subalgebra. Then $W$ is also strongly local.

The following is [31, corollary 8.2].
Theorem 4.37. Let $V_1, V_2$ be simple unitary strongly local vertex operator algebras. Then $V_1 \otimes V_2$ is also strongly local.

We list some examples of strongly local vertex operator algebras following [31].

Example 4.38. The unitary vertex algebra $L(c, 0)$ is strongly local.

Example 4.39. Let $g$ be a complex simple Lie algebra and let $V^k_g$ be the corresponding level $k$ unitary vertex operator algebra. Then $V^k_g$ is generated by $(V^k_g)_1$ and hence it is strongly local.

The following is [31, theorem 8.15]. This construction was first made in [96].

Example 4.40. The moonshine vertex operator algebra $V^\natural$ is a simple unitary strongly local vertex operator algebra. Hence the automorphism group of the corresponding local conformal net is the Monster group.

The following is [31, conjecture 8.17].

Conjecture 4.41. Let $\Lambda$ be an even lattice and $V_{\Lambda}$ be the corresponding unitary vertex operator algebra. Then $V_{\Lambda}$ is strongly local.

The following is (a part of ) [31, theorem 9.2] which is given by extending the methods in [53].

Theorem 4.42. Let $V$ be a simple unitary strongly local vertex operator algebra and $\{A(V_{\Lambda})(I)\}$ be the corresponding local conformal net. Then one can recover the vertex operator algebra structure on $V$, which is an algebraic direct sum of the eigenspaces of the conformal Hamiltonian, from the local conformal net $\{A(V_{\Lambda})(I)\}$.

This is proven by constructing the smeared vertex operators from abstract considerations using only the local conformal net $\{A(V_{\Lambda})(I)\}$.

We note the $C_2$-cofiniteness has some formal similarity to complete rationality as follows. By considering $V/\{V_{\Lambda}\}^w w \in V$, we can define $C_\mu$-cofiniteness, but $C_1$-cofiniteness is trivial with codimension 0 and $C_\mu$-cofiniteness for $\mu > 2$ follows from $C_2$-cofiniteness. In the definition of the $\mu$-index, we can split $S^1$ into $2\mu$ intervals, and consider a subfactor generated by the factors corresponding to alternating intervals which is contained into the commutant of the factor corresponding to the other alternating intervals. Let $\mu_{\mu}$ be the index of this subfactor. The Haag duality implies we always have $\mu_1 = 1$. The finiteness of $\mu_2$ implies finiteness of all other $\mu_{\mu}$. (See [97] for more details.) Based on this analogy, we have the following conjecture.

Conjecture 4.43. We have a bijective correspondence between completely rational local conformal nets and unitary $C_2$-cofinite vertex operator algebras. We also have equivalence of unitary fusion categories for finite dimensional representations of a completely rational local conformal net and modules of the corresponding vertex operator algebras. We further have coincidence of the corresponding characters of the finite dimensional representations of a completely rational local conformal net and modules of the corresponding vertex operator algebra.
We remark that a relation between local conformal nets and unitary vertex operator algebras is somehow similar to that between Lie groups and Lie algebras. The relation between loop groups and Kac–Moody Lie algebras is somewhere between the two relations.

Based on the above conjectured correspondence, we also list the following problem.

Problem 4.44. For a given finite group $G$, construct a local conformal net whose automorphism group is $G$. The construction should be ‘as natural as possible’.

We believe this problem has a positive solution and it would produce the moonshine local conformal net if the group $G$ is the Monster group. This is based on the same reason as the one given after problem 3.22.

## 5. Other types of conformal field theory

Besides the 2-dimensional chiral conformal field theory, we also have other types of 2-dimensional conformal field theory. Here we briefly mention other settings.

### 5.1. Full conformal field theory

A full conformal field theory is a quantum field theory on 2-dimensional Minkowski space $\mathcal{M} = \{(x, t) | x, t \in \mathbb{R}\}$. We present the formulation following [94].

We have light ray coordinates $\xi = t \pm x$, and set $\mathcal{L}_\pm = \{\xi \mid \xi = 0\}$, the two light ray lines. A double cone is an open subset of $\mathcal{M}$ of the form $\mathcal{C} = I_+ \times I_-$ where $I_+ \subset \mathcal{L}_+$ are bounded intervals. We set $\mathcal{K}$ to be the set of double cones. The group $\text{PSL}(2, \mathbb{R})$ acts on $\mathcal{K}$ by fractional linear transformations, so the actions restricts to a local action on $\mathcal{M}$ as in [21]. In particular, if $F \subset \mathcal{M}$ has a compact closure, then there exists a connected neighborhood $\mathcal{C}$ of the identity in $\text{PSL}(2, \mathbb{R})$ such that we have $\mathcal{C}F \subset \mathcal{K}$ for all $\mathcal{C} \in \mathcal{K}$. We regard this as a local action of the universal covering group $\text{PSL}(2, \mathbb{R})$ on $\mathcal{M} = \mathcal{L}_+ \times \mathcal{L}_-$. We then have a local product action of $\text{PSL}(2, \mathbb{R}) \times \text{PSL}(2, \mathbb{R})$ on $\mathcal{M} = \mathcal{L}_+ \times \mathcal{L}_-$.

**Definition 5.1.** A local Möbius covariant net $\{\mathcal{A}(\mathcal{C})\}$ is an assignment of a von Neumann algebra $\mathcal{A}(\mathcal{C})$ on a fixed Hilbert space $H$ to $\mathcal{C} \in \mathcal{K}$ satisfying the following properties.

1. (Isotony) for $\mathcal{C}_1 \subset \mathcal{C}_2$, we have $\mathcal{A}(\mathcal{C}_1) \subset \mathcal{A}(\mathcal{C}_2)$.
2. (Möbius covariance) there exists a unitary representation $U$ of $\text{PSL}(2, \mathbb{R}) \times \text{PSL}(2, \mathbb{R})$ on $H$ such that for every double cone $\mathcal{C}$, we have
   
   $$U(g) \mathcal{A}(\mathcal{C}) U(g)^* = \mathcal{A}(g \mathcal{C}), \quad g \in U,$$

   where $U \subset \text{PSL}(2, \mathbb{R}) \times \text{PSL}(2, \mathbb{R})$ is any connected neighborhood of the identity with $g \mathcal{C} \subset \mathcal{M}$ for all $g \in U$.
3. (Locality) if $\mathcal{C}_1$ and $\mathcal{C}_2$ are spacelike separated, then we have $[\mathcal{A}(\mathcal{C}_1), \mathcal{A}(\mathcal{C}_2)] = 0$.
4. (Positive energy condition) the one-parameter unitary subgroup of $U$ corresponding to time translations has a positive generator.
5. (Existence of the vacuum vector) there exists a unit $U$-invariant vector $\Omega$ with $\bigcup_{\mathcal{C} \in \mathcal{K}} \mathcal{A}(\mathcal{C}) \Omega = H$.
6. (Irreducibility) The von Neumann algebra generated by all $\mathcal{A}(\mathcal{C})$ is $B(H)$.

Let $\mathcal{G}$ be the quotient of $\text{PSL}(2, \mathbb{R}) \times \text{PSL}(2, \mathbb{R})$ modulo the relation $(r_{2\pi}, r_{-2\pi}) = (\text{id}, \text{id})$, where $r_{\theta}$ is a rotation by $\theta$. We then see that the representation $U$ of...
A local Möbius covariant net \( \{ A(\mathcal{O}) \} \) extends to a local \( G \)-covariant net on the Einstein cylinder \( \mathcal{E} = \mathbb{R} \times S^1 \), the cover of \( S^1 \times S^1 \) obtained by lifting the time coordinate from \( S^1 \) to \( \mathbb{R} \). We use the same symbol \( \{ A(\mathcal{O}) \} \) for the net on \( M \) and its extension on \( \mathcal{E} \). We have various properties for local Möbius covariant nets as in [94, proposition 2.2]. We can then define a local conformal net on \( M \) by requiring appropriate conformal covariance as in [94, pages 68–69].

For a local Möbius covariant net \( \{ A(\mathcal{O}) \} \), and a bounded interval \( I \subset L \), we define \( A_+ (I) = \bigcap_{\partial \rightarrow I} A(\mathcal{O}) \), where the intersection is taken over all intervals \( I \subset L \). We also have \( A_-(I) \). We can restrict \( A_\pm (I) \) to the Hilbert spaces \( H_\pm = A_\pm (I) \mathbb{H} \), which is independent of \( I \). Then we can regard \( A_\pm (I) \) as local conformal nets and \( \{ A(\mathcal{O}) \} \) is an extension of \( \{ A_+ (I) \otimes A_-(J) \} \) on \( H_+ \otimes H_- \cong H \) as in [94, proposition 2.3, corollary 2.4]. We also write \( A_\pm \) for \( A_+ \) and \( A_- \), respectively. We also define a notion of a representation of \( \{ A(\mathcal{O}) \} \), and the \( \mu \)-index and complete rationality of \( \{ A(\mathcal{O}) \} \) as in [94, section 2.1].

Suppose we have completely rational local conformal nets \( \{ A_L (I) \} \) and \( \{ A_R (I) \} \) and an extension \( \{ B(I \times J) \} \) with \( A_L (I) \otimes A_R (J) \subset B(I \times J) \). Let \( \theta = \sum_{i,j} Z_{ij} \lambda_i^L \otimes \lambda_j^R \) be the dual canonical endomorphism of the subfactor \( A_L (I) \otimes A_R (J) \subset B(I \times J) \) where \( \{ \lambda_i^L \} \) and \( \{ \lambda_j^R \} \) are the irreducible representations of \( \{ A_L (I) \} \) and \( \{ A_R (I) \} \), respectively. Let \( S_L \), \( S_R \) be the S-matrices of \( \{ A_L (I) \} \) and \( \{ A_R (I) \} \), respectively, and let \( T_L \), \( T_R \) be the T-matrices of \( \{ A_L (I) \} \) and \( \{ A_R (I) \} \), respectively. Then Müger has the following. (See also [10, proposition 6.6].)

**Theorem 5.2.** The following are equivalent.

1. The local conformal net \( \{ B(\mathcal{O}) \} \) has only the trivial irreducible representation.
2. The \( \mu \)-index of \( \{ B(\mathcal{O}) \} \) is 1.
3. We have \( T_L Z = Z T_R \) and \( S_L Z = Z S_R \).

In particular, if we can naturally identify \( \{ A_L (I) \} \) and \( \{ A_R (I) \} \) and have \( \mu \)-index equal to 1, then the matrix \( Z \) is a modular invariant. If both \( \{ A_L (I) \} \) and \( \{ A_R (I) \} \) have the same central charge, we call it the central charge of \( \{ B(\mathcal{O}) \} \). Study of a local conformal net \( \{ B(\mathcal{O}) \} \) is reduced to that of the following.

1. The two local conformal nets \( \{ A_L (I) \} \) and \( \{ A_R (I) \} \).
2. The Q-system corresponding to the dual canonical endomorphism for the subfactor \( A_L (I) \otimes A_R (J) \subset B(I \times J) \).

The Q-system above is one for the representation category \( \text{Rep}(A_L) \boxtimes \text{Rep}(A_R)^{opp} \), where \( \text{opp} \) means the unitary modular tensor category where the braiding is reversed.

For the case the central charge is less than 1, we have the following classification theorem. (See [94, theorem 5.5].)

**Theorem 5.3.** The local conformal nets \( \{ B(\mathcal{O}) \} \) with central charge less than 1 which are maximal with respect to inclusions are in a bijective correspondence the modular invariants listed in [25].

The modular invariants in [25] are labelled with pairs of the A-D-E Dynkin diagrams with Coxeter numbers differing by 1. That is we have the pairs \( (A_{n-1}, A_n) \), \( (D_{2n+1}, A_{2n}) \), \( (A_{4n}, D_{2n+2}) \), \( (D_{2n+2}, A_{4n+2}) \), \( (A_{4n+2}, D_{2n+3}) \), \( (A_{10}, E_6) \), \( (E_6, A_{11}) \), \( (A_{16}, E_7) \), \( (E_7, A_{18}) \), \( (A_{28}, E_8) \), and \( (E_8, A_{30}) \). The uniqueness for each pair follows from [94, theorem 5.3].
For realization of these modular invariants, we appeal to the following result [129, corollary 1.6].

**Theorem 5.4.** Let \( \{ A(I) \} \) be a completely rational local conformal net. Let \( (\theta, v, w) \) be a Q-system (without locality assumed) where \( \theta \) is an object in the representation category \( \text{Rep}(A) \). Let \( (Z_{\lambda, \mu}, \lambda \oplus \mu) \) be the modular invariant arising from the \( \alpha \)-induction associated with \( (\theta, v, w) \) as in theorem 3.27. Then we have a Q-system for the representation \( Z(\lambda, \mu) \), hence a 2-dimensional local conformal net extending \( \{ A(I) \otimes A(J) \} \).

It is expected that the \( N = 2 \) full superconformal field theory is related to Calabi–Yau manifolds [70], so we also have the following problem.

**Problem 5.5.** Construct an operator algebraic object corresponding to a Calabi–Yau manifold in the setting of \( N = 2 \) full superconformal field theory and study the mirror symmetry in this context.

### 5.2. Boundary conformal field theory

We present our setting for boundary conformal field theory based on [112]. (Also see [30, 10, section 6.4].)

Let \( \mathcal{M}_+ = \{ (t, x) \in \mathcal{M} | x > 0 \} \) be the half 2-dimensional Minkowski space. Let \( \mathcal{K}_+ \) be the set of double cones \( \mathcal{O} \) whose closures are contained in \( \mathcal{M}_+ \). A double cone \( \mathcal{O} \in \mathcal{K}_+ \) is represented as \( I \times J \) where \( I, J \) are bounded intervals in \( \mathbb{R} \) with \( I < J \). We fix a completely rational local conformal net \( \{ A(I) \} \) and restrict it to a net on \( \mathbb{R} \) by removing the point \( \infty \).

The universal cover \( \text{PSL}(2, \mathbb{R}) \) acts globally in the universal cover of \( S^1 \). The product action on the chiral lines of \( \mathcal{M} \) gives a local action of \( \text{PSL}(2, \mathbb{R}) \) on \( \mathcal{M} \). We deal with the local action of \( \text{PSL}(2, \mathbb{R}) \) obtained by restricting the local action of \( \text{PSL}(2, \mathbb{R}) \times \text{PSL}(2, \mathbb{R}) \) to the diagonal. This action restricts to local actions of \( \text{PSL}(2, \mathbb{R}) \) on \( \mathcal{M}_+ \) and its boundary, the time-axis.

**Definition 5.6.** An assignment of a von Neumann algebra \( B_{\mathcal{O}}(\mathcal{O}) \) on a fixed Hilbert space \( H_{\mathcal{O}} \) to each double cone \( \mathcal{O} \in \mathcal{K}_+ \) is called a boundary net if it satisfies the following.

1. (Isotony) for \( \mathcal{O}_1 \subset \mathcal{O}_2 \), we have \( B_{\mathcal{O}_1}(\mathcal{O}_1) \subset B_{\mathcal{O}_2}(\mathcal{O}_2) \).
2. (Locality) if \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \) are spacelike separated in \( \mathcal{M}_+ \), then we have \( [B_{\mathcal{O}_1}(\mathcal{O}_1), B_{\mathcal{O}_2}(\mathcal{O}_2)] = 0 \).
3. (Möbius covariance) there exists a unitary representation \( U \) of \( \text{PSL}(2, \mathbb{R}) \) on the Hilbert space \( H_{\mathcal{O}} \) such that we have \( U(g)B_{\mathcal{O}}(\mathcal{O})U(g)^* = B_{\mathcal{O}}(g\mathcal{O}) \) for every \( \mathcal{O} \in \mathcal{K}_+ \) with \( g \in \text{PSL}(2, \mathbb{R}) \) having a path of elements \( g_s \in \text{PSL}(2, \mathbb{R}) \) connecting the identity of \( \text{PSL}(2, \mathbb{R}) \) and \( g \) satisfying \( g_s\mathcal{O} \in \mathcal{K}_+ \) for all \( s \).
4. (Positive energy condition) the generator of the translation one-parameter subgroup of \( U \) is positive.
5. (Existence of the vacuum vector) we have a unit vector \( \Omega \in H_{\mathcal{O}} \) such that \( C\Omega \) are the \( U \)-invariant vectors and we have \( B_{\mathcal{O}}(\mathcal{O})\Omega = H_{\mathcal{O}} \) for each \( \mathcal{O} \in \mathcal{K}_+ \).

Furthermore, we have the following.
Definition 5.7. Let \( \{A_+(\mathcal{O})\} \) be the boundary net on \( \mathcal{M}_+ \) given by \( A_+(\mathcal{O}) = A(I) \cup A(J) \), where \( \mathcal{O} = I \times J \). Then a boundary net \( \{B_+(\mathcal{O})\} \) associated with \( \{A(I)\} \) is a boundary net \( \{B_+(\mathcal{O})\} \) satisfying the following conditions.

1. (Joint irreducibility) there is a representation \( \pi \) of \( \{A(I)\} \) on \( H_B \) such that we have \( \pi(A_+(\mathcal{O})) \subset B_+(\mathcal{O}) \) and \( U(g)\pi(A_+(\mathcal{O}))U(g)^* = \pi(A_+(g\mathcal{O})) \) for doubles cones \( \mathcal{O}, g\mathcal{O} \in K_+ \).

2. For each double cone \( \mathcal{O} \), the von Neumann algebra generated by \( B_+(\mathcal{O}) \) and all algebras \( \pi(A(I)) \) is \( B(H_B) \).

The above axioms imply that the inclusion \( \pi(A_+(\mathcal{O})) \subset B_+(\mathcal{O}) \) is irreducible.

Starting with a boundary net \( \{B_+(\mathcal{O}) \supset \pi(A_+(\mathcal{O}))\} \), we define the generated net \( B^{\text{gen}}(I) \supset \pi(A(I)) \) with \( I \subset \mathbb{R} \) by
\[
B^{\text{gen}}(I) = \bigvee_{\mathcal{O} \in K_+ \cap \mathcal{O} \subset W_I} B_+(\mathcal{O}) \supset \pi(A(I)),
\]
where \( W_I = \{(t, x) | t + x \in I\} \) is the left wedge such that its intersection with the \( t \)-axis is \( I \). The following is [112, proposition 2.5].

Theorem 5.8. The net \( \{B^{\text{gen}}(I)\} \) is isotonous and covariant in the sense that we have \( U(g)B^{\text{gen}}(I)U(g)^* = B^{\text{gen}}(gI) \). We also have \( \pi(A(I)) \subset B^{\text{gen}}(I) \subset \pi(A(I'))' \).

The net \( \{B^{\text{gen}}(I)\} \) may not satisfy locality, though we have relative locality in the sense \( \pi(A(I_1)), B^{\text{gen}}(I_2) = 0 \) for \( I_1, I_2 \) with \( I_1 \cap I_2 = \emptyset \). We say \( \{B^{\text{gen}}(I)\} \) is a non-local extension of \( \{A(I)\} \) in this case. (The name ‘non-local’ means ‘possibly non-local’.)

For a given non-local extension \( \{B(I) \supset A(I)\} \) on \( \mathbb{R} \), we define \( B^{\text{ind}}_+(\mathcal{O}) = B(L) \cap B(K)' \), where \( \mathcal{O} = I \times J \) and \( L \sqsubset K \) with \( L \cap K' = I \cup J \), or equivalently \( \mathcal{O} = W_L \cap W_K' \).

The dual net is defined by \( B^{\text{dual}}_+(\mathcal{O}) = B_+(\mathcal{O})' \) and we have \( B^{\text{dual}}_+(\mathcal{O}) = B_+(\mathcal{O}) \) if and only if \( \{B_+(\mathcal{O})\} \) satisfies the Haag duality, where \( \mathcal{O}' \) is the causal complement of \( \mathcal{O} \).

We have \( (B^{\text{dual}})^{\text{ind}}_+(\mathcal{O}) = B(I) \) and \( (B^{\text{dual}})^{\text{ind}}_+(\mathcal{O}) = B^{\text{dual}}_+(\mathcal{O}) = B_+(\mathcal{O}) \) if \( \{B_+(\mathcal{O})\} \) already has the Haag duality. We then have a bijective correspondence between boundary nets \( \{B_+(\mathcal{O})\} \) associated with \( \{A(I)\} \) with Haag duality and non-local extensions \( \{B(I) \supset A(I)\} \) of \( \{A(I)\} \).

When \( \{A(I)\} \) is the Virasoro net with \( c < 1 \), we can classify all irreducible non-local extensions, hence all boundary nets associated with \( \{\text{Vir},(I)\} \) with Haag duality as follows.

Let \( G_i \) be one of the \( A-D-E \) Dynkin diagrams with Coxeter number \( m \). Let \( G_2 \) be one of the \( A-D-E \) Dynkin diagrams with Coxeter number \( m + 1 \). Let \( v_1, v_2 \) be vertices of \( G_1, G_2 \), respectively. For a vertex \( v \) of a graph, we denote the orbit of \( v \) under the graph automorphisms by \([v]\). Then we have the following theorem ([98, theorem 3.1].)

Theorem 5.9. Irreducible non-local extensions of the Virasoro nets \( \{\text{Vir},(I)\} \) with \( c < 1 \) are in a bijective correspondences to the quadruples \( (G_i, [v_1], G_2, [v_2]) \) as above.

We can pass from a boundary conformal field theory to a full conformal field theory and also back by removing and adding the boundary. See [10, 30, 113] for these relations between full and boundary conformal field theories.

We also have results on the phase boundaries and topological defects in the operator algebraic setting. See [11, 12] for details. See [64, 122] for earlier works on topological defects.
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