Projectors separating spectra
for $L^2$ on symmetric spaces $\text{GL}(n, \mathbb{C})/\text{GL}(n, \mathbb{R})$

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The Plancherel decomposition of $L^2$ on a pseudo-Riemannian symmetric space $\text{GL}(n, \mathbb{C})/\text{GL}(n, \mathbb{R})$ has spectrum of $[n/2]$ types. We write explicitly orthogonal projectors separating spectrum into uniform pieces.

1 Formulas for the projectors

1.1. Problem. It is well-known that in various problems of non-commutative harmonic analysis spectra split into pieces of different nature.

In 1977 I. M. Gelfand and S. G. Gindikin [5] formulated a problem about an explicit description of a decomposition of $L^2$ on a semisimple Lie group $G$ (and, more generally, on a semi-simple pseudo-Riemannian symmetric space) into a direct sum of representations having uniform spectra. They also gave an answer for $G = \text{SL}(2, \mathbb{R})$ in the terms of boundary values of holomorphic functions. After this work the question became a topic of intensive discussions and series of works.

The picture is quite nice for $L^2(\text{SL}(2, \mathbb{R}))$ and $L^2(\text{SL}(2, \mathbb{R})/H)$, where $H$ is the diagonal subgroup, see [7], [8], [1], and [14], [17] respectively. However, up to now the situation for higher groups is far to be well-understood.

1.2. Known approaches.

a) G. I. Olshanski [21] proposed a way to split off holomorphic discrete series using boundary values of holomorphic functions, this approach was used in several works, see, e.g. [13], [12].

S. G. Gindikin, B. Krötz and G. ´Olafsson [9] showed that in some cases an integral of the most continuous series can be splitted off in a similar way.

b) V. F. Molchanov [15] and S. G. Gindikin [8] in different way obtained the desired decomposition for $L^2$ on multi-dimensional hyperboloids $\text{O}(p, q)/\text{O}(p, q−1)$ (this includes the cases $L^2(\text{SL}(2, \mathbb{R}))$ and $L^2(\text{SL}(2, \mathbb{R})/H)$ mentioned above).

c) In [16] there was proposed a way for separation of summands of complementary series, see more in [18], [20].

1.3. Purposes of the present work.

For a pseudo-Riemannian symmetric space $\text{GL}(n, \mathbb{C})/\text{GL}(n, \mathbb{R})$ consider a decomposition of $L^2$ into a direct sum of subspaces, in which $\text{GL}(n, \mathbb{C})$ has uniform spectra:

$$L^2(\text{GL}(n, \mathbb{C})/\text{GL}(n, \mathbb{R})) = L_0 \oplus L_1 \oplus \cdots \oplus L_{[n/2]}.$$  \hfill (1.1)

We also have a corresponding decomposition of the identity operator:

$$E = \Pi_0 \oplus \Pi_1 \oplus \cdots \oplus \Pi_{[n/2]}.$$ \hfill (1.2)

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\footnote{Here $[n/2]$ denotes the integer part of $n/2$.}
where $\Pi_r$ is the orthogonal projector to a subspace $L_r$. We intend to obtain explicit expressions for the projectors $\Pi_r$.

The present work is based on both results and auxiliary calculations of Shigeru Sano [22]. The formulas for the projectors are given by Theorem 2.1.

1.4. The spaces $\text{GL}(n,\mathbb{C})/\text{GL}(n,\mathbb{R})$. Below

$$G := \text{GL}(n,\mathbb{C}), \quad H := \text{GL}(n,\mathbb{R}).$$

We realize the symmetric space $G/H$ as the space $M$ of all matrices $x \in \text{GL}(n,\mathbb{C})$ satisfying the condition

$$x^T = 1.$$

The group $G$ acts on $M$ by transformations

$$g : x \mapsto x \circ g := g^{-1}xg,$$

the stabilizer of $x = e$ is $H$. Notice that $H$ acts on $M$ by conjugations.

Denote by $C_c^\infty(G/H)$ the space of $C^\infty$-smooth functions on $G/H$ with compact support. Let $\chi$ be an $H$-invariant distribution on $G/H$. It determines a $G$-intertwining operator

$$A[\chi] : C_c^\infty(G/H) \to C_c^\infty(G/H)$$

by the pairing

$$A[\chi]f(g) = \langle f(x \circ g, \chi)_{G/H},$$

a function on $G$ obtained in this way is $H$-invariant and therefore it is a function on $G/H$.

Our purpose is to write $H$-invariant distributions on $G/H$ determining the projectors $\Pi_r$.

1.5. Notation. By $dx$ we denote a $G$-invariant measure on $G/H$ (it is unique up to a constant factor). Denote by $S_m$ the symmetric group of order $m$, by $\mathbb{T}^p$ the torus $(\mathbb{R}/2\pi\mathbb{Z})^p$. Denote by $\Delta$ the Vandermonde expression:

$$\Delta(s) = \Delta(s_1,\ldots,s_n) := \prod_{1 \leq p < q \leq n} (s_p - s_q). \quad (1.3)$$

1.6. Cartan subspaces and the Weyl integration formula. Let $\varphi$, $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, $t \in \mathbb{R}$. Denote by $v(\theta,t)$ a $2 \times 2$ matrix given by

$$v(\theta,t) = e^{i\theta} \begin{pmatrix} \cosh t & i \sinh t \\ -i \sinh t & \cosh t \end{pmatrix} = \begin{pmatrix} e^{i\theta} \cosh t & ie^{i\theta} \sinh t \\ -ie^{i\theta} \sinh t & e^{i\theta} \cosh t \end{pmatrix},$$

its eigenvalues are

$$e^z := e^{t+i\theta}, \quad e^{-z} := e^{-t+i\theta}.$$

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3By $(f,\chi)_L$ we denote a pairing of a test function $f$ and a distribution $\chi$ on a manifold $L$. 

2
We define a Cartan subspace $A_k$, where $k = 1, 2, \ldots, \lfloor n/2 \rfloor$, as the set of matrices $a \in M$ having the following block-diagonal form

$$a^{(k)} := \begin{pmatrix} e^{i\varphi_1} & & & & \varepsilon \theta_k, t_k \\ & \ddots & & & \varepsilon \theta_{k}, t_{k} \\ & & \ddots & & \varepsilon \theta_{1}, t_{1} \end{pmatrix} \quad (1.4)$$

We equip $A_k$ with the standard Lebesgue measure

$$da^{(k)} := \prod_{l=1}^{n-2k} d\varphi_l \prod_{m=1}^{k} (d\theta_m dt_m).$$

The relative Weyl group $W_k$ corresponding to the Cartan subspace $A_k$ is

$$W_k \simeq S_{n-2k} \times (S_k \rtimes \mathbb{Z}_2^k).$$

The factor $S_{n-2k}$ acts on $A_k$ by permutations of $e^{i\varphi_2}, \ldots, e^{i\varphi_{n-2k}}$, the group $S_k$ acts by permutations of pairs $(\theta_1, t_1), \ldots, (\theta_k, t_k)$. The group $\mathbb{Z}_2^k$ is generated by reflections

$$R_m : (t_1, \ldots, t_{m-1}, t_m, t_{m+1}, \ldots, t_k) \mapsto (t_1, \ldots, t_{m-1}, -t_m, t_{m+1}, \ldots, t_k) \quad (1.5)$$

(all the coordinates $\varphi$ and $\theta$ remain to be fixed). We denote

$$\gamma_k := \frac{1}{\#W_k} = \frac{1}{k!(n-2k)!2^k}. \quad (1.6)$$

1.7. Averaging operators. An element of $M$ having different eigenvalues can be reduced to one of subalgebras $A_k$ by a conjugation by some $h \in H$. This element $a \in A_k$ is defined up to the action of $W_k$.

Define a function $\Delta_k$ on $A_k$ by the Vandermonde expression

$$\Delta_k(a^{(k)}) = \Delta(e^{i\varphi_1}, \ldots, e^{i\varphi_{n-2k}}, e^{z_k}, e^{-z_k}, \ldots, e^{z_1}, e^{-z_1}).$$

Notice that $\Delta_k(a^{(k)})$ are pure real or pure imaginary depending on $n$, $k$; also the sign of $\Delta_k(a^{(k)})$ depends on the choice of order of eigenvalues.

On the other hand, denote by $B_k \subset H$ the subgroup fixing all elements of $A_k$. It consists of block diagonal real matrices with $n-2k$ blocks of size $1 \times 1$ and $k$ blocks of size $2 \times 2$ having the form $\begin{pmatrix} c & d \\ -d & c \end{pmatrix}$, $c^2 + d^2 \neq 0$. Consider a map $A_k \times (H/B_k) \rightarrow M$ given by

$$(a^{(k)}, y) \mapsto ya^{(k)}y^{-1}. \quad (1.7)$$

An element $y \in H$ is a matrix determined up to an equivalence $y \sim yb$, where $b \in B_k$. Therefore the map is well-defined. Equip $H/B_k$ with an invariant
measure \( d_k y \), which will be normalized several lines below. For any function \( f \in C_c^\infty(G) \) we define a collection of functions \( I_k f \) on \( A_k \) (integrals of \( f \) over orbits of \( H \)):

\[
I_k f(a^{(k)}) = \int_{H/B_k} f(ya^{(k)}y^{-1}) \, d_k y
\]

(functions \( I_k f \) are well-defined on the set \( \Delta_k(a^{(k)} \neq 0) \). Then under a certain normalization of measures \( dx \) and \( d_k y \) we have the identity

\[
\int_{G/H} f(x) \, dx = \sum_{k=0}^{\lfloor n/2 \rfloor} \gamma_k \int_{A_k} I_k f(a^{(k)}) |\Delta(a^{(k)})|^2 \, da^{(k)}. \tag{1.8}
\]

This is a version of the Weyl integration formula. At each point \( a^{(k)} \in A_k \) the volume form on \( G/H \) is a product of forms \( |\Delta(a^{(k)})|^2 \, da^{(k)} \wedge d_k y \) and this explains a normalization of \( d_k y \). The factors \( \gamma_k = 1/\#W_k \) arise because the map (1.7) covers each point of its image \( \#W_k \) times. See [22], Sect.4.

1.8. Normalized averaging operators. For each \( k \) we define the expression \( \varepsilon_k(a^{(k)}) \) on \( A_k \) by

\[
\varepsilon_k(a^{(k)}) := \prod_{1 \leq p < q \leq n-2k} \text{sign} \sin \left( \frac{\varphi_p - \varphi_q}{2} \right).
\]

For each \( k \) we define an averaging operator from \( C_c^\infty(G) \) to the space of functions on \( A_k \) by

\[
\Xi_k f(a^{(k)}) := \varepsilon_k(a^{(k)}) (\det a^{(k)})^{-(n-1)/2} \frac{\Delta_k(a^{(k)})}{\Delta_k(a^{(k)})} \int_{H/B_k} f(ya^{(k)}y^{-1}) \, d_k y. \tag{1.9}
\]

The functions \( \Xi_k f \) are invariant with respect to the action of the subgroups \( S_k, S_{n-2k} \subset W_k \) and change signs under the reflections \( R_m \), see [17]. The hypersurfaces \( t_m = 0 \) divide \( A_k \) into a union of \( 2^k \) ‘octants’

\[
\left\{ \begin{array}{c}
t_1 \leq 0, \quad \ldots \quad t_k \leq 0.
\end{array} \right.
\]

(1.10)

Generally, \( \Xi_k f(a^{(k)}) \) is discontinuous on these hypersurfaces. A function \( \Xi_k f(a^{(k)}) \) admits a smooth continuation to the closure of each ‘octant’ and the operator \( \Xi_k \) is a bounded operator in a natural sense ([25], Corollary 8.5.1.2).

An intersection \( A_k \cap A_{k+1} \) of Cartan subspaces is a hypersurface in each subspace and \( \Xi_k f, \Xi_{k+1} f \) satisfy some boundary conditions on this hypersurface (see Subsect. 3.2).

Thus we get an operator \( \Xi \), which sends a function \( f \in C_c^\infty(G/H) \) to a collection of functions

\[
\Xi : f \mapsto (\Xi_0 f, \ldots, \Xi_{\lfloor n/2 \rfloor} f).
\]
An $H$-invariant distribution on $G/H$ determines a linear functional on the image of $\Xi$.

1.9. Differential Vandermonde. Consider the following $n$-tuple of first order differential operators on $A_k$:

$$\frac{\partial}{i\partial \varphi_1}, \ldots, \frac{\partial}{i\partial \varphi_{n-2k}}, \frac{1}{2}\left(\frac{\partial}{\partial t_k} + \frac{\partial}{i\partial \theta_k}\right), \frac{1}{2}\left(\frac{\partial}{\partial t_k} - \frac{\partial}{i\partial \theta_k}\right), \ldots, \frac{1}{2}\left(\frac{\partial}{\partial t_1} + \frac{\partial}{i\partial \theta_1}\right), \frac{1}{2}\left(\frac{\partial}{\partial t_1} - \frac{\partial}{i\partial \theta_1}\right).$$

(1.11)

Denote them by $X_1, \ldots, X_n$. We define the differential operator

$$\Delta(\partial) = \Delta_k(\partial) := \Delta(X_1, \ldots, X_n).$$

Any function $F = \Delta_k(\partial) \Xi_k f(a(k))$ on $A_k$ satisfies the following properties (see [22], (8.2)):

A°. $F$ is skew-symmetric with respect to the subgroup $S_{n-2k} \subset W_k$ and invariant with respect to $S_k \ltimes \mathbb{Z}_k^2$.

B°. $F$ admits a continuous extension to the whole $A_k$;

C°. $F$ is smooth on the closure of each 'octant' (1.10).

There is a constant $\gamma_*$ depending on a normalization of the measure on $G/H$ such that

$$\Delta_{[n/2]} \Xi_{[n/2]} f(0) = \gamma_* f(e)$$

(1.12)

for any $f \in C^\infty_c(G/H)$.

1.10. Distributions $\Lambda_p$. Recall that a function $\cot \varphi/2$ determines a distribution on the circle $\mathbb{R}/2\pi \mathbb{Z}$ by the formula

$$f \mapsto \text{p.v.} \int_{-\pi}^\pi \cot \frac{\varphi}{2} f(\varphi) d\varphi.$$

Consider an even-dimensional torus $\mathbb{T}^{2m}$ with standard coordinates $e^{i\varphi_1}, \ldots, e^{i\varphi_{2m}}$. Define the distribution $\Lambda_{2m}$ by

$$\Lambda_{2m}(\varphi) := \frac{(-i)^m}{(2\pi)^m 2^m m!} \sum_{\sigma \in S_{2m}} (-1)^\sigma \prod_{j=1}^m \left(\cot \frac{\varphi_\sigma(2j-1)}{2} \cdot \delta\left(\varphi_\sigma(2j-1) + \varphi_\sigma(2j)\right)\right).$$

(1.13)

For an odd-dimensional torus $\mathbb{T}^{2m+1}$ we define the distribution $\Lambda_{2m+1}$ by

$$\Lambda_{2m+1}(\varphi) := \frac{(-i)^m}{(2\pi)^m 2^m m!} \sum_{\sigma \in S_{2m+1}} (-1)^\sigma \times$$

$$\times \delta\left(\varphi_\sigma(2m+1)\right) \cdot \prod_{j=1}^m \left(\cot \frac{\varphi_\sigma(2j-1)}{2} \cdot \delta\left(\varphi_\sigma(2j-1) + \varphi_\sigma(2j)\right)\right).$$

(1.14)

1.11. Formulas for projectors. The purpose of the present paper is the following formula.
Theorem 1.1  Invariant distributions $\Theta_r : C^\infty_c(G/H) \to \mathbb{C}$ determining the projectors $\Pi_0, \ldots, \Pi_{[n/2]}$ in (1.1)–(1.2) are given by the formula

$$\gamma_* (f, \Theta_r)_G = \frac{(-1)^{n(n-1)/2(n-2r)!}}{([n/2] - r)!} \sum_{k=0}^{r} (-1)^k 4^{r-k} \times$$

$$\times \gamma_k \langle \Delta_k(\theta)\Xi_k f(\varphi, \theta, t), \Lambda_{n-2k}(\varphi) \cdot \prod_{j=1}^{k} \delta(t_j)\delta(\theta_j) \rangle_{Ak}.$$  \quad (1.15)

In particular, the projector corresponding to the discrete series is determined by the distribution

$$\gamma_* \Theta_0(f) = \frac{(-1)^{n(n-1)/2(n-2r)!}}{[n/2]!} \langle \Delta_0\Xi_0 f, \Lambda_n \rangle_{Ak}.$$  

1.12. Remarks on the general problem for pseudo-Riemannian symmetric spaces. The problem of separation of spectra is reduced to integration of spherical distributions as functions of parameters with respect to the Plancherel measure.

The spaces $\text{GL}(n, \mathbb{C})/\text{GL}(n, \mathbb{R})$ are representatives of spaces $G(\mathbb{C})/G(\mathbb{R})$, where $G(\mathbb{C})$ is a complex semisimple (reductive) Lie group and $G(\mathbb{R})$ is its real form. The Plancherel formula for such spaces was obtained by Sh. Sano, N. Bopp, and P. Harink [23], [3], [10]. The spaces $G(\mathbb{C})/G(\mathbb{R})$ are close relatives and spherical distributions admit elementary expressions similar to (3.10)-(3.11).

In a recent preprint [19] the similar problem was solved for $L^2$ on the pseudo-unitary group $U(p, q)$ (formulas for projectors have another form but calculations are similar). In the case of real semisimple groups spherical distributions are characters, according Harish-Chandra characters are locally integrable functions admitting elementary expressions.

For general pseudo-Riemannian semisimple symmetric spaces the way used here and in [19] is impossible. On the other hand, calculations of V. F. Molchanov in rank one case [15] indicate that a general formulation of the problem requires an improvement.

1.13. Structure of the paper. The proof of Theorem 1.1 is based on the Plancherel formula obtained by Sano [22] and also on his proof. For this reason, we must expose numerous elements of the paper [22]. In the Section 2 we establish a skew-symmetric counterpart of the formula

$$\sum_{n=-\infty}^{\infty} e^{in\varphi} = 2\pi \delta(\varphi).$$

Section 3 contains preliminaries and Section 4 evaluation of the projectors.

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2 A skew-symmetric analog of the delta-function

2.1. Distributions Λ. For integers $a_1 > \cdots > a_p$ denote
\[\delta_p^a = \delta_p^{a_1, \ldots, a_p}(e^{i\varphi_1}, \ldots, e^{i\varphi_p}) := \sum_{\sigma \in S_p} (-1)^{s} e^{i \sum_m a_{\pi(m)} \varphi_m}. \quad (2.1)\]

Denote by $\Lambda_p$ the following distribution on a torus $\mathbb{T}^p$:
\[\mathcal{L}_p(e^{i\varphi_1}, \ldots, e^{i\varphi_p}) := \frac{1}{(2\pi)^p} \sum_{a_1 > \cdots > a_p} \delta_p^a. \quad (2.2)\]

**Theorem 2.1** We have $\mathcal{L}_p = \Lambda_p$, where $\Lambda_p$ is given by (1.13), (1.14).

For a proof of this statement we need some combinatorial lemmas.

2.2. Matchings. Consider a $p$-element ordered set $C$, it is convenient to assume $C = \{p, p-1, \ldots, 1\}$. If $p$ is even, we say that a matching of $C$ is a partition
\[\zeta : C = \{c_1, c_2\} \cup \{c_3, c_4\} \cup \cdots \quad (2.3)\]
of $C$ into two-element subsets. If $p$ is odd, then a matching is a partition of $C$ into $(p-1)/2$ two-element subsets and one single point subset,
\[\zeta : C = \{c_1, c_2\} \cup \cdots \cup \{c_{p-2}, c_{p-1}\} \cup \{c_p\} \quad (2.4)\]

We draw matchings as diagrams with arcs, see Figure 1. Denote by $\text{Match}(C) = \text{Match}_p$ the set of all matchings. Define the standard matching $\zeta_0$ by
\[\zeta_0 = \begin{cases} \{p, p-1\} \cup \{p-2, p-3\} \cup \cdots \cup \{2, 1\}, & \text{if } p \text{ is even;} \\ \{p, p-1\} \cup \{p-2, p-3\} \cup \cdots \cup \{3, 2\} \cup \{1\}, & \text{if } p \text{ is odd.} \end{cases}\]

The symmetric group $S_p$ acts on $\text{Match}_p$ in a natural way. It is convenient to imagine this action as on Figure 2 (we glue a diagram of matching with a diagram of a permutation).

Consider a matching (2.3) or (2.4). We say that pairs $\{c_\alpha, c_{\alpha+1}\}$ and $\{c_\beta, c_{\beta+1}\}$ are interlacing if precisely one point $c_\beta$ or $c_{\beta+1}$ lies between $c_\alpha$ and $c_{\alpha+1}$ (in the sense of the ordering of $C$). On Figure 1 an interlacing of pairs corresponds to

![Figure 1: A matching. Even case and odd case.](image-url)
Figure 2: Action of symmetric group on Match_p.

an intersection of arcs. If \#C = p is odd, we say that a distinguished element \( \{c_p\} \) of a matching *interlaces* a pair \( \{c_\alpha, c_{\alpha+1}\} \) if \( c_p \) lies between \( c_\alpha \) and \( c_{\alpha+1} \).

We say that a matching \( \zeta \) is even (respectively, odd) if the number of pairs of interlacing elements of \( \zeta \) is even (respectively, odd). Denote the parity by \((-1)^\zeta\).

**Lemma 2.2** For any \( p \),

\[
\Sigma(p) := \sum_{\zeta \in \text{Match}_p} (-1)^\zeta = 1. \tag{2.5}
\]

**Proof.** First, let \( p \) be even. Define an involution \( J \) of Match_p in the following way.

— If \( \{p, p-1\} \) is not an element of a matching \( \zeta \), we apply to \( \zeta \) the transposition \( (p, p-1) \in S_p \).

— If \( \{p, p-1\} \) is an element of \( \zeta \) and \( \{p-2, p-3\} \) not, we apply to \( \zeta \) the transposition \( (p-2, p-3) \in S_p \). Etc.

— \( J \) fixes the matching \( \zeta_0 \).

The involution \( J \) changes parity of all matchings \( \zeta \neq \zeta_0 \). This implies our statement.

Next, let \( p \) be odd. Forgetting a singleton \( \{c_p\} \) in \( \zeta \in \text{Match}_p \), we get a map \( h : \text{Match}_p \to \text{Match}_{p-1} \). Moreover,

\[
(-1)^\zeta = (-1)^{c_p+1}(-1)^{h(\zeta)}. 
\]

For \( c_p = 1, 3, 5, \ldots, p \), we have \((-1)^\zeta = (-1)^{h(\zeta)}\), for \( c_p = 2, 4, \ldots, 2p-1 \), the signs are different. Therefore \( \Sigma(p) = \Sigma(p-1) \).

**Lemma 2.3** For any permutation \( \sigma \in S_p \)

\[
(-1)^{\sigma \zeta_0} = (-1)^\sigma \cdot \text{sign}(\sigma(p) - \sigma(p-1)) \cdot \text{sign}(\sigma(p-2) - \sigma(p-3)) \ldots
\]

**Proof.** We imagine \( \sigma \in S_p \) as a bipartite graph as on Figure 2. Inversions in permutations correspond to intersections of arcs. Parity of a permutation is parity of number of intersections. A matching \( \sigma \zeta_0 \) is obtained by gluing the
matching $\zeta_0$ and the permutation $\sigma$. Multiplication of $(-1)^\sigma$ by $\prod_{j=0}^{p-1} \text{sign}(\sigma(p-2j) - \sigma(p-2j-1))$ means that we do not take to account possible inversions in pairs $(p-2j, p-2j-1)$. Now the statement must be clear from Figure 2. \hfill \Box

2.3. A transformation of the expression for $\Lambda_p$. Now we can write formulas (1.13), (1.14) in the form

$$\Lambda_p(e^{i\phi_1}, \ldots, e^{i\phi_p}) = (-i)^{[p/2]} \sum_{\zeta \in \text{Match}_p} (-1)^\zeta \prod_{\{\alpha, \beta\} \in \zeta} \text{cot}(\varphi_{2\alpha}/2) \left(\delta(\varphi_\alpha) + \delta(\varphi_\beta)\right) \times \prod_{\{\gamma\} \in \zeta} \delta(\varphi_{2\gamma}). \quad (2.6)$$

The product $\prod_{\{\gamma\} \in \zeta}$ is taken over the set consisting of 0 or 1 elements, i.e., for even $p$ the product equals 1 and for odd $p$ it consists of 1 factor.

2.4. Proof of Theorem 2.1.

Lemma 2.4 Let $a_1 > \cdots > a_p$, and $\delta^a_p$ be given by (3.9).

a) $p = 2m$ be even. Then

$$i^m \left\langle \delta^a_p, \prod_{l=1}^{m} \left(\cot \frac{\varphi_{2l-1}}{2} \cdot \delta(\varphi_{2l-1} + \varphi_{2l})\right) \right\rangle_{\mathbb{T}^{2m}} = (2\pi)^{2m} 2^m m!. \quad (2.7)$$

b) $p = 2m + 1$ be odd. Then

$$i^m \left\langle \delta^a_p, \delta(\varphi_{2m+1}) \prod_{l=1}^{m} \left(\cot \frac{\varphi_{2l-1}}{2} \cdot \delta(\varphi_{2l-1} + \varphi_{2l})\right) \right\rangle_{\mathbb{T}^{2m}} = (2\pi)^{2m+1} 2^m m!. \quad (2.8)$$

**Proof.** We prove a), a proof of b) is same. Recall that

$$i \cot \varphi/2 = \sum_{n>0} (e^{-in\varphi} - e^{in\varphi}).$$

Therefore

$$i^m \left\langle (-1)^\sigma e^{ia_1 \varphi_1 + i(a_3-a_2)\varphi_2 + \cdots} \prod_{l=1}^{m} \left(\cot \frac{\varphi_{2l-1}}{2} \cdot \delta(\varphi_{2l-1} + \varphi_{2l})\right) \right\rangle = \quad (2.9)$$

$$= i^m (2\pi)^m (-1)^\sigma \left\langle e^{i(a_1-a_2)\varphi_1 + i(a_3-a_4)\varphi_2 + \cdots} \prod_{l=1}^{m} \cot \frac{\varphi_{2l-1}}{2} \right\rangle =$$

$$= (2\pi)^m (-1)^\sigma \prod_{l=1}^{m} \text{sign}(a_{\sigma(2l-1)} - a_{\sigma(2l)}) = (-1)^\sigma \zeta_0 (2\pi)^{2m}. \quad (2.10)$$

Thus (2.10) is

$$(2\pi)^{2m} \sum_{\sigma \in S_p} (-1)^\sigma \zeta_0 = 2^m m! (2\pi)^{2m} \sum_{\zeta \in \text{Match}_p} (-1)^\zeta.$$
The latter sum was evaluated in Lemma 2.3. □

To be definite, assume that \( p \) is even. Since \( \delta_p \) is skew-symmetric, replacing

\[
I(\varphi) := \prod \cot \frac{\varphi_{2l-1}}{2} \cdot \delta(\varphi_{2l-1} + \varphi_{2l})
\]

in (2.7) by its skew-symmetrization

\[
J := \sum_{\sigma \in S_p} (-1)^{\sigma} I(\varphi_{\sigma(1)}, \ldots, \varphi_{\sigma(p)})
\]

we get the same right-hand side multiplied by \( p! \). Next, we replace \( \delta_p \) by \( \exp(\sum i a_k \varphi_k) \), after this the right hand side is divided by \( p! \). Thus we get Fourier coefficients of the complex conjugate distribution \( J \). This is Theorem 2.1.

3 The Plancherel formula

This section contains preliminaries from [22].

3.1. Choose of a sign in the average operator. For even \( n \) the right hand side of formula (1.9) is defined up to a sign. Let us fix it. Denote

\[
\nu_k(a^{(k)}) = \prod_{1 \leq p < q \leq n-2k} \sin \left( \frac{\varphi_p - \varphi_q}{2} \right).
\]

We have

\[
\nu_k(a^{(k)}) (\det a^{(k)})^{-(n-1)/2} = \prod_{1 \leq p < q \leq n-2k} (2i)^{-1} e^{-i(\varphi_p - \varphi_q)/2} \left( e^{i(\varphi_p - \varphi_q)} - 1 \right) \times \prod_{p=1}^{n-2k} e^{i\varphi_p(n-1)/2} \prod_{r=1}^{k} e^{i\theta_r(n-1)}.
\]

Collecting factors \( e^{i\varphi_m} \) we get a continuous expression

\[
e^{-i(\varphi_2 + 2\varphi_3 + \cdots + (n-2k-1)\varphi_{n-2k})} \prod e^{i\theta_r(n-1)} \prod (e^{i(\varphi_p - \varphi_q)} - 1),
\]

which is well-defined up to global change of a sign. We fix its sign assuming that \( \nu_k(a^{(k)}) \) is positive if the order \( e^{i\varphi_1}, \ldots, e^{i\varphi_{n-2k}} \) are located clock-wise. It remains to set

\[
\varepsilon_k(a^{(k)}) (\det a^{(k)})^{-(n-1)/2} := \frac{\nu_k(a^{(k)}) (\det a^{(k)})^{-(n-1)/2}}{|\nu_k(a^{(k)})|}.
\]

3.2. Conditions of glueing. Consider a Cartan subspace \( A_k \) (see (1.4) with the standard coordinates

\[
\varphi_1, \ldots, \varphi_{n-2k-2}, \varphi_{n-2k-1}, \varphi_{n-2k}, t_k, \theta_k, \ldots, t_1, \theta_1
\]

(3.1)
and the Cartan subspace \( A_{k+1} \) with coordinates
\[
\varphi_1, \ldots, \varphi_{n-2k-2}, t_{k+1}, \theta_{k+1}, t_k, \theta_k, \ldots, t_1, \theta_1.
\] (3.2)

The intersection \( A_k \cap A_{k+1} \) in \( A_{k+1} \) is given by the equation \( t_{k+1} = 0 \); in \( A_k \) it is defined by the equation \( \varphi_{n-2k-1} = \varphi_{n-2k} \). Let us change coordinates in \( A_k \) and in \( A_{k+1} \) in the following way. In the both cases we leave coordinates
\[
\varphi_1, \ldots, \varphi_{n-2k-3}, t_k, \theta_k, \ldots, t_1, \theta_1.
\] (3.3)

being the same. Also:
- in \( A_{k+1} \), we rename \( t := t_{k+1}, \theta := \theta_{k+1} \);
- in \( A_k \) we set \( \theta := (\varphi_{n-2k-1} + \varphi_{n-2k})/2, \tau := \varphi_{n-2k-1} - \varphi_{n-2k} \).

Take a point \( b \) of the hypersurface \( t = 0 \) such that other \( t_j \neq 0 \). Then for any \( N > 0 \) there are smooth functions \( u_j(\cdot) \) depending on coordinates (3.3) and \( \theta \) such that in a small neighborhood of \( b \) we have asymptotic expansions of the form
\[
\Xi_{k+1} f(t, \ldots) = \sum_{j=0}^{N} u_j t^j + o(t^N), \quad \text{for } t > 0;
\]
\[
\Xi_{k+1} f(t, \ldots) = \sum_{j=0}^{N} (-1)^{j+1} u_j t^j + o(t^N), \quad \text{for } t < 0;
\]
\[
\Xi_k f(\tau, \ldots) = i \sum_{m=0}^{[N/2]} (-1)^m u_{2m} \tau^{2m} + o(\tau^N),
\]
see [22], Theorem 4.4.

**Remark.** For \( t \geq 0 \) denote
\[
\Xi_{k+1} f(t, \ldots) = (\Xi_{k+1} f(t, \ldots) + \Xi_{k+1} f(-t, \ldots))/2
\]

Then the function
\[
\Xi f(s) := \begin{cases} 
\Xi_{k+1} f(\sqrt{s}, \ldots), & \text{for } s \geq 0; \\
i\Xi_k f(\sqrt{-s}, \ldots), & \text{for } s \leq 0
\end{cases}
\]
is a smooth function near \( s = 0 \). Appearance of the factor \( i \) is artificial, it is related to the normalization of the factor \( \Delta(a^{(k)}) \) in (1.9). See also [24] for elementary explanations, our asymptotics can be reduced to the case \( p = 1, q = 2 \) of that paper.

**3.3. Spherical distributions**, see [22], Sect. 7. Points of the spectrum of \( L^2(G/H) \) are enumerated by an integer \( r = 0, \ldots, [n/2] \) and signatures of type \( r \):
\[
(c, l) = (c_1, \ldots, c_{n-2r}, l_1, \ld_1, \ldots, l_r, \ld_r).
\] (3.4)
Here $c_1 > c_2 > \cdots > c_{n-2r}$ are integers, and

$$l_p = (m_p - i\lambda_p)/2,$$

where $m_p \in \mathbb{Z}$, $\lambda_p \in \mathbb{R}$, and $\lambda_1 > \lambda_2 > \cdots > \lambda_r > 0$. To write a distribution corresponding to a given signature $(c, l)$, we need some notation.

Let $\varphi_1, \varphi_2$ be defined modulo $2\pi$. Then $(\varphi_1 - \varphi_2)/2$ is defined modulo $\pi$. We will use two ways to define $(\varphi_1 - \varphi_2)/2$ modulo $2\pi$ setting

$$\left\lfloor \frac{\varphi_1 - \varphi_2}{2} \right\rfloor \in (-\pi/2, \pi/2), \quad \left\lceil \frac{\varphi_1 - \varphi_2}{2} \right\rceil \in (0, \pi).$$

Next, for $l = (m - i\lambda)/2$ we define a function

$$\xi(l; e^{i\varphi_1}, e^{i\varphi_2}) := e^{im\theta} (e^{i\lambda t} - e^{-i\lambda t}), \quad \text{where} \quad e^z = e^{t+\imath\theta},$$

and a function $D(l; e^{i\varphi_1}, e^{i\varphi_2})$. If $m \in 2\mathbb{Z}$, we set

$$D(l; e^{i\varphi_1}, e^{i\varphi_2}) := 2e^{im(\varphi_1 + \varphi_2)/2} \frac{\cosh \lambda [(\varphi_1 - \varphi_2)/2] - \pi/2)}{\sinh \pi\lambda/2}.$$ \hspace{1cm} (3.7)

If $m \in 2\mathbb{Z} + 1$, then (formula (8.6) in [22] contains a typo).

$$D(l; e^{i\varphi_1}, e^{i\varphi_2}) := -2e^{i(m-1)(\varphi_1 + \varphi_2)/2} \times \frac{\sinh \lambda [(\varphi_1 - \varphi_2)/2] - \pi/2)}{\cosh \pi\lambda/2}.$$ \hspace{1cm} (3.8)

Functions $D_l$ are continuous on the torus $\mathbb{T}^2$, smooth outside the diagonal $\varphi_1 = \varphi_2$, and have a kink on the diagonal.

We write a spherical distribution $\Phi^r_{c,l}$ corresponding to a signature (3.4) in the form

$$\langle f, \Phi^r_{c,l} \rangle = \sum_k \gamma_k \int_{A_k} \Xi_k f(a^{(k)}) \varphi_k(c, l; a^{(k)}) \, da^{(k)},$$ \hspace{1cm} (3.9)

where $\gamma_k$ are given by (1.6) and $\varphi_k(c, l; a^{(k)})$ will be defined now.

Let $k \leq r$. We consider diagrams $\mathcal{G}$ of the form shown on Fig. 3. The upper row consists of $n-2r$ white circles corresponding to the parameters $c_1,
of a given diagram \( S \) that we can do not take to account arcs of Type 3. The number is also a parity of the number of intersections of the vertical arcs, notice formula:

\[
\kappa_k(c, l; a^{(k)}) := 0 \quad \text{for } k > r;
\]

\[
\kappa_k(c, l; a^{(k)}) = \sum_{\mathcal{S} \in \Omega(r, k)} \prod_{[e^i\varphi_\alpha] \in \mathcal{S}} e^{i\varphi_\alpha} \prod_{[l, z]} \xi(l; e^{i\varphi}) \times \prod_{[l, z]} D(l; e^{i\varphi}, e^{i\varphi}) \quad \text{for } k \leq r,
\]

the summation is taken over all diagrams \( \mathcal{S} \), the product is taken over all pieces of a given diagram \( \mathcal{S} \) (see [22], equation (7.6), and the expression \( \kappa \) at the last row of \( \S 7 \)).

**Remark.** The expressions \( \kappa_k(c, l; a^{(k)}) \) are invariant with respect to the subgroups \( S_{n-2k}, S_k \subset W_k \) acting on \( A_k \) and and changes a sign under reflections \( \{l, . . . , T, . . . , l, \} \). Also, they are invariant with respect to permutations of \( c_1, . . . , c_{n-2r} \) and with respect to permutations of \( l_1, . . . , l_r \). They change a sign under the reflections

\[
(\ldots, l_{j-1}, l_j, l_{j+1}, . . .) \mapsto (\ldots, l_{j-1}, T_j, l_{j+1}, . . .).
\]

**3.4. Functions** \( \kappa(c, l; a^{(k)}) \) **as eigenfunctions of symmetric differential operators.** Notice, that the functions \( \xi, D \) can be written as

\[
\xi(l; z) = e^{zl} - e^{-zl};
\]

\[
D(l; e^{i\varphi_1}, e^{i\varphi_2}) = \frac{2}{e^{2\pi i l} - 1} e^{i\varphi_1 l + i\varphi_2 l} - \frac{2}{e^{2\pi i l} - 1} e^{i\varphi_1 l + i\varphi_2 l},
\]
in the second expression we choose $\varphi_1/2 > \varphi_2/2$.

Therefore locally the functions $\varphi_k(\varphi, z)$ are linear combinations of exponents of linear functions.

Consider a symmetric polynomial $S(x_1, \ldots, x_n)$. Substituting the first order differential operators (1.11) to $S(\cdot)$ we come to the identity

$$S(X_1, \ldots, X_n) \varphi_k(c, l; a^{(k)}) = S(\lambda_1, \ldots, \lambda_n) \varphi_k(c, l; a^{(k)}),$$

which holds outside hypersurfaces $\varphi_p = \varphi_q$. Moreover, this identity is valid in a distributional sense. Precisely, for any $f \in C_\infty(G)$

$$\sum_{k=0}^{[n/2]} \gamma_k \int_{A_k} S(-X)\Xi_k f(a^{(k)}) \cdot \varphi_k(a^{(k)}) \, da^{(k)} =$$

$$= \sum_{k=0}^{[n/2]} \gamma_k \int_{A_k} \Xi_k f(a^{(k)}) \cdot S(X)\varphi_k(a^{(k)}) \, da^{(k)}.$$  

Boundary terms, which appear after integration by parts, cancel due the gluing conditions, see [22], Lemma 6.2. Sano also establishes a more general integration by parts identity (it will be used below). Let $S(x_1, \ldots, x_n)$, $T(x_1, \ldots, x_n)$ be homogeneous polynomials, which are both symmetric or both skew-symmetric in $x_1, \ldots, x_n$. Consider differential operators $S(X_1, \ldots, X_n)$, $T(X_1, \ldots, X_n)$, where $X_j$ are given by (1.11). Then

$$\sum_{k=0}^{[n/2]} \gamma_k \int_{A_k} \Xi_k f(a^{(k)}) \cdot S(X)T(X)\varphi_k(a^{(k)}) \, da^{(k)} =$$

$$= \sum_{k=0}^{[n/2]} \gamma_k \int_{A_k} S(-X)\Xi_k f(a^{(k)}) \cdot T(X)\varphi_k(a^{(k)}) \, da^{(k)}. \quad (3.15)$$

3.5. The Plancherel Theorem. Consider the distributions $\Phi_{c,l}^r$ on $G/H$ given by (3.9). Denote (see formula (1.3))

$$\Delta(c, l) := \Delta(c_1, \ldots, c_{n-2r}, l_1, \ldots, l_r, \tilde{l}_r).$$

Then the following Plancherel formula holds. For any $f \in C_\infty(G/H)$,

$$\gamma_x f(e) = \frac{1}{(2\pi)^n} \sum_{r=0}^{[n/2]} \frac{(n-2r)!}{([n/2]-r)!} \times$$

$$\times \sum_{c_1 > \cdots > c_{n-2r}} \sum_{m_1, \ldots, m_r} \int_{\lambda_1 > \cdots > \lambda_r > 0} \langle f, \Phi_{c,l}^r \rangle \Delta(c, l) \, d\lambda_1 \cdots d\lambda_r. \quad (3.16)$$

where $\Phi_{c,l}^r$ are spherical distributions given be (3.9) and $\gamma_x$ is the same constant as in (1.12). See [22], Theorem 8.7. The formula assumes that summands of the exterior sum $\sum_{r}$ are absolutely convergent (as integrals over measures on spaces $\mathbb{Z}^{n-2r} \times \mathbb{R}^r \times \mathbb{R}_+^r$).
Thus, we must evaluate summands of the exterior sum in (3.16), i.e. we must find a distribution given by
\[
\gamma^\ast (2\pi)^n (f, \Theta_r) = \frac{(n - 2r)!}{([n/2] - r)!} \sum_{c_1 > \cdots > c_{n-2r}} \sum_{m_1, \ldots, m_r} \times \\
\times \int_{\lambda_1 > \cdots > \lambda_r > 0} \left( \sum_{k=0}^{r} \gamma_k \int_{A_k} \Xi_k f(a^{(k)}) : \varkappa_k(c, l; a^{(k)}) \, da^{(k)} \right) \Delta(c, l) \, d\lambda_1 \cdots d\lambda_r.
\]

(4.1)

4.1. Integration by parts. Let \( l \) be as above, \( l = (m - i\lambda/2). \) Define functions
\[
\xi'(l; e^\pm) := -e^{im\theta}(e^{i\lambda\theta} + e^{-i\lambda\theta}).
\]
If \( m \in 2\mathbb{Z}, \) we set
\[
D'(l; e^{i\varphi_1}, e^{i\varphi_2}) := 2e^{i(m+1)(\varphi_1+\varphi_2)/2} \times \\
\times \frac{\sinh \lambda(\lfloor (\varphi_1 - \varphi_2)/2 \rfloor - \pi/2)}{\sinh \pi \lambda/2} \cdot \text{sign}(\lfloor (\varphi_1 - \varphi_2)/2 \rfloor).
\]
If \( m \in 2\mathbb{Z} + 1, \) then
\[
D'(l; e^{i\varphi_1}, e^{i\varphi_2}) := -2e^{i(m-1)(\varphi_1+\varphi_2)/2} \times \\
\times e^{i\varphi_2+i((\varphi_1-\varphi_2)/2)} \frac{\cosh \lambda(\lfloor (\varphi_1 - \varphi_2)/2 \rfloor - \pi/2)}{\cosh \pi/2}.
\]
(4.2)

A function \( D(l; e^{i\varphi_1}, e^{i\varphi_2}) \) has a jump on the diagonal \( e^{i\varphi_1} = e^{i\varphi_2} \) and \( C^\infty \)-smooth outside the diagonal.

Following [22], consider functions \( \varkappa'(c, l; a^{(k)}) \) on the union of \( A_k \) given by the formula
\[
\varkappa'_k(c, l; a^{(k)}) := 0 \quad \text{for } k > r.
\]
(4.3)

\[
\varkappa'_k(c, l; a^{(k)}) = \\
= \sum_{\mathcal{S} \in \Omega(r, k)} (-1)^{\mathcal{S}} \prod_{[c_p, \varphi_{\alpha}] \in \mathcal{S}} e^{ic_p \varphi_{\alpha}} \prod_{[l_q, \varphi_{\beta}] \in \mathcal{S}} \xi'(l_q; e^{\varphi_{\beta}}) \times \\
\times \prod_{[l_q, \varphi_{\beta}] : [l_q, \varphi_{\beta}] \in \mathcal{S}} D'(l_q; e^{i\varphi_{\alpha}}, e^{i\varphi_{\beta}}) \quad \text{for } k \leq r.
\]
(4.4)

Remark. These functions are skew-symmetric with respect to the variables \( \varphi_m \) and invariant with respect to the subgroup \( S_k \rtimes \mathbb{Z}_k^2 \subset W_k. \) They are skew-symmetric with respect to the parameters \( c_m \) and invariant with respect to the permutations of \( l_{\alpha} \) and the reflections [3.12] \( \Box \)
We have
\[ \Delta_k(\partial) \varphi_k(c, l; a^{(k)}) = \Delta(c, l) \varphi'_k(c, l; a^{(k)}); \quad (4.5) \]
\[ \Delta_k(\partial) \varphi'_k(c, l; a^{(k)}) = \Delta(c, l) \varphi_k(c, l; a^{(k)}). \quad (4.6) \]

These identities hold pointwise outside diagonals \( e^{i\varphi} = e^{i\varphi'}, \) this easily follows from (3.13). Moreover, we have an identity:
\[ \sum_{k \leq r} \gamma_k \int_{A_k} \Xi_k f(a^{(k)}) \cdot \Delta(c, l) \varphi_k(c, l; a^{(k)}) \, da^{(k)} = \]
\[ = (-1)^{n(n-1)/2} \sum_{k \leq r} \gamma_k \int_{A_k} \Delta_k(\partial) \varphi'_k(c, l; a^{(k)}) \, da^{(k)} \]

To obtain this, we apply (3.15) with \( S, T = \Delta. \)

Thus,
\[ \gamma_\ast \cdot (2\pi)^n \langle f, \Pi_r \rangle = \frac{(n-2r)! [n/2]-r(-1)^r(-1)^{n(n-1)/2}}{[n/2]-r!} \sum_{k} \gamma_k U_{r,k}, \]

where
\[ U_{r,k} = \sum_{c_1 > \cdots > c_{n-2r}, m_1, \ldots, m_r} \int_{\lambda_1 > \cdots > \lambda_r > 0} \int_{A_k} h_k(a^{(k)}) \cdot \varphi'_k(c, l; a^{(k)}) \, da^{(k)} \, d\lambda_1 \ldots d\lambda_r \quad (4.7) \]

where \( h_k(a^{(k)}) = \Delta_k(\partial) \Xi_k f(a^{(k)}). \)

4.2. Summation of distributions.

**Lemma 4.1** Let \( h_k(a^{(k)}) \) be skew-symmetric with respect to \( S_{n-2k} \subset W_k, \) symmetric with respect to \( S_k \times Z^k_2 \) and smooth in the domain \( t_1 \geq 0, \ldots, t_k \geq 0. \) Then the sum given by (4.7) equals to
\[ \frac{(2\pi)^n(n-2r)!(-2)^{-k}(-i)^{[n/2]-r}}{(n-2r)!} \frac{\langle h_k, A_{n-2k}(\varphi) \prod_{m=1}^{k} \delta(\theta_m) \delta(t_m) \rangle_{A_k}}{A_k}. \]

**Notation for a proof.** For a diagram \( \mathcal{S} \in \Omega(r, k) \) consider a diagram \( \tilde{\mathcal{S}} \) obtained by removing vertical arcs of Type 1, i.e., arcs \([c_p, \varphi_a], \) see Figure 4a. Next, we define the diagram \( \mathcal{S}^\circ \in \Omega(r, k) \) obtained from \( \tilde{\mathcal{S}} \) by adding a collection of nonintersect arcs from white circles to free black boxes. Denote by \( \Omega^\circ(r, k) \) the set of diagrams of this form, \( \Omega^\circ(r, k) \subset \Omega(r, k). \)

For \( \mathcal{S} \in \Omega^\circ(r, k) \) we denote by \( u_1 < \cdots < u_{n-2k}, \) which are connected with white boxes.
a) The diagram $\tilde{\mathcal{S}}$ for $\mathcal{S}$ given be Fig.3.

b) The diagram $\mathfrak{T} = \mathcal{S}^o \in \Omega^o(r, k)$

c) The diagram $\mathfrak{Q} = \mathfrak{T}^\square \in \Omega^\square(r, k)$.

Figure 4: .

Also for $\mathfrak{T} \in \Omega^o(r, k)$ we define two parites, $\varepsilon_1(\mathfrak{T})$, $\varepsilon_2(\mathfrak{T})$: let $\varepsilon_1(\mathfrak{T})$ be the parity of number of intersections between arcs of Type 1 and arcs of Type 2. Respectively, $\varepsilon_2(\mathfrak{T})$ is the parity of the number of intersections of arcs of Type 2,

$(-1)^\mathfrak{T} = (-1)^{\varepsilon_1(\mathfrak{T})}(-1)^{\varepsilon_2(\mathfrak{T})}$.

Finally, for $\mathfrak{T} \in \Omega^o(r, k)$ we consider the diagram $\mathfrak{T}^\square$ obtained by the following operation: we forget black circles and black boxes, we forget arcs between black circles and black boxes; we transform any pairs of arcs $[l_q, \varphi_\alpha]$, $[\bar{l}_q, \varphi_\beta]$ to an arc between $\varphi_\alpha$, $\varphi_\beta$. We denote the set of such diagrams by $\Omega^\square(r, k)$.

In the same way, for $Q \in \Omega(r, k)$ we define two numbers $(-1)^{\varepsilon_1(\Omega)}$, $(-1)^{\varepsilon_2(\Omega)}$ and the collection $u_1 < \cdots < u_{n-2k}$.

PROOF. By the symmetry of the expressions $\kappa'_k$ in the parameters with
with respect to \( S_r \ltimes \mathbb{Z}_2^r \) we can represent this as

\[
U_{r,k} = \frac{1}{r! 2^r} \sum_{c_1 > \cdots > c_{a_2 - 2r}} \sum_{m_1, \ldots, m_r} \int \int_{A_k} h_k(a^{(k)}) x_k(c, l; a^{(k)}) \, da_k \, d\lambda_1 \ldots d\lambda_r =
\]

\[
\frac{1}{r! 2^r} \sum_{S \in \Omega(r,k)} (-1)^S \sum_{c_1 > \cdots > c_{a_2 - 2r}} \sum_{m_1, \ldots, m_r} \int \int_{A_k} h_k(a^{(k)}) \times
\]

\[
\prod_{|c_p, \varphi_a| \in \mathcal{S}} e^{i c_p \varphi_a} \prod_{|l_q, z_z| \in \mathcal{S}} \xi'(l_q; e^{z_i}) \times
\]

\[
\prod_{|l_q; \varphi, \varphi_a, l_q; \varphi, \varphi_b| \in \mathcal{S}} D'(l_q; e^{i \varphi_a}, e^{i \varphi_b}) \, da_k \, d\lambda_1 \ldots d\lambda_r. \quad (4.8)
\]

We will use the following identities for distributions. Obviously,

\[
\sum_m \int_{\lambda > 0} \xi'(l; z) \, d\lambda = -(2\pi)^2 \delta(t) \delta(\theta). \quad (4.9)
\]

By [22], Lemma 8.4, we have

\[
\sum_m \int_{\lambda > 0} D'(l; e^{i \varphi_1}, e^{i \varphi_2}) \, d\lambda = 4i \sum_{a_1 > a_2} \left( e^{i a_1 \varphi_1 + i a_2 \varphi_2} - e^{i a_2 \varphi_1 + i a_1 \varphi_2} \right) =
\]

\[
4i L_2(e^{i \varphi_1}, e^{i \varphi_2}) = (-8\pi) \cot(\varphi_1/2) \delta(\varphi_1 + \varphi_2). \quad (4.10)
\]

According [22], we can change order of summation and integration (4.8) in an arbitrary way. We successively integrate with respect to the following groups of variables:

- for each \([l_q, z_z] \in \mathcal{S}\) we integrate

\[
\int d\theta_\delta \int dt_\delta \sum_{m_q} \int d\lambda_q(\ldots)
\]

applying (4.9);

- for each pair \([l_q; \varphi, \varphi_a], [l_q; \varphi, \varphi_b] \in \mathcal{S}\) we integrate

\[
\int d\varphi \alpha \int d\varphi \beta \sum_{m_q} \int d\lambda_q(\ldots)
\]

applying (4.9).
We come to

\[ U_{r,k} = \frac{(2\pi)^{n-2r}(8\pi^2)^k(-8\pi)^{r-k}}{r! 2^r} \sum_{T \in \Omega^\square(r,k)} \times \]
\[ \times \left\langle h_k, (-1)^{\varepsilon_1(T)} \Lambda_{n-2r}(e^{i\varphi_{n_1}}, \ldots, e^{i\varphi_{n-2r}}) \times \right. \]
\[ \left. \times (-1)^{\varepsilon_2(T)} \prod_{\{l_1,z_1\} \in \mathcal{T}} \delta(t_j) \delta(\theta_j) \prod_{\{l_1,\varphi\} \in \mathcal{T}} \cot(\varphi_\alpha/2) \delta(\varphi_\alpha + \varphi_\beta) \right\rangle. \]

A summand corresponding to \( \mathcal{T} \square \) actually depends on \( \mathcal{T} \square \), and there are \( r! \) different \( \mathcal{T} \square \) for a given \( \mathcal{T} \square \). We also apply the expression (2.6) for \( \Lambda \) and get

\[ U_{r,k} = \frac{(2\pi)^{n-2r}(8\pi^2)^k(-8\pi)^{r-k}r!(-i)^{[n-2r]/2}}{r! 2^r} \times \left\langle h_k, \prod_{j=1}^k \delta(t_j) \delta(\theta_j) \times \right. \]
\[ \left. \sum_{\Omega \in \Omega^\square(r,k)} \sum_{\zeta \in \text{Match}(\{u_1, \ldots, u_{n-2r}\})} (-1)^\zeta \prod_{\{\alpha, \beta\} \in \zeta} \cot(\varphi_\alpha/2) \delta(\varphi_\alpha + \varphi_\beta) \right. \]
\[ \left. \times (-1)^{\varepsilon_2(\Omega)} \prod_{\{l_1,\varphi\} \in \Omega} \delta(\varphi_\gamma) \cdot (-1)^{\varepsilon_1(\Omega)} \prod_{\{l_2,\varphi\} \in \Omega} \cot(\varphi_\alpha/2) \delta(\varphi_\alpha + \varphi_\beta) \right\rangle. \]

The sum in the big curly brackets is \( \Lambda_{n-2k}(e^{i\varphi_{n_1}}, \ldots, e^{i\varphi_{n-2k}}) \). This completes the calculation. \( \square \)

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