A BERNSTEIN RESULT AND COUNTEREXAMPLE FOR ENTIRE SOLUTIONS TO DONALDSON’S EQUATION

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Abstract. We show that convex entire solutions to Donaldson’s equation are quadratic, using a result of Weiyong He. We also exhibit entire solutions to the Donaldson equation that are not of the form discussed by He. In the process we discover some non-trivial entire solutions to complex Monge-Ampère equations.

1. Introduction

In this note we show the following.

Theorem 1. Suppose that $u$ is a convex solution to the Donaldson equation on $\mathbb{R} \times \mathbb{R}^{n-1} = (t, x_2, ..., x_n)$

$$\bar{\sigma}_2(D^2u) = u_{11}(u_{22} + u_{33} + ... + u_{nn}) - u_{12}^2 - ... - u_{1n}^2 = 1.$$ (1)

Then $u$ is a quadratic function.

Donaldson introduced the operator

$$Q(D^2u) = u_{tt}\Delta u - |\nabla u_t|^2$$

arising in the study of the geometry of the space of volume forms on compact Riemannian manifolds [1]. On Euclidean space, (1) becomes an interesting non-symmetric fully nonlinear equation. Weiyong He has studied aspects of entire solutions on Euclidean space, and was able to show that [2, Theorem 2.1] if $u_{11} = \text{const}$, then the solution can be written in terms of solutions to Laplace equations.

Here we show that any convex solution must also satisfy $u_{11} = \text{const}$. It follows quickly that the solution must be quadratic. We also show that, in the absence of the convexity constraint, solutions exists for which $u_{11} = \text{const}$ fails.

Theorem 2. There exists solutions to the Donaldson equation which are not of the form given by He.

In real dimension 3 we note that solutions of (1) can be extended to solutions of the complex Monge-Ampère equation on $\mathbb{C}^2$

$$\det (\partial \bar{\partial} u) = 1$$ (2)

and we can conclude the following.

Corollary 3. There exist a nonflat solution of the complex Monge-Ampère equation (2) on $\mathbb{C}^2$ whose potential depends on only three real variables.

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2. Proof of Theorem 1

Lemma 4. Suppose that \( K_h \) is the sublevel set \( u \leq h \) of a nonnegative solution to
\[
\tilde{\sigma}_2(D^2u) = u_{11}(u_{22} + u_{33} + \ldots + u_{nn}) - u_{12}^2 - \ldots - u_{1n}^2 = 1.
\]
Then for all ellipsoids \( E \subset K_h \) such that if \( A : E \to B_1 \) is affine diffeomorphism with
\[
A = Mx + \vec{b},
\]
we have
\[
\tilde{\sigma}_2(M^2) \geq \frac{1}{4h^2}.
\]

Proof. Consider the function \( v \) on \( \mathbb{R}^n \) defined by
\[
v(x) = h|A(x)|^2.
\]
On the boundary of \( E \), we have
\[
v(x) = h \geq u.
\]
We have
\[
Dv = 2hM \left( Mx + \vec{b} \right)
\]
\[
D^2v = 2hM^2.
\]
Thus
\[
\tilde{\sigma}_2(D^2v) = 4h^2\tilde{\sigma}_2(M^2).
\]
Now suppose that
\[
\tilde{\sigma}_2(M^2) < \frac{1}{4h^2}.
\]
Then
\[
\tilde{\sigma}_2(D^2v) < 1,
\]
so \( v \) is a supersolution to the equation, and must lie strictly above the solution \( u \). But \( v \) must vanish at \( A^{-1}(0) \). Because \( u \) is nonnegative, this is a contradiction of the strong maximum principle. \(\square\)

Proposition 5. Suppose that \( u \) is an entire convex solution to
\[
\tilde{\sigma}_2(D^2u) = u_{11}(u_{22} + u_{33} + \ldots + u_{nn}) - u_{12}^2 - \ldots - u_{1n}^2 = 1
\]
Then
\[
\lim_{t \to \infty} u_1(t, 0, \ldots, 0) = \infty.
\]

Proof. Assume not. Instead assume that \( u_1 \leq A \). Assume that \( u(0) = 0 \) and \( Du(0) = 0 \), adjusting \( A \) if necessary. Then
\[
u(t, 0, \ldots, 0) = \int_0^t u_1(s)ds \leq \int_0^t Ads \leq At.
\]
Now consider the convex sublevel set \( u \leq h \). This must contain the point
\[
\left( \frac{h}{A}, 0, \ldots, 0 \right).
\]
The level set $u = h$ intersect the other axes at

$$(0, a_2(h), 0, ..., )$$
$$(0, 0, a_3(h), ..., 0)$$

e tc.

This level set is convex. It must contain the simplex with the above points as vertices, and this simplex must contain an ellipsoid $E$ which has an affine transformation to the unit ball of the following form

$$A = Mx + b$$
$$M = c_n \begin{pmatrix} \frac{A}{h} & \frac{1}{a_2} & \frac{1}{a_3} & \cdots \end{pmatrix}.$$

Thus

$$M^2 = c_n^2 \begin{pmatrix} (\frac{A}{h})^2 & \left(\frac{1}{a_2}\right)^2 & \left(\frac{1}{a_3}\right)^2 & \cdots \end{pmatrix}$$

and

$$\tilde{\sigma}_2(M^2) = c_n^2 \left(\frac{A}{h}\right)^2 \left(\frac{1}{a_2}\right)^2 + ... + \left(\frac{1}{a_n}\right)^2 \geq \frac{1}{4} \frac{1}{h^2}$$

with the latter inequality following from the previous lemma.

Thus

$$\left(\frac{1}{a_2}\right)^2 + \left(\frac{1}{a_3}\right)^2 + ... + \left(\frac{1}{a_n}\right)^2 \geq \frac{1}{c_n^2 A^2}.$$ 

It follows that for some $i$,

$$\frac{1}{a_i^2} \geq \frac{1}{4(n-1)c_n^2 A^2}.$$ 

That is

$$a_i \leq 2\sqrt{n-1}c_n A.$$ 

Now to finish the argument, let

$$R = 2\sqrt{n-1}c_n A.$$ 

On a ball of radius $R$, there is some bound on the function (not a priori but depending on $u$) say $\bar{U}$. That is

$$u(x) \leq \bar{U}$$ on $B_R$.

Now by convexity for any large enough $h$ the level set $u = h$ is non-empty and convex. Choose $h > \bar{U}$. According to the above argument, this level set must intersect some axis at a point less than $R$ from the origin, which is a contradiction. $\square$
Now using this Proposition, we may repeat the argument of He [2, section 3]:

Letting $z = u_1(t, x)$ the map

$$\Phi : \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R} \times \mathbb{R}^{n-1}$$

$$\Phi(t, x) = (z, x)$$

is a diffeomorphism. Thus for $x$ fixed, there exists a unique $t = t(z, x)$ such that $z = u_1(t, x)$. Defining

$$\theta(z, x) = t(z, x)$$

the computations in [2, section 3] yield that $\theta$ is a harmonic function. It follows that $\frac{\partial \theta}{\partial z} = \frac{1}{u_{11}}$ is a positive harmonic function, so must be constant. Now we have

$$u(t, x) = at^2 + tb(x) + g(x)$$

which satisfies [2, section 2]

$$\Delta b = 0$$

$$\Delta g = \frac{1}{2a} \left(1 + |\nabla b|^2\right).$$

Letting $t = 0$ we conclude that $g$ is convex. Letting $t \to \pm \infty$ we conclude that $b$ is convex and concave, so must be linear. It follows that $|\nabla b|$ is constant, and

$$\Delta g - c |x|^2$$

is a semi-convex harmonic function, which must be a quadratic.

3. COUNTEREXAMPLES

We use the method described in [3] and restrict to $n = 3$. Consider

$$u(t, x) = r^2 e^t + h(t)$$

where $r = (x_2^2 + x_3^2)^{1/2}$. At any point we may rotate $\mathbb{R}^2$ so that $x_2 = r$ and get

$$D^2 u = \begin{pmatrix}
 r^2 e^t + h''(t) & 2re^t & 0 \\
 2re^t & 2e^t & 0 \\
 0 & 0 & 2e^t
\end{pmatrix}.$$

We compute

$$\tilde{\sigma}_2 \left(D^2 u\right) = 4e^t \left(r^2 e^t + h''(t)\right) - 4r^2 e^{2t} = 4e^t h''(t).$$

Then

$$u = r^2 e^t + \frac{1}{4} e^{-t}$$

is a solution.

Now defining complex variables

$$z_1 = t + is$$

$$z_2 = x + iy$$

we can consider the function

$$u = (x^2 + y^2) e^t + \frac{1}{4} e^{-t}.$$  

The function [3] satisfies the equation complex Monge-Ampère equation

$$(\partial_{z_1 \bar{z}_1} u) \left(\partial_{z_2 \bar{z}_2} u\right) - (\partial_{z_1 \bar{z}_1} u) \left(\partial_{z_2 \bar{z}_2} u\right) = 1.$$
One can check that the induced Ricci-flat complex metric
\[ g_{i\bar{j}} = \partial_{z_i} \partial_{z_j} u \]
on \mathbb{C}^2 is neither complete nor flat.

References

[1] Simon K. Donaldson. Nahm’s equations and free-boundary problems. In *The many facets of geometry*, pages 71–91. Oxford Univ. Press, Oxford, 2010.
[2] Weiyong He. Entire solutions of Donaldson’s equation. *Pacific J. Math.*, 256(2):359–363, 2012.
[3] Micah Warren. Non-polynomial entire solutions to $\sigma_k$ equations. *arXiv:1214118*.

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