YANGIANS AND BAXTER’S RELATIONS

HUAFENG ZHANG

Abstract. We study a category $\mathcal{O}$ of representations of the Yangian associated to an arbitrary finite-dimensional complex simple Lie algebra. We obtain asymptotic modules as analytic continuation of a family of finite-dimensional modules, the Kirillov–Reshetikhin modules. In the Grothendieck ring we establish the three-term Baxter’s TQ relations for the asymptotic modules.

Introduction

Fix $\mathfrak{g}$ to be a finite-dimensional complex simple Lie algebra, and $\hbar$ a non-zero complex number. The universal enveloping algebra of the current algebra $\mathfrak{g} \otimes \mathbb{C}[t]$, as a co-commutative Hopf algebra, can be deformed to the Yangian $Y_{\hbar}(\mathfrak{g})$. The latter is a Hopf algebra neither commutative nor co-commutative, and it contains the universal enveloping algebra of $\mathfrak{g}$ as a sub-Hopf-algebra.

In this paper we are interested in a category $\mathcal{O}$ of representations of $Y_{\hbar}(\mathfrak{g})$. Its objects are $Y_{\hbar}(\mathfrak{g})$-modules which, viewed as $\mathfrak{g}$-modules, belong to the category of Bernstein–Gelfand–Gelfand (BGG for short) without integrability assumption [26]. Category $\mathcal{O}$ contains all finite-dimensional modules, it is abelian and monoidal, and its Grothendieck ring is commutative thanks to a highest weight classification of irreducible objects, a common phenomenon in Lie Theory.

As a prototype, consider the Lie algebra $\mathfrak{sl}_2$ of traceless two-by-two matrices. It admits the vector representation on $\mathbb{C}^2$ by matrix multiplication, and the infinite-dimensional Verma module $\mathcal{M}_x$ of highest weight $x \in \mathbb{C}$. In the Grothendieck ring of the category BGG of $\mathfrak{sl}_2$-modules one has

$$[\mathbb{C}^2][\mathcal{M}_x] = [\mathcal{M}_{x+1}] + [\mathcal{M}_{x-1}].$$

Indeed, the Verma module $\mathcal{M}_{x+1}$ is embedded in the tensor product $\mathbb{C}^2 \otimes \mathcal{M}_x$ via highest weight vectors, the quotient of which is isomorphic to $\mathcal{M}_{x-1}$.

Our main result of this paper, Theorem 20, describes similar three-term relations for the Yangian $Y_{\hbar}(\mathfrak{g})$. Namely, we find Yangian analogs of:

- the Verma modules $\leadsto$ the asymptotic modules $\mathcal{L}(\frac{y}{k,x})$ in Definition 17;
- the two-dimensional module $\leadsto$ the modules $\mathcal{M}_{k,x}^{(i)}$ above Equation (23).

Here $k, x, y$ are complex numbers with the condition that $k$ is not a half integer, $i$ is an arbitrary Dynkin node of the underlying Lie algebra $\mathfrak{g}$, and the $\mathcal{M}_{k,x}$ in the Yangian situation are tensor products of asymptotic modules. As a byproduct, we obtain three-term identities in the Grothendieck ring of the category BGG of $\mathfrak{g}$-modules, which in the case $\mathfrak{g} = \mathfrak{sl}_2$ reduce to Equation (1).

One of our motivations behind these relations lies in the spectral problem of exactly solvable models. For such a model, it is essential to determine the common spectra of a family $T(z)$, with a complex parameter $z$, of commuting endomorphisms of a vector space. An important idea of R. Baxter [1] is to construct auxiliary endomorphisms $Q(z)$ that mutually commute with the $T(z)$ and obey a functional
relation, the Baxter’s TQ relation:
\begin{equation}
T(z)Q(z) = f(z)Q(z+h) + g(z)Q(z-h)
\end{equation}
for certain scalar functions \(f(z)\) and \(g(z)\). Consider the Heisenberg spin chain as an example. The \(T(z)\) and \(Q(z)\) are polynomials in \(z\), so that the eigenvalues of \(Q(z)\) have the form \(\prod_{i=1}^{n}(z - z_i)\). Under genericity assumption on the roots, by inserting \(z = z_i\) in the TQ relation we obtain the Bethe Ansatz Equations
\[
\prod_{j: j \neq i} \frac{z_i - z_j + h}{z_i - z_j - h} = \frac{g(z_i)}{f(z_i)}, \quad 1 \leq i \leq n.
\]
Eigenvalues of \(T(z)\) are expressed in terms of the roots \(z_i\) based on the TQ relation. This idea was first applied to the eight-vertex model \([1]\) for which \(T(z)\) and \(Q(z)\) are entire functions of \(z\) with theta function-like double periodicity properties.

After the pioneer works of Bazhanov–Lukyanov–Zamolodchikov \([4, 5]\) and the recent work of Frenkel–Hernandez \([14]\), Baxter’s TQ relation receives a representation theory meaning. One starts from a quantum affine algebra \(U_q(\hat{g})\) with a fixed representation \(V\), referred to as quantum space. Hernandez–Jimbo \([23]\) introduced a category \(\mathcal{O}_{\mathcal{H}I}\) of representations of the upper Borel subalgebra whose Grothendieck ring \(K_0(\mathcal{O}_{\mathcal{H}I})\) is commutative. The standard transfer matrix construction based on the universal R-matrix endows \(V\) with an action of \(K_0(\mathcal{O}_{\mathcal{H}I})\).

This already recovers the well-known six-vertex model: the Lie algebra \(g\) is \(\mathfrak{sl}_2\); \(T(z)\) is the action of the isomorphism class of \(C_2^2\), a two-dimensional irreducible module with a non-zero complex parameter; \(Q(z)\) is the action of the isomorphism class of the so-called prefundamental module \(L^+_a\); Baxter’s TQ relation is a consequence of the following identity in \(K_0(\mathcal{O}_{\mathcal{H}I})\),
\[
[C_q^2][L^+_a] = [C_q^2][L^+_{a+2}] + [C_q^{-1}][L^+_{a+1}] \quad \forall a \in \mathbb{C}^\times.
\]
Here \(C_q\) and \(C_q^{-1}\) are one-dimensional weight modules in category \(\mathcal{O}_{\mathcal{H}I}\). Their transfer matrices are scalar products by \(f(z)\) and \(g(z)\) in the TQ relation.

For higher rank Lie algebras, various three-term identities in \(K_0(\mathcal{O}_{\mathcal{H}I})\) have been established \([9, 15, 24]\), the right-hand side of which is a sum of two monomials in the prefundamental modules. This leads to a proof of Bethe Ansatz Equations for quantum integrable system attached to \(U_q(\hat{g})\).

Let us return to Heisenberg spin chain. The quantum group in question is the Yangian \(Y_h(\mathfrak{g}_{\mathfrak{l}_2})\). One does not have Borel subalgebras to define category \(\mathcal{O}_{\mathcal{H}I}\). Indeed, from the viewpoint of Yangian double \([25]\), the Yangian itself is a Borel subalgebra. Still there is the prefundamental module \(L^+_a\) over the degenerate Yangian \([2, 3]\). Baxter’s TQ relation \([2]\) comes from a similar identity
\begin{equation}
[C^2_a][L^+_a] = [\theta_a][L^+_{a+h}] + [\theta'_a][L^+_{a-h}] \quad \forall a \in \mathbb{C},
\end{equation}
where \(C^2_a\) is a two-dimensional irreducible module over \(Y_h(\mathfrak{g}_{\mathfrak{l}_2})\) indexed by \(a \in \mathbb{C}\), and \(\theta_a\) and \(\theta'_a\) are one-dimensional modules over the degenerate Yangian.

Equations \([1]\) and \([3]\) bear strong resemblance. In this paper we argue that the prefundamental modules over degenerate Yangian can be replaced by the asymptotic modules over the ordinary Yangian. One main reason is that the right-hand side of the three-term identity in Theorem \([24]\) is a sum of two monomials in asymptotic modules. This should eventually lead to Bethe Ansatz Equations for quantum integrable system attached to \(Y_h(\mathfrak{g})\), upon identification of the Q-operators with transfer matrices of asymptotic modules. See \([12]\) Appendix A] for the \(\mathfrak{sl}_2\) case.

We give a few comments on the asymptotic modules of Yangians. Let \(I\) be the set of Dynkin nodes of \(\mathfrak{g}\). It is a classical result of V. Drinfeld \([8]\) that finite-dimensional irreducible \(Y_h(\mathfrak{g})\)-modules are parametrized by \(I\)-tuples of rational functions of \(u\).
satisfying a polynomiality property. Let \( i \in I \) and \( x \in \mathbb{C} \). For \( k \) a positive integer, one has the Kirillov–Reshetikhin module \([27]\) corresponding to the \( I \)-tuple

\[
(1, \ldots, 1, \frac{u + kd_i h + x h}{u + x h}, 1, \ldots, 1)
\]

with the non-trivial function at the \( i \)-th position. The asymptotic module \( \mathcal{L}'(\frac{y_i}{x_i}) \) for \( y \in \mathbb{C} \) is an analytic continuation of these modules: as the integer \( k \) goes to infinity, one replaces \( kd_i \) with \( y - x \). This is inspired by earlier limit construction for quantum affine algebras \([23]\) where \( q^{-k} \) specializes to 0 as \( k \) goes to infinity.

We shall also write down the three-term identities for asymptotic modules of quantum affine algebras. The proof is the same as for Yangians and is omitted. Here we use a category of \( U_q(\mathfrak{g}) \)-modules introduced by D. Hernandez \([22]\) and further studied by Mukhin–Young \([30]\); it is a subcategory of \( \mathcal{O}_W \).

Our approach has the advantage that all representations are defined over the full quantum group. It was initiated in a joint work with G. Felder \([12]\) and further developed in \([38]\) on Baxter’s TQ relations for elliptic quantum groups of type A, which again do not have Borel subalgebras. We expect the asymptotic modules as well as their three-term relations to exist for more general elliptic quantum groups \([14, 19, 35]\), quantum toroidal algebras \([10]\) and affine Yangians. These relations should have cluster algebra interpretations as in \([24]\).

Based on the R-matrix realization \([7, 20, 25]\), one should be able to construct the prefundamental modules over degenerate Yangian of general type.

This paper is structured as follows. In Section 1 we define the category \( \mathcal{O} \) of \( Y_h(\mathfrak{g}) \)-modules and study its basic properties: highest-weight classification in terms of rational functions and q-character map of H. Knight. Section 2 deduces some combinatorial properties of Kirillov–Reshetikhin modules from those in the case of quantum affine algebras via the functor of Gautam–Toledano Laredo. In Section 3 we construct asymptotic modules and prove the three-term identities. In the appendix we provide the three-term identities for quantum affine algebras.

1. Preliminaries

We collect basic facts on Yangians and their representations. All vector spaces are over \( \mathbb{C} \). Let \( \mathfrak{h} \) be the Cartan subalgebra of \( \mathfrak{g} \), and \( I := \{1, 2, \ldots, r\} \) the set of Dynkin nodes. The dual space \( \mathfrak{h}^\ast \) admits a basis of simple roots \( (\alpha_i)_{i \in I} \) and a non-degenerate symmetric bilinear form \( (,): \mathfrak{h}^\ast \times \mathfrak{h}^\ast \rightarrow \mathbb{C} \). For \( i, j \in I \) set

\[
c_{ij} := \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \in \mathbb{Z}, \quad d_{ij} := \frac{(\alpha_i, \alpha_j)}{2}, \quad d_i := d_{ii}.
\]

After normalization \( d_i \in \{1, 2, 3\} \) and their greatest common divisor is 1. (If \( \mathfrak{g} \) is simply laced, then \( d_i = 1 \) for all \( i \).) Let \( \mathcal{Q} := \oplus_{i \in I} \mathbb{Z} \alpha_i \subseteq \mathfrak{h}^\ast \) be the root lattice and set \( \mathcal{Q}^- := \oplus_{i \in I} \mathbb{Z}_{\leq 0} \alpha_i \). Define the fundamental weights \( \varpi_i \in \mathfrak{h}^\ast \) for \( i \in I \) by the equations \( (\alpha_i, \varpi_j) = d_i \delta_{ij} \). Define the height function \( h: \mathcal{Q}^- \rightarrow \mathbb{Z} \) to be the abelian group morphism sending each \( \alpha_i \) to 1.

The Yangian \( Y_h(\mathfrak{g}) \) is an algebra generated by elements \( \{x^\pm_{i,n}, \xi_{i,n}\}_{i \in I, n \in \mathbb{Z}_{\geq 0}} \) subject to the following relations for \( i, j \in I \) and \( m, n \in \mathbb{Z}_{\geq 0} \):

\[
(4) \quad [\xi_{i,m}, \xi_{j,n}] = 0, \quad [\xi_{i,0}, x^\pm_{j,n}] = \pm 2d_{ij} x^\pm_{j,n}, \quad [x^+_{i,m}, x^-_{j,n}] = \delta_{ij} \xi_{i,m+n},
\]

\[
(5) \quad [\xi_{i,m+1}, x^-_{j,n}] - [\xi_{i,m}, x^-_{j,n+1}] = \pm d_{ij} h(\xi_{i,m} x^+_{j,n} + x^-_{j,n} \xi_{i,m}),
\]

\[
(6) \quad [x^\pm_{i,m+1}, x^\pm_{j,n}] - [x^\pm_{i,m}, x^\pm_{j,n+1}] = \pm d_{ij} h(x^\pm_{i,m} x^\pm_{j,n} + x^\pm_{j,n} x^\pm_{i,m}),
\]

\[
(7) \quad \text{ad}_{x^\pm_{i,0}}^{1,\pm}(x^\pm_{j,n}) = 0 \quad \text{if} \quad i \neq j.
\]
Here $ad_x(y) := xy - yx$. Define the generating series in $u^{-1}$ for $i \in I$:

$$(8) \quad x_i^\pm(u) := h \sum_{n=0}^{\infty} x_i^{\pm n} u^{-n-1}, \quad \xi_i(u) := 1 + h \sum_{n=0}^{\infty} \xi_{i,n} u^{-n-1}.$$ 

Algebra $Y_h(g)$ is $\mathbb{Q}$-graded (weight grading): the elements $\xi_{i,n}$ are of weight 0, while the $x_i^{\pm n}$ are of weight $\pm \alpha_i$ respectively. For $\beta \in \mathbb{Q}$ let $Y_h(g)_\beta$ denote the subspace of elements in $Y_h(g)$ of weight $\beta$. By Relation (4) an element $x \in Y_h(g)$ is of weight $\beta$ if and only if $[\xi_{i,0}, x] = (\alpha_i, \beta)x$ for all $i \in I$.

Let $Y_h^+(g), Y_h^-(g)$ be the subalgebras of $Y_h(g)$ generated by the $x_{i,n}^+, x_{i,n}^-$ respectively. Let $Y_h^0(g)$ be the commutative subalgebra of $Y_h(g)$ generated by the $\xi_{i,n}$.

We have the triangular decomposition $Y_h^+(g) Y_h^0(g) Y_h^-(g) = Y_h(g)$.

The subalgebra generated by the $x_{i,0}^\pm, \xi_{i,0}$ for $i \in I$ is isomorphic to the enveloping algebra $U(g)$ of $g$. Relations (4) and (7) for $m = n = 0$ identified with the Serre presentation. $Y_h(g)$ admits a co-multiplication $\Delta : Y_h(g) \rightarrow Y_h(g)^{\otimes 2}$ extending that on the enveloping algebra: $\Delta(x) = 1 \otimes x + x \otimes 1$ for $x \in \{x_{i,0}^\pm, \xi_{i,0}\}$.

Lemma 1. [20] For $i \in I$ we have (summations run over $\beta \in \mathbb{Q}$):

$$\Delta(x_i^+(u)) - x_i^+(u) \otimes 1 - \xi_i(u) \otimes x_i^+(u) \in \sum_{h(\beta) \geq 2} Y_h(g)_{\alpha_i - \beta} \otimes Y_h(g)_\beta [[u^{-1}]],$$

$$\Delta(x_i^-(u)) - 1 \otimes x_i^-(u) - x_i^-(u) \otimes \xi_i(u) \in \sum_{h(\beta) \geq 2} Y_h(g)_{-\beta} \otimes Y_h(g)_{\beta - \alpha_i} [[u^{-1}]],$$

$$\Delta(\xi_i(u)) - \xi_i(u) \otimes \xi_i(u) \in \sum_{h(\beta) \geq 1} Y_h(g)_{-\beta} \otimes Y_h(g)_\beta [[u^{-1}]].$$

Remark 2. Let $V$ and $W$ be $Y_h(g)$-modules. Suppose $\omega \in W$ is annihilated by all the $x_i^+(u)$. By triangular decomposition, $Y_h(g)\omega = Y_h^+(g) Y_h^0(g) \omega$, so $Y_h(g)_\beta \omega = 0$ whenever $h(\beta) > 0$. In the tensor product module $V \otimes W$, the subspace $V \otimes \omega$ is annihilated by the extra summations of Lemma 4.

Let $V$ be a $Y_h(g)$-module. For $\beta \in \mathfrak{h}^*$, the following subspace, if non-zero, is called a weight space (and $\beta$ a weight of $V$):

$$V_\beta := \{ v \in V \mid \xi_{i,0} v = (\alpha_i, \beta)v \text{ for } i \in I \}. $$

We have $Y_h(g)_\alpha V_\beta \subseteq V_{\alpha + \beta}$ for $\alpha \in \mathbb{Q}$. Let $\text{wt}(V) \subseteq \mathfrak{h}^*$ be the set of weights of $V$.

Definition 3. $\mathcal{O}$ is a full subcategory of the category of $Y_h(g)$-modules. An object of $\mathcal{O}$ is a $Y_h(g)$-module $V$ subject to the following conditions:

1. it is a direct sum of finite-dimensional weight spaces;
2. there exist $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathfrak{h}^*$ such that $\text{wt}(V) \subseteq \bigcup_{j=1}^{n}(\lambda_j + \mathbb{Q} \cdot \lambda_j)$.

Every finite-dimensional $Y_h(g)$-module is in category $\mathcal{O}$, its weight grading guaranteed by the $g$-module structure. Category $\mathcal{O}$ is abelian and equipped with a monoidal structure by the co-multiplication of $Y_h(g)$.

We need a refinement of weights based on spectral decomposition of the action of the commutative subalgebra $Y_h^0(g)$. Introduce the group $\mathcal{L} := (1 + u^{-1} \mathbb{C}[u^{-1}])^I$ of $I$-tuple of Laurent series in $u^{-1}$ with component-wise multiplication. Its elements are written typically as $d, e, f$ etc. For such an element $e$ and for $(i, n) \in I \times \mathbb{Z}_{\geq 0}$, let $e_{i,n}(u) \in \mathbb{C}(u^{-1})$ denote the $i$-th component of $e$, and $e_{i,n} \in \mathbb{C}$ the coefficient of $u^{-n-1}$ in $h^{-1} e_{i,n}(u)$. So $e_{i,n}(u) = 1 + h^{-1} \sum_{n \geq 0} e_{i,n} u^{-n-1}$.

The following map, called weight projection, is an abelian group morphism:

$$\varpi : \mathcal{L} \rightarrow \mathfrak{h}^*, \quad e \mapsto \sum_{i \in I} \frac{e_{i,0}}{d_i} \omega_i.$$
Let $V$ be a $Y_h(\mathfrak{g})$-module. For $e \in \mathcal{L}$, the following subspace, if non-zero, is called an $\ell$-weight space (and $e$ an $\ell$-weight of $V$)

$$V_e := \{ v \in V \mid \forall (i, n) \in I \times \mathbb{Z}_{\geq 0} \exists m \in \mathbb{Z}_{>0} \text{ such that } (\xi_{i,n} - e_{i,n})^{m}v = 0 \}.$$  

Let $\text{wt}(V) \subseteq \mathcal{L}$ denote the set of $\ell$-weights of $V$.

A non-zero vector $v \in V$ is called a highest $\ell$-weight vector if it is a common eigenvector of the $\xi_{i,n}$ and is annihilated by the $x_{i,n}^+$. If furthermore $V = Y_h(\mathfrak{g})v$, then $V$ is called a highest $\ell$-weight module. In this situation, by Remark $\mathcal{E}_\ell$ $v$ belongs to a one-dimensional $\ell$-weight space $V_\ell$ which is also the weight space $V_{e}(d)$ (call $d$ the highest $\ell$-weight of $V$), and we have $\text{wt}(V) = \varpi(d) + Q_\ast$. The tensor product of two highest $\ell$-weight vectors is a highest $\ell$-weight vector.

If $V$ is in category $\mathcal{O}$, then each weight space $V_\beta$ is a direct sum of the $V_e$ where $e \in \text{wt}(V)$ and $\varpi(e) = \beta$. Following H. Knight $\mathcal{E}_\ell$ define its $q$-character by

$$\chi_0(V) := \sum_{e \in \text{wt}(V)} \dim(V_e)e \in \mathcal{E}_\ell.$$  

The target $\mathcal{E}_\ell$ is the set of formal sums $\sum_{e \in \mathcal{L}} n_e e$ of the symbols $e$ with integer coefficients $n_e$ subject to the following conditions $\mathcal{E}$ [3.4].

(E1) for each $\beta \in \mathfrak{h}^*$ the set $\{ e \in \mathcal{L} \mid n_e \neq 0, \varpi(e) = \beta \}$ is finite;

(E2) there exist $\lambda_1, \lambda_2, \cdots, \lambda_m \in \mathfrak{h}^*$ such that $\varpi(e) \in \cup_{j=1}^{m} (\lambda_j + Q_\ast)$ if $n_e \neq 0$.

It is a ring: addition is the usual one of formal sums; multiplication is induced by that of $\mathcal{L}$. (One views $\mathcal{E}_\ell$ as a completion of the group ring $\mathbb{Z}[[\mathcal{L}]]$.)

**Remark 4.** To $x \in \mathbb{C}$ one attaches a Hopf algebra automorphism:

$$(9) \quad \tau_x : Y_h(\mathfrak{g}) \rightarrow Y_h(\mathfrak{g}), \quad x_i^\pm(u) \mapsto x_i^\pm(u + xh), \quad \xi_i(u) \mapsto \xi_i(u + xh)$$
called spectral parameter shift. The pullback of a module $V$ by $\tau_x$ is another module whose q-character is obtained from $\chi_0(V)$ by replacing each of the $\ell$-weights $(e_i(u))_{i \in I}$ of $V$ with $(e_i(u + xh))_{i \in I}$.

Define $\mathcal{R}'$ to be the subset of $\mathcal{L}$ whose elements are highest $\ell$-weights of highest $\ell$-weight modules in category $\mathcal{O}$. For $d \in \mathcal{R}'$, there exists a unique (up to isomorphism) irreducible module in category $\mathcal{O}$ of highest $\ell$-weight $d$, denoted by $L(d)$. Based on the triangular decomposition, the $L(d)$ for $d \in \mathcal{R}'$ form the set of (mutually non-isomorphic) irreducible modules in category $\mathcal{O}$.

**Lemma 5.** Fix $d \in \mathcal{R}'$. Take a highest $\ell$-weight vector $\omega$ from $L(d)$.

(a) We have $d \in \mathcal{R}$. Fix $i \in I$ and let $u^s + c_1 u^{s-1} + \cdots + c_{s-1} u + c_s$ be the denominator of the rational function $d_i(u)$.

(b) The $L(d)$-valued Laurent series $(u^s + c_1 u^{s-1} + \cdots + c_{s-1} u + c_s)x_i^-(u)\omega$ is a polynomial in $u$ of degree $s-1$ with leading term $hx_i^\pm(u)\omega u^{s-1}$.

In the special case $d_i(u) = 1$, we have $s = 0$ and $x_i^-(u)\omega = 0$.

**Proof.** The finite-dimensional weight space $L(d)_{\varpi(d)} - \omega$, being spanned by infinitely many vectors $x_{i,n}^\pm \omega$ for $n \in \mathbb{Z}_{\geq 0}$, one finds $a_1, \cdots, a_m \in \mathbb{C}$ such that

$$x_{i,m}^+ \omega + a_1 x_{i,m-1}^+ \omega + \cdots + a_{m-1} x_{i,1}^+ \omega + a_m x_{i,0}^+ \omega = 0.$$  

Applying the $x_{i,n}^+$ to this identity, from Relation $\mathcal{E}$ we obtain that the Laurent series $(u^m + a_1 u^{m-1} + \cdots + a_{m-1} u + a_m)dx_i(u)$ in $u^{-1}$ is a monic polynomial in $u$ of degree $m$, so $d_i(u)$ is a ratio of two monic polynomials of the same degree. This proves part (a). For part (b), one checks that for $n \in \mathbb{Z}_{\geq 0}$ the vector

$$x_{i,n}^{-1} \omega + c_1 x_{i,n+1}^{-1} \omega + \cdots + c_{s-1} x_{i,n+s-1}^{-1} \omega + c_s x_{i,n}^{-1} \omega \in L(d)_{\varpi(d)} - \omega,$$

is annihilated by all the $x_j^+(u)$ and must be zero because of irreducibility. \qed
Let us view $\mathcal{L}$ as a subgroup of the monoid $\mathbb{C}((u^{-1}))^I$. For $(i, x) \in I \times \mathbb{C}$, the following three invertible elements of $\mathbb{C}((u^{-1}))^I$ will play a crucial role:

$$
\Psi_{i, x} := \left(1, \cdots, 1, u + xh, 1, \cdots, 1\right)
$$

prefundamental weight,

$$
A_{i, x} := \prod_{j \in I} \frac{\Psi_{j, x + d_{j, i}}}{\Psi_{j, x - d_{j, i}}}
$$

generalized simple root,

$$
Y_{i, x} := \frac{\Psi_{i, x + \frac{1}{2}d_{i}}}{\Psi_{i, x - \frac{1}{2}d_{i}}}
$$

fundamental loop weight.

The ratios $\frac{\Psi_{i, x}}{\Psi_{i, y}}$ for $(i, x, y) \in I \times \mathbb{C}^2$ belong to $\mathcal{L}$ and they generate a subgroup, denoted by $\mathcal{R}$. Equivalently, an element $e \in \mathcal{L}$ belongs to $\mathcal{R}$ if and only if each of the $e_i(u)$ is the Taylor expansion around $u = \infty$ of a rational function.

**Proposition 6.** Fix $j \in I$. Let $V$ be in category $\mathcal{O}$ and $e, f \in \text{wt}_\ell(V)$. Consider the projection $\text{pr}_f : V \rightarrow V_f$ with respect to the $\ell$-weight space decomposition. If $\text{pr}_f(x_j^+V_e) \neq 0$ for certain $n \in \mathbb{Z}_{\geq 0}$, then $f = A_{j, x}e$ for a unique $x \in \mathbb{C}$.

**Proof.** We follow the proof of [20 Proposition 3.8]. Choose ordered bases $(v_k)_{1 \leq k \leq s}$ and $(w_l)_{1 \leq l \leq t}$ of $V_e$ and $V_f$ respectively such that for $i \in I$:

- the series $\xi_i(u)v_k$ is $e_i(u)v_k$ plus a sum of the $v_l$ for $1 \leq l < k$;
- the series $\xi_i(u)w_k$ is $f_i(u)w_k$ plus a sum of the $w_l$ for $k < l \leq t$.

Since $\text{pr}_f(x_j^+V_e) \neq 0$, at least one of the $\text{pr}_f(x_j^+(z)v_k)$ is nonzero. One can find $1 \leq K \leq s, 1 \leq L \leq t$ and $0 \neq \lambda(z) \in \mathbb{C}[[z^{-1}]]$ such that:

- the series $\text{pr}_f(x_j^+(z)v_k)$ is $\lambda(z)w_L$ plus a sum of the $w_l$ for $L < l \leq t$;
- the series $\text{pr}_f(x_j^+(z)v_k)$ is zero for $1 \leq k < K$.

Let $Z$ be the subspace of $V_f$ spanned by the $w_l$ for $L < l \leq t$. Then modulo the space $Z[[z^{-1}, u^{-1}]]$ we have $\text{pr}_f(x_j^+(z)\xi_i(u)v_K) \equiv e_i(u)\lambda(z)$.

Let us rewrite the relation (5) in the form of generating series [18 §2.3]:

$$(u - z - d_{ij}h)\xi_i(u)x_j^+(z) = (u - z - d_{ij}h)x_j^+(z)\xi_i(u) - 2d_{ij}hz_j^+(u - d_{ij}h)\xi_i(u).$$

Apply this relation to $v_K$ and then project the resulting identity to $V_f$. Since the projection commutes with $\xi_i(u)$, modulo the space $Z[[z^{-1}, u^{-1}]]$, we have

$$\text{LHS} = (u - z - d_{ij}h)\xi_i(u)\text{pr}_f(x_j^+(z)v_K) \equiv (u - z - d_{ij}h)f_i(u)\lambda(z)w_L,$$

$$\text{RHS} = (u - z + d_{ij}h)\text{pr}_f(x_j^+(z)\xi_i(u)v_K) - 2d_{ij}h\text{pr}_f(x_j^+(u - d_{ij}h)\xi_i(u)v_K)$$

$$\equiv [(u - z + d_{ij}h)\lambda(z) - 2d_{ij}h\lambda(u - d_{ij}h)]e_i(u)w_L,$$

$$(u - z - d_{ij}h)f_i(u)\lambda(z) = (u - z + d_{ij}h)e_i(u)\lambda(z) - 2d_{ij}hе_i(u)\lambda(u - d_{ij}h).$$

Write $\lambda(z) = \lambda_m z^{-m-1} + \lambda_{m+1} z^{-m-2} + \cdots$ with $\lambda_m \neq 0$. We take coefficients of $z^{-m-1}$ in the last equation. The third term does not contribute, and we get

$$\frac{f_i(u)}{e_i(u)} = \frac{u + d_{ij}h - \lambda_{m+1} \lambda_m^{-1}}{u - d_{ij}h - \lambda_{m+1} \lambda_m^{-1}}$$

for $i \in I$.

This implies $f = A_{j, x}e$ with $x = -\lambda_{m+1} \lambda_m^{-1}h^{-1}$. □

Let $\mathcal{P}_+$ and $\mathcal{Q}_-$ be the submonoids of $\mathcal{R}$ generated by the $Y_{i, x}$ and the $A_{i, x}^{-1}$ for $(i, x) \in I \times \mathbb{C}$ respectively. We shall also need the subgroup $\mathcal{P}$ of $\mathcal{R}$ generated by the $Y_{i, x}$. By [5], $\mathcal{P}_+ \subseteq \mathcal{R}$ and the irreducible module $L(d)$ is finite-dimensional if and only if $d \in \mathcal{P}_+$. Elements of $\mathcal{P}_+$ are called dominant $\ell$-weights.

One checks that $\varpi(A_{i, x}) = \alpha_i$ and $\varpi(Y_{i, x}) = \varpi$. So $\varpi(\mathcal{Q}_-) = \mathcal{Q}_-$. Consider an irreducible module $S := L(d)$ in category $\mathcal{O}$. The height of an $\ell$-weight $f$ of $S$, defined as $h(\varpi(f^{-1}d))$, is a non-negative integer since $\varpi(d^{-1}f) \in \mathcal{Q}_-$. 


For \( n \in \mathbb{Z}_{\geq 0} \), let \( \text{wt}^n_\ell(S) \) be the set of \( \ell \)-weights of \( S \) of height \( n \). Then \( \text{wt}_\ell(S) \) is a disjoint union of these \( \text{wt}^n_\ell(S) \). In particular, \( \{d\} = \text{wt}_\ell^0(S) \).

**Corollary 7.** Let \( S \) be an irreducible module in category \( O \). We have
\[
\text{wt}^{n+1}_\ell(S) \subseteq \text{wt}_\ell^n(S)\{A_{2,x}^{-1} | (j, x) \in I \times \mathbb{C}\} \quad \text{for } n \in \mathbb{Z}_{\geq 0}.
\]
In particular, \( \text{wt}_\ell(S) \subseteq d\mathcal{Q} \), where \( d \) is the highest \( \ell \)-weight of \( S \).

**Proof.** Let \( e \in \text{wt}_\ell^{n+1}(S) \). Since \( e \neq d \), the \( \ell \)-weight space \( S_e \) is not the highest \( \ell \)-weight space. There exists \( j \in I \) such that \( x_j^+(u)S_e \neq 0 \). Such a non-zero space must project non-trivially to certain \( \ell \)-weight space \( S_r \). By Proposition 6 there exists \( x \in \mathbb{C} \) such that \( f = A_{j,x}e \). We have \( e = A_{j,x}^{-1}f \) and \( f \in \text{wt}_\ell^n(S) \). \( \square \)

Let \( V \) be a module in category \( O \) which admits a one-dimensional weight space \( V_\lambda \) such that \( \text{wt}(V) \subset \lambda + \mathbb{Q}_- \). Then \( V_\lambda \) is also an \( \ell \)-weight space of \( \text{wt}_\ell \), and we define the **normalized \( q \)-character** by \( \chi_\ell(V) := \chi(V) \times d^{-1} \in \mathcal{E}_\ell \). If \( V \) is irreducible, then Corollary 7 implies that \( \chi_\ell(V) \) is a (possibly infinite) sum of monomials in the \( A_{j,x}^{-1} \) with leading term 1.

We introduce the completed Grothendieck ring \( K_0(O) \) as in [24, §3.2]. Its elements are formal sums \( \sum_{d \in \mathbb{Z}^+} n_d[L(d)] \) of the symbols \([L(d)]\) with integer coefficients \( n_d \) such that the direct sum of \( Y_q(\mathfrak{g}) \)-modules \( \oplus_{d \in \mathbb{Z}^+} [L(d)]^{[n_d]} \) is in category \( O \). Addition is the usual one of formal sums.

Let \( V \) be in category \( O \). In general \( V \) may not admit a Jordan–Hölder series of finite length. Still, as in the case of Kac–Moody algebras [20, §9.3], for \( d \in \mathbb{N} \) the multiplicity \( m_{d,V} \in \mathbb{Z}_{\geq 0} \) of the irreducible module \( L(d) \) in \( V \) makes sense, and \([V] := \sum_{d \in \mathbb{N}^+} m_{d,V} [L(d)]\) is a well-defined element in \( K_0(O) \). Multiplication in \( K_0(O) \) is induced by \([V][W] = [V \otimes W]\) for \( V, W \) in category \( O \).

Since \( \chi_q \) respects exact sequences, the assignment \([V] \mapsto \chi_q(V)\) extends uniquely to a group homomorphism \( \chi_q : K_0(O) \rightarrow \mathcal{E}_\ell \), called the \( q \)-character map.

**Theorem 8.** [20] The \( q \)-character map is an injective homomorphism of rings.

We conclude as in [23, Remark 3.13] that \( K_0(O) \) is a commutative ring.

**Example 9.** Take \( q = \mathfrak{sl}_2 \), so that \( I = \{1\} \) and \( c_{11} = 2 = 2d_1 \). For \( k, x \in \mathbb{C} \), there is a representation \( \mathcal{L}^{x+k} \) of \( Y_q(\mathfrak{sl}_2) \) on \( \bigoplus_{i=0}^\infty \mathcal{C}v_i \) defined by [6, Proposition 2.6]:
\[
\begin{align*}
x_{1,n}v_i & = (x + 1 - i)^n v_{i-1}, \quad x_{-1,n}v_i = (x - i)^n (i + 1)(k + 1)v_{i+1}, \\
\xi_1(u)v_i & = \frac{(u + (x - 1)h)(u + (x + 1)h)}{(u + (x + i - 1)h)(u + (x + i)h)}v_i.
\end{align*}
\]

It is in category \( O \) with \( q \)-character \( \chi_q(\mathcal{L}^{x+k}) = \frac{\Psi_{x+k}}{\Psi_{1,x}} \chi_q(\mathcal{L}^x) \) and
\[
\chi_q(\mathcal{L}^x) = 1 + A_{-1,x}^{-1} + A_{1,x}^{-1} A_{1,x+1}^{-1} + A_{1,x}^{-1} A_{1,x+1}^{-1} A_{1,x+2}^{-1} + \cdots.
\]
If \( k \in \mathbb{Z}_{\geq 0} \), then \( v_0, v_1, \ldots, v_k \) span a submodule \( \mathcal{C}_k^{x+1} \cong L(\frac{\Psi_{1,x+k}}{\Psi_{1,x}}) \). Comparing \( q \)-characters gives \([C_2]^y[\mathcal{L}^x] = [\mathcal{L}^{x+1}] + [\mathcal{L}^{x-1}]\) for \( y \in \mathbb{C} \). Restricted to \( \mathfrak{sl}_2 \) this is Equation (11). Indeed, \( C_2^y \cong \mathbb{C}^2 \), and \( \mathcal{L}_y^x \) is the graded dual of the Verma module \( M_{x-y} \) twisted by Cartan involution so that their isomorphism classes coincide.

## 2. Kirillov–Reshetikhin modules

In this section we study two families of finite-dimensional irreducible modules and their \( q \)-character properties. For \( (i, k, x) \in I \times \mathbb{Z}_{\geq 0} \times \mathbb{C} \) define
\[
\mathbf{w}^{(i)}_{k,x} := Y_{1,x+i} Y_{1,x+i+1} \cdots Y_{1,x+(k-\frac{1}{2})d_i} = \frac{\Psi_{1,x+kd_i}}{\Psi_{1,x}} \in \mathcal{P}_+.
\]
The finite-dimensional irreducible module \( L(w_{k,x}^{(i)}) \) is denoted by \( W_{k,x}^{(i)} \) and called Kirillov–Reshetikhin module \([27]\), KR module for short.

Let \( Q^- \) be the submonoid of \( Q_+ \) generated by the \( A_{j,w} \) for \( j \in I \) and \( w \in x + \frac{1}{2} \mathbb{Z} \).

**Theorem 10.** \([21, 32]\) Let \((i, x) \in I \times \mathbb{C} \). For \( k \in \mathbb{Z}_{>0} \) and \( 0 \leq l < k \):

(a) \( w_{k,x}^{(i)} A_{i,x}^{-1} A_{i,x+d_1}^{-1} \cdots A_{i,x+l d_i}^{-1} \) is an \( \ell \)-weight of \( W_{k,x}^{(i)} \) of multiplicity one;

(b) any \( \ell \)-weight of \( W_{k,x}^{(i)} \) different from (1) and from \( w_{k,x}^{(i)} \) must be of the form \( w_{k,x}^{(i')} A_{i',x}^{-1} e \) where \( e \in Q^- \), \( i' \in I \setminus \{i\} \), and \( z \) belongs to the subset \( \mathcal{F}_{k,x}^{(i')} \) of \( x + \{-1/2, -1, -1/2, 0, 1/2\} \) defined at the second column of the table:

| conditions on \((i', k)\) | the set \( \mathcal{F}_{k,x}^{(i')} \) |
|---------------------------|-----------------|
| \( c_{i'i} = 0 \)         | \( \emptyset \) |
| \( c_{i'i} = -1 \) or \( k = 1 \) | \( \{x - d_{i'i'}\} \) |
| \( c_{i'i} = -2 \) and \( k > 1 \) | \( \{x - 1, x\} \) |
| \( c_{i'i} = -3 \) and \( k = 2 \) | \( \{x - \frac{1}{2}, x - \frac{1}{2}, x + \frac{1}{2}\} \) |
| \( c_{i'i} = -3 \) and \( k > 2 \) | \( \{x - \frac{1}{2}, x - \frac{1}{2}, x + \frac{1}{2}\} \) |

As \( k \) tends to infinity, \( \tilde{\chi}_q(W_{k,x}^{(i)}) \) converges to a power series of the \( A_{j,w}^{-1} \) in \( Q^- \).

This is a translation of q-character property of KR modules over the quantum affine algebra \( U_q(\hat{g}) \) due to H. Nakajima \([32]\) and D. Hernandez \([21]\). Here \( q = e^{2\pi i h} \) and \( h \notin \mathbb{Q} \). The irrationality assumption is inessential for Yangians since the Hopf algebras \( Y_h(\hat{g}) \) and \( Y_1(\hat{g}) \) are isomorphic for all \( h \in \mathbb{C}^\times \). One applies the inverse functor of Gautam–Toledano Laredo \([18, 6]\), which sends finite-dimensional irreducible \( U_q(L\hat{g})\)-modules to irreducible \( Y_h(\hat{g})\)-modules, and matches the q-character maps of Frankel–Reshetikhin \([17]\) and H. Knight \([29]\). Our \( Y_{i,a} \) is equal to \( X_{i,a} \) in \([18, 7.6]\), and the map \( e_{i,a} \) in \([18, 7.7]\) sends our \( A_{i,a+b} \) for \( a \in C \) and \( m \in Z \) to the generalized simple root \( A_{i,a+b} \) for \( U_q(\hat{g}) \). The condition \( w \in x + \frac{1}{2} \mathbb{Z} \) of the theorem corresponds to \( e^{2\pi i w h} \in q^{2 \pi i h q^2} \) in \([21, 3.4(2)]\).

Part (a) of Theorem 10 generalizes Example 9. To explain part (b), let \( 1 \leq m \leq k \).

There is a unique decomposition of \( \ell \)-weights in terms of ratios of \( \Psi \):

\[
\frac{w_{k,x}^{(i)} A_{i,x}^{-1} A_{i,x+d_1}^{-1} \cdots A_{i,x+(m-1)d_i}^{-1} \Psi_{i,x-d_i} \Psi_{i,x+m d_i}}{\Psi_{i,x+(m-1)d_i} \Psi_{i,x+md_i}} \times \prod_{c_{i'i} < 0} \prod_{z \in \mathcal{F}_{k,x}^{(i')}} \Psi_{i',z}^{c_{i'i}}
\]

where each \( m_z \) is an integer such that \( 1 \leq m_z d_i \leq m \). The presence of \( \Psi_{i',z} \) in the denominator gives rise to the factor \( A_{2i,z}^{-1} \) in part (b). See \([21, \text{Lemma 5.5}])\).

We recall an important notion due to Frenkel–Mukhin \([10, 6]\). Since the abelian group \( \mathcal{P} \) is freely generated by the \( Y_{j,y} \), each element \( m \in \mathcal{P} \) is factorized uniquely as a finite product \( \prod_{(i,x) \in I \times C} Y_{i,x}^{c_{i,x}(m)} \) where the exponents \( c_{i,x}(m) \) are integers. Call \( m \) right-negative if \( m \neq 1 \) and

(RN) if \( c_{i,x}(m) > 0 \) for certain \((i, x) \in I \times C \), then there exists \((j, y) \in I \times C \) such that \( c_{j,y}(m) < 0 \) and \( y - x \in \frac{1}{2} \mathbb{Z} < 0 \).

A right-negative \( \ell \)-weight is never dominant. The product of two (and hence finitely many) right-negative \( \ell \)-weights is still right-negative.

**Example 11.** \([16]\) The inverse of a generalized simple root is right-negative:

\[
A_{j,x}^{-1} = Y_{j,x}^{-1} \prod_{i,c_{ij} = 1} Y_{i,x} \prod_{i} Y_{i,x+1} Y_{i,x} \prod_{i} Y_{i,x+1} Y_{i,x}^{-1}.
\]

Consequently, if \( m \in \mathcal{P} \) is right-negative, then so is any element of \( mQ^- \).
Now we discuss a second family of finite-dimensional irreducible modules. For \((i, x, k, t) \in I \times \mathbb{C} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}\), the following is a dominant \(\ell\)-weight:

\[
d^{(i, t)}_{k, x} = w^{(i)}_{k, x} \cdot w^{(i)}_{k + x - (k + 1)d_i} A^{-1}_{i, x} \cdot A^{-1}_{i, x + 2d_i} \cdots A^{-1}_{i, x - kd_i},
\]

The finite-dimensional irreducible module \(L(d^{(i, t)}_{k, x})\) is denoted by \(D^{(i, t)}_{k, x}\).

**Theorem 12.** \([13]\) Let \((i, k, t) \in I \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}\). Then in \(K_0(\mathcal{O})\) holds

\[
[W^{(i)}_{k, 0} \otimes W^{(i)}_{k + t, d_i}] = [W^{(i)}_{k - 1, d_i} \otimes W^{(i)}_{k + t + 1, 0}] + [D^{(i, t)}_{k, (k + 1)d_i}].
\]

**Proof.** This is \([13\, \text{Theorem 4.1}]\) applied to the by the inverse functor of \([13\, \text{§6}]\). Or, one can copy the proof of \([13]\). First one checks that the following \(\ell\)-weights

\[
w^{(i)}_{k, 0} A^{-1}_{i, 0} w^{(i)}_{k + t, d_i}, \quad w^{(i)}_{k, 0} w^{(i)}_{k + t, d_i} A^{-1}_{i, 0} A^{-1}_{i, 2} \quad \text{for } i' \in I \setminus \{i\} \text{ and } z \in \mathbb{Z}^{(i')}_d,
\]

are right-negative. Next \(W^{(i)}_{k - 1, d_i} \otimes W^{(i)}_{k + t + 1, 0}\) is shown to be irreducible by restriction to the subalgebra of \(Y_k(g)\) generated by the \(x^\pm_{i, n} \xi_{i, n}\) for \(n \in \mathbb{Z}\). Last, a dominant \(\ell\)-weight of \(W^{(i)}_{k, 0} \otimes W^{(i)}_{k + t, d_i}\) either appears in \(W^{(i)}_{k - 1, d_i} \otimes W^{(i)}_{k + t + 1, 0}\), or is \(d^{(i, t)}_{k, k + 1}\). Notably, \(D^{(i, t)}_{k, (k + 1)d_i}\) is special \([22]\) since it admits a unique dominant \(\ell\)-weight.

**Corollary 13.** For \((i, t) \in I \times \mathbb{Z}_{\geq 0}\), there exists (up to scalar multiple) a unique short exact sequence of finite-dimensional \(Y_k(g)\)-modules

\[
0 \longrightarrow D^{(i, t)}_{1, 2d_i} \longrightarrow W^{(i)}_{1, 0} \otimes W^{(i)}_{t + 1, d_i} \longrightarrow W^{(i)}_{t + 2, 0} \longrightarrow 0.
\]

**Proof.** The idea of the proof of \([38\, \text{Lemma 3.6}]\) works here. Let \(\omega_1\) and \(\omega_2\) be highest \(\ell\)-weight vectors of \(W^{(i)}_{1, 0}\) and \(W^{(i)}_{t + 1, d_i}\) respectively. It suffices to prove that the subspace \(Z\) of \(W^{(i)}_{1, 0} \otimes W^{(i)}_{t + 1, d_i}\) spanned by the \(x^\pm_{i, n} (\omega_1 \otimes \omega_2)\) for \(n \in \mathbb{Z}_{\geq 0}\) is at least two-dimensional. (This implies the non-irreducibility of \(Z\), and a comparison of Jordan–Hölder series forces \(Z\) to be the entire tensor product.)

Since \(\omega_2\) is a highest \(\ell\)-weight vector, by Remark \([2]\) we have

\[
x_i^- (u)(\omega_1 \otimes \omega_2) = \omega_1 \otimes x_i^- (u) \omega_2 + x_i^- (u) \omega_1 \otimes \xi_i (u) \omega_2
\]

\[
= \omega_1 \otimes x_i^- (u) \omega_2 + \frac{u + (t + 2)d_i}{u + d_i h} x_i^- (u) \omega_1 \otimes \omega_2.
\]

The \(i\)-th components of \(w^{(i)}_{1, 0}\) and \(w^{(i)}_{t + 1, d_i}\) are \(\frac{d_i h}{u + d_i h}\) and \(\frac{u + (t + 2)d_i h}{u + d_i h}\). Their denominators are \(u\) and \(u + d_i h\). It follows from Lemma \([26\, (b)]\) that

\[
x_i^- (u) \omega_1 = h x_i^-_0 \omega_1, \quad (u + d_i h) x_i^- (u) \omega_2 = h x_i^-_0 \omega_2.
\]

Multiplying \(x_i^- (u)(\omega_1 \otimes \omega_2)\) by \(u(u + d_i h)\), we obtain a \(Z\)-valued polynomial

\[
u_0 \omega_1 \otimes h x_i^-_0 \omega_2 + h x_i^-_0 \omega_1 \otimes (u + (t + 2)d_i h) \omega_2.
\]

Its constant term being \((t + 2)d_i h^2 x_i^-_0 \omega_1 \otimes \omega_2\), we have \(x_i^-_0 \omega_1 \otimes \omega_2 \in Z\). This implies in turn that \(\omega_1 \otimes x_i^-_0 \omega_2 \in Z\) and so \(\text{dim } Z \geq 2\), as desired.

**Corollary 14.** Let \((i, k, x) \in I \times \mathbb{Z}_{\geq 0} \times \mathbb{C}\). If \(d^{(i, t)}_{k, x}\) is an \(\ell\)-weight of \(D^{(i, 1)}_{k, x}\), then either \(e \in \{1, A^{-1}_{i, x}\}\) or \(e \in A^{-1}_{i, x - kd_i + z} Q^x_{i, x - kd_i + z}\) for certain \((i', z) \in I \times \frac{1}{2} \mathbb{Z}\) with

\[
either \ (i' = i, \ z = -d_i) \text{ or } (c_{i'} < 0, -\frac{3}{2} \leq z \leq \frac{1}{2})\]
Proof. We have the pullback formula \( \tau_{a}^{*}(D_{k,x}^{(i,t)}) = D_{k,x+a}^{(i,t)} \) by Remark 4. Assume without loss of generality \( x = kd \). By Theorem 12, \( D_{l,k}^{(i,t)} \) is an irreducible subquotient of \( W_{k-d_i}^{(i)} \otimes W_{k+1,0}^{(i)} \). It follows from Equation (15) that
\[
A_{i}^{-1}(k-1)d_iA_{i}^{-1}(k-2)d_i \cdots A_{i}^{-1}d_iA_{i}^{-1} = e' e''
\]
where \( e' \) and \( e'' \) are monomials in the normalized q-characters of the first and the second KR modules respectively. Note that \( e, e', e'' \in Q_{0} \). Suppose \( e \neq 1 \).

If \( e' \neq 1 \), then \( e' \in A_{i}^{-1}(k-1)d_iQ_{0} \), which implies \( e \in A_{i}^{-1}(k-1)d_iQ_{0} \).

Assume \( e' = 1 \). Then \( e'' \in A_{i}^{-1}(k-1)d_iA_{i}^{-1}(k-2)d_i \cdots A_{i}^{-1}d_iA_{i}^{-1}Q_{0} \). Applying Theorem 10 to the KR module \( W_{k+1,0}^{(i)} \) we have:
- either \( e'' = A_{i}^{-1}(k-1)d_iA_{i}^{-1}(k-2)d_i \cdots A_{i}^{-1}d_iA_{i}^{-1} = A_{i}^{-1}d_i \);
- or \( e'' \in A_{i}^{-1}Q_{0} \) where \( c_{i,i'} < 0 \) and \( z \in \mathfrak{A}_{k+1,0} \subseteq \{-\frac{3}{2}, -1, -\frac{1}{2}, 0, \frac{1}{2} \} \), implying \( e \in A_{i,i'}Q_{0} \) for such a pair \((i', z)\).

The proof of the corollary is completed. \( \square \)

Similar short exact sequence as Corollary 13 exists for all \( D_{k,(k+1)i}^{(i,t)} \), because the tensor product \( W_{k,0}^{(i)} \otimes W_{k+1,0}^{(i)} \) is of highest \( \ell \)-weight 33 34.

For latter purpose, let us write \( d_{k,x}^{(i,t)} \) in terms of the \( \Psi \) based on Equation (11). Assume \( k \in 6Z_{>0} \) so that \( k \) is divisible by all the \( d_j \). We have
\[
d_{k,x}^{(i,t)} = \frac{\psi_{x+(kd)}^{i} \prod_{j<i, j \neq 0} \psi_{x+jd} \prod_{j<i} \psi_{x+jd-kd}}{\psi_{x}^{i} \prod_{j<i, j \neq 0} \psi_{x+jd} \prod_{j<i} \psi_{x+jd-kd}}
\]
\[
\times \frac{\psi_{x+1}^{i} \psi_{x-1} \psi_{x-k}^{i} \psi_{x-k}^{i} \psi_{x+1}^{i} \psi_{x-1} \psi_{x-k}^{i} \psi_{x-k}^{i}}{\psi_{x}^{i} \psi_{x-k}^{i} \psi_{x-k}^{i}}.
\]
One may view \( d_{k,0}^{(i,t)} \) as the Yangian analog of \( \tilde{\Psi}_{k}^{(i,t)} \) in [15 §4.3].

3. ASYMMETRIC REPRESENTATIONS

We construct infinite-dimensional asymptotic modules in category \( \mathcal{O} \) as limits of KR modules and provide three-term identities in the Grothendieck ring.

From now on up to Definition 17 \( i \in I \) is fixed, and when writing \( k > l \) we implicitly assume that \( k, l \) are non-negative integers.

The limit construction is in the spirit of Hernandez-Jimbo [23], with simplified arguments from the case of elliptic quantum groups [35 §4]: to produce an inductive system of vector spaces \( W_{m,0}^{(i)} \subset W_{l,0}^{(i)} \subset W_{t,0}^{(i)} \subset \cdots \), and then to establish asymptotic properties for this system.

For \( k > l \) let \( Z_{k} \) be the KR module \( W_{k,l}^{(i)} \), with a fixed highest \( \ell \)-weight vector \( \omega_{k} \). (This differs from the notation \( Z_{k} \) in [35 §4] due to the opposite tensor products taken in Corollary 13 and in [35 Lemma 3.6].) In the case \( l = 0 \) we have \( Z_{0k} = W_{k,0}^{(i)} \) and write simply \( \omega_{k} := \omega_{0k} \).

The proof of [35 Lemma 4.1] almost goes without change. For \( m < k \) there exists a unique \( Y_{k}(q) \)-module morphism \( Z_{ml}^{(i)} \otimes Z_{l} \rightarrow Z_{mk} \) which sends \( \omega_{ml} \otimes \omega_{l} \) to \( \omega_{mk} \). As two special cases, for \( l < k \) and \( l < t - 1 \) we have
\[
F_{k,l} : W_{l,0}^{(i)} \otimes Z_{l} \rightarrow W_{k,0}^{(i)}, \quad F_{l,t} : Z_{l,t+1} \otimes Z_{l+1,t} \rightarrow Z_{lt}.
\]
Let us define the restriction map as in [23 §4.2]:
\[
F_{k,l} : W_{l,0}^{(i)} \rightarrow W_{k,0}^{(i)}, \quad v \mapsto F_{k,l}(v \otimes \omega_{l}).
\]
Then $F_{k,l}(\omega_l) = \omega_k$, and $(W_{t,0}^{(i)}, F_{k,l})$ forms an inductive system of vector spaces: $F_{k,l}F_{m,n} = F_{m,n}$ for $m < l < k$. If $l < k - 1$ then for $w \otimes v \in W_{t,0}^{(i)} \otimes Z_{l+1}$ we have:
\begin{equation}
F_{k,l}((w \otimes g_{k,l}(v \otimes \omega_{l+1,k}))) = F_{k,l+1}F_{l+1,i}(w \otimes v).
\end{equation}
(There is a typo at the right-hand side of [38, Eq.(4.19)]: $F_{k,l} should be $F_{k,l+1}$.)

**Lemma 15.** For $l < k$ the map $F_{k,l}$ is an injective morphism of $Y_h^+(\mathfrak{g})$-modules.

*Proof.* Since $\omega_{l,k}$ is a highest $l$-weight vector, by Remark [2] in $W_{t,0}^{(i)} \otimes Z_{l,k}$:
\begin{equation}
x_j^+(u)(w \otimes \omega_{l,k}) = x_j^+(u)w \otimes \omega_{l,k} \quad \text{for } w \in W_{t,0}^{(i)} \text{ and } j \in I.
\end{equation}
It follows from the $Y_h(\mathfrak{g})$-linearity of $F_{k,l}$ that
\begin{equation}
x_j^+(u)F_{k,l}(w) = F_{k,l}(x_j^+(u)(w \otimes \omega_{l,k})) = F_{k,l}x_j^+(u)(w \otimes \omega_{l,k})
= F_{k,l}(x_j^+(u)w \otimes \omega_{l,k}) = F_{k,l}x_j^+(u)w.
\end{equation}
So $x_j^+(u)F_{k,l} = F_{k,l}x_j^+(u)$ for all $j \in I$, meaning that $F_{k,l}$ is $Y_h^+(\mathfrak{g})$-linear. The injectivity is proved similarly as [38 Proposition 4.2 (2)], with the RLL generators therein replaced by the Drinfeld generators; see also [38 Lemma 4.2]. □

Remark [2] also gives the following commutation relations for $l < k$ and $j \in I$:
\begin{equation}
\xi_j(u)F_{k,l} = F_{k,l+1}(A_j^1(u) + B_j^1(u)kd_i),
\end{equation}
\begin{equation}
x_j^-(u)F_{k,l} = F_{k,l+1}(A_j^1(u) + B_j^1(u)kd_i)
\end{equation}
for the last equation we used $x_j^-(u)\omega_{l,k} = 0$ for $j \neq i$, based on Lemma 4

**Lemma 16.** For $l < k - 1$ we have
\begin{equation}
x_i^-(u)F_{k,l}(w) = F_{k,l+1}(A_i^1(u) + B_i^1(u)kd_i),
\end{equation}
where $A_i^1(u)$ and $B_i^1(u)$ are $\text{Hom}_C(W_{t,0}^{(i)}, W_{l+1,0}^{(i)})$-valued power series in $u^{-1}$ independent of $k$ and defined by the following formulas. For $w \in W_{t,0}^{(i)}$,
\begin{align*}
A_i^1(u) : \quad w &\mapsto \frac{u}{u+ld_i} F_{l+1,i}x_i^-(u)w - \frac{th}{u+ld_i} x_i^-(u)(w \otimes x_i^0\omega_{l+1,l+1}), \\
B_i^1(u) : \quad w &\mapsto \frac{h}{u+ld_i} F_{l+1,i}x_i^-(u)w + \frac{h}{(u+ld_i)d_i} x_i^-(u)(w \otimes x_i^0\omega_{l+1,l+1}).
\end{align*}
*Proof.* First we prove the following vector is in the kernel of $g_{k,l}$:
\begin{equation}
v := \omega_{l+1,l+1} \otimes x_i^0\omega_{l+1,k} - (k - l - 1)x_i^0\omega_{l+1,k} \otimes \omega_{l+1,k} \in Z_{l+1,l+1} \otimes Z_{l+1,k}.
\end{equation}
One checks that $x_i^0v = 0$ based on the identity
\begin{equation}
x_i^0x_i^0\omega_{l,k} = [x_i^0,x_i^0]\omega_{l,k} = \xi_i\omega_{l,k} = (k - l)d_i\omega_{l,k}.
\end{equation}
Note that $g_{k,l}(v)$ is in the weight space $(Z_{l+1})_{(k-l)\gamma_{-\alpha_i}}$, which by Theorem 10 is spanned by $x_i^0\omega_{l,k}$. Now $x_i^0x_i^0\omega_{l,k} \neq 0$ and $x_i^0g_{k,l}(v) = 0$ force $g_{k,l}(v) = 0$.

Applying Lemma 4(b) to $Z_{l,k}$ we get (u+ld_i)x_i^-(u)\omega_{l,k} = h x_i^0\omega_{l,k} and
\begin{align*}
x_i^0\omega_{l,k} &= x_i^0g_{k,l}(\omega_{l+1,l+1} \otimes \omega_{l+1,k}) = g_{k,l}x_i^0(\omega_{l+1,l+1} \otimes \omega_{l+1,k}) \\
&= g_{k,l}(\omega_{l+1,l+1} \otimes x_i^0\omega_{l+1,k} + x_i^0\omega_{l+1,l+1} \otimes \omega_{l+1,k} - v) \\
&= (k - l)g_{k,l}(x_i^0\omega_{l+1,l+1} \otimes \omega_{l+1,k}),
\end{align*}
\begin{align*}
x_i^-(u)\omega_{l,k} &= (k - l)\frac{h}{u+ld_i} g_{k,l}(x_i^0\omega_{l+1,l+1} \otimes \omega_{l+1,k}).
\end{align*}
We compute \( x_i^{-}(u)F_{k,t}(w) \) for \( w \in W_{t,0}^{(i)} \), as in the proof of Lemma 15.
\[
x_i^{-}(u)F_{k,t}(w) = F_{k,t}(x_i^{-}(u)w \otimes \omega_k) = F_{k,t}(x_i^{-}(u)w + x_i^{-}(u)w \otimes \xi(u)\omega_k)
\]
\[
= \frac{(k-l)\hbar}{u + ld_t\hbar} F_{k,t}(w \otimes F_{k,t}(x_i^{-}(u)\omega_{t+1} \otimes \omega_{t+1,k})) + \frac{u + kd_t\hbar}{u + ld_t\hbar} F_{k,t}(x_i^{-}(u)w)
\]
\[
= F_{k,t+1} \left( \frac{(k-l)\hbar}{u + ld_t\hbar} \mathcal{F}_{k,t+1}(w \otimes x_i^{-}(u)\omega_{t+1}) + \frac{u + kd_t\hbar}{u + ld_t\hbar} F_{k,t+1}(x_i^{-}(u)w) \right).
\]
The last row is Equation (17) applied to \( w \otimes x_i^{-}\omega_{t+1} \in W_{t,0}^{(i)} \otimes Z_{t,t+1} \). Equation (20) is a reformulation of it. \( \square \)

Let \( S := \{x_{j,n}^{-}, \xi_{j,n}, x_{j,n}^{+} \mid (j,n) \in I \times \mathbb{Z}_{\geq 0} \} \) denote the set of Drinfeld generators of \( Y_h(g) \). As a summary of Lemma 15 and Equations (18)–(20), to \( s \in S \) and \( l \in \mathbb{Z}_{\geq 0} \) are attached linear maps \( A_s^t \) and \( B_s^t \) from \( W_{t,0}^{(i)} \) to \( W_{t+1,0}^{(i)} \) such that
\[
sF_{k,t} = F_{k,t+1}(A_s^t + B_s^t \times kd_t) \quad \text{for} \quad k > l + 1.
\]
As in [36, §2], the \( A_s^t \) and \( B_s^t \) are unique and form morphisms of inductive systems,
\[
F_{k+1,t+1}A_s^t = A_s^tF_{k,t}, \quad F_{k+1,t+1}B_s^t = B_s^tF_{k,t} \quad \text{for} \quad k > l,
\]
so they admit inductive limits, denoted by \( A_s \) and \( B_s \) respectively, which are linear operators on the inductive limit \( W_{\infty}^{(i)} := \lim_{\longrightarrow} W_{t,0}^{(i)} \).

**Definition 17.** For \( y \in \mathbb{C} \) the assignment \( s \mapsto A_s + yB_s \) as \( s \in S \) defines a \( Y_h(g) \)-module structure on \( W_{\infty}^{(i)} \), denoted by \( \mathcal{L}(\Psi_{i,y}) \). More generally, for \( x \in \mathbb{C} \) let \( \mathcal{L}(\frac{\Psi_{i,x}}{\Psi_{i,0}}) \) be the pullback of \( \mathcal{L}(\frac{\Psi_{i,y}}{\Psi_{i,0}}) \) by the spectral parameter shift \( \tau_x \) from Equation (20), and call it an asymptotic module. (Even if \( y = x \) we still use \( \mathcal{L}(\frac{\Psi_{i,x}}{\Psi_{i,0}}) \) to emphasize the dependence on \( i \) and \( x \).)

This definition was made in [31] for \( g = sl_2 \). It is inspired from the case of quantum affine algebra \( U_q(g) \), where one replaces the \( q^{-k} \), as \( k \) tends to infinity, by zero [23, §4.2] or by a non-zero complex number [30, Claim 2.1].

**Theorem 18.** The asymptotic modules are in category \( \mathcal{O} \) with \( q \)-characters
\[
\chi_q(\mathcal{L}(\frac{\Psi_{i,y}}{\Psi_{i,0}})) = \frac{\Psi_{i,y}}{\Psi_{i,0}} \chi_q(W_{t,0}^{(i)}) = \lim_{\longrightarrow} \chi_q(W_{t,0}^{(i)}) \quad \text{for} \quad (i,y, x) \in I \times \mathbb{C}^2.
\]
Furthermore, the following statements hold in category \( \mathcal{O} \).

(a) The set \( R' \) of highest \( \ell \)-weights equals \( R \).
(b) Let \( V \) be a finite-dimensional \( \ell \)-weight module. In the \( q \)-character formula of \( V \), replace \( \chi_q(V) \) by \([V]_q\), write each of its \( \ell \)-weight as a product of the \( \Psi_{i,x} \) and replace the \( \Psi_{i,x} \) by the ratios \( \frac{\mathcal{L}(\frac{\Psi_{i,x}}{\Psi_{i,0}})}{\mathcal{L}(\frac{\Psi_{i,y}}{\Psi_{i,0}})} \). Then we get a relation in the fractional ring of the Grothendieck ring \( K_0(\mathcal{O}) \).
(c) For \((i,a,b,x,y) \in I \times \mathbb{C}^2 \) we have in the Grothendieck ring
\[
[\mathcal{L}(\frac{\Psi_{i,y}}{\Psi_{i,a}})][\mathcal{L}(\frac{\Psi_{i,b}}{\Psi_{i,x}})] = [\mathcal{L}(\frac{\Psi_{i,y}}{\Psi_{i,a}})][\mathcal{L}(\frac{\Psi_{i,b}}{\Psi_{i,x}})].
\]

**Proof.** It suffices to prove Equation (21): its right-hand side converges according to Theorem 10 and implies the cone conditions of weights; parts (a)–(c) are then proved in the same way as [23, Theorem 3.11] and [14, Theorem 4.8], by replacing the isomorphism class \( [V]_q \) with \( q \)-character \( \chi_q(V) \). Since the \( q \)-character map is compatible with the spectral parameter shifts, one may assume \( x = 0 \).
Consider the module $\mathcal{L}(\frac{\varphi}{w_{i,x}})$ in Definition 17 with the injective structural maps $F_l : W_{l,0}^{(i)} \rightarrow \mathcal{L}(\frac{\varphi}{w_{i,x}})$ for $l \geq 0$. Since the pre-factor at the right-hand side of Equation (18) is a polynomial of $kd$, by definition of asymptotic representation, when taking inductive limit $k \rightarrow \infty$ in the equation one replaces $kd$ by $y$:

$$\xi_j(u)F_l = \left(\frac{u + y\hbar}{u + (kd\hbar)}\right)\delta_{ij}\xi_j(u).$$

So $F_l$ sends an $\ell$-weight vector to another $\ell$-weight vector with $\ell$-weight multiplied by $\frac{\varphi}{w_{i,x}}$. This proves the limit formula (21). □

By Equation (21), it makes sense to talk about the normalized $q$-characters of tensor products of asymptotic modules and irreducible modules. The normalized $q$-character of $\mathcal{L}(\frac{\varphi}{w_{i,x}})$ is independent of $y$. Also, any $\ell$-weight of this module different from $\frac{\varphi}{w_{i,x}}$ must belong to $\mathcal{L}(\frac{\varphi}{w_{i,x}})\mathcal{A}_i^{-1}Q_x^{-}$. For $(i, k, x) \in I \times \mathbb{C}$, let $\mathcal{A}_k^{(i)}$ be the irreducible module of highest $\ell$-weight

$$\mathbf{m}_{k,x}^{(i)} := \frac{\Psi_{i,x} + d_i}{\Psi_{i,x}} \prod_{j: c_{ij} < 0} \frac{\Psi_{j,x + d_i - kd_i}}{\Psi_{j,x + d_i - kd_i}}.$$

If $k \in 6\mathbb{Z}_{>0}$, then $\mathbf{m}_{k,x}^{(i)}$ is dominant and $\mathcal{A}_k^{(i)}$ is finite-dimensional.

**Corollary 19.** Let $k \in 6\mathbb{Z}_{>0}$ and $(i, x) \in I \times \mathbb{C}$. If $\mathbf{m}_{k,x}^{(i)}$ is an $\ell$-weight of $\mathcal{A}_k^{(i)}$, then either $e \in \{1, A_i^{-1}\}$, or $e \in A_{j,x + d_i - kd_i}^{-1}Q_x^{-}$ for certain $j \in I$ with $c_{ij} < 0$.

**Proof.** View $\mathcal{A}_k^{(i)}$ as an irreducible sub-quotient of $W_{1,x}^{(i)} \otimes M$ where

$$M := \bigotimes_{j: c_{ij} < 0} W_{j,x}^{(j)} = \bigotimes_{j: c_{ij} < 0} \frac{\Psi_{j,x + d_i - kd_i}}{\Psi_{j,x + d_i - kd_i}}.$$

is a tensor product of KR modules of arbitrary order. Then $e = e'e''$ with $e'$ and $e''$ being monomials in the normalized $q$-characters of $W_{1,x}^{(i)}$ and $M$ respectively. So $e \in Q_x^{-}$. Together with Corollary 17 we have $\text{wt}^2(\mathcal{A}_k^{(i)}) \subset \text{wt}^2(\mathcal{A}_k^{(i)})Q_x^{-}$ for $n \geq 2$.

We may therefore assume that the $\ell$-weight $\mathbf{m}_{k,x}^{(i)}e$ of $\mathcal{A}_k^{(i)}$ is of height at most 2.

If $e'' \neq 1$, Theorem 10 applied to $M$, there exists $j \in I$ such that $c_{ij} < 0$ and $e'' \in A_{j,x + d_i - kd_i}^{-1}Q_x^{-}$ that implies $e \in A_{j,x + d_i - kd_i}^{-1}Q_x^{-}$.

Assume $e'' = 1$, so that $e = e'$ and $W_{1,x}^{(i)}$ is an $\ell$-weight of $W_{1,x}^{(i)}$ of height at most 2. By Theorem 10, either $e \in \{1, A_i^{-1}\}$, or $e = A_{A_{i',x + d_{i'}}}^{-1}A_{i',x + d_{i'}}$ for certain $i' \in I$ with $c_{i'i'} < 0$, because $\mathcal{A}_1^{(i')} = \{x + d_{i'}\}$. Let us assume that the latter case happens and arrive at a contradiction. View $\mathcal{A}_{k,x}^{(i)}$ as an irreducible sub-quotient of $D_{k,x}^{(1)} \otimes A$ where $A$ is the following tensor product of asymptotic modules

$$A := \bigotimes_{j: c_{ij} = -2} \mathcal{L}(\frac{\Psi_{j,x-k}}{\Psi_{j,x}}) \otimes \bigotimes_{j: c_{ij} = -3} \mathcal{L}(\frac{\Psi_{j,x-k}^{-1}}{\Psi_{j,x-k}}) \otimes \mathcal{L}(\frac{\Psi_{j,x-k}^{-1}}{\Psi_{j,x+k}}).$$

Then $A_{i',x + d_{i'}}^{-1}A_{i',x + d_{i'}} = e^De^A$ where $e^D$ and $e^A$ are monomials in $\bar{\chi}_{\mathcal{A}}(D_{k,x}^{(1)})$ and $\bar{\chi}_{\mathcal{A}}(A)$ respectively. Since $d_{i'} \geq \frac{-3}{2}$ and $k \geq 6$, we have $x + d_{i'} \neq x - kd_i + s$ for any half integer $s$ between $\frac{-3}{2}$ and $\frac{1}{2}$. By Corollary 14, neither $A_{i',x + d_{i'}}^{-1}$ nor $A_{i',x + d_{i'}}^{-1}$ is a monomial in $\bar{\chi}_{\mathcal{A}}(D_{k,x}^{(1)})$, so at least one of them is a monomial in $\bar{\chi}_{\mathcal{A}}(A)$. Then $A_{i',x + d_{i'}}^{-1}$ has to be a monomial in $\bar{\chi}_{\mathcal{A}}(A)$. One obtains $j$ such that $c_{ij} = -3$ and $d_{i'i'} = -\frac{1}{2}$. This implies that $g$ is of type $G_2$ and $d_i = 1$, but then $c_{i'i'} = 2d_{i'i'} = -1$ is an entry of the Cartan matrix of $g$, absurd. □
We are ready to prove the main result of this paper.

**Theorem 20.** Let $(i,k,x,y) \in I \times \mathbb{C}$. If $k \notin \frac{1}{2}\mathbb{Z}$, then in $K_0(O)$ holds

$$
\left[ \mathcal{M}_{k,x}^{(1)}(\frac{\Psi_{i,x} \cdot d_i}{\Psi_{i,y}}) \right] = \left[ \mathcal{L}( \Psi_{i,x} ) \right] \prod_{j : c_{ij} < 0} \left[ \mathcal{L}( \frac{\Psi_{j,x + d_j}}{\Psi_{j,y}} ) \right] \prod_{j : c_{ij} < 0} \left[ \mathcal{L}( \frac{\Psi_{y,x - d_j}}{\Psi_{y,i}} ) \right] + \left[ \mathcal{L}( \frac{\Psi_{y,x - d_j}}{\Psi_{y,i}} ) \right] \prod_{j : c_{ij} < 0} \left[ \mathcal{L}( \frac{\Psi_{j,x + d_j}}{\Psi_{j,y}} ) \right].
$$

When $g = \mathfrak{sl}_2$, we recover Example 10 as $\mathcal{M}_{k,x}^{(1)} \cong \mathbb{C}$ and $\mathcal{L}( \frac{\Psi_{y,x}}{\Psi_{y,i}} ) \cong \mathcal{L}_y$.

**Proof.** Equation (24) is equivalent to the normalized $q$-character formula:

$$
\chi_{\mathcal{M}_{k,x}^{(1)}}(1) = (1 + A_{i,x}^{-1}) \prod_{j : c_{ij} < 0} \chi_{\mathcal{L}( \frac{\Psi_{j,x + d_j}}{\Psi_{j,y}} )}.
$$

This formula does not involve $y \in \mathbb{C}$. So we may assume with loss of generality that neither $y - x$ nor $y - x + kd$ belongs to $\frac{1}{2}\mathbb{Z}$.

We follow closely the proof of [38, Theorem 5.1]. Introduce tensor products of asymptotic/KR modules (arbitrary order of tensor products):

$$
S^\pm := \mathcal{L}( \frac{\Psi_{i,x + d_i}}{\Psi_{i,y}} ) \otimes \mathcal{L}( \frac{\Psi_{j,x + d_j}}{\Psi_{j,y}} ) \otimes \mathcal{L}( \frac{\Psi_{k,x + d_k}}{\Psi_{k,y}} ),
$$

$$
T_n := \bigotimes_{j : c_{ij} < 0} W^{(i)}_{n,x + d_i + kd_i}, \quad S_n := W^{(i)}_{n,y} \otimes T_n \text{ for } n \in \mathbb{Z}_{>0}.
$$

One views $S^+$, $S^-$ and $S_n$ as analogs of $S^0$, $S^1$ and $S'_n$ in the proof of Claim 1 in [38, Theorem 5.1]. Let $s^\pm$, $t_n$ and $s_n$ be the obvious highest $\ell$-weights of $S^\pm$, $T_n$ and $S_n$ as products of highest $\ell$-weights of asymptotic/KR modules. Then $s^- = s^+ A_{i,x}^{-1}$.

**Step 1: irreducibility of the tensor products.**

Any $\ell$-weight of $S_n$ is of the form $e' e''$ where $e'$ and $e''$ are $\ell$-weights of $W^{(i)}_{n,y}$ and $T_n$ respectively. By Theorem 10, $e'$ is a monomial in the $Y_{j,b}^+$ with $b \in y + \frac{1}{2}\mathbb{Z}$, while $e''$ is such a monomial but with $b \in x - kd_i + \frac{1}{2}\mathbb{Z}$. Since $x - kd_i - y \notin \frac{1}{2}\mathbb{Z}$, the $\ell$-weight $e' e''$ is dominant if and only if both $e'$ and $e''$ are dominant.

Suppose $e'' \neq t_n$. We prove that it is right-negative. By Theorem 10 there exists $i' \in I$ such that $c_{i'j} < 0$ and $e'' \in t_n A_{i',x+d_i'+kd_i}^{-1} Q_{x-kd_i}$. It suffices to show that $t_n A_{i',x+d_i'+kd_i}^{-1}$ is right-negative. By Equation (17), $t_n$ is of the form

$$
\prod_{j : c_{ij} < 0} Y^\dagger_{j,x+d_j + \frac{1}{2}d_j - kd_j},
$$

Here $Y^\dagger_{j,a}$ for $(j,a) \in I \times \mathbb{C}$ means a certain product of $Y_{j,b}$ with the $Y_{j,b}$ such that $b \in a + \frac{1}{2}\mathbb{Z}_{>0}$. By Example 11 the right-negativity of $A_{i',a}^{-1}$ for $a \in \mathbb{C}$ arises from the negative power $Y_{i',a}^{-1} - \frac{1}{2}d_i$. So we are led to prove that for all $j \in I$ with $c_{ij} < 0$ the following half integer is strictly positive:

$$
(x + d_j + \frac{1}{2}d_j - kd_j) - (x + d_i' - kd_i' - \frac{1}{2}d_i') = d_{ij} + \frac{1}{2}d_j - d_i' + \frac{1}{2}d_i' > 0.
$$

Assume the contrary: the left-hand side as a half integer is non-positive. Since $d_j$ and $d_i'$ are strictly positive integers, we must have $i' \neq j$. Next,

$$
\frac{1}{2}d_j (c_{ij} + 1) = d_{ij} + \frac{1}{2}d_j \leq d_i' - \frac{1}{2}d_i' = \frac{1}{2}d_i (c_{i'i} - 1) \leq -d_i' \leq -1.
$$

So $c_{ij} < -1$. This implies $d_j = 1$ and $c_{ij} \leq -3$. The Lie algebra $g$ must be of rank two and of type $G_2$, but $i, i', j$ are two-by-two distinct, absurd.
Similarly, if $e' \neq w_{i,y}^{(i)}$, then $e'$ is right-negative. $S_n$ admits only one dominant $\ell$-weight and must be irreducible. View it as an irreducible sub-quotient of

$$L(s^+) \otimes L\left(\frac{\Psi_{j,y+nd_j}}{\Psi_{j,x+nd_j}}\right) \otimes \bigotimes_{j : c_{ij} < 0} L\left(\frac{\Psi_{j,x+di_j-kd_j}}{\Psi_{j,x+di_j}}\right).$$

By assumption, $Q^+_x$ and $Q^+_x Q^+_{-kd_j}$ have trivial intersection. We obtain as in the proof of Claim 1 in \[35\] Theorem 5.1 that all monomials (counted with multiplicity) in the normalized $q$-character $S_n$ must appear in that of $L(s^+)$, i.e., $\tilde{\chi}_q(S_n)$ is bounded above by $\tilde{\chi}_q(L(s^+))$. As $n$ goes to infinity, $\tilde{\chi}_q(S_n)$ converges to $\tilde{\chi}_q(S^+)$, and so the latter is bounded above by $\tilde{\chi}_q(L(s^+))$. Comparing highest $\ell$-weights, we conclude that $L(s^+) \cong S^+$, namely, $S^+$ are also irreducible.

Claim 2 of \[35\] Theorem 5.1 adapted directly to the present situation, $S^+$ appear as irreducible sub-quotients of the tensor product $\mathcal{M}_{k,x}^{(i)} \otimes L\left(\frac{\Psi_{1,1}}{\Psi_{1,0}}\right)$, and $\tilde{\chi}_q(\mathcal{M}_{k,x}^{(i)})$ is bounded below by the right-hand side of Equation (25).

**Step 2: upper bound for the normalized $q$-character.**

For $n \in 6\mathbb{Z}_{>0}$, view $\mathcal{M}_{k,x}^{(i)}$ as an irreducible sub-quotient of $\mathcal{M}_{n,x}^{(i)} \otimes B_n$, where $B_n$ is the following tensor product of asymptotic modules:

$$B_n = \bigotimes_{j : c_{ij} < 0} L\left(\frac{\Psi_{j,x+d_j-kd_j}}{\Psi_{j,x+di_j}}\right).$$

Fix an $\ell$-weight $m_{k,x}^{(i)} e$ of $\mathcal{M}_{k,x}^{(i)}$. It is of the form $e = e^M e^B$ such that $e^M \in Q^+_x$ and $e^B \in Q^+_{-kd_j}$ are monomials in $\tilde{\chi}_q(\mathcal{M}_{n,x}^{(i)})$ and $\tilde{\chi}_q(B_n)$ respectively; such a factorization is unique because $Q^+_x$ and $Q^+_{-kd_j}$ intersect trivially. Let $n$ be so large that none of the $A^{-1}_{j,x+d_j-kd_j}$ for $j \in I$ appears as a factor of the monomial $e$. By Corollary \[13\] either $e^M = 1$ or $e^M = A_{i,x}^{-1}$. In both cases, the $\ell$-weight space of $\mathcal{M}_{n,x}^{(i)}$ of $\ell$-weight $m_{k,x}^{(i)} e^M$ is one-dimensional, as seen from the proof of Corollary \[13\] This implies that $\tilde{\chi}_q(\mathcal{M}_{k,x}^{(i)})$ is bounded above by $\tilde{\chi}_q(B_n(1+A_{i,x}^{-1}))$, which is the right-hand side of Equation (25). Together with the lower bound at Step 1, we obtain Equation (25). The proof of Theorem 20 is completed. □

**Remark 21.** For $(i,k,x) \in I \times \mathbb{C}^2$ define

$$m_{k,x}^{(i)} = \prod_{j : c_{ij} = -2} \Psi_{j,x-k} - \prod_{j : c_{ij} = -3} \Psi_{j,x-k} \times \Psi_{j,x} - \Psi_{j,x-k} \Psi_{j,x-k} \Psi_{j,x-k}. $$

If $k \in 6\mathbb{Z}_{>0}$, then $m^{(i)}_{k,x} n^{(i)}_{k,x} = d^{(i)}_{k,x}$ by Equation (16). Using Corollary \[14\] one can modify the proof of Theorem 20 to obtain a tensor product factorization:

$$L(m_{k,x}^{(i)} n_{k,x}^{(i)}) \cong L(m_{k,x}^{(i)}) \otimes L(n_{k,x}^{(i)}) \quad \text{if either } k \notin \frac{1}{2} \mathbb{Z} \text{ or } k \in 6\mathbb{Z}_{>0}.$$

**Example 22.** Let $g = s_3$ and $i = 1$. Suppose $k \in \mathbb{C} \setminus \frac{1}{2} \mathbb{Z}$. The KR module $W_{1,0}^{(1)}$ is a vector representation of $Y_{k}(s_3)$ on $\mathbb{C}^3$. From Theorem 10 we get:

$$\chi_q(W_{1,0}^{(1)}) = \Psi_{1,1} \frac{1}{\Psi_{1,0}} (1 + A_{1,0}^{-1} A_{1,0}^{-1} A_{1,0}^{-1} A_{1,0}^{-1}) = \Psi_{1,1} \frac{1}{\Psi_{1,0}} + \Psi_{1,1} \frac{1}{\Psi_{1,0}} + \Psi_{2,0} \frac{1}{\Psi_{2,0}} + \Psi_{2,0} \frac{1}{\Psi_{2,0}}.$$

$$[W_{1,0}^{(1)}] = L\left(\frac{\Psi_{1,1}}{\Psi_{1,0}}\right) + L\left(\frac{\Psi_{1,1}}{\Psi_{1,0}}\right) + L\left(\frac{\Psi_{2,0}}{\Psi_{2,0}}\right) + L\left(\frac{\Psi_{2,0}}{\Psi_{2,0}}\right).$$

$$\mathcal{M}_{k,x}^{(1)} L\left(\frac{\Psi_{1,x}}{\Psi_{1,y}}\right) = L\left(\frac{\Psi_{1,x+1}}{\Psi_{1,y}}\right) L\left(\frac{\Psi_{2,x+1}}{\Psi_{2,y}}\right) + L\left(\frac{\Psi_{1,x+1}}{\Psi_{1,y}}\right) L\left(\frac{\Psi_{2,x+1}}{\Psi_{2,y}}\right).$$
One can show that as $sl_3$-module $\mathcal{M}_a^{(1)}$ is irreducible of highest weight $\varpi_1 + k\varpi_2$.

**Appendix A. Three-term relations for quantum affine algebras**

Assume $h \notin \mathbb{Q}$, so that $q := e^{-\pi h} \in \mathbb{C}^\times$ is not a root of unity. For $x \in \mathbb{C}$ and $i,j \in I$, set $q^x := e^{-\pi hx}$, $q_{ij} := q^{d_{ij}}$ and $q_i := q^{d_i}$. We produce three-term relations in category $\mathcal{O}$ of representations of the quantum affine algebra $U_q(\hat{\mathfrak{g}})$ in [30, Definition 3.3]. Such a category appeared first in [22].

Let $x_{i,r}^\pm, \varphi_{ij}^\pm$, for $(i,r,n) \in I \times \mathbb{Z} \times \mathbb{Z}_{\geq 0}$, be the Drinfeld generators of $U_q(\hat{\mathfrak{g}})$. The $\varphi_{ij}^\pm$ mutually commute and $\varphi_{1,0}\varphi_{1,0}^{-1} = 1$, whose spectral decomposition on a module defines the notion of $\ell$-weight.

As in [30, Definition 3.5], let $\mathcal{R}$ be the set of $I$-tuples $(e_i(u))_{i \in I}$ of rational functions of $u$ such that each $e_i(u)$ is regular at $0, \infty$ and $e_i(0)e_i(\infty) = 1$. Equivalently, each $e_i(u)$ is a finite product of rational functions of the form $\frac{\sum x_i C_i}{\sum u^{n_i}}$ for $c, a \in \mathbb{C}^\times$.

We take Taylor expansions around $0, \infty$ to obtain formal power series in $u^{\pm 1}$:

$$\sum_{n \geq 0} e_{i,n}u^n = e_i(u) = \sum_{n \geq 0} e_{i,-n}u^{-n}.$$ 

View $\mathbb{C}[[u]]^I$ as a monoid by component-wise multiplication. For $(i,a) \in I \times \mathbb{C}^\times$ we define an invertible element $\Phi_{i,a}$ of this monoid by

$$(\Phi_{i,a})_j(u) = 1 \text{ if } j \neq i, \quad (\Phi_{i,a})_i(u) = a - u^{-1}. $$

This is a scalar multiple of the prefundamental weight $\Psi_{i,a}$ in [23, Definition 3.7], and equals to the variable $X_{i,a-1}$ in [9, §4.1]. Now $\mathcal{R}$ is characterized as the subgroup of $\mathbb{C}[[u]]^I$ generated by the ratios $\frac{\Phi_{i,a}}{\Phi_{j,b}}$ for $(i,a,c) \in I \times \mathbb{C}^\times \times \mathbb{C}^\times$.

By [30, Theorem 3.6], $\mathcal{R}$ is in bijection with the isomorphism classes of irreducible modules in category $\mathcal{O}$. To $d \in \mathcal{R}$ is attached an irreducible $U_q(\hat{\mathfrak{g}})$-module, denoted by $L(d)$, which is generated by a vector $\omega$ subject to relations:

$$x_{i,r}^\pm, \omega = 0, \quad \varphi_{ij}^\pm, \omega = d_{ij}^\pm \omega \text{ for } (i,r,n) \in I \times \mathbb{Z} \times \mathbb{Z}_{\geq 0}.$$ 

Fix $(i,a,c) \in I \times \mathbb{C}^\times \times \mathbb{C}^\times$. There is a series of finite-dimensional irreducible modules $L(\frac{\Phi_{i,a}}{\Phi_{j,b}})$ for $k \in \mathbb{Z}_{\geq 0}$, called Kirillov–Reshetikhin modules. It is explained in [30, Appendix A] that this series admits analytic continuation: as $k$ tends to infinity by replacing $q_k$ with $c$ one obtains an asymptotic module denoted by $\mathcal{L}(\frac{\Phi_{i,a}}{\Phi_{j,b}})$.

**Theorem 23.** Let $(i,a,c,k) \in I \times \mathbb{C}^\times \times \mathbb{C}^\times \times \mathbb{C}$ such that $q_k \notin q^{\pm 1}$. In the Grothendieck ring of category $\mathcal{O}$ we have

$$[L(\frac{\Phi_{i,a}}{\Phi_{j,b}} \prod_{j,c,j < 0} \frac{\Phi_{j,a,q_{ij}}}{\Phi_{j,a,q_{ij}^k}})][\mathcal{L}(\frac{\Phi_{i,a}}{\Phi_{i,c}})] =$$

$$\mathcal{L}(\frac{\Phi_{i,a}}{\Phi_{i,c}}) \prod_{j,c,j < 0} \mathcal{L}(\frac{\Phi_{j,a,q_{ij}}}{\Phi_{j,a,q_{ij}^k}}) + [\mathcal{L}(\frac{\Phi_{i,a,q_{ij}^{-1}}}{\Phi_{i,c}})] \prod_{j,c,j < 0} \mathcal{L}(\frac{\Phi_{j,a,q_{ij}^{-1}}}{\Phi_{j,a,q_{ij}^{-1}^k}}).$$

One may view $\mathcal{O}$ as a subcategory of the category $\mathcal{O}$ of modules over the upper Borel subalgebra introduced by Hernandez–Jimbo [23, Definition 3.8]. Category $\mathcal{O}$ admits irreducible modules of highest $\ell$-weights $\Phi_{i,a}$ for all $(i,a) \in I \times \mathbb{C}^\times$. Equation (26) becomes [24, Eq. (6.13)] and [9, Eq. (1.3)] when one removes all the $\Phi_{i,c}$ and $\Phi_{j,a,q_{ij}^k}$ in the denominators and replaces $\mathcal{L}$ by $L$.

Similar three-term relations have been established for prefundamental modules over quantum toroidal $\mathfrak{gl}_1$ [10, Eq. (4.24)] and for asymptotic modules over quantum affine superalgebras of type $A$ [37, Eq. (5.30)].
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