Jagged-time-step technique improving convergence order of Fernandez’s Explicit Robin-Neumann scheme for the coupling of incompressible fluid with thin-walled structure

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Abstract

Inspired by Rybak’s multiple-time-step technique, jagged-time-step technique is proposed and applied to Fernandez’s Explicit Robin-Neumann scheme. For some instances, numerical experiments demonstrate higher convergence orders and accuracy with lower computation cost as time and space get refined.

*Notes: the work described in this article was done more than two years ago. This article is being written and extended. More numerical results, including but not limited to those at refinement rate = 5, are to present. The ideas described might be applicable to other algorithms. On the other hand, for easier implementation, the errors were computed in an approximate way at that time, which is to correct.

1 Introduction

For the coupling of incompressible fluid with thin-walled structure, [2] proposes Explicit Robin-Neumann scheme. The scheme with first-order extrapolation yields unconditional stability and optimal accuracy of first-order in time.

Rybak [4] developes a multiple-time-step technique and applies it to a decoupled scheme for coupled free flow and porous medium systems. The whole time interval is partitioned into some fine grids. Multiple fine grids constitute a coarse grid. Within each coarse time grid, the free flow solutions are computed at each fine time steps using information from the porous medium at the beginning of current coarse time grid. When it reaches the end of current coarse time grid, the porous medium solutions are computed using information from the free flow solutions. The technique improves efficiency of computation, preserves orders of convergence and proves to be stable. Compared with algorithms that do not adopt the technique (namely the monolithic approach and decoupled scheme with single time step), the only disadvantage is it is a bit less accurate.

It is thus meaningful to design a technique that improves both efficiency and accuracy. This work is devoted to investigation of such a technique and its application to decoupled algorithms, such as Fernandez’s Explicit Robin-Neumann scheme for the coupling of incompressible fluid with thin-walled structure. For convenience, the technique is named jagged-time-step technique and described as follows.

The whole time interval is partitioned into some fine grids. A coarse grid consists of 10 fine grids. Both the fluid and structure are computed for multiple steps (the number of such steps are not necessarily equal to 10 ) within each coarse grid using latest information from each other. Let \( N_f \) denote the number of steps that the fluid is solved within each coarse grid, while \( N_s \) for the structure.

The original decoupled algorithm, Fernandez’s Explicit Robin-Neumann scheme, runs at all fine grids. Equivalently, both the fluid and structure are computed 10 steps during each coarse
grid. Therefore, taking $N_f = 10, N_s = 10$ and applying the jagged-time-step technique to the algorithm does not make any change.

Because the fluid domain is of one dimension higher than that of the structure (in this work, for the simplified problem considered, the fluid domain is two-dimensional, while the structure is one-dimensional), it takes much more cost to compute a step of fluid than that of structure. Hence, for sake of efficiency, there should be constraints $N_f < 10$, and $N_f + N_s \leq 20$.

In what follows, the simplified problem and Fernandez’s Explicit Robin-Neumann scheme from [2] are cited. The jagged-time-step technique is applied to the scheme. Numerical experiments, conclusions and possible extensions follows.

2 The simplified problem and Fernandez’s Explicit Robin-Neumann scheme

2.1 The simplified problem

Consider the simplified problem studied in [2] where the fluid is governed by the Stokes equations in a $d$-dimensional ($d = 2, 3$) domain $\Omega$ and the structure is assumed to be a linear thin-solid defined on a $(d - 1)$-manifold $\Sigma$, with $\partial \Omega = \Sigma \cup \Gamma^d \cup \Gamma^n$. The coupled simplified problem reads: find the fluid velocity $u : \Omega \times \mathbb{R}^+ \to \mathbb{R}^d$, the fluid pressure $p : \Omega \times \mathbb{R}^+ \to \mathbb{R}$, the structuredisplacement $d : \Sigma \times \mathbb{R}^+ \to \mathbb{R}^d$ such that

$$
\begin{align*}
\rho_f \partial_t u - \text{div} \sigma(u, p) &= 0 \quad \text{in} \quad \Omega, \\
\text{div} u &= 0 \quad \text{in} \quad \Omega, \\
u &= 0 \quad \text{on} \quad \Gamma^d, \\
sigma(u, p)n &= f^\Gamma \quad \text{on} \quad \Gamma^n,
\end{align*}
$$

(1)

$$
\begin{align*}
u &= \dot{d} \quad \text{on} \quad \Sigma, \\
\rho_s \epsilon \partial_t \dot{d} + L^e \dot{d} + L^v \dot{d} &= -\sigma(u, p)n \quad \text{on} \quad \Sigma, \\
\dot{d} &= \partial_t d \quad \text{on} \quad \Sigma, \\
d &= 0 \quad \text{on} \quad \partial \Sigma,
\end{align*}
$$

(2)

with initial conditions

$$
u(0) = u^0, \quad d(0) = d^0, \quad \dot{d}(0) = \dot{d}^0,$$

where $\rho_f$ denotes the fluid density, $\rho_s$ the structure density, $\epsilon$ the structure thickness, $\dot{d}$ the structure velocity, $n$ the exterior unit normal vector to $\partial \Omega$, $f^\Gamma$ a given surface force on $\Gamma^n$, and

$$
\sigma(u, p) \equiv -pI + 2\mu \varepsilon(u), \quad \varepsilon(u) \equiv \frac{1}{2}(\nabla u + \nabla u^T),
$$

where $\mu$ denotes the fluid dynamic viscosity. $L^e$ and $L^v$ stand for the elastic and viscous contributions respectively.

2.2 Notations

For all the algorithms mentioned in this work, $\tau$ denotes time step, while $h$ stands for space discretization parameter.

Given arbitrary variable $x$, the notation

$$
x^{n, \star} \equiv \begin{cases} 
0 & \text{if } r = 0, \\
x^{n-1} & \text{if } r = 1, \\
2x^{n-1} - x^{n-2} & \text{if } r = 2,
\end{cases}
$$

(3)
Explicit Robin-Neumann scheme

Fernandez’s Explicit Robin-Neumann scheme [2] is used for interface extrapolations of order $r$. 

2.3 Fernandez’s Explicit Robin-Neumann scheme

Fernandez’s Explicit Robin-Neumann scheme (intrinsic formulation)

\begin{align*}
\rho^n \partial_t u^n - \text{div} \sigma(u^n, p^n) &= 0 & \text{in} & \Omega, \\
\text{div} u^n &= 0 & \text{in} & \Omega, \\
u^n &= 0 & \text{on} & \Gamma^d, \\
\sigma(u^n, p^n) n &= f^\Gamma & \text{on} & \Gamma^u, \\
\sigma(u^n, p^n) n + \frac{\rho^n \epsilon}{\tau} u^n &= \frac{\rho^n \epsilon}{\tau} d^{n-1} - L^c d^* - L^v \dot{d}^* & \text{on} & \Sigma.
\end{align*}

2. Solid step: find $d^n : \Sigma \times \mathbb{R}^+ \rightarrow \mathbb{R}^d$ such that

\begin{align*}
\rho^n \epsilon \partial_t d^n + L^c d + L^v \dot{d} &= -\sigma(u^n, p^n) n & \text{on} & \Sigma, \\
\dot{d} &= \partial_\nu d^n & \text{on} & \Sigma, \\
d^n &= 0 & \text{on} & \partial \Sigma,
\end{align*}

Fernandez’s Explicit Robin-Neumann scheme (time semi-discrete) says that in each time step, first solve the fluid with Robin condition [4] on interface with data of structure from last time step (e.g. $d^{n-1}$) or by certain extrapolation strategies (e.g. $d^*$, see section 3.1 in [2] for details) and then solve the structure with Neumann condition [5], on interface with latest data computed from fluid. It is the Robin-Neumann conditions on interface that guarantee the stability (free of added-mass effect). With finite element discretization in space, involving variational residuals of fluid stresses on the interface, the fully discrete version of the preceding algorithm is detailed as follows (see Algorithm 5 in [2]).

Fernandez Explicit Robin-Neumann scheme (intrinsic formulation)

1. Fluid step: Find $(u^n_h, p^n_h) \in V_h \times Q_h$ such that

\begin{align*}
\rho^f (\partial_t u^n_h, v_h) + a(u^n_h, v_h) + b(p^n_h, v_h) - b(q_h, u^n_h) + s_h(p_h, q_h) \\
+ \rho^\epsilon \tau (u^n_h, v_h)_\Sigma &= \frac{\rho^\epsilon}{\tau} (d^{n-1}_h + \tau \partial_\nu d^n_h, v_h)_\Sigma \\
+ \rho^f (\partial_t u^n_h, L_h v_h)_{\Omega^f} + a(u^n_h, L_h v_h) + b(p^n_h, L_h v_h) + l(v_h)
\end{align*}

for all $(u_h, q_h) \in V_h \times Q_h$ with $v_h|_\Sigma \in W_h$.

2. Solid step: Find $(d^n_h, d^n_h) \in W_h \times W_h$, such that

\begin{align*}
\dot{d}^n_h &= \partial_\nu d^n_h, \\
\rho^\epsilon (\partial_t d^n_h, w_h)_\Sigma + a^c(d^n_h, w_h) + a^v(\dot{d}^n_h, w_h) = -\rho^f (\partial_t u^n_h, L_h w_h) - a(u^n_h, L_h w_h) - b(p^n_h, L_h w_h)
\end{align*}
3 Application of jagged-time-step technique

3.1 New notations
Let the expression solveFluid((u_h^n, p_h^n); (u_h^{n-1}, p_h^{n-1}), \tau_f, (d_h^m, d_h^m), extr = r) denote the procedure solving the fluid part at fluid time step n with known data of fluid from fluid time step n − 1 and data of structure from step m, where (u_h^n, p_h^n) are the unknowns, (u_h^{n-1}, p_h^{n-1}) are known fluid velocity and pressure from fluid time step n − 1, \tau_f is the length of fluid time step, (d_h^m, d_h^m) are the known structure velocity and displacement from structure time step m, extr stands for the order of extrapolation for structure velocity and displacement (see section 3.1 in [2] for details).

Analogously, solveSolid((d_h^m, d_h^m); (d_h^{m-1}, d_h^{m-1}), \tau_s, (u_h^n, p_h^n), extr = r) denotes the procedure solving the structure part at structure time step m with known data of structure from step m − 1 and data of fluid from fluid time step n.

3.2 Explicit Robin-Neumann scheme rewritten
With the above notations, Fernandez’s Explicit Robin-Neumann scheme (intrinsic formulation) can be rewritten as

(Fernandez) Explicit Robin-Neumann scheme (intrinsic formulation)

Given final time T_{final}, time step length \tau_3.
Let N = \frac{T_{final}}{\tau_3}.
For n = 1, 2, ..., N, do
  solveFluid((u_h^n, p_h^n); (u_h^{n-1}, p_h^{n-1}), \tau_3, (d_h^n, d_h^n), extr = r);
  solveSolid((d_h^n, d_h^n); (d_h^{n-1}, d_h^{n-1}), \tau_3, (u_h^n, p_h^n), extr = r);
end for

3.3 Explicit Robin-Neumann scheme with jagged-time-step technique

Algorithm 1 Explicit Robin-Neumann scheme with jagged-time-step technique

Given final time T_{final}, coarse time step length \tau_{coarse} = 10 * \tau_3, number of fluid steps within each coarse time interval N_f, number of structure steps within each coarse time interval N_s, N_f < 10, N_f + N_s ≤ 20.
Let N_{coarse} = \frac{T_{final}}{\tau_{coarse}}, \tau_f = \frac{\tau_{coarse}}{N_f}, \tau_s = \frac{\tau_{coarse}}{N_s}, n_{global} = 1.
For i = 1, 2, ..., N_{coarse}, within the coarse time interval \((i - 1) * \tau_{coarse}, i * \tau_{coarse}\), do
  For (integer m = 1; m ≤ N_s; m + +), do
    For (integer n = n_{global}; n ≤ N_f; n + +), do
      If \((m − 1) * \tau_s < n * \tau_f ≤ m * \tau_s\), do
        solveFluid((u_h^n, p_h^n); (u_h^{n-1}, p_h^{n-1}), \tau_f, (d_h^{m-1}, d_h^{m-1}), extr = r);
        n_{global} = n + 1; // remember past steps
      else do
        break the for loop;
      end if
    end for
    solveSolid((d_h^m, d_h^m); (d_h^{m-1}, d_h^{m-1}), \tau_s, (u_h^{n_{global}-1}, p_h^{n_{global}-1}), extr = r);
  end for
end for

for all w_h \in W_h.
end for
\[ n_{global} = 1; \]
end for

3.4 Some instances of Algorithm 1

For short, let the expression "'F \( N_f \) S \( N_s \)' Algorithm 1" stand for Algorithm 1 with given \( N_f \) and \( N_s \). Figure 1 and 2 describe the procedure of "'F 2 S 3' Algorithm 1" and "'F 3 S 2' Algorithm 1", respectively. Arrows therein point to the time when either the fluid or the structure is solved.

![Figure 1: "F 2 S 3" Algorithm 1](image1)

![Figure 2: "F 3 S 2" Algorithm 1](image2)

3.5 Two special cases of Algorithm 1

The two cases of Algorithm 1 with \( N_f = 1 \) or \( N_s = 1 \) are of special interests because they directly adopt the multiple-time-technique from [4]. They are named Algorithm 1.1 and Algorithm 1.2, respectively.

5
Algorithm 1.1 A special case of Algorithm 1

Set $N_f = 1$ in Algorithm 1

Algorithm 1.2 The other special case of Algorithm 1

Set $N_s = 1$ in Algorithm 1

4 Numerical experiments

4.1 Configuration

The test-case as Section 6.1 in [2] is adopted, except that in [2] set

$$L^v \ddot{d} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$ 

Everything else remains intact. Namely, the fluid is defined on $\Omega = [0, L] \times [0, R]$, where $L = 6, R = 0.5$ (all the quantities are under CGS system), with $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Sigma \cup \Gamma_4$ (see Figure 3).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{Geometrical configuration}
\end{figure}

On $\Gamma_2$ a sinusoidal pressure $P(t) = P_{\text{max}}(1 - \cos(2t\pi/T^*))/2$ is prescribed, with $P_{\text{max}} = 2 \times 10^4$ when $0 \leq t \leq T^*$, $P_{\text{max}} = 0$ when $t > T^*, T^* = 5 \times 10^{-3}$. Zero pressure is imposed on $\Gamma_4$ and a symmetry condition is applied on $\Gamma_1$. Physical parameters for the fluid are

$$\rho_f = 1.0, \quad \mu = 0.035.$$ 

The structure is assumed to be a generalized string defined on $\Sigma$ with the two end points ($x = 0, L$) fixed, which therefore has lower dimension than the fluid. In [2], take

$$d = \begin{pmatrix} 0 \\ d_y \end{pmatrix}, \quad L^c d = \begin{pmatrix} 0 \\ -\lambda_1 \partial_x d_y + \lambda_0 d_y \end{pmatrix},$$

with

$$\lambda_1 = \frac{E \epsilon}{2(1 + \nu)}, \quad \lambda_0 = \frac{E \epsilon}{R^2(1 - \nu^2)}.$$ 

Physical parameters for the structure are

$$E = 0.75 \times 10^6, \quad \epsilon = 0.1, \quad \nu = 0.5, \quad \rho^s = 1.1.$$ 

6
All algorithms mentioned above are implemented using FreeFem++ [3]. The Lagrange $P_1$ finite element is employed for both the fluid and structure, with symmetric pressure stabilization method introduced in [1]. The order of extrapolation is set to 1 ($\text{extr} = r = 1$). All run from time 0 to the final $T_{\text{final}} = 0.015(s)$.

To demonstrate both the $h$-uniformity and the order of convergence in time, the time and space are refined at the same rate. Particularly, in Fernandez’s Explicit Robin-Neumann scheme (intrinsic formulation), set

$$(\tau_3, h) = \left(\frac{5 \times 10^{-4}, 0.1}{2^{\text{rate}}}, \quad \text{rate} = 0, 1, 2, 3, 4.\right)$$

In Algorithm 1, 1.1 and 1.2, set

$$(\tau_{\text{coarse}}, h) = (10 \times \tau_3, h) = \left(\frac{5 \times 10^{-3}, 0.1}{2^{\text{rate}}}, \quad \text{rate} = 0, 1, 2, 3, 4.\right)$$

### 4.2 Numerical results

By comparing solutions of the above scheme to the reference solution generated by a fully implicit scheme with high space-time grid resolution ($\tau = 10^{-6}, h = 3.125 \times 10^{-3}$), relative errors $E_{\text{rate}}$ ($\text{rate} = 0, 1, 2, 3, 4$) in elastic energy-norm (see [2]) and time-convergence orders $O_{\text{rate}}$ ($\text{rate} = 1, 2, 3, 4$) defined as

$$O_{\text{rate}} = \frac{\log E_{\text{rate}}}{\log 2}$$

for structure displacement corresponding to different refinement rates are computed.

Table 1 displays numerical results of Fernandez’s Explicit Robin-Neumann scheme. As predicted by the theoretical analysis in [2], time-convergence order of this scheme approaches 1 as both time and space get refined and is expected to reaches 1 as the refinement continues.

Table 2 reports some instances of Algorithm 1 with $N_f + N_s = 20, N_f < 10$.

Table 3 exhibits some other instances of Algorithm 1 with $N_f + N_s < 20, N_f < 10$.

Tables above demonstrate that as refinement $\text{rate}$ increases, instances of Algorithm 1 reported in Table 2 and Table 3 obtain higher and higher convergence orders. Starting from certain refinement rates, their convergence orders and accuracy become higher than those of Fernandez’s Explicit Robin-Neumann scheme.

There are some other observations. For example, with $N_f$ fixed, increasing $N_s$ generally leads to lower relative errors and higher time-convergence orders. For example, “F 6 S 13” Algorithm 1 works slightly better than “F 6 S 12” Algorithm 1 in accuracy and convergence orders.

Neither Algorithm 1.1 nor Algorithm 1.2 yield satisfactory results. Some instances are not even stable. Among the stable instances, ”F 1 S 20” performs best, but still much worse than Fernandez’s Explicit Robin-Neumann. Table 4 reports results of ”F 1 S 20” Algorithm 1.1”, namely ”F 1 S 20” Algorithm 1”.

| rate | $E_{\text{rate}}$ | $O_{\text{rate}}$ |
|------|-----------------|-----------------|
| 0    | 0.959089        | 0.415238        |
| 1    | 0.719217        | 0.725292        |
| 2    | 0.435036        | 0.847834        |
| 3    | 0.128601        | 0.910399        |

Table 1: Numerical results of Explicit Robin-Neumann scheme ("F 10 S 10” Algorithm 1)
Table 2: Numerical results of some instances of Algorithm 1 ($N_f + N_s = 20, N_f < 10$)

4.3 Graphs of structure displacements

Figures 4, 5 and 6 display structure displacements at time $T_{\text{final}}$ and refinement rate = 4 for all instances presented in Tables 2, 3 and 4 in comparison with those of the reference and Fernandez’s Explicit Robin-Neumann.

5 Conclusions

As refinement rate increases, some instances of Algorithm 1 obtain higher convergence orders and accuracy than the original Explicit Robin-Neumann scheme with lower cost.

6 Discussion and future work

The ideas of jagged-time-step technique might applicable to other algorithms or problems.

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The ideas as well as algorithms described in this work (except those cited from other sources explicit stated), numerical experiments, results and conclusions are completed entirely independently, without any assistance from anyone else.

More than two years ago, Dr. Mingchao CAI found the article [4] and suggested applying directly the multiple-time-technique therein to the coupling of incompressible fluid with thin-walled structure. Dr. Lian ZHANG adopted his ideas. However, this work has nothing to do with their ideas or work.
| rate | $E_{rate}$ | $O_{rate}$ |
|------|-----------|-----------|
| 0    | 1.07127   |           |
| 1    | 1.04302   | 0.0385553 |
| 2    | 0.688734  | 0.598748  |
| 3    | 0.30437   | 1.17812   |
| 4    | 0.121974  | 1.31925   |

| rate | $E_{rate}$ | $O_{rate}$ |
|------|-----------|-----------|
| 0    | 1.0744    |           |
| 1    | 1.0464    | 0.0380968 |
| 2    | 0.68863   | 0.603634  |
| 3    | 0.302521  | 1.18669   |
| 4    | 0.120325  | 1.3301    |

| rate | $E_{rate}$ | $O_{rate}$ |
|------|-----------|-----------|
| 0    | 1.07029   |           |
| 1    | 0.971898  | 0.139125  |
| 2    | 0.593959  | 0.710442  |
| 3    | 0.275873  | 1.10636   |
| 4    | 0.125772  | 1.13319   |

| rate | $E_{rate}$ | $O_{rate}$ |
|------|-----------|-----------|
| 0    | 1.0723    |           |
| 1    | 0.964902  | 0.152254  |
| 2    | 0.575995  | 0.744326  |
| 3    | 0.257087  | 1.1638    |
| 4    | 0.11213   | 1.19708   |

| rate | $E_{rate}$ | $O_{rate}$ |
|------|-----------|-----------|
| 0    | 1.05435   |           |
| 1    | 0.879988  | 0.260798  |
| 2    | 0.504881  | 0.80154   |
| 3    | 0.239873  | 1.0863    |
| 4    | 0.114482  | 1.06715   |

| rate | $E_{rate}$ | $O_{rate}$ |
|------|-----------|-----------|
| 0    | 1.05739   |           |
| 1    | 0.834026  | 0.315878  |
| 2    | 0.482387  | 0.789901  |
| 3    | 0.242821  | 0.99351   |
| 4    | 0.120704  | 1.00521   |

Table 3: Numerical results of some other Algorithm 1 instances ( $N_f + N_s < 20$, $N_f < 10$ )

| rate | $E_{rate}$ | $O_{rate}$ |
|------|-----------|-----------|
| 0    | 1.04684   | -0.066041 |
| 1    | 1.12592   | -0.105063 |
| 2    | 1.07841   | 0.0621985 |
| 3    | 0.639951  | 0.752872  |

Table 4: Numerical results of "F 1 S 20" Algorithm 1.1 (namely "F 1 S 20" Algorithm 1)
Figure 4: Structure displacements of each instance in Table 2, the reference and Fernandez’s Explicit Robin-Neumann at $T_{\text{final}}$
Figure 5: Structure displacements of each instance in Table 3, the reference and Fernandez's Explicit Robin-Neumann at $T_{\text{final}}$. 
Figure 5: Structure displacements of each instance in Table 3, the reference and Fernandez’s Explicit Robin-Neumann at $T_{final}$

Figure 6: Structure displacements of the instance in Table 4, the reference and Fernandez’s Explicit Robin-Neumann at $T_{final}$
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