A COMBINATORIAL PROOF OF THE HOMOLOGY COBORDISM CLASSIFICATION OF LENS SPACES

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Abstract. It follows implicitly from recent work in Heegaard Floer theory that lens spaces are homology cobordant exactly when they are oriented homeomorphic. We provide a new combinatorial proof using the Heegaard Floer d-invariants, which themselves may be defined combinatorially for lens spaces.

INTRODUCTION

An integer homology cobordism between two closed, oriented 3-manifolds \( Y_1 \) and \( Y_2 \) is a compact, oriented 4-manifold \( W \) whose boundary is \( \partial W = Y_1 \cup -Y_2 \) such that the inclusion maps induce isomorphisms \( H_i(Y_1; Z) \cong H_i(W; Z) \cong H_i(Y_2; Z) \) for all homology groups; homology cobordism gives an equivalence relation. There are also corresponding definitions of rational homology cobordisms and spin-c rational homology cobordisms.

The homology cobordism classification of the lens spaces was only recently completed. In 1983, Gilmer and Livingston demonstrated that the lens spaces \( L(p,q) \) for prime \( p \) are homology cobordant iff they are diffeomorphic \([GL83]\). Fintushel and Stern extended this result in 1988 for odd \( p \) \([FS87]\). Nicolaescu proved in 2001 that the Ozsváth-Szabó d-invariant recovers Reidemeister-Franz torsion \([Nic04\text{ Section 5}]\), which, in turn, recovers homeomorphism type for lens spaces by results of Brody and Reidemeister \([Bro60, Rei35]\) (technically, Nicolaescu showed the Ozsváth-Szabó theta divisor recovers the sum of the Casson-Walker invariant and Reidemeister-Franz torsion, but the Casson-Walker invariant of a lens space is the sum of its d-invariants, by a result of Rasmussen \([Ras04\text{ Lemma 2.2}]\), and the theta divisor is the precursor of the d-invariant \([OS]\)). In 2011, Greene showed that 2-bridge links are mutants iff their branched double covers (recall, all lens spaces are branched double covers of 2-bridge links) are homeomorphic iff the covers’ \( \widehat{HF} \) are the same \([Gre13]\), but \( \widehat{HF} \) recovers Reidemeister-Franz torsion by Rustamov \([Rus\text{ Theorem 3.4}]\).

There are many known cobordism invariants, including some from Heegaard Floer homology. Ozsváth and Szabó associated the d-invariants to a manifold and spin-c structure which is invariant under spin-c rational homology cobordism, and the d-invariant function on the torsor of spin-c structures is likewise invariant under rational or integral homology cobordism \([OS03\text{ Theorem 1.2}]\). We provide a combinatorial proof that two lens spaces \( L(p,q_1) \) and \( L(p,q_2) \) share the same d-invariant function precisely when they are oriented homeomorphic. Since the d-invariants are defined combinatorially for lens spaces, this produces a proof of the homology cobordism classification of lens spaces which is entirely combinatorial (modulo the proof that the d-invariants are spin-c homology cobordism invariants; in fact, there is a proof of this invariance for lens spaces which is combinatorial except for its use of Donaldson’s Theorem \([Gre13]\)).

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Theorem 1. Two lens spaces are cobordant by an integral homology cobordism exactly when they are oriented homeomorphic.

We begin with a review of facts about d-invariants and spin-c structures and their behavior under homology cobordism. We also define a type of relative d-invariant $f(s,n)$ which carries all the information we need about the d-invariants. Next, we show that, if $\text{Spin}^c(L(p,q_1))$ and $d(L(p,q_1),\cdot)$ are isomorphic to $\text{Spin}^c(L(p,q_2))$ and $d(L(p,q_2),\cdot)$ in the category of torsors and functions, then $q_1 = q_2$ or $q_1 q_2 \equiv 1 \pmod{p}$. Finally, we derive a more explicit description of the d-invariants modulo $\mathbb{Z}$ in the special case where $p$ is prime.

**Notation**

Throughout this paper, let $[a]_p$ denote a representative of the class in the interval $[0,p)$. Let $a \equiv_p b$ mean $a$ and $b$ are equivalent modulo $p$. Let $a'$ denote the inverse of $a$ (if it exists), so $aa' \equiv_p 1$.

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**D-Invariants and Spin-C Structures**

Heegaard Floer homology assigns several flavors of invariants (including $HF^\infty$ and $HF^+$) to a closed, connected, oriented 3-manifold and a choice of spin-c structure using a Heegaard decomposition of the manifold [OS04b, OS04a]. The generators come with a relative $\mathbb{Z}$-grading. A spin-c cobordism $(W,\mathfrak{s})$ from $(Y_1,\mathfrak{s}|_{Y_1})$ to $(Y_2,\mathfrak{s}|_{Y_2})$ produces a map

$$F^+_{W,\mathfrak{s}} : CF^+(Y_1,\mathfrak{s}|_{Y_1}) \rightarrow CF^+(Y_2,\mathfrak{s}|_{Y_2})$$

which induces a relative grading between generators for the two manifolds:

$$gr(F^+_{W,\mathfrak{s}}(x)) - gr(x) = \frac{c_1(\mathfrak{s})^2 - 2\chi(W) - 3\text{sign}(W)}{4}.$$  

For an appropriate choice of spin-c manifold, including a rational homology sphere with its unique spin-c structure, this grading shift allows a lift of the relative $\mathbb{Z}$-grading to an absolute $\mathbb{Q}$-grading by fixing a canonical grading for $S^3$ with its unique spin-c structure.

Derived from this absolute grading is the correction term or d-invariant $d(Y,\sigma)$, the minimal grading of any non-torsion element in $HF^+(Y,\sigma)$ inherited from $HF^\infty(Y,\sigma)$ [OS03]. It is invariant under spin-c rational homology cobordism (if $W$ is a rational homology cobordism, then the right side of Equation (1) is 0 for both $W$ and $-W$). The d-invariants, as a function on a torsor over $H^2(Y) \cong H_1(Y)$, is also invariant under integral homology cobordism in the following fashion:

**Proposition 2.** If $Y_1$ and $Y_2$ are integrally homology cobordant, then $\text{Spin}^c(Y_1)$ and $d(Y_1,\cdot)$ are isomorphic to $\text{Spin}^c(Y_2)$ and $d(Y_2,\cdot)$ in the category of torsors and functions.

**Proof.** Let $W$ be the 4-manifold cobordism with $\partial W = Y_1 \cup -Y_2$.

$\text{Spin}^c(Y_1)$ is a torsor over $H^2(Y_1) \cong H_1(Y_1)$, and $\text{Spin}^c(W)$ is a torsor over $H^2(W) \cong H_2(W,\partial W)$. The long exact sequence for the pair $(W,\partial W)$ splits:

$$0 \rightarrow H_2(W,\partial W) \xrightarrow{r_1} H_1(Y_1) \oplus H_1(-Y_2) \xrightarrow{r_2} H_1(W) \rightarrow 0.$$
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where \( r_i \) is the restriction map to \( L(p, q)_i \). This short sequence induces isomorphisms

\[
H_1(Y_1) \xleftarrow{r_1^*} H_2(W, \partial W) \xrightarrow{r_2^*} H_1(Y_2)
\]

which in turn induce the required torsor isomorphism

\[
Spin^c(Y_1) \xrightarrow{r_2 r_1^{-1}} Spin^c(Y_2).
\]

There is a \( \mathbb{Z}/2\mathbb{Z} \) conjugation action \( t \mapsto \bar{t} \) on the spin-c structures which fixes the spin structures. The restrictions maps and so also this isomorphism respect it.

Because it is invariant under spin-c homology cobordism, \( d(Y_1, r_1(t)) = d(Y_2, r_2(t)) \), and the functions \( d(y, \cdot) \) are isomorphic. \( \square \)

The lens space \(-L(p, q)\) has a pointed Heegaard diagram \((T^2, \alpha, \beta, z)\) with a single \( \alpha \) curve and \( \beta \) curve and exactly \( p \) intersection points \( \alpha \cap \beta \), one in each of the \( p \) spin-c structures. For example, the Heegaard decomposition of \(-L(5, 2)\) looks like:

We have chosen the orientation on \( L(p, q) \) so that the manifold is \(-p/q\) surgery on the unknot. Choose an identification of \( Spin^c(L(p, q)) \) by labelling the intersection points \( 0, 1, \ldots, p-1 \) from left to right across the bottom of the diagram, beginning with the 0 for the bottom right corner of the domain containing the basepoint \( z \) [OS03, Proposition 4.8]. To see the difference of two spin-c structures \( i - j \in H_1(L(p, q)) \) under this identification, observe the curve \( \gamma \), which is a generator of \( H_1(L(p, q)) \) and connects \( i \) to \( i + q \) along the \( \alpha \) curve and \( i + q \) to \( i \) along the \( \beta \) curve, so we say \( (i + q) - i = [\gamma] \). Any other \( i - j \) gives a multiple of \( [\gamma] \).

There is a combinatorial description of the \( d \)-invariants of a lens space based on the grading shift in Equation (1) derived in [OS03, Proposition 4.8]. Assuming \( 0 < q < p \),

\[
d(L(p, q), i) = \frac{1}{4} - \frac{(2[i]_p + 1 - p - q)^2}{4pq} - d(L(p, q), i).
\]

Derived from this recursive formula is a more direct formula for how the \( d \)-invariants change under the \( \gamma \)-action [LL08, Corollary 5.2]:

\[
(2) \quad d(L(p, q), i + q) - d(L(p, q), i) = \frac{p - 1 - 2[i]_p}{p}.
\]

The spin structures are exactly the integers among the following:

\[
(3) \quad \frac{q - 1}{2} \quad \text{and} \quad \frac{p + q - 1}{2}.
\]

This result may be deduced from Equation (2): The conjugation action which fixes a spin structure \( s \) must identify \( s + n \) and \( s - n \), and \( d(L(p, q), i + q) = d(L(p, q), i - q) \) implies \( \frac{p - 1 - 2i}{p} = \frac{p - 1 - 2(i - q)}{p} \), or \( 2i \equiv_p q - 1 \). For alternative explanations, Cf [Ue09] p. 134] or
Lemma 3. The function $f_{L}$ obeys

$$f(s, 0) = 0$$

$$f(s, n + 1) = f(s, n) + p - 1 - 2[s + nq]_p$$

$$f(s, n) \equiv_p -n^2q$$

If $L(p, q_1)$ and $L(p, q_2)$ are homology cobordant by $W$, and if $f_1$ and $f_2$ are the corresponding functions for some compatible choice of spin structure $s_1$ and $s_2$ which restrict the same spin structure on $W$,

$$f_2(s_2, n) = f_1(s_1, nu)$$

$$q_2 \equiv_p u^2q_1$$

for some unit $u \in \mathbb{Z}/p\mathbb{Z}$.

A cobordism between two lens spaces tells us about the torsor structure defined above.

Proof. The first two equalities follow from Equation (2) and the definitions of $f$ and $s$. The third equality holds because $f(s, n + 1) \equiv_p f(s, n) - (2n + 1)q$ and $f(s, 0) \equiv_p 0$. The fourth follows from Proposition 2 assuming that the spin structures were chosen so that $r_2r_1^{-1}(s_1) = s_2$, and the last equality follows from the third and fourth. \qed

Proof of Theorem 1

We will now prove the main theorem.

Proof of Theorem 1 By Lemma 3 there is a unit $u$ such that

$$q_1 = q \quad \text{and} \quad q_2 \equiv_p u^2q.$$
and
\[ g(m + uq) = g(m) + p - 1 - 2[s_2 + mu]_p. \]

Since the above relations must hold for all \( m \), we can compute \( g(m + uq + q) \) in two ways, as \( g((m + uq) + u) \) or as \( g((m + u) + uq) \). Since the results must be the same, we get
\[ [s_1 + m]_p + [s_2 + (m + q)u]_p = [s_2 + mu]_p + [s_1 + m + uq]_p. \]

Now recall that
\[ [X + Y]_p = \begin{cases} [X]_p + [Y]_p & \text{if } [X]_p < p - [Y]_p, \\ [X]_p + [Y]_p - p & \text{if } [X]_p \geq p - [Y]_p. \end{cases} \]

Equation (4) is therefore equivalent to the condition that
\[ [s_1 + m]_p < p - [uq]_p \iff [s_2 + mu]_p < p - [uq]_p \]
for all \( m \in \mathbb{Z}/p\mathbb{Z} \).

By Lemma 5 below, Condition (5) can only be satisfied for all \( m \in \mathbb{Z}/p\mathbb{Z} \) if either \( u \equiv_p \pm 1 \) or \( uq \equiv_p \pm 1 \). That is, either \( q_2 = q_1 \) or \( q_1 q_2 \equiv_p 1 \).

Note that we did not use any information in the above proof about the explicit forms the \( s_i \) take, merely the fact that there exist \( s_1 \) and \( s_2 \) which are restrictions of some spin structure on \( W \); in particular, the parity of \( p \) is irrelevant.

We now address two technical lemmata required for the proof above.

**Lemma 4.** Let
\[ H : \{0, 1, \cdots, p - 1\} \to \{0, 1, \cdots, p - 1\} \]
be a function such that \( H(i) \equiv_p H(0) + i \). If
\[ H(i) < C \iff i < C \]
where \( 2 \leq C \leq p - 2 \), then
\[ H(i) = i \text{ or } H(i) = C - 1 - i. \]

A few experiments will quickly convince the reader that this lemma should be true. For the sake of completeness, we prove:

**Proof.** Choose \(-p/2 \leq n \leq p/2\). Assume, for the moment, that \( C \leq p/2 \).

If \( n = 1 \), then \( H(0) = 0 \) and \( H(i) = i \).

If \( n = -1 \), then \( H(0) = C - 1 \) and \( H(i) = C - i - 1 \).

For any other \( n \), there will eventually be an \( i < C \) with \( H(i) \geq C \). For example, for \( n \geq 2 \), take
\[ i_0 = \left\lfloor \frac{C - H(0)}{n} \right\rfloor + 1 \]
Note that \( 0 < i_0 < C \) since \( C \geq 2 \), and
\[ 0 < H(0) + ni_0 \leq H(0) + n \left( \frac{C - H(0)}{n} + 1 \right) = C + n \leq p \]
so we may remove the \( \equiv_p \) in the definition of \( H(i_0) \):
\[ H(i_0) = H(0) + n i_0 \geq H(0) + n \left( \frac{C - H(0)}{n} \right) = C, \]
as desired.

Similarly, for \( n \leq -2 \), take
\[ i_0 = \left\lfloor \frac{H(0)}{|n|} \right\rfloor + 1. \]
Now $-p < H(0) + ni_0 < 0$, so

$$H(i_0) = H(0) + ni_0 + p \geq H(0) - |n| \left( \frac{H(0)}{|n|} + 1 \right) + p \geq p - |n| \geq p/2 \geq C$$

It is easy to adjust the above proof to accommodate $C \geq p/2$. The key is that some (at least two and at most $p - 2$) adjacent values of $i$ map to (the same number of) adjacent values of $H(i)$. The following are equivalent:

$$H(i) < C \iff i < C$$

$$0 \leq H(i) \leq C - 1 \iff 0 \leq i \leq C - 1$$

$$p - C \leq H(i) + p - C \leq p - 1 \iff p - C \leq i + p - C \leq p - 1$$

$$p - C \leq H(i - p + C) + p - C \leq p - 1 \iff p - C \leq i \leq p - 1$$

$$H(i + C) - C < p - C \iff i < p - C$$

and $\tilde{H}(i) = H(i + C) - C$ also obeys the rule $\tilde{H}(i) \equiv_p H(0) + in$. □

**Lemma 5.** Let

$$f(m) = [x + my]_p$$

$$F(m) = [X + mY]_p.$$

with $y$ and $Y$ units modulo $p$. If

$$f(m) < C \iff F(m) < C$$

for some $2 \leq C \leq p - 2$, then

$$Y = \pm y.$$

**Proof.** Rescale $m$ by precomposing $f$ and $F$ with

$$m(i) = (i - x)y'.$$

Then

$$h(i) := f(m(i)) = [i]_p$$

and

$$H(i) := F(m(i)) = [(X - xy'Y) + i(y'Y)]_p.$$

The lemma statement is equivalent to

$$h(i) < C \iff H(i) < C,$$

which is equivalent to

$$H(i) < C \iff i < C.$$

Note that $H(i) \equiv_p H(0) + in$, and apply Lemma 4.

If $H(i) = i$, then $y'Y = 1$, or $Y = y$, and $X - xy'Y = 0$, or $X = x$.

If $H(i) = C - 1 - i$, then $H(0) = C - 1 \equiv_p X - xy'Y$ and $H(C - 1) = 0 \equiv_p X - xy'Y + (C - 1)y'Y \equiv_p C - 1 + (C - 1)y'Y$, so $Y \equiv_p -y$ and $X \equiv_p -x + C - 1$. □
If $p$ is prime

In the special case where $p$ is a prime, we have a more precise description of the $d$-invariants modulo $\mathbb{Z}$. Consider the reduction of $f$ modulo $p$, $\overline{f}(s,\cdot) : \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$. We denote by

$$\overline{\mathcal{S}}(L(p,q)) \subseteq \mathbb{Z}/p\mathbb{Z}$$

the image of $\overline{f}$.

**Theorem 6.** Let $p$ be a prime number and $q$ coprime to $p$.

(a) If $q$ is a quadratic residue modulo $p$, then

$$\overline{\mathcal{S}}(L(p,q)) = \{a \in \mathbb{Z}/p\mathbb{Z} \mid -a \text{ is a square in } \mathbb{Z}/p\mathbb{Z}\}.$$  

(b) If $q$ is a quadratic non-residue modulo $p$, then

$$\overline{\mathcal{S}}(L(p,q)) = \{a \in \mathbb{Z}/p\mathbb{Z} \mid -a \text{ is not a square in } \mathbb{Z}/p\mathbb{Z}\} \cup \{0\}.$$  

In the residue case, a more explicit description of the $d$-invariants is possible:

**Corollary 7.** Let $p$ be an odd prime number and $q$ a residue coprime to $p$.

(a) There is only one $n$ such that $\overline{f}(s,n) = 0$, namely, $n = 0$.

(b) For every $a \in \overline{\mathcal{S}}(L(p,q)) \setminus \{0\}$, there are exactly two $n$ such that $\overline{f}(s,n) = a$.

(c) $\overline{\mathcal{S}}(L(p,q))$ contains exactly $(p + 1)/2$ elements.

**Proof of Theorem 6.** Since $f(s,n) \equiv p - n^2q$,

$$\overline{\mathcal{S}}(L(p,q)) = \{a \in \mathbb{Z}/p\mathbb{Z} \mid a \text{ satisfies } a \equiv -n^2q \text{ for some } n\}.$$  

If $a = 0$, then $n = 0$.

Let $\left( \frac{m}{p} \right)$ denote the Legendre symbol of $m$ and $p$, defined by

$$\left( \frac{m}{p} \right) := \begin{cases} 1 & \text{if } m \text{ is a quadratic residue modulo } p, \\ -1 & \text{if } m \text{ is a quadratic non-residue modulo } p, \\ 0 & \text{if } m \text{ is zero modulo } p. \end{cases}$$

Assume $a \neq 0$. Then the condition $a \equiv -n^2q$ can be written as $-aq' \equiv n^2q$, or

$$\left( \frac{-aq'}{p} \right) = 1.$$  

Since the Legendre symbol is multiplicative in the first argument, we can write the condition as

$$\left( \frac{-a}{p} \right) \left( \frac{q'}{p} \right) = 1,$$

and, multiplying both sides by $\left( \frac{q}{p} \right)$, we get

$$\left( \frac{-a}{p} \right) = \left( \frac{q}{p} \right),$$

where we have used that $\left( \frac{q'}{p} \right) \left( \frac{q}{p} \right) = \left( \frac{aq'}{p} \right) = \left( \frac{1}{p} \right) = 1$. We can thus write $\overline{\mathcal{S}}(L(p,q))$ as

$$\overline{\mathcal{S}}(L(p,q)) = \{a \in \mathbb{Z}/p\mathbb{Z} \mid a = 0 \text{ or } \left( \frac{-a}{p} \right) = \left( \frac{q}{p} \right) \}.$$  

\[\square\]
Proof of Corollary 7. If $p$ is prime, $(\mathbb{Z}/p\mathbb{Z})[x]$ is a unique factorization domain. If $p \neq 2$, this means every equation $n^2 \equiv_p -aq'$ with $a \neq 0$ has exactly two solutions. Part (c) follows because the total number of $d$-invariants, counted with multiplicities, is equal to $p$. \hfill $\Box$

Note that $\overline{S}(L(2, 1)) = \mathbb{Z}/2\mathbb{Z}$, so (b) and (c) are false for $p = 2$.

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