Comparing the Parameter Complexity of Hypernetworks and the Embedding-Based Alternative

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Abstract

In the context of learning to map an input $I$ to a function $h_I : X \rightarrow \mathbb{R}$, we compare two alternative methods: (i) an embedding-based method, which learns a fixed function in which $I$ is encoded as a conditioning signal $e(I)$ and the learned function takes the form $h_I(x) = q(x,e(I))$, and (ii) hypernetworks, in which the weights $\theta_I$ of the function $h_I(x) = g(x; \theta_I)$ are given by a hypernetwork $f$ as $\theta_I = f(I)$.

We extend the theory of [8] and provide a lower bound on the complexity of neural networks as function approximators, i.e., the number of trainable parameters. This extension, eliminates the requirements for the approximation method to be robust. Our results are then used to compare the complexities of $q$ and $g$, showing that under certain conditions and when letting the functions $e$ and $f$ be as large as we wish, $g$ can be smaller than $q$ by orders of magnitude. In addition, we show that for typical assumptions on the function to be approximated, the overall number of trainable parameters in a hypernetwork is smaller by orders of magnitude than the number of trainable parameters of a standard neural network and an embedding method.

1 Introduction

Conditioning often refers to the existence of an additional input signal. For example, in an autoregressive model, where the input is the current hidden state or the output of the previous time step, a conditioning signal can drive the process in a desired direction. For example, in conditioned WaveNets [39] used for text to speech, the autoregressive signal is concatenated to the signal arising from the language features.

While conditioning is a straightforward form of adding two inputs, other forms are less intuitive. For example, in Style GANs [19], conditioning takes place by changing the weights of the normalization layers according to the desired style.

When the two signals are nested, i.e., one may want to apply multiple inputs $x$ in the context of the same conditioning input $I$, a natural solution is to encode the latter by some network $e$ and to concatenate it to the input, when performing inference $q(x, e(I))$. Here $q$ is the primary network.

An alternative solution, which is commonly refereed to as a hypernetwork is to use a primary network $g$ whose weights are not directly learned. Instead $g$ has a fixed architecture, and another network $f$ generates its weights based on the conditioning input as $\theta_I = f(I)$. The network $g$, with the weights $\theta_I$ can then be applied to any input $x$. 

In this paper, we consider the two alternatives: the one based on the embedding function \( e \) and the hypernetwork one. Since networks often have millions of weights while embedding vectors have a dimension that is seldom larger than a few thousands, it may seem that \( f \) is required to be much more complex than \( e \). However, in hypernetworks, often the layer before the output one is a bottleneck. More importantly, it is often the case that the function \( g \) can be small, and it is the adaptive nature (where \( g \) changes according to \( I \)) that enables the entire hypernetwork (\( f \) and \( g \) together) to be expressive.

The existence of multiple alternatives, calls for an analysis of the differences between them. The case of hypernetworks is especially interesting, since a network that learns the weights of another network can be thought of as a way of obtaining abstraction, and since they are repeatedly shown to lead to state of the art results across multiple application domains.

In this paper we theoretically study the expressiveness of hypernetworks, in comparison to the embedding method. The central contributions in this paper are as follows: (1) Thm. 2 extends the theory of [8] and provides a lower bound on the number of trainable parameters of a neural networks, when approximating smooth functions. In contrast to previous work, our result does not require that the approximation method is robust. (2) In Sec. 5.1 we compare the complexities of the primary functions under the two methods (\( q \) and \( g \)) and show that for a large enough embedding method, the hypernetwork primary network \( g \) can be smaller than \( q \) by orders of magnitude. (3) In Sec. 5.2 we show that under common assumptions on the function to be approximated, the overall number of trainable parameters in a hypernetwork is much smaller than the number of trainable parameters of a standard neural network, even when an embedding is used.

1.1 Related Work

**Hypernetworks** The so-called dynamic layers, in which the convolution weights are determined by a separate neural network based on the input, appeared as a way to adapt the lower layers to the motion or illumination of the image input [20, 34]. This was extended for multiple layers by [18], for video frame and stereo view prediction.

The term hypernetwork originated from the work of [12], where a RNN was used to generate the weights of the RNN used to perform the actual task, called the primary network. The Bayesian formulation of [21] introduces variational inference that involves both the parameter generating network and a primary network.

Hypernetworks are especially suited for meta-learning. [3] use it for few-shot learning tasks, making use of the knowledge sharing ability of the weights generating network. Knowledge sharing in hypernetworks was recently used for continual learning by [41].

Predicting the weights instead of performing backpropagation can lead to efficient neural architecture search, as was demonstrated by [5, 42]. An application for parameter selection was presented in [24], which applies hypernetworks for hyperparameters selection.

Hypernetworks are currently state of the art in a diverse set of fields, ranging from the fundamental computer vision problem of 3D reconstruction from a single image [23] to various molecule prediction tasks in bioinformatics, based on graph hypernetworks [32].

A recent paper studies the role of multiplicative interaction within a unifying framework to describe a range of classical and modern neural network architectural motifs, such as gating, attention layers, hypernetworks, and dynamic convolutions amongst others [17]. It is shown that standard neural networks are a strict subset of neural networks with multiplicative interactions, however, they do not provide any theoretical quantitative guarantees that support the claim that interactions are beneficial.

Despite their success and increasing prominence, little theoretical work was done in order to better understand hypernetworks and their behavior. [6] showed that applying standard initializations on a hypernetwork produces sub-optimal initialization of the primary network. A principled technique for weight initialization in hypernetworks is then developed.

**Approximation theory** In the last few decades, there were various attempts to understand the capabilities of neural networks as function approximators. Several publications, such as [7, 16] show that shallow neural networks serve as universal approximators of smooth target functions. Multiple extensions of these results [31, 25, 13, 22, 50] quantify tradeoffs between the number of trainable
parameters, width and depth of the neural networks as universal approximators. In particular, [31] suggested upper bounds on the size of the neural networks of order $O(\epsilon^{-n/r})$, where $n$ is the input dimension, $r$ is the order of smoothness of the target functions and $\epsilon > 0$ is the approximation accuracy.

In another contribution, [32] prove a lower bound on the complexity of the neural network that matches the upper bound $\Omega(\epsilon^{-n/r})$. However, their analysis assumes that the approximation is robust in some sense (see Sec. 4 for details). In Sec. 4 we show that for certain activation functions and under reasonable conditions, there exists a robust approximator and, therefore, the lower bound of the complexity is $\Omega(\epsilon^{-n/r})$. In [27, 29, 28] a similar lower bound is shown. However, this result only applies for shallow neural networks.

In an attempt to understand the benefits of locality in convolutional neural networks, [30] shows that when the target function is a hierarchical function, it can be approximated by a hierarchic neural network of smaller complexity, compared to the worst-case complexity for approximating arbitrary functions. In our Thm. 1, we take a similar approach. We show that under standard assumptions in meta-learning, the overall number of trainable parameters in a hypernetwork necessary to approximate the target function is smaller by orders of magnitude, compared to approximating arbitrary functions with neural networks and the embedding method in particular.

**Identifiability** Neural network identifiability is the property in which the input-output map realized by a feed-forward neural network with respect to a given activation function uniquely specifies the network architecture, weights, and biases of the neural network up to neural network isomorphisms (i.e., re-ordering the neurons in the hidden layers). Several publications investigate this property. For instance, [2, 37] show that shallow neural networks are identifiable. The main result of [10] considers feed-forward neural networks with the tanh activation functions are shows that these are identifiable when the networks satisfy certain “genericity assumptions”. In [40] it is shown that for a wide class of activation functions, one can find an arbitrarily close function that induces identifiability (see Thm. 1). Throughout the proofs of our Thm. 2, we make use of this last result in order to construct a robust approximator for the target functions of interest.

### 2 Problem Setup

In various meta-learning settings, we have an unknown target function $y: \mathcal{X} \times \mathcal{I} \to \mathbb{R}$ that we would like to model. Here, $x \in \mathcal{X}$ and $I \in \mathcal{I}$ are two different inputs of $y$. The two inputs have different roles, as the input $I$ is “task” specific and $x$ is independent of the task. Typically, the modeling of $y$ is done in the following manner:

$$H(x, I) = G(x, E(I)) \approx y(x, I)$$

(1)

where $E$ is an embedding function and $G$ is a predictor on top of it. The distinction between different embedding methods stems from the architectural relationship between $E$ and $G$. In this work, we compare two task embedding methods: (i) neural embedding methods and (ii) hypernetworks.

**A neural embedding method** is a network of the form:

$$h(x, I; \theta_e, \theta_q) = q(x, e(I; \theta_e); \theta_q)$$

(2)

It consists of a composition of neural networks $q$ and $e$ parameterized with real-valued vectors $\theta_q \in \Theta_q$ and $\theta_e \in \Theta_e$ (resp.). The term $e(I; \theta_e)$ serves as an embedding of $I$, see Fig. 1 for an illustration. For given two families $q := \{q(x, z; \theta_q) \mid \theta_q \in \Theta_q\}$ and $e := \{e(I; \theta_e) \mid \theta_e \in \Theta_e\}$ of functions, we denote by $E_{e,q} := \{q(x, e(I; \theta_e); \theta_q) \mid \theta_q \in \Theta_q, \theta_e \in \Theta_e\}$ the neural embedding method that is formed by them.

A special case of neural embedding methods is the family of the conditional neural processes models [11]. In such processes, $\mathcal{I}$ consists of a the set of $d$ images $I = \{I_i\}_{i=1}^d \in \mathcal{I}$, and the embedding is computed as an average of the embeddings over the batch, $e(I; \theta_e) := \frac{1}{d} \sum_{i=1}^d e(I_i; \theta_e)$.

**A hypernetwork** $h(x, I) = g(x; f(x; \theta_f))$ is a pair of collaborating neural networks, $f: \mathcal{I} \to \Theta_q$ and $g: \mathcal{X} \to \mathbb{R}$, such that for an input $I$, $f$ produces the weights of $g$, i.e.,

$$\theta_f = f(I; \theta_f)$$

(3a)

$$h(x, I; \theta_f) = g(x; \theta_f)$$

(3b)
Figure 1: An embedding method: the embedding function $e$ takes the input $I$ and returns an embedding $e(I)$. The network $q$ takes the concatenation of $x$ and $e(I)$ as input and returns the output.

Figure 2: A hypernetwork: the network $f$ takes the input $I$ and produces the weights $\theta_f := f(I)$ of the primary $g$. The primary takes the input $x$ and returns the output of the model. A hypernetwork typically takes a hourglass form, where the weights of $g$ are obtained as a linear function (or a shallow MLP) of a low-dimension representation of $I$.

where $\theta_f \in \Theta_f$ consists of the weights of $f$, see Fig. 2 for an illustration. The function $f(I; \theta_f)$ takes a conditioning input $I$ and returns the parameters $\theta_f \in \Theta_f$ for $g$. The network $g$ takes an input $x$ and returns an output $g(x; \theta_f)$ that depends on both $x$ and the task specific input $I$. In practice, $f$ is typically a large neural network and $g$ is a small neural network.

The entire prediction process for hypernetworks is denoted by $h(x, I; \theta_f)$, and the set of functions $h(x, I; \theta_f)$ that are formed by two families $f := \{f(I; \theta_f) | \theta_f \in \Theta_f\}$ and $g := \{g(x; \theta_g) | \theta_g \in \Theta_g\}$ as a hypernetwork is denoted by $\mathcal{H}_{f,g} := \{g(x; f(I; \theta_f)) | \theta_f \in \Theta_f\}$.

### 2.1 Terminology and notations

We start with a review of some necessary notations.

Throughout the paper, we assume that $X = [-1, 1]^{m_1}$ and $I = [-1, 1]^{m_2}$ and denote, $m := m_1 + m_2$.

For a closed set $X \subset \mathbb{R}^n$, we denote by $C^r(X)$ the linear space of all $r$-continuously differentiable functions $h : X \rightarrow \mathbb{R}$ on $X$ equipped with the supremum norm $\|h\|_\infty := \max_{x \in X} \|h(x)\|_1$.

Throughout the paper, we denote parametric classes of functions by calligraphic lower letters, e.g., $f = \{f(\cdot; \theta_f) | \theta_f \in \Theta_f\}$. A specific function from the class is denoted by the non-calligraphic lower case version of the letter $f$ or $f(x; \theta_f)$.

Frequently, we will use the notation $f(\cdot; \theta_f)$ to specify a function $f$ and its parameters $\theta_f$ without specifying a concrete input of this function. The set $\Theta_f$ is closed a subset of $\mathbb{R}^{N_f}$ and consists of the various parameterizations of members of $f$. Here, $N_f$ is the number of parameters in $f$ and is referred as the complexity of $f$. As part of the definition of parametric classes, we assume that $\lim_{\theta \rightarrow \theta_0} \|f(\cdot; \theta) - f(\cdot; \theta_0)\|_\infty = 0$. 


In this paper, we will focus specifically on classes of neural networks. A class of neural networks \( f \) is a set of functions of the form:

\[
f(x; [W, b]) := W^k \cdot \sigma(W^{k-1} \ldots \sigma(W^1 x + b^1) + b^{k-1})
\]

(4)

with weights \( W^i \in \mathbb{R}^{h_i+1 \times h_i} \) and biases \( b^i \in \mathbb{R}^{h_i+1} \), for some \( h_i \in \mathbb{N} \). In addition, \( W = [W^1, \ldots, W^k] \) and \( b = [b^1, \ldots, b^{k-1}] \) accumulate the weights and biases of the neural network. For simplicity, when considering neural networks, we denote \( \theta := [W, b] \) to aggregate the parameters of \( f \). The function \( \sigma : \mathbb{R} \rightarrow \mathbb{R} \) is a non-linear activation function, typically ReLU(\( x \)) := max(0, \( x \)), logistic function \( \frac{1}{1+exp(-x)} \) or the hyperbolic tangent \( \tanh(x) \).

We define the spectral complexity of a neural network \( f := f(\cdot; [W, b]) \) as follows:

\[
C(f) := C([W, b]) := L^{k-1} \prod_{i=1}^{k} ||W^i||_1
\]

(5)

where \( ||W||_1 := \sup_{x \neq 0}(||W x||_1/||x||_1) \) is the induced \( L_1 \) norm of the matrix \( W \in \mathbb{R}^{m \times n} \) and \( L \) is the Lipschitz constant of \( \sigma \). In general, \( C(f) \) upper bounds the Lipschitz constant of \( f \) (see Lem. 3 in the supplementary material).

We will make use of the notation \( \mathcal{W}_{r,n} \) to refer to the set of functions \( h : [-1,1]^n \rightarrow \mathbb{R} \) with continuous partial derivatives of orders up to \( r \), such that, the Sobolev norm is bounded, \( ||h||_s := ||h||_{\infty} + \sum_{1 \leq |k| \leq r} ||D^k h||_{\infty} \leq 1 \), where \( D^k \) denotes the partial derivative indicated by the multi–integer \( k \geq 1 \), and \( |k|_1 \) is the sum of the components of \( k \).

We define \( \mathcal{P}_{r,w,c}^{k_1,k_2} \) to be the set of functions \( h : \mathbb{R}^{k_1} \rightarrow \mathbb{R}^{k_2} \) of the form \( h(x) = M \cdot P(x) \), where \( P : \mathbb{R}^{k_1} \rightarrow \mathbb{R}^w \) and \( W \in \mathbb{R}^{k_2 \times w} \) is some matrix of bounded induced \( L_1 \) norm \( ||W||_1 \leq c \). Each output coordinate \( P_i \) of \( P \) is a member of \( \mathcal{W}_{r,k_i} \). The linear transformation on top of these functions serves to enable blowing up the dimension of the produced output. However, the “effective” dimensionality of the output is bounded by \( w \). For simplicity, when \( k_1 \) and \( k_2 \) are clear from context, we simply denote \( \mathcal{P}_{r,w,c} := \mathcal{P}_{r,w,c}^{k_1,k_2} \). We can think of the functions in this set as linear projections of a set of features of size \( w \).

### 2.2 Identifiability

We recall the terminology of identifiability from \([10,40]\).

**Definition 1** (Identifiability). A class \( \mathcal{F} = \{ f(\cdot; \theta_f) : A \rightarrow B \mid \theta_f \in \Theta_f \} \) is identifiable up to (invariance) continuous functions \( \Pi = \{ \pi : \Theta_f \rightarrow \Theta_f \} \), if

\[
f(\cdot; \theta_f) \equiv_A f(\cdot; \theta'_{f}) \iff \exists \pi \in \Pi \ s.t \ \theta'_{f} = \pi(\theta_f)
\]

(6)

where the equivalence \( \equiv_A \) is equality for all \( x \in A \).

A special case of identifiability is identifiability up to isomorphisms. Informally, we say that two neural networks are isomorphic, if they share the same architecture and they are equivalent up to permuting the neurons in each layer (excluding the input and output layers).

**Definition 2** (Isomorphism). Let \( \mathcal{F} \) be a class of neural networks. Two neural networks \( f(x; [W, b]) \) and \( f(x; [V, d]) \) of the same class \( \mathcal{F} \) are isomorphic if: there are permutations \( \gamma_i : [h_i] \rightarrow [h_i] \), such that:

1. \( \gamma_i \) is the identity permutation for \( i = 1 \) and \( i = k + 1 \).
2. For all \( i \in [k], j \in [h_{i+1}] \) and \( l \in [h_i] \),

\[
V^i_{j,l} = W^i_{\gamma_{i+1}(j),\gamma_i(l)} \quad \text{and} \quad d^i_j = b^i_{\gamma_{i+1}(j)}
\]

(7)

An isomorphism \( \pi \) is specified by permutation functions \( \gamma_1, \ldots, \gamma_{k+1} \) that satisfy conditions (1), (2) and (3). For a given neural network \( f(x; [W, b]) \) and isomorphism \( \pi \), we denote by \( \pi \circ [W, b] \) the parameters of a neural network produced by the isomorphism \( \pi \).

As noted by \([10,40]\), for a given class of neural networks, \( \mathcal{F} \), there are several ways to construct pairs of non-isomorphic neural networks that are equivalent as functions.
In the first approach, suppose that we have a neural network with depth \( k \geq 2 \), and there exist indices \( i, j_1, j_2 \) with \( 1 \leq i \leq k - 1 \) and \( 1 \leq j_1 < j_2 \leq h_{i+1} \), such that, \( b_{j_1}^i = b_{j_2}^i \) and \( W_{j_1, t}^i = W_{j_2, t}^i \) for all \( t \in [h_i] \). Then, if we construct a second neural network that shares the same weights and biases, except replacing \( W_{i+1, j_1}^i \) and \( W_{i+1, j_2}^i \) with a pair \( W_{i+1, j_1}^i \) and \( W_{i+1, j_2}^i \), such that, \( W_{i+1, j_1}^i + W_{i+1, j_2}^i = W_{i+1, j_1}^i + W_{i+1, j_2}^i \). Then, the two neural networks are equivalent, regardless of the activation function. The \( j_1 \) and \( j_2 \) neurons in the \( i \)th layer are called clones and are defined formally in the following manner.

**Theorem 1** ([40]). There exists a closed ball \( B \) around 0, such that, for any considered identifiability inducing activation function \( \sigma \), if \( f \) is a class of neural networks with \( \sigma \) activations and \( y \in Y \), there is a unique function \( f^* \in \{ f(\cdot; \theta) \mid \theta \in B \} \), such that: \( \| f - y \|_{\infty} = \inf_{\theta \in \Theta} \| f(\cdot; \theta) - y \|_{\infty} \).

**3 Assumptions**

In this section, we introduce the assumptions made in order to obtain the theoretical results. The first assumption is not strictly necessary, but greatly reduces the complexity of the proofs: we assume the existence of a unique function \( f \in f \) that best approximates a given target function \( y \).

**Assumption 1** (Unique Approximation). There exists a closed ball \( B \) around 0, such that, for any considered identifiability inducing activation function \( \sigma \), if \( f \) is a class of neural networks with \( \sigma \) activations and \( y \in Y \), there is a unique function \( f^* \in \{ f(\cdot; \theta) \mid \theta \in B \} \), such that: \( \| f - y \|_{\infty} = \inf_{\theta \in \Theta} \| f(\cdot; \theta) - y \|_{\infty} \).
Assumption 1 and 2, for any function \( y / \). Let Assumption 2.

Theorem 2. We note that \( \tilde{f} \) is already perfectly approximated by \( f \), for any function \( y / \) that, is close enough to \( f \), and find the continuous selector with respect to the class \( g \) that is the same architecture as \( f \), except the activations are \( \rho \).

The following lemma shows that under Assumption 2, any function \( y / \) that has a best approximator, then, the approximator is normal.

Lemma 1. Let \( f / \) be a class of neural networks. Let \( y / \) be a target function. Assume that \( y / \) has a best approximator \( f / \in \ f / \). If \( y / \notin \ f / \), then, \( f / \in \ f / n / \).

(All proofs are presented in the supplementary material.) As can be seen from Lem. 1, by combining Assumptions 1 and 2 for any function \( y / \notin \ f / \), there is a unique solution \( \theta / \) for the equation, \( \| f / ( ; / \theta / ) - y / \|_{\infty} / \), up to isomorphisms.

4 Degrees of Approximation

In this section, we describe the approximation properties of function classes. The paradigm is as follows. We are interested in determining how complex a model ought to be to theoretically guarantee approximation of an unknown target function \( y / \) up to a given accuracy \( \epsilon / > 0 / \).

Formally, let \( \mathcal{Y} / \) be a set of target functions to be approximated. For a set \( \mathcal{P} / \) of candidate approximators, we measure its ability to approximate \( \mathcal{Y} / \) as \( d(\mathcal{P} ; / \mathcal{Y}) := \sup_{p / \in / \mathcal{Y}} \inf_{p / \in / \mathcal{P}} \| y - p /\|_{\infty} / \).

Typical approximation results show that the class \( \mathcal{Y} = \mathcal{W}_{r,m} / \) can be approximated using classes of neural networks \( \ f / \) of sizes \( \mathcal{O}(\epsilon^{-m/r}) / \), where \( \epsilon / \) is an upper bound on \( d(f ; / \mathcal{Y}) / \). For instance, in \( [31] / \) this property is shown for neural networks with activations \( \sigma / \) that are infinitely differentiable and not polynomial on any interval; \( [13] / \) prove this property for ReLU neural networks. We call activation functions with this property universal.

Definition 7 (Universal activation). Let an activation function \( \sigma / : / \mathbb{R} \rightarrow / \mathbb{R} / \), \( \sigma / \) is universal if for any \( r, / n / \in / \mathbb{N} / \) and \( \epsilon / > 0 / \), there is a class of neural networks \( \ f / \) with \( \sigma / \) activations, of size \( \mathcal{O}(\epsilon^{-n/r}) / \), such that, \( d(f : / \mathcal{W}_{r,n} / \leq / \epsilon / \).

An interesting question is whether this bound is tight. We recall the \( N / \)-width framework of \( [8] / \) (see also \( [13] / \)). Let \( \ f / \) be a class of functions and \( \mathcal{S} : \mathcal{Y} \rightarrow \mathbb{R}^N / \) be a continuous mapping between a function \( y / \) and its approximation, where with \( N := / N_f / \). In this setting, we approximate \( y / \) using \( f / ( ; / S(y) ) / \), where the continuity of \( S / \) means that the selection of parameters is robust with respect to perturbations in \( y / \). The nonlinear \( N / \)-width of the compact set \( \mathcal{Y} = \mathcal{W}_{r,m} / \) is defined as follows: \( \tilde{d}_N(\mathcal{Y}) := \inf_{\mathcal{f}} \inf_{\mathcal{S}} \sup_{y / \in / \mathcal{Y}} \| f / ( ; / S(y)) - y /\|_{\infty} / \), where the infimum is taken over \( \mathcal{f} / \), such that, \( N_f = N / \) and \( \mathcal{S} / \) is continuous. As shown by \( [8] / \), if there exists \( f / \), such that, \( \tilde{d}(f ; / \mathcal{Y}) / \leq / \epsilon / \) (i.e., \( \tilde{d}_N(\mathcal{Y}) \leq / \epsilon / \)), then \( N_f = \Omega(\epsilon^{-m/r}) / \). We note that \( d(f ; / \mathcal{Y}) / \leq / \tilde{d}(f ; / \mathcal{Y}) / \) and, therefore, this analysis does not provide a full solution for this question.

In the following theorem, we show that under certain conditions, the lower bound holds, even when removing the assumption that the selection is robust.

Theorem 2. Let \( \sigma / : / \mathbb{R} \rightarrow \mathbb{R} / \) be a piece-wise \( C^1 / (\mathbb{R}) / \) activation function with \( \sigma^r \in / \text{BV}(\mathbb{R}) / \). Let \( \ f / \) be a class of neural networks with \( \sigma / \). Let \( \mathcal{Y} = \mathcal{W}_{r,m} / \). Assume that any non-constant \( y / \in / \mathcal{Y} / \) cannot be represented as a neural network with \( \sigma / \) activations. Then, if \( d(f ; / \mathcal{Y}) / \leq / \epsilon / \), we have \( N_f = \Omega(\epsilon^{-m/r}) / \).

To prove this theorem, we show the existence of continuous selector \( \mathcal{S} : / \mathcal{Y} \rightarrow \Theta_f / \) for approximating the class \( \mathcal{Y} / \), i.e., there is a function exists a constant \( \alpha / > 0 / \), such that, for all \( y / \in / \mathcal{Y} / \), we have: \( \| f / ( ; / S(y)) - y /\|_{\infty} \leq / \alpha / \cdot / \inf_{\theta / \in / \Theta_f} \| f / ( ; / \theta) - y /\|_{\infty} / \). In order to do so, we take an identifiability inducing activation function \( \rho / \) that is close enough to \( \sigma / \) and find the continuous selector with respect to the class \( g / \) that is the same architecture as \( f / \), except the activations are \( \rho / \).
5 Expressivity of Hypernetworks

In this section, we study the expressive power of hypernetworks. In the first part, we compare the complexities of \( q \) and \( e \). We show that when letting \( e \) and \( f \) be large enough, one can approximate it using a hypernetwork where \( q \) is smaller than \( q \) by orders of magnitude. In the second part, we show that under typical assumptions on \( y \), one can approximate \( y \) using a hypernetwork with overall much fewer parameters than the number of parameters required for a neural embedding method.

5.1 Comparing the complexities of \( q \) and \( e \)

We recall that for an arbitrary \( r \)-smooth function \( y \in \mathcal{W}_{r,m} \), the complexity for approximating it is \( O(e^{-n/r}) \). We show that hypernetwork models can effectively learn a different function for each input instance \( I \). Specifically, that the hypernetwork model is able to capture a separate approximator \( h_I = g(\cdot; f(I; \theta_f)) \) for each \( y_I \) that has a minimal complexity \( O(e^{-m_1/r}) \). On the other hand, we show that for a smoothness order of \( r = 1 \), under certain constraints, when applying an embedding method, it is impossible to provide a separate approximator \( h_I = q(\cdot, e(I; \theta_e); \theta_q) \) of complexity \( O(e^{-m_1}) \). Therefore, the embedding method does not enjoy the same compositional properties of hypernetworks.

The next result shows that the complexity of the main-network \( q \) in any embedding method has to be of non-optimal complexity and that this holds regardless of the size of \( e \), as long as the functions \( e \in e \) are of bounded Lipschitzness.

**Theorem 3.** Let \( \sigma : \mathbb{R} \rightarrow \mathbb{R} \) be a universal, piece-wise \( C^1(\mathbb{R}) \) activation function with \( \sigma' \in BV(\mathbb{R}) \) and \( \sigma(0) = 0 \). Let \( \mathcal{E}_{e,q} \) be an neural embedding method. Assume that \( e \) is a class of continuously differentiable neural network \( e \) with zero biases, output dimension \( k = O(1) \) and bounded spectral complexity \( C(e) \leq \ell_1 \) and \( q \) is a class of neural networks \( q \) with \( \sigma \) activations, bounded spectral complexity \( C(q) \leq \ell_2 \). Let \( \mathcal{Y} := \mathcal{W}_{l,m} \). Assume that any non-constant \( y \in \mathcal{Y} \) cannot be represented as a neural network with \( \sigma \) activations. If the embedding method achieves error \( d(\mathcal{E}_{e,q}, \mathcal{Y}) \leq e \), then, the complexity of \( q \) is:

\[
N_q = \Omega\left(e^{-(m_1+m_2)}\right)
\]  

(9)

The following theorem extends Thm. 3 to the case where the output dimension of \( e \) depends on \( e \). In this case too, the parameter complexity is not optimal.

**Theorem 4.** In the setting of Thm. 3 except \( k \) is not necessarily \( O(1) \). Assume that the first layer of any \( q \in q \) is bounded \( \|W^1\|_1 \leq c \), for some constant \( c > 0 \). If the embedding method achieves error \( d(\mathcal{E}_{e,q}, \mathcal{Y}) \leq e \), then, the complexity of \( q \) is:

\[
N_q = \Omega\left(e^{-\min(m, 2m_1)}\right)
\]  

(10)

The following theorem shows that for any function \( y \in \mathcal{W}_{r,m} \), there is a large enough hypernetwork, that maps between \( I \) and an approximator of \( y_I \) of approximating error \( \epsilon \).

**Theorem 5.** Let \( \sigma : \mathbb{R} \rightarrow \mathbb{R} \) be a universal, piece-wise \( C^1(\mathbb{R}) \) activation function with \( \sigma' \in BV(\mathbb{R}) \). Let \( \mathcal{Y} := \mathcal{W}_{l,m} \). Assume that any non-constant \( y \in \mathcal{Y} \) cannot be represented as a neural network with \( \sigma \) activations. Then, there is a class \( q \) of neural networks with \( \sigma \) activations, such that, for any \( y \in \mathcal{Y} \), there is a large enough neural network \( f \), such that, the hypernetwork \( h(x, I) = g(x; f(I; \theta_f)) \) achieves error \( \leq \epsilon \) in approximating \( y \) and:

\[
N_q = O\left(e^{-m_1/r}\right)
\]  

(11)

When comparing the results in Thms. 3, 4 and 5 in the case of \( r = 1 \), we notice that in the hypernetworks case, \( q \) can be of complexity \( O(e^{-m_1}) \) in order to achieve approximation error \( \leq \epsilon \). On the other hand, for the embedding method case, the complexity of the primary-network \( q \) is at least \( \Omega(e^{-\min(m_1+m_2)}) \) when the embedding dimension is of constant size and at least \( \Omega\left(e^{-\min(m, 2m_1)}\right) \) when it is unbounded to achieve approximation error \( \leq \epsilon \). In both cases, the primary network of the embedding method is larger by orders of magnitude than the primary network of the hypernetwork.
Figure 3: (a-c) The error obtained by hypernetworks and the embedding method with varying number of layers (x-axis). The MSE (y-axis) is computed between the learned function and the target function at test time. The blue curve stands for the performance of the hypernetwork model and the orange one for the neural embedding method. (a) Target functions of neural network type, (b) Functions of the form \( y(x, I) = h(x \circ I) \), where \( h \) is a neural network, (c) Target functions of the form \( y(x, I) = \langle x, h(I) \rangle \), where \( h \) is a neural network. (d-f) Measuring the performance for the same three target functions when varying the size of the embedding layer to be 100 times the value on the x-axis. The error bars depict the variance across 100 repetitions of the experiment.

5.2 Parameter Complexity of Hypernetworks

In Sec. 4, we show that under certain conditions, for \( Y := \{y_I\}_{I \in \mathcal{I}} \subset \mathcal{W}_{r,m} \), there is a continuous selector \( \hat{S} : Y \rightarrow \Theta_g \) that takes a function \( y \) and returns the parameters \( \hat{S}(I) \) of a network \( g \) that approximates \( y \). In particular, since \( I \mapsto y_I \) is a continuous function, we can define a continuous function \( S : \mathcal{I} \rightarrow \Theta_g \) that takes an input instance \( I \) and returns parameters \( S(I) = \hat{S}(y_I) \), such that, \( g(\cdot; S(I)) \) well approximates \( y_I \). For further details, see Lem. 16 in the supplementary material.

A common structure of hypernetworks is such that in common practical scenarios, the typical assumption regarding the selection function \( S \) is that it takes the form \( W \cdot h \), for some continuous function \( h : \mathcal{I} \rightarrow \mathbb{R}^{w} \) for some relatively small \( w > 0 \) and \( W \) is a linear mapping \([38, 24, 6, 23]\). In this section, we show that for functions \( y \) with a continuous selector \( S \) of this type, the complexity of the function \( f \) can be reduced to \( O(\epsilon^{-m_2/r} + \epsilon^{-m_1/r}) \).

**Theorem 6.** Let \( \sigma : \mathbb{R} \rightarrow \mathbb{R} \) be a universal Lipschitz continuous activation function, such that, \( \sigma(0) = 0 \). Let \( g \) be a class of neural networks with \( \sigma \) activations. Let \( y \in Y := \mathcal{W}_{r,m} \) be a target function. Assume that there is a continuous selector \( S \in \mathcal{P}_{r,w,c} \) for the class \( \{y_I\}_{I \in \mathcal{I}} \) within \( g \). Then, there is a hypernetwork \( \mathcal{H}_{f,g} \) that achieves error \( \leq \epsilon \) in approximating \( y \), such that:

\[
N_f = \mathcal{O}(w^{1+m_2/r} \cdot \epsilon^{-m_2/r} + w \cdot N_g)
\]

\[
= \mathcal{O}(\epsilon^{-m_2/r} + \epsilon^{-m_1/r})
\]

(12)

We note that the number of learned parameters in a hypernetwork is measured by \( N_f \). By Thm. 2, the number of trainable parameters in a neural network is \( \Omega(\epsilon^{-(m_1+m_2)/r}) \) in order to be able to approximate any function \( y \in \mathcal{W}_{r,m} \). Thm. 4 shows that in the case of the common hypernetwork structure, the number of trainable parameters of the hypernetwork is reduced to \( \mathcal{O}(\epsilon^{-m_2/r} + \epsilon^{-m_1/r}) \).

For embedding methods, where the total number of parameters combines those of both \( q \) and \( e \), it is evident that the overall trainable parameters of an embedding method is \( \Omega(\epsilon^{-(m_1+m_2)/r}) \).
6 Experiments

To validate the prediction in Sec. 5.1, we conducted an experiment comparing the ability of hypernetworks and embedding methods of similar complexities in approximating an unknown target function.

We experimented with three types of target functions. In all three cases, \( x \) and \( I \) are vectors of dimension 1000. The first group of functions consists of randomly initialized fully connected neural networks \( y(x, I) \). The neural network has four layers of dimensions \( 2000 \to 100 \to 50 \to 50 \to 1 \) and applies ELU activations. The second type of target functions \( y(x, I) := h(x \circ I) \) consists of fully-connected neural network applied on top of the element-wise multiplication between \( x \) and \( I \). The neural network consists of four layers of dimensions \( 1000 \to 100 \to 100 \to 50 \to 1 \) and applies ELU activations. The third type of target functions is of the form \( y(x, I) := \langle x, h(I) \rangle \), where \( h \) is a fully-connected neural network. The neural network has three layers of dimensions \( 1000 \to 300 \to 300 \to 1000 \) and applies sigmoid activations within the two hidden layers and softmax on top of the network. The reason we apply softmax on top of the network is to restrict its output to be bounded.

In all of the experiments, the weights of \( y \) are set using the He uniform initialization \[15\].

**Varying the number of layers** To compare between the two models, we took the primary-networks \( g \) and \( q \) to be neural networks with two layers of dimensions \( d_{in} \to 10 \to 1 \) and ReLU activation within the hidden layer. The input dimension of \( g \) is \( d_{in} = 1000 \) and for \( q \) is \( d_{in} = 1100 \). In addition, the \( f \) and \( e \) are neural networks with a varying number of layers between 2 and 9. Each layer in \( e \) and \( f \) is of dimension 100. The output dimension of \( e \) is 100.

We compared the MSE losses at test time of the hypernetwork and the embedding method in approximating the target function \( y \). The training was done over 30000 samples of pairs \((x, I)\) taken from a standard normal distribution. The samples are divided into batches of size 200 and the learning rate is \( \mu = 0.01 \).

As can be seen in Fig. 3(a-c), when the number of layers of \( f \) and \( e \) are \( \geq 3 \), the hypernetwork model outperforms the embedding method in terms of minimizing the approximation error. It is also evident that the approximation error of hypernetworks improves, as long as we increase the number of layers of \( f \). This is in contrast to the case of the neural embedding method, where increasing the number of layers of \( e \) does not improve the approximation error significantly.

These results are very much in line with the theorems in Sec. 5.2. As be be seen in Thms. 3 and 5 when fixing the sizes of \( g \) and \( q \), while letting \( f \) and \( e \) be as large as we wish we can achieve a much better approximation with the hypernetwork model.

**Varying the embedding dimension** We next investigate the effect of varying the embedding dimension in both models to be 100i, for \( i = 1..8 \). The primary-networks \( g \) and \( q \) are set to be neural networks with two layers of dimensions \( d_{in} \to 10 \to 1 \) and ReLU activation in the hidden layer. The input dimension of \( g \) is \( d_{in} = 1000 \) and for \( q \) is \( d_{in} = 1100 \). The networks \( f \) and \( e \) are taken to be fully connected networks with three layers. The dimensions of \( f \) are \( 1000 \to 100 \to 100i \to N_y \) and the dimensions of \( e \) are \( 1000 \to 100 \to 100 \to 100i \). As can be seen from Fig. 3(d-f), by increasing the embedding dimension of both models, the performance improves only slightly. Also, the overall performance is much worse and much less stable than the performance of hypernetworks with deeper \( f \), as presented in Fig. 3(a-c). This result verifies the claim in Thm. 4 that by increasing the embedding dimension the embedding model is unable to achieve the same rate of approximation as the hypernetwork model.

7 Conclusions

We aim to understand the success of hypernetworks from a theoretical standpoint and compared the complexity of hypernetworks and embedding methods in terms of the number of trainable parameters. In order to achieve error \( \leq \epsilon \) when modeling a function \( y(x, I) \) using hypernetworks, the primary-network can be selected to be of a much smaller family of networks then the primary-network of an embedding method. This result manifests the ability of hypernetworks to effectively learn distinct functions for each \( y_I \) separately.
8 Acknowledgements

This project has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant ERC CoG 725974). The contribution of Tomer Galanti is part of Ph.D. thesis research conducted at Tel Aviv University.
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9 Preliminaries

9.1 Multi-valued Functions

Throughout the proofs, we will make use of the notion of multi-valued functions and their continuity. A multi-valued function is a mapping \( F : A \to \mathcal{P}(B) \) from a set \( A \) to the power set \( \mathcal{P}(B) \) of some set \( B \).

To define the continuity of \( F \), we define distance measure between sets. Let \( d_B \) be a distance function over a set \( B \). The Hausdorff distance \( d_H \) between subsets of \( B \) is defined as follows:

\[
   d_H(E_1, E_2) := \max \left\{ \sup_{b_1 \in E_1} \inf_{b_2 \in E_2} d_B(b_1, b_2), \sup_{b_2 \in E_2} \inf_{b_1 \in E_1} d_B(b_1, b_2) \right\}
\]  

(13)

In general, the Hausdorff distance serves as an extended pseudo-metric, i.e., satisfies \( d_H(E, E) = 0 \) for all \( E \), is symmetric and satisfies the triangle inequality, however, it can attain infinite values and there might be \( E_1 \neq E_2 \), such that, \( d_H(E_1, E_2) = 0 \). When considering the space \( \mathcal{C}(B) \) of non-empty compact subsets of \( B \), the Hausdorff distance becomes a metric.

**Definition 8** (Continuous multi-valued functions). Let metric spaces \( (A, d_A) \) and \( (B, d_B) \) and multi-valued function \( F : A \to \mathcal{P}(B) \). Then, we define:

- **Convergence**: we denote \( E = \lim_{a \to a_0} F(a) \), if \( E \) is a compact subset of \( B \) and it satisfies:
  \[
  \lim_{a \to a_0} d_H(F(a), E) = 0
  \]
  (14)

- **Continuity**: we say that \( F \) is continuous in \( a_0 \), if \( \lim_{a \to a_0} F(a) = F(a_0) \).

9.2 Lemmas

In this section, we provide several lemmas that will be useful throughout the proofs of the main results.

Let \( [W^1, b^1] \) and \( [W^2, b^2] \) be two parameters. We denote by \( [W^1, b^1] - [W^2, b^2] = [W^1 - W^2, b^1 - b^2] \) the element-wise subtraction between the two parameters. In addition, we define the \( L_2 \)-norm of \( [W, b] \) to be:

\[
   \| [W, b] \|_2 := \sqrt{\sum_{i=1}^{k} (\|W^i\|_2^2 + \|b^i\|_2^2)}
\]  

(15)

**Lemma 2.** Let \( f(x; [W^1, b^1]) \) and \( f(x; [W^2, b^2]) \) be two neural networks. Then, for a given isomorphism \( \pi \), we have:

\[
   \pi \circ [W^1, b^1] - \pi \circ [W^2, b^2] = \pi \circ [W^1 - W^2, b^1 - b^2]
\]  

(16)

and

\[
   \| \pi \circ [W, b] \|_2 = \| [W, b] \|_2
\]  

(17)

**Proof.** Follows immediately from the definition of isomorphisms.

**Lemma 3.** Let \( \sigma : \mathbb{R} \to \mathbb{R} \) be a \( L \)-Lipschitz continuous activation function, such that, \( \sigma(0) = 0 \). Let \( f(\cdot; [W, 0]) : \mathbb{R}^m \to \mathbb{R} \) be a neural network with zero biases. Then, for any \( x \in \mathbb{R}^m \), we have:

\[
   \|f(x; [W, 0])\|_1 \leq L^{k-1} \cdot \|x\|_1 \prod_{i=1}^{k} \|W^i\|_1
\]  

(18)

**Proof.** Let \( z = W^{k-1} \cdot \sigma(\ldots \sigma(W^1 x)) \). We have:

\[
   \|f(x; [W, 0])\|_1 \leq \|W^k \cdot \sigma(z)\|_1
   \leq \|W^k \cdot \sigma(z)\|_1
   = \|W^k\|_1 \cdot \|\sigma(z) - \sigma(0)\|_1
   \leq \|W^k\|_1 \cdot L \cdot \|z\|_1
\]  

(19)

by induction we have the desired. \( \square \)
Lemma 4. Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a $L$-Lipschitz continuous activation function, such that, $\sigma(0) = 0$. Let $f(\cdot; [W, b])$ be a neural network. Then, the Lipschitzness of $f(\cdot; [W, b])$ is given by:

$$\text{Lip}(f(\cdot; [W, b])) \leq L^{k-1} \cdot \prod_{i=1}^{k} \|W_i\|_1$$  \hspace{1cm} (20)

Proof. Let $z_i = W^{k-1} \cdot \sigma(\ldots \sigma(W^1 x_1 + b^1))$ for some $x_1$ and $x_2$. We have:

$$\|f(x_1; [W, b]) - f(x_2; [W, b])\|_1 \leq \|W^k \cdot \sigma(z_1) - W^k \cdot \sigma(z_2)\|_1$$

$$\leq \|W^k \cdot (\sigma(z_1 + b^{k-1}) - \sigma(z_2 + b^{k-1}))\|_1$$

$$= \|W^k\|_1 \cdot \|\sigma(z_1 + b^{k-1}) - \sigma(z_2 + b^{k-1})\|_1$$

and by induction we have the desired.

Proof. Let $z_i = W^{k-1} \cdot \sigma(\ldots \sigma(W^1 x_1 + b^1))$ for some $x_1$ and $x_2$. We have:

$$\|f(x_1; [W, b]) - f(x_2; [W, b])\|_1 \leq \|W^k \cdot \sigma(z_1) - W^k \cdot \sigma(z_2)\|_1$$

$$\leq \|W^k \cdot (\sigma(z_1 + b^{k-1}) - \sigma(z_2 + b^{k-1}))\|_1$$

$$= \|W^k\|_1 \cdot \|\sigma(z_1 + b^{k-1}) - \sigma(z_2 + b^{k-1})\|_1$$

$$\leq \|W^k\|_1 \cdot \|z_1 - z_2\|_1$$

Lemma 5. Let $f$ be a class of neural networks. Let $y$ be a target function. Assume that $y$ has a best approximator $f \in f$. If $y \notin f$, then, $f \in f_n$.

Proof. Let $f(\cdot; [W, b]) \in f$ be the best approximator of $y$. Assume it is not normal. Then, $f(\cdot; [W, b])$ has at least one zero neuron or at least one pair of clone neurons. Assume it has a zero neuron. Hence, by removing the specified neuron, we achieve a neural network of architecture smaller than $f$ that achieves the same approximation error as $f^1$ does. This is in contradiction to Assumption 2. For clone neurons, we can simply merge them into one neuron and obtain a smaller architecture that achieves the same approximation error, again, in contradiction to Assumption 2.

Lemma 6. Let $f$ be a class of functions. Let $\mathcal{Y}$ be a class of target functions. Then, the function $\|f(\cdot; \theta) - y\|_\infty$ is continuous with respect to both $\theta$ and $y$.

Proof. Let sequences $\theta_n \to \theta_0$ and $y_n \to y_0$. By the triangle inequality, we have:

$$\|f(\cdot; \theta_n) - y_n\|_\infty - \|f(\cdot; \theta_0) - y_0\|_\infty \leq \|f(\cdot; \theta_n) - f(\cdot; \theta_0)\|_\infty + \|y_n - y_0\|_\infty$$

Since $\theta_n \to \theta_0$, we have: $\|f(\cdot; \theta_n) - f(\cdot; \theta_0)\|_\infty \to 0$. Hence, the upper bound tends to 0.

Lemma 7. Let $\sigma$ be an identifiability inducing activation function. Let $f$ be a class of neural networks with $\sigma$ activations and $\Theta_f = \mathbb{B}$ be the closed ball in Assumption 1. Let $\mathcal{Y}$ be a class of target functions. Denote by $f_y$ the unique approximator of $y$ within $f$. Then, $f_y$ is continuous with respect to $y$.  

$$\text{M}[y; f] := \arg \min_{\theta \in \Theta_f} \|f(\cdot; \theta) - y\|_\infty$$  \hspace{1cm} (25)
Proof. Let \( y_0 \in \mathbb{Y} \) be some function. Assume by contradiction that there is a sequence \( y_n \to y_0 \), such that, \( g_n := f_{y_n} \not\to f_{y_0} \). Then, \( g_n \) has a sub-sequence that has no cluster points or it has a cluster point \( h \neq f_{y_0} \).

**Case 1:** Let \( g_{n_k} \) be a sub-sequence of \( g_n \) that has no cluster points. By Assumption 1, there is a sequence \( \theta_{n_k} \in \bigcup_{k=1}^{\infty} M[y_{n_k}; f] \) that is bounded in \( \mathbb{E} \). By the Bolzano-Weierstrass' theorem, it has a sub-sequence \( \theta_{n_{k_j}} \) that converges to some vector \( \theta_0 \). Therefore, we have:

\[
\|f(\cdot; \theta_{n_{k_j}}) - f(\cdot; \theta_0)\|_\infty \to 0 \quad (26)
\]

Hence, \( g_{n_k} \) has a cluster point \( f(\cdot; \theta_0) \) in contradiction.

**Case 2:** Let sub-sequence \( f_{y_{n_k}} \) that converge to a function \( h \neq f_{y_0} \). We have:

\[
\|h - y_0\|_\infty \leq \|f_{y_{n_k}} - h\|_\infty + \|f_{y_{n_k}} - y_{n_k}\|_\infty + \|y_{n_k} - y_0\|_\infty \quad (27)
\]

In addition, by Lem. 6

\[
\|f_{y_{n_k}} - y_{n_k}\|_\infty \to \|f_{y_0} - y_0\|_\infty \quad (28)
\]

and also \( y_{n_k} \to y_0, f_{y_{n_k}} \to h \). Therefore, we have:

\[
\|h - y_0\|_\infty \leq \|f_{y_0} - y_0\|_\infty \quad (29)
\]

Hence, since \( f_{y_0} \) is the unique minimizer, we conclude that \( h = f_{y_0} \) in contradiction. Therefore, we conclude that \( f_{y_n} \) converges and by the analysis in Case 2 it converges to \( f_{y_0} \). \( \square \)

10 Proofs of the Main Results

10.1 Existence of a continuous selector

**Lemma 8.** Let \( \sigma \) be an identifiability inducing activation function. Let \( f \) be a class of neural networks with \( \sigma \) activations and \( \Theta_f = \mathbb{E} \) be the closed ball in Assumption 1. Let \( \mathbb{Y} \) be a class of normal target functions. Then, \( M[y; f] := \arg\min_{\theta \in \mathbb{E}} \|f(\cdot; \theta) - y\|_\infty \) is a continuous multi-valued function of \( y \).

**Proof.** Assume by contradiction that \( M \) is not continuous. We distinguish between two cases:

1. There exists a sequence \( y_n \to y \) and constant \( c > 0 \), such that,

\[
\sup_{\theta \in M[y;f]} \inf_{\theta' \in M[y;f]} \|\theta_1 - \theta_2\|_2 > c > 0 \quad (30)
\]

2. There exists a sequence \( y_n \to y \) and constant \( c > 0 \), such that,

\[
\sup_{\theta_1 \in M[y;f]} \inf_{\theta_2 \in M[y;f]} \|\theta_1 - \theta_2\|_2 > c > 0 \quad (31)
\]

**Case 1:** We denote by \( \theta_1 \) a member of \( M[y; f] \) that satisfies:

\[
\forall n \in \mathbb{N} : \inf_{\theta \in M[y;f]} \|\theta_1 - \theta_2\|_2 > c > 0 \quad (32)
\]

The set \( \bigcup_{n=1}^{\infty} M[y_n; f] \subset \Theta_f \) is a bounded subset of \( \mathbb{R}^N \), and therefore by the Bolzano-Weierstrass theorem, for any sequence \( \{\theta_{n_k}^2\}_{n=1}^{\infty} \), such that, \( \theta_{n_k}^2 \in M[y_n; f] \), there is a sub-sequence \( \{\theta_{n_{k_j}}^2\}_{k=1}^{\infty} \) that converges to some \( \theta_2^* \). We notice that:

\[
\|f(\cdot; \theta_{n_{k_j}}^2) - y_{n_{k_j}}\|_\infty = \min_{\theta \in \Theta_f} \|f(\cdot; \theta) - y_{n_{k_j}}\|_\infty = F(y_{n_{k_j}}) \quad (33)
\]

In addition, by the continuity of \( F \), we have:

\[
\lim_{k \to \infty} F(y_{n_{k_j}}) = F(y) \quad (34)
\]

This yields that \( \theta_{n_{k_j}}^2 \) is a member of \( M[y; f] \). Since \( f_y := \arg\min_{f \in \mathbb{F}} \|f - y\|_\infty \) is unique and normal, by the identifiability hypothesis, there is a function \( \pi \in \Pi \), such that, \( \pi(\theta_{n_{k_j}}^2) = \theta_1 \). Since the function \( \pi \) is continuous

\[
\lim_{k \to \infty} \|\pi(\theta_{n_{k_j}}^2) - \theta_1\|_2 = \lim_{k \to \infty} \|\pi(\theta_{n_{k_j}}^2) - \pi(\theta_{n_k}^2)\|_2 = 0 \quad (35)
\]

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We notice that $\pi(\theta_{\alpha}^{n_k}) \in M[y_{nk}; f]$. Therefore, we have:

$$\lim_{k \to \infty} \inf_{\theta_2 \in M[y_{nk}; f]} \|\theta_1 - \theta_2\| = 0$$

in contradiction to Eq. \[32\]

**Case 2:** Let $\theta_0^0 \in M[y_0; f]$ be a sequence, such that,

$$\inf_{\theta_2 \in M[y_0; f]} \|\theta_0^0 - \theta_2\| > c$$

The set $\cup_{n=1}^{\infty} M[y_n; f] \subset \Theta_f$ is a bounded subset of $\mathbb{R}^N$, and therefore by the Bolzano-Weierstrass theorem, there is a sub-sequence $\theta_0^{n_k}$ that converges to some vector $\theta_0$. The function $\|f(\cdot; \theta) - y\|_\infty$ is continuous with respect to $\theta$ and $y$. Therefore,

$$\lim_{k \to \infty} \min_{\theta \in \Theta_f} \|f(\cdot; \theta) - y_{n_k}\|_\infty = \lim_{k \to \infty} \|f(\cdot; \theta_0^{n_k}) - y_{n_k}\|_\infty = \|f(\cdot; \theta_0) - y\|_\infty$$

By Lem. 6, $\|f(\cdot; \theta_0) - y\|_\infty = \min_{\theta \in \Theta_f} \|f(\cdot; \theta) - y\|_\infty$. In particular, $\theta_0 \in M[y; f]$, in contradiction to Eq. \[37\] \[ \square \]

**Theorem 7.** Let $\sigma$ be an identifiability inducing activation function. Let $f$ be a class of neural networks with $\sigma$ activations and $\Theta_f = \mathbb{B}$ be the closed ball in Assumption 1. Let $\mathbb{Y}$ be a compact class of normal target functions. Then, there is a continuous selector $S : \mathbb{Y} \to \Theta_f$, such that, $S(y) \in M[y; f]$.

**Proof.** Let $y_0$ be a member of $\mathbb{Y}$. We notice that $M[y_0; f]$ is a finite set. We denote its members by: $M[y_0; f] = \{\theta_0^0, \ldots, \theta_0^N\}$. Then, we claim that there is a small enough $\epsilon := \epsilon(y_0) > 0$ (depending on $y_0$), such that, $S(y_0) = \theta_0^0$ and $S(y) = \arg \min_{\theta \in M[y_0; f]} \|\theta - \theta_0\|_2$ for all $y \in \mathbb{B}_\epsilon(y_0)$, is continuous in $\mathbb{B}_\epsilon(y_0)$. The set $\mathbb{B}_\epsilon(y_0) := \{y \mid \|y - y_0\|_\infty < \epsilon\}$ is the open ball of radius $\epsilon$ around $y_0$. We denote

$$c := \min_{\pi_1 \neq \pi_2 \in \Pi} \|\pi_1 \circ S(y_0) - \pi_2 \circ S(y_0)\|_2 > 0$$

This constant exists since $\Pi$ is a finite set of transformations and $\mathbb{Y}$ is a class of normal functions. In addition, we select $\epsilon$ to be small enough to suffice that:

$$\max_{y \in \mathbb{B}_\epsilon(y_0)} \|S(y) - S(y_0)\|_2 < c/4$$

Assume by contradiction that there is no such $\epsilon$. Then, for each $\epsilon_n = 1/n$ there is a function $y_n \in \mathbb{B}_{\epsilon_n}(y_0)$, such that,

$$\|S(y) - S(y_0)\|_2 \geq c/4$$

Therefore, we found a sequence $y_n \to y_0$ that satisfies:

$$M[y_0; f] \not\to M[y_0; f]$$

in contradiction to the continuity of $M$.

For any given $y_1, y_2 \in \mathbb{B}_{\epsilon}(y_0)$ and $\pi_1 \neq \pi_2 \in \Pi$, by the triangle inequality, we have:

$$\|\pi_1 \circ S(y_1) - \pi_2 \circ S(y_2)\|_2 \geq \|\pi_1 \circ S(y_0) - \pi_2 \circ S(y_0)\|_2 - \|\pi_1 \circ S(y_1) - \pi_1 \circ S(y_0)\|_2$$

$$\geq \|\pi_1 \circ S(y_0) - \pi_2 \circ S(y_0)\|_2 - \|\pi_1 \circ S(y_0) - \pi_1 \circ S(y_0)\|_2$$

$$= \|\pi_1 \circ S(y_0) - \pi_2 \circ S(y_0)\|_2 - \|\pi_1 \circ S(y_0) - \pi_1 \circ S(y_0)\|_2$$

$$\geq \|\pi \circ S(y_0) - \pi \circ S(y_0)\|_2 > c/2$$

In particular, $\|\pi \circ S(y_1) - S(y_2)\|_2 > c/2$ for every $\pi \neq \text{Id}$.

Since $M$ is continuous, for any sequence $y_n \to y \in \mathbb{B}_{\epsilon}(y_0)$, there are $\pi_n \in \Pi$, such that:

$$\lim_{n \to \infty} \pi_n \circ S(y_n) = S(y)$$

Therefore, by the above inequality, we address that for any large enough $n$, $\pi_n = \text{Id}$. In particular, for any sequence $y_n \to y$, we have:

$$\lim_{n \to \infty} S(y_n) = S(y)$$
This implies that $S$ is continuous in any $y \in B_{\epsilon}(y_0)$.

We note that $\{B_{\epsilon(y_0)}(y_0)\}_{y_0 \in \mathcal{Y}}$ is an open cover of $\mathcal{Y}$. In particular, since $\mathcal{Y}$ is compact, there is a finite sub-cover $\{C_i\}_{i=1}^T$ of $\mathcal{Y}$. In addition, we denote by $\{c_i\}_{i=1}^T$ the corresponding constants in Eq. (39).

Next, we construct the continuous function $S$ inductively. We denote by $S$ the locally continuous function that corresponds to $C_i$. For a given pair of sets $C_{i_1}$ and $C_{i_2}$ that intersect, we would like to construct a continuous function over $C_{i_1} \cup C_{i_2}$. First, we would like to show that there is an isomorphism $\pi$, such that, $\pi \circ S_{i_2}(y) = S_{i_1}(y)$ for all $y \in C_{i_1} \cap C_{i_2}$. Assume by contradiction that there is no such $\pi$. Then, let $y_1 \in C_{i_1} \cap C_{i_2}$ and $\pi_1$, such that, $\pi_1 \circ S_{i_2}(y_1) = S_{i_1}(y_1)$. We denote by $y_2 \in C_{i_1} \cap C_{i_2}$ a member, such that, $\pi_1 \circ S_{i_2}(y_2) \neq S_{i_1}(y_2)$. Therefore, we take a isomorphism $\pi_2 \neq \pi_1$, that satisfies $\pi_2 \circ S_{i_2}(y_2) = S_{i_1}(y_2)$. We note that:

$$\|\pi_1 \circ S_{i_2}(y_1) - \pi_2 \circ S_{i_2}(y_2)\|_2 > \max\{c_1, c_2\}/2$$

(46)

on the other hand:

$$\|\pi_1 \circ S_{i_2}(y_1) - \pi_2 \circ S_{i_2}(y_2)\|_2 = \|S_{i_1}(y_1) - S_{i_1}(y_2)\|_2 < c_1/4$$

(47)

in contradiction.

Hence, let $\pi$ be such isomorphism. To construct a continuous function over $C_{i_1} \cup C_{i_2}$, we proceed as follows. First, we replace $S_{i_2}$ with $\pi \circ S_{i_2}$ and define a selection function $S_{i_1,i_2}$ over $C_{i_1} \cup C_{i_2}$ to be:

$$S_{i_1,i_2}(y) := \begin{cases} S_{i_1}(y) & \text{if } y \in C_{i_1} \\ \pi \circ S_{i_2}(y) & \text{if } y \in C_{i_2} \end{cases}$$

(48)

Since each one of the functions $S_{i_1}$ and $\pi \circ S_{i_2}$ are continuous, they conform on $C_{i_1} \cap C_{i_2}$ and the sets $C_{i_1}$ and $C_{i_2}$ are open, $S_{i_1,i_2}$ is continuous over $C_{i_1} \cup C_{i_2}$. We define a new cover $\{\{C_i\}_{i=1}^T \setminus \{C_{i_1}, C_{i_2}\}\} \cup \{C_{i_1} \cup C_{i_2}\}$ of size $T - 1$ with locally continuous selection functions $S_1', \ldots, S_{T-1}'$.

By induction, we can construct $S$ over $\mathcal{Y}$.

**Lemma 9.** Let $f$ be a class of neural networks with $\sigma$ activations. Let $\mathcal{Y}$ be a compact class of target functions. Assume that any $y \in \mathcal{Y}$ cannot be represented as a neural network with $\sigma$ activations. Then,

$$\inf_{y \in \mathcal{Y}} \inf_{\theta \in \Theta} \|f(\cdot; \theta) - y\|_\infty > c_2$$

(49)

for some constant $c_2 > 0$.

**Proof.** Assume by contradiction that:

$$\inf_{y \in \mathcal{Y}} \min_{\theta \in \Theta} \|f(\cdot; \theta) - y\|_\infty = 0$$

(50)

Then, there is a sequence $y_n \in \mathcal{Y}$, such that:

$$\min_{\theta \in \Theta} \|f(\cdot; \theta) - y_n\|_\infty \to 0$$

(51)

Since $\mathcal{Y}$ is compact, there exists a converging sub-sequence $y_{n_k} \to y_0 \in \mathcal{Y}$. By Lem. 5, we have:

$$\min_{\theta \in \Theta} \|f(\cdot; \theta) - y_0\|_\infty = 0$$

(52)

This is in contradiction to the assumption that any $y \in \mathcal{Y}$ cannot be represented as a neural network with $\sigma$ activations.

**Lemma 10.** Let $f$ be a class of neural networks with $\sigma$ activations. Let $\mathcal{Y}$ be a compact class of target functions. Assume that any $y \in \mathcal{Y}$ cannot be represented as a neural network with $\sigma$ activations. Then, there exists a closed ball $B$ around 0 in the Euclidean space $\mathbb{R}^N$, such that:

$$\min_{\theta \in \Theta} \|f(\cdot; \theta) - y\|_\infty \leq 2 \inf_{\theta \in \Theta} \|f(\cdot; \theta) - y\|_\infty$$

(53)

**Proof.** Let $c_2 > 0$ be the constant from Lem. 9. By Lem. 9 and Lem. 6, $f_y$ is continuous over the compact set $\mathcal{Y}$. Therefore, there is a small enough $\delta > 0$, such that, for any $y_1, y_2 \in \mathcal{Y}$, such that, $\|y_1 - y_2\|_\infty < \delta$, we have: $\|f_{y_1} - f_{y_2}\|_\infty < c_2/2$. For each $y \in \mathcal{Y}$ we define $B(y) :=$
Therefore, if we take $y'$, we note that:

$$\parallel y - y'\parallel_\infty < \min\{c_2/2, \delta\}.$$  

The sets $\{B(y)\}_{y \in \mathcal{Y}}$ form an open cover to $\mathcal{Y}$. Since $\mathcal{Y}$ is a compact set, it has a finite sub-cover $\{B(y_1), \ldots, B(y_k)\}$. For each $y' \in B(y_i)$, we have:

$$\parallel f_{y_i} - y'\parallel_\infty \leq \parallel f_{y_i} - f_{y'}\parallel_\infty + \parallel f_{y'} - y'\parallel_\infty$$

$$\leq c_2/2 + \parallel f_{y'} - y'\parallel_\infty$$

$$\leq 2\parallel f_{y'} - y'\parallel_\infty$$

Therefore, if we take $H = \{\theta_i\}_{i=1}^k$ for $\theta_i$, such that, $f(\cdot; \theta_i) = f_{y_i}$, we have:

$$\min_{i \in [n]} \parallel f(\cdot; \theta_i) - y\parallel_\infty \leq 2 \inf_{\theta \in \mathcal{Y}} \parallel f(\cdot; \theta) - y\parallel_\infty$$

In particular, if we take $B$ to be the closed ball around 0 that contains $H$, we have the desired.  

**Lemma 11.** Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a $L$-Lipschitz continuous activation function. Let $f$ be a class of neural networks with $\sigma$ activations. Let $\mathcal{Y}$ be a compact class of normal target functions. Let $\rho$ be an activation function, such that, $\parallel \sigma - \rho\parallel_\infty < \delta$. Let $B = \mathcal{B}_1 \cup \mathcal{B}_2$ be the closed ball around 0, where $\mathcal{B}_1$ is the ball in Assumption 1 and $\mathcal{B}_2$ is the ball from Lem. 10. In addition, let $q$ be the class of neural networks of the same architecture as $f$ except the activations are $\rho$. Then, for any $\theta \in \mathcal{B}$, we have:

$$\parallel f(\cdot; \theta) - q(\cdot; \theta)\parallel_\infty \leq c_1 \cdot \delta$$

for some constant $c_1 > 0$ independent of $\delta$.

**Proof.** First, we note that by Lem. 10 there exists such a ball $\mathcal{B}$. We prove by induction that for any input $x \in \mathcal{X}$ the outputs the $i$'th layer of $f(\cdot; \theta)$ and $q(\cdot; \theta)$ are $O(\delta)$-close to each other.

**Base case:** we note that:

$$\parallel \sigma(W^1 \cdot x + b^1) - \rho(W^1 \cdot x + b^1)\parallel_1 \leq \sum_{i=1}^{h_2} \parallel \sigma((W^1_i \cdot x) + b^1_i) - \rho((W^1_i \cdot x) + b^1_i)\parallel_1$$

$$\leq h_2 \cdot \delta =: c^1 \cdot \delta$$

Hence, the first layer’s activations are $O(\delta)$-close to each other.

**Induction step:** assume that for any two vectors of activations $x_1$ and $x_2$ in the $i$'th layer of the neural networks, we have:

$$\parallel x_1 - x_2\parallel_1 \leq c^i \cdot \delta$$

By the triangle inequality:

$$\parallel \sigma(W^{i+1} \cdot x_1 + b^{i+1}) - \rho(W^{i+1} \cdot x_2 + b^{i+1})\parallel_1$$

$$\leq \parallel \sigma(W^{i+1} \cdot x_1 + b^{i+1}) - \sigma(W^{i+1} \cdot x_2 + b^{i+1})\parallel_1$$

$$+ \parallel \sigma(W^{i+1} \cdot x_2 + b^{i+1}) - \rho(W^{i+1} \cdot x_2 + b^{i+1})\parallel_1$$

$$\leq L \cdot \parallel (W^{i+1} \cdot x_1 + b^{i+1}) - (W^{i+1} \cdot x_2 + b^{i+1})\parallel_1$$

$$+ \sum_{j=1}^{h_{i+2}} \parallel \sigma((W^{i+1}_j \cdot x_1) + b^{i+1}_j) - \rho((W^{i+1}_j \cdot x_1) + b^{i+1}_j)\parallel_1$$

$$= L \cdot \parallel W^{i+1} (x_1 - x_2)\parallel_1 + h_{i+2} \cdot \delta$$

$$\leq L \cdot \parallel W^{i+1}\parallel_1 \cdot \parallel x_1 - x_2\parallel_1 + h_{i+2} \cdot \delta$$

$$\leq L \cdot \parallel W^{i+1}\parallel_1 \cdot c^i \cdot \delta + h_{i+2} \cdot \delta$$

$$\leq (h_{i+2} + L \cdot \parallel W^{i+1}\parallel_1 \cdot c^i) \cdot \delta$$

Since $\theta \in \mathcal{B}$ is bounded, each $\parallel W^{i+1}\parallel_1$ is bounded (for all $i \leq k$ and $\theta$). Hence, Eq. [56] holds for some constant $c_1 > 0$ independent of $\delta$.  

**Lemma 12.** Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a $L$-Lipschitz continuous activation function. Let $f$ be a class of neural networks with $\sigma$ activations. Let $\mathcal{Y}$ be a compact class of target functions. Assume that any $y \in \mathcal{Y}$ cannot be represented as a neural network with $\sigma$ activations. Let $\rho$ be an activation function, such that, $\parallel \sigma - \rho\parallel_\infty < \delta$. Let $B$ be the closed ball from Lem. 17. In addition, let $q$ be the class of
neural networks of the same architecture as $f$ except the activations are $\rho$. Then, for any $y \in \mathcal{Y}$, we have:
\[
\left| \min_{\theta \in \Theta} \|f(\cdot; \theta) - y\|_{\infty} - \min_{\theta \in \Theta} \|g(\cdot; \theta) - y\|_{\infty} \right| \leq c_1 \cdot \delta
\]  
(60)

for $c_1$ from Lem. [11]

**Proof.** By Lem. [11] for all $\theta \in \Theta$, we have:
\[
\|f(\cdot; \theta) - y\|_{\infty} \leq \|g(\cdot; \theta) - y\|_{\infty} + c_1 \cdot \delta
\]  
(61)

In particular,
\[
\min_{\theta \in \Theta} \|f(\cdot; \theta) - y\|_{\infty} \leq \min_{\theta \in \Theta} \|g(\cdot; \theta) - y\|_{\infty} + c_1 \cdot \delta
\]  
(62)

By a similar argument, we also have:
\[
\min_{\theta \in \Theta} \|g(\cdot; \theta) - y\|_{\infty} \leq \min_{\theta \in \Theta} \|f(\cdot; \theta) - y\|_{\infty} + c_1 \cdot \delta
\]  
(63)

Hence,
\[
\left| \min_{\theta \in \Theta} \|f(\cdot; \theta) - y\|_{\infty} - \min_{\theta \in \Theta} \|g(\cdot; \theta) - y\|_{\infty} \right| \leq c_1 \cdot \delta
\]  
(64)

**Lemma 13.** Let $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ be a $L$-Lipschitz continuous activation function. Let $f$ be a class of neural networks with $\sigma$ activations. Let $\mathcal{Y}$ be a compact set of target functions. Assume that any $y \in \mathcal{Y}$ cannot be represented as a neural network with $\sigma$ activations. Then, for every $\hat{\epsilon} > 0$ there is a continuous selector $S : \mathcal{Y} \rightarrow \Theta_f$, such that, for all $y \in \mathcal{Y}$, we have:
\[
\|f(\cdot; S(y)) - y\|_{\infty} \leq 2 \inf_{\theta \in \Theta_f} \|f(\cdot; \theta) - y\|_{\infty} + \hat{\epsilon}
\]  
(65)

**Proof.** Let $\rho$ be an identifiability inducing activation function, such that, $\|\sigma - \rho\|_{\infty} < \frac{1}{2c_2} \min(\hat{\epsilon}, c_1)$, where $c_1$ and $c_2$ are the constants in Lems. [11] and [9]. We note that by Lems. [11] and [12] for any $y \in \mathcal{Y}$, we have:
\[
\min_{\theta \in \Theta} \|g(\cdot; \theta) - y\|_{\infty} > c_2 - c_1 \cdot \delta > 0
\]  
(66)

where $\Theta$ is the ball from Lem. [11]. Therefore, by Lem. [11] each $y \in \mathcal{Y}$ is normal with respect to the class $g$. Hence, by Thm. [7] there is a continuous selector $S : \mathcal{Y} \rightarrow \Theta_f$, such that,
\[
\|g(\cdot; S(y)) - y\|_{\infty} = \min_{\theta \in \Theta} \|g(\cdot; \theta) - y\|_{\infty}
\]  
(67)

By Lem. [12] we have:
\[
\left| \min_{\theta \in \Theta} \|f(\cdot; \theta) - y\|_{\infty} - \|g(\cdot; S(y)) - y\|_{\infty} \right| \leq c_1 \cdot \delta
\]  
(68)

By the triangle inequality:
\[
\left| \min_{\theta \in \Theta} \|f(\cdot; \theta) - y\|_{\infty} - \|f(\cdot; S(y)) - y\|_{\infty} \right| \leq \|f(\cdot; S(y)) - y\|_{\infty} - \|g(\cdot; S(y)) - y\|_{\infty} + \left| \min_{\theta \in \Theta} \|f(\cdot; \theta) - y\|_{\infty} - \|g(\cdot; S(y)) - y\|_{\infty} \right|
\]  
(69)

By Eq. (68) and Lem. [11] we have:
\[
\left| \min_{\theta \in \Theta} \|f(\cdot; \theta) - y\|_{\infty} - \|f(\cdot; S(y)) - y\|_{\infty} \right| \leq 2c_1 \cdot \delta
\]  
(70)

Since $\delta < \hat{\epsilon}/2c_2$, we obtain the desired inequality:
\[
\|f(\cdot; S(y)) - y\|_{\infty} \leq \min_{\theta \in \Theta} \|f(\cdot; \theta) - y\|_{\infty} + \hat{\epsilon}
\]  
\[
\leq 2 \min_{\theta \in \Theta_f} \|f(\cdot; \theta) - y\|_{\infty} + \hat{\epsilon}
\]  
(71)
10.2 Proof of Thm. 2

Before we provide a formal statement of the proof, we introduce an informal outline of it.

In Lem. [3] we showed that for a compact class $\mathcal{Y}$ of target functions that cannot be represented as neural networks with $\sigma$ activations, (for $\hat{c} := \inf_{\theta \in \Theta} \| f(\cdot; \theta) - y \|_\infty$) there is a continuous selector $S(y)$ of parameters, such that,

$$\| f(\cdot; S(y)) - y \|_\infty \leq 3 \inf_{\theta \in \Theta} \| f(\cdot; \theta) - y \|_\infty$$  \hspace{1cm} (72)

Therefore, in this case, we have: $d_N(f; \mathcal{Y}) = \Theta(\hat{d}_N(f; \mathcal{Y}))$. As a next step, we would like to apply this claim on $\mathcal{Y} := \mathcal{W}_{r,m}$ and apply the lower bound of $\hat{d}_N(f; \mathcal{W}_{r,m}) = \Omega(N^{-r/m})$ to lower bound $d_N(f; \mathcal{Y})$. However, both of the classes $f$ and $\mathcal{Y}$ include constant functions, and therefore, we have: $\mathcal{Y} \cap \mathcal{Y} = \emptyset$ which contradicts the assumption that any $y \in \mathcal{Y}$ cannot be represented as a neural network with $\sigma$ activations.

To solve this issue, we consider a compact subset $\mathcal{W}_{r,m}^\gamma$ of $\mathcal{W}_{r,m}$ that does not include any constant functions but still satisfies $\hat{d}_N(f; \mathcal{W}_{r,m}^\gamma) = \Omega(N^{-r/m})$. Then, assuming that any non-constant function $y \in \mathcal{W}_{r,m}$ cannot be represented as a neural network with $\sigma$ activations, implies that any $y \in \mathcal{W}_{r,m}^\gamma$ cannot be represented as a neural network with $\sigma$ activations. In particular, by Lem. [13] we have the desired lower bound: $d_N(f; \mathcal{W}_{r,m}) \geq d_N(f; \mathcal{W}_{r,m}^\gamma) = \Theta(\hat{d}_N(f; \mathcal{W}_{r,m}^\gamma)) = \Omega(N^{-r/m})$.

For this purpose, we provide some technical notations. For a given function $f : [-1, 1]^m \to \mathbb{R}$, we define by:

$$\| h \|_{r,m}^\gamma := \sum_{1 \leq |k| \leq r} \| D^k h \|_\infty$$  \hspace{1cm} (73)

the Sobolev norm of $h$ excluding the $L_\infty$ norm on $h$. In addition, we define the Sobolev space of functions with derivatives $\geq \gamma$, as follows:

$$\mathcal{W}_{r,m}^\gamma := \left\{ f : [-1, 1]^m \to \mathbb{R} \mid \| f \|_{r,m}^\gamma \leq 1 \text{ and } \| f \|_{r,m}^\gamma \geq \gamma \right\}$$  \hspace{1cm} (74)

We notice that this set is compact, since it is closed and subset to the compact set $\mathcal{W}_{r,m}$ (see [3]).

Next, we would like to produce a lower bound for the $N$-width of $\mathcal{W}_{r,m}^\gamma$. In [8, 33], in order to achieve a lower bound for the $N$-width of $\mathcal{W}_{r,m}$, two steps are taken. First, they prove that for any $K \subset L^\infty([-1, 1]^m)$, we have: $\hat{d}_N(K) \geq b_N(K)$. Here, $b_N(K) := \sup_{X_N+1} \sup \left\{ \rho \mid \rho \cdot U(X_N+1) \subset K \right\}$ is the Bernstein $N$-width of $K$. The supremum is taken over all $N+1$ dimensional linear subspaces $X_{N+1}$ of $L^\infty([-1, 1]^m)$ and $U(X) := \{ f \in X \mid \| f \|_\infty \leq 1 \}$ stands for the unit ball of $X$. As a second step, they show that the Bernstein $N$-width of $\mathcal{W}_{r,m}$ is larger than $\Omega(N^{-r/m})$.

Unfortunately, in the general case, Bernstein’s $N$-width is very limited in its ability to estimate the nonlinear $N$-width. When considering a set $K$ that is not centered around 0, Bernstein’s $N$-width can be arbitrarily smaller than the actual nonlinear $N$-width of $K$. For example, if all of the members of $K$ are distant from 0, then, the Bernstein’s $N$-width of $K$ is zero but the nonlinear $N$-width of $K$ that might be large. Specifically, the Bernstein $N$-width of $\mathcal{W}_{r,m}^\gamma$ is small even though intuitively, this set should have a similar width as the standard Sobolev space (at least for a small enough $\gamma > 0$). Therefore, for the purpose of measuring the width of $\mathcal{W}_{r,m}^\gamma$, we define the extended Bernstein $N$-width of a set $K$,

$$\tilde{b}_N(K) := \sup_{X_{N+1}} \sup \left\{ \rho \mid \exists \beta < \rho \text{ s.t. } \rho \cdot U(X_{N+1}) \setminus \beta \cdot U(X_{N+1}) \subset K \right\}$$  \hspace{1cm} (75)

with the supremum taken over all $N+1$ dimensional linear subspaces $X_{N+1}$ of $L^\infty([-1, 1]^m)$.

The following lemma extends Lem. 3.1 in [8] and shows that the extended Bernstein $N$-width of a set $K$ is a lower bound of the nonlinear $N$-width of $K$.

**Lemma 14.** Let $K \subset L^\infty([-1, 1]^m)$. Then, $\tilde{d}_N(K) \geq \tilde{b}_N(K)$.

**Proof.** The proof is based on the proof of Lem. 3.1 in [8]. For completeness, we re-write the proof with minor modifications. Let $\rho < \tilde{b}_N(K)$ and let $X_{N+1}$ be an $N+1$ dimensional subspace of
we let \( \hat{S}(y) := S(y) - S(-y) \). We notice that, \( \hat{S}(y) \) is an odd continuous mapping of \( \partial(\rho \cdot U(X_{N+1})) \) into \( \mathbb{R}^N \). Hence, by the Borsuk-Ulam antipodality theorem \([4][26]\) (see also \([29]\)), there is a function \( y_0 \) in \( \partial(\rho \cdot U(X_{N+1})) \) for which \( S(y_0) = 0 \), i.e. \( S(-y_0) = S(y_0) \). We write

\[
2y_0 = (y_0 - f(\cdot ; S(y_0))) - (-y_0 - f(\cdot ; S(-y_0))
\]

and by the triangle inequality:

\[
2\rho = 2\|y_0\|_\infty \leq \|y_0 - f(\cdot ; S(y_0))\|_\infty + \|y_0 - f(\cdot ; S(-y_0))\|_\infty
\]

It follows that one of the two functions \( y_0, -y_0 \) are approximately by \( f(\cdot ; S(y_0)) \) with an error \( \geq \rho \). Therefore, we have: \( \alpha \geq \rho \). Since the lower bound holds uniformly for all continuous selections \( S \), we have: \( \tilde{d}_N(K) \geq \rho \).

**Lemma 15.** Let \( \gamma \in (0, 1) \) and \( r, m, N \in \mathbb{N} \). We have:

\[
\tilde{d}_N(W^r_{r,m}) \geq C \cdot N^{r/m}
\]

for some constant \( C > 0 \) that depends only on \( r \).

**Proof.** Similar to the proof of Thm. 4.2 in \([8]\) with additional modifications. We fix the integer \( r \) and let \( \phi \) be a \( C^\infty(\mathbb{R}^m) \) function which is one on the cube \([1/4, 3/4]^m\) and vanishes outside of \([-1, 1]^m\). Furthermore, let \( C_0 \) be such that \( 1 < \|D^k \phi\|_\infty \leq C_0 \), for all \( |k| < r \). With no loss of generality, we consider integers \( N \) of the form \( N = d^m \) for some positive integer \( d \) and we let \( Q_1, \ldots, Q_N \) be the partition of \([-1, 1]^m\) into closed cubes of side length \( 1/d \). Then, by applying a linear change of variables which takes \( Q_j \) to \([-1, 1]^m\), we obtain functions \( \phi_1, \ldots, \phi_N \) with \( \phi_j \) supported on \( Q_j \), such that:

\[
\forall k \text{ s.t } |k| \leq r : d^{|k|} \leq \|D^k \phi_j\|_\infty \leq C_0 \cdot \|k\|
\]

Let \( y = \sum_{j=1}^N c_j \cdot \phi_j \). By Lem. 4.1 in \([8]\), for \( p = q = \infty \), we have:

\[
\|y\|_\infty \leq C_1 \cdot N^{r/m} \cdot \max_{j \in [N]} |c_j|
\]

for some constant \( C_1 > 0 \) depending only on \( r \). By definition, for any \( x \in Q_j \), we have: \( y(x) = c_j \cdot \phi_j(x) \). In particular,

\[
\|y\|_\infty = \max_{j \in [N]} \max_{x \in Q_j} |c_j| \cdot \|\phi_j(x)\|_\infty
\]

Therefore, by Eq. 80, we have:

\[
\max_{j \in [N]} |c_j| \leq \|y\|_\infty \leq C_0 \cdot \max_{j \in [N]} |c_j|
\]

Hence,

\[
\|y\|_p^* \leq C_1 \cdot N^{r/m} \cdot \|y\|_\infty
\]

Then, by taking \( \rho := C_1^{-1} \cdot N^{-r/m} \), any \( y \in \rho \cdot U(X_N) \) satisfies \( \|y\|_p^* \leq 1 \). Again, by Lem. 4.1 and Eq. 80, we also have:

\[
\|y\|_p^* \geq C_2 \cdot \|y\|_p^* \geq C_3 \cdot N^{r/m} \cdot \max_{j \in [N]} |c_j|
\]

For some constants \( C_2, C_3 > 0 \) depending only on \( r \). By Eq. 83 we obtain:

\[
\|y\|_p^* \geq \|y\|_\infty \cdot \frac{C_3}{C_0} \cdot N^{r/m}
\]

Then, for any \( \beta > 0 \), such that,

\[
\gamma < \frac{\beta \cdot C_3}{C_0} \cdot N^{r/m} < 1
\]

we have: \( \rho \cdot U(X_N) \backslash \beta \cdot U(X_N) \subset W^r_{r,m} \). Hence, we have:

\[
\tilde{d}_N(W^r_{r,m}) \geq \tilde{b}_N(W^r_{r,m}) \geq \rho = C_1^{-1} \cdot N^{-r/m}
\]
**Theorem 8.** Let $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ be a piece-wise $C^1(\mathbb{R})$ activation function with $\sigma' \in BV(\mathbb{R})$. Let $f$ be a class of neural networks with $\sigma$ activations. Let $\mathcal{Y} = \mathcal{W}_{r,m}$. Assume that any non-constant $y \in \mathcal{Y}$ cannot be represented as a neural network with $\sigma$ activations. Then, if $d(f; \mathcal{Y}) \leq \epsilon$, we have:

$$N_f = \Omega(\epsilon^{-m/r})$$

**Proof.** Let $\hat{Y} = \mathcal{W}_{r,m}^{0,1,1} \subset \mathcal{W}_{r,m}$ (the selection of $\gamma = 0.1$ is arbitrary). We note that any $y \in \mathcal{Y}$ is non-constant. By Lem. [13] for $\epsilon = \epsilon$, there is a continuous selector $S : \hat{Y} \rightarrow \Theta_f$, such that,

$$\forall y \in \hat{Y} : \|f(\cdot; S(y)) - y\|_\infty \leq 2 \min_{\theta \in \Theta_f} \|f(\cdot; \theta) - y\|_\infty + \epsilon$$

Since $d(f; \mathcal{Y}) \leq \epsilon$, we have:

$$\forall y \in \hat{Y} : \|f(\cdot; S(y)) - y\|_\infty \leq 3\epsilon$$

By Lem. [15] we have:

$$3\epsilon \geq \hat{d}_N(\hat{Y}) \geq C \cdot N^{-r/m}$$

for some constant $C > 0$ and $N = N_f$. Therefore, we conclude that: $N_f = \Omega(\epsilon^{-m/r})$. \qed

### 10.3 Proof of Thms. 3 and 4

**Lemma 16.** Let $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ be universal, piece-wise $C^1(\mathbb{R})$ activation function with $\sigma' \in BV(\mathbb{R})$. Let $\mathcal{E}_{e,q}$ be an neural embedding method. Assume that $\|e\|_1 \leq \ell_1$ for every $e \in e$ and $q$ is a class of $\ell_2$-Lipschitz neural networks with $\sigma$ activations and bounded first layer $\|W_q^1\|_1 \leq c$. Let $\mathcal{Y} := \mathcal{W}_{1,m}$. Assume that any non-constant $y \in \mathcal{Y}$ cannot be represented as a neural network with $\sigma$ activations. If the embedding method achieves error $d(\mathcal{E}_{e,q}, \mathcal{Y}) \leq \epsilon$, then, the complexity of $q$ is:

$$N_q = \Omega\left(\epsilon^{-\min(m,2,m_1)} \right)$$

where the constant depends only on the parameters $c$, $\ell_1$, $\ell_2$, $m_1$ and $m_2$.

**Proof.** Assume that $N_q = o(\epsilon^{-(m_1+m_2)})$. For every $y \in \mathcal{Y}$, we have:

$$\inf_{\theta_e, \theta_q} \left\| y - q(x, e(I; \theta_e); \theta_q) \right\|_\infty \leq \epsilon$$

We denote by $k$ the output dimension of $e$. Let $\sigma \circ W_q^1$ be the first layer of $q$. We consider that $W_q^1 \in \mathbb{R}^{w_1 \times (m_1 + k)}$, where $w_1$ is the size of the first layer of $q$. One can partition the layer into two parts:

$$\sigma(W_q^1(x, e(x; \theta_e))) = \sigma(W_q^{1,1}_q x + W_q^{1,2}_q e(I; \theta_e))$$

where $W_q^{1,1} \in \mathbb{R}^{w_1 \times m_1}$ and $W_q^{1,2} \in \mathbb{R}^{w_1 \times k}$. We divide into two cases.

#### Case 1

Assume that $w_1 = \Omega(\epsilon^{-m_1})$. Then, by the universality of $\sigma$, we can approximate the class of functions $e$ with a class $d$ of neural networks of size $O(k \cdot \epsilon^{-m_2})$ with $\sigma$ activations. To show it, we can simply take $k$ neural networks of sizes $O((\epsilon/\ell_1)^{-m_2}) = O(\epsilon^{-m_2})$ to approximate the $i$th coordinate of $e$ separately. By the triangle inequality, for all $y \in \mathcal{Y}$, we have:

$$\inf_{\theta_e, \theta_d} \left\| y - q(x, d(I; \theta_d); \theta_q) \right\|_\infty \leq \inf_{\theta_e, \theta_d, \theta_q} \left\{ \left\| y - q(x, e(I; \theta_e); \theta_q) \right\|_\infty + \left\| q(x, d(I; \theta_d); \theta_q) - q(x, e(I; \theta_e); \theta_q) \right\|_\infty \right\}$$

$$\leq \sup_{\theta_d} \inf_{\theta_e} \left\{ \left\| y - q(x, e(I; \theta_e); \theta_q^*) \right\|_\infty + \left\| q(x, d(I; \theta_d); \theta_q^*) - q(x, e(I; \theta_e); \theta_q^*) \right\|_\infty \right\}$$

$$\leq \sup_{\theta_d} \inf_{\theta_e} \left\{ q(x, d(I; \theta_d); \theta_q^*) - q(x, e(I; \theta_e); \theta_q^*) \right\}_\infty + \epsilon$$

where $\theta_q^*$, $\theta_e^*$ are the minimizers of $\left\| y - q(x, e(I; \theta_e); \theta_q) \right\|_\infty$. Next, by the Lipschitzness of $q$, we have:

$$\inf_{\theta_d} \left\| q(x, d(I; \theta_d); \theta_q^*) - q(x, e(I; \theta_e); \theta_q^*) \right\|_\infty \leq \ell_2 \cdot \inf_{\theta_d} \left\| d(I; \theta_d) - e(I; \theta_e) \right\|_\infty \leq \ell_2 \cdot \epsilon$$
In particular,
\[ \inf_{\delta_d,\theta_q} \left\| y - q(x,d(I;\theta_d);\theta_q) \right\|_\infty \leq (\ell_2 + 1) \cdot \epsilon \quad (98) \]

By Thm. 3, the size of the architecture \( q(x,d(I;\theta_d);\theta_q) \) is \( \Omega(\epsilon^{-m}) \). Since \( N_q = o(\epsilon^{-(m_1+m_2)}) \), we must have \( k = \Omega(\epsilon^{-m_1}) \). Otherwise, the overall size of the neural network \( q(x,d(I;\theta_d);\theta_q) \) is \( o(\epsilon^{-(m_1+m_2)}) + O(\epsilon \cdot \epsilon^{-m_2}) = o(\epsilon^{-m}) \) in contradiction. Therefore, the size of \( q \) is at least \( w_1 \cdot k = \Omega(\epsilon^{-2m_1}) \).

**Case 2** Assume that \( w_1 = o(\epsilon^{-m_1}) \). In this case we approximate the class \( W_q^{1,2} \cdot e \), where \( W_q^{1,2} \in \mathbb{R}^{w_1 \times k} \), where \( \|W_q^{1,2}\|_1 \leq c \). The approximation is done using a class \( d \) of neural networks of size \( O(w_1 \cdot \epsilon^{-m_2}) \). By the same analysis of Case 1, we have:
\[ \inf_{\delta_d,\theta_q} \left\| y - \tilde{q}(x,d(I;\theta_d);\theta_q) \right\|_\infty \leq (\ell_2 + 1) \cdot \epsilon \quad (99) \]
where \( \tilde{q} = q'(W_q^{1,1} x + 1 \cdot d(I;\theta_d)) \) and \( q' \) consists of the layers of \( q \) excluding the first layer. We notice that \( W_q^{1,1} x + 1 \cdot d(I;\theta_d) \) can be represented as a matrix multiplication \( M \cdot (x,d(I;\theta_d)) \), where \( M \) is a block diagonal matrix with blocks \( W_q^{1,1} \) and \( 1 \). Therefore, we achieved a neural network that approximates \( y \). However, the overall size of \( q(x,d(I;\theta_d);\theta_q) \) is \( o(\epsilon^{-(m_1+m_2)}) + O(w_1 \cdot \epsilon^{-m_2}) = o(\epsilon^{-m}) \) in contradiction.

**Lemma 17.** Let \( \sigma \) be a universal piece-wise \( C^1(\mathbb{R}) \) activation function with \( \sigma' \in BV(\mathbb{R}) \). Let neural embedding method \( E_{e,q} \). Assume that \( \|e\|_1 \leq \ell_1 \) and the output dimension of \( e \) is \( k = O(1) \) for every \( e \in e \). Assume that \( q \) is a class of \( \ell_2 \)-Lipschitz neural networks with \( \sigma \) activations. Let \( \mathcal{Y} := W_{1,m} \). Assume that any non-constant \( y \in \mathcal{Y} \) cannot be represented as neural networks with \( \sigma \) activations. If the embedding method achieves error \( d(E_{e,q},\mathcal{Y}) \leq \epsilon \), then, the complexity of \( q \) is:
\[ N_q = \Omega(\epsilon^{-m}) \quad (100) \]
where the constant depends only on the parameters \( \ell_1, \ell_2, m_1 \) and \( m_2 \).

**Proof.** Follows from the analysis in Case 1 of the proof of Lem. 16.

**Theorem 4.** In the setting of Thm. 3 except \( k \) is not necessarily \( O(1) \). Assume that the first layer of any \( q \in q \) is bounded \( \|W_1^1\|_1 \leq c \), for some constant \( c > 0 \). If the embedding method achieves error \( d(E_{e,q},\mathcal{Y}) \leq \epsilon \), then, the complexity of \( q \) is:
\[ N_q = \Omega(\epsilon^{-\min(m,2m_1)}) \quad (10) \]

**Proof.** First, we note that since \( \sigma' \in BV(\mathbb{R}) \), we have: \( \|\sigma'\|_\infty < \infty \). In addition, \( \sigma \) is piece-wise \( C^1(\mathbb{R}) \), and therefore, by combining the two, it is Lipschitz continuous as well. Let \( e := e(I;\theta_e) \) and \( q := q(x,z;\theta_q) \) be members of \( e \) and \( q \) respectively. By Lems. 3 and 4 we have:
\[ \|e\|_\infty = \sup_{I \in I} \|e(I;\theta_e)\|_1 \leq \ell_1 \cdot \|I\|_1 \leq m_2 \cdot \ell_1 \quad (101) \]
and also
\[ \text{Lip}(e) \leq \ell_1 \quad (102) \]
Since the functions \( e \) are continuously differentiable, we have:
\[ \sum_{1 \leq |k|_1 \leq 1} \|D^k e\|_\infty \leq \|\nabla e\|_\infty \leq \text{Lip}(e) \leq \ell_1 \quad (103) \]
Hence,
\[ \|e\|_1^* \leq (m_2 + 1) \cdot \ell_1 \quad (104) \]
By similar considerations, we have: \( \text{Lip}(q) \leq \ell_2 \). Therefore, by Lem. 16 we have the desired. \( \square \)
Theorem 3. Let $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ be a universal, piece-wise $C^1(\mathbb{R})$ activation function with $\sigma' \in BV(\mathbb{R})$ and $\sigma(0) = 0$. Let $\mathcal{E}_{e,q}$ be an neural embedding method. Assume that $\epsilon$ is a class of continuously differentiable neural network $e$ with zero biases, output dimension $k = \mathcal{O}(1)$ and bounded spectral complexity $C(e) \leq \ell_1$ and $q$ is a class of neural networks $q$ with $\sigma$ activations, bounded spectral complexity $C(q) \leq \ell_2$. Let $\mathcal{Y} := \mathcal{W}_{1,m}$. Assume that any non-constant $y \in \mathcal{Y}$ cannot be represented as a neural network with $\sigma$ activations. If the embedding method achieves error $d(\mathcal{E}_{e,q}, \mathcal{Y}) \leq \epsilon$, then, the complexity of $q$ is:

$$N_q = \Omega \left( \epsilon^{-(m_1+m_2)} \right)$$  

(9)

Proof. Follows from Lem. 17 and the proof of Thm. 3.

10.4 Proof of Thm. 5

Lemma 18. Let $y \in \mathcal{W}_{r,m}$. Then, $\{y_I\}_{I \in \mathcal{I}}$ is compact and $F : I \rightarrow y_I$ is a continuous function.

Proof. First, we note that the set $\mathcal{X} \times \mathcal{I}$ is compact, since it is a closed and bounded subset of a Euclidean space. Since $y$ is continuous, it is uniformly continuous over $\mathcal{X} \times \mathcal{I}$. Therefore, the function $F : I \rightarrow y_I$ is a continuous function,

$$\lim_{I \rightarrow I_0} \|y_I - y_{I_0}\|_\infty = \lim_{I \rightarrow I_0} \sup_{x \in \mathcal{X}} \|y(x, I) - y(x, I_0)\| = 0$$  

(105)

In addition, $\mathcal{I}$ is compact since it is a closed and bounded subset of a Euclidean space as well. Hence, the image $\{y_I\}_{I \in \mathcal{I}}$ of $F$ is compact.

Theorem 9. Let $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ be a universal, piece-wise $C^1(\mathbb{R})$ activation function with $\sigma' \in BV(\mathbb{R})$. Let $\mathcal{Y} := \mathcal{W}_{r,m}$. Assume that any non-constant $y \in \mathcal{Y}$ cannot be represented as a neural network with $\sigma$ activations. Then, there is a class of neural networks with $\sigma$ activations, such that, for any $y \in \mathcal{Y}$, there is a large enough neural network $f$, such that, the hypernetwork $h(x, I) = g(x; f(I; \theta_f))$ achieves error $\leq \epsilon$ in approximating $y$ and:

$$N_q = \mathcal{O} \left( \epsilon^{-m_1/r} \right)$$  

(106)

where the constant depends on $m_1$, $m_2$ and $r$.

Proof. By the universality of $\sigma$, there is a class of neural networks $q$ with $\sigma$ activations of size:

$$N_q = \mathcal{O} \left( \epsilon^{-m_1/r} \right)$$  

(107)

such that,

$$\forall p \in \mathcal{W}_{m_1,r} : \inf_{\theta_g \in \Theta_q} \|g(\cdot; \theta_g) - p\|_\infty \leq \epsilon$$  

(108)

We note that, for each $I \in \mathcal{I}$, $y_I \in \mathcal{W}_{m_1,r}$. Therefore,

$$\forall I \in \mathcal{I} : \inf_{\theta_g \in \Theta_q} \|g(\cdot; \theta_g) - y_I\|_\infty \leq \epsilon$$  

(109)

By Lem. 13 there is a continuous selector $S : \mathcal{I} \rightarrow \Theta_q$, such that, for any $I \in \mathcal{I}$, we have:

$$\|g(\cdot; S(I)) - y_I\|_\infty \leq \inf_{\theta_g \in \Theta_q} \|g(\cdot; \theta_g) - y_I\|_\infty + \epsilon \leq 2\epsilon$$  

(110)

Since $S$ is a continuous over the set $\mathcal{I} = [-1, 1]^{m_2}$, by 13, one can approximate $S$ up to any accuracy $\hat{\epsilon} > 0$ using a large enough ReLU neural network $f$. The set $\mathcal{I}$ is compact, and $S$ is continuous. Therefore, $\{S(I)\}_{I \in \mathcal{I}}$ is compact as well. Therefore, there exists a closed ball $B$ around 0 that contains $\{S(I)\}_{I \in \mathcal{I}}$. We notice that $g$ is uniformly continuous with respect to $\theta_g \in B$, and therefore, for a small enough $\tilde{\epsilon}$, we have:

$$\forall I \in \mathcal{I} : \|g(\cdot; f(I)) - y_I\|_\infty \leq 3\epsilon$$  

(111)

as desired.
10.5 Proof of Thm. 5

**Theorem 10.** Let $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ be a universal Lipschitz continuous activation function, such that, $\sigma(0) = 0$. Let $g$ be a class of neural networks with $\sigma$ activations. Let $y \in \mathbb{Y} := \mathcal{W}_{r,m}$ be a target function. Assume that there is a continuous selector $S \in \mathcal{P}_{r,w,c}$ for the class $\{y_x\}_{x \in X}$ within $g$. Then, there is a hypernetwork $\mathcal{H}_{f,g}$ that achieves error $\leq \epsilon$ in approximating $y$, such that:

$$N_f = O(w^{1+m_2/r} \cdot e^{-m_2/r} + w \cdot N_g)$$

$$= O(e^{-m_2/r} + \epsilon^{-m_1/r})$$

**(112)**

**Proof.** We would like to approximate the function $S$ using a neural network $f$ of the specified complexity. Since $S \in \mathcal{P}_{r,w,c}$, we can represent $S$ in the following manner:

$$S(I) = M \cdot P(I)$$

**(113)**

Here, $P : \mathbb{R}^{m_2} \rightarrow \mathbb{R}^w$ and $M \in \mathbb{R}^{N_x \times w}$ is some matrix of bounded norm $\|M\|_1 \leq c$. We recall that any constituent function $P_i$ are in $\mathcal{W}_{r,m_2}$. By [31], such functions can be approximated by neural networks of sizes $O(e^{-m_2/r})$ up to accuracy $\epsilon > 0$. Hence, we can approximate $S(I)$ using a neural network $f(I) := M \cdot H(I)$, where $H : \mathbb{R}^{m_2} \rightarrow \mathbb{R}^w$, such that, each coordinate $H_i$ is of size $O(e^{-m_2/r})$. The error of $f$ in approximating $S$ is therefore upper bounded as follows:

$$\|M \cdot H(I) - M \cdot P(I)\|_1 \leq \|M\|_1 \cdot \|H(I) - P(I)\|_1$$

$$\leq c \cdot \sum_{i=1}^{w} |H_i(I) - P_i(I)|$$

**(114)**

$$\leq c \cdot w \cdot \epsilon$$

In addition,

$$\|M \cdot P(I)\|_1 \leq \|M\|_1 \cdot \|P(I)\|_1 \leq c \cdot w$$

**(115)**

Therefore, each one of the output matrices and biases in $S(I)$ is of norm bounded by $c \cdot w$.

Next, we denote by $W^i$ and $b^i$ the weight matrices and biases in $S(I)$ and by $V^i$ and $d^i$ the weight matrices and biases in $f(I)$. We would like to prove by induction that for any $x \in \mathcal{X}$ and $I \in \mathcal{I}$, the activations of $g(x; S(I))$ and $g(x; f(I))$ are at most $O(\epsilon)$ distant from each other and the norm of these activations is $O(1)$.

**Base case:** let $x \in \mathcal{X}$. Since $\mathcal{X}$ is bounded, $\|x\|_1 \leq m_1 =: \alpha^1$. In addition, we have:

$$\|\sigma(W^1 x + b^1) - \sigma(V^1 x + d^1)\|_1 \leq L\|W^1 x + b^1 - (V^1 x + d^1)\|_1$$

$$\leq L\|W^1 - V^1\|_1 \|x\|_1 + \|b^1 - d^1\|_1$$

$$\leq m_1 \cdot L \cdot c \cdot w \cdot \epsilon + c \cdot w \cdot \epsilon$$

$$=: \beta^1 \cdot \epsilon$$

**(116)**

Here, $L$ is the Lipschitz constant of $\sigma$.

**Induction step:** let $x_1$ and $x_2$ be the activations of $g(x; S(I))$ and $g(x; f(I))$ in the $i$'th layer. Assume that there are constants $\alpha^i, \beta^i > 0$ (independent of the size of $g$, $x_1$ and $x_2$), such that, $\|x_1 - x_2\|_1 \leq \beta^i \cdot \epsilon$ and $\|x_1\|_1 \leq \alpha^i$. Then, we have:

$$\|\sigma(W^{i+1} x_1 + b^{i+1})\|_1 = \|\sigma(W^{i+1} x_1 + b^{i+1}) - \sigma(0)\|_1$$

$$\leq L \cdot \|W^{i+1} x_1 + b^{i+1} - 0\|_1$$

$$\leq L \cdot \|W^{i+1} x_1\|_1 + L \cdot \|b^{i+1}\|_1$$

$$\leq L \cdot \|W^{i+1}\|_1 \cdot \|x_1\|_1 + L \cdot c \cdot w$$

$$\leq L \cdot c \cdot w (1 + \alpha^i) =: \alpha^{i+1}$$

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and also:
\[
\| \sigma(W^{i+1} \cdot x_1 + b^{i+1}) - \sigma(V^{i+1}x_2 + d^{i+1}) \|_1 \\
\leq L \cdot \| (W^{i+1} \cdot x_1 + b^{i+1}) - (V^{i+1}x_2 + d^{i+1}) \|_1 \\
\leq L \cdot \| W^{i+1}x_1 - V^{i+1}x_2 \|_1 + L \cdot \| b^{i+1} - d^{i+1} \|_1 \\
\leq L \cdot \| W^{i+1}x_1 - V^{i+1}x_2 \|_1 + L \cdot \epsilon \\
\leq L \cdot (\| W^{i+1} \|_1 \cdot \| x_1 - x_2 \|_1 + \| W^{i+1} - V^{i+1} \|_1 \cdot \| x_2 \|_1 ) + L \cdot \epsilon \\
\leq L \cdot (c \cdot \| x_1 - x_2 \|_1 + c \cdot \| x_2 \|_1 ) + L \cdot \epsilon \\
\leq L \cdot (c \cdot \| x_1 - x_2 \|_1 + c \cdot \| x_2 \|_1 ) + L \cdot \epsilon \\
\leq L \cdot (c \cdot \| x_1 - x_2 \|_1 ) + L \cdot \epsilon \\
= \beta^{i+1} \cdot \epsilon
\]

If \(i+1\) is the last layer, than the application of \(\sigma\) is not present. In this case, \(\alpha^{i+1}\) and \(\beta^{i+1}\) are the same except the multiplication by \(L\). Therefore, we conclude that \(\| g(\cdot; S(I)) - g(x; f(I))\|_\infty = O(\epsilon)\).

Since \(f\) consists of \(w\) hidden functions \(H\) and a matrix \(M\) of size \(w \cdot N_g\), the total number of trainable parameters of \(f\) is: \(N_f = O(w^{1+m_2/r} \cdot e^{-m_2/r} + w \cdot N_g)\) as desired. \(\square\)