A General Robust Bayes Pseudo-Posterior: Exponential Convergence results with Applications

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Abstract

Although Bayesian inference is an immensely popular paradigm among a large segment of scientists including statisticians, most of the applications consider the objective priors and need critical investigations (Efron, 2013). And although it has several optimal properties, one major drawback of Bayesian inference is the lack of robustness against data contamination and model misspecification, which becomes pernicious in the use of objective priors. This paper presents the general formulation of a Bayes pseudo-posterior distribution yielding robust inference. Exponential convergence results related to the new pseudo-posterior and the corresponding Bayes estimators are established under the general parametric set-up and illustrations are provided for the independent stationary models and the independent non-homogenous models. For the first case, the discrete priors and the corresponding maximum posterior estimators are discussed with additional details. We further apply this new pseudo-posterior to propose robust versions of the Bayes predictive density estimators and the expected Bayes estimator for the fixed-design (normal) linear regression models; their properties are illustrated both theoretically as well as empirically.

Keywords: Robust Bayes Estimator; Pseudo-Posterior; Density Power Divergence; Exponential Convergence; Predictive Density Estimator; Bayesian Linear Regression.

1 Introduction

Bayesian analysis is arguably one of the most popular statistical paradigms with increasing applications across different disciplines of science and industry. It is widely preferred by many non-statisticians due to its nice interpretability and incorporation of the prior knowledge about experimental quantities. From a statistical point of view, it is widely accepted even among many non-Bayesians, because of its nice optimal (asymptotic) properties. Bayesian inference is built on the famous ‘Bayes theorem’, the celebrated 1763 paper of Thomas Bayes, which combines prior knowledge with experimental evidence to produce the posterior conclusion. However, over these 250 years of applications, Bayesian inference has also been subject to several criticisms and some of these debates are still ongoing; Efron (2013) termed the Bayes’ theorem as a “controversial theorem”. Other than the controversies about its internal logic (Halpern, 1999a,b; Arnborg and Sjödin, 2001; Dupre and Tipler, 2009), a major practical drawback of Bayesian inference is its non-robust nature against
misspecification in models (including data contamination and outliers) and the priors, as has been extensively observed in the literature; see Berk (1966), McCulloch (1989), Weiss (1996), Weiss and Cho (1998), Millar and Stewart (2007), De Blasi and Walker (2012), Walker (2013), Owhadi and Scovel (2014, 2015), Owhadi et al. (2013, 2015a,b) and the references therein. However, the optimal solution to this problem has been developed mainly for prior misspecifications and has a long history (Berger 1984, 1994; Berger and Berliner, 1986; Wasserman, 1990; Gelfand and Dey, 1991; Dey and Birmiwal, 1994; Delampady and Dey, 1994; Dey et al., 1996; Gustafson and Wasserman, 1995; Ghosh et al., 2006; Martin et al., 2009); this is because the Bayesians traditionally suggest the model to be prefixed perfectly. Thus the possibility of model misspecification has been generally ignored.

In applying Bayesian inference to different complicated datasets of the present era, however, we need to use many complex and sophisticated models which are highly prone to misspecification. In reality, where “All models are wrong” (Box, 1976), the Bayesian philosophy of refining the fixed model adaptively (Gelman et al., 1996) often fails to handle complex modern scenarios or leads to ‘a model as complex as the data’ (Wang and Blei, 2016). These certainly led to severe consequences in posterior conclusions having erroneous implications. The problem becomes more clear but pernicious in case of inference with objective or reference priors. As a simple example, the Bayes estimate of the mean of a normal model, with any objective prior and symmetric loss function, is the extremely non-robust sample mean. What is a matter of greater concern, as noted by Efron (2013), is that most of the recent applications of Bayesian inference hinge on objective priors and so they always need to be scrutinized carefully, sometime even from a frequentist perspective. The posterior non-robustness against model misspecification makes the process vulnerable and we surely need an appropriate solution to this problem which gives due regard to the robustness issue.

From a true Bayesian perspective, there are only few solutions to the problem of model misspecification (Ritov, 1985, 1987; Sivaganesan, 1993; Dey et al., 1996; Shyamalkumar, 2000). However, most of them, if not all, assume that the perturbation in the model is known beforehand, such as gross error contaminated models with known contamination proportion $\epsilon$. This is rarely possible in practice, specially with the modern complex datasets and so it restricts their real-life applications. Some attempts have been made to develop alternative solutions by linking Bayesian inference suitably with the frequentist concept of robustness. Note that, in the frequentist sense, there are two major approaches to achieve robustness, namely the use of heavy tailed distributions (e.g., $t$-distribution in place of normal), or new (robust) inference methodologies (Hampel et al., 1986; Basu et al., 2011). The first one has been adapted by some Bayesian scientists; see Andrade and O’Hagan (2006, 2011) and Desgagne (2013) among others. However, the difficulty with this approach is the availability of appropriate heavy tailed alternatives in complex scenarios and it indeed does not solve the non-robustness of Bayesian inference for a specified model (which might be of a lighter tail). The second approach of frequentist robustness, namely the modified
inference methodologies, serves the purpose but differs in the strictest probabilistic sense from the Bayesian philosophy, since one needs to alter the posterior density appropriately to achieve robustness against data contamination or model misspecification; the resulting modified posteriors are generally referred to as pseudo-posterior densities. Different such pseudo-posteriors have been proposed by Greco et al. (2008), Agostinelli and Greco (2013), Cabras et al. (2014), Hooker and Vidyashankar (2014), Ghosh and Basu (2016), Danesi et al. (2016), Atkinson et al. (2017) and Nakagawa and Hashimoto (2017); all of them have primarily considered independent stationary models and have different pros and cons.

Another recent attempt, in the borderline of these two approaches, has been proposed by Wang and Blei (2016), who have transformed the given model to a localized model involving hyperparameters to be estimated through the empirical Bayes approach.

Here, we consider a particular pseudo-posterior originally proposed by Ghosh and Basu (2016) in the independently and identically distributed (IID) set-up. The choice of this pseudo-posterior has been motivated by the several particularly nice properties that it has been observed to possess, and its potential for extension to more general set-ups. As a brief description, consider \( n \) IID random variables \( X_1, \ldots, X_n \) taking values in a measurable space \((\chi, \mathcal{B})\). Assume that there is an underlying true probability space \((\Omega, \mathcal{B}_\Omega, P)\) such that, for \( i = 1, \ldots, n \), \( X_i \) is \( \mathcal{B}/\Omega \) measurable, independent with respect to \( P \) and its induced distribution \( G(x) \) has an absolutely continuous density \( g(x) \) with respect to some dominating \( \sigma \)-finite measure \( \lambda(dx) \). We model \( G \) by a parametric family of distributions \( \{F_\theta : \theta \in \Theta \subseteq \mathbb{R}^p\} \) which is assumed to be absolutely continuous with respect to \( \lambda \) having density \( f_\theta \). Consider a prior density for \( \theta \) over the parameter space \( \Theta \) given by \( \pi(\theta) \).

Following Ghosh and Basu (2016), the robust pseudo-posterior density, namely the \( R(\alpha) \)-posterior density of \( \theta \), given the sample observation \( x_n = (x_1, \ldots, x_n)^T \) on the random variable \( X_n = (X_1, \ldots, X_n)^T \), is defined as

\[
\pi_n(\alpha|\theta|x_n) = \frac{\exp(q_n(\alpha)(x_n|\theta))\pi(\theta)}{\int \exp(q_n(\alpha)(x_n|\theta'))\pi(\theta')d\theta'}, \quad \alpha \geq 0,
\]

where \( q_n(\alpha)(x_n|\theta) \) is the \( \alpha \)-likelihood of \( x_n \) given by

\[
q_n(\alpha)(x_n|\theta) = \frac{n}{1+\alpha} \left[ \frac{1+\alpha}{\alpha} \int f_\theta^\alpha dG_n - \int f_\theta^{1+\alpha} \right] - \frac{n}{\alpha} = \frac{1}{\alpha} \sum_{i=1}^n f_\theta^\alpha(x_i) - \frac{n}{1+\alpha} \int f_\theta^{1+\alpha} - \frac{n}{\alpha} = \sum_{i=1}^n q_\theta(\alpha)(x_i),
\]

with \( G_n \) being the empirical distribution based on the data and

\[
q_\theta(\alpha)(y) = \frac{1}{\alpha} (f_\theta^\alpha(y) - 1) - \frac{1}{1+\alpha} \int f_\theta^{1+\alpha}.
\]

Note that, in a limiting sense, \( q_n(0)(x_n|\theta) = n \int \log(f_\theta)dG_n - n = \sum_{i=1}^n (\log(f_\theta(x_i)) - 1), \)
which is simply the usual log-likelihood (plus a constant); so the $R^{(0)}$-posterior is just the ordinary Bayes posterior. The idea came from a frequentist robust estimator, namely the minimum density power divergence (DPD) estimator (MDPDE) of Basu et al. (1998), which has proven to be a useful robust generalization of the classical maximum likelihood estimator (MLE); see Ghosh and Basu (2016) for more details. The important similarity of this approach (at $\alpha > 0$) with the usual Bayes posterior (at $\alpha = 0$) is that, it does not require nonparametric smoothing like some other pseudo-posteriors and it is additive in the data so that the posterior update is easily possible with new observations. Ghosh and Basu (2016) extensively illustrated its robustness with respect to the model and prior misspecification and proved a Bernstein-von Mises type limiting result under the IID set-up.

In this paper, we provide a generalization of the $R^{(\alpha)}$-posterior density for a completely general model set-up beyond IID data, by imposing a suitable structure and conditions to define the $\alpha$-likelihood function. Their forms are explicitly derived for several important applications like the independent non-homogeneous data including regressions, time series and Markov model data, diffusion processes etc. The concept of the associated $R^{(\alpha)}$-marginal density of data has also been introduced. In addition to the Ghosh and Basu (2016) interpretations, we show that the $R^{(\alpha)}$-posterior density can also be thought of as an ordinary Bayes posterior with a suitably modified model and modified prior (see Remark 2.1), and hence we are also able to retain the conditional probability interpretation through this proposed approach of robust Bayes analysis.

As a second major contribution of the present paper, we derive the exponential convergence results associated with the new $R^{(\alpha)}$-posterior probabilities under a completely general set-up. This, in fact, generalizes the corresponding results for the usual Bayes posterior (Barron, 1988) and is seen to hold under appropriate assumptions based on the concepts of merging of distributions in probability, (modified) prior negligibility and existence of uniform exponential consistent tests. Some simpler conditions indicating the merging in probability phenomenon are derived for the $R^{(\alpha)}$-posterior under the general set-up. The required assumptions are explicitly studied and simplified for the two most common set-ups, namely for the IID data and the independent non-homogeneous data including fixed-design regressions; the results provide some significant insights about their convergence rates. Further, as an application, the exponential consistency of the Bayes estimates associated with the $R^{(\alpha)}$-posterior is shown for a general class of “bounded” loss functions. The interesting cases of discrete priors are discussed separately, along with the exponential consistency of the corresponding maximum $R^{(\alpha)}$-posterior estimator under the IID set-up. These optimality results further justify the usefulness of the proposed $R^{(\alpha)}$-Bayes estimators besides their already proved robustness advantages.

An important application of the Bayesian inference is in predictive density estimation for future data. As a third contribution, we propose a robust version of the Bayesian predictive density estimator based on the $R^{(\alpha)}$-posterior density and prove its exponential consistency for the loss functions like squared error loss, absolute error loss, Hellinger loss and 0-1 loss.
Their robustness properties are also illustrated empirically.

The rest of the paper is organized as follows. In Section 2, we extend the form of the $R^{(\alpha)}$-posterior density for a completely general set-up of parametric estimation along with several examples. Section 3 presents the main results on the exponential consistency of the $R^{(\alpha)}$-posterior probabilities and the $R^{(\alpha)}$-Bayes estimators. In Section 4, we apply these general results to the independent stationary models. The cases of discrete priors and the robust Bayes predictive density estimators are discussed in Sections 4.2 and 4.3, respectively. Section 5 presents the applications to the independent non-homogeneous models including the fixed design regressions. The usefulness of the proposed $R^{(\alpha)}$-posterior are also examined numerically in the linear regression models in this section. Finally the paper ends with a brief concluding discussion. The issues of merging in probability and the associated results are presented in Appendix A. Proofs of all theorems are given in the Online Supplement.

2 A general form of the $R^{(\alpha)}$-posterior distribution

In order to extend the $R^{(\alpha)}$-posterior density to a more general set-up, let us assume that the random variable $\mathbf{X}_n$ is defined on a general measurable space $(\chi_n, B_n)$ for each $n$. Also assume that there is an underlying true probability space $(\Omega, B_{\Omega}, \mathbb{P})$ such that, for each $n \geq 1$, $\mathbf{X}_n$ is $B_n/\Omega$ measurable and its induced distribution $G^n(\mathbf{x}_n)$ is absolutely continuous with respect to some $\sigma$-finite measure $\lambda^n(dx_n)$ having “true” probability density $g^n(\mathbf{x}_n)$. We model it by a parametric family of distributions $\mathcal{F}_n = \{F^n(\cdot|\theta) : \theta \in \Theta_n \subseteq \mathbb{R}^p\}$ where the elements of $\mathcal{F}_n$ are assumed to be absolutely continuous with respect to $\lambda^n$ having density $f^n(\mathbf{x}_n|\theta)$. Note that, we have not assumed the parameter space $\Theta_n$ to be independent of the sample size $n$. Similarly, the prior measure $\pi_n(\theta)$ on $\Theta_n$ may be $n$-dependent with $\pi_n(\Theta_n) \leq 1$. Consider a $\sigma$-field $B_{\Theta_n}$ on the parameter space $\Theta_n$. Generalizing from (2), we suitably define the $\alpha$-likelihood function $q^{(\alpha)}_n(\mathbf{x}_n|\theta)$ in such a way that ensures

$$q^{(0)}_n(\mathbf{x}_n|\theta) := \lim_{\alpha \downarrow 0} q^{(\alpha)}_n(\mathbf{x}_n|\theta) = \log f^n(\mathbf{x}_n|\theta) - n, \text{ for all } \mathbf{x}_n \in \chi_n. \quad (4)$$

Our definition should guarantee that the $\alpha$-likelihood, as a function of $\theta$, is $B_{\Theta_n}$ measurable for each $\mathbf{x}_n$ and jointly $B_n \times B_{\Theta_n}$ measurable when both $\mathbf{X}_n$ and $\theta$ are random. Then, for this general set-up, we define the corresponding $R^{(\alpha)}$-posterior probabilities following (1) as

$$\pi^{(\alpha)}_n(A_n|\mathbf{x}_n) = \frac{\int_{A_n} \exp(q^{(\alpha)}_n(\mathbf{x}_n|\theta)) \pi_n(\theta)d\theta}{\int_{\Theta_n} \exp(q^{(\alpha)}_n(\mathbf{x}_n|\theta)) \pi_n(\theta)d\theta}, \quad A_n \in B_{\Theta_n}, \quad (5)$$

whenever the denominator is finitely defined and is positive; otherwise we may define it arbitrarily, e.g., $\pi^{(\alpha)}_n(A_n|\mathbf{x}_n) = \pi_n(A_n)$. (4) ensures that $\pi^{(0)}_n$ is the usual Bayes posterior.

For an alternative representation, we define $Q^{(\alpha)}_n(S_n|\theta) := \int_{S_n} \exp(q^{(\alpha)}_n(\mathbf{x}_n|\theta))d\mathbf{x}_n$, $M^{(\alpha)}_n(S_n, A_n) := \int_{A_n} Q^{(\alpha)}_n(S_n|\theta) \pi_n(\theta)d\theta$ and $M^{(\alpha)}_n(S_n) := M^{(\alpha)}_n(S_n, \Theta_n)/M^{(\alpha)}_n(\chi_n, \Theta_n)$ for $S_n \in B_n$ and $A_n \in B_{\Theta_n}$. Throughout this paper, we will assume that the model and priors
are chosen so as to satisfy \(0 < M_n^{(\alpha)}(\chi_n, \Theta_n) < \infty\). Then, the last two measures have density functions with respect to \(\lambda^{(n)}(d\chi_n)\) given by 

\[
m_n^{(\alpha)}(\chi_n, A_n) = \int_{A_n} \exp(\varphi_n^{(\alpha)}(\chi_n)) \pi_n(\theta) d\theta
\]

and

\[
m_n^{(\alpha)}(\chi_n) = m_n^{(\alpha)}(\chi_n, \Theta_n)/M_n^{(\alpha)}(\chi_n, \Theta_n)
\]

respectively. Clearly, \(m_n^{(\alpha)}(\chi_n)\) is a proper probability density function, which we refer to as the \(R^{(\alpha)}\)-marginal density of \(X_n\); the corresponding \(R^{(\alpha)}\)-marginal distribution is \(M_n^{(\alpha)}(\cdot)\). It gives a robust version of the ordinary Bayes marginal for \(\alpha > 0\); \(m_n^{(0)}(\chi_n)\) corresponds to the usual marginal density of \(X_n\) since

\[
Q_n^{(0)}(S_n|\theta) = e^{-n}F^{(n)}(S_n|\theta) \quad \text{and} \quad M_n^{(0)}(\chi_n, \Theta_n) = Q_n^{(0)}(\chi_n|\theta) = e^{-n}
\]

by (4). In terms of the \(R^{(\alpha)}\)-marginal density, we can re-express the \(R^{(\alpha)}\)-posterior probabilities from [5] as

\[
\pi_n^{(\alpha)}(A_n | \chi_n) = \frac{m_n^{(\alpha)}(\chi_n, A_n)}{m_n^{(\alpha)}(\chi_n)} = \frac{m_n^{(\alpha)}(\chi_n, A_n)/M_n^{(\alpha)}(\chi_n, \Theta_n)}{m_n^{(\alpha)}(\chi_n)} = \frac{m_n^{(\alpha)}(d\chi_n, A_n)}{M_n^{(\alpha)}(\chi_n, \Theta_n)}, \quad A_n \in B_{\Theta_n},
\]

whenever \(0 < m_n^{(\alpha)}(\chi_n) < \infty\). Then the \(R^{(\alpha)}\)-Bayes joint posterior distribution of the parameter \(\theta\) and the data \(X_n\) is defined as

\[
\pi_n^{(\alpha)}(d\theta, d\chi_n) = \pi_n^{(\alpha)}(d\theta | \chi_n) m_n^{(\alpha)}(d\chi_n) = m_n^{(\alpha)}(d\chi_n, d\theta)/M_n^{(\alpha)}(\chi_n, \Theta_n).
\]

This gives a nice interpretation of the deduced measure \(M_n^{(\alpha)}(S_n, A_n)\) which is, after suitable standardization, the product measure associated with the \(R^{(\alpha)}\)-Bayes joint posterior distribution of \(\theta\) and \(X_n\). At \(\alpha = 0\), all these again simplify to the ordinary Bayes measures.

Example 2.1 [Independent Stationary Data]:

The simplest possible set-up is that of IID observations as described in Section 1. In terms of the general notation presented above, we have \(X_n = (X_1, \ldots, X_n)\) with its observed value \(\chi_n = (x_1, \ldots, x_n)\) and the general measurable space \((\chi, \mathcal{B})\). Assuming the same true probability space \((\Omega, \mathcal{B}_\Omega, P)\), the induced data distribution is \(G^{(n)}(\chi_n) = \prod_{i=1}^{n} G(x_i)\) which has an absolutely continuous density \(g^{(n)}(\chi_n) = \prod_{i=1}^{n} g(x_i)\) with respect to the product \(\sigma\)-finite measure \(\lambda^{(n)}(d\chi_n) = \prod_{i=1}^{n} \lambda(x_i)\). Similarly the model distribution and density, in their general notation, also have the product forms: \(F^{(n)}(\chi_n | \theta) = \prod_{i=1}^{n} F_{\theta}(x_i)\), \(f^{(n)}(\chi_n | \theta) = \prod_{i=1}^{n} f_{\theta}(x_i)\) and so \(F_n\) is the \(n\)-fold product of the family of individual distributions \(F_{\theta}\). Under these notations, the \(\alpha\)-likelihood \(Q_n^{(\alpha)}(\chi_n | \theta)\) is given by [2] which satisfies the required measurability assumptions along with the condition in [4].

Then, under suitable assumptions on the prior distribution as before, the corresponding \(R^{(\alpha)}\)-posterior distribution is defined by [3] which is now equivalent to [1] and can be written as a product of stationary independent terms corresponding to each \(x_i\). Also, the deduced measure becomes \(Q_n^{(\alpha)}(S_n | \theta) = \prod_{i=1}^{n} Q^{(\alpha)}(S^i | \theta)\) for any \(S_n = S^1 \times S^2 \times \cdots \times S^n \in \mathcal{B}_n\) with \(S^i \in \mathcal{B}\) for all \(i\) and \(Q^{(\alpha)}(S^i | \theta) = \int_{S^i} \exp(q^{(\alpha)}_{\theta}(y)) dy\), where \(q^{(\alpha)}_{\theta}(y)\) is as defined in [3]. Other related measures can be defined from these; details are discussed in Section 2.

Example 2.2 [Independent Non-homogeneous Data]:

Suppose \(X_1, \ldots, X_n\) are independently but not identically distributed random variables, where each \(X_i\) is defined on a measurable space \((\chi^i, \mathcal{B}^i)\) for \(i = 1, \ldots, n\). Considering an
underlying common probability space \((\Omega, \mathcal{B}_\Omega, P)\), the random variable \(X_i\) is assumed to be \(B^i/\Omega\) measurable, independent with respect to \(P\) and its induced distribution \(G_i(x)\) has an absolutely continuous density \(g_i(x)\) with respect to some common dominating \(\sigma\)-finite measure \(\lambda(dx)\), for each \(i = 1, \ldots, n\). The true distributions \(G_i\) is to be modeled by a parametric family \(\mathcal{F} = \{F_{i, \theta} : \theta \in \Theta \subseteq \mathbb{R}^p\}\) which is absolutely continuous with respect to \(\lambda\) having density \(f_{i, \theta}\). Note that, although the densities are potentially different for each \(i\), they are assumed to share the common unknown parameter \(\theta\) leaving us with enough degrees of freedom for “good” estimation of this parameter \(\theta\).

This set-up of independent non-homogeneous observations, which we refer to as the I-NH set-up, covers many interesting practical problems, the most common one being the regression with fixed design. Suppose \(t_1, \ldots, t_n\) be \(n\) fixed, \(k\)-variate design points. For each \(i = 1, \ldots, n\), given \(t_i\) we independently observe \(x_i\) which has the parametric model density \(f_{i, \theta}(x_i) = f(x_i; t_i, \theta)\) depending on \(t_i\) through a regression structure. For example,

\[
E(X_i) = \psi(t_i, \beta), \quad i = 1, \ldots, n, \tag{8}
\]

where \(\beta \subseteq \theta\) is the unknown regression coefficients and \(\psi\) is a suitable link function.

In general, the unknown parameter \(\theta = (\beta, \sigma)\) may additionally contain some variance parameter \(\sigma\). For the subclass of generalized linear models, we take \(\psi(t_i, \beta) = \psi(t_i^T \beta)\) and \(f\) from the exponential family of distributions. For normal linear regression, we have \(\psi(t_i, \beta) = t_i^T \beta\) and \(f\) is the normal density with mean \(t_i^T \beta\) and variance \(\sigma^2\). Here, the underlying random variables \(X_i\)'s, associated with observations \(x_i\)'s, have the I-NH structure with the common parameter \(\theta = (\beta, \sigma)\) and the different densities \(f_{i, \theta}\). We can further extend this set-up to include the heterogeneous variances (by taking different \(\sigma_i\) for different \(f_{i, \theta}\) but involving common unknown parameters) which is again a part of our I-NH set-up.

In terms of the general notation, the random variable \(X_n = (X_1, \ldots, X_n)\) is defined on the measurable space \((\mathcal{X}_n, \mathcal{B}_n) = \otimes_{i=1}^n (\mathcal{X}_i, \mathcal{B}_i)\). The true induced distribution is \(G^n(x_n) = \prod_{i=1}^n G_i(x_i)\) and its absolutely continuous density has the form \(g^n(x_n) = \prod_{i=1}^n g_i(x_i)\) with respect to \(\lambda^n(dx_n) = \prod_{i=1}^n \lambda(x_i)\). The model distribution and density are given by \(F^n(x_n|\theta) = \prod_{i=1}^n F_{i, \theta}(x_i)\) and \(f^n(x_n|\theta) = \prod_{i=1}^n f_{i, \theta}(x_i)\) so that \(\mathcal{F}_n = \otimes_{i=1}^n \mathcal{F}_i\).

Now, under this I-NH set-up, we can define the \(R(\alpha)\)-posterior by suitably extending the definition of the \(\alpha\)-likelihood function \(q_n^{(\alpha)}(x_n|\theta)\) from its IID version in (4) keeping in mind the general requirement (4). Borrowing ideas from Ghosh and Basu (2013), who have developed the MDPDE for the I-NH set-up, and following the intuition behind the construction of the \(\alpha\)-likelihood (4) of Ghosh and Basu (2016), one possible extended definition for the \(\alpha\)-likelihood in the I-NH case can be given by

\[
q_n^{(\alpha)}(x_n|\theta) = \sum_{i=1}^n \left[ \frac{1}{\alpha} f_{i, \theta}^{\alpha}(x_i) - \frac{1}{1 + \alpha} \int f_{i, \theta}^{1+\alpha} \right] - \frac{n}{\alpha} \sum_{i=1}^n q_{i, \theta}^{(\alpha)}(x_i), \tag{9}
\]

with \(q_{i, \theta}^{(\alpha)}(y) = \frac{1}{\alpha} \left( f_{i, \theta}^{\alpha}(y) - 1 \right) - \frac{1}{1 + \alpha} \int f_{i, \theta}^{1+\alpha} \). Note that, \(q_n^{(0)}(x_n|\theta) = \sum_{i=1}^n (\log(f_{i, \theta}(x_i)) - 1), \)
satisfying the required condition in (4). So, assuming a suitable prior distribution for \( \theta \), the \( R^{(\alpha)} \)-posterior for the I-NH observations is defined through (5) with \( q_n^{(\alpha)}(x_n|\theta) \) being given by (9). Note that, the resulting posterior is again a product of independent but non-homogeneous terms. Similarly, we can also define the deduced measures \( Q^{(\alpha)}_n(S_n|\theta) \), \( M^{(\alpha)}_n(\cdot, A_n) \) and \( M^{(\alpha)}_n(\cdot) \), along with the important \( R^{(\alpha)} \)-marginal density function \( m_n^{(\alpha)}(x_n) \). We discuss their properties in more detail in Section 5. \( \square \)

**Remark 2.1** In the first introduction of the \( R^{(\alpha)} \)-posterior under IID set-up, it has been noted that its only drawback is the lose of the probabilistic interpretation \( (\text{Ghosh and Basu 2016}) \). Here also we have developed the general \( R^{(\alpha)} \)-posterior differently from the conditional probability approach of usual Bayesian inference and referred it as a pseudo-posterior. But, in fact, it can also be interpreted as an ordinary Bayes posterior under a suitably modified model and prior densities, which becomes more prominent under the general set-up and definition of this section. To see this, let use define an \( \alpha \)-modified model density \( \tilde{q}_n^{(\alpha)}(x_n|\theta) = \exp\{q_n^{(\alpha)}(x_n|\theta)\} \) and the \( \alpha \)-modified prior density \( \tilde{\pi}_n^{(\alpha)}(\theta) = \frac{Q^{(\alpha)}_n(x_n|\theta)\pi_n(\theta)}{M^{(\alpha)}_n(x_n,\theta_n)} \). Both are proper densities and satisfy the required meaurability assumptions whenever the relevant quantities are assumed to exist finitely. Further, \( \tilde{\pi}_n^{(\alpha)}(\theta) \) is a function of \( \theta \) only (independent of the data) and hence may be used as a prior density in Bayesian inference; but it depends on \( \alpha \) and the model. In particular, at \( \alpha = 0, \tilde{\pi}_n^{(0)}(\theta) = \pi_n(\theta) \) and \( \tilde{q}_n^{(0)}(x_n|\theta) = q_n^{(0)}(x_n|\theta) = f_n(x_n|\theta) \) so that they indeed represent a modification of the model and the prior, respectively, in order to achieve robustness against data contamination. Now, for any measurable event \( A_n \subset B_{\Theta_n} \), the standard Bayes (conditional) posterior probability of \( A_n \) with respect to the (\( \alpha \)-modified) model family \( \mathcal{F}_{n,\alpha} = \{ q_n^{(\alpha)}(\cdot|\theta) : \theta \in \Theta_n \} \) and the (\( \alpha \)-modified) prior density \( \tilde{\pi}_n^{(\alpha)}(\theta) \) is given by \( \int_{A_n} \tilde{q}_n^{(\alpha)}(x_n|\theta)\tilde{\pi}_n^{(\alpha)}(\theta)d\theta \), which simplifies to \( \pi_n^{(\alpha)}(A_n|x_n) \), the proposed \( R^{(\alpha)} \)-posterior probability as defined in (5).

In the following we briefly present the possible forms of the \( \alpha \)-likelihood function for some other statistical models, but their detailed investigations are kept for the future.

**Example 2.3** [Time Series Data]:
Consider the true probability space \( (\Omega, \mathcal{B}_\Omega, P) \) and an index set \( T \). A measurable time series \( X_t(\omega) \) is a function defined on \( T \times \Omega \), which is a random variable on \( (\Omega, \mathcal{B}_\Omega, P) \) for each \( t \in T \). Given a time series \( \{X_t(\omega) : t \in T\} \), they are assumed to be associated with an increasing sequence of sub \( \sigma \)-fields \( \{\mathcal{G}_t\} \) and have absolute continuous densities \( g(X_t|\mathcal{G}_t) \) for \( t \in T \). For a stationary time series, one might take \( \mathcal{G}_t = \mathcal{F}_{t-1} \), the \( \sigma \)-field generated by \( \{X_{t-1}, X_{t-2}, \ldots\} \), for each \( t \in T \).

In parametric inference, we model \( g(X_t|\mathcal{G}_t) \) by a parametric density \( f_\theta(X_t|\mathcal{F}_{t-1}) \) and try to infer about the unknown parameter \( \theta \) from an observed sample \( x_n = \{x_t : t \in T = \{1, 2, \ldots, n\}\} \) of size \( n \). For example, in a Poisson autoregressive model, we assume \( f_\theta(x_t|\mathcal{F}_{t-1}) \) to be a Poisson density with mean \( \lambda_t \) and \( \lambda_t = h_\theta(\lambda_{t-1}, X_{t-1}) \) for all \( t \in T = \mathbb{Z} \) and some known function \( h_\theta \) involving the unknown parameter \( \theta \in \Theta \subseteq \mathbb{R}^p \). In the Bayesian paradigm,
we additionally assume a prior density \( \pi(\theta) \) and update it to get inference based on the posterior density of \( \theta \) given the observed sample data.

We can develop the robust Bayesian inference for any such time series model through the proposed \( R(\alpha) \)-posterior density provided a suitable \( \alpha \)-likelihood function can be defined. Following the construction of the MDPDE in such time series models (Kim and Lee 2011; Kang and Lee 2013; Kang and Lee 2014, among others), we define the \( \alpha \)-likelihood function under the time series models as

\[
q_n^{(\alpha)}(x_n|\theta) = \sum_{t=1}^{n} \left[ \frac{1}{\alpha} f_\theta^0(x_t|F_{t-1}) - \frac{1}{1+\alpha} \int f_\theta^{1+\alpha}(x|F_{t-1}) dx \right] - \frac{n}{\alpha} = \sum_{t=1}^{n} q_{t,\theta}^{(\alpha)}(x_t),
\]

with \( q_{t,\theta}^{(\alpha)}(y) = \frac{1}{\alpha} (f_\theta^0(y|F_{t-1}) - 1) - \frac{1}{1+\alpha} \int f_\theta^{1+\alpha}(x|F_{t-1}) dx \). Note that, \( q_n^{(0)}(x_n|\theta) = \sum_{t=1}^{n} (\log(f_\theta(x_t|F_{t-1})) - 1) \), which clearly satisfies the required condition (4). The robust \( R(\alpha) \)-posterior inference about \( \theta \) can then be developed based on this \( \alpha \)-likelihood function.

**Example 2.4 [Markov Process]:**

Example 2.3 can be easily generalized to Markov processes with stationary transitions. Consider the random variables \( X_1, \ldots, X_n \) defined on the underlying true probability space \((\Omega, \mathcal{B}_\Omega, P)\) having true transition probabilities \( g(X_{k+1}|X_k), k = 0, 1, 2, \ldots, n, \) with \( X_0 \) being the initial value of the process. We model it by a parametric family of stationary probabilities \( f_\theta(X_{k+1}|X_k) \) depending on the unknown parameter \( \theta \in \Theta \subseteq \mathbb{R}^p \). Then, the required \( \alpha \)-likelihood function given the observed sample \( x_n = (x_1, \ldots, x_n) \) can be defined as \( q_n^{(\alpha)}(x_n|\theta) = \sum_{k=1}^{n} \left[ \frac{1}{\alpha} f_\theta^0(x_{k+1}|x_k) - \frac{1}{1+\alpha} \int f_\theta^{1+\alpha}(x|x_k) dx \right] - \frac{n}{\alpha} \). Clearly it satisfies the required condition (4) and it is possible to perform robust \( R(\alpha) \)-posterior inference about \( \theta \) under this set-up.

**Example 2.5 [Diffusion Process]:**

Consider the true probability space \((\Omega, \mathcal{B}_\Omega, P)\) and an index set \( T \). A measurable random variable \( X_t \) defined on \( T \) follows a diffusion process if it satisfies

\[
dX_t = a(X_t, \mu) dt + b(X_t, \sigma) dW_t, \quad t \geq 0,
\]

with \( X_0 = x_0 \) and two known functions \( a \) and \( b \), where \( \{W_t : t \geq 0\} \) is a standard Wiener process and the parameter of interest is \( \theta = (\mu, \sigma)^T \in \Theta \), a convex compact subset of \( \mathbb{R}^p \times \mathbb{R}^+ \). This model has important practical applications in finance, where some inference about \( \theta \) is desired based on discretized observations \( X^n_t, t = 1, \ldots, n \), from the diffusion process (10). We generally assume \( t^n_i = i h_n \) with \( h_n \to 0 \) and \( n h_n \to \infty \) as \( n \to \infty \). Freqtintist’s robust MDPDEs of \( \theta \) based on such observations are developed for two of its special cases, \( a(X_t, \mu) = a(X_t) \) and \( b(X_t, \sigma) = \sigma \), respectively, by Song et al. (2007) and Lee and Song (2013).

However, whenever we have some prior knowledge about \( \theta \), quantified through a prior density \( \pi(\theta) \), one would like to apply the Bayesian approach of inference. A robust Bayes inference can be developed for such models by using the proposed \( R(\alpha) \)-posterior with a
suitably defined $\alpha$-likelihood function. For this purpose, we note that

$$X_{i_{i-1}} = X_{i_{i-1}} = a(X_{i_{i-1}}, \mu)h_n + b(X_{i_{i-1}}, \sigma)\sqrt{h_n}Z_{n,i} + \Delta_{n,i}, \quad i = 1, \ldots, n,$$

where we define $Z_{n,i} = h_n^{-1/2}(W_{i_{i-1}} - W_{i_{i-1}})$ and $\Delta_{n,i} = \int_{i_{i-1}}^{t_n} [a(X_s, \mu) - a(X_{i_{i-1}}, \mu)] ds + \int_{i_{i-1}}^{t_n} [b(X_s, \sigma) - b(X_{i_{i-1}}, \sigma)] dW_s$. Clearly, $Z_{n,i}$ are IID standard normal variables for $i = 1, \ldots, n$. Therefore, whenever $\Delta_{n,i}$ can be ignored in $P$-probability, for large enough $n$, the random variables $X_{i_{i-1}} \mid G_{i_{i-1}}$, $i = 1, \ldots, n$, behave as I-NH variables with densities $f_i, \theta(\cdot \mid G_{i_{i-1}}) \equiv N \left( X_{i_{i-1}} + a(X_{i_{i-1}}, \mu)h_n, b(X_{i_{i-1}}, \sigma)^2h_n \right)$, where $G_{i_{i-1}}$ is the $\sigma$-field generated by $\{W_s : s \leq t_n \}$. Then, the corresponding $\alpha$-likelihood function based on the observed data $\underline{\alpha}_n = (x_{i_{i-1}}, \ldots, x_{i_{i-1}})$ can be derived as in Example 2.3 using (10). It clearly satisfies the general requirement (4) and has the simplified form, $q_n^{(\alpha)}(\underline{\alpha}_n | \theta) = \sum_{i=1}^{n} q_{i, \theta}^{(\alpha)}(x_{i_{i-1}})$, with

$$q_{i, \theta}^{(\alpha)}(x_{i_{i-1}}) = \begin{cases} \frac{1}{(2\pi b(x_{i_{i-1}}, \sigma)^2h_n)^{\alpha/2}} \left[ \frac{-a(x_{i_{i-1}} - x_{i_{i-1}}, \sigma)h_n}{2b(x_{i_{i-1}}, \sigma)^2h_n} \right] \frac{1}{\alpha}, & \text{if } \alpha > 0, \\ -\frac{1}{\alpha} \log \left( 2\pi b(x_{i_{i-1}}, \sigma)^2h_n \right) - 1, & \text{if } \alpha = 0, \end{cases}$$

The robust $R^{(\alpha)}$-posterior and the corresponding estimates of $\theta$ can be easily obtained using this $\alpha$-likelihood function. The details are left for the reader. \hfill $\Box$

## 3 Exponential Convergence Results under General Set-up

### 3.1 Consistency of the $R^{(\alpha)}$-Posterior Probabilities

Barron (1988) first demonstrated the useful exponential convergence results for the usual Bayes posterior probabilities under suitable assumptions about “merging in probability” (see Definition A.1 and A.2 in Appendix A) and existence of uniformly consistent tests. These results have later been refined by several authors (see Ghosal et al., 1995; 2000; Walker, 2004; Ghosal and van der Vaart, 2007; Walker et al., 2007; Martin and Hong, 2012 among others). In this paper, we follow the approach of Barron (1988) to show the exponential consistency of our new robust $R^{(\alpha)}$-posterior probabilities under suitable extended assumptions. In particular, for some sequences of measurable sets $A_n, B_n, C_n \subseteq \Theta_n$ and sequences of constants $b_n, c_n$, we consider the following assumptions.

(A1) $A_n, B_n$ and $C_n$ together complete $\Theta_n$, i.e., $A_n \cup B_n \cup C_n = \Theta_n$.

(A2) $B_n$ satisfies $\pi_n^{(\alpha)}(B_n) = \frac{M_n^{(\alpha)}(B_n)}{M_n^{(\alpha)}(\Theta_n)} \leq b_n$.

(A3) $C_n$ are such that there exists sets $S_n \in B_n$ satisfying $\lim_{n \to \infty} G^n(S_n) = 0$ and $\sup_{\theta \in C_n} \frac{Q_n^{(\alpha)}(S_n | \theta)}{Q_n^{(\alpha)}(\underline{\alpha}_n | \theta)} \leq c_n$. 

10
(A3)* $C_n$ are such that there exists sets $S_n \in \mathcal{B}_n$ satisfying $P(\mathbf{X}_n \in S_n \text{ i.o.}) = 0$ and 
\[ \sup_{\theta \in C_n} \frac{Q_n(\theta)}{Q_n(\alpha_n(\theta))} \leq c_n, \text{ where i.o. denotes “infinitely often”}. \]

Here we need either of Condition (A3) or Condition (A3)* which, respectively, help us to prove the convergence results in probability or with probability one. Clearly, Condition (A3)* is stronger and imply (A3), but (A3) is sufficient in most practices yielding a convergence in probability type result. Also, if Condition (A3) holds with $c_n = e^{-nr}$ for some $r > 0$, then it indeed ensures the existence of a uniformly exponentially consistent (UEC) test for $G^n$ against the family of $\alpha$-modified probability distributions 
\[ \left\{ \frac{Q_n(\theta)}{Q_n(\alpha_n(\theta))} : \theta \in C_n \right\} \]
corresponding to the $\alpha$-modified model density $\tilde{q}_n(\theta)$ defined in Remark 2.1. Further, at $\alpha = 0$, these conditions simplify to those used by Barron (1988) for proving the exponential convergence of ordinary Bayes posterior probabilities as proved in Barron (1988). Our theorem generalizes it for the robust $R^{(\alpha)}$-posterior probabilities, besides yielding robust results under data contamination, is asymptotically optimal in exactly the same exponential rate as the ordinary posterior for all $\alpha \geq 0$. We will further illustrate the required assumptions for the IID data and the I-NH data in Sections 4 and 5 respectively. Some useful sufficient conditions for merging of $R^{(\alpha)}$-marginal distribution with $G^n$ are also developed under the general case and are presented in Appendix A.

**Theorem 3.1 (Exponential Consistency of the $R^{(\alpha)}$-posterior probabilities)**

1. Suppose that the true distribution $G^n$ and the $R^{(\alpha)}$-marginal distribution $M_n^{(\alpha)}(\cdot)$ merge in probability according to Definition A.1 and let $A_n \in \mathcal{B}_{\Theta_n}$ be any sequence of sets. Then, $\limsup_{n \to \infty} P\left( \pi_n(\alpha_n^c(X_n) < e^{-nr}) = 1, \text{ for some } r > 0, \text{ if and only if there exists constants } r_1, r_2 > 0 \text{ and sets } B_n, C_n \in \mathcal{B}_{\Theta_n} \text{ such that Assumptions (A1)–(A3) are satisfied with } b_n = e^{-nr_1} \text{ and } c_n = e^{-nr_2} \text{ respectively.} \]

2. Suppose the true distribution $G^n$ and the $R^{(\alpha)}$-marginal distribution $M_n^{(\alpha)}(\cdot)$ merge with probability one according to Definition A.2 and let $A_n \in \mathcal{B}_{\Theta_n}$ be any sequence of sets. Then, $P\left( \pi_n(\alpha_n^c(X_n) \geq e^{-nr} \text{ i.o.}) = 0, \text{ for some } r > 0, \text{ if and only if there exists constants } r_1, r_2 > 0 \text{ and sets } B_n, C_n \in \mathcal{B}_{\Theta_n} \text{ such that Assumptions (A1), (A2) and (A3)* are satisfied with } b_n = e^{-nr_1} \text{ and } c_n = e^{-nr_2} \text{ respectively.} \]

Note that, for $\alpha = 0$, Theorem 3.1 coincides with the classical exponential convergence results of ordinary Bayes posterior probabilities as proved in Barron (1988). Our theorem generalizes it for the robust $R^{(\alpha)}$-posterior probabilities under suitable conditions. Hence, the $R^{(\alpha)}$-posterior distribution, besides yielding robust results under data contamination, is asymptotically optimal in exactly the same exponential rate as the ordinary posterior for all $\alpha \geq 0$. We will further illustrate the required assumptions for the IID data and the I-NH data in Sections 4 and 5 respectively. Some useful sufficient conditions for merging of $R^{(\alpha)}$-marginal distribution with $G^n$ are also developed under the general case and are presented in Appendix A.
Let us now examine the asymptotic properties of the \( R^{(\alpha)} \)-Bayes estimators associated with the new robust \( R^{(\alpha)} \)-posterior distribution \( \tilde{F}_n \) under the general set-up of Section 2. In the decision-theoretic framework, we consider the problem of estimation of a functional \( \phi_P := \phi(P) \) of the true probability \( P \); for example \( \phi_P \) could be the probability density of \( P \), or any summary measure (like mean) of \( P \). For the given parametric family \( F^n(\cdot|\theta) \), let us denote \( \phi_\theta := \phi_{F^n(\cdot|\theta)} \). Then, our action space is \( \Phi = \{ \phi_Q : Q \) is a probability measure on \((\Omega, \mathcal{B}_\Omega)\} \); consider a non-negative loss function \( L_n(\phi, \hat{\phi}) \) on \( \Phi \times \Phi \) denoting the loss in estimating \( \phi \) by \( \hat{\phi} \). The general \( R^{(\alpha)} \)-Bayes estimator \( \hat{\phi} = \hat{\phi}(\cdot|\hat{x}_n) \) of \( \phi \) is then defined as

\[
\hat{\phi} = \arg \min_{\phi \in \Phi} \int L_n(\phi_\theta, \phi) \pi_n^{(\alpha)}(d\theta|\hat{x}_n),
\]

provided the minimum is attained; we need the loss function \( L_n(\phi_\theta, \phi) \) to be \( \mathcal{B}_{\Theta_n} \) measurable for each \( \phi \in \Phi \). In particular, the \( R^{(\alpha)} \)-Bayes estimator of \( \phi_\theta = \theta \) is the mean of the \( R^{(\alpha)} \)-posterior distribution for squared error loss if it exists finitely, or a median of the \( R^{(\alpha)} \)-posterior distribution for absolute error loss. Similarly, when the loss function \( L_n \) is the Dirac delta function (0-1 loss), the corresponding \( R^{(\alpha)} \)-Bayes estimator of \( \theta \) is the mode of the \( R^{(\alpha)} \)-posterior distribution, if it is uniquely defined.

However, if the minimum in (12) is not attained, we may define the approximate \( R^{(\alpha)} \)-Bayes estimator \( \tilde{\phi} \) of \( \phi \) through the relation

\[
\int L_n(\phi_\theta, \tilde{\phi}) \pi_n^{(\alpha)}(d\theta|\hat{x}_n) \leq \inf_{\phi \in \Phi} \int L_n(\phi_\theta, \phi) \pi_n^{(\alpha)}(d\theta|\hat{x}_n) + \delta_n, \quad \text{with } \lim_{n \to \infty} \delta_n = 0.
\]

An useful example is the approximate mode of the \( R^{(\alpha)} \)-posterior for discrete parameter space, which is an approximate \( R^{(\alpha)} \)-Bayes estimator under 0-1 loss. Also, note that, if the \( R^{(\alpha)} \)-Bayes estimator exists, it is also an approximate \( R^{(\alpha)} \)-Bayes estimator.

In the following, we assume the loss function \( L_n \) to be bounded and equivalent to a pseudo-metric \( d_n \) on \( \Phi \times \Phi \) as defined below.

**Definition 3.1** A loss function \( L_n \) on \( \Phi \times \Phi \) is said to be bounded if there exists \( \bar{L} < \infty \) such that \( L_n(\phi_\theta, \phi_P) \leq \bar{L} \) for all \( n \) and all \( \theta \in \Theta_n \), where \( \phi_P \) is the value of the target functional \( \phi \) under the true probability distribution \( P \).

**Definition 3.2** A loss \( L_n \) on \( \Phi \times \Phi \) is said to be equivalent to a pseudo-metric \( d_n \) on \( \Phi \times \Phi \) if there exist two strictly increasing functions \( h_1 \) and \( h_2 \) on \([0, \infty)\) that are continuous at \( 0 \) with \( h_1(0) = h_2(0) = 0 \) and satisfy \( L_n \leq h_1(d_n) \) and \( d_n \leq h_2(L_n) \) on \( \Phi \times \Phi \) and for all \( n \).

Note that, Definition 3.2 is equivalent to saying \( \lim_{n \to \infty} L_n(\phi_n, \hat{\phi}_n) = 0 \) if and only if \( \lim_{n \to \infty} d_n(\phi_n, \hat{\phi}_n) = 0 \). As an example, the squared Hellinger loss is bounded and equivalent to the \( L_1 \)-distance metric. Also, the absolute error \( (L_1) \) loss is equivalent to itself and bounded by twice the Hellinger loss.
Our next theorem states the asymptotic consistency of \( R^{(\alpha)} \)-Bayes and approximate \( R^{(\alpha)} \)-Bayes estimators of \( \phi_\theta \) to the true value \( \phi_P \) for such loss functions. The proof follows along the lines of the proof of Lemma 12 in [Barron, 1988] and is hence omitted for brevity.

**Theorem 3.2 (Consistency of \( R^{(\alpha)} \)-Bayes Estimators)** Given any sample data \( \mathcal{X}_n \), let \( \hat{\phi}_n = \hat{\phi}(\cdot; \mathcal{X}_n) \) be an approximate \( R^{(\alpha)} \)-Bayes estimator (or the \( R^{(\alpha)} \)-Bayes estimator) of \( \phi_P \) with respect to a loss function \( L_n \) that is bounded as in Definition 3.1 and is equivalent to a pseudo-metric \( d_n \) as in Definition 3.2. Also, for any \( \epsilon > 0 \), define \( A_{\epsilon,n} = \{ \theta : d_n(\phi_P, \phi_\theta) \leq \epsilon \} \). Then, we have \( \lim \pi_n^{(\alpha)}(A_{\epsilon,n}) = 0 \) in probability or with probability one for all \( \epsilon > 0 \), then \( \lim \pi_n^{(\alpha)}(A_{\epsilon,n}) = 0 \) in probability or with probability one, respectively.

In simple language, Theorem 3.2 states that whenever the target \( \phi_P \) is close enough to the model value \( \phi_\theta \) in the pseudo-metric \( d_n \), asymptotically under the \( R^{(\alpha)} \)-posterior probability, the corresponding \( R^{(\alpha)} \)-Bayes estimator with respect to \( L_n \) is asymptotically consistent for \( \phi_P \) in \( d_n \). But, Theorem 3.1 yields \( \lim \pi_n^{(\alpha)}(A_{\epsilon,n}) = 0 \) under appropriate conditions and hence the corresponding \( R^{(\alpha)} \)-Bayes estimators are consistent in a suitable pseudo-metric \( d_n \). In particular, Theorem 3.2 applies to the \( R^{(\alpha)} \)-Bayes estimators with respect to the squared Hellinger loss and the \( L_1 \)-loss to deduce their \( L_1 \) consistency.

## 4 Application (I): Independent Stationary Models

### 4.1 \( R^{(\alpha)} \)-Posterior convergence

Consider the set-up of the independent stationary model as in Example 2.1. Let us study the conditions required for the exponential convergence of the \( R^{(\alpha)} \)-posterior for this particular set-up. First, to verify the merging of \( G^n \) and \( M_n^{(\alpha)} \), we define the individual \( \alpha \)-modified density as \( \tilde{q}^{(\alpha)}(\cdot|\theta) = \exp\left(q^{(\alpha)}(\cdot)\right)/Q^{(\alpha)}(\chi|\theta) \) and the \( \alpha \)-modified prior \( \tilde{\pi}^{(\alpha)}(\cdot) \) as in Remark 2.1 with \( \pi_n = \pi \). Consider the Kullback-Leibler divergence measure between two absolutely continuous densities \( f_1 \) and \( f_2 \) with respect to the common \( \sigma \)-finite measure \( \lambda \) defined as

\[
KLD(f_1, f_2) = \int f_1 \log \left( \frac{f_1}{f_2} \right) d\lambda.
\]  

Then we define the information denseness of the prior \( \pi \) under independent stationary models with respect to \( \mathcal{F}_\alpha = \{ \tilde{q}^{(\alpha)}(\cdot|\theta) : \theta \in \Theta \} \) as follows.

**Definition 4.1** The prior \( \pi \) under the independent stationary model is said to be information dense at \( G \) with respect to \( \mathcal{F}_\alpha \) if there exists a finite measure \( \bar{\pi} \) such that

\[
\liminf_{n \to \infty} e^{nr} \frac{d\pi_n^{(\alpha)}}{d\bar{\pi}}(\theta) \geq 1, \quad \text{for all } r > 0, \theta \in \Theta,
\]  

(15)
and $\pi \left( \{ \theta : \text{KLD}(g, \tilde{q}^{(\alpha)}(\cdot | \theta)) < \epsilon \} \right) > 0$ for all $\epsilon > 0$.

Note that, in view of Remark A.1 and Theorem A.2 of Appendix A, Definition 4.1 implies that $G^n$ and $M^{(\alpha)}_n$ merge in probability for the independent stationary models. Then, our main Theorem 3.1 may be restated as follows.

**Proposition 4.1** Consider the set-up of independent stationary models and assume that the prior $\pi$ is independent of $n$ and is information dense at $g$ with respect to $F_\alpha$ as per Definition 4.1. Take any sequence of measurable parameter sets $A_n \subset \Theta$. Then, $\pi_n^{(\alpha)}(A \cap |X_n) \subset \Theta$. Then, $\pi_n^{(\alpha)}(A \cap |X_n)$ is exponentially small with $P$-probability one, if and only if there exists constants $r_1, r_2 > 0$ and sequences of sets $B_n, C_n \in B_\Theta$ such that (1) $A_n \cup B_n \cup C_n = \Theta$, (2) $B_n$ satisfies (A2) with $b_n = e^{-nr_1}$ and (3) $C_n$ satisfies $(A3)^*$ with $c_n = e^{-nr_2}$.

Further, Condition 3 in Proposition 4.1 indeed holds under the assumption of the existence of a UEC test for $G^n$ against the family of $(\alpha$-modified) distributions 
\[
\left\{ \frac{Q_n^{(\alpha)}(\cdot | \theta)}{Q_n^{(\alpha)}(\cdot | \theta)} : \theta \in C_n \right\},
\]
or equivalently under the existence of a UEC test for $G$ against the family 
\[
\left\{ \frac{Q_n^{(\alpha)}(\cdot | \theta)}{Q_n^{(\alpha)}(\cdot | \theta)} : \theta \in C_n \right\}
\]
by the form of $Q_n^{(\alpha)}(\cdot | \theta$) in IID models. We can further simplify this assumption by using a necessary and sufficient condition for the existence of UEC from [Barron 1989] which states that, “for every $\epsilon > 0$ there exists a sequence of UEC tests for the hypothesized distribution $P$ versus the family of distributions $\{Q : d_T(P, Q) > \epsilon/2 \}$ if and only if the sequence of partitions $T_n$ has effective cardinality of order $n$ with respect to $P$”; here, for any measurable partition $T$, $d_T$ denotes the $T$-variation norm given by $d_T(P, Q) = \sum_{A \in T} |P(A) - Q(A)|$.

Using this, one can show that the $R^{(\alpha)}$-posterior asymptotically concentrate on the $L_1$ model neighborhood of the true density $g$. Let us define, for any density $p$ and any partition $T$, the “theoretical histogram” density $p_T$ as $p_T(x) = \frac{1}{\lambda(A)} \int_A p(y) \lambda(dy)$, for $x \in A \in T$, whenever $\lambda(A) \neq 0$, and $p_T = 0$ otherwise. We call a sequence of partitions $T_n$ to be “rich” if the corresponding sequence of densities $g^{T_n}$ converges to the true density $g$ in $L_1$-distance. Also, define $B^{T_n} = \{ \theta : d_1 \left( f_\theta, \tilde{q}^{(\alpha)}(\cdot | \theta) \right) > \epsilon \}$ for any $\epsilon > 0$ and sequence of partition $T_n$, where $d_1$ denotes the $L_1$ distance, and consider the following assumption.

**Assumption (B):** For any $\epsilon > 0$, $\frac{\pi_n^{(\alpha)}}{\pi_n^{(\alpha)}}(B^{T_n}_\epsilon) = \frac{M_n^{(\alpha)}(X_n, B^{T_n}_\epsilon)}{M_n^{(\alpha)}(X_n, \theta)}$ is exponentially small for some rich sequence of partitions $T_n$ with effective cardinality of order $n$.

Note that, Assumption (B) implies Condition 2 of Proposition 4.1 for $B^{T_n}_\epsilon$, or any smaller subset of it. So, applying it with $B_n = \{ \theta : d_1(g, f_\theta) \geq \epsilon, d_T \left( G, \frac{Q_n^{(\alpha)}(\cdot | \theta)}{Q_n^{(\alpha)}(\cdot | \theta)} \right) < \epsilon/2 \} \subset B^{T_n}_\epsilon$ and the existence result of UEC tests with $C_n = \{ \theta : d_T \left( G, \frac{Q_n^{(\alpha)}(\cdot | \theta)}{Q_n^{(\alpha)}(\cdot | \theta)} \right) > \epsilon/2 \}$, Proposition 4.1 yields the asymptotic exponential concentration of the $R^{(\alpha)}$-posterior probability in the $L_1$-neighborhood $A_n = \{ \theta : d_1(g, f_\theta) < \epsilon \}$.

**Proposition 4.2** Consider the set-up of independent stationary models and assume that the prior $\pi$ is independent of $n$ and information dense at $g$ with respect to $F_\alpha$ as per Definition 4.1. If Assumption (B) holds then, for every $\epsilon > 0$, the $R^{(\alpha)}$-posterior probability $\pi_n^{(\alpha)} \left( \{ \theta : d_1(g, f_\theta) \geq \epsilon \} |X_n \right)$ is exponentially small with $P$-probability one.
However, if Assumption (B) does not hold, we can deduce a weaker conclusion in terms of $T_n$-variance distance in place of the $L_1$ distance. The idea goes back to Barron (1988) for a similar result in case of the ordinary posterior; an extended version for the $R^{(\alpha)}$-posterior is presented in the following proposition.

**Proposition 4.3** Consider the set-up of independent stationary models and assume that the prior $\pi$ is independent of $n$ and information dense at $g$ with respect to $F_\alpha$ as per Definition 4.1. Then, for any sequence of partitions $T_n$ with effective cardinality of order $n$, the $R^{(\alpha)}$-posterior probability $\pi_n^{(\alpha)} \left( \left\{ \theta : d_{T_n}(G, \frac{Q_n^{(\alpha)}(\cdot | \theta)}{Q_n^{(\alpha)}(\chi_n | \theta)}) \geq \epsilon \right\} \bigg| X_n \right)$ is exponentially small with $P$-probability one.

### 4.2 The cases of Discrete priors: Maximum $R^{(\alpha)}$-posterior estimator

We can derive the exponential consistency of the $R^{(\alpha)}$-posterior Bayes estimators with respect to the bounded loss functions from Theorem 3.2 along with Proposition 4.1–4.3 of the previous subsection. Here, we consider the particular cases of discrete priors and the most intuitive Bayes estimator under this case, namely the maximum $R^{(\alpha)}$-posterior estimator, in more detail.

Consider the set-up of independent stationary models as before, but now with a countable parameter space $\Theta$. On this countable parameter space, we consider a sequence of discrete priors $\pi_n(\theta)$ which are sub-probability mass functions, i.e., $\sum_\theta \pi_n(\theta) \leq 1$. The most common loss-function to consider under this set-up is the 0-1 loss function, for which the resulting $R^{(\alpha)}$-Bayes estimator is the (global) mode of the $R^{(\alpha)}$-posterior density; we call this estimator of $\theta$ as the “maximum $R^{(\alpha)}$-posterior estimator (MRPE)”. However, when this mode is not attained, we consider an approximate version $\hat{\theta}_n^{(\alpha)}$, to be referred to as an “approximate maximum $R^{(\alpha)}$-posterior estimator (AMRPE)”, defined by the relation

$$\tilde{\pi}_n^{(\alpha)}(\hat{\theta}_n^{(\alpha)} | x_n) > \sup_{\theta} \pi_n^{(\alpha)}(\theta) \tilde{q}_n^{(\alpha)}(x_n | \theta)e^{-n\delta_n},$$

with $\lim_{n \to \infty} \delta_n = 0$, (16)

where $\tilde{q}_n^{(\alpha)}(\cdot | \theta)$ and $\pi_n^{(\alpha)}(\theta)$ are the $\alpha$-modified model and prior densities as defined in Remark 2.1. This definition follows from the fact that the $R^{(\alpha)}$-posterior density is proportional to $\pi_n^{(\alpha)}(\theta) q_n^{(\alpha)}(x_n | \theta)$. Note that, if the MRPE exists, then it is also an AMRPE. Assume that this estimator $\hat{\theta}_n^{(\alpha)} = \hat{\theta}_n^{(\alpha)}(x_n)$, as a function of data $x_n$, is measurable.

To derive the properties of the AMRPE, we assume that the sequence of priors satisfies

$$\liminf_{n \to \infty} e^{nr} \pi_n^{(\alpha)}(\theta) \geq 1, \quad \text{for all } r > 0, \theta \in \Theta.$$  

This Assumption (17) indeed signifies that the ($\alpha$-modified) prior probabilities are not exponentially small anywhere in $\Theta$. Then, we have the following theorem.

**Theorem 4.4** Consider the set-up of stationary independent models with fixed countable parameter space $\Theta_n = \Theta$ and discrete prior sequence $\pi_n$ satisfying Assumption (17). Sup-
pose \( \pi_n \) is information dense at the true probability mass function \( g \) with respect to \( \mathcal{F}_\alpha \) as in Definition 4.1. Then \( \pi_n^{(\alpha)}(A_n|\mathbf{X}_n) \) is exponentially small with probability one for a sequence of measurable subsets \( A_n \subseteq \Theta \). Then any approximate maximum \( R^{(\alpha)} \)-posterior estimator \( \tilde{\theta}_\alpha \in A_n \) for all sufficiently large \( n \) with probability one.

The next theorem gives some sufficient condition for the exponential convergence of the \( R^{(\alpha)} \)-posterior in the present case of countable \( \Theta \), simplified from our main Theorem 3.1.

**Theorem 4.5** Consider the set-up of stationary independent models with fixed countable parameter space \( \Theta_n = \Theta \) and a discrete prior sequence \( \pi_n \) satisfying Assumption (17). Then, for any true density \( g \) which is an information limit of the (countable) family \( \{ \tilde{g}^{(\alpha)}(\cdot|\theta) : \theta \in \Theta \} \), we have \( \pi_n^{(\alpha)}(\{ \theta : d_1(g,f_\theta) \geq \epsilon \}|\mathbf{X}_n) \) is exponentially small with probability one, for each \( \epsilon > 0 \). Therefore, any AMRPE \( \tilde{\theta}_\alpha \) satisfies: \( \lim_{n \to \infty} d_1(g,f_{\tilde{\theta}_\alpha}) = 0 \), with probability one.

**Remark 4.1** Theorem 4.5, in a special case \( \alpha = 0 \), yields a stronger version of Theorem 15 of Barron (1988). Our result requires fewer assumptions than required by Barron’s result.

## 4.3 Robust \( R^{(\alpha)} \)-Bayes Predictive Density Estimators

We now present a further application of the general theory to develop a robust version of Bayes predictive density estimators that are also exponentially consistent. Considering the set-up of IID observation as in Subsection 4.1, we are interested in estimating the true (predictive) density \( g \) based on observed data \( \mathbf{x}_n = (x_1, \ldots, x_n) \). A robust version of the ordinary Bayes predictive density estimator of \( g \) given \( \mathbf{x}_n \) can be defined based on the \( R^{(\alpha)} \)-posterior probability \( \pi_n^{(\alpha)}(\theta|\mathbf{x}_n) \) as given by (1). Let us denote by \( \mathcal{G} \) the set of all probability densities on \((\Omega, B_{\Omega}, P)\) that are absolutely continuous with respect to \( \lambda \). For a suitably chosen loss function \( L(\cdot, \cdot) \) on \( \mathcal{G} \times \mathcal{G} \), we define the \( R^{(\alpha)} \)-Bayes predictive density estimate of \( g \) as given by \( \tilde{g}_\alpha^E(z) = \arg \min_{\theta \in \Theta} \int_\Theta L(f_\theta,g)\pi_n^{(\alpha)}(\theta|\mathbf{x}_n)d\theta \), whenever the minimum exists. If the minimum does not exist, we may extend this to define the approximate \( R^{(\alpha)} \)-predictive density estimator \( \tilde{g}_\alpha^{A,L} \) through the relation

\[
\int_\Theta L(f_\theta,\tilde{g}_\alpha^{A,L})\pi_n^{(\alpha)}(\theta|\mathbf{x}_n)d\theta < \inf_{g \in \mathcal{G}} \int_\Theta L(f_\theta,g)\pi_n^{(\alpha)}(\theta|\mathbf{x}_n)d\theta + \delta_n, \quad \text{with} \quad \lim_{n \to \infty} \delta_n = 0. \tag{18}
\]

The most popular and useful loss function is the the squared error loss, under which the corresponding predictive density estimator is the mean of the model density with respect to the \( R^{(\alpha)} \)-posterior distribution and is given by \( \tilde{g}_\alpha^E(z) = \int_\Theta f_\theta(z)\pi_n^{(\alpha)}(\theta|\mathbf{x}_n)d\theta \). We refer to it as the “expected \( R^{(\alpha)} \)-posterior predictive density estimator” (ERPDE) of \( g \). The corresponding ERPDE of the underlying true probability \( P \), given the data \( \mathbf{x}_n \), is

\[
\hat{P}_\alpha^E(B;\mathbf{x}_n) = \int_\Theta F_\theta(B)\pi_n^{(\alpha)}(\theta|\mathbf{x}_n)d\theta, \quad \text{for all} \ B \in \mathcal{B}, \tag{19}
\]

where we use the same notation \( F_\theta(\cdot) \) for both the distribution function and the corresponding measure. The accuracy of this estimator may be measured by a sequence of loss...
functions $L_n(P, \hat{P}_\alpha^E)$ on $\Phi \times \Phi$, where $\Phi$ is the set of probability measures on $(\chi, \mathcal{B})$. Considering the extension of $L_n$ on $\Phi \times \tilde{\Phi}$, with $\tilde{\Phi}$ being the set of sub-probability measures on $(\chi, \mathcal{B})$, we assume $L_n(P, aF_\theta)$ to be $\mathcal{B}_\Phi$ measurable for each $0 < a \leq 1$ and each $P \in \Phi$ along with the following conditions.

**Assumption (L1):** For each $n \geq 1, a \in [0, 1]$ and for all $P, Q, Q_1, Q_2 \in \Phi$, $L_n(\cdot, \cdot)$ satisfies

(i) **Monotonicity:** $Q_2 \geq Q_1 \Rightarrow L_n(P, Q_2) \leq L_n(P, Q_1)$.

(ii) **Convexity:** $L_n(P, aQ_1 + (1 - a)Q_2) \leq aL_n(P, Q_1) + (1 - a)L_n(P, Q_2)$.

(iii) **Scaling:** $L_n(P, aQ) \leq L_n(P, Q) + \rho(a)$, with $\rho(a)$ independent of $n$ and $\lim_{a \to 1} \rho(a) = 0$.

Assumption (L1) is satisfied by several common loss functions, including the relative entropy loss, the $L_1$-distance and the Hellinger distance loss. Then, we have the following consistency result for the ERPDE defined in (19); see the Online Supplement for its proof.

**Proposition 4.6** Suppose the loss functions $L_n$ satisfy Assumption (L1). Then, for any $\epsilon > 0$, we have $L_n(P, \hat{P}_\alpha^E) \leq \epsilon + \rho(\pi_n(\cdot)(A_{n, \epsilon}|x_n))$, where $A_{n, \epsilon} = \{\theta : L_n(P, F_\theta) < \epsilon\}$ and $\rho(\cdot)$ is as defined in item (iii) of Assumption (L1). Therefore, whenever $\pi_n(\alpha)(A_{n, \epsilon}|x_n) \to 1$ exponentially for every $\epsilon > 0$, $L_n(P, \hat{P}_\alpha^E)$ is exponentially small as $n \to 0$, i.e., the ERPDE is exponentially consistent in $L_n$.

Some other loss functions used in the Bayesian analyses are the $L_1$-distance loss and the (squared) Hellinger loss function. The $R^{(\alpha)}$-Bayes estimator of $g$ with respect to the first one is a measurable version of median, say $\tilde{g}_\alpha^A(z)$, of the $R^{(\alpha)}$-posterior distribution of $f_\theta(z)$; we refer to it as the “absolute $R^{(\alpha)}$-posterior predictive density estimator” (ARPDE) of $g$ given the data $x_n$. On the other hand, the $R^{(\alpha)}$-predictive density estimator of $g$ under the (squared) Hellinger loss function is given by $\tilde{g}_\alpha^H(z) \propto \left[\int_{\Theta} \sqrt{f_\theta(z)} \pi_n(\cdot)(\theta|x_n)d\theta\right]^2$, which we refer as the “Hellinger $R^{(\alpha)}$-posterior predictive density estimator” (HRPDE) of $g$ given $x_n$. Note that both the underlying loss functions are bounded and equivalent to the $L_1$-distance. Hence, Theorem 3.2 can be applied to get the following consistency result for these predictive density estimators; here the action space is $\Phi = \mathcal{G}$ and $\phi_P = g$, the true density under the probability $P$.

**Proposition 4.7** If $\pi_n(\cdot)(\{\theta : d_1(g, f_\theta) \leq \epsilon\}|x_n) \to 1$ exponentially for every $\epsilon > 0$, then the ARPDE and HRPDE are exponentially consistent in $L_1$-distance.

Combining Propositions 4.6 and 4.7 with Proposition 4.2, we get the following corollary.

**Corollary 4.8** Under the assumptions of Proposition 4.4, the robust predictive density estimators ERPDE, ARPDE and HRPDE are all exponentially consistent in $L_1$-distance.

However, when the parameter space $\Theta$ is discrete as in Subsection 4.2, the most natural choice of loss function is the 0-1 loss function; the resulting predictive density estimator, to
be referred to as the “maximum \( R^{(\alpha)} \)-posterior predictive density estimate” (MRPDE) of \( g \),
is given by \( \hat{g}^{M}_{\alpha}(z) = f_{\hat{\theta}_{\alpha}}(z) \), where \( \hat{\theta}_{\alpha} \) is the (global) mode of the \( R^{(\alpha)} \)-posterior density. We may also consider the “approximate maximum \( R^{(\alpha)} \)-posterior predictive density estimator” (AMRPDE) of \( g \) given by \( \hat{g}^{A,M}_{\alpha}(z) = f_{\hat{\theta}_{\alpha}}(z) \), where \( \hat{\theta}_{\alpha} \) is an approximate maximum \( R^{(\alpha)} \)-posterior estimator defined in \([16]\). The MRPDE, if it exists, is also an AMRPDE. Their \( L_1 \) consistency at the exponential rate has already been shown in Theorem 4.5.

**Example 4.1 (Robust ERPDE of the Normal Mean with Known Variance):** Let us consider the normal model with unknown mean \( \theta \) and known variance \( \sigma^2 \), i.e., the model density \( f_{\theta} \equiv N(\theta, \sigma^2) \) with \( \theta \in \Theta = \mathbb{R} \). We have an observed sample \( x_n = (x_1, \ldots, x_n)^T \) of size \( n \) from the true density \( g \equiv N(\theta_0, \sigma^2) \). Further, we have a prior density \( \pi(\theta) \); for simplicity, let us first assume the prior is uniform over \( \mathbb{R} \) (improper), i.e., \( \pi(\theta) = 1 \) for all \( \theta \in \mathbb{R} \). The standard Bayes posterior distribution of \( \theta \) is then \( N(\bar{X}, \sigma^2/n) \) where \( \bar{X} \) is the sample mean. It is well-known that \( \bar{X} \) is highly non-robust against data contamination/outliers and hence any inference based on this Bayes posterior \( N(\bar{X}, \sigma^2/n) \) is also highly sensitive to outlying data points. In particular, the posterior estimate of \( \theta \) with respect to any symmetric loss function is \( \bar{X} \) having zero breakdown point and unbounded influence function.

However, we can perform robust inference based on the \( R^{(\alpha)} \)-posterior density with \( \alpha > 0 \) which, in this case, simplifies to \( s_{\alpha}^{(n)}(\theta|\mathbf{x}_n) \propto \exp \left[ \frac{1}{\alpha (\sqrt{2\pi\sigma})} \sum_{i=1}^{n} e^{-\frac{\alpha (\theta-x_i)^2}{2\sigma^2}} \right] \). This is because, we now have \( q_{\alpha}^{(n)}(\mathbf{x}_n|\theta) = \frac{1}{\alpha (\sqrt{2\pi\sigma})} \sum_{i=1}^{n} e^{-\frac{\alpha (\theta-x_i)^2}{2\sigma^2}} - n\zeta_{\alpha} \), with \( \zeta_{\alpha} = (\sqrt{2\pi\sigma})^{-\alpha} (1 + \alpha)^{-3/2} \). Ghosh and Basu (2016) illustrated that the expectation of this \( R^{(\alpha)} \)-posterior (ERPE) is a very good robust estimator of \( \theta \) for moderately large \( \alpha \approx 0.5 \).

Here we illustrate the effectiveness of the predictive density estimator, the ERPDE, \( \hat{g}_\alpha^E(z) \). Note that, at \( \alpha = 0 \), the usual Bayes predictive density estimate is \( N(\bar{X}, \frac{n+1}{n}\sigma^2) \) which is again a function of the non-robust sample mean \( \bar{X} \). We expect to overcome this lack of robustness through the \( R^{(\alpha)} \)-posterior based predictive density estimator \( \hat{g}_\alpha^E(z) \) with \( \alpha > 0 \). We empirically compute and compare the KLD between \( \hat{g}_\alpha^E(z) \) and the true density \( g \) which measures the information loss in our estimation.

Note that, there is no simplified expression for the ERPDE with \( \alpha > 0 \) and so we need to compute it numerically. We use importance sampling Monte Carlo for this purpose with the proposal distribution as \( N(\bar{X}, s_\alpha^2) \) and 20000 steps. We simulate 1000 samples from the \( N(5,1) \) distribution (i.e., \( \theta_0 = 5 \) and \( \sigma^2 = 1 \)). For each given sample, we compute the ERPDE empirically using the importance sampling MC scheme and obtain the KLD between the ERPDE \( \hat{g}_\alpha^E \) and the true \( N(5,1) \) density for different \( \alpha \geq 0 \). The average KLD values over 1000 replications are reported for different sample sizes \( n = 20, 50, 100 \) (see Table 1). Clearly, in this case of pure data, the estimated predictive density \( \hat{g}_\alpha^E \) is closest to the true density in the KLD sense at \( \alpha = 0 \), the ordinary Bayes predictive density estimator, as expected. However, as \( \alpha \) increases, the increase in the corresponding average KLD values is not quite significant at small positive \( \alpha \) and this becomes even smaller for
larger sample sizes (where all KLD values are much closer to zero).

Table 1: Average KLD values between $\hat{g}_E^{\alpha}$, with uniform prior, and the true density $N(5, 1)$ for different $\alpha$, sample sizes $n$ and contamination proportions $\epsilon$.

| $n$ | $\epsilon$ | 0   | 0.2 | 0.3 | 0.5 | 0.7 | 0.8 | 1          |
|-----|------------|-----|-----|-----|-----|-----|-----|------------|
| 20  | 0          | 0.0221 | 0.0253 | 0.0264 | 0.0289 | 0.0359 | 0.0417 | 0.0575   |
|     | 0.05       | 0.0330 | 0.0298 | 0.0300 | 0.0327 | 0.0398 | 0.0473 | 0.0934   |
|     | 0.1        | 0.0655 | 0.0460 | 0.0424 | 0.0416 | 0.0481 | 0.0569 | 0.1171   |
|     | 0.2        | 0.1915 | 0.1230 | 0.0988 | 0.0775 | 0.0832 | 0.1006 | 0.1904   |
| 50  | 0          | 0.0093 | 0.0096 | 0.0102 | 0.0115 | 0.0133 | 0.0142 | 0.0173   |
|     | 0.05       | 0.0173 | 0.0138 | 0.0132 | 0.0138 | 0.0150 | 0.0159 | 0.0191   |
|     | 0.1        | 0.0544 | 0.0304 | 0.0224 | 0.0189 | 0.0187 | 0.0192 | 0.0229   |
|     | 0.2        | 0.1818 | 0.1054 | 0.0737 | 0.0389 | 0.0293 | 0.0287 | 0.0341   |
| 100 | 0          | 0.0045 | 0.0047 | 0.0049 | 0.0054 | 0.0062 | 0.0065 | 0.0076   |
|     | 0.05       | 0.0156 | 0.0091 | 0.0078 | 0.0070 | 0.0074 | 0.0077 | 0.0087   |
|     | 0.1        | 0.0471 | 0.0219 | 0.0153 | 0.0098 | 0.0088 | 0.0089 | 0.0099   |
|     | 0.2        | 0.1808 | 0.1020 | 0.0666 | 0.0268 | 0.0151 | 0.0134 | 0.0129   |

Next, to illustrate the advantages of the proposed ERPDE at $\alpha > 0$ in terms of robustness, let us repeat the above simulation exercise by replacing 100% of each sample by a moderate outlying value of $x = 8$. We have considered the contamination proportions as $\epsilon = 0.05, 0.10, 0.20$ and the corresponding average KLD values are reported in Table 1. Note that, in the presence of contamination, the usual Bayes predictive density estimator $\hat{g}_0^E$ goes away from the true density yielding larger KLD values. But, the ERPDEs $\hat{g}_E^\alpha$ with $\alpha > 0$ remain much closer to the true density in the KLD sense; their KL distances decrease as $\alpha$ increases up to a suitable value and then increases slightly (due to increase in the variance part) although always being significantly smaller than that for $\alpha = 0$ ($\hat{g}_0^E$). The minimum KL distance shows up at $\alpha \approx 0.5$ for moderate amount of contamination and moderate sample sizes; this optimum $\alpha$-value increases with increase in both contamination proportion and sample size. This clearly illustrate the significant robustness advantages of the proposed ERPDE with moderately large $\alpha > 0$ over the existing estimate at $\alpha = 0$, at the price of only a slight loss in case of pure data.

Table 2: Average KLD values between $\hat{g}_E^{\alpha}$, with conjugate $N(5, 9)$ prior, and the true density $N(5, 1)$ for different $\alpha$, sample sizes $n$ and contamination proportions $\epsilon$.

| $n$ | $\epsilon$ | 0   | 0.2 | 0.3 | 0.5 | 0.7 | 0.8 | 1          |
|-----|------------|-----|-----|-----|-----|-----|-----|------------|
| 20  | 0          | 0.0239 | 0.0249 | 0.0259 | 0.0289 | 0.0341 | 0.0386 | 0.0592   |
|     | 0.05       | 0.0322 | 0.0295 | 0.0300 | 0.0329 | 0.0387 | 0.0439 | 0.0669   |
|     | 0.1        | 0.0614 | 0.0419 | 0.0382 | 0.0373 | 0.0427 | 0.0490 | 0.0784   |
|     | 0.2        | 0.1908 | 0.1217 | 0.0970 | 0.0739 | 0.0752 | 0.0845 | 0.1276   |
| 50  | 0          | 0.0093 | 0.0095 | 0.0101 | 0.0113 | 0.0130 | 0.0139 | 0.0167   |
|     | 0.05       | 0.0158 | 0.0129 | 0.0127 | 0.0135 | 0.0152 | 0.0164 | 0.0194   |
|     | 0.1        | 0.0502 | 0.0267 | 0.0210 | 0.0165 | 0.0165 | 0.0173 | 0.0205   |
|     | 0.2        | 0.1826 | 0.1060 | 0.0740 | 0.0388 | 0.0283 | 0.0274 | 0.0312   |
| 100 | 0          | 0.0045 | 0.0047 | 0.0049 | 0.0054 | 0.0061 | 0.0065 | 0.0074   |
|     | 0.05       | 0.0156 | 0.0090 | 0.0077 | 0.0070 | 0.0073 | 0.0076 | 0.0085   |
|     | 0.1        | 0.0470 | 0.0219 | 0.0153 | 0.0097 | 0.0087 | 0.0088 | 0.0097   |
|     | 0.2        | 0.1800 | 0.1007 | 0.0659 | 0.0262 | 0.0145 | 0.0127 | 0.0121   |

Similar significant improvement in the robustness of the ERPDE can also be observed.
in the cases of proper subjective priors. As an illustration, suppose we have some vague prior belief about the true parameter value $\theta_0$, quantified through a $N(\theta_0, \tau^2)$ prior density, where $\tau$ measures the strength of our belief. Note that, it is the conjugate prior to the normal model family and the resulting $R^{(\alpha)}$-posterior density with $\alpha > 0$ is given by

$$
\pi_n^{(\alpha)}(\theta | x_n) \propto \exp \left[ \frac{1}{\alpha (\sqrt{2\pi\sigma})^\alpha} \sum_{i=1}^n \exp \left( -\frac{(\theta_i - q_i(x_i))^2}{2\sigma^2} - \frac{(\theta_i - \theta_0)^2}{2\tau^2} \right) \right].
$$

(20)

We repeat the previous simulation exercise, but now considering the conjugate normal prior with $\tau = 3$ (moderately strong prior belief) and the $R^{(\alpha)}$-posterior as in (20). The resulting values of average KLD measures are reported in Table 2, the robustness of the ERPDE with moderate $\alpha > 0$ is again clearly observed as in the previous case.

5 Application (II): Independent Non-homogeneous Models

5.1 Convergences of $R^{(\alpha)}$-Posterior and $R^{(\alpha)}$-Bayes estimators

Consider the set-up of independent but non-homogeneous (I-NH) models as described in Example 2.2 of Section 2. We simplify the exponential convergence results for the $R^{(\alpha)}$-posterior probabilities under this I-NH set-up from the general results of Section 3. Note that, in this case, $q_n^{(\alpha)}(x_i | \theta) = \sum_{i=1}^n q_i^{(\alpha)}(x_i)$ for any observed data $x_n = (x_1, \ldots, x_n)$, and hence $Q_n^{(\alpha)}(S_n | \theta) = \prod_{i=1}^n Q(i,\alpha)(S^i | \theta)$ for any $S_n = S_1 \times S_2 \times \cdots \times S_n \in B_n$ with $S^i \in B^i$ for all i and $Q(i,\alpha)(S^i | \theta) = \int_{S^i} \exp(q_i^{(\alpha)}(y))dy$. Assume that $\Theta_n = \Theta$ and $\pi_n = \pi$ are independent of $n$. Then, we have

$$
\tilde{q}_n^{(\alpha)}(x_i | \theta) = \frac{\prod_{i=1}^n \exp(q_i^{(\alpha)}(x_i))}{Q_n^{(\alpha)}(S_n | \theta)} = \prod_{i=1}^n \tilde{q}(i,\alpha)(x_i | \theta) + \exp(q_i^{(\alpha)}(x_i))
$$

so that, in the notation of Appendix A $D_n^{(\alpha)}(\theta) = \frac{1}{n} \sum_{i=1}^n KLD(g_i, \tilde{q}(i,\alpha)(\cdot | \theta))$. We define the information denseness for the I-NH models as follows:

Definition 5.1 The prior $\pi$ under the I-NH model is said to be information dense at $G_n = (G_1, \ldots, G_n)$ with respect to $F_{n,\alpha} = \otimes_{i=1}^n F_{G_i}$, if there exists a finite measure $\pi\tilde{\pi}$ satisfying (13) such that $\pi\tilde{\pi}\left(\{\theta : \limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^n KLD(g_i, \tilde{q}(i,\alpha)(\cdot | \theta)) < \epsilon\} \right) > 0$, for all $\epsilon > 0$.

When $f_{i,\theta} = f_\theta$ is independent of $i$, then the I-NH set-up coincides with the IID set-up and the information denseness in Definition 5.1 coincides with that in Definition 4.1. Further, Definition 5.1 implies the general Definition A.3 of Appendix A and hence Theorem A.2 yields $G^n$ and $M_n^{(\alpha)}$ merge in probability for the I-NH model. Then, our main exponential convergence results for the I-NH set-up simplifies as follows.

Proposition 5.1 Consider the set-up of I-NH models with $\Theta_n = \Theta$ and assume that the prior $\pi$ is independent of $n$ and information dense at $G_n$ with respect to $F_{n,\alpha}$ as per Definition 5.1. Then, for any sequence of measurable parameter sets $A_n \subset \Theta$, the $R^{(\alpha)}$-posterior probabilities $\pi_n^{(\alpha)}(A_n | X_n)$ is exponentially small with $P$-probability one, if and only if there
exists sequences of measurable parameter sets \( B_n, C_n \subset \Theta \) such that \( A_n \cup B_n \cup C_n = \Theta \), 
\[
\frac{M^{(\alpha)}_n(x_n, B_n)}{M^{(\alpha)}_n(x_n, \Theta_n)} \leq e^{-nr} \text{ for } r > 0 \text{ and a UEC test for } G_n \] 
against \( \left\{ \frac{Q^{(\alpha)}_n(\theta)}{Q^{(\alpha)}_n(x_n|\theta)} : \theta \in C_n \right\} \) exists.

However, the existence of the required UEC in Proposition 5.1 is equivalent to the existence of a UEC test for \( G_1 \) against \( \left\{ \frac{Q^{(\alpha)}_n(\theta)}{Q^{(\alpha)}_n(x_n|\theta)} : \theta \in C_n \right\} \) uniformly over \( i = 1, \ldots, n \).

Following the discussions of Section 4.1, this holds if Assumption (B) is satisfied for \( B^{T_n}_\epsilon \) in place of \( B^{T_n}_\epsilon \). This leads to the following simplification.

**Proposition 5.2** Consider the set-up of the I-NH models with \( \Theta_n = \Theta \) and assume that the prior \( \pi \) is independent of \( n \) and information dense at \( G_n \) with respect to \( F_{n, \alpha} \) as per Definition 5.1. If Assumption (B) holds for \( B^{T_n}_\epsilon \) in place of \( B^{T_n}_\epsilon \) for every \( \epsilon > 0 \), the \( R^{(\alpha)} \)-posterior probability \( \pi^{(\alpha)}_n \left( \{ \theta : \frac{1}{n} \sum_{i=1}^n d_1(q_i, f_{i, \theta}) \geq \epsilon \} \mid X_n \right) \) is also exponentially small with \( P \)-probability one for each \( \epsilon > 0 \).

We can then derive the consistency of any \( R^{(\alpha)} \)-Bayes estimator with respect to suitable bounded loss functions from Theorem 3.2 along with the simplified Proposition 5.2.

### 5.2 Robust Bayes Estimation under Fixed Design Regression Models

As noted in Example 2.2 of Section 2, the most common and useful example of the general I-NH set-up is the regression models with fixed design. Let us consider an important example of the regression model [8] with \( n \) fixed \( k \)-variate design points \( t_1, \ldots, t_n \) and \( f_{i, \theta}(x) = \frac{1}{f} \left( \frac{x - \psi(t_i, \theta)}{\sigma} \right) \) for some univariate density \( f \). The corresponding \( \alpha \)-likelihood is given by
\[
q^{(\alpha)}_n(x_n|\beta, \sigma) = \sum_{i=1}^n q^{(\alpha)}_{i, (\beta, \sigma)}(x_i) \quad \text{with} \quad q^{(\alpha)}_{i, (\beta, \sigma)}(x_i) = \frac{1}{\alpha \sigma} f \left( \frac{x_i - \psi(t_i, \theta)}{\sigma} \right)^\alpha - \frac{M_{f, \alpha}}{(1+\alpha)\sigma^\alpha} - \frac{1}{\alpha},
\]
where \( M_{f, \alpha} = \int f^{1+\alpha} \). Consider a prior density \( \pi(\beta, \sigma) \) for the parameters \( (\beta, \sigma) \) over the parameter space \( \Theta = \mathbb{R}^k \times (0, \infty) \) \( [p = k + 1] \). Note that this prior can be chosen to be the conjugate prior or any subjective or objective priors; a common objective prior is the Jeffrey’s prior given by \( \pi(\beta, \sigma) = \sigma^{-1} \). Then, the \( R^{(\alpha)} \)-posterior density of \( (\beta, \sigma) \) is given by [5] which simplifies in this case as
\[
\pi^{(\alpha)}_n(\beta, \sigma|X_n) = \frac{\prod_{i=1}^n \exp \left[ \frac{1}{\alpha \sigma} f \left( \frac{x_i - \psi(t_i, \beta)}{\sigma} \right)^\alpha - \frac{M_{f, \alpha}}{(1+\alpha)\sigma^\alpha} \right] \pi(\beta, \sigma)}{\int \prod_{i=1}^n \exp \left[ \frac{1}{\alpha \sigma} f \left( \frac{x_i - \psi(t_i, \beta)}{\sigma} \right)^\alpha - \frac{M_{f, \alpha}}{(1+\alpha)\sigma^\alpha} \right] \pi(\beta, \sigma)d\beta d\sigma}. \quad (21)
\]

If \( \sigma \) is known as in the Poisson or logistic regression models (or can be assumed to be known with properly scaled variables), we consider a prior only on \( \beta \) given by, say, \( \pi(\beta) \) which is either the objective uniform prior or the conjugate prior or some other proper prior. In such cases, we can get the simplified form for the \( R^{(\alpha)} \)-posterior density of \( \beta \) as given by
\[
\pi^{(\alpha)}_n(\beta|X_n) = \frac{\prod_{i=1}^n \exp \left[ \frac{1}{\alpha \sigma} f \left( \frac{x_i - \psi(t_i, \beta)}{\sigma} \right)^\alpha \right] \pi(\beta)}{\int \prod_{i=1}^n \exp \left[ \frac{1}{\alpha \sigma} f \left( \frac{x_i - \psi(t_i, \beta)}{\sigma} \right)^\alpha \right] \pi(\beta)d\beta}. \quad (22)
\]
We can easily obtain the $R^{(\alpha)}$-Bayes estimators of $\beta$ and $\sigma$ under any suitable loss function.

Example 5.1 (Normal Linear Regression Model):
Here we have $\psi(t, \beta) = t^T \beta$ and $f$ is the standard normal density. For simplicity, let us assume that the data are properly scaled so that $\sigma$ can be assumed to be known and equal to 1; the unknown $\sigma$ case can be considered similarly having the same robustness implications. When $\sigma = 1$, we can simplify the $R^{(\alpha)}$-posterior from (22) and compute the expected $R^{(\alpha)}$-posterior estimator (ERPE) of $\beta$ through an importance sampling Monte-Carlo. Let us denote $D = [t_1, \ldots, t_n]^T$, the fixed-design matrix, and $x = (x_1, \ldots, x_n)^T$; then the ordinary least square estimate of $\beta$ is $\hat{\beta} = (D^T D)^{-1} D^T x$, which is also the ordinary Bayes estimator under the uniform prior and has the variance $n^{-1}(D^T D)^{-1}$.

In our simulation, we use 20000 steps in the importance sampling Monte-Carlo with the proposal density $N_k(\hat{\beta}, n^{-1}(D^T D)^{-1})$ and empirically compute the bias and MSE of the ERPE over 1000 replications for two different priors, different contamination proportions $\epsilon_C = 0\%$ (pure data), 5\%, 10\%, 20\% and different sample sizes $n = 20, 50, 100$. For each case, we simulate $n$ observations $t_{11}, \ldots, t_{1n}$ independently from $N(5, 1)$ to fix the predictor values $t_i = (1, t_{1i})^T$ and $n$ independent error values $\epsilon_1, \ldots, \epsilon_n$ from $N(0, 1)$ (note $\sigma = 1$); then the responses are generated through the linear regression structure $x_i = t_i^T \beta + \epsilon_i$ for $i = 1, \ldots, n$, where the true value of $\beta$ is taken as $\beta_0 = (5, 2)^T$. For contaminated samples, $[n\epsilon_C]$ error values are contaminated by generating them from $N(5, 1)$ instead of $N(0, 1)$. As the first choice of the prior $\pi(\beta)$, we consider the non-informative uniform prior $\pi(\beta) \equiv 1$; the corresponding results are shown in Figure 1. Secondly, we consider the conjugate normal prior for $\beta$ given by $\pi(\beta) \equiv N_k(\beta_0, \tau^2 I_k)$ which signifies that the prior belief about our true parameter value is quantified by a symmetric structure with uncertainty quantified by $\tau$. The empirical biases and MSEs for the latter case are presented in Figure 2.

It is clearly observed from the figures that, under pure data, the bias and the MSE are the least for the usual Bayes estimator of $\beta$ at $\alpha = 0$, but their increments are not quite significant for the ERPEs with moderate $\alpha > 0$. On the other hand, in presence of contaminations, the usual Bayes estimator (at $\alpha = 0$) has severely inflated bias and MSE and becomes highly unstable. The proposed ERPEs with $\alpha > 0$ are much more stable under contamination in terms of both bias and MSE; the maximum stability is observed for tuning parameters $\alpha \in [0.4, 0.6]$ yielding significantly improved robust Bayes estimators.

6 Concluding Remark
This paper presents a general Bayes pseudo-posterior under general parametric set-up that produces pseudo-Bayes estimators which incorporate prior belief in the general spirit of Bayesian philosophy but are also robust against data contamination. The exponential consistency of the proposed pseudo-posterior probabilities and the corresponding estimators are proved and illustrated for the cases of independent stationary and non-homogeneous models; separate attention is given to the case of discrete priors with stationary models.
Further applications of the proposed pseudo-Bayes estimators are described in the context of predictive density estimation and linear regression models. All the results of Barron (1988) turn out to be special cases of our results when the tuning parameter $\alpha$ is set to 0.

On the whole, we trust that this paper opens up a new and interesting area of research on robust hybrid inference that has the flexibility to incorporate prior belief and inherits optimal properties from the Bayesian paradigm along with the frequentists’ robustness against data contamination and hence could be very helpful in different complex practical problems. In this sense, all Bayesian inference methodologies can be extended with this new pseudo-posterior in future research. In particular, a detailed study of the examples discussed in Section 2 should be an interesting future work for different applications. Extended versions of the Bayes testing and model selection criteria based on this new pseudo-posterior can also be developed to achieve greater robustness against data contamination. Also it will be practically helpful to develop a data-driven rule for the selection of the appropriate tuning parameter $\alpha$. We hope to pursue some of these extensions in the future.
Figure 2: Empirical Bias and MSE of the ERPE of $\beta$ in the linear regression model with Conjugate normal prior. [Dotted line: $\epsilon_C = 0\%$, Dash-Dotted line: $\epsilon_C = 5\%$, Dashed line: $\epsilon_C = 10\%$, Solid line: $\epsilon_C = 20\%$]

A Conditions for Merging of $R^{(\alpha)}$-marginal Distribution

Definition A.1 Two probability distributions $G^n_1$ and $G^n_2$ of the random variable $X_n$ are said to merge in probability if for every $\epsilon > 0$, \( \lim_{n \to \infty} P \left( \frac{g^n_2(X_n)}{g^n_1(X_n)} > e^{-n\epsilon} \right) = 1 \), where $g^n_i$ is the density function of $G^n_i$ with respect to $\lambda^n$ for $i = 1, 2$.

Definition A.2 Two probability distributions $G^n_1$ and $G^n_2$ of the random variable $X_n$ are said to merge with probability one if for every $\epsilon > 0$, \( P \left( \frac{g^n_2(X_n)}{g^n_1(X_n)} > e^{-n\epsilon} \right. \) for all large $n$ = 1.

Barron (1988) described several useful conditions under which two distributions merge in probability (or, with probability one). In particular, an application of Markov’s inequality yields that Definitions A.1 and A.2 are equivalent to the conditions \( \lim_{n \to \infty} \frac{1}{n} \log \frac{g^n_2(X_n)}{g^n_1(X_n)} = 0 \) in probability or with probability one, respectively. See Barron (1988, Section 4) for more results and discussions.
We have seen that a crucial condition needed for the exponential convergence of the R(\alpha)-Bayes estimators is the merging of G_n and M_n(\alpha) in probability. Now we present some sufficient conditions under which this merging holds.

Consider the general set-up of Section 2. We first derive a frequentist large-deviation approximation to the joint R(\alpha)-Bayes distribution of \theta and X_n that merge in (Kullback-Leibler) information; this in turn implies the merging of G_n and M_n(\alpha) in information and hence in probability. Let us consider the \alpha-modified model and prior densities \tilde{\pi}_n(\cdot|\theta) and \tilde{\pi}_n^{(\alpha)}(\theta) as defined in Remark A.2. Extending Barron (1988), we define the required joint frequentist distribution of \theta and X as given by \bar{L}_n^{(\alpha)}(d\theta, dx_n) = \pi_n^{(\alpha)}(d\theta) G_n(dx_n), where the probability distribution \pi_n^{(\alpha)} of \theta on \Theta_n is defined as \pi_n^{(\alpha)}(d\theta) = \frac{e^{-nD_n^{(\alpha)}(\theta)\tilde{\pi}_n^{(\alpha)}(d\theta)}}{c_n}, with D_n^{(\alpha)}(\theta) = \frac{1}{n} KLD\left(g_n(\cdot), \tilde{\pi}_n^{(\alpha)}(\cdot|\theta) \right) and c_n = \int e^{-nD_n^{(\alpha)}(\theta)\tilde{\pi}_n^{(\alpha)}(d\theta)}.

Assumption (M1): For any \epsilon, r > 0, there exists a positive integer N such that
\[ \tilde{\pi}_n^{(\alpha)}\left(\left\{ \theta : D_n^{(\alpha)}(\theta) < \epsilon \right\} \right) \geq e^{-nr}, \text{ for all } n \geq N. \]

Theorem A.1 Under Assumption (M1), we have \( \lim_{n \to \infty} \frac{1}{n} KLD\left(\bar{L}_n^{(\alpha)}, L_n^{(\alpha)}\right) = 0 \), i.e., the R(\alpha)-Bayes joint distribution \bar{L}_n^{(\alpha)} merge in information with the frequentist approximation \bar{L}_n^{(\alpha)}. Further, \( \lim_{n \to \infty} \frac{1}{n} E_{\Theta_n} \left[ KLD\left(\pi_n^{(\alpha)}(\cdot), \bar{\pi}_n^{(\alpha)}(\cdot|X_n) \right) \right] = 0 \), and \( \lim_{n \to \infty} \frac{1}{n} KLD\left(g_n, m_n^{(\alpha)}\right) = 0 \). Also, the second one implies that the distributions G^n and M_n^{(\alpha)} merge in probability.

Our next result simplifies Assumption (M1) further in terms of a suitable extended notion of the information denseness of priors \pi_n with respect to the family of \alpha-modified model densities \mathcal{F}_{n,\alpha} = \left\{ \tilde{\pi}_n(\cdot|\theta) : \theta \in \Theta_n \right\}.

Definition A.3 Suppose \Theta_n = \Theta is independent of n and define \( \bar{D}^{(\alpha)}(\theta) = \limsup_{n \to \infty} D_n^{(\alpha)}(\theta) \), the relative entropy rate of the sequence of measures G^n and \( \frac{Q_n^{(\alpha)}(\cdot)}{Q_n^{(\alpha)}(\Theta)} \). Then, the prior sequence \pi_n is said to be information dense at G^n with respect to \mathcal{F}_{n,\alpha} if there exists a finite measure \tilde{\pi} satisfying (17) such that \( \tilde{\pi}\left(\left\{ \theta : \bar{D}^{(\alpha)}(\theta) < \epsilon \right\} \right) > 0 \), for all \epsilon > 0.

Under this definition of information denseness for the general set-up, we have the required result on the merging of G^n and M_n^{(\alpha)} as presented in the next theorem.

Theorem A.2 If the prior is information dense with respect to \mathcal{F}_{n,\alpha} as in definition A.3, then G^n and M_n^{(\alpha)} merge in information and hence they also merge in probability.

Remark A.1 Under the independent stationary models considered in Section 4, Definition A.3 coincides with the simplified Definition A.4.

Remark A.2 Under the countable parameter space with the independent stationary model as considered in Subsection 4.2, Assumption (17) on the prior sequence \pi_n simplifies Definition A.3 further, so that we can take any strictly positive probability mass function on \Theta as
the measure $\tilde{\pi}$ and hence the condition in Definition A.3 reduces to \( \inf \{ D^{(\alpha)}(\theta) : \theta \in \Theta \} = 0. \) Thus, for any density $g$ which is an information limit of the (countable) family \( \{ q^{(\alpha)}(\cdot | \theta) : \theta \in \Theta_n \} \), the prior sequence $\pi_n$ will be information dense at $g$ with respect to $F_{n,\alpha}$.

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B Online Supplement: Proofs of the Results

B.1 Proof of Theorem 3.1

We use an argument similar to that used by Barron (1988). Let us consider the following two assumptions in addition to Assumptions (A1)–(A3) and (A3)*.

(A4) The true distribution $G^n$ and the $R(\alpha)$-marginal distribution $M_n^{(\alpha)}$ satisfy
\[
\lim_{n \to \infty} P \left( \frac{m_n^{(\alpha)}(X_n)}{g^n(X_n)} \geq a_n \right) = 1.
\]

(A4)* The true distribution $G^n$ and the $R(\alpha)$-marginal distribution $M_n^{(\alpha)}$ satisfy
\[
P \left( \frac{m_n^{(\alpha)}(X_n)}{g^n(X_n)} < a_n \text{ i.o.} \right) = 0.
\]

Note that, if Conditions (A4) and (A4)* hold with $a_n = e^{-n\epsilon}$ for every $\epsilon > 0$, they indicate that the true distribution $G^n$ and the $R(\alpha)$-marginal distribution $M_n^{(\alpha)}$ merge in probability or with probability one respectively.

Now, we start with two primary results on the convergence of the $R(\alpha)$-posterior probabilities.

Lemma B.1 Suppose Assumptions (A1)–(A3) and (A4) hold with $\lim b_n = \lim c_n = 0$ such that $r_n := (b_n + c_n)/a_n$ is finitely defined. Then, for all $\delta > 0$, we have
\[
\limsup_{n \to \infty} P \left( \pi_n^{(\alpha)}(A_n^c | X_n) > \frac{r_n}{\delta} \right) \leq \delta. \tag{23}
\]

Further, if additionally Assumptions (A3)* and (A4)* are satisfied, then for any summable sequence $\delta_n > 0$ we have
\[
P \left( \pi_n^{(\alpha)}(A_n^c | X_n) > \frac{r_n}{\delta_n} \text{ i.o.} \right) = 0. \tag{24}
\]

Proof: Note that, with $G^\infty$ probability one, the $R(\alpha)$-posterior probability can be re-expressed
\[
\pi_n^{(\alpha)}(A_n^c | X_n) = \frac{m_n^{(\alpha)}(X_n, A_n^c)}{m_n^{(\alpha)}(X_n)} \frac{M_n^{(\alpha)}(X_n, \Theta_n) g^n(X_n)}{m_n^{(\alpha)}(X_n) g^n(X_n)}, \tag{25}
\]

since $g^n(X_n)$ is non-zero for each $n$ with $G^\infty$ probability one. Let us first consider the numerator in (25) and define $E_n$ to be the event that the numerator is greater than $(b_n + c_n)/\delta$. Note that, $G^n(E_n) \leq G^n(E_n \cap S_n^c) + G^n(S_n)$ for any sequence of measurable sets...
Hence, \( G^n(E_n \cap S_n^c) = \int_{E_n \cap S_n^c} G^n(d\mathcal{F}_n) \)
\[
\leq \frac{\delta}{(b_n + c_n)} \int_{S_n^c} \frac{m_n(a_n)}{M_n^a(x_n, \Theta_n)g^n(x_n)} G^n(d\mathcal{F}_n) \quad \text{[by Markov’s inequality and definition of } E_n] \\
= \frac{\delta}{(b_n + c_n)M_n(x_n, \Theta_n)} \int_{S_n} m_n^a(x_n, A_n^c) d\mathcal{F}_n \\
= \frac{\delta}{(b_n + c_n)M_n(x_n, \Theta_n)} \int_{S_n} \int_{A_n^c} \exp(q_n^a(x_n(\theta))) \pi_n(\theta) d\theta d\mathcal{F}_n \\
= \frac{\delta}{(b_n + c_n)M_n(x_n, \Theta_n)} \int_{A_n^c} Q_n^a(S_n^c | \theta) \pi_n(\theta) d\theta \\
\leq \frac{\delta}{(b_n + c_n)M_n(x_n, \Theta_n)} \left[ \int_{B_n} Q_n^a(x_n | \theta) \pi_n(\theta) d\theta + \int_{C_n} Q_n^a(S_n^c | \theta) \pi_n(\theta) d\theta \right] \\
\leq \frac{\delta}{(b_n + c_n)M_n(x_n, \Theta_n)} \left[ M_n(x_n, B_n) + \sup_{\theta \in C_n} Q_n^a(S_n^c | \theta) M_n^a(x_n, C_n) \right] \\
\leq \frac{\delta}{(b_n + c_n)} \left[ b_n + c_n \frac{M_n(x_n, C_n)}{M_n(x_n, \Theta_n)} \right] \quad \text{[by Assumptions (A2) and (A3)]} \\
\leq \delta.
\]

Hence, \( G^n(E_n) \leq \delta + G^n(S_n) \) and using Assumption (A3) we get \( \limsup_{n \to \infty} G^n(E_n) \leq \delta \).

Further, by Assumption (A4) the denominator in (25) is less than \( a_n \) has probability tending to zero. Combining the numerator and denominator probabilities (using the bound by the union of events related to numerator and denominator), we get the desired result (23).

To prove the second part (24), we proceed as before by noting that \( P(\mathbf{X}_n \in E_n \text{ i.o.}) \leq P(\mathbf{X}_n \in E_n \cap S_n^c \text{ i.o.}) + P(\mathbf{X}_n \in S_n \text{ i.o.}) \). Then, defining \( E_n \) with any summable sequence \( \delta_n \) and proceeding as before, we get \( P(\mathbf{X}_n \in E_n \cap S_n^c \text{ i.o.}) = 0 \) by Borel-Cantelli Lemma. Next, by Assumption (A3)*, we have \( P(\mathbf{X}_n \in S_n \text{ i.o.}) = 0 \) and hence \( P(\mathbf{X}_n \in E_n \text{ i.o.}) = 0 \). Then, the desired result (24) follows by noting that the denominator in (25) is less than \( a_n \) infinitely often with probability zero by Assumption (A4)*.

**Lemma B.2** Suppose, for some sequence of constants \( r_n \), we have
\[
\lim_{n \to \infty} P \left( \pi_n^a(A_n^c | X_n) \leq r_n \right) = 1.
\]

Then, for any sequences \( b_n \) and \( c_n \) satisfying \( b_nc_n \geq r_n \), there exists parameter sets \( B_n, C_n \subset \Theta_n \) such that Conditions (A1)–(A3) hold.
Moreover, if additionally we have

\[ P\left(\pi_n^{(a)}(A_n|X_n) > r_n \ i.o.\right) = 0, \quad (27) \]

then Conditions (A1), (A2) and (A3)* hold.

**Proof:** Let us define \( S_n = \left\{ x_n : \pi_n^{(a)}(A_n^c|x_n) > r_n \right\} \) so that \( \lim_{n \to \infty} C^n(S_n) = 0 \) by Assumption (26). Next, for any sequence \( c_n \), we construct the parameter sets

\[
C_n = \left\{ \theta : \frac{Q_n^{(a)}(S_n^c|\theta)}{Q_n^{(a)}(X_n|\theta)} \leq c_n \right\}, \quad B_n = \left\{ \theta \in A_n^c : \frac{Q_n^{(a)}(S_n^c|\theta)}{Q_n^{(a)}(X_n|\theta)} > c_n \right\}.
\]

Then, Conditions (A2) and (A3) hold by constructions of \( C_n \) and \( B_n \). Finally, to show Condition (A2), note that \( m_n^{(a)}(x_n, A_n^c) \leq r_n M_n^{(a)}(X_n, \Theta_n) m_n^{(a)}(x_n) \) for all \( x_n \in S_n^c \) by its definition. Then,

\[
\frac{M_n^{(a)}(X_n, B_n)}{M_n^{(a)}(X_n, \Theta_n)} = \frac{1}{M_n^{(a)}(X_n, \Theta_n)} \int_{B_n} Q_n^{(a)}(X_n|\theta) \pi_n(\theta) d\theta \\
\leq \frac{1}{M_n^{(a)}(X_n, \Theta_n) c_n} \int_{A_n^c} Q_n^{(a)}(S_n^c|\theta) \pi_n(\theta) d\theta \\
\quad \text{[by Definition of } B_n \text{ and Markov’s inequality]} \\
\leq \frac{1}{M_n^{(a)}(X_n, \Theta_n) c_n} \int_{A_n^c} \exp(q_n^{(a)}(x_n|\theta)) d\pi_n x_n \pi_n(\theta) d\theta \\
\leq \frac{1}{M_n^{(a)}(X_n, \Theta_n) c_n} \int_{S_n} m_n^{(a)}(x_n, A_n^c) d\pi_n x_n \quad \text{[by Fubini Theorem]} \\
\leq \frac{1}{M_n^{(a)}(X_n, \Theta_n) c_n} \int_{S_n} r_n M_n^{(a)}(X_n, \Theta_n) m_n^{(a)}(x_n) d\pi_n x_n \quad \text{[by the construction of } S_n \text{]} \\
\leq \frac{r_n}{c_n} \int_{S_n} m_n^{(a)}(x_n) d\pi_n x_n \quad \text{[for any sequence } b_n \text{ satisfying } b_n c_n \geq r_n \text{]}
\]

For the second part of the Lemma, we use the same definitions of sets as above. Then, by Assumption (27), we have \( P(X_n \in S_i \ i.o.) = 0 \) and hence Condition (A3)* holds by the construction of \( C_n \). Other two conditions then hold similarly as before. \( \square \)

**Proof of Theorem 3.1:**

Theorem 3.1 now follows directly from the above two lemmas.

The sufficiency part of the theorem follows from Lemma B.1 by taking \( b_n = e^{-nr_1} \), \( c_n = e^{-nr_2} \), \( a_n = e^{-n\epsilon} \) and \( \delta_n = e^{-n\Delta} \) (for Part 2) with \( \epsilon, \Delta > 0 \) and \( \epsilon + \Delta < \min\{r_1, r_2\} \)

Then, \( r_n \) and \( r_n' = r_n/\delta_n \) tend to zero exponentially fast.

The Necessity part of the theorem follows from Lemma B.2 with \( r_n = e^{-nr} \) and then
letting \( b_n = e^{-nr_1}, c_n = e^{-nr_2} \) for any \( r_1, r_2 > 0 \) with \( r_1 + r_2 \leq r \).

\[ \square \]

### B.2 Proof of Theorem 4.4

Note that, by the definition of \( \hat{d}_{\alpha} \), it is sufficient to show that

\[
\sup_{\theta \in \mathcal{A}_n} \frac{\tilde{\pi}_n^{(a)}(\theta) q_n^{(a)}(\mathbf{X}_n | \theta)}{\tilde{\pi}_n^{(a)}(\theta) q_n^{(a)}(\mathbf{X}_n | \theta)} < \sup_{\theta} \frac{\tilde{\pi}_n^{(a)}(\theta) \tilde{q}_n^{(a)}(\mathbf{X}_n | \theta)}{\tilde{\pi}_n^{(a)}(\theta) \tilde{q}_n^{(a)}(\mathbf{X}_n | \theta)} e^{-nd_n} \text{ a.s.}[G], \quad \text{for all large } n. \quad (28)
\]

Now, by the information denseness assumption, Remark A.1 and Theorem A.2 of Appendix A imply that \( G_n^\alpha \) and \( M_n^{(a)} \) merge in probability. Therefore, the exponential convergence of \( \tilde{\pi}_n^{(a)}(A_n^* | \mathbf{X}_n) \) is equivalent to

\[
\sum_{\theta \in \mathcal{A}_n} \tilde{\pi}_n^{(a)}(\theta) \tilde{q}_n^{(a)}(\mathbf{X}_n | \theta) \leq m_n^{(a)}(\mathbf{X}_n) e^{-nr_1} < g_n(\mathbf{X}_n) e^{-nr} \text{ a.s.}[G], \quad \text{for all large } n, \text{ for some } r_1, r > 0.
\]

Let us now choose a \( \theta^* \in \Theta \) such that \( KLD(g, \tilde{q}^{(a)}(\cdot | \theta^*)) < r/4 \). Then, using SLLN along with Assumption (17), we get

\[
g_n(\mathbf{X}_n) < \frac{\pi_n^{(a)}(\theta^*) q_n^{(a)}(\mathbf{X}_n | \theta^*)}{\pi_n^{(a)}(\theta) q_n^{(a)}(\mathbf{X}_n | \theta)} e^{nr/2} \text{ a.s.}[G], \quad \text{for all large } n.
\]

Therefore, for all large \( n \), we have with a.s.\([G]\),

\[
\sup_{\theta \in \mathcal{A}_n} \tilde{\pi}_n^{(a)}(\theta) q_n^{(a)}(\mathbf{X}_n | \theta) \leq \sum_{\theta \in \mathcal{A}_n} \tilde{\pi}_n^{(a)}(\theta) \tilde{q}_n^{(a)}(\mathbf{X}_n | \theta) < g_n(\mathbf{X}_n) e^{-nr} < \tilde{\pi}_n^{(a)}(\theta^*) \tilde{q}_n^{(a)}(\mathbf{X}_n | \theta^*) e^{-nr/2} < \sup_{\theta} \tilde{\pi}_n^{(a)}(\theta) \tilde{q}_n^{(a)}(\mathbf{X}_n | \theta) e^{-nd_n}.
\]

This completes the proof that \( \hat{d}_{\alpha} \in A_n \text{ a.s.}[G] \), for all sufficiently large \( n \). \( \square \)

### B.3 Proof of Theorem 4.5

Using the equivalence of \( d_1 \) and \( d_H \), it is enough to show that \( \pi_n^{(a)}(A^c | \mathbf{X}_n) \) is exponentially small with probability one, with \( A = \{ \theta : d_H(g, f_{\theta}) \geq \epsilon \} \) for each fixed \( \epsilon > 0 \). Note that, \( G_n^\alpha \) and \( M_n^{(a)} \) merge in probability by applying Theorem A.2 and Remark A.2 of Appendix A. So, we will use Theorem 3.1 by constructing suitable parameter sets \( B_n \) and \( C_n \) with \( A \cup B_n \cup C_n = \Theta \).

Put \( B_n = \{ \theta : \pi_n(\theta) < e^{-nc/4} \} \) and \( C_n = \{ \theta \in A_n^c : \pi_n(\theta) \geq e^{-nc/4} \} \). Then, clearly \( A \cup B_n \cup C_n = \Theta \). Further, for some \( \tau \in (0, 1) \),

\[
\frac{M_n^{(a)}(\mathbf{X}_n, B_n)}{M_n^{(a)}(\mathbf{X}_n, \Theta_n)} = \sum_{\theta \in B_n} \frac{Q_n^{(a)}(\chi_n | \theta)}{M_n^{(a)}(\mathbf{X}_n, \Theta_n)} \pi_n(\theta) \leq \frac{e^{-nc(1-\tau)/4}}{M_n^{(a)}(\mathbf{X}_n, \Theta_n)} \sum_{\theta \in B_n} Q_n^{(a)}(\chi_n | \theta) \pi_n(\theta)^\tau
\]

34
But, since the prior sequence \( \pi_n \) satisfies Assumption (17), we get, for all sufficiently large \( n \), (assuming all the relevant quantities exists finitely)

\[
M_n(\alpha_n, \Theta_n) = \sum_{\theta \in \Theta} Q_n(\alpha_n | \theta) \pi_n(\theta)
\geq e^{-n(1-\tau)\epsilon/8} \sum_{\theta \in \Theta} Q_n(\alpha_n | \theta) \pi_n(\theta)^\tau
\geq e^{-n(1-\tau)\epsilon/8} \sum_{\theta \in B_n} Q_n(\alpha_n | \theta) \pi_n(\theta)^\tau,
\]

and hence

\[
\frac{M_n(\alpha_n, B_n)}{M_n(\alpha_n, \Theta_n)} \leq e^{-n(1-\tau)\epsilon/8}.
\]

Thus, the first two conditions of Theorem 3.1 hold. For the third condition related to \( C_n \), note that \( \sum_{\theta \in C_n} \pi_n(\theta) \leq 1 \) and so the number of points in \( C_n \) is less than \( e^{n\epsilon/4} \). Then, consider the likelihood ratio test for \( g_n \) against \( \left\{ \frac{\exp(q_n(\alpha_n \cdot | \theta))}{Q_n(\alpha_n | \theta)} : \theta \in C_n \right\} \) having the critical sets

\[
S_n = \left\{ x_n : \max_{\theta \in C_n} \frac{\exp(q_n(\alpha_n(\theta)))}{Q_n(\alpha_n | \theta)} > g_n(x_n) \right\}.
\]

We will show that this \( S_n \) serves as the desired set in the required condition (A3) on \( C_n \). For note that, \( S_n = \bigcup_{\theta \in C_n} S_{n, \theta} \), where \( S_{n, \theta} = \left\{ x_n : \left[ \frac{\exp(q_n(\alpha_n(\theta)))}{Q_n(\alpha_n | \theta)} \right]^{1/2} > g_n(x_n)^{1/2} \right\} \). But for each of these sets, we get from Markov inequality that,

\[
G_n(S_{n, \theta}) \leq \left[ 1 - \frac{1}{2} d_H(g, f_\theta) \right]^n < e^{-n\epsilon/2},
\]

and hence \( G_n(S_n) < e^{-n\epsilon/4} \). Similarly, we can also show that

\[
\frac{Q_n(\alpha_n(S_n^c | \theta))}{Q_n(\alpha_n | \theta)} \leq \frac{Q_n(\alpha_n(S_n^c | \theta_{n, \theta}))}{Q_n(\alpha_n | \theta)} < e^{-n\epsilon/2},
\]

uniformly over \( \theta \in C_n \). Hence, all the required conditions of Theorem 3.1 hold and we get the first part of the present theorem.

The second part follows from Theorem 4.4 and Remark A.2 \( \Box \)

**B.4 Proof of Proposition 4.6**

Fix \( \epsilon > 0 \) and \( x_n \). Let \( a = \pi_n^a(A_n, \epsilon | x_n) \). If \( a = 0 \), the result is trivial. So, assume \( a > 0 \) and consider the distribution \( \pi_n^a(d\theta | x_n, A_n, \epsilon) \) obtained from \( \pi_n^a(A_n, \epsilon | x_n) \) by conditioning.
on $\theta \in A_{n,\epsilon}$. Then, we have

\[
L_n \left( P, \hat{P}_\alpha \right) \leq L_n \left( P, \int_{A_{n,\epsilon}} F_\theta \pi_n^{(a)} (d\theta | x_n) \right) \quad \text{[By Monotonicity]}
\]

\[
= L_n \left( P, \int_{A_{n,\epsilon}} (aF_\theta) \pi_n^{(a)} (d\theta | x_n, A_{n,\epsilon}) \right)
\]

\[
\leq \int_{A_{n,\epsilon}} L_n (P, aF_\theta) \pi_n^{(a)} (d\theta | x_n) \quad \text{[By Convexity]}
\]

\[
\leq \int_{A_{n,\epsilon}} [L_n (P, F_\theta) + \rho(a)] \pi_n^{(a)} (d\theta | x_n) \quad \text{[By Scaling]}
\]

\[
\leq \epsilon + \rho(a).
\]

\[\Box\]

B.5 Proof of Theorem A.1

First note that, by help of Equation (6), we can rewrite the $R^{(\alpha)}$-Bayes joint distribution as

\[
L_n^{(\alpha)}(d\theta, dx_n) = \pi_n^{(a)} (d\theta | x_n) n_n^{(a)} (dx_n) = \frac{m_n^{(a)} (dx_n, d\theta)}{M_n^{(a)} (X_n, \Theta_n)} = \frac{q_n^{(a)} (dx_n | \theta) \pi_n^{(a)} (d\theta)}{c_n},
\]

which has a density $\frac{q_n^{(a)} (x_n | \theta)}{c_n}$ with respect to $\pi_n^{(a)} \times \lambda_n$. On the other hand, the frequentist approximation $L_n^{(a)}$ has the density function $e^{-nD_n^{(\alpha)}(\theta)} g_n (x_n) / c_n$ and hence we get

\[
KLD \left( L_n^{(\alpha)}, L_n^{(\alpha)} \right) = E_{L_n^{(\alpha)}} \left[ \log \frac{e^{-nD_n^{(\alpha)}(\theta)} g_n (x_n) / c_n}{\pi_n^{(a)} (x_n | \theta)} \right]
\]

\[
= E_{\pi_n^{(a)}} [E_{G_n} \left[ -nD_n^{(\alpha)}(\theta) + \log \frac{g_n (x_n)}{q_n^{(a)} (x_n | \theta)} \right] - \log c_n]
\]

\[
= E_{\pi_n^{(a)}} \left[ -nD_n^{(\alpha)}(\theta) + E_{G_n} \log \frac{g_n (x_n)}{q_n^{(a)} (x_n | \theta)} \right] - \log c_n
\]

\[
= E_{\pi_n^{(a)}} \left[ -nD_n^{(\alpha)}(\theta) + nD_n^{(\alpha)}(\theta) \right] - \log c_n
\]

\[
= - \log c_n = - \log \left[ \int e^{-nD_n^{(\alpha)}(\theta) \pi_n^{(a)} (d\theta)} \right]
\]

Therefore, for any $\epsilon > 0$, we get

\[
\frac{1}{n} KLD \left( L_n^{(\alpha)}, L_n^{(\alpha)} \right) = - \frac{1}{n} \log \int e^{-nD_n^{(\alpha)}(\theta) \pi_n^{(a)} (d\theta)} \leq \frac{\epsilon}{2} - \frac{1}{n} \log \pi_n^{(a)} \left( \left\{ \theta : D_n^{(\alpha)}(\theta) < \epsilon \right\} \right)
\]

\[
\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \quad \text{for all but finitely many } n,
\]

(30)
by applying Assumption (M1) with \( r = \frac{\epsilon}{2} \). Since \( \epsilon > 0 \) is arbitrary, this completes the proof of the first part of Theorem.

The second part of the theorem follows by the relation

\[
\frac{1}{n} KLD \left( L_n^{*(\alpha)}, L_n^{(\alpha)Bayes} \right) = \frac{1}{n} E_{G^n} \left[ KLD \left( \pi_n^{*(\alpha)}(\cdot), \pi_n^{(\alpha)}(\cdot | X_n) \right) \right] + \frac{1}{n} KLD \left( g^n, m_n^{(\alpha)} \right).
\]

To proof the the last part of theorem, note that the Kullback-Leibler divergence satisfies the relation

\[
E \left| \log \frac{g_n(X_n)}{m_n^{(\alpha)}(X_n)} \right| \leq KLD \left( g^n, m_n^{(\alpha)} \right) + \frac{2}{e}.
\]

Therefore, by the second part of theorem, we get \( \lim_{n \to \infty} E \left| \log \frac{g_n(X_n)}{m_n^{(\alpha)}(X_n)} \right| = 0 \) and hence \( G_n \) and \( M_n^{(\alpha)} \) merge in probability by using Markov inequality.

\[\square\]

**B.6 Proof of Theorem [A.2]**

We will show that Assumption (M1) holds and then the present theorem will follow by Theorem [A.1]. To show Assumption (M1), let us fix \( \epsilon, r > 0 \) and define \( \rho_n(\theta) = e^{nr \frac{d\pi_n}{d\bar{\pi}}(\theta)} \). Then, using Fatou’s Lemma, we get

\[
\lim_{n \to \infty} e^{nr} \pi_n \left( \left\{ \theta : D_n^{(\alpha)}(\theta) < \epsilon \right\} \right) = \lim_{n \to \infty} e^{nr} \int I \left( \left\{ \theta : D_n^{(\alpha)}(\theta) < \epsilon \right\} \right) \pi_n(d\theta)
\]

\[
= \lim_{n \to \infty} \int I \left( \left\{ \theta : D_n^{(\alpha)}(\theta) < \epsilon \right\} \right) \rho_n(\theta) \bar{\pi}(d\theta)
\]

\[
= \int \lim_{n \to \infty} I \left( \left\{ \theta : D_n^{(\alpha)}(\theta) < \epsilon \right\} \right) \rho_n(\theta) \bar{\pi}(d\theta)
\]

\[
= \int I \left( \left\{ \theta : \bar{D}^{(\alpha)}(\theta) < \epsilon \right\} \right) \bar{\pi}(d\theta)
\]

\[
= \bar{\pi} \left( \left\{ \theta : D_n^{(\alpha)}(\theta) < \epsilon \right\} \right),
\]

which is strictly positive by the information denseness with respect to \( F_{n,\alpha} \) (Definition [A.3]). This implies Assumption (M1) and we are done. \(\square\)