Magnetic reconnection with anomalous resistivity in two-and-a-half dimensions I: Quasi-stationary case

Leonid M. Malyshev and Timur Linde

Department of Astronomy, The University of Chicago (The Center for Magnetic SelfOrganization (CMSO), Chicago, IL 60637.

Russell M. Kulsrud

Princeton Plasma Physics Laboratory (The Center for Magnetic Self-Organization (CMSO), Princeton, NJ 08543.

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In this paper quasi-stationary, two-and-a-half-dimensional magnetic reconnection is studied in the framework of incompressible resistive magnetohydrodynamics (MHD). A new theoretical approach for calculation of the reconnection rate is presented. This approach is based on local analytical derivations in a thin reconnection layer, and it is applicable to the case when resistivity is anomalous and is an arbitrary function of the electric current and the spatial coordinates. It is found that a quasi-stationary reconnection rate is fully determined by a particular functional form of the anomalous resistivity and by the local configuration of the magnetic field just outside the reconnection layer. It is also found that in the special case of constant resistivity reconnection is Sweet-Parker and not Petschek.

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I. INTRODUCTION

Magnetic reconnection is the physical process of breaking and rearrangement of magnetic field lines, which changes the topology of the field. It is one of the most fundamental processes of plasma physics and is believed to be at the core of many dynamical phenomena in laboratory experiments and in cosmic space. Unfortunately, in spite of being so important, magnetic reconnection is still relatively poorly understood from the theoretical point of view. The reason is that plasma usually have very high temperatures and low densities. In such plasma, as the Spitzer resistivity is extremely small and magnetic fields are almost perfectly frozen into the plasma. As a result, simple theoretical models, such as the Sweet-Parker reconnection model [1, 2] predict that the magnetic reconnection processes are extremely slow and insignificant throughout the universe. On the other hand, astrophysical observations show magnetic reconnection tends to be fast and is likely to be the primary driver of any highly energetic cosmic processes, such as solar flares and geomagnetic storms. This contradiction between theoretical estimates and astrophysical observations triggered multiple attempts to build a theoretical model of fast magnetic reconnection.

First, in 1964 Petschek proposed a fast reconnection model [3], in which fast reconnection is achieved by introducing switch-on magnetohydrodynamic (MHD) shocks attached to the ends of the reconnection layer in the downstream regions and by choosing the reconnection layer length to be equal to its minimal possible value under the condition of no significant disruption to the plasma flow. However, later numerical simulations and theoretical derivations did not confirm the Petschek theoretical picture for the geometry of a reconnection layer [4-6]. Second, numerical simulations of anomalous magnetic reconnection, for which resistivity is enhanced locally in the reconnection layer, were pioneered by Ugalde and Tsuda [7, 8], by Hayashi and Sato [9, 10], and by Scholer [11]. Third, Lazarian and Vishniac proposed that fast reconnection can occur in turbulent plasma [12], although the back-reaction of magnetic fields can slow down reconnection in this case [13]. Finally, recently there have been number of attempts to explain fast magnetic reconnection by considering non-MHD effects [14-21].

Most previous theoretical and numerical studies concentrated on reconnection processes in two-dimensions or in two-and-a-half dimensions. The later is the term used for a problem in which physical scalars and all three components of physical vectors depend only on two spatial coordinates (e.g., x and y) and are independent of the third coordinate (z).

In this paper we consider two-and-a-half dimensional magnetic reconnection with anomalous resistivity in the classical Sweet-Parker-Petschek reconnection layer, which is shown in the left plot in Fig. 1. The reconnection layer is in the x-y plane with the y-axis being along the layer and the x-axis being perpendicular to the layer. The length of the layer is equal to 2L. Note that L is approximately equal to or smaller than the global magnetic field scale, which we denote as L. The thickness of the classical reconnection layer, 2δ, is much smaller than its length, i.e., 2δ ≈ 2L. The classical Sweet-Parker-Petschek reconnection layer is assumed to possess a point symmetry with respect to its geometric center.

Electronic address: leonal@flash.uchicago.edu
Electronic address: lindel@flash.uchicago.edu
Electronic address: rmk@pppl.gov

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center point \(O\) and reflection symmetries with respect to the axes \(x\) and \(y\) (refer to Fig. 1). Thus, for example, the \(x\)- and \(y\)-components of the plasma velocities \(V\) and of the magnetic \(B\) have the following simple symmetries:

\[
V_x(x; y) = V_k(x; y), \quad V_y(x; y) = V_k(x; y),
\]

\[
B_x(x; y) = B_k(x; y) \quad \text{and} \quad B_y(x; y) = B_k(x; y).
\]

There might be a pair of Petschek shocks attached to each of the two reconnection layers in the downstream regions (see Fig. 1). Because of the MHD jump conditions on the Petschek shocks, the presence of these shocks requires the presence of a significant perpendicular magnetic \(B\) at the reconnection layer ends [5, 22]. The plasma outflow velocity from the reconnection layer is approximately equal to the Alfvén velocity \(V_A\) (if the plasma viscosity is not very large). The plasma in outflow velocity \(V_R\) outside of the reconnection layer, at point \(M\) in Fig. 1, is much smaller than the outflow velocity, \(V_R \approx V_A\). Finally, the magnetic \(B\) outside the reconnection layer is mostly in the direction of the layer (i.e., in the \(y\)-axis direction).

The problem of quasi-stationary anomalous magnetic reconnection in the classical Sweet-Parker-Petschek reconnection layer was recently theoretically addressed by Kulsrud [5] for the special case of zero guide \(B\) (\(B = 0\)), zero plasma viscosity, and anomalous resistivity and an arbitrary function of the electric current and the two spatial coordinates. Second, it gives a new important insight into the reconnection problem, such as dependence of the reconnection rate on magnetic field configuration just outside the reconnection layer. Third, our approach, based on local calculations, is applicable to cases when there is no well-defined global magnetic field structure, such as the case of multiple current sheets in a turbulent plasma. In our calculations we make only a few assumptions, which are described in detail in the beginning of Sec. III.

This paper is organized as follows. Because our derivations are rather complicated in the general case of reconnection with anomalous resistivity, we nd it useful and instructive to consider the Sweet-Parker reconnection rst and to compare our theoretical approach to the classical Sweet-Parker calculations in the next section. In Sec. III, we derive our general equations for magnetic reconnection with anomalous resistivity, including the equation for the reconnection rate. In Sec. IV we consider and analyze different special cases of magnetic reconnec-

![Diagram of Petschek shocks](image)
tition, in which our equations simplify and become easier for analysis and comparison to the previous theoretical and simulation results. In Sec. V we present the results of our numerical simulations of unforced anomalous magnetic reconnection. These simulations are intended for a demonstration of the predictions of our theoretical reconnection model for the special case of the piecewise linear resistivity function that was considered by Kulsrud [5]. Finally, in Sec. V we give our conclusions and discuss our results. Some derivations are given in the appendices of the paper.

II. THE SWEET-PARKER MODEL OF MAGNETIC RECONNECTION

For simplicity and brevity, hereafter in this paper for all electron magnetic variables we use physical units in which the speed of light and four times are replaced unity, c = 1 and 4 = 1. To rewrite our equations in the standard CGS units, one needs to make the following substitutions for electron magnetic variables in the equations: magnetic field B = 4/2, electric field E = 4, electric current j = 4, resistivity = (does not change), magnetic vector potential A = 4. 

In this section we consider Sweet-Parker reconnection [1, 2]. We assume that resistivity is constant, const = ρ, the plasma viscosity is zero, the guide field is zero (Bz = 0), and the geometry of the reconnection layer is the classical Sweet-Parker geometry with the layer half-thickness L and the layer half-length L0 = L, as shown in Fig. 1. The purpose of this section is to introduce and explain our new theoretical approach to calculation of the reconnection rate and to compare it with the classical Sweet-Parker calculations. For this purpose, we first present the classical, conventional derivation of the Sweet-Parker formula for the reconnection rate, and after we present our new derivation of the formula and explain the difference between the two derivations.

In the case of a quasi-stationary reconnection the magnetic field in the reconnection region changes slowly in time, @B = @t = 0. Therefore, in the two-and-a-half-dimensional geometry (B = @x = 0) the Faraday’s law equation @E = @B = @t = 0 results in the z-component of the electric field being constant in the reconnection region, @Ez = 0 and Ez = Ez(t) is a function of time only. On the other hand, in two-and-a-half-dimensional geometry the x- and y-components of Faraday’s law equation @E = @t = r r V E = (j V j) reduce to the following equation for the x-component of the magnetic vector potential A:

\[ E_x(t) = @A_x = @t \rightarrow \nabla \times E_x = \nabla \times V \times B \]

and is valid even if resistivity is anomalous and non-constant.

Now, the left- and right-hand-sides of equation (1) are constant in space. Therefore, the right-hand-side of equation (1) is constant across the reconnection layer (i.e., along the x-axis). We equate its values at points O and M, which are on the x-axis and are shown in Fig. 1. As a result, we immediately obtain

\[ \phi_B = \phi_M = \phi \]

where we use the following notations: Point O is the geometric center of the reconnection layer, where the z-component of the current is \( j_z = j_x (x = 0, y = 0) \), and the plasma velocity is zero. Point M is a point on the x-axis just outside the reconnection layer, where the resistivity term can be neglected in equation (1), see Fig. 1. We also use the notations \( B_y = B_y \) and \( V_o = V_o \) for the y-component of the field and x-component of the plasma velocity at point M, and take the reconnection velocity \( V_o \) positive. Note that \( V_y = B_x = 0 \) at point M because of the symmetry of the problem with respect to the x-axis. Next, we estimate current \( j_z \) at the central point \( O \) of the reconnection layer as

\[ j_z B_m = \phi \]

where \( \phi \) is the reconnection layer half-thickness, and we use Ampere’s law, \( \phi = \phi = \phi = \phi = \phi = \phi \), in which we drop the \( B_z = \phi y \) term because the reconnection layer is thin [36].

Next, consider the x- and y-components of the equation of plasma motion. We will see below that the Sweet-Parker reconnection is slow, \( V_o = V_y = 0 \), and therefore, in the equation of plasma motion along the x-axis (i.e., across the reconnection layer) the inertial term can be neglected, and this equation becomes the force balance equation, \( \phi = \phi = \phi = \phi = \phi = \phi \), resulting in \( F_0 = F_m + B_x z = 2 \), \( F_0 = F_m + B_x z = 2 \), where \( F_0 = P_0 \) and \( F_m = P_m \) are the values of the plasma pressure at points O and M. Here we use the fact that the magnetic field is zero at the central point O because of the symmetry of the problem. As far as the equation of plasma motion along the y-axis (i.e., along the layer) is concerned, the x- and y-components of the magnetic tension force and the pressure gradient force are approximately equal in the Sweet-Parker reconnection case. Indeed, on the y-axis the tension force can be approximated as \( \phi \) \( \phi = \phi = \phi = \phi = \phi = \phi \) because the \( B_x = \phi y \) is produced by the rotation of the \( B_y \) field component in the reconnection layer [5], and in the Sweet-Parker model \( \phi \) is assumed to be the downstream pressure is equal to the upstream pressure \( P_0 \) (see [1, 2]). The pressure and magnetic tension forces accelerate plasma along the reconnection layer (i.e., along the y-axis) up to the downstream speed \( V_{out} \), which can be estimated from the energy conservation equation

\[ \phi(V_{out}^2 B_m^2 = 2) V_{out} = \phi \phi = \phi = \phi = \phi = \phi \]
This equation means that the work done by the pressure and magnetic tension forces along the entire reconnection layer is equal to the kinetic energy of the plasma in the downstream regions.

Finally, in the Sweet-Parker reconnection model, the plasma is assumed to be incompressible. Therefore, the mass conservation condition for the entire reconnection layer results in

$$LV_R \cdot \nabla \rho_{\text{out}};$$

(5)

where $\rho_{\text{out}}$ is the velocity of plasma out along the downstream regions, as given by equation (4). Using equations (2)-(6), we obtain the formula for the Sweet-Parker reconnection velocity $L$, $L^2$:

$$V_R = (V_L L)^2;$$

(6)

Equations (2)-(6) are the classical Sweet-Parker equations. Note that equations (2), (3) and (6) are local, i.e., they are written for a small region of space, which is localized at the geometric center of a thin reconnection layer (point O in Fig. 1) and has a size of order of the layer half-thickness $\delta$. All physical quantities that enter these three equations are defined in this small region of space. At the same time, equations (4) and (5) are global, i.e., they result from consideration of the entire reconnection layer and include the plasma out along velocity $\rho_{\text{out}}$, which is a physical quantity in the downstream regions at the ends of the reconnection layer.

In our new theoretical approach to the calculation of the reconnection rate, we intend to use only local equations. Our intent and the derivations will be justified by the new and important results that we obtain in the next section of this paper and discussed in Sec. VI. At present, let us explain our theoretical approach for the simple case of Sweet-Parker reconnection that we consider in this section.

In the derivation of our theoretical model for magnetic reconnection, we keep equations (2) and (3) unchanged because these equations are local. However, we rewrite equations (4) and (5) because they are global. To rewrite these two global equations in a local form, we consider a point $O = (x = 0; y = 0)$ on the x-axis in a thin layer near the reconnection layer in the centerpoint $O$ (see Fig. 1) and have an approximately small value of its y-coordinate $y$. Since $y > 0$, along the O'C interval we can use the up-to-the $r^2$-order Taylor expansion $B_x (0; y) = y \cdot V_R (0; y) + \frac{1}{2} \cdot (0; y)^2 = \frac{1}{2} \cdot (0; y)^2$ for the values of the perpendicular magnetic field $B_x$, plasma velocity $V_R$ and z-component of the current $\pi$.

These expansions are along the y-axis, and, of course, $(\partial B_x = \partial y)$, $(\partial \rho_{\text{out}} = \partial y)$, are the $r^2$-order partial derivatives of $B_x$ and $V_R$ at point $O$ (note that $(\partial \pi = \partial y)_0 = 0)$. Now, for equation (4) for the plasma acceleration along the y-axis, we can easily be rewritten in a local form at point $O$ as

$$\frac{1}{2} \cdot (y)^2 = \frac{1}{2} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \·
III. MAGNETIC RECONNECTION WITH ANOMALOUS RESISTIVITY

In this section we study reconnection with anomalous resistivity and derive a simple and accurate estimate of the reconnection rate in the classical Sweet-Parker-Petschek two-and-a-half dimensional reconnection layer shown in Fig. 1. We consider resistivity to be a given arbitrary function of the z-componet of the electric current and the two-dimensional coordinates, \( j = (j_x, j_y) \), which has nine derivatives in \( y \) up to the second order and in \( x \) and \( y \) up to the fourth order.

Let us list the assumptions that we make. First, we assume that the characteristic Lundquist number of the problem is large (by our definition) equivalent to the assumption that resistivity is negligible outside the reconnection layer and the non-resistive MHD equations apply there. Second, we assume that the plasma is incompressible, \( \nabla \cdot \mathbf{V} = 0 \). In the limit of very high Lundquist numbers and slow reconnection rates the incompressibility condition is a very good approximation in a reconnection layer even for compressible plasma as [23]. Third, we assume that the reconnection process is quasi-stationary. This can only be the case if the reconnection rate is small \( \dot{\gamma} = \nabla A \nabla \cdot \mathbf{V} = 0 \). If the plasma is incompressible, \( \nabla \cdot \mathbf{V} = 0 \) and see Eq. (2), there are no plasma instabilities in the reconnection layer. This means that in our model the reconnection rate can still be much faster than the Sweet-Parker rate. Fourth, we assume that the reconnection layer is thin, \( \dot{\gamma} = \frac{1}{L_0} \). If plasma a kinematic viscosity is small (in comparison with resistivity), then the plasma a outflow velocity in the downstream regions is equal to the Alfvén velocity \( V_A \), we have plasma mass conservation condition \( \dot{\gamma} = \nabla A \cdot \mathbf{V} = 0 \), and this assumption of a thin reconnection layer is fully equivalent to the previous assumption of a small all reconnection rate. However, in plasma is very viscous, then the plasma a outflow velocity is smaller than \( V_A \) and assumption \( \dot{\gamma} = \frac{1}{L_0} \) is stronger than assumption \( V_R = V_A \). Finally, note that we make no assumption about the values of the guide \( B_0 \) and the plasma a viscosity. However, below we will see that our assumption of a quasi-stationary reconnection process in a thin current sheet layer results in a necessary condition \( j = (j_x, j_y, \frac{\dot{\gamma}}{L_0}) \) for the plasma a kinematic viscosity. Refer to Eq. (27) for more details.

Now note that several equations that we derived in the previous section for the case of the Sweet-Parker reconnection with constant resistivity stay the same in the case of anomalous resistivity. Indeed, equation (1) stays valid when the resistivity is constant. Therefore, equation (2) also stays valid, except that \( \dot{\gamma} \) now is the value of resistivity at the reconnection layer central point \( O \) (see Fig. 1), i.e. \( \dot{\gamma} = (j_x, j_y, \frac{\dot{\gamma}}{L_0}) \). Equations (3) and (8), which result from the Ampère's law and the plasma a incompressibility respectively, obviously stay valid too [39]. At the same time, in the general case of anomalous resistivity that we consider in this section, we need to re-derive equations (7) and (9), which are the equations of the plasma a acceleration and spatial homogeneity of the electric \( z \)-componet along the reconnection layer (i.e. along the \( y \)-axis). However, before we re-derive these two equations, we would first like to derive the equation of magnetic energy conservation. Using Eqs. (2), (3) and (8), we immediately obtain

\[
B^2_0 = \frac{\dot{\gamma}^2}{L_0^2} + 1;
\]

where for the purpose of comparison of our theoretical results to our numerical simulations we introduce a constant \( \dot{\gamma} \), which is of order of unity. Equation (10) is the equation for magnetic energy conservation. The rate of the supply of magnetic energy into the reconnection layer is equal to the rate of Ohm's dissipation inside the layer.

Next we derive the equation of plasma a acceleration along the reconnection layer (i.e. along the \( y \)-axis), taking into consideration all forces acting on the plasma. The MHD equations for the \( y \)-component of the plasma a velocity \( V_y \), assuming the quasi-stationarity of the reconnection \( \dot{\gamma} = 0 \) and plasma a incompressibility \( \nabla \cdot \mathbf{V} = 0 \), is [24]

\[
\{ V \cdot \mathbf{r} \} V_y = \left( \frac{\partial \mathbf{V}_y}{\partial y} + \frac{\partial \mathbf{B}_y}{\partial y} + \frac{\partial \mathbf{B}_z}{\partial y} \right) V_y \; ;
\]

where \( \mathbf{V} \) is the plasma a density, \( \mathbf{B} \) is the plasma a kinematic viscosity (assumed to be constant) and \( \mathbf{P} \) is the sum of the plasma a pressure and the guide \( B_0^2 \) pressure. Taking the \( r \)th order partial derivative \( \frac{\partial \mathbf{V}_y}{\partial y} \) of equation (11) at the central point \( O \), we obtain

\[
\frac{2}{\mathbf{B}_0} = \frac{\partial \mathbf{V}_y}{\partial y} \; \; \; \text{at point} \; O
\]

where we use param eter \( \frac{\partial \mathbf{V}_y}{\partial y} \) and the Ampère's law \( j = (B_y, 0, 0) \). At point \( O \), we also use the formulas \( V_y = 0 \) on the \( y \)-axis, and \( V_y = B_x \) at point \( O \), which follow from the symmetry of the problem with respect to the \( x \)- and \( y \)-axes.
The pressure term \((\beta^2 P=\gamma^2)\), on the right-hand-side of equation (12) can be precisely calculated in analogy with the Sweet-Parker derivation of the pressure decline along the reconnection layer, which employs the force balance condition for the plasma across the reconnection layer and leads to equation (4). The viscosity term \(\beta^2 (\nabla V_{r}=y)^o\) in equation (12) can be calculated approximately by using estimates for the \(V_{r}\) velocity derivatives. In Appendix A we carry out these calculations and show that the pressure and viscosity terms are equal to
\[
(\beta^2 P=\gamma^2)^o = B_m (\beta^2 B_y=\gamma^2)^m + \text{of} \; \frac{1}{2}, \quad j_o = \frac{2}{\beta} g; \quad (13)
\]
\[
\beta^2 (\nabla V_{r}=y)^o = \frac{2}{\beta} j_o; \quad (14)
\]
where \(B_m (\beta^2 B_y=\gamma^2)^m\) is calculated just outside the reconnection layer at point \(M\) (see Fig.1), and expression of \(\gamma^2\); \(j_o = \frac{2}{\beta} g\) denotes terms that are small compared to \(\beta^2 B_y\) or \(\beta^2 y\) in the limit of a slow reconnection rate in a thin reconnection layer, see Ref. [60]. Substituting equations (13) and (14) into equation (12) and using equation (3) for \(\phi\), we obtain
\[
\beta^2 (\beta^2 B_y=\gamma^2)^m + \frac{j_o}{\beta} = \beta^2 B_m; \quad (15)
\]
where \((\beta B_y=\gamma)^0\) at point \(0\).

Note that this equation is exact if plasma viscosity can be neglected \((\beta=0)\).

Now we use the condition of spatial homogeneity of the electric field component along the reconnection layer, i.e., along the \(y\)-axis. We take the second order partial derivatives \(\beta^2=\gamma^2\) of the left- and right-hand-sides of equation (1) at the central point \(O\) (note that the rst order partial derivatives are identical zero). Taking into account the symmetry of the problem, so that \(V_x = B_y = 0\) on the \(y\)-axis, and \(V_y = B_x = \beta j_o = \gamma = 0\) at point \(O\), we obtain
\[
(\phi + \frac{j_o}{\beta} (\beta y)=\beta y)^o) (\beta^2 j_o=\gamma^2)^o = 2 (\beta B_y=\gamma)^o, (\beta^2 B_y=\gamma^2)^o = 2; \quad (16)
\]
where we use formulas \((\beta V_y=\gamma)^o\) and \((\beta B_x=\gamma)^o\). Finally, we need to estimate the \(\beta^2 j_o=\gamma^2\) term, which enters the left-hand-side of equation (16). This estimation can be done by taking the second order partial derivative \(\beta^2=\gamma^2\) of equation (3), while keeping \(\gamma^2\) constant because the partial derivative in \(y\) is to be taken at a constant value \(x = \text{const} = \beta\). In Appendix B we give the detailed derivations and note that the \(y\)-scale of the current \(j_o\) is about the same as the \(x\)-scale of the outside magnetic field, i.e., \(\beta^2 j_o=\gamma^2\). \(B_m (\beta^2 B_y=\gamma^2)^m\). However, for the purpose of comparison of our theoretical results to numerical simulations in Sec. V, we find it convenient to write
\[
(\beta^2 j_o=\gamma^2)^o = B_m (\beta^2 B_y=\gamma^2)^m; \quad 1; \quad (17)
\]
where we introduce the coefficient \(B_m\), which is of order unity.

Let us take the dimensionless coefficients \(B_m\), \(B_m\), which enter equations (10) and (17), and are of order unity, \(B_m\) and \(B_m\). Now we have all the equations necessary to determine all unknown physical parameters. In particular, using Eqs. (10), (15), (16) and (17), we easily obtain the following approximate algebraic equation for the second order derivative \(j_o\) at the reconnection layer central point \(O\):
\[
3 + \frac{j_o}{\beta} (\beta y)=\beta y)^o + \frac{B_m}{o (\beta^2 B_y=\gamma^2)^m} + \frac{2B_m}{(\beta^2 B_y=\gamma^2)^m}; \quad (18)
\]
where \(B_m\) and \(B_m\) is defined as \(B_m = \beta y\), and \(B_m = (\beta = \beta x = 0; y = 0)\) is the resistivity at point \(O\). Given the resistivity function \(\beta = (\beta x; y); \beta m\) as well as the magnetic field \(B_m\) and its second order derivative \(\beta^2 B_y=\gamma^2\), outside the reconnection layer, we can solve equation (18) for the current \(j_o\) and nd the reconnection rate, which is the rate of destruction of magnetic field at point \(O\), equal to \((\beta A=\beta)^o\). Using Eq. (1), we nd that the reconnection rate is equal to \(j_o = \beta^2 B_m\). Note that for the classical reconnection layer that we consider (see Fig. 1) the right-hand-side of equation (18) is positive because \(2B_m = \beta^2 B_y=\gamma^2\), \(B_m = 0\), where \(L\) is the global scale of the magnetic field outside the reconnection layer. Now the current \(j_o\) is calculated by means of equation (18), we can easily calculate all other reconnection parameters, using Eqs. (2), (3), (8) and (15),
\[
V_R o j_o = B_m, \quad (19)
\]
\[
\frac{j_o}{B_m^2} = j_o; \quad (20)
\]
\[
\beta j_o = \frac{(1 + \beta)}{\beta} (\beta j_o = \beta B_m)^2 \quad + \frac{\beta j_o}{\beta} (\beta B_y=\gamma^2)^m; \quad (21)
\]
\[
B_m = j_o, \quad \beta j_o; \quad (22)
\]
Equations (18)-(22) are the most general result of magnetic reconnection that we obtain in this paper. Restoring the coefficients \(1\) and \(1\), equation (18) becomes
\[
(1 + 2) + \frac{j_o}{\beta (\beta y)=\beta y)^o} + \frac{B_m}{o (\beta^2 B_y=\gamma^2)^m} = \frac{2B_m}{(\beta^2 B_y=\gamma^2)^m}; \quad (23)
\]
Hereafter we will consider the natural case when \(\beta = \beta^2 B_m\), \(0\) and \(\beta = \beta^2 B_m\), \(0\) because plasma conductivity decreases as the current increases and we are interested in fast anomalous reconnection (i.e., faster than the Sweet-Parker reconnection). In this case the first, second and third terms on the left-hand-side of equation (18) are all positive. It is easy to see that the first term is related to Sweet-Parker reconnection with constant resistivity equal to \(o\), the second term is related to fast reconnection associated with the dependence of
anomalous resistivity on the current, and the third term
is related to fast reconnection associated with an ad hoc
localization of resistivity in space (see the next section for
details). Also note that if the plasma kinetic viscosity is
larger than the resistivity $\nu$, then, according to
equation (18), the current $j_B$ and reconnection rate $j_{Bz}$
become smaller as $\nu$ grows, i.e., the reconnection slows
down for viscous plasma as one expects.

We postpone the analysis of equations (18)–(22) until
Sec. VI. Let us now make an estimate of the half-length of
the reconnection layer $L^0$ (see Fig. 1). Note that $L^0$
is not needed for the calculation of the reconnection rate
$j_{Bz}$ by means of equation (18). Nevertheless, we are
still interested in such an estimate of $L^0$, in particular,
because we need to check our assumption that the reconnec-
tion layer is thin, i.e., that the condition $L^0$ is satis-
ed. It is clear that $L^0$ cannot be much larger than
the global scale of the magnetic field outside the layer $L$
Therefore we have the condition $L^0 < L$. However, $L^0$
can be much smaller than $L$, in which case the reconnec-
tion layer has a pair of the Petschek switch-off MHD shocks
attached to each end of the layer in the downstream re-
A

where we use the first-order Taylor expansion for an
estimate $B_x (0, y = L^0) = B_x = \theta y$. Now, using Eqs. (19), (21)
and (24), we can easily nd $L^0$. Before we write the
explicit formula for $L^0$, note that the absolute value of the
$B_m = +$ term in equation (21) is equal or smaller than the
$B_m = -$ term. This follows from the fact that the left-hand-side of equa-
tion (18) is equal or greater than unity (see our com-
ments in the paragraph that follows Eq. (23)). Therefore,
the $B_m = -$ term can be omitted in equation (21) for the purpose of estimating $L^0$. As a result, we obtain the following rough estimates for the reconnec-
tion layer half-length $L^0$ and the velocity of plasma a out ow in the downstream regions $V_{out}$:

\[ V_{out} = \frac{L^0}{L^0} \quad \text{and} \quad \text{velocity of plasma a out ow} \]

where we use equations (19), (20), (21) and (24). Note
that the condition $L^0 < L$ is always satis ed because
$B_m = \theta y^2$ and the left- and right-hand-sides of equation (18) are equal or greater than unity. However, the condition that the reconnection layer is thin, $L^0$, is satis ed only if plasma a viscosity is not too large,

\[ \nu = \text{const} \]

where, to derive this formula, we use Eqs. (19), (22)
and (25). In other words, to be able to form a thin
reconnection layer, the plasma a viscosity should not be too vis-
cous. Note that in the case of constant resistivity and
large viscosity, $\nu = \text{const}$ and $\nu$, the reconnec-
tion velocity is $V_R = V_A (\nu = V_A L)^1 \approx \nu = \nu^4$ (see [26])
and condition (27) reduces to $\nu = V_A L \approx V_A L (\nu = V_A L)^{13}$. Finally, if the plasma a viscosity is small in com parison to the resistivity, $\nu$, then from equa-
tion (26) we immediately nd an important and well-
known result that in this case the velocity of the plasma a
out ow in the downstream regions at the ends of the reconnec-
tion layer is approximately equal to the Alfvén velocity $V_A$: (see Fig. 1).

At the end of this section we would like to discuss se-
veral assumptions that we used in our derivations. First,
the solution of equation (18) is valid only if it gives
$V_R B_m$, which is our assumption of a slow quas-
stationary reconnection. Because of equation (19), condi-
tion $V_R B_m$ is equivalent to $V_R B_m$, i.e., the reconnec-
tion velocity, which is the velocity of the inc-
coming plasma, must be small in comparison to the Alfvén velocity in the upstream region. Second, the co-
cient $(B_m = \theta y)_{\nu}$, given by Eq. (21), must be much smaller
than the current $j_B = (B_m = \theta y)_{\nu}$ because the reconnec-
tion layer is assumed to be thin. It is easy to see that this condition is satis ed. Indeed, the
current $j_B$ in the brackets $[\ldots]$ in equation (21) is much
smaller than unity because of the upper limit for plasma a viscosity given by equation (27). The second term in the brackets in equation (21) is also much smaller than unity because $j_B$ is much larger than the electric current outside
the reconnection layer due to our assumption of a large characteristic Lundquist number of the system.

IV. SPECIAL CASES OF MAGNETIC RECONNECTION

In this section we focus on the cases for the reconnec-
tion rate, which arise when one of the three terms on
the left-hand-side of equation (18) dominates over
the other two. We consider the classical Sweet-Parker-
Petschek reconnection layer shown in Fig. 1 and de-
note the global scale of the magnetic field outside the reconnec-
tion layer as $L \approx B_m = (B_m = \theta y^2)_{\nu}$ (see [42]).
In addition, for the purpose of clarity, in this section we
focus only on resistivity efects and neglect plasma a vis-
cosity, assuming that $\nu = \text{const}$.

A. Sweet-Parker reconnection $= \text{const}$

First, consider the case when resistivity is constant,
$(\rho_{\nu}, x; y) = \rho_{\nu} = \text{const}$. In this case only the \text{first}
\text{term on the left-hand-side of equation (18) is nonzero}
and equations (18)-(22) and (25) reduce to

$$3^{1/2} \quad \alpha \frac{\partial^2 L}{\partial V_m B_n^2} \quad \beta \quad m_l = L S_0, \quad S_0 = \frac{V_m}{L} \quad 1,$$

$$V_m \quad \frac{V_m}{L} \quad 1; \quad \frac{V_m}{L} \quad \frac{S_0}{1/2} \quad \frac{L}{L}; \quad \frac{L}{L}; \quad \frac{L}{L}; \quad \frac{L}{L}; \quad \frac{L}{L};$$

where we set the plasma kinematic viscosity to zero ($\nu = 0$), use our definition of the global eddyscale length $L = B_m = B_m$, introduce the Lundquist number $S_0$, and assume for our estimates that $3^{1/2} = 1$. The above equations are the Sweet-Parker reconnection equations with constant resistivity equal to $\omega$. Thus, we see that if resistivity is constant, then the reconnection must be Sweet-Parker and not Petschek [5, 22]. We discuss this important result in Sec. VI. Note that in this section, contrary to Sec. II, we do not assume the Sweet-Parker geometric configuration for the reconnection layer, but derive it together with the reconnection rate from our general equations of the previous section. A typical configuration of the reconnection layer in the case of Sweet-Parker reconnection is shown in the left-bottom plot in Fig. 2. This plot is marked by letters 'S-P'.

**B. Petschek-Kulsrud reconnection, = $\beta$**

Now let us consider the case when resistivity is anomalous and is a monotonically increasing function of the electric current only, = $\alpha$. Let us further assume that this dependence of resistivity on the current is very strong, so that $\beta = \alpha$ ($\alpha = d_1 \beta$). In this case the second term on the left-hand-side of equation (18) is domi-

nant, and equations (18) and (25) reduce to

$$(d = d_1 \beta) \quad \frac{3^{1/2} L^2}{\alpha \beta} \quad \frac{V_m^2}{B_m^2} \quad \frac{1}{1/2} \quad \frac{d}{d_1 \beta}; \quad \frac{L}{L}; \quad \frac{L}{L}; \quad \frac{L}{L}; \quad \frac{L}{L}; \quad \frac{L}{L}; \quad \frac{L}{L}; \quad \frac{L}{L}; \quad \frac{L}{L}; \quad \frac{L}{L};$$

Here again we set $\alpha = 0$, and we use the formula $L_0 = 2 B_m = 2 B_m^2 = 2 B_m^2 \frac{1}{0}$ and Eqs. (19) and (22). From equation (30) we see that the half-length of the reconnection layer $L_0$ is much less than the global eddyscale length $L$ in the case of a strong dependence of resistivity on the current ($\beta = \alpha$ ($\alpha = d_1 \beta$)). This means that in this case the geometry of the reconnection layer is Petschek with a pair of shocks attached to each end of the layer in the downstream regions, as shown in Fig. 1. Equation (29) for the reconnection rate was not analytically derived by Kulsrud [5] for a special case when $d = d_1 \beta$. const. = $\gamma = \beta$. In the Petschek-Kulsrud reconnection case the reconnection rate, given by Eq. (29), can be considerably faster than the Sweet-Parker reconnection rate. This fast reconnection has been previously observed in many numerical simulations done with anomalous resistivity = $\alpha$ (e.g., see [9, 10, 27]). The typical configuration of the reconnection layer in the case of Petschek-Kulsrud reconnection is shown in the left-bottom plot in Fig. 2. This plot is marked by letters 'P-K'.

**C. Spatially localized reconnection, = (x;y)**

Finally let us consider the special case when resistivity is given by = (x;y) and it is spatially localized around the central point 0 of the reconnection layer (see Fig. 1), so that $2 = 2 \frac{1}{0} = 2 \frac{1}{0}$. In other words, we assume that resistivity is anomalous and is localized on scale 1 that is much smaller than the global eddyscale length $L$. In this case the second term on the left-hand-side of equation (18) is dominant, and equations (18)-(22) and (25) reduce to

$$(d = d_1 \beta) \quad \frac{3^{1/2} L^2}{\alpha \beta} \quad \frac{V_m^2}{B_m^2} \quad \frac{1}{1/2} \quad \frac{d}{d_1 \beta}; \quad \frac{L}{L}; \quad \frac{L}{L}; \quad \frac{L}{L}; \quad \frac{L}{L}; \quad \frac{L}{L}; \quad \frac{L}{L}; \quad \frac{L}{L}; \quad \frac{L}{L}; \quad \frac{L}{L}; \quad \frac{L}{L};$$

where we again set $\alpha = 0$. The above equations are the same as the Sweet-Parker equations (28) with the global eddyscale length $L$ replaced by the resistivity scale length $L$. Note that, when resistivity is localized, the reconnection rate becomes faster than the Sweet-Parker rate by a factor $\frac{L}{L}$.
V. NUMERICAL SIMULATIONS OF MAGNETIC RECONNECTION

In this section we present the results of our numerical simulations of unforced reconnection of two cylindrical magnetic flux tubes. These simulations are not intended as a check or a proof of our theoretical results for magnetic reconnection. Our equations (18)-(22) have been derived analytically and are very general. A comprehensive testing of them would require extensive computational work, which is beyond the scope of this paper. Instead, we present our simulations as a demonstration of our reconnection model predictions.

Following Kulcsár [5], we assume that plasma resistivity is given by the following piecewise linear function of the z-component of the electric current:

\[
\left\{ \begin{array}{ll}
\eta = \gamma & \text{if } j_z > 0; \\
\eta = \gamma - \gamma_0 & \text{if } j_z < 0;
\end{array} \right. 
\]

(32)

where \( \gamma \) is the Spitzer resistivity, which is assumed to be very small, \( \gamma_0 \) is the anomalous resistivity parameter and \( j_z \) is the current density parameter. The Kulcsár model's prediction for the reconnection rate in the case \( j_z = 0 \) (determined by equation (29)), has already been checked and confirmed numerically by Breslau and Jaridin [27]. Here we simulate reconnection with anomalous resistivity given by equation (32) for a different computational setup, a higher Lundquist number, and a wider range of parameters \( \gamma \) and \( \gamma_0 \). Our intent is to see how our general formula (18) for the reconnection rate works in this case.

We consider an unforced reconnection of two cylindrical magnetic flux tubes with the initial z-component of the magnetic field vector potential equal to

\[
A_z (x; y) = A_0 \exp \left( -\frac{x^2}{2R_0^2} \right) + \exp \left( -\frac{x^2}{2R_0^2} \right), \quad A_0 = B_0 R_0 \frac{e^{-x^2/y^2}}{y};
\]

(33)

see the top-left plot in Fig. 2. This convenient computational setup was suggested to us by M. Kics and Vainstein [29]. We choose the parameters in equation (33) as \( R_0 = 1 \) (the global scale of the axis is unity), \( d = 2.7162 \) (5% of initially connected flux), and \( B_0 = 1 \) (the maximal initial axis is unity). Thus, \( A_0 = 0 \).

In addition, for convenience we choose the plasma a density \( n_0 = 1 \), so that the typical Alfven velocity and time are unity, \( V_{A_0} = B_0 R_0 = 1 \) and \( t_0 = V_{A_0} = 1 \). The guide field is chosen to be zero, \( B_0 = 0 \), and the initial plasma velocities are zero. The initial gas pressure \( P \) is chosen in such a way that each of the two cylindrical magnetic flux tubes (33) would initially be in complete equilibrium, \( P + B_0^2/2 + B_0^2/2 = \) const., if there were no magnetic forces from the other tube. The plasma kinematic viscosity is chosen to be equal to the Spitzer resistivity, \( \gamma \). The boundary conditions are placed at \( x; y = 25 \), which are virtually at infinity (the magnetic vector potential (33) drops to less than \( 10^{-10} \) at the boundaries). Because of the symmetry of the problem, in the case of a quasi-stationary reconnection considered here, it is enough to run simulations only in the upper-right quadrant of the full computational box. We use the FLASH code for our simulations. This is a compressible adaptive-mesh refinement (AMR) code written and supported at the ASC Center of the University of Chicago. (For a comprehensive description of the FLASH code see [30, 31]).

The MHD module of the code uses central differences to properly resolve all resistive and viscous scales. Comparing numerical results obtained by simulations done with a compressible code to our approximate theoretical formulas derived for incompressible fluids is a case of a very high Lundquist number. This is because in this case the incompressibility condition is a very good approximation in a reconnection layer even for compressible plasma as [23]. Indeed, in our simulations the plasma density varies by no more than 15% inside the reconnection layer. The biggest advantage of the FLASH code for our purposes is that it is an already existing, well tested code with the AMR feature, which allows us to place the boundary conditions at infinity. The size of the smallest element grid cell in our two-dimentional simulations was chosen typically to vary from \( 25 \times 25 \) to \( 25 \times 25 \), which was sufficient to resolve the resistive reconnection layer.

The two cylindrical magnetic flux tubes, initially set up according to equation (33), attract each other (in a similar way as two wires with collinear currents do). As a result, as time goes on, the tubes move toward each other, form a thin reconnection layer along the y-axis and eventually completely merge together by reconnection. This merging process is displayed in the three top plots in Fig. 2, which show the electric potential \( A_z \) in a central region of the full computational box. The two bottom-left plots in Fig. 2 show the electric current \( j_z \) in a central region that includes the reconnection layer. These plots clearly show the reconnection layer formation, which is formed during the reconnection process in the case of the Sweet-Parker and Petschek-Kulsár reconnection (refer to Secs. IV A and IV B). The bottom-right plot in Fig. 2 demonstrates the functional dependence on time of the normalized reconnection rate \( R = A_z B_0/B_0 \) and \( c_d = \omega = \omega_0 B_0/V_{A_0} \) at the reconnection layer central point. (This point is shown in Fig. 1). Next we compare the maximum peak reconnection rate observed in the numerical simulations with the theoretical rate predicted by our reconnection model.

When resistivity is given by equation (32) and \( \gamma = \gamma_0 \), our theoretical formula (18) for the reconnection rate reduces to

\[
3 + \frac{\gamma}{\gamma_0} \left( 1 + \delta \right)^2 \left( \frac{L^2}{V_{A_0} B_0 R_0} \right) \leq \frac{\gamma}{\gamma_0} \left( 1 + \delta \right)^2 \left( \frac{L^2}{V_{A_0} B_0 R_0} \right); \quad \gamma_0 = \gamma_0 \left( 1 + \delta \right)^2 \left( \frac{L^2}{V_{A_0} B_0 R_0} \right);
\]

(34)
where, as explained above, in our numerical simulations we choose\( B_0 = 1, R_0 = 1, V_{A0} = 1\) and by definition the global field scale is \( L = 2B_0 = (\Theta B^2 = \Theta y^2)_{0} \). We also assume that \( j_z > j_x \), which is the case that we consider in our numerical simulations. As we can see from equation (34), the theoretical reconnect rate \( \sigma_{j_z} \) is rather sensitive to \( B_{m} \) and \( L \), which are the strength and scale of the magnetic field at point \( M \) outside the reconnection layer (see Fig. 1). Therefore, in order to compare the theoretical results with the results of our simulations, we need to accurately calculate the \( B_{m} \) and \( L \) observed in the simulations. As a result, the choice of the exact position of point \( M \), at which \( B_{m} \) and \( L \) are calculated, is important. First, in our simulations we choose the point \( M \) to be the point on the x-axis at which the observed resistivity term \( j_z \) is three times smaller than that observed at the central point \( O \), i.e. \( (\frac{1}{j_z})_{m} = (1 - 3) (\frac{1}{j_z})_{o} \). The three plots on the left in Fig. 3 demonstrate our results for this choice. The top plot on the left shows the reconnect rate. The crosses are a log-log plot of the maximal (peak) reconnection rate observed in our simulations as a function of parameter \( \gamma \) for \( xed \ = 0.0002 \) and \( j_z = 17.100 \). The boxes are the theoretical reconnection rate, which is given by equation (34) with the appropriate values of \( B_{m}, L \) and density observed in the simulations. The solid horizontal line (simulations) and the dashed horizontal line (theory) correspond to the \( \gamma = 0 \) case. The inclined dotted line demarcates the \( 1^{\text{st}} \) scaling (refer to Eq. (29)). The crosses and the boxes do not follow the \( 1^{\text{st}} \) scaling for large values of \( \gamma \), simply because in this case the reconnection rate becomes relatively fast and the magnetic field outside of the reconnection layer is not piled up as much as in the case when \( \gamma \) is small and the reconnection rate is relatively slow. As a result, our rate curves attain at large values of \( \gamma \). The central and the bottom plots on the left in Fig. 3 show the observed values of \( \sigma_{j_z} \) and \( \sigma_{L} \), which are directly calculated from the simulation data by using equations (10) and (17). As we can see, and are of order of unity, as one expects. The three plots on the right in Fig. 3 are the same as the three plots on the left except point \( M \) is now chosen as the point on the x-axis at which \( (\frac{1}{j_z})_{m} = (1 - 10) (\frac{1}{j_z})_{o} \). Comparing the plots on the left and the plots on the right, we see that the choice of point \( M \) is indeed important. Note that for the choice \( (\frac{1}{j_z})_{m} = (1 - 3) (\frac{1}{j_z})_{o} \) for the position of point \( M \), the three plots on the left are similar to the three plots on the right in Fig. 3.
M the half-thickness of the reconnection layer \( o \), defined as the abscissa of point M, increases from 0.011 to 0.050 as \( \gamma \) increases from zero to its maximal value shown on the plots in Fig. 3. At the same time, for the choice \(( j_0)_m = (l=10) \), the position of point M ranges from 0.018 to 0.256, which is noticeably larger. Perhaps simulations of forced reconnection with a strict control of position of point M together with control of the outside \( B_x \) and its scale \( L \), could be better suited for comparison to our theoretical model. Such simulations are beyond the scope of this paper. However, see more discussion of forced reconnection in the next section. Finally, Figure 4 shows the same results as Fig. 3, except the former has plots for a smaller value of the critical current, \( j_0 = 4 \).  

We believe that the results presented in Figs. 3 and 4 generally confirm our theoretical model. In particular, in all cases the theoretical reconnection rates and the rates observed in the simulations do not differ by more than 33%. The observed relatively small discrepancy between the theoretical and simulated rates is mainly due to the coe cient not being precisely constant in the simulations, while the variations of coe cient are somewhat less important (see Eq. (23) for the theoretical reconnection rate). This discrepancy can be due to two causes. First, our theoretical model is general, but approximate, and second, the plasma is compressible in the simulations, while it is assumed to be incompressible in the theoretical model.

VI. DISCUSSION AND CONCLUSIONS  

Let us summarize our main results. In this paper we take a new theoretical approach to the calculation of the rate of quasi-stationary magnetic reconnection. Our approach is based on analytical derivations of the reconnection rate from the resistive MHD equations in a small region of space that is localized about the center of a thin reconnection layer and has its size equal to the layer thickness. Our local-equations approach turns out to be feasible and insightful. It allows us to consider magnetic reconnection with an arbitrary anomalous resistivity and to calculate the reconnection rate for this general case (see Eq. (18)). We end the interesting and important result that if plasma is incompressible and reconnection is quasi-stationary, then the reconnection rate is determined by the anomalous resistivity function \(( j_x j_y)\) and the structure of the magnetic field just outside of the reconnection layer (i.e. at point M in Fig. 1). Thus, we end that the global magnetic field and its configuration are not directly relevant for the purpose of calculation of a quasi-stationary reconnection rate, although, of course, the local magnetic field outside the reconnection layer depends on the global field.

One of the major results of this paper is that in the case of constant resistivity, \( \gamma = 0 \), the magnetic reconnection rate is the slow West-Parker reconnection rate and not the fast Petschek reconnection rate [refer to Eqs. (28)]. This result agrees with numerical simulations and at the same time contradicts the result of the original Petschek theoretical model. In the framework of our theoretical approach, based on local equations, the reason for this contradiction can be understood as follows: In the Petschek model our parameter \(( V_y = \theta y)_0 = ( V_y = \theta x)_0 \), which is equal to the first order partial derivatives of the incompressible plasma densities at the reconnection layer center, is basically treated as a free parameter. This is because in the Petschek model can be estimated as the ratio of the plasma out flow velocity (equal to the Alfvén velocity for viscosity-free plasma) and the reconnection layer length \( L = ( V_y = \theta y)_0 \). In this case \( L = \theta y \) is treated as a free parameter by Petschek. In his model \( L \) is taken to be equal to the minimal possible value, that is, the Petschek shocks do not seriously perturb the magnetic field in the upstream region. This is \( L = S_0 \), where \( S_0 = V_A L = \frac{V_A L}{S_0} \) is the Lundquist number \( [9, 5] \). In this case \( S_0 \) is equal to \( V_A L = \frac{V_A L}{S_0} \) and, according to equations (19) and (20), the reconnection velocity in this case is equal to \( V_A L = \frac{V_A L}{S_0} \), which is the Petschek result. On the other hand, in our theoretical model the parameter \( L \) is not treated as a free parameter. In fact, our three physical parameters \(( V_y = \theta y)_0 \), \( ( V_x = \theta x)_0 \) and \( ( j_0 = \theta x)_0 \), are connected to each other and must be calculated from equations (10), (15) and (16). Let us discuss the meaning of these equations. Equation (10) is the equation of magnetic energy conservation. It says that the rate of supply of the magnetic field into the reconnection layer \( B_m \) must be equal to the rate of the resistive dissipation of this energy \( S_0 \). In the Petschek model and, accordingly, the rate of the magnetic energy supply \( B_m \) is basically prescribed by hand (resulting in an ad hoc fast reconnection), while in our model they are self-consistently calculated from the MHD equations. Equation (15) is the equation of plasma acceleration along the reconnection layer. It says that the magnetic tension force \( j_0 \) must be large enough in order to be able to push out all the plasma along the layer, which is supplied into the layer. Finally, equation (16) is the equation of spatial homogeneity of the electric field \( z \)-component along the reconnection layer. This equation sets an upper limit on the product \( D \) in the case of a quasi-stationary reconnection and is directly related to the calculations and arguments given by Kulsrud [5] in the framework of the global equations theoretical approach, see Ref. [43]. As a result, none of parameters \( j_0 \) and \( j_0 \) can be treated as free parameters, and all of them must be self-consistently determined from the MHD equations.

It is very instructive to briefly examine our results from the two distinct points of view in connection with the reconnection problem, which are expressed in numerous papers on computer simulation of magnetic reconnection. These two points of view are: unforced (free) magnetic reconnection and forced magnetic reconnection. In the
case of unforced reconnection, one should solve our main equation (18) for the current \( j_z \) at the reconnection layer center and then solve for the reconnection velocity \( V_R \) by using equation (19). The solution for \( j_z \) and \( V_R \) will depend on \( B_m \), which is the strength of the magnetic field outside the reconnection layer and which enters the right-hand-side of equation (18). On the other hand, in the case of forced magnetic reconnection the reconnection velocity \( V_R \) is prescribed and \( j_z \) is known. In this case the magnetic field outside the reconnection layer \( B_m \) should be treated as an unknown quantity, and equations (18) and (19) should be solved together in order to nd the correct quasi-stationary values of \( j_z \) and \( B_m \). In other words, in the forced reconnection case an initially weak outside magnetic field \( B_m \) gets piled up to higher and higher values until the resulting current \( j_z \) in the reconnection layer becomes large enough to be able to exactly match the prescribed reconnection velocity \( V_R \) and to be able to reconnect all magnetic flux and magnetic energy, which are supplied into the reconnection region in the quasi-stationary reconnection regime.

Finally, a couple of words about plasma viscosity and guide field and their effect on magnetic reconnection. First, according to our equations (18) and (19), in the case when the resistivity is constant, \( \eta = \) const, and the plasma viscosity is much larger than resistivity, \( \eta \), the reconnection velocity becomes \( V_R = V_A \), which is \( \eta = \) const times an order of magnitude smaller than the Sweet-Parker reconnection velocity given by formula (28), see [26]. Thus, we see that the reconnection rate becomes smaller when the plasma viscosity becomes large. However, note that in many astrophysical and laboratory applications plasma is very hot and highly rarefied. Under these conditions the ion gyro-radius becomes much shorter than the ion mean-free-path, and the plasma becomes strongly magnetized. As a result, the plasma viscosity becomes the Bagnold viscosity, which is dominated by magnetized ions [32]. In this case in all our equations above the isotropic viscosity, \( \eta \), which is proportional to the ion mean-free-path, should be replaced by the Bagnold perpendicular viscosity, which is proportional to the ion gyro-radius and is much smaller than the perpendicular viscosity in a strongly magnetized plasma. Second, according to our results, in two-and-a-half dimensional geometry the guide field \( B_z \) has no effect on the quasi-stationary reconnection rate. Indeed, in our derivations the guide field appears only as a magnetic pressure \( B_z^2 = 2 \) term in addition to the plasma pressure. The combined pressure \( P_{com} \) enters equations (11) and (12) of plasma acceleration along the reconnection layer and the value of spatial derivative of \( P \) is given by equation (13), which does not involve \( B_z \). Thus, the guide field gets eliminated and does not enter into our final equation (13) for the reconnection rate. However, if one assumes that the anomalous resistivity depends on \( x \)-and \( y \)-components of the current \( j_x = \partial B_z / \partial y \) and \( j_y = \partial B_z / \partial x \) in addition to its dependence on the total component of the current \( j_z \), then the reconnection rate will depend on the guide field \( B_z \).

In this paper we consider quasi-stationary magnetic reconnection in a thin reconnection layer. We leave a study of tearing modes instability in a reconnection layer and non-quasi-stationary reconnection for a future paper.

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**Appendix A: Derivation of Equations (13) and (14)**

Below, for brevity, we assume that spatial derivatives are to be taken with respect to all indexes that are listed after the comma signs in the subscripts, e.g. \( V_{x,y} \) \( \partial^2 V_x = \partial y^2 \).

We derive equation (13) first. The derivation is somewhat analogous to the Sweet-Parker derivation of the pressure decrease along the reconnection layer, which leads to equation (4). Namely, to nd the pressure decrease and the pressure second derivative along the layer, we integrate the pressure gradient vector along the contour \( \Gamma' \) shown in Fig. 1 and use the force balance condition for the plasma across the reconnection layer. Now we carry out these calculations in a mathematically precise way.

For infinitesimally small values of the \( y \)-coordinate, taking into account the symmetry of the reconnection layer with respect to the \( x \)-and \( y \)-axes and plasma incompressibility, we use the following Taylor expansions in \( y \) for the plasma velocity \( V(x,y) \) and for the magnetic field \( B(x,y) \):

\[
\begin{align*}
V_x &= V_x^{(0)} + (y^2 = 2)V_{x,y}^{(0)} (x) + (y^4 = 24)V_{x,y,y}^{(0)} (x); \\
B_x &= yB_{x,y}^{(0)} (x) + (y^2 = 6)B_{x,y,y}^{(0)} (x); \\
V_y &= yV_{x,y}^{(0)} (x) + (y^2 = 6)V_{x,y,y}^{(0)} (x); \\
B_y &= B_{y,y}^{(0)} (x) + (y^2 = 2)B_{y,y,y}^{(0)} (x); \\
\end{align*}
\]

(A1)
where the variables with the superscripts \(^{(m)}\) are taken at \(y = 0\) and depend only on \(x\). Assuming quasi-stationarity of reconnection (\(\Theta = \Theta t = 0\)) and plasma incompressibility, the MHD equation for the plasma velocity \(v\) is

\[
rP + r (B_x^2 + B_y^2) = 2 = (V r) V + (\partial r) B + r^2 V; \quad (A 2)
\]

where \(P\) is the sum of the plasma pressure and the z-component \(\Theta\) magnetic pressure \(B_z^2 = 2\). Let us calculate line integrals of the left- and right-hand-sides of equation (A 2) along the contour \(O!M!M!O\) shown in Fig. 1. First, the line integral of the left-hand-side is obviously

\[
\begin{align*}
P (0; y) & = (0; 0) + (1=2)B^2_x (0; y) \\
& = (y^2 = 2)F_y (0; 0) + 2; \quad (A 3)
\end{align*}
\]

where \(y\) is the y-coordinate of points \(M\) and \(O\), and we use the formulas \(B_y = 0\) on the y-axis, \(B_x = 0\) at point \(O\), and \(B_x (0; y) = y\) for small \(y\) [see the definition of \(B_x (0; y)\) in Eq. (15)]. Second, using expansion formulas (A 1), we calculate the following line integrals along the contour \(O!M!M!O\), up to the second order in \(y\):

\[
\begin{align*}
Z \{ (V r) V \} & = \frac{y^2}{2} V_x (0) + V_y (0) + V_y (0) \\
& + 2 V_x (0) V_y (0) dx; \quad (A 4)
\end{align*}
\]

\[
\begin{align*}
Z \{ (\partial r) B \} & = \frac{y^2}{2} B^2_y (0) + 2 B_x (0; y) + B_x (0; y) dx; \quad (A 5)
\end{align*}
\]

\[
\begin{align*}
Z \{ r^2 V \} & = \frac{y^2}{2} 2 V_y (0) + V_x (0) + V_y (0) \\
& + 2 V_x (0) V_y (0) dx; \quad (A 6)
\end{align*}
\]

where the variables with the superscripts \(^{(m)}\) are calculated at point \(M\) (see Fig. 1).

Next, we estimate the terms on the right-hand-sides of Eqs. (A 4)–(A 6). Recall the definitions \(L^0\) and \(o\) for the half-length and half-thickness of the reconnection layer (see Fig. 1), and that \(o = L^0 = 1\) (which is our assumption of a thin reconnection layer). The z-derivative inside the reconnection layer is approximately equal to \(\dot{\Theta}_o B_m = o\) [see Eq. (3)] whereas \(B_m = B^y (0)\) is the \(\Theta\) magnetic field at \(y = 0\), while the z-derivative outside the layer is \(B_m = L^0\). The last two terms on the right-hand-side of equation (A 5) can be estimated as \(B_x V_x (0)\).

\[
B_m = L^0, \quad \dot{\Theta}_m = \dot{\Theta}_o, \quad o = L^0 = 1.
\]

Next, from equation (1) with the resistivity term dropped we see that outside the reconnection layer the typical scale of the plasma velocity \(v\) is about the same as the typical scale of the magnetic field \(B\) and, therefore, cannot be smaller than \(L^0\). Thus, the estimations of the three terms on the right-hand-side of equation (A 4) are \(V_x (0) L^0\) and \(V_y (0) L^0\) and \(V_x (0) L^0\).

The estimations of the second derivative of the magnetic pressure at point \(M\) are \(B_y = (\Theta \Theta y) dy = (\Theta^2 \Theta y^2)\) and \(B_x = 2 B^2 = 2\), \(B_x = 2 + B_x = 2\). Note that here we use \(L^0 \) for estimation of \(y\)-derivatives. In fact, using the global scale \(L^0\) of the magnetic field outside the reconnection layer would have been more appropriate for some of the estimations (as shown in Appendix B).

However, using \(L^0\) for the upper estimations of the \(\Theta\) and \(y\)-derivatives is perfectly fine for the purposes in this appendix.

Next, calculating the line integral of the right-hand-side of Eq. (A 2) by using formulas (A 4)–(A 6) and our estimations above, we obtain equation (13). Note that the 1st term on the right-hand-side of equation (13) can also be written in terms of the second derivative of the magnetic pressure at point \(M\): \(B_m = (\Theta \Theta y) dy = (\Theta^2 \Theta y^2)\) and \(B_x = 2 B^2 = 2\), \(B_x = 2 + B_x = 2\). Note that \(\dot{\Theta}_o\) because the reconnection layer is thin. Therefore, as noted by Zweibel [33], equation (13) is similar to Bernoulli's equation.

Now we derive equation (14), which gives an approximate estimate of the viscosity term \(r^2 \dot{\Theta} V_x (0)\). In equation (12), we also estimate this term as follows: Note that \(V_x (0) = 0\) on the y-axis because of the symmetry of the problem relative to the x-axis. Therefore, from the second order Taylor expansion of \(V_x (0; y)\) in \(y\), we obtain an approximate formula for \(V_x (0; 0) = \hat{V}_x (0; 0)\) from \(V_x (0; 0) = \hat{V}_x (0; 0)\) and \(\hat{V}_x (0; 0) = \hat{V}_x (0; 0)\), where we take into account that \(\hat{V}_x (0; 0) = \hat{V}_x (0; 0) = \hat{V}_x (0; 0) = \hat{V}_x (0; 0)\). We can rewrite this approximate formula for \(V_x (0; 0)\) as the following exact formula: \(V_x (0; 0) = \hat{V}_x (0; 0)\), where \(C\) is an unknown coefficient of order unity. Our numerical simulations of reconnection with constant resistivity show that \(C\) is indeed about unity if \(o\) is estimated by equation (3). Thus we immediately find that \(r \dot{\Theta} V_x (0) = \hat{V}_x (0; 0)\), which is equation (14). Here we also use \(V_x (0; 0)\) instead of \(V_x (0)\).

### Appendix B: Derivation of Equation (17)

Here as in Appendix A, we assume that spatial derivatives are treated with respect to all indexes that are listed after the comma signs in the subscripts, e.g., \(B_{x y} = \Theta B_{x y}\). We derive equation (17) in Appendix B.
two steps.

First, we estimate \( B_{xy} \) at points O and M (see Fig. 1), since we will need these estimates below. Consider the formula \( B_{xy} = B_{yy} \), which represents the fact that the magnetic field is divergence-free. Take the \( \theta = \theta y \) and \( \theta z = \theta y^2 \) derivatives of this formula and integrate the resulting equations along the interval OM shown in Fig. 1. We obtain

\[
\begin{align*}
B_{xy}(m) &= \frac{Z_M}{Z_M} B_{xyx} \, dx = \frac{Z_M}{Z_M} B_{xyy} \, dx = \frac{Z_M}{Z_M} B_{yxy} ; \\
B_{xyy}(m) &= B_{xyy} \, dx \quad \text{as shown in Fig. 1}. 
\end{align*}
\]

where the variables with the superscripts \((m)\) and \((o)\) are taken at points O and M respectively and \( = B_{yy}(o) \) see Eq. (15). Note that \( \theta z \) is the half-thickness of the reconnection layer, equal to the abscissa of point M (see Fig. 1).

In making the estimates of the integrals on the right-hand-sides of Eqs. (B1) and (B2), we take into account that \( B_{yy}(o) = B_{xyy}(o) = 0 \) because \( B_{yy} \) on the y-axis. Now, using Eq. (B1), we estimate that \( B_{xyy}(m) \). Let \( o \) be the global scale of the magnetic field outside the reconnection layer. Then, since point \( M \) is located outside the reconnection layer, we have \( B_{xyy}(m) = L^2 \) and \( B_{xxy}(m) = L^2 \). Next, using these estimates, the above estimate for \( B_{xy} \) and Eq. (B2), we obtain the following formula:

\[
\begin{align*}
B_{xyy}(m) &= L^2 \quad \text{(o=L^2)} B_{xxy}(m) ; \\
B_{xxy}(m) &= B_{xyy} \, dx. 
\end{align*}
\]

Second, we estimate \( j_{xy} \) at point O. Consider Ampere's law formula \( \oint j_{xy} = B_{xy} \cdot B_{xy} \). We take the \( \theta z = \theta y^2 \) derivative of this equation and integrate the result along the interval OM shown in Fig. 1. We need that

\[
\begin{align*}
j_{xy} \, dx &= \frac{Z_M}{Z_M} B_{xy} \, dx = \frac{Z_M}{Z_M} B_{xy} \, dx = \frac{Z_M}{Z_M} B_{xy} \, dx. 
\end{align*}
\]

The integral \( \int_{O}^{M} B_{xyy} \, dx \) can be estimated by using formula (B3). The integral of \( j_{xy} \) can be estimated as \( \int_{O}^{M} j_{xy} \, dx \). As a result, we obtain

\[
\int_{O}^{M} j_{xy} \, dx = L^2 \int_{O}^{M} B_{xyy} \, dx. (B5)
\]

where we use \( L \) and \( \theta z = (B_{yy} = L^2) \) \( B_{yy} \) \( (B_{yy} = L^2) \) \( B_{yy} \) \( \theta z = \theta y^2 \) \( B_{yy} = \theta y^2 \) \( B_{yy} \) \( L \). In the second case the reconnection layer half-length is much smaller than the global scale, \( L \), and the reconnection is fast (relative to the Sweet-Parker reconnection). In this case, the z-current \( j_z \) is a smooth function on the global scale \( L \) along the lines that lie inside the reconnection layer and extend into the shock separatrices. Thus, in this case, despite \( L \), the y-scale of the z-current at the reconnection layer central point O is still \( L \). This graphical interpretation is well demonstrated by the bottom-left plot of the current for the Petschek-Kulsrud (F-K) reconnection case in Fig. 2.

[1] P. A. Sweet, in Electron magnetic Phenomena in Ionized Gases, edited by B. Lehnert (Cambridge University Press, New York, 1958), p. 123.
[2] E. N. Parker, Astrophys. J. Suppl. Ser. 8, 177 (1963).
[3] H. E. Petschek, in A.A.S.-NASA Symposium on Solar Flares NASA SP 50 National Aeronautics and Space Administration, Washington, DC, 1964), p. 425.
[4] D. R. Low, Phys. Fluids 26, 1520 (1986).
[5] R. M. Kulsrud, Earth, Planets and Space 53, 417 (2001); astro-ph/0007075.
[6] D. A. Uzdensky and R. M. Kulsrud, Phys. Plasm. A 7, 4018 (2000).
[7] M. Udagawa and T. Tsuda, J. Plasm. Phys. 17, 537 (1977).
[8] T. Tsuda and M. Udagawa, J. Plasm. Phys. 18, 451 (1977).
[9] T. Hayashi and T. Sato, J. Geophys. Res. 83, 217 (1978).
[10] T. Sato and T. Hayashi, Phys. Fluids 22, 1189 (1979).
[11] M. Scholer, J. Geophys. Res. 94, 8805 (1989).
[12] A. Lazarro and E. T. Vlahos, A Astrophys. J. 517, 700 (1999).
[13] E. Kim and P. H. Diamond, Astrophys. J. 556, 1052 (2001).
[14] D. Biskamp, E. Schwarz, and J. F. Drake, Phys. Rev. 4018 (2000).
Not that if we adopt the global derivations approach, then our additional unknown paramenter \( B_z = \theta y \), can be estimated as \( B_z = B_a = \text{L} \), the paramenter \( V_{out} = \text{L} \), and our equation (9) reduces to the Sweet-Parker equation (2), as one expects.

We assume \( \theta \) to be a function of \( j \) instead of the total current \( j = x y z \). This is because the connection process proceeds due to the z-component of the electric field, see Eq. (1), and it is reasonable to assum e that the electrical conductivity in the z-direction can be reduced by plasma instabilities due to large values of \( j \).

Note that Eqs. (3) and (8) are exact for the Hakim model of reconnection sheet [35], which has \( B_y = B_a \) tahn \( (x = \omega) \), \( B_x = 0, B_y = B_a \) cosh \( (x = \omega) \), \( B_z = \text{const} \) and \( V_x = (x = \omega) \) tahn \( (x = \omega) \).

Note that the of \( \hat{z} j \); \( \hat{y} g \) term scan still bemuch larger than the \( B_y = B_a \) tahn \( (x = \omega) \) term in equation (13), in which case the pressure term \( \theta y y \) is unimportant and negligible in equation (12). This happens when reconnection with anomalous resistivity is much faster than Sweet-Parker reconnection.

The jump condition (24) was used by Kulmud [5] for the case of a viscosity-free plasma. It can be shown from the full non-ideal MHD equations that this condition is unchanged in the case of a viscous plasma.

Following Kulmud [5], for the definition of the global magnetic field \( \text{L} \) we use formula \( B_y = B_a \) (1) \( y^2 = L^2 \) for the electric y-component along the interval MM that is outside the reconne cction layer as shown in Fig. 1.

The spatial homogeneity of the electric field-z component and the resulting upper limit on are directly related to the explanation of why the Petschek reconnection model does not work, as shown by Kulmud [5] in the framework of the global equations. Kulmud's argument is that the length of the Petschek reconnection layer \( L \) is not a free parameter, but must be determined by the condition that the perpendicular magnetic 
\( \text{L} \) component \( B_z \) has to be generated by the rotation of the parallel electric component \( B_y \) at the same rate as it is being swept away by the downstream current. It is easy to see that the integration of \( \text{L} \) without the resistivity term of the are a of the contour \( O' \) \( N' \) \( M' \) \( O' \) shown in Fig. 1 will result in the same \( \text{L} \) equation for the balance of the \( B_z \) field, which was used by Kulmud in his work [5].
