A general variational formulation for relativistic mechanics based on fundamentals of differential geometry

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Abstract

The first part of this article develops a variational formulation for relativistic mechanics. The results are established through standard tools of variational analysis and differential geometry. The novelty here is that the main motion manifold has a $n + 1$ dimensional range. It is worth emphasizing in a first approximation we have neglected the self-interaction energy part. In its second part, this article develops some formalism concerning the causal structure in a general space-time manifold. Finally, the last article section presents a result concerning the existence of a generalized solution for the world sheet manifold variational formulation.

1 Introduction

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded, connected set with a smooth boundary (at least $C^1$ class) denoted by $\partial \Omega$ and let $[0, T]$ be a time interval. Consider a relativistic motion given by a position field

$$(r \circ \hat{u}) : \Omega \times [0, T] \rightarrow \mathbb{R}^{n+1}.$$ 

Here, for an open, bounded and connected set $D$ with a smooth boundary, we consider a world sheet smooth ($C^3$ class) manifold $r : D \subset \mathbb{R}^{m+1} \rightarrow \mathbb{R}^{n+1}$, where point-wise

$$r(\hat{u}) = (ct, X_1(u), \ldots, X_n(u))$$

and where

$$r(\hat{u}(x, t)) = (u_0(x, t), X_1(u(x, t)), \ldots, X_n(u(x, t))),$$

$$\hat{u}(x, t) = (u_0(x, t), u_1(x, t), \ldots, u_m(x, t)),$$

$1 \leq m < n$ and

$$u_0(x, t) = ct.$$
Consider also a density scalar field given by
\[ m|\phi(u)|^2 : \Omega \times [0, T] \to \mathbb{R}^+, \]
where \( m \) is the total system mass and
\[ \phi : D \subset \mathbb{R}^{m+1} \to \mathbb{C} \]
is a wave function.

At this point we highlight that
\[
\frac{dr(u(x, t))}{dt} \cdot \frac{dr(u(x, t))}{dt} = -c^2 + \sum_{j=1}^{n} \left( \frac{dX_j(u(x, t))}{dt} \right)^2 = -c^2 + v^2,
\]
where \( c \) denotes the speed of light at vacuum and
\[
v = \sqrt{\sum_{j=1}^{n} \left( \frac{dX_j(u(x, t))}{dt} \right)^2}.
\]

We also emphasize that generically, for \( a = (\hat{x}_0, x_1, x_2, x_3) \in \mathbb{R}^4 \) and \( b = (\hat{y}_0, y_1, y_2, y_3) \in \mathbb{R}^4 \) we have
\[
a \cdot b = -\hat{x}_0 \hat{y}_0 + \sum_{j=1}^{3} x_j y_j.
\]
Moreover \( x_0 = t, \ x = (x_1, x_2, x_3) \) and
\[
dx = dx_1 dx_2 dx_3.
\]
Finally, we generically refer to
\[(r \circ \hat{u}) : \Omega \times [0, T] \to \mathbb{R}^{n+1}\]
as a space-time manifold. Furthermore, with such a notation in mind we denote
\[
ds^2 = \frac{dr(u(x, t)) \cdot dr(u(x, t))}{-c^2 dt^2 + ([dX_1(u(x, t))|^2 + [dX_2(u(x, t))|^2 + \ldots + [dX_n(u(x, t))]|^2)}.
\]

**Remark 1.1.** About the references, the mathematical background necessary may be found in [3, 7]. For the part on relativistic physics, we follow at some extent, the references [3, 7].

## 2 The system energy

Consider first the mass differential, given by,
\[
dm = \frac{m|\phi(u(x, t))|^2}{\sqrt{1 - v^2}} \sqrt{-g} \sqrt{U} \ dx,
\]

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so that the kinetics energy differential is defined by

\[
\begin{align*}
    dE_c &= \frac{dr}{dt} \cdot \frac{dr}{dt} \, dm \\
    &= -\frac{c^2 - v^2}{\sqrt{1 - \frac{v^2}{c^2}}} m |\phi|^2 \sqrt{g} \sqrt{U} \, dx \\
    &= -mc \sqrt{c^2 - v^2} |\phi|^2 \sqrt{g} \sqrt{U} \, dx \\
    &= -mc \sqrt{-\frac{dr}{dt} \cdot \frac{dr}{dt}} |\phi|^2 \sqrt{-g} \sqrt{U} \, dx \\
    &= -mc |\phi|^2 \sqrt{-g_{jk} \frac{\partial u_j}{\partial t} \frac{\partial u_k}{\partial t}} \sqrt{-g} \sqrt{U} \, dx.
\end{align*}
\]

Where

\[
\begin{align*}
g_j &= \frac{\partial r(u)}{\partial u_j}, \forall j \in \{0, \ldots, m\}, \\
g_{jk} &= g_j \cdot g_k, \forall j, k \in \{0, \ldots, m\}, \\
\{g^{jk}\} &= \{g_{jk}\}^{-1}, \\
g &= \det\{g_{ij}\}
\end{align*}
\]

and

\[
U_{ij} = \frac{\partial u(x, t)}{\partial x_i} \cdot \frac{\partial u(x, t)}{\partial x_j}, \forall i, j \in \{0, 1, 2, 3\}.
\]

Moreover, we define

\[
U = |\det\{U_{ij}\}|.
\]

At this point, we assume there exists a smooth normal field \( \mathbf{n} \) such that

\[
\text{Span} \left\{ \left\{ \frac{\partial r(u)}{\partial u_j}, \forall j \in \{0, \ldots, m\} \right\}, \mathbf{n}(u) \right\} \subset \mathbb{R}^{n+1}, \forall \mathbf{u} \in D
\]

and

\[
\frac{\partial^2 r(u)}{\partial u_j \partial u_k} = \Gamma^l_{jk}(u) \frac{\partial r(u)}{\partial u_l} + b_{jk}(u) \mathbf{n}(u), \forall \mathbf{u} \in D,
\]

where \( \{\Gamma^l_{jk}\} \) are the Christoffel symbols and the concerning normal field \( \mathbf{n}(u) \) is also such that

\[
\mathbf{n}(u) \cdot \mathbf{n}(u) = 1, \forall \mathbf{u} \in D,
\]

\[
\frac{\partial r(u)}{\partial u_l} \cdot \mathbf{n}(u) = 0, \text{ in } D, \forall l \in \{0, \ldots, m\}
\]

and

\[
b_{jk}(u) = \frac{\partial^2 r(u)}{\partial u_j \partial u_k} \cdot \mathbf{n}(u), \forall \mathbf{u} \in D, \forall j, k \in \{0, \ldots, m\}.
\]
Suppose also the concerning world sheet position field is such that there exist smooth normal fields
\[ \hat{n}_1, \ldots, \hat{n}_s \]
where \( m + 1 + s \geq n + 1 \) such that
\[
\text{Span} \left\{ \frac{\partial \mathbf{r}(u)}{\partial u_j}, \forall j \in \{0, \ldots, m\} \right\} \cdot \hat{n}_1(u), \ldots, \hat{n}_s(u) = \mathbb{R}^{n+1}, \forall u \in D
\]
so that
\[
\mathbf{n}(u) = f_q(u) \hat{n}_q(u), \forall u \in D
\]
for an appropriate field \( \{f_q\}_{q=1}^s \).

Moreover, we assume
\[
\hat{n}_j(u) \cdot \hat{n}_k(u) = \delta_{jk}, \forall u \in D, j, k \in \{1, \ldots, s\}
\]
and
\[
\frac{\partial \mathbf{r}(u)}{\partial u_j} \cdot \hat{n}_k(u) = 0, \forall u \in D, \forall j \in \{0, \ldots, m\}, k \in \{1, \ldots, s\}.
\]
Here we recall that
\[
\mathbf{n}(u) \cdot \frac{\partial \mathbf{r}(u)}{\partial u_k} = 0, \text{ in } D.
\]
Hence,
\[
\frac{\partial \mathbf{n}(u)}{\partial u_j} \cdot \frac{\partial \mathbf{r}(u)}{\partial u_k} + \mathbf{n}(u) \cdot \frac{\partial^2 \mathbf{r}(u)}{\partial u_j \partial u_k} = 0,
\]
that is,
\[
\frac{\partial \mathbf{n}(u)}{\partial u_j} \cdot \frac{\partial \mathbf{r}(u)}{\partial u_k} = -b_{jk}.
\]
(3)
We may also denote
\[
\frac{\partial \mathbf{n}(u)}{\partial u_j} = c_j^s \frac{\partial \mathbf{r}(u)}{\partial u_s} + c_j^q \hat{n}_q,
\]
for an appropriate \( \{c_j^q\} \) and where
\[
c_j^q = \frac{\partial \mathbf{n}(u)}{\partial u_j} \cdot \hat{n}_q.
\]
From this and (3), we obtain
\[
\frac{\partial \mathbf{r}(u)}{\partial u_s} \cdot \mathbf{g}_k = c_j^s g_{sk} = -b_{jk},
\]
so that
\[
c_j^s g_{sk} g^{kl} = -b_{jk} g^{kl} = -b_j^l,
\]
that is,
\[
c_j^l = c_j^s \delta_s^l = -\delta_j^l,
\]
where
\[
b_j^l = b_{jk} g^{kl}.
Summarizing, we have got
\[ \frac{\partial n(u)}{\partial u_j} = -b^l_j \frac{\partial r(u)}{\partial u_l} + e^q_j n_q. \]

Observe now that
\[
\frac{\partial^3 r(u)}{\partial u_i \partial u_j \partial u_k} = \frac{\partial}{\partial u_i} \left( \Gamma^l_{jk} \frac{\partial r(u)}{\partial u_l} + b_{jk} n \right) \\
= \left( \frac{\partial \Gamma^l_{jk}}{\partial u_i} + \Gamma^p_{jk} \Gamma^l_{pi} \right) \frac{\partial r(u)}{\partial u_l} \\
+ \Gamma^p_{jk} b_{pi} n + \frac{\partial b_{jk}}{\partial u_i} n - b_{jk} \frac{\partial r(u)}{\partial u_l} \\
+ b_{jk} e^l_j n_l. \tag{4}
\]

Similarly
\[
\frac{\partial^3 r(u)}{\partial u_j \partial u_i \partial u_k} = \frac{\partial}{\partial u_j} \left( \Gamma^l_{ik} \frac{\partial r(u)}{\partial u_l} + b_{ik} n \right) \\
= \left( \frac{\partial \Gamma^l_{ik}}{\partial u_j} + \Gamma^p_{ik} \Gamma^l_{pj} \right) \frac{\partial r(u)}{\partial u_l} \\
+ \Gamma^p_{ik} b_{pj} n + \frac{\partial b_{ik}}{\partial u_j} n - b_{ik} \frac{\partial r(u)}{\partial u_l} \\
+ b_{ik} e^l_j n_l. \tag{5}
\]

Thus, for such a smooth \((C^3\) class\) manifold, from
\[
\frac{\partial^3 r(u)}{\partial u_i \partial u_j \partial u_k} = \frac{\partial^3 r(u)}{\partial u_j \partial u_i \partial u_k},
\]
assuming a concerning linear independence and equating the terms in
\[ \frac{\partial r(u)}{\partial u_l}, \]
we get
\[ W^l_{ijk} = b_{jk} b^l_i \\
= \frac{\partial \Gamma^l_{jk}}{\partial u_i} \frac{\partial \Gamma^l_{ik}}{\partial u_j} + \Gamma^p_{jk} \Gamma^l_{pi} - \Gamma^p_{ik} \Gamma^l_{pj} + b_{ik} b^l_j. \tag{6}\]

Defining the Riemann curvature tensor by
\[
R^l_{ijk} = \frac{\partial \Gamma^l_{jk}}{\partial u_i} - \frac{\partial \Gamma^l_{ik}}{\partial u_j} + \Gamma^p_{jk} \Gamma^l_{pi} - \Gamma^p_{ik} \Gamma^l_{pj}, \tag{7}
\]
Thus, for such a smooth \((C^3\) class\) manifold, from
\[ \frac{\partial^3 r(u)}{\partial u_i \partial u_j \partial u_k} = \frac{\partial^3 r(u)}{\partial u_j \partial u_i \partial u_k},
\]
we also define the energy part \( J_1(\phi, r, u, n) \) as
\[
J_1(\phi, r, u, n) = \frac{1}{2} \int_0^T \int_{\Omega} |\phi|^2 g^{jk} b_{jl} b_{lk} \sqrt{-g} \sqrt{U} \, dx \, dt \\
= \frac{1}{2} \int_0^T \int_{\Omega} |\phi|^2 g^{jk} R_{jk} \sqrt{-g} \sqrt{U} \, dx \, dt \\
+ \frac{1}{2} \int_0^T \int_{\Omega} |\phi|^2 g^{jk} b_{jl} b_{lk} \sqrt{-g} \sqrt{U} \, dx \, dt.
\]
(8)

The next energy part is defined through the tensor \( S_{ijk} \) which, considering the Levi-Civita connection \( \nabla \) and the standard Lie Bracket \([\cdot, \cdot]\) (see [2, 3] for more details), is such that
\[
\nabla \left[ \phi \partial_{\phi} \partial_{u_j} \partial_{u_k} \right] \left( \phi^* \frac{\partial r(u)}{\partial u_l} \right) = S_{ijkl} \frac{\partial r(u)}{\partial u_l} + \hat{b}_{ijk} n.
\]
Observe that
\[
\nabla \left( \frac{\partial \phi \partial_{\phi} \partial_{u_j} \partial_{u_k}}{\partial_{\phi} \partial_{u_l} \partial_{u_i}} \right) \left( \phi^* \frac{\partial r(u)}{\partial u_l} \right) \\
= \frac{\partial \phi \partial_{\phi} \partial_{u_j} \partial_{u_i} \partial_{u_k}}{\partial_{\phi} \partial_{u_j} \partial_{u_i} \partial_{u_k}} + \frac{\partial \phi \partial_{\phi} \partial_{u_j} \partial_{u_i} \partial_{u_l} \partial_{u_k}}{\partial_{\phi} \partial_{u_j} \partial_{u_i} \partial_{u_l} \partial_{u_k}} \delta_{lk} \\
+ \frac{\partial \phi \partial_{\phi} \partial_{u_j} \partial_{u_i} \partial_{u_l} \partial_{u_k}}{\partial_{\phi} \partial_{u_j} \partial_{u_i} \partial_{u_l} \partial_{u_k}} \Theta_{lkl} + \frac{\partial \phi \partial_{\phi} \partial_{u_j} \partial_{u_i} \partial_{u_l} \partial_{u_k}}{\partial_{\phi} \partial_{u_j} \partial_{u_i} \partial_{u_l} \partial_{u_k}} + \hat{b}_{ijk} n
\]
(9)

Thus,
\[
S_{ijkl} = \frac{\partial \phi \partial_{\phi} \partial_{u_j} \partial_{u_i} \partial_{u_l} \partial_{u_k}}{\partial_{\phi} \partial_{u_j} \partial_{u_i} \partial_{u_l} \partial_{u_k}} \delta_{lk} + \frac{\partial \phi \partial_{\phi} \partial_{u_j} \partial_{u_i} \partial_{u_l} \partial_{u_k}}{\partial_{\phi} \partial_{u_j} \partial_{u_i} \partial_{u_l} \partial_{u_k}} \Theta_{lkl} + \frac{\partial \phi \partial_{\phi} \partial_{u_j} \partial_{u_i} \partial_{u_l} \partial_{u_k}}{\partial_{\phi} \partial_{u_j} \partial_{u_i} \partial_{u_l} \partial_{u_k}} + \hat{b}_{ijk} n.
\]

With such results in mind, we define this energy part as
\[
J_2(\phi, r, u, n) = \frac{1}{2} \int_0^T \int_{\Omega} g^{jk} \text{Re}[S_{ijkl}] \sqrt{-g} \sqrt{U} \, dx \, dt,
\]
where generically \( \text{Re}[z] \) and \( z^* \) denote the real part and complex conjugation, respectively, of \( z \in \mathbb{C} \).

### 3 The final energy expression

The expression for the energy, already including the Lagrange multiplier concerning the mass restriction, is given by
\[
J(\phi, r, u, n, E) = - \int_0^T \int_{\Omega} dE_c \, dt + J_1(\phi, r, u, n) + J_2(\phi, r, u, n) \\
- \int_0^T E(t) \left( \int_{\Omega} |\phi|^2 \sqrt{-g} \sqrt{U} \, dx - 1 \right) \, dt,
\]
(10)
so that

\[
J(\phi, r, u, E) = \int_0^T \int_\Omega mc|\phi|^2 \sqrt{-g_{jk}} \frac{\partial u_j}{\partial t} \frac{\partial u_k}{\partial t} \sqrt{-gU} \, dx \, dt \\
+ \frac{1}{2} \int_0^T \int_\Omega |\phi|^2 g^{jk} b_{bj} b_{lk} \sqrt{-gU} \, dx \, dt \\
+ \frac{1}{2} \int_0^T \int_\Omega g^{jk} \frac{\partial \phi}{\partial u_j} \frac{\partial \phi^*}{\partial u_k} \sqrt{-gU} \, dx \, dt \\
+ \frac{1}{4} \int_0^T \int_\Omega \left( \frac{\partial \phi}{\partial u_l} \phi^* + \frac{\partial \phi^*}{\partial u_l} \phi \right) \Gamma^l_{jk} g^{lk} \sqrt{-gU} \, dx \, dt \\
- \int_0^T E(t) \left( \int_\Omega |\phi|^2 \sqrt{-gU} \, dx - 1 \right) \, dt
\]

We shall look for critical points subject to

\[
n(u(x, t)) \cdot n(u(x, t)) = 1, \quad \text{in } \Omega \times [0, T]
\]

and

\[
\frac{\partial r(u(x, t))}{\partial u_j} \cdot n(u(x, t)) = 0, \quad \text{in } \Omega \times [0, T], \quad \forall j \in \{0, \ldots, m\}.
\]

Already including the concerning Lagrange multipliers, the final functional expression would be

\[
J(\phi, r, u, n, E, \lambda) = \int_0^T \int_\Omega mc|\phi|^2 \sqrt{-g_{jk}} \frac{\partial u_j}{\partial t} \frac{\partial u_k}{\partial t} \sqrt{-gU} \, dx \, dt \\
+ \frac{1}{2} \int_0^T \int_\Omega |\phi|^2 g^{jk} b_{bj} b_{lk} \sqrt{-gU} \, dx \, dt \\
+ \frac{1}{2} \int_0^T \int_\Omega g^{jk} \frac{\partial \phi}{\partial u_j} \frac{\partial \phi^*}{\partial u_k} \sqrt{-gU} \, dx \, dt \\
+ \frac{1}{4} \int_0^T \int_\Omega \left( \frac{\partial \phi}{\partial u_l} \phi^* + \frac{\partial \phi^*}{\partial u_l} \phi \right) \Gamma^l_{jk} g^{lk} \sqrt{-gU} \, dx \, dt \\
- \int_0^T E(t) \left( \int_\Omega |\phi|^2 \sqrt{-gU} \, dx - 1 \right) \, dt \\
+ m \sum_{j=0}^m \int_0^T \int_\Omega \lambda_j(x, t) \frac{\partial r(u(x, t))}{\partial u_j} \cdot n(u(x, t)) \sqrt{-gU} \, dx \, dt \\
+ \int_0^T \int_\Omega \lambda_{m+1}(x, t) (n(u(x, t)) \cdot n(u(x, t)) - 1) \sqrt{-gU} \, dx \, dt
\]

Remark 3.1. We must consider such a functional defined on a space of sufficiently smooth functions with appropriate boundary and initial conditions prescribed.

Finally, the main difference concerning standard differential geometry in \(\mathbb{R}^3\) is that, since

\[
1 \leq m < n,
\]

we have to obtain through the variation of \(J\), the optimal normal field \(n\). Summarizing, at first we do not have an explicit expression for such a field.
4 Causal structure

In this section we develop some formalism concerning the causal structure in a space-time manifold defined by a function

$$(r \circ \hat{u}) : \Omega \times (-\infty, +\infty) \to \mathbb{R}^{n+1},$$

where $t \in (-\infty, +\infty)$ denotes time.

We follow at some extent, the content in the Wald’s book [4], where more details may be found.

**Definition 4.1.** Let $M$ be a space-time manifold time oriented, in the sense that the light cone related to the tangent spaces varies smoothly along $M$. A $C^1$ class curve $\lambda : [a, b] \to M$ is said to be time-like future directed if for each $p \in \lambda$ the respective tangent vector is time-like future directed, that is,

$$\frac{d\lambda(s)}{ds} \cdot \frac{d\lambda(s)}{ds} < 0, \forall s \in [a, b], \text{ (time-like condition)}$$

and

$$\frac{dt(s)}{ds} > 0, \forall s \in [a, b], \text{ (future directed condition)}.$$

Here

$$\lambda(s) = r(\hat{u}(\mathbf{x}(s), t(s)))$$

for appropriate smooth functions

$$\mathbf{x}(s), t(s).$$

Similarly, we say that such a curve is causal future directed, if the tangent vector is a time-like future directed or is a null vector, $\forall s \in [a, b]$.

Finally, in an analogous fashion we may define a continuous and piece-wise $C^1$ class time-like future directed curve.

**Remark 4.2.** At this point we highlight that in the next lines the norm $\| \cdot \|$ refers to the standard Euclidean one in $\mathbb{R}^{n+1}$.

**Definition 4.3.** The chronological future of $p \in M$, denoted by $I^+(p)$, is defined as

$$I^+(p) = \{ q \in M : \text{there exists a continuous and piece-wise } C^1 \text{ class time-like future directed curve } \lambda : [a, b] \to M \text{ such that } \lambda(a) = p \text{ and } \lambda(b) = q \}.$$ (13)

Observe that, if $M$ is smooth (as previously indicated, the world sheet manifold in question is at least $C^3$ class) by continuity, if $q \in I^+(p)$ there exists a neighborhood $O(q)$ such that

$$O(q) \cap M \subset I^+(p).$$

From now and on we always assume any space-time mentioned is always smooth and time-oriented.

Also, for $S \subset M$, we define

$$I^+(S) = \cup_{p \in S} I^+(p),$$

so that since $I^+(p)$ is open for each $p \in M$, we may infer that $I^+(S)$ is open.
Remark 4.4. Similarly, we define the chronological pasts $I^{-}(p)$ and $I^{-}(S)$.
Moreover the causal future of $p \in M$, denoted by $J^{+}(p)$ is defined as
\[
J^{+}(p) = \{ q \in M : \\
\text{there exists a continuous and piece-wise } C^1 \text{ class} \\
\text{casual future directed curve } \lambda : [a, b] \to M \\
such that } \lambda(a) = p \text{ and } \lambda(b) = q \}.
\] (14)

Also, we define $J^{+}(S) = \cup_{p \in S} J^{+}(p)$, and similarly define the causal pasts $J^{-}(p)$ and $J^{-}(S)$.

Definition 4.5. Let $M$ be a space time manifold. We say that $M$ is normal if for each connected set $S \subset M$, there exists $r > 0$ such that if $p, q \in I^{+}(S)$ and $0 < \| p - q \| < r$, then, interchanging the roles of $p$ and $q$ if necessary, there exists a smooth time-like future directed curve $\lambda : [a, b] \to I^{+}(S)$ such that
\[
\lambda(a) = p \\
\lambda(b) = q.
\]
Moreover for each $U \subset M$ open in $M$, $I^{+}(p)|_{U}$ consists of all points reach by time-like future directed geodesics starting in $p$ and contained in $U$, so that $I^{+}(p)|_{U}$ denotes the chronological future of the space-time $U \subset M$.

Definition 4.6. A set $S \subset M$ is said to be achronal if does not exist $p, q \in S$ such that $q \in I^{+}(p)$, that is if
\[
I^{+}(S) \cap S = \emptyset.
\]

Theorem 4.7. Let $M$ be a space-time manifold. Let $S \subset M$. Under such assumptions $\partial I^{+}(S)$ is achronal.

Proof. Let $q \in \partial I^{+}(S)$. Assume $p \in I^{+}(q)$. Thus $q \in I^{-}(p)$ and since $I^{-}(p)$ is open in $M$ there exists $U$ open in $M$, such that $U \subset I^{-}(p)$ and also such that $q \in U$.

Note that since $q \in \partial I^{+}(S)$ we have that
\[
U \cap I^{+}(S) \neq \emptyset.
\]

Let $q_{1} \in U \cap I^{+}(S)$.

From this, there exists $p_{1} \in S$ and a continuous and piece-wise $C^1$ class time-like future directed curve $\lambda : [a, b] \to M$ such that $\lambda(a) = p_{1}$ and $\lambda(b) = q_{1} \in U \subset I^{-}(p)$.

From such a result we may obtain a continuous and piece-wise $C^1$ class time-like future directed curve $\lambda_{1} : [b, c] \to M$ such that $\lambda_{1}(b) = q_{1}$ and $\lambda_{1}(c) = p$ so that $\lambda_{2} : [a, c] \to M$ such that
\[
\lambda_{2}(s) = \begin{cases} 
\lambda(s), & \text{if } s \in [a, b] \\
\lambda_{1}(s), & \text{if } s \in [b, c]
\end{cases}
\] (15)
is a continuous and piece-wise $C^1$ time-like future directed curve such that $\lambda_{2}(a) = p_{1} \in S$ and $\lambda_{2}(c) = p$. 

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Therefore, we may infer that $p \in I^+(S)$, $\forall p \in I^+(q)$, so that
\[ I^+(q) \subset I^+(S). \]

Suppose, to obtain contradiction, that $\partial I^+(S)$ is not achronal. Thus, there exist $q, r \in \partial I^+(S)$ such that
\[ r \in I^+(q) \subset I^+(S). \]

From this, we may infer that
\[ r \in \partial I^+(S) \cap I^+(S), \]
which contradicts $I^+(S)$ to be open.
Therefore, $\partial I^+(S)$ is achronal.

**Definition 4.8.** Let $M$ be a space-time manifold and let $\lambda \subset M$ be a causal future directed curve. We say that a point $p \in M$ is a final point of $\lambda$ if for each open set $U$ such that $p \in U$, there exists $s_0 \in \mathbb{R}$ such that if $s > s_0$, then
\[ \lambda(s) \in U. \]

Moreover, we say that a curve is inextensible if does not have any final point. Past inextensibility is defined similarly.

**Theorem 4.9.** Let $M$ be a closed space-time manifold. Let
\[ \lambda_n : (-\infty, b] \to M \]
be a sequence of differentiable past inextensible curves such that for each $m \in \mathbb{N}$ there exists $K_m, \hat{K}_m \in \mathbb{R}^+$ such that
\[ \|\lambda_n(s)\| \leq K_m, \forall s \in [-m, b], \]
and
\[ \|\lambda_n'(s)\| \leq \hat{K}_m, \forall s \in [-m, b]. \]

Assume there exists $p \in M$ that for each open $U$ such that $p \in U$, there exists $n_0 \in \mathbb{N}$ such that if $n > n_0$ then there exists $s_n \in [-\infty, b)$, such that
\[ \lambda_n(s) \subset U, \forall s \in (s_n, b]. \]

Under such hypotheses, there exist a subsequence $\{\lambda_{n_k}\}$ of $\{\lambda_n\}$ and a continuous curve $\lambda : (-\infty, b] \to M$ such that
\[ \lambda_{n_k} \to \lambda, \text{ uniformly in } [-m, b], \forall m \in \mathbb{N} \]
and
\[ \lambda(b) = p. \]
Proof. Let \[ \{\alpha_n\} = \{q \in \mathbb{Q} : q \leq b\}. \]

Observe that, from the hypotheses \( \{\lambda_n(\alpha_1)\} \subset M \) is a bounded sequence, so that there exists a subsequence \( \{\lambda_{n_k}(\alpha_1)\} \)

and a vector which we shall denote by \( \lambda(\alpha_1) \) such that \[ \lambda_{n_k}(\alpha_1) \to \lambda(\alpha_1), \text{ as } k \to \infty. \]

We shall also denote \[ \lambda_{n_k}(\alpha_1) = L_k^1(\alpha_1). \]

Similarly \( \{L_k^k(\alpha_2)\} \) is bounded so that there exists a subsequence \( \{L_{n_k}^1\} \) of \( \{L_k^1\} \) and a vector in \( M \), which we will denote by \( \lambda(\alpha_2) \) such that \[ L_{n_k}^1(\alpha_2) \to \lambda(\alpha_2), \text{ as } k \to \infty. \]

Denoting \( L_{n_k}^1 = L_k^2 \) we have obtained \[ L_k^2(\alpha_1) \to \lambda(\alpha_1), \]

and \[ L_k^2(\alpha_2) \to \lambda(\alpha_2), \text{ as } k \to \infty. \]

Proceeding in this fashion, we may inductively obtain a subsequences \( \{L_k^j\}_{k \in \mathbb{N}} \) of \( \lambda_n \) such that \[ L_k^j(\alpha_l) \to \lambda(\alpha_l) \text{ as } k \to \infty, \forall l \in \{1, \ldots, j\}. \]

Let \( \varepsilon > 0, \ l \in \mathbb{N} \) and \( j \geq l. \) Hence there exists \( K_j \in \mathbb{N} \) such that if \( k \geq K_j \) then \[ \|L_k^j(\alpha_l) - \lambda(\alpha_l)\| < \varepsilon. \]

In particular \[ \|L_{K_j}^j(\alpha_l) - \lambda(\alpha_l)\| < \varepsilon, \forall j > l. \]

Hence, denoting \( \Lambda_j = L_{K_j}^j, \)

we have obtained that \( \{\Lambda_j\} \) is a subsequence of \( \{\lambda_n\} \) such that \[ \Lambda_k(\alpha_j) \to \lambda(\alpha_j), \forall j \in \mathbb{N}. \]

Fix \( m \in \mathbb{N} \) such that \( -m < b \) and let \( s \in [-m, b]. \) We are going to prove that \[ \{\Lambda_k(s)\} \]

is a Cauchy sequence.

Let \( \{\alpha_{n_l}\} \) be a subsequence of \( \{\alpha_n\} \) such that \[ \alpha_{n_l} \to s, \text{ as } l \to \infty. \]
Hence, there exists \( l_0 \in \mathbb{N} \) such that if \( l > l_0 \), then
\[
|\alpha_{n_l} - s| < \frac{\varepsilon}{3\hat{K}_m}.
\]

Choose \( l > l_0 \). Since \( \{\Lambda_k(\alpha_{n_l})\} \) is a Cauchy sequence, there exists \( k_0 \in \mathbb{N} \) such that if \( k, p > k_0 \), then
\[
\|\Lambda_k(\alpha_{n_l}) - \Lambda_p(\alpha_{n_l})\| < \frac{\varepsilon}{3}.
\]

Thus, if \( k, p > k_0 \), we obtain
\[
\|\Lambda_k(s) - \Lambda_p(s)\| \leq \|\Lambda_k(s) - \Lambda_k(\alpha_{n_l}) + \Lambda_k(\alpha_{n_l}) - \Lambda_p(\alpha_{n_l}) + \Lambda_p(\alpha_{n_l}) - \Lambda_p(s)\| \\
\leq \|\Lambda_k(s) - \Lambda_k(\alpha_{n_l})\| + \|\Lambda_k(\alpha_{n_l}) - \Lambda_p(\alpha_{n_l})\| + \|\Lambda_p(\alpha_{n_l}) - \Lambda_p(s)\| \\
\leq \hat{K}_m|s - \alpha_{n_l}| + \frac{\varepsilon}{3} + \hat{K}_m|s - \alpha_{n_l}| \\
< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
= \varepsilon.
\]

(16)

From this we may infer that \( \{\Lambda_k(s)\} \) is a Cauchy sequence so that we may define
\[
\lambda(s) = \lim_{k \to \infty} \Lambda_k(s), \quad \forall s \in [-m, b].
\]

We claim that this last convergence, up to a subsequence, is uniform on \([-m, b]\).
Indeed, let
\[
c_k = \sup_{s \in [-m, b]} \{\|\Lambda_k(s) - \lambda(s)\|\}.
\]

Let \( s_k \in [-m, b] \) be such that
\[
c_k - 1/k < \|\Lambda_k(s_k) - \lambda(s_k)\| \leq c_k.
\]

Since \([-m, b]\) is compact, there exist a subsequence \( \{s_{k_l}\} \) of \( \{s_k\} \) and \( s \in [-m, b] \) such that
\[
s_{k_l} \to s, \quad \text{as} \quad l \to \infty.
\]

At this point we shall prove that
\[
\|\lambda(s_{k_l}) - \lambda(s)\| \to 0.
\]

Indeed, there exists \( l_0 \in \mathbb{N} \) such that if \( l > l_0 \), then
\[
|s_{k_l} - s| < \frac{\varepsilon}{\hat{K}}.
\]

\[
\|\Lambda_p(s_{k_l}) - \Lambda_p(s)\| \leq \hat{K}|s_{k_l} - s| < \varepsilon, \quad \forall l > l_0, \quad \forall p \in \mathbb{N}.
\]

From this, we get
\[
\|\lambda(s_{k_l}) - \lambda(s)\| = \lim_{p \to \infty} \|\Lambda_p(s_{k_l}) - \Lambda_p(s)\| \leq \varepsilon, \quad \forall l > l_0.
\]

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Observe that from such a result we may, in a similar fashion, infer that \( \lambda \) is continuous.

From these last results, observing that there exists \( l_1 \in \mathbb{N} \) such that if \( l > l_1 \), then

\[
\|\Lambda_{k_l}(s) - \lambda(s)\| < \varepsilon,
\]

we have that

\[
\begin{align*}
\|\Lambda_{k_l}(s_{k_l}) - \lambda(s_{k_l})\| \\
= \|\Lambda_{k_l}(s_{k_l}) - \Lambda_{k_l}(s) + \Lambda_{k_l}(s) - \lambda(s) + \lambda(s) - \lambda(s_{k_l})\| \\
\leq \|\Lambda_{k_l}(s_{k_l}) - \Lambda_{k_l}(s)\| + \|\Lambda_{k_l}(s) - \lambda(s)\| + \|\lambda(s) - \lambda(s_{k_l})\| \\
\leq \varepsilon + \varepsilon + \varepsilon \\
= 3\varepsilon, \quad \forall l > \max\{l_0, l_1\}.
\end{align*}
\]

(17)

From this we may infer that \( c_{k_l} \to 0 \) as \( l \to \infty \), so that the convergence in question of the subsequence \( \{\Lambda_{k_l}\} \) of \( \{\lambda_n\} \) is uniform. We claim now that \( c_k \to 0 \) as \( k \to \infty \).

Suppose, to be contradiction, that the claim is false. Thus, \( \{c_k\} \) does not converge to 0.

Hence, there exists \( \varepsilon_0 > 0 \) such that for each \( k \in \mathbb{N} \) there exists \( k_l > k \) such that

\[
c_{k_l} \geq \varepsilon_0.
\]

(18)

However, exactly as we have done with \( \{c_k\} \) in the lines above, we may obtain a subsequence of \( \{c_{k_l}\} \) which converges to 0. This contradicts (18).

Therefore

\[
c_k \to 0, \quad \text{as } k \to \infty.
\]

From this we may infer that

\[
\Lambda_k \to \lambda, \quad \text{uniformly in } [-m, b], \quad \forall m \in \mathbb{N} \text{ such that } -m < b.
\]

The proof is complete. \( \square \)

**Theorem 4.10.** Let \( M \) be a space time manifold. Assume that \( \lambda : (-\infty, b] \to M \) is a causal future directed past inextensible curve.

Under such hypotheses,

\[
\lambda(s) \in \overline{I^+(\lambda)}, \quad \forall s \in (-\infty, b].
\]

Proof. Let \( s \in (-\infty, b] \) and choose \( s_1 < s \).

Thus, \( \lambda\res_{[s_1, s]} \) is a causal future directed curve such that denoting \( p = \lambda(s_1) \) and \( q = \lambda(s) \), we have that

\[
q \in \overline{I^+(p)} \subset \overline{I^+(\lambda)}, \quad \forall s \in (-\infty, b].
\]

The proof is complete. \( \square \)

**Theorem 4.11.** Let \( M \) be a normal space time manifold. Assume \( \lambda : (-\infty, c] \to M \) is a causal future directed past inextensible curve which passes through a point \( p \in M \).

Under such hypotheses, for each \( q \in I^+(p) \) there exists a continuous and piece-wise \( C^1 \) class time-like future directed past inextensible curve \( \gamma : (-\infty, b] \to M \), such that

\[
\gamma \subset I^+(\lambda)
\]

and

\[
\gamma(b) = q.
\]

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Proof. Let $\hat{\lambda} : [a, b] \rightarrow I^+(\lambda)$ be a time-like future directed curve such that 
\[ \hat{\lambda}(a) = p \]
and
\[ \hat{\lambda}(b) = q. \]

We claim that $\hat{\lambda} \subset I^+(\lambda)$.
Indeed, let $s \in (a, b]$. Denoting $q_1 = \hat{\lambda}(s)$ we have
\[ \hat{\lambda}(s) = q_1 \in I^+(p) \subset I^+(\lambda), \; \forall s \in (a, b]. \]

So the concerning claim holds.
Let $\lambda_1 : (-\infty, b] \rightarrow M$ be the curve defined by
\[ \lambda_1(s) = \begin{cases} \lambda(s), & \text{if } s \in (-\infty, a] \\
\hat{\lambda}(s), & \text{if } a \leq s \leq b \end{cases} \quad (19) \]

Since the graph of $\lambda$ is connected and $M$ is normal, there exists $r > 0$ such that if $\tilde{p}, \tilde{q} \in I^+(\lambda)$ and
\[ 0 < \|\tilde{p} - \tilde{q}\| < r, \]
then renaming $\tilde{p}, \tilde{q}$ if necessary, there exists a time-like future directed curve $\tilde{\lambda} : [c, d] \rightarrow I^+(\lambda)$ such that
\[ \tilde{\lambda}(c) = \tilde{p} \]
and
\[ \tilde{\lambda}(d) = \tilde{q}. \]

Let $\{s_n\} \subset (-\infty, b]$ be a real sequence such that $s_1 = b, s_n > s_{n+1}, \forall n \in \mathbb{N}$,
\[ \lim_{n \to \infty} s_n = -\infty. \]

and also such that
\[ \|\lambda_1(s_{n+1}) - \lambda_1(s_n)\| < \frac{r}{3}, \forall n \in \mathbb{N}. \]

Define $p_1 = q$. Since
\[ \lambda_1 \subset \overline{I^+(\lambda)} \]
and $I^+(\lambda)$ is open, for each $n > 1$ we may select $p_n \in I^+(\lambda)$ such that
\[ 0 < \|p_n - \lambda_1(s_n)\| < \frac{r}{3}. \]

Observe that in such a case,
\[
\|p_{n+1} - p_n\| = \|p_{n+1} - \lambda_1(s_{n+1}) + \lambda_1(s_{n+1}) - \lambda_1(s_n) + \lambda_1(s_n) - p_n\|
\leq \|p_{n+1} - \lambda_1(s_{n+1})\| + \|\lambda_1(s_{n+1}) - \lambda_1(s_n)\| + \|\lambda_1(s_n) - p_n\|
\leq \frac{r}{3} + \frac{r}{3} + \frac{r}{3}
= r, \forall n \in \mathbb{N}. \quad (20)
\]
Moreover, \( \{p_n\} \) may be chosen such that

\[
0 < d(p_n, \lambda) < \frac{C}{1 + \sqrt{n}}, \forall n \in \mathbb{N},
\]

for some appropriate constant \( C > 0 \).

Thus from [20] and from the fact that \( M \) is normal, concerning such \( r > 0 \), we may obtain a smooth time-like future directed curve

\[
\tilde{\lambda}_n : [s_{n+1}, s_n] \to I^+(\lambda)
\]

such that

\[
\tilde{\lambda}_n(s_{n+1}) = p_{n+1},
\]

and

\[
\tilde{\lambda}_n(s_n) = p_n.
\]

Therefore, we may define \( \gamma : (-\infty, b] \to M \) such that

\[
\gamma = \{\tilde{\lambda}_n : [s_{n+1}, s_n] \to I^+(\lambda) : n \in \mathbb{N}\},
\]

which is a continuous and piece-wise \( C^1 \) class time-like future directed past inextensible curve such that

\[
\gamma(b) = q,
\]

and

\[
\gamma \subset I^+(\lambda).
\]

The proof is complete. \( \square \)

**Definition 4.12.** Let \( M \) be a space time manifold. We say that \( M \) is strongly causal if for each \( p \in M \) and each neighborhood \( U \) of \( p \), there exists a neighborhood \( V \) of \( p \) such that \( V \subset U \) and no causal curve intersects \( V \) more than one time.

**Theorem 4.13.** Let \( M \) be a space-time manifold strongly causal. Let \( K \subset M \) be a compact set. Under such hypotheses, each causal curve \( \lambda \) contained in \( K \) must have past and future final points.

**Proof.** Let \( \lambda : [-\infty, +\infty] \to M \) be a causal curve contained in \( K \). Let \( \{s_j\} \subset \mathbb{R} \) be such that \( s_j < s_{j+1} \) and

\[
\lim_{j \to \infty} s_j = +\infty.
\]

Observe that

\[
\{\lambda(s_j) = p_j\} \subset K
\]

and \( K \) is compact. Hence, there exists a subsequence

\[
\{p_{j_k}\}
\]

and \( p \in K \) such that

\[
p_{j_k} \to p, \ \text{as} \ k \to \infty.
\]
Suppose, to obtain contradiction, we may obtain an open set $U$ such that $p \in U$ and such that for each $s_0 \in \mathbb{R}$ there exists $s > s_0$ such that $\lambda(s) \notin U$. Thus we have the same for all $V \subset U$ such that $p \in V$. Fixing an arbitrary $V \subset U$ with $p \in V$, we have that $\lambda$ enters and leaves $V$ more than one time, because each time $\lambda$ enters $V$ it does not remain completely in $V$. Since $V \subset U$ has been arbitrary, this contradicts the strong causality of $M$.

Thus, $p$ is a future final point for $\lambda$. Similarly we may prove that $\lambda$ has a past final point. This completes the proof.

5 Dependence domains and hyperbolicity

**Definition 5.1.** Let $S$ be a closed and achronal set. We define the domain of future dependence of $S$, denoted by $D^+(S)$, by

$$D^+(S) = \{p \in M : \text{each piece-wise smooth causal future directed past inextensible curve which passes through } p \text{ intercepts } S\}.$$  

Observe that

$$S \subset D^+(S) \subset J^+(S),$$

and since $S$ is achronal, we have that

$$D^+(S) \cap I^-(S) = \emptyset.$$

The domain of past dependence of $S$, denoted by $D^-(S)$ is defined similarly.

We also define

$$D(S) = D^+(S) \cup D^-(S),$$

Finally, an achronal set $\Sigma$ for which $D(\Sigma) = M$ is said to be a Cauchy surface for $M$. Observe that, in such a case,

$$\partial \Sigma = \emptyset.$$

Finally, a space-time manifold which has a Cauchy surface is said to be globally hyperbolic.

**Theorem 5.2.** Let $M$ be a normal space-time manifold and let $S \subset M$ be a set closed in $M$. Under such hypotheses, Let $p \in D^+(S)$ if, and only if, each time-like future directed past inextensible curve which passes through $p$ intercepts $S$.

Proof. Suppose there exists a time-like future directed past inextensible curve which does not intercept $S$.

Hence there exists a set $U$ open in $M$ such that $p \in U$ with such a propriety. Thus $U \cap D^+(S) = \emptyset$, so that $p \notin D^+(S)$.

Reciprocally, suppose each time-like future direct past inextensible curve which passes through $p$ intercepts $S$.

Thus, either $p \in S \subset D^+(S) \subset D^+(S)$, and in such a case the proof would be finished, or $p \in I^+(S) \setminus S$.

In this latter case, let $q \in I^+(p) \cap I^+(S)$.

Suppose, to obtain contradiction, that $q \notin D^+(S)$.

Thus there exists a causal future directed past inextensible curve $\lambda$ which passes through $q$ and does not intercept $S$. 

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Note that
\[ \lambda \subset I^+(\lambda) \setminus S, \]
so that, in such a case, since \( M \) is normal, similarly as in the proof of Theorem 4.11, we may obtain a piece-wise smooth time-like future directed past inextensible curve \( \gamma \) such that
\[ \gamma \subset I^+(\lambda), \]
also such that \( \gamma \cap S = \emptyset \) and \( \gamma \) passes through \( p \), which contradicts the hypotheses in question.

Hence, if \( q \in I^-(p) \cap I^+(p) \), then \( q \in D^+(S) \), so that
\[ I^-(p) \cap I^+(S) \subset D^+(S). \]
Since each neighborhood of \( p \in I^+(S) \) intercepts
\[ I^-(p) \cap I^+(S) \subset D^+(S), \]
we have that \( p \in \overline{D^+(S)} \).

The proof is complete.

**Theorem 5.3.** Let \( M \) be a space-time manifold. Let \( S \subset M \) and let \( \lambda : [a, b] \to M \) be a \( C^1 \) class future directed curve such that \( \lambda(s) \in \partial I^+(S) \), \( \forall s \in [a, b] \).

Under such hypotheses, \( \lambda \) is a null geodesics, that is,
\[ \frac{d\lambda(s)}{ds} \cdot \frac{d\lambda(s)}{ds} = 0, \forall s \in [a, b]. \]

**Proof.** Suppose, to obtain contradiction, that there exists \( s_0 \in (a, b) \) such that
\[ \frac{d\lambda(s_0)}{ds} \cdot \frac{d\lambda(s_0)}{ds} < 0. \]
By continuity, there exists \( \delta > 0 \) such that
\[ \frac{d\lambda(s)}{ds} \cdot \frac{d\lambda(s)}{ds} < 0, \forall s \in (s_0 - \delta, s_0 + \delta). \]
Define
\[ p_1 = \lambda(s_0), \]
and
\[ p_2 = \lambda \left( s_0 + \frac{\delta}{2} \right). \]
Thus, \( p_2 \in I^+(p_1) \), and \( p_1, p_2 \in \partial I^+(S) \), which contradicts \( \partial I^+(S) \) to be achronal. Hence,
\[ \frac{d\lambda(s)}{ds} \cdot \frac{d\lambda(s)}{ds} = 0, \forall s \in [a, b]. \]
The proof is complete.

**Theorem 5.4.** Let \( M \) be a space-time manifold. Let \( S \subset M \) and suppose \( \lambda : [a, b] \to \overline{I^+(S)} \) is a future directed null geodesics.

Under such hypotheses,
\[ \lambda(s) \in \partial I^+(S), \forall s \in [a, b]. \]

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Proof. Since $\lambda$ is future directed null geodesics, we have that
\[
\frac{d\lambda(s)}{ds} \cdot \frac{d\lambda(s)}{ds} = 0, \forall s \in [a, b].
\]
From this, since $\lambda(s) \in I^+(S)$, we get
\[
\lambda(s) \in \partial I^+(S), \forall s \in [a, b].
\]
The proof is complete. $\square$

**Theorem 5.5.** Let $M$ be a normal space-time manifold. Let $\Sigma \subset M$ be a Cauchy surface and let $\lambda : [-\infty, +\infty] \to M$ be a causal inextensible curve.
Under such hypotheses, $\lambda$ intercepts $\Sigma$, $I^+(\Sigma)$ and $I^-(\Sigma)$.

**Proof.** Suppose, to obtain contradiction, that $\lambda$ does not intercept $I^-(\Sigma)$. Similarly as in the proof of Theorem 4.11, we may obtain a time-like past inextensible curve such that
\[
\gamma \subset I^+(\lambda) \subset I^+(\Sigma \cup I^+(\Sigma)) = I^+(\Sigma).
\]
Extending $\gamma$ to the future indefinitely (if possible), such a curve cannot intercept $\Sigma$, because in such a case $\Sigma$ would not be achronal, which is contradiction.
However, since each causal inextensible curve must intercept $\Sigma$, we have got a final contradiction (that is, such a $\gamma$ does not exists).
From this we may infer that $\lambda$ intercepts $I^-(\Sigma)$.
Similarly, we may show that $\lambda$ intercepts $I^+(\Sigma)$.
The proof is complete. $\square$

6 Existence of solution for the previous general functional

In this section, under some conditions, we prove the existence of solution for the general functional presented in the previous sections. Specifically, we will be concerned with the existence of a kind of generalized solution for the main world sheet manifold.

We start with the following remark.

**Remark 6.1.** Considering the position field given by
\[
r : D = [0, T] \times D_1 \to \mathbb{R}^{N+1}
\]
and fixing a small $\varepsilon > 0$, define
\[
U = \{ \tilde{u} = (r, \phi, n) \in C^2(\overline{D}; \mathbb{R}^{N+1}) \times C^1(\overline{D}; \mathbb{C}) \times C^1(\overline{D}; \mathbb{R}^4)
\text{ such that } |\phi|^2 \geq \varepsilon \text{ in } D, \forall k \in \{0, \ldots, m\},
\text{r}(0, u) = \hat{r}_0, \text{ in } D_1, \text{r}(T, u) = \hat{r}_1, \text{ in } D_1,
\text{r}(t, u) = \hat{r}_2, \text{ on } \partial D_1 \times [0, T]
\text{\phi}(0, u) = \hat{\phi}_0, \text{ in } D_1, \phi(cT, u) = \hat{\phi}_1, \text{ in } D_1,
\phi(ct, u) = \hat{\phi}_2, \text{ on } \partial D_1 \times [0, T]\},
\]
\[ U = \{(r, \phi, n) \in W^{2,2}(D; \mathbb{R}^{N+1}) \times W^{1,2}(D; \mathbb{C}) \times W^{1,2}(D; \mathbb{R}^4) \} \]

such that \( r(0, u) = \hat{r}_0 \), in \( D_1 \), \( r(T, u) = \hat{r}_1 \), in \( D_1 \),
\( r(t, u) = \hat{r}_2 \), on \( \partial D_1 \times [0, T] \)
\( \phi(0, u) = \hat{\phi}_0 \), in \( \Omega \), \( \phi(cT, u) = \hat{\phi}_1 \), in \( D_1 \),
\( \phi(ct, u) = \hat{\phi}_2 \), on \( \partial D_1 \times [0, T] \), \( (23) \)

\[ U_1 = \left\{ \tilde{u} \in U : \int_{D_1} |\phi(ct, u)|^2 \sqrt{\gamma_1} du = 1, \text{ on } [0, T] \right\}, \]
\[ U_2 = \left\{ \tilde{u} \in U : \frac{\partial r(u)}{\partial u_j} \cdot n(u) = 0, \text{ in } D \right\}, \]
and
\[ U_3 = \{ \tilde{u} \in U_1 \times U_2 : n(u) \cdot n(u) = 1, \text{ in } D \}. \]

Finally, we define also,
\[ A = U \cap U_1 \cap U_2 \]
and
\[ A_1 = \tilde{U} \cap U_1 \cap U_2. \]

With such definitions in mind we state and prove the following existence theorem.

**Theorem 6.2.** For \( 5 \leq m \leq 8 \) and \( m < N \), let \( J_K : U \to \mathbb{R} \) be defined by

\[ J_K(\tilde{u}) = J(\tilde{u}) + K \int_0^T \left( \int_{D_1} |\phi|^2 \sqrt{\gamma_1} du - 1 \right)^2 dt \]
\[ + \frac{K}{2} \sum_{j=0}^m \int_D \left( \frac{\partial r(u)}{\partial u_j} \cdot n(u) \right)^2 \sqrt{\gamma_1} \sqrt{U} du dt \]
\[ + \frac{K}{2} \int_D (n(u) \cdot n(u) - 1)^2 \sqrt{\gamma_1} du dt, \quad (24) \]

where

\[ J(\tilde{u}) = \frac{1}{2} \int_D |\phi|^2 g_{ik} b_j b_j \sqrt{\gamma_1} du dt \]
\[ + \frac{1}{2} \int_D g_{ik} \frac{\partial \phi}{\partial u_j} \frac{\partial \phi^*}{\partial u_k} \sqrt{\gamma_1} du dt \]
\[ + \frac{1}{4} \int_D \left( \frac{\partial \phi}{\partial u} \phi + \frac{\partial \phi^*}{\partial u} \phi^* \right) \Gamma_{jk} g_{jk} \sqrt{\gamma_1} du dt, \quad (25) \]

and where \( K \in \mathbb{N} \) is a large constant. Let \( \{ \tilde{v}^K_n \} \) be a minimizing sequence for \( J_K \), such that

\[ \alpha \leq J_K(\tilde{v}^K_n) < \alpha + \frac{1}{n}, \]

where

\[ \alpha = \inf_{\tilde{v} \in U} J_K(\tilde{u}). \]

Suppose such a sequence is such that
1. There exists $c_0 > 0$ such that 
\[
(g^{jk})_n^{K} y_j y_k \geq c_0 y_j y_j, \quad \forall y = \{y_j\} \in \mathbb{R}^2, \forall n \in \mathbb{N}, \text{ in } D.
\]

2. There exists $c_1 > 0$ such that 
\[
\left| \phi^K_n \right|^2 (g^{ik})_n^K z_i \cdot (g^{is})_n^K z_s \cdot (g^{is})_n^K \geq c_1 z_j \cdot z_j, \\
\forall \{z_j\} \in \mathbb{R}^{(N+1)(m+1)}, \text{ in } D, \forall n \in \mathbb{N},
\]
so that
\[
\left| \phi^K_n \right|^2 (g^{ik})_n^K (b^{jl})_n^K (b^{lk})_n^K \geq (c_2)_{ij} \left| \frac{\partial r^K_n}{\partial u_i} \cdot \frac{\partial r^K_n}{\partial u_j} \right|^2, \text{ in } D, \forall n \in \mathbb{N}.
\]

3. There exists $\{(c_2)_{ij}\}$ such that $(c_2)_{ij} > 0$, $\forall i, j \in \{0, \ldots, m\}$, so that
\[
\left| \phi^K_n \right|^2 (g^{ik})_n^K (b^{jl})_n^K (b^{lk})_n^K \geq (c_2)_{ij} \left| \frac{\partial r^K_n}{\partial u_i} \cdot \frac{\partial r^K_n}{\partial u_j} \right|^2, \text{ in } D, \forall n \in \mathbb{N}.
\]

4. 
\[
\| (g^{ik})_n^K \|_{C^{1, \nu} (\overline{D})} \leq \hat{K}, \forall k \in \{0, \ldots, m\}, \forall n \in \mathbb{N},
\]
for some $\hat{K} \in \mathbb{R}^+$ and some $0 < \nu < 1$.

Moreover, assume there exists $K_0 \in \mathbb{N}$, such that if $K > K_0$, then there exists $n_0 \in \mathbb{N}$ such that if $n > n_0$, then
\[
\int_{D_1} \left| \phi^K_n \right|^2 (\sqrt{g})_n^K d\mathbf{u} \geq \frac{1}{4}, \text{ on } [0, T]
\]
and
\[
\mathbf{u}_n^K \cdot \mathbf{u}_n^K \geq \frac{1}{4}, \text{ in } D.
\]

Under such hypotheses, there exists $\tilde{u}_0^K \in \tilde{U}$ such that
\[
J_K(\tilde{u}_0^K) = \inf_{\tilde{u} \in U} J_K(\tilde{u}).
\]

Finally, there exists a subsequence $\{K_j\}$ of $\mathbb{N}$ and $\tilde{u}_0 \in A_1$ such that
\[
J(\tilde{u}_0) = \lim_{j \to \infty} J_{K_j}(\tilde{u}_0^{K_j}) = \inf_{\tilde{u} \in A} J(\tilde{u}).
\]

Proof. From the hypotheses we may infer that there exists $K_1 \in \mathbb{R}^+$ such that

1. 
\[
\left\| \frac{\partial^2 r^K_n}{\partial u_i \partial u_j} \right\|_2 \leq K_1, \forall n \in \mathbb{N}, \forall i, j \in \{0, \ldots, m\}.
\]

2. 
\[
\| \phi^K_n \|_2 \leq K_1, \forall n \in \mathbb{N}.
\]

3. 
\[
\left\| \frac{\partial \phi^K_n}{\partial u_j} \right\|_2 \leq K_1, \forall n \in \mathbb{N}, \forall j \in \{0, \ldots, m\}.
\]
Observe that $J_K$ is lower semi-continuous so that, from the Ekeland variational principle there exists a sequence $\{\tilde{u}_n^K\} \in U$ such that

$$\alpha \leq J_K(\tilde{u}_n^K) < \alpha + \frac{1}{n},$$

and

$$\|\delta J_K(\tilde{u}_n^K)\|_U \leq \frac{1}{\sqrt{n}}, \forall n \in \mathbb{N},$$

From such a result and from the variation of $J_K$ in $n$ we obtain that

$$\frac{\partial}{\partial u_j} \left( (a_{ij})_n^K \frac{\partial (n)_n^K}{\partial u_i} \right) = (f_l)_n^K, \text{ in } D,$$

for appropriate positive definite $\{(a_{ij})_n^K\}$ of $C^1$ class and $(f_l)_n^K \in L^2, \forall l \in \{1, \ldots, n+1\}, i, j \in \{0, \ldots, m\}$.

Thus, from the Theory of Elliptic Partial Differential Equations, we have that $n_n^K \in W^{2,2}$ and, since $\{(a_{ij})_n^K\}$ and $\{(f_l)_n^K\}$ are uniformly bounded in $C^1$ and $L^2$, respectively, there exists $K_3 \in \mathbb{R}^+$ such that

$$\|n_n^K\|_{2,2} \leq K_3, \forall n \in \mathbb{N}.$$

With such results, we may similarly obtain that

$$\|\phi_n^K\|_{2,2} \leq K_4, \forall l \in \mathbb{N},$$

for some $K_4 \in \mathbb{R}^+$.

From such results and the Rellich-Kondrachov Theorem, we may obtain a subsequence $\{n_l\}$ of $\mathbb{N}$ and $\tilde{u}_0^K \in \tilde{U}$ such that

1. $\phi_{n_l}^K \rightharpoonup \phi_0^K$, as $l \to \infty$, weakly in $W^{2,2}$.
2. $\phi_{n_l}^K \to \phi_0^K$, as $l \to \infty$, strongly in $W^{1,q}$,
3. $r_{n_l}^K \rightharpoonup r_0^K$, as $l \to \infty$, weakly in $W^{2,2}$.
4. $r_{n_l}^K \to r_0^K$, as $l \to \infty$, strongly in $W^{1,q}$.
5. $n_{n_l}^K \rightharpoonup n_0^K$, as $l \to \infty$, weakly in $W^{2,2}$.
6. $n_{n_l}^K \to n_0^K$, as $l \to \infty$, strongly in $W^{1,q}$,
\( \forall 1 \leq q \leq \frac{2m}{m-4} \equiv p^* \). At this point, firstly we highlight that, up to a not relabeled subsequence

\[
\left| (\sqrt{-g})_{n_l}^K - (\sqrt{-g})_0^K \right|^4 \to 0, \text{ as } l \to \infty, \text{ a.e. in } D,
\]

and

\[
\left\| (\sqrt{-g})_{n_l}^K - (\sqrt{-g})_0^K \right\|_\infty < \hat{K}_1, \forall l \in \mathbb{N},
\]

for some appropriate \( \hat{K}_1 \in \mathbb{R}^+ \), so that, from the Lebesgue Dominated Convergence Theorem, we have

\[
\left\| (\sqrt{-g})_{n_l}^K - (\sqrt{-g}_0)^K \right\|_4 \to 0, \text{ as } l \to \infty.
\]
Thus,

\[
\begin{align*}
\left| \int_D \frac{\partial \phi^K_{n_t}}{\partial u_j} & \frac{\partial (\phi_0^K)_{n_t}}{\partial u_k} (g^{jk}_{n_t})^K_{n_t}(\sqrt{-g})^K_{n_t} \, du \right| \\
- & \int_D \frac{\partial \phi^K_0}{\partial u_j} \frac{\partial (\phi_0^K)_{0_t}}{\partial u_k} (g^{jk}_{0_t})^K_{0_t}(\sqrt{-g})^K_{0_t} \, du \\
\leq & \int_D \left| \frac{\partial \phi^K_0}{\partial u_j} \frac{\partial (\phi_0^K)_{0_t}}{\partial u_k} (g^{jk}_{0_t})^K_{0_t}(\sqrt{-g})^K_{0_t} \right| \, du \\
\leq & K_1 \| \phi_0^K \|_{1,4} \| (\phi_0^K)_{1,4} \| (g^{jk}_{1,4})^K_{1,4} \| (\sqrt{-g})^K_{1,4} \\
& + K_1 \| \phi_0^K \|_{1,4} \| (\phi_0^K)_{1,4} \| (g^{jk}_{1,4})^K_{1,4} \| (\sqrt{-g})^K_{1,4} \\
& + K_1 \| \phi_0^K \|_{1,4} \| (g^{jk}_{1,4})^K_{1,4} \| (\sqrt{-g})^K_{1,4} \\
& + K_1 \| \phi_0^K \|_{1,4} \| (\sqrt{-g})^K_{1,4} - (\sqrt{-g})^K_{1,4} \\
\to & 0, \text{ as } l \to \infty.
\end{align*}
\]

Similarly we may prove the continuity of the remaining functional parts, so that

\[ J_K(\tilde{u}_n^K) \to J(\tilde{u}_0^K) = \min_{u \in U} J_K(\tilde{u}) \]

At this point we observe that, through the Euler-Lagrange equations, the hypotheses and the limit process, we have obtained

\[ \int_{D_t} |\phi_0^K|^2 \sqrt{-g_0^K} \, du - 1 = \mathcal{O}(1/K), \text{ on } [0,T], \]

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\[
\int_D \left( \frac{\partial r^K}{\partial u_j} \cdot n^K_0 \right)^2 \sqrt{-g^K_0} \, du \, dt = O(1/K)
\]
and
\[
\int_D (n^K_0 \cdot n^K_0 - 1)^2 \sqrt{-g^K_0} \, du \, dt = O(1/K).
\]

Observe also that the previous estimates are valid also for the sequence \( \{\tilde{u}^K_0\} \) (the concerning constants do not depend on \( K \)) so that there exists \( \tilde{u}_0 \in \tilde{U} \) such that, up to a not relabeled subsequence,

1. \( \phi^K_0 \rightharpoonup \phi_0 \), as \( K \to \infty \), weakly in \( W^{2,2} \).
2. \( \phi^K_0 \to \phi_0 \), as \( K \to \infty \), strongly in \( W^{1,q} \).
3. \( r^K_0 \rightharpoonup r_0 \), as \( K \to \infty \), weakly in \( W^{2,2} \).
4. \( r^K_0 \to r_0 \), as \( K \to \infty \), strongly in \( W^{1,q} \).
5. \( n^K_0 \rightharpoonup n_0 \), as \( K \to \infty \), weakly in \( W^{2,2} \).
6. \( n^K_0 \to n_0 \), as \( K \to \infty \), strongly in \( W^{1,q} \),

where as previously indicated, \( \forall 1 \leq q \leq p^* \).

Moreover from the previous estimates and concerning limits (obtained similarly as above indicated),
\[
\int_{D_1} |\phi_0|^2 \sqrt{-g_0} \, du - 1 = 0, \text{ on } [0, T],
\]
\[
\int_D \left( \frac{\partial r_0}{\partial u_j} \cdot n_0 \right)^2 \sqrt{-g_0} \, du \, dt = 0
\]
and
\[
\int_D (n_0 \cdot n_0 - 1)^2 \sqrt{-g_0} \, du \, dt = 0.
\]

From this we get,
\[ \tilde{u}_0 \in A_1, \]
so that
\[ J(\tilde{u}_0) = \lim_{K \to \infty} J_K(\tilde{u}^K_0) = \inf_{\tilde{u} \in A} J(\tilde{u}). \]

The proof is complete. \( \square \)
7 Conclusion

In this article we have obtained a variational formulation for relativistic mechanics based
on standard tools of differential geometry. The novelty here is that the main manifold has its
range in a space of dimension $n + 1$. In such a formulation the concept of normal field plays a
fundamental role.

In the second article part, we have presented some formalism concerning the causal structure
in a general space-time manifold defined by a function

$$(r \circ \hat{u}) : \Omega \times (-\infty, +\infty) \to \mathbb{R}^{n+1}.$$

It is worth highlighting the main reference for this second part is the book [4].

Finally, in the last section, we develop an existence result of a kind of generalized solution
for the main manifold variational formulation.

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