Generalized Kaluza-Klein reduction and scalar-tensor theories

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Abstract

We investigate the generalized description of the Kaluza-Klein (KK) dimensional reduction from higher-dimensional space-time. We also explore the properties of the resultant scalar-tensor theories. We demonstrate that there exists the anti-de Sitter (AdS) background solution, and examine its stability through the analysis of the perturbations around the background AdS solution. Moreover, we derive the conditions for the dimensional reduction to successfully be executed and present the KK compactification mechanism.

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I. INTRODUCTION

According to recent cosmological observations, not only inflation in the early universe but also the late-time cosmic acceleration (the so-called dark energy problem) has been supported. As a possible approach to realize these cosmic accelerated expansions, recently, scalar-tensor theories including the Brans-Dicke (BD) theory\(^1\) has widely been studied in the literature. This is because modified gravity theories such as \(F(R)\) gravity can be rewritten to a kind of the BD type theory (for reviews on issues of dark energy and modified gravity, see, for instance, \([2,3]\)). It is known that scalar-tensor theories can be constructed as four-dimensional effective theories through the Kaluza-Klein (KK) dimensional reduction (for reviews, see, e.g., \([4]\)).

Moreover, in the framework of the anti-de Sitter (AdS)/the conformal field theory (CFT) correspondence\(^2\), with the KK dimensional reduction, the relation between the solutions of general relativity plus a negative cosmological constant and those in the Minkowski space-time has been explored \([6]\). Furthermore, very recently, by developing the above considerations, the connection between the solutions of general relativity with a positive cosmological constant and those of that with a negative one in Ref.\(^7\).

In this paper, we generalize the procedure of the KK dimensional reduction from higher-dimensional space-time. In particular, we examine the natures of the scalar-tensor theories obtained through the KK compactification. Concretely, by starting with the Einstein-Hilbert action in the \(D\)-dimensional space-time \((D \geq 5)\) and adopting the compactification of the coordinate, we find the BD theory in \((D - 1)\)-dimension. The resultant theory is different from the usual BD one, because instead of the usual BD theory, we also have an auxiliary scalar field, \(\Psi\), which is non-minimally coupled to the BD scalar field\(^1\). It is remarkable to explicitly state that the reduction process from higher-dimensional theories performed in Ref.\(^7\) corresponds to a specific class of our investigations. The methodology in this work seems to be similar to the KK compactification, but there exists the difference between our approach and the KK procedure in terms of the technique and result. Our model of lower-dimensional gravity includes two types of the auxiliary fields. A non-minimal coupling between the fields appears in a natural way and in a systematic form. Here, such a non-

\(^1\) In Ref.\(^8\), cosmology in the theory with two scalar fields non-minimally coupled to the Ricci scalar has been examined in detail.
minimal interaction is not introduced ad hoc. Another motivation of the present work is to extend the KK compactification to more than one compactified direction. As a natural extension, if we start with the $(D+4)$-dimensional static metric and we compactified $D$-coordinates $y^i \sim y^i + 2\pi R^i$ with $1 \leq i \leq D$, the reduction of the Einstein-Hilbert action to the lower-dimensional space-time is a type of the BD theory with a non-minimally coupling in $(D=4)$-dimensional space-time. We here emphasize that our dimensional reduction scheme is different from the original reductions of KK, Scherk-Schwarz (I, II). One more reason for the study of such dimensional reductions of gravity in higher-dimensional space-time is that all of the most reliable unified models of the fundamental four forces in the nature such as superstring $(D=10)$ and M-theory $(D=11)$ live in the higher-dimensional space-time. We use units of $k_B = c = \hbar = 1$ and denote the gravitational constant, $G$, by $\kappa^2 \equiv 8\pi G$, so that $G = 1/M_{Pl}^2$ with $M_{Pl} = 1.2 \times 10^{19}$ GeV the Planck mass.

The organization of the paper is the followings. In Sec. II, we explain the formulation of the KK reduction mechanism from and explore the natures of the resultant theories. The field equations are also derived. In Sec. III, we examine the stability of the AdS solution through the analysis of the perturbations around the background AdS solution. In addition, in Sec. IV, we explore the conditions for the procedure of the dimensional reduction to be performed successfully and mention the KK compactification mechanism. In Sec. V, conclusions are described.

II. REDUCTION FORMALISM

We first present the formalism of the KK dimensional reduction.

A. Formulation

We use the following notations for indices. The Greek indices run as $\mu, \nu = 0, \ldots, n$ and the Latin indices do like $i, j, \ldots = 1, \ldots, n$, where $n$ is a natural number. The coordinate frame is defined by $x^\mu \equiv \{x^0, x^i\}$. Metric of the spacetime is defined by the following representation:

$$g = g_{\mu\nu} dx^\mu \otimes dx^\nu.$$  \hspace{1cm} (2.1)

Suppose that the metric $g_{\mu\nu}$ of the space-time is static with respect to a coordinate,
we call it as \( x^0 = t \). In general, it does not mean the physical time, but it can be angle of only a coordinate without any meaning of length. However, since we usually use the terminology “static” for the time-independent and irrotational sources of matters, we use the same notation as time. For the static metric in this sense, it satisfies the two conditions
\[
\frac{\partial g_{\mu\nu}}{\partial t} = 0, \quad g_{\mu t} = 0. \tag{2.2}
\]
Here, owing to the nature of being non-stationary, there is no gauge field \( A_\mu = A_t \) like in the case of the KK reduction, and therefore the model has no \( U(1) \) gauge field. In the language of the \((1+3)\) time-space decomposition \[9\], we have only gravitoelectric field. In the original KK reduction, we have gravitoelectromagnetic fields.

We adopt the static metric to the parameterization
\[
V_n : \quad g = g_{\mu\nu} dx^\mu \otimes dx^\nu = -e^{2\gamma} dt \otimes dt + e^{2\sigma} h_{ij} dx^i \otimes dx^j. \tag{2.3}
\]
We assume that \( h_{ij} \) is also static and metric functions are time-independent. In this case, it defines the space-time
\[
V_{n-1} : \quad dt^2 = h_{ij} dx^i \otimes dx^j. \tag{2.4}
\]
Clearly, we see that
\[
V_{n-1} \subset V_n. \tag{2.5}
\]
Our aim here is to reduce the Einstein-Hilbert action in terms of \( n \)-dimensional metric to a lower dimensional action. The action in the \( n \)-dimensional space-time is described as
\[
S_n = \int \sqrt{-g} d^n x \frac{R_n}{2\kappa_n^2}, \tag{2.6}
\]
where \( R_n \) is the Ricci scalar in the \( n \)-dimensional space-time and \( \kappa_n^2 \equiv 8\pi G_n \) with \( G_n \) the gravitational constant in the \( n \)-dimensional space-time. In what follows, the subscription \( n ( (n-1) ) \) denotes the quantities in the \( n \) \(( (n-1) )\)-dimensional space-time. With the metric in Eq. (2.3), the Ricci scalar \( R(V_n) = R_n \) is given by
\[
R_n = e^{-2\sigma} \left[ R_{n-1} + 2(n-2) \triangle \sigma + 2 \triangle \gamma + 2(n-3) \partial_i \sigma \partial^i \gamma \right.
+ (n-2)(n-3) \partial_i \sigma \partial^i \sigma + 2 \partial_i \gamma \partial^i \gamma \]. \tag{2.7}
\]
This is a recursion relation between \( R_n \) and \( R_{n-1} \). For a given set of the metric functions, it is not easy to find \( R_{n-1} \). The only simple solvable case is happen when \( \sigma = 0 \). If we rewrite
this recursion relation to the alternative form

\[ R_n = e^{-2\sigma} + f_{n,n-1}, \]  \hspace{1cm} (2.8)

the following formal solution is obtained for \( R_n \):

\[ R_n = e^{-2\sum_{k=0}^{n-1} \sigma_{n-k}} R_{n-k} + f_{n,n-1} + e^{-2\sigma} f_{n-1,n-2} + e^{-2(\sigma_n + \sigma_{n-1})} f_{n-2,n-3} + \ldots + e^{-2(\sigma_n + \sigma_{n-1} + \ldots + \sigma_{n-k})} f_{n-(k-1),n-k} + \ldots . \]  \hspace{1cm} (2.9)

We can truncate it at \( k = n-4 > 0 \). However, still this form is too much complicated to work out well for our theory. Thus, we leave it as an interesting mathematical problem.

Here, all the derivatives are with respect to \( V_{n-1} \) : \( h_{ij} dx^i \otimes dx^j \) in Eq. (2.4), where

\[ \partial_i f = \frac{\partial f}{\partial x^i}, \hspace{1cm} \Delta = h^{-1/2} \partial_i (h^{ik} h^{1/2} \partial_k). \]  \hspace{1cm} (2.10)

It is necessary to find the determinant of the \( V_n \), form which we have

\[ \sqrt{-g} = e^{\gamma+(n-1)\sigma} \sqrt{h}. \]  \hspace{1cm} (2.11)

Now, provided that in the same terminology as in the KK reduction, \( t \) is called as the compactified direction

\[ t \sim t + 2\pi R. \]  \hspace{1cm} (2.12)

We define the two new metric functions

\[ \lambda = n-3/2 \sigma, \hspace{1cm} \mu = \gamma + \lambda. \]  \hspace{1cm} (2.13)

Consequently, the action in Eq. (2.6) is rewritten in terms of these functions as

\[ S_{n-1} = \int \frac{\sqrt{h}e^{\mu+\lambda}}{2\kappa_n^2} d^{n-1}x \left[ R_{n-1} + 2 \Delta \mu + \frac{2(n-1)}{n-3} \Delta \lambda + \frac{2(n-1)}{n-3} \partial_i \lambda \partial^i \lambda + 2 \partial_i \mu \partial^i \mu \right]. \]  \hspace{1cm} (2.14)

This is the \((n-1)\)-dimensional action with two scalar degrees of freedom. With the partial integration over the static boundary \( \partial M \), the action is simplified as

\[ \int \sqrt{h}e^{\mu+\lambda} \Delta \mu d^{n-1}x = \int_{\partial M} \sqrt{h}h^{ij} n_i \partial_j \mu e^{\mu+\lambda} d^{n-2}x - \int h^{1/2} h^{ij} \partial_j \mu (\partial_i \mu + \partial_i \lambda) e^{\mu+\lambda} d^{n-1}x, \hspace{1cm} (2.15) \]
where $n_i$ is a unit vector normal to the hypersurface defined by $x^{n-1} \equiv \text{constant}$. As a result, we obtain

$$S_{n-1} = \int \sqrt{he^{\mu+\lambda}d^{n-1}x} \left[ \frac{R_{n-1}}{2\kappa_n^2} - \frac{2(n-2)}{\kappa_n^2(n-3)} \partial_i \mu \partial^i \lambda \right]$$

$$+ \kappa_n^{-2} \int_{\partial M} \sqrt{h} h^{ij} n_i e^{\mu+\lambda} d^{n-2}x \left( \partial_j \mu + \frac{n-1}{n-3} \partial_j \lambda \right),$$

(2.16)

### B. Features of the resultant lower-dimensional theories

Furthermore, we define the following two new functions $\Phi$ and $\Psi$ by $\mu + \lambda \equiv \ln \Phi$ and $\mu - \lambda = \ln \Psi$. By using these two scalar fields, the action $S_{n-1}$ can eventually be represented as

$$S_{n-1} = \int \frac{\sqrt{h} d^{n-1}x}{2\kappa_n^2} \left[ \Phi \left( R_{n-1} - \frac{\zeta}{4} \frac{\Phi_i \Phi^i}{\Phi^2} \right) + \frac{\zeta}{4} \frac{\Phi_i \Psi^i \Psi^i}{\Psi^2} \right],$$

(2.17)

where

$$\zeta = \frac{2(n-2)}{n-3}. \quad (2.18)$$

This is a new type of the non minimally coupled Lagrangian of the BD theory with the BD parameter $\omega_{BD} = \frac{\zeta}{4}$. To avoid the ghost, the condition $2 \leq n < 3$ has to be satisfied. However, we here keep the value of $n$ arbitrary.

If we relax the static condition, by adding a gauge field $A_i$ and using $n = D + 1$, $\sigma = \phi$, and $\gamma = (D - 2)\phi$, we acquire the KK action. Our proposed model is a static analogous of the KK unified scenario of electromagnetism and gravity. The symmetries of the action in Eq. (2.17) are summarized as follows.

**Diffeomorphism invariance:** The theory described by the action in Eq. (2.17) is invariant under the $(n-1)$-dimensional diffeomorphism transformations:

$$x^i \rightarrow x^i + \zeta (x^k). \quad (2.19)$$

Under this transformation, the metric tensor $h_{ij}(\subseteq V_{n-1})$ behaves like a rank 2 tensor.

**Gauge transformations along the compactified coordinate:** In addition, this theory is invariant under the following gauge transformations

$$t \rightarrow t + \beta (x^k). \quad (2.20)$$

Thanks to this symmetry, there always exists a wide local freedom to choice the origin along the compactified direction.
Moreover, we investigate the two specific cases.

**Case (i) General relativity (GR) with a massless scalar field:** If we set \( \mu = -\lambda \), we find
\[
S_{n-1} = \int \sqrt{h} d^{n-1}x \left( \frac{R_{n-1}}{2\kappa^2_n} + \frac{2(n-2)}{\kappa^2_n(n-3)} \mu_{,i} \mu^{,i} \right) - \frac{2\kappa_{n-2}}{n-3} \int_{\partial M} \sqrt{h} h^{ij} n_i \mu_{,j} d^{n-2}x. \tag{2.21}
\]
It is just the GR in the presence of a massless scalar field with the right sign of the kinetic term.

**Case (ii) The KK theory in the Einstein frame without the U(1) gauge field:** With \( n = D + 1, \sigma = \phi, \gamma = (D - 2)\phi \), and \( A_\mu = 0 \), we get
\[
S_{n-1} = \int d^Dx \sqrt{h} \phi \left( \frac{R_D}{2\kappa^2_D} - \frac{3}{2} (D-1)(D-2) \partial_i \phi \partial^i \phi \right). \tag{2.22}
\]
In this case, the scalar field \( \phi \) is called the dilaton. To understand its meaning in a better way, we assume that there exists an \( S^1 \) in the vicinity of any point \( x^\mu \) of \( V_{n-1} \). The reduced metric on \( S^1 \) reads
\[
ds^2|_{S^1} = e^{2\gamma} dt^2. \tag{2.23}
\]
The size of the circle is defined by
\[
\int_{S^1} \equiv \int_0^{2\pi R} ds_{S^1} = 2\pi R e^\gamma \tag{2.24}
\]
According to the terminology in the KK theory, we observe that the effective radius of \( S^1 \) at any point \( x^\mu \) is \( R e^\gamma \). Here, the dilaton field \( \gamma \) defines the size of the \( S^1 \), similar to that in the KK theory.

**C. Equations of motion**

To find the equation of motion of the gravitational field for the model in Eq. (2.17), we perform the variation of \( S_{n-1} \) with respect to \( h^{ij} \). The equations of motions are derived as
\[
\rho_{ij} - \frac{1}{2} \rho_{ii} = \frac{1}{\Phi} \omega_{BD} \Phi \left( 2\Psi_{,i} \Psi_{,j} - h_{ij} \Psi_{,k} \Psi_{,k} \right)
+ \frac{\omega_{BD}}{\Phi^2} \left( \Phi_{,i} \Phi_{,j} - \frac{1}{2} h_{ij} \Phi_{,k} \Phi_{,k} \right) + \frac{1}{\Phi} (\Phi_{,i,j} - h_{ij} \triangle \Phi), \tag{2.25}
\]
\[
\triangle \Phi = \frac{\kappa^2_n}{3 + 2\omega} \left[ \omega_{BD} (3-n) \frac{2}{4} \left( \frac{\zeta \partial_k \Psi \partial^k \Psi}{\Psi^2} + \frac{\partial_k \Phi \partial^k \Phi}{\Phi^2} \right) + (2-n) \frac{\triangle \Phi}{\Phi} \right], \tag{2.26}
\]
\[
\frac{1}{\sqrt{h}} \partial_i \left( \frac{\sqrt{h} h^{ij} \partial_j \Phi}{\Psi^2} \right) + \frac{2\Phi}{\Psi^3} \partial_i \Psi \partial^i \Psi = 0. \tag{2.27}
\]
Here, \( r_{ij} = R_{ij}(h) \) corresponds to the Ricci tensor, which is constructed by \( h_{ij} \), \( r = R(h) = R(h)_{ij} \), and \( \omega_{BD} = \frac{c}{4} \).

### III. Perturbations of AdS\(_n\)

In this section, we study the non-linear stability of the AdS\(_n\) in the framework of our reduced scalar-tensor model \( \{h_{ij}, \Psi, \Phi\} \). The description “AdS\(_n\)” denotes an exact solution, the so-called “the anti-de Sitter solution”, of the \( n \)-dimensional Einstein gravity with negative cosmological constant. The end point of the instability of AdS is a Schwarzschild-AdS black hole, which is asymptotically stable. The (in)stability of AdS under the perturbations has recently been investigated from different points of the view \(^{10}\).

We describe the unperturbed (i.e., background) AdS solution by the following set of the functions

\[
\Phi = \Phi^{(0)}(y), \quad \Psi = \Psi^{(0)}(y), \quad h_{ij}^{(0)} = \text{diag}(1, 1, \ldots, 1), \quad (n-1) \text{ times}, \quad (3.1)
\]

where \( y(>0) \) is the radial coordinate of the half plane (one patch description of the AdS solution) and the metric is \((n-1)\)-static. The perturbations are expressed as

\[
\Psi^{(1)} = \Psi^{(0)} + \delta \Psi(y, \vec{x}),
\]

\[
\Phi^{(1)} = \Phi^{(0)} + \delta \Phi(y, \vec{x}),
\]

\[
h_{ij}^{(1)} = h_{ij}^{(0)} + \delta h_{ij}(y, \vec{x}),
\]

\[
\delta h_{ij}(y, \vec{x}) = \text{diag} \left[ \delta h_{yy}(y, \vec{x}), \delta h_{11}(y, \vec{x}), \ldots, \delta h_{(n-2)(n-2)}(y, \vec{x}) \right]. \quad (3.5)
\]

By computing the \( \delta S_{n-1} = 0 \), we find the perturbations equations

\[
\frac{\zeta \Phi^{(0)}}{8} \left( \partial_i \ln \Psi^{(0)} \partial^i \ln \Psi^{(0)} - \partial_i \ln \Phi^{(0)} \partial^i \ln \Phi^{(0)} \right) h_{ij}^{(0)} \delta h_{ij}^{(0)}
\]

\[
- \partial_k \Phi^{(0)} \left( h_{ij}^{(0)} \delta \Gamma_k^{ij} - h_{ik}^{(0)} \delta \Gamma_{mj}^{m} \right) = 0,
\]

\[
\frac{1}{2} \left( \partial_i \Psi^{(0)} \partial^i \Psi^{(0)} - \partial_i \Phi^{(0)} \partial^i \Phi^{(0)} \right) \delta \Phi - \Phi^{(0)} \partial_i (\ln \Phi^{(0)}) \delta (\partial^i (\ln \Phi)) = 0,
\]

\[
\partial_i (\ln \Psi^{(0)}) \delta \left[ \partial^i (\ln \Psi) \right] = 0,
\]

with

\[
\delta \Gamma_{ij}^{k} = \frac{1}{2} \delta^{kd} \left( \partial_j (\delta h_{id}) + \partial_i (\delta h_{jd}) - \partial_d (\delta h_{ij}) \right). \quad (3.9)
\]
Here, we have simplified the equations for the AdS solution due to the facts that for the AdS solution, we have $r_{ij} = 0$, $r = 0$, and $h = 1$.

Equations (3.7) and (3.8) for the AdS solution give us the exact solutions in terms of the unperturbed fields

$$\delta \Psi(y, \vec{x}) = c_0(\vec{x})\Psi^{(0)}(y),$$

$$\delta \Phi(y, \vec{x}) = c_1(\vec{x}) \exp \left[ \frac{1}{2} \int dy \frac{\partial_i \Psi^{(0)} \partial_i \Phi^{(0)}}{\Phi^{(0)^2}} + \frac{\partial_i \Phi^{(0)} \partial_i \Phi^{(0)}}{\Phi^{(0)^2}} (\Phi^{(0)})^2 + \partial_y \ln \Phi^{(0)} \right].$$

The form of Eq. (3.6) reads

$$\frac{\zeta}{8} \left[ \frac{\partial_i \Psi^{(0)} \partial_i \Psi^{(0)}}{\Psi^{(0)^2}} - \frac{\partial_i \Phi^{(0)} \partial_i \Phi^{(0)}}{\Phi^{(0)^2}} \right] \delta_{ij} \delta h^{ij} - [\partial_y (\delta h_{iy} - \delta h_{ii}) + \partial_x (\delta h_{xy})] = 0.$$  

(3.12)

Thanks to the planar symmetry, we use the Fourier decomposition of the different components of the metric as

$$\delta h_{ij}(y, \vec{x}) = \int d^{n-2}k e^{i\vec{k} \cdot \vec{x}} \tilde{\delta} h_{ij}(y, \vec{k}).$$

(3.13)

Substituting this expression into Eq. (3.12), we find

$$g(y) \delta_{ij} \tilde{\delta} h^{ij}(y, \vec{k}) - [\partial_y (\tilde{\delta} h_{iy}(y, \vec{k}) - \tilde{\delta} h_{ii}(y, \vec{k})) + ik \tilde{\delta} h_{xy}(y, \vec{k})] = 0,$$

where

$$g(y) = \frac{\zeta}{8} \left[ \frac{\partial_i \Psi^{(0)} \partial_i \Psi^{(0)}}{\Psi^{(0)^2}} - \frac{\partial_i \Phi^{(0)} \partial_i \Phi^{(0)}}{\Phi^{(0)^2}} \right].$$

(3.15)

With diagonal metric $\delta h_{ij}(y, \vec{k})$, we obtain

$$g(y) \delta_{ij} \tilde{\delta} h^{ij}(y, \vec{k}) + \partial_y (\tilde{\delta} h_{ii}(y, \vec{k})) = 0.$$

(3.16)

A simple integration leads to

$$\delta_{ij} \tilde{\delta} h^{ij}(y, \vec{k}) = c_2(\vec{k}) e^{-\int g(y) dy}.$$  

(3.17)

Consequently, we get

$$\delta h_{ij}(y, \vec{x}) = e^{-\int g(y) dy} \int d^{n-2}k e^{i\vec{k} \cdot \vec{x}} c_2(\vec{k}) = e^{-\int g(y) dy} \tilde{c}_{ij}(\vec{x}).$$

(3.18)

To complete the stability of the AdS solution, we make use of the relations $\Psi^{(0)} = y^{-1}$, $\Phi^{(0)} = y^{2-n}$, and $g(y) = \frac{\zeta(1-n^2+4n-3)}{8(2-n)} y^{-1}$. As a result, we acquire

$$\delta h_{ij}(y, \vec{x}) = y^{\frac{\zeta(1-n^2+4n-3)}{8(2-n)}} \tilde{c}_{ij}(\vec{x}).$$

(3.19)
In order for the AdS solution to be stable, the perturbations have to tend to zero at the asymptotic boundary \( y \to 0 \), where \( |\tilde{c}_{ij}(\vec{x})| \to 0 \). Thus, the following conditions are required
\[
n > 1.
\] (3.20)

However, since in our reduction, we have \( n > 5 \), for this range we find
\[
\lim_{y \to 0} \delta h_{ij}(y, \vec{x}) = \tilde{c}_{ij}(\vec{x}) \lim_{y \to 0} y^{\frac{4(n-2)}{n^2-4n+5}} = 0,
\] (3.21)
\[
\lim_{y \to 0} \delta \Psi(y, \vec{x}) = c_0(\vec{x}) y^{-1} = \infty,
\] (3.22)
\[
\lim_{y \to 0} \delta \Phi(y, \vec{x}) = c_1(\vec{x}) y^{\frac{1}{2} \frac{n^2-4n+5}{2-n}} = \infty.
\] (3.23)

Accordingly, we conclude that \( AdS_n \) is unstable in our scalar model. This result is supported by different methods in Ref. [10]. Indeed, \( AdS_{d+1} \), where \( d \geq 3 \), is unstable against a large class of arbitrarily small perturbations. Our stability analysis has been performed for one patch of the AdS space. To complete the analysis of the AdS solution, we suppose that the unperturbated \( AdS_n \) is parametrized by the following set of the functions
\[
\Phi^{(0)}(x) = \left( \frac{l}{\cos x} \right)^{n-2},
\] (3.24)
\[
\Psi^{(0)}(x) = \left( \frac{l}{\cos x} \right),
\] (3.25)
\[
h^{(0)}_{ij} = \text{diag}(1, \sin^2 x \omega_{n-2}),
\] (3.26)
where \( (t, x) \in \mathcal{R} \times [0, \pi/2] \), and \( \omega_{n-2} \) is the metric of a unit sphere \( S^{n-2} \). The perturbations are expressed as
\[
\Psi^{(1)} = \Psi^{(0)} + \delta \Psi(t, x, \omega),
\] (3.27)
\[
\Phi^{(1)} = \Phi^{(0)} + \delta \Phi(t, x, \omega),
\] (3.28)
\[
h^{(1)}_{ij} = h^{(0)}_{ij} + \delta h_{ij}(t, x, \omega),
\] (3.29)
\[
\delta h_{ij}(t, x, \omega) = \text{diag} \left[ \delta h_{xx}(t, x, \omega), \delta h_{11}(t, x, \omega), \ldots, \delta h_{(n-2)(n-2)}(t, x, \omega) \right].
\] (3.30)

We represent all the perturbations in terms of the harmonics \( Y_k(\omega) \) on the \( S^{n-2} \) as
\[
\Delta_\omega Y_k(\omega) = -l(l + d - 2) Y_k(\omega).
\] (3.31)

Hence, we rewrite Eqs. (3.27)–(3.29) to
\[
\Psi^{(1)} = \Psi^{(0)} + \Sigma_k \delta \Psi_k(t, x) Y_k(\omega),
\] (3.32)
\[
\Phi^{(1)} = \Phi^{(0)} + \Sigma_k \delta \Phi_k(t, x) Y_k(\omega),
\] (3.33)
\[
h^{(1)}_{ij} = h^{(0)}_{ij} + \Sigma_k \delta h^k_{ij}(t, x) Y_k(\omega).
\] (3.34)
The solution of perturbations is given by

$$\delta \Psi_k(t, x) = c_k^+(t) \frac{l \sin x}{\cos^2 x} \cdot$$

(3.35)

This result is valid for the electromagnetic and gravitational perturbations. If we define $y \equiv \pi/2 - x$, Eq. (3.35) is described as

$$\delta \Psi_k(t, y) = c_k^+(t) \frac{l \cos y}{\sin y} \cdot$$

(3.36)

We need to obtain $\lim_{y \to 0} \delta \Psi_k(t, y) \to 0$. In fact, we acquire

$$\lim_{y \to 0} \delta \Psi_k(t, y) = c_k^+(t) \exp \left[ \frac{l}{2y^2} + \frac{l}{6} + \frac{l}{2} \left( \ln y - \ln 2 - \frac{1}{2} \right) + \mathcal{O}(y^2) \right] \to \infty .$$

(3.37)

Furthermore, we get

$$\delta \Phi_k(t, x) = c_k^-(t) \exp \left[ \frac{1}{2} \int dx \frac{\partial \Phi^{(0)} \partial \Phi^{(0)}}{(\Phi^{(0)})^2} + \frac{\partial \Phi^{(0)} \partial \Phi^{(0)}}{(\Phi^{(0)})^2} \right]$$

$$= c_k^-(t) \left( \cos x \right)^{-1} n^{\frac{3n+4}{n-2}} = c_k^-(t) \left( \sin y \right)^{-1} n^{\frac{3n+4}{n-2}} .$$

(3.38)

The only possibility to have $\lim_{y \to 0} \delta \Phi_k(t, y) = 0$ is $n < 2$, the condition of which is physically unacceptable. It is possible to integrate and find $\delta h^k_{ij}(t, x)$. Thus, the AdS solution is unstable under the non-linear perturbations in terms of $\Psi$. This consequence proves that in a similar sense, the former results on the instability of the AdS under the non-linear perturbations are obtained.

IV. SUCCESSFUL REDUCTIONS

A. General description

Now, we suppose that the metric $g_{\mu \nu}$ is defined in the $(D + p)$-dimensional space-time as

$$V_{D+p} = \bigcup_{i=1}^{p} V_i \oplus \bigcup_{j=p}^{D+p} V_j ,$$

(4.1)

$$ds_{D+p}^2 = g_{\mu \nu} dx^\mu \otimes dx^\nu = -\Sigma_{j=1}^{p} e^{2\gamma_j} dt_j \otimes dt_j + e^{2\Sigma_{j=1}^{p} \sigma_j} h_{AB} dx^A \otimes dx^B ,$$

(4.2)

$$\{A, B\} = \{p, p + 1, ..., p + D\}, \quad \alpha_j = 2 \left( \Sigma_{i=1}^{j} \gamma_i + \Sigma_{i=1}^{j-1} \sigma_i \right) .$$

(4.3)
We define the concept of \( p \)-static metric. We call that the metric \( g_{\mu \nu} \) of the space-time is \( p \)-static, if there is a set of coordinates \( x^a \) with \( 1 \leq a \leq p \) such that

\[
g_{\mu,a} = 0, \quad \frac{\partial g_{\mu \nu}}{\partial x^a} = 0. \tag{4.4}
\]

Equivalently, there exists a set of the commutative Killing vectors \( \zeta^a \) as

\[
\zeta^a \zeta^b - \zeta^b \zeta^a = 0. \tag{4.5}
\]

These vector fields are generators of the symmetries of the metric \( g_{\mu \nu} \).

**B. Concrete examples**

We examine the following two examples.

**Example (i) Cosmological Bianchi-I models:** The Bianchi-I metric in four-dimensions is \( p = 3 \)-static in \( n = D + p = 4 \)

\[
ds^2 = -dt^2 + \Sigma_{i=1}^3 A_i(t)^2 (dx^i)^2. \tag{4.6}
\]

This is because we have a set of coordinates \( x^a = \{x^1, x^2, x^3\} \) satisfying the \( p \)-static condition in Eq. (4.4). The Killing vectors correspond to these coordinates are

\[
\zeta^a = \{\partial_x, \partial_y, \partial_z\}. \tag{4.7}
\]

These vectors meets

\[
[\zeta^x, \zeta^y] = [\zeta^y, \zeta^z] = [\zeta^z, \zeta^x] = 0. \tag{4.8}
\]

**Example (ii) Static-spherically symmetric space-time:** The generally static-spherically symmetric metric in four-dimensions is \( p = 2 \)-static in \( n = D + p = 4 \)

\[
ds^2 = -A(r)dt^2 + B(r)dr^2 + C(r)(d\theta^2 + \sin \theta d\varphi^2), \tag{4.9}
\]

because we have a set of coordinates \( x^a = \{t, \varphi\} \) meeting the \( p \)-static condition in Eq. (4.4). The set of commutative Killing vectors are

\[
\zeta^1 = \partial_t, \quad \zeta^2 = \partial_\varphi. \tag{4.10}
\]

Using Eq. (4.2), we are able to reduce the action \( S_{p+D} \) to the lower \( D \) dimensional one. We assume that by Eq. (4.2), the metric \( h_{AB} \) is static with respect to all the “time” coordinates \( t_j \) as follows

\[
h_{A,t_j} = 0, \quad \frac{\partial h_{AB}}{\partial t_j} = 0, \quad 1 \leq j \leq p. \tag{4.11}
\]
In the so-called string frame (the Jordan frame), when $\sigma_l = 0$ with $l = 1, \ldots, p$ the level of $p$-static, the Ricci scalar of $V_{D+p}$ and that of $V_D$ satisfy the equation

$$R_{D+p} = R_D + 2\Gamma_{\cdot A}\Gamma^A, \quad \Gamma_{\cdot A}\Gamma^A = \sum_{l=1}^{p} \partial_{\cdot A}\gamma_l \partial^{\cdot A}\gamma_l .$$

(4.12)

Hence, we obtain the following reduction from $S_{D+p}$ to $S_D$

$$S_D = \frac{S_{D+p}}{\prod_{j=1}^{p} dt_j} = \int \frac{\sqrt{h}d^Dx}{2\kappa_{D+p}^2} \left( R_D + 2\Gamma_{\cdot A}\Gamma^A \right) ,$$

(4.13)

where in the second equality, the Lagrangian function in $S_D$ is independent of the coordinates $t_j$ with $1 \leq j \leq p$. Another equivalent form is given by

$$S_D = \int \frac{e^{\sum_{j=1}^{p} \alpha_j} \sqrt{h}d^Dx}{2\kappa_D^2} \left( R_D + 2\Gamma_{\cdot A}\Gamma^A \right), \quad \kappa_D^2 = \frac{\kappa_{D+p}^2}{\prod_{j=1}^{p} dt_j} .$$

(4.14)

If $\sum_{j=1}^{p} \alpha_j = 0$, the model becomes a good and simple example. In this case, since the action of $S_D$ equals to the Einstein gravity plus the auxiliary scalar field, all live on the $D$-dimensional sub space-time

$$S_D = \int \frac{\sqrt{h}d^Dx}{2\kappa_D^2} \left( R_D + 2\Gamma_{\cdot A}\Gamma^A \right).$$

(4.15)

Thus, we conclude that if we start with a curved spec-time in $(D + p)$-dimensions, and if this space-time is $p$-static, after $p$-times reduction, by the suitable choices of the metric functions, we obtain the Einstein gravity with the scalar field in $D$-dimensions.

C. Compactification mechanism

For $p$-static metrics, provided that the following compactification has already been carried out

$$t_j \rightarrow t_j + \beta_j(x^k), \quad 1 \leq j \leq p .$$

(4.16)

This indicates that we have $p$ gauge freedoms to select the origin along the compactified directions for the simple reduction of $V_{D+p} \rightarrow V_D$.

V. CONCLUSIONS

In the present paper, we have presented the generalized formulation of the KK dimensional reduction from higher-dimensional space-time. Particularly, we have explored the
resultant scalar-tensor theories. As a result, we have found the AdS background solution. Moreover, we have investigated the stability of the AdS solution by examining the perturbations from the background AdS solution. Furthermore, we have derived the conditions that the dimensional reduction can successfully be realized. Also, the consequence of the KK compactification has been stated.

We finally mention that according to Ref. [11], the quantum fluctuations violate the weak equivalence principle in the BD theory. This means that the Jordan frame can no longer be interpreted as physical. In our model, the AdS space-time exists, and it is unstable under the perturbations.

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