ALGEBRAIC COLIMIT CALCULATIONS IN HOMOTOPY THEORY USING FIBRED AND COFIBRED CATEGORIES

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ABSTRACT. Higher Homotopy van Kampen Theorems allow the computation as colimits of certain homotopical invariants of glued spaces. One corollary is to describe homotopical excision in critical dimensions in terms of induced modules and crossed modules over groupoids. This paper shows how fibred and cofibred categories give an overall context for discussing and computing such constructions, allowing one result to cover many cases. A useful general result is that the inclusion of a fibre of a fibred category preserves connected colimits. The main homotopical application are to pairs of spaces with several base points, but we also describe briefly the situation for triads.

1. Introduction

One of our aims is to give the framework of fibred and cofibred categories to certain colimit calculations of algebraic homotopical invariants for spaces with parts glued together: the data here is information on the invariants of the parts, and of the gluing process.

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The second aim is to advertise the possibility of such calculations, which are based on various Higher Homotopy van Kampen Theorems (HHvKTs)\(^2\), proved in 1978-87. These are of the form that a *homotopically defined* functor

\[ \Pi : \text{(topological data)} \rightarrow \text{(algebraic data)} \]

preserves certain colimits [BH78, BH81b, BL87a], and where the algebraic data contains non-Abelian information.

The antecedent for dimension 1 of such functors as \( \Pi \) is the fundamental group \( \pi_1(X,a) \) of based spaces: the van Kampen theorem in its form due to Crowell, [Cro59], gives that this \( \pi_1 \) preserves certain pushouts.

The next advance was the fundamental groupoid functor \( \pi_1(X,A) \) from spaces \( X \) with a *set* \( A \) of base points to groupoids: groupoids in effect carry information in dimensions 0 and 1. For example, the groupoid van Kampen theorem [Bro67, Bro06] gives the fundamental group of the circle as the infinite cyclic group \( \mathbb{C} \) obtained in the category of groupoids from the (finite) groupoid \( I \cong \pi_1([0,1],\{0,1\}) \) by identifying 0, 1. That is, we have the analogous pushouts

\[
\begin{array}{ccc}
\{0,1\} & \rightarrow & \{0\} \\
\downarrow & & \downarrow & & \downarrow \\
[0,1] & \rightarrow & S^1 & \rightarrow & \pi_1 \\
\downarrow & & \downarrow & & \downarrow \\
I & \rightarrow & C
\end{array}
\]

(1)

the first in the category of spaces, the second in the category of groupoids, and \( \pi_1 \) takes the first with the appropriate sets of base points, to the second. The aim is to have a similar argument at the module level for \( \pi_n, n > 1 \).

One of the problems of obtaining such results in higher homotopy theory is that low dimensional identifications of spaces usually affect high dimensional homotopy invariants. To cope with this fact, the algebraic data for the values of the functor \( \Pi \) must have structure interacting from low to high dimensions, in order to model how the spaces are glued together.

An example of such dimensional interaction is the operation of the fundamental group on the higher homotopy groups, which fascinated the early workers in homotopy theory (private communication, J.H.C. Whitehead, 1957). This operation can be seen as necessitated by the dependence of these groups on the base point, but is somewhat neglected in algebraic topology, perhaps for perceived deficiencies in modes of calculations.

A recognised reason for the difficulty of traditional invariants such as homotopy groups in dealing with gluing spaces, and so obtaining colimit calculations, is the failure of excision. If \( X \) is the union of open sets \( U, V \) with intersection \( W \) then the inclusion of pairs

\[ \varepsilon : (V,W) \rightarrow (X,U) \]

\(^2\)This name for certain generalisations of van Kampen’s Theorem was recently suggested by Jim Stasheff.
does not induce an isomorphism of relative homotopy groups as it does for singular homology. However there is a residue of homotopical excision which can be deduced from a HHvKT, as in \cite{BH81b}; it requires connectivity conditions and gives information in only the critical dimension, but this information involves the additional structure of operations. We state it at this stage for a single base point, as follows:

**Homotopical Excision Theorem (HET)**

If $U, V$ and $W = U \cap V$ are path connected with base point $a \in W$, and $(V, W)$ is $(n - 1)$-connected, $n \geq 3$, then $(X, U)$ is $(n - 1)$-connected and the excision map

$$\varepsilon_* : \pi_n(V, W, a) \to \pi_n(X, U, a)$$

presents the $\pi_1(U, a)$-module $\pi_n(X, U, a)$ as induced by the morphism of fundamental groups $\pi_1(W, a) \to \pi_1(U, a)$ from the $\pi_1(W, a)$-module $\pi_n(V, W, a)$. The same holds for $n = 2$ with ‘module’ replaced by ‘crossed module’.

We show that this theorem implies the classical Relative Hurewicz Theorem, (Corollary 9.11) for which in more usual terms see for example \cite{Whi78}. The case $n = 2$ implies a deep theorem of Whitehead on free crossed modules (Corollary 9.8) (of which \cite{Bro80} explains the original proof), and indeed allows the more general calculation of $\pi_2(X \cup CA, X, a)$ as a crossed $\pi_1(X, a)$-module (Corollary 9.13). That result is applied in \cite{BW95, BW03} to calculate crossed modules representing some homotopy 2-types of mapping cones of maps of classifying spaces of groups.

The notion of **inducing** used here is both an indication of the existence of some universal properties in higher homotopy theory, and also of the tools needed to exploit such properties as there are. In the case $n > 2$ the inducing construction is well known for modules over groups: if $M$ is a module over the group $G$, and $f : G \to H$ is a morphism of groups, then the induced module $f_*(M)$ is isomorphic to $M \otimes_{\mathbb{Z}G} \mathbb{Z}H$. For $n = 2$, the analogous construction is non-Abelian.

We give the approach of fibred and cofibred categories to this inducing construction. Also the following simple example of algebraic modelling of a gluing process suggests a further generalisation is needed. Consider the following maps involving the $n$-sphere $S^n$, the first an inclusion and the second an identification:

$$S^n \xrightarrow{i} S^n \vee [0, 1] \xrightarrow{p} S^n \vee S^1. \quad (2)$$

Here $n \geq 2$, the base point $0$ of $S^n$ is in the second space identified with $0$, and the map $p$ identifies $0, 1$ to give the circle $S^1$.

This clearly requires an algebraic theory dealing with many base points, and so a generalisation from groups to groupoids. The geometry of the base points and their interconnections can then play a role in the theory and applications.

This programme has proved successful for the algebraic data of crossed complexes over groupoids, or equivalent structures, and for $cat^n$-groups, or, equivalently, crossed $n$-cubes of groups. In both categories there is a HHvKT asserting that a homotopically defined functor $\Pi$ preserves certain colimits, and this leads to new calculations of homotopical invariants.
The calculation of colimits in these algebraic categories usually requires working up in dimensions. So it is useful to consider for these hierarchical structures the forgetful functors from higher complex structures to lower structures.

The prototype again is groupoids, with the object functor from groupoids to sets. The fibre of this functor over a set \( I \) is written \( \text{Gpd}_I \): the objects of this category are groupoids with object set \( I \) and morphisms are functors which are the identity on \( I \). This fibre has nice properties, and is what is called protomodular; in rough terms it has properties analogous to those of the category of groups. In particular, there is a notion of normal subgroupoid, and of coproduct, so that colimits can be calculated in \( \text{Gpd}_I \) as quotients of a coproduct by a normal subobject.

However the whole category \( \text{Gpd} \) is of interest for homotopical modelling. The functor \( \text{Ob} : \text{Gpd} \to \text{Set} \) has the properties of being a fibration and a cofibration in the sense of Grothendieck, \([\text{Gro}68]\). So a function \( u : I \to J \) of sets gives rise to a pair of functors \( u^* : \text{Gpd}_J \to \text{Gpd}_I \), known often as pullback, and \( u_* : \text{Gpd}_I \to \text{Gpd}_J \), which could be called ‘pushin’ or ‘push forward’, such that \( u_* \) is left adjoint to \( u^* \). So for \( G \in \text{Gpd}_I \) we have a morphism \( u' : G \to u_* G \) over \( u \) with a universal property which is traditionally called ‘cocartesian’, and it is also said that \( u_* G \) is the object induced from \( X \) by \( u \).

It is interesting to see that the main parts of the pattern for groupoids goes over to the general case. Our categorical results are for a fibration of categories \( \Phi : \text{X} \to \text{B} \). The first and a main result (Theorem 2.7) is that the inclusion \( X_I \to X \) of a fibre \( X_I \) of \( \Phi \) preserves colimits of connected diagrams. Our second set of results relates pushouts in \( X \) to the construction of the functors \( u_* : X_I \to X_J \) for \( u \) a morphism in \( \text{B} \) (Proposition 4.2). Finally, we show how the combination of these results uses the computation of colimits in \( \text{B} \) and in each \( X_I \) to give the computation of colimits in \( X \) (Theorem 4.4).

These general results are developed in the spirit of ‘categories for the working mathematician’ in sections 2 to 4. We illustrate the use of these results for homotopical calculations not only in groupoids (section 5), but also for crossed modules (section 7) and for modules, in both cases over groupoids. Finally we give a brief account of some relevance to crossed squares (section 10), as an indication of a more extensive theory, and in which these ideas need development.

We are grateful to Thomas Streicher for his Lecture Notes \([\text{Str}99]\), on which the following account of fibred categories is based, and for helpful comments leading to improvements in the proofs. For further accounts of fibred and cofibred categories, see \([\text{Gra}66, \text{Bor}94, \text{Vis}04]\) and the references there. The first paper gives analogies between fibrations of categories and Hurewicz fibrations of spaces.

2. Fibrations of categories

We recall the definition of fibration of categories.

2.1. Definition. Let \( \Phi : \text{X} \to \text{B} \) be a functor. A morphism \( \varphi : Y \to X \) in \( \text{X} \) over \( u := \Phi(\varphi) \) is called cartesian if and only if for all \( v : K \to J \) in \( \text{B} \) and \( \theta : Z \to X \) with \( \Phi(\theta) = uv \) there is a unique morphism \( \psi : Z \to Y \) with \( \Phi(\psi) = v \) and \( \theta = \varphi \psi \).
This is illustrated by the following diagram:

\[
\begin{array}{c}
Z \xrightarrow{\psi} Y \xrightarrow{\varphi} X \\
\downarrow{\theta} \quad \varphi \downarrow \\
K \xrightarrow{v} J \xrightarrow{u} I
\end{array}
\]

\[
\Phi
\]

It is straightforward to check that cartesian morphisms are closed under composition, and that \(\varphi\) is an isomorphism if and only if \(\varphi\) is a cartesian morphism over an isomorphism.

A morphism \(\alpha : Z \to Y\) is called \textit{vertical} (with respect to \(\Phi\)) if and only if \(\Phi(\alpha)\) is an identity morphism in \(\mathcal{B}\). In particular, for \(I \in \mathcal{B}\) we write \(X_I\), called the \textit{fibre over} \(I\), for the subcategory of \(\mathcal{X}\) consisting of those morphisms \(\alpha\) with \(\Phi(\alpha) = \text{id}_I\).

2.2. Definition. The functor \(\Phi : \mathcal{X} \to \mathcal{B}\) is a \textit{fibration} or \textit{category fibred over} \(\mathcal{B}\) if and only if for all \(u : J \to I\) in \(\mathcal{B}\) and \(X \in \mathcal{X}_I\) there is a cartesian morphism \(\varphi : Y \to X\) over \(u\): such a \(\varphi\) is called a \textit{cartesian lifting} of \(X\) along \(u\).

Notice that cartesian liftings of \(X \in \mathcal{X}_I\) along \(u : J \to I\) are unique up to vertical isomorphism: if \(\varphi : Y \to X\) and \(\psi : Z \to X\) are cartesian over \(u\), then there exist vertical arrows \(\alpha : Z \to Y\) and \(\beta : Y \to Z\) with \(\varphi \alpha = \psi\) and \(\psi \beta = \varphi\) respectively, from which it follows by cartesianness of \(\varphi\) and \(\psi\) that \(\alpha \beta = \text{id}_Y\) and \(\beta \alpha = \text{id}_Z\) as \(\psi \beta \alpha = \varphi \alpha = \psi \psi \text{id}_Y\) and similarly \(\varphi \beta \alpha = \varphi \text{id}_Y\).

2.3. Example. The forgetful functor, \(\text{Ob} : \text{Gpd} \to \text{Set}\), from the category of groupoids to the category of sets is a fibration. We can for a groupoid \(G\) over \(I\) and function \(u : J \to I\) define the cartesian lifting \(\varphi : H \to G\) as follows: for \(j, j' \in J\) set

\[
H(j, j') = \{(j, g, j') \mid g \in G(uj, uj')\}
\]

with composition

\[
(j_1, g_1, j'_1)(j, g, j') = (j_1, g_1g, j'),
\]

with \(\varphi\) given by \(\varphi(j, g, j') = g\). The universal property is easily verified. The groupoid \(H\) is usually called the \textit{pullback} of \(G\) by \(u\). This is a well known construction (see for example [Mac05, §2.3], where pullback by \(u\) is written \(u^{\perp}\)).

2.4. Definition. If \(\Phi : \mathcal{X} \to \mathcal{B}\) is a fibration, then using the axiom of choice for classes we may select for every \(u : J \to I\) in \(\mathcal{B}\) and \(X \in \mathcal{X}_I\) a cartesian lifting of \(X\) along \(u\)

\[
u^X : u^*X \to X.
\]

Such a choice of cartesian liftings is called a \textit{cleavage} or \textit{splitting} of \(\Phi\).
If we fix the morphism \( u : J \to I \) in \( \mathcal{B} \), the splitting gives a so-called *reindexing functor* 

\[
u^* : X_I \to X_J
\]

defined on objects by \( X \mapsto u^* X \) and the image of a morphism \( \alpha : X \to Y \) is \( u^* \alpha \) the unique vertical arrow commuting the diagram:

\[
\begin{array}{ccc}
u^* X & \xrightarrow{u^* X} & X \\
\downarrow \alpha & & \downarrow \alpha \\
u^* Y & \xrightarrow{u^* Y} & Y
\end{array}
\]

We can use this reindexing functor to get an adjoint situation for each \( u : J \to I \) in \( \mathcal{B} \).

**2.5. Proposition.** Suppose \( \Phi : X \to \mathcal{B} \) is a fibration of categories, \( u : J \to I \) in \( \mathcal{B} \), and a reindexing functor \( u^* : X_I \to X_J \) is chosen. Then there is a bijection

\[
X_J(Y, u^* X) \cong X_u(Y, X)
\]

natural in \( Y \in X_J \), \( X \in X_I \) where \( X_u(Y, X) \) consists of those morphisms \( \alpha \in X(Y, X) \) with \( \Phi(\alpha) = u \).

**Proof** This is just a restatement of the universal properties concerned.

In general for composable maps \( u : J \to I \) and \( v : K \to J \) in \( \mathcal{B} \) it does not hold that

\[
v^* u^* = (uv)^*
\]

as may be seen with the fibration of Example 2.3. Nevertheless there is a natural equivalence \( c_{u,v} : v^* u^* \cong (uv)^* \) as shown in the following diagram in which the full arrows are cartesian and where \( (c_{u,v})_X \) is the unique vertical arrow making the diagram commute:

\[
\begin{array}{ccc}
u^* u^* X & \xrightarrow{v^* u^* X} & u^* X \\
\downarrow (c_{u,v})_X \cong & & \downarrow u^* X \\
(uv)^* X & \xrightarrow{(uv)^* X} & X
\end{array}
\]

Let us consider this phenomenon for our main examples:
2.6. Example. 1.- Typically, for $\Phi_B = \partial_1 : B^2 \to B$, the fundamental fibration for a category with pullbacks, we do not know how to choose pullbacks in a functorial way.

2.- In considering groupoids as a fibration over sets, if $u : J \to I$ is a map, we have a reindexing functor, also called pullback, $u^* : \text{Gpd}_I \to \text{Gpd}_J$. We notice that $v^*u^*Q$ is naturally isomorphic to, but not identical to $(uv)^*Q$. □

A result which aids understanding of our calculation of pushouts and some other colimits of groupoids, modules, crossed complexes and higher categories is the following. Recall that a category $C$ is connected if for any $c, c' \in C$ there is a sequence of objects $c_0 = c, c_1, \ldots, c_{n-1}, c_n = c'$ such that for each $i = 0, \ldots, n-1$ there is a morphism $c_i \to c_{i+1}$ or $c_{i+1} \to c_i$ in $C$. The sequence of morphisms arising in this way is called a zig-zag from $c$ to $c'$ of length $n$.

2.7. Theorem. Let $\Phi : X \to B$ be a fibration, and let $J \in B$. Then the inclusion $i_J : X_J \to X$ preserves colimits of connected diagrams.

Proof We will need the following diagrams:

Let $T : C \to X_J$ be a functor from a small connected category $C$ and suppose $T$ has a colimit $L \in X_J$. So we regard $L$ as a constant functor $L : C \to X_J$ which comes with a universal cocone $\gamma : T \Rightarrow L$ in $X_J$. Let $i_J : X_J \to X$ be the inclusion. We prove that the natural transformation $i_J\gamma : i_JT \Rightarrow i_JL$ is a colimit cocone in $X$.

We use the following lemma.

2.8. Lemma. Let $X \in X$, with $\Phi(X) = I$, be regarded as a constant functor $X : C \to X$ and let $\theta : i_JT \Rightarrow X$ be a natural transformation, i.e. a cocone. Then
(i) for all $c \in C$, $u = \Phi(\theta(c)) : J \to I$ in $B$ is independent of $c$, and
(ii) the cartesian lifting $Y \to X$ of $u$ determines a cocone $\psi : T \Rightarrow Y$.

Proof The natural transformation $\theta$ gives for each object the morphism $\theta(c) : T(c) \to X$ in $X$. Since $C$ is connected, induction on the length of a zig-zag shows it is sufficient to
prove (i) when there is a morphism $f : c \to c'$ in $C$. By naturality of $\theta$, $\theta(c')T(f) = \theta(c)$. But $\Phi T(f)$ is the identity, since $T$ has values in $X_J$, and so $\Phi(\theta(c)) = \Phi(\theta(c'))$. Write $u = \Phi(\theta(c))$.

Since $\Phi$ is a fibration, there is a $Y \in X_J$ and a cartesian lifting $\varphi : Y \to X$ of $u$. Hence for each $c \in C$ there is a unique vertical morphism $\psi(c) : T(c) \to Y$ in $X_J$ such that $\varphi \psi(c) = \theta(c)$. We now prove that $\psi$ is a natural transformation $T \Rightarrow Y$ in $X_J$, where $Y$ is regarded as a constant functor.

To this end, let $f : c \to c'$ be a morphism in $X_J$. We need to prove $\psi(c) = \psi(c')T(f)$.

The outer part of diagram (a) commutes, since $\theta$ is a natural transformation. The upper and lower triangles commute, by construction of $\varphi$. Hence $\varphi \psi(c) = \theta(c) = \theta(c')T(f) = \varphi \psi(c')T(f)$.

Now $T(f)$, $\psi(c)$ and $\psi(c')$ are all vertical. By the universal property of $\varphi$, $\psi(c) = \psi(c')T(f)$, i.e. the left hand cell commutes. That is, $\psi$ is a natural transformation $T \Rightarrow Y$ in $X_J$.

To return to the theorem, since $L$ is a colimit in $X_J$, there is a unique vertical morphism $\psi' : L \to Y$ in the right hand diagram (b) such that for all $c \in C$, $\psi' \gamma(c) = \psi(c)$. Let $\theta' = \varphi \psi' : L \to X$. This gives a morphism $\theta' : L \to X$ such that $\theta' \gamma(c) = \theta(c)$ for all $c$, and, again using universality of $\varphi$, this morphism is unique. □

2.9. Remark. The connectedness assumption is essential in the Theorem. Any small category $C$ is the disjoint union of its connected components. If $T : C \to X$ is a functor, and $X$ has colimits, then $\mathrm{colim} T$ is the coproduct (in $X$) of the $T_i$ where $T_i$ is the restriction of $T$ to a component $C_i$. But given two objects in the same fibre of $\Phi : X \to B$, their coproduct in that fibre is in general not the same as their coproduct in $X$. For example, the coproduct of two groups in the category of groups is the free product of groups, while their coproduct as groupoids is their disjoint union. □

2.10. Remark. A common application of the theorem is that the inclusion $X_J \to X$ preserves pushouts. This is relevant to our application of pushouts in section 4. Pushouts are used to construct free crossed modules as a special case of induced crossed modules, [BH78], and to construct free crossed complexes as explained in [BH91, BG89]. □

2.11. Remark. George Janelidze has pointed out a short proof of Theorem 2.7 in the case $\Phi$ has a right adjoint, and so preserves colimits, which applies to our main examples here. If the image of $T$ is inside $\Phi(b)$, then $\Phi T$ is the constant diagram whose value is $\{b, 1_b\}$, and if $C$ is connected this implies that $\mathrm{colim} \Phi T = b$. But if $\Phi(\mathrm{colim} T) = \mathrm{colim} \Phi T = b$, then colim $T$ is inside $\Phi(b)$. □

3. Cofibrations of categories

We now give the duals of the above results.
3.1. **Definition.** Let $\Phi : X \to B$ be a functor. A morphism $\psi : Z \to Y$ in $X$ over $v := \Phi(\psi)$ is called *cocartesian* if and only if for all $u : J \to I$ in $B$ and $\theta : Z \to X$ with $\Phi(\theta) = uv$ there is a unique morphism $\varphi : Y \to X$ with $\Phi(\varphi) = u$ and $\theta = \varphi \psi$.

This is illustrated by the following diagram:

\[
\begin{array}{ccc}
Z & \xrightarrow{\psi} & Y \\
\downarrow{\theta} & & \xrightarrow{\varphi} \downarrow{\Phi} \\
K & \xrightarrow{v} & J \\
\end{array}
\]

\[
\begin{array}{ccc}
I & \xrightarrow{u} & X \\
\downarrow{\text{uv}} & & \\
\end{array}
\]

It is straightforward to check that cocartesian morphisms are closed under composition, and that $\psi$ is an isomorphism if and only if $\psi$ is a cocartesian morphism over an isomorphism.

3.2. **Definition.** The functor $\Phi : X \to B$ is a *cofibration* or *category cofibred over* $B$ if and only if for all $v : K \to J$ in $B$ and $Z \in X_K$ there is a cartesian morphism $\psi : Z \to Z'$ over $v$: such a $\psi$ is called a *cocartesian lifting* of $Z$ along $v$.

The cocartesian liftings of $Z \in X_K$ along $v : K \to J$ are also unique up to vertical isomorphism.

3.3. **Remark.** As in Definition 2.4, if $\Phi : X \to B$ is a cofibration, then using the axiom of choice for classes we may select for every $v : K \to J$ in $B$ and $Z \in X_K$ a cocartesian lifting of $Z$ along $v$

$$v_Z : Z \to v_*Z.$$

Under these conditions, the functor $v_*$ is commonly said to give the objects *induced* by $v$. Examples of induced crossed modules of groups are developed in [BW03], following on from the first definition of these in [BH78].

We now have the dual of Proposition 2.5.

3.4. **Proposition.** For a cofibration $\Phi : X \to B$, a choice of cocartesian liftings of $v : K \to J$ in $B$ yields a functor $v_* : X_K \to X_J$, and an adjointness

$$X_J(v_*Z, Y) \cong X_v(Z, Y)$$

for all $Y \in X_J$, $Z \in X_K$.

We now state the dual of Theorem 2.7.

3.5. **Theorem.** Let $\Phi : X \to B$ be a category cofibred over $B$. Then the inclusion of each fibre of $\Phi$ into $X$ preserves limits of connected diagrams.

Many of the examples we are interested in are both fibred and cofibred. For them we have an adjoint situation.
3.6. **Proposition.** For a functor $\Phi : X \to B$ which is both a fibration and cofibration, and a morphism $u : J \to I$ in $B$, a choice of cartesian and cocartesian liftings of $u$ gives an adjointness

$$X_J(Y, u^*X) \cong X_I(u_*Y, X)$$

for $Y \in X_J$, $X \in X_I$.

It is interesting to get a characterisation of the cofibration property for a functor that already is a fibration. The following is a useful weakening of the condition for cocartesian in the case of a fibration of categories.

3.7. **Proposition.** Let $\Phi : X \to B$ be a fibration of categories. Then $\psi : Z \to Y$ in $X$ over $v : K \to J$ in $B$ is cocartesian if only if for all $\theta' : Z \to X'$ over $v$ there is a unique morphism $\psi' : Y \to X'$ in $X_J$ with $\theta' = \psi' \psi$.

**Proof** The ‘only if’ part is trivial. So to prove ‘if’ we have to prove that for any $u : J \to I$ and $\theta : Z \to X$ such that $\Phi(\theta) = uv$, there exists a unique $\varphi : Y \to X$ over $u$ completing the diagram

$$Z \xrightarrow{\theta} Y \xrightarrow{\varphi} X$$

$$K \xrightarrow{v} J \xrightarrow{u} I.$$

Since $\Phi$ is a fibration there is a cartesian morphism $\kappa : X' \to X$ over $u$. By the cartesian property, there is a unique morphism $\theta' : Z \to X'$ over $v$ such that $\kappa \theta' = \theta$, as in the diagram

$$Z \xrightarrow{\theta} X' \xrightarrow{\kappa} X.$$

Now, suppose $\varphi : Y \to X$ over $u : J \to I$ satisfies $\varphi \psi = \theta$, as in the diagram:

$$Z \xrightarrow{\psi} Y \xrightarrow{\varphi} X$$

$$X' \xrightarrow{\kappa} X.$$

By the given property of $\psi$ there is a unique morphism $\psi' : Y \to X'$ in $X_J$ such that $\psi' \psi = \theta'$. By the cartesian property of $\kappa$, there is a unique morphism $\varphi' : X' \to X_J$ such that $\kappa \varphi' = \varphi$. Then

$$\kappa \psi' \psi = \kappa \theta' = \theta = \varphi \psi = \kappa \varphi' \psi.$$
By the cartesian property of $\kappa$, and since $\psi' \psi$, $\phi' \psi$ are over $uv$, we have $\psi' \psi = \phi' \psi$. By the given property of $\psi$, and since $\phi'$, $\psi'$ are in $X_J$, we have $\phi' = \psi'$. So $\phi = \kappa \psi'$, and this proves uniqueness.

But we have already checked that $\kappa \psi' \psi = \theta$, so we are done. □

The following Proposition allows us to prove that a fibration is also a cofibration by constructing the adjoints $u_*$ of $u^*$ for every $u$.

3.8. Proposition. Let $\Phi : X \to B$ be a fibration of categories. Let $u : J \to I$ have reindexing functor $u^* : X_I \to X_J$. Then the following are equivalent:

(i) For all $Y \in X_J$, there is a morphism $u_Y : Y \to u_* Y$ which is cocartesian over $u$;

(ii) there is a functor $u_* : X_J \to X_I$ which is left adjoint to $u^*$.

Proof That (ii) implies (i) is clear, using Proposition 3.7, since the adjointness gives the bijection required for the cocartesian property.

To prove that (i) implies (ii) we have to check that the allocation $Y \mapsto u_*(Y)$ gives a functor that is adjoint to $u^*$. As before the adjointness comes from the cocartesian property.

We leave to the reader the check the details of the functoriality of $u_*$.

To end this section, we give a useful result on compositions.

3.9. Proposition. The composition of fibrations, (cofibrations), is also a fibration (cofibration).

Proof We leave this as an exercise. □

4. Pushouts and cocartesian morphisms

Here is a small result which we use in this section and section 9, as it applies to many examples, such as the fibration $\text{Ob} : \text{Gpd} \to \text{Set}$.

4.1. Proposition. Let $\Phi : X \to B$ be a functor that has a left adjoint $D$. Then for each $K \in \text{Ob} \, B$, $D(K)$ is initial in $X_K$. In fact if $u : K \to J$ in $B$, then for any $X \in X_J$ there is a unique morphism $\varepsilon_K : DK \to X$ over $u$.

Proof This follows immediately from the adjoint relation $X_u(DK, X) \cong B(K, \Phi X)$ for all $X \in \text{Ob} \, X_J$. □

Special cases of cocartesian morphisms are used in [Bro06, BH78, BH81b], and we review these in section 9. A construction which arises naturally from the various Higher Homotopy van Kampen theorems is given a general setting as follows:
4.2. Theorem. Let \( \Phi : X \rightarrow B \) be a fibration of categories which has a left adjoint \( D \).
Suppose that \( X \) admits pushouts. Let \( v : K \rightarrow J \) be a morphism in \( B \), and let \( Z \in X_K \).
Then a cocartesian lifting \( \psi : Z \rightarrow Y \) of \( v \) is given precisely by the pushout in \( X \):

\[
\begin{array}{ccc}
D(K) & \xrightarrow{D(v)} & D(J) \\
\downarrow{\varepsilon_K} & & \downarrow{\varepsilon_J} \\
Z & \xrightarrow{\psi} & Y
\end{array}
\]

(*)

Proof  Suppose first that diagram (*) is a pushout in \( X \). Let \( u : J \rightarrow I \) in \( B \) and let \( \theta : Z \rightarrow X \) satisfy \( \Phi(\theta) = uv \), so that \( \Phi(X) = I \). Let \( f : D(J) \rightarrow X \) be the adjoint of \( u : J \rightarrow \Phi(X) \).

\[
\begin{array}{ccc}
D(K) & \xrightarrow{D(v)} & D(J) \\
\downarrow{\varepsilon_K} & & \downarrow{\varepsilon_J} \\
Z & \xrightarrow{\psi} & Y \\
\downarrow{\theta} & & \downarrow{f} \\
X
\end{array}
\]

(**)

Then \( \Phi(fD(v)) = uv = \Phi(\theta\varepsilon_K) \) and so by Proposition 4.1, \( fD(v) = \theta\varepsilon_K \). The pushout property implies there is a unique \( \varphi : Y \rightarrow X \) such that \( \varphi\psi = \theta \) and \( \varphi\varepsilon_J = f \). This last condition gives \( \Phi(\varphi) = u \) since \( u = \Phi(f) = \Phi(\varphi\varepsilon_J) = \Phi(\varphi) \text{id}_J = \Phi(\varphi) \).

For the converse, we suppose given \( f : D(J) \rightarrow X \) and \( \theta : Z \rightarrow X \) such that \( \theta\varepsilon_K = fD(v) \). Then \( \Phi(\theta) = uv \) and so there is a cocartesian lifting \( \varphi : Y \rightarrow X \) of \( u \). The additional condition \( \varphi\varepsilon_J = f \) is immediate by Proposition 4.1.

\[\square\]

4.3. Corollary. Let \( \Phi : X \rightarrow B \) be a fibration which has a left adjoint and suppose that \( X \) admits pushouts. Then \( \Phi \) is also a cofibration.

In view of the construction of hierarchical homotopical invariants as colimits from the HHvKTs in [BH81b] and [BL87a], the following is worth recording, as a consequence of Theorem 2.7.

4.4. Theorem. Let \( \Phi : X \rightarrow B \) be fibred and cofibred. Assume \( B \) and all fibres \( X_I \) are cocomplete. Let \( T : C \rightarrow X \) be a functor from a small connected category. Then \( \text{colim}T \) exists and may be calculated as follows:

(i) First calculate \( I = \text{colim}(\Phi T) \), with cocone \( \gamma : \Phi T \Rightarrow I \);

(ii) for each \( c \in C \) choose cocartesian morphisms \( \gamma'(c) : T(c) \rightarrow X(c) \), over \( \gamma(c) \) where \( X(c) \in X_I \);
(iii) make \(c \mapsto X(c)\) into a functor \(X : C \to X_I\), so that \(\gamma' : T \Rightarrow X\);

(iv) form \(Y = \text{colim } X \in X_I\) with cocone \(\mu : X \Rightarrow Y\).

Then \(Y\) with \(\mu \gamma' : T \Rightarrow Y\) is \(\text{colim } T\).

**Proof** We first explain how to make \(X\) into a functor.

We will in stages build up the following diagram:

\[
\begin{array}{cccccc}
T(c) & \xrightarrow{\gamma'(c)} & X(c) & \xrightarrow{\mu(c)} & Y & \xrightarrow{\tau} & Z \\
\downarrow \Phi & & \downarrow \mu(c) & & \downarrow \gamma' & & \downarrow \tau \\
K & \xrightarrow{\Phi T(f)} & J & \xrightarrow{\gamma'(c')} & I & \xrightarrow{1} & H
\end{array}
\]

Let \(f : c \to c'\) be a morphism in \(C\), \(K = \Phi T(c), J = \Phi T(c')\). By cocartesianness of \(\gamma'(c)\), there is a unique vertical morphism \(X(f) : X(c) \to X(c')\) such that \(X(f)\gamma'(c) = \gamma'(c')T(f)\). It is easy to check, again using cocartesianness, that if further \(g : c' \to c''\), then \(X(gf) = X(g)X(f)\), and \(X(1) = 1\). So \(X\) is a functor and the above diagram shows that \(\gamma'\) becomes a natural transformation \(T \Rightarrow X\).

Let \(\eta : T \Rightarrow Z\) be a natural transformation to a constant functor \(Z\), and let \(\Phi(Z) = H\). Since \(I = \text{colim}(\Phi T)\), there is a unique morphism \(w : I \to H\) such that \(w\gamma = \Phi(\eta)\).

By the cocartesian property of \(\gamma'\), there is a natural transformation \(\eta' : X \Rightarrow Z\) such that \(\eta'\gamma' = \eta\).

Since \(Y\) is also a colimit in \(X\) of \(X\), we obtain a morphism \(\tau : Y \to Z\) in \(X\) such that \(\tau\mu = \eta'\). Then \(\tau\mu\gamma' = \eta\). Let \(\tau' : Y \to Z\) be another morphism such that \(\tau'\mu\gamma' = \eta\). Then \(\Phi(\tau) = \Phi(\tau') = w\), since \(I\) is a colimit. Again by cocartesianness, \(\tau'\mu = \tau\mu\). By the colimit property of \(Y\), \(\tau = \tau'\).

This with Theorem 4.4 shows how to compute colimits of connected diagrams in the examples we discuss in sections 5 to 10, and in all of which a van Kampen type theorem is available giving colimits of algebraic data for some glued topological data.

4.5. **Corollary.** Let \(\Phi : X \to B\) be a functor satisfying the assumptions of theorem 4.4. Then \(X\) is connected complete, i.e. admits colimits of all connected diagrams.

5. Groupoids bifibred over sets

We have already seen in Example 2.3 that the functor \(\text{Ob} : \text{Gpd} \to \text{Set}\) is a fibration. It also has a left adjoint \(D\) assigning to a set \(I\) the discrete groupoid on \(I\), and a right
adjoint assigning to a set \( I \) the codiscrete groupoid on \( I \).

It follows from general theorems on algebraic theories that the category \( \text{Gpd} \) is co-complete, and in particular admits pushouts, and so it follows from previous results that \( \text{Ob} : \text{Gpd} \to \text{Set} \) is also a cofibration. A construction of the cocartesian liftings of \( u : I \to J \) for \( G \) a groupoid over \( I \) is given in terms of words, generalising the construction of free groups and free products of groups, in [Hig71, Bro06]. In these references the cocartesian lifting of \( u \) to \( G \) is called a universal morphism, and is written \( u_* : G \to U_u(G) \). This construction is of interest as it yields a normal form for the elements of \( U_u(G) \), and hence \( u_* \) is injective on the set of non-identity elements of \( G \).

A homotopical application of this cocartesian lifting is the following theorem on the fundamental groupoid. It shows how identification of points of a discrete subset of a space can lead to ‘identifications of the objects’ of the fundamental groupoid:

5.1. THEOREM. Let \((X, A)\) be a pair of spaces such that \( A \) is discrete and the inclusion \( A \to X \) is a closed cofibration. Let \( f : A \to B \) be a function to a discrete space \( B \). Then the induced morphism
\[
\pi_1(X, A) \to \pi_1(B \cup_f X, B)
\]
is the cocartesian lifting of \( f \).

This theorem immediately gives the fundamental group of the circle \( S^1 \) as the infinite cyclic group \( \mathbb{C} \), since \( S^1 \) is obtained from the unit interval \([0, 1]\) by identifying 0 and 1, as shown in the Introduction in diagram (1). The theorem is a translation of [Bro06, 9.2.1], where the words ‘universal morphism’ are used instead of ‘cocartesian lifting’. Section 8.2 of [Bro06] shows how free groupoids on directed graphs are obtained by a generalisation of this example.

The calculation of colimits in a fibre \( \text{Gpd}_I \) is similar to that in the category of groups, since both categories are protomodular, [BB04]. Thus a colimit is calculated as a quotient of a coproduct, where quotients are themselves obtained by factoring by a normal subgroupoid. Quotients are discussed in [Hig71, Bro06].

Theorem 4.4 now shows how to compute general colimits of groupoids.

We refer again to [Hig71, Bro06] for further developments and applications of the algebra of groupoids; we generalise some aspects to modules, crossed modules and crossed complexes in the next sections.

6. Groupoid modules bifibred over groupoids

Modules over groupoids are a useful generalisation of modules over groups, and also form part of the basic structure of crossed complexes. Homotopy groups \( \pi_n(X; X_0), n \geq 2 \), of a space \( X \) with a set \( X_0 \) of base points form a module over the fundamental groupoid \( \pi_1(X, X_0) \), as do the homotopy groups \( \pi_n(Y, X : X_0), n \geq 3 \), of a pair \((Y, X)\).

6.1. DEFINITION. A module over a groupoid is a pair \((M, G)\), where \( G \) is a groupoid with set of objects \( G_0 \), \( M \) is a totally disconnected abelian groupoid with the same set of
objects as 

for all \( x, y \in G_0 \). These maps are denoted by \((m, p) \mapsto m^p\) and satisfy the usual properties, i.e. \(m^1 = m\), \((m^p)^{p'} = m^{pp'}\) and \((m + m')^p = m^p + m'^p\), whenever these are defined. In particular, any \( p \in G(x, y) \) induces an isomorphism \( m \mapsto m^p \) from \( M(x) \) to \( M(y) \).

A morphism of modules is a pair \((\theta, f) : (M, G) \to (N, H)\), where \( f : G \to H \) and \( \theta : M \to N \) are morphisms of groupoids and preserve the action. That is, \( \theta \) is given by a family of morphisms

\[
\theta(x) : M(x) \to N(f(x))
\]

for all \( x \in G_0 \) satisfying \( \theta(y)(m^p) = \left( \theta(x)(m) \right)^{f(p)} \), for all \( p \in G(x, y), m \in M(x) \).

This defines the category \( \text{Mod} \) having modules over groupoids as objects and the morphisms of modules as morphisms. If \((M, G)\) is a module, then \((M, G)_0\) is defined to be \( G_0 \).

We have a forgetful functor \( \Phi_{M} : \text{Mod} \to \text{Gpd} \) in which \((M, G) \mapsto G\).

**6.2. Proposition.** The forgetful functor \( \Phi_{M} : \text{Mod} \to \text{Gpd} \) has a left adjoint and is fibred and cofibred.

**Proof** The left adjoint of \( \Phi_{M} \) assigns to a groupoid \( G \) the module written \( 0 \to G \) which has only the trivial group over each \( x \in G_0 \).

Next, we give the pullback construction to prove that \( \Phi_{M} \) is fibred. This is entirely analogous to the group case, but taking account of the geometry of the groupoid.

So let \( v : G \to H \) be a morphism of groupoids, and let \((N, H)\) be a module. We define \((M, G) = v^*(N, H)\) as follows. For \( x \in G_0 \) we set \( M(x) = \{x\} \times N(vx) \) with addition given by that in \( N(vx) \). The operation is given by \((x, n)^p = (y, n^vp)\) for \( p \in G(x, y) \).

The construction of \( N = v_*(M, G) \) for \((M, G)\) a \( G \)-module is as follows.

For \( y \in H_0 \) we let \( N(y) \) be the abelian group generated by pairs \((m, q)\) with \( m \in M, q \in H \), and \( t(q) = y, s(q) = v(t(m)) \), so that \( N(y) = 0 \) if no such pairs exist. The operation of \( H \) on \( N \) is given by \((m, q)^{p'} = (m, qq')\), addition is \((m, q) + (m', q) = (m + m', q)\) and the relations imposed are \((m^p, q) = (m, v(p)q)\) when these make sense. The cocartesian morphism over \( v \) is given by \( \psi : m \mapsto (m, 1_{v(t(m)}) \).}

**6.3. Remark.** We will discuss the relation between a module over a groupoid and the restriction to the vertex groups in section 8 in the general context of crossed complexes. However it is useful to give the general situation of many base points to describe the relative homotopy group \( \pi_n(X, A, a_0) \) when \( X \) is obtained from \( A \) by adding \( n \)-cells at various base points. The natural invariant to consider is then \( \pi_n(X, A, A_0) \) where \( A_0 \) is an appropriate set of base points.

We now describe free modules over groupoids in terms of the inducing construction. The interest of this is two fold. Firstly, induced modules arise in homotopy theory from a
HHvKT, and we get new proofs of results on free modules in homotopy theory. Secondly, this indicates the power of the HHvKT since it gives new results.

6.4. Definition. Let $Q$ be a groupoid. A free basis for a module $(N, Q)$ consists of a pair of functions $t_B : B \to Q_0$, $i : B \to N$ such that $t_N i = t_B$ and with the universal property that if $(L, Q)$ is a module and $f : B \to L$ is a function such that $t_L f = t_N$ then there is a unique $Q$-module morphism $\varphi : N \to L$ such that $\varphi i = f$.

6.5. Proposition. Let $B$ be an indexing set, and $Q$ a groupoid. The free $Q$-module $(FM(t), Q)$ on $t : B \to Q_0$ may be described as the $Q$-module induced by $t : B \to Q$ from the discrete free module $\mathbb{Z}B = (\mathbb{Z} \times B, B)$ on $B$, where $B$ denotes also the discrete groupoid on $B$.

Proof This is a direct verification of the universal property. □

6.6. Remark. Proposition 3.7 shows that the universal property for a free module can also be expressed in terms of morphisms of modules $(FM(t), Q) \to (L, R)$.

7. Crossed modules bifibred over groupoids

Out homotopical example here is the family of second relative homotopy groups of a pair of spaces with many base points.

A crossed module over a groupoid, [BH81a], consists first of a morphism of groupoids $\mu : M \to P$ of groupoids with the same set $P_0$ of objects such that $\mu$ is the identity on objects, and $M$ is a family of groups $M(x), x \in P_0$; second, there is an action of $P$ on the family of groups $M$ so that if $m \in M(x)$ and $p \in P(x, y)$ then $m^p \in M(y)$. This action must satisfy the usual axioms for an action with the additional properties:

CM1) $\mu(m^p) = p^{-1} \mu(m)p$, and
CM2) $m^{-1}nm = n^m$

for all $p \in P$, $m, n \in M$ such that the equations make sense. These form the objects of the category $\text{XMod}$ in which a morphism is a commutative square of morphisms of groups

$$
\begin{array}{ccc}
M & \xrightarrow{g} & N \\
\downarrow{\mu} & & \downarrow{\nu} \\
P & \xrightarrow{f} & Q
\end{array}
$$

which preserve the action in the sense that $g(m^p) = (gm)^f p$ whenever this makes sense.

The category $\text{XMod}$ is equivalent to the well known category $\text{2-Gpd}$ of 2-groupoids, [BH81c]. However the advantages of $\text{XMod}$ over 2-groupoids are:

- crossed modules are closer to the classical invariants of relative homotopy groups;
- the notion of freeness is clearer in $\text{XMod}$ and models a standard topological situation, that of attaching 1- and 2-cells;
• the category $\text{XMod}$ has a monoidal closed structure which helps to define a notion of homotopy; these constructions are simpler to describe in detail than those for 2-groupoids, and they extend to all dimensions.

We have a forgetful functor $\Phi_1 : \text{XMod} \to \text{Gpd}$ which sends $(M \to P) \mapsto P$. Our first main result is:

7.1. **Proposition.** The forgetful functor $\Phi_1 : \text{XMod} \to \text{Gpd}$ is fibred and has a left adjoint.

**Proof** The left adjoint of $\Phi_1$ assigns to a groupoid $P$ the crossed module $0 \to P$ which has only the trivial group over each $x \in P_0$.

Next, we give the pullback construction to prove that $\Phi_1$ is fibred. So let $f : P \to Q$ be a morphism of groupoids, and let $\nu : N \to Q$ be a crossed module. We define $M = \nu^*(N)$ as follows.

For $x \in P_0$ we set $M(x)$ to be the subgroup of $P(x) \times N(f(x))$ of elements $(p,n)$ such that $fp = \nu n$. If $p_1 \in P(x,x'), n \in N(f(x))$ we set $(p,n)^{p_1} = (p_1^{-1}pp_1, n^{f(p_1)})$, and let $\mu : (p,n) \mapsto p$. We leave the reader to verify that this gives a crossed module, and that the morphism $(p,n) \mapsto n$ is cartesian. □

The following result in the case of crossed modules of groups appeared in [BH78], described in terms of the crossed module $\partial : u^*(M) \to Q$ induced from the crossed module $\mu : M \to P$ by a morphism $u : P \to Q$.

7.2. **Proposition.** The forgetful functor $\Phi_1 : \text{XMod} \to \text{Gpd}$ is cofibred.

**Proof** We prove this by a direct construction.

Let $\mu : M \to P$ be a crossed module, and let $f : P \to Q$ be a morphism of groupoids. The construction of $N = f_*(M)$ and of $\partial : N \to Q$ requires just care to the geometry of the partial action in addition to the construction for the group case initiated in [BH78] and pursued in [BW03] and the papers referred to there.

Let $y \in Q_0$. If there is no $q \in Q$ from a point of $f(P_0)$ to $y$, then we set $N(y)$ to be the trivial group.

Otherwise, we define $F(y)$ to be the free group on the set of pairs $(m,q)$ such that $m \in M(x)$ for some $x \in P_0$ and $q \in Q(f(x),y)$. If $q' \in Q(y,y')$ we set $(m,q)^{q'} = (m,qq')$. We define $\partial' : F(y) \to Q(y)$ to be $(m,q) \mapsto q^{-1}(fm)q$. This gives a precrossed module over $\partial : F \to Q$, with function $i : M \to F$ given by $m \mapsto (m,1)$ where if $m \in M(x)$ then $1$ here is the identity in $Q(f(x))$.

We now wish to change the function $i : M \to F$ to make it an operator morphism. For this, factor $F$ out by the relations

$$(m,q)(m',q) = (mm',q),$$

$$(mp,q) = (m,(fp)q),$$

whenever these are defined, to give a projection $F \to F'$ and $i' : M \to F'$. As in the group case, we have to check that $\partial' : F \to Q$ induces $\partial'' : F' \to H$ making this a
precrossed module. To make this a crossed module involves factoring out Peiffer elements, whose theory is as for the group case in [BH82]. This gives a crossed module morphism \((\varphi, f) : (M, P) \to (N, Q)\) which is cocartesian. \(\square\)

We recall the algebraic origin of free crossed modules, but in the groupoid context.

Let \(P\) be a groupoid, with source and target functions written \(s, t : P \to P_0\). A subgroupoid \(N\) of \(P\) is said to be normal in \(P\), written \(N \triangleleft P\), if \(N\) is wide in \(P\), i.e. \(N_0 = P_0\), and for all \(x, y \in P_0\) and \(a \in P(x, y)\), \(a^{-1} N(x)a = N(y)\). If \(N\) is also totally intransitive, i.e. \(N(x, y) = \emptyset\) when \(x \neq y\), as we now assume, then the quotient groupoid \(P/N\) is easy to define. (It may also be defined in general but we will need only this case.)

Now suppose given a family \(R(x), x \in P_0\) of subsets of \(P(x)\). Then the normaliser \(N_P(R)\) of \(R\) is well defined as the smallest normal subgroupoid of \(P\) containing all the sets \(R(x)\). Note that the elements of \(N_P(R)\) are all consequences of \(R\) in \(P\), i.e. all well defined products of the form

\[ c = (r_1^{\varepsilon_1})^{a_1} \ldots (r_n^{\varepsilon_n})^{a_n}, \quad \varepsilon_i = \pm 1, a_i \in P, n \geq 0 \tag{4} \]

and where \(b^a\) denotes \(a^{-1}ba\). The quotient \(P/N_P(R)\) is also written \(P/R\), and called the quotient of \(P\) by the relations \(r = 1, r \in R\).

As in group theory, we need also to allow for repeated relations. So we suppose given a set \(R\) and a function \(\omega : R \to P\) such that \(s\omega = t\omega = \beta\), say. This ‘base point function’, saying where the relations are placed, is a useful part of the information.

We now wish to obtain ‘syzygies’ by replacing the normal subgroupoid by a ‘free object’ on the relations \(\omega : R \to P\). As in the group case, this is done using free crossed modules.

7.3. Remark. There is a subtle reason for this use of crossed modules. A normal subgroupoid \(N\) of \(P\) (as defined above) gives a quotient object \(P/N\) in the category \(\text{Gpd}_X\) of groupoids with object set \(X = P_0\). Alternatively, \(N\) defines a congruence on \(P\), which is a particular kind of equivalence relation. Now an equivalence relation is in general a particular kind of subobject of a product, but in this case, we must take the product in the category \(\text{Gpd}_X\). As a generalisation of this, one should take a groupoid object in the category \(\text{Gpd}_X\). Since these totally disconnected normal subgroupoids determine equivalence relations on each \(P(x, y)\) which are congruences, it seems clear that a groupoid object internal to \(\text{Gpd}_X\) is equivalent to a 2-groupoid with object set \(X\).

7.4. Definition. A free basis for a crossed module \(\partial : C \to P\) over a groupoid \(P\) is a set \(R\), function \(\beta : R \to P_0\) and function \(i : R \to C\) such that \(i(r) \in C(\beta r), r \in R\), with the universal property that if \(\mu : M \to P\) is a crossed module and \(\theta : R \to M\) a function over the identity on \(P_0\) such that \(\mu \theta = \partial i\), then there is a unique morphism of crossed \(P\)-modules \(\varphi : C \to M\) such that \(\varphi i = \theta\).

7.5. Example. Let \(R\) be a set and \(\beta : R \to P_0\) a function. Let \(\text{id} : F_1(R) \to F_2(R)\) be the identity crossed module on two copies of \(F(R)\), the disjoint union of copies \(\mathbb{C}(r)\) of the infinite cyclic group \(\mathbb{C}\) with generator \(c_r \in \mathbb{C}(r)\). Thus \(F_2(R)\) is a totally intransitive
groupoid with object set \( R \). Let \( i : R \to F_1(R) \) be the function \( r \mapsto c_r \). Let \( \beta : R \to R \) be the identity function. Then \( \text{id} : F_1(R) \to F_2(R) \) is the free crossed module on \( i \). The verification of this is simple from the diagram

\[
\begin{array}{ccc}
R & \xrightarrow{i} & F_1(R) \\
\downarrow \text{id} & & \downarrow f \\
F_2(R) & \xrightarrow{\text{id}} & F_2(R)
\end{array}
\]

The morphism \( f \) simply maps the generator \( c_r \) to \( \theta r \).

7.6. **Proposition.** Let \( R \) be a set, and \( \mu : M \to P \) a crossed module over the groupoid \( P \). Let \( \beta : R \to P_0 \) be a function. Then the functions \( i : R \to M \) such that \( s\mu = t\mu = \beta \) are bijective with the crossed module morphisms \( (f, g) \)

\[
\begin{array}{ccc}
F_1(R) & \xrightarrow{f} & M \\
\downarrow \text{id} & & \downarrow \mu \\
F_2(R) & \xrightarrow{g} & P
\end{array}
\]

such that \( sg = \beta \).

Further, the free crossed module \( \partial : C(\omega) \to P \) on a function \( \omega : R \to P \) such that \( s\omega = t\omega = \beta \) is determined as the crossed module induced from \( \text{id} : F_1(R) \to F_2(R) \) by the extension of \( \omega \) to the groupoid morphism \( F_2(R) \to P \).

**Proof** The first part is clear since \( g = \mu f \) and \( f \) and \( i \) are related by \( f(c_r) = i(r), r \in R \).

The second part follows from the first part and the universal property of induced crossed modules as shown in the following diagram:

\[
\begin{array}{ccc}
F_1(R) & \xrightarrow{f} & C(\omega) \\
\downarrow \text{id} & & \downarrow \partial \\
F_2(R) & \xrightarrow{g} & P
\end{array}
\]

8. **Crossed complexes and an HHvKT**

Crossed complexes are analogous to chain complexes but also generalise groupoids to all dimensions and with their base points and operations relate dimensions 0, 1 and \( n \). The structure and axioms for a crossed complex are those universally satisfied by the main
topological example, the fundamental crossed complex \( \Pi X_* \) of a filtered space \( X_* \), where \( (\Pi X_*)_1 \) is the fundamental groupoid \( \pi_1(X_1, X_0) \) and for \( n \geq 2 \) \( (\Pi X_*)_n \) is the family of relative homotopy groups \( \pi_n(X_n, X_{n-1}, x_0) \) for all \( x_0 \in X_0 \), with associated operations of the fundamental groupoid and boundaries.

Crossed complexes fit into our scheme of algebraic structures over a range of dimensions satisfying a HHvKT in that the fundamental crossed complex functor

\[
\Pi : (\text{filtered spaces}) \rightarrow (\text{crossed complexes})
\]

preserves certain colimits. We state a precise version below.

A crossed complex \( C \) is in part a sequence of the form

\[
\cdots \xrightarrow{\delta_n} C_n \xrightarrow{\delta_{n-1}} C_{n-1} \xrightarrow{\delta_{n-2}} \cdots \xrightarrow{\delta_3} C_2 \xrightarrow{\delta_2} C_1
\]

where all the \( C_n, n \geq 1 \), are groupoids over \( C_0 \). Here \( \delta_2 : C_2 \rightarrow C_1 \) is a crossed module and for \( n \geq 3 \) \( (C_n, C_1) \) is a module. The further axioms are that \( \delta_n \) is an operator morphism for \( n \geq 2 \) and that \( \delta_2 c \) operates trivially on \( C_n \) for \( c \in C_2 \) and \( n \geq 3 \). We assume the basic facts on crossed complexes as surveyed in for example [Bro99, Bro04]. The category of crossed complexes is written \( \text{Crs} \). A full exposition of the theory of crossed complexes will be given in [BHS08].

To state the Higher Homotopy van Kampen Theorem for relative homotopy groups, namely Theorem C of [BH81b, Section 5], we need the following definition:

8.1. Definition. A filtered space \( X_* \) is said to be connected if the following conditions hold for each \( n \geq 0 \):

- \( \varphi(X_*, 0) \) : If \( r > 0 \), the map \( \pi_0 X_0 \rightarrow \pi_0 X_r \), induced by inclusion, is surjective; i.e. \( X_0 \) meets all path connected components of all stages of the filtration \( X_* \).
- \( \varphi(X_*, n) \) (for \( n \geq 1 \)) : If \( r > n \) and \( x \in X_0 \), then \( \pi_n(X_r, X_n, x) = 0 \).

8.2. Theorem. [Higher Homotopy van Kampen Theorem] Let \( X_* \) be a filtered space and suppose:

(i) \( X \) is the union of the interiors of subspaces \( U, V \);

(ii) the filtrations \( U_*, V_* \) and \( W_* \), formed by intersection with \( X_* \), and where \( W = U \cap V \), are connected filtrations. Then

(Conn) the filtration \( X_* \) is connected, and

(Pushout) the following diagram of morphisms of crossed complexes induced by inclusions

\[
\begin{array}{ccc}
\Pi W_* & \longrightarrow & \Pi U_* \\
\downarrow & & \downarrow \\
\Pi V_* & \longrightarrow & \Pi X_*
\end{array}
\]

is a pushout of crossed complexes.
8.3. Remark. The connectivity conclusion is significant, but not as important as the algebraic conclusion. This theorem is proved without recourse to traditional methods of algebraic topology such as homology and simplicial approximation. Indeed, the implications in dimension 2 are in general nonabelian and so unreachable by the traditional abelian methods. Instead the theorem is proved using the construction of $\rho X_\ast$, the cubical homotopy $\omega$–groupoid of the filtered space $X_\ast$, defined in dimension $n$ to be the set of filter homotopy classes of maps $I^n_\ast \to X_\ast$. The properties of this construction enable the proof of the 1-dimensional theorem van Kampen theorem to be generalised to higher dimensions, and the theorem on crossed complexes is deduced using a non trivial equivalence between the two constructions.

The paper [BH81b] also proves a more general theorem, in which arbitrary unions lead to a coequaliser rather than a pushout. The paper also assumes a $J_0$ condition on the filtered spaces; but this can be relaxed by the refined definition of making filter homotopies of maps $I^n_\ast \to X_\ast$ to be rel vertices, as has been advertised in [BH91].

8.4. Remark. Colimits of crossed complexes may be computed from the colimits of the groupoids, crossed modules and modules from which they are constituted.

8.5. Remark. A warning has to be given that some of the algebra is not as straightforward as that in traditional homological and homotopical algebra. For example in an abelian category, a pushout of the form

\[
\begin{array}{ccc}
A & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
B & \longrightarrow & C
\end{array}
\]

is equivalent to an exact sequence

\[ A \xrightarrow{i} B \xrightarrow{p} C \rightarrow 0. \]

However in the category $\text{Mod}_\ast$ of modules over groups a pushout of the form

\[
\begin{array}{ccc}
(M, G) & \longrightarrow & (0, 1) \\
\downarrow & & \downarrow \\
(N, H) & \longrightarrow & (P, K)
\end{array}
\]

is equivalent to a pair of an exact sequence of groups

\[ G \xrightarrow{i} H \xrightarrow{p} K \rightarrow 1, \]

and an exact sequence of induced modules over $K$

\[ M \otimes_{ZG} ZK \xrightarrow{i} N \otimes_{ZH} ZK \xrightarrow{p} P \rightarrow 0. \]

This shows that pushouts give much more information in our case, but also shows that handling the information may be more difficult, and that misinterpretations could lead to false conjectures or proofs.
8.6. PROPOSITION. The truncation functor $tr^1 : \text{Crs} \to \text{Gpd}$, $C \mapsto C_1$, is a bifibration.

Proof The previous results give the constructions on modules and crossed modules. The functoriality of these constructions give the construction of the boundary maps, and the axioms for all these follow. □

We will also need for later applications (Proposition 9.14) the relation of a connected crossed complex to the full, reduced (single vertex) crossed complex it contains, analogous to the well known relation of a connected groupoid to any of its vertex groups.

Recall that a codiscrete groupoid $T$ is one on which $T(x, y)$ is a singleton for all objects $x, y \in T_0$. This is called a tree groupoid in [Bro06]. Similarly, a codiscrete crossed complex $T$ is one in which the groupoid $T_1$ is codiscrete and which is trivial in higher dimensions.

We follow similar conventions for crossed complexes as for groupoids in [Bro06]. Thus if $D$ and $E$ are crossed complexes, and $S = D_0 \cap E_0$ then by the free product $D * E$ we mean the crossed complex given by the pushout of crossed complexes

\begin{equation}
\begin{array}{ccc}
S & \to & E \\
\downarrow & & \downarrow j \\
D & \to & D * E
\end{array}
\end{equation}

where the set $S$ is identified with the discrete crossed complex which is $S$ in dimension 0 and trivial in higher dimensions. The following result is analogous to and indeed includes standard facts for groupoids (cf. [Bro06, 6.7.3, 8.1.5]).

8.7. PROPOSITION. Let $C$ be a connected crossed complex, let $x_0 \in C_0$ and let $T$ be codiscrete wide subcrossed complex of $C$. Let $C(x_0)$ be the subcrossed complex of $C$ at the base point $x_0$. Then the natural morphism

$$\varphi : C(x_0) * T \to C$$

determined by the inclusions is an isomorphism, and $T$ determines a strong deformation retraction

$$r : C \to C(x_0).$$

Further, if $f : C \to D$ is a morphism of crossed complexes which is the identity on $C_0 \to D_0$ then we can find a retraction $s : D \to D(x_0)$ giving rise to a pushout square

\begin{equation}
\begin{array}{ccc}
& & C(x_0) \\
\downarrow f' & & \downarrow \leftarrow \\
D & \to & D(x_0)
\end{array}
\end{equation}

in which $f'$ is the restriction of $f$.

Proof Let $i : C(x_0) \to C$, $j : T \to C$ be the inclusions. We verify the universal property of the free product. Let $\alpha : C(x_0) \to E$, $\beta : T \to E$ be morphisms of crossed complexes
agreeing on $x_0$. Suppose $g : C \to E$ satisfies $gi = \alpha, gj = \beta$. Then $g$ is determined on $C_0$. Let $c \in C_1(x, y)$. Then

$$c = (\tau x)((\tau x)^{-1}c(\tau y))(\tau y)^{-1}$$

and so

$$gc = g(\tau x)g((\tau x)^{-1}c(\tau y))g(\tau y)^{-1} = \beta(\tau x)\alpha((\tau x)^{-1}c(\tau y))\beta(\tau y)^{-1}.$$ 

If $c \in C_n(x), n \geq 2$, then

$$c = (c^{tx})(\tau x)^{-1}$$

and so

$$g(c) = \alpha(c^{tx})\beta(\tau x)^{-1}.$$ 

This proves uniqueness of $g$, and conversely one checks that this formula defines a morphism $g$ as required.

In effect, equations (*) and (**) give for the elements of $C$ normal forms in terms of elements of $C(x_0)$ and of $T$.

This isomorphism and the constant map $T \to \{x_0\}$ determine the strong deformation retraction $r : C \to C(x_0)$.

The retraction $s$ is defined by the elements $f\tau(x), x \in C_0$, and then the diagram (6) is a pushout since it is a retract of the pushout square

$$\begin{array}{ccc}
C & \xrightarrow{1} & C \\
\downarrow f & & \downarrow f \\
D & \xrightarrow{1} & D 
\end{array}$$

9. Homotopical excision and induced constructions

We now interpret the HHvKT (Theorem 8.2) when the filtrations have essentially just two stages.

9.1. Definition. By a based pair $X_\bullet = (X, X_1; X_0)$ of spaces we mean a pair $(X, X_1)$ of spaces together with a set $X_0 \subseteq X_1$ of base points. For such a based pair and $n \geq 2$ we have an associated filtered space $X_\bullet^{[n]}$ which is $X_0$ in dimension 0, $X_1$ in dimensions $1 \leq i < n$ and $X$ for dimensions $i \geq n$. We write $\Pi X_\bullet^{[n]}$ for the crossed complex $\Pi X_\bullet^{[n]}$. This crossed complex is trivial in all dimensions $\neq 1, n$, and in dimension $n$ is the family of relative homotopy groups $\pi_n(X, X_1, x_0)$ for $x_0 \in X_0$, considered as a module (crossed module if $n = 2$) over the fundamental groupoid $\pi_1(X_1, X_0)$. Colimits of such crossed complexes are equivalent to colimits of the corresponding module or crossed module. □

The following is clear.
9.2. Proposition. If \( X_\bullet = (X, X_1; X_0) \) is a based pair, and \( n \geq 2 \) then the associated filtered space \( X^{[n]}_\bullet \) is connected if and only if the based pair \( X_\bullet \) is \((n - 1)\)-connected, i.e. if:

- \( X_0 \) meets each path component of \( X_1 \) and of \( X \);
- each path in \( X \) joining points of \( X_0 \) is deformable in \( X \) rel end points to a path in \( X_1 \); and
- \( \pi_r(X, X_1, x_0) = 0 \) for all \( x_0 \in X_0 \) and \( 1 < r < n \).

The last part of the condition is of course vacuous if \( n = 2 \).

With this in mind, Theorem 8.2 may be restated as:

9.3. Theorem. Let \( (X, X_1; X_0) \) be a based pair of spaces and suppose:

(i) \( X \) is the union of the interiors of subspaces \( U, V \);

(ii) the based pairs \((U, U_1; U_0), (V, V_1; V_0)\) and \((W, W_1; W_0)\) formed by intersection with \((X, X_1; X_0)\), and where \( W = U \cap V \), are \((n - 1)\)-connected. Then

(Conn) \((X, X_1; X_0)\) is \((n - 1)\)-connected, and

(Pushout) the following diagram of morphisms

\[
\begin{array}{ccc}
\Pi_n(W, W_1; W_0) & \longrightarrow & \Pi_n(U, U_1; U_0) \\
\downarrow & & \downarrow \\
\Pi_n(V, V_1; V_0) & \longrightarrow & \Pi_n(X, X_1; X_0)
\end{array}
\]

is a pushout of modules if \( n \geq 3 \) and of crossed modules if \( n = 2 \).

Proof. The important point is the equivalence between colimits of crossed complexes which for a given \( n > 2 \) have \( C_i = 0 \) for \( i \neq 0, 1, n \) or which are trivial in dimensions > 2, and colimits of the corresponding modules or crossed modules.

9.4. Remark. It is not easy to see for Theorem 9.3 a direct proof in terms of modules or crossed modules, since one needs the intermediate structure between 0 and \( n \) to use the connectivity conditions. The cubical homotopy \( \omega \)-groupoid with connections \( \rho X_\bullet \) is found convenient for this inductive process in [BH81b]. There are two reasons for this: cubical methods are convenient for constructing homotopies, and also for algebraic inverses to subdivision.

We now concentrate on excision, since this gives rise to cocartesian morphisms and so induced modules and crossed modules.
9.5. Theorem. [Homotopy Excision Theorem] Let the topological space $X$ be the union of the interiors of sets $U, V$, and let $W = U \cap V$. Let $n \geq 2$. Let $W_0 \subseteq U_0 \subseteq U$ be such that the based pair $(V, W; W_0)$ is $(n - 1)$-connected and $U_0$ meets each path component of $U$. Then $(X, U; U_0)$ is $(n - 1)$-connected, and the morphism of modules (crossed if $n = 2$)

$$\pi_n(V, W; W_0) \rightarrow \pi_n(X, U; U_0)$$

induced by inclusions is cocartesian over the morphism of fundamental groupoids

$$\pi_1(W, W_0) \rightarrow \pi_1(U, U_0)$$

induced by inclusion.

**Proof** We deduce this excision theorem from the pushout theorem 9.3, applied to the based pair $(X, U; U_0)$ and the following diagram of morphisms:

$$\begin{array}{ccc}
\Pi_n(W, W; W_0) & \longrightarrow & \Pi_n(U, U; U_0) \\
\downarrow & & \downarrow \\
\Pi_n(V, W; W_0) & \longrightarrow & \Pi_n(X, U; U_0)
\end{array} \quad (7)$$

This is a pushout of modules if $n \geq 3$ and of crossed modules if $n = 2$, by Theorem 9.3. However

$$\Pi_n(W, W; W_0) = (0, \pi_1(W, W_0)), \quad \Pi_n(U, U; U_0) = (0, \pi_1(U, U_0)).$$

So the theorem follows from Theorem 4.2 and our discussion of the examples of modules and crossed modules. □

9.6. Corollary. [Homotopical excision for an adjunction] Let $i : W \rightarrow V$ be a closed cofibration and $f : W \rightarrow U$ a map. Let $W_0$ be a subset of $W$ meeting each path component of $W$ and $V$, and let $U_0$ be a subset of $U$ meeting each path component of $U$ such that $f(W_0) \subseteq U_0$. Suppose that the based pair $(V, W)$ is $(n - 1)$-connected. Let $X = U \cup_f V$. Then the based pair $(X, U)$ is $(n - 1)$-connected and the induced morphism of modules (crossed if $n = 2$)

$$\pi_n(V, W; W_0) \rightarrow \pi_n(X, U; U_0)$$

is cocartesian over the induced morphism of fundamental groupoids

$$\pi_1(W, W_0) \rightarrow \pi_1(U, U_0).$$

**Proof** This follows from Theorem 9.5 using mapping cylinders in a similar manner to the proof of a corresponding result for the fundamental groupoid [Bro06, 9.1.2]. That is, we form the mapping cylinder $Y = M(f) \cup W$. The closed cofibration assumption ensures that the projection from $Y$ to $X = U \cup_f V$ is a homotopy equivalence. □
9.7. Remark. These methods were used in [BH78].

9.8. Corollary. [Attaching n-cells] Let the space $Y$ be obtained from the space $X$ by attaching $n$-cells, $n \geq 2$, at a set of base points $A$ of $X$, so that $Y = X \cup \bigcup f_\lambda e^n_\lambda, \lambda \in \Lambda$, where $f_\lambda : (S^{n-1}, 0) \to (X, A)$. Then $\pi_n(Y, X; A)$ is isomorphic to the free $\pi_1(X, A)$-module (crossed if $n = 2$) on the characteristic maps of the $n$-cells.

9.9. Remark. The previous corollary for $n = 2$ was a theorem of J.H.C. Whitehead. An account of Whitehead's proof is given in [Bro80]. There are several other proofs in the literature but none give the more general homotopical excision result, theorem 9.5.

9.10. Example. We can now explain the example in the Introduction. That $S^n$ is $(n-1)$-connected and $\pi_n(S^n, 0) \cong \mathbb{Z}$ follows by induction in the usual way from the homotopical excision theorem and the calculation $\pi_1(S^1, 0) \cong \mathbb{Z}$ by the groupoid van Kampen theorem. Applying the HET to writing $S^n \vee [0, 1]$ as a union of $S^n$ and $[0, 1]$ we get that $\pi_n(S^n \vee [0, 1], [0, 1]; \{0, 1\})$ is the free $\pi_1([0, 1], \{0, 1\})$-module on one generator. Again applying the HET but now identifying 0, 1 we get that $\pi_1(S^n \vee S^1, 0)$ is the free $\pi_1(S^1, 0)$-module in one generator.

9.11. Corollary. [Relative Hurewicz Theorem] Let $A \to X$ be a closed cofibration and suppose $A$ is path connected and $(X, A)$ is $(n-1)$-connected. Then $X \cup CA$ is $(n-1)$-connected and $\pi_n(X \cup CA, x)$ is isomorphic to $\pi_n(X, A, x)$ factored by the action of $\pi_1(A, x)$.

We now point out that a generalisation of a famous result of Hopf, [Hop42], is a corollary of the relative Hurewicz theorem. The following for $n = 2$ is part of Hopf's result. The algebraic description of $H_2(G)$ which he gives for $G$ a group is shown in [BH78] to follow from the HHvKT.

9.12. Proposition. [Hopf's theorem] Let $(V, A)$ be a pair of pointed spaces such that:

(i) $\pi_i(A) = 0$ for $1 < i < n$;

(ii) $\pi_i(V) = 0$ for $1 < i \leq n$;

(iii) the inclusion $A \to V$ induces an isomorphism on fundamental groups.

Then the pair $(V, A)$ is $n$-connected, and the inclusion $A \to V$ induces an epimorphism $H_n A \to H_n V$ whose kernel consists of spherical elements, i.e. of the image of $\pi_n A$ under the Hurewicz morphism $\omega_n : \pi_n(A) \to H_n(A)$.

Proof That $(V, A)$ is $n$-connected follows immediately from the homotopy exact sequence of the pair $(V, A)$ up to $\pi_n(V)$. We now consider the next part of the exact homotopy sequence and its relation to the homology exact sequence as shown in the
The Relative Hurewicz Theorem implies that $H_n(V, A) = 0$, and that $\omega_{n+1}$ is surjective. Also $\partial$ in the top row is surjective, since $\pi_n(V) = 0$. It follows easily that the sequence $\pi_n(A) \to H_n(A) \to H_n(V) \to 0$ is exact. □

There is a nice generalisation of Corollary 9.8, which so far has been proved only as a deduction from a HHvKT.

9.13. COROLLARY. [Attaching cones] Let $A$ be a space and let $S$ be a set consisting of one point in each path component of $A$. By $CA$, the cone on $A$, we mean the union of cones on each path component of $A$. Let $f : A \to X$ be a map, and let $S'$ be the image of $S$ by $f$. Then $\pi_2(X \cup CA, X; S')$ is isomorphic to the $\pi_1(X, S')$-crossed module induced from the identity crossed module $\pi_1(A, S) \to \pi_1(A, S)$ by the induced morphism $f_* : \pi_1(A, S) \to \pi_1(X, S')$.

The paper [BW03] uses this result to give explicit calculations for the crossed modules representing the homotopy 2-types of certain mapping cones.

We now explain the relevance to free crossed modules of Proposition 8.7, leaving the module and other cases to the reader.

9.14. PROPOSITION. Let $X$ be a path connected space with base point $a$, and let $Y = X \cup f_{x} \{e^1_{x} \}$ be obtained by attaching cells by means of pointed maps $f_{x} : (S^1, 0) \to (X, a_x)$, determining elements $x_{\lambda} \in \pi_1(X, a_{\lambda})$, $\lambda \in \Lambda$. Let $A = \{a\} \cup \{a_{\lambda} \mid \lambda \in \Lambda\}$. Let $T$ be a tree groupoid in $\pi_1(X, A)$ determining a retraction $r : \pi_1(X, A) \to \pi_1(X, a)$. Then $\pi_2(Y, X, a)$ is isomorphic to the free crossed $\pi_1(X, a)$-module on the elements $r(x_{\lambda})$.

Proof We consider the following diagram:

\[
\begin{array}{ccccccccc}
0 & \to & \bigcup_{\lambda} \mathbb{Z} & \to & 0 & \to & P & \to & 0 & \to & P(a) \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
& & (\bigcup_{\lambda} \mathbb{Z} \to \bigcup_{\lambda} \mathbb{Z}) & \to & (C(\Lambda) \to P) & \xrightarrow{s} & (F \to P(a))
\end{array}
\] (8)

The left hand square is the pushout defining the free crossed module $C(\Lambda) \to P$ as an induced crossed module. The right hand square is the special case of crossed modules of the retraction of Proposition 8.7, and so is also a pushout. Hence the composite square is a pushout. Hence the crossed module $F \to P(a)$ is the free crossed module as described. □
9.15. Remark. An examination of Whitehead’s paper [Whi41], and the exposition of part of it in [Bro80], shows that the geometrical side of the last proposition is intrinsic to his approach. Of course a good proportion of Whitehead’s work was devoted to extending ideas of combinatorial group theory to higher dimensions in combinatorial homotopy theory. The argument here is that this extension requires combinatorial groupoid theory for good modelling of the geometry.

10. Crossed squares and triad homotopy groups

In this section we give a brief sketch of the theory of triad homotopy groups, including the exact sequence relating them to homotopical excision, and show that the third triad group forms part of a crossed square which, as an algebraic structure with links over several dimensions, in this case dimensions 1,2,3, fits our criteria for a HHvKT. Finally we indicate a bifibration from crossed squares, so leading to the notion of induced crossed square, which is relevant to a triadic Hurewicz theorem in dimension 3.

A triad of spaces \((X : A, B; x)\) consists of a pointed space \((X, x)\) and two pointed subspaces \((A, x), (B, x)\). Then \(\pi_n(X : A, B; x)\) is defined for \(n \geq 2\) as the set of homotopy classes of maps

\[
(I^n : \partial^-_1 I^n, \partial^-_2 I^n; J_{1,2}^{n-1}) \to (X : A, B; x)
\]

where \(J_{1,2}^{n-1}\) denotes the union of the faces of \(I^n\) other than \(\partial^-_1 I^n, \partial^-_2 I^n\). For \(n \geq 3\) this set obtains a group structure, using the direction 3, say, which is Abelian for \(n \geq 4\). Further there is an exact sequence

\[
\to \pi_{n+1}(X : A, B; x) \to \pi_n(A, C, x) \xrightarrow{\epsilon} \pi_n(X, B, x) \to \pi_n(X : A, B; x) \to
\]

where \(C = A \cap B\), and \(\epsilon\) is the excision map. It was the fact that these groups measure the failure of excision that was their main interest. However they do not shed light on the above Homotopical Excision Theorem 9.5.

The third triad homotopy group fits into a diagram of possibly non-Abelian groups

\[
\begin{array}{ccc}
\pi_3(X; A, B, x) & \longrightarrow & \pi_2(B, C, x) \\
\Pi(X; A, B, x) := & & \\
\pi_2(A, C, x) & \longrightarrow & \pi_1(C, x)
\end{array}
\]

in which \(\pi_1(C, x)\) operates on the other groups and there is also a function

\[
\pi_2(A, C, x) \times \pi_2(B, C, x) \to \pi_3(X : A, B; x)
\]

known as the generalised Whitehead product.

This diagram has structure and properties which are known as those of a crossed square, [GWL81, BL87a], explained below, and so this gives a homotopical functor

\[
\Pi : (based\ triads) \to (crossed\ squares).
\]
A crossed square is a commutative diagram of morphisms of groups

\[
\begin{array}{ccc}
L & \xrightarrow{\lambda} & M \\
\downarrow{\lambda'} & & \downarrow{\mu} \\
N & \xrightarrow{\nu} & P
\end{array}
\]

(12)

together with left actions of \(P\) on \(L, M, N\) and a function \(h : M \times N \rightarrow L\) satisfying a number of axioms which we do not give in full here. Suffice it to say that the morphisms in the square preserve the action of \(P\), which acts on itself by conjugation; \(M, N\) act on each other and on \(L\) via \(P\); \(\lambda, \lambda', \mu, \nu\) and \(\mu \lambda\) are crossed modules; and \(h\) satisfies axioms reminiscent of commutator rules, summarised by saying it is a biderivation. Morphisms of crossed squares are defined in the obvious way, giving a category \(\text{XSq}\) of crossed squares.

Let \(\text{XMod}^2\) be the category of pairs of crossed modules \(\mu : M \rightarrow P, \nu : N \rightarrow P\) (with \(P\) and \(\mu, \nu\) variable), and with the obvious notion of morphism. There is a forgetful functor \(\Phi : \text{XSq} \rightarrow \text{XMod}^2\). This functor has a right adjoint \(D\) which completes the pair \(\mu : M \rightarrow P, \nu : N \rightarrow P\) with \(L = M \times_P N\) and \(\lambda, \lambda'\) given by the projections and \(h : M \times N \rightarrow L\) given by \(h(m, n) = (\pi_m m^{-1}, \pi_n n^{-1}), m \in M, n \in N\). More interestingly, it has a left adjoint which to the above pair of crossed \(P\)-modules yields the ‘universal crossed square’

\[
\begin{array}{ccc}
M \otimes N & \xrightarrow{\alpha \otimes \beta} & N \\
\downarrow\mu & & \downarrow\nu \\
M & \xrightarrow{\delta \otimes \beta} & P
\end{array}
\]

(13)

where \(M \otimes N\), as defined in [BL87a], is the nonabelian tensor product of groups which act on each other.

Then \(\Phi\) is a fibration of categories and also a cofibration. Thus we have a notion of induced crossed square, which according to Proposition 4.2 is given by a pushout of the form

\[
\begin{array}{ccc}
(M \otimes N) & \xrightarrow{(\alpha \otimes \beta, \beta, \gamma)} & (R \otimes S) \\
\downarrow{\alpha} & & \downarrow{\gamma} \\
(M \otimes N) & \xrightarrow{(\delta \otimes \beta, \delta, \gamma)} & (T \otimes S)
\end{array}
\]

in the category of crossed squares, given morphisms \((\alpha, \gamma) : (M \rightarrow P) \rightarrow (R \rightarrow Q), (\beta, \gamma) : (N \rightarrow P) \rightarrow (S \rightarrow Q)\) of crossed modules.
The functor \( \Pi \) is exploited in [BL87a] for an HHvKT implying some calculations of the non-Abelian group \( \pi_3(X; A, B; x) \). The applications are developed in [BL87b] for a triadic Hurewicz Theorem, and for the notion of free crossed square, both based on ‘induced crossed squares’. Free crossed squares are exploited in [Ell93] for homotopy type calculations.

In fact the HHvKT works in all dimensions and in the more general setting of \( n \)-cubes of spaces, although not in a many base point situation. For a recent application, see [EM08].

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