NEUMANN BOUNDARY FEEDBACK STABILIZATION FOR A NONLINEAR WAVE EQUATION: A STRICT $H^2$-LYAPUNOV FUNCTION

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Abstract. For a system that is governed by the isothermal Euler equations with friction for ideal gas, the corresponding field of characteristic curves is determined by the velocity of the flow. This velocity is determined by a second-order quasilinear hyperbolic equation. For the corresponding initial-boundary value problem with Neumann-boundary feedback, we consider non-stationary solutions locally around a stationary state on a finite time interval and discuss the well-posedness of this kind of problem. We introduce a strict $H^2$-Lyapunov function and show that the boundary feedback constant can be chosen such that the $H^2$-Lyapunov function and hence also the $H^2$-norm of the difference between the non-stationary and the stationary state decays exponentially with time.

1. Introduction. The flow of gas through a pipeline is modelled by the isothermal Euler equations with friction. In the operation of gas pipelines, it is essential that the velocities remain below critical values where vibrations occur and noise is created, see [38]. We study a quasilinear wave equation for the gas velocity in the case of ideal gas which is derived from the isothermal Euler equations with friction. Using Neumann feedback at one end of the pipe, we stabilize the solution of the corresponding initial-boundary value problem with homogeneous Dirichlet boundary conditions at the other end of the pipe to a desired subsonic stationary state. Except for its nonlinearity, this system is of a similar form as the system with the linear wave equation which has been studied for example in [26].

The first results on the boundary feedback stabilization for a quasilinear wave equation have been obtained by M. Slemrod in [34] and J. Greenberg & T. Li in [14] by using the method of characteristics. In [9], J.-M. Coron, B. d’Andréa-Novel & G. Bastin constructed a strict $H^2$-Lyapunov function for the boundary control of hyperbolic systems of conservation laws without source term. In [10],

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they constructed a strict $H^2$-Lyapunov function for quasilinear hyperbolic systems with dissipative boundary conditions without source term. More recently in [8], Coron and Bastin study the Lyapunov stability of the $C^1$-norm for quasilinear hyperbolic systems of the first order. They consider $W^1_p$-Lyapunov functions for $p < \infty$ and look at the limit for $p \to \infty$.

Based upon [9], M. Dick, M. Gugat & G. Leugering considered the isothermal Euler equations with friction with Dirichlet boundary feedback at both ends of the system and introduced a strict $H^1$-Lyapunov function, which is a weighted and squared $H^1$-norm of the difference between the nonstationary and the stationary state. They developed Dirichlet boundary feedback conditions which guarantee that the $H^1$-norm of the difference between the non-stationary and the stationary state decays exponentially with time (see [12]). In [19], we have defined a strict $H^2$-Lyapunov function for this stabilization problem. In contrast to [9], [12] and [19] in the present paper a Neumann boundary feedback law is used at one end of the interval for the stabilization of the system. This is motivated by the nice properties of the corresponding Neumann feedback for the linear wave equation that leads to finite-time stabilization for a certain feedback parameter, see [26], [1].

In our paper, by constructing a strict $H^2$-Lyapunov function and choosing suitable boundary feedback conditions, we give results about the boundary feedback stabilization for a second-order quasilinear hyperbolic equation with source term. The exponential decay of the solution of a second-order quasilinear hyperbolic equation is established. This solution measures the difference between the present state and a desired stationary state, which is in general not constant for our system.

This paper is organized as follows: In Section 2 we consider the isothermal Euler equations both in physical variables and in terms of Riemann invariants. Then we transform the isothermal Euler equations to a second-order quasilinear hyperbolic equation. In Section 3 we state a result about the well-posedness of general second-order quasilinear hyperbolic systems on a finite time interval (see Lemma 3.1). Our main results about the exponential decay of the $H^2$-norm and $C^1$-norm are presented in Theorem 4.5 and Corollary 4.7 in Section 4.2. The proofs of Theorem 4.5 and Corollary 4.7 are given in Section 5. The infinite time horizon case is studied in Section 6. We show that due to the stabilization, the solution exists globally in time.

2. The isothermal Euler equations and a quasilinear wave equation. In this section, we present the isothermal Euler equations with friction for a single pipe both in terms of the physical variables and in terms of Riemann invariants.

Let a finite time $T > 0$ be given. The system dynamics for the gas flow in a single pipe can be modeled by a hyperbolic system, which is described by the isothermal Euler equations (see [4],[5],[12],[22]):

\[
\rho_t + q_x = 0, \quad (1)
\]

\[
q_t + \left( \frac{q^2}{\rho} + a^2 \rho \right)_x = -\frac{f_g}{2\delta} \frac{q|q|}{\rho}, \quad (2)
\]

where $\rho = \rho(t,x) > 0$ is the density of the gas, $q = q(t,x)$ is the mass flux, the constant $f_g > 0$ is a friction factor, $\delta > 0$ is the diameter of the pipe and $a > 0$ is the sonic velocity in the gas. We consider the equations on the domain $\Omega := [0,T] \times [0,L]$. Equation (1) states the conservation of mass and equation (2)
is the momentum equation. We use the notation

\[ \theta = \frac{f_\rho}{\delta}. \]

In this paper, we consider positive gas flow in subsonic or subcritical states, that is,

\[ 0 < \frac{q}{\rho} < a. \]

The isothermal Euler equations (1) and (2) give rise to the second-order equation

\[ \ddot{u}_{tt} + 2 \dot{u} \ddot{u}_{tx} - (a^2 - \dot{u}^2) \ddot{u}_{xx} = \ddot{F}(\dot{u}, \dot{u}_x, \ddot{u}_t), \]

(3)

where \( \dot{u} \) is the unknown function and satisfies

\[ \dot{u} = \frac{q}{\rho}, \]

(4)

that is \( \dot{u} \) is the velocity of the gas. The lower order term is

\[ \ddot{F}(\dot{u}, \dot{u}_x, \ddot{u}_t) = -2 \dot{u}_t \dot{u} \dot{u}_x - 2 \dot{u} \ddot{u}^2_x - \frac{3}{2} \theta \dot{u} |\dot{u}| \dot{u}_x - \theta |\ddot{u}| \dot{u}_t. \]

(5)

From the velocity \( \dot{u} \), the density \( \rho \) can be obtained from the initial value \( \rho(0, \cdot) \) and the differential equation

\[ (\ln \rho)_t = \frac{1}{a^2} \left( \ddot{u} \dot{u}_t + (\dot{u}^2 - a^2) \ddot{u}_x + \frac{1}{2} \theta |\dot{u}| \ddot{u}_t^2 \right). \]

(6)

Then \( q \) can be obtained from the equation \( q = \rho \dot{u} \).

To stabilize the system governed by the quasilinear wave equation (3) locally around a given stationary state \( \bar{u}(x) \), we use the boundary feedback law

\[ \ddot{u}_x(t, 0) = \bar{u}_x(0) + k \dot{u}_t(t, 0), \]

\[ \dot{u}(t, L) = \bar{u}(L), \]

with a feedback parameter \( k \in (0, \infty) \).

In terms of the physical variables \((q, \rho)\), the boundary feedback law is

at \( x = 0 \) : \( q_x - (\ln(\rho))_x q = \rho \ddot{u}_x(0) + k [q_t - (\ln(\rho))_t q] \),

at \( x = L \) : \( q = \bar{u}(L) \rho \).

Sufficient conditions for the exponential stability of this system will be presented in Theorem 4.5 in Section 4.2.

2.1. The Riemann invariants and a differential equation for \( \rho \) in terms of the velocity. For classical solutions the isothermal Euler equations (1) and (2) can be equivalently written as the following system

\[ \partial_t \begin{pmatrix} \rho \\ q \end{pmatrix} + \hat{A}(\rho, q) \partial_x \begin{pmatrix} \rho \\ q \end{pmatrix} = \hat{G}(\rho, q) \]

(7)

with the matrix

\[ \hat{A}(\rho, q) := \begin{pmatrix} 0 & 1 \\ \frac{a^2 - \dot{u}^2}{\rho^2} & \frac{2}{\rho} \end{pmatrix} \]

and the source term

\[ \hat{G}(\rho, q) := \begin{pmatrix} 0 \\ -\frac{\theta |\dot{u}| q}{\rho} \end{pmatrix}. \]
System (7) has two eigenvalues \( \hat{\lambda}_-(\rho, q), \hat{\lambda}_+(\rho, q) \) and in the subsonic case we have
\[
\hat{\lambda}_-(\rho, q) = \frac{q}{\rho} - a < 0 < \hat{\lambda}_+(\rho, q) = \frac{q}{\rho} + a.
\]
In terms of the Riemann invariants \( R_{\pm} = R_{\pm}(\rho, q) = -\frac{q}{\rho} \mp a \ln(\rho) \) the system (7) has the diagonal form
\[
\partial_t \begin{pmatrix} R_+ \\ R_- \end{pmatrix} + \hat{D}(R_+, R_-) \partial_x \begin{pmatrix} R_+ \\ R_- \end{pmatrix} = \hat{S}(R_+, R_-),
\]
where
\[
\hat{D}(R_+, R_-) := \begin{pmatrix} \hat{\lambda}_+ & 0 \\ 0 & \hat{\lambda}_- \end{pmatrix} = \begin{pmatrix} -\frac{R_+ + R_-}{2} + a & 0 \\ 0 & -\frac{R_+ + R_-}{2} - a \end{pmatrix},
\]
\[
\hat{S}(R_+, R_-) := -\frac{\theta}{8} (R_+ + R_-) |R_+ + R_-| \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\]
In terms of \( R_{\pm} \), for the physical variables \( \rho \) and \( q \) we have
\[
\rho = \exp \left( \frac{R_- - R_+}{2a} \right),
\]
\[
q = -\frac{R_+ + R_-}{2} \exp \left( \frac{R_- - R_+}{2a} \right).
\]
A gas flow is positive and subsonic (i.e. \( 0 < q/\rho < a \)) if and only if
\[
-2a < R_+(t, x) + R_-(t, x) < 0 \text{ for all } (t, x) \in \Omega.
\]
For the velocity \( \tilde{u} = \tilde{u}(\rho, q) \) defined in (4) we have
\[
\tilde{u} = \frac{R_+ + R_-}{-2a}.
\]
Due to (8), we can express the velocity in terms of the eigenvalues as
\[
\tilde{u} = \frac{\hat{\lambda}_+ + \hat{\lambda}_-}{2}.
\]
Due to equation (10), (9) yields the second-order equation (3). A detailed derivation can be found in [23]. The second-order quasilinear equation (3) is hyperbolic with the eigenvalues
\[
\hat{\lambda}_- = \tilde{u} - a < 0 < \hat{\lambda}_+ = \tilde{u} + a.
\]
Using the isothermal Euler equations (1) and (2), we obtain the partial derivatives of \( \tilde{u} \) with respect to \( t \) and \( x \), respectively,
\[
\tilde{u}_t = \frac{q_t}{\rho} - \frac{q \rho_t}{\rho^2}
\]
\[
= -\frac{1}{\rho} \left( \frac{q^2}{\rho} + a^2 \rho \right)_x - \frac{q \rho_t}{\rho^2} - \frac{\theta}{2} \frac{q |q|}{\rho^2}
\]
\[
= \tilde{u} \frac{\rho_t}{\rho} + (\tilde{u}^2 - a^2) \frac{\rho_x}{\rho} - \frac{\theta}{2} \tilde{u} |\tilde{u}|
\]
and
\[
\tilde{u}_x = \frac{q_x}{\rho} - \frac{q \rho_x}{\rho^2} = -\frac{\rho_t}{\rho} - \tilde{u} \frac{\rho_x}{\rho}.
\]
Multiplying \( \tilde{u}_t \) and \( \tilde{u}_x \) by \( \tilde{u} \) and \( \tilde{u}^2 - a^2 \), respectively, by adding the two equations we obtain (6), which means that \( \rho \) and \( q \) can be obtained from \( \tilde{u} \) and the initial data. Note that since \( \tilde{u} = \frac{q}{\rho} \), we have the same value for \( \tilde{u} \) for \( \lambda q \) and \( \lambda \rho \) where
\( \lambda \in (0, 1) \). So we cannot expect to recover the values of \((q, \rho)\) from \(\tilde{u}\) without additional information on \((q, \rho)\). In a similar way as (6), we obtain the equation

\[
\ln(\rho)_x = -\frac{1}{a^2} \left( \tilde{u}_t + \tilde{u} \tilde{u}_x + \frac{\theta}{2} |\tilde{u}| \tilde{u} \right).
\]

(11)

Thus if \(\tilde{u}\) is known, the values of \(\rho\) can be determined from the value of \(\rho\) at a boundary point \((x = 0 \text{ or } x = L)\) and (11) by integration.

2.2. Stationary states of the system. In [13] the existence, uniqueness and the properties of stationary subsonic \(C^1\)-solutions \((\bar{p}(x), \bar{q}(x))\) of the isothermal Euler equations have been discussed. The stationary states of the system on networks are studied in [17].

Here we focus on the stationary states of (3). Let \(\bar{u} = \tilde{u}(x)\) denote a stationary state for the second-order equation (3). Then (3) yields the following second-order ordinary differential equations for \(\bar{u}(x)\):

\[
(a^2 - \bar{u}^2(x)) \frac{d^2}{dx^2} \bar{u}(x) = 2 \bar{u}(x) \left( \frac{\theta}{2} |\bar{u}(x)| \right) \frac{d}{dx} \bar{u}(x).
\]

(12)

This implies that equation (3) has constant stationary states \(\bar{u} \in (-\infty, \infty)\) that can attain arbitrary real values. In contrast to this situation, the isothermal Euler equations with friction (that is (1), (2)) do not have constant stationary states except for the case of velocity zero. The stationary states of (1), (2) have been discussed. The stationary states of the system on networks are studied in [17].

Now we consider the question: Given a constant state \(\bar{u} = \lambda \in (0, \infty)\), is there a solution \((q, \rho)\) of (1), (2) that corresponds to the constant velocity \(\bar{u}\)? For \(\lambda = 0\) we obtain the constant solution of (1), (2) where \(q = 0\). For \(\lambda > 0\) there is a corresponding solution of travelling wave type (in particular the corresponding solution of (1), (2) is not stationary), namely

\[
(q(t, x), \rho(t, x)) = (\lambda \alpha(\lambda t - x), \alpha(\lambda t - x))
\]

(13)

where the function \(\alpha\) is given by

\[
\alpha(z) = C \exp \left( \frac{\lambda^2 \theta z}{2 a^2} \right)
\]

(14)

and \(C > 0\) is a positive constant. Equation (12) can be rewritten in the form

\[
\frac{d}{dx} \left( (a^2 - \bar{u}^2(x)) \bar{u}_x(x) - \frac{\theta}{2} |\bar{u}(x)| \bar{u}^2(x) \right) = 0.
\]

(15)

Thus for every stationary state \(\bar{u}\) of (3) there exists a constant \(\lambda \in (-\infty, \infty)\) such that \(\bar{u}\) satisfies the first order ordinary differential equation

\[
(a^2 - \bar{u}^2(x)) \bar{u}_x(x) = \lambda + \frac{\theta}{2} |\bar{u}| \bar{u}^2(x).
\]

(16)

We use the notation \(\bar{u}_0 := \bar{u}(0)\). Assume that \(\bar{u}_0 \in (0, a)\). Let \([0, x_0)\) denote the maximal existence interval of the solution. For the solutions that are not constant, we have two cases:

If \(\lambda + \frac{\theta}{2} \bar{u}_0^3 > 0\), \(\bar{u}\) is strictly increasing on \([0, x_0)\) and

\[
\lim_{x \to x_0^-} \bar{u}(x) = a, \quad \lim_{x \to x_0^-} \frac{d}{dx} \bar{u}(x) = +\infty, \quad \lim_{x \to x_0^-} \frac{d^2}{dx^2} \bar{u}(x) = +\infty.
\]

If \(\lambda + \frac{\theta}{2} \bar{u}_0^3 < 0\), \(\bar{u}\) is strictly decreasing on \([0, x_0)\) and

\[
\lim_{x \to x_0^-} \bar{u}(x) = -a, \quad \lim_{x \to x_0^-} \frac{d}{dx} \bar{u}(x) = -\infty, \quad \lim_{x \to x_0^-} \frac{d^2}{dx^2} \bar{u}(x) = -\infty.
\]
For stationary \( \rho \) and \( \bar{\rho} \), equation (6) implies (16) with \( \lambda = 0 \). The stationary states that correspond to \( \lambda \neq 0 \) cannot be deduced from the stationary states of (1), (2). Thus all the stationary solutions of (3) that correspond to a stationary state of (1), (2) must satisfy the equation

\[
\bar{u}'(0) = \frac{\theta |\bar{u}_0| \bar{u}_0^2}{2 a^2 - \bar{u}_0^2}.
\]

(17)

Lemma 2.1 contains an explicit representation for these stationary velocities.

**Lemma 2.1.** Let a subsonic stationary state \( \bar{u}(x) > 0 \) for \( x \in [0, L] \) that is not constant and satisfies (17) be given. Let \( W_{-1}(x) \) denote the real branch of the Lambert W-function (see [7, 27]) with \( W_{-1}(x) \leq -1 \). Then the following equation holds for all \( x \in [0, L] \):

\[
(\bar{u}(x))^2 = a^2 \frac{-W_{-1}(-\exp(\theta x + C))}{1 - \theta L},
\]

(18)

where \( C \) is a real constant such that \( C \leq -1 - \theta L \).

**Proof.** Separation of variables yields

\[
x + \bar{C} = \int \frac{a^2 - \bar{u}(x)^2}{a^2 \bar{u}(x)^3} \bar{u}'(x) dx = -\frac{1}{\theta} \left[ \ln(a^2) + \ln \left( \frac{\bar{u}(x)^2}{a^2} \right) + \frac{a^2}{\bar{u}(x)^2} \right].
\]

Define \( \xi = \frac{a^2}{\bar{u}(x)^2} \in (1, \infty) \). We have \( \xi + \ln(1/\xi) = \xi - \ln(\xi) \). Thus

\[
-\exp(\theta x + \theta \bar{C} + \ln(a^2)) = -\exp(-\xi + \ln(\xi)) = (-\xi) \exp(-\xi).
\]

Now the definition of \( W_{-1} \) as the inverse function of \( z \exp(z) \) for \( z \in (-\infty, -1) \) yields the assertion.

Since for the stationary states \( (q, \rho) \) of (1), (2) the flow rate \( q \) is constant, by (4) we get the corresponding density as \( \rho(x) = \frac{2}{\bar{u}(x)^2} \).

**3. Well-posedness of the system locally around stationary states.** Now we consider non-stationary solutions locally around a subsonic stationary state \( \bar{u}(x) > 0 \) on \( \Omega \) that satisfies (16) with \( \lambda = 0 \), that is that corresponds to a stationary state of (1), (2). For a solution \( \bar{u}(t,x) \) of (3), define

\[
u(t,x) = \bar{u}(t,x) - \bar{u}(x).
\]

(19)

Then (3), (12) and (16) yield the equation

\[
u_{tt} + 2(\bar{u} + u) \nu_{xx} - \left( a^2 - (\bar{u} + u)^2 \right) \nu_{xx} = F(x, u, u_x, u_t),
\]

(20)

where \( F := F(x, u, u_x, u_t) \) satisfies

\[
F = \tilde{F}(u + \bar{u}, u_x + \bar{u}_x, u_t) + \frac{a^2 - (\bar{u} + u)^2}{a^2 - \bar{u}^2} \bar{u} \left( 2(\bar{u}_x)^2 + \frac{3}{2} \theta |\bar{u}| \bar{u}_x \right),
\]

(21)

\[
= \tilde{F}(u + \bar{u}, u_x + \bar{u}_x, u_t) - \frac{a^2 - (\bar{u} + u)^2}{a^2 - \bar{u}^2} \tilde{F}(\bar{u}, \bar{u}_x, 0).
\]

(22)

If \( \bar{u} \geq 0 \) and \( \bar{u} + u \geq 0 \), we have

\[
\tilde{F}(u + \bar{u}, u_x + \bar{u}_x, u_t) = \tilde{F}(\bar{u}, \bar{u}_x, 0) + \tilde{F}(u, u_x, u_t)
\]

\[
-2 \bar{u} u_x^2 - 2 \bar{u} u_{tx} u_t - 4 \bar{u}_x u u_x - 2 \bar{u}_x^2 u - 4 \bar{u} \bar{u}_x u_x
\]

\[
-3 \theta \bar{u} u u_x - \frac{3}{2} \theta \bar{u}^2 u_x^2 - \frac{3}{2} \theta \bar{u}_x u^2 - 3 \theta \bar{u}_x \bar{u} u - \theta \bar{u}_t u_t.
\]
Using
\[ \bar{u}_x = \frac{\theta}{2} \frac{1}{a^2 - \bar{u}^2} \bar{u}^3 \]
this yields
\[
F = \tilde{F}(u, u_x, u_t) - \theta^2 \frac{3a^4 u^4 - 2a^2 \bar{u}^6 + \bar{u}^8}{2(a^2 - \bar{u}^2)^3} u - \theta \frac{a^2 \bar{u}}{a^2 - \bar{u}^2} u_t - \theta \frac{\bar{u}^4 + 3a^2 \bar{u}^2}{2(a^2 - \bar{u}^2)} u_x
- \theta^2 \frac{2 \bar{u}^7 - 3a^2 \bar{u}^5 + 3a^4 \bar{u}^3}{4(a^2 - \bar{u}^2)^3} u^2 - 2 \bar{u} u_x^2 - \theta \frac{3a^2 \bar{u} - \bar{u}^3}{a^2 - \bar{u}^2} u u_x
\]
with \( \tilde{F} \) as defined in (5). For the second-order quasilinear hyperbolic equation (20), we consider the initial conditions
\[
t = 0: \ u = \varphi(x), \ u_t = \psi(x), \ x \in [0, L] \tag{24}
\]
and the boundary feedback conditions
\[
x = 0: \ u_x = ku_t, \tag{25}
\]
\[
x = L: \ u = 0, \tag{26}
\]
where \( k > 0 \) is a real constant. We work in the framework of classical semi-global solutions. To apply the theory presented in [37], the second order equation is written as a first order system (see the proof of Theorem 1 in [30]). In this way the following result can be obtained (see Lemma 1 in [30]):

**Lemma 3.1.** Let a subsonic stationary state \( \bar{u}(x) > 0 \) as in Lemma 2.1 be given. Choose \( T > 0 \) arbitrarily large.

There exist constants \( \varepsilon_0(T) > 0 \) and \( C_T > 0 \), such that if the initial data \( (\varphi(x), \psi(x)) \in C^2([0, L]) \times C^1([0, L]) \) satisfies
\[
\max \{ \| \varphi(x) \|_{C^2([0, L])}, \| \psi(x) \|_{C^1([0, L])} \} \leq \varepsilon_0(T) \tag{27}
\]
and the \( C^2 \)-compatibility conditions are satisfied at the points \( (t, x) = (0, 0) \) and \( (0, L) \), then the initial-boundary problem (20), (24), (25)-(26) has a unique solution \( u(t, x) \in C^2([0, T] \times [0, L]) \). Moreover the following a priori estimate holds:
\[
\| u \|_{C^2([0, T] \times [0, L])} \leq C_T \max \{ \| \varphi(x) \|_{C^2([0, L])}, \| \psi(x) \|_{C^1([0, L])} \}. \tag{28}
\]

For less regular initial data that are only continuously differentiable, we also obtain a less regular solution where only first order derivatives appear:

**Lemma 3.2.** Let a subsonic stationary state \( \bar{u}(x) > 0 \) as in Lemma 2.1 be given. Choose \( T > 0 \) arbitrarily large.

There exist constants \( \varepsilon_0(T) > 0 \) and \( C_T > 0 \), such that if the initial data \( (\varphi(x), \psi(x)) \in C^1([0, L]) \times C([0, L]) \) satisfies the inequality
\[
\max \{ \| \varphi(x) \|_{C^1([0, L])}, \| \psi(x) \|_{C([0, L])} \} \leq \varepsilon_0(T) \tag{29}
\]
and the \( C^1 \)-compatibility conditions are satisfied at the points \( (t, x) = (0, 0) \) and \( (0, L) \), then the initial-boundary problem (20), (24), (25)-(26) has a unique solution \( u(t, x) \in C^1([0, T] \times [0, L]) \) in the sense that \( u \) satisfies (24), (25)-(26) and \( \bar{u} = \bar{u} + u \) satisfies (3) in the weak sense that for all test functions \( \varphi \in C^\infty([0, T] \times [0, L]) \) with compact support in \( (0, T) \times (0, L) \) we have
\[
\int_0^T \int_0^L \left( \bar{u}_t + \bar{u} u_x + \frac{\theta}{2} |\bar{u}| \bar{u} \right) (\varphi_t + \bar{u} \varphi_x) \, dx \, dt = \int_0^T \int_0^L a^2 \bar{u}_x \varphi_x \, dx \, dt. \tag{30}
\]
In addition, equation (9) holds with \( \lambda_\pm = \pm a + \bar{u} \) and
\[
R_\pm = -\bar{u} \pm \frac{1}{a} \int_0^x \bar{u}_t + \bar{u} \bar{u}_x + \frac{\theta}{2} |\bar{u}| \bar{u} \, dx.
\]
(31)

Moreover the following a priori estimate holds:
\[
\|u\|_{C^1([0,T] \times [0,L])} \leq C_T^3 \max \left\{ \|\varphi(x)\|_{C^1([0,L])}, \|\psi(x)\|_{C^1([0,L])} \right\}.
\]
(32)

**Proof.** For \( x \in [0, L] \), define the function
\[
h_{0,\varphi,\psi}(x) = -\frac{1}{a^2} \int_0^x \psi + (\bar{u} + \varphi) (\bar{u}_x + \varphi_x) + \frac{\theta}{2} |\bar{u} + \varphi| (\bar{u} + \varphi) \, dx.
\]
Then \( h_{0,\varphi,\psi}(\cdot) \in C^1([0, L]) \). Define
\[
R_{\pm,0,\varphi,\psi} = -(\bar{u} + \varphi) + a h_{0,\varphi,\psi}.
\]

Consider the initial boundary value problem for \((R_+, R_-)\) with the initial conditions \( R_\pm(0, x) = R_{\pm,0,\varphi,\psi}(x) \), the hyperbolic system of partial differential equations (9) that is in diagonal form and the boundary conditions
\[
-\frac{1}{2} \left[ (R_+)_x(t, 0) + (R_-)_x(t, 0) \right] = \bar{u}_x(0) - \frac{1}{2} k \left[ (R_+)_t(t, 0) + (R_-)_t(t, 0) \right]
\]
at \( x = 0 \) and \( R_+(t, L) + R_-(t, L) = -2 \bar{L}(t) \) at \( x = L \).

Note that for \((\varphi, \psi) = (0, 0)\), the initial boundary value problem has the stationary (that is, time–independent) solution
\[
(\bar{R}_+, \bar{R}_-) := (R_{+,0,0,0}, R_{-,0,0,0}).
\]

Therefore (by considering the initial boundary value problem for the difference \((R_+ - \bar{R}_+, R_- - \bar{R}_-)\)) the theory of semi-global solutions (see [28]) implies that if \( \varepsilon_3(T) \) in (29) is chosen sufficiently small, there exists a classical solution \((R_+, R_-)\) of the initial boundary value problem on the time interval \([0, T]\). To be precise we define \((r_+, r_-) = (R_+ - \bar{R}_+, R_- - \bar{R}_-)\). The boundary condition at \( x = L \) is equivalent to
\[
r_-(t, L) = -r_+(t, L)
\]
(33)
and the boundary condition at \( x = 0 \) is equivalent to
\[
r_+(t, 0) = -r_-(t, 0) + (r_+(0, 0) + r_-(0, 0)) + \frac{1}{k} \int_0^t (r_+)_x(s, 0) + (r_-)_x(s, 0) \, ds.
\]
(34)

Due to (9), \((r_+, r_-)\) satisfies the system in diagonal form
\[
\partial_t \begin{pmatrix} r_+ \\ r_- \end{pmatrix} + \hat{D}(\bar{R}_+ + r_+, \bar{R}_- + r_-) \partial_x \begin{pmatrix} r_+ \\ r_- \end{pmatrix} = \hat{S}(\bar{R}_+ + r_+, \bar{R}_- + r_-) - \hat{S}(\bar{R}_+, \bar{R}_-)
\]
(35)
\[
+ \left[ \hat{D}(\bar{R}_+, \bar{R}_-) - \hat{D}(\bar{R}_+ + r_+, \bar{R}_- + r_-) \right] \partial_x \begin{pmatrix} \bar{R}_+ \\ \bar{R}_- \end{pmatrix}.
\]
(36)

With initial data for \((r_+, r_-)\) in \([C^1([0,L])]^2\) at \( t = 0 \) that are sufficiently small (with respect to the \( C^1\)-norm), and satisfy the \( C^1\)-compatibility conditions for (33), (34), as in [29] we obtain a semi-global classical solution \((r_+, r_-) \in C^1([0,T] \times [0, L])\) that satisfies an a priori bound. Define
\[
\bar{u} = -\frac{R_+ + R_-}{2}.
\]
(38)
Then we have
\[ \tilde{u}(0, x) = -\frac{R_{+0}(x) + R_{-0}(x)}{2} = \tilde{u}(x) + \varphi(x). \]

Moreover, (9) implies
\[ \begin{align*}
(R_{\pm})_t (0, \cdot) &= -\lambda_{\pm} \partial_x R_{\pm}(0, \cdot) + \frac{\theta}{2} |\tilde{u}(0, \cdot)| \tilde{u}(0, \cdot) \\
&= -(\pm a + \tilde{u}(0, \cdot)) \{ -\tilde{u}_x - \varphi_x \\
&\pm \frac{1}{a} \left[ \psi + (\tilde{u} + \varphi)(\tilde{u}_x + \varphi_x) + \frac{\theta}{2} |\tilde{u} + \varphi| (\tilde{u} + \varphi) \right] \\
&+ \frac{\theta}{2} |\tilde{u}(0, \cdot)| \tilde{u}(0, \cdot) \\
&= -\psi \pm a (\tilde{u}_x - \varphi_x) \\
&\mp \tilde{u}(0, \cdot) \left[ \psi + (\tilde{u} + \varphi)(\tilde{u}_x + \varphi_x) + \frac{\theta}{2} |\tilde{u} + \varphi| (\tilde{u} + \varphi) \right]
\end{align*} \]

Thus we have
\[ \tilde{u}_t(0, x) = -\frac{(R_{+})_t(0, x) + (R_{-})_t(0, x)}{2} = \psi(x). \]

Hence \( u = \tilde{u} - \bar{u} \) satisfies the initial conditions (24). Moreover, \( \tilde{u} \) satisfies the boundary conditions (25)-(26). Now we show that \( \tilde{u} \) satisfies (30). Equation (9) implies that we have
\[ \begin{align*}
\tilde{u}_t &= -\frac{1}{2} [(R_{+})_t + (R_{-})_t] \\
&= -\frac{1}{2} [(-\lambda_{+} (R_{+})_x - \lambda_{-}(R_{-})_x + \theta |\tilde{u}| \tilde{u}] \\
&= -\frac{1}{2} [(a - \tilde{u}) (R_{+})_x + (a - \tilde{u})(R_{-})_x + \theta |\tilde{u}| \tilde{u}] \\
&= -\tilde{u} \left( -\frac{(R_{+})_x + (R_{-})_x}{2} \right) - \frac{\theta}{2} |\tilde{u}| \tilde{u} + \frac{a}{2} [(R_{+})_x - (R_{-})_x] \\
&= -\tilde{u} \tilde{u}_x - \frac{\theta}{2} |\tilde{u}| \tilde{u} + \frac{a}{2} [(R_{+})_x - (R_{-})_x].
\end{align*} \]

Thus we have
\[ \tilde{u}_t + \tilde{u} \tilde{u}_x + \frac{\theta}{2} |\tilde{u}| \tilde{u}^2 = \frac{a}{2} [(R_{+})_x - (R_{-})_x]. \quad (39) \]

By multiplication with \( \tilde{u} \) this implies
\[ \begin{align*}
\tilde{u} \tilde{u}_t + \tilde{u}^2 \tilde{u}_x + \frac{\theta}{2} |\tilde{u}| \tilde{u}^2 &= \frac{a}{2} \tilde{u} [(R_{+})_x - (R_{-})_x] \\
&= \frac{a}{2} [(a + \tilde{u})(R_{+})_x - (a + \tilde{u})(R_{-})_x] - \frac{a^2}{2} [(R_{+})_x + (R_{-})_x] \\
&= a^2 \tilde{u}_x + \frac{a}{2} [- (R_{+})_t + (R_{-})_t].
\end{align*} \]

This yields
\[ \tilde{u} \tilde{u}_t + (\tilde{u}^2 - a^2) \tilde{u}_x + \frac{\theta}{2} |\tilde{u}| \tilde{u}^2 = -\frac{a}{2} [(R_{+})_t - (R_{-})_t]. \quad (40) \]
For all test functions $\varphi$ we have
\[ \int_0^T \int_0^L \frac{a}{2} \, [(R_+)_x - (R_-)_x] \, \varphi_t \, dx \, dt \quad (41) \]
\[ = - \int_0^T \int_0^L -\frac{a}{2} \, [(R_+)_t - (R_-)_t] \, \varphi_x \, dx \, dt. \]
Hence (39) and (40) imply that (30) holds. The construction of $\tilde{u}$ implies that $R_\pm$ defined by (31) satisfy (9).

The theory of classical semi-global solutions yields a $C^1$ a priori estimate for $(r_+, r_-)$. The definition (38) of $\tilde{u}$ yields
\[ u = -\frac{1}{2} (r_+ + r_-) \quad (42) \]
which implies the a priori estimate (32).

**Remark 3.3.** For initial data $(\varphi, \psi) \in H^2([0, L]) \times H^1([0, L])$ such that $\|\varphi\|_{H^2([0, L])} + \|\psi\|_{H^1([0, L])}$ is sufficiently small, we obtain a solution $u$ with
\[ u \in C([0, T]; H^2([0, L])). \quad (43) \]
This can be seen as follows: For the initial boundary value problem for $(r_+, r_-)$ from the proof of Lemma 3.2 with initial data $(r_+(0, \cdot), r_-(0, \cdot)) \in H^2([0, L])^2$, as in Theorem B.1. in [6] we obtain a solution $(r_+, r_-) \in C([0, T]; H^2([0, L]))^2$. Then equation (42) yields the assertion.

4. **Exponential stability.** In this section, we introduce a strict $H^2$-Lyapunov function for the closed-loop system consisting of the quasilinear wave equation (20) and the boundary conditions (25), (26). To motivate the choice of the Lyapunov function, let us reconsider the classical energy for systems governed by the linear wave equation $u_{tt} - c^2 u_{xx} = 0$, which is $\int_0^L c^2 (u_x)^2 + (u_t)^2 \, dx$. In our quasilinear wave equation (20), instead of the square of the wave speed $c^2$ the term $(a^2 - (\bar{u} + u)^2)$ appears as a factor in front of $u_{xx}$, so it makes sense to replace $c^2$ by this expression in the definition of our Lyapunov function. In the same line of reasoning, if our quasilinear equation would be
\[ u_{tt} - (a^2 - (\bar{u} + u)^2) u_{xx} = F, \]
the integral $\int_0^L (a^2 - (\bar{u} + u)^2) (u_x)^2 + (u_t)^2 \, dx$ would be a candidate for a Lyapunov function. However, in our wave equation also the term $2(\bar{u} + u) u_{tx}$ appears. In order to deal with this term, we introduce an additional quantity in our Lyapunov function in such a way that, via equation (20), we can find an upper bound for its time-derivative. For this purpose, it makes sense to introduce a term that contains the product $u_t u_x$ in the integral defining the first part of our Lyapunov function. As a further motivation, we return to the linear wave equation $u_{tt} - u_{xx} = 0$ with the associated boundary conditions $u(t, 0) = 0$ and $u_t(t, L) = -k u_x(t, L)$ with $k > 0$. For a number $\lambda \in (0, \frac{2k}{(a^2 + k^2)})$, the quantity
\[ E(t) = \int_0^L (u_x)^2 + (u_t)^2 + \lambda x u_x u_t \, dx \]
can be used to show the exponential decay, since $E'(t) \leq -\lambda (1 - \lambda L) E(t)$. 
For many hyperbolic systems exponential weights in the Lyapunov function have been used successfully, see various examples in [11]. We define the weights

\[ h_1(x) = |k|, \]  
\[ h_2(x) = \exp\left(-\frac{x}{L}\right). \]  

(44) \hspace{1cm} (45)

In the sequel we consider

\[ E_1(t) = \int_0^L h_1(x) \left( (a^2 - (\bar{u} + u)^2) u_x^2 + u_t^2 \right) - 2 h_2(x) \left( (\bar{u} + u)u_x^2 + u_t u_x \right) dx \]

since according to the previous considerations, this is a natural candidate to define a Lyapunov function for our system.

To show the exponential decay with respect to the \( H^2 \)-norm, it is necessary to deal with the second order derivatives. Therefore we also introduce \( E_2(t) \) which is defined analogously to \( E_1 \) to show the decay of the partial derivatives of second order. We define

\[ E_2(t) = \int_0^L h_1(x) \left( (a^2 - (\bar{u} + u)^2) u_{xx}^2 + u_{tx}^2 \right) - 2 h_2(x) \left( (\bar{u} + u)u_{xx}^2 + u_{tx} u_{xx} \right) dx. \]

We define the Lyapunov function \( E(t) \) as

\[ E(t) = E_1(t) + E_2(t). \]  

(46)

In the following subsection we show that our Lyapunov function \( E(t) \) as defined in (46) is bounded above and below by the product of appropriate constants and the square of the \( H^2 \)-norm of \( u \).

4.1. **Equivalence of \( \sqrt{E(t)} \) with \( E(t) \) as in (46) and the \( H^2 \)-norm of the state.** In this section we show that \( \sqrt{E(t)} \) with \( E(t) \) as in (46) is equivalent to the \( H^2 \)-norm of the state. This is an essential property of a Lyapunov function since we want to use it to show the exponential decay of the \( H^2 \)-norm. Note that the constants in Lemma 4.1 are independent of the length \( T \) of the time interval.

**Lemma 4.1.** Let a real number

\[ \gamma \in (0, \frac{1}{2}] \]  

(47)

be given. Choose a real number \( k > 0 \) such that

\[ \frac{1}{k} \in (0, (1 - \gamma) a). \]  

(48)

Assume that \( \bar{u} \) is such that we have

\[ \bar{u} \in (0, \gamma a). \]  

(49)

Then for the weights defined in (44), (45) on the interval \([0, L]\) we have the strict inequality

\[ h_2 < (a - \bar{u}) h_1. \]  

(50)

In addition, we assume that \( \bar{u} \) is sufficiently small in the sense that

\[ \sup_{x \in [0,L]} \frac{\bar{u}(a^2 - \bar{u}^2)}{a^2 + 3 \bar{u}^2} < \frac{1}{2 e k}. \]  

(51)

Then for the weights we have the inequality

\[ h_2 > \frac{\bar{u}(a^2 - \bar{u}^2)}{a^2 + 3 \bar{u}^2} 2 h_1. \]  

(52)
For a real number $z$ define
\[ b_{11}(z) = 1 + 2z k - \frac{(1 + 2z k)^2}{k^2 (a^2 - z^2)}. \] (53)
Assume that
\[ v > k^2. \] (54)
Define the matrix
\[ \hat{B}_3(z) = \begin{pmatrix} b_{11}(z) & \frac{1 + 2z k}{k(a^2 - z^2)} - k \\ \frac{1 + 2z k}{k(a^2 - z^2)} - k & v - \frac{1}{a^2} \end{pmatrix}. \] (55)
For a real number $z$ define
\[ C_g(z) = \frac{a^2 - z^2}{2 + \frac{3}{2} \theta z + \frac{2 \theta z^3}{a^2 - z^2}}. \] (56)
Define the matrix
\[ \hat{A}_3(z) = \begin{pmatrix} \frac{1}{e} \frac{a^2 + 3z^2}{k^2} & 2z k & k - \frac{1}{e} \frac{2z^2}{a^2 - z^2} \\ \frac{1}{e} \frac{2z}{a^2 - z^2} & C_g(z) & \frac{1}{e} \frac{2z}{a^2 - z^2} \end{pmatrix}. \] (57)
Then there exists $\varepsilon_1(v) > 0$ such that for all $z$ with $|z| \leq 2\varepsilon_1(v)$ the matrix $\hat{B}_3(z)$ is positive definite and the matrix $\hat{A}_3(z)$ is positive definite.

**Proof.** First we show that (50) holds. This is equivalent to the inequality
\[ \frac{1}{k} < \inf_{x \in [0, L]} \exp \left( \frac{1}{L} x \right) \left( a - \bar{u}(x) \right). \]
Our assumptions (48) and (49) imply that
\[ \frac{1}{k} < (1 - \gamma) a \leq \inf_{x \in [0, L]} (a - \bar{u}(x)) \leq \inf_{x \in [0, L]} \exp \left( \frac{1}{L} x \right) \left( a - \bar{u}(x) \right), \]
and (50) follows. If (51) holds, we have
\[ \left( \frac{h_2}{h_1} \right) \geq \frac{1}{k} > \sup_{x \in [0, L]} \frac{2 \bar{u} (a^2 - \bar{u}^2)}{a^2 + 3\bar{u}^2} \]
and (52) follows.

Now we come to the assertion for the symmetric matrix $\hat{B}_3$. Due to (48) we have $b_{11}(0) = 1 - \frac{1}{k^2 a^2} > 0$. Due to the continuity of $b_{11}(\cdot)$ this implies that there exists a constant $\varepsilon_1 > 0$ such that for all $|z| \leq 2 \varepsilon_1$ we have $b_{11}(z) > 0$. We have
\[ \det \hat{B}_3(0) = \det \left( 1 - \frac{1}{k^2 a^2} \frac{1}{v - \frac{1}{a^2}} \right) = \left( 1 - \frac{1}{k^2 a^2} \right) (v - k^2). \]
Hence (54) implies $\det \hat{B}_3(0) > 0$. Due to the continuity of $\det \hat{B}_3(\cdot)$ this implies that we can choose the constant $\varepsilon_1 > 0$ in such a way that for all $|z| \leq 2 \varepsilon_1$ we have $\det \hat{B}_3(z) > 0$, and thus $\hat{B}_3(z)$ is positive definite. We can choose the constant $\varepsilon_1 > 0$ in such a way that for all $|z| \leq 2 \varepsilon_1$ for the $2 \times 2$ matrix $\hat{A}_3(z)$ the upper left element in the matrix is greater than zero. We have
\[ \lim_{z \to 0^+} C_g(z) = \infty. \] (58)
Due to (58) we can assume that $\varepsilon_1 > 0$ is sufficiently small such that for all $|z| \leq 2 \varepsilon_1$ we have $\det \hat{A}_3(z) > 0$, and thus $\hat{A}_3(z)$ is positive definite.
In Lemma 4.2 we show several inequalities that we need to show that $E(t)$ as in (46) can be bounded above and below by the squared $H^2$-norm. Note that also in Lemma 4.2 the constants are independent of the length $T$ of the time interval.

**Lemma 4.2.** Let all assumptions of Lemma 4.1 hold. In particular, let $k$ such that (48) holds be given. Let a stationary subsonic state $\bar{u}(x) \in C^2(0, L)$ be given. Assume that $\bar{u}$ is sufficiently small in the sense that (49) and (51) hold.

For $x \in [0, L]$ and real numbers $v_0$ define the real function

$$k_1(x, v_0) = h_1(x) \left( a^2 - (\bar{u}(x) + v_0)^2 \right) - 2 h_2(x) (\bar{u}(x) + v_0).$$

(59)

If $\varepsilon_2 > 0$ is chosen sufficiently small, we have

$$k_1(x, v_0) = \min_{x \in [0, L], |v_0| \leq \varepsilon_2} \min_k \frac{h_2^2(x)}{h_1(x)} > 0,$$

(60)

$$\bar{k}_1 := \min_{x \in [0, L], |v_0| \leq \varepsilon_2} \min_k \frac{h_1(x) k_1(x, v_0) - h_2^2(x)}{k_1(x, v_0)} > 0.$$

(61)

Assume in the sequel that $\varepsilon_2 > 0$ is chosen such that (60) and (61) hold. For $x \in [0, L]$ and real numbers $v_0, v_1, v_2$ define the real function

$$\chi^x(v_0, v_1, v_2) = h_1(x) \left( (a^2 - (\bar{u}(x) + v_0)^2) v_1^2 + v_2^2 \right) - 2 h_2(x) \left( (\bar{u}(x) + v_0) v_1^2 + v_1 v_2 \right).$$

(62)

Then $\chi^x$ can be represented in the form

$$\chi^x(v_0, v_1, v_2) = \left( k_1(x, v_0) - \frac{h_2^2(x)}{h_1(x)} \right) v_1^2 + \left( \sqrt{h_1(x)} v_2 - \frac{h_2(x)}{\sqrt{h_1(x)}} v_1 \right)^2$$

(63)

$$= \frac{h_1(x) k_1(x, v_0) - h_2^2(x)}{k_1(x, v_0)} v_1^2 + \frac{1}{k_1(x, v_0)} \left( k_1(x, v_0) v_1 - h_2(x) v_2 \right)^2.$$ 

(64)

**Proof.** Inequality (50) implies that, if $\varepsilon_2 > 0$ is sufficiently small and $|v_0| < \varepsilon_2$ we have

$$k_1(x, v_0) = h_1(x) \left[ a - (\bar{u}(x) + v_0) \right] \left[ a + (\bar{u}(x) + v_0) \right] - 2 h_2(x) (\bar{u}(x) + v_0)$$

$$> (h_2(x) - h_1(x) v_0) [a + \bar{u}(x) + v_0] - 2 h_2(x) (\bar{u}(x) + v_0)$$

$$= h_2(x) [a - \bar{u}(x)] - [(a + \bar{u}(x)) h_1(x) + h_2(x)] v_0 - h_1(x) v_0^2$$

$$> h_2^2(x) \frac{h_2(x)}{h_1(x)}$$

which implies (60). This in turn implies (61).

The representations (65) and (66) follow directly from the definition of $\chi^x$. 

In the sequel we assume that the assumptions from Lemma 4.1 hold. With $\chi^x$ as defined in Lemma 4.2 we have

$$E_1(t) = \int_0^L \chi^x(u(t, x), u_x(t, x), u_{xt}(t, x)) \, dx,$$

(67)

$$E_2(t) = \int_0^L \chi^x(u(t, x), u_{xx}(t, x), u_{xxt}(t, x)) \, dx.$$

(68)

With these representations of $E_1$ and $E_2$, Lemma 4.2 yields lower and upper bounds for $E_1(t)$ and $E_2(t)$. 

Lemma 4.3. Assume that

$$\max_{x \in [0,L]} |u(t, x)| \leq \varepsilon_2 $$ (69)

where $\varepsilon_2$ is chosen as in Lemma 4.2. For $E_1$ defined in (67) and $k_1$ defined in (59) we have the lower bounds

$$E_1(t) \geq \int_0^L \left( k_1(x, u(x)) - \frac{h_2^2(x)}{h_1(x)} \right) u_x^2(x) \, dx \geq K_1 \int_0^L u_x^2 \, dx, $$ (70)

and

$$E_1(t) \geq \int_0^L h_1(x) k_1(x, u) - h_2^2(x) \, u_t^2(x) \, dx \geq K_1 \int_0^L u_t^2 \, dx. $$ (71)

Moreover, we have the upper bounds

$$E_1(t) \leq \int_0^L \left( k_1(x, u(t, x)) + \frac{h_2^2(x)}{h_1(x)} \right) u_x^2(t, x) + 2 h_1(x) u_t^2(t, x) \, dx $$ (72)

and

$$E_1(t) \leq \int_0^L 2 h_1(x) \left( a^2 - (\bar{u} + u)^2 \right) u_x^2(t, x) + 2 h_1(x) u_t^2(t, x) \, dx. $$ (73)

For $E_2$ defined in (68), we have the lower bounds

$$E_2(t) \geq \int_0^L \left( k_1(x, u(x)) - \frac{h_2^2(x)}{h_1(x)} \right) u_{xx}^2(x) \, dx \geq K_1 \int_0^L u_{xx}^2 \, dx, $$ (74)

and

$$E_2(t) \geq \int_0^L h_1(x) k_1(x, u) - h_2^2(x) \, u_{xx}^2(x) \, dx \geq K_1 \int_0^L u_{xx}^2 \, dx. $$ (75)

Moreover, we have the upper bounds

$$E_2(t) \leq \int_0^L \left( k_1(x, u(t, x)) + \frac{h_2^2(x)}{h_1(x)} \right) u_{xx}^2(t, x) + 2 h_1(x) u_{xx}^2(t, x) \, dx. $$ (76)

$$E_2(t) \leq \int_0^L 2 h_1(x) \left( a^2 - (\bar{u} + u)^2 \right) u_{xx}^2(t, x) + 2 h_1(x) u_{xx}^2(t, x) \, dx. $$ (77)

Proof. Equation (65) and (60) imply the lower bound (70) for $E_1$.

The representation (66) and (61) imply the lower bound (71). The upper bound (72) follows from (65) and Young’s inequality. The upper bound (73) follows from (72) using $\frac{h_2^2}{h_1} < k_1$ and the definition of $k_1$.

The representations (65) and (60) also imply the lower bound (74) for $E_2$, and (66) and (61) imply the lower bound (75). The upper bound (76) again follows from (65) and Young’s inequality. The upper bound (77) follows from (76) using $\frac{h_2^2}{h_1} < k_1$ and the definition of $k_1$. □

Now we can show that $E(t)$ can be bounded above and below by the squared $H^2$-norm. Define the number

$$K_{\text{max}} = \max \left\{ 2k, \max_{x \in [0,L]} \max_{v_0 \leq \varepsilon_2} k_1(x, v_0) + \frac{h_2^2(x)}{h_1(x)} \right\}. $$ (78)

If (69) holds, by (72) and (76) Lemma 4.3 implies the inequality

$$E(t) \leq K_{\text{max}} \int_0^L u_x^2(t, x) + u_x^2(t, x) + u_t^2(t, x) + u_{ xx}^2(t, x) + u_{ xx}^2(t, x) \, dx. $$ (79)
Define
\[ K_{\min} = \frac{1}{2} \min\{ K_1, \tilde{K}_1 \}. \]  
(80)
By the definition of \( E \) and (70), (71), (74), (75) we also have the lower bound
\[ E(t) \geq K_{\min} \int_0^L u_x^2(t, x) + u_t^2(t, x) + u_{tx}^2(t, x) + u_{xx}^2(t, x) \, dx. \]  
(81)
The Poincaré-Friedrichs inequality states that if (26) holds, we have
\[ \int_0^L u^2(t, x) \, dx \leq 2 L^2 \int_0^L u_x^2(t, x) \, dx. \]  
(82)
Using this inequality and (20), inequality (81) implies that if \( E(t) \) is small, also the \( H^2 \)-norm of \( u(t, x) \) is small. Similarly \( E_1(t) \) can be bounded above and below by the squared \( H^1 \)-norm.

4.2. Exponential decay of the \( H^2 \)-Lyapunov function. In this section we present our main result about the exponential decay of the Lyapunov function that we have introduced in (46). Consider the system
\[
\begin{cases}
\dot{u}_{tt} + 2 \dot{u} \dot{u}_{tx} - (a^2 - \ddot{u}^2) \ddot{u}_{xx} = \tilde{F}(\dot{u}, \ddot{u}, \dot{u}), \\
\ddot{u}_x(t, 0) = \ddot{u}_x(0) + k \ddot{u}_t(t, 0), \quad t \in [0, T], \\
\ddot{u}(t, L) = \ddot{u}(L), \quad t \in [0, T], \\
t = 0: \quad \ddot{u} = \varphi(x) + \bar{u}(x), \quad \dot{u}_t = \psi(x), \quad x \in [0, L]
\end{cases}
\]  
(83)
with \( \tilde{F} \) as defined in (5). In Theorem 4.5 we present our main result about the stabilization of (83) for \( \ddot{u} \). For the analysis we use the fact that (83) is equivalent to (20),(24),(25),(26) that is stated in terms of \( u \) which is defined in (19) as the difference between \( \ddot{u} \) and the stationary state \( \bar{u} \). In Theorem 4.5 we state that the function \( E(t) \) defined in (46) is a strict Lyapunov function. In Theorem 4.5 it is assumed that \( \ddot{u} > 0 \) is sufficiently small and \( k \) is sufficiently large. Before we state the theorem, in the following remark we comment on condition (86) that appears in the statement of the Theorem and explain why it can be satisfied for all \( a > 0 \) if \( \ddot{u} > 0 \) is sufficiently small and \( k \) is sufficiently large.

Remark 4.4. For \( k > 0 \) and \( \bar{u}_0 > 0 \) define
\[
K_\theta(k, \bar{u}_0) = 2 \left[ \frac{4}{k^2} + \frac{2 \bar{u}_0}{k} + \theta \frac{\bar{u}_0^4 + 3 a^2 \bar{u}_0^2 + 2 \bar{u}_0}{2(a^2 - \bar{u}_0^2)} \right] + \frac{5}{2} \frac{\theta}{k^2} + \frac{\theta}{k} \frac{3 a^2 \bar{u}_0 - \bar{u}_0^3}{a^2 - \bar{u}_0^2}. 
\]  
(84)
Then (84) implies
\[
\lim_{\bar{u}_0 \to 0^+} K_\theta(k, \bar{u}_0) = \lim_{\bar{u}_0 \to 0^+} 2 \left[ \frac{4}{k^2} + \frac{5}{2} \frac{\theta}{k^2} \right] = \frac{2}{k^2} \left[ 4 + \frac{5}{2} \theta \right].
\]
This in turn implies that
\[
\lim_{\bar{u}_0 \to 0^+} 2 k^2 K_\theta(k, \bar{u}_0) = \left[ a^2 - \left( \bar{u}_0 + \frac{2}{k} \right)^2 \right] = 4 \left[ a^2 + \frac{5}{2} \theta \right] - a^2 + \frac{4}{k^2}. 
\]
Hence we have
\[
\lim_{k \to \infty} \lim_{\bar{u}_0 \to 0^+} 2 k^2 K_\theta(k, \bar{u}_0) = -a^2 < 0. 
\]
This implies that if \( \bar{u}_0 > 0 \) is sufficiently small and \( k \) is sufficiently large, then condition (86) with \( \bar{u}(0) = \bar{u}_0 \) in Theorem 4.5 below holds. In fact, if \( \bar{u}_0 > 0 \) is sufficiently small, for \( k = \frac{1}{\bar{u}_0} \) condition (86) holds, since

\[
\lim_{k \to \infty} 2k^2 K_0 \left( k, \frac{1}{k} \right) - \left[ a^2 - \left( \frac{1}{k} + \frac{2}{k} \right)^2 \right] = -a^2.
\]

**Theorem 4.5. (Exponential Decay of the \( H^2 \)-Lyapunov Function).** Let a real number \( \gamma \in (0, \frac{1}{2}] \) be given. Choose a real number \( k > 0 \) such that

\[
a(1 - \gamma)k > 1.
\]

Let a stationary subsonic state \( \bar{u}(x) \in C^2(0, L) \) be given that satisfies (17). Assume that for all \( x \in L \) we have \( \bar{u}(x) \in (0, \gamma a) \). Assume that for \( K_0(k, \bar{u}_0) \) as defined in (84) we have

\[
2k^2 K_0(k, \bar{u}(0)) \leq a^2 - \left( \bar{u}(0) + \frac{2}{k} \right)^2.
\]

Assume that \( \|\bar{u}\|_{C^2([0,L])} \) is sufficiently small such that \( \|\bar{u}\|_{C([0,L])} < \varepsilon_1(2k^2) \) (with \( \varepsilon_1 \) from Lemma 4.1) and (51) holds.

Let \( T > 0 \) be given. If the initial data satisfies

\[
\| (\varphi(x), \psi(x)) \|_{C^2([0,L]) \times C^1([0,L])} \leq \varepsilon_0(T)
\]

and the \( C^2 \)-compatibility conditions at the points \((t,x)=(0,0)\) and \((t,x)=(0,L)\), the initial-boundary value problem (83) for \( \bar{u} \) has a unique classical solution \( \bar{u} \in C^2([0,T] \times [0,L]) \). Problem (20), (24), (25), (26) has a unique classical solution \( u \in C^2([0,T] \times [0,L]) \) that satisfies the a priori estimate (28). Since \( \varepsilon_1(u) \) from Lemma 4.1 with \( v = 2k^2 \) and \( \varepsilon_2 \) from Lemma 4.2 are independent of \( T \), we can choose the constant \( \varepsilon_0(T) > 0 \) from Lemma 3.1 sufficiently small such that the a priori estimate (28) implies that the inequality

\[
\|u\|_{C([0,T]) \times [0,L]} \leq \min\{\varepsilon_1(2k^2), \varepsilon_2\}
\]

holds and for all \( t \in [0,T], x \in [0,L] \) we have the inequalities

\[
|u(t,x)| \leq \min \left\{ \gamma a - |\bar{u}(x)|, |\bar{u}(x)|, \frac{1}{k} \right\} \text{ and } |u_x(t,x)| \leq \min \left\{ 1, \frac{1}{k} \right\}.
\]

Moreover, choose \( \varepsilon_0(T) \) and \( \bar{u} \) sufficiently small such that

\[
\kappa := \max_{t \in [0,T]} (P_0(\Theta(t)) + P_1(\Theta(t)))(1 + L^2) \left( \frac{1}{K_1} + \frac{1}{K_1} \right) + 2k^2 C_{E_1}(\bar{u}(0)) \frac{L}{K_1} \leq \frac{1}{4e L k}
\]

with the functions \( P_0, P_1 \) and \( C_{E_1} \) defined in (111), (120) and (127) and

\[
\Theta(t) = \max_{x \in [0,L]} \{ |u(t,x)|, |u_x(t,x)|, |u_t(t,x)|, |\bar{u}(x)|, |\bar{u}'(x)| \}.
\]

Define the number

\[
\mu = \frac{1}{2e L k} - \kappa \geq \frac{1}{4e L k}.
\]

Then we have

\[
E_1(t) \leq E_1(0) \exp(-\mu t) \text{ for all } t \in [0,T],
\]

\[
E(t) \leq E(0) \exp(-\mu t) \text{ for all } t \in [0,T]
\]
that is $E_1(t)$ and $E(t)$ as defined in (46) are strict Lyapunov functions for our control system (83).

**Remark 4.6.** Theorem 4.5 states that if $\bar{u} > 0$ is sufficiently small and $k$ is sufficiently large for sufficiently small initial data the Lyapunov function decays exponentially and the decay rate is at least $\mu_0 = \frac{1}{4eLk}$ which is independent of $\bar{u}$ and $T$, since the conditions on $k$ do not depend on $T$. For arbitrarily large $T$, we can always achieve this decay rate $\mu_0$ for sufficiently small initial data. With this decay rate, it is possible to determine a time $T_0 > 0$ when the size of the $H^2 \times H^1$-norm of the solution is reduced at least by a factor 1/3. In fact let

$$T_0 = 4eLk \ln \left( 9 \left( 1 + 2L^2 \right) \frac{K_{\max}}{K_{\min}} \right) \quad (95)$$

with $K_{\max}$ from (78) and $K_{\min}$ from (80). Then due to (81) and (82) we have

$$\| (u(T_0, \cdot), u_t(T_0, \cdot)) \|^2_{H^2(0,L) \times H^1(0,L)} = \int_0^L u^2(T_0, x) + u_x^2(T_0, x) + u_{xx}^2(T_0, x) + u_{tx}^2(T_0, x) \, dx \leq \frac{1 + 2L^2}{K_{\min}} E(T_0).$$

If the assumption of Theorem 4.5 hold for the time interval $[0, T_0]$, by (94) and (79) this yields

$$\| (u(T_0, \cdot), u_t(T_0, \cdot)) \|^2_{H^2(0,L) \times H^1(0,L)} \leq \frac{1 + 2L^2}{K_{\min}} E(0) \exp(-\mu_0 T_0)$$

$$\leq \left( 1 + 2L^2 \right) \frac{K_{\max}}{K_{\min}} \exp(-\mu_0 T_0) \| (\varphi(x), \psi(x)) \|^2_{H^2(0,L) \times H^1(0,L)}$$

$$= \frac{1}{9} \| (\varphi(x), \psi(x)) \|^2_{H^2(0,L) \times H^1(0,L)}$$

$$= \frac{1}{9} \| (u(0, \cdot), u_t(0, \cdot)) \|^2_{H^2(0,L) \times H^1(0,L)}.$$

**Corollary 4.7.** (Exponential Decay of the $H^2$-Norm and the $C^1$-Norm)

Under the assumptions of Theorem 4.5, for the semi-global classical solution $u$ of the mixed initial-boundary value problem (20), (24), (25), (26) the $H^2$-norm decays exponentially with time on $[0, T]$. More precisely, there exists a constant $\eta_1 > 0$ that is independent of $T$ such that for any $t \in [0, T]$ the inequality

$$\| (u(t, \cdot), u_t(t, \cdot)) \|^2_{H^2(0,L) \times H^1(0,L)} \leq \eta_1 \| (\varphi(x), \psi(x)) \|^2_{H^2(0,L) \times H^1(0,L)} \exp \left( -\frac{\mu}{2} t \right) \quad (96)$$

holds. Furthermore, there exists a constant $\eta_2 > 0$ that is independent of $T$ such that for any $t \in [0, T]$ the $C^1$-norm of the solution satisfies

$$\| (u(t, \cdot), u_t(t, \cdot)) \|^2_{C^1[0,L] \times C^0[0,L]} \leq \eta_2 \| (\varphi(x), \psi(x)) \|^2_{H^2(0,L) \times H^1(0,L)} \exp \left( -\frac{\mu}{2} t \right). \quad (97)$$

Due to (92), this implies that for $T$ sufficiently large we have

$$\| (u(T, \cdot), u_t(T, \cdot)) \|^2_{H^2(0,L) \times H^1(0,L)} \leq \frac{1}{2} \| (\varphi(x), \psi(x)) \|^2_{H^2(0,L) \times H^1(0,L)} \quad (98)$$
Proof. With the notation \( \hat{u} \), \((u(T, \cdot), u_t(T, \cdot))\) \(C^1[0, L] \times C^0[0, L] \leq \frac{1}{2} \| (\varphi(x), \psi(x)) \| H^2(0, L) \times H^1(0, L). \) (99)

The proofs of Theorem 4.5 and Corollary 4.7 are given in Section 5.

5. Proofs of Theorem 4.5 and Corollary 4.7. In this section we prove Theorem 4.5 and Corollary 4.7 from Section 4.2. For the proof, we consider the time derivative of the Lyapunov function \( E(t) \).

5.1. Time derivative of the Lyapunov function. First we consider the evaluation of the time derivative of the Lyapunov function \( E(t) \).

Lemma 5.1. Let the assumptions of Theorem 4.5 hold. Then the time-derivative of \( E_1 \) is given by the following equation:

\[
\frac{d}{dt} E_1(t) = I_1 + I_2 + I_3
\]

with

\[
I_1 = \int_0^L h_{2x} (a^2 - (\ddot{u} + u)^2) u_x^2 + h_{2x} u_t^2 \, dx,
\]

\[
I_2 = \int_0^L 2 h_1 (\ddot{u}' + u_x) u_t^2 - 2 h_1 (\ddot{u} + u) u_t u_x^2 + 4 h_1 (\ddot{u} + u) (\ddot{u}' + u_x) u_t u_x,
\]

\[
I_3 = \left( (a^2 - (\ddot{u} + u)^2) (2 h_1 u_x u_t - 2 h_2 F u_x) - (2 h_1 (\ddot{u} + u) + h_2) u_t^2 \right)_{x=0}.
\]

Proof. With the notation \( \hat{d} = a^2 - (\ddot{u} + u)^2 \) we have \( \hat{d}_t = -2(\ddot{u} + u) u_t \), \( \hat{d}_x = -2(\ddot{u} + u) (\ddot{u}' + u_x) \) and

\[
E_1(t) = \int_0^L h_1 \left( \hat{d} u_x^2 + u_t^2 \right) - 2 h_2 \left( (\ddot{u} + u) u_x^2 + u_t u_x \right) \, dx.
\]

Hence differentiation yields

\[
\frac{d}{dt} E_1(t) = \int_0^L 2 h_1 \left( (u_{tt} - (\ddot{u} + u) u_t^2) u_t + \hat{d} u_x u_{xt} \right) - 2 h_2 \left( u_t u_x^2 + (u_{tt} + 2(\ddot{u} + u) u_{xt}) u_x + u_t u_{xt} \right) \, dx.
\]

Now integration by parts for the term \( \hat{d} u_x u_{xt} = (\hat{d} u_x) (u_t)_x \) yields the equation

\[
\frac{d}{dt} E_1(t) = \int_0^L 2 h_1 \left[ u_{tt} - \hat{d} u_{xx} - \hat{d}_x u_x - (\ddot{u} + u) u_x^2 \right] u_t
\]

\[
- 2 h_2 \left[ u_t u_x^2 + (u_{tt} + 2(\ddot{u} + u) u_{xt}) u_x + u_t u_{xt} \right] \, dx + \left[ 2 h_1 \hat{d} u_x u_t \right]_{x=0}.
\]

Hence we get the equation

\[
\frac{d}{dt} E_1(t) = \int_0^L 2 h_1 \left[ (u_{tt} - \hat{d} u_{xx}) u_t - \hat{d}_x u_x u_t - (\ddot{u} + u) u_x^2 u_t \right]
\]

\[
- 2 h_2 \left[ u_t u_x^2 + (u_{tt} + 2(\ddot{u} + u) u_{xt}) u_x + u_t u_{xt} \right] \, dx
\]

\[
+ \left[ 2 h_1 \hat{d} u_x u_t \right]_{x=0}.
\]
By the partial differential equation (20) we have \( u_{tx} - \hat{d} u_{xx} = F - 2(\bar{u} + u)u_{tx} \) and obtain
\[
\frac{d}{dt} E_1(t) = \int_0^L 2h_1 \left[ (F - 2(\bar{u} + u)u_{tx})u_t + 2(\bar{u} + u)(u' + u_x)u_t u_x - (\bar{u} + u)u_t u_x^2 \right] - 2h_2 \left[ F + \hat{d} u_{xx} \right] u_t + u_t^2 + u_t u_{tx} \, dx + \left[ 2h_1 \hat{d} u_t u_x \right]_{x=0}^L.
\]

Using integration by parts we obtain the identities
\[
\int_0^L -4h_1(\bar{u} + u)u_t u_{tx} \, dx = \int_0^L -2h_1(\bar{u} + u)(u_t^2) x \, dx = \left[ -2h_1 u_t^2(\bar{u} + u) \right]_{x=0}^L + \int_0^L 2 (h_1(\bar{u} + u)) u_t^2 \, dx
\]
and
\[
\int_0^L -2h_2 \hat{d} u_x u_{tx} - 2h_2 u_t u_{tx} \, dx = \int_0^L -h_2 \hat{d}(u_t^2) x - h_2 (u_t^2) \, dx
\]
\[
= \left[ -h_2 \hat{d} u_t^2 - h_2 u_t^2 \right]_{x=0}^L + \int_0^L h_2 \hat{d} u_t^2 - 2h_2 (\bar{u} + u)(\bar{u}' + u_x) u_t^2 + h_2 x (u_t^2) \, dx.
\]
Using these identities we obtain the equation \( \frac{d}{dt} E_1(t) = I_1 + I_2 + I_3 \). Here, \( I_3 \) contains all the terms coming from the boundary and \( I_1 = \int_0^L h_2 \hat{d} u_t^2 + h_2 x u_t^2 \, dx \) contains all the terms where \( h_2 x \) appears. The remaining terms appear in \( I_2 \).

Similarly the next lemma is proved, where the time derivative of \( E_2 \) is considered.

**Lemma 5.2.** Let the assumption of Theorem 4.5 hold. Then the following equation holds:
\[
\frac{d}{dt} E_2(t) = \hat{I}_1 + \hat{I}_2 + \hat{I}_3
\]
with
\[
\hat{I}_1 = \int_0^L h_2 x (a^2 - (\bar{u} + u)^2) u_{xx}^2 + h_2 x u_{tx}^2 \, dx,
\]
\[
\hat{I}_2 = \int_0^L 4 h_2 (\bar{u}' + u_x) u_{xx} u_{tx} - 2h_2 u_t u_{xx}^2 + 2h_2 (\bar{u} + u)(\bar{u}' + u_x) u_{xx}^2 - 2h_2 F_x u_{xx} - 2h_1 (\bar{u} + u) u_t u_{xx}^2 - 2h_1 (\bar{u}' + u_x) u_{tx}^2 + 2h_1 F_x u_{tx} \, dx,
\]
\[
\hat{I}_3 = [(a^2 - (\bar{u} + u)^2)(2h_1 u_{xx} u_{tx} - h_2 u_{xx}^2) - (2h_1 (\bar{u} + u) + h_2) u_{tx}^2]_{x=0}^L.
\]

**Proof.** Again using the notation \( \hat{d} = a^2 - (\bar{u} + u)^2 \) we have
\[
\frac{d}{dt} E_2(t) = \int_0^L 2h_1 \left[ u_{ttt} u_{tx} - (\bar{u} + u) u_{xx}^2 u_t + \hat{d} u_{xx} u_{xxx} \right] - 2h_2 \left[ u_t u_{xx}^2 + (u_{ttt} + 2(\bar{u} + u) u_{xxx}) u_{xx} + u_{tx} u_{xxx} \right] \, dx.
\]
Integration by parts for the term \( \hat{d} u_{xx} u_{xxt} = (\hat{d} u_{xx}) (u_{tx}) \) yields the equation

\[
\frac{d}{dt} E_2(t) = \int_0^L 2 h_1 \left[ u_{tx} u_{tx} - (\ddot{u} + u) u_{xx}^2 u_t - \hat{d} u_{xxx} u_{tx} - \hat{d}_x u_{xx} u_{tx} \right] dx
\]

\[
+ \left[ 2 h_1 \hat{d} u_{xx} u_{tx} \right]_{x=0}^L
\]

\[
- \int_0^L 2 h_2 \left[ u_t u_{xx}^2 + (u_{tx} + 2(\ddot{u} + u) u_{xx}) u_{xx} + u_{tx} u_{txx} \right] dx.
\]

Hence we get the equation

\[
\frac{d}{dt} E_2(t) = \int_0^L 2 h_1 \left[ (u_{tx} - \hat{d} u_{xxx}) u_{tx} - (\ddot{u} + u) u_{xx}^2 u_t - \hat{d}_x u_{xx} u_{tx} \right] dx
\]

\[
+ \left[ 2 h_1 \hat{d} u_{xx} u_{tx} \right]_{x=0}^L
\]

\[
- \int_0^L 2 h_2 \left[ u_t u_{xx}^2 + (u_{tx} + 2(\ddot{u} + u) u_{xx}) u_{xx} + u_{tx} u_{txx} \right] dx.
\]

By the partial differential equation (20) we have

\[ u_{ttx} - \hat{d} u_{xxx} = \hat{d}_x u_{xx} + F_x - 2(\ddot{u}' + u_x) u_{tx} - 2(\ddot{u} + u) u_{txx} \]

and obtain

\[
\frac{d}{dt} E_2(t)
\]

\[
= \int_0^L 2 h_1 \left[ (\hat{d}_x u_{xx} + F_x - 2(\ddot{u}' + u_x) u_{tx} - 2(\ddot{u} + u) u_{txx}) \right] u_{tx}
\]

\[
- (\ddot{u} + u) u_{xx}^2 u_t - \hat{d}_x u_{xx} u_{tx} \right] dx + \left[ 2 h_1 \hat{d} u_{xx} u_{tx} \right]_{x=0}^L
\]

\[
- \int_0^L 2 h_2 \left[ u_t u_{xx}^2 + (\hat{d} u_{xxx} + \hat{d}_x u_{xx} + F_x - 2(\ddot{u}' + u_x)) u_{xx} \right] u_{xx} + u_{tx} u_{txx} \right] dx
\]

\[
= \int_0^L 2 h_1 \left[ (F_x - 2(\ddot{u}' + u_x) u_{tx} - 2(\ddot{u} + u) u_{txx}) \right] u_{tx} - (\ddot{u} + u) u_{xx}^2 u_t \right] dx
\]

\[
+ \left[ 2 h_1 \hat{d} u_{xx} u_{tx} \right]_{x=0}^L
\]

\[
- \int_0^L 2 h_2 \left[ u_t u_{xx}^2 + (\hat{d} u_{xxx} + \hat{d}_x u_{xx} + F_x - 2(\ddot{u}' + u_x)) u_{xx} \right] u_{xx} + u_{tx} u_{txx} \right] dx.
\]

Using integration by parts we obtain the identities

\[
\int_0^L -2 h_2 (\hat{d} u_{xxx} + \hat{d}_x u_{xx}) u_{xx} - 2 h_2 u_{tx} u_{txx} dx
\]

\[
= \int_0^L -h_2 \hat{d}_x (u_{xx})_x - 2 h_2 \hat{d}_x (u_{xx})^2 - h_2 (u_{tx})_x^2 dx
\]

\[
= \int_0^L h_2 \hat{d}_x u_{xx}^2 + h_2 u_{xx}^2 + 2 h_2 (\ddot{u} + u)(\ddot{u}' + u_x) u_{xx}^2 dx
\]

\[
+ \left[ -h_2 \hat{d} u_{xx}^2 - h_2 u_{xx}^2 \right]_{x=0}^L
\]
and
\[ \int_0^L -4h_1(\bar{u} + u)u_{tx}u_{txx} \, dx = [-2h_1(\bar{u} + u)u_{tx}]_{x=0}^L + \int_0^L 2h_1(\bar{u}' + u_x)u_{tx}^2 \, dx. \]

Using these identities we obtain \( \frac{d}{dt}E(t) = I_1 + I_2 + I_3 \) where \( I_3 \) contains all the terms coming from the boundary and \( I_1 = \int_0^L h_2x \bar{u}_{xx} + h_2x u_{tx}^2 \, dx \) contains all the terms where \( h_{2x} \) appears.

5.2. Proof of Theorem 4.5.

Proof. In the proof, we use Lemma 4.1. Therefore we assume that \( \bar{u} \) is sufficiently small in the sense that (51) holds. Moreover, we use Lemma 4.2. Therefore we assume that \( \varepsilon_0(T) > 0 \) is sufficiently small such that (88) holds. We have

\[ \frac{d}{dt}E(t) = \frac{d}{dt}E_1(t) + \frac{d}{dt}E_2(t). \] (108)

First we consider \( \frac{d}{dt}E_1(t) = I_1 + I_2 + I_3 \). Define \( h_2 = \frac{1}{L} \). By the definition of \( h_2 \) in (45) we have \( (h_2)_x = -\mu_2 h_2 \) and thus

\[ I_1 = -\int_0^L (a^2 - (\bar{u} + u)^2) \mu_2 h_2 u_x^2 + \mu_2 h_2 u_{tx}^2 \, dx. \] (109)

For all \( x \in [0, L] \) we have \( \mu_2 h_2(x) \geq \frac{1}{eL} \) hence we have

\[ I_1 \leq -\frac{1}{2eLk} \int_0^L 2h_1(a^2 - (\bar{u} + u)^2) u_x^2 + 2h_1 u_{tx}^2 \, dx. \]

Thus, by (73) we have

\[ I_1 \leq -\frac{1}{2eLk} E_1(t). \] (110)

Now we consider the term \( I_2 \) as defined in (101). Note that due to (23), each of the terms that are added in \( I_2 \), in particular \( F \) \( u_x \) and \( F u_{tx} \), contains a second order term of \( u, u_x, u_{tx} \) as a factor, that is \( u u_x, u u_{tx}, u_x^2 \) or \( u_{tx}^2 \).

More precisely, the terms that appear as factors are either third order terms \( u_x u_{tx}^2, u_x u_{tx}, u_x u_{tx}, u_{tx}^2 \) or terms of the form \( \theta uu_x \), \( \theta uu_{tx} \), \( \theta uu_{tx} \) or \( \theta uu_{tx} \). Since we have \( h_1(x) = k \) and \( \max_{x \in [0, L]} |h_2(x)| = 1 \), the definition of \( I_2 \) implies that there exists a continuous function \( P_0 \) with \( P_0(0) = 0 \) such that we have an estimate of the form

\[ I_2 \leq P_0 \left( \max_{x \in [0, L]} \{|u(t,x)|, |u_x(t,x)|, |u_t(t,x)|, |\bar{u}(x)|\} \right) \int_0^L (u_x^2 + u_{tx}^2 + u_{txx}^2) \, dx. \]

In fact, the definitions of \( I_2, F \) and \( F \) imply that we can choose

\[ P_0(t) = 4k^2 t^2 + 2k \left( 1 + \frac{\theta}{2} \right) t + 4k \left( 1 + \frac{\theta}{2} \right) t^2 + 2t + 4 \left( 1 + \frac{\theta}{2} \right) t^3 + \theta t^4 + 2t^2 + 3\theta t^3 \]

\[ + \theta t^4 + 3\theta t^3 + \theta t^4 + 3\theta t^3 \]

\[ + 2t + 2t^2 + \frac{3}{2} \theta t^2 + \theta t. \] (111)
Using (82), and then (70), (71) we obtain the inequality
\[ I_2 \leq P_0(\Theta(t))(1 + L^2) \int_0^L (u_x^2 + u_{tx}^2) \, dx \]
\[ \leq P_0(\Theta(t))(1 + L^2) \left( \frac{1}{K_1} + \frac{1}{K_1} \right) E_1(t). \tag{112} \]

Now we focus on the boundary term \( I_3 \). We use the notation \( \bar{u}_0 := \bar{u}(0) \) and \( \bar{u}_L := \bar{u}(L) \). Since \( k > 0 \), by the boundary conditions (25), (26) we have
\[ I_3 = I_3^L - I_3^0 \tag{113} \]
where
\[ I_3^L = -(a^2 - \bar{u}_0^2) e^{-1} u_x^2(t, L) \tag{114} \]
\[ I_3^0 = \left( a^2 - \left( (\bar{u}_0 + u(t, 0)) + \frac{1}{k} \right)^2 \right) u_x^2(t, 0). \tag{115} \]

Since (89) holds, we have \( |\bar{u}_0 + u(t, 0)| \leq |\bar{u}_0| + \gamma a - |\bar{u}_0| = \gamma a \leq a - \frac{1}{k} \). Hence we have \( I_3^0 \geq 0 \). Since \( I_3^L \leq 0 \), due to (113) this implies
\[ I_3 \leq 0. \tag{116} \]

Then inequalities (110), (112) and (116) yield
\[
\frac{d}{dt} E_1(t) \leq -\frac{1}{2e L k} E_1(t) + P_0(\Theta(t))(1 + L^2) \left( \frac{1}{K_1} + \frac{1}{K_1} \right) E_1(t) \\
= - \left( \frac{1}{2e L k} - P_0(\Theta(t))(1 + L^2) \left( \frac{1}{K_1} + \frac{1}{K_1} \right) \right) E_1(t). \tag{117} \]

With (90) this implies that \( E_1(t) \) is a strict Lyapunov function and (93) holds.

Similarly, for \( \tilde{I}_1 \), we infer
\[ \tilde{I}_1 = - \int_0^L (a^2 - (\bar{u} + u)^2) \mu_2 h_2 u_{xx}^2 + \mu_2 h_2 u_{tx}^2 \, dx \leq - \frac{1}{2e L k} \int_0^L (a^2 - (\bar{u} + u)^2) 2 h_1 u_{xx}^2 + 2 h_1 u_{tx}^2 \, dx. \]

Hence (77) yields
\[ \tilde{I}_1 \leq - \frac{1}{2e L k} E_2(t). \tag{118} \]

Now we consider \( \tilde{I}_2 \) as defined in (106). All the terms that are added in \( \tilde{I}_2 \) are contain factors \( u_{xx}, u_{tx}, u_{ttx}, F_x, u_{tx} \). Except for \( F_x u_{xx}, F_x u_{tx} \), it can easily be seen that the coefficients that are multiplied with these quadratic terms become arbitrarily small if if \( \Theta(t) \) as defined in (91) is sufficiently small.

Now we have a closer look at \( F_x \). From (23), we have
\[
F_x = -2u_x u_{tx} - 2u_t u_{xx} - 2u_x^3 - 4u_x u_{xx} - 3\theta u x u_x^2 - \frac{3}{2} \theta u_x u_{xx} - \theta u_x u_t - \theta uu_{tx} \]
\[ - \frac{\theta^3}{8(a^2 - \bar{u}^2)^4} \bar{u}^3 - \frac{3a^2 \bar{u}^3 - 3a^2 \bar{u}^3}{2(a^2 - \bar{u}^2)^3} u_{tx} \]
\[ - \frac{\theta^3}{a^2 - \bar{u}^2} u_{x}^2 - 4\bar{u} u x u_{xx} - \theta \frac{3a^2 \bar{u} - \bar{u}^3}{a^2 - \bar{u}^2} (u_x^2 + uu_{xx}) \]
Also in \(F_x u_{xx}, F_x u_{tx}\) all the terms that are added contain quadratic factors \(u_{xx} u_{tx}, u_{xx}, u_{tx}, u_x u_{tx}, u_x u_{xx}, u_t u_{tx}, u_t u_{xx}, u u_{xx}\) and the coefficients that are multiplied with these factors become arbitrarily small if if \(\Theta(t)\) as defined in (91) is sufficiently small. Thus similar as in the estimate of \(I_2\), we can find a continuous function \(P_1(t)\) with \(P_1(0) = 0\) such that using (82), and then (70), (71) we obtain the inequality

\[
\dot{I}_2 \leq P_1(\Theta(t)) (1 + L^2) \int_0^L \left( u_x^2 + u_{xx}^2 + u_{tx}^2 + u_{xxx}^2 \right) dx
\]

\[
\leq P_1(\Theta(t)) (1 + L^2) \left( \frac{1}{K_1} + \frac{1}{K_1} \right) (E_1(t) + E_2(t)).
\]  

(119)

In fact if we replace in the representation of \(F_x\) in each of the terms that are added except for one factor the expressions \(u, u_x, u_t, u_{xx}, u_{tx}, \bar{u}\) by \(t\), and treat the other terms from the definition of \(I_2\) in a similar way since \(h_1 = k\) and \(|h_2| \leq 1\) we can choose

\[
P_1(t) = 8 t + 2 t + 8 t^2 + 4 k t^2 + 4 k t
\]

\[
+ 2(k + 1) \left[ \theta^3 \frac{9a^6 t^2 + 2t^9 + 6a^4 t^5 + 11a^2 t^7}{8(a^2 - t^2)^3} t + \theta^2 t^7 + 6a^4 t^3 + 3a^7 t^5 \right]
\]

\[
+ \theta \frac{3a^3 t + t^3 + 3a^2 t + t^5}{a^2 - t^2} t + \theta^3 \frac{3a^2 t^5 + 4a^2 t^3 + 2t^7 + 2t^5}{2(a^2 - t^2) t} \right]
\]

\[
+ 2t + 2t + 2t^2 + 4t^2 + 3\theta t^2 + \frac{3}{2} \theta t^2 + \theta t + \theta t
\].

(120)

For the boundary term \(\dot{I}_3\), we use (20) in the form

\[
(a^2 - (\bar{u} + u)^2) u_{xx} = u_{tt} + 2(\bar{u} + u) u_{tx} - F.
\]

In particular, for \(x = L\) due to (26) we have \(u(t, L) = u_t(t, L) = u_{tt}(t, L) = 0\), hence at \(x = L\) we get

\[
(a^2 - \bar{u}_L^2) u_{xx}(t, L) = 0 + 2 \bar{u}_L u_{tx}(t, L) - F(t, L).
\]

Using (23) and the definition of \(\bar{F}\) we obtain

\[
F(t, L) = -\bar{u}_L \left( \frac{3}{2} \theta \bar{u}_L + 4 \bar{u}_x(L) \right) u_x(t, L) - 2 \bar{u}_L u_{xx}^2(t, L).
\]

Due to (89) this yields

\[
|F(t, L)| \leq \bar{u}_L \left( \frac{3}{2} \theta \bar{u}_L + 4|\bar{u}_x(L)| + 2 \right) |u_x(t, L)|.
\]  

(121)

We have

\[
\dot{I}_3 = \dot{I}_3^L - \dot{I}_3^0
\]
where \( \tilde{I}_3^0 \) is given in (128) and

\[
\tilde{I}_3^L = \left[ 2k u_{tx}(t, L) - \frac{1}{e} u_{xx}(t, L) \right] [2 \bar{u}_L u_{tx}(t, L) - F(t, L)]
\]

\[
- \left( 2k \bar{u}_L + \frac{1}{e} \right) u_{tx}^2(t, L)
\]

\[
= \left[ 2k u_{tx}(t, L) - \frac{1}{e} \frac{2 \bar{u}_L u_{tx}(t, L) - F(t, L)}{a^2 - \bar{u}_L^2} \right] [2 \bar{u}_L u_{tx}(t, L) - F(t, L)]
\]

\[
- \left( 2k \bar{u}_L + \frac{1}{e} \right) u_{tx}^2(t, L)
\]

\[
= \left[ 2k \bar{u}_L - \frac{1}{e} \frac{a^2 + 3 \bar{u}_L^2}{a^2 - \bar{u}_L^2} \right] u_{tx}^2(t, L)
\]

\[
+ \frac{1}{e} \frac{4 \bar{u}_L - 2k}{a^2 - \bar{u}_L^2} u_{tx}(t, L) F(t, L) - \frac{1}{e} \frac{1}{a^2 - \bar{u}_L^2} F(t, L)^2.
\]

With \( C_g(\bar{u}_L) \) as defined in (56) we have

\[
C_g(\bar{u}_L) = \frac{a^2 - \bar{u}_L^2}{e \bar{u}_L^2} \left( \frac{3}{2} \theta \bar{u}_L + 4 |u_x(L)| + 2 \right) > \frac{a^2 - \bar{u}_L^2}{e \bar{u}_L^2} \left( 2 + \frac{3}{2} \theta \bar{u}_L + \frac{2 \theta \bar{u}_L^3}{a^2 - \bar{u}_L^2} \right)^2 > 0.
\]

Then due to (121) we have \( C_g(\bar{u}_L) F(t, L)^2 \leq (a^2 - \bar{u}_L^2) e^{-1} u_x^2(t, L) \). Hence (114) implies

\[
I_3^L + C_g(\bar{u}_L) F(t, L)^2 \leq \left[ - \frac{1}{e} (a^2 - \bar{u}_L^2) + \frac{1}{e} (a^2 - \bar{u}_L^2) \right] u_x^2(t, L) = 0.
\]

Consider the matrix \( \tilde{A}_3(\bar{u}_L) \) as defined in (57). With the notation \( F = F(t, L) \) and \( u_{tx} = u_{tx}(t, L) \) we have

\[
\tilde{I}_3^L - C_g F^2 = -(u_{tx}, F) \tilde{A}_3(\bar{u}_L) \left( \begin{array}{c} u_{tx} \\ F \end{array} \right).
\]

Due to (88), Lemma 4.1 implies that the matrix \( \tilde{A}_3(\bar{u}_L) \) is positive definite. Thus \( \tilde{I}_3^L - C_g F^2 \leq 0 \). Due to (123) this yields

\[
I_3^L + \tilde{I}_3^L = I_3^L + C_g F^2(t, L) + \tilde{I}_3^L - C_g F^2(t, L) \leq 0.
\]

Now we look at \( \tilde{I}_3^0 \) that depends on the values at \( x = 0 \) where the Neumann condition (25) is prescribed and we have \( u_t(t, 0) = \frac{1}{k} u_x(t, 0), u_{tx}(t, 0) = \frac{1}{k} u_{xx}(t, 0) \). Hence with the notation \( F(t, 0) = F(u(t, 0), u_x(t, 0), u_{tx}(t, 0)) \) and \( \bar{u}(t, 0) = \bar{u}_0 \) for \( x = 0 \), (20) yields

\[
u_{xx}(t, 0) = \frac{1 + 2k (\bar{u}_0 + u(t, 0))}{k (a^2 - (\bar{u}_0 + u(t, 0))^2)} u_{tx}(t, 0) - \frac{1}{(a^2 - (\bar{u}_0 + u(t, 0))^2)} F(t, 0).
\]

Due to (23) we have for \( x = 0 \)

\[
F(t, 0) = -2 \left[ \frac{1}{k} + \bar{u}_0 + u \right] u_x^2
\]

\[- \left[ \frac{\bar{u}_0^4 + 3a^2 \bar{u}_0^2 + 2a^2 \bar{u}_0}{2(a^2 - \bar{u}_0^2)} + \theta \frac{u}{k} + \frac{3}{2} \theta u^2 + \theta \frac{3a^2 \bar{u}_0 - \bar{u}_0^3}{a^2 - \bar{u}_0^2} u \right] u_x \]
Due to (89) this yields

$$|F(t, 0)| \leq 2 \left[ \frac{2}{k^2} + \frac{\bar{u}_0}{k} \right] |u_x(t, 0)|$$

$$+ \left[ \theta \frac{\bar{u}_0^4 + 3a^2 \bar{u}_0^2 + \frac{2}{a^2} \bar{u}_0}{2(a^2 - \bar{u}_0^2)} + \frac{5}{2} \frac{\theta}{k^2} \frac{3a^2 \bar{u}_0 - \bar{u}_0^3}{a^2 - \bar{u}_0^2} \right] |u_x(t, 0)|$$

$$+ \theta \frac{k^2}{2} \frac{6ka^4 \bar{u}_0^3 - 4ka^2 \bar{u}_0^2 + 2k \bar{u}_0 + 2 \bar{u}_0^2 - 3a^2 \bar{u}_0^2 + 3a^4 \bar{u}_0^3}{4(a^2 - \bar{u}_0^2)^3} |u(t, 0)|.$$  

With $K_{d_1}(k, \bar{u}_0)$ as defined in (84) due to Young’s inequality we have

$$F(t, 0)^2 \leq K_{d_1}(k, \bar{u}_0) u_x^2(t, 0) + C_{E1}(\bar{u}_0) u(t, 0)^2$$  \hspace{1cm} (126)$$

with the constant

$$C_{E1}(\bar{u}_0) = 2 \theta \frac{k^4}{k^2} \left[ \frac{6ka^4 \bar{u}_0^3 - 4ka^2 \bar{u}_0^2 + 2k \bar{u}_0 + 2 \bar{u}_0^2 - 3a^2 \bar{u}_0^2 + 3a^4 \bar{u}_0^3}{4(a^2 - \bar{u}_0^2)^3} \right]^2.$$  \hspace{1cm} (127)$$

For $v := 2k^2 > k^2$ we have

$$\tilde{P}_3^0 + v F(t, 0)^2 = (u_x(t, 0), F(t, 0)) \tilde{B}_3(h_{\tilde{B}_3}(t)) \left( \begin{array}{c} u_x(t, 0) \\ F(t, 0) \end{array} \right)$$

with the matrix $\tilde{B}_3(h_{\tilde{B}_3}(t))$ from (55). Since we have assumed that $|\bar{u}_0| < \varepsilon_1(2k^2)$ and $|u(t, 0)| < \varepsilon_1(2k^2)$, by the definition of $h_{\tilde{B}_3}(t)$ this implies $|h_{\tilde{B}_3}(t)| < 2 \varepsilon_1(2k^2)$ hence Lemma 4.1 implies that the matrix $\tilde{B}_3(h_{\tilde{B}_3}(t))$ is positive definite. Thus we have $\tilde{P}_3^0 + v F(t, 0)^2 \geq 0$. Hence due to (115), (89), (126) and (86) we have

$$I_3^0 + \tilde{I}_3^0 = I_3^0 - 2k^2 F(t, 0) + \tilde{I}_3^0 + 2k^2 F(t, 0)$$

$$\geq I_3^0 - 2k^2 F(t, 0)$$

$$\geq \left( a^2 - \left( \bar{u}_0 + \frac{2}{K} \right)^2 - 2k^2 K_{d_1}(k, \bar{u}_0) \right) u_x^2(t, 0) - 2k^2 C_{E1}(\bar{u}_0) u(t, 0)^2$$

$$\geq -2k^2 C_{E1}(\bar{u}_0) u(t, 0)^2.$$  

Due to the Dirichlet boundary condition (26) we have

$$u(t, 0)^2 = |u(t, 0) - u(t, L)|^2 = \left| \int_0^L u_x(t, s) \, ds \right|^2 \leq L \int_0^L u_x^2(s) \, ds \leq \frac{L}{K_1} E_1(t)$$

where the last inequality follows from (70). This yields

$$I_3^0 + \tilde{I}_3^0 \geq -2k^2 C_{E1}(\bar{u}_0) \frac{L}{K_1} E_1(t).$$  \hspace{1cm} (129)$$
We have
\[ \frac{d}{dt} E(t) = I_1 + I_2 + \tilde{I}_1 + \tilde{I}_2 + (I^L_3 + \tilde{I}^L_3) - (I^0_3 + \tilde{I}^0_3). \]

Then inequalities (110), (112), (118), (119), (124) and (129) yield
\[ \frac{d}{dt} E(t) \leq -\left[ \frac{1}{2 e L k} - P_0 (\Theta(t)) (1 + L^2) \left( \frac{1}{K_1} + \frac{1}{\tilde{K}_1} \right) \right] E_1(t) \]
\[ - \left[ \frac{1}{2 e L k} - P_1 (\Theta(t)) (1 + L^2) \left( \frac{1}{K_1} + \frac{1}{\tilde{K}_1} \right) \right] E_2(t) \]
\[ + \left[ P_1 (\Theta(t)) (1 + L^2) \left( \frac{1}{K_1} + \frac{1}{\tilde{K}_1} \right) + 2 k^2 C_{E1}(\tilde{u}_0) \frac{L}{K_1} \right] E_1(t). \]

Note that due to (28), \( \Theta(t) \) becomes arbitrarily small if the norm of the initial data and of \( \tilde{u} \) is sufficiently small. Define the number \( \kappa > 0 \) as in (90) and \( \mu \) as in (92). Then (130) and the definition of \( \mu \) yield
\[ \frac{d}{dt} E(t) \leq -\mu E(t) \text{ for all } t \in [0, T], \]
which implies inequality (94).

5.3. **Proof of Corollary 4.7.** Now we present the proof of Corollary 4.7.

**Proof.** For all \( t \in [0, T] \) the inequalities (81) and (82) for the \( H^2 \)-Lyapunov function \( E(t) \) defined in (46) imply the following inequalities
\[ \| u(t, \cdot) \|_{H^2(0,L)} \leq \sqrt{1 + 2L^2} \sqrt{E(t)}, \]
\[ \| u_t(t, \cdot) \|_{H^1(0,L)} \leq \sqrt{1 - K_{\text{min}}} \sqrt{E(t)}. \]

Inequality (79) implies for \( t = 0 \):
\[ \sqrt{K_{\text{max}}} \| (u(0, \cdot), u_t(0, \cdot)) \|_{H^2(0,L) \times H^1(0,L)} \geq \sqrt{E(0)}. \]

With the positive constants
\[ \tau_1 := \frac{K_{\text{min}}}{1 + 2L^2}, \quad \tau_2 := K_{\text{max}}, \quad \eta_1 := 2 \sqrt{\tau_2 / \tau_1}, \]
the inequalities (132), (133), (94), (134) imply the estimate (96).

The inequality (97) follows from (96) and the Sobolev embedding \( H^2((0,L)) \hookrightarrow C^1([0,L]) \) and \( H^1((0,L)) \hookrightarrow C^0([0,L]), \) see ([3],[36]).

Theorem 4.5 implies that for all \( T > 0 \) the exponential decay rate \( \mu_0 = \frac{1}{4 e L k} \) (see Remark 4.6) can be achieved for sufficiently small initial data. With this decay rate, for
\[ T \geq \max_{i \in \{1, 2\}} 8 e L k \ln(2 \eta_i), \]
and \( i \in \{1, 2\} \) we have the inequality \( \eta_i \exp(-\frac{\mu}{2}T) \leq \frac{1}{2} \). Hence the inequalities (96) and (97) imply the inequalities (98) and (99).
6. Global solutions. In this section we show that the exponential decay of the Lyapunov function defined in (46) implies that the solution exists globally in time without losing regularity, that is it keeps the regularity of the initial state.

Theorem 6.1. (Global Exponential Decay of the \( H^2 \)-Lyapunov Function).
Let a stationary subsonic state \( \bar{u}(x) \in C^2(0, L) \) be given that satisfies (18). Let \( \gamma \in (0, 1/2] \) be given. Assume that for all \( x \in L \) we have \( u(x) \in (0, \gamma a) \). Choose a real number \( k \) such that \( a(1 - \gamma)k > 1 \). Assume that \( \bar{u} \) is sufficiently small and \( k \) sufficiently large such that for \( K_0(k, \bar{u}(0)) \) as defined in (84) condition (86) holds. Assume that \( \| \bar{u} \|_{C^2([0, L])} \) is sufficiently small such that \( \| \bar{u} \|_{C^1([0, L])} < \varepsilon_1(2k^2) \) with \( \varepsilon_1 \) from Lemma 4.1 and (51) holds. Choose \( \varepsilon_2 \) as in Lemma 4.2. Define \( T \) as
\[
T = \max_{i \in \{1, 2\}} 8eL/k \ln(2\eta_i)
\]
with \( \eta_1 \) and \( \eta_2 \) from Corollary 4.7. Choose a real number \( \varepsilon_0 \in (0, \varepsilon_0(T)) \) sufficiently small such that for all initial data that satisfies
\[
\|(\varphi(x), \psi(x))\|_{H^2([0, L]) \times H^1([0, L])} \leq \varepsilon_0 \tag{137}
\]
and the \( C^1 \)-compatibility conditions at the points \( t, x = (0,0) \) and \( t, x \) = (0, L) the solution of problem (24), (25), (26), (30) exists on the time-interval \([0, T]\) and (43) holds. Moreover, assume that \( \varepsilon_0 > 0 \) and \( \bar{u} \) are sufficiently small such that (88), (89) and (90) hold.

If the initial data satisfies (137), the mixed initial-boundary value problem (83) ((20), (24), (25), (26) respectively) has a unique solution \( u \in L^\infty(0, \infty), H^2([0, L]) \) with \( u_t \in L^\infty(0, \infty), H^1([0, L]) \). For the number \( \mu \in [\frac{1}{4Lk}, \frac{1}{2Lk}] \) defined in (92) we have the inequality
\[
E(t) \leq E(0) \exp(-\mu t) \text{ for all } t > 0 \tag{138}
\]
with the strict Lyapunov function \( E(t) \) as defined in (46). Moreover, we have
\[
E_1(t) \leq E_1(0) \exp(-\mu t) \text{ for all } t > 0.
\]

Proof. Due to (137), if \( \varepsilon_0 \) is chosen sufficiently small, (29) holds. Hence Lemma 3.2 implies the existence of \( \bar{u} \in C^2([0, T_0] \times [0, L]) \) that solves the initial-boundary value problem problem (24), (25), (26), (30). Moreover, \( \bar{u} \) satisfies the a priori bound (32). Thus if \( \tilde{\varepsilon}_0 \) and \( \bar{u} \) are chosen sufficiently small, (88), (89) and (90) hold. In addition, (43) holds, see Remark 3.3.

If assumption (87) is weakened to assumption (29) and instead of the \( C^2 \) compatibility conditions only the \( C^1 \) compatibility conditions are assumed, Theorem 4.5 and Corollary 4.7 still hold if in the statement of Theorem 4.5 the \( C^2 \) solution is replaced by the weak solution that satisfies (30), (43) and the a priori inequality (32).

Since \( T \) satisfies (136), the proof of Corollary 4.7 implies that inequality (99) holds. Define the new initial data
\[
(\varphi_1, \psi_1) = (u(T, \cdot), u_t(T, \cdot)).
\]
Then (99) implies
\[
\|(\varphi_1(x), \psi_1(x))\|_{H^2([0, L]) \times H^1([0, L])} \leq \frac{1}{2} \|(\varphi(x), \psi(x))\|_{H^2([0, L]) \times H^1([0, L])} \leq \frac{1}{2} \varepsilon_0.
\]
Therefore \( (\varphi_1, \psi_1) \) satisfy the assumptions of Lemma 3.2, that yields the existence of the solution of the initial boundary value problem on a time interval of the length \( T \). This yields the function \( u(T + s, x) \) for \((s, x) \in [0, T] \times [0, L] \). In this way we
obtain the solution $u$ on the time interval $[T, 2T]$. If we concatenate the solution on $[0, T]$ and the solution on $[T, 2T]$, we obtain the solution on $[0, 2T]$. Similarly as before, due to (99) now we can continue with the initial data

$$(\varphi_2, \psi_2) = \tilde{u}((2T, \cdot), \tilde{u}_t(2T, \cdot))$$

and Lemma 3.2 yields the solution on the interval $[2T, 3T]$ and thus on $[0, 3T]$. By using this construction again and again inductively, this yields the solution on the time intervals $[0, nT]$ for all $n \in \{1, 2, 3, \ldots\}$ and thus for all times $t > 0$. For each time interval $[nT, (n + 1)T]$ the inequalities (93) and (94) from Theorem 4.5 hold. Thus we have for all $n \in \{0, 1, 2, 3, \ldots\}$ and $t \in [0, T]$

$$E_1(nT + t) \leq E_1(nT) \exp(-\mu t), \ E(nT + t) \leq E(nT) \exp(-\mu t).$$

(139)

In particular, for all $n \in \{0, 1, 2, 3, \ldots\}$ we have

$$E_1((n + 1)T) \leq E_1(nT) \exp(-\mu T), \ E((n + 1)T) \leq E(nT) \exp(-\mu T).$$

By induction, we obtain

$$E_1(nT) \leq E_1(0) \exp(-\mu nT), \ E(nT) \leq E(0) \exp(-\mu nT).$$

With (139) this yields for $t \in [0, T]$

$$E_1(nT + t) \leq E_1(0) \exp(-\mu nT) \exp(-\mu t) = E_1(0) \exp(-\mu (nT + t))$$

and similarly $E(nT + t) \leq E(0) \exp(-\mu (nT + t))$. This yield the exponential decay of the Lyapunov functions $E(t)$ and $E_1(t)$. In particular (138) follows.

7. Summary and outlook. In this paper we have considered a quasilinear wave equation for the velocity of a gas flow that is governed by the isothermal Euler equations for ideal gas with friction. We have presented a method of boundary feedback stabilization to stabilize the velocity locally around a given stationary state. For the proof, we have introduced a strict $H^2$-Lyapunov function (see (46)).

We have shown that, for initial conditions with sufficiently small $C^2 \times C^1$-norm and for appropriate boundary feedback conditions, the $H^2 \times H^1$-norm of the solution decays exponentially with time. In addition, we have shown that with our velocity feedback law, for initial data with sufficiently small $H^2 \times H^1$-norm the solution exists globally in time and the $H^2 \times H^1$-norm of the solution decays exponentially.

In this paper, the strict $H^2$-Lyapunov function is used to prove the stability of the solution. It would also be interesting to consider other types of Lyapunov functions, such as weak Lyapunov functions. Moreover, when a disturbance is considered, Input-to-State Stability Lyapunov functions should be studied (see [31],[32]).

We have presented our stabilization method for a single pipe applying an active control at one end of the pipe. Some additional work is required to extend this method to more complicated gas networks. For the stabilization of networks it is often necessary to apply an active control in the interior of the networks. The well-posedness of systems of balance laws on networks is studied in [18]. For a star-shaped network of vibrating strings governed by the wave equation, a method of boundary feedback stabilization is presented in [20], where not for each string an active control is necessary. A related open problem is the feedback stabilization of more complicated pipe networks with leaks. Moreover, also feedback stabilization of second-order hyperbolic equations with time-delayed controls is worth to be studied. For wave equations, this has been done in [15] and in [33] and for the isothermal Euler equations with an $L^2$-Lyapunov function in [16]. In the current paper we have
considered an ideal gas with constant sound speed. It would be interesting to look
at more realistic models of gas where the sound speed also depends on the pressure.

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