Fermion-induced quantum critical point in Landau-Devonshire theory

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Fluctuations can change the phase transition properties drastically. An example is the fermion-induced quantum critical point (FIQCP), in which fluctuations of the massless Dirac fermions turn a putative Landau-de Gennes first-order phase transition with a cubic boson interaction into a continuous one. Here, we report a new type of FIQCP. In this kind of FIQCP, the boson part falls into the Landau-Devonshire model with a negative quartic interaction and also hosts a first-order phase transition. By using the functional renormalization group analysis, we show that this negative quartic boson coupling can be changed to be positive by fluctuations of the Dirac fermions, resulting in a continuous phase transition, which belongs to the Gross-Neveu-Yukawa universality class. Moreover, we point out that the present FIQCP can be a supersymmetric critical point, which has been excluded in the previous case. We then show that the low-temperature phase diagram can provide sharp experimental evidences to detect the FIQCP.

Introduction. — Understanding universal behaviors in phase transitions is one of the central and challenging issues in modern condensed matter physics and statistical mechanics [1, 2]. According to the divergences of the derivation of the free energy at the transition point, phase transitions can be classified into first-order phase transitions (FOPTs) and continuous ones [3]. The order of the phase transition is reflected in the behavior of the order parameter, which was introduced by Landau and Ginzburg to characterize different phases on different sides of the transition point [4]. At an FOPT the order parameter jumps discontinuously, while at a continuous phase transition the order parameter changes continuously. Based on the symmetry consideration, Landau and Ginzburg proposed that the free energy can be expanded as a series of the order parameter. Moreover, taking the long-wavelength fluctuation of the order parameter into account, Wilson’s renormalization group (RG) theory [5] supplemented the continuous phase transition theory and has made great success in various systems by classifying their critical properties according to the corresponding fixed point of the RG flow [1, 2, 6, 7].

In stark contrast, to what extent the fluctuations can modify the picture of the FOPT is ambiguous. Except for the (1 + 1)-dimensional (D) Potts model [8], usually it is believed that the fluctuations of the order parameter cannot change FOPT qualitatively, since there is no long-wavelength fluctuations even at the transition point of FOPT [9, 10]. Recently, this paradigm has been challenged by various examples, in which the FOPTs can be rounded into continuous ones by the fluctuations from other degrees of freedom, rather than from the order parameter itself. The first example is the deconfined quantum critical point [11–24], which happens at the transition between two ordered phases. According to the Landau-Ginzburg theory, this transition should be an FOPT [4]. But fluctuations of emergent spinons and gauge fields dominate the physics near the transition point and result in a continuous change of the order parameter [11, 12]. Another example is the fermion-induced quantum critical point (FIQCP) [25–31], in which the additional fluctuations come from massless Dirac fermions. Evidence for an FIQCP has been discovered in the phase transition from the Dirac semi-metal to the Kekule valence bond solid (KVBS) in (2 + 1)D honeycomb lattice [25]. Since the $Z_3$-symmetric properties of the KVBS, the free energy includes cubic terms of the order parameter according to the Landau-Ginzburg theory. So the pure boson free energy should have hosted a Landau-de Gennes-type FOPT. However, the story has been changed qualitatively by the massless Dirac fermions. The $1/N$-RG analysis shows that the $Z_3$ term is irrelevant near the Gross-Neveu-Yukawa (GNY) fixed point and the phase transition is changed to be continuous by the Dirac fermion fluctuations [25, 28, 29]. This conclusion has been verified by numerical results [25, 31, 32], $4 - \varepsilon$ expansion [26], and the functional renormalization group (FRG) [27, 30]. In the following, this type of FIQCP will be referred to as type-I FIQCP.

Besides the transition between two ordered phases and the Landau-de Gennes theory discussed above, another very common kind of the FOPT is the Landau-Devonshire model [33]. In the Landau-Devonshire theory, the coefficient of the quartic term is negative and an energy barrier appears at the transition point to separate the ordered phase from the disordered phase. This model describes phase transitions in lots of condensed matter and synthetic cold-atom systems, including the ferroelectric phase transitions [34], magnetic phase transitions of the Blume-Capel model [35, 36], critical non-fermi liquids in itinerant fermionic systems [37], and superfluid phase transitions in optical lattices [38]. Furthermore, investigations on critical behaviors in correlated Dirac systems have attracted persistent attentions [39–51] since their theoretic importance in reflecting some mysteries in high-energy physics and potential applications in electron devices [52–54]. These inspire us to explore the fate of the Landau-Devonshire FOPT in the presence of the
massless Dirac fermions.

In this paper, we explore the effects induced by the coupling between the Landau-Devonshire model and Dirac fermions. Since both first-order and continuous phase transitions are involved, we resort to the non-perturbation FRG method [55, 56], which has been proved to be a very powerful tool in investigations on both classical and quantum phase transitions [26, 27, 57–60]. We find that the long-range fluctuation of the Dirac fermion changes the phase transition drastically. In particular, we find that the negative coefficient of the quartic boson coupling can flow to a positive value with the aid of the fermion fluctuations, giving rise to a new type of the FIQCP, as illustrated in Fig. 1, which will be termed as type-II FIQCP [33], which is separated from the Wilson-Fisher type continuous phase transition by a tricritical point with $\lambda_4 = 0$ in the mean-field level. At the FOPT transition point, the free energies are equal for the ordered phase and disordered phase, as shown in Fig. 1 (a). However, between them, there is an energy barrier proportional to $|\lambda_4|^3$ and the order parameter jumps discontinuously to cross this barrier via nucleation and growth.

**Heuristic analysis.**—Before the full FRG derivation, Let us begin with a heuristic analysis. Physically, the FIQCP describes the continuous phase transition in which the energy barrier at the phase transition point can be erased out by the Dirac fermions. Since the energy barrier in the Landau-Devonshire model is induced by the negative $\lambda_4$, the possibility of the FIQCP is reflected in the impact of the coupling with fermions on $\lambda_4$. To see this, we study the RG flow of $\lambda_4 \equiv k^{D-4} \tilde{\lambda}_4$. For instance, when $S = 1$ and $\lambda_2 = 0$, the one-loop flow equation reads [63, 64]

$$\frac{d\lambda_4}{dt} = \lambda_4 - \frac{6}{\pi^2} \lambda_4^2 + \frac{8N_f g^2}{3\pi^2},$$

in which $t = \ln(k/\Lambda)$ with $\Lambda$ being the UV cutoff from the lattice spacing and variables without tilde are the corresponding dimensionless ones. In Eq. (3), a general flavor number $N_f$ is introduced, although $N_f = 2$ has been chosen in Eq. (1). In the right hand side of Eq. (3), the first two terms come from the boson contribution, while the third term comes from the Dirac fermion fluctuation.
Without coupling to fermions, Eq. (3) shows that a negative $\lambda_4$ in the ultraviolet (UV) length scale will decrease monotonously without a lower bound as the energy scale goes down to the infrared (IR) direction. This corresponds to the Landau-Devonshire FOPT. But, remarkably, this situation can be changed when the coupling to the fermion is introduced. Due to the fermion statistics, the last term in Eq. (3) makes opposite contribution in contrast to the second one. Accordingly, the right hand side of Eq. (3) can be positive for large enough $g$. Thus, this fermion term can propel $\lambda_4$ to flow to positive values and what follows are the disappearance of the energy barrier at the FOPT and the appearance of the FIQCP in the IR length scale. To qualitatively investigate the critical properties of this type-II FIQCP, we employ the FRG analysis in the following.

**FRG results.** The FRG flow of the effective action Eq. (1) satisfies the Wetterich equation [55, 65]

\[
\frac{\partial \Gamma_k}{\partial t} = \frac{1}{2} \text{STr} \left[ (\Gamma_k^{(2)} + R_k)^{-1} \frac{\partial R_k}{\partial t} \right],
\]

in which $R_k$ is the regulator function, STr denotes the supertrace, and $\Gamma_k^{(2)} = \frac{1}{2} \Gamma_k \Gamma_k$ with $\Phi = (\phi_a, \bar{\psi}, \psi)$. In addition, the dimensionless variables, defined as

\[
\rho \equiv Z_{\phi,k} k^{2-D} \bar{\rho}, \quad \psi \equiv \sqrt{Z_{\psi,k}} k^{1-D} \bar{\psi}, \quad g^2 \equiv k^{D-4} Z_{\phi,k} Z_{\psi,k} g^2, \quad u(\rho) \equiv k^{-D} U(\bar{\rho}),
\]

are needed to explore the fixed point information.

To solve Eq. (4), we employ the modified local potential approximation (LPA') [55] with the Litim regulators [66]. The RG equations in the symmetric side read

\[
\partial_t u(\rho) = -D u(\rho) + (D - 2 + \eta_{b,k}) \rho u'(\rho) + 2(S - 1) v_D l_0^{(B)} (u'(\rho), \eta_{b,k}) + 2 v_D l_0^{(R)} (u'(\rho) + 2 \rho u''(\rho), \eta_{b,k}) - 2 v_D N f_d l_0^{(F)} (2 \rho^2, \eta_{f,k}) - 8 (S - 2) v_D l_1^{(F)} (u'(0), \eta_{b,k}, \eta_{f,k}),
\]

that when $g \neq 0$, or more accurately, $g$ is larger than some value $g_{tr}$ (See Fig. 2 (e) and discussions below), the RG flow runs towards a fixed point when $\lambda_2(0)$ is finely tuned. This indicates an emergent FIQCP since $\lambda_2$ is the only relevant scaling variable. Moreover, Fig. 2 (b) shows that $\lambda_4$ changes its sign from negative to positive at some intermediate scale, consistent with the one-loop RG arguments. Also, $g^2$ tends to a nonzero value, showing that the interaction between the Dirac fermionic field and the boson field plays indispensable roles. We then calculate the critical exponents. The correlation length exponent $\nu$ is estimated from the eigenvalue of the stability matrix, while the boson and fermion anomalous dimensions are determined from their definitions. We list the exponents in Table. 1. By comparing these exponents with known results [44–46, 57–60], we find that the type-II FIQCP belongs to the GNY universality class.

Intuitively, for small $g$, the coupling between fermionic and bosonic modes is too weak to soften the FOPT. Thus, there should be a tricritical point, $g_{tr}$, only when $g > g_{tr}$ can the FIQCP arise. Quantitatively, to prevent $\lambda_4$ from
falling, a finite value of $g_{tr}$ is needed, since the UV dimensions of $g^2(0)$ and $\lambda_4(0)$ are equal. We identify $g_{tr}$ by affirming that for $g < g_{tr}$ a stable RG flow cannot exist even though $\lambda_2(0)$ is finely tuned in a large scope. For two sets of bare parameters, $g_{tr}$ is determined in this way as shown in the caption of Fig. 2. In addition, from Eq. (3), we conclude that $g^2_{tr}$ increases as the absolute value of $\lambda_4$ increases and is inversely proportional to $N_f$. These deductions are verified in supplemental material [67]. Furthermore, by noting the essential roles played by the Dirac fermions, we assert this tricritical point is just the chiral tricritical point [68].

For the case of $N_f = 1/2$ and $S = 2$, the GNY universality class corresponds to an emergent supersymmetric fixed point described by the Wess-Zumino theory [69]. Scaling analyses show the type-I FIQCP cannot occur in these systems, since the cubic term is always relevant near the supersymmetric fixed point [25–27]. In contrast, here we show that the type-II FIQCP can exhibit this supersymmetric fixed point [67] (See also caption in Fig. 2 and Table 1). The reason is that the cubic interaction is more relevant than the negative quartic interaction. As a result, the type-I FIQCP can only be realized for large enough $N_f$, independent of $g$. In contrast, the type-II FIQCP can arise for any $N_f$ as long as $g$ is large enough.

**Discussion.**—A crucial question is how to distinguish the type-II FIQCP from usual GNY phase transition with $\lambda_4 > 0$ in experiments. Since any experiment is implemented at finite temperatures, we discuss the thermal effects near the FIQCP. The schematic phase diagram is shown in Fig. 3. We focus on the case for $S = 1$ and compare the FIQCP with the case for $\lambda_4 > 0$. Besides the quantum critical region, which is still controlled by the quantum critical point, there is a classical phase transition. For $\lambda_4 > 0$, the classical phase transition is continuous and fermionic fluctuations are completely decoupled in low frequencies as a result of the finite Matsubara-frequency-gap at finite temperatures [70, 82]. In contrast, for the FIQCP, at finite temperatures, in the light of reductio ad absurdum, fermionic modes cannot change the bosonic FOPT to be continuous. Thus, at the FOPT point, fermion fluctuations may play significant roles in the nucleation and growth. Moreover, the latent energy between two spinodal points should satisfy scaling theory of the FIQCP, as shown in Fig. 3. This can provide sharply detectable evidence for the emergence of the FIQCP in experiments.

Although our discussion begins with a field-theoretic model, the type-II FIQCP can be realized in lattice models. It was shown that FOPT and continuous phase transition have been observed in a spin model with both nearest-neighbor and next-nearest neighbor interactions [71]. The effective theory for this model is just the Landau-Devonshire theory [71]. We can consider a similar model in graphene, whose elementary excitation is the Dirac fermion with $N_f = 2$ and the spin freedom is naturally coupled to the Dirac fermion. Actually, the chiral tricritical point therein is proposed [68]. So its continuous phase transition side should be the type-II FIQCP. For the case of $N_f = 1/2$ and $S = 2$ case, in which the supersymmetric type-II FIQCP appears, it can be realized in the surface of the 3D topological insulator [73–80]. Moreover, various synthetic cold atoms platforms, which host Dirac fermions, have been realized with tunable interactions [72, 81]. It is expected that the type-II FIQCP can be observed therein.

**Summary and outlook.**—In summary, we have studied a type-II FIQCP, in which fermion fluctuations round the bosonic Landau-Devonshire FOPT to a continuous phase transition by changing the negative sign of $\lambda_4$ to be positive. Accordingly, $\lambda_4$ is converted from a relevant scaling variable into a marginal one. The FRG analysis has shown that this continuous phase transition belongs to the GNY universality class. We have also pointed out that to realize the type-II FIQCP, the Yukawa coupling should be larger than a tricritical value. The type-II FIQCP for $N_f = 1/2$ and $S = 2$ has been shown to be a supersymmetric critical point, which has been excluded in the type-I FIQCP. We have also argued that at finite temperatures the classical phase transition is an FOPT, affected by the FIQCP. This can provide distinctly evidences of the FIQCP in experiments. The lattice models

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| $N_f$ = 2, $S$ = 1 | $N_f$ = 2, $S$ = 2 | $N_f$ = 2, $S$ = 3 | $N_f$ = 1/2, $S$ = 2 |
|-------------------|-------------------|-------------------|-------------------|
|                   | $1/\nu$           | $\eta_\nu$        | $\eta_f$          |
| $N_f$ = 2, $S$ = 1 | 0.981             | 0.759             | 0.032             |
| $N_f$ = 2, $S$ = 2 | 0.862             | 0.875             | 0.062             |
| $N_f$ = 2, $S$ = 3 | 0.771             | 1.015             | 0.084             |
| $N_f$ = 1/2, $S$ = 2 | 1.062             | 0.353             | 0.323             |

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![FIG. 3. Schematic phase diagram near an FIQCP. The fan of the quantum critical region is dominated by the zero-temperature FIQCP. Besides, we show the classical phase transition is an FOPT, whose transition point is denoted by $T_1$ (solid). We also argue that the FOPT at low temperatures is also affected by the FIQCP. For instance, the latent heat between two spinodal curves, denoted by $T_s$ (dotted) and $T_c$ (dashed), should satisfy the scaling theory controlled by the FIQCP.](image)
have also been discussed.

In addition, it is also interesting to study the FIQCP in systems with itinerant fermions. The tendency to turn the first-order transitions within the bare model into continuous phase transitions was found in some pioneer works [37, 83]. However, these studies began with a pure boson action in which the fermion fluctuations had been integrated out completely beforehand. Thus, it is instructive to study the action with the fermionic and the bosonic freedoms on the same foot. This is left for further studies.

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SUPPLEMENTAL MATERIAL

A. Threshold functions

In the main text, the explicit formulation of the FRG flow equation depends on the detailed form of the regulator. Using the Litim’s regulator \( R_k(q) = Z_{\phi,k}(k^2 - q^2)\Theta(k^2 - q^2) \) for bosonic fields and \( R_k(q) = Z_{\phi,k}iq_\mu\gamma_\mu(k/q - 1)\Theta(k^2 - q^2) \) for fermion fields, in which \( \Theta \) is the step function, one obtains the threshold functions as follows [3]

\[
\begin{align*}
    l_0^{(B)}(w, \eta_b) &= \frac{2}{D} \left( 1 - \frac{\eta_b}{D + 2} \right) \frac{1}{1 + w}, \\
    l_0^{(F)}(w, \eta_b) &= \frac{2}{D} \left( 1 - \frac{\eta_f}{D + 1} \right) \frac{1}{1 + w}, \\
    l_1^{(F)}(w, \eta_b, \eta_f) &= \frac{2}{D} \left[ \left( 1 - \frac{\eta_f}{D + 1} \right) + \left( 1 - \frac{\eta_b}{D + 2} \right) \frac{1}{1 + w} \right] \frac{1}{1 + w}, \\
    m_4^{(F)}(\eta_f) &= \frac{3}{4} + \frac{1}{2(D - 2)}, \\
    m_{12}^{(FB)}(w, \eta_b) &= \left( 1 - \frac{\eta_b}{D + 1} \right) \frac{1}{(1 + w)^2},
\end{align*}
\]

in which \( w \) corresponds to the boson mass \( u'(\rho) \) and the fermion mass is zero in the symmetric phase. The first three functions are used in the RG flows of the bosonic potential and the Yukawa coupling, and the last two appear in the running anomalous dimensions, which are solved as

\[
\begin{align*}
    \eta_b, k &= \frac{4\nu D}{D} 2N_f d_f h^2 m_4^{(F)}(\eta_f, k), \\
    \eta_f, k &= \frac{8\nu D}{D} S h^2 m_{12}^{(FB)}(u'(0), \eta_b, k),
\end{align*}
\]

for bosonic and the fermionic field, respectively. When \( k \to 0 \) and the parameters therein are chosen as their fixed values, Eqs (S6) and (S7) give the anomalous dimensions of the bosonic and fermionic fields, respectively.

B. Type-II FIQCP for other \( N_f \) and \( S \)

In this section, we discuss the type-II FIQCP for other \( N_f \) and \( S \) by studying the corresponding RG flows. For \( N_f = 2 \) and \( S = 3 \), Fig. S1 shows the RG flows for different parameters. We find that \( \lambda_3 \) is the only relevant parameter, similar to the case of \( S = 1 \). Also, we find that \( \lambda_3 \) also changes from a negative value to a positive fixed point. This indicates that the energy barrier at the transition point of the FOPT is smeared out by the fermion fluctuations and the Landau-Devonshire FOPT is changed to be a continuous one. The corresponding critical exponents \( \nu, \eta_b, \) and \( \eta_f \) are then calculated according to the parameters at the fixed point and have been listed in Table I. One finds that for \( S = 3 \), the type-II FIQCP belongs to the chiral Heisenberg universality class. For the case of \( N_f = 2 \) and \( S = 2 \), the RG flows are quite similar. And, its critical exponents are also listed in Table I. From their values, one finds that the corresponding type-II FIQCP belongs to the chiral XY universality class.

The type-II FIQCP can happen in the case of \( N_f = 1/2 \) and \( S = 2 \), which has been excluded as a type-I FIQCP. Figure S2 shows the RG flows run towards a fixed point by tuning \( \lambda_3(0) \). This fixed point is characterized by the Wess-Zumino supersymmetric field theory [2]. Both the fermionic and the bosonic anomalous dimensions equal \( 1/3 \). The FRG calculation gives an approximated solution, as shown in Table I. This result was also reported in Ref. [3]. Also, Fig. S2 shows again that the quartic coupling \( \lambda_4 \) change from a negative bare value to a positive one, demonstrating the appearance of a continuous phase transition. Besides, \( \lambda_3 \) runs towards a negative fixed point. The reason is that the fermion fluctuations for \( N_f = 1/2 \) are relatively weak.

C. Tricritical point for different \( N_f \)

In this section, we study the dependence of the \( g_{tr} \) on \( N_f \). We take the case of \( S = 1 \) as an example. Directly determining \( g_{tr} \) from the RG flow equations is very complex as a result of the fine-tuning of \( \lambda_2 \). Here, we resort to the
FIG. S1. For $N_f = 2$ and $S = 3$, the RG flows run from $t = 0$ (UV) to $t \to -\infty$ (IR) are shown in (a-d). The bare parameters are chosen as $\lambda_4(0) = -0.2$, $g^2(0) = 0.9$, $\lambda_6(0) = 0.05$, $\lambda_8(0) = \lambda_{10}(0) = \lambda_{12}(0) = 0.1$. By tuning $\lambda_2$, a fixed point can be found at $\lambda_2(0) = r_C = 0.2204655093005$. (a) shows the RG flow for $\lambda_2$ at and near the critical point, as marked. (b-d) show the RG flow at $\lambda_2(0) = r_C$ for $\lambda_4$, $g^2$, and $\lambda_6$, respectively. In particular, $\lambda_4$ runs from a negative value to a positive one as shown in (b). The arrow in (b) denotes the zero point.

FIG. S2. For $N_f = 1/2$ and $S = 2$, the RG flows run from $t = 0$ (UV) to $t \to -\infty$ (IR) are shown in (a-d). The bare parameters are chosen as $\lambda_4(0) = -0.2$, $g^2(0) = 0.9$, $\lambda_6(0) = 0.05$, $\lambda_8(0) = \lambda_{10}(0) = \lambda_{12}(0) = 0.1$. By tuning $\lambda_2$, a fixed point can be found at $\lambda_2(0) = r_C = 0.05919299198090491$. (a) shows the RG flows for $\lambda_2$ at and near the critical point, as marked. (b-d) show the RG flow at $\lambda_2(0) = r_C$ for $\lambda_4$, $g^2$, and $\lambda_6$, respectively. In particular, $\lambda_4$ runs from a negative value to a positive one as shown in (b). The arrow in (b) denotes the zero point.
FIG. S3. Curves of the tricritical point $g_{tr}$ versus $N_f$. Other parameters are chosen as $\lambda_2(0) = 0.1997273432490756$, $\lambda_6(0) = 0.05$, and $\lambda_8(0) = \lambda_{10}(0) = \lambda_{12}(0) = 0.1$. Double logarithmic scales are used. Power fitting shows that $g_{tr}^2 \propto 1/N_f^{0.959}$ for $\lambda_4(0) = -2$.

The method of “virtual” RG flow, which is constructed by multiplying a minus sign before the beta function of the $\lambda_2$ to change artificially $\lambda_2$ into an irrelevant variable while keeps other beta functions unchanged. This method is based on the following thoughts: With an arbitrary set of parameters, if it is controlled by a continuous phase transition with $\lambda_2$ being its only relevant direction, the resulting virtual RG flow will run towards the finite fixed point characterizing this continuous phase transition; In contrast, if it is controlled by an FOPT, there are more than one relevant direction, the virtual flow will go towards infinity, driven by other relevant directions. Therefore, this method can be employed to classify the phase space. However, since the virtual flow is essentially different from the real one, the quantitative performance is needed to be further checked. We have checked that for the parameters chosen in Fig. 2, $g_{tr} \simeq 0.224$ for $\lambda_2(0) = 0.1997273432490756$. We also find that the minimum value of $g_{tr}$ is $g_{tr} \simeq 0.18$ for $\lambda_2(0) < 3$. This value is close to the one obtained directly from the real flow. Also, we find that in the case of $N_f = 1/2$ and $S = 2$ the minimum value of $g_{tr}$ is $g_{tr} \simeq 0.58$ for $\lambda_2(0) < 3$. This value is also close to the one obtained from the real flow.

Figure S3 shows the results for other cases. From Fig. S3 one finds that for a bare fixed $\lambda_2(0)$, the tricritical point $g_{tr}$ decreases as $N_f$ increases. The reason is that for large $N_f$, fermion fluctuations are stronger and the watershed $g_{tr}$ is not needed to be very large to induce a continuous phase transition. In double logarithmic scales, the curve of $g_{tr}^2$ versus $N_f$ is almost a straight line. By power fitting, one finds that $g_{tr}^2$ satisfies $g_{tr}^2 \propto 1/N_f$ approximatively. Also one finds that this rule is also mainly applicable for other values of $\lambda_4(0)$. The reason can be found by inspecting Eq. (3). From Eq. (3), one finds that for $\lambda_4(0) < 0$, the flow direction changes when

$$g^2(0) > \frac{3\lambda_4(0)(6\lambda_4(0) - \pi^2)}{8N_f}.$$  \hspace{1cm} (S8)

Accordingly one obtains $g_{tr}^2(0) \propto 1/N_f$. However, from Fig. S3, one finds that this formula is slightly violated when the absolute value of $\lambda_4$ is small. The reason may be that other parameters, such as $\lambda_6$, would also play roles in determining $g_{tr}$ for small $\lambda_4$.

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