On the separation of correlation-assisted sum capacities of multiple access channels

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Abstract

The capacity of a channel characterizes the maximum rate at which information can be transmitted through the channel asymptotically faithfully. For a channel with multiple senders and a single receiver, computing its sum capacity is possible in theory, but challenging in practice because of the nonconvex optimization involved. In this work, we study the sum capacity of a family of multiple access channels (MACs) obtained from nonlocal games. For any MAC in this family, we obtain an upper bound on the sum rate that depends only on the properties of the game when allowing assistance from an arbitrary set of correlations between the senders. This approach can be used to prove separations between sum capacities when the senders are allowed to share different sets of correlations, such as classical, quantum or no-signalling correlations. We also construct a specific nonlocal game to show that the approach of bounding the sum capacity by relaxing the nonconvex optimization can give arbitrarily loose bounds. Towards a potential solution to this problem, we first prove a Lipschitz-like property for the mutual information. Using a modification of existing algorithms for optimizing Lipschitz-continuous functions, we then show that it is possible to compute the sum capacity of an arbitrary two-sender MAC to a fixed additive precision in quasi-polynomial time. We showcase our method by efficiently computing the sum capacity of a family of two-sender MACs for which one of the input alphabets has size two. Furthermore, we demonstrate with an example that our algorithm may compute the sum capacity to a higher precision than using the convex relaxation.

1 Introduction

Studying information transmission over a noisy channel is of fundamental importance in communication theory. In his landmark paper, Shannon studied the rate of transmission over a channel having a single sender and a single receiver [1]. The maximum number of bits per use of channel that can be transmitted through the channel with asymptotically vanishing error is known as the capacity of the channel [2]. Shannon showed that the channel capacity can be calculated by maximizing the mutual information

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between the input and the output of the channel over all possible input probability distributions [1, 2]. This is a convex optimization problem that can be solved using standard optimization techniques [3] or more specialized methods like the Arimoto-Blahut algorithm [4, 5].

The importance of Shannon’s work was soon recognized, and consequently, Shannon’s ideas were generalized in various ways. In this study, we focus on one such generalization, namely a channel that has multiple senders and a single receiver, commonly known as a multiple access channel (MAC). For such a channel, a tuple of rates is called achievable if each sender can send information through their respective input at their respective rate such that the total error of transmission vanishes asymptotically. The set of all such achievable rate tuples is called the capacity region, and Ahlswede [6] and Liao [7] were the first to give an entropic characterization of it. The total rate at which asymptotically error-free transmission through a MAC is possible is called the sum capacity of the MAC.

While there has been much research on MACs since the work of Ahlswede & Liao, there are no efficient (polynomial time) algorithms to date that compute the sum capacity of a MAC. This difficulty stems from the fact that, unlike the computation of the capacity of a point-to-point channel, the optimization involved in computing the sum capacity is nonconvex [8, 9]. Proposals to solve this nonconvex problem efficiently were found to be unsuitable. On the contrary, it was recently shown that computing the sum capacity to a precision that scales inversely with the cube of the input alphabet size is an NP-hard problem [10]. Therefore, one should not expect efficient general-purpose algorithms for computing the capacity region or the sum capacity of a MAC.

Since computing the sum capacity of a MAC is a hard nonconvex problem, a common approach that is adopted to circumvent this optimization is to relax the optimization to obtain a convex problem. The relaxation gives an upper bound on the sum capacity which can be efficiently computed. Such an approach was adopted, for example, by Calvo et al. [8]. Since we generally expect that there are no efficient algorithms to compute the sum capacity, quantifying the performance of such an upper bound becomes important and essential.

We undertake this task of elucidating the performance of the upper bound on sum capacity obtained through a convex relaxation. For this purpose, we consider MACs that are constructed from nonlocal games. Such MACs were introduced by Leditzky et al. [10] based on previous work of Quek & Shor [11], and subsequently generalized in [12]. We present an analytical upper bound on the sum capacity of such MACs. Our bound extends to cases where the senders of these MACs can share arbitrary correlations, such as classical, quantum, and no-signalling correlations. Our upper bound depends only on the number of question tuples in the nonlocal game and the maximum winning probability of the game when the questions are drawn uniformly and answers are obtained using the strategies allowed by the shared correlations. Using these bounds, we obtain separations between the sum rate obtained from using different sets of correlations. These separations help distinguish the ability of these correlations to assist communication in MAC coding scenarios. In particular, our bound gives a larger separation between the sum capacity and the entanglement-assisted sum rate for the MAC obtained from the magic square game than the previously reported separation [10]. Furthermore, using our bounds, we show how prior bounds on the sum capacity obtained using convex relaxation [8] can be arbitrarily loose.

Our result highlights the need to find better techniques to bound the sum capacity from above. We take a step in this direction by showing that computing the sum capacity is equivalent to optimizing a Lipschitz-like function. Thereupon, we present some algorithms for optimization of Lipschitz-like functions. We show that these algorithms can compute the sum capacity of two-sender MACs to a given additive precision in quasi-polynomial time. Instead of a fixed precision, one of our algorithms can also accept a fixed number of iterations as an input and output an upper bound on the sum capacity.
capacity. In particular, for a specific family of two-sender MACs that includes binary MACs, the number of iterations required for convergence grows at most polynomially with the dimensions and inverse precision. Thus, while it might not be possible to efficiently compute the sum capacity for an arbitrary MAC in practice, we can nevertheless efficiently compute the sum capacity for a large family of MACs.

The rest of the paper is structured as follows. We start in Section 2 with a brief review of quantum states and measurements, nonlocal games, and multiple access channels. Our main study is then divided into two parts. In the first part, we focus on obtaining separations between the sum rate of a MAC assisted by different sets of correlations, and show that convex relaxations of the sum capacity computation can perform rather poorly. Specifically, in Section 3 we obtain an upper bound on the sum capacity of MACs constructed from nonlocal games. We then show in Section 4 that the upper bound on the sum capacity obtained by relaxing the optimization problem can be arbitrarily loose. The second part consists of Section 5, in which we present some algorithms to optimize Lipschitz-like functions and the corresponding convergence analysis. This culminates in algorithms that can compute or upper-bound the sum capacity of any two-sender MAC. The overlap between the two parts has been kept to a minimum, so that the reader can read either one of the parts without loss of continuity.

2 Preliminaries

We briefly review some concepts that are used later. A short summary of the notation used throughout is given below.

We denote the set of natural numbers as \( \mathbb{N} = \{0,1,\ldots\} \) and the set of positive integers as \( \mathbb{N}_+ = \{1,2,\ldots\} \). For any integer \( N \geq 1 \), let \( [N] := \{1,\ldots,N\} \). We denote the \((n-1)\)-dimensional standard simplex by \( \Delta_n = \{ x \in \mathbb{R}^n \mid x \geq 0, \sum_{i=1}^{n} x_i = 1 \} \). When \( x \in \mathbb{R}^n \) is a vector, we interpret the inequality \( x \geq 0 \) component-wise, i.e., \( x_i \geq 0 \) for all \( i \in [n] \). We denote the non-negative orthant in \( n \)-dimensional Euclidean space by \( \mathbb{R}_+^n = \{ x \in \mathbb{R}^n \mid x \geq 0 \} \). The Euclidean inner product between two vectors \( x, y \in \mathbb{R}^n \) is denoted by \( \langle x, y \rangle \). The space of \((m \times n)\)-matrices with entries in \( \mathbb{C} \) is denoted by \( \mathbb{C}^{m \times n} \). The tensor product between two matrices \( A \) and \( B \) is denoted by \( A \otimes B \).

For a random variable \( X \) taking values in a finite alphabet \( \mathcal{X} \), we denote by \( p_X(x) \) the probability of \( X \) taking the value \( x \in \mathcal{X} \). The Shannon entropy of the random variable \( X \) is denoted by \( H(X) = -\sum_{x \in \mathcal{X}} p_X(x) \log(p_X(x)) \). The unit of entropy is referred to as bits when the base of the logarithm is \( e \). The mutual information between two random variables \( X \) and \( Y \) is defined as \( I(X; Y) = H(X) + H(Y) - H(X, Y) \).

A single-input single-output channel is a triple \((\mathcal{X}, \mathcal{Z}, \mathcal{N})\) consisting of an input alphabet \( \mathcal{X} \), an output alphabet \( \mathcal{Z} \) and a probability transition matrix \( \mathcal{N}(z|x) \) giving the probability of transmitting \( z \in \mathcal{Z} \) given the input \( x \in \mathcal{X} \). The capacity of the channel \( \mathcal{N} \) is given by the single-letter formula \( C(\mathcal{N}) = \max_{p_X} I(X; Z) \), where \( X \) and \( Z \) are the random variables describing the input and output of \( \mathcal{N} \), respectively [2].

2.1 Quantum states and measurements

A quantum state (or density matrix) \( \rho \in \mathbb{C}^{d \times d} \) is a self-adjoint, positive semi-definite (PSD) matrix with unit trace [13]. A measurement of this quantum state can be described by a positive operator-valued measure (POVM), a collection of PSD matrices \( \{F_1,\ldots,F_M\} \) of size \( d \times d \) satisfying \( \sum_{m=1}^{M} F_m = \mathbb{I} \), where \( \mathbb{I} \) is the identity matrix on the Hilbert space \( \mathbb{C}^d \). Each POVM element \( F_m \) is associated to a measurement outcome \( m \in [M] \), which is obtained with probability \( \text{Prob}(m) = \text{Tr}(\rho F_m) \) [13].
In quantum mechanics, a Hermitian operator $O \in \mathbb{C}^{d \times d}$ is often called an observable. It can be measured in the following sense. First, we write the spectral decomposition of $O$ as $O = \sum_{m=1}^{M} \lambda_{m} P_{m}$, where $P_{m} \in \mathbb{C}^{d \times d}$ is the projector onto the eigenspace of $O$ corresponding to the eigenvalue $\lambda_{m} \in \mathbb{R}$ for $m = 1, \ldots, M$. Then, the POVM associated with measuring the observable $O$ is given by $\{P_{m} \mid m \in [M]\}$ [13]. For qubits, i.e., two-dimensional quantum systems described by quantum states $\rho \in \mathbb{C}^{2 \times 2}$, the Pauli matrices

$$
\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
$$

represent three commonly used observables associated to the spin of an electron along different axes.

Let two parties, Alice and Bob, share a quantum state $\rho$. If they perform local measurements with POVMs $\{A_1, \ldots, A_J\}$ and $\{B_1, \ldots, B_J\}$, respectively, then the probability that Alice observes the outcome $i$ and Bob observes the outcome $j$ is given by $\text{Prob}(i,j) = \text{Tr}[(\rho A_i \otimes B_j)]$. In other words, the overall POVM for the measurement is given by $\{A_i \otimes B_j \mid i \in [I], j \in [J]\}$.

### 2.2 Nonlocal games

A nonlocal game is played between two players who each receive a question from a referee that they need to answer. The players are not allowed to communicate with each other during the game. Prior to starting the game, one fixes the set of questions and answers and a winning criterion, which are known to everyone. A referee then randomly draws questions and hands them out to the players. If the answers of the players satisfy the winning condition, they win the game.

Formally, an $N$-player promise-free nonlocal game $G$ is a tuple $G = (\mathcal{X}, \mathcal{Y}, \mathcal{W})$, where $\mathcal{X}$ and $\mathcal{Y}$ are the question and answer set for the $i$th player, respectively, and the winning condition $W \subseteq (\mathcal{X}_1 \times \cdots \times \mathcal{X}_N) \times (\mathcal{Y}_1 \times \cdots \times \mathcal{Y}_N)$ determines the tuples of questions and answers that win the game [14]. Throughout this study, we restrict our attention to the case when $\mathcal{X}_i$ and $\mathcal{Y}_i$ are finite sets for all $i \in [N]$. Unless stated otherwise, we will always refer to promise-free nonlocal games as nonlocal games.\(^1\)

For convenience, we denote the question set as $\mathcal{X} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_N$ and the answer set as $\mathcal{Y} = \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_N$. Given any question tuple $x = (x_1, \ldots, x_N) \in \mathcal{X}$, the $i$th question is given by $x_i \in \mathcal{X}_i$, and a similar notation is used for answers.

A strategy for the game $G = (\mathcal{X}, \mathcal{Y}, \mathcal{W})$ is any conditional probability distribution $p_{\mathcal{Y} \mid \mathcal{X}}(y \mid x)$ on the answer tuples $y \in \mathcal{Y}$ given a question tuple $x \in \mathcal{X}$. There are different types of strategies that one can consider, depending on whether the players are allowed to use shared correlations after the game starts. We list a few strategies that are central to our study.

1. **Classical strategy:** This is the typical setup of a nonlocal game. Given a question $x_i \in \mathcal{X}_i$, the $i$th player decides an answer as per the probability distribution $p_{\mathcal{Y}_i \mid \mathcal{X}_i}(y_i \mid x_i)$, $i \in [N]$. This gives rise to a classical strategy

$$
p_{\mathcal{Y} \mid \mathcal{X}}(y_1, \ldots, y_N \mid x_1, \ldots, x_N) = p_{\mathcal{Y}_1 \mid \mathcal{X}_1}(y_1 \mid x_1) \cdots p_{\mathcal{Y}_N \mid \mathcal{X}_N}(y_N \mid x_N)
$$

\(^1\)One may consider nonlocal games for which the possible question tuples are restricted to a subset $P \subseteq \mathcal{X}_1 \times \cdots \times \mathcal{X}_N$ called a promise. A promise-free nonlocal game as defined above is one with $P = \mathcal{X}_1 \times \cdots \times \mathcal{X}_N$. Any nonlocal game $G = (\mathcal{X}, \mathcal{Y}, \mathcal{W})$ with promise $P$ can be turned into a promise-free game by defining a new winning condition $W' = W \cup [(\mathcal{X}_1 \times \cdots \times \mathcal{X}_N) \setminus P] \times (\mathcal{Y}_1 \times \cdots \times \mathcal{Y}_N)$, i.e., the players automatically win on question tuples not contained in the promise.
In the general setting the players are allowed to choose such a strategy on the basis of the outcome of a random variable shared by all players, which is called a probabilistic strategy. Therefore, the set of classical strategies $\mathcal{S}_{cl}$ corresponds to the convex hull of conditional probability distributions of the form given in Eq. (2) [15]. Operationally, convex combinations of product distributions correspond to shared randomness, and hence the set of classical correlations we use is similar to those defined by local hidden variable theories [15, 16]. However, when the questions are drawn uniformly at random, the classical strategy maximizing the probability of winning the game is of the form given in Eq. (2), with $p_{yx_i|x_i} = \delta_{y_i,f_i(x_i)}$ for $i \in [N]$ [14]. Here, $f_i : \mathcal{X}_i \rightarrow \mathcal{Y}_i$ is a function that outputs an answer given a question. Such a strategy is called a deterministic strategy.

2. Quantum strategy: The players have access to a shared quantum state $\rho$, but cannot communicate otherwise. Given a question $x_i \in \mathcal{X}_i$, the $i$th player performs a local measurement with some POVM $\{E^{(x_i)}_y \mid y \in \mathcal{Y}_i\}$. Subsequently, a quantum strategy is described by the probability distribution

$$p_{Y \mid X}(y_1, \ldots, y_N | x_1, \ldots, x_N) = \text{Tr} \left[ \rho \left( E^{(x_1)}_{y_1} \otimes \cdots \otimes E^{(x_N)}_{y_N} \right) \right]$$

We denote the set of quantum strategies by $\mathcal{S}_Q$. For any given classical strategy, one can construct a quantum state and POVMs so that the quantum strategy reduces to the given classical strategy, and therefore, $\mathcal{S}_{cl} \subseteq \mathcal{S}_Q$ [15].

3. No-signalling strategy: A strategy $p_{Y \mid X}(y_1, \ldots, y_N | x_1, \ldots, x_N)$ is said to be no-signalling if

$$p_{Y_i \mid X}(y_i | x_i, \ldots, x_N) = p_{Y_i \mid X}(y_i | x_i), \quad i \in [N]$$

for all $x_1, \ldots, x_N$ [17]. Informally, this means that players must respect locality, i.e., no information can be transmitted between players “faster than light”. As a consequence, the strategy used by each player cannot depend on the questions received by the other players. We denote the set of no-signalling strategies by $\mathcal{S}_{NS}$. Both classical and quantum strategies are no-signalling, and therefore, $\mathcal{S}_{cl} \subseteq \mathcal{S}_Q \subseteq \mathcal{S}_{NS}$.

4. Full communication: When we impose no restriction on the distribution $p_{Y \mid X}(y | x)$, we are implicitly allowing for communication between the players after they have received the questions. The set of all possible conditional probability distributions $p_{Y \mid X}(y | x)$ is denoted by $\mathcal{S}_{all}$, which corresponds to allowing full communication between the players. This contains all classical, quantum and no-signalling strategies. Communication between the players is usually not allowed in nonlocal games, but we introduce this setting here for later use nevertheless.

In summary, we have the following hierarchy of correlations: $\mathcal{S}_{cl} \subseteq \mathcal{S}_Q \subseteq \mathcal{S}_{NS} \subseteq \mathcal{S}_{all}$.

Suppose that the questions are drawn randomly from the set $\mathcal{X}$ as per the probability distribution $\pi(x)$. Say the players obtain their answers using strategies in some set $\mathcal{S}$. Then the maximum winning probability of the game $G$ is given by

$$\omega^\mathcal{S}_\pi(G) = \sup_{p_{Y \mid X} \in \mathcal{S}} \sum_{(x,y) \in \mathcal{Y}} \pi(x) p_{Y \mid X}(y | x).$$

Notice in particular that, if $\mathcal{S} \subseteq \mathcal{S}'$, then $\omega^\mathcal{S}_\pi(G) \leq \omega^\mathcal{S'}_\pi(G)$. Hence, as we go from classical to quantum to no-signalling strategies, the winning probability never decreases.

We will mainly be concerned with the scenario where the questions are drawn uniformly at random, and so $\pi = \pi_U$ is the uniform distribution on $\mathcal{X}$ with $\pi_U(x) = 1/|\mathcal{X}|$ for all $x \in \mathcal{X}$. In this case, we
use the notation \( \omega^\mathcal{C}_n(G) = \omega^\mathcal{C}(G) \). The maximum winning probability when the questions are drawn uniformly and answers are generated using classical strategies is written as \( \omega^\mathcal{C}(G) \). Similarly, when using quantum strategies, we write \( \omega^\mathcal{Q}(G) \), and when using no-signalling strategies, we write \( \omega^\mathcal{NS}(G) \).

### 2.3 Multiple Access Channels

A discrete memoryless multiple access channel without feedback, simply referred to as multiple access channel in this study, is a tuple \((\mathcal{B}_1, \ldots, \mathcal{B}_N, \mathcal{Z}, \mathcal{N}(z|b_1, \ldots, b_N))\) consisting of input alphabets \(\mathcal{B}_1, \ldots, \mathcal{B}_N\), an output alphabet \(\mathcal{Z}\), and a probability transition matrix \(\mathcal{N}(z|b_1, \ldots, b_N)\) that gives the probability that the channel output is \(z \in \mathcal{Z}\) when the input is \(b_1 \in \mathcal{B}_1, \ldots, b_N \in \mathcal{B}_N\) \cite{2}. For simplicity of notation, we will denote a MAC by the probability transition matrix \(\mathcal{N}\) when the input and output alphabets are understood.

Since transmission over the channel is error-prone, one usually encodes the messages and transmits them over multiple uses of the channel, and subsequently, the transmitted symbols are decoded to reconstruct the original message. Formally, an \((\mathcal{M}_1, \ldots, \mathcal{M}_N, n)\)-code for a multiple access channel consists of message sets \(\mathcal{M}_i = \{1, \ldots, M_i\}\) for \(i \in [N]\), encoding functions \(g_i^{(e)} : \mathcal{M}_i \to \mathcal{B}_i^{\times n}\) for \(i \in [N]\), and a decoding function \(g_i^{(d)} : \mathcal{Z}^n \to \mathcal{M}_1 \times \cdots \times \mathcal{M}_N \cup \{\text{err}\}\), where err is an error symbol \cite{2}. For convenience, we denote the message set as \(\mathcal{M} = \mathcal{M}_1 \times \cdots \times \mathcal{M}_N\). The performance of the code can be quantified by the average probability of error in reconstructing the message,

\[
P_e(n) = \frac{1}{|\mathcal{M}|} \sum_{m \in \mathcal{M}} \text{Prob}\{g_i^{(d)}(Z^n) \neq m \mid \text{message } m \text{ was sent}\}.
\]

Here we assume that the messages are chosen uniformly at random and transmitted through the channel. Note that the above code uses the channel \(n\) times. We say that a rate tuple \((R_1, \ldots, R_N)\) is achievable if there exists a sequence of codes \(([2^{nR_1}], \ldots, [2^{nR_N}]), (n)\) such that \(P_e(n) \to 0\) as \(n \to \infty\) \cite{2}.

The capacity region of a multiple access channel is defined as the closure of the set of achievable rate tuples \((R_1, \ldots, R_N)\). For \(N = 2\) one obtains a two-sender MAC. Ahlswede \cite{6} and Liao \cite{7} were the first to give a so-called single-letter characterization of the capacity region of a two-sender MAC. The capacity region for an \(N\)-sender MAC \(\mathcal{N}\) can be written as

\[
\text{Cap}(\mathcal{N}) = \text{conv} \left\{ (R_1, \ldots, R_N) \mid 0 \leq \sum_{j \in J} R_i \leq I(B_j; Z|B_{\bar{J}}) \forall \emptyset \neq J \subseteq [N], \quad p(b_1, \ldots, b_N) = p(b_1) \cdots p(b_N) \right\}
\] (6)

where \(p(b_1, \ldots, b_N)\) is a probability distribution over the joint random variable \(B_1, \ldots, B_N\) describing the channel input, \(p(b_i)\) is a probability distribution for the random variable \(B_i\) corresponding to the \(i\)-th sender’s input, random variable \(Z\) describes the channel output, and for any set \(J \subseteq [N]\), \(B_J = \{B_j \mid j \in J\}\) \cite{8}. Crucially, the optimization in Eq. (6) restricts the joint input random variable to have a product distribution, i.e., \(p(b_1, \ldots, b_N) = p(b_1) \cdots p(b_N)\). The main quantity of interest in our study is the sum capacity of a MAC \cite{8},

\[
S(\mathcal{N}) = \sup \left\{ \sum_{i=1}^N R_i \mid (R_1, \ldots, R_N) \in \text{Cap}(\mathcal{N}) \right\}
= \max_{p(b_1, \ldots, b_N)} I(B_1, \ldots, B_N; Z) \quad \text{such that } p(b_1, \ldots, b_N) = p(b_1) \cdots p(b_N).
\] (7)
Informally, the sum capacity represents the maximum sum rate at which the senders can send information through the MAC such that the transmission error vanishes asymptotically. Because the maximization involved in computing the sum capacity is constrained to be over product distributions on the input, the resulting optimization problem is nonconvex. This nonconvexity is the main source of difficulty in computing the sum capacity of a MAC in practice [9, 10]. A common approach to avoid this difficulty is to relax the nonconvex constraint and maximize over all possible probability distributions on the input. Such an approach was adopted by Calvo et al. [8], leading to the upper bound

\[ C(\mathcal{N}) = \max_{p(b_1, \ldots, b_N)} I(B_1, \ldots, B_N; Z) \] (8)

on the sum capacity \( S(\mathcal{N}) \), where we now maximize over arbitrary input probability distributions. Note that \( C(\mathcal{N}) \) is the capacity of the channel \( \mathcal{N} \) when we think about it as a single-input single-output channel. Since it is a relaxation of the sum capacity corresponding to the capacity region, we call \( C \) the relaxed sum capacity.

3 Bounding the sum capacity of MACs from nonlocal games

Suppose we are given a promise-free nonlocal game \( G \) with question sets \( \mathcal{X}_1, \ldots, \mathcal{X}_N \), answer sets \( \mathcal{Y}_1, \ldots, \mathcal{Y}_N \), and a winning condition \( W \subseteq (\mathcal{X}_1 \times \cdots \times \mathcal{X}_N) \times (\mathcal{Y}_1 \times \cdots \times \mathcal{Y}_N) \). Following Leditzky et al. [10], we construct a MAC \( \mathcal{N}_G \) with input alphabets \( \mathcal{B}_i = \mathcal{X}_i \times \mathcal{Y}_i \) for \( i \in [N] \), output alphabet \( \mathcal{Z} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_N \), and a probability transition matrix

\[
\mathcal{N}_G(\hat{x}_1, \ldots, \hat{x}_N | x_1, y_1, \ldots, x_N, y_N) = \begin{cases} 
\delta_{\hat{x}_1, x_1} \cdots \delta_{\hat{x}_N, x_N} (x_1, \ldots, x_N, y_1, \ldots, y_N) \in W \\
\frac{1}{d} & (x_1, \ldots, x_N, y_1, \ldots, y_N) \notin W,
\end{cases}
\] (9)

where \( d = |\mathcal{Z}| = |\mathcal{X}_1 \times \cdots \times \mathcal{X}_N| \).

In words, the channel \( \mathcal{N}_G \) takes question-answer pairs from each player as input, and if they win the game, then the questions are transmitted without any noise. However, if they lose the game, a question tuple chosen uniformly at random is output by the channel. For convenience, we will denote the input to the MAC \( \mathcal{N}_G \) as \( \mathcal{X}^{\mathcal{Y}} = (\mathcal{X}_1 \times \mathcal{Y}_1) \times \cdots \times (\mathcal{X}_N \times \mathcal{Y}_N) \), and write \( xy = (x_1, y_1, \ldots, x_N, y_N) \). We will usually abbreviate the phrase “MAC obtained from a nonlocal game” to “nonlocal game MAC”.

Before diving into any technical details, we explain why such MACs are suitable for obtaining bounds on the sum capacity that are better than the relaxed sum capacity. We begin by noting that the sum capacity of the MAC \( \mathcal{N}_G \) can be written as

\[ S(\mathcal{N}_G) = \max_{p^{(1)}(x_1, y_1) \ldots p^{(N)}(x_N, y_N)} I(X_1, Y_1, \ldots, X_N, Y_N; Z) \]

where \((X_i, Y_i)\) is the random variable (with distribution \( p^{(i)} \)) describing the \( i \)th input and \( Z \) is the random variable describing the output of \( \mathcal{N}_G \). Note that \( p^{(i)} \) is a probability distribution over the question-answer pairs of the \( i \)th player. By writing \( p^{(i)}(x_i, y_i) = \pi^{(i)}(x_i)p_{Y_i|X_i}(y_i|x_i) \), we can break \( p^{(i)} \) into a distribution \( \pi^{(i)} \) over questions and a strategy \( p_{Y_i|X_i} \), chosen by the \( i \)th player. As a result, we can write \( p^{(1)} \cdots p^{(N)} = \pi p_{Y|X} \), where \( \pi = \pi^{(1)} \cdots \pi^{(N)} \) is some distribution over the questions and \( p_{Y|X} = p_{Y_1|X_1} \cdots p_{Y_N|X_N} \) is a classical strategy chosen by the players. Such a decomposition allows us to optimize separately over questions and strategies. By performing suitable relaxations, we can obtain a bound on the sum capacity that depends only on the winning probability of the game (see Thm. (5) for a precise statement). In fact, such a proof technique allows us to bound the sum capacity even when the players are allowed to use different sets of strategies such as those obtained.
from quantum or no-signalling correlations. The resulting bound is helpful in obtaining separations between the communication capabilities of different sets of correlations.

In contrast, the relaxed sum capacity $C(N_G)$ is computed by maximizing over all possible probability distributions. For a nonlocal game MAC, this amounts to maximizing over all distributions over the questions and all possible strategies (allowing full communication between the players). Using such strategies, the players will always win the game, assuming that the game has at least one correct answer for every question. This results in the trivial bound $C(N_G) = \ln(d)$ on the sum capacity, since the players can always win the game. On the other extreme, if the winning probability of the game is zero, then we have $S(N_G) = 0$. Following this line of thinking, we construct a game in Section 4 that allows us to give an arbitrarily large separation between $S(N_G)$ and $C(N_G)$.

The above discussion suggests that the sum capacity of the MAC $N_G$ increases with the winning probability of the game $G$, an observation that was also noted by Leditzky et al. [10]. On the other hand, we know that the winning probability of the game can increase if we allow the players to use a larger set of strategies. This motivates us to allow the senders to share some set of correlations to play the game so as to increase the sum capacity of the MAC.

### 3.1 Correlation assistance

By a correlation, we mean a probability distribution $P(\mathbf{y} \mid \mathbf{x})$, where $\mathbf{xy} \in \mathcal{X}\mathcal{Y}$ is the input to the MAC, while $\mathbf{y} \in \mathcal{Y}$ is an answer to the nonlocal game $G$.\footnote{The correlation $P$ may depend on the answers $\mathbf{y} \in \mathcal{Y}$ in addition to the questions $\mathbf{x} \in \mathcal{X}$ because we wish to construct a larger MAC that has the same structure as the MAC $N_G$ and has assistance from the correlation $P$. See Fig. 1 for example. In practice, the correlation will often produce the answer tuple $\mathbf{y}'$ solely from the question tuple $\mathbf{x}$, using some strategy for the non-local game $G$.} We allow the $N$ senders to share the correlation $P$ and perform local post-processing of the answer generated by $P$ to obtain their final answer for the game for the input questions. This post-processing can be expressed as a probability distribution $f_i(\mathbf{y}_i \mid x_i, y_i, y'_i)$ over the answers $\mathbf{y}_i \in \mathcal{Y}_i$ given the input question-answer pair $(x_i, y_i) \in \mathcal{X}_i \times \mathcal{Y}_i$ and the answer $y'_i \in \mathcal{Y}_i$ generated by the correlation $P$. For convenience, denote

$$f(\mathbf{y} \mid \mathbf{x}, \mathbf{y}') = \prod_{i=1}^{N} f_i(\mathbf{y}_i \mid x_i, y_i, y'_i),$$

which is a distribution over the answers $\mathbf{y} \in \mathcal{Y}$ given input question-answer pairs $\mathbf{xy} \in \mathcal{X}\mathcal{Y}$ and answers $\mathbf{y}' \in \mathcal{Y}$ generated by the correlation $P$.

This gives rise to the channel $A_{P,f}$ having input and output alphabets $\mathcal{X}\mathcal{Y}$ and the probability transition matrix

$$A_{P,f}(\mathbf{xy} \mid \mathbf{xy}) = \delta_{\mathbf{x}, \mathbf{x}} \sum_{\mathbf{y}' \in \mathcal{Y}} f(\mathbf{y} \mid \mathbf{xy}, \mathbf{y}') P(\mathbf{y}' \mid \mathbf{xy}).$$

(11)

This probability transition matrix gives a probability distribution over the question-answer pairs $\mathbf{xy} \in \mathcal{X}\mathcal{Y}$ given some input question-answer pair $\mathbf{xy} \in \mathcal{X}\mathcal{Y}$. The probability distributions $f_i$ denote local post-processing by the $i$th sender to generate the final answer $\mathbf{y}_i$. Note that the definition of $A_{P,f}$ in Eq. (11) ensures that the input questions $\mathbf{x} \in \mathcal{X}$ are transmitted without any change, i.e., $\mathbf{x} = \mathbf{x}$. The post-processings are only used to generate the final answers. This definition is slightly more restrictive than the one given in Leditzky et al. [10], where post-processing of both questions and answers is considered. We impose such a restriction on post-processing so as to ensure that the set of strategies induced by the channel $A_{P,f}$ has a close relation to the set of correlations $C$ shared by the senders. For example, classical or quantum correlations shared by the senders give rise to classical or quantum strategies to play the nonlocal game $G$. We make this idea precise in the following discussion.
An input probability distribution \( p(xy) \) over the question-answer pairs \( \mathcal{X} \mathcal{Y} \) can be written as \( p(xy) = \pi(x)p_{\mathcal{Y}|x}(y|x) \), a product of probability distribution over the questions \( \pi(x) \) and a strategy \( p_{\mathcal{Y}|x}(y|x) \). This input probability distribution is modified by the channel \( A_{P,f} \) to give

\[
\bar{p}(xy) = \sum_{x,y \in \mathcal{X} \mathcal{Y}} A_{P,f}(xy) p(xy) = \pi(x)\bar{p}_{\mathcal{Y}|x}(y|x),
\]

(12)

where the new strategy \( \bar{p}_{\mathcal{Y}|x}(y|x) \) is given by

\[
A_{P,f}(p_{\mathcal{Y}|x}) := \bar{p}_{\mathcal{Y}|x}(y|x) = \sum_{y \in \mathcal{Y}} \sum_{y' \in \mathcal{Y}} f(y|x,y') P(y'|xy) p_{\mathcal{Y}|x}(y|x).
\]

(13)

We use the notation \( A_{P,f}(p_{\mathcal{Y}|x}) \) to emphasize that \( \bar{p}_{\mathcal{Y}|x}(y|x) \) is the strategy induced by the action of \( A_{P,f} \) on \( p_{\mathcal{Y}|x} \). Therefore, the channel \( A_{P,f} \) modifies the input strategy \( p_{\mathcal{Y}|x} \) by incorporating assistance from the correlation \( P \) and local post-processings \( f \). For this reason, we call \( A_{P,f} \) the correlation-assistance channel.

If the senders have access to some set of correlations \( \mathcal{C} \), we can define the set of strategies induced by these correlations as

\[
\mathcal{S}_C = \left\{ A_{P,f}(p_{\mathcal{Y}|x}) \mid p_{\mathcal{Y}|x} = \prod_{i=1}^{N} p_{\mathcal{Y}|X_i}, \ P \in \mathcal{C}, \ f \in \text{PP} \right\}.\tag{14}
\]

where PP is the set of all local post-processings of the answers generated by the correlation (as defined in Eq. (10)). We now show that there is a close relation between the correlations \( \mathcal{C} \) shared by the senders and the strategies \( \mathcal{S}_C \) induced by these correlations.

If \( \mathcal{C}_{cl} \) is the set of classical correlations, then any \( P \in \mathcal{C}_{cl} \) can be written as \( P(y'|xy) = \prod_{i=1}^{N} P_i(y_i'|x_i, y_i) \) and convex combinations thereof, where \( P_i \) are some distributions over \( \mathcal{Y}_i \) given a question-answer pair from \( (X_i, Y_i) \) for \( i \in [N] \). Then, from Eq. (13), we find that the strategy \( A_{P,f}(p_{\mathcal{Y}|x}) \) induced by the correlation assistance channel is also a classical strategy (since the input \( p_{\mathcal{Y}|x} \) is a classical strategy). Therefore, the set of strategies induced by classical correlations (as defined in Eq. (14)) is the set of classical strategies \( \mathcal{S}_{cl} \).

On the other hand, if \( \mathcal{C}_Q \) is the set of quantum correlations, then any \( P \in \mathcal{C}_Q \) can be written as

\[
P(y'|xy) = \text{Tr} \left[ \rho \left( E_{y_1}^{(x_1,y_1)} \otimes \cdots \otimes E_{y_N}^{(x_N,y_N)} \right) \right]
\]

where \( \rho \) is the quantum state shared between the senders and \( \{ E_{y_i}^{(x_i,y_i)} \mid y_i \in \mathcal{Y}_i \} \) is the local measurement implemented by the \( i \)-th player upon receiving the input \((x_i, y_i)\) for each \( i \in [N] \). Now, we define new local measurement operators

\[
E_{y_i}^{(x_i,y_i)} = \sum_{y_i, y_i' \in \mathcal{Y}_i} p_{\mathcal{Y}_i|X_i}(y_i|x_i)f_i(y_i|x_i, y_i') E_{y_i'}^{(x_i, y_i')},
\]

so that the strategy induced by the correlation \( P \) can be written using Eq. (13) as

\[
\bar{p}_{\mathcal{Y}|x}(y|x) = \text{Tr} \left[ \rho \left( E_{y_1}^{(x_1)} \otimes \cdots \otimes E_{y_N}^{(x_N)} \right) \right].
\]

This shows that the set of strategies induced by quantum correlations, as defined in Eq. (14), is the set of quantum strategies \( \mathcal{S}_Q \). Similarly, one can verify that the set of strategies induced by no-signalling correlations \( \mathcal{C}_{NS} \) is the set of no-signalling strategies \( \mathcal{S}_{NS} \).

We now elaborate on how one can use the correlation-assistance channel to boost the sum capacity of the nonlocal games MAC. Given the MAC \( \mathcal{N}_G \) obtained from a nonlocal game \( G \), some correlation \( P \)
shared by the senders and local post-processings $f$, we define the correlation-assisted MAC $N_G \circ A_{P,f}$.
That is, the input question-answer pair is first passed through the correlation-assistance channel $A_{P,f}$, which tries to improve the strategy for playing the game, and the modified question-answer pair is passed on to the MAC $N_G$. A schematic of this procedure for the case of two senders is shown in Fig. 1. If the local post-processing $f$ discards information about the input questions as well as the answers generated by the correlation $P$, i.e., $f(y|xy, y') = \delta_{y,y'}$, then $A_{P,f}$ becomes the identity channel. Therefore, the correlation assisted MAC $N_G \circ A_{P,f}$ is at least as powerful as the MAC $N_G$ if we allow the senders to perform any local post-processing.

Suppose that the senders share a set of correlations $C$. The $C$-assisted achievable rate region of the MAC $N_G$ is defined as

$$\text{Cap}^{(1)}_C(N_G) = \bigcup_{P \in C, f \in FP} \text{Cap}(N_G \circ A_{P,f})$$

where $\text{Cap}(N_G \circ A_{P,f})$ is the capacity region defined in Eq. (6) evaluated for the correlation-assisted MAC $N_G \circ A_{P,f}$. The $C$-assisted achievable sum rate of the MAC $N_G$ is

$$S_C(N_G) = \sup \left\{ \sum_{i=1}^N R_i \mid (R_1, \ldots, R_N) \in \text{Cap}^{(1)}_C(N_G) \right\}.$$  

We now derive an alternate expression of $S_C(N_G)$ that is more convenient for computation. Prior to obtaining this expression, note that for any given family of sets $\{F_i\}_{i \in I}$ indexed by some set $I$ and any function $g: \cup_{i \in I} F_i \to \mathbb{R}$, we have

$$\sup_{r \in \cup_{i \in I} F_i} g(r) = \sup_{i \in I} \sup_{r \in F_i} g(r).$$

Using this equation along with Eq. (7), we can write

$$S_C(N_G) = \sup_{P \in C, f \in FP} S(N_G \circ A_{P,f}) = \sup_{P \in C, f \in FP} \sup_{p^{(1)}, \ldots, p^{(N)}} I(X_1, Y_1, \ldots, X_N, Y_N; Z),$$

where $(X_i, Y_i)$ is the random variable (with distribution $p^{(i)}$) describing the input for $i \in [N]$, and $Z$ is the random variable describing the output of the MAC $N_G \circ A_{P,f}$.

Figure 1: Correlation-assisted MAC $N_G \circ A_{P,f}$ for the case of two senders, obtained from the nonlocal games MAC $N_G$ defined in Eq. (9) and correlation-assistance channel $A_{P,f}$ defined in Eq. (11).
Since \( S_C(N_G) \) corresponds to a maximization over all possible local post-processings, we must have
\[
S(N_G) \leq S_C(N_G)
\]
for any set of correlations \( C \). Furthermore, if \( C_1 \subseteq C_2 \), then \( \text{Cap}^{(1)}_{C_1}(N_G) \leq \text{Cap}^{(1)}_{C_2}(N_G) \), and consequently also \( S_{C_1}(N_G) \leq S_{C_2}(N_G) \). Finally, note that we compute the relaxed sum capacity \( C(N_G) \) by maximizing over all possible distributions over the questions and all possible strategies \( \mathcal{S}_{\text{all}} \) that the players can use (since the maximization in Eq. (8) is over all input probability distributions). Because \( \mathcal{S}_C \subseteq \mathcal{S}_{\text{all}} \) for any set of correlations \( C \), we have \( S_C(N_G) \leq C(N_G) \). Therefore, we obtain a hierarchy,
\[
S(N_G) \leq S_{\text{cl}}(N_G) \leq S_Q(N_G) \leq S_{NS}(N_G) \leq C(N_G),
\]
where “cl”, “Q”, and “NS” denote classical, quantum, and no-signalling correlations, respectively. Note that the sum capacity \( S(N_G) \) might not be equal to \( S_{\text{cl}}(N_G) \) because classical correlations can be convex combinations of product distributions.

We now proceed to obtaining a bound on the \( C \)-assisted achievable sum rate.

### 3.2 Bounding the correlation-assisted sum rate

Let \( C \) be any set of correlations shared between the senders. In order to bound \( S_C(N_G) \), we first obtain an optimization problem in terms of distributions over questions and strategies induced by the shared correlations. For a given correlation \( P \in C \), the input and output of \( N_G \circ A_{P,f} \) can be described as follows. The channel \( A_{P,f} \) takes the input random variable \( (X_1, Y_1, \ldots, X_N, Y_N) \) and outputs \( (\overline{X}_1, \overline{Y}_1, \ldots, \overline{X}_N, \overline{Y}_N) \). The output of \( A_{P,f} \) becomes the input to \( N_G \) that returns \( Z \), i.e.,
\[
(X_1, Y_1, \ldots, X_N, Y_N) \xrightarrow{A_{P,f}} (\overline{X}_1, \overline{Y}_1, \ldots, \overline{X}_N, \overline{Y}_N) \xrightarrow{N_G} Z
\]
forms a Markov chain. From the data processing inequality [2], we obtain
\[
I(X_1, Y_1, \ldots, X_N, Y_N; Z) \leq I(\overline{X}_1, \overline{Y}_1, \ldots, \overline{X}_N, \overline{Y}_N; Z). \tag{19}
\]
Then, using Eq. (17) and Eq. (19), we get
\[
S_C(N_G) \leq \sup_{p} I(\overline{X}_1, \overline{Y}_1, \ldots, \overline{X}_N, \overline{Y}_N; Z) \tag{20}
\]
where the probability distribution \( \overline{p} \) defined in Eq. (12) is obtained by varying product distributions
\[
p(x, y) = \prod_{i=1}^{N} p^{(i)}(x_i, y_i)
\]
input to \( A_{P,f} \), the correlation \( P \in C \), and the post-processing \( f \in \text{PP} \).

We can reinterpret the above equation as a maximum over distributions over questions and strategies for playing the game \( G \) induced by the correlations \( C \). First, we write \( p^{(i)} = \pi^{(i)} p_{Y | X_i} \), where \( \pi^{(i)} \) is a distribution over the questions \( X_i \) and \( p_{Y | X_i} \) is a strategy chosen by \( i \)-th player for \( i \in [N] \). Therefore, the input probability distribution in Eq. (17) can be written as \( p^{(1)} \cdots p^{(N)} = \pi p_{Y | X} \), where \( \pi = \pi^{(1)} \cdots \pi^{(N)} \) is a distribution over questions \( X \), and \( p_{Y | X} = \prod_{i=1}^{N} p_{Y | X_i} \) is a classical strategy chosen by the players. In particular, the input strategy \( p_{Y | X} \) is always a classical strategy. As noted in Eq. (13), the channel \( A_{P,f} \) takes this input strategy and returns a new strategy \( A_{P,f}(p_{Y | X}) \) that incorporates assistance from the shared correlation \( P \). Since the senders have access to the set of correlations \( C \), we can write Eq. (20) as
\[
S_C(N_G) \leq \sup_{\pi^{(1)} \cdots \pi^{(N)}} \sup_{p_{Y | X} \in \mathcal{S}_C} I(\overline{X}_1, \overline{Y}_1, \ldots, \overline{X}_N, \overline{Y}_N; Z), \tag{21}
\]
where \( \mathcal{S}_C \) is the set of strategies induced by \( C \) as defined in Eq. (14). To obtain an upper bound, we will perform relaxations of the RHS of the above equation, and solve the resulting optimization problems.
We begin by writing the RHS of Eq. (21) in a form that is amenable for calculations. To that end, note that given an input probability distribution \( \overline{p}(x, y) = \pi(x) \overline{p}_{Y|X}(y|x) \), the probability distribution corresponding to the output of the channel \( N_G \) is given by

\[
p(z) = \sum_{x,y \in \mathcal{W}} N_G(z|x,y) \overline{p}(x,y) = \sum_{x,y \in \mathcal{W}} N_G(z|x,y) \overline{p}(x,y) = \pi(z) \sum_{y:z,y \in \mathcal{W}} \overline{p}_{Y|X}(y|z) + \frac{1}{d} p_L
\]

where \( d = |Z| \) denotes the number of question pairs, while \( p_L = \sum_{\overline{x} \notin \mathcal{W}} \overline{p}(\overline{x}) \) denotes the probability of losing the game when questions are drawn as per the probability distribution \( \pi \).

Note that \( Z = \mathcal{X}_1 \times \cdots \times \mathcal{X}_N = \mathcal{X} \) is the set of question tuples output by the MAC \( N_G \). Since \( Z \) is a finite set of size \( d \), we can fix a labelling for the elements of \( Z \) and write \( Z = \{z_1, \ldots, z_d\} \). Each \( z_i \) corresponds to a particular question tuple. Then, we can define the contribution of a given strategy \( \overline{p}_{Y|X} \) towards winning the game for each question tuple. We use \( Z \) and \( \mathcal{X} \) interchangeably in the following discussion.

**Definition 1** (Winning vector). Given a strategy \( \overline{p}_{Y|X} \) for playing the game \( G = (\mathcal{X}, \mathcal{Y}, \mathcal{W}) \), we let

\[
w_i = \sum_{y:z_i,y \in \mathcal{W}} \overline{p}_{Y|X}(y|z_i) \quad \text{for } i \in [d]
\]

denote the contribution of the strategy towards winning the game \( G \) for question \( z_i \). We call the vector \( w = (w_1, \ldots, w_d) \) the winning vector corresponding to the strategy \( \overline{p}_{Y|X} \). Let \( \mathcal{W}_C \) denote the set of winning vectors allowed by the strategies \( \mathcal{E}_C \),

\[
\mathcal{W}_C = \left\{ w \in [0, 1]^d \mid w_i = \sum_{y:z_i,y \in \mathcal{W}} \overline{p}_{Y|X}(y|z_i), \text{ for } \overline{p}_{Y|X} \in \mathcal{E}_C \text{ and } i \in [d] \right\}.
\]

Observe that \( w_i \in [0, 1] \) for all \( i \in [d] \), so that \( w \) is an element of the unit hypercube in \( \mathbb{R}^d \). Note that we have \( w_i = 1 \) for a fixed strategy \( \overline{p}_{Y|X} \) if and only if the players always win the game \( G \) when asked the question \( z_i \) using the strategy \( \overline{p}_{Y|X} \). On the other extreme, \( w_i = 0 \) if and only if the players always lose the game \( G \) when asked the question \( z_i \) using the strategy \( \overline{p}_{Y|X} \). Generally, questions are drawn with probability \( \pi \) over \( \mathcal{X} \). The probability of winning the game for question \( i \) is \( \pi_i w_i \), where \( \pi_i = \pi(z_i) \) is the probability of drawing the question tuple \( z_i \). The total probability of winning the game is \( p_W = \sum_{i=1}^d \pi_i w_i \) and the probability of losing the game is

\[
p_L = 1 - \sum_{i=1}^d \pi_i w_i = 1 - \langle \pi, w \rangle.
\]

Defining the matrix \( \overline{W} \) with components

\[
\overline{W}_{ij} = w_i \delta_{ij} + \frac{1 - w_j}{d},
\]

one may write the output probability \( p(\overline{z}) \) in Eq. (22) as

\[
p = \overline{W} \pi.
\]

The mutual information \( I(\overline{X}_1, \overline{Y}_1, \ldots, \overline{X}_N, \overline{Y}_N; Z) \) can be written as

\[
I(\overline{X}_1, \overline{Y}_1, \ldots, \overline{X}_N, \overline{Y}_N; Z) = H(Z) - H(Z|\overline{X}_1, \overline{Y}_1, \ldots, \overline{X}_N, \overline{Y}_N) = H(Z) - \ln(d).
\]
where we used the fact that $H(Z|x,y) = 0$ when $x,y \in W$ whereas $H(Z|x,y) = \ln(d)$ when $x,y \notin W$. Note that the formula (28) was first derived in Ref. [10] for nonlocal games MAC with two senders. Using Eq. (27) and Eq. (25), we obtain

$$ I_w(\pi) := I(X_1,Y_1,\ldots,X_N,Y_N;Z) = H(W) + \langle \pi, w \rangle \ln(d) - \ln(d) $$  \hspace{1cm} (29)

where the notation, $I_w(\pi)$, for the mutual information emphasizes that it is only a function of the distribution $\pi$ over questions and the winning vector $w$.

The RHS in Eq. (21) can be written as

$$ \sup_{\pi(1)\cdots\pi(N)} \sup_{P_{X|Y}} I(X_1,Y_1,\ldots,X_N,Y_N;Z) = \sup_{\pi(1)\cdots\pi(N)} \sup_{w \in W_C} I_w(\pi(1)\cdots\pi(N)) $$ \hspace{1cm} \leq \sup_{\pi \in \Delta_d} \sup_{w \in W_C} I_w(\pi), \hspace{1cm} (30)

where $W_C$ is the set of winning vectors defined in Eq. (24). To obtain Eq. (30), we relax the product distribution constraint $\pi(1)\cdots\pi(N)$ over the questions to obtain a maximization over all distribution $\pi \in \Delta_d$ over the questions, where $\Delta_d$ denotes the $(d-1)$-dimensional standard simplex. This relaxation differs from that of Eq. (8) used in obtaining $C(N_G)$ in that we only relax the distribution over the questions, but not the whole probability distribution.

For a fixed $w$, the function $I_w(\pi)$ is continuous in $\pi$ over the compact set $\Delta_d$. Thus, the maximization in Eq. (31) can be written as

$$ \sup_{w \in W_C} \max_{\pi \in \Delta_d} I_w(\pi). $$  \hspace{1cm} (31)

The inner optimization in Eq. (31) is a convex problem since $I_w(\pi)$ is concave in $\pi$ and $\Delta_d$ is a convex set. However, $I_w(\pi)$ is not jointly concave in $\pi$ and $w$, and moreover, $W_C$ need not be a convex set. Therefore, the optimization in Eq. (31) is generally nonconvex.

Our goal is to obtain an upper bound on the optimization in Eq. (31). To give a general idea of our approach to obtaining this bound, we list the main steps we will carry out.

**Step 1:** For a fixed $w$, we obtain an upper bound on the inner optimization in Eq. (31). This bound is tight when either $w \in \{0,1\}^d$ or $w > 0$ component-wise.

**Step 2:** We relax the set of allowed $w$ values to bound the outer optimization in Eq. (31) from above.

This procedure will result in the upper bound noted in Eq. (39). We explain the steps in detail in the following subsections.

**Step 1: Bounding the inner optimization over question distributions**

We obtain an upper bound on $\max_{\pi \in \Delta_d} I_w(\pi)$ by considering two cases. First, we perform this optimization exactly for the case when $w \in \{0,1\}^d$. Next, for any $w \in W_C$, we find an upper bound on $\max_{\pi \in \Delta_d} I_w(\pi)$ using the result of case 1. The upper bound obtained in case 2 is tight when $w > 0$.

**Case 1: optimizing $\max_{\pi \in \Delta_d} I_w(\pi)$ for fixed $w \in \{0,1\}^d$**

Winning vectors $w \in \{0,1\}^d$ arise from strategies that either always win or always lose the game for any given question. Deterministic strategies, for example, give rise to such winning vectors. Recall that a classical deterministic strategy corresponds to functions $f_i : X_i \rightarrow Y_i$, $i \in [N]$, chosen by the players. Such functions give rise to the classical strategy $p_{Y_i|X_i}(y|x) = \delta_{y,f_i(x)}$ that is 1 at $y = f_i(x)$ and zero...
The quantity \( \mathcal{I}_K^* \) is given by the expression

\[
\mathcal{I}_K^* = \ln \left( K + (d - K)d^{-\frac{d-1}{d}} \right).
\]

**Proof.** See Appendix A. \( \square \)

Observe that the maximum only depends on the total number \( d \) of questions, as well as the number \( K \) of questions that can be answered correctly using the deterministic strategy.

**Case 2: Bounding \( \max_{\pi \in \Delta_d} \mathcal{I}_w(\pi) \) for fixed \( w \in \mathcal{W}_C \)**

When we work with arbitrary winning vectors, it is more challenging to maximize the mutual information over distributions on the questions. To make this maximization easier, we first show that the maximum mutual information \( \mathcal{I}_{d-1}^* \) corresponding to a winning vector that can answer exactly \( d-1 \) questions correctly will always be larger than the maximum mutual information \( \max_{\pi \in \Delta_d} \mathcal{I}_w(\pi) \) for any winning vector \( w \) that answers no more than \( d-1 \) questions correctly. We therefore turn our attention to \( w \) that doesn’t necessarily do worse than this case, and obtain an expression for the maximum mutual information in terms of such \( w \).

**Proposition 3.** Let \( G \) be a nonlocal game, and let \( \mathcal{N}_G \) be the MAC obtained from this nonlocal game. Suppose that the senders of \( \mathcal{N}_G \) share a set of correlations \( C \). Let \( w \in \mathcal{W}_C \) be any winning vector allowed by the correlations \( C \) as defined in Eq. (24). Let \( K = \{ i \in [d] \mid w_i \neq 0 \} \) be the set of questions with non-zero probability of winning the game using this strategy, and denote \( K = |K| \). Then, the following statements hold.

1) Suppose that \( \max_{\pi \in \Delta_d} \mathcal{I}_w(\pi) \) is achieved at \( \pi^* \). Denote \( K^* = \{ i \in [d] \mid w_i\pi^*_i \neq 0 \} \) and \( K^* = |K^*| \) (we have \( K^* \subseteq K \)). Then, if \( K^* < d \), we have \( \max_{\pi \in \Delta_d} \mathcal{I}_w(\pi) \leq \mathcal{I}_{d-1}^* \), where \( \mathcal{I}_{d-1}^* \) is given by Eq. (33).

2) As a consequence of the above result, we restrict our attention to strategies with \( K^* = K = d \). In that case, we have

\[
\max_{\pi \in \Delta_d} \mathcal{I}_w(\pi) = \mathcal{I}^*(w) = \ln \left( \sum_{j=1}^{d} \exp \left[ dw_{\text{eff}} \ln d \left( 1 - \frac{1}{w_j} \right) \right] \right)
\]

(34)
where

\[ w_{\text{eff}} = \left( \sum_{i=1}^{d} \frac{1}{w_i} \right)^{-1}. \] (35)

**Proof.** See Appendix A. \(\square\)

Owing to the above result, we only need to focus on maximizing \(\mathcal{J}^*(w)\) for those \(w \in \mathcal{W}_C\) with \(w_i > 0\) for all \(i \in [d]\). This is done in the next step.

**Step 2: Bounding the outer optimization over winning vectors**

As noted in the previous step, our goal is to maximize \(\mathcal{J}^*(w)\) with respect to the feasible winning vectors \(w \in \mathcal{W}_C\) with \(w_i > 0\) for all \(i \in [d]\). The set of (feasible) winning vectors \(\mathcal{W}_C\) was defined in Eq. (24). Note that \(\mathcal{W}_C\) depends on the winning condition \(W\) of the game as well as the set of correlations \(C\) shared by the senders. Since we make no assumptions about the game or the set of correlations, it is difficult to optimize over \(\mathcal{W}_C\). For this reason, we obtain a relaxation of the set \(\mathcal{W}_C\), over which we optimize \(\mathcal{J}^*(w)\). We will do this in two steps: (1) relate \(w\) to the winning probability when the questions are drawn uniformly, and (2) use the maximum winning probability \(\omega^{\mathcal{S}_C}(G)\) of the game (assumed to be known) corresponding to the strategies \(\mathcal{S}_C\) when the questions are drawn uniformly in order to obtain a convex set containing \(\mathcal{W}_C\).

(1) From the definition of winning vector given in Eq. (23), we know that

\[ w_i = \sum_{y,z:y \in W} p_{Y|X}(y|z_i). \]

Recall that the winning probability of the game can be written as \(p_W = \sum_{i=1}^{d} \pi_i w_i\) when the questions are drawn as per probability \(\pi \in \Delta_d\). If the questions are drawn uniformly, then \(\pi_U(z) = 1/d\) for all questions \(z_1, \ldots, z_d\). Therefore, \(p_W = \sum_{i=1}^{d} w_i/d\) is the winning probability determined by the winning vector \(w\) when the questions are drawn uniformly.

(2) We now look for a convex relaxation \(\overline{\mathcal{W}}_C\) of \(\mathcal{W}_C\). We want to make \(\overline{\mathcal{W}}_C\) fairly independent of the winning set, except for dependence on \(\omega^{\mathcal{S}_C}(G)\) and the number of question tuples \(d\) in the game.

Since \(\omega^{\mathcal{S}_C}(G)\) is the maximum winning probability using the set of strategies \(\mathcal{S}_C\) under consideration, we must have

\[ \frac{1}{d} \sum_{i=1}^{d} w_i \leq \omega^{\mathcal{S}_C}(G) \] (36)

where \(w \in \mathcal{W}_C\). Now we make the relaxation that we allow any winning vector that satisfies Eq. (36).

Consequently, we define

\[ \overline{\mathcal{W}}_C = \left\{ w \in [0,1]^d \mid \frac{1}{d} \sum_{i=1}^{d} w_i \leq \omega^{\mathcal{S}_C}(G) \right\}. \] (37)

Since any \(w \in \mathcal{W}_C\) will satisfy Eq. (36), we have \(\mathcal{W}_C \subseteq \overline{\mathcal{W}}_C\), confirming that \(\overline{\mathcal{W}}_C\) is a relaxation of \(\mathcal{W}_C\). Such a relaxation may allow for strategies not described by \(\mathcal{S}_C\). Note that \(\overline{\mathcal{W}}_C\) is a compact and convex set, and it depends only on the maximum winning probability and the number of questions in the game. Using this relaxation, we compute an upper bound on \(\mathcal{J}^*(w)\) maximized over \(w \in \overline{\mathcal{W}}_C\) satisfying \(w > 0\) componentwise.
Proposition 4. Let $G$ be a nonlocal game and let $N_G$ be the MAC constructed from $G$, as defined in Eq. (9). Suppose that the senders share the set of correlations $C$, and let $\mathcal{W}_C$ be the corresponding set of winning vectors as defined in Eq. (24). Let $\overline{\mathcal{W}}_C$ be the convex relaxation of $\mathcal{W}_C$ defined in Eq. (37) that depends only on the number of question tuples $d$ in the game and the maximum winning probability $\omega^{\mathcal{C}}(G)$ when the questions are drawn uniformly and answers given using strategies in $\mathcal{C}$. Let $\mathcal{I}^*(w)$ be the function defined in Eq. (34). Then the maximum of $\mathcal{I}^*(w)$ over winning vectors $w > 0$ in $\overline{\mathcal{W}}_C$ is bounded from above by

$$\sup_{w \in \overline{\mathcal{W}}_C, w > 0} \mathcal{I}^*(w) \leq \ln \left( d - 1 + d^{-(1-\omega^{\mathcal{C}}(G))d} \right)$$ (38)

Proof. See Appendix A.

Bound on the correlation-assisted achievable sum rate

We put all the above steps together to obtain a bound on $S_C(N_G)$.

Theorem 5. Let $G$ be an $N$-player promise-free nonlocal game with $d$ question tuples, and let $N_G$ be the MAC obtained from $G$ as defined in Eq. (9). Suppose that the senders share a set of correlations $C$. Let $\mathcal{S}_C$ be the set of strategies induced by the correlations as defined in Eq. (14). Let $\omega^{\mathcal{C}}(G)$ denote the maximum winning probability of the game when the questions are drawn uniformly and answers are obtained using strategies in $\mathcal{C}$. Let $S_C(N_G)$ denote the $C$-assisted achievable sum rate of the MAC $N_G$ as defined in Eq. (17). Then, we have

$$S_C(N_G) \leq \ln \left( d - 1 + d^{-(1-\omega^{\mathcal{C}}(G))d} \right)$$ (39)

with entropy measured in nats.

Proof. To obtain an upper bound on $S_C(N_G)$, we start with Eq. (21). The RHS of Eq. (21) can be bounded by performing the maximization $\sup_{w \in \mathcal{W}_C} \max_{\pi \in \Delta_d} \mathcal{I}_w(\pi)$, where $\mathcal{I}_w(\pi)$ is the mutual information defined in Eq. (29). The set $\Delta_d$ denote the $(d-1)$-dimensional standard simple, while $\mathcal{W}_C$ denotes the set of winning vectors induced by the correlations $C$ as defined in Eq. (24).

In Prop. 3, we show that if $w \in \mathcal{W}_C$ has one or more zero entries, then $\max_{\pi \in \Delta_d} \mathcal{I}_w(\pi) \leq \mathcal{I}^*_d - 1$, where $\mathcal{I}^*_d$ is given by Eq. (33). Therefore, we only maximize $\mathcal{I}^*_d(w) = \max_{\pi \in \Delta_d} \mathcal{I}_w(\pi)$ over winning vectors $w \in \mathcal{W}_C$ satisfying $w > 0$. The expression for $\mathcal{I}^*_d(w)$ in this case is given by Eq. (34). We relax the set $\mathcal{W}_C$ to the compact and convex set $\overline{\mathcal{W}}_C$ defined in Eq. (37). Then, we give an upper bound on $\sup_{w \in \overline{\mathcal{W}}_C, w > 0} \mathcal{I}^*_d(w)$ in Eq. (38).

By preceding remarks, we have

$$S_C(N_G) \leq \max \left\{ \mathcal{I}^*_d - 1, \ln \left( d - 1 + d^{-(1-\omega^{\mathcal{C}}(G))d} \right) \right\}$$ (40)

while from Eq. (33) we have

$$\mathcal{I}^*_d = \ln \left( d - 1 + d^{-d} \right).$$

Using $\ln \left( d - 1 + d^{-(1-\omega^{\mathcal{C}}(G))d} \right) \geq \mathcal{I}^*_d - 1$ in Eq. (40), we obtain Eq. (39).

Corollary 6. Let $G$ be an $N$-player promise-free nonlocal game with $d$ question tuples, and let $N_G$ be the MAC obtained from $G$ as defined in Eq. (9). Let $\omega^{cl}(G)$ denote the maximum winning probability of
the game when the questions are drawn uniformly and answers are obtained using classical strategies. Let \( S(N_G) \) denote the sum capacity of the MAC \( N_G \). Then, we have

\[
S(N_G) \leq \ln \left( d - 1 + d^{-(1-\omega_{cl}(G))d} \right)
\]

with entropy measured in nats.

**Proof.** From Eq. (18), we know that \( S(N_G) \leq S_{cl}(N_G) \). Then, using Thm. 5, we obtain Eq. (41). \( \Box \)

Note that the bounds on \( S_{cl}(N_G) \) and \( S(N_G) \) given by Eq. (39) and Eq. (41), respectively, lie between \( \ln(d-1) \) and \( \ln(d) \). For sufficiently large \( d \), when \( \omega_{cl}(G) \) is not close to 1, the sum capacity is bounded above by \( \approx \ln(d-1) \). On the other hand, for \( \omega_{cl}(G) = 1 \), we obtain an upper bound of \( \ln(d) \), which can be achieved by \( w = (1, \ldots, 1)^T \) as seen from Prop. 2. Using this, we can obtain separations between the correlation-assisted achievable sum rate corresponding to two different sets of correlations.

### 3.3 Separation between sum rates with assistance from different sets of correlations

If \( C_1 \) and \( C_2 \) are two sets of correlations such that \( \omega_{S_{cl}}(G) < 1 \) while \( \omega_{S_{cl}}(G) = 1 \), then \( S_{C_1}(N_G) < S_{C_2}(N_G) = \ln(d) \). We use this idea to provide separations of correlation-assisted achievable sum rate using classical, quantum and no-signalling correlations.

#### 3.3.1 Separating \( S_{Q}(N_G) \) from \( S(N_G) \) for two-sender MACs

Consider the Magic Square Game, \( G_{MS} \), used previously in [18, 19, 20, 14] to obtain a separation between \( S(N_G) \) and \( S_{Q}(N_G) \). In this game, the referee selects a row \( r \in \{1, 2, 3\} \) and column \( c \in \{1, 2, 3\} \) from a \( 3 \times 3 \) grid uniformly at random. The row is handed over to Alice while the column is given to Bob. Without communicating with each other, Alice & Bob need to fill bits in the given row and column such that the total parity of bits in the row is even, total parity of bits in the column is odd, and the bit at the intersection of the given row and column match.

There are \( d = 9 \) possible question pairs corresponding to the indices \((r, c)\). Classically, Alice & Bob can win the game at least 8 out of 9 times by implementing for example the following strategy:

\[
\begin{array}{ccc}
1 & 0 & 1 \\
1 & 1 & 0 \\
1 & 0 & ? \\
\end{array}
\]

where the entry in each box indicates bits filled by Alice and Bob. It can be shown that this strategy is optimal, therefore \( \omega_{cl}(g_{ms}) = 8/9 \) [14].

On the other hand, if Alice & Bob are allowed to use a quantum strategy, then they can share two copies of a maximally entangled Bell state,

\[
\rho_{Bell} = \frac{1}{2} \begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 \\
\end{pmatrix},
\]

and submit answers to the referee based on a set of measurements given in Table 1. Alice and Bob answer 0 if their measurement yields an eigenvector with eigenvalue 1, else they answer 1. Answers
obtained this way can be shown to always satisfy the winning condition of the magic square game, i.e., \( \omega^Q(G_{MS}) = 1 \) \cite{18, 19}.

Since \( \omega^{cl}(G_{MS}) = \frac{8}{9} \), we may use Cor. (6) to obtain a bound \( S(N_{G_{MS}}) \leq 3.02 \) bits. At the same time, a perfect quantum strategy is available, i.e., \( \omega^Q(G_{MS}) = 1 \), and thus \( S_Q(N_{G_{MS}}) = 3.17 \) bits \cite{10}. This tighter bound shows how we obtain new non-trivial bounds on the sum capacity for a small game.

Since every quantum strategy is also a no-signalling strategy, we automatically obtain a separation between \( S(N_{G_{MS}}) \) and \( S_{NS}(N_{G_{MS}}) \). However, \( S_{NS}(N_{G_{MS}}) = S_Q(N_{G_{MS}}) \). In the following section, we use a game different from \( G_{MS} \) to obtain a separation between the quantum and no-signalling assisted sum rates.

### 3.3.2 Separating \( S_{NS}(N_{G}) \) from \( S(N_{G}) \) and \( S_Q(N_{G}) \) for two-sender MACs

In order to obtain a separation between the quantum-assisted sum rate and the no-signalling assisted sum rate, we consider the Clauser-Horne-Shimony-Holt (CHSH) game \( G_{CHSH} \) \cite{21, 22}. In this game, a referee selects bits \( x_1, x_2 \in \{0, 1\} \) uniformly at random, and gives them to Alice and Bob, respectively. Upon receiving these question bits, Alice answers with the bit \( y_1 \in \{0, 1\} \) and Bob with \( y_2 \in \{0, 1\} \). Alice and Bob chose their answers without communicating with each other. They win the game if

\[
x_1 \wedge x_2 = y_1 \oplus y_2,
\]

where \( \wedge \) and \( \oplus \) represent logical AND and bitwise addition modulo 2. This game has a total of \( d = 4 \) question pairs.

It is known that the best classical strategy can answer only 3 out of the 4 question pairs correctly, i.e., \( \omega^{cl}(G_{CHSH}) = 3/4 \) \cite{22}. The optimal quantum strategy achieves a winning probability of \( \omega^Q(G_{CHSH}) = (1 + 1/\sqrt{2})/2 \approx 85.4\% \) \cite{22}. While there is no classical or quantum strategy that can always win the game, one can construct a no-signalling distribution,

\[
P_{PR}(y_1, y_2|x_1, x_2) = \frac{1}{2} \delta_{x_1 \wedge x_2, y_1 \oplus y_2},
\]

usually called the Popescu-Rohrlich (PR) box \cite{23}, which represents a perfect strategy for winning the CHSH game. Therefore \( \omega^{NS}(G_{CHSH}) = 1 \).

Using \( \omega^{cl}(G_{CHSH}) = 3/4 \) in Cor. (6) gives \( S(N_{G_{CHSH}}) \leq 1.7 \) in the classical case. On the other hand, using \( \omega^Q(G_{CHSH}) = (1 + 1/\sqrt{2})/2 \) in Thm. 5, gives an upper bound, \( S_Q(N_{G_{CHSH}}) \leq 1.78 \) bits, in the quantum case. In the case of no-signalling, a perfect strategy is possible and thus we have \( S_{NS}(N_{G_{CHSH}}) = 2 \) bits. In this way, we obtain a separation between the quantum and no-signalling assisted achievable sum rate.
3.3.3 Separating $S_Q(N_{G})$ from $S(N_{G})$ for $N$-sender MACs

We now consider a game $G_{MPP}$ that we call the multiparty parity game, which was first introduced by Brassard et al. [24]. In this game, $N$ players are each handed a bit and they each answer by returning a bit. The players have a promise: the total number of ones in the $N$-bit string handed to them is even. If this even number is divisible by 4, then the winning condition is that the total bit string returned by the players have an even number of ones. Otherwise, the winning condition is to return a string with an odd number of ones.

Formally, we have $X_i = \{0, 1\}$ and $Y_i = \{0, 1\}$ for $i \in [N]$. As before, we denote $X = X_1 \times \cdots \times X_N$ as the set of questions and $Y = Y_1 \times \cdots \times Y_N$ as the set of answers for the $N$ players. The promise,

$$P = \left\{ x \in X \mid \sum_i x_i = 0 \pmod{2} \right\}$$

is a subset of $X$ from which the questions are draw. The winning condition for the game can be described by the set

$$W = \left\{ (x, y) \in P \times Y \mid \sum_i y_i - \frac{1}{2} \sum_i x_i = 0 \pmod{2} \right\}.$$

Brassard et al. [24] demonstrated that classical strategies can win this game with a probability of at most $\omega^{cl}(G_{MPP}) = 1/2 + 2^{-[N/2]}$ when the questions are drawn uniformly from the promise set. In contrast, a perfect quantum strategy is possible [24].

Now we consider the following promise-free version of this game. Herein, the question and answer set remain the same, but the winning condition is defined as the set $W = W^P \cup (P^c \times Y)$. That is, the players win automatically if a question from outside the promise set is presented to them. To apply the bound obtained in Thm. 5, we need to compute the maximum classical winning probability $\omega^{cl}(G_{MPP}^P)$ for the promise-free case when questions are drawn uniformly. To that end, we show how to compute the maximum winning probability $\omega^{cl}(G)$ when we convert any game with a promise $P$ to a promise-free game, assuming that the question are drawn uniformly and answers are given using strategies in $S$. Since $W^P \cap (P^c \times Y) = \emptyset$, we have

$$p_W = \sum_{(x,y) \in W^P} \frac{1}{|X|} p(y|x) + \sum_{x \in P^c} p(x) = \frac{|P|}{|X|} p_W^P + \left(1 - \frac{|P|}{|X|}\right).$$

Since the set of strategies $S$ chosen by the players have a maximum probability $\omega^{S,P}(G)$ of winning when the questions are drawn uniformly from $P$, we can infer that

$$\omega^{S}(G) = \frac{|P|}{|X|} \omega^{S,P}(G) + \left(1 - \frac{|P|}{|X|}\right).$$

For the multiparty parity game, since half the $N$-bit strings are in $P$ and the other half are in $P^c$, we get

$$\omega^{cl}(G_{MPP}) = \frac{3}{4} + 2^{-([N/2]+1)}.$$  \hspace{1cm} (42)

Then, by Cor. (6), we find that the sum capacity for the MAC obtained from the multiparty parity game is bounded above as $S(N_{G_{MPP}}) \leq \log(d - 1 + 2^{-(1-\omega^{cl}(G))d})$ bits, where $d = 2^N$ and $\omega^{cl}(G)$ is given by Eq. (42). In particular, $S(N_{G_{MPP}}) < \log(d)$. In contrast, since a perfect quantum strategy is available, we have $S_Q(N_{G_{MPP}}) = \log(d)$, thus giving a separation between the sum capacity and the quantum-assisted sum rate for $N$-sender MACs. For example, when we have $N = 3$ senders, we obtain $S(N_{G_{MPP}}) \leq 2.84$ bits. In contrast, $S_Q(N_{G_{MPP}}) = 3$ bits in the quantum case.
4 Looseness of convex relaxation of the sum capacity

In the previous section, we looked at separations between the sum rates with assistance from classical, quantum and no-signalling strategies. In this section, we construct a game such that one can obtain an arbitrarily large separation between the sum capacity and the relaxed sum capacity. Recall that the relaxed sum capacity corresponds to dropping the product distribution constraint in the maximization problem:

$$C(N_G) = \max_{\overline{p}(x,y)} I(\overline{X}_1, \overline{Y}_1, \ldots, \overline{X}_N, \overline{Y}_N; Z),$$

where $\overline{X}_1, \overline{Y}_1, \ldots, \overline{X}_N, \overline{Y}_N$ are random variables describing the input to the MAC $N_G$, while $Z$ is the random variable describing the output. We maximize over all possible input probability distributions $\overline{p}$, so that the resulting quantity is the capacity of $N_G$ when we think of it as a single-input single-output channel. As noted in Sec. 3.1, we have $S(N_G) \leq S_C(N_G) \leq C(N_G)$ for any set of correlations $C$. Indeed, since we maximize over all probability distributions over the input to $N_G$, we can write the relaxed sum capacity as

$$C(N_G) = \max_{\overline{p}(x,y) \in \mathcal{S}_{\text{all}}} \max_{\pi \in \Delta_d} I(\overline{X}_1, \overline{Y}_1, \ldots, \overline{X}_N, \overline{Y}_N; Z),$$

where $\mathcal{S}_{\text{all}}$ denotes the set of all possible strategies that the players can use to play the game. In particular, this amounts to allowing the players to communicate after the questions have been handed over to them.

To analyze $C(N_G)$, we study some properties of $\mathcal{S}_{\text{all}}$. It can be verified that $\mathcal{S}_{\text{all}}$ is a convex set. The extreme points of this set correspond to deterministic strategies $f: X \rightarrow Y$ that allow for communication between the players (see Prop. 10 in App. B). This implies that the maximum winning probability of the game $\omega_{\mathcal{S}_{\text{all}}}(G)$, when the questions are drawn uniformly and answers are obtained using the strategies in $\mathcal{S}_{\text{all}}$, is always achieved by a deterministic strategy of the form mentioned above. We now give an explicit description of a deterministic strategy (not necessarily unique) achieving the maximum winning probability.

The best deterministic strategy $f^{(D)}: X \rightarrow Y$ can be written as

$$f^{(D)}(x) = \begin{cases} y & \exists y \in Y \text{ such that } (x, y) \in W \\ y_o & \text{otherwise,} \end{cases}$$

(43)

where $y_o \in Y$ is an arbitrary element chosen beforehand. We note that for each $x \in X$, some element $y \in Y$ satisfying $(x, y) \in W$ is chosen apriori (if it exists), so that the function is well-defined, though not necessarily unique. In other words, $f^{(D)}$ gives the correct answer if a correct answer for the given question exists, and if not, it gives an arbitrary answer that is necessarily incorrect. It can, therefore, be inferred that the maximum winning probability can be written as

$$\omega_{\mathcal{S}_{\text{all}}}(G) = \frac{|\{x \in X | \exists y \in Y \text{ such that } (x, y) \in W\}|}{d}.$$  (44)

This is the best that one can do given any nonlocal game $G$. Note also that $\omega_{\mathcal{S}_{\text{all}}}(G)$ can be directly computed from the winning condition $W$.

The above observation directly leads to upper and lower bounds on $C(N_G)$. The upper bound is obtained by using $\omega_{\mathcal{S}_{\text{all}}}(G)$ from Eq. (44) in Thm. 5. Here, we implicitly use the fact that our upper bound is valid even when the questions are drawn arbitrarily. Let $w^{(D)} \in \{0, 1\}^d$ be the winning vector corresponding to the best deterministic strategy $f^{(D)}$ given in Eq. (43). Note that we maximize over all distributions over the questions when computing $C(N_G)$. Therefore, using $w^{(D)}$ in Prop. 2, we obtain
a lower bound on $C(N_G)$. In particular, if there is at least one correct answer for every question, then $f^{(D)}$ is a perfect strategy and $C(N_G) = \ln(d)$.

Now, we obtain a separation between $C(N_G)$ and $S(N_G)$. To that end, we construct a game called the signalling game.

### 4.1 Signalling game $G_s$ and separation of $C(N_{G_s})$ from $S(N_{G_s})$ and $S_{NS}(N_{G_s})$

Consider a game where Alice & Bob are each given a question from some set of questions. They win the game if they can correctly guess the question handed over to the other person. Since the game can be won if Alice & Bob “signal” their question to each other, we call this the signalling game $G_s$.

Formally, we consider question sets $X_1, X_2$ and answer sets $Y_1 = X_2$ and $Y_2 = X_1$, and the winning condition is defined as

$$W = \{(x_1, x_2, y_1, y_2) \in (X_1 \times X_2) \times (Y_1 \times Y_2) \mid y_1 = x_2, \ y_2 = x_1\}.$$

Note that there is exactly one correct answer for each question pair $(x_1, x_2) \in X_1 \times X_2$.

Since we bound the sum capacity of the MAC $N_{G_s}$ obtained from the signaling game $G_s$ using the maximum winning probability, we analyze the winning strategies for this game. To that end, consider some set of strategies $S$ that Alice and Bob use to play the game. For simplicity, we assume that this is a compact set (thinking of strategies as vectors as in Prop. 10), which holds, for example, when $S$ is the set of no-signalling strategies (see Prop. 11). Suppose that $\omega^S(G) = 1$ corresponding to this set of strategies, and let $p^*_{Y|X}$ be the strategy that achieves this maximum winning probability. When the questions are drawn uniformly, the winning probability for strategy $p^*_{Y|X}$ is given by $p^*_W = (\sum_i w^*_i) / d$, where $w^* = (w^*_1, \ldots, w^*_d)$ is the winning vector corresponding to $p^*_{Y|X}$, as defined in Eq. (23). Since $p^*_W = \omega^S(G) = 1$ by assumption and $w^* \in [0,1]^d$, we must have $w^*_i = 1$ for all $i \in [d]$.

For convenience, denote $X = X_1 \times X_2$ as the set of questions. Using Def. 23, we can write the winning vector $w^*$ as

$$w^*_x = \sum_{y: y \in W} p^*_{Y|X}(y|x)$$

where we label the components of $w^*$ using the questions $x \in X$. Since $w^*_x = 1$ for all $x \in X$ and because there is exactly one correct answer for each question $x$, we can infer that $p^*_Y|X$ must be a deterministic strategy. Written explicitly, we have

$$p^*_{Y|X}(y_1, y_2|x_1, x_2) = \delta_{y_1, x_2} \delta_{y_2, x_1}.$$

Note that $p^*_{Y|X}$ cannot be a no-signalling strategy. Indeed,

$$p^*_{Y|X}(y_1|x_1, x_2) = \sum_{y_2 \in Y_2} p^*_{Y|X}(y_1, y_2|x_1, x_2) = \delta_{y_1, x_2}. \tag{45}$$

This cannot satisfy the no-signalling condition $p_{Y|X}(y_1|x_1, x_2) = p_{Y|X}(y_1|x_1)$ given in Eq. (4) because the RHS in Eq. (45) depends on $x_2$. In other words, we cannot have $w^*_x = 1$ for any question $x \in X$ using a no-signalling strategy. In particular, the perfect strategy is not no-signalling, and therefore, $\omega^{NS}(G_s) < 1$ for no-signalling strategies. Subsequently, we also have $\omega^Q(G_s) < 1$ and $\omega^d(G_s) < 1$, because the set of classical and quantum strategies are contained in the set of no-signalling strategies. It then follows from Thm. 5 and Cor. 6 that each of $S(N_{G_s}), S_Q(N_{G_s}), S_{NS}(N_{G_s})$ is strictly less than $\ln(d)$. On the other hand, since a perfect strategy is possible allowing communication between Alice
& Bob, we have $C(N_{G_s}) = \ln(d)$. Therefore, we have obtained a separation between $C(N_{G_s})$ and $S(N_{G_s}), S_Q(N_{G_s}), S_{NS}(N_{G_s})$.

Below, we argue that this separation becomes arbitrarily large as the number of questions increases. To that end, we compute $\omega^{cl}(G_s)$. Since the maximum winning probability obtained using classical strategies when questions are drawn uniformly is achieved by a deterministic strategy, it is sufficient to restrict our attention to classical deterministic strategies chosen by Alice & Bob. Recall that a classical deterministic strategy corresponds to two functions $f_1 : \mathcal{X}_1 \to \mathcal{Y}_1$ and $f_2 : \mathcal{X}_2 \to \mathcal{Y}_2$ chosen by Alice & Bob, respectively. This translates to the probability distribution $p^{(D)}_{Y|X}(y_1, y_2 | x_1, x_2) = \delta_{y_1, f_1(x)} \delta_{y_2, f_2(x)}$. Then, the winning probability using a classical deterministic strategy when the questions are drawn uniformly is given by

$$\omega^{cl}(G_s) = \frac{1}{d} \sum_{x_1, x_2 \in \mathcal{X}} \delta_{x_2, f_1(x_1)} \delta_{x_2, f_2(x_2)}$$

where we used the fact that the signalling game has only one correct answer $(x_2, x_1)$ corresponding to each question $(x_1, x_2)$. It can be seen from Eq. (46) that, for achieving maximum winning probability, the function $f_1$ must be able to invert the action of the function $f_2$ or vice-versa.

If $|\mathcal{X}_2| \leq |\mathcal{X}_1|$, then the set $f_2(\mathcal{X}_2)$ can cover at most $|\mathcal{X}_2|$ elements of $\mathcal{X}_1$. Subsequently, $\delta_{x_2, f_2(x_2)} \neq 0$ for at most $|\mathcal{X}_2|$ elements of $\mathcal{X}_1$. We can then infer from Eq. (46) that $p^{(D)}_W \leq \frac{|\mathcal{X}_2|}{d} = \frac{1}{|\mathcal{X}_1|}$ since $d = |\mathcal{X}_1||\mathcal{X}_2|$. Using a similar reasoning when $|\mathcal{X}_1| \leq |\mathcal{X}_2|$, we get $\omega^{cl}(G_s) = \frac{1}{\max(|\mathcal{X}_1|, |\mathcal{X}_2|)}$.

In particular, we have $d \to \infty$ if either of $|\mathcal{X}_1|$ or $|\mathcal{X}_2|$ diverges, while the winning probability $\omega^{cl}(G) \to 0$. Subsequently, $C(N_{G_s}) \to \infty$ but $S(N_{G_s}) \to 0$. Therefore, we get an arbitrarily large separation between $C(N_{G_s})$ and $S(N_{G_s})$.

In fact, we verify through numerical simulations that the situation is equally bad for the no-signalling assisted sum rate. To that end, we compute the maximum winning probability $\omega^{NS}(G_s)$ numerically. We show in Prop. 11 that the set of no-signalling strategies for $N$-player games is a compact and convex set (specifically, a convex polytope). Therefore, computing $\omega^{NS}(G_s)$ amounts to solving a linear program (this fact is well-known for 2-player games [25]). For $2 \leq |\mathcal{X}_1|, |\mathcal{X}_2| \leq 10$, we verify that the numerically computed value for $\omega^{NS}(G_s)$ matches $\omega^{cl}(N_{G_s}) = 1/ \max(|\mathcal{X}_1|, |\mathcal{X}_2|)$. Thus, we expect $\omega^{cl}(N_{G_s}) = \omega^{Q}(N_{G_s}) = \omega^{NS}(N_{G_s})$ for the signalling game. This would imply that $S_Q(N_{G_s}), S_{NS}(N_{G_s}) \to 0$ as $d \to \infty$ but $C(N_{G_s}) \to \infty$. In other words, even with quantum or no-signalling assistance, the sum rate is far lower than the bound given by the relaxed sum capacity.

This example highlights the importance of finding better methods to upper bound the sum capacity of MACs. In the next section, we take a step in this direction by providing an algorithm that can upper-bound, and even compute to a desired accuracy in certain situations, the sum capacity for a family of two-sender MACs.

5 Computing the sum capacity of two-sender MACs

As discussed in Sec. 2.3, a two-sender MAC with input alphabets $B_1$ and $B_2$ and output alphabet $Z$ is described by a transition matrix $N$. Each entry of this matrix, $N(z|b_1, b_2)$, where $z \in Z, b_1 \in B_1$, and $b_2 \in B_2$, represents the probability that the channel output takes a value $z$ when the first and second
channel inputs take values $b_1$ and $b_2$, respectively. We introduce new notation for this sub-section. Let $d_1, d_2,$ and $d_o$ denote $|B_1|, |B_2|,$ and $|Z|$ respectively. We write $p(b_1) \in \Delta_{d_1}$, but refer to $p(b_2)$ by $q(b_2)$ where $q(b_2) \in \Delta_{d_2}$. In this notation, the sum capacity (7) of the two-sender MAC $\mathcal{N}$ takes the form

$$S(\mathcal{N}) = \max_{p(b_1, b_2)} I(B_1, B_2; Z) \quad \text{such that} \quad p(b_1, b_2) = p(b_1)q(b_2),$$

(47)

where $B_1, B_2,$ and $Z$ are random variables that describe the channel’s first input, second input, and output respectively. For fixed $\mathcal{N}(z|b_1, b_2)$, the mutual information $I(B_1, B_2; Z)$ function is concave in the argument $p(b_1, b_2)$. On the other hand, the set of joint distributions which satisfy the product constraint, $p(b_1, b_2) = p(b_1)p(b_2)$, is not convex. This lack of convexity turns the maximization in Eq. (47) into a non-convex problem.

Our approach to solving the non-convex problem (47) is to move the non-convexity from the constraint to the objective function. Instead of maximizing over the set of product distributions, $p(b_1, b_2) = p(b_1)q(b_2)$, we maximize sequentially, that is, we write

$$S(\mathcal{N}) = \max_{q \in \Delta_{d_2}} \max_{p \in \Delta_{d_1}} I(B_1, B_2; Z).$$

(48)

Now, both the inner and outer maximization in Eq. (48) are carried out over convex sets $\Delta_{d_1}$ and $\Delta_{d_2}$, respectively. To carry out these optimizations we derive certain convenient expressions. The output probability distribution $p^Z$ over $Z$ can be written as

$$p^Z(z) = \sum_{b_1 \in B_1} A_q(z, b_1)p(b_1)$$

(49)

where

$$A_q(z, b_1) = \sum_{b_2 \in B_2} \mathcal{N}(z|b_1, b_2)q(b_2)$$

(50)

for $z \in Z$ and $b_1 \in B_1$. Note that $A_q$ can be considered as a left stochastic matrix of size $d_o \times d_1$, i.e., every entry of $A_q$ is non-negative and the columns sum to 1. One can view Eq. (49) as a vector equation $p^Z = A_q p$, where $p \in \Delta_{d_1}$ and $p^Z \in \Delta_{d_o}$. The mutual information $I(B_1, B_2; Z)$ can be written as

$$I(B_1, B_2; Z) = H(Z) - \sum_{b_1, b_2} p(b_1)q(b_2)H(Z|B_1 = b_1, B_2 = b_2).$$

To express the mutual information in terms of vectors and matrices, we define a $d_1$-dimensional vector $b_q$ with non-negative components $b_q(b_1)$ for $b_1 = 1, \ldots, d_1$, where

$$b_q(b_1) = -\sum_{b_2 \in B_2} q(b_2) \sum_{z \in Z} \mathcal{N}(z|b_1, b_2) \log(\mathcal{N}(z|b_1, b_2)).$$

(51)

This allows us to express the mutual information compactly as

$$I(p, q) \equiv I(B_1, B_2; Z) = H(A_q p) - \langle b_q, p \rangle,$$

(52)

and the sum capacity as

$$S(\mathcal{N}) = \max_{q \in \Delta_{d_2}} \max_{p \in \Delta_{d_1}} \{ H(A_q p) - \langle b_q, p \rangle \}.$$ 

(53)

The inner optimization over $p \in \Delta_{d_1}$ is a convex optimization problem because $I(p, q)$ is a concave function of $p$ for fixed $q$. Since $I(p, q)$ is not jointly concave over $(p, q)$, the function

$$I^*(q) = \max_{p \in \Delta_{d_1}} (H(A_q p) - \langle b_q, p \rangle)$$

(54)
is not concave in general. Therefore, the outer optimization of $I^*(q)$ over $q \in \Delta_d$ is in general a nonconvex problem.

The non-concave function $I^*(q)$ is Lipschitz-like, by which we mean that there is a non-negative, continuous and monotonically increasing function $\beta_I$ with $\beta_I(0) = 0$, such that

$$|I^*(q) - I^*(q')| \leq \beta_I \left( \|q - q'\|_1 \right)$$

(55)

where $\|\cdot\|_1$ denotes the $l_1$-norm. Such an inequality is sometimes referred to as a continuity bound in information theory. In our case, the function $\beta_I$ is given by

$$\beta_I(x) = \left( \frac{1}{2} \log(d_0 - 1) + H_N^{\text{max}} \right) x + \overline{h} \left( \frac{x}{2} \right)$$

(56)

where

$$H_N^{\text{max}} = \max_{a_1 \in A_1, a_2 \in A_2} \left\{ - \sum_{z \in \mathcal{Z}} \mathcal{N}(z|a_1, a_2) \log(\mathcal{N}(z|a_1, a_2)) \right\}$$

(57)

and

$$\overline{h}(x) = \begin{cases} -x \log(x) - (1 - x) \log(1 - x) & \text{if } x \leq \frac{1}{2} \\ \log(2) & \text{if } x \geq \frac{1}{2} \end{cases}$$

(58)

is the modified binary entropy. The Lipschitz-like property (55) is proved in Prop. 19 in App. D.

This observation leads us to propose an adaption of the Piyavskii-Shubert algorithm [26, 27] for solving such a nonconvex problem. This algorithm, sometimes called the sawtooth method, is designed to optimize Lipschitz continuous functions over an interval. We also show how this algorithm can be generalized to optimize Lipschitz-like functions over compact and convex domains. Our primary focus will be on optimization of Lipschitz-like functions over the standard simplex.

5.1 Optimization of Lipschitz-like functions

We begin our study by defining a Lipschitz-like function.

**Definition 7 (Lipschitz-like function).** Let $\beta : \mathbb{R}_+ \to \mathbb{R}$ be a non-negative, continuous, monotonically increasing function such that $\beta(0) = 0$. Let $\mathcal{D}$ be a subset of $\mathbb{R}^d$ for some dimension $d \in \mathbb{N}_+$. Then a function $f : \mathcal{D} \to \mathbb{R}$ is said to be Lipschitz-like or $\beta$-Lipschitz-like if it satisfies

$$|f(x) - f(x')| \leq \beta(\|x - x'\|) \quad \forall x, x' \in \mathcal{D},$$

(59)

where $\|\cdot\|$ is some norm on $\mathbb{R}^d$.

Since $\beta$ is (right-)continuous at 0, it follows from Eq. (59) that $f$ is a continuous function. If $\beta(x) = Lx$ for some $L > 0$, then $f$ is a Lipschitz continuous function with Lipschitz constant $L$, and if $\beta(x) = Lx^\gamma$ for $L, \gamma > 0$, then $f$ is a Hölder continuous function with constant $L$ [28]. Lipschitz-like functions are therefore a generalization of Lipschitz and Hölder continuous functions. Such functions are of relevance in information theory because some entropic quantities are Lipschitz-like but not Lipschitz. We encountered such a function $I^*$ in Sec. 5, which satisfies the Lipschitz-like property as per Eq. (55). For the remainder this section, we use the $l_1$-norm in Eq. (59) because that is the relevant norm for computing the sum capacity. This assumption is mainly for convenience and it can be relaxed (see the discussion in App. C.4). Furthermore, we assume that $\beta$ does not explicitly depend on the dimension $d$.\footnote{This assumption is made for convenience. If $\beta$ depends explicitly on the dimension, then statements about the scaling of the number of iterations (with respect to the dimension) required for computing the optimum to a fixed precision must be appropriately updated.}
All the algorithms we propose in this study are designed to optimize any $\beta$-Lipschitz-like function $f$. The function $\beta$ is assumed to be known beforehand, but the function $f$ is unknown and we can only query it at a specified point. The algorithms can then use the knowledge of the domain, the function $\beta$, the queried points and corresponding values of the objective function $f$ to approximate the maximum of $f$ to an additive precision $\epsilon > 0$. Such a setting is commonly used to study the performance of optimization algorithms [28]. By an efficient algorithm for optimization, we mean an algorithm that computes the optimum of $f$ to a given additive precision $\epsilon > 0$ in time polynomial in the dimension and inverse precision $1/\epsilon$. We sometimes informally use the phrase “practically efficient” and variations thereof to mean that the algorithm runs reasonably fast in practice, e.g., to exclude situations where the scaling of runtime with dimension is too large (for example, $O(d^{10})$).

Before discussing technical details of the algorithms presented in this study, we give a high-level overview of the main ideas. We begin by presenting an algorithm that can optimize any $\beta$-Lipschitz-like function $f$ when the domain $D = [a, b]$ is a closed interval. For all the algorithms discussed in this study, we assume that $\beta$ is known beforehand. This algorithm is a generalization of the Piyavskii-Shubert algorithm [26, 27], and it focuses on constructing successively better upper-bounding functions by using the Lipschitz-like property of the objective function. At each iteration, the maximum of the upper-bounding function is computed, which is an easier problem because we only need to maximize the function $\beta$ that is known to be monotonically increasing. The computed maximum of the upper-bounding function at each iteration generates a sequence of points that partitions the interval. When the distance between any two of these points becomes sufficiently small, one can show that the maximum of the upper-bounding function is close to the maximum of the original objective function. Our algorithm is guaranteed to converge to the optimal solution within a precision of $\epsilon > 0$ in $[(b - a)/\delta]$ time steps in the worst-case, where $\delta > 0$ is the largest number satisfying $\beta(\delta) \leq \epsilon/2$ (see Prop. 12). We refer to this algorithm as modified Piyavskii-Shubert algorithm. We remark that several extensions of the Piyavskii-Shubert algorithm have been presented in the literature (see, for example, Ref. [29, 30, 31, 32]). It would be interesting to undertake a more detailed study comparing such algorithms with our method in the future.

For the higher dimensional case, the objective $f$ is a $\beta$-Lipschitz-like function over a compact and convex domain $D \subseteq \mathbb{R}^d$. When $D$ is the standard simplex, we resort to a straightforward grid search. Using the results of Ref. [33], one can show that a grid of size $O(d^{1/\delta^2})$ suffices to converge to a specified precision of $\epsilon > 0$, where $\delta$ is the largest number satisfying $\beta(\delta) \leq \epsilon/2$ (see Prop. 14).\(^5\) These results stand to demonstrate that one can, in principle, compute the maximum of $f$ in polynomial time for a fixed precision. Moreover, a simple observation about ordering the elements of the grid allows for efficient construction of the grid, along with the possibility of parallelizing the grid search. Despite the possibility of polynomial complexity (in dimension) and numerical improvements, grid search is still too inefficient to be of practical use except for very small dimensions.

Another strategy we propose is to construct a “dense” Lipschitz continuous curve that gets close to each point of $D$ to within some specified distance. Such a strategy was adopted by Ref. [34] for optimizing Lipschitz continuous functions over a hypercube, and is referred to as Alienor method in the literature. When $D$ is the standard simplex in $d$ dimensions, we give a time and memory efficient algorithm to construct such a curve. This allows us to reduce the $d$-dimensional problem of optimizing $f$ over $D$ to the one-dimensional problem of optimizing it over an interval using the generated curve. If $\alpha > 0$ is the largest number satisfying $\beta(\alpha) \leq \epsilon/2$, this method takes $O(\alpha^{1-d}/d)$ iterations in the worst case for large dimensions. While this is much worse than a grid search in large dimensions, in small

\(^5\)Note the different scaling in $\delta$ in Prop. 14 compared to that of $O(d^{1/\sqrt{\delta}})$ mentioned in Ref. [33], which we believe to be erroneous.
dimensions this takes fewer iterations to converge than grid search when the tolerance \( \epsilon \) is small.

For cases where it is non-trivial to construct a curve filling \( \mathcal{D} \), we embed \( \mathcal{D} \) inside a hypercube (which is possible because \( \mathcal{D} \) is compact), and fill the hypercube with a curve following Ref. [34]. Because the domain of the objective function \( f \) is \( \mathcal{D} \) and not the hypercube, we show how to extend the function \( f \) to the full Euclidean space \( \mathbb{R}^d \) while retaining the Lipschitz-like property when \( \beta \) is itself Lipschitz-like (see Def. 16 and Prop. 17). The extended function \( \overline{f} \) has the property that its optimum coincides with the optimum of \( f \) when optimized over any set containing \( \mathcal{D} \). Since we have the freedom to choose which set to optimize \( \overline{f} \) over, there are different algorithms one can potentially use for this optimization. In this study, we find the maximum value of \( f \) by optimizing its extension \( \overline{f} \) over a hypercube containing \( \mathcal{D} \) as noted above. To perform this optimization, we resort to using \( \alpha \)-dense curves because the convergence analysis is very similar to the case of \( \mathcal{D} = \Delta_d \). The effective problem, as before, is one-dimensional and can be solved using the modified Piyavskii-Shubert algorithm. In general, this algorithm needs exponentially many iterations (with dimension) to find the optimal solution to within a specified precision of \( \epsilon > 0 \) (see Prop. 18 for a precise statement). This exponential scaling with the dimension stems from the fact that we do not use any structure of \( \mathcal{D} \) to construct the dense curve and instead rely on a curve generated for a hypercube. The \( O(1/\epsilon^d) \) complexity cannot be improved in general without additional assumptions on \( \mathcal{D} \) or the class of functions we optimize (see Prop. 18).

However, it might be possible to improve the other factors that scale exponentially in the algorithm as noted in App. C.4. An advantage of using this algorithm is that one can specify a fixed number of iterations to obtain an upper bound on the maximum. Because optimizing \( f \) over an arbitrary compact and convex domain is not directly relevant to our study of MACs, we relegate this discussion to App. C.4.

For the remainder of this section, we will focus our attention on the case when \( \mathcal{D} = \Delta_d \) is the standard simplex in \( \mathbb{R}^d \). For \( d = 2 \) this is just an interval and the modified Piyavskii-Shubert algorithm is both efficient in theory and in practice. It also has the advantage that one could specify a fixed number of iterations to obtain an upper bound on the maximum value of the objective function. For \( d \geq 3 \), the grid search has the advantage that the complexity scales polynomially with the dimensions, whereas the method of using dense curves has exponential scaling with the dimension in the worst-case scenario. However, using dense curves to perform the optimization has the advantage that we can find successively better upper bounds on the function, so one could stop the computation after a fixed number of time steps. Despite this, we remark that both of these methods are impractical for even moderately large dimensions. We detail these methods here in the hope that these engender the development of more practical algorithms for optimizing Lipschitz-like functions over the standard simplex. Such algorithms could be of interest in information theory because some entropic quantities are Lipschitz-like functions of the underlying probability distribution.

We now dive into details of the proposed algorithms.

### 5.1.1 Optimizing Lipschitz-like functions over an interval using modified Piyavskii-Shubert algorithm

We begin our study by presenting an algorithm for computing the maximum of any \( \beta \)-Lipschitz-like function \( f \) over a closed interval \( \mathcal{D} = [a, b] \). A pseudocode for this algorithm is given in Alg. 1.

We refer to the function \( F_i \) defined in line 8 of Alg. 1 as a bounding function, since \( f(q) \leq F_i(q) \) for all \( q \in [a, b] \) and all \( i \). Note that \( F_i(q^{(i)}) = f(q^{(i)}) \) for all \( i \). The bounding function \( F_i \) depends non-trivially on the argument \( q \) only through the function \( \beta(|q - q^{(i)}|) \). We compute the optimum of the function \( f \) by maximizing these bounding functions, which is an easier problem because \( \beta \) is continuous.
Algorithm 1: Computing the maximum of a function $f$ satisfying Eq. (59) for $D = [a, b]$, given $\epsilon > 0$

1: function MAXIMIZE_LIPTHSCHITZ-LIKE_FUNCTION_1D(\epsilon)
2:     Initialize $q^{(0)} = a$
3:     Set $F_0(q) = f(q^{(0)}) + \beta(|q - q^{(0)}|)$ for $q \in [a, b]$
4:     Set $F \leftarrow F_0$ and $q^* \leftarrow b$
5:     Set $q^{(1)} \leftarrow q^*$, $k \leftarrow 1$
6:     while $F(q^*) - f(q^*) > \epsilon$ do
7:         Sort $\{q^{(0)}, \ldots, q^{(k)}\}$ from smallest to largest and relabel the points in ascending order.
8:         Define $F_i(q) = f(q^{(i)}) + \beta(|q - q^{(i)}|)$ for $0 \leq i \leq k$
9:         for $i = 0, \ldots, k - 1$ do
10:             Set $g_i(q) = F_i(q) - F_{i+1}(q)$
11:             Find $\overline{q}^{(i)} \in [q^{(i)}, q^{(i+1)}]$ such that $g_i(\overline{q}^{(i)}) = 0$ using any root finding method.
12:         end for
13:         Pick an index $m \in \arg\max_{0 \leq i \leq k-1} F_i(\overline{q}^{(i)})$ and set $q^* = \overline{q}^{(m)}$.
14:         Update $F \leftarrow F_m$
15:         Set $q^{(k+1)} \leftarrow q^*$, $k \leftarrow k + 1$
16:     end while
17:     return $f(q^*)$
18: end function

and monotonic. Essentially, the algorithm does the following. Suppose that at the $K$th time step, we have the iterates $q^{(0)}, \ldots, q^{(K)}$ sorted in the ascending order $a = q^{(0)} \leq q^{(1)} \leq \cdots \leq q^{(K)} = b$. Then in each of the intervals $[q^{(k)}, q^{(k+1)}]$ for $k \in \{0, \ldots, K - 1\}$ we compute the point $\overline{q}^{(k)}$ attaining the maximum of the bounding function $\min\{F_k, F_{k+1}\}$ (see lines 10, 11 in Alg. 1). Our next iterate $q^{(K+1)}$ is chosen to be in $\arg\max\{\overline{q}^{(0)}, \ldots, \overline{q}^{(K-1)}\}$. In the next iterate, since $F_{K+1}(q^{(K+1)}) = f(q^{(K+1)})$, we have essentially tightened the upper bound on the function $f$. Proceeding this way, one can verify that the algorithm eventually approximates the function $f$ from above well enough. A schematic of this procedure is shown in Fig. 2.

![Figure 2: A schematic of iterations of Alg. 1 for optimizing a Lipschitz-like function $f$.](image)

Indeed, we show in Prop. 12 that Alg. 1 is guaranteed to converge to the global maximum within an error of $\epsilon$. This convergence takes at most $\lceil (b - a) / \delta \rceil$ in the worst-case, where $\delta > 0$ is the largest number satisfying $\beta(\delta) \leq \epsilon / 2$. Since we find successively better upper bounds on the objective, Alg. 1 can be modified so that it accepts a fixed number of iterations instead of a precision, and outputs the upper bound $F(q^*)$ on the maximum of $f$. This upper bound has an error of at most $F(q^*) - f(q^*)$. 

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When the domain $D = \Delta_2$ is the standard simplex in 2-dimensions, we can parameterize any $x \in \Delta_2$ as $x = (q, 1-q)$ with $q \in [0, 1]$. Furthermore, we have $\|x - y\|_1 = 2|p - q|$ for $x, y \in \Delta_2$ parametrized as $x = (q, 1-q)$ and $y = (p, 1-p)$. Thus, taking $a = 0$, $b = 1$, and replacing $|q - q^{(i)}|$ with $2|q - q^{(i)}|$ in Alg. 1, we get an algorithm for optimizing $f$ over $\Delta_2$.

Next, we present two different generalizations to higher dimensions, wherein we optimize $f$ over $D = \Delta_d$ for $d \geq 3$. The first generalization is a simple grid search, whereas the second generalization uses dense curves to fill $D$.

### 5.1.2 Optimizing Lipschitz-like functions over the standard simplex using grid search

Based on the results of Ref. [33], we present a grid search method to compute the maximum of $f$ over $D = \Delta_d$. We begin by defining the integer grid

$$I_{d,N} = \left\{ n \in \mathbb{N}^d \mid \sum_{i=1}^{d} n_i = N \right\},$$

where $d \in \mathbb{N}_+$ denotes the dimension and $N \in \mathbb{N}_+$ denotes the size of the integer grid. This grid has

$$N_{\text{grid}} = \binom{N + d - 1}{d - 1}$$

elements because each element of $I_{d,N}$ can be obtained by arranging $N$ ones into $d$ coordinates. From $I_{d,N}$, we obtain the grid

$$\Delta_{d,N} = \left\{ \frac{n}{N} \mid n \in I_{d,N} \right\}$$

for the standard simplex $\Delta_d$. Note that $\Delta_{d,N}$ is just a rescaled version of $I_{d,N}$, i.e., $\Delta_{d,N} = I_{d,N}/N$.

The authors of Ref. [33] propose to search the grid $\Delta_{d,N}$ in order to optimize Hölder continuous functions over the standard simplex. Their results are based on approximations of the function $f$ using Bernstein polynomials, and are in fact general enough to handle the optimization of Lipschitz-like functions (see Prop. 14 for details). Such an approach to compute the maximum of $f$ to a specified precision $\epsilon > 0$ is summarized in Alg. 2.

**Algorithm 2** Computing the maximum of a function $f$ satisfying Eq. (59) for $D = \Delta_d$; given $\epsilon > 0$

1: function MAXIMIZE_LIPSCHITZ-LIKE_FUNCTION_SIMPLEX($d$, $\beta$, $\epsilon$)
2:  Set $N = \lceil 1/\delta^2 \rceil$, where $\delta$ is the largest number satisfying $\beta(\delta) \leq \epsilon/2$
3:  Construct the grid $\Delta_{d,N}$
4:  return $f^* = \max\{f(x) \mid x \in \Delta_{d,N}\}$
5: end function

In Prop. 14, we show that Alg. 2 computes the maximum of $f$ to a precision $\epsilon > 0$ in $N_{\text{grid}} = \binom{N + d - 1}{d - 1}$ time steps, where $N = \lceil 1/\delta^2 \rceil$ and $\delta$ is the largest number satisfying $\beta(\delta) \leq \epsilon/2$. For fixed $\epsilon > 0$ and $d \gg N$, this amounts to $O(d^{1/\delta^2})$ iterations. In other words, we can find the optimum of $f$ in polynomial time for a fixed precision. We note that if $\beta$ achieves the value $\epsilon/2$, we can obtain $\delta$ by solving $\beta(\delta) = \epsilon/2$ using bisection (or any other root finding method) because $\beta$ is continuous and monotonically increasing.

The crucial step in implementing the above algorithm is computing the grid $\Delta_{d,N}$ efficiently. For that purpose, note that we can write any element $x \in \Delta_{d,N}$ as $x = (N-n_{d-1}, n_{d-1}-n_{d-2}, \ldots, n_2-n_1, n_1)/N$, where $0 \leq n_i \leq n_{i+1} \leq N$ for $i \in [d-2]$ (see Prop. 13 for a proof). Thus, the elements of $\Delta_{d,N}$ can
be computed iteratively, with the total number of iterations equalling $N_{\text{grid}}$. This also allows for parallelizing the search over the grid. The exact algorithm we use to query the elements of the grid orders the elements such that every consecutive element is equidistant with respect to $l_1$-norm. This approach is explained in the next section and allows for easy parallelization.

5.1.3 Optimizing Lipschitz-like functions over the standard simplex using $\alpha$-dense curves

Next, we outline a method to optimize Lipschitz-like functions over $D = \Delta_n$ by filling $D$ with an $\alpha$-dense Lipschitz curve. Such a strategy was outlined in Ref. [34] to optimize Lipschitz continuous functions over a hypercube.

Definition 8 ($\alpha$-dense curve). Given a number $\alpha > 0$ and a nonempty set $S \subseteq \mathbb{R}^n$, a function $\gamma: [a, b] \rightarrow S$ is said to be an $\alpha$-dense curve in the norm $\|\cdot\|$ if for any $x \in S$, we can find some $\theta \in [a, b]$ such that $\|\gamma(\theta) - x\| \leq \alpha$ [34].

The curve $\gamma$ is said to be $\beta_\gamma$-Lipschitz-like if there is some non-negative, continuous, monotonically increasing function $\beta_\gamma: \mathbb{R}_+ \rightarrow \mathbb{R}$ with $\beta_\gamma(0) = 0$ such that for any $\theta, \theta' \in [a, b]$, we have $\|\gamma(\theta) - \gamma(\theta')\| \leq \beta_\gamma(|\theta - \theta'|)$.

The numbers $a, b \in \mathbb{R}$ are the end points of the interval over which the curve is defined. As noted previously, we will focus on the case when the norm in Def. 8 is the $l_1$-norm. We outline an algorithm to construct an $\alpha$-dense curve for the case of $\alpha = 2(d - 1)/N$, where $N \in \mathbb{N}_+$. For this purpose, we need to order the grid $\mathcal{I}_{d,N}$ defined in Eq. (60) as follows.

Definition 9 (Equidistant ordering of $\mathcal{I}_{d,N}$). Given $d, N \in \mathbb{N}_+$ with $d \geq 2$, let $\mathcal{I}_{d,N}$ be the grid defined in Eq. (62). For $d = 2$, define the forward ordering $\mathcal{I}_{2, N} = \{(0, N), (N, 0), (N - 1, N - 1), \ldots, (N - 1, 1), (N, 0)\}$ and the reverse ordering $\mathcal{I}_{2, N} = \{N, N - 1, \ldots, (0, N), (1, N - 1), \ldots, (N - 1, 1), (N, 0)\}$. For $d \geq 3$, define the forward ordering inductively as follows. Start with $\mathcal{I}_{d, N} = \emptyset$. For each $n_{d-1} \in [N]$, forward order the elements of $\mathcal{I}_{d-1, n_{d-1}}$ if $n_{d-1}$ is odd, and reverse order them if $n_{d-1}$ is even. Append the elements $(N - n_{d-1}, n_{d-1} - n_{d-2}, \ldots, n_2 - n_1, n_1)$ with $(n_{d-1} - n_{d-2}, \ldots, n_2 - n_1, n_1) \in \mathcal{I}_{d-1, n_{d-1}}$ as ordered above to the (ordered) set $\mathcal{I}_{d, N}$. Reverse ordering of $\mathcal{I}_{d, N}$ corresponds to writing the forward ordered set in the reverse order.

We remark that $d = 2$ is just a special case of the definition showing the basis step for induction. It can be seen that the first element of $\mathcal{I}_{d, N}$ is $(N, 0, \ldots, 0)$ and the last element is $(0, \ldots, 0, N)$ if $\mathcal{I}_{d, N}$ is forward ordered (and the opposite is true if the elements are reverse ordered). In Prop. 13, we show that $\mathcal{I}_{d, N}$ can be ordered in this manner and the distance between any two consecutive elements of $\mathcal{I}_{d, N}$ according to this ordering is 2 measured in $l_1$-norm. By default, we work with forward ordering unless specified otherwise. As an example, the (forward) ordering of $\mathcal{I}_{3, 3}$ is given by

$$\mathcal{I}_{3, 3} = \{(3, 0, 0), (2, 1, 0), (2, 0, 1), (1, 0, 2), (1, 1, 1), (1, 2, 0), (0, 3, 0), (0, 2, 1), (0, 1, 2), (0, 0, 3)\}.$$  

It can be seen that any two consecutive elements have an $l_1$-norm distance of 2. Since the grid $\Delta_{d, N}$ is just a scaled version of $\mathcal{I}_{d, N}$, this ordering also applies to $\Delta_{d, N}$.

To generate a dense curve filling $\Delta_{d, N}$, we make use of the grid $\Delta_{d, N}$ along with the ordering described above. However, $\Delta_{d, N}$ has $N_{\text{grid}} = \binom{N + d - 1}{d - 1}$ elements, and therefore, it can get very expensive to store this grid in memory even if $N$ and $d$ are only moderately large (for example, $N = d = 20$ gives $N_{\text{grid}} \approx 7 \times 10^{10}$). For this purpose, we develop an algorithm that can efficiently query the elements of ordered $\Delta_{d, N}$ without explicitly constructing the set. We skip the details of this algorithm in the study, but include an implementation of the algorithm on Github (see Sec. 8). Owing to such an approach,
the resulting construction is both time and memory efficient, i.e., \( \gamma(\theta) \) can be computed efficiently for a curve \( \gamma \). We describe the construction of the curve below.

**Algorithm 3** Construct an \( \alpha \)-dense curve that fills the simplex \( \Delta_d \) for \( \alpha = 2(d-1)/N \), \( N \in \mathbb{N}_+ \) and \( \theta \in [0, L_{\text{curve}}] \), where \( L_{\text{curve}} \) is given in Eq. (63)

1: \textbf{function} \text{CONSTRUCT\_DENSE\_CURVE\_STANDARD\_SIMPLEX}(d, N, \theta)
2: \hspace{1em} Compute the index \( k = \lfloor \theta N/2 \rfloor \)
3: \hspace{1em} Compute \( t = 1 + k - \theta N/2 \)
4: \hspace{1em} Obtain the grid points \( x_k, x_{k+1} \in \Delta_{d,N} \), where the elements of \( \Delta_{d,N} \) are ordered as per Def. (9)
5: \hspace{1em} \textbf{return} \( tx_k + (1-t)x_{k+1} \)
6: \textbf{end function}

We now explain the reasoning behind Alg. 3. We construct the \((2(d-1)/N)\)-dense curve \( \gamma \) by joining the consecutive points of the ordered grid \( \Delta_{d,N} \). Since every two consecutive points in the ordered grid \( \Delta_{d,N} \) are \( 2/N \) distance apart in \( l_1 \)-norm, the total length of this curve measured in \( l_1 \)-norm is

\[
L_{\text{curve}} = \frac{2}{N} N_{\text{grid}} = \frac{2}{N} \left( \frac{N + d - 1}{d - 1} \right). \tag{63}
\]

In order to obtain \( \gamma(\theta) \) efficiently for a given \( \theta \in [0, L_{\text{curve}}] \), we first compute the grid points \( x_k \) and \( x_{k+1} \) for which \( \gamma(\theta) \in [x_k, x_{k+1}] \). In particular, we have \( \gamma(\theta) = tx_k + (1-t)x_{k+1} \) for some \( t \in [0,1] \). In order to compute \( t \), we note that \( \|\gamma(\theta) - x_k\|_1 = \theta - 2k/N \), where the RHS is the length of the curve at \( \theta \) minus the length of the curve at grid point \( x_k \) (which is \( 2k/N \)). Since \( \|\gamma(\theta) - x_k\|_1 = (1-t)\|x_{k+1} - x_k\|_1 = 2(1-t)/N \), we obtain \( t = 1 + k - \theta N/2 \). Since \( \theta N/2 - 1 \leq k \leq \theta N/2 \), we have \( 0 \leq t \leq 1 \). In Prop. 15, we show that the curve constructed using Alg. 3 is \((2(d-1)/N)\)-dense and satisfies the property

\[
\|\gamma(\theta) - \gamma(\theta')\|_1 \leq \min\{\|\theta - \theta'\|, 2\}. \tag{64}
\]

That is, the curve \( \gamma \) is \( \beta_\gamma \)-Lipschitz-like with \( \beta_\gamma(x) = \min\{x, 2\} \). In particular, \( \gamma \) is Lipschitz continuous with Lipschitz constant 1.

The essential idea behind the optimization of Lipschitz-like functions using an \( \alpha \)-dense curve is as follows. If the objective function \( f : \mathcal{D} \to \mathbb{R} \) is Lipschitz-like and \( \gamma : [a, b] \to \mathcal{D} \) is an \( \alpha \)-dense Lipschitz-like curve, then the function \( f \circ \gamma : [a, b] \to \mathbb{R} \) is also Lipschitz-like. Therefore, one can optimize the function \( f \circ \gamma \) using Alg. 1. In order to converge to the optimum within a precision of \( \epsilon > 0 \), the constant \( \alpha \) must be chosen appropriately. The exact procedure is outlined in Alg. 4.

**Algorithm 4** Computing the maximum of a \( \beta \)-Lipschitz-like function \( f \) satisfying Eq. (59) for \( \mathcal{D} = \Delta_d \), given \( \epsilon > 0 \)

1: \textbf{function} \text{MAXIMIZE\_LIPSCHITZ-LIKE\_FUNCTION\_SIMPLEX}(d, \beta, \epsilon)
2: \hspace{1em} Compute the largest number \( \alpha > 0 \) such that \( \beta(\alpha) \leq \epsilon/2 \)
3: \hspace{1em} Set \( N = \lceil 2(d-1)/\alpha \rceil \)
4: \hspace{1em} Construct the \((2(d-1)/N)\)-dense curve \( \gamma : [0, L_{\text{curve}}] \to \mathcal{D} \) as per Alg. 3
5: \hspace{1em} Compute the maximum \( g^* \) of \( g = f \circ \gamma \) over \([0, L_{\text{curve}}]\) to a precision of \( \epsilon/2 \) using Alg. 1
6: \hspace{1em} \textbf{return} \( g^* \)
7: \textbf{end function}

In Prop. 15, we show that Alg. 4 is guaranteed to converge to the maximum of \( f \) to within a precision of \( \epsilon > 0 \). This takes \( \lceil 2(N+\beta^{-1}) \alpha \rceil \) iterations in the worst case. For large dimensions and fixed precision, this number scales as \( O(\alpha^{1-d}/d) \) with the dimension.
For verifying the performance of the grid search algorithm (Alg. 2) and the algorithm based on dense curves and the modified Piyavskii-Shubert method (Alg. 4), we present two numerical examples in Tab. 2 where the maximum over the simplex can be computed exactly. The functions considered in both these examples are Lipschitz continuous with respect to the $l_1$-norm. It can be seen that both these methods compute the maximum to within the specified precision. For small dimensions, we find that the algorithm based on dense curves requires fewer iterations when the precision $\epsilon$ is small. We remark that Alg. 4 can sometimes be slower to run because one needs to perform sorting and root-finding as opposed to simply searching the points of the grid in Alg. 2. However, when the dimension and the precision are small enough, like in the examples given below, the runtime of Alg. 4 is comparable to or faster than Alg. 2. As the dimension $d$ increases, both these algorithms quickly become infeasible to implement in practice.

| function  | algorithm    | true maximum | computed maximum | iterations |
|-----------|--------------|--------------|------------------|------------|
| $f(x) = \sin(||x||_2)$ | grid search | 0.841        | 0.841            | 16290      |
|           | dense curve  | 0.841        | 0.836            | 155        |
| $f(x) = -\frac{||x||_2^2}{6} + \frac{||x||_2^2}{4} - \frac{||x||_2}{6\pi}$ | grid search | 0.033        | 0.033            | 19900      |
|           | dense curve  | 0.033        | 0.033            | 480        |

Table 2: Computing the maximum of Lipschitz-continuous functions $f$ over the standard simplex $\Delta_d \subseteq \mathbb{R}^d$ in dimension $d = 3$ to a precision of $\epsilon = 0.15$. Grid search (Alg. 2) and the algorithm based on dense curves (Alg. 4) are used to compute the maximum numerically. Numerical values are rounded to three decimal places.

Details on how one can use $\alpha$-dense curves to optimize a Lipschitz-like function over any compact and convex domain is given in App. C.4.

5.2 Computing the sum capacity

We now return to the problem of computing the sum capacity of a two-sender MAC $\mathcal{N}$. Recall that $\mathcal{B}_1$ and $\mathcal{B}_2$ denote input alphabets of $\mathcal{N}$ with size $d_1$ and $d_2$, respectively. $\mathcal{Z}$ denotes the output alphabet of the MAC, with size $d_o$. We focus on the case $d_1, d_2, d_o \geq 2$, and assume that $d_2 \leq d_1$ without loss of generality. Let $I(p,q)$ denote the mutual information between the inputs and the output of the MAC $\mathcal{N}$, given input probability distributions $p$ over $\mathcal{B}_1$ and $q$ over $\mathcal{B}_2$, as defined in Eq. (52). Using Eq. (53) and Eq. (54), we can write the sum capacity of $\mathcal{N}$ as

$$S(\mathcal{N}) = \max_{q \in \Delta_{d_2}} I^*(q) \quad (65)$$

where $I^*(q) = \max_{p \in \Delta_{d_1}} I(p,q)$.

Since $I^*$ is a $\beta_I$-Lipschitz-like function as noted in Eq. (55), we can perform the optimization (65) over the standard simplex using the algorithms described in the previous section. In particular, one can use either grid search detailed in Alg. 2 or dense curves detailed in Alg. 4 to compute the sum capacity of the MAC $\mathcal{N}$. Recall that the number of iterations needed by grid search to compute the maximum of a Lipschitz-like function over the standard simplex to a fixed precision scales polynomially with the dimension (see Prop. 14). Therefore, we use this approach to get an upper bound on the complexity of computing the sum capacity to a fixed precision. In Prop. 21, we show that the total cost of computing the sum capacity to a fixed precision $0 < \epsilon \leq 1$ is roughly bounded above by
poly(d_1, d_2, 1/\epsilon)O(d_2^{6\log^2(d_2)/\epsilon^2+2}) \) (see Eq. (112) for a more precise statement). Therefore, the sum capacity can be computed to a fixed precision \( \epsilon > 0 \) in quasi-polynomial time. On the other hand, Alg. 4 can be much costlier for large dimensions as noted in Sec. 5.1.3. However, for small dimensions and small \( \epsilon \), the method using dense curves can converge in fewer number of iterations compared to grid search. In any case, the algorithms based on grid search and dense curves are not practically implementable even for a fairly small dimension \( d_2 \).

Nevertheless, if we restrict our attention to the family of two-sender MACs where one of the input alphabets is of size 2, then a theoretically and practically efficient algorithm is available for computing the sum capacity. Note that all binary MACs are members of this family. Since \( d_2 \leq d_1 \) by assumption, we take \( d_2 = 2 \). Since any \( q \in \Delta_2 \) can be expressed as \( q = (s, 1-s) \) for some \( 0 \leq s \leq 1 \), the maximization over \( q \in \Delta_2 \) is essentially one-dimensional. Thus, the algorithm based on dense curves can be essentially replaced by modified Piyavskii-Shubert algorithm as noted in Sec. 5.1.1. This is explained below in more detail.

Denoting \( q_s = (s, 1-s) \), we have \( \|q_s - q_{s'}\|_1 = 2|s - s'| \). Then, considering the objective \( I^*(s) := I^*(q_s) \) as a function of \( s \), we can write the sum capacity for any MAC \( \mathcal{N} \) in this family as

\[
S(\mathcal{N}) = \max_{s \in [0,1]} I^*(s).
\]

We can perform this maximization using modified Piyavskii-Shubert algorithm described in Alg. 1. Here, we take \( \beta(x) = \beta_I(2x) \), where the factor of 2 stems from the conversion of the \( l_1 \)-norm for \( q_s \) to an absolute value for \( s \). For the convenience of the reader, we rewrite Alg. 1 specifically for the purpose of computing the sum capacity when \( d_2 = 2 \) in Alg. 5 below. The channel probability matrix \( \mathcal{N} \) is used to compute the objective function \( I^*(s) \). This computation can be done using standard convex optimization techniques. We remark that similar to Alg. 1, one can modify Alg. 5 so that it accepts a fixed number of iterations and outputs the value \( F(s^*) \), which is an upper bound on the sum capacity. This upper bound exceeds the sum capacity by at most \( F(s^*) - I^*(s^*) \).

**Algorithm 5** Computing the sum capacity of a two-sender MAC \( \mathcal{N} \) to a precision \( \epsilon > 0 \), when one of the input alphabets has size 2.

1. **function** COMPUTE_SUM_CAPACITY(\( \mathcal{N} \), \( d_1 \), \( \epsilon \))
2. | **Initialize** \( s^{(0)} = 0 \)
3. | **Define** \( F_0(s) = I^*(s^{(0)}) + \beta_I(2|s - s^{(0)}|) \) for \( s \in [0,1] \)
4. | **Set** \( F \leftarrow F_0 \) and \( s^* \leftarrow 1 \)
5. | **Set** \( s^{(1)} \leftarrow s^* \), \( k \leftarrow 1 \)
6. **while** \( F(s^*) - I^*(s^*) > \epsilon \)
7. | **Sort** \( \{s^{(0)}, \ldots, s^{(k)}\} \) from smallest to largest and relabel the points in ascending order.
8. | **Set** \( F_i(s) = I^*(s^{(i)}) + \beta_I(2|s - s^{(i)}|) \) for \( 0 \leq i \leq k \)
9. **for** \( i = 0, \ldots, k-1 \)
10. | **Set** \( g_i(s) = F_i(s) - F_{i+1}(s) \)
11. | **Find** \( \bar{s}_i \in [s^{(i)}, s^{(i+1)}] \) such that \( g_i(\bar{s}_i) = 0 \) using any root finding method.
12. **end for**
13. **Pick** an index \( m \in \arg\max_{0 \leq i \leq k-1} F_i(\bar{s}_i) \) and set \( s^* = \bar{s}_m \).
14. **Update** \( F \leftarrow F_m \)
15. **Set** \( s^{(k+1)} \leftarrow s^* \), \( k \leftarrow k + 1 \)
16. **end while**
17. **return** \( I^*(s^*) \)
18. **end function**
In Prop. 20, we show that the number of iterations required by Alg. 5 to converge to the sum capacity within a tolerance of $0 < \epsilon \leq 3$ is bounded above as $O(\log(d_0)/\epsilon)$, where $d_0 \geq 2$ is the size of the output alphabet and $\epsilon$ is chosen independent of $d$. Note that this bound does not account for the number of iterations required to compute $I^*$, for sorting or for root-finding. We show in Prop. 20 that the total cost involved is at most polynomial in $d_1$, $d_0$, and $1/\epsilon$.

To conclude this section, we compare the performance of modified Piyavskii-Shubert algorithm described in Alg. 5 to that of grid search for $d_2 = 2$. When the dimensions $d_1$, $d_0$ are small and the precision $\epsilon$ is reasonably large, grid search can sometimes be a better option compared to Alg. 5. This is because the objective function $I^*$ for sum capacity computation can be computed quickly in practice (since $d_1$, $d_0$ are small), the total number of iterations required by grid search is not too large (since $\epsilon$ is large), and we avoid the cost of sorting and root-finding required by the modified Piyavskii-Shubert algorithm. On the other hand, when the dimensions $d_1$, $d_0$ are large and the precision is small, the modified Piyavskii-Shubert algorithm is expected to work better. The reason is that the number of iterations for convergence is smaller compared to grid search when $\epsilon$ is small. Together with the fact that computing $I^*$ gets costlier as the dimensions $d_1$ and $d_0$ increase, this results in an advantage for the modified Piyavskii-Shubert algorithm over grid search. In practice, we find that the modified Piyavskii-Shubert algorithm usually converges in comparable time or faster even for small dimensions $d_1,d_0$ (as small as $d_1 = d_0 = 2$). We demonstrate this through a numerical example. We randomly generate a MAC with $d_1 = 10$, $d_2 = 2$, and $d_0 = 20$. Then to compute the sum capacity of this MAC to a precision of $\epsilon = 0.15$, grid search takes just over a minute on a personal computer (average time of 3 repetitions). On the other hand, the same problem is solved by modified Piyavskii-Shubert algorithm in a few seconds (average time of 3 repetitions). Thus, for this particular example, modified Piyavskii-Shubert algorithm is faster than grid search roughly by a factor of 30.

### 5.3 Comparison with relaxed sum capacity

In order to compare Alg. 5 with the relaxed sum capacity $C$, we consider two examples. These examples are constructed from a family of MACs that we call the noise-free subspace MAC. To that end, let $\mathcal{A}$ and $\mathcal{B}$ be input alphabets and let $\mathcal{Z}$ by the output alphabet. We assume that $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{Z}$ are finite sets. The noise-free subspace corresponds to a set $\mathcal{W} \subseteq \mathcal{A} \times \mathcal{B}$. Consider the mapping $n_F: \mathcal{W} \rightarrow \mathcal{Z}$ that determines the symbol that is deterministically output by the channel when the input is in the noise-free subspace. Then, given the tuple $(\mathcal{A}, \mathcal{B}, \mathcal{Z}, \mathcal{W}, n_F)$, the MAC $\mathcal{N}_F$ has the probability transition matrix

$$\mathcal{N}_F(z|a,b) = \frac{\delta_{z,n_F(a,b)}}{|\mathcal{Z}|} \begin{cases} (a,b) \in \mathcal{W} \\ (a,b) \notin \mathcal{W} \end{cases}$$

When the input is in the noise-free subspace, the channel deterministically outputs the symbol selected by $n_F$. On the other hand, when the input is not in the noise-free subspace, the channel outputs a symbol uniformly at random. This channel can be understood as a generalization of the MAC obtained from a nonlocal game.

We construct two examples using the noise-free subspace MAC. For both of these examples, we consider the alphabets $\mathcal{A} = \{a_1, a_2\}$, $\mathcal{B} = \{b_1, b_2\}$, and $\mathcal{Z} = \{z_1, z_2\}$. Therefore, $\mathcal{N}_F$ is a binary MAC in either example.

For the first example, we take $\mathcal{W} = \{(a_1, b_1)\}$ and $n_F(a_1, b_1) = z_1$. That is, only a single input $(a_1,b_1)$ is transmitted noise-free. For this example, we can write the probability transition matrix as

$$\mathcal{N}_F^{(1)} = \begin{pmatrix} 1 & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{pmatrix}$$
where the rows correspond to \( z \in \mathcal{Z} \), while the columns correspond to \((a, b) \in \mathcal{A} \times \mathcal{B}\). In App. E, we show that the sum capacity in nats is

\[
S(N_F^{(1)}) = h \left( \frac{4}{5} \right) - \frac{2}{5} \ln(2) \approx 0.223
\]

where \( h \) is the binary entropy measured in nats. We furthermore show that the relaxed sum capacity in this case is also equal to \( C(N_F^{(1)}) = 0.223 \) nats. Thus, \( C \) gives a tight bound on the sum capacity for this example. Now, we compute the sum capacity numerically using Alg. 5. We find that for a precision of \( \epsilon = 0.1 \), the algorithm outputs a value of \( \approx 0.22 \) nats. As we increase the precision to \( \epsilon = 0.01 \), we find that the algorithm outputs a sum capacity of \( \approx 0.223 \) nats. In either case, we find that the Alg. 5 computes the sum capacity to the desired precision.

For the second example, we take \( W = \{(a_1, b_1), (a_2, b_2)\} \) as well as \( n_F(a_1, b_1) = z_1 \) and \( n_F(a_2, b_2) = z_2 \). That is, the inputs \((a_1, b_1)\) and \((a_2, b_2)\) are transmitted noise-free. The probability transition matrix in this case can be written as

\[
N_F^{(2)} = \begin{pmatrix}
1 & 0.5 & 0.5 & 0 \\
0 & 0.5 & 0.5 & 1
\end{pmatrix}
\]

when the rows correspond to \( z \in \mathcal{Z} \), while the columns correspond to \((a, b) \in \mathcal{A} \times \mathcal{B}\). In App. E, we show that the sum capacity of the MAC \( N_F^{(2)} \) in nats is equal to

\[
S(N_F^{(2)}) = 0.5 \ln(2) \approx 0.3466.
\]

On the other hand, we show that in this example the relaxed sum capacity takes the maximum possible value \( C(N_F^{(2)}) = \ln(2) \) nats, thus significantly overestimating the sum capacity.

On the other hand, we find that Alg. 5 gives a value of \( \approx 0.3459 \) nats for the sum capacity corresponding to \( \epsilon = 0.1 \), and a value of \( \approx 0.3466 \) nats corresponding to \( \epsilon = 0.01 \). As before, we find that Alg. 5 computes the sum capacity to the desired precision, and performs much better than the relaxed sum capacity for the second example.

6 Conclusion and future directions of research

Computing the sum capacity of a multiple access channel is a nonconvex optimization problem. For MACs obtained from nonlocal games, we obtained an analytical upper bound on the sum capacity that depends only on the number of question tuples in the game and the maximum winning probability of the game when the questions are drawn uniformly at random. Our formula is an upper bound on the achievable sum rate even when the senders of the MAC can share an arbitrary set of correlations. Using this formula, we found a separation between the sum capacity and the entanglement-assisted sum rate for the 2-sender MAC obtained from the Magic Square game that is larger than the previously reported value. We also obtained separations in some other relevant scenarios using the CHSH game and multiparty parity game.

Furthermore, we studied the performance of the upper bound on the sum capacity obtained by relaxing the nonconvex problem to a convex optimization problem. By constructing the signalling game, we showed that one can obtain an arbitrarily large separation between the sum capacity and the relaxed sum capacity. With the help of numerical simulations, we argued that this separation holds even when the senders are allowed to share no-signalling correlations. These results indicate that the relaxed sum capacity can be a very poor upper bound on the sum capacity. In a recent work, Fawzi & Fermé [35] compute the no-signalling assisted sum rate for MACs, allowing feedback. In their study,
they pose the question as to whether the no-signalling assisted sum rate (allowing feedback) is equal to the relaxed sum capacity. It would be interesting to see if the MAC obtained from the signalling game can be used to answer this question in the negative.

In response to the above observations, we studied algorithms to compute the sum capacity. First, we identified that the mutual information occurring in the computation of the sum capacity satisfies a Lipschitz-like property. We subsequently proposed a few algorithms one can use to optimize such functions, by appropriately modifying and generalizing existing algorithms for optimizing Lipschitz-continuous functions. Using this, we were able to show that the sum capacity of any two-sender MAC can be computed to a fixed precision in quasi-polynomial time. Our algorithms are practically efficient for computing the sum capacity of a family of two-sender MACs that have at least one input alphabet of size two.

We remark that some other entropic quantities also satisfy a Lipschitz-like property. This is also true in the quantum setting, for example for the von Neumann entropy. Therefore, further investigation of algorithms to optimize such Lipschitz-like functions might be helpful in solving nonconvex problems in both classical and quantum information theory. In particular, it would be interesting to see if there are other practically relevant problems in information theory which would benefit from such an approach. In any case, the algorithms we present in this study for optimizing Lipschitz-like functions over the standard simplex suffer from the drawback that they are not scalable, i.e., the optimization is very costly to perform in practice as the dimension increases. Therefore, finding more practical algorithms to perform this optimization is an interesting direction for future research. In particular, the algorithm for optimization using dense curves has a scope for improvement because we only prove sub-optimal convergence guarantees.

Another avenue for performing such nonconvex optimization is to devise randomized algorithms, which might allow for faster convergence (with high probability). We remark that using randomized algorithms to optimize Lipschitz-like functions over an arbitrary compact and convex domain is not expected to give significant improvements over the deterministic algorithm we presented in this study. This is because the number of iterations needed for convergence will scale exponentially with the dimension in both the deterministic and stochastic setting [36]. It is therefore important to use information about the domain (for example, standard simplex) or impose some additional restrictions on the objective functions while designing such algorithms.

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8 Code availability

An open source implementation of the codes used in this study can be found at https://github.com/akshayseshadri/sum-capacity-computation.

This repository contains code to perform the following tasks: 1) Maximize Lipschitz-like functions
over the interval (Alg. 1) and standard simplex using grid search (Alg. 2) and dense curves (Alg. 4),
2) Compute the sum capacity of two-sender MACs by implementing the aforementioned algorithms, and
3) Find the maximum winning probability of an $N$-player non-local game using no-signalling strategies
and a corresponding optimal NS strategy.

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A Bounding the correlation-assisted achievable sum rate of MACs from nonlocal games

Proof of Proposition 2. We solve the minimization problem \( \min_{\pi \in \Delta_d} -J(\pi) \), which is equivalent to the given maximization problem. Note that since the entries of \( w \) are either 0 or 1, we can write \( K = \{ i \in [d] \mid w_i \neq 0 \} \). Then we can write the Lagrangian for this minimization problem as

\[
\mathcal{L}(\pi; \lambda, \nu) = -J_w(\pi) - \langle \lambda, \pi \rangle + \nu \left( \sum_{i=1}^{d} \pi_i - 1 \right)
\]

where

\[
p_i = (W \pi)_i = \pi_i w_i + \frac{1}{d} - \frac{1}{d} \sum_{j=1}^{d} \pi_j w_j.
\]

The expression for the probability distribution \( p \), which is the output of the channel \( N_{G} \), is obtained from Eqs. (26) and (27). The variables \( \lambda_1, \ldots, \lambda_d \) are the dual variables corresponding to the inequality constraint \( \pi_i \geq 0 \) for all \( i \in [d] \), and \( \nu \) is the dual variable corresponding to the equality constraint \( \sum_{i=1}^{d} \pi_i = 1 \). We also write \( K = |K| \).

Case 1: 0 < K < d
We can write

\[
p_i = \begin{cases} 
\pi_i + \frac{1}{d} - \frac{1}{d} \sum_{j \in K} \pi_j & \text{for } i \in K, \\
\frac{1}{d} - \frac{1}{d} \sum_{j \in K} \pi_j & \text{for } i \notin K.
\end{cases}
\]

Since \( p_i \geq 0 \) for all \( i \), we must have

\[
\sum_{j \in K} \pi_j \leq 1,
\]
Then, we can consider two sub-cases: either (a) \( \sum_{j \in K} \pi_j = 1 \) or (b) \( \sum_{j \in K} \pi_j < 1 \) at the optimum.

(a) If \( \sum_{j \in K} \pi_j = 1 \) at the optimum, then from Eq. (69), we get
\[
p_i = \begin{cases} \pi_i & \text{for } i \in K \\ 0 & \text{for } i \notin K. \end{cases}
\] (70)

Furthermore, since \( \sum_{j \in K} \pi_j = 1 \), we also have \( \pi_i = 0 \) for \( i \notin K \). It follows from Eq. (29) and Eq. (70) that \( \mathcal{I}_w(\pi) = H(p) \), where \( p \) is as given in Eq. (70). Thus, \( p \) can be taken as a probability distribution on the indices \( K \), and subsequently, we obtain
\[
\max_{\pi} \mathcal{I}_w(\pi) = \ln K.
\] (71)

(b) If \( \sum_{j \in K} \pi_j < 1 \) at the optimum, we can infer from Eq. (69) that \( p_i > 0 \) for all \( i \in [d] \) at the optimum. Thus, the entropy \( H(p) \) is differentiable at the optimum, and consequently, we can differentiate the Lagrangian given in Eq. (67). The gradient of the Lagrangian is given as
\[
\frac{\partial L}{\partial \pi_j} = \begin{cases} \ln p_j - \frac{1}{d} \sum_{i=1}^{d} \ln p_i - \ln d - \lambda_j + \nu & \text{for } j \in K \\ -\lambda_j + \nu & \text{for } j \notin K. \end{cases}
\]

Note that we are solving a convex optimization problem and Slater’s condition holds because the constraint set is a simplex [37]. Therefore, the KKT conditions are necessary and sufficient for optimality (see Ch. 5.5 in Ref. [37]). Subsequently, the optimal distribution \( \pi \) satisfies [37]

(Primal feasibility) \( \pi \in \Delta_d \),

(Dual feasibility) \( \lambda_i \geq 0 \) for all \( i \in [d] \),

(Complementary slackness) \( \lambda_i \pi_i = 0 \) for all \( i \in [d] \), and

(Stationarity) \( \nabla_\pi L = 0 \).

For \( j \notin K \), the condition \( \nabla_\pi L = 0 \) gives
\[
\lambda_j = \nu. \tag{72}
\]

Since \( \sum_{j \in K} \pi_j < 1 \) by assumption and \( \pi \in \Delta_d \), we must have \( \pi_{j'} > 0 \) for some \( j' \notin K \). Then, by complementary slackness, we obtain \( \lambda_{j'} = 0 \). Using this in Eq. (72), we get \( \nu = 0 \).

On the other hand, for \( j \in K \) the condition \( \nabla_\pi L = 0 \) gives
\[
\ln p_j - \frac{1}{d} \sum_{i=1}^{d} \ln p_i = \lambda_j + \ln d,
\]

where we used the fact that \( \nu = 0 \). To simplify this equation further, we make the observation that
\[
p_i = \begin{cases} \pi_i + \frac{p_L}{d} & \text{for } i \in K \\ \frac{p_L}{d} & \text{for } i \notin K. \end{cases}
\]

with \( p_L > 0 \), which follows from Eq. (25) and Eq. (69). Therefore,
\[
\sum_{i=1}^{d} \ln p_i = \sum_{i \in K} \ln p_i + (d - K) \ln(p_L/d),
\]

39
where $d-K = |[d] \setminus K|$. Thus, for $j \in K$, we obtain

$$\ln p_j - \frac{1}{d} \sum_{i \in K} \ln p_i = \lambda_j + \ln d + \left(1 - \frac{K}{d}\right) \ln \frac{p_L}{d}.$$ 

If we label the indices in $K$ as $j_1, \ldots, j_K$, we can write the above as the following matrix equation:

$$\begin{pmatrix}
1 - \frac{1}{d} & -\frac{1}{d} & \cdots & -\frac{1}{d} \\
\vdots & \ddots & \ddots & \vdots \\
-\frac{1}{d} & -\frac{1}{d} & \cdots & 1 - \frac{1}{d}
\end{pmatrix}
\begin{pmatrix}
\ln p_{j_1} \\
\vdots \\
\ln p_{j_K}
\end{pmatrix}
= \begin{pmatrix}
\lambda_{j_1} + \ln d + \left(1 - \frac{K}{d}\right) \ln \frac{p_L}{d} \\
\vdots \\
\lambda_{j_K} + \ln d + \left(1 - \frac{K}{d}\right) \ln \frac{p_L}{d}
\end{pmatrix},$$

(73)

Let $o = \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix}^T$ denote the vector of ones. Then, the matrix appearing in the LHS of (73) can be expressed as $I - oo^T/d$. By the matrix determinant lemma (see Cor. (18.1.3) in Ref. [38]), we know that this matrix is invertible iff $1 - o^T o/d = 1 - K/d \neq 0$, which is always true by our assumption that $K < d$. Its inverse is given by the Sherman-Morrison formula [39, 40] as follows:

$$\left(I - \frac{1}{d} oo^T\right)^{-1} = I + \frac{1}{d - K} oo^T.$$ 

Using this inverse in Eq. (73), for $j \in K$, we obtain

$$p_j = E_j \frac{p_L}{d},$$

where

$$E_j = \exp \left(\lambda_j + \frac{1}{d-K} \sum_{i \in K} \lambda_i + \frac{d}{d-K} \ln d\right).$$

Since $\lambda_i \geq 0$ for all $i \in [d]$ due to dual feasibility, we have $E_j > 1$ for all $j \in K$.

Using $p_j = \pi_j + p_L/d$ for $j \in K$, we obtain

$$\pi_j = (E_j - 1) \frac{p_L}{d}.$$

(74)

Because $E_j > 1$, we have $\pi_j > 0$ for each $j \in K$. Then, by complementary slackness, we have $\lambda_j = 0$ for all $j \in K$. Therefore, we obtain

$$E_j = d^{\frac{d-K}{d}} \forall j \in K.$$ 

(75)

Now, we will solve for $\pi_j$ for $j \in K$. For this purpose, note that

$$e_j^{-1} = \frac{d}{E_j - 1}$$

is a positive number for all $j \in K$. Then, using $p_L = 1 - \sum_{i \in K} \pi_i$ and $\pi_j = e_j p_L$ for $j \in K$ (see Eq. (74)), we can write

$$e_j^{-1} \pi_j + \sum_{i \in K} \pi_i = 1,$$

which can be written in matrix form as

$$\begin{pmatrix}
e_j^{-1} + 1 & 1 & \cdots & 1 \\
\vdots & \ddots & \ddots & \vdots \\
1 & 1 & \cdots & e_j^{-1} + 1
\end{pmatrix}
\begin{pmatrix}
\pi_{j_1} \\
\vdots \\
\pi_{j_K}
\end{pmatrix}
= \begin{pmatrix} 1 \\
\vdots \\
1\end{pmatrix}.$$
The matrix appearing in the LHS of the above equation can be written as \( \text{diag}(e_j^{-1}) + oo^T \), which is invertible iff \( 1 + \sum_{j \in K} e_j \neq 0 \). This condition always holds because \( e_j > 0 \) for each \( j \in K \). Denoting \( A = \text{diag}(e_j) \), we can use the Sherman-Morrison formula [39, 40] to write

\[
\begin{pmatrix}
\pi_{j_1} \\
\vdots \\
\pi_{j_K}
\end{pmatrix} = A o - o^T A o = \frac{e_{j_1}}{1 + \sum_{j \in K} e_j} \quad \frac{e_{j_K}}{1 + \sum_{j \in K} e_j}.
\]

(76)

Then, using \( e_j = (E_j - 1)/d \) for \( j \in K \) along with Eq. (75) and Eq. (76), we obtain

\[
\pi_j = \frac{d \pi - 1}{d + K (d \pi - 1)} \quad \forall j \in K
\]

and

\[
p_L = \frac{d}{d + K (d \pi - 1)}.
\]

Subsequently, we obtain

\[
\mathcal{I}_K^* \equiv \max_{\pi \in \Delta_d} \mathcal{I}_w(\pi) = - \frac{K d \pi - 1}{d + K (d \pi - 1)} \ln \left( \frac{d \pi - 1}{d + K (d \pi - 1)} \right) - \frac{d}{d + K (d \pi - 1)} \ln d.
\]

(77)

While the expression for \( \mathcal{I}_K^* \) looks complicated, it can be greatly simplified. Denoting

\[
\delta_K = \frac{d - K}{d \pi - 1},
\]

one can rearrange terms in \( \mathcal{I}_K^* \) to obtain

\[
\mathcal{I}_K^* = \ln (K + \delta_K).
\]

(78)

From Eq. (71) and Eq. (77) and the fact that \( \delta_K > 0 \) for \( 0 < K < d \), we can infer that

\[
\max_{\pi \in \Delta_d} \mathcal{I}_w(\pi) = \mathcal{I}_K^*.
\]

Case 2: \( K = 0 \) or \( K = d \)

\( K = 0 \) implies \( w_i = 0 \) for all \( i \in [d] \). Thus, \( \langle \pi, w \rangle = 0 \) for all \( \pi \in \Delta_d \), i.e., we will always lose the game. In that case, it can be verified that we must have \( \mathcal{I}_w(\pi) = 0 \) for all \( \pi \in \Delta_d \).

On the other hand, \( K = d \) implies \( w_i = 1 \) for all \( i \in [d] \), meaning that the strategy is perfect. In particular, we have \( \langle \pi, w \rangle = 1 \) for all \( \pi \in \Delta_d \), i.e., \( p_L = 0 \) for any distribution on questions. It can be verified that \( \mathcal{I}_w(\pi_U) = \ln d \) for the uniform distribution \( \pi_U \in \Delta_d \).

Proof of Proposition 3: 1) Let \( w^{(D)} \in \{0, 1\}^d \) denote a binary deterministic strategy that can answer \( d - 1 \) of the \( d \) questions correctly. Then, there is exactly one index \( k \in [d] \) such that \( w_k^{(D)} = 0 \). We show that we can achieve \( \max_{\pi} \mathcal{I}_w(\pi) \) using the deterministic strategy \( \mathcal{I}_{w^{(D)}}(\pi) \) for some appropriate distribution \( \tilde{\pi} \in \Delta_d \). Note that the winning vector \( w^{(D)} \) may not be permitted by the game, but it still gives an upper bound on \( \max_{\pi} \mathcal{I}_w(\pi) \).
Since by assumption $K^* < d$, we can find an injective function $\sigma : K^* \to [d] \setminus \{k\}$. The map $\sigma$ will be used to label the indices. We construct the distribution $\tilde{\pi} \in \Delta_d$ as follows:

$$
\tilde{\pi}_i = w_{\sigma^{-1}(i)} \pi_{\sigma^{-1}(i)}^*, \quad \text{for } i \in \sigma(K^*) \\
\tilde{\pi}_i = 0, \quad \text{for } i \notin \sigma(K^*) \setminus \{k\} \\
\tilde{\pi}_k = 1 - \sum_{i \in \sigma(K^*)} \tilde{\pi}_i. \tag{79}
$$

Since $\sigma$ is an injective function on $K^*$, it is invertible when its co-domain is restricted to its range $\sigma(K^*)$. Thus, $\sigma^{-1}(i)$ is well-defined for $i \in \sigma(K^*)$.

Since $w_j^{(D)} = 1$ for $j \in [d] \setminus \{k\}$ and $w_k^{(D)} = 0$, we have

$$
\langle w^{(D)}, \tilde{\pi} \rangle = \sum_{i \in \sigma(K^*)} w_{\sigma^{-1}(i)} \pi_{\sigma^{-1}(i)}^* = \sum_{j \in K^*} w_j \pi_j^* = \langle w, \pi^* \rangle. \tag{80}
$$

That is, the winning probabilities obtained from $w, \pi^*$ and $w^{(D)}, \tilde{\pi}$ are the same.

Next, we will look at the output probability of the channel in the two cases. For $w, \pi^*$, the output probability is given as (see Eq. (27))

$$
\hat{p}_i^* = \begin{cases} 
  w_i \pi_i^* + \frac{1}{d} - \frac{1}{d} \sum_{j \in K^*} w_j \pi_j^* & \text{for } i \in K^* \\
  \frac{1}{d} - \frac{1}{d} \sum_{j \in K^*} w_j \pi_j^* & \text{for } i \notin K^*,
\end{cases}
$$

whereas for $w^{(D)}, \tilde{\pi}$, the output probability is given as

$$
\hat{\tilde{p}}_i = \begin{cases} 
  w_{\sigma^{-1}(i)} \pi_{\sigma^{-1}(i)}^* + \frac{1}{d} - \frac{1}{d} \sum_{j \in K^*} w_j \pi_j^* & \text{for } i \in \sigma(K^*) \\
  \frac{1}{d} - \frac{1}{d} \sum_{j \in K^*} w_j \pi_j^* & \text{for } i \notin \sigma(K^*).
\end{cases}
$$

Here we used Eq. (80) to obtain the expression for $\tilde{\pi}$. Since $|K^*| = |\sigma(K^*)|$, the probability $\tilde{\pi}$ is just a permutation of $p^*$. Furthermore, because Shannon entropy is invariant under permutation of the entries of the probability distribution, we have $H(\tilde{\pi}) = H(p^*)$. Then from Eq. (27), Eq. (29), and Eq. (80), we have

$$
\max_{\pi \in \Delta_d} I_w(\pi) = I_w(p^*) = I_{w^{(D)}}(\tilde{\pi}) \leq \max_{\pi \in \Delta_d} I_{w^{(D)}}(\pi) = I_{d-1}^*
$$

where $I_{d-1}^*$ is given by Eq. (33).

2) We focus on the case where $K^* = K = d$. This implies $w_i \neq 0$ and $\pi_i^* \neq 0$ for all $i \in [d]$. Our goal is to solve the optimization problem $\max_{\pi \in \Delta_d} I_w(\pi)$. Following Eq. (67), we write the Lagrangian for the problem

$$
\mathcal{L}(\pi; \lambda, \nu) = \sum_{i=1}^{d} p_i \ln p_i - \sum_{i=1}^{d} \pi_j w_j \ln d + \ln d - \sum_{i=1}^{d} \lambda_i \pi_i + \nu \left( \sum_{i=1}^{d} \pi_i - 1 \right), \tag{81}
$$

where

$$
p_i = \pi_i w_i + \frac{1}{d} - \frac{1}{d} \sum_{j=1}^{d} \pi_j w_j. \tag{82}
$$

Since $K^* = d$, the probabilities $p_i$ are non-zero at the optimum, and hence we can differentiate the entropy at the optimum.
Note that we are solving a convex optimization problem and Slater’s condition holds [37]. Therefore, KKT conditions are necessary and sufficient for optimality [37]. From KKT conditions, it follows that
\[ \nabla_{\pi} L = 0, \]
which gives
\[ \ln p_j - \frac{1}{d} \sum_{i=1}^{d} \ln p_i = \frac{\lambda_j}{w_j} - \frac{\nu}{w_j} + \ln d, \quad j \in [d]. \]

Since by assumption \( \pi_i \neq 0 \) at the optimum for all \( i \in [d] \), we have \( \lambda_i = 0 \) by complementary slackness. This gives
\[ p_j = d \left( \prod_{i=1}^{d} p_i \right)^{1/d} \exp \left( -\frac{\nu}{w_j} \right) \]
for \( j \in [d] \). By multiplying Eq. (83) for \( j = 1, \ldots, d \), we can infer that
\[ \nu = dw_{\text{eff}} \ln d \]
since \( p_i \neq 0 \) at the optimum. Next, summing over \( j \in [d] \) in Eq. (83), we obtain
\[ d \left( \prod_{i=1}^{d} p_i \right)^{1/d} = \frac{1}{\sum_{j=1}^{d} \exp \left( -\frac{\nu}{w_j} \right)}. \] Combining this with Eq. (83) and Eq. (84), we obtain
\[ p_i = \frac{\exp \left( -\frac{dw_{\text{eff}} \ln d}{w_i} \right)}{\sum_{j=1}^{d} \exp \left( -\frac{dw_{\text{eff}} \ln d}{w_j} \right)}. \]

Therefore, at the optimum, we have a Boltzmann distribution for the probability of outcomes of the channel.

Now, instead of solving for \( \pi \) from Eq. (86), we find a suitable expression for the objective function \( J_w(\pi) \) in terms of \( p \) and \( w \) (instead of \( \pi \) and \( w \)). To this end, subtracting Eq. (82) corresponding to two different indices \( i \) and \( j \) and summing over the index \( i \), we obtain
\[ \sum_{i=1}^{d} \pi_i w_i = 1 + d(\pi_j w_j - p_j). \]
Dividing this equation by \( w_j \) and summing over \( j \), we get
\[ \sum_{j=1}^{d} \frac{1}{w_j} \sum_{i=1}^{d} \pi_i w_i = \sum_{j=1}^{d} \frac{1}{w_j} + d \left( 1 - \sum_{j=1}^{d} \frac{p_j}{w_j} \right) \]
\[ \sum_{i=1}^{d} \pi_i w_i = 1 - dw_{\text{eff}} \left( \sum_{j=1}^{d} \frac{p_j}{w_j} - 1 \right), \]
where \( w_{\text{eff}} \) is as defined in Eq. (35). Substituting this in Eq. (29), we find that the mutual information \( J_w(\pi) \) can be written as a function of the outcome probability \( p \) as follows:
\[ J_w(\pi) = H(p) - dw_{\text{eff}} \left( \sum_{j=1}^{d} \frac{p_j}{w_j} - 1 \right) \ln d. \]
To obtain the value of $\mathcal{I}_w(\pi)$ at the optimum, we substitute Eq. (86) in Eq. (87). Denoting $\max_{\pi \in \Delta_d} \mathcal{I}_w(\pi) = \mathcal{I}^*(w)$, one can rearrange terms to obtain

$$\mathcal{I}^*(w) = dw_{\text{eff}} \ln d + \ln \left( \sum_{j=1}^{d} \exp \left( - \frac{dw_{\text{eff}} \ln d}{w_j} \right) \right)$$

(88)

$$= \ln \left( \sum_{j=1}^{d} \exp \left( dw_{\text{eff}} \ln d \left( 1 - \frac{1}{w_j} \right) \right) \right),$$

(89)

where we get the last equation by noting that $dw_{\text{eff}} \ln d = \ln \exp(dw_{\text{eff}} \ln d)$.

Proof of Proposition 4. To obtain an upper bound on

$$\sup_{w \in \mathcal{W}_C, w > 0} \mathcal{I}^*(w)$$

given in Eq. (38), we solve a relaxation of this optimization problem. For this purpose, note that the set over which we optimize can be written as

$$\mathcal{W}_C^\succ = \left\{ w \in [0, 1]^d \mid w > 0, \frac{\sum_{i=1}^{d} w_i}{d} \leq \omega^c(G) \right\}.$$

Let $\text{HM}(w_1, \ldots, w_d)$ be the harmonic mean and $\text{AM}(w_1, \ldots, w_d)$ be the arithmetic mean of $w_1, \ldots, w_d$, respectively. Observe that

$$dw_{\text{eff}} = \text{HM}(w_1, \ldots, w_d).$$

Then, since $\text{HM}(w_1, \ldots, w_d) \leq \text{AM}(w_1, \ldots, w_d)$ for $w_1, \ldots, w_d > 0$ (with equality iff $w_1 = \cdots = w_d$), we have

$$dw_{\text{eff}} \leq \frac{\sum_{i=1}^{d} w_i}{d} \leq \omega^{\text{SC}}(G).$$

(90)

With this in mind, we define the set

$$\mathcal{W}_{\text{eff}} = \left\{ w \in [0, 1]^d \mid w > 0, \ dw_{\text{eff}} \leq \omega^{\text{SC}}(G) \right\}.$$

Then, from Eq. (90), it follows that $\mathcal{W}_C^\succ \subseteq \mathcal{W}_{\text{eff}}$. Therefore, we have

$$\sup_{w \in \mathcal{W}_C^\succ} \mathcal{I}^*(w) \leq \sup_{w \in \mathcal{W}_{\text{eff}}} \mathcal{I}^*(w).$$

Subsequently, we solve the optimization problem $\sup_{w \in \mathcal{W}_{\text{eff}}} \mathcal{I}^*(w)$. To that end, we make the change of variables

$$t_i = \frac{1}{w_i}$$

for $i \in [d]$. The optimization problem then becomes

$$\sup_{t \geq 1} \ln \left( \sum_{i=1}^{d} \exp \left( \frac{d}{\sum_{j=1}^{d} t_j} \ln d \left( 1 - t_i \right) \right) \right)$$

s.t. $t \geq 1$

$$\frac{\sum_{i=1}^{d} t_i}{d} \geq \frac{1}{\omega^{\text{SC}}(G)}.$$

(91)
We solve Eq. (91) by splitting it into two maximizations. To that end, define

\[ s = \sum_{i=1}^{d} t_i, \tag{92} \]

so that the objective of the optimization in Eq. (91) can be written as

\[ f(t, s) = f(t, s) = \text{LSE} \left( -\frac{\ln d}{s} t_1, \ldots, -\frac{\ln d}{s} t_d \right) + \frac{\ln d}{s}, \tag{93} \]

where \( \text{LSE}(y_1, \ldots, y_d) = \ln \left( \sum_{j=1}^{d} e^{y_j} \right) \) is the log-sum-exp function. Since \( \text{LSE} \) is a convex function, we can conclude that \( f(t, s) \) is convex in \( t \) for a fixed \( s \). Then, Eq. (91) can be expressed as

\[ \sup_{t \in \mathbb{W}_{\text{eff}}^{(s)}} f(t) = \sup_{s \geq (\omega S_{\text{eff}}(G))^{-1}} \max_{t \in \mathbb{W}_{\text{eff}}^{(s)}} f(t, s), \]

where

\[ \mathbb{W}_{\text{eff}}^{(s)} = \left\{ t \in \mathbb{R}^d \mid t \geq 1, \sum_{i=1}^{d} t_i = s \right\} \]

is a translated and scaled simplex. The extreme points of \( \mathbb{W}_{\text{eff}}^{(s)} \) are given as

\[ t_s^{(1)} = \left( 1, \ldots, 1, 1 + (s - 1)d \right)^T, \tag{94} \]

and its permutations are denoted by \( t_s^{(2)}, \ldots, t_s^{(d)} \). Note that the constraints in Eq. (91) imply \( s \geq 1 \).

Now, we are seeking to maximize the convex function \( f(t, s) \) over the set \( \mathbb{W}_{\text{eff}}^{(s)} \) (for a fixed \( s \)). However, since \( f \) is convex and any point \( t \in \mathbb{W}_{\text{eff}}^{(s)} \) can be written as \( t = \sum_{i=1}^{d} \lambda_i t_s^{(i)} \), we have

\[ f(t, s) \leq \sum_{i=1}^{d} \lambda_i f(t_s^{(i)}, s). \]

Then, because \( f(t, s) \) is invariant under the permutation of the components of \( t \), we have \( f(t_s^{(i)}, s) = f(t_s^{(j)}, s) \) for any \( i, j \in [d] \). Therefore,

\[ \max_{t \in \mathbb{W}_{\text{eff}}^{(s)}} f(t, s) = f(t_s^{(1)}, s) = \ln \left( d - 1 + \exp \left[ -d \ln d \left( \frac{s - 1}{s} \right) \right] \right) \]

where we substituted Eq. (94) in Eq. (93) in the last equation. Then, since \( f(t^{(1)}, s) \) is a decreasing function of \( s \), we can infer that

\[ \sup_{s \geq (\omega S_{\text{eff}}(G))^{-1}} \max_{t \in \mathbb{W}_{\text{eff}}^{(s)}} f(t, s) = f \left( t_s^{(1)}, \frac{1}{\omega S_{\text{eff}}(G)} \right) = \ln \left( d - 1 + d^{-(1-\omega S_{\text{eff}}(G))d} \right), \]

giving the desired bound.

### B Some notes on strategies for nonlocal games

We begin by characterizing the extreme points of the set of conditional distributions (over finite sets).

**Proposition 10.** Let \( \mathcal{X} \) and \( \mathcal{Y} \) be finite sets, and let \( \mathcal{C} \) denote the set of conditional probability distributions on \( \mathcal{Y} \) given \( \mathcal{X} \). Then, the extreme points of \( \mathcal{C} \) correspond to conditional probabilities \( p_{Y|X}^{f}(y|x) = \delta_{y,f(x)} \) obtained from functions \( f : \mathcal{X} \to \mathcal{Y} \). In particular, any conditional probability distribution \( p_{Y|X} \in \mathcal{C} \) can be written as a convex combination of \( p_{Y|X}^{f} \) corresponding to functions \( f \).
Proof. For convenience, let us denote $|X| = m$ and $|Y| = n$ for some $m, n \in \mathbb{N}$. We also fix a labelling of elements $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$. The set of conditional probability distributions $\mathcal{C}$ can be thought of as the set of functions from $X \rightarrow \Delta_n$. That is, for each $x \in X$, we have a probability distribution over $Y$. Thus, we identify $\mathcal{C}$ with the set $(\Delta_n)^m$.

Since $\Delta_n$ is a simplex in $\mathbb{R}^n$, its extreme points are $e_1, \ldots, e_n$, where $e_i$ is the standard Euclidean basis vector for the $i$th coordinate. Therefore, the extreme points of $\mathcal{C} = (\Delta_n)^m$ are $(e_{i_1}, \ldots, e_{i_m})$ for $i_1, \ldots, i_m \in [n]$. We argue that such extreme points can be obtained from functions $f: X \rightarrow Y$.

To that end, given an extreme point $(e_{i_1}, \ldots, e_{i_m})$ of $\mathcal{C}$, construct the function

$$f(x_j) = y_{i_j} \quad j \in [m].$$

Observe that the conditional probability distribution determined by the function $f$ is

$$p^f_{Y|X}(\cdot, x_j) = \delta_{j, f(x_j)} = \delta_{j, e_{i_j}}$$

for $j \in [m]$. Since there are a total of $|Y|^{|X|}$ extreme points of $\mathcal{C}$, and just as many functions from $X$ to $Y$, the above construction gives a bijective mapping between the extreme points and these functions. Since $\mathcal{C}$ is a compact and convex set with a finite number of extreme points, it is generated as the convex hull of its extreme points by the Krein-Milman theorem [41].

In the following proposition, we show that the set of no-signalling strategies is a compact and convex set. The proof of this proposition is constructive, and it can therefore be used to construct the set of no-signalling distributions numerically.

**Proposition 11.** Let $X_1, \ldots, X_N$ denote the question set for $N$ players, and let $Y_1, \ldots, Y_N$ denote the answer set. Let $\mathcal{S}_{NS}$ denote the set of no-signalling strategies used by the players. That is, $p_{Y|X} \in \mathcal{S}_{NS}$ iff

$$p_{Y|X}(y_i|x_1, \ldots, x_N) = p_{X}^{Y_i}(y_i|x_i) \forall x_k \in X_k, \ k \in [N] \setminus \{i\}$$

for all $y_i \in Y_i, x_i \in X_i, i \in [N]$. Then, $\mathcal{S}_{NS}$ is a compact and convex set. Specifically, $\mathcal{S}_{NS}$ is a convex polytope obtained as the intersection of hyperplanes and halfspaces.

**Proof.** For convenience, we denote $X = X_1 \times \cdots \times X_N$ as the question set and $Y = Y_1 \times \cdots \times Y_N$ as the answer set. The set of all strategies $p_{Y|X}(y_1, \ldots, y_N|x_1, \ldots, x_N)$ can be written as a product of simplices $(\Delta_{|Y|})^{|X|}$ (see Prop. 10), and is therefore a compact set. Also note that we can write the no-signalling condition given in Eq. (95) as

$$\sum_{y_j \in Y_j \atop j \neq i} p_{Y|X}(y_1, \ldots, y_i, \ldots, y_N|x_1, \ldots, x_i, \ldots, x_N)$$

$$= \sum_{y_j \in Y_j \atop j \neq i} p_{Y|X}(y_1, \ldots, y_i, \ldots, y_N|x_1', \ldots, x_i', \ldots, x_N') \forall x_k, x'_k \in X_k, \ k \in [N] \setminus \{i\}$$

(96)

for all $y_i \in Y_i, x_i \in X_i, i \in [N]$. In the above equation, the sum over $y_j \in Y_j$ is a shorthand for the sum over $y_1 \in Y_1, \ldots, y_N \in Y_N$.

We will show that the set of no-signalling strategies $\mathcal{S}_{NS}$ is a closed set. First, note that $\mathcal{S}_{NS} \subseteq (\Delta_{|Y|})^{|X|}$, i.e., the set of no-signalling strategies is contained in the set of all strategies. Thus, we can write the elements of $\mathcal{S}_{NS}$ as vectors $v = (v^{(1)}, \ldots, v^{(|X|)})$, where $v^{(i)} \in \Delta_{|Y|}$ is a probability vector. Note that $v^{(i)}$ is a $|Y|$-dimensional vector whereas $v$ is a $|X||Y|$-dimensional vector. Essentially, the vector
\( v^{(i)} \) denotes a probability distribution \( p_Y|X(y|x^{(i)}) \) over \( Y \) for a fixed \( x^{(i)} \in X \). Using this vectorial representation of a strategy, we will write the no-signalling condition given in Eq. (96) in matrix form.

To that end, fix an ordering for the elements of \( Y \). Then, we can index the elements of \( v^{(i)} \) as \( v_{(y_1, \ldots, y_N)}^{(i)} \) for \((y_1, \ldots, y_N) \in Y\) corresponding to that ordering. For each \( y_i \in Y_i \) \( (i \in [d]) \), let \( s_{y_i} \) denote the \(|Y|\)-dimensional vector with 1 at each index \((y_1', \ldots, y_N') \in Y\) with \( y_i' = y_i \) and 0 elsewhere. For example, if \( Y_1 = \{a, b\} \) and \( Y_2 = \{c, d\} \) and we write the elements of \( Y = \{a, c, (a, d), (b, c), (b, d)\} \) in that order, then, \( s_a = (1, 1, 0, 0)^T \), \( s_c = (0, 1, 0, 1)^T \), and so forth. Observe also that, for each \( k \in [|X|] \),

\[
S_{y_i}^{(k)} = \sum_{y_j \in Y_j} p_{Y|X}(y_1, \ldots, y_i, \ldots, y_N|x^{(k)}).
\]

Similarly, fix an ordering for \( X \). For each \( x_i \in X_i \) with \( i \in [N] \), let \( I_{x_i} \) be the set that contains the indices \((x'_1, \ldots, x'_N) \in X\) with \( x'_i = x_i \). Then, define the \([|X|] \times [|Y|] \) matrix \( S_{(x_i,y_i)} \) in block form as follows. Imagine each row of \( S_{(x_i,y_i)} \) being split into \(|X|\) blocks of \(|Y|\)-dimensional vectors, i.e., \((b_1 \cdots b_{|Y|})\) with \( b_i \) a \(|Y|\)-dimensional (row) vector. We label the rows of \( S_{(x_i,y_i)} \) with \( I_{x_i} \). Define the row \( k \in I_{x_i} \) to be the block \( (0_{1 \times |Y|} \cdots s_{y_i}^T \cdots 0_{1 \times |Y|}) \) with \( s_{y_i}^T \) in the \( k\)-th block.

Let us label the blocks \( v^{(i)} \) of the vector \( v = (v^{(1)}, \ldots, v^{(|X|)}) \in (\Delta_{|Y|})^{|X|} \) as \( v^{(x_1, \ldots, x_N)} \) for \((x_1, \ldots, x_N) \in X\) using the ordering of \( X \) that we have fixed. Then, we have \((S_{(x_i,y_i)} v)^{(x_1, \ldots, x_N)} = s_{y_i}^T v^{(x_1, \ldots, x_N)} \) for \((x_1, \ldots, x_N) \in I_{x_i} \). That is, \( S_{(x_i,y_i)} v \) is a \([|X_i|] \times |Y|\)-dimensional vector with the entries

\[
(S_{(x_i,y_i)} v)^{(x_1, \ldots, x_N)} = \sum_{y_j \not\in Y_j} p_{Y|X}(y_1, \ldots, y_i, \ldots, y_N|x_1, \ldots, x_i, \ldots, x_N)
\]

for \((x_1, \ldots, x_N) \in I_{x_i} \). Since the \( i\)-th component of \((x_1, \ldots, x_N) \in I_{x_i} \) is fixed to be \( x_i \), the no-signalling condition given in Eq. (96) says that all the components of \( S_{(x_i,y_i)} v \) are equal.

Therefore, we enforce the no-signalling condition as follows. Define a \([|X_i| - 1] \times |I_{x_i}| \) matrix

\[
D_{x_i} = \begin{pmatrix}
1 & -1 & 0 & \cdots & 0 & 0 \\
0 & 1 & -1 & \cdots & 0 & 0 \\
0 & 0 & \ddots & \cdots & 1 & -1
\end{pmatrix}
\]

and observe that if \( r = \begin{pmatrix}
 r_1 & \cdots & r_{|I_{x_i}|}
\end{pmatrix}^T \) is any \(|I_{x_i}|\)-dimensional vector, then \( D_{x_i} r \) is the \((|I_{x_i}| - 1)\)-dimensional vector \( \begin{pmatrix}
 r_1 - r_2 & \cdots & r_{|I_{x_i}| - 1} - r_{|I_{x_i}|}
\end{pmatrix}^T \). Then, by the preceding remarks, the no-signalling condition can be written as

\[
D_{x_i} S_{(x_i,y_i)} v = 0 \quad \text{for all } x_i \in X_i, \ y_i \in Y_i, \ i \in [N]. \tag{97}
\]

Since \( D_{x_i} S_{(x_i,y_i)} \) is a \([|I_{x_i}| - 1] \times |X| \) matrix and \( v \) is a \(|X| \times |Y|\)-dimensional vector, the equation \( D_{x_i} S_{(x_i,y_i)} v = 0 \) encodes \(|X| \) \((|I_{x_i}| - 1)\) hyperplanes (thinking of \( v \) as an arbitrary \(|X| \times |Y|\)-dimensional vector). Therefore, the set of no-signalling strategies can be written as

\[
\mathcal{S}_{NS} = \{ v \in (\Delta_{|Y|})^{|X|} \mid D_{x_i} S_{(x_i,y_i)} v = 0 \ \forall x_i \in X_i, \ y_i \in Y_i, \ i \in [N] \}.
\]

This is the intersection of the compact set \((\Delta_{|Y|})^{|X|}\) with the hyperplanes defined by \( D_{x_i} S_{(x_i,y_i)} v = 0 \). Since hyperplanes are closed, the set \( \mathcal{S}_{NS} \) is closed as well.

Since \((\Delta_{|Y|})^{|X|}\) is bounded, \( \mathcal{S}_{NS} \) is also bounded, so that \( \mathcal{S}_{NS} \) is compact. The convexity of \( \mathcal{S}_{NS} \) follows from the fact that \((\Delta_{|Y|})^{|X|}\) and hyperplanes are convex.

\[
\square
\]
C Analysis of algorithms for optimization of Lipschitz-like function

In this section, we present convergence analysis for algorithms used for Lipschitz-like optimization. We also present an algorithm for performing optimization of Lipschitz-like functions over an arbitrary compact & convex domain.

C.1 Optimizing Lipschitz-like functions over an interval using modified Piyavskii-Shubert algorithm

We first present a convergence analysis for modified Piyavskii-Shubert algorithm that finds the global maximum of Lipschitz-like functions over a closed interval.

Proposition 12. Let \( \beta : \mathbb{R}_+ \to \mathbb{R} \) be a non-negative, continuous, monotonically increasing function with \( \beta(0) = 0 \). Let \( D = [a, b] \) be a closed interval, where \( a, b \in \mathbb{R} \). Let \( f : D \to \mathbb{R} \) be a real-valued function satisfying

\[
|f(q) - f(q')| \leq \beta(|q - q'|)
\]

Then, for each choice of tolerance \( \epsilon > 0 \), Alg. 1 terminates in a finite number of time steps. If \( f(q^*) \) denotes the output of Alg. 1 corresponding to a tolerance of \( \epsilon > 0 \), we have \( \max_{q \in D} f(q) - f(q^*) \leq \epsilon \).

The number of time steps required to converge with a tolerance of \( \epsilon > 0 \) is bounded above by \( \lfloor (b-a)/\delta \rfloor \), where \( \delta = \sup \{ \delta' > 0 \mid \beta(x) < \epsilon/2 \forall 0 \leq x \leq \delta' \} \).

Proof. Let \( i \in \mathbb{N} \) be a natural number. Given any \( q^{(i)} \in [a, b] \), define the function

\[
F_i(q) = f(q^{(i)}) + \beta(|q - q^{(i)}|)
\]

for \( q \in [a, b] \). Since \( f(q^{(i)}) - f(q) \geq -\beta(|q - q^{(i)}|) \) by assumption, we have \( f(q) \leq F_i(q) \) for all \( q \in [a, b] \). Given that we initialize \( q^{(0)} = a \), the maximum of \( F_0(q) \) occurs at \( b \) because \( \beta \) is a monotonically increasing function with \( \beta(0) = 0 \). This justifies the assignment \( q^{(1)} = b \) at the start of the algorithm.

Now, suppose that Alg. 1 terminates at the \( K \)th time step. Then, the algorithm has generated points \( q^{(0)}, \ldots, q^{(K)} \in [a, b] \) and a point \( q^* \in [a, b] \) satisfying \( F(q^*) - f(q^*) \leq \epsilon \). The function \( F \) and the point \( q^* \) are obtained as follows. The points \( q^{(0)}, \ldots, q^{(K)} \) are sorted at the beginning of the \( K \)th iteration, so that \( a = q^{(0)} \leq q^{(1)} \leq \cdots \leq q^{(K)} = b \). Then, points \( \bar{q}^{(i)} \in \arg\max_{q \in [q^{(i)}, q^{(i+1)}]} \min\{F_i(q), F_{i+1}(q)\} \) are obtained through root finding for \( 0 \leq i \leq K - 1 \). We choose \( m \in \arg\max_{0 \leq i \leq K-1} F_i(\bar{q}^{(i)}) \) and set \( q^* = \bar{q}^{(m)} \) and \( F = F_m \).

Next, we elaborate on how \( \bar{q}^{(i)} \in \arg\max_{q \in [q^{(i)}, q^{(i+1)}]} \min\{F_i(q), F_{i+1}(q)\} \) is computed using root finding. First, consider the function \( g_i(q) = F_i(q) - F_{i+1}(q) \). From the non-negativity and monotonicity of \( \beta \), it follows that \( F_i(q) \) is a monotonically increasing function in the interval \( q \in [q^{(i)}, q^{(i+1)}] \), whereas \( F_{i+1}(q) \) is a monotonically decreasing function in the interval \( q \in [q^{(i)}, q^{(i+1)}] \). Therefore, \( g_i(q) \) is a continuous and monotonically increasing function in the interval \( q \in [q^{(i)}, q^{(i+1)}] \), where the continuity of \( g_i \) follows from that of \( F_i \) and \( F_{i+1} \). Since \( \beta(0) = 0 \) and \( f \leq F_i \) for all \( 0 \leq i \leq K \), we have \( g_i(q^{(i)}) \leq 0 \) and \( g_i(q^{(i+1)}) \geq 0 \). Therefore, the function \( g_i \) has a root \( \bar{q}^{(i)} \) in the interval \( [q^{(i)}, q^{(i+1)}] \), and since \( g_i = F_i - F_{i+1} \), we have \( F_i(\bar{q}^{(i)}) = F_{i+1}(\bar{q}^{(i)}) \). From the monotonicity properties of \( F_i \) and \( F_{i+1} \), we can infer that \( \bar{q}^{(i)} \) maximizes the bounding function \( \min\{F_i, F_{i+1}\} \) in the interval \( [q^{(i)}, q^{(i+1)}] \), and this maximum value is equal to \( F_i(\bar{q}^{(i)}) = F_{i+1}(\bar{q}^{(i)}) \).

Then, because \( m \in \arg\max_{0 \leq i \leq K-1} F_i(\bar{q}^{(i)}) \), \( q^* = \bar{q}^{(m)} \) and \( F = F_m \), we can infer that for all \( q \in [q^{(i)}, q^{(i+1)}] \), we have

\[
f(q) \leq \min\{F_i(q), F_{i+1}(q)\} \leq F_i(\bar{q}^{(i)}) \leq F(q^*) \leq f(q^*) + \epsilon
\]
Since the above equation holds for every $0 \leq i \leq K - 1$, and the intervals $[q(0), q(1)], \ldots, [q(K-1), q(K)]$ cover $[a, b]$, we can infer that
\[
\max_{q \in [a, b]} f(q) \leq f(q^*) + \epsilon
\]

It remains to show that Alg. 1 terminates in a finite number of time steps. To that end, note that $\beta(x)$ is a continuous function of $x \in \mathbb{R}_+$ with $\beta(0) = 0$. Therefore, for any given $\epsilon > 0$, we can find a $\delta > 0$ such that $\beta(x) \leq \epsilon/2$ whenever $0 \leq x \leq \delta$. Let $K \in \mathbb{N}$ denote the current time step. For $0 \leq i \leq K - 1$, let $\mathcal{I}_i$ denote a root of $F_i - F_{i+1}$ in the interval $[q(i), q(i+1)]$. Let $m \in \text{argmax}_{0 \leq i \leq K-1} F_i(q(i))$, $q^* = \mathcal{I}_m$, and $F = F_m$ as before. Since $f(q(m)) - f(q^*) \leq \beta(|q^* - q(m)|)$, we have $F(q^*) - f(q^*) \leq 2\beta(|q^* - q(m)|)$. Therefore, when $|q^* - q(m)| \leq \delta$, we have $F(q^*) - f(q^*) \leq \epsilon$. Since $F(q^*) = F_m(q^*) = F_{m+1}(q^*)$, we can similarly infer that $F(q^*) - f(q^*) \leq \epsilon$ whenever $|q^* - q(m+1)| \leq \delta$. Therefore, the algorithm terminates if either $|q^* - q(m)| \leq \delta$ or $|q^* - q(m+1)| \leq \delta$, where $q^* \in [q(m), q(m+1)]$.

As per the procedure outlined in Alg. 1, the point $q^*$ will join the iterates $q(0), \ldots, q(K)$ at the $(K + 1)$th time step. When this new iterate $q^*$ is added, the updated intervals include $[q(m), q^*]$ and $[q^*, q(m+1)]$. Since $q^* \in [q(m), q(m+1)]$, we either have $|q^* - q(m)| \leq |q(m+1) - q(m)|/2$ or $|q^* - q(m+1)| \leq |q(m+1) - q(m)|/2$. Since there are only a finite number of intervals at each time step and the length of one of the sides of the interval where the new iterate falls is at least halved at each time step, at some large enough time step $K$, we will have $|q^* - q(m)| \leq \delta$ or $|q^* - q(m+1)| \leq \delta$. Thus, the algorithm terminates in a finite number of time steps.

The worst case scenario corresponds to the situation where $|q(i) - q(i+1)| = \delta$ for all $0 \leq i \leq K - 1$. In this case, the algorithm terminates at the $K$th time step, wherein $q^*$ falls within one of these intervals. Taking $\delta = \sup\{\delta' > 0 \mid \beta(\delta') \leq \epsilon/2\}$, there is a sequence $\delta \rightarrow \delta$ with $\beta(\delta_n) \leq \epsilon/2 \forall n$, so that by continuity of $\beta$, we have $\beta(\delta) = \lim_{n \rightarrow \infty} \beta(\delta_n) \leq \epsilon/2$. Therefore, the number of time steps required for the algorithm to terminate is bounded above by the number $\lceil (b - a)/\delta \rceil$, where $\delta > 0$ can be taken as the largest number that satisfies $\beta(x) \leq \epsilon/2$ for $0 \leq x \leq \delta$.

Next, we present details of constructing and searching over the grid for Lipschitz-like optimization.

### C.2 Lipschitz-like optimization over the standard simplex using grid search

We make the following observations about the integer grid defined in Eq. (60). For the grid $\Delta_{d,N}$ defined in Eq. (62) over the standard simplex, we have $\Delta_{d,N} = \mathcal{I}_{d,N}/N := \{n/N \mid n \in \mathcal{I}_{d,N}\}$. Therefore, the observations noted below also apply to $\Delta_{d,N}$ with appropriate modifications.

**Proposition 13.** Given $d, N \in \mathbb{N}_+$, let $\mathcal{I}_{d,N}$ denote the integer grid defined in Eq. (60). Then the following hold.

1) Any element $n \in \mathcal{I}_{d,N}$ can be written as $n = (N - \ell_{d-1}, \ell_{d-1} - \ell_{d-2}, \ldots, \ell_2 - \ell_1, \ell_1)$ for some integers $0 \leq \ell_i \leq \ell_{i+1} \leq N$, $i \in [d - 2]$.

2) The elements of $\mathcal{I}_{n,N}$ can be ordered such that any two consecutive elements are distance 2 apart in $l_1$-norm. This ordering is constructive and can be implemented algorithmically.

3) The grid $\Delta_{d,N}$ defined in Eq. (62) is a $(2(d - 1)/N)$-net in the $l_1$-norm for the standard simplex in dimension $d$. That is, given any $x \in \Delta_d$, there is some $z \in \Delta_{d,N}$ such that $\|x - z\|_1 \leq 2(d - 1)/N$.

**Proof.** 1) We prove this statement by induction on the dimension. For $d = 2$, any $n \in \mathcal{I}_{d,M}$ satisfies $n_1 + n_2 = M$, and therefore, $n = (M - n_1, n_1)$ holds for all $M \in \mathbb{N}_+$. Now, assume that for any
\( M \in \mathbb{N}_+ \), we can write each \( m \in I_{d-1,M} \) in dimension \( d - 1 \) as \( m = (M - s_{d-2}, \ldots, s_2 - s_1, s_1) \), where \( s_1, \ldots, s_{d-2} \in \mathbb{N} \) satisfy \( 0 \leq s_i \leq s_{i+1} \leq M \) for \( i \in [d-3] \). Then, given any \( n \in I_{d,N} \), we can write \( n_d + \sum_{i=1}^{d-1} n_i = N \). Denote \( \ell_{d-1} = \sum_{i=1}^{d-1} n_i \), so that \( n_d = N - \ell_{d-1} \) and \( 0 \leq \ell_{d-1} \leq N \). Then, since \( \sum_{i=1}^{d-1} n_i = \ell_{d-1} \), we can find some numbers \( \ell_1, \ldots, \ell_{d-2} \) such that \( 0 \leq \ell_i \leq \ell_{i+1} \leq \ell_{d-1} \leq N \) for \( i \in [d-3] \) and \( (n_1, \ldots, n_{d-1}) = (\ell_{d-1} - \ell_{d-2}, \ldots, \ell_2 - \ell_1, \ell_1) \) by assumption. Subsequently, we can write \( n = (N - \ell_{d-1}, \ldots, \ell_2 - \ell_1, \ell_1) \). Thus, by induction, the statement holds for all dimensions.

2) We explicitly construct an ordering to prove the assertion. The proof uses induction to obtain the desired result. Suppose that for dimension \( d \in \mathbb{N}_+ \) and any \( N \in \mathbb{N}_+ \), the elements of \( I_{d,N} \) can be arranged such that the first element is \( (N, 0, \ldots, 0) \), the last element is \( (0, \ldots, 0, 0) \), and the \( l_1 \)-norm distance between any two consecutive elements is 2. We call this forward ordering of elements. Writing the elements of a forward ordered set gives us reverse ordering, i.e., the first element is \( (0, \ldots, 0, N) \), the last element is \( (N, 0, \ldots, 0) \), and the \( l_1 \)-norm distance between any two consecutive elements is 2.

For \( d = 2 \), the elements of \( I_{2,N} \) can be ordered as either \( \{(N, 0), (N - 1, 1), \ldots, (1, N - 1), (0, N)\} \) (forward ordering) or \( \{(0, N), (1, N - 1), \ldots, (N - 1, 1), (N, 0)\} \) (reverse ordering), so that \( l_1 \)-norm distance between any two consecutive elements is 2.

Assuming that this statement (induction hypothesis) holds for dimension \( d \), we show that it also holds for \( d + 1 \). To that end, we note using the previous result that every element of \( I_{d+1,N} \) can be written as \( (N - s_d, \ldots, s_2 - s_1, s_1) \) with \( 0 \leq s_i \leq s_{i+1} \leq N \) for \( i \in [d-1] \). Then, we (forward) order the elements of \( I_{d+1,N} \) as follows. Let the first element be \( (N, 0, \ldots, 0) \). Choose the next sequence of elements as follows. For elements of the form \( (N - 1, 1 - s_d, \ldots, s_2 - s_1, s_1) \) arrange the elements \( (1 - s_d, \ldots, s_2 - s_1, s_1) \) in forward order, which is possible by assumption. For elements of the form \( (N - 2, 2 - s_d, \ldots, s_2 - s_1) \) arrange the elements \( (2 - s_d, \ldots, s_2 - s_1) \) in reverse order. Continuing this way, given elements of the form \( (N - s_d, s_2 - s_d - s_{d-1}, \ldots, s_2 - s_1, s_1) \) for fixed \( s_d \), arrange the elements \( (s_d - s_d - s_{d-1}, \ldots, s_1) \) in forward order if \( s_d \) is odd, and in reverse order if \( s_d \) is even.

Then, for a fixed \( 0 \leq s_d \leq N \), if \( (N - s_d, s_d - s_{d-1}, \ldots, s_1) \) is the element after \( (N - s_d, s_d - s_{d-1}, \ldots, s_1) \) as per the above ordering, we have \( \| (N - s_d, s_d - s_{d-1}, \ldots, s_1) - (N - s_d, s_d - s_{d-1}, \ldots, s_1) \|_1 = \| (s_d - s_{d-1}, \ldots, s_1) - (s_d - s_{d-1}, \ldots, s_1) \|_1 = 2 \) by induction hypothesis. Next, we consider the case when \( s_d \) increases by 1. In this case, the last element of \( (N - s_d, s_2 - s_1) \) is \( (N - s_d, s_d, 0, \ldots, 0) \) if \( s_d \) is even and it is \( (N - s_d, 0, \ldots, 0, s_d) \) if \( s_d \) is odd. Then, the first element of the next sequence \( (N - s_d - 1, 1 - s_d - 1, \ldots, s_1) \) is \( (N - s_d - 1, 1 - s_d - 1, 0, \ldots, 0) \) if \( s_d \) is even and it is \( (N - s_d - 1, 0, \ldots, 0, s_d + 1) \) if \( s_d \) is odd. Therefore, the \( l_1 \)-norm distance between consecutive elements when \( s_d \) increases by 1 is equal to 2. Therefore, the elements of \( I_{d+1,N} \) can be ordered as described above. By induction, the result holds for any dimension.

3) We prove this by induction. Given dimension \( d \in \mathbb{N}_+ \), assume that for every \( x \in \Delta_d \), there is some \( z \in \Delta_d, N \) such that \( \| x - z \|_1 \leq 2(d - 1)/N \) for all \( N \in \mathbb{N}_+ \). For \( d = 2 \), we can write any \( x \in \Delta_2 \) as \( x = (1 - x_1, x_1) \) for some \( x_1 \in [0, 1] \). Let \( n_1 \in \arg\min \{ |x_1 - m_1|/N \mid 0 \leq m_1 \leq N \} \), so that have \( |x_1 - n_1|/N \leq 1/2N \). Choosing \( z = (1 - n_1/N, n_1/N) \), we obtain \( \| x - z \|_1 \leq 1/N \leq 2/N \).

Now, suppose that the assumption holds for dimension \( d \). We show that it also holds for dimension \( d + 1 \). To that end, let \( x \in \Delta_{d+1} \) be written as \( x = (1 - s, x_2, \ldots, x_{d+1}) \), where \( s = \sum_{i=2}^{d+1} x_i \). If \( s = 0 \), then \( x = (1, 0, \ldots, 0) \) and the result follows. Therefore, let \( s > 0 \) and consider the vector \( x' = (x_2, \ldots, x_{d+1})/s \), so that \( x' \in \Delta_d \). Let \( M = \lfloor sN \rfloor \), and by assumption, there is some \( z' \in \Delta_{d,M} \) such that \( \| x' - z' \|_1 \leq 2(d - 1)/M \). Denote \( z' = (n_2, \ldots, n_{d+1})/M \) and define \( n_1 = N - \sum_{i=2}^{d+1} n_i = N - M \). Then, \( |(1 - s) - n_1|/N = |sN - M|/N \leq 1/N \) since \( sN \leq M \leq sN + 1 \). Then, denoting
Prop. 13.1 helps in iterative construction of the grid, while Prop. 13.2,3 will be used in construction of dense curve. In practice, the ordering of elements given in Prop. 13.2 can be done efficiently for moderately small dimensions. We use this ordering in practice to generate the grid, and this allows for easy parallelization.

Next, based on the results of Ref. [33], we calculate the value of \( N \) that needs to be used in the grid search to converge to a precision of \( \epsilon > 0 \).

**Proposition 14.** Let \( f: \mathcal{D} \to \mathbb{R} \) be a \( \beta \)-Lipschitz-like function satisfying Eq. (59), where \( \mathcal{D} = \Delta_d \) is the standard simplex in \( \mathbb{R}^d \). Let \( N = [1/\delta^2] \), where \( \delta = \text{sup}\{\delta' > 0 \mid \beta(\delta') \leq \epsilon/2\} \). Let \( \Delta_d,N \) be the grid defined in Eq. (62), and let \( f^* = \max\{f(z) \mid z \in \Delta_d,N\} \). Then, we have \( \max_{x \in \Delta_d} f(x) - f^* \leq \epsilon \).

**Proof.** This result is implied by techniques developed in Ref. [33]. For the sake of completeness, we give a proof here for the specific case of Lipschitz-like functions.

Given \( f: \mathcal{D} \to \mathbb{R} \), the Bernstein polynomial approximation of \( f \) of order \( N \in \mathbb{N}_+ \) is defined as

\[
B_N(f)(x) = \sum_{n \in \mathbb{I}_d,N} f\left(\frac{n}{N}\right) \frac{N!}{n!} x^n
\]

for \( x \in \Delta_d \), where we use the multi-index notation \( n! = n_1! \cdots n_d! \), \( x^n = x_1^{n_1} \cdots x_d^{n_d} \) for \( n \in \mathbb{I}_d,N \) [33]. The sup-norm of a continuous function \( f: \mathcal{D} \to \mathbb{R} \) is given by \( \|f\|_{\infty} = \max_{x \in \mathcal{D}} |f(x)| \), where we use the fact that \( \mathcal{D} \) is compact. Next, given a continuous function \( f: \mathcal{D} \to \mathbb{R} \), the modulus of continuity of \( f \) is defined as

\[
\omega(f, \delta) = \max_{x,y \in \mathcal{D}, \|x-y\| \leq \delta} |f(x) - f(y)|
\]

with respect to a suitable norm \( \|\cdot\| \) on \( \mathbb{R}^n \) [33]. If \( f \) is Lipschitz-like, then we use the norm with respect to which \( f \) satisfies the Lipschitz-like property. For a \( \beta \)-Lipschitz-like function \( f \), using Eq. (59), we have

\[
\omega(f, \delta) \leq \beta(\delta),
\]

where we used the fact that \( \beta \) is a monotonically increasing function.

Then, given a \( \beta \)-Lipschitz-like function \( f: \mathcal{D} \to \mathbb{R} \), using Thm. (3.2) of Ref. [33], we have

\[
\|B_N(f) - f\|_{\infty} \leq 2\omega\left(f, \frac{1}{\sqrt{N}}\right) \leq 2\beta\left(\frac{1}{\sqrt{N}}\right)
\]

where we used Eq. (100) to obtain the last inequality. In particular, this implies that \( \max_{x \in \mathcal{D}} B_N(f) \geq \max_{x \in \mathcal{D}} f(x) - 2\beta(1/\sqrt{N}) \). Using this result along with Lemma (3.1) of Ref. [33], we obtain

\[
\max_{x \in \mathcal{D}} f(x) - 2\beta\left(\frac{1}{\sqrt{N}}\right) \leq \max_{x \in \mathcal{D}} B_N(f)(x) \leq \max_{x \in \Delta_d,N} f(x).
\]
Therefore, choosing 
\[ \beta \left( \frac{1}{\sqrt{N}} \right) \leq \frac{\epsilon}{2}, \]
we obtain 
\[ \max_{x \in \mathcal{D}} f(x) \leq \max_{x \in \Delta_{d,N}} f(x) + \epsilon \]
giving us the desired result. In other words, to compute the maximum of \( f \) to within a precision of \( \epsilon > 0 \), it suffices to search the grid \( \Delta_{d,N} \) with \( N = \lceil 1/\delta^2 \rceil \), where \( \delta \) is the largest number satisfying \( \beta(\delta) \leq \epsilon/2 \).

C.3 Lipschitz-like optimization over the standard simplex using dense curves

We show how to optimize Lipschitz-like functions over the standard simplex using \( \alpha \)-dense curves. Such curves get within a distance \( \alpha \) of all points in the simplex (see Def. (8)). As we show below, by appropriately choosing \( \alpha \), one can perform a one-dimensional optimization to obtain the maximum of \( f \) to the desired precision.

**Proposition 15.** Suppose that \( f : \Delta_d \to \mathbb{R} \) is a \( \beta \)-Lipschitz-like function satisfying Eq. (59). Then the following hold.

1. Given \( N \in \mathbb{N}_+ \), let \( \gamma : [0, L_{\text{curve}}] \to \Delta_d \) be a Lipschitz-like curve constructed as per Alg. 3. Then the curve \( \gamma \) is \( (2(d-1)/N) \)-dense in the simplex \( \Delta_d \) and satisfies Eq. (64).

2. Alg. 4 computes the maximum of \( f \) to within a precision of \( \epsilon > 0 \) for any \( \beta \)-Lipschitz-like, \( (2(d-1)/N) \)-dense curve \( \gamma \). Here, \( N = \lceil 2(d-1)/\alpha \rceil \) with \( \alpha = \sup \{ \alpha' > 0 \mid \beta(\alpha') \leq \epsilon/2 \} \) as noted in Alg. 4. In the worst case, the algorithm takes
\[
\left\lceil \frac{2}{N\delta} \left( \frac{N + d - 1}{d - 1} \right) \right\rceil
\]
time steps to converge to the maximum within a precision of \( \epsilon > 0 \), where \( \delta = \sup \{ \delta' > 0 \mid \beta(\beta(\delta')) \leq \epsilon/2 \}. \) When \( \gamma \) is the curve generated by Alg. 3, \( \alpha < 1 \) and \( d \gg 1 \), this amounts to \( O(\alpha^{1-d}/d) \) time steps in the worst case.

**Proof.** 1. From Prop. 13, we know that \( \Delta_{d,N} \) is a \( (2(d-1)/N) \)-net. Since \( \Delta_{d,N} \subseteq \text{Range}(\gamma) \), for any \( x \in \Delta_d \), there is some \( z \in \text{Range}(\gamma) \) such that \( \| x - z \|_1 \leq 2(d-1)/N \). In other words, \( \gamma \) is a \( (2(d-1)/N) \)-dense curve in \( \Delta_d \). Next, we show that \( \gamma \) is Lipschitz-like function. To that end, given \( \theta, \theta' \in \mathbb{R} \), let \( k = \lceil \theta N/2 \rceil \) and \( k' = \lceil \theta' N/2 \rceil \). Without loss of generality, take \( \theta \leq \theta' \). Given \( i \in [N_{\text{grid}}] \), where \( N_{\text{grid}} \) is defined in Eq. (61), denote \( x_i \in \Delta_{d,N} \) to be the \( i \)th element of \( \Delta_{d,N} \) ordered as per Eq. (9). Now, if \( k = k' \), then we can write \( \gamma(\theta) = tx_k + (1-t)x_{k+1} \) and \( \gamma(\theta') = t'x_k + (1-t')x_{k+1} \), where \( t = 1 + k - \theta N/2 \) and \( t' = 1 + k - \theta' N/2 \) (see Alg. 3). Therefore, \( \| \gamma(\theta') - \gamma(\theta) \|_1 \leq \| t - t' \| \| x_{k+1} - x_k \|_1 = | \theta' - \theta | \), where we used the fact that \( \| x_{k+1} - x_k \|_1 = 2/N \) (see Prop. 13). Thus, consider the case \( k' > k \), so that
\[
\| \gamma(\theta') - \gamma(\theta) \|_1 \leq \| \gamma(\theta') - x_k \|_1 + \sum_{i=k+1}^{k'-1} \| x_{i+1} - x_i \|_1 + \| x_{k+1} - \gamma(\theta) \|_1 = | \theta' - \theta | \]
where in the last line we used the fact that \( \theta \) (and \( \theta' \)) is defined as the length along obtained by joining consecutive points of the grid until we reach \( \theta \) (and \( \theta' \) respectively). Since \( \| x - y \|_1 \leq 2 \) for all \( x, y \in \Delta_d \), Eq. (64) follows.

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2. Consider the real-valued function \( g = f \circ \gamma \) defined on the interval \([0, L_{\text{curve}}]\). Here, \( \gamma \) is a \((2(d-1)/N)\)-dense \( \beta_{\gamma} \)-Lipschitz-like curve with \( N = [2(d-1)/\alpha] \) and \( \alpha = \sup\{\alpha' > 0 \mid \beta(\alpha) \leq \epsilon/2\} \). Since \( 2(d-1)/N \leq \alpha \), \( \gamma \) is also an \( \alpha \)-dense curve. Let \( \theta^* \in \text{argmax}\{g(\theta) \mid \theta \in [0, L_{\text{curve}}]\} \), and let \( \theta^*_x \in [0, L_{\text{curve}}] \) be the point output by Alg. 1. Since \( f \) is \( \beta \)-Lipschitz-like and \( \gamma \) is \( \beta_{\gamma} \)-Lipschitz-like, \( g = f \circ \gamma \) is \( \beta \circ \beta_{\gamma}\)-Lipschitz-like, so by Prop. 12, we know that \( g(\theta^*) - g(\theta^*_x) \leq \epsilon/2 \). Let \( x^* \in \text{argmax}\{f(x) \mid x \in \Delta_d\} \) denote a point achieving the maximum of \( f \). Then, since \( \gamma \) is an \( \alpha \)-dense curve, there is some point \( \theta_0 \in [0, L_{\text{curve}}] \) such that \( \|x^* - \gamma(\theta_0)\|_1 \leq \alpha \). Noting that \( g(\theta_0) \leq g(\theta^*) \), we have

\[
\begin{align*}
 f(x^*) - g(\theta^*_x) & \leq f(x^*) - g(\theta^*) + \frac{\epsilon}{2} \\
 & \leq f(x^*) - g(\theta_0) + \frac{\epsilon}{2} \\
 & \leq \beta(\|x^* - \gamma(\theta_0)\|_1) + \frac{\epsilon}{2} \\
 & \leq \epsilon
\end{align*}
\]

To obtain the penultimate inequality we used the Lipschitz-like property of \( f \) along with the fact that \( g(\theta_0) = f(\gamma(\theta_0)) \). The last inequality follows by noting that \( \beta(\|x^* - \gamma(\theta_0)\|_1) \leq \beta(\alpha) \leq \epsilon/2 \). The number of iterations needed to compute \( \theta^*_x \) in the worst-case is given by

\[
N_{\text{maxiter}} = \left\lceil \frac{L_{\text{curve}}}{\delta} \right\rceil = \left\lceil \frac{2N + 2d - 1}{\delta(d - 1)} \right\rceil
\]

where \( \delta = \sup\{\delta' > 0 \mid \beta(\beta_{\gamma}(\delta')) \leq \epsilon/2\} \), which follows from Prop. 12. Using the fact that \((n/k)^k \leq (en/k)^k \) and \( 2(d-1)/\alpha \leq N \leq 2(d-1)/\alpha + 1 \), the number of iterations is bounded as

\[
\frac{\alpha}{(d-1)\delta} \left(\frac{2(d-1)/\alpha + 1}{d - 1}\right)^{d-1} \leq N_{\text{maxiter}} \leq \frac{\alpha}{(d-1)\delta} \left(\frac{\epsilon(2(d-1)/\alpha + d)}{d - 1}\right)^{d-1}.
\]

Thus, assuming \( \alpha < 1 \) and \( d \gg 1 \), we need \( O(\alpha^{2-d}\delta^{-1}/d) \) time steps for convergence. For the case when \( \gamma \) is the curve generated by Alg. 3, we can take \( \beta_{\gamma}(x) = x \) owing to Eq. (64), and therefore, the statement of the proposition follows.

\[\square\]

### C.4 Lipschitz-like optimization over a compact & convex domain

We now consider the general problem of optimizing a \( \beta \)-Lipschitz-like function \( f : D \to \mathbb{R} \) satisfying Eq. (59), where \( D \subseteq \mathbb{R}^d \) is a nonempty compact and convex domain. Many techniques have been developed for optimizing Lipschitz or Hölder continuous functions when \( D \) is a specific set such as hypercube, simplex or is the full Euclidean space \( \mathbb{R}^n \) as noted in Sec. 5.1. However, when dealing with an arbitrary compact and convex domain, it is not always obvious how to encode the constraints defining the domain in the algorithm.

We circumvent this problem by looking for a function \( \overline{f} : \mathbb{R}^d \to \mathbb{R} \) satisfying the following properties:

1. The function \( \overline{f} \) is an extension of \( f \) in the sense that \( \overline{f} \) is Lipschitz-like and \( \overline{f}|_D = f \).

2. The maximum of \( \overline{f} \) and \( f \) coincide. More specifically, \( \sup_{x \in K} \overline{f} = \max_{x \in D} f \) for every set \( D \subseteq K \subseteq \mathbb{R}^d \).

3. The function \( \overline{f} \) can be efficiently computed whenever \( f \) and \( \beta \) can be efficiently computed.
The first property ensures that we are able to use existing Lipschitz-like optimization algorithms. The second property ensures that we can compute the maximum of $f$ by computing the maximum of $\overline{f}$ over a convenient set $\mathcal{K}$ containing the domain $\mathcal{D}$. The choice of the set $\mathcal{K}$ will usually depend on the actual algorithm used to perform the optimization. The last property ensures that the function $\overline{f}$ can be efficiently evaluated in practice.

We show that one can construct a function $\overline{f}$ satisfying all the above properties for the case when $\beta$ is itself a Lipschitz-like function. That is, there is some non-negative, continuous, monotonically increasing function $\kappa: \mathbb{R}_+ \to \mathbb{R}$ with $\kappa(0) = 0$ such that, for all $x, y \in \mathbb{R}_+$,

$$|\beta(x) - \beta(y)| \leq \kappa(|x - y|). \quad (101)$$

This assumption is not too restrictive for our purposes because Lipschitz continuous functions (i.e., $\beta(x) = x$), and functions $\beta$ relevant to entropic quantities like the one in Eq. (56) satisfy Eq. (101) (see Prop. 17). In this case, we can define the extension of $f$ as follows.

**Definition 16** (Lipschitz-like extension). Let $f: \mathcal{D} \to \mathbb{R}$ be a $\beta$-Lipschitz-like function over a compact and convex domain $\mathcal{D} \subseteq \mathbb{R}^d$. Let $\|\cdot\|$ be the norm on $\mathbb{R}^d$ with respect to which Eq. (59) holds. Suppose that $\beta$ is $\kappa$-Lipschitz-like. Then, the Lipschitz-like extension of $f$ is the function $\overline{f}: \mathbb{R}^d \to \mathbb{R}$ defined as

$$\overline{f}(x) = f(\Pi_\mathcal{D}(x)) - \beta(\|x - \Pi_\mathcal{D}(x)\|), \quad (102)$$

where $\Pi_\mathcal{D} = \arg\min_{z \in \mathcal{D}}\|z - x\|_2$ is the projection of $x$ onto $\mathcal{D}$ with respect to the Euclidean norm.

We remark that the choice of norm in Eq. (59), and subsequently in Eq. (102), is flexible. Moreover, because all norms on $\mathbb{R}^d$ are equivalent, one can change the function $\beta$ so as to get a Lipschitz-like property with respect to a different norm. Recall that two norms $\|\cdot\|_a$ and $\|\cdot\|_b$ on $\mathbb{R}^d$ are said to be equivalent if there exist constants $c_1, c_2 > 0$ (possibly dimension dependent) such that $c_1 \|\cdot\|_a \leq \|\cdot\|_b \leq c_2 \|\cdot\|_a$. The caveat of using equivalence of norms to change the function $\beta$ is that the modified $\beta$ might end up depending on the dimension. This could negatively impact the convergence rate (for example, polynomial time convergence guarantee for optimization over the simplex using grid search might be lost if $\beta$ depends on the dimension). This observation is also of relevance in Def. 16 because we define the projection with respect to the Euclidean norm, while allow an arbitrary choice of norm in Eq. (102) (also see Prop. 17).

Note that $\Pi_\mathcal{D}$ is well-defined since $\mathcal{D}$ is compact and convex (see Thm. (2.5) in Ref. [41]). Since $\Pi_\mathcal{D}(x) = x$ for $x \in \mathcal{D}$, we can see that $\overline{f}$ is indeed an extension of $f$. Roughly speaking, the extension is also Lipschitz-like because both $f$ and $\beta$ are Lipschitz-like. Furthermore, since $\beta$ is a non-negative monotonically increasing function, the value of $\overline{f}$ decreases as we move away from $\mathcal{D}$. Thus, the maximum of $\overline{f}$ occurs over $\mathcal{D}$. Finally, so long as the projection $\Pi_\mathcal{D}$ can be efficiently computed, the function $\overline{f}$ can be efficiently computed (assuming $f$ and $\beta$ can be efficiently computed).

We formalize the observations made above in the following proposition.

**Proposition 17.** Let $\beta: \mathbb{R}_+ \to \mathbb{R}$ be a non-negative, continuous, monotonically increasing function with $\beta(0) = 0$. Suppose that $\beta$ is $\kappa$-Lipschitz-like in the sense of Eq. (101) for some appropriate $\kappa$. Let $f: \mathcal{D} \to \mathbb{R}$ be a $\beta$-Lipschitz-like function over some compact and convex domain $\mathcal{D} \subseteq \mathbb{R}^d$. Let $\overline{f}$ be the extension of $f$ as defined in Eq. (102). Then the following statements hold.

1. Define the constant $C = c_2/c_1$, where $c_1, c_2 > 0$ are obtained from equivalence of $\|\cdot\|, \|\cdot\|_2$ on $\mathbb{R}^d$, i.e., $c_1 \|\cdot\|_a \leq \|\cdot\|_2 \leq c_2 \|\cdot\|_a$. Then, the extension $\overline{f}$ is a $\overline{\beta}$-Lipschitz-like function, where

$$\overline{\beta}(x) = \beta(Cx) + \kappa((C + 1)x). \quad (103)$$
2. Given any set \( D \subseteq K \subseteq \mathbb{R}^d \), we have \( \sup_{x \in K} \bar{f}(x) = \max_{x \in D} f(x) \). Moreover, any point achieving the maximum of \( f \) achieves the maximum of \( \bar{f} \).

3. The modified binary entropy \( \bar{h} \) defined in Eq. (58) is an \( \bar{h} \)-Lipschitz-like function, that is,

\[
|\bar{h}(x) - \bar{h}(y)| \leq \bar{h}(|x - y|)
\]

for all \( x, y \in \mathbb{R}_+ \). Subsequently, \( \beta_I \) defined in Eq. (56) is a \( \beta_I \)-Lipschitz-like function.

Proof. 1. We first note that \( \Pi_D \) is a non-expansive mapping on \( \mathbb{R}^d \), i.e., for all \( x, y \in \mathbb{R}^d \), we have

\[
\|\Pi_D(x) - \Pi_D(y)\|_2 \leq \|x - y\|_2 \ [42].
\]

By equivalence of norms, we can find a (possibly dimension-dependent) constant \( C > 0 \) noted in the statement of the proposition, such that

\[
\|\Pi_D(x) - \Pi_D(y)\| \leq C\|x - y\|.\]

From this, we can infer the following list of inequalities:

\[
|\bar{f}(x) - \bar{f}(y)| \leq |f(\Pi_D(x)) - f(\Pi_D(y))| + |\beta(\|x - \Pi_D(x)\|) - \beta(\|y - \Pi_D(y)\|)|
\]

\[
\leq \beta(\|\Pi_D(x) - \Pi_D(y)\|) + C(\|x - \Pi_D(x)\| - \|y - \Pi_D(y)\|)
\]

\[
\leq \beta(C\|x - y\|) + \kappa(\|x - y + \Pi_D(y) - \Pi_D(x)\|)
\]

\[
\leq \beta(C\|x - y\|) + \kappa(\|x - y\| + \|\Pi_D(x) - \Pi_D(y)\|)
\]

\[
\leq \beta(C\|x - y\|) + \kappa((C + 1)\|x - y\|)
\]

\[
= \beta(|x - y|)
\]

To obtain the second inequality, we use the fact that \( f \) is \( \beta \)-Lipschitz-like and \( \beta \) is \( \kappa \)-Lipschitz-like. To obtain the third inequality, we use the reverse triangle inequality, the fact that \( \Pi_D \) is non-expansive, and that \( \beta \) and \( \kappa \) are monotonically increasing functions. The last two inequalities follow from similar observations. Finally, we note that \( \bar{h}(x) = \beta(Cx) + \kappa((C + 1)x) \) is a non-negative, continuous, monotonically increasing function with \( \bar{h}(0) = 0 \).

2. Since \( \beta \) is a non-negative function, we have \( \bar{f}(x) \leq f(\Pi_D(x)) \) for all \( x \in \mathbb{R}^d \). Therefore, given any set \( D \subseteq K \subseteq \mathbb{R}^d \), we have \( \sup_{x \in K} \bar{f}(x) \leq \sup_{x \in K} f(\Pi_D(x)) = \max_{x \in D} f(x) \). The last equality follows by noting that \( \Pi_D(x) \in D \) for all \( x \in \mathbb{R}^d \), \( f \) is continuous and \( D \) is compact. If \( x^* \in D \) achieves the maximum of \( f \), then \( \sup_{x \in K} \bar{f}(x) = \max_{x \in D} f(x) = f(x^*) = \bar{f}(x^*) \), where in the last step, we used the fact that \( f = \bar{f} \) on \( D \).

3. Let \( \bar{h} \) be the modified binary entropy defined in Eq. (58). If \( x \in [0, 1/2] \) then, \( \bar{h}(x) = h(x) \). Furthermore, we have \( h(x) = H((x, 1 - x)) \) for \( x \in [0, 1] \), where \( H \) is the Shannon entropy. Therefore, given \( x, y \in [0, 1/2] \), we have \( |\bar{h}(x) - \bar{h}(y)| = |h(x) - h(y)| = |H((x, 1 - x)) - H((y, 1 - y))| \leq h(|x - y|) = \bar{h}(|x - y|) \), where the inequality follows from the results of Ref. [43]. Here, we used the fact that \( |x - y| \leq 1/2 \) when \( x, y \in [0, 1/2] \). For \( x, y \geq 1/2 \), we have \( \bar{h}(x) = \bar{h}(y) = \ln(2) \), and thus

\[
|\bar{h}(x) - \bar{h}(y)| = 0 \leq \bar{h}(|x - y|).
\]

Finally, we consider the case when \( x \in [0, 1/2] \) and \( y \geq 1/2 \). Here, we have \( |\bar{h}(x) - \bar{h}(y)| = \ln(2) - h(x) = h(1/2) - h(x) \leq h(1/2 - x) = \bar{h}(1/2 - x) \leq \bar{h}(|y - x|) \). In the last step, we used the fact that \( |1/2 - x| \leq |y - x| \) and \( \bar{h} \) is monotonically increasing. Therefore, we have \( |\bar{h}(x) - \bar{h}(y)| \leq \bar{h}(|x - y|) \) for all \( x, y \in \mathbb{R}_+ \).

The above extension gives some freedom in determining what algorithm to use to perform the maximization of \( \bar{f} \), especially for the case when \( f \) is Lipschitz or Hölder continuous. One can, for example, use an unconstrained optimization method developed for optimization of Lipschitz or Hölder continuous function. Alternatively, one can embed \( D \) inside a hypercube \( K \), and use an algorithm that can perform global Lipschitz optimization [44, 45, 34]. In this study, we generalize the method
Lipschitz-like compact function $g$ with $\beta$ and $\eta > 0$, we used the $\alpha$-dense curve for filling the hypercube $\mathcal{K} = \prod_{i=1}^{d}[a_i, b_i]$ used in Ref. [34], which we reproduce below for convenience. Let $d \geq 2$, $\eta > 0$, $\eta_1 = 1$ and define

$$\eta_i = \left(\frac{\eta}{\pi}\right)^{i-1} \prod_{j=2}^{i} \frac{1}{|a_j| + |b_j|}$$

(104)

for $i = 2, \ldots, d$. Then, the curve $\gamma : [0, \pi/\eta_d] \to \mathcal{K}$ defined as

$$\gamma_i(\theta) = \frac{a_i - b_i}{2} \cos(\eta_i \theta) + \frac{a_i + b_i}{2}$$

(105)

for $i \in [d]$ is a $(\sqrt{d-1} \eta)$-dense curve with respect to the Euclidean norm [34]. Furthermore, $\gamma$ is Lipschitz continuous (with respect to the Euclidean norm) with Lipschitz constant

$$L_\gamma = \frac{1}{2} \left( \sum_{i=1}^{n} (|a_i| + |b_i|)^2 \eta_i^2 \right)^{1/2}$$

(106)

Algorithm 6 Computing the maximum of a $\beta$-Lipschitz-like function $f$ satisfying Eq. (59) for $\mathcal{D} = \Delta_d$, given $\epsilon > 0$

1: function MAXIMIZE\_LIPSHCITZ\_LIKE\_FUNCTION\_COMPACT\_CONVEX\_DOMAIN($d$, $\beta$, $\epsilon$)
2: Find $\mathcal{K} = \prod_{i=1}^{d}[a_i, b_i] \subseteq \mathbb{R}^d$ such that $\mathcal{D} \subseteq \mathcal{K}$
3: Find $c_1 > 0$ such that $c_1 \|\cdot\| \leq \|\cdot\|_2$
4: Construct the curve $\gamma$ as per Eq. (105) for $\eta = c_1 \alpha / \sqrt{d-1}$, where $\alpha = \sup\{\alpha' > 0 \mid \beta(\alpha') \leq \epsilon/2\}$
5: Construct the extension $\mathcal{F}$ as per Eq. (102)
6: Compute the maximum $\mathcal{F}^*$ of $\mathcal{F} = \mathcal{F} \circ \gamma$ over $[0, \pi/\eta_d]$ to a precision of $\epsilon/2$ using Alg. 1
7: return $\mathcal{F}^*$
8: end function

Below, we show that the above algorithm will converge to the maximum within precision $\epsilon > 0$. The proof essentially adapts the ideas used to prove Prop. 15.

Proposition 18. Let $f : \mathcal{D} \to \mathbb{R}$ be a $\beta$-Lipschitz-like function with respect to the norm $\|\cdot\|$ on $\mathbb{R}^d$, where $\mathcal{D} \subseteq \mathbb{R}^d$ is a compact $\ell^1$ convex set. Suppose that $\beta$ is $\kappa$-Lipschitz-like. Let $\mathcal{K} \subseteq \mathbb{R}^d$ be a bounded set containing $\mathcal{D}$. Let $\gamma$ be any $\alpha$-dense, $\beta\gamma$-Lipschitz-like curve from the interval $[a, b]$ into the set $\mathcal{K}$. Let $\mathcal{F}$ be a $\beta$-Lipschitz-like extension of $f$ defined in Eq. (102), where $\mathcal{F}$ is given in Eq. (103). Suppose that we compute the maximum of $f$ using an appropriate generalization of Alg. 6 with the curve $\gamma$. Then, for $\alpha = \sup\{\alpha' > 0 \mid \beta(\alpha') \leq \epsilon/2\}$, the algorithm computes the maximum of $f$ to within a precision of $\epsilon > 0$. In the worst case, this algorithm takes $[(b - a)/\delta]$ time steps to converge, where $\delta = \sup\{\delta' > 0 \mid \beta(\delta') \leq \epsilon/2\}$.

In particular, if $\mathcal{K}$ is a hypercube containing $\mathcal{D}$ and $\gamma$ is the curve generated as per Eq. (105), a value of $\eta = c_1 \alpha / \sqrt{d-1}$ suffices to converge to the maximum of $f$ within a precision of $\epsilon > 0$. Here, $\alpha$ is as defined above and $c_1 > 0$ is a (possibly dimension dependent) constant satisfying $c_1 \|\cdot\| \leq \|\cdot\|_2$. The algorithm takes $\lceil \frac{\pi}{\delta \eta_1^d} \rceil$ time steps to converge in the worst case, where $\eta_d$ is defined in Eq. (104) and $\delta$ is as defined above. For the case when $f$ is a Lipschitz continuous function with Lipschitz constant $L$ with respect to the Euclidean norm, Alg. 6 takes $O((6\pi L \sqrt{d}/\epsilon)^d)$ iterations to converge to a precision of $\epsilon > 0$. The exponential scaling with $L$ and $\epsilon$ cannot be improved without additional assumptions on the functions or the domain.
Proof. Since \( \mathcal{D} \subseteq \mathbb{R}^d \) is compact, it is bounded, and can therefore be embedded into a bounded set \( \mathcal{K} \subseteq \mathbb{R}^d \). Let \( \overline{f} \) be the \( \beta \)-Lipschitz-like extension of \( f \) defined in Eq. (102), where \( \overline{f} \) is given in Eq. (103). Let \( \gamma : [a, b] \rightarrow \mathcal{K} \) be an \( \alpha \)-dense, \( \beta \)-Lipschitz-like curve, where \( \alpha = \sup\{\alpha' > 0 \mid \overline{f}(\alpha') \leq \epsilon/2\} \). Consider the real-valued function \( \overline{g} = \overline{f} \circ \gamma \) defined on the interval \([a, b] \). Let \( \theta^* \in \arg\max\{\overline{g}(\theta) \mid \theta \in [a, b]\} \), and let \( \theta^* \in [a, b] \) be the point output by Alg. 1. Since \( \overline{f} \) is \( \beta \)-Lipschitz-like and \( \gamma \) is \( \beta \)-Lipschitz-like, \( \overline{g} = \overline{f} \circ \gamma \) is \( \beta \circ \beta \)-Lipschitz-like. So, by Prop. 12, we know that \( \overline{g}(\theta^*) - \overline{g}(\theta^*) \leq \epsilon/2 \).

Let \( x^* \in \arg\max\{f(x) \mid x \in \Delta_d\} \) denote a point achieving the maximum of \( f \). From Prop. 17, we know that \( \overline{f}(x^*) = \sup_{x \in \mathcal{D}} \overline{f}(x) = \sup_{x \in \mathcal{D}} f(x) = f(x^*) \). Then, since \( \gamma \) is an \( \alpha \)-dense curve and \( \mathcal{D} \subseteq \mathcal{K} \), there is some point \( \theta_0 \in [a, b] \) such that \( \|x^* - \gamma(\theta_0)\|_1 \leq \alpha \). Noting that \( \overline{g}(\theta_0) \leq \overline{g}(\theta^*) \), we have

\[
\begin{align*}
f(x^*) - \overline{g}(\theta^*) & \leq \overline{f}(x^*) - \overline{g}(\theta^*) + \frac{\epsilon}{2} \\
& \leq \overline{f}(x^*) - \overline{g}(\theta_0) + \frac{\epsilon}{2} \\
& \leq \overline{f}(\|x^* - \gamma(\theta_0)\|_1) + \frac{\epsilon}{2} \\
& \leq \epsilon
\end{align*}
\]

To obtain the penultimate inequality we used the Lipschitz-like property of \( \overline{f} \) along with the fact that \( \overline{g}(\theta_0) = \overline{f}(\gamma(\theta_0)) \). The last inequality follows by noting that \( \overline{f}(\|x^* - \gamma(\theta_0)\|_1) \leq \overline{f}(\alpha) \leq \epsilon/2 \). The number of iterations needed to compute \( \theta^* \) in the worst-case is given by

\[
N_{\text{maxiter}} = \left\lceil \frac{b - a}{\delta} \right\rceil
\]

where \( \delta = \sup\{\delta' > 0 \mid \overline{f}(\beta(\delta')) \leq \epsilon/2\} \), which follows from Prop. 12.

Now suppose that \( \mathcal{K} \) is a hypercube containing \( \mathcal{D} \) and \( \gamma \) is the curve given in Eq. (105). By equivalence of norms on \( \mathbb{R}^d \), we have \( c_1 \|\cdot\| \leq \|\cdot\| \leq c_2 \|\cdot\| \) for some (possibly dimension dependent) constants \( c_1, c_2 > 0 \). Therefore, \( \gamma \) is \( (\sqrt{d} - 1/\eta)\)-dense and \( (L_\gamma/c_1) \)-Lipschitz continuous with respect to the norm \( \|\cdot\| \), where \( L_\gamma \) is given in Eq. (106). By the above results, we can choose \( \eta = c_1 \alpha / \sqrt{d} - 1 \) for convergence, where \( \alpha = \{\alpha' > 0 \mid \overline{f}(\alpha') \leq \epsilon/2\} \). In this worst case, this algorithm takes \( \lceil \pi/(\eta \delta) \rceil \), where \( \eta \delta \) is defined in Eq. (104) and \( \delta = \sup\{\delta' > 0 \mid \overline{f}(\beta(\delta')) \leq \epsilon/2\} \).

Consider the case when \( d \geq 2, \mathcal{D} = [0, 1]^d, \mathcal{K} = \mathcal{D} \), and \( f \) is Lipschitz continuous with Lipschitz constant \( L \) (independent of the dimension) with respect to the Euclidean norm \( \|\cdot\|_2 \). Then, \( c_1 = c_2 = 1 \) as defined above, and subsequently, \( C \) defined in Prop. 17 is equal to 1. Then, since \( \kappa(x) = \beta(x) = Lx \), we have \( \overline{f}(x) = 3Lx \). Subsequently, we have \( \alpha = \epsilon/6L \), and for \( \epsilon < 6L \), we have \( \alpha < 1 \). For the curve \( \gamma \), we have \( \eta = \alpha / \sqrt{d} - 1 = \epsilon / 6L \sqrt{d - 1} \), \( \eta = (\pi/\eta \delta)^d - 1 \), \( \delta = \epsilon / 6LL_\gamma \), and \( L_\gamma \leq \sqrt{d} / 2 \) assuming that \( \alpha < 1 \). Thus, the Alg. 6 takes

\[
\left\lceil \frac{6\pi^2 dL_\gamma}{\eta \delta - 1} \right\rceil \leq \left\lceil \frac{(6\pi L \sqrt{d} d^d)}{\epsilon d} \right\rceil
\]

iterations to converge. It is known that any algorithm (in the sense of a black box model) needs at least \( O((L/2\epsilon)^d) \) iterations to compute the optimum of a \( L \)-Lipschitz function over the unit hypercube to a precision of \( \epsilon > 0 \) (see Ref. [28] for details). Thus, one cannot, in general improve the \( 1/\epsilon^d \) scaling for Alg. 6 without additional assumptions on \( \mathcal{D} \) or the class of functions that are optimized by the algorithm.

The above result shows that the exponential scaling with the dimension cannot be improved without additional assumptions on the functions or the domain. It is likely the case that the \( \sqrt{d} \) factor is sub-optimal and comes from the choice of the curve \( \gamma \) in Eq. (105), so one might be able to improve that factor by using better constructions for the curve or by resorting to a different approach altogether. \( \square \)
We remark that the above algorithm will perform worse than Alg. 2 and Alg. 4 in practice when \( D \) is the standard simplex. This is because we essentially use no information about the domain except for the projection \( \Pi_D \) in constructing the extension \( \tilde{f} \) (see Eq. (102)). Indeed, when using grid search specially designed for \( D = \Delta_d \), we can get polynomial scaling with the dimension for a fixed precision (see Prop. 14). Thus, it might be preferable to resort to methods designed specifically for the particular domain and function class of interest, as opposed to general algorithms like Alg. 6. That said, an advantage of Alg. 6 is that the maximum is computed by finding successively better approximations. Therefore, one can decide to terminate the computation after some fixed number of iterations in order to obtain an upper bound on the maximum.

D Analysis of algorithms for computing the sum capacity for two-sender MACs

Proposition 19. Let \( \mathcal{N} \) be any two-sender MAC with input alphabets \( B_1, B_2 \) of size \( d_1, d_2 \), and output alphabet \( Z \) of size \( d_o \). Assume that \( d_1, d_2, d_o \geq 2 \). Given input probability distributions \( p \) over \( B_1 \) and \( q \) over \( B_2 \), the mutual information between the inputs and the output of the MAC can be written as

\[
I(p, q) = H(A_q p) - \langle b_q, p \rangle
\]

where the matrix \( A_q \) and the vector \( b_p \) are defined in Eq. (50) and Eq. (51) respectively. Define the function

\[
I^*(q) = \max_{p \in \Delta_{d_1}} I(p, q)
\]

Then, for any \( p \in \Delta_{d_1} \), and any \( q, q' \in \Delta_{d_2} \), we have

\[
|I(p, q) - I(p, q')| \leq \beta_I \left( \|q - q'\|_1 \right),
\]

and subsequently,

\[
|I^*(q) - I^*(q')| \leq \beta_I \left( \|q - q'\|_1 \right)
\]

where the function \( \beta_I \) is defined in Eq. (56).

Proof. For any probability distributions \( p_1^Z, p_2^Z \in \Delta_{d_o} \), we have

\[
|H(p_1^Z) - H(p_2^Z)| \leq \theta \log(d_o - 1) + \theta h(\theta),
\]

where \( \theta = \|p_1^Z - p_2^Z\|_1 / 2 \) and \( h \) is the binary entropy function [43]. In particular, if we define \( \overline{h} \) as in Eq. (58), then we have

\[
|H(p_1^Z) - H(p_2^Z)| \leq \frac{1}{2} \log(d_o - 1) \|p_1^Z - p_2^Z\|_1 + \overline{h} \left( \frac{1}{2} \|p_1^Z - p_2^Z\|_1 \right)
\]

because \( h(x) \leq \overline{h}(x) \) for all \( x \in [0, 1] \).

Now, note that using the same ideas employed in obtaining \( I(p, q) \) given in Eq. (52), we can interchange the roles of \( p \in \Delta_{d_1} \) and \( q \in \Delta_{d_2} \) to obtain

\[
I(p, q) = H(A_p q) - \langle b_p, q \rangle
\]

where

\[
A_p(z, b_2) = \sum_{b_1 \in B_1} N(z|b_1, b_2)p(b_1) \quad \text{and} \quad b_p(b_2) = - \sum_{b_1 \in B_1} p(b_1) \sum_{z \in Z} N(z|b_1, b_2) \log(N(z|b_1, b_2)).
\]

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Then, for any \( q,q' \in \Delta_{d_2} \) and a fixed \( p \in \Delta_{d_1} \), we have
\[
|I(p,q) - I(p,q')| \leq |H(A_{pq}) - H(A_{pq'})| + |\langle b_p, q - q' \rangle|
\leq \frac{1}{2} \log(d_o - 1) \|A_{pq} - A_{pq'}\|_1 + \frac{\mu}{2} \|b_p\|_\infty \|q - q'\|_1
\]  
(108)
where in the second step, we used Eq. (107) and Hölder’s inequality.

Now, we note that \( \|A_{pq} - A_{pq'}\|_1 \leq \|A_p\|_{1 \to 1} \|q - q'\|_1 \), where \( \|A_p\|_{1 \to 1} \) is the induced matrix norm with respect to \( l_1 \)-norm on the domain and the co-domain of \( A_p \). This norm is equal to the maximum (absolute value) column sum of \( A_p \). Since \( A_p \) is a left stochastic matrix, all of its entries are non-negative and all of its columns sum to 1, and therefore, \( \|A_p\|_{1 \to 1} = 1 \). Similarly, we have \( 0 \leq b_p(b_2) \leq H_N^{\max} \) for all \( b_2 \in B_2 \) (see Eq. (57) for definition of \( H_N^{\max} \)). Therefore, we have \( \|b_p\|_\infty \leq H_N^{\max} \). Then, since \( h \) a monotonically increasing function, from Eq. (108), we obtain
\[
|I(p,q) - I(p,q')| \leq \frac{1}{2} \log(d_o - 1) \|q - q'\|_1 + \frac{\mu}{2} \|q - q'\|_1 + H_N^{\max} \|q - q'\|_1 = \beta_I \|q - q'\|_1
\]  
(109)
Since Eq. (109) holds for every \( p \in \Delta_{d_1} \), we have for fixed \( q,q' \in \Delta_{d_2} \)
\[
\max_{p \in \Delta_{d_1}} I(p,q) \leq \beta_I \|q - q'\|_1 \text{ and } \max_{p \in \Delta_{d_1}} I(p,q') \leq \beta_I \|q - q'\|_1
\]
Since \( I^*(q) = \max_{p \in \Delta_{d_1}} I(p,q) \), we obtain \( |I^*(q) - I^*(q')| \leq \beta_I \|q - q'\|_1 \).

**Proposition 20.** Let \( \mathcal{N} \) be any two-sender MAC with input alphabets of size \( d_1, d_2 \), and output alphabet of size \( d_o \). Suppose that that \( d_1, d_o \geq 2 \) and \( d_2 = 2 \). Then, for \( 0 < \epsilon \leq 3 \log(d_o) \), the number of iterations required by the while loop in Alg. 5 to compute the sum capacity of the MAC \( \mathcal{N} \) to a precision \( \epsilon > 0 \) is bounded above by
\[
\left[ \frac{2(\mu^2 + 4)}{(\epsilon \mu + 4) - \sqrt{16 + 8 \epsilon \mu - 4 \epsilon^2}} \right]
\]  
(110)
where \( \mu = 3 \log(d_o) \). In particular, for a precision \( 0 < \epsilon \leq 3 \) chosen independent of \( d_o \), the number of iterations is bounded above by \( O(\log(d_o) / \epsilon) \) for \( \log(d_o) \gg 1 \).

The total cost of computing the sum capacity to a precision \( \epsilon > 0 \), including the costs of performing intermediate steps such as computing \( I^* \), sorting and root-finding, is at most polynomial in \( d_1, d_o \) and \( 1/\epsilon \).

**Proof.** To compute the total number of iterations required to compute the sum capacity to a precision \( \epsilon > 0 \), we break the analysis into the following steps.

1. Number of iterations required for the while loop to converge in Alg. 5.
2. Cost of computing \( I^*(s) \) in Alg. 5 to a precision \( \epsilon_I > 0 \).
3. Cost of sorting the points \( s^{(0)}, \ldots, s^{(k)} \) in Alg. 5.
4. Cost for finding a root of \( q_i(s) \) in Alg. 5 to a precision \( \epsilon_r > 0 \).

From these, we can compute the total cost of converging to a precision \( \epsilon > 0 \).

1. We begin by computing the number of iterations required for Alg. 5 to converge to a precision \( \epsilon > 0 \). To that end, note that for obtaining Alg. 5 from Alg. 1, we used the fact that \( \|q_s - q_{s'}\|_1 = 2|s - s'| \), where given any \( s \in [0,1] \), we define \( q_s = (s,1-s) \). Therefore, we have an additional factor
of 2 in $\beta_1(2, s - s')$ in Alg. 5. Therefore, to make Alg. 5 the same as Alg. 1, we define $\beta(x) = \beta_1(2x)$. Then, from Prop. 12, we know that the number of iterations required to converge to the maximum within an error of $\epsilon > 0$ is bounded above by $[1/\delta]$, where we used the fact that $D = [0, 1]$ for the optimization. The number $\delta > 0$ is chosen such that $\beta(x) \leq \epsilon/2$ whenever $0 \leq x \leq \delta$.

In the following analysis, we measure entropy in bits. To proceed, we note that the binary entropy $h$ satisfies the inequality $h(x) \leq 2\sqrt{x(1-x)}$ [46]. Further, for any probability transition matrix $N$, the quantity $H_{\text{max}}^N$ defined in Eq. (57) is bounded above by $\log(d_o)$, where $d_o$ is the size of the output alphabet. For $x \leq 1/2$, we have $h(x) = \overline{h}(x)$, where $\overline{h}$ is the modified binary entropy function defined in Eq. (58). Then, from Eq. (56) and the definition $\beta(x) = \beta_1(2x)$, it follows that

$$\beta(x) < \mu x + 2\sqrt{x(1-x)}$$

for $x \leq 1/2$, where $\mu = 3\log(d_o)$. Then, solving the inequality $\mu x + 2\sqrt{x(1-x)} \leq \epsilon/2$, we obtain

$$x \leq \frac{\epsilon(\mu + 4) - \sqrt{16 + 8\epsilon\mu - 4\epsilon^2}}{2(\mu^2 + 4)}$$

which holds whenever $\epsilon \leq \mu$. Therefore, we choose

$$\delta = \frac{\epsilon(\mu + 4) - \sqrt{16 + 8\epsilon\mu - 4\epsilon^2}}{2(\mu^2 + 4)}$$

which is a positive number for $0 < \epsilon \leq \mu$. Then, from Prop. 12, we know that the number of iterations to converge to a precision $\epsilon > 0$ is bounded above by

$$K_{PS} = \left\lceil \frac{2(\mu^2 + 4)}{(\epsilon(\mu + 4) - \sqrt{16 + 8\epsilon\mu - 4\epsilon^2})} \right\rceil$$

where the subscript PS stands for Piyavskii-Shubert.

2. Next, we calculate the cost of computing $I^*(s)$ to a precision $\epsilon_I > 0$. This is a non-trivial cost because is obtained by solving a convex optimization problem. Specifically, $I^*(s) = \max_{p \in \Delta_d} H(A_q, p) - \langle b_q, p \rangle$, where $q_s = (s, 1 - s)$. This cost depends on the algorithm one uses to solve the optimization problem. For example, if one uses an interior point method based on the log-barrier, then one needs at most $O(d_1^3 \sqrt{d_1 + d_0} \log((d_1 + d_0)/\epsilon_I))$ flops to converge to the optimum within a precision of $\epsilon_I > 0$ (see Ch. (11.5) in Ref. [37]). We denote the cost of computing $I^*$ as $K_I$.

3. If we use quicksort to sort the numbers $s^{(0)}, \ldots, s^{(k)}$, then in the worst case, one needs $O(k^2)$ operations to perform this sorting.

4. If we compute the root to a precision $\epsilon_r > 0$ using bisection, then one needs at most $O(\log(1/\epsilon_r))$ iterations to find this root.

The total cost of sorting (including all $K_{PS}$ iterations) is bounded above by $O(K_{PS}^3)$. The number of times the root needs to be computed is bounded above by $O(K_{PS}^2)$ and the number of times the function $I^*$ is calculated in the algorithm is $O(K_{PS} \log(1/\epsilon_r))$. Since we can only compute $I^*$ to a precision $\epsilon_I$ and find the root to a precision $\epsilon_r$, we need to choose these small enough so that the total error is below $\epsilon$. In particular, if we choose $\epsilon_I = \epsilon_r$ such that $O(K_{PS}^2 \log(1/\epsilon_I)) \epsilon_I = \epsilon/2$, then using the fact that $-x \ln(x) \geq x - x^2$ for $x > 0$, we can infer that it is sufficient to choose $\epsilon_I = O(\epsilon/K_{PS}^2)$. Then running the while loop in Alg. 5 to a precision $\epsilon/2$ so that the total error is within a tolerance of $\epsilon$, we can infer that the total number of iterations needed is bounded above by

$$O(K_I K_{PS}^2 \log(K_{PS}^2/\epsilon))$$

where we used the fact that $K_I$ is at least as large as $K_{PS}$. Therefore, the total cost to converge to the optimum is at most polynomial in $d_1, d_0$, and $1/\epsilon$. \qed
Next, we obtain an upper bound on the number of iterations needed to compute the sum capacity of an arbitrary two-sender MAC to a fixed precision. The algorithm we use to analyze this is grid search explained in Sec. 5.1.2, and many of the calculations follow the ideas given in Prop. 20.

**Proposition 21.** Let \( N \) be any two-sender MAC with input alphabets of size \( d_1, d_2 \), and output alphabet of size \( d_o \). Suppose that that \( d_1, d_2, d_o \geq 2 \) with \( d_1 \geq d_2 \). Suppose that all the entropies are measured in bits and denote \( \mu = 3 \log(d_o) \). Then, for a fixed \( 0 < \epsilon \leq 3 \), the number of iterations required by the grid search in Alg. 5 to compute the sum capacity of the MAC \( N \) to a precision \( \epsilon > 0 \) is bounded above by

\[
\left( \frac{e \epsilon^2}{\mu^2} d_2 + e \right)^{\frac{4 \epsilon^2}{\mu^2} + 1}
\]

(111)

when \( \mu \geq \max\{(8/\epsilon) + 2\sqrt{(16/\epsilon) - 1}, \epsilon/2\} \).

The total cost of computing the sum capacity to a precision \( \epsilon > 0 \), including the cost of computing \( I^* \), is at most

\[
\text{poly} \left( d_1, d_2, \frac{1}{\epsilon} \right) \left( \frac{e \epsilon^2}{36} d_2 + \epsilon \right)^{\frac{90(\log(d_o))^2 + 2}{\epsilon^2}}
\]

(112)

for \( 0 < \epsilon \leq 3 \) and \( \mu \geq \max\{(16/\epsilon) + 2\sqrt{(32/\epsilon) - 1}, \epsilon/4\} \).

**Proof.** The sum capacity of a two sender MAC can be expressed as \( S(N) = \max_{q \in \Delta d_2} I^*(q) \), where \( I^* \) is defined in Eq. (54). Note that \( I^* \) is \( \beta_l \)-Lipschitz-like, where \( \beta_l \) is defined in Eq. (56). By Prop. 14, we know that the number of iterations needed for the grid search algorithm to compute the sum capacity is equal to

\[
K_{\text{GS}} = \left( N + d_2 - 1 \right) \left( d_2 - 1 \right)
\]

where \( N = \lceil 1/\delta^2 \rceil \) and \( \delta = \sup\{\delta' > 0 \mid \beta_l(\delta') \leq \epsilon/2\} \). In particular, the number \( \delta > 0 \) can be chosen such that \( \beta_l(x) \leq \epsilon/2 \) whenever \( 0 \leq x \leq \delta \) to obtain an upper bound on the number of iterations.

As noted in the proof of Prop. 20, the binary entropy \( h(x) \) (measured in bits) satisfies the inequality \( h(x) \leq 2\sqrt{x(1-x)} \) [46]. Further, the quantity \( H_N^{\text{max}} \) defined in Eq. (57) is bounded above by \( \log(d_o) \). For \( x \leq 1/2 \), we have \( h(x) = \tilde{h}(x) \), where \( \tilde{h} \) is the modified binary entropy function defined in Eq. (58). Then, from Eq. (56), it follows that

\[
\beta_l(x) < \frac{\mu}{2} x + \sqrt{x(2-x)}
\]

for \( x \leq 1/2 \), where \( \mu = 3 \log(d_o) \). Then, solving the inequality \( (\mu/2)x + \sqrt{x(2-x)} \leq \epsilon/2 \), we obtain

\[
x \leq \frac{(\epsilon\mu + 4) - 2\sqrt{4 + 2\epsilon\mu - \epsilon^2}}{\mu^2 + 4}
\]

which holds whenever \( \epsilon \leq \mu \). Therefore, we choose

\[
\delta = \frac{(\epsilon\mu + 4) - 2\sqrt{4 + 2\epsilon\mu - \epsilon^2}}{\mu^2 + 4}
\]

which is a positive number for \( 0 < \epsilon \leq \mu \). For a fixed \( 0 < \epsilon \leq \min\{16, \mu\} \) and \( \mu \geq \max\{(8/\epsilon) + 2\sqrt{(16/\epsilon) - 1}, \epsilon/2\} \), it can be verified that

\[
\frac{\mu}{\epsilon} \leq \frac{1}{\delta} \leq \frac{2\mu}{\epsilon}.
\]
Further, since \( \binom{n}{k} \leq (en/k)^k \), we can write \( \binom{N+d_2-1}{d_2-1} = \binom{N+d_2-1}{N} \leq (e(N+d_2-1)/N)^N \). Then, noting that \( 1/\delta^2 \leq N \leq 1/\delta^2 + 1 \), the number of iterations needed for grid search to converge to a precision \( 0 < \epsilon \leq \min\{\mu, 16\} \) is bounded above by

\[
K_{GS} \leq \left( \frac{e\epsilon^2}{\mu^2} d_2 + e \right)^{\frac{4e^2}{\epsilon^2} + 1}
\]

when \( \mu \geq \max\{(8/\epsilon) + 2\sqrt{(16/\epsilon) - 1}, \epsilon/2\} \). In order to avoid specifying simultaneous conditions on both \( \epsilon \) and \( \mu \), we note that \( \mu = 3\log(d_o) \geq 3 \) for \( d_o \geq 2 \). Therefore, we can simply assume \( 0 < \epsilon \leq 3 \) in the above equations.

Next, as noted in the proof of Prop. 20, the cost of computing \( I^* \) to a precision of \( \epsilon_I > 0 \) is at most \( O(d_1^3 \sqrt{d_1 + d_o} \log((d_1 + d_o)/\epsilon_I)) \). Choosing \( \epsilon_I = \epsilon/2K_{GS} \), we can ensure that after \( K_{GS} \) calls to \( I^* \), the error is at most \( \epsilon/2 \). Then, solving the grid search to a precision of \( \epsilon/2 \), we can infer that the total cost of computing the sum capacity is at most

\[
O\left(d_1^3 \sqrt{d_1 + d_o} \log\left(\frac{2(d_1 + d_o)}{\epsilon}K_{GS}\right)\right) \left( \frac{e\epsilon^2}{4\mu^2} d_2 + e \right)^{\frac{16\epsilon^2}{\mu^2} + 1}
\]

\[
\leq O\left(d_1^3 \sqrt{d_1 + d_o} \log\left(\frac{2(d_1 + d_o)}{\epsilon}\right)\right) \left( \frac{e\epsilon^2}{4\mu^2} d_2 + e \right)^{\frac{32\epsilon^2}{\mu^2} + 2}
\]

when \( 0 < \epsilon \leq 6 \) and \( \mu \geq \max\{(16/\epsilon) + 2\sqrt{(32/\epsilon) - 1}, \epsilon/4\} \). To obtain the last inequality, we used the fact that \( \log(K_{GS}) \leq K_{GS} \) for \( K_{GS} > 0 \).

\[\square\]

E Noise-free subspace MAC

We compute the sum capacity and the relaxed sum capacity for the examples constructed using the noise-free subspace MAC defined in Eq. (66). For both the examples, we consider the input alphabets \( \mathcal{A} = \{a_1, a_2\}, \mathcal{B} = \{b_1, b_2\} \) and the output alphabet \( \mathcal{Z} = \{z_1, z_2\} \).

E.1 Example 1

For the first example, the MAC \( \mathcal{N}_F^{(1)} \) has the probability transition matrix

\[
\mathcal{N}_F^{(1)} = \begin{pmatrix} 1 & 0.5 & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 & 0.5 \end{pmatrix}
\]

Let us denote the input probability distribution of sender \( A \) as \( (p, 1-p) \) corresponding to the symbols \( (a_1, a_2) \). Similarly, denote the input probability distribution of sender \( B \) as \( (q, 1-q) \) corresponding to the symbols \( (b_1, b_2) \). When the input to the MAC is a product distribution, the output probability distribution determined by the channel is

\[
p^Z(z_1) = \frac{1 + pq}{2} \quad \text{and} \quad p^Z(z_2) = \frac{1 - pq}{2}.
\]

The mutual information \( I(A, B; Z) \) between the senders and the receiver is given by

\[
I(A, B; Z) = -\frac{1 + pq}{2} \ln\left(\frac{1 + pq}{2}\right) - \frac{1 - pq}{2} \ln\left(\frac{1 - pq}{2}\right) - (1 - pq) \log(2).
\]

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We wish to compute the sum capacity

\[ S(N_F^{(1)}) = \max_{0 \leq p, q \leq 1} I(A, B; Z). \]

For \((p, q) \in \{(0, 0), (0, 1), (1, 0), (1, 1)\}\), we have \(I(A, B; Z) = 0\), and hence we can focus on the interior of the domain. For maximization over \(p\) for a fixed \(q\), by complementary slackness (Lagrangian not written here), we can simply set the derivative of \(I(A, B; Z)\) with respect to \(p\) equal to 0. The derivative is given as

\[ \frac{\partial I}{\partial p} = \frac{q}{2} \ln \left( \frac{1 - pq}{1 + pq} \right). \]

Setting this equal to zero gives \(pq = 3/5\). In this case, no maximization over \(q\) is necessary, and subsequently, we find that \(S(N_F^{(1)}) = h(4/5) - (2/5) \ln(2)\), where \(h\) is the binary entropy function.

For computing \(C(N_F^{(1)})\), we maximize over all probability distributions over the inputs. For an arbitrary input probability distribution \(p(a, b)\), we can write the mutual information between the inputs and the output as

\[ I(A, B; Z) = -\frac{1 + p(a_1, b_1)}{2} \ln \left( \frac{1 + p(a_1, b_1)}{2} \right) - \frac{1 - p(a_1, b_1)}{2} \ln \left( \frac{1 - p(a_1, b_1)}{2} \right) - (1 - p(a_1, b_1)) \ln(2). \]

In this case, we again find that the maximum is attained at \(p(a_1, b_1) = 3/5\), and subsequently, the maximum value of the mutual information is equal to \(h(4/5) - (2/5) \ln(2)\) as before. Thus, the capacity corresponding to the relaxed sum region matches the actual sum capacity.

### E.2 Example 2

For the second example, the probability transition matrix is given as

\[ N_F^{(2)} = \begin{pmatrix} 1 & 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 & 1 \end{pmatrix}. \]

As before, denote the input probability distribution of sender \(A\) as \((p, 1 - p)\) and that of sender \(B\) as \((q, 1 - q)\). Then, given an input probability distribution to the MAC \(N_F^{(2)}\), the output probability distribution is given by

\[ p(z_1) = \frac{p + q}{2} \text{ and } p(z_2) = 1 - \frac{(p + q)}{2}. \]

Then, the mutual information \(I(A, B; Z)\) between the senders and the receiver can be written as

\[ I(A, B; Z) = -\frac{p + q}{2} \ln \left( \frac{p + q}{2} \right) - \left( 1 - \frac{(p + q)}{2} \right) \ln \left( 1 - \frac{(p + q)}{2} \right) - (p + q - 2pq) \ln(2). \]

We wish to compute the sum capacity \(S(N_F^{(1)}) = \max_{0 \leq p, q \leq 1} I(A, B; Z)\). The edge cases \((p, q = 0, 1)\) will be handled separately. First, we perform the maximization over \(p\) for each fixed \(q\),

\[ I^*(q) = \max_{0 \leq p \leq 1} I(A, B; Z). \]

We look for maxima in the interior \((0, 1)\), which can be obtained through \(\frac{\partial I}{\partial p} I(A, B; Z) = 0\). Note that the output probability is \((1, 0)\) or \((0, 1)\) only when \((p, q) \in \{(0, 0), (1, 1)\}\), and therefore, the derivative of \(I(A, B; Z)\) is well-defined in the interior. This derivative is given as

\[ \frac{\partial I}{\partial p} = -\frac{1}{2} \ln \left( \frac{2 - (p + q)}{p + q} \right) - (1 - 2q) \ln(2). \]
Setting the derivative to zero, we find that
\[ p + q = 2/(\kappa_q + 1), \]
where \( \kappa_q = 2^{2-q} \). Therefore, the function \( I^*(q) \) can be written as
\[
I^*(q) = h\left(\frac{1}{\kappa_q + 1}\right) - \left(\frac{2}{\kappa_q + 1} - 2\left(\frac{2}{\kappa_q + 1} - q\right)\right) \ln(2),
\]
where \( h \) is the binary entropy. Now, to compute the sum capacity, we wish to perform the maximization \( S(\mathcal{F}_F^{(2)}) = \max_{0 \leq q \leq 1} I^*(q) \). Since \( I^*(q) \) is a continuous function of \( q \), the maximum is either attained at the interior or at the boundaries. The maxima in the interior can be found via
\[
\frac{\partial}{\partial q} I^*(q) = 0:
\]
\[
\ln\left(\frac{1}{\kappa_q}\right) = \ln(2) \left(4q - 2 - \frac{1}{\ln(2)} \left(\frac{\kappa_q + 1}{\kappa_q}\right) (q(\kappa_q + 1) - 1)\right) \implies q = \frac{1}{\kappa_q + 1}.
\]
It can be verified that \( q = 1/2 \) is a solution to the above equation, corresponding to which we have \( p = 1/2 \). Furthermore, this solution is unique. Thus, in the interior of the domain, the maximum occurs at \( p = 1/2, q = 1/2 \). Correspondingly, the maximum value of \( I(A, B; Z) \) in the interior is equal to \( 0.5 \ln(2) \).

Now, we check the edge cases. For \( p = 0 \), we need to maximize \( g(q) = I(p = 0, q) = h(q/2) - q \ln(2) \) over \( q \). At \( q = 0, 1 \), we have \( g(0) = g(1) = 0 \). Then, the maximum in the interior can be obtained by setting the derivative with respect to \( q \) to zero. This gives \( q = 2/5 \), and since \( g(2/5) \approx 0.223 < 0.5 \log(2) \), this is not the global maximum. On the other hand, for \( p = 1 \), we need to maximize \( g(q) = I(p = 1, q) = h(1 + q)/2) - (1 - q) \ln(2) \). At \( q = 0, 1 \), we have \( g(0) = g(1) = 0 \). The maximum in the interior can be obtained by setting the derivative of \( g(q) \) with respect to \( q \) equal to zero. This gives \( q = 3/5 \), and correspondingly, we have \( g(3/5) \approx 0.223 < 0.5 \ln(2) \). Since \( I(p, q) \) is symmetric under interchange of \( p \) and \( q \), the same results will be obtained when beginning with \( q = 0 \) or \( q = 1 \) and then maximizing over \( p \). Therefore, we find that the sum capacity is equal to \( S(\mathcal{F}_F^{(2)}) = 0.5 \ln(2) \).

Now, we compute the relaxed sum capacity. For this, note that output probability distribution of the MAC \( \mathcal{F}_F^{(1)} \) when using an arbitrary input probability distribution \( p(a, b) \) is given as
\[
p(z_1) = \frac{1}{2} + \frac{p(a_1, b_1) - p(a_2, b_2)}{2} \quad \text{and} \quad p(z_2) = \frac{1}{2} - \frac{(p(a_1, b_1) - p(a_2, b_2))}{2}.
\]

Correspondingly, the mutual information between the senders and the receivers is given by
\[
I(A, B; Z) = h\left(\frac{1}{2} + \frac{p(a_1, b_1) - p(a_2, b_2)}{2}\right) - (1 - p(a_1, b_1) - p(a_2, b_2)) \ln(2).
\]
It can be verified that for \( p(a_1, b_1) = p(a_2, b_2) = 1/2 \) we have \( I(A, B; Z) = \ln(2) \), which is the maximum possible value for the mutual information. Therefore, \( C(\mathcal{F}_F^{(2)}) = \ln(2) \).