MONOTONICITY-BASED RECONSTRUCTION OF EXTREME INCLUSIONS IN ELECTRICAL IMPEDANCE TOMOGRAPHY

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Abstract. The monotonicity-based approach has become one of the fundamental methods for reconstructing inclusions in the inverse problem of electrical impedance tomography. Thus far the method has not been proven to be able to handle extreme inclusions that correspond to some parts of the studied domain becoming either perfectly conducting or perfectly insulating. The main obstacle has arguably been establishing suitable monotonicity principles for the corresponding Neumann-to-Dirichlet boundary maps. In this work, we tackle this shortcoming by first giving a convergence result in the operator norm for the Neumann-to-Dirichlet map when the conductivity coefficient decays to zero and/or grows to infinity in some given parts of the domain. This allows passing the necessary monotonicity principles to the limiting case. Subsequently, we show how the monotonicity method generalizes to the definite case of reconstructing either perfectly conducting or perfectly insulating inclusions, as well as to the indefinite case where the perturbed conductivity can take any values between, and including, zero and infinity.

Keywords: electrical impedance tomography, perfectly insulating, perfectly conducting, monotonicity method.

2010 Mathematics Subject Classification: 35R30, 35R05, 47H05.

1. Introduction

Electrical impedance tomography (EIT) is an imaging technology for retrieving information about the interior electrical conductivity of a physical object through measurements of electric current and voltage at electrodes placed on its boundary. The corresponding mathematical inverse problem is known as Calderón’s problem or the inverse conductivity problem. In mathematical terms, the inverse conductivity problem consists of reconstructing a coefficient inside a divergence term of a second order elliptic partial differential equation, based on information about the Neumann-to-Dirichlet (ND) map that encodes all possible Cauchy data (pairs of current densities and voltages) at the domain boundary. For more information on the inverse conductivity problem, we refer to [2, 1, 39, 9] and the references therein.

In this paper we consider the related, easier, problem of inclusion detection with the goal of retrieving information on an unknown conductivity perturbation in a known background. Typically, inclusion detection algorithms aim at reconstructing the (outer shape of the) support of the perturbation, often referred to as inclusions. We distinguish between definite inclusions that correspond to conductivity perturbations that are either only positive or only negative and indefinite inclusions defined by perturbations that may take both positive and negative values.

In the standard setting of EIT, where the conductivity is bounded away from zero and infinity, there are several approaches to solving the inclusion detection problem. Among some of the popular techniques are the factorization method [7, 8, 29, 18, 19, 17, 24, 34, 33] and the enclosure method [26, 27, 5]. A more recent approach, which we consider in this article, is the monotonicity method [38, 21, 22] that relies on the monotonicity properties of the map \( \sigma \mapsto \Lambda(\sigma) \), where \( \sigma \) represents a conductivity coefficient and \( \Lambda(\sigma) \) the corresponding ND map. The monotonicity method is flexible in the sense that it directly handles both definite and indefinite inclusions, even when only having access to a part of the object boundary, with non-homogeneous background conductivity, with inclusions having several connected components, and without a priori information about the locations of the positive and negative inclusions. In the definite case the method gives
a criterion on whether a given open set is enclosed by the (outer shape of the) inclusions or not; in the indefinite case the roles are reversed and the criterion is on whether a given set encloses the inclusions or not. By applying either of these criteria to various test sets, the outer shape of the inclusions can be reconstructed. Furthermore, due to the simplicity of the method, it adapts in a natural way to more complicated and realistic electrode models allowing a straightforward numerical implementation [14, 15, 23, 11]. The ideas behind the monotonicity method have also been used to obtain stability estimates from finitely many measurements [20], and the method has very recently been applied to the reconstruction of piecewise constant conductivities with a layered structure [12].

For extreme inclusions, meaning that the conductivity formally takes the value zero (perfectly insulating) or infinity (perfectly conducting) in some parts of the domain, there exist variants of both the factorization method (e.g. [6, Section 4.3.1] and [37, Section 2.2.2]) and the enclosure method [26, 4]. Furthermore, there are a number of reconstruction methods designed specifically for extreme inclusions; see [10, 36, 32, 31, 3, 35] for information on some of them. A common feature of all these methods is that they have not been proven to be functional if the examined body is simultaneously contaminated by both perfectly conducting and perfectly insulating inclusions.

The monotonicity method has not previously been generalized to allow extreme inclusions inside the domain of interest; fixing this flaw is the main topic of this paper. The main obstacle for such a generalization has arguably been establishing suitable monotonicity principles for the ND map in the presence of extreme inclusions. To this end, we first show that any ND map corresponding to extreme inclusions can be reached in the natural operator topology as a limit of a sequence of standard ND maps when the conductivity coefficient decays to zero and/or grows to infinity in the appropriate parts of the domain. In fact, the convergence rate equals the square root of the truncation parameter $\epsilon > 0$ that via multiplication/division controls the convergence of the conductivity to zero/infinity. Utilizing the standard monotonicity relations for the ND maps corresponding to the truncated conductivities, we show that both definite and indefinite inclusions can be reconstructed with the monotonicity method in the presence of extreme inclusions, that is, we obtain natural generalizations of the results in [22, 15]. In particular, our result on the indefinite case covers a wide class of inclusions: the perturbed conductivity can simultaneously have parts that are perfectly insulating, parts that are perfectly conducting, as well as other perturbed parts bounded away from zero and infinity.

This article is organized as follows. Section 2 gives a brief introduction to the mathematical formulation of the forward problem of EIT in the presence of extreme inclusions. Section 3 lists the main results of the paper, while Sections 4–6 are dedicated to the proofs of these main results. We also include an Appendix A on some generic monotonicity estimates for ND maps with various combinations of extreme inclusions.

To be somewhat more precise, Section 4 proves the convergence of the electric potential corresponding to a conductivity bounded away from zero and infinity toward its counterpart with extreme inclusions as the conductivity decays to zero and grows to infinity in the appropriate parts of the domain. The actual result is stated in Theorem 3.1, and it also covers the convergence of the associated ND maps. Sections 5 and 6 prove the functionality of the monotonicity method in a framework that also allows extreme inclusions. In the definite case, the result for the linearized version of the method is given in Theorem 3.6 and for the nonlinear version in Theorem 3.7. Our main result on the characterization of indefinite inclusions, incorporating the extreme cases, is given as Theorem 3.10.

1.1. Notational remarks. Let $\mathscr{L}(X, Y)$ denote the space of bounded linear operators between Banach spaces $X$ and $Y$, with $\mathscr{L}(X) := \mathscr{L}(X, X)$. For a Hilbert space $X$ and self-adjoint operators $A, B \in \mathscr{L}(X)$, we write $A \geq B$ if $A - B$ is positive semidefinite.

We denote the interior of a set $E \subseteq \mathbb{R}^d$ by $E^\circ$ and its characteristic function by $\chi_E$. For a real measurable essentially bounded function $\varsigma$, we write $\inf_{E}(\varsigma)$ and $\sup_{E}(\varsigma)$ in place of $\text{ess inf}(\varsigma|_{E})$ and $\text{ess sup}(\varsigma|_{E})$, and avoid the subscript notation if no restriction is used.

$K$ will denote a generic positive constant.
2. The continuum model

Assume $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is a bounded Lipschitz domain with $\mathbb{R}^d \setminus \overline{\Omega}$ connected. Let $\Gamma \subseteq \partial \Omega$ be a relatively open non-empty boundary piece; $(\cdot, \cdot)$ denotes the associated inner product on $L^2(\Gamma)$. For any measurable set $\Omega \subseteq \Omega$, we define

$$L_+^\infty(\Omega) := \{ \varsigma \in L^\infty(\Omega; \mathbb{R}) \mid \inf(\varsigma) > 0 \}.$$  

If $\tilde{\Omega}$ is a Lipschitz domain with $\Gamma \subseteq \partial \tilde{\Omega}$, we define the “$\Gamma$-mean free spaces”:

$$H^1_0(\tilde{\Omega}) := \{ w \in H^1(\tilde{\Omega}) \mid (1, w|_\Gamma) = 0 \},$$
$$L^2_0(\Gamma) := \{ g \in L^2(\Gamma) \mid (1, g) = 0 \}.$$  

The family of admissible inclusions is introduced as

$$\mathcal{A} := \{ C \subseteq \Omega \mid C \text{ is the closure of an open set,}$$

has connected complement, 
and has Lipschitz boundary $\partial C \}. $$

We often use the notation $C = \mathcal{A}(C_0, C_\infty)$ to indicate that $C := C_0 \cup C_\infty$ where $C_0, C_\infty \in \mathcal{A}$ and $C_0 \cap C_\infty = \emptyset$, i.e. that $C$ is the union of an admissible pair of non-overlapping inclusions. Moreover, we note that $C_0$ and $C_\infty$ can consist of several connected components.

In the following, $C_0$ will represent a part of the domain that is perfectly insulating (zero conductivity) and $C_\infty$ will represent a part of the domain that is perfectly conducting (infinite conductivity). For $\varsigma \in L_+^\infty(\Omega)$, $\sigma = \sigma(\varsigma, C_0, C_\infty)$ denotes the non-negative conductivity

$$\sigma := \begin{cases} 
\varsigma & \text{in } \Omega \setminus C, \\
0 & \text{in } C_0, \\
\infty & \text{in } C_\infty.
\end{cases}$$

Throughout this work we are interested in the following elliptic boundary value problem:

$$-\nabla \cdot (\sigma \nabla u) = 0 \text{ in } \Omega \setminus C,$$
$$\nu \cdot \sigma \nabla u = \begin{cases} 
f & \text{on } \Gamma, \\
0 & \text{on } \partial(\Omega \setminus C_0) \setminus \Gamma,
\end{cases}$$
$$\nabla u = 0 \text{ in } C_\infty,$$

where $\nu$ is the outer unit normal to $\Omega \setminus C$ and $f \in L^2_0(\Gamma)$ represents an applied current density on $\Gamma$, while $u$ is the corresponding electric potential in $\Omega \setminus C_0$.

Formally, we write the above problem as

$$-\nabla \cdot (\sigma \nabla u) = 0 \text{ in } \Omega, \quad \nu \cdot \sigma \nabla u = \begin{cases} 
f & \text{on } \Gamma, \\
0 & \text{on } \partial \Omega \setminus \Gamma,
\end{cases} \tag{2.1}$$

where the values 0 and $\infty$ of $\sigma$ are to be interpreted via the appropriate boundary conditions for $C_0$ and $C_\infty$, and $u$ is only defined in $\Omega \setminus C_0$. If $C_0 = C_\infty = \emptyset$, then (2.1) reduces to the standard conductivity equation for a conductivity coefficient in $L_+^\infty(\Omega)$.

We may equip $H^1_0(\Omega \setminus C_0)$ with either of the inner products

$$\langle v, w \rangle_\varsigma := \int_{\Omega \setminus C_0} \varsigma \nabla v \cdot \nabla w \, dx, \tag{2.2}$$
$$\langle v, w \rangle* := \langle \nabla v, \nabla w \rangle_{L^2(\Omega; C_0; C_\infty)}.$$  

The corresponding norms $\| \cdot \|_\varsigma$ and $\| \cdot \|*$ are equivalent to the usual $H^1$-norm on $H^1_0(\Omega \setminus C_0)$, due to a Poincaré-type inequality related to the zero-mean condition on $\Gamma$:

$$\frac{1}{\sup(\varsigma)^{1/2}} \| v \|_\varsigma \leq \| v \|* \leq \| v \|_{H^1(\Omega \setminus C_0)} \leq K \| v \|* \leq \frac{K}{\inf(\varsigma)^{1/2}} \| v \|_\varsigma, \quad \forall v \in H^1_0(\Omega \setminus C_0).$$

We will often make use of the above equivalences, and will do so without explicitly mentioning it.
In order to rigorously define the variational problem associated to (2.1), we introduce the function space
\[ \mathcal{H}_\sigma := \{ w \in H^1_0(\Omega \setminus C_0) \mid \nabla w = 0 \text{ in } C^\infty_\sigma \}. \]
It follows straightforwardly that \( \mathcal{H}_\sigma \) is a closed subspace (thus itself a Hilbert space) of \( H^1_0(\Omega \setminus C_0) \) when equipped with either of the above inner products. In particular, we denote by \( P(\varsigma, C_0, C_\infty) \) the orthogonal projection of \( H^1_0(\Omega \setminus C_0) \) onto \( \mathcal{H}_\sigma \) in the inner product \( \langle \cdot, \cdot \rangle_\varsigma \) (cf. (2.2)), while \( P_L(\varsigma, C_0, C_\infty) \) is the orthogonal projection onto the orthogonal complement \( \mathcal{H}_\sigma^\perp \).

Employing the Lax–Milgram lemma, it easily follows that (2.1) has a unique weak solution
\[ u = u^\gamma_j \in \mathcal{H}_\sigma \]
satisfying
\[ \int_{\Omega \setminus C_0} \sigma \nabla u \cdot \nabla v \, dx = \langle f, v \rangle, \quad \forall v \in \mathcal{H}_\sigma, \]
for conductivity coefficients with other choices of background conductivity, perfectly conducting and non-negative conductivity \( \sigma \). Similar notation will also be used for conductivity coefficients with other choices of background conductivity, perfectly conducting parts, and perfectly insulating parts. In particular, for \( \varsigma \in L^\infty_+(\Omega) \) we note that \( \varsigma = \sigma(\varsigma, \emptyset, \emptyset) \), and therefore \( u^\gamma_j \in H^1_0(\Omega) \) satisfies
\[ \int_{\Omega} \varsigma \nabla u^\gamma_j \cdot \nabla v \, dx = \langle f, v \rangle, \quad \forall v \in H^1_0(\Omega). \]

Another useful characterization of \( u^\gamma_j \) relates it to the inclusion-free potential \( u^\gamma_0 \).

**Proposition 2.2.** Let \( C = \mathcal{A}(C_0, C_\infty) \), \( \sigma = \sigma(\varsigma, C_0, C_\infty) \), and \( P(\varsigma, C_0, C_\infty) \). For any \( f \in L^2_0(\Gamma) \), \( u^\gamma_j \) can be uniquely characterized as follows:
\[ u^\gamma_j = P(u^\gamma_0|_{\Omega \setminus C_0} - w), \]
where \( w \) is the unique solution in \( H^1_0(\Omega \setminus C_0) \) to
\[ -\nabla \cdot (\varsigma \nabla w) = 0 \text{ in } \Omega \setminus C_0, \]
\[ \nu \cdot \nabla w = \nu \cdot (\varsigma \nabla u^\gamma_0|_{\Omega \setminus C_0} \text{ on } \partial C_0, \]
\[ \nu \cdot \varsigma \nabla w = 0 \text{ on } \partial\Omega. \]

**Proof.** It is clear from the boundary value problems defining \( u^\gamma_0 \) and \( w \) that \( u^\gamma_j = u^\gamma_j|_{\Omega \setminus C_0} - w \) where \( \hat{\sigma} = \sigma(\varsigma, C_0, \emptyset) \). Recall that \( P \) is an orthogonal projection and hence self-adjoint with respect to the inner product (2.2). Due to the weak formulation for \( u^\gamma_j \), the following holds for all \( v = Pv \in \mathcal{H}_\sigma(C_0, C_\infty) \):
\[ \langle f, v \rangle = \int_{\Omega \setminus C_0} \varsigma \nabla u^\gamma_j \cdot \nabla P v \, dx = \int_{\Omega \setminus C_0} \varsigma \nabla P u^\gamma_0 \cdot \nabla P v \, dx. \]
By the unique solvability of the variational problem (2.3), we conclude that \( u^\gamma_j = Pu^\gamma_0 \). \( \square \)

Finally, we define the local Neumann-to-Dirichlet (ND) map as \( \Lambda(\sigma) : f \mapsto u^\gamma_j|_{\Gamma} \), which is a compact self-adjoint operator in \( \mathcal{L}(L^2_0(\Gamma)) \).

### 3. Main results

Our first result is on retrieving the electric potential, and thereby also the ND map, in the case of perfectly insulating and perfectly conducting inclusions via a limit of a sequence of truncated conductivities that belong to \( L^\infty_+(\Omega) \). This theorem will be used for the proofs of all the other main results.
Theorem 3.1. Let $C = \mathcal{A}(C_0, C_{\infty})$, $\sigma = \sigma(\varsigma, C_0, C_{\infty})$, and define the $\epsilon$-truncated version of $\sigma$, with $\epsilon > 0$, by
\[
\sigma_\epsilon := \begin{cases} 
\varsigma & \text{in } \Omega \setminus C, \\
\epsilon \varsigma & \text{in } C_0, \\
\epsilon^{-1} \varsigma & \text{in } C_{\infty}.
\end{cases}
\]

Then the following estimate holds
\[
\|u_\sigma^\epsilon - u_\sigma^0\|_{H^1(\Omega \setminus C_0)} \leq K \epsilon^{1/2} \|f\|_{L^2(\Gamma)},
\]
with $K > 0$ independent of $f$ and $\epsilon$. As a direct consequence
\[
\|\Lambda(\sigma) - \Lambda(\sigma)\|_{L^2(\Gamma)} \leq K \epsilon^{1/2}.
\]

Remark 3.2. For the sake of keeping the presentation of Theorem 3.1 and its proof clean, we have not included the precise dependence of the constant $K$ on $\varsigma$. Actually, $K$ only depends on $\varsigma$ as a multiple of a positive power of $\sup(\varsigma)$ and a negative power of $\inf(\varsigma)$. Furthermore, $K$ depends on $\Omega$, $C_0$, and $C_{\infty}$ via the constants that appear in the corresponding trace theorems and Poincaré inequalities.

As a side note for Theorem 3.1, while there exist several $H^1$-extensions of $u_\sigma^0$ into $C_0$, there is a natural extension that coincides with the limiting behavior of $u_\sigma^\epsilon|_{C_0}$ for $\epsilon \to 0$.

Corollary 3.3. Let $C$, $\sigma$, and $\sigma_\epsilon$ be as in Theorem 3.1. Consider the $H^1$-extension operator $E = E(\varsigma, C_0) : H^1(\Omega \setminus C_0) \to H^1(\Omega)$, defined for $w \in H^1(\Omega \setminus C_0)$ as
\[
Ew := \begin{cases} 
w & \text{in } \Omega \setminus C_0, \\
\tilde{w} & \text{in } C_0,
\end{cases}
\]
where $\tilde{w}$ is the unique solution in $H^1(C_0^0)$ to the Dirichlet problem
\[
-\nabla \cdot (\varsigma \nabla \tilde{w}) = 0 \text{ in } C_0^0, \quad \tilde{w} = w \text{ on } \partial C_0.
\]
Then it holds
\[
\|u_\sigma^\epsilon - Eu_\sigma^0\|_{H^1(\Omega)} \leq K \epsilon^{1/2} \|f\|_{L^2(\Gamma)}.
\]

Proof. Since $\nabla \cdot (\varsigma \nabla u_\sigma^0) = \nabla \cdot (\sigma_\epsilon \nabla u_\sigma^0) = 0$ in $C_0^0$, the restriction $v := (u_\sigma^\epsilon - Eu_\sigma^0)|_{C_0}$ satisfies the Dirichlet problem
\[
-\nabla \cdot (\varsigma \nabla v) = 0 \text{ in } C_0^0, \quad v = u_\sigma^\epsilon - u_\sigma^0 \text{ on } \partial C_0.
\]
Hence, by the continuous dependence on the Dirichlet boundary value, the trace theorem in $\Omega \setminus C_0$, and the result of Theorem 3.1, we have
\[
\|v\|_{H^1(C_0^0)} \leq K \|(u_\sigma^\epsilon - u_\sigma^0)\|_{H^{1/2}(\partial C_0)} \leq K \|u_\sigma^\epsilon - u_\sigma^0\|_{H^1(\Omega \setminus C_0)} \leq K \epsilon^{1/2} \|f\|_{L^2(\Gamma)}.
\]
This estimate, combined with the result of Theorem 3.1, concludes the proof.

Next we will consider the reconstruction of inclusions via the monotonicity method. Before discussing the more general indefinite case, we first assume $D \in \mathcal{A}$ represents either perfectly insulating or perfectly conducting inclusions that we seek to recover. These partial results are stated as separate theorems because they allow to recover $D$ based on positive semidefiniteness tests in comparison to conductivities that belong to $L^+_\infty(\Omega)$ (see Theorem 3.7), which is straightforward from the standpoint of a numerical implementation. We also introduce a linearized version of the reconstruction method that only relies on the ND map and its Fréchet derivative at the background conductivity, hence leading to a very fast algorithm.

Let $\gamma_0 \in L^\infty(\Omega)$ denote the so-called background conductivity and let $C \in \mathcal{A}$. We use the following notation:

(i) $D\Lambda(\gamma_0, \cdot) \in \mathcal{L}(L^\infty(\Omega; \mathbb{R}), \mathcal{L}(L^2_0(\Gamma)))$ denotes the Fréchet derivative of $\Lambda|_{L^\infty(\Omega)}$ evaluated at $\gamma_0$, characterized by the well-known quadratic formula
\[
(D\Lambda(\gamma_0; \eta)f, f) = -\int_\Omega \eta|\nabla u_\gamma^0|^2 \, dx, \quad \forall \eta \in L^\infty(\Omega; \mathbb{R}), \quad f \in L^2_0(\Gamma).
\]
(ii) \( \Lambda_0(C) := \Lambda(\sigma(\gamma_0, C, \emptyset)) \) denotes the ND map for the conductivity characterized by perfectly insulating inclusions in \( C \) and \( \gamma_0 \) in \( \Omega \). 
(iii) \( \Lambda_\infty(C) := \Lambda(\sigma(\gamma_0, \emptyset, C)) \) denotes the ND map for the conductivity characterized by perfectly conducting inclusions in \( C \) and \( \gamma_0 \) in \( \Omega \). 

We will furthermore require that \( \gamma_0 \) satisfies a unique continuation principle.

**Definition 3.4.** Suppose \( U \subseteq \overline{\Omega} \) is a relatively open and connected set that intersects \( \Gamma \). We say that \( \varsigma \in L^\infty_+ (\Omega) \) satisfies the weak unique continuation principle (UCP) in \( U \) for the conductivity equation, provided that only the trivial solution of 
\[
-\nabla \cdot (\varsigma \nabla v) = 0 \text{ in } U^\circ
\]
can be identically zero in a non-empty open subset of \( U \), and likewise only the trivial solution has vanishing Cauchy data on \( \partial U \cap \Gamma \). If \( U = \overline{\Omega} \) we simply say that \( \varsigma \) satisfies the UCP.

**Remark 3.5.** As a simple example, a piecewise analytic conductivity coefficient \( \varsigma \in L^\infty_+ (\Omega) \) satisfies the UCP; see, e.g., [30] for the case \( \partial \Omega \in \mathcal{C}^\infty \).

We are now ready to introduce the linearized and nonlinear reconstruction methods for recovering \( D \) based on boundary data given either as \( \Lambda_0(D) \) or \( \Lambda_\infty(D) \).

**Theorem 3.6.** Let \( \gamma_0 \in L^\infty_+ (\Omega) \) satisfy the UCP, let \( 0 < \beta \leq \inf(\gamma_0) \), and let \( B \subseteq \Omega \) be a non-empty open set. For \( D \in \mathcal{A} \), it holds
\[
B \subseteq D \quad \text{if and only if} \quad \Lambda_0(D) \geq \Lambda(\gamma_0) - \beta \Lambda(\gamma_0; \chi_B)
\quad \text{if and only if} \quad \Lambda(\gamma_0) + \beta \Lambda(\gamma_0; \chi_B) \geq \Lambda_\infty(D).
\]

**Theorem 3.7.** Let \( \gamma_0 \in L^\infty_+ (\Omega) \) satisfy the UCP and let \( B \subseteq \Omega \) be a non-empty open set. For \( D \in \mathcal{A} \), it holds:
\[
\text{If } 0 < \beta < \inf(\gamma_0), \text{ then } B \subseteq D \quad \text{if and only if} \quad \Lambda_0(D) \geq \Lambda(\gamma_0 - \beta \chi_B).
\]
\[
\text{If } \beta > 0, \text{ then } B \subseteq D \quad \text{if and only if} \quad \Lambda(\gamma_0 + \beta \chi_B) \geq \Lambda_\infty(D).
\]

To summarize, if \( B \) is the set of open balls in \( \Omega \), for the appropriate values of \( \beta \) in Theorem 3.6 and Theorem 3.7, we have
\[
D^\circ = \cup \{ B \in B \mid \Lambda_0(D) \geq \Lambda(\gamma_0) - \beta \Lambda(\gamma_0; \chi_B) \}
= \cup \{ B \in B \mid \Lambda_0(D) \geq \Lambda(\gamma_0 - \beta \chi_B) \}
= \cup \{ B \in B \mid \Lambda(\gamma_0) + \beta \Lambda(\gamma_0; \chi_B) \geq \Lambda_\infty(D) \}
= \cup \{ B \in B \mid \Lambda(\gamma_0 + \beta \chi_B) \geq \Lambda_\infty(D) \}.
\]

Based on these characterizations, we readily obtain \( D \) as the closure of \( D^\circ \), if necessary. In a numerical implementation, one may wish to replace “balls” by “pixels/voxels/triangulation” or other appropriate open sets.

Finally, we consider the setting where the conductivity may be both larger and smaller than the background conductivity \( \gamma_0 \), and it may also have parts that are perfectly insulating and parts that are perfectly conducting.

**Definition 3.8 (indefinite inclusion).** Suppose a set \( D \subseteq \Omega \) can be decomposed as
\[
D = D^+ \cup D^-, \quad D^- = D^- F \cup D_0, \quad \text{and} \quad D^+ = D^+ F \cup D_\infty,
\]
where we assume:
(i) \( D \in \mathcal{A} \).
(ii) \( D_0, D_\infty \in \mathcal{A} \) and \( D_0^\circ \cap D_\infty^\circ = \emptyset \).
(iii) \( D_0^\circ \) are measurable and mutually disjoint with the set \( D_0 \cup D_\infty \).
(iv) For every open neighbourhood \( W \) of \( x \in \partial D \), there exists a relatively open set \( V \subseteq D \) that intersects \( \partial D \) and either \( V \subseteq D^- \cap W \) or \( V \subseteq D^+ \cap W \).
Acknowledgments. This work was supported by the Academy of Finland (decision 312124) and the Aalto Science Institute (AScI). The authors thank Antti Hannukainen (Aalto University) and Arne Jensen (Aalborg University) for useful discussions.

4. PROOF OF THEOREM 3.1

For sake of brevity, we write $u = u^0$ and $u_\varepsilon = u^\varepsilon$. Furthermore, we use the shorthand notation $\mathcal{H}_\varepsilon = \mathcal{H}_\varepsilon(C_0, C_\infty)$ for the function space related to $u$; the space related to $u_\varepsilon$ is $H_\varepsilon^1(\Omega)$ since the corresponding conductivity is bounded away from zero and infinity in $\Omega$. As in (2.2), the inner product $\langle v, w \rangle_\varepsilon$ is defined via an integral over $\Omega \setminus C_0$. To more easily distinguish between the weak formulations of $u$ and $u_\varepsilon$, we write down (2.3) specifically for $u_\varepsilon$ here:

$$\int_\Omega \sigma_\varepsilon \nabla u_\varepsilon \cdot \nabla v \, dx = \langle f, v \rangle_{\varepsilon}, \quad \forall v \in H_\varepsilon^1(\Omega). \quad (4.1)$$

First we give some estimates related to $u_\varepsilon$ in terms of the $L^2(\Gamma)$-norm of $f$. Using (4.1), the “Cauchy inequality with $\varepsilon$” with $\alpha > 0$ taking the role of $\varepsilon$, and the trace theorem in $\Omega \setminus C$ leads to

$$\int_\Omega \sigma_\varepsilon |\nabla u_\varepsilon|^2 \, dx = \langle f, u_\varepsilon \rangle_{\varepsilon} \leq \alpha \|f\|_{L^2(\Gamma)}^2 + \frac{1}{4\alpha} \|u_\varepsilon \|_{H_\varepsilon^1(\partial\Omega)}^2 \lesssim \frac{K}{4\alpha} \int_{\Omega \setminus C} |\nabla u_\varepsilon|^2 \, dx.$$
Choosing $\alpha = K$ and restructuring gives
\[ \frac{3}{4} \int_{\Omega \setminus C} \varsigma |\nabla u_\epsilon|^2 \, dx + \epsilon^{-1} \int_{C_\infty} \varsigma |\nabla u_\epsilon|^2 \, dx + \epsilon \int_{C_0} \varsigma |\nabla u_\epsilon|^2 \, dx \leq K \|f\|_{L^2(\Gamma)}^2. \]

As direct consequences,
\[ \|\nabla u_\epsilon\|_{L^2(C_{\infty};C^4)} \leq \epsilon^{1/2} K \|f\|_{L^2(\Gamma)}, \tag{4.2} \]
\[ \|\epsilon^{1/2} \varsigma \nabla u_\epsilon\|_{L^2(C_0;\mathbb{C}^4)} \leq K \|f\|_{L^2(\Gamma)}, \tag{4.3} \]
where $K > 0$ remains a generic constant independent of $f$ and $\epsilon$. These bounds will be used later to finalize the proof.

We proceed now to prove the first part of Theorem 3.1. Let $P = P(\varsigma, C_0, C_\infty)$ and $P_\perp = P_\perp(\varsigma, C_0, C_\infty)$. Since the projections $P$ and $P_\perp$ are self-adjoint and $u \in H_\varsigma$, Cauchy–Schwarz’ inequality yields
\[ \|u - u_\epsilon\|_\varsigma^2 = \|P(u - u_\epsilon)\|_\varsigma^2 + \|P_\perp(u - u_\epsilon)\|_\varsigma^2 \leq \langle u - u_\epsilon, P(u - u_\epsilon) \rangle_\varsigma + \|P_\perp u_\epsilon\|_\varsigma \|u - u_\epsilon\|_\varsigma, \tag{4.4} \]
where we have suppressed the notation indicating that the projections are applied to the restriction of $u_\epsilon$ onto $\Omega \setminus C_0$. The rest of the proof will focus on estimating the two terms on the right-hand side of (4.4), with the aim of finally obtaining the desired convergence result by dividing with $\|u - u_\epsilon\|_\varsigma$.

### 4.1. Estimating the First Term in (4.4)

By using the fact $\nabla P(u - u_\epsilon) = 0$ in $C_0^\omega$, it follows that
\[ \langle u - u_\epsilon, P(u - u_\epsilon) \rangle_\varsigma = \int_{\Omega \setminus C} \varsigma \nabla u \cdot \nabla P(u - u_\epsilon) \, dx - \int_{\Omega \setminus C_0} \sigma \nabla u_\epsilon \cdot \nabla P(u - u_\epsilon) \, dx \]
\[ = \int_{C_0} \epsilon \varsigma \nabla u_\epsilon \cdot \nabla P(u - u_\epsilon) \, dx, \tag{4.5} \]
where the second step is based on the weak formulations (2.3) and (4.1) for $u$ and $u_\epsilon$, respectively.

We adopt the notation that $\varsigma \nabla u_\epsilon$ equals $(\varsigma \nabla u_\epsilon)_\text{int}$ in $C_0^\omega$ and $(\varsigma \nabla u_\epsilon)_\text{ext}$ in $\Omega \setminus C_0$ to respect the jump-relation $\nu \cdot (\varsigma \nabla u_\epsilon)_\text{ext} = \nu \cdot (\varsigma \nabla u_\epsilon)_\text{int}$ on $\partial C_0$ induced by (2.1). Since $u_\epsilon$ satisfies the conductivity equation in $C_0^\omega$, we may rewrite (4.5) as
\[ \langle u - u_\epsilon, P(u - u_\epsilon) \rangle_\varsigma = \epsilon^{1/2} \langle \epsilon^{1/2} \nu \cdot (\varsigma \nabla u_\epsilon)_\text{int} |_{\partial C_0}, P(u - u_\epsilon) |_{\partial C_0} \rangle_{\partial C_0}, \tag{4.6} \]
where $\langle \cdot, \cdot \rangle_{\partial C_0} = \langle \cdot, \cdot \rangle_{H^{-1/2}(\partial C_0), H^{1/2}(\partial C_0)}$ is the dual bracket on $\partial C_0$. Now we can straightforwardly estimate (4.6) using the boundedness of the dual bracket, the trace theorem in $\Omega \setminus C_0$, and the boundedness of $P$,
\[ \langle u - u_\epsilon, P(u - u_\epsilon) \rangle_\varsigma \leq \epsilon^{1/2} \|\epsilon^{1/2} \nu \cdot (\varsigma \nabla u_\epsilon)_\text{int} |_{\partial C_0} \|_{H^{-1/2}(\partial C_0)} \| P(u - u_\epsilon) |_{\partial C_0} \|_{H^{1/2}(\partial C_0)} \]
\[ \leq K \epsilon^{1/2} \|\epsilon^{1/2} \nu \cdot (\varsigma \nabla u_\epsilon)_\text{int} |_{\partial C_0} \|_{H^{-1/2}(\partial C_0)} \|u - u_\epsilon\|_\varsigma. \tag{4.7} \]

Taking the normal component is a bounded map from $H_{\text{div}}(C_0^\omega)$ to $H^{-1/2}(\partial C_0)$. Hence, the equality $\nabla \cdot (\epsilon^{1/2} \varsigma \nabla u_\epsilon) = \nabla \cdot (\epsilon \varsigma \nabla u_\epsilon) = 0$ in $C_0^\omega$ implies with (4.3) that
\[ \|\epsilon^{1/2} \nu \cdot (\varsigma \nabla u_\epsilon)_\text{int} |_{\partial C_0} \|_{H^{-1/2}(\partial C_0)} \leq K \left( \|\epsilon^{1/2} \varsigma \nabla u_\epsilon\|_{L^2(C_\infty;\mathbb{C}^4)}^2 + \|\nabla \cdot (\epsilon^{1/2} \varsigma \nabla u_\epsilon)\|_{L^2(C_0)}^2 \right)^{1/2} \]
\[ \leq K \|f\|_{L^2(\Gamma)}. \tag{4.8} \]
Combining (4.7) and (4.8) handles the first term of (4.4).
4.2. Estimating the second term in (4.4). The following lemma gives the needed estimate for $\|P_\perp u_\epsilon\|_\varsigma$ in the second term on the right-hand side of (4.4). However, the lemma will also be useful in proving some other results in what follows.

**Lemma 4.1.** Let $C = \mathcal{A}(C_0, C_\infty)$, $\varsigma \in L^\infty(\Omega)$, and $P_\perp = P_\perp(\varsigma, C_0, C_\infty)$. If $v \in H^1_0(\Omega \setminus C_0)$ satisfies $\nabla \cdot (\varsigma \nabla v) = 0$ in $\Omega \setminus C$, then there is a constant $K > 0$ (independent of $v$) such that

$$\int_{\Omega \setminus C_0} \varsigma |\nabla P_\perp v|^2 \, dx \leq K \int_{\partial C_\infty} |\nabla v|^2 \, dx.$$ 

**Proof.** Let $v^{C_\infty}$ denote the piecewise constant function in $H^1(C_\infty^\infty)$ that in each connected component $C_i$ of $C_\infty$ equals the mean value of the corresponding restriction $v|_{C_i}$. We define $w$ as the unique solution in $\mathcal{H}_\epsilon = \mathcal{H}_\epsilon(C_0, C_\infty)$ to the auxiliary Dirichlet problem

$$-\nabla \cdot (\varsigma \nabla w) = 0 \text{ in } \Omega \setminus C,$$

$$w = v \text{ on } \partial \Omega \cup \partial C_0,$$

$$w = v^{C_\infty} \text{ in } C_\infty^\infty.$$

It easily follows that $v - w$ satisfies a similar boundary value problem in $\Omega \setminus C$ with the only non-trivial boundary condition being a Dirichlet condition on $\partial C_\infty$. Using the continuous dependence on the Dirichlet condition for an elliptic equation, the trace theorem in $C_\infty$, and the Poincaré inequality in each connected component of $C_\infty$, we obtain the bound

$$\|v - w\|_{H^1(\Omega \setminus C_0)}^2 = \|v - w\|_{H^1(\Omega \setminus C)}^2 + \|v - v^{C_\infty}\|_{H^1(C_\infty^\infty)}^2$$

$$\leq K \left( \|v - v^{C_\infty}\|_{\partial C_\infty}^2 |_{H^{1/2}(\partial C_\infty)}^2 + \|v - v^{C_\infty}\|_{H^1(C_\infty^\infty)}^2 \right)$$

$$\leq K \|v - v^{C_\infty}\|_{H^1(C_\infty^\infty)}^2.$$ 

The projection theorem and (4.9) finally yield

$$\|P_\perp v\|_\varsigma = \inf_{\hat{v} \in \mathcal{H}_\epsilon} \|v - \hat{v}\|_\varsigma \leq \|v - w\|_\varsigma \leq K \|\nabla v\|_{L^2(C_\infty^\infty)}$$

hence concluding the proof.

Applying Lemma 4.1 to $u_\epsilon|_{\Omega \setminus C_0}$ and subsequently using (4.2) gives the bound

$$\|P_\perp u_\epsilon\|_\varsigma \leq \epsilon^{1/2} K \|f\|_{L^2(\Gamma)}$$

that we were looking for.

4.3. Completing the proof. Combining the obtained bounds for the two terms on the right-hand side of (4.4) leads to (3.1) and proves the first part of Theorem 3.1. To prove the second part, we write

$$\|\Lambda(\sigma_\epsilon) - \Lambda(\sigma)\|_{\mathcal{L}(L^2(\Gamma))} = \sup_{\|f\|_{L^2(\Gamma)} = 1} \|u_\epsilon^\sigma - u_\sigma\|_{L^2(\Gamma)}$$

$$\leq \sup_{\|f\|_{L^2(\Gamma)} = 1} K \|u_\epsilon^\sigma - u_\sigma\|_{H^1(\Omega \setminus C_0)} \leq K \epsilon^{1/2},$$

where we used the trace theorem in $\Omega \setminus C_0$.

5. Proofs of Theorem 3.6 and Theorem 3.7

We denote by $\Lambda_\alpha(C) := \Lambda(\gamma_\alpha, C), \alpha > 0$, the ND map for the conductivity

$$\gamma_{\alpha, C} := \begin{cases} \gamma_0 & \text{in } \Omega \setminus C, \\ \alpha \gamma_0 & \text{in } C. \end{cases}$$ (5.1)

By Theorem 3.1, we have $\Lambda_\alpha(C) \to \Lambda_0(C)$ in $\mathcal{L}(L^2_{\normalfont \mathcal{A}}(\Gamma))$ as $\alpha \to 0$ and $\Lambda_\alpha(C) \to \Lambda_\infty(C)$ in $\mathcal{L}(L^2_{\normalfont \mathcal{A}}(\Gamma))$ as $\alpha \to \infty$ for any $C \in \mathcal{A}$. This will allow us to apply to $\Lambda_\infty(C)$ and $\Lambda_0(C)$ the well known monotonicity properties of non-extreme ND maps via a limiting procedure.
To this end, we recall some monotonicity principles for $A|_{L^\infty_T(\Omega)}$ as well as a fundamental result on so-called localizing solutions to (2.1).

**Lemma 5.1** (Monotonicity principles). For $f \in L^2_\Gamma(\Gamma)$ and $c_1, c_2 \in L^\infty_\Omega(\Omega)$, it holds

$$
\int_{\Omega} \frac{c_2}{c_1} (c_1 - c_2) |\nabla u_f^i|^2 \, dx \leq \langle (A(c_2) - A(c_1)) f, f \rangle \leq \int_{\Omega} (c_1 - c_2) |\nabla u_f^i|^2 \, dx.
$$

**Proof.** This type of results go back to [28, 25]. See [22, Lemma 3.1] or [21, Lemma 2.1] for a proof of this particular version with $\Gamma = \partial \Omega$, readily modifiable to the case of local ND maps using the weak formulation (2.3). See also [22, Section 4.3] for remarks on such an extension. \hfill \Box

**Lemma 5.2** (Localized potentials). Let $U \subset \overline{\Omega}$ be a relatively open connected set that intersects $\Gamma$ and has connected complement. Let $B \subset U$ be a non-empty open set and assume $\zeta \in L^\infty_T(\Omega)$ satisfies the UCP in $U$. Then there are sequences $(f_i) \subset L^2_\Gamma(\Gamma)$ and $(u_i) \subset H^L_1(\Omega)$ with $u_i = u_f^i$, such that

$$
\lim_{i \to \infty} \int_{B} |\nabla u_i|^2 \, dx = \infty \quad \text{and} \quad \lim_{i \to \infty} \int_{\Omega \setminus U} |\nabla u_i|^2 \, dx = 0.
$$

**Proof.** This result is a special case of [16, Theorem 2.7]. \hfill \Box

We note that one also obtains localized potentials in the case of extreme inclusions, with the same sequence of current densities $(f_i)$ as for the background conductivity, as long as the extreme inclusions are located in the part of the domain where the gradient tends to zero. In the following result, it may be worth recalling the definition of the extension operator into the insulating parts from Corollary 3.3.

**Lemma 5.3.** Let the sets $U$ and $B$ be given as in Lemma 5.2. For $\zeta \in L^\infty_T(\Omega)$ and $(f_i) \subset L^2_\Gamma(\Gamma)$, suppose that $u_i = u_f^i$, satisfies

$$
\lim_{i \to \infty} \int_{B} |\nabla u_i|^2 \, dx = \infty \quad \text{and} \quad \lim_{i \to \infty} \int_{\Omega \setminus U} |\nabla u_i|^2 \, dx = 0.
$$

If $C = \sigma\sigma(C_0, C_\infty)$ with $C \subset \Omega \setminus \overline{U}$ and $\xi_i = E(\zeta, C_0)u_f^\sigma_i$ with $\sigma = \sigma(\zeta, C_0, C_\infty)$, then it also holds

$$
\lim_{i \to \infty} \int_{B} |\nabla \xi_i|^2 \, dx = \infty \quad \text{and} \quad \lim_{i \to \infty} \int_{\Omega \setminus U} |\nabla \xi_i|^2 \, dx = 0.
$$

**Proof.** Set $\tilde{v}_i = u_f^\sigma_i$, and use Proposition 2.2 to write

$$
\tilde{v}_i = P_{v_i},
$$

where $v_i = u_i|_{\Omega \setminus C_0} - w_i$ and $w_i \in H^L_1(\Omega \setminus C_0)$ solves the auxiliary problem in Proposition 2.2 with $u_f^\sigma$ replaced by $u_i$. In the following we denote $(\xi \nabla u_i)|_{\Omega \setminus C_0} := \xi \nabla u_i|_{\Omega \setminus C_0}$.

By the continuous dependence of $v_i$ on the Neumann condition and a bound similar to (4.8) involving $H^L_1(\Omega \setminus (V \cup C_0))$ instead of $H^L(\Omega \setminus C_0)$, we have

$$
\|\nabla v_i\|_{L^2(T \setminus (\partial \Omega \cup C_\Omega); C^\delta)} \leq \|\nabla v_i\|_{H^L(T \setminus (\partial \Omega \cup C_\Omega); C^\delta)} \leq K \|\nabla v_i\|_{L^2(\Omega \setminus (\partial \Omega \cup C_\Omega); C^\delta)} \leq K \|\nabla v_i\|_{L^2(T \setminus (\partial \Omega \cup C_\Omega); C^\delta)},
$$

where $V$ is chosen such that $C_0 \subset \Omega \setminus \overline{V} \subset \Omega \setminus \overline{U}$ and $\partial (\Omega \setminus \overline{V})$ is Lipschitz continuous. From (5.3) and (5.5), and because $K > 0$ is independent of $f_i$, we have that $\nabla \tilde{v}_i$ converges to a zero-vector in $L^2(\Omega \setminus C_0; C^\delta)$. Therefore the limiting behavior of $\nabla v_i$ coincides with the limiting behavior of $\nabla u_i|_{\Omega \setminus C_0}$. By applying (5.3) again we deduce that

$$
\lim_{i \to \infty} \int_{B} |\nabla v_i|^2 \, dx = \infty \quad \text{and} \quad \lim_{i \to \infty} \int_{\Omega \setminus (U \cup C_0)} |\nabla v_i|^2 \, dx = 0.
$$

We write $\hat{v}_i = v_i - P_{\perp} v_i$ and focus on the latter term. Lemma 4.1 gives the bound

$$
\|\nabla P_{\perp} v_i\|_{L^2(\Omega \setminus C_0; C^\delta)} \leq K \|P_{\perp} v_i\|_{L^2(\Omega \setminus C_0; C^\delta)} \leq K \|v_i\|_{L^2(C_\infty; C^\delta)}. \quad (5.7)
$$
Since $C_\infty \subset \Omega \setminus (\overline{U} \cup C_0)$, the combination of (5.6) and (5.7) gives
\[
\lim_{i \to \infty} \int_B |\nabla \hat{u}_i|^2 \, dx = \infty \quad \text{and} \quad \lim_{i \to \infty} \int_{\Omega \setminus (U \cup C_0)} |\nabla \hat{u}_i|^2 \, dx = 0.
\]
As $\hat{u}_i|_{\Omega \setminus C_0} = \hat{u}_i$, what remains is to investigate $\hat{u}_i|_{C_0}$.

Since $\hat{u}_i|_{\Omega \setminus C_0}$ converges to zero in the quotient space $H^1(\Omega \setminus (\overline{U} \cup C_0))/\mathbb{C}$, it follows from an obvious extension of the trace theorem that $\hat{u}_i|_{\partial C_0}$ tends to zero in $H^{1/2}(\partial C_0)/\mathbb{C}$. Note that
\[
\nabla \hat{u}_i = \nabla \hat{u}_i \in C_0^0, \text{ where } \hat{u}_i \text{ is the unique solution in } H^1(C_0^0)/\mathbb{C} \text{ to } \nabla \cdot (\chi \nabla \hat{u}_i) = 0 \text{ in } C_0^0 \text{ with its Dirichlet trace on } \partial C_0 \text{ being the element of } H^{1/2}(\partial C_0)/\mathbb{C} \text{ defined by } \hat{u}_i|_{\partial C_0} \in H^{1/2}(\partial C_0).
\]
Due to the continuous dependence of $\hat{u}_i$ on its Dirichlet trace in the respective quotient topologies, we have
\[
\lim_{i \to \infty} \int_{C_0} |\nabla \hat{u}_i|^2 \, dx = \lim_{i \to \infty} \int_{C_0} |\nabla \hat{u}_i|^2 \, dx = 0,
\]
thereby concluding the proof. \hfill \Box

\textbf{Remark 5.4.} In the proofs of Theorem 3.6, Theorem 3.7, and Theorem 3.10, we first employ the monotonicity principles from Lemma 5.1 for truncated conductivities, and then show “$\Rightarrow$” of the theorem statements by passing to the limit with the help of Theorem 3.1. It is tempting to also try something similar in the opposite direction “$\Leftarrow$” by proving the contrapositive of the theorem statements. Indeed, the $\epsilon$-truncated conductivities define an analytic (see e.g. [13, Theorem A.2]) family of compact self-adjoint operators, converging in the operator norm, such that for each small enough $\epsilon > 0$ one of the listed inequalities is contradicted. Hence, for the associated operators, the bottom of the spectrum is strictly negative for $\epsilon > 0$. Going to the limit $\epsilon \to 0$ implies that the bottom of the spectrum, say $\lambda$, is less than or equal to zero for the limiting compact self-adjoint operator as well. If one could guarantee that $\lambda$ is an eigenvalue of the limiting operator, then one could also easily establish that actually $\lambda < 0$, which would prove the direction “$\Leftarrow$” of the theorems. For Theorem 3.6 and Theorem 3.7 this could be done by comparing with the analogous operator with a smaller $\beta$, and for Theorem 3.10 by slightly magnifying the test inclusion $C$. However, it turns out that it is indeed possible to construct families of compact self-adjoint operators, depending analytically on a positive parameter and converging in the operator norm as this parameter tends to zero, such that the bottom of the spectrum converges from negative to zero, but without zero being an eigenvalue of the limiting operator. To circumvent this problem, we will prove the direction “$\Leftarrow$” of the aforementioned theorems by an alternative technique relying on the tighter monotonicity estimates established in Appendix A.

We now state a straightforward corollary that will be useful specifically in proving the direction “$\Leftarrow$” in Theorem 3.6 and Theorem 3.7.

\textbf{Corollary 5.5.} Let $f \in L^2_\sharp(\Gamma)$, $\varsigma \in L^\infty_\sharp(\Omega)$, and $\gamma_0 = \sigma(\gamma_0, D, \emptyset)$. By denoting $\hat{u}_i = E(\gamma_0, D)u_\gamma^{\gamma_0}$ and $u = u_\gamma^{\gamma_0}$, it holds
\[
\langle (\Lambda_0(D) - \Lambda(\varsigma))f, f \rangle \leq \int_{\Omega \setminus D} (\varsigma - \gamma_0)|\nabla \hat{u}_i|^2 \, dx + \int_D \varsigma|\nabla \hat{u}_i|^2 \, dx,
\]
\[
\langle (\Lambda(\varsigma) - \Lambda_{\infty}(D))f, f \rangle \leq \int_\Omega \frac{\gamma_0}{\varsigma}(\gamma_0 - \varsigma)|\nabla u|^2 \, dx + K \int_D |\nabla u|^2 \, dx,
\]
where $K > 0$ is independent of $f$.

\textbf{Proof.} Applying Lemma 5.1 to the truncated conductivity $\gamma_{\alpha, D}$ from (5.1) gives
\[
\langle (\Lambda(\gamma_0, D) - \Lambda(\sigma))f, f \rangle \leq \int_\Omega (\sigma - \gamma_{\alpha, D})|\nabla u|_{\gamma_0}^{\gamma_{\alpha, D}}|^2 \, dx. \tag{5.8}
\]
Since (5.8) holds for all $\alpha > 0$, taking the limit $\alpha \to 0$ proves the first inequality by virtue of Theorem 3.1 and Corollary 3.3.

To prove the second inequality, consider
\[
\langle (\Lambda(\varsigma) - \Lambda_{\infty}(D))f, f \rangle = \langle (\Lambda(\varsigma) - \Lambda(\gamma_0))f, f \rangle + \langle (\Lambda(\gamma_0) - \Lambda_{\infty}(D))f, f \rangle. \tag{5.9}
\]
The proof is completed by a direct application of Lemma 5.1 to the first term on the right-hand side of (5.9) and of Lemma A.1(ii) to the second term on the right-hand side of (5.9).

5.1. Proof of Theorem 3.6. Let $\alpha_0 > 0$ be fixed and assume $0 < \beta \leq \frac{\alpha_0}{1 + \alpha_0} \inf(\gamma_0)$. At the end of the proof, we will let $\alpha_0 \to \infty$, and hence allow $\beta$ to take values up to and including $\inf(\gamma_0)$.

We start by considering the first “if and only if” statement.

Proof of “$\Rightarrow$”. Assume $B \subset D$ and consider values of $\alpha$ in the range $0 < \alpha \leq 1 - \frac{\alpha_0}{1 + \alpha_0}$. Applying Lemma 5.1 and (3.2) leads to

\[
((\Lambda_0(D) - \Lambda(\gamma_0) + \beta D \Lambda(\gamma_0; \chi_B)) f, f) \geq \int_D [(1 - \alpha)\gamma_0 - \beta \chi_B] |\nabla u_\gamma^0|^2 \, dx \geq \frac{\alpha_0}{1 + \alpha_0} \int_B [\gamma_0 - \inf(\gamma_0)] |\nabla u_\gamma^0|^2 \, dx \geq 0
\]

for all $f \in L^2_\gamma(\Gamma)$. Hence $B \subset D \implies \Lambda_0(D) \geq \Lambda(\gamma_0) - \beta D \Lambda(\gamma_0; \chi_B)$ for any $0 < \alpha \leq 1 - \frac{\alpha_0}{1 + \alpha_0}$, i.e. letting $\alpha \to 0$ gives $B \subset D \implies \Lambda_0(D) \geq \Lambda(\gamma_0) - \beta D \Lambda(\gamma_0; \chi_B)$.

Proof of “$\Leftarrow$”. Assume $B \not\subset D$ so that $B \setminus D$ contains an open ball $\hat{B}$, in which we may concentrate the localized potentials via a suitable set $U$. To this end, let $U \subset \overline{U}$ be a relatively open connected set that intersects $\Gamma$, has connected complement, contains $\hat{B}$, and satisfies $\overline{U} \cap D = \emptyset$. Annotating $\gamma_0 = \sigma(\gamma_0; D, \emptyset)$, we now pick $(f_i) \subset L^2_\gamma(\Gamma)$ via Lemma 5.2 and Lemma 5.3 such that the corresponding potentials $u_i = E(\gamma_0; D) u_i^\sigma$ and $u_i = u_i^\sigma$ satisfy

\[
\lim_{i \to \infty} \int_D |\nabla u_i|^2 \, dx \leq \lim_{i \to \infty} \int_{\Omega \setminus U} |\nabla u_i|^2 \, dx = 0, \quad \lim_{i \to \infty} \int_B |\nabla u_i|^2 \, dx = \infty.
\]

By Corollary 5.5 and (3.2), we have

\[
((\Lambda_0(D) - \Lambda(\gamma_0) + \beta D \Lambda(\gamma_0; \chi_B)) f_i, f_i) \leq \sup_D |\nabla u_i|^2 \, dx - \beta \int_B |\nabla u_i|^2 \, dx \to -\infty,
\]

and thus $B \not\subset D \implies \Lambda_0(D) \not\geq \Lambda(\gamma_0) - \beta D \Lambda(\gamma_0; \chi_B)$.

Now consider the second “if and only if” statement, that is, “$B \subset D \iff \Lambda(\gamma_0) + \beta D \Lambda(\gamma_0; \chi_B) \geq \Lambda_\infty(D)$”.

Proof of “$\Rightarrow$”. Assume $B \subset D$ and $\alpha \geq 1 + \alpha_0$. Lemma 5.1 and (3.2) imply

\[
((\Lambda(\gamma_0) + \beta D \Lambda(\gamma_0; \chi_B) - \Lambda_\alpha(D)) f, f) \geq \int_D [(1 - \alpha^{-1})\gamma_0 - \beta \chi_B] |\nabla u_\gamma^0|^2 \, dx \geq \frac{\alpha_0}{1 + \alpha_0} \int_B [\gamma_0 - \inf(\gamma_0)] |\nabla u_\gamma^0|^2 \, dx \geq 0
\]

for all $f \in L^2_\gamma(\Gamma)$. Taking the limit $\alpha \to \infty$ leads to $B \subset D \implies \Lambda(\gamma_0) + \beta D \Lambda(\gamma_0; \chi_B) \geq \Lambda_\infty(D)$.

Proof of “$\Leftarrow$”. Now assume $B \not\subset D$ and let us localize $u_i$ in the same manner as in the insulating case, i.e.

\[
\lim_{i \to \infty} \int_D |\nabla u_i|^2 \, dx \leq \lim_{i \to \infty} \int_{\Omega \setminus U} |\nabla u_i|^2 \, dx = 0, \quad \lim_{i \to \infty} \int_B |\nabla u_i|^2 \, dx \geq \lim_{i \to \infty} \int_B |\nabla u_i|^2 \, dx = \infty.
\]

Corollary 5.5 and (3.2) thus yield

\[
((\Lambda(\gamma_0) + \beta D \Lambda(\gamma_0; \chi_B) - \Lambda_\infty(D)) f_i, f_i) \leq K \int_D |\nabla u_i|^2 \, dx - \beta \int_B |\nabla u_i|^2 \, dx \to -\infty,
\]

and we conclude that $B \not\subset D \implies \Lambda(\gamma_0) + \beta D \Lambda(\gamma_0; \chi_B) \not\geq \Lambda_\infty(D)$.

Since all above results hold for any $\alpha_0 > 0$, we may actually choose $0 < \beta \leq \inf(\gamma_0)$ as in the theorem statement by letting $\alpha_0 \to \infty$. \hfill \Box
5.2. Proof of Theorem 3.7. Let us consider the first “if and only if” statement. Hence, we assume $0 < \beta < \inf(\gamma_0)$ and choose $0 < \alpha_0 < 1$ to be small enough so that $\alpha_0\gamma_0 \leq \gamma_0 - \beta\chi_B$.

Proof of “$\Rightarrow$”. Assume $B \subset D$, which implies $\gamma_{\alpha,D} \leq \gamma_0 - \beta\chi_B$ for $0 < \alpha \leq \alpha_0$. Using the monotonicity principles in Lemma 5.1, we directly have $\Lambda_\alpha(D) \geq \Lambda(\gamma_0 - \beta\chi_B)$ for all $0 < \alpha \leq \alpha_0$. Letting $\alpha \to 0$ gives $B \subset D \Rightarrow \Lambda_0(D) \geq \Lambda(\gamma_0 - \beta\chi_B)$.

Proof of “$\Leftarrow$”. Now assume $B \not\subset D$. We pick $U$ and $\tilde{B}$ as in the proof of Theorem 3.6 and localize $(\tilde{u}_i)$ in precisely the same manner as for the insulating case in the proof of Theorem 3.6:

$$\lim_{i \to \infty} \int_{\Omega \setminus U} \left| \nabla \tilde{u}_i \right|^2 \, dx = 0 \quad \text{and} \quad \lim_{i \to \infty} \int_B \left| \nabla \tilde{u}_i \right|^2 \, dx = \infty.$$

Hence, by Corollary 5.5,

$$\left( (\Lambda_0(D) - \Lambda(\gamma_0 - \beta\chi_B))f_i, f_i \right) \leq -\beta \int_{B \setminus D} \left| \nabla \tilde{u}_i \right|^2 \, dx + \int_D (\gamma_0 - \beta\chi_B) \left| \nabla \tilde{u}_i \right|^2 \, dx$$

$$\leq -\beta \int_B \left| \nabla \tilde{u}_i \right|^2 \, dx + \sup_D (\gamma_0 - \beta\chi_B) \int_{\Omega \setminus U} \left| \nabla \tilde{u}_i \right|^2 \, dx.$$

Letting $i \to \infty$ implies $B \not\subset D \Rightarrow \Lambda_0(D) \not\geq \Lambda(\gamma_0 - \beta\chi_B)$.

Let us then consider the second “if and only if” statement. We therefore assume $\beta > 0$ and choose $\alpha_0 > 0$ large enough so that $\alpha_0\gamma_0 \geq \gamma_0 + \beta\chi_B$.

Proof of “$\Rightarrow$”. Assume $B \subset D$, which implies $\gamma_0 + \beta\chi_B \leq \gamma_{\alpha,D}$ for $\alpha \geq \alpha_0$. Due to the monotonicity principles in Lemma 5.1, it follows that $\Lambda(\gamma_0 + \beta\chi_B) \geq \Lambda_\alpha(D)$ for all $\alpha \geq \alpha_0$, so letting $\alpha \to \infty$ gives $B \subset D \Rightarrow \Lambda(\gamma_0 + \beta\chi_B) \geq \Lambda_\infty(D)$.

Proof of “$\Leftarrow$”. Now assume $B \not\subset D$ and localize $(u_i)$ as in the proof of Theorem 3.6, i.e.

$$\lim_{i \to \infty} \int_{\Omega \setminus U} \left| \nabla u_i \right|^2 \, dx = 0 \quad \text{and} \quad \lim_{i \to \infty} \int_B \left| \nabla u_i \right|^2 \, dx = \infty.$$

Once again we apply Corollary 5.5 to write

$$\left( (\Lambda(\gamma_0 + \beta\chi_B) - \Lambda_\infty(D))f_i, f_i \right) \leq -\beta \int_B \frac{\gamma_0}{\gamma_0 + \beta} \left| \nabla u_i \right|^2 \, dx + K \int_D \left| \nabla u_i \right|^2 \, dx$$

$$\leq -\beta \inf_B \left( \frac{\gamma_0}{\gamma_0 + \beta} \right) \int_B \left| \nabla u_i \right|^2 \, dx + K \int_{\Omega \setminus U} \left| \nabla u_i \right|^2 \, dx.$$

Letting $i \to \infty$ implies $B \not\subset D \Rightarrow \Lambda(\gamma_0 + \beta\chi_B) \not\geq \Lambda_\infty(D)$, thereby concluding the proof. \qed

6. Proof of Theorem 3.10

Proof of “$\Rightarrow$”. Define the $\epsilon$-truncation of $\gamma$, with $\epsilon > 0$, as

$$\gamma_\epsilon := \begin{cases} 
\epsilon \gamma_0 & \text{in } D_0, \\
\epsilon^{-1} \gamma_0 & \text{in } D_\infty, \\
\gamma_- & \text{in } D_\epsilon^-, \\
\gamma_+ & \text{in } D_\epsilon^+, \\
\gamma_0 & \text{in } \Omega \setminus D.
\end{cases}$$

In particular, we have $\Lambda(\gamma_\epsilon) \to \Lambda(\gamma)$ in $\mathcal{L}(L_2^2(\Gamma))$ as $\epsilon \to 0$ by Theorem 3.1. In the following we assume $0 < \epsilon_0 < 1$ is small enough so that $\epsilon_0\gamma_0 \leq \gamma$ in $D^-\epsilon$ and $\epsilon_0^{-1} \gamma_0 \geq \gamma$ in $D^+\epsilon$.

Assume $D \subseteq C$ and let $0 < \epsilon \leq \epsilon_0$. Then it holds (cf. (5.1))

$$\gamma_\epsilon,C \leq \gamma_\epsilon \leq \gamma_{\epsilon^{-1},C}.$$

In particular, by virtue of the monotonicity principles in Lemma 5.1, we have $\Lambda_\epsilon(C) \geq \Lambda(\gamma_\epsilon) \geq \Lambda_{\epsilon^{-1}}(C)$ for all $0 < \epsilon \leq \epsilon_0$. Letting $\epsilon \to 0$ gives $D \subseteq C \Rightarrow \Lambda_0(C) \geq \Lambda(\gamma) \geq \Lambda_\infty(C)$.
Proof of “⇐”. Now assume \( D \not\subseteq C \). Due to Definition 3.8, there exist an open ball \( B \subseteq D \setminus C \) and a relatively open connected set \( U \subseteq \overline{U} \) such that \( U \) has connected complement, \( U \) intersects \( \Gamma, B \subseteq U, \overline{U} \cap C = \emptyset \), and one of the following four options holds:

\[
\begin{align*}
(a): & \quad \overline{U} \cap (D \setminus D^+_F) = \emptyset, \\
(b): & \quad \overline{U} \cap (D \setminus D^-_F) = \emptyset, \\
(c): & \quad \overline{U} \cap (D \setminus D_\infty) = \emptyset, \\
(d): & \quad \overline{U} \cap (D \setminus D_0) = \emptyset.
\end{align*}
\]

We will consider these four cases separately. However, first we need to introduce some auxiliary conductivity coefficients and potentials.

Let us define \( \gamma^F \in L^\infty_\sigma(\Omega) \) as

\[
\gamma^F := \begin{cases} 
\gamma_- & \text{in } D^-_F, \\
\gamma_+ & \text{in } D^+_F, \\
\gamma_0 & \text{in } \Omega \setminus (D^-_F \cup D^+_F)
\end{cases}
\]

and introduce six (extreme) conductivity coefficients:

\[
\begin{align*}
\gamma_1 &= \sigma(\gamma_0, C, \emptyset), \\
\gamma_2 &= \sigma(\gamma_0, D_0, \emptyset), \\
\gamma_3 &= \sigma(\gamma_0, \emptyset, D_\infty), \\
\gamma_4 &= \sigma(\gamma_0, D_0, D_\infty).
\end{align*}
\]

Furthermore, for \((f_i), (g_i) \subseteq L^2_\sigma(\Gamma)\), we define eight auxiliary potential sequences via

\[
\begin{align*}
u_{0,i} &= u^{g_0}_{f_i}, & \tilde{u}_{0,i} &= u^{g_0}_{g_i}, \\
u_{1,i} &= E(\gamma_0, C)u^{\gamma_1}_{f_i}, & \tilde{u}_{1,i} &= E(\gamma_0, C)u^{\gamma_1}_{g_i}, \\
u_{2,i} &= E(\gamma_0, D_0)u^{\gamma_2}_{f_i}, & \tilde{u}_{2,i} &= E(\gamma^F, D_0)u^{\gamma^F}_{g_i}, \\
u_{3,i} &= u^{\gamma_3}_{f_i}, & \tilde{u}_{3,i} &= u^{\gamma_3}_{g_i}, \\
u_{4,i} &= u^{\gamma_4}_{f_i}, & \tilde{u}_{4,i} &= u^{\gamma_4}_{g_i}.
\end{align*}
\]

In cases (a) and (b) we pick \((f_i), (g_i)\), via Lemma 5.2 and Lemma 5.3, such that for \( j \in \{0, 1, 2, 3, 4\} \) it holds

\[
\lim_{i \to \infty} \int_{\Omega \setminus U} |\nabla u_{j,i}|^2 \, dx = 0 \quad \text{and} \quad \lim_{i \to \infty} \int_{B} |\nabla u_{j,i}|^2 \, dx = \infty. \tag{6.1}
\]

Recall that \( \gamma^F \) equals \( \gamma_0 \) away from \( D^\pm_F \) and thus satisfies the UCP there. According to [22, Lemma 3.7], \((\tilde{u}_{0,i})\) and \((\tilde{u}_{4,i})\) can thus be localized in \( B \) along the set \( U \) by resorting to the same sequence of current densities \((g_i)\) in cases (c) and (d). In consequence, we may pick \((g_i)\), via Lemma 5.2 and Lemma 5.3, such that

\[
\lim_{i \to \infty} \int_{\Omega \setminus U} |\nabla \tilde{u}_{j,i}|^2 \, dx = 0 \quad \text{and} \quad \lim_{i \to \infty} \int_{B} |\nabla \tilde{u}_{j,i}|^2 \, dx = \infty \tag{6.2}
\]

holds for \( j \in \{0, 1, 2, 4\} \) in case (c) and for \( j \in \{0, 1, 3, 4\} \) in case (d).

We will go in turns through the four cases (a)–(d) and contradict one of the two inequalities in Theorem 3.10. More precisely, cases (b) and (d) contradict the first inequality, whereas cases (a) and (c) contradict the second inequality.

**Case (a).** As noted above, the aim is to contradict the second inequality in Theorem 3.10. To this end, we write

\[
\langle (\Lambda(\gamma) - \Lambda_\infty(C))f_i, f_i \rangle = \langle (\Lambda(\gamma) - \Lambda(\gamma_4))f_i, f_i \rangle + \langle (\Lambda(\gamma_4) - \Lambda(\gamma_2))f_i, f_i \rangle + \langle (\Lambda(\gamma_2) - \Lambda(\gamma_0))f_i, f_i \rangle + \langle (\Lambda(\gamma_0) - \Lambda_\infty(C))f_i, f_i \rangle. \tag{6.3}
\]
We now apply (i), (ii), (iii), and (ii) of Lemma A.1, respectively, to the four terms on the right-hand side of (6.3) to arrive at

\[
\begin{align*}
\langle (\Lambda(\gamma) - \Lambda(\gamma_4))f_i, f_i \rangle &\leq \int_{D_+^i} \frac{\gamma_0}{\gamma_-} (\gamma_0 - \gamma_-) |\nabla u_{4,i}|^2 dx + \int_{D_-^i} \frac{\gamma_0}{\gamma_+} (\gamma_0 - \gamma_+) |\nabla u_{4,i}|^2 dx, \\
\langle (\Lambda(\gamma) - \Lambda(\gamma_2))f_i, f_i \rangle &\leq - \inf_{D_+^i} \int_{D_+^i} |\nabla u_{2,i}|^2 dx, \\
\langle (\Lambda(\gamma) - \Lambda(\gamma_1))f_i, f_i \rangle &\leq \sup_{D_0} \int_{D_0} |\nabla u_{2,i}|^2 dx, \\
\langle (\Lambda(\gamma) - \Lambda_\infty(C))f_i, f_i \rangle &\leq K \int_{C_\infty} |\nabla u_{0,i}|^2 dx.
\end{align*}
\] (6.4)

Using (6.1), it follows that (6.5)–(6.7) tend to zero as \(i \to \infty\). Since \(\gamma_+ \geq \gamma_0\) in \(D_+^x \supseteq B\) and \(\gamma_- \leq \gamma_0\) in \(D_-^x \subseteq \Omega \setminus U\), we obtain from (6.4) that

\[
\langle (\Lambda(\gamma) - \Lambda(\gamma_4))f_i, f_i \rangle \leq \sup_{D_+^i} \left( \frac{\gamma_0}{\gamma_-} (\gamma_0 - \gamma_-) \right) \int_{\Omega \setminus U} |\nabla u_{4,i}|^2 dx \\
- \inf_{D_+^i} \left( \frac{\gamma_0}{\gamma_+} (\gamma_0 - \gamma_+) \right) \int_{B} |\nabla u_{4,i}|^2 dx.
\] (6.8)

According to Definition 3.8, it holds that \(\inf_{D_+^i} (\gamma_+ - \gamma_0) > 0\). Since \(\gamma_0 \in L^\infty_+(\Omega)\) and \(\gamma_\pm \in L^\infty_+(D_+^x)\), (6.8) and (6.1) yield

\[
\lim_{i \to \infty} \langle (\Lambda(\gamma) - \Lambda_\infty(C))f_i, f_i \rangle = \lim_{i \to \infty} \langle (\Lambda(\gamma) - \Lambda(\gamma_4))f_i, f_i \rangle = -\infty.
\]

**Case (b).** In this case, the goal is to contradict the first inequality in Theorem 3.10. For this purpose, we write

\[
\begin{align*}
\langle (\Lambda(\gamma) - \Lambda(\gamma_4))f_i, f_i \rangle &= (\langle \Lambda(\gamma) - \Lambda(\gamma_0) \rangle f_i, f_i \rangle + \langle \Lambda(\gamma_0) - \Lambda(\gamma_3) \rangle f_i, f_i \rangle + \\
&\quad + \langle \Lambda(\gamma_3) - \Lambda(\gamma_4) \rangle f_i, f_i \rangle + \langle \Lambda(\gamma_4) - \Lambda(\gamma) \rangle f_i, f_i \rangle.
\end{align*}
\] (6.9)

We now apply (iii), (ii), (iii), and (i) of Lemma A.1, respectively, to the four terms on the right-hand side of (6.9) to conclude

\[
\begin{align*}
\langle (\Lambda(\gamma) - \Lambda(\gamma_0))f_i, f_i \rangle &\leq \sup_{C} \int_{C} |\nabla u_{1,i}|^2 dx, \\
\langle (\Lambda(\gamma) - \Lambda(\gamma_3))f_i, f_i \rangle &\leq K \int_{D_\infty} |\nabla u_{0,i}|^2 dx, \\
\langle (\Lambda(\gamma) - \Lambda(\gamma_3))f_i, f_i \rangle &\leq - \inf_{D_0} \int_{D_0} |\nabla u_{3,i}|^2 dx, \\
\langle (\Lambda(\gamma) - \Lambda(\gamma_4))f_i, f_i \rangle &\leq \int_{D^+} (\gamma_- - \gamma_0) |\nabla u_{4,i}|^2 dx + \int_{D^+} (\gamma_+ - \gamma_0) |\nabla u_{4,i}|^2 dx.
\end{align*}
\] (6.10)

Due to (6.1), it is clear that (6.10)–(6.12) tend to zero as \(i \to \infty\). Since \(\gamma_- \leq \gamma_0\) in \(D_+^x \supseteq B\) and \(\gamma_+ \geq \gamma_0\) in \(D_-^x \subseteq \Omega \setminus U\), the estimate (6.13) leads to

\[
\langle (\Lambda(\gamma) - \Lambda(\gamma_4))f_i, f_i \rangle \leq \sup_{D^+_0} (\gamma_- - \gamma_0) \int_{B} |\nabla u_{4,i}|^2 dx + \sup_{D^+_0} (\gamma_+ - \gamma_0) \int_{\Omega \setminus U} |\nabla u_{4,i}|^2 dx.
\] (6.14)

According to Definition 3.8, it holds that \(\sup_{D^+_0} (\gamma_- - \gamma_0) < 0\). As a consequence, (6.14) and (6.1) guarantee that

\[
\lim_{i \to \infty} \langle (\Lambda(\gamma) - \Lambda(\gamma_4))f_i, f_i \rangle = \lim_{i \to \infty} \langle (\Lambda(\gamma) - \Lambda(\gamma))f_i, f_i \rangle = -\infty.
\]
Since resorting once again to Lemma A.1, we apply its subresults (ii), (iii), (i), and (ii), respectively,

\[ 16 \text{ V. Candiani, J. Dardé, H. Garde, and N. Hyvönen} \]

\[ \text{Case (c). This time around we will contradict the second inequality in Theorem 3.10. We start} \]
\[ \text{by writing} \]
\[ \langle (\Lambda(\gamma) - \Lambda_{\infty}(C)) g_i, g_i \rangle = \langle (\Lambda(\gamma) - \Lambda_{\infty}(\gamma_2^F)) g_i, g_i \rangle + \langle (\Lambda(\gamma_2^F) - \Lambda(\gamma)) g_i, g_i \rangle \]
\[ + \langle (\Lambda(\gamma) - \Lambda(\gamma_0)) g_i, g_i \rangle + \langle (\Lambda(\gamma_0) - \Lambda_{\infty}(C)) g_i, g_i \rangle. \]  

(6.15)

Resorting once again to Lemma A.1, we apply its subresults (ii), (iii), (i), and (ii), respectively,
to the four terms on the right-hand side of (6.15) to demonstrate that

\[ \langle (\Lambda(\gamma) - \Lambda_{\infty}(C)) g_i, g_i \rangle \leq \inf_{D_0^{\infty}} \| \nabla \bar{u}_{2,i} \|^2_{L^2}, \]  

(6.16)

\[ \langle (\Lambda(\gamma_2^F) - \Lambda(\gamma)) g_i, g_i \rangle \leq \sup_{D_0} \langle \nabla \bar{u}_{2,i} \|^2_{L^2}, \]

(6.17)

\[ \langle (\Lambda(\gamma) - \Lambda(\gamma_0)) g_i, g_i \rangle \leq \sup_{D_0^{\infty} \cup D_0^{\infty}} \langle \gamma_0 - \gamma \rangle \int_{D_0^{\infty} \cup D_0^{\infty}} |\nabla \bar{u}_{4,i}|^2 dx, \]

(6.18)

\[ \langle (\Lambda(\gamma_0) - \Lambda_{\infty}(C)) g_i, g_i \rangle \leq K \int_{C_\infty} |\nabla \bar{u}_{0,i}|^2 dx. \]

(6.19)

Since \( D_0^{\infty} \subset \Omega \setminus \overline{U} \), it follows directly from (6.2) that (6.17)–(6.19) tend to zero and (6.16) goes to minus infinity as \( i \to \infty \), i.e.

\[ \lim_{i \to \infty} \langle (\Lambda(\gamma) - \Lambda_{\infty}(C)) g_i, g_i \rangle = \lim_{i \to \infty} \langle (\Lambda(\gamma) - \Lambda(\gamma_2^F)) g_i, g_i \rangle = -\infty. \]

Case (d). In this final case, we will contradict the first inequality in Theorem 3.10. To achieve this objective, we first write

\[ \langle (\Lambda_0(C) - \Lambda(\gamma)) g_i, g_i \rangle = \langle (\Lambda_0(C) - \Lambda(\gamma_0)) g_i, g_i \rangle + \langle (\Lambda(\gamma_0) - \Lambda(\gamma)) g_i, g_i \rangle \]
\[ + \langle (\Lambda(\gamma) - \Lambda(\gamma_0^F)) g_i, g_i \rangle + \langle (\Lambda(\gamma_0^F) - \Lambda(\gamma)) g_i, g_i \rangle. \]  

(6.20)

Analogously to the previous three cases, we apply (iii), (i), (ii), and (iii) of Lemma A.1, in this order, to the four terms on the right-hand side of (6.20) to reach the estimates

\[ \langle (\Lambda_0(C) - \Lambda(\gamma_0)) g_i, g_i \rangle \leq \sup_{C_0} \langle \nabla \bar{u}_{1,i} \rangle^2 dx, \]  

(6.21)

\[ \langle (\Lambda(\gamma_0) - \Lambda(\gamma^F)) g_i, g_i \rangle \leq \sup_{D_0^{\infty} \cup D_0^{\infty}} \langle \gamma^F - \gamma \rangle \int_{D_0^{\infty} \cup D_0^{\infty}} |\nabla \bar{u}_{0,i}|^2 dx, \]

(6.22)

\[ \langle (\Lambda(\gamma) - \Lambda(\gamma_0^F)) g_i, g_i \rangle \leq K \int_{D_0^{\infty}} |\nabla \bar{u}_{4,i}|^2 dx, \]

(6.23)

\[ \langle (\Lambda(\gamma_0^F) - \Lambda(\gamma)) g_i, g_i \rangle \leq -\inf_{D_0^{\infty}} \langle \nabla \bar{u}_{3,i} \rangle^2 dx. \]

(6.24)

Since \( D_0^{\infty} \subset \Omega \setminus \overline{U} \), it follows directly from (6.2) that (6.21)–(6.23) tend to zero and (6.24) converges to minus infinity as \( i \to \infty \), leading to the sought-for conclusion

\[ \lim_{i \to \infty} \langle (\Lambda_0(C) - \Lambda(\gamma)) g_i, g_i \rangle = \lim_{i \to \infty} \langle (\Lambda(\gamma_0^F) - \Lambda(\gamma)) g_i, g_i \rangle = -\infty. \]

The results for the cases (a)–(d) complete the proof of “⇐”, that is, \( D \not\subseteq C \) indeed implies either \( \Lambda_0(C) \not\geq \Lambda(\gamma) \) or \( \Lambda(\gamma) \not\geq \Lambda_{\infty}(C) \).

To conclude the proof of Theorem 3.10, and simultaneously the whole paper, the established “if and only if” statement guarantees \( D \subseteq C \) for all sets \( C \) belonging to

\[ \mathcal{M} := \{ C \in \mathcal{A} \mid \Lambda_0(C) \geq \Lambda(\gamma) \geq \Lambda_{\infty}(C) \}. \]

Since \( D \) itself is also a member of \( \mathcal{M} \), this proves \( D = \cap \mathcal{M} \).
APPENDIX A. SOME MONOTONICITY ESTIMATES

In this appendix, we will give some generic monotonicity estimates that are extensively utilized in the proof Theorem 3.10. For more information on the notation and definitions, we refer to Section 2. See also Corollary 3.3 for the definition of the extension operator into the perfectly insulating parts.

In the following $f \in L^2(\Gamma)$ is arbitrary but fixed. We denote $u_{\varsigma, C_0, \varsigma_0} = E(\varsigma, C_0)u_0^\varsigma$ with $\sigma = \sigma(\varsigma, C_0, C_\infty)$, and we analogously set $\Lambda_{\varsigma, C_0, \varsigma_0} = \Lambda_{\sigma}$. To motivate such notation, take note that we will consider several cases where $C_0$, $C_\infty$, or both are empty sets and we also choose $\varsigma$ in several ways.

**Lemma A.1.** Let $C = \mathscr{A}(C_0, C_\infty)$ and $\varsigma, \varsigma_1, \varsigma_2 \in L^\infty(\Omega)$.

(i) Different background conductivity, but with the same extreme inclusions:
\[
\int_{\Omega \setminus C} \frac{\varsigma_2}{\varsigma_1} (\varsigma_1 - \varsigma_2) |\nabla u_{\varsigma_1, C_0, \varsigma_0}|^2 \, dx \leq \langle (\Lambda_{\varsigma_2, C_0, \varsigma_0} - \Lambda_{\varsigma_1, C_0, \varsigma_0}) f, f \rangle \leq \int_{\Omega \setminus C} (\varsigma_1 - \varsigma_2) |\nabla u_{\varsigma_1, C_0, \varsigma_0}|^2 \, dx.
\]

(ii) Same background conductivity and perfectly insulating inclusions, but with and without perfectly conducting inclusions:
\[
\int_{C_\infty} \varsigma |\nabla u_{\varsigma, C_0, \varsigma_0}|^2 \, dx \leq \langle (\Lambda_{\varsigma, C_0, \varsigma_0} - \Lambda_{\varsigma, C_0, C_\infty}) f, f \rangle \leq K \int_{C_\infty} |\nabla u_{\varsigma, C_0, \varsigma_0}|^2 \, dx,
\]
where $K > 0$ is independent of $f$.

(iii) Same background conductivity and perfectly conducting inclusions, but with and without perfectly insulating inclusions:
\[
\int_{C_0} \varsigma |\nabla u_{\varsigma, C_0, \varsigma_0}|^2 \, dx \leq \langle (\Lambda_{\varsigma, C_0, \varsigma_0} - \Lambda_{\varsigma, \emptyset, C_\infty}) f, f \rangle \leq \int_{C_0} \varsigma |\nabla u_{\varsigma, C_0, \varsigma_0}|^2 \, dx.
\]

Proof of (i). Abbreviate $\sigma_j = \sigma(\varsigma_j, C_0, C_\infty)$, $u_j = u_j^{\sigma_j}$, and $\Lambda_j = \Lambda(\sigma_j)$ for $j \in \{1, 2\}$.

Using the weak forms (2.3) for $u_1$ and $u_2$ as well as the self-adjointness of $\Lambda_1$, we have
\[
\langle (\Lambda_2 - \Lambda_1) f, f \rangle = \int_{\Omega \setminus C} (\varsigma_1 |\nabla u_1|^2 + \varsigma_2 |\nabla u_2|^2 - \varsigma_2 \nabla u_2 \cdot \nabla u_1 - \varsigma_1 \nabla u_1 \cdot \nabla u_2) \, dx
\]
\[
= \int_{\Omega \setminus C} \varsigma_1 |\nabla u_1 - \frac{\varsigma_2}{\varsigma_1} \nabla u_2|^2 \, dx + \int_{\Omega \setminus C} \frac{\varsigma_2}{\varsigma_1} (\varsigma_1 - \varsigma_2) |\nabla u_2|^2 \, dx. \tag{A.1}
\]

Estimating the first term on the right-hand side of (A.1) from below by zero gives the left-hand inequality in (i).

Using this time around the self-adjointness of $\Lambda_2$, we obtain
\[
\langle (\Lambda_2 - \Lambda_1) f, f \rangle = \int_{\Omega \setminus C} ((\varsigma_1 - \varsigma_2) |\nabla u_2|^2 - \varsigma_1 |\nabla u_1|^2 - \varsigma_1 |\nabla u_2|^2 + \varsigma_1 \nabla u_1 \cdot \nabla u_2 + \varsigma_1 \nabla u_2 \cdot \nabla u_1) \, dx
\]
\[
= \int_{\Omega \setminus C} (\varsigma_1 - \varsigma_2) |\nabla u_2|^2 \, dx - \int_{\Omega \setminus C} \varsigma_1 |\nabla (u_1 - u_2)|^2 \, dx. \tag{A.2}
\]

Estimating the second term on the right-hand side of (A.2) from above by zero gives the right-hand inequality in (i).

Proof of (ii). Abbreviate $\sigma_1 = \sigma(\varsigma, C_0, C_\infty)$ and $\sigma_2 = \sigma(\varsigma, C_0, \emptyset)$, and in the same way as above, set $u_j = u_j^{\sigma_j}$ and $\Lambda_j = \Lambda(\sigma_j)$ for $j \in \{1, 2\}$. We also abbreviate $P = P(\varsigma, C_0, C_\infty)$ and $P_\perp = P_\perp(\varsigma, C_0, C_\infty)$.

Let us then introduce an $\epsilon$-truncation of $\sigma_1$,
\[
\sigma_\epsilon = \begin{cases} 
\varsigma & \text{in } \Omega \setminus C, \\
\epsilon^{-1} \varsigma & \text{in } C_\infty.
\end{cases}
\]

Using the lower bound of Lemma 5.1, with $\Omega$ replaced by $\Omega \setminus C_0$, we get
\[
(1 - \epsilon) \int_{C_\infty} \varsigma |\nabla u_2|^2 \, dx \leq \langle (\Lambda_2 - \Lambda(\sigma_\epsilon)) f, f \rangle.
\]
Due to Theorem 3.1, the left-hand inequality in (ii) follows by taking the limit \( \epsilon \to 0 \).

To obtain the right-hand inequality in (ii), note first that \( u_1 = Pu_2 \) due to Proposition 2.2. In particular, it holds that \( u_2 = u_1 + P_\perp u_2 \), and thus the weak formulation for \( u_2 \) gives

\[
\langle (A_2 - A_1)f, f \rangle = \langle f, (A_2 - A_1)f \rangle = \langle f, u_2 |_{\Gamma} \rangle - \langle f, u_1 |_{\Gamma} \rangle = \langle f, (P_\perp u_2) |_{\Gamma} \rangle
\]

\[
= \int_{\Omega \setminus C_0} \varsigma \nabla u_2 \cdot \nabla P_\perp u_2 \, dx = \int_{\Omega \setminus C_0} \varsigma |\nabla P_\perp u_2|^2 \, dx.
\]

The right-hand inequality of (ii) is now a simple consequence Lemma 4.1.

**Proof of (iii).** Abbreviate \( \sigma_1 = \sigma(\varsigma, C_0, C_\infty) \) and \( \sigma_2 = \sigma(\varsigma, \emptyset, C_\infty) \), set \( u_1 = E(\varsigma, C_0)u_{j_1}^\varsigma \) and \( u_2 = u_{j_2}^\varsigma \), and introduce the corresponding ND maps \( A_j = A(\sigma_j) \) for \( j \in \{1, 2\} \).

To prove the left-hand inequality of (iii), we introduce a truncation of \( \sigma_1 \),

\[
\sigma_\epsilon = \begin{cases} 
\varsigma & \text{in } \Omega \setminus C, \\
\varsigma & \text{in } C_0, \\
\infty & \text{in } C_\infty.
\end{cases}
\]

When there are no perfectly insulating inclusions, the upper bound of (i) gives

\[
\langle (A_2 - A(\sigma_\epsilon))f, f \rangle \leq (\epsilon - 1) \int_{C_0} \varsigma |\nabla u_2|^2 \, dx.
\]  

(A.3)

Due to the convergence result of Theorem 3.1, taking the limit \( \epsilon \to 0 \) gives the left-hand inequality of (iii). (In fact, it is enough to consider the first term on the right-hand side of (4.4).)

Using the weak formulations for \( u_1 \) and \( u_2 \), along with the self-adjointness of \( A_1 \), we have

\[
\langle A_1 f, f \rangle = \int_{\Omega \setminus C} \varsigma |\nabla u_1|^2 \, dx = \int_{\Omega \setminus C_\infty} \varsigma |\nabla u_2|^2 \, dx = \int_{\Omega \setminus C_\infty} \varsigma |\nabla u_1 \cdot \nabla u_2| \, dx.
\]

It follows that

\[
\int_{\Omega \setminus C_\infty} \varsigma |(u_1 - u_2)|^2 \, dx = \int_{\Omega \setminus C_\infty} (\varsigma |\nabla u_1|^2 + \varsigma |\nabla u_2|^2 - \varsigma |\nabla u_1 \cdot \nabla u_2| - \varsigma |\nabla u_2 \cdot \nabla u_1|) \, dx
\]

\[
= \langle (A_2 - A_1)f, f \rangle + \int_{C_0} \varsigma |\nabla u_1|^2 \, dx.
\]  

(A.4)

Since the left-hand side of (A.4) is non-negative, we have proven the right-hand inequality in (iii) and thereby completed the whole proof.

\[\square\]

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