Supplementary Materials to
“Bootstrap methods for
multivariate hypothesis testing”

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1 Results of simulation studies

Tables 1–14 contain the results of the Monte Carlo simulation studies considered in Section 5 of the paper. Empirical sizes (as percentages) of all testing procedures are given in Tables 1–8 for all distributions of $e_i$ under consideration. Tables 9–14 contain empirical powers (as percentages) of all tests for Laplace and $\chi^2_{20}$ distributions of $e_i$. Since the asymptotic tests based on $Q_n(S_{n,B,\varepsilon}^{-2})$, $Q_n(S_{n,B,\varepsilon}^{-3})$ and $Q_n(S_{n,\varepsilon}^+)\text{ are usually too liberal,}$ their empirical powers are not really comparable, but they are included for illustration and completeness. In Tables 1–14, the three methods for approximating the null distribution of test statistics based on the asymptotic distribution, the nonparametric and parametric bootstrap are labeled as AD, NB and PB, respectively. Table 15 presents the numbers of rejected nonparametric bootstrap samples for which $\text{rank}(S_n^*) \neq \text{rank}(S_n)$ per 1000 for some data generated similarly as in Section 5 of the paper.
Table 1: Empirical sizes (as percentages) of the tests versus different covariance matrices and numbers of observations under normal distribution of errors ($\mu_0 = 0_p$, $p = 6$, $r = 4$, $\alpha = 5\%$, $B = 1000$, $nr = 1000$).

| $\Sigma$ | n     | Method | $Q_n(I_p)$ | $Q_n(S_{n,B,\varepsilon}^{-1})$ | $Q_n(S_{n,B,\varepsilon}^{-2})$ | $Q_n(S_{n,B,\varepsilon}^{-3})$ | $Q_n(S_{n,\varepsilon}^+)$ |
|---------|-------|--------|------------|-------------------------------|-------------------------------|-------------------------------|-----------------|
| $\Sigma_{(1)}$ | 7 | AD    | 6.7        | 2.9                           | 11.8                         | 18.3                         | 44.5            |
|         |     | NB    | 24.2       | 12.5                          | 17.7                         | 13.3                         | 2.5             |
|         |     | PB    | 7.6        | 3.4                           | 5.9                          | 3.5                          | 4.8             |
|         | 10  | AD    | 6.6        | 2.7                           | 10.9                         | 15.0                         | 30.6            |
|         |     | NB    | 7.5        | 3.7                           | 4.8                          | 2.3                          | 0.1             |
|         |     | PB    | 6.3        | 3.7                           | 5.4                          | 5.0                          | 5.1             |
|         | 14  | AD    | 6.0        | 3.6                           | 8.0                          | 9.6                          | 20.5            |
|         |     | NB    | 6.3        | 4.6                           | 4.0                          | 2.1                          | 1.0             |
|         |     | PB    | 6.0        | 4.7                           | 4.5                          | 3.7                          | 5.1             |
|         | 20  | AD    | 7.3        | 2.6                           | 10.1                         | 10.4                         | 15.2            |
|         |     | NB    | 7.7        | 3.6                           | 6.4                          | 4.3                          | 3.1             |
|         |     | PB    | 7.0        | 3.6                           | 7.5                          | 6.1                          | 6.0             |
| $\Sigma_{(2)}$ | 7 | AD    | 8.3        | 7.3                           | 11.8                         | 26.1                         | 48.5            |
|         |     | NB    | 21.2       | 12.7                          | 15.1                         | 13.5                         | 3.4             |
|         |     | PB    | 8.6        | 5.0                           | 4.6                          | 6.1                          | 5.7             |
|         | 10  | AD    | 7.2        | 7.2                           | 10.4                         | 18.9                         | 28.9            |
|         |     | NB    | 8.3        | 3.9                           | 3.4                          | 1.8                          | 0.1             |
|         |     | PB    | 7.5        | 4.7                           | 5.1                          | 4.8                          | 5.2             |
|         | 14  | AD    | 6.9        | 6.5                           | 8.2                          | 14.1                         | 20.0            |
|         |     | NB    | 6.3        | 3.6                           | 3.7                          | 2.1                          | 0.4             |
|         |     | PB    | 6.7        | 4.6                           | 5.2                          | 5.1                          | 4.2             |
|         | 20  | AD    | 8.0        | 7.6                           | 8.1                          | 11.3                         | 15.3            |
|         |     | NB    | 7.9        | 5.4                           | 4.2                          | 2.8                          | 1.9             |
|         |     | PB    | 7.8        | 6.2                           | 5.9                          | 5.0                          | 5.1             |
| $\Sigma_{(3)}$ | 7 | AD    | 8.6        | 7.9                           | 10.9                         | 25.5                         | 49.6            |
|         |     | NB    | 21.7       | 13.4                          | 14.5                         | 14.9                         | 3.1             |
|         |     | PB    | 8.7        | 4.9                           | 3.6                          | 5.7                          | 5.4             |
|         | 10  | AD    | 8.5        | 8.2                           | 10.0                         | 17.9                         | 30.3            |
|         |     | NB    | 9.0        | 5.8                           | 5.0                          | 2.1                          | 0.1             |
|         |     | PB    | 8.3        | 6.4                           | 6.2                          | 5.8                          | 5.6             |
|         | 14  | AD    | 6.2        | 6.0                           | 7.3                           | 13.4                         | 20.0            |
|         |     | NB    | 6.4        | 3.4                           | 2.7                           | 2.8                          | 0.5             |
|         |     | PB    | 6.3        | 4.9                           | 4.2                           | 4.1                          | 4.3             |
|         | 20  | AD    | 5.8        | 5.8                           | 6.7                           | 10.3                         | 13.9            |
|         |     | NB    | 6.0        | 4.4                           | 4.3                           | 3.7                          | 1.6             |
|         |     | PB    | 5.9        | 4.8                           | 5.1                           | 5.3                          | 4.3             |
Table 2: Empirical sizes (as percentages) of the tests versus different covariance matrices and numbers of observations under Laplace distribution of errors ($\mu_0 = 0_p$, $p = 6$, $r = 4$, $\alpha = 5\%$, $B = 1000$, $nr = 1000$).

| $\Sigma$     | $n$ | Method | $Q_n(I_p)$ | $Q_n(S_{n,B,\varepsilon}^{-1})$ | $Q_n(S_{n,B,\varepsilon}^{-2})$ | $Q_n(S_{n,B,\varepsilon}^{-3})$ | $Q_n(S_{n,\varepsilon}^{+})$ |
|--------------|-----|--------|------------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|
| $\Sigma_{(1)}$ | 7   | AD     | 6.9        | 3.7                           | 11.1                          | 17.6                          | 45.4                          |
|              |     | NB     | 23.4       | 13.6                          | 14.9                          | 12.2                          | 2.1                           |
|              |     | PB     | 7.3        | 4.2                           | 3.4                           | 2.6                           | 3.9                           |
|              | 10  | AD     | 5.7        | 3.8                           | 8.7                           | 14.1                          | 29.6                          |
|              |     | NB     | 6.9        | 3.7                           | 3.7                           | 1.5                           | 0                             |
|              |     | PB     | 5.6        | 3.4                           | 3.9                           | 3.4                           | 3.7                           |
|              | 14  | AD     | 4.9        | 3.3                           | 6.6                           | 10.3                          | 22.2                          |
|              |     | NB     | 5.3        | 4.3                           | 3.2                           | 2.2                           | 0.5                           |
|              |     | PB     | 5.3        | 3.9                           | 4.6                           | 4.7                           | 5.1                           |
|              | 20  | AD     | 4.8        | 3.4                           | 7.3                           | 7.6                           | 13.8                          |
|              |     | NB     | 4.5        | 4.0                           | 3.6                           | 2.7                           | 1.1                           |
|              |     | PB     | 5.0        | 4.2                           | 4.6                           | 4.5                           | 4.4                           |
| $\Sigma_{(2)}$ | 7   | AD     | 8.8        | 8.5                           | 12.2                          | 24.6                          | 47.1                          |
|              |     | NB     | 19.4       | 12.9                          | 14.3                          | 12.8                          | 3.0                           |
|              |     | PB     | 8.9        | 5.3                           | 4.1                           | 4.2                           | 4.7                           |
|              | 10  | AD     | 7.8        | 7.2                           | 10.6                          | 19.1                          | 31.0                          |
|              |     | NB     | 8.4        | 4.0                           | 3.0                           | 2.2                           | 0                             |
|              |     | PB     | 6.9        | 5.0                           | 4.8                           | 5.4                           | 6.3                           |
|              | 14  | AD     | 7.3        | 7.3                           | 9.4                           | 14.1                          | 19.1                          |
|              |     | NB     | 7.6        | 4.3                           | 3.7                           | 2.7                           | 1.0                           |
|              |     | PB     | 7.4        | 5.6                           | 5.2                           | 5.6                           | 5.6                           |
|              | 20  | AD     | 6.3        | 6.4                           | 7.3                           | 10.7                          | 16.3                          |
|              |     | NB     | 6.5        | 4.9                           | 3.4                           | 2.7                           | 2.1                           |
|              |     | PB     | 6.7        | 5.6                           | 4.9                           | 5.1                           | 4.6                           |
| $\Sigma_{(3)}$ | 7   | AD     | 9.2        | 8.7                           | 11.8                          | 23.7                          | 45.4                          |
|              |     | NB     | 17.5       | 12.4                          | 15.8                          | 13.8                          | 2.3                           |
|              |     | PB     | 9.2        | 6.6                           | 5.2                           | 5.1                           | 5.0                           |
|              | 10  | AD     | 7.6        | 7.2                           | 7.8                           | 17.7                          | 29.0                          |
|              |     | NB     | 8.6        | 4.6                           | 3.8                           | 2.4                           | 0                             |
|              |     | PB     | 7.3        | 5.4                           | 4.4                           | 4.7                           | 5.3                           |
|              | 14  | AD     | 8.0        | 7.8                           | 7.1                           | 14.7                          | 20.1                          |
|              |     | NB     | 8.1        | 4.9                           | 3.3                           | 2.7                           | 0.9                           |
|              |     | PB     | 8.4        | 6.5                           | 4.0                           | 5.0                           | 5.4                           |
|              | 20  | AD     | 7.8        | 7.7                           | 7.3                           | 10.6                          | 14.4                          |
|              |     | NB     | 7.6        | 5.0                           | 4.3                           | 2.0                           | 1.7                           |
|              |     | PB     | 8.2        | 6.2                           | 5.3                           | 4.9                           | 4.5                           |
Table 3: Empirical sizes (as percentages) of the tests versus different covariance matrices and numbers of observations under $\chi^2_{20}$ distribution of errors ($\mu_0 = 0_p$, $p = 6$, $r = 4$, $\alpha = 5\%$, $B = 1000$, $nr = 1000$).

| $\Sigma$ | $n$  | Method | $Q_n(I_p)$ | $Q_n(S_{n,B,\varepsilon}^{-1})$ | $Q_n(S_{n,B,\varepsilon}^{-2})$ | $Q_n(S_{n,B,\varepsilon}^{-3})$ | $Q_n(S_{n,B,\varepsilon}^+)$ |
|----------|------|--------|-------------|-------------------------------|-------------------------------|-------------------------------|----------------------|
| $\Sigma_{(1)}$ | 7    | AD     | 6.1         | 2.7                           | 10.9                          | 18.9                          | 45.8                 |
|          |      | NB     | 22.3        | 11.4                          | 16.7                          | 14.4                          | 2.7                  |
|          |      | PB     | 6.2         | 3.0                           | 4.9                           | 3.4                           | 4.2                  |
|          | 10   | AD     | 7.0         | 3.1                           | 10.1                          | 14.2                          | 29.6                 |
|          |      | NB     | 8.2         | 4.0                           | 4.8                           | 2.6                           | 0.1                  |
|          |      | PB     | 7.1         | 3.5                           | 5.4                           | 5.3                           | 5.5                  |
|          | 14   | AD     | 6.2         | 3.3                           | 9.6                           | 10.3                          | 21.1                 |
|          |      | NB     | 6.8         | 4.1                           | 4.9                           | 2.6                           | 1.0                  |
|          |      | PB     | 6.4         | 4.2                           | 6.0                           | 4.9                           | 6.0                  |
|          | 20   | AD     | 5.3         | 2.3                           | 8.1                           | 8.1                           | 13.7                 |
|          |      | NB     | 5.5         | 2.5                           | 4.2                           | 3.7                           | 2.3                  |
|          |      | PB     | 5.4         | 2.6                           | 5.4                           | 5.0                           | 5.6                  |
| $\Sigma_{(2)}$ | 7    | AD     | 9.1         | 8.6                           | 14.1                          | 25.5                          | 47.5                 |
|          |      | NB     | 20.0        | 13.1                          | 14.5                          | 12.0                          | 2.9                  |
|          |      | PB     | 9.3         | 5.4                           | 4.6                           | 5.4                           | 5.7                  |
|          | 10   | AD     | 8.4         | 8.1                           | 10.3                          | 19.0                          | 30.6                 |
|          |      | NB     | 9.5         | 4.3                           | 2.9                           | 2.9                           | 0.2                  |
|          |      | PB     | 8.2         | 5.1                           | 5.4                           | 6.2                           | 6.5                  |
|          | 14   | AD     | 7.3         | 7.4                           | 8.4                           | 13.8                          | 21.1                 |
|          |      | NB     | 7.5         | 4.3                           | 3.7                           | 1.5                           | 0.7                  |
|          |      | PB     | 7.3         | 5.8                           | 5.3                           | 4.5                           | 5.7                  |
|          | 20   | AD     | 5.8         | 6.0                           | 8.6                           | 12.5                          | 15.1                 |
|          |      | NB     | 6.1         | 4.8                           | 5.4                           | 3.0                           | 1.7                  |
|          |      | PB     | 6.1         | 5.2                           | 6.5                           | 5.9                           | 5.1                  |
| $\Sigma_{(3)}$ | 7    | AD     | 8.4         | 7.6                           | 10.8                          | 24.5                          | 46.2                 |
|          |      | NB     | 19.1        | 12.2                          | 14.1                          | 13.9                          | 2.9                  |
|          |      | PB     | 8.7         | 4.7                           | 3.6                           | 4.8                           | 5.9                  |
|          | 10   | AD     | 7.4         | 7.7                           | 8.4                           | 17.3                          | 26.4                 |
|          |      | NB     | 7.9         | 4.3                           | 3.0                           | 2.1                           | 0.1                  |
|          |      | PB     | 7.4         | 5.3                           | 4.6                           | 5.2                           | 4.1                  |
|          | 14   | AD     | 7.4         | 7.2                           | 7.5                           | 14.0                          | 22.4                 |
|          |      | NB     | 7.6         | 4.2                           | 4.2                           | 3.0                           | 1.6                  |
|          |      | PB     | 7.3         | 5.0                           | 5.8                           | 6.7                           | 6.3                  |
|          | 20   | AD     | 6.3         | 6.3                           | 5.8                           | 10.4                          | 13.8                 |
|          |      | NB     | 6.5         | 4.7                           | 3.8                           | 3.3                           | 2.1                  |
|          |      | PB     | 6.6         | 5.4                           | 4.5                           | 5.5                           | 5.5                  |
Table 4: Empirical sizes (as percentages) of the tests versus different covariance matrices and numbers of observations under log-normal distribution of errors ($\mu_0 = 0_p$, $p = 6$, $r = 4$, $\alpha = 5\%$, $B = 1000$, $nr = 1000$).

| $\Sigma$  | $n$ | Method | $Q_n(I_p)$ | $Q_n(S_{n,B,\epsilon}^{-1})$ | $Q_n(S_{n,B,\epsilon}^{-2})$ | $Q_n(S_{n,B,\epsilon}^{-3})$ | $Q_n(S_{n,B,\epsilon}^+)$ |
|-----------|-----|--------|------------|-----------------|-----------------|-----------------|-----------------|
| $\Sigma_{(1)}$ | 7   | AD     | 10.7       | 3.9             | 16.0            | 34.8            | 70.3            |
|           |     | NB     | 27.0       | 9.8             | 18.5            | 19.1            | 8.2             |
|           |     | PB     | 11.0       | 3.8             | 9.5             | 14.2            | 20.2            |
| 10        | AD  | 11.4    | 3.3        | 17.5            | 30.2            | 58.8            |
|           | NB  | 12.2    | 4.1        | 7.6             | 6.0             | 0.5             |
|           | PB  | 11.1    | 3.7        | 11.9            | 17.1            | 28.5            |
| 14        | AD  | 8.4     | 2.8        | 15.1            | 23.5            | 45.7            |
|           | NB  | 9.3     | 3.6        | 7.8             | 6.8             | 3.1             |
|           | PB  | 8.7     | 3.3        | 11.5            | 15.5            | 27.2            |
| 20        | AD  | 7.9     | 2.0        | 14.2            | 19.3            | 39.6            |
|           | NB  | 8.6     | 3.0        | 9.0             | 6.5             | 5.3             |
|           | PB  | 8.3     | 2.5        | 12.8            | 14.6            | 25.3            |
| $\Sigma_{(2)}$ | 7   | AD     | 12.8       | 11.7            | 21.4            | 41.0            | 67.8            |
|           |     | NB     | 22.6       | 15.0            | 19.2            | 18.6            | 7.5             |
|           |     | PB     | 12.8       | 8.7             | 12.8            | 15.4            | 20.2            |
| 10        | AD  | 13.8    | 13.2       | 20.5            | 31.7            | 53.2            |
|           | NB  | 14.8    | 8.6        | 9.0             | 5.7             | 0.3             |
|           | PB  | 13.5    | 10.5       | 11.8            | 16.7            | 23.6            |
| 14        | AD  | 11.0    | 10.4       | 16.1            | 27.6            | 44.0            |
|           | NB  | 11.2    | 7.2        | 7.4             | 6.2             | 2.3             |
|           | PB  | 11.0    | 8.9        | 12.1            | 17.3            | 23.2            |
| 20        | AD  | 7.5     | 7.1        | 11.4            | 21.0            | 33.0            |
|           | NB  | 7.6     | 5.0        | 5.9             | 5.6             | 4.1             |
|           | PB  | 7.3     | 5.7        | 9.3             | 14.2            | 20.0            |
| $\Sigma_{(3)}$ | 7   | AD     | 13.5       | 12.8            | 19.7            | 39.5            | 65.9            |
|           |     | NB     | 23.1       | 15.2            | 18.9            | 18.7            | 6.5             |
|           |     | PB     | 13.6       | 9.7             | 10.6            | 14.3            | 19.0            |
| 10        | AD  | 10.2    | 10.2       | 15.8            | 30.7            | 52.4            |
|           | NB  | 11.1    | 6.5        | 5.8             | 4.6             | 0.5             |
|           | PB  | 10.4    | 7.9        | 9.9             | 15.5            | 22.2            |
| 14        | AD  | 11.3    | 11.0       | 14.2            | 27.7            | 45.4            |
|           | NB  | 11.9    | 7.2        | 7.1             | 6.2             | 2.6             |
|           | PB  | 11.7    | 8.7        | 10.0            | 17.7            | 24.5            |
| 20        | AD  | 6.2     | 6.0        | 10.5            | 20.5            | 33.0            |
|           | NB  | 6.4     | 4.4        | 5.6             | 4.8             | 4.1             |
|           | PB  | 6.4     | 5.5        | 9.2             | 14.2            | 20.0            |
Table 5: Empirical sizes (as percentages) of the tests versus different covariance matrices and “greater” numbers of observations under normal distribution of errors ($\boldsymbol{\mu}_0 = \mathbf{0}_p$, $p = 6$, $r = 4$, $\alpha = 5\%$, $B = 1000$, $nr = 1000$).

| $\Sigma$  | n  | Method | Test statistic | $Q_n(I_p)$ | $Q_n(S_{n,B,\varepsilon}^{-1})$ | $Q_n(S_{n,B,\varepsilon}^{-2})$ | $Q_n(S_{n,B,\varepsilon}^{-3})$ | $Q_n(S_{n,\varepsilon}^+)$ |
|-----------|----|--------|----------------|------------|--------------------------------|-------------------------------|-------------------------------|--------------------------|
| $\Sigma_1$ | 30 | AD     | $Q_n(I_p)$     | 6.6        | 3.6                             | 8.4                           | 8.1                           | 11.4                     |
|           |    | NB     | $Q_n(S_{n,B,\varepsilon}^{-1})$ | 6.5        | 4.7                             | 5.6                           | 5.0                           | 3.3                      |
|           |    | PB     | $Q_n(S_{n,B,\varepsilon}^{-2})$ | 6.7        | 4.5                             | 6.4                           | 5.8                           | 5.2                      |
|           | 40 | AD     | $Q_n(I_p)$     | 5.1        | 2.9                             | 6.3                           | 6.8                           | 8.4                      |
|           |    | NB     | $Q_n(S_{n,B,\varepsilon}^{-1})$ | 5.5        | 4.0                             | 4.7                           | 4.5                           | 3.8                      |
|           |    | PB     | $Q_n(S_{n,B,\varepsilon}^{-2})$ | 5.4        | 4.1                             | 5.2                           | 5.0                           | 5.2                      |
|           | 50 | AD     | $Q_n(I_p)$     | 5.1        | 3.4                             | 5.1                           | 5.5                           | 7.7                      |
|           |    | NB     | $Q_n(S_{n,B,\varepsilon}^{-1})$ | 5.3        | 4.2                             | 3.9                           | 3.6                           | 3.8                      |
|           |    | PB     | $Q_n(S_{n,B,\varepsilon}^{-2})$ | 5.5        | 4.5                             | 4.2                           | 4.5                           | 5.0                      |
|           | 60 | AD     | $Q_n(I_p)$     | 5.4        | 3.2                             | 5.8                           | 6.0                           | 8.6                      |
|           |    | NB     | $Q_n(S_{n,B,\varepsilon}^{-1})$ | 5.4        | 4.2                             | 4.8                           | 4.7                           | 6.0                      |
|           |    | PB     | $Q_n(S_{n,B,\varepsilon}^{-2})$ | 5.4        | 4.3                             | 5.1                           | 5.4                           | 6.4                      |
| $\Sigma_2$ | 30 | AD     | $Q_n(I_p)$     | 6.8        | 6.9                             | 8.3                           | 9.4                           | 11.4                     |
|           |    | NB     | $Q_n(S_{n,B,\varepsilon}^{-1})$ | 7.0        | 5.7                             | 5.7                           | 4.7                           | 3.3                      |
|           |    | PB     | $Q_n(S_{n,B,\varepsilon}^{-2})$ | 7.2        | 6.5                             | 6.5                           | 6.4                           | 5.2                      |
|           | 40 | AD     | $Q_n(I_p)$     | 6.5        | 6.1                             | 6.1                           | 7.3                           | 8.4                      |
|           |    | NB     | $Q_n(S_{n,B,\varepsilon}^{-1})$ | 5.7        | 5.2                             | 4.8                           | 3.8                           | 3.8                      |
|           |    | PB     | $Q_n(S_{n,B,\varepsilon}^{-2})$ | 6.4        | 5.8                             | 5.2                           | 5.0                           | 5.2                      |
|           | 50 | AD     | $Q_n(I_p)$     | 5.4        | 5.3                             | 4.8                           | 6.4                           | 7.7                      |
|           |    | NB     | $Q_n(S_{n,B,\varepsilon}^{-1})$ | 5.1        | 4.4                             | 3.9                           | 3.8                           | 3.8                      |
|           |    | PB     | $Q_n(S_{n,B,\varepsilon}^{-2})$ | 5.3        | 4.7                             | 4.4                           | 4.6                           | 5.0                      |
|           | 60 | AD     | $Q_n(I_p)$     | 5.2        | 5.0                             | 5.9                           | 7.2                           | 8.6                      |
|           |    | NB     | $Q_n(S_{n,B,\varepsilon}^{-1})$ | 5.6        | 4.6                             | 4.8                           | 5.0                           | 6.0                      |
|           |    | PB     | $Q_n(S_{n,B,\varepsilon}^{-2})$ | 5.2        | 4.8                             | 5.0                           | 5.8                           | 6.4                      |
Table 6: Empirical sizes (as percentages) of the tests versus different covariance matrices and “greater” numbers of observations under Laplace distribution of errors ($\mu_0 = 0_p$, $p = 6$, $r = 4$, $\alpha = 5\%$, $B = 1000$, $nr = 1000$).

| $\Sigma$ | n  | Method | Test statistic | $Q_n(I_p)$ | $Q_n(S_{n,B,\varepsilon}^{-1})$ | $Q_n(S_{n,B,\varepsilon}^{-2})$ | $Q_n(S_{n,B,\varepsilon}^{-3})$ | $Q_n(S_{n,\varepsilon}^+)$ |
|----------|----|--------|----------------|------------|---------------------------------|---------------------------------|---------------------------------|-------------------------------|
| $\Sigma_{(1)}$ | 30 | AD     |                | 6.1        | 4.4                             | 6.5                             | 7.6                             | 9.9                           |
|          |    | NB     |                | 6.2        | 5.9                             | 4.6                             | 3.9                             | 3.8                           |
|          |    | PB     |                | 6.3        | 5.4                             | 5.1                             | 5.2                             | 5.3                           |
|          | 40 | AD     |                | 4.8        | 4.0                             | 5.1                             | 6.8                             | 8.1                           |
|          |    | NB     |                | 5.0        | 4.7                             | 3.0                             | 3.8                             | 2.6                           |
|          |    | PB     |                | 5.0        | 4.4                             | 4.0                             | 4.7                             | 4.6                           |
|          | 50 | AD     |                | 5.0        | 4.3                             | 5.5                             | 5.6                             | 7.1                           |
|          |    | NB     |                | 5.2        | 4.8                             | 3.6                             | 3.8                             | 3.6                           |
|          |    | PB     |                | 5.4        | 4.5                             | 4.5                             | 4.5                             | 4.8                           |
|          | 60 | AD     |                | 4.7        | 3.5                             | 5.2                             | 6.6                             | 8.3                           |
|          |    | NB     |                | 4.9        | 4.7                             | 4.1                             | 4.6                             | 4.1                           |
|          |    | PB     |                | 4.7        | 4.3                             | 4.7                             | 5.2                             | 5.4                           |
| $\Sigma_{(2)}$ | 30 | AD     |                | 5.4        | 5.5                             | 5.7                             | 8.4                             | 10.2                          |
|          |    | NB     |                | 6.0        | 4.7                             | 3.3                             | 3.6                             | 3.0                           |
|          |    | PB     |                | 5.8        | 4.9                             | 4.0                             | 4.7                             | 4.6                           |
|          | 40 | AD     |                | 4.7        | 4.9                             | 6.0                             | 6.9                             | 8.6                           |
|          |    | NB     |                | 5.2        | 4.0                             | 3.9                             | 3.5                             | 3.7                           |
|          |    | PB     |                | 4.8        | 4.1                             | 5.0                             | 4.1                             | 4.8                           |
|          | 50 | AD     |                | 5.7        | 5.8                             | 6.2                             | 7.2                             | 7.6                           |
|          |    | NB     |                | 6.1        | 5.3                             | 5.2                             | 3.9                             | 3.3                           |
|          |    | PB     |                | 6.1        | 5.8                             | 5.4                             | 4.6                             | 4.5                           |
|          | 60 | AD     |                | 4.7        | 4.6                             | 4.7                             | 6.2                             | 7.0                           |
|          |    | NB     |                | 4.8        | 4.3                             | 3.9                             | 3.9                             | 3.8                           |
|          |    | PB     |                | 4.5        | 4.4                             | 4.2                             | 4.3                             | 4.8                           |
Table 7: Empirical sizes (as percentages) of the tests versus different covariance matrices and “greater” numbers of observations under $\chi^2_{20}$ distribution of errors ($\mu_0 = 0_p$, $p = 6$, $r = 4$, $\alpha = 5\%$, $B = 1000$, $nr = 1000$).

| $\Sigma$ | n  | Method | $Q_n(I_p)$ | $Q_n(S_n^{-1}_{n,B,\varepsilon})$ | $Q_n(S_n^{-2}_{n,B,\varepsilon})$ | $Q_n(S_n^{-3}_{n,B,\varepsilon})$ | $Q_n(S_n^{+}_{n,\varepsilon})$ |
|----------|-----|--------|------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| $\Sigma_{(1)}$ | 30  | AD     | 5.8        | 2.4                             | 8.6                             | 6.8                             | 12.6                             |
|          |     | NB     | 6.0        | 3.6                             | 5.4                             | 4.3                             | 4.5                             |
|          |     | PB     | 6.3        | 3.3                             | 6.9                             | 5.0                             | 6.5                             |
|          | 40  | AD     | 6.4        | 3.8                             | 7.6                             | 7.1                             | 10.0                             |
|          |     | NB     | 6.2        | 5.0                             | 5.6                             | 4.8                             | 4.4                             |
|          |     | PB     | 6.5        | 4.4                             | 6.9                             | 5.6                             | 6.1                             |
|          | 50  | AD     | 4.9        | 3.2                             | 5.8                             | 5.2                             | 8.3                             |
|          |     | NB     | 4.8        | 4.1                             | 4.4                             | 4.1                             | 4.0                             |
|          |     | PB     | 5.1        | 3.6                             | 4.9                             | 4.5                             | 5.4                             |
|          | 60  | AD     | 4.9        | 3.3                             | 5.8                             | 5.3                             | 6.7                             |
|          |     | NB     | 5.3        | 4.4                             | 4.6                             | 4.0                             | 3.7                             |
|          |     | PB     | 5.2        | 3.7                             | 5.2                             | 4.2                             | 4.4                             |
| $\Sigma_{(2)}$ | 30  | AD     | 6.5        | 6.7                             | 7.1                             | 8.9                             | 10.1                             |
|          |     | NB     | 6.7        | 5.7                             | 4.8                             | 4.7                             | 3.9                             |
|          |     | PB     | 6.4        | 5.6                             | 5.8                             | 6.1                             | 5.7                             |
|          | 40  | AD     | 6.2        | 5.9                             | 6.2                             | 7.3                             | 9.0                             |
|          |     | NB     | 6.5        | 5.3                             | 5.2                             | 4.1                             | 4.2                             |
|          |     | PB     | 6.0        | 5.5                             | 5.8                             | 5.0                             | 5.7                             |
|          | 50  | AD     | 5.2        | 5.1                             | 5.9                             | 6.5                             | 8.8                             |
|          |     | NB     | 5.1        | 4.5                             | 3.9                             | 4.3                             | 4.6                             |
|          |     | PB     | 5.2        | 4.7                             | 5.0                             | 4.7                             | 5.1                             |
|          | 60  | AD     | 6.1        | 5.9                             | 6.4                             | 7.0                             | 8.7                             |
|          |     | NB     | 6.1        | 5.2                             | 4.7                             | 4.6                             | 5.4                             |
|          |     | PB     | 5.9        | 5.2                             | 5.8                             | 4.8                             | 6.3                             |
Table 8: Empirical sizes (as percentages) of the tests versus different covariance matrices and “greater” numbers of observations under log-normal distribution of errors ($\mu_0 = 0_p$, $p = 6$, $r = 4$, $\alpha = 5\%$, $B = 1000$, $nr = 1000$).

| $\Sigma$  | $n$  | Method | $Q_n(I_p)$ | $Q_n(S_{n,B,\varepsilon}^{-1})$ | $Q_n(S_{n,B,\varepsilon}^{-2})$ | $Q_n(S_{n,B,\varepsilon}^{-3})$ | $Q_n(S_{n,\varepsilon}^+)$. |
|-----------|------|--------|------------|-----------------|-----------------|-----------------|-------------------|
| $\Sigma_{(1)}$ | 30   | AD     | 5.7        | 1.6             | 13.9            | 16.2            | 28.6              |
|           |      | NB     | 5.3        | 2.4             | 8.4             | 6.8             | 5.8               |
|           |      | PB     | 5.8        | 2.0             | 12.3            | 13.0            | 20.7              |
|           | 40   | AD     | 6.9        | 2.3             | 13.1            | 14.7            | 23.8              |
|           |      | NB     | 6.9        | 2.9             | 8.4             | 7.6             | 6.8               |
|           |      | PB     | 7.3        | 2.4             | 12.5            | 12.4            | 19.2              |
|           | 50   | AD     | 6.8        | 2.0             | 13.6            | 13.9            | 23.5              |
|           |      | NB     | 6.8        | 3.0             | 9.7             | 6.9             | 5.3               |
|           |      | PB     | 6.9        | 2.4             | 13.6            | 12.1            | 19.7              |
|           | 60   | AD     | 6.2        | 1.9             | 11.1            | 11.1            | 20.8              |
|           |      | NB     | 6.4        | 2.6             | 7.6             | 5.3             | 5.9               |
|           |      | PB     | 6.4        | 2.1             | 10.4            | 10.0            | 16.9              |
| $\Sigma_{(2)}$ | 30   | AD     | 7.7        | 7.4             | 10.1            | 16.0            | 25.1              |
|           |      | NB     | 7.8        | 6.2             | 5.8             | 5.1             | 5.5               |
|           |      | PB     | 7.7        | 6.7             | 8.4             | 12.0            | 17.6              |
|           | 40   | AD     | 6.2        | 5.7             | 9.4             | 14.6            | 20.8              |
|           |      | NB     | 6.3        | 4.0             | 5.7             | 6.1             | 6.4               |
|           |      | PB     | 6.5        | 4.9             | 8.4             | 12.0            | 17.4              |
|           | 50   | AD     | 5.2        | 5.3             | 7.8             | 12.9            | 18.3              |
|           |      | NB     | 5.4        | 4.6             | 4.6             | 5.1             | 5.0               |
|           |      | PB     | 5.5        | 4.6             | 6.9             | 10.2            | 14.0              |
|           | 60   | AD     | 7.9        | 7.4             | 8.2             | 12.2            | 17.6              |
|           |      | NB     | 8.0        | 5.7             | 5.3             | 5.8             | 5.9               |
|           |      | PB     | 7.5        | 7.1             | 7.4             | 10.4            | 14.4              |
Table 9: Empirical powers (as percentages) of the tests under Laplace distribution of errors, $\Sigma = \Sigma_{(1)}$ and $n = 20$ ($\mu_0 = 0_p$, $p = 6$, $r = 4$, $\alpha = 5\%$, $B = 1000$, $nr = 1000$). The empirical powers of the asymptotic tests based on $Q_n(S_{n,B,\varepsilon}^{-2})$, $Q_n(S_{n,B,\varepsilon}^{-3})$ and $Q_n(S_{n,\varepsilon}^+)$ are included for illustration and completeness only, since they are usually too liberal.

| $\mu'$   | Method | Test statistic | $Q_n(I_p)$ | $Q_n(S_{n,B,\varepsilon}^{-1})$ | $Q_n(S_{n,B,\varepsilon}^{-2})$ | $Q_n(S_{n,B,\varepsilon}^{-3})$ | $Q_n(S_{n,\varepsilon}^+)$ |
|----------|--------|----------------|------------|--------------------------------|-------------------------------|-------------------------------|-------------------------|
| $(2, 0’_5)$ | AD     | 70.8           | 38.1       | 74.0                           | 71.3                          | 72.9                          |
|          | NB     | 71.2           | 39.9       | 62.2                           | 51.6                          | 32.6                          |
|          | PB     | 70.6           | 39.3       | 68.3                           | 59.3                          | 52.1                          |
| $(0, 2, 0’_4)$ | AD     | 70.5           | 39.2       | 73.5                           | 70.0                          | 71.8                          |
|          | NB     | 70.6           | 41.4       | 62.4                           | 51.7                          | 29.3                          |
|          | PB     | 71.5           | 39.9       | 66.9                           | 59.9                          | 50.8                          |
| $(1.51’_2, 0’_4)$ | AD     | 74.9           | 43.4       | 78.8                           | 77.4                          | 78.0                          |
|          | NB     | 74.9           | 45.0       | 69.2                           | 55.7                          | 34.7                          |
|          | PB     | 75.3           | 44.7       | 74.3                           | 67.5                          | 58.4                          |
| $(0’_2, 1, 0’_3)$ | AD     | 16.7           | 3.3        | 8.4                            | 56.2                          | 96.3                          |
|          | NB     | 18.3           | 4.2        | 4.4                            | 43.5                          | 69.1                          |
|          | PB     | 17.4           | 4.0        | 5.6                            | 49.3                          | 88.5                          |
| $(0’_2, 1’_2, 0’_2)$ | AD     | 44.7           | 3.2        | 8.6                            | 77.6                          | 99.9                          |
|          | NB     | 46.8           | 4.2        | 5.0                            | 65.6                          | 95.7                          |
|          | PB     | 44.5           | 3.9        | 6.2                            | 71.3                          | 99.1                          |
| $(1’_3, 0’_3)$ | AD     | 60.1           | 23.8       | 49.3                           | 77.7                          | 99.6                          |
|          | NB     | 61.0           | 25.8       | 38.0                           | 64.0                          | 80.0                          |
|          | PB     | 59.7           | 25.7       | 41.4                           | 69.8                          | 95.4                          |
| $(1’_4, 0’_2)$ | AD     | 75.6           | 23.8       | 49.3                           | 88.8                          | 100                           |
|          | NB     | 76.5           | 25.7       | 37.2                           | 80.1                          | 97.5                          |
|          | PB     | 75.4           | 26.2       | 41.5                           | 84.4                          | 99.7                          |
Table 10: Empirical powers (as percentages) of the tests under Laplace distribution of errors, \( \Sigma = \Sigma_{(2)} \) and \( n = 20 \) (\( \mu_0 = 0_p, \ p = 6, \ r = 4, \ \alpha = 5\%, \ B = 1000, \ nr = 1000, \ a_1 = v_1 + v_2, \ a_2 = v_3 + v_4, \ a_3 = v_1 + v_2 + v_3, \ a_4 = v_1 + \cdots + v_4, \) where \( v_1, \ldots, v_6 \) are the eigenvectors of \( \Sigma_{(2)} \) corresponding to the eigenvalues \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_6 \) respectively). The empirical powers of the asymptotic tests based on \( Q_n(S_{n,B,\varepsilon}^{-1}) \), \( Q_n(S_{n,B,\varepsilon}^{-2}) \) and \( Q_n(S_{n,B,\varepsilon}^+) \) are included for illustration and completeness only, since they are usually too liberal.

| \( \mu \) | Method | Test statistic |
|----------|--------|----------------|
| | | \( Q_n(L_p) \) & \( Q_n(S_{n,B,\varepsilon}^{-1}) \) & \( Q_n(S_{n,B,\varepsilon}^{-2}) \) & \( Q_n(S_{n,B,\varepsilon}^+) \) |
| 2v₁ | AD | 86.8 | 86.8 | 81.2 | 80.7 | 80.1 |
| | NB | 87.1 | 81.5 | 70.7 | 56.1 | 40.0 |
| | PB | 86.3 | 83.6 | 75.0 | 68.5 | 61.0 |
| v₂/2 | AD | 9.4 | 5.3 | 73.3 | 75.8 | 75.0 |
| | NB | 9.6 | 3.9 | 60.1 | 49.8 | 33.5 |
| | PB | 9.4 | 4.5 | 66.0 | 61.4 | 55.7 |
| a₁/2 | AD | 20.7 | 15.2 | 75.6 | 79.0 | 78.4 |
| | NB | 21.2 | 11.8 | 63.8 | 53.2 | 37.8 |
| | PB | 20.9 | 13.2 | 69.7 | 65.5 | 59.5 |
| v₃/3 | AD | 7.2 | 5.3 | 9.7 | 88.4 | 88.6 |
| | NB | 7.9 | 4.3 | 5.8 | 68.3 | 53.2 |
| | PB | 7.0 | 4.4 | 7.1 | 80.1 | 73.7 |
| a₂/6 | AD | 6.7 | 5.4 | 7.3 | 40.1 | 98.9 |
| | NB | 6.9 | 4.2 | 4.7 | 18.8 | 85.0 |
| | PB | 6.4 | 4.3 | 5.3 | 26.9 | 95.3 |
| a₃/3 | AD | 14.1 | 10.7 | 47.4 | 93.1 | 92.8 |
| | NB | 14.5 | 7.8 | 34.0 | 79.2 | 64.7 |
| | PB | 14.2 | 8.7 | 40.2 | 86.7 | 81.5 |
| a₄/8 | AD | 7.1 | 6.5 | 12.2 | 34.1 | 93.0 |
| | NB | 7.7 | 5.2 | 6.2 | 14.2 | 59.4 |
| | PB | 7.4 | 5.5 | 8.2 | 22.8 | 78.5 |
Table 11: Empirical powers (as percentages) of the tests under Laplace distribution of errors, $\Sigma = \Sigma(3)$ and $n = 20$ ($\mu_0 = 0_p$, $p = 6$, $r = 4$, $\alpha = 5\%$, $B = 1000$, $n_r = 1000$, $w = 1000$, $n_r = 1000$, $v_1, \ldots, v_6$ are the eigenvectors of $\Sigma(3)$ corresponding to the eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_6$ respectively).

The empirical powers of the asymptotic tests based on $Q_n(S_{n,B,\varepsilon}^{-1})$, $Q_n(S_{n,B,\varepsilon}^{-2})$ and $Q_n(S_{n,\varepsilon}^{+})$ are included for illustration and completeness only, since they are usually too liberal.

| $\mu$ | Method | $Q_n(L_p)$ | $Q_n(S_{n,B,\varepsilon}^{-1})$ | $Q_n(S_{n,B,\varepsilon}^{-2})$ | $Q_n(S_{n,B,\varepsilon}^{-3})$ | $Q_n(S_{n,\varepsilon}^{+})$ |
|-------|--------|------------|-------------------------------|-------------------------------|-------------------------------|--------------------------|
| $2v_1$ | AD     | 92.4       | 92.2                          | 89.0                          | 87.0                          | 85.9                     |
|       | NB     | 92.8       | 89.5                          | 82.9                          | 67.3                          | 46.7                     |
|       | PB     | 92.2       | 90.8                          | 85.2                          | 76.7                          | 71.4                     |
| $v_2/2$ | AD     | 9.8        | 5.6                           | 80.9                          | 90.6                          | 91.3                     |
|       | NB     | 9.9        | 4.1                           | 74.0                          | 74.0                          | 55.7                     |
|       | PB     | 10.1       | 4.6                           | 76.3                          | 84.1                          | 78.4                     |
| $a_1/2$ | AD     | 23.1       | 15.8                          | 83.6                          | 93.0                          | 92.2                     |
|       | NB     | 23.5       | 11.1                          | 76.5                          | 77.5                          | 56.7                     |
|       | PB     | 22.7       | 12.5                          | 79.3                          | 86.2                          | 80.9                     |
| $v_3/3$ | AD     | 6.8        | 5.5                           | 19.5                          | 83.0                          | 84.3                     |
|       | NB     | 6.7        | 4.2                           | 14.5                          | 60.4                          | 45.1                     |
|       | PB     | 7.1        | 4.5                           | 15.6                          | 71.9                          | 66.2                     |
| $a_2/6$ | AD     | 6.4        | 5.5                           | 9.5                           | 35.3                          | 96.9                     |
|       | NB     | 6.3        | 4.2                           | 5.9                           | 14.5                          | 71.6                     |
|       | PB     | 6.1        | 4.5                           | 7.1                           | 21.9                          | 87.6                     |
| $a_3/3$ | AD     | 13.6       | 8.6                           | 57.0                          | 95.0                          | 94.9                     |
|       | NB     | 14.7       | 5.8                           | 46.7                          | 81.6                          | 67.1                     |
|       | PB     | 13.4       | 7.4                           | 52.1                          | 89.9                          | 85.0                     |
| $a_4/8$ | AD     | 5.8        | 5.6                           | 13.3                          | 31.5                          | 85.1                     |
|       | NB     | 5.9        | 3.9                           | 9.2                           | 12.8                          | 45.2                     |
|       | PB     | 5.6        | 4.1                           | 10.2                          | 19.2                          | 67.1                     |
Table 12: Empirical powers (as percentages) of the tests under $\chi^2_{20}$ distribution of errors, $\mathbf{\Sigma} = \mathbf{\Sigma}_{(4)}$ and $n = 20$ ($\mu_0 = 0_p$, $p = 6$, $r = 4$, $\alpha = 5\%$, $B = 1000$, $nr = 1000$). The empirical powers of the asymptotic tests based on $Q_n(S^{-2}_{n,B,\epsilon})$, $Q_n(S^{-3}_{n,B,\epsilon})$ and $Q_n(S^+_{n,\epsilon})$ are included for illustration and completeness only, since they are usually too liberal.

| $\mu'$ | Method | Test statistic |
|--------|--------|---------------|
|        |        | $Q_n(I_p)$    | $Q_n(S^{-1}_{n,B,\epsilon})$ | $Q_n(S^{-2}_{n,B,\epsilon})$ | $Q_n(S^{-3}_{n,B,\epsilon})$ | $Q_n(S^+_{n,\epsilon})$ |
| (2, $0'_5$) | AD 71.8 | 42.9 | 75.0 | 70.9 | 72.2 |
|         | NB 72.0 | 45.3 | 58.9 | 47.6 | 24.2 |
|         | PB 71.8 | 45.3 | 67.2 | 57.6 | 45.9 |
| (0, 2, $0'_4$) | AD 71.4 | 41.5 | 74.7 | 70.7 | 72.1 |
|         | NB 72.1 | 44.0 | 60.3 | 48.0 | 25.1 |
|         | PB 72.0 | 43.9 | 66.2 | 58.1 | 46.2 |
| (1.51'_2, $0'_4$) | AD 76.5 | 51.1 | 77.5 | 72.7 | 73.8 |
|         | NB 77.2 | 53.3 | 61.7 | 51.6 | 29.4 |
|         | PB 76.9 | 54.1 | 68.5 | 60.8 | 49.8 |
| (0'_2, 1, $0'_3$) | AD 16.4 | 2.2 | 8.7 | 59.1 | 98.7 |
|         | NB 17.2 | 2.9 | 4.6 | 46.4 | 76.5 |
|         | PB 16.4 | 2.8 | 6.1 | 51.9 | 93.2 |
| (0'_2, 1'_2, $0'_2$) | AD 41.2 | 2.1 | 8.6 | 79.2 | 100 |
|         | NB 43.3 | 3.0 | 5.6 | 69.7 | 99.0 |
|         | PB 42.1 | 2.9 | 6.4 | 73.6 | 100 |
| (1'_3, $0'_4$) | AD 59.7 | 25.9 | 41.7 | 75.7 | 99.6 |
|         | NB 60.2 | 29.3 | 26.1 | 60.8 | 82.7 |
|         | PB 61.0 | 28.8 | 33.4 | 67.1 | 97.0 |
| (1'_4, $0'_2$) | AD 80.6 | 26.4 | 41.3 | 87.5 | 100 |
|         | NB 81.4 | 28.8 | 26.9 | 78.9 | 99.2 |
|         | PB 80.3 | 29.1 | 33.3 | 83.8 | 100 |
Table 13: Empirical powers (as percentages) of the tests under $\chi^2_{20}$ distribution of errors, $\Sigma = \Sigma^{(2)}$ and $n = 20$ ($\mu_0 = 0_p$, $p = 6$, $r = 4$, $\alpha = 5\%$, $B = 1000$, $nr = 1000$, $a_1 = v_1 + v_2$, $a_2 = v_3 + v_4$, $a_3 = v_1 + v_2 + v_3$, $a_4 = v_1 + \cdots + v_4$, where $v_1, \ldots, v_6$ are the eigenvectors of $\Sigma^{(2)}$ corresponding to the eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_6$ respectively). The empirical powers of the asymptotic tests based on $Q_n(S_{n,B,\varepsilon}^{-1})$, $Q_n(S_{n,B,\varepsilon}^{-2})$ and $Q_n(S_{n,\varepsilon}^{-)}$ are included for illustration and completeness only, since they are usually too liberal.

| $\mu$ | Method | $Q_n(I_p)$ | $Q_n(S_{n,B,\varepsilon}^{-1})$ | $Q_n(S_{n,B,\varepsilon}^{-2})$ | $Q_n(S_{n,B,\varepsilon}^{-3})$ | $Q_n(S_{n,\varepsilon}^{-})$ |
|-------|--------|------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| $2v_1$ | AD     | 84.0       | 83.8                            | 80.0                            | 76.4                            | 77.2                            |
|       | NB     | 84.2       | 79.4                            | 68.9                            | 56.4                            | 40.2                            |
|       | PB     | 84.4       | 81.1                            | 73.5                            | 66.0                            | 61.0                            |
| $v_2/2$ | AD     | 9.0        | 6.6                             | 70.4                            | 72.9                            | 72.9                            |
|       | NB     | 9.9        | 4.6                             | 58.9                            | 47.0                            | 31.8                            |
|       | PB     | 9.3        | 5.5                             | 64.3                            | 59.8                            | 51.3                            |
| $a_1/2$ | AD     | 19.3       | 14.3                            | 72.9                            | 75.5                            | 74.8                            |
|       | NB     | 19.3       | 10.5                            | 60.7                            | 51.1                            | 36.8                            |
|       | PB     | 19.4       | 11.9                            | 65.6                            | 61.0                            | 54.6                            |
| $v_3/3$ | AD     | 7.3        | 6.3                             | 9.5                             | 86.7                            | 85.4                            |
|       | NB     | 7.5        | 4.4                             | 5.6                             | 64.2                            | 48.3                            |
|       | PB     | 7.4        | 5.5                             | 6.7                             | 75.6                            | 68.2                            |
| $a_2/6$ | AD     | 6.6        | 6.2                             | 7.6                             | 38.7                            | 99.0                            |
|       | NB     | 7.0        | 4.4                             | 4.2                             | 19.0                            | 83.5                            |
|       | PB     | 7.1        | 5.4                             | 5.6                             | 26.4                            | 95.0                            |
| $a_3/3$ | AD     | 13.3       | 9.5                             | 44.6                            | 90.9                            | 91.0                            |
|       | NB     | 13.4       | 7.3                             | 31.3                            | 76.2                            | 62.1                            |
|       | PB     | 13.3       | 8.4                             | 35.9                            | 85.0                            | 78.6                            |
| $a_4/8$ | AD     | 7.4        | 6.9                             | 11.8                            | 32.7                            | 90.7                            |
|       | NB     | 7.5        | 4.9                             | 7.0                             | 15.9                            | 54.7                            |
|       | PB     | 7.7        | 5.9                             | 8.3                             | 21.6                            | 75.2                            |
Table 14: Empirical powers (as percentages) of the tests under $\chi^2_{20}$ distribution of errors, $\Sigma = \Sigma_{(3)}$ and $n = 20$ ($\mu_0 = 0$, $p = 6$, $r = 4$, $\alpha = 5\%$, $B = 1000$, $nx = 1000$, $a_1 = v_1 + v_2$, $a_2 = v_3 + v_4$, $a_3 = v_1 + v_2 + v_3$, $a_4 = v_1 + \cdots + v_4$, where $v_1, \ldots, v_6$ are the eigenvectors of $\Sigma_{(3)}$ corresponding to the eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_6$ respectively). The empirical powers of the asymptotic tests based on $Q_n(S_{n,B,\varepsilon}^{-1})$, $Q_n(S_{n,B,\varepsilon}^{-2})$ and $Q_n(S_{n,\varepsilon}^{+})$ are included for illustration and completeness only, since they are usually too liberal.

| $\mu$ | Method | Test statistic |
|-------|--------|----------------|
|       | $Q_n(I_p)$ | $Q_n(S_{n,B,\varepsilon}^{-1})$ | $Q_n(S_{n,B,\varepsilon}^{-2})$ | $Q_n(S_{n,B,\varepsilon}^{-3})$ | $Q_n(S_{n,\varepsilon}^{+})$ |
| $2v_1$ | AD | 95.2 | 95.3 | 90.6 | 87.7 | 84.9 |
|       | NB | 95.2 | 91.0 | 82.9 | 64.2 | 41.8 |
|       | PB | 94.8 | 93.0 | 86.9 | 75.4 | 64.7 |
| $v_2/2$ | AD | 11.4 | 6.4 | 83.9 | 92.1 | 91.4 |
|       | NB | 11.8 | 4.6 | 76.2 | 73.3 | 53.7 |
|       | PB | 12.6 | 5.2 | 80.7 | 83.8 | 76.0 |
| $a_1/2$ | AD | 22.7 | 16.9 | 87.9 | 94.5 | 93.6 |
|       | NB | 23.8 | 12.2 | 80.8 | 77.1 | 57.0 |
|       | PB | 23.6 | 13.6 | 84.7 | 87.1 | 79.4 |
| $v_3/3$ | AD | 8.1 | 6.5 | 18.4 | 81.9 | 83.1 |
|       | NB | 8.6 | 4.6 | 13.8 | 58.1 | 41.3 |
|       | PB | 8.4 | 5.2 | 16.0 | 68.8 | 61.7 |
| $a_2/6$ | AD | 7.1 | 6.3 | 9.6 | 32.1 | 96.1 |
|       | NB | 7.9 | 4.5 | 5.3 | 15.2 | 65.8 |
|       | PB | 7.6 | 5.3 | 7.2 | 21.0 | 85.8 |
| $a_3/3$ | AD | 15.7 | 11.7 | 57.4 | 97.5 | 97.7 |
|       | NB | 16.3 | 8.7 | 48.4 | 85.2 | 69.7 |
|       | PB | 16.1 | 9.6 | 51.4 | 92.9 | 88.7 |
| $a_4/8$ | AD | 8.1 | 7.1 | 13.7 | 30.1 | 84.0 |
|       | NB | 8.3 | 5.2 | 8.6 | 11.5 | 42.2 |
|       | PB | 8.3 | 5.8 | 10.5 | 18.7 | 63.2 |
Table 15: Numbers of rejected nonparametric bootstrap samples for which \( \text{rank}(S^*_n) \neq \text{rank}(S_n) \) per 1000 for data generated similarly as in Section 5 of the paper \((p = 6, r = 4)\). For \( n = 15, \ldots, 20 \), the numbers of rejected samples were always equal to zero.

| \( \Sigma \) | \( n \) | Normal | Laplace | \( \chi^2_{20} \) | Log-normal |
|---|---|---|---|---|---|
| \( \Sigma_{(1)} \) | 7 | 595 | 567 | 580 | 609 |
| | 8 | 304 | 292 | 287 | 299 |
| | 9 | 117 | 120 | 119 | 106 |
| | 10 | 32 | 34 | 37 | 28 |
| | 11 | 4 | 8 | 6 | 7 |
| | 12 | 1 | 0 | 2 | 0 |
| | 13 | 2 | 1 | 0 | 0 |
| | 14 | 0 | 0 | 0 | 0 |
| \( \Sigma_{(2)} \) | 7 | 569 | 603 | 596 | 613 |
| | 8 | 291 | 295 | 276 | 279 |
| | 9 | 121 | 110 | 110 | 118 |
| | 10 | 37 | 31 | 35 | 29 |
| | 11 | 9 | 7 | 3 | 3 |
| | 12 | 0 | 2 | 1 | 4 |
| | 13 | 1 | 0 | 0 | 0 |
| | 14 | 0 | 0 | 0 | 0 |
| \( \Sigma_{(3)} \) | 7 | 577 | 604 | 567 | 612 |
| | 8 | 278 | 273 | 290 | 277 |
| | 9 | 127 | 97 | 116 | 117 |
| | 10 | 25 | 26 | 38 | 32 |
| | 11 | 8 | 6 | 4 | 4 |
| | 12 | 0 | 1 | 1 | 2 |
| | 13 | 1 | 0 | 0 | 0 |
| | 14 | 0 | 0 | 0 | 1 |