GLOBAL WEAK SOLUTION TO THE INHOMOGENEOUS NAVIER-STOKES-VLASOV EQUATIONS

DEHUA WANG AND CHENG YU

Abstract. The inhomogeneous Navier-Stokes-Vlasov equations for fluid-particle flows are considered in the three-dimensional space. The coupling in the fluid-particle system arises from the drag force in the fluid equations and the acceleration in the Vlasov equation. An initial-boundary value problem is studied in a bounded domain with large initial data. The existence of global weak solution is established through an approximation scheme, a fixed point argument, energy estimates, and a weak convergence method.

1. Introduction

We are concerned with the Navier-Stokes-Vlasov equations for particles dispersed in a density-dependent incompressible viscous fluid (cf. [20, 26, 29]):

\[ \rho_t + \text{div}(\rho \mathbf{u}) = 0, \]

\[ (\rho \mathbf{u})_t + \text{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p - \mu \Delta \mathbf{u} = \rho F_f \]

\[ \text{div} \mathbf{u} = 0, \]

\[ f_t + \mathbf{v} \cdot \nabla_x f + \text{div}_v(\rho F_df) = 0, \]

for \((t, x, \mathbf{v})\) in \((0, T) \times \Omega \times \mathbb{R}^3\), where \(\Omega \subset \mathbb{R}^3\) is a bounded domain, \(\rho\) is the density, \(\mathbf{u}\) is the velocity, \(p\) is the pressure, \(\mu\) is the kinematic viscosity of the fluid. The distribution function \(f(t, x, \mathbf{v})\) of particles depends on the time \(t \in [0, T]\), the physical position \(x \in \Omega\) and velocity \(\mathbf{v} \in \mathbb{R}^3\) of particles. The interaction of the fluid and particles is through the drag force exerted by the fluid onto the particles. This force is assumed to be proportional to the relative velocity of the fluid and the particles:

\[ F_d = F_0(\mathbf{u} - \mathbf{v}), \]

where \(F_0\) is a positive constant. \(F_f\) is given by

\[ F_f = -\int_{\mathbb{R}^3} F_d f \, d\mathbf{v} = F_0 \int_{\mathbb{R}^3} (\mathbf{u} - \mathbf{v}) f \, d\mathbf{v}, \]

and hence the right side of (1.2) is

\[ \rho F_f = F_0 \int_{\mathbb{R}^3} \rho(\mathbf{u} - \mathbf{v}) f \, d\mathbf{v}. \]

Without loss of generality we take \(\mu = F_0 = 1\) throughout the paper. The fluid-particle system arises in many applications such as sprays, aerosols, sedimentation of solid grain by external forces, fuel-droplets in combustion theory (such as in the study of engines),

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chemical engineering, bio-sprays in medicine, waste water treatment, pollutants in the air, and so on. We refer the readers to [1,5,9,12,13,16,20,26–29,31] and the references therein for more physical background, applications and discussions of the fluid-particle systems.

The aim of this paper is to establish the global existence of weak solution to the initial-boundary value problem for the system (1.1)-(1.6) subject to the following initial data:

\[ \rho|_{t=0} = \rho_0(x) \geq 0, \ x \in \Omega, \]  
\[ (\rho u)|_{t=0} = m_0(x), \ x \in \Omega, \]  
\[ f|_{t=0} = f_0(x,v), \ x \in \Omega, \ v \in \mathbb{R}^3, \]  
and the following boundary conditions:

\[ u(t,x) = 0 \text{ on } \partial \Omega, \]  
\[ f(t,x,v) = f(t,x,v^*) \text{ for } x \in \partial \Omega, \ v \cdot \nu(x) < 0, \]  
where

\[ v^* = v - 2(v \cdot \nu(x))\nu(x) \]

is the specular velocity, and \( \nu(x) \) is the outward normal vector to \( \Omega \). The second boundary condition in (1.11) means that the particles are reflected by the boundary, as in Hamdache [19] according to the specular reflection laws. When the drag force is assumed independent of density in (1.7), the hydrodynamic limits and the global existence of weak solution to the Navier-Stokes and Vlasov-Fokker-Planck equations were studied in Goudon-Jabin-Vasseur [17,18] and Mellet-Vasseur [24,25]. When the drag force depends on the density as in (1.7), it is hard to obtain a priori lower bounds of the density in mathematical analysis due to vacuum. In this paper we shall establish the existence of global weak solution to the initial-boundary value problem (1.1)-(1.11) with large data in certain functional spaces. In general, the analysis of the fluid-particle system is challenging since the density distribution function \( f \) depends on more variables than the fluid density \( \rho \) and velocity \( u \). The existence of global weak solution to the Stokes-Vlasov equations in a bounded domain was studied in Hamdache [19]. In Boudin-Desvillettes-Grandmont-Moussa [3] the convection term was included and the incompressible Navier-Stokes-Vlasov equations were considered in periodic domains. A similar system with thermal diffusion acting on the particles, that is, the incompressible Navier-Stokes equations coupled with the Vlasov-Fokker-Planck equations, has been studied in Goudon-He-Moussa-Zhang [16] where the authors established the existence of classical solutions with small data, and in Chae-Kang-Lee [7] where they established the global existence of weak solution in the three-dimensional space and the global regularity in the two-dimensional space, as well as the existence of global strong solution to the Stokes-Vlasov equations in the three-dimensional space. Lin-Liu-Zhang [23] proved a global existence and uniqueness of classical solution to a micro-macro model for polymeric fluids with the initial data near the hydrodynamic equilibrium. Carrillo-Choi-Karper [6] proved the global existence, hydrodynamic limit, and large-time behavior of weak solution to a kinetic flocking model coupled with the incompressible Navier-Stokes equations. Recently, the existence of global solutions to the incompressible Navier-Stokes-Vlasov equations in a bounded domain or in the whole space was obtained in [32,33]. The paper [15] studied a similar problem numerically. In addition, there have been a lot of studies on hydrodynamic limits, and we refer the reader to [5,17,18,25] (and the references therein) where some scaling and convergence methods such as the compactness and relative entropy method were applied to investigate
the hydrodynamic limits. A key idea in Goudon-Jabin-Vasseur [17, 18] is to control the dissipation rate of certain free energy associated with the whole space. The global existence of weak solution to the compressible Navier-Stokes equations coupled with the Vlasov-Fokker-Planck equations was established in Mellet-Vasseur [24]. The coupled system (1.1)-(1.6) has extra difficulties due to the appearance of density in the interaction and in the Vlasov equation as well as the lack of diffusion in the Vlasov equation. We note that the local classical solution to the Euler-Vlasov equations was obtained in Baranger-Desvillettes [1] when the drag force is assumed to be in the form of (1.7).

When \( f \equiv 0 \), the system (1.1)-(1.4) is reduced to the density-dependent Navier-Stokes equations. We refer the reader to Lions [22] for the compactness, existence of global weak solution, and more discussions for the density-dependent Navier-Stokes equations. For the Navier-Stokes equations, it is necessary for the external force to be in the functional space \( L^2(\Omega \times (0, T)) \) in order to obtain the global existence. However, for the system (1.1)-(1.6) the term \(-\int (u - v) \rho f dv\) does not have enough regularity. To overcome this difficulty, we decompose the term into two components:

\[
- \int_{\mathbb{R}^3} (u - v) \rho f dv = -\rho u \int_{\mathbb{R}^3} f dv + \rho \int_{\mathbb{R}^3} v f dv,
\]

and we can view \( \rho \int_{\mathbb{R}^3} v f dv \) as the external force of the Navier-Stokes equations, and \( \rho u \int_{\mathbb{R}^3} f dv \) as the internal force on the left side of the Navier-Stokes equations. Meanwhile, we introduce a regularization function \( R_\delta \) (see Section 3 for the definition) as in Hamdache [19] to construct an approximation of (1.12):

\[
-R_\delta \int_{\mathbb{R}^3} (u - v) \rho f dv = -\rho u R_\delta \int_{\mathbb{R}^3} f dv + \rho R_\delta \int_{\mathbb{R}^3} v f dv.
\]

To keep a similar energy inequality for the approximation scheme, we need to add the regularized acceleration in the Vlasov equation too. Then we see that the external force term is in \( L^2(0, T; \Omega) \) and the internal forces is finite when \( \delta \) is fixed, hence we can solve the regularized Navier-Stokes equations. The uniqueness and existence of the Vlasov equation can be obtained when \( (\rho, u) \) is smooth, see [10, 19]. The compactness of \( f \) will be obtained by an approach motivated by the recent work [4]. In fact, the DiPerna-Lions compactness in [10] will be helpful when we use the fixed point argument to solve our approximate system. The next step is to pass the limit to recover the original system from the approximation scheme. We shall see that the \( L^p \) regularity of velocity average [11] and the fine compactness of the system guarantee the existence of global weak solution to (1.1)-(1.11).

We organize the rest of the paper as follows. In Section 2, we deduce a priori estimates from (1.1)-(1.6), give the definition of weak solution, and also state our main result. In Section 3, we construct an approximation scheme for (1.1)-(1.6), establish the global existence of the approximate solution, and use the uniform estimates and \( L^p \) average velocity lemma to recover the original system.

2. A Priori Estimates and Main Result

In this section, we shall derive some fundamental a priori estimates and then state our main result. These estimates will play an important role in the compactness analysis later.
since they will allow us to deduce the global existence upon passing to the limit in the regularized approximation scheme.

We take any function $\beta \in C^1([0, \infty); \Omega)$, multiply (1.1) by $\beta'(\rho)$, and use the incompressibility condition and integration by parts over $\Omega$ to obtain

$$\frac{d}{dt} \int_{\Omega} \beta(\rho) \, dx = 0.$$  

Applying the maximum principle to the transport equations (1.1) and (1.3), one deduces that

$$\|\rho\|_{L^\infty} \leq \|\rho_0\|_{L^\infty},$$

and also $\rho \geq 0$, so we have

$$0 \leq \rho(t, x) \leq \|\rho_0\|_{L^\infty}$$

for almost every $t$.

We now multiply (1.2) by $u$ and integrate over $\Omega$, and use (1.1), (1.3), and (1.6) to deduce that

$$\frac{d}{dt} \int_{\Omega} \rho |u|^2 \, dx + 2 \int_{\Omega} |\nabla u|^2 \, dx = -2 \int_{\Omega} \int_{\mathbb{R}^3} \rho f(u - v) \cdot u \, dv \, dx.$$  

On the other hand, we multiply the Vlasov equation (1.4) by $\frac{1}{2} |v|^2$, integrate over $\Omega \times \mathbb{R}^3$, and use integration by parts to obtain

$$\frac{d}{dt} \int_{\Omega} \int_{\mathbb{R}^3} f |v|^2 \, dv \, dx - \int_{\partial\Omega \times \mathbb{R}^3} (v \cdot \nu) \frac{|v|^2}{2} \rho f \, dv \, dx = \int_{\Omega} \int_{\mathbb{R}^3} \rho f(u - v) \, dv \, dx$$

$$-2 \int_{\Omega} \int_{\mathbb{R}^3} \rho f |u - v|^2 \, dv \, dx + 2 \int_{\Omega} \int_{\mathbb{R}^3} \rho f(u - v)u \, dv \, dx.$$  

We can rewrite $v^*$ as follows

$$v^* = Rv, \quad \text{where } R = I - 2\nu
$$

where $I$ is the $3 \times 3$ identity matrix, and $\nu$ is the outward unit normal vector to $\partial\Omega$. By direct computation, one obtains that $|v^*|^2 = |v|^2$. Thus, we can treat the boundary term in (2.3) as follows:

$$\int_{\partial\Omega \times \mathbb{R}^3} (v \cdot \nu) \frac{|v|^2}{2} \rho f \, dv \, dx = \int_{v \cdot \nu > 0} (v \cdot \nu) \frac{|v|^2}{2} \rho f \, dv \, dx + \int_{v \cdot \nu < 0} (v \cdot \nu) \frac{|v|^2}{2} \rho f \, dv \, dx$$

$$= \int_{v \cdot \nu > 0} (v \cdot \nu) \frac{|v|^2}{2} \rho f \, dv \, dx - \int_{v \cdot \nu > 0} (v^* \cdot \nu) \frac{|v^*|^2}{2} \rho f \, dv^* \, dx$$

$$= 0.$$  

We integrate (1.4) with respect to $x$ and $v$, and use the integration by parts to get

$$\frac{d}{dt} \int_{\Omega} \int_{\mathbb{R}^3} f \, dv \, dx - \int_{\partial\Omega \times \mathbb{R}^3} (v \cdot \nu) \rho f \, dv \, dx = 0,$$
and then treat the boundary term as follows:

\[
\int_{\partial \Omega \times \mathbb{R}^3} (\mathbf{v} \cdot \nu) \rho f \, d\mathbf{v} \, dx = \int_{\mathbf{v} \cdot \nu > 0} (\mathbf{v} \cdot \nu) \rho f \, d\mathbf{v} \, dx + \int_{\mathbf{v} \cdot \nu < 0} (\mathbf{v} \cdot \nu) \rho f \, d\mathbf{v} \, dx
\]

\[
= \int_{\mathbf{v} \cdot \nu > 0} (\mathbf{v} \cdot \nu) \rho f \, d\mathbf{v} \, dx - \int_{\mathbf{v} \cdot \nu > 0} (\mathbf{v}^* \cdot \nu) \rho f(t, x, \mathbf{v}^*) \, d\mathbf{v}^* \, dx = 0,
\]

which yields the conservation of mass:

\[
\frac{d}{dt} \int_{\Omega} \int_{\mathbb{R}^3} f \, d\mathbf{v} \, dx = 0. \tag{2.5}
\]

From (2.2)-(2.5) we obtain the following energy equality for the system (1.1)-(1.6):

\[
\frac{d}{dt} \int_{\Omega} \rho |\mathbf{u}|^2 \, dx + \frac{d}{dt} \int_{\mathbb{R}^3} f(1 + |\mathbf{v}|^2) \, d\mathbf{v} \, dx + 2 \int_{\Omega} \int_{\mathbb{R}^3} \rho f |\mathbf{u} - \mathbf{v}|^2 \, d\mathbf{v} \, dx + 2 \int_{\Omega} |\nabla \mathbf{u}|^2 \, dx \leq 0. \tag{2.6}
\]

Integrating (2.6) with respect to \( t \), we obtain, for all \( t \),

\[
\int_{\Omega} \rho |\mathbf{u}|^2 \, dx + \int_{\mathbb{R}^3} f(1 + |\mathbf{v}|^2) \, d\mathbf{v} \, dx + 2 \int_{0}^{t} \int_{\Omega} \int_{\mathbb{R}^3} \rho f |\mathbf{u} - \mathbf{v}|^2 \, d\mathbf{v} \, dx \, dt + 2 \int_{0}^{t} \int_{\Omega} |\nabla \mathbf{u}|^2 \, dx \, dt \leq \int_{\Omega} \frac{|\mathbf{m}_0|^2}{\rho_0} \, dx + \int_{\mathbb{R}^3} f_0(1 + |\mathbf{v}|^2) \, d\mathbf{v} \, dx. \tag{2.7}
\]

From (2.6) one also sees that the global energy is non-increasing with respect to \( t \):

\[
\frac{d}{dt} \left( \int_{\Omega} \rho |\mathbf{u}|^2 \, dx + \int_{\mathbb{R}^3} f(1 + |\mathbf{v}|^2) \, d\mathbf{v} \, dx \right) \leq 0.
\]

Assume

\[
\int_{\Omega} \frac{|\mathbf{m}_0|^2}{\rho_0} \, dx + \int_{\mathbb{R}^3} f_0(1 + |\mathbf{v}|^2) \, d\mathbf{v} \, dx < \infty,
\]

then

\[
\int_{0}^{t} \int_{\mathbb{R}^3} \rho f |\mathbf{u} - \mathbf{v}|^2 \, d\mathbf{v} \, dx \, dt \leq C,
\]

and

\[
\|\nabla \mathbf{u}\|_{L^2((0,T) \times \Omega)} \leq C, \tag{2.8}
\]

\[
\sup_{0 \leq t \leq T} \|\rho |\mathbf{u}|^2\|_{L^1(\Omega)} \leq C, \tag{2.9}
\]

for any given \( T > 0 \) and some generic positive constant \( C \). Moreover, by the Poincaré inequality we obtain

\[
\|\mathbf{u}\|_{L^2(0,T;H^1_0(\Omega))} \leq C. \tag{2.10}
\]
For given smooth $u$ and $\rho$, the characteristic curves $(X(s;x,v,t), V(s;x,v,t))$ of (1.4) are defined by the following:

$$
\frac{dX}{dt} = V, \\
\frac{dV}{dt} = \rho(u - V), \\
X(s;x,v,t) = x, \\
V(s;x,v,t) = v.
$$

Rewriting (1.4) as

$$
f_t + v \cdot \nabla_x f + \rho(u - v) \cdot \nabla_v f = 3\rho f,
$$

we have

$$
f(t, x, v) = f_0(X(0, x, v, t), V(0, x, v, t))e^{3\int_0^t \rho(X(s), s) ds},
$$

and then

$$
\|f\|_{L^\infty} \leq e^{3T\|\rho\|_{L^\infty((0,T) \times \Omega)}} \|f_0\|_{L^\infty} \tag{2.11}
$$

for all $t \in [0, T]$. Moreover, $f_0 \geq 0$ implies $f \geq 0$ for almost every $(t, x, v)$. Then, by the conservation of mass (2.5) and (2.11), one has the following estimate:

$$
\|f\|_{L^\infty((0,T) \times \Omega \times \mathbb{R}^3)} + \|f\|_{L^\infty((0,T;L^1(\Omega \times \mathbb{R}^3)))}
\leq C(T) \left( \|f_0\|_{L^\infty((0,T) \times \Omega \times \mathbb{R}^3)} + \|f_0\|_{L^\infty((0,T;L^1(\Omega \times \mathbb{R}^3)))} \right). \tag{2.12}
$$

Let $w(t, x)$ be a smooth vector field in $\mathbb{R}^3$ and let $f$ be a solution to the following kinetic equation:

$$
\partial_t f + v \cdot \nabla_x f + \text{div}_v((w - v)f) = 0, \\
f(0, x, v) = f_0(x, v), \quad f(t, x, v) = f(t, x, v^*) \quad \text{for } x \in \partial \Omega, \quad v \cdot v(x) < 0, \tag{2.13}
$$

in $\Omega \times \mathbb{R}^3$. DiPerna-Lions [10] obtained the existence and uniqueness of solution to (2.13) when $w$ is not smooth. Denote the moments of $f$ by

$$
m_{k}f(t, x) = \int_{\mathbb{R}^3} f(t, x, v)|v|^k dv, \\
M_{k}f(t) = \int_{\Omega} \int_{\mathbb{R}^3} f(t, x, v)|v|^k dv dx,
$$

for any $t \in [0, T]$, $x \in \Omega$, and integer $k \geq 0$. It is clear that

$$
M_{k}f(t) = \int_{\Omega} m_{k}f(t, x) dx.
$$

We first recall the following lemma from Hamdache [19]:

**Lemma 2.1.** Let $w \in L^p(0,T;L^{3+k}(\Omega))$ with $1 \leq p \leq \infty$ and $k \geq 1$. Assume that $f_0 \in (L^\infty \cap L^1) (\Omega \times \mathbb{R}^3)$ and $|v|^k f_0 \in L^1(\Omega \times \mathbb{R}^3)$. Then, the solution $f$ of (2.13) has the following estimates:

$$
M_k f \leq C \left( (M_k f_0)^{1/(3+k)} + (|f_0|_{L^\infty} + 1) \|w\|_{L^p(0,T;L^{3+k}(\Omega))} \right)^{3+k}
$$

for all $0 \leq t \leq T$, where the constant $C$ depends only on $T$. 

We also recall the average compactness result for the Vlasov equation due to DiPerna-Lions-Meyer [11]:

**Lemma 2.2.** Suppose
\[
\frac{\partial f^n}{\partial t} + v \cdot \nabla_x f^n = \text{div}_v(F^n f^n) \quad \text{in } \mathcal{D}'(\Omega \times \mathbb{R}^3 \times (0, \infty)),
\]
where \( f^n \) is bounded in \( L^\infty(0, \infty; L^2(\Omega \times \mathbb{R}^3)) \) and \( f^n(1 + |v|^2) \) is bounded in \( L^\infty(0, \infty; L^1(\Omega \times \mathbb{R}^3)) \), \( \frac{F^n}{1 + |v|^2} \) is bounded in \( L^\infty((0,\infty) \times \mathbb{R}^3; L^2(\Omega)) \). Then \( \int_{\mathbb{R}^3} f^n \eta(v) dv \) is relatively compact in \( L^q(0,T; L^p(\Omega)) \) for \( 1 \leq q < \infty, 1 \leq p < 2 \) and for \( \eta \) such that \( \frac{\eta}{(1 + |v|)^\sigma} \in L^1 + L^\infty, \sigma \in [0,2) \).

**Remark 2.1.** We shall use this lemma for the Vlasov equation to obtain the compactness of \( m_0 f \) and \( m_1 f \), which will allow us to pass the limit when \( \varepsilon \) and \( \delta \) go to zero in the approximation in Section 3.

In this paper, we assume that
\[
\begin{cases}
\rho_0 \geq 0 \text{ almost everywhere in } \Omega, \quad \rho_0 \in L^\infty(\Omega), \\
m_0 \in L^2(\Omega), \quad m_0 = 0 \text{ almost everywhere on } \{ \rho_0 = 0 \}, \quad \frac{|m_0|}{\rho_0} \in L^1(\Omega), \\
f_0 \in L^\infty(0,T; L^1(\Omega \times \mathbb{R}^3)), \quad |v|^3 f_0 \in L^1(\Omega \times \mathbb{R}^3).
\end{cases}
\tag{2.14}
\]

**Definition 2.1.** \((\rho, u, f)\) is a global weak solution to problem (1.1)-(1.11) if, for any \( T > 0, \)
- \( \rho \geq 0, \rho \in L^\infty([0,T] \times \Omega), \quad \rho \in C([0,T]; L^p(\Omega)), \quad 1 \leq p < \infty; \)
- \( u \in L^2(0,T; H^1_0(\Omega)); \)
- \( |\rho|^2 \in L^\infty(0,T; L^1(\Omega)); \)
- \( f(t,x,v) \geq 0, \text{ for any } (t,x,v) \in (0,T) \times \Omega \times \mathbb{R}^3; \)
- \( f \in L^\infty(0,T; L^\infty(\Omega \times \mathbb{R}^3) \cap L^1(\Omega \times \mathbb{R}^3)); \)
- \( |v|^3 f \in L^\infty(0,T; L^1(\Omega \times \mathbb{R}^3)); \)
- For any \( \varphi \in C^1([0,T] \times \Omega), \) such that \( \text{div}_x \varphi = 0, \) for almost everywhere \( t, \)
\[
- \int_\Omega m_0 \cdot \varphi(0,x) \, dx + \int_0^T \int_\Omega \left( -\rho u \cdot \partial_t \varphi - (\rho u \otimes u) \cdot \nabla \varphi + \mu \nabla u \cdot \nabla \varphi + \mu \rho \int_{\mathbb{R}^3} f(\rho u - v) \cdot \varphi \, dv \right) \, dx \, dt = 0; \tag{2.15}
\]
- For any \( \phi \in C^1([0,T] \times \Omega \times \mathbb{R}^3) \) with compact support in \( v, \) such that \( \phi(T,\cdot,\cdot) = 0, \)
\[
- \int_0^T \int_\Omega \int_{\mathbb{R}^3} f \left( \partial_t \phi + v \cdot \nabla_x \phi + \rho(u - v) \cdot \nabla_v \phi \right) \, dx \, dv \, ds = \int_\Omega \int_{\mathbb{R}^3} f_0 \phi(0,\cdot,\cdot) \, dx \, dv; \tag{2.16}
\]
- The energy inequality
\[
\begin{align*}
&\int_{\Omega} \rho |u|^2 \, dx + \int_{\Omega} \int_{\mathbb{R}^3} f (1 + |v|^2) \, dv \, dx + 2 \int_0^T \int_\Omega \int_{\mathbb{R}^3} f |u - v|^2 \, dv \, dx \, dt \\
&+ 2 \int_0^T \int_\Omega |\nabla u|^2 \, dx \, dt \\
&\leq \int_{\Omega} \frac{|m_0|^2}{\rho_0} \, dx + \int_{\Omega} \int_{\mathbb{R}^3} (1 + |v|^2) f_0 \, dv \, dx \tag{2.17}
\end{align*}
\]
holds for almost everywhere \( t \in [0, T] \).

Our main result on global weak solution reads as follows.

**Theorem 2.1.** Under the assumption (2.14), there exists a global weak solution \((\rho, u, f)\) to the initial-boundary value problem (1.1)-(1.11) for any \( T > 0 \).

### 3. Existence of Global Weak Solution

In this section, we shall prove Theorem 2.1 in two steps. First, we build a regularized approximation system for the original system, and solve this approximation system. Then, we recover the original system from the approximation scheme by passing to the limit of the sequence of solutions obtained in the first step.

#### 3.1. Construction of approximation solutions.

By (2.1), the density \( \rho \) is bounded uniformly. If one considers weak solution, the term \(- \int_{\mathbb{R}^3} (u - v) \rho f dv\) does not have enough regularity to solve the Navier-Stokes equations. Inspired by the work of [19], we introduce the following regularization function:

\[
R_\delta = R_\delta(m_0 f, m_1 f) = \frac{1}{1 + \delta \int_{\mathbb{R}^3} f dv + \delta \left| \int_{\mathbb{R}^3} \frac{v}{|v|} dv \right|},
\]

for any fixed \( \delta > 0 \). Clearly

\[
0 < R_\delta(m_0 f, m_1 f) < 1
\]

for any \( \delta > 0 \), and

\[
R_\delta(m_0 f, m_1 f) \to 1
\]

as \( \delta \to 0 \). For any fixed \( \delta > 0 \), as mentioned in the introduction, the regularized force term

\[
\rho R_\delta \int_{\mathbb{R}^3} (u - v) f dv
\]

consists of two terms:

\[
\rho u R_\delta \int_{\mathbb{R}^3} f dv \quad \text{and} \quad \rho R_\delta \int_{\mathbb{R}^3} v f dv,
\]

the first one can be viewed as the internal force, and the second one can be viewed as the external force. The regularized external force is in \( L^2((0, T) \times \Omega) \) for any given \( \delta \) and \( f \), which ensures that the regularized Navier-Stokes equations with the internal force have a smooth solution. To keep a similar energy inequality for the approximation scheme, we need to regularize the acceleration term as

\[
(u - v) R_\delta \rho f
\]

in the Vlasov equation. Thus, inspired by Lions’ work on the density-dependent Navier-Stokes equations [22] as well as Hamdache’s work on the Vlasov-Stokes equations [19] and [3], we consider the following approximation problem:

\[
\rho_t + \text{div}(\rho u_\varepsilon) = 0, \quad (3.1)
\]

\[
\frac{\partial (\rho u)}{\partial t} + \text{div}(\rho u_{\varepsilon} \otimes u) - \mu \Delta u + \nabla p + \rho \rho R_\delta \int_{\mathbb{R}^3} f dv = \rho R_\delta \int_{\mathbb{R}^3} v f dv, \quad (3.2)
\]

\[
\text{div} u = 0, \quad (3.3)
\]
\[
\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \text{div}_\mathbf{v}((\mathbf{u}_\varepsilon - \mathbf{v})R_\delta f) = 0,
\]
where \( \mathbf{u}_\varepsilon \) is defined as follows. First we set
\[
\Omega_\varepsilon = \{ x \in \Omega : \text{dist}(x, \partial \Omega) > \varepsilon \}
\]
for any \( \varepsilon > 0 \) if \( \Omega \) is smooth; otherwise, we can choose a smooth connected domain \( \Omega_\varepsilon \) such that
\[
\{ x \in \Omega, \text{dist}(x, \partial \Omega) > \varepsilon \} \subset \Omega_\varepsilon \subset \overline{\Omega_\varepsilon} \subset \Omega.
\]
Then we let \( \hat{\mathbf{u}}^\varepsilon \) be the truncation in \( \Omega_\varepsilon \) of \( \mathbf{u} \), and extend it by 0 to \( \Omega \). Let \( \theta \) be the standard mollifier satisfying
\[
\theta \in C^\infty(\mathbb{R}^3), \quad \theta \geq 0, \quad \int_{\mathbb{R}^3} \theta \, dx = 1, \quad \text{supp } \theta \subset B(0,1) := \{ x \in \mathbb{R}^3 : |x| < 1 \},
\]
and for each \( \varepsilon > 0 \) set
\[
\theta_\varepsilon := \varepsilon^{-3} \theta \left( \frac{x}{\varepsilon} \right).
\]
Finally we define
\[
\mathbf{u}_\varepsilon = \hat{\mathbf{u}}^\varepsilon \ast \theta_\varepsilon.
\]
We note that \( \mathbf{u}_\varepsilon \) is a function which is smooth with respect to \( x \), and
\[
\mathbf{u}_\varepsilon = 0 \quad \text{on } \partial \Omega \quad \text{and } \text{div} \mathbf{u}_\varepsilon = 0 \quad \text{in } \mathbb{R}^3.
\]

To impose the initial value for our approximate system, we need the following version of the Hodge-de Rham decomposition (see Lions [22]):

**Lemma 3.1.** Let \( N \geq 2, \rho \in L^\infty(\mathbb{R}^N) \) such that \( \rho \geq \rho_0 \geq 0 \) almost everywhere on \( \mathbb{R}^N \) for some \( \rho_0 \in (0, \infty) \). Then there exists two bounded operators \( P_\delta, Q_\delta \) on \( L^2(\mathbb{R}^N) \) such that for all \( \mathbf{m} \in L^2(\mathbb{R}^N) \), \( (\mathbf{m}_p, \mathbf{m}_q) = (P_\rho \mathbf{m}, Q_\rho \mathbf{m}) \) is the unique solution in \( L^2(\mathbb{R}^N) \) of
\[
\mathbf{m} = \mathbf{m}_p + \mathbf{m}_q, \quad (\nabla)^{-1/2} \text{div} (\rho^{-1} \mathbf{m}_p) = 0, \quad (\nabla)^{-1/2} \text{rot} (\mathbf{m}_q) = 0.
\]
Furthermore, if \( \rho_n \in L^\infty(\mathbb{R}^N) \), \( \rho \leq \rho_n \leq \rho_0 \) almost everywhere on \( \mathbb{R}^N \) for some \( 0 < \rho \leq \rho_0 \) converges almost everywhere to \( \rho \), then \( P_{\rho_n} \mathbf{m}_n, Q_{\rho_n} \mathbf{m}_n \) converges weakly in \( L^2(\mathbb{R}^N) \) to \( (P_\rho \mathbf{m}, Q_\rho \mathbf{m}) \) whenever \( \mathbf{m}_n \) converges weakly to \( \mathbf{m} \).

We now discuss the initial conditions for the approximation scheme (3.1)-(3.4). Since an initial vacuum may exist, we cannot directly impose initial data on the velocity \( \mathbf{u} \). We adopt the idea from Lions [22] to define
\[
\hat{\rho}_0 = \begin{cases} 
\rho_0, & \text{if } x \text{ is in } \Omega, \\
1, & \text{if } x \text{ is in } \Omega^c,
\end{cases}
\]
and then set
\[
(\rho_0)_\varepsilon = \hat{\rho}_0 \ast \theta_\varepsilon|_{\Omega},
\]
\[
(\rho_0^2 - \frac{1}{\varepsilon})_\varepsilon = \hat{\rho}_0^2 \ast \theta_\varepsilon|_{\Omega},
\]
\[
(m_0 \rho_0^2 - \frac{1}{\varepsilon})_\varepsilon = \left( m_0 \rho_0^2 - \frac{1}{\varepsilon} 1_{\{d > 2\varepsilon\}} \right) \ast \theta_\varepsilon,
\]
where \( d = \text{dist}(x, \partial \Omega) \) and \( 1_{\{d > 2\varepsilon\}} \) is the characteristic function. Now we define
\[
\rho|_{t=0} = \rho_0^\varepsilon = (\rho_0)_\varepsilon + \varepsilon,
\]
which implies
\[ \varepsilon \leq \rho_0^\varepsilon \leq C_0, \] (3.6)
where \( C_0 = \|\rho_0\|_{L^\infty(\Omega)} + 2 \) which is independent of \( \varepsilon \in (0, 1) \). Clearly, \( \rho_0^\varepsilon \in C^k(\Omega) \) for all \( k \geq 0 \), and
\[ \rho_0^\varepsilon \rightarrow \rho_0 \quad \text{in} \quad L^p(\Omega) \quad \text{for all} \quad 1 \leq p < \infty. \] (3.7)
To impose the initial value of the velocity, we first define
\[ \bar{\mathbf{m}}_0^\varepsilon = (\mathbf{m}_0\rho_0^{-1/2})\varepsilon(\tilde{\rho}_0^{1/2})\varepsilon \in C_0^\infty(\Omega), \]
satisfying obviously the following:
\[ \bar{\mathbf{m}}_0^\varepsilon \rightarrow \bar{\mathbf{m}}_0 \quad \text{in} \quad L^2(\Omega), \quad \bar{\mathbf{m}}_0^\varepsilon(\rho_0^{\varepsilon})^{-1/2} \rightarrow \bar{\mathbf{m}}_0\rho_0^{-1/2} \quad \text{in} \quad L^2(\Omega). \] (3.8)
Using Lemma 3.1, we decompose \( \bar{\mathbf{m}}_0^\varepsilon \) as
\[ \bar{\mathbf{m}}_0^\varepsilon = \rho_0^\varepsilon \bar{\mathbf{u}}_0^\varepsilon + \nabla q_0^\varepsilon, \quad \text{where} \quad \bar{\mathbf{u}}_0^\varepsilon, \quad q_0^\varepsilon \in C^k(\Omega), \]
\[ \text{div} \bar{\mathbf{u}}_0^\varepsilon = 0 \quad \text{in} \quad \Omega. \]
We choose \( \mathbf{u}_0^\varepsilon \in C_0^\infty(\Omega) \) such that \( \text{div} \mathbf{u}_0^\varepsilon = 0 \) and
\[ \| \mathbf{u}_0^\varepsilon - \bar{\mathbf{u}}_0^\varepsilon \|_{L^2} \leq \varepsilon. \]
Let
\[ \mathbf{m}_0^\varepsilon = \rho_0^\varepsilon \mathbf{u}_0^\varepsilon + \nabla q_0^\varepsilon, \] (3.9)
we have
\[ \mathbf{m}_0^\varepsilon \rightarrow \mathbf{m}_0 \quad \text{in} \quad L^2(\Omega), \quad \mathbf{m}_0^\varepsilon(\rho_0^{\varepsilon})^{-1/2} \rightarrow \mathbf{m}_0\rho_0^{-1/2} \quad \text{in} \quad L^2(\Omega). \] (3.10)
Thus
\[ \rho \mathbf{u}|_{t=0} = \mathbf{m}_0^\varepsilon = \mathbf{m}_0^\varepsilon + \rho_0^\varepsilon(\mathbf{u}_0^\varepsilon - \bar{\mathbf{u}}_0^\varepsilon), \]
and we can impose the initial condition of \( \mathbf{u} \) as
\[ \mathbf{u}|_{t=0} = \mathbf{u}_0^\varepsilon. \] (3.11)
Finally, we impose the initial condition for \( f \) as
\[ f|_{t=0} = f_0. \] (3.12)
We now state and prove the following existence result.

**Theorem 3.1.** With the above notations and assumptions, there exists a solution \((\rho, \mathbf{u}, f)\) of (3.1)-(3.4) with the initial conditions (3.5), (3.11) and (3.12), and the boundary conditions (1.11), such that \( \rho \in C^\infty([0, T] \times \bar{\Omega}), \quad \mathbf{u} \in C^\infty([0, T] \times \bar{\Omega}), \quad \text{and} \quad f \in C^\infty([0, T] \times \bar{\Omega} \times \mathbb{R}^3). \)

**Proof.** We define \( M \) as the convex set in \( L^2(0, T; H^1_0(\Omega)) \) by
\[ M = \left\{ \bar{\mathbf{u}} \in L^2(0, T; H^1_0(\Omega)) : \text{div} \bar{\mathbf{u}} = 0 \text{ almost everywhere on} \ (0, T) \times \Omega, \right. \]
\[ \left. \| \bar{\mathbf{u}} \|_{L^2(0, T; H^1_0(\Omega))} \leq K \right\}, \]
where \( K > 0 \) is to be determined. We now define a map \( \mathcal{T} : M \rightarrow M : \)
\[ \mathcal{T}(\bar{\mathbf{u}}) = \mathbf{u}, \]
in the following three steps:
Step 1: In the first step, we consider the following initial-value problem:
\[
\frac{\partial \rho}{\partial t} + \text{div}(\mathbf{u}_\varepsilon \rho) = 0, \quad \rho|_{t=0} = \rho_0^\varepsilon, \tag{3.13}
\]
in \((0, T) \times \Omega\), with \(\mathbf{u}_\varepsilon = \mathbf{u} * \theta_\varepsilon\), here we extend \(\mathbf{u}\) to \(\mathbb{R}^3\) by taking \(\mathbf{u} = 0\) for all \(x \in \mathbb{R}^3 \setminus \Omega\). The construction of \(\mathbf{u}_\varepsilon\) implies that \(\mathbf{u}_\varepsilon \in L^2(0, T; H^k(\Omega))\) for all \(k \geq 0\).

The solution of (3.13) can be written in terms of characteristics:
\[
\frac{dX(s; x, t)}{ds} = \mathbf{u}_\varepsilon(X, s), \quad X(t; x, t) = x, \quad x \in \Omega, \quad t \in [0, T]. \tag{3.14}
\]
By the properties of \(\mathbf{u}_\varepsilon \in L^2(0, T; H^k(\Omega))\) for all \(k \geq 0\), and the basic theory of ordinary differential equations, we know that there exists a unique solution \(X\) of (3.14), which is continuous with respect to \((s, t) \in [0, T]^2\), smooth in \(x \in \Omega\) such that \(\partial_s^p X \in C([0, t] \times \Omega \times [0, T])\) for all \(p \geq 1\) and \(X(s; x, t) \in \Omega\) for all \((s, t) \in [0, T]^2\), \(x \in \Omega\). Therefore, the solution \(\bar{\rho}\) to (3.13) is given by
\[
\bar{\rho}(t, x) = \rho_0^\varepsilon(X(0; t, x)), \quad \text{for all } t \in [0, T], \quad x \in \Omega.
\]
It is clear that \(\varepsilon \leq \bar{\rho} \leq C_0\) in \([0, T] \times \Omega\), and \(\bar{\rho} \in C([0, T]; H^k(\Omega))\) for all \(k \geq 0\). By (3.13) and the properties of \(\mathbf{u}_\varepsilon\), we have
\[
\frac{\partial \bar{\rho}}{\partial t} \in L^2(0, T; H^k(\Omega)), \quad \text{for all } k \geq 0.
\]
Thus, \(\bar{\rho}\) and \(\frac{\partial \bar{\rho}}{\partial t}\) are bounded in these associated spaces uniformly with respect to \(\mathbf{u} \in M\). In particular, by the Arzela-Ascoli lemma, the set of \(\bar{\rho}\) built in this way is clearly compact in \(C([0, T] \times \Omega)\).

Step 2: The second step is to solve the Vlasov equation as follows:
\[
\begin{align*}
f_t + \mathbf{v} \cdot \nabla_x f + \text{div}R_\delta(\mathbf{u}_\varepsilon - \mathbf{v})\bar{\rho}f &= 0, \\
f|_{t=0} = f_0(x, \mathbf{v}), \quad f(t, x, \mathbf{v}) = f(t, x, \mathbf{v}^*) \quad \text{for } x \in \partial \Omega, \ \mathbf{v} \cdot \nu(x) < 0. \tag{3.15}
\end{align*}
\]
Due to the regularity of \(\bar{\rho}, \mathbf{u}_\varepsilon\) and the initial data, the existence and uniqueness for the above Vlasov equation can be obtained as in [2, 10]. We use \(\hat{f}\) to denote the solution to (3.15). We follow the same idea in [4] for the compactness. Thanks to the maximum principle, we have
\[
\|\hat{f}\|_{L^\infty} \leq C\|f_0\|_{L^\infty},
\]
where \(C\) depends on \(T\) and \(\rho_0\). Thus, this solution is a renormalized solution in the DiPerna-Lions sense introduced in [10]. By the definition of \(\hat{f}, \bar{\rho}\) and \(\mathbf{u}_\varepsilon\), we conclude that the vector fields \(\mathbf{v}\) and \(\bar{\rho}(\mathbf{u}_\varepsilon - \mathbf{v})\) satisfy all the desired assumptions in [10]. Applying the DiPerna-Lions stability in [10], we conclude that the set of \(\hat{f}\) built in this way is compact in \(L^p_{\text{loc}}\) for any \(p > 1\). On the other hand, \(\hat{f}\) is bounded in the space \(C([0, T], C^\infty_{x,v}(\Omega \times \mathbb{R}^3))\) due to the regularity of \(\mathbf{u}_\varepsilon\).

Step 3: The third step is to build \(\mathbf{u}\) by solving the following problem:
\[
\begin{align*}
\frac{\partial \mathbf{u}}{\partial t} + \bar{\rho} \mathbf{u}_\varepsilon \nabla \mathbf{u} - \Delta \mathbf{u} + \nabla p + \bar{\rho} \left( R_\delta \int_{\mathbb{R}^3} f \, dv \right) \mathbf{u} &= \bar{\rho} \left( R_\delta \int_{\mathbb{R}^3} \mathbf{v} f \, dv \right), \\
\text{div} \mathbf{u} &= 0, \quad \mathbf{u}|_{t=0} = \mathbf{u}_0^\varepsilon, \quad \text{div} \mathbf{u}_0^\varepsilon = 0, \tag{3.16}
\end{align*}
\]
\( e = R_\delta \int_{\mathbb{R}^3} f \, dv \geq 0, \quad g = R_\delta \int_{\mathbb{R}^3} v \, f \, dv. \)

By the definition of the function \( R_\delta \), \( g \) is bounded in \( L^\infty(0, T; L^2(\Omega)) \) for any given \( \delta > 0 \).

Multiplying by \( u \) the both sides of (3.16), one obtains the following energy equality:

\[
\partial_t \int_\Omega \frac{1}{2} \overline{\rho} |u|^2 \, dx + \int_\Omega |\nabla u|^2 \, dx + \int_\Omega e \overline{\rho} |u|^2 \, dx = \int_\Omega \overline{\rho} g u \, dx.
\]

The right-hand side of above energy equality is bounded by

\[
\int_\Omega e \overline{\rho} |u|^2 \, dx \leq \left( \int_\Omega |g|^2 \, dx \right)^{\frac{1}{2}} \left( \int_\Omega |u|^2 \, dx \right)^{\frac{1}{2}} \leq \| \rho_0 \|_{L^\infty(\Omega)}^2 \| g \|_{L^2(\Omega)} \| \sqrt{\overline{\rho}} u \|_{L^2(\Omega)}.
\]

Thus, we obtain, for all \( t \in (0, T) \),

\[
\begin{align*}
\frac{1}{2} \int_0^t \int_\Omega |\overline{\rho} u|^2 \, dx \, dt + \frac{1}{2} \int_0^t \int_\Omega |\nabla u|^2 \, dx \, dt + \frac{1}{2} \int_0^t \int_\Omega e |\overline{\rho} u|^2 \, dx \, dt \\
\leq C \int_0^t \int_\Omega |g|^2 \, dx \, dt + C \int_0^t \int_\Omega \overline{\rho} |u|^2 \, dx \, dt + \frac{1}{2} \int_\Omega \frac{|m_0|^2}{\rho_0} \, dx.
\end{align*}
\]

Applying Gronwall’s inequality, we obtain

\[
\sup_{t \in (0, T)} \| \overline{\rho} u \|_{L^1(\Omega)} \leq C, \quad \| \sqrt{e} \overline{\rho} u \|_{L^2(0, T; \Omega)} \leq C, \quad \| u \|_{L^2(0, T; H^2(\Omega))} \leq C;
\]

where \( C \) denotes a generic constant which depends only on \( T, \Omega, \varepsilon, \delta \), \( \| \rho_0 \|_{L^\infty(\Omega)} \), and \( \| \rho_0 |u|^2 \|_{L^1(\Omega)} \).

Rewriting (3.16) as

\[
c \frac{\partial u}{\partial t} + b \cdot \nabla u - \Delta u + \nabla p + au = h, \quad \text{div} u = 0, \quad u|_{t=0} = u_0^\varepsilon, \quad \text{div} u_0^\varepsilon = 0,
\]

in \((0, T) \times \Omega\), where

\[
c \in L^\infty((0, T) \times \Omega), \quad b \in L^2(0, T; L^\infty(\Omega)), \quad a \in L^\infty((0, T) \times \Omega),
\]
\[
h \in L^\infty((0, T) \times \Omega), \quad c \geq \varepsilon > 0.
\]

We now prove the following lemma which will be used to continue the proof:

**Lemma 3.2.** There exists a unique solution \( u \) to (3.18) with the following regularity for any given \( \varepsilon > 0 \):

\[
u \in L^2(0, T; H^2(\Omega)) \cap C([0, T]; H^1_0(\Omega)); \quad \nabla p, \frac{\partial u}{\partial t} \in L^2((0, T) \times \Omega).
\]
Proof. First, we multiply \((3.18)\) by \(\frac{\partial u}{\partial t}\) and use integration by parts over \(\Omega\) to obtain:

\[
\varepsilon \int_{\Omega} \left| \frac{\partial u}{\partial t} \right|^2 dx + \frac{d}{dt} \int_{\Omega} \frac{1}{2} |\nabla u|^2 dx \\
\leq \int_{\Omega} \left( |h| |u_t| + |b||\nabla u| |u_t| + |a||u| |u_t| \right) dx.
\]

Using the Cauchy-Schwarz inequality and embedding inequality, one deduces that

\[
\frac{\varepsilon}{2} \int_{\Omega} \left| \frac{\partial u}{\partial t} \right|^2 dx + \frac{d}{dt} \int_{\Omega} \frac{1}{2} |\nabla u|^2 dx \leq C \left( 1 + \|b\|_{L^\infty(\Omega)}^2 + \|a\|_{L^\infty(\Omega)}^2 \lambda_0 \right) \int_{\Omega} |\nabla u|^2 dx,
\]

where \(\lambda_0\) is a constant from the Sobloev inequality. By the regularity of \(a, b\) and Gronwall’s inequality, we deduce that \(u \in L^\infty(0, T; H^1_0(\Omega))\), \(\frac{\partial u}{\partial t} \in L^2(0, T \times \Omega)\).

From \((3.18)\), one has

\[
-\Delta u + \nabla p = h - cu_t - b \cdot \nabla u - au,
\]
\[
\text{div}\ u = 0,
\]

in \(\Omega \times (0, T)\), and \(u \in H^1_0(\Omega)\). Let \(\tilde{h} = h - cu_t - b \cdot \nabla u - au\), then \(\tilde{h} \in L^2(0, T; \Omega)\),

and

\[
-\Delta u + \nabla p = \tilde{h},
\]
\[
\text{div}\ u = 0.
\]

By the regularity of \(u\) and \(\tilde{h}\), we conclude that \(p\) is bounded in \(L^2((0, T); H^{-1}(\Omega))\). We deduce that \(p\) is bounded in \(L^2((0, T) \times \Omega)\) and

\[
\int_{\Omega} p \, dx = 0.
\]

For this detail, we refer readers to Lemma 2.1 as in [14]. Also we conclude that \(u\) is bounded in \(L^2(0, T; H^2(\Omega))\) by the classical regularity on the Stokes equation. Thus, we have proved the regularity in \((3.19)\). The existence and uniqueness of \((3.18)\) follows from the Lax-Milgram theorem, see for example [8].

By Lemma 3.2, there exists a unique solution to \((3.16)\) with the regularity in \((3.19)\). By the Arzela-Ascoli lemma, \(u\) is compact in \(L^2(0, T; H^1_0(\Omega))\), which, with the help of the compactness of \(\bar{\rho}\) and \(\bar{f}\) uniformly with respect to \(u_x \in M\), implies that the subset \(\mathcal{S}(M)\) is a relatively compact subset of \(M\). By \((3.17)\), we have

\[
\|\mathcal{S}(u)\|_{L^2(0, T; H^1_0(\Omega))} = \|u\|_{L^2(0, T; H^1_0(\Omega))} \leq K,
\]

for some constant \(K = K(\varepsilon) > 0\).

Here we need the following lemma on the continuity of the operator \(\mathcal{S}(u)\).

**Lemma 3.3.** The operator \(\mathcal{S}\) is continuous from \(M\) to \(M\).
Proof. Let \( \{\tilde{u}_n\} \) be a sequence of \( M \) such that \( \tilde{u}_n \to \tilde{u} \) strongly in \( M \). For each given \( \tilde{u}_n \) in \( M \), there exist solutions \((\rho_n, u_n, f_n)\) to (3.13), (3.15), (3.16) constructed in the previous three steps, where \( u_n = \Theta(\tilde{u}_n) \). By Lemma 3.2, for any \( n > 0 \), we have

\[
\frac{\partial u_n}{\partial t} \in L^2(0, T; H^2(\Omega)) \cap C([0, T]; H^1_0(\Omega)); \quad \frac{\partial u_n}{\partial t} \in L^2((0, T) \times \Omega).
\] (3.22)

By the Aubin-Lions Lemma, the sequence \( \{\Theta(\tilde{u}_n)\} \) has a convergent subsequence. To prove that the whole sequence of \( \{\Theta(\tilde{u}_n)\} \) converges to \( \Theta(\tilde{u}) \), we only need to show that it has only one accumulation point \( \Theta(\tilde{u}) \). Indeed, assume that \( w \in M \) is an accumulation point of \( \Theta(\tilde{u}_n) \), and \( \{\tilde{u}_{n_i}\} \) is a subsequence of \( \{\tilde{u}_n\} \) such that \( \Theta(\tilde{u}_{n_i}) \to w \) in \( M \). For the subsequence \( \{\tilde{u}_{n_i}\} \), there exists solutions \((\rho_{n_i}, u_{n_i}, f_{n_i})\) to (3.13), (3.15), (3.16). In particular, for each \( n_i \), \( u_{n_i} = \Theta(\tilde{u}_{n_i}) \) satisfies (3.22). Note that we still have \( u_{n_i} \to \tilde{u} \) in \( M \). By the standard compactness argument with the help of the compactness of \( \rho_{n_i} \) and \( f_{n_i} \) (from the construction in the above Step 1 and Step 2), we conclude that the limit function \( w \) solves (3.16). Thus, we have \( w = \Theta(\tilde{u}) \) by the uniqueness of solutions proved in Lemma 3.2, which implies the continuity of \( \Theta \).

□

Now we are ready to use the Schauder fixed point argument, which yields \( \tilde{u} = u \), \( \tilde{\rho} = \rho \) and \( \tilde{f} = f \). We observe that \( u_\varepsilon \in C([0, T]; C^k(\Omega)), \rho_\varepsilon \in C^{0, \frac{1}{2}}([0, T]; C^k(\Omega)) \) and \( f \in C^{0, \frac{1}{2}}(0, T, C^k(\Omega); C^k(\mathbb{R}^3)) \) for all \( k \geq 0 \). Applying the \( L^p \)-theory, we deduce that \( u \in L^p(0, T; W^{2, p}(\Omega)) \) and \( \frac{\partial u}{\partial t} \in L^p(0, T \times \Omega) \) for all \( 1 < p < \infty \). With such regularity on \( u \), we can bootstrap and obtain higher regularity on \( u_\varepsilon \), then \( \rho \) and \( f \) have more regularity in \( x \).

Thus we have proved Theorem 3.1. □

Remark 3.1. The solutions \((\rho, u, f)\) obtained in Theorem 3.1 is a smooth solution, and hence we have the following energy inequality:

\[
\frac{d}{dt} \left( \int_{\Omega} \frac{1}{2} \rho |u|^2 \, dx + \int_{\Omega} \int_{\mathbb{R}^3} \frac{1}{2} f(1 + |v|^2) \, dv \, dx \right) + \int_{\Omega} |\nabla u|^2 \, dx + \int_{\Omega} \int_{\mathbb{R}^3} R \delta \rho f(u - v)^2 \, dv \, dx + \int_{\Omega} \int_{\mathbb{R}^3} R \delta \rho f(u - u_\varepsilon) v \, dv \, dx = - \int_{\Omega} \int_{\mathbb{R}^3} R \delta \rho f(u - u_\varepsilon) v \, dv \, dx.
\] (3.23)

We control the right side as

\[
\left| - \int_{\Omega} \int_{\mathbb{R}^3} R \delta \rho f(u - u_\varepsilon) v \, dv \, dx \right| \leq 2C(\delta) \int_{\Omega} \sqrt{\rho} \sqrt{|u|} \, dx
\]

\[
\leq 2C(\delta) \left( \int_{\Omega} \rho \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} \rho |u|^2 \, dx \right)^{\frac{1}{2}}
\]

\[
\leq C(\delta) \int_{\Omega} \rho \, dx + C(\delta) \int_{\Omega} \rho |u|^2 \, dx,
\]

and use the Gronwall inequality in (3.23) to obtain

\[
\int_{\Omega} \frac{1}{2} \rho |u|^2 \, dx + \int_{\Omega} \int_{\mathbb{R}^3} \frac{1}{2} f(1 + |v|^2) \, dv \, dx + \int_0^T \int_{\Omega} |\nabla u|^2 \, dx + \int_0^T \int_{\Omega} R \delta \rho f(u - v)^2 \, dv \, dx \leq C(\delta) \left( \int_{\Omega} \rho_0 |u_0|^2 \, dx + T \int_{\Omega} \rho \, dx \right) e^T.
\] (3.24)
The energy inequality will be crucial in deriving a priori estimates on the solutions \((\rho, u, f)\) of the approximate system.

3.2. **Pass to the limit as** \(\varepsilon \to 0\). The objective of this subsection is to recover the original system from the approximation scheme (3.1)-(3.4) upon letting \(\varepsilon\) go to 0. For this purpose, we denote by \((\rho^\varepsilon, u^\varepsilon, f^\varepsilon)\) the solution constructed in Theorem 3.1. In this subsection, we fix \(\delta > 0\) and all estimates in this subsection are uniformly in \(\varepsilon\).

We take \(\beta \in C(\Omega, \mathbb{R}^3)\), use (3.1) and (3.3) to find that
\[
\int_\Omega \beta(\rho^\varepsilon) dx
\]
is independent of time \(t\), that is,
\[
\int_\Omega \beta(\rho^\varepsilon) dx = \int_\Omega \beta(\rho^\varepsilon_0) dx \quad \text{for all } t \in (0, \infty).
\]
Observing that \((\rho^\varepsilon, u^\varepsilon, f^\varepsilon)\) satisfies (3.24), one obtains
\[
\int_\Omega \frac{1}{2} \rho^\varepsilon |u^\varepsilon|^2 dx + \int_\Omega \int_{\mathbb{R}^3} \frac{1}{2} f^\varepsilon (1 + |v|^2) dxdv
\]
\[
+ \int_0^t \int_\Omega |\nabla u^\varepsilon|^2 dxdt + \int_0^t \int_\Omega \int_{\mathbb{R}^3} R_\delta \rho^\varepsilon f^\varepsilon |u^\varepsilon - v|^2 dxdvdt
\]
\[
\leq C(\delta) \left( \int_\Omega \rho^\varepsilon_0 |u^\varepsilon_0|^2 dx + T \int_\Omega \rho^\varepsilon_0 dx \right) e^T
\]
for all \(t > 0\), where we used \(\int_\Omega \rho^\varepsilon dx = \int_\Omega \rho^\varepsilon_0 dx\).

By (3.10), we have
\[
\frac{1}{2} \int_\Omega \rho^\varepsilon_0 |u^\varepsilon_0|^2 dx = \frac{1}{2} \int_\Omega \frac{|m^\varepsilon_0|^2}{\rho^\varepsilon_0} < \infty.
\]
Thus, with the help of (3.26), one obtains the following estimates:
\[
\|u^\varepsilon\|_{L^2(0,T;H_0^1(\Omega))} \leq C,
\]
\[
\sup_{0 \leq t \leq T} \|\rho^\varepsilon |u^\varepsilon|^2\|_{L^1(\Omega)} \leq C,
\]
where \(C = C(\delta)\) denotes a generic positive constant independent of \(\varepsilon\).

By (3.5), (3.6), (3.7), (3.8), together with Theorem 2.4 in [22], we assume that, up to a subsequence,
\[
\rho^\varepsilon \to \rho \quad \text{weakly in } L^p((0,T) \times \Omega) \text{ for any } 1 \leq p < \infty.
\]
We denote by \(u\) the weak limit of \(u^\varepsilon\) in \(L^2(0,T;H_0^1(\Omega))\) due to (3.27). By the compactness of the embedding \(L^p(\Omega) \hookrightarrow H^{-1}(\Omega)\) for any \(p > 6/5\), one deduces from (3.28):
\[
\rho^\varepsilon \to \rho \quad \text{in } C([0,T];H^{-1}(\Omega)),
\]
which, together with (3.27), yields
\[
\rho^\varepsilon u^\varepsilon \to \rho u \quad \text{in } \mathcal{D}'((0,T) \times \Omega).
\]
Here if we equip \(L^p(\Omega)\) for any \(1 < p < \infty\) with the weak topology and the associated distance over a large ball containing all values \(\rho^\varepsilon(t)\) for \(t \in [0,T]\), we can deduce that
\[
\rho^\varepsilon \to \rho \quad \text{in } C([0,T];L^p_w(\Omega)),
\]
and also we have
\[
\rho(0) = \rho_0, \quad \text{a.e in } \Omega.
\]
Let a function \( g \in C([0, T]; L^p_w(\Omega)) \) for any \( 1 < p < \infty \) satisfy \( g(0) = 0 \) on \( \Omega \) and
\[
\frac{\partial g}{\partial t} + \text{div}(g \mathbf{u}) = 0 \quad \text{in} \quad \mathcal{D}'((0, T) \times \Omega),
\]
then \( g \equiv 0 \), which implies the uniqueness of the density \( \rho \) when \( \mathbf{u} \) is fixed. Thus we have proved that \( \rho \) is the solution to (1.1). We now estimate \( m_0 f^\varepsilon \):
\[
m_0 f^\varepsilon = \int_{\mathbb{R}^3} f^\varepsilon \, d\mathbf{v} = \int_{|\mathbf{v}| < r} f^\varepsilon \, d\mathbf{v} + \int_{|\mathbf{v}| \geq r} f^\varepsilon \, d\mathbf{v}
\leq C \| f^\varepsilon \|_{L^\infty} r^3 + \frac{1}{r^k} \int_{|\mathbf{v}| \geq r} |\mathbf{v}|^k f^\varepsilon \, d\mathbf{v}
\]
for all \( k \geq 0 \). Taking \( r = \left( \int_{\mathbb{R}^3} |\mathbf{v}|^k f^\varepsilon \, d\mathbf{v} \right)^{\frac{1}{k+3}} \), we have
\[
m_0 f^\varepsilon \leq C \| f^\varepsilon \|_{L^\infty} \left( \int_{\mathbb{R}^3} |\mathbf{v}|^k f^\varepsilon \, d\mathbf{v} \right)^{\frac{3}{k+3}} + \left( \int_{\mathbb{R}^3} |\mathbf{v}|^k f^\varepsilon \, d\mathbf{v} \right)^{\frac{3}{k+3}}.
\]
Letting \( k = 3 \), then
\[
\| m_0 f^\varepsilon \|_{L^2(\Omega)} \leq C(\| f^\varepsilon \|_{L^\infty} + 1) \left( \int_{\Omega} \int_{\mathbb{R}^3} |\mathbf{v}|^3 f^\varepsilon \, d\mathbf{v} \right)^{1/2}.
\]
Thanks to Lemma 2.1, we conclude that \( m_3 f^\varepsilon \) is bounded in \( L^\infty(0, T; L^1(\Omega)) \). Thus
\[
\| m_0 f^\varepsilon \|_{L^2(0, T; L^3/2(\Omega))} \leq C(\delta).
\]
(3.30)
Following the same argument, one deduces that
\[
\| m_1 f^\varepsilon \|_{L^2(0, T; L^3/2(\Omega))} \leq C(\delta).
\]
(3.31)
Using the fact \( R_\delta \leq 1 \), we see that
\[
\| \rho R_\delta m_0 f^\varepsilon \mathbf{u}^\varepsilon \|_{L^2(0, T; L^{3/2}(\Omega))} \leq C \| \rho_0 \|_{L^\infty(0, T) \times \Omega} \| m_0 f^\varepsilon \|_{L^\infty(0, T; L^3(\Omega))} \cdot \| \mathbf{u}^\varepsilon \|_{L^2(0, T; L^6(\Omega))},
\]
(3.32)
and
\[
\| \rho R_\delta m_1 f^\varepsilon \|_{L^\infty(0, T; L^{3/2}(\Omega))} \leq C \| \rho_0 \|_{L^\infty(0, T) \times \Omega} \| m_1 f^\varepsilon \|_{L^\infty(0, T; L^3/2(\Omega))}.
\]
(3.33)
Observing
\[
\rho^\varepsilon R_\delta \int_{\mathbb{R}^3} (\mathbf{u}^\varepsilon - \mathbf{v}) f^\varepsilon \, d\mathbf{v} = \rho^\varepsilon R_\delta m_0 f^\varepsilon \mathbf{u}^\varepsilon - \rho R_\delta m_1 f^\varepsilon,
\]
and using (3.32) and (3.33), we obtain that
\[
\rho^\varepsilon R_\delta \int_{\mathbb{R}^3} (\mathbf{u}^\varepsilon - \mathbf{v}) f^\varepsilon \, d\mathbf{v}
\]
is bounded in \( L^2(0, T; L^{3/2}(\Omega)) \).
Since
\[
\frac{\partial (\rho^\varepsilon \mathbf{u}^\varepsilon)}{\partial t} = -\text{div}(\rho^\varepsilon \mathbf{u}^\varepsilon \otimes \mathbf{u}^\varepsilon) + \Delta \mathbf{u}^\varepsilon + \nabla \rho + \rho R_\delta \int_{\mathbb{R}^3} (\mathbf{u}^\varepsilon - \mathbf{v}) f^\varepsilon \, d\mathbf{v},
\]
and in particular, \( \nabla \mathbf{u}^\varepsilon \) is bounded in \( L^2((0, T) \times \Omega) \) and
\[
\rho^\varepsilon R_\delta \int_{\mathbb{R}^3} (\mathbf{u}^\varepsilon - \mathbf{v}) f^\varepsilon \, d\mathbf{v}
\]
is bounded in \( L^2(0, T; L^{3/2}(\Omega)) \),
while $\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon$ is bounded in $L^2(0, T; L^2(\Omega))$, one concludes that
\[
\frac{\partial (\rho^\varepsilon u^\varepsilon)}{\partial t} \text{ is bounded in } L^2(0, T; H^{-1}(\Omega)).
\]
By Theorem 2.4 of [22], we obtain
\[
\sqrt{\rho^\varepsilon u^\varepsilon} \to \sqrt{\rho u} \text{ in } L^p(0, T; L^r(\Omega))
\]
for $2 < p < \infty$ and $1 \leq r < \frac{6p}{3p-4}$, and thus
\[
\rho^\varepsilon u^\varepsilon \to \rho u \text{ in } L^p(0, T; L^r(\Omega)) \tag{3.34}
\]
for the same values of $p, r$.

Applying Lemma 2.2 to (3.4), we obtain
\[
m_0f^\varepsilon \to m_0f, \ m_1f^\varepsilon \to m_1f, \text{ a.e. } (t,x). \tag{3.35}
\]
By (3.30) and (3.31), the relation (3.35) can be strengthened to the following statement:
\[
m_0f^\varepsilon \to m_0f \text{ strongly in } L^2(0, T; L^2(\Omega)),
m_1f^\varepsilon \to m_1f \text{ strongly in } L^2(0, T; L^{3/2}(\Omega)). \tag{3.36}
\]
By (3.28), we have
\[
\rho^\varepsilon m_0f^\varepsilon \to \rho m_0f \text{ strongly in } L^2(0, T; L^{\frac{2p}{3p+3}}(\Omega)), \tag{3.37}
\]
and
\[
\rho^\varepsilon m_1f^\varepsilon \to \rho m_1f \text{ strongly in } L^2(0, T; L^{\frac{3p}{3p+3}}(\Omega)). \tag{3.38}
\]
Thanks to (3.37)-(3.38) and the weak convergence of $u^\varepsilon$ in $L^2(0, T; H^1_0(\Omega))$, one has
\[
R_\delta \int_{\mathbb{R}^3} (u^\varepsilon - v)\rho^\varepsilon f^\varepsilon \, dv \to R_\delta \int_{\mathbb{R}^3} (u - v)\rho f \, dv \text{ in the sense of distributions.} \tag{3.39}
\]
The next step is to deal with the convergence of $\text{div}_v(R_\delta \rho^\varepsilon (u^\varepsilon - v)f^\varepsilon)$. Let $\phi(v) \in D(\mathbb{R}^3)$ be a test function, then
\[
\int_{\mathbb{R}^3} f^\varepsilon \nabla_v \phi \, dv \to \int_{\mathbb{R}^3} f \nabla \phi \, dv
\]
a.e. for $(t,x)$ due to the average compactness results of DiPerna-Lions-Meyer [11]. This, together with (3.34), implies that
\[
\lim_{\varepsilon \to 0} \left( \int_{\Omega} \int_{\mathbb{R}^3} R_\delta \rho^\varepsilon u^\varepsilon \cdot \nabla_v \phi \, dv \, dx \right) = \int_{\Omega} \int_{\mathbb{R}^3} R_\delta \rho u \cdot \nabla_v \phi \, dv \, dx, \tag{3.40}
\]
while we have
\[
\lim_{\varepsilon \to 0} \left( \int_{\Omega} \int_{\mathbb{R}^3} R_\delta \rho^\varepsilon f^\varepsilon \nabla_v \phi \, dv \, dx \right) = \int_{\Omega} \int_{\mathbb{R}^3} R_\delta \rho f \nabla_v \phi \, dv \, dx. \tag{3.41}
\]
By (3.40) and (3.41), we have
\[
\lim_{\varepsilon \to 0} \left( \int_{\Omega} \int_{\mathbb{R}^3} (R_\delta \rho^\varepsilon (u^\varepsilon - v) f^\varepsilon) \cdot \nabla_v \phi \, dv \, dx \right) = \int_{\Omega} \int_{\mathbb{R}^3} (R_\delta \rho (u - v) f) \cdot \nabla_v \phi \, dv \, dx. \tag{3.42}
\]
We consider a test function $\varphi \in C^1_0([0,T] \times \Omega)$ such that $\text{div}\varphi = 0$, and a test function $\phi \in C^1([0,T] \times \Omega \times \mathbb{R}^3)$ with compact support in $\mathbf{v}$, such that $\phi(T,\cdot,\cdot) = 0$. The weak formulation associated with the approximation scheme (3.1)-(3.4) is the following:

\[
- \int_{\Omega} \rho_0^\varepsilon \mathbf{u}_0^\varepsilon \cdot \varphi(0,x) \, dx + \int_0^T \int_{\Omega} \left\{ - \rho^\varepsilon \mathbf{u}^\varepsilon \cdot \partial_t \varphi - (\rho^\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}^\varepsilon) \cdot \nabla \varphi \\
+ \nabla \mathbf{u}^\varepsilon \cdot \nabla \varphi + \varphi \cdot R_\delta \int_{\mathbb{R}^3} (\mathbf{u}^\varepsilon - \mathbf{v}) \rho^\varepsilon \, f^\varepsilon \, dv \right\} \, dx \, dt = 0;
\]

(3.43)

and

\[
- \int_0^T \int_{\Omega} \int_{\mathbb{R}^3} f^\varepsilon (\partial_t \phi + \mathbf{v} \cdot \nabla_x \phi + R_\delta (\mathbf{u}_\varepsilon - \mathbf{v}) \rho^\varepsilon \cdot \nabla \varphi) \, dx \, dv \, ds \\
= \int_\Omega \int_{\mathbb{R}^3} f_0 \phi(0,\cdot,\cdot) \, dx \, dv.
\]

(3.44)

By (3.9)-(3.10), we have

\[
\int_{\Omega} \rho_0^\varepsilon \mathbf{u}_0^\varepsilon \cdot \varphi \, dx = \int_{\Omega} \mathbf{m}_0^\varepsilon \cdot \varphi \, dx \to \int_{\Omega} \mathbf{m}_0 \cdot \varphi \, dx \quad \text{as} \ \varepsilon \to 0,
\]

for all test functions $\varphi$.

All the above convergence results in this subsection allow us to recover (2.15)-(2.16) by passing to the limits in (3.43) and (3.44) as $\varepsilon \to 0$.

By (3.23), the solution $(\rho^\varepsilon, \mathbf{u}^\varepsilon, f^\varepsilon)$ satisfies the following:

\[
\int_{\Omega} \frac{1}{2} \rho^\varepsilon |\mathbf{u}^\varepsilon|^2 \, dx + \int_{\Omega} \int_{\mathbb{R}^3} \frac{1}{2} f^\varepsilon (1 + |\mathbf{v}|^2) \, dv \, dx + \int_0^T \int_{\Omega} |\nabla \mathbf{u}^\varepsilon|^2 \, dx \, dt \\
+ \int_0^T \int_{\Omega} \int_{\mathbb{R}^3} R_\delta \rho^\varepsilon f^\varepsilon |\mathbf{u}^\varepsilon - \mathbf{v}|^2 \, dv \, dx \, dt \\
= \frac{1}{2} \int_{\Omega} \rho_0^\varepsilon |\mathbf{u}_0^\varepsilon|^2 \, dx + \frac{1}{2} \int_{\Omega} \int_{\mathbb{R}^3} (1 + |\mathbf{v}|^2) f_0 \, dv \, dx + \int_0^T \int_{\mathbb{R}^3} R_\delta \rho^\varepsilon (\mathbf{u}_\varepsilon - \mathbf{u}^\varepsilon) v \, dv \, dx \, dt.
\]

It is clear that

\[
\int_0^T \int_{\mathbb{R}^3} R_\delta \rho^\varepsilon f^\varepsilon (\mathbf{u}_\varepsilon - \mathbf{u}^\varepsilon) v \, dv \, dx \, dt \to 0 \quad \text{as} \ \varepsilon \to 0.
\]

(3.45)

The difficulty of passing the limit for the energy inequality is the convergence of the term $\int_0^T \int_{\Omega \times \mathbb{R}^3} R_\delta \rho^\varepsilon f^\varepsilon |\mathbf{u}^\varepsilon - \mathbf{v}|^2 \, dv \, dx \, dt$. We follow the same idea as in [19, 32] to treat this term as:

\[
\int_0^T \int_{\Omega \times \mathbb{R}^3} R_\delta \rho^\varepsilon f^\varepsilon |\mathbf{u}^\varepsilon - \mathbf{v}|^2 \, dv \, dx \, dt \\
= \int_0^T \int_{\Omega \times \mathbb{R}^3} (R_\delta \rho^\varepsilon f^\varepsilon |\mathbf{u}^\varepsilon|^2 - 2 R_\delta \rho^\varepsilon f^\varepsilon \mathbf{u}^\varepsilon \cdot \mathbf{v} + R_\delta \rho^\varepsilon f^\varepsilon |\mathbf{v}|^2) \, dx \, dv \, dt.
\]

(3.46)

By the embedding inequality, we have

\[
\mathbf{u}^\varepsilon \to \mathbf{u} \quad \text{weakly in} \quad L^2(0,T; L^6(\Omega)).
\]

(3.47)

By (3.37), (3.47), we deduce that

\[
R_\delta \rho^\varepsilon m_0 f^\varepsilon |\mathbf{u}^\varepsilon|^2 \to R_\delta \rho m_0 f |\mathbf{u}|^2 \quad \text{weakly in} \quad L^1((0,T) \times \Omega)
\]
as \( \varepsilon \to \infty \). Similarly,
\[
R_\delta \rho \varepsilon m_1 f^\varepsilon u^\varepsilon \to R_\delta \rho m_1 f^\varepsilon u \quad \text{weakly in } L^1((0, T) \times \Omega)
\]
as \( \varepsilon \to 0 \).

Finally, let us look at the terms:
\[
\left| \int_0^T \int_{\mathbb{R}^3} R_\delta \rho \varepsilon f^\varepsilon |\nabla|^2 dv \, dx \, dt - \int_0^T \int_{\mathbb{R}^3} R_\delta \rho f |\nabla|^2 dv \, dx \, dt \right| 
\leq \int_0^T \int_{\Omega} |\rho - \rho^\varepsilon| m_2 f^\varepsilon dx \, dt + C\|\rho\|_{L^\infty} \int_0^T \int_{\Omega} |m_2 f - m_2 f^\varepsilon| dx \, dt
= I_1 + I_2.
\]

Similarly to (3.30), we can show that
\[
\|m_2 f^\varepsilon\|_{L^2((0, T); L^2(\Omega))} < \infty,
\]
by the convergence of \( \rho^\varepsilon \to \rho \in L^\infty(0, T; L^p(\Omega)) \), and we conclude \( I_1 \to 0 \) as \( \varepsilon \to 0 \). For the term \( I_2 \), because
\[
f^\varepsilon \to f \quad \text{weak star in } L^\infty(0, T; L^p(\Omega \times \mathbb{R}^3))
\]
for all \( p \in (1, \infty) \) and \( m_3 f^\varepsilon \) is bounded in \( L^\infty(0, T; L^1(\Omega)) \), then for any fixed \( r > 0 \), we have
\[
\int_0^T \int_{\mathbb{R}^3} f^\varepsilon |\nabla|^2 dx \, dv \, dt = \int_0^T \int_{\mathbb{R}^3} \chi(|\nabla| < r) |\nabla|^2 f^\varepsilon dx \, dv \, dt + O(r^{-1})
\]
uniformly in \( \varepsilon \) where \( \chi \) is the characteristic function of the ball of \( \mathbb{R}^3 \) of radius \( r \). Letting \( \varepsilon \to 0 \), then \( r \to \infty \), we find
\[
\int_0^T \int_{\mathbb{R}^3} f^\varepsilon |\nabla|^2 dx \, dv \, dt \to \int_0^T \int_{\mathbb{R}^3} f |\nabla|^2 dx \, dv \, dt,
\]
which means \( I_2 \to 0 \) as \( \varepsilon \to 0 \). Thus, we have proved
\[
\int_0^T \int_{\mathbb{R}^3} R_\delta \rho \varepsilon f^\varepsilon |u^\varepsilon - v|^2 dv \, dx \, dt \to \int_0^T \int_{\mathbb{R}^3} R_\delta \rho f |u - v|^2 dv \, dx \, dt \quad (3.48)
\]
as \( \varepsilon \to \infty \).

We observe that
\[
\int_\Omega |\rho_0^\varepsilon u_0^\varepsilon|^2 dx = \int_\Omega \frac{1}{\rho_0^\varepsilon} |m_0^\varepsilon - \nabla q_0^\varepsilon|^2 dx 
= \int_\Omega \left( \frac{|m_0^\varepsilon|^2}{\rho_0^\varepsilon} + \frac{|\nabla q_0^\varepsilon|^2}{\rho_0^\varepsilon} - \frac{2}{\rho_0^\varepsilon} (\rho_0^\varepsilon u_0^\varepsilon + \nabla q_0^\varepsilon) \cdot \nabla q_0^\varepsilon \right) dx 
= \int_\Omega \left( \frac{|m_0^\varepsilon|^2}{\rho_0^\varepsilon} - 2 u_0^\varepsilon \cdot \nabla q_0^\varepsilon \right) \frac{|\nabla q_0^\varepsilon|^2}{\rho_0^\varepsilon} dx,
\]
by Lemma 3.1. Using \( \text{div} u_0^\varepsilon = 0 \), one obtains,
\[
\int_\Omega \rho_0^\varepsilon |u_0^\varepsilon|^2 dx + \int_\Omega |\nabla q_0^\varepsilon|^2 \frac{1}{\rho_0^\varepsilon} dx = \int_\Omega \frac{|m_0^\varepsilon|^2}{\rho_0^\varepsilon} dx. \quad (3.50)
\]
Integrating (3.23) with respect to time $t$, letting $\varepsilon \to 0$, using (3.10), (3.45), (3.48), (3.50) and the weak convergence of $(\rho^\varepsilon, u^\varepsilon, f^\varepsilon)$, we obtain,

$$
\int_\Omega \frac{1}{2} \rho |u|^2 \, dx + \int_\Omega \int_{\mathbb{R}^3} \frac{1}{2} f(1 + |v|^2) \, dx \, dv \\
+ \int_0^t \int_\Omega |\nabla u|^2 \, dx \, dt + \int_0^t \int_{\mathbb{R}^3} R_\delta \rho f |u - v|^2 \, dx \, dv \, dt \\
\leq \frac{1}{2} \int_\Omega \frac{|m_0|^2}{\rho_0} \, dx + \frac{1}{2} \int_\Omega \int_{\mathbb{R}^3} (1 + |v|^2) f_0 \, dx \, dv.
$$

So far, we have proved the following result:

**Proposition 3.1.** For any $T > 0$, there is a weak solution $(\rho^\delta, u^\delta, f^\delta)$ to the following system:

$$
\rho_t + \text{div}(\rho u) = 0, \\
\frac{\partial (\rho u)}{\partial t} + \text{div}(\rho u \otimes u) - \mu \Delta u + \nabla p = \rho R_\delta \int_{\mathbb{R}^3} (u - v) f \, dv, \\
\text{div} u = 0, \\
\frac{\partial f}{\partial t} + v \cdot \nabla f + \text{div}_v (R_\delta (u - v) \rho f) = 0.
$$

with the initial data $u(0, x) = u_0$ and $f(0, x, v) = f_0(x, v)$, and the boundary conditions:

$$
u \cdot n = 0 \quad \text{on} \quad \partial \Omega, \\
f(t, x, v) = f(t, x, v^*) \quad \text{for} \quad x \in \partial \Omega, \quad v \cdot n(x) < 0.
$$

In addition, the solution satisfies the following energy inequality:

$$
\int_\Omega \frac{1}{2} \rho |u|^2 \, dx + \int_\Omega \int_{\mathbb{R}^3} \frac{1}{2} f(1 + |v|^2) \, dx \, dv \\
+ \int_0^t \int_\Omega |\nabla u|^2 \, dx \, dt + \int_0^t \int_{\mathbb{R}^3} R_\delta \rho f |u - v|^2 \, dx \, dv \, dt \\
\leq \frac{1}{2} \int_\Omega \frac{|m_0|^2}{\rho_0} \, dx + \frac{1}{2} \int_\Omega \int_{\mathbb{R}^3} (1 + |v|^2) f_0 \, dx \, dv.
$$

### 3.3. Pass the limit as $\delta \to 0$

The last step of showing the existence of global weak solution is to pass the limit as $\delta$ goes to zero. First, we let $(\rho^\delta, f^\delta, u^\delta)$ be a solution constructed in Proposition 3.1. We remark that all estimates of this subsection are uniformly in $\delta$. By Proposition 3.1, we have the following estimates for $(\rho^\delta, f^\delta, u^\delta)$:

$$
\sqrt{\rho_0} u^\delta \quad \text{is bounded in} \quad L^\infty(0, T; L^2(\Omega)); \\
u^\delta \quad \text{is bounded in} \quad L^2(0, T; H^1_0(\Omega)).
$$

Also, we have

$$
0 \leq \rho_\delta(t, x) \leq \|\rho_0\|_{L^\infty}.
$$

By the above estimates, we can treat the terms related to the Navier-Stokes part as before. It remains to show the convergence of the terms:

$$
\int_{\mathbb{R}^3} R_\delta \rho^\delta f^\delta (u^\delta - v) \, dv, \quad \text{and} \quad \text{div}(R_\delta \rho^\delta (u^\delta - v)).
$$
The next step is to deal with the convergence of \( \text{div}(R_\delta (u^\delta - v) \rho^\delta f^\delta) \). Let \( \phi(v) \in \mathcal{D}(\mathbb{R}^3) \) be a test function, we want to show

\[
\lim_{\delta \to 0} \left( \int_{\Omega} R_\delta \rho^\delta u^\delta \left( \int_{\mathbb{R}^3} f^\delta \nabla_v \phi dv \right) - \int_{\mathbb{R}^3} R_\delta \rho^\delta f^\delta \nabla_v \phi \right) = \int_{\Omega} \rho u \left( \int_{\mathbb{R}^3} f \nabla_v \phi dv \right) \quad (3.51)
\]

To prove (3.51), we follow the same idea as in [19] to define a function \( Q_\delta = 1 - R_\delta \). It is easy to see that

\[
Q_\delta \to 0 \quad \text{as} \quad \delta \to 0.
\]

Note that

\[
\int_{\Omega} R_\delta \rho^\delta u^\delta \left( \int_{\mathbb{R}^3} f^\delta \nabla_v \phi dv \right) dx = \int_{\mathbb{R}^3} \rho^\delta u^\delta \left( \int_{\mathbb{R}^3} f^\delta \nabla_v \phi dv \right) dx - \int_{\Omega} Q_\delta \rho^\delta u^\delta \left( \int_{\mathbb{R}^3} f^\delta \nabla_v \phi dv \right) dx. \quad (3.52)
\]

On one hand, applying Lemma 2.2 to (3.4), we have

\[
\int_{\mathbb{R}^3} \rho^\delta \nabla_v \phi d\nu \to \int_{\mathbb{R}^3} \nabla_v \phi d\nu \quad \text{almost everywhere} \quad (t, x). \quad (3.53)
\]

It is easy to see

\[
\left| \int_{\mathbb{R}^3} \rho^\delta \nabla_v \phi d\nu \right| \leq C|m_0f^\delta|, \quad (3.54)
\]

which, together with (3.36), strengthens (3.53) as

\[
\int_{\mathbb{R}^3} \rho^\delta \nabla_v \phi d\nu \to \int_{\mathbb{R}^3} \nabla_v \phi d\nu \quad \text{strongly in} \quad L^2(0, T; L^2(\Omega)). \quad (3.55)
\]

By the convergence of \( \rho^\delta \), (3.55) and the weak convergence of \( u^\delta \) in \( L^2(0, T; H_0^1(\Omega)) \), one deduces

\[
\int_{\Omega} \rho^\delta u^\delta \left( \int_{\mathbb{R}^3} f^\delta \nabla_v \phi dv \right) dx \to \int_{\Omega} \rho u \left( \int_{\mathbb{R}^3} f \nabla_v \phi dv \right) dx.
\]

On the other hand,

\[
\left| \int_{\Omega} Q_\delta \rho^\delta u^\delta \left( \int_{\mathbb{R}^3} f^\delta \nabla_v \phi dv \right) dx \right| \leq C \int_{\Omega} Q_\delta m_0f|u^\delta| \ dx \ dt
\]

which yields

\[
\left| \int_{\Omega} Q_\delta \rho^\delta u^\delta \left( \int_{\mathbb{R}^3} f^\delta \nabla_v \phi dv \right) \right| \to 0 \quad \text{as} \quad \delta \to 0,
\]

by the fact that \( m_0f^\delta \) is bounded in \( L^\infty(0, T; L^2(\Omega)) \), \( u^\delta \) is bounded in \( L^2(0, T; L^6(\Omega)) \), and \( Q_\delta \to 0 \) strongly in \( L^2(0, T; L^3(\Omega)) \). Thus, we have proved the convergence of the first
integral on the left side of (3.51). We can treat similarly the convergence of the second integral of (3.51). Thus, we finish the proof of (3.51).

To complete the proof of Theorem 2.1, it only remains to check that \((\rho, u, f)\) satisfies the energy inequality (2.17). In order to verify the energy inequality (2.17), we need to show

\[
\int_0^t \int_{\Omega} \mathcal{R}_\delta f^\delta |u^\delta - v|^2 \, dx \, dt \to \int_0^t \int_{\Omega} \rho f |u - v|^2 \, dx \, dt
\]

as \(\delta \to 0\). Denote

\[
E_1^\delta = \int_{\Omega} \int_{\mathbb{R}^3} \rho^\delta f^\delta |u^\delta - v|^2 \, dv \, dx = E_1^\delta - 2E_2^\delta + E_3^\delta,
\]

where

\[
E_1^\delta = \int_{\Omega} \int_{\mathbb{R}^3} \rho^\delta f^\delta |u^\delta|^2 \, dv \, dx = \int_{\Omega} \rho^\delta m_0 f^\delta |u^\delta|^2 \, dx,
\]

\[
E_2^\delta = \int_{\Omega} \int_{\mathbb{R}^3} \rho^\delta f^\delta v^\delta \, dv \, dx = \int_{\Omega} \rho^\delta m_1 f^\delta u^\delta \, dx,
\]

and

\[
E_3^\delta = \int_{\Omega} \int_{\mathbb{R}^3} \rho^\delta f^\delta |v|^2 \, dv \, dx = \int_{\Omega} \rho^\delta m_2 f^\delta \, dx.
\]

Write \(R_\delta E^\delta = E^\delta - Q_\delta E^\delta\), we consider the convergence of \(E_1^\delta\) first. Since

\[
\left| \int_0^T \int_{\Omega \times \mathbb{R}^3} \rho^\delta f^\delta |u^\delta|^2 \, dv \, dx \, dt - \int_0^T \int_{\Omega \times \mathbb{R}^3} \rho f |u|^2 \, dv \, dx \, dt \right|
\]

\[
\leq \int_0^T \int_{\Omega} \left( |\rho^\delta - \rho| \right) m_0 f^\delta |u^\delta|^2 \, dx \, dt + \int_0^T \int_{\Omega} \rho (m_0 f^\delta - m_0 f) |u^\delta|^2 \, dx \, dt
\]

\[
+ \int_0^T \int_{\Omega} \rho m_0 f \left( |u^\delta|^2 - |u|^2 \right) \, dx \, dt,
\]

then

\[
\int_0^t E_1^\delta \, dt \to \int_0^t \int_{\Omega} \int_{\mathbb{R}^3} \rho f |u|^2 \, dv \, dx \, dt \quad \text{as} \quad \delta \to 0
\]

for all \(t > 0\). Similarly, we obtain

\[
\int_0^t E_2^\delta \, dt \to \int_0^t \int_{\Omega} \int_{\mathbb{R}^3} \rho f u \, dv \, dx \, dt \quad \text{as} \quad \delta \to 0
\]

for all \(t > 0\).

Next, let us examine

\[
\left| \int_0^t \int_{\Omega} \int_{\mathbb{R}^3} \rho^\delta f^\delta |v|^2 \, dv \, dx \, dt - \int_0^t \int_{\Omega} \int_{\mathbb{R}^3} \rho f |v|^2 \, dv \, dx \, dt \right|
\]

\[
\leq ||\rho^\delta - \rho||_{L^\infty} \int_0^T \int_{\Omega} m_2 f^\delta \, dx \, dt + C||\rho||_{L^\infty} \int_0^T \int_{\Omega} (m_2 f - m_2 f^\delta) \, dx \, dt
\]

\[
= I_1 + I_2.
\]

It is clear that \(I_1 \to 0\) as \(\delta \to 0\). For the term \(I_2\), because

\[
f^\delta \rightharpoonup f \quad \text{weak star in} \; L^\infty(0, T; L^p(\Omega \times \mathbb{R}^3))
\]
for all \( p \in (1, \infty) \) and \( m_3 f^\delta \) is bounded in \( L^\infty(0, T; L^1(\Omega)) \), then for any fixed \( r > 0 \), we have

\[
\int_0^T \int_{\Omega \times \mathbb{R}^3} f^\delta |v|^2 \, dx \, dv \, dt = \int_0^T \int_{\Omega \times \mathbb{R}^3} \chi(|v| < r) |v|^2 f^\delta \, dx \, dv \, dt + O(r^{-1})
\]

uniformly in \( \delta \) where \( \chi \) is the characteristic function of the ball of \( \mathbb{R}^3 \) of radius \( r \). Letting \( \delta \to 0 \), then \( r \to \infty \), we find

\[
\int_0^T \int_{\Omega \times \mathbb{R}^3} f^\delta |v|^2 \, dx \, dv \, dt \to \int_0^T \int_{\Omega \times \mathbb{R}^3} f|v|^2 \, dx \, dv \, dt,
\]

which means \( I_2 \to 0 \) as \( \delta \to 0 \). Thus, we have proved

\[
\int_0^T \int_{\Omega \times \mathbb{R}^3} \chi(|v| < r) |v|^2 f^\delta \, dx \, dv \, dt + O(r^{-1})
\]

uniformly in \( \delta \), where \( \chi \) is the characteristic function of the ball of \( \mathbb{R}^3 \) of radius \( r \). Letting \( \delta \to 0 \), then \( r \to \infty \), we find

\[
\int_0^T \int_{\Omega \times \mathbb{R}^3} \chi(|v| < r) |v|^2 f^\delta \, dx \, dv \, dt = \int_0^T \int_{\Omega \times \mathbb{R}^3} f|v|^2 \, dx \, dv \, dt + O(r^{-1})
\]

uniformly in \( \delta \), where \( \chi(x) \) is a characterized function. Then

\[
\rho^\delta \to \rho \quad \text{in } C([0, T]; L^p(\Omega)) \text{ for any } 1 \leq p < \infty,
\]

and by the definition of \( Q^\delta \), we have

\[
\chi(|v| < r) Q^\delta \rho^\delta \to 0 \quad \text{strongly in } L^p(0, T; L^q(\Omega)) \text{ for any } 1 \leq p, q < \infty.
\]

Thus

\[
\int_0^T \int_{\Omega \times \mathbb{R}^3} Q^\delta \rho^\delta |v|^2 f^\delta \, dx \, dv \, dt \to 0
\]
as $\delta \to 0$ and $r \to \infty$. Hence we have proved (3.58) and hence (3.57). Thanks to the convergence facts and the convexity of the energy inequality, we deduce (2.17) from the energy inequality in Proposition 3.1.

The proof of Theorem 2.1 is complete.

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