Extension complexities of Cartesian products involving a pyramid

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Abstract

It is an open question whether the linear extension complexity of the Cartesian product of two polytopes \( P, Q \) is the sum of the extension complexities of \( P \) and \( Q \). We give an affirmative answer to this question for the case that one of the two polytopes is a pyramid.

1 Introduction

For a non-empty polytope \( P \), the linear extension complexity of \( P \) is defined as the smallest number of facets of any polytope that can be affinely projected onto \( P \), and is denoted by \( \text{xc}(P) \). Given any non-empty polytopes \( P \) and \( Q \), one can easily observe that \( \text{xc}(P \times Q) \leq \text{xc}(P) + \text{xc}(Q) \), while it is an open question whether this inequality actually holds as an equality, i.e., whether

\[
\text{xc}(P \times Q) = \text{xc}(P) + \text{xc}(Q)
\]

holds in general. This question has been asked at several occasions (see, e.g., [3, Conj. 1] or [5, Prob. 3]) but it seems that the most general case in which it is known that (1) holds is when one of the two polytopes is a simplex. The latter fact has been observed by several authors and can be explicitly found in [3, Cor. 10]. In this note, we prove that (1) holds whenever one of the two polytopes is a pyramid (in Section 2 we recall the definition of a pyramid):

**Theorem 1.** Let \( P, Q \) be non-empty polytopes such that one of the two polytopes is a pyramid. Then we have \( \text{xc}(P \times Q) = \text{xc}(P) + \text{xc}(Q) \).

While pyramids are still very special polytopes, with respect to linear extensions they are closely related to their bases, which can be arbitrary polytopes. Indeed, given a pyramid \( P \) with base \( B \) it is easy to see that \( \text{xc}(P) = \text{xc}(B) + 1 \) holds. Thus, although our proof crucially exploits the structure of Cartesian products involving a pyramid, we hope that our result opens doors for further generalizations.

In the next section, we discuss basic ingredients needed for the proof of Theorem 1 while the proof itself is given in Section 3.

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2 Preliminaries

A polytope $P \subseteq \mathbb{R}^d$ is called a pyramid with base $B \subseteq \mathbb{R}^d$ and apex $v \in \mathbb{R}^d$ if $P = \text{conv}(B \cup \{v\})$ and $v$ is not contained in the affine hull of $B$. Note that $v$ is contained in every facet of $P$ except for one which contains all remaining vertices of $P$.

Let $P = \{x \in \mathbb{R}^d : \langle a_i, x \rangle \leq b_i, i = 1, \ldots, m \} = \text{conv}\{v_1, \ldots, v_m\}$ for some $a_1, \ldots, a_m \in \mathbb{R}^d$, $b_1, \ldots, b_m \in \mathbb{R}$, and $v_1, \ldots, v_m \in \mathbb{R}^d$, where $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product of $\mathbb{R}^d$. Then the matrix $S \in \mathbb{R}_{\geq 0}^{m \times n}$ defined via $S_{i,j} := b_i - \langle a_i, v_j \rangle$ is called a slack matrix of $S$. A well-known result of Yannakakis [6] states that the linear extension complexity of $P$ is equal to the nonnegative rank of $S$, which is defined as the smallest number $r_+(S)$ such that $S$ can be written as the sum of $r_+(S)$ nonnegative rank-one matrices. The nonnegative rank $r_+(S)$ of a polytope is indeed well defined despite the fact its definition relies on the slack matrix $S$ which, in turn, is defined by a particular linear description of $P$. This follows from the fact that $r_+(S)$ neither depends on the scaling of the constraints used to describe $P$ nor on the potential presence of redundant constraints.

Although not needed for this work, the interested reader may consider the surveys [4, 1] and the book chapter [2, Chap. 4] as excellent sources for background information and recent developments on linear extended formulations.

In our proof, we make use of two simple facts about decompositions into nonnegative rank-one matrices: Let $S = R^1 + \cdots + R^k$ where $R^1, \ldots, R^k$ are nonnegative rank-one matrices and suppose that $S_{i,j} = 0$ holds. First, since all $R^\ell$ are nonnegative, this implies $(R^\ell)_{i,j} = 0$ for all $\ell$. Second, since all $R^\ell$ have rank one, for every pair of indices $(i', j')$ and every $\ell$ we must have $(R^\ell)^{i',j'} = 0$ or $(R^\ell)^{i,j} = 0$.

Given two polytopes $P, Q$ with

$$P = \{x \in \mathbb{R}^{d_P} : \langle a_i^P, x \rangle \leq b_i^P, i = 1, \ldots, m_P \} = \text{conv}\{v_1^P, \ldots, v_{m_P}^P\}$$

and

$$Q = \{y \in \mathbb{R}^{d_Q} : \langle a_i^Q, y \rangle \leq b_i^Q, i = 1, \ldots, m_Q \} = \text{conv}\{v_1^Q, \ldots, v_{m_Q}^Q\},$$

one immediately obtains

$$P \times Q = \{(x, y) \in \mathbb{R}^{d_P} \times \mathbb{R}^{d_Q} : \langle a_i^P, x \rangle \leq b_i^P, i = 1, \ldots, m_P, \langle a_i^Q, y \rangle \leq b_i^Q, i = 1, \ldots, m_Q\}$$

$$= \text{conv}\{(v_i^P, v_j^Q) : i \in [m_P], j \in [m_Q]\}.$$ 

Thus, if $S \in \mathbb{R}_{\geq 0}^{m_P \times m_Q}$ and $T = [t_1 \cdots t_{m_Q}] \in \mathbb{R}_{\geq 0}^{m_Q \times m_Q}$ are slack matrices of $P$ and $Q$, respectively, then the matrix

| $S$ | $S$ | $\cdots$ | $S$ |
|-----|-----|---------|-----|
| $t_1 \cdots t_1$ | $t_2 \cdots t_2$ | $\cdots$ | $t_{m_Q} \cdots t_{m_Q}$ |

\[ \in \mathbb{R}_{\geq 0}^{(m_P+m_Q) \times (m_P+m_Q)} \]
is a slack matrix of $P \times Q$, where $t_1, \ldots, t_{n_Q} \in \mathbb{R}_{\geq 0}^{m_Q}$ denote the columns of $T$. The columns of the above slack matrix correspond, from left to right, to the vertices $(v^P_1, v^Q_1), (v^P_2, v^Q_1), \ldots, (v^P_{n_p}, v^Q_{n_Q})$. Moreover, the first block of rows correspond to the constraints of $P$ and the second block of rows to the constraints of $Q$.

### 3 Proof of Theorem 1

We may assume that $Q$ is a pyramid. First, note that there exists a slack matrix $S \in \mathbb{R}_{\geq 0}^{m_P \times n_P}$ of $P$ such that every row contains at least one entry being zero. Indeed, every row containing no entry being zero corresponds to a redundant inequality and hence can be removed from the description of $P$. Second, by assuming that the description of $Q$ does not contain any redundant inequalities, the slack matrix $T \in \mathbb{R}_{\geq 0}^{m_Q \times n_Q}$ of $Q$ has the form

$$T = \begin{pmatrix} T' \bigcirc \\ \bigcirc 1 \end{pmatrix}$$

where $T' \in \mathbb{R}^{(m_Q-1) \times (n_Q-1)}$. Thus, the matrix $A \in \mathbb{R}_{\geq 0}^{(m_P+m_Q) \times (n_P \cdot n_Q)}$ defined via

\[
A := \begin{array}{c|c|c|c|c}
S & S & \cdots & S & S \\
\hline
t'_1 \cdots t'_1 & t'_2 \cdots t'_2 & \cdots & t'_k \cdots t'_k & \bigcirc \\
\bigcirc & \bigcirc & \cdots & \bigcirc & 1 \cdots 1
\end{array}
\]

is a slack matrix of $P \times Q$, where $t'_1, \ldots, t'_k \in \mathbb{R}_{\geq 0}^{m_Q-1}$ are the columns of $T'$ (here $k = n_Q - 1$). Recall that we have $xc(P \times Q) = r_+(A)$, $xc(P) = r_+(S)$, and $xc(Q) = r_+(T)$. Furthermore, it is straightforward to check that $r_+(T) = r_+(T') + 1$ holds. Thus, it remains to show that

$$r_+(A) \geq r_+(S) + r_+(T') + 1$$

holds. For the sake of contradiction, let us assume that we have

$$r_+(A) \leq r_+(S) + r_+(T'),$$

i.e., there exists a set $\mathcal{R}$ of nonnegative rank-one matrices in $\mathbb{R}_{\geq 0}^{(m_P+m_Q) \times (n_P \cdot n_Q)}$ with $|\mathcal{R}| \leq r_+(S) + r_+(T')$ whose sum is equal to $A$. Let $\mathcal{R}'$ and $\mathcal{R}''$ denote the set of matrices in $\mathcal{R}$ that have support in the red and blue parts of $A$, respectively.

Claim 1: The sets $\mathcal{R}'$ and $\mathcal{R}''$ form a partition of $\mathcal{R}$ satisfying $|\mathcal{R}'| = r_+(T')$ and $|\mathcal{R}''| = r_+(S)$. 

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First, observe that $R'$ and $R''$ are disjoint due to the O-block within $A$ that is below the blue $S$-block. Since the red part of $A$ contains $T'$ as a submatrix, we must have $|R'| \geq r_+(T')$, and since the blue part contains $S$ as a submatrix, we must have $|R''| \geq r_+(S)$, which yields the claim.

**Claim 2:** There exists at least one matrix in $R'$ that has support in the green part of $A$.

Since the nonnegative rank of the green submatrix of $A$ is equal to the nonnegative rank of $S$, at least $r_+(S)$ matrices in $R$ must have support in this part. Note that at least one matrix in $R''$ has support in the last row of the blue part of $A$ and hence it cannot have support in the green part of $A$. The claim follows since $|R''| = r_+(S)$.

**Claim 3:** Let $R \in R'$ and pick exactly one column of each of the $k$ red submatrices of $A$. Then $R$ has support in at least one of these columns.

Suppose the contrary. Then we can pick exactly one column of each of the $k$ red submatrices of $A$ such that $R$ has no support on any of these columns. Restricting to the submatrix formed by these columns, observe that this submatrix is identical to $T'$ but can be written as the sum of all matrices in $R' \setminus \{R\}$ and hence $r_+(T') \leq |R'| - 1 = r_+(T') - 1$, a contradiction.

**Claim 4:** No matrix in $R'$ can have support in the green part of $A$ (a contradiction to Claim 2).

Assume that there is some $R \in R'$ that has a positive entry $e_1$ in the green part of $A$. By our choice of $S$, every of the first $k$ blocks of $A$ contains a column of $A$ in which this row has a zero entry. By the previous claim, $R$ has a positive entry $e_2$ in the red part of one of these columns. Restricting $R$ to the two-by-two submatrix containing the entries $e_1, e_2$, it looks as follows (up to swapping its columns):

|     | 0 |
|-----|---|
| $*$ | $e_2 > 0$ |

However, there is no rank-one matrix with such a sign pattern.

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