Solar system constraints of a polymer black hole in loop quantum gravity

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(Dated: April 19, 2022)

Abstract

A new polymer black hole solution in loop quantum gravity was proposed recently. The difference between the polymer black hole and Schwarzschild black hole is captured by a quantum parameter $A$. In order to get the constraints on parameter $A$, we consider the observational constraints imposed on $A$ by using the Solar System experiments and calculate the deflection of light, Shapiro time delay, perihelion precession and obtain the effects associated with parameter $A$. Moreover, the parameterized post-Newtonian approach of this loop quantum gravity black hole was also carried out. It turns out the tightest constraint on $A$ can be improved to $0 < A < 4.0 \times 10^{-6}$.

PACS numbers:
I. INTRODUCTION

Einstein’s general relativity (GR) has been proposed for more than a century and still being an exciting and prosperous field. It has undergone more and more sophisticated tests with the development of tools like atomic clocks, radio telescopes. Currently, the gravity test projects such as the deflection of light, the perihelion advance, and Shapiro time Delay have been carried out in the weak field zone especially in solar system \[1\]. While in the strong field regime, the gravity tests include the observation of binary pulsars \[2–4\], the direct detections of gravitational waves \[5\] and the image of the black hole shadow \[6, 7\]. Remarkably, the predictions of GR are all consistent with these observations up to now.

However, despite all of these successes, GR is still far from being perfect. From theoretical consideration, unifying GR with quantum mechanics into a consistent quantum gravity theory remains the biggest theoretical challenge to fundamental physics. Among various approaches to quantum gravity, loop quantum gravity (LQG) is notable with its background independence and non-perturbative features \[8–11\].

Inspired by the full LQG, different models of black holes have been constructed to solve the singularity in the black hole interior. Without loss of generality \[12, 13\], in the effective Hamiltonian, one usually replace the \(b\) and \(c\) which are the components of Ashtekar connection with their holonomies

\[b \rightarrow \frac{\sin(\delta_b)}{\delta_b}, \quad c \rightarrow \frac{\sin(\delta_c)}{\delta_c}, \quad (1.1)\]

where parameters \(\delta_b\) and \(\delta_c\) correspond to the fundamental discreteness of LQG. Under different choices of \(\delta_b\) and \(\delta_c\), the current quantization schemes can be divided into three classes \[14, 15\]: 1. \(\mu_0\)-scheme \[16, 17\], 2. \(\bar{\mu}\)-scheme \[18, 19\], 3. generalised \(\mu_0\)-scheme \[20–23\]. For example, in Ref. \[16, 17\], the \(\mu_0\)-scheme is studied where \(\delta_b\) and \(\delta_c\) are considered as the constants. Recently, an attractive polymerized black hole \[14, 24\] solution that corresponds to a specific \(\bar{\mu}\)-scheme is constructed with

\[\delta_b = \pm \frac{4\lambda_j}{\gamma |p_b|}, \quad \delta_c = \pm \frac{8\lambda_k}{\gamma \sqrt{|p_c|}}.\quad (1.2)\]

where \(p_b\) and \(p_c\) are conjugate pairs correspond to \(b\) and \(c\), \(\lambda_j\) and \(\lambda_k\) are the polymerized constants that related to the inverse Planck curvature and Planck length after rescaling of the fiducial cell \[24\], respectively. This solution results in quantum extensions of the Schwarzschild black hole. The classical singularity of the Schwarzschild black hole has been resolved by connecting the black hole region with the white hole region through a bounce. Some aspects of this Schwarzschild-like metric have already been studied. For example, the thermodynamic properties of the effective polymer black hole and the corresponding quantum corrections as functions of black and white hole masses have been investigated in Ref. \[25\]. In Ref. \[26\], the influences of the quantum effects on the weak and strong bending angles of light rays have been studied. More investigations on the polymerized black holes can be found in Ref. \[15, 27, 28\].

Moreover, the rotating black holes are typically found in nature. In recent Ref. \[29\], the authors found a rotational solution of loop quantum gravity black hole (LQGBH) by the Newman-Janis algorithm. Then they investigate the test of loop quantum gravity by the shadow cast of the rotating black hole as well as other experiments. And they obtain a constraint on the quantum correction parameter \(A\) as \(0 < A < 7.7 \times 10^{-5}\). Note that...
the more precise experiment constraints on the quantum parameter may help us to get a deeper understanding of LQG and consider that the gravitational experiment in the Solar system provides very high precision [30]. In this paper, we will analyze the influence of the polymerized spacetime solution of the classical observations in the Solar system such as the light deflection, Shapiro time delay, perihelion advance, and the Parameterized Post-Newtonian (PPN) approach. In addition, in this paper, we ignore the effects of the angular momentum of the spacetime, the reason for this is two folds. On one hand, although the rotation of the Sun or the Earth can affect the Shapiro time delay and the light deflection, and have a further impact on the constraint upper bound of parameter $A$. However, the impact is very weak and can be safely ignored. On the other hand, the data from the MESSENGER mission [33] or LAGEOS II [34] has already taken into account and removed the rotational effect (also named “Lense-Thirring effect”). Therefore, in this paper we need only focus on the non-rotating quantum corrected space-time.

The structure of this paper is organized as follows. In Sec. II, we give a brief introduction to review the metric of LQGBH. Then in Sec. III, we obtain the conserved quantities and the motion equations of a test body in the nonrotating loop quantum gravity spacetime. We study the observational tests including deflection of light, Shapiro time delay and perihelion precession as well as the PPN approach in Sec. IV, and we obtain the constraints on the dimensionless parameter $A$. The main conclusions and some discussions are given in Sec. V.

II. NONROTATING LOOP QUANTUM GRAVITY BLACK HOLE

Starting with the non-rotating loop quantum gravity black hole (LQGBH) [14, 24, 29], the Schwarzschild-like metric in the loop quantum gravity reads

$$ds^2 = -8AM_b^2A(r)dt^2 + \frac{dr^2}{8AM_b^2A(r)} + B(r)(d\theta^2 + \sin^2(\theta)d\phi^2).$$  \hspace{1cm} (2.1)

The metric functions $A(r)$ and $B(r)$ are given by

$$A(r) = \frac{1}{B(r)}\left(\frac{r^2}{8AM_w^2} + 1\right)\left(1 - \frac{2M_b}{\sqrt{8AM_b^2 + r^2}}\right),$$  \hspace{1cm} (2.2)

$$B(r) = \frac{512A^3M_b^4M_w^2 + \left(\sqrt{8AM_b^2 + r^2} + r\right)^6}{8\sqrt{8AM_b^2 + r^2}\left(\sqrt{8AM_b^2 + r^2} + r\right)^3},$$  \hspace{1cm} (2.3)

where $M_b$ and $M_w$ is the mass of asymptotically Schwarzschild black hole and white hole respectively, and the dimensionless parameter $A$ is defined by $A = (\lambda_j/M_bM_w)^{2/3}/2$. Notice that the parameter $\lambda_j$ will be eliminated during fixing the integration constants and introducing $M_b$ and $M_w$ for solving the effective equations [24]. Hence, $\lambda_j$ doesn’t appear in the metric (2.1).

Without loss of generality, we consider the most interesting and meaningful scheme that $M_b = M_w = M$ [14, 24, 29]. It displays a symmetric spacetime reflecting at the transition surface ($r = 0$). In this scheme, the metric functions $A(r)$ and $B(r)$ reduce to

$$A(r) = \frac{1}{B(r)}\left(\frac{r^2}{8AM^2} + 1\right)\left(1 - \frac{2M}{\sqrt{8AM^2 + r^2}}\right),$$  \hspace{1cm} (2.4)

$$B(r) = 2AM^2 + r^2.$$  \hspace{1cm} (2.5)
This metric solves the interior singularity of black hole. While the positive solutions of \( A(r) = 0 \) corresponds to the black hole horizon of LQGBH as
\[
    r_+ = 2\sqrt{M^2 - 2AM^2}.
\]  
(2.6)

It should be noted that horizons will disappear when \( A > 1/2 \). In the \( A \to 0 \) limit, the expressions of \( 8AM^2A(r) \) and \( B(r) \) can be simplified to
\[
    8AM^2A(r) \to 1 - \frac{2M}{r},
\]  
(2.7)
\[
    B(r) \to r^2.
\]  
(2.8)

Hence, the metric will go back to Schwarzschild case in the classical limit.

III. MOTION EQUATIONS OF TEST PARTICLES

Let us consider the evolution of a test particle in the LQGBH spacetime. Note that we have two Killing vectors \( K_t^\mu = (\frac{\partial}{\partial t})^\mu \) and \( K_\phi^\mu = (\frac{\partial}{\partial \phi})^\mu \). Then the corresponding conserved quantities can be obtained as
\[
    E = -p_tK_t^\mu = -p_t = -mg_{tt}\dot{t},
\]  
(3.1)
\[
    J = p_\phi K_\phi^\mu = p_\phi = mg_{\phi\phi}\dot{\phi},
\]  
(3.2)

where the point \( \dot{\cdot} \) represents the covariant derivative with respect to proper time or affine parameter \( \tau \) and \( m \) is the mass of the test body. \( E \) and \( J \) denote the conserved energy and the angular momentum of the test body, respectively. In addition, the Hamilton-Jacobi equation reads
\[
    \frac{\partial S}{\partial \tau} + H = 0,
\]  
(3.3)

where \( S \) and \( H = g_{\mu\nu}p^\mu p^\nu/2m \) is the Jacobi action and the Hamiltonian, respectively. Due to the Eq.(3.1) and Eq.(3.2), the Jacobi action \( S \) can be assumed as
\[
    S = J\phi - Et + \frac{k\tau}{2} + rS_r(r) + S_\theta(\theta).
\]  
(3.4)

Then, substituting the Ansatz (3.4) into Eq.(3.3), the equation can be separated with Carter constant \( Q \) as follows
\[
    -8AM^2A(r)B(r) \left( \frac{\partial S_r(r)}{\partial r} \right)^2 + \frac{E^2B(r)}{8AM^2A(r)} - kB(r) = \left( \frac{\partial S_\theta(\theta)}{\partial \theta} \right)^2 + J^2 \csc^2(\theta) = Q,
\]  
(3.5)

with massive particles\( (k \neq 0) \) and massless particles\( (k = 0) \). Taking into account the relation between the momentum \( p^\mu \) and the Jacobi action that \( \partial S/\partial x^\mu = p_\mu \), the Eqs. (3.1), (3.2),
and the variables with unity mass ($\bar{E} = E/m$, $\bar{J} = J/m$, $\bar{Q} = Q/m^2$), equations of motion for test body read

$$\frac{dt}{d\tau} = \frac{\bar{E}}{8AM^2A(r)}, \quad (3.6)$$

$$\left(\frac{dr}{d\tau}\right)^2 = E^2 - \frac{8AM^2A(r)\bar{Q}}{B(r)} - 8AM^2A(r)\bar{k}, \quad \text{ (3.7)}$$

$$\left(\frac{d\theta}{d\tau}\right)^2 = \frac{\bar{Q}}{B(r)^2} - \frac{J^2\csc^2(\theta)}{B(r)^2}, \quad (3.8)$$

$$\frac{d\phi}{d\tau} = \frac{\bar{J}}{B(r)}, \quad (3.9)$$

with $\bar{k} = k/m^2$ (massive particles $\bar{k} = 1$) and massless one ($\bar{k} = 0$). In the following, to simplify the equation of motion, we can always choose a new coordinate system so that the initial position and velocity of the test object are in the equatorial plane ($\theta_0 = \pi/2$, $\dot{\theta}_0 = 0 \Rightarrow \bar{Q} = \bar{J}^2$). With the initial conditions, the solution to the Eq. (3.8) is $\theta = \pi/2$ and $\dot{\theta} = 0$. For simplicity, the $\bar{}$ is eliminated, and we directly use the $E, J$ and $Q$ to denote $\bar{E}, \bar{J}$ and $\bar{Q}$, respectively.

IV. OBSERVATIONAL TESTS IN THE LOOP QUANTUM GRAVITY

Using the motion of the test body that we have discussed above, the classical tests including the deflection of light, Shapiro time delay, perihelion advance, and geodesic precession of the spinning object will be studied in this section.

A. Deflection of Light

Starting from the motion equation of massless particle ($k = 0$), Eqs. (3.7)-(3.9) implies

$$\frac{dr}{d\phi} = \left(\frac{dr}{d\tau}\right) / \left(\frac{d\phi}{d\tau}\right) = \sigma \sqrt{\frac{B(r)}{b^2} - \frac{8AM^2A(r)}} \quad (4.1)$$

where an “impact” parameter $b$ represents the ratio of $J$ and $E$ ($b = J/E$) is introduced. Moreover, the parameter $\sigma = \pm 1$ correspond to the outgoing and ingoing trajectories, respectively.

The light deflection angle can be expressed as

$$\Delta \phi = 2 \int_{r_0}^{\infty} \frac{1}{\sqrt{B(r)} \left( \frac{B(r)}{b^2} - 8AM^2A(r) \right)} dr - \pi, \quad (4.2)$$

where $r_0$ is the turning point of the trajectories which is given by $\frac{dr}{d\phi}|_{r=r_0} = 0$. Combine this condition with Eq. (4.1), the relation between impact parameter $b$ and the closest approach $r_0$ reads

$$b = \sqrt{\frac{B(r_0)}{8AM^2A(r_0)}} \quad (4.3)$$
To simplify the integration in Eq. (4.1), we do the transformation

\[ r = \frac{r_0}{u}. \]  

(4.4)

Then the light deflection angle formula (4.1) can be reexpressed as

\[ \Delta \phi = 2 \int_0^1 \frac{r_0}{u^2} \sqrt{\frac{1}{u^2} B(r_0/u) (B(r_0/u) - 8AM^2A(r_0/u)b^2)} \, du - \pi. \]  

(4.5)

In the weak field limit, the magnitude of \( \epsilon = M/r_0 \) is a small quantity. By replacing \( r_0 \) in Eq. (4.5) with \( \epsilon \) and expanding the integrand, the approximations of the integration in terms of \( \epsilon \) is given by

\[ \Delta \phi = 2 \int_0^1 \left( \frac{1}{\sqrt{1-u^2}} + \epsilon \left( \frac{u^2 + u + 1}{(u+1)\sqrt{1-u^2}} \right) + \frac{\epsilon^2 \left(3(u^2 + u + 1)^2 - 4A(u+1)^2(2u^2 + 1)\right)}{2(u+1)^2\sqrt{1-u^2}} \right) du 
- \pi + O(\epsilon^3). \]  

(4.6)

Then one obtains the deflection angle of the light as

\[ \Delta \phi \approx \Delta \phi_{GR} - 4\pi \left( \frac{M}{r_0} \right)^2 A \approx \Delta \phi_{GR} \left(1 - \pi A \frac{M_0}{r_0} \right), \]  

(4.7)

where \( \Delta \phi_{GR} \) is the light deflection in general relativity.

In the solar system, we assume that the closest distance approaching the Sun \( r_0 \) of the electromagnetic wave signal is approximately being the radius of the Sun \( R_S \). Hence, the effect of parameter \( A \) on light deflection can be described as \( \Delta \phi_{GR} \pi AM_S/R_S \) with \( M_S \) being the mass of the Sun. Using the measurement data of the radio wave deflection among four quasars sources with the Very Long Baseline Array (VLBA) \[31\] in the solar system and considering \( A > 0 \), the constraint of the quantum parameter \( A \) can be obtained for

\[ 0 < A < 74.9836 \quad (68\% \, C.L.). \]  

(4.8)

**B. Shapiro time Delay**

In this section, we still consider the case of a massless test body moving in curved space-time. Considering Eqs. (3.6) and (3.7) when the parameter \( k \) equals to zero \( (k = 0) \), the differential equation of massless particles between \( t \) and \( r \) reads

\[ \frac{dt}{dr} = \left( \frac{dt}{d\tau} \right) \left( \frac{dr}{d\tau} \right) = \pm \sqrt{\frac{B(r)}{64A^2M^4A(r)^2 (B(r) - 8Ab^2M^2A(r))}}. \]  

(4.9)

Then the time difference of the electromagnetic wave signal moving from the closest approach point \( P_0 \) of the Sun to the point \( X \) of satellite or planet as

\[ \Delta t(r_X) = \int_{r_0}^{r_X} \sqrt{\frac{B(r)}{64A^2M^4A(r)^2 (B(r) - 8Ab^2M^2A(r))}} \, dr. \]  

(4.10)
where \( r_0 \) is the radius of the closest approach point \( P_0 \) and \( r_X \) is the radius of the point \( X \). Doing the same transformation as in Eq. [4.4] and expanding the expression of time \( \Delta t(r_X) \), with the perturbation quantity \( \epsilon = M/r_0 \) in the weak field, we found that

\[
\Delta t(r_X) = M \int_{r_0}^{r_X} \sqrt{\frac{1}{1 - u^2}} \left( \frac{1}{u^2 \epsilon} + \frac{(3u + 2)}{u(u + 1)} + \frac{(-16A(u + 1)^2 + 3u(5u + 8) + 12) \epsilon}{2(u + 1)^2} \right) du
\]

\[+ O(\epsilon^2). \quad (4.11)\]

The time \( \Delta t \) up to the sub-leading order can be expressed as

\[
\Delta t(r_X) \approx \sqrt{\frac{r_X^2 - r_0^2}{r_X^2}} + M \left( \sqrt{\frac{r_X - r_0}{r_X + r_0}} + 2 \cos^{-1} \left( \frac{r_X}{r_0} \right) \right) + \frac{1}{2} M \epsilon \left( \frac{r_X}{r_X + r_0} \right)^{3/2}
\]

\[\times \left( \sqrt{\frac{r_X - r_0}{r_X^2}} + 2(16A - 15) \left( \frac{r_X + r_0}{r_X} \right)^{3/2} \sin^{-1} \left( \frac{r_X - r_0}{\sqrt{2r_X^2}} \right) \right). \quad (4.12)\]

Now, we consider the Shapiro time delay by sending an electromagnetic wave signal out from a source (satellite or Earth) \( X \), and receiving the signal reflected by another reflection body (satellite or planet) \( Y \). Whether the electromagnetic wave signal passes perihelion (the turning point of the trajectories \( P_0 \)) or not, the calculation of time delay can be divided into two categories, the inferior conjunction case and superior one.

In the inferior conjunction case, in which the object \( Y \) that reflects the radar signal is located between Earth (or spacecraft, denoted by \( X \)) and the Sun. The time delay with the effect of the parameter \( A \) reads

\[
\delta t_I \approx 4 \log \left( \frac{r_X}{r_Y} \right) M + \frac{2r_0(r_X - r_Y)}{r_Xr_Y} M + \frac{2(8A - 6)(r_X - r_Y)}{r_Xr_Y} M^2
\]

\[= \delta t_{I}^{GR} + \frac{16A(r_X - r_Y)}{r_Xr_Y} M^2, \quad (4.13)\]

where \( \delta t_{I}^{GR} \) referred to as the Shapiro time delay in Einstein’s general relativity.

In the superior conjunction case the object \( Y \) reflects the radar signal and the object \( X \) is on opposite sides of the Sun. By taking the similar procedures that in Eq \( \text{(4.13)} \). The time delay in this case is then given by

\[
\delta t_S \approx 4M + 4 \log \left( \frac{4r_Xr_Y}{r_0^2} \right) M - \frac{2(r_0(r_X + r_Y))}{r_Xr_Y} M
\]

\[+ \left( \frac{16\pi A - 15\pi + 8}{r_0} - \frac{4(4A - 3)(r_X + r_Y)}{r_Xr_Y} \right) M^2
\]

\[= \delta t_{S}^{GR} + 16A \left( \frac{\pi}{r_0} - \frac{r_X + r_Y}{r_Xr_Y} \right) M^2. \quad (4.14)\]

In astronomical measurements of the Cassini experiment, the researchers always measure the relative change of the radar signal frequency in the superior conjunction case, rather than measuring the detail time delays directly. Combining with the time delay expression as Eq \( \text{(4.14)} \) in the superior conjunction case, the relative change of the frequency of the
radar signal reads

\[ \delta \nu = \frac{d}{dt} \delta t_S = \frac{d}{dt} \left( \delta t_S^{GR} + 16A \left( \frac{\pi}{r_0} - \frac{r_X + r_Y}{r_Xr_Y} \right) M^2 \right) \]

\[ \approx \delta \nu_S^{GR} - \frac{16AM^2 \pi r_0'(t)}{r_0^2}. \] (4.15)

According to the Cassini experiment\[32\], the frequency shift caused by quantum gravity effect is

\[ \delta \nu_A \approx \frac{4096\pi}{729} \left( \frac{M_S}{R_S} \right)^2 v_E A < 10^{-14}, \] (4.16)

where \( v_E = r_0'(t) \) is the Earth’s average orbit velocity, \( M_S \) and \( R_S \) are the mass and the radius of the Sun respectively. By transferring geometric units\((c = G = 1)\) to SI one and use the experiment data, the constraint on parameter \( A \) reads

\[ 0 < A < 1.27. \] (4.17)

### C. Perihelion Advance

Considering the motion equation for massive particle \((k = -1)\). Eqs. (3.7)-(3.9) imply a differential equation of massive particles between \( r \) and \( \phi \) as

\[ \frac{dr}{d\phi} = \left( \frac{dr}{d\tau} \right) \left( \frac{d\phi}{dr} \right) = \sigma \sqrt{\frac{B(r) (E^2B(r) - 8AM^2A(r) (J^2 + B(r)))}{J^2}}, \] (4.18)

where the parameter \( \sigma = \pm 1 \) corresponds to the outgoing and ingoing moving, respectively. By doing the standard transformation \( u = r_0/r \), Eq.(4.18) can be restructured to

\[ \left( \frac{du}{d\phi} \right)^2 = \frac{u^4B(r_0/u) r_0^2}{r_0^2} \left( \frac{E^2B(r_0/u)}{J^2} - 8AM^2A(r_0/u) \left( \frac{B(r_0/u)}{J^2} + 1 \right) \right), \] (4.19)

By introducing the small parameter \( \epsilon = M/r_0 \), the above equation can be expressed as:

\[ u''(\phi) + u(\phi) = f, \] (4.20)

where \( f \) is the function of \( u(\phi) \) as

\[ f = \frac{1}{J^2 I \epsilon} \left( M^2 + 2 \left( 2E^2 - 5 \right) AM^2 I u(\phi) + (22AM^2 + 3J^2) \epsilon^2 u(\phi)^2 \right. \]

\[ + 8AI \left( (E^2 - 4) AM^2 - 2J^2 \right) \epsilon^3 u(\phi)^3 + 32A \left( 2AM^2 + J^2 \right) \epsilon^4 u(\phi)^4 \right), \] (4.21)

and

\[ I = \sqrt{8A\epsilon^2 u(\phi)^2 + 1}. \] (4.22)

Next, we use the perturbation method to obtain the approximate solution of the above equation. Hence, we need to consider the approximations of the differential equation in
terms of $\epsilon$ in the weak field where the magnitude of $\epsilon = M/r_0$ is small, and the above differential equation can be truncated as

$$u''(\phi) + u(\phi) - \frac{Mr_0}{J^2} = \frac{2 (2E^2 - 5) AM^2 u(\phi)}{J^2} + \frac{3 (6AM^2 + J^2) u(\phi)\epsilon}{4J^2} + \frac{A ((E^2 - 4) AM^2 - 2J^2) u(\phi)\epsilon^2}{2J^2} + O(\epsilon^3). \quad (4.23)$$

The unperturbed solution of Eq. (4.23) can be obtained by solving the equation

$$u''_0(\phi) + u_0(\phi) - \frac{Mr_0}{J^2} = 0, \quad (4.24)$$

the solution of Eq. (4.24) reads

$$u_0(\phi) = \frac{Mr_0}{J^2} (1 + e \cos(\phi)). \quad (4.25)$$

Second, consider the first-order correction $u_1$ of the unperturbed orbit $u_0$. The approximate solution can be expanded as $u \approx u_0 + u_1$, where the $u_1$ should satisfy

$$u''_1(\phi) + u_1(\phi) - \frac{Mr_0}{J^2} \approx \frac{2 (2E^2 - 5) AM^2 u_0(\phi)}{J^2} + \frac{3 (6AM^2 + J^2) u_0(\phi)\epsilon}{4J^2} + \frac{A ((E^2 - 4) AM^2 - 2J^2) u_0(\phi)\epsilon^2}{2J^2}. \quad (4.26)$$

Plugging the Eq. (4.25) into Eq. (4.26), we can find that

$$u''_1(\phi) + u_1(\phi) = \sum_{n=0}^{3} A_n \cos^n(\phi), \quad (4.27)$$

where

$$A_0 = \frac{1}{f^8 \epsilon} \left( 8 (E^2 - 4) A^2 M^8 + 2A J^2 M^6 + (2 (2E^2 - 5) A + 3) J^4 M^4 \right), \quad (4.28)$$

$$A_1 = \frac{1}{f^8 \epsilon} \left( 2e (12 (E^2 - 4) A^2 M^8 - 6A J^2 M^6 + ((2E^2 - 5) A + 3) J^4 M^4) \right), \quad (4.29)$$

$$A_2 = \frac{1}{f^8 \epsilon} \left( 3e^2 M^4 (8 (E^2 - 4) A^2 M^4 - 10AJ^2 M^2 + J^4) \right), \quad (4.30)$$

$$A_3 = \frac{1}{f^8 \epsilon} \left( 8A e^3 M^6 \left( ((E^2 - 4) AM^2 - 2J^2) \right) \right). \quad (4.31)$$

Considering the initial conditions

$$u_1(0) = 0, \quad u'_1(0) = 0. \quad (4.32)$$

The solution $u_1$ reads

$$u_1(\phi) = \sum_{n=0}^{3} C_n \cos(n\phi) + S_1 \phi \sin(\phi), \quad (4.33)$$
where

\[ C_0 = A_0 + \frac{A_2}{2}, \quad (4.34) \]
\[ C_1 = -\left( A_0 + \frac{A_2}{3} - \frac{A_3}{32} \right), \quad (4.35) \]
\[ C_2 = -\frac{1}{6} A_2, \quad (4.36) \]
\[ C_3 = -\frac{1}{32} A_3, \quad (4.37) \]
\[ S_1 = \frac{A_1 \phi}{2} + \frac{3A_3 \phi}{8}. \quad (4.38) \]

It is easy to see that the perihelion advance only depends on the \( S_1 \phi \sin(\phi) \) term in \( u_1 \). Hence, the \( \sum_{n=0}^3 C_n \cos(n\phi) \) term can be ignored, and the approximation solution of \( u \) reads

\[ u(\phi) \approx \frac{Mr_0}{J^2} (\mathcal{P} \sin(\phi) + \cos(\phi)) = \frac{Mr_0}{J^2} \left( 1 + e \sqrt{1 + \mathcal{P}^2} \cos(\phi - \phi_0) \right), \quad (4.39) \]

where

\[ \mathcal{P} = (2E^2 A - 5A + 3) \frac{M^2 \phi}{J^2} - 6A \left( e^2 + 1 \right) \frac{M^4 \phi}{J^4} + 3 \left( E^2 - 4 \right) A^2 \left( e^2 + 4 \right) \frac{M^6 \phi}{J^6}, \quad (4.40) \]

\[ \phi_0 = \frac{\delta \phi_0}{2\pi} = \arctan(P). \quad (4.41) \]

Now, considering the orbit of the solution (4.39), the perihelion radius \( r_- \) and aphelion radius \( r_+ \) of the orbit read respectively

\[ \frac{r_0}{r_+} = \frac{(1 - e) r_0 M^2}{M J^2}, \quad (4.42) \]
\[ \frac{r_0}{r_-} = \frac{(1 + e) r_0 M^2}{M J^2}. \quad (4.43) \]

Combining the Eqs. (4.42) and (4.43), we found that

\[ \frac{M^2}{J^2} = \frac{2M}{(1 - e^2) (r_- + r_+)} = \frac{M}{(1 - e^2) \kappa} \sim \frac{M}{r_0} = \epsilon, \quad (4.44) \]

where \( \kappa := (r_+ + r_-)/2 \) is the semi-major axis of the orbit. Therefore \( M^2/J^2 \) and \( \epsilon = M/r_0 \) have the same order of magnitude. Hence, the angular \( \phi_0 \) in Eq. (4.41) can be expanded in terms of \( M^2/J^2 \), and the angular shift of the perihelia per orbit \( \delta \phi_0 \) reads

\[ \delta \phi_0 \approx 2\pi \left( 3 + (2E^2 - 5) A \right) \frac{M^2}{J^2} + O \left( \left( \frac{M^2}{J^2} \right)^2 \right). \quad (4.45) \]

By solving \( dr/d\phi|_{r_0} = 0 \) with Eq. (4.18), the relation between the energy \( E \) and the closest approach \( r_0 \) reads

\[ E^2 = 8AM^2A(r_0) \left( \frac{J^2}{B(r_0)} + 1 \right) = 1 - 2\epsilon + O(\epsilon^2). \quad (4.46) \]
Therefore, in the weak field approximation, the Eq. (4.45) can be simplified as
\[ \Delta \phi = \delta \phi_0 \approx \frac{6\pi M}{(1 - e^2) \kappa} (1 - A + O(\epsilon^1)) \approx \Delta \phi^{GR}(1 - A). \] (4.47)

Now, we could get an upper bound of the parameter \( A \) by using the observational data. For the experimental data of the anomalous Mercury perihelion advance from the MESSENGER mission \[33\], the precession rate of perihelion caused by the gravitoelectric effect reads
\[ \Delta \phi = (42.9799 \pm 0.0009)''/\text{century}. \] (4.48)

For the motion of Mercury around the Sun, the observed error of anomalous perihelion advance is 0.0009''/century. The contribution of LQG is expected to be less than the observational error. Therefore, the constraint range of parameters \( A \) turns out to be
\[ 0 < A < 2.09 \times 10^{-5}. \] (4.49)

Analogically, using the perihelion advance experimental data of the LAGEOS satellites that move around the Earth \[34\]
\[ \Delta \phi = \Delta \phi^{GR}(1 + (0.28 \pm 2.14) \times 10^{-3}), \] (4.50)
we can obtain the constraint range of parameters \( A \) as
\[ 0 < A < 1.86 \times 10^{-3}. \] (4.51)

D. Parameterized Post-Newtonian (PPN) approach

Now we are going to calculate the PPN parameters of the Schwarzschild like metric (2.1) in the loop quantum gravity and obtain the relation between the PPN parameter and the parameter \( A \) that contains the LQG effect. First, we perform the following transformations \[37\]
\[ r = \sqrt{r^2 - 2AM^2}. \] (4.52)
The metric (2.1) then can be reformulated as
\[ ds^2 = -N^2(\bar{r})d\bar{t}^2 + \frac{B^2(\bar{r})}{N^2(\bar{r})}d\bar{r}^2 + \bar{r}^2(d\theta^2 + \sin^2 \theta d\phi^2), \] (4.53)
where
\[ N^2(\bar{r}) = 1 - \frac{2M^2}{\bar{r}^2}(-3A + \sqrt{A} \sqrt{6 + \frac{\bar{r}^2}{AM^2}}), \] (4.54)
\[ B^2(\bar{r}) = \frac{\bar{r}^2}{-2AM^2 + \bar{r}^2}. \] (4.55)

From the transformation (4.52), we know that \( \bar{r} > \sqrt{2AM} \) or \( \bar{r} < -\sqrt{2AM} \) must be satisfied to insure that \( B^2(\bar{r}) > 0 \). On the one hand, the metric component of (4.53) can be expanded in terms of \( M/\bar{r} \) as \[36, 37\]
\[ N^2(\bar{r}) = 1 - \frac{2M}{\bar{r}} + 6A \frac{M^2}{\bar{r}^2} + O\left((M/\bar{r})^3\right), \] (4.56)
\[ \frac{B^2(\bar{r})}{N^2(\bar{r})} = 1 + \frac{2M}{\bar{r}} + O\left((M/\bar{r})^2\right). \] (4.57)
On the other hand, the PPN approximation of the asymptotic spacetime can be described as

\[ ds^2 = -G^2(\bar{r})dt^2 + F^2(\bar{r})d\bar{r}^2 + \bar{r}^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{4.58} \]

where

\[ G^2(\bar{r}) = 1 - 2\frac{M}{\bar{r}} + 2(\beta - \gamma)\frac{M^2}{\bar{r}^2}, \tag{4.59} \]

\[ F^2(\bar{r}) = 1 + 2\gamma M\bar{r}. \tag{4.60} \]

Here \( \gamma \) and \( \beta \) are the PPN parameters. Comparing equations (4.56)-(4.57) and (4.59)-(4.60), we immediately obtain the corresponding equations as follows

\[ \beta - \gamma = 3A, \quad \gamma = 1 \tag{4.61} \]

or equivalently

\[ \gamma = 1, \quad \beta = 3A + 1. \tag{4.62} \]

Next, we consider observational constraints that imposed to PPN parameter \( \beta \) by the MESSENGER mission \[33\] which reads

\[ -6.6 \times 10^{-5} < \beta - 1 < 1.2 \times 10^{-5}, \tag{4.63} \]

and this immediately in turn implies

\[ 0 < A < 4 \times 10^{-6}. \tag{4.64} \]

V. CONCLUSIONS

Cosmological and black holes models inspired by LQG provide elegant solutions to black holes and big bang singularities have been constructed. Along this line, recently, a new polymer black hole model has been proposed \[14, 24, 29\]. In this paper, we study the classical tests of polymerized black hole in effective loop quantum gravity including the

| Experiments/Observations                  | A                  | Datasets                          |
|----------------------------------------|--------------------|-----------------------------------|
| Light deflection                       | 74.9836            | VLBI observation of quasars\[31\] |
| Time delay                             | 1.27               | Cassini experiment\[32\]          |
| Perihelion advance                     | \(2.09 \times 10^{-5}\) | MESSENGER mission\[33\]          |
|                                        | \(1.86 \times 10^{-3}\) | LAGEOS satellites\[34\]          |
| PPN approach(\(\beta = 3A + 1\))      | \(4.0 \times 10^{-6}\) | MESSENGER mission\[33\]          |
| Shadow of black hole\[29\]            | 0.24               | Shadow of M87*\[6\]              |
| Tests strong equivalence principle\[29\]| \(7.7 \times 10^{-5}\) | Lunar laser ranging data \[35\]   |
light deflection, Shapiro time delay, perihelion advance, and PPN methods. Based on these classical observations, we calculate the influences of the parameter $A$ and then obtain the constraints on the parameter $A$ using the latest astronomical observations in the Solar System.

The upper bounds of the parameter $A$ are summarized in Table I. We interestingly observed that the MESSENGER mission gives a nice constraint on the parameter $A$ through perihelion advance as $0 < A < 2.09 \times 10^{-5}$. Moreover, the best constraint comes from the PPN method which gives rise $0 < A < 4.0 \times 10^{-6}$.

Note that the observations such as the light deflection and Shapiro time delay do not impose tight constraints on parameter $A$. The reason is that parameter $A$ will only appear at the “nonlinearity” $(M^2/r^2)$ term that related to the PPN parameter $\beta$ in the Eq. (A.3). In contrast, the quantum parameter $P$ in Ref. [30] appears at the “linearity” term $(M/r)$ and the “nonlinearity” $(M^2/r^2)$ term that will relate to the $\gamma$ and effective gravitational “constant” $\bar{G}$ in Appendix A. Moreover, in the MESSENGER mission experiment [33], constraints on the PPN parameter $\beta$ and un-normalized solar quadrupole moment $J_2$ are given directly. However, the constraint of perihelion advance is determined by the estimated uncertainty of other parameters such as $\beta$, $J_2$, and $\gamma$ from Cassini. Hence, using the constraint on PPN parameter $\beta$ in MESSENGER mission get the better constraints on $A$ than perihelion advance one.

It is worth noting that in 2018, the joint European-Japanese BepiColombo project launched two spacecrafts that will explore Mercury [38, 39]. Through the BepiColombo spacecrafts, the accuracy of Mercury’s perihelion advance measurements will be further improved by an order of magnitude compared to the MESSENGER mission and the thus tighter constraint on the parameter $A$ will be obtained accordingly. We would like to leave this for future study.

**Acknowledgments**

This work is supported by NSFC with No.11775082.

**Appendix A: PPN approach for the self-dual spacetime in loop quantum gravity**

Starting with the self-dual spacetime in LQG [17, 30], the Schwarzschild-like metric reads

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{g(r)} + h(r)d\Omega^2,$$

(A1)

where

$$h(r) = \frac{a_0^2}{r^2} + r^2,$$

(A2)

$$f(r) = \frac{(r-r_-)(r-r_+)(r+r_*)^2}{a_0^2 + r^4},$$

(A3)

$$g(r) = \frac{r^4(r-r_-)(r-r_+)}{(a_0^2 + r^4)(r+r_*)^2}.$$
and
\[
\begin{align*}
r_- &= \frac{2MP^2}{(P+1)^2}, & r_+ &= \frac{2M}{(P+1)^2}, & r_* &= \frac{2MP}{(P+1)^2}. \quad (A5)
\end{align*}
\]
Assume that \( a_0 = 0 \) \([17, 30]\), and expand the function \( f(r) \) and \( 1/g(r) \)
\[
\begin{align*}
f(r) &= 1 - \frac{2(P-1)^2}{(P+1)^2} \frac{M}{r} - \frac{8P(P^2 - P + 1)}{(P+1)^4} \frac{M^2}{r^2} + O\left(\left(\frac{M}{r}\right)^3\right), \quad (A6)
\end{align*}
\]
\[
\begin{align*}
\frac{1}{g(r)} &= 1 + 2 \frac{M}{r} - \frac{4(P^2 + 1)}{(P+1)^2} \frac{M^2}{r^2} + O\left(\left(\frac{M}{r}\right)^3\right). \quad (A7)
\end{align*}
\]
Comparing equations (4.59) and (4.60) with (A6) and (A7), the parameter \( P \) does not
appear in the linear term of Eq.(A7) that corresponds to the PPN parameter \( \gamma \) in the
Eq.(4.60). In order to obtain the relation between PPN parameter and \( P \), we must do some
transformations. Inspired by Eq.(202) in Ref. [40], we take the transformation as
\[
\bar{G} = \frac{(P-1)^2}{(P+1)^2} G. \quad (A8)
\]
The function \( f(r) \) and \( 1/g(r) \) can be reformulated as
\[
\begin{align*}
f(r) &= 1 - 2 \frac{\bar{G}M}{c^2 r} - \frac{8P(P^2 - P + 1)}{(P+1)^4} \frac{\bar{G}^2 M^2}{c^4 r^2} + O\left(\frac{\bar{G}M/(c^2 r)}{(P+1)^3}\right), \quad (A9)
\end{align*}
\]
\[
\begin{align*}
\frac{1}{g(r)} &= 1 + 2 \frac{(P+1)^2}{(P-1)^2} \frac{\bar{G}M}{c^2 r} + \frac{4(P+1)^2(P^2 + 1)}{(P+1)^4} \frac{\bar{G}^2 M^2}{c^4 r^2} + O\left(\frac{(\bar{G}M/(c^2 r))^3}{(P+1)^3}\right). \quad (A10)
\end{align*}
\]
Comparing equations (4.59)-(4.60) with (A9)-(A10), we immediately read off the relation
between PPN coefficients and parameter \( P \) as
\[
\begin{align*}
\gamma &= \frac{(P+1)^2}{(P-1)^2} = 1 + 4P + 8P^2 + O\left(P^3\right), \quad (A11)
\end{align*}
\]
\[
\begin{align*}
\beta &= \frac{(P-4)P + 1}{(P+1)^4} = 1 - 4P^2 + O\left(P^3\right). \quad (A12)
\end{align*}
\]
Using this relation, we can calculation the light deflection, Shapiro time delay and perihelion
advance in the self-dual spacetime directly. For example, perihelion advance per orbit can
be expressed as
\[
\Delta \phi = \frac{6\pi \bar{G}M}{(1-e^2)c^2 \kappa} \left(\frac{1}{3} (2 + 2\gamma - \beta)\right) = \frac{6\pi GM}{(1-e^2)c^2 \kappa} \left(\frac{1 - 4P}{3} + O\left(P^2\right)\right). \quad (A13)
\]
This replicates the result of Ref. [30]. We find that \( \gamma \) and the effective gravitational “constant” \( \bar{G} \) contribute \( P \) to the leading order of light deflection, Shapiro time delay and perihelion advance, while \( \beta \) only contributes \( P^2 \) that can be ignored. On the contrary, in
our work, \( A \) is contributed by \( \gamma \) in the leading order of the perihelion advance, and \( A \) only
appears in the second order of the light deflection and Shapiro time delay and hence it is
easy to understand that they are not of the same order of magnitude.

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