An Exponential Cubic B-spline Finite Element Method for Solving the Nonlinear Coupled Burger Equation

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Abstract

The exponential cubic B-spline functions together with Crank Nicolson are used to solve numerically the nonlinear coupled Burgers’ equation using collocation method. This method has been tested by three different problems. The proposed scheme is compared with some existing methods. We have noticed that proposed scheme produced a highly accurate results.

1 Introduction

The purpose of this paper is to apply the exponential B-spline collocation method to the coupled Burgers equation system. The Coupled Burger equation system in the following form

\[
\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2} + k_1 U \frac{\partial U}{\partial x} + k_2 (UV)_x = 0
\]
\[
\frac{\partial V}{\partial t} - \frac{\partial^2 V}{\partial x^2} + k_1 V \frac{\partial V}{\partial x} + k_3 (UV)_x = 0
\]

(1)

where \(k_1, k_2\) and \(k_3\) are reel constants and subscripts \(x\) and \(t\) denote differentiation, \(x\) distance and \(t\) time, is considered. Boundary conditions

\[
U(a, t) = f_1(a, t), \quad U(b, t) = f_2(b, t)
\]
\[
V(a, t) = g_1(a, t), \quad V(b, t) = g_2(b, t), \quad t > 0
\]

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and initial conditions

\[ U(x, 0) = f(x) \]
\[ V(x, 0) = g(x), \quad a \leq x \leq b \]

will be decided in the later sections according to test problem.

Various methods are used solve the nonlinear coupled Burgers’ equation numerically; which is suggested by [1] firstly. Fourth order accurate compact ADI scheme [2], A chebyshev spectral collocation method [3], A meshfree technique [4], the Fourier pseudospectral method [5], the generalized two-dimensional differential transform method [6], cubic B-spline collocation method [7], generalized differential quadrature method [8], a robust technique for solving optimal control of coupled Burgers’ equations [9], a differential quadrature method [10], Galerkin quadratic B-spline finite element method [11], a fully implicit finite-difference method [12], a composite numerical scheme based on finite difference [13], an implicit logarithmic finite difference method [14], modified cubic B-spline collocation method [15] was applied to obtain numerical solution of nonlinear Coupled Burgers system. There are not many articles about exponential cubic B-spline method for the solving nonlinear differential equation system.

The exponential splines and exponential B-splines are defined as a generalization of the well-known splines and B-splines by McCartin [16, 17, 18]. He has also showed a reliable algorithm by using the exponential spline functions to solve the hyperbolic conservation laws McCartin [19]. The exponential B-splines include a free parameter which cause to have different bell like piece wise polynomial. The best free parameter is determined for the exponential B-spline functions for solving the differential equations. The use of the exponential B-splines is not as common as the well known B-splines. There are a few studies existing to use exponential B-splines for build up numerical methods. The singularly perturbed boundary value problem has been solved based on the collocation methods with the exponential B-splines [20, 21, 22]. Very recently, Exponential B-spline collocation method is applied to compute numerical solution of the convection diffusion equation [23].

The paper is organized as follows. In Section 2, some details about exponential cubic B-spline collocation method are provided. In Section 3, the initial states are documented. In section 4, numerical results for three different problems and some related figures are given in order to show the efficiency as well as the accuracy of the proposed method. Finally, conclusions are followed in Section 5.

## 2 Exponential Cubic B-spline Collocation Method

Let \( \pi \) be partition of the problem domain \([a, b]\) defined at the knots

\[ \pi : a = x_0 < x_1 < \ldots < x_N = b \]

with mesh spacing \( h = (b - a)/N \). The exponential B-splines, \( B_i(x) \), with knots at the points of \( \pi \) can be defined as
\[ B_i(x) = \begin{cases} 
\frac{b_2}{p} \left( (x_{i-2} - x) - \frac{1}{p} (\sinh(p(x_{i-2} - x))) \right) & \text{for } [x_{i-2}, x_{i-1}], \\
a_1 + b_1(x_i - x) + c_1 \exp(p(x_i - x)) + d_1 \exp(-p(x_i - x)) & \text{for } [x_{i-1}, x_i], \\
a_1 + b_1(x - x_i) + c_1 \exp(p(x - x_i)) + d_1 \exp(-p(x - x_i)) & \text{for } [x_i, x_{i+1}], \\
\frac{b_2}{p} \left( (x - x_{i+2}) - \frac{1}{p} (\sinh(p(x - x_{i+2}))) \right) & \text{for } [x_{i+1}, x_{i+2}], \\
0 & \text{otherwise.}
\end{cases} \]

where

\[ a_1 = \frac{phc}{phc - s}, \quad b_1 = \frac{p}{2} \left( \frac{c(c - 1) + s^2}{(phc - s)(1 - c)} \right), \quad b_2 = \frac{p}{2(phc - s)}, \]

\[ c_1 = \frac{1}{4} \left( \frac{\exp(-ph)(1 - c) + s(\exp(-ph) - 1)}{(phc - s)(1 - c)} \right), \]

\[ d_1 = \frac{1}{4} \left( \frac{\exp(ph)(c - 1) + s(\exp(ph) - 1)}{(phc - s)(1 - c)} \right). \]

and \( s = \sinh(ph), \ c = \cosh(ph), \ p \) is a free parameter. When \( p = 1 \), graph of the exponential cubic B-splines over the interval \([0.1]\) is depicted in Fig. 1.

Fig.1: Exponential cubic B-splines for \( p = 1 \) over the interval \([0.1]\)

An additional knots outside the problem domain, positioned at \( x_{-1} < x_0 \) and \( x_N < x_{N+1} \) are necessary to define all exponential splines. So that \( \{B_{-1}(x), B_0(x), \ldots, B_{N+1}(x)\} \) forms a basis for the functions defined over the interval. Each sub interval \([x_i, x_{i+1}]\) is covered by four consecutive exponential B-splines. The exponential B-splines and its first and second derivatives vanish outside its support interval \([x_{i-2}, x_{i+2}]\). Each basis function \( B_i(x) \) is twice continuously differentiable. The values of \( B_i(x), B'_i(x) \) and \( B''_i(x) \) at the knots \( x_i \) can be computed from Eq. (4) are shown Table 1.
Table 1: Values of $B_i(x)$ and its principle two 
derivatives at the knot points

| $x$ | $x_{i-2}$ | $x_{i-1}$ | $x_i$ | $x_{i+1}$ | $x_{i+2}$ |
|-----|----------|----------|------|----------|----------|
| $B_i$ | 0 | $s - ph$ | 1 | $s - ph$ | 0 |
| $B'_i$ | 0 | $\frac{2(phc - s)}{p(1 - c)}$ | 0 | $\frac{2(phc - s)}{p(c - 1)}$ | 0 |
| $B''_i$ | 0 | $\frac{p^2 s}{2(phc - s)}$ | $\frac{p^2 s}{phc - s}$ | $\frac{p^2 s}{2(phc - s)}$ | 0 |

An approximate solution $U_N(x, t)$ and $V_N(x, t)$ to the analytical solution $U(x, t)$ and $V(x, t)$ can be assumed of the forms

$$U_N(x, t) = \sum_{i=1}^{N+1} \delta_i B_i(x), \quad V_N(x, t) = \sum_{i=1}^{N+1} \phi_i B_i(x) \quad (5)$$

where $\delta_i$ are time dependent parameters to be determined from the collocation method. The first and second derivatives also can be defined by

$$U'_N(x, t) = \sum_{i=1}^{N+1} \delta_i B'_i(x), \quad V'_N(x, t) = \sum_{i=1}^{N+1} \phi_i B'_i(x) \quad (6)$$

$$U''_N(x, t) = \sum_{i=1}^{N+1} \delta_i B''_i(x), \quad V''_N(x, t) = \sum_{i=1}^{N+1} \phi_i B''_i(x) \quad (7)$$

Using the Eq. (5), (6) and Table 1, we see that the nodal values $U_i, V_i$, their first derivatives $U'_i, V'_i$ and second derivatives $U''_i, V''_i$ at the knots are given in terms of parameters by the following relations

$$U_i = U(x_i, t) = \frac{s - ph}{2(phc - s)} \delta_{i-1} + \delta_i + \frac{s - ph}{2(phc - s)} \delta_{i+1},$$

$$U'_i = U'(x_i, t) = \frac{p(1 - c)}{2(phc - s)} \delta_{i-1} + \frac{p(c - 1)}{2(phc - s)} \delta_{i+1},$$

$$U''_i = U''(x_i, t) = \frac{p^2 s}{2(phc - s)} \delta_{i-1} - \frac{p^2 s}{phc - s} \delta_i - \frac{p^2 s}{2(phc - s)} \delta_{i+1},$$

$$V_i = V(x_i, t) = \frac{s - ph}{2(phc - s)} \phi_{i-1} + \phi_i + \frac{s - ph}{2(phc - s)} \phi_{i+1},$$

$$V'_i = V'(x_i, t) = \frac{p(1 - c)}{2(phc - s)} \phi_{i-1} + \frac{p(c - 1)}{2(phc - s)} \phi_{i+1},$$

$$V''_i = V''(x_i, t) = \frac{p^2 s}{2(phc - s)} \phi_{i-1} - \frac{p^2 s}{phc - s} \phi_i + \frac{p^2 s}{2(phc - s)} \phi_{i+1}.$$ 

The Crank-Nicolson scheme is used to discretize time variables of the unknown $U$ and $V$ in the Coupled Burger equation system which is given (1), we obtain the time
discretized form of the equation as

$$
\begin{align*}
\frac{U^{n+1} - U^n}{\Delta t} &= \frac{U^{n+1} + U^n}{2} + k_1 \frac{(UU_x)^{n+1} + (UU_x^n)}{2} + k_2 \frac{(UV)^{n+1} + (UV)_x^n}{2} \\
\frac{V^{n+1} - V^n}{\Delta t} &= \frac{V^{n+1} + V^n}{2} + k_1 \frac{(VV_x)^{n+1} + (VV_x^n)}{2} + k_3 \frac{(UV)^{n+1} + (UV)_x^n}{2}
\end{align*}
$$

(8)

where \( U^{n+1} = U(x, t_n + \Delta t) \) and \( V^{n+1} = V(x, t_n + \Delta t) \). The nonlinear term \((UU_x)^{n+1}, (VV_x)^{n+1}\) and \((UV)_x^{n+1}\) in Eq. (8) is linearized by using the following form [24]:

$$
\begin{align*}
(UU_x)^{n+1} &= U^{n+1}U_x^n + U^nU_x^{n+1} - U^nU_x^n \\
(VV_x)^{n+1} &= V^{n+1}V_x^n + V^nV_x^{n+1} - V^nV_x^n \\
(UU)_x^{n+1} &= (U_xV)^{n+1} + (UV)_x^{n+1} \\
&= U_x^{n+1}V^n + U^nV_x^{n+1} - U^nV^n + U^{n+1}V_x^n + U^nV_x^{n+1} - U^nV_x^n
\end{align*}
$$

(9)

Substitution the approximate solution [7] into (8) and evaluating the resulting equations at the knots yields the system of the fully-discretized equations

$$
\nu_{m1}\delta_{m-1}^{n+1} + \nu_{m2}\phi_{m-1}^{n+1} + \nu_{m3}\phi_{m}^{n+1} + \nu_{m4}\phi_{m+1}^{n+1} + \nu_{m5}\delta_{m+1}^{n+1} + \nu_{m6}\phi_{m+1}^{n+1} = \nu_{m7}\delta_{m-1}^{n} + \nu_{m8}\delta_{m}^{n} + \nu_{m9}\phi_{m+1}^{n+1}
$$

(10)

and

$$
\nu_{m10}\delta_{m-1}^{n+1} + \nu_{m11}\phi_{m-1}^{n+1} + \nu_{m12}\delta_{m}^{n+1} + \nu_{m13}\delta_{m+1}^{n+1} + \nu_{m14}\phi_{m+1}^{n+1} + \nu_{m15}\phi_{m+1}^{n+1} = \nu_{m7}\delta_{m-1}^{n} + \nu_{m8}\phi_{m}^{n} + \nu_{m9}\phi_{m+1}^{n+1}
$$

(11)

where

$$
\begin{align*}
\nu_{m1} &= \left( \frac{2}{\Delta t} + k_1 K_2 + k_2 L_2 \right) \alpha_1 + \left( k_1 K_1 + k_2 L_1 \right) \beta_1 - \gamma_1 \\
\nu_{m2} &= \left( k_1 K_2 \right) \alpha_1 + \left( k_2 K_1 \right) \beta_1 \\
\nu_{m3} &= \left( \frac{2}{\Delta t} + k_1 K_2 + k_2 L_2 \right) \alpha_2 - \gamma_2 \\
\nu_{m4} &= \left( k_1 K_2 \right) \alpha_2 \\
\nu_{m5} &= \left( \frac{2}{\Delta t} + k_1 K_2 + k_2 L_2 \right) \alpha_1 - \left( k_1 K_1 + k_2 L_1 \right) \beta_1 - \gamma_1 \\
\nu_{m6} &= \left( k_1 K_2 \right) \alpha_1 - \left( k_2 K_1 \right) \beta_1 \\
\nu_{m7} &= \frac{2}{\Delta t} \alpha_1 + \gamma_1 \\
\nu_{m8} &= \frac{2}{\Delta t} \alpha_2 + \gamma_2 \\
\nu_{m9} &= \frac{2}{\Delta t} \alpha_1 + \gamma_1 \\
\nu_{m10} &= \left( k_3 L_2 \right) \alpha_1 + \left( k_3 L_1 \right) \beta_1 \\
\nu_{m11} &= \left( \frac{2}{\Delta t} + k_1 L_2 + k_3 K_2 \right) \alpha_1 + \left( k_1 L_1 + k_3 K_1 \right) \beta_1 - \gamma_1 \\
\nu_{m12} &= \left( k_3 L_2 \right) \alpha_2 \\
\nu_{m13} &= \left( \frac{2}{\Delta t} + k_1 L_2 + k_3 K_2 \right) \alpha_2 - \gamma_2 \\
\nu_{m14} &= \left( k_3 L_2 \right) \alpha_1 - \left( k_3 L_1 \right) \beta_1 \\
\nu_{m15} &= \left( \frac{2}{\Delta t} + k_1 L_2 + k_3 K_2 \right) \alpha_1 - \left( k_1 L_1 + k_3 K_1 \right) \beta_1 - \gamma_1
\end{align*}
$$

5
The system with (10) and (11) can be converted to the following matrices system;

\[
K_1 = \alpha_1 \delta_{m-1}^n + \alpha_2 \delta_m^n + \alpha_3 \delta_{m+1}^n \quad L_1 = \alpha_1 \delta_{m-1}^n + \alpha_2 \delta_m^n + \alpha_3 \delta_{m+1}^n
\]

\[
K_2 = \beta_1 \delta_{m-1}^n + \beta_2 \delta_m^n + \beta_3 \delta_{m+1}^n \quad L_2 = \beta_1 \delta_{m-1}^n + \beta_2 \delta_m^n + \beta_3 \delta_{m+1}^n
\]

\[
\alpha_1 = \frac{s - ph}{2(phc - s)}, \quad \alpha_2 = 1,
\]

\[
\beta_1 = \frac{p(1 - c)}{2(phc - s)}, \quad \beta_2 = \frac{p(c - 1)}{2(phc - s)}
\]

\[
\gamma_1 = \frac{p^2s}{2(phc - s)}, \quad \gamma_2 = -\frac{p^2s}{phc - s}
\]

The system with (10) and (11) can be converted to the following matrices system;

\[
\begin{align*}
K_1 &= \alpha_1 \delta_{m-1}^n + \alpha_2 \delta_m^n + \alpha_3 \delta_{m+1}^n \\
K_2 &= \beta_1 \delta_{m-1}^n + \beta_2 \delta_m^n + \beta_3 \delta_{m+1}^n
\end{align*}
\]

\[
\alpha_1 = \frac{s - ph}{2(phc - s)}, \quad \alpha_2 = 1,
\]

\[
\beta_1 = \frac{p(1 - c)}{2(phc - s)}, \quad \beta_2 = \frac{p(c - 1)}{2(phc - s)}
\]

\[
\gamma_1 = \frac{p^2s}{2(phc - s)}, \quad \gamma_2 = -\frac{p^2s}{phc - s}
\]

The system (12) consist of 2N + 2 linear equation in 2N + 6 unknown parameters \( \mathbf{x}^{n+1} = (\delta_{-1}^{n+1}, \delta_0^{n+1}, \delta_1^{n+1}, \ldots, \delta_{n+1}^{n+1}) \). To obtain a unique solution an additional four constraints are needed. By imposing the Dirichlet boundary conditions this will lead us to the following relations;

\[
\begin{align*}
\delta_{-1} &= (f_1(a, t) - \alpha_2 \delta_0 - \alpha_3 \delta_1) / \alpha_1 \\
\phi_{-1} &= (g_1(a, t) - \alpha_2 \phi_0 - \alpha_3 \phi_1) / \alpha_1 \\
\delta_{N+1} &= (f_2(b, t) - \alpha_1 \delta_{N-1} - \alpha_2 \delta_N) / \alpha_3 \\
\phi_{N+1} &= (g_2(b, t) - \alpha_1 \phi_{N-1} + \alpha_2 \phi_N) / \alpha_3
\end{align*}
\]
3 The Initial State

Initial parameters $\delta_{-1}^0, \phi_{-1}^0, \delta_0^0, \phi_0^0, \ldots, \delta_{N+1}^0, \phi_{N+1}^0$ can be determined from the initial condition and first space derivative of the initial conditions at the boundaries as the following:

\[
U^0(a, 0) = \frac{s - ph}{2(phc - s)} \delta_{-1}^0 + \frac{s - ph}{2(phc - s)} \delta_1^0
\]

\[
U^0(x_m, 0) = \frac{s - ph}{2(phc - s)} \delta_{m-1}^0 + \frac{s - ph}{2(phc - s)} \delta_m^0 + \frac{s - ph}{2(phc - s)} \delta_{m+1}^0, \quad m = 1, 2, \ldots, N - 1
\]

\[
U^0(b, 0) = U_N^0 = \frac{s - ph}{2(phc - s)} \delta_{N-1}^0 + \frac{s - ph}{2(phc - s)} \delta_N^0 + \frac{s - ph}{2(phc - s)} \delta_{N+1}^0
\]

and

\[
V^0(a, 0) = \frac{s - ph}{2(phc - s)} \phi_{-1}^0 + \frac{s - ph}{2(phc - s)} \phi_1^0
\]

\[
V^0(x_m, 0) = \frac{s - ph}{2(phc - s)} \phi_{m-1}^0 + \frac{s - ph}{2(phc - s)} \phi_m^0 + \frac{s - ph}{2(phc - s)} \phi_{m+1}^0, \quad m = 1, 2, \ldots, N - 1
\]

\[
V^0(b, 0) = \frac{s - ph}{2(phc - s)} \phi_{N-1}^0 + \frac{s - ph}{2(phc - s)} \phi_N^0 + \frac{s - ph}{2(phc - s)} \phi_{N+1}^0
\]

The system (14) which is constituted for initial conditions consists $N + 1$ equations and $N + 3$ unknown, so we have to eliminate $\delta_{-1}^0$ and $\delta_{N+1}^0$ for solving this system using following derivatives conditions

\[
\delta_{-1}^0 = \delta_1^0 + \frac{2(phc - s)}{p(1-c)} U'_0, \quad \delta_{N+1}^0 = \delta_{N-1}^0 - \frac{2(phc - s)}{p(1-c)} U'_N,
\]

and if the equations system is rearranged for the above conditions, then following form is obtained

\[
\begin{bmatrix}
1 & \frac{s - ph}{phc - s} \\
\frac{s - ph}{phc - s} & 1 & \frac{s - ph}{phc - s} \\
\vdots & \ddots & \ddots & \ddots \\
\frac{s - ph}{phc - s} & \frac{s - ph}{phc - s} & 1 & \frac{s - ph}{phc - s}
\end{bmatrix}
\begin{bmatrix}
\delta_0^0 \\
\delta_1^0 \\
\vdots \\
\delta_{N-1}^0 \\
\delta_N^0
\end{bmatrix}
= \begin{bmatrix}
U'_0 - \frac{s - ph}{p(1-c)} U'_0 \\
U'_1 \\
\vdots \\
U'_{N-1} \\
U'_N - \frac{s - ph}{p(c-1)} U'_N
\end{bmatrix}
\]

which can also be solved using a variant of the Thomas algorithm. As the same way, from the system (15), $\phi_{-1}^0$ and $\phi_{N+1}^0$ can be eliminated using

\[
\phi_{-1}^0 = \phi_1^0 + \frac{2(phc - s)}{p(1-c)} V'_0, \quad \phi_{N+1}^0 = \phi_{N-1}^0 - \frac{2(phc - s)}{p(1-c)} V'_N,
\]
conditions and the following three bounded matrix is obtained.
\[
\begin{bmatrix}
1 & \frac{s-ph}{2(ph-c-s)} & \cdots & \frac{1}{2(ph-c-s)} \\
\frac{s-ph}{2(ph-c-s)} & 1 & \cdots & \frac{s-ph}{2(ph-c-s)} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{s-ph}{2(ph-c-s)} & \cdots & \frac{1}{2(ph-c-s)} & 1
\end{bmatrix}
\begin{bmatrix}
\phi_0 \\
\phi_1 \\
\vdots \\
\phi_{N-1} \\
\phi_N
\end{bmatrix}
= 
\begin{bmatrix}
V_0 - \frac{s-ph}{p(1-c)} V'_0 \\
V'_1 \\
\vdots \\
V'_{N-1} \\
V'_N - \frac{s-ph}{p(c-1)} V'_N
\end{bmatrix}
\]

4 Numerical Tests

In this section, we will compare the efficiency and accuracy of suggested method problem. The obtained results will compare with [7], [10], [15] and [25], while \( p \) changes. The accuracy of the schemes is measured in terms of the following discrete error norm \( L_\infty \)
\[
L_\infty = |U - U_N|_\infty = \max_j |U_j - (U_N)_j^n|.
\]

Problem 1) Consider the Coupled Burgers equation system (1) with the following initial and boundary conditions
\[
U(x, 0) = \sin(x), \quad V(x, 0) = \sin(x)
\]
and
\[
U(-\pi, t) = U(\pi, t) = V(-\pi, t) = V(\pi, t) = 0
\]
The exact solution is
\[
U(x, t) = V(x, t) = e^{-t} \sin(x)
\]

We compute the numerical solutions using the selected values \( k_1 = -2, k_2 = 1 \) and \( k_3 = 1 \) with different values of time step length \( \Delta t \). In our first computation, we take \( t = 0.1, \Delta t = 0.001 \) while the number of partition \( N \) changes. The corresponding results are presented in Table 2 a. In our computation, we compute the maximum absolute errors at time level \( t = 1 \) for the parameters with different decreasing values of \( t \). The corresponding results are reported in Table 2 b. In both computations, the results are same for \( U(x, t) \) and \( V(x, t) \) because of symmetric initial and boundary conditions. And also we correspond the obtained numerical solutions by different settings of parameters, specifically for those taken by [15] in Table 2 c for \( N = 50, \Delta t = 0.01 \) and increasing \( t \). And also in Table 2, we present the rate of convergence in space which is clearly of second order.

| Table 2 a: \( L_\infty \) Error norms for \( t = 0.1, \Delta t = 0.001, U(x, t) = V(x, t) \) |
|----------------|----------------|----------------|----------------|
| \( N = 200 \)  | 0.01489 \times 10^{-5} | 0.00121 \times 10^{-5} | 0.74326 \times 10^{-5} |
| \( N = 400 \)  | 0.00372 \times 10^{-5} | 0.00044 \times 10^{-5} | 0.18534 \times 10^{-5} |

Present (Various \( p \))

25, (\( \lambda = 0 \))

25 (Various \( \lambda \))

25, (\( \lambda = 0 \))

25 (Various \( \lambda \))

25, (\( \lambda = 0 \))

25 (Various \( \lambda \))

25, (\( \lambda = 0 \))

25 (Various \( \lambda \))

25, (\( \lambda = 0 \))

25 (Various \( \lambda \))

25, (\( \lambda = 0 \))

25 (Various \( \lambda \))
The corresponding graphical illustrations are presented in Figures 2 for $k_1 = -2$, $k_2 = 1$, $k_3 = 1$, $N = 400$ and $\Delta t = 0.001$ at different $t$ for best parameter $p = 0.0002166$. In Figure 3-4, computed solutions of $v$ different time levels for $k_1, k_2$ fixed and $k_1, k_3$ fixed respectively.

**Table 2 b:** $L_\infty$ Error norms for $t = 1$, $N = 400$, $U(x,t) = V(x,t)$

| $\Delta t$ | Present ($p = 1$) | Present (Various $p$) | [25] $(\lambda = 0)$ | [25] (Various $\lambda$) |
|------------|--------------------|-----------------------|----------------------|--------------------------|
|            | $1.8194 \times 10^{-5}$ | $0.00247 \times 10^{-5}$ | $1.08691 \times 10^{-5}$ | $0.00131 \times 10^{-5}$ |
|            | $0.00010099997$ | $(p = 0.00010099997)$ | $(\lambda = -5.896 \times 10^{-5})$ | $(\lambda = -5.992 \times 10^{-5})$ |

**Table 2 c:** $L_\infty$ Error norms for $\Delta t = 0.01$, $N = 50$, $U(x,t) = V(x,t)$ different $t$.

| $t$ | Present ($p = 1$) | Present (Various $p$) | [10] | [11] | [15] |
|-----|--------------------|----------------------|------|------|------|
| 0.5 | $7.9881 \times 10^{-4}$ | $3.4770 \times 10^{-4}$ | $1.51688 \times 10^{-4}$ | $2.26627 \times 10^{-5}$ | $1.103080984 \times 10^{-4}$ |
| 1.0 | $9.6837 \times 10^{-4}$ | $4.2166 \times 10^{-4}$ | $1.83970 \times 10^{-4}$ | $1.46179 \times 10^{-5}$ | $1.33680384 \times 10^{-4}$ |
| 2.0 | $7.1154 \times 10^{-4}$ | $3.1006 \times 10^{-4}$ | $1.35250 \times 10^{-4}$ | $0.73805 \times 10^{-5}$ | $9.818252567 \times 10^{-5}$ |
| 3.0 | $3.9213 \times 10^{-4}$ | $1.7100 \times 10^{-4}$ | $7.46014 \times 10^{-4}$ | $0.40272 \times 10^{-5}$ | $1.029870405 \times 10^{-5}$ |

**Table 2 d:** The rate of convergence for $\Delta t = 0.01$ and $\Delta t = 0.0001$ respectively

| $\Delta t$ | Present ($p = 1$) | order | $\Delta t = 0.0001$ | Present ($p = 1$) | order |
|------------|--------------------|-------|----------------------|--------------------|-------|
| $N = 50$   | $3.9213 \times 10^{-4}$ |       | $N = 50$             | $3.9213 \times 10^{-4}$ |       |
| $N = 100$  | $9.9430 \times 10^{-5}$ | $1.9773$ | $N = 100$           | $9.9430 \times 10^{-5}$ | $1.9909$ |
| $N = 150$  | $4.4895 \times 10^{-5}$ | $1.9649$ | $N = 150$           | $4.4895 \times 10^{-5}$ | $2.0032$ |
| $N = 200$  | $2.5808 \times 10^{-5}$ | $1.9192$ | $N = 200$           | $2.5808 \times 10^{-5}$ | $1.9931$ |
| $N = 250$  | $1.6965 \times 10^{-5}$ | $1.8734$ | $N = 250$           | $1.6965 \times 10^{-5}$ | $1.9929$ |

The corresponding graphical illustrations are presented in Figures 2 for $k_1 = -2$, $k_2 = 1$, $k_3 = 1$, $N = 400$ and $\Delta t = 0.001$ at different $t$ for best parameter $p = 0.0002166$. In Figure 3-4, computed solutions of $v$ different time levels for $k_1, k_2$ fixed and $k_1, k_3$ fixed respectively.

**Figure 2:** Numerical Solutions at various $t$ for $N = 400$, $\Delta t = 0.001$, $p = 0.0002166$
Problem 2) Numerical solutions of considered coupled Burgers’ equations are obtained for $k_1 = 2$ with different values of $k_2$ and $k_3$ at different time levels. In this situation the exact solution is

$$U(x, t) = a_0 - 2A\left(\frac{2k_2-1}{4k_2k_3-1}\right) \tanh(A(x - 2At))$$

$$V(x, t) = a_0\left(\frac{2k_3-1}{2k_2-1}\right) - 2A\left(\frac{2k_2-1}{4k_2k_3-1}\right) \tanh(A(x - 2At))$$

Figure 3: Computed solutions of $V$ Problem 1 for different time levels ($k_1, k_2$ fixed)

Figure 4: Computed solutions of $V$ Problem 1 for different time levels ($k_1, k_3$ fixed)

Figure 5: Computed solutions of $V$ Problem 1 for different time levels ($k_2, k_3$ fixed)
Thus, the initial and boundary conditions are taken from the exact solution is

\[ U(x, 0) = a_0 - 2A \left( \frac{2k_2 - 1}{4k_2k_3 - 1} \right) \tanh(Ax) \]

\[ V(x, 0) = a_0 \left( \frac{2k_3 - 1}{2k_2 - 1} \right) - 2A \left( \frac{2k_2 - 1}{4k_2k_3 - 1} \right) \tanh(Ax) \]

Thus, the initial and boundary conditions are extracted from the exact solution. Where \( a_0 = 0.05 \) and \( A = \frac{1}{2} \left( \frac{a_0(4k_2k_3-1)}{2k_2-1} \right) \). The numerical solutions have been computed for the domain \( x \in [0, 1] \), \( \Delta t = 0.001 \) and number of partitions \( N = 10 \) and 100. The maximum absolute errors have been computed and compared in Tables 3 a-3 b for \( t = 1 \) with those available in the literature [25].

| Table 3 a: \( L_\infty \) Error norms for \( t = 1, \Delta t = 0.001, U(x, t), k_1 = 2, k_2 = 1 \) and \( k_3 = 0.3 \) |
|---------------------------------|----------------|----------------|
| Present \((p = 1)\) [25], \((\lambda = 0)\) [25] (Various \( \lambda \)) | \( N = 10 \) | 3.7323 \times 10^{-6} | 3.73505 \times 10^{-5} |
| | 3.7350 \times 10^{-6} | 3.73503 \times 10^{-5} |
| | 0.00077 \times 10^{-5}(\lambda = 6 \times 10^{-5}) | 0.00078 \times 10^{-5}(\lambda = -4.087 \times 10^{-5}) |
| \( N = 100 \) | 3.7350 \times 10^{-6} | 3.73503 \times 10^{-5} |
| | 0.00077 \times 10^{-5}(\lambda = 6 \times 10^{-5}) | 0.00078 \times 10^{-5}(\lambda = -4.087 \times 10^{-5}) |

| Table 3 b: \( L_\infty \) Error norms for \( t = 1, \Delta t = 0.001, V(x, t), k_1 = 2, k_2 = 1 \) and \( k_3 = 0.3 \) |
|---------------------------------|----------------|----------------|
| Present \((p = 1)\) [25], \((\lambda = 0)\) [25] (Various \( \lambda \)) | \( N = 10 \) | 1.2569 \times 10^{-6} | 1.29030 \times 10^{-5} |
| | 1.2871 \times 10^{-6} | 1.29038 \times 10^{-5} |
| | 0.00079 \times 10^{-5}(\lambda = -6 \times 10^{-5}) | 0.00079 \times 10^{-5}(\lambda = -4.087 \times 10^{-4}) |
| \( N = 100 \) | 1.2871 \times 10^{-6} | 1.29038 \times 10^{-5} |
| | 0.00079 \times 10^{-5}(\lambda = -6 \times 10^{-5}) | 0.00079 \times 10^{-5}(\lambda = -4.087 \times 10^{-4}) |

| Table 3 c: Maximum error norms for \( U(x, t) \) in Problem 2 \((N = 21, \Delta t = 0.01, k_1 = 2)\) |
|---------------------------------|----------------|----------------|
| \( t \) | \( k_2 \) | \( k_3 \) | Present \((p = 1)\) [10] | [15] |
| 0.5 | 0.1 | 0.3 | 8.8160 \times 10^{-6} | 4.173 \times 10^{-5} |
| | 0.03 | 9.2556 \times 10^{-6} | 4.585 \times 10^{-5} |
| 1.0 | 0.1 | 0.3 | 8.8878 \times 10^{-6} | 8.275 \times 10^{-5} |
| | 0.03 | 9.3324 \times 10^{-6} | 9.167 \times 10^{-5} |
| 3.0 | 0.1 | 0.3 | 8.9174 \times 10^{-6} | 2.408 \times 10^{-4} |
| | 0.03 | 9.3691 \times 10^{-6} | 2.747 \times 10^{-4} |

| Table 3 d: Maximum error norms for \( V(x, t) \) in Problem 2 \((N = 21, \Delta t = 0.01, k_1 = 2)\) |
|---------------------------------|----------------|----------------|
| \( t \) | \( k_2 \) | \( k_3 \) | Present \((p = 1)\) [10] | [15] |
| 0.5 | 0.1 | 0.3 | 2.8380 \times 10^{-6} | 5.418 \times 10^{-5} |
| | 0.03 | 1.1179 \times 10^{-5} | 2.826 \times 10^{-5} |
| 1.0 | 0.1 | 0.3 | 2.8686 \times 10^{-6} | 1.074 \times 10^{-4} |
| | 0.03 | 1.1269 \times 10^{-5} | 5.673 \times 10^{-5} |
| 3.0 | 0.1 | 0.3 | 2.9081 \times 10^{-6} | 3.119 \times 10^{-4} |
| | 0.03 | 1.1301 \times 10^{-5} | 1.663 \times 10^{-4} |
Problem 3) Consider the Coupled Burger Equation system \( \{ \) with the following initial conditions

\[
U(x, 0) = \begin{cases} 
\sin(2\pi x), & x \in [0, 0.5] \\
0, & x \in (0.5, 1]
\end{cases}
\]

\[
V(x, 0) = \begin{cases} 
0, & x \in [0, 0.5] \\
-\sin(2\pi x), & x \in (0.5, 1]
\end{cases}
\]

and zero boundary conditions. In the Problem 3, the solutions have been carried out on \( x \in [0, 1] \) with \( \Delta t = 0.001 \) and number of partitions as 50. Maximum values of \( u \) and \( v \) at different time levels for \( k_2 = k_3 = 10 \) have been given in Table 4a and 4 b, while the Tables 4 c and 4 d represent the maximum values for \( k_2 = k_3 = 100 \).

| Table 4 a: Maximum values of \( U \) at different time levels for \( k_2 = k_3 = 10 \) |
|---|---|---|---|---|
| \( t \) | Present \((p = 1)\) | 7 | 15 | at point |
| 0.1 | 0.144501 | 0.14456 | 0.14491495800 | 0.58 |
| 0.2 | 0.052353 | 0.05237 | 0.052356151890 | 0.54 |
| 0.3 | 0.019317 | 0.01932 | 0.019318838080 | 0.52 |
| 0.4 | 0.007183 | 0.00718 | 0.007184856672 | 0.50 |

| Table 4 b: Maximum values of \( V \) at different time levels for \( k_2 = k_3 = 10 \) |
|---|---|---|---|---|
| \( t \) | Present \((p = 1)\) | 7 | 15 | at point |
| 0.1 | 0.143155 | 0.14306 | 0.143141957500 | 0.66 |
| 0.2 | 0.047003 | 0.04697 | 0.047006446750 | 0.56 |
| 0.3 | 0.017258 | 0.01725 | 0.017260356430 | 0.52 |
| 0.4 | 0.006415 | 0.00641 | 0.006416614856 | 0.50 |
Table 4 c: Maximum values of $U$ at different time levels for $k_2 = k_3 = 100$

| $t$  | Present ($p = 1$) |  |  | at point |
|------|-------------------|---|---|----------|
| 0.1  | 0.04168           | 0.04175 | 0.041682987260 | 0.46 |
| 0.2  | 0.01476           | 0.01479 | 0.014770415340 | 0.58 |
| 0.3  | 0.00533           | 0.00534 | 0.005337325631 | 0.54 |
| 0.4  | 0.00197           | 0.00198 | 0.001978065014 | 0.52 |

Table 4 d: Maximum values of $V$ at different time levels for $k_2 = k_3 = 100$

| $t$  | Present ($p = 1$) |  |  | at point |
|------|-------------------|---|---|----------|
| 0.1  | 0.05074           | 0.05065 | 0.050737669860 | 0.76 |
| 0.2  | 0.01035           | 0.01033 | 0.010356602970 | 0.64 |
| 0.3  | 0.00351           | 0.00350 | 0.003517189432 | 0.56 |
| 0.4  | 0.00129           | 0.00129 | 0.001294450199 | 0.52 |

Figs. 7, 8 and 9 show the numerical results obtained for different time levels $t \in [0, 1]$ at $k_2 = k_3 = 10$ for $U$ and $V$ with different values of $k_1$. From the Figs. 7-9, it can be easily seen that the numerical solutions $U^n$ and $V^n$ decay to zero as $t$ and $k_1$ increased.

Fig 7: Num. Sol. $U(x,t)$ and $V(x,t)$ of Problem 3 at different time levels for $k_2 = k_3 = 10$ while $k_1 = 1$

Fig 8: Num. Sol. $U(x,t)$ and $V(x,t)$ of Problem 3 at different time levels for $k_2 = k_3 = 10$ while $k_1 = 10$
Fig 9: Num. Sol. $U(x, t)$ and $V(x, t)$ of Problem 3 at different time levels for $k_2 = k_3 = 10$ while $k_1 = 50$

5 Conclusion

The collocation method together with the exponential B-spline as trial functions has presented to get the numerical solutions of the coupled Burgers’ equation system. The free parameter in the exponential B-splines is searched experimentally to get the best numerical solution for the first problem. In the other problem, results are documented for $p = 1$ as an example. The proposed method has produced less error than the methods listed in the tables for some text problems. The results are satisfactory and competent with some available solutions in the literature. Another advantages is that the method can be used without the complex calculations. to solve the system of differential equations reliably. And also the collocation method together with B-spline approximations represents an economical alternative since it only requires the evaluation of the unknown parameters at the grid points.

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