On minimal prime graphs and posets

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Abstract

We show that there are four infinite prime graphs such that every infinite prime graph with no infinite clique embeds one of these graphs. We derive a similar result for infinite prime posets with no infinite chain or no infinite antichain.

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1 Presentation of the results

This paper is about prime graphs and prime posets. Our notations and terminology mostly follow [1]. The graphs we consider are undirected, simple and have no loops. That is, a graph is a pair $G := (V, E)$, where $E$ is a subset of $[V]^2$, the set of 2-element subsets of $V$. Elements of $V$ are the vertices of $G$ and elements of $E$ its edges. The complement of $G$ is

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the graph \( G \) whose vertex set is \( V \) and edge set \( \mathcal{E} := |V|^2 \setminus \mathcal{E} \). If \( A \) is a subset of \( V \), the pair \( G|_A := (A, \mathcal{E} \cap [A]^2) \) is the graph induced by \( G \) on \( A \). The graph \( G \) embeds a graph \( G' \) and we set \( G' \leq G \) if \( G' \) is isomorphic to an induced subgraph of \( G \). A subset \( A \) of \( V \) is called autonomous in \( G \) if for every \( v \notin A \), either \( v \) is adjacent to all vertices of \( A \) or \( v \) is not adjacent to any vertex of \( A \). Clearly, the empty set, the singletons in \( V \) and the whole set \( V \) are autonomous in \( G \); they are called trivial. An undirected graph is called indecomposable if all its autonomous sets are trivial. With this definition, graphs on a set of size at most two are indecomposable. Also, there are no indecomposable graph on a three-element set. An indecomposable graph with more than three elements will be said prime.

The graph \( P_4 \), the path on four vertices, is prime. In fact, as it is well known, every prime graph contains an induced \( P_4 \). Sumner [15] for finite graphs and Kelly [10] for infinite graphs. Furthermore, every infinite prime graph contains an induced countable prime graph [9]. This leads to the question: Which countable prime graphs occur necessarily as induced subgraphs of infinite prime graphs?

More specifically, let us say that a graph \( G \) is minimal prime if \( G \) is prime and every prime induced subgraph with the same cardinality as \( G \) embeds a copy of \( G \). One could ask then the following:

Questions 1. (a) Does every infinite prime graph embed a countable minimal prime graph?

(b) Are there only finitely many infinite countable minimal prime graphs?

These questions are the motivation behind this paper. We give a positive answer for graphs not containing an infinite clique or an infinite independent set.

In order to state our result, let \( \mathcal{G} := \{G_i : i < 4\} \) be the set of graphs defined as follows. All these graphs are bipartite, all but \( G_3 \) have the same set of vertices which decomposes into two disjoint independent sets \( A := \{a_i : i \in \mathbb{N}\} \) and \( B := \{b_i : i \in \mathbb{N}\} \). A pair \( \{a_i, b_j\} \) is an edge in \( G_0 \) if \( i \neq j \), an edge in \( G_1 \) if \( i < j \), an edge in \( G_2 \) if \( j = i \) or \( j = i + 1 \) and, finally, an edge in \( G_3 \) if \( j = i \). For \( G_3 \), a new vertex \( c \) adjacent to every element of \( B \) is added to \( A \cup B \). The graph \( G_0 \) is the comparability graph of \( D_{\aleph_0} \) (the so-called standard poset, made of the atoms and co-atoms of the Boolean algebra \( \mathcal{P}(\mathbb{N}) \) of the subsets of \( \mathbb{N} \)), whereas the graph \( G_1 \) is the half complete bipartite graph. The graph \( G_2 \) is the one-way infinite path \( P_{\aleph_0} \), whereas the graph \( G_3 \) is a tree made of countably infinitely many disjoint edges connected to a single vertex (namely \( c \)). These graphs are represented Figure 1.

These graphs are prime. A fact which follows from the next proposition (the proof is easy and let to the reader).

Proposition 1. A bipartite graph on more than three vertices is prime if and only if it is connected and distinct vertices have distinct neighborhoods.

Moreover, none of these graphs embed in another. To see that one may observe that for each pair \( (i, j) \) with \( 0 \leq i \neq j \leq 3 \), there is a finite graph \( H_{ij} \) which embeds into \( G_i \) and not \( G_j \) (eg take for \( H_{01} \) the union of two disjoint edges).

Theorem 2. An infinite prime graph which does not contain an infinite clique embeds a member of \( \mathcal{G} \).
An immediate consequence of Theorem 2 (which can be obtained directly) is that the members of $\mathcal{G}$ are countable minimal prime graphs.

From Theorem 2 we derive two consequences for prime posets.

Throughout, $P := (V, \leq)$ denotes an ordered set (poset), that is a set $V$ equipped with a binary relation $\leq$ on $V$ which is reflexive, antisymmetric and transitive. The dual of $P$ denoted $P^*$ is the order defined on $V$ as follows: if $x, y \in V$, then $x \leq y$ in $P^*$ if and only if $y \leq x$ in $P$. A subset $A$ of $V$ is called autonomous in $P$ if for all $v \notin A$ and for all $a, a' \in A$ $(v < a \Rightarrow v < a')$ and $(a < v \Rightarrow a' < v)$.

(1)

As for graphs, the empty set, the singletons and the whole set $V$ are autonomous and are said to be trivial. A poset is indecomposable if all its autonomous sets are trivial, it is prime if it is indecomposable with more than three elements.

The comparability graph, respectively the incomparability graph, of $P := (V, \leq)$ is the undirected graph, denoted by $\text{Comp}(P)$, respectively $\text{Inc}(P)$, with vertex set $V$ and edges the pairs $\{u, v\}$ of comparable distinct vertices (that is, either $u < v$ or $v < u$) respectively incomparable vertices. We recall the following result (see [10]).

**Theorem 3.** A poset $P$ is prime if and only if $\text{Comp}(P)$ is prime. Moreover, if $\text{Comp}(P)$ is prime then it has exactly two transitive orientations, namely $P$ and $P^*$.

From this, we have readily:

**Proposition 4.** A poset $P$ is minimal prime if and only if $\text{Comp}(P)$ is minimal prime.

**Proof.** The fact that $\text{Comp}(P)$ is minimal prime whenever $P$ is minimal prime follows directly from the first part of Theorem 3.
The proof of the converse requires also the second part. Indeed, let \( P := (V, \leq) \) such that \( \text{Comp}(P) \) is minimal prime. From the first part of Theorem 3 we deduce that \( P \) is prime. Let \( V' \subseteq V \) such that \( |V'| = |V| \) and \( P_{\mid V'} \) is prime. Then again \( \text{Comp}(P_{\mid V'}) \) is prime. Since \( \text{Comp}(P_{\mid V'}) = \text{Comp}(P)_{\mid V'} \) and \( \text{Comp}(P) \) is minimal prime, there is an embedding \( f \) of \( \text{Comp}(P) \) into \( \text{Comp}(P)_{\mid V'} \). If \( f \) is not an embedding of \( P \) into \( P_{\mid V'} \), then from the second part of Theorem 3, \( f \) must be order reversing, hence \( g = f \circ f \) is an order preserving map of \( P \) into \( P_{\mid V'} \). \( \square \)

As illustrated in Figure 1 the four members of \( G \) are comparability graphs. Since from Theorem 2 they are minimal prime, Proposition 4 asserts that their orientations are minimal prime, whereas Theorem 3 ensures that each one has exactly two orientations. Deciding \( a_0 < b_0 \) is each of these graphs we obtain four posets \( Q_0, Q_1, Q_2, Q_3 \). The posets \( Q_0 \) and \( Q_0^* \) are isomorphic to the standard poset \( D_{\aleph_0} \). The posets \( Q_1 \) and \( Q_1^* \) are interval orders, they do not embed in each other. The posets \( Q_2 \) and \( Q_2^* \) are two one-way infinite fences, they are not isomorphic but they do embed in each other. The posets \( Q_3 \) and \( Q_3^* \) do not embed in each other. Hence, no member of \( Q := \{Q_0, Q_1, Q_1^*, Q_2, Q_3, Q_3^*\} \) embeds in another. From Theorem 2 we obtain immediately the following.

**Theorem 5.** Every infinite prime poset with no infinite chain embeds a member of \( Q \).

Theorem 2 applies also to incomparability graphs. Indeed, since a graph is prime if and only if its complement is prime, a poset is prime if and only if its incomparability graph is prime. We may note that only \( G_1 \) and \( G_2 \) are incomparability graphs of posets. Indeed, as it is well known the comparability graph of a poset is an incomparability graph if and only if the poset has dimension at most two \( \aleph_0 \). Since \( G_0 = \text{Comp}(D_{\aleph_0}) \) and \( D_{\aleph_0} \) has infinite dimension and since \( G_3 = \text{Comp}(Q_3) \) and \( Q_3 \) has dimension 3, neither \( G_0 \) nor \( G_3 \) are incomparability graphs. We recall that if a poset \( P \) has dimension 2, an order complement of \( P \) is a transitive orientation of its incomparability graph. Note that from Theorem 3 a prime poset has two order complements. The complements \( G_1^* \) and \( G_1' \) of \( G_1 \) and \( G_2 \) have two orientations which are respectively the order complements \( P_1 \) and \( P_1^* \) of \( Q_1 \) (as well as \( Q_1^* \)), respectively the order complements \( P_2 \) and \( P_2^* \) of \( Q_2 \) (as well as \( Q_2^* \)). See \( P_1 \) and \( P_2 \) in Figure 2. Let \( \mathcal{L} \) be the set of these four posets. These posets are minimal prime and none embeds in another. From Theorem 2 we obtain:

**Theorem 6.** Every infinite prime poset with no infinite antichain embeds a member of \( \mathcal{L} \).

Theorem 2 is a consequence of two properties of the neighborhood lattice of a graph. To a graph \( G := (V, \mathcal{E}) \) we associate a complete lattice \( \overline{\mathcal{N}(G)} \), the neighborhood lattice of \( G \). It is made of intersections of subsets of \( \mathcal{N}(G) := \{N_G(x) : x \in V\} \) where \( N_G(x) := \{y : \{x, y\} \in \mathcal{E}\} \). Thus ordered by inclusion this is a complete lattice. As we will see, if \( G \) is prime, then \( \overline{\mathcal{N}(G)} \) is infinite. Under the condition that \( G \) contains no infinite clique, we prove that if \( \overline{\mathcal{N}(G)} \) contains an infinite chain, then \( G \) embeds \( G_0 \) or \( G_1 \). On the other hand if \( \overline{\mathcal{N}(G)} \) contains no infinite chain we prove that \( G \) embeds \( G_2 \) or \( G_3 \). Precisely, we prove:
Theorem 7. Let $G$ be a graph with no infinite clique. Then $\hat{N}(G)$ contains an infinite chain if and only if $G$ contains an induced subgraph isomorphic to $G_0$ or to $G_1$.

Theorem 8. Let $G$ be an infinite prime graph. If all chains in $\hat{N}(G)$ are finite, then $G$ embeds $G_2$ or $G_3$.

The proofs of Theorem 7 and Theorem 8 rely on properties of incidence structures and on Ramsey’s theorem, with a technique which appeared in [5] and [11]. They are given in Section 3 and Section 4. The properties we need in order to prove Theorems 7 and 8 are given in the next section.

We should note that there are other minimal prime graphs and posets. Up to now, we have shown that:

Theorem 9. There are at least sixteen, respectively twenty two, countable minimal prime graphs, resp. posets, none embedding in an other. Furthermore, for every uncountable cardinal $\kappa$, there are at least fourteen, respectively nineteen, minimal prime graphs, respectively posets of size $\kappa$, none embedding in an other.

The examples leading to Theorem 9 and the proof are presented in Section 5. In the last section, we present some questions.

2 The neighborhood lattice

Properties of the neighborhood lattice are better understood in terms of incidence structure and Galois lattices. In the following subsection, we recall some fundamental properties of these objcts.

2.1 Incidence structures, Galois lattices and coding

Let $E, F$ be two sets. A binary relation from $E$ to $F$ is any subset $\rho$ of the cartesian product $E \times F$. As usual, we denote by $x\rho y$ the fact that $(x, y) \in \rho$ and by $x\not\rho y$ the negation. The
triple $R := (E, \rho, F)$ is an incidence structure; its complement is $\neg R := (E, \rho, F)$, where $\neg \rho := (E \times F) \setminus \rho$, whereas its dual is $R^{-1} := (F, \rho^{-1}, E)$, where $\rho^{-1} := \{(y, x) : (x, y) \in \rho\}$. For $x \in E$ we set $R(x) := \{y \in F : x \rho y\}$. Hence for $y \in F$, $R^{-1}(y) = \{x \in X : y \rho^{-1} x\} = \{x \in X : x \rho y\}$. Let $R[E] := \{R(x) : x \in E\}$ and $R^{-1}[F] := \{R^{-1}(y) : y \in F\}$. We denote by $Gal(R)$ the set of all intersections of members of $R^{-1}[F]$ (with the convention that $E \in Gal(R)$). Ordered by inclusion, $Gal(R)$ is a complete lattice, called the Galois lattice of $R$. The Galois lattice of $R^{-1}$ is the set $Gal(R^{-1})$ of all intersections of members of $R[E]$, ordered by inclusion. A fundamental result about Galois lattices is:

**Theorem 10.** $Gal(R^{-1})$ is isomorphic to $Gal(R)^*$, the dual of $Gal(R)$.

We recall that an incidence structure $R := (E, \rho, F)$ is Ferrers if $x \rho y$ and $x' \rho y'$ imply $x \rho y'$ or $x' \rho y$ for all $x, x' \in E, y, y' \in F$ [13]. Equivalently, $Gal(R)$ is a chain. We also recall that a poset $P$ is an interval order iff $(P, <, P)$ is Ferrers.

Let $R := (E, \rho, F)$, $R' := (E', \rho', F')$ be two incidence structures, a coding from $R$ into $R'$ is a pair of maps $f : E \to E'$, $g : F \to F'$ such that

$$x \rho y \iff f(x) \rho' g(y).$$

When such a pair exists, we say that $R$ has a coding into $R'$.

Bouchet’s Coding theorem ([2], see also [3]) relates the notions of coding and embedding. A straightforward consequence is this.

**Lemma 11.** If an incidence structure $R$ has a coding into $R'$, then $Gal(R)$ embeds into $Gal(R')$.

For an example, $Gal((\mathbb{N}, \neq, \mathbb{N})) = \mathcal{P}(\mathbb{N})$, whereas $Gal((\mathbb{N}, <, \mathbb{N}))$ is the set $I(\mathbb{N})$ of initial segments of $\mathbb{N}$ ordered by inclusion and $Gal((\mathbb{N}, >, \mathbb{N}))$ is the set of final segments of $\mathbb{N}$ ordered by inclusion. Hence, from Lemma [11] if one of these structures has a coding in an incidence structure $R$, the Galois lattice of $R$ embeds the corresponding Galois lattice, thus contains an infinite antichain. The converse was proved in [11] (see Theorem 2.9).

**Theorem 12.** The Galois lattice $Gal(R)$ contains an infinite chain if and only if there is a coding of one of the following incidence structures: $(\mathbb{N}, =, \mathbb{N})$, $(\mathbb{N}, <, \mathbb{N})$ or $(\mathbb{N}, >, \mathbb{N})$ into $R$.

Our proof of Theorem [7] follows similar lines. In fact, [11] contains part of Theorem [7] (see Corollary 2.15).

### 2.2 Basic facts about the neighborhood lattice

Let $G := (V, \mathcal{E})$ be an undirected graph without loops. If $x, y \in V$ we denote by $x \sim y$ the fact that $\{x, y\} \in \mathcal{E}$ and $x \sim y$ otherwise. We set $N_G(y) := \{x \in V : x \sim y\}$. This is the neighborhood of $y$. We insist on the fact that $y \not\in N_G(y)$. The degree of $y$ in $G$ is $d_G(y) := |N_G(y)|$, the cardinality of $N_G(y)$. We set $N(G) := \{N_G(y) : y \in V\}$. Let $\hat{N}(G)$ be the set of intersections of subsets of $N(G)$. We make the convention that $V$ is the intersection
of the empty set, hence $V \in \widehat{N}(G)$. Since $\widehat{N}(G)$ is closed under intersections, once ordered by inclusion this is a complete lattice. We call it the **neighborhood lattice** of $G$.

Identifying $\mathcal{E}$ to a subset of $V \times V$, or more precisely setting $\mathcal{E} = \{(x, y) : \{x, y\} \in \mathcal{E}\}$, we have $R^{-1}(y) = N_G(y)$.

**Lemma 13.** *The lattice $\widehat{N}(G)$ is the Galois lattice of $R := (V, \mathcal{E}, V)$.*

Since $R = R^{-1}$, Theorem 10 yields:

**Lemma 14.** *The lattice $\widehat{N}(G)$ is isomorphic to its dual.*

**Corollary 15.** *Let $R' := (E', \rho', F')$ be an incidence structure. If $R'$ has a coding into $(V, \mathcal{E}, V)$, where $G := (V, \mathcal{E})$ is a graph, then $\text{Gal}(R')$ embeds into $\widehat{N}(G)$.*

If $f$ is an embedding from $G'$ into $G$ then $(f, f)$ is a coding from $R'$ to $R$. Thus:

**Corollary 16.** *If a graph $G'$ embeds into $G$, then $\widehat{N}(G')$ embeds into $\widehat{N}(G)$.***

**Lemma 15** yields:

**Corollary 17.** *If a graph $G$ contains an infinite clique or an induced subgraph isomorphic to $G_0$, then $\widehat{N}(G')$ contains an induced poset isomorphic to $\mathcal{P}(\mathbb{N})$ ordered by inclusion. If it contains an induced subgraph isomorphic to $G_1$ then it contains a chain of type $\omega$ and a chain of type $\omega^*$.***

**Proof.** Let $G := (V, \mathcal{E})$. If $G$ contains an infinite clique or an induced subgraph isomorphic to $G_0$ then there is a coding from $(\mathbb{N}, \neq, \mathbb{N})$ in $R := (V, \mathcal{E}, V)$, whereas if it contains an induced subgraph isomorphic to $G_1$ there is coding from $(\mathbb{N}, \leq \mathbb{N})$ into $R$ and then a coding from $(\mathbb{N}, < \mathbb{N})$ into $R$. According to Lemma 15, in the first case, $\widehat{N}(\mathbb{N}) = \text{Gal}(R)$ embeds $\text{Gal}((\mathbb{N}, \neq, \mathbb{N}))$ whereas in the second case, $\text{Gal}(G)$ embeds $\text{Gal}((\mathbb{N}, < \mathbb{N}))$. Since $\text{Gal}((\mathbb{N}, \neq, \mathbb{N})) = \mathcal{P}(\mathbb{N})$ and $\text{Gal}((\mathbb{N}, < \mathbb{N})) = I(\mathbb{N})$, the conclusion follows.

In the sequel, given a subset $X$ of $V$, we set $X^+ := \cap\{N_G(x) : x \in X\}$; eg. $\{x\}^+ = N_G(x)$. With the convention above, if $X = \emptyset$, then $X^+ = V$. Clearly $\widehat{N}(G) = \{X^+ : X \subseteq V\}$. We also set $X^{++} := (X^+)^+$. 

Figure 3: The neighborhood lattices of $G_2$ and $G_3$. 
Lemma 18. The empty set is the least element of \( \widehat{N}(G) \).

Proof. \( \emptyset = V^+ \). \qed

A graph \( G := (V, E) \) is point determining if \( x \neq y \) implies \( N_G(x) \neq N_G(y) \) for all \( x, y \in V \) (cf. [14]). We reduce our study to the case of point determining graphs. Indeed, let \( x, y \in V \). Set \( x \equiv y \) if \( N_G(x) = N_G(y) \). The relation \( \equiv \) is an equivalence relation. Since \( x \notin N_G(x) \) for all \( x \), \( N_G(x) = N_G(y) \) implies \( x \sim y \). Hence, each equivalence class is an independent subset of \( V \). In fact, each equivalence class is also an autonomous subset in \( G \). Set \( V/\equiv \) be the set of these equivalence classes and \( p : V \to V/\equiv \) the map associating to each vertex \( x \) its equivalence class \( p(x) \). Since the equivalence classes are independent sets, \( G/\equiv \) is an undirected graph with no loops. Furthermore, \( G \) is the lexicographical sum of its equivalence classes indexed by \( G/\equiv \). From this fact follows readily that \( G/\equiv \) is point determining (in fact \( \equiv \) is the unique equivalence relation on \( V \) for which the equivalence classes are independent and autonomous and \( G/\equiv \) is point determining). Furthermore, the map \( p \) induces an order isomorphism from \( \widehat{N}(G) \) onto \( \widehat{N}(G/\equiv) \). In conclusion,

Lemma 19. For every graph \( G \), \( \widehat{N}(G) \) is isomorphic to \( \widehat{N}(G') \) where \( G' \) is point determining. In particular, \( \widehat{N}(G) \) and \( \widehat{N}(G') \) have the same cardinality.

In the remainder of this section, we consider a point determining graph \( G := (V, E) \).

Lemma 20. If \( X \) is minimal above \( \emptyset \) in \( \widehat{N}(G) \), then \( X \) is a singleton.

Proof. Claim 1: For every \( x \in X \) and for every \( y \in V \) with \( y \sim x \) we have \( X \subseteq N(y) \).

Indeed, otherwise set \( Y := X \cap N(y) \). We have \( X > Y \cap N(y) \in \widehat{N}(G) \setminus \{\emptyset\} \) contradicting the minimality of \( X \).

From Claim 1 we get:

Claim 2: \( X \) is an independent and autonomous set.

We may now conclude that \( X \) is a singleton. Suppose the contrary and let \( x \neq x' \) in \( X \). Since \( G \) is point determining, \( N(x) \neq N(x') \). Let \( y \) be a vertex witnessing this fact. Without loss of generality we may suppose that \( y \sim x \) and \( y \sim x' \). This contradicts Claim 1. \qed

Let \( X \in \widehat{N}(G) \), we denote by \( \uparrow X \) the final segment generated by \( X \), that is, \( \uparrow X := \{X' \in \widehat{N}(G) : X \subseteq X'\} \).

Lemma 21. Let \( X \in \widehat{N}(G) \) such that

(1) \( \uparrow X \) is infinite.

(2) \( \uparrow X' \) is finite for all \( X' \in \widehat{N}(G) \) which contains strictly \( X \).
Then

(1') \( X^+ \) is finite.

(2') \( (X \cup \{x\})^+ \) is finite for every \( x \not\in X \).

**Proof.** Let \( x \not\in X \) and set \( X' := (X \cup \{x\})^+ \). Since \( X \subseteq X \cup \{x\} \), we have \( X \subseteq X' \) and since \( x \in X' \setminus X \), \( X \neq X' \). Thus from (2) \( \uparrow X' \) is finite. Since \( X' \subseteq \{y\}^+ = N_G(y) \) for all \( y \in (\{x\} \cup X)^+ \), the set \( \{N_G(y) : y \in (\{x\} \cup X)^+\} \) is finite. Since \( G \) is point determining, \( (\{x\} \cup X)^+ \) is finite as required.

\( \square \)

**Corollary 22.** The following properties are equivalent.

1. For every \( X \in \overline{N(G)} \setminus \{\emptyset\} \), \( \uparrow X \) is finite,
2. \( N_G(x) \) is finite for every \( x \in V \).

**Proof.** (1) \( \Rightarrow \) (2) Apply Lemma \ref{lemma21} with \( X := \emptyset \).

(2) \( \Rightarrow \) (1) Let \( X \) be non empty. The set \( \uparrow X \) is finite if there are only finitely many \( N_G(y) \) containing \( X \). Pick \( x \in X \). Since \( N_G(x) \) is finite, the numbers of \( N_G(y) \) such that \( y \in N_G(x) \) is finite. In particular the number of \( N_G(y) \) containing \( X \) is finite.

\( \square \)

3. **Proof of Theorem 7**

If \( G \) contains an induced subgraph isomorphic to \( G_0 \) or to \( G_1 \) then, according to Corollary \ref{corollary17} \( \overline{N(G)} \) contains an infinite chain. Conversely, suppose that \( \overline{N(G)} \) contains an infinite chain. Then it contains a chain of type \( \omega \) or \( \omega^* \). Since the lattice \( \overline{N(G)} \) is selfdual (Lemma \ref{lemma14}), it contains a chain of type \( \omega \), meaning that there exists a strictly increasing sequence \( (X_n)_{n \geq 0} \) of members of \( \overline{N(G)} \). From this, we may define two maps \( f_0 : \mathbb{N} \rightarrow V(G) \) and \( f_1 : \mathbb{N} \rightarrow V(G) \) such that for all \( n \in \mathbb{N} \):

1. \( f_0(n) \in X_{n+1} \) and \( f_1(n) \in X_n^+ \).
2. \( f_0(n) \sim f_1(n) \).
   Indeed, since \( X_{n+1} \not\subseteq X_n = X_n^{++} \), there are \( a \in X_{n+1} \) and \( b \in X_n^+ \) such that \( a \sim b \).
   Set \( f_0(n) := a \) and \( f_1(n) := b \).
   Beyond that, the maps \( f_0 \) and \( f_1 \) has the following properties:
3. \( f_0(n) \sim f_1(m) \) for all \( n < m \).
   Indeed \( X_{n+1} \subseteq X_m \subseteq X_{m+1} \) thus \( X_{m+1}^+ \subseteq X_m^+ \subseteq X_{n+1}^+ \). Since \( f_1(m) \in X_m^+ \) we have \( b_m \in X_{n+1}^+ \). Since \( f_0(n) \in X_{n+1} \) this yields \( f_0(n) \sim f_1(m) \).
4. \( f_1(n) \neq f_1(m) \) for all \( n < m \).
5. \( f_0(n) \neq f_0(m) \) for all \( n < m \).
Indeed, from (1) we have \( f_0(n) \sim f_1(m) \), thus (3) holds. Similarly (1) yields \( f_0(m) \sim f_1(m) \) and (2) yields \( f_0(n) \sim f_1(m) \). Thus (4) holds. The proof of (5) is similar.

Let \( [\mathbb{N}]^2 \) be the set of two element subsets of \( \mathbb{N} \) identified with ordered pairs \((n,m)\) such that \( n < m \). Divide \( [\mathbb{N}]^2 \) into blocks such that two such pairs \( u := (n_0,n_1) \) and \( u' := (n'_0,n'_1) \) are in the same block if

\[
f_i(n_k)\rho f_j(n_l) \Leftrightarrow f_i(n'_k)\rho f_j(n'_l)
\]

holds for all \( i, j, k, l \in \{0, 1\} \) and \( \rho \in \{=, \sim\} \). As it is easy to see the number of blocks is finite. Indeed, it is bounded by \( 2^{24} \) (each block can be coded by a relational structure made of six binary relations on a two element set).

Ramsey’s theorem on pairs ensures that there is an infinite subset \( I \) of \( \mathbb{N} \) such that all pairs belong to the same block. Let \( \phi(0) < \phi(1) < ... < \phi(n)... \) be an enumeration of \( I \) and \( f_i := f_i \circ \phi \) (\( i < 2 \)). Then equivalence (2) holds with \( f_i \) and \( f_j \) replaced by \( f_i \) and \( f_j \), meaning that all pairs of \( [\mathbb{N}]^2 \) are in the same block. Thus, without loss of generality, we may choose \( f_0 \) and \( f_1 \) such that equivalence (2) holds. We say then that the pair \((f_0,f_1)\) behaves uniformly on \( \mathbb{N} \). In this case we have the following additional properties:

(6) \( f_0(n) \sim f_0(m) \)

and

(7) \( f_1(n) \sim f_1(m) \)

for all \( n < m \). Indeed, if \( f_0(n_0) \sim f_0(m_0) \) for some pair \((n_0,m_0)\) then since \((f_0,f_1)\) behaves uniformly, \( f_0(n'_0) \sim f_0(m'_0) \) holds for all other pairs, and thus \( G \) contains an infinite clique. The proof of (7) is similar.

(8) \( f_0(n) \neq f_1(m) \) for all \( n \neq m \).

Indeed, if \( n < m \) this follows from (2). If \( f_0(n) = f_1(n) \) for some \( n \), then since the pair \((f_0,f_1)\) behaves uniformly, \( f_0(n) = f_1(n) \) for all \( n \). But for \( n < m \) we get \( f_0(n) \sim f_1(m) \sim f_0(m) \) contradicting (5). If \( f_0(n) = f_1(m) \) for some \( m < n \), then \( f_0(n) = f_1(m) \) for all \( m < n \). Taking \( m < n < n + 1 \) we get \( f_0(n) = a_{n+1} \) contradicting (4).

So far, the sets \( A' := \{f_0(n) : n < \omega\} \) and \( B' := \{f_1(m) : m < \omega\} \) are two disjoint independent subsets of \( G \) for which (1) and (2) hold. We consider two cases.

**Case 1.** (9) \( f_1(n) \sim f_0(m) \) for some \( n < m \).

Since \((f_0,f_1)\) behaves uniformly, this property holds for all \( n < m \). In this case \( G_{I \cup A' \cup B'} \) is isomorphic to \( G_0 \).

**Case 2.** (10) \( f_1(n) \sim f_0(m) \) for some \( n < m \). Again, this property holds for all \( n < m \). In this case \( G_{I \cup A' \cup B' \setminus \{b_0\}} \) is isomorphic to \( G_1 \), via the map \( \phi \) from \( G_1 \) to \( G \) defined by \( \phi(a_n) = f_0(n) \) and \( \phi(b_n) = f_1(n+1) \).

\( \square \)
4 Proof of Theorem \[8\]

Let \( G \) be an infinite prime graph. Since its is prime, it is point determining. This allows us to apply Lemma \[21\]. Let \( \mathcal{X} \) be the set of \( X \in \overline{\mathcal{N}(G)} \) such that the final segment \( \uparrow X \) of \( \overline{\mathcal{N}(G)} \) is infinite. Since \( \overline{\mathcal{N}(G)} \) is infinite, \( \emptyset \in \mathcal{X} \). Since \( \overline{\mathcal{N}(G)} \) contains no infinite chain, \( \mathcal{X} \) has a maximal element (this may require the axiom of dependent choices). Let \( X \) be such an element. Then for each \( X' \in \overline{\mathcal{N}(G)} \) containing strictly \( X \), the final segment \( \uparrow X' \) is finite. According to Lemma \[21\] \( X^+ \) is infinite. Let \( G' := G_{\uparrow X^+} \).

**Case 1.** \( G' \) contains an infinite connected component.

In this case, \( G' \) contains an infinite path. Indeed, according to Lemma \[21\] for each \( x \in X^+ \), the degree of \( x \) in \( G' \) is finite.

**Case 2.** All connected components of \( G' \) are finite.

In this case, since \( G \) is prime, \( G \) is connected hence \( G' \neq G \), that is, \( X \neq \emptyset \).

**Claim 1.** For every connected component \( C \), but perhaps one, there are \( a_C \in C \) and \( b_C \in V \setminus X^+ \) such that \( a_C \sim b_C \).

**Proof of Claim 1.** Let \( C \) be a connected component of \( G' \). Since \( C \subseteq X^+ \), we have \( X \subseteq F(a) := N_C(a) \cap (X \setminus X^+) \). If \( X = F(a) \) for every \( a \), then \( C \) is autonomous in \( G \). Since \( G \) is prime, \( C \) is a singleton, that is, \( C = \{a\} \) for some \( a \). There is no other connected component \( C' \) reduced to a singleton, because otherwise the set \( \{a, a'\} \), where \( A' = \{a'\} \), is autonomous in \( G \). Thus, all connected components \( C \) but one contain an element \( a_C \) such that \( X \neq F(a_C) \). Pick \( b_C \in F(a_C) \setminus X^+ \).

We define inductively two maps \( f_0 : \mathbb{N} \to V(G) \) and \( f_1 : \mathbb{N} \to V(G) \). Suppose that \((f_0(i), f_1(i))_{i<n}\) has been defined. According to Lemma \[21\] \((\cup_{i<n} N_G(f_0(i)) \cap X^+) \) is finite. Pick a connected component \( C \) of \( G' \) which does not meet \( \cup_{i<n} N_G(f_1(i)) \) and set \( f_0(n) := a_C \) and \( f_1(n) := b_C \). The sequence \((f_0(n), f_1(n))_{n \in \mathbb{N}}\) has the following properties:

1. \( f_1(n) \neq f_0(m) \) for all \( n \neq m \).
   Indeed, \( f_0(m) \in X^+ \) and \( f_1(n) \notin X^+ \).

2. \( f_1(n) \sim f_0(m) \) for all \( n < m \).
   Indeed, \( f_0(m) \) has been selected in a connected component which does not meet \( N_G(f_1(n)) \).

3. \( f_0(n) \neq f_0(m) \) for all \( n \neq m \).
   Indeed, these elements are chosen in different connected components of \( G' \).

4. \( f_1(n) \sim f_1(m) \) for all \( n < m \).
   Otherwise, we would have \( f_1(n) = f_1(m) \) for some \( n < m \). Since \( f_1(m) \sim f_0(m) \) we would have \( f_1(n) \sim f_0(m) \), contradicting (2).

Apply Ramsey Theorem as in the proof of Theorem \[14\]. There is an infinite subset \( I \) of \( \mathbb{N} \) on which the pair \((f_0, f_1)\) behaves uniformly. Without loss of generality we may suppose that \( I = \mathbb{N} \) (otherwise relabel \( I \) with the integers). From the fact that \( N(G) \) contains no infinite chain, \( G \) contains no infinite clique. This excludes \( f_1(n) \sim f_1(m) \). Again the fact that \( \overline{\mathcal{N}(G)} \) contains no infinite chain excludes \( f_0(n) \sim f_1(m) \) for \( n < m \). Let \( C := \{c\} \cup \{f_0(n), f_1(n) : n \in \mathbb{N}\} \) where \( c \in X \). Then \( G_{|C} \) is isomorphic to \( G_3 \). \(\square\)
5 Examples of minimal prime graphs and posets

All the minimal prime graphs that we have been able to obtain so far have, at the exception of $G_2$ and its complement, a common feature, that we present in full generality. As a byproduct, we obtain examples of minimal prime graphs of arbitrarily cardinality. Then we identify those which are comparability graphs.

5.1 Uniform graphs

A graph $G := (V, E)$ is uniform (II) if $V$ is the disjoint union of a finite set $K$ and a set of the form $E \times \{0, 1\}$. The set $E$ is equipped with a linear order $\le$. For two distinct vertices $u$ and $v$, the fact that they form an edge or not only depends upon how $x$ and $y$ are related by the order and upon the values of $i$ and $j$ if $u := (x, i)$ and $v := (y, j)$ or upon the value of $i$ if $u := (x, i)$ and $v \in K$. Formally, this translates to:

1. $(x_k, i) \rho (x_l, j) \iff (x'_k, i) \rho (x'_l, j)$.
2. $(x_k, i) \rho y \iff (x'_k, i) \rho y$.

for all $x_0 < x_1$, $x'_0 < x'_1$ in $E$, $y$ in $F$, $i, j, k, l \in \{0, 1\}$, $\rho \in \{\sim, =\}$.

Examples 1. Let $C := (E, \le)$. Set $G_k := (V_k, E_k)$ for $k \in \{0, 1, 3, 4\}$ with $V_k := K_k \cup E \times \{0, 1\}$, $K_k = \emptyset$ in case $k \neq 3$ and $K_3 = \{c\}$. Set

1. $(x, i) \sim_0 (x', i')$ if $i \neq i'$ and $x \neq x'$.
2. $(x, i) \sim_1 (x', i')$ if $i \neq i'$ and either $i < i'$ and $x \le x'$ or $i > i'$ and $x' \le x$.
3. $(x, i) \sim_3 (x', i')$ if $i \neq i'$ and $x = x'$, $c \sim (x, i)$ if $i = 1$.
4. $(x, i) \sim_4 (x', i')$ if either $i = i' = 0$ and $x \neq x'$ or $i \neq i'$ and $x = x'$.

We introduce three more graphs, in a more informal way.

1. The graph $K(C)$ has the same set of vertices as $G_1$ and edge set $E_1$ augmented with the set of unordered pairs of distinct elements of $E \times \{0\}$.
2. The graph $G_c(C)$ is obtained from $K(C)$ by adding a new vertex $c$ adjacent to all elements of $E \times \{1\}$.
3. $G_{ab}(C)$ is obtained from $K(C)$ by adding two extra vertices $a$ and $b$ and an edge between $a$ and all elements of $\{b\} \cup E \times \{0\}$.
5.2 Examples of minimal prime graphs

Let us recall that a chain $C$ isomorphic to the chain of nonnegative integers has order type $\omega$. More generally, if $C$ is well ordered its order type is the unique ordinal to which $C$ is isomorphic. In the remainder of this section we will mostly consider initial ordinals, e.g. $\omega, \omega_1, \omega_2, ..., \omega_n, ...$. These are cardinal numbers, the aleph’s. With the axiom of choice, there are no others.

In the definition of $G_i(C)$’s, $i \in \{0, 3, 4\}$, the order of $C$ is irrelevant. Replacing $C$ by $\omega$, we obtain $G_i = G_i(\omega)$ for $i \in \{0, 1, 3, 4\}$. The order is quite relevant in the other cases.

**Theorem 23.** For every infinite initial ordinal $\kappa$, the graphs $G_i(\kappa)$ for $i < 5, i \neq 2$, their complements, and the graphs $G_c(C), G_{ab}(C), \overline{G}_{ab}(C)$ for $C \in \{\kappa, \kappa^*\}$ are minimal prime. Furthermore, none of these fourteen graphs embeds in another.

The fact that those graphs are minimal prime is an immediate consequence of the following lemma.

**Lemma 24.** Let $C := (E, \leq)$ be a chain.

1. The graphs $G_i(C)$, $i < 5, i \neq 2$, $G_c(C)$ and $G_{ab}(C)$ are prime, provided that $|E| \geq 3$ if $i = 0$ and $|E| \geq 2$ in all other cases.

2. If $\kappa$ is an infinite initial ordinal and $C$ is a chain of order type $\kappa$ or $\kappa^*$, these graphs are minimal prime.

**Proof.** Let $G$ be one of the six graphs listed in Theorem 23. (1) The graph $G$ is prime. If $G \in \{G_i(C) : i \in \{0, 1, 3\}\}$, apply Proposition 1

Let $G \in \{G_4(C), G_c(C), G_{ab}(C)\}$. Let $X$ be an autonomous subset in $G$ with $|X| \geq 2$. To prove that $G$ is prime we need to prove that $X = V$. For that we make a repeated use of the observation that if a vertex $v$ separates two vertices $u$ and $u'$ of $X$ (that is, $v \sim u$ and $v \sim u'$ or $v \sim u$ and $v \sim u'$), then $v \in X$.

Set $X_i := X \cap E \times \{i\}$ for $i \in \{0, 1\}$.

**Claim 1.** Let $i, j \in \{0, 1\}$ with $i \neq j$. If $|X_i| \geq 2$ then $X_j \neq \emptyset$.

Indeed, let $u := (x, i), u' := (x, i)$ be two distinct elements of $X_i$. We may suppose that $x < x'$. Let $v := (x', 0)$ if $i = 1, v := (x, 1)$ if $i = 0$. Then $v$ separates $u$ and $u'$. Thus $v \in X_j$, proving our claim.

**Claim 2.** If $X_0$ and $X_1$ are non empty then $X = V$.

Note first that there is some $z \in E$ such that $(z, i) \in X$ for all $i \in \{0, 1\}$. Indeed, let $(x, 0), (x', 1) \in X$. If $x = x'$ we are done. If $x \neq x'$ then $(x, 1)$ separates $(x, 0)$ and $(x', 1)$ thus $(x, 1) \in X$ and we are done. Let $z$ be such an element. Let $x \in E$. If $G = G_4(C)$, then $(x, 0)$ separates $(z, 0)$ and $(z, 1)$, thus belongs to $X$. Furthermore, since $(x, 1)$ separates $(z, 0)$ and $(x, 1), (x, 1) \in X$, proving that $X = V$. Suppose that $G \in \{G_c(C), G_{ab}(C)\}$. Let $\gamma \in \{c, a\}$. Since $\gamma$ separates $(z, 0)$ and $(z, 1)$, it belongs to $X$. If $\gamma = c$ then $(x, 0)$ separates $\gamma$ and $(z, 0)$, hence belongs to $X$; furthermore, since $(x, 1)$ separates $\gamma$ and $(x, 0)$, it belongs to $X$, proving that $X = V$. If $\gamma = a$ then, since it separates $(z, 0)$ and $(z, 1)$, it belongs to
X. Since \((x, 0)\) separates \(a\) and \(b\) and \((x, 1)\) separates \(a\) and \((x, 0)\), the vertices \((x, 0)\) and \((x, 1)\) belong to \(X\), proving that \(X = V\).

If \(X \neq V\), it follows from Claim 1 and Claim 2 that \(X\) contains at most one element \((x, i)\) of \(E \times \{0, 1\}\). Since \(|X| \geq 2\), this is impossible if \(G = G_d(C)\). Suppose that \(G \in \{G_c(C), G_{ab}(C)\}\) and let \(\gamma \in X \setminus E \times \{0, 1\}\). If \(\gamma \in \{c, b\}\) then \(\gamma\) separates \((x, i)\) and \((x, j)\), with \(j \neq i\), \((x, j) \in X\) and \(\gamma = a\). We may then suppose that \(G = G_{ab}(C)\) and \(\gamma = a\). In this case, \(b\) separates \(a\) and \((x, i)\), \(b \in X\), and we are lead to the previous case, which yield a contradiction. In all cases \(X = V\), hence \(G\) is prime.

\((2)\) The graph \(G\) is minimal prime. Let \(V' \subseteq V\) such that \(|V'| = |V|\) and \(G_{|V'|}\) is prime. Our goal is to define an embedding from \(G\) into \(G'\). Set \(E'(i) := \{x \in E : (x, i) \in V'\}\) for \(i \in \{0, 1\}\). We will define \(f_i : E \rightarrow E'(i)\) \((i < 2)\) such that the map \(F\) defined by \(F(x) := x\) for \(x \in K\) and \(F(x, i) := (f_i(x), i)\) is an embedding. In order to do so, it will be enough that:

- (i) \(f_i(x) < f_j(y)\) for all \(x < y\) and \(i < 2\);
- (ii) \(f_0(x) \leq f_1(y)\) if and only if \(x \leq y\).

Suppose that \(G = G_i(C)\) for \(i \in \{0, 3, 4\}\). Set \(E' := E'(0) \cap E'(1)\). Observe that the symmetric difference \(E'(0) \Delta E'(1)\) has at most two elements. This yields \(|E'| = |E|\). Let \(f : E \rightarrow E'\) be one to one (in this case, we do not need to impose that \(f\) is order preserving). Set \(f_i(x) := f(x)\). Suppose that \(G \in \{G_1(C), G_c(C), G_{ab}(C)\}\). Notice first that if \(G = G_c(C)\) or \(G_{ab}(C)\) then \(G'\) must contain \(c\) or \{\(a, b\}\), hence our goal reduces to define the \(f_i\)'s. To do so, we use some properties of Galois lattices (in order to avoid a transfinite enumeration, which requires care if \(\kappa\) is singular). Let \(R' := (E'(0), \rho, E'(1))\) where \(\rho := \{(x, y) \in E'(0) \times E'(1) : (x, 0) \sim (y, 1)\}\). Since \(G'\) is prime, it is point determining, hence (iii) \(R'(x) \neq R'(x)\) whenever \(x \neq x'\) and similarly (iv) \(R'^{-1}(y) \neq R'^{-1}(y)\) whenever \(y \neq y'\). Next, \(R'\) is Ferrers, hence \(Gal(R')\) is a chain. Due to condition (iii), this chain is isomorphic to a subchain of \(I(C'_0)\); similarly \(Gal(R'^{-1})\) is isomorphic to a subchain of \(F(C'_1)\), where \(C'_1 := C'|_{E_i}\). For the simplicity of the exposition, suppose that \(C\) has order type \(\kappa\). In this case, \(Gal(R')\) is a well ordered chain of order type \(\kappa' + 1\) with \(\kappa' \leq \kappa\). Since \(|V| = \kappa\), \(\kappa' = \kappa\). Let \(f_0\) be the unique order isomorphism from \(C\) onto \(C'_0\). Define \(f_1\) by choosing for \(f_1(y)\) the least element of \(E'_1\) which is greater or equal to \(f_0(y)\).

\(\square\)

**Lemma 25.** Let \(\kappa\) be an infinite initial ordinal and \(C := (E, \leq)\) be a chain of order type \(\kappa\) or \(\kappa^*\). Then:

- (a) \(K(C) \leq K(C)\) and \(G_c(C) \leq G_c(C)\).
- (b) \(K(C) \not\leq K(C^*)\), \(G_c(C) \not\leq G_c(C^*)\) and \(G_{ab}(C) \not\leq G_{ab}(C^*)\).
- (c) \(G_c(C), G_{ab}(C)\) and \(G_{ab}(C)\) form an antichain with respect to embeddability.

**Proof.** (a) We suppose that \(C\) as type \(\kappa\) and in fact is equal to \(\kappa\). Let \(\varphi : E \times \{0, 1\} \rightarrow E \times \{0, 1\}\) defined by \(\varphi(x, 0) := (x, 1)\) and \(f(y, 1) := (y + 1, 0)\). Then \(\varphi\) embeds \(K(C)\) into \(K(C)\). Let \(\tilde{\varphi}\) be defined by \(\tilde{\varphi}((x, i)) := \varphi((x, i))\) and \(\tilde{\varphi}(c) := c\). This map embeds \(G_c(C)\) into \(G_c(C)\).
(b) $N(K(C))$ contains a chain made of cliques of order type $\kappa + 1$. This is not the case for $N(K(C^*))$, hence $K(C) \not\leq K(C^*)$. The rest follows.

(c) Enough to observe that for every pair $(H_i, H_j)$ of distinct graphs in $\{G_e(C), G_{ab}(C), \overline{G}_{ab}(C)\}$ there is a finite graph $H_{ij}$ with $H_{ij} \leq H_i$ and $H_{ij} \not\leq H_j$. A simple inspection shows that this can be achieved with graphs of size at most 6. $\square$

Since $G_e(C) \leq \overline{G}_e(C)$ (cf. Lemma 25), we do not need to add $\overline{G}_e(C)$ to the set $\mathcal{M}$ of graphs listed in Theorem 23. To complete the proof of Theorem 23 we need only to prove that $\mathcal{M}$ forms an antichain with respect to embeddability. We divide it into three subsets, namely $\mathcal{B}$ made of those graphs which are bipartite, $\overline{\mathcal{B}}$ made of the complements of these graphs and $\mathcal{R}$ made of the remaining graphs. Clearly, $\mathcal{B}$ is an antichain, hence $\mathcal{B} \cup \overline{\mathcal{B}}$ is an antichain. Each member of $\mathcal{R}$ is the union of an infinite independent set and an infinite clique (plus, possibly, an extra element), hence is incomparable to all members of $\mathcal{B} \cup \overline{\mathcal{B}}$. To conclude it remains to show that $\mathcal{R}$ is an antichain. Since in $G_4(C)$ the members of the independent set have degree 1, $G_4(C)$ is incomparable to the other members of $\mathcal{R}$. Since the complement of a member of $\mathcal{R}$ embeds into an other member, the same holds true for $\overline{G}_4(C)$. Thus, we are left to show that the six remaining graphs $G_e(C), G_{ab}(C), \overline{G}_{ab}(C)$ for $C \in \{\kappa, \kappa^*\}$ form an antichain. We may apply Lemma 25. But, as we will see in the next section, these graphs are comparability graphs. From their pictorial representation it is easy to see that the twelve transitive orientations of these graphs form an antichain (with respect to the embeddability relation between posets); in particular these graphs form an antichain.

5.3 Examples of minimal prime posets

Let $C := (E, \leq)$ be an infinite chain. The graphs $G_0(C)$ and $G_3(C)$ are comparability graphs, exactly as $G_0$ and $G_3$ are. Indeed, $G_0(C) = \text{Comp}(Q_0(C))$, where $Q_0(C)$ is the set of atoms and coatoms of $\mathcal{P}(E)$ ordered by inclusion, whereas $G_3(C) = \text{Comp}(Q_3(C))$, where $Q_3(C)$ is the set $E \times \{0, 1\}$ augmented of an element $c$ and ordered so that $(x, i) < (x, j)$ if $i = 0, j = 1$ and $c < (x, 1)$ for all $x \in E$. The graphs $G_0(C), G_3(C), G_4(C)$ and $\overline{G}_4(C)$ are not comparability graphs. The graph $G_1(C)$ and its complement are comparability graphs. In fact, $G_1(C) = \text{Comp}(Q_1(C))$, where $Q_1(C)$ is the set $E \times \{0, 1\}$ ordered so that $(x, i) < (y, j)$ if $i = 0, j = 1$ and $x \leq y$, whereas $\overline{G}_1(C) = \text{Comp}(P_1(C))$, where $P_1(C)$ is the set $E \times \{0, 1\}$ ordered so that $(x, i) < (y, j)$ if $i \geq j$ and $x > y$.

Lemma 26. If $\kappa$ is an infinite initial ordinal, the seven posets $Q_0(\kappa), Q_1(\kappa), Q_1(\kappa)^*, Q_3(\kappa), Q_3(\kappa)^*, P_1(\kappa)$ and $P_1(\kappa)^*$ are minimal prime. If $\kappa = \omega$, then with the one way infinite fence $Q_2$ and the transitive orientations $P_2$ and $P_2^*$ of $Q_2$ represented Figure 4, they form an antichain of ten minimal prime posets.

In order to obtain more prime posets, we order $E \times \{0, 1\}$ by setting $(x, i) < (y, j)$ whenever $i = 0$ and $x \leq y$. Let $P_1(C)$ be the poset obtained by adding to $E \times \{0, 1\}$ an extra element $c$ in such a way that $c < (x, 1)$ for all $x \in E$ and $P_{ab}(C)$ be the poset obtained by adding two extra elements $a$ and $b$ to $E \times \{0, 1\}$ in such a way that 1) $(x, 0) < a$
for all $x \in E$ and $2) b < a$. Let $P_{a/b}(C)$ be the poset obtained from $P_{ab}(C)$ by removing the comparabilities $((x,0), (x,1))$ for all $x \in E$ and $(b,a)$ and adding the comparabilities $(x,1) < b$ and $(x,0) < b$ for all $x \in E$. The posets obtained by taking $C$ to be the chain $\omega$ and then its dual $\omega^*$ are represented in Figure 4. As it is easy to check, we have:

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure4}
\caption{Minimal prime posets of dimension 2.}
\end{figure}

**Lemma 27.** $G_c(C) = \text{Comp}(P_c(C))$, $G_{ab}(C) = \text{Comp}(P_{ab}(C))$ and $\overline{G_{ab}(C)}$ is isomorphic to $\text{Comp}(P_{a/b}(C))$, via the map $\varphi$ defined by $\varphi(a) := a$, $\varphi(b) := b$ and $\varphi((x,i)) := (x,i+1)$ (where $0+1 = 1$ and $1+1 = 0$).

Note that, from (a) of Lemma 25, $G_c(C) \leq G_c(C)$, hence $G_c(C)$ is a comparability graph.

**Lemma 28.** The posets $P_c(C)$, $P_{ab}(C)$ and $P_{a/b}(C)$, where $C \in \{\kappa, \kappa^*\}$, are minimal prime of dimension 2. With their dual, they form an antichain of twelve minimal prime graphs.

**Proof.** The fact that they are minimal prime follows from Lemma 24 and Theorem 23. Since the complement of their comparability graph is a comparability graph, they have dimension 2. The fact that they form an antichain follows from Lemma 25 and a careful examination of Figure 4. \qed

### 5.4 Proof of Theorem 9

Let $\kappa$ be an infinite cardinal. Theorem 23 yields fourteen minimal prime graphs which are pairwise incomparable. The inventory made in Lemma 26 and Lemma 28 of those which are comparability graphs yields nineteen minimal prime posets. If $\kappa = \omega$, we may add to the list of minimal prime graphs the infinite fence and its complement and to the list of prime graphs the infinite fence and the two transitive orientations of its complement.
6 Open questions

The countable minimal prime graphs described in Section 5 consist of the $G_i$'s, for $i < 5$, and their complements plus six graphs obtained from $K(\omega)$ and $K(\omega^*)$ by adding one or two vertices.

**Question 2.** Does these sixteen graphs are the only countable minimal prime graphs?

A preliminary question is:

**Question 3.** Does a countable prime graph embedding neither $K(\omega)$ nor $K(\omega^*)$, necessarily embeds one of the graphs $G_i$ or $\overline{G_i}$ for $i < 5$.

In this paper, we have described some countable minimal prime graphs and posets. All our examples, except one, the path, extend to arbitrary infinite cardinality. And so far we have obtained fourteen minimal prime graph in each uncountable cardinality. One could try to characterize minimal prime graphs and posets of any cardinality.

Another possible direction for future research on this subject is the study of minimal prime relational structures. The notion of an autonomous set for general relational structures was introduced by Fraïssé [7] who used the term ”interval” rather than autonomous set. We can therefore define prime relational structures in a similar way as for prime graphs and posets. But it must be noticed that even in the case of directed graphs without circuits the number of those which are countable and minimal prime is at least countable. Moreover there are infinite prime directed graphs without circuits which do not embed a countable minimal prime directed graph. To illustrate observe that all orientations of a one way directed graph are prime. These orientations being coded by an infinite word on a two letter alphabet, the minimal ones are coded by periodic words, whereas those embedding a minimal prime graph are coded by eventually periodic words.

Still, for posets and tournaments, we ask:

**Questions 4.** (a) Does every infinite prime poset, respectively tournament, embeds a countable minimal prime poset, respectively tournament?

(b) Are there only finitely many infinite countable minimal prime posets, respectively tournament?

Some countable minimal tournaments have been identified in [4].

In the special case of posets, Theorems 4 and 5 yield respectively six and four countable prime minimal posets. Among these posets seven have dimension two. Posets depicted in Figures 4 and their dual yield twelve countable minimal prime posets with dimension 2. We do not know whether this list is complete. A preliminary question is this.

**Questions 5.** Do the nineteen posets of dimension 2 mentioned above are the only countable minimal prime posets with dimension 2?
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