An exact sequence for contact- and symplectic homology

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Abstract. A symplectic manifold $W$ with contact type boundary $M = \partial W$ induces a linearization of the contact homology of $M$ with corresponding linearized contact homology $HC(M)$. We establish a Gysin-type exact sequence in which the symplectic homology $SH(W)$ of $W$ maps to $HC(M)$, which in turn maps to $HC(M)$, by a map of degree $−2$, which then maps to $SH(W)$. Furthermore, we give a description of the degree $−2$ map in terms of rational holomorphic curves with constrained asymptotic markers, in the symplectization of $M$.

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1. Introduction

Let $(W, \omega)$ be a compact symplectic manifold with contact type boundary $M := \partial W$. This means that there exists a vector field $X$ defined in a neighbourhood of $M$, transverse and pointing outwards along $M$, and such that

$$\mathcal{L}_X \omega = \omega.$$
Such an $X$ is called a \textbf{Liouville vector field}. The 1-form $\lambda := (\iota_X \omega)|_M$ is a contact form on $M$. We denote by $\xi$ the contact distribution defined by $\lambda$ and we call $(W, \omega)$ a \textbf{filling} of $(M, \xi)$.

We assume throughout the paper that $(W, \omega)$ satisfies the condition

$$\int_{T^2} f^* \omega = 0 \quad \text{for all smooth } f : T^2 \to W, \quad (1)$$

where $T^2$ is the 2-torus. This condition guarantees that the energy of a Floer trajectory (for a definition, see for example [7, Sect. 2]) does not depend on its homology class, but only on its endpoints. Our main class of examples is provided by exact symplectic forms.

Theorem 1 ties together the symplectic homology groups of $(W, \omega)$ and the linearized contact homology groups of $(M, \xi)$. Both these invariants encode algebraically the dynamics of the same vector field, the \textbf{Reeb vector field} $R_\lambda$ defined by $\ker \omega|_M = \langle R_\lambda \rangle$ and $\lambda(R_\lambda) = 1$. But their natures are quite different: the former belongs to the realm of Floer theory [17,32], whereas the latter belongs to the realm of symplectic field theory (SFT) [16]. Our result can be read as a way to make symplectic homology fit into SFT.

Let us introduce some relevant notation. Given a free homotopy class $a$ of loops in $W$ we denote by $SH^a_*(W, \omega)$ the symplectic homology groups of $(W, \omega)$ in the homotopy class $a$. The free homotopy class of the constant loop will be denoted by $0$. We also denote by $SH^+_*(W, \omega)$ the symplectic homology groups in the trivial homotopy class truncated at a small positive value of the action functional. We refer to Sect. 2 for the definitions.

Let $i : M \hookrightarrow W$ be the inclusion. Given a free homotopy class $a$ of loops in $W$ we denote by $i^{-1}(a)$ the set of free homotopy classes in $M$ which are mapped to $a$ via $i$, and we use the convention $i^{-1}(+) := i^{-1}(0)$. We denote by $HC^{-1}_*(M, \xi)$ the linearized contact homology groups of $(M, \xi)$ based on closed Reeb orbits whose free homotopy class belongs to $i^{-1}(a)$. We refer to Sect. 3.1 for the definition.

Both $SH^a_*$ and $HC^{-1}_*(M, \xi)$ are defined over the Novikov ring $\Lambda_\omega$ with $\mathbb{Q}$-coefficients consisting of formal combinations $\lambda := \sum_{A \in H_2(W; \mathbb{Z})} \lambda_A e^A$, $\lambda_A \in \mathbb{Q}$ such that

$$\# \{ A \mid \lambda_A \neq 0, \omega(A) \leq c \} < \infty$$

for all $c > 0$. The multiplication in $\Lambda_\omega$ is given by the convolution product.

We assume the existence of an almost complex structure $J$ such that linearized contact homology is defined. This means that $J$ needs to be regular for rigid holomorphic planes in the symplectic completion of $W$, as well as for rational holomorphic curves with one positive puncture in the symplectization of $M$ satisfying the following property. These curves are asymptotic, at all negative punctures except at most one, to Reeb orbits which can be capped with rigid holomorphic planes in the symplectic completion of $W$. We refer to Sect. 3.1, Remark 9 for a discussion of these regularity assumptions. We expect this technical assumption to be completely removed.
using the new ongoing approach to transversality by Cieliebak and Mohnke (see [13] for the symplectic case), or using the polyfold theory developed by Hofer, Wysocki and Zehnder [18,21].

**Theorem 1.** If $a \neq 0$ or $a = +$ there exists a long exact sequence

$$
\cdots \to SH^a_{k-(n-3)}(W, \omega) \to HC^{i-1}(M, \xi) \xrightarrow{D} HC^{i-1}(M, \xi) \\
\to SH^a_{k-1-(n-3)}(W, \omega) \to \cdots
$$

Moreover, the map $D$ can be described exclusively in terms of rational holomorphic curves with constrained asymptotic markers in the symplectization of $(M, \xi)$, and of rigid holomorphic planes in the symplectic completion of $W$.

The description of the map $D$ is given in Proposition 8 of Sect. 7.2. We emphasize the fact that, in general, the linearized contact homology groups and the map $D$ depend on the filling $(W, \omega)$. The example of Riemann surfaces in Sect. 8.1 shows that this is the case even in the simple situation $M = S^1$. On the other hand, there are classes of contact manifolds for which the linearized contact homology groups and the map $D$ only depend on $(M, \xi)$. This is illustrated in Sect. 8.2 by subcritically Stein fillable contact manifolds $(M, \xi)$ of dimension $\geq 3$ with $c_1(\xi) = 0$.

Besides the Floer/SFT distinction mentioned above, the groups $SH_*$ and $HC_*$ differ in a more subtle way, related to the fact that the loop space naturally carries an $S^1$-action. The construction of contact homology groups is intrinsically $S^1$-equivariant in the sense that the generators of the complex are unparametrized Reeb orbits and $S^1$ acts on the relevant spaces of solutions, whereas the construction of symplectic homology groups is non-equivariant, i.e. the Hamiltonian is time-dependent and the generators of the complex are parametrized Hamiltonian orbits.

One should recall at this point the Gysin exact sequence relating ordinary and $S^1$-equivariant homology of an $S^1$-space $X$, which reads

$$
\cdots \to H_*(X) \to H_*^{S^1}(X) \xrightarrow{e_*} H_{*-2}^{S^1}(X) \to H_{*-1}(X) \to \cdots
$$

(3)

The analogy – modulo shifts in the grading – between (2) and (3) is by no means formal. We prove in [8] that an $S^1$-equivariant version of symplectic homology is isomorphic to linearized contact homology and that (2) is the corresponding Gysin exact sequence, whereas the paper [14] constructs a non-equivariant version of (linearized) contact homology fitting into a Gysin exact sequence with the usual contact homology groups.

From this point of view, contact- and symplectic homology are closely related complementary theories, linked via a Gysin exact sequence. This perspective on symplectic homology also relates to a recent conjecture of Seidel [31] predicting that symplectic homology is isomorphic to the Hochschild homology $HH_*(\mathcal{C})$ of a suitable $A_\infty$-category $\mathcal{C}$. Then
$S^1$-equivariant symplectic homology should be isomorphic to the cyclic homology $HC_\ast(C)$ and the Gysin exact sequences mentioned above should be isomorphic to the standard Connes exact sequence connecting Hochschild and cyclic homology

$$\cdots \to HH_\ast(C) \to HC_\ast(C) \to HC_{\ast-2}(C) \to HH_{\ast-1}(C) \to \cdots$$

As far as the connecting map $D$ is concerned, the analogy with the finite dimensional case is again fertile. It can be thought of as a cap product with an Euler class, just as the map $H^{S^1}_2(X) \to H^{S^1}_{\ast-2}(X)$ in (3) is the cap product with the Euler class of the $S^1$-bundle over the homotopy quotient $X \times_{S^1} ES^1$.

This paper essentially consists of a proof of Theorem 1 and we now give an overview of the proof. We draw the reader’s attention to Sect. 4, where we have concentrated the key statements in rigorous form. The preliminary constructions are given in Sects. 2 and 3.

The main technical tool for our proof is the Morse–Bott complex developed in [7] following ideas from [3]. The construction, which is summarized in Sect. 2.2, gives a recipe to compute the symplectic homology groups in terms of the moduli spaces of Floer trajectories for a time-independent Hamiltonian $H$ under the assumption – generic for autonomous Hamiltonians – that the 1-periodic orbits of the latter are either constant and non-degenerate, or nonconstant and transversally nondegenerate. The Morse–Bott complex mimicks a time-dependent perturbation of $H$ via the choice of a perfect Morse function $f_\gamma$ along the geometric image of each nonconstant orbit $\gamma$, which is a circle. Each (unparametrized) nonconstant orbit $\gamma$ gives rise to two generators in the Morse–Bott complex, one for the minimum and one for the maximum of $f_\gamma$.

For the special type of autonomous Hamiltonians used to define symplectic homology the nonconstant orbits $\gamma$ are in one-to-one correspondence with closed Reeb orbits $\gamma'$ on $M$. A nice feature of the Morse–Bott complex is that it is naturally filtered by the Maslov index $\mu(\gamma')$, and this filtration gives rise to a spectral sequence supported in two lines. As seen in Sect. 7.1, any such spectral sequence gives rise to a long exact sequence of the type

$$\cdots \to E^\infty \to E^2_{k,0} \xrightarrow{d^2} E^2_{k-2,1} \to E^\infty \to \cdots$$

By definition we have $E^\infty \simeq SH$. In order to establish Theorem 1 we prove that $E^2_{k,i} \simeq HC_{k+(n-3), i} = 0, 1$ and identify the differential $d^2$ in the following way. In Sect. 3.2 we define a non-equivariant version of contact homology, inspired by [14], by means of a construction of a Morse–Bott complex which we call the $S^1$-parametrized contact complex, analogous to the one in Sect. 2.2. This complex is also filtered by $\mu(\gamma')$ and the $E^2$-term of the associated spectral sequence is trivially identified with $HC_\ast$. We obtain thus two filtered complexes, one for symplectic homology and another one for contact homology, which we prove in Sect. 6 to be isomorphic. This automatically implies that the $E^2$-terms of the corresponding spectral
sequences are isomorphic, and also that the corresponding $d^2$-differentials coincide.

The isomorphism between the two filtered complexes is established by considering “mixed” moduli spaces consisting of punctured curves defined on the cylinder $\mathbb{R} \times S^1$ and taking values in the symplectization $M \times \mathbb{R}$. Near $-\infty$, these curves are holomorphic and asymptotic to a Reeb orbit; near $+\infty$, they satisfy Floer’s equation and are asymptotic to a 1-periodic orbit of $H$. Since the contact action decreases along such curves the resulting chain map has upper triangular form, and we show, by constructing solutions to the mixed problem described above and showing they are unique, that the entries on the diagonal are $\pm 1$. This method of establishing an isomorphism at the chain level by using mixed moduli spaces is reminiscent of [1].

We note at this point the fact that the Morse–Bott construction of the $S^1$-parametrized contact complex is necessary only in order to identify the differential $d^2$ in the exact sequence. The isomorphism between the $E^1$-term of the symplectic homology spectral sequence and the linearized contact complex can be established directly by using the mixed moduli spaces described above.

Two more remarks are in order. The first one concerns the fact that the Floer trajectories for $H$ might wander deep inside the filling $W$, whereas the isomorphism between the two filtered complexes is constructed in the symplectization $M \times \mathbb{R}$. The basic technique in Sect. 5.2 is to stretch the neck near the boundary of $W$ and show that, when the stretching parameter is large enough, the Floer trajectories in $W$ are in bijective correspondence with punctured Floer trajectories in $M \times \mathbb{R}$, capped at the punctures with rigid holomorphic planes in the symplectization of $W$. The maximum principle plays a crucial role for showing that the limit building contains no curve with more than one positive puncture.

The second remark concerns the problem of good and bad orbits (see Sect. 3.1 for the definition). One of the most pleasant features of the spectral sequences described above is that, although the starting complex contains two generators for each Reeb orbit, only the generators corresponding to good Reeb orbits survive to $E^1$. At this point it is crucial to use $\mathbb{Q}$-coefficients for $\Lambda_\omega$ rather than $\mathbb{Z}$-coefficients. Besides the analysis of signs borrowed from [7], we need to show that, for a suitable choice of the Hamiltonian $H$, certain rigid Floer trajectories do not appear and hence do not contribute to the expression of $d^0$ in the case of symplectic homology (see Sect. 4 for details). This is done in Sect. 5.1 by slowing down the rate of variation of $H$ and using the regularity assumptions on the time-independent almost complex structure $J$ for contact homology.

The paper ends with Sect. 8 in which we treat four examples: Riemann surfaces with one boundary component, subcritical Stein domains, negative disc bundles and unit cotangent bundles.

**Note on pictorial conventions.** We use several different types of moduli spaces in the paper, the most important of which are shown in Fig. 3
on p. 659. There and throughout the paper we use the following conventions (cf. Fig. 1): gradient trajectories of Morse functions are represented by horizontal lines, solutions of Floer’s equation by vertical lines, and holomorphic curves in a symplectization by dashed vertical lines. Vertical dots stand for holomorphic curves in a symplectization going to $\pm \infty$ at a puncture.

Fig. 1. Pictorial conventions

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2. Symplectic homology

2.1. Construction and basic properties of symplectic homology

Let $(W, \omega)$ be a compact symplectic manifold with boundary $M := \partial W$ of contact type, satisfying (1). The Liouville vector field is denoted by $X$, the induced contact form on $M$ is $\lambda$, the contact distribution is $\xi$ and the Reeb vector field is $R_\lambda$.

We now construct the symplectic homology groups of $(W, \omega)$. For more details, we refer to [7, Sect. 2]. Let $\phi$ be the flow of $X$. We parametrize a neighbourhood $U$ of $M$ by $G : M \times [-\delta, 0] \to U$, $(p, t) \mapsto \phi^t(p)$. Then $d(e^t \lambda)$ is a symplectic form on $M \times \mathbb{R}^+$ and $G^* \omega = d(e^t \lambda)$. The symplectic
An exact sequence for contact- and symplectic homology completion \((\hat{W}, \hat{\omega})\) is defined by

\[
\hat{W} := W \bigcup_G M \times \mathbb{R}^+, \quad \hat{\omega} := \begin{cases} \omega, & \text{on } W, \\ d(e^t \lambda), & \text{on } M \times \mathbb{R}^+. \end{cases}
\]

Given a time-dependent Hamiltonian \(H : S^1 \times \hat{W} \to \mathbb{R}\), we define the Hamiltonian vector field \(X^\theta_H\) by

\[
\hat{\omega}(X^\theta_H, \cdot) = dH_\theta, \quad \theta \in S^1 = \mathbb{R}/\mathbb{Z},
\]

where \(H_\theta := H(\theta, \cdot)\). We denote by \(\phi_H\) the flow of \(X^\theta_H\), defined by \(\phi^0_H = \text{Id}\) and

\[
d \frac{d}{d\theta} \phi^\theta_H(x) = X^\theta_H(\phi^\theta_H(x)), \quad \theta \in \mathbb{R}.
\]

We denote by \(\mathcal{P}(H)\) the set of 1-periodic orbits of \(X^\theta_H\). Given a free homotopy class \(a\) of loops in \(W\), we denote by \(\mathcal{P}^a(H)\) the set of 1-periodic orbits of \(X^\theta_H\) in the class \(a\). The symplectic homology groups of \((W, \omega)\) are defined as the direct limit

\[
SH^a_* := \lim_{\to} H \in \mathcal{H} SH^a_*(H).
\]

Here \(\mathcal{H}\) is a suitable class of Hamiltonians and \(SH^a_*(H)\) are the Floer homology groups of \(H\) in the class \(a\). The underlying complex \(SC^a_*(H, J)\) is generated by the elements of \(\mathcal{P}^a(H)\), and the differential \(d\) is defined using a time-dependent almost complex structure \(J\) on \(\hat{W}\) which is regular for \(H\) and has a special behaviour at infinity. The resulting homology groups do not depend on \(J\) and we omit it from the notation.

Let now \(a = 0\) be the trivial homotopy class. In this case we denote the symplectic homology groups by \(SH_*(W, \omega)\). We define the reduced Hamiltonian action functional

\[
\mathcal{A}^0_H : C^\infty_{\text{contr}}(S^1, \hat{W}) \to \mathbb{R}
\]

by

\[
\mathcal{A}^0_H(\gamma) := -\int_{D^2} \sigma^* \hat{\omega} - \int_{S^1} H(\theta, \gamma(\theta)) d\theta.
\]

Here \(C^\infty_{\text{contr}}(S^1, \hat{W})\) denotes the space of smooth contractible loops in \(\hat{W}\) and \(\sigma : D^2 \to \hat{W}\) is a smooth extension of \(\gamma\). Note that \(\mathcal{A}^0_H\) is well-defined thanks to condition (1) and that \(\mathcal{A}^0_H\) is decreasing along Floer trajectories. We define the action spectrum of \((M, \lambda)\) by

\[
\text{Spec}(M, \lambda) := \{ T \in \mathbb{R}^+ \mid \text{there is a closed } R_\lambda\text{-orbit of period } T \}.
\]
We fix $\epsilon > 0$ such that $\epsilon < T$ for all $T \in \text{Spec}(M, \lambda)$. For a regular almost complex structure $J$ we define the chain complexes

$$SC^{-}_*(H, J) := \bigoplus_{\gamma \in P^0(H)} \Lambda_\omega(\gamma) \subset SC_*(H, J)$$  \hspace{1cm} (4)

and

$$SC^+_*(H, J) := SC_*(H) / SC^-_*(H, J).$$

The differential on $SC^\pm_*(H, J)$ is induced by $d$. The groups $SC^\pm_*(H) := H_\ast(SC^\pm_*(H, J), d)$ neither depend on $J$ nor on $\epsilon$. We define $SC^\pm_*(W, \omega) := \lim_{H \to H} SC^\pm_*(H)$. We call $SH^+_*(W, \omega)$ the **positive symplectic homology group** of $(W, \omega)$.

**Remark 1.** For contractible orbits condition (1) can be replaced by the weaker **symplectic asphericity** condition $\langle \omega, \pi_2(W) \rangle = 0$.

Let us assume that $M$ is of **positive contact type** [26, §4.3]. This means that every closed $R_\lambda$-orbit $\gamma$ on $M$ which is contractible in $W$ has positive action $A_\omega(\gamma)$ bounded away from zero, where

$$A_\omega(\gamma) := \int_{D^2} \sigma^* \omega$$

for some extension $\sigma : D^2 \to W$ of $\gamma$. This condition is automatically satisfied if the boundary $M$ is of restricted contact type, i.e. the vector field $X$ is globally defined on $W$. Under the positive contact type assumption we have [32, Proposition 1.4] (see also [26])

$$SH^-_*(W, \omega) = H_{*+n}(W, \partial W; \Lambda_\omega), \quad n = \frac{1}{2} \dim W.$$

Moreover, the short exact sequence of complexes $SC^-_*(H) \to SC_*(H) \to SC^+_*(H)$ induces the long exact sequence [32]

$$\cdots \to SH^+_*(W, \omega) \to H_{*+n}(W, \partial W; \Lambda_\omega) \to SH_*(W, \omega) \to SH^+_*(W, \omega) \to \cdots$$  \hspace{1cm} (5)

2.2. **Morse–Bott description of symplectic homology**

In this section we recall the Morse–Bott formalism of [7]. We assume in this section that the closed $R_\lambda$-orbits are transversally nondegenerate in $M$. We denote by $\phi_\lambda$ the flow of $R_\lambda$. 
In [7, §3] we used a class $\mathcal{H}'$ of admissible Hamiltonians consisting of elements $H : \hat{W} \to \mathbb{R}$ such that

(i) $H|_W$ is a $C^2$-small Morse function and $H < 0$ on $W$;
(ii) $H(p, t) = h(t)$ outside $W$, where $h(t)$ is a strictly increasing function with $h(t) = \alpha e^t + \beta$, $\alpha, \beta \in \mathbb{R}$, $\alpha \notin \text{Spec}(M, \lambda)$ for $t$ bigger than some $t_0$, and such that $h'' - h' > 0$ on $[0, t_0]$.

Note that the 1-periodic orbits of $X_H$ in $W$ are constant and nondegenerate by assumption (i). A direct computation shows that

$$X_H(p, t) = -e^{-t}h'(t)R_\lambda,$$

for $(p, t) \in M \times [0, \infty]$. (6)

The 1-periodic orbits of $X_H$ fall in two classes:

(1) critical points of $H$ in $W$;
(2) nonconstant 1-periodic orbits of $X_h$, located on levels $M \times \{t\}$, $t \in [0, t_0[$, which are in one-to-one correspondence with closed $-R_\lambda$-orbits of period $e^{-t}h'(t)$.

Let $\alpha := \lim_{t \to \infty} e^{-t} H(p, t)$. Let $\mathcal{P}_\lambda$ be the set of closed unparametrized $R_\lambda$-orbits in $M$. We denote by $\mathcal{P}_{\lambda}^{\leq \alpha}$ the set of all $\gamma' \in \mathcal{P}_\lambda$ such that $\mathcal{A}_\lambda(\gamma') \leq \alpha$. Because $H$ is independent of $\theta$, every orbit $\gamma' \in \mathcal{P}_{\lambda}^{\leq \alpha}$ gives rise to a whole circle of nonconstant 1-periodic orbits $\gamma$ of $X_H$, which are transversally nondegenerate and whose parametrizations differ by a shift $\theta \in S^1$. We denote by $S_\gamma$ the set of such orbits, so that $S_\gamma = S_{\gamma(+\cdot \theta)}$ for all $\theta \in S^1$.

Let $\mathcal{J} = \mathcal{J}(\hat{W}, \hat{\omega})$ be the space of $\theta$-dependent almost complex structures $J$ such that

(i) $J$ is compatible with $\hat{\omega}$;
(ii) for $t$ large enough, $J$ is independent of $\theta$;
(iii) $J$ preserves the contact distribution $\xi$;
(iv) $J\frac{\partial}{\partial t} = R_\lambda$.

Given $\mathcal{F}$, $\gamma \in \mathcal{P}(H)$, $\tilde{q} \in \text{Crit}(H)$ and $J \in \mathcal{J}$, we denote by

$$\widehat{\mathcal{M}}^A(S_{\mathcal{F}}, S_\gamma; H, J), \quad \widehat{\mathcal{M}}^A(S_{\mathcal{F}}, \tilde{q}; H, J)$$

the spaces of Floer trajectories for $(H, J)$ starting at $S_{\mathcal{F}}$ and ending at $S_\gamma$ or $\tilde{q}$, respectively. Such a Floer trajectory is a map $u : \mathbb{R} \times S^1 \to \hat{W}$ satisfying

$$\partial_s u + J(\partial_\theta u - X_H) = 0 \quad \text{for all } (s, \theta) \in \mathbb{R} \times S^1,$$

as well as the conditions

$$\lim_{s \to -\infty} u(s, \cdot) \in S_{\mathcal{F}}$$

(8)
and, respectively,
\[
\lim_{s \to \infty} u(s, \cdot) \in S'_\gamma \quad \text{or} \quad \lim_{s \to \infty} u(s, \cdot) = \tilde{q}.
\]  
(9)

The *Morse–Bott moduli spaces of Floer trajectories* are defined by
\[
\mathcal{M}^A(S\gamma, S'_\gamma; H, J) := \hat{\mathcal{M}}^A(S\gamma, S'_\gamma; H, J)/\mathbb{R}
\]
and
\[
\mathcal{M}^A(S\gamma, \tilde{q}; H, J) := \hat{\mathcal{M}}^A(S\gamma, \tilde{q}; H, J)/\mathbb{R}.
\]

Let \( \mathcal{J}_{\text{reg}}(H) \subset \mathcal{J} \) be the set of those almost complex structures for which the linearization of (7) at its solutions is surjective, so that \( \mathcal{J}_{\text{reg}}(H) \) is dense in \( \mathcal{J} \) [7, Proposition 3.5]. Given \( J \in \mathcal{J}_{\text{reg}}(H) \) the Morse–Bott moduli spaces of Floer trajectories are smooth manifolds, and their respective dimensions are
\[
\dim \mathcal{M}^A(S\gamma, S'_\gamma; H, J) = \mu(\gamma) - \mu(\gamma') + 2(c(1(TW), A)),
\]
\[
\dim \mathcal{M}^A(S\gamma, \tilde{q}; H, J) = \mu(\gamma) - \mu(\tilde{q}) + 2(c(1(TW), A)).
\]  
(10)

Here \( \mu(\tilde{q}) = \text{ind}(\tilde{q}; -H) - n \) is the Conley–Zehnder index of the constant orbit \( \tilde{q} \), whereas \( \mu(\gamma), \mu(\gamma') \) denote the Conley–Zehnder indexes of the linearized Hamiltonian flows restricted to \( \xi \).

We have natural evaluation maps
\[
\overline{ev} : \mathcal{M}^A(S\gamma, S'_\gamma; H, J) \to S\gamma, \quad \underline{ev} : \mathcal{M}^A(S\gamma, S'_\gamma; H, J) \to S'_\gamma
\]
and
\[
\overline{ev} : \mathcal{M}^A(S\gamma, \tilde{q}; H, J) \to S\gamma,
\]

defined by
\[
\overline{ev}([u]) := \lim_{s \to -\infty} u(s, \cdot), \quad \underline{ev}([u]) := \lim_{s \to \infty} u(s, \cdot).
\]

For each \( S_\gamma, \gamma \in \mathcal{P}(H) \) we choose a Morse function \( f_{S_\gamma} : S_\gamma \to \mathbb{R} \) with exactly one maximum \( M \) and one minimum \( m \). To simplify notation, we shall write in the sequel \( f_\gamma \) instead of \( f_{S_\gamma} \), so that \( f_\gamma = f_\gamma(\cdot + \theta) \) for all \( \theta \in S^1 \). We denote by \( \gamma_m, \gamma_M \) the orbits in \( S_\gamma \), starting at the minimum and the maximum of \( f_\gamma \) respectively. For a generic choice of these Morse functions [7, Lemma 3.6], all the maps \( \overline{ev} \) are transverse to the unstable manifolds \( W^u(p), p \in \text{Crit}(f_\gamma) \), all the maps \( \underline{ev} \) are transverse to the stable manifolds \( W^s(p), p \in \text{Crit}(f_\gamma) \) and all pairs
\[
(\overline{ev}, \underline{ev}) : \mathcal{M}^A(S\gamma, S'_\gamma; H, J) \to S\gamma \times S'_\gamma,
\]
\[
(\overline{ev}, \underline{ev}) : \mathcal{M}^{A_1}(S\gamma, S_{\gamma_1}; H, J) \times \overline{ev} \times \underline{ev} \mathcal{M}^{A_2}(S_{\gamma_1}, S'_\gamma; H, J) \to S\gamma \times S'_\gamma
\]  
(11)
are transverse to products $W^u(p) \times W^s(q)$, $p \in \text{Crit}(f_\gamma)$, $q \in \text{Crit}(f_\gamma)$. The unstable and stable manifolds are understood with respect to $\nabla f_\gamma$, so that $W^u(M) = \{ M \}, W^s(M) = S_\gamma \setminus \{ m \}$, $W^u(m) = S_\gamma \setminus \{ M \}$ and $W^s(m) = \{ m \}$. We denote by $\mathcal{F}_{\text{reg}}(H, J)$ the set consisting of collections $\{ f_\gamma \}$ of Morse functions that satisfy the above transversality conditions.

Let now $J \in \mathcal{F}_{\text{reg}}(H)$ and $\{ f_\gamma \} \in \mathcal{F}_{\text{reg}}(H, J)$. For $p \in \text{Crit}(f_\gamma)$ we denote the Morse index by

$$\text{ind}(p) := \dim W^u(p; \nabla f_\gamma).$$

Let $\gamma, \gamma' \in \mathcal{P}(H)$ and $p \in \text{Crit}(f_\gamma), q \in \text{Crit}(f_\gamma')$. For $m \geq 0$ we denote by

$$\mathcal{M}_m^A(p, q; H, \{ f_\gamma \}, J)$$

the union for $\widetilde{\gamma}_1, \ldots, \widetilde{\gamma}_{m-1} \in \mathcal{P}(H)$ and $A_1 + \cdots + A_m = A$ of the fibered products

$$W^u(p) \times_{\mathcal{P}(H)} \big( \mathcal{M}_1^{A_1}(S_{\gamma_1}, S_{\gamma_1'}) \times \mathbb{R}^+ \big) \times_{\mathcal{P}(H)} \cdots \times_{\mathcal{P}(H)} \big( \mathcal{M}_m^{A_m}(S_{\gamma_{m-1}}, S_{\gamma'}) \times \mathbb{R}^+ \big)$$

This is a smooth manifold of dimension

$$\dim \mathcal{M}_m^A(p, q; H, \{ f_\gamma \}, J) = \mu(\gamma) + \text{ind}(p) - \mu(\gamma') - \text{ind}(q) + 2\langle c_1(TW), A \rangle - 1,$$

as shown in [7]. Note that $\mathcal{M}_0^A(p, q; H, \{ f_\gamma \}, J)$ is a submanifold of $\mathcal{M}(S_{\gamma}, S_{\gamma'}; H, J)$. We denote

$$\mathcal{M}(p, q; H, \{ f_\gamma \}, J) := \bigcup_{m \geq 0} \mathcal{M}_m^A(p, q; H, \{ f_\gamma \}, J)$$

and we call this the moduli space of Morse–Bott broken trajectories, whereas $\mathcal{M}_m^A(p, q; H, \{ f_\gamma \}, J)$ is called the moduli space of Morse–Bott broken trajectories with $m$ sublevels. We refer to Fig. 3a on p. 659 for a visual representation of the elements of these moduli spaces.

Although in the sequel we use only 0-dimensional moduli spaces $\mathcal{M}(p, q; H, \{ f_\gamma \}, J)$, we give now a brief description of the topology of the compactification $\overline{\mathcal{M}(p, q; H, \{ f_\gamma \}, J)}$ in the general case. We first start with the compactification $\overline{\mathcal{M}_m^A(p, q; H, \{ f_\gamma \}, J)}$. There are three types of codimension 1 degeneracies for sequences $u_n$ of elements in $\mathcal{M}_m^A(p, q; H, \{ f_\gamma \}, J)$. Firstly, one of the $m$ Floer trajectories composing $u_n$ can break in two Floer trajectories as $n \to \infty$. Secondly, the flow time of one of the $m - 1$ finite gradient trajectories can shrink to 0. Thirdly, one of the $m + 1$ gradient trajectories can break in two gradient trajectories. Higher codimension degeneracies of sequences of elements of $\mathcal{M}_m^A(p, q; H, \{ f_\gamma \}, J)$ are obtained by combining the above three types of degeneracies. The space $\overline{\mathcal{M}(p, q; H, \{ f_\gamma \}, J)}$ is obtained by gluing the moduli spaces $\overline{\mathcal{M}_m^A(p, q; H, \{ f_\gamma \}, J)}$ along their common boundary strata.

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involving only degenerations of the first two types. For simplicity, we just describe the case of codimension 1 boundary strata: these are common boundary strata for $\overline{M}_m^A(p, q; H, \{f_\gamma\}, J)$ and $\overline{M}_m^A(p, q; H, \{f_\gamma\}, J)$ and correspond to degenerations of the first and second type respectively. The boundary (and the corners) of $\overline{M}_m^A(p, q; H, \{f_\gamma\}, J)$ correspond to at least one degeneracy of the third type.

Given $\overline{\gamma} \in \overline{\mathcal{P}}(H)$, $p \in \text{Crit}(f_{\overline{\gamma}})$, $\widetilde{q} \in \text{Crit}(H)$, we define the moduli spaces $\mathcal{M}_m^A(p, \widetilde{q}; H, \{f_\gamma\}, J)$, $m \geq 0$ of Morse–Bott broken trajectories by replacing the last term $\mathcal{M}_m^A(S_{\overline{\gamma}_m-1}, S_\gamma)$ ev $W^s(q)$ in (12) with $\mathcal{M}_m^A(S_{\overline{\gamma}_m-1}, \widetilde{q}; H, J)$. The union over $m \geq 0$ of these spaces is denoted by $\mathcal{M}_m^A(p, \widetilde{q}; H, \{f_\gamma\}, J)$. This is again well defined as a smooth manifold of dimension

$$\dim \mathcal{M}_m^A(p, \widetilde{q}; H, \{f_\gamma\}, J) = \mu(\overline{\gamma}) + \text{ind}(p) - \text{ind}(\widetilde{q}; -H) + n + 2(c_1(TW), A) - 1.$$ 

Again, $\mathcal{M}_m^A(p, \widetilde{q}; H, \{f_\gamma\}, J)$ is naturally a submanifold of the space $\mathcal{M}_m^A(S_{\overline{\gamma}}, \widetilde{q}; H, J)$.

**Remark 2.** Since $H$ is $C^2$-small, the moduli spaces of Floer trajectories $\mathcal{M}_m^A(p, \widetilde{q}; H, J)$, $p, \widetilde{q} \in \text{Crit}(H)$ of expected dimension

$$\text{ind}(p; -H) - \text{ind}(\widetilde{q}; -H) + 2(c_1(TW), A)) - 1 = 0$$

consist exclusively of gradient trajectories of $H$ in $W$ [19, Theorem 6.1] (see also [29, Theorem 7.3]). As a consequence, these moduli spaces are empty whenever $A \neq 0$.

For each $[u] \in \mathcal{M}_m^A(p, q; H, \{f_\gamma\}, J)$ or $[u] \in \mathcal{M}_m^A(p, \widetilde{q}; H, \{f_\gamma\}, J)$ we have defined in [7] a sign $\bar{\epsilon}(u)$. Let $a$ be a free homotopy class of loops in $W$. We define the **Morse–Bott chain groups** by

\begin{align}
BC^a_*(H) := & \bigoplus_{\gamma \in \overline{\mathcal{P}}^a(H)} \Lambda_{\omega} \langle \gamma_m, \gamma_M \rangle, \quad a \neq 0, \\
BC^0_*(H) := & \bigoplus_{p \in \text{Crit}(H)} \Lambda_{\omega} \langle \widetilde{p} \rangle \oplus \bigoplus_{\gamma \in \overline{\mathcal{P}}^0(H)} \Lambda_{\omega} \langle \gamma_m, \gamma_M \rangle.
\end{align}

The grading is defined by

$$|e^A \widetilde{p}| := \text{ind}(\widetilde{p}; -H) - n - 2(c_1(TW), A),$$

$$|e^A \gamma_m| := \mu(\gamma) + 1 - 2(c_1(TW), A),$$

$$|e^A \gamma_M| := \mu(\gamma) - 2(c_1(TW), A).$$

We define the **Morse–Bott differential**

$$d : BC^a_*(H) \to BC^a_{*-1}(H)$$
by

\[ d\widehat{p} := \sum_{\tilde{q} \in \text{Crit}(H)} \sum_{[\tilde{p}] - [\tilde{q}] = 1} \tilde{\epsilon}(u)\tilde{q}, \quad (16) \]

\[ dy_p := \sum_{\tilde{q} \in \text{Crit}(H)} \sum_{[\gamma_p] - |e^A\tilde{q}| = 1} \tilde{\epsilon}(u)e^A\tilde{q} \]

\[ + \sum_{\gamma \in \mathcal{P}(H), q \in \text{Crit}(f_\gamma)} \sum_{[u] \in \mathcal{M}^A(p,q;H,\{f_\gamma\},J)} \tilde{\epsilon}(u)e^A\gamma_q, \quad p \in \text{Crit}(f_\gamma). \quad (17) \]

The sums (16) and (17) clearly involve only periodic orbits in the same free homotopy class as that of \( \widehat{p} \) or \( y_p \) respectively.

The Correspondence Theorem 3.7 in [7] implies \( d \circ d = 0 \) and

\[ \lim_{H \to H'} H_*(BC_*^a(H), d) = SH_*^a. \]

We shall denote in the sequel

\[ SH_*^a(H, J) := H_*(BC_*^a(H), d). \]

Moreover, if we define the subcomplex

\[ BC_*^{-}(H) := \bigoplus_{\widehat{p} \in \text{Crit}(H)} \Lambda_\omega\langle \widehat{p} \rangle \]

and the quotient

\[ BC_*^{+}(H) := BC_*^0(H)/BC_*^{-}(H), \]

we have

\[ \lim_{H \to H'} H_*(BC_*^{+}(H), d) = SH_*^{+}(W, \omega). \]

**Notation.** From now on the letter \( a \) will denote either a free homotopy class in \( W \) or one of the symbols \( \pm \). The notation \( i^{-1}(\pm) \) stands for \( i^{-1}(0) \).

The previous description of the Floer differential can be generalized to the case of an \( s \)-dependent family of autonomous Hamiltonians. More precisely, let \( H_s, s \in \mathbb{R} \) be a homotopy of autonomous Hamiltonians satisfying the following conditions:

(i) \( H_s \) is increasing with respect to \( s \), and constant for \( |s| \) large enough;

(ii) \( H_s(p, t) = h_s(t) \) outside \( W \), where \( h_s \) is a strictly increasing function with \( h_s(t) = \alpha(s)e^t + \beta(s) \) for \( t \) bigger than some \( t_0 \);

(iii) \( H_{\pm} := \lim_{s \to \pm\infty} H_s \) belong to the class \( \mathcal{H}' \) of admissible Hamiltonians.
We call this an admissible homotopy of Hamiltonians. Similarly, we define an admissible homotopy of almost complex structures to be a family $J_s$, $s \in \mathbb{R}$ of elements of $\mathcal{J}$, which is constant for $|s|$ large enough.

Given $A \in H_2(W, \mathbb{Z})$, $\gamma \in \mathcal{P}(H_-)$, $\gamma_+ \in \mathcal{P}(H_+)$, and $\tilde{q} \in \text{Crit}(H_+)$, we denote by
\[ \mathcal{M}^A(S_\gamma, S_{\gamma_+}; H_s, J_s), \quad \mathcal{M}^A(S_\gamma, \tilde{q}; H_s, J_s) \]
the spaces of Floer trajectories $u : \mathbb{R} \times S^1 \rightarrow \tilde{W}$ satisfying
\[ \partial_s u + J_s \partial_\theta u = X_{H_s}(u) \quad \text{for all } (s, \theta) \in \mathbb{R} \times S^1, \quad (18) \]
as well as conditions (8) and (9). Since (18) is $s$-dependent, the additive group $\mathbb{R}$ does not act on the spaces of solutions, and we shall refer to these as the Morse–Bott moduli spaces of $s$-dependent Floer trajectories. One shows as in [7, Sect. 4.1] that, for a generic choice of the homotopy $J_s$, these are smooth manifolds whose respective dimensions are given by (10).

Let us now choose for each $\gamma_i$, $\gamma \in \mathcal{P}(H_-)$ a perfect Morse function $f_\gamma^\pm : S_\gamma \rightarrow \mathbb{R}$. Given $\gamma \in \mathcal{P}(H_-)$, $\gamma_+ \in \mathcal{P}(H_+)$, $p \in \text{Crit}(f_\gamma^-$), $q \in \text{Crit}(f_\gamma^+)$, $m \geq 1$, and $i \in \{1, \ldots, m\}$, we define
\[ \mathcal{M}^A_m(p, q; H_s, \{f_\gamma^\pm\}, J_s) \]
as the (disjoint) union over $\gamma_1, \ldots, \gamma_{i-1} \in \mathcal{P}(H_-)$, $\gamma_i, \ldots, \gamma_{m-1} \in \mathcal{P}(H_+)$, and $A_1 + \cdots + A_m = A$ of the following fibered products (with the convention $\gamma_0 := \gamma, \gamma_m := \gamma$)
\[ W^u(p) \times_{\mathcal{E}V} \left( \mathcal{M}^{A_1}(S_{\gamma_0}, S_{\gamma_1}; H_-, J_-) \times \mathbb{R}^+ \right) \]
\[ \varphi_{f^+_{\gamma_{i-1}}} \circ \mathcal{E}V \times \mathcal{E}V \quad \cdots \quad \varphi_{f^+_{\gamma_{i+1}}} \circ \mathcal{E}V \times \mathcal{E}V \left( \mathcal{M}^{A_i-1}(S_{\gamma_{i-2}}, S_{\gamma_{i-1}}; H_-, J_-) \times \mathbb{R}^+ \right) \]
\[ \varphi_{f^-_{\gamma_{i-1}}} \circ \mathcal{E}V \times \mathcal{E}V \left( \mathcal{M}^{A_i^1}(S_{\gamma_{i-1}}, S_{\gamma_i}; H_s, J_s) \times \mathbb{R}^+ \right) \]
\[ \varphi_{f^+_{\gamma_i}} \circ \mathcal{E}V \times \mathcal{E}V \left( \mathcal{M}^{A_i+1}(S_{\gamma_i}, S_{\gamma_{i+1}}; H_+, J_+) \times \mathbb{R}^+ \right) \]
\[ \varphi_{f^+_{\gamma_{i+1}}} \circ \mathcal{E}V \times \mathcal{E}V \quad \cdots \quad \varphi_{f^+_{\gamma_m}} \circ \mathcal{E}V \times \mathcal{E}V \mathcal{M}^{A_m}(S_{\gamma_{m-1}}, S_{\gamma_m}; H_+, J_+) \mathcal{E}V \times W^s(q). \]

We note the similarity to the fibered product defining (12), with the term $\mathcal{M}^{A_i}(S_{\gamma_{i-1}}, S_{\gamma_i}; H, J)$ being replaced by $\mathcal{M}^{A_i}(S_{\gamma_{i-1}}, S_{\gamma_i}; H_s, J_s)$. One shows as in [7, Lemma 3.6] that, for a generic choice of the collection of Morse functions $\{f_\gamma^\pm\}$, each space $\mathcal{M}^A_m(p, q; H_s, \{f_\gamma^\pm\}, J_s)$ is a smooth manifold of dimension
\[ \dim \mathcal{M}^A_m(p, q; H_s, \{f_\gamma^\pm\}, J_s) = \mu(\gamma) + \text{ind}(p) - \mu(\gamma_0) - \text{ind}(q) + 2(c_1(TW), A). \]
The union over $m \geq 1$ of the spaces $\mathcal{M}^A_m(p, q; H_s, \{f_\gamma^\pm\}, J_s)$ is denoted by $\mathcal{M}^A(p, q; H_s, \{f_\gamma^\pm\}, J_s)$ and is called the moduli space of $s$-dependent
Morse–Bott broken trajectories. The topology of the compactification $\mathcal{M}^A(p, q; H_s, \{f^\pm_r\}, J_s)$ is described similarly to the $s$-independent case.

An increasing homotopy as above defines a continuation morphism

$$\sigma_{H_+, H_-} : BC^a_*(H_-) \to BC^a_*(H_+),$$

which preserves the degree and is obtained by a count of rigid configurations in $\mathcal{M}^A(p, q; H_s, \{f^\pm_r\}, J_s)$. Via the identification proved in [7, Theorem 3.7] between the Morse–Bott complexes of $H_{\pm}$ and the Floer complexes of suitable time-dependent perturbation of these Hamiltonians, the continuation morphism $\sigma_{H_+, H_-}$ coincides with the usual continuation morphism in Floer homology [17]. This can be seen for example by repeating the gluing arguments of [7] in the context of $s$-dependent families.

3. Contact homology

3.1. Linearized contact homology

In this section we define linearized contact homology of the contact manifold $(M, \xi)$ with the symplectic filling $(W, \omega)$ following [4].

For each free homotopy class of loops $b$ in $M$ we denote by $\mathcal{P}_\lambda^b$ the set of all $\gamma' \in \mathcal{P}_\lambda$ in the homotopy class $b$. The inclusion $i : M \hookrightarrow W$ induces a map (still denoted by $i$) between the sets of free homotopy classes of loops in $M$ and $W$ respectively. For each free homotopy class $a$ in $W$ we denote

$$\mathcal{P}_\lambda^{-1}(a) := \bigcup_{b \in i^{-1}(a)} \mathcal{P}_\lambda^b.$$

We assume in this section that all the closed Reeb orbits are transversally nondegenerate in $M$. This means that, for every orbit $\gamma'$ of period $T > 0$, we have

$$\det \left( 1 - d\phi^T_\lambda (\gamma'(0)) |_{\xi} \right) \neq 0.$$

This can always be achieved by an arbitrarily small perturbation of $\lambda$ or, equivalently, of $X$, which does not change the symplectic homology groups. In this situation one can assign to each $\gamma' \in \mathcal{P}_\lambda$ a Conley–Zehnder index $\mu_{CZ}(\gamma')$ according to the following recipe.

We fix a reference loop $l_a : S^1 \to \hat{W}$ for each free homotopy class $a$ in $\hat{W}$ such that $[l_a] = a$. If $a$ is the trivial homotopy class we choose $l_a$ to be a constant loop, and we require that $l_a^{-1}$ coincides with the loop $l_a$ with the opposite orientation. We also choose symplectic trivializations $\Phi_a : S^1 \times \mathbb{R}^{2n} \to l_a^* T\hat{W}$ for each class $a$. If $a$ is the trivial homotopy class we choose the trivialization to be constant, and we require that $\Phi_{a^{-1}}(\theta, \cdot) = \Phi_a(-\theta, \cdot)$, $\theta \in S^1 = \mathbb{R}/\mathbb{Z}$. 

We fix a reference loop \( l_b : S^1 \to M \) for each free homotopy class \( b \) in \( M \) such that \([l_b] = b\). If \( b \) is the trivial homotopy class we choose \( l_b \) to be a constant loop and we require that \( l_{b^{-1}} \) coincides with \( l_b \) with the opposite orientation. We define symplectic trivializations

\[
\Phi_b : S^1 \times \mathbb{R}^{2n-2} \to l_b^n \xi
\]

as follows. For each class \( b \) we choose a homotopy \( h_{ab} : S^1 \times [0, 1] \to W \) from \( l_a, a = i(b) \) to \( l_b \) such that

\[
h_{a^{-1}b^{-1}}(\tau, \cdot) = h_{ab}(-\tau, \cdot). \tag{19}
\]

We extend the trivialization \( \Phi_a : S^1 \times \mathbb{R}^{2n} \to l_a^*T\tilde{W} \) over the homotopy \( h_{ab} \) to get a trivialization \( \Phi'_b : S^1 \times \mathbb{R}^{2n} \to l_b^*T\tilde{W} \). This trivialization is homotopic to another one, still denoted \( \Phi'_b \), such that

\[
\Phi'_b(S^1 \times \{0\} \times \mathbb{R}) = l_b^*\langle R_\lambda \rangle.
\]

We define \( \Phi_b := \Phi'_b|_{S^1 \times \mathbb{R}^{2n-2} \times \{0\} \times \{0\}} \). If \( b \) is the trivial homotopy class we choose \( h_{ab} \) to be a path of constant loops, so that \( \Phi_b \) is constant.

We fix for each \( \gamma' \in \mathcal{P}_\lambda \) a map \( \sigma_{\gamma'} : \Sigma \to M \), with \( \Sigma \) a Riemann surface with two boundary components \( \partial_0 \Sigma \) (with the opposite boundary orientation) and \( \partial_1 \Sigma \) (with the boundary orientation), satisfying

\[
\sigma|_{\partial_0 \Sigma} = l_{|\gamma'|}, \quad \sigma|_{\partial_1 \Sigma} = \gamma'. \tag{21}
\]

For each \( \gamma' \in \mathcal{P}_\lambda \) there exists a unique (up to homotopy) trivialization

\[
\Phi : \Sigma \times \mathbb{R}^{2n-2} \to \sigma_{\gamma'}^*\xi
\]

such that \( \Phi = \Phi_{\gamma'}|_{\partial_0 \Sigma \times \mathbb{R}^{2n-2}} \). Let

\[
\Psi : [0, T] \to \text{Sp}(2n-2), \quad \Psi(\tau) := \Phi^{-1} \circ d\phi^\tau_\lambda(p) \circ \Phi, \quad p \in \gamma'([0, T]). \tag{22}
\]

Because \( \gamma' \) is nondegenerate we can define the **Conley–Zehnder index** \( \mu(\gamma') \) by

\[
\mu(\gamma') := \mu(\gamma', \sigma_{\gamma'}) := \mu_{\text{CZ}}(\Psi), \tag{23}
\]

where \( \mu_{\text{CZ}}(\Psi) \) is the Conley–Zehnder index of a path of symplectic matrices [27].
Remark 3. Given $B \in H_2(M; \mathbb{Z})$, we define a map $\sigma_{\gamma} \# B$ up to homology as the connected sum of $\sigma_{\gamma}$ with a surface representing $J'_\partial$ up to homology $= \gamma'_i$. The positive $|\times M$ and $\gamma$ on $\times B$ which are not bad are called $T = \mu(\gamma_B(\lambda))$ $k_2$ are $-L$ punctured $A(24)$ $|M_2 + \gamma = : = \gamma_\xi$ is $\xi$ denote the set of $B$ and $p$ good orbits $' \mu(\gamma \times \lambda))$ is any compatible complex structure on the symplectic $' \gamma$ in $A$ admissible almost complex (which is independent of $' \gamma$ of $\gamma$ reduced Conley–Zehnder index. The grading is given by $B$. We define a grading on $\sigma_{\gamma}$ which are not bad are called good orbits. We define a grading on the space of loops $a$ in $W$, the contact chain group with coefficients in $\Lambda_\omega$ is denoted by $C_*^{\Lambda_\omega}(\lambda)$ and is defined as the free $\Lambda_\omega$-module generated by all good orbits $' \gamma \in \mathcal{P}_\lambda$ and is well-defined independently of the trivialization of $\xi$ along $' \gamma$.

For each simple orbit $' \gamma \in \mathcal{P}_\lambda$ we denote by $' \gamma^k, k \in \mathbb{Z}^+$ its positive iterates. The parity of the Conley–Zehnder index of all the odd, respectively even iterates is the same. If these two parities differ we say that all even iterates $' \gamma^{2k}, k \in \mathbb{Z}^+$ are bad orbits. It can be seen that the even iterates of a simple orbit $' \gamma$ of period $T$ are bad if and only if $d\phi_1^T(p)|_\xi, p \in ' \gamma([0, T])$ has an odd number of real negative eigenvalues strictly smaller than $-1$. The orbits in $\mathcal{P}_\lambda$ which are not bad are called good orbits.

We define a grading on $\Lambda_\omega$ by $|e^A| := -2\langle c_1(TW), A \rangle$. Note that, if $A = i_*(B), B \in H_2(M; \mathbb{Z})$ then $|e^A| = -2\langle c_1(\xi), B \rangle$. For each free homotopy class of loops $a$ in $W$, the contact chain group with coefficients in $\Lambda_\omega$ is denoted by $C_*^{\Lambda_\omega}(\lambda)$ and is defined as the free $\Lambda_\omega$-module generated by all good orbits $' \gamma \in \mathcal{P}_\lambda$. The grading is given by

$$|e^A' \gamma| := \mu(' \gamma) - 2\langle c_1(TW), A \rangle + n - 3.$$ 

We define the reduced Conley–Zehnder index $\overline{\mu}(\gamma) := \mu(\gamma) + n - 3$, so that the grading is $|e^A' \gamma| = \overline{\mu}(\gamma) - 2\langle c_1(TW), A \rangle$.

We call the symplectic manifold $(M \times \mathbb{R}, d(e^\lambda))$ the symplectization of $(M, \xi)$. Its symplectomorphic type does not depend on $\lambda$, but only on the isotopy class of $\xi$. Let $J(\lambda)$ denote the set of admissible almost complex structures on $M \times \mathbb{R}$, consisting of elements $J_\infty$ satisfying

$$\begin{align*}
J_{\infty}(p, t)|_{\xi} &= J_0, \\
J_{\infty}(p, t) \frac{\partial}{\partial t} &= R_\lambda
\end{align*}$$

on $M \times \mathbb{R}$. Here $J_0$ is any compatible complex structure on the symplectic bundle $(\xi, d\lambda)$ which is independent of $\theta$ and $t$.

From now on, we choose for each $' \gamma \in \mathcal{P}_\lambda$ a point $P_{' \gamma}$ on the geometric image of $' \gamma$.

Let $J_{\infty} \in J(\lambda), \overline{\gamma}', \gamma'_1, \ldots, \gamma'_k \in \mathcal{P}_\lambda$ and $B \in H_2(M; \mathbb{Z})$. We define the space

$$\hat{M}^B(\overline{\gamma}', \gamma'_1, \ldots, \gamma'_k; J_{\infty})$$

of punctured $J_{\infty}$-holomorphic cylinders as the set of tuples

$$(F, \overline{L}, \underline{L}, L_1, \ldots, L_k),$$
consisting of a solution
\[ F = (f, a) : \mathbb{R} \times S^1 \setminus \{z_1, \ldots, z_k\} = \mathbb{CP}^1 \setminus \{0, \infty, z_1, \ldots, z_k\} \to M \times \mathbb{R} \]
of the Cauchy–Riemann equation
\[ \partial_s F + J_\infty \partial_\theta F = 0, \quad (26) \]
and of half-lines \( L_i \subset T_{z_i}(\mathbb{R} \times S^1), \overline{L} \subset T_0\mathbb{CP}^1, \underline{L} \subset T_\infty\mathbb{CP}^1, \) subject to the asymptotic conditions
\[ \lim_{s \to \pm \infty} a(s, \theta) = \mp \infty, \quad (27) \]
\[ \lim_{s \to -\infty} f(s, \theta) = \overline{\gamma}(\overline{-T}\theta), \quad \lim_{s \to +\infty} f(s, \theta) = \underline{\gamma}(\underline{-T}\theta), \quad (28) \]
uniformly in \( \theta, \) and
\[ \lim_{z \to z_i, z \in L_i} f(z) = P_{\overline{\gamma}'}, \quad \lim_{z \to \infty, z \in \underline{L}} f(z) = P_{\underline{\gamma'}}. \quad (29) \]
Moreover, we require that, given polar coordinates \((\rho_i, \theta_i) \in ]0, 1] \times \mathbb{R}/\mathbb{Z}\) around \( z_i, \) we have
\[ \lim_{z \to z_i} a(z) = -\infty, \quad \lim_{\rho_i \to 0} f(\rho_i, \theta_i) = \gamma'_i(T_i\theta_i), \quad i = 1, \ldots, k, \quad (30) \]
\[ \lim_{z \to z_i, z \in L_i} f(z) = P_{\gamma'_i}, \quad i = 1, \ldots, k. \quad (31) \]
In addition, we require that
\[ [(\sigma_{\overline{\gamma}'} \cup \sigma_{\gamma'_1} \cup \cdots \cup \sigma_{\gamma'_k})#f] = [\sigma_{\gamma'}#B]. \quad (32) \]
These convergence conditions are to be understood as holding for some \( R_\lambda\)-parametrized representatives of \( \gamma'_i, \) respectively \( \overline{\gamma}', \, \underline{\gamma}'. \) By the conditions \( z \in L_i \) we mean that \( z \) belongs to some curve with endpoint \( z_i \) and asymptotically tangent to \( L_i \) (similarly for \( \overline{L}, \underline{L} \)). The half-lines \( \overline{L}, \underline{L}, L_1, \ldots, L_k \) are called asymptotic markers. We refer to Fig. 3b on p. 659 for a representation of these objects (see also the notion of a capped punctured \( J\)-holomorphic cylinder at the end of this section).

**Notational convention.** To simplify, we shall use the shorthand notation \( F \in \hat{\mathcal{M}}^B(\overline{\gamma}', \gamma'_1, \ldots, \gamma'_k; J_\infty) \) for a tuple \((F, \overline{L}, \underline{L}, L_1, \ldots, L_k)\). The same convention applies for all subsequent (moduli) spaces.

**Remark 4.** Our convention for the asymptotic behaviour is different from the usual one in contact homology and is motivated by the usual conventions for symplectic homology.

**Remark 5.** Under the nondegeneracy assumption on \( \overline{\gamma}', \gamma'_1, \ldots, \gamma'_k \) conditions (27), (28) and (30) are equivalent to the finiteness of the Hofer
energy [20, Theorem 1.2] 
\[ \mathcal{E}(F) := \sup_{\phi \in \mathcal{C}} \int_{\mathbb{R} \times S^1 \setminus \{z_1, \ldots, z_k\}} F^* d(\phi \lambda), \]
where \( \mathcal{C} := \{ \phi \in C^\infty(\mathbb{R}, [0, 1]) | \phi' \geq 0 \} \). We define the contact action functional
\[ \mathcal{A}_\lambda : C^\infty(S^1, M) \to \mathbb{R} : \gamma' \mapsto \int_{S^1} \gamma'^* \lambda. \]
For every \( F \in \tilde{\mathcal{M}}^B(\overline{\gamma}', \gamma', \gamma'_1, \ldots, \gamma'_k; J_\infty) \) we have \( \mathcal{E}(F) = \mathcal{A}_\lambda(\overline{\gamma'}) \).

The group of biholomorphisms on the domain of \( F \) is \( \mathbb{R} \times S^1 \) and it acts freely on the space \( \tilde{\mathcal{M}}^B(\overline{\gamma}', \gamma', \gamma'_1, \ldots, \gamma'_k; J_\infty) \) by
\[ h \cdot F := F \circ h^{-1}, \quad h \cdot L := h \cdot L \]
with \( L \in \{ \overline{L}, L, L_1, \ldots, L_k \} \). The moduli space of punctured \( J_\infty \)-holomorphic cylinders is defined by
\[ \mathcal{M}^B(\overline{\gamma}', \gamma', \gamma'_1, \ldots, \gamma'_k; J_\infty) := \tilde{\mathcal{M}}^B(\overline{\gamma}', \gamma', \gamma'_1, \ldots, \gamma'_k; J_\infty)/(\mathbb{R} \times S^1). \]
It follows from (34) below that the virtual dimension of this moduli space is
\[ \mu(\overline{\gamma'}) - \mu(\gamma') + 2(c_1(\xi), B) - \sum_{i=1}^k \mu(\gamma'_i). \]

An almost complex structure \( J_\infty \in \mathcal{J}(\lambda) \) is called regular for cylinders if the linearized operator below is surjective for all \( \overline{\gamma}', \gamma' \in \mathcal{P}_\lambda, B \in H_2(M; \mathbb{Z}) \) and \( F \in \tilde{\mathcal{M}}^B(\overline{\gamma}', \gamma'; J_\infty) \). The linearized operator is
\[ D_F : W^{1,p,d}(\mathbb{R} \times S^1, F^* T(M \times \mathbb{R})) \oplus \mathbb{R}^4 \to L^{p,d}(\mathbb{R} \times S^1, F^* T(M \times \mathbb{R})), \]
\[ D_F \xi := \nabla_s \xi + J_\infty \nabla_\theta \xi + (\nabla_\xi J_\infty) \partial_\theta F, \]
where \( \xi := \xi_0 + v_1^- \xi_1^- + v_2^- \xi_2^- + v_1^+ \xi_1^+ + v_2^+ \xi_2^+ \), \( \xi_0 \in W^{1,p,d}(\mathbb{R} \times S^1, F^* T(M \times \mathbb{R})), v_i^\pm \in \mathbb{R}, i = 1, 2, p > 2, d > 0 \) small enough. The sections \( \xi_i^\pm, i = 1, 2 \) are asymptotically constant with the following asymptotic values:
\[ \xi_1^-(s, \theta) = R_\lambda, \quad \xi_2^-(s, \theta) = \partial/\partial t \quad \text{for} \ s \leq -1, \]
\[ \xi_i^-(s, \theta) = 0 \quad \text{for} \ s \geq 1, \]
\[ \xi_i^+(s, \theta) = \xi_i^-(-s, \theta), \quad i = 1, 2. \]
The spaces \( W^{1,p,d}(\mathbb{R} \times S^1, F^* T(M \times \mathbb{R})) \) and \( L^{p,d}(\mathbb{R} \times S^1, F^* T(M \times \mathbb{R})) \) are the completions of \( C^\infty(\mathbb{R} \times S^1, F^* T(M \times \mathbb{R})) \) with respect to the norms
\[ \| \xi \|_{1,p,d} := \left( \int_{\mathbb{R} \times S^1} \left( \| \xi \|^p + \| \nabla_\theta \xi \|^p + \| \nabla \xi \|^p e^{d|x|} \right) ds d\theta \right)^{1/p}, \]
\[ \| \xi \|_{p,d} := \left( \int_{\mathbb{R} \times S^1} \| \xi \|^p e^{d|x|} ds d\theta \right)^{1/p}. \]
Because the orbits \( \gamma', \gamma'' \) are transversally nondegenerate the operator \( D_F \) is Fredholm with index \([6, \text{Proposition 4}]\)

\[
\text{ind}(D_F) = \mu(\gamma') - \mu(\gamma'') + 2\langle c_1(\xi), B \rangle + 2,
\]

\( F \in \hat{M}^B(\gamma', \gamma''; J_\infty). \) \hfill (33)

so that the virtual dimension of \( \mathcal{M}^B(\gamma', \gamma''; J_\infty) \) is \( \mu(\gamma') - \mu(\gamma'') + 2\langle c_1(\xi), B \rangle \).

The above discussion can be generalized in a fairly straightforward way in order to define \textbf{regular almost complex structures for punctured holomorphic cylinders}. The relevant operator now has index

\[
\text{ind}(D_F) = \mu(\gamma') - \mu(\gamma'') + 2\langle c_1(\xi), B \rangle - \sum_{i=1}^{k} \mu(\gamma'_i) + 2
\]

for \( F \in \hat{M}^B(\gamma', \gamma'_1, \gamma'_2, \ldots, \gamma'_k; J_\infty). \) The point is that, unlike in Floer homology, the existence of almost complex structures which are regular for punctured cylinders is not guaranteed. Indeed, since the almost complex structure is not domain dependent, a ramified covering of a punctured holomorphic cylinder is again holomorphic, and its index may be smaller than the dimension of the space of ramified coverings. In that case the corresponding linearized operator cannot be surjective.

Let now \( J \) be a time-independent almost complex structure on \( \hat{W} \) which is compatible with \( \hat{\omega} \) and whose restriction \( J_\infty \) to \( M \times \mathbb{R}^+ \) is translation invariant and corresponds to an element of \( \mathcal{J}(\lambda). \) A \textbf{J-holomorphic plane in} \( \hat{W} \) is a \( J \)-holomorphic map \( F : \mathbb{C} = \mathbb{C}P^1 \setminus \{\infty\} \to \hat{W} \) such that, for \( |z| \) large enough \( F(z) \in M \times \mathbb{R}^+ \) and, writing \( F = (f, a) \), we have \( a(z) \to \infty, |z| \to \infty \) and there exist \( \gamma' \in \mathcal{P}_\lambda \) such that \( f(\hat{r}e^{2\pi i \theta}) \to \gamma'(T\theta) \), \( r \to \infty \) uniformly in \( \theta \). As for cylinders, this convergence condition has to be understood with respect to some \( R_x \)-parametrized representative of \( \gamma' \).

Given \( \gamma' \in \mathcal{P}_\lambda, A \in H_2(W; \mathbb{Z}) \) we define the \textbf{space} \( \mathcal{M}^A(\gamma', \emptyset; J) \) of \textbf{J-holomorphic planes} as the set of pairs \((F, L)\) with \( F \) as above and \( L \subset T_{\infty} \mathbb{C}P^1 \) a half-line, such that \( \lim_{\hat{r} \to \infty, z \in L} f(z) = P_{\gamma'} \) and \( \{F\} = [\sigma_\gamma # A] \). The group of biholomorphisms of the domain \( \mathbb{C} \) consists of affine transformations and has real dimension 4. It acts by

\[
h \cdot F := F \circ h^{-1}, \quad h \cdot L := h \cdot L.
\]

The quotient is denoted by \( \mathcal{M}^A(\gamma', \emptyset; J) \) and is called the \textbf{moduli space of J-holomorphic planes}.

The relevant linearized operator now has index

\[
\text{ind}(D_F) = \overline{\text{ind}}(\gamma') + 2\langle c_1(TW), A \rangle + 4,
\]

\( F \in \hat{M}^A(\gamma', \emptyset; J), \) so that the virtual dimension of the moduli space of \( J \)-holomorphic planes \( \mathcal{M}^A(\gamma', \emptyset; J) \) is \( \overline{\text{ind}}(\gamma') + 2\langle c_1(TW), A \rangle. \) We say that an almost complex structure \( J \) on \( \hat{W} \) is \textbf{regular for holomorphic planes} if the linearized operator \( D_F \) is surjective for every \( \gamma' \in \mathcal{P}_\lambda, A \in H_2(W; \mathbb{Z}) \) and \( F \in \hat{M}^A(\gamma', \emptyset; J) \). Like for punctured holomorphic cylinders, the existence of such regular almost complex structures is not guaranteed.
For the definitions that follow we assume that $J$ is regular for the relevant holomorphic curves. A list of examples in which this assumption is satisfied is given in Remark 9 below.

One can associate a sign $\epsilon(F)$ to each element $[F] \in M^A(\gamma', \emptyset; J)$ such that $\pi_*(\gamma') + 2c_1(TW), A) = 0$ (see [6]). We define a homomorphism

$$e : C_*^{-1}(a) (\lambda) \to \Lambda_{w}$$

by

$$e(\gamma') := \sum_{\Lambda \in H_2(W;\mathbb{Z})} \left( \sum_{|e^A| = |\gamma'|} \epsilon(F) \right) e^A.$$

**Remark 6.** The homomorphism $e$ is obtained from the natural **augmentation** on the differential graded algebra for the full contact homology of $(M, \gamma)$ defined by the symplectic filling $(W, \omega)$ (see [4] for details on augmentations).

Given $\gamma' \in P_\lambda$, we denote by $\kappa_{\gamma'} \in \mathbb{Z}^+$ its **multiplicity**. It is the largest integer such that $\gamma'(\theta + \frac{1}{\kappa_{\gamma'}}) = \gamma'(\theta), \theta \in \mathbb{R}/\mathbb{Z}$.

If $k \neq 0$, $\overline{\gamma'} \neq \gamma'$ or $0 \neq B \in H_2(M; \mathbb{Z})$, the additive group $\mathbb{R}$ acts freely on the moduli space $\overline{M^B(\gamma', \gamma', \gamma', \ldots, \gamma'; J_\infty)}$ by translations in the $t$ direction

$$t_0 \cdot [(f, a)] := [(f, a + t_0)].$$

We denote the quotient by $\overline{M^B(\gamma', \gamma', \gamma', \ldots, \gamma'; J_\infty)} / \mathbb{R}$. This space can be compactified to $\overline{M^B(\gamma', \gamma', \gamma', \ldots, \gamma'; J_\infty)}$, a space consisting of $J_\infty$-holomorphic buildings with a tree structure [5, Theorem 10.1]. If $k = 0, \overline{\gamma'} = \gamma'$ and $B = 0$, the moduli space consists of $\kappa := \kappa_{\overline{\gamma'}} = \kappa_{\gamma'}$ points. The underlying holomorphic curve is the constant cylinder over the orbit $\overline{\gamma'} = \gamma'$, and the equivalence class of the pair of asymptotic markers $(\overline{L}, \overline{L})$ is determined by the difference $(\text{Arg}(\overline{L}) - \text{Arg}(\overline{L}))/2\pi \in \mathbb{R}/\mathbb{Z}$, which is a multiple of $1/\kappa$. The action of $\mathbb{R}$ is in this case trivial.

When $\mu(\overline{\gamma'}) - \mu(\gamma') + 2c_1(\xi, B) - \sum_{i=1}^{k} \pi(\gamma'_i) = 1$, the moduli space $\overline{M^B(\gamma', \gamma', \gamma', \ldots, \gamma'; J_\infty)} / \mathbb{R}$ is compact and therefore consists of a finite number of points. One can associate a sign $\epsilon(F)$ to each element $[F]$ of this moduli space [6] and we define the **linearized contact differential**

$$\partial : C_*^{-1}(a) (\lambda) \to C_*^{-1}(a) (\lambda)$$

by

$$\partial \overline{\gamma'} := \sum_{\gamma'_1, \gamma'_2, \ldots, \gamma'_k, B} \frac{n^B(\gamma', \gamma', \gamma', \ldots, \gamma'; J_\infty)}{\prod_{i=1}^{k} \kappa_{\gamma'_i}} \sum_{|e^A| + \sum |\gamma'_i| - |\gamma'| - 1} e(\gamma'_1) \ldots e(\gamma'_k) e^A \overline{\gamma'},$$

(35)
where
\[ n^B(\mathcal{P}', \gamma', \gamma'_1, \ldots, \gamma'_k; J_{\infty}) := \sum_{[F] \in \mathcal{M}^B(\mathcal{P}', \gamma', \gamma'_1, \ldots, \gamma'_k; J_{\infty})/\mathbb{R}} \epsilon(F). \]

The reader is warned that the classes \( B \) in the above sum live in \( H_2(M; \mathbb{Z}) \), but nevertheless the coefficient in front of \( \gamma' \) is an element of \( \Lambda_\omega \) due to the factors \( \epsilon(\gamma'_i), i = 1, \ldots, k \). As a matter of fact, the fraction in (35) is an integer because our moduli spaces involve asymptotic markers.

Since \( e \) comes from the natural augmentation defined by \((W, \omega)\), it follows that (see [4])
\[ \partial \circ \partial = 0 \quad \text{and} \quad e \circ \partial = 0. \] (36)

We define the **linearized contact homology groups** of the pair \((\lambda, J)\) by
\[ HC^{i-1}_{*}(\lambda, J) := H_*(C^{i-1}_{*}(\lambda), \partial). \]

The linearized contact homology groups \( HC^{i-1}_{*}(\lambda, J) \) depend only on the symplectic filling \((W, \omega)\) of \((M, \xi)\). Since the former is part of the data in the context of the present paper, we simply denote the resulting homology groups by
\[ HC^{i-1}_{*}(M, \xi) \]
without reference to \( W \).

We shall need in the sequel the following alternative description of the differential \( \partial \) for linearized contact homology. Given \( A \in H_2(W; \mathbb{Z}) \), \( \mathcal{P}' \), \( \gamma' \in \mathcal{P}_k \) we define the **moduli space of capped punctured \( J \)-holomorphic cylinders**
\[ \mathcal{M}^A_\epsilon(\mathcal{P}', \gamma'; J) \]
as the set of equivalence classes of pairs \( F = (F', F'') \), where \( F' \in \mathcal{M}^B(\mathcal{P}', \gamma', \gamma'_1, \ldots, \gamma'_k; J_{\infty}), \gamma'_1, \ldots, \gamma'_k \in \mathcal{P}_k, B \in H_2(M; \mathbb{Z}), F'' \) is a collection of \( k \) \( J \)-holomorphic planes in \( \tilde{W} \), of total homology class \( A - B \in H_2(W; \mathbb{Z}) \), and with asymptotics at their positive punctures corresponding to \( \gamma'_1, \ldots, \gamma'_k \). Recall that \( F' \) and \( F'' \) are endowed with asymptotic markers \((L', L'_0, L'_1)\) and \((L''_0)\), \( i = 1, \ldots, k \) which determine conformal identifications of the tangent spaces at the punctures with asymptotes \( \gamma'_i \). Two pairs \(((L'_0, L'_0, L'_{i,0}), (L''_{i,0})), ((L'_1, L'_1, L'_{i,1}), (L''_{i,1}))\) corresponding to the same maps \( F', F'' \) are equivalent if \( L'_0 = L'_1, L'_0 = L'_1, L'_{i,0} = L'_{i,1}, L''_{i,0} = L''_{i,1} \). This last condition is equivalent to the equality
\[ \text{Arg}(L'_{i,0}) - \text{Arg}(L'_{i,1}) = \text{Arg}(L''_{i,0}) - \text{Arg}(L''_{i,1}), \quad i = 1, \ldots, k \] (37)
with respect to fixed conformal charts at the corresponding punctures. The dimension of this moduli space is

\[ \mu(\gamma') - \mu(\gamma') + 2\langle c_1(TW), A \rangle. \]

We refer to Fig. 3b for a representation of such objects. The differential for linearized contact homology can then be rewritten as

\[ \partial \gamma' = \sum_{\gamma', A} \frac{1}{k_{\gamma'}} \sum_{[F] \in \mathcal{M}_A^\delta(\gamma', \gamma'; J)/\mathbb{R}} \epsilon(F) e^A \gamma'. \quad (38) \]

The sign \( \epsilon(F) \) is defined as the product of the signs of the components of \( F \).

**Remark 7.** The proof of the invariance of \( HC_{i-1}((a)^\ast(\lambda, J)) \) with respect to \( \lambda \) and \( J \) makes use of the polyfold formalism currently being developed by Hofer, Wysocki and Zehnder [18, 21].

**Remark 8.** Note that the linearized contact homology is related to the cylindrical contact homology defined in [16], in the following situation. Let us assume that \( c_1(\xi) = 0 \), and that there are no Reeb orbits \( \gamma' \) with reduced Conley–Zehnder index \( \overline{\mu}(\gamma') = -1, 0, 1 \) and which are contractible in \( M \), so that cylindrical contact homology is well-defined. Then, if \( c_1(TW) = 0 \) and if there are no Reeb orbits \( \gamma' \) of reduced Conley–Zehnder index \( \overline{\mu}(\gamma') = 0 \) which are contractible in \( W \), the linearized contact homology groups are isomorphic to the cylindrical contact homology groups.

**Remark 9 (Note on transversality).** Let \( a \) be a free homotopy class of loops in \( W \). In order for the linearized contact differential to be well-defined on \( i^{-1}(a) \) we need that the almost complex structure \( J \) on \( \hat{W} \) satisfies the following conditions.

(A) \( J \) is regular for holomorphic planes belonging to moduli spaces \( \mathcal{M}^A(\gamma', \emptyset; J) \) of virtual dimension \( \leq 0 \);

(B_0) \( J_\infty \) is regular for punctured holomorphic cylinders asymptotic at \( \pm \infty \) to closed Reeb orbits in \( i^{-1}(a) \), belonging to moduli spaces of virtual dimension \( \leq 2 \), and which are asymptotic at the punctures to closed Reeb orbits \( \gamma' \) such that \( \mathcal{M}^A(\gamma', \emptyset; J) \neq \emptyset \) and has virtual dimension 0.

We expect these technical assumptions to be completely removed using the new ongoing approach to transversality by Cieliebak and Mohnke (see [13] for the symplectic case), or using the polyfold theory developed by Hofer, Wysocki and Zehnder [18, 21]. We give below the list of examples known to us in which both these conditions are satisfied. In many cases condition (A) is empty, so that linearized contact homology reduces to cylindrical contact homology.
(i) Stabilizations $W := V \times D^2$ of subcritical Stein manifolds $V$ with $c_1(V) = 0$, and in particular the standard balls $B^{2n}$, $n \geq 2$, for the trivial free homotopy class. In this case, if one chooses on $W$ a symplectic form corresponding to sufficiently thin handles, then there are no $J$-holomorphic planes of index $\leq 0$ since $\pi_c(\gamma') \geq n \geq 2$ for all closed Reeb orbits $\gamma'$ [34, Theorem 3.1 (III), Lemma 4.2]. Thus condition (A) is empty and condition $(B_0)$ has to be checked only for cylinders without punctures. It is proved in [34, Lemma 7.5] that, for each bound $\alpha$ on the contact action, there exists a symplectic structure on $W$ such that condition $(B_0)$ is satisfied for cylinders whose asymptotes have action at most $\alpha$. This is enough in order to define linearized contact homology in the trivial free homotopy class.

(ii) Negative disc bundles $W$ over symplectically aspherical manifolds $(B, \beta)$, for the trivial free homotopy class. Here $\mathcal{L} \xrightarrow{\pi} B$ is a Hermitian line bundle with $c_1(\mathcal{L}) = -[\beta]$ and $W = \{v \in \mathcal{L} : |v| \leq 1\}$. The symplectic form is $\hat{\omega} = \pi^*\beta + d(r^2\theta)$, where $r$ is the radial coordinate in the fibers and $\theta$ is the angular form, and we choose compatible almost complex structures $J_B$ on $B$, $J$ on $\hat{W}$ such that $\pi$ is $(J, J_B)$-holomorphic and $J$ satisfies (25). The Reeb orbits on $M := \partial W$ are the circles in the fibers and, to achieve nondegeneracy, we choose a Morse function $f : B \to \mathbb{R}$, we perturb $\hat{\omega}$ to $\hat{\omega}_\varepsilon = \hat{\omega} + \varepsilon fd(r^2\theta)$, $\varepsilon > 0$ and $J$ to $J_\varepsilon$ so that $J_\varepsilon$ satisfies (25) for the perturbed Reeb vector field. For each bound $\alpha$ on the contact action, there exists $\varepsilon > 0$ such that the closed Reeb orbits with period $\leq \alpha$ are the circles in the fibers over $\text{Crit}(f)$. We claim that $J_\varepsilon$ satisfies conditions (A) and $(B_0)$ for curves with asymptotes of period $\leq \alpha$, which is enough for our purposes.

The main point is that all punctured holomorphic curves in $(\hat{W}, J)$ and $(M \times \mathbb{R}, J)$ are contained in the fibers of $\mathcal{L}$, due to the asphericity of $B$. The index of a curve with one positive puncture and $m$ negative punctures is $\dim(B) + 2(m - 1)$, and a direct computation shows that they are all regular (obvious elements in the kernel of the linearized operator correspond to varying the basepoint and the $m - 1$ ramification points). The Morse–Bott analysis in [3, Chap. 5] shows that, after a slight perturbation of the evaluation maps on the above moduli spaces, regularity still holds after $\varepsilon$-perturbation. The $J_\varepsilon$-holomorphic curves involved in condition (A) are the fibers over minima of $f$, and the $J_\varepsilon$-holomorphic curves involved in $(B_0)$ are trivial cylinders contained in the fibers over $\text{Crit}(f)$, cylinders over gradient trajectories of $f$ of index 1 or 2, and once punctured cylinders contained in the fibers over the minima of $f$.

(iii) Unit cotangent bundles $W = DT^*L = \{v \in T^*L : |v| \leq 1\}$ of closed Riemannian manifolds $L$ such that (A) is empty, for a free homotopy class $a$ containing only simple closed geodesics. Condition (A) is empty if either $\dim L \geq 4$ or $L$ carries no contractible closed
geodesics. In the first case the lift $\gamma'_{\alpha}$ of a closed geodesic $\alpha$ satisfies $\overline{p}(\gamma'_{\alpha}) = \text{ind}_{\text{Morse}}(\alpha) + \dim L - 3 \geq 1$, whereas the second case includes manifolds with strictly negative sectional curvature and flat tori. Condition $(B_\alpha)$ is satisfied because any cylinder asymptotic at $\pm \infty$ to a simple orbit is somewhere injective, and the set of almost complex structures which are regular for somewhere injective curves is of the second Baire category in $\mathcal{J}(\lambda)$ [15, Theorem 1.8].

3.2. A non-equivariant construction

In this section we apply the Morse–Bott formalism of [7] in the context of contact homology. The result is a chain complex closely related to the non-equivariant contact homology construction of [14].

For $\gamma' \in \mathcal{P}_\lambda$ we denote by $S'_{\gamma}$ the circle of parametrized closed Reeb orbits representing $\gamma'$. Given $\overline{\gamma}, \gamma'_1, \ldots, \gamma'_k \in \mathcal{P}_\lambda$ and $B \in H_2(M; \mathbb{Z})$ we denote by

$$\hat{M}^B(S'_\overline{\gamma}, S'_\gamma, \gamma'_1, \ldots, \gamma'_k; J_\infty)$$

the **space of punctured $S^1$-parametrized $J_\infty$-holomorphic cylinders**, consisting of tuples $(u, L_1, \ldots, L_k)$ such that

$$u = (f, a) : \mathbb{R} \times S^1 \setminus \{z_1, \ldots, z_k\} \to M \times \mathbb{R}$$

satisfies

$$\partial_s u + J_\infty \partial_\theta u = 0,$$

the $L_i$’s are asymptotic markers at the punctures $z_i$, and we require

$$\lim_{s \to \pm \infty} a(s, \theta) = \mp \infty, \quad \lim_{s \to -\infty} f(s, \cdot) \in S'_\overline{\gamma}, \quad \lim_{s \to \infty} u(s, \cdot) \in S'_\gamma,$$

as well as (30)–(32) at $z_1, \ldots, z_k$. The asymptotic conditions on $f$ have to be understood in the sense of (28). Note that the space of punctured $J_\infty$-holomorphic cylinders $\hat{M}^B(S'_\overline{\gamma}, S'_\gamma, \gamma'_1, \ldots, \gamma'_k; J_\infty)$ defined in the previous section consists of $k_{\overline{\gamma}} k_{\gamma'}$ copies of the space $\hat{M}^B(S'_\overline{\gamma}, S'_\gamma, \gamma'_1, \ldots, \gamma'_k; J_\infty)$.

The group $\mathbb{R}$ acts on $\hat{M}^B(S'_\overline{\gamma}, S'_\gamma, \gamma'_1, \ldots, \gamma'_k; J_\infty)$ by translations in the $s$-variable and we denote by

$$M^B(S'_\overline{\gamma}, S'_\gamma, \gamma'_1, \ldots, \gamma'_k; J_\infty) := \hat{M}^B(S'_\overline{\gamma}, S'_\gamma, \gamma'_1, \ldots, \gamma'_k; J_\infty)/\mathbb{R}$$

the **moduli space of punctured $S^1$-parametrized $J_\infty$-holomorphic cylinders**.

Since the almost complex structure $J_\infty$ satisfies assumption $(B_\alpha)$, the moduli space $M^B(S'_\overline{\gamma}, S'_\gamma, \gamma'_1, \ldots, \gamma'_k; J_\infty), \overline{\gamma}, \gamma' \in i^{-1}(a)$ is a smooth
manifold of dimension

\[ \mu(\gamma') - \mu(\gamma') + 2\langle c_1(\xi), B \rangle - \sum_{i=1}^{k} r(\gamma'_i) + 1. \]

Note that this differs by 1 from the dimension of the moduli space \( M^B(\gamma', \gamma', \gamma'_1, \ldots, \gamma'_k; J_\infty) \), because the \( S^1 \)-symmetry is now broken.

If \( \gamma' \neq \gamma' \) the additive group \( \mathbb{R} \) acts freely on the above moduli space by translations in the \( t \) direction, and we denote the quotient by \( M^B(\gamma'_1, \gamma'_2, \gamma'_1, \ldots, \gamma'_k; J_\infty)/\mathbb{R} \). Each such quotient is equipped with smooth evaluation maps \( \tilde{\varphi}, \tilde{\eta} \) with target \( S'_\gamma \) respectively \( S'_{\gamma'} \).

Given \( \gamma' \in \mathcal{P}_\lambda \) we choose a Morse function \( f' : S'_{\gamma'} \to \mathbb{R} \) having two critical points \( m' \) and \( M' \). We denote by \( \phi_{f'_\gamma} \) the flow of \( \nabla f'_\gamma \) with respect to some Riemannian metric on \( S'_{\gamma'} \). For \( p' \in \text{Crit}(f'_{\gamma'}) \) we denote the Morse index by

\[ \text{ind}(p') := \dim W^u(p', f'_{\gamma'}). \]

We denote by \( \gamma' \) the Reeb orbit in \( S'_{\gamma'} \) which corresponds to the critical point \( p' \in \text{Crit}(f'_{\gamma'}) \). We define the grading by

\[ |\gamma'_p| := |\gamma'| + \text{ind}(p'). \]  \hfill (39)

Let \( \gamma', \gamma'_1, \ldots, \gamma'_k, \ldots, \gamma'_{m_1}, \ldots, \gamma'_{m_k} \in \mathcal{P}_\lambda, p' \in \text{Crit}(f'_{\gamma'}), q'_1 \in \text{Crit}(f'_\gamma) \) and \( B \in H_2(M; \mathbb{Z}) \). For \( m \geq 0 \) we denote by

\[ M^B_m(p', q'_1, \gamma'_1, \ldots, \gamma'_{k_1}, \ldots, \gamma'_{m_1}, \ldots, \gamma'_{k_m}, \{ f'_\gamma \}, J_\infty) \]

the union over \( \gamma'_1, \ldots, \gamma'_{m-1} \in \mathcal{P}_\lambda \) and \( B_1 + \cdots + B_m = B \) of the fibered products

\[ W^u(p') \times_{\tilde{\varphi}} \left( (M^{B_1}(\gamma'_{1_1}, \ldots, \gamma'_{k_1})/\mathbb{R}) \times \mathbb{R}^+ \right) \]

\[ \varphi_{f'_{\gamma_1}} \circ \tilde{\varphi} \times_{\tilde{\varphi}} \left( (M^{B_2}(\gamma'_{1_2}, \ldots, \gamma'_{k_2})/\mathbb{R}) \times \mathbb{R}^+ \right) \]

\[ \cdots \varphi_{f'_{\gamma_{m-1}}} \circ \tilde{\varphi} \times_{\tilde{\varphi}} M^{B_m}(\gamma'_{1_m}, \ldots, \gamma'_{k_m}, \{ f'_\gamma \}, J_\infty) / \mathbb{R} \] \times \mathbb{R}^+ \times W^u(q'). \] \hfill (40)

In this fibered product, the factors \( \mathbb{R}^+ \) play the role of flow times for \( \varphi_{f'_{\gamma_1}}, \ldots, \varphi_{f'_{\gamma_{m-1}}} \). By our transversality assumptions, this is a smooth manifold of dimension

\[ \dim M^B_m(p', q'_1, \gamma'_1, \ldots, \gamma'_{k_1}, \ldots, \gamma'_{m_1}, \ldots, \gamma'_{k_m}; \{ f'_\gamma \}, J_\infty) \]

\[ = \text{ind}(p') - 1 + (\dim M^{B_1}(\gamma'_{1_1}, \ldots, \gamma'_{k_1}; J_\infty)/\mathbb{R} + 1) - 1 \]

\[ + \cdots + \dim M^{B_m}(\gamma'_{1_m}, \ldots, \gamma'_{k_m}; J_\infty)/\mathbb{R} \]

\[ - 1 + (1 - \text{ind}(q')). \]
\[
\mu(\overline{\gamma'}) - \mu(\gamma') + \text{ind}(p') - \text{ind}(q') - 1 \\
+ 2\langle c_1(\xi), B_1 + \ldots + B_m \rangle - \sum_{i=1}^{m} \sum_{j=1}^{k_i} \mathcal{P}(\gamma'^{i}_{j}) \\
= |\mathcal{V}_p| - |\mathcal{V}_q'| - \sum_{i=1}^{m} \sum_{j=1}^{k_i} \mathcal{P}(\gamma'^{i}_{j}) + 2\langle c_1(\xi), B \rangle - 1.
\]

We denote

\[
\mathcal{M}^B(p', q', \gamma'_1, \ldots, \gamma'_k; \{ f'_{\gamma'} \}, J_\infty)
:= \bigcup_{m \geq 0} \mathcal{M}^B_m(p', q', \gamma'^{1}_1, \ldots, \gamma'^{1}_{k_1}, \ldots, \gamma'^{m}_{k_m}; \{ f'_{\gamma'} \}, J_\infty)
\]

with \{ \gamma'^{1}_1, \ldots, \gamma'^{1}_{k_1}, \ldots, \gamma'^{m}_{k_m} \} = \{ \gamma'_1, \ldots, \gamma'_k \}, and we call this the moduli space of punctured \(S^1\)-parametrized broken \(J_\infty\)-holomorphic cylinders, whereas

\[
\mathcal{M}^B_m(p', q', \gamma'_1, \ldots, \gamma'^{1}_{k_1}, \ldots, \gamma'^{m}_{k_m}; \{ f'_{\gamma'} \}, J_\infty)
\]

will be called the moduli space of punctured \(S^1\)-parametrized broken \(J_\infty\)-holomorphic cylinders with \(m\) sublevels. We refer to Fig. 3c on p. 659 for a representation of the elements of these moduli spaces (see also the capped version at the end of this section).

The compactified moduli spaces \(\overline{\mathcal{M}}^B(p', q', \gamma'_1, \ldots, \gamma'_k; \{ f'_{\gamma'} \}, J_\infty)\) admit a topology which is described similarly to that of the compactified moduli spaces \(\overline{\mathcal{M}}^A(p, q; H, \{ f_{\gamma} \}, J)\) in Sect. 2.2. The boundary (and corners) of \(\mathcal{M}^B(p', q', \gamma'_1, \ldots, \gamma'_k; \{ f'_{\gamma'} \}, J_\infty)\) contain configurations which involve broken gradient trajectories, like for the compactification \(\overline{\mathcal{M}}^A(p, q; H, \{ f_{\gamma} \}, J)\), but also configurations which involve punctured \(S^1\)-parametrized broken \(J_\infty\)-holomorphic cylinders and genus zero \(J_\infty\)-holomorphic buildings of arbitrary height (in the sense of [5, Sect. 7.2]), with exactly one positive puncture.

Remark 10. Since we are considering parametrized Reeb orbits \(\overline{\gamma'}, \gamma'\), all moduli spaces under consideration are orientable provided that the asymptotes \(\gamma'_1, \ldots, \gamma'_k\) are good. In the sequel we shall restrict ourselves to such asymptotes at the punctures. In this case one can associate a sign \(\epsilon(F)\) to each \([F] \in \mathcal{M}^B(p', q', \gamma'_1, \ldots, \gamma'_k; \{ f'_{\gamma'} \}, J_\infty)\) by using coherent orientations [6] and the fiber-sum rule [7, §4].

Let \(a\) be a free homotopy class of loops in \(W\). We define a differential complex, which we call the \(S^1\)-parametrized contact complex, by setting

\[
BC^i_{\ast}(a) := \bigoplus_{\gamma' \in \mathcal{P}^i_{\ast}(a)} \Lambda_\ast\langle \gamma'_M, \gamma'_M \rangle.
\]
We define the $S^1$-parametrized contact differential

$$\delta : BC_{*}^{i-1}(a)(\lambda) \to BC_{*-1}^{i-1}(a)(\lambda)$$

by

$$\delta \gamma_p := \sum_{y_q, y'_1, \ldots, y'_k, B} \frac{n^B (p', q', y'_1, \ldots, y'_k; \{ f'_\gamma \} \cup \{ J_\infty \})}{\prod_{i=1}^{k} \kappa_{y'_i}} \times e(y'_i) \cdot e_B(y'_k),$$

where

$$n^B (p', q', y'_1, \ldots, y'_k; \{ f'_\gamma \} \cup \{ J_\infty \}) := \sum_{[F] \in M^B (p', q', y'_1, \ldots, y'_k; \{ f'_\gamma \} \cup \{ J_\infty \})} \epsilon(F).$$

The classes $B$ in the above sum live in $H_2(M; \mathbb{Z})$, but nevertheless the coefficient in front of $\sum_{q}$ is an element of $\Lambda_\omega$ due to the factors $e(y'_i)$, $i = 1, \ldots, k$. Moreover, the fraction in (41) is an integer.

The map $\delta$ satisfies $\delta \circ \delta = 0$. As explained above, the boundary of the 1-dimensional moduli spaces $M^B (p', q', y'_1, \ldots, y'_k; \{ f'_\gamma \} \cup \{ J_\infty \})$ consists of configurations with a single broken gradient trajectory, and of configurations involving a rigid punctured $J_\infty$-holomorphic cylinder. The count of configurations of the first type using the augmentation $e$ is equal to $\delta \circ \delta$, and the count of configurations of the second type using the augmentation $e$ involves $e \circ \partial = 0$ and hence vanishes. As a consequence, the composition $\delta \circ \delta$ vanishes as well. The homology groups $H_*(BC_*^{i-1}(a)(\lambda), \delta)$ are actually isomorphic to the non-equivariant contact homology groups of [14].

**Remark 11.** Let $\alpha \in \mathbb{R}^+$ be such that $\alpha \notin \text{Spec}(M, \lambda)$. One can define subcomplexes

$$BC_{*}^{i-1}(a, \leq \alpha)(\lambda) := \bigoplus_{y'_m, y'_M} \Lambda_\omega (y'_m, y'_M) \subseteq BC_{*}^{i-1}(a)(\lambda)$$

and, for $\alpha \to \infty$, we obviously have

$$\lim_{\alpha \to \infty} H_*(BC_{*}^{i-1}(a, \leq \alpha)(\lambda), \delta) = H_*(BC_{*}^{i-1}(a)(\lambda), \delta).$$

We can give an alternative description of the $S^1$-parametrized contact differential as follows. Given $A \in H_2(W; \mathbb{Z})$, $y'_m, y'_M \in \mathcal{P}_\lambda$, $p' \in \text{Crit}(f'_{y'_m})$, $q' \in \text{Crit}(f'_{y'_M})$ we define the moduli space of capped punctured $S^1$-parametrized broken $J$-holomorphic cylinders

$$\mathcal{M}_c^A (p', q'; \{ f'_{y'_m} \} \cup \{ f'_{y'_M} \}).$$
as the set of equivalence classes of pairs \( u = (u', u'') \), with \( u' \) an element of the moduli space \( \mathcal{M}^B(p', q', \gamma_1, \ldots, \gamma_k; \{ f_{\gamma} \}, J, \infty) \), \( \gamma_1, \ldots, \gamma_k \in \mathcal{P}_k \), \( B \in H_2(M; \mathbb{Z}) \), and \( u'' \) a collection of \( J \)-holomorphic planes in \( \hat{W} \), of total homology class \( A - B \in H_2(W; \mathbb{Z}) \), and with asymptotics at their positive punctures corresponding to \( \gamma_1, \ldots, \gamma_k \). The elements \( u', u'' \) are endowed with asymptotic markers \( L'_{i}, L''_{i} \), \( i = 1, \ldots, k \). Two collections of asymptotic markers \( (L'_{i,0}, L''_{i,0}), (L'_{i,1}, L''_{i,1}) \) are equivalent if they satisfy (37). The dimension of this moduli space is

\[
\left| \gamma_p \right| - \left| \gamma_q' \right| + 2\langle c_1(TW), A \rangle - 1.
\]

We refer to Fig. 3c on p. 659 for a pictorial representation of these objects. The \( S^1 \)-parametrized contact differential can then be rewritten as

\[
\delta \gamma_p = \sum_{u \in \mathcal{M}^A(p', q'; \{ f_{\gamma} \}, J)} \sum_{|\gamma_q A| = |\gamma_p| - 1} \epsilon(u)e^A \gamma_q'.
\] (42)

The sign \( \epsilon(u) \) is defined as the product of the signs of the components of \( u \).

**Remark 12.** One can define in an obvious manner moduli spaces \( \mathcal{M}_c^A(S^1_{\gamma}, S^1_{\gamma}; J) \) of capped punctured \( S^1 \)-parametrized \( J_\infty \)-holomorphic cylinders, of dimension \( \mu(\gamma') - \mu(\gamma') + 2\langle c_1(TW), A \rangle + 1 \), so that the moduli spaces \( \mathcal{M}_c^A(p', q'; \{ f_{\gamma} \}, J) \) are obtained from \( \mathcal{M}_c^A(S^1_{\gamma}, S^1_{\gamma}; J) \) via a fibered product construction analogous to (40).

### 4. Filtrations

This section is organized as follows. We first exhibit natural filtrations on the Morse–Bott complex for symplectic homology, as well as on the \( S^1 \)-parametrized contact complex. We then describe the differential on the 0-th page of the associated spectral sequences. The crucial result in that direction is Proposition 3. We then define an isomorphism of filtered complexes in Proposition 4, and explain the main steps for the proof of Theorem 1.

Let \( a \) be a free homotopy class of loops in \( W \). For \( a \neq 0 \) or \( a = + \) we define

\[
B_k C^a_s(H) := \bigoplus_{\gamma \in \mathcal{P}^a(H)} \{ e^A \gamma_m, e^A \gamma_M \}, \quad (43)
\]

We claim that

\[
\bigoplus_{k \leq \ell} B_k C^a_s(H), \quad \ell \in \mathbb{Z}
\]
forms an increasing filtration on $BC^a_*(H)$. Indeed, the sum (17) involves elements $\gamma \in \mathcal{P}$ such that $\mu(\gamma) + \text{ind}(p) - \mu(\gamma) - \text{ind}(q) + 2\langle c_1(TW), A \rangle = 1$. Since

$$\text{ind}(p) - \text{ind}(q) \in \{-1, 0, 1\}$$

it follows that

$$\mu(\gamma) - \mu(\gamma) + 2\langle c_1(TW), A \rangle \in \{0, 1, 2\}.$$

As a consequence, the differential $d$ on $BC^a_*(H)$ can be written as

$$d = d^0 + d^1 + d^2,$$

with

$$d^i : B_kC^a_*(H) \to B_{k-i}C^a_*(H), \quad i = 0, 1, 2.$$

Similarly, we define a filtration on $BC^{-1}_*(a, \leq a(\lambda))$

$$B_kC^{-1}_*(a, \leq a(\lambda)) := \bigoplus_{\gamma' \in \mathcal{P}_k^{-1}(a, \leq a, A \in H_2(W; \mathbb{Z})} \langle e^A \gamma'_m, e^A \gamma'_M \rangle.$$ (45)

The same argument as above shows that

$$\bigoplus_{k \leq \ell} B_kC^a_*(\lambda), \quad \ell \in \mathbb{Z}$$

forms an increasing filtration on $BC_{i-1}(a, \leq a(\lambda))$, and the differential $\delta$ can be written as

$$\delta = \delta^0 + \delta^1 + \delta^2$$

with

$$\delta^i : B_kC_{i-1}(a, \leq a(\lambda)) \to B_{k-i}C_{i-1}(a, \leq a(\lambda)), \quad i = 0, 1, 2.$$

**Remark 13 (on the pictorial representation of the moduli spaces).** We have drawn in Fig. 3 on p. 659 elements of the moduli spaces involved in $\delta^i$, $i = 0, 1, 2$. The justification for $\delta^0$ and $\delta^1$ is given below, whereas the justification for $\delta^2$ is given in Sect. 7.2. It is a consequence of Proposition 3 and Remark 14 that we can choose the Hamiltonian $H$ so that the same type of moduli spaces are involved also in the differentials $d^i$, $i = 0, 1, 2$. 
We now further elaborate on the differential \( \delta^0 \), corresponding to orbits \( \overline{\gamma}, \gamma' \in \mathcal{P}_\lambda^{i^{-1}(a), \leq a} \) such that \( \mu(\overline{\gamma}) - \mu(\gamma') + 2\langle c_1(TW), A \rangle = 0 \). The decomposition (45) of \( B_k C_*^{-1}(a), \leq a(\lambda) \) as a direct sum induces a splitting of the differential \( \delta^0 \) as

\[
    \delta^0 = \sum_{\gamma' \in \mathcal{P}_\lambda^{i^{-1}(a), \leq a}} \delta^0_{\gamma'},
\]

with \( \text{im}(\delta^0_{\gamma'}) \subset \Lambda_0(\gamma_m, \gamma'_M) \).

We claim that \( \delta^0_{\gamma'}(\overline{\gamma}) = 0 \) if \( \overline{\gamma} \neq \overline{\gamma}' \). Let us recall for that purpose the moduli spaces of capped punctured \( S^1 \)-parametrized \( J_\infty \)-holomorphic cylinders \( \mathcal{M}_c^A(\overline{\gamma}_p, \gamma'_q; J) \) defined in Remark 12. Each such space necessarily has dimension 1. However, it carries a nontrivial \( S^1 \)-action, as well as a free \( \mathbb{R} \)-action on the target if \( \overline{\gamma} \neq \overline{\gamma}' \), in which case it must be empty. As a consequence, we must have \( \delta^0(\gamma') = \delta^0_{\gamma'}(\gamma') \). Since the difference of actions of the asymptotes at \( \pm \infty \) for elements of \( \mathcal{M}_c^A(\overline{\gamma}_p, \gamma'_q; J_\infty) \) is zero, the latter spaces are nonempty only if \( A = 0 \) and there are no punctures. Hence \( \delta^0 \) counts gradient trajectories and, in particular, we have \( \delta^0(\gamma'_M) = 0 \). The next result is a straightforward adaptation of Lemma 4.25 in [7].

**Proposition 1.** If \( \gamma' \in \mathcal{P}_\lambda^{i^{-1}(a), \leq a} \) is a good orbit, then

\[
    \delta^0(\gamma'_m) = 0.
\]

If \( \gamma' \in \mathcal{P}_\lambda^{i^{-1}(a), \leq a} \) is a bad orbit, then

\[
    \delta^0(\gamma'_m) = \pm 2\gamma'_M.
\]

Let us now discuss the differential \( d^0 \), which corresponds to orbits \( \overline{\gamma}, \gamma \in \mathcal{P}^a(H) \) such that \( \mu(\overline{\gamma}) - \mu(\gamma) + 2\langle c_1(TW), A \rangle = 0 \). As above, it also splits as

\[
    d^0 = \sum_{\gamma \in \mathcal{P}^a(H)} d^0_{\gamma'}
\]

with \( \text{im}(d^0_{\gamma'}) \subset \Lambda_0(\gamma_m, \gamma_M) \).

One important situation is \( \overline{\gamma} = \gamma \). In this case the moduli spaces of Floer trajectories \( \mathcal{M}^A(\overline{\gamma}, \overline{\gamma}; H, J) \), \( A \neq 0 \) are empty, whereas the space \( \mathcal{M}^0(\overline{\gamma}_p, \overline{\gamma}'_q; H, J) \) consists of constant cylinders and is naturally parametrized by \( \overline{\gamma}_p \). The Morse–Bott differential \( d^0_{\overline{\gamma}}(\overline{\gamma}_p), p \in \text{Crit}(f_{\overline{\gamma}}) \) is given by a count of gradient trajectories and therefore

\[
    d^0_{\overline{\gamma}}(\overline{\gamma}_M) = 0,
\]

while \( d^0_{\overline{\gamma}}(\overline{\gamma}_M) \) is a multiple of \( \overline{\gamma}_M \). The next statement is a reformulation of Lemma 4.25 in [7].
Proposition 2. If $\overline{\gamma} \in \mathcal{P}^<_{\lambda}$ is a good orbit then
$$d^0_{\overline{\gamma}m} = 0.$$  
If $\overline{\gamma} \in \mathcal{P}^<_{\lambda}$ is a bad orbit then
$$d^0_{\overline{\gamma}m} = \pm 2\overline{\gamma}_M.$$  

We shall describe in Sect. 5.1 a procedure for slowing down the Hamiltonian $H$ which produces a family $H^R$, $R > 0$.  

Proposition 3. Assume the almost complex structure $\widehat{J}$ on $\widehat{W}$ satisfies conditions (A) and (B) in Sect. 3.1. If $R > 0$ is large enough, then we have $\mathcal{M}^A(S_{\overline{\gamma}}, S_{\gamma}; H^R, \widehat{J}) = \emptyset$ if $\overline{\gamma} \neq \gamma \in \mathcal{P}^<_{\lambda}$, $\overline{\gamma}, \gamma \in i^{-1}(a)$ and $\mu(\overline{\gamma}) - \mu(\gamma) + 2(c_1(TW), A) = \pm 2\overline{\gamma}$.  

The proof of Proposition 3 is given at the end of Sect. 5.1.  

Remark 14. By a limiting argument, one sees that the conclusion of Proposition 3 still holds if $J \in \mathcal{J}_{\text{reg}}(H^R)$ is a small time-dependent perturbation of $\widehat{J}$. As a consequence, one can find regular almost complex structures for which $d^0_p(\overline{\gamma}_p) = d^0_{\overline{\gamma}p}(\overline{\gamma}_p)$, $\overline{\gamma} \in \mathcal{P}^<_{\lambda}$, $p \in \text{Crit}(f_{\overline{\gamma}})$ and Proposition 2 holds with $d^0_p$ replaced by $d^0$.  

Remark 15. We assume in this remark that the almost complex structure $\widehat{J}$ on $\widehat{W}$ satisfies conditions (A) and (B) in Sect. 3.1. Let $H^R_- \leq H^R_+$ be slow Hamiltonians as constructed in Sect. 5.1, and let $H^R_s$, $s \in \mathbb{R}$ be an increasing homotopy. If $R > 0$ is large enough and if $\partial_s H^R_s$ is small enough in $C^0$-norm, we have $\mathcal{M}^A(S_{\overline{\gamma}}, S_{\gamma}; H^R_s, \widehat{J}) = \emptyset$ for $\overline{\gamma} \in \mathcal{P}^<_{\lambda}(H^R_s)$, $\gamma \in \mathcal{P}^<_{\lambda}(H^R_+)$ such that $\overline{\gamma}, \gamma \in i^{-1}(a)$ and $\mu(\overline{\gamma}) - \mu(\gamma) + 2(c_1(TW), A) = -1$. This is essentially a consequence of the fact that the homotopy $H^R_s$ is independent of $\theta$, and is proved using the arguments for the proof of Proposition 3 in Sect. 5.1 (the dimensions of the corresponding moduli spaces have to be shifted by 1 due to the $s$-dependence of the equation). Moreover, the conclusion of Remark 14 above continues to hold in the $s$-dependent situation.  

This implies that the continuation morphism
$$\sigma_{H^R_-, H^R_+} : \text{BC}^a_s(H^-_+) \rightarrow \text{BC}^a_s(H^R_+)$$
defined in Sect. 2.2 preserves the filtrations. Indeed, this map preserves the degree and is obtained by a count of rigid configurations in $\mathcal{M}^A(p, q; H^R_s, \{f^\pm_{\gamma}\}, J_s)$, which can be of the following types:

1. $\mathcal{M}^A(m, M; H^R_s, \{f^\pm_{\gamma}\}, J_s)$, with $\mu(\overline{\gamma}) - \mu(\gamma) + 2(c_1(TW), A) = -1$;
2. $\mathcal{M}^A(m, m; H^R_s, \{f^\pm_{\gamma}\}, J_s)$ or $\mathcal{M}^A(M, M; H^R_s, \{f^\pm_{\gamma}\}, J_s)$, such that $\mu(\overline{\gamma}) - \mu(\gamma) + 2(c_1(TW), A) = 0$;
3. $\mathcal{M}^A(M, m; H^R_s, \{f^\pm_{\gamma}\}, J_s)$, with $\mu(\overline{\gamma}) - \mu(\gamma) + 2(c_1(TW), A) = 1$.  

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Our discussion shows that rigid configurations of type 1 do not exist, hence \( \sigma_{H^+, H^\#} \) preserves the filtrations.

We shall describe in Sect. 5.2 a procedure for stretching the neck in the neighbourhood of \( \partial W \) which produces a deformed almost complex structure \( J^\tau \) and a deformed Hamiltonian \( H^{\tau, R} \), \( \tau > 0 \) on \( \tilde{W} \), so that \( J^\tau \in \mathcal{J}_{\text{reg}}(H^{\tau, R}) \), the conclusion of Proposition 3 still holds and the Floer trajectories of \( H^{\tau, R} \) are “close” to punctured Floer trajectories in the symplectization, capped with rigid holomorphic planes in \( \tilde{W} \). From now on we denote by \( H, J \) the Hamiltonian \( H^{\tau, R} \) and the almost complex structure \( J^\tau \) with the above properties.

Let \((E'_d, \tilde{d}^r)\) and \((E'_d, \tilde{d}^r), r \geq 0\) be the spectral sequences corresponding to the above filtrations on the complexes \((BC^a_*(H), d)\) and \((BC^a_{i-1}(a), \leq a(\lambda), \delta)\) respectively. As a consequence of the previous discussion on \( \delta^0 \) and \( d^0 \) we infer that

\[
E^1_\delta := H_*(BC^{-1}_{i-1}(a), \leq a(\lambda), \delta^0) = C^i_{*}(a), \leq a(\lambda) \otimes H_*(S^1) \tag{46}
\]

and

\[
E^1_d := H_*(BC^a_*(H), d^0) = C^i_{*}(a), \leq a(\lambda) \otimes H_*(S^1) \tag{47}
\]

as \( \Lambda_\omega \)-modules. As a consequence we have \( E^1_\delta \simeq E^1_d \) as \( \Lambda_\omega \)-modules.

The next statement implies in particular that we have an isomorphism of differential complexes \((E^1_\delta, \tilde{d}^1) \simeq (E^1_d, \tilde{d}^1)\).

**Proposition 4.** There is an isomorphism of filtered complexes

\[
\Phi : BC^{-1}_{i-1}(a), \leq a(\lambda) \xrightarrow{\sim} BC^a_*(H)
\]

which decreases the degree by \( n - 3 \).

The proof of Proposition 4 is given in Sect. 6.

**Corollary 1.** The map \( \Phi \) induces an isomorphism of spectral sequences

\[
(E'_d, \tilde{d}^r) \xrightarrow{\sim} (E'_d, \tilde{d}^r), \quad r \geq 0.
\]

The differential \( \tilde{d}^1 \) is closely connected to the differential \( \partial \) for linearized contact homology. More precisely, \( \tilde{d}^1 \) is given by a count of elements in moduli spaces \( \mathcal{M}^A_c(p', q'; \{ f'_\gamma \}, J) \), \( p' \in \text{Crit}(f'_\gamma), q' \in \text{Crit}(f'_\gamma) \) with \( \text{ind}(p') = \text{ind}(q') \), so that either \( W^u(p') \) or \( W^s(q') \) is reduced to one point. If \( p' = m' \) is a maximum, then \( W^u(p') \) is reduced to one point and the moduli space \( \mathcal{M}^A_c(p', q'; \{ f'_\gamma \}, J) \) is diffeomorphic to \( \kappa_{p'} \) copies of the quotient \( \mathcal{M}^A_c(S_{p'}, S_{p'}^\gamma; J)/S^1 \). If \( q' = m' \) is a minimum, then \( W^s(q') \) is reduced to one point and the moduli space \( \mathcal{M}^A_c(p', q'; \{ f'_\gamma \}, J) \) is diffeomorphic to \( \kappa_{p'} \) copies of the quotient \( \mathcal{M}^A_c(S_{p'}, S_{p'}^\gamma; J)/S^1 \). Recalling that \( \mathcal{M}^A_c(\gamma', \gamma'; J) \)
is diffeomorphic to $\kappa_{\gamma'} \gamma'$ copies of $\mathcal{M}_c^A(S_{\gamma'}, S_{\gamma'}; J)/S^1$, and denoting by $\# \mathcal{M}_c^A(S_{\gamma'}, S_{\gamma'}; J)/S^1$ the algebraic count of elements of the moduli space $\mathcal{M}_c^A(S_{\gamma'}, S_{\gamma'}; J)/S^1$, we obtain

$$\partial(\gamma') = \sum_{A, \gamma'} \kappa_{\gamma'} \# \mathcal{M}_c^A(S_{\gamma'}, S_{\gamma'}; J)/S^1 e^{A_{\gamma'}},$$

$$\bar{\delta}^1(\gamma'M) = \sum_{A, \gamma'} \kappa_{\gamma'} \# \mathcal{M}_c^A(S_{\gamma'}, S_{\gamma'}; J)/S^1 e^{A_{\gamma'M}},$$

$$\bar{\delta}^1(\gamma'm) = \sum_{A, \gamma'} \kappa_{\gamma'} \# \mathcal{M}_c^A(S_{\gamma'}, S_{\gamma'}; J)/S^1 e^{A_{\gamma'm}}.$$

We define an automorphism $\Theta$ of $C_*^{i-1}(a) \leq a(\lambda) \otimes H_*(S^1)$ by

$$\Theta(\gamma' \otimes M) := \gamma' \otimes M, \quad \Theta(\gamma' \otimes m) := \frac{1}{\kappa_{\gamma'}} \gamma' \otimes m.$$  \hspace{1cm} (48)

Here $M, m$ are the generators of $H_0(S^1)$, respectively $H_1(S^1)$. Then

$$\bar{\delta}^1 = \Theta^{-1} \circ \partial \circ \Theta.$$  

In particular, $\Theta$ induces an isomorphism

$$\tilde{\Theta} : E^2_{d; *, 0} \tilde{\sim} HC_*^{i-1}(a) \leq a(\lambda, J) \otimes H_*(S^1).$$  \hspace{1cm} (49)

Theorem 1 follows from the previous considerations and the commutative diagram below, with vertical arrows being isomorphisms (see Sect. 7.1 for the full details).

5. Floer- and holomorphic cylinders

The goal of this section is to prove Proposition 3 and to prepare the proof of Proposition 4. For Proposition 3 we modify the Hamiltonian to another one with the same asymptotic slope but which varies very slowly in $M \times \mathbb{R}$, and analyze the limit of Floer trajectories as the rate of variation goes to zero. For Proposition 4 we must, loosely speaking, confine Floer trajectories in $\hat{W}$ near the boundary, so that we can view them in the symplectization $M \times \mathbb{R}$. In Sect. 5.2 we stretch the neck near $M$ and show, by a compactness
argument, that the rigid Floer trajectories connecting nonconstant orbits of $X_H$ are in bijective correspondence with punctured solutions of Floer’s equation in the symplectization $M \times \mathbb{R}$, capped at the punctures with rigid holomorphic planes in $\hat{W}$.

We recall that we work under the standing assumptions (A) and (B_a) in Sect. 3.1, Remark 9.

5.1. Slowing down the Hamiltonian

We start with the following two perturbative lemmas.

**Lemma 1.** Let $c : \mathbb{R} \to \mathbb{R}_+$ be a smooth increasing function with the property that $c'(s) \neq 0$ if $c(s) \in \text{Spec}(M, \lambda)$. Let $u = (f, a) : \mathbb{R} \times S^1 \setminus \{z_1, \ldots, z_k\} \to M \times \mathbb{R}$ be a solution of the equation

$$\partial_s u + \hat{J}_\infty \partial_\theta u - c(a(s, \theta)) \frac{\partial}{\partial t} = 0$$

which, in polar coordinates $(\rho_i, \theta_i)$ around $z_i$, satisfies

$$\lim_{z \to z_i} a(z) = -\infty, \quad \lim_{\rho_i \to 0} f(\rho_i, \theta_i) = \gamma_i'(T_i \theta_i),$$

for some $\gamma_1', \ldots, \gamma_k' \in \mathcal{P}_\lambda$, as well as one of the following asymptotic conditions at $\pm \infty$.

1. $$\lim_{s \to \pm \infty} a(s, \cdot) = \mp \infty, \quad \lim_{s \to \pm \infty} f(s, \theta) = \gamma_\pm'(T \pm \theta),$$

2. $$\lim_{s \to -\infty} a(s, \cdot) = +\infty, \quad \lim_{s \to +\infty} a(s, \cdot) = a_0, \quad \lim_{s \to \pm \infty} f(s, \theta) = \gamma_\pm'(T \pm \theta),$$

3. $$\lim_{s \to -\infty} a(s, \cdot) = a_0, \quad \lim_{s \to +\infty} a(s, \cdot) = -\infty, \quad \lim_{s \to \pm \infty} f(s, \theta) = \gamma_\pm'(T \pm \theta),$$

for $\gamma_\pm' \in \mathcal{P}^{\leq \alpha}_\lambda$. Given $c_0 > 0$ we denote $u_{c_0} := (f, a - c_0 s)$, so that $u_{c_0}$ satisfies the equation

$$\partial_s u + \hat{J}_\infty \partial_\theta u - (c(a(s, \theta)) - c_0) \frac{\partial}{\partial t} = 0.$$  

Assume $c_0$ is such that $c'(s) \neq 0$ if $c(s) - c_0 \in \text{Spec}(M, \lambda)$. Then the linearized operator $D_{u, c}$ corresponding to (50) is surjective if and only if the linearized operator $D_{u_{c_0}, c - c_0}$ corresponding to (55) is surjective.

**Proof.** We first treat case 1. The asymptotic conditions at $z_1, \ldots, z_k$ are obviously preserved, whereas the asymptotic conditions at $\pm \infty$ are still
satisfied because \( c_0 > 0 \). The two linearized operators have the same expressions, and their domain and target are canonically identified since the components of \( u \) and \( u_{c_0} \) along \( M \) are the same. The conclusion follows.

Cases 2 and 3 are similar so we treat only case 2. There is one asymptotic condition which changes, namely \( \lim_{n \to \infty} a(s, \cdot) = -\infty \), and this causes a change in the domain of the linearized operators. The domain of \( D_{u_{c_0}, c-c_0} \) contains an additional 1-dimensional summand \( V \) corresponding to a degeneracy of the asymptote \( \gamma' \) in the direction \( \frac{\partial}{\partial t} \). However, since (55) is invariant under translation in the \( s \)-variable, we have \( \partial_s u_{c_0} \in \ker D_{u_{c_0}, c-c_0} \) and \( \partial_s u_{c_0} \) has a nontrivial component along this additional 1-dimensional summand \( V \). Since the quotient \( \text{dom}(D_{u_{c_0}, c-c_0})/V \) is canonically identified with \( \text{dom}(D_{u,c}) \), the conclusion follows.

\[ \square \]

**Lemma 2.** For any \( C > 0 \) there exists \( \epsilon = \epsilon(C) > 0 \) such that, for any function \( c : \mathbb{R} \to [0, C] \) with \( \text{Supp}(c') \subset [-1, 1] \), such that \( c'(s) \neq 0 \) if \( c(s) \in \text{Spec}(M, \lambda) \), and such that

\[
0 \leq c' < \epsilon,
\]

the almost complex structure \( \widehat{J}_\infty \) is regular for all solutions \( u = (f, a) : \mathbb{R} \times S^1 \setminus \{z_1, \ldots, z_k\} \to M \times \mathbb{R} \) of (50) satisfying (51) and one of the additional asymptotic conditions (52), (53) or (54).

**Proof.** We prove the lemma by contradiction. Without loss of generality we can assume that there exist sequences \( \epsilon_n \to 0 \), \( c_n \) and \( u_n \) as in the statement of the lemma, where all the \( u_n \)'s have the same asymptotes, such that the linearized operators \( D_{u_n, c_n} \) are not surjective. The sequence \( c_n \) converges to some constant \( c \in [0, C] \) and we choose \( c_0 > 0 \) such that \( c - c_0 \notin \text{Spec}(M, \lambda) \). In particular, the constant \( c_0 \) satisfies the conditions of Lemma 1 for each \( c_n \) if \( n \) is large enough. Renaming the sequences \( u_n, c_0 \) and \( c_n - c_0 \) as \( u_n, c_n \), the operators \( D_{u_n, c_n} \) are not surjective by Lemma 1 and the maps \( u_n \) satisfy the asymptotic conditions (51) and (52). We also rename \( c - c_0 \) as \( c \).

After passing to a subsequence [5, Sect. 10.2] (see also the proof of step 1 in Proposition 5), \( u_n \) converges to a broken curve \( u \) whose components either solve \( \partial_s w + \widehat{J}_\infty \partial_\theta w - c_1 \frac{\partial}{\partial t} = 0 \) or are \( \widehat{J}_\infty \)-holomorphic. The linearized operator at each such component is surjective by Lemma 1 and the regularity assumption \( (B_n) \) for \( \widehat{J}_\infty \). For \( n \) large enough, \( u_n \) is approximated by a gluing construction as follows. Let us denote \( \widehat{\partial}_{c_n} := \partial_s + \widehat{J}_\infty \partial_\theta - c_n \frac{\partial}{\partial t} \). We replace each holomorphic component \( v = (f, a) \) by \( \psi_{c_n} := (f, a + c_n (-\infty) s) \) and each component \( w = (f, a) \) solving \( \widehat{\partial}_{c} w = 0 \) by \( w_{c_n} := (f, a + (c_n (a) - c) s) \). The first remark is that \( \| \widehat{\partial}_{c_n} (w_{c_n}) \| \) is arbitrarily small for \( n \) large enough. Here we use an \( L^p \)-norm with exponential weights for a metric which has cylindrical ends near the punctures \( z_i \). These weights do not play any role in estimating \( \| \widehat{\partial}_{c_n} (w_{c_n}) \| \) because \( \text{supp}(c'_n) \subset [-1, 1] \) and \( \widehat{\partial}_{c_n} (w_{c_n}) \) vanishes in a fixed neighborhood of the punctures \( z_i \) in view of the asymptotic condition (51). In fact, the function \( \widehat{\partial}_{c_n} (w_{c_n}) \) has compact
support because it satisfies (52). Decomposing $T(M \times \mathbb{R}) = (\frac{\partial}{\partial r}) \oplus \langle R_\lambda \rangle \oplus \xi$ we obtain

$$\tilde{\delta}_{c_n}w_{c_n} = \begin{pmatrix}
(\partial_s(a + (c_n(a) - c)s) - \lambda(\partial_\theta)f - c_n(a)) & \lambda(\partial_s f) + \partial_\theta(a + (c_n(a) - c)s) \\
0 & 0
\end{pmatrix} = \begin{pmatrix}
c_n'(a)s \partial_s a & c_n'(a)s \partial_\theta a \\
0 & 0
\end{pmatrix},$$

and this quantity goes to zero when $n \to \infty$ since it is bounded by a constant multiple of $\epsilon_n$. The second remark is that each linearized operator $D_{w_{c_n}}$ is close to the linearized operator $D_{w_{c}}$ because they again differ by a term involving $c_n'(a)$. Hence each $D_{w_{c_n}}$ is surjective, whereas each linearized operator $D_{v_{c_n},c_n(-\infty)}$ is surjective by the argument in Lemma 1. We now apply the standard gluing construction to the broken curve with components $v_{c_n}, w_{c_n}$ and get a solution $\tilde{u}_n$ solving $\tilde{\delta}_{c_n}(\tilde{u}_n) = 0$ and having a surjective linearized operator $D_{\tilde{u}_n,c_n}$ with uniformly bounded right inverse with respect to $n$. Moreover, $\tilde{u}_n$ is arbitrarily close to $u_n$ as $n \to \infty$. As a conclusion, the linearized operator $D_{u_n,c_n}$ is surjective for $n$ large enough, a contradiction. \qed

We now describe a family of admissible Hamiltonians $H^R : \hat{W} \to \mathbb{R}$ such that, on $M \times [0, \infty]$, we have $H^R(p, t) = h^R(t) = \rho(t)e^t + \tilde{C}$ for some $\tilde{C} \in \mathbb{R}$, with $\rho(t) = \alpha_0, \alpha_0 < \min \text{Spec}(M, \lambda)$ if $t$ is close to 0 and $\rho(t) = \alpha$ for $t \geq t_R$ large enough.

Let $\epsilon = \epsilon(\alpha)$ be such that Lemma 2 holds for $C = \alpha$. After possibly choosing a smaller value for $\epsilon$, we can assume without loss of generality that, for all $T_1 \neq T_2 \in \text{Spec}(M, \lambda)$ such that $T_1, T_2 \leq \alpha$ we have $|T_1 - T_2| \geq 2\epsilon$. We require the function $h^R$ to have the following property:

The interval $[0, t_R]$ is the concatenation of $N = \lceil \frac{2\epsilon}{\alpha} \rceil$ intervals $[-\frac{R}{2}, \frac{R}{2}]$ on which the function $c = \rho + \rho'$ satisfies the hypotheses of Lemma 2.

**Proof of Proposition 3.** We proceed by contradiction and we assume there exist sequences $R_n \to \infty$ and $u_n \in \mathcal{M}^A(S_\gamma, S_\gamma; H^{R_n}, J_\gamma, Y)$, $\gamma \in \mathcal{P}(H)$ with $\gamma \neq \gamma$ and $\mu(\gamma) - \mu(\gamma') + 2(c_1(TW), A) = 0$. Note that the index of each of the operators $D_{u_n}$ is equal to 1. After passing to a subsequence [5, Sect. 10.2] (see also the proof of Step 1 in Proposition 5), $\mu_n$ converges to a broken curve whose components in $M \times \mathbb{R}$ are either $J_\infty$ holomorphic curves, or satisfy (50) and condition (51), as well as one of the asymptotic conditions (52), (53) or (54). Moreover, the components in $\hat{W}$ are $\hat{J}$-holomorphic planes. By the regularity assumptions $(A)$ and $(B_\gamma)$ on $\hat{J}$ and Lemma 2 we know that the almost complex structures $\hat{J}_\infty$ and $\hat{J}$ are regular for all these components. As a consequence, the Fredholm index of the linearized operators at the components satisfying (50) is at most 1, and the same holds for the dimension of their kernel. On the other hand, since $\gamma \neq \gamma$ there is at least one such component which is not a (reparametrized) vertical cylinder. Since (50) is invariant under reparametrizations in both variables $s$ and $\theta$, we deduce that the kernel of the corresponding linearized operator is at least 2-dimensional, a contradiction. \qed
5.2. Stretching the neck near the boundary

In this section we assume without loss of generality that $H \in \mathcal{H}'$ is of the form $H = \alpha_0 e^t + \beta_0$ on a neighbourhood $\partial W \times [-\epsilon, \epsilon]$ of $\partial W$ in $\hat{W}$, where $0 < \alpha_0 < \min \text{Spec}(M, \lambda)$. Moreover, we assume that

$$\left| \frac{d}{dt}(e^{-t}h'(t)) \right| < 1. \quad (56)$$

We define a deformation $(\hat{W}^\tau, \hat{\omega}^\tau)$, $\tau \geq 0$ by

$$\hat{W}^\tau := W \bigcup (M \times [-\tau, 0]) \bigcup (M \times \mathbb{R}^+)$$

and

$$\hat{\omega}^\tau := \begin{cases} e^{-\tau} \omega, & \text{on } W, \\ d(e^t \lambda), & \text{on } M \times [-\tau, \infty[. \end{cases}$$

Let $\beta : \mathbb{R} \to [0, 1]$ be a smooth function such that $\beta(t) = 1$ for $t \leq -C$, $\beta(t) = 0$ for $t \geq 0$, where $C > 0$ is some large constant. We define a family of Hamiltonians $\{H^\tau\}$, $H^\tau : \hat{W}^\tau \to \mathbb{R}$ by

$$H^\tau := \begin{cases} e^{-\tau} H, & \text{on } W, \\ (e^{-\tau} \beta(t) + 1 - \beta(t))(\alpha_0 e^t + \beta_0), & \text{on } M \times [-\tau, 0], \\ h(t), & \text{on } M \times \mathbb{R}^+. \end{cases} \quad (57)$$

We define $H_\infty : M \times \mathbb{R} \to \mathbb{R}$ by $H_\infty := \lim_{\tau \to \infty} H^\tau|_{M \times [-\tau, \infty]}$. We then have $H_\infty(t) = 0$ for $t \leq -C$ and, if $C$ is large enough, we can choose $\beta$ so that the functions $e^{-t} \frac{d}{dt} H^\tau(t)$, $\tau \geq 0$ are increasing on $[-\tau, \infty[$.

Let $\hat{J}$ be an almost complex structure satisfying the regularity assumptions $(A)$ and $(B_\alpha)$. Similarly to Proposition 3.5 in [7], we can choose a small time-dependent perturbation $J_\infty$ of $\hat{J}_\infty$ which is localized in an
arbitrarily small neighbourhood of the 1-periodic orbits of $X_H$ and away from the orbits themselves, and which is regular for punctured Floer trajectories

$$u = (f, a) : \mathbb{R} \times S^1 \setminus \{z_1, \ldots, z_k\} \rightarrow M \times \mathbb{R}$$

satisfying

$$\partial_s u + J_\infty(\partial_\theta u - X_H(u)) = 0,$$

$$\lim_{s \rightarrow \pm \infty} u(s, \cdot) = \gamma_\pm(\cdot), \quad \gamma_\pm \in \mathcal{P}(H),$$

as well as (30) for some $\gamma_1' \in \mathcal{P}_{<\alpha}^{\leq}$, $i = 1, \ldots, k$. We denote by $J$ the resulting almost complex structure on $\hat{W}$, which coincides with $\hat{J}$ on $W$ and with $J_\infty$ on $M \times \mathbb{R}^+$. We define a family $\{\hat{J}^\tau\}$, $\tau \geq 0$ of $\hat{\omega}^*$-compatible almost complex structures on $\hat{W}^\tau$ by

$$\hat{J}^\tau := \begin{cases} \hat{J}, & \text{on } W, \\ J_\infty, & \text{on } M \times [-\tau, \infty[. \end{cases}$$

As above, we can choose a small time-dependent perturbation $\{J^\tau\}$ of the family $\{\hat{J}^\tau\}$ which is supported in an arbitrarily small neighbourhood of the 1-periodic orbits of $X_H$, which satisfies

$$J^\tau|_{M \times \mathbb{R}^+} \rightarrow J_\infty, \quad \tau \rightarrow \infty,$$

and such that each $J^\tau$ is regular for $H^\tau$.

The sets $\mathcal{P}(H)$ and $\mathcal{P}(H^\tau)$, $\tau \geq 0$ are naturally identified. These sets also contain constant elements (critical points of $H$ and $H^\tau$), but we shall tacitly consider in the rest of this section only nonconstant elements. The same convention applies to $\mathcal{P}(H_\infty)$.

For any $\tau \geq 0$, the set $\mathcal{F}_{\text{reg}}(H^\tau, J^\tau)$ of regular collections of perfect Morse functions $\{f^\tau_\gamma : S_\gamma \rightarrow \mathbb{R}, \gamma \in \mathcal{P}(H)\}$ defined in Sect. 2.2 is of the second Baire category in the space of collections of perfect Morse functions [7, Lemma 3.6]. Hence, given a sequence $\tau_\nu \rightarrow \infty$, $\nu \rightarrow \infty$, there exists a collection $\{f^\tau_\gamma\}$ which belongs to $\mathcal{F}_{\text{reg}}(H^\tau_\nu, J^\tau_\nu)$ for any $\nu$. The moduli spaces of Morse–Bott broken trajectories

$$\mathcal{M}^A(p, q; H^\tau_\nu, \{f^\tau_\gamma\}, J^\tau_\nu)$$

are then well-defined, for any $\gamma, \gamma' \in \mathcal{P}(H)$, $p \in \text{Crit}(f^\tau_\gamma)$, $q \in \text{Crit}(f^\tau_{\gamma'})$, and $A \in H_2(W; \mathbb{Z})$.

Our goal in this section is to establish a bijective correspondence with the moduli spaces $\mathcal{M}^A_c(p, q; H_\infty, \{f_\gamma\}, J_\infty)$ of capped punctured Morse–Bott broken trajectories which we now define.

Given elements $\gamma, \gamma' \in \mathcal{P}(H)$, $\gamma'_1, \ldots, \gamma'_k \in \mathcal{P}_{<\alpha}^{\leq}$ and $B \in H_2(M; \mathbb{Z})$, we denote by

$$\hat{\mathcal{M}}^B(S_\gamma, S_{\gamma'}, \gamma'_1, \ldots, \gamma'_k; H_\infty, J_\infty)$$
the space of punctured Floer trajectories, consisting of tuples \((u, L_1, \ldots, L_k)\) such that

\[
u = (f, a) : \mathbb{R} \times S^1 \setminus \{z_1, \ldots, z_k\} \to M \times \mathbb{R}
\]
satisfies (58) and (59) for some \(\gamma_- \in S_\mathcal{F}, \gamma_+ \in S_\mathcal{L}\), as well as (30) and (31) and the relation

\[
[(\sigma_{\mathcal{F}} \cup \sigma_{\gamma_1} \cup \cdots \cup \sigma_{\gamma_k}) \# f] = [\sigma_{\mathcal{F}} \# B].
\]

(60)

The additive group \(\mathbb{R}\) acts freely by shifts on the domain and we define the moduli space of punctured Floer trajectories by

\[
\mathcal{M}^B(S_\mathcal{F}, S_\mathcal{L}, \gamma_1', \ldots, \gamma_k'; H_\infty, J_\infty)
\]

\[
:= \hat{\mathcal{M}}^B(S_\mathcal{F}, S_\mathcal{L}, \gamma_1', \ldots, \gamma_k'; H_\infty, J_\infty)/\mathbb{R}.
\]

For a choice of regular \(J_\infty\) as above, this is a smooth manifold of dimension (see also [30, §3.3])

\[
\mu(\overline{\gamma}) - \mu(\overline{\gamma}) + 2\langle c_1(TW), B \rangle - \sum_{i=1}^k \overline{\mu}(\gamma_i') .
\]

Moreover, there are smooth evaluation maps

\[
\overline{ev}, ev : \mathcal{M}^B(S_\mathcal{F}, S_\mathcal{L}, \gamma_1', \ldots, \gamma_k'; H_\infty, J_\infty) \to S_\mathcal{F}, S_\mathcal{L}.
\]

(61)

Given \(A \in H_2(W; \mathbb{Z})\) we define the moduli space of capped punctured Floer trajectories

\[
\mathcal{M}_c^A(S_\mathcal{F}, S_\mathcal{L}; H_\infty, J)
\]
as the set of equivalence classes of pairs \((u, F)\), where \(u\) belongs to \(\mathcal{M}^B(S_\mathcal{F}, S_\mathcal{L}, \gamma_1', \ldots, \gamma_k'; H_\infty, J_\infty)\), \(\gamma_1', \ldots, \gamma_k' \in \mathcal{P}^{\text{ext}}, B \in H_2(M; \mathbb{Z})\), \(F\) is a collection of \(J\)-holomorphic planes in \(\hat{W}\) of total homology class \(A - B \in H_2(W; \mathbb{Z})\), and with top asymptotes \(\gamma_1', \ldots, \gamma_k'\). Two sets of asymptotic markers at the punctures asymptotic to \(\gamma_1', \ldots, \gamma_k'\) are equivalent if they satisfy (37). This moduli space is a smooth manifold of dimension

\[
\mu(\overline{\gamma}) - \mu(\overline{\gamma}) + 2\langle c_1(TW), A \rangle .
\]

Again, there are smooth evaluation maps

\[
\overline{ev}, ev : \mathcal{M}_c^A(S_\mathcal{F}, S_\mathcal{L}; H_\infty, J) \to S_\mathcal{F}, S_\mathcal{L}.
\]

(62)

The nonconstant elements of \(\mathcal{P}(H_\infty)\) and \(\mathcal{P}(H)\) are the same. We denote by \(\mathcal{F}_{\text{reg}}(H_\infty, J_\infty)\) the set whose elements are collections of perfect Morse functions \(\{f_\gamma : S_\gamma \to \mathbb{R}, \gamma \in \mathcal{P}(H)\}\) which satisfy the transversality properties in (11) with respect to the evaluation maps in (61) (or, equivalently, in (62)). The proof of Lemma 3.6 in [7] carries over verbatim and shows that
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$F_{\text{reg}}(H_\infty, J_\infty)$ is of the second Baire category in the space of collections of perfect Morse functions.

Given $\{f_\gamma\} \in F_{\text{reg}}(H_\infty, J_\infty), \overline{\gamma}, \gamma \in \mathcal{P}(H), p \in \text{Crit}(f_\overline{\gamma}), q \in \text{Crit}(f_\gamma), A \in H_2(W; \mathbb{Z})$, and $m \geq 0$, we define the moduli space $\mathcal{M}^A_{c,m}(p, q; H_\infty, \{f_\gamma\}, J)$ of capped punctured Morse–Bott broken trajectories with $m$ sublevels as the union for $\gamma_1, \ldots, \gamma_m \in \mathcal{P}(H)$ and $A_1 + \cdots + A_m = A$ of the fibered products

$$W^s(p) \times_{\overline{\gamma}} (\mathcal{M}^A_{c,1}(S_{\gamma_1}, S_{\gamma_2}) \times \mathbb{R}^+) \times_{\overline{\gamma}} (\mathcal{M}^A_{c,2}(S_{\gamma_1}, S_{\gamma_2}) \times \mathbb{R}^+) \times_{\overline{\gamma}} \cdots \times_{\overline{\gamma}} (\mathcal{M}^A_{c,m}(S_{\gamma_m-1}, S_{\gamma_m}) \times \mathbb{R}^+) \times_{\overline{\gamma}} W^s(q).$$

In the above formula one has to read

$$\mathcal{M}^A_{c,1}(S_{\gamma_1}, S_{\gamma_2}) = \mathcal{M}^A_{c,1}(S_{\gamma_2}, S_{\gamma_1}; H_\infty, J).$$

The moduli space of capped punctured Morse–Bott broken trajectories is

$$\mathcal{M}^A_c(p, q; H_\infty, \{f_\gamma\}, J) := \bigcup_{m \geq 0} \mathcal{M}^A_{c,m}(p, q; H_\infty, \{f_\gamma\}, J).$$

This is a smooth manifold of dimension

$$\dim \mathcal{M}^A_c(p, q; H_\infty, \{f_\gamma\}, J) = \mu(\gamma) + \text{ind}(p) - \mu(\overline{\gamma}) - \text{ind}(q) + 2\langle c_1(TW), A \rangle - 1.$$  

In the statement of the next result we fix a sequence $\tau^v \to \infty, \nu \to \infty$, and assume that

$$\{f_\gamma\} \in \bigcap_{\nu} F_{\text{reg}}(H^{\tau^v}, J^{\tau^v}) \cap F_{\text{reg}}(H_\infty, J_\infty). \tag{63}$$

**Proposition 5.** There exists $\nu_0$ such that the following holds for any $\nu \geq \nu_0$. For any $\overline{\gamma}, \gamma \in \mathcal{P}(H), p \in \text{Crit}(f_\overline{\gamma}), q \in \text{Crit}(f_\gamma)$, and $A \in H_2(W; \mathbb{Z})$ such that

$$\mu(\overline{\gamma}) + \text{ind}(p) - \mu(\gamma) - \text{ind}(q) + 2\langle c_1(TW), A \rangle - 1 = 0, \tag{64}$$

the (discrete and finite) moduli spaces

$$\mathcal{M}^A_c(p, q; H^{\tau^v}, \{f_\gamma\}, J^{\tau^v}) \text{ and } \mathcal{M}^A_c(p, q; H_\infty, \{f_\gamma\}, J)$$

are in natural bijective correspondence which preserves the signs of their elements.

Our proof of Proposition 5 involves an extension of the Correspondence Theorem 3.7 in [7] which states that, in dimension 0, the moduli spaces of Morse–Bott broken trajectories are in sign-preserving bijective correspondence with the moduli spaces of Floer trajectories for a suitable time-dependent perturbation of the Hamiltonian. This time-dependent perturbation is constructed as follows [11, Proposition 2.2]. For each $\gamma \in \mathcal{P}(H)$
we denote by $\ell_\gamma \in \mathbb{Z}^+$ the maximal positive integer such that $\gamma(\theta + 1/\ell_\gamma) = \gamma(\theta)$, $\theta \in S^1$. For each $S_\gamma$, $\gamma \in \mathcal{P}(H)$ we choose a chart $S^1 \times \mathbb{R}^{2n-1} \ni (\tilde{\theta}, p)$ in a neighbourhood of the geometric image of $\gamma$ such that $\tilde{\theta} \circ \gamma(\theta) = \ell_\gamma \theta$ and $p \circ \gamma(\theta) = 0$. We also choose a smooth cut-off function $\rho_\gamma = \rho_{S_\gamma} : S^1 \times \mathbb{R}^{2n-1} \to \mathbb{R}$. For $\delta > 0$ small enough the time-dependent Hamiltonian

$$H_\delta(\theta, \tilde{\theta}, p) := H + \delta \sum_{S_\gamma} \rho_\gamma(\tilde{\theta}, p) f_\gamma(\tilde{\theta} - \ell_\gamma \theta)$$  \hfill (65)

has only nondegenerate orbits. More precisely, out of each circle $S_\gamma$ of elements of $\mathcal{P}(H)$ survive precisely the orbits $\gamma_m$ and $\gamma_M$ starting at the minimum, respectively maximum of $f_\gamma$, with Conley–Zehnder index $\mu(\gamma_\nu) = \mu(\gamma) + \text{ind}(p)$.

**Proof of Proposition 5.** The Correspondence Theorem 3.7 in [7] implies the following. For each $\nu$, there exists $\delta_\nu > 0$ such that, for any $\delta \in [0, \delta_\nu]$, the moduli space $\mathcal{M}_1^A(p, q; H^{\tau_\nu}, \{f_\gamma\}, J^{\tau_\nu})$ is in sign-preserving bijective correspondence with the moduli space of Floer trajectories $\mathcal{M}_1^A(\tilde{\gamma}_p, \tilde{\gamma}_q; H^{\tau_\nu}, J^{\tau_\nu})$ for the time-dependent perturbation described by (65). Here the standing assumptions are (63) and (64), and we can assume without loss of generality that $\delta_\nu \to 0$, $\nu \to \infty$.

Let $H^\nu := H^{\tau_\nu}_\delta$, $J^\nu := J^{\tau_\nu}$. We prove Proposition 5 in three steps.

**Step 1.** We show that, after passing to a subsequence, any sequence $\nu, \nu \in \mathcal{M}_1^A(\tilde{\gamma}_p, \tilde{\gamma}_q; H^\nu, J^\nu)$ converges to an element of the moduli space $\mathcal{M}_1^A(p, q; H^\infty, \{f_\gamma\}, J)$.

There are two types of degenerations for the Floer trajectories $\nu$ as $\nu \to \infty$. The first type, due to the fact that the perturbation $\delta_\nu \to 0$, is that $\nu$ can spend an amount of time $T_\nu \simeq T/\delta_\nu \to \infty$ in the neighbourhood of a nontrivial periodic orbit of $H^\infty$. The second type, due to the fact that we stretch the neck, is that $\|d\nu\|_{L^\infty}$ can become unbounded on suitable sequences $(s_\nu, \theta_\nu) \in \mathbb{R} \times S^1$.

The first type of degeneration is dealt with in [7, Proposition 4.7]. The second type of degeneration is the object of the compactness theorems for SFT [5, Sect. 10.2], applied in the following setting. By the mapping cylinder construction, the elements of the moduli space $\mathcal{M}_1^A(\tilde{\gamma}_p, \tilde{\gamma}_q; H^\nu, J^\nu)$ can be interpreted as holomorphic sections of a symplectic fibration over $\mathbb{R} \times S^1$ with fiber $\tilde{W}^{\tau_\nu}$ for some almost complex structure $\tilde{J}^\nu$. This is an almost complex manifold with symmetric cylindrical ends in the sense of [5, Sects. 2.2 and 3.2]. Although the case of a non-compact fiber is not explicitly considered in [5], this causes no problem here because $\tilde{W}^{\tau_\nu}$ is convex at infinity.

The important observation now is that these two types of degenerations happen for regions of $\mathbb{R} \times S^1$ which have disjoint images via $\nu$. More precisely, for the second type of degeneration, the image of a neighbourhood of $(s_\nu, \theta_\nu)$ is necessarily contained in the region $W \cup (\tilde{M} \times [-\tau_\nu, -C]) \subset \tilde{W}^{\tau_\nu}$ for $\nu$ large enough (otherwise it would give rise to a nonconstant $J$-holo-
morphing sphere in \( M \times \mathbb{R} \)). On the other hand, for the first type of\n\( \text{degeneration} \), the image of a cylinder of size \( T_v \) staying close to a nonconstant\nperiodic orbit of \( H_\infty \) is contained in \( M \times \mathbb{R}_+ \). As a consequence, the\narguments used to deal with the first type of degeneration do not interfere\nwith those used to deal with the second type of degeneration. We obtain,

\[
(c_m, [\tilde{u}_m, F_m], c_{m-1}, [\tilde{u}_{m-1}, F_{m-1}], \ldots, [\tilde{u}_1, F_1], c_0), \quad m \geq 0
\]

with the following properties (see also [7, Definition 4.1]):

- \( \tilde{u}_i = (\tilde{f}_i, \tilde{a}_i) : \Sigma_i \setminus \{ z_i^1, \ldots, z_i^{k_i} \} \rightarrow M \times \mathbb{R}, i = 1, \ldots, m, k_i \geq 0 \) with

\[
\Sigma_i = \mathbb{R} \times S^1 \quad \text{or} \quad \Sigma_i = \mathbb{R} \times S^1 \sqcup \mathbb{R} \times S^1 =: \Sigma_i \sqcup \Sigma_i,
\]

satisfying \( \partial_s \tilde{u}_i + J_{\infty} (\partial_\theta \tilde{u}_i - X_{H_\infty} (\tilde{u}_i)) = 0 \) and the asymptotic conditions

- if \( \Sigma_i = \mathbb{R} \times S^1 \), then \( \lim_{s \rightarrow -\infty} \tilde{u}_i(s, \cdot) \in S_{\gamma_i} \), \( \lim_{s \rightarrow \infty} \tilde{u}_i(s, \cdot) \in S_{\gamma_i-1} \),

- if \( \Sigma_i = \Sigma_i \sqcup \Sigma_i \), then

\[
\begin{align*}
\lim_{s \rightarrow -\infty} \tilde{u}_i|_{\Sigma_i}(s, \cdot) &\in S_{\gamma_i}, & \lim_{s \rightarrow \infty} \tilde{u}_i|_{\Sigma_i}(s, \cdot) &\in S_{\gamma_i-1}, \\
\lim_{s \rightarrow -\infty} \tilde{f}_i|_{\Sigma_i}(s, \theta) &\in \gamma_i(-T_i \theta), & \lim_{s \rightarrow \infty} \tilde{f}_i|_{\Sigma_i}(s, \theta) &\in \gamma_i'(T_i \theta), \\
\lim_{s \rightarrow -\infty} \tilde{a}_i|_{\Sigma_i}(s, \cdot) &\in -\infty, & \lim_{s \rightarrow \infty} \tilde{a}_i|_{\Sigma_i}(s, \cdot) &\in -\infty,
\end{align*}
\]

as well as (30) at \( z_i^1, \ldots, z_i^{k_i} \) with asymptotes \( \gamma_1^i, \ldots, \gamma_{k_i}^i \).

Here \( \gamma_i \in \mathcal{P}(H_\infty), i = 0, \ldots, m \) with \( \gamma_m = \gamma, \gamma_0 = \gamma \), the orbits \( \gamma_i, \gamma_i' \in \mathcal{P}_\lambda^{\leq \alpha}, i = 1, \ldots, m \) have periods \( T_i, T_i \), and we have \( \gamma_1^i, \ldots, \gamma_{k_i}^i \in \mathcal{P}_\lambda^{\leq \alpha} \).

- \( c_0 : [-1, +\infty[ \rightarrow S_{\gamma_0}, c_i : [-T_i/2, T_i/2] \rightarrow S_{\gamma_i}, i = 1, \ldots, m - 1 \) for some \( T_i > 0 \), and \( c_m : ]-\infty, 1] \rightarrow S_{\gamma_m} \), satisfy \( \dot{c}_i = \nabla f_i \circ c_i, i = 0, \ldots, m \).

- \( \overline{c} \nu(\tilde{u}_i) = \overline{c} \nu(c_i), \overline{c} \nu(\tilde{u}_i) = \overline{c} \nu(c_{i-1}), i = 1, \ldots, m, \) and \( c_0(+\infty) = q \),

\( c_m(-\infty) = p \).

- \( F_i, i = 1, \ldots, m \) is a genus zero \( J \)-holomorphic building in \( \hat{W} \) of height \( 0|1|k_+, k_+ \geq 0 \), whose top asymptotes are \( \gamma_1^i, \ldots, \gamma_{k_i}^i \) and, possibly, \( \gamma_i, \gamma_i' \) if \( \Sigma_i = \Sigma_i \sqcup \Sigma_i \), and whose underlying nodal Riemann surface has exactly \( k_i \), respectively \( k_i + 1 \) connected components.

- if \( \Sigma_i = \Sigma_i \sqcup \Sigma_i \), the following additional matching condition has to be satisfied. There exists a finite sequence \( (F_1^i, \ldots, F_{n_i}^i), n_i \geq 1 \) of connected components of the building \( F_i \), on which we have marked punctures \( \overline{z}_j^i, \overline{z}_j^i \), \( j = 1, \ldots, n_i \) such that

- \( F_j^i \) is asymptotic to \( \overline{z}_j^i \) at \( \overline{z}_j^i, F_{j+1}^i \) is asymptotic to \( \gamma_j^i \) at \( \overline{z}_j^i \),

- the component \( F_j^i, j = 1, \ldots, n_i - 1 \) can be glued to \( F_{j+1}^i \) at \( \overline{z}_j^i, \overline{z}_{j+1}^i \),

- upon gluing in this way all the \( F_j^i, j = 1, \ldots, n_i \) and capping all the

remaining punctures except \( \overline{z}_1^i, \overline{z}_{n_i}^i \), the asymptotic markers at \( \overline{z}_1^i, \overline{z}_{n_i}^i \),
are opposite to each other with respect to the conformal structure on the cylinder.

Note that this last condition is present because the almost complex structure in Floer’s equation (58) depends on \( \theta \) (also, it ensures that pregluing produces an approximate solution).

We must show that any such tuple belongs to the moduli space \( \mathcal{M}^A_c(p, q; H_\infty, \{ f_\gamma \}, J) \). For that purpose we notice that it is enough to show that the domain of each \( \tilde{u}_i \) is connected. If this is the case, the dimension condition (64) implies that all the holomorphic buildings \( F_i \) have height \( k_+ = 0 \), and we recover the definition of an element of \( \mathcal{M}^A_c(p, q; H_\infty, \{ f_\gamma \}, J) \).

Let us assume by contradiction that the domain of some \( \tilde{u}_i \) has two components. Let \( \tilde{u}_i = (f_i, a_i) \) and let \( t_{i-1} \) be the level on which the asymptote \( \gamma_{i-1} \) is located. Then the period \( T_{i-1} \) of \( \gamma_{i-1} \) is given by \( T_{i-1} = e^{-t_{i-1}}h'(t_{i-1}) \). Let \( s_0 \in \mathbb{R} \) be such that \( \tilde{u}_i \big|_{\Sigma}([s_0, \infty[ \times S^1) \) is contained in a neighbourhood of \( \gamma_{i-1} \) where \( J_\infty = \tilde{J}_\infty \) and therefore preserves \( \xi \).

We claim that there exists a point \( (s, \theta) \in [s_0, \infty[ \times S^1 \subset \Sigma_i \) such that \( a_i(s, \theta) > t_{i-1} \). Let \( \tilde{a}_i(s) := \int_{S^1} a_i(s, \theta) d\theta \). For \( s \geq s_0 \), Floer’s equation \( \partial_s \tilde{u}_i + J_\infty \partial_\theta \tilde{u}_i - e^{-a_i} h'(a_i) \frac{\partial}{\partial s} = 0 \) can be rewritten as

\[
\partial_s a_i - \lambda (\partial_\theta f_i) - e^{-a_i} h'(a_i) = 0, \tag{66}
\]

\[
\lambda (\partial_s f_i) + \partial_\theta a_i = 0,
\]

\[
\pi_\xi \circ df_i \circ j - J_\infty \pi_\xi \circ df_i = 0, \tag{67}
\]

where \( \pi_\xi : T\Sigma \to \xi \) is the projection along \( R_\lambda \). By (66) and using the fact that the function \( t \mapsto e^{-t}h'(t) \) is increasing we get that

\[
\partial_s \tilde{a}_i(s) = \int_{S^1} \lambda (\partial_\theta f_i(s, \theta)) d\theta + \int_{S^1} e^{-a_i(s, \theta)} h'(a_i(s, \theta)) d\theta
\]

\[
\leq \int_{S^1} \lambda (\partial_\theta f_i(s, \theta)) d\theta + T_{i-1}.
\]

On the other hand, for each \( s_0 \leq s \leq s' \) we have by Stokes’ theorem

\[
\int_{S^1} \lambda (\partial_\theta f_i(s', \theta)) d\theta - \int_{S^1} \lambda (\partial_\theta f_i(s, \theta)) d\theta = \int_{[s, s'] \times S^1} f^* d\lambda \geq 0, \tag{68}
\]

where the last inequality follows from (67) and from the compatibility of \( J_\infty \) with \( d\lambda \). Because \( \lim_{s \to \infty} f_i(s, \cdot) = \gamma_{i-1} \), it follows that

\[
\lim_{s \to \infty} \int_{S^1} \lambda (\partial_\theta f_i(s, \theta)) d\theta = -T_{i-1}
\]

and, by the monotonicity relation (68), we get

\[
\partial_s \tilde{a}_i(s) \leq 0, \quad s \geq s_0.
\]

Since \( \lim_{s \to \infty} a_i(s, \theta) = t_{i-1} \), we have \( \tilde{a}_i(s) \geq t_{i-1} \) for \( s \geq s_0 \). But if \( a_i(s, \theta) \) is constant equal to \( t_{i-1} \), then \( \int f^* d\lambda = 0 \), which means that \( \tilde{u}_i \).
is constant, a contradiction. Hence we must have \( a_i(s, \theta) > t_{i-1} \) for some \((s, \theta) \in \{s_0, \infty\} \times S^1\) as claimed.

On the other hand, \( \lim_{s \to -\infty} a_i(s, \theta) = -\infty \), so that \( a_i \) has a local maximum on \( \mathbb{R} \times S^1 \), which is impossible by the maximum principle. This final contradiction shows that each \( \tilde{u}_i \) has a domain which is connected.

**Step 2.** We show that, for \( \nu > 0 \) large enough, the elements of \( \mathcal{M}(\overline{\gamma}_p, \gamma_q; H^\tau, J^\nu) \) can be obtained by a gluing construction from the elements of \( \mathcal{M}_{c}\langle p, q; H_\infty, \{f_y\}, J \rangle \).

For \( \tau > 0 \) large enough let \( H_\infty^\tau \) be the trivial extension to \( \tilde{W}^\tau \) of the Hamiltonian \( H_\infty |_{M \times [-\tau, \infty]} \). For \( \delta > 0 \) small enough, let \( H_\infty^\tau, \delta \) be the perturbation of \( H_\infty^\tau \) described by (65).

Combining the gluing arguments from Proposition 4.22 in [7] with arguments for gluing holomorphic curves along nondegenerate Reeb orbits in SFT ([3] and [6, Proposition 5]), we produce a Floer trajectory in \( \tilde{W}^\tau \) for \( H_\infty^\tau, \delta \), for \( \tau > 0 \) large enough and \( \delta > 0 \) small enough. As in Step 1, the important observation is that the gluing constructions take place in disjoint regions of \( \tilde{W}^\tau \). More precisely, the gluing construction of punctured Floer trajectories with gradient trajectories along nonconstant periodic orbits of \( H_\infty \) takes place in \( M \times \mathbb{R}_+ \), while the gluing construction of punctured Floer trajectories which are holomorphic near the punctures with holomorphic planes takes place in \( M \times [-\tau, -C] \). Therefore, the gluing estimates do not interfere and we obtain the desired Floer trajectory for \( H_\infty^\tau, \delta \). Since the elements of \( \mathcal{M}(p, q; H_\infty, \{f_y\}, J) \) are rigid, the same arguments as in the uniqueness part of [7, Proposition 4.22] show that we obtain in this way all elements in \( \mathcal{M}(\overline{\gamma}_p, \gamma_q; H_\infty^\tau, \delta, J^\nu) \).

On the other hand, for \( \tau > 0 \) large enough the Floer trajectories for \( H_\infty^\tau, \delta \) and for \( H_\delta^\tau \) are in bijective correspondence. The claim in Step 2 follows.

**Step 3.** Combining Steps 1 and 2, we obtain a sign-preserving bijective correspondence between the 0-dimensional moduli spaces \( \mathcal{M}(\overline{\gamma}_p, \gamma_q; H^\nu, J^\nu) \) and \( \mathcal{M}_{c}\langle p, q; H_\infty, \{f_y\}, J \rangle \), when \( \nu > 0 \) is large enough. As explained in the beginning of the proof, the former moduli space is in sign-preserving bijective correspondence with the moduli space \( \mathcal{M}(p, q; H^\nu, \{f_y\}, J^\nu) \), and the conclusion follows.

**Remark 16.** Given an increasing homotopy \( H_t \) between admissible Hamiltonians \( H_- \leq H_+ \), the continuation morphism

\[
\sigma_{H_+, H_-} : BC_*^a(H_-) \to BC_*^a(H_+)
\]

can also be described in terms of capped punctured Floer trajectories. More precisely, (57) applied to the homotopy \( H_t \) gives rise to increasing homotopies \( H_\tau^\tau \) from \( H_-^\tau \) to \( H_+^\tau \) and, in the limit \( \tau \to \infty \), to an increasing homotopy \( H_{t, \infty} \) from \( H_-, \infty \) to \( H_+, \infty \) on \( M \times \mathbb{R} \). The latter homotopy is supported in the region \( t > -C \) for some large enough constant \( C > 0 \). Given a generic homotopy \( J_t \) of almost complex structures on \( \hat{W} \), we denote by \( J_\tau^\tau \), respectively \( J_{t, \infty} \), the corresponding homotopies of almost complex structures on \( \hat{W} \).
structures on $\hat{W}_\tau$, respectively $M \times \mathbb{R}$. The moduli space
\[
\mathcal{M}_\ast^A(p, q; H_{s, \infty}, \{f_\gamma^\pm\}, J_{s, \infty})
\]
of capped punctured $s$-dependent Morse–Bott broken trajectories is
defined similarly to the moduli space $\mathcal{M}_\ast^A(p, q; H_T^\pm, \{f_\gamma^\pm\}, J_T^\pm)$ of $s$-dependent Morse–Bott broken trajectories introduced in Sect. 2.2. The modification of the definition is straightforward, with Floer trajectories for $H_T^\pm$, $H_T^\pm$ being replaced by capped punctured Floer trajectories for $H_{s, \infty}$, $H_{s, \infty}$, and respectively $H_{s, \infty}$. The obvious $s$-dependent version of Proposition 5 still holds true with the same proof, so that the moduli spaces $\mathcal{M}_\ast^A(p, q; H_{s, \infty}, \{f_\gamma^\pm\}, J_{s, \infty})$ and $\mathcal{M}_\ast^A(p, q; H_T^\pm, \{f_\gamma^\pm\}, J_T^\pm)$ of dimension 0 are in sign-preserving bijective correspondence.

6. A filtered isomorphism

In this section we construct the chain complex isomorphism in Proposition 4 by counting “mixed moduli spaces” consisting of rigid punctured cylinders which are asymptotic to a Reeb orbit at one end and to a Hamiltonian orbit at the other end. We prove in Proposition 7 that this isomorphism is compatible with the continuation maps in Floer homology.

For this construction, we omit the free homotopy classes of loops from the notation. Let $\rho : \mathbb{R} \to [0, 1]$ be a smooth increasing function such that $\rho(s) = 0$ if $s \ll 0$ and $\rho(s) = 1$ if $s \gg 0$. We define

\[
H_\rho^c : \mathbb{R} \times (M \times \mathbb{R}) \to \mathbb{R}, \quad (s, p, t) \mapsto \rho(s)H_\infty(t).
\]

Given $\gamma', \gamma'_1, \ldots, \gamma'_k \in \mathcal{P}_{\lambda}^{\leq \alpha}$, $\gamma \in \mathcal{P}(H)$, $B \in H_2(M; \mathbb{Z})$ we define the space of punctured interpolating trajectories
\[
\mathcal{M}^B(S_{\gamma'}, S_{\gamma'_1}, \ldots, S_{\gamma'_k}; H_\rho^c, J_\infty)
\]
to consist of tuples $(u, L_1, \ldots, L_k)$ such that $u = (f, a) : \mathbb{R} \times S^1 \setminus \{z_1, \ldots, z_k\} \to M \times \mathbb{R}$ satisfies

\[
\partial_s u + J_\infty(\partial_\theta u - X_{H_\rho^c}(u)) = 0, \quad \lim_{s \to -\infty} a(s, \theta) = +\infty, \quad \lim_{s \to -\infty} f(s, \cdot, \cdot) \in S_{\gamma'},
\]

\[
\lim_{s \to -\infty} u(s, \cdot, \cdot) = \gamma(\cdot + \theta), \quad \text{for some } \theta \in S^1,
\]

and (30), (31) and (60). Note that we have natural evaluation maps

\[
\text{ev} : \mathcal{M}^B(S_{\gamma'}, S_{\gamma'_1}, \ldots, S_{\gamma'_k}; H_\rho^c, J_\infty) \to S_{\gamma'}
\]

and

\[
\overline{\text{ev}} : \mathcal{M}^B(S_{\gamma'}, S_{\gamma'_1}, \ldots, S_{\gamma'_k}; H_\rho^c, J_\infty) \to S_{\gamma'}.
\]

As in Proposition 3.5 in [7] we can choose $J_\infty$ regular for all the elements of $\mathcal{M}^B(S_{\gamma'}, S_{\gamma'_1}, \ldots, S_{\gamma'_k}; H_\rho^c, J_\infty)$ and so that it still satisfies all the previous regularity assumptions. Note that, due to the last term in the left hand side
of (69), the additive group \( \mathbb{R} \) does not act on the space \( \mathcal{M}^B \), and therefore we call it from now on the **moduli space of punctured interpolating trajectories**. This is a smooth manifold of dimension [30, §3.3]

\[
\dim \mathcal{M}^B(S'_\varphi, S'_\gamma, \gamma'_1, \ldots, \gamma'_k; H^p, J) = \mu(\gamma') - \mu(\gamma) + 2(c_1(\xi), B) - \sum_{i=1}^k \tilde{\mu}(\gamma'_i) + 1.
\]

Given \( m, \ell \geq 0 \), let \( \gamma', \gamma'_1, \ldots, \gamma'_k \), \( \gamma'_{m+\ell+1}, \ldots, \gamma'_{k_{m+\ell+1}} \in \mathcal{P}_\mathcal{K}^{<\alpha} \), \( \gamma \in \mathcal{P}(H) \), \( p' \in \text{Crit}(f'_\varphi) \), \( q \in \text{Crit}(f'_\gamma) \) and \( B \in H_2(M, \mathbb{Z}) \). We denote by

\[
\mathcal{M}^B_{m, \ell}(p', q, \gamma'_1, \ldots, \gamma'_m; \gamma', \gamma'_1, \ldots, \gamma'_k; \gamma'_{m+\ell+1}, \ldots, \gamma'_{m+\ell+1}; H^p, \{ f'_\gamma, f''_\gamma \}, J)
\]

the union over \( \gamma'_1, \ldots, \gamma'_m \in \mathcal{P}_\mathcal{K}, \gamma'_m+1, \ldots, \gamma'_{m+\ell} \in \mathcal{P}(H) \) and \( B_1 + \cdots + B_{m+\ell+1} = B \) of the fibered products

\[
W^u(p') \times_\mathbb{R} \left( \mathcal{M}^B_{1}(S'_\varphi, S'_\gamma, \gamma'_1, \ldots, \gamma'_1; J) / \mathbb{R} \right) \times \mathbb{R}^+
\]

\[
\varphi_{f'_\varphi} \circ \text{ev} \times \text{ev} \cdots \left( \mathcal{M}^B_{m, \ell}(S'_\varphi, S'_\gamma, \gamma'_1, \ldots, \gamma'_k; J) / \mathbb{R} \right) \times \mathbb{R}^+ \cup
\]

\[
\varphi_{f'_\varphi} \circ \text{ev} \times \text{ev} \left( \mathcal{M}^B_{m, \ell}(S'_\varphi, S'_\gamma, \gamma'_1, \ldots, \gamma'_k; J) / \mathbb{R} \right) \times \mathbb{R}^+ \cup
\]

\[
\varphi_{f'_\gamma} \circ \text{ev} \times \text{ev} \left( \mathcal{M}^B_{m+1}(S'_\varphi, S'_\gamma, \gamma'_1, \ldots, \gamma'_k; J) / \mathbb{R} \right) \times \mathbb{R}^+ \cup
\]

\[
\varphi_{f'_\gamma} \circ \text{ev} \times \text{ev} \left( \mathcal{M}^B_{m+2}(S'_\varphi, S'_\gamma, \gamma'_1, \ldots, \gamma'_k; J) / \mathbb{R} \right) \times \mathbb{R}^+ \cup
\]

\[
\varphi_{f'_\varphi} \circ \text{ev} \times \text{ev} \left( \mathcal{M}^B_{m+\ell}(S'_\varphi, S'_\gamma, \gamma'_1, \ldots, \gamma'_k; J) / \mathbb{R} \right) \times \mathbb{R}^+ \cup
\]

\[
\text{ev} \times W^s(q).
\]

By our transversality assumptions, this is a smooth manifold of dimension

\[
\dim \mathcal{M}^B_{m, \ell}(p', q, \gamma'_1, \ldots, \gamma'_k; \gamma'_{m+\ell+1}, \ldots, \gamma'_{k_{m+\ell+1}}; H^p, \{ f'_\gamma, f''_\gamma \}, J) = \text{ind}(p') - 1 + \left( \dim \mathcal{M}^B_{m, \ell}(S'_\varphi, S'_\gamma; J) / \mathbb{R} + 1 \right) - 1
\]

\[
+ \ldots + \left( \dim \mathcal{M}^B_{m, \ell}(S'_\varphi, S'_\gamma; J) / \mathbb{R} + 1 \right) - 1
\]

\[
+ \left( \dim \mathcal{M}^B_{m+1}(S'_\varphi, S'_\gamma; J) / \mathbb{R} + 1 \right) - 1
\]

\[
+ \left( \dim \mathcal{M}^B_{m+2}(S'_\varphi, S'_\gamma; J) / \mathbb{R} + 1 \right) - 1
\]

\[
+ \ldots + \sum \dim \mathcal{M}^B_{m+\ell+1}(S'_\varphi, S'_\gamma; J) - 1 + (1 - \text{ind}(q))
\]

\[
= \mu(\gamma'_p) - \mu(\gamma'_q) + \text{ind}(p') - \text{ind}(q)
\]

\[
+ 2(c_1(\xi), B_1 + \cdots + B_{m+\ell+1}) - \sum_{i=1}^{m+\ell+1} \sum_{j=1}^{k_i} \tilde{\mu}(\gamma'_i)
\]

\[
= \mu(\gamma'_p) - \sum_{i=1}^{m+\ell+1} \sum_{j=1}^{k_i} \tilde{\mu}(\gamma'_i) + 2(c_1(\xi), B).
\]
We denote
\[
\mathcal{M}^B(p', q, \gamma_1', \ldots, \gamma_k', H^\infty, \{f_\gamma, f'_\gamma\}, J_\infty) := 
\bigcup_{m, \ell \geq 0} \mathcal{M}^B_{m, \ell}(p', q, \gamma_1', \ldots, \gamma_k', \ldots, \gamma_{m+\ell+1}, H^\infty, \{f_\gamma, f'_\gamma\}, J_\infty)
\]
with \(\{\gamma_1', \ldots, \gamma_k', \ldots, \gamma_{m+\ell+1}\} = \{\gamma_1', \ldots, \gamma_k'\}\), and we call this the moduli space of punctured interpolating Morse–Bott broken trajectories, whereas the spaces \(\mathcal{M}^B_{m, \ell}\) are called the moduli spaces of punctured interpolating Morse–Bott broken trajectories with \(m\) holomorphic sublevels and \(\ell\) Floer sublevels. We refer to Fig. 3 for a representation of the elements of these moduli spaces (see also their capped version below). The description of the topology of the compactified moduli spaces \(\overline{\mathcal{M}}^B(p', q, \gamma_1', \ldots, \gamma_k'; H^\infty, \{f_\gamma, f'_\gamma\}, J_\infty)\) is entirely similar to that of the compactified moduli spaces \(\overline{\mathcal{M}}^A(p, q; H, \{f_\gamma\}, J)\) in Sect. 2.2 and \(\overline{\mathcal{M}}^B(p', q', \gamma_1', \ldots, \gamma_k'; \{f'_\gamma\}, J_\infty)\) in Sect. 3.2.

The system of coherent orientations in [7, Sect. 4.4] orients zero-dimensional moduli spaces as the set of equivalence classes of pairs \(A \in H_2(W; \mathbb{Z})\), \(\mathcal{P}^{\leq \alpha}_\gamma \in \mathcal{P}^{\leq \alpha}_\lambda\), \(\gamma \in \mathcal{P}(H)\), \(p' \in \text{Crit}(f'_\gamma)\), \(q \in \text{Crit}(f_\gamma)\) we define the moduli space of capped punctured interpolating Morse–Bott broken trajectories
\[
\mathcal{M}^A_c(p', q; H^\infty, \{f_\gamma, f'_\gamma\}, J)
\]
as the set of equivalence classes of pairs \(F = (F', F'')\), where \(F'\) is an element of the moduli space \(\mathcal{M}^B(p', q, \gamma_1', \ldots, \gamma_k', H^\infty, \{f_\gamma, f'_\gamma\}, J_\infty)\), \(\gamma_1', \ldots, \gamma_k' \in \mathcal{P}_\lambda\), \(B \in H_2(M; \mathbb{Z})\) and \(F''\) a collection of \(J\)-holomorphic planes in \(\hat{W}\), of total homology class \(A - B \in H_2(W; \mathbb{Z})\), and whose top asymptotes are \(\gamma_1', \ldots, \gamma_k'\). Two sets of asymptotic markers at the punctures asymptotic to \(\gamma_1', \ldots, \gamma_k'\) are equivalent if they satisfy (37). The dimension of this moduli space is
\[
\mu(\gamma_p') - \mu(\gamma_Q) + 2\langle c_1(TW), A \rangle.
\]
An exact sequence for contact- and symplectic homology

(a) : Morse–Bott broken trajectory in Sect. 2.2

(b) : Capped punctured $J$-hol. cylinder in Sect. 3.1

(c) : Capped punctured $S^1$-parametrized broken $J$-hol. cylinder in Sect. 3.2

\[ \Phi \left( \gamma' \right) := \sum_{F \in \mathcal{M}_c^A(p',q;H^A,\{f_{\gamma'},\{f'_{\gamma'}\},J)} \bar{\epsilon}(F)e^{A_{\gamma}q}. \]
We shall also need in the proof of Proposition 4 the following variant of
the moduli space of capped punctured interpolating Morse–Bott broken tra-
jectories. Given $A \in H_2(W; \mathbb{Z})$, $\gamma', \gamma_1' \in \mathcal{P}_{\leq}^{\leq} \gamma \in \mathcal{P}(H)$, $p' \in \text{Crit}(f_w^r)$, $q \in \text{Crit}(f_w^\gamma)$ we denote by

$$\mathcal{M}_{c,1}^A(p', q, \gamma_0' ; H_\infty^0, \{ f_\gamma, f_\gamma' \}, J)$$

the set of equivalence classes of pairs $(F', F'')$, where $F'$ is an element of the
moduli space $\mathcal{M}^B(p', q, \gamma_1', \gamma_2', \ldots, \gamma_k', H_\infty^0, \{ f_\gamma, f_\gamma' \}, J)$ with $\gamma_2', \ldots, \gamma_k' \in \mathcal{P}_{\leq}^{\leq}$, $B \in H_2(M; \mathbb{Z})$, $F''$ is a collection of $J$-holomorphic planes in $\hat{\mathcal{W}}$
of total homology class $A - B \in H_2(W; \mathbb{Z})$, and with top asymptotes
$\gamma_2', \ldots, \gamma_k'$. Two sets of asymptotic markers at the punctures asymptotic
to $\gamma_2', \ldots, \gamma_k'$ are equivalent if they satisfy (37). One should think of this
as being the moduli space of punctured interpolating Morse–Bott broken tra-
jectories which are capped at all but one of the punctures.

**Proof of Proposition 4.** The map $\Phi$ preserves filtrations since all the moduli
spaces in the definition of $\mathcal{M}_{c,1}^A(p', q; H_\infty^0, \{ f_\gamma, f_\gamma' \}, J)$ are regular and hence
$\mu(\mathcal{Y}) - \mu(e^A \gamma) \geq 0$. As a matter of fact, since (75) involves only moduli
spaces such that $\mu(\mathcal{Y}) - \mu(\gamma_q) + 2\langle c_1(TW), A \rangle = 0$, and since $\text{ind}(p) - \text{ind}(q) \in \{1, 0, -1\}$, we have $\mu(\mathcal{Y}) - \mu(e^A \gamma) \in \{0, 1\}$. Moreover, the
relevant moduli spaces are empty if $p'$ is a minimum and $q$ is a maximum.

We now prove that $\Phi$ is a morphism of differential complexes, i.e. that
it satisfies the relation

$$\Phi \circ \delta = d \circ \Phi.$$  \hspace{1cm} (76)

For this, we consider the 1-dimensional components of the moduli space
$\mathcal{M}_{c}^A(\mathcal{Y}, \gamma_q; H_\infty^0, \{ f_\gamma, f_\gamma' \}, J)$ of capped punctured interpolating Morse–Bott broken trajectories. We claim that its boundary has the form

$$\left. \bigcup_{\gamma' \in \mathcal{P}_{\leq}^{\leq}} \mathcal{M}_{c}^A(p', r'; \{ f_\gamma' \}, J) \times \mathcal{M}_{c}^{A-\lambda}(r', q; H_\infty^0, \{ f_\gamma, f_\gamma' \}, J) \right|_{r' \in \text{Crit}(f_\gamma')\gamma'}$$

$$\left. \bigcup_{\gamma \in \mathcal{P}(H)} \mathcal{M}_{c}^{A}(p', r; H_\infty^0, \{ f_\gamma, f_\gamma' \}, J) \times \mathcal{M}_{c}^{A-\lambda}(r, q; H_\infty^0, \{ f_\gamma \}, J) \right|_{r \in \text{Crit}(f_\gamma)}$$

$$\left. \bigcup_{\gamma_1', \gamma_2' \in \mathcal{P}_{\leq}^{\leq}} \left[ \mathcal{M}_{c,1}^{A}(p', q, \gamma_1'; H_\infty^0, \{ f_\gamma, f_\gamma' \}, J) \times \mathcal{M}_{c}^{A-\lambda}(\gamma_1', \gamma_2'; J) / \mathbb{R} \times \mathcal{M}_{c}^{A-\lambda}(\gamma_2', \emptyset; J) \right] \right|_{J} \right. ,$$

with $[\ldots]$ denoting triples modulo the equivalence relation given by (37)
for the asymptotic markers at the punctures with common asymptotics $\gamma_1', \gamma_2'$.
Indeed, the compactness theorem for SFT [5, Theorem 10.2] ensures that
the boundary has the form above with all moduli spaces replaced by their
compactifications. In our situation the boundary is 0-dimensional and the
above moduli spaces are already compact.

Let us discuss the contribution of the three types of boundary components
above. The count of the elements of the first type corresponds to \( \Phi \circ \delta \),
whereas the count of the elements of the second type corresponds to \( d \circ \Phi \).

We claim that the count of the elements of the third type is equal to 0.
Indeed, the count of elements of
\[ \mathcal{M}^{A_2}((\gamma'_1, \gamma'_2); J)/\mathbb{R} \times \mathcal{M}^{A_1-A_2}((\gamma'_2, \emptyset); J) \]
is equal to \( e \circ \partial(\gamma'_1) = 0 \) by (36). Here \([\ldots]\) denotes pairs modulo the equiva-
lence relation given by (37) for the asymptotic markers at the punctures with
common asymptote \( \gamma'_2 \). This shows that \( \Phi \circ \delta \pm d \circ \Phi = 0 \). Relation (76)
follows from the fact that orientations are coherent with the gluing operation,
and we refer to [17, p. 69] for details (the situation is analogous to the one
in Floer homology involving a homotopy of Hamiltonians, i.e. an equation
which is not translation invariant).

We now prove that \( \Phi \) is an isomorphism. The first important observation
is that \( \Phi \) increases the contact action \( \gamma \mapsto \int \gamma^* \lambda \). This is due to the
fact that \( \int u^* d\lambda \geq 0 \) if \( u \) satisfies (69). Indeed, \( i(X_{H^\rho \infty})d\lambda = 0 \) so that
\( d\lambda(\partial_\nu u, \partial_\theta u) = d\lambda(\partial_\nu u, J_\infty \partial_\nu u) \geq 0 \).

We arrange the generators of the complex \( BC^{i-1}(a), \leq \sigma(a) \) in increasing
order according to their action. Then the matrix of \( \Phi \) in this basis is lower
triangular and we claim that the entries on the diagonal are all equal to \( \pm 1 \),
which implies that \( \Phi \) is an isomorphism.

Given \( \gamma \in \mathcal{P}_\leq \sigma(\mathcal{P}(H)), p' \in \text{Crit}(f'_\gamma), p \in \text{Crit}(f_\gamma) \) with \( \text{ind}(p) = \text{ind}(p') \), we therefore have to determine the elements \( u \) of the moduli space
\( \mathcal{M}^B(p', p, \gamma'_1, \ldots, \gamma'_k; H^\rho \infty, \{f_\gamma, f'_\gamma\}, J_\infty) \). Since
\[ \int u^* d\lambda = \mathcal{A}(\gamma) - \mathcal{A}(\gamma) - \sum_{i=1}^k \mathcal{A}((\gamma'_i) \geq 0 \]
we infer that \( k = 0 \). Moreover \( \int u^* d\lambda = 0 \), which implies that \( u(s, \theta) \) is
a vertical cylinder of the form
\[ u(s, \theta) = (a(s, \theta), \gamma \circ b(s, \theta)). \]

Equation (69) is then equivalent to the system
\[ \begin{align*}
\partial_s a - \partial_\theta b - \rho(s) H'_\infty(a) e^{-a} &= 0, \\
\partial_\theta a + \partial_\nu b &= 0,
\end{align*} \tag{78} \]
and we claim that any solution satisfying
\[ \begin{align*}
\lim_{s \to -\infty} a(s, \cdot) &= +\infty, & \lim_{s \to +\infty} a(s, \cdot) &= a_0, \\
\lim_{s \to \pm \infty} b(s, \theta) &= -T \theta + \theta_{\pm}, & T &= e^{-a_0} h'(a_0)
\end{align*} \]
has the following property:
(i) if \( p', p \) are both minima and \( \theta_+ \) corresponds to the orbit \( \gamma_p \) (and \( \theta_- \) is arbitrary), then
\[
a(s, \theta) = a(s), \quad b(s, \theta) = -T\theta + \theta_+;
\]
(ii) if \( p', p \) are both maxima and \( \theta_- \) corresponds to the orbit \( \gamma'_p \) (and \( \theta_+ \) is arbitrary), then
\[
a(s, \theta) = a(s), \quad b(s, \theta) = -T\theta + \theta_-.
\]

We treat only the first case since the other case is entirely analogous. We linearize (78) and obtain
\[
\partial_s \zeta + J_0 \partial_\theta \zeta + S \zeta = 0, \quad \zeta : \mathbb{R} \times S^1 \to \mathbb{R}^2,
\]
(79)
with \( S(s, \theta) = -\rho(s) \left( \begin{array}{c} \partial_a(e^{-a} H'_{\infty}(a)) \\ 0 \end{array} \right) \). Our hypothesis (56) on \( h \) ensures that \( \|S\| < 1 \).

By [29, Proposition 4.2] applied with \( \tau = 1 \) we infer that every solution \( \zeta \) of (79) is independent of \( \theta \). Hence solutions of (79) satisfy
\[
\partial_s \zeta_1 = \rho(s) \partial_a(e^{-a} H'_{\infty}(a)) \zeta_1, \quad \partial_s \zeta_2 = 0.
\]

In particular we see that, if \( \lim_{s \to +\infty} \zeta_1(s, \cdot) = 0 \), we necessarily have \( \zeta_1 \equiv 0 \) because \( \rho(s) \partial_a(e^{-a} H'_{\infty}(a)) \) is strictly positive for \( s \) large.

Since (78) is independent of \( \theta \), we know that, for any solution \( (a, b) \) the vector \( \zeta = (\partial_a a, \partial_\theta b) \) solves (79). In particular \( \partial_\theta b = -T \) (by the asymptotic conditions on \( b \)) and \( \partial_s b = -\partial_\theta a = 0 \), so that \( b(s, \theta) = -T\theta + \theta_+ \) as claimed.

We have already proved that \( \partial_\theta a = 0 \), hence \( a(s, \theta) = a(s) \) solves the differential equation \( a' + e^{-a_0} H'_{\infty}(a_0) - \rho(s)(e^{-a} H'_{\infty}(a)) = 0 \). Since \( e^{-a} H'_{\infty}(a) \) is increasing with \( a \), we must have \( a(s) = a_0 \) when \( \rho(s) = 1 \). Hence the solution \( a(s) \) is unique.

As a conclusion, we have showed that \( \mathcal{M}_{c_1}(p', p; H_{\infty}, \{ f_{\gamma}, f_{\gamma}' \}, J) \) contains a unique element, which is the “trivial” cylinder over \( \gamma \) described above.

\begin{remark} (An inverse for \( \Phi \)). \end{remark}

We chose to construct the morphism \( \Phi \) by counting mixed moduli spaces of curves with asymptotes \( \overline{\gamma}_p \) at \(-\infty\) and \( \gamma_q \) at \(+\infty\). In heuristic terms, we considered moduli spaces of curves “going down” from Reeb orbits to Hamiltonian orbits, and for that purpose we needed to consider the cut-off Hamiltonian \( H_{\infty}^{\rho} \).

We might have as well proceeded the other way around, by constructing a filtered isomorphism
\[
\Psi : BC^{a}_{\leq a}(H) \xrightarrow{\sim} BC^{r-1}(a), \leq a
\]
obtained by counting mixed moduli spaces of curves with asymptotes \( \overline{\gamma}_p \) at \(-\infty\) and \( \gamma'_q \) at \(+\infty\), i.e. “going down” from Hamiltonian orbits to Reeb
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orbits. For that purpose (69) would have had to be replaced by

$$\partial_s u + J_\infty (\partial_\theta u - X_{H_\infty}(u)) = 0.$$  

The one difference with respect to (69) is that this equation does not depend on the variable $s$ and therefore $\mathbb{R}$ acts by translations on the spaces of solutions. Nevertheless, the same dimension formula holds for the corresponding moduli space after the quotient by this $\mathbb{R}$-action.

One can prove that $\Psi$ is an inverse for $\Phi$ up to (filtered) homotopy.

We introduce now an algebraic concept which is useful when working with spectral sequences. Let $(C_*, \delta), (D_*, d)$ be differential complexes endowed with increasing filtrations $F_\ell C_*$ and $F_\ell D_*$ with $\ell \in \mathbb{Z}$. Let $f, g : C_* \rightarrow D_*$ be filtration preserving chain maps and $K : C_* \rightarrow D_{*+1}$ be a chain homotopy between $f$ and $g$, so that $f - g = K\delta + dK$. We say that $K$ is a chain homotopy of order $k \geq 0$ if $K(F_\ell C_*) \subset F_{\ell+k} D_{*+1}$ for all $\ell \in \mathbb{Z}$.

One can prove that $\Psi$ is an inverse for $\Phi$ up to (filtered) homotopy.

We introduce now an algebraic concept which is useful when working with spectral sequences. Let $(C_*, \delta), (D_*, d)$ be differential complexes endowed with increasing filtrations $F_\ell C_*$ and $F_\ell D_*$ with $\ell \in \mathbb{Z}$. Let $f, g : C_* \rightarrow D_*$ be filtration preserving chain maps and $K : C_* \rightarrow D_{*+1}$ be a chain homotopy between $f$ and $g$, so that $f - g = K\delta + dK$. We say that $K$ is a chain homotopy of order $k \geq 0$ if $K(F_\ell C_*) \subset F_{\ell+k} D_{*+1}$ for all $\ell \in \mathbb{Z}$.

Let $(E^r_\delta, \tilde{d}^r)$ and $(E^r_d, \tilde{d}^r)$, $r \geq 0$ be the spectral sequences associated to the given filtrations on $(C_*, \delta)$ and respectively $(D_*, d)$. Then $f$ and $g$ induce chain maps $f_r, g_r : E^r_\delta \rightarrow E^r_d$ for all $r \geq 0$. The next result motivates the concept of a chain homotopy of order $k$.

**Proposition 6 ([22, Exercise 3.8, p. 87]).** If $K : C_* \rightarrow D_{*+1}$ is a chain homotopy of order $k$ between the filtered chain maps $f, g : C_* \rightarrow D_*$, then $f_k, g_k : E^k_\delta \rightarrow E^k_d$ are chain homotopic, and the maps $f_r, g_r : E^r_\delta \rightarrow E^r_d$ coincide for any $k < r \leq \infty$.

The next proposition shows that the filtered isomorphism $\Phi$ of Proposition 4 is compatible with the continuation morphisms in Floer homology. Let $\tau > 0$ be large enough and, with the notations of Sect. 5.2, let $H^\alpha = H^{\alpha, \tau} \leq H^{\alpha'} = H^{\alpha', \tau}$ be admissible Hamiltonians on $\hat{W}^\tau$ with asymptotic slopes $\alpha, \alpha' \notin \text{Spec}(M, \lambda)$. Let $H_s = H^\tau, s \in \mathbb{R}$ be an increasing homotopy from $H^\alpha$ to $H^{\alpha'}$, such that $\partial_s H_s$ is small enough in $C^0$-norm. Let $J_s = J^\tau_s$ be a generic regular homotopy of admissible almost complex structures on $\hat{W}^\tau$ which is a small perturbation of the time independent almost complex structure postulated in our standing assumptions (A) and (Ba). Thus, Proposition 5 and Remark 16 apply to the Hamiltonians $H^\alpha, H^{\alpha'}$, and respectively $H_s$.

**Proposition 7.** The diagram below commutes up to a chain homotopy of order 1

$$
\begin{array}{ccc}
BC^{i-1}_*(\alpha), \leq \alpha(\lambda) & \xrightarrow{\Phi^\alpha} & BC^\alpha_*(H^\alpha) \\
\kappa_{\alpha', \alpha} \downarrow & & \downarrow \sigma_{\alpha', \alpha} \\
BC^{i-1}_*(\alpha), \leq \alpha'(\lambda) & \xrightarrow{\Phi^{\alpha'}} & BC^\alpha_*(H^{\alpha'}). 
\end{array}
$$
Here $\Phi^\alpha$, $\Phi^\alpha'$ are the filtered isomorphisms of Proposition 4 for the Hamiltonians $H^\alpha$, respectively $H^\alpha'$, the map $\sigma_{\alpha',\alpha}$ is the continuation morphism defined in Sect. 2.2, and $\kappa_{\alpha',\alpha}$ is the inclusion.

Proof. All morphisms in the diagram (80) preserve the filtrations on the corresponding chain complexes. This was proved in Proposition 4 for $\Phi^\alpha$ and $\Phi^\alpha'$, it was proved in Remark 15 for the continuation morphism $\sigma_{\alpha',\alpha}$, and it follows directly from the definition for the inclusion $\kappa_{\alpha',\alpha}$.

We denote by $H_{s,\infty}^\rho : M \times \mathbb{R} \to \mathbb{R}$ the homotopy from $H_{\infty}^\alpha$ to $H_{\infty}^\alpha'$ defined via formula (57) applied to $H_s(t)$. As above, let $\rho : \mathbb{R} \to [0, 1]$ be a smooth increasing function such that $\rho(s) = 0$ if $s < 0$ and $\rho(s) = 1$ if $s \gg 0$. Given $r \in \mathbb{R}$ we define the Hamiltonian family $H_{s,\infty}^{\rho, r}, \ s \in \mathbb{R}$ by

$$H_{s,\infty}^{\rho, r} : \mathbb{R} \times (M \times \mathbb{R}) \to \mathbb{R}, \ \ (s, p, t) \mapsto \rho(s-r) H_{s,\infty}(t).$$

Given $\gamma' \in \mathcal{P}_\lambda \subseteq \alpha, \ \gamma \in \mathcal{P}(H^\alpha'), \ p' \in \text{Crit}(f_{\gamma'}), \ q \in \text{Crit}(f_{\gamma})$, and an $s$-dependent family of admissible almost complex structures $J^\gamma = J_s$, the moduli space of capped punctured $s$-dependent interpolating Morse–Bott broken trajectories

$$\mathcal{M}_c^A(p', q; H_{s,\infty}^{\rho, r}, \{ f_{\gamma'}, f_{\gamma} \}, J^\gamma),$$

is defined similarly to the moduli space of capped punctured interpolating Morse–Bott broken trajectories (73), using $H_{s,\infty}^{\rho, r}$ instead of $H_{s,\infty}^\rho$. For a generic choice of the collection of perfect Morse functions $\{ f_{\gamma'}, f_{\gamma} \}$ and of the $s$-dependent almost complex structure $J^\gamma$, this is a smooth manifold of dimension

$$\mu(\gamma_p') - \mu(\gamma_q) + 2 \langle c_1(TW), A \rangle.$$

Let us now consider the moduli space

$$\mathcal{M}_c^A := \bigcup_{r \in \mathbb{R}} \mathcal{M}_c^A(p', q; H_{s,\infty}^{\rho, r}, \{ f_{\gamma'}, f_{\gamma} \}, J^\gamma)$$

for $\mu(\gamma_p') - \mu(\gamma_q) + 2 \langle c_1(TW), A \rangle = 0$ and $\gamma' \in \mathcal{P}_\lambda \subseteq \alpha, \ \gamma \in \mathcal{P}(H^\alpha')$. For a generic choice of a smooth 1-parameter family of almost complex structures $J^\gamma$, the space $\mathcal{M}_c^A$ is a smooth 1-dimensional manifold, and its boundary splits into a disjoint union

$$\partial \mathcal{M}_c^A = \partial^+ \mathcal{M}_c^A \cup \partial^- \mathcal{M}_c^A \cup \partial^0 \mathcal{M}_c^A,$$

where $\partial^\pm \mathcal{M}_c^A$ correspond to $r \to \pm \infty$ and $\partial^0 \mathcal{M}_c^A$ corresponds to finite values of $r$. Since $H_{s,\infty}^{\rho, r} = H_{\infty}^{\alpha',\rho(-r)}$ for $r \gg 0$, the set $\partial^+ \mathcal{M}_c^A$ is in bijective correspondence with

$$\mathcal{M}_c^A(p', q; H_{\infty}^{\alpha',\rho(-r)}, \{ f_{\gamma'}, f_{\gamma} \}, J^\gamma).$$
and the count of its elements gives rise to a morphism which is chain homotopic to $\Phi^{\alpha'} \circ \kappa_{\alpha',\alpha}$ in (80) (the chain homotopy comes from the need to further deform $\rho(\cdot - r)$ to $\rho$). On the other hand, for $r \ll 0$ the count of elements in $\partial^- \mathcal{M}_c^A$ gives rise to a morphism which is chain homotopic to the morphism obtained by counting the elements of

$$\mathcal{M}_c^{A_1}(p', q_1; H_{s,\infty}^{\alpha'}(\cdot - r), \{ f_y, f'_y \}, J) \times \mathcal{M}_c^{A-A_1}(q_1, q; H_{s,\infty}^\alpha, \{ f_y \}, J).$$

Here $q_1 \in \text{Crit}(f_{y_1})$ satisfies the equation $\mu(\gamma_1) - \mu(\gamma_1) + \text{ind}(q_1) + 2\langle c_1(TW), A_1 \rangle = 0$. This last morphism is in turn chain homotopic to $\sigma_{\alpha',\alpha} \circ \Phi^\alpha$. Finally, the set $\partial^0 \mathcal{M}_c^A$ is in bijective correspondence with the union of

$$\mathcal{M}_c^{A_1}(p', q_1; H_{s,\infty}^{\alpha}, \{ f_y, f'_y \}, J') \times \mathcal{M}_c^{A-A_1}(q_1, q; H_{s,\infty}^{\alpha'}, \{ f_y \}, J)$$

and

$$\mathcal{M}_c^{A_1}(p', q'_1; \{ f'_y \}, J_{\infty}) \times \mathcal{M}_c^{A-A_1}(q'_1, q; H_{s,\infty}^{\alpha'}, \{ f_y \}, J'),$$

for $r \in \mathbb{R}$, $q_1 \in \text{Crit}(f_{y_1})$, $q'_1 \in \text{Crit}(f_{y'_1})$ such that $\mu(\gamma_1) - \mu(\gamma_1) + \text{ind}(q_1) + 2\langle c_1(TW), A_1 \rangle = -1$, respectively $\mu(\gamma'_1) - \mu(\gamma'_1) + \text{ind}(q'_1) - \text{ind}(q) + 2\langle c_1(TW), A - A_1 \rangle = -1$.

The count of the elements of the above moduli spaces of index $-1$ gives rise to a chain homotopy between the morphisms corresponding to the count of elements of $\partial^+ \mathcal{M}_c^A$ and $\partial^- \mathcal{M}_c^A$ respectively. More precisely, let

$$K : BC_{*-1}^{(\alpha), \leq \alpha}(\ell) \rightarrow BC_{*-1}^{\alpha}(H^\alpha)$$

be defined by

$$K(\gamma_p) = \sum_{r \in \mathbb{R}} \sum_{F \in \mathcal{M}_c^{A_1}(p', q; H_{s,\infty}^{\alpha}, \{ f_y, f'_y \}, J')} \bar{e}(F) e^A \gamma_p q,$$

where the second sum runs over elements such that $\mu(\gamma_p) - \mu(\gamma'_p) + \text{ind}(p') - \text{ind}(q) + 2\langle c_1(TW), A \rangle = -1$. It follows from our transversality assumptions and the discussion above that there are only a finite number of values of the parameter $r \in \mathbb{R}$ for which the second sum is nonempty.

We claim that the chain homotopies constructed above have order 1. We argue only for $K$, the other cases being similar. We need to show that $\mu(\gamma_p) - \mu(\gamma'_p) \geq -1$. The moduli spaces involved in the definition of $\mathcal{M}_c^{A_1}(p', q; H_{s,\infty}^{\alpha'}, \{ f_y, f'_y \}, J')$ are of one of the following three types: $\mathcal{M}_c^{A_1}(S'_{y_1-1}, S'_y; J_{\infty})$ ($i < 0$), or $\mathcal{M}_c^{A_0}(S'_{y_1-1}, S_{y_1}; H_{s,\infty}^{\alpha'}, J')$, or $\mathcal{M}_c^{A_1}(S_{y_1-1}, S_{y_1}; H_{s,\infty}^{\alpha'}, J)$ ($j > 0$). The moduli spaces of the first and third type are regular, so that $\mu(\gamma_{y_1-1}) - \mu(e^{A_1}\gamma'_1) \geq 0$, and $\mu(\gamma_{y_1-1}) - \mu(e^{A_1}\gamma'_1) \geq 0$. On the other hand, due to the presence of the parameter $r \in \mathbb{R}$, the index of the Fredholm problem for the moduli space of the second type is 1 bigger than the index of the Fredholm problem obtained by replacing $H_{s,\infty}^{\alpha'}$ with $H_{s,\infty}^{\alpha}$ (see also the
definition of $\Phi$). Since our moduli spaces are regular and have dimension at least 1 (see Remark 14), we infer that $\mu(\gamma_{\alpha/0}) - \mu(e^{A0}\gamma_0) \geq -1$. This proves the claim, and the proposition. \qed

7. Proof of the main theorem

7.1. The long exact sequence

Proof of Theorem 1. Let $\alpha > 0$ be such that $\alpha \notin \text{Spec}(M, \lambda)$. Let $H \in \mathcal{H}'$ be an admissible Hamiltonian of maximal slope $\alpha$. By Sects. 5.1 and 5.2 and with notation as there, we modify $H$ to $H^{R,\tau}$ and the almost complex structure $J$ to $J^{\tau}$ which is regular for Floer’s equation so that Proposition 3 holds, and so that the Floer trajectories are close to punctured Floer trajectories in the symplectization, capped with rigid holomorphic planes in $\hat{W}$. We have performed the “slowing down” and “stretch of the neck” operations separately in order to emphasize the key ideas for each of them, but it is clear that they can be performed simultaneously in order to obtain such a $H^{R,\tau}$. We denote in the sequel $H = H^{R,\tau}$ and $J = J^{\tau}$, with both parameters $R, \tau$ being large enough.

The spectral sequence $(E^r_d, d^r)$ for symplectic homology is supported in two lines. Its $E^2$ page has the form

\[
\begin{array}{c}
\bullet \bullet \bullet \bullet \\
p \\
q \\
\end{array}
\]

and the only possibly nontrivial differentials are $d^2 : E^2_{k,0} \to E^2_{k-2,1}$. By definition of $E^3$ we have exact sequences

\[
0 \to E^3_{k,0} \to E^2_{k,0} \xrightarrow{d^2} E^2_{k-2,1} \to E^3_{k-2,1} \to 0.
\]

The spectral sequence converges to $SH^a_*(H, J)$ and $E^3_d = E^\infty_0$ for dimensional reasons, so that we have exact sequences

\[
0 \to E^3_{k-1,1} \to SH^a_k(H, J) \to E^3_{k,0} \to 0
\]

by definition of convergence. This information can be put together into a long exact sequence by discarding the $E^3$ terms

\[
\begin{array}{c}
E^3_{k-1,1} \to SH^a_k(H, J) \to E^2_{k,0} \xrightarrow{d^2} E^2_{k-2,1} \to SH^a_{k-1}(H, J)
\end{array}
\]

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\[
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by definition of convergence. This information can be put together into a long exact sequence by discarding the $E^3$ terms

\[
\begin{array}{c}
E^3_{k-1,1} \to SH^a_k(H, J) \to E^2_{k,0} \xrightarrow{d^2} E^2_{k-2,1} \to SH^a_{k-1}(H, J)
\end{array}
\]
We have already seen in Corollary 1 that the map $\Phi$ induces an isomorphism of spectral sequences $\Phi : (E^r_\delta, \delta^r) \xrightarrow{\sim} (E^r_d, d^r)$, $r \geq 0$. In particular we have the following commutative diagram, with vertical arrows being isomorphisms

$$
\begin{array}{ccc}
E^2_\delta & \xrightarrow{\delta^2} & E^2_\delta \\
\Phi \simeq & & \Phi \\
E^2_d & \xrightarrow{d^2} & E^2_d
\end{array}
$$

Combining the isomorphism (49)

$$
\Theta : E^2_\delta \xrightarrow{\sim} HC^{i-1}(a, \leq \alpha) (\lambda, J) \otimes H_\ast(S^1)
$$

with the two previous diagrams we get a long exact sequence

$$
\cdots \to SH^a_k(\lambda, J) \to HC^{i-1}(a, \leq \alpha) (\lambda, J) \xrightarrow{D} HC^{i-1}(a, \leq \alpha) (\lambda, J) \\
\xrightarrow{\Theta} SH^{a-1}_{k-1}(\lambda, J) \to \cdots
$$

(81)

with $D = \Theta \circ \delta^2 \circ \Theta^{-1}$. The shift in degree is due to the fact that $\Phi$ decreases degrees by $n - 3$. Since the limiting slope of $H$ equals $\alpha$ we have

$$
SH^a_\ast(\lambda, J) \simeq SH^{a-\alpha}(W, \omega),
$$

where the latter notation stands for a direct limit on $\mathcal{H}^\prime$ of Floer homology groups truncated by the values of the action functional in the range $[-\infty, \alpha]$.

We claim that the exact sequences (81) form a natural directed system, i.e. for $\alpha < \alpha^\prime \notin \text{Spec}(M, \lambda)$ the continuation morphisms $\sigma_{\alpha^\prime, \alpha}$ in symplectic homology induced by an increasing homotopy of Hamiltonians as in Sect. 2.2, and the extension morphisms $\kappa_{\alpha^\prime, \alpha}$ in linearized contact homology induced by inclusion, fit into a commutative diagram

$$
\begin{array}{ccc}
\cdots & \cdots & \cdots \\
SH^a_k(\lambda, J^\alpha) & \xrightarrow{\sigma_{\alpha^\prime, \alpha}} & HC^{i-1}(a, \leq \alpha) (\lambda, J) \\
\downarrow & & \downarrow \\
\cdots & \cdots & \cdots
\end{array}
$$

(82)

$$
\begin{array}{ccc}
\cdots & \cdots & \cdots \\
HC^{i-1}(a, \leq \alpha) (\lambda, J) & \xrightarrow{\kappa_{\alpha^\prime, \alpha}} & HC^{i-1}(a, \leq \alpha) (\lambda, J) \\
\downarrow & & \downarrow \\
\cdots & \cdots & \cdots
\end{array}
$$

Here $H^\alpha = H^{R(\alpha), \tau(\alpha)}$, $H^{\alpha^\prime} = H^{R(\alpha^\prime), \tau(\alpha^\prime)}$, $J^\alpha = J^{\tau(\alpha)}$ and $J^{\alpha^\prime} = J^{\tau(\alpha^\prime)}$ are as explained in the beginning of the proof. The asymptotic slope of $H^\alpha$ is equal to $\alpha$, the asymptotic slope of $H^{\alpha^\prime}$ is equal to $\alpha^\prime$ and, by Proposition 5, we can assume without loss of generality that $\tau(\alpha) = \tau(\alpha^\prime) = \tau$ so that the Hamiltonians and almost complex structures are defined on $\hat{W}^\tau$ as in Sect. 5.2. Moreover, we can also assume without loss of generality that $H^\alpha \leq H^{\alpha^\prime}$.
By Remark 15 in Sect. 4, the continuation morphism

$$\sigma_{a',a} : BC^a_*(H^\alpha) \to BC^a_*(H^{\alpha'})$$

preserves the filtrations, and hence induces a morphism of spectral sequences

$$\sigma_{a',a} : \left( E^r_d(H^\alpha), \tilde{d}^r \right) \to \left( E^r_d(H^{\alpha'}), \tilde{d}^r \right), \quad r \geq 0.$$

The inclusion $$\kappa_{a',a} : BC^{i-1(a), \leq \alpha}(\lambda) \to BC^{i-1(a), \leq \alpha'}(\lambda)$$ preserves the filtrations by definition, and therefore also induces a morphism between the associated spectral sequences

$$\kappa_{a',a} : \left( E^r_\delta^{\leq \alpha}, \tilde{\delta}^r \right) \to \left( E^r_\delta^{\leq \alpha'}, \tilde{\delta}^r \right), \quad r \geq 0.$$

In view of Proposition 6, commutativity of (82) follows from the commutativity up to a chain homotopy of order 1 of the following diagram of morphisms of filtered complexes

$$\begin{array}{ccc}
BC^{i-1(a), \leq \alpha}(\lambda) & \xrightarrow{\Phi^a} & BC^a_*(H^\alpha) \\
\downarrow {\kappa_{a',a}} & & \downarrow {\sigma_{a',a}} \\
BC^{i-1(a), \leq \alpha'}(\lambda) & \xrightarrow{\Phi^{a'}} & BC^a_*(H^{\alpha'}). \\
\end{array}$$

(83)

Here $$\Phi^a$$ and $$\Phi^{a'}$$ are the filtered isomorphisms constructed in Proposition 4, for the Hamiltonians $$H^\alpha$$, respectively $$H^{\alpha'}$$.

The commutativity of (83) up to a chain homotopy of order 1 is precisely the content of Proposition 7 above. Therefore (82) is commutative and, passing to the direct limit on $$\alpha$$, we obtain an exact sequence

$$\cdots \to SH_k^a(W, \omega) \to HC^{i-1(a)}_{k+(n-3)}(\lambda, J) \xrightarrow{D} HC^{i-1(a)}_{k-2+(n-3)}(\lambda, J) \xrightarrow{\Phi_{\lambda, J}} SH_{k-1}^a(W, \omega) \to \cdots$$

(84)

Here we used the fact that the direct limit functor is exact. By the invariance of linearized contact homology with respect to the pair $$(\lambda, J)$$ (see Remark 7) we obtain the exact sequence in the statement of Theorem 1.

Finally, the description of the differential $$D$$ claimed in the statement of Theorem 1 is the content of Proposition 8 below. This finishes the proof. □

Remark 18. What we have actually proved is that $$\Phi$$ induces an isomorphism of degree $$3 - n$$ between the exact sequences
In order to establish Theorem 1 we have only used the undotted arrows in the above diagram. We shall not explain in this paper the significance of the discarded terms $E_d^2$ and $H_s(BC^{-1}_s(\lambda), \delta)$. This will be the topic of the forthcoming papers [8] and [14] respectively.

7.2. The differential $D$

The purpose of this section is to give a description of $D$ which does not make use of the auxiliary Morse functions $f', f''$, and thus complete the proof of Theorem 1.

Given half-lines $\overline{L} \subseteq T_0\mathbb{C}P^1$, $L \subseteq T_\infty\mathbb{C}P^1$, we define half-lines $\overline{L}_\infty \subseteq T_\infty\mathbb{C}P^1$, $L_0 \subseteq T_0\mathbb{C}P^1$ by choosing a global polar coordinate on $\mathbb{C}P^1 \setminus \{0, \infty\}$ and requiring

$$\text{Arg}(\overline{L}_\infty) = \text{Arg}(\overline{L}), \quad \text{Arg}(L_0) = \text{Arg}(L).$$

Given half-lines $L_0 \subseteq T_0\mathbb{C}P^1$, $L_\infty \subseteq T_\infty\mathbb{C}P^1$, and a map $F = (f, a) : \mathbb{R} \times S^1 = \mathbb{C}P^1 \setminus \{0, \infty\} \to M \times \mathbb{R}$ satisfying (27) and (28), we define $\text{ev}(L_0) = \lim_{z \to 0, z \in L_0} f(z)$, $\text{ev}(L_\infty) = \lim_{z \to \infty, z \in L_\infty} f(z)$, so that $\text{ev}(L_0)$ belongs to the geometric image of $\overline{\gamma}$ and $\text{ev}(L_\infty)$ belongs to the geometric image of $\gamma'$. We also recall that we have chosen a point $P_{\gamma'}$ on the geometric image of each $\gamma' \in \mathcal{P}_\lambda$.

Given $\overline{\gamma}, \gamma' \in \mathcal{P}_\lambda$, $A \in H_2(W; \mathbb{Z})$ we denote by

$$\mathcal{M}^A_{1,c}(P_{\gamma}, P_{\gamma'}; J) \subseteq \mathcal{M}^A_{c}(\overline{\gamma}, \gamma'; J)$$

the subset of equivalence classes of pairs $[u, F] \in \mathcal{M}^A_{c}(\overline{\gamma}, \gamma'; J)$ such that the asymptotic markers $\overline{L}$ at 0 and $\overline{L}$ at $\infty$ satisfy

$$\overline{L}_\infty = L, \quad \text{or equivalently } L_0 = \overline{L}.$$ 

The decoration “1” for the moduli space is motivated by the fact that it consists of curves with one sublevel.

Given $\overline{\gamma}, \gamma' \in \mathcal{P}_\lambda$, $A \in H_2(W; \mathbb{Z})$ we denote by

$$\mathcal{M}^{A,+}_{2,c}(P_{\gamma}, P_{\gamma'}; J)$$

$$\subseteq \bigcup_{\gamma' \in \mathcal{P}_\lambda, A_1 \in H_2(W; \mathbb{Z})} \left[ \mathcal{M}^{A_1}_{c}(\overline{\gamma}, \gamma'; J) \times \mathcal{M}^{A-A_1}_{c}(\gamma', \gamma'%; J) \right]$$

the subset of pairs of equivalence classes $([u], [u''])$ for the equivalence relation given by ignoring the asymptotic markers $\overline{L}^\prime, \overline{L}''$ corresponding to the common asymptote $\gamma'$, such that the cyclic order of the points $(P_{\gamma'}, \text{ev}(\overline{L}_\infty), \text{ev}(L_0))$ is the same as the natural orientation of the geometric image of $\gamma'$. The decorations “2” and “+” for the moduli space are motivated by the fact that it consists of curves with two sublevels and
satisfying an additional “positive” cyclic order condition. In the situation \(\mu(\mathcal{W}) - \mu(\gamma') + 2(c_1(TW), A) = 2\) and for a generic choice of the points \(P_{\gamma'}\) the moduli spaces \(\mathcal{M}^{1,c}(P_{\mathcal{W}}, P_{\gamma'}; J)\) and \(\mathcal{M}^{2,c}(P_{\mathcal{W}}, P_{\gamma'}; J)\) are rigid and one can associate a sign \(\epsilon(u)\) to each of their elements via coherent orientations and fibered products.

For each free homotopy class \(a\) in \(W\) we define a map

\[
\Delta : C^{-1}(\lambda) \to C^{-1}(\lambda),
\]

\[
\Delta(\gamma') = \sum_{\gamma' : \mathcal{A}} \frac{1}{\kappa_{\gamma'}} \sum_{u \in \mathcal{M}^{1,c}(P_{\mathcal{W}}, P_{\gamma'}; J) \cup \mathcal{M}^{2,c}(P_{\mathcal{W}}, P_{\gamma'}; J)} \epsilon(u) e^A \gamma'.
\]  

(85)

**Proposition 8.** The map \(\Delta\) defined by (85) is a chain map, and induces in homology the map \(D\) in the long exact sequence of Theorem 1.

**Proof.** Let us first reinterpret the previous moduli spaces in terms of moduli spaces of capped punctured \(S^1\)-parametrized holomorphic cylinders. Given \(\mathcal{W}, \gamma' \in \mathcal{P}_\lambda, A \in H_2(W; \mathbb{Z})\) we denote by

\[
\tilde{\mathcal{M}}^{1,c}(P_{\mathcal{W}}, P_{\gamma'}; J) \subset \mathcal{M}^{1,c}(S_{\mathcal{W}}, S_{\gamma'}; J)
\]

the subset consisting of pairs \(u = (u', F') \in \mathcal{M}^{1,c}(S_{\mathcal{W}}, S_{\gamma'}; J)\) such that

\[
\mathcal{ev}(u') = P_{\mathcal{W}}, \quad \mathcal{ev}(u') = P_{\gamma'}.
\]

It follows from the definition that there is a bijective correspondence

\[
\mathcal{M}^{1,c}(P_{\mathcal{W}}, P_{\gamma'}; J) \simeq \tilde{\mathcal{M}}^{1,c}(P_{\mathcal{W}}, P_{\gamma'}; J).
\]

Given \(\mathcal{W}, \gamma' \in \mathcal{P}_\lambda, A \in H_2(W; \mathbb{Z})\) we denote by

\[
\tilde{\mathcal{M}}^{1,c}(P_{\mathcal{W}}, S_{\gamma'}; J), \tilde{\mathcal{M}}^{1,c}(S_{\mathcal{W}}, P_{\gamma'}; J) \subset \mathcal{M}^{1,c}(S_{\mathcal{W}}, S_{\gamma'}; J)
\]

the subsets consisting of pairs \(u = (u', F') \in \mathcal{M}^{1,c}(S_{\mathcal{W}}, S_{\gamma'}; J)\) such that

\[
\mathcal{ev}(u') = P_{\mathcal{W}}, \quad \mathcal{ev}(u') = P_{\gamma'}.
\]

If \(\mu(\mathcal{W}) - \mu(\gamma') + 2(c_1(TW), A) = 1\) these moduli spaces are rigid. Let

\[
\tilde{\mathcal{M}}^{2,c}(P_{\mathcal{W}}, P_{\gamma'}; J) \subset \bigcup_{\gamma' \in \mathcal{P}_\lambda, A \in H_2(W; \mathbb{Z})} \tilde{\mathcal{M}}^{1,c}(P_{\mathcal{W}}, S_{\gamma'}; J) \times \tilde{\mathcal{M}}^{1,c}(S_{\mathcal{W}}, P_{\gamma'}; J)
\]

be the subset consisting of pairs \((\overline{u}, u)\) such that the cyclic order of the points \((P_{\gamma'}, \mathcal{ev}(\overline{u}), \mathcal{ev}(u))\) is the same as the one induced by the chosen
orientation of $S'_y$. It follows from the definition that there is a bijective correspondence

$$\mathcal{M}_{2,c}^{A,+}(P_{\gamma'}, P_{\gamma''}; J) \simeq \widetilde{\mathcal{M}}_{2,c}^{A,+}(P_{\gamma'}, P_{\gamma''}; J).$$

Hence $\widetilde{\Delta} := \Theta^{-1} \circ \Delta \circ \Theta : C_{*}^{-1}(\lambda) \otimes H_0(S^1) \to C_{*}^{2}(\lambda) \otimes H_1(S^1)$, where $\Theta$ is defined in (48), acts by

$$\widetilde{\Delta}(\gamma_M) = \sum_{\gamma', A} \sum_{u \in \tilde{\mathcal{M}}_{1,c}(P_{\gamma'}, P_{\gamma''}; J) \cup \tilde{\mathcal{M}}_{2,c}^{A,+}(P_{\gamma'}, P_{\gamma''}; J)} e(u)e^A \gamma'_m.$$

For a generic choice of Morse functions $f'_y$, $\gamma' \in \mathcal{P}$, the map $\delta^2$ is induced in homology by the map $\delta^2$ in the decomposition of the $S^1$-parametrized differential $\delta$, and does not depend on the choice of the collection $\{f'_y\}$. It is therefore enough to show that, for a suitable choice of this collection, the map induced by $\delta^2$ on $(E^1, \delta^1)$ is $\widetilde{\Delta}$ itself.

Let us fix $\alpha > 0$ such that $\alpha \notin \text{Spec}(M, \lambda)$. Let $\widetilde{\Delta}^\alpha$ and $\delta^{2,\alpha}$ be the truncations of $\widetilde{\Delta}$ and $\delta^2$ to action less than $\alpha$. It is enough to show that $\widetilde{\Delta}^\alpha = \delta^{2,\alpha}$ for a suitable choice of the collection $\{f'_y\}$ which depends on $\alpha$.

By letting $\alpha \to \infty$ we then get $\widetilde{\Delta} = \delta^2$.

The set $\mathcal{P}^{\leq \alpha}$ is finite and, for each pair $\gamma', \gamma'' \in \mathcal{P}^{\leq \alpha}$, the moduli space of holomorphic curves asymptotic to $\gamma', \gamma''$ is compact and therefore involves only a finite number of homology classes. As a consequence, for each $\gamma' \in \mathcal{P}^{\leq \alpha}$ we can choose an open neighbourhood $U_{\gamma', \alpha}$ of $P_{\gamma'}$ in $S'_y$ such that every collection $\{q_{\gamma'}\} \in \prod_{\gamma'' \in \mathcal{P}^{\leq \alpha}} U_{\gamma', \alpha}$ is regular and the map $\widetilde{\Delta}^\alpha$ associated to $\{q_{\gamma'}\}$ is equal to the map $\widetilde{\Delta}^\alpha$ associated to $\{P_{\gamma'}\}$.

By choosing a generic collection $\{q_{\gamma'}\}, q_{\gamma'} \in U_{\gamma', \alpha}, \gamma' \in \mathcal{P}^{\leq \alpha}$ and small neighbourhoods $V_{\gamma', \alpha} \subset U_{\gamma', \alpha}$ of $q_{\gamma'}$, we can further assume that the evaluation maps $e_{\gamma'} \circ \delta^2$ defined on the spaces $\mathcal{M}_{c}(q_{\gamma'}, S'_{\lambda}; J)$ and $\mathcal{M}_{c}(S'_{\gamma'}, q_{\gamma'}; J)$ respectively, with $|\gamma'| - |e^A \gamma'| = 1$, miss the neighbourhoods $V_{\gamma', \alpha}$.

We choose the Morse functions $f'_y$, $\gamma' \in \mathcal{P}^{\leq \alpha}$ generically so that both critical points $m'$ and $M'$ of $f'_y$ lie inside $V_{\gamma', \alpha}$ and so that the “long arc” in $S'_y$ running from $m'$ to $M'$ and exiting $V_{\gamma', \alpha}$ has the same orientation as the chosen orientation of $S'_y$. Let us show that in this situation we have $\widetilde{\Delta}^\alpha = \delta^{2,\alpha}$.

We first note that $\delta^{2,\alpha}$ is built out of two kinds of moduli spaces of capped punctured $S^1$-parametrized broken $J_\infty$-holomorphic cylinders, namely having either one or two sublevels. Indeed, we cannot have more than two sublevels since the difference of indices at the extremities is equal to two, and each sublevel introduces a difference of index of at least one due to the fact that the moduli spaces of punctured $S^1$-parametrized $J_\infty$-holomorphic cylinders are regular and carry a one-dimensional symmetry given by the action of $S^1$. 

Let us fix $\gamma', \gamma' \in \mathcal{P}_{z,0}^\alpha$ and denote by $M'$ the maximum of $f_{\gamma'}$ and by $m'$ the minimum of $f_{\gamma'}$. Since $M' \in \mathcal{V}_{\gamma'}^\alpha \subset \mathcal{U}_{\gamma'}^\alpha$ and $m' \in \mathcal{V}_{\gamma'}^\alpha \subset \mathcal{U}_{\gamma'}^\alpha$, the elements of $\mathcal{M}_c^A(M', m'; \{ f_{\gamma'} \}, J)$ having one sublevel are in one-to-one correspondence with elements of $\tilde{\mathcal{M}}_{1,c}^A(q_{\gamma'}', q_{\gamma'}'; J)$ and their signs coincide.

We claim now that the subset of $\mathcal{M}_c^A(M', m'; \{ f_{\gamma'} \}, J)$ consisting of elements with two sublevels is in bijective correspondence with $\tilde{\mathcal{M}}_{2,c}^A(q_{\gamma'}', q_{\gamma'}'; J)$, with preservation of signs. Indeed, such elements are of the form

$$(\bar{u}, \underline{u}) \in \mathcal{M}_c^{A,1}(M', S_{\gamma'}'; J) \times \mathcal{M}_c^{A-A,1}(S_{\gamma'}', m'; J)$$

so that there is a gradient trajectory of $f_{\gamma'}$ running from $\text{ev}(\bar{u})$ to $\text{ev}(\underline{u})$. Since these evaluation maps miss the neighbourhood $\mathcal{V}_{\gamma'}^\alpha$ and by our choice of order for the two critical points of $f_{\gamma'}$, this is equivalent to saying that the cyclic order on the triple $(q_{\gamma'}', \text{ev}(\bar{u}), \text{ev}(\underline{u}))$ is the same as the one induced by the chosen orientation of $S_{\gamma'}'$. Since $M'$ is close to $q_{\gamma'}'$ and $m'$ is close to $q_{\gamma'}$ there is a unique element in $\tilde{\mathcal{M}}_{2,c}^A(q_{\gamma'}', q_{\gamma'}'; J)$ corresponding to such a pair $(\bar{u}, \underline{u})$. Their signs coincide for continuity reasons and this proves our claim.

We have shown that $\delta^{2,\alpha} = \hat{\Delta}^\alpha$ on $(E^1_\delta, \hat{\delta}^1)$, as desired. \qed

8. Examples

8.1. Riemann surfaces

We compute in this section the exact sequence (2) for genus $g$ Riemann surfaces $\Sigma = \Sigma_{g,1}$ with one boundary component. We shall see that the cases $g = 0$ and $g \geq 1$ are fundamentally different: although the boundary $M = \partial \Sigma$ is the same, i.e. the circle $S^1$, the linearized contact homology groups $HC_c(M)$ differ, and so do the corresponding maps $D$. Note that regularity is automatic when the target manifold is a Riemann surface.

Let us note that free homotopy classes of loops in $M = S^1$ are indexed by $\mathbb{Z}$ via the degree of the corresponding maps $S^1 \to S^1$. Given a free homotopy class $b$ of loops we denote by the same symbol $b \in \mathbb{Z}$ its degree. There are no closed Reeb orbits in any class $b \leq 0$, whereas each class $b \in \mathbb{Z}^+$ contains exactly one closed Reeb orbit $\gamma_b$. Since the contact distribution is zero-dimensional we need to use the special convention $\mu(\gamma_b) = 2b$ for the Maslov index, corresponding to the index of the linearized Reeb flow in the symplectization.

Let us first consider the case $g \geq 1$. The inclusion $i$ of free homotopy classes of loops from the boundary to $\Sigma$ is injective and we denote $i(b)$ by $b$. For each $b \in \mathbb{Z}^+$ we have $HC_c^b(M) = \mathbb{Q}$ if $\ast = 2b + (1 - 3) = 2b - 2$ and 0 otherwise. The Reeb orbit $\gamma_b$ gives rise to two Hamiltonian orbits of indices $2b$ and $2b + 1$ and we have $SH^b_\ast(\Sigma) = \mathbb{Q}$ if $\ast = 2b$, $2b + 1$ and 0 otherwise. The interesting portion of the exact sequence is there-
fore

$$0 \xrightarrow{D} HC_{2b-2}^b \xrightarrow{0} SH_{2b+1}^b(\Sigma) \xrightarrow{D} 0 \xrightarrow{0} SH_{2b}^b(\Sigma) \xrightarrow{HC_{2b-2}^b(M)} 0.$$  

We see in particular that $D$ vanishes.

We now consider the case $g = 0$, so that we can assume without loss of generality that $\Sigma = D^2$, the unit disc in the complex plane. Note that all closed Reeb orbits are contractible in $D^2$. Since $SH_*^0(D^2) = 0$ we have $SH_*^1(D^2) \simeq H_*(D^2, \partial D^2) = \mathbb{Q}$ if $* = 2$ and 0 otherwise. The linearized contact complex is lacunary with generators of index $2b-2$, $b \geq 1$ and therefore $HC_*^{i-1}(0) = \mathbb{Q}$ if $* = 2b-2$, $b \geq 1$ and 0 otherwise. The long exact sequence (2) therefore splits into short exact sequences of which the nontrivial ones are

$$0 \xrightarrow{SH_2^+(D^2)} HC_0(S^1) \xrightarrow{D} HC_{-2}(S^1) \xrightarrow{0}$$

and

$$0 \xrightarrow{SH_{2b}^+(D^2)} HC_{2b-2}(S^1) \xrightarrow{D} HC_{2b-4}(S^1) \xrightarrow{0}, \quad b \geq 2.$$  

We see in particular that the map $D : HC_{2b-2}(S^1) \sim HC_{2b-4}(S^1)$ does not vanish for $b \geq 2$. We can actually describe it explicitly as follows. The only contractible Reeb orbit of normalized index 0 is $\gamma_1$ in the class $b = 1$ and, up to reparametrization, there is a unique holomorphic plane in $\mathbb{C}^2 \equiv \hat{D}^2$ asymptotic to it, namely $z \mapsto cz + d$, $c \in \mathbb{C}^*$, $d \in \mathbb{C}$. Since there are no rigid nontrivial holomorphic cylinders in $S^1 \times \mathbb{R}$ the map $D$ is obtained by a count of punctured curves in $S^1 \times \mathbb{R}$ with only one sublevel. These must necessarily have three punctures: a positive one asymptotic to $\gamma_b$ and two negative ones asymptotic to $\gamma_{b-1}$ and $\gamma_1$. Note that the puncture asymptotic to $\gamma_1$ corresponds to the augmentation of Remark 6 and the count of these curves gives the coefficient of $\gamma_{b-1}$ in $D(\gamma_b)$. Such curves correspond to meromorphic functions on the Riemann sphere with one pole of order $b$ and two zeroes of order $b - 1$ and 1, respectively. Meromorphic functions with those properties are unique up to reparametrization and thus the sum defining $D$ reduces to $D(\gamma_b) = \gamma_{b-1}$, so that $D$ is the obvious isomorphism.

### 8.2. Subcritical Stein manifolds

A **Stein manifold** $\hat{W}$ is a triple $(\hat{W}, J, \phi)$, where $J$ is a complex structure on $\hat{W}$ and $\phi : \hat{W} \rightarrow \mathbb{R}$ is an exhausting plurisubharmonic function. That $\phi$ is
exhausting means that $\phi$ is proper and bounded from below. That $\phi$ is plurisubharmonic means that $\omega \phi := -dJ^* d\phi$ is a symplectic form and $J$ is compatible with $\omega_\phi$. We say that $\hat{W}$ is of finite type if we can choose $\phi$ such that the set of its critical points is compact. In this situation we can assume without loss of generality that $\phi$ is Morse [2, Theorem 8.1.C]. All its critical points have index at most $\frac{1}{2} \dim \hat{W}$, and we call $\hat{W}$ subcritical if all critical points have index strictly smaller than $\frac{1}{2} \dim \hat{W}$. We assume in this section that $\hat{W}$ is a subcritical Stein manifold of finite type.

A Stein domain $W \subset \hat{W}$ is a domain such that $W = \{ \phi \leq c \}$ for some $c \in \mathbb{R}$ large enough. In particular $c$ is a regular value of $\phi$ and $W$ is a smooth manifold with boundary $\partial W = \{ \phi = c \}$. Actually $W$ is an exact symplectic manifold with boundary of contact type and Liouville vector field $\nabla \phi$, the gradient with respect to the metric $\omega_\phi(\cdot, J\cdot)$. Moreover, the symplectic completion of $W$ is symplectomorphic to $(\hat{W}, \omega_\phi)$.

The isotopy class of the contact structure $\xi_\phi$ induced on $\partial W$ does not depend on $\phi$ and $c$. A contact manifold $(M, \xi)$ is called subcritically Stein fillable if it can be realized as $(\partial W, \xi_\phi)$ for some subcritical Stein manifold of finite type $(\hat{W}, J, \phi)$. We call the Stein domain $W$ a filling of $M$.

Mei-Lin Yau has computed in [34] the cylindrical contact homology groups with rational coefficients in the trivial homotopy class for a subcritically Stein fillable contact manifold $(M^{2n-1}, \xi)$, $n \geq 2$ satisfying $c_1(\xi) = 0$. A crucial ingredient in the proof is an estimate on the reduced Conley–Zehnder index of Reeb orbits running between different handles [34, Lemma 4.2], which implies in particular that every Reeb orbit $\gamma$ which is contractible in $W$ satisfies $\mu(\gamma) \geq 1$. Hence there are no rigid holomorphic planes in $\hat{W}$ and cylindrical contact homology is tautologically isomorphic to linearized contact homology $HC_0(M)$.

The next theorem states Mei-Lin Yau’s result in a form which is equivalent to the original one of [34].

**Theorem 2 (M.-L. Yau [34]).** Let $(M^{2n-1}, \xi)$, $n \geq 2$ be a subcritically Stein fillable contact manifold with $c_1(\xi) = 0$, and let $W$ be any subcritical Stein filling of $M$. Then, using rational coefficients, we have

$$HC_k^0(M) \simeq \bigoplus_{m \geq 0} H_{k-2m+2}(W, \partial W), \quad k \in \mathbb{Z}. \tag{86}$$

Denote by $i$ the map associating to the free homotopy class of a loop in $\partial W$ the free homotopy class of the same loop in $W$. We claim that subcriticality implies $i^{-1}(0) = 0$, i.e. a loop in $\partial W$ which is contractible in $W$ is actually contractible in $\partial W$. Indeed, the homotopy to a point defines a chain of dimension 2. Since $n \geq 2$ and the isotropic skeleton of $W$ is of dimension at most $n-1$, we can perturb the homotopy so as to avoid it. Finally we push the homotopy to $\partial W$ by the Liouville vector field $\nabla \phi$.

We recall now the discussion on regularity in Remark 9. If $W$ is a stabilization of a subcritical Stein manifold, i.e. $W = W' \times D^2$ with $W'$ subcritical, then all necessary regularity assumptions are met. In fact, unpublished
work of Cieliebak [10] shows that $W$ is deformation equivalent to such a stabilization if and only if it has the homotopy type of a complex of dimension at most $n - 2$. The exact sequence (2) becomes

$$
\cdots \to SH_{k-(n-3)}^+(W) \to HC_k^0(M) \xrightarrow{D} HC_{k-2}^0(M) \to SH_{k-1-(n-3)}^+(W) \to \cdots
$$

**Proposition 9.** Let $W$ be a stabilization of a subcritical Stein manifold, and denote $M = \partial W$. The exact sequence relating contact and symplectic homology in the trivial homotopy class is isomorphic to an exact sequence of the form

$$
\cdots \to 0 \to H_{k+2}(W, \partial W) \to \bigoplus_{m \geq 0} H_{k-2m+2}(W, \partial W) \to \bigoplus_{m \geq 1} H_{k-2m+2}(W, \partial W) \to 0 \to H_{k+1}(W, \partial W) \to \cdots
$$

**Proof.** The symplectic homology groups $SH_*(W)$ of the subcritical Stein domain $W$ vanish for the trivial homotopy class [9,25], and the tautological long exact sequence (5) implies

$$
SH_{k-(n-3)}^+(W) \simeq H_{k+2}(W, \partial W), \quad k \in \mathbb{Z}.
$$

We prove by induction that the maps $HC_k^0(M) \to H_{k+3}(W, \partial W), \ k \in \mathbb{Z}$ vanish. This holds for $k \leq n - 3$ or $k \geq 2n - 2$ because $W$ is an oriented manifold of dimension $2n$ having the homotopy type of a CW-complex of dimension $\leq n - 1$, and therefore $H_{k+3}(W, \partial W) = 0$. Assuming that the map $HC_k^0(M) \to H_{k+3}(W, \partial W)$ vanishes for some $k \in \mathbb{Z}$, the exactness of the sequence

$$
HC_{k+1}^0(M) \to H_{k+4}(W, \partial W) \xrightarrow{i} \bigoplus_{m \geq 0} H_{k-2m+4}(W, \partial W)
\xrightarrow{} \bigoplus_{m \geq 1} H_{k-2m+4}(W, \partial W) \to 0
$$

shows, for dimensional reasons, that $i$ is necessarily injective. Hence the map $HC_{k+1}^0(M) \to H_{k+4}(W, \partial W)$ vanishes as well and the induction step is completed. \qed

**Remark 19.** Proposition 9 shows that our long exact sequence splits in the case under study into short exact sequences of the form

$$
0 \to H_{k+2}(W, \partial W) \xrightarrow{i} \bigoplus_{m \geq 0} H_{k-2m+2}(W, \partial W)
\xrightarrow{p} \bigoplus_{m \geq 1} H_{k-2m+2}(W, \partial W) \to 0.
$$
It is actually the case that $i$ and $p$ are, respectively, the obvious injection and projection. This follows from the results in [8].

Although for transversality reasons we stated Proposition 9 only for stabilizations of subcritical Stein manifolds, we expect it to hold for arbitrary subcritical Stein manifolds (see Remark 9).

8.3. Negative disc bundles

Let $\mathcal{L} \rightarrow B$ be a Hermitian line bundle over a compact, symplectically aspherical manifold $(B, \beta)$ with $c_1(\mathcal{L}) = -[\beta]$. Let $W = \{v \in \mathcal{L} : |v| \leq 1\}$ be the corresponding disc bundle, with symplectic form $\omega = \pi^* \beta + d(r^2 \theta)$, where $r$ is the radial coordinate in the fibers and $\theta$ is the angular form. The boundary $M = \partial W$ is a contact manifold with contact form $\theta$. The closed Reeb orbits on $M$ are the fibers of the natural projection to $B$.

Let us assume $\dim_{\mathbb{R}} B = 2n - 2$, so that $\dim_{\mathbb{R}} W = 2n$. It follows from [26, Theorem D] that $SH_*(W) = 0$, and therefore

$$SH^+_*(W) \simeq H_{s+n-1}(W, \partial W) \simeq H_{s+n-3}(B).$$

**Proposition 10 ([3]).** The (linearized) contact homology of the prequantization bundle $M = \partial W$ is well-defined and equal to

$$HC_*(M, \xi) = \bigoplus_{m=0}^{\infty} H_{s-2m}(B). \quad (88)$$

**Proof.** As in Remark 9(ii), we choose compatible almost complex structures $J$ on $W$ and $J_B$ on $B$, and a generic Morse function $f : B \rightarrow \mathbb{R}$. The Morse–Bott contact complex [3, §8.1] is generated by closed Reeb orbits $\gamma_{p,k}$ of multiplicity $k$ above critical points $p \in \text{Crit}(f)$. The grading of $\gamma_{p,k}$ with respect to the symplectic trivialization given by the fiber is $\text{ind}(p; f) + 2k - \frac{1}{2}(2n - 2) + n - 3 = \text{ind}(p; f) + 2k - 2 \ [3, \text{Lemma 2.4}].$ All closed orbits are good, because the parity of the grading of $\gamma_{p,k}$ does not depend on the multiplicity $k$. The differential in the Morse–Bott complex counts rigid configurations consisting of $J$-holomorphic curves with gradient fragments of $f$ (for this reason, the underlying gluing theorem is similar to [7, Theorem 3.7]). Since $J$-holomorphic curves in the symplectization $M \times \mathbb{R}$ are branched covers of vertical cylinders over closed Reeb orbits, and since branch points can always be displaced along the corresponding vertical cylinder, it follows that the only such rigid configurations are rigid gradient trajectories of $f$. Hence, for each multiplicity $k \geq 1$, the contact differential coincides with the Morse differential of $f$ on $B$, which proves (88). \qed

**Proposition 11.** Let $W$ be a disc bundle over a compact, symplectically aspherical manifold $(B, \beta)$ with $c_1(\mathcal{L}) = -[\beta]$. The exact sequence relating
contact and symplectic homology is isomorphic to an exact sequence of the form

\[ \cdots \to H_k(B) \to \bigoplus_{m \geq 0} H_{k-2m}(B) \to \bigoplus_{m \geq 1} H_{k-2m}(B) \to H_{k-1}(B) \to \cdots \]

**Proof.** Similarly to Proposition 9, we prove by descending induction on \( k \in \mathbb{Z} \) that the maps \( H_{C_k}(M) \to H_{k+1}(B) \) vanish. The claim is true for \( k \geq 2n - 2 \) for dimensional reasons. Assuming the claim to be true for some \( k \in \mathbb{Z} \), the exactness of the sequence

\[
\begin{array}{c}
H_{C_k}(M) \to H_{k+1}(B) \to \bigoplus_{m \geq 0} H_{k+1-2m}(B) \to \bigoplus_{m \geq 1} H_{k+1-2m}(B) \to H_k(B) \\
\end{array}
\]

implies, for dimensional reasons, that the last map vanishes, so that the claim is true for \( k - 1 \). \( \square \)

**Remark 20.** The similarity between Proposition 11 and Proposition 9 is best explained via the \( S^1 \)-equivariant approach in [8]. The spectral sequence in [26] admits an \( S^1 \)-equivariant version which implies that positive \( S^1 \)-equivariant symplectic homology is isomorphic to \( H_*(B) \otimes H_*(\mathbb{C}P^\infty) \). The isomorphism between contact homology and positive \( S^1 \)-equivariant symplectic homology therefore implies (88). Moreover, the exact sequence in Proposition 11 is the tensor product of \( H_*(B) \) with the Gysin exact sequence of the subcritical pair \((D^2, S^1)\).

### 8.4. Cotangent bundles

Our next example are unit cotangent bundles

\[ W = DT^*L := \{ p \in T^*L : |p| \leq 1 \} \]

of closed Riemannian manifolds \( L \). We recall the transversality discussion in Remark 9 where we imposed conditions \((A)\) and \((B_a)\) in all statements involving linearized contact homology (we assume either \( \dim L \geq 5 \) or \( L \) has no contractible closed geodesics, and we work in a free homotopy class \( a \) containing only simple closed geodesics). As mentioned in Remark 9, we expect these two conditions to be completely removed in the future, and we do not mention them anymore in the discussion that follows.

The symplectic manifold \( W = DT^*L \) is exact with boundary of restricted contact type

\[ M = ST^*L := \{ p \in T^*L : |p| = 1 \}. \]
The Liouville form determines a contact structure on $M$ whose isotopy class does not depend on the choice of metric, since the space of Riemannian metrics is convex.

The first ingredient involved in our long exact sequence are the symplectic homology groups $SH^a(W)$ in a free homotopy class $a$. These have been computed by Viterbo [33], Salamon–Weber [28] and Abbondandolo–Schwarz [1]:

$$SH^a_k(DT^*L) \simeq H_k(\Lambda^a L), \quad k \in \mathbb{Z},$$

where $\Lambda^a L \subset \Lambda L$ is the connected component $a$ in the free loop space $\Lambda L$, i.e. the space of continuous maps from $S^1$ to $L$. The space $\Lambda L$ is endowed with the canonical $S^1$-action $(\theta, \gamma(\cdot)) \mapsto \gamma(\cdot + \theta), \theta \in S^1 = \mathbb{R}/\mathbb{Z}, \gamma \in \Lambda L$.

The above isomorphism actually holds in an improved version involving, on the left hand side, the symplectic homology groups truncated by the values of the action functional and, on the right hand side, the relative homology groups of sublevel sets of the energy functional on the loop space [1,28]. When $a = 0$, in particular, the tautological exact sequence (5) is identified with the exact sequence of the pair $(\Lambda^0 L, L)$ and we have

$$SH_k^+(DT^*L) \simeq H_k(\Lambda^0 L, L).$$

The second ingredient in our long exact sequence are the linearized contact homology groups $HC^i_{k-1}(a)(ST^*L)$, where $i$ is the map associating to the free homotopy class of a loop in $ST^*L$ its free homotopy class as a loop in $DT^*L$. An argument similar to that of Sect. 8.2 shows that, if dim $L \geq 3$, the map $i$ is bijective and $i^{-1}(0) = 0$. An ongoing project of K. Cieliebak and J. Latschev [12] aims at computing the entire SFT of $ST^*L$. The theorem below is a particular case of their more general results.

**Theorem 3 (Cieliebak–Latschev [12]).** Let $L$ be a closed oriented Riemannian manifold. Given a free homotopy class of loops $a$ in $DT^*L$ (hence in $L$), the following isomorphisms hold for $k \in \mathbb{Z}$:

$$HC^i_{k-1}(a)(ST^*L) \simeq H_{k-(n-3)}(\Lambda^a L/S^1), \quad a \neq 0,$$

$$HC^i_{k-1}(0)(ST^*L) \simeq H_{k-(n-3)}(\Lambda^0 L/S^1, L). \quad (89)$$

**Remark 21.** One can rephrase Theorem 3 within the setting of $S^1$-equivariant homology. Given a topological space $M$ endowed with an $S^1$-action, the $S^1$-equivariant homology groups are defined as

$$H^S_*(M) := H_*(M \times_{S^1} ES^1),$$

where $ES^1$ is a contractible space on which $S^1$ acts freely (for example $ES^1 = S^\infty$, the infinite dimensional sphere). If we work with rational
coefficients and $S^1$ acts with finite – hence cyclic – isotropy groups, then $S^1$-equivariant homology is isomorphic to the homology of the quotient. The reason is that the map $M \times_{S^1} ES^1 \to M/S^1$ induced by the projection on the first factor behaves like a fibration with fibers $B\mathbb{Z}/k\mathbb{Z}, k \geq 1$ and the latter are $\mathbb{Q}$-acyclic. As a consequence we obtain that

$$H_*(\Lambda^0 L/S^1, L) \cong H^*_\ast(\Lambda^0 L, L).$$

The above discussion can be summarized as follows.

**Proposition 12.** Given a closed oriented Riemannian manifold $L$, the long exact sequence relating contact and symplectic homology for $DT^\ast L$ in the free homotopy class $a$ is isomorphic to an exact sequence of the form

$$\cdots \to H_k(\Lambda^a L) \to H^s\ast_k(\Lambda^a L) \to H^s_{k-2}(\Lambda^a L) \to H_{k-1}(\Lambda^a L) \to \cdots$$

if $a \neq 0$, respectively

$$\cdots \to H_k(\Lambda^0 L, L) \to H^s\ast_k(\Lambda^0 L, L) \to H^s_{k-2}(\Lambda^0 L, L) \to H_{k-1}(\Lambda^0 L, L) \to \cdots$$

if $a = 0$.

It follows from the results in [8] that these are the classical Gysin exact sequences for $S^1$-equivariant homology.

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