TORIC CO-HIGGS SHEAVES

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Abstract. We characterise and investigate co-Higgs sheaves and associated algebraic and combinatorial invariants on toric varieties. In particular, we compute explicit examples.

1. Introduction

1.1. Higgs and co-Higgs bundles. In [Sim94] Simpson studied the moduli space of (semistable) $\Lambda$-modules, where $\Lambda$ is a sheaf of rings of differential operators on a scheme of finite type over a noetherian scheme over $\mathbb{C}$. On one hand side, this notion encapsulates Higgs bundles as introduced by Hitchin [Hit87], where $\Lambda$ is induced by the cotangent sheaf. On the other hand, and more importantly for us, it also embraces the notion of co-Higgs bundles which goes back to [Hit11] and [Ray11] in the context of Hitchin’s generalised geometries. Here, $\Lambda$ is induced by the tangent sheaf. Concretely, let $X$ be a normal variety over $\mathbb{C}$ and consider a reflexive sheaf of $\mathcal{O}_X$-modules $E$. Moreover, consider an $\mathcal{O}_X$-linear map $\Phi : E \to E \otimes \mathcal{O}_X T_X$. We define the homomorphism $\Phi \wedge \Phi : E \to E \otimes \mathcal{O}_X \Lambda^2 T_X$ as the composition

$$E \xrightarrow{\Phi} E \otimes \mathcal{T}_X \xrightarrow{\Phi \otimes \text{id}} E \otimes T_X \otimes T_X \xrightarrow{\text{id} \otimes \Lambda^2} E \otimes \Lambda^2 T_X.$$ 

Definition 1. If $\Phi \wedge \Phi = 0$ holds, $(E, \Phi)$ is called a co-Higgs sheaf; $\Phi : E \to E \otimes T_X$ is referred to as the co-Higgs field.

In contrast, a Higgs sheaf is given by an $\mathcal{O}_X$-linear map $\Psi : E \to E \otimes \mathcal{O}_X \Omega^1_X$ with $\Psi \wedge \Psi = 0$. In fact, the duality between Higgs and co-Higgs sheaves goes somewhat deeper. Simpson showed that the moduli space of stable Higgs bundles is isomorphic to the representation variety of $X$. In particular there are no nontrivial stable Higgs bundles on Fano or toric varieties (cf. also Remark 9).

1.2. Co-Higgs bundles on toric varieties. Starting with a complete toric variety we are thus naturally led to the investigation of co-Higgs bundles. As does the very recent article [BDPR20], we appeal to Klyachko’s description of toric vector bundles, cf. Section 2. However, in contrast to [BDPR20] we do not only focus on invariant, i.e. $M$-homogeneous co-Higgs fields, but on general ones on toric sheaves. This leads to the notion of Higgs polytopes and Higgs ranges reflecting the position of possible multidegrees of co-Higgs fields. A particularly interesting instance is the intrinsic case where $E$ is the tangent sheaf itself. Here, the co-Higgs sheaves have a natural interpretation in terms of generalised complex structures on the tangent sheaf [Gua11, Hit11], and we will for instance study this case on smooth complete toric surfaces. The projective plane has already been considered in [Ray14] albeit from a non-toric point of view.

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1.3. Plan of the paper. We briefly summarise the content of the paper and our main results.

As a start, we first use Klyachko’s formalism to give in Theorem 8 a combinatorial description of homogeneous (co-) Higgs sheaves which is different from [BDPR20, Theorem 3.1]. Moreover, we first neglect the integrability condition $\Phi \wedge \Phi = 0$ and focus on the structure of all toric maps $\Phi : E \to E \otimes T_X$ which we call pre-co-Higgs fields. The reason for this is that pre-co-Higgs fields behave well under sums, i.e. under decomposition into their homogeneous components.

Afterwards, we use the integrability condition $\Phi \wedge \Phi = 0$ to define a family of endomorphism algebras parametrised by the torus (Proposition 12). Section 4 introduces two combinatorial invariants. First, a toric Higgs field gives rise to the Higgs polytope in the character lattice by taking the convex hull of its homogeneous degrees. Second, the totality of degrees of all possible toric pre-Higgs fields defines another polytope, the Higgs range.

The computation of this Higgs range for the case of smooth and complete toric surfaces will be the endeavour for the rest of this paper. In Section 5 we explain the computation for $\mathbb{P}^2$ in some detail which yields Theorem 21, namely the complete description of the Higgs range. Further, we sketch the case of Hirzebruch and Fano surfaces in Section 6. Finally, we exhibit explicit Higgs polytopes on del Pezzos of degree 6 and 7 in Subsection 6.6. In these examples every subpolytope of the Higgs range can be realized as the Higgs polytope of some toric co-Higgs field.

1.4. Convention. As we have pointed out it makes no sense to consider Higgs sheaves on toric varieties, hence we work only with co-Higgs sheaves. However, to simplify language, we drop the qualifier “co-” in the sequel and simply speak about toric Higgs sheaves when actually meaning toric co-Higgs sheaves (with Remark 9 as sole exception). Hopefully no confusion will arise.

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2. Klyachko’s formalism

Klyachko’s description of toric vector bundles or, more generally, of toric reflexive sheaves appeared in [Kly90]. See also [Kly02] for his 2002 ICM talk on this subject, and a short summary can be found in [Pay08]. Further, more recent approaches can be found in [RJS18] and [KM].

2.1. Klyachko’s description of toric sheaves. Consider a toric variety $X = TV(\Sigma)$ given by a fan $\Sigma$ in $N_\mathbb{R} = N \otimes \mathbb{R}$, where $N$ is a lattice of rank $q$. As usual, its dual $M = \text{Hom}_\mathbb{Z}(N, \mathbb{Z})$ denotes the character lattice. Then $X$ contains the torus $T = N \otimes \mathbb{C}^*$ and we may pick the neutral element $1 \in T \subseteq X$. Each $\mathcal{O}_X$-module $\mathcal{E}$ gives rise to a $\mathbb{C}$-vector space

$$E := \mathcal{E}(1) := \mathcal{E}_1/m_{X,1}\mathcal{E}_1,$$

where $\mathcal{E}_1$ denotes the stalk of $\mathcal{E}$ at $1 \in X$ and $m_{X,1}$ the maximal ideal of 1. If $\mathcal{E}$ is a $T$-equivariant, i.e., $T$-linearized, torsion free sheaf on $X$, the global sections of $\mathcal{E}$ are an
$M$-graded subset of $E \otimes \mathbb{C} [M]$. If, in addition, $E$ is reflexive, then $E$ is already determined by its restriction to open subsets whose complements are of codimension equal or greater than two. We briefly refer to $E$ as a \textit{toric sheaf}.

Via Klyachko’s description \cite{Kly90}, a toric sheaf $E$ corresponds to a set of decreasing $\mathbb{Z}$-filtrations

$$E^*_\rho = [\ldots \supseteq E^\ell_{\rho-1} \supseteq E^\ell_{\rho} \supseteq E^\ell_{\rho+1} \supseteq \ldots] \quad (\ell \in \mathbb{Z})$$

of the vector space $E$ which are parametrized by the rays or one-dimensional cones $\rho \in \Sigma(1)$. By abuse of notation we use $\rho$ for both the ray and its primitive generator. The filtrations encode the sections of $E$ on the $T$-invariant open subsets $U_{\rho} = TV(\rho) \subseteq X$ defined by $\rho$. Namely, for $r \in M$,

$$e \otimes \chi^r \in \Gamma(U_{\rho}, E) \iff e \in E^{-\langle r, \rho \rangle}_{\rho}.$$

Since $\bigcup_{\rho \in \Sigma(1)} U_{\rho}$ is an open set of codimension at least two,

$$e \otimes \chi^r \in \Gamma(X, E) \iff e \in \bigcap_{\rho \in \Sigma(1)} E^{-\langle r, \rho \rangle}_{\rho}.$$

\textbf{Remark 2.} The reflexive sheaf $E$ defines a toric \textit{vector bundle} if it is subject to Klyachko’s compatibility condition \cite{Kly90}: For each cone $\sigma \in \Sigma$ there exists a decomposition $E = \bigoplus_{[u] \in M/M \cap \sigma^\perp} E_{[u]}$ such that $E^\ell_{\rho} = \sum_{\langle u, \rho \rangle \geq \ell} E_{[u]}$ for each ray $\rho$ contained in $\sigma$.

\textbf{Example 3.} For a smooth toric variety line bundles, the tangent and the cotangent bundle are toric with filtrations as follows (cf. \cite[Example 2.3]{Kly90}):

(i) Let $D_{\rho} = \text{orb}(\rho)$ be the closure of the orbit defined by the ray $\rho$. For $D = \sum_{\rho} \lambda_{\rho} D_{\rho}$, the invertible sheaf $\mathcal{O}(D)$ is encoded by

$$E^\ell_{\rho} := \begin{cases} \mathbb{C} & \text{if } \ell \leq \lambda_{\rho} \\ 0 & \text{if } \ell \geq \lambda_{\rho} + 1 \end{cases} \subseteq \mathbb{C} =: E.$$

(ii) The cotangent sheaf $\Omega^1_X$ corresponds to the filtration

$$E^\ell_{\rho} := \begin{cases} \mathcal{M}_{\rho} & \text{if } \ell \leq -1 \\ \rho^\perp & \text{if } \ell = 0 \\ 0 & \text{if } \ell \geq 1 \end{cases} \subseteq \mathcal{M}_{\rho} =: E.$$

(iii) On the other hand, the tangent sheaf $T_X$ corresponds to the filtration

$$T^\ell_{\rho} := \begin{cases} \mathcal{N}_{\rho} & \text{if } \ell \leq 0 \\ \text{span}(\rho) & \text{if } \ell = 1 \\ 0 & \text{if } \ell \geq 2 \end{cases} \subseteq \mathcal{N}_{\rho} =: E.$$

For instance, for $\mathbb{P}^1$ we recover the first example since $T_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(D_{[1,0]} + D_{[0,1]})$. In fact, Examples (ii) and (iii) are related via the general formula relating the filtrations of an equivariant reflexive sheaf with its dual sheaf.

We can use the description of Example \textbf{3} to calculate the global sections of various toric sheaves using \cite{2}.
Example 4. For further use we consider the twisted tangent sheaf $\mathcal{T}(d)$ over $\mathbb{P}^2$. The Euler sequence immediately yields that $\Gamma(\mathbb{P}^2, \mathcal{T}_{\mathbb{P}^2}(d))$ is a $h(d) = (d^2 + 6d + 8)$-dimensional complex vector space if $d \geq -1$ and trivial otherwise. To derive this from a toric point of view, we let
\begin{equation}
(3) \quad a = (1, 0), \quad b = (0, 1) \quad \text{and} \quad c = (-1, -1)
\end{equation}
denote the rays of the fan of $\mathbb{P}^2$. The filtration is given by $F^k_\rho = \sum_{i+j=\ell} E^i_\rho \otimes T^j_\rho$ where $T^k_\rho$ and $E^k_\rho$ are the filtrations of the tangent sheaf $\mathcal{T}_{\mathbb{P}^2}$ and of the line bundle $\mathcal{O}(dD_a)$ respectively. Hence
\begin{equation}
F^k_a = \begin{cases}
\mathbb{C}^2 & \text{if } \ell \leq d \\
\text{span}(a) & \text{if } \ell = d + 1 \\
0 & \text{if } \ell \geq d + 2
\end{cases}
F^k_{b,c} = \begin{cases}
\mathbb{C}^2 & \text{if } \ell \leq 0 \\
\text{span}(b, \text{span}(c)) & \text{if } \ell = 1 \\
0 & \text{if } \ell \geq 2
\end{cases}
\end{equation}
Then $0 \neq f \in \bigcap_{\rho \in \Sigma(1)} F^{-(r, \rho)}$ implies the inequalities $r_1 \geq -d - 1$, $r_2 \geq -1$, and $r_1 + r_2 \leq 1$. Now the vertices of this polytope cannot give rise to nontrivial sections as $\text{span}(\rho) \cap \text{span}(\rho') = 0$ for rays $\rho \neq \rho'$. The $3 \cdot (d + 2)$ lattice points in the facets span a one-dimensional space. On the other hand, the $(d + 1)(d + 2)/2$ interior lattice points span a two-dimensional space each. Hence $h(d) = 3(d + 2) + (d + 2)(d + 1) = d^2 + 6d + 8$ in accordance with the Euler formula.

2.2. Klyachko’s description of morphisms between toric sheaves. Next assume that $\mathcal{E}$ and $\mathcal{F}$ are two toric sheaves over $X = TV(\Sigma)$ given by filtrations $E^k_\rho \subseteq E$ and $F^k_\rho \subseteq F$, $\rho \in \Sigma(1)$, respectively. The space of homomorphisms $\mathcal{E} \rightarrow \mathcal{F}$ is therefore graded over $M$,
\begin{equation}
\text{Hom}(\mathcal{E}, \mathcal{F}) = \bigoplus_{r \in M} \text{Hom}_T(\mathcal{E}, \mathcal{F}[r]) \subseteq \text{Hom}(E, \mathcal{F}) \otimes \mathbb{C}[M].
\end{equation}
Here, $\text{Hom}_T(\cdot, \cdot)$ denotes the $T$-equivariant morphisms and $\mathcal{F}[r]$ is the toric sheaf $\mathcal{F}$ with the new $T$-action obtained by twisting with the character $\chi^r$. In particular, an equivariant $\Phi \in \text{Hom}_T(\mathcal{E}, \mathcal{F})$ corresponds to a linear map $\phi \in \text{Hom}(E, \mathcal{F})$ which satisfies $\phi(E^k_\rho) \subseteq F^k_\rho$ for all $\rho \in \Sigma(1)$ and $\ell \in \mathbb{Z}$. Since the filtration of $\mathcal{F}[r]$ is given by
\begin{equation}
F^{-(r, \rho)}_\rho \subseteq F, \quad \rho \in \Sigma(1),
\end{equation}
an equivariant $\Phi \in \text{Hom}_T(\mathcal{E}, \mathcal{F}[r])$ is given by $\phi \otimes \chi^r$ with
\begin{equation}
\phi(E^k_\rho) \subseteq F^{-(r, \rho)}_\rho \quad \text{for all } \rho \in \Sigma(1) \text{ and } \ell \in \mathbb{Z}.
\end{equation}
A general homomorphism $\Phi \in \text{Hom}(\mathcal{E}, \mathcal{F})$ is the sum $\Phi = \sum_{r \in M} \Phi^r$ of homogeneous morphisms of degree $r$ with $\Phi^r = \phi^r \otimes \chi^r$, where $\phi^r \in \text{Hom}(E, \mathcal{F})$ is the associated $\mathbb{C}$-linear map.

Example 5. Let us compute a basis for $\text{Hom}(\mathcal{O}(1), \mathcal{T}_{\mathbb{P}^2}) \cong \Gamma(\mathbb{P}^2, \mathcal{T}_{\mathbb{P}^2}(-1))$ using the notation from Example 4. The line bundle $\mathcal{O}(1) = \mathcal{O}(D_a)$ is given by
\begin{equation}
E^k_a = \begin{cases}
\mathbb{C} & \text{if } \ell \leq 1 \\
0 & \text{if } \ell \geq 2
\end{cases}, \quad E^k_{\rho = b,c} = \begin{cases}
\mathbb{C} & \text{if } \ell \leq 0 \\
\text{span}(\rho) & \text{if } \ell = 1 \\
0 & \text{if } \ell \geq 2
\end{cases}
\end{equation}
while we find
\begin{equation}
T^k_{\rho = a,b,c} = \begin{cases}
\mathbb{C}^2 & \text{if } \ell \leq 0 \\
\text{span}(\rho) & \text{if } \ell = 1 \\
0 & \text{if } \ell \geq 2
\end{cases}
\end{equation}
for the tangent bundle. If \( 0 \neq \Phi^r = \phi^r \otimes \chi^r \in \text{Hom}(\mathcal{O}_{\mathbb{P}^2}(1), T_{\mathbb{P}^2}) \) is nontrivial, then \( \phi^r(E^\ell_\rho) \) is nontrivial if \( \ell - (r, \rho) \leq 1 \) for \( \ell \leq 1 \) if \( \rho = a \) and \( \ell \leq 0 \) if \( \rho = b, c \). This is equivalent to the inequalities \( r_1 \geq 0, r_1 \geq -1 \) and \( r_1 + r_2 \leq 1 \). Excluding the vertices of the resulting polytope we find the same result as in Example 4.

3. Toric Higgs sheaves

3.1. Toric pre-Higgs fields. We now focus on the case \( \mathcal{F} = \mathcal{E} \otimes_{\mathcal{O}_X} T_X \). We recall our convention from the introduction that we drop the qualifier “co-” although we tensor with \( T_X \), not \( \Omega^1_X \).

Definition 6. A pre-Higgs field \( \Phi \) on \( \mathcal{E} \) is a morphism in

\[
\text{Hom}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{O}_X} T_X) = \bigoplus_{r \in M} \text{Hom}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{O}_X} T_X)[r].
\]

It is thus the direct sum \( \Phi = \sum \Phi^r \) of \( M \)-homogeneous maps \( \Phi^r : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} T_X \) of degree \( r \in M \). The \( \Phi^r \) are called homogeneous pre-Higgs fields. Writing \( \Phi^r = \phi^r \otimes \chi^r \) we obtain the associated \( \mathbb{C} \)-linear map \( \phi^r : E \to E \otimes N_\mathbb{C} \). The pair \( (\mathcal{E}, \Phi) \) is called a toric pre-Higgs sheaf.

We need to analyse the filtrations \( F^*_\rho \) of \( \mathcal{F} = \mathcal{E} \otimes T_X \) next. These fit into the following exact sequence:

Lemma 7. For \( \rho \in \Sigma(1) \) the sequence

\[
0 \longrightarrow F^\ell_\rho \longrightarrow E^\ell_{\rho-1} \otimes N_\mathbb{C} \longrightarrow \pi_\rho ((E^\ell_{\rho-1}/E^\ell_\rho) \otimes (N_\mathbb{C}/\text{span}(\rho))) \longrightarrow 0,
\]

where \( \pi_\rho \) denotes the natural projection, is exact.

Proof. If \( T^j_\rho \) denotes the filtration of the tangent sheaf (cf. Example 3 (iii)), then \( F^\ell_\rho = \sum_{i+j=\ell} E^i_\rho \otimes T^j_\rho \). Since \( E^i_\rho \otimes T^j_\rho \subset E^\ell_\rho \otimes N_\mathbb{C} \) whenever \( i = \ell - j, j \leq 0 \), we have \( F^\ell_\rho = E^\ell_\rho \otimes N_\mathbb{C} + E^\ell_{\rho-1} \otimes \text{span}(\rho) \). In particular, we have a natural injection \( F^\ell_\rho \to E^\ell_{\rho-1} \otimes N_\mathbb{C} \). Clearly, \( F^\ell_\rho \subset \ker \pi_\rho \); equality follows on dimensional grounds.

Given a map \( \phi : \mathcal{E} \to \mathcal{E} \otimes N_\mathbb{C} \) it will be convenient to consider the contraction of \( \phi \) by \( s \in M \), namely

\[
\phi_s := (s, \phi) \in \text{End}(E),
\]

and to regard \( \phi \) as a \( \mathbb{Z} \)-linear map \( M \to \text{End}(E), s \mapsto \phi_s \).

Theorem 8. A \( \mathbb{C} \)-linear map \( \phi : \mathcal{E} \to \mathcal{E} \otimes N_\mathbb{C} \) induces a homogeneous pre-Higgs field of degree \( r \) if and only if the associated contractions satisfy

\[
\phi_s(E^\ell_\rho) \subseteq \left\{ \begin{array}{ll}
E^\ell_{\rho-(r, \rho)} & \text{if } s \in \rho^+ \\
E^\ell_{\rho-1-(r, \rho)} & \text{if } s \not\in \rho^+ \end{array} \right\} \subseteq E^\ell_{\rho-(r, \rho)}
\]

for all \( s \in M, \rho \in \Sigma(1) \) and \( \ell \in \mathbb{Z} \).

Proof. If \( \phi \) is \( \mathbb{C} \)-linear, then \( \phi(E^\ell_\rho) \subset E^\ell_{\rho-(r, \rho)} = E^\ell_\rho \otimes N_\mathbb{C} + E^\ell_{\rho-1} \otimes \text{span}(\rho) \). Hence this is a necessary condition. Conversely, we already know that \( \phi(E^\ell_\rho) \subseteq E^\ell_{\rho-1} \otimes N \) for \( \phi_s(E^\ell_\rho) \subseteq E^\ell_{\rho-1-(r, \rho)} \). To show that the image actually lies in \( F^\ell_{\rho-(r, \rho)} \otimes N \) we need to prove...
that $\pi_\rho \circ \phi = 0$. Considering this as a map $(N_C / \text{span}(\rho))^* \rightarrow E^{t-1-\langle r, \rho \rangle} / E^{t-\langle r, \rho \rangle}$ this must vanish. Since $\phi_s(E^\rho) = (s, \phi(E^\rho))$ this holds by assumption. 

**Remark 9.** It is instructive to compare Theorem 3 with the corresponding result for toric pre-(non-co)-Higgs fields in the usual sense, i.e., for morphisms $\Psi : E \rightarrow E \otimes_{O_X} \Omega^1_X$. As mentioned in the introduction it follows from the general theory that there are no nontrivial (that is, stable) examples at least for complete toric varieties.

From the toric point of view we can support this as follows. A $\mathbb{C}$-linear map $\psi : E \rightarrow E \otimes_{C} M_C$ corresponds to a homogeneous morphism $\Psi : E \rightarrow E \otimes_{O_X} \Omega^1_X$ of degree $r$ if and only if for all $a, b \in \Sigma(1)$ and $\ell \in \mathbb{Z}$ we have $\psi_b(E^\ell_a) \subseteq E^{\ell-\langle r, a \rangle + \delta_{ab}}$, where $\delta_{ab} = 1$ if and only if $a = b$. And it is just the sign in $\pm \delta_{ab}$ making the difference between (non-co)-Higgs and co-Higgs.

Consequently, if $r = 0$ or $r \in M \setminus \vert \Sigma \vert$ (i.e. for all $r \in M$ if $\Sigma$ is complete), the associated $\mathbb{C}$-linear pre-(non-co)-Higgs fields $\psi : E \rightarrow E \otimes_{C} M_C$ gives rise to nilpotent endomorphisms $\psi_a$, $a \in \Sigma(1)$. Indeed, for $r \in M \setminus \vert \Sigma \vert$ there is a $b \in \Sigma(1)$ such that $\langle r, b \rangle < 0$. Then $\psi_a(E^\ell_b) \subseteq E^{\ell(r, b)} \subseteq E^{\ell+1}$. It follows, for instance, that the tangent bundle of $\mathbb{P}^2$ does not admit any nontrivial toric pre-(non-co)-Higgs field $\Psi : E \rightarrow E \otimes_{O_X} \Omega^1_X$ (as we knew before of course).

3.2. Toric Higgs fields. We return to our Convention 1.4) and come to the central definition of this paper. It introduces the equivariant versions of Definition 1.

**Definition 10.**

(i) A toric Higgs sheaf $(\mathcal{E}, \Phi)$ consists of a toric sheaf $\mathcal{E}$ over the toric variety $X = \mathbb{T} \setminus \Sigma$, and an arbitrary, not necessarily homogeneous pre Higgs field $\Phi : \mathcal{E} \rightarrow \mathcal{E} \otimes T_X$ which satisfies $\Phi \wedge \Phi = 0$, cf. [1]. We refer to $\Phi$ as the Higgs field of the underlying toric sheaf $\mathcal{E}$.

(ii) A homogeneous Higgs sheaf $(\mathcal{E}, \Phi)$ is a toric Higgs sheaf with homogeneous Higgs field $\Phi$ of given degree $r \in M$. It corresponds to a $\mathbb{C}$-linear map $\phi : E \rightarrow E \otimes M_C$.

We speak of Higgs bundles instead of sheaves if $\mathcal{E}$ is actually locally free.

**Remark 11.** The condition $\Phi \wedge \Phi = 0$ is not inherited by the homogeneous components of $\Phi$, so that $\Phi = \sum \Phi^r$ is merely a decomposition into homogeneous pre-Higgs fields. On the other hand, the direct sum of homogeneous Higgs fields $\Phi = \sum \Phi^r$ is not necessarily a Higgs field as it might not satisfy $\Phi \wedge \Phi = 0$.

3.3. The Higgs algebra. A pre-Higgs field can be considered as an element of $\text{End}(E) \otimes N_C \otimes \mathbb{C}[M]$ via $\Phi = \sum_{r \in M} \Phi^r = \sum_{r \in M} \phi^r \otimes \chi^r$. In particular, we can contract $\Phi$ with $s \in M$ and $t \in T$ to obtain

$$\Phi_s := (s, \Phi) = \sum_{r \in M} \phi^r_s \otimes \chi^r \in \text{End}(E) \otimes \mathbb{C}[M] \quad \text{and} \quad \Phi_s(t) \in \text{End}(E).$$

The condition $\Phi \wedge \Phi = 0$ translates into

**Proposition 12.** For any $s, s' \in M$ we have $[\Phi_s, \Phi_s'] = 0$ in $\text{End}(E) \otimes \mathbb{C}[M]$. In particular, every Higgs field defines a family

$$A(t) = \mathbb{C} [\Phi_s(t) = \sum_{r \in M} \chi^r(t) \phi^r_s \mid s \in M] \subset \text{End}(E), \quad t \in T$$
of (commutative) finitely generated subalgebras with \( \text{id}_E \) as unit.

Proof. This is actually a non-toric property in the following sense. Consider a general Higgs field \( \Phi \) as in Definition 1. Locally, we can fix a base of vector fields \( \{ \nu_i \mid i \in I \} \subset T_X \) with \( \#I = q \) over \( U \) and write \( \Phi|_U = \sum_{i \in I} \Phi_i \otimes \nu_i \) with \( \Phi_i \in \Gamma(U, \text{End}(\mathcal{E})) \). It follows that \( [\Phi_i, \Phi_j] = 0 \) for all \( i, j \in I \), for \( 0 = \Phi \wedge \Phi = \sum_{i<j} [\Phi_i, \Phi_j] \nu_i \wedge \nu_j \). In fact, for every local section \( \omega \in \Omega^1_X \) we can evaluate so that \( \nu_i(\omega) \in \mathcal{O}_X \) and end up with a set of commuting endomorphisms \( \Phi_s = \sum_{r \in M} \phi_r^s \otimes \chi^r \). □

Definition 13. We call the finitely generated, commutative \( \mathbb{C}[M] \)-subalgebra \( A = A(\Phi) := \mathbb{C}[M][\Phi_s = \sum_{r \in M} \phi_r^s \otimes \chi^r \mid s \in M] \) the Higgs algebra with \( \text{id}_E \otimes \chi^0 \) as unit element.

Remark 14. Alternatively, the construction from the previous proof yields in the non-toric setting a sheaf of commutative subalgebras of \( \text{End}(\mathcal{E}) \). The associated relative spectrum \( \tilde{X} \rightarrow X \) relates to the Hitchin fibration. Note, however, that in contrast to the latter one, our algebra involves the minimal polynomial instead of the characteristic one. Implicitly, we are using the isospectral decomposition of \( E \) via characters providing the eigenvalues. Since there might be summands of dimension \( >1 \), this obstructs the construction of an honest fibration over \( X \) (the spectral variety). In the toric case it would be interesting to see how this sheaf relates to the algebra \( A \) which is just the restriction to \( T \).

For each \( t \in T \), we obtain a surjection \( A \rightarrow A(t) \) within \( \text{End}(E) \otimes \mathbb{C}[M] \rightarrow \text{End}(E) \). We obtain the following commutative diagram where only the rightmost column is non-commutative:

\[
\begin{array}{cccc}
\Phi_s & \mathbb{C}[M] \xrightarrow{\alpha} & A \xrightarrow{\psi} & \text{End}(E) \otimes \mathbb{C}[M] \\
\downarrow_{t \in T} & \downarrow & \downarrow & \downarrow \\
\mathbb{C} & A \otimes_t \mathbb{C} \xrightarrow{\psi_t} & (\text{End}(E) \otimes \mathbb{C}[M]) \otimes_t \mathbb{C} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\mathbb{C} & A(t) \xrightarrow{\sim} & \text{End}(E) \\
\end{array}
\]

The injectivity of \( \psi_t \) is equivalent to \( A \otimes_t \mathbb{C} \rightarrow A(t) \) being an isomorphism which corresponds to the flatness of \( a \).

4. COMBINATORIAL INVARIANTS

4.1. The Higgs polytope. For a given toric pre-Higgs sheaf \( (\mathcal{E}, \Phi) \) we can define a combinatorial invariant as follows. Let \( \text{supp} \Phi = \{ r \in M \mid \Phi^r = \phi^r \otimes \chi^r \neq 0 \} \) be the support of the pre-Higgs field \( \Phi \).

Definition 15. The convex lattice polytope in \( M_{\mathbb{R}} \) defined by

\[
\nabla(\Phi) := \text{conv supp} \Phi,
\]

is the Higgs polytope.
is called the \textit{Higgs polytope} of $(E, \Phi)$.

This combinatorial invariant does heavily depend on the toric data. Whenever $\nabla' \subseteq \nabla$ is a subpolytope, e.g., $\nabla' = \{r\}$ for a single $r \in \nabla \cap M$, then we define the restriction of a pre-Higgs field $\Phi$ to $\nabla'$ by

$$\Phi|_{\nabla'} := \sum_{r \in \nabla'} \Phi^r.$$ 

Obviously, this defines again a pre-Higgs field. In contrast, even for an honest Higgs field $\Phi$, the restriction $\Phi|_{\nabla'}$ is merely a pre-Higgs field in general, for it does not need to satisfy $\Phi|_{\nabla'} \wedge \Phi|_{\nabla'} = 0$, cf. Remark 11. However, we have the following

**Proposition 16.** Let $\Phi$ be a Higgs field and $F \leq \nabla(\Phi)$ be a face. Then the restriction $\Phi|_F$ is a Higgs field, too. In particular, the $\C$-linear pre-Higgs field $\phi^r$ arising from a vertex $v \in \nabla(\Phi)$ via $\Phi|_v = \Phi^v = \phi^v \otimes \chi^v$ is an honest $\C$-linear Higgs field of degree $r$.

**Proof.** Let $a \in N$ be an integral vector defining the face $F$, i.e., $F = \{r \in M_\R \mid \langle r, a \rangle = \min(\nabla(\Phi), a)\}$. From $\Phi = \sum_{r \in \nabla(\Phi)} \Phi^r$ we obtain $\Phi \wedge \Phi = \sum_{r,s \in \nabla(\Phi)} \Phi^r \wedge \Phi^s$ where the $(r,s)$-summand has degree $r + s \in M$. Contracting the $M$-degrees via the linear map $\langle \cdot, a \rangle : M \to \Z$ exhibits the pairs $(r,s) \in F \cap F$ exactly as those with minimal $\Z$-degree. Thus, $\Phi|_F \wedge \Phi|_F = (\Phi \wedge \Phi)|_{F \cap F} = 0$. \hfill $\square$

4.2. \textbf{The Higgs range.} In order to see what kind of polytopes can arise for a given toric sheaf $E$, we call $r \in M$ \textit{admissible} for $E$, if there exists a homogeneous pre-Higgs field $\Phi$ of degree $r$.

**Definition 17.** Let $E$ be a toric sheaf. The \textit{Higgs range} of $E$ is the convex hull $\mathcal{H}(E)$ in $M_\Z$ defined by the admissible points. Moreover, for any $r \in \mathcal{H}(E)$ we let $V_r(E)$ denote the complex vector space of maps $\phi : E \to E \otimes N_\C$ which are associated to some homogeneous pre-Higgs field $\Phi$ of degree $r$. One can think of $V_r(E)$ as a kind of multiplicity of the lattice point $r \in \mathcal{H}(E)$.

It follows from Proposition 16 that the Higgs polytope $\nabla(\Phi)$ of every toric pre-Higgs field $\Phi$ on $E$ has to be contained in $\mathcal{H}(E)$. Moreover, $\mathcal{H}(E)$ is a polytope itself by Proposition 18, hence there exists a maximal toric pre-Higgs field $\Phi$ satisfying $\nabla(\Phi) = \mathcal{H}(E)$. It is an immediate question whether one can even find a true Higgs field $\Phi$ with this property. The answer seems to be “no” or at least non-trivial as it is indicated from the example of Subsection 5.5.2. Even more elementary is the question whether every admissible $r \in M$ always allows a true homogeneous Higgs field $\Phi^r$ of degree $r$.

**Proposition 18.** Let $E$ be a toric sheaf over a complete toric variety. Then the Higgs range $\mathcal{H}(E)$ is bounded, that is, it is a (possibly empty) convex polytope.

**Proof.** Recall from Theorem 8 that for all $\ell \in \Z$, we have at least

$$\phi^r(E^\ell_\rho) \subseteq E^{\ell-1-(r,\rho)}_\rho.$$ 

Denote by $N \in \N$ the maximum length of the filtrations $E^\rho_\rho$ for $\rho \in \Sigma(1)$. Then for each $\rho$ there exists an index $\ell(\rho)$ such that $E^{\ell(\rho)}_\rho = E$, but $E^{\ell(\rho)+N}_\rho = 0$. In particular,

$$\mathcal{H}(E) \subseteq \{r \in M_\R \mid \langle r, \rho \rangle \geq -N \text{ for all } \rho \in \Sigma(1)\},$$
where the latter set is bounded by completeness. Indeed, if one of these inequalities is violated, say \( \langle r, \rho \rangle < -N \), then \(-1 - \langle r, \rho \rangle \geq N\), so that \( \phi^r_s = 0 \) for every \( s \in M \) by (5), and thus \( \phi^r = 0 \).

Example 19. Recall from Example 1(iii) that the tangent sheaf \( T_X \) is encoded by the filtrations \( E^r_\rho \) of \( N_\mathbb{C} \) with \( E^1_\rho = \text{span}(\rho), \rho \in \Sigma(1) \) as the only nontrivial subspace whence \( N = 2 \) and \( \ell(\rho) = 0 \) for all \( \rho \in \Sigma(1) \). On the other hand, the fan of the projective plane \( \mathbb{P}^2 \) is the inner normal fan of the polytope \( \Delta \) cut out by the equations \( \langle m, \rho_0 \rangle \geq 1 \) and \( \langle m, \rho_{1,2} \rangle \geq 0 \). Then, the proof of Proposition 18 shows that \( H(T_{\mathbb{P}^2}) \) is contained in the polytope whose facets are at distance two from the origin are parallel to \( \Delta \), see the red lines in the figure below. However, the true Higgs range \( H(T_{\mathbb{P}^2}) \) is even smaller; it is given by the yellow polytope. See Section 5 for the computation; the result for \( H \) can be found in Subsection 5.3.

5. Tracefree \( T_X \)-Higgs fields on \( \mathbb{P}^2 \)

The computation of \( H(E) \) and \( \{ V_r(E) \}_{r \in H(E)} \) for the intrinsic case \( E = T_X \) will occupy us for the remainder of this paper. For simplicity, we write \( H(X) \) and \( V_r(X) \) in this case, see Definition 17. Moreover, we will restrict to tracefree Higgs fields from now – the reason is that we can decompose any Higgs field into a tracefree one and a vector field. The corresponding subspaces we will denote by \( V^0_r(\mathbb{P}^2) \subseteq V_r(\mathbb{P}^2) \). In the present section we focus on \( \mathbb{P}^2 \) to illustrate the key ideas.

5.1. Encoding endomorphisms. In the particular case of \( \mathbb{P}^2 \), we try to keep the symmetry via understanding

\[
N = \mathbb{Z}^3 / \mathbb{Z} \quad \text{and} \quad M = \mathbb{Z}^4 = \{ r \in \mathbb{Z}^3 | r_0 + r_1 + r_2 = 0 \} \subseteq \mathbb{Z}^3.
\]

Thus, any \( \phi := \phi^r_s : N_\mathbb{C} \to N_\mathbb{C} \) becomes a map \( \mathbb{C}^3 \to \mathbb{C}^3 \) sending \( 1 \) into \( \text{span}(1) \). Then, \( \phi \) is represented by a \((3 \times 3)\)-matrix

\[
\tilde{\phi} = \begin{pmatrix} c_{00} & c_{01} & c_{02} \\ c_{10} & c_{11} & c_{12} \\ c_{20} & c_{21} & c_{22} \end{pmatrix}
\]

with \( \sum_{j=0}^2 c_{ij} \) being independent on the row \( i \).

Altering \( \tilde{\phi} \) into \( \tilde{\phi} + (a b c) \) (adding 3 equal rows) does not change \( \phi \). Hence, we obtain a canonical representative (also called \( \phi \)) by asking for \( c_{11} = c_{22} = c_{33} = 0 \). We will
Theorem 8 says that this property has to be strengthened by one, i.e. we ask for $E(\parallel)$ to the corresponding facets of the polytope $\Delta = \text{conv}\{v_i\}$. Thus, the affine hyperplanes $F$ from filtrations to facets. The determinant (as an endomorphism of $N$ with endomorphisms. They can be written as sometimes use the isomorphism $\mathbb{Z}^2 \cong N = \mathbb{Z}^3 / \mathbb{Z}$ and its inverse $\mathbb{Z}^3 \to N \cong \mathbb{Z}^2$ given by the matrices 

$$
\begin{pmatrix}
0 & 0 \\
1 & 0 \\
0 & 1
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
-1 & 1 & 0 \\
1 & 0 & 1
\end{pmatrix},
$$

respectively, to understand $\phi$ or $\tilde{\phi}$ as a linear map $\mathbb{C}^2 \to \mathbb{C}^2$. Doing so, $\tilde{\phi}$ transforms into 

$$
\left(
-1 & 1 & 0 \\
1 & 0 & 1
\right) \cdot
\begin{pmatrix}
c_{00} & c_{01} & c_{02} \\
c_{10} & c_{11} & c_{12} \\
c_{20} & c_{21} & c_{22}
\end{pmatrix}
\cdot
\begin{pmatrix}
0 & 0 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}
= 
\begin{pmatrix}
c_{11} - c_{01} & c_{12} - c_{02} \\
c_{21} - c_{01} & c_{22} - c_{02}
\end{pmatrix}.
$$

This yields $\text{tr}(\phi) = (c_{00} + c_{11} + c_{22}) - \sum_{j=0}^{2} c_{ij}$ (for each $i = 0, 1, 2$). In particular, the normal form of $\phi$ equals 

$$
\phi = 
\begin{pmatrix}
0 & c_{01} & c_{02} \\
c_{10} & 0 & c_{12} \\
c_{20} & c_{21} & 0
\end{pmatrix}
\quad \text{with} \quad c_{01} + c_{02} = c_{10} + c_{12} = c_{20} + c_{21} = -\text{tr}(\phi).
$$

Note that $\text{tr}(\phi)$ does not refer to the literal interpretation as the trace of the representing $(3 \times 3)$-matrix, but of $\phi \in \text{End}(N)$. As already announced, we will focus on trace free endomorphisms. They can be written as 

$$
\phi = 
\begin{pmatrix}
0 & x & -x \\
-y & 0 & y \\
z & -z & 0
\end{pmatrix}
= x A_0 + y A_1 + z A_2 \sim \phi = 
\begin{pmatrix}
-x & x + y \\
-(x + z) & x
\end{pmatrix}
$$

with 

$$
A_0 = 
\begin{pmatrix}
0 & 1 & -1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
A_1 = 
\begin{pmatrix}
0 & 0 & 0 \\
-1 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix},
A_2 = 
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & -1 & 0
\end{pmatrix}.
$$

The determinant (as an endomorphism of $N_\mathbb{C}$) is $\det(\phi) = xy + yz + zx$.

5.2. From filtrations to facets. Recall from Example 3(iii) and Example 19 that $E_\rho^1 = \text{span}(\rho)$ with $\rho \in \Sigma(1)$ are the only non-trivial parts of the filtrations encoding $T_{F_2}$. In the proof of Proposition 13 we have already used the general property $\phi^s(E^\ell_\rho) \subseteq E^{\ell-1-(r,\rho)}_\rho$ ($\forall \ell \in \mathbb{Z}$) characterizing a pre-Higgs field. For the special relation $s \in \rho^1$, however, Theorem 8 says that this property has to be strengthened by one, i.e. we ask for $\phi^s(E^\ell_\rho) \subseteq E^{\ell-(r,\rho)}_\rho$. For $\rho \in \Sigma(1)$ and $c \in \mathbb{Z}_{\geq 0}$ we denote 

$$
F^c_\rho := [(\ast, -\rho) = c] \quad \text{and} \quad F^c_{\geq c} := [(\ast, -\rho) \geq c].
$$

Thus, the affine hyperplanes $F^c_\rho$ have lattice distance $c$ from the origin, and they are parallel to the corresponding facets of the polytope $\Delta = \text{conv}\{[0,0], [1,0], [0,1]\}$ from
Example 19. The second notion $F^\rho_{\geq c}$ points to the outside area, i.e. beyond $F^\rho_c$.

The more “$\rho$-outside” the degrees $r$ are, i.e. the larger $c$ with $r \in F^\rho_{\geq c}$, the larger have to be the $\rho$-jumps $j$ with $\phi^*_s(E^\rho_r) \subseteq E^\rho_{r+j}$, i.e. the more restricted is $\phi^r$. To make this precise we introduce the following notation: An endomorphism $\phi \in \text{End}(E)$ is said to belong to the classes

(i)_{\rho} if $\phi(\rho) \in \text{span}(\rho)$, and

(ii)_{\rho} if $\phi(E) \subseteq \text{span}(\rho) \subseteq \ker(\phi)$. Note that the latter implies nilpotency.

Obviously, [(ii)_{\rho} \Rightarrow (i)_{\rho}], and in the language of Subsection (5.1), these conditions mean the following for trace free endomorphisms $\phi$ (here explained for $\rho = \rho_0$):

(i)_{\rho} $\phi$ is a linear combination

$$
\phi = \begin{pmatrix}
0 & x & -x \\
z & 0 & -z \\
z & -z & 0 \\
\end{pmatrix} = xA_0 + z(A_2 - A_1),
$$

and

(ii)_{\rho} $\phi = \begin{pmatrix}
0 & x & -x \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix} = x \cdot A_0$ is nilpotent.

Lemma 20. Let $r, s \in M$. Then $\phi^*_s$ satisfies the general Higgs condition $\phi^*_s(E^\rho_r) \subseteq E^\rho_{r^1-\langle r, \rho \rangle}$ if and only if

$$
\begin{cases}
 r \in F^\rho_{\geq 3} \Rightarrow \phi^*_s = 0 \\
r \in F^\rho_2 \Rightarrow \phi^*_s \in (ii)_{\rho} \\
r \in F^\rho_1 \Rightarrow \phi^*_s \in (i)_{\rho^\perp}.
\end{cases}
$$

Moreover, if $s \in \rho^\perp$, then the associated stronger condition arises from replacing $F^\rho_{\geq 3}$ by $F^\rho_{\geq 2}$ and $F^\rho_i$ by $F^\rho_{i-1}$ for $i = 1, 2$.

(The proof is straightforward.)

In the following figures we will indicate the conditions (i)_{\rho} and (ii)_{\rho} by the colors blue and red, respectively. Moreover, we put black (or green) dots on all lattice points $r \in M$ where a non-vanishing $\phi^*_s$ is (still) possible. For general $s \in M$, we start with a blue 3$\Delta$ and a red 6$\Delta$ (shifted into central position). For $s \in \rho^\perp$, the $\rho$-facets will be shifted towards the origin yielding blue 2$\Delta$ and red 5$\Delta$ in different positions. Note that for linearly independent $\rho, \rho'$ the intersection (i)_{\rho} \cap (ii)_{\rho'} leads to $\phi^*_s = 0$. Hence, we can exclude
red-red and red-blue intersections. The blue-blue intersection \((i)_{\nu-1} \cap (i)_{\nu+1}\) leads to the unique \(\phi = A_{\nu-1} + A_{\nu+1} - A_{\nu} \ (\nu \in \mathbb{Z}/3\mathbb{Z})\).

5.3. Linear dependence on \(s\). For a fixed \(r \in M\), the endomorphisms \(\phi^r_s\) do linearly depend on \(s \in M\). For the present \(\mathbb{P}^2\) example, we choose \(s^0 := [0, 1, -1]\), \(s^1 := [-1, 0, 1]\) and \(s^2 := [1, -1, 0]\) being contained in \(\rho^0_2\), \(\rho^1_2\), and \(\rho^2_2\), respectively. Denoting \(\phi_i := \phi^r_s\), the relation \(s^0 + s^1 + s^2 = 0\) implies \(\phi_0 + \phi_1 + \phi_2 = 0\). Moreover, since any two of \(\{\phi_0, \phi_1, \phi_2\}\) span \(\{\phi_s \mid s \in M\}\), the vanishing of two \(\phi_i\) implies the vanishing of \(\phi^r_s\) for all \(s \in M\). This leads to the following improvement of the previous figures:

This leads to the following

**Theorem 21.** The Higgs range \(\mathcal{H}(\mathbb{P}^2)\) equals the convex hull of the points \(r^0 = [-2, 1, 1]\), \(r^1 = [1, -2, 1]\), and \(r^2 = [1, 1, -2]\). Moreover, for each \(\nu \in \mathbb{Z}/3\mathbb{Z}\) we have \(\phi^\nu_r = 0\). The dimensions of the vector space \(V^0_r(\mathbb{P}^2)\) equal 1 if \(r\) is a vertex of \(\mathcal{H}(\mathbb{P}^2)\), equal 2 if \(r\) is among the six remaining lattice points on the boundary, and equal 3 for \(r = 0\).

**Proof.** So far we have seen that \(\mathcal{H}(\mathbb{P}^2)\) is contained in the announced convex hull, and we do also know that \(\phi^\nu_r = 0\). It remains to check the dimensions of \(V^0_r(\mathbb{P}^2)\) – but this will be done in Subsection (5.4).
5.4. Analysing the endomorphisms for each knot. We have three type of degrees \( r \in M \). Let us take a closer look to all of them and their associated \( \phi'_r \). Recall from Subsection \((5.3)\) that we abreviate \( \phi_{\nu} := \phi'_{r_{\nu}} \) with \( s^0 = [0,1,-1] \), \( s^1 = [-1,0,1] \), and \( s^2 = [1,-1,0] \). Moreover, we will use, for a general \( s = [s_0,s_1,s_2] \in \mathbb{1}^2 = M \), the representation \( s = [s_0,s_1,s_2] = s_1 \cdot s^0 - s_0 \cdot s^1 = s_0 \cdot s^2 - s_2 \cdot s^0 \). And from Subsection \((5.1)\) we keep the trace less endomorphisms \( A_0 \), \( A_1 \), and \( A_2 \). Here comes the three classes of degrees capable for carrying pre-Higgs fields:

(i) \( r \in \{r^0,r^1,r^2\} \) is a vertex of \( \mathcal{H} \). Let us assume that \( r = r^0 \).

For all \( s \in M \), \( \phi^0_s \) fits into condition (ii)\(_0\) of Subsection \((5.2)\). Moreover, for \( s = [s_0,s_1,s_2] \in M \) we use \( s = s_1 \cdot s^0 - s_0 \cdot s^1 \); hence, \( \phi^0_s = 0 \) implies that \( \phi^0_s = -s_0 \cdot \phi^0_1 = s_0 \cdot \phi^0_2 \). Altogether, this means that

\[
\phi^0_s = \begin{pmatrix}
0 & c_0 s_0 & -c_0 s_0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} = c_0 \cdot (s, \rho_0) \cdot A_0, \text{ i.e. } \phi^0_s = c_0 \cdot A_0 \otimes \rho_0
\]

for some parameter \( c_0 \in \mathbb{C} \). In other words, \( \{A_0 \otimes \rho_0\} \) is a \( \mathbb{C} \)-basis for all possible \( \phi^{(-2,1,1)} \).

(ii) \( r \in M \cap \partial \mathcal{H} \) being not a vertex of \( \mathcal{H} \). Let us assume that \( r = [-1,1,0] \).

In the coordinates of the previous figures, \( r \) equals \([1,0]\), i.e. in the figures for \( \rho_0^1 \), \( \rho_1^1 \), and \( \rho_2^1 \). We sit on the 0-red line, the 0-blue line, and the intersection of the 0-blue and the 2-blue lines, respectively. This information translates into \( \phi^{[-1,1,0]}_0 \in (ii)_0 \), \( \phi^{[-1,1,0]}_1 \in (i)_0 \), and \( \phi^{[-1,1,0]}_2 \in (i)_0 \cap (i)_2 \).

These three classes equal \( \text{span}(A_0) \), \( \text{span}(A_0, A_2 - A_1) \), and \( \text{span}(A_0 - A_1 + A_2) \), respectively. Thus, the relation \( \phi_0 + \phi_1 + \phi_2 = 0 \) leads to the following basis for the vector space of possible trace free \( \phi^{[-1,1,0]} \):

\[
\{A_0 \otimes \rho_2, (A_0 - A_1 + A_2) \otimes \rho_0\}.
\]

(iii) \( r = 0 \) is the origin.

Here we know that \( \phi^0_\nu \in (i)_\nu \) for \( \nu = 0,1,2 \). The trace less part of these classes is \( \text{span}(A_0, A_2 - A_1), \text{span}(A_1, A_0 - A_2) \), and \( \text{span}(A_2, A_1 - A_0) \), respectively. This implies that

\[
\phi^0_\nu = (c_{\nu-1} - c_{\nu+1})A_\nu + c_\nu (A_{\nu-1} - A_{\nu+1}) = (c_{\nu-1}A_\nu + c_\nu A_{\nu-1}) - (c_{\nu+1}A_\nu + c_\nu A_{\nu+1})
\]

for \( \nu \in \mathbb{Z}/3\mathbb{Z} \) and some \( c \in \mathbb{C}^3 \). This leads to the following basis for the vector space of all possible trace less \( \phi^0 \):

\[
\{(A_2 \otimes \rho_1 + A_1 \otimes \rho_2), (A_0 \otimes \rho_2 + A_2 \otimes \rho_0), (A_1 \otimes \rho_0 + A_0 \otimes \rho_1)\}.
\]

Here comes the summary of all possible trace free pre-Higgs fields on \( \mathbb{P}^2 \). Most important, we have altogether \((3 \times 1) + (6 \times 2) + (1 \times 3) = 18 \) dimensions of them. For a better orientation we have kept the Higgs range polytope \( \mathcal{H} \) (indicated in black) and the original reflexive polytope \( 3\Delta \) (indicated in blue). And as before, the origin is visible as a green
5.5. From pre-Higgs to Higgs. The commutators of the trace free matrices $A_i$ ($i \in \mathbb{Z}/3\mathbb{Z}$) can be expressed as

$$[A_{i-1}, A_{i+1}] = (A_{i-1} + A_{i+1}) - A_i \quad (i \in \mathbb{Z}/3\mathbb{Z}).$$

In Subsection (5.4) we got an 18-dimensional space of trace-free pre-Higgs fields on $\mathbb{P}^2$. Using SINGULAR, i.e. [DGPS19], we have incorporated the commutator vanishing $[\Phi_s, \Phi_t] = 0$ for all $s, t \in M$. This leads to an ideal $I$ in 18 variables with 100 generators (increasing to 435 generators after calculating a dp-Gröbner basis). The dimension of $V(I) \subset \mathbb{C}^{18}$ is 8, hence it becomes 7 if understood as a projective subvariety of $\mathbb{P}_{\mathbb{C}}^{17}$. However, it is not clear, if $V(I)$ is smooth or at least irreducible – SINGULAR crashed when calculating the 10-minors, and it timed-out when trying the primary decomposition.

5.5.1. Higgs facets. On the other hand, it is easily possible to calculate the commutator property for the three facets of the Higgs range polytope $\mathcal{H} = \mathcal{H}(T_{\mathbb{P}^2})$. This leads to all Higgs fields $\Phi$ with maximal one-dimensional Higgs polytopes $\nabla(\Psi) \subset \mathcal{H}$. According to the figure at the end of Subsection (5.4), we have named the 18 coordinates in the following way:

- $c_{30}$
- $d_{20}$
- $c_{10}$
- $d_{10}$
- $c_{00}$
- $c_{21}$
- $d_{21}$
- $e_0$
- $e_1$
- $e_2$
- $c_{01}$
- $d_{01}$
- $c_{12}$
- $d_{12}$
- $c_{02}$
- $d_{02}$
- $c_{03}$
Then, the three facet ideals are
\[ I_0 = (-c_{03}d_{21} + c_{12}d_{12}, -c_{21}d_{21} + c_{30}d_{12}, -c_{21}c_{12} + c_{30}c_{03}) \]
\[ I_1 = (-c_{03}d_{01} + c_{02}d_{02}, -c_{01}d_{01} + c_{00}d_{02}, -c_{01}c_{02} + c_{00}c_{03}) \]
\[ I_2 = (-c_{30}d_{10} + c_{20}d_{20}, -c_{10}d_{10} + c_{30}d_{20}, -c_{10}c_{20} + c_{30}c_{00}). \]

All of them define a specific toric variety described, in each case, by a 3-dimensional, triangular prism. This can be seen by rewriting the binomial equations over the respective tori as
\[ \frac{c_{12}}{c_{03}} = \frac{d_{11}}{d_{12}} = \frac{c_{00}}{c_{10}}, \quad \frac{c_{11}}{c_{02}} = \frac{d_{02}}{d_{01}} = \frac{c_{01}}{c_{02}}, \quad \frac{c_{00}}{c_{10}} = \frac{d_{10}}{d_{20}} = c_{20}. \]

5.5.2. **Involving the center.** Here we will do the opposit of Subsection 5.5.1 – we keep the central variables \(e_0, e_1, e_2\) of degree 0 and the three corner degrees, i.e. the variables \(c_{00}, c_{30}, c_{03}\). This allows to approach the question raised in Subsection 4.2: Is there always a true Higgs bundle having \(H\) as its associated Higgs polytope? If this were true, then all the corner degrees have to be involved.

At this point we will additionally assume that the intermediate degrees do not occur, which is a non-trivial restriction though. Thus, we have only six variables left, and a \texttt{SINGULAR} calculation yields that the resulting Higgs variety consists of three projectively one-dimensional components
\[ V(e_0 - e_1, e_2, c_{00}, c_{30}), \quad V(e_0, e_1 - e_2, c_{30}, c_{03}), \quad V(e_1, e_0 - e_2, c_{00}, c_{03}) \]
inside \(\mathbb{P}^5\), and that there are three embedded components, too. In any case, the associated Higgs polytopes are the line segments connecting a vertex of \(H\) with the central point 0.

6. **Tracefree \(T_X\)-Higgs fields on smooth complete surfaces**

Next we sketch how the techniques for \(\mathbb{P}^2\) generalise to any smooth, complete toric surface \(X\) for the computation of the Higgs range \(H(X)\) and the associated vector spaces \(V_r^U(X)\).

6.1. **Encoding endomorphisms.** The fan of the Hirzebruch surface \(\mathbb{H}_a\), \(a \geq 2\), is induced by the primitive generators \(\rho_0(a) = (-1, -a)\), \(\rho_1 = (1, 0)\), \(\rho_2 = (0, 1)\) and \(\rho_3 = (0, -1)\). In the same vein as in the previous section we consider the lattice \(\mathbb{Z}^3 = \mathbb{Z}\rho_0(a) \oplus \mathbb{Z}\rho_1 \oplus \mathbb{Z}\rho_2\), \(a \geq 1\), and identify
\[ N \cong \mathbb{Z}^3 / \mathbb{Z}(1, 1, a) \quad \text{and} \quad M \cong (1, 1, a)^{\perp} = \{ r \in \mathbb{Z}^3 \mid r_0 + r_1 + ar_2 = 0 \} \subseteq \mathbb{Z}^3. \]

While we do not make use of this, we note in passing that the rays \(\rho_0(a)\), \(\rho_1\) and \(\rho_2\) provide the fan of the singular weighted projective plane \(\mathbb{P}(1, 1, a)\). Proceeding as above yields the representation
\[ \tilde{\phi} = \begin{pmatrix} 0 & ax & -x \\ -ay & 0 & y \\ z & -z & 0 \end{pmatrix} = xA_0(a) + yA_1(a) + zA_2 \sim \phi_0 = \begin{pmatrix} -ax & x + y \\ -a^2x - z & ax \end{pmatrix}, \]
where
\[ A_0(a) = \begin{pmatrix} 0 & a & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_1(a) = \begin{pmatrix} 0 & 0 & 0 \\ -a & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix}. \]

Under this representation, the determinant is given by \(\det(\phi_0) = a^2xy + yz + zx\).
6.2. From filtrations to facets. To compute a basis for the vector spaces $V^0_r(X)$, we recall from Section 5.2 that an endomorphism $\varphi \in \text{End}(E)$ belongs to the class (i)$_\rho$ if $\varphi(\rho) \in \text{span}(\rho)$, and to the class (ii)$_\rho$ if $\varphi(E) \subseteq \text{span}(\rho) \subseteq \ker(\varphi)$. For instance, for $a \geq 1$ an endomorphism of class (i)$_{\rho(a)}$ is determined by the eigenvector equation $(xA_0(a) + yA_1(a) + zA_2)\rho_0(a) = \lambda \rho_0(a)$, or equivalently, by the matrix equation

$$
\begin{pmatrix}
0 & ax & -x \\
-ay & 0 & y \\
z & -z & 0
\end{pmatrix} \cdot \begin{pmatrix}
0 \\
1 \\
a
\end{pmatrix} = \begin{pmatrix}
0 \\
\lambda \\
a\lambda
\end{pmatrix}
$$

which implies $z = -a^2y$. Any such endomorphism is thus of the form $I_{\rho_0(a)}(x, y) = xA_0(a) + y(A_1(a) - a^2A_2)$ for $x, y \in \mathbb{C}$. Similarly, any endomorphism of class (ii)$_{\rho_0(a)}$ is given by $II_{\rho_0(a)}(x) = xA_0(a)$. Table 1 displays the endomorphisms $I_{\rho}$ and $II_{\rho}$ of type (i)$_{\rho}$ and (ii)$_{\rho}$ for the primitive generators $\rho_0(a)$, $\rho_1$ and $\rho_2$ of $\mathbb{H}_2$.

| $\rho \in \Sigma_a(1)$ | Basis of all $I_{\rho}$ | Basis of all $II_{\rho}$ |
|------------------------|-------------------------|-------------------------|
| $\rho_0(a)$            | $A_0(a), A_1(a) - a^2A_2$ | $A_0(a)$               |
| $\rho_1$               | $A_0(a) - a^2A_2, A_1(a)$ | $A_1(a)$               |
| $\rho_2$               | $A_0(a) - A_1(a), A_2$   | $A_2(a)$               |

Table 1. The endomorphisms $I_{\rho}$ and $II_{\rho}$ for $a \geq 1$.

6.3. The Hirzebruch surfaces $\mathbb{H}_a$. With Table 1 at hand we can now determine $\mathcal{H}(\mathbb{H}_a)$ with associated $V^0_r(X)$ exactly in the same way as for the projective space.

Example 22. For $\mathcal{H}(\mathbb{H}_2)$ the Higgs range is the convex hull of the points $r^0 = (1, 0)$, $r^1 = (-1, 0)$, $r^2 = (1, -2)$ and $r^3 = (3, -2)$ given in the figure below:
A basis for the vector spaces \( V^0_r(\mathbb{H}_2) \) is given as follows:

\[
\begin{aligned}
&A_0(2) - A_1(2) - 4A_2) \otimes \rho_0(2) + A_1(2) \otimes \rho_2 \\
&A_0(2) \otimes \rho_2 + A_2 \otimes \rho_1 \\
&A_0(2) \otimes \rho_1 + A_1(2) \otimes \rho_0 \\
&A_0(2) - A_1(2) - 4A_2) \otimes \rho_2 \\
&A_0(2) - A_1(2)) \otimes \rho_2 \\
&A_0(2) + 4A_1(2) - 16A_2) \otimes \rho_2 \\
&A_0(2) \otimes \rho_0(2) \\
A_1(2) \otimes \rho_2 \\
A_2 \otimes \rho_2 \\
\end{aligned}
\]

For general \( a \) we obtain the Higgs range of \( \mathbb{H}_a \) as follows:

(i) We keep the lattice points \((-1,0), (0,0), (1,0), (0,-1), (1,-1) \) and \((1,-2)\) together with \( V^0_r(\mathbb{H}_a) = V^0_r(\mathbb{H}_2) \).

(ii) We add \( a-1 \) points \((2,-1), \ldots, (a,-1)\) with \( V^0_{(x,-1)}(\mathbb{H}_a) = V^0_{(1,-1)}(\mathbb{H}_2) \) for \( x \leq a-1 \) and \( V^0_{(a,-1)} = \text{span}(A_0(a) + a^2(A_1(a) - a^2A_2)) \otimes \rho_2, A_2 \otimes \rho_0(a)\).

(iii) Finally, we add \( 2(a-1) \) points \((2,-2), \ldots, (2a-1, -2)\) with vector space \( V^0_r(\mathbb{H}_a) = \text{span}(A_2 \otimes \rho_2)\).

**Example 23.** For instance, we find the following Higgs range for \( \mathbb{H}_4 \) (where we only indicated the dimensions of \( V^0_r \):

\[
\begin{array}{cccc}
2 & 3 & 2 \\
\bullet & 2 & 3 & 3 & 3 & 2 \\
1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}
\]

**Remark 24.** Similarly, we can deal with \( \mathbb{P}^1 \times \mathbb{P}^1 \) after we compute the corresponding basis for endomorphisms of type (i), \( \rho \) and (ii), \( \rho \).

### 6.4. The Higgs range under blow-ups

A general smooth and complete toric surface \( X \) is obtained from a minimal surface by a finite sequence of blow-ups at fixed points of the torus action, so we need to discuss how such blow-ups affect \( \mathcal{H}(X) \).

Combinatorically, the blow-up \( X' \to X \) of a surface \( X \) arises from subdividing a maximal cone given by \( \rho, \tau \in \Sigma(1) \) by inserting the primitive generator \( \sigma = \rho + \tau \). This yields new lines \( F^\sigma_i, i = 0,1,2 \) which possibly exclude further points of \( \mathcal{H}(X) \) or decrease the dimension of the corresponding vector spaces \( V^0_r \) by adding further linear dependencies, cf. Subsection 5.3. We therefore have the

**Proposition 25.** If \( X' \to X \) is the blow-up of a smooth toric surface, then we have natural inclusions \( \mathcal{H}(X') \subset \mathcal{H}(X) \) and \( V^0_r(X') \subset V^0_r(X) \) whenever \( r \in \mathcal{H}(X') \).
6.5. Fano surfaces. The precise shape of $H(X')$ depends of course on the combinatorics of $H(X)$ and on the fixed point we blow up. For illustration, consider the five smooth toric Fano surfaces given by the reflexive del Pezzo polytopes:

![Fano surfaces](image)

Their Higgs ranges together with the dimension of $V_\rho^0(X)$ are given as follows:

![Higgs ranges](image)

In particular, whenever this makes sense we find $V_\rho^0(X') = V_\rho^0(P^2)$ for $X' = P^{2'}, P^{2''}$ and $P^{2'''}$.

6.6. Higgs fields and their Higgs algebras on $P^{2''}$ and $P^{2'''}$. Using Subsection (5.5.2), we exhibit some explicit Higgs fields on the del Pezzos $X = P^{2''}$ and $P^{2'''}$ and compute their associated invariants.

We start with the degree six del Pezzo $P^{2''}$ where $H(P^{2''}) = \{(0,0)\}$. From Subsection (5.5.2), we gather that the space of Higgs fields is given by the three components $V(e_0 - e_1, e_2)$, $V(e_0, e_1 - e_2)$, and $V(e_1, e_0 - e_2)$ with corresponding Higgs fields

$$
\begin{align*}
\Phi_1 &= A_2 \otimes \rho_0 + A_2 \otimes \rho_1 + (A_0 + A_1) \otimes \rho_2 \\
\Phi_2 &= (A_1 + A_2) \otimes \rho_0 + A_0 \otimes \rho_1 + A_0 \otimes \rho_2 \\
\Phi_3 &= A_1 \otimes \rho_0 + (A_0 + A_2) \otimes \rho_1 + A_1 \otimes \rho_2.
\end{align*}
$$

The resulting Higgs polytopes $\nabla(\Phi_i), i = 1, 2, 3$, coincide trivially with $H(P^{2''}) = \{(0,0)\}$. Since $\det \Phi_i = -d^2$ for $i = 1, 2, 3$, the Higgs algebras $A(\Phi_i)$ are all isomorphic to $\mathbb{C}[M]/(z^2 - d^2) \cong \mathbb{C}[M] \times \mathbb{C}[M]$. It follows that $A \otimes_\mathbb{C} \mathbb{C} \cong \mathbb{C} \cong A(t)$ is the product ring. The fibre of the spectral variety $\text{Spec} A(\Phi_i) \to T$ correspond thus of the two distinct eigenvalues of $\Phi_i$.

Next we turn to the degree seven del Pezzo $P^{2'''}$. A SINGULAR aided computation yields for instance the three dimensional component $V(e_0 - e_1, c_{12}, d_{21}, e_2)$ with corresponding Higgs field

$$
\Phi = (e_1(A_0 - A_2)\chi^{(0,0)} + d_{12}(A_0 - A_2)\chi^{(0,-1)} + c_{21}A_1\chi^{(-1,0)}) \otimes \rho_2.
$$

Depending on the concrete choice of the coefficients, the Higgs polytope $\nabla(\Phi)$ realises every subpolytope of the Higgs range $H(P^{2''''}) = \{(-1,0), (0,0), (0,-1)\}$; generically, both polytopes coincide.
The Higgs algebras $\mathcal{A}(\Phi)$ is again generated by a single Higgs field with minimal polynomial $\mu(z) = z^2 + \det \Phi$. Since $\det \Phi = -d^2$ is a square in $\mathbb{C}[M]$, the resulting algebra $\mathcal{A}(\Phi)$ is again isomorphic to $\mathbb{C}[M] \times \mathbb{C}[M]$. Furthermore, $\mathcal{A}(\Phi) \otimes_\mathbb{C} \mathbb{C} \cong \mathcal{A}(t)$. The difference to the previous case is that up to isomorphism we have now two isomorphism types of $\mathcal{A}(t)$: If $t \in T$ is a zero of $\delta$, then $\mathcal{A}(t) = \mathbb{C}[\Phi(t)] \cong \mathbb{C}[x]/(x^2)$ for $\Phi$ is nilpotent. Otherwise, $\mathcal{A}(t_0)$ is just the product ring $\mathbb{C} \times \mathbb{C}$.

**Remark 26.** We can also consider the determinant as a map from $\text{End}(E) \otimes \mathbb{C}[M] \otimes N \to \mathbb{C}[M] \otimes S^*N$ (the toric version of the Hitchin map [Hit87]). For our special examples the generators $\Phi_i$ have a triangular form which implies that the determinant $\det \Phi_i \in S^*N \otimes \mathbb{C}[M]$ admits a square root in $N \otimes \mathbb{C}[M]$. The latter defines two vector fields on $T$ whose image in the tangent bundle represent the spectral variety.

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