On C-embedded subspaces of the Sorgenfrey plane
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1. Introduction

Recall that a subset $A$ of a topological space $X$ is called functionally open (functionally closed) in $X$ if there exists a continuous function $f : X \rightarrow [0,1]$ such that $A = f^{-1}((0,1])$ ($A = f^{-1}(0)$). Sets $A$ and $B$ are completely separated in $X$ if there exists a continuous function $f : X \rightarrow [0,1]$ such that $A \subseteq f^{-1}(0)$ and $B \subseteq f^{-1}(1)$.

A subspace $E$ of a topological space $X$ is

- **$C$-embedded** ($C^*$-embedded) in $X$ if every (bounded) continuous function $f : E \rightarrow \mathbb{R}$ can be continuously extended on $X$;
- **$z$-embedded** in $X$ if every functionally closed set in $E$ is the restriction of a functionally closed set in $X$ to $E$;
- **well-embedded in $X$** if $E$ is completely separated from any functionally closed set of $X$ disjoint from $E$.

Clearly, every $C$-embedded subspace of $X$ is $C^*$-embedded in $X$. The converse is not true. Indeed, if $E = \mathbb{N}$ and $X = \beta \mathbb{N}$, then $E$ is $C^*$-embedded in $X$ (see [4, 3.6.3]), but the function $f : E \rightarrow \mathbb{R}$, $f(x) = x$ for every $x \in E$, does not extend to a continuous function $f : X \rightarrow \mathbb{R}$.

A space $X$ has the property $(C^* = C)$ if every closed $C^*$-embedded subset of $X$ is $C$-embedded in $X$. The classical Tietze-Urysohn Extension Theorem says that if $X$ is a normal space, then every closed subset of $X$ is $C^*$-embedded and $X$ has the property $(C^* = C)$. Moreover, a space $X$ is normal if and only if every its closed subset is $z$-embedded (see [3, Proposition 3.7]).

The following theorem was proved by Blair and Hager in [2, Corollary 3.6].

**Theorem 1.1.** A subset $E$ of a topological space $X$ is $C^*$-embedded in $X$ if and only if $E$ is $z$-embedded and well-embedded in $X$.

A space $X$ is said to be $\delta$-normally separated if every closed subset of $X$ is well-embedded in $X$. The class of $\delta$-normally separated spaces includes all normal spaces and all countably compact spaces. Theorem 1.1 implies the following result.

**Corollary 1.2.** Every $\delta$-normally separated space has the property $(C^* = C)$.

According to [15] every $C^*$-embedded subspace of a completely regular first countable space is closed. The following problem is still open:

**Problem 1.3.** [12] Does there exist a first countable completely regular space without property $(C^* = C)$?

H. Ohta in [11] proved that the Niemytzki plane has the property $(C^* = C)$ and asked does the Sorgenfrey plane $S^2$ (i.e., the square of the Sorgenfrey line $S$) have the property $(C^* = C)$?

In the given paper we obtain some necessary conditions on a set $E \subseteq S^2$ to be $C^*$-embedded. We prove that every $C^*$-embedded subset of $S^2$ is a hereditarily Baire subspace of $\mathbb{R}^2$. We also characterize $C^*$- and $C^*$-embedded subspaces of the anti-diagonal $D = \{(x, -x) : x \in \mathbb{R}\}$ of $S^2$. Namely, we prove that for a subspace $E \subseteq D$ of $S^2$ the following conditions are equivalent: (i) $E$ is $C^*$-embedded in $S^2$; (ii) $E$ is $C^*$-embedded in $S^2$; (iii) $E$ is a countable $G_\delta$-subspace of $\mathbb{R}^2$ and (iv) $E$ is a countable functionally closed subspace of $S^2$.

2. Every finite power of the Sorgenfrey line is a hereditarily $\alpha$-favorable space

Recall the definition of the Choquet game on a topological space $X$ between two players $\alpha$ and $\beta$. Player $\beta$ goes first and chooses a nonempty open subset $U_0$ of $X$. Player $\alpha$ chooses a nonempty open subset $V_1$ of $X$ such that $V_1 \subseteq U_0$. Following this player $\beta$ must select another nonempty open subset $U_1 \subseteq V_1$ of $X$ and $\alpha$ must select a nonempty open subset $V_2 \subseteq U_1$. Acting in this way, the players $\alpha$ and $\beta$ obtain sequences of nonempty open sets $(U_n)_{n=0}^{\infty}$ and $(V_n)_{n=1}^{\infty}$ such that $U_{n+1} \subseteq V_n \subseteq U_n$ for every $n \in \mathbb{N}$. The player $\alpha$ wins if $\bigcap_{n=1}^{\infty} V_n \neq \emptyset$. Otherwise, the player $\beta$ wins. If there exists a rule (a strategy) such that $\alpha$ wins if he plays according to this rule, then $X$ is called $\alpha$-favorable. Respectively, $X$ is called $\beta$-unfavorable if the player $\beta$ has no winning strategy. Clearly, every $\alpha$-favorable space $X$ is $\beta$-unfavorable. Moreover, it is known [13] that a topological space $X$ is Baire if and only if it is $\beta$-unfavorable in the Choquet game.

If $A$ is a subspace of a topological space $X$, then $\overline{A}$ and $\text{int}A$ mean the closure and the interior of $A$ in $X$, respectively.
Lemma 2.1. Let $X = \bigcup_{k=1}^{n} X_k$, where $X_k$ is an $\alpha$-favorable subspace of $X$ for every $k = 1, \ldots, n$. Then $X$ is an $\alpha$-favorable space.

Proof. We prove the lemma for $n = 2$. Let $G = G_1 \cup G_2$, where $G_i = \text{int} X_i$, $i = 1, 2$. We notice that for every $i = 1, 2$ the space $X_i$ is $\alpha$-favorable, since it contains dense $\alpha$-favorable subspace. Then $G_i$ is $\alpha$-favorable as an open subspace of the $\alpha$-favorable space $X_i$. It is easy to see that the union $G$ of two open $\alpha$-favorable subspaces is an $\alpha$-favorable space. Therefore, $X$ is $\alpha$-favorable, since $G$ is dense in $X$. \hfill \Box

Let $p = (x, y) \in \mathbb{R}^2$ and $\varepsilon > 0$. We write

$$B(p; \varepsilon) = [x, x + \varepsilon] \times [y, y + \varepsilon],$$

$$B(p; \varepsilon) = (x - \varepsilon, x + \varepsilon) \times (y - \varepsilon, y + \varepsilon).$$

If $A \subseteq \mathbb{S}^2$ then the symbol $cl_{\mathbb{S}^2} A$ ($cl_{\mathbb{R}^2} A$) means the closure of $A$ in the space $\mathbb{S}^2$ ($\mathbb{R}^2$).

We say that a space $X$ is hereditarily $\alpha$-favorable if every its closed subspace is $\alpha$-favorable.

Theorem 2.2. For every $n \in \mathbb{N}$ the space $\mathbb{S}^n$ is hereditarily $\alpha$-favorable.

Proof. Let $n = 1$ and $\emptyset \neq F \subseteq S$. Assume that $\beta$ chose a nonempty open in $F$ set $U_0 = (a_0, b_0) \cap F$, $a_0 \in F$. If $U_0$ has an isolated point $x \in S$, then $\alpha$ chooses $V_1 = \{x\}$ and wins. Otherwise, $\alpha$ put $V_1 = [a_0, c_0] \cap F$, where $c_0 \in (a_0, b_0) \cap F$ and $c_0 - a_0 < 1$. Now let $U_1 = (a_1, b_1) \cap F \subseteq V_1$ be the second turn of $\beta$ such that $a_1 \in F$ and the set $(a_1, b_1) \cap F$ has no isolated points in $S$. Then there exists $c_1 \in (a_1, b_1) \cap F$ such that $c_1 - a_1 < \frac{1}{2}$.

Let $V_2 = [a_1, c_1] \cap F$. Repeating this process, we obtain sequences $(U_m)_{m=0}^{\infty}$, $(V_m)_{m=1}^{\infty}$ of open subsets of $F$ and sequences of points $(a_m)_{m=0}^{\infty}$, $(b_m)_{m=0}^{\infty}$ and $(c_m)_{m=1}^{\infty}$ such that $(a_m, b_m) \supseteq [a_m, c_m) \supseteq [a_{m+1}, b_{m+1})$, $c_m - a_m < \frac{1}{m+1}$, $c_m \in F$, $U_m = (a_m, b_m) \cap F$ and $V_{m+1} = [a_m, c_m] \cap F$ for every $m = 0, 1, \ldots$. According to the Nested Interval Theorem, the sequence $(c_m)_{m=1}^{\infty}$ is convergent in $S$ to a point $x^* \in \bigcap_{m=0}^{\infty} V_m$. Since $F$ is closed in $S$, $x^* \in F$. Hence, $F \cap \bigcap_{m=0}^{\infty} V_m \neq \emptyset$. Consequently, $F$ is $\alpha$-favorable.

Suppose that the theorem is true for all $1 \leq k \leq n$ and prove it for $k = n + 1$.

Consider a set $\emptyset \neq F \subseteq \mathbb{S}^{n+1}$. Let the player $\beta$ chooses a set $U_0 = [a_0, b_0, k_0] \cap \mathbb{S}^n$ with $a_0 = (a_0, k_0)_{k=1}^{n+1} \in F$. Denote $U_0^+ = \prod_{k=1}^{n+1} (a_0, k_0, b_0, k_0)$ and consider the case $U_0^+ \cap F = \emptyset$. For every $k = 1, \ldots, n + 1$ we set $U_0, k = (a_0, k) \times \prod_{i \neq k}^{n} (a_i, b_i, k)$ and $F_0, k = F \cap U_0, k$. Since $U_0, k$ is homeomorphic to $\mathbb{S}^n$, by the inductive assumption the space $F_0, k$ is $\alpha$-favorable for every $k = 1, \ldots, n + 1$. Then $F$ is $\alpha$-favorable according to Lemma 2.1. Now let $U_0^+ \cap F \neq \emptyset$. If there exists an isolated point $x \in U_0^+$, then $\alpha$ put $V_1 = \{x\}$ and wins. Assume $U_0$ has no isolated points in $\mathbb{S}^{n+1}$. Then there is $c_0 = (c_0, k_0)_{k=1}^{n+1} \in U_0^+ \cap F$ such that $\text{diam}(\prod_{k=1}^{n+1} (a_0, k_0, k)) < 1$. We put

$V_1 = F \cap \prod_{k=1}^{n+1} (a_0, k_0, k)$. Let $U_1 = F \cap \prod_{k=1}^{n+1} (a_1, k_1, k)$. Then, using the inductive assumption, we obtain that $U_1 \subseteq V_1$. Again, if $U_1 \cap F = \emptyset$, where $U_1^+ = \prod_{k=1}^{n+1} (a_1, k_1, b_1, k)$, then, using the inductive assumption, we obtain that for every $k = 1, \ldots, n + 1$ the space $F \cap (\prod_{i \neq k}^{n+1} (a_i, b_i, k))$ is $\alpha$-favorable. Then $\alpha$ has a winning strategy in $F$ by Lemma 2.1. If $U_1^+ \cap F \neq \emptyset$ and $U_1$ has no isolated points in $\mathbb{S}^{n+1}$, the player $\beta$ chooses a point $c_1 = (c_1, k_1)_{k=1}^{n+1} \in U_1^+ \cap F$ such that $\text{diam}(\prod_{k=1}^{n+1} (a_1, k_1, c_1)) < 1/2$ and put $V_2 = F \cap \prod_{k=1}^{n+1} (a_1, k_1, c_1)$. Repeating this process, we obtain sequences of points $(a_m)_{m=0}^{\infty}$, $(b_m)_{m=0}^{\infty}$ and $(c_m)_{m=0}^{\infty}$, and of sets $(U_m)_{m=0}^{\infty}$ and $(V_m)_{m=0}^{\infty}$, which satisfy the following properties:

1. $U_m = F \cap \prod_{k=1}^{n+1} (a_m, k, b_m, k)$;
2. $a_m \in F$, $c_m \in U_m \cap F$;
3. $V_{m+1} = F \cap \prod_{k=1}^{n+1} (a_m, k, c_m, k)$;
4. $V_{m+1} \subseteq U_m \subseteq V_m$;
5. $\text{diam}(V_{m+1}) < \frac{1}{m+1}$. 


for every \( m = 0, 1, \ldots \). We observe that the sequence \((c_m)_{m=0}^{\infty}\) is convergent in \( \mathbb{R}^{n+1} \) and \( x^* = \lim_{m \to \infty} c_m \in \bigcap_{m=0}^{\infty} V_m \). Since \( c_m \to x^* \) in \( S^{n+1} \), \( c_m \in F \) and \( F \) is closed in \( S^{n+1} \), \( x^* \in F \cap (\bigcap_{m=0}^{\infty} V_m) \). Hence, \( F \) is \( \alpha \)-favorable.

3. Every \( C^* \)-embedded subspace of \( S^2 \) is a hereditarily Baire subspace of \( \mathbb{R}^2 \).

Lemma 3.1. A set \( E \subseteq \mathbb{R}^2 \) is functionally closed in \( S^2 \) if and only if

1) \( E \) is \( G_\delta \) in \( \mathbb{R}^2 \); and

2) if \( F \) is \( \mathbb{R}^2 \)-closed set disjoint from \( E \), then \( F \) and \( E \) are completely separated in \( S^2 \).

Proof. Necessity. Let \( f : S^2 \to \mathbb{R} \) be a continuous function such that \( E = f^{-1}(0) \). According to [10] Theorem 2.1, \( f \) is a Baire-one function on \( \mathbb{R}^2 \). Consequently, \( E \) is a \( G_\delta \) subset of \( \mathbb{R}^2 \).

Condition (2) follows from the fact that every \( \mathbb{R}^2 \)-closed set is, evidently, a functionally closed subset of \( S^2 \).

Sufficiency. Since \( E \) is \( G_\delta \) in \( \mathbb{R}^2 \), there exists a sequence of \( \mathbb{R}^2 \)-closed sets \( F_n \) such that \( X \setminus E = \bigcup_{n=1}^{\infty} F_n \).

Clearly, \( E \cap F_n = \emptyset \). Then condition (2) implies that for every \( n \in \mathbb{N} \) there exists a continuous function \( f_n : S^2 \to \mathbb{R} \) such that \( E \subseteq f_n^{-1}(0) \) and \( F_n \subseteq f_n^{-1}(1) \). Then \( E = \bigcap_{n=1}^{\infty} f_n^{-1}(0) \). Hence, \( E \) is functionally closed in \( S^2 \). \( \square \)

Lemma 3.2. Let \( X \) be a metrizable space, \( A \subseteq X \) be a set without isolated points and let \( B \subseteq X \) be a countable set such that \( A \cap B = \emptyset \). Then there exists a set \( C \subseteq A \) without isolated points such that \( \overline{C} \cap B = \emptyset \).

Proof. Let \( d \) be a metric on \( X \), which generates its topological structure. For \( x_0 \in X \) and \( r > 0 \) we denote \( B(x_0, r) = \{ x \in X : d(x, x_0) < r \} \) and \( B[x_0, r] = \{ x \in X : d(x, x_0) \leq r \} \). Let \( B = \{ b_n : n \in \mathbb{N} \} \). We put \( A_0 = \emptyset \) and construct sequences \((A_n)_{n=1}^{\infty}\) and \((V_n)_{n=1}^{\infty}\) of nonempty finite sets \( A_n \subseteq A \) and open neighborhoods \( V_n \) of \( b_n \) which for every \( n \in \mathbb{N} \) satisfy the following conditions:

(1) \( A_{n-1} \subseteq A_n \);

(2) \( \forall x \in A_n \exists y \in A_n \setminus \{ x \} \text{ with } d(x, y) \leq \frac{1}{n} \);

(3) \( d(A_n, \bigcup_{1 \leq i \leq n} V_i) > 0 \).

Let \( A_1 = \{ x_1, y_1 \} \), where \( d(x_1, y_1) \leq 1 \) and \( x_1 \neq y_1 \). We take \( \varepsilon > 0 \) such that \( A_1 \cap B[1, \varepsilon] = \emptyset \) and put \( V_1 = B(b_1, \varepsilon) \). Assume that we have already defined finite sets \( A_1, \ldots, A_k \) and neighborhoods \( V_1, \ldots, V_k \) of \( b_1, \ldots, b_k \) respectively, which satisfy conditions (1)–(3) for every \( n = 1, \ldots, k \). Let \( A_k = \{ a_1, \ldots, a_m \} \), \( m \in \mathbb{N} \). Taking into account that the set \( D = A \setminus \bigcup_{1 \leq i \leq k} V_i \) has no isolated points, for every \( i = 1, \ldots, m \) we take \( c_i \in D \) with \( c_i \neq a_i \) and \( d(a_i, c_i) \leq \frac{1}{k+1} \). Put \( A_{k+1} = A_k \cup \{ c_1, \ldots, c_m \} \). Take \( \delta > 0 \) such that \( A_{k+1} \cap B[b_k+1, \delta] = \emptyset \). Let \( V_{k+1} = B(b_{k+1}, \delta) \). Repeating this process, we obtain needed sequences \((A_n)_{n=1}^{\infty}\) and \((V_n)_{n=1}^{\infty}\). \( \square \)

The following results will be useful.

Theorem 3.3 (8). A subspace \( E \) of a topological space \( X \) is \( C^* \)-embedded in \( X \) if and only if every two disjoint functionally closed subsets of \( E \) are completely separated in \( X \).

Theorem 3.4 (16). The Sorgenfrey plane \( S^2 \) is strongly zero-dimensional, i.e., for any completely separated sets \( A \) and \( B \) in \( S^2 \) there exists a clopen set \( U \subseteq S^2 \) such that \( A \subseteq U \subseteq S^2 \setminus B \).

Recall that a space \( X \) is hereditarily Baire if every its closed subspace is Baire.

Theorem 3.5. Let \( E \) be a \( C^* \)-embedded subspace of \( S^2 \). Then \( E \) is a hereditarily Baire subspace of \( \mathbb{R}^2 \).

Proof. Assume that \( E \) is not \( \mathbb{R}^2 \)-hereditarily Baire space and take an \( \mathbb{R}^2 \)-closed countable subspace \( E_0 \) without \( \mathbb{R}^2 \)-isolated point (see [3]). Notice that \( E \) is \( S^2 \)-closed according to [15] Corollary 2.3]. Therefore, \( E_0 \) is \( S^2 \)-closed set. By Theorem 2.2 the space \( E_0 \) is \( \alpha \)-favorable, and, consequently, \( E_0 \) is a Baire subspace of \( S^2 \).

Let \( E_0' \) be a set of all \( S^2 \)-nonisolated points of \( E_0 \). Since \( E_0' \) is the set of the first category in \( S^2 \)-Baire space \( E_0 \), the set \( G = E_0 \setminus E_0' \) is \( S^2 \)-dense open discrete subspace of \( E_0 \). We notice that \( G \) is \( \mathbb{R}^2 \)-dense subspace of
that A and B be any $\mathbb{R}^2$-dense in $F$ disjoint sets such that $F = A \cup B$. Evidently A and B are clopen subsets of $F$, since $F$ is $\mathbb{S}^2$-discrete space. Notice that $F$ is $z$-embedded in $E$, because $F$ is countable. Moreover, $F$ is $\mathbb{R}^2$-closed in $E$. Hence, $F$ is $\mathbb{S}^2$-functionally closed in $E$. By Theorem 4.3.1 the set $F$ is $C^*$-embedded in $\mathbb{S}^2$ set $E$. Consequently, $F$ is $C^*$-closed in $\mathbb{S}^2$. Therefore, Theorem 3.3 and Theorem 3.4 imply that there exist disjoint clopen set $U, V \subseteq \mathbb{S}^2$ such that $A = U \cap F$ and $B = V \cap F$. According to Lemma 3.1 the sets $U$ and $V$ are $G_\delta$ in $\mathbb{R}^2$. Let $D = cl_{\mathbb{R}^2}F$. Then $U \cap D$ and $V \cap D$ are $\mathbb{R}^2$-dense in $D$ disjoint $G_\delta$-sets, which contradicts to the baireness of $D$.

\section*{4. Every discrete $C^*$-embedded subspace of $\mathbb{S}^2$ is a countable $G_\delta$-subspace of $\mathbb{R}^2$.}

**Lemma 4.1.** Let $X$ be a metrizable separable space and $A \subseteq X$ be an uncountable set. Then there exists a set $Q \subseteq A$ which is homeomorphic to the set $Q$ of all rational numbers.

**Proof.** Let $A_0$ be the set of all points of $A$ which are not condensation points $A$ (a point $a \in A$ is called a condensation point of $A$ in $X$ if every neighborhood of $a$ contains uncountably many elements of $A$). Notice that $A_0$ is countable, since $A$ has a countable base. Put $B = A \setminus A_0$. Then the inequality $|A| > \aleph_0$ implies that every point of $B$ is a condensation point of $B$. Take a countable subset $Q \subseteq B$ which is dense in $B$. Clearly, every point of $Q$ is not isolated. Hence, $Q$ is homeomorphic to $Q$ by the Sierpiński Theorem [14].

**Lemma 4.2.** Let $E$ be an $\mathbb{R}^2$-hereditarily Baire $z$-embedded subspace of $\mathbb{S}^2$. Then the set $E^0$ of all isolated points of $E$ is at most countable.

**Proof.** Assume $E^0$ is uncountable. Notice that $E^0$ is an $F_\sigma$-subset of $E$, since $E^0$ is an open subset of $E$ and $\mathbb{S}^2$ is a perfect space by [6]. Then $E^0 = \bigcup_{n=1}^{\infty} E_n$, where every set $E_n$ is closed in $E$. Take $N \in \mathbb{N}$ such that $E_N$ is uncountable. According to Lemma 3.1 there exists a set $Q \subseteq E_N$ which is homeomorphic to $Q$. Since $Q$ is clopen in $E_N$ and $E_N$ is a clopen subset of a $z$-embedded in $\mathbb{S}^2$ set $E$, there exists a functionally closed subset $Q_1$ of $\mathbb{S}^2$ such that $Q = E \cap Q_1$. By Lemma 4.1 the set $Q_1$ is a $G_\delta$-set in $\mathbb{R}^2$. Then $Q$ is a $G_\delta$-subset of a hereditarily Baire space $E$. Hence, $Q$ is a Baire space, a contradiction.

**Theorem 4.3.** If $E$ is a discrete $C^*$-embedded subspace of $\mathbb{S}^2$, then $E$ is a countable $G_\delta$-subspace of $\mathbb{R}^2$.

**Proof.** Theorem 3.3 and Lemma 4.2 imply that $E$ is a countable hereditarily Baire subspace of $\mathbb{R}^2$. According to Proposition 12 the set $E$ is $G_\delta$ in $\mathbb{R}^2$.

The converse implication in Theorem 4.3 is not valid as Theorem 4.5 shows.

**Lemma 4.4.** Let $A$ be an $\mathbb{S}^2$-closed set, $\varepsilon > 0$ and $L(A; \varepsilon) = \{p \in \mathbb{S}^2 : B[p; \varepsilon] \subseteq A\}$. Then $L(A; \varepsilon)$ is $\mathbb{R}^2$-closed.

**Proof.** We take $p_0 = (x_0, y_0) \in cl_{\mathbb{R}^2}L(A; \varepsilon)$ and show that $p_0 \in L(A; \varepsilon)$. We consider $U = int_{\mathbb{R}^2}B[p_0; \varepsilon]$ and prove that $U \subseteq A$. Take $p = (x, y) \in U$ and put $\delta = \min\{(x - x_0)/2, (y - y_0)/2, (x_0 + \varepsilon - x)/2, (y_0 + \varepsilon - y)/2\}$. Let $p_1 \in B(p_0; \delta) \cap L(A; \varepsilon)$. It is easy to see that $p \in B[p_1; \varepsilon]$. Then $p \in A$, since $p_1 \in L(A; \varepsilon)$. Hence, $U \subseteq A$. Then $B[p_0; \varepsilon] = cl_{\mathbb{R}^2}U \subseteq cl_{\mathbb{R}^2}A = A$, which implies that $p_0 \in L(A; \varepsilon)$. Therefore, $L(A; \varepsilon)$ is closed in $\mathbb{R}^2$.

**Theorem 4.5.** There exists an $\mathbb{S}^2$-closed countable discrete $G_\delta$-subspace $E$ of $\mathbb{R}^2$ which is not $C^*$-embedded in $\mathbb{S}^2$.

**Proof.** Let $C$ be the standard Cantor set on $[0, 1]$ and let $(I_n)_{n=1}^{\infty}$ be a sequence of all complementary intervals $I_n = (a_n, b_n)$ to $C$ such that $\text{diam}(I_{n+1}) \leq \text{diam}(I_n)$ for every $n \geq 1$. We put $p_n = (b_n, 1 - a_n)$, $E = \{p_n : n \in \mathbb{N}\}$ and $F = \{(x, 1 - x) : x \in \mathbb{R}\} \cap (C \times [0, 1])$. Notice that $E$ is a closed subset of $\mathbb{S}^2$, $F$ is functionally closed in $\mathbb{S}^2$ and $E \cap F = \emptyset$.

Let $N' \subseteq \mathbb{N}$ be a set such that $\{b_n : n \in N'\}$ and $\{b_n : n \in \mathbb{N} \setminus N'\}$ are dense subsets of $C$. To show that $E$ is not $C^*$-embedded in $\mathbb{S}^2$ we verify that disjoint clopen subsets

$E_1 = \{p_n : n \in N'\}$ and $E_2 = \{p_n : n \in \mathbb{N} \setminus N'\}$

of $E$ can not be separated by disjoint clopen subsets in $\mathbb{S}^2$. Assume the contrary and take disjoint clopen subsets $W_1$ and $W_2$ of $\mathbb{S}^2$ such that $W_i \cap E = E_i$ for $i = 1, 2$.

We prove that $W_1 \cap F$ is $\mathbb{R}^2$-dense in $F$. To obtain a contradiction we take an $\mathbb{R}^2$-open set $O$ such that $O \cap F \cap W_1 = \emptyset$. Since the set $U = \mathbb{S}^3 \setminus W_1$ is clopen, $U = \bigcup_{n=1}^{\infty} L(U; \frac{1}{n})$, where $L(U; \frac{1}{n}) = \{p \in \mathbb{S}^2 : B[p; 1/n] \subseteq U\}$ and the set $F_n = L(U; \frac{1}{n})$ is $\mathbb{R}^2$-closed by Lemma 4.3 for every $n \in \mathbb{N}$. Since $O \cap F$ is a Baire subspace of $\mathbb{R}^2$,
there exist $N \in \mathbb{N}$ and an $\mathbb{R}^2$-open in $F$ subset $I \subseteq F$ such that $I \cap O \subseteq F_N \cap F \subseteq S^2 \setminus E_1$. Taking into account that $\text{diam}(I_n) \to 0$, we choose $n_1 > N$ such that $b_n - a_n < \frac{1}{n} n_1$ for all $n \geq n_1$. Since the set $\{a_n : n \in N\}$ is dense in $C$, there exists $n_2 \in N$ such that $n_2 > n_1$ and $p = (a_{n_2}; 1 - a_{n_2}) \in I$. Clearly, $p \in F$. Consequently, $B[p; \frac{1}{n_2}] \cap E_1 = \emptyset$. But $p_{n_2} \in B[p; \frac{1}{n_2}] \cap E_1$, a contradiction.

Similarly we can show that $W_1 \cap F$ is also $\mathbb{R}^2$-dense in $F$.

Notice that $W_1$ and $W_2$ are $G_\delta$ in $\mathbb{R}^2$ by Lemma 3.1. Hence, $W_1 \cap F$ and $W_2 \cap F$ are disjoint dense $G_\delta$-subsets of a Baire space $F$, which implies a contradiction. Therefore, $E$ is not $C^*$-embedded in $S^2$. □

5. A CHARACTERIZATION OF $C$-EMBEDDED SUBSETS OF THE ANTI-DIAGONAL OF $S^2$.

By $\mathbb{D}$ we denote the anti-diagonal $\{(x, -x) : x \in \mathbb{R}\}$ of the Sorgenfrey plane. Notice that $\mathbb{D}$ is a closed discrete subspace of $S^2$.

Theorem 5.1. For a set $E \subseteq \mathbb{D}$ the following conditions are equivalent:

1) $E$ is $C$-embedded in $S^2$;
2) $E$ is $C^*$-embedded in $S^2$;
3) $E$ is a countable $G_\delta$-subspace of $\mathbb{R}^2$;
4) $E$ is a countably functionally closed subspace of $S^2$.

Proof. The implication (1) $\Rightarrow$ (2) is obvious. The implication (2) $\Rightarrow$ (3) follows from Theorem 4.1.

We prove (3) $\Rightarrow$ (4). To do this we verify condition (2) from Lemma 4.1. Let $F$ be an $\mathbb{R}^2$-closed set disjoint from $E$. Denote \( D = F \cap \mathbb{D} \) and \( U = \bigcup_{p \in D} B[p; 1] \). We show that $U$ is clopen in $S^2$. Clearly, $U$ is open in $S^2$. Take a point $p_0 \in cl_{S^2} U$ and show that $p_0 \in U$. Choose a sequence $p_n \in U$ such that $p_n \to p_0$ in $S^2$. For every $n$ there exists $q_n \in D$ such that $p_n \in B(q_n, 1)$. Notice that the sequence $(q_n)_{n=1}^\infty$ is bounded in $\mathbb{R}^2$ and take a convergent in $\mathbb{R}^2$ sequence $(y_n)_{n=1}^\infty$ of $(q_n)_{n=1}^\infty$. Since $D$ is $\mathbb{R}^2$-closed, $q_0 = \lim_{k \to \infty} y_n \in D$. Then $p_0 \in cl_{S^2} B(q_0, 1)$. If $p_0 \in B(q_0, 1)$, then $p_0 \in U$. Assume $p_0 \notin B(q_0, 1)$ and let $q_0 = (x_0, y_0)$. Without loss of generality we may suppose that $p_0 \in [x_0, x_0 + 1] \times (y_0 + 1)$. Since $p_{n_k} \to p_0$ in $S^2$, $q_{n_k} \in (-\infty, x_0] \times [y_0, +\infty)$ for all $k \geq k_0$ and $p_0 \in [x_0, x_0 + 1] \times \{y_0 + 1\}$. Then $p_0 \in \bigcup_{k=1}^\infty B(q_{n_k}, 1) \subseteq U$. Hence, $U$ is clopen and $D = U \cap \mathbb{D}$. Since $\mathbb{D}$ and $F \setminus U$ are disjoint functionally closed subsets of $S^2$, there exists a clopen set $V$ such that $\mathbb{D} \cap V = \emptyset$ and $F \setminus U \subseteq V$. Then $F \subseteq U \cup V \subseteq S^2 \setminus E$. Consequently, $F$ and $E$ are completely separated in $S^2$. Therefore, $E$ is functionally closed in $S^2$ by Lemma 3.1.

(4) $\Rightarrow$ (1). Notice that $E$ satisfies the conditions of Theorem 1.1. Indeed, $E$ is $z$-embedded in $S^2$, since $|E| \leq \aleph_0$. Moreover, $E$ is well-embedded in $S^2$, since $E$ is functionally closed.

□

Remark 5.2. Notice that a subset $E$ of $\mathbb{R}^2$ is countable $G_\delta$ if and only if it is scattered in $\mathbb{R}^2$. Indeed, assume that $E$ is countable $G_\delta$-set which contains a set $Q$ without isolated points. Then $Q$ is a $G_\delta$-subset of $\mathbb{R}^2$ which is homeomorphic to $Q$, a contradiction. On the other hand, if $E$ is scattered, then Lemma 4.1 implies that $E$ is countable. Since $E$ is hereditarily Baire and countable, $E$ is $G_\delta$ in $\mathbb{R}^2$.

Finally, we show that the Sorgenfrey plane is not a $\delta$-normally separated space. Let $E = \{(x, -x) : x \in \mathbb{Q}\}$ and $F = \mathbb{D} \setminus E$. Then $E$ is closed and $F$ is functionally closed in $S^2$, since $F$ is the difference of the functionally closed set $\mathbb{D}$ and the functionally open set $\bigcup_{p \in E} B[p, 1]$. But $E$ and $F$ cannot be separated by disjoint clopen sets in $S^2$, because $E$ is not $G_\delta$-subset of $\mathbb{D}$ in $\mathbb{R}^2$.

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