Renormalization constants of local operators for Wilson type improved fermions

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Perturbative and non-perturbative results are presented on the renormalization constants of the quark field and the vector, axial-vector, pseudoscalar, scalar and tensor currents. The perturbative computation, carried out at one-loop level and up to second order in the lattice spacing, is performed for a fermion action, which includes the clover term and the twisted mass parameter yielding results that are applicable for unimproved Wilson fermions, as well as for improved clover and twisted mass fermions. We consider ten variants of the Symanzik improved gauge action corresponding to ten different values of the plaquette coefficients. Non-perturbative results are obtained using the twisted mass Wilson fermion formulation employing two degenerate dynamical quarks and the tree-level Symanzik improved gluon action. The simulations are performed for pion masses in the range of 480 MeV to 260 MeV and at three values of the lattice spacing, $a$, corresponding to $\beta = 3.9, 4.05, 4.20$. For each renormalization factor computed non-perturbatively we subtract its perturbative $O(a^2)$ terms so that we eliminate part of the cut-off artifacts. The renormalization constants are converted to $\overline{\text{MS}}$ at a scale of $\mu = 2$ GeV. The perturbative results depend on a large number of parameters and are made easily accessible to the reader by including them in the distribution package of this paper, as a Mathematica input file.

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I. INTRODUCTION

Simulations of Quantum Chromodynamics (QCD) are nowadays being carried out at almost physical parameters. Therefore studies of hadron structure within lattice QCD are beginning to yield results that can be connected to experiment more reliably than ever before. In these lattice QCD studies one calculates matrix elements of local operators between hadron states. Unless these operators correspond to a conserved current they have to be renormalized. Calculation of renormalization factors can be carried out using lattice perturbation theory. Although perturbation theory on the lattice is computationally more complex than in the continuum these calculations can be extended beyond one-loop order [1–3]. Various methods to improve the convergence of lattice perturbation theory have been introduced [4, 5] yielding valuable first input to the values of the renormalization constants. In this work we will use the perturbative results to improve the non-perturbative evaluation of the renormalization constants. We use the Rome-Southampton method (also known as the RI-MOM scheme) [6] to compute renormalization coefficients of arbitrary quark-antiquark operators non-perturbatively. In this approach the procedure is similar to that used in continuum perturbation theory. In particular, the renormalization conditions are defined similarly in perturbative and non-perturbative calculations. The renormalization factors, obtained for different values of the renormalization scale, are evolved perturbatively to a reference scale $\mu = 2$ GeV. In addition, they are translated to $\overline{\text{MS}}$ at 2 GeV using 3-loop perturbative results for the conversion factors. Since in the end one wants to make contact with phenomenological studies, which almost exclusively refer to operators renormalized in the $\overline{\text{MS}}$ scheme of dimensional regularization, one needs the renormalization factors leading from the bare operators on the lattice to the $\overline{\text{MS}}$ operators in the continuum.

A number of lattice groups are producing results on nucleon form factors and first moments of structure functions closer to the physical regime both in terms of pion mass as well as in terms of the continuum limit [7–13]. In these lattice QCD computations one calculates hadron matrix elements of bilocal operators. In order to compare hadron matrix elements of local operators to experiment one needs to renormalize them. The aim of this paper is to calculate non-perturbatively the renormalization factors of the vector, axial-vector, scalar, pseudoscalar and tensor currents within the twisted mass formulation of Wilson lattice QCD [14]. We show that, although the lattice spacings considered in this work are smaller than 1 fm, $O(a^2 p^2)$ terms are non-negligible and are significantly larger than statistical errors. We therefore compute the $O(a^2 g^0)$-terms perturbatively and subtract them from the non-perturbative results. This subtraction suppresses lattice artifacts considerably depending on the operator under study and leads to a more accurate determination of the renormalization constants. This approach was applied to evaluate the renormalization constants for one-derivative bilinear operators in Ref. [15].

The paper is organized as follows: in Section II we give the expressions for the fermion and gluon actions we employed, and define the operators. Sections III and IV concentrate on the perturbative procedure, and the $O(a^2)$-corrected expressions for the renormalization constants $Z_V$ and $Z_G$. In Section V we provide the renormalization prescription of the RI$'$-MOM scheme, and we discuss alternative ways for its application, while in Section VI we provide all necessary formulae for the conversion to $\overline{\text{MS}}$ and the evolution to a reference scale of 2 GeV. Section VII focuses on the non-perturbative computation, where we explain the different steps of the calculation. The main results of this work are presented in Section VIII: the reader can find numerical values for the $Z$-factors of the fermion field and fermion operators, which are computed non-perturbatively and corrected using the perturbative $O(a^2)$ terms presented in Sections III and IV. For comparison with phenomenological and experimental results, we convert the $Z$-factors to the $\overline{\text{MS}}$ scheme at 2 GeV. In Section IX we give our conclusions.

II. FORMULATION

A. Lattice actions

Our perturbative calculation makes use of the twisted mass fermion action including the usual clover (SW) term with a clover parameter that is left free. For $N_F$ flavor species and using standard notation, this action reads

$$S_F = -\frac{a^3}{2} \sum_{x,f,\mu} \left[ \bar{\psi}_f(x) (r - \gamma_\mu) U_{x,x+a,\mu} \psi_f(x + a, \mu) + \bar{\psi}_f(x + a, \mu) (r + \gamma_\mu) U_{x+a,\mu,x} \psi_f(x) \right]$$

$$+ a^4 \sum_{x,f} \left( \frac{4r}{a} + m_f^0 + i\mu_0^0 \gamma_5 \tau^3 \right) \bar{\psi}_f(x) \psi_f(x)$$

$$- \frac{a^5}{4} \sum_{x,f,\mu,\nu} \rho c_{SW} \bar{\psi}_f(x) \sigma_{\mu\nu} F_{\mu\nu}(x) \psi_f(x),$$

(1)
Therefore, with no loss of generality we can set parameters: $c$ here we employ the tree-level Symanzik action \( S_{\mu \nu} \) as well as the twisted mass parameter $\mu_0^f$ are kept as free parameters throughout. $F_{\mu \nu}$ is the standard clover discretization of the gluon field tensor \[16\].

For the non-perturbative calculation, we consider the purely twisted mass fermion action (no clover term), which for two degenerate flavors of quarks is given by:

$$ S_f = a^4 \sum_x \overline{\chi}(x) \left( \frac{1}{2} \gamma_\mu (\nabla_\mu + \nabla^*_\mu) - \frac{a r}{2} \nabla_\mu \nabla^*_\mu + m_0 + i \mu_0 \gamma_5 r^3 \right) \chi(x), $$

(2)

with $r^3$ the Pauli matrix acting in the isospin space, and $\mu_0$ the bare twisted mass. Maximally twisted Wilson quarks are obtained by setting the untwisted bare quark mass $m_0$ to its critical value $m_{cr}$, while the twisted quark mass parameter $\mu_0$ is kept non-vanishing in order to give the light quarks their mass \[14, 17\]. The physical quantities extracted from lattice simulations employing maximally twisted Wilson quarks are automatically $O(a)$ improved \[17\]. In Eq. (2) the quark fields $\chi$ are in the so-called “twisted basis”. The “physical basis” is obtained for maximal twist by the simple transformation:

$$ \psi(x) = \exp \left( \frac{i \omega}{2} \gamma_5 r^3 \right) \chi(x), \quad \overline{\psi}(x) = \overline{\chi}(x) \exp \left( \frac{i \omega}{2} \gamma_5 r^3 \right), \quad \omega = \frac{\pi}{2}. $$

(3)

In terms of the physical fields the action is given by:

$$ S_f = a^4 \sum_x \overline{\psi}(x) \left( \frac{1}{2} \gamma_\mu [\nabla_\mu + \nabla^*_\mu] - i \gamma_5 r^3 \left( - \frac{a}{2} \nabla_\mu \nabla^*_\mu + m_{cr} \right) + \mu_0 \right) \psi(x). $$

(4)

One can check that this action is equivalent to the action in the twisted basis given by Eq. (2), by performing the rotations defined in Eq. (3) and identifying $m_0 = m_{cr}$.

For gluons we employ the Symanzik improved action, involving Wilson loops with 4 and 6 links (\(1 \times 1\) plaquette, \(1 \times 2\) rectangle, \(1 \times 2\) chair, and \(1 \times 1 \times 1\) parallelogram wrapped around an elementary 3-d cube), which is given by

$$ S_G = \frac{2}{g_0^2} \left[ c_0 \sum_{\text{plaq.}} \text{Re Tr} \{1 - U_{\text{plaq.}} \} + c_1 \sum_{\text{rect.}} \text{Re Tr} \{1 - U_{\text{rect.}} \} \right. $$

$$ + \left. c_2 \sum_{\text{chair}} \text{Re Tr} \{1 - U_{\text{chair}} \} + c_3 \sum_{\text{paral.}} \text{Re Tr} \{1 - U_{\text{paral.}} \} \right]. $$

(5)

The coefficients $c_i$ can in principle be chosen arbitrarily, subject to the following normalization condition, which ensures the correct classical continuum limit of the action:

$$ c_0 + 8c_1 + 16c_2 + 8c_3 = 1. $$

(6)

Some popular choices of values for $c_i$ used in numerical simulations will be considered in this work, and are itemized in Table I (the acronym TILW represent the Tadpole Improved Lüscher-Weisz action): they are normally tuned in a way as to ensure $O(a^2)$ improvement in the pure gluon sector. In our non-perturbative computation presented here we employ the tree-level Symanzik action ($c_0 = 5/3$, $c_1 = -1/12$, $c_2 = c_3 = 0$). Our 1-loop Feynman diagrams do not involve pure gluon vertices, and the gluon propagator depends only on three combinations of the Symanzik parameters:

$$ C_0 \equiv c_0 + 8c_1 + 16c_2 + 8c_3 = 1, $$

$$ C_1 \equiv c_2 + c_3, $$

$$ C_2 \equiv c_1 - c_2 - c_3. $$

(7)

Therefore, with no loss of generality we can set $c_2 = 0$. 
of the isospin index \( a \) corresponding \( Z \) constants are defined in the chiral limit, where iso-spin symmetry is recovered. Hence in the physical limit.

For convenience we have included \( O \) computed in perturbation theory to \( O \) following conditions in the massless theory, i.e., at critical mass and vanishing twisted mass.

The renormalization constants are computed both perturbatively and non-perturbatively in the RI- \( \overline{\text{MS}} \) scheme at \( \mu = 2 \text{ GeV} \) using a conversion factor computed in perturbation theory to \( O(g^6) \) as described in Section VI. The \( Z \)-factors are determined by imposing the following conditions in the massless theory, i.e., at critical mass and vanishing twisted mass

\[
Z_q = \frac{1}{12} \text{Tr} \left[ (S^L(p))^{-1} S^{(0)}(p) \right]_{p^2 = \mu^2}
\]

\[
Z_q^{-1} Z_O = \frac{1}{12} \text{Tr} \left[ \Gamma^L(p) \Gamma^{(0)-1}(p) \right]_{p^2 = \mu^2} = 1,
\]

B. Definition of operators and Renormalization condition

The ultra-local bi-fermion operators considered in this work are the following:

\[
O^a_S = \bar{\chi} \tau^a \chi = \begin{cases} 
\bar{\psi} \gamma^a \psi & a = 1, 2 \\
- i \bar{\psi} \gamma_5 \psi & a = 3 
\end{cases}
\]

\[
O^b_p = \bar{\chi} \gamma_5 \tau^a \chi = \begin{cases} 
\bar{\psi} \gamma_5 \tau^a \psi & a = 1, 2 \\
- i \bar{\psi} \tau^1 \psi & a = 3 
\end{cases}
\]

\[
O^c_v = \bar{\chi} \gamma_\mu \tau^a \chi = \begin{cases} 
\bar{\psi} \gamma_5 \gamma_\mu \tau^a \psi & a = 1 \\
- i \bar{\psi} \gamma_5 \gamma_\mu \tau^1 \psi & a = 2 \\
\bar{\psi} \gamma_\mu \tau^3 \psi & a = 3 
\end{cases}
\]

\[
O^d_A = \bar{\chi} \gamma_5 \gamma_\mu \tau^a \chi = \begin{cases} 
\bar{\psi} \gamma_5 \gamma_\mu \tau^a \psi & a = 1 \\
- i \bar{\psi} \gamma_5 \gamma_\mu \tau^1 \psi & a = 2 \\
\bar{\psi} \gamma_5 \gamma_\mu \tau^3 \psi & a = 3 
\end{cases}
\]

\[
O^e_T = \bar{\chi} \sigma_{\mu\nu} \tau^a \chi = \begin{cases} 
\bar{\psi} \gamma_{\mu\nu} \tau^a \psi & a = 1, 2 \\
- i \bar{\psi} \gamma_{\mu\nu} \tau^1 \psi & a = 3 
\end{cases}
\]

\[
O^f_{Tp} = \bar{\chi} \gamma_5 \gamma_5 \tau^a \chi = \begin{cases} 
\bar{\psi} \gamma_5 \gamma_5 \tau^a \psi & a = 1, 2 \\
- i \bar{\psi} \gamma_5 \gamma_5 \tau^1 \psi & a = 3 
\end{cases}
\]

For convenience we have included \( O^a_{Tp} \) even though its components are related to those of \( O^e_T \). We denote the corresponding \( Z \)-factors by \( Z^a_S, Z^b_p, Z^c_v, Z^d_A, Z^e_T, Z^f_{Tp} \). In a massless renormalization scheme the renormalization constants are defined in the chiral limit, where iso-spin symmetry is recovered. Hence \( Z \)-factors become independent of the isospin index \( a = 1, 2, 3 \) and we drop the \( a \) index on the \( Z \)-factors from here on. Still note that, for instance, the physical \( \bar{\psi} \gamma_\mu \tau^1 \psi \) is renormalized with \( Z_A \) while \( \bar{\psi} \gamma_\mu \tau^3 \psi \) needs \( Z_V \), which differ from each other even in the chiral limit.

\[
\begin{array}{|c|c|c|c|}
\hline
\text{Action} & c_0 & c_1 & c_3 \\
\hline
\text{Plaquette} & 1.0 & 0 & 0 \\
\text{Symanszik} & 5/3 & -1/12 & 0 \\
\text{TILW, } \beta_{c_0} = 8.60 & 2.3168064 & -0.151791 & -0.0128098 \\
\text{TILW, } \beta_{c_0} = 8.45 & 2.3460240 & -0.154846 & -0.0134070 \\
\text{TILW, } \beta_{c_0} = 8.30 & 2.3869776 & -0.159128 & -0.0142442 \\
\text{TILW, } \beta_{c_0} = 8.20 & 2.4127840 & -0.161827 & -0.0147710 \\
\text{TILW, } \beta_{c_0} = 8.10 & 2.4465400 & -0.165353 & -0.0154645 \\
\text{TILW, } \beta_{c_0} = 8.00 & 2.4891712 & -0.169805 & -0.0163414 \\
\text{Iwasaki} & 3.648 & -0.331 & 0 \\
\text{DBW2} & 12.2688 & -1.4086 & 0 \\
\hline
\end{array}
\]

\text{TABLE I: Input parameters } c_0, c_1, c_3.
where the trace is taken over spin and color indices, $\mu$ is the renormalization scale, while $S^L$ and $\Gamma^L$ correspond to the perturbative or non-perturbative results, and $S^{(0)}$ is the tree-level result for the propagator defined as:

$$S^{(0)}(p) = \frac{-i \sum_\rho \gamma_\rho \sin(p_\rho)}{\sum_\rho \sin(p_\rho)^2},$$

while $\Gamma^{(0)}$ is the tree-level value for the fermion operators $S$, $P$, $V$, $A$, $T$, $T'$, that is

$$\Gamma^{(0)}(p) = \mathbb{1}, \gamma_5, \gamma_\mu, \gamma_5 \gamma_\mu, \gamma_5 \sigma_{\mu\nu}, \sigma_{\mu\nu},$$

respectively. The trace is taken over spin and color indices. For alternative renormalization prescriptions the reader can refer to Ref. [15].

The choices for $S^{(0)}$ and $\Gamma^{(0)}$ given in Eqs. (16) - (17) are optimal, since we obtain $Z_q = 1, Z_\mathcal{O} = 1$ when the gauge field is set to unity. Similarly, in the perturbative computation this condition leads to $Z_q = 1, and Z_\mathcal{O} = 1$ at tree-level.

### III. CORRECTIONS TO THE FERMION PROPAGATOR

The fermion propagator of the interacting theory is given by the following 2-point correlation function (Green’s function), with the various quantities computed in perturbation theory:

$$\langle \chi^{a,f}_\alpha(x)\bar{\chi}^{b,g}_\beta(y) \rangle = \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} \left( S_{\text{tree}} \cdot \sum_{n=0}^{\infty} (-S^{-1}_{\text{amp}}(p) \cdot S_{\text{tree}})^n \right)^{fg}_{\alpha\beta},$$

with $S^{-1}_{\text{amp}}(p)$ being the amputated, 1PI 2-point function in momentum space, computed perturbatively up to a desired order. $S_{\text{tree}}$ is the tree-level propagator for the twisted mass action and it is given by

$$S_{\text{tree}} = \frac{1}{i\not{p} + M(p) + i\mu_0 \gamma_5}, \quad M(p) \equiv m_0^f + \frac{2}{a} \sum_\mu \sin^2(a p_\mu/2), \quad \not{p} \equiv \sum_\mu \gamma_\mu \frac{1}{a} \sin(a p_\mu),$$

where $\alpha, \beta$ are Dirac indices, $f_1, f_2$ are flavor indices in the fundamental representation of $SU(N_F)$, and $a, b$ are color indices in the fundamental representation of $SU(N_c)$. The dot product runs over flavor and Dirac indices. Due to the diagonal form of the $\tau^3$ matrix, and since we are studying the case of only two degenerate quarks (up/down) we can simplify the expression of $S_{\text{tree}}$ and omit $\tau^3$ by giving a flavor index to the twisted mass parameter, and at the same time we take $m_0^f \rightarrow m_0$:

$$S_{\text{tree}} = \frac{1}{i\not{p} + M(p) + i\mu_0^{(1)} \gamma_5},$$

where now $\mu_0^{(1)} = +\mu_0$ for the up quark propagator and $\mu_0^{(2)} = -\mu_0$ for the down quark propagator. The 1-loop Feynman diagrams that enter our 2-point amputated Green’s function calculation ($S^{-1}_{\text{amp}}$), are depicted in Fig. 1.

![FIG. 1: One-loop diagrams contributing to the fermion propagator. Wavy (solid) lines represent gluons (fermions).](image)
For the algebraic operations involved in evaluating Feynman diagrams, we make use of our symbolic package in Mathematica. In a nutshell, the required steps for the computation of a Feynman diagram are the following (the reader can find more details in Ref. [18]):

- A preliminary expression for each diagram can be obtained by contracting the appropriate vertices, which is performed automatically once the algebraic expression of the vertices and the topology (‘incidence matrix’) of the diagram are specified. To limit the proliferation of the algebraic expressions we exploit symmetries of the theory, and we simplify the color dependence, Dirac matrices and tensor structures.

- The \( O(a^2) \) computation introduces several complications, especially when isolating logarithms and Lorentz non-invariant terms, which leads to a whole family of infrared divergent integrals. These can be reduced to a minimal set of approximately 250 ‘basis’ integrals. This is achieved by converting all propagator denominators to a standard form \((\hat{q}^2 + M^2)^{-1}\) using two kinds of subtractions, one for the fermion propagator

\[
\frac{1}{q^2} = \frac{1}{\tilde{q}^2} + \left\{ \frac{4 \sum_{\mu} \sin^2(q_{\mu}/2) - 4 \left( \sum_{\mu} \sin^2(q_{\mu}/2) \right)^2 - 4 m_0 \sum_{\mu} \sin^2(q_{\mu}/2)}{\tilde{q}^2} \right\}
\]

(21)

where the denominator of the fermion propagator, \( \tilde{q}^2 \), is defined as

\[
\tilde{q}^2 = \sum_{\mu} \sin^2(q_{\mu}) + \left( m_0 + \frac{1}{2} q^2 \right)^2 + (\mu_{f}^{(f)})^2, \quad q^2 = 4 \sum_{\mu} \sin^2(q_{\mu}/2), \quad M^2 = m_0^2 + \mu_0^2,
\]

(22)

and one for the gluon propagator:

\[
D(q) = D_{\text{plaq}}(q) + \left\{ D(q) - D_{\text{plaq}}(q) \right\} = D_{\text{plaq}}(q) + D_{\text{plaq}}(q) \left\{ D_{\text{plaq}}^{-1}(q) - D^{-1}(q) \right\} D(q).
\]

(23)

\( D \) is the \( 4 \times 4 \) Symanzik gluon propagator; the expression for the matrix \( D_{\text{plaq}}^{-1}(q) - D^{-1}(q) \), which is \( O(q^4) \), is independent of the gauge parameter, \( \lambda \), and it can be easily obtained in closed form. Moreover, we have

\[
(D_{\text{plaq}}(q))_{\mu\nu} = \frac{\delta_{\mu\nu}}{\tilde{q}^2} - (1 - \lambda) \frac{\hat{q}_{\mu} \hat{q}_{\nu}}{\tilde{q}^2}
\]

(24)

Terms in curly brackets of Eqs. (21) and (23) are less IR divergent than their unsubtracted counterparts, by one or two powers in \( a \). These subtractions are performed iteratively until all divergent integrals (initially depending on the fermion and the Symanzik propagator) are expressed in terms of the gluon propagator, \((\hat{q}^2 + M^2)^{-1}\). The computation of the divergent integrals is performed in a non-integer number of dimensions \( D > 4 \). Ultraviolet divergences are explicitly isolated à la Zimmermann and evaluated as in the continuum. The remainders are \( D \)-dimensional, parameter-free, zero external momentum lattice integrals which can be recast in terms of Bessel functions, and finally expressed as sums of a pole part plus numerical constants.

A small subset of the infrared divergent integrals, shown in Appendix A, contains the most demanding ones in the list; they must be evaluated to two further orders in \( a \), beyond the order at which an IR divergence initially sets in. As a consequence, their evaluation requires going to \( D \geq 6 \) dimensions. A correct way to evaluate strong divergent integrals is given in detail in a previous publication [18].

- The required numerical integrations of the algebraic expressions for the loop integrands are performed by highly optimized Fortran programs; these are generated by our Mathematica ‘integrator’ routine. Each integral is expressed as a sum over the discrete Brillouin zone of finite lattices, with varying size \( L (4^4 \leq L^4 \leq 128^4) \), and evaluated for all values of the Symanzik coefficients listed in Table I.

- The last part of the evaluation is the extrapolation of the numerical results to infinite lattice size. This procedure entails a systematic error, which is reliably estimated, using a complex inference technique; for one-loop quantities we expect a relative error smaller than \( 10^{-7} \).

Next, we provide the total expression for the inverse fermion propagator \( S_{\text{pert}}^{-1}(p) \), computed up to 1-loop in perturbation theory. Here we should point out that for dimensional reasons, there is a global prefactor \( 1/a \) multiplying our expressions for the inverse propagator, and thus, the \( O(a^2) \) correction is achieved by considering all terms up to \( O(a^3) \). The most general expression for the inverse propagator appears in the Mathematica file ZFactors.m (see Appendix D for notation). In the main text we provide a more compact expression, for special values of the various parameters, that is tree-level Symanzik improved gluon action, \( c_{SW} = 0 \), Landau gauge (\( \lambda = 0 \)), but we keep the Lagrangian mass and the twisted mass parameter general.
\[ S^{-1}_{\text{pert}} = m + i \mu \gamma^5 + i \not{p} + \frac{\alpha p^2}{2} - \frac{\alpha^2}{6} i \not{p}^3 \]

\[ + \hat{g}^2 \left\{ -13.0232725(2) i \not{p} + m \left( 0.5834586(2) - 3 \ln[a^2 M^2 + a^2 p^2] - \frac{3 M^2 \ln[1 + \frac{p^2}{M^2}]}{p^2} \right) \right. \]

\[ + i \mu \gamma^5 \left( 8.7100834(2) - 3 \ln[a^2 M^2 + a^2 p^2] - \frac{3 M^2 \ln[1 + \frac{p^2}{M^2}]}{p^2} \right) + a \left[ \left( -10.69642965(5) p^2 \right. \right. \]

\[ - 0.8530378(3) m^2 - 1.842911859(4) M^2 + \frac{6 M^2 m^2 \ln[1 + \frac{p^2}{M^2}]}{p^2} + \left( \frac{3 p^2}{2} + 3 m^2 + \frac{3 M^2}{2} \right) \ln[a^2 M^2 + a^2 p^2] \]

\[ + i m \not{p} \left( 0.3393996(2) + \frac{3 M^2}{2 p^2} + \frac{3}{2} \ln[a^2 M^2 + a^2 p^2] - \frac{3 M^4 \ln[1 + \frac{p^2}{M^2}]}{2(p^2)^2} \right) \]

\[ + i \mu m \gamma^5 \left( -6.68582372(4) + 3 \ln[a^2 M^2 + a^2 p^2] + \frac{6 M^2 \ln[1 + \frac{p^2}{M^2}]}{p^2} \right) \right\} \]

\[ + a^2 \left\{ m \left( 2.3547298(1) p^2 + 2.3562747(1) m^2 + 3.46524146(4) M^2 - \frac{M^4}{6 p^2} + \frac{M^6}{3(p^2)^2} - \frac{3 m^2 p^2}{2(M^2 + p^2)} \right. \right. \]

\[ - \left( \frac{p^2}{4} + 3 m^2 + \frac{11 M^2}{3} \right) \ln[a^2 M^2 + a^2 p^2] + \left( \frac{p^2}{3} - 9 m^2 - 2 M^2 - \frac{M^6}{3(p^2)^2} \right) \frac{M^2 \ln[1 + \frac{p^2}{M^2}]}{p^2} \]

\[ + i \mu \gamma^5 \left( 0.70640552(8) p^2 + 6.79538844(2) m^2 + 1.16985307(3) M^2 - \frac{M^4}{6 p^2} + \frac{M^6}{3(p^2)^2} - \frac{3 m^2 p^2}{2(M^2 + p^2)} \right. \]

\[ - \left( \frac{p^2}{4} + 3 m^2 + \frac{2 M^2}{3} \right) \ln[a^2 M^2 + a^2 p^2] + \left( \frac{p^2}{3} - 9 m^2 - \frac{M^2}{2} - \frac{M^6}{3(p^2)^2} \right) \frac{M^2 \ln[1 + \frac{p^2}{M^2}]}{p^2} \right\} \]

\[ + \left( \sum_{\rho} \frac{p^4_{\rho}}{p^2} \right) \left( m + i \mu \gamma^5 \right) \left( \frac{1}{2} - \frac{2 M^2}{9 p^2} + \frac{M^4}{3(p^2)^2} - \frac{2 M^6}{3(p^2)^3} + \left( - \frac{1}{3} + \frac{2 M^6}{3(p^2)^3} \right) \frac{M^2 \ln[1 + \frac{p^2}{M^2}]}{p^2} \right) \]

\[ + i \not{p} \left( 1.1471643(7) p^2 - 0.2145514(2) m^2 + 1.15904388(6) M^2 - \frac{9 m^2 M^2}{2 p^2} - \frac{209 M^4}{360 p^2} - \frac{M^6}{240(p^2)^2} + \frac{7 M^8}{40(p^2)^3} \right. \]

\[ - \left( \frac{73 p^2}{360} + \frac{3 m^2}{2} + \frac{2 M^2}{3} \right) \ln[a^2 M^2 + a^2 p^2] + \left( \frac{1}{2} + \frac{9 m^2}{2 p^2} + \frac{43 M^2}{72 p^2} - \frac{M^4}{12(p^2)^2} - \frac{7 M^6}{40(p^2)^3} \right) \frac{M^4 \ln[1 + \frac{p^2}{M^2}]}{p^2} \]

\[ + i \not{p}^2 \left( 4.2478764(2) - \frac{67 M^2}{120 p^2} + \frac{M^4}{120(p^2)^2} - \frac{8 M^6}{15(p^2)^3} + \frac{7 M^8}{30(p^2)^4} - \frac{157}{180} \ln[a^2 M^2 + a^2 p^2] \right. \]

\[ + \left( \frac{1}{2} \right) - \frac{5 M^2}{18 p^2} + \frac{5 M^2}{12(p^2)^2} - \frac{7 M^4}{30(p^2)^3} \right) \frac{M^4 \ln[1 + \frac{p^2}{M^2}]}{(p^2)^2} \right) + i \left( \sum_{\rho} \frac{p^4_{\rho}}{p^2} \right) \not{p}^2 \left( \frac{7}{240} + \frac{M^2}{48 p^2} + \frac{67 M^4}{72(p^2)^2} \right. \]

\[ + \frac{13 M^6}{24(p^2)^3} - \frac{7 M^8}{12(p^2)^4} + \left( \frac{5}{12} - \frac{5 M^2}{4 p^2} - \frac{M^4}{4(p^2)^2} + \frac{7 M^6}{12(p^2)^3} \right) \frac{M^4 \ln[1 + \frac{p^2}{M^2}]}{(p^2)^2} \right) \right\} + \mathcal{O}\left( a^3, g^4 \right)
where \( \mathbf{p}' = \sum_\rho \gamma^\rho p^\rho \) and \( \mathbf{p}^3 = \sum_\rho \gamma^\rho p^\rho \). To make the above expressions less complicated we defined \( m \equiv m_0 \) and \( M^2 = m_0^2 + \mu_0^2 \). We would like to point out that the up quark propagator is obtained by the choice \( \mu^{(1)} = +\mu_0 \), while for the down quark propagator one should choose \( \mu^{(2)} = -\mu_0 \). Moreover, \( \tilde{g}^2 \equiv \frac{g^2 C_F}{16 \pi^2} \) and \( C_F \equiv \frac{N_c^2 - 1}{2 N_c} \).

Another byproduct of this part of the computation is the additive critical fermion mass; its general expression depends on \( c_{SW} \) and the Symanzik parameters. These are terms proportional to \( 1/a \) that have been left out of Eq. (25) for conciseness:

\[
m_{cr} = -\frac{\tilde{g}^2}{a} \left[ c_{m}^{(1)} + c_{m}^{(2)} c_{SW} + c_{m}^{(3)} c_{SW}^2 \right] + \frac{1}{a} \mathcal{O}(\tilde{g}^4). \tag{26}
\]

The quantities \( \varepsilon_m^{(i)} \) (listed in Table II) are numerical coefficients depending on the Symanzik parameters.

| Action       | \( \varepsilon_m^{(1)} \)  | \( \varepsilon_m^{(2)} \)  | \( \varepsilon_m^{(3)} \)  |
|--------------|-----------------------------|-----------------------------|-----------------------------|
| Plaquette    | -51.4347118(2)              | 13.73313097(5)              | 5.71513853(1)               |
| Symanzik     | -40.44324019(7)             | 11.94821988(5)              | 4.6626712112(4)             |
| TILW (8.60)  | -34.17747288(3)             | 10.76516514(3)              | 3.99348778(8)               |
| TILW (8.45)  | -33.9488671(1)              | 10.71947605(3)              | 3.9735187(1)                |
| TILW (8.30)  | -33.6344391(1)              | 10.65629321(4)              | 3.939135834(8)              |
| TILW (8.20)  | -33.43979350(6)             | 10.61705314(7)              | 3.917851255(1)              |
| TILW (8.10)  | -33.1892274(1)              | 10.56629305(3)              | 3.890401337(1)              |
| TILW (8.00)  | -32.87904072(9)             | 10.50313393(3)              | 3.856345868(2)              |
| Iwasaki      | -26.07292275(7)             | 9.01533524(3)               | 3.1061330684(3)             |
| DBW2         | -11.5127475(2)              | 4.9953066(1)                | 1.351772367(3)              |

TABLE II: Numerical results for the coefficients \( \varepsilon_m^{(1)}, \varepsilon_m^{(2)}, \varepsilon_m^{(3)} \) (Eq. (26)) for different actions. The systematic errors in parentheses come from the extrapolation over finite lattice size, \( L \to \infty \).

**IV. CORRECTIONS TO FERMION BILINEAR OPERATORS**

In the context of this work we also study the perturbative \( \mathcal{O}(a^2) \) corrections to Green’s functions of local fermion operators (currents) that have the form:

\[
O_{\Gamma} = \sum_z \sum_{\alpha'} \left( \bar{\chi}_{\alpha'}^f (z) \Gamma_{\alpha' \beta' \mu' \nu'} \chi_{\beta'}^g (z) \right). \tag{27}
\]

We restrict ourselves to forward matrix elements (i.e. 2-point Green’s functions, zero momentum operator insertions). The symbol \( \Gamma \) corresponds to the following set of products of the Euclidean Dirac matrices:

\[
\Gamma \in \{ S, P, V, A, T, T' \} \equiv \{ 1, \gamma^5, \gamma_\mu, \gamma^5 \gamma_\mu, \gamma^5 \sigma_{\mu \nu}, \sigma_{\mu \nu} \}; \quad \sigma_{\mu \nu} = \frac{1}{2} [\gamma_\mu, \gamma_\nu], \tag{28}
\]

for the scalar \( O_S \), pseudoscalar \( O_P \), vector \( O_V \), axial \( O_A \), tensor \( O_T \) and tensor prime \( O_T' \) operator, respectively. The matrix elements of \( O_{T'} \) can be related to those of \( O_T \); this is a nontrivial check for our calculational procedure [18]. The relationship between the amputated 2-point Green functions \( \Lambda_T \) and \( \Lambda_{T'} \) is:

\[
\Lambda_T^{\mu \nu} = -\frac{1}{2} \sum_{\mu' \nu'} \epsilon_{\mu \nu \mu' \nu'} \Lambda^{\mu' \nu'}_{T'}. \tag{29}
\]

The matrix elements of the above set of fermion bilinear operators can be obtained as:

\[
\langle \chi_{\alpha}^f (x) O_{\Gamma} \tilde{\chi}_{\beta}^g (y) \rangle = \int_{-\frac{2\pi}{a}}^{\frac{2\pi}{a}} \frac{d^4 p}{(2\pi)^4} e^{ip(x-y)} \left( S \cdot \Lambda_{\Gamma}^{\text{pert.}} (p) \cdot S \right)_{\alpha \beta}^{fg}. \tag{30}
\]
the renormalization condition is implemented by requiring that in the chiral limit
the amputated Greens functions of the operators $O_{\Gamma}$, in momentum space, which upon
contraction of indices becomes:

$$\Lambda_{\Gamma}^{\text{pert}}(p) = \sum_{\alpha' \beta'} \langle \chi_{\alpha}'(p) \Gamma_{\alpha' \beta'} \chi_{\beta}'(p) \rangle \chi_{\beta}'(p) \rangle^{\text{amp}}.$$  \hfill (31)

The only 1-particle irreducible Feynman diagram that enters the calculation of the above Green’s function is shown in Fig. 2.

$$\begin{array}{c}
\text{FIG. 2: One-loop diagram contributed to the bilinear operators. A} \\
wavy (solid) line represents gluons (fermions). A cross denotes the \\
Dirac matrix $\Gamma$.
\end{array}$$

In this diagram there are two fermion propagators, for which we allowed different $\mu$-values, in order to have more
general results. In other words, each of the two internal fermion lines on the left and on the right of the operator
insertion (see Fig. 2) can independently represent the up or down propagator. For the evaluation of the $Z$-factors, we
keep the two flavors independent. The amputated Greens functions of the operators $O_{\Gamma}$ are given in the Mathematica file Zfactors.m. As mentioned above, one may choose the two fermion propagators of the diagram to correspond either
to the up or down quark, thus there are two twisted mass parameters $\mu^{(1)}$, and $\mu^{(2)}$. These can have any sign and
the only restriction is: $|\mu^{(1)}| = |\mu^{(2)}|$.

V. QUARK FIELD AND QUARK BILINEAR RENORMALIZATION CONSTANTS IN THE RI’-MOM SCHEME

An operator renormalization constant (RC) can be thought of as the link between its matrix element, regularized on
the lattice, and its renormalized continuum counterpart. The RCs of lattice operators are necessary ingredients in the
prediction of physical probability amplitudes from lattice matrix elements. In this section we present the multiplicative
RCs, in the RI’-MOM scheme, of the quark field ($Z_{q_{\text{RI’-MOM}}}$) and quark bilinear operators ($Z_{\Gamma_{\text{RI’-MOM}}}$), obtained by using the perturbative expressions of $S^{-1}(p)$ and $\Lambda_{\Gamma}(p)$.

The RI’-MOM renormalization scheme consists in imposing that the forward amputated Green function $\Lambda_{\Gamma}(p)$, computed in the chiral limit and at a given (large Euclidean) scale $p^2 = \mu^2$, is equal to its tree-level value. In practice, the renormalization condition is implemented by requiring that in the chiral limit:

$$Z^{-1}_{q} Z_{\Gamma} V_{\Gamma}(p)|_{p_{\mu}=\mu_{\mu}} = 1, \quad V_{\Gamma}(p) \equiv \frac{1}{4} \text{Tr} \left[ \Lambda_{\Gamma}(p) \cdot P_{\Gamma} \right].$$  \hfill (33)

where $P_{\Gamma}$ are the Dirac projectors defined as follows:

$$P_{\Gamma} \in \{ P_{S}, P_{P}, P_{V}, P_{A}, P_{T}, P_{T^*} \} \equiv \{ 1, \gamma^5, \gamma_{\mu}, -\gamma^5\gamma_{\mu}, -\gamma^5\gamma_{\mu}, -\sigma_{\mu\nu}, -\sigma_{\mu\nu} \};$$  \hfill (34)

they are chosen to obey the relation $\text{Tr}[\Gamma \cdot P_{\Gamma}] = 4$. The traces are always taken over the spin indices. The quark field
RC $Z_{q}$, which enters Eq. (33), is obtained by imposing, again in the chiral limit, the condition$^2$:

$$Z^{-1}_{q} \frac{1}{4} \text{Tr} \left[ \Lambda_{\Gamma}(p) \cdot \Lambda_{\Gamma}^{\text{tree}} \right]_{p_{\mu}=\mu_{\mu}} \equiv \frac{1}{4} \text{Tr} \left[ \Lambda_{\Gamma}^{\text{tree}} \cdot \Lambda_{\Gamma}^{\text{tree}} \right],$$  \hfill (32)

where $\Lambda_{\Gamma}^{\text{tree}}$ is the tree-level value of $\Lambda_{\Gamma}(p)$.

$^1$ A simpler version of Eq. (33) is given by the relation:

$$Z^{-1}_{q} \frac{1}{4} \text{Tr} \left[ \Lambda_{\Gamma}(p) \cdot \Lambda_{\Gamma}^{\text{tree}} \right]_{p_{\mu}=\mu_{\mu}} \equiv \frac{1}{4} \text{Tr} \left[ \Lambda_{\Gamma}^{\text{tree}} \cdot \Lambda_{\Gamma}^{\text{tree}} \right],$$  \hfill (32)

where $\Lambda_{\Gamma}^{\text{tree}}$ is the tree-level value of $\Lambda_{\Gamma}(p)$.

$^2$ Strictly speaking, the renormalization condition of Eq. (36) defines the so called RI’ scheme. In the original RI-MOM scheme the quark field renormalization condition reads:

$$Z^{-1}_{q} \frac{1}{4} \text{Tr} \left[ \gamma_{\mu} \frac{\partial S_{q}(p)}{\partial p_{\mu}} \right]_{p^2=\mu^2} = 1.$$  \hfill (35)

The two schemes differ in the Landau gauge at the N$^{2}$LO.
\[ Z^{-1}_q \mathcal{V}_q(p)|_{p_\mu = \mu_\rho} = 1, \quad \mathcal{V}_q(p) = -\frac{i}{4} \text{Tr} \left[ \frac{1}{a} \sum_{\rho} \gamma_\rho \sin(a p_\rho) \cdot S^{-1}_1(p) \right]. \] (36)

We compute \( Z_q \) in the RI’-MOM renormalization scheme, defined in Eq. (14), which can be Taylor expanded up to \( \mathcal{O}(a^2) \) terms. This leads to:

\[
Z_q = -\frac{i}{4} \text{Tr} \left[ \frac{\sum_{\rho} \gamma_\rho (p_{\rho} - a^2 \frac{e^2 p^3_\rho}{6})}{\sum_{\rho} p^2_\rho} \left( 1 + \frac{a^2 \sum_{\rho} p^4_\rho}{3 \sum_{\rho} p^2_\rho} \right) \cdot S^{-1}_{1\text{-loop}}(p) \right] + \mathcal{O}(a^4 g^2, g^4)
\]

\[
= -\frac{i}{4} \text{Tr} \left[ \frac{p^0}{p^2} \cdot S^{-1}_{1\text{-loop}}(p) - \frac{a^2}{3} \left( \frac{1}{2} \frac{p^3}{p^2} - \frac{p^4}{(p^2)^2} \right) \cdot S^{-1}_{1\text{-loop}}(p) \right] + \mathcal{O}(a^4 g^2, g^4).
\] (37)

The trace is taken only over spin indices and \( S^{-1}_{1\text{-loop}} \) is the inverse fermion propagator that we computed up to 1-loop and up to \( \mathcal{O}(a^2) \). We make the following definitions for convenience: \( p^2 \equiv \sum_{\rho} p^2_\rho \), \( p^4 \equiv \sum_{\rho} p^4_\rho \), \( \gamma = \sum_{\rho} \gamma_\rho p_\rho \) and \( \gamma^3 \equiv \sum_{\rho} \gamma_\rho p^3_\rho \).

A very important issue is that the \( \mathcal{O}(a^2) \) terms in \( Z_q \) depend not only on \( |p| \), but also on the direction of the renormalization scale, \( p_\mu = \mu_\rho \), as manifested by the presence of \( \sum_{\rho} p^4_\rho \):

\[
\mathcal{V}_q^{\text{pert.}}(p) = -\frac{i}{4} \text{Tr} \left[ \frac{p^0}{p^2} \left( 1 + \frac{a^2 \sum_{\rho} p^4_\rho}{3 \sum_{\rho} p^2_\rho} \right) \cdot S^{-1}_{\text{pert.}}(p) \right] + \mathcal{O}(a^4 g^2, g^4).
\] (38)

As a consequence, alternative renormalization prescriptions, involving different directions of the renormalization scale \( \mu_\rho = p_\rho \), treat lattice artifacts differently.

By implementing the perturbative expressions of \( S^{-1}(p) \) and \( \Lambda_\rho(p) \) in Eqs. (33) and (36), we obtain the corresponding RCs. For the following special choices (independent): tree-level Symanzik gauge action, \( c_{SW} = 0 \), Landau gauge, and general mass \( m \) the results of the RCs under study are (for \( Z_q \) we have also kept \( \mu \) and \( M = \sqrt{m^2 + \mu^2} \) as free parameters):

\[
Z^{\text{pert.}}_q = 1
\]

\[
+ g^2 \left\{ -13.0232725(2) + a m \left[ 0.3393996(2) + \frac{3 \ln[a^2 M^2 + a^2 p^2]}{2} + \frac{3 M^2}{2 p^2} - \frac{3 M^4 \ln[1 + \frac{p^2}{M^2}]}{2 p^2} \right] + a^2 \left[ 1.1471643(7) p^2 ight. \\
+ \frac{1.6045466(12) M^2 + 2.10649577(2) p^4 - 0.9 m^2 M^2}{p^2} - \frac{209 M^4}{360 p^2} - \frac{M^6}{240 p^2} - \frac{7 M^8}{40 p^2} \\
- \frac{\left( \frac{73 p^2}{360} + \frac{3 m^2}{2} + \frac{2 M^2}{3} + \frac{157 p^4}{180 p^2} \right) \ln[a^2 M^2 + a^2 p^2]}{p^2} + \left( \frac{1}{24} - \frac{9 m^2}{2 p^2} + \frac{43 M^2}{72 p^2} - \frac{4 M^4}{12 p^2} - \frac{7 M^6}{40 p^2} \right) \ln[1 + \frac{p^2}{M^2}] \\
+ \frac{p^4}{p^2} \left( \frac{43 M^2}{80 p^2} + \frac{169 M^4}{120 p^2} + \frac{M^6}{20 p^2} - \frac{7 M^8}{20 p^2} + \left( \frac{1}{12} - \frac{35 M^2}{36 p^2} + \frac{M^4}{6 p^2} + \frac{7 M^6}{20 p^2} \right) \ln[1 + \frac{p^2}{M^2}] \right) \right\} \right\} _{\mu_\rho = \mu_\rho} + \mathcal{O}(a^3 g^2, g^4).
\]

The RCs of the bilinear operators have lengthy expressions and are not shown in the main text; they are presented in Appendix B. We also include Appendix C which is related to the perturbative results appearing in our publication for RCs of one-derivative operators [15].
VI. CONVERSION TO THE CONTINUUM $\overline{\text{MS}}$ SCHEME AT A REFERENCE RENORMALIZATION SCALE

A. Conversion factors

In this section we provide the expressions for the conversion factors to the $\overline{\text{MS}}$ scheme, as adapted from Ref. [19]. In our analysis we use 2-loop formulae; 3-loop corrections for the particular expressions are at the one per cent level. We use the following definitions for the conversion factors: $Z_{q}^{\overline{\text{MS}}} = C_{q} Z_{q}^{\text{RI'-MOM}}$ (note we use $C_{q}$ in contrast to Ref. [19] where the same quantity was denoted with $C_{q}^{-1}$), and $Z_{C}^{\overline{\text{MS}}} = C_{C}^{-1} Z_{C}^{\text{RI'-MOM}}$.

\[
C_{q} = 1 + \lambda \frac{g^{2} C_{F}}{16 \pi^{2}} - \left[ (8 \lambda^{2} + 5) C_{F} \right. \\
- \left. (9 \lambda^{2} - 24 \zeta(3) \lambda + 52 \lambda - 24 \zeta(3) + 82) N_{c} + 14 N_{F} - \lambda^{2} C_{F}^{2} \right] \frac{C_{F}}{8} \left( \frac{g^{2}}{16 \pi^{2}} \right)^{2} \tag{40}
\]

\[
C_{S,P} = 1 - (\lambda + 4) C_{F} \frac{g^{2}}{16 \pi^{2}} + \left[ (24 \lambda^{2} + 96 \lambda - 288 \zeta(3) + 57) C_{F} \\
+ 166 N_{F} - (18 \lambda^{2} + 84 \lambda - 432 \zeta(3) + 1285) N_{c} \right] \frac{C_{F}}{24} \left( \frac{g^{2}}{16 \pi^{2}} \right)^{2} \tag{41}
\]

\[
C_{A,V} = 1 + O(g^{6}) \tag{42}
\]

\[
C_{T,T'} = 1 + \lambda C_{F} \frac{g^{2}}{16 \pi^{2}} + \left[ (216 \lambda^{2} + 4320 \zeta(3) - 4815) C_{F} + 626 N_{F} \\
+ (162 \lambda^{2} + 756 \lambda - 3024 \zeta(3) + 5987) N_{c} \right] \frac{C_{F}}{216} \left( \frac{g^{2}}{16 \pi^{2}} \right)^{2} \tag{43}
\]

The variables $g$, $\lambda$ correspond to the RI’ scheme coupling constant and covariant gauge parameter (defined in Ref. [19]); in the Landau gauge, $\lambda = 0$. $\zeta(n)$ is the Riemann zeta function. The coupling constant, $g$, is related to the bare coupling, $g_{0}$, and up to $O(g^{6})$ the relation takes the form

\[
\frac{g^{2}}{4 \pi} = \frac{g_{0}^{2}}{4 \pi} + d_{1}(a_{\mu}) \left( \frac{g_{0}^{2}}{4 \pi} \right)^{2} + d_{2}(a_{\mu}) \left( \frac{g_{0}^{2}}{4 \pi} \right)^{3}. \tag{44}
\]

The coefficients $d_{1}$ and $d_{2}$ depend on the renormalization scale $a_{\mu}$ and are given by [20]:

\[
d_{1}(a_{\mu}) = - \frac{1}{2 \pi} \left( \frac{11}{3} N_{c} - \frac{2}{3} N_{F} \right) \ln(a_{\mu}) - \frac{\pi}{2 N_{c}} + 2.13573007 N_{c} - 0.08414443(8) N_{F},
\]

\[
d_{2}(a_{\mu}) = d_{1}(a_{\mu})^{2} - \frac{1}{24 \pi^{2}} \left[ 34 N_{c}^{2} - N_{F} \left( 13 N_{c} - \frac{3}{N_{c}} \right) \right] \ln(a_{\mu})
\]

\[
+ \frac{3 \pi^{2}}{8 N_{c}^{2}} - 2.8626216 + 1.2491158 N_{c}^{2} + N_{F} \left[ 0.18898(22) \frac{1}{N_{c}} - 0.15789(26) N_{c} \right].
\]

B. Evolution to a reference scale

All our $Z$-factors have been evaluated for a range of renormalization scales. In this subsection we use 3-loop perturbative expressions to extrapolate to a scale $\mu = 2$ GeV. Thus, each result is extrapolated to 2 GeV, maintaining information of the initial renormalization scale at which it was computed.

The scale dependence is predicted by the renormalization group equation (at fixed bare parameters), that is [21]

\[
Z_{C}^{\overline{\text{MS}}} (\mu) = R_{C}(\mu, \mu_{0}) Z_{C}^{\overline{\text{MS}}} (\mu_{0}) \tag{45}
\]
where
\[
R_{\mathcal{O}}(\mu, \mu_0) = \exp \frac{F(\bar{g}^2(\mu^2))}{\exp F(\bar{g}^2(\mu_0^2))}
\] (46)

with
\[
F(x) = \frac{\gamma_0}{2\beta_0} \ln(x) + \frac{\beta_0 \gamma_2 - \beta_2 \gamma_0}{4\beta_0 \beta_2} \ln((\beta_0 + \beta_1 x + \beta_2 x^2)) + \frac{2\beta_0 \beta_2 \gamma_1 - \beta_1 \beta_2 \gamma_0 - \beta_0 \beta_1 \gamma_2}{2\beta_0 \beta_2 \sqrt{4\beta_0 \beta_2 - \beta_1^2}} \arctan \left( \frac{\beta_1 + 2\beta_2 x}{\sqrt{4\beta_0 \beta_2 - \beta_1^2}} \right).
\] (47)

To 3 loops, the running coupling, $\beta$-function and anomalous dimension $\gamma$ are as follows [21–25], for $N_c = 3$:

\[
\frac{g^2(\mu^2)}{16\pi^2} = \frac{1}{\beta_0 \ln(\mu^2/\Lambda^2)} - \frac{\beta_1 \ln(\ln(\mu^2/\Lambda^2))}{\beta_0^2 \ln^2(\mu^2/\Lambda^2)} + \frac{1}{\beta_0^3 \ln^3(\mu^2/\Lambda^2)} \left( \beta_1^2 \ln^2(\ln(\mu^2/\Lambda^2)) - \beta_1 \ln(\ln(\mu^2/\Lambda^2)) \right) + \cdots
\] (48)

\[
\beta_0 = 11 - \frac{2}{3} N_F
\] (49)

\[
\beta_1 = 102 - \frac{38}{3} N_F
\] (50)

\[
\beta_2 = \frac{2857}{2} - \frac{5033 N_F}{18} + \frac{325 N_F^2}{54}
\] (51)

\[
\gamma_0^q = 0
\] (52)

\[
\gamma_1^q = -2 \left( \frac{67}{3} - \frac{4}{3} N_F \right)
\] (53)

\[
\gamma_2^q = -2 \left( \frac{20729}{36} - \frac{79}{2} \zeta(3) - \frac{550}{9} N_F + \frac{20}{27} N_F^2 \right)
\] (54)

\[
\gamma_0^{S,P} = -2 \frac{3 C_F}{4}
\] (55)

\[
\gamma_1^{S,P} = -2 \left( \frac{202}{3} - \frac{20}{9} N_F \right)
\] (56)

\[
\gamma_2^{S,P} = -2498 + \left( \frac{4432}{27} + \frac{320}{3} \zeta(3) \right) N_F + \frac{280}{81} N_F^2
\] (57)

\[
\gamma_0^{V,A} = \gamma_1^{V,A} = \gamma_2^{V,A} = 0
\] (58)

\[
\gamma_0^{T,T'} = \frac{8}{3}
\] (59)

\[
\gamma_1^{T,T'} = -\frac{4}{27} (26 N_F - 543)
\] (60)

\[
\gamma_2^{T,T'} = -\frac{2}{81} (36 N_F^2 + 1440 \zeta(3) N_F + 5240 N_F + 2784 \zeta(3) - 52555)
\] (61)
Eqs. (52) - (61) differ by numerical factors compared to Refs. [21–25] due to alternative definitions of the factor $R_O(\mu, \mu_0)$.

VII. NON-PERTURBATIVE CALCULATION

In the literature there are two main approaches that have been employed for the non-perturbative evaluation of the renormalization constants. They both start by considering that the operators can all be written in the form

$$O(z) = \sum_{z'} \overline{\pi}(z) J(z, z') d(z'),$$

where $u$ and $d$ denote quark fields in the physical basis and $J$ denotes the operator we are interested in, e.g. $J(z, z') = \delta_{z,z'} \gamma_{\mu}$ would correspond to the local vector current. For each operator we define a bare vertex function given by

$$G(p) = \frac{a^{12}}{V} \sum_{x,y,z,z'} e^{-ip(x-y)} \langle u(x)\overline{\pi}(y) J(z, z') d(z')d(y) \rangle,$$

where $p$ is a momentum allowed by the boundary conditions, $V$ is the lattice volume, and the gauge average, denoted by the brackets, is performed over gauge-fixed configurations. We have suppressed the Dirac and color indices of $G(p)$.

The first approach relies on translation invariance to shift the coordinates of the correlators in Eq. (63) to position $z = 0$ [26–28]. Having shifted to $z = 0$, one can calculate the amputated vertex function for a given operator $J$ for any momentum with one inversion per quark flavor.

In this work we explore the second approach, introduced in Ref. [21], which uses directly Eq. (63) without employing translation invariance. One must now use a source that is momentum dependent but can couple to any operator. For twisted mass fermions, we use the symmetry $S^u(x,y) = \gamma_5 S^d(y, x) \gamma_5$ between the $u$- and $d$-quark propagators. Therefore with a single inversion one can extract the vertex function for a single momentum. The advantage of this approach is a high statistical accuracy and the evaluation of the vertex for any operator including extended operators at no significant additional computational cost. Since we are interested in a number of operators with their associated renormalization constants we use the second approach. We fix to Landau gauge using a stochastic over-relaxation algorithm [29], converging to a gauge transformation which minimizes the functional

$$F = \sum_{x,\mu} \text{Re Tr} \left[ U_\mu(x) + U_\mu^\dagger(x-\hat{\mu}) \right].$$

Questions related to the Gribov ambiguity will not be addressed in this work. The propagator in momentum space, in the physical basis, is defined by

$$S^u(p) = \frac{a^8}{V} \sum_{x,y} e^{-ip(x-y)} \langle u(x)\overline{\pi}(y) \rangle, \quad S^d(p) = \frac{a^8}{V} \sum_{x,y} e^{-ip(x-y)} \langle d(x)\overline{d}(y) \rangle.$$

An amputated vertex function is given by

$$\Gamma(p) = (S^u(p))^{-1} G(p) (S^d(p))^{-1}.$$

and the corresponding renormalized quantities are assigned the values

$$S_{R}(p) = Z_q S(p), \quad \Gamma_{R}(p) = Z_1^{-1} Z_0 \Gamma(p).$$

In the twisted basis at maximal twist, Eq. (63) takes the form

$$G(p) = \frac{a^{12}}{4V} \sum_{x,y,z,z'} e^{-ip(x-y)} \langle (1 + i\gamma_5)u(x)\overline{\pi}(y) (1 + i\gamma_5) J(z, z') (1 - i\gamma_5) d(z') \overline{d}(y) (1 - i\gamma_5) \rangle.$$

After integration over the fermion fields, and using $S^u(x, z) = \gamma_5 S^d(z, x) \gamma_5$ this becomes

$$G(p) = -\frac{a^{12}}{4V} \sum_{x,z'} \left( \langle 1 - i\gamma_5 \rangle S^d(z, p) (1 - i\gamma_5) J(z, z') (1 - i\gamma_5) S^d(z', p) (1 - i\gamma_5) \right)_G,$$
where \( \langle \ldots \rangle^G \) denotes the integration over gluon fields, and \( \bar{S}(z,p) = \sum_y e^{ipy}S(z,y) \) is the Fourier transformed propagator on one of its argument on a particular gauge background. It can be obtained by inversion using the Fourier source

\[
b^a_\alpha(x) = e^{ipx} \delta_\alpha \delta_{ab},
\]

for all Dirac \( \alpha \) and color \( a \) indices. The propagators in the physical basis given in Eq. (65) can be obtained from

\[
S^d(p) = \frac{1}{4} \sum_z e^{-ipz} \langle (1 - i\gamma_5)\bar{S}(z,p)(1 - i\gamma_5) \rangle^G,
\]

\[
S^u(p) = \frac{1}{4} \sum_z e^{ipz} \langle (1 - i\gamma_5)\bar{S}^+(z,p)(1 - i\gamma_5) \rangle^G,
\]

which evidently only need 12 inversions despite the occurrence of both \( u \) and \( d \) quarks in the original expression.

We evaluate Eq. (68) and Eq. (71) for each momentum separately employing Fourier sources over a range of \( a^2 p^2 \) for which perturbative results can be trusted and finite \( a \) corrections are reasonably small.

### VIII. NON-PERTURBATIVE RESULTS

We perform the non-perturbative calculation of renormalization constants for three values of the lattice spacing, corresponding to \( \beta = 3.9, 4.05 \) and \( 4.20 \) [30]. In this work we use the lattice spacing as determined from the nucleon mass. The values we obtained are \( 0.089(1)(5) \) fm, \( 0.070(1)(4) \) fm and \( 0.056(2)(3) \) fm for \( \beta = 3.9, 4.05 \) and \( 4.20 \), respectively [31] and they are in agreement with the ones determined from the pion sector. To extract the renormalization constants reliably one needs to consider momenta in the range \( \Lambda_{QCD} < p < 1/a \). We relax the upper bound to be \( \sim 2/a \) to \( 5/a \), which is justified by the linear dependence of our results on \( a^2 \). Therefore, we consider momenta spanning the range \( 0.5 < a^2 p^2 < 5 \) for which perturbation theory is trustworthy and lattice artifacts are still small enough. It is important to note that the extrapolation to the continuum limit, \( (a p)^2 \rightarrow 0 \), is performed for a fixed momentum range in physical units. In Table III we summarize the various parameters of the action, that we used in our simulations, and in Table IV we present the values we used for the momenta \( (p_t, p_x, p_y, p_z) = (2\pi/L_t n_t, 2\pi/L_x n_x, 2\pi/L_y n_y, 2\pi/L_z n_z) \).

| \( \beta \) (fm) | \( \alpha \mu_0 \) | \( m_\pi \) (GeV) | \( L^3 \times T \) |
|-----------------|-------------------|------------------|-----------------|
| 3.9             | 0.089             | 0.0040           | 0.3021(14)      | \( 24^3 \times 48 \) |
| 3.9             | 0.089             | 0.0064           | 0.37553(80)     | \( 24^3 \times 48 \) |
| 3.9             | 0.089             | 0.0085           | 0.4302(11)      | \( 24^3 \times 48 \) |
| 3.9             | 0.089             | 0.01             | 0.4675(12)      | \( 32^3 \times 64 \) |
| 4.05            | 0.070             | 0.003            | 0.2925(18)      | \( 32^3 \times 64 \) |
| 4.05            | 0.070             | 0.006            | 0.4082(31)      | \( 32^3 \times 64 \) |
| 4.05            | 0.070             | 0.006            | 0.404(2)        | \( 32^3 \times 64 \) |
| 4.05            | 0.070             | 0.008            | 0.465(1)        | \( 32^3 \times 64 \) |
| 4.20            | 0.056             | 0.002            | 0.2622(11)      | \( 32^3 \times 64 \) |
| 4.20            | 0.056             | 0.0065           | 0.476(2)        | \( 32^3 \times 64 \) |

**TABLE III:** \( \beta \)-values and lattice size used in the simulations are given in the first and last columns respectively. The lattice spacing \( a \) in fm is determined from the nucleon mass. We also give the bare light quark mass \( \alpha \mu_0 \) and pion mass.

The number of configuration in each ensemble varies between 10 to 100. Using even 10 configurations leads to results with very high statistical accuracy, easily below 0.5%. Thus, in the plots presented here the statistical errors are too small to be visible. We mostly use in our computation democratic momenta, in the sense that they have the same \( p_x, p_y, p_z \). We have also tested a few non-democratic momentum, which turn out to behave similarly to democratic ones (e.g. \( (n_t, n_x, n_y, n_z) = (3,3,3,2) \) is similar to \( (3,3,3,3) \)). We would like to point out that the non-perturbative results have a significant dependence on the value of the momentum in the spatial direction, indicating large lattice artifacts in some cases. Such a study appeared in Ref. [15].
\[ \beta = 3.9 \]

| \( n_t,2,2,2 \), \( n_t : 4 - 8, 10 - 14 \) | \( n_t,3,3,3 \), \( n_t : 2 - 6, 8 - 9 \) | \( n_t,4,4,4 \), \( n_t : 4 - 9 \) | \( 3,3,2 \) |
|---|---|---|---|
| \( \beta = 4.05 \) | \( \beta = 4.05 \) | \( \beta = 4.05 \) | \( \beta = 4.05 \) |
| \( n_t,2,2,2 \), \( n_t : 4 - 8, 10 - 14 \) | \( n_t,3,3,3 \), \( n_t : 2 - 6, 8 - 11, 13 \) | \( n_t,4,4,4 \), \( n_t : 8 - 10 \) | \( 3,3,2 \) |
| \( \beta = 4.20 \) | \( \beta = 4.20 \) | \( \beta = 4.20 \) | \( \beta = 4.20 \) |
| \( n_t,2,2,2 \), \( n_t : 4 - 8, 10 - 14 \) | \( n_t,3,3,3 \), \( n_t : 2 - 6, 8 - 11, 13 \) | \( n_t,4,4,4 \), \( n_t : 7 - 11 \) | \( 3,3,2 \) |

TABLE IV: Values of momentum used for the various ensembles at \( \beta = 3.9, 4.05, 4.20 \).

A. Pion mass dependence

In Table III we give the number of pion mass that we studied for each of the three \( \beta \) values. These ensembles have been produced by the ETM Collaboration [30, 32–34]. The pion mass dependence of \( Z_q, Z_V, Z_A, Z_T \) is displayed in Fig. 3 and is not significant. A linear extrapolation to the data shown in Fig. 3 yields a slope consistent with zero. This behavior is also observed at the other \( \beta \) values. Thus, it would be sufficient to obtain the results on the aforementioned RCs at one pion mass value, although we perform the chiral extrapolation with all available data on different pion masses. The need of having simulations at a number of pion masses comes from the fact that one has to perform the subtraction of the pion pole contribution. This is discussed in Subsection VIII C.

![Fig. 3: \( Z_q, Z_V, Z_A, Z_T \) at \( \beta = 3.9 \), as a function of the pion mass. Computations were performed at pion masses of \( m_\pi = 0.302 \text{ GeV} (\mu_0 = 0.004) \), \( m_\pi = 0.375 \text{ GeV} (\mu_0 = 0.0064) \), \( m_\pi = 0.429 \text{ GeV} (\mu_0 = 0.0085) \) and \( m_\pi = 0.468 \text{ GeV} (\mu_0 = 0.01) \).](attachment:fig3.png)

B. Volume dependence

We perform the evaluation of the RCs at \( \beta = 4.05 \) and \( \mu = 0.006 \) for two volumes, \( 24^3 \times 48 \) and \( 32^3 \times 64 \) in order to check for finite volume effects. For this comparison we used momenta that correspond to the same renormalization scale. For the small lattice we use \( a_p = 2\pi(3/48,3/24,3/24,3/24) \), in lattice units, whereas for the larger one we employ \( a_p = 2\pi(4/64,4/32,4/32,4/32) \). The volume effects appear to be in the worst case \( \sim 0.1\% \), as can be seen from Table V. We would like to point out that the \( Z_P \) estimator shows the largest volume dependence, which however tends to decrease after the pion pole subtraction (Subsection VIII C, Fig. 4).

C. Pion-pole subtraction

The correlation functions of the pseudoscalar operator have pion-pole contamination and therefore need to be treated carefully. In order to subtract the pole contribution we use the following Ansatz for the pseudoscalar amputated vertex
function, $\Lambda_P$,

$$\Lambda_P = a_P + b_P m^2_\pi + \frac{c_P}{m^2_\pi},$$

which we apply to data produced at a given value of $\beta$. Once we have the fitting parameters we subtract the pion-pole using the value of $c_P$, determined from the fitting, i.e. we take

$$\Lambda_{P,sub} = \Lambda_P - \frac{c_P}{m^2_\pi}.\quad (73)$$

To reliably obtain the three fitting parameters of Eq. (72) we need the RC of the pseudoscalar operator for at least 4 pion masses; this is feasible for $\beta = 3.9$. On the contrary, for $\beta = 4.05$ we have data for three pion masses, and for $\beta = 4.20$ only for two pion masses. At $\beta = 3.9$ we determine the parameters using results at three of the four pion masses $\beta = 3.9$ and compare them with the fit resulting when using all available data. The conclusion is that the values obtained are compatible. Therefore at $\beta = 4.05$ we determine the parameters using results at the three pion masses that are available. One may observe the effectiveness of the pion-pole subtraction in Fig. 4, where we show results before and after the pion pole subtraction. After the subtraction results obtained at different pion mass fall on each other. While the pion-pole term has an appreciable contribution, the quadratic term with the $b_P$ coefficient is expected to be small. The values extracted for $b_P$ at $\beta = 3.9$ and $\beta = 4.05$ are indeed small showing a very weak pion mass dependence of the $b_P$ coefficient. This is consistent with the weak pion mass dependence observed for the vector and axial-vector RCs (see Subsection VIII A for the other RCs). It is also verified by our data: after subtracting the pion-pole term determined from fitting to the data, the remaining pion mass dependence ($b_P m^2_\pi$) is negligible for all the ensembles. This allows us to perform a two parameter fit at $\beta = 4.2$ of the form:

$$\Lambda_P = a_P + \frac{c_P}{m^2_\pi},$$

using data on the two pion masses $m_\pi = 476$ and 262 MeV. The two sets correspond to different lattice size, $24^3 \times 48$ ($32^3 \times 64$) for the lower (higher) pion mass. As a result, momenta with the same values for $(n_t, n_x, n_y, n_z)$ correspond to different $(a_P)^2$. Thus, in order to perform the fit using the Ansatz of Eq. (74) we carefully choose the momenta in the two ensembles to have almost the same $(a_P)^2$. In general, this could lead to additional uncertainties, but we have already checked that volume effects are negligible. Indeed the fit using the Ansatz of Eq. (74) yields a value for $c_{P(S)}$ that accurately removes the pion pole as demonstrated in Fig. 5.

The errors shown in Figs. 4 - 5 are computed in two ways: using super jackknife error analysis [35, 36] and requiring that a correlated change of the fit parameters increases the minimum value of $\chi$ by one. We find that both methods lead to similar errors.

Our data for the ratio $Z_P/Z_S$ also show dependence on the pion mass, and thus to form this ratio we used the subtracted data of Fig. 4 which we compute in the chiral limit. This procedure leads to the values shown in Fig. 6. With black circles we show the non-perturbative results after subtracting the pion pole from $Z_P$ using Eq. (73). If one further subtracts from $Z_S$ and $Z_P$ the perturbative $O(a^2)$ contributions, presented in Sections III - IV, one obtains the values shown with the magenta diamonds in Fig. 6. The ratio $Z_P/Z_S$ is renormalization scale independent and therefore one can directly use the $O(a^2)$-perturbatively subtracted non-perturbative results to extrapolate to the continuum limit, eliminating any remaining cut-off effects.
In this section we present results in the RI’-MOM scheme for \( Z_q, Z_S, Z_P \), as well as for the scale-independent RCs \( Z_V \) and \( Z_A \). We have also performed a computation of \( Z_T \) and its results are presented in the next section. In all cases we subtract the leading discretization effects of \( \mathcal{O}(a^2) \) computed to one-loop in perturbation theory from the non-perturbative results. All renormalization constants are evaluated at the three \( \beta \) values, where the simulations were carried out. For all \( \beta \) values we perform a chiral extrapolation using results at different pion masses; the results have negligible dependence on the quark mass as demonstrated in Fig. 3 and therefore we use a constant fit to extrapolate to the chiral limit.

The renormalization constant of the fermion field is needed as an input in various expressions, and the results obtained are displayed in Fig. 7.

The non-perturbative values of \( Z_q \) are obtained for all available momenta and they reveal a non-smooth behavior as a function of the momentum (see Fig. 7), which becomes smoother once we subtract the \( \mathcal{O}(a^2 g^2) \) perturbative terms. We would like to point out that in all our non-perturbative results before subtraction we have some data that fluctuate several standard deviations around the mean value and these correspond to momenta that lead to large non-Lorentz invariant contributions in the perturbative expressions of Sections III-IV. These terms are of the form \( (\sum \rho p^4)/(\sum \rho p^2) \). After subtraction these non-Lorentz invariant contributions are removed (to \( \mathcal{O}(a^2 g^2) \)), resulting in the much smoother behavior of the subtracted data. The unsubtracted data of \( Z_q \) that suffer from large non-Lorentz
FIG. 6: $Z_P/Z_s$ at $\beta = 3.9$ (upper plot), $\beta = 4.05$ (middle plot) and $\beta = 4.20$ (lower plot) as a function of $(a\,p)^2$. In each plot we demonstrate the effect of subtracting the $O(a^2)$-terms by plotting the non-perturbative results before (black circles) and after (magenta diamonds) subtraction. The pion-pole term has been removed from all the data that we show here.

Invariant contributions are show in Fig. 7 with filled black circles. Note that as the lattice spacing decreases, the discrepancy between unsubtracted and subtracted data becomes smaller.

The RCs $Z_V$ and $Z_A$ are scale-independent and therefore there is no need to evolve them. In Fig. 8 we show results on $Z_V$ and $Z_A$ before and after subtraction of the perturbatively determined $O(a^2)$-terms. As can be seen, the subtraction weakens the dependence on $(a\,p)^2$. In fact, fitting the subtracted data to a straight line of the form $z + s(a\,p)^2$ results in a value of the slope $s$ consistent with zero. This shows that leading cut-off effects are effectively removed by the subtraction of perturbatively determined $O(a^2)$-terms. The small remaining lattice artifacts are removed by extrapolating linearly to the continuum line. The unsubtracted data can also be extrapolated linearly but, in this case, the slope is generally non-zero as can be seen in Fig. 8. As the lattice spacing decreases, the deviation between subtracted and unsubtracted data decreases. We note that, although the value found at the continuum limit for the unsubtracted data approaches that extracted for the subtracted data, small differences still remain. This is an indication that the systematic error due to cut-off effects is larger than the statistical error and therefore the subtraction of $O(a^2)$-terms helps in diminishing the uncertainty in the choice of the fit range.

In order to perform the continuum extrapolation we choose the same momentum range in physical units for all $\beta$ values and we thus extract all renormalization constants using the same physical momentum range, $p^2 \sim 15 - 32$ (GeV)$^2$. This momentum range is in line with what has been chosen in our previous work on the RCs for one-derivative
FIG. 7: Non-perturbative results on $Z_q$ for $\beta = 3.9$ (upper left plot), $\beta = 4.05$ (upper right plot) and $\beta = 4.20$ (lower plot). In all plots we show chirally extrapolated results. Black circles (magenta diamonds) represent the non-perturbative data before (after) subtracting the $O(a^2)$-terms. The unsubtracted data that suffer from large non-Lorentz invariant contributions are show with filled black circles.

bilinear operators [15], ensuring that we use data in a region where an approximate plateau exists. The momentum range in lattice units at each $\beta$-value is as follows: $\beta = 3.9$: $(ap)^2 = 3 − 5$, $\beta = 4.05$: $(ap)^2 = 1.8 − 3.1$, $\beta = 4.20$: $(ap)^2 = 1.2 − 2.5$. The choice for the momentum range is not so relevant for $Z_V$ and $Z_A$ as it was for the case of the RCs for one-derivative operators, since the subtracted data are almost constant. However, for consistency we use the same range as in Ref. [15].

E. MS scheme

In this Section we present our results on $Z_q$, $Z_S$, $Z_P$ and $Z_T$ converted to the continuum MS scheme and at a reference scale of $\mu = 2$ GeV. For the conversion from RI’-MOM to MS we use the formulae given in Eqs. (40) - (43). We use the 3-loop formulae of Eqs. (45) - (46) to evolve the scale from $\mu$ to 2 GeV.

As already discussed, a “renormalization window” should exist for $\Lambda_{QCD}^2 << \mu^2 << 1/\alpha^2$ where perturbation theory holds and finite $\alpha$ artifacts are small, leading to scale-independent results (plateau). In practice such a condition is hard to satisfy. The right inequality is extended to $(2 - 5)/\alpha^2$ leading to lattice artifacts in our results that are of $O(a^2p^2)$. Fortunately our perturbative calculations allow us to subtract the leading perturbative $O(a^2)$ lattice artifacts which alleviates the problem. To remove the remaining $O(a^2p^2)$ artifacts we extrapolate linearly to $(ap)^2 = 0$ as demonstrated in Figs. 8 -11. The statistical errors are negligible, however an estimate of the systematic errors is important. The largest systematic error comes from the choice of the momentum range to use for the extrapolation to $(ap)^2 = 0$. One way to estimate this systematic error is to vary the momentum range where we perform the fit. Another approach is to fix a range and then eliminate a given momentum in the fit range and refit. The spread of the results about the mean gives an estimate of the systematic error. In the final results we give as systematic the largest of the two, which is the one obtained by modifying the fit range. As already mentioned we choose the same momentum range in physical units for the three $\beta$-values and extract all renormalization constants using the same physical momentum range, $p^2 \sim 15 - 32$ (GeV)$^2$; within this range the data fall on a straight line of a small slope. We note that the $O(a^2)$ perturbative terms which we subtract, tend to decrease with increasing $\beta$, as expected. The error bars in Fig. 10 are due to the fit uncertainties in performing the pion pole subtraction.
FIG. 8: The renormalization constants for the vector and axial-vector operators, $Z_V$ and $Z_A$, for $\beta = 3.9$ (upper left plot), $\beta = 4.05$ (upper right plot) and $\beta = 4.20$ (lower plot). In all plots we show chirally extrapolated results. Black circles (magenta diamonds) represent the non-perturbative data before after subtracting the $O(a^2)$-terms.

FIG. 9: Results on $Z_q$ for $\beta = 3.9$ (upper left plot), $\beta = 4.05$ (upper right plot) and $\beta = 4.20$ (lower plot). In all plots we show chirally extrapolated results. Black circles (magenta diamonds) represent the non-perturbative data before (after) subtracting the $O(a^2)$-terms. The corresponding dashed lines show the extrapolation to the continuum limit and the filled diamond shows the final value in the continuum. Statistical errors are smaller than the size of the symbols.
Our final results for the $Z$-factors in the $\overline{\text{MS}}$-scheme at $\mu = 2$ GeV are given in Table VI. As pointed out, these are obtained in the continuum limit by extrapolating linearly in $(ap)^2$ using data in a fixed momentum range $p^2 \approx 15 - 32$ (GeV)$^2$. The continuum extrapolation was carried out at the chiral limit. The systematic error due to the continuum extrapolation, is estimated from the difference between results using the fit range $p^2 \approx 15 - 32$ (GeV)$^2$ and the range $p^2 \approx 17 - 24$ (GeV)$^2$. The results at $\beta = 3.9$ and $\beta = 4.05$ agree within error bars with the results of Ref. [26]. Since their evaluation procedure differs, this agreement provides a nice confirmation of the values obtained. In Ref. [26] the vertex computation employs translation invariance to evaluate the correlation functions for all values of the momentum, whereas we calculate the vertex for a given momentum dependent source, leading to smaller statistical errors. More importantly, the two procedures differ in the analysis of the lattice data, both in the way the chiral extrapolation of the renormalization constants at fixed $p^2$ is carried out as well as in the way the systematic errors...
associated with the extrapolation $p^2 \to 0$ are estimated. Due to the different approaches used, the statistical and systematic errors between the two computations is somewhat different. In this work, we additionally compute the renormalization constants at $\beta = 4.2$ but not at $\beta = 3.8$, which were included in Ref. [26]. Given the consistency between our values of the renormalization constant and those of Ref. [26] at $\beta = 3.9$ and $\beta = 4.05$ consolidates our value at $\beta = 4.20^3$.

| $\beta$ | $Z_q$       | $Z_S$       | $Z_P$       | $Z_P/Z_S$ | $Z_V$       | $Z_A$       | $Z_T$       |
|---------|-------------|-------------|-------------|-----------|-------------|-------------|-------------|
| 3.90    | 0.754(9)(9) | 0.726(5)(11)| 0.457(10)(16)| 0.639(3)(1)| 0.627(1)(3) | 0.758(1)(1) | 0.750(9)(10) |
| 4.05    | 0.775(4)(5) | 0.691(9)(16)| 0.497(8)(15)| 0.682(2)(1)| 0.662(1)(3) | 0.773(1)(1) | 0.798(7)(8)  |
| 4.20    | 0.798(4)(9) | 0.695(10)(13)| 0.501(8)(10)| 0.713(2)(2)| 0.686(1)(1) | 0.789(1)(2) | 0.822(4)(6)  |

TABLE VI: Final results of the renormalization constants $Z_q$, $Z_S$, $Z_P$, $Z_T$ in the $\overline{\text{MS}}$ scheme, as well as for the scale-independent $Z_P/Z_S$, $Z_V$ and $Z_A$. Statistical errors are shown in the first parenthesis. The number in the second parenthesis is the systematic error due to the continuum extrapolation, taken as the difference between results using the fit range $p^2 \approx 15 - 32 \,(\text{GeV})^2$ and the range $p^2 \approx 17 - 24 \,(\text{GeV})^2$.

IX. CONCLUSIONS

The values of the renormalization factor for the fermion field $Z_q$, and for the scalar, pseudoscalar, vector, axial-vector and tensor local operators, $Z_S$, $Z_P$, $Z_V$, $Z_A$, $Z_T$, have been calculated non-perturbatively. The method of choice is to use a momentum dependent source and extract the renormalization factors for all the relevant operators. This leads to a very accurate evaluation of these factors using a small ensemble of gauge configurations. The precision of the results allows us to reliably investigate the light quark mass dependence. For most of the renormalization constants studied in this work we do not find any light quark mass dependence within our small statistical errors. For all $\beta$ values we obtain the value at the chiral limit by fitting the data to a constant. For the RC of the pseudoscalar operator, $Z_P$, we find a quark mass dependence due to the pion-pole, which we subtract. Once the pole is subtracted, the behavior of the data show a weak dependence on the light quark mass and therefore we again compute the value at the chiral limit by fitting the pion pole subtracted data to a constant. We also show that, despite using a lattice spacing smaller than 0.1 fm, $O(a^2 p^2)$ cut-off effects are visible given the high precision with which the RCs are calculated. Thus we perform a perturbative subtraction of $O(a^2 g^2)$ terms. This leads to a milder dependence of the renormalization constants on $(a p)^2$. Residual $O(a^2 p^2)$ effects are removed by extrapolating to zero. In this way we can accurately determine the renormalization constants in the $\text{RI}'$-MOM scheme. In order to compare with experiment we convert our values to the $\overline{\text{MS}}$ scheme at a scale of 2 GeV. The statistical errors are in general smaller than the systematic ones. The latter are estimated by changing the window of values of the momentum used to extrapolate to $a^2 p^2 = 0$. Our final values are given in Table VI.

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Appendix A: Strong IR divergent integrals

The integrals, with strong IR divergences (convergent only beyond $D > 6$), encountered in the present calculation are listed below with their results. For completeness we also include the integrals that appeared in our related publication [15] for the matrix elements of twist-2 operators. All these integrals can be found in electronic form in the Mathematica file Zfactors.m, with the names: IntegralPropagator1 - IntegralPropagator3, IntegralBilinears1 - IntegralBilinears6, and IntegralExtendedBilinears1 - IntegralExtendedBilinears2. To avoid heavy notation we define:

\[
M^2 = (m_0)^2 + \mu^2, \quad M_2^2 = (m_0')^2 + \mu'^2,
\]

\[
p_2 = \sum_\rho p_\rho^2, \quad p_4 = \sum_\rho p_\rho^4,
\]

\[
\hat{q}_\nu = 2 \sin\left(\frac{q_\nu}{2}\right), \quad \check{q}^2 = 4 \sum_\rho \sin^2\left(\frac{q_\nu}{2}\right),
\]

where $q$ stands for $k$ or $k + ap$, while $k$ is the loop momentum and $p$ is the external momentum. No summation over the indices $\nu_i$ is implied.

**Propagator**

\[
\bullet \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \frac{1}{\vec{k}^2 \left( k + \frac{1}{2} a p + a^2 M^2 \right)} = \text{IntegralPropagator1} + \mathcal{O}(a^4) \quad (A1)
\]

\[
= 0.03667832907475711(1) - \frac{\ln[a^2 M^2 + a^2 p^2]}{16\pi^2} - \frac{M^2 \ln[1 + \frac{a^2}{16\pi^2}]}{16\pi^2 a^2} + a^2 \left(0.00007524033(9)p_2
\right)
\]

\[- 0.00396328514(4)M^2 + \frac{M^2 \ln[a^2 M^2 + a^2 p^2]}{128\pi^2} - \frac{M^4}{128\pi^2 p_2} + \left(\frac{1}{64\pi^2} + \frac{M^2}{128\pi^2 p_2}\right) \frac{M^4 \ln[1 + \frac{a^2}{16\pi^2}]}{p_2}
\]

\[+ \frac{p_4}{p_2} \left(\frac{1}{384\pi^2} + \frac{M^4}{128\pi^2 p_2} + \frac{M^4}{64\pi^2 p_2} - \left(\frac{1}{192\pi^2} + \frac{M^2}{64\pi^2 p_2} + \frac{M^4}{64\pi^2 p_2^2}\right) \frac{M^4 \ln[1 + \frac{a^2}{16\pi^2}]}{p_2}\right) + \mathcal{O}(a^4)
\]

\[
\bullet \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \frac{\sin k_{\nu_1}}{\vec{k}^2 \left( k + \frac{1}{2} a p + a^2 M^2 \right)} = \text{IntegralPropagator2} + \mathcal{O}(a^3) \quad (A2)
\]

\[
= a p_{\nu_1} \left(-0.008655827647937295(1) + \frac{\ln[a^2 M^2 + a^2 p^2]}{32\pi^2} - \frac{M^2}{32\pi^2 p_2} + \left(\frac{1}{16\pi^2} + \frac{M^2}{32\pi^2 p_2}\right) \frac{M^2 \ln[1 + \frac{a^2}{16\pi^2}]}{p_2}\right)
\]

\[+ a^3 \left(\frac{p_{\nu_1} p_4}{p_2^2} \left(-\frac{1}{768\pi^2} - \frac{M^2}{96\pi^2 p_2} - \frac{3M^4}{128\pi^2 p_2^2} - \frac{M^6}{64\pi^2 p_2^4}
\right)
\]

\[+ \left(\frac{1}{192\pi^2} + \frac{M^2}{48\pi^2 p_2} + \frac{M^4}{32\pi^2 p_2^2} + \frac{M^6}{64\pi^2 p_2^4}\right) \frac{M^2 \ln[1 + \frac{a^2}{16\pi^2}]}{p_2}\right)
\]

\[+ \frac{p_{\nu_1}}{p_2} \left(-0.0005107794(2)p_2 + \frac{p_2 \ln[a^2 M^2 + a^2 p^2]}{768\pi^2} - 0.00028240872(9)M^2 + \frac{13M^4}{1536\pi^2 p_2} + \frac{M^6}{256\pi^2 p_2^2}
\right)
\]

\[+ \left(\frac{1}{128\pi^2} + \frac{M^2}{96\pi^2 p_2} + \frac{M^4}{256\pi^2 p_2^2}\right) \frac{M^4 \ln[1 + \frac{a^2}{16\pi^2}]}{p_2}\right)
\]

\[+ 5\frac{M^2}{384\pi^2 p_2} + \frac{13M^4}{768\pi^2 p_2^2} + \frac{M^6}{128\pi^2 p_2^4} - \left(\frac{1}{96\pi^2} + \frac{M^2}{48\pi^2 p_2} + \frac{M^4}{48\pi^2 p_2^2} + \frac{M^6}{128\pi^2 p_2^4}\right) \frac{M^2 \ln[1 + \frac{a^2}{16\pi^2}]}{p_2}\right) + \mathcal{O}(a^3)
\]
\[
\int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \frac{\sin k_{\nu_1} \sin k_{\nu_2}}{(k^2 + a p^2 + a^2 M^2)^2} = \text{IntegralPropagator} + \mathcal{O}(a^4) \\
= \delta_{\nu_1 \nu_2} \left( 0.004327913823968469(1) - \frac{\ln[a^2 M^2 + a^2 p^2] + M^2}{64\pi^2} - \frac{M^4}{64\pi^2 p^2} + \frac{M^4 \ln[1 + \frac{p^2}{M^2}]}{64\pi^2 p^2} \right) \\
+ \frac{p_{\nu_1} p_{\nu_2}}{p^2} \left( \frac{1}{16\pi^2} + \frac{M^2}{16\pi^2 p^2} - \left( \frac{1}{16\pi^2} + \frac{M^2}{16\pi^2 p^2} \right) \frac{M^2 \ln[1 + \frac{p^2}{M^2}]}{p^2} \right) \\
+ a^2 \left( \delta_{\nu_1 \nu_2} \left( 0.00025539124(4)p^2 - \frac{p^2 \ln[a^2 M^2 + a^2 p^2]}{1536\pi^2} + 0.000103584432(2) M^2 + \frac{5M^4}{3072\pi^2 p^2} + \frac{M^6}{512\pi^2 p^2} \right) - \frac{23M^4}{1536\pi^2 p^2} - \frac{3M^6}{256\pi^2 p^2} + \left( \frac{1}{128\pi^2} + \frac{M^2}{48\pi^2 p^2} + \frac{3M^4}{256\pi^2 p^2} \right) \frac{M^4 \ln[1 + \frac{p^2}{M^2}]}{p^2} \right) \\
+ \delta_{\nu_1 \nu_2} p^4 \left( -0.00013565411323666763(1) + \frac{\ln[a^2 M^2 + a^2 p^2]}{768\pi^2} + \frac{5M^2}{768\pi^2 p^2} + \frac{31M^4}{1536\pi^2 p^2} + \frac{3M^6}{256\pi^2 p^2} \right) - \frac{5M^2}{192\pi^2 p^2} + \frac{3M^4}{256\pi^2 p^2} \right) \frac{M^4 \ln[1 + \frac{p^2}{M^2}]}{p^2} \right) \\
+ \delta_{\nu_1 \nu_2} \frac{p^4}{p^2} \left( \frac{1}{1536\pi^2} - \frac{M^2}{768\pi^2 p^2} - \frac{M^4}{256\pi^2 p^2} + \frac{M^6}{128\pi^2 p^2} + \left( \frac{1}{384\pi^2} + \frac{M^2}{128\pi^2 p^2} + \frac{M^4}{128\pi^2 p^2} \right) \frac{M^4 \ln[1 + \frac{p^2}{M^2}]}{p^2} \right) \\
+ \frac{p_{\nu_1} p_{\nu_2} p^4}{p^2} \left( \frac{1}{768\pi^2} + \frac{M^2}{192\pi^2 p^2} + \frac{11M^4}{128\pi^2 p^2} + \frac{5M^6}{64\pi^2 p^2} \right) - \left( \frac{1}{96\pi^2} + \frac{M^2}{16\pi^2 p^2} + \frac{M^4}{8\pi^2 p^2} + \frac{5M^6}{64\pi^2 p^2} \right) \frac{M^4 \ln[1 + \frac{p^2}{M^2}]}{p^2} \right) \right) + \mathcal{O}(a^4)
\]

**Bilinears**

\[
\int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \frac{k}{k^2 + a p^2 + a^2 M^2} = \text{IntegralBilinears} + \mathcal{O}(a^2) \\
= 0.0039632853(1) - \frac{p^4}{128\pi^2 p^2} + \sum_{j=1}^{2} \left\{ (-1)^{j+1} \frac{\ln[a^2 M^2 + a^2 p^2]}{16\pi^2 (M^2 - M_j^2)} + \frac{M_j^2 \ln[1 + \frac{p^2}{M_j^2}]}{16\pi^2 p^2 (M^2 - M_j^2)} + \frac{M_j^2}{128\pi^2 p^2} \right\} \\
+ \left\{ (-1)^{j+1} \frac{M_j^2 \ln[1 + \frac{p^2}{M_j^2}]}{128\pi^2 p^2} + (-1)^{j+1} \left( \frac{1}{64\pi^2} + \frac{M_j^2}{128\pi^2 p^2} \right) \frac{M_j^2 \ln[1 + \frac{p^2}{M_j^2}]}{p^2 (M^2 - M_j^2)} \right\} + \mathcal{O}(a^2)
\]
\[ \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \frac{\sin(k_{\nu_1} + a p_{\nu_1})}{k^2 (k + a p + a^2 M^2)^2} = \text{IntegralBilinears2} + O(a^3) \] (A5)

\[ = \frac{1}{a} p_{\nu_1} + \frac{p_{\nu_1}^4}{p^2} \left( \frac{1}{384\pi^2} + \frac{M^2 M_2^2}{64\pi^2 p^2} \right) - p_{\nu_1} \left( \frac{0.0002071688(1) + M^2 M_2^2}{256\pi^2 p^2} \right) - \frac{p_{\nu_1}^3}{p^2} \left( \frac{1}{384\pi^2} + \frac{M^2 M_2^2}{128\pi^2 p^2} \right) \]

\[ + \sum_{j=1}^{2} \left\{ (-1)^{j+1} p_{\nu_1} \left( \frac{\ln[a^2 M_1^2 + a^2 p^2]}{32\pi^2 (M^2 - M_1^2)} - \frac{M_j^4 \ln[1 + \frac{p^2}{M_j^2}]}{32\pi^2 p^2 (M^2 - M_1^2)} \right) \right. \]

\[ + \left. a \left( \frac{p_{\nu_1}^4}{p^2} \left( \frac{M_j^4}{128\pi^2 p^2} + \frac{M_j^4}{64\pi^2 p^2} \right) - (-1)^j \left( \frac{1}{192\pi^2} + \frac{M_j^4}{64\pi^2 p^2} + \frac{M_j^4}{4\pi^2 p^2} \right) \right) \right. \]

\[ \left. + p_{\nu_1} \left( \frac{5M_j^4}{1536\pi^2 p^2} \right) \right) \}

\[ + O(a^3) \]

\[ \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \frac{\sin(k_{\nu_1} + a p_{\nu_1}) \sin(k_{\nu_2} + a p_{\nu_2})}{k^2 (k + a p + a^2 M^2)^2} = \text{IntegralBilinears3} + O(a^4) \] (A6)

\[ \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \frac{\sin k_{\nu_1} \sin k_{\nu_2}}{k^2 (k + a p + a^2 M^2)^2} = \text{IntegralBilinears4} + O(a^2) \] (A7)

\[ \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \frac{\sin k_{\nu_1} \sin k_{\nu_2}}{k^2 (k + a p + a^2 M^2)^2} = \text{IntegralBilinears5} + O(a^3) \] (A8)

\[ \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \frac{\sin k_{\nu_3} \sin k_{\nu_4} (k_{\nu_3} + a p_{\nu_3}) \sin (k_{\nu_4} + a p_{\nu_4})}{k^2 (k + a p + a^2 M^2)^2} = \text{IntegralBilinears6} + O(a^3) \] (A9)

**Extended Bilinears**

\[ \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \frac{\sin (k_{\nu_1} + a p_{\nu_1}) \sin (k_{\nu_2} + a p_{\nu_2})}{k^2 (k + a p + a^2 M^2)^2} = \text{IntegralExtendedBilinears1} + O(a^5) \] (A10)

\[ \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \frac{\sin (k_{\nu_1} + a p_{\nu_1}) \sin (k_{\nu_2} + a p_{\nu_2}) \sin (k_{\nu_3} + a p_{\nu_3}) \sin (k_{\nu_4} + a p_{\nu_4})}{k^2 (k + a p + a^2 M^2)^2} = \text{IntegralExtendedBilinears2} + O(a^5) \] (A11)
Appendix B: Analytic expressions for RCs of bilinear operators

In this Appendix we provide the analytic expressions for the RCs of the ultra-local bilinears, as defined in Eq. (33):

\[ Z_{S}^{G_{\text{pert}}} = 1 \]

\[ + g^2 \left\{ -13.606731(1) + 3 \ln[a^2 m^2 + a^2 p^2] + \frac{9 m^2 \ln[1 + \frac{m^2}{ap^2}]}{p^2} \right\} \]

\[ + 2.7312983(2) - \frac{15}{2} \ln[a^2 m^2 + a^2 p^2] + \frac{3 m^2}{2 p^2} + \frac{6 m^2}{m^2 + p^2} \left( 6 + \frac{3 m^2}{2 p^2} \right) \left( \frac{m^2 \ln[1 + \frac{m^2}{ap^2}]}{p^2} \right) \] \]

\[ + 10.853390(2) m^2 - \frac{1289 m^4}{360 p^2} + \frac{721 m^6}{240 p^2} + \frac{7 m^8}{40 p^2} - \frac{18 m^4}{m^2 + p^2} + \frac{3 m^6}{(m^2 + p^2)^2} + \left( \frac{107 m^2}{6} + \frac{17 p^2}{360} \right) \ln[a^2 m^2 + a^2 p^2] \]

\[ + \left( -1 + \frac{1321 m^2}{24 p^2} + \frac{367 m^4}{72 p^2} + \frac{35 m^6}{12 p^2} + \frac{7 m^8}{40 p^2} \right) m^2 \ln[1 + \frac{p^2}{m^2}] + \frac{p^4}{p^2} \left( -0.3935023(2) - \frac{157 \ln[a^2 m^2 + a^2 p^2]}{180} \right) \] \]

\[ + O(a^3, g^4) \]

\[ Z_{\rho}^{G_{\text{pert}}} = 1 \]

\[ + g^2 \left\{ -21.733356(1) + 3 \ln[a^2 m^2 + a^2 p^2] + \frac{3 m^2 \ln[1 + \frac{m^2}{ap^2}]}{p^2} \right\} \]

\[ + 7.0252230(2) - \frac{3}{2} \ln[a^2 m^2 + a^2 p^2] + \frac{3 m^2}{2 p^2} - \left( 6 + \frac{3 m^2}{2 p^2} \right) \ln[1 + \frac{m^2}{ap^2}] \]

\[ + a^2 \left[ 0.440762(2) m^2 - 5.520750(2) m^2 - \frac{1769 m^4}{360 p^2} - \frac{27 m^6}{80 p^2} + \frac{7 m^8}{40 p^2} - \frac{3 m^4}{2 (m^2 + p^2)} + \left( \frac{3 m^2}{2} + \frac{17 p^2}{360} \right) \ln[a^2 m^2 + a^2 p^2] \right] \]

\[ + \left( \frac{1}{3} + \frac{239 m^2}{24 p^2} + \frac{367 m^4}{72 p^2} + \frac{m^6}{4 p^2} - \frac{7 m^8}{40 p^2} \right) m^2 \ln[1 + \frac{p^2}{m^2}] + \frac{p^4}{p^2} \left( 1.6064977(2) - \frac{157 \ln[a^2 m^2 + a^2 p^2]}{180} - \frac{227 m^2}{720 p^2} \right) \] \]

\[ + \frac{109 m^4}{180 p^2} + \frac{27 m^6}{40 p^2} - \frac{7 m^8}{20 p^2} + \left( \frac{1}{3} + \frac{m^2}{12 p^2} - \frac{35 m^4}{36 p^2} - \frac{m^6}{2 p^2} + \frac{7 m^8}{20 p^2} \right) \frac{m^2 \ln[1 + \frac{m^2}{ap^2}]}{p^2} \] \]

\[ + O(a^3, g^4) \]
\[ Z_{\nu}^{\text{pert.}} = 1 \]
\[ + \tilde{g}^2 \left\{ -16.6028865(8) + a \left[ 2.2261230(2) + 3 \ln[a^2 m^2 + a^2 p^2] + \frac{p_{\nu}}{p^2} \left( -3 + \frac{6m^2}{p^2} \right) + \left( 3 - \frac{6m^2 p_{\nu}^2}{p^2} \right) \frac{m^2 \ln[1 + \frac{p_{\nu}^2}{m^2}]}{p^2} \right] \right\} \]
\[ + a^2 \left\{ 1.125750(1) p^2 + 1.102770(2) m^2 + \frac{65m^4}{48p^2} + \frac{m^6}{8p^2} + \left( -\frac{25m^2}{4} + \frac{76p_{\nu}^2}{45} - \frac{7p^2}{24} \right) \ln[a^2 m^2 + a^2 p^2] \right. \]
\[ + p_{\nu}^2 \left( -2.714031(1) + \frac{5017m^2}{360p^2} - \frac{9401m^4}{360p^2} - \frac{m^6}{10p^2} + \frac{7m^8}{10p^2} - \frac{6m^2}{m^2 + p^2} \right) \]
\[ + \frac{p_{\nu}^4}{p^2} \left( \frac{323}{180} - \frac{59m^2}{18p^2} - \frac{35m^4}{18p^2} - \frac{14m^6}{3p^2} + \frac{14m^8}{3p^2} \right) \]
\[ + \left( -\frac{41}{4} - \frac{17m^2}{12p^2} - \frac{m^4}{8p^2} + \frac{p_{\nu}^2}{p^2} \left( -\frac{11}{12} + \frac{236m^2}{9p^2} - \frac{m^4}{4p^2} - \frac{7m^6}{10p^2} \right) + \frac{p_{\nu}^4}{p^2} \left( \frac{11}{3} + \frac{14m^2}{3p^2} + \frac{7m^4}{3p^2} - \frac{14m^6}{3p^2} \right) \right) \frac{m^2 \ln[1 + \frac{p_{\nu}^2}{m^2}]}{p^2} \]
\[ + \frac{p^4}{p^2} \left( 2.0773310(2) - \frac{157 \ln[a^2 m^2 + a^2 p^2]}{180} - \frac{67m^2}{120p^2} + \frac{m^4}{120p^2} \right) \]
\[ - \frac{8m^6}{15p^2} + \frac{7m^8}{30p^2} + \frac{p_{\nu}^2}{p^2} \left( \frac{7}{120} + \frac{m^2}{12p^2} + \frac{77m^4}{12p^2} + \frac{6m^6}{p^2} - \frac{7m^8}{p^2} \right) \]
\[ + \left( \frac{1}{2} + \frac{5m^2}{18p^2} + \frac{5m^4}{12p^2} - \frac{7m^6}{30p^2} + \frac{p_{\nu}^2}{p^2} \left( -\frac{5}{2} - \frac{10m^2}{p^2} - \frac{5m^4}{2p^2} + \frac{7m^6}{p^2} \right) \right) \frac{m^2 \ln[1 + \frac{p_{\nu}^2}{m^2}]}{p^2} \right\} \]
\[ \bigg|_{p_{\nu} = \mu_{\nu}} + \mathcal{O}(a^3, g^4) \]
\[ Z^\text{pert.}_A = 1 \]

\[ 1 + \frac{\bar{g}^2}{2} \left\{ -12.5395741(8) + \frac{2m^2}{p^2} - \frac{8p^2 m^2}{p^2} + \left( 2 - \frac{2m^2}{p^2} + \frac{p^2}{p^2} \left( 4 + \frac{8m^2}{p^2} \right) \right) \frac{m^2 \ln[1 + \frac{p^2}{m^2}]}{p^2} \right\} \]

\[ + \alpha m \left\{-1.4208394(2) - \frac{5m^2}{p^2} + \frac{4m^2}{m^2 + p^2} + \frac{p^2}{p^2} \left( -1 + \frac{26m^2}{p^2} + \frac{4p^2}{m^2 + p^2} \right) \right\} \]

\[ + \left( -3m + \frac{5m^2}{p^2} - \frac{p^2}{p^2} \left( 12 + \frac{26m^2}{p^2} \right) \right) \frac{m^2 \ln[1 + \frac{p^2}{m^2}]}{p^2} \]

\[ + \alpha^2 \left[-0.153718(1) p^2 + 1.290617(2) m^2 \right] \]

\[ + \frac{557m^4}{48p^2} - \frac{13m^6}{12p^2} + \frac{3m^8}{4p^2} - \frac{10m^4}{m^2 + p^2} + \frac{2m^6}{(m^2 + p^2)^2} + \left( \frac{23m^2}{12} - \frac{14p^2}{45} + \frac{5p^2}{24} \right) \ln[a^2 m^2 + a^2 p^2] \]

\[ + \frac{p^2}{p^2} \left(-0.892808(1) + \frac{2707m^2}{360p^2} - \frac{23201m^4}{360p^2} - \frac{3m^6}{5p^2} - \frac{23m^8}{10p^2} - \frac{11m^2}{m^2 + p^2} + \frac{2m^4}{(m^2 + p)^2} \right) \]

\[ + \frac{p^4}{p^2} \left( \frac{323}{180} + \frac{5m^2}{18p^2} + \frac{145m^4}{18p^2} + \frac{8m^6}{3p^2} + \frac{46m^8}{3p^2} \right) + \left( -\frac{2}{3} + \frac{73m^2}{12p^2} - \frac{11m^4}{p^2} + \frac{17m^6}{24p^2} - \frac{3m^8}{4p^2} \right) \]

\[ + \frac{p^4}{p^2} \left( \frac{4}{3} + \frac{9m^2}{4p^2} + \frac{581m^4}{9p^2} + \frac{7m^6}{10p^2} + \frac{23m^8}{10p^2} \right) + \frac{p^4}{p^2} \left( -2 - \frac{3m^2}{p^2} - \frac{40m^4}{3p^2} - \frac{m^6}{3p^2} + \frac{46m^8}{3p^2} \right) \]

\[ + \frac{p^4}{p^2} \left( \frac{0.7439977(2)}{180} - \frac{157}{180} \ln[a^2 m^2 + a^2 p^2] + \frac{19m^2}{60p^2} - \frac{13m^4}{40p^2} + \frac{163m^6}{60p^2} - \frac{34m^8}{15p^2} + \frac{4p^2}{3(m^2 + p^2)} + \left( \frac{2}{3} \right) \right) \]

\[ + \frac{m^2}{6p^2} - \frac{11m^4}{9p^2} - \frac{19m^6}{12p^2} + \frac{34m^8}{15p^2} + \frac{p^2}{p^2} \left( \frac{2}{3} + \frac{11m^2}{2p^2} + \frac{14m^4}{p^2} - \frac{5m^6}{2p^2} - \frac{23m^8}{p^2} \right) \]

\[ + \frac{p^2}{p^2} \left( \frac{167}{120} - \frac{4p^2}{3m^2} - \frac{41m^2}{12p^2} - \frac{91m^4}{12p^2} - \frac{9m^6}{p^2} + \frac{23m^8}{p^2} + \frac{4p^2}{3(m^2 + p^2)} \right) \]

\[ \right\} \bigg|_{p^2 = \mu^2} + O(a^3, g^4) \]
\[
(Z_R^{\text{pert}})_{\nu_1 \neq \nu_2} = 1
\]

\[
1 + \rho^2 \left\{ -13.538296(8) - \ln[a^2m^2 + a^2p^2] - \frac{2m^2}{p^2} + \left(1 + \frac{2m^2}{p^2}\right) \frac{m^2 \ln[1 + \frac{\rho^2}{m^2}]}{p^2} \right. \\
+ \frac{(p_{\nu_1}^2 + p_{\nu_2}^2)}{p^2} \left(4\frac{m^2}{p^2} - \left(2 + 4\frac{m^2}{p^2}\right) \frac{m^2 \ln[1 + \frac{\rho^2}{m^2}]}{p^2}\right) + \alpha m \left[0.4107689(2) + \frac{7}{2} \ln[a^2m^2 + a^2p^2]\right] \\
+ \frac{13m^2}{2p^2} - \frac{13m^2 \ln[1 + \frac{\rho^2}{m^2}]}{2p^2} + \frac{(p_{\nu_1}^2 + p_{\nu_2}^2)}{p^2} \left(-1 - \frac{10m^2}{p^2} - \frac{2p^2}{m^2 + p^2} + \left(6 + \frac{10m^2}{p^2}\right) \frac{m^2 \ln[1 + \frac{\rho^2}{m^2}]}{p^2}\right) \right. \\
+ \alpha^2 \left[1.000358(2)p^2 + 1.509337(2)m^2 - \frac{628m^4}{45p^2} + \frac{407m^6}{720p^2} - \frac{23m^8}{40p^2} + \frac{3m^4}{2(m^2 + p^2)}\right] \\
- \left(\frac{55m^2}{9} + \frac{41p^2}{120}\right) \ln[a^2m^2 + a^2p^2] + \frac{p_{\nu_1}^2p_{\nu_2}^2}{p^2} \left(\frac{20}{9} - \frac{38m^2}{9p^2} + \frac{4m^4}{p^2} + \frac{16m^6}{3p^2}\right) \\
+ \left(\frac{1}{3} - \frac{65m^2}{24p^2} + \frac{109m^4}{8p^2} - \frac{5m^6}{18p^2} + \frac{23m^8}{40p^2} + \frac{p_{\nu_1}^2p_{\nu_2}^2}{p^2} \left(\frac{8m^2}{3p^2} - \frac{20m^4}{3p^2} - \frac{16m^6}{3p^2}\right)\right) m^2 \ln[1 + \frac{p^2}{m^2}] \\
+ \frac{(p_{\nu_1}^2 + p_{\nu_2}^2)}{p^2} \left(-2.0217225(2) + \ln[a^2m^2 + a^2p^2] + \frac{383m^4}{72p^2} + \frac{103m^6}{6p^2} - \frac{29m^8}{12p^2} + \frac{3m^8}{2p^2} + \frac{5m^2}{2(m^2 + p^2)} - \frac{m^4}{(m^2 + p^2)^2}\right) \\
+ \left(\frac{1}{3} - \frac{85m^2}{6p^2} - \frac{95m^4}{6p^2} + \frac{5m^6}{3p^2} - \frac{3m^8}{2p^2}\right) \frac{m^2 \ln[1 + \frac{\rho^2}{m^2}]}{p^2} \right) + \frac{(p_{\nu_1}^2 + p_{\nu_2}^2)}{p^2} \left(\frac{10}{9} - \frac{35m^2}{9p^2} - \frac{3m^4}{p^2} - \frac{11m^6}{3p^2} + \frac{10m^8}{p^2}\right) \\
+ \left(1 + \frac{14m^2}{3p^2} + \frac{17m^4}{3p^2} - \frac{4m^6}{3p^2} - \frac{10m^8}{p^2}\right) \frac{m^2 \ln[1 + \frac{\rho^2}{m^2}]}{p^2} \right) + \frac{p^4}{p^2} \left(2.4814977(2) - \frac{157 \ln[a^2m^2 + a^2p^2]}{180} - \frac{497m^2}{720p^2}\right) \\
- \frac{73m^4}{90p^2} - \frac{43m^6}{40p^2} + \frac{43m^8}{20p^2} + \left(-\frac{3}{3} + \frac{11m^2}{12p^2} + \frac{55m^4}{36p^2} + \frac{43m^8}{20p^2}\right) \frac{m^2 \ln[1 + \frac{\rho^2}{m^2}]}{p^2} \right. \\
+ \frac{(p_{\nu_1}^2 + p_{\nu_2}^2)}{p^2} \left(-\frac{2}{3} + \frac{2p^2}{3m^2} + \frac{7m^2}{4p^2}\right) \right. \\
+ \frac{7m^4}{p^2} - \frac{15m^4}{2p^2} - \frac{15m^6}{p^2} - \frac{2p^2}{3(m^2 + p^2)} + \left(-\frac{3}{3} - \frac{4m^2}{p^2} - \frac{12m^4}{p^2} + \frac{15m^8}{p^2}\right) \frac{m^2 \ln[1 + \frac{\rho^2}{m^2}]}{p^2} \right) \right) \right) \right) \bigg|_{\rho = \mu, \nu} + \mathcal{O}(a^3, g^4)
\begin{align}
(\mathcal{Z}_{\nu}^{\text{pert.}})_{\nu_1 \neq \nu_2} &= 1 \\
1 + g^2 \left\{ -13.5382926(8) - \ln[a^2 m^2 + a^2 \rho] + \frac{2 m^2}{\rho} - \left(1 + \frac{2 m^2}{\rho} \right) \frac{m^2 \ln[1 + \frac{\rho^2}{m^2}]}{\rho} \\
+ \frac{\left(p_{\nu_1}^2 + p_{\nu_2}^2\right)}{\rho^2} \left( -\frac{4 m^2}{\rho} + \left(2 + \frac{4 m^2}{\rho} \right) \frac{m^2 \ln[1 + \frac{\rho^2}{m^2}]}{\rho} \right) + a m \left[ -2.5892311(2) + \frac{7}{2} \ln[a^2 m^2 + a^2 \rho] - \frac{7 m^2}{2 \rho} \\
+ \frac{2 m^2}{\rho^2} + \left(6 + \frac{7 m^2}{\rho} \right) \frac{m^2 \ln[1 + \frac{\rho^2}{m^2}]}{\rho} + \frac{p_{\nu_1}^2 + p_{\nu_2}^2}{\rho^2} \left(1 + \frac{10 m^2}{\rho^2} + \frac{2 m^2}{\rho^2} - \left(6 + \frac{10 m^2}{\rho^2} \right) \frac{m^2 \ln[1 + \frac{\rho^2}{m^2}]}{\rho} \right) \right] \right\} \\
+ a^2 \left[ 0.089747(2) \rho + 7.217670(2) \frac{m^2}{\rho} + \frac{469 m^4}{90 \rho^2} + \frac{587 m^6}{720 \rho^2} + \frac{37 m^8}{40 \rho^2} - \frac{2 m^4}{\rho^2} \\
+ \frac{m^6}{(m^2 + \rho^2)^2} + \left(79 \rho^2 - \frac{55 m^2}{9} \right) \ln[a^2 m^2 + a^2 \rho] + \frac{p_{\nu_1}^2 + p_{\nu_2}^2}{\rho^2} \left(20 \frac{9}{9 \rho^2} + \frac{4 m^4}{\rho^2} + \frac{16 m^6}{3 \rho^2} \right) \\
+ \left(-\frac{1}{3} - \frac{37 \rho^2}{24 \rho^2} - \frac{133 \rho^2}{24 \rho^2} - \frac{23 m^6}{18 \rho^2} - \frac{37 m^8}{40 \rho^2} + \frac{p_{\nu_1}^2 + p_{\nu_2}^2}{\rho^2} \left(8 m^2 - \frac{20 m^4}{3 \rho^2} - \frac{16 m^6}{3 \rho^2} \right) \right) m^2 \ln[1 + \frac{\rho^2}{m^2}] + \left(p_{\nu_1}^2 + p_{\nu_2}^2\right) \right] \right\} \\
- 0.2004998(2) \ln[a^2 m^2 + a^2 \rho] - \frac{79 m^2}{72 \rho^2} - \frac{127 m^4}{6 \rho^2} - \frac{35 m^6}{12 \rho^2} - \frac{3 m^8}{2 \rho^2} - \frac{5 m^2}{2 \rho^2} + \frac{m^4}{(m^2 + \rho^2)^2} \\
+ \left(\frac{2}{3} + \frac{23 m^2}{2 \rho^2} + \frac{45 m^4}{2 \rho^2} + \frac{11 m^6}{3 \rho^2} + \frac{3 m^8}{2 \rho^2} \right) \frac{m^2 \ln[1 + \frac{\rho^2}{m^2}]}{\rho^2} + \frac{p_{\nu_1}^2 + p_{\nu_2}^2}{\rho^2} \left(10 \frac{2}{3} - \frac{m^2}{3 \rho^2} + \frac{7 m^4}{2 \rho^2} + \frac{9 m^6}{2 \rho^2} - \frac{10 m^8}{2 \rho^2} \right) \\
+ \left(-\frac{1}{3} - \frac{2 m^2}{\rho^2} - \frac{37 \rho^2}{3 \rho^2} - \frac{4 m^6}{3 \rho^2} + \frac{10 m^8}{3 \rho^2} \right) \frac{m^2 \ln[1 + \frac{\rho^2}{m^2}]}{\rho^2} + \frac{p_{\nu_1}^2 + p_{\nu_2}^2}{\rho^2} \left(2 \frac{2}{3} - \frac{2 m^2}{3 \rho^2} - \frac{7 m^2}{4 \rho^2} \\
+ \frac{11 m^6}{720 \rho^2} \frac{57 m^8}{3 \rho^2} + \frac{2 m^2}{2 \rho^2} + \frac{2 m^2}{2 \rho^2} \right) + \left(\frac{1}{3} + \frac{m^2}{4 \rho^2} - \frac{5 \rho^2}{36 \rho^2} + \frac{4 m^6}{5 \rho^2} + \frac{57 m^8}{3 \rho^2} \right) \frac{m^2 \ln[1 + \frac{\rho^2}{m^2}]}{\rho^2} + \frac{p_{\nu_1}^2 + p_{\nu_2}^2}{\rho^2} \left(2 \frac{2}{3} - \frac{2 m^2}{3 \rho^2} - \frac{7 m^2}{4 \rho^2} \\
- \frac{7 m^4}{2 \rho^2} + \frac{15 m^6}{2 \rho^2} + \frac{15 m^8}{2 \rho^2} + \frac{12 m^2}{2 \rho^2} \right) \frac{m^2 \ln[1 + \frac{\rho^2}{m^2}]}{\rho^2} \right] \right\} \bigg|_{p_\rho = \mu_\rho} + \mathcal{O}(a^3, g^4) \\
\end{align}

where

\begin{align}
M^2 &= m_0^2 + \mu_0^2, \\
\nu_1 &= \mu, \\
\nu_2 &= \nu.
\end{align}
Appendix C: Analytic expressions for one-derivative operators

Here we present the $Z$-factors for the one-derivative vector, axial and tensor operators published separately in Ref. [15], defined as follows

\[
O_{\text{DV}}^{(\mu \nu)} = \gamma_5 \gamma_\mu \gamma_\nu \tau^a \chi = \begin{cases} \bar{\psi} \gamma_5 \gamma_\mu \gamma_\nu \tau^2 \psi & a = 1 \\ -\bar{\psi} \gamma_5 \gamma_\mu \gamma_\nu \tau^1 \psi & a = 2 \\ \bar{\psi} \gamma_5 \gamma_\mu \gamma_\nu \tau^3 \psi & a = 3 \end{cases} \quad (C1)
\]

\[
O_{\text{DA}}^{(\mu \nu)} = \gamma_5 \gamma_\mu \gamma_\nu \tau^a \chi = \begin{cases} \bar{\psi} \gamma_\mu \gamma_\nu \tau^2 \psi & a = 1 \\ -\bar{\psi} \gamma_\mu \gamma_\nu \tau^1 \psi & a = 2 \\ \bar{\psi} \gamma_\mu \gamma_\nu \tau^3 \psi & a = 3 \end{cases} \quad (C2)
\]

\[
O_{\text{DT}}^{(\nu \rho)} = \gamma_5 \sigma_\mu \sigma_\rho \gamma_\nu \gamma_\rho \tau^a \chi = \begin{cases} \bar{\psi} \gamma_5 \sigma_\mu \sigma_\rho \gamma_\nu \gamma_\rho \tau^a \psi & a = 1, 2 \\ -i \bar{\psi} \sigma_\mu \sigma_\rho \gamma_\nu \gamma_\rho \psi & a = 3 \end{cases} \quad (C3)
\]

The above operators are symmetrized over two Lorentz indices and are made traceless

\[
O^{(\sigma \tau)} = \frac{1}{2} \left( O^{\sigma \tau} + O^{\tau \sigma} \right) - \frac{1}{4} \delta^{\sigma \tau} \sum_{\chi} O^{\chi \chi}.
\]

The one derivative operators fall into different irreducible representations of the hypercubic group, depending on the choice of indices:

- $O_{\text{DV}1} = O_{\text{DV}}$ with $\mu = \nu$
- $O_{\text{DV}2} = O_{\text{DV}}$ with $\mu \neq \nu$
- $O_{\text{DA}1} = O_{\text{DA}}$ with $\mu = \nu$
- $O_{\text{DA}2} = O_{\text{DA}}$ with $\mu \neq \nu$
- $O_{\text{DT}1} = O_{\text{DT}}$ with $\mu \neq \nu = \rho$
- $O_{\text{DT}2} = O_{\text{DT}}$ with $\mu \neq \nu \neq \rho \neq \mu$.

Thus, $Z_{\text{DV}1}$, $Z_{\text{DA}1}$ will be different from $Z_{\text{DV}2}$, $Z_{\text{DA}2}$, respectively. More details on the one-derivative renormalization factors can be found in Ref. [15].

We have computed, to $O(a^2)$, the forward matrix elements of these operators for general external indices $\mu$, $\nu$, and $\rho$ for the tensor operator, external momentum $p$, $m$, $g$, $N_c$, $a$, $csw$ and gauge fixing. Our final results were obtained for the 10 sets of Symanzik coefficients given in Table I.

The amputated Greens functions of the $O_{\text{DT}}$ operator appear in the Mathematica file Zfactors.m as below:

\[
\begin{align*}
L_{\text{DV}}^{\text{pert.}} &= \text{LDV[Action, csw, beta, g2tilde, a, m]} + O(a^3, g^4), \\
L_{\text{DA}}^{\text{pert.}} &= \text{LDA[Action, csw, beta, g2tilde, a, m]} + O(a^3, g^4), \\
L_{\text{DT}}^{\text{pert.}} &= \text{LDT[Action, csw, beta, g2tilde, a, m]} + O(a^3, g^4).
\end{align*}
\]

In order to define $Z_{\mathcal{O}}$, we have used a renormalization prescription which is most amenable to non-perturbative treatment:

\[
Z_{\mathcal{O}}^{-1} Z_{\mathcal{O}} \text{Tr} \left[ L_{\text{tree}}^{\mathcal{O}}(p) \cdot L_{\text{tree}}^{\mathcal{O}}(p) \right]_{p_{\lambda} = \mu_{\lambda}} = \text{Tr} \left[ L_{\text{tree}}^{\mathcal{O}}(p) \cdot L_{\text{tree}}^{\mathcal{O}}(p) \right]_{p_{\lambda} = \mu_{\lambda}} \quad (C4)
\]

where $L^O$ denotes the amputated 2-point Green’s function of the operators up to 1-loop and up to $O(a^2)$. These
Z-factors appear in electronic form with the name:

\[
\begin{align*}
(Z_{DV1}^{pert.})_{\nu_1=\nu_2} &= ZDV1[\text{Action, csw, beta, g2tilde, a, m}] + \mathcal{O}(a^4, g^4), \\
(Z_{DV2}^{pert.})_{\nu_1\neq\nu_2} &= ZDV2[\text{Action, csw, beta, g2tilde, a, m}] + \mathcal{O}(a^3, g^4), \\
(Z_{DA1}^{pert.})_{\nu_1=\nu_2} &= ZDA1[\text{Action, csw, beta, g2tilde, a, m}] + \mathcal{O}(a^3, g^4), \\
(Z_{DA2}^{pert.})_{\nu_1\neq\nu_2} &= ZDA2[\text{Action, csw, beta, g2tilde, a, m}] + \mathcal{O}(a^3, g^4), \\
(Z_{DT1}^{pert.})_{\nu_1=\nu_3\neq\nu_2} &= ZDT1[\text{Action, csw, beta, g2tilde, a, m}] + \mathcal{O}(a^3, g^4), \\
(Z_{DT2}^{pert.})_{\nu_1\neq\nu_2\neq\nu_3} &= ZDT2[\text{Action, csw, beta, g2tilde, a, m}] + \mathcal{O}(a^3, g^4).
\end{align*}
\]

Due to very lengthy expressions we only show the results for specific choices of the action parameters, that is Landau gauge, tree-level Symanzik gluons, \(c_{SW} = 0\), \(m = 0\):

\[
\begin{align*}
(Z_{DV1}^{pert.})_{\nu_1=\nu_2} &= ZDV1[2, 0.1, g2tilde, a, 0] + \mathcal{O}(a^4, g^4) \\
&= \delta_{\nu_1} \left(1 + \tilde{g}^2 \right) \left(1.14698(1) - \frac{8}{3} \ln[a^2 p^2] + \frac{2p_{\nu_1}^2}{3p_1} - \frac{6p_{\nu_1}^2}{8p_{c1}^2 + p^2} + a^2 \left(1.62067(6)p_2 - 6.4175(7)p_{\nu_1}^2 + \frac{21p_{\nu_1}^4}{10p_2} + \frac{23.328(6)p_{\nu_1}^2}{8p_{c1}^2 + p^2} \right) \right) + \mathcal{O}(a^4, g^4) \quad (C5) \\
(Z_{DV2}^{pert.})_{\nu_1\neq\nu_2} &= ZDV2[2, 0.1, g2tilde, a, 0] + \mathcal{O}(a^4, g^4) \\
&= \delta_{\nu_1} \delta_{\nu_2} \left(1 + \tilde{g}^2 \right) \left(2.02248(1) - \frac{8}{3} \ln[a^2 p^2] + \frac{4p_{\nu_1}^2 p_{\nu_2}^2}{3p_1 (p_{c1}^2 + p_{c2}^2)} + a^2 \left(1.01505(3)p_2 + \frac{209p_{\nu_1}^2 p_{c1}^2}{90p_2} - 2.1276(1) (p_{\nu_1}^2 + p_{\nu_2}^2) \right) \right) + \mathcal{O}(a^4, g^4) \quad (C6) \\
(Z_{DA1}^{pert.})_{\nu_1=\nu_2} &= ZDA1[2, 0.1, g2tilde, a, 0] + \mathcal{O}(a^4, g^4) \\
&= \delta_{\nu_1} \left(1 + \tilde{g}^2 \right) \left(3.48606(1) - \frac{8}{3} \ln[a^2 p^2] + \frac{2p_{\nu_1}^2}{3p_1} - \frac{6p_{\nu_1}^2}{8p_{c1}^2 + p^2} + a^2 \left(0.46577(6)p_2 + 3.8584(7)p_{\nu_1}^2 + \frac{21p_{\nu_1}^4}{10p_2} - \frac{62.787(6)p_{\nu_1}^2}{8p_{c1}^2 + p^2} \right) \right) + \mathcal{O}(a^4, g^4) \quad (C7)
\end{align*}
\]
\[
\begin{align*}
\left( Z_{\text{DTA}}^{\text{pert}} \right)_{\nu_1 \neq \nu_2}^{\nu_1 \neq \nu_2} &= Z\text{DA2}[2, 0, 1, g\tilde{t}t\text{id}, a, 0] + O(a^4, g^4) \\
&= \delta_{\nu_1 \nu_1} \delta_{\nu_2 \nu_2} \left( 1 + g^2 \right) \left( 3.07868(1) - \frac{8}{3} \ln[a^2 p^2] + \frac{4p_{v_1}^2 p_{v_2}^2}{3p_2 (p_{v_1}^2 + p_{v_2}^2)} + a^2 \left( 0.38848(3)p^2 + \frac{209p_{v_1}^2 p_{v_2}^2}{90p_2} - 1.8613(1) (p_{v_1}^2 + p_{v_2}^2) \right) \\
&+ \frac{-1.1283(3)p_{v_1}^2 p_{v_2}^2}{p_{v_1}^2 + p_{v_2}^2} - \frac{8p_{v_1}^2 p_{v_2}^2}{9p_2 (p_{v_1}^2 + p_{v_2}^2)} + \frac{8p_2}{45} + \frac{19}{40} (p_{v_1}^2 + p_{v_2}^2) - \frac{17p_{v_1}^2 p_{v_2}^2}{90 (p_{v_1}^2 + p_{v_2}^2)} \right) \ln[a^2 p^2] \\
&+ p^4 \left( \frac{2.59467755(2)}{p^2} + \frac{157 \ln[a^2 p^2]}{180p^2} - \frac{29p_{v_2}^2 p_{v_2}^2}{90p_2 (p_{v_1}^2 + p_{v_2}^2)} - \frac{29p_{v_2}^2 p_{v_2}^2}{90p_2 (p_{v_1}^2 + p_{v_2}^2)} \right) ) + O(a^4, g^4)
\end{align*}
\]

\[
\begin{align*}
\left( Z_{\text{DTT}}^{\text{pert}} \right)_{\nu_1 = \nu_3 \neq \nu_2}^{\nu_1 = \nu_3 \neq \nu_2} &= Z\text{DT1}[2, 0, 1, g\tilde{t}t\text{id}, a, 0] + O(a^4, g^4) \\
&= \delta_{\nu_1 \nu_1} \delta_{\nu_2 \nu_2} \delta_{\nu_3 \nu_3} \left( 1 + g^2 \right) \left( 3.882961(1) - \frac{18.7832(1)p_{v_1}^2}{5p_2} + \frac{12p_{v_1}^2}{8p_{v_1}^2 - p_{v_2}^2 + p^2} \ln[a^2 p^2] \\
&+ a^2 \left( -0.7730(3)p^2 + 26.269(8)p_{v_2}^2 - 3.3610(1)p_{v_2}^2 \right) \frac{78p_{v_1}^2 p_{v_2}^2}{5p_2} - \frac{7058p_{v_1}^2}{45p_2} + \frac{28.163(3)p_{v_1}^2 p_2}{8p_{v_1}^2 - p_{v_2}^2 + p^2} + \frac{2.4703(2)p^2}{90p_2} \\
&+ \frac{302.33(1)p_{v_1}^4}{8p_{v_1}^2 - p_{v_2}^2 + p^2} + \frac{302.33(1)p_{v_1}^4}{8p_{v_1}^2 - p_{v_2}^2 + p^2} + \frac{2.4703(2)p^2}{90p_2} \\
&+ \frac{302.33(1)p_{v_1}^4}{8p_{v_1}^2 - p_{v_2}^2 + p^2} + \frac{302.33(1)p_{v_1}^4}{8p_{v_1}^2 - p_{v_2}^2 + p^2} + \frac{2.4703(2)p^2}{90p_2} \\
&+ \frac{8p_{v_1}^2}{45} - \frac{739p_{v_1}^2}{180} - \frac{292p_{v_1}^2}{180} - \frac{192p_{v_1}^2}{180} + \frac{658p_{v_1}^2}{9 (8p_{v_1}^2 - p_{v_2}^2 + p^2)} + \frac{4p_{v_1}^2}{(8p_{v_1}^2 - p_{v_2}^2 + p^2)^2} \\
&- \frac{64p_{v_1}^2}{(8p_{v_1}^2 - p_{v_2}^2 + p^2)^2} \frac{224p_{v_1}^2}{(8p_{v_1}^2 - p_{v_2}^2 + p^2)^2} \ln[a^2 p^2] + p^4 \left( \frac{2.46066643(2)}{p^2} - \frac{2.4703(1)}{8p_{v_1}^2 - p_{v_2}^2 + p^2} - \frac{131p_{v_1}^2}{180p^2} (8p_{v_1}^2 - p_{v_2}^2 + p^2) \\
&- \frac{6.26108(4)p_{v_1}^2}{(8p_{v_1}^2 - p_{v_2}^2 + p^2)^2} + \left( -\frac{157}{180p^2} + \frac{193}{180} (8p_{v_1}^2 - p_{v_2}^2 + p^2) + \frac{4p_{v_1}^2}{(8p_{v_1}^2 - p_{v_2}^2 + p^2)^2} \right) \ln[a^2 p^2] \right) ) + O(a^4, g^4)
\end{align*}
\]

\[
\begin{align*}
\left( Z_{\text{DT2}}^{\text{pert}} \right)_{\nu_1 \neq \nu_2 \neq \nu_3 \neq \nu_1}^{\nu_1 \neq \nu_2 \neq \nu_3 \neq \nu_1} &= Z\text{DT2}[2, 0, 1, g\tilde{t}t\text{id}, a, 0] + O(a^4, g^4) \\
&= \delta_{\nu_1 \nu_1} \delta_{\nu_2 \nu_2} \delta_{\nu_3 \nu_3} \left( 1 + g^2 \right) \left( 2.82413(1) - 3 \ln[a^2 p^2] + a^2 \left( 0.92582(3)p^2 + 0.73604(2)p_{v_2}^2 + \frac{67p_{v_1}^2 p_{v_3}^2}{45p_2} - 2.1124(1) (p_{v_1}^2 + p_{v_2}^2) \right) \\
&+ \frac{67p_{v_1}^4}{90p_2} + \frac{1.2403(3)p_{v_1}^2 p_{v_3}^2}{p_{v_1}^2 + p_{v_3}^2} + \frac{67p_{v_1}^2 p_{v_3}^2 p_{v_2}^2}{15p_2 (p_{v_1}^2 + p_{v_2}^2)} + \left( \frac{p_2}{2} + \frac{301p_{v_2}^2}{360} + \frac{331}{720} (p_{v_1}^2 + p_{v_2}^2) + \frac{71p_{v_1}^2 p_{v_3}^2}{20 (p_{v_1}^2 + p_{v_3}^2)} \right) \ln[a^2 p^2] \\
&+ p^4 \left( \frac{2.1064977(2)}{p^2} + \frac{41}{60p^2} + \frac{157 \ln[a^2 p^2]}{180p^2} \right) ) + O(a^4, g^4)
\end{align*}
\]

where for ZDV1, ZDV2, ZDA1, ZDA2:

\[
\begin{align*}
\nu_1 &= \mu, \\
\nu_2 &= \nu,
\end{align*}
\]

and for ZDT1, ZDT2:

\[
\begin{align*}
\nu_1 &= \rho, \\
\nu_2 &= \mu, \\
\nu_3 &= \nu.
\end{align*}
\]
Appendix D: Notation in Mathematica file: Zfactors.m

The full body of our results can be accessed online through the Mathematica file Zfactors.m, which is a Mathematica input file. It includes the expressions for the amputated Green’s functions of the inverse propagator:

\[ S_{\text{pert.}}^{-1} = \delta^{g^f} \left( \text{propagator[Action, csw, beta, g2tilde, a, m, mu] + } O(a^3, g^4) \right), \]  

(D1)

from which one can construct the fermion field renormalization constant for any renormalization scheme. This expression depends on the variables:

- action: Selection of improved gauge action as follows, 1 → Plaquette, 2 → Tree Level Symanzik, 3 → TILW \((\beta c_0 = 8.60)\), 4 → TILW \((\beta c_0 = 8.45)\), 5 → TILW \((\beta c_0 = 8.30)\), 6 → TILW \((\beta c_0 = 8.20)\), 7 → TILW \((\beta c_0 = 8.10)\), 8 → TILW \((\beta c_0 = 8.00)\), 9 → Iwasaki, 10 → DBW2
- csw: clover parameter
- beta: gauge parameter \((\text{Landau/Feynman/Generic correspond to } 1/0/beta)\)
- g2tilde=\(\frac{a^2C_F}{16\pi}\), g: coupling constant
- a: lattice spacing
- m: Lagrangian mass
- mu: twisted mass parameter

The expression for the critical mass is defined in the variable mcritical:

\[ m_{c} = m_{\text{critical}}[\text{Action, csw, g2tilde, aL}] + 4O(g^4). \]  

The reader may also find the amputated Green’s functions relevant to the ultra-local operators:

\[ A_{\text{pert.}}^{\text{scalar}} = [\text{Action, csw, beta, g2tilde, a, m, mu1, mu2}] + O(a^3, g^4), \]  

(D2)

\[ A_{\text{pert.}}^{\text{pseudoscalar}} = [\text{Action, csw, beta, g2tilde, a, m, mu1, mu2}] + O(a^3, g^4), \]  

(D3)

\[ A_{\text{pert.}}^{\text{vector}} = [\text{Action, csw, beta, g2tilde, a, m, mu1, mu2}] + O(a^3, g^4), \]  

(D4)

\[ A_{\text{pert.}}^{\text{axial}} = [\text{Action, csw, beta, g2tilde, a, m, mu1, mu2}] + O(a^3, g^4), \]  

(D5)

\[ A_{\text{pert.}}^{\text{tensor}} = [\text{Action, csw, beta, g2tilde, a, m, mu1, mu2}] + O(a^3, g^4), \]  

(D6)

\[ A_{T_\text{pert.}}^{\text{tensorprime}} = [\text{Action, csw, beta, g2tilde, a, m, mu1, mu2}] + O(a^3, g^4), \]  

(D7)

as well as the Green’s functions of the one-derivative vector, axial and tensor operators:

\[ A_{\text{pert.}}^{\text{DV}} = \text{LDV} [\text{Action, csw, beta, g2tilde, a, m}] + O(a^3, g^4), \]  

(D8)

\[ A_{\text{pert.}}^{\text{DA}} = \text{LDA} [\text{Action, csw, beta, g2tilde, a, m}] + O(a^3, g^4), \]  

(D9)

\[ A_{\text{pert.}}^{\text{DT}} = \text{LDT} [\text{Action, csw, beta, g2tilde, a, m}] + O(a^3, g^4). \]  

(D10)

We note that Eqs. (D2) - (D7) hold for quarks with the same Lagrangian mass and \(\mu_1 = \pm \mu_2\), while Eqs. (D8) - (D10) correspond to zero twisted mass parameters, \(\mu_1, \mu_2\), and non-zero Lagrangian mass.

The RCs are interesting quantities for other studies and are also provided in the Mathematica file Zfactors.m at the RF-MOM scheme by employing Eq. (36) and Eq. (33), for the fermion field and fermion operator RCs, respectively.

\[ Z_q^{\text{pert.}} = zq[2, 0, 1, g2tilde, a, m, mu, p2, p4] + O(a^3 g^2, g^4) \]  

(D11)

\[ Z_s^{\text{pert.}} = zs[2, 0, 1, g2tilde, a, m, mu1, mu2] + O(a^3, g^4) \]  

(D12)

\[ Z_{p+}^{\text{pert.}} = zp[2, 0, 1, g2tilde, a, m, mu1, mu2] + O(a^3, g^4) \]  

(D13)

\[ Z_{p-}^{\text{pert.}} = zp[2, 0, 1, g2tilde, a, m, mu1, mu2] + O(a^3, g^4) \]  

(D14)

\[ Z_A^{\text{pert.}} = za[2, 0, 1, g2tilde, a, m, mu1, mu2] + O(a^3, g^4) \]  

(D15)

\[ (Z_T^{\text{pert.}})_{\nu_1 \neq \nu_2} = zt[2, 0, 1, g2tilde, a, m, mu1, mu2] + O(a^3, g^4) \]  

(D16)

\[ (Z_{T_1}^{\text{pert.}})_{\nu_1 \neq \nu_2} = ztp[2, 0, 1, g2tilde, a, m, mu1, mu2] + O(a^3, g^4) \]  

(D17)

The additional variables are
• $p_2: \sum_{i=1}^{4} p_i^2$
• $p_4: \sum_{i=1}^{4} p_i^4$

For completeness we include in the Mathematica file Zfactors.m the conversion factors to the $\overline{\text{MS}}$ scheme for the one-derivative RCs, studied in Ref. [15].

\begin{align*}
C_{DV1,DA1} &= \text{CDV1}[\alpha_R, \lambda, C_A, C_F, N_f, T_f] + O(g^8) \quad \text{(D18)} \\
C_{DV2,DA2} &= \text{CDV2}[\alpha_R, \lambda, C_A, C_F, N_f, T_f] + O(g^8) \quad \text{(D19)}
\end{align*}

where

\begin{align*}
\alpha_R &= \frac{g^2}{(16\pi^2)} \\
\lambda &= 1 - \beta
\end{align*}
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