POSITIVE SOLUTIONS OF PERTURBED ELLIPTIC PROBLEMS INVOLVING HARDY POTENTIAL AND CRITICAL SOBOLEV EXPONENT

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(Communicated by Susanna Terracini)

Abstract. In this paper, we are concerned with the following nonlinear Schrödinger equations with hardy potential and critical Sobolev exponent
\[
\begin{aligned}
-\Delta u + \lambda a(x)u^q &= \mu \frac{u}{|x|^2} + |u|^{2^*-2}u, & \text{in } \mathbb{R}^N, \\
u > 0, & \text{in } D^{1,2}(\mathbb{R}^N),
\end{aligned}
\]
where \(2^* = \frac{2N}{N-2}\) is the critical Sobolev exponent, \(0 \leq \mu < \mu = \frac{(N-2)^2}{4}\), \(a(x) \in C(\mathbb{R}^N)\). We first use an abstract perturbation method in critical point theory to obtain the existence of positive solutions of (1) for small value of \(|\lambda|\).

Secondly, we focus on an anisotropic elliptic equation of the form
\[
-\text{div}(B_\lambda(x)\nabla u) + \lambda a(x)u^q = \mu \frac{u}{|x|^2} + |u|^{2^*-2}u, x \in \mathbb{R}^N.
\]
The same abstract method is used to yield existence result of positive solutions of (2) for small value of \(|\lambda|\).

1. Introduction. In this paper, we shall study the existence of positive solutions of the critical elliptic problem
\[
\begin{aligned}
-\Delta u + \lambda a(x)u^q &= \mu \frac{u}{|x|^2} + |u|^{2^*-2}u, & \text{in } \mathbb{R}^N, \\
u > 0, & \text{in } D^{1,2}(\mathbb{R}^N),
\end{aligned}
\]
where \(\lambda \in \mathbb{R}, 1 \leq q < 2^*, 2^* = \frac{2N}{N-2}, N \geq 3\), \(a(x)\) is a real, nonnegative function on \(\mathbb{R}^N\), \(0 \leq \mu < \mu = \frac{(N-2)^2}{4}\), and \(D^{1,2}(\mathbb{R}^N)\) is the usual Sobolev space.

2010 Mathematics Subject Classification. Primary: 35B33, 35B20; Secondary: 35B09.
Key words and phrases. Positive solutions, critical exponent, perturbation method, anisotropic problem.

Research supported by the Specialized Fund for the Doctoral Program of Higher Education and the National Natural Science Foundation of China (Grant No 11571187).
We notice that by a variant of the Pohozaev identity (see [6]), for any solution \( u \) of problem (3), we have
\[
N \left( \frac{1}{q+1} - \frac{1}{2^*} \right) \int_{\mathbb{R}^N} a(x) u^{q+1} dx + \frac{1}{q+1} \int_{\mathbb{R}^N} (\nabla u(x)|x) u^{q+1} dx = 0, \tag{4}
\]
where \((.,.)\) denotes the scalar product in \( \mathbb{R}^N \). As a consequence, if \( a(x) \equiv 1 \), (4) gives \( \int_{\mathbb{R}^N} u^{q+1} dx = 0 \), and thus problem (3) has no solution if \( \lambda \neq 0 \).

There are several elliptic problems on \( \mathbb{R}^N \) which are perturbative in nature. For these perturbative problems a specific approach, that takes advantage of such a perturbative setting, seems the most appropriate. These abstract tools are provided by perturbation methods in critical point theory. Actually, it turns out that such a framework can be used to handle a large variety of equations, usually considered different in nature.

The main reason of interest in Hardy term relies in their criticality, indeed they have the same homogeneity as the Laplacian and the critical Sobolev exponent and don’t belong to the Kato class, hence they cannot be regarded as a lower order perturbation term.

Another reason why we investigate (3), in addition to the inverse square potential, is the presence of the critical Sobolev exponent and the unbounded domain \( \mathbb{R}^N \), which cause the loss of compactness of embedding \( \mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N) \) and \( H^1(\mathbb{R}^N) \hookrightarrow L^{q}(\mathbb{R}^N) \). Hence, including the non-compactness of the imbedding \( \mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N, |x|^{-2} dx) \), we face a type of triple loss of compactness whose interacting each other will result in some new difficulties. In last two decades, loss of compactness leads to many interesting existence and nonexistence phenomena for elliptic equations.

Ambrosetti et al. [2] studied the equation
\[
-\Delta u + \lambda a(x) u^q = (1 + \lambda K(x))|u|^{2^*-2}u, \quad x \in \mathbb{R}^N, \tag{5}
\]
by applying a perturbative approach, which relies on a suitable use of an abstract perturbation method in critical point theory discussed in [1, 3, 4]. Cingolani [9] used perturbative approach to study the positive solutions to perturbed elliptic problems in \( \mathbb{R}^N \) involving critical Sobolev exponent,
\[
-\Delta u + \lambda a(x) u^q = |u|^{2^*-2} u, \quad \text{in} \quad \mathbb{R}^N,
\]
then obtained that the equation has a positive solution.

In this paper, we shall prove the existence of positive solutions of (3), assuming that \( a(x) \) satisfies the following assumptions:

\((a_1)\) \( a(x) \in L^\infty(\mathbb{R}^N) \cap L^r(\mathbb{R}^N) \) with \( r < \frac{2^*}{2^*(q+1)} \);
\((a_2)\) \( a(x) \geq 0, a(x) \geq \nu > 0 \) in a subset \( \Omega \) of \( \mathbb{R}^N \) with positive measure.

As in [9], we shall apply the perturbative approach discussed in [1, 3, 4]. Roughly speaking, for small value of \(|\lambda|\), problem (3) can be seen as perturbation of the critical problem on \( \mathbb{R}^N \):
\[
\begin{cases}
-\Delta u = \mu \frac{u}{|x|^m} + |u|^{2^*-2} u, & \text{in} \quad \mathbb{R}^N, \\
u > 0, & \text{in} \quad \mathcal{D}^{1,2}(\mathbb{R}^N). 
\end{cases} \tag{6}
\]

Denote \( \beta = \sqrt{\mu} - \mu \), Catrina and Wang [8] proved that for \( 0 < \mu < \mu, \varepsilon > 0 \), all positive solutions of (6) are of the form \( z_\varepsilon(x) = \varepsilon^{-\frac{r^2}{2^*}} z(\frac{x}{\varepsilon}) \), where
\[
z(x) = \frac{C_N}{|x|^\frac{N-2}{2} - \beta (1 + |x|^{\frac{4m}{N-2}}) \frac{N-2}{2}}
\]
for an appropriate constant \( C_N > 0 \). These solutions achieve \( S_\mu \), where

\[
S_\mu = \inf \left\{ \int_{\mathbb{R}^N} |\nabla u|^2 - \frac{\mu u^2}{|x|^2} : u \in \mathcal{D}^{1,2} (\mathbb{R}^N), |u|_{2^*} = 1 \right\}.
\]

Firstly, we are motivated by a paper due to Silvia Cingolani [9] and obtain the following result:

**Theorem 1.1.** Assume \((a_1)-(a_2)\). Moreover assume that

\[
N \geq 5, \quad 1 \leq q < 2^* - 1, \quad 0 \leq \mu < \overline{\lambda} - 1.
\]  

Then there exist \( \overline{\lambda} > 0 \) and \( \varepsilon > 0 \) such that for any \( \lambda \in \mathbb{R}, |\lambda| \leq \overline{\lambda} \), problem (3) has a positive solution \( u_\lambda \) and \( u_\lambda \rightarrow w \) in \( \mathcal{D}^{1,2} (\mathbb{R}^N) \) as \( \lambda \rightarrow 0 \).

Secondly, we focus on an anisotropic elliptic equation of the form

\[
- \text{div} (B_\lambda(x) \nabla u) + \lambda a(x) u^q = \mu \frac{u}{|x|^2} + |u|^{2^* - 2} u, \quad x \in \mathbb{R}^N,
\]  

where \( N \geq 3 \) and \( B_\lambda(x) = I + \lambda B(x) \), \( B(x) \) is a symmetric matrix with positive bounded coefficients and \( a(x) \) satisfies \((a_1)-(a_2)\). The same abstract method is used to yield existence result of positive solutions of (8) for small value of \( |\lambda| \). We point out that a perturbation result for an anisotropic Schrödinger equation is obtained in [5] in the subcritical case.

### 2. Critical points for perturbed functionals

In this section, we shall recall an abstract perturbative method in critical point theory, which provides an abstract tool to deal with several perturbed semilinear equations and to find multiple homoclinic orbits to a class of second-order Hamiltonian systems (for example, see [3, 4, 9]).

We consider a family of \( C^2 \) functionals \( f_\lambda \) defined on a Hilbert space \( E \) of the form

\[
f_\lambda(u) = f_0(u) + \lambda g(u),
\]

where \( f_0, g \in C^2 (E, \mathbb{R}) \) and \( \lambda > 0 \). On the unperturbed functional \( f_0 \) we shall make the following assumption:

(I) there exists a \( d \)-dimensional \( C^2 \) manifold \( Z \), \( d \geq 1 \), consisting of critical points of \( f_0 \).

Such \( Z \) is called critical manifold of \( f_0 \). The presence of a critical manifold is usually due to the fact that the unperturbed functional \( f_0 \) is invariant under the action of a symmetry group: for example, in the case discussed in the next sections it will be invariant under scale changes.

Let \( T_zZ \) denote the tangent space to \( Z \). We further suppose:

(II) \( \forall z \in Z, D^2 f_0(z) \) is a Fredholm operator of index zero;

(III) \( \forall z \in Z, T_zZ = \ker D^2 f_0(z) \).

Assumption (III) is a sort of nondegeneracy condition. Actually, when \( Z \) is an isolated point \( z \), (III) becomes \( \ker D^2 f_0(z) = \{0\} \), which just means that \( z \) is a nondegenerate critical point of \( f_0 \).

If the preceding assumptions hold true, one can use the Implicit Function Theorem to find \( w = w(\lambda, z) \) such that

\[
f_\lambda(z + w) \in T_zZ.
\]

Letting \( Z_\lambda = \{z + w(\lambda, z)\} \), it turns out that \( Z_\lambda \) is locally diffeomorphic to \( Z \) and any critical point of \( f_\lambda \) constrained on \( Z_\lambda \) is a stationary point of \( f_\lambda \). A manifold
with this property will be called a natural constraint for \( f_{\lambda} \).

We find that the behaviour of \( f_{\lambda} \) on \( Z_{\lambda} \) is well approximated by the behaviour of the function \( \Gamma \equiv g_{|z|} \). In particular, critical points of \( \Gamma \) on \( Z \) give rise to critical points of \( f_{\lambda} \) on \( Z_{\lambda} \).

Assume that

\[
 f_{\lambda}(u) = f_0(z) + \lambda \Gamma(z) + o(\lambda).
\]

for any \( u \in Z_{\lambda} \) and \( |\lambda| \) small enough. The following result was proved in [2, 9].

**Theorem 2.1.** Assume that (1)–(3) hold and that there exists a critical point \( \tau \in Z \) of \( \Gamma \) such that one of the following conditions holds:

(i) \( \tau \) is nondegenerate;
(ii) \( \tau \) is a local proper minimum or maximum;
(iii) \( \tau \) is isolated and the local topological degree of \( \Gamma' \) at \( \tau \), \( \deg_{\text{loc}}(\Gamma', 0) \neq 0 \).

Then for \( |\lambda| \) small enough, the functional \( f_{\lambda} \) has a critical point \( u_{\lambda} \) such that \( u_{\lambda} \to \tau \) as \( \lambda \to 0 \).

3. **The variational setting.** It is well known that the space \( D^{1,2}(\mathbb{R}^N) = \{ u \in L^2(\mathbb{R}^N) : |\nabla u| \in L^2(\mathbb{R}^N) \} \) with the inner product

\[
 \int_{\mathbb{R}^N} \nabla u \cdot \nabla v,
\]

and the corresponding norm

\[
 ||u||^2 = \int_{\mathbb{R}^N} |\nabla u|^2 dx,
\]

is a Hilbert space and is the closure of \( C_0^\infty(\mathbb{R}^N) \). The energy functional associated with problem (3) is defined by \( f_{\lambda} : D^{1,2}(\mathbb{R}^N) \to \mathbb{R} \)

\[
 f_{\lambda}(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 - \mu \frac{u^2}{|x|^2}) dx - \frac{1}{2} \int_{\mathbb{R}^N} u_+^2 dx + \frac{\lambda}{q+1} \int_{\mathbb{R}^N} a(x) u_+^{q+1} dx,
\]

where \( u_+ = \max\{0,u\} \). By \((a_2)\) the functional \( f_{\lambda} \) is well defined and \( C^2 \) on \( D^{1,2}(\mathbb{R}^N) \), except for the case \( q = 1 \).

If \( q = 1 \), we shall consider the \( C^2 \) energy functional \( I_{\lambda} \) associated with problem (3) and is defined by \( I_{\lambda} : D^{1,2}(\mathbb{R}^N) \to \mathbb{R} \)

\[
 I_{\lambda}(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 - \mu \frac{u^2}{|x|^2}) dx - \frac{1}{2} \int_{\mathbb{R}^N} |u|^2 dx + \frac{\lambda}{2} \int_{\mathbb{R}^N} a(x) u^2 dx.
\]

Solutions of problem (3) can be found as critical points of \( f_{\lambda} \) and \( I_{\lambda} \), respectively, in the case \( 1 < q < 2^* - 1 \) and \( q = 1 \).

Now, let us recall some well known facts. Considering the the variational problem

\[
 S_{\mu} = \inf_{u \in D^{1,2}(\mathbb{R}^N), u \neq 0} \frac{\int_{\mathbb{R}^N} (|\nabla u|^2 - \frac{\mu u^2}{|x|^2}) dx}{\left(\int_{\mathbb{R}^N} |u|^2 dx\right)^{\frac{2}{q}}}. 
\]

All the minimizers of \( S_{\mu} \) are given by the functions

\[
 z_{\varepsilon}(x) = \varepsilon^{\frac{N-2}{2}} \frac{x}{|x|^2},
\]

with \( \varepsilon > 0 \).

Letting \( Z = \{ z_{\varepsilon} : \varepsilon > 0 \} \subset D^{1,2}(\mathbb{R}^N) \), \( Z \) is a 1-dimensional manifold of critical points of \( f_0 \) and \( I_0 \), diffeomorphic to \((0, +\infty)\). It is worth pointing out that \( Z \subset \)
Lemma 3.1. \( T_{\varepsilon}Z = \ker D^2 f_0(z_{\varepsilon}) \subset D^{1,2}(\mathbb{R}^N). \)

By applying the abstract method in Section 2, we construct the perturbed manifold

\[ Z_\lambda = \{ z_\varepsilon + w(\lambda, \varepsilon) \}, \]

where \( w(\lambda, \varepsilon) \in (T_{\varepsilon}Z)^\perp \) and \( \| w(\lambda, \varepsilon) \| = o(1) \) for small \( |\lambda| \).

At this point, let us introduce the auxiliary function \( \Gamma : Z \to \mathbb{R} \) defined by setting

\[ \Gamma(\varepsilon) = \Gamma(z_\varepsilon) = \frac{1}{q + 1} \int_{\mathbb{R}^N} a(x) z_\varepsilon^{q+1}(x) \, dx, \]

with \( 1 \leq q < 2^* - 1 \).

By the well known abstract method we then have

\[ f_\lambda(u) = \frac{1}{N} S^{N}_{\mu} + \lambda \Gamma(\varepsilon) + o(\lambda), \]

for any \( u = z_\varepsilon + w(\lambda, \varepsilon) \in Z_\lambda \) and \( |\lambda| \) sufficiently small, where \( S^{N}_{\mu} = f_0(z_\varepsilon) \).

Similarly, we have

\[ I_\lambda(u) = \frac{1}{N} S^{N}_{\mu} + \lambda \Gamma(\varepsilon) + o(\lambda), \]

for any \( u = z_\varepsilon + w(\lambda, \varepsilon) \in Z_\lambda \) and \( |\lambda| \) sufficiently small, where \( S^{N}_{\mu} = I_0(z_\varepsilon) \).

4. Proof of the main result. In this section, we shall prove Theorem 1.1. In what follows, we always assume that (a1)-(a2) and (7) hold. For simplicity, throughout the remainder of the paper, we also denote by \( C > 1 \) a universal positive constant.

Lemma 4.1. \( z^{q+1}(x) \in L^1(\mathbb{R}^N). \)

Proof. First of all, we have

\[
\int_{B_1(0)} z^{q+1}(x) \, dx \leq \int_{B_1(0)} \frac{1}{|x|^{N+\beta}} \, dx = C \int_0^1 \rho^{N+\beta} \, d\rho
\]

\[
= \frac{C}{N + \beta + 1} < +\infty.
\]

Secondly, according to Kelvin translation, we obtain

\[
\int_{B_1(0)} z^{q+1}(x) \, dx \leq C \int_{B_1(0)} \left( \frac{1}{1 + |x| (\sqrt{N} + \sqrt{\rho - \mu})} \right)^{q+1} \, dx
\]

\[
\leq C \int_1^{+\infty} \rho^{-(q+1)(\sqrt{N} + \sqrt{\rho - \mu}) + N-1} \, d\rho = C \rho^{N-(q+1)(\sqrt{N} + \sqrt{\rho - \mu})} \big|_1^{+\infty}.
\]

By (7), we have \( N - (q + 1)(\sqrt{N} + \sqrt{\rho - \mu}) < 0 \). The last inequality shows that \( \int_{B_1(0)} z^{q+1}(x) \, dx < +\infty \). Therefore, we have

\[
\int_{\mathbb{R}^N} z^{q+1}(x) \, dx = \int_{B_1(0)} z^{q+1}(x) \, dx + \int_{B_1(0)} z^{q+1}(x) \, dx < +\infty,
\]

and hence \( z^{q+1}(x) \in L^1(\mathbb{R}^N). \quad \Box \)
Lemma 4.2. There exists $\epsilon \in (0, +\infty)$ such that $\Gamma$ has a global maximum at $\epsilon$.

Proof. First of all, by a change of variable, we have

$$\Gamma(\epsilon) = \frac{\epsilon^{-\alpha}}{q + 1} \int_{\mathbb{R}^N} a(x)z^{q+1}(\frac{z}{\epsilon})dx = \frac{\epsilon^{N-\alpha}}{q + 1} \int_{\mathbb{R}^N} a(\epsilon x)z^{q+1}(x)dx, \quad (9)$$

where $\alpha = \frac{(q+1)(N-2)}{2}.$

Now, we notice that if (7) holds, by Lemma 4.1, we have $z^{q+1}(x) \in L^1(\mathbb{R}^N).$ Moreover, since $a(x)$ is bounded we get $\int_{\mathbb{R}^N} a(\epsilon x)z^{q+1}(x)dx \leq \|a\|_{\infty} \int_{\mathbb{R}^N} z^{q+1}(x)dx.$ As a consequence, since $N > \alpha$, by (9) we infer $\Gamma(\epsilon) \to 0$ as $\epsilon \to 0^+.$

Secondly, by assumption $(a_1)$, we can fix $1 < r < \frac{2^*}{2^* - (q+1)}$ such that $a(x) \in L^r(\mathbb{R}^N).$ Moreover, let be $s = \frac{\alpha}{q+1}$. It is immediately check up that $(q+1)s > 2^*$ and then $z^{(q+1)s} \in L^1(\mathbb{R}^N).$ By $(a_1)$ and Hölder inequality, we deduce that

$$\int_{\mathbb{R}^N} a(\epsilon x)|z|^{q+1}(x)dx \leq \left(\int_{\mathbb{R}^N} |a(\epsilon x)|^r dx\right)^{\frac{1}{r}} \left(\int_{\mathbb{R}^N} |z|^{q+1+s}(x)dx\right)^{\frac{1}{s}} \leq \epsilon^{\frac{N-\alpha}{2}} \int_{\mathbb{R}^N} |a(x)|^r dx \left(\int_{\mathbb{R}^N} |z|^{q+1+s}(x)dx\right)^{\frac{1}{s}}.$$

As a consequence, by the above inequality, we infer for $\epsilon$ large enough

$$\Gamma(\epsilon) = \frac{\epsilon^{N-\alpha}}{q + 1} \int_{\mathbb{R}^N} a(\epsilon x)|z|^{q+1}(x)dx \leq \frac{\epsilon^{N-\alpha - \frac{N}{r}}}{q + 1} \int_{\mathbb{R}^N} |a(x)|^r dx \left(\int_{\mathbb{R}^N} |z|^{q+1+s}(x)dx\right)^{\frac{1}{s}}.$$

Now, we notice that $r < \frac{2^*}{2^* - (q+1)}$ implies $N - \alpha - \frac{N}{r} < 0$ and thus by the above inequality we can conclude that $\Gamma(\epsilon)$ tends to $0$ as $\epsilon \to +\infty.$

In the end, since $\Gamma$ is continuous function on $Z$ such that $\Gamma(\epsilon) \to 0$ as $\epsilon \to 0^+$ and $\epsilon \to +\infty$, then $\Gamma$ has a global maximum at some $\epsilon$ with $\epsilon > 0.$

Proof of Theorem 1.1. We firstly consider the case $q > 1$. By virtue of Lemma 4.2 and Theorem 2.1, we conclude that the functional $f_{\lambda}$ has a critical point $u_{\lambda} = z\tau + w(\lambda, \tau) \in Z_{\lambda}$ for $\lambda$ small enough and $u_{\lambda} \to z\tau$ in $D^{1,2}_{\text{loc}}(\mathbb{R}^N)$ as $\lambda \to 0.$ Clearly $u_{\lambda} \geq 0$. Finally, we can apply the Harnack-type inequality in Theorem 1.1 of [11] to prove that $u_{\lambda} > 0$.

Now, we aim to focus on the case $q = 1$. We shall certify that $u_{\lambda}$ is positive for $|\lambda|$ sufficiently small. Firstly, we aim to prove that $u_{\lambda}$ cannot change its sign. By contradiction, we assume that $u_{\lambda} = u_{\lambda}^+ + u_{\lambda}^-$ with $u_{\lambda}^+ \neq 0$ and $u_{\lambda}^- \neq 0$. For convenience, we denote $u^+ = u_{\lambda}^+$ and $u^- = u_{\lambda}^-$. Since $u_{\lambda}$ solves equation (3), we deduce that

$$\|u^\pm\|^2 = -\lambda \int_{\mathbb{R}^N} a(x)(u^\pm)^2 dx + \int_{\mathbb{R}^N} |u^\pm|^2 dx + \int_{\mathbb{R}^N} \mu \frac{(u^\pm)^2}{|x|^2} dx. \quad (10)$$

At this moment we define

$$C_{\lambda, \mu} = \inf_{u \in D^{1,2}_{\text{loc}}(\mathbb{R}^N), u \neq 0} \frac{\int_{\mathbb{R}^N} (|\nabla u|^2 - \mu \frac{u^2}{|x|^2}) dx + \lambda \int_{\mathbb{R}^N} a(x)u^2 dx}{(\int_{\mathbb{R}^N} |u|^2 dx)^{\frac{2}{2^*}}}.$$

Since $a(x) \in L^{\frac{N}{N-2}}(\mathbb{R}^N)$ and the embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N, |x|^{-2} dx)$ is continuous, the real constant $C_{\lambda, \mu}$ is well defined. Moreover, for $|\lambda|$ small enough, $C_{\lambda, \mu} > 0$.
and $C_{\lambda,\mu} \to S_{\mu}$ as $\lambda \to 0$. Moreover, since $u^\pm \neq 0$, we have
\[
\int_{\mathbb{R}^N} (|\nabla u^\pm|^2 - \frac{\mu(u^\pm)^2}{|x|^2})dx + \lambda \int_{\mathbb{R}^N} a(x)(u^\pm)^2dx \geq C_{\lambda,\mu}(\int_{\mathbb{R}^N} |u^\pm|^2 dx)^{\frac{2}{2^*}},
\]
for $|\lambda|$ small enough. By (10) and (11) we can deduce
\[
\int_{\mathbb{R}^N} |u^\pm|^2 = \|u\|^2 - \int_{\mathbb{R}^N} \frac{\mu(u^\pm)^2}{|x|^2}dx + \lambda \int_{\mathbb{R}^N} a(x)(u^\pm)^2dx
\geq C_{\lambda,\mu}(\int_{\mathbb{R}^N} |u^\pm|^2 dx)^{\frac{2}{2^*}},
\]
and hence by (12) it follows that
\[
\int_{\mathbb{R}^N} |u^\pm|^2 \geq C_{\lambda,\mu}^\frac{2}{2^*}.
\]
Finally, by (10) and (13), we have
\[
I_{\lambda}(u_\lambda) = I_{\lambda}(u^+ + u^-)
= \left(\frac{1}{2} - \frac{1}{2^*}\right)(\int_{\mathbb{R}^N} |u^+|^2 dx + \int_{\mathbb{R}^N} |u^-|^2 dx)
\geq 2\left(\frac{1}{2} - \frac{1}{2^*}\right)C_{\lambda,\mu}^\frac{2}{2^*},
\]
for $|\lambda|$ small enough. On the other hand, since $u_\lambda \to z_\sigma$ in $D^{1,2}(\mathbb{R}^N)$ as $\lambda \to 0$, we get
\[
I_{\lambda}(u_\lambda) \to (\frac{1}{2} - \frac{1}{2^*})S_\mu^\frac{N}{2}, \quad \lambda \to 0,
\]
which contradicts (14). This contradiction shows that $u_\lambda$ can not change its sign for $|\lambda|$ sufficiently small. Without loss of generality, we assume that $u_\lambda \geq 0$. Then we can apply the Harnack-type inequality in [11] to prove that $u_\lambda > 0$. \hfill $\square$

5. Anisotropic Schrödinger equations. In this section, we shall consider the existence of positive solutions of the elliptic problem
\[
\begin{cases}
-\text{div}(B_\lambda(x)\nabla u) + \lambda a(x)u^q = \mu \frac{u}{|x|^2} + |u|^{2^*-2}u, & \text{in } \mathbb{R}^N, \\
u > 0, & \text{in } D^{1,2}(\mathbb{R}^N),
\end{cases}
\]
where $N \geq 3, \lambda \in \mathbb{R}, 0 \leq \mu < \mu_0 = \frac{(N-2)^2}{4}, a(x)$ satisfies assumptions $(a_1)-(a_2)$. Furthermore, we shall consider $B_\lambda(x) = I + \lambda B(x)$, where $I$ is the identity matrix and $B(x) = \{b_{i,j}(x)\}$ is a real, $N \times N$ matrix which satisfies
$(B_1)$ for any $i, j = 1, \ldots, N$, $b_{i,j} \in C(\mathbb{R}^N)$, $b_{i,j} \geq 0$, positive somewhere and $\lim_{|x| \to +\infty} b_{i,j}(x) = 0$.

Now, let us denote $\|\cdot\|$ the usual norm on the space of real, $N \times N$ matrices, $A = \{a_{i,j}\}$, defined by $\|A\| = (\sum_j \sum_i a_{i,j}^2)^{\frac{1}{2}}$.

Furthermore, we shall assume that
$(B_2)$ $\lim_{|x| \to 0} |x|^\alpha \|B(x) - B(0)\| = 0$, where $\alpha = \frac{(N-2)(q+1)}{2}$.

We shall prove the following result:

**Theorem 5.1.** Assume $(a_1)-(a_2)$ and $(B_1)-(B_2)$. Moreover; suppose that (7) holds. Then there exist $\lambda > 0$ and $\sigma \in (0, +\infty)$ such that for any $\lambda \in \mathbb{R}, |\lambda| \leq \lambda$, problem (15) has a positive solution $u_\lambda$ and
\[
u_{\lambda} \to z_\sigma.
Theorem 5.2. Assume \((a_1) - (a_2)\) and suppose that \((7)\) holds. Moreover, suppose that
\begin{enumerate}
\item \(b \in C(\mathbb{R}^N), b(x) \geq 0, \text{ positive somewhere and } \lim_{|x| \to +\infty} b(x) = 0;\)
\item \(\lim_{x \to 0} |x|^{N-\alpha}|b(x) - b(0)| = 0, \text{ where } \alpha = \frac{(N-2)(q+1)}{2}.\)
\end{enumerate}
Then there exist \(\overline{X} > 0\) and \(\epsilon \in (0, +\infty)\) such that for any \(\lambda \in \mathbb{R}, |\lambda| \leq \overline{X}\), problem \((16)\) has a positive solution \(u_\lambda\) and
\[ u_\lambda \to z_\epsilon \]
in \(D^{1,2}(\mathbb{R}^N)\) as \(\lambda \to 0.\)

As in Section 3, we distinguish the different cases \(1 < q < 2^* - 1\) and \(q = 1.\) If \(1 < q < 2^* - 1,\) solutions of \((15)\) are found as critical points of the energy functional
\[ g_\lambda : D^{1,2}(\mathbb{R}^N) \to \mathbb{R} \]
defined by
\[
g_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 - \mu \frac{u^2}{|x|^2}) dx - \frac{1}{2^*} \int_{\mathbb{R}^N} |u|^{2^*} dx
+ \lambda \int_{\mathbb{R}^N} (B(x)|\nabla u|^2 + \mu \frac{u^2}{|x|^2}) dx + \frac{\lambda}{q + 1} \int_{\mathbb{R}^N} a(x)u^{q+1} dx.
\]
The functional \(g_\lambda\) is well defined and \(C^2\) on \(D^{1,2}(\mathbb{R}^N).\)

Otherwise if \(q = 1,\) we shall consider the energy functional
\[ J_\lambda : D^{1,2}(\mathbb{R}^N) \to \mathbb{R} \]
defined by
\[
J_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 - \mu \frac{u^2}{|x|^2}) dx - \frac{1}{2^*} \int_{\mathbb{R}^N} |u|^{2^*} dx
+ \lambda \int_{\mathbb{R}^N} (B(x)|\nabla u|^2 + \mu \frac{u^2}{|x|^2}) dx + \frac{\lambda}{q + 1} \int_{\mathbb{R}^N} a(x)u^2 dx.
\]
The functional \(J_\lambda\) is well defined and \(C^2\) on \(D^{1,2}(\mathbb{R}^N).\)

Now, let us introduce the auxiliary function \(\Gamma : Z \to \mathbb{R}\) defined by setting
\[
\Gamma(z_\epsilon) = \Gamma(\epsilon) = \Gamma_1(\epsilon) + \Gamma_2(\epsilon),
\]
where
\[
\Gamma_1(\epsilon) = \frac{1}{2} \int_{\mathbb{R}^N} (B(x)\nabla z_\epsilon |\nabla z_\epsilon)| dx = \frac{1}{2} \int_{\mathbb{R}^N} (B(\epsilon x)\nabla z |\nabla z)| dx,
\]
and
\[
\Gamma_2(\epsilon) = \frac{1}{q + 1} \int_{\mathbb{R}^N} a(\epsilon x)z_\epsilon^{q+1} dx = \frac{\epsilon^{N-\alpha}}{q + 1} \int_{\mathbb{R}^N} a(\epsilon x)z^{q+1} dx, \quad 1 \leq q < 2^* - 1.\]

It is not hard to check that
\[
g_\lambda(u) = \frac{1}{N} S_{\mu}^{\frac{N}{2}} + \lambda \Gamma(\epsilon) + o(\lambda)
\]
for any \(u = z_\epsilon + w(\lambda, \epsilon) \in Z_\lambda\) and \(|\lambda|\) sufficiently small, where \(S_{\mu}^{\frac{N}{2}} = g_0(z_\epsilon).\)
Analogously, we have

\[ J_\lambda(u) = \frac{1}{N} S_\mu^N + \lambda \Gamma(\varepsilon) + o(\lambda) \]

for any \( u = z_\varepsilon + w(\lambda, \varepsilon) \in Z_\lambda \) and \( |\lambda| \) sufficiently small, where \( S_\mu^N = J_0(z_\varepsilon) \).

Before proving Theorem 5.1, we study the qualitative behaviour of the function \( \Gamma \). In particular, we shall prove that \( \Gamma \) has a local maximum at \( \varepsilon > 0 \) for small \( b \).

Moreover, since \( \mu \) is a suitable positive constant. Furthermore, by the continuity of the coefficients \( b_{i,j}(x) \) and by applying Dominated Convergence Theorem, we obtain

\[ \lim_{\varepsilon \to 0^+} \Gamma_{1}(\varepsilon) = \lim_{\varepsilon \to 0^+} \frac{1}{2} \int_{\mathbb{R}^N} (B(\varepsilon x) \nabla z)(\nabla z) dx = \frac{1}{2} \int_{\mathbb{R}^N} (B(0) \nabla z)(\nabla z) dx. \]

Moreover, thanks to Lemma 4.2, it yields that

\[ \lim_{\varepsilon \to 0^+} \Gamma_{1}(\varepsilon) = \frac{1}{2} \int_{\mathbb{R}^N} (B(0) \nabla z)(\nabla z) dx. \]

Next, by Lemma 4.2 we have

\[ \lim_{\varepsilon \to +\infty} \Gamma_{2}(\varepsilon) = 0. \]

Moreover, since \( b_{i,j}(x) \to 0 \) as \( |x| \to +\infty \), we can check that \( (B(\varepsilon x) \nabla z)(\nabla z) \to 0 \) as \( \varepsilon \to +\infty \). Moreover, by (17) and by applying Dominated Convergence Theorem, we deduce that

\[ \lim_{\varepsilon \to +\infty} \Gamma_{1}(\varepsilon) = \lim_{\varepsilon \to +\infty} \frac{1}{2} \int_{\mathbb{R}^N} (B(\varepsilon x) \nabla z)(\nabla z) dx = 0. \]

As a consequence, we have \( \Gamma(\varepsilon) \to 0 \) as \( \varepsilon \to +\infty \).

Last of all, it is sufficient to prove that \( \Gamma(\varepsilon) \geq \Gamma(0) := \frac{1}{2} \int_{\mathbb{R}^N} (B(0) \nabla z)(\nabla z) dx \), for \( \varepsilon > 0 \) sufficiently small.

Firstly, we notice that

\[ \Gamma_{2}(\varepsilon) = \frac{N - \alpha}{q + 1} \int_{\mathbb{R}^N} a(\varepsilon x) z^{q+1}(x) dx \geq \frac{N - \alpha}{q + 1} \int_{\mathbb{R}^N} a(\varepsilon x) z^{q+1}(x) dx \geq \frac{N - \alpha}{q + 1} \nu \int_{\mathbb{R}^N} z^{q+1}(x) dx. \]

Since \( z^{q+1} \in L^1(\mathbb{R}^N) \) and \( \int_{\mathbb{R}^N} z^{q+1}(x) dx \to \int_{\mathbb{R}^N} z^{q+1}(x) dx > 0 \) as \( \varepsilon \to 0^+ \), we deduce for small \( \varepsilon > 0 \) and some constant \( c_1 > 0 \)

\[ \Gamma_{2}(\varepsilon) \geq c_1 \varepsilon^{N-\alpha}. \]

On the other hand, we have

\[ \frac{|\Gamma_{1}(\varepsilon) - \Gamma_{1}(0)|}{\varepsilon^{N-\alpha}} \leq C \int_{\mathbb{R}^N} \frac{||B(\varepsilon x) - B(0)||}{\varepsilon^{N-\alpha}} |\nabla z(x)|^2 dx \]
By assumption (B2), we infer for \( \varepsilon \) small enough
\[
|\frac{\|B(\varepsilon x) - B(0)\|}{\varepsilon^{N-\alpha}}|\nabla z(x)|^2 \leq |x|^{N-\alpha}|\nabla z(x)|^2.
\]
We notice that
\[
|x|^{N-\alpha}|\nabla z(x)|^2
= |x|^{N-\alpha}C^2_N\frac{|x|^{2(\frac{N-2}{2}-\beta)-1}(1 + |x|^{\frac{4\beta}{N-2}})^{N-4}(\frac{N-2}{2} - \beta)(1 + |x|^{\frac{4\beta}{N-2}}) + 2|x|^{\frac{4\beta}{N-2}}|^2}{|x|^{4(\frac{N-2}{2}-\beta)}(1 + |x|^{\frac{4\beta}{N-2}})^2(N-2)}
= |x|^{N-\alpha}C^2_N\frac{((\frac{N-2}{2} - \beta)(1 + |x|^{\frac{4\beta}{N-2}}) + 2|x|^{\frac{4\beta}{N-2}})^2}{|x|^{N-2\beta}(1 + |x|^{\frac{4\beta}{N-2}})^N},
\]
and thus \( \alpha > 2\beta \) and then we infer \( |x|^{N-\alpha}|\nabla z(x)|^2 \in L^1(\mathbb{R}^N) \). As a consequence, by (18) and by applying the Dominated Convergence Theorem, we deduce that
\[
\lim_{\varepsilon \to 0^+} \frac{|\Gamma_1(\varepsilon) - \Gamma_1(0)|}{\varepsilon^{N-\alpha}} = 0.
\]
Hence, we have
\[
|\Gamma_1(\varepsilon) - \Gamma_1(0)| = o(\varepsilon^{N-\alpha}),
\]
as \( \varepsilon \to 0^+ \).

We can conclude that \( \Gamma(\varepsilon) \) achieves its maximum at some point \( \varepsilon \) with \( \varepsilon > 0 \). \( \square \)

**Proof of Theorem 5.1.** We first consider the case \( q > 1 \). By Lemma 5.3 and Theorem 2.1, we infer that the functional \( g_\lambda \) has a critical point \( u_\lambda = z_\varepsilon + w(\lambda, \varepsilon) \in Z_\lambda \) for \( |\lambda| \) small enough and \( u_\lambda \to z_\varepsilon \) in \( D^{1,2}(\mathbb{R}^N) \) as \( \lambda \to 0 \). Clearly \( u_\lambda \geq 0 \). Moreover, by applying the Harnack type inequality in [11] we get \( u_\lambda > 0 \).

We aim to focus on the case \( q = 1 \). As before by Lemma 5.3 and Theorem 2.1, we infer that the functional \( J_\lambda \) has a critical point \( u_\lambda = z_\varepsilon + w(\lambda, \varepsilon) \in Z_\lambda \) for \( |\lambda| \) small enough. Arguing as in the proof of the Theorem 1.1 for the case \( q = 1 \), it is possible to show that \( u_\lambda \) cannot change its sign. Therefore we can assume \( u_\lambda \geq 0 \). Finally, we can apply the Harnack-type inequality to prove that \( u_\lambda > 0 \). \( \square \)

**Acknowledgments.** We would like to thank the editor and the referee for their valuable comments which have led to an improvement of the presentation of this paper.

**REFERENCES**

[1] A. Ambrosetti and Andrea Malchiodi, *Perturbation Methods and Semilinear Elliptic Problems on \( \mathbb{R}^N \)*, Birkhäuser Verlag, 2006.

[2] A. Ambrosetti, J. García Azorero and I. Peral, *Perturbation of \( \Delta u + u^{\frac{N+2}{N-2}} = 0 \). The scalar curvature problems in \( \mathbb{R}^N \) and related topics*, J. Funct. Anal., **165** (1999), 117–149.

[3] A. Ambrosetti and M. Badiale, *Variational perturbative methods and bifurcation of bounds states from the essential spectrum*, Proc. Roy. Soc. Edinburgh A, **128** (1998), 1131–1161.

[4] A. Ambrosetti, M. Badiale and S. Cingolani, *Semiclassical states of nonlinear Schrödinger equations*, Arch. Rational Mech. Anal., **140** (1997), 285–300.

[5] M. Badiale, J. García Azorero and I. Peral, *Perturbation results for an anisotropic Schrödinger equation via a variational form*, NoDEA, **7** (2000), 201–230.

[6] H. Berestycki and P. L. Lions, *Nonlinear scalar field equations I – existence of a ground state*, Arch. Rational Mech. Anal., **82** (1983), 313–345.

[7] K. J. Brown and N. Stavrakakis, *Global bifurcation results for a semilinear elliptic equation on all \( \mathbb{R}^N \)*, Duke Math. J., **85** (1996), 77–94.
[8] F. Catrina and Z. Q. Wang, On the Caffarelli-Kohn-Nirenberg inequalities: Sharp constants, existence (and nonexistence), and symmetry of external functions, *Comm. Pure Appl. Math.*, 54 (2001), 229–258.

[9] S. Cingolani, Positive solutions to perturbed elliptic problems in $\mathbb{R}^N$ involving critical Sobolev exponent, *Nonlinear Analysis*, 48 (2002), 1165–1178.

[10] O. Rey, The role of the Green’s function in a non-linear elliptic equation involving the critical Sobolev exponent, *J. Funct. Anal.*, 89 (1990), 1–52.

[11] N. S. Trudinger, On Harnack type inequalities and their application to quasilinear elliptic equations, *Comm. Pure Appl. Math.*, 20 (1967), 721–747.

Received January 2014; revised April 2016.

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