Perfect Fluid FRW with Time-varying Constants Revisited.

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In this paper we revise a perfect fluid FRW model with time-varying constants “but” taking into account the effects of a “c-variable” into the curvature tensor. We study the model under the following assumptions, \( \text{div}(T) = 0 \) and \( \text{div}(T) \neq 0 \), and in each case the flat and the non-flat cases are studied. Once we have outlined the new field equations, it is showed in the flat case i.e. \( K = 0 \), that there is a non-trivial homothetic vector field i.e. that this case is self-similar. In this way, we find that there is only one symmetry, the scaling one, which induces the same solution that the obtained one in our previous model. At the same time we find that “constants” \( G \) and \( c \) must verify, as integration condition of the field equations, the relationship \( G/c^2 = \text{const.} \) in spite of that both “constants” vary. We also find that there is a narrow relationship between the equation of state and the behavior of the time functions \( G, c \) and the sign of \( \Lambda \) in such a way that these functions may be growing as well as decreasing functions on time \( t \), while \( \Lambda \) may be a positive or negative decreasing function on time \( t \). In the non-flat case it will be showed that there is not any symmetry. For the case \( \text{div}(T) \neq 0 \), it will be studied again the flat and the non-flat cases. In order to solve this case it is necessary to make some assumptions on the behavior of the time functions \( G, c \) and \( \Lambda \). We also find the flat case with \( \text{div}(T) = 0 \), is a particular solution of the general case \( \text{div}(T) \neq 0 \).

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I. INTRODUCTION

Since the pioneering work of Dirac ([1]), who proposed, motivated by the occurrence of large numbers in Universe, a theory with a time variable gravitational coupling constant \( G \), cosmological models with variable \( G \) and nonvanishing and variable cosmological term have been intensively investigated in the physical literature (see for example [2]–[14]).

Recently, the cosmological implications of a variable speed of light during the early evolution of the Universe have been considered. Varying speed of light (VSL) models proposed by Moffat ([15]) and Albrecht and Magueijo ([16]), in which light was travelling faster in the early periods of the existence of the Universe, might solve the same problems as inflation. Hence they could become a valuable alternative explanation of the dynamics and evolution of our Universe and provide an explanation for the problem of the variation of the physical “constants”. Einstein’s field equations (EFE) for Friedmann–Robertson–Walker (FRW) spacetime in the VSL theory have been solved by Barrow ([17]), who also obtained the rate of variation of the speed of light required to solve the flatness and cosmological constant problems (see J. Magueijo ([18]) for a review of these theories).

This model is formulated under the strong assumption that a \( c \) variable (where \( c \) stands for the speed of light) does not introduce any corrections into the curvature tensor, furthermore, such formulation does not verify the covariance and the Lorentz invariance as well as the resulting field equations do not verify the Bianchi identities either (see Bassett et al [19]).

Nevertheless, some authors (T. Harko and M. K. Mak [20] and P.P. Avelino and C.J.A.P. Martins [21]) have proposed a new generalization of General Relativity which also allows arbitrary changes in the speed of light, \( c \), and the gravitational constant, \( G \), but in such a way that variations in the speed of light introduce corrections to the curvature tensor in the Einstein equations in the cosmological frame. This new formulation is both covariant and Lorentz invariant and as we will show the resulting field equations (FE) verify the Bianchi identities. We will be able to obtain the energy conservation equation from the field equations as in the standard case.

The purpose of this paper is to revise a FRW perfect fluid model with time varying constants (see [22] and [23]) but taking into account the effects of a \( c \) variable into the field equations. In our previous models ([22]–[23]) we worked under the assumption that a \( c \)-variable does not induce corrections into the curvature tensor and hence the classical Friedman equations remain valid. We will show that such effect is minimum but exists. In section 2, once we have
studied two particular cases, the flat case, i.e. $K = 0$, and the non-flat case, $K \neq 0$. Since with the proposed method we are not able to solve the so-called flat problem we need to study separately both cases (the flat and the non-flat cases) introducing such conditions as an assumption.

In this approach, in the case $K = 0$, we will show that the model is self-similar since it is found a non-trivial homothetic vector field. In order to integrate the FE under the assumptions $div(T) = 0$ and $K = 0$, we use the Lie group tactic which allows us to find a particular form of $G$ and $c$ for which our FE admit symmetries i.e. are integrable. As we will show under these assumptions the FE, only admit one symmetry, the scaling symmetry. This is the main difference with respect to our previous approach ([22]–[23]) where we did not consider the effects of a $c$-variable into the curvature tensor in such a way that the resulting FE admitted more symmetries. We also obtain as integration condition that the “constants” $G$ and $c$ must verify the relationship $G/c^2 = const.$ in spite of that both “constants” vary. In this work we have found three solutions. The first one is very similar to the de-Sitter solution i.e. the energy density vanishes, the cosmological constant is a true constant while $G$ and $c$ follow an exponential law as the scale factor. The second and third solutions behave as the standard FRW where all the quantities follow power law with respect to time $t$, nevertheless the third solution is non-singular. In the second studied case, the non-flat case $K \neq 0$, we will show that the FE do not admit any symmetry. Nevertheless we will try to find a particular solution imposing some restrictions.

In section 4 we consider the possibility that $div(T) \neq 0$, and we will study again two particular cases, the flat case, i.e. $K = 0$, and the non-flat case, $K \neq 0$. The possibility that the covariant conservation condition $div(T) = 0$ be relaxed has been advanced by Rastall ([24]), who pointed out that a non-zero divergence of the energy–momentum tensor has not been ruled out experimentally at all yet. In order to integrate the resulting FE we will need to make some assumptions (scaling assumptions) about the behavior of the “constants” $G$, $c$ and $\Lambda$, which allow us to find particular solutions to the FE. This scaling assumptions work well in the flat (homothetic) case, as it is expected, but they seem very restrictive in the non-flat case. It is also showed that we can recover the solution obtained in the $div(T) = 0$ as a particular solution of this model in such a way that $G$ and $c$ must verify the relationship $G/c^2 = const.$ etc... as it is expected.

In section 5 we end with a brief conclusions.

II. THE MODEL

We will use the field equations in the form:

$$R_{ij} - \frac{1}{2} g_{ij} R = \frac{8\pi G(t)}{c^4(t)} T_{ij} + \Lambda(t) g_{ij}, \quad (1)$$

where arbitrary variations in $c$ and $G$ will be allowed. We assume that variations in the speed of light introduce corrections to the curvature terms in the Einstein equations in the cosmological frame. In our model variations in the velocity of light are always allowed to contribute to the curvature terms. These contributions are computed from the metric tensor in the usual way. The line element is defined by (we are following the O'Neill’s notation [25]):

$$ds^2 = -c(t)^2 dt^2 + f^2(t) \left[ \frac{dv^2}{1 - Kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right], \quad (2)$$

and the energy momentum tensor is:

$$T_{ij} = (\rho + p) u_i u_j - pg_{ij}, \quad (3)$$

where $p$ and $\rho$ satsify the usual equation of state, $p = \omega \rho$ in such a way that $\omega = const.$, usually $\omega$ is taken such that, $\omega \in (-1, 1)$, that is to say, our universe is modeled by a perfect fluid. The 4–velocity $u^i$ is defined as follows $u^i = (c^{-1}(t), 0, 0, 0)$ such that $u_i u^i = -1$.

The field equations are as follows:

$$2 \frac{f''}{f} - 2 \frac{c^2 f'}{f} + \frac{f'^2}{f^2} + \frac{Kc^2}{f^2} = \frac{8\pi G}{c^2} p + \Lambda c^2, \quad (4)$$

$$\frac{f'^2}{f^2} + \frac{Kc^2}{f^2} = \frac{8\pi G}{3c^2} \rho + \frac{1}{3} \Lambda c^2, \quad (5)$$
where as we can see, the only difference with respect to the usual one is the factor $-\frac{2c^2 f'}{f^2}$ in eq. (4). Here a prime denotes differentiation with respect to time $t$.

Since the divergence of $(R_{ij} - \frac{1}{2}g_{ij}R)$ vanishes then we impose that the right hand of equation (1) has zero divergence too. Therefore applying the covariant divergence to the second member of the field equation we get:

$$\text{div} \left( \frac{8\pi G R_{ij}}{c^4} + \delta^{ij} \Lambda \right) = 0,$$  

(6)

where we are considering that $\frac{8\pi G}{c^4}$ is a function on time $t$. Hence simplifying it yields:

$$\rho' + 3(\omega + 1) \rho H = - \left( \frac{G'}{G} - 4\frac{c'}{c} \right) \rho - \frac{\Lambda' c^4}{8\pi G},$$

(7)

where $H = f'/f$.

Therefore the new field equations are as follows:

$$2H' - 2\frac{c'}{c} H + 3H^2 + \frac{Kc^2}{f^2} = - \frac{8\pi G}{c^4} \rho + \Lambda c^2,$$

(8)

$$H^2 + \frac{Kc^2}{f^2} = \frac{8\pi G}{3c^2} \rho + \frac{1}{3} \Lambda c^2,$$

(9)

$$\rho' + 3(\omega + 1) \rho H = - \left( \frac{G'}{G} - 4\frac{c'}{c} \right) \rho - \frac{\Lambda' c^4}{8\pi G},$$

(10)

As we have mentioned in the introduction these field equations are not new in the literature. They have been outlined by T. Harko and M. Mak ([20]) where they study a perfect fluid model and the influence of the time variation of the constants in the matter creation. Later they study Bianchi I and V model with time varying constants (taking into account the influence of a $c$-variable into the curvature tensor). Other author are P.P. Avelino and C.J.A.P. Martins ([21]) and H. Shojaie and M. Farhoudi, have obtained similar equations ([26]) since these authors consider $G$ as a true constant.

We would like to emphasize that deriving eq. (9) and substituting this result into eq. (8) it is obtained eq.(7) i.e. the covariant divergence of the right hand of our field equation in the same way as in the standard cosmological model.

Curvature is described by the tensor field $R_{ikl}$. It is well known that if one uses the singular behavior of the tensor components or its derivates as a criterion for singularities, one gets into trouble since the singular behavior of the coordinates or the tetrad basis rather than the curvature tensor. In order to avoid this problem, one should examine the scalars formed out of the curvature. The invariants $\text{Riem}S$ and $\text{Ricc}S$ (the Kretschmann scalars) are very useful for the study of the singular behavior, being these as follows:

$$\text{Riem}S = R_{ijlm}R^{ijkl},$$

(11)

$$\text{Riem}S := \frac{12}{c^4} \left[ \left( \frac{f''}{f} \right)^2 - 2 \frac{f''}{f} f' \frac{c'}{c} + \left( \frac{f'}{f} \right)^2 \left( \frac{c'}{c} \right)^2 + \left( \frac{f'}{f} \right)^4 + 2 \left( \frac{f'}{f} \right)^2 \frac{c^2 K}{f^2} + \frac{c^4 K}{f^4} \right],$$

(12)

and

$$\text{Ricc}S := R_{ij}R^{ij},$$

$$\text{Ricc}S := \frac{12}{c^4} \left[ \left( \frac{f''}{f} \right)^2 - 2 \frac{f''}{f} f' \frac{c'}{c} + \left( \frac{f'}{f} \right)^2 \left( \frac{c'}{c} \right)^2 + \left( \frac{f'}{f} \right)^4 + 2 \left( \frac{f'}{f} \right)^2 \frac{c^2 K}{f^2} + \frac{c^4 K}{f^4} \right],$$

$$\frac{f''}{f} \left( \frac{c^2 K}{f^2} + \left( \frac{f'}{f} \right)^2 \right) - \frac{f'}{f} \frac{c'^2 K}{f c f^2} - \frac{c'}{c} \left( \frac{f'}{f} \right)^3 \right].$$

(13)

In order to try to solve eqs. (8)-(10) we need to make some hypotheses on the behavior of the quantities.
In this first solution we make the following assumption

\[ \text{div} T = 0 \]  

and we will consider two subclasses, the flat and the non-flat cases.

Our tactic consists in studying the field equations through the Lie method (see [27]–[28]–[29]). In particular we seek the forms of \( G \) and \( c \) for which our field equations admit symmetries i.e. are integrable. We will consider the following assumption \( \text{div} T = 0 \), which transforms eq. (10) into these two new equations:

\[ \rho' + 3(\omega + 1)\rho H = 0, \]  

\[ \frac{G'}{G} - \frac{4c'}{c} = -\frac{Nc^4}{8\pi G\rho}. \]  

In order to use the Lie method, we rewrite the field equations as follows. From (8 and 9) we obtain

\[ 2H' - \frac{2c'}{c}H - 2\frac{Kc^2}{f^2} = -\frac{8\pi G}{c^2} (p + \rho), \]  

From equation (15), we can obtain

\[ H = -\frac{1}{3(\omega + 1)} \frac{\rho'}{\rho} \Rightarrow H = -C_1 \frac{\rho'}{\rho}, \]  

where \( C_1 = \frac{1}{3(\omega + 1)} \), therefore

\[ -2C_1 \left( \frac{\rho''}{\rho} - \frac{\rho'^2}{\rho^2} \right) + 2C_1 \frac{\rho'}{\rho} \frac{c'}{c} - 2\frac{Kc^2}{f^2} = -\frac{8\pi G}{c^2} (\omega + 1)\rho, \]  

hence

\[ \rho'' = \frac{\rho'^2}{\rho} + \rho' \frac{c'}{c} - 2\frac{Kc^2}{f^2}\rho + A \frac{G}{c^2}\rho^2, \]  

where \( A = 12\pi (\omega + 1)^2 > 0, \forall \omega. \) Following this tactic we try to make the smallest hypothesis number and to obtain the exact behavior of the “constants” \( G, c \) and \( \Lambda \). Following other tactics we are obliged to make assumptions that could be unphysical.

### A. Flat case. Self-similar approach.

As have been pointed out by Carr and Coley ([30]), the existence of self-similar solutions (Barenblatt and Zeldovich ([31])) is related to conservation laws and to the invariance of the problem with respect to the group of similarity transformations of quantities with independent dimensions. This can be characterized within general relativity by the existence of a homothetic vector field and for this reason one must distinguish between geometrical and physical self-similarity. Geometrical similarity is a property of the spacetime metric, whereas physical similarity is a property of the matter fields (our case). In the case of perfect fluid solutions admitting a homothetic vector, geometrical self-similarity implies physical self-similarity.

As we show in this section as well as in previous works, the assumption of self-similarity reduces the mathematical complexity of the governing differential equations. This makes such solutions easier to study mathematically. Indeed self-similarity in the broadest Lie sense refers to an invariance which allows such a reduction.

Perfect fluid space-times admitting a homothetic vector within general relativity have been studied by Eardley ([32]). In such space-times, all physical transformations occur according to their respective dimensions, in such a way that geometric and physical self-similarity coincide. It is said that these space-times admit a transitive similarity group and space-times admitting a non-trivial similarity group are called self-similar. Our model i.e. a flat FRW model with a perfect fluid stress-energy tensor has this property and as already have been pointed out by Wainwright ([33]), this model has a power law solution.
Under the action of a similarity group, each physical quantity \( \phi \) transforms according to its dimension \( q \) under the scale transformation. For spacetime with a transitive similarity group, dimensionless quantities are therefore spacetime constants. This implies that the ratio of the pressure of the energy density is constant so that the only possible equation of state is the usual one in cosmology i.e. \( p = \omega \rho \), where \( \omega \) is a constant. In the same way, the existence of homothetic vector implies the existence of conserved quantities.

In the first place we would like to emphasize that under the hypothesis \( K = 0 \), we have found that the space-time \((M, g)\) is self-similar since we have been able to calculate a homothetic vector field, \( X_H \in \mathfrak{X}(M) \), that is to say

\[
L_X g = 2g,
\]

where

\[
X_H = \left( \frac{\int c(t)dt + C_1}{c(t)} \right) \partial_t + \left( 1 - \frac{\int c(t)dt + C_1}{c(t)} \right) H \left( x\partial_x + y\partial_y + z\partial_z \right).
\]

This kind of spacetimes have been studied by (Eardly ([32]), Wainwright ([33]), K. Rosquits and R. Jantzen ([34]) and for reviews see Carr and Coley ([30]) and Duggal et al ([35])).

As we have mentioned above we are only interested in the case \( K = 0 \) for this reason eq. (20) yields

\[
\rho'' = \frac{\rho'^2}{\rho} + \rho \frac{c'}{c} + A\frac{G'}{c^2}\rho^2.
\]

Now, we apply the standard Lie procedure to this equation. A vector field \( X = \xi(t, \rho)\partial_t + \eta(t, \rho)\partial_\rho \), is a symmetry of (23) iff

\[
\begin{align*}
\xi_\rho + \rho \xi_{\rho\rho} &= 0, \\
\eta c - 2c' \rho^2 \xi_\rho - c \rho \eta_\rho + c \rho^2 \eta_{\rho\rho} - 2c^2 \rho \xi_{\rho \rho} &= 0, \\
-cc' \rho \xi_t - 3c^2 AG \xi_\rho + \left( c^2 \rho - cc' \rho \right) \xi_t + 2c^2 \rho \eta_\rho - c^2 \rho \xi_{\rho \rho} - 2c^2 \eta_t &= 0, \\
\rho^2 AG \eta_\rho + c^2 \eta_{\rho\rho} - cc' \eta_t - 2\rho AG \eta - 2\rho^2 AG \xi_t + \rho^2 A\xi \left( 2G' - Gc' \right) &= 0,
\end{align*}
\]

Solving (24-27), we find that

\[
\xi = at + b, \quad \eta = -2a\rho,
\]

subject to the following constrains, from eq. (26):

\[
c'' = \frac{c'^2}{c} - \frac{ac'}{at + b},
\]

and from eq. (27)

\[
\frac{G'}{G} = 2\frac{c'}{c},
\]

where \( a \) and \( b \) are numerical constants.

We would like to emphasize that this is the main difference with respect to our previous work ([22]) where we found that the model admitted more symmetries. Note that in this paper we are considering that \( c \)-variable introduces corrections into the curvature tensor, this possibility brings us to obtain a new eq. (23) which contains an extra term \( \left( \rho'\frac{c'}{c} \right) \) with respect to the employed one in our previous paper ([22]).

In this situation we have found that this model only admits two symmetries \( X_1 = \partial_t \), i.e. a movement and \( X_2 = t\partial_t - 2\rho\partial_\rho \), i.e. a scaling symmetry as it is expected in this kind of models (self-similar model). In such a way that \([X_1, X_2] = X_1\) i.e. they form a \( L_2 \) Lie-algebra, see ([28]).

From (30) we can see that

\[
G \approx c^2 \quad i.e. \quad \frac{G}{c^2} = \text{const} := B,
\]
(where we will assume that $B > 0$) that is to say, that both constants vary but in such a way that the relationship $\frac{G}{c^2}$ remains constant for any $t$ and independently of the any value of the constants $a$ and $b$. For this reason our equation (23) may be rewritten as follows

$$\rho'' = \frac{\rho'^2}{\rho} + \rho' \frac{c'}{c} + A \rho^2,$$

where $A = 12\pi (\omega + 1)^2 B = const > 0, \forall \omega, (\omega \neq -1)$, in such a way that all the restrictions come from eq. (29).

The knowledge of one symmetry $X$ might suggest the form of a particular solution as an invariant of the operator $X$, i.e. as solution of

$$\frac{dt}{\xi} = \frac{d\rho}{\eta},$$

this particular solution is known as an invariant solution (generalization of similarity solution) furthermore an invariant solution is in fact a particular singular solution.

In order to solve (32), we consider the following cases.

1. Case I.

Taking $a = 0, b \neq 0$, we get

$$c'' = \frac{c^2}{c}, \implies c(t) = K_2 e^{K_1 t},$$

where the $(K_i)_{i=1}$ are integration constants. In this way we find that from eq. (30) $G$ behaves as:

$$G = K_2^2 e^{2K_1 t}.$$

Therefore the equation (32) yields:

$$\rho'' = \frac{\rho'^2}{\rho} + \rho' K_1 + A \rho^2,$$

where $X_1 = \partial_t$. The use of the canonical variables brings us to an Abel ode:

$$y' = -Ax^2 y^3 - K_1 y^2 - \frac{y}{x},$$

where $x = \rho$ and $y = 1/\rho'$, since we cannot solve it, then we try to obtain a solution through the invariants

This method brings us to the following relationship

$$\frac{dt}{\xi} = \frac{d\rho}{\eta} \implies \rho \approx const := \rho_0,$$

which seems not to be physical. This particular solution must satisfy eq. (32) which means that $\rho = 0$. With this solution we go back to eq. (9)

$$3H^2 = \Lambda c^2,$$

and from eq. (16) it is obtained the behavior of “constant” $\Lambda$

$$\left(\frac{G'}{G} - 4 \frac{c'}{c}\right) \rho = \frac{N c^4}{8\pi G} = 0 \implies \Lambda = \Lambda_0 = const.,$$

therefore

$$3H^2 = \Lambda_0 K_2^2 e^{2K_1 t} \implies f = C_1 \exp \left(\frac{\sqrt{\frac{3H^2}{K_1}}}{\exp(K_1 t)}\right),$$
This symmetry has brought us to obtain a super de Sitter solution, with a energy density vanishing and with a scale factor growing like a power of exponential functions, while $G$ and $c$ follow an exponential functions that depends of the constant $K_1$. If for example we fix $K_1 = 1$, then there is a sudden singularity in a finite time, in this case both “constants” $G$ and $c$ are growing functions on time $t$. While if $K_1 = -1$, the scale factor reaches an asymptotic behavior with $G$ and $c$ decreasing on time $t$. In any case $\Lambda = \text{const.} > 0$, i.e. is a true positive constant.

2. Case II.

Taking $b = 0, a \neq 0$, we get that the infinitesimal $X$ is $X_1 = t\partial_t - 2\rho \partial_\rho$, which is precisely the generator of the scaling symmetries. Therefore the invariant solution will be the same than the obtained one with the dimensional method.

With these values of $a$ and $b$ equation (29) yields

$$c'' = \frac{c^2}{c} - \frac{c'}{t}, \quad \Rightarrow \quad c(t) = K_2 t^{K_1},$$

as we expected, $c(t)$ follows a power-law solution (self-similar solution), where $K_1$ and $K_2$ are numerical constants, $K_1, K_2 \neq 0$. We would like to emphasize that with this behavior for $c$ the homothetic vector field behaves as:

$$X_H = \frac{t}{K_1 + 1} \partial_t + \left(1 - \frac{tH}{K_1 + 1}\right) (x\partial_x + y\partial_y + z\partial_z)$$

as in the standard model (see for example Eardley ([32]) and Wainwrith ([33])).

In this way $G$ behaves as:

$$G = K_2^2 t^{2K_1}.$$  

Therefore equation (32) yields:

$$\rho'' = \rho'^2 \rho + \rho \frac{K_1}{t} + A \rho^2,$$  

we expect that this ode admits a scaling solution (i.e. that the energy density follows a power law solution) as it is expected in this kind of models.

The canonical variables bring us to obtain an Abel ode

$$y' = x (2 + 2K_1 - Ax) y^3 - (1 + K_1)y^2 - \frac{y}{x},$$

where $x = \rho t^2$ and $y = \frac{1}{\rho^{2(1+2K_1)}}$, and which solution is unknown.

The solution obtained through invariants is:

$$\frac{dt}{\xi} = \frac{d\rho}{\eta} \Rightarrow \rho \approx t^{-2}, \quad \rho = \frac{2(1 + K_1)}{At^2},$$

finding that $K_1 > -1$ from physical considerations i.e. $\rho > 0$ iff $K_1 > -1$. This is the kind of solution expected in a self-similar model (see for example Wainwrith ([33]) and Jantzen ([34])).

From eq. (16) it is obtained the behavior of “constant” $\Lambda$

$$\Lambda' = 16\pi B \rho c^\prime c^\prime \Rightarrow \Lambda = -\frac{16\pi BK_1}{K_2^2 t^{2(1+K_1)}},$$

as it is observed if $K_1 > 0$ then $\Lambda$ is a negative decreasing function on time $t$, while if $K_1 < 0$ then $\Lambda$ is a decreasing function on time $t$, but with the restriction $K_1 \in (-1, 0)$. We would like to emphasize that the self-similar relationship

$$\Lambda \approx \frac{1}{c^2 t^2} \approx \frac{1}{K_2^2 t^{2(1+K_1)}},$$
is trivially verified as it is expected in this kind of solutions.

Now, we will calculate the scale factor $f$. In order to do that we may follow two ways, in the first one, from eq. (15) we find that

$$f = f_0 t^{2/3(\omega+1)},$$

where $f_0 = \left(\frac{A_0 A}{2(1 + K_1)}\right)^{1/(\omega+1)}$, observing again that necessarily $K_1 > -1$.

If we follow our second way it is found that from eq. (9 with $K = 0$

$$3H^2 = \frac{16\pi B}{At^2} \implies f = f_0 t^{\sqrt{\pi}},$$

where $H_0 = \frac{16\pi B}{3A}$, but taking into account the value of $A = 12\pi (\omega + 1)^2 B$, then $H_0 = \frac{4}{9(\omega + 1)}$, and hence $\sqrt{H_0} = \frac{2}{3(\omega + 1)}$, as it is expected from eq. (51). Therefore this new way does not add any more information.

In this way we found that

$$H = \sqrt{H_0} t^{-1}, \quad \text{and} \quad q = \frac{d}{dt} \left(\frac{1}{H}\right) - 1 = \frac{1}{\sqrt{H_0}} - 1.$$

therefore $q < 0$ iff $\omega \in (-1, -\frac{1}{3})$, note that if $\omega < -1$ (phantom cosmology), then $f(t)$ is a decreasing function on time $t$.

If we make the assumption (scaling symmetry) on the scale factor

$$f \approx ct \implies f \approx t^{K_1 + 1}$$

in such a way that equating this expression with eq. (51) then

$$K_1 = -\frac{1 + 3\omega}{3(\omega + 1)}$$

finding that $K_1 > 0 \iff \omega \in (-1, -\frac{1}{3})$. Hence, if $K_1 > 0 \iff \omega \in (-1, -\frac{1}{3})$, $G$ and $c$ are growing functions on time $t$, while $A < 0$, i.e. is a negative decreasing function on time $t$. In other case, if $K_1 < 0$, $G$ and $c$ are decreasing functions on time $t$, while $A > 0$, i.e. is a decreasing function on time $t$.

The Kretschmann scalars behave as:

$$RiemS \approx \frac{1}{K_2 t^{4(1+K_1)}}, \quad RiccS \approx \frac{1}{K_3 t^{4(1+K_1)}},$$

finding in this way that if $K_1 < -1$ (forbidden possibility) then both scalars tend to zero while if $K_1 > -1$, then both scalars tend to infinity i.e. there is a true singularity.

In the first place we would like to emphasize that this is the solution that we have obtained in our previous works using D.A. (see [23] and [36]). This is due to two reasons. The first one, because D.A. is a powerful tool that may be used even when the FE are not well outlined. The second one, because in the previous works we were using only three of the FE, ignoring eq. (8) and using eqs. (9,15 and 16). Therefore the solution found in those papers work well and we refer to them for all the physical considerations.

In this approach we are supposing that $q < 0$ i.e. the universe accelerates due to a equation of state $\omega \in (-1, -\frac{1}{3})$, but as it has been pointed out by R.G.Vishwakarma (see [37]) the acceleration of the universe may be explained through different mechanisms in such a way that $q < 0$ while $\omega \in [0, 1]$. In this way we would like to stress that to solve some of the cosmological problems that present the standard model, we have found that $K_1 > -1$, with $K_1 = -\frac{1 + 3\omega}{3(\omega + 1)}$ in such a way that if $K_1 > 0$ iff $\omega \in (-1, -\frac{1}{3})$ then $q < 0$ and $G$ and $c$ are growing functions on time $t$, while $A$ is a negative decreasing function on time $t$. With $K_1 < 0$, $q > 0$ and $G$ and $c$ are decreasing functions on time $t$, while $A$ is a positive decreasing function on time $t$, i.e. there is a narrow relationship between the behavior of $G$ and $c$ and the sign of $A$ controlled by $\omega$. 
This case is a generalization of the above case, simply, we will avoid the singular case. Taking \( a, b \neq 0 \) we get

\[
e'' = \frac{c^2}{c} - \frac{ac'}{at+b} \implies c(t) = K_2 (at+b)^{\frac{K_1}{a}},
\]

obtaining in this case that \( G \) behaves as:

\[
G = K_2^2 (at+b)^{2K_1},
\]

hence equation (23) yields:

\[
\rho'' = \frac{\rho'^2}{\rho} + \rho' \frac{K_1}{at+b} + Ap^2,
\]

The canonical variables bring us to an Abel ode

\[
y' = x \left( 2a^2 + 2aK_1 - Ax \right) y^3 - (a + K_1)y^2 - \frac{y}{x}
\]

where \( x = \rho (at+b)^2 \) and \( y = \frac{1}{(at+b)^2(\rho + 2a)} \), that is to say, we obtain a very complicate ode which at this time we do not know how to solve. Nevertheless, as we have pointed out in (23) the general solution of this kind of equations are unphysical, for this reason it is sufficient consider particular solutions obtained through the invariants.

The solution obtained through invariants is:

\[
dt = \frac{d\rho}{\eta} \implies \rho = \frac{2a(a + K_1)}{A(at+b)},
\]

where it is observed that \( K_1 > -a \) (but if you choose \( a = b = 1 \), then we have the same result as before i.e. \( K_1 > -1 \)). Therefore the scale factor behaves as:

\[
f = \left( \frac{A \omega A (at+b)^2}{2a(a + K_1)} \right)^{1/(3(\omega+1))}, \implies f = f_0 (at+b)^{2/(3(\omega+1))},
\]

which is very similar to the last result (see eq. (51) but in this occasion this solution is non-singular).

We end finding the behavior of \( \Lambda \), as in previous cases taking into account eq. (16) it is obtained:

\[
\Lambda' = 16\pi B \rho \frac{c'}{c^3} \implies \Lambda = -\frac{16\pi}{AK_2^2} \frac{BaK_1}{(at+b)^{2(\frac{\omega}{\omega+1})}},
\]

as we can see in this case if \( K_1 > 0 \) then \( \Lambda \) is a negative decreasing function on time \( t \).

The Kretschmann scalars behaves as:

\[
RimS \approx \frac{1}{K_2^4 (at+b)^{4(1+K_1)}}, \quad RiccS \approx \frac{1}{K_2^4 (at+b)^{4(1+K_1)}},
\]

showing a non-singular state when \( t \) runs to zero.

This solution is very similar to the previous one except that this solution is non-singular. In fact, the above solution is a particular solution of this one.

### B. The non-flat case.

In this subsection we go next to study the particular case \( K \neq 0 \). One of the drawbacks of the above approach is that we need to make the assumption \( K = 0 \) i.e. our approach is unable of solving the so-called flatness problem. In order to research if it is possible to solve such problem we go next to study eq. (20) through the Lie method, seeking symmetries that allow us to obtain any solution in closed from. But as we will see eq. (20) does not admit any symmetry in such a way that in order to obtain a particular solution we will impose a concrete symmetry, but
this is precisely the method that we are trying to avoid, to make assumptions or at least to make the minor number of assumptions or to make assumptions under any physical or mathematical (symmetries) well founded reasons. We only explore one case.

Therefore the equation under study is:

\[ \rho'' = \rho^2 + \rho \frac{c'}{c} \left( -2Kc^2 \rho f^2 + A \frac{G}{c^2} \rho^2 \right), \]  

but as

\[ \text{div} T = 0 \iff \rho = A\omega f^{3(\omega+1)}, \implies f = \left( \frac{\rho}{A\omega} \right)^{\frac{1}{3(\omega+1)}}, \]  

and hence

\[ \rho'' = \frac{\rho^2}{\rho} + \rho \frac{c'}{c} \left( -2Kc^2 \rho f^2 + A \frac{G}{c^2} \rho^2 \right), \]

where \( a = \frac{3(\omega+1)}{3(\omega+1)!} \), and for simplicity we have adopted the case \( K = 1 \).

The Lie group method brings us to obtain the following system of pdes

\[ \rho \xi_{\rho \rho} + \xi_{\rho} = 0, \]  

\[ -\rho^{-1} \eta_{\rho} - 2 \frac{c'}{c} \xi_{\rho \rho} + \eta_{\rho \rho} - 2 \xi_{t \rho} + \rho^{-2} \eta = 0, \]  

\[ \left( \left( \frac{c'}{c} \right)^2 - \frac{c''}{c} \right) \xi + 3 \rho^2 \left( 2c^2 \rho^{\alpha-2} - A \frac{G}{c^2} \right) \xi_{\rho} - 2 \rho^{-1} \eta_{t} - \xi_{tt} + 2 \eta_{t} - 2 \frac{c'}{c} \xi_{t} = 0, \]  

\[ \eta_{tt} - \frac{c'}{c} \eta_{t} + \rho^2 \left( A \frac{G}{c^2} - 2c^2 \rho^{\alpha-2} \right) \eta_{\rho} + 2 \rho^2 \left( 2c^2 \rho^{\alpha-2} - A \frac{G}{c^2} \right) \xi_t \]

\[ + A \rho^2 \frac{G}{c^2} \left( 4\rho^{\alpha-2} \frac{c^4}{AG} \frac{c'}{G} - 2 \frac{c'}{c} \right) \xi + 2 \rho \left( ac^{\rho^{\alpha-2}} - A \frac{G}{c^2} \right) \eta = 0 \]

which has no solution, that is to say, eq. (67) does not admit any symmetry, for this reason we will need to follow other approaches.

For example, if we “impose” any particular symmetry \( X \), maybe we may found some restrictions for the behavior of the quantities \( G, c \) and \( \rho \). We will explore such possibility.

1. Case I.

In this case, we choose \( (\xi = 1, \eta = 0) \), i.e. \( X = \partial_t \), in such a way that from eq.(70) it is obtained the following restriction

\[ c'' = \frac{c^2}{c} \implies c(t) = K_2 e^{K_1 t}, \]  

and from eq.(71) it is obtained the following one

\[ 2 \left( 2\rho^{\alpha-2} \frac{c^4}{AG} + 1 \right) \frac{c'}{c} = \frac{G'}{G}, \]  

hence

\[ 2 \left( 2\rho^{\alpha-2} \frac{K_2^2 e^{AK_1 t}}{AG} + 1 \right) K_1 = \frac{G'}{G}, \]
as we can see from eq. (73), we cannot obtain the condition $G = Bc^2$ (as in the flat solution) since such condition means that
\[ 4\rho a^2 \frac{C^4}{AGc} \frac{\rho'}{c} = \frac{G'}{G} - 2\frac{\rho'}{c} = 0 \iff \rho = 0, \] (75)
that is to say, the energy density vanishes.

In order to find a particular solution to eq. (74) we impose the condition $a = 2$ (as mathematical condition) which means that $\omega = -\frac{5}{3} \ll -1$, although such possibility is very restrictive (and maybe unphysical, the ultra phantom equation of state). In this way it is found that
\[ G' = 2K_1G + \frac{4K_1K_2^4e^{4K_1t}}{A}, \implies G(t) = C_1e^{2tK_1} + \frac{2}{A}K_2^4e^{4(tK_1)}, \] (76)
where $C_1$ is an integration constant, therefore eq. (67) yields
\[ \rho'' = \frac{\rho'^2}{\rho} + \rho'K_1 + \left( A \frac{C_1}{K_2^2} \right) \rho^2, \] (77)
since this equation has not an explicit (analytical) solution, note that we have obtained the same eq. as in section 3.1 case I eq. (36), we find again that a particular solution is $\rho = 0$, therefore from the field equation (9)
\[ H^2 + \frac{K_2^2e^{2K_1t}}{f^2} = \frac{\Lambda_0}{3}K_2^2e^{2K_1t}, \] (78)
and hence in this case we have found that a solution is:
\[ f = \frac{\sqrt{3}}{\sqrt{\Lambda_0}} \left( \frac{2}{3} + \frac{1}{6} \exp \left( \frac{2\sqrt{3} \sqrt{\Lambda_0}K_2}{K_1} (\exp(K_1t) + C_1) \right) \exp \left( -\frac{\sqrt{3} \sqrt{\Lambda_0}K_2}{K_1} (\exp(K_1t) - C_1) \right) \right), \] (79)
since
\[ 0 = \left( \frac{G'}{G} - \frac{4\rho'}{c} \right) \rho = \frac{\Lambda'c^4}{8\pi G} \implies \Lambda = \Lambda_0 = \text{const.} > 0. \] (80)

This solution looks very unphysical or at least very restrictive, a vanishing energy density and a constant cosmological constant and is very similar to the obtained one in section 3.1 case I the super de-Sitter solution, with the same behavior and the same restriction for the numerical constant $K_1$, i.e. sudden singularities or asymptotic behavior depending of the sign of $K_1$. In this case, the employed method does not allow us to obtain the behavior of $G$, $c$ and $\Lambda$ since the equation under study does not admit any symmetry.

**IV. SOLUTION II.** $div T \neq 0$.

In the previous section we have made the assumption that $div(T) = 0$ i.e. the divergence of the energy–momentum tensor vanishes. Nevertheless we have obtained as general conservation equation eq. (10) i.e.
\[ \rho' + 3(\omega + 1)\rho H = -\left( \frac{G'}{G} - \frac{4\rho'}{c} \right) \rho - \frac{\Lambda'c^4}{8\pi G}, \] (81)
i.e. $div(T) = \rho' + 3(\omega + 1)\rho H \neq 0$ and we have assumed a particular case (with perfect mathematical sense) $div(T) = 0$. In this section we will study the general case $div(T) \neq 0$.

The possibility that cosmological and physical considerations may require that the covariant conservation condition $div(T) = 0$ be relaxed has been advanced by Rastall ([24]), who pointed out that a non-zero divergence of the energy–momentum tensor has as yet not been ruled out experimentally at all. In Rastall’s theory ([24]), the divergence of $T$ is assumed to be proportional to the gradient of the scalar curvature $S$, $div(T) = \lambda \text{grad}(S)$, where $\lambda = \text{constant}$, and in fact the modified field equations are equivalent to standard general relativity with an additional variable $\Lambda$ term. We refer to the reader to the Harko and Mark work ([20]) to see a matter creation and thermodynamical approach in this context.
In this case we are going to consider the field equations “but” without the condition $\text{div}(T) = 0$, i.e. the field equations will be

$$2H' - 2\frac{c'}{c}H + 3H^2 = -\frac{8\pi G}{c^2}p + \Lambda c^2,$$

$$3H^2 = \frac{8\pi G}{c^2}p + \Lambda c^2,$$

$$\rho' + 3(\omega + 1)pH = -\left(\frac{G'}{G} - 4\frac{c'}{c}\right)\rho - \frac{\Lambda'c^4}{8\pi G},$$

but as we can see we have 2 equations with 5 unknowns, therefore it is necessary to make some assumptions, for example in the behavior of $G, c$ and $\Lambda$ as follows:

$$G = G_0H^a, \quad c = c_0H^b, \quad \Lambda = \Lambda_0c^{-2}H^2,$$

where $G_0, c_0$ and $\Lambda_0$ are dimensional constants and $a, b \in \mathbb{R}$, without any restriction i.e. we do not need to assume any concrete sign or value for these numerical constants. Furthermore, we must stress that the conditions $K = 0$, together to power law assumptions bring us to a scaling solution.

Taking into account these assumptions and form eq. (83) we obtain $\rho$

$$\rho = \left(\frac{3 - \Lambda_0}{d_0}\right)H^{2(1+b)-a}, \quad i.e. \quad \rho = \rho_0H^{2(1+b)-a},$$

where $d_0 = 8\pi G_0/c_0^2$, and taking this relationship into eq. (84), we obtain the following ode in quadrature

$$\frac{\alpha H'}{H} + 3(\omega + 1)H = -a\frac{H'}{H} + 4b\frac{H'}{H} - \tilde{K}\frac{H'}{H},$$

and therefore

$$\left(\alpha + a - 4b + \tilde{K}\right)\frac{H'}{H^2} = -3(\omega + 1),$$

where $\alpha = 2(1+b) - a$, $\tilde{K} = (2(1-b)c_0^2\Lambda_0/8\pi G_0\rho_0)$, and therefore

$$\frac{H'}{H^2} = -\frac{(\omega + 1)(3 - \Lambda_0)}{2(1-b)} \quad \Rightarrow \quad \text{H = h_0nt}^{-1}, \quad \Rightarrow \quad f = f_0h_0n^{1},$$

where $n = \frac{(\omega + 1)(3 - \Lambda_0)}{2(1-b)}$, and $h_0 = \text{const} > 0$, is an integration constant and we impose that $b \neq 1, \Lambda_0 \neq 3$ and $\omega \neq -1$. As we can see, there are some restrictions, for example, if $\omega \in (-1, 1)$ then $b, \Lambda_0$ must verify at the same time that $b < 1$ and $\Lambda_0 < 3$ or $b > 1$ and $\Lambda_0 > 3$. Now if $\omega < -1$ then it should exist a combination between the signs of $b, \Lambda_0$ such that $n > 0$, in other way, the radius of the Universe decreases. It is observed that the behavior of the scale factor does not depend of the constant $a$, i.e. of the behavior of the gravitational “constant”.

Once we have obtained the behavior of $f$ i.e. of $H$ we go next to complete our calculations of the rest of the quantities i.e.

$$\rho = \rho_0H^{2(1+b)-a} \quad \Rightarrow \rho = \rho_0(h_0nt^{-1})^{2(1+b)-a} \approx t^{a-2(1+b)},$$

with the restrictions

$$\rho_0(h_0n)^{2(1+b)-a} > 0, \quad \text{and} \quad a - 2(1+b) < 0,$$

i.e. we are assuming that the energy density is a positive decreasing function on time $t$.

It is observed too that if $2b = a$, then we obtain the particular solution $\rho \approx t^{-2}$, as well as the relationship $G/c^2 = \text{const}$ as in the above cases, i.e. the obtained solution under the assumption $\text{div}(T) = 0$. In this way we can see that the $\text{div}(T) = 0$ case is a particular solution of the $\text{div}(T) \neq 0$ case, as one may expected, but we have not any physical or mathematical (symmetry or integrability condition) reason to assume such relationship.
We end calculating the behavior of the “constants” $G, c$ and $\Lambda$ i.e.

$$G = G_0 h_0^a n^a t^{-a}, \quad c = c_0 (h_0 n)^b t^{-b}, \quad \Lambda = \Lambda_0 c_0^{-2} (h_0 n)^{2(b-1)} t^{2(b-1)},$$

in such a way that $\Lambda$ will be a decreasing function on time iff $b < 1$.

In this way we found that

$$H = h_0 n t^{-1}, \quad \text{and} \quad q = \frac{d}{dt} \left( \frac{1}{H} \right) - 1 = \frac{1}{h_0 n} - 1 = \frac{2(1-b)}{h_0 (\omega+1)(3-\Lambda_0)} - 1,$$

therefore $q < 0$ iff $|2(1-b)| < |h_0 (\omega+1)(3-\Lambda_0)|$.

The Kretschmann scalars behave as:

$$\text{RiemS} \approx \frac{1}{c_0^{3+4(1-b)}}, \quad \text{RiccS} \approx \frac{1}{c_0^{3+4(1-b)}},$$

finding in this way that there is a true singularity.

In this case we have found a scaling solution as it is expected for this kind of models ($K = 0$). As we can see the scale factor $f$ does not depend of $G$ only of $c$ and $\Lambda$. The energy density $\rho$ depends of $G$ and $c$. Nevertheless we have not been able to find better restrictions for the introduced (ad hoc) numerical constants $a$ and $b$ such that they give us more information about the behavior of the time functions $G, c$ and $\Lambda$.

### B. The non-flat case.

As in the above section we will study the case $K \neq 0$ separately. For this purpose the FE are now:

$$2H' - 2\frac{c'}{c}H + 3H^2 + \frac{Kc^2}{f^2} = -\frac{8\pi G}{c^2} \rho + \Lambda c^2,$$

$$H' + \frac{Kc^2}{f^2} = \frac{8\pi G}{3c^2} \rho + \frac{1}{3} \Lambda c^2,$$

$$\rho' + 3 (\omega + 1) \rho H = - \left( \frac{G'}{G} - \frac{4c'}{c} \right) \rho - \frac{\Lambda' c^4}{8\pi G},$$

but as we can see we have 2 equations with 5 unknowns, therefore it is necessary to make some assumptions, for example in the behavior of $G, c$ and $\Lambda$ as follows:

$$G = G_0 H^a, \quad c = c_0 H^b, \quad \Lambda = \Lambda_0 c^{-2} H^2,$$

where $G_0, c_0$ and $\Lambda_0$ are numerical constants. We must stress that in this occasion these hypotheses could be unphysical since $K \neq 0$ and we are imposing a scaling behavior typical of the flat case.

Taking into account these assumptions and form eq. (96) we obtain $\rho$

$$\rho = \left( \frac{3 - \Lambda_0}{d_0} \right) H^{2(1+b)-a} + \frac{Kc_0 H^{2b}}{f^2}, \quad \text{i.e.} \quad \rho = \rho_0 H^{2(1+b)-a} + \frac{Kc_0 H^{2b}}{f^2},$$

where $d_0 = 8\pi G_0 / c_0^2$, and taking this relationship into eq. (97), we obtain the following second order for $f$

$$f'' = -D_1 \frac{(f')^{2b}}{f(1+2b)} + D_2 \frac{(f')^2}{f},$$

where

$$D_1 = \frac{(3\omega + 1) Kc_0^2}{2 (1-b)}, \quad D_2 = \left( 1 - \frac{(\omega + 1)(3 - \Lambda_0)}{2(1-b)} \right).$$

In the first place we may note that $(b \neq 1)$ and that if $\omega = -1/3 \implies D_1 = 0$, independently of the value of constant $K$, in this case we are mainly interested in the $K \neq 0$ case. If $K = 0$, or $D_1 = 0$ then we obtain again the solution already obtained in the latter (last) case.
Calculation of Eq. (100). making the following change of variables it is obtained the first order ode

\[
(x = f, \quad y = \frac{1}{f^2}) \implies y' = D_1 x^{-(1+2b)} y^{3-2b} - D_2 x^{-1} y, \tag{102}
\]

and which solution is:

\[
y = \left( C_1 u^{2D_2(1-b)} + \frac{2(b-1)D_1}{(2bD_2 - 2D_2 - 2b)} u^{-2b} \right)^{\frac{1}{2}} \tag{103}
\]

therefore

\[
f' = \left( C_1 f^{2D_2(1-b)} + \frac{2(b-1)D_1}{(2bD_2 - 2D_2 - 2b)} f^{-2b} \right)^{\frac{1}{2}} \tag{104}
\]

and hence

\[
t = \int f \left( C_1 u^{2D_2(1-b)} + \frac{2(b-1)D_1}{(2bD_2 - 2D_2 - 2b)} u^{-2b} \right)^{\frac{1}{2}} du + C_2, \tag{105}
\]

Since we have not obtain information about the behavior of \( f \) then we try to find a particular solution. For this purpose, we observe that eq. (100) admits the following symmetries:

\[
X_1 = \partial_t, \quad X_2 = t \partial_t + (1-b) f \partial_f, \tag{106}
\]

where we would emphasize that \( X_2 \) is a scaling symmetry, maybe induced by the hypotheses about the behavior of \( G, c \) and \( \Lambda \), (scaling relationships).

The invariant solution (particular solution) that induces \( X_2 \) is the following one:

\[
\frac{dt}{t} = \frac{df}{(1-b)f} \quad \implies \quad f = f_0 t^{(1-b)}, \tag{107}
\]

with

\[
f_0 = \frac{D_1(b-1)}{D_2(b-1) - b} \sqrt{\frac{D_2(b-1) - b}{D_1(b-1)}} = \frac{K c_0^2 (1 + 3\omega)}{(3\omega + 1 - \Lambda_0(\omega + 1))} \sqrt{\frac{(3\omega + 1 - \Lambda_0(\omega + 1))}{K c_0^2 (1 + 3\omega)}} = \text{const.}, \tag{108}
\]

and we necessarily impose that \( b < 1 \) and \( b \neq 0 \), this means that if \( b \to 1 \) then \( f \to f_0 \). As we can observe \( K \neq 0 \), and we must be careful with the signs since if \( K = -1 \) then it should be satisfied the relationship \( 3\omega + 1 < \Lambda_0(\omega + 1) \).

In this way we found that

\[
H = (1-b)t^{-1}, \quad \text{and} \quad q = \frac{d}{dt} \left( \frac{1}{H} \right) - 1 = \frac{1}{1-b} - 1 = \frac{b}{1-b}. \tag{109}
\]

therefore \( q < 0 \) iff \( b < 0 \), which implies that \( c \) is a growing function on time \( t \).

If we make the assumption (scaling-symmetry) on the scale factor

\[
f \approx ct \quad \implies \quad f \approx t^{(1-b)}, \tag{110}
\]

The Kretschmann scalars behave as:

\[
\text{RiemS} \approx \frac{1}{C^4 t^{4(b-1)}}, \quad \text{RiccS} \approx \frac{1}{C^4 t^{4(b-1)}}, \tag{111}
\]

finding in this way that there is a true singularity.

Calculation of \( \rho \), and the possible restrictions for the constants \( a \) and \( b \).

\[
\rho = \rho_0 (1-b)^{2b-a+2} t^{a-2(b+1)} + \frac{K c_0}{f_0^2} (1-b)^{2b} t^{-2}, \tag{112}
\]

and

\[
G \approx G_0 t^{-a}, \quad c \approx c_0 t^{-b}, \quad \Lambda \approx \Lambda_0 t^{2(b-1)}. \tag{113}
\]

As we have seen, we have only been able to obtain a particular solution. Imposing so many hypotheses, we lose information and we do not know how to recover it in such a way that we are not able to know better the behavior of the time functions \( G, c \) and \( \Lambda \).
V. CONCLUSIONS.

In this paper we have studied a perfect fluid FRW model with time-varying constants “but” taking into account the possible effects of a $c-$variable into the curvature tensor. In this way, as other authors have already pointed out, such effects are minimum in the field equations but they exist and are very restrictive. Under the made hypotheses, we have seen that the Einstein tensor has covariant divergence zero, in this way we have imposed that the right hand of the field equations has a vanishing divergence too i.e.

$$\text{div} \left( \frac{8\pi G}{c^4} T + g\Lambda \right) = 0. \quad (114)$$

In this way we have obtained the set of the new FE, emphasizing the fact that we can recover eq. (114) from the field equations as in the standard case i.e. deriving one of them and simplifying with the other one.

In other to solve the resulting FE we have considered the following cases. In the first case we have imposed the condition $\text{div}(T) = 0$, as a particular case of eq. (114) and we have studied the flat and non-flat cases. We have needed to make such distinction because with the employed method we are not able of solving the so called flatness problem, for this reason we have needed to study separately then.

The flat case under this hypothesis is the already studied one in our previous works (see [22]) and under this new considerations i.e. taking into account the effects of a $c$-variable into the curvature tensor, we have shown that the scaling solutions the only one while in ([22]), without this new assumption, we obtained more solutions, i.e. we obtained other solutions apart from the scaling solution. We would like to emphasize that it has been obtained as integration condition that the “constants” $G$ and $c$ must verify the relationship $G/c^2 = \text{const.}$ in spite of the fact that both “constants” vary. This result is in agreement with our scaling solution obtained in our previous work (see [22]) but in those works we needed to make such relationship as assumption.

We have also shown that this model is self-similar since we have been able of obtaining a non-trivial homothetic vector field. This result is agreement with the obtained scaling solution as is well known.

With the obtained solution, if we want to solve the acceleration of the universe i.e. $q < 0$, then the time function $G$ and $c$ are growing function on time $t$, while $\Lambda$ is a negative decreasing function on time $t$, in this case the equation of state belong to the interval: $\omega \in (-1, -1/3)$. In other cases $G$ and $c$ are decreasing function while $\Lambda$ is a decreasing function and in this case the equation of state belongs to the interval: $\omega \in [-1/3, 1]$.

The non-flat case does not admit any symmetry and the particular studied case looks very unphysical.

The second class of studied models verifies the condition $\text{div}(T) \neq 0$, i.e. without imposing any restriction to eq. (114) and we have studied again the flat and non-flat cases. To solve these cases we have needed to make scaling assumptions on the behavior of the time functions $G$, $c$ and $\Lambda$. These assumptions work well in the flat case (self-similar case) but in the non-flat case seem very restrictive. In the flat case, it is found for the made assumptions, that there is a relationship between the numerical constants that determine the behavior of $G$ and $c$ that brings us to obtain again the particular case $\text{div}(T) = 0$, i.e. such case could be seen as a particular solution of the more general case $\text{div}(T) \neq 0$.

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