Determination of moving load characteristics by output-only identification over the Pescara beams

A Bellino¹, L Garibaldi¹, S Marchesiello¹

¹ Dynamics & Identification Research Group, Dipartimento di Meccanica, Politecnico di Torino, C.so Duca degli Abruzzi 24, I-10129 Torino, Italy

andrea.bellino@polito.it, luigi.garibaldi@polito.it

Abstract. The determination of the characteristics of moving loads over bridges and beams is a topic that only recently has gained the interest of the researchers. In real applications, in fact, as for the case of bridges, it is not always possible to know the load and speed of the trains which are travelling over the bridge. Moreover, in real applications the systems analyzed cannot be always considered linear. Because of these difficulties, the present paper proposes firstly a technique for the identification of the nonlinearity, secondly a procedure to subtract its effect on the modal parameters and finally a method based on them to extract the information on the mass and the speed of the moving load crossing a beam.

For this study, some reinforced concrete beams have been tested in the framework of a vast project titled "Monitoring and diagnostics of railway bridges by means of the analysis of the dynamics response due to train crossing", financed by Italian Ministry of Research. These beams show a clear softening nonlinear behaviour during the crossing of a moving carriage. The method is able to detect the load characteristics after having eliminated the nonlinear influence.

1. Introduction

In literature, the study of nonlinear and time-varying systems are usually treated separately, mainly due to the articulated approaches needed to study these topics. Just few researchers have treated both; for example Barthels and Wauer [1], who dealt with a mechanical device including these two effects. Allison et al. [2], instead, applied the proper orthogonal decomposition to both nonlinear and time-varying systems, in order to look for the different characteristics. In general, however, the nonlinear systems are identified with specific tools, as in [3,4,5]. On the other side, the time-varying systems are usually analyzed with parametric methods [6] or techniques extended from the linear time-invariant case, such as the “frozen technique” [7,8].

The idea of this article is to apply the identification procedures proper of the time-varying case to the nonlinear ones. The frozen technique, able to estimate the modal parameters in successive signal windows, allows to find the relationship between frequency and time. For this reason, it can be applied to nonlinear systems, where the frequency is amplitude-dependent. If a nonlinear time-varying system is considered, then the frequency variation has two different sources, so that they must be separated by subtracting the effect of the nonlinearity, previously estimated from the analysis in time-invariant conditions.

One of the most important cases of time-variability is the train crossing a railway bridges. This system has been simulated in the laboratory adopting different reinforced concrete beams. Since the
beams show a nonlinear softening behaviour, firstly the contribution of the nonlinearity is estimated and hence it is subtracted from the first natural frequency, such that it depends only on the load characteristics. Successively, it is possible to estimate the mass and the velocity of the carriage by using the minimum value and the trend of the considered frequency.

2. Estimation of the nonlinearity
This section starts with a mathematical background on the nonlinear equation of a single d.o.f. and continues with the introduction of the correction function, which allows to estimate the nonlinearity.

2.1. Mathematical background
The method is based on the homogeneous undamped Duffing equation, with a cubic nonlinearity:

\[ m\ddot{x} + k_L x + k_{NL} x^3 = 0 \]  

with initial conditions

\[ x(0) = A \quad \dot{x}(0) = 0 \]  

In the equation, \( m \) is the system mass, \( k_L \) the linear stiffness and \( k_{NL} \) the nonlinear stiffness.

The solution of this equation is given in terms of elliptic functions:

\[ x(t) = A cn(\omega t, M) \]  

where \( cn \) is the Jacobi elliptic function, with frequency \( \omega \) and modulus \( M \) [9]:

\[ \omega^2 = \frac{k_L}{m} + \frac{k_{NL}}{m} A^2 \] 

\[ M = \frac{k_{NL} A^2}{2(k_L + k_{NL} A^2)} \]  

The Jacobi elliptic function is defined as follows:

\[ u = \int_0^\varphi \frac{d\varphi}{\sqrt{1 - M \sin^2 \varphi}} \] 

\[ cn(u, M) = \cos(\phi) \]  

2.2. The correction function
Actually, while \( \omega_L = \sqrt{k/m} \) is the angular frequency of the underlying linear system, \( \phi \) represents the angular frequency of the nonlinear system and it is varying with the signal amplitude. The frequency \( \omega \) has a known expression but it must be linked to \( \phi \), in order to finally estimate both the linear frequency and the nonlinearity.

If a simple numerical example is built, based on eq. (1), with the parameters shown in Table 1, then it is possible to simulate the displacements (figure 1.a) with eq. (3) and then to calculate the natural frequencies by a subspace identification method (figure 1.b).

In Figure 1.b, the dotted line corresponds to the natural frequency of the underlying linear system. It is possible to note that, when increasing the signal level, the natural frequencies increase, as it happens for hardening systems. The results coincide only when the level of the displacement tends to zero.

### Table 1. Parameters chosen.

| Parameter          | Value     |
|--------------------|-----------|
| Mass               | \( m = 1.3 \text{ kg} \) |
| Linear stiffness   | \( k_L = 800 \text{ N/m} \) |
| Cubic Stiffness    | \( k_{NL} = 150000 \text{ N/m}^3 \) |
| Damping            | \( c = 2 \text{ kg/s} \) |
Therefore, since the characteristics of the system are included in $\omega$, a correction function $h(M)$ must be introduced such that $f = hF$, where $f = \omega / 2\pi$ and $F$ is the identified frequency.

This operation is one of the central points of the analysis, because it is useful to have some analytical information about the dynamics of the system. Indeed, since $f$ and $F$ are known, it is possible to estimate the general trend of the function $h(M)$.

Since $h$ depends on $M = M(k_L, k_{NL}A^2)$, then $h = h(k_L, S)$, where $S = k_{NL}A^2$. If $h(k_L, S)$ is fitted with a polynomial as a function of $S$, then the coefficients will depend exclusively on $k_L$:

$$h(k_L, S) = \sum_i a_i(k_L)S^i$$

(8)

The total operation can be done by the use of two successive regressions, which allow to perform the variable separation. The first regression isolate the part depending on $S$, and the coefficients $a_i$.

Since all these coefficients show a hyperbolic trend, the second regression will be performed by considering as independent variable the inverse of the linear stiffness. In both the regression, a polynomial order at least equal to four is required. The final operation is to write:

$$f(k_L, S) = h(k_L, S)F$$

(9)

$$\sqrt{\frac{k_L + S^i}{m}} = F\sum_i a_i(k_L)S^i$$

(10)

$$\sqrt{\frac{k_L + k_{NL}A^2}{m}} = 2\pi F\sum_i a_i(k_L)k_{NL}A^{2i}$$

(11)

The unique unknown present in the equation is the nonlinear term $k_{NL}$. The solution can be found by using a method for solving nonlinear systems. The value of the displacement $A$ can be obtained by considering the positive peaks of the displacements.

2.3. The algorithm

The formula expressed in eq. (11) is the basis for the extraction of the nonlinear stiffness. In this subsection, the algorithm is proposed by listing all the successive steps.

I. Extraction of the natural frequencies by means of the ST-SSI method [8], already used for the identification of time-varying systems. This aspect is crucial because the method allows to link directly the natural frequencies to the values of displacement.
II. Estimation of the coefficients \( a_i \), which depend only on the linear stiffness. If this is not available, then calculate it by inverting the expression of the natural frequency of the underlying linear system (supposing the mass being known).

III. Within a cycle on all the available time values, evaluate
   a. The value of the displacement corresponding to the present time. Usually only the values above a fixed threshold are considered because for the low values of displacements the nonlinearity is less in evidence.
   b. Search the zero of the function
      \[
      f(k_L, k_{NL}, A^3) - h(k_L, k_{NL}, A^3)F = 0
      \]
      where the unique unknown is the nonlinear stiffness \( k_{NL} \).

IV. Collect all the results: that is the estimation of the nonlinear stiffness, indicated by \( \tilde{k}_{NL} \).

The advisable excitation is an impulse, for example produced by a hammer, and the time history considered should be the free decay. In this way, it is possible (by means of a regression process) to associate the following physical quantities to every time instant, such that they have a monotone trend:
- The displacements peaks
- The natural frequency considered
- The nonlinear stiffness

2.4. Applicability to different nonlinearities
Since the nonlinear stiffness is calculated at each displacement value, it means that the results are disposed along a constant line only if the nonlinearity is cubic, otherwise they are disposed along a curve. The general way to obtain the nonlinear force acting on the system under analysis is to multiply the results for the nonlinear stiffness by the third power of the displacements. The validity field of this operation is limited to the range of the displacement values available.

![Figure 2](image_url) Numerical example with cubic nonlinearity. (a.) Simulated displacements. (b.) Displacement peaks. (c.) Identified natural frequencies. (d.) Estimation of the nonlinearity.
3. Numerical examples

In order to understand how the method works, two numerical examples are here presented. The first one concerns a cubic nonlinearity, while the second concerns a seventh-order polynomial. The most simple case of nonlinearity is the cubic one. By assuming the values shown in Table 1, but with \( k_{NL} = 5 \cdot 10^4 \), it is possible first to extract the displacement peaks, then to identify the natural frequencies and finally to estimate the quantity \( k_{NL} \). In Figure 2, the entire process from the system simulation to the final results has been represented.

The system has a hardening nonlinearity and this is very clear from Figure 2.c. The results for the nonlinear stiffness are included in a small range of values, such that they can be considered part of a constant line, obtained as mean of all the values. The approximation is very good, because the relative error is equal to 0.64%.

If the nonlinearity is represented by a seventh-order polynomial:

\[
m \ddot{x} + c \dot{x} + k_L x + k_7 x^7 = 0
\]

with \( k_7 = 10^8 \), then the results are values forming a curve (see figure 3.a), and the nonlinear force \( k_7 x^7 \) can be estimated by multiplying these ones by the term \( x^3 \). The original time history for the displacement can be reconstructed by simulating a system with the characteristics just extracted: the result is depicted in figure 3.b, where the reconstructed displacement is very close to the original one.

![Figure 3](a) Estimation of the nonlinearity as function of the displacements. (b) Reconstruction of the original displacement.

4. Elimination of the nonlinearity

One of the innovative aspects of the present article is the elimination of the nonlinear contribution to the dynamical characteristics of the system under analysis. In particular, since the case that will be considered is a beam crossed by a moving load, it is important to purify the natural frequencies from the nonlinearity effect, such that they depend only on the load properties.

The process starts by considering a generic time history referred to a system showing a time-varying behaviour. As for the previous methodology, the first step is to consider the maxima of the displacement, bearing in mind that now the analysis is no more performed on a free decay. The second step is the identification of the first natural frequency. It is inserted, for each time instants, in the eq. (9), as well as the nonlinear stiffness, in order to estimate the correction function. Successively, the frequency \( f_{NL} \) depending only on the nonlinearity is calculated as:

\[
f_{NL}(k_L,S) = \frac{f(k_L,S)}{h(k_L,S)}
\]

The contribution of the nonlinearity is the equal to

\[
e_{NL} = f_{NL} - f_{II}
\]
where \( f_{NI} \) is the natural frequency of the underlying linear system. The frequency \( f_{NL} \) is equal to \( F \) if the system under analysis is time-invariant. Otherwise the frequency \( f_{TV} \) depending only on the time-varying effects (for example the transit of a moving load over a beam) is calculated as:

\[
f_{TV} = F - c_{NL}
\]  

(15)

During this technique, it is advisable to consider a time history with displacement peaks not significantly different from the time history considered for the estimation of the nonlinearity.

5. Estimation of load characteristics

The determination of the load characteristics must be divided in two parts. The first one is dedicated to the velocity and the second one to the mass.

The calculations are done for a simply supported Euler-Bernoulli beam. The beam is crossed by a moving load, modeled by a punctual mass moving on the beam. The first natural frequency can be calculated through an analytical formula, neglecting the dynamical effects of the velocity [10]:

\[
f_j = f(x_j) = \frac{f_{BEAM}}{\sqrt{1 + 2 \frac{m}{\mu L^2} \sin\left(\frac{\pi}{L} x_j\right)^2}}
\]

(16)

where \( L \) is the beam length, \( \mu \) is the mass per unit length, \( f_{BEAM} \) is the first natural frequency of the beam, \( x_j = x_j(v, t_j) \) is mass location at time \( t_j \) and \( m \) is the load mass.

Both the velocity and the mass of the load can be extracted directly from the knowledge of the first natural frequency, with a comparison with the theoretical one.

5.1. Velocity estimation

The carriage velocity, in general, cannot be considered constant, therefore it must be estimated by inspecting the shape of the first natural frequency. The velocity \( v \) does not alter the frequency values but only the symmetry of the frequency.

By dividing the total beam length \( L \) in a fixed number \( N \) of parts, it is possible to obtain \( N + 1 \) points, called \( x_j, j = 1, \ldots, N + 1 \). For each point, consider the value \( f_j \) of the theoretical frequency, from eq. (16). Successively, find the time instant \( T_j \) that corresponds to each value of \( f_j \), on the trend of the frequency depending only on the mass. Once collected all the \( T_j \), plot \( x_j \) in function of \( T_j \): the displacement as function of time are obtained. The velocity is estimated as the tangent, in every point, to the curve just extracted.

5.2. Mass estimation

The estimation of the mass is very simple because it is sufficient to invert the relationship expressed in eq. (16). It is convenient to consider the case when the mass is in the middle of the beam, where the first natural frequency is minimum. This operation is done because the minimum of the frequency extracted \( F_{MIN} \) is the most simple value to extract, therefore the estimation will be more precise:

\[
m = \frac{\mu L}{2} \left( \frac{f_{BEAM}}{F_{MIN}} \right)^2 - 1
\]

(16)

6. Experimental examples: the Pescara beams

In the framework of a huge project titled "Monitoring and diagnostics of railway bridges by means of the analysis of the dynamics response due to train crossing", financed by Italian Ministry of Research, some experimental tests on several reinforced concrete beams have been performed in Pescara, Italy.
In figure 4, a simple scheme of the simply supported beams analyzed is shown. In table 2, the characteristics of the beam are presented. These values, obtained experimentally, can be considered as a general reference but every beam, even of different types, shows values slightly different.

Since these beams show a clear softening behaviour, two different types of tests have been conducted:

- the beam has been lifted a few centimetres by one end and then it has been released, such that it bumped against the support
- a moving load has been placed on the beam and pulled by means of a rope.

Before starting the tests with the moving load, two further beams have been added in order to have a zone in which the load can reach a certain velocity and a zone in which this quantity is decreased until to stop the mass.

In figure 5, one of the beams analyzed and the moving load travelling on it are shown.

In this article, two experimental cases are studied on the beam called T7-3 (series 7, number 3): the first one is obtained by the first type of excitation, while the second is obtained by means of the carriage transit on the beam. Moreover, in this acquisition, three impulses produced by a hammer have been added. The time histories of the two cases are represented in figure 6, by zooming in order to highlight the decay of the accelerations.
Figure 6. Time history of the cases considered, with a magnification for the decay visualization. Case with the beam release (a). Case with the load transit, and three impulses (b).

In figure 6.a, the decay is very clear and smooth, while in figure 6.b the decay can be seen only after the end of the load passage. Since the excitation level produced by the transit is very low compared with the hammer impulse, then it is possible to use the nonlinearity, identifiable from the first case, to obtain the frequency depending only on the mass, because the signal levels are quite similar in both the cases. The sampling frequency is equal to 4096 Hz, while the duration of the acquisitions varies depending on the test. Since the quantities recorded are the accelerations in seven different points along the beam, an integration process is used to extract the displacement values. In order to reduce integration errors, the measured accelerations have been first filtered around the lowest natural frequency. For this reason high order modes have been neglected. Moreover, displacement signals have been assumed having zero mean. For simplicity, only the signal coming from the accelerometer placed in the middle of the beam is considered.

6.1. Identification

As already discussed, the first frequency is sufficient to perform the analysis, and the method ST-SSI is used to extract it from the data. It is represented in figure 7, together with the first damping factor: it is easy to note that the natural frequency increases with the time, so it decreases with the displacement peaks, this denoting a softening effect. The frequency of the underlying linear system can be calculated as the asymptote of the curve obtained by the values extracted; in our case it is assumed equal to 19.62 Hz.

Figure 7. Results of the identification on the case with the release of the beam. First natural frequency (a). First damping factor (b).
6.2. Estimation of the nonlinearity
Once the modal parameters have been identified, the method described in session 2 is applied in order to have an estimation of the nonlinearity. To show the reliability of the method, it is applied three times over tests of the same type, i.e. with the release of the beam in correspondence of one support. The results, displayed in figure 8, demonstrate that the estimation is almost the same, for all the cases.

![Figure 8](image)

**Figure 8.** Estimation of the nonlinearity for three “release” tests.

6.3. Estimation of the load characteristics
Now, the case of the transit over the beam is considered. The carriage mass is equal to $m = 11.65$ kg. In figure 9, the natural frequency identified by means of the ST-SSI method and the frequency after the elimination of the nonlinear effect are depicted. There are two main comments:

![Figure 9](image)

**Figure 9.** Identified time-varying frequency (a) and frequency dependent only on the mass (b).

![Figure 10](image)

**Figure 10.** Identified non-constant velocity.
The minimum of frequency dependent only on the mass is greater than the minimum of the identified frequency. This happens because of the softening nonlinearity.

In figure 9.a the time-invariant part, just after the end of the transit, is estimated with a slightly increasing frequency, but in figure 9.b, after the nonlinearity removal, it is estimated with an horizontal line, as it happens for linear systems. The velocity is estimated based on the procedure explained in session 5.1 and the result is proposed in figure 10. There are zones in which the carriage is quicker and other parts in which it is quite slow: this is due to the rope pulling the mass. Finally, by inserting the minimum value of the frequency, it is possible to estimate the load mass, which is equal to \( m = 11.50 \), with a relative error of 1.23%, with respect to the measured one.

7. Conclusions
The method proposed is based on three main techniques:
1. The identification of the nonlinearity, by means of the connections between the natural frequencies and the displacement peaks.
2. The elimination of the nonlinear effect; it is useful where analyzing a time-varying system.
3. Since the experimental problem is a moving mass over a beam, an algorithm for the estimation of the load characteristics (mass and velocity) has been constructed, based only on the knowledge of the first natural frequencies.

The method results are very satisfying for all the different parts. Moreover, the procedure requires only the knowledge of the displacements and of the system mass, and it does not need the input or the knowledge of the nonlinearity type.

References
[1] Barthels P, Wauer J 2008 Non-smooth and time-varying systems of geometrically nonlinear beams, Journal of Sound and Vibration, Vol.315, pp. 455-466.
[2] Allison C A, Miller A K, Inman D J 2009 A time-varying identification method for mixed response measurements, Journal of Sound and Vibration, Vol. 319, pp. 850-868.
[3] Marchesiello S, Garibaldi L 2007 Subspace-based identification of nonlinear structures, Shock and Vibration, Vol. 14, pp. 1-10.
[4] Worden K, Hickey D, Haroon M, Adams D E 2009 Nonlinear identification of automotive dampers: a time and frequency-domain analysis, Mechanical System and Signal Processing, Vol. 23, pp. 104-126.
[5] Richards C M, Singh R 1998 Identification of Multi-degree-of-freedom non-linear systems under random excitations by the reverse path spectral method, Journal of Sound and Vibration, Vol.13, pp. 673-708.
[6] Poulimenos A G, Fassois S D 2006 Parametric time-domain methods for non-stationary random vibration modelling and analysis – A critical survey and comparison, Mechanical System and Signal Processing, Vol. 20, pp. 763-816.
[7] Liu K 1997 Identification of linear time-varying systems, Journal of Sound and Vibration, Vol. 206, pp. 487-505.
[8] Bellino A, Garibaldi L, Marchesiello S 2009 Time-Varying Output-Only Identification of a cracked beam, Key Engineering Materials, Vol. , pp.413-414.
[9] Cvetcicanin L 2006 Homotopy-perturbation method for pure nonlinear differential equation, Chaos, Solitons and fractals, Vol. 30, pp. 1221-1230.
[10] Bellino A, Fasana A, Garibaldi L, Marchesiello S 2010 PCA-based detection of damage in time-varying systems, Mechanical System and Signal Processing, Vol. 24, pp. 2250-2260.