Deconstructing triplet nucleon-nucleon scattering

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Nucleon-nucleon scattering in spin-triplet channels is analysed within an effective field theory where one-pion exchange is treated nonperturbatively. Justifying this requires the identification of an additional low-energy scale in the strength of that potential. Short-range interactions are organised according to the resulting power counting, in which the leading term is promoted to significantly lower order than in the usual perturbative counting. In each channel there is a critical momentum above which the waves probe the singular core of the tensor potential and the new counting is necessary. When the effects of one- and two-pion exchange have been removed using a distorted-wave Born approximation, the residual scattering in waves with \( L \leq 2 \) is well described by the first three terms in the new counting. In contrast, the scattering in waves with \( L \geq 3 \) is consistent with the perturbative counting, at least for energies up to 300 MeV. This pattern is in agreement with estimates of the critical momenta in these channels.

I. INTRODUCTION

Over the last fifteen years, considerable effort has gone into trying to analyse nuclear forces using the systematic tools of effective field theory (EFT).\(^1\) The starting point was Weinberg’s original proposal \(^3\) that these forces could be described within the framework of chiral perturbation theory (ChPT). This approach organises the terms in the effective Lagrangian or Hamiltonian according to powers of low-energy scales they contain. These scales, generically denoted \( Q \), include momenta and the pion mass. In this “Weinberg” power counting, the leading terms of the nucleon-nucleon potential are one-pion exchange (OPE) and an energy-independent contact interaction, both of which are of order \( Q^0 \).

Weinberg also noted the enhancement of the nonrelativistic two-nucleon propagator near threshold and proposed that the leading terms in the potential should be iterated to all orders in order to generate nonperturbative effects, such as the deuteron bound state. This approach, referred to here as the “Weinberg–van-Kolck” (WvK) scheme, has been widely applied by van Kolck and collaborators and by many others.\(^2\) However, even with this enhancement, nucleon-nucleon loop integrals are of order \( Q \). Although this is lower than the order \( Q^2 \) expected in a relativistic theory, it means that each interaction of the leading potential in a scattering equation raises the order by one power of \( Q \). Hence the resulting amplitude should still be perturbatively expandable in powers of the scales \( Q \).

In order to justify treating the leading terms in the potential nonperturbatively, a further IR enhancement is needed, to promote them to order \( Q^{-1} \). Such a promotion is only possible within a consistent power counting if we can identify additional low-energy scales in the nucleon-nucleon system. In the case of \( S \)-wave scattering, the large scattering lengths provide such scales, and these lead to an EFT in which the leading, energy-independent contact terms are treated nonperturbatively \(^4\). At low energies, where the finite-range of OPE is not resolved, the resulting expansion of the potential is simply the effective-range expansion \(^5\).

Although a similar systematic justification for the iteration of OPE was not provided, the WvK scheme has been successfully used to describe a variety of few-nucleon systems and their interactions. Nonetheless, its validity in the \(^3S_1–^3D_1 \) channel has been questioned \(^6\) and, more recently, several groups have observed that Weinberg power counting can break down for nucleon-nucleon scattering in spin-triplet channels with nonzero orbital angular momentum \(^7\). In particular, Nogga, Timmermans and van Kolck \(^8\) find that the leading contact interactions can be substantially promoted in channels where tensor OPE is attractive, although this conclusion does depend on the choice of cut-off, as stressed by Epelbaum and Meissner \(^9\).

To establish a quantitative form for this new power counting, we need first to identify a low-energy scale that would justify iterating OPE, and then to analyse the scale dependence of the associated short-range interactions.

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\(^1\) For reviews, from various points of view, see Refs. \(^3\), \(^4\).

\(^2\) Examples of successful applications can be found the reviews cited in footnote 1.

\(^3\) Closely related observations can be found in the work of Pavón Valderrama and Ruiz Arriola \(^5\).
The renormalisation group (RG) \[15\] provides the natural tool for such an analysis. In its Wilsonian version, it has been applied to two-body scattering by short-range forces, showing that the effective-range expansion is based on a nontrivial fixed point of the RG flow \[16\]. Distorted-wave methods have been used to extend the approach to systems with known long-range forces \[17, 18\].

Once a factor of $1/M_N$ has been divided out of the Hamiltonian, the strength of the OPE potential can be expressed in terms of the momentum scale

$$
\lambda_\pi = \frac{m_\pi^2}{f_{\pi NN}^2 M_N} \simeq 290 \text{ MeV}, \tag{1}
$$

where $f_{\pi NN}$ is the pseudovector $\pi N$ coupling constant. In the chiral limit, it can be written

$$
\lambda_\pi = \frac{16\pi F^2}{g_A^2 M_N}, \tag{2}
$$

where $M_N$ the nucleon mass, $g_A$ is the axial coupling of the nucleon and $F_\pi$ the pion decay constant. In strict ChPT, $\lambda_\pi$ is therefore a high-energy scale, built out of $4\pi F_\pi$ and $M_N$. None-the-less its numerical value is small, only about twice $m_\pi$. As a result, perturbative treatments of OPE (as advocated by Kaplan, Savage and Wise \[7\]) fail to converge or converge only slowly \[14, 15\]. This suggests that we should explore the consequences of identifying $\lambda_\pi$ as a low-energy scale, counting it as of order $Q$. Since $\lambda_\pi$ is proportional to $1/M_N$, this can be thought of as a concrete version of Weinberg’s suggestion that $1/M_N$ should be treated as if it were of order $Q$ \[3\].

In Ref. \[12\], I applied a renormalisation group analysis to the short-range potential in the presence of tensor OPE. This made use of the distorted waves (DW’s) of a $1/r^3$ potential (the chiral limit of tensor OPE). In the resulting power counting, the leading short-range potential is of order $Q^{-1/2}$, independently of the orbital angular momentum. This is quite different from Weinberg counting, where the leading term in the $L$-th partial wave is of order $Q^{2L}$. Subleading terms containing powers of the energy appear at orders $Q^{3/2}, Q^{7/2}$, and so on. This promotion of short-range terms confirms the numerical observations of Nogga, Timmermans and van Kolck \[11\] and makes quantitative the new counting proposed there. The terms in the resulting potential can be directly related to a DW Born expansion, similar to that in Refs. \[17, 18\].

The validity of this counting does depend on the energies considered, since in each channel there is critical momentum above which waves penetrate the centrifugal barrier and reach the region where the $1/r^3$ singularity dominates. The analyses of Refs. \[11, 12\] show that a nonperturbative treatment of OPE, and hence the new counting, is needed in the $S$, $P$ and $D$ waves for momenta of order $m_\pi$. In contrast, waves with $L \geq 3$ do not probe the singularity until momenta of $\sim 2$ GeV are reached. In these higher partial waves OPE can be treated as a perturbation and short-distance interactions can be organised according to the usual Weinberg power counting.

The results of the analysis of Ref. \[12\] were purely formal, leading to the power counting that governs the importance of the terms in the expansion of the short-range potential. In the present paper, I explore its practical consequences by analysing nucleon-nucleon scattering in spin-triplet channels, with an extension of the method applied to singlet channels in Ref. \[20\]. For simplicity I consider only the uncoupled waves: $^3P_{0,1}, ^3D_2, ^3F_3$ and $^3G_4$. The extension of the method to coupled waves such as $^3S_1$-$^3D_1$ or $^3P_2$-$^3F_2$ is very similar in principle, but is technically more complicated because of the matrix nature of the equations.

The RG analysis relies on the forms of the DWs at small radii, where they tend to asymptotic forms that are independent of energy. These waves are obtained by solving the Schrödinger equation, as described in Sec. II. At small enough radii they show nonperturbative behaviour controlled by the $1/r^3$ singularity of the tensor potential. In the case of waves with $L \leq 2$, this region extends out to about 1 fm. For lab kinetic energies up to 300 MeV the waves reach their asymptotic forms only for radii less than about 0.6 fm, and there they are dominated by the $1/r^3$ potential. This nonperturbative behaviour is present in waves with $L \geq 3$, but only for only for radii less than about 0.2 fm. In the range 0.2–0.6 fm they have the normal power-law forms associated with the centrifugal barrier. This confirms the expectations in Refs. \[11, 12\] that low partial waves need the new power counting for energies in this range, whereas the higher waves can still be described perturbatively using Weinberg counting.

I then use DW methods to “deconstruct” empirical scattering amplitudes by removing the effects of known long-range-forces. The residual amplitude can then be interpreted directly in terms of an effective short-range potential. This technique can provide a better indication of how well the known forces are able to describe the scattering, compared to simply plotting phase shifts. Such plots can be misleading since they tend to hide small differences in peripheral waves at low energies, which is just where the long-range forces should dominate. If the resulting potential still shows strongly nonlinear energy dependence at low energies, then this implies that long-range forces are still making important contributions. Short-range forces lead to a smooth energy dependence that can be expanded as a power series.

As in the similar treatment of singlet channels \[21\], I take several Nijmegen PW A’s or potentials \[21\], to give an indication of the uncertainties involved in these analyses of the data. In Sec. III, I use them to construct scattering
amplitudes between the DW’s of the OPE potential and from these I extract short-range potentials in the uncoupled spin-triplet channels. The resulting potentials show rapid energy dependence at low energies, indicating that important long-range physics is still present.

The most obvious long-range forces that need to be removed next are two-pion exchange (TPE) and relativistic corrections to OPE. These appear at orders $Q^2$ and $Q^3$ (in Weinberg counting). I use here the forms of the TPE potentials given in Refs. [22, 23] and the corresponding order-$Q^2$ correction to OPE [24]. At this order, there is also a $\gamma \pi$-exchange potential, calculated in Ref. [22]. These potentials can all be subtracted perturbatively using the DWBA. The residual short-distance interactions shown in Sec. IV are consistent with the new power counting in the $^3P_{0,1}$ and $^3D_2$ waves. In higher waves, $^3P_3$ and $^3G_4$, the uncertainties in the Nijmegen PWA’s make it hard to draw very strong conclusions but the residual short-range potentials are smaller after removal of TPE, and similar to those in the singlet channels [20].

II. DISTORTED WAVES

The radial Schrödinger equation that describes the relative motion of two nucleons interacting through the long-range OPE potential is

$$- \frac{1}{M_N} \left[ \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{L(L+1)}{r^2} \right] \psi(r) + \left[ V_{\pi c}(r) + V_{\pi T}(r) \right] \psi(r) = \frac{p^2}{M_N} \psi(r), \quad (3)$$

where the central piece of the lowest-order potential is

$$V_{\pi c}(r) = \frac{1}{3} f_{\pi N N}^2 \frac{e^{-m_\pi r}}{r} (\sigma_1 \cdot \sigma_2)(\tau_1 \cdot \tau_2), \quad (4)$$

and its tensor piece is

$$V_{\pi T}(r) = \frac{1}{3} f_{\pi N N}^2 \frac{e^{-m_\pi r}}{r^3} \left( 3 + 3m_\pi r + m_\pi^2 r^2 \right) \frac{e^{-m_\pi r}}{r^3} S_{12}(\tau_1 \cdot \tau_2). \quad (5)$$

Here the tensor operator, $S_{12} = 3(\sigma_1 \cdot \vec{r})(\sigma_2 \cdot \vec{r}) - \sigma_1 \cdot \sigma_2$, takes the value $+2$ in the uncoupled $^3P_1$, $^3D_2$, . . . channels, and $-4$ in the $^3P_0$ channel. The isospin factor, $\tau_1 \cdot \tau_2$, is $+1$ for channels with odd $L$ and $-3$ for even $L$. The on-shell momentum in the centre-of-mass frame, denoted by $p$, is related to the lab kinetic energy, $T$, by $T = 2p^2/M_N$.

At small enough radii, all the solutions in a given channel tend to a common, energy-independent form which is determined by the $1/r^3$ term of the tensor potential and the $1/r^2$ centrifugal barrier. This form can be found by solving Eq. (5) at zero energy in the chiral limit ($m_\pi = 0$) where it can be written

$$\left[ \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{L(L+1)}{r^2} - \frac{\beta_{LJ}}{r^3} \right] \psi_0(r) = 0, \quad (6)$$

Here I have specialised to the case of uncoupled triplet channels and I have introduced the length scale

$$\beta_{LJ} = \begin{cases} -4/\lambda_\pi, & L = 1, J = 0, \\ +2/\lambda_\pi, & L = J \text{ odd,} \\ -6/\lambda_\pi, & L = J \text{ even.} \end{cases} \quad (7)$$

The solutions in this limit can be expressed in terms of Bessel functions of order $2L + 1$. This can be seen by defining the variable $x = \sqrt{\beta_{LJ}/r}$ and the function $\phi(x) = x^{-1/2} \psi_0(|\beta_{LJ}|/x^2)$, so that the equation becomes

$$\left[ \frac{d^2}{dx^2} + \frac{1}{x} \frac{d}{dx} - \frac{(2L + 1)^2}{x^2} \pm 4 \right] \phi(x) = 0,$$

where the plus sign applies to channels with even $J$ (where the tensor potential is attractive) and the minus sign to odd $J$.

The solutions in the attractive channels are oscillatory:

$$\psi_0(r) = A \sqrt{\frac{\beta_{LJ}}{r}} \left[ \sin \alpha J_{2L+1} \left( 2 \sqrt{\frac{|\beta_{LJ}|}{r}} \right) + \cos \alpha Y_{2L+1} \left( 2 \sqrt{\frac{|\beta_{LJ}|}{r}} \right) \right], \quad (9)$$
where $J_{2L+1}$ and $Y_{2L+1}$ denote the regular and irregular Bessel functions. In the limit $r \to 0$, these solutions tend to the WKB form of a sinusoidal function of $2\sqrt{|\beta_{LJ}|/r}$ times $r^{-1/4}$. They depend on a free parameter $\alpha$. This angle fixes the phase of the small-$r$ oscillations or, equivalently, it specifies a self-adjoint extension of the original Hamiltonian (see Refs. [12, 18] for further references). This is necessary since both Bessel functions give acceptable solutions for small $r$. There is a redundancy between $\alpha$ and the leading term in the effective short-range potential since both have the effect of fixing the phase of the wave function for small $r$. In waves where the scattering is weak, it is simplest to set $\alpha = 0$ and use the potential to represent short-distance physics. This leads to solutions to Eq. (6) that grow like $r^L$ for large $r$. In channels where the OPE potential can be treated perturbatively, this allows the waves to match on to their usual short-distance forms at larger radii where the centrifugal barrier dominates over the tensor potential.

The special choice $\alpha = \pi/2$ gives a solution to Eq. (6) that decays like $r^{-(L+1)}$ for large $r$. Imposing this boundary condition on the full OPE problem would lead to a wave that was very large inside the attractive well of the $1/r^3$ potential. This would correspond to a system with a low-energy bound state or resonance. Since none of the NN channels with $L > 0$ has such a low-energy state, values of $\alpha$ close to $\pi/2$ should be avoided. In practice, $\alpha = 0$ is a good choice for all but one of the waves studied here.

In the repulsive channels, the solutions are given by modified Bessel functions, and the regular one has the form

$$\psi_0(r) = A\sqrt{\frac{|\beta_{LJ}|}{r}} K_{2L+1} \left( 2\sqrt{\frac{|\beta_{LJ}|}{r}} \right).$$

This vanishes exponentially with $2\sqrt{|\beta_{LJ}|/r}$ as $r \to 0$ but, like Eq. (6), grows as $r^L$ for large $r$.

It is convenient to normalise these solutions so that, as $r$ increases, they match on to the expected short-distance behaviour of the free solutions, $j_L(pr)/p^L$. (Since the asymptotic solutions are defined at zero energy, I have divided out the $p^L$ energy dependence from the spherical Bessel functions.) With this normalisation,

$$\psi_0(r) \sim \frac{p^L}{(2L + 1)!!} \quad \text{as } r \to \infty,$$

the asymptotic solutions become

$$\psi_0(r) = \begin{cases} 
- \frac{\pi|\beta_{LJ}|}{(2L)!!(2L + 1)!!} \cos \alpha \sqrt{\frac{|\beta_{LJ}|}{r}} \left[ \sin \alpha J_{2L+1} \left( 2\sqrt{\frac{|\beta_{LJ}|}{r}} \right) + \cos \alpha Y_{2L+1} \left( 2\sqrt{\frac{|\beta_{LJ}|}{r}} \right) \right] & J \text{ even}, \\
\frac{2|\beta_{LJ}|}{(2L)!!(2L + 1)!!} \sqrt{\frac{|\beta_{LJ}|}{r}} K_{2L+1} \left( 2\sqrt{\frac{|\beta_{LJ}|}{r}} \right) & J \text{ odd}. 
\end{cases}$$

Fig. 1 shows the solutions to the full Schrödinger equation (3) for the channels $^3P_0$, $^3P_1$, $^3D_2$ and $^3G_4$ at lab kinetic energies of 5 and 300 MeV, and compares them to the zero-energy, chiral-limit solutions of Eq. (12). To make this comparison easier, I have also divided out the $p^L$ energy dependence from the full solutions. The waves shown for the attractive channels all have the short-distance phase $\alpha = 0$.

Although there is still some energy dependence in their normalisations, the shapes of the solutions in these channels reach their energy-independent forms for radii smaller than about 0.8 fm. The waves in channels where the tensor potential is attractive all oscillate for small enough radii. In the lowest waves, $^3P_1$ and $^3D_2$, this behaviour covers the whole energy-independent region. Indeed the first node of the $^3P_0$ wave function appears at the edge of or, depending on the choice of $\alpha$, within the domain of the low-energy EFT. Moreover the normalisation of the short-distance wave functions shows significant energy-dependence beyond the $p^L$ expected from the free solutions.

In contrast, the $^3G_4$ wave becomes oscillatory only for radii smaller than about 0.25 fm, beyond the scope of our EFT. In the rest of its energy-independent region it has power-law behaviour as expected from the centrifugal potential. More importantly from the EFT point of view, the energy dependence of the short-distance normalisation is almost entirely given by the $p^L$ factor expected for a free solution. These results are consistent with the estimates in Ref. [12] based on the chiral limit of the tensor potential. There, the critical momentum scales for the breakdown of perturbation theory were found to be of the order of $m_\pi$ or $\lambda_\pi$ in the $S$, $P$ and $D$ waves, and much larger ($\sim 2$ GeV) in the $F$ waves and above.

The $^3D_2$ and $^3G_4$ waves have low-energy bound states or narrow resonances if $\alpha$ is taken to be close to $\pi/2$. The phase shifts can depend significantly on the choice of $\alpha$ in this region which, in the $^3D_2$ case, is roughly $\pi/4 \lesssim \alpha \lesssim 3\pi/4$. The high centrifugal barrier in the $^3G_4$ wave means its phase shift is very weakly dependent on $\alpha$, outside a very narrow band around $\pi/2$. In contrast the $^3P_0$ phase shift shows a strong dependence on $\alpha$ over the whole range $0 \leq \alpha < \pi$. 
FIG. 1: Plots of the wave functions $\psi(r)/p^L$ (in arbitrary units) against $r$ (in fm) for the channels (a) $^3P_0$, (b) $^3P_1$, (c) $^3D_2$, and (d) $^3G_4$. Short-dashed lines: $T = 5$ MeV; long-dashed lines: $T = 300$ MeV; solid lines: energy-independent asymptotic form from Eq. (6).

The wave functions in the repulsive channels $^3P_1$ and $^3F_3$ both have qualitatively similar forms, and so only one is shown in Fig. 1. They switch from power-law behaviour at larger radii, where the centrifugal barrier dominates, to exponential at small radii, where the $1/r^3$ potential wins out. In both cases the energy dependence of the short-distance normalisation is quite well described by the free $p^L$ form, at least for energies below about 300 MeV. At higher energies than considered here, 500 MeV or above, the $^3P_1$ normalisation does develop a much stronger energy dependence (whereas the $^3F_3$ does not). These numerical observations indicate that the effect of the finite pion mass has been to shift the critical momentum scale in $^3P_1$ channel to a somewhat higher value than the estimate in Ref. [12]. From the wave functions alone, it is thus not clear whether tensor OPE is better treated nonperturbatively in this channel.

III. ONE-PION-EXCHANGE EFFECTIVE POTENTIAL

The RG method developed in Refs. [17] shows that the terms in the short-range effective potential are directly related to an expansion of the DW $K$-matrix in powers of the energy and other low-energy scales. A DW approach like this is obviously needed in channels where OPE must be treated nonperturbatively. I will also use it in the channels where a perturbative treatment of OPE would be valid, since it provides a convenient alternative to fourth-order perturbation theory. The scattering between DW’s of the long-range potential can be described by a reactance matrix $\tilde{K}$. In the uncoupled channels, its on-shell matrix element is related to the difference between the observed and OPE phase shifts by

$$\tilde{K}(p) = -\frac{4\pi}{Mp} \tan\left(\delta_{\text{PWA}}(p) - \delta_{\text{OPE}}(p)\right),$$

where $\delta_{\text{PWA}}(p)$ is the empirical phase shift (from the Nijmegen group’s PWA or one of their potentials that have been fitted directly to data) and $\delta_{\text{OPE}}(p)$ is that obtained from the solutions of Eq. (6). None of the channels examined
here contains a low-energy bound state or resonance, and so the residual scattering can be represented by an EFT expanded around a trivial fixed point. The amplitude $\tilde{K}(p)$ is then given by the distorted-wave Born approximation (DWBA): the matrix element of the short-range interaction between the DW’s of the long-range potential.

Since these DW’s are either vanishing or singular as $r \to 0$, an ordinary δ-function at the origin cannot be used to represent the potential. Instead, I take a δ-shell potential, as in Refs. [12 17 20]. If the radius of this, $R_0$, is chosen to be smaller than about 0.7 fm, in the region where the waves have reached their common energy-independent forms, then the extracted potential will be independent of $R_0$ to a good approximation. The results shown below are for $R_0 = 0.1$ fm, the same radial cut-off as used in Ref. [20], but I return to the question of the choice of cut-off at the end of Sec. IV, after examining the subtraction of TPE.

To remove the dependence on the arbitrary radius $R_0$, I divide out the square of the asymptotic radial function, Eqs. (9) or (10), from the strength of the δ-shell potential. The resulting potential is defined by

$$V_S(p, r) = \frac{1}{4\pi R_0^2 |\psi_0(R_0)|^2} \tilde{V}(p) \delta(r - R_0),$$

(14)

as in the RG analysis of Refs. [12 17]. Equating $\tilde{K}(p)$ to the DWBA matrix element of this potential, we can deduce the strength of the potential directly from the residual scattering amplitude, as in Ref. [20]:

$$\tilde{V}(p) = \frac{|\psi_0(R_0)|^2}{|\psi(p, R_0)|^2} \tilde{K}(p),$$

(15)

where $\psi(p, R_0)$ is the solution to the Schrödinger equation with the known long-range potential, and $\psi_0(R_0)$ is its energy-independent short-distance form.

The leading term in the short-distance interaction represents only short-distance physics, and so it should be independent of any low-energy (long-distance) scales such as $p$, $m_\pi$ or $\lambda_\pi$. To ensure this, any dependence on these scales should be factored out of the normalisation of the asymptotic solutions in Eqs. (9) and (10). In waves where the OPE potential can be treated perturbatively, this means normalising these functions by dividing out the energy-dependent factor of $p^L$, as in Eq. (14). The resulting potentials in the $^3F_3$ and $^3G_4$ channels are the defined in the same way as the one used to analyse the spin-singlet channels in Ref. [20]. At LO their matrix elements are proportional to $p^{2L}$, showing that they are equivalent to $2L$-th derivatives of δ-functions (the more conventional representations for the effective interactions in these partial waves).

As discussed above, $\lambda_\pi$ should be regarded as an additional low-energy scale in the channels where OPE must be treated nonperturbatively. Since $\beta_{LJ} \propto 1/\lambda_\pi$, the wave functions normalised as in Eq. (12) still contain powers of $\lambda_\pi$. This can be removed by dividing out the factors of $|\beta_{LJ}|^{L+1/4}$ from the zero-energy solutions. Using solutions with this normalisation has the effect of multiplying $\tilde{V}(p)$ by $\lambda_\pi^{2L+1/2}$. This obviously changes the magnitudes of the short-range interactions, but it does not affect their energy dependences. In the plots below, I just show results for $\tilde{V}(p)$ defined using the forms given in Eq. (12), but the extra factors would be needed if one wanted to estimate the momentum scales in this potential.

Once the effects of leading OPE have been removed, the residual potential has at order $Q^2$ in the standard power-counting notation. Contributions at this order include the leading TPE potential [22 23], and a relativistic correction to OPE [24]. Since the power counting for the long-range forces is not affected by the renormalisation of the short-range ones, I denote this potential by $\tilde{V}^{(2)}(p)$. In Weinberg’s counting for a partial wave with angular momentum $L$, the leading short-range term appears at the usual order, $Q^{2L}$. However, if OPE is treated nonperturbatively, we need to take account of the fact that this term also contains a factor of $\lambda_\pi^{-2L+1/2}$. Since $\lambda_\pi$ should be now regarded as a low-energy scale, we see that the net order of such a term is $Q^{-1/2}$, for any $L$. This half-integer power agrees with the RG analysis in Ref. [12], which shows that the leading term has an RG eigenvalue of 1/2. After extracting the effects of tensor OPE we are therefore left with an interaction that starts at order $Q^{-1/2}$. This also contains a term proportional to energy ($p^2$) at order $Q^{3/2}$.

Fig. 2 shows the short-distance interactions $\tilde{V}^{(2)}(p)$ extracted directly from various Nijmegen analyses [21], using Eq. (15) with $R_0 = 0.1$ fm. The partial-wave analysis, PWA93, and three potentials, NijmegenI, NijmegenII and Reid93, all fit the np data with similarly good values of $\chi^2$. They can thus be regarded as alternative partial-wave analyses. Using results from all of them gives an indication of the systematic uncertainties associated with the different choices of parametrisation. This is particularly important in higher partial waves where each fit contains only a small
FIG. 2: Plots of the short-distance interaction $\tilde{V}^{(2)}(p)$, in fm$^{2L+2}$, against lab kinetic energy $T$, in MeV. These have been extracted from Nijmegen PWA’s or potentials using Eq. (15) with $R_0 = 0.1$ fm, for the $np$ channels (a) $^3P_1$, (b) $^3D_2$, (c) $^3F_3$, and (d) $^3G_4$.

number of parameters. In the $^3D_2$ and $^3G_4$ channels where the tensor OPE is attractive, I have taken the phase $\alpha = 0$ since, as already noted, their OPE phase shifts depend only weakly on $\alpha$, provided the region around $\pi/2$ is avoided.

One immediate observation is that the residual interactions all show rapid, nonlinear dependences on energy below about 150 MeV. This is similar to what was found for the corresponding spin-singlet channels in Ref. [20]. Experience with those channels suggests that TPE is responsible for much of this rapid energy dependence. Therefore these forces need to be subtracted before any conclusions can be drawn about short-range ones.

FIG. 3: Plots of the $^3P_0$ short-distance interaction $\tilde{V}^{(2)}(p)$, in fm$^4$, against lab kinetic energy $T$, in MeV. Results for two choices of short-distance phase are shown: (a) $\alpha = 0$, (b) $\alpha = 0.54$.

Finally I turn to the $^3P_0$ wave, shown in Fig. 3. The left-hand plot was obtained using $\alpha = 0$, as in the $^3D_2$ and $^3G_4$ cases. As in the attractive $^3D_2$ channel, this shows a rapid, non-monotonic energy dependence. In addition,
its overall strength is much larger than in the $^3P_1$ channel, by a factor of 30 or more. This implies that the leading (order-$Q^{-1/2}$) interaction in this channel is too strong to be treated in the DWBA. It should either be iterated to higher orders or, possibly, treated nonperturbatively.

It practice, it is most convenient to include this term to all orders using the equivalent short-distance parameter $\alpha$. As discussed above, this defines a self-adjoint extension of the the OPE Hamiltonian in Eq. (3) by fixing the phase of the oscillations of the wave functions as $r \to 0$. In the right-hand plot, I show the residual interaction for $\alpha = 0.54$. This value was chosen so that the interaction vanishes at zero energy, within the uncertainties of the PWA’s. Removing the effects of the energy-independent term in this way leaves a residual interaction that is consistent with a monotonic piece plus one linear in the energy. Again, more definite conclusions require subtraction of the effects of other long-range forces, which I now turn to.

IV. TWO-PION EXCHANGE

The DW method described above could also be used to separate off the scattering produced by other known long-range forces, most importantly those arising from two-pion exchange. However, since such forces start at order $Q^2$, their effects can just be subtracted perturbatively from the residual interactions left after removal of OPE [20]. Although iterations of TPE will contribute at higher orders, this DWBA treatment is adequate up to order $Q^4$.

The relevant TPE potentials have been calculated at orders $Q^2$ and $Q^3$ and their forms can be found in Refs. [22,23]. At order $Q^2$, the long-range interactions also include a term from expanding the relativistic correction to OPE [24],

$$V^{(2)}_{1\pi}(r) = -\frac{p^2}{2M^2} [V_{2\pi}(r) + V_{1\pi}(r)].$$

Lastly, there is an electromagnetic interaction, generated by $\pi\gamma$ exchange, whose form is derived in [25].

If all of these are subtracted from $\tilde{K}(p)$ using the DWBA, then any residual long-ranged contributions to the scattering start at order $Q^4$. The resulting potential is

$$\tilde{V}^{(4)}(p) = \frac{|\psi_0(R_0)|^2}{|\psi(p,R_0)|^2} \left( \tilde{K}(p) - \langle \psi(p) | V^{(2)}_{1\pi} + V^{(2,3)}_{2\pi} + V_{2\gamma} | \psi(p) \rangle \right).$$

Note that although this potential is denoted $\tilde{V}^{(4)}(p)$ because it contains long-range contributions of order $Q^4$ and higher, it may contain short-range terms of lower order. Such terms appear first at order $Q^2$ for $P$-waves in the usual power counting, and at order $Q^{-1/2}$ for waves where tensor OPE is treated nonperturbatively.

The long-range potentials are singular and so their matrix elements can be divergent. I therefore cut the integrals off, using the same radius, $R_0$, as in the definition of the short-range interaction. For example, the most divergent part of the order-$Q^3$ TPE potential at small radii is proportional to $1/r^6$ [22,23]. In a perturbative treatment of OPE, the wave functions behave like $1/r^6$ as $r \to 0$. Together these lead to a $1/R_0$ divergence in the $P$-wave matrix elements which can be renormalised by introducing the order-$Q^2$ counterterm just mentioned. Other partial waves do not give rise to divergences to this order.

When OPE is treated nonperturbatively the small-$r$ forms of the wave functions are quite different, as described above. The $r^{-1/4}$ behaviour of these leads to a $1/R_0^{7/2}$ divergence in the matrix element of the order-$Q^4$ potential, at least in the channels where tensor OPE is attractive. This can be renormalised by the leading short-distance potential, of order $Q^{-1/2}$. In addition, there are other, weaker divergences which appear multiplied by powers of $\lambda_\pi$ or $m_\pi$ and which can be renormalised by higher-order counterterms. However these energy-independent terms can not be disentangled phenomenologically from the leading one. The relativistic correction to OPE, Eq. (16), also leads to a divergence and this can be renormalised by an order-$Q^{3/2}$ term, proportional to $p^2$. The complete set of terms with orders below $Q^4$ also includes one proportional to the square of the energy, $p^4$. This term is of order $Q^{7/2}$ and it has a finite coefficient at the current level of approximation. However, when long-range forces of order $Q^4$ are included, it will be needed to renormalise divergences from, for example, the next relativistic correction to OPE.

The short-range terms up to order $Q^{7/2}$ provide a smooth, quadratic energy-dependence. Subtracting them should leave a residual interaction consisting only of terms of order $Q^4$ or higher. I do this by fitting a quadratic form,

$$\tilde{V}^{(7/2)}(p) = C_0 + C_2 p^2 + C_4 p^4,$$

(18)

to $\tilde{V}^{(4)}(p)$ in the range $T = 100$ to 200 MeV. The lower end of this range is chosen to lie above the region where OPE and TPE can lead to rapid energy dependence. The upper limit corresponds to a relative momentum $p \simeq 300$ MeV. Above this point, higher powers of the the energy could start to become noticeable, representing physics that has been integrated out, such as excitation of the $\Delta$ resonance.
Let me start with the two peripheral waves, where the breakdown scale for perturbation theory is so high that there is no question about the validity of the perturbative treatment of OPE or the standard power counting. As in Ref. [20], I use the full DW solutions for these, although these are not strictly necessary, to avoid the complications of fourth-order perturbation theory. The residual interactions \( \tilde{V}^{(4)}(p) \) for these are shown in Fig. 4. In both cases, \( \tilde{V}^{(4)}(p) \) was extracted at \( R_0 = 0.1 \) fm, which is small enough that the wave functions have attained their energy-independent forms and that the \( ^3F_3 \) radial integral has converged. The \( ^3G_4 \) integral has reached a plateau here; the divergences associated with the oscillatory region do not appear until \( R_0 \) is less than about 0.03 fm.

As in the corresponding spin-singlet waves [20], the uncertainties in the PWA’s make it hard to draw strong conclusions. The \( ^3G_4 \) residual interactions, \( \tilde{V}^{(2)}(p) \) and \( \tilde{V}^{(4)}(p) \), are both consistent with zero, given the spread of the PWA’s. In the \( ^3F_3 \) channel, the energy dependence in \( \tilde{V}^{(2)}(p) \) below 100 MeV is somewhat reduced by subtraction of the order-\( Q^2 \) long-range forces.

The interaction \( \tilde{V}^{(4)}(p) \) in the \( ^3P_0 \) channel is shown in Fig. 5. The unsubtracted interaction, on the left, is very large, with a strong linear energy dependence, as a result of the divergences in the integrals of the order-\( Q^2 \) potentials. The residual interaction after subtracting a quadratic fit is shown on the right. This shows that the quadratic form can provide a good account of the energy dependence of \( \tilde{V}^{(4)}(p) \) in the \( ^3P_0 \) channel over the whole range of \( T \) up to 300 MeV. Any contributions from forces of order \( Q^4 \) and higher lie within the uncertainties of the PWA’s.

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5 In this channel, I have omitted the Reid93 results from the fit since they lie systematically below the ones from the other three Nijmegen analyses. Including this potential would simply shift the fitted constant, and significantly increase the uncertainty associated with the PWA’s.
Fig. 6 shows the results of a similar analysis of scattering in the \( ^3D_2 \) channel. Here, the quadratic fit can account well for the energy dependence from \( T \simeq 80 \text{ MeV} \) to about 250 MeV. Below 80 MeV, the PWA’s require a significant extra attractive interaction with a long range in view of its rapid energy dependence. It is not obvious what could be responsible for this, since no appropriate piece seems to be present in the order-\( Q^4 \) pion-exchange potentials \[26, 27\]. However it should be noted that its size is comparable to the uncertainties in the PWA’s in this region.

\[
V^{(4)}(p), \text{ in fm}^6, \text{ against lab kinetic energy } T, \text{ in MeV, for the channel } ^3D_2 \text{ with } \alpha = 0, \text{ (a) unsubtracted, (b) quadratic fit to } T = 100 - 200 \text{ MeV subtracted.}
\]

Finally, \( V^{(4)}(p) \) for the \( ^3P_1 \) channel has a smooth, approximately linear energy dependence, as can be seen in Fig. 7. The value at zero energy increases as the cut-off radius \( R_0 \) is decreased from 0.6 fm to about 0.2 fm. This is consistent with the standard power counting, where a divergence is present at order \( Q^2 \). In that counting, this is the only divergence below order \( Q^4 \). However the coefficient of the linear energy dependence in the results here also increases significantly over this range of \( R_0 \). This suggests that the \( ^3P_1 \) channel may in fact be better analysed using the new power counting, a conclusion that is supported by the low breakdown scale for the perturbative treatment of OPE found in Ref. \[12\].

\[
V^{(4)}(p), \text{ in fm}^4, \text{ against lab kinetic energy } T, \text{ in MeV, for the channel } ^3P_1, \text{ (a) unsubtracted, (b) quadratic fit to } T = 100 - 200 \text{ MeV subtracted.}
\]

The results shown so far are all obtained with a radial cut-off \( R_0 = 0.1 \text{ fm} \). This is well inside the region where the dependence on \( R_0 \) is very small for the range of energies considered but it does correspond to momentum scales greater than 1 GeV, far beyond the range of validity of our EFT. This small radial cut-off was used in Ref. \[20\] to avoid introducing artefacts proportional to positive powers of \( R_0 \). The coefficients in the residual potential could then be related directly to scales of the underlying physics. The connection is more complicated here, as one would first need to renormalise the coefficients to remove the divergences proportional to powers of \( 1/R_0 \) (and there are several of these in the energy-independent term, involving different powers of \( m_\pi \) and \( \lambda_\pi \)). As a result, I do not attempt to make such an interpretation of the coefficients here and hence the choice of such a small radius is not crucial.

In this case, one can ask what happens if a larger cut-off radius is chosen. The question is pertinent since Epelbaum and Meissner \[13\] have shown that a single extra term, as required by the new counting, can give a fairly good description of \( P \)- and \( D \)-wave phase shifts for momentum cut-offs as low as \( \sim 3 \text{ fm}^{-1} \). Such values would also avoid
regions where TPE might need to be treated nonperturbatively. I have therefore repeated these analyses with larger cut-off radii. The results are essentially indistinguishable from the ones shown above for radii up to about 0.6 fm (except in regions around zeros of the wave functions). Beyond that point, the wave functions do show noticeable energy dependence over the range up to \( T = 300 \) MeV, as already noted from Fig. 1. This leads to a dependence on \( R_0 \) of the residual interactions, especially at higher energies.

If these artefacts of a finite radial cut-off are expanded as a power series in \( R_0^2 p^2 \), the first three terms (up to order \( p^4 \)) can be absorbed in the coefficients of the quadratic fit. They are therefore removed when this fitted potential is subtracted as described above. However, they will generate terms in the renormalised coefficients of the short-range potential where the momentum scale is set by \( 1/R_0 \) and not by the underlying physics. This would make it difficult to interpret these coefficients in terms of physical scales.

This cut-off dependence increases with energy and so it is most prominent at higher energies. A measure of how much the shape of the waves changes is provided by the ratio of ratios,

\[
\rho = \frac{\psi(p_{\text{max}}, R_0)\psi(0, R_1)}{\psi(p_{\text{max}}, R_1)\psi(0, R_0)}^2,
\]

in which the energy-dependent normalisation of the short distance wave functions cancels out. For \( R_0 = 0.6 \) fm, \( R_1 = 0.1 \) fm and \( p_{\text{max}} = 375 \) MeV (corresponding to \( T = 300 \) MeV), \( \rho \) is greater than 0.8 for most of the waves considered here. The \( \sim 20\% \) changes that this induces in the the residual interactions can be removed by the subtraction of the quadratic fit. The resulting subtracted interactions are almost indistinguishable from the ones shown in Figs. 6 (b) and 7 (b), except for the \( ^3P_0 \) which shows larger effects as a result of a nearby zero in its wave functions.

For \( R_0 = 1.0 \) fm, the ratio \( \rho \) is around 0.6 for these waves. The residual interaction \( \tilde{V}^{(4)}(p) \) for the \( ^3D_2 \) channel is shown in Fig. 8. The divergent terms are much smaller and so the overall magnitude of the potential is greatly reduced compared to the one shown in Fig. 6 (a). However the bulk of the change lies in the terms up to order \( p^4 \). After these have been subtracted, the differences between Figs. 6 (b) and 8 (b) are small, at least for energies below about 250 MeV. The pattern in the \( ^3P_1 \) channel is similar. The \( ^3P_0 \) is complicated by the fact that the wave functions pass through zero near 0.8 fm. Cut-off-independent results for the subtracted interaction can be obtained, but only for \( R_0 \lesssim 0.5 \) fm or \( R_0 \simeq 1.2 \) fm. In the higher partial waves, \( ^3F_3 \) and \( ^3G_4 \), the unsubtracted residual interactions show very little cut-off dependence over the range 0.02 fm \( \lesssim R_0 \lesssim 1 \) fm.

**V. CONCLUSIONS**

Nogga, Timmermans and van Kolck \[11\] have found that a new power counting is needed to organise the EFT describing nucleon-nucleon scattering in spin-triplet channels. In particular, the leading short-distance terms in \( P \) and \( D \) waves are significantly promoted compared to the perturbative (Weinberg) counting. In Ref. \[12\] I obtained a quantitative statement of this new counting by identifying the scale of the OPE potential, \( \lambda_\pi \), as an additional low-energy scale, and then treating OPE nonperturbatively. This leads to an expansion of the short-range interaction describing scattering between the DW’s of the OPE potential. Its terms correspond to those of an expansion of the DW Born amplitude in powers of the energy. The leading term is of order \( Q^{-1/2} \), for any orbital angular momentum.
The forms of the DW’s show that a nonperturbative treatment of tensor OPE, and hence the new power counting, is required for energies that are large enough for the waves to probe the region where the $1/r^3$ core of the potential dominates over the centrifugal barrier. Otherwise the short-distance wave functions have the normal $r^L$ behaviour and perturbative counting remains valid. The analytic estimates in Ref. [12] and the numerical wave functions in Sec. II both indicate that the nonperturbative approach is required in waves with $L \leq 2$ for momenta of the order of $m_N$ or larger. In $F$ waves and above, a perturbative treatment is expected to remain valid up to energies well beyond the validity of the EFT.

Here I have “deconstructed” empirical phase shifts by using DW methods to remove the effects of long-range pion-exchange forces. This generates a residual short-range interaction directly from the observed phase shifts. Unlike comparisons of phase-shift plots, this approach emphasises the low-energy region, where the EFT description ought to work best. The use of several Nijmegen partial wave analyses [21] allows estimates of the uncertainties in these fits to the data. These can be large at low energies, implying that it may be misleading to fit the coefficients of an EFT to very low-energy data.

Removing only the effects of OPE leaves residual interactions with strong energy dependences at low energies. This suggests that higher-order long-range interactions are also important. After removing the effects of OPE, I therefore use the DWBA to subtract the contributions of other long-range potentials up to order $Q^3$. These include TPE and a relativistic correction to OPE, all of which highly singular at the origin. Their DWBA matrix elements diverge as inverse powers of the cut-off radius but these divergences can be cancelled using counterterms at the orders required by the new power counting. In contrast, Weinberg counting at order $Q^3$ would provide only one, energy-independent counterterm in each $P$ wave, and none in any higher wave.

TPE and other long-range forces up to order $Q^3$ are able to account for much of rapid energy dependence seen below 100 MeV in the $P$ and $D$ waves. When these are subtracted the residual scattering amplitudes can, in general, be well fitted by three contact terms up to order $Q^7/2$ in the new counting. The only exception is the $^3D_2$ channel, which seems to require an additional long-range attraction. It is not clear where this could arise, given the forms of the order-$Q^3$ chiral potentials [26, 27], but one should note that the uncertainties in the PWA’s for this channel are significant below about 70 MeV. Otherwise, TPE and the short-range forces up to order $Q^7/2$ are able to give a good description of the scattering in these triplet waves up to energies of about 250 MeV.

In the more peripheral $F$ and $G$ waves, the arguments of Ref. [12] suggest that the standard Weinberg counting should be adequate for the energies considered here. The numerical wave functions support this, their short-range forms having the expected $p^L$ dependence driven by the centrifugal barrier. Subtraction of the long-range forces leaves small residual interactions, as in the corresponding singlet channels studied in Ref. [20]. The DWBA cut-off radius is larger than about 0.1 fm, there is no sign of any divergences whose renormalisation would require the new power counting.

These results indicate that the spin-triplet waves with $L \leq 2$ can be analysed consistently using the nonperturbative power counting developed in Refs. [11, 12]. In addition, the results show that deconstructing scattering amplitudes, using the approach of Ref. [20], can provide a very useful tool for determining effective potentials directly from empirical phase shifts. It should be straightforward to extend the method to coupled channels, such as $^3S_1 – ^3D_1$, despite the more complicated matrix equations involved. It would also be interesting to use it to subtract two- and three-pion exchanges at order $Q^3$ [26, 27]. However the approach also demonstrates that there are significant uncertainties in the currently available Nijmegen PWA’s, particularly for energies below about 80 MeV where the scattering is most sensitive to long-range forces. As a result, attempts to study the importance of higher-order forces may require the newer PWA’s of Refs. [23, 28], when these become available.

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