EXACT VERSUS APPROXIMATE BEAMING FORMULAE IN GAMMA-RAY BURST AFTERGLOWS

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ABSTRACT

Assuming the emitted gamma-ray burst (GRB) radiation is not spherically symmetric but is confined into a narrow jet, we present the exact analytic expressions to compute the value of the detector arrival time at which we start to “see” the sides of the jet, in both the fully radiative and adiabatic regimes. We obtain this result using our exact analytic expressions for the equitemporal surfaces in GRB afterglows. We reexamine the validity of three different approximate formulae currently adopted for the adiabatic regime in the GRB literature. We also present an empirical fit of the numerical solutions of the exact equations, compared and contrasted with these three approximate formulae. The extent of the differences is such as to require a reassessment of the existence and entity of beaming in the cases considered in the current literature, as well as of its consequences on the GRB energetics.

Subject headings: gamma rays: bursts — ISM: kinematics and dynamics — relativity

1. INTRODUCTION

After the work by Mao & Yi (1994), the possibility that gamma-ray bursts (GRBs) originate from a beamed emission has been one of the most debated issues concerning the nature of the GRB sources in the current literature (see, e.g., Piran 2005, Mészáros 2006, and references therein). In particular, on the grounds of the theoretical considerations by Sari et al. (1999), it was conjectured that, within the framework of a conical jet model, one may find that the gamma-ray energy released in all GRBs is narrowly clustered around $5 \times 10^{50}$ ergs (Frail et al. 2001).

In a recent Letter (Bianco & Ruffini 2005b), we analyzed the approximate power-law relations between the Lorentz gamma factor and the radial coordinate usually adopted in the current GRB literature. We pointed out how such relations are found to be mathematically correct but only approximately valid in a very limited range of physical and astrophysical parameters and in an asymptotic regime that is reached only for a very short time, if any. Therefore, such relations were shown to be nonapplicable to GRBs. Instead, the exact analytic solutions of the equations for the equitemporal thin and uniform shell expanding in the interstellar medium (ISM) in the fully radiative and adiabatic regimes were presented.

This program of identifying the exact analytic solutions, instead of an approximate power-law solution, is in this Letter carried one step further. Using the above exact solutions, we here introduce the exact analytic expressions of the relations between the detector arrival time $t_{\text{jet}}^{d}$ of the GRB afterglow radiation and the corresponding half-opening angle $\theta$ of the expanding source visible area due to relativistic beaming (see, e.g., Ruffini et al. 2004). Such visible area must be computed not over the spherical surface of the shell but over the equitemporal surface (EQTS) of the detector arrival time $t_{\text{jet}}^{d}$, i.e., over the surface locus of points that is the source of the radiation reaching the observer at the same arrival time $t_{\text{jet}}^{d}$ (see Bianco & Ruffini 2004, 2005a for details). The exact analytic expressions for the EQTS in GRB afterglows, which have been presented in Bianco & Ruffini (2005a), are therefore crucial in our present derivation. This approach clearly differs from the ones in the current literature, which usually neglect the contributions of the radiation emitted from the entire EQTS.

The analytic relations between $t_{\text{jet}}^{d}$ and $\theta$ presented in this Letter allow to compute the value $(t_{\text{jet}}^{d})_{\text{mol}}$ of the detector arrival time at which we start to “see” the sides of the jet, assuming that the expanding shell is not spherically symmetric but is confined into a narrow jet with half-opening angle $\theta$. A corresponding “break” in the observed light curve should occur later than $(t_{\text{jet}}^{d})_{\text{mol}}$ (see, e.g., Sari et al. 1999). In the current literature, $(t_{\text{jet}}^{d})_{\text{mol}}$ is usually defined as the detector arrival time at which $\gamma \sim 1/\delta_{\theta}$, where $\gamma$ is the Lorentz factor of the expanding shell (see, e.g., Sari et al. 1999 and also our eq. [2]). In our formulation, we do not consider the effects of the lateral spreading of the jet.

In the current literature, in the case of the adiabatic regime, different approximate power-law relations between $(t_{\text{jet}}^{d})_{\text{mol}}$ and $\delta_{\theta}$ have been presented, in contrast to each other (see, e.g., Sari et al. 1999, Panaitescu & Mészáros 1999, and Panaitescu 2006). We show here that in four specific cases of GRBs, encompassing more than 5 orders of magnitude in energy and more than 2 orders of magnitude in ISM density, both the power-law relation by Panaitescu & Mészáros (1999) and the one by Sari et al. (1999) overestimate the exact analytic result. A third relation presented by Panaitescu (2006) slightly underestimates the exact analytic result. We also present an empirical fit of the numerical solutions of the exact equations for the adiabatic regime, compared and contrasted with the three above approximate relations. In the fully radiative regime, and therefore in the general case, no simple power-law relation of the kind found in the adiabatic regime can be established, and the general approach we have outlined has to be followed.

Although evidence for spherically symmetric emission in GRBs is emerging from observations (Soderberg et al. 2006) and from theoretical arguments (Ruffini et al. 2004a, 2004b), it is appropriate to develop here an exact theoretical treatment of the relation between $(t_{\text{jet}}^{d})_{\text{mol}}$ and $\delta_{\theta}$. This will allow us to make an assessment on the existence and, in the positive case, on the extent of beaming in GRBs, which in turn is going to be essential for establishing their correct energetics.

2. ANALYTIC FORMULAE FOR THE BEAMING ANGLE

The boundary of the visible region of a relativistic thin and uniform shell expanding in the ISM is defined by (see, e.g., Ruffini et al. 2003 and references therein)

$$\cos \theta = \frac{v}{c},$$

(1)
where $\theta$ is the angle between the line of sight and the radial expansion velocity of a point on the shell surface, $v$ is the velocity of the expanding shell, and $c$ is the speed of light. To find the value of the half-opening beaming angle $\theta_0$ corresponding to an observed arrival time $(t'_{\text{jet}})_{\text{obs}}$, this equation must be solved together with the equation describing the EQTS of the arrival time $(t'_{\text{jet}})_{\text{obs}}$ (Bianco & Ruffini 2005a). In other words, we must solve the following system:

$$
\cos \theta_0 = \frac{v(r)}{c},
$$

$$
\cos \theta_0 = \cos \left\{ \delta[r(t'_{\text{jet}})_{\text{EQTS}}(\theta_0)] \right\}.
$$

(2)

It should be noted that, in the limit of $\theta_0 \to 0$ and $v \to c$, this definition of $(t'_{\text{jet}})_{\text{obs}}$ is equivalent to the one usually adopted in the current literature (see § 1).

### 2.1. The Fully Radiative Regime

In this case, the analytic solution of the equations of motion gives (see Bianco & Ruffini 2005a, 2005b)

$$
\frac{v}{c} = \sqrt{\left(1 - \gamma_0^2 \right) \left[ 1 + (M_{\text{ism}}/M_p) + (M_{\text{ism}}/M_B)^2 \right]} / \left[ 1 + (M_{\text{ism}}/M_p)(1 + \gamma_0^{-1})[1 + 1/(M_{\text{ism}}/M_B)] \right],
$$

(3)

where $\gamma_0$ and $M_p$ are, respectively, the values of the Lorentz gamma factor and of the mass of the accelerated baryons at the beginning of the afterglow phase, and $M_{\text{ism}}$ is the value of the ISM matter swept up to radius $r$: $M_{\text{ism}} = (4\pi/3) m_{\text{p}} n_{\text{ism}} (r - r_0)$, where $r_0$ is the starting radius of the baryonic matter shell, $m_p$ is the proton mass, and $n_{\text{ism}}$ is the ISM number density. Using the analytic expression for the EQTS given in Bianco & Ruffini (2005a), equation (2) takes the form

$$
\cos \theta_0 = \sqrt{\left(1 - \gamma_0^2 \right) \left[ 1 + (M_{\text{ism}}/M_p) + (M_{\text{ism}}/M_B)^2 \right]} / \left[ 1 + (M_{\text{ism}}/M_p)(1 + \gamma_0^{-1})[1 + 1/(M_{\text{ism}}/M_B)] \right] - \frac{M_p - m_p}{2 r_c \sqrt{C}} (r - r_0) + \frac{m_p^2 r_0}{8 \sqrt{C} \left( \frac{r_0}{r} \right)^2 - 1} + \frac{r_0 \sqrt{C}}{12 r m_p A^2} \ln \left\{ \frac{[A + (r r_0)^3] A^3}{[A^2 + (r r_0)^3] A + 1} \right\} + \frac{c t_0}{r} - \frac{c(t'_{\text{jet}})_{\text{jet}}}{r} + \frac{r}{r(1 + z)} + \frac{r_0 \sqrt{C}}{6 r m_p A^2} \left[ \arctan \frac{2(r r_0) - A}{A^2} - \arctan \frac{2 - A}{A^2} \right].
$$

(4)

where $t_0$ is the value of the time $t$ at the beginning of the afterglow phase, $m_p^0 = (4\pi/3) \pi m_{\text{p}} n_{\text{ism}} r_0^3$, $r^*$ is the initial size of the expanding source, $A = [(M_p - m_p^0)/m_p]^3$, $C = M_p^2(\gamma_0 - 1)/(\gamma_0 + 1)$, and $z$ is the cosmological redshift of the source.

### 2.2. The Adiabatic Regime

In this case, the analytic solution of the equations of motion (see Bianco & Ruffini 2005a, 2005b)

$$
\frac{v}{c} = \frac{\gamma_0 M_{\text{ism}}}{M_p}
$$

Using the analytic expression for the EQTS given in Bianco & Ruffini (2005a), equation (2) takes the form

$$
\cos \theta_0 = \frac{\gamma_0 M_{\text{ism}}}{M_p}
$$

$$
\cos \theta_0 = \frac{m_p^0}{4 M_p \gamma_0^2 - 1} \left[ \frac{r_0}{r} \right] + \frac{c t_0}{r} - \frac{c(t'_{\text{jet}})_{\text{jet}}}{r} + \frac{r}{r(1 + z)} + \frac{\gamma_0 - (m_p^0/M_p)}{\gamma_0 - 1} \left( \frac{r_0}{r} - 1 \right),
$$

where all the quantities have the same definition as in equation (4).

### 2.3. The Comparison between the Two Solutions

In Figure 1, we plot the numerical solutions of both equation (4), corresponding to the fully radiative regime, and equation (6), corresponding to the adiabatic one. Both curves have been plotted assuming the same initial conditions, namely, the ones of GRB 991216 (see Ruffini et al. 2003).

### 3. COMPARISON WITH THE EXISTING LITERATURE

Three different approximate formulae for the relation between $(t'_{\text{jet}})_{\text{jet}}$ and $\theta_0$ have been given in the current literature, all assuming the adiabatic regime. Panaitescu & Mészáros (1999) proposed

$$
\cos \theta_0 = 1 - 5.9 \times 10^7 \left( \frac{M_{\text{ism}}}{E} \right)^{1/4} \left( \frac{t'_{\text{jet}}}_{\text{jet}} \right)^{1/4}. \tag{7}
$$

![Graph showing comparison between the numerical solution of eq. (4) assuming a fully radiative regime (blue curve) and the corresponding solution of eq. (6) assuming an adiabatic regime (red curve). The departure from power-law behavior at small arrival times follows from the constant Lorentz $\gamma$ factor regime, while that at large angles follows from the approach to the nonrelativistic regime (see details in § 4 and Fig. 4, as well as in Bianco & Ruffini 2005b).](image)
Fig. 2.—Comparison between the numerical solution of eq. (6) (red curve), and the corresponding approximate formulae given in eq. (8) (blue curve), in eq. (7) (black curve), and in eq. (9) (green curve). All four curves have been plotted for four different GRBs: (a) GRB 991216 (see Ruffini et al. 2003), (b) GRB 980519 (R. Ruffini et al. 2006, in preparation), (c) GRB 031203 (see Bernardini et al. 2005), and (d) GRB 980425 (see Ruffini et al. 2004b). The ranges of the two axes have been chosen so as to focus on the sole domains of the application of the approximate treatments in the current literature.

Sari et al. (1999), instead, advanced

\[ \theta_0 = 7.4 \times 10^3 \left( \frac{n_{\text{ISM}}}{E} \right)^{1/8} \left( \frac{t_0^e}{t_0} \right)^{3/8} \left( \frac{c}{H_1} \right)^{1/2} \]  

In both equations (7) and (8), \( t_0^e \) is measured in seconds, \( E \) is the source initial energy measured in units of ergs, and \( n_{\text{ISM}} \) is the ISM number density in units of particles \( \text{cm}^{-3} \). The formula by Sari et al. (1999) has been applied quite often in the current literature (see, e.g., Frail et al. 2001, Ghirlanda et al. 2004, and Fox et al. 2005).

Both equations (7) and (8) compute the arrival time of the photons at the detector by assuming that all the radiation is emitted at \( \theta = 0 \) (i.e., on the line of sight) by neglecting the full shape of the EQTSs. Recently, a new expression has been proposed by Panaitescu (2006), again neglecting the full shape of the EQTSs but assuming that all the radiation is emitted from \( \theta = 1/\gamma \), i.e., from the boundary of the visible region. Such an expression is as follows:

\[ \theta_0 = 5.4 \times 10^3 \left( \frac{n_{\text{ISM}}}{E} \right)^{1/8} \left( \frac{t_0^e}{t_0} \right)^{3/8} \left( \frac{c}{H_1} \right)^{1/2} \]  

In Figure 2, we plot equations (7)–(9) together with the numerical solution of equation (6) relative to the adiabatic regime. All four curves have been plotted by assuming the same initial conditions for four different GRBs, encompassing more than 5 orders of magnitude in energy and more than 2 orders of magnitude in ISM density: (a) GRB 991216 (see Ruffini et al. 2003), (b) GRB 980519 (R. Ruffini et al. 2006, in preparation), (c) GRB 031203 (see Bernardini et al. 2005), and (d) GRB 980425 (see Ruffini et al. 2004b). The approximate equation (8) by Sari et al. (1999) and equation (9) by Panaitescu (2006) both imply a power-law relation between \( \theta_0 \) and \( t_0^e \), with constant index for any value of \( \theta_0 \), while equation (7) by Panaitescu & Mészáros (1999) implies a power-law relation with constant index \( 1/2 \) only for \( \theta_0 \to 0 \) (for greater \( \theta_0 \) values, the relation is trigonometric).

All the above three approximate treatments are based on the approximate power-law solutions of the GRB afterglow dynamics that have been shown in Bianco & Ruffini (2005b) to be nonapplicable to GRBs. They also do not fully take into account the structure of the EQTSs, although in different ways. Both equations (7) and (8), which assume all the radiation coming from \( \theta_0 = 0 \), overestimate the behavior of the exact solution. On the other hand, equation (9), which assumes all the radiation coming from \( \theta = 1/\gamma \), is a better approximation than the previous two but still slightly underestimates the exact solution.

4. AN EMPIRICAL FIT OF THE NUMERICAL SOLUTION

For completeness, we now fit our exact solution with a suitable explicit functional form in the four cases considered in
an approximate empirical fitting formula can only be applied to their full extension (see Fig. 1), we see that such relations differ from the approximate ones presented in Figure 2 (see Fig. 3). However, if we enlarge the axis ranges to their full extension (see Fig. 1), we see that such an approximate empirical fitting function can only be applied for $\theta_0 < 25^\circ$ and $(t_a^*)_{\text{jet}} > 10^2$ s (see the gray dashed rectangle in Fig. 4).

An equivalent empirical fit in the fully radiative regime is not possible. In this case, indeed, there is a domain in the $[(t_a^*)_{\text{jet}}, \theta_0]$-plane where the numerical solution shows a power-law dependence on time, with an index $\sim 0.423$ (see Fig. 1). However, the dependence on the energy cannot be factorized out with a simple power law. Therefore, in the fully radiative regime, which is the relevant one for our GRB model (see, e.g., Ruffini et al. 2003), the application of the full equation (4) does not appear to be avoidable.

5. CONCLUSIONS

We have presented in equations (4) and (6) the exact analytic relations between the jet half-opening angle $\theta_0$ and the detector arrival time $(t_a^*)_{\text{jet}}$ at which we start to “see” the sides of the jet, and these relations may be used in GRB sources in which an achromatic light-curve break is observed. The limiting cases of fully radiative and adiabatic regimes have been outlined. Such relations differ from the approximate ones presented in the current literature in the adiabatic regime: both the relation just presented by Panaitescu (2006) slightly underestimates it. However, in the fully radiative regime, such a simple empirical power-law fit does not exist, and the application of the exact equation (4) is needed. This same situation is also expected to occur in the general case.

In light of the above results, the assertion that the gamma-ray energy released in all GRBs is narrowly clustered around $5 \times 10^{50}$ ergs (Frail et al. 2001) should be reconsidered. In addition, the high-quality data by Swift, going without gaps from the “prompt emission” all the way to the latest afterglow phases, will help in uniquely identifying the equations of motion of the GRB sources and the emission regimes. Consequently, on the grounds of the results presented in this Letter, which encompass the different dynamical and emission regimes in GRB afterglow, an assessment of the existence and, in the positive case, on the extent of beaming in GRBs will be possible. This is a first step in the determination of their energetics.

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