Entanglement and nonlocality are inequivalent for any number of particles

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Understanding the relation between nonlocality and entanglement is one of the fundamental problems in quantum physics. In the bipartite case, it is known that the correlations observed for some entangled quantum states can be explained within the framework of local models, thus proving that these resources are inequivalent in this scenario. However, except for a single example of an entangled three-qubit state that has a local model, almost nothing is known about such relation in multipartite systems. We provide a general construction of genuinely multipartite entangled states that do not display genuinely multipartite nonlocality, thus proving that entanglement and nonlocality are inequivalent for any number of particles.

Introduction. Nonlocality – the phenomenon of the impossibility of describing by local hidden variable (LHV) theories the correlations arising from measuring quantum states – is a fundamental characteristics of quantum mechanics. It was Bell who first pointed out that there exist quantum states for which underlying classical variables cannot account for the measurement statistics on them [1]. Such states are called nonlocal and they violate Bell inequalities (see [2]). Having been confronted with the result of Bell one might be tempted to identify nonlocality with entanglement, another essential resource of quantum information theory. This intuition is, however, not correct as the relationship between these two notions is more involved: while LHV models trivially exist for separable states, not all entangled states are nonlocal.

The first step in the exploration of this inequivalence was taken by Werner in Ref. [3]. There, he introduced a family of highly symmetric states, nowadays known as the Werner states, and provided an explicit LHV model reproducing the measurement statistics obtained when two parties perform local projective measurements on some states from this family. Building on this model, Barrett proved that there are entangled Werner states that remain local even if general measurements are performed [4]. Both these models were later adapted to mixtures of any state ρ and the white noise [5], such as the isotropic states for which ρ is the maximally entangled state [6]. The nonlocal properties of noisy states have also been related to an important mathematical notion known as the Grothendieck constant in [7].

Very little is known about the relation between entanglement and nonlocality in the general multipartite scenario. Here, this question becomes much subtler as the multiparty scenario offers a richer variety of different types of entanglement and nonlocality. For instance, for any number of parties, it is trivial to construct a non-separable, and thus entangled but a manifestation of the inequivalence between entanglement and nonlocality for two parties. Thus, the most natural question in the multipartite scenario is whether for an arbitrary number of parties N, there always exists a gap between genuinely N-party entanglement and genuinely N-party non-locality. Our main goal is to show that this is the case, thus proving that entanglement and nonlocality are inequivalent for any number of parties.

The departure point of our proof are bipartite entangled states with local models for general measurements. Basing on them, we provide a general construction of genuinely multipartite entangled (GME) states of N parties that do not display genuinely multipartite nonlocality. In fact, our construction is such that the resulting N-party entangled state has a bilocal model, in which the parties are divided into two groups, inherited from the local model of the initial bipartite state (see Fig. 1). We also generalize our construction to map any N-party GME state with a K-party local model to an N’-party GME state, with N’ > N, that also has a K-local model.

Before moving to the proof of our results, it is worth mentioning that for three parties Tóth and Acín found a fully local model for arbitrary projective measurements on a genuinely entangled three–qubit state [8]. This model has been later extended to general measurements [9], proving ultimately that there are GME states of three parties that do not display any form of nonlocality. However, it is unknown whether this model can be extended to more parties.

Preliminaries. Let us start by introducing some notation and terminology. Consider N parties $\mathbf{A} := A_1, \ldots, A_N$ (for...
small $N$ also denoted $A$, $B$, etc.) sharing an $N$-partite quantum state $\rho_A \in \mathcal{B}(\mathcal{H}_{N,d})$, where $\mathcal{H}_{N,d} = (\mathbb{C}^d)^{\otimes N}$ and $\mathcal{B}(\mathcal{H})$ stands for the set of bounded linear operators acting on the Hilbert space $\mathcal{H}$. Moreover, by $\mathcal{S}_{M,d}$ and $P^X_{\text{sym}}$ we denote, respectively, the symmetric subspace of $\mathcal{H}_{M,d}$ and the projector onto the symmetric subspace of the subsystem $X$.

Let us then partition the parties into $K$ pairwise disjoint groups $S_i$ such that by adding them one recovers $A$, and call it a $K$-partition; for $K = 2$ we call it a bipartition and denote $S|\bar{S}$ with $S$ being the complement of $S$ in $A$. Denoting by $\mathcal{S}_K$ the set of all $K$-partitions, we say that $\rho_A$ is $K$-separable (biseparable for $K = 2$) if it is a probabilistic mixture of $N$-partite states separable with respect to some $K$-partitions, i.e.,

$$\rho_A = \sum_{S \in \mathcal{S}_K} p_S \sum_{i} q^S_i \otimes |\psi^S_i\rangle \langle \psi^S_i|.$$  \hspace{1cm} (1)

Here, $p_S$ and $q^S_i$ are probability distributions and $|\psi^S_i\rangle$ are pure states defined on the $S_k$ subsystem. One then calls $\rho_A$ fully separable if it is $N$-separable, and, genuinely multipartite entangled (GME) if it does not admit any of the above forms of separability; in particular, it is not biseparable.

Analogously to the notions of separability one introduces those of locality. Imagine that on their share of the state $\rho_A$, each party $A_i$ performs a measurement $M_i = \{M^{(i)}_{a_j}\}$, where $a_j$ enumerate the outcomes of $M_i$, while $M^{(i)}_{a_j}$ are the measurement operators, i.e., $M^{(i)}_{a_j} \geq 0$ and $\sum_{a_j} M^{(i)}_{a_j} = \mathbb{I}_d$. If, additionally, $M^{(i)}_{a_j}$ are supported on orthogonal subspaces, we call the corresponding measurement projective (PM) and generalized (GM; also called POVM) otherwise. Now, adopting the definition from Ref. [10], one says that the state $\rho_A$ is $K$-local for GMs, or shortly $K$-local, if for any choice of measurements $\mathcal{M} := \{M_1, \ldots, M_N\}$, the probability of obtaining the outcomes $a := a_1, \ldots, a_N$ decomposes as

$$p(a|M) = \sum_{S \in \mathcal{S}_K} p_S \int d\lambda \omega_S(\lambda) \prod_{k=1}^K p_k(a_{S_k}|M_{S_k}, \lambda).$$  \hspace{1cm} (2)

Here, the sum goes over all possible $K$-partitions of $A$, $p_S$ and $\omega_S$ are probability distributions, and $p_k(\cdot | \cdot)$ is the probability (also called response function) that the parties belonging to $S_k$ obtain $a_{S_k}$ upon measuring $M_{S_k}$, while having the classical information $\lambda$. Accordingly, a state $\rho_A$ admitting (2) is said to have a $K$-local model. In particular, if $K = N$ we say that the state is fully local, while if $K = 2$ — bilocal. Notice that there are also local models reproducing projective measurements only, and mixed ones, i.e., those reproducing GMs for some parties and PMs for the rest. By comparing (1) and (2), it is direct to realize that every $K$–separable state is $K$–local. Multipartite states which do not admit any form of bilocality are called genuinely multipartite nonlocal (GMN).

It should be noted that such definition of $K$-locality, Eq. (2), has been shown to be inconsistent with an operational interpretation of nonlocality given in Refs [11, 12]. However, we use it here because of its direct analogy to entanglement. Moreover, it allows us to state our construction in a general way and facilitates the proof of our result. Nevertheless, as we argue below, the inequivalence between entanglement and nonlocality also holds for these operational definitions.

**Main result.** Entanglement and nonlocality are inequivalent for any number of parties $N$, as for any $N$ there exist genuinely entangled $N$–partite states with bilocal models.

To prove the result we proceed in two steps. First, we show that any bipartite local state can be converted into a multipartite state with a bilocal model. Then, we argue that such construction may lead to GME states for any $N$.

As to the first step, we generalize the observation made by Barrett [4]. Let $\mathcal{q}_{AB} \in \mathcal{B}(\mathcal{H}_{2,d})$ be arbitrary and let

$$\Lambda_{A \rightarrow S} : \mathcal{B}(\mathbb{C}^d) \rightarrow \mathcal{B}((\mathbb{C}^d)^{\otimes L})$$  \hspace{1cm} (3)

and

$$\Lambda_{B \rightarrow \bar{S}} : \mathcal{B}(\mathbb{C}^{d'}) \rightarrow \mathcal{B}((\mathbb{C}^{d'})^{\otimes N-L})$$  \hspace{1cm} (4)

be a pair of quantum channels sending operators acting on a single-party Hilbert space $\mathbb{C}^d$ to operators acting on $L$-partite and $(N - L)$-partite Hilbert spaces of local dimension $d'$, respectively, with $S = A_1 \ldots A_L$ and $\bar{S} = A_{L+1} \ldots A_N$. Now, one can prove the following lemma.

**Lemma 1.** If $\mathcal{q}_{AB}$ has a local model for generalized measurements, then, for any pair of quantum channels $\Lambda_{A \rightarrow S}$ and $\Lambda_{B \rightarrow \bar{S}}$ defined above, the $N$-partite state

$$\sigma_A = (\Lambda_{A \rightarrow S} \otimes \Lambda_{B \rightarrow \bar{S}})(\rho_{AB})$$  \hspace{1cm} (5)

has a bilocal model for any measurements.

**Proof.** The reasoning is analogous to the one by Barrett from Ref. [4], but for completeness we present it here.

The fact that $\rho_{AB}$ has a local model for generalized measurements means that the probabilities of obtaining results $a, b$ upon performing measurements $M_A = \{M^A_{a_j}\}$ and $M_B = \{M^B_{a_j}\}$, respectively, by the parties $A$ and $B$, assume the "local" form (2), which for $N = 2$ simplifies to

$$p(a, b|M_A, M_B) = \int d\lambda \omega(\lambda) p_a(a|M_A, \lambda) p_b(b|M_B, \lambda).$$  \hspace{1cm} (6)

Here we have used the subscript $p$ to emphasize that the probabilities correspond to $\rho_{AB}$. Exploiting this model, we will now demonstrate that $\rho_A$ is bilocal with respect to the bipartition $S|\bar{S} = A_1 \ldots A_L|A_{L+1} \ldots A_N$. To this end, let us assume that the parties perform measurements $M_i = \{M^{(i)}_{a_j}\}, i = 1, \ldots, N$, on their shares of the state $\sigma_A$. Then, denoting by $\Lambda_{S \rightarrow A}$ and $\Lambda_{S \rightarrow \bar{A}}$ the dual maps of $\Lambda_{A \rightarrow S}$ and $\Lambda_{B \rightarrow \bar{S}}$ [13], respectively, we define the following operators

$$\bar{M}^A_{\bar{S}} = \Lambda_{S \rightarrow A} \left( \bigotimes_{i=1}^L M^{(i)}_{a_i} \right), \quad \bar{M}^B_{\bar{S}} = \Lambda_{S \rightarrow B} \left( \bigotimes_{i=L+1}^N M^{(i)}_{a_i} \right)$$  \hspace{1cm} (7)
acting on $\mathbb{C}^d$ and indexed by the outcomes $a_S := a_1, \ldots, a_L$ and $a_{\bar{S}} := a_{L+1}, \ldots, a_N$. Since the dual map of a quantum channel is positive and unital (it preserves the identity operator), it is direct to see that the operators (7) form generalized measurements, denoted $M_A$ and $M_B$.

With their aid, let us now define the response functions for the state $\sigma_A$ corresponding to the parties $A_1, \ldots, A_L$ and $A_{L+1}, \ldots, A_N$, respectively, as $p_{\sigma}(a_S|\rho, \lambda) = p_{\rho}(a_S|M_A, \lambda)$ and $p_{\rho}(a_{\bar{S}}|\rho, \lambda) = p_{\rho}(a_{\bar{S}}|M_B, \lambda)$. Then,

$$p(a|M) = \text{Tr}[(M_{a_1} \otimes \cdots \otimes M_{a_N})\sigma_A] = \text{Tr}[(M_{a_1} \otimes \cdots \otimes M_{a_N})(\Lambda_{A-S} \otimes \Lambda_{B-\bar{S}})(\rho_{AB})] = \text{Tr}[\bar{M}_{a_1, \ldots, a_N} \otimes \bar{M}_{a_{L+1}, \ldots, a_N}\rho_{AB}] = \int d\lambda \omega(\lambda)p_{\rho}(a_S|M_A, \lambda)p_{\rho}(a_{\bar{S}}|M_B, \lambda)$$

where we have utilized Eqs. (7) and the definition of $\sigma_A$. It thus follows that $\sigma_A$ has a bilocal model for GMs with respect to $A_1, \ldots, A_L, A_{L+1}, \ldots, A_N$.

The critical point of our approach will be to observe that the above mapping of local bipartite states to bilocal multipartite ones may lead to GME states. To argue this, we need a technical result concerning genuine multipartite entanglement.

**Lemma 2.** Consider an $N$-partite state $\sigma_A \in B(\mathcal{H}_{N,d})$ and assume that with respect to some bipartition $S|\bar{S}$, the subsystems $S$ and $\bar{S}$ are symmetric, that is, $P^S_{\text{sym}} \otimes P^\bar{S}_{\text{sym}}\sigma_A P^S_{\text{sym}} \otimes P^\bar{S}_{\text{sym}} = \sigma_A$ holds. If $\sigma_A$ is not GME, then it is biseparable with respect to this bipartition, i.e.,

$$\sigma_A = \sum_i p_i (\sigma^i_S \otimes \sigma^i_{\bar{S}})$$

with $\sigma^i_S$ and $\sigma^i_{\bar{S}}$ being states defined on subsystems $S$ and $\bar{S}$.

**Proof.** As the proof is rather technical and lengthy, here we present its sketch moving the details to Appendix D.

The assumption that $\sigma_A$ is not GME means that it admits the decomposition (1) with $K = 2$, i.e., $\sigma_A = \sum_{T|\bar{T} \in S_2} p_{\rho_{T|\bar{T}}} \rho_{T|\bar{T}}$. The sum goes over all bipartitions $T|\bar{T}$ of $A$ and $\rho_{T|\bar{T}}$ is some state separable with respect to $T|\bar{T}$, i.e., $\rho_{T|\bar{T}} = \sum_i q_{T|i} |e^i_T\rangle |f^i_{\bar{T}}\rangle \otimes |f^i_{\bar{T}}\rangle |e^i_T\rangle$. Now, exploiting the assumption that the subspace $S$ and $\bar{S}$ of $\sigma_A$ are symmetric, one can prove that each $\rho_{T|\bar{T}}$ with $T \neq \bar{T}$ in the decomposition of $\sigma_A$ is of the form (9) ($\sigma_{S|\bar{S}}$ is already of this form). To this aim, it is enough to realize that every pure state $|e^i_T\rangle |f^i_{\bar{T}}\rangle$ must obey $P^{S}_{\text{sym}}(e^i_T) |f^i_{\bar{T}}\rangle = |e^i_T\rangle |f^i_{\bar{T}}\rangle$. This, after some algebra, implies that it must also be product with respect to $S|\bar{S}$. ⊓⊔

We are now in position to prove our main result. A straightforward corollary of Lemma 2 is that any $N$-partite state $\sigma_A$, which does not admit the form (9), i.e., is entangled across some cut $S|\bar{S}$, and whose subsystems $S$ and $\bar{S}$ are symmetric, is GME. Take now a bipartite entangled state $\rho_{AB} \in B(\mathbb{C}^d \otimes \mathbb{C}^d)$ and the quantum channels $\Lambda_{A-S} : B(\mathbb{C}^d) \rightarrow B(S_{d},d^d)$ and $\Lambda_{B-\bar{S}} : B(\mathbb{C}^d) \rightarrow B(S_{N-L,d},d^d)$ that are invertible in the sense that for both of them there exists a channel $\Lambda$ such that $\Lambda \otimes \Lambda$ is the identity map on $B(\mathbb{C}^d)$. Note that now these channels output states acting on the corresponding $L$ and $(N-L)$-partite symmetric subspaces. Clearly, the $N$-partite state $\sigma_A$ resulting from the application of $\Lambda_{A-S}$ and $\Lambda_{B-\bar{S}}$ to $\rho_{AB}$ is symmetric on the subspaces $S$ and $\bar{S}$, and, as $\rho_{AB}$ is entangled, must be GME; if $\sigma_A$ is not GME, then, as the two channels are invertible, $\rho_{AB}$ must be separable. If we further assume that $\rho_{AB}$ is local, the resulting state $\sigma_A$ will have, according to Lemma 1, a bilocal model, proving the desired.

As a result we have a general method for constructing bilocal genuinely entangled $N$-partite states with an arbitrary $N$.

**Applications.** Let us now see how our method works in practice. We consider for this purpose two paradigmatic classes of states: the isotropic and the Werner states [3, 6]. The quantum channels are chosen to be $\Lambda_{A-S}(\cdot) = V_L(\cdot)V_L^\dagger$ and $\Lambda_{B-\bar{S}}(\cdot) = V_{N-L}(\cdot)V_{N-L}^\dagger$, with $V_M : \mathbb{C}^d \rightarrow S_{d,M} \otimes \mathbb{C}^d$ being an isometry defined through $V_M|i\rangle = |i\rangle \otimes |\psi\rangle$ for any element of the standard basis in $\mathbb{C}^d$.

Let us begin with the two-qudit isotropic states which are given by $\rho_{iso}(p) = |\psi^+_d\rangle\langle\psi^+_d| + (1-p) |\psi^-\rangle\langle\psi^-|$, where $|\psi^\pm_d\rangle = (1/\sqrt{d}) \sum_{i=0}^{d-1} |ii\rangle$ is the maximally entangled state. Application of the isometries to $\rho_{iso}(p)$ leads us to the mixture of the well-known GHZ state of $N$ qudits $|\text{GHZ}_{N,d}\rangle = (1/\sqrt{d}) \sum_{i=0}^{d-1} |ii\rangle \otimes |\psi^d\rangle$ and some coloured noise:

$$\sigma_A(p) = p|\text{GHZ}_{N,d}\rangle\langle\text{GHZ}_{N,d}| + (1-p) \frac{P_{L,d} \otimes P_{N-L,d}}{d^2},$$

where $P_{L,d} = \sum_{i=0}^{d-1} |i\rangle|\psi^d_{i\rangle}^\perp \otimes \text{with } 1 \leq L \leq N - 1$. Now, as the isotropic states are local for $p \leq (3d-1)(d-1)^{-1}/d^d(d+1)$ [5], it stems from Lemma 1 that for the same range of $p$ and $L = 1, \ldots, N - 1$, the state $\sigma_A(p)$ is bilocal with respect to the bipartition $A_1, \ldots, A_L|A_{L+1}, \ldots, A_N$. Further, isometric channels are always invertible ($V_M^2V_M = 1_d$) and thus, as required, preserve entanglement. Hence, the states $\sigma_A(p)$ are GME for the same range of $p$ as $\rho_{iso}(p)$ are entangled, i.e., for $p > 1/(d + 1)$.

Concluding, the states (10) constitute our first example of GME states with a bilocal model for any $N$.

Let us now consider the Werner states which read $\rho_{W}(p) = |\psi^d_{2/d}(d-1)\rangle|\psi^d_{2/d}(d-1)^{-1}/d^d(d+1)|^\langle\psi^d_{2/d}(d-1)|$, where $\rho_{asy}

Moreover, $\rho_W(p)$ are entangled for $p > 1/(d+1)$, thus $\sigma_A^p(p)$ are GME for the same range of $p$.

Generalizing the construction. Interestingly, our construction can be generalized to the case when the initial bipartite state is replaced by a multipartite genuinely entangled state with a local model for GMs. To be precise, let us first consider a $K$-partite state $\rho_{A_1...A_K}$ acting on $\mathcal{H}_{d_1...d_K}$ and a collection of $K$ quantum channels $\Lambda_{d_{A_1}...d_{A_K}} : B(\mathbf{C}^{d_1}) \rightarrow B(\mathcal{H}_{d_1...d_K})$ with $L_i \geq 1$ such that $L_1 + \ldots + L_K = N > K$. By definition, each channel “expands” a single-particle Hilbert space to an $L_k$-partite one of local dimension $d'_k$ corresponding to parties from the group $S_k = \{A_1+\sum_{i=1}^{K}L_i-1, \ldots, A_1+\sum_{i=1}^{K}L_i\}$, with $k = 1, \ldots, K$ and $L_0 = 0$. By applying these channels to subsystems of $\rho_{A_1...A_K}$, one obtains an $N$-partite state

$$\sigma_{A} = \Lambda_{d_{A_1}...d_{A_K}}(\rho_{A_1...A_K})$$

acting on $\mathcal{H}_{N,d'}$ (the subsystem $A_1$ of $\rho_{A_1...A_K}$ is mapped to $S_1 = A_1 \ldots A_{L_1}$, of $\sigma_{A}$ etc.). Now, following the same arguments as in the proof of Lemma 1 (see Appendix), one shows that if $\rho_{A_1...A_K}$ has a fully local model for GMs, then $\sigma_{A}$ has a $K$-local model for GMs with respect to the $K$-partition determined by the groups $S_k$. Furthermore, generalizing Lemma 2 (see Appendix), one finds that with a proper choice of the channels $\Lambda_{A_k} \rightarrow S_k$ we can guarantee that the resulting state is GME. Thus, any genuinely entangled $K$-partite state admitting a fully local model gives rise to a genuinely entangled $N$-partite state, with any $N > K$, having $K$-local model.

Note that this generalization, when applied to the existing example of a tripartite GME state with a local model for generalized measurements [9], implies the existence of GME $N$-partite states with three-local models for any $N$.

Finally, let us comment on the operational definitions of $K$-locality given in Refs. [11, 12]. As shown there, a definition of $K$-locality which is operationally consistent looks like (2) with the additional constraint that all the probability distributions $p_k(s_{A_k}, |M_{S_k}, \lambda)$ satisfy the no-signalling principle. Importantly, this condition can be easily met in our construction. Namely, it is enough to extend the part of a state on which the response function in a local model is quantum, i.e., given by the Born rule, as the resulting response on this part will be automatically no-signalling. The isotropic and the Werner states do have local models with one of the response functions being quantum, so the states $\sigma_{A}$ and $\sigma_{A}^p$ constructed above with $L = 1$ have bilocal models in which the response functions corresponding to $N - 1$ parties are no-signalling. Thus, the inequivalence between entanglement and nonlocality holds even when using the operational definitions of $K$-locality.

Conclusions and discussion. We have provided a general method of deriving from $N$-party GME states with a $K$-local models $N'$-party GME states with the same type of locality for any $N' > N$. Our construction implies then that entanglement and nonlocality are inequivalent for any number of parties, even if the operational definitions of multipartite locality are considered.

The most interesting open problem following from our work is to understand the extent to which the inequivalence between entanglement and nonlocality holds. With the current state of knowledge, our results show that there exist GME $N$-party states that have a 2- and even 3-local model. Now, what is the maximum value of $K$ such that there exist $N$-party states with a $K$-local model for any $N$? In particular, are there genuinely entangled $N$-party states with a fully local model? This happens to be the case for $N = 2, 3$, but no results are known beyond these two cases. A related question is whether there exists some threshold value of $N$ above which the GME states are too entangled to allow for a fully local model.

We then note that a number of different operationally meaningful nonlocality scenarios beyond the one considered in the present work have been introduced. These are: the network approach [14], Bell scenarios defined on copies of a state [15] (see also Ref. [16]) or sequential measurements [17–19]. In these more general approaches, states that are local in the standard setup may display nonlocal properties. Nevertheless, it remains open whether in such scenarios the equivalence between nonlocality and entanglement holds. It would be thus of interest to verify whether nonlocality of the states introduced here could be revealed in one of these more general setups.

Let us conclude by pointing out that our construction also implies that genuine multipartite entanglement is inequivalent to steering—another intriguing phenomenon of quantum information theory [20]. That is, by applying it to a bipartite state that has a local model with quantum response function, the construction produces a GME state which is unsteerable (in at least one direction) across the same bipartition with respect to which it is bilocal.

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[13] For a linear map $\Lambda : B(\mathcal{H}) \rightarrow B(\mathcal{K})$ we define the dual map as a linear map $\Lambda^* : B(\mathcal{K}) \rightarrow B(\mathcal{H})$ that satisfies $\text{Tr}[\Lambda^*(Y)X] = \text{Tr}[\Lambda(X)Y]$ for any $X \in B(\mathcal{K})$ and $Y \in B(\mathcal{H})$. Recall that if $\Lambda$ is positive (in particular completely positive), its dual...
\( \Lambda^1 \) is also positive. Moreover, if \( \Lambda \) is trace-preserving, i.e., 
\[ \text{Tr}[\Lambda(X)] = \text{Tr}X \] for any \( X \), the dual map \( \Lambda^1 \) is unital, i.e., 
\[ \Lambda^1(\mathbb{1}_K) = \mathbb{1}_K \] with \( \mathbb{1}_X \) being the identity operator acting on \( X \).

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**APPENDICES**

Here we state and prove generalizations of Lemma 1 and 2 from the main text. We also formulate our main result as a theorem.

We start with a generalization of Lemma 1. For this purpose, consider a \( K \)-partite state \( \rho_{A_1 \ldots A_K} \) acting on \( \mathcal{H}_{K,d} = (\mathbb{C}^d)^{\otimes K} \) and a collection of \( K \) quantum channels

\[ \Lambda_{A_k \rightarrow S_k} : \mathcal{B}(\mathbb{C}^{d^d}) \rightarrow \mathcal{B}(\mathbb{C}^{d^d} \otimes L_k) \quad (i = 1, \ldots, K), \]

such that \( L_i \geq 1 \) for any \( i \) and \( L_1 + \ldots + L_K = N > K \). Each channel \( \Lambda_{A_k \rightarrow S_k} \) maps operators acting on the Hilbert space \( \mathbb{C}^{d^d} \) corresponding to the party \( A_k \) to operators acting on \( L_i \)-partite Hilbert space \( (\mathbb{C}^{d^d})^{\otimes L_i} \) corresponding to the group of parties denoted

\[ S_k = \{ A_{1+\sum_{i=1}^{k-1} L_i}, \ldots, A_{\sum_{i=1}^{k} L_i} \}. \]

with \( k = 1, \ldots, K \) and \( L_0 = 0 \). Notice that in the latter Hilbert space the local dimension \( d^d \) may be different than \( d \). Consequently, application of these channels to the subsystems of \( \rho_{A_1 \ldots A_K} \) gives rise to an \( N \)-partite state

\[ \sigma_A = (\Lambda_{A_1 \rightarrow S_1} \otimes \ldots \otimes \Lambda_{A_K \rightarrow S_K})(\rho_{A_1 \ldots A_K}) \]

that now acts on a larger \( N \)-partite Hilbert space \( \mathcal{H}_{N,d^d} \). This mapping induces naturally the \( K \)-partition of \( A \) determined by the groups \( S_k \) given in (14).

We are now ready to generalize Lemma 1 from the main text.

**Lemma 3.** If a \( K \)-partite state \( \rho_{A_1 \ldots A_K} \) acting on \( \mathcal{H}_{K,d} \) has a fully local model for GMs, then the \( N \)-partite state \( \sigma_A \) in Eq. (15) with \( L_i \geq 1 \) and \( L_1 + \ldots + L_K = N > K \) has a \( K \)-local model for GMs with respect to the \( K \)-partition defined by (14).

**Proof.** We start by noting the fact that \( \rho_{A_1 \ldots A_K} \) has a fully local model for generalized measurements means that the probabilities of obtaining results \( a_1, \ldots, a_K \) upon performing measurements \( M_i = \{ M^{a_i}_i \} \) \((i = 1, \ldots, K)\) take the following form [cf. Eq. (2) in the main text]

\[ p(a_1, \ldots, a_K | M_1, \ldots, M_K) \]
\[ = \text{Tr} \left( \left( M^{a_1}_1 \otimes \ldots \otimes M^{a_K}_K \right) \rho_{A_1 \ldots A_K} \right) \]
\[ = \int_{\Omega} d\lambda \langle \omega(\lambda) \rangle P_\rho(a_1 | M^{a_1}_1, \lambda) \ldots P_\rho(a_K | M^{a_K}_K, \lambda), \]

where \( \Omega \) denotes the set over which the classical information \( \lambda \) is distributed with probability distribution \( \omega \), and we have used the subscript \( \rho \) to emphasize that the probabilities correspond to \( \rho_{A_1 \ldots A_K} \). Exploiting (16), we can now construct a local model for \( \sigma_A \) with respect to the \( K \)-partition defined by the groups \( S_k \) given in Eq. (14). To this end, let us assume that the parties \( A_1, \ldots, A_N \) perform measurements \( \tilde{M}_i = \{ \tilde{M}^{a_i}_i \} \) \((i = 1, \ldots, N)\) on their share of the state \( \sigma_A \). Then, we define the following operators

\[ \tilde{M}^{a_1}_1 = \Lambda_{S_1 \rightarrow A_1}^\dagger \left( \tilde{M}^{a_1}_1 \otimes \ldots \otimes \tilde{M}^{a_{L_1}}_{L_1} \right) \]
\[ \tilde{M}^{a_2}_2 = \Lambda_{S_2 \rightarrow A_2}^\dagger \left( \tilde{M}^{a_{L_1}+1}_{L_1+1} \otimes \ldots \otimes \tilde{M}^{a_{L_1+L_2}}_{L_1+L_2} \right) \]
\[ \vdots \]
\[ \tilde{M}^{a_N}_N = \Lambda_{S_N \rightarrow A_K}^\dagger \left( \tilde{M}^{a_{L_1+\ldots+L_K-1}+1}_{L_1+\ldots+L_K-1+1} \otimes \ldots \otimes \tilde{M}^{a_{L_N}}_{L_N} \right) \]

acting on \( \mathbb{C}^{d^d} \) and indexed by the collections of outcomes \( \tilde{S}_k \). Here, \( \Lambda_{S_k \rightarrow A_k} : \mathcal{B}(\mathbb{C}^{d^d} \otimes L_k) \rightarrow \mathcal{B}(\mathbb{C}^{d^d}) \) stands for the dual map of \( \Lambda_{A_k \rightarrow S_k} \). Due to the fact that the dual map of a quantum channel is unital, i.e., it preserves the identity, and positive, it is fairly easy to see that for each \( k = 1, \ldots, K \), the set of operators \( \tilde{M}_k = \{ \tilde{M}^{a_k}_{a_k} \}_{a_k} \) forms a generalized measurement, that is, for any \( k = 1, \ldots, K \), \( \tilde{M}^{a_k}_{a_k} \geq 0 \) for any \( a_k \) and

\[ \sum_{a_k} \tilde{M}^{a_k}_{a_k} = \mathbb{1}_d. \]

Using the measurements \( \tilde{M}_k \), let us now define the response functions corresponding to the sets of parties \( S_k \) as

\[ p_\sigma(a_{S_k} | M_{S_k}, \lambda) = p_\rho(a_{S_k} | M_{S_k}, \lambda) \]

with \( k = 1, \ldots, K \). We now have

\[ p(a | M) = \text{Tr} \left[ \left( M^{(1)}_{a_1} \otimes \ldots \otimes M^{(N)}_{a_N} \right) \sigma_A \right] \]
\[ = \text{Tr} \left[ \left( M^{(1)}_{a_1} \otimes \ldots \otimes M^{(N)}_{a_N} \right) \otimes \Lambda_{A_k \rightarrow S_k} (\rho_{A_1 \ldots A_K}) \right] \]

where we have employed the definition of \( \sigma_A \). This, with the aid of Eqs. (17) can be further rewritten as

\[ p(a | M) = \text{Tr} \left[ \left( \tilde{M}^{(1)}_{a_1} \otimes \ldots \otimes \tilde{M}^{(K)}_{a_K} \right) \rho_{A_1 \ldots A_K} \right]. \]
Since the state $\rho_{A_1...A_K}$ is fully local, we finally obtain
\[
p(a|\mathcal{M}) = \int_{\Omega} d\lambda \omega(\lambda) \prod_{k=1}^{K} P_{\rho}(a_{S_k}|\mathcal{M}_k,\lambda)
= \int_{\Omega} d\lambda \omega(\lambda) \prod_{k=1}^{K} P_{\sigma}(a_{S_k}|\mathcal{M}_k,\lambda),
\]
where the last equality stems from the definitions of the response functions for the state $\sigma_{\mathcal{A}}$ given in Eq. (19). As a result, one sees that $\sigma_{\mathcal{A}}$ has a $K$-local model for generalized measurement with respect to the $K$-partition defined by the groups $S_k$ given in (14). Notice that $\Omega$ and $\omega$ are the same as in the local model for $\rho_{A_1...A_K}$.

Let us remark that it is fairly easy to see that Lemma 3 can be further generalized to the case when the initial Hilbert spaces $\mathcal{H}_{K,d}$ (and analogously the final one $\mathcal{H}_{N,d}$) have different local dimensions.

We can now move to the discussion on genuine multipartite entanglement. For this purpose let us consider again $N$ parties $A_1,\ldots,A_N$ sharing some $N$-partite state $\rho_{\mathcal{A}}$ acting on $\mathcal{H}_{N,d} = (\mathbb{C}^d)^\otimes N$. Let us then consider some $K$-partition of the parties $A_1,\ldots,A_N$, into $K$ pairwise disjoint sets $S_k$ such that they together contain all the parties. Finally, by $P^X_{\text{sym}}$ we denote the projector onto the symmetric subspace of the Hilbert space corresponding to the subsystem $X$.

One then proves the following generalization of Lemma 2 from the main text.

**Lemma 4.** Let $\rho_{\mathcal{A}}$ be an $N$-partite state acting on $\mathcal{H}_{N,d}$ such that with respect to some $K$-partition given by the groups $S_k$, its subsystems corresponding to $S_k$ are defined on symmetric subspaces, i.e.,
\[
P^{S_k}_{\text{sym}}\rho_{\mathcal{A}}P^{S_k}_{\text{sym}} = \rho_{\mathcal{A}}
\]
with $k = 1,\ldots,K$. If $\rho$ is not GME, then it takes the biseparable form
\[
\rho_{\mathcal{A}} = \sum_{T\bar{T}} p_{T\bar{T}} \rho_{T\bar{T}}
\]
where $p_{T\bar{T}}$ is some probability distribution and every $\rho_{T\bar{T}}$ is a state separable across the bipartition $T\bar{T}$ with $T$ and $\bar{T}$ being unions of $S_k$.

**Proof.** From the fact that $\rho_{\mathcal{A}}$ is not GME it follows that it can be written as [cf. Eq. (1) in the main text]
\[
\rho_{\mathcal{A}} = \sum_{T\bar{T} : T \subseteq S_2} p'_{T\bar{T}} \varrho_{T\bar{T}}, \quad p'_{T\bar{T}} \geq 0, \quad \sum_{T\bar{T}} p'_{T\bar{T}} = 1,
\]
where the sum goes over all bipartitions $T\bar{T}$ and $\varrho_{T\bar{T}}$ is some state that is separable with respect to the bipartition $T\bar{T}$, i.e., it admits the form
\[
\varrho_{T\bar{T}} = \sum_i q^i_{T\bar{T}} |e^i_T\rangle |f^i_{\bar{T}}\rangle,
\]
with $q^i_{T\bar{T}}$ being some probability distribution for any $T\bar{T}$, and $|e^i_T\rangle$ and $|f^i_{\bar{T}}\rangle$ denoting some pure states from the Hilbert spaces corresponding to subsystems $T$ and $\bar{T}$.

We will now prove, using the assumption that the state $\rho_{\mathcal{A}}$ is symmetric on the subsystems $S_k$, that any $T$ and $\bar{T}$ appearing in Eq. (25) must be a union of the sets $S_k$. To this end, it is enough to show for each $T$ that every pure state $|e^i_T\rangle |f^i_{\bar{T}}\rangle$ appearing in (25) must also be product across a bipartition $T\bar{T}$ in which $T$ and $\bar{T}$ are unions of the sets $S_k$.

We first notice that the assumption that the subsystems $S_k$ of $\rho_{\mathcal{A}}$ are defined on the corresponding symmetric subspaces implies that any pure state $|e^i_T\rangle |f^i_{\bar{T}}\rangle$ appearing in (25) for any bipartition $T$ must obey the following set of conditions
\[
P^{S_k}_{\text{sym}} |e^i_T\rangle |f^i_{\bar{T}}\rangle = |e^i_T\rangle |f^i_{\bar{T}}\rangle
\]
with $k = 1,\ldots,K$. This in particular means that for any pair of parties $A_m$ and $A_n$ belonging to the same set $S_k$,
\[
V_{A_mA_n} |e^i_T\rangle |f^i_{\bar{T}}\rangle = |e^i_T\rangle |f^i_{\bar{T}}\rangle,
\]
where $V$ is the swap operator defined through the condition
\[
V|\psi\rangle|\phi\rangle = |\psi\rangle|\phi\rangle \quad \text{for any pair of vectors } |\psi\rangle, |\phi\rangle \in \mathbb{C}^d.
\]

Let us now consider a particular bipartition $T\bar{T}$ in Eq. (24) for which $T$ and $\bar{T}$ are not unions of the sets $S_k$. Then, there exists a pair of parties $A_m$, $A_n$ belonging to one of the sets $S_k$ (the same one) such that $A_m \in T$ and $A_n \in \bar{T}$. For such a pair we use the Schmidt decompositions of the vectors $|e^i_T\rangle$ and $|f^i_{\bar{T}}\rangle$,
\[
|e^i_T\rangle = \sum_j \sqrt{\mu^j} |e^i_{A_m}\rangle |e^j_{T\setminus A_m}\rangle,
\]
and
\[
|f^i_{\bar{T}}\rangle = \sum_j \sqrt{\nu^j} |f^i_{\bar{A}_n}\rangle |f^j_{\bar{T}\setminus A_n}\rangle,
\]
where for simplicity we have skipped the upper index $i$ and the subscript $T\setminus A_m$ means the subsystem $T$ but the single-party subsystem $A_m$. Then, the condition (27) implies
\[
\sum_{i,j,j'} \sqrt{\mu^j} \sqrt{\nu^{j'}} |e^i_{A_m}\rangle |e^{j'}_{T\setminus A_m}\rangle |f^i_{\bar{A}_n}\rangle |f^{j'}_{\bar{T}\setminus A_n}\rangle = \sum_{i,j} \sqrt{\mu^j} \sqrt{\nu^{j}} |e^i_{A_m}\rangle |e^j_{T\setminus A_m}\rangle |f^i_{\bar{A}_n}\rangle |f^j_{\bar{T}\setminus A_n}\rangle,
\]
which by virtue of the orthogonality of the vectors in (28) and (29), implies that $|e^i_{A_m}\rangle = |f^i_{A_n}\rangle$ for any pair of indices $j,j'$. Denoting then $|g\rangle = |e^i_{A_m}\rangle = |f^i_{A_n}\rangle$, one finally finds that $|e^i_T\rangle |f^i_{\bar{T}}\rangle = |g_{A_m}\rangle |e^i_{T\setminus A_m}\rangle |g_{A_n}\rangle |f^i_{\bar{T}\setminus A_n}\rangle$, i.e., every vector in the decomposition (25) must be product with respect to the parties $A_m$ and $A_n$. By repeating this procedure for all pairs of parties $A_m$, $A_n$ such that both belong to one of the sets $S_k$, but $A_m \in T$ and $A_n \in \bar{T}$ (actually, not all pairs are necessary), one finds that every $|e^i_T\rangle |f^i_{\bar{T}}\rangle$ in the decomposition of $\varrho_{T\bar{T}}$ is product with respect to some bipartition $T\bar{T}$ with $T$ and $\bar{T}$ being unions of groups $S_k$. 

\[
\int_{\Omega} d\lambda \omega(\lambda) \prod_{k=1}^{K} P_{\sigma}(a_{S_k}|\mathcal{M}_k,\lambda).
\]
By applying exactly the same argument to the remaining bipartitions, one shows that any $\varrho_T|\bar{T}$ in (24) is separable with respect to some bipartition $\mathcal{T}|\bar{\mathcal{T}}$ with $\mathcal{T}$ and $\bar{\mathcal{T}}$ being unions of $S_k$, giving us the form (23) and completing the proof. 

Clearly, Lemma 2 from the main text follows directly from the above one. Precisely, assuming that with respect to some bipartition $S|\bar{S}$, the state $\rho_A$ obeys

$$P^S_{\text{sym}}\rho_A P^S_{\text{sym}} = \rho_A$$

and

$$P^\bar{S}_{\text{sym}}\rho_A P^\bar{S}_{\text{sym}} = \rho_A,$$

it follows that if it is not GME, then $\rho_A$ must be separable across the bipartition $S|\bar{S}$, that is,

$$\rho_A = \sum_i p_i \rho_i^S \otimes \rho_i^\bar{S},$$

with $\rho_i^S$ and $\rho_i^\bar{S}$ being some states corresponding to the groups $S$ and $\bar{S}$.

The following corollary straightforwardly stems from Lemma 4. For any $N$-partite state $\rho_A$ such that with respect to some $K$-partition its subsystems corresponding to the sets $S_k$ ($k = 1, \ldots, K$) are symmetric, if $\rho$ does not admit the decomposition (24) with $\rho_{T|\bar{T}}$ being separable across bipartitions $T|\bar{T}$ for which $T$ and $\bar{T}$ are unions of the sets $S_k$, then $\rho$ is GME. This fact gives rise to the following theorem, being a generalization of the main result of our work.

**Theorem.** Let $\rho_{A_1 \ldots A_K}$ be an entangled state acting on $\mathcal{H}_{K,d}$ that has a fully local model for generalized measurements. Then, for any collection of $K$ invertible quantum channels

$$\Lambda_{A_k \rightarrow S_k} : \mathcal{B}(\mathbb{C}^{d^\prime}) \rightarrow \mathcal{B}(S_{L_k,d^\prime}),$$

(34)

with $S_{L_k,d^\prime}$ denoting the symmetric subspace of $(\mathbb{C}^{d^\prime})^\otimes L_k$, the state

$$\sigma_A = (\Lambda_{A_1 \rightarrow S_1} \otimes \ldots \otimes \Lambda_{A_K \rightarrow S_K})(\rho_{A_1 \ldots A_K})$$

(35)

has a $K$-local model with respect to the $K$-partition with the groups $S_k$ defined in Eq. (14), and it is GME.

**Proof.** From Lemma 3 it follows that the state $\sigma_A$ has a $K$-local model for generalized measurements with respect to the given $K$-partition. Then, the fact that $\rho_{A_1 \ldots A_K}$ is genuinely multipartite entangled implies that so is $\sigma_A$. To make it more explicit, let us assume, in contrary, that $\sigma_A$ is not GME. Due to Lemma 4, this means that it admits the decomposition (23), with all $T$ and $\bar{T}$ being unions of the sets $S_k$ (by the definition of the channels $\Lambda_{A_k \rightarrow S_k}$, the $S_k$ subsystems of $\rho_A$ are symmetric). Since all channels $\Lambda_{A_k \rightarrow S_k}$ ($k = 1, \ldots, K$) are invertible, this implies that $\rho_{A_1 \ldots A_K}$ is not GME, contradicting the assumption. 

$\Box$