Well-posed constrained evolution of 3+1 formulations of General Relativity

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We present an analysis of well-posedness of constrained evolution of 3+1 formulations of GR. In this analysis we explicitly take into account the energy and momentum constraints as well as possible algebraic constraints on the evolution of high-frequency perturbations of solutions of Einstein’s equations. In this respect, our approach is principally different from standard analyses of well-posedness of free evolution in general relativity. Our study reveals the existence of subsets of the linearized Einstein’s equations that control the well-posedness of constrained evolution. It is demonstrated that the well-posedness of ADM, BSSN and other 3+1 formulations derived from ADM by adding combinations of constraints to the right-hand-side of ADM and/or by linear transformation of the dynamical ADM variables depends entirely on the properties of the gauge. For certain classes of gauges we formulate conditions for well-posedness of constrained evolution. This provides a new basis for constructing stable numerical integration schemes for a classical Arnowitt–Deser–Misner (ADM) and many other 3+1 formulations of general relativity.

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I. INTRODUCTION

An outstanding problem of numerical general relativity (GR) is achieving long-term stable numerical integration of Einstein’s equations. Until very recently, problems such as the general case of colliding black holes (BH) could not be solved due to instability of numerical integration [1–6]. During the last year, several groups have succeeded in simulating certain binary black hole (BBH) spacetimes. Collisions of non-rotating black holes were simulated using BSSN [2] with a specific choice of “1 + log” slicing and modified gamma-freezing shift conditions in [7] and [8]. A three-dimensional collision of two black holes originated from the collapse of a scalar field was simulated using a generalized harmonic decomposition of the GR field equations in [9]. In addition to using a special gauge, the authors of [7] and [8] enforced some of the BSSN constraints and used some of the constraints to control the constraint violating modes. The author of [9], in addition to special coordinates he used the constraints to damp constraint violating modes.

In spite of this remarkable success the general problem of long-term and stable integration of the Einstein equations remains unsolved. There is no general method to choose an appropriate formulation and appropriate coordinates to guarantee well-posedness and stability of numerical integration for a general strong field case. In particular, one does not know how the approaches used in [7–9] will behave in other strong vacuum field astrophysical cases, or cases where matter or even matter and magnetic fields are present. Perhaps the final word to this problem would be the development of schemes which implement fully constrained evolution of well-posed formulations. The purpose of this paper is to formulate a general approach to study the well-posedness of constrained evolution of 3+1 formulations of GR.

Generally, a 3+1 formulation is comprised of the evolutionary part and the constraints. The standard approach to solving a 3+1 system consists of integration of the evolutionary part in time (free evolution) starting with constrained initial conditions. If the constraints are satisfied initially, they should automatically be satisfied throughout the evolution due to the mathematical properties of Einstein’s equations.

Well-posedness of free evolution of 3+1 formulations has been analyzed in [3, 15]. For example, a classical ADM 3+1 formulation [1] is usually ill-posed. Ill-posedness of free evolution precludes stable numerical integration. We note, however, that well-posedness can not guarantee global in time existence of solutions, but only local existence. However, it is a necessary condition for stable integration.

There has been a number of attempts to overcome the instability of numerical integration. Using a harmonic gauge makes the ADM 3+1 system well-posed [10]. However, this gauge is not convenient for many physical problems. Introduction of a conformal factor and the trace of extrinsic curvature as additional unknown variables into the system allows to increase the duration of stable integration [2]. The evolutionary part of a 3+1 system can also be modified by adding a combination of constraints to its right-hand side. Choosing a special gauge (densitized lapse and zero shift) and addition of certain combinations of constraints to the right hand sides (RHS) of the ADM evolution equations makes the modified system well-posed [3]. However, all these modifications did not lead to a complete elimination of instabilities. Numerical experiments show that in general three-dimensional problems of GR the constraint equations are eventually violated even for a well-posed free evolution, and this terminates computations. Little progress has been made to-
wards a theoretical understanding of this behavior. Possible explanations are given in [6, 15].

Recently, attempts have been made to improve the behavior of modified 3+1 systems by enforcing the constraints after every integration time step of a free evolution [4, 5]. According to [4] this procedure allows integration of an isolated spherical black hole space-time for extended periods of time. We also mention that for certain cases perturbative approaches provide an alternative to straightforward numerical integration, e.g., for forming initial conditions for BH collisions [12]. Yet, the general problem of long-term stable integration remains open.

In a high-frequency perturbation analysis of a free evolution it is possible to separate perturbations on three parts: (1) perturbations of space-time itself, (2) perturbations of a coordinate system, and (3) perturbations describing deviations from constraints. If the behavior of space-time at a given point does not depend on future, then we must associate ill-posedness with coordinate and constraint-violating modes of perturbations. To achieve stable numerical integration, we must (A) use a gauge that does not lead to ill-posedness, and (B) eliminate or suppress ill-posedness caused by constraint violating modes.

A general theory of gauge stability (problem A) has been formulated, and well-posedness of gauges has been analyzed in [13]. It was demonstrated that coordinate perturbations in the evolution of the metric can be separated from the other two types of perturbations and the study of gauge stability can be reduced to a study of a general quasi-linear system of eight coupled partial differential equations for perturbations of lapse $\alpha$, shift $\beta_i$, $i = 1, \ldots, 3$, and perturbations of space-time coordinates $x^a$, $a = 0, \ldots, 3$. Conditions for well-posedness have been formulated in [13] for several types of gauges. We will repeatedly return to this subject in subsequent sections.

Constraint-violating modes of perturbations (problem B) are not fully understood at present. Recently, attempts have been made to stabilize numerical integration by enforcing the constraint equations after every integration time step of a hyperbolic free evolution [4, 5]. Such enforcement is not a unique procedure. Several possibilities are discussed in [5]. According to [4], constraint enforcement improves integration of an isolated spherical black hole space-time. Analysis of well-posedness of constraint enforcing procedure for a hyperbolic 3+1 formulation, densitized lapse, zero shift, and flat Minkowski space-time is given in [11].

An alternative to enforcement of constraints after a free evolution time step may be the construction of numerical schemes for constrained evolution in which growing constraint-violating modes are explicitly removed. In order to achieve this goal we must understand the nature of evolution of perturbations which satisfy constraints.

In this paper we present a general analysis of constraint-satisfying perturbations and address the issue of well-posedness of constrained evolution of 3+1 formulations of GR. We explicitly take into consideration the energy and momentum constraints on the evolution of high-frequency perturbations of solutions of Einstein’s equations. In this respect, our analysis is principally different from standard analyses of well-posedness of a free evolution in general relativity. Our study reveals the existence of subsets of the linearized version of Einstein’s equations that control well-posedness of constrained evolution. We demonstrate that the well-posedness of ADM, BSSN and other 3+1 formulations derived from ADM by adding combinations of constraints to the right-hand-side (RHS) of ADM and/or by linear transformation of the dynamical ADM variables depends entirely on the properties of the gauge. For certain classes of gauges we formulate conditions of well-posedness. The existence of these subsets provides a basis for constructing stable numerical integration schemes that incorporate the constraints directly.

The paper is organized as follows. We begin with the ADM 3+1 formulation and gauge classification (Section II). In Section III we give a general theory of well-posedness of constrained evolution of ADM. In Section IV we extend our theory to other 3+1 formulations including the Kidder-Scheel-Teukolsky (KST) and the Baumgarte-Shapiro-Shibata-Nakamura (BSSN). Discussion and conclusions are given in Section V.

II. ADM 3+1 FORMULATION AND GAUGES

A general form of the ADM 3+1 formulation consists of the evolutionary part

$$\frac{\partial \gamma_{ij}}{\partial t} = -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i, \quad (1)$$

$$\frac{\partial K_{ij}}{\partial t} = \alpha \left( (3)R_{ij} + KK_{ij} - 2\gamma^{mn}K_{im}K_{jn} \right) - \nabla_i \nabla_j \alpha + (\nabla_i \beta^m)K_{mj} + (\nabla_j \beta^m)K_{mi} + \beta^m \nabla_m K_{ij}, \quad (2)$$

and the energy and momentum constraints which we will call the kinematic constraints,

$$\mathcal{H} : \quad (3)R + K^2 - K_{mn}K^{mn} = 0, \quad (3)$$

$$\mathcal{M} : \quad \nabla_m K^{mi} - \nabla_i K = 0, \quad i = 1, 2, 3, \quad (4)$$

where $K = \gamma^{mn}K_{mn}$, $\gamma_{ij}$ and $K_{ij}$ are the three-dimensional metric of a space-like hypersurface and the extrinsic curvature respectively, $\alpha$ is the lapse function, $\beta^i$ is the shift vector (these are gauge functions), and $(3)R_{ij}$ is the three-dimensional Ricci tensor. $(3)R = \gamma^{ij}(3)R_{ij}$. We must add a specification of gauge (lapse and shift) in order to close the system (1), (2).

In this paper we use a general gauge specification similar to that introduced in [13] with one modification. Instead of working with the dual shift vector $\beta_k$ here we
work with the shift vector $\beta^k = \gamma^{kj} \beta_j$. A general gauge can be specified as

$$F_a \left( x^b, \alpha, \beta^i, \partial \alpha, \frac{\partial^2 \alpha}{\partial x^b \partial x^c}, \cdots, \frac{\partial \beta^i}{\partial x^b}, \cdots \gamma_{ij}, \frac{\partial \gamma_{ij}}{\partial x^b}, \cdots \right) = 0,$$

$$a, b, c = 0, \ldots, 3, \quad i, j = 1, 2, 3,$$

where $\gamma_{ij}$ are the components of the metric tensor.

### Section III. Analysis of Well-Posedness

We begin with a brief description of our general approach to the analysis of well-posedness of constrained sets of partial differential equations (PDEs). Let

$$\partial_\ell \vec{u} = \hat{M}^\ell \vec{u} + \hat{M}^\ell \vec{u}, \quad \ell = 1, 2, 3$$

be a set of $n$ first order quasi linear partial differential equations, where $\vec{u}$ is the column vector of the $n$ unknown variables, $\hat{M}^\ell$ are $n \times n$ matrices and $\hat{M}^\ell$ is a $n$-component column vector.

The concept of mathematical well-posedness is often related to strong hyperbolicity. For system (8) we present the following theorem from [23] without proof.

**Theorem 1.** The Cauchy problem for a first-order system of quasi-linear PDEs (8) is well posed if and only if the following two conditions hold:

1. For all unit one forms $v_i$, all eigenvalues of the characteristic matrix, $M = v_i \hat{M}^i$ are purely real.

2. There is a constant $K$, and for each $v_i$, there is a transformation $S(v_i)$ with

$$|\hat{S}(v_i)| + |\hat{S}^{-1}(v_i)| \leq K,$$

such that the transformed matrix $\hat{S}(v_i)\hat{M}(u_i)\hat{S}^{-1}(v_i)$ is diagonal.

**Definition 1.** A first-order system of quasi-linear PDEs (8) is called strongly hyperbolic if all conditions of the theorem above are met.

**Definition 2.** A first-order system of quasi-linear PDEs (8) is called weakly hyperbolic if it satisfies only the first condition of Theorem 1 and does not satisfy the second condition. Weakly hyperbolic systems are ill-posed.

For a constrained system (8), the dynamical variables satisfy a set of $m < n$ quasi linear constraint equations of the form:

$$\hat{C}^\ell(\vec{u}) \partial_\ell \vec{u} + \hat{C}^o(\vec{u}) = 0, \quad \ell = 1, 2, 3$$

where $\hat{C}^\ell$ are $m \times n$ matrices and $\hat{C}^o$ is an $m$-component column vector. We call a constrained system a collection of all solutions of (8) which satisfy the constraints. If evolution starts on the constrained surface it will remain on this surface.

In order to study well-posedness one drops the zeroth-order terms $\hat{M}^o$, freezes the coefficients $\hat{A}^i$ of (8) and studies the characteristic matrix of the system for all frozen in problems. This is equivalent to considering 1) high frequency and 2) small amplitude planar perturbations on the dynamical variables along a line locally specified by a unit vector $\vec{v}$ and parameterized by $\lambda$, so that for an arbitrary function $u(x^1, x^2, x^3)$

$$\frac{\partial u}{\partial x^\ell} = v_i \frac{\partial u}{\partial \lambda},$$

where $v_i$ is the dual vector to $v^i$, given by $v_i = \gamma_{ij} v^j$ and $\gamma_{ij}$ is the 3-metric of a spacelike hypersurface embedded in the manifold carrying the background solution $\vec{u}$ about which we perturb. The two aforementioned properties are the properties of all perturbations considered in this paper and those will be implied whenever the word "perturbation" is used. For perturbations $\delta \vec{u}$ on $\vec{u}$, combination of (8) and (11) yields

$$\partial_\ell \delta \vec{u} = \hat{M}^\ell v_i \frac{\partial \delta \vec{u}}{\partial \lambda} \equiv \hat{M} \frac{\partial \delta \vec{u}}{\partial \lambda}.$$  

(12)

For perturbations of $\vec{u}$ to remain on the constraint surface, they must satisfy the linearized constraint equations. After linearizing equations (10) we obtain

$$\hat{C}^\ell v_i \frac{\partial \delta \vec{u}}{\partial \lambda} \equiv \hat{C} \frac{\partial \delta \vec{u}}{\partial \lambda} = 0,$$

(13)
where $\hat{M}$ and $\hat{C}$ are the principal matrices of equations (8) and (10) respectively. Equations (13) is a set of $m$ equations for the spatial derivatives of the $n$ unknown variables, which in general can be solved for $m$ of the $n$ spatial derivatives of variables $\vec{u}$. Substitution of (13) into (12), leads to a set of $n = n - m$ linear partial differential equations for $q$ of the initial $n$ variables. This is schematically given by

$$\frac{\partial \vec{a}}{\partial \lambda} = \hat{A}_q(\vec{u}, v_i) \frac{\partial \vec{a}_q}{\partial \lambda},$$

where $\hat{A}_q$ is a $(q \times q)$ matrix. We will refer to (14) as the minimal set. The solution of (14) completely determines the solution of the entire linearized system (12). Therefore, the well-posedness of the minimal set determines the well-posedness of the entire constrained system. Theorem 1 and Definitions 1, 2 apply to (14).

We should note here that the description above is for first-order systems of PDEs. When one deals with second-order systems, then it is straightforward to obtain the equivalent first-order system of PDEs, by simply defining the first-order derivatives as new variables. Courant and Hilbert [21] show that if one derives the first-order form in the fashion described above, then the totality of solutions of the two systems coincide, for given Cauchy data. In addition to that in [22] it is shown that the hyperbolic properties of the second-order system and its equivalent first-order counterpart are the same. Although a reduction to a first-order system is not necessary, it makes it easier to analyze well-posedness, since in second-order systems one has to carefully study the behavior of the first order terms.

### A. Linearized equations of ADM

We want to study well-posedness of an ADM 3+1 formulation (evolutionary part (1), (2) plus constraints (3), (4)). For the analysis of well-posedness it is convenient to rewrite equations (1) - (4) in first-order form. We introduce new variables

$$D_{ij;k} = \frac{\partial \gamma_{ij}^{k}}{\partial x^{k}}$$

and drop all low-order terms, which do not contribute to the principal part of the equations. In terms of these variables, the ADM equations linearized with respect to a certain unperturbed solution (not necessarily a flat Minkowski spacetime)

$$\gamma_{ij}, \quad K_{ij}, \quad D_{ij;k}, \quad \alpha, \quad \beta^{k}$$

can be written as

$$\frac{\partial \delta \gamma_{ij}}{\partial t} = 2 \gamma_{ij}^{k}(\partial_{j}\delta \beta^{k}),$$

$$\frac{\partial \delta K_{ij}}{\partial t} = \alpha \left( - \frac{1}{\alpha} \partial_{i}\delta \alpha + R_{ij}^{1} + \beta^{k} \frac{\partial \delta D_{ij;k}}{\partial x^{k}} \right)$$

$$+ \Gamma_{ij}^{k} \partial_{k} \delta \alpha + 2 K_{k(i} \partial_{j)} \delta \beta^{k},$$

where $\delta \gamma_{ij}, \delta K_{ij}, \delta D_{ij;k}, \delta \alpha, \delta \beta^{m}$, are perturbations of (16), $R_{ij}^{1}$ is the principal part of the Ricci tensor

$$R_{ij}^{1} = \frac{1}{2} \gamma^{s k} \left( \frac{\partial \delta D_{k;j}}{\partial x^{s}} + \frac{\partial \delta D_{j;k}}{\partial x^{s}} - \frac{\partial \delta D_{ij;k}}{\partial x^{s}} \right)$$

and

$$\Gamma_{ij}^{k} = \frac{1}{2} \gamma^{k n} (D_{in;j} + D_{jn;i} - D_{ij;n}).$$

To close the system equations (17) - (23) must be supplemented with the linearized version of gauge equations (5).

We notice that the evolution equations for the 3-metric (17) are decoupled from the evolution equations for $K_{ij}, D_{ij;k}$ and the linearized gauge equations. It is the subsystem (18), (19) and the linearized gauge equations which determine the well-posedness of the entire system. The solution of (17) is completely determined by the solution of the above subsystem.

The linearized constraint equations in new variables are

$$\mathcal{H} : \quad R_{ij}^{1} = \gamma^{nm} R_{nm}^{1} = 0,$$

and

$$\mathcal{M}_{i} : \quad \gamma^{ms} \frac{\partial \delta K_{mi}}{\partial x^{s}} - \delta_{i}^{s} \gamma^{mn} \frac{\partial \delta K_{mn}}{\partial x^{s}} = 0,$$

where $\delta_{i}^{s}$ is the Kronecker symbol. The introduction of extra variables imposes new linear constraints on the system, which, for perturbations of (16), can be written as follows:

$$\frac{\partial \delta D_{ij;k}}{\partial x^{m}} = \frac{\partial \delta D_{ij;m}}{\partial x^{k}},$$

where (24) is derived by use of (15) and the fact that partial derivatives commute.

### B. Analysis of well-posedness of constrained evolution and well-posed subsets

We consider planar perturbations

$$\delta \gamma_{ij}, \quad \delta K_{ij}, \quad \delta D_{ij;k}, \quad \delta \alpha, \quad \delta \beta^{m},$$

moving in an arbitrary direction locally specified by a unit vector $\nu^{i}, \nu^{i}v_{i} = 1$. Substitution of (25) into (17) - (19) gives a set of thirty linear PDEs for perturbations (25). Substitution of (25) into the eighteen in number PDEs (24) gives twelve independent linear PDEs for
those perturbations. This means that along the direction which we are probing only six components of the eighteen $\delta D_{ij,k}$ are independent. The linearized energy and momentum constraints (22) and (23) give four additional linear PDEs for perturbations (25). None of these equations contain time derivatives of (25), because the constraints do not involve time derivatives of the perturbations.

Here we note that in the context of the 1st-order formulation equations (15) are constraints on the initial data. They are not involved in the analysis of well-posedness but they serve to guarantee the coincidence of the solutions of the 1st order and second order ADM equations. Since the evolution of the perturbations of the 3-metric is decoupled from and determined by the evolution of the perturbations of the subset variables $K_{ij}$, $D_{ij,k}$ and the linearized gauge equations, the study of well-posedness of a constrained evolution reduces to analyzing the subset of variables $K_{ij}$, $D_{ij,k}$, and all remaining constraints (22), (23), (24). There are twenty four evolution equations and sixteen constraints involved that leads to eight degrees of freedom.

By elimination of 16 spatial derivatives of the dynamical variables from the subset of twenty four equations, we obtain eight linearly independent equations, which form the minimal set. This can be schematically written as

$$\frac{\partial \hat{a}_k}{\partial t} = \hat{A}_8(\bar{u}, v_i) \frac{\partial \hat{a}_k}{\partial \lambda}, \quad (26)$$

where $\hat{A}_8$ is an $8 \times 8$ matrix which depends on the direction of planar perturbations and the background solution, $a_8$ are the independent perturbation amplitudes.

To study the well-posedness of the Cauchy problem of system (26) we first find the eigenvalues $\omega_k$ of the principal matrix of the system for every direction $v_i$. If non-zero imaginary parts are present in some of those eigenvalues, this indicates that part of the system forms an elliptic subset and the Cauchy problem is ill-posed. If all $\omega_k$ are real, then we investigate whether $\hat{A}_8$ is diagonalizable for every direction $v_i$ (uniformly diagonalizable), and whether the second condition (9) of Theorem 1 is satisfied (transformation $\hat{S}$ is uniformly bounded).

A convenient way to investigate properties of $\hat{A}_8$ is to use a reduction of $\hat{A}_8$ to a Jordan canonical form [17],

$$\hat{A}_8 = \hat{S} \hat{J} \hat{S}^{-1}, \quad (27)$$

so that

$$\frac{\partial}{\partial t} \left( \hat{S}^{-1} \hat{a}_8 \right) = \hat{J} \frac{\partial}{\partial \lambda} \left( \hat{S}^{-1} \hat{a}_8 \right), \quad (28)$$

where $\hat{S}$ is a non-singular matrix of a similarity transformation (27) and $\hat{J}$ is a block-diagonal Jordan canonical matrix

$$\hat{J} = \begin{bmatrix} \hat{J}_1 & 0 & \ldots & 0 \\ 0 & \hat{J}_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \hat{J}_N \end{bmatrix}. \quad (29)$$

In general, the number of canonical blocks $N$ may vary from one to eight and the size of each square block can vary from one to eight as well. The diagonal elements of $\hat{J}$ are eigenvalues of $\hat{A}_8$. If for any and all possible directions $v_i$ all Jordan blocks have size one (i.e. all off-diagonal elements of $\hat{J}$ are zero), then $\hat{A}_8$ has a complete set of eigenvectors and hence $\hat{A}_8$ is uniformly diagonalizable.

If some of off-diagonal elements of $\hat{J}$ are non-zero at least in one direction $v_k$, the set of eigenvectors in this direction is not complete and hence (26) cannot admit a well-posed Cauchy problem. The system (26) is then weakly hyperbolic.

In general, uniform diagonalizability of $\hat{A}_8$ may not be enough to guarantee the existence of a uniformly bounded transformation $\hat{S}$ required by Theorem 1. We show in Appendix A, however, that if the eigenvalues $\omega_k$ of $\hat{A}_8$ are analytic functions and the elements of $\hat{A}_8$ are ratios of analytic functions, then uniform diagonalizability guarantees the existence of a uniformly bounded transformation matrix $\hat{S}$. For the general relativity cases studied in the paper, the elements of the matrices of all minimal sets are always ratios of polynomials in $v_k$. Furthermore, those matrices have real eigenvalues which are analytic functions of $v_k$. Thus, for those systems studied here uniform diagonalizability is sufficient in order to satisfy the conditions for well-posedness.

The Jordan decomposition also provides information about combinations of independent variables that evolve according to the corresponding eigenfrequencies $\omega_k$. These combinations can be determined by applying a similarity transformation $\hat{U}^{-1}$ to the vector of original variables $\bar{a}_8$ (28). In the case of week hyperbolicity or elliptic behavior, this allows one to find a combination of variables whose time evolution is responsible for this behavior.

The above discussion is strictly valid for ADM and any 3+1 formulation of general relativity derived from ADM by a linear transformation of variables and addition of combinations of constraints on the RHS of the ADM equations, provided that they are coupled with fixed or algebraic gauges. If a gauge is differential (specified by a general (5)), then the number of linearized ADM equations may be greater than thirty and the final system of independent perturbations (26) will contain more than eight components. Analogous consideration must then be applied to this larger reduced system. A detailed discussion of well-posedness of ADM 3+1 coupled with differential gauges is given in section III.D. Next
section discusses well-posedness of ADM with fixed and algebraic gauges.

C. Well-posed subsets for ADM with fixed and algebraic gauges

For fixed and algebraic gauges, the study of (26) can be carried out analytically but resulting expressions for an arbitrary direction \( v_k \) are extremely complicated. We present here only results for algebraic gauges of a simple form

\[
\alpha = \alpha(t, x^k, \gamma_{ik}), \quad \beta^i = \beta^i(t, x^k)
\]

(31)

and

\[
\alpha = \alpha(t, x^k, \gamma_{ik}), \quad \beta_i = \beta_i(t, x^k).
\]

(32)

For gauge (31) the perturbation frequencies obtained from (26) are

\[
\omega_{1,2} = \beta^i v_i, \quad \omega_{3,\ldots,6} = \beta^i v_i \pm \alpha,
\]

\[
\omega_{7,8} = \beta^i v_i \pm \sqrt{\frac{\partial \alpha^2}{\partial \gamma_{ij}} v_i v_j}
\]

(33)

and for gauge (32) they are

\[
\omega_{1,\ldots,6} = \pm \sqrt{\alpha^2 - \beta_i \beta^i},
\]

\[
\omega_{7,8} = \frac{\beta^i v_i \pm \sqrt{\frac{\partial \alpha^2}{\partial \gamma_{ij}} + \beta^i \beta^j}}{\alpha}
\]

(34)

The difference in formulas (33) and (34) arises because (31) fixes the shift vector whereas (32) fixes its dual counterpart. Metric-independent contravariant shift corresponds to metric-dependent covariant shift and vice versa. The results for gauge (32) coincide exactly with those of our analysis of this gauge in [13].

Six eigen-frequencies \( \omega_{1,\ldots,6} \) in (33) and (34) are real. Eigenfrequencies \( \omega_{7,8} \) in (33) and (34) are real, if

\[
\left( \frac{\partial \alpha^2}{\partial \gamma_{ij}} v_i v_j > 0 \right) \text{ and } \left( \frac{\partial \alpha^2}{\partial \gamma_{ij}} + \beta^i \beta^j \right) v_i v_j > 0.
\]

The eigenvectors of \( \hat{A}_8 \) in (26) are uniformly linearly independent, for this particular choice of gauges. For the case of fixed gauges, \( \frac{\partial \alpha^2}{\partial \gamma_{ij}} = 0 \), the eigenvalues are still real, but \( \hat{A}_8 \) has only seven linearly independent eigenvectors. That is, all fixed gauges lead to ill-posed constrained evolution. The general conditions for well-posedness (strong hyperbolicity) of the constrained ADM 3+1 formulation with algebraic gauges (31) and (32) therefore are

\[
\left( \frac{\partial \alpha^2}{\partial \gamma_{ij}} \right) v_i v_j > 0
\]

(35)

and

\[
\left( \frac{\partial \alpha^2}{\partial \gamma_{ij}} + \beta^i \beta^j \right) v_i v_j > 0
\]

(36)

for an arbitrary \( v_i \), respectively. Again, (36) is the result obtained in [13].

The analysis of matrix \( \hat{A}_8 \) in (26) for algebraic gauges shows that the minimal set (26) separates into four independent subsets each consisting of two equations. Among these subsets there are three well-posed subsets corresponding to pairs of eigenvalues \( \omega_{1,\ldots,6} \). These three subsets describe propagation of gravitational waves and a gauge wave. We call these wave subsets. The fourth subset, which corresponds to eigenvalues \( \omega_{7,8} \), describes another two gauge modes. The solution of the fourth subset depends on solutions of the strongly hyperbolic wave subsets and the gauge choice. Those four subsets completely describe the behavior of Einstein’s equations (evolutionary part + constraints) in the high frequency limit. It is therefore evident that the posedness of Einstein’s equations depends entirely upon the properties of the gauge.

As a simple illustration, we explicitly present the form of these subsets for a general metric \( \gamma_{ij} \), extrinsic curvature \( K_{ij} \), gauge \( \alpha = \alpha(\gamma_{ij}, x^a, t) \) and \( \beta^i = \beta^i(x^a, t) \), and propagation of perturbations along \( x^3 \) coordinate direction, \( v_k = (v_1, 0, 0) \) and \( \gamma_{11} v_1 v_1 = 1 \). The eigenvalues for this case are:

\[
\{ \beta^1 v_1 - \alpha, \beta^1 v_1 - \alpha, \beta^1 v_1 + \alpha, \beta^1 v_1 + \alpha, \beta^1 v_1, \beta^1 v_1 + \alpha \sqrt{2A^{11} v_1 v_1}, \beta^1 v_1 - \alpha \sqrt{2A^{11} v_1 v_1} \}
\]

and the minimal set is explicitly given by the following four subsets

\[
\frac{\partial \delta K_{23}}{\partial t} = \alpha v_1 \left[ -\frac{1}{2} \gamma_{11} \frac{\partial \delta D_{23;1}}{\partial \alpha} + \beta^1 \frac{\partial \delta K_{23}}{\partial \alpha} \right],
\]

(37)

\[
\frac{\partial \delta D_{23;1}}{\partial t} = \alpha v_1 \left[ -\frac{1}{2} \gamma_{11} \frac{\partial \delta D_{23;1}}{\partial \alpha} + \beta^1 \frac{\partial \delta D_{23;1}}{\partial \alpha} \right]
\]

\[
\frac{\partial \delta K_{33}}{\partial t} = \alpha v_1 \left[ -\frac{1}{2} \gamma_{11} \frac{\partial \delta D_{33;1}}{\partial \alpha} + \beta^1 \frac{\partial \delta K_{33}}{\partial \alpha} \right],
\]

(38)
\[
\frac{\partial \delta D_{ij;1}}{\partial t} = \alpha v_1 \left\{ -2 \left( \frac{\gamma^{ij,12} \gamma^{13} + \gamma^{11} \gamma^{13,22} - 2 \gamma^{11} \gamma^{12,23}}{\gamma^{11} (\gamma^{12,12} - \gamma^{11,22})} \right) \frac{\partial \delta K_{23}}{\partial \lambda} - 2 \left( \frac{\gamma^{12} \gamma^{13} - \gamma^{11} \gamma^{12,23}}{\gamma^{11} (\gamma^{12,12} - \gamma^{11,22})} \right) \frac{\partial \delta K_{33}}{\partial \lambda} \right\}, \\
\frac{\partial \delta D_{13;1}}{\partial t} = \alpha v_1 \left\{ \frac{2}{\gamma^{11}} \left[ \gamma^{12} \frac{\partial \delta K_{23}}{\partial \lambda} + \gamma^{13} \frac{\partial \delta K_{33}}{\partial \lambda} \right] + \frac{\beta^1}{\alpha} \frac{\partial \delta D_{13;1}}{\partial \lambda} \right\}, \\
\frac{\partial \delta K_{11}}{\partial t} = \alpha v_1 \left[ -A^{11} \frac{\partial \delta D_{11;1}}{\partial \lambda} - 2A^{12} \frac{\partial \delta D_{12;1}}{\partial \lambda} - 2A^{13} \frac{\partial \delta D_{13;1}}{\partial \lambda} - \left( \frac{1}{2} \gamma^{22} d^{23} + \gamma^{23} + 2A^{23} + A^{22} d^{23} \right) \frac{\partial \delta D_{23;1}}{\partial \lambda} \right], \\
\frac{\partial \delta D_{11;1}}{\partial t} = \alpha v_1 \left[ -2 \frac{\partial \delta K_{11}}{\partial \lambda} + \frac{\beta^1}{\alpha} \frac{\partial \delta D_{11;1}}{\partial \lambda} \right].
\]

(39)

where

\[ A^{ij} = \frac{\partial \ln \alpha}{\partial \gamma_{ij}}, \quad d^{23} = -2 \frac{\gamma^{12} \gamma^{13} - \gamma^{11} \gamma^{23}}{\gamma^{12} (\gamma^{12,12} - \gamma^{11,22})}, \]

\[ d^{33} = -\frac{\gamma^{13} \gamma^{13} - \gamma^{11} \gamma^{33}}{\gamma^{12} (\gamma^{12,12} - \gamma^{11,22})}. \]

(41)

As one can easily see these equations are valid only when \( \gamma^{12,12} - \gamma^{11,22} \neq 0 \). This has been our assumption to solve the momentum and Hamiltonian constraints for the derivatives of the dynamical variables which we wanted to eliminate. Although it may seem that this is not a general result we point out that the kinematic constraints can always be used to eliminate 4 of the dynamical variables provided that certain conditions are held true. If the condition above is not satisfied then there will be another set of variables that we will be able to eliminate and thus obtain the minimally coupled set of partial differential equations. In this respect we have not lost generality.

Subsets I, II, and III describe wave propagation and are well-posed. The first two propagate with the shift plus the fundamental speed (the lapse function) and the third one with the shift velocity. Subset IV will be well posed, if the lapse satisfies \( \frac{\partial \ln \alpha}{\partial \gamma_{ij}} v_1 > 0 \Rightarrow \frac{\partial \ln \alpha}{\partial \gamma_{ij}} > 0 \), which is a particular case of the general condition (35).

If \( \frac{\partial \ln \alpha}{\partial \gamma_{ij}} = 0 \) (fixed gauge), the fourth subset takes the form

\[ \frac{\partial \delta K_{11}}{\partial t} = \alpha v_1 \left[ -\left( \frac{1}{2} \gamma^{22} d^{23} + \gamma^{23} \right) \frac{\partial \delta D_{23;1}}{\partial \lambda} + \left( \frac{1}{2} \gamma^{12} \gamma^{13} - \gamma^{11} \gamma^{12,23} \right) \frac{\partial \delta D_{23;1}}{\partial \lambda} \right], \]

IV:

\[ \frac{\partial \delta K_{11}}{\partial t} = \alpha v_1 \left[ -\left( \frac{1}{2} \gamma^{22} d^{23} + \gamma^{23} \right) \frac{\partial \delta D_{23;1}}{\partial \lambda} + \frac{\beta^1}{\alpha} \frac{\partial \delta D_{11;1}}{\partial \lambda} \right], \]

IV:

\[ \frac{\partial \delta D_{11;1}}{\partial t} = \alpha v_1 \left[ -2 \frac{\partial \delta K_{11}}{\partial \lambda} + \frac{\beta^1}{\alpha} \frac{\partial \delta D_{11;1}}{\partial \lambda} \right]. \]

(42)

This subset is weakly hyperbolic and ill-posed. This can be most easily seen if we consider a simplest case with \( \delta D_{23;1} = \delta D_{33;1} = 0 \) and \( \beta^1 = 0 \). Then the solution of (42) will be \( \delta K_{11} = \delta K_{11}(\lambda, 0) \) and \( \delta D_{11;1}(\lambda, t) = \delta K_{11}(\lambda, 0) + \left( \frac{\partial \ln \alpha}{\partial \gamma_{ij}} (\lambda, 0) \right) t \). The linear growth of \( \delta D_{11;1} \) depends on initial conditions and may be arbitrarily fast. Since in the high frequency limit we can treat the gauge functions as constants, the physical acceleration can be neglected. Therefore, the linear growth of \( \delta D_{11;1} \) physically describes the deformation of a synchronous reference frame with time, which in a general non-linear case when the perturbations are not small, leads to the formation of caustics.

Constrained evolution of perturbations of all other variables is completely determined by the solution of subsets I - IV. As an example we present the evolution of \( \delta K_{13} \) and \( \delta D_{22;1} \) when, for simplicity, the shift vector is zero:

\[ \frac{\partial \delta K_{13}}{\partial t} = \alpha v_1 \left( \frac{1}{2} \gamma^{12} \frac{\partial \delta D_{23;1}}{\partial \lambda} + \frac{1}{2} \gamma^{13} \frac{\partial \delta D_{33;1}}{\partial \lambda} \right), \]

\[ \frac{\partial \delta D_{22;1}}{\partial t} = -2 \alpha v_1 \left[ 2 \left( \frac{\gamma^{11} \gamma^{12} - \gamma^{12} \gamma^{13}}{\gamma^{12} (\gamma^{12,12} - \gamma^{11,22})} \right) \frac{\partial \delta K_{33}}{\partial \lambda} + \frac{\gamma^{11} \gamma^{33} - \gamma^{13} \gamma^{13}}{\gamma^{12} (\gamma^{12,12} - \gamma^{11,22})} \frac{\partial \delta K_{33}}{\partial \lambda} \right]. \]

(43)

The amplitudes of \( \delta K_{13} \) and \( \delta D_{22;1} \) will not grow. Equations for \( \delta K_{22} \) and \( \delta K_{12} \) are analogous but more complicated. All other perturbations satisfy equations

\[ \frac{\partial \delta D_{ij;2}}{\partial t} = \frac{\partial \delta D_{ij;3}}{\partial t} = 0. \]

(44)

We found that the behavior described above is similar to that of a general case of algebraic gauges and any arbitrary direction of propagation of perturbations.
Examples of algebraic gauges are the widely used gauge $\alpha = C(x^i) \gamma^{1/2}$ often referred to as the “harmonic” gauge [2], the “1+log” gauge $\alpha = 1 + \log(\gamma)$ [20], and the densitized lapse gauge $\alpha = C(x^i) \gamma^{\beta}$ [3], all depending on the determinant of the three-metric, $\gamma = \text{det}(\gamma_{ij})$. For these gauges, condition (35) can be written as

$$\frac{\partial \ln \alpha}{\partial \gamma_{ij}} v_i v_j = \frac{\partial \ln \alpha}{\partial \gamma} \frac{\partial \gamma}{\partial \gamma_{ij}} v_i v_j = \frac{\partial \ln \alpha}{\partial \gamma} \gamma^{ij} v_i v_j = \frac{\partial \ln \alpha}{\partial \gamma} > 0.$$  \hfill (45)

It can be readily seen that both “harmonic” and “1+log” gauges satisfy this condition and lead to a well posed constrained evolution. The densitized lapse will provide a well-posed constrained evolution only if

$$\sigma > 0.$$  \hfill (46)

D. Well-posedness of ADM with differential gauges

Similar approach to posedness of 3+1 formulations can be carried out for more complex gauges involving non-zero shift and general elliptic, parabolic, or hyperbolic differential gauges. As examples of elliptic and parabolic differential gauges we will consider the maximal slicing condition and its parabolic extension. Although these two gauges are believed to prevent coordinate singularities, here we demonstrate that both of them produce weakly hyperbolic minimal sets and thus produce ill-posed constrained evolution.

The maximal slicing condition [16] is $K = 0$, where $K$ is the trace of the extrinsic curvature. In the following analysis we need the evolution equations for $K$ and the determinant $\gamma$ of the 3-metric in vacuum. Those can be derived by taking the traces of (1) and (2) and they are

$$\partial_t \ln \gamma^{1/2} = -\alpha K + \nabla_i \beta^i$$  \hfill (47)

$$\partial_t K = -\gamma^{ij} \nabla_i \nabla_j \alpha + \alpha K_{ij} K^{ij} + \beta^i \nabla_i K$$  \hfill (48)

If the $K = 0$ condition is imposed then (48) results in the following elliptic differential equation for the lapse function

$$\gamma^{ij} \nabla_i \nabla_j \alpha - \alpha K_{ij} K^{ij} = 0.$$  \hfill (49)

If the Hamiltonian constraint (3) is satisfied (49) yields

$$\gamma^{ij} \nabla_i \nabla_j \alpha - \alpha R = 0.$$  \hfill (50)

In the limit of high frequency perturbations the Ricci scalar vanishes on the surface of constraints and equation (50) reduces to

$$\gamma^{ij} \partial_i \partial_j \delta \alpha = 0,$$

which, if written along a given direction $v^i$, yields

$$\gamma^{ij} v_i v_j \frac{\partial \delta \alpha}{\partial \gamma} = \frac{\partial \delta \alpha}{\partial \gamma} \frac{\partial \gamma}{\partial \gamma_{ij}} v_i v_j = \frac{\partial \delta \alpha}{\partial \gamma} \gamma^{ij} v_i v_j = \frac{\partial \delta \alpha}{\partial \gamma} > 0.$$  \hfill (52)

Then the principal term of $\nabla_i \nabla_j \delta \alpha = v_i v_j \frac{\partial ^2 \delta \alpha}{\partial \gamma ^2}$ vanishes and therefore the ADM + Maximal Slicing equations have the same properties as ADM + Fixed Gauges, which means that the Cauchy problem for ADM+Maximal Slicing is ill posed. We should keep in mind that result (52) is not only valid for a constrained evolution, but also for an unconstrained one, since it can be derived from (49), too. Equation (52) results because of the high frequency perturbations we are considering here.

Here we ought to resolve the apparent contradiction between our analysis and the fact that maximal slicing prevents the formation of coordinate singularities [16]. This can be seen if (47) is written as

$$K = L_n \ln(\gamma^{1/2}),$$  \hfill (53)

where $L_n$ denotes the Lie derivative along the unit normal vector $\hat{n}$ to the spacelike hypersurfaces with 3 metric $\gamma_{ij}$. If we set $K = 0$ then from (53) the local volume element $\gamma^{1/2}$ is proper-time independent and cannot shrink to zero. This means that a coordinate singularity cannot be formed.

However, the well-posedness properties of algebraic $K = 0$ slicing condition are different from those of (50) with initial conditions $K = 0$. If maximal slicing is imposed by using $K = 0$ at all times by eliminating one of the components, for example $K_{11} = -\frac{\gamma^{ij} \delta \alpha _{,ij}}{\gamma} K_{ij}$ this reduces the number of dynamical degrees of freedom in the minimal set (26) from eight to seven [27]. In this case the perturbations of the trace of the extrinsic curvature are identically zero, $\delta K = 0$, and the minimal set of seven equations is well posed. On the other hand, if we use (50) and initial conditions $K(t = 0) = 0$ alone, the minimal set of eight equations is ill-posed because $\delta K$ are not necessarily zero. If we write equation (48) in the limit of high frequency perturbations, keeping (52) in mind, we obtain

$$\partial_t \delta K = \beta^i \partial_i \delta K.$$  \hfill (54)

Although this equation is well-posed and hence the perturbations of $K$ do not grow, we notice that the $K = 0$ condition may now be violated by perturbations of $K$. It is this violation which is the root to the ill-posedness associated with (50).

Let us illustrate this with the following example. Consider planar high frequency perturbations along $x^i$ about Minkowski spacetime, which means that the unperturbed lapse is $\alpha = 1$ and the unperturbed shift is $\beta^i = 0$. In this case, the minimal set for the differential maximal slicing condition contains (42), which for this case can be written as

$$\frac{\partial \delta K_{11}}{\partial t} = 0,$$

$$\frac{\partial \delta K_{11}}{\partial \gamma} = -2v_1 \frac{\partial \delta K_{11}}{\partial \lambda}.$$  \hfill (55)

This subset (55) is weakly hyperbolic and hence ill-posed. In addition, if we impose the algebraic maximal slicing
condition at all times, then in the high frequency limit
\[
\frac{\partial \delta K_{11}}{\partial \lambda} = -\left(\frac{\partial \delta K_{22}}{\partial \lambda} + \frac{\partial \delta K_{33}}{\partial \lambda}\right). \tag{56}
\]
This equation eliminates \(\delta K_{11}\) from the minimal set. The linearized momentum constraints for this particular case also read
\[
\frac{\partial \delta K_{22}}{\partial \lambda} + \frac{\partial \delta K_{33}}{\partial \lambda} = 0. \tag{57}
\]
Combining (56) and (57) we obtain \(\frac{\partial \delta K_{11}}{\partial \lambda} = 0\) and (55) reduces to one equation
\[
IV: \frac{\partial \delta D_{1111}}{\partial t} = 0. \tag{58}
\]
Equation (58) is well posed, and this eliminates the possibility of formation of coordinate singularities.

The parabolic extension of maximal slicing [19] has the following form
\[
\frac{\partial \alpha}{\partial t} = \frac{1}{\epsilon} \left(\gamma^{ij} D_i D_j \alpha - K_{ij} K^{ij} \alpha - cK\right), \tag{59}
\]
where \(\epsilon\) is a positive constant. Since the lower order term \(-K_{ij} K^{ij} \alpha - cK\) does not belong to the principal part, application of high frequency perturbations along any arbitrary direction yields
\[
\frac{\partial \delta \alpha}{\partial t} = \frac{1}{\epsilon} \frac{\partial^2 \delta \alpha}{\partial \lambda^2}. \tag{60}
\]
The linearized evolution equations for variables belonging to the minimal set for this particular gauge are given by (see (18) and (19)):
\[
\frac{\partial \delta K_{ij}}{\partial t} = \alpha \left(-\frac{1}{\alpha} v_i v_j \frac{\partial^2 \delta \alpha}{\partial \lambda^2} + R_{ij}^1\right) + \Gamma_{ij}^{kl} \frac{\partial \delta \alpha}{\partial \lambda}, \tag{61}
\]
\[
\frac{\partial \delta D_{ijkl}}{\partial t} = -2\alpha v_k \frac{\partial \delta K_{ij}}{\partial \lambda} - 2K_{ij} v_k \frac{\partial \delta \alpha}{\partial \lambda}. \tag{62}
\]
In the high frequency limit we are considering here, the evolution equation for the lapse (60) is completely decoupled from (61) and (62) whatsoever. But, the linearized “parabolic maximal slicing” is a diffusion equation, which is well-posed and it dictates that the amplitude of the perturbation of the lapse function in this case diffuses out and hence it does not grow. This means that as time passes the entire system will asymptotically resemble the case of fixed lapse and zero shift, which is a special case of a fixed gauge. Therefore the constrained evolution of ADM with “parabolic maximal slicing” is ill-posed. In appendix B we present a more rigorous proof of this fact. These results for maximal slicing and its parabolic extension agree with those obtained in [13] for the same slicing conditions.

Following Bona et al. [14] hyperbolic gauges can be given in general by the following conditions:
\[
\partial_t \alpha = -\alpha^2 Q, \quad \partial_t \beta^i = -\alpha Q^i, \tag{63}
\]
where \(Q, Q^i\) will be given by either algebraic or differential equations relating them with other variables of the system and will be chosen accordingly in order to obtain hyperbolic equations for the lapse and/or the shift. For such slicings, one has to modify the ADM equations by defining new variables in order to obtain the 1st order form. Therefore we define \(A_i = \partial_t \ln \alpha, B^i_j = \partial_j \beta^i\) and with these definitions we compute: \(\partial_t \partial_i \alpha = \alpha (A_i A_j + \partial_i \partial_j)\) and \(\partial_t \partial_i \beta^i = \partial_i B^i_j\). Thus, the linearized principal part of the ADM equations can be written as
\[
\frac{\partial \delta \gamma_{ij}}{\partial t} = 2\gamma_{ij} (\delta B^j_{\ell} \ell), \tag{64}
\]
\[
\frac{\partial \delta K_{ij}}{\partial t} = -\alpha \left(\frac{R_{ij}^1}{\alpha} + \frac{\partial K_{ij}}{\partial x^\ell} - \frac{\partial A_j}{\partial x^\ell}\right), \tag{65}
\]
\[
\frac{\partial \delta D_{ijkl}}{\partial t} = -2\alpha v_k \frac{\partial \delta K_{ij}}{\partial \lambda} - 2K_{ij} v_k \frac{\partial \delta \alpha}{\partial \lambda} + \frac{1}{\alpha} \left[ -\frac{\partial \delta B^i_{\ell j}}{\partial x^\ell} + \frac{\partial \delta E^i_{\ell j}}{\partial x^\ell} \right]. \tag{66}
\]
We have introduced twelve new variables and therefore we need twelve additional evolution equations to describe them. One could expect that the number of differential equations of the minimal set would be \(8+12=20\) equations. However, we must impose eight new linear constraints arising due to the introduction of new variables \(B^i_k\) and \(A_i\),
\[
\frac{\partial B^i_k}{\partial x^l} = \frac{\partial B^j_{\ell k}}{\partial x^\ell} \tag{67}
\]
and
\[
\frac{\partial A_i}{\partial x^l} = \frac{\partial A_j}{\partial x^l}. \tag{68}
\]
Equations (67) and (68) mean that there are 3 linearly independent \(B^i_k\) and 1 independent \(A_i\). Therefore the minimal set will in principle consist of only twelve evolution equations for our dynamical variables.

As an example we will consider the Bona-Masso family of slicing conditions [18] which we write in terms of the new variables as
\[
\partial_t \ln \alpha = \beta^i A_i - \alpha f(\alpha) (K - K_0), \tag{69}
\]
where \(f(\alpha)\) is a strictly positive function of the lapse, \(K\) is the trace of the extrinsic curvature and \(K_0 = K(t = t_0)\).
Then the principal part of the evolution equations of the perturbations of the new variables is

$$\partial_t A_i = \beta^k \partial_k \delta A_i - \alpha f(\alpha) \gamma^k \partial \delta K_{kl} \partial x^l.$$  \hspace{3cm} (70)

However, because of constraints (68) only one of the above equations will be a part of the minimal set. Our analysis of the minimal set, carried out for this gauge, shows that the constrained evolution is well posed.

As an illustration of the latter, we present the linearized constrained evolution equations for the zero shift vector (also known as the $K$-driver condition). The minimal set then consists of only nine partial differential equations (three equations get eliminated due to $\beta^i = 0$) for planar perturbations of the dynamical variables $K_{11}$, $K_{33}$, $D_{11;1}, D_{12;1}, D_{13;1}, D_{23;1}, D_{33;1},$ and $A_1$ along the $x^1$-axis, which we group into the following subsets

\[I: \quad \frac{\partial \delta K_{23}}{\partial t} = -\frac{1}{2} \alpha v_1 \gamma_{11} \frac{\partial \delta D_{23;1}}{\partial \lambda}, \]

\[\frac{\partial \delta D_{23;1}}{\partial t} = -2\alpha v_1 \frac{\partial \delta K_{23}}{\partial \lambda}, \hspace{2cm} (71)\]

\[II: \quad \frac{\partial \delta K_{33}}{\partial t} = -\frac{1}{2} \alpha v_1 \gamma_{11} \frac{\partial \delta D_{33;1}}{\partial \lambda}, \]

\[\frac{\partial \delta D_{33;1}}{\partial t} = -2\alpha v_1 \frac{\partial \delta K_{33}}{\partial \lambda}, \hspace{2cm} (72)\]

\[\frac{\partial \delta D_{12;1}}{\partial t} = -2\alpha v_1 \frac{\partial \delta K_{11}}{\partial \lambda}. \hspace{2cm} (73)\]

and an equation for the evolution of $\delta D_{11;1}$

\[\frac{\partial \delta D_{11;1}}{\partial t} = -2\alpha v_1 \frac{\partial \delta K_{11}}{\partial \lambda}. \hspace{2cm} (75)\]

Subsets I and II describe gravitational waves propagating with the fundamental speed. The eigenvalues which correspond to these two subsets are $\{-\alpha, -\alpha, \alpha, \alpha\}$. These are well-posed and completely decoupled from the rest of the system. The perturbations which correspond to subset III are completely determined by the solution of the first two subsets and they do not grow. Two zero eigenvalues correspond to the third subset and therefore it corresponds to static modes. The fourth subset is coupled to the first two, but it is not completely defined by the solution of I and II. Subset IV is also well posed and it describes a gauge wave propagating with speed $\alpha \sqrt{f(\alpha)}$. Finally equation (75), which describes a static gauge mode, is completely determined by the solution of the fourth subset and the perturbation for $\delta D_{11;1}$ does not grow. It is clear therefore that this system of equations has a well-posed Cauchy problem. We found exactly the same behavior in the most general case of perturbations of arbitrary direction. The Jordan matrix is always diagonal and hence the system is well-posed. If the shift vector is fixed, but non vanishing, its presence does not affect the well-posedness of the constrained system, since the Jordan matrix is still diagonal and the 9 non zero real eigenvalues are

$$\beta^i v_i, \beta^i v_i, \beta^i v_i, \beta^i v_i \pm \alpha, \beta^i v_i \pm \alpha, \beta^i v_i \pm \alpha \sqrt{f(\alpha)}.$$  \hspace{3cm} (76)

One can demonstrate, as in the previous section, that the solution of the entire linearized system can be retrieved once the minimal set is solved and that its posedness is completely dependent on the posedness of the minimal set. Thus, we conclude that the Bona-Masso family of slicing conditions gives rise to a well-posed constrained ADM evolution.

IV. ANALYSIS OF EXTENDED 3+1 FORMULATIONS

The analysis of the standard ADM 3+1 formulation presented in the previous section can be applied to other 3+1 formulations of GR. In this section we will explicitly study two of those re-formulations of GR, namely the Kidder-Scheel-Teukolsky (KST) [3] (and all KST-like formulations) and the Baumgarte-Shapiro-Shibata-Nakamura (BSSN) [2].
A. The Kidder-Scheel-Teukolsky formulation of 3+1 GR

Kidder et al. [3] suggested a new formulation of Einstein’s equations with a strongly (or even symmetric) hyperbolic set of evolution equations. They obtained this formulation by adding terms proportional to the constraint equations to the RHS of the ADM evolution equations. This does not change the physics the equations describe but changes the character of partial differential equations which describe the free evolution. The modified set they suggested is (using their notation for this section only)

\[ \partial_t K_{ij} = (. . .) + \gamma N g_{ij} C + \zeta N g^{ab} C_{a(ij)b}, \]
\[ \partial_t d_{kij} = (. . .) + \eta N g_{k(i} C_{j)} + \chi N g_{ij} C_k, \]

where \((. . .)\) stands for the RHS of the ADM evolution equations, \(N\) is the lapse function, \(g_{ab}\) is the 3-metric, \(K_{ij}\) the extrinsic curvature, \(d_{kij}\) are the same variables as our \(D_{ij;k}\), \(C\) and \(C_k\) are the hamiltonian and momentum constraints, respectively and

\[ C_{klij} \equiv \partial_k d_{lij} = 0. \]  

Finally, \(\{\gamma, \zeta, \eta, \chi\}\) are arbitrary constants.

The system was closed with the densitized lapse

\[ Q = \ln(N g^{-\sigma}), \]

where \(Q\) is a function independent of the dynamical fields, \(g\) is the determinant of the three-metric and \(\sigma\) is the densitization parameter, and was found that \(\sigma > 0\) is essential for obtaining a well-posed set of evolution equations. This is exactly what we obtain by our analysis (46), without adding constraints to the ADM equations, but explicitly imposing them.

If we apply our constrained perturbation analysis to the KST formulation we will cancel the added constraints on the RHS of their formulation. As result the KST formulation has exactly the same analysis as the ADM formulation.

According to [3] any transformation of dynamical variables does not change the hyperbolic classification of a set of PDEs, if this transformation satisfies the following conditions

1. The transformation is linear in all dynamical variables except possible the metric
2. The transformation is invertible
3. Time and space derivatives of the metric can be written as a sum of only the non-principal terms.

Their redefinition of variables and the introduction of “kinematical” ones is a transformation which satisfies the aforementioned criteria and thus it does not affect the hyperbolic properties of the set of evolution equations. Hence, the constrained perturbation analysis of the KST formulation and all KST-like formulations (i.e. all those formulations which are derived by addition of constraints to the RHS of ADM and perhaps a transformation of variables with the aforementioned properties) is equivalent to that of the ADM equations.

This argument may also be used to conclude that any 3+1 system directly obtained from ADM using the above transformation will be equivalent to ADM.

B. The Baumgarte-Shapiro-Shibata-Nakamura formulation of 3+1 GR

The BSSN formulation was initially introduced by Nakamura et al. [2], then modified by Shibata and Nakamura [2], and it was later reintroduced slightly modified by T. Baumgarte and S. Shapiro [2]. Before we proceed with our analysis let us review the BSSN formulation first. In what follows, we will use the notation introduced by Baumgarte and Shapiro [2].

The fundamental dynamical variables of BSSN are \((\varphi, \tilde{\gamma}_{ij}, K, \tilde{A}_{ij}, \tilde{\Gamma}^i)\) instead of \((\gamma_{ij}, K_{ij})\), where

\[ \varphi = (1/12) \log(\det \gamma_{ij}), \]
\[ \tilde{\gamma}_{ij} = e^{-4\varphi} \gamma_{ij}, \]
\[ K = \gamma_{ij} K_{ij}, \]
\[ \tilde{A}_{ij} = e^{-4\varphi} (K_{ij} - (1/3) \gamma_{ij} K), \]
\[ \tilde{\Gamma}^i = \tilde{\Gamma}^i_{jk} \tilde{\gamma}^j_k. \]

The “connection” symbols \(\tilde{\Gamma}^i_{jk}\) are the Christoffell symbols associated with the conformal three-metric \(\tilde{\gamma}_{ij}\). In the BSSN formulation, the Ricci curvature tensor is calculated as

\[ R^\text{BSSN}_{ij} = R^\varphi_{ij} + \tilde{R}_{ij}, \]
\[ R^\varphi_{ij} = -2 D_i D_j \varphi - 2 \tilde{\gamma}_{ij} \tilde{D}^k \tilde{D}_k \varphi + 4 (\tilde{D}_i \varphi) (\tilde{D}_j \varphi) - 4 \tilde{\gamma}_{ij} (\tilde{D}^k \varphi) (\tilde{D}_k \varphi), \]
\[ \tilde{R}_{ij} = - (1/2) \tilde{\gamma}^{lk} \partial_l \tilde{\gamma}_{ij} + \tilde{\gamma}_{ij} (\tilde{D}_j \varphi) \tilde{\Gamma}^k + \tilde{\Gamma}^k \tilde{\Gamma}^{(ij)k} + 2 \tilde{\gamma}_l^{lm} \tilde{\Gamma}^{(ij)klm} + \tilde{\gamma}_l^{lm} \tilde{\Gamma}^{(ij)k} \tilde{\Gamma}^k, \]

where \(\tilde{D}_i\) is the covariant derivative associated with \(\tilde{\gamma}_{ij}\). The evolution equations for these dynamical variables
\[\partial_t \varphi = -(1/6) \alpha K + (1/6) \beta^i (\partial_i \varphi) + (\partial_i \beta^i),\]
\[\partial_t \tilde{\gamma}_{ij} = -2 \alpha \dot{A}_{ij} + \tilde{\gamma}_{ik} (\partial_k \beta^j) + \tilde{\gamma}_{jk} (\partial_i \beta^k)
- (2/3) \tilde{\gamma}_{ij} (\partial_k \beta^k) + \beta^i (\partial_k \tilde{\gamma}_{ij})\]
\[\partial_t K = -2 D^i D_j \alpha + \alpha \dot{A}_{ij} \dot{A}^{ij} + (1/3) \alpha K^2 + \beta^i (\partial_i K),\]
\[\partial_t \dot{A}_{ij} = -e^{-4\varphi} (D_i D_j \alpha)_{TF} + e^{-4\varphi} \alpha (R^B_{ij})_{TF}
+ \alpha K \dot{A}_{ij} - 2 \alpha \dot{A}_{ik} \dot{A}_{kj} + (\partial_j \beta^i) \dot{A}_{kj} + (\partial_j \beta^k) \dot{A}_{ij} + \beta^i (\partial_k \dot{A}_{ij})\]
\[\partial_t \tilde{\Gamma}^i = -2 (\partial_j \alpha) \dot{A}^{ij} + 2 \alpha (\tilde{\Gamma}_{jk} \dot{A}^{kj} - (2/3) \tilde{\gamma}^{ij} \partial_j \dot{K})
+ 6 \dot{A}^{ij} (\partial_j \varphi) - \partial_j (\beta^k (\partial_k \tilde{\gamma}^{ij}) - \tilde{\gamma}^{ij} (\partial_k \beta^i))
- \tilde{\gamma}^{ij} (\partial_k \beta^j) + (2/3) \tilde{\gamma}^{ij} (\partial_k \beta^k),\]
\[(82)\]

where all trace free (TF) two index quantities \(T_{ij}\) are given by
\[T_{ij}^{TF} = T_{ij} - \frac{1}{3} \tilde{\gamma}_{ij} T, \quad T = \gamma^{kl} T_{kl}.\]
\[(83)\]

The constraint equations are
\[H^BSSN = R^BSSN + K^2 - K_{ij} K^{ij} = 0,\]
\[(84)\]
\[M_i^{BSSN} = M_i^{ADM} = 0,\]
\[(85)\]
\[G^i = \tilde{\Gamma}^i - \tilde{\gamma}^{jk} \tilde{\Gamma}^{ij}_{jk} = 0,\]
\[(86)\]
\[A = \dot{A}_{ij} \tilde{\gamma}^{ij} = 0,\]
\[(87)\]
\[S = \tilde{\gamma} = 1 = 0.\]
\[(88)\]

\(H^BSSN\) and \(M_i^{BSSN}\) are the Hamiltonian and momentum constraints (the kinematic constraints) respectively, while the latter three are algebraic constraints due to the requirements of BSSN formulation.

### 2. Linearized equations of BSSN

The formulation, as given above, is first order in time and second order in spatial derivatives. The second order derivatives occur in the evolution equations for the conformal traceless extrinsic curvature. In order to apply our analysis we need the first order form. Therefore, we define new variables
\[\partial_m \varphi = \phi_m \quad \text{and} \quad \partial_m \tilde{\gamma}_{ij} = \dot{D}_{ij;m},\]
\[(89)\]

where \(\cdot :\cdot\) does not imply a covariant derivative, but it separates indices of different nature. In terms of those variables the BSSN equations linearized with respect to a certain background spacetime solution (not necessarily a Minkowski spacetime)
\[\varphi, \quad \tilde{\gamma}_{ij}, \quad \dot{A}_{ij}, \quad \tilde{\Gamma}^i, \quad \dot{D}_{ij;k}, \quad \phi_i\]
\[(90)\]

can be written as
\[\partial_t \delta \varphi = \partial_t \delta \beta^i,\]
\[\partial_t \delta \tilde{\gamma}_{ij} = \tilde{\gamma}_{ik} (\partial_k \delta \beta^j) + \tilde{\gamma}_{jk} (\partial_i \delta \beta^k) - (2/3) \tilde{\gamma}_{ij} (\partial_k \delta \beta^k),\]
\[\partial_t \delta K = -e^{-4\varphi} \tilde{\gamma}_{ik} \partial_t \delta \alpha + \beta^i (\partial_i \delta K),\]
\[\partial_t \delta \dot{A}_{ij} = -e^{-4\varphi} (\partial_i \partial_j \delta \alpha)_{TF} + e^{-4\varphi} \alpha (\delta R^B_{ij})_{TF}
+ (\partial_i \delta \beta^k) \dot{A}_{kj} + (\partial_j \delta \beta^k) \dot{A}_{ij}
- (2/3) \partial_k (\partial_k \delta \beta^k) \dot{A}_{ij} + \beta^i (\partial_k \dot{A}_{ij}),\]
\[\partial_t \delta \tilde{\Gamma}^i = -2 (\partial_j \delta \alpha) \dot{A}^{ij} + 2 \alpha (\tilde{\Gamma}_{jk} \dot{A}^{kj} - (2/3) \tilde{\gamma}^{ij} (\partial_j \delta K)
- \beta^k (\partial_j \partial_k \delta \beta^i) + (2/3) \tilde{\gamma}^{ij} (\partial_j \partial_k \delta \beta^k),\]
\[(91)\]
\[\partial_t \delta \dot{D}_{ij;k} = -2 \alpha \partial_{\ell} \delta \dot{A}_{ij} + \beta^\ell \partial_{\ell} \delta \dot{D}_{ij;k} + \tilde{\gamma}_{ij} \partial_{\ell} \delta \beta^\ell
+ \tilde{\gamma}^\ell \partial_{\ell} \delta \beta^\ell - 2 \tilde{\gamma}^{ij} \partial_{\ell} \delta \alpha
+ \dot{D}_{ij;\ell} + 2 (\partial_i \delta \beta^\ell) \dot{D}_{ij;k} - \frac{1}{3} \partial (\partial_i \delta \beta^\ell) \dot{D}_{ij;k}\]
\[(92)\]

where
\[\delta \varphi, \quad \delta \tilde{\gamma}_{ij}, \quad \delta \dot{A}_{ij}, \quad \delta K, \quad \delta \tilde{\Gamma}^i, \quad \delta \dot{D}_{ij;k}, \quad \delta \phi_i\]
\[(93)\]

are small amplitude and high frequency perturbations of (90) and
\[\delta R^B_{ij} = -2 \partial_t \delta \phi_i - 2 \tilde{\gamma}^{\ell k} \partial_t \delta \phi_k - \frac{1}{2} \tilde{\gamma}^{\ell k} \partial_t \delta \dot{D}_{ij;k}
+ \frac{1}{2} \left( \tilde{\gamma}^{\ell k} \partial_t \delta \dot{D}_{ij;k} + \tilde{\gamma}^{\ell k} \partial_t \delta \dot{D}_{ij;k} \right)\]
\[(94)\]

In the context of the first order formulation, since the evolution of the conformal 3-metric is decoupled from the evolution of the rest of the system (just like in ADM), then the derivative of constraint (88) provides a constraint for the \(\dot{D}_{ij;k}\) variables, which has to be taken into consideration and is
\[\tilde{\gamma}^{ij} \dot{D}_{ij;k} = 0\]
\[(95)\]

Then, the linearized constraint equations in new variables are
\[\delta R^BSSN = R_{ij}^{BSSN} = 0,\]
\[(96)\]
\[\tilde{\gamma}^{kl} \partial_{\ell} \delta \dot{A}_{kl;k} - \frac{2}{3} \partial_{\ell} \delta K \partial_{\ell} \delta \dot{D}_{ij;k} = 0,\]
\[(97)\]
\[
\frac{\partial \delta \tilde{\Gamma}^i}{\partial x^s} = \tilde{\gamma}^{ik} \tilde{\gamma}^{jm} \frac{\partial \delta \tilde{D}_{km;j}}{\partial x^s},
\]
(98)

\[
\tilde{\gamma}^{ij} \frac{\partial \delta \tilde{A}_{ij}}{\partial x^k} = 0,
\]
(99)

\[
\tilde{\gamma}^{ij} \frac{\partial \delta \tilde{D}_{ij;k}}{\partial x^s} = 0.
\]
(100)

The introduction of additional variables implies the introduction of new linear constraint equations which for perturbations of (90) can be written as follows

\[
\partial_m \delta \tilde{D}_{ij;k} = \partial_k \delta \tilde{D}_{ij;m} \quad \text{and} \quad \partial_m \delta \phi_k = \partial_k \delta \phi_m
\]
(101)

and

\[
\partial_m \delta \tilde{\gamma}_{ij} = 0 \quad \text{and} \quad \partial_m \delta \varphi = 0.
\]
(102)

Finally, equations (91)-(92) have to be supplemented with the linearized gauge equations (5).

3. Analysis of well posedness of BSSN with fixed and algebraic gauges

Constraints (101) dictate that there is only one independent \( \phi_k \) and 6 independent \( \tilde{D}_{ij;k} \). Equation (102) tells us (just like in the case of the ADM formulation) that the evolution of \( \delta \varphi \) and \( \delta \tilde{\gamma}_{ij} \) is decoupled from the evolution of the perturbations of the remaining dynamical variables. For BSSN the Hamiltonian constraint can always be solved for the derivative of that \( \phi_k \) involved in it, thus eliminating \( \phi_k \) from the minimal set. The momentum constraints can be solved for the spatial derivatives of two \( \tilde{A}_{ij} \)'s and the spatial derivative of \( K \). Constraint (87) can be used for the elimination of one of the components of \( \tilde{A}_{ij} \). Finally, constraints (98) completely eliminate the \( \tilde{\Gamma}^i \) variables and (100) can eliminate one more of the \( \tilde{D}_{ij;k} \) variables. This means that fully imposing the linearized constraint equations results in a set of linear PDEs for three \( \tilde{A}_{ij} \) and five \( \tilde{D}_{ij;k} \), the well-posedness of which will determine the well-posedness of the entire linearized system (91)-(92).

In this section we mainly analyze gauges for which the lapse function is dependent on the coordinates and the dynamical variables and the shift vector is function of only the coordinates. With this in mind, the linearized principal part of the set of variables which will be part of the BSSN minimal set can be written as follows

\[
\partial_t \delta \tilde{A}_{ij} = -e^{-4\varphi} \left( \partial_t \partial_j \delta \alpha - \frac{1}{3} \tilde{\gamma}_{ij} \gamma^{kl} \partial_k \partial_l \delta \alpha \right) + e^{-4\varphi} \alpha \delta R^{BSSN}_{ij} + \beta^k \partial_k \delta \tilde{A}_{ij},
\]
(103)

\[
\partial_t \delta \tilde{D}_{ij;k} = -2 \alpha \partial_k \delta \tilde{A}_{ij} + \beta^l \partial_l \delta \tilde{D}_{ij;k}.
\]

To obtain the minimal set one has to fully impose constraints (96)-(100) on equations (103).

The equations for the constrained evolution of BSSN are extremely complicated. For a general background solution, we were able to carry out an analytic analysis of BSSN with fixed gauges only. Similarly to ADM, we found that the constrained evolution of BSSN with fixed gauges is ill-posed.

For algebraic gauges, the simplest case possible is perturbations about a flat space \( \gamma_{ij} = \delta_{ij} \). If we define \( \Delta^{ij} = \frac{\partial \alpha}{\partial \gamma_{ij}} \) and \( \Delta = \frac{\partial \alpha}{\partial \varphi} \), our analysis shows that the minimal set has eigenvalues \( \{ \beta^i v_1, \beta^j v_2, \beta^i v_1 \pm \alpha, \beta^j v_2 \pm \sqrt{\Delta} \} \), where

\[
\tilde{D} = \Delta + 12 \Delta^{ij} v_i v_j - 4 \Delta^{km} \delta_{km} > 0
\]
(104)

is the necessary condition for all eigenvalues to be real. Since the Jordan matrix for this case is diagonal the BSSN constrained evolution will be well-posed if (104) is satisfied. As in the ADM case one could show that solving the minimal set of BSSN is adequate to obtain the solution of the entire linearized system.

As an illustration we present a set of constrained evolution equations for perturbations of variables

\[
\delta \tilde{A}_{11}, \delta \tilde{A}_{22}, \delta \tilde{A}_{23}, \delta \tilde{D}_{11;1}, \delta \tilde{D}_{12;1}, \delta \tilde{D}_{13;1}, \delta \tilde{D}_{22;1}, \delta \tilde{D}_{23;1}
\]
(105)

propagating along the \( x^1 \) direction and perturbed about flat space.

I:

\[
\frac{\partial \delta \tilde{A}_{23}}{\partial t} = -\frac{1}{2} \Delta^{23} \frac{\partial \delta \tilde{D}_{23;1}}{\partial \lambda} + \beta^1 \frac{\partial \delta \tilde{A}_{23}}{\partial \lambda},
\]
(106)

\[
\frac{\partial \delta \tilde{D}_{23;1}}{\partial t} = -2 \alpha \frac{\partial \delta \tilde{A}_{23}}{\partial \lambda} + \beta^1 \frac{\partial \delta \tilde{D}_{23;1}}{\partial \lambda},
\]

II:

\[
\frac{\partial \delta \tilde{D}_{12;1}}{\partial t} = \beta^1 \frac{\partial \delta \tilde{D}_{12;1}}{\partial \lambda},
\]
(107)

\[
\frac{\partial \delta \tilde{D}_{13;1}}{\partial t} = \beta^1 \frac{\partial \delta \tilde{D}_{13;1}}{\partial \lambda}.
\]
\[ \frac{\partial \tilde{\delta} \tilde{A}_{11}}{\partial t} = \alpha \left\{ -\frac{2}{3} \left[ (\Delta^{11} - \Delta^{33} + \frac{\Delta}{8}) \frac{\partial \delta \tilde{D}_{11;1}}{\partial \lambda} \right. \\
+ 2\Delta^{12} \frac{\partial \delta \tilde{D}_{12;1}}{\partial \lambda} + 2\Delta^{13} \frac{\partial \delta \tilde{D}_{13;1}}{\partial \lambda} \right. \\
+ (\Delta^{22} - \Delta^{33}) \frac{\partial \delta \tilde{D}_{22;1}}{\partial \lambda} \\
+ 2\Delta^{23} \frac{\partial \delta \tilde{D}_{23;1}}{\partial \lambda} \left. \right\} + \frac{\beta}{\alpha} \frac{\partial \tilde{A}_{11}}{\partial \lambda}, \]

\[ \frac{\partial \delta \tilde{A}_{22}}{\partial t} = \alpha \left\{ \frac{1}{3} \left[ (\Delta^{11} - \Delta^{33} + \frac{\Delta}{8} - \frac{3}{4}) \frac{\partial \delta \tilde{D}_{11;1}}{\partial \lambda} \right. \\
+ 2\Delta^{12} \frac{\partial \delta \tilde{D}_{12;1}}{\partial \lambda} + 2\Delta^{13} \frac{\partial \delta \tilde{D}_{13;1}}{\partial \lambda} \right. \\
+ (\Delta^{22} - \Delta^{33} - \frac{3}{2}) \frac{\partial \delta \tilde{D}_{22;1}}{\partial \lambda} \\
+ 2\Delta^{23} \frac{\partial \delta \tilde{D}_{23;1}}{\partial \lambda} \left. \right\} + \frac{\beta}{\alpha} \frac{\partial \tilde{A}_{22}}{\partial \lambda} \right\}, \]

\[ \frac{\partial \delta \tilde{D}_{11;1}}{\partial t} = -2\alpha \frac{\partial \delta \tilde{A}_{11}}{\partial \lambda} + \beta \frac{\partial \delta \tilde{D}_{11;1}}{\partial \lambda}, \]

\[ \frac{\partial \delta \tilde{D}_{22;1}}{\partial t} = -2\alpha \frac{\partial \delta \tilde{A}_{22}}{\partial \lambda} + \beta \frac{\partial \delta \tilde{D}_{22;1}}{\partial \lambda}. \] (108)

Subset I corresponds to a gravitational wave propagating with the shift + fundamental speed along \( x^1 \). It is decoupled from the other subsets and it is well-posed. Subset II describes propagation of two waves, which travel with the shift vector speed. This subset just like the first one is decoupled from the rest of the system. It is a wave subset and therefore it is well-posed. Subset III is coupled to the first two subsets and hence its solution depends on the solutions of I and II. At a first glance it may seem as a contradiction that there are 3 subsets when for ADM we had 4. However, subset III consists of two independent. One of them describes a gravitational wave travelling with the shift + fundamental speed and the other is a gauge wave travelling with speed \( \beta^3 \pm \sqrt{D} \), if \( \tilde{D} > 0 \), where for this case

\[ \tilde{D} = \Delta - 4(\Delta^{11} + \Delta^{22} + \Delta^{33}) + 12\Delta^{11}, \] (109)

which is a special case of (104).

The constrained evolution of BSSN with fixed gauges, that is, \( \Delta = \Delta^{ij} = 0 \) has the same subsets, but III is now decoupled from I and II:

\[ \frac{\partial \delta \tilde{A}_{11}}{\partial t} = \beta_1 \frac{\partial \delta \tilde{A}_{11}}{\partial \lambda}, \]

\[ \frac{\partial \delta \tilde{A}_{22}}{\partial t} = -\alpha \frac{\partial \delta \tilde{D}_{11;1}}{\partial \lambda} - \frac{\alpha}{2} \frac{\partial \delta \tilde{D}_{22;1}}{\partial \lambda} + \beta_1 \frac{\partial \delta \tilde{A}_{22}}{\partial \lambda}, \]

\[ \frac{\partial \delta \tilde{D}_{11;1}}{\partial t} = -2\alpha \frac{\partial \delta \tilde{A}_{11}}{\partial \lambda} + \beta_1 \frac{\partial \delta \tilde{D}_{11;1}}{\partial \lambda}, \]

\[ \frac{\partial \delta \tilde{D}_{22;1}}{\partial t} = -2\alpha \frac{\partial \delta \tilde{A}_{22}}{\partial \lambda} + \beta_1 \frac{\partial \delta \tilde{D}_{22;1}}{\partial \lambda}. \] (110)

The four eigenfrequencies of this system are \( \{ \beta^3, \beta^3, \beta^1 \pm \alpha \} \), which are all real. However, the Jordan matrix is not diagonal, which implies that the principal matrix of (110) does not have a complete set of eigenvectors, hence the constrained evolution is weakly hyperbolic, that is, ill-posed. To assign a physical meaning to weak hyperbolicity it is instructive to consider the case of vanishing shift. Then one can easily see that subset III breaks into two subsets. The ill-posed subset is:

\[ \frac{\partial \delta \tilde{A}_{11}}{\partial t} = 0, \quad \frac{\partial \delta \tilde{D}_{11;1}}{\partial t} = -2\alpha \frac{\partial \delta \tilde{A}_{11}}{\partial \lambda} \] (111)

and its solution is \( \delta \tilde{A}_{11} = \delta \tilde{A}_{11}(\lambda, t = 0), \) and \( \delta \tilde{D}_{11;1}(\lambda, t) = \delta \tilde{D}_{11;1}(\lambda, 0) + \left( \frac{\partial \delta \tilde{A}_{11}}{\partial \lambda} \right) t \). The linear growth of \( \delta \tilde{D}_{11;1} \) depends on initial conditions and may be arbitrarily fast. As in the ADM formulation, physically the linear growth of \( \delta \tilde{D}_{11;1} \) describes the inertial deformation of a synchronous reference frame with time which, in general non-linear case when perturbations are not small leads, to the formation of caustics.

A special class of algebraic gauges is that with the lapse function depending on the determinant of the 3-metric, that is, \( \alpha = \alpha(\gamma) \), which in the BSSN formulation obtains the form \( \alpha = \alpha(\gamma) \), and therefore the lapse does not depend on the conformal 3-metric, but on the conformal factor only. Thus the strong hyperbolicity condition (104) reduces to \( \tilde{D} = \Delta > 0 \). For “harmonic” slicing \( \alpha = c(x^i)e^{6\gamma} \) and “1 + log” slicing \( \alpha = 1 + 2\varphi \) already discussed in the previous section it is easy to show that they produce a well-constrained BSSN formulation. Similarly, for a densitized lapse \( Q = \ln(\alpha e^{-12\varphi}) \), we find that the requirement for well-posedness \( \tilde{D} = \Delta = 12\sigma \alpha > 0 \) yields a necessary condition for \( \sigma \), which is of course the same as (46).

To conclude this sub-section we note that the results of the analysis presented above do not depend on the order of linearization and enforcement of algebraic constraints of BSSN. Instead of linearizing the unconstrained BSSN first, one could have chosen to reduce the number of variables of BSSN, by eliminating as many variables as there are algebraic constraints and then linearize the reduced system. It is straightforward to check that both ways lead exactly to the same minimal sets.
C. BSSN and Differential Gauges

One can perform the same constrained perturbation analysis for the BSSN formulation in conjunction with the differential gauges considered in the ADM analysis above. However, the numerical relativity community has recently been resorting to non-trivial shift conditions, for example [24, 25], as an attempt to accurately evolve black hole binaries. Therefore, instead of analyzing the same gauges as in the ADM analysis above, here we focus on the elliptic “Gamma freezing” condition [24], which was formulated for the BSSN formulation

\[ \partial_t \tilde{\Gamma}^i = 0 \iff \partial_j \partial_k \tilde{\gamma}^{ij} = 0. \]  

(112)

This condition obviously “freezes” the evolution of the \( \Gamma^i \) variables, hence the name. By use of (112) and the evolution equations of \( \Gamma^i \), from (82), one obtains the following elliptic equations which the shift vector has to satisfy.

\[
-2(\partial_j \alpha) \tilde{A}^{ij} + 2\alpha (\tilde{\Gamma}^{ij}_k \tilde{A}^{kj} - (2/3)\tilde{\gamma}^{ij}(\partial_j K)) \\
+ 6\tilde{A}^{ij}(\partial_j \varphi) - \partial_j (\beta^k(\partial_k \tilde{\gamma}^{ij}) - \tilde{\gamma}^{kj}(\partial_k \beta^j)) - \tilde{\gamma}^{kj}(\partial_k \beta^j) \\
- \frac{1}{2}\tilde{\gamma}^{kj}(\partial_k \beta^j) + (2/3)\tilde{\gamma}^{ij}(\partial_k \beta^k) = 0.
\]

(113)

Here we consider only a 1D perturbation approach about flat space to show that this gauge is good at least in the case considered.

To reduce the system to first order form we define the derivatives of the shift vector as new variables \( B^i_j = \partial_j \beta^i \). The derivatives of the shift satisfy (65) as in the ADM analysis. Of course the introduction of those 9 new variables leads to the introduction of 9 new constraints which have to be fully imposed. In the high frequency limit of small amplitude perturbations those constraints read \( \partial_j \partial_k \beta^j = 0 \) together with (67). In terms of the new variables the linearized principal part of (113) becomes

\[
\tilde{\gamma}^{kj}(\partial_j \partial_k \delta B^i_j + 1/3 \tilde{\gamma}^{ij}(\partial_k \delta B^j_k - 2\alpha 2/3 \tilde{\gamma}^{ij}(\partial_j K + \beta^j(\partial_j \partial_k \tilde{\Gamma}^i)).
\]

(114)

Equation (114) can be treated as 3 more constraints in our approach, allowing us hence to eliminate the three independent perturbations \( \delta B^i_j \). Thus, after imposing all available constraints the minimal set consists of 8 equations, as it was expected. To simplify the analysis further we study this shift condition in conjunction with a lapse function of the form \( \alpha = \alpha(\varphi) \), that is the lapse depends only on the determinant of the 3 metric. This simplification results in the evolution equations of \( \delta \tilde{A}_{ij} \) in (106)-(108) being the same, but where \( \Delta^{ij} = 0 \), so we will not write them here. However, the evolution equations of \( \delta \tilde{D}_{ij,1} \) in (106)-(108) are different, since we have to deal with the shift terms now. Those equations change as it is dictated by the linearized evolution equations of \( \delta \tilde{D}_{ij,k} \) in (92), which in new variables read

\[
\partial_t \delta \tilde{D}_{ij,1} = \ldots + \tilde{\gamma}^{\ell t} \frac{\partial \delta B^t_{ij}}{\partial x^\ell} + \tilde{\gamma}^{\ell t} \frac{\partial \delta B^t_{ij}}{\partial x^\ell} - \frac{2}{3} \tilde{\gamma}^{ij} \frac{\partial \delta B^t_{ij}}{\partial x^\ell}.
\]

(115)

Where \((\ldots)\) stands for the RHS of the evolution equations of \( \delta D_{ij,1} \) in (106)-(108). All other terms in (92) contribute to the low-order part only. If one imposes all possible constraints then the evolution equations of \( \delta \tilde{D}_{ij,1} \) yield

\[
\begin{align*}
\partial_t \tilde{D}_{11,1} &= 0, \\
\partial_t \tilde{D}_{12,1} &= 0, \\
\partial_t \tilde{D}_{13,1} &= 0, \\
\partial_t \tilde{D}_{22,1} &= \ldots + \frac{1}{2} \beta^1 \frac{\partial \tilde{D}_{11,1}}{\partial \lambda} + \alpha \frac{\partial \tilde{A}_{11}}{\partial \lambda}, \\
\partial_t \tilde{D}_{23,1} &= \ldots
\end{align*}
\]

(116)

The Jordan decomposition of the resulting system shows that the Jordan matrix is diagonal with the following eigenvalues on the diagonal \{0, 0, 0, \beta^1, \beta^1 + \alpha, \beta^1 - \alpha, \beta^1 + \alpha, \beta^1 - \alpha\}. All eigenvalues are real and hence the system is well behaved. Surprisingly, the eigenvalues which correspond to the functional form of the lapse, that is, \( \pm \alpha \Delta/\Delta \) are missing. Therefore, the “G-freezing” condition gives rise to a well behaved constrained 1D evolution, even if the lapse chosen is fixed or calculated by the maximal slicing condition, because in 1D perturbations about flat space this shift condition eliminates the gauge waves which correspond to the algebraic lapse considered when the shift vector is fixed. This is what one truly obtains, if one carries out the analysis of the “G-freezing” shift in conjunction with a fixed lapse or maximal slicing.

Those results constitute a good indication of the well-posedness of the constrained BSSN evolution with this gauge condition. However, a definite answer requires a complete analysis, that is, consideration of planar perturbations about an arbitrary spacetime. This is a very complicated task. We will address this in a future paper along with the other popular shift conditions, the parabolic and hyperbolic “G-Driven” conditions [24] in conjunction with the Bona-Maso family of slicing conditions.

D. Equivalence of conditions for well-posedness of constrained evolution of ADM and BSSN

We will now demonstrate that, in the limit of high frequency perturbations about flat space, the ADM condition (35) which can be written as

\[
A = \frac{\partial \ln \alpha}{\partial \varphi_{ij}} \tilde{v}_i v_j > 0
\]

(117)

is equivalent to that of BSSN, equation (104). We remind that

\[
\tilde{\gamma}_{ij} = e^{-4\varphi} \gamma_{ij} \quad \text{and} \quad \varphi = \frac{1}{12} \ln |\gamma|.
\]

(118)

Now using that \( d \gamma = \gamma \tilde{\gamma}_{ij} d \gamma_{ij} \), we obtain

\[
\frac{\partial \varphi}{\partial \tilde{\gamma}_{ij}} = \frac{1}{12} \tilde{\gamma}_{ij}
\]

(119)
and

\[
\frac{\partial \tilde{\gamma}_{km}}{\partial \gamma_{ij}} = e^{-4\varphi} \delta_k^i \delta_m^j - \gamma_{km} e^{-4\varphi} \frac{1}{12} \gamma^{ij}. \tag{120}
\]

Since,

\[
\frac{\partial \ln \alpha}{\partial \gamma_{ij}} = \frac{\partial \ln \alpha}{\partial \varphi} \frac{\partial \varphi}{\partial \gamma_{ij}} + \frac{\partial \ln \alpha}{\partial \tilde{\gamma}_{km}} \frac{\partial \tilde{\gamma}_{km}}{\partial \gamma_{ij}},
\]

we obtain:

\[
\frac{\partial \ln \alpha}{\partial \gamma_{ij}} = \Delta \frac{1}{12} \gamma^{ij} + e^{-4\varphi} \Delta^{ij} - e^{-4\varphi} \Delta \frac{\delta_{km}}{3} \delta^{ij}. \tag{122}
\]

However, we are considering flat space so the latter becomes

\[
\frac{\partial \ln \alpha}{\partial \gamma_{ij}} = \frac{1}{12} \Delta^{ij} - \Delta \frac{\delta_{km}}{3} \delta^{ij}. \tag{123}
\]

Therefore, if we consider that the vector along which we perturb is unit, the strong hyperbolicity condition of the ADM formulation yields:

\[
\frac{\partial \ln \alpha}{\partial \gamma_{ij}} v_i v_j = \frac{1}{12} (\Delta + 12 \Delta^{ij} v_i v_j - 4 \delta_{km} \delta_{jm}) > 0 \tag{124}
\]

This last one is the same as condition (109).

If one considers gauges for which the lapse function depends on the determinant of the three-metric and/or spacetime coordinates, as we have already shown the condition for well-posedness, reduces to

\[
A = \frac{\partial \ln \alpha}{\partial \gamma} = \frac{1}{12} \Delta = \frac{1}{12} \frac{\partial \ln \alpha}{\partial \varphi} > 0. \tag{125}
\]

which for both the ADM and BSSN language depends on the functional form of the lapse only.

\[\text{V. DISCUSSION AND CONCLUSIONS}\]

We have presented a general theory to study the well-posedness of constrained evolution of systems of quasi linear partial differential equations, which consist of a set of \(\mathcal{A}\) first order evolution equations and a set of \(\mathcal{M}\) first order constraint equations with \(m < n\). We applied this theory to constrained evolution of 3+1 formulations of GR. In our analysis we explicitly took into account the Hamiltonian and momentum constraints as well as possible algebraic constraints on the evolution of high-frequency perturbations of solutions of Einstein’s equations. Our analysis revealed the existence of subsets of the linearized Einstein’s equations that control the well-posedness of constrained evolution.

We demonstrated that the well-posedness of ADM and 3+1 formulations derived from ADM by adding combinations of constraints to the right-hand-side (RHS) of ADM and/or by a linear transformation of the dynamical ADM variables, depends entirely on the properties of the gauge and are equivalent to ADM on the surface on constraints. We note that our method concerns the constraint satisfying modes only. Those are present in free evolution schemes, too. Therefore, a bad choice of gauge, which we define as one that produces ill-posed constrained evolutions, is bad for a free evolution, as well. However, a good choice of gauge for a constrained evolution scheme cannot in principle guarantee the well-posedness of a free evolution with the same gauge, due to the existence of constraint violating modes.

Even on the surface of constraints we do not expect that all 3+1 formulation of GR which are derived from ADM by non-linear transformations and addition of extra variables to have equivalent well-posedness properties when using the same gauges. For example the analysis of the exponential stretch rotation (ESR) formulation [26], which is derived by a general non-linear exponential transformation of the ADM variables, shows that in the simplest case of geodesic slicing its behavior is elliptic, whereas ADM with the same gauge is weakly hyperbolic. The analysis of ESR will be the subject of a future paper.

In this paper we also analyzed the BSSN 3+1 formulation which is derived by a non-linear transformation of the ADM variables and addition of extra dynamical variables. We were able to show that the well-posedness properties of BSSN and ADM on the surface of constraints are similar for fixed and algebraic gauges. The results seem to indicate that, in general, the non-linear transformation of variables leading from ADM to BSSN does not change the well-posedness properties of the constrained evolution when the same gauge is used. However, the proof that the fully constrained evolution of BSSN is well-posed if and only if the constrained evolution of ADM is well-posed, if such proof exists, is out of the scope of this paper.

Our study shows that fixed gauges, that is, when the lapse function and the shift vector depend only on the spacetime coordinates, result in an ill-posed Cauchy problem for the constrained evolution of both ADM and BSSN as well as many other 3+1 formulations of GR. Algebraic gauges on the other hand can give rise to a well-posed constrained evolution provided that they satisfy (35), (36) or (104). In particular, fixed shift with the “harmonic” and “1 + log” slicing conditions, as well as with a densitized lapse having \(\sigma > 0\) are all well behaved gauges. Our study of the Bona-Masso family of hyperbolic slicing conditions showed that it provides us with a well-posed constrained evolution. The study of well-posedness of constrained evolution with maximal slicing and fixed shift shows that it depends on the way this gauge is implemented. The algebraic implementation \(\gamma^{ij} K_{ij} = 0\) leads to a well-posed evolution whereas the often used differential implementation (49) or (50) is ill-posed. The parabolic extension of maximal slicing with fixed shift leads to an ill-posed evolution. Finally, we demonstrated evidence that the constrained evolution in conjunction with the ”\(T\)-freezing” shift condition and an algebraic lapse leads to a well behaved constrained
evolution at least in the case of 1D perturbations about flat space. However, a complete well-posedness analysis is still required and this will be a subject of a future paper.

Our analysis demonstrates that the weak hyperbolicity associated with fixed gauges is directly related to the inertial deformation of a synchronous reference frame with time, which, in a general non-linear case when the perturbations are not small, leads to the formation of caustics (see equation (42) and discussion following it).

Finally, we note that gauge stability may be investigated more directly by considering variations of gauge degrees of freedom only. In general, this requires the analysis of a system of eight quasi-linear partial differential equations presented in [13]. The main advantage of the method outlined in this paper is that, in addition to gauge conditions, it provides us with subsets which control the constrained evolution of spacetime. The method is also able to provide sufficient conditions of well-posedness, whereas the analysis in [13] gives only the necessary conditions. The subsets controlling the constrained evolution can be used for construction of stable numerical schemes for 3+1 formulations of GR. This will be the subject of our future paper.

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APPENDIX A: STRONG HYPERBOLICITY FOR CONSTRAINED EVOLUTION

In this appendix we show that for uniformly diagonalizable systems there exists a uniformly bounded similarity transformation Ŝ which diagonalizes the principal matrix Ā, provided that the following two conditions are met: 1) Ā has real eigenvalues, which are analytic functions of v_k, and 2) The elements of Ā can be represented as ratios of analytic functions of v_k.

To study well-posedness of a constrained evolution as outlined in this paper we use the m linearized constraint equations (13) to eliminate some of the dynamical variables. Equations (13) are an under-determined set of m algebraic equations for the spatial derivatives of the n unknown variables, which in general can be solved for m of the n spatial derivatives of variables u. We can write equations (13) schematically as follows

$$\hat{C}(v_i) \frac{\partial u_m}{\partial \lambda} + \hat{F}_m(v_i, \frac{\partial u_q}{\partial \lambda}) = 0,$$

(A1)

where u_m is a column vector of m of the n dynamical variables of the formulation, matrix \( \hat{C} \) is m x m and depends on the direction v_i along which we perturb, and \( \hat{F}_m(v_i, \frac{\partial u_q}{\partial \lambda}) \) is a column vector with m components which are functions of the direction v_i and the spatial derivatives of the q = n - m dynamical variables left, and solve them as

$$\frac{\partial u_m}{\partial \lambda} = -(\hat{C})^{-1} \hat{F}_m(v_i, \frac{\partial u_q}{\partial \lambda}).$$

(A2)

Substitution of (A2) in equations (12) leads to a set of q = n - m linear partial differential equations for q of the initial n variables, which is schematically given by (14). This elimination process includes inversion of \( \hat{C}(v_i) \), and thus division by its determinant, which is a polynomial in the components of the unit one form v_i. This may not be possible for every direction because there may be directions which make matrix \( \hat{C} \) singular. In order to obtain the minimal set, the determinant |\( \hat{C} \)| has to be non-vanishing. The domain of a minimal set consists of all directions for which |\( \hat{C} \)| ≠ 0. For singular directions v_s we must use another set of m dynamical variables for which |\( \hat{C} \)| ≠ 0 in the singular direction v_s. It can be shown that this is always possible for the GR equations.

A transformation matrix S(v_i) which diagonalizes Ā in (14) has the same domain as Ā and has non-zero determinant in its domain. However, when we approach a singular direction, the determinant of S or its inverse may tend to zero (or infinity) and then (9) may not be satisfied. However, the choice of eigenvectors and the corresponding transformation matrix Ŝ are not unique. The eigenvectors can be rescaled and this will change Ŝ.

The systems analyzed in this paper for which the eigenvalues are real and for which there exists a complete set of eigenvectors for all directions, we find that the eigenvalues are analytic functions of v_k (see (33), (34), (76)). For such systems we show below that it is always possible to rescale the eigenvectors in such a way that all rescaled eigenvectors will be analytic functions of v_k. Then, according to [23] the transformation matrix Ŝ will satisfy (9) and thus the system will be strongly hyperbolic and by definition well-posed.

First consider matrix Ā. Its coefficients may be ratios of polynomials due to substitution of constraints. This is the case with the Einstein equations and all gauges we have studied in this paper. We write this schematically as

$$\left( \hat{A}(v_k) \right)_{ij} = \frac{p_{ij}(v_k)}{q_{ij}(v_k)} = \left( \frac{p_{11}}{q_{11}} \ldots \frac{p_{1q}}{q_{1q}} \right) \ldots \left( \frac{p_{11}}{q_{1q}} \ldots \frac{p_{1q}}{q_{1q}} \right),$$

(A3)

where \( p_{ij}(v_k) \) and \( q_{ij}(v_k) \) are polynomial (and hence analytic) functions of v_k. We further assume that Ā_q has
real eigenvalues and a complete set of eigenvectors for all possible directions \( v_k \) in its domain. This is the case for algebraic gauges and the Bona-Masso hyperbolic gauges considered in this paper.

Let \( \vec{V}_1 \) be a set of eigenvectors corresponding to the eigenvalues \( \omega_1 \). They satisfy

\[
(\hat{A}_q - \omega_1 \hat{I}_q) \vec{V}_1 = \hat{A}_i(v_k) \vec{V}_1 = 0,
\]

where \( \hat{I}_q \) is the \( q \times q \) identity matrix. The coefficients of newly defined matrices \( \hat{A}_i \) are ratios of analytic functions because the eigenvalues of \( \hat{A}_q \) are analytic functions.

\[
(\hat{A}(v_k))_{ij} = \frac{P_{ij}(v_k)}{q_{ij}(v_k)} = \left( \begin{array}{c} \frac{p_{11}}{q_{11}} \; v_1 + \frac{p_{12}}{q_{12}} \; v_2 + \ldots + \frac{p_{1q}}{q_{1q}} \; v_q \\ \ldots \\ \frac{p_{q1}}{q_{q1}} \; v_1 + \frac{p_{q2}}{q_{q2}} \; v_2 + \ldots + \frac{p_{qq}}{q_{qq}} \; v_q \end{array} \right).
\]

Consider now a particular eigenvalue, e.g., \( \omega_1 \) and a particular eigenvector \( \vec{V} \). Dropping the subscript 1 we can write (A4) as

\[
\begin{align*}
\frac{P_{11}}{q_{11}} & \; v_1 + \frac{P_{12}}{q_{12}} \; v_2 + \ldots + \frac{P_{1q}}{q_{1q}} \; v_q = 0 \\
\frac{P_{q1}}{q_{q1}} & \; v_1 + \frac{P_{q2}}{q_{q2}} \; v_2 + \ldots + \frac{P_{qq}}{q_{qq}} \; v_q = 0
\end{align*}
\]

(A6)

where subscripts of \( V \) indicate components of a particular eigenvector \( \vec{V} = \vec{V}_1 \) we consider. If the rank of the matrix \( \hat{A}_q \) is \( r \), where \( r < q \) then there are only \( r \) linearly independent equations in (A6). This means that we can solve the algebraic set for \( r \) of the \( q \) components of the eigenvector \( \vec{V} \), which we denote by \( \vec{V}_r \). The remaining \( s = q - r \) components of \( \vec{V} \) form a vector \( \vec{V}_s \). We write the reduced system schematically as

\[
\hat{D}_r \vec{V}_r = \hat{D}_s \vec{V}_s = \vec{B}_r,
\]

(A7)

where \( \hat{D}_r \) is a \( r \times r \) matrix and \( \hat{D}_s \) is a \( r \times s \) matrix, which are both formed by elements of \( \hat{A}_q \). Therefore, both \( \hat{D}_r \) and \( \hat{D}_s \) have elements which are ratios of analytic functions of \( v_k \). The non-zero components of \( \vec{V}_s \) can be chosen freely, for example as constants, so that components of \( \vec{B}_r \) will be ratios of analytic functions in \( v_k \). By Cramer’s rule the solution of system (A7) for the \( r \) unknown eigenvector components is

\[
(\vec{V}_r)_i = \frac{|(\vec{D}_r)_i|}{|\vec{D}_r|},
\]

(A8)

where \( (\vec{V}_r)_i \) is the \( i \)-th component of \( \vec{V}_r \) and \( (\vec{D}_r)_i \) is the matrix formed by replacing the \( i \)-th column of \( \vec{D}_r \) by the column vector \( \vec{B}_r \). Therefore, all components of the eigenvector \( \vec{V} \) can be expressed as ratios of analytic functions, since the sums and products of analytic functions are analytic, which we write schematically as

\[
(\vec{V}_r)_i = \frac{P_i(v_k)}{Q_i(v_k)},
\]

(A9)

If we multiply the eigenvector \( \vec{V} \) by the product \( Q = \Pi_i Q_i(v_k) \) the rescaled eigenvector will be an analytic function of \( v_k \). We can carry out such rescaling for all eigenvectors. Then according to [23] (A1) is strongly hyperbolic because \( \hat{A} \) has real eigenvalues and a complete set of eigenvectors which are analytic functions of \( v_k \) for any direction.

We now illustrate an example of rescaling that leads to a uniformly bounded \( \hat{S} \). Consider perturbations in the \( x^1, x^2 \) plane about flat spacetime for the ADM + densitized lapse system \( \gamma = C(x) \gamma^0 \), for which the components \( v_1 \) and \( v_2 \), of the unit one-form along which we perturb, are assumed non-vanishing. For this simple case \( q=30-22=8 \). The linearized evolution equations of a minimal set are in matrix notation:

\[
\frac{\partial \vec{u}}{\partial t} = A_8 \frac{\partial \vec{u}}{\partial \lambda},
\]

(A10)

where

\[
\vec{a}^T = \left( K_{11} \; K_{23} \; K_{33} \; D_{11;1} \; D_{13;1} \; D_{22;1} \; D_{23;1} \; D_{33;1} \right)
\]

(A11)

and

\[
A_8 = \left( \begin{array}{cccccccc}
0 & 0 & 0 & -\alpha v_1 \sigma & 0 & -\alpha v_1 \sigma & 0 & \alpha [1 - v_1^2 (2 \sigma + 1)] \\
0 & 0 & 0 & 0 & \alpha v_1^2 & 0 & -2 \alpha v_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\alpha v_1 & 0 \\
-2 \alpha v_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 \alpha v_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 \alpha v_1^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -2 \alpha v_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 \alpha v_1 & 0 & 0 & 0 & 0 & 0 \\
\end{array} \right),
\]

(A12)

where \( \vec{a}^T \) is the transpose of \( \vec{a} \). When \( \sigma > 0 \), this matrix always has real eigenvalues.
\{0, 0, -\alpha, \alpha, -\alpha, \alpha, -\alpha\sqrt{2\sigma}, \alpha\sqrt{2\sigma}\} and complete set of eigenvectors in its domain, i.e. \( v_1 \neq 0, v_2 \neq 0 \). This is also the domain of the matrix \((S)\) of eigenvectors of \(A_8\) as columns, which is a matrix that diagonalizes \(A_8\) via a similarity transformation. The determinant of this matrix turns out to be:

\[ |S| = -\frac{\sqrt{2\sigma}}{v_1^2 v_2^2} S. \quad (A13) \]

As we approach the singular points \( v_1 = 0, v_2 = 0 \) this determinant blows up and one cannot obtain an upper bound to satisfy (9). However, if we define a new matrix (within the same domain) by

\[ \tilde{S} = v_1^{2/8} v_2^{5/8} S, \quad (A14) \]

the determinant of this new matrix is

\[ |\tilde{S}| = -\sqrt{2\sigma}. \quad (A15) \]

This is a well behaved non-singular transformation that satisfies (9).

\[ \frac{\partial \delta A_k}{\partial t} = \frac{1}{\epsilon} \left[ \frac{1}{v_1} v_1 \gamma_i^{\ell} \gamma_j^{\ell} v_k v_\ell \frac{\partial \delta A_1}{\partial \lambda} - \frac{2}{v_1} v_1 \gamma_i^{\ell} \gamma_j^{\ell} v_k \partial D_{m;j} - \frac{\partial D_{m;i}}{\partial \lambda} \right] + v_1 \gamma_i^{\ell} v_k \frac{\partial \delta K_{ij}}{\partial \lambda}. \quad (B1) \]

Now define \( K_{ij} \equiv K_{ij} + \epsilon v_1 v_k v_j \alpha \). Then equations (59)-(62) yield for the evolution of high frequency perturbations of \(\alpha, K_{ij}\) and \(D_{ij;k}\):

\[ \frac{\partial \delta \alpha}{\partial t} = \frac{1}{\epsilon} \frac{\partial \delta A_1}{\partial \lambda}, \quad (B2) \]

\[ \frac{\partial \delta K_{ij}}{\partial t} = \alpha \frac{\partial \delta K_{ij}}{\partial \lambda}, \quad (B3) \]

\[ \frac{\partial \delta D_{ij;k}}{\partial t} = -2\alpha \frac{\partial \delta K_{ij}}{\partial \lambda}. \]

The momentum constraints (23) have exactly the same form for this newly defined “extrinsic curvature” variable \(K_{ij}\) in the limit of high frequency perturbations, i.e.

\[ \gamma^{mx} v_x \frac{\partial \delta K_{mi}}{\partial \lambda} - v_1 \gamma^{mx} \frac{\partial \delta K_{mn}}{\partial \lambda} = 0. \quad (B4) \]

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