Cosmological dynamics of ‘exponential gravity’

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Received 7 December 2007, in final form 10 April 2008
Published 11 June 2008
Online at stacks.iop.org/CQG/25/135002

Abstract
We present a detailed investigation of the cosmological dynamics based on \(\exp(-R/A)\) gravity. We apply the dynamical system approach to both the vacuum and matter cases and obtain exact solutions and their stability in the finite and asymptotic regimes. The results show that cosmic histories exist which admit a double de Sitter phase which could be useful for describing the early- and the late-time accelerating universe.

PACS number: 04.50.Kd

1. Introduction

In recent years several observational surveys appear to show that the universe is currently undergoing an accelerated expansion phase (for a review, see [1]). On the other hand, astrophysical observations on galactic scales gave us a clear indication that the amount of the luminous matter is not enough to account for the rotation curves of galaxies. These remarkable observations have radically changed our ideas about the evolution of the universe. In fact, the only way in which this behavior can be explained within the standard Friedmann general relativistic cosmology framework is to invoke two dark matter components, one with a conventional dust equation of state (\(w = 0\)) needed to fit astrophysical data on galactic scales (dark matter) and the other with a more exotic equation of state (\(w < -\frac{1}{3}\)) needed to explain the current accelerated expansion phase of the universe (dark energy). There is little doubt that unraveling the nature of dark matter and dark energy is currently one of the most important problems in theoretical physics.

One approach to the problem of dark energy that has received considerable attention in recent years is the modification of general relativity on cosmological scales. In this approach one supposes that dark energy is a manifestation of a non-Einsteinian behavior of the gravitational interaction rather than a new form of energy density. The introduction of corrections to the Hilbert–Einstein action and their effects have been studied for long time and are believed to be unavoidable when the quantum nature of the universe is introduced in general relativity (GR).
In the recent years many different extended versions of the Einstein theory of gravity have been proposed. One of the most studied approaches is higher-order theories of gravity [13–42], in which the gravitational action is nonlinear in the Ricci curvature and/or its derivatives [6–8]. These theories have a number of interesting features on cosmological and astrophysical scales. In fact they are known to admit natural inflation phases [4] and to explain the flattening of the galactic rotation curves [9]. Another very interesting feature of these models is that the higher-order corrections to the Hilbert–Einstein action can be viewed as an effective fluid which can mimic the properties of dark energy [5].

One of the main problems that occurs in the study of higher-order theories of gravity is that the high degree of nonlinearity makes the derivation of exact cosmological solutions extremely difficult. This problem can be partially addressed using a suitable choice of generalized coordinates, together with the assumption of homogeneity. In this case the field equations can be written as a system of first-order autonomous differential equations together with a constraint equation [10]. In this way, we can exploit the methods of dynamical systems theory [2] in order to both understand the qualitative behavior of the cosmological dynamics and obtain special exact solutions of the cosmological equations. The general approach allowing one to analyze higher-order gravity with dynamical system techniques has been presented elsewhere (see [3]).

In this paper we will apply this approach to the homogeneous and isotropic Friedmann cosmologies, within a class of theories characterized by the action

\[ A = \int d^4x L = \int d^4x \sqrt{-g} \left[ e^{-\frac{R}{\Lambda_1}} + L_M \right], \]  

(1)

where \( \Lambda_1 \) is the cosmological constant, \( R \) is the Ricci scalar and \( L_M \) is the Lagrangian of standard matter. This action has a number of interesting features. Firstly, given the form of the Taylor expansion of the exponential function, its field equations are much simpler than those found for any other combination of terms of the type \( R^n \). This provides a relatively easy way of investigating some of the properties resulting from a combination of powers of the Ricci scalar. The information obtained from studying this model can then be used as a guide for the treatment of more popular models like \( R + \alpha R^2 \). Secondly, in the small curvature limit this action reduces to

\[ \exp \left( -\frac{R}{\Lambda_1} \right) = 1 - \frac{R}{\Lambda_1} + O[R^2], \]  

(2)

so in the small curvature limit this action reduces to the Hilbert–Einstein one (plus cosmological constant). This means that in every phase of a cosmic history in which the value of the Ricci scalar is small this theory is equivalent to a GR+\( \Lambda_1 \) model, although, as we will see, it also exhibits features typical of fourth-order gravity models.

This paper has been arranged as follows. In section 2, we present the basic equations of the model. In sections 3 and 4 we find exact solutions and their stability in the vacuum and matter cases respectively. Finally in section 5 we present a discussion of the results and present our conclusions.

In what follows we will use natural units (\( \hbar = c = k_B = 8\pi G = 1 \)) and the signature (+, −, −, −).

2. Basic equations

The general action for a fourth-order theory of gravity in a homogeneous isotropic spacetime is:

\[ A = \int d^4x \sqrt{-g} \left[ f(R) + L_M \right]. \]  

(3)
where $f(R)$ is a function of Ricci scalar $R$. Varying this action with respect to the metric gives

$$G_{\mu\nu} = T^{\text{TOT}}_{\mu\nu} = \tilde{T}^M_{\mu\nu} + T^R_{\mu\nu},$$

where $G_{\mu\nu}$ is the usual Einstein tensor, and

$$\tilde{T}^M_{\mu\nu} = \frac{1}{f'(R)} T^M_{\mu\nu}, \quad T^R_{\mu\nu} = \frac{1}{f'(R)} \left[ \frac{1}{2} g_{\mu\nu} (f(R) - R f'(R)) + f'(R) (g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\beta} g_{\mu\nu}) \right].$$

the prime denotes the derivative with respect to $R$, $T^M_{\mu\nu}$ is the energy–momentum tensor for standard matter, which is assumed to be a perfect fluid and $T^R_{\mu\nu}$ is the stress–energy tensor of an effective fluid (sometimes referred to as the curvature fluid), which represents the non-Einsteinian part of the gravitational interaction and constitutes an additional source term of purely geometrical origin [11]. By assuming $f(R) = \exp(-R/\Lambda)$ we obtain the field equations:

$$G_{\mu\nu} = -\Lambda e^{R/\Lambda} \tilde{T}^M_{\mu\nu} = e^{-R/\Lambda} \left[ \frac{1}{2} g_{\mu\nu} (\Lambda + R) + \frac{1}{\Lambda^2} (R^{\alpha\beta} R^{\mu\nu} + \Lambda R^{\alpha\beta}) (g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\beta} g_{\mu\nu}) \right].$$

In the case of the Friedmann–Lemaître–Robertson–Walker (FLRW) metric, the above equations reduce to:

$$H^2 + \frac{k}{a^2} - \frac{H}{\Lambda} \frac{\dot{R}}{a} + \frac{R}{6} + \frac{\Lambda}{6} + \frac{\Lambda \rho}{3e^{-R/\Lambda}} = 0,$$

$$\frac{\ddot{a}}{a} + \frac{\dot{R}}{3} - \frac{H}{\Lambda} + \frac{1}{\Lambda} \frac{\dot{R}^2}{a^2} - \frac{1}{\Lambda} \frac{\dot{R}}{a} - \frac{\Lambda \rho}{3e^{-R/\Lambda}} (1 + 3w) + \frac{\Lambda}{3} = 0,$$

with

$$R = -6 \left( \frac{\dot{a}}{a} + H^2 + \frac{k}{a^2} \right).$$

where $H = \frac{\dot{a}}{a}$ is the Hubble parameter, $a$ is the usual scale factor, $k$ is the spatial curvature, $\rho$ is the energy density of standard matter and $w$ is its barotropic factor. The Bianchi identities applied to the total stress–energy tensor $T^{\text{TOT}}_{\mu\nu}$ lead to the energy conservation equation for standard matter [2]:

$$\dot{\rho} + 3H \rho (1 + w) = 0.$$

3. The vacuum case

In the vacuum case ($\rho = 0$) equations (8) and (9) can be written as a closed system of first-order differential equations using the dimensionless variables:

$$x = \frac{\dot{R}}{\Lambda H}, \quad y = \frac{R}{6H^2}, \quad z = \frac{\Lambda}{6H^2}, \quad K = \frac{k}{a^2 H^2}.$$

Here the variables $y$ and $z$ are a measure of the expansion normalized Ricci curvature and the cosmological constant respectively, $K$ is the spatial curvature parameter of the Friedmann
model, while $x$ is a measure of the time rate of change of the Ricci curvature. The evolution equations for variables (11) are given by

\[
\begin{align*}
\frac{dx}{dN} &= \varepsilon(2z + 2K - 2) + x\varepsilon(1 + x + y + K), \\
\frac{dy}{dN} &= x\varepsilon z + 2y\varepsilon(2 + y + K), \\
\frac{dz}{dN} &= 2z\varepsilon(2 + y + K), \\
\frac{dK}{dN} &= 2K\varepsilon(y + 1 + K),
\end{align*}
\]

(12)

where the prime represents the derivatives with respect to the time variable $N = |\ln a|$ and $\varepsilon = |H|/H$. This system is completed with the Friedmann constraint,

\[1 + K + y + z - x = 0,\]

(13)

which defines a hyperplane in the total phase space of the system. Consequently, all solutions of the dynamical system will be located in a non-compact submanifold of the phase space associated with (12). The time derivative of (13) is nothing other than the Raychaudhuri equation.

3.1. Finite analysis

The dimensionality of the state space of system (12) can be reduced by eliminating any one of the four variables using the constraint equation (13). If we choose to eliminate $x$ the dynamical equations become:

\[
\begin{align*}
\frac{dy}{dN} &= y\varepsilon(4 + 2K + 2y + z) + z\varepsilon(1 + K + z), \\
\frac{dz}{dN} &= 2z\varepsilon(2 + K + y), \\
\frac{dK}{dN} &= 2K\varepsilon(1 + K + y),
\end{align*}
\]

(14)

System (14) admit two invariant submanifolds: $z = 0$ and $K = 0$. This means that the points contained in these submanifolds form an invariant set under the transformation defined by system (14), i.e. (14) sends points on these sets only to points of the same set. As a consequence, if we choose as an initial condition for an orbit $z = 0$ ($K = 0$) this orbit will never leave the plane $z = 0$ ($K = 0$), and for orbits with initial condition $z \neq 0$ ($K \neq 0$), the only way to smoothly approach $z = 0$ ($K = 0$) is to approach it asymptotically. This implies that no orbit crosses the $z = 0$ plane and consequently no global attractor can exist, because the phase space is divided into independent sectors which contain complete cosmological histories.

Setting $K' = 0$, $y' = 0$, $z' = 0$, we obtain four fixed points that we will label with capital letters and the subscript $v$ to indicate that are found for the vacuum case (see table 1).

We can obtain exact cosmological solutions at these points using the Raychaudhuri equation,

\[\ddot{H} = -(y + K + 2)H^2.\]

(15)

Of course we could have chosen to eliminate any other variable of the system; our choice is motivated by the fact that the equation for $x$ is by far the most complicated one to solve and that with this choice the number of invariant submanifolds is maximized. As we will see this will help in the investigation of the properties of the cosmology, particularly in the matter case.
Table 1. Coordinates of the fixed points, eigenvalues, stability and solutions for \( \exp(-\frac{R}{\Lambda_1}) \) gravity in vacuum.

| Point | Coordinates \((y, z, K)\) | Eigenvalues | Stability      | Solution                      |
|-------|---------------------------|-------------|----------------|-------------------------------|
| \(A_v\) | \([0, 0, 0]\) | \([2, 4, 4]\) | Repeller       | \(a = a_0(t - t_0)^2\)          |
| \(B_v\) | \([0, 0, 1]\) | \([-2, 2, 2]\) | Saddle         | \(a = a_0(t - t_0)\)            |
| \(C_v\) | \([-2, 0, 0]\) | \([-4, -2, 0]\) | Saddle-node    | \(a = a_0e^{y(t-t_0)}\)         |
| \(D_v\) | \([-2, 1, 0]\) | \([-\frac{3+\sqrt{17}}{2}, -2, \frac{3+\sqrt{17}}{2}]\) | Saddle       | \(a = a_0e^{y(t-t_0)}\)         |

In fact, at any fixed point with \(y + K + 2 \neq 0\), equation (15) reduces to

\[
\dot{H} = -\frac{1}{\alpha}H^2, \quad \alpha = (y + K + 2)^{-1},
\]

where the subscript ‘*’ indicates that a quantity has been calculated at the fixed point. Equation (15) applies to both the matter and vacuum cases and describes a general power-law evolution of the scale factor. In addition, integrating with respect to time we obtain

\[
a = a_0(t - t_0)^\alpha, \quad \alpha \neq 0.
\]

This means that by finding the value of \(\alpha\) at a given fixed point, we can obtain the solutions associated with it using equation (15), and this solution will be given by (17) for \(\alpha \neq 0\).

In this way, points \(A_v\) and \(B_v\) are found to represent Milne and power-law evolutions respectively (see table 1). However, by direct substitution into the cosmological equations it can be shown that these fixed points cannot be considered physical because in order to satisfy (8), (9) one needs to violate the weak energy condition \(\rho \geq 0\). This does not constitute a problem because, as we will see below these points are always unstable, which means that we can choose initial conditions as close as we want to these points.

For the points \(C_v\) and \(D_v\) we have \(\alpha = 0\) so that (15) reduces to \(\dot{H} = 0\) and the scale factor is given by

\[
a = a_0e^{y(t-t_0)}. \quad (18)
\]

The value of the constant \(y\) can be obtained by direct substitution into equations (8) and (9). For both \(C_v\) and \(D_v\) we obtain

\[
y = \pm \sqrt{\frac{\Lambda}{6}}, \quad (19)
\]

so they represent an exponential evolution. The contracting or expanding nature of this solution depends on the direction of approach of the orbits with respect to the hypersurface \(y + K + 2 = 0\). This hypersurface divides the phase space into two hypervolumes characterized by a contracting or expanding evolution. In particular, for \(y < -K - 2\) the orbits describe a contracting universe, while for \(y > -K - 2\) they represent an expanding one.

The stability of the hyperbolic fixed points \(A_v\), \(B_v\) and \(D_v\) is obtained by using Hartman–Grobman theorem. The point \(C_v\), instead, is non-hyperbolic and we have to use the local center manifold theorem to find its stability. A brief review of this theorem can be found in [12] (see also appendix A.1). In our case, using the transformation

\[
\begin{align*}
y &= u_1 - 2u_2 - m, \\
z &= 4m, \\
k &= u_2
\end{align*}
\]
system (14) can be written in the form
\begin{align}
\dot{u}_1 &= -4u_1\varepsilon + \varepsilon(12m^2 + 2mu_1 + 2u_1^2 - 4mu_2 - 2u_1), \\
\dot{u}_2 &= -2u_2\varepsilon + \varepsilon(-2mu_2 + 2u_1u_2 - 2u_2^2), \\
\dot{m} &= \varepsilon(-2m^2 + 2mu_1 - 2mu_2),
\end{align}
where
\begin{align}
F_{s1}(u_1, u_2, m) &= \varepsilon(12m^2 + 2mu_1 + 2u_1^2 - 4mu_2 - 2u_1), \\
F_{s2}(u_1, u_2, m) &= \varepsilon(-2mu_2 + 2u_1u_2 - 2u_2^2), \\
F_m(u_1, u_2, m) &= \varepsilon(-2m^2 + 2mu_1 - 2mu_2),
\end{align}
represent the nonlinear terms. By substituting the expansions
\begin{align}
h_1(m) &= am^2 + bm^3 + O(m^4), \\
h_2(m) &= cm^2 + dm^3 + O(m^4)
\end{align}
into equations (A.3) and (A.4) and then solving for the coefficients $a$, $b$, $c$ and $d$, we obtain
\begin{align}
h_1(m) &= 3m^2 + \frac{9}{2}m^3 + O(m^4), \\
h_2(m) &= O(m^4).
\end{align}
Substituting this result into equation (23) then yields
\begin{align}
\dot{m} &= -2m^2 + O(m^3)
\end{align}
on the center manifold $W^c(0)$, around the point $C_v$. This implies that the point $C_v$ is a saddle-node, i.e. it behaves like a saddle or an attractor depending on the direction from which the orbit approaches. The local phase portrait in the neighborhood of $C_v$ is shown in figure 1.
Figure 2. The invariant submanifold $K = 0$ for $\exp(-R/\Lambda)$ gravity in vacuum.

If one now considers transformation (20), one realizes that $m \propto z$, so that $C_v$ is an attractor for $z > 0$ and a saddle for $z < 0$. This is also clear from figure 2 in which the invariant submanifold $K = 0$ is depicted.

Finally, it is useful to derive an expression for the deceleration parameter $q$ in terms of the dynamical variables:

$$q = -\frac{\dot{H}}{H^2} - 1 = -(y + K + 1).$$

(31)

This equation holds for both the vacuum and matter cases. Note that $q > 0$ is realized only when $(y + K + 1) < 0$. This condition is satisfied only for the point $C_v$ as expected by looking at the solution associated with this fixed point (see table [1]). In figure 3 we give the location of the $q = 0$ plane relative to the fixed points $A_v, C_v$ and $B_v$.

3.2. Asymptotic analysis

In this section we will determine the fixed points at infinity and study their stability. In order to simplify the asymptotic analysis we will compactify the phase space using the Poincaré method. Transforming to polar coordinates $(r, \theta, \phi)$:

$$z \to r \cos \theta, \quad K \to r \sin \theta \cos \phi, \quad y \to r \sin \theta \sin \phi$$

and substituting $r \to \frac{R}{1 - R}$, the regime $r \to \infty$ corresponds to $R \to 1$. Using this coordinate transformation and taking the limit $R \to 1$, system (14) can be written as

$$R' \to \frac{1}{2}(8 \cos \phi \sin \theta - \sin \theta[-7 \sin \theta + \sin 3\theta] 8A \cos^2 \theta \sin \theta + 4A \cos \theta \sin^2 \theta \sin \phi),$$

(32)

$$R\theta' \to -\frac{\varepsilon \cos^2 \theta \sin \phi \sin \theta}{R - 1} (A + \cot \theta),$$

(33)

$$R\phi' \to \frac{\varepsilon \cos \phi \cot \theta}{R - 1} (A + \cot \theta),$$

(34)
Figure 3. The invariant submanifold \( z = 0 \). We show explicitly the location of the \( q = 0 \) plane relative to the fixed points \( A_v, C_v \) and \( B_v \) for \( \exp(-R/\Lambda) \) gravity in vacuum.

Table 2. Coordinates, eigenvalues and the stability of the fixed points in the asymptotic regime for \( \exp(-R/\Lambda) \) gravity in vacuum. Here \( L_1 \) and \( L_2 \) are functions of \( \phi \) which are too complicated to be written here (see figure 4 for their plots), \( E = [2 + 2 \cot \phi_0 + \cot \theta \csc \phi_0 + A_0 \csc \phi_0^2(1 + A_0^{-1} \cos \phi_0)] \) and \( A_0 \) is the value of \( A \) in \( \phi_0 \).

| Point          | (\( \theta, \phi \))           | Eigenvalues                                                                 | Solution                                                                 |
|---------------|--------------------------------|----------------------------------------------------------------------------|--------------------------------------------------------------------------|
| \( I_\infty \) | \( [\pi/2, \phi] \)              | \( 0, -A \cos \phi \)                                                      | \( (N - N_\infty) = \left[ \frac{1}{2}(c_1 \pm c_0(t_1 - t_0)) \right]^2 \) |
| \( O_\infty \) | \( [-\arccot A, \phi] \)          | \( L_1(\phi) < 0 \forall \phi, L_2(\phi) > 0 \forall \phi \)             | \( (N - N_\infty) = \left[ \frac{E^{L_2}(c_1 \pm c_0(t_1 - t_0))}{E^{L_2}} \right]^{\pi/2} \) |

where \( A = \cos \phi + \sin \phi \). Since equation (32) does not depend on the coordinate \( \mathcal{R} \), we can find the fixed points of the above system using equations (33) and (34) only. From equations (33) and (34) if

\[
A + \cot \theta = 0, \tag{35}
\]

then \( \theta' = \phi' = 0 \), which means that \( O_\infty \) = \( [-\arccot A, \phi] \) represent a fixed subspace. The other fixed subspace is given by \( I_\infty = [\pi/2, \phi] \) and there are no single fixed points (see table 2).

Let us now derive the solution for \( I_\infty \). In this subspace the first equation of system (14) reduces to

\[
\frac{dy}{dN} = 2\epsilon y^2(\cot \phi_0 + 1), \quad \text{where} \quad \cot \phi_0 = K/y, \tag{36}
\]

and equation (15) becomes

\[
\dot{H} = -y(\cot \phi_0 + 1)H^2. \tag{37}
\]

Integrating equation (15) we obtain

\[
y = \frac{-1}{2\epsilon (1 + \cot \phi_0)(N - N_\infty)}, \tag{38}
\]
where $N_{\infty}$ is an integration constant. Substituting $y$ back into equation (37) and solving for $N$, we obtain

$$N - N_{\infty} = \left[ \frac{1}{2\epsilon} (c_1 \pm c_2(t - t_0)) \right]^2. \tag{39}$$

The same procedure can be used to obtain solutions for $O_{\infty}$ (see table 2 for the result).

Using (32) to take into account the radial behavior of the orbits, the stability of $I_{\infty}$ is:

- $-\pi/4 < \phi < \pi/2$ stable,
- $\pi/2 < \phi < 3\pi/4$ unstable,
- $3\pi/4 < \phi < 3\pi/2$ stable,
- $3\pi/2 < \phi < 7\pi/4$ unstable.

We obtain the stability of $O_{\infty}$ in the same way and it turns out that these points are never stable. For the values of $\phi$ for which $L_1(\phi)$ and $L_2(\phi)$ are both positive the points in $O_{\infty}$ are repellers; for the values of $\phi$ for which these functions have opposite signs they are saddles.

In the following section we will see how the introduction of matter modifies the picture we obtained in the vacuum case.

4. The matter case

In this case we can use the same dynamical variables we used for the vacuum case together with one additional variable $D$, that is related to the matter energy density:

$$x = \frac{\dot{R}}{\Lambda H}, \quad y = \frac{R}{6H^2}, \quad z = \frac{\Lambda}{6H^2}, \quad K = \frac{k}{a^2H^2}, \quad D = \frac{\Lambda \rho}{3H^2 e^{R/H}}. \tag{40}$$

The definition of the variables reveals that not all of the phase space corresponds to physical situations. This becomes clear if we divide $D$ by $z$. We obtain

$$\frac{D}{z} = 2\rho \exp\left(\frac{-R}{\Lambda}\right). \tag{41}$$
which has the same sign as $\rho$. This means that the sectors in the phase space for which the sign of $D$ is different from the sign of $z$ contain orbits in which standard matter violates the weak energy condition $\rho > 0$, and have to be discarded as not physical. As we will see this implies restrictions on the cosmic evolution scenarios allowed in this model.

Following the same procedure we used in the vacuum case, we obtain an autonomous system equivalent to the cosmological equations with non-zero matter density:

$$\begin{align*}
\frac{dx}{dN} &= 2\varepsilon(z - K - 1) + x\varepsilon(1 + x + y + K) - D\varepsilon(1 + 3w), \\
\frac{dy}{dN} &= xz\varepsilon + 2y\varepsilon(2 + y + K), \\
\frac{dz}{dN} &= 2z\varepsilon(2 + y + K), \\
\frac{dK}{dN} &= 2K\varepsilon(y + 1 + K), \\
\frac{dD}{dN} &= D\varepsilon(1 - 3w + 2y + 2K + x),
\end{align*}$$

(42)

(together with the constraint equation

$$1 + K - x + y + z + D = 0,$$

(43)

where the prime again denotes the derivative with respect to the logarithmic time variable $N$.

4.1. Finite analysis

System (42) can be further simplified using the constraint equation (43) to eliminate $x$. We obtain:

$$\begin{align*}
\frac{dy}{dN} &= y\varepsilon(4 + 2K + 2y + z) + z\varepsilon(1 + K + D + z), \\
\frac{dK}{dN} &= 2K\varepsilon(1 + K + y), \\
\frac{dz}{dN} &= 2z\varepsilon(2 + y + K), \\
\frac{dD}{dN} &= D\varepsilon(2 - 3w + 3K + D + 3y + z).
\end{align*}$$

(44)

The structure of (44) reveals that in this case we have three invariant submanifolds: $K = 0, z = 0$ and $D = 0$, so also in this case no global attractor can exist. Setting $K' = 0, y' = 0, z' = 0$ and $D' = 0$ we obtain seven fixed points (see table 3).

As in the vacuum case, we can use the coordinates of these fixed points and equation (15) to find the behavior of the scale factor at these points. In addition, the behavior of the energy density $\rho$ can be obtained from equation (10), which at a fixed point reads

$$\dot{\rho} = -3(1 + w)$$

(45)

where $\alpha$ is defined by (16). However, direct substitution in the cosmological equations reveals that all the fixed points correspond to vacuum states.

The exact solutions at the fixed points are summarized in table 3. As in the vacuum case, we use the Hartman–Grobman theorem together with the center manifold theorem to analyze the stability of all the fixed points. The results are shown in table 4.
4.2. Asymptotic analysis

We complete the analysis for the matter case by investigating the asymptotic behavior of system (44). In order to achieve this we compactify the phase space by transforming to 4D polar coordinates. The transformation equations are

\[ D \rightarrow r \cos \delta, \quad z \rightarrow r \sin \delta \cos \theta, \]
\[ K \rightarrow r \sin \delta \sin \theta \cos \phi, \quad y \rightarrow r \sin \theta \sin \delta \sin \phi, \]

where \( r \in [0, \infty], \delta \in [0, \pi], \theta \in [0, \pi] \) and \( \phi \in [0, 2\pi] \). We then transform the radial coordinate \( r \rightarrow \frac{R}{\sqrt{\delta}} \) and in the limit \( R \rightarrow 1 \), system (44) reduces to

\[ \mathcal{R}' \rightarrow \cos^3 \delta + \cos \delta \cos \theta \sin^2 \delta \sin \phi + \cos^2 \delta \sin \delta (\cos \theta + 3 \sin \theta) \]
\[ + \sin^3 \delta \sin \theta [\cos \phi (2 + B) + \sin \phi (3 \cos^2 \theta + 2 \sin^2 \theta + B)], \quad \text{(46)} \]
\[ \mathcal{R} \delta' \rightarrow \frac{\sin \delta \cos \delta}{8(R - 1)} [8 \cos \delta (B - 1) - \sin \delta \cos 3 \theta + 8 \cos \phi \sin \theta \]
\[ + 8 \sin^3 \theta \sin \phi + 7 \cos \theta + 4 \cos \theta \sin^2 \phi (\cos 2 \phi - \sin 2 \phi)]]], \quad \text{(47)} \]
\[ \mathcal{R} \theta' \rightarrow \frac{\cos^2 \theta}{2(R - 1)} [2 \cos \delta \sin \phi + \sin \delta [2 \cos \theta \sin \phi + \sin \theta (1 - \cos 2 \phi + \sin 2 \phi)]], \quad \text{(48)} \]
1.0 

Figure 5. The graph of the function $f_1(\theta)$.

Table 5. Coordinates, eigenvalues and the solutions for fixed points in the asymptotic regime for the $\exp(-R/A)$ gravity in the matter case. Here $f_1$ and $f_2$ are functions of $\theta$ while $g$ is a function of $\phi$ (see figures 5 and 6). $S = \{2 + \cot \theta + \cot^2 \theta(1 + \cot \delta \sec \theta)\}$ and $\tilde{S} = \{-2 + \cot \theta - \cot^2 \theta(1 + \cot \delta \sec \theta)\}$.

| Point | $(\delta, \theta, \phi)$ | Eigenvalues | Solution |
|-------|--------------------------|-------------|----------|
| $A_m^\infty$ | $\arccot(-\sin \theta - \cos \theta), \theta, \frac{\pi}{4}$ | $[0, 0, f_1(\theta)]$ | $(N-N_m) = \left\{ \left[ \frac{1}{2}(c_1 \pm c_0(t_1 - t_0)) \right] \right\}$ |
| $B_m^\infty$ | $\arccot(\sin \theta - \cos \theta), \theta, \frac{\pi}{4}$ | $[0, 0, f_2(\theta)]$ | $(N-N_m) = \left\{ \left[ \frac{1}{2}(c_1 \pm c_0(t_1 - t_0)) \right] \right\}$ |
| $C_m^\infty$ | $\arccot(-\Lambda), \frac{3\pi}{4}, \phi$ | $[0, 0, g(\phi) > 0 \ \forall \phi]$ | $(N-N_m) = \left\{ \left[ \frac{1}{2}(c_1 \pm c_0(t_1 - t_0)) \right] \right\}$ |
| $D_m^\infty$ | $\arccot(\Lambda), \frac{3\pi}{4}, \phi$ | $[0, 0, g(\phi) > 0 \ \forall \phi]$ | $(N-N_m) = \left\{ \left[ \frac{1}{2}(c_1 \pm c_0(t_1 - t_0)) \right] \right\}$ |
| $E_m^\infty$ | $\arccot(-\cos \theta), \theta, \frac{3\pi}{4}$ | $[0, 0, 0]$ | $a = e^{\epsilon(\gamma - \delta)}$ |

where $B = \cos \theta \sin \theta \sin \phi$. Note that the first equation of the previous system does not depend on $\mathcal{R}$, which means that the fixed points of this system can be determined by the angular equations alone. Like in the vacuum case there are no isolated fixed points (see table 5).

The solutions at the fixed points can be obtained by following the same procedure we used in the vacuum case.

Taking into account the radial behavior of the orbits we can deduce the stability of the first two fixed subspaces. We have that $A_m^\infty$ and $B_m^\infty$ attractors for $0 < \theta < 3\pi/4$ and $0 < \theta < \pi/4$ respectively. The fixed subspaces $C_m^\infty$ and $D_m^\infty$ are unstable for all $\phi$ (see figure 7).

The stability analysis of the subspace $E_m^\infty$ is complicated by the fact that the eigenvalues are all zero. This means that the system for these points is not structurally stable or, in other words, that their stability is determined mainly by the nonlinear terms. One can get an idea of the stability of $E_m^\infty$ by studying second-order small perturbations around this fixed subspace. To second order the evolution equations for the perturbations $(\delta', \tilde{\delta'}, \phi', \tilde{\phi'})$ around $E_m^\infty$ are

$$
R \phi' \rightarrow \frac{\cos \delta \cot \theta + \cos \theta \sin \delta [A + \cot \theta]}{R - 1},
$$

(49)

where $B = \cos \theta \sin \theta \sin \phi$. Note that the first equation of the previous system does not depend on $\mathcal{R}$, which means that the fixed points of this system can be determined by the angular equations alone. Like in the vacuum case there are no isolated fixed points (see table 5).

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$$
\delta' = \epsilon (a_1 \delta + a_2 \tilde{\delta}^2 + a_3 \tilde{\delta} \phi + a_4 \tilde{\phi} + a_5 \tilde{\phi}^2 + a_6 \phi + a_7 \phi^2 + a_8 \phi \tilde{\phi} + a_9 \phi \tilde{\phi} \tilde{\phi}),
$$

$$
\tilde{\delta}' = \epsilon (b_1 \tilde{\delta} + b_2 \delta \phi + b_3 \delta \phi + b_4 \tilde{\phi} + b_5 \tilde{\phi} + b_6 \phi + b_7 \phi \tilde{\phi} + b_8 \phi \tilde{\phi} \tilde{\phi}),
$$

$$
\phi' = \epsilon (c_1 \delta + c_2 \phi + c_3 \tilde{\phi} + c_4 \phi \tilde{\phi} + c_5 \phi \tilde{\phi} \tilde{\phi} + c_6 \phi \tilde{\phi} + c_7 \phi \tilde{\phi} \tilde{\phi} + c_8 \phi \tilde{\phi} \tilde{\phi} + c_9 \phi \tilde{\phi} \tilde{\phi} \tilde{\phi}),
$$

(50)

$$
\bar{\mathcal{R}}' = \epsilon (d_1 \delta^2 + \theta d_2 \delta + \phi d_3 \delta + \phi d_4 + \phi d_5 + \phi d_6 + \phi d_7 \phi + \phi d_8 \phi + \phi d_9 \phi + \delta d_2 + \delta d_3 + \delta d_4).
$$
where the coefficients are shown in appendix A.2. The system above is very difficult to solve exactly, so we will limit ourselves to plotting the solutions for specific values of $\theta$ and to deduce the stability from them. Figure 8 shows the solutions of the perturbed system (50) for specific values of $\theta$ (i.e. $\theta = \pi/6$, $\pi/4$, $\pi/3$, $3\pi/4$ and $5\pi/6$), it is clear from the graphs that all the solutions are unstable.

5. Discussion and conclusions

In this paper we have applied the dynamical system approach to the exponential gravity cosmological model, and found exact solutions together with their stability for both the vacuum and matter cases.

In the vacuum case we found four finite critical points $A_v$, $B_v$, $C_v$ and $D_v$, of which only two $C_v$ and $D_v$ are found to be physical. These last points were found to represent a solution whose nature depends on the parameter $\gamma(\Lambda)$; for $\Lambda > 0$ we can have either exponential expansion ($\gamma > 0$) or exponential contraction ($\gamma < 0$) and for $\Lambda < 0$ the solution oscillates.

From the stability point of view, the point $C_v$, which sits in the invariant submanifold $z = 0$, is of particular interest because, since it is non-hyperbolic, it represents an attractor for $z > 0$ and saddle for $z < 0$, while the other physical point $D_v$ is found to be a saddle.
On the other hand, the solution connected with the non-physical points $A_v$ and $B_v$ are found to correspond to power-law evolution and are also interesting because the orbits can approach arbitrarily close to them.

The invariant submanifold $z = 0$ divides the phase space into two regions, $z > 0$ and $z < 0$ which correspond to $\Lambda > 0$ and $\Lambda < 0$ respectively. The fact that no orbit can cross the plane $z = 0$ is then consistent with the fact that $\Lambda$ has to have always the same sign during a cosmic history.

In the vacuum case, we found that the region $z < 0$ does not contain any finite critical point. Thus, the only attractors in the region $z < 0$ are asymptotic, which means that all the models that begin their evolution in this region will re-collapse. However, in the plane $z = 0$, we have the physical point $C_v$ which represents a saddle and the non-physical points $A_v$ and $B_v$ are a repeller and saddle point respectively. This means that during the evolution toward the asymptotic attractors a transient de Sitter or a power-law phase(s) might be present depending on the initial conditions.

From a physical point of view, the region $z > 0$ appears to be more interesting because in this region the point $C_v$ represents a de Sitter attractor for which might be associated with a DE/inflation era. The same region also contains the point $D_v$, which represents an unstable
de Sitter phase (see figure 1). This implies that the subset of the orbits that converge to \( C_v \) can also contain cosmic histories that present a second, unstable, de Sitter phase. In addition orbits that evolve near the non-physical point \( B_v \) can also present an intermediate power-law phase.

Finally, looking at figure 3 it is clear that the de Sitter phases \( C_v \) and \( D_v \) are separated from the past attractor \( A_v \) by the plane \( q = 0 \), therefore any model with initial conditions near the past attractor \( A_v \) and evolving toward the future de Sitter attractor \( C_v \) will cross the plane \( q = 0 \), indicating a transition from an accelerating evolution to a decelerating one.

In the asymptotic regime we found no isolated fixed point, but only two fixed subspaces: \( I^\infty_v \) and \( O^\infty_v \). The stability analysis reveals that the only asymptotic attractors are in \( I^\infty_v \). This means that all the models that evolve toward an asymptotic attractor are bound to recollapse.

The introduction of matter into this model increases the dimensionality of the phase space, making it more difficult to visualize. By a direct substitution into the field equations we found that all the fixed points in the matter case do not represent physical solutions. This suggests that the exponential Lagrangian does not present a Friedmann-like phase. On the other hand, since there is evidence that a fourth-order Lagrangian in the form of polynomials can admit such phases [44] one can conclude that somehow the presence of Friedmann-like phase is related to power-law terms in the gravitational Lagrangian.

In the finite regime (see table 3), we found that the four points \( A_m, B_m, F_m \) and \( E_m \) are generalizations of the vacuum fixed points \( A_v, B_v, C_v \) and \( D_v \) respectively and present the same solutions.

The points \( A_m \) and \( B_m \) are found to represent Milne solutions while \( C_m \) and \( D_m \) represent a power-law evolution. For points \( E_m, F_m \) and \( G_m \) we find that \( \dot{H} = 0 \), which means that these points represent Einstein–de Sitter solutions.

In the asymptotic regime we found three different classes of solutions (see table 5). The first class contains the fixed points \( A^\infty_m \) and \( B^\infty_m \), and the solutions at these points depend on the value of \( S(\delta, \theta) \) and \( \tilde{S}(\delta, \theta) \). The solutions at these points represent an expansion if \( S, \tilde{S} > 1 \), a contraction if \( S, \tilde{S} < 1 \). These fixed subspaces contain attractive parts (specifically \( 0 < \theta < 3\pi/4 \) for \( A^\infty_m \) and \( 0 < \theta < \pi/4 \) for \( B^\infty_m \)).

The second class of solutions contains the two fixed points \( C^\infty_m \) and \( D^\infty_m \). These subspaces represent a recollapsing evolution, but they are always unstable.

The third class contains a single fixed subspace \( E^\infty_m \) and the solution at this point depends on the parameter \( \gamma \). By substituting this solution into the field equations (8), and by ignoring the subdominant terms we obtain

\[
\gamma^2 = 0,
\]

\[
\gamma^2 \left[ -1 + \frac{144}{\Lambda} \left( \frac{k}{a^2} \right)^2 + 24 \left( \frac{k}{a^2} \right) + 3 \left( \frac{k}{a^2} \right) \right] = 0.
\]

It is then clear that the only consistent solution is \( \gamma = 0 \) when the spatial part of the spacetime is flat \( k = 0 \). The stability of the points in this subspace cannot be performed in general due to their non-hyperbolic character. We limited ourselves to investigate few specific cases analyzing the evolution of the second-order perturbations around a general point of \( e^\infty_m \). These points seem to be always unstable. Such a behavior is interesting because it points toward the presence of bounce solutions for this cosmological model.

In conclusion, \( \exp(-R/\sqrt{\Lambda}) \) gravity has a very rich structure that includes a series of diverse and interesting cosmological histories. Particularly important are those including multiple de Sitter phases because they could provide us with natural models describing the early- and late-time acceleration of the universe. Unfortunately, as is clear from figure 2, this scenario does
not include a decelerated expansion phase between these two de Sitter phases. This implies that these cosmic histories will not, in general, admit a standard structure formation scenario. This result seems to be consistent with that found in [43]. However great care must be taken in making this statement. In fact our technique is different and not necessarily compatible with that given in [43]. For example, the point \( \mathcal{F}_m \equiv (x = 3, y = -2, z = 0, D = 0) \) does not exist in [43] and there is also no fixed point in our analysis that corresponds to \( P_4 \) and \( P_6 \) of [43]4.

Finally, another important point to consider is that, as found in [44], the evolution of scalar perturbations in fourth-order gravity does not necessarily need the presence of a decelerated expansion phase. This means that only a detailed numerical analysis of these specific orbits (and the perturbation evolution along them) will be able to clarify whether or not the cosmic histories in this model are compatible with the observations.

**Acknowledgments**

This work was supported by the National Research Foundation (South Africa) and the Ministero degli Affari Esteri-DG per la Promozione e Cooperazione Culturale (Italy) under the joint Italy/South Africa Science and Technology agreement. MA thanks the African Institute for Mathematical Sciences (AIMS) for financial support.

**Appendix**

**A.1. Center manifold theorem**

In section 3.1 we used the center manifold theorem to analyze the stability of the non-hyperbolic fixed point \( C_v \) in the vacuum case, here we will give a brief review of this approach [12]. Consider the nonlinear system (bold letters represent vectors),

\[
\dot{x} = f(x). 
\]

(A.1)

For simplicity we shall assume that the origin is a non-hyperbolic fixed point for this system5, i.e. \( f(0) = 0 \). If \( f \in C^1 \), (A.1) can be written in the diagonal form

\[
\dot{x} = Ax + F(x), 
\]

(A.2)

where \( A = DF(0), F(x) = x - Ax \) and \( F(0) = DF(0) = 0 \). We can find a linear transformation \( T \) which transform \( A \) into a diagonal form system (A.2) will then take the form

\[
\begin{pmatrix}
\dot{u} \\
\dot{v} \\
\dot{m}
\end{pmatrix} =
\begin{pmatrix}
A_s & 0 & 0 \\
0 & A_l & 0 \\
0 & 0 & A_c
\end{pmatrix}
\begin{pmatrix}
u \\
v \\
m
\end{pmatrix} +
\begin{pmatrix}
F_s(u, v, m) \\
F_l(u, v, m) \\
F_c(u, v, m)
\end{pmatrix},
\]

where \((u, v, m) \in R^s \times R^l \times R^c, s + l + c = n\), \(A_s\) is a diagonal \( s \times s \) matrix having eigenvalues with negative real part, \(A_l\) is a diagonal \( l \times l \) matrix having eigenvalues with positive real part, \(A_c\) is a diagonal \( c \times c \) matrix having eigenvalues with zero real part, and

4 The reason for this difference is that in [43], a parameter \( m \) is introduced in order to obtain a general analysis of the phase space of fourth-order theories of gravity. This parameter is effectively treated as a numerical constant in the sense that all the results are expressed in terms of \( m \). However \( m \) is in fact a function of the dynamical variables and a dynamical system analysis which ignores this point cannot be considered formally correct. It is interesting to note that if one substitutes \( m \) in terms of the dynamical variables into the general dynamical system of [43], one reproduces our results rather than those given in [43].

5 This assumption does not affect the generality of our treatment because it is always possible to change the coordinates to make the fixed point the origin of the new coordinate system.
\( F_s(u, v, m), F_l(u, v, m) \) and \( F_c(u, v, m) \) are the \( s, l \) and \( c \) components, respectively of the vector \( T^{-1}F(T(u, v, m)) \).

Suppose (A.1) is \( C'\), \( r \geq 2 \). Then there exists a \( C'\)-dimensional local stable manifold \( W^s_{\text{loc}}(0) \) tangent to the stable subspace \( E^s \) at the origin, a \( C'\)-dimensional local stable manifold \( W^s_{\text{loc}}(0) \) tangent to the unstable subspace \( E^u \) at the origin and \( C'\)-dimensional local center manifold \( W^c_{\text{loc}}(0) \) tangent to the center subspace \( E^c \) at the origin. In particular we have

\[
W^s_{\text{loc}}(0) = \{ (u, v, m) \in R^r \times R^r \times R^r | v = h^s_m(u), m = h^s_m(u); Dh^s_m = 0, Dh^s_m = 0; |u| < \delta \},
\]

\[
W^s_{\text{loc}}(0) = \{ (u, v, m) \in R^r \times R^r \times R^r | u = h^s_m(v), m = h^s_m(v); Dh^s_m = 0, Dh^s_m = 0; |v| < \delta \},
\]

\[
W^c_{\text{loc}}(0) = \{ (u, v, m) \in R^r \times R^r \times R^r | u = h^c_m(v), v = h^c_m(m); Dh^c_m = 0, Dh^c_m = 0; |m| < \delta \},
\]

where \( h^s_m(u), h^s_m(v), h^c_m(v), h^c_m(m) \) and \( h^c_m(m) \) are \( C'\). In order to find the center manifold we need to solve

\[
\begin{align*}
Dh^c_m(m)(A + m + F_c(h^c_m(m), h^c_m(m), m)) - A_1h^c_m(m) - F_c(h^c_m(m), h^c_m(m), m),
\end{align*}
\]

(A.3)

\[
\begin{align*}
Dh^c_m(m)(A + m + F_c(h^c_m(m), h^c_m(m), m)) - A_1h^c_m(m) - F_c(h^c_m(m), h^c_m(m), m).
\end{align*}
\]

(A.4)

With the boundary conditions

\[
\begin{align*}
h^c_m(0) = Dh^c_m(0) = 0, \quad h^c_m(0) = Dh^c_m(0) = 0.
\end{align*}
\]

We can approximate \( h^c_m(0) \) and \( h^c_m(0) \) as

\[
\begin{align*}
h^c_m(m) &= a_0 + a_1 m + a_2 m^2 + a_3 m^3 + \cdots, \\
\end{align*}
\]

(A.5)

\[
\begin{align*}
h^c_m(m) &= b_0 + b_1 m + b_2 m^2 + b_3 m^3 + \cdots.
\end{align*}
\]

(A.6)

Form the boundary conditions \( a_0 = b_0 = a_1 = b_1 = 0 \). To find the other coefficient substitute these polynomials into (A.3) and (A.4), and match the coefficients. Now the flow on the center manifold \( W^c(0) \) in the neighborhood of the origin is defined by the system

\[
\begin{align*}
m = A_1 m + F_c(m, h^c_m(m), h^c_m(m)).
\end{align*}
\]

For all \( m \in R^r \). In general the flow on the center manifold near the fixed point takes the form

\[
\begin{align*}
m = a m^r + \cdots.
\end{align*}
\]

If \( r \geq 2 \) and \( a_r \neq 0 \), then for \( r \) even we have saddle-node at the fixed point, for \( r \) odd and \( a_r > 0 \) we have unstable node and for \( r \) odd and \( a_r < 0 \) we have a topological saddle.

A.2. Coefficients of the second order system

The coefficients of the evolution equations (50) are

\[
\begin{align*}
a_1 &= -\frac{(3 + \cos 2\theta_0) \sec \theta_0^2(-4+\sqrt{2} \sin 2\theta_0)}{8(1 + \sec \theta_0^2)^2}, \\
a_2 &= -\frac{(3 + \cos 2\theta_0) \sec \theta_0(-4+\sqrt{2} \sin 2\theta_0) \tan \theta_0^2}{8(1 + \sec \theta_0^2)^3}, \\
a_3 &= -\frac{\sqrt{2} \cos 2\theta_0 + (-2 + \sec \theta_0^2) \tan \theta_0}{(1 + \sec \theta_0^2)^2}, \\
a_4 &= \frac{(4 \cos 2\theta_0 - \sqrt{2}(\sin 2\theta_0 - 4 \tan \theta_0)) \tan \theta_0^2}{4(1 + \sec \theta_0^2)^2}.
\end{align*}
\]
\[
a_5 = \frac{(2 \sec\theta_0 - \sqrt{2} \sin\theta_0) \tan\theta_0}{2(1 + \sec\theta_0^2)^2},
\]
\[
a_6 = \frac{-5\sqrt{2} \sin\theta_0 + 2 \sec\theta_0(1 + \sqrt{2} \tan\theta_0)}{4(1 + \sec\theta_0^2)^2},
\]
\[
a_7 = \frac{\sin\theta_0 \tan\theta_0}{(1 + \sec\theta_0^2)^2}, \quad a_8 = \frac{\sqrt{2} \sec\theta_0 - \sin\theta_0 \tan\theta_0}{(1 + \sec\theta_0^2)^2},
\]
\[
a_9 = \frac{-\sqrt{2} \cos\theta_0 - 6 \sin\theta_0 + \sec\theta_0(3\sqrt{2} \tan\theta_0)}{2(1 + \sec\theta_0^2)^2},
\]
\[
b_1 = \frac{\sin\theta_0 \cos\theta_0}{\sqrt{2}(\sqrt{1 + \sec\theta_0^2})}, \quad b_2 = \frac{-2 - 6 \cos 2\theta_0 + \sqrt{2} \sin 2\theta_0}{4(\sqrt{1 + \sec\theta_0^2})},
\]
\[
b_3 = \frac{\cos\theta_0 + \cos\theta_0^2}{\sqrt{2}(\sqrt{1 + \sec\theta_0^2})}, \quad b_4 = \frac{-\cos\theta_0(\sqrt{2} + \sqrt{2} \cos\theta_0^2 + \sin 2\theta_0)}{2(\sqrt{1 + \sec\theta_0^2})},
\]
\[
b_5 = \frac{-(2 + 3 \cos\theta_0^2) \sin\theta_0}{\sqrt{2}(\sqrt{1 + \sec\theta_0^2})}, \quad b_6 = \frac{\cos\theta_0 \sin\theta_0}{\sqrt{1 + \sec\theta_0^2}},
\]
\[
b_7 = \frac{-3 + 5 \cos 2\theta_0}{4\sqrt{2}(\sqrt{1 + \sec\theta_0^2})}, \quad b_8 = \frac{-\cos\theta_0 \sin\theta_0}{\sqrt{1 + \sec\theta_0^2}},
\]
\[
c_1 = \frac{-2 \sec\theta_0 + \sin\theta_0}{\sqrt{2}(\sqrt{1 + \sec\theta_0^2})}, \quad c_2 = \frac{2 \cos\theta_0 + \sqrt{2}(-2 \csc\theta_0 + \sin\theta_0)}{2(\sqrt{1 + \sec\theta_0^2})},
\]
\[
c_3 = \frac{\cos\theta_0 + 2 \cot\theta_0 \csc\theta_0 + \sec\theta_0}{\sqrt{2}(\sqrt{1 + \sec\theta_0^2})}, \quad c_4 = \frac{2(\cos\theta_0^2 + \sin\theta_0^2) + \sqrt{2} \cot\theta_0 \csc\theta_0 - \sin\theta_0 - \sqrt{2} - 2 \tan\theta_0}{\sqrt{1 + \sec\theta_0^2}},
\]
\[
c_5 = -\frac{1}{\sqrt{2}(\sqrt{1 + \sec\theta_0^2})}, \quad c_6 = -\frac{1}{\sqrt{1 + \sec\theta_0^2}},
\]
\[
c_7 = -\frac{1}{\sqrt{1 + \sec\theta_0^2}}, \quad c_8 = \frac{\cot\theta_0 + 2 \tan\theta_0}{2\sqrt{2}\sqrt{1 + \sec\theta_0^2}},
\]
\[
d_1 = -\frac{(\cos(2\theta_0) + 3) \sec^2(\theta_0)(\sqrt{2} \sin(2\theta_0) - 4)}{4(\sec^2(\theta_0) + 1)^{3/2}},
\]
\[
d_2 = -\frac{(\cos(2\theta_0) + 3) \sec(\theta_0)(\sqrt{2} \tan(\theta_0) + 2)}{4(\sec^2(\theta_0) + 1)^{3/2}},
\]
\[
d_3 = -\frac{\sqrt{2} \sec^3(\theta_0) + 2(\sqrt{2} - 2 \tan(\theta_0)) \sec(\theta_0) - 5 \sqrt{2} \cos(\theta_0) + 2 \sin(\theta_0)}{2(\sec^2(\theta_0) + 1)^{3/2}},
\]
\[
d_4 = \frac{\tan(\theta_0)(\sqrt{2} \tan(\theta_0) + 2)}{2(\sec^2(\theta_0) + 1)^{3/2}}.
\]
\[ d_5 = \frac{\sqrt{2} (5 - 2 \sec^2(\theta_0)) \tan(\theta_0) + 2}{4(\sec^2(\theta_0) + 1)^{3/2}} \]
\[ d_6 = \frac{\sec(\theta_0)(3\sqrt{2} \cos(2\theta_0) + \sin(2\theta_0) + 7\sqrt{2} \tan(\theta_0))}{2(\sec^2(\theta_0) + 1)^{3/2}} \]
\[ d_7 = -\frac{\sec(\theta_0)(7\sqrt{2} \cos(2\theta_0) + 6 \sin(2\theta_0) + 9\sqrt{2} \tan(\theta_0))}{4(\sec^2(\theta_0) + 1)^{3/2}} \]
\[ d_8 = \frac{(3\sqrt{2} - 2 \tan(\theta_0)) \sec(\theta_0) + 6 \tan(\theta_0) + 7\sqrt{2}}{2(\sec^2(\theta_0) + 1)^{3/2}} \]
\[ d_9 = -\frac{\tan^2(\theta_0)}{(\sec^2(\theta_0) + 1)^{3/2}}. \]

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