Quantum networks with coherent routing of information through multiple nodes

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Large-scale communication networks, such as the Internet, rely on routing packets of data through multiple intermediate nodes to transmit information from a sender to a receiver. In this paper, we develop a model of a quantum communication network that routes information simultaneously along multiple paths passing through intermediate stations. We demonstrate that a quantum routing approach can in principle extend the distance over which information can be transmitted reliably. Surprisingly, the benefit of quantum routing also applies to the transmission of classical information: even if the transmitted data is purely classical, delocalising it on multiple routes can enhance the achievable transmission distance. Our findings highlight the potential of a future quantum internet not only for achieving secure quantum communication and distributed quantum computing but also for extending the range of classical data transmission.

Quantum communication has the potential to revolutionise traditional telecommunication networks, enabling the transmission of private messages through quantum key distribution and the implementation of distributed quantum computing. However, noise and decoherence pose a major challenge, exponentially suppressing the transmission of quantum states over large distances. To address this challenge, quantum repeaters have been proposed as intermediate stations that achieve long-distance entanglement through entanglement swapping combined with entanglement purification and error correction techniques. Nevertheless, the rate of quantum communication remains an issue, as repeated rounds of transmission are required to establish long-distance entanglement. This issue has led to a search for methods to improve the quality of long-distance communication, either by directly reducing noise in transmission lines or by using encoding strategies to approach the optimal achievable rates.

A promising way to enhance quantum communication is to exploit the ability of quantum particles to propagate simultaneously through multiple paths, as in the iconic double-slit experiment. Groundbreaking work by Gisin, Linden, Massar, and Popescu has demonstrated that path superposition could be used to filter out errors, in principle enabling secure key distribution in highly noisy environments. More recent studies have shown enhancements in classical and quantum communication capacities.

While these studies focused on direct communication between a sender and receiver, in real-world classical communication networks information is typically routed through intermediate nodes. In this setting, a crucial problem is to find the optimal route. With the emergence of quantum communication networks, such as the quantum internet, the path of information carriers can be coherently controlled by quantum routers, enabling the superposition of multiple routes. However, despite the potential benefits, the potential of quantum routing through multiple intermediate nodes remains largely unexplored.

In this paper, we introduce a model for quantum communication networks that allows for coherent information routing assisted by local operations at multiple intermediate nodes. Our model is fully characterised by the operational description of the communication devices available to experimenters, in contrast to previous models which rely on knowledge of noisy interactions with an environment. We demonstrate that this model can increase the distance over which information can be transmitted reliably in a number of scenarios. In the idealised case where the path degree of freedom is not affected by decoherence, we find that the model even enables classical communication at a finite rate over arbitrarily long distances. Our findings show that a quantum internet could be used not only for achieving distributed quantum computing and secure quantum communication, but also for extending the range of classical communication: even if the data is purely classical, the ability to delocalise its route can benefit the transmission.

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Results
Quantum networks with coherent control

Here we formulate a model of communication networks with quantum control over the trajectories connecting a sender to a receiver through intermediate nodes.

In the basic scenario, illustrated in Fig. 1, the communication network is described by a directed graph, where the vertices represent communicating parties and the edges represent communication channels between them. When a communication channel is not used, we will model its input as being the ‘vacuum state’, a state orthogonal to all the states used to encode information. This modelling is inspired by quantum communication with polarisation qubits, where the information is encoded in the subspace generated by the one-photon states $|\psi\rangle_{k,h} \otimes |0\rangle_{k,v}$ and $|0\rangle_{k,h} \otimes |\psi\rangle_{k,v}$, while the absence of inputs is described by the vacuum state $|0\rangle_{k,h} \otimes |0\rangle_{k,v}$, where $k$ denotes the wave vector and $H(V)$ horizontal (vertical) polarisation.

The action of a communication channel on the extended, particle-plus-vacuum system is specified by a quantum channel (completely positive trace-preserving map) $\hat{E}$, acting on density matrices on the Hilbert space $\mathcal{H} := \mathcal{H} \otimes \mathcal{H}_{\text{vac}}$, where $\mathcal{H}$ is the single-particle Hilbert space, and $\mathcal{H}_{\text{vac}}$ the one-dimensional Hilbert space spanned by the vacuum state $|\text{vac}\rangle$. The channel $\hat{E}$ is an extension of a channel $E$, acting only on the single-particle subspace, which has typically been considered in single-particle communication. This extension, called the vacuum extension, has Kraus operators of the form $[\hat{E}_i := E_i + a_i |\text{vac}\rangle\langle\text{vac}|, \{E_i\}$ are Kraus operators of the channel $E$, and $\{a_i\}$ are complex amplitudes satisfying the normalisation condition $\sum |a_i|^2 = 1$. In a given physical communication scenario, the vacuum extension is the complete description of the physical device accessible to the experimenter, when restricted to act on single-particle and vacuum states, i.e., the device is completely characterised by its input-output description on the Hilbert space $\mathcal{H} := \mathcal{H} \otimes \mathcal{H}_{\text{vac}}$. For example, in a quantum-optics scenario, the choice of the $\{a_i\}$ is completely determined by the given Hamiltonian of the electromagnetic field\cite{4.13,4.29}. As such, the vacuum extension can be found using quantum process tomography on the space spanned by the vacuum and single-particle states $|\psi\rangle_{k,h} \otimes |0\rangle_{k,v}$ and $|\psi\rangle_{k,h} \otimes |1\rangle_{k,v}$, where the subscripts 1 and 2 refer to the input ports of the channels $\hat{E}^{(1)}$ and $\hat{E}^{(2)}$. Equivalently, the transmission can be modelled in terms of an external degree of freedom, which controls the particle’s path. In this picture, the states $|\psi\rangle_{k,h} \otimes |0\rangle_{k,v}$ and $|\psi\rangle_{k,h} \otimes |1\rangle_{k,v}$ are represented as $|\psi\rangle \otimes |0\rangle$ and $|\psi\rangle \otimes |1\rangle$, where $|0\rangle$ and $|1\rangle$ are orthogonal states of the path degree of freedom. The evolution of a particle travelling from node $A$ to node $B$ of Fig. 1 is described by a quantum channel $S_{\rho^{(1)}\rho^{(2)}}$ with Kraus operators\cite{4.14}

$$S_{\rho} = E^{(1)}_{\rho^{(1)}}a^{(1)}_{\rho^{(1)}} \otimes |0\rangle\langle0| + E^{(2)}_{\rho^{(2)}}a^{(2)}_{\rho^{(2)}} \otimes |1\rangle\langle1|,$$  

(1)

where $E^{(k)}_{\rho^{(k)}}$, $a^{(k)}_{\rho^{(k)}}$ are the Kraus operators and vacuum amplitudes associated with channel $\hat{E}^{(k)}$, for $k = 1, 2$, and $|0\rangle$ and $|1\rangle$ are orthogonal states of the path degree of freedom.

If the external degree of freedom is initially in the superposition state $|\pm\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$, the effective evolution from node $A$ to node $B$ is described by the channel $S_{\rho^{(0)}\rho^{(1)}}(\rho) := S_{\rho^{(0)}\rho^{(1)}}(\rho \otimes |\pm\rangle\langle\pm|)$, which can be regarded as a quantum superposition of the two channels $\hat{E}^{(1)}$ and $\hat{E}^{(2)}$, specified by the vacuum extensions $\hat{E}^{1}$ and $\hat{E}^{2}$. When the two channels are identical ($\hat{E}^{1} = \hat{E}^{2}$), the particle’s state after the transmission is

$$S_{\rho}^{(1)\pm}(|\pm\rangle) = E^{(1)}_{\rho^{(0)}}a^{(1)}_{\rho^{(0)}} \otimes |\pm\rangle\langle\pm| + E^{(2)}_{\rho^{(0)}}a^{(2)}_{\rho^{(0)}} \otimes |\pm\rangle\langle\pm|,$$  

(2)

where we use the shorthand $S_{\rho}^{(1)\pm} := S_{\rho}^{(1)\pm}$, the first (second) factor of each tensor product corresponds to the message (path), and $F := \sum \beta_i E_i$ is called the vacuum interference operator\cite{4.25} (a summary of the derivation is also provided in the Methods section IV).

Given two nodes of a communication network, the problem is to find the optimal way to transfer information from one node to the other. This amounts to optimising the route through the intermediate nodes, the encoding operations performed by the sender, the decoding operations performed by the receiver, and the operations performed by the intermediate parties.

With the general framework in place, in the next sections, we will show that the superposition of network paths travelling through multiple intermediate nodes can enhance the rate for the transmission of classical data.

Communication through asymptotically long sequences of binary asymmetric channels

An important type of classical channel is the binary asymmetric channel, which transmits a bit with different probabilities of error depending on whether the bit value is 0 or 1. The quantum realisation of this channel is given by the map

$$E(\rho) = (1 - q)|0\rangle\langle0|\rho + q|1\rangle\langle1|\rho|0\rangle\langle0| + p|0\rangle\langle1|\rho|1\rangle\langle0| + (1 - p)|1\rangle\langle1|\rho|1\rangle\langle1|.$$  

(3)

In the following, we shall consider first the special case of binary asymmetric channel in which the value 0 is transmitted without error, namely $q = 0$. The resulting channel is known as the Z-channel\cite{4.34} and has several applications in optical communications\cite{4.45}. The quantum realisation of the Z-channel is the map

$$E(\rho) = p|0\rangle\langle0|Tr(\rho) + (1 - p)\rho_{\text{diag}},$$  

(4)

with $\rho_{\text{diag}} := |0\rangle\langle0|\rho|0\rangle\langle0| + |1\rangle\langle1|\rho|1\rangle\langle1|$. Now, suppose that a quantum particle has to traverse $n$ Z-channels on the way from the sender to the receiver. If the particle follows a definite path, its state after the transmission will be

$$E(\rho) = [1 - (1 - p)^n]|0\rangle\langle0|Tr(\rho) + (1 - p)^n\rho_{\text{diag}},$$  

(5)

which is equivalent to another Z-channel with effective error probability $1 - (1 - p)^n$. For large $n$, the dependence of the output on the input vanishes exponentially, resulting in an exponential decay of the transmission rate.

Now, suppose that two alternative paths are available. If a quantum particle travels them in a quantum superposition, with the path initialised in the $|+\rangle$ state, then the output state is

$$S_{E}^{(1)\pm}(\rho) = \frac{E^{(1)}(\rho) + E^{(1)}(\rho)^{+} + E^{(2)}(\rho)^{-}}{2} \otimes |+\rangle\langle+| + \frac{E^{(2)}(\rho) - E^{(2)}(\rho)^{+} + E^{(2)}(\rho)^{-}}{2} \otimes |-\rangle\langle-|.$$  

(6)

This is depicted in Fig. 2. In the large-$n$ limit, the operator $F^*$ tends to the projector $\Pi$ on the eigenspace of $F$ associated to eigenvalues of modulus 1 (if $F$ does not have an eigenvalue of modulus 1, then $\Pi = 0$). Hence, the output state converges to

$$S_{E}^{(1)\pm}(\rho) := \lim_{n \to \infty} S_{E}^{(1)\pm}(\rho) = \frac{Tr(\rho)(|0\rangle\langle0| + |1\rangle\langle1|)}{2} \otimes |+\rangle\langle+| + \frac{Tr(\rho)(|0\rangle\langle0| - |1\rangle\langle1|)}{2} \otimes |-\rangle\langle-|.$$  

(7)

For $\Pi \neq 0$, this state has a non-trivial dependence on the input state $\rho$ and therefore permits a transmission of classical data, despite the fact that the paths between the sender and receiver are asymptotically long.

The crucial question, at this point, is whether or not the operator $F$ has a non-trivial eigenspace with eigenvalue(s) of norm 1. We now show that the
answer is affirmative in a large class of physical realisations of the Z-channel, and provide an example of how one such realisation could arise in the setting of single-photon quantum optics.

A simple class of vacuum extensions that have an eigenspace of \( \lvert \text{vac} \rangle \) are the relative phase between the vacuum and (some subspace of) the one-particle sector. Such a term is given by a Kraus operator of the form \( P \otimes e^{i\theta} \left| \text{vac} \right\rangle \langle \text{vac} \right| \), where \( P \) is a projector.

One such vacuum-extended channel can arise as the result of the following two steps: first, complete decoherence in the computational basis \( \{0\}, \{1\} \), and then a random operation that replaces the state \( |0\rangle \) with probability \( p \), or leaves the input state invariant with probability \( 1 - p \). For the physical realisation of the decoherence map, we take a controlled-NOT interaction with an environment \( E_i \), initially in the state \( |0\rangle \). In this realisation, the vacuum extension is easily found: if the system is not present, then the environment remains unaffected. Overall, this first interaction is described by the unitary gate

\[
\hat{U}_{M_i} = |0\rangle \langle 0| \otimes I_{E_i} + |1\rangle \langle 1| \otimes X_{E_i} + |\text{vac}\rangle \langle \text{vac}| \otimes I_{E_i},
\]

where the vectors \( |0\rangle, |1\rangle, |\text{vac}\rangle \) form a basis for the vacuum-extended message system \( M \), the subscript \( E_i \) denotes operators acting on the environment, and \( X := |0\rangle \langle 1| + |1\rangle \langle 0| \). Similarly, the interaction that resets the input state to \( |0\rangle \) can be modelled as a SWAP gate, where the state of the system is swapped with the state of an environment \( E_2 \), initially in the state \( |0\rangle \). The vacuum extension of the SWAP gate is given by the unitary gate

\[
\hat{V}_{M_i E_2} = \text{SWAP} + |\text{vac}\rangle \langle \text{vac}| \otimes I_{E_i}.
\]

Once the environments \( E_i \) and \( E_2 \) are discarded, the above interactions give rise to an extended channel \( \hat{C}_i \) with Kraus operators \( \hat{C}_{10} = |0\rangle \langle 0| \otimes |\text{vac}\rangle \langle \text{vac}| \) and \( \hat{C}_{11} = |1\rangle \langle 1| \). This channel is applied to the input with probability \( p \). With probability \( 1 - p \), instead, only the first interaction takes place, resulting in an extended channel \( \hat{C}_i \) with Kraus operators \( \hat{C}_{00} = |0\rangle \langle 0| \otimes |\text{vac}\rangle \langle \text{vac}| \) and \( \hat{C}_{01} = |1\rangle \langle 1| \). Overall, the evolution of the system is described by the extended channel \( \hat{C} = p \hat{C}_i + (1 - p) \hat{C}_i \). This channel has a vacuum interference operator \( F = |0\rangle \langle 0| \), and the limit channel (7) becomes

\[
S_{\text{eff}}^{(k)}(\rho) = \text{Tr}(\rho) |0\rangle \langle 0| + (q_+ |+\rangle \langle +| + q_- |-\rangle \langle -|),
\]

with \( q_+ := (1 \pm (0|p|0))/2 \). This channel is equivalent to a classical Z-channel with error probability \( p = 1/2 \), whose classical capacity is known to be \( \log_2(5/4) \approx 0.32 \). Hence, reliable classical communication through asymptotically long distances is possible, by using a coherent routing of two paths.

In the Supplementary Material, we consider the case of \( K \geq 3 \) paths, showing that the overall classical capacity increases with the number of coherent paths. In the (purely theoretical) asymptotic limit of infinitely many coherent paths, we find that the classical capacity tends to 1, meaning that perfect classical communication is possible. In the following, we restrict ourselves to the more practical scenarios of only two coherent paths.

**Extension to variable bases**

So far, we have considered only networks whose errors arise from identical Z-channels, all defined with respect to the computational basis, and all with the same error probability \( p \). However, in practice, it is reasonable to expect variations within the error parameters between different parts of the network. For any chosen path through the network, the sequence of errors acting on the particle could be modelled more realistically as a sequence of possibly non-identical Z-channels \( \hat{C}_{n} \otimes \hat{C}_{n-1} \otimes \cdots \hat{C}_{1} \), with

\[
\hat{C}_k(\rho) = p_k |\eta_k\rangle \langle \eta_k| \text{Tr}(\rho) + (1 - p_k) \rho_{\text{diag}},
\]

where \( k \in \{1, \ldots, n\}, p_k \) is a probability, \(|\eta_k\rangle, |\eta_{k+1}\rangle\rangle \) is an orthonormal basis, and \( \rho_{\text{diag}} := \langle \eta_k| \rho |\eta_k\rangle |\eta_k\rangle \langle \eta_k| + |\eta_{k+1}\rangle \langle \eta_{k+1}| \). In this case, the overall vacuum interference operator of each sequence is given by \( F = F_{n} F_{n-1} \cdots F_{1} \), which approaches zero as \( n \to \infty \), assuming \((0|\eta_{k+1}|\eta_k\rangle < 1\) for all \( k \). Nevertheless, by inserting at each node in the network, an intermediate operation \( R_k \) engineered to have a vacuum interference operator \( G_k = |\eta_k\rangle \langle \eta_k| + G_{k,\text{est}} \), with \( G_{k,\text{est}} |\eta_k\rangle \langle \eta_k| = 0 \), the effective vacuum interference operator \( F_{\text{eff}} = F_n F_{n-1} \cdots F_1 G_{1,\text{est}} \) has a singular value \( 1 \). (An example is \( R_k(\cdot) = R_{k}(\cdot) R_k \), with \( R_k = |\eta_{k+1}\rangle \langle \eta_k| + |\eta_k\rangle \langle \eta_{k+1}| \). Note that \( F \) having a singular value \( 1 \) is sufficient to avoid \( F_{\text{eff}} \) decaying to zero; an eigenvalue of modulus \( 1 \) in the previous examples is not required in general.

Then, the superposition of the two paths gives

\[
S_{\text{eff}}^{(k)}(\rho) = \hat{C}_k(\rho) = \frac{e_{\text{eff}} R_{n+1} \cdots e_{n+1} \cdots e_{1} \rho + F_{\text{eff}} \rho F_{\text{eff}}^{\dagger}}{2} \otimes (+)(+) + \frac{e_{\text{eff}} R_{n+1} \cdots e_{n+1} \cdots e_{1} \rho - F_{\text{eff}} \rho F_{\text{eff}}^{\dagger}}{2} \otimes (-)(-),
\]
where above theorem, the channels that permit a transmission of classical representation \( f \) is possible to achieve a non-zero communication capacity even with in

The example shown in the previous sections showed that it is sometimes possible to achieve a non-zero communication capacity even with infinitely long sequences of noisy channels. We now characterise completely the set of channels giving rise to this phenomenon.

The following theorem characterises completely the set of channels that have zero capacity when used asymptotically many times in a dephased superposition of two identical sequences of channels is

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\[
\text{Tr}(\rho)|\eta_0\rangle\langle\eta_1| \oplus (q_+ |+\rangle + q_- |-\rangle) \quad (13)
\]

where \( q_{\pm} = (1 \pm \langle \eta |p|\eta \rangle)/2 \). This expression has the same form as Eq. (10), enabling classical communication at a non-zero rate through asymptotically long sequences of non-identical Z-channels.

**Theorem 1.** Let \( \mathcal{E} : \mathcal{L}(\mathcal{H}_\chi) \to \mathcal{L}(\mathcal{H}_\psi) \) be a quantum channel. Let \( \tilde{\mathcal{E}} \) be a vacuum extension of \( \mathcal{E} \), with vacuum interference operator \( F \) and Kraus representation \( |E_i\rangle \equiv E_i + a_i|\text{vac}\rangle|\text{vac}\rangle_{n-1} \), where \( |E_i\rangle_{n-1} \) are Kraus operators of \( \mathcal{E} \). Assume that for every intermediate operation \( Q : \mathcal{L}(\mathcal{H}_\psi) \to \mathcal{L}(\mathcal{H}_\chi) \), the classical capacity of the concatenated channel \( \mathcal{E}_n(\mathcal{Q}\mathcal{E})^{n-1} \) tends to zero as \( n \to \infty \). Then, the following are equivalent:

1. There exists an intermediate operation \( \mathcal{R} : \mathcal{L}(\mathcal{H}_\psi) \to \mathcal{L}(\mathcal{H}_\chi) \) with vacuum extension \( \tilde{\mathcal{R}} \) and vacuum interference operator \( G \), such that the superposition of two independent identical sequences of channels \( \mathcal{E}_n(\mathcal{R}\mathcal{E})^{n-1} : \mathcal{L}(\mathcal{H}_\psi) \to \mathcal{L}(\mathcal{H}_\psi \otimes \mathcal{H}_\rho) \) has classical capacity strictly greater than zero as \( n \to \infty \).

2. The vacuum interference operator \( F \) has singular value 1.

3. There exist two pure states \( |\psi\rangle \in \mathcal{H}_\chi, |\theta\rangle \in \mathcal{H}_\psi \) and \( \theta \in [0, 2\pi) \) such that for all \( j \in \{0, \ldots, r-1\} \): (a) \( a_j = e^{i\theta}\sqrt{\langle \theta | E_j | \theta \rangle} \) and (b) \( E_j | \theta \rangle = a_j | \theta \rangle \) [and hence \( \mathcal{E}(| \psi \rangle \langle \psi |) = | \theta \rangle \langle \theta | \) ].

The proof is given in the Supplementary Material. As shown by the above theorem, the channels that permit a transmission of classical information even in the asymptotic limit of infinitely many sequential repetitions are a very special subset of the set of all quantum channels. In the following, we will show that, nevertheless, the superposition of paths can generically offer advantages in non-asymptotic scenarios involving a finite number of repetitions.

**Decoherence on the path and general channels**

The possibility of communication at a finite rate through asymptotically long paths is important as a proof of principle. On the other hand, its practical applicability is limited by two crucial assumptions: (a) that the path degree of freedom remains completely noiseless throughout the whole sequence, and (b) that the quantum channels along the paths have vacuum interference operators with a singular value equal to 1. However, in practice, these assumptions are only justified as idealisations of more complex scenarios; realistic transmission lines do not in general retain perfect coherence on the path degree of freedom, nor will they always correspond to channels with unit singular values of their vacuum interference operators.

In the following, we examine what happens in the realistic case where the above assumptions are relaxed. We perform numerical simulations of various superpositions of binary asymmetric channels to calculate lower bounds to their classical capacity. To this purpose, we evaluate the maximum Holevo information of the superposition of channels over all input ensembles consisting of two orthogonal states. This gives a lower bound to the Holevo capacity, which in turn gives a lower bound to the classical capacity. We compare this with the analytically known expression for the classical capacity of the corresponding sequence of binary asymmetric channels without a superposition of paths. For the Z-channel with probability \( p \), the capacity is given by \( \log(1 + (1 - p)p^{\ell(1-p)^{1-n}}) \), while the capacity of the binary asymmetric channel with error probabilities \( q \) and \( p \) is given by a similar but longer formula, which can be found in Ref. 35, Eq. (19).

Let us start by relaxing the assumption that the path is completely noiseless. In particular, we consider a dephasing error on the path. That is, between each pair of nodes, the path qubit undergoes the dephasing channel

\[
D(\omega) = s 2Z\omega Z + (1 - s)\omega,
\]

where \( s \in (0, 1/2] \) is a probability and \( \omega \in \mathcal{L}(\mathcal{H}_\rho) \) is the initial state of the path.

We consider again two independent identical sequences of a quantum channel \( \mathcal{E} \), with vacuum extension \( \tilde{\mathcal{E}} \) and vacuum interference operator \( F \), interleaved with independent and identical dephasing channels on the path. The state of a particle after travelling through a superposition of two such sequences of channels is

\[
\mathcal{S}_{\mathcal{E}'}^{(\ell)}(p) = \frac{e^{(\rho + pF^\ell F^\ell \omega)}}{2} |+\rangle \langle +| + \frac{e^{(\rho - pF^\ell F^\ell \omega)}}{2} |-\rangle \langle -|,
\]

where \( \ell = (1 - 2s)^n \) (see the Methods Section IV for the derivation).

This shows that the magnitude of the coherence terms \( e^{\ell F^\ell F^\ell \omega} \) decreases exponentially with sequence length \( n \). However, for finite \( n \) and small enough dephasing probability \( s \), communication at a non-zero rate is still possible even if the channel \( \mathcal{E} \) has zero capacity — in stark contrast to using the channels in a classical configuration without a superposition of paths.

To illustrate the effect of dephasing, Fig. 4 shows a numerical plot of (lower bounds to) the classical capacity against sequence length \( n \) for a dephased superposition of two identical sequences of Z-channels with error probability \( p = 0.5 \), for various dephasing parameters \( s \) on the path. Note that \( s = 0 \) corresponds to no dephasing, while \( s = 0.5 \) corresponds to complete dephasing on the path so that the final state of the path is simply the maximally mixed state (this is equivalent to using a single sequence of n channels without a superposition of paths). We see that for every value of \( n \), the lower bound to the capacity
monotonically increases as $s$ decreases from 0.5 to 0, and an observable advantage over no superposition of paths is still present for small $n$ when $s \leq 0.2$.

The second relaxation of our initial assumptions is to consider quantum channels with non-unit singular values of their vacuum interference operator. While in this case, it is not possible to achieve communication through asymptotically many uses of the channels, we now show that a communication advantage is still present for a finite number of uses.

To illustrate the advantage, we consider a binary asymmetric channel $E$ with $q > 0$, whose vacuum extension does not give rise to unit singular values. As in the above, we assume that the channel is implemented via a controlled-NCZ interaction with an environment initially in the state $|0\rangle$, followed by a choice of two random operations that either replace the input state with the state $|0\rangle$, with probability $p$, or with the state $|1\rangle$, with probability $q$ (where $p + q = 1$). For this implementation, we obtain a vacuum extension of the binary asymmetric channel described by the Kraus operators

\begin{equation}
\begin{aligned}
E_{1,0} &= \sqrt{p} |0\rangle \langle 0| \oplus |\text{vac}\rangle \langle \text{vac}| \\
E_{1,1} &= \sqrt{p} |1\rangle \langle 1| \\
E_{2,0} &= \sqrt{q} |1\rangle \langle 0| \\
E_{2,1} &= \sqrt{q} |0\rangle \langle 1| \\
E_{3,0} &= \sqrt{1 - q} |0\rangle \langle 0| \oplus |\text{vac}\rangle \langle \text{vac}| \\
E_{3,1} &= \sqrt{1 - q} |1\rangle \langle 1|,
\end{aligned}
\end{equation}

which gives a vacuum interference operator of the form $F = (1 - q) |0\rangle \langle 0|$.

Figure 5 shows (lower bounds to) the capacity as a function of $n$, for a superposition of two identical sequences of such binary asymmetric channels with error probabilities $p = 0.2$ and $p = 0.5$, and various values of $q$.

In order to compare the capacity to the case without a superposition of paths, we note that the concatenation of $n$ identical binary asymmetric channels $E^n$ with error probabilities $p$ is itself a binary asymmetric channel with error probabilities $(q(p), P(0))$, where

\begin{equation}
\begin{aligned}
q(0) &= q^{1 - (1 - p^2)q^2} - p^2 \frac{q^2}{p} q^2 \\
P(0) &= p^{1 - (1 - p^2)q^2}.
\end{aligned}
\end{equation}

This enables the capacity to be calculated exactly. For each sequence of channels shown in Fig. 5, the corresponding capacity without a superposition of paths is also shown.

For every choice of error probabilities $p$ and $q$, the coherent routing of paths provides a non-zero advantage in the classical capacity for small $n$. The gap between the capacities of the implementations with and without a superposition of paths increases as $q$ tends to 0, in the limit of which we recover the ideal case of the Z-channel.

**Discussion**

Our work provides a new paradigm of quantum communication networks with coherent routing of information through multiple paths. This model includes earlier results that showed improvements in communication through individual channels superposed either in space\(^{15,16}\) or in time\(^{14}\). These works considered direct communication from a sender to a receiver, possibly through multiple communication channels. Here, instead, we address the network scenario where information is sent through multiple intermediate nodes, undergoing a noisy channel between every two successive nodes. In this scenario, the main question is how the rate of information transmission scales with the number of nodes between the sender and the receiver.

One of our key findings is that coherent control can sometimes be used to suppress the exponential decay of information through asymptotically long paths in the network, provided that the path is immune from decoherence. In the ideal case, this phenomenon can enable communication at a non-zero rate through asymptotically long sequences of noisy channels coherently routed over a finite number of paths. Remarkably, we find that in the double asymptotic limit of coherent control over asymptotically many paths, each consisting of an asymptotically long sequence of channels, it is even possible to achieve perfect classical capacity, despite each individual channel being noisy. In more realistic scenarios with coherent control over only two paths, each consisting of a finite sequence of channels with small deviations from the ideal case, we still observe enhancements in the classical capacity for finite sequence lengths, compared to the case where the particle is sent on a classical trajectory.

**Methods**

**Superposition of quantum channels**

Here we summarise the notion of superpositions of quantum channels\(^{14,15,16,29}\) and the framework of Ref. 14.

Consider a quantum channel (completely positive trace-preserving map) $E : L(H) \to L(H)$, representing one use of a communication device, which can transmit a message encoded in a quantum particle. (Here $L(H)$ denotes the space of linear operators over a Hilbert space $H$). When a quantum particle is not transmitted, the device is described by an identity channel acting on the vacuum state. Overall, we can describe the action of
the device by a vacuum extension of the original channel: a channel $\tilde{E} : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})$ is a vacuum extension of the channel $E : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})$ if it satisfies

$$
\begin{aligned}
\tilde{E}(\rho) &= E(\rho) \quad \forall \rho \in \mathcal{L}(\mathcal{H}) \\
\tilde{E}(\text{vac} \langle \text{vac} \rangle) &= \text{vac} \langle \text{vac} \rangle ,
\end{aligned}
$$

(18)

where $\mathcal{H}$ is the single-particle Hilbert space, $\mathcal{H}_{\text{vac}}$ is the one-dimensional Hilbert space spanned by the vacuum state $|\text{vac}\rangle$, and $\tilde{\mathcal{H}} := \mathcal{H} \otimes \mathcal{H}_{\text{vac}}$. This gives the Kraus decomposition of a vacuum extension as

$$
[E_{i} := E_{i} \otimes \alpha_{i} |\text{vac}\rangle \langle \text{vac}|]_{i=1}^{n-1} ,
$$

(19)

where $\{E_{i}\}_{i=1}^{n-1}$ are Kraus operators of the original channel, and $\{\alpha_{i}\}_{i=1}^{n-1}$ are complex vacuum amplitudes, satisfying $\sum_{i=1}^{n-1} |\alpha_{i}|^2 = 1$. Mathematically, for a given channel $E$, the choice of vacuum extension $\tilde{E}$ is non-unique. However, the particular vacuum extension corresponding to a given physical transmission line is completely determined by its physics, for example, the Hamiltonian of the electromagnetic field.

Consider now the scenario where we want to coherently control whether a particle is transmitted through a channel $E^{(1)}$ or a channel $E^{(2)}$, depending on the state of a control system $P$, the 'path'. Given two channels $E^{(1)}, E^{(2)}$, with vacuum extensions $\bar{E}^{(1)}, \bar{E}^{(2)}$, we define a superposition of the two channels, specified by the vacuum extensions $E^{(1)}, E^{(2)}$, with the path fixed in the state $\omega$, as the channel

$$
S_{\bar{E}^{(1)} \bar{E}^{(2)}}^{(\omega)}(\rho) := U^{\dagger} \otimes E^{(1)} \otimes E^{(2)} \otimes U(\rho \otimes \omega) .
$$

(20)

Here, $U(\cdot) := U(\cdot)U^{\dagger}$ is the isomorphism $\mathcal{H} \otimes \mathcal{H}_{\rho} \rightarrow (\mathcal{H}^{(1)} \otimes \mathcal{H}_{\rho}^{(1)}) \otimes (\mathcal{H}^{(2)} \otimes \mathcal{H}_{\rho}^{(2)})$ that identifies the message $M$ and path $P$ in the 'particle picture' with the one-particle sector of the composite system $(\mathcal{H}^{(1)} \otimes \mathcal{H}_{\rho}^{(1)}) \otimes (\mathcal{H}^{(2)} \otimes \mathcal{H}_{\rho}^{(2)})$ in the 'mode picture'. Explicitly, $U$ is defined by

$$
\begin{aligned}
U(|\psi\rangle_{M} \otimes |0\rangle_{\rho}) &:= |\psi\rangle_{M^{\otimes 2} \otimes \text{Vac}} \otimes |\text{vac}\rangle_{M^{\otimes 2} \otimes \text{Vac}} \\
U(|\psi\rangle_{M} \otimes |1\rangle_{\rho}) &:= |\text{vac}\rangle_{M^{\otimes 2} \otimes \text{Vac}} \otimes |\psi\rangle_{M^{\otimes 2} \otimes \text{Vac}} .
\end{aligned}
$$

(21)

Substituting Eq. (19) into Eq. (20) gives the Kraus decomposition of Eq. (1)

for the superposition channel $S_{\bar{E}^{(1)} \bar{E}^{(2)}}^{(\omega)}$.

**Dephasing on the superposition of channels**

Here we extend the framework of the superposition of channels to more realistic scenarios where the path degree of freedom is subject to dephasing errors (14). As an application, we then derive Eq. (15).

Mathematically, the presence of dephasing on the path degree of freedom is equivalent to dephasing between the one-particle and vacuum sectors on one branch of the superposition:

$$
D_{s}(\rho) = s(I \otimes -|\text{vac}\rangle \langle \text{vac}|)\rho(I \otimes -|\text{vac}\rangle \langle \text{vac}|) + (1 - s)\rho ,
$$

(22)

where $s \in [0, 1/2]$. Then, the dephased superposition of channels is given by

$$
S_{\bar{E}^{(1)} \bar{E}^{(2)}}^{(\omega), s}(\rho) := U^{\dagger} \otimes (D_{s} \otimes \bar{E}^{(2)}) \otimes U(\rho \otimes \omega) .
$$

(23)

Note that $D_{s}$ commutes with any vacuum extended channel $\bar{E}^{(2)}$, because any vacuum extended channel is block diagonal with respect to the one-particle/vacuum sector partition.

Now consider the case where the message travels in a superposition of two identical channels $E$, with vacuum extensions $\bar{E}$, and the path qubit is initialised in the $|+\rangle$ state. A direct calculation (e.g. by substituting the Kraus operators $[E_{i} \otimes \alpha_{i} |\text{vac}\rangle \langle \text{vac}|]_{i=1}^{n-1}$ of $\tilde{E}$ into Eq. (23)), reveals that

$$
S_{\bar{E}^{(1)} \bar{E}^{(2)}}^{(\omega), s}(\rho) = \frac{\bar{E}(\rho) + (1 - 2s)\bar{E}(\rho) \otimes |+\rangle \langle +| + \bar{E}(\rho) \otimes |+\rangle \langle -| + \langle -| \langle +| \bar{E}(\rho) \otimes |+\rangle \langle +| \bar{E}(\rho) \otimes |+\rangle \langle -|}{2} + \bar{E}(\rho) \otimes |+\rangle \langle +|. (24)
$$

Applying Eq. (23) $n$ times, we obtain the result in Eq. (15), where we use the shorthand $S_{\bar{E}^{(1)} \bar{E}^{(2)}}^{(\omega), s}(\rho) = \sum_{i=0}^{n-1} \bar{E}^{i} \otimes \bar{E}^{i}$.

**Data availability**

This is a theoretical paper and no experimental data is available beyond the numerical calculations described in the paper.

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Additional information

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