A Gauge-Fixing Action for Lattice Gauge Theories

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ABSTRACT

We present a lattice gauge-fixing action $S_{gf}$ with the following properties: (a) $S_{gf}$ is proportional to the trace of $(\sum_\mu \partial_\mu A_\mu)^2$, plus irrelevant terms of dimension six and higher; (b) $S_{gf}$ has a unique absolute minimum at $U_{x,\mu} = I$. Noting that the gauge-fixed action is not BRST invariant on the lattice, we discuss some important aspects of the phase diagram.

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1. Gauge theories are presently being investigated in two ways: in the continuum (mostly) using perturbation theory, and on the lattice, which is the method of choice for nonperturbative calculations. In the continuum, the classical action needs to be changed in order to define the quantum theory; this is done by gauge fixing, which takes care of the otherwise ill-defined integration over gauge orbits. In the usual lattice approach the integration over the gauge group is well defined due to the compactness of the gauge group, and gauge fixing is not necessary. However, it is interesting to ask whether the continuum approach can be successfully implemented nonperturbatively, i.e. on the lattice.

A concrete proposal on how to do this was put forward in ref. [1] (henceforth called I) in an attempt to define lattice chiral gauge theories. Because of the fact that gauge invariance is broken when one regulates chiral gauge theories, it may in fact be necessary to gauge-fix the theory on the lattice, as has been suggested some years ago in ref. [2]. (See also ref. [3] for a recent review.) The central observation is that, due to the lack of gauge invariance, the longitudinal component of the gauge field couples to the fermions. It therefore becomes important to have a good control over the dynamics of the longitudinal degree of freedom. A natural way to achieve this is via gauge fixing, which can provide a kinetic term (as well as possible interaction terms) for the longitudinal component.

In I a lattice gauge-fixing action was proposed which in the continuum limit leads to a nonlinear gauge-fixing action of the form \((\partial \cdot A + gA^2)^2\). This gauge is not suitable for \(SU(N)\) theories, and is also less familiar. In this paper, we propose a lattice version of the usual covariant gauge-fixing term \((\partial \cdot A)^2\). In addition to presenting the form of the lattice action, we show that it has a single absolute minimum at \(U_{x,\mu} = I\) (\(U_{x,\mu}\) is the compact lattice gauge field), and we discuss some aspects of the phase diagram.

A central feature of our gauge-fixing approach is that the gauge-fixed lattice action will not be invariant under BRST transformations. This is important, because it was shown that, nonperturbatively, a BRST-invariant partition function, as well as expectation values of BRST-invariant operators, vanish as a consequence of the existence of lattice Gribov copies [4]. (If one is interested in perturbation theory only, BRST symmetry can actually be maintained on the lattice, see for instance ref. [5] and references therein.) This implies that the vector boson mass term (along with other BRST symmetry violating relevant and marginal operators) must be tuned to zero by hand [4]. The continuum limit is therefore characterized by a vanishing second derivative at the minimum of the potential for the vector field. This condition defines the boundary between a conventional Higgs or Higgs-confinement phase, and a new phase (denoted FMD) which is characterized by the condensation of a vector field.
Below, the transition between the rotationally invariant phase and the FMD phase will be denoted as “the FMD transition.”

A condensate which breaks rotational symmetry appears strange at first. However, perturbation theory with a regulator that is not gauge invariant already points at such a phenomenon. Because of the absence of gauge invariance of the regulated theory, a gauge-field mass counterterm will be needed, with a parameter that needs to be tuned. In the lattice version of the theory, one can envisage choosing this parameter “too small.” The resulting negative value for the renormalized squared gauge-boson mass suggests that spontaneous symmetry breakdown occurs.

2. In this section, we will construct a gauge-fixed lattice action $S(U_{x,\mu})$ which is invariant under \textit{global} gauge transformations in the gauge group, $G$ (but of course not under local transformations in this group). We will then show in the next section that this action has the desired properties.

It will be convenient to separate out the longitudinal degree of freedom by introducing an additional group-valued scalar field $\phi_x$, and replacing $U_{x,\mu} \rightarrow \phi_x \dagger U_{x,\mu} \phi_x + \hat{\mu}$. (This is a standard trick \cite{6} that leads to a new, mathematically equivalent, formulation of the theory.) The result is a Higher Derivative (HD) action $S(\phi_x \dagger U_{x,\mu} \phi_{x+\hat{\mu}})$, and the symmetry group is enlarged to $G_{\text{local}} \times G_{\text{global}}$:

$$U_{x,\mu} \rightarrow h_x U_{x,\mu} h_x \dagger, \quad \phi_x \rightarrow h_x \phi_x g \dagger, \quad (1)$$

where $g \in G_{\text{global}}$ represents the original global symmetry. One can regain the original formulation by choosing $\phi_x = I$, which amounts to gauge-fixing the enlarged, unphysical local symmetry. The only symmetry present in that case is the global symmetry

$$U_{x,\mu} \rightarrow g U_{x,\mu} g \dagger, \quad (2)$$

(setting $h_x = g$ in eq. (1)). This is the symmetry that we require to enlarge to a local symmetry in the continuum limit. The field $\phi_x$ explicitly represents the unphysical gauge degrees of freedom, which couple to the transversal degrees of freedom because the lattice action $S(U)$ is not invariant under local $g$-transformations \cite{3}.

One starts with a simple model that gives rise to the FMD transition described above. The action (see Sect. 3.a and 4.b of $I$), which borrows from previous work on higher derivative actions \cite{7}, is given by

$$S_H = \text{tr} \sum \left( -\kappa \phi \dagger \Box (U) \phi + \tilde{\kappa} \phi \dagger \Box^2 (U) \phi \right), \quad (3)$$

where

$$\Box_{xy} (U) = \sum_{\mu} (\delta_{x+\hat{\mu},y} U_{x,\mu} + \delta_{x-\hat{\mu},y} U_{y,\mu} \dagger) - 8\delta_{x,y}, \quad (4)$$
is the standard nearest-neighbor covariant laplacian, and the lattice spacing $a$ is set equal to one. For the gauge field we will assume the standard plaquette action. (Because $\phi_x$ is unitary, $S_H$ is in fact a function of $\phi^+_x U_{x,\mu} \phi_{x+\hat{\mu}}$, in agreement with the discussion above.)

Since we are interested in taking the gauge coupling $g_0$ to be small, it is relevant to consider the reduced model, which is obtained by turning off the gauge field in eq. (3) altogether. The symmetry of the reduced model is $G_{\text{global}} \times G_{\text{global}}$, which corresponds to setting $h_x = h$ in eq. (I). Here we will focus on the region of the phase diagram with $\tilde{\kappa} \to \infty$. The idea is that in the model with no gauge field $\tilde{\kappa} \to \infty$ is a zero temperature limit where $v \equiv \langle \phi \rangle \to I$. (Note that $\langle \phi \rangle$ is well-defined in the reduced model.) This breaks the symmetry $G_{\text{global}} \times G_{\text{global}} \to G_{\text{global}}$ (with $h = g$ in eq. (II)) consistent with the symmetry of the formulation with the scalar field fixed to $\phi_x = I$ (cf. eq. (2)).

Had we chosen $\tilde{\kappa}$ such that we were close to a phase transition where $v$ becomes small, $\phi_x$ would develop a radial mode dynamically, and new, undesired excitations would be present in the continuum limit. All this implies that the continuum limit must be taken well inside some broken phase of the reduced model. (We note that – in the case of chiral gauge theories – an alternative explanation as to why the continuum limit must be taken in a broken phase is provided by a generalized No-Go theorem [8], which asserts (modulo some delicate loopholes) that the fermion spectrum in any symmetric phase is vector-like.)

The continuum limit, then, will be defined by approaching the gaussian critical point $g_0 = 1/\tilde{\kappa} = 0$ on the FMD phase boundary. Since $v \to I$ for $\tilde{\kappa} \to \infty$, setting $\phi_x = I$ (which amounts to fixing the gauge for $G_{\text{local}}$ in eq. (IV)) provides the starting point for a valid perturbative expansion around this critical point. (For some additional discussion of the HD version of our action, we refer to the conclusion.) The usual weak coupling expansion $U_{x,\mu} = \exp(ig_0 A^\mu_{x,\mu} + \cdots)$ then gives

$$S_H|_{\phi_x = I} = \kappa g_0^2 \tr \left( \sum_\mu \left( A^2_\mu - \frac{g_0^2}{12} A^4_\mu + \cdots \right) \right)$$

$$+ \tilde{\kappa} g_0^2 \tr \left( \left( \sum_\mu \Delta^-_\mu A_\mu \right)^2 + g_0^2 \left( \sum_\mu A^2_\mu \right)^2 + \cdots \right),$$

(where we have suppressed coordinates summations). Note in particular the presence of the Lorentz symmetry violating term $\sum_\mu A^4_\mu$. The dots in eq. (3) stand for irrelevant operators. $\Delta^-_\mu$ is the backward lattice derivative, defined for an arbitrary function $f_x$ as $\Delta^-_\mu f_x = f_x - f_{x-\hat{\mu}}$.

As can be seen from eq. (3), the $\tilde{\kappa}$-term in the action eq. (3) leads to a kinetic term for the longitudinal part of the vector field. Motivated by this observation, we
set
\[ \tilde{\kappa} g_0^2 \equiv (2\alpha_0)^{-1}, \tag{6} \]
where \( \alpha_0 \) is assumed to be a parameter of order one. Thanks to the presence of kinetic terms for all polarizations, the vector lagrangian that governs the critical region is manifestly renormalizable. The \( \kappa \)-term in eq. (5) is seen to lead to a mass term for the vector field. As shown in more detail in I and below, at tree level the FMD transition occurs at \( \kappa = 0 \), with a non-zero vector condensate for \( \kappa < 0 \).

Now, we are interested in recovering a Yang–Mills theory in the continuum limit. To this end a renormalizable, but otherwise arbitrary, vector lagrangian will not suffice. What we need first is that the tree-level lagrangian will agree with the continuum lagrangian of a gauge theory, when the latter is quantized in a renormalizable gauge. Moreover, the lattice-regularized perturbation expansion explicitly breaks the gauge invariance of the target continuum theory. Therefore, the BRST identities must be enforced order by order in perturbation theory (the issue of nonperturbative tuning will not be addressed in this letter). A major role is played by the BRST identity that requires the renormalized vector-boson mass to vanish; this defines the location of the FMD transition. This fact is at the heart of our approach. Thus, \( \kappa \) is tuned to \( \kappa_{c.l.} \), where in perturbation theory the latter is given as a power series
\[ \kappa_{c.l.}(g_0, \alpha_0) = \sum_{n \geq 1} c_n(\alpha_0) g_0^{2(n-1)}. \tag{7} \]
Note the absence of an \( O(g_0^{-2}) \) term on the righthand side of eq. (7), in accordance with the requirement that the tree-level vector-boson mass vanish.

In order to obtain a gauge-fixed continuum action, the marginal terms on the second line of eq. (5) should be of the form
\[ \frac{1}{2\alpha_0} (\text{gauge condition})^2, \tag{8} \]
for some gauge condition. Hence we need an additional term that, without spoiling the phase diagram, will bring the marginal gauge-symmetry violating terms in the vector lagrangian into the form (8).

Clearly, one has two options. The new marginal term can be chosen to cancel the quartic term on the second row of eq. (5). In this case only the bilinear term will remain, which corresponds to the standard covariant gauge \( \partial \cdot A = 0 \). Alternatively, the new marginal term can be a mixed term proportional to \( (\sum \mu \partial_\mu A_\mu)(\sum \nu A_\nu^2) \). In this case one recovers the nonlinear gauge \( (\partial \cdot A + gA^2) = 0 \), which was used in I. The main disadvantage of this choice is that this nonlinear gauge condition is consistent only for \( U(1) \) or \( SU(N) \times U(1) \).
We will now present a new HD action $S_{HD}$, with corresponding gauge-fixing action

$$S_g(U) \equiv S_{HD}(\phi, U)\big|_{\phi_x = I},$$

that enjoys the following properties:

- $S_g$ admits the expansion $S_g = \frac{1}{2\alpha_0} \text{tr} \left( \sum_\mu \partial_\mu A_\mu \right)^2 + \text{irrelevant terms}$.
- $S_g$ has a unique absolute minimum at $U_{x,\mu} = I$.
- The important features of the phase diagram are unchanged.

The new HD action is given by

$$S_{HD} = \frac{1}{2\alpha_0 g_0^2} \text{tr} \sum_\mu \left( \phi^\dagger \square^2(U) \phi - B^2 \right),$$

$$B_x = \sum_\mu \left( \frac{V_{x-\mu,\mu} + V_{x,\mu}}{2} \right)^2,$$

$$V_{x,\mu} = \frac{1}{2i} \left( \phi^\dagger_x U_{x,\mu} \phi_{x+\mu} - \text{h.c.} \right).$$

(Note that the above definition of $V_\mu$ leaves out a $g_0^{-1}$ factor present in the corresponding definition in $I$.)

From the point of view of the weak coupling expansion, one has $V_\mu \rightarrow g_0 A_\mu + O(g_0^2)$. Thus, the reader can easily check that the unwanted $\left( \sum_\mu A_\mu^2 \right)^2$ term in eq. (5) is canceled by the new term.

3. Let us now discuss the properties of $S_g$ in more detail. Introducing

$$C_x = -\sum_y \square_{xy}(U),$$

one can write

$$S_g = \frac{1}{2\alpha_0 g_0^2} \sum_x S_x,$$

where

$$S_x = \text{tr} \left( C_x^\dagger C_x - B_x^2 \right).$$

Decomposing $C_x$ into its hermitian and anti-hermitian parts and using cyclicity of the trace, one has $S_x = S_x^{(1)} + S_x^{(2)}$ where

$$S_x^{(1)} = \text{tr} \left( \frac{C_x^\dagger - C_x}{2i} \right)^2,$$

$$S_x^{(2)} = \text{tr} \left( \frac{C_x^\dagger + C_x}{2} + B_x \right) \left( \frac{C_x^\dagger + C_x}{2} - B_x \right).$$
Substituting eq. (13) into eq. (16) leads to

\[ S_x^{(1)} = \text{tr} \left( \sum_{\mu} \Delta_\mu V_{x,\mu} \right)^2. \]  

(18)

The expression inside the brackets is recognized as a lattice transcription of the continuum \( \sum_\mu \partial_\mu A_\mu \). Thus, \( S_x^{(1)} \) provides the desired longitudinal kinetic term, up to irrelevant operators. These irrelevant terms are innocuous as long as one stays near the classical vacuum \( U_{x,\mu} = I \). However, \( S_x^{(1)} \) is known to have a host of other zeros along the trivial orbit. These minima are lattice artifact Gribov copies of the classical vacuum. As argued in I, if one were to use only \( S_x^{(1)} \) in the gauge-fixing action, one would end up with a phase diagram that differs qualitatively from the desired one.

(We note in passing that the zeros of \( \sum_\mu \Delta_\mu V_{x,\mu} \) correspond to extrema of the functional \( \text{Re tr} \sum_{x,\mu} U_{x,\mu} \) on a given gauge orbit. The choice of \( S_x^{(1)} \) as the gauge-fixing action would assign equal probability to all extrema of this functional. On the other hand, when one speaks about the lattice Landau gauge, one usually refers to picking the global maximum of that functional. For a manifestly gauge-invariant theory, such as lattice QCD, this is believed to be a valid gauge-fixing procedure. But here we want to be able to use an action to generate configurations which is not gauge invariant. Hence, the lattice Landau gauge method and other nonlocal methods such as the use of the laplacian gauge \([9]\) introduce a genuine nonlocality. It is very difficult to check whether this nonlocality disappears in the continuum limit. If it does not, this may entail some inconsistency in the analytic continuation back to Minkowski space. Our method avoids all these difficulties because locality is manifestly preserved.)

The role of \( S_x^{(2)} \) is to cure the above problem. As we will now show, \( S_x^{(2)} \) contains only irrelevant operators, and its unique absolute minimum is at \( U_{x,\mu} = I \). This validates weak coupling perturbation theory, and lattice artifact Gribov copies are suppressed by \( S_x^{(2)} \sim \text{constant}/(\alpha_0 g_0^2) \). \( S_x^{(2)} \) breaks BRST invariance explicitly, since \( S_x^{(1)} + S_x^{(2)} \) cannot be written as the square of a local gauge-fixing condition on the lattice.

Our aim is to prove that \( S_x^{(2)} \) is nonnegative, and that it vanishes only for \( U_{x,\mu} = I \). The trace of the product of two positive matrices is positive, and the positivity of \( (C_x^\dagger + C_x)/2 + B_x \) is obvious. Consequently, the positivity of \( S_x^{(2)} \) will follow once we show that \( (C_x^\dagger + C_x)/2 - B_x \) is a positive matrix too. It is a straightforward exercise to check that

\[ (C_x^\dagger + C_x)/2 - B_x = \sum_\mu \left( D_{x,\mu}^{(1)} + D_{x,\mu}^{(2)} \right), \]  

(19)
where

\[ D^{(1)}_{x,\mu} = \left( I - \frac{1}{4} \left( U_{x,\mu} + U_{x-\bar{\mu},\mu} + \text{h.c.} \right) \right)^2, \] (20)

\[ D^{(2)}_{x,\mu} = \frac{1}{2} I - \frac{1}{8} \left( U_{x,\mu} U_{x-\bar{\mu},\mu} + U_{x,\mu} U_{x-\bar{\mu},\mu} + \text{h.c.} \right). \] (21)

The positivity of \( D^{(1)}_{x,\mu} \) is manifest, whereas the positivity of \( D^{(2)}_{x,\mu} \) follows from the unitarity of the link variables.

We next show that \( S^{(2)}_x = 0 \) iff \( U_{x,\mu} = I \). For the abelian case this statement is trivial to check. In the nonabelian case, the condition \( S^{(2)}_x = 0 \) requires that there exists an orthogonal basis, such that each basis vector is a zero eigenvector of \((C^\dagger_x + C_x)/2 + B_x\) and/or \((C^\dagger_x + C_x)/2 - B_x\). Now, a zero eigenvector of the sum of two positive matrices must be a common zero eigenvector \((v^\dagger M v = 0 \iff M v = 0 \text{ for any positive hermitian matrix } M)\). Note that \((C^\dagger_x + C_x)/2 + B_x\) is explicitly the sum of two positive matrices and, in view of eq. (19), a similar statement applies to \((C^\dagger_x + C_x)/2 - B_x\). Therefore, each of the above basis vectors must in particular be a zero eigenvector of \(C^\dagger_x + C_x\) and/or \(\sum_\mu D^{(1)}_{x,\mu}\). It is easy to check that the zero eigenvectors of \(C^\dagger_x + C_x\) and \(\sum_\mu D^{(1)}_{x,\mu}\) are in fact common. They occur iff for all \(\mu\), \(U_{x,\mu}\) and \(U_{x-\bar{\mu},\mu}\) have a common submatrix equal to the identity. Thus, the condition \( S^{(2)}_x = 0 \) requires that both \(C^\dagger_x + C_x\) and \(\sum_\mu D^{(1)}_{x,\mu}\) be zero simultaneously, which is true iff \(U_{x,\mu} = U_{x-\bar{\mu},\mu} = I\). The proof is valid for unitary and orthogonal groups. It can probably be generalized to any compact group.

Lastly, we wish to check that \( S^{(2)}_x \) contains only irrelevant operators. One has the following expansion

\[ (C^\dagger_x + C_x)/2 + B_x = 2g_0^2 \sum_\mu A_\mu^2 + \cdots, \] (22)

where only the lowest dimensional operator is shown. Similarly,

\[ D^{(1)}_{x,\mu} = \frac{g_0^4}{4} A_\mu^4 + \cdots, \] (23)

\[ D^{(2)}_{x,\mu} = \frac{g_0^2}{4} \left( \Delta_\mu A_\mu \right)^2 + \cdots. \] (24)

From these expansions it follows that \( S^{(2)}_x \) only contains operators of dimension six and higher.

We now digress momentarily to close a gap in the formulation of the nonlinear gauge presented in \( I \). While the minimum of the classical potential was shown to be \( A_\mu = 0 \), it was not established that \( A_\mu = 0 \) remains the absolute minimum when \( A_\mu \) is allowed not to be constant. A lattice action that features the same properties as
eq. (10), except that the marginal terms correspond to the continuum gauge-fixing action \((\partial \cdot A + gA^2)^2\), is given by

\[
S_{n.l.}^{HD} = \frac{1}{2\alpha_0 g_0^2} tr \sum \left( \phi \Box^2 (U) \phi + 2B \sum \Delta \mu \Delta V \phi \right).
\] (25)

The corresponding gauge-fixing action density is

\[
S_{n.l.}^x = tr \left( \sum \Delta \mu \Delta V_{x,\mu} + B_x \right)^2 + S^{(2)}_x,
\] (26)

where \(S^{(2)}_x\) is the same as in the linear case (cf. eq. (17)). Thus, in both cases the same irrelevant operator \(S^{(2)}_x\) is used to protect the uniqueness of the absolute minimum at \(U_{x,\mu} = I\).

4. We will now discuss some properties of the phase diagram of the theory defined by the new action eq. (10). (We plan to present a more complete analysis elsewhere.) For large \(\tilde{\kappa}\) (and small \(g_0\)) one is in a broken phase, which could be an ordinary broken phase or an FMD phase. As we will now see, the latter is characterized by a vectorial order parameter that defines a preferred direction. (The large-\(\tilde{\kappa}\) rotationally-invariant region of the phase diagram is a Higgs or Higgs-confinement phase. With “ordinary broken phase,” we refer to the large-\(\tilde{\kappa}\) properties of this phase.)

In order to look for the FMD transition, we set \(\phi_x = I\) and take \(U_{\mu} = \exp(iA_{\mu})\) constant (assuming that translation invariance is not broken), and we minimize the free energy with respect to \(A_{\mu}\). (Here we rescaled \(g_0 A_{\mu} \rightarrow A_{\mu}\).) The task is simplified in the limit \(\tilde{\kappa} \rightarrow \infty\), or equivalently \(g_0 \rightarrow 0\) (cf. eq. (11)), which is the region of the phase diagram where we want to be anyway. In that case, the free energy is just the classical potential for \(A_{\mu}\). As we will now show, \(\kappa = 0\) is the location of the FMD transition classically, and a nonzero \(\langle A_{\mu} \rangle\) develops for \(\kappa < 0\).

In order to find the FMD transition for large \(\tilde{\kappa}\), we only have to keep the lowest dimensional terms in the classical potential separately for the \(\kappa\)- and \(\tilde{\kappa}\)-terms. This leads to

\[
V_{cl} \approx \kappa \text{tr} \sum \mu A_{\mu}^2 + \tilde{\kappa} \text{tr} \left( \sum \mu A_{\mu}^2 \right) \left( \sum \nu A_{\nu}^4 \right).
\] (27)

(For \(\kappa/\tilde{\kappa}\) small, it is consistent to keep only the quadratic part of the \(\kappa\)-term.) Minimizing this with respect to \(A_{\nu}\), and taking the gauge group to be \(U(1)\), we obtain

\[
\left[ 2\kappa + \tilde{\kappa} \left( \sum \mu A_{\mu}^4 + 2 \left( \sum \mu A_{\mu}^2 \right) A_{\nu}^2 \right) \right] A_{\nu} = 0, \quad \text{for all } \nu.
\] (28)

Assuming \(\kappa < 0\), the minimum is found to be

\[
\langle A_{\mu} \rangle = \pm \left( \frac{|\kappa|}{6\tilde{\kappa}} \right)^{\frac{1}{4}}, \quad \text{all } \mu.
\] (29)
In the nonabelian case, one also has to take into account the contribution from the plaquette term. For $SU(2)$ this leads to the requirement that the $\langle A_\mu \rangle$ commute. Up to a similarity transformation, $\langle A_\mu \rangle$ is equal to $\sigma_3$ times the righthand side of eq. (29).

Note that the expectation value points in one of the sixteen directions defined by the lattice vectors $(\pm 1, \pm 1, \pm 1, \pm 1)$. This is not surprising since the classical potential eq. (27) is invariant only under the lattice rotation group, but not under an arbitrary $O(4)$ rotation. The vectorial expectation value leaves unbroken the subgroup of lattice rotations in the hyperplane perpendicular to $\langle A_\mu \rangle$.

As follows from eq. (29), the mean-field critical exponent now is $1/4$ rather than $1/2$ as found in I in the nonlinear case. This suggests that the new critical point is in fact a tricritical point in some larger parameter space. This is indeed the case. Quantum corrections will require the addition of counterterms, and for constant $A_\mu$ the only possible ones are $(\sum_\mu A_\mu^2)^2$ and $\sum_\mu A_\mu^4$ (we consider again the $U(1)$ case for simplicity). So let us consider a more general potential of the form

$$V = \kappa \sum_\mu A_\mu^2 + \beta (\sum_\mu A_\mu^2)^2 + \gamma \sum_\mu A_\mu^4 + \frac{\tilde{\kappa}}{2} (\sum_\mu A_\mu^2) (\sum_\nu A_\nu^4). $$

(30)

We assume $\tilde{\kappa} > 0$. Again, this approximation is self-consistent for $\tilde{\kappa}$ large relative to the other couplings. The minimization conditions now become

$$\kappa + 2\beta \sum_\mu A_\mu^2 + 2\gamma A_\mu^2 + \frac{\tilde{\kappa}}{2} \left( \sum_\mu A_\mu^4 + 2 \left( \sum_\mu A_\mu^2 \right) A_\nu^2 \right) = 0,$$

(31)

for all components of the vector field that do not vanish. All nonvanishing components have to be equal (up to signs), and if we set those all equal to $\mathcal{A}$, assuming that $n \in \{1, 2, 3, 4\}$ of them do not vanish, we obtain

$$\frac{3}{2} n \tilde{\kappa} A^4 + 2(n\beta + \gamma) A^2 + \kappa = 0.$$  

(32)

The value $V_0$ of the potential at any extremum point can be written as

$$V_0 = \frac{n A^2}{3} \left( 2\kappa + (n\beta + \gamma) A^2 \right).$$

(33)

This expression is useful in studying the order of the transition. Note that we are dealing here with a three parameter phase diagram, spanned by $\beta$, $\gamma$ and $\kappa$ ($\tilde{\kappa}$ is large and can be scaled away by absorbing it into $\mathcal{A}$). We will now show that for $\gamma + \min(\beta, 4\beta) > 0$ there is a second order transition, whereas for $\gamma + \min(\beta, 4\beta) < 0$ the transition is first order. Assume first $\gamma + \min(\beta, 4\beta) > 0$. For any $\kappa > 0$, the lefthand side of eq. (32) is greater than zero, whereas for $\kappa = 0$ the potential has a quartic zero at the origin. Hence, $\kappa = 0$ is a second order transition point. Now
assume $\gamma + \min(\beta, 4\beta) < 0$. For $\kappa = 0$, eq. (32) has a solution $A^2 > 0$ for which $V_0 < 0$ at least for one $n$. This implies that a first order transition has already occurred at some $\kappa > 0$. The first order surface joins the second order surface smoothly at the tricritical line $\gamma + \min(\beta, 4\beta) = 0$, $\kappa = 0$, separating a rotationally-invariant phase ($A = 0$) from an FMD phase ($A \neq 0$). Close to the tricritical line the first order surface is $\kappa = (n\beta + \gamma)^2/(2n\tilde{\kappa})$ where $n\beta = \min(\beta, 4\beta)$, corresponding to $n = 1$ for $\beta > 0$ and $n = 4$ for $\beta < 0$. Away from the tricritical line (and in the quadrant $\beta < 0$, $\gamma < 0$) the value of $n$ may be different.

5. In this letter we have addressed the question as to how a nonperturbative (i.e. lattice) definition can be given of a Lorentz gauge-fixed Yang–Mills theory. In particular, we proposed a lattice version of the gauge-fixing action that has a unique global minimum at $U_{x,\mu} = I$, and that has the correct classical continuum limit. The model can be studied in weak coupling perturbation theory. We also expect that lattice artifact Gribov copies will be suppressed in the continuum limit.

Because the lattice gauge-fixed action is not BRST invariant, the integration over gauge orbits leads to nontrivial dynamics. We argued that a new second order phase transition is expected between a Higgs or Higgs-confinement phase, and an FMD phase which is characterized by the condensation of the vector field. As explained in the introduction, standard perturbative arguments already indicate that such a phase transition is unavoidable if one regulates a gauge theory in a gauge noninvariant way. A mass term for the gauge field will generically appear, and will have to be subtracted in order to keep the gauge field massless. From the lattice point of view, this corresponds to tuning to a continuous phase transition, and the desired continuum theory corresponds to the critical theory (for $g_0 \to 0$). (With a gauge-invariant regulator, the gauge symmetry guarantees the theory to be at this critical point.)

The details of this symmetry breaking, such as the critical exponent and the allowed directions of the condensate, depend on the gauge condition and the specific regulator employed. As we showed, our lattice discretization of the Lorentz gauge-fixing action corresponds to a tricritical point. In practice, one should be so close to the critical point that, within a given accuracy, the nonphysical effects from being in either the Higgs or the FMD phase are small enough. This should be possible, since the phase transition is a continuous one.

There are two equivalent formulations for the lattice model we presented in this letter. The HD version makes the gauge degrees of freedom explicit through the (unphysical) group-valued scalar field $\phi_x$, while the gauge-fixed version is obtained by setting $\phi_x = I$. This is not specific to the lattice formulation of the model: the HD
version of any gauge-fixed continuum gauge theory can be defined in a similar way. In perturbation theory, of course, one usually does not introduce the scalar field. On the lattice, however, it is useful to do this, since that makes it possible to first consider the reduced model – the pure scalar theory obtained by setting $U_{x,\mu} = I$ in the HD action. The latter is more easily amenable to nonperturbative techniques such as numerical simulation. This brings up an interesting point: the quadratic part of the scalar action contains four-derivative terms, and therefore raises the specter of infrared behavior divergences. This was discussed in some detail in $I$, where arguments were given that infrared divergences in fact do not arise. We expect this because of the intimate relation between perturbation theory in the reduced model and in the full model, and since in the full model one has standard IR behavior. We intend to report on a more detailed investigation of this point in the near future.

In order to complete the definition of the model, a Faddeev–Popov term (in the nonabelian case), and counterterms (of which the $\kappa$-term and the $\beta$- and $\gamma$-terms in eq. (30) already are examples) will have to be added. The coefficients of the counterterms are calculable in perturbation theory. Note that the divergent as well as the finite parts of the counterterms are needed to recover the BRST identities. The $\phi_x$ dependence can be made explicit by replacing $g_0A_\mu$ with $V_\mu$ defined in eq. (12). Once the complete action is constructed, we may again study the phase diagram. The Faddeev–Popov ghosts are of course crucial for unitarity of the target continuum gauge theory. However, their effects only come into play at one loop (where the optical theorem would be violated without ghosts), i.e. at order $g_0^2 \sim 1/\tilde{\kappa}$ (eq. (12)). The interaction of the ghosts with $\phi_x$ will therefore be suppressed by $1/\tilde{\kappa}$, and hence we expect that the ghosts will not change the essential features of the FMD transition at large $\tilde{\kappa}$. The effect of counterterms on the potential for $A_\mu$ has already been discussed above.

First, however, a detailed investigation of the phase diagram(s) of the actions given in eqs. (3) and (10) with $U_{x,\mu} = I$ is in order. (In the reduced model, the FMD phase is characterized by a nonzero momentum of the ferromagnetic groundstate $[1]$, and the FMD transition is actually an FM-FMD transition in the relevant part of the phase diagram.) The nature of the FMD transition should be studied in more detail in order to find out whether this approach to lattice gauge theories may lead to the same results as the standard (perturbative) continuum version and the usual gauge-invariant lattice approach. Of course, after that many issues remain, such as the explicit construction of ghost- and counterterms, and the inclusion of the full gauge field. Lattice artifact Gribov copies should be investigated in more detail, and then the problem of continuum Gribov copies should be addressed. If this program is
successful, it may lead to a method for constructing nonperturbative versions of gauge
theories for which no gauge-invariant formulation is known. Chiral gauge theories
constitute an example where gauge fixing appears to address the essential problems
that sofar have hampered attempts to define them on the lattice.

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