Mirror Transform and String Theory

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Some aspects of Mirror symmetry are reviewed, with an emphasis on more recent results extending mirror transform to higher genus Riemann surfaces and its relation to the Kodaira-Spencer theory of gravity\textsuperscript{1}. 

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One of the most beautiful aspects of string theory is that strings moving on one manifold may behave identically with strings moving on a different manifold. Any pair of manifolds which behave in this way are called mirror pairs. In this paper I will review some aspects of this phenomenon (see [1] for a collection of articles on this subject).

One of the most successful methods in solving difficult questions in mathematical physics has been to transform the problems from their initial setup to a more trivial setup where everything can be understood in simple terms. One of the most classical examples of this is Fourier transform which transforms a differential equation to a polynomial equation which is easy to solve. A more recent such example is the twistor transform which maps the classification of self-dual geometries and self-dual Yang-Mills fields in four dimensions to the question of classification of appropriate complex manifolds and holomorphic vector bundles in six (real) dimensions which is more manageable. The most recent example of such a transformation is the mirror transform which is the subject of this paper. The idea is that one maps certain difficult questions of interest in algebraic geometry on one manifold to simple questions of variation of Hodge structures on a different (mirror) manifold. As we shall see, however, the status of this transformation is not as clearly understood as the other two cases we mentioned.

We first need to recall some basic aspects of string theory. The main ingredient in a string theory is a two dimensional quantum field theory. An interesting class of 2d QFTs are sigma models, which may be viewed as path integrals over the space of maps \( \phi \) from a Riemann surface \( \Sigma^g \) (of genus \( g \)) to a manifold \( M \):

\[
\phi : \Sigma^g \longrightarrow M
\]

The integral over the space of maps is written abstractly as

\[
\int D\phi e^{-S(\phi)} = F_g(M)
\]

where \( S(\phi) \) is the energy functional \( S(\phi) = \int_{\Sigma^g} |d\phi|^2 \) and \( F_g(M) \) is called the genus \( g \) partition function of strings propagating on \( M \). In most cases of interest in superstrings one considers supersymmetric sigma models which means that in addition to maps from the Riemann surface to \( M \) one considers fermionic fields which take their values in the tangent bundle to \( M \) and one modifies \( S \) in an appropriate way to obtain a supersymmetric theory.
We say that strings cannot distinguish between $M$ and $M'$, or $M \equiv M'$ if and only if

$$F_g(M) = F_g(M') \lambda^{g-1} \quad \text{for all } g$$

for some constant $\lambda$ (playing the role of string coupling constant). The simplest example of mirror symmetry corresponds to choosing $M$ to be a circle of circumference $L$ and $M'$ to be a circle of circumference $1/L$. This case is particularly easy because $F_g(M)$ can be explicitly computed (this is seldom the case in more interesting examples). Identifying the map $\phi$ from $\Sigma^g$ to the circle with the coordinate on the circle, we have

$$\phi \sim \phi + L \quad \text{for } M_1$$

$$\phi \sim \phi + \frac{1}{L} \quad \text{for } M_2$$

The space of maps from the Riemann surface to the circle decomposes into infinitely many components depending on how the surface wraps around the circle. To be concrete, let $a_i, b_i$ as $i$ runs from 1 to $g$ denote a canonical basis for $H_1(\Sigma^g)$, and let $\alpha^i$ and $\beta^i$ be the corresponding harmonic one forms. Then for each set of integers $(n_i, m_i)$ we get a component of the map of the surface to $M$ characterized by the condition that

$$d\phi = (n_i \alpha^i + m_i \beta^i) L + d\tilde{\phi}$$

where $\tilde{\phi}$ is a univalued function on $\Sigma$. Expanding the action $S[\phi]$ for this component we get

$$S[\phi] \to L^2 \int (n\alpha + m\beta) \wedge * (n\alpha + m\beta) + S[\tilde{\phi}]$$

Now we can consider the $L$ dependence of $F_g$. We will have to sum over all disconnected components of such maps. However the fact that for each component $S$ decomposes in the way described above we see that

$$F_g(L) = \left( \sum_{n,m} \exp\left[\int (n\alpha + m\beta) \wedge * (n\alpha + m\beta)\right]\right) \cdot \int D\tilde{\phi} \exp[S[\tilde{\phi}]]$$

Note that the path integral we are left to perform is independent of $n, m$. Even though the path integral over $\tilde{\phi}$ can be computed easily the important aspect to emphasize in our

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2 I am giving a definition of equivalence which is necessary but has not been proven to be sufficient, though I believe it is also sufficient. The strict definition of equivalence will have to include not just the partition function but all the correlation functions.
case is its dependence on $L$. The only dependence on $L$ comes from the constant maps $\tilde{\phi}$ where it gives a factor of $L$ (the rest of it just gives us $(det' \Delta/ \int \sqrt{g})^{-1/2}$ where $\Delta$ is the Laplacian on the Riemann surface). We thus immediately deduce that

$$\frac{F_g(L_1)}{F_g(L_2)} = \frac{L_1 \sum_{n,m} \exp[-L_1^2 \int (n\alpha + m\beta) \wedge *(n\alpha + m\beta)]}{L_2 \sum_{n',m'} \exp[-L_2^2 \int (n'\alpha + m'\beta) \wedge *(n'\alpha + m'\beta)]}$$

Note that if $L_2 = 1/L_1$ then this ratio is equal to $(1/L_1^2)^{g-1}$ by using Poisson resummation on $(n, m) \to (n', m')$. We have thus seen that

$$F_g(L) = F_g(1/L)\lambda^{g-1}$$

where $\lambda = 1/L^2$. This implies that two circles with circumferences which are inverse to each other are mirror pairs. One can easily extend this example to the target being a $d$-dimensional torus. This mirror symmetry has actually been used to give a model of string cosmology 2. One of the main troubles of early universe cosmology for point particle theories is that the universe would be singular at the time of the big bang. This gives rise to many unphysical things such as infinite temperature and infinitely strong gravitational fields. In the context of strings a toroidal universe will not have these difficulties since it maps a singular universe (a torus with zero size) to infinite size universe which is manifestly non-singular.

In the context of superstrings one usually considers target spaces being a $d$-complex dimensional Kähler manifold which admits a Ricci-flat metric. Such Kähler manifolds are known as Calabi-Yau manifolds (Kähler manifolds with trivial canonical line bundle which admit a nowhere vanishing holomorphic $d$-form). Of particular importance in studying sigma models on Calabi-Yau is the structure of the moduli space. There are two types of deformations to consider: Complex deformations and Kähler deformations. The complex deformations of Calabi-Yau relevant for sigma models is the classical one (studied in 3). The dimension of this moduli space is equal to the dimension of $H^1(M, T)$ which for Calabi-Yau manifolds is the hodge number $h^{1, d-1}$. The Kähler deformations in the context of sigma models is more complicated than the classical picture and in particular belong to the complexified $H^{1,1}(M)$. The real part of it plays the usual role of a Kähler class whereas the complex part plays the role of introducing a phase in the measure of the sigma model. The basic idea is that if $b$ denotes the complex part of the Kähler class this means that we modify the path integral measure $D\phi$ by

$$D\phi \to D\phi \exp(i \int \Delta \phi(b))$$
where $\phi^*(b)$ is the pullback of the two form $b$ to the Riemann surface. The reason it is called the complexified Kähler class is that it can be unified with the real part to write the action in a way which automatically includes the above twisting of the path integral measure. Let $\tilde{g}_{ij} = g_{ij} + i b_{ij}$. Let $\phi^i$ denote the map $\phi$ in component form. Then the action can be written as

$$S = \int \tilde{g}_{ij} \partial \phi^i \bar{\partial} \phi^j + \tilde{g}^*_{ij} \bar{\partial} \phi^i \partial \phi^j$$

It often is convenient to formally think of $\tilde{g}$ and $\tilde{g}^*$ as independent parameters and take the $\tilde{g}^* \to \infty$. In this case the second piece of the action blows up for generic maps and the path integral will be finite only for holomorphic maps where the second piece vanishes. In this way the asymmetric limit of fixing $g$ and sending $\tilde{g}^* \to \infty$ gives a path-integral which basically measures how many holomorphic maps there are from the Riemann surface to the Kähler manifold.

Let us consider a simple example of one dimensional Calabi-Yau manifolds. This is nothing but the elliptic curve. The moduli space of this manifold as far as the complex deformation is concerned is parametrized by one complex parameter $\tau$ on the upper half plane. One represents the torus in the standard fashion as a parallelogram with sides 1 and $\tau$ imbedded in the complex plane. To get the moduli space we have to recall that the $\tau$’s differing by the $PSL(2, \mathbb{Z})$ action represent the same torus. As far as the Ricci-flat Kähler metrics are concerned, the area $A$ of the flat torus is the only parameter. However as discussed above this is complexified to $A - iB$. Let us define the complex parameter $\rho$ parametrizing complexified Kähler deformations by

$$\rho = iA + B$$

Note that since $B$ affects the path-integral only by multiplication by phases it is a periodic variable as far as the moduli space of the theory is concerned. So we have to identify

$$B \sim B + 1$$

which means that

$$\rho \sim \rho + 1$$

There is in fact a larger symmetry to mod out in order to get the moduli space: Let us consider a rectangular torus of size $(R_1, R_2)$ on each side. Setting $B = 0$ this gives

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3 Strictly speaking one needs to consider the topologically twisted sigma model to get this \[4\].
\[ \rho = iR_1 R_2. \]

From the discussion about the duality of the circle, it should be clear that if we consider a rectangular torus with sides \((1/R_1, 1/R_2)\) we get an equivalent theory: This gives \(\rho = i/(R_1 R_2)\). So we see that in this case we have the following identification

\[ \rho \sim \frac{-1}{\rho} \]

It can be shown that this applies even to more general configurations of the torus, and so the moduli space of Kähler deformation, is identified with \(\rho\) up to the group generated by these two transformations which magically enough is again \(PSL(2, \mathbb{Z})\). We therefore see that \(\tau\) and \(\rho\) play very similar roles in this example, even though one parametrizes the complex structure and the other parametrizes the Kähler structure. In fact as it turns out to get the full moduli space there is an extra symmetry to mod out and that is the \(\tau \leftrightarrow \rho\) exchange. To see this consider the special case of rectangular torus discussed above. To begin with we have

\[ (R_1, R_2) \to \tau = \frac{iR_2}{R_1} \quad \rho = iR_1 R_2 \]

We can consider an equivalent theory by taking the duality transformation on the first circle. This leads to

\[ \left( \frac{1}{R_1}, R_2 \right) \to \tau' = iR_2 R_1 \quad \rho' = \frac{iR_2}{R_1} \]

which means that there is an exchange symmetry of \(\tau\) and \(\rho\). This can be shown to be more generally valid for any torus and leads to the first example of the mirror phenomenon. Below we shall see that this is a special case of the general phenomenon of Calabi-Yau mirrors where the role of complex and Kähler deformations are exchanged.

Even though the above example is very instructive it is a bit too special and one would like to find more non-trivial examples of the above phenomenon. It turns out there is at least a hint: In distinguishing manifolds we consider topological invariants. For example the Euler characteristic \(\chi\). We conclude that if

\[ \chi(M_1) \neq \chi(M_2) \to M_1 \neq M_2 \]

So we can easily disprove the existence of the isomorphism between \(M_1\) and \(M_2\) if their Euler characteristic is different. Similarly for string theory we should try to construct invariants of 2d QFT in order to distinguish them. If two QFT’s have different such invariants then they cannot be mirror pairs. These invariants are to be constructed out of objects which canonically make sense in the algebraic formulation of QFT’s without any
recourse to a target manifold. For supersymmetric QFT’s which are the ones which arise in considering superstrings propagating on Kähler manifolds, there is a basic invariant which is Witten’s index

\[ I = Tr(-1)^F \]

where \((-1)^F\) is a mod 2 gradation of the Hilbert space of the QFT (by using the fermion number) and the trace is over the full Hilbert space. \((-1)^F\) is characterized by the fact that it squares to one and it anticommutes with the operator which implements supersymmetry \(Q\). However this makes \((-1)^F\) well defined only up to an overall sign change

\[ (-1)^F \rightarrow -(-1)^F \]

We thus see that only \(|I|\) is an invariant for the QFT. If two QFT’s arising from sigma models on two manifolds have different \(|I|’s\) we can deduce that they cannot be mirror pairs. In the case of a supersymmetric sigma model on a manifold \(M\) it turns out that

\[ |I| = |\chi(M)| \]

We therefore deduce that if \(|\chi(M_1)| \neq |\chi(M_2)|\) then \(M_1\) and \(M_2\) cannot be mirror pairs. This however leaves open the door for two manifolds being topologically distinct and having \(\chi(M_1) = -\chi(M_2)\) but which are nevertheless mirror pairs; at least the index \(|I|\) does not distinguish them. Note that this condition for finding mirror pairs fits very well with the example of torus considered above because as we saw torus was its own mirror and on the other hand the Euler characteristic of torus is zero.

We could try to ask if there are analogs of more refined invariants such as the betti numbers, or for the case of the Calabi-Yau manifolds, which is of most interest to us, the hodge number \(h^{p,q}\), which can distinguish different sigma models. It turns out that \(h^{p,q}\) can be defined as the dimension of a canonical subspace of the Hilbert space of the QFT (the ground state subsector with a particular \(U(1) \times U(1)\) charge). But there is still an ambiguity. Let \(d\) be the complex dimension of the Calabi-Yau manifold. Define

\[ \hat{p} = -\frac{d}{2} + p \]

Then there is a canonical \(U(1) \times U(1)\) gradation of the Hilbert space of the corresponding QFT. Let \(\mathcal{H}^{\hat{p},\hat{q}}\) denote the subspace of the Hilbert space consisting of ground states with \(U(1) \times U(1)\) gradation given by \(\hat{p}, \hat{q}\). Then

\[ h^{p,q} = \dim \mathcal{H}^{\hat{p},\hat{q}} \]
However again it turns out that there is a $Z_2$ ambiguity in canonically assigning the $U(1) \times U(1)$ gradation which is obtained by flipping the relative sign of the two $U(1)$’s: $(\hat{p}, \hat{q}) \rightarrow (-\hat{p}, \hat{q})$. This in particular means that the dimension of a given $\mathcal{H}_{\hat{p}, \hat{q}}$ could in principle correspond either to $h^{p,q}$ or $h^{d-p,q}$. Just as before the existence of this ambiguity leaves the door open for having mirror pairs $(M_1, M_2)$ for which in particular

$$h^{p,q}(M_1) = h^{d-p,q}(M_2)$$

$$\chi(M_1) = (-1)^d\chi(M_2)$$

It was conjectured based on the existence of simple toroidal examples and the lack of any method to resolve the $Z_2$ ambiguity in computing $h^{p,q}$ that this kind of mirror pairs always exist [6].

Note that for a Calabi-Yau manifold the dimension of moduli space of complex deformations is given by $h^{d-1,1}$ (this is proven in [3]) and the Kähler deformations by $h^{1,1}$. Therefore for mirror pairs the two moduli spaces interchange their roles: The complex deformations for one correspond to Kähler deformations for the other and vice-versa.

The mirror conjecture as stated cannot be possibly true because there exist rigid Calabi-Yau manifolds with no complex deformations. So their mirrors will have no Kähler deformations, i.e., it will not even be a Kähler manifold. However there is a generalization of the conjecture (involving supermanifolds) which can take care of such cases [7].

An important element in relating $(N = 2)$ supersymmetric 2d QFT’s (and in particular conformal field theories) with geometry of Calabi-Yau was Gepner’s construction [8]. These were CFT’s which were constructed by taking the tensor product of simplest representations of $(N = 2)$ Virasoro algebra (minimal models) which had an unexpected similarity with what one would expect for CFT’s coming from sigma models on certain Calabi-Yau manifolds. Subsequently it was discovered that the Landau-Ginzburg description of $(N = 2)$ minimal models [9] associates an isolated singularity (taken as the superpotential of Landau-Ginzburg model) to each minimal model. The simplest ones correspond to monomials $x^n$. Thus the Gepner’s construction was reinterpreted in this context as taking combinations of these monomials and constructing quasi-homogeneous polynomials to be identified with the defining equation of the associated Calabi-Yau manifold in weighted projective space [10] (see also the recent work [11]).

From this point on one could use amazing properties of conformal field theories to construct mirror pairs. In particular if one considers the minimal model given by $x^n$, the
theory has a $Z_n$ symmetry given by multiplying $x$ by an $n$-th root of unity. It turns out that if we divide out this $Z_n$ symmetry we end up with the original theory again! This is somewhat unexpected as usually one expects that if we divide out a theory by a symmetry we should get a theory with fewer degrees of freedom. This is not the case for conformal theories because dividing by a symmetry gets rid of some states which are not invariant but at the same time introduces new states coming from loops which are closed only up to the group action. So the total degrees of freedom remain constant. That we end up with the same theory in this case can be seen by realizing that this model has an associated circle corresponding to phases of $x$ and modding out by $Z_n$ changes its radius from $R$ to $1/R$, and thus the previous argument applies to show that we end up with the original theory.

First non-trivial examples of Calabi-Yau mirrors were constructed in [12] using this symmetry. In particular it was shown there that if we consider the quintic given by $\sum_{i=1}^{5} x_i^5 = 0$ in $CP^4$ and modding out by the maximal subgroup of $Z_5$ phase multiplications of each monomial consistent with preserving the holomorphic 3-form $(Z_5 \times Z_5 \times Z_5)$ one gets the same theory back but now interpreted as a sigma model on a quotient of the quintic three-fold. There was also a large class of examples suggested by computation of the Euler class of Calabi-Yau threefolds (as hypersurfaces in weighted projective space) in which for almost all manifolds searched of Euler class $\chi$ there was one with Euler class $-\chi$ [13]. There has been further work giving a conjectured construction for mirror pairs for a very wide class of examples [14].

The existence of mirror pairs as discussed up to now seems like a pure curiosity as far as string theory is concerned. It simply points out that topological invariants are not necessarily good stringy invariants. However what makes the existence of mirror pairs mathematically interesting is that certain computations of QFT correlation functions in one manifold which have one mathematical interpretation for one manifold have a completely different interpretation for the other. Therefore the existence of mirror symmetry implies in particular that two different mathematical computations on two distinct manifolds are unexpectedly equal. Moreover it turns out that on the one side the computation is simple and on the other side no one knows how to perform it! The mirror pairs thus serve in this context as a way to transform a difficult computation to an easy one.

The difficult computation is what is known as the computation of quantum cohomology ring [4], which is a deformation of the classical cohomology ring. The deformation parameters are in one to one correspondence with Kähler deformations of the manifold. In
the limit we take the size of the manifold to infinity the quantum cohomolgy ring reduces to the classical one. The simplest way to describe it is to note that classical cohomology ring can be described by counting the intersection points of dual cycles representing the cohomology classes. The quantum cohomology ring is simply a blurred version of classical intersection theory where the intersection point is replaced by a rational curve in the manifold. We say three cycles intersect if they all intersect a given rational curve. We weigh each such intersection by the number of points they intersect the rational curve and in addition by \( \exp(-\int k) \) where \( k \) is the (complexified) Kähler class and it is integrated over the rational curve in question. Note that if we let \( k \to \infty \) the rational curves are suppressed in this quantum intersection theory and so we are left with the degree zero rational curves which are just points, i.e. we get back the classical intersection theory.

The hard computation is thus a coholomolgy ring deformation which is a function of Kähler moduli. The general structure of mirror pairs discussed above suggests that the mirror computation should involve rings which are functions of complex moduli. It turns out that such objects have already been encountered in the mathematics under the general subject of studying variations of Hodge structure. For the particular case of the Calabi-Yau threefold the mirror of the quantum triple intersection defined above can be computed as follows: For each cycle dual to the Kähler class on one manifold there is a direction for the variation of complex structure on the mirror manifold. So fixing three classes in the previous computation will correspond to fixing three directions for the variation of complex structure on the mirror, which we denote by \((i, j, k)\). Let \( \omega \) be a non-vanishing holomorphic 3-form of the Calabi-Yau manifold. The mirror computation is simply written as

\[
C_{ijk} = \int \omega \wedge \partial_i \partial_j \partial_k \omega
\]

This is easy to compute and so we have managed to transform a difficult computation of quantum triple intersection on one manifold to classical questions of variations of Hodge structure on the other. The first explicit example was carried out in [15] for the quintic three fold and led to the impressive computation of the number of rational curves of arbitrary degree on the quintic.

One can continue this line of correspondence in the context of arbitrary genus Riemann surface. On the difficult side one has to compute the number of holomorphic maps from a genus \( g \) Riemann surface to the Calabi-Yau threefold; more precisely the natural question is to allow the moduli of the Riemann surface to vary and ask how many holomorphic maps
there are for some point on the moduli space of the Riemann surface. Actually for high enough genus one expects not to get isolated holomorphic maps but families of them. In such cases the natural thing to compute is the Euler character of a certain bundle over the moduli space of holomorphic maps from Riemann surfaces of genus $g$ to Calabi-Yau manifolds [16].

This is the hard part of the question and we would like to discuss what is the mirror computation which is expected to be simple? This question has recently been answered [17] and we shall now summarize some of the results of this work.

At the level of sphere the easy side of the computation corresponded to classical computations which means ordinary integrals which arise even for point particle theories. In other words the sphere computation, which is the tree level for strings, leads to classical computations (ordinary integrals) which in principle could arise from point particle theories (the usual overlap of wave functions). Therefore it is natural to expect that the higher genus computations which are quantum corrections for string theory will be mapped to quantum corrections for the would be point particle theory. This expectation is borne out, and the corresponding point particle theory which quantizes the variations of Hodge structure is called the Kodaira-Spencer theory of gravity [17].

The basic idea of this theory is to start with the classical equations of the theory which is the variations of complex structure of a Calabi-Yau threefold $M$. According to Kodaira-Spencer theory, the variation of the complex structure can be encoded by saying how $\bar{\partial}$ varies:

$$\bar{\partial} \rightarrow \bar{\partial} + A \partial$$

where $A$ is a section of $TM \times T^*M$, which I will denote, counting the vectors with negative weight as opposed to forms by a (-1,1) ‘vector-form’. The Kodaira-Spencer equation which is the consistency condition for the new $\bar{\partial}$ for a finite shift of complex structure holomorphic tangent bundle is

$$\bar{\partial} A + \frac{1}{2} [A, A] = 0$$

where the commutator takes the commutator of the $A$ as a vector and wedges the anti-holomorphic forms. To solve this equation we follow [3]. We use the notation that when we contract a vector-form with the holomorphic three form $\Omega$ we put a prime over the vector-form. Contracting $A$ with $\Omega$ gives us a (2,1) form $A'$. One chooses variations $A$ which in addition respect the three form, which means we choose $A$ such that

$$\partial A' = 0$$
Then the lemma of [3] shows that in such a case

\[ [A, A]' = \partial(A \wedge A)' \]

By contracting the KS equation with \( \Omega \) (which is nowhere zero) we get the equivalent equation

\[ \bar{\partial}A' + \frac{1}{2} \partial(A \wedge A)' = 0 \]

For a physical realization, this has to be the classical equations of an action which is indeed the case. The action is given by

\[ S = \frac{1}{2} \int_M A' \frac{1}{\partial} \bar{\partial}A' + \frac{1}{6} \int_M A' \wedge (A \wedge A)' \]

This action including the condition \( \partial A' = 0 \) arises naturally in string field theory [17]. The action as written above is formal because of \( \frac{1}{2} \) but it can be made precise sense of [17].

Tree level expansions of this theory naturally gives rise to perturbative solutions of the KS equation which had been studied before [3]. The fact that the tree level computations of this theory leads to the computations which arise in the variations of Hodge structure is not surprising—after all the tree level computations are equivalent to classical equations which in this case is the KS equation.

One should then go on to compute the quantum corrections for the above action. In such cases one typically encounters divergencies which should be regularized. The first such case is the one-loop correction. For the Kodaira-Spencer action the one-loop correction turns out to be a certain particular combination of holomorphic Ray-Singer torsions. Let \( \Delta_{p,q} \) be the Laplacian acting on \((p, q)\) forms on \(M\). The one-loop correction turns out to be

\[ F_1 = \frac{1}{2} \log \prod_{p,q} \det \Delta_{p,q}^{\frac{-1}{p+q}} \]

of course for such an object to make sense one needs to delete the zero modes and in addition it needs to be regularized. This kind of determinant has been rigorously studied in [19] and

\[ F_1 = \frac{1}{2} \int \sum_{n=0}^{\infty} \frac{(\tau_1, \tau_2)_{n+1}}{n!} \log \prod_{p,q} \det \Delta_{p,q}^{\frac{-1}{p+q}} \]

\[ \text{where the integral is over the moduli space of torus, } q \text{ is the modular parameter, } H_{L,R} \text{ denote the left- and right- moving Hamiltonians, and } F_{L,R} \text{ denote the } U(1) \times U(1) \text{ left-right fermion number gradation. This object was defined in [18] as a generalization of Ray-Singer torsion to the loop space of Kähler manifolds.} \]
regularized using the zeta function regularization techniques\footnote{\text{\textsuperscript{5}}}. In order to compute $F_1$ one generally studies how $F_1$ varies as a function of moduli of complex structures of Calabi-Yau manifold. Formally one argues that $F_1$ is a sum of a holomorphic and an antiholomorphic function on moduli space. But as is well known there is an anomaly, the Quillen anomaly, which falsifies the formal argument. This anomaly is captured by studying the curvature $\partial \bar{\partial} F_1$. This can be computed either using string techniques [20] or the more conventional techniques [19] with the result that\footnote{\text{\textsuperscript{6}}}

$$\partial \bar{\partial} F_1 = \frac{1}{2} \partial \bar{\partial} \left[ \sum_{p,q} (-1)^{p+q} q d_{p,q} \right] + \frac{1}{2} \int_M Td(T) \sum_p (-1)^p p \ Ch(\wedge^{n-p} T^*) \big|_{(1,1)-\text{part}}$$

$$= \frac{1}{2} \partial \bar{\partial} \left[ \sum_{p,q} (-1)^{p+q} q d_{p,q} \right] - \frac{\chi(M)}{24} G$$

where $d_{p,q}$ denotes the determinant of harmonic $(p,q)$ forms (in a holomorphic basis), $Td, Ch$ denote the Todd and chern classes, $T$ and $T^*$ denote the holomorphic tangent and cotangent bundles, $\chi(M)$ denotes the Euler characteristic of $M$, and $G$ is the Kähler form on moduli space (which is the same as $c_1(L)$ where $L$ is the line bundle over the moduli space whose section is a holomorphic 3-form). In evaluating the index integral above use has been made of the fact that Calabi-Yau manifolds have vanishing first chern class. Integrating the anomaly equation to get $F_1$ is now an easy task with the additional boundary condition of how $F_1$ behaves near the boundaries of moduli space. It should be noted that $F_1$ defined above makes sense for CY manifolds of arbitrary dimension.

Let us consider the simplest example of mirror phenomenon, namely the elliptic curve, and compute $F_1$ in either language. In the easy side of the computation as we discussed above we are going to get the analytic torsion which is simply

$$F_1(q) = -\log \sqrt{\tau_2 \eta(q) \eta(q)}$$

\footnote{\text{To the best of my knowledge the particular combination of torsions appearing here which leads to a magically simple anomaly formula for the Calabi-Yau manifolds has not been studied in the mathematics literature.}}

\footnote{\text{In the stringy computation the $\frac{1}{24}$ in the second term arises in computation as a result of the volume of the moduli space of torus whereas in the index derivation it appears in the expansion of todd class.}}

12
where \( q = \exp(2\pi i \tau) \) specifies the complex structure of the torus and \( \eta \) is the Dedekind \( \eta \) function given by

\[
\eta = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n).
\]

Of course the easy way to show this is to study Quillen anomaly as discussed above: Applying the analysis to this case is rather simple and we end up with the anomaly equation that

\[
\partial_\tau \bar{\partial}_\bar{\tau} F_1 = \frac{-1}{2(\tau - \bar{\tau})^2}.
\]

Modular invariance and regularity in the interior gives rise to the \( \eta \) function as we integrate the above anomaly equation. On the ‘difficult’ mirror to this as we mentioned we would expect to be counting the number of holomorphic curves from the torus to the manifold which in this case is again a torus where as discussed before now the role of \( \tau \) and \( \rho \) are exchanged–in other words we now think of \( \tau \) as parameterizing the complexified Kähler class of the torus. Fixing a point on the torus and mapping it to a particular point on the target torus (dual to the Kähler class) is equivalent to studying \( q \partial F_1 / \partial q \). Note that in the dictionary for the mirror map \( q \) counts the degree of the holomorphic map to the manifold, so the coefficient in front of \( q^n \) signifies the number of maps of degree \( n \). However the existence of anomaly means that we have an inevitable mixture of \( q \) and \( \bar{q} \). To simply count the holomorphic curves we should consider an asymmetric limit fixing \( g \) but taking \( g^* \to \infty \) which concentrates the path integral on holomorphic maps as discussed before. In this case this means keeping \( \tau \) fixed and sending \( \bar{\tau} \to \infty \). We thus have (expanding the logarithm of the Dedekind \( \eta \) function)

\[
q \frac{\partial F_1}{\partial q} \bigg|_{\bar{\tau} \to \infty} = \frac{-1}{4\pi i (\tau - \bar{\tau})} \bigg|_{\bar{\tau} \to \infty} + \sum_{n,m=1}^{\infty} nq^{nm} - \frac{1}{24} = \sum_{n,m=1}^{\infty} nq^{nm} - \frac{1}{24} \]

Let us check whether this is what we expect by directly counting the number of holomorphic maps from torus to torus: At degree zero, every elliptic curve can be mapped holomorphically to the torus by the constant map. So at degree zero we have a continuous family parameterized by the moduli space of tori. The physics computes (up to sign) roughly the volume of this space which in this case is \( \frac{1}{24} \). The coefficients in front of \( q^{nm} \) should signify the number of holomorphic curves which cover the target torus \( nm \) times. Let us start with \( n = m = 1 \). For a fixed target torus, there is clearly a unique torus for which there exists a holomorphic map, namely the torus with exactly the same modulus.
as the target torus and the map being the identity map. This explains the coefficient of $q$ being 1. We can consider an $nm$-fold covering of this torus, by going $n$ times covering in one direction and $m$ times covering in another direction. By tilting the parallelogram representing the torus in one direction we can actually get $n$ inequivalent such tori. One can show by a careful check that up to $SL(2, \mathbb{Z})$ transformation there are no more (in particular the number of holomorphic maps of degree $r$ is equal to $\sum_{n \text{ divides } r} n$) in agreement with the above formula derived by using mirror symmetry.

We can also ask how higher quantum corrections can be computed. As remarked before higher loop corrections correspond to maps from higher genus Riemann surfaces to the Calabi-Yau manifold. It is at this point that threefolds play a very distinguished role, in that the quantum corrections vanish for all Calabi-Yau $n$-folds except for $n = 3$. On the ‘easy side’, which for higher loops are not so easy, one is computing the quantum corrections to the Kodaira-Spencer theory. This will have to be studied very carefully with an eye on regularization of the theory. On the hard side, one is computing the number of holomorphic maps (or the Euler character of an appropriate bundle on moduli space of holomorphic maps) from genus $g$ Riemann surface to the three fold.

So the question is how one does the computation? The method followed in [17] makes use of the fact that the answer is expected to be essentially a holomorphic function of moduli (or more precisely a holomorphic section of an appropriate line bundle on moduli space). If it were exactly holomorphic to completely fix it one would only need to know its behavior near the boundaries of moduli space which in general would require only a finite data. However the holomorphicity is not exactly right at higher genus just as it was not exactly right for genus one, where one encountered Quillen’s anomaly. In the genus one case, to get a well defined answer as mentioned above we have to take the holomorphic derivative of $F_1$ in the direction of moduli and the Quillen anomaly is the statement that this function is not a holomorphic function of moduli. For higher genus $F_g$ is well defined without having to take any derivatives, because the higher genus does not have any isometries, so the anomaly is going to be captured by a statement involving $\bar{\partial}F_g \neq 0$. In order to compute this anomaly, one has in principle two options: Either use the Kodaira-Spencer theory directly and study its anomaly at higher loops (which is the generalization of what has been studied in the case of one loop in [19]), or study it directly in string theory where $F_g$ is defined as the integral of a particular measure on moduli space of Riemann surfaces of genus $g$. Since the regularization of Kodaira-Spencer theory has not been studied at higher loops the first approach seems difficult at present. It turns out
that the second approach, i.e. using string theory techniques, is extremely easy. Even if the regularization of Kodaira-Spencer theory had been studied it is hard to believe the derivation of the anomaly would have been as simple as the stringy derivation. In the context of string theory one finds that

\[ \partial_i F_g = \int_{\mathcal{M}_g} \partial \bar{\partial} \rho_i \]

where \( \mathcal{M}_g \) is the moduli space of Riemann surfaces of genus \( g \), and \( \rho_i \) is a well defined form on moduli space. Using this structure the computation reduces to the contributions from the boundary of moduli space of Riemann surfaces. Naturally objects one encounters there would be related to lower genus computations. The answer one obtains turns out to be

\[ \partial_i F_g = \frac{1}{2} \overline{C}_{ijk} g^{ij} g^{kk} \left( D_j D_k F_{g-1} + \sum_{r=1}^{g-1} D_j F_r D_k F_{g-r} \right) \]

The two boundary contributions on the right hand side come from the two distinct type of degenerations of the Riemann surface (from the handle degeneration and splitting of the surface respectively). Without going to too much detail let us just note that \( F_g \) is a section of \( \mathcal{L}^{2-2g} \) where \( \mathcal{L} \) is the line bundle of \( H^{3,0}(M) \) on the moduli space, and \( D_i \) represent covariant derivatives with respect to the natural connection and \( g_{ij} = \int_M \omega_i \wedge \overline{\omega}_j \) represent the canonical metric on the \((2,1)\) forms \( \omega_i \). This anomaly equation can be naturally captured in the master form including all genera at once by considering \( Z = \exp \sum_{g=1}^{\infty} \lambda^{2g-2} F_g \) and we have

\[ \left[ \overline{\partial}_i - \partial_i F_1 - \frac{\lambda^2}{2} \overline{C}_{ijk} g^{ij} g^{kk} D_j D_k \right] Z = 0 \]

So the basic strategy to compute \( F_g \) at higher genus is by induction. Suppose we know \( F_g \) up to a given genus. To get the next one, we integrate the above anomaly formula using the lower genus \( F_g \) and the behaviour of \( F_g \) near the boundaries of moduli space. Using this strategy the case of the quintic threefold was studied in [17] with the result indicated in the table below.
| Degree | $g = 0$ | $g = 1$ |
|--------|--------|--------|
| n=0    | 5      | 50/12  |
| n=1    | 2875   | 0      |
| n=2    | 609250 | 0      |
| n=3    | 317206375 | 609250 |
| n=4    | 242467530000 | 3721431625 |
| n=5    | 229305888887625 | 12129909700200 |
| n=6    | 248249742118022000 | 31147299732677250 |
| n=7    | 295091050570845659250 | 71578406022880761750 |
| n=8    | 37563216093747660355000 | 154990541752957846986500 |
| n=9    | 503840510416985243645106250 | 324064464310279585656399500 |
| ...    | ...    | ...    |
| large n| $a_0 n^{-3} (\log n)^{-2} e^{2\pi n\alpha}$ | $a_1 n^{-1} e^{2\pi n\alpha}$ |

| Degree | $g = 2$ | $g = 3$ |
|--------|--------|--------|
| n=0    | -5/144 | -100 \cdot [c^3_{g-1}] |
| n=1    | 0      | 0      |
| n=2    | 0      | 0      |
| n=3    | 0      | 0      |
| n=4    | 534750 | 0      |
| n=5    | 75478987900 | 0      |
| n=6    | 871708139638250 | 0      |
| n=7    | 5185462556617269625 | 0      |
| n=8    | 90067364252423675345000 | 0      |
| n=9    | 325859687147358266010240500 | 0      |
| ...    | ...    | ...    |
| large n| $a_2 n (\log n)^2 e^{2\pi n\alpha}$ | $a_3 n^{2g-3} (\log n)^{2g-2} e^{2\pi n\alpha}$ |

Table 1. # curves of genus $g$ on quintic hypersurface
In this table we have included also the genus 0 answer which was computed in \cite{15} and
the genus 1 result which was computed using the anomaly in \cite{20}. As far as the numbers
which have been checked mathematically, up to now the genus 0 answer up to degree 3 has
been confirmed. For higher genus, the direct computations are even more difficult. The
only non-trivial one which has been confirmed is by Stromme et. al. \cite{21} (using a formula
of Bott) and it is for the \( n = 4 \) for \( g = 1 \) (the \( n = 3 \) for \( g = 1 \) can be easily argued to
be equal to \( n = 2 \) for \( g = 0 \) which agrees with the physical computation). For \( g > 2 \) we
have determined only the asymptotic behavior of the number of holomorphic curves (to
fix all the numbers we have to have a little bit more precise knowledge about the finite
number of coefficients which dominate the divergence behavior of \( F_g \) near the boundaries
of moduli space of the quintic and fixes the holomorphic ambiguity in integrating the
anomaly equation).

Mirror symmetry is a reflection of the rich structure in the loop space of Calabi-Yau
manifolds in that they give rise to 2 dimensional conformal theories with \( N = 2 \) supersym-
metry. As is well known these are only very special examples of \( N = 2 \) superconformal
theories, many of which have no target manifold interpretation. We have learned from the
existence of mirror symmetry that classical concepts of invariants are to be modified when
going to loop space, and in fact we are led to ask more generally what are the invariants
of the \( N = 2 \) conformal field theories which arise for CY sigma models. Having a good
understanding of these invariants will be a necessary tool in classifying them. So far only
very limited set of invariants of the \( N = 2 \) theories have been constructed. There could
very well be more refined invariants (by invariants we mean aspects of the theory which
are unaffected by moving in the moduli space of these theories). In particular defining
invariants over integers which is natural for manifolds has not yet been defined for the
\( N = 2 \) CFT’s. This is a challenging and important area to develop.

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