Research Article

Discrete Fourier-Jacobi transform

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ABSTRACT
Discrete analogues of the classical Fourier-Jacobi transform are introduced and investigated. It involves series and integrals with respect to parameters of the Gauss hypergeometric function $2F_1\left(a + \frac{in}{2}, a - \frac{in}{2}; c; -x^2\right)$, $x > 0$, $n \in \mathbb{N}$, $a, c > 0$, $i$ is the imaginary unit. The corresponding inversion formulas for suitable functions and sequences in terms of these series and integrals are established.

1. Introduction and preliminary results

As is known (cf. [1]), the Fourier-Jacobi or Jacobi transform of an arbitrary function $f$ with respect to parameters of the Gauss hypergeometric function can be defined by the following integral

$$F(\tau) = \int_0^\infty 2F_1\left(a + \frac{i\tau}{2}, a - \frac{i\tau}{2}; c; -x^2\right)f(x) \, dx, \quad \tau > 0, \quad (1.1)$$

where $a, c > 0$, and $\Gamma(z)$ is the Euler gamma function [2, Vol. III]. It was introduced by Olevskii [3] in 1949 who proved the inversion formula in the space of integrable functions, having the form (cf. [4, formula (7.51)])

$$f(x) = \frac{x^{2c-1}}{2\pi} \int_0^\infty \left| \frac{\Gamma(c - a + i\tau/2)\Gamma(a + i\tau/2)}{\Gamma(c)\Gamma(i\tau)} \right|^2 2F_1\left(c - a + \frac{i\tau}{2}, c - a - \frac{i\tau}{2}; c; -x^2\right) \times F(\tau) \, d\tau, \quad x > 0. \quad (1.2)$$

This is why operator (1.1) is also called the Olevskii transform [4,5], or the index hypergeometric transform, or the $2F_1$-index transform. Transformation (1.1) represents a class of the so-called index transforms which were investigated by the author in [5].

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The main goal of this paper is to investigate two discrete analogues of the Fourier-Jacobi transform in the series and integral form, respectively,

\[
f(x) = \sum_{n=1}^{\infty} a_n \, 2F_1 \left( a + \frac{in}{2}, a - \frac{in}{2}; -x^2 \right), \quad x > 0,
\]

\[
a_n = \int_0^{\infty} 2F_1 \left( a + \frac{in}{2}, a - \frac{in}{2}; -x^2 \right) f(x) \, dx, \quad n \in \mathbb{N},
\]

for suitable function \( f \) and sequence \( \{a_n\}_{n=1}^{\infty} \). Our approach suggests using the classical Fourier series and some integrals, involving the Gauss hypergeometric function. In fact, the crucial result in our investigation will be the following lemma.

**Lemma 1.1:** Let \( a > 0, \ c > \max(0, a - 1/2), \ u \in \mathbb{R}, \ n \in \mathbb{N} \). Then the equality holds

\[
\Gamma(2(c-a)+1) \left| \frac{\Gamma(a+in/2)}{\Gamma(c)} \right|^2 \int_0^{\infty} y^{c-1} \, 2F_1 \left( c - a + \frac{1}{2}, c - a + 1; -y \right) \frac{\sinh(y)}{y} \, dy = \frac{2^{2(c-a)+1} \pi}{\sinh(y)} \sinh(\pi n).
\]

**Proof:** Since the Gauss hypergeometric functions in (1.5) behaves as (see [2, Vol. III])

\[
2F_1 \left( a + \frac{in}{2}, a - \frac{in}{2}; -y \right) = O(y^{-a}), \quad y \rightarrow \infty,
\]

\[
2F_1 \left( c - a + \frac{1}{2}, c - a + 1; -\frac{y}{\cosh^2(u)} \right) = O(y^{a-c-1/2}), \quad y \rightarrow \infty,
\]

integral (1.5) converges absolutely. Then, appealing to the Mellin–Barnes representation of the function (1.7) (see [2, Vol. III, Entry 8.4.49.13])

\[
x^{c-1} \, 2F_1 \left( c - a + \frac{1}{2}, c - a + 1; -\frac{x}{\cosh^2(u)} \right) = \frac{4^{c-a} [\cosh(u)]^{2(c-1)} \Gamma(c)}{\Gamma(2(c-a)+1) \, 2 \pi^{3/2} i} \Gamma(3/2-a-s) \Gamma(2-a-s) \frac{\Gamma(s+c-1)}{\Gamma(1-s)} \left( \frac{x}{\cosh^2(u)} \right)^{-s} ds,
\]

where \( \max(1-c, 1-a) < \gamma < \min(1, 3/2-a) \), we substitute the right-hand side of (1.8) into (1.5) and change the order of integration by Fubini’s theorem due to the estimate

\[
\int_0^{\infty} y^{-\gamma} \left| 2F_1 \left( a + \frac{in}{2}, a - \frac{in}{2}; -y \right) \right| dy
\]

\[
\times \int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma(3/2-a-s) \Gamma(2-a-s) \frac{\Gamma(s+c-1)}{\Gamma(1-s)} \left| \frac{x}{\cosh^2(u)} \right|^{-s} ds < \infty.
\]

This estimate, indeed, holds via (1.6) and the Stirling asymptotic formula for the gamma function [4]. Consequently, recalling the formula in [2, Vol. III, Entry 8.4.49.13] and the duplication formula for the gamma function, we derive

\[
\Gamma(2(c-a)+1) \left| \frac{\Gamma(a+in/2)}{\Gamma(c)} \right|^2 \int_0^{\infty} y^{c-1} \, 2F_1 \left( c - a + \frac{1}{2}, c - a + 1; -\frac{y}{\cosh^2(u)} \right) \frac{\sinh(y)}{y} \, dy
\]
where $\mu = 2(\gamma + a - 1)$. The latter integral can be written, employing the Parseval equality for the Mellin transform [4] and formulas in [2, Vol. III, Entries 8.4.3.1, 8.4.23.1]. Hence we get from (1.9)

\[
\begin{align*}
\Gamma(2(c - a) + 1) \left( \frac{\Gamma(a + in/2)}{\Gamma(c)} \right)^2 &\int_0^\infty y^{c-1} _2F_1 \left( c - a + 1/2, c - a + 1; c; \frac{y}{\cosh^2(u)} \right) \\
\times _2F_1 \left( a + in/2, a - in/2; c; -y \right) dy \\
&= 4^{c-a+1/2} [\cosh(u)]^{2(c-a)} \int_0^\infty e^{-x} K_{in} \left( \frac{x}{\cosh(u)} \right) dx,
\end{align*}
\]

(1.10)

where $K_{in}(z)$ is the modified Bessel function [5]. But the integral on the right-hand side of (1.10) is calculated via [2, Vol. II, Entry 2.16.6.1]

\[
\int_0^\infty e^{-x \cosh(u)} K_{in}(x) dx = \frac{\pi \sin(nu)}{\sinh(u) \sinh(\pi n)}.
\]

(1.11)

Therefore, making a simple change of variables, we end up with (1.5), completing the proof of Lemma 1.1.

\[\square\]

Remark 1.1: Formula (1.5) is a special case of [2, Vol. III, Entry 2.21.9.7] which is a task to retrieve.

2. Inversion theorems

We begin with

\[\text{Theorem 2.1: Let } a > 0, \max(1/2, 2a - 1/2) < c < 2a + 1/2 \text{ and } 0 < a < 1/2 \text{ when } c = 1/2. \text{ If the sequence } \{a_n\}_{n\geq1} \text{ satisfies the condition}
\]

\[
\sum_{m=1}^\infty \frac{|a_m| e^{-\delta m}}{\Gamma(a + im/2)^2} < \infty, \quad \delta \in \left[0, \frac{\pi}{2}\right),
\]

(2.1)
Then the discrete transformation
\[ f(x) = \sum_{n=1}^{\infty} a_n \, _2F_1 \left( a + \frac{in}{2}, a - \frac{in}{2}; c; -x^2 \right), \quad x > 0, \]
can be inverted by the formula
\[ a_n = \frac{4a-c}{\pi^2} \Gamma(2(c - a) + 1) \left| \frac{\Gamma(a + in/2)}{\Gamma(c)} \right|^2 \sinh(\pi n) \int_0^{\infty} x^{2c-1} \Phi_n(x) f(x) \, dx, \quad (2.2) \]
where
\[ \Phi_n(x) = \int_{-\pi}^{\pi} _2F_1 \left( c - a + \frac{1}{2}, c - a + 1; c; -\frac{x^2}{\cosh^2(u)} \right) \tanh(u) \sin(nu) \left( \frac{\sinh(\pi c)}{\cosh(\pi u)^{2(c-a)}} \right)^{\nu}, \quad (2.3) \]
and integral (2.2) converges absolutely.

**Proof:** In order to proceed with the derivation of inversion formula (2.3), we will need an estimation of the Gauss hypergeometric function. To do this, we employ its integral representation in terms of the product of Bessel functions (see \([2, \text{Vol. II, Entry 2.16.21.1}]\)
\[ _2F_1 \left( a + \frac{in}{2}, a - \frac{in}{2}; c; -x^2 \right) = \frac{2^{1+c-2a} x^{1-c} \Gamma(c)}{\Gamma(a + in/2)} \int_0^{\infty} y^{2a-c} J_{c-1}(xy) K_{\nu}(y) \, dy, \quad (2.4) \]
Then, appealing to the inequality for the modified Bessel function (see \([5, \text{p. 15}]\))
\[ |K_{i\tau}(x)| \leq e^{-\delta \tau} K_0(x \cos(\delta)), \quad x, \tau > 0, \quad \delta \in \left[ 0, \frac{\pi}{2} \right), \quad (2.5) \]
and the inequality \(x^{1/2} |J_{\nu}(x)| < C_{\nu}, \quad \nu \geq -1/2, \) where \(C_{\nu} > 0\) is a constant, we deduce
\[ \left| _2F_1 \left( a + \frac{in}{2}, a - \frac{in}{2}; c; -x^2 \right) \right| \leq \frac{2^{1+c-2a} x^{1-c} e^{-\delta n} \Gamma(c)}{|\Gamma(a + in/2)|^2} \times \int_0^{\infty} y^{2a-c} |J_{c-1}(xy)| K_0(y \cos(\delta)) \, dy \]
\[ \leq \frac{2^{1+c-2a} x^{1/2-c} e^{-\delta n} \Gamma(c) A_c}{|\Gamma(a + in/2)|^2} \int_0^{\infty} y^{2a-c-1/2} K_0(y \cos(\delta)) \, dy \]
\[ = \frac{\Gamma(c) A_c x^{1/2-c} e^{-\delta n}}{\sqrt{2} |\cos(\delta)|^{2a-c+1/2}} \left| \frac{\Gamma(a - c/2 + 1/4)}{\Gamma(a + in/2)} \right|^2, \quad \frac{1}{2} \leq c < 2a + \frac{1}{2}, \quad (2.6) \]
where \(A_c > 0\) is a constant. Thus, substituting the value of \(f\) by formula (1.3) on the right-hand side of (2.2), we change the order of integration and summation due to assumption (2.1), inequality (2.6) and the estimate
\[ \int_0^{\infty} x^{-1/2} \int_0^{\pi} \left| _2F_1 \left( c - a + \frac{1}{2}, c - a + 1; c; -\frac{x^2}{\cosh^2(u)} \right) \right| \sinh(u) \left( \frac{\sinh(\pi c)}{\cosh(\pi u)^{2(c-a)}} \right)^{\nu} \, du \, dx \sum_{m=1}^{\infty} \left| a_m \right| \frac{e^{-\delta m}}{\Gamma(a + im/2)^2} < \infty \quad (2.7) \]
under the condition $c > 2a - 1/2$ (see (1.7)). Consequently, appealing to Lemma 1.1, we obtain

$$\frac{4^{a-c}}{\pi^2} \Gamma(2c-a+1) \left| \frac{\Gamma(a+in/2)}{\Gamma(c)} \right|^2 \sinh(\pi n) \int_0^\infty x^{2c-1} \Phi_n(x)f(x) \, dx$$

$$= \frac{1}{\pi} \left| \frac{\Gamma(a+in/2)}{\Gamma(c)} \right|^2 \sinh(\pi n) \int_\pi^{\pi} \sin(nu) \sum_{m=1}^\infty a_m \frac{\sin(mu)}{|\Gamma(a+im/2)|^2 \sinh(\pi m)} \, du = a_n.$$ 

Theorem 2.1 is proved. ■

**Remark 2.1:** The summation in (1.3) can be from $n = 0$ since $a_0 = 0$ via (2.2). Then by the parity it can spread over $\mathbb{Z}$.

Concerning Fourier-Jacobi transform (1.4), we have the following result.

**Theorem 2.2:** Let parameters $a, c$ satisfy conditions of Theorem 2.1 and $f$ be a complex-valued function on $\mathbb{R}_+$ which is represented by the integral

$$f(x) = x^{2c-1} \int_\pi^{\pi} 2F_1 \left( c - a + \frac{1}{2}, c - a + 1; c; -\frac{x^2}{\cosh^2(u)} \right)$$

$$\times \frac{\varphi(u)}{[\cosh(u)]^{2(c-a)+1}} \, du, \quad x > 0,$$

where $\varphi(u) = \psi(u) \sinh(u)$ and $\psi$ is a $2\pi$-periodic function, satisfying the Lipschitz condition on $[-\pi, \pi]$, i.e.

$$|\psi(u) - \psi(v)| \leq C|u - v|, \quad \forall u, v \in [-\pi, \pi],$$

where $C > 0$ is an absolute constant. Then for the transformation

$$a_n = \int_0^\infty 2F_1 \left( a + \frac{in}{2}, a - \frac{in}{2}; c; -x^2 \right) f(x) \, dx, \quad n \in \mathbb{N},$$

the following inversion formula holds:

$$f(x) = \frac{4^{a-c-1/2} \Gamma(2c-a+1) x^{2c-1}}{[\pi \Gamma(c)]^2} \sum_{n=1}^\infty \sinh(\pi n) \left| \frac{\Gamma(a+in/2)}{\Gamma(c)} \right|^2 \Phi_n(x)a_n, \quad x > 0,$$

where $\Phi_n(x)$ is defined by (2.3).

**Proof:** The proof is based on the integral for the product of the Bessel and Gauss hypergeometric functions (see [2, Vol. III, Entry 2.21.4.2])

$$\int_0^\infty y^c J_{c-1}(xy) 2F_1 \left( c - a + \frac{1}{2}, c - a + 1; c; -\frac{y^2}{\cosh^2(u)} \right) \, dy$$
\[ a_n = \frac{2^{1+2a} c^{c-2a}}{\Gamma(a + in/2)^2} \int_0^\infty \int_0^\infty y^{2a-c} K_n(y) x^{c-1} I_{c-1}(xy) f(x) \, dx \, dy. \] 

(2.12)

The interchange follows via Fubini's theorem by virtue of the estimate (see (2.5), (2.7), (2.8) and relation 2.16.2.2 in [2, Vol. II])

\[
\int_0^\infty y^{2a-c} |K_n(y)| \int_0^\infty x^{c-1} |I_{c-1}(xy)f(x)| \, dx \, dy \\
\leq 2 \int_0^\infty y^{2a-c-1/2} K_0(y) \, dy \int_0^\infty x^{c-1/2} \left| \frac{\varphi(u)}{[\cosh(u)]^{2(c-a)+1}} \right| \, du \, dx \\
\times 2F_1 \left( \frac{c-a+1/2, c-a+1; c-\frac{x^2}{\cosh^2(u)}}{2} \right) \int_0^\infty x^{-1/2} 2F_1 \left( \frac{c-a+1/2, c-a+1; c-\frac{x^2}{\cosh^2(u)}}{2} \right) \, dx \\
\times \int_0^\pi \frac{\varphi(u)}{[\cosh(u)]^{1/2+c-2a}} \, du < \infty.
\]

Hence, returning to (2.12), we substitute \( f(x) \) by formula (2.8) and using values of integrals (1.11) and (2.11), it becomes

\[ a_n = \frac{2^{1+2(c-a)} \pi \Gamma^2(c)}{\Gamma(2(c-a) + 1) \sinh(\pi n) \Gamma(a + in/2)^2} \int_{-\pi}^\pi \varphi(u) \sin(nu) \sinh(u) \, du. \] 

(2.13)

Therefore, following the same scheme as in the proof of Theorem 5 in [6], we substitute the value of \( a_n \) by (2.13) and \( \Phi_n(x) \) by (2.3) into the partial sum of series (2.10). Then, calculating this sum via the known identity, we invoke the definition of \( \varphi \) to obtain

\[ S_N(x) = \frac{4^{a-c-1/2} \Gamma(2(c-a) + 1) x^{2c-1}}{[\pi \Gamma(c)]^2} \sum_{n=1}^N \sinh(\pi n) \frac{\Gamma(a + in/2)^2}{\sinh(\pi n)} \Phi_n(x) a_n \\
\]

\[ = \frac{1}{\pi} \sum_{n=1}^N \int_{-\pi}^\pi 2F_1 \left( \frac{c-a+1/2, c-a+1; c-\frac{x^2}{\cosh^2(t)}}{2} \right) \frac{\tanh(t) \sin(nt)}{[\cosh(t)]^{2(c-a)}} \, dt \\
\times \int_{-\pi}^\pi \frac{\varphi(u)}{\sinh(u)} \, du \\
= \frac{x^{2c-1}}{4\pi} \sum_{n=1}^N \int_{-\pi}^\pi 2F_1 \left( \frac{c-a+1/2, c-a+1; c-\frac{x^2}{\cosh^2(t)}}{2} \right) \frac{\tanh(t)}{[\cosh(t)]^{2(c-a)}} \\
\times \int_{-\pi}^\pi \left[ \psi(u) - \psi(-u) \right] \frac{\sin((2N+1)(u-t)/2)}{\sin((u-t)/2)} \, du \, dt. \] 

(2.14)
Since $\psi$ is $2\pi$-periodic, we treat the latter integral with respect to $u$ as follows:

$$
\int_{-\pi}^{\pi} [\psi(u) - \psi(-u)] \frac{\sin((2N + 1)(u-t)/2)}{\sin(u/2)} \, du
= \int_{-\pi}^{\pi} [\psi(u) - \psi(-u)] \frac{\sin((2N + 1)(u-t)/2)}{\sin(u/2)} \, du
= \int_{-\pi}^{\pi} [\psi(u + t) - \psi(-u - t)] \frac{\sin((2N + 1)u/2)}{\sin(u/2)} \, du.
$$

Moreover,

$$
\frac{1}{2\pi} \int_{-\pi}^{\pi} [\psi(u + t) - \psi(-u - t)] \frac{\sin((2N + 1)u/2)}{\sin(u/2)} \, du - [\psi(t) - \psi(-t)]
= \frac{1}{2\pi} \int_{-\pi}^{\pi} [\psi(u + t) - \psi(t) - \psi(-u - t)] \frac{\sin((2N + 1)u/2)}{\sin(u/2)} \, du.
$$

When $u + t > \pi$ or $u + t < -\pi$ then we interpret the value $\psi(u + t) - \psi(t)$ by formulas

$$
\psi(u + t) - \psi(t) = \psi(u + t - 2\pi) - \psi(t - 2\pi),
$$

$$
\psi(u + t) - \psi(t) = \psi(u + t + 2\pi) - \psi(t + 2\pi),
$$

respectively. Analogously, the value $\psi(-u - t) - \psi(-t)$ can be treated. Then due to Lipschitz condition (2.9), we have the uniform estimate for any $t \in [-\pi, \pi]$

$$
|\psi(u + t) - \psi(t) + \psi(-t) - \psi(-u - t)| \leq 2C \left|\frac{u}{\sin(u/2)}\right|.
$$

Therefore, owing to the Riemann–Lebesgue lemma

$$
\lim_{N \to \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} [\psi(u + t) - \psi(-u - t) - \psi(t) + \psi(-t)] \frac{\sin((2N + 1)u/2)}{\sin(u/2)} \, du = 0
$$

(2.15)

for all $t \in [-\pi, \pi]$. Besides, returning to (2.14), we estimate the iterated integral

$$
\int_{-\pi}^{\pi} \left|2F_1 \left( c - a + \frac{1}{2}, c - a + 1; c; - \frac{x^2}{\cosh^2(t)} \right) \right| \frac{|\tanh(t)|}{[\cosh(t)]^{2(c-a)}}
\times \int_{-\pi}^{\pi} \left|[\psi(u + t) - \psi(-u - t) - \psi(t) + \psi(-t)] \frac{\sin((2N + 1)u/2)}{\sin(u/2)} \right| \, du \, df
\leq 4C \int_{0}^{\pi} \left|2F_1 \left( c - a + \frac{1}{2}, c - a + 1; c; - \frac{x^2}{\cosh^2(t)} \right) \right| \frac{\tanh(t)}{[\cosh(t)]^{2(c-a)}} \, dt
\times \int_{-\pi}^{\pi} \left|\frac{u}{\sin(u/2)}\right| \, du < \infty, \quad x > 0.
$$

Consequently, via the dominated convergence theorem, it is possible to pass to the limit when $N \to \infty$ under the integral sign, and recalling (2.15), we derive

$$
\lim_{N \to \infty} \frac{x^{2c-1}}{4\pi} \int_{-\pi}^{\pi} 2F_1 \left( c - a + \frac{1}{2}, c - a + 1; c; - \frac{x^2}{\cosh^2(t)} \right) \frac{\tanh(t)}{[\cosh(t)]^{2(c-a)}}
$$
\[
\times \int_{-\pi}^{\pi} \left[ \psi(u + t) - \psi(-u - t) - \psi(t) + \psi(-t) \right] \\
\times \frac{\sin ((2N + 1)u/2)}{\sin(u/2)} \, du \, dt \\
= \frac{x^{2c-1}}{4\pi} \int_{-\pi}^{\pi} 2F_1 \left( c - a + \frac{1}{2}, c - a + 1; -\frac{x^2}{\cosh^2(t)} \right) \frac{\tanh(t)}{[\cosh(t)]^{2(c-a)}} \\
\times \lim_{N \to \infty} \int_{-\pi}^{\pi} \left[ \psi(u + t) - \psi(-u - t) - \psi(t) + \psi(-t) \right] \\
\times \frac{\sin ((2N + 1)u/2)}{\sin(u/2)} \, du \, dt = 0.
\]

Hence, combining with (2.14), we obtain by virtue of the definition of \( \varphi \) and \( f \)
\[
\lim_{N \to \infty} S_N(x) = \frac{x^{2c-1}}{2} \int_{-\pi}^{\pi} 2F_1 \left( c - a + \frac{1}{2}, c - a + 1; -\frac{x^2}{\cosh^2(t)} \right) \\
\times \frac{\varphi(t) + \varphi(-t)}{[\cosh(t)]^{2(c-a)+1}} \, dt = f(x),
\]

where integral (2.8) converges since \( \varphi \in C[-\pi, \pi] \). Thus we established (2.10), completing
the proof of Theorem 2.2.

\[\square\]

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