Josephson Effects in a Bose-Einstein Condensate of Magnons

Roberto E. Troncoso\textsuperscript{a,b}, Álvaro S. Núñez\textsuperscript{b}

\textit{a}Institute for Theoretical Physics, Utrecht University, Leuvenlaan 4, 3584 CE Utrecht, The Netherlands
\textit{b}Departamento de Física, Facultad de Ciencias Físicas y Matemáticas, Universidad de Chile, Casilla 487-3, Santiago, Chile

Abstract

A phenomenological theory is developed, that accounts for the collective dynamics of a Bose-Einstein condensate of magnons. In terms of such description we discuss the nature of spontaneous macroscopic interference between magnon clouds, highlighting the close relation between such effects and the well known Josephson effects. Using those ideas we present a detailed calculation of the Josephson oscillations between two magnon clouds, spatially separated in a magnonic Josephson junction.

Keywords: Josephson Effect, Bose-Einstein Condensation, Magnons

1. Introduction

Efforts to improve our understanding and ability to manipulate magnonic excitations in ferromagnetic thin films have received great attention in recent years\cite{1,2}. A special role in this context is played by Ytrium-Iron garnet (YIG) based systems. Characterized by a magnon spectrum greatly isolated from other degrees of freedom\cite{3,4}, YIG systems have allowed the observation of several novel effects (magnonic lattices, spin pumping, etc.). In this work we focus on several reports on the Bose-Einstein condensation (BEC) of magnons\cite{5,6,7}.

Along with the uncontroversial evidence of macroscopic occupation of the lowest lying state, several questions arose regarding the appropriateness

Email addresses: r.troncoso.c@gmail.com (Roberto E. Troncoso), alnunez@dfi.uchile.cl (Álvaro S. Núñez)

Preprint submitted to Physics Letter May 21, 2013
of the concept of BEC to refer to collective magnon behavior [8]. One of the most striking phenomena in the nature of BECs is the emergence of a macroscopic wavefunction that displays phase coherence over macroscopic length-scales. Finding effects associated with such coherence is important in clarifying the true nature of cloud of condensed magnons. Recently in [28] was reported the existence of the two-component Bose-Einstein condensate of magnons as the linear superposition between two spatially non-uniform macroscopic wavefunctions, describing each one the magnon-BEC state at the doubly degenerate lowest-energy. This interference results in a spatially non-uniform ground state with a periodic modulation at the density profile of the condensate, the so-called spin density wave (SDW), as correctly predicted early in [15] and recently studied in [27]. Additionally in [28] were observed vortices-like excitations in the gas of condensed magnons that turns out in a dislocation at the SDW pattern. This type of topological excitations was first studied in [15] based on a microscopic model for the condensed of magnons.

An interesting discussion was carried out in [27] where the authors study the relation between the magnon-BEC phase transition and the contrast of the experimentally observed periodic modulation. In particular it is suggested that the phase transitions may be identified by measuring the contrast of the spatial interference pattern for different values of the thickness of the sample $d$ and the in-plane magnetic field $H$. Additionally they addressed a type of collective mode referred as zero sound in analogy to the Landau’s Fermi liquid theory. This oscillation results from the coupling of the relative phase between both components of the condensate and its imbalance density.

In the present work we start from a phenomenological stand-point and proceed to explore the physical nature of the non-linear dynamics of the condensate. To exploit the occurrence of the macroscopic coherence is necessary to analyze and perform the macroscopic interference effect between magnon condensates. The interference phenomena of such states is referred as the Josephson effect.

Discovered and observed early in superconductivity [9,10], the Josephson effect has been observed in superfluid helium $^3$He [11] and $^4$He [12], and in Bose-Einstein condensates of alkali atomic gases in double well traps [13]. In the last case, the Josephson dynamics between weakly coupled BEC’s manifests itself in several novel phenomena, mainly due to the nonlinearity that stems from the interaction among bosons. The observation of such phenomena in the context of magnon condensate will provide irrefutable evidence of the spontaneous macroscopic coherence.
The realization of the magnonic Josephson junction (MJJ) will consist principally of two stages. The first is based in the usual way for modeling the splitting of condensed clouds in alkali atomic gases, i.e. introducing a potential well inside the trap that splits the single trapped condensate into two parts. The partitioning leads to two weakly coupled condensates, where the tunneling can be tuned modifying the parameters of the system. The second point is related to the spin-wave tunneling effect on ferromagnetic thin films. In that situation a magnetic field inhomogeneity is induced over the thin film by a conductor placed transversely this. The magnon condensate created on the film will be divided in two parts when, by the wire conductor, across a dc current in such direction that the locally increases the magnetization saturation.

Within this work the dynamics of condensed magnons, in a double-well potential, will be described by a Gross-Pitaevskii-like equation \[14\]. This equation will be derived phenomenologically. It turns out to be essentially the same as the one that can be derived microscopically \[15\].

2. Phenomenological Description of the Magnon Condensate

In a related work \[15\] we have discussed theoretically the existence of phase coherence starting from the standard microscopic description of the magnon gas dynamics. Our theoretical description predicts the existence of such coherence when the magnon density reaches a certain critical density. Interactions between magnons turned out to be essential in the creation of such coherence. In this work we pursue a phenomenological approach, based on the basic microscopic features that gave rise to our previous treatment. The basic conclusions reached at the end of both treatments are essentially equivalent. The basic processes that have been found \[16, 17\] to dominate magnon dynamics are:

(i) a dipolar interaction-renormalized dispersion relation, that shifts the states of minimum energy away from the \( \mathbf{k} = 0 \), that is expected solely on account of the exchange term, to \( \mathbf{k} = \pm \mathbf{k}^0 \), this degenerate minima is depicted in Fig.\([1]\);
(ii) a so-called 3-magnon confluence (resp. splitting) term that reduces (resp. increases) the magnon number. These processes are consequence of the long wave length contributions of the dipolar energy;
(iii) a magnon-magnon scattering term that comprises contributions of both the exchange and the dipolar interactions,
(iv) parametric excitation of magnons, through a pumping field that creates magnons at a rate, $P$. Magnon condensation, in the form of macroscopic occupation of the lowest energy state, is observed when $P$ exceeds a critical value, $P_c$.

We note that magnon excitations can be treated effectively as bosonic excitations. Indeed, we can use bosonic operators directly related to the magnetization through the well known Holstein-Primakoff transformation \[18, 19\]. In this representation the spin ladder operators are mapped into bosonic creation and annihilation operators. In this way the spin raising operator is associated with the annihilation of a bosonic excitation $S^+_i \sim b$, while the spin lowering operator is correspondingly associated with the creation of a bosonic excitation $S^-_i \sim b^\dagger$. The dispersion relation can be written, in the vicinity of the base states, in terms of effective masses,

$$\hbar \omega_k = \hbar \omega_0 + \hbar^2 q^2_{||}/2m_{||} + \hbar^2 q^2_\perp/2m_{\perp},$$  \hspace{1cm} (1)
where \( \mathbf{q} = \mathbf{k} \pm \mathbf{k}^0 \). We remark the explicit global \( U(1) \) symmetry breaking induced by magnon decay processes on this model. This peculiar behavior reflects the fact that the full dipolar interaction term does not conserve the net magnon number. It will be shown that this fact does not pose any obstacle to a proper interpretation of the system’s behavior in terms of spontaneous coherence phenomena. This fact is in direct analogy with the case of the magneto-crystalline anisotropy in ferromagnets. As in that case, the weak anisotropy is of relevance only after a condensate state is achieved.

To take into account macroscopic coherence over macroscopic length scales an envelope wavefunction approach can be envisaged. From this picture, the system is described in terms of the two collective wave-functions associated with each minima. Using the collective field \( \Phi_\sigma(x, t) \), whose absolute value corresponds to the local density of magnons in states \( \sigma = -1, 1 \), see Fig. 1, while its phase correspond to the local collective phase. The energy associated with this state can be written in a compact form by using the following notation, \( (x, t, \sigma) \) labels will be summarized in a single subindex. For an homogeneous system the energy can be expressed in terms of the series expansion

\[
\mathcal{E} = \sum_{M=m+n} \Gamma_{\eta,y,\tau}^{\sigma,x,t} \Phi_{\sigma_1}(x_1, t_1) \cdots \Phi_{\sigma_n}(x_n, t) \Phi^*_{\eta_1}(y_1, \tau_1) \cdots \Phi^*_{\eta_m}(y_m, \tau_m) \tag{2}
\]

Terms unbalanced in the field and it conjugate are explicit violation of overall \( U(1) \) symmetry associated with conservation of the number of particles. In general, this argument forces them to cancel. However, the microscopic dynamics of magnons does not manifest invariance under such symmetry, reflecting the inherent lack of magnon conservation, and in principle such terms must be considered. By restricting our attention to low momentum, we need to focus only in those terms for which \( (\sigma_1 + \cdots + \sigma_n) = (\eta_1 + \cdots + \eta_m) \).

In particular we can discard from the contribution to the energy, terms proportional to odd powers of the fields. Such reduction is as far as one can get due to the \( U(1) \) symmetry-breaking terms. The ”anomalous”-terms are restricted to valley-mixing terms. Requiring that: (1) in the limit of vanishing density the system recovers the magnon spectrum Eq. 1; (2) the net momentum of the magnons be zero, and (3) the system is symmetric with respect to valley indices; it is possible to simplify the energy into:

\[
\mathcal{E}[\Phi, \Phi^*] = \int d\mathbf{r} \left( \hat{\hbar} \omega(\partial_\mathbf{r}) \Phi_1 \Phi^*_1 + \hat{\hbar} \omega(\partial_\mathbf{r}) \Phi_2 \Phi^*_2 \right) + \mu \left( \Phi^*_1 \Phi_1 + \Phi^*_2 \Phi_2 \right) \tag{3}
\]
\[ + \nu \Phi_1^\dagger \Phi_2 + \nu^* \Phi_1 \Phi_2 + \frac{\gamma_1}{2} \left( \Phi_1^\dagger \Phi_1 + \Phi_2^\dagger \Phi_2 \right)^2 + \frac{\gamma_2}{2} \left( \Phi_1^\dagger \Phi_1 - \Phi_2^\dagger \Phi_2 \right)^2 \]

where \( \nu, \gamma_1 \) and \( \gamma_2 \) are phenomenological parameters that should be determined from the experiment. Despite the explicit breakdown of the \( U(1) \) symmetry, as reflected by the terms proportional to \( \nu \), this energy is invariant under the residual symmetry transformation:

\[ \Phi_1 \rightarrow e^{i\delta} \Phi_1 \quad \text{and} \quad \Phi_2 \rightarrow e^{-i\delta} \Phi_2. \]  

(4)

The parameters \( \mu, \bar{\nu}, \gamma_1 \) can be obtained from experimental data as follows. First we set \( \nu = \bar{\nu} e^{i\psi} \). If \( \gamma_2 > 0 \) the energy is easily minimized by equally populating both valleys. Let \( \Phi_\sigma = \sqrt{n} e^{i\psi_\sigma} \), the energy density becomes

\[ \frac{\mathcal{E}}{\mathcal{A}} = 2n \left( \mu + \bar{\nu} \cos (\psi_1 + \psi_2 + \psi) \right) + 2\gamma_1 n^2 \]  

(5)

From the last equation, we find a condensation transition at \( \mu - \bar{\nu} < 0 \). We identify this symmetry breaking transition with the transition towards a macroscopically occupied lowest energy state reported in the experiments [5, 6, 7]. We can use this fact to associate

\[ \mu - \nu = \lambda (P_c - P), \]  

(6)

for a positive value phenomenological parameter \( \lambda \) and where \( P_c \) is the critical pumping power to reach the condensation transition. We can rewrite the Eq. (6) as \( \bar{\nu} = \nu + \lambda P \) and interpret the anomalous coefficient as composed of \( \nu \), coming from the intrinsic dynamics of the gas, and \( P \) the external flow of magnons into the condensate, i.e. the parametric pumping. The stationary density of magnons in such regime is:

\[ n_{\text{BEC}} = 2n = \frac{\lambda}{\gamma_1} (P - P_c). \]  

(7)

Remarkably, this behavior, linear in \( (P - P_c) \) is a natural consequence of the phenomenological approach together with the identity in Eq. (6). Its agreement with experimental data [29] can be readily verified and constitutes a non trivial correct prediction of the present phenomenological model. Additionally, following [14] a healing length can be calculated:

\[ \xi^2 = \frac{\hbar^2}{2 \sqrt{m_\parallel m_\perp} \lambda (P - P_c)^{-1}}. \]  

(8)
with a measurement of $P_c, n_{\text{BEC}}$ and $\zeta$ the phenomenological parameters can be determined.

The main result of this section is to provide a phenomenological picture of the collective dynamics of the magnons. The dissipation mechanisms can be encoded, within the phenomenological approach, in terms of a Rayleigh dissipation function [20]. In principle this function must be expanded in powers of $\partial_t \Phi_i$, this expansion to lowest order becomes:

$$\mathcal{R} = \alpha \int \, dr \left( |\partial_t \Phi_1|^2 + |\partial_t \Phi_2|^2 \right).$$

(9)

where $\alpha$ characterizes the damping constant as a phenomenological parameter.

The condensate consists, roughly speaking, of two magnon condensates lying the vicinity of the two points of minimum energy in momentum space, and magnetic interactions introduce a coupling between them. The basic phenomenological description of the dynamics is obtained, using the energy functional $\mathcal{E}$ together with a kinetic term

$$S = \int d\mathbf{r} dt \left( \Phi_1^\dagger i\hbar \partial_t \Phi_1 + \Phi_2^\dagger i\hbar \partial_t \Phi_2 \right) - \mathcal{E}[\Phi, \Phi^\dagger].$$

(10)

In the magnon condensate, the equations of motion correspond to the Euler-Lagrange equations:

$$\frac{\delta S}{\delta \Phi_i^\dagger} = \frac{\delta \mathcal{R}}{\delta (\partial_t \Phi_i^\dagger)}.$$  

(11)

Straightforward calculations lead to the conclusion that the dynamics of the two condensates follow can be described by the following generalized pseudo-spin GPE:

$$i\hbar (1 + i\alpha) \partial_t |\Psi\rangle = -\frac{\hbar^2}{2m||} \nabla^2 || |\Psi\rangle - \frac{\hbar^2}{2m_\perp} \nabla^2 _\perp |\Psi\rangle + \mu |\Psi\rangle + \bar{\nu} \sigma_x |\Psi^*\rangle + \gamma_1 |\Psi|^2 |\Psi\rangle + \gamma_2 \langle \Psi | \sigma_z |\Psi\rangle \sigma_z |\Psi\rangle$$

(12)

where the pseudo spin $|\Psi\rangle = (\Phi_1, \Phi_2)^\dagger$, refers to valley degeneracy in momentum space, while $\alpha, m, \mu, \bar{\nu}$ and $\gamma_i$ are real parameters characterizing the dynamics. The only term in the equation that breaks the time reversal symmetry (associated with the transformation $|\Psi(t)\rangle \rightarrow \sigma_x |\Psi^*(-t)\rangle$) is the term proportional to $\alpha$. This term plays the role of a damping constant.
much in the same way as the Gilbert damping term in magnetism [21, 22, 23].

Before closing this section we comment on other phenomenological approaches that have been taken in the literature. Gross-Pitaevskii equations have been constructed to describe the dynamics of magnon condensates in the works of [24, 26] and recently in [27]. We emphasize that this form of the equation is essentially different than the phenomenological ones proposed in those works, since Eq. (12) has a different form for the dissipation term and an explicitly gauge symmetry breaking term proportional to $\bar{\nu}$. In the next section a discussion concerning with the experimental relevance of Eq. (12) will be carried out.

3. Stationary states and comparison with experiment

Several predictions that can be drawn from the phenomenological model just described have been experimentally ratified. We start with the prediction of a spin density wave whose period is determined by the interference between the condensates at different valley.

The existence of periodic modulation of the condensate density, i.e. spin density wave, as well as its dislocation (equivalent to the vortices of the condensate) was recently observed in the experimental work [28]. This detection clearly shows the phase blocking between the two condensates located at $\pm k^0$ leading by the breaking of the residual symmetry discussed above. In what follows we will discuss the experimental results in the light of our phenomenological model. We will show that several features of the experiment can be given a simple explanation within the language presented in the previous section.

As well described in the last section the magnons spontaneously occupy macroscopically the doubly degenerate lowest energy when the condition Eq. (1) is satisfied. The two-component wave function for the ground state is directly associated in space by the periodic modulation of the magnetization deviation $\delta M = \sqrt{\rho} (\cos \theta, \sin \theta, 0) \cos (k^0 x + \delta)$. The phase $\theta$ define the plane of polarization of the SDW while its position is fixed by the relative phase $\delta$ and, according to the previous discussion, determined by the spontaneously breaking symmetry mechanism. The wavevector of the condensed magnons determine the wavelength of the SDW and this is given by $\lambda_{SDW} = 2\pi/k^0$.

For the experimental parameters used in [28], i.e. a YIG’s thin film of $5.1 \mu m$ thickness and placed into a static magnetic field $H_0 = 0.1 T$, we estimate for
SDW wavelength $\lambda_{SDW} \approx 1.8[\mu m]$ which is very similar to the experimental result (See Fig. 1 of Ref. [28]).

A consequence of long-scale phase coherence is the emergence of vortex-like excitations from the ground state of the condensate which is a natural feature inherited from the breaking of the $U(1)$ symmetry. This is observed as dislocation-like defects in the SDW pattern [28]. This kind of excitations were studied in [15] where the vortex is described by a velocity field $v = \frac{\hbar}{\sqrt{m_{||}m_{\perp}}} \nabla \delta(x)$, with $\delta(x)$ the phase of the condensed magnons. The single-valuedness of the wavefunction for the vortex, given by $\Phi_{(1,2)}(x) = \sqrt{\rho_0 R(x)} e^{\pm i\ell \delta(x)/2}$, requires that $\ell$ be an integer ($\ell$ the winding number) with $\sqrt{\rho_0}$ the condensate density of the ground state. The profile density $R(x)$ is vanishes in the origin, since the velocity field is singular at $|x| \to 0$, and becoming constant in the bulk. The size of the vortex is determined by a characteristic length, the so-called healing length. Such description is equivalent to the phenomenology presented in the previous section and it is fully characterized by the Eq. (8).

![Vortex structure with an elliptic cross section](image)

Figure 2: Vortex structure with an elliptic cross section of aspect ratio $\gamma = \sqrt{m_{||}/m_{\perp}} \sim 5$. Such structure emerge as a dislocation over the spin density wave, and with a Burgers vector proportional to the winding number of the vortex. The presented figure correspond to the square of magnetization deviation from saturation $|\delta \vec{M}/M_0|^2$ normalized by the magnetization saturation $M_0$.

In the experiment in Ref. [28] the critical chemical potential necessary to achieve condensation is $\mu_C = 1.5 \times 10^{-2}meV$. Here we take the critical pumping power (in units of $eV$) as $P_c = \mu_C$, the coefficient $\nu = 0.1\mu_C$ (the equivalent of $\nu$ in Eq. (1) of [28] is denoted by $J$) and the effective magnon’s mass $m_{||} \approx 50m_{\perp} \approx 5 \times 10^{-3}m_{He}$. We estimate for the healing length $\zeta_{HL} \approx 0.5[\mu m]$, i.e. the size of the vortex is about four times smaller than
the wavelength of the condensed magnons. This value is in agreement with the measurements and the model proposed in [28].

For the values $\lambda_{SDW}$ and $\zeta_{HL}$ obtained above we solve Eq. (12) for a vortex described by $\Phi_{(1,2)}(x)$. The vortex-solution is shown in the Fig. 2 where clearly it can be seen as a dislocation in the SDW pattern. Note that this solution correspond to a pair of vortices with opposite topologic charge and located in the same position. Each vortex exists in the $\pm k^0$—component respectively. Due to the different longitudinal and transverse masses, the vortex is anisotropic, with an elliptic cross section of aspect ratio $\gamma = \sqrt{m ||/m \perp} \sim 5$, for in-plane magnetic field $\sim 0.1[T]$, see Fig. 2.

It is worth emphasizing that in order to simulate the experiment the authors in [28] were forced to include a term in the Ginzburg-Landau equations that makes it equivalent with the features of Eq. (12). Both equations are peculiar due to the anomalous term proportional to $|\Psi^*|$, which coefficient is denoted $J$ in [28] and $\nu$ in Eq. (12).

4. Internal Josephson Effect

The notion that a condensate formed by two components (such the valley in our present case) might display Josephson oscillations can be traced back to work in cold atom gases [30]. As a first step we study the so-called internal Josephson effect in $\vec{k}$-space. If we separate the phase difference, $\phi$, between the valleys from Eq. (12), doing

$$|\Psi\rangle = \left( \begin{array}{c} e^{i\phi(t)} \sqrt{n_1(t)} \\ e^{-i\phi(t)} \sqrt{n_2(t)} \end{array} \right)$$

where $n_{(1,2)}(t)$ are the density in each valley, we can easily find a Josephson-like relationship. Defining the imbalance of magnons density between valleys,

$$p = \frac{\langle \Psi | \sigma_z | \Psi \rangle}{\langle \Psi | \Psi \rangle} \equiv \frac{n_1 - n_2}{n_1 + n_2},$$

the phase difference displays a damped behavior due to the dissipation coefficient $\alpha$. This coefficient couples the equations for $\phi$ and $p$

$$\hbar \dot{\phi} = -(\mu + (\gamma_1 + \gamma_2)n) p - \hbar \alpha \dot{p},$$

$$\dot{p} = -\alpha \dot{\phi}$$
that can be derived from Eq. (12). In this case, Eq. (15) have an interesting implication concerning the damping of the sliding modes. The equations can be rearranged in the form \( \dot{p} = -\frac{p}{\tau_p} \), where \( \tau_p = \hbar/2\alpha (\mu + (\gamma_1 + \gamma_2)n) \), and correspond to a simple dependence of the damping rate of such modes on the net magnon density of the system. There is a profound relationship between Eq. (15), and the semiclassical equations of motion for the collective dynamics of a Josephson junction [31] and a single-domain easy plane ferromagnet in an in-plane field [32]. This phase relationship over the relative phase of the condensate indicates that Josephson-related effects should be displayed by the system. Those effects will link the dynamics of magnon population and the spatial configuration of the magnetic patterns.

5. Magnonic Josephson Junction

We propose a magnonic Josephson junction (MJJ) on ferromagnetic YIG thin films for weakly linked magnon condensates. The splitting of the cloud condensate can be implemented by applying a direct current, where the tunneling between both states is adjusted by varying the current and geometric
parameters of the setup. This approach to achieved the desired spatial splitting the condensed of cloud naturally arises from the experimental studies of spin wave tunneling in a nonuniform magnetized thin films [33, 34]. In this section we take out the realization of MJJ and will build, from a phenomenological point of view, the semiclassical equations of motion for the collective dynamics between the condensate states.

Figure 4: (a) Sketch of the experimental setup for the magnon Josephson’s effect realization. The spatial fragmentation of the cloud magnon condensate over the YIG thin film is allowed by means of a wire conductor, for which crosses a DC current j and producing an local inhomogeneity, $H_j(x||)$, in the magnetization. The current direction determines the sign of the potential barrier which magnons feel. (b) Cartoon of the two-mode approximation over the full macroscopic wave function of the condensate state. The local inhomogeneity in the magnetization produced by a DC current through a wire conductor, corresponds to a potential barrier to the magnons.

Once the population of magnons, created through a parametric pumping in a YIG thin film, surpasses a critical level, the systems develops a phase coherence. When the macroscopic state is partitioned in two condensate clouds, the low-energy dynamics between them is completely described by two macroscopic observables, namely, population imbalance $\eta$ and the relative phase $\phi$. We introduce a local inhomogeneity $H_j(x||)$ in the magnetization, produced by means of a current that goes through a conductor and placed transversely to the YIG sample. In Fig. 4 a cartoon of the experimental setup is depicted, where the spatial separation of the magnon cloud condensate can be achieved.

As shown above the condensate state of magnons has two components that belong to the vicinity of energy minimum due to the double valley degeneracy, $\pm k_0$ in the spectrum, Fig. 4. The dynamics follow the pseudo-
spin GP equation,

\[
    i\hbar(1 + i\alpha)\partial_t|\Psi\rangle = -\frac{\hbar^2}{2m_||}\nabla_{||}^2|\Psi\rangle - \frac{\hbar^2}{2m_\perp}\nabla_\perp^2|\Psi\rangle + \mu|\Psi\rangle + V_{\text{ext}}(r)|\Psi\rangle + \nu\sigma_z|\Psi\rangle + \text{\gamma}_1|\Psi|^2|\Psi\rangle + \text{\gamma}_2\langle\Psi|\sigma_z|\Psi\rangle\sigma_z|\Psi\rangle.
\]

(16)

The potential barrier produced by the current crossing the wire conductor have the simple form,

\[
    V_j(x||) = \frac{\hat{\gamma}\hbar H_j}{\sqrt{\delta^2 + x||^2}}
\]

(17)

where \(H_j\) is the Oersted magnetic field produced by the dc current, \(\delta\) the separation between the wire and the YIG film, and \(\hat{\gamma}\) the effective coupling between the magnons and the magnetic field. That external potential introduce an additional energetic gap than the magnons in the cloud condensate must overcome to get in the regions right beneath the wire introducing a depletion of magnons, this will play the role of a weak link between the magnon condensates at each side of the barrier.

To be more specific we write the full macroscopic wave function as the addition of a two spatially separated time-dependent states,

\[
    \Psi(x, t) = \psi_L(t)(x)\Phi_L(x) + \psi_R(t)(x)\Phi_R(x),
\]

(18)

This approximation, so-called the two-mode approximation [35, 36], in the Gross-Pitaevskii equation has proven to be a successful description to predict the existence of a Josephson tunneling phenomena in clouds of bosonic system confined in a double-well potential. The left and right modes can be obtained from

\[
    \Phi_{L,R}(x) = \frac{1}{\sqrt{2}}(\Phi_g \pm \Phi_x),
\]

(19)

corresponding to the symmetric and antisymmetric functions, are constructed from the ground-state \(\Phi_g\), and the first excited state \(\Phi_x\) satisfying the stationary Gross-Pitaevskii equation. Uniformity in the direction parallel to the wire allows a further simplification, the wavefunction depends just on the longitudinal coordinate and the stationary Gross-Pitaevskii equation can be reduced to the one-dimensional nonlinear Schrödinger equation with a external potential given by Eq. (17). To determine this functions we have modified the barrier shape into a piecewise constant potential, and solved the non-linear equations in terms of Jacobi functions [36]. Outside the barrier
the elliptic function \( \text{sn} \) was used, while inside the barrier the solutions where written in terms of the elliptic function \( \text{nc} \). After matching the boundary conditions we were able to determine \( \Phi_g \) and \( \Phi_x \). Putting the Eq. (18) in the full Gross-Pitaevskii equation Eq. (16), using the mentioning above statements we find the Josephson equations for the two dynamical modes to obeys

\[
i \hbar (1 + i\alpha) \partial_t \psi_L(t) = \left[ E_L + U_L \left( \gamma_1 |\psi_L|^2 + \gamma_2 \left( \psi_L^* \sigma_z \psi_L \right) \sigma_z \right) \right] \psi_L(t) + \nu \sigma_x \psi_L^*(t) + K \psi_R(t) \\
i \hbar (1 + i\alpha) \partial_t \psi_R(t) = \left[ E_R + U_R \left( \gamma_1 |\psi_R|^2 + \gamma_2 \left( \psi_R^* \sigma_z \psi_R \right) \sigma_z \right) \right] \psi_R(t) + \nu \sigma_x \psi_R^*(t) + K \psi_L(t)
\] (20)

where the spatial dependence was integrated utilizing the orthogonality condition for \( \Phi_{L,R}(x) \). These systems of non-linear equations represents the dynamics between two magnon condensate states with a coupling factor, proportional to the wave function overlap.

The information about of the spatial dependence is contained in the coefficients \( E_i, U_i \) and \( K \). The meaning of such parameters are the following: the coefficient \( E_i \) represent the zero point energy in each region, \( U_i n_i^\pm k m \) are proportional to the self-interaction energies, while \( K \) describe the amplitude of the tunneling between both condensates. Those coefficients can be written in terms of \( \Phi_{g,x}(r) \) wave-functions overlaps and the effective parameters which characterize the condensed phase determined by the experimental realization displayed in the Fig. (4). The expressions for each one of this coefficients, where the weakly linked approximations was used, correspond to

\[
E_i = \int dr \Phi_i(r) \left[ -\frac{\hbar^2}{2m} \nabla^2 + \mu + V_j(r) \right] \Phi_i(r) \quad (21)
\]

\[
U_i = \int dr |\Phi_i|^4 (r) \quad (22)
\]

\[
K = \int dr \left( \frac{\hbar^2}{2m} \nabla \Phi_L(r) \nabla \Phi_R(r) + V_{\text{ext}}(r) \Phi_L(r) \Phi_R(r) \right) \quad (23)
\]

where the relation between those Josephson coefficients and the effective microscopic parameters, which describe the condensed phase, is clear. From this motivation the wave functions \( \psi_i, \ i = L, R \), can be written in occupation.
density-phase representation, i.e.

\[ \psi_i = \left( \frac{\sqrt{n_i(t)}e^{i\phi_i(t)}}{\sqrt{n_i(t)}e^{-i\phi_i(t)}} \right) \] (24)

where we are limiting ourselves to consider the in-phase oscillations with the total number of magnons associated to each valley is the same.

6. Phase dynamics in a Magnonic Josephson Junction

In the last section we have established several properties, characteristics of the magnon condensate system fragmented by a potential well. For now we will restrict ourselves to the case where the internal oscillations are frozen and just we considered, as a relevant variables, the oscillations between both left and right states, i.e. the external magnon Josephson effect. The interesting case on the dynamics of the interplay between the ±k-valleys and the spatially separated clouds is will leave for a further study.

Writing the two-mode dynamical equation, Eq. (20), by considering the expression for the wave functions Eq. (24), the coupled magnon condensates system can be driven to the equations

\[ \dot{\eta} = -\alpha \Gamma \eta - \sqrt{1 - \eta^2} \sin \phi \] (25)

\[ \dot{\phi} = \Lambda \eta + \frac{1}{\sqrt{1 - \eta^2}} (\eta \cos \phi - \alpha \sin \phi) \]

where the magnon population imbalance and relative phase are defined as

\[ \phi = \phi_R(t) - \phi_L(t) \]

\[ \eta = \frac{(n_L(t) - n_R(t))}{n_T}. \]

The pair of equations Eq. (25) represent the macroscopic interference between the cloud of magnons. The time is rescaled to a dimensionless characteristic time \( t_c = \hbar/2K \). Such time has been estimated, using the values for the coefficients as \( \sim 10 \) ns. The phenomenological parameters enter in the following combined form

\[ \Lambda = \gamma_1 U \rho_c / 2K, \]

\[ \Gamma = (2E + 2\gamma_1 U \rho_c + \bar{\nu} - \mu) / 2K. \]
The total conserved energy, i.e. without dissipation, can be written as

$$H[\phi, \eta] = \frac{\Lambda}{2} \eta^2 - \sqrt{1 - \eta^2} \cos \phi$$  \hspace{1cm} (26)

and which give rise to the equations Eq. (25) for $\alpha = 0$. The dimensionless parameters $\Lambda, \Gamma$ and the dissipation determine the dynamic regimes of the magnon condensate tunneling. Is evident that if there is not dissipation, only the $\Lambda$ parameter determine the type of oscillation in the Eq. (25) being, in turn, characterized by the in-plane magnetic field and the size of the inhomogeneity.

Let us start with the most elemental case, i.e. the Josephson’s oscillation of small amplitude. The oscillatory behavior of the relative phase reflect the macroscopic interference of the magnon condensate states.

The solutions for magnon current and relative phase are calculated for several values of the potential well, i.e. to different intensities of the dc current applied. In the Fig. (6) we illustrate experimentally accessible solutions for those macroscopic observables of the system, when the inhomogeneity in the magnetization take values in the range $250 < H_0 < 500$[Oe]. For small amplitude of oscillations, the phase and population imbalance oscillate with frequency, $\omega_{ac} = \sqrt{1 + \Lambda}$. In fact in a realistic scenery the frequency is about

![Figure 5: (a) Energy levels in phase space for the magnonic Josephson oscillations. The close or open nature of the iso-energy curves delimits two, qualitative different, Josephson’s oscillation regimes. The pictures represents the case $\Lambda \sim 10$. (b) Parameter $\Lambda$ calculated from our model for the magnonic junction as a function of the Oersted field generated by the current.](image)
\[ \omega_{ac} \sim 0.6\text{[GHz]} \text{ for } 250 < H_0 < 500\text{[Oe]} \]. It is worth noting that the critical magnon current is related to the amplitude which is determined by the initial population imbalance between the condensate states, while the frequency of oscillation \( \omega_{ac} \) is determined by the dc current.

The solutions displayed in the Fig. (6) are calculated taking into account the dissipation and considering small Fig. (6a) and long-amplitude oscillations Fig. (6b), where we considered a in-plane magnetic field \( H_0 = 1\text{[KOe]} \) and dissipation coefficient \( \alpha = 10^{-4} \). The small amplitude oscillation is determined by the values of parameters \( \Lambda = 20(H_j = 250\text{[Oe]}) \) and initial conditions \( \eta_0 = 0.1, \phi_0 = 0.1 \), while for long-amplitude oscillations we use \( \Lambda = 70(H_j = 500\text{[Oe]}) \) and initial conditions, \( \eta_0 = 0.1, \phi_0 = \pi/2 \). It is worth point than that solutions, near of critical value, \( \Lambda_c \), which separates the two regimes presents in the Eq. (25), the period of oscillation is sensitive to initial conditions since the nonlinear contributions begin to be relevant. A feature that distinguishes this regime of oscillation is the mean value of the magnon current \( \langle \eta(t) \rangle = 0 \), i.e. the system of two macroscopic states oscillate around an equilibrium value set in \( \eta_{eq} = 0 \). Moreover, it follows from Fig. (6b) that the large-amplitude oscillations are quickly damped, respect to the small amplitude oscillations, due to the nonlinearity of system. In this sense, the dissipation prevents the spread of nonlinearity more efficiently than the small amplitude solutions, driving rapidly the nonlinear solution to the linear regime.

The solutions presented here correspond to the Josephson effect for the condensed magnons. The frequency of oscillations of the magnon current is directly related to the macroscopic relative phase of the condensates, which are within the typical experimental resolution range. Therefore we provide a precise scheme for the observation of ac Josephson’s oscillations, over the YIG thin films, as a prove of the spontaneous macroscopic coherence of the condensate of magnons.

7. Macroscopic self-trapping of magnons

The magnon Josephson’s oscillations discussed so far correspond to an oscillating current of magnons crossing the potential barrier, they are characterized by symmetric oscillations of the occupation density \( \langle \eta(t) \rangle = 0 \). However, this scenery change drastically when the interaction surpasses a critical value \( \Lambda > \Lambda_c \), where the Josephson’s oscillations follow a qualitatively different behavior. In this stage the evolution of magnon population
Figure 6: Dynamical behavior of both relative phase $\phi$ and population imbalance $\eta$ between the clouds magnons condensate. The solutions are calculated for the typical experimental conditions over the YIG thin film, a in-plane magnetic field $H_0 = 1$[KOe] and dissipation $\alpha = 10^{-4}$. In (a) and (b) the small amplitude and long wave oscillations are displayed for $\Lambda(H_a = 400[\text{Oe}]) = 30$ and initial conditions, $\phi(0) = 0.1\pi$-$\eta(0) = 0.1$ and $\phi(0) = 0.7\pi$-$\eta(0) = 0.1$, respectively. While in (c) the MQST solutions are obtained for $\Lambda(H_a = 250[\text{Oe}]) = 50$ and initial conditions $\phi(0) = \pi$, $\eta(0) = 0.1$. 

---

| Time, ns | Time, ns | Time, ns |
|----------|----------|----------|
| 0        | 20       | 40       |
| 0.1      | 0.0      | 0.0      |
| -0.1     | 0.0      | 0.0      |

| Time, ns | Time, ns | Time, ns |
|----------|----------|----------|
| 0        | 15       | 30       |
| 0.2      | 0.0      | 0.0      |
| -0.2     | 0.0      | 0.0      |

| Time, ns | Time, ns | Time, ns |
|----------|----------|----------|
| 0        | 20       | 40       |
| 0.1      | 0.0      | 0.0      |
| -0.1     | 0.0      | 0.0      |

| Time, ns | Time, ns | Time, ns |
|----------|----------|----------|
| 0        | 15       | 30       |
| 0.2      | 0.0      | 0.0      |
| -0.2     | 0.0      | 0.0      |

| Time, ns | Time, ns | Time, ns |
|----------|----------|----------|
| 0        | 20       | 40       |
| 0.2      | 0.0      | 0.0      |
| -0.2     | 0.0      | 0.0      |

| Time, ns | Time, ns | Time, ns |
|----------|----------|----------|
| 0        | 15       | 30       |
| 0.2      | 0.0      | 0.0      |
| -0.2     | 0.0      | 0.0      |

| Time, ns | Time, ns | Time, ns |
|----------|----------|----------|
| 0        | 15       | 30       |
| 0.2      | 0.0      | 0.0      |
| -0.2     | 0.0      | 0.0      |
imbalance is characteristic of a nonzero time-average \( \langle \eta(t) \rangle \neq 0 \), i.e. there is a self-trapping of magnons at one side of the wire conductor, where the dynamics evolution is strongly influenced by the initial conditions.

This nonlinear phenomena, discovered by [36, 13] in the context of BEC’s of alkali gases and coined as macroscopic quantum self-trapping (MQST), which quantum nature involves the coherence of a macroscopic number of bosons in the two condensates. Here we show that the magnons condensate manifest themselves in a macroscopic self-trapping state (MST), for certain values of the self-interaction and where the dissipation play a important role in the dynamics of such state. That regime is achieved when the parameter \( \Lambda \) exceed the critical value \( \Lambda_c \), which is in turn, specified by the initial conditions \( \eta_0 \) and \( \phi_0 \), through:

\[
\Lambda_c = \frac{1 + \sqrt{1 - \eta_0^2 \cos \phi_0}}{\eta_0^2 / 2}.
\]  

Once \( \Lambda > \Lambda_c \), is satisfied, the Josephson’s oscillations are driven to the MST regime. Due to the effects of damping, it can be seen that the condensed magnons remain in the MST state for some time until decay into a long-amplitude oscillation, see Fig. 6(c). Indeed the dissipation quickly destroys such state giving way to the Josephson’s oscillations studied before, i.e. for \( \Lambda > \Lambda_c \) fixed but increasing the initial energy \( H_0 \), the MST solutions are more quickly damped, and then the lifetime of the magnons self-trapped, is smaller. That point can be visualized in the breakdown of the MST state, establishing a characteristic life-time mean for this state, and for later give rise to the Josephson’s oscillations.

8. Discussion and summary

Before closing it is worth commenting on the experimental signals that might be expected from the effects discussed in this work. Magnetization oscillations can be measured in several ways. The basic mechanism used so far in the context of magnon condensates, is the Brillouin light scattering technique (BLS) [40]. Such technique probes the magnons system by studying their effect on microwave radiation reflected by the sample. In this way it might be expected that the oscillations in magnon density between the two magnonic clouds might be detected. Since our predictions involve oscillation periods on the order of \( 5 - 20\text{[ns]} \). Such oscillations are however shorter than the characteristic resolution of the BLS measurement. As an alternative the
magnon dynamics can be mapped into spin currents pumped into a metallic sample in contact with the system [11]. Such currents have been measured by means of the inverse spin Hall effect in Pt [42], that converts them in charge currents. In the present case, it is easy to show that the presence of magnon-condensate implies a constant current. Oscillations in such current can be detected and interpreted as signatures of the underlying oscillations.

In conclusion we have presented a phenomenological theory, that focusing only on the low-energy and momentum projections of the magnon spectrum, accounts for the collective dynamics of a Bose-Einstein condensate of magnons. Such theory has allowed us to provide a simple understanding of the mechanisms behind the condensation of magnons and to establish a clear understanding of the meaning of the collective wave function used to describe it. Despite these efforts to understand the BEC phase in YIG a systematic study concerning the interaction between the condensed and thermal magnons is still lacking. In terms of such description we discuss the nature of macroscopic interference between magnon clouds. Starting with the discussion of the internal Josephson oscillations, that correspond to oscillations between the $\pm k^0$ components of the condensed cloud, we have highlighted the close relation between such effects and the well-known Josephson effect. Using those ideas we presented a detailed calculation of the Josephson oscillations between two magnon clouds, spatially separated in a magnonic Josephson junction. Among the results we remark the clear and distinctive oscillations that characterized common Josephson oscillations and also a regime that corresponds to the so-called macroscopic self-trapping, that locks the oscillations favoring one side of the junction over the other.

9. Acknowledgements

ASN would like to thank Professor R. A. Duine for helpful comments. This work was partially funded by Proyecto Fondecyt numbers 11070008 and 1110271, Proyecto Basal FB0807-CEDENNA, Núcleo Científico Milenio P06022-F and Proyecto Anillo de Ciencia y Tecnologa, ACT 1117.

References

[1] V. V. Kruglyak, et al. J. Phys. D Appl. Phys. 43 (2010) 264001.
[2] A. G. Gurevich and G. A. Melkov, *Magnetization Oscillation and Waves* CRC-Press, 1996.

[3] V. Cherepanov, et al. Phys. Rep *229*, 81 1993.

[4] A. A. Serga, et al. J. Phys. D: Appl. Phys. 43 (2010) 264002.

[5] S. O. Demokritov et al. Nature (London) *443*, 403 (2006).

[6] V. E. Demidov et al. Phys. Rev. Lett. *99*, 037205 (2007); Phys. Rev. Letters *100*, 047205 (2008).

[7] V.E. Demidov, et al. Phys. Usp. 53, 853 (2010).

[8] D. Snoke, Nature 443, 403 (2006).

[9] B. D. Josephson, Phys. Letters 1, 251 (1962).

[10] P. W. Anderson, et al. Phys. Rev. Letters *10*, 230 (1963); S. Shapiro, Phys. Rev. Letters *11*, 80 (1963); J. M. Rowell, Phys. Rev. Letters *11*, 200 (1963).

[11] S. Backhaus, et al. Science *278*, 1435 1997.

[12] K. Sukhatme, et al. Nature *411*, 280 2001).

[13] S. Levy, et al. Nature *449*, 579 (2007); M. Albiez, et al. Phys. Rev. Letters *95*, 010402 (2005)

[14] L. Pitaevskii and S. Stringari, *Bose-Einstein Condensation*, Oxford University Press, Oxford, 2003.

[15] R. E. Troncoso and A.S. Núñez, J. Phys.: Condens. Matter *24* 036006 (2012).

[16] A. I. Akhiezer and S.V. Peletminskii, *Spin Waves* (North-Holand, 1968).

[17] B. A. Kalinikos and A. N. Slavin, J. Phys. C *19*, 7013 (1986).

[18] T. Holstein and H. Primakoff, Phys. Rev. B*58*, 1098(1940).

[19] A. Auerbach, *Interacting electrons and quantum magnetism*, Springer-Verlag, New York, 1994.
[20] L. D. Landau and E.M. Lifshitz, *Mechanics*, Pergamon Press, Oxford, 1987.

[21] Z. Qian and G. Vignale, Phys. Rev. Letters 88, 56404 (2002).

[22] J. Fernández-Rossier, et al. Phys. Rev. B, 69, 12 (2004).

[23] E. Rossi, et al. Phys. Rev. B, 72, 174412 (2005).

[24] B. A. Malomed, et al. Phys. Rev. B 81, 024418 (2010).

[25] S. M. Rezende, Phys. Rev. B 79, 060410(R) (2009), S. M. Rezende, Phys. Rev. B 79, 174411 (2009).

[26] S. M. Rezende, Phys. Rev. B, 81, 020414(R) (2010).

[27] F. Li, et al. Sci. Rep. 3, 1372 (2013).

[28] Nowik-Boltyk, P., et al. Sci. Rep. 2, 482 (2012).

[29] S. O. Demokritov, et al., New Journal of Physics 10, 045029 (2008).

[30] Hall, D. S., Matthews, M. R., Wieman, C. E. and Cornell, E. Phys. Rev. Lett. 81, 1543(1998).

[31] See for example, Y. Makhlin et al. Rev. Mod. Phys. 73, 357, (2001), and work cited therein.

[32] Enrico Rossi, et al. Phys. Rev. Letters 95, 266804 (2005), and work cited therein.

[33] U.H. Hansen, et al. Phys. Rev. Letters 99 (2007).

[34] T. Schneider, et al. App. Phys. Lett. 92(2008).

[35] S. Giovanazzi, et al. Phys. Rev. Letters 84, 4521 (2000).

[36] S. Raghavan, et al. Phys. Rev. A 59, 620 (1999).

[37] L. Cruzeirohansson, et al. Phys. Rev. B 37, 7896 (1988).

[38] O. Dzyapko, et al. Phys. Usp. 53, 853 (2010)

[39] M. O. J. Heikkinen, et al. Phys. Rev. Letters 105 (2010).
[40] S. Demokritov, et al. Phys. Rep 348, 442 (2001).

[41] Y. Tserkovnyak et al. Phys. Rev. Letters 88, 117601 (2002), Rev. Mod. Phys. 77, 1375 (2005).

[42] E. Saitoh, et al. Appl. Phys. Lett. 88, 182509 (2006)