GENERIC SYZYGY SCHEMES

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Abstract. For a finite dimensional vector space $G$ we define the $k$-th generic syzygy scheme $\text{Gensyz}_k(G)$ by explicit equations. If $X \subset \mathbb{P}^n$ is cut out by quadrics and $f$ is a $p$-th syzygy of rank $p + k + 1$ we show that the syzygy scheme $\text{Syz}(f)$ of $f$ is a cone over a linear section of $\text{Gensyz}_k(G)$. We also give a geometric description of $\text{Gensyz}_k(G)$ for $k = 0, 1, 2$, in particular $\text{Gensyz}_2(G)$ is the union of a Plücker embedded Grassmannian and a linear space. From this we deduce that every smooth, non-degenerate projective curve $C \subset \mathbb{P}^n$ which is cut out by quadrics and has a $p$-th linear syzygy of rank $p + 3$ admits a rank 2 vector bundle $\mathcal{E}$ with $\text{det} \mathcal{E} = \mathcal{O}_C(1)$ and $h^0(\mathcal{E}) \geq p + 4$.

1. Introduction

Let $X \subset \mathbb{P}^n$ be a projective variety that is cut out by quadrics. One can then look at the linear strand of its minimal free resolution and ask whether a $p$-th linear syzygy $f$ carries some geometric information about $X$. For this purpose Ehbauer [Ehb94] introduced the syzygy scheme $\text{Syz}(f)$, which is cut out by the quadrics involved in $f$. The syzygy scheme always contains $X$ and can be explicitly calculated in some cases. Ehbauer studied this construction when $X$ is a set of points in uniform position.

Another geometric invariant of a $p$-th syzygy $f$ is the space $G^*$ of linear forms involved in $f$. Its dimension is called the rank of $f$. Interesting syzygy varieties often arise from syzygies of low rank.

In [Sch91] Schreyer observed that for $p = 1$ the syzygy scheme $\text{Syz}(f)$ is always a cone over a linear section of a generic syzygy scheme $\text{Gensyz}_k$ with $k = \text{rank } f - 2$ and gave explicit equations for $\text{Gensyz}_k$ in this case. Eusen and Schreyer found a geometric description of these schemes for $k \in \{0, \ldots, 4\}$ and $p = 1$ in [ES94].

In this paper we define more general generic syzygy schemes $\text{Gensyz}_k(G)$ by explicit equations depending on a finite dimensional vector space $G$. With these schemes we prove:

Theorem 3.4 Let $I \subset R$ be a homogeneous ideal generated by quadrics and $f$ a $p$-th rank $p + k + 1$ linear syzygy of $I$. Then the syzygy scheme $\text{Syz}(f)$ is isomorphic to a cone over a linear section of $\text{Gensyz}_k(G)$ where $G$ is the space of $(p - 1)$-st syzygies involved in $f$.

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We also obtain a geometric description of Gensyz\(_k\)(G) for \(k = 0, 1, 2\) and arbitrary \(G\). We show that Gensyz\(_0\)(G) is always the union of a hypersurface with a point and that Gensyz\(_1\)(G) is a Segre-embedded \(\mathbb{P}^1 \times \mathbb{P}^{\text{rank} f-1}\). The main new result of this paper is

**Theorem 6.1.** Let \(G\) be a \(g\) dimensional vector space, then

\[
\text{Gensyz}_2(G) = \mathcal{G}(\mathbb{C} \oplus G^*, 2) \cup \mathbb{P}\left(\bigwedge^2 G^*\right) \subset \mathbb{P}(G^* \oplus \bigwedge^2 G^*),
\]

where \(\mathcal{G}(\mathbb{C} \oplus G^*, 2)\) is the Grassmannian of two dimensional quotient spaces of \(\mathbb{C} \oplus G^*\). Moreover the second generic syzygy ideal \(I\) of \(G\) is reduced and saturated.

The geometric descriptions of Gensyz\(_k\)(G) allow us to draw a number of conclusions:

**Corollary 4.2.** Let \(X \subset \mathbb{P}^n\) be a projective variety, \(I_X\) generated by quadrics and \(f \in F_p\) a \(p\)-th syzygy of rank \(p + 1\). Then \(X\) is either contained in a hyperplane or reducible.

This result seems to be well known, but we include it since it follows directly from our methods.

**Corollary 5.2.** Let \(X \subset \mathbb{P}^n\) be a non-degenerate irreducible projective variety, \(I_X\) generated by quadrics and \(f \in F_p\) a \(p\)-th syzygy of rank \(p + 2\). Then the syzygy scheme \(\text{Syz}(f)\) of \(f\) is a scroll of degree \(p + 2\) and codimension \(p + 1\).

In particular a \(p\)-th syzygy of rank \(p + 1\) implies the existence of a special pencil \(|D|\) on \(X\) cut out by the fibers of the scroll. If \(X\) is a canonical curve \(|D|\) has low Clifford index. These pencils are the ones that play a role in Green’s conjecture [Gre84]. Our corollary above is therefore probably well known to experts in this field.

Our main new geometric result is

**Theorem 6.7.** Let \(C \subset \mathbb{P}^n\) be a smooth, irreducible non-degenerate curve. If \(C\) is cut out by quadrics and has a \(p\)-th syzygy \(f\) of rank \(p + 3\), then there exists a rank 2 vector bundle \(\mathcal{E}\) on \(C\) with \(\det \mathcal{E} = \mathcal{O}_C(1)\) and \(h^0(\mathcal{E}) \geq p + 4\).

In the case of a canonical curve these are rank 2 bundles with canonical determinant.

One can also use the methods of this paper to construct the Mukai-Lazarsfeld bundle on a \(K3\) surface directly from a syzygy \(f\). This is the vector bundle that played a central role in Voisin’s proof of Green’s conjecture [Voi02], [Voi03]. The Grassmannian used by Voisin in her proof is dual to the Grassmannian obtained as the generic syzygy scheme of \(f\).

This paper is structured as follows. In Section 2 we recall what we need about syzygies, syzygy ideals and syzygy schemes. In Section 3 we define the generic syzygy varieties and show that every syzygy scheme is a cone over a linear section of a generic syzygy scheme. In the last three sections we describe the \(k\)-th generic syzygy varieties for \(k = 0, 1, 2\) geometrically and study syzygies of rank \(p + 1, p + 2\) and \(p + 3\).
2. Syzygies, Syzygy Ideals and Syzygy Schemes

For the purpose of this paper let \( R = \mathbb{C}[x_0, \ldots, x_n] \) be the homogeneous coordinate ring of \( \mathbb{P}^n \). With \( R(-i) \) we denote \( R \) with its grading shifted, i.e. \( R(-i) = R_j \). Often we abbreviate the space of linear polynomials \( R_1 \subset R \) by \( V \) and write \( \mathbb{P}^n = \mathbb{P}(V) \) using the Grothendieck notation.

**Definition 2.1.** Let \( I \subset R \) be a homogeneous ideal, generated by quadrics, and

\[
F_\bullet : I \leftarrow F_0 \otimes R(-2) \leftarrow \cdots \leftarrow F_p \otimes R(-r - 2)
\]

the linear part of the minimal free resolution of \( I \). The elements of \( F_i \) are called \( i \)-th linear syzygies of \( I \).

**Definition 2.2.** Let \( I \subset R \) be a homogeneous ideal, generated by quadrics and \( f \in F_p \) a \( p \)-th linear syzygy. We define the space of \((p - 1)\)-st linear syzygies involved in \( f \) as the smallest vector space \( G \subset F_{p-1} \) such that there is a commutative diagram

\[
\begin{array}{cccccccc}
F_{p-1} \otimes R(-p - 1) & \leftarrow & F_p \otimes R(-p - 2) \\
| & & | \\
G \otimes R(-p - 1) & \leftarrow & f \otimes R(-p - 2).
\end{array}
\]

We define the rank of \( f \) as the dimension of \( G \).

The above diagram extends to a map from the Koszul complex of \( G \) to the linear strand of \( I \):

\[
\begin{array}{cccccccc}
I & \leftarrow & F_0 \otimes R(-2) & \leftarrow & \cdots & \leftarrow & F_p \otimes R(-p - 2) \\
| & & | & & | & & | \\
\wedge^{p+1} G \otimes R(-1) & \leftarrow & \wedge^p G \otimes R(-2) & \leftarrow & \cdots & \leftarrow & f \otimes R(-p - 2).
\end{array}
\]

The image of \( \wedge^p G \) in \( I \) is called the syzygy ideal \( I_f \) of \( f \).

**Remark 2.3.** Observe that by dualizing and twisting the morphism

\[
G \otimes R(-p - 1) \leftarrow f \otimes R(-p - 2)
\]

from above, \( G^* \) is exhibited as a space of linear forms on \( \mathbb{P}^n \). We therefore call \( G^* \) the space of linear forms involved in \( f \).

**Lemma 2.4.** In the map of complexes of Definition 2.2 all vertical maps are nonzero.

**Proof.** Suppose there exists an integer \( k \) such that in the diagram

\[
\begin{array}{cccccccc}
F_{k-1} \otimes R(-k - 1) & \leftarrow & F_k \otimes R(-k - 2) \\
| & & \varphi_{k-1} & & \varphi_k & & | \\
\wedge^{p-k+1} G \otimes R(-k - 1) & \leftarrow & \wedge^{p-k} G \otimes R(-k - 2)
\end{array}
\]
the morphism $\varphi_{k-1}$ is zero, but $\varphi_k$ is nonzero. Then the image of $\varphi_k$ is a free summand of $F_k \otimes R(-k - 2)$ which maps to zero in the linear strand of the minimal free resolution of $I$. This contradicts the minimality of the resolution. \hfill \Box

**Corollary 2.5.** Let $f$ be a $p$-th linear syzygy of $I \subset R$. Then $\text{rank } f \geq p+1$.

*Proof.* If $\text{rank } f \leq p$ then $\bigwedge^{p+1} G$ vanishes and the first vertical map of the map of complexes in Definition 2.2 would have to be zero. \hfill \Box

**Definition 2.6.** Let $I \subset R$ be an ideal generated by quadrics, $f \in F_p$ a $p$-th linear syzygy and $I_f$ the syzygy ideal of $f$. Then the vanishing set $\text{Syz}(f) = V(I_f)$ is called the syzygy scheme associated to $f$.

**Remark 2.7.** Observe that $\text{Syz}(f) \subset \mathbb{P}^n$ is always a strict subset, since the syzygy ideal $I_f$ is never empty by Lemma 2.4.

### 3. Generic Syzygy Schemes

**Definition 3.1.** Let $G$ be a vector space of dimension $g$ and consider the ring $S = \mathbb{C}[G^* \oplus \bigwedge^k G^*]$. The ideal $I$ defined by the natural inclusion

$$I = \bigwedge^{k+1} G^* \subset G^* \otimes \bigwedge^k G^* \subset S^2 \left( G^* \oplus \bigwedge^k G^* \right) \subset S$$

is called the $k$-th generic syzygy ideal of $G$. Its vanishing set $\text{Gensyz}_k(G)$ is called the $k$-th generic syzygy scheme of $G$.

**Proposition 3.2.** Let $I$ be the $k$-th generic syzygy ideal of $G$. Then the linear strand of $I$ has the last $g - k$ steps of the Koszul complex associated to $G^*$ as a natural subcomplex, i.e. we have a commutative diagram:

$$\begin{array}{cccccc}
I & \leftarrow & F_0 \otimes S(-2) & \leftarrow & \cdots & \leftarrow & F_{g-k-1} \otimes S(-g + k - 1) \\
\bigwedge^{k+1} G^* \otimes S(-2) & \leftarrow & \cdots & \leftarrow & \bigwedge^g G^* \otimes S(-g + k - 1) \\
\end{array}$$

*Proof.* The inclusion $I = \bigwedge^{k+1} G^* \subset G^* \otimes \bigwedge^k G^* \subset S^2 \left( G^* \oplus \bigwedge^k G^* \right) \subset S$ induces a commutative diagram of free $S$-modules

$$\begin{array}{cccc}
I & \leftarrow & F_0 \otimes S(-2) & \\
\bigwedge^k G^* \otimes S(-1) & \leftarrow & \bigwedge^{k+1} G^* \otimes S(-2). \\
\end{array}$$

The top arrow is resolved by the minimal free resolution of $I$ and the bottom arrow by the rest of the Koszul complex. Since both complexes are exact and minimal, the maps above lift to a map of complexes. This map is injective in each new step since it is injective in the $F_0$ step. For degree reasons, the image of this map of complexes must lie in the linear strand of $I$. \hfill \Box
Corollary 3.3. The $k$-th generic syzygy scheme of $G$ has a natural 1-dimensional space of rank $g$ linear syzygies in step $g - k - 1$. The space of $(g - k - 2)$-nd syzygies involved in anyone of these is isomorphic to $G$.

Proof. The $(g - k - 1)$-st syzygies given by Proposition 3.2 have rank at most $g$ since $\wedge^{g-1} G^* \cong G$ has dimension $g$. The rank of these syzygies cannot be smaller, since the last map of the Koszul complex is surjective in degree $g - k$. \qed

Theorem 3.4. Let $I \subset R$ be a homogeneous ideal generated by quadrics and $f$ a $p$-th rank $p + k + 1$ linear syzygy of $I$. Then the syzygy scheme $\text{Syz}(f)$ is isomorphic to a cone over a linear section of $\text{Gensyz}_k(G)$ where $G$ is the space of $(p - 1)$-st syzygies involved in $f$.

Proof. We have the map of complexes

$$
\begin{array}{c}
R & & F_0 \otimes R(-2) & & \cdots & & F_p \otimes R(-p - 2) \\
\alpha & & \downarrow & & \downarrow & & \downarrow \\
\wedge^{p+1} G \otimes R(-1) & & \wedge^p G \otimes R(-2) & & \cdots & & f \otimes R(-p - 2)
\end{array}
$$

from Definition 2.2. Consider the map

$$\varphi: G^* \oplus \wedge^k G^* \to V$$

given by mapping the elements of $G^*$ to their corresponding linear forms and the elements of $\wedge^k G^* = \wedge^{p+1} G$ to their images under the map $\alpha$. The induced diagram

$$
\begin{array}{c}
R & & F_0 \otimes R(-2) & & \cdots & & F_p \otimes R(-p - 2) \\
\alpha & & \downarrow & & \downarrow & & \downarrow \\
\wedge^{p+1} G \otimes R(-1) & & \wedge^p G \otimes R(-2) & & \cdots & & S & & \wedge^{k+1} G^* \otimes S(-2) \\
\wedge^k G^* \otimes S(-1) & & \wedge^{k+1} G^* \otimes S(-2)
\end{array}
$$

and its degree 2 part

$$
\begin{array}{c}
S^2 V & & F_0 \wedge & & \cdots & & F_p \wedge \\
\alpha & & \downarrow & & \downarrow & & \downarrow \\
\wedge^{p+1} G \otimes V & & \wedge^p G & & S^2 (G^* \oplus \wedge^k G^*) & & \wedge^{k+1} G^* \\
\wedge^k G^* \otimes (G^* \oplus \wedge^k G^*) & & \wedge^{k+1} G^*
\end{array}
$$
shows that $\varphi$ maps the $k$-th generic syzygy ideal surjectively to the syzygy ideal $I_f$ of $f$. Projectively the image of $\varphi$ defines a linear subspace

$$\mathbb{P}(\text{Im} \varphi) \subset \mathbb{P}(G^* \oplus \bigwedge^k G^*).$$

The calculation above shows that $\text{Syz}(f)$ is a cone over $\mathbb{P}(\text{Im} \varphi) \cap \text{Gensyz}_k(G)$ with vertex $V(\text{Im} \varphi) \subset \mathbb{P}(V)$. □

4. Reducible Syzygies

**Proposition 4.1.** Let $G$ be a $g$ dimensional vector space, then

$$\text{Gensyz}_0(G) \cong \mathbb{P}(G^*) \cup \mathbb{P}(\mathbb{C}) \subset \mathbb{P}(G^* \oplus \mathbb{C}),$$

i.e $\text{Gensyz}_0(G)$ is the union of a hyperplane and a point. Moreover the generic syzygy ideal of $I$ of $\text{Gensyz}_0(G)$ is reduced and saturated.

**Proof.** The ideal of the hyperplane $\mathbb{P}(G^*) \cong \mathbb{P}^{g-1}$ is generated by the linear forms in $\bigwedge^0 G^* \cong \mathbb{C}$. The ideal of the point $\mathbb{P}(\bigwedge^0 G^*) \cong \mathbb{P}^0$ is generated by the linear forms in $G^*$. Since the two ideals involve different sets of variables, their intersection is the same as their product:

$$I_{\mathbb{P}^{g-1}} \cap I_{\mathbb{P}^0} = (G^*) \cap (\bigwedge^0 G^*) = (G^*) \cdot (\bigwedge^0 G^*) = (G^* \otimes \bigwedge^0 G^*) = (\bigwedge^1 G^*)$$

This is the 0-th generic syzygy ideal of $G$. □

**Corollary 4.2.** Let $X \subset \mathbb{P}^n$ be a projective variety, $I_X$ generated by quadrics and $f \in F_p$ a $p$-th syzygy of rank $p+1$. Then $X$ is either contained in a hyperplane or reducible.

**Proof.** By Theorem 3.4 and Proposition 4.1 $\text{Syz}(f)$ is a cone over a linear section of a hyperplane and a point. Since $\text{Syz}(f)$ can not contain all of $\mathbb{P}^n$ by Remark 2.7 $\text{Syz}(f) \subset \mathbb{P}^n$ must be the union of a hyperplane and possibly a second linear subspace. Since $X$ is contained in $\text{Syz}(f)$ it must be either reducible or contained in one of the two linear subspaces. □

**Definition 4.3.** Let $X \subset \mathbb{P}^n$ be a projective scheme, whose ideal is cut out by quadrics. A $p$-th linear syzygy of $X$ is called reducible, if it has rank $p+1$.

5. Scrollar Syzygies

**Theorem 5.1.** Let $G$ be a $g$ dimensional vector space, then

$$\text{Gensyz}_1(G) = \mathbb{P}(G^*) \times \mathbb{P}^1 \subset \mathbb{P}(G^* \oplus G^*).$$

Moreover the second generic syzygy ideal $I$ of $G$ is reduced and saturated.

**Proof.** Observe that $G^* \otimes (\mathbb{C} \oplus \mathbb{C}) = G^* \oplus G^*$. We can therefore consider the Segre embedding

$$\mathbb{P}^{g-1} \times \mathbb{P}^1 = \mathbb{P}(G^*) \times \mathbb{P}(\mathbb{C} \oplus \mathbb{C}) \subset \mathbb{P}(G^* \oplus G^*).$$
The ideal of $\mathbb{P}^{g-1} \times \mathbb{P}^1$ is generated by the Segre quadrics:

$$I_{\mathbb{P}^{g-1} \times \mathbb{P}^1} = \left( \bigwedge^2 G^* \otimes \bigwedge^2 (\mathbb{C} \oplus \mathbb{C}) \right) = \left( \bigwedge^2 G^* \right)$$

This is the first generic syzygy ideal of $G$.

**Corollary 5.2.** Let $X \subset \mathbb{P}^n$ be a non degenerate irreducible projective variety, $I_X$ generated by quadrics and $f \in F_p$ a $p$-th syzygy of rank $p+2$. Then the syzygy scheme $\text{Syz}(f)$ of $f$ is a scroll of degree $p+2$ and codimension $p+1$.

**Proof.** Let $G$ be the $g = p + 2$ dimensional space of $(p - 1)$-st syzygies involved in $f$. By theorem 5.1, the syzygy scheme $\text{Syz}(f)$ is a linear section of a cone over $\mathbb{P}^{p+1} \times \mathbb{P}^1$. Since $\mathbb{P}^{p+1} \times \mathbb{P}^1$ has codimension $p+1$ and degree $p+2$ in $\mathbb{P}(G^* \oplus \Lambda^1 G^*)$ we only have to prove that this intersection is of expected codimension. By Eisenbud [Eis95, Ex. A2.19] this is the case if the matrix $M$ whose $2 \times 2$-minors cut out $\mathbb{P}^{p+1} \times \mathbb{P}^1$ remains 1-generic after we apply the map

$$\varphi: G^* \oplus \Lambda^1 G^* \to V$$

from the proof of Theorem 3.4.

If $\varphi(M)$ is not 1-generic, we can choose bases of $G^*$ and $\mathbb{C} \oplus \mathbb{C}$ such that $\varphi(M)$ has the form

$$M = \begin{pmatrix} l_1 & \ldots & l_i & l_{i+1} & \ldots & l_g \\ a_1 & \ldots & a_i & 0 & \ldots & 0 \end{pmatrix}$$

with $l_1, \ldots, l_{p+1}$ a basis of $G^*$ and $a_1, \ldots, a_i$ linearly independent. Since the syzygy ideal $I_f$ cannot be empty by Lemma 2.4, it has to be at least 1. In this situation $I_f$ contains the $2 \times 2$ minor

$$\det \begin{pmatrix} l_1 & l_g \\ a_1 & 0 \end{pmatrix} = l_g \cdot a_1$$

which implies that $X$ must be reducible or degenerate. This contradicts our assumptions.

**Definition 5.3.** Let $X \subset \mathbb{P}^n$ be a projective scheme, whose ideal is cut out by quadrics. A $p$-th linear syzygy of $X$ is called scrollar, if it has rank $p+2$.

**Example 5.4.** Let $C \subset \mathbb{P}^{g-1}$ be a non hyperelliptic canonical curve of genus $g$ and $|D|$ a pencil of Clifford index $\text{cliff}(D) = g - p - 3$. The $p$-th syzygy of $C$ constructed by the method of Green and Lazarsfeld in [GL84] is scrollar.

With the above geometric description of scrollar syzygy varieties one can prove the following well known converse of the Green-Lazarsfeld construction:

**Proposition 5.5.** Let $C \subset \mathbb{P}^{g-1}$ be a non hyperelliptic canonical curve of genus $g$ and $f \in F_p$ a $p$-th scrollar syzygy. Then there exists a linear system $|D|$ on $C$ with Clifford index $\text{cliff}(D) \leq g - p - 3$. 
Proof. Let $G^*$ be the $p + 2$ dimensional space of linear forms involved in $f$. Then the syzygy scheme $\text{Syz}(f)$ of $f$ is a scroll that contains $C$ and has the vanishing set $V(G^*)$ as a fiber. Set $D = C \cap V(G^*)$. Since $C \subset \mathbb{P}^{g-1}$ is non-degenerate, $D$ is a divisor on $C$. We consider the linear system $|D|$. Since $D$ is cut out by the ruling of $\text{Syz}(f)$ we have $h^0(D) \geq 2$. Also $h^0(K - D) \geq p + 2$ since the linear forms in $G^*$ cut out canonical divisors of $C$ that contain $D$. Riemann-Roch now gives:

$$\text{cliff } D := d - 2r = (h^0(D) - h^0(K - D) - 1 + g) - 2h^0(D) + 2 =$$

$$= g + 1 - h^0(D) - h^0(K - D) \geq g + 1 - 2 - (p + 2) = g - p - 3.$$

\[\square\]

Remark 5.6. For general $k$-gonal canonical curves $C$ Green’s conjecture is equivalent to the claim that every step of the linear strand $C$ contains at least one scrollar syzygy. This was recently shown by Voisin [Voi02], [Voi03].

More generally one can make the following conjecture

Conjecture 5.7 (Generic Geometric Syzygy Conjecture). Let $C \subset \mathbb{P}^{g-1}$ be a general canonical curve of genus $g$. Then for every $p$ the space of $p$-th linear syzygies of $C$ is spanned by scrollar syzygies.

This conjecture is known for for $p = 1$ when $g \neq 8$ and for $p = 2$ when $g = 8$ by [vB00] and [vB02].

6. Grassmannian Syzygies

Theorem 6.1. Let $G$ be a $g$ dimensional vector space, then

$$\text{Gensyz}_2(G) = \mathbb{G}(\mathbb{C} \oplus G^*, 2) \cup \mathbb{P}(\bigwedge^2 G^*) \subset \mathbb{P}(G^* \oplus \bigwedge^2 G^*),$$

where $\mathbb{G}(\mathbb{C} \oplus G^*, 2)$ is the Grassmannian of two dimensional quotient spaces of $\mathbb{C} \oplus G^*$. Moreover the second generic syzygy ideal $I$ of $G$ is reduced and saturated.

Proof. Observe that $\bigwedge^2 (\mathbb{C} \oplus G^*) = G^* \oplus \bigwedge^2 G^*$. We can therefore consider the Plücker embedding

$$\mathbb{G} := \mathbb{G}(\mathbb{C} \oplus G^*, 2) \subset \mathbb{P}(G^* \oplus \bigwedge^2 G^*)$$

and the ideal of the Grassmannian $\mathbb{G}$ which is generated by $4 \times 4$-pfaffians of a skew symmetric matrix. More precisely:

$$I_\mathbb{G} = (\bigwedge^2 (\mathbb{C} \oplus G^*)) = (\bigwedge^4 G^* \oplus \bigwedge^4 G^*) \subset S^2(G^* \oplus \bigwedge^2 G^*).$$

On the other hand $\mathbb{P}(\bigwedge^2 G^*) \cong \mathbb{P}(\mathbb{G})^{-1} =: \mathbb{P}$ is cut out by the linear forms in $G^*$, so $I_\mathbb{P} = (G^*)$. To prove the theorem we calculate the intersection of
these two irreducible ideals:

$$I_P \cap I_G = (G^*) \cap \left( \bigwedge^3 G^* \oplus \bigwedge^4 G^* \right) \cap \left( \bigwedge^3 G^* \right) = (G^*) \cap \left( \bigwedge^3 G^* \right) + (G^*) \cap \left( \bigwedge^4 G^* \right).$$

Now the quadrics in the ideal $G^*$ are given by the image of

$$G^* \otimes \left( G^* \bigwedge^2 G^* \right) \rightarrow S^2 \left( G^* \bigwedge^2 G^* \right),$$

i.e

$$(I_P)_2 = S^2 G^* \otimes G^* \bigwedge^2 G^* = \left( \bigwedge^3 G^* \right) \bigwedge^2 G^*.$$ 

This shows that $\left( \bigwedge^3 G^* \right)$ is contained in $G^*$. For the second intersection of ideals notice that $\bigwedge^4 G^*$ is contained in $S^2(\bigwedge^2 G^*)$. So the generators of $\left( \bigwedge^4 G^* \right)$ and $G^*$ involve different sets of variables and the intersection of the two ideals is the same as their product:

$$(G^*) \cap \left( \bigwedge^4 G^* \right) = (G^*) \cdot \left( \bigwedge^4 G^* \right) = (G^*) \otimes \left( \bigwedge^4 G^* \right) = \left( \bigwedge^5 G^* \oplus \bigwedge^{4,1} G^* \right) \subseteq G^* \otimes S^2 \left( \bigwedge^2 G^* \right) \subset S^3 \left( G^* \bigwedge^2 G^* \right).$$

On the other hand the cubics of $\left( \bigwedge^3 G^* \right)$ contain

$$\bigwedge^3 G^* \otimes G^* \bigwedge^2 G^* = \bigwedge^5 G^* \oplus \bigwedge^{4,1} G^* \subset G^* \otimes S^2 \left( \bigwedge^2 G^* \right).$$

Since these representations occur only once in $G^* \otimes S^2(\bigwedge^2 G^*)$ they must be the ones that generate the product of ideals above. In total we have shown

$$I_P \cap I_G = \left( \bigwedge^3 G^* \right)$$

which is the second generic syzygy ideal of $G$. \hfill \Box

**Definition 6.2.** Let $X \subset \mathbb{P}^n$ be a projective scheme, whose ideal is cut out by quadrics. A $p$-th linear syzygy of $X$ is called grassmannian, if it has rank $p + 3$.

**Example 6.3.** Let $X$ be a $K3$-surface of sectional genus $g$ in $\mathbb{P}^g$ with Picard group generated by a general hyperplane section $H$. Then $X$ has grassmannian $p$-th syzygies for $p \leq \frac{g-4}{2}$.

**Proof.** $X$ is cut out by quadrics. Since $X$ is irreducible and non-degenerate, $X$ has no reducible syzygies and does not lie on quadrics of rank 2 or 1. $X$ can also not lie on a quadrics of rank 4 or 3, since in this case the rulings of the quadrics would cut out divisors of degree smaller than $H$ on $X$. Hence, because scrolls are cut out by $2 \times 2$ minors of rank at most 4, $X$ can have no scrollar syzygies.

Now intersect $X$ with a general hyperplane $H$. Then $X \cap H = C \subset \mathbb{P}^{g-1}$ is a canonical curve whose minimal free resolution is the restriction of the minimal free resolution of $X$ to $H$. By the construction of Green and Lazarsfeld
C has scrollar $p$-th syzygies for $p \leq \frac{q-4}{2}$. The rank of a syzygy $f$ can fall by at most one when restricting to a general hyperplane (i.e. when the linear form defining $H$ is involved in $f$). Since $X$ has no scrollar syzygies, the scrollar syzygies of $C$ must come from grassmannian syzygies of $X$. □

We now describe some geometric consequences of grassmannian syzygies. For this let $Q$ be the universal rank 2 quotient bundle on the Grassmannian $G = G(\mathbb{C} \oplus G^*, 2)$. The global sections of $Q$ are given by $H^0(G, Q) = \mathbb{C} \oplus G^*$.

**Lemma 6.4.** Let $s \in H^0(G, Q)$ be a global section and $I_s$ the ideal of its vanishing locus on $G$. Then $I_s$ is generated by hyperplane sections of $G$, more precisely

$$I_s = (s \wedge H^0(G, Q)).$$

**Proof.** Consider the Koszul complex associated to $s$:

$$0 \to \mathcal{O}_G \to Q \to I_s \otimes H^2 \to 0$$

Taking cohomology shows $(s \wedge H^0(G, Q)) \subset I_s$. Since $Q$ is globally generated, the converse also follows. □

**Remark 6.5.** Observe that for a section $s \in \mathbb{C} \subset \mathbb{C} \oplus G^* = H^0(G, Q)$ we have $I_G + I_P = I_s$. In other words a grassmannian syzygy $f$ defines up to a constant a section of $Q$.

**Lemma 6.6.** Let $X \subset \mathbb{P}(V)$ be a projective variety cut out by quadrics, $f$ a $p$-th grassmannian syzygy of $X$, $G$ the space of $(p - 1)$-st syzygies involved in $f$, and $\varphi : G^* \oplus \bigwedge^2 G^* \to V$ the induced map. Then the natural map

$$H^0(G, Q) \to H^0(G \cap \mathbb{P}(\text{Im } \varphi), Q|_{G \cap \mathbb{P}(\text{Im } \varphi)})$$

is injective.

**Proof.** By construction $\text{Im } \varphi$ contains $G^*$ so the non-zero elements of $G^*$ are not contained in $I_{\mathbb{P}(\text{Im } \varphi)}$. On the other hand the vanishing ideal

$$I_s = (s \wedge (\mathbb{C} \oplus G^*))$$

contains the whole space $\mathbb{C} \wedge G^* = G^*$ if $s \in \mathbb{C}$, or a non-zero element of $G^* \wedge \mathbb{C} = G^*$ if $s \in G^*$. So $I_s$ can never be contained in $I_{\mathbb{P}(\text{Im } \varphi)}$ and

$$H^0(Q \otimes I_{Gr \cap \mathbb{P}(\text{Im } \varphi)}/G) = 0.$$ The proposition then follows from the exact sequence

$$0 \to Q \otimes I_{Gr \cap \mathbb{P}(\text{Im } \varphi)}/G \to Q \to Q|_{G \cap \mathbb{P}(\text{Im } \varphi)}/G \to 0.$$ □

**Theorem 6.7.** Let $C \subset \mathbb{P}^n$ be a smooth, irreducible non-degenerate curve. If $C$ is cut out by quadrics and has a $p$-th grassmannian syzygy $f$, then there exists a rank 2 vector bundle $E$ on $C$ with $\det E = \mathcal{O}_C(1)$ and $h^0(E) \geq p + 4$.

**Proof.** Let $\text{Syz}(f)$ be the syzygy scheme of $f$. By Theorems 3.4 and 6.7 $\text{Syz}(f)$ is a cone over a linear section of $G(p + 4, 2) \cup \mathbb{P}^{p+3}$. Now $\text{Syz}(f)$ contains $C$ and $C$ is irreducible and non-degenerate, so $C$ must be contained in a cone $Y$ over a linear section of $G$. The universal quotient bundle $Q$ on
\( G \) restricts to \( G \cap \mathbb{P}(\text{Im} \, \phi) \) and pulls back to a rank 2 vector bundle \( Q_{Y^o} \) on \( Y^o = Y \setminus V(\text{Im} \, \phi) \). If \( C \) does not intersect the vertex \( V(\text{Im} \, \phi) \) of \( Y \) the restriction of \( Q_{Y^o} \) to \( C \) is a vector bundle \( \mathcal{E} \).

If \( C \) intersects the vertex of \( Y \) in a divisor, we consider the blowup \( \tilde{Y} \) of \( Y \) in the vertex. \( Q \) then pulls back to a rank 2 vector bundle \( Q_{\tilde{Y}} \) on \( \tilde{Y} \). Since \( C \) is smooth the strict transform \( \tilde{C} \) of \( C \) is isomorphic to \( C \) and \( Q_{\tilde{Y}} \) restricts to a rank 2 vector bundle \( \mathcal{E} \) on \( \tilde{C} \cong C \).

Finally \( C \) can not be contained in the vertex of \( Y \) since \( C \) is non-degenerate.

By Lemma 6.6 we have \( h^0(Q|_{C \cap \mathbb{P}(\text{Im} \, \phi)}) \geq p + 4 \). These sections extend to \( Y^o \). By Lemma 6.4 the zero loci of sections of \( Q \) are cut out by linear forms and their closures contain the vertex of \( Y \). Since \( X \) is non-degenerate it can not lie in one of these zero loci, so all sections of \( Q \) descend to sections of \( \mathcal{E} \). \( \square \)

**Example 6.8.** Our method can is some cases also be used to obtain vector bundles on varieties of higher dimension. Let for example \( X \subset \mathbb{P}^g \) be a \( K3 \) surface of even sectional genus \( g = 2k \) whose Picard group is generated by a general hyperplane section. Then \( X \) has a grassmannian \((k - 2)\)-nd syzygy by the argument of Example 6.3. One can show that in this case the map

\[ \varphi: G^* \oplus S^2 G^* \to V \]

is surjective. Therfore \( \text{Syz}(f) \) is not a cone, and \( Q \) restricts to a rank 2 vector bundle \( \mathcal{E} \) on \( X \) with \( \det \mathcal{E} = \mathcal{O}_X(1) \) and \( h^0(\mathcal{E}) \geq k + 2 \). This is the Mukai-Lazarsfeld bundle used by Voisin in her proof of Green’s conjecture \[Vo02\].

This example leads us to ask

**Question 6.9.** Let \( X \subset \mathbb{P}^n \) be a surface cut out by quadrics whose Picard group is generated by a general hyperplane section. Does every step of the linear strand of \( X \) contain a grassmannian syzygy?

**Remark 6.10.** Voisin’s Theorem about the syzygies of \( K3 \) surfaces in \[Vo02\] prove that the answer to this question is “yes” in the case of \( K3 \) surfaces \( X \subset \mathbb{P}^g \) with sectional genus \( g = 2k \).

Even more generally we ask

**Question 6.11.** Let \( X \subset \mathbb{P}^n \) be a surface cut out by quadrics whose Picard group is generated by a general hyperplane section. Is the space of \( p \)-th linear syzygies of \( X \) spanned by grassmannian syzygies?

**Remark 6.12.** The answer to this question is ”yes” for general \( K3 \) surfaces \( X \subset \mathbb{P}^g \) with sectional genus \( g \leq 8 \) by the methods of \[VdB02\].

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