A TABLEAU FORMULA FOR ETA POLYNOMIALS

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ABSTRACT. We use raising operators to prove a tableau formula for the eta polynomials of \[BKT3\] and the Stanley symmetric functions which correspond to Grassmannian elements of the Weyl group \( \tilde{W}_n \) of type \( D_n \). We define the skew elements of \( \tilde{W}_n \) and exhibit a bijection between the set of reduced words for any skew \( w \in \tilde{W}_n \) and a set of certain standard typed tableaux on a skew shape \( \lambda/\mu \) associated to \( w \).

0. Introduction

Let \( k \) be any positive integer and \( OG = OG(n-k, 2n) \) be the even orthogonal Grassmannian which parametrizes isotropic subspaces of dimension \( n-k \) in a complex vector space of dimension \( 2n \), equipped with a nondegenerate symmetric bilinear form. Following \[BKT1\], the Schubert classes \( \sigma_\lambda \) in \( H^*(OG, \mathbb{Z}) \) are indexed by typed \( k \)-strict partitions \( \lambda \). Recall that a \( k \)-strict partition is an integer partition \( \lambda = (\lambda_1, \ldots, \lambda_\ell) \) such that all parts \( \lambda_i \) greater than \( k \) are distinct. A typed \( k \)-strict partition is a pair consisting of a \( k \)-strict partition \( \lambda \) together with an integer \( \text{type}(\lambda) \in \{0, 1, 2\} \) which is positive if and only if \( \lambda_i = k \) for some index \( i \).

In \[BKT3\], we discovered a remarkable connection between the cohomology of even (type D) and odd (type B) orthogonal Grassmannians, which allows a uniform approach to the Schubert calculus on both varieties. We used this together with our earlier work \[BKT2\] to obtain a Giambelli formula for the Schubert class \( \sigma_\lambda \), which expresses it as a polynomial in certain special Schubert classes. Moreover, we employed Young’s raising operators to define and study the eta polynomials \( H_\lambda(x; y) \) – a family of polynomials which multiply like the Schubert classes on \( OG(n-k, 2n) \) when \( n \) is sufficiently large. We proved that the polynomial \( H_\lambda(x; y) \) is equal to the Billey-Haiman Schubert polynomial \( D_w(\lambda; x; y) \) indexed by the corresponding \( k \)-Grassmannian element \( w_\lambda \) in the Weyl group \( \tilde{W}_n \) for the root system of type \( D_n \).

The first goal of the present paper is to combine the above theory with the raising operator approach to tableau formulas developed in \[T\]. Our main theorem (Theorem 3) is a formula for the eta polynomial \( H_\lambda(x; y) \), which writes it as a sum of monomials \( (xy)^U \) over certain fillings \( U \) of the Young diagram of \( \lambda \) called typed \( k' \)-bitableaux. The analysis turns out to be rather subtle, since a direct combination of the results of \[BKT3\] and \[T\] does not lead to positive formulas.

We call an element \( w \) of \( \tilde{W}_n \) skew if there exists a \( w' \in \tilde{W}_n \) and a \( k \)-Grassmannian element \( w_\lambda \) such that \( ww' = w_\lambda \) and \( \ell(w) + \ell(w') = |\lambda| \). The skew elements of the symmetric group are precisely the 321-avoiding permutations or fully commutative elements, which were introduced in \[BJS\] and explored further in \[S1, S2\].
other classical Lie types, although every fully commutative element is skew, the converse is false. The skew elements of the the hyperoctahedral group were studied in [11], and we extend this theory here to $\tilde{W}_n$.

Let $\lambda$ and $\mu$ typed $k$-strict partitions such that the diagram of $\mu$ is contained in the diagram of $\lambda$. Our approach leads naturally to the definition of certain symmetric functions $E_{\lambda/\mu}(x)$, given as a sum of monomials $x^T$ corresponding to typed $k'$-tableaux $T$ on the skew shape $\lambda/\mu$. We find that the function $E_{\lambda/\mu}(x)$, when non-zero, is equal to a type D Stanley symmetric function $E_w(x)$ indexed by a skew element $w$ in $\tilde{W}_n$. As in [BJS] and [11], there is an explicit bijection between the standard typed $k'$-tableaux on the skew shape $\lambda/\mu$ and the reduced words for a corresponding skew element $w \in \tilde{W}_n$.

This paper is organized as follows. Section 1 reviews the Giambelli and Pieri formulas which hold in the Chern subring of the stable cohomology ring $\text{C}^{(k)}$ of $\text{OG}(n-k, 2n)$ as $n \to \infty$. We also establish the mirror identity in $\text{C}^{(k)}$, a key technical tool which provides a bridge between the Pieri rule and our tableau formulas. Section 2 introduces the eta polynomials and proves various reduction formulas which are then combined to obtain our main tableau formula for $H_\lambda(x; y)$. Finally, in Section 3 we relate this theory to the type D Schubert polynomials and Stanley symmetric functions of [BH] [L], and study the skew elements of $\tilde{W}_n$.

1. Preliminary results

1.1. Raising operators. An integer sequence is a sequence of integers $\alpha = \{\alpha_i\}_{i \geq 1}$, only finitely many of which are non-zero. The largest integer $\ell \geq 0$ such that $\alpha_\ell \neq 0$ is called the length of $\alpha$, denoted $\ell(\alpha)$; we will identify an integer sequence of length $\ell$ with the vector consisting of its first $\ell$ terms. We let $|\alpha| = \sum \alpha_i$ and $\#\alpha$ equal the number of non-zero parts $\alpha_i$ of $\alpha$. Write $\alpha \geq \beta$ if $\alpha_i \geq \beta_i$ for each $i$. An integer sequence $\alpha$ is a composition if $\alpha_i \geq 0$ for all $i$ and a partition if $\alpha_i \geq \alpha_i + 1 \geq 0$ for all $i$. As is customary, we represent a partition $\lambda$ by its Young diagram of boxes, which has $\lambda_i$ boxes in row $i$ for each $i \geq 1$. We write $\mu \subset \lambda$ instead of $\mu \leq \lambda$ for the containment relation between two Young diagrams; in this case the set-theoretic difference $\lambda \setminus \mu$ is called a skew diagram and is denoted by $\lambda/\mu$.

Given any integer sequence $\alpha = (\alpha_1, \alpha_2, \ldots)$ and $i < j$, we define

$$R_{ij}(\alpha) = (\alpha_1, \ldots, \alpha_i + 1, \ldots, \alpha_j - 1, \ldots);$$

a raising operator $R$ is any monomial in these $R_{ij}$’s. Given any formal power series $\sum_{t \geq 0} c_t t^t$ in the variable $t$ and an integer sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$, we write $c_\alpha = c_{\alpha_1} c_{\alpha_2} \cdots c_{\alpha_\ell}$ and set $R c_\alpha = c_{R\alpha}$ for any raising operator $R$. We will always work with power series with constant term 1, so that $c_0 = 1$ and $c_t = 0$ for $i < 0$.

1.2. The Giambelli formula. Fix an integer $k > 0$. We consider an infinite family $w_1, w_2, \ldots$ of commuting variables, with $w_i$ of degree $i$ for all $i$, and set $w_0 = 1$, $w_r = 0$ for $r < 0$, and $w_\alpha = \prod_i w_{\alpha_i}$. Let $I^{(k)} \subset \mathbb{Z}[w_1, w_2, \ldots]$ be the ideal generated by the relations

$$\frac{1 - R_{12}}{1 + R_{12}} w_{(r,r)} = w_r^2 + 2 \sum_{i=1}^r (-1)^i w_{r+i} w_{r-i} = 0 \quad \text{for} \quad r > k.$$  

Define the graded ring $C^{(k)} = \mathbb{Z}[w_1, w_2, \ldots]/I^{(k)}$. 
Let $\Delta^\circ = \{(i, j) \in \mathbb{N} \times \mathbb{N} \mid 1 \leq i < j\}$, equipped with the partial order $\leq$ defined by $(i', j') \leq (i, j)$ if and only if $i' \leq i$ and $j' \leq j$. A finite subset $D$ of $\Delta^\circ$ is a valid set of pairs if it is an order ideal, i.e., $(i, j) \in D$ implies $(i', j') \in D$ for all $(i', j') \in \Delta^\circ$ with $(i', j') \leq (i, j)$. For any valid set of pairs $D$, we define the raising operator

$$R^D = \prod_{i<j} (1 - R_{ij}) \prod_{i<j; (i,j) \in D} (1 + R_{ij})^{-1}.\label{1.3}$$

A partition $\lambda$ is $k$-strict if all its parts greater than $k$ are distinct. The $k$-length of $\lambda$, denoted $\ell_k(\lambda)$, is the cardinality of the set $\{i \mid \lambda_i > k\}$. We say that $\lambda$ has positive type if $\lambda_i = k$ for some index $i$. The monomials $w_\lambda$ for $\lambda$ a $k$-strict partition form a $\mathbb{Z}$-basis of $C^{(k)}$.

Given a $k$-strict partition $\lambda$, we define a valid set of pairs $C(\lambda)$ by

$$(2) \quad C(\lambda) = \{(i, j) \in \Delta^\circ \mid \lambda_i + \lambda_j \geq 2k + j - i \text{ and } j \leq \ell(\lambda)\}$$

and set $R^\lambda := R^{C(\lambda)}$. Furthermore, define $W_\lambda \in C^{(k)}$ by the Giambelli formula

$$W_\lambda := R^\lambda w_\lambda.\label{1.4}$$

Since

$$W_\lambda = w_\lambda + \sum_{\mu > \lambda} b_{\lambda\mu} w_\mu$$

with the sum over $k$-strict partitions $\mu$ which strictly dominate $\lambda$, it follows that the $W_\lambda$ as $\lambda$ runs over $k$-strict partitions form another $\mathbb{Z}$-basis of $C^{(k)}$. More generally, given any valid set of pairs $D$ and an integer sequence $\alpha$, we denote $R^D w_\alpha$ by $W^{D, \alpha}$.\label{1.5}

**Example 1.** In the ring $C^{(2)}$ we have

$$W_{322} = \frac{1 - R_{12}}{1 + R_{12}} (1 - R_{13})(1 - R_{23}) w_{322} = (1 - 2R_{12} + 2R_{12}^2 - \cdots)(1 - R_{13})(1 - R_{23}) w_{322} = (1 - 2R_{12} + 2R_{12}^2 - 2R_{12}^3)(1 - R_{13} - R_{23} + R_{13}R_{23}) w_{322} = w_{322} - w_{321} - 2w_7 + 2w_{61} - w_{331} + w_{43} = w_3 w_2^2 - w_4 w_2 w_1 - 2w_7 + 2w_6 w_1 - w_3^2 w_1 + w_4 w_3.\label{1.6}$$

### 1.3. Cohomology of even orthogonal Grassmannians

Let the vector space $V = \mathbb{C}^{2n+2}$ be equipped with a nondegenerate symmetric bilinear form. A subspace $\Sigma$ of $V$ is called isotropic if the form vanishes when restricted to $\Sigma$. The dimensions of such isotropic subspaces $\Sigma$ range from 0 to $n+1$.

Choose $k$ with $0 < k \leq n$ and let $OG = OG(n+1-k, 2n+2)$ denote the Grassmannian parametrizing isotropic subspaces of $V$ of dimension $n+1-k$. Let $Q$ denote the universal quotient vector bundle of rank $n+1+k$ over $OG$, and $H^*(OG, \mathbb{Z})$ denote the subring of the cohomology ring $H^*(OG, \mathbb{Z})$ which is generated by the Chern classes of $Q$. There is a ring epimorphism $\psi : C^{(k)} \to H^*(OG, \mathbb{Z})$ which maps the generators $w_\mu$ to the Chern classes $c_\mu(Q)$ for each $\mu \geq 1$; in particular we have $\psi(w_p) = 0$ if $p > n+k$.

Let $R(n, k)$ denote a rectangle of size $(n+1-k) \times (n+k)$. It follows from [BKT3, Theorem 1] that for any $k$-strict partition $\lambda$, we have

$$\psi(2^{-\ell_k(\lambda)} W_\lambda) = [Y_\lambda].\label{1.7}$$
if the diagram of \( \lambda \) fits inside \( R(n, k) \), and \( \psi(W_\lambda) = 0 \) otherwise. Here \( Y_\lambda \) is a certain Zariski closed subset of pure codimension \( |\lambda| \) in OG, which is a Schubert variety in OG, if \( \lambda \) does not have positive type, and a union of two Schubert varieties in OG, if \( \lambda \) has positive type. The cohomology classes \( [Y_\lambda] \) for \( \lambda \) a \( k \)-strict partition contained in \( R(n, k) \) form a \( \mathbb{Z} \)-basis for \( H^*(OG, \mathbb{Z})_1 \). More details and the proofs of these facts are provided in [BKT3].

1.4. The Pieri rule. We let \([r, c]\) denote the box in row \( r \) and column \( c \) of a Young diagram. Suppose that \( c \leq k < c' \). We say that the boxes \([r, c]\) and \([r', c']\) are \( k' \)-related if \( c + c' = 2k + 1 + r - r' \). In the diagram of Figure 1, the two grey boxes are \( k' \)-related.

Given two partitions \( \lambda \) and \( \mu \) with \( \lambda \subset \mu \), the skew Young diagram \( \mu/\lambda \) is called a horizontal strip (respectively, vertical strip) if it does not contain two boxes in the same column (respectively, row). For any two \( k \)-strict partitions \( \lambda \) and \( \mu \), let \( \alpha_i \) (respectively \( \beta_i \)) denote the number of boxes of \( \lambda \) (respectively \( \mu \)) in column \( i \), for \( 1 \leq i \leq k \). We have a relation \( \lambda \to \mu \) if \( \mu \) can be obtained by removing a vertical strip from the first \( k \) columns of \( \lambda \) and adding a horizontal strip to the result, so that for each \( i \) with \( 1 \leq i \leq k \),

1. If \( \beta_i = \alpha_i \), then the box \([\alpha_i, i]\) is \( k' \)-related to at most one box of \( \mu \setminus \lambda \); and

2. If \( \beta_i < \alpha_i \), then the boxes \([\beta_i, i], \ldots, [\alpha_i, i]\) must each be \( k' \)-related to exactly one box of \( \mu \setminus \lambda \), and these boxes of \( \mu \setminus \lambda \) must all lie in the same row.

If \( \lambda \to \mu \), we let \( \mathbb{A} \) be the set of boxes of \( \mu \setminus \lambda \) in columns \( k + 1 \) and higher which are not mentioned in (1) or (2). Define the connected components of \( \mathbb{A} \) by agreeing that two boxes in \( \mathbb{A} \) are connected if they share at least a vertex. Then define \( N(\lambda, \mu) \) to be the number of connected components of \( \mathbb{A} \), and set

\[
M(\lambda, \mu) = \ell_k(\lambda) - \ell_k(\mu) + \begin{cases} N(\lambda, \mu) + 1 & \text{if } \lambda \text{ has positive type and } \mu \text{ does not,} \\ N(\lambda, \mu) & \text{otherwise.} \end{cases}
\]

We deduce from [BKT3 Theorem 5] that the following Pieri rule holds: For any \( k \)-strict partition \( \lambda \) and integer \( p \geq 0 \),

\[
w_p \cdot W_\lambda = \sum_{\substack{|\mu| = |\lambda| + p \\
\lambda \to \mu}} 2^{M(\lambda, \mu)} W_\mu.
\]

For any \( d \geq 1 \) define the raising operator \( R_d^\lambda \) by

\[
R_d^\lambda = \prod_{1 \leq i < j \leq d} (1 - R_{ij}) \prod_{i < j : (i, j) \in C(\lambda)} (1 + R_{ij})^{-1}.
\]
We compute that
\[
wp \cdot W_\lambda = w_p \cdot R_\lambda^\ell w_\lambda = R_\lambda^\ell \cdot \prod_{i=1}^\ell (1 - R_i, \ell + 1)^{-1} w_{\lambda, p}
\]
\[
= R_\lambda^\ell \cdot \prod_{i=1}^\ell (1 + R_i, \ell + 1 + R_i^2, \ell + 1 + \cdots) w_{\lambda, p} = \sum_{\nu \in N(\lambda, p)} W^\nu_{C(\lambda)} ,
\]
where \( N = N(\lambda, p) \) is the set of all compositions \( \nu \geq \lambda \) such that \( |\nu| = |\lambda| + p \) and \( \nu_j = 0 \) for \( j > \ell + 1 \), where \( \ell \) denotes the length of \( \lambda \). Equation (4) is therefore equivalent to the identity
\[
(5) \sum_{\nu \in N(\lambda, p)} W^\nu_{C(\lambda)} = \sum_{\lambda \rightarrow \mu} 2^{M(\lambda, \mu)} W^\nu_{C(\mu)} ,
\]
which was proved in [BKT3].

1.5. The mirror identity. Let \( \lambda \) and \( \nu \) be \( k \)-strict partitions such that
\[
\nu_1 > \max(\lambda_1, \ell(\lambda) + 2k - 1)
\]
and \( p, m \geq 0 \). Then the Pieri rule (4) implies that the coefficient of \( W_\nu \) in the Pieri product \( w_p \cdot W_\lambda \) is equal to the coefficient of \( W_{(\nu_1 + m, \nu_2, \nu_3, \ldots)} \) in the product \( w_{p+m} \cdot W_\lambda \). We apply this to make the following important definition.

**Definition 1.** Let \( \lambda \) and \( \mu \) be \( k \)-strict partitions with \( \mu \subset \lambda \), and choose any \( p \geq \max(\lambda_1 + 1, \ell(\lambda) + 2k - 1) \). If \( |\lambda| = |\mu| + r \) and \( \lambda \rightarrow (p + r, \mu) \), then we write \( \mu \rightarrow_\lambda \) and say that \( \lambda/\mu \) is a \( k' \)-horizontal strip. We define \( m(\lambda/\mu) := M(\lambda, (p + r, \mu)) \); in other words, the numbers \( m(\lambda/\mu) \) are the exponents that appear in the Pieri product
\[
(6) w_p \cdot W_\lambda = \sum_{r, \mu} 2^{m(\lambda/\mu)} W_{(p + r, \mu)}
\]
with the sum over integers \( r \geq 0 \) and \( k \)-strict partitions \( \mu \subset \lambda \) with \( |\mu| = |\lambda| - r \).

Note that a \( k' \)-horizontal strip \( \lambda/\mu \) is a pair of partitions \( \lambda \) and \( \mu \) with \( \mu \rightarrow_\lambda \). As such it depends on \( \lambda \) and \( \mu \) and not only on the difference \( \lambda \setminus \mu \).

**Lemma 1.** Let \( \Psi = \prod_{j=2}^{\ell+1} \frac{1 - R_{1j}}{1 + R_{1j}} \), and suppose that we have an equation
\[
\sum_\nu a_\nu w_\nu = \sum_\nu b_\nu w_\nu
\]
in \( C^{(k)} \), where the sums are over all \( \nu = (\nu_1, \ldots, \nu_\ell) \), while \( a_\nu \) and \( b_\nu \) are integers only finitely many of which are non-zero. Then we have
\[
\sum_\nu a_\nu \Psi w_{(p, \nu)} = \sum_\nu b_\nu \Psi w_{(p, \nu)}
\]
in the ring \( C^{(k)} \), for any integer \( p \).

**Proof.** The proof is the same as [T, Proposition 2]. \( \square \)
Let $\lambda$ be any $k$-strict partition of length $\ell$. Consider the following version of (5):

$$(7) \quad \sum_{\alpha \geq 0} W_{\lambda+\alpha}^{C(\lambda)} = \sum_{\lambda \rightarrow \mu} 2^{M(\lambda,\mu)} W_{\mu}^{C(\mu)}$$

where the first sum is over all compositions $\alpha$ of length at most $\ell + 1$, and the second over $k$-strict partitions $\mu$ with $\lambda \rightarrow \mu$. Following [T, §4], the next result is the mirror identity of (7).

**Theorem 1.** For $\lambda$ any $k$-strict partition we have

$$(8) \quad \sum_{\alpha \geq 0} 2^{\#\alpha} W_{\lambda-\alpha}^{C(\lambda)} = \sum_{\mu \rightarrow \lambda} 2^{m(\lambda/\mu)} W_{\mu},$$

where the first sum is over all compositions $\alpha$.

**Proof.** Choose $p \geq |\lambda| + 2k$ and let $\ell = \ell(\lambda)$. Expanding the Giambelli formula with respect to the first row gives

$$w_p \cdot W_\lambda = R^C(p,\lambda) \prod_{j=2}^{\ell+1} \frac{1 + R_{ij}}{1 - R_{ij}} w_{(p,\lambda)} = R^C(p,\lambda) \prod_{j=2}^{\ell+1} (1 + 2R_{ij} + 2R_{ij}^2 + \cdots) w_{(p,\lambda)}$$

$$= \sum_{\alpha \geq 0} 2^{\#\alpha} W_{(p,\lambda)}^{C(p,\lambda)} W_{(p+|\alpha|,\lambda-\alpha)}.$$

Comparing this with (8), we deduce that

$$(9) \quad \sum_{\alpha \geq 0} 2^{\#\alpha} W_{(p+|\alpha|,\lambda-\alpha)}^{C(p,\lambda)} = \sum_{\mu \rightarrow \lambda} 2^{m(\lambda/\mu)} W_{(p+|\mu|,\mu)}.$$ 

We claim that for every integer $r \geq 0$,

$$(10) \quad \sum_{|\alpha| = r} 2^{\#\alpha} W_{(p+r,r,\lambda-\alpha)}^{C(p,\lambda)} = \sum_{|\mu| = r} 2^{m(\lambda/\mu)} W_{(p+r,\mu)}.$$ 

The proof is by induction on $r$. The base case $r = 0$ is clearly true. For the induction step, suppose that we have for some $r > 0$ that

$$(11) \quad \sum_{s \geq r} \sum_{|\alpha| = s} 2^{\#\alpha} W_{(p+s,\lambda-\alpha)}^{C(p,\lambda)} = \sum_{s \geq r} \sum_{|\mu| = s} 2^{m(\lambda/\mu)} W_{(p+s,\mu)}.$$ 

Expanding the Giambelli formula with respect to the first component, we obtain $w_{p+s} W_{\lambda-\alpha}$ as the leading term of $W_{(p+s,\lambda-\alpha)}$, while $w_p W_{\mu}$ is the leading term of $W_{(p+s,\mu)}$. Since the set of all products $w_d W_{\nu}$ for which $(d, \nu)$ is a $k$-strict partition is linearly independent in $C(k)$, we deduce from (11) that

$$(12) \quad \sum_{|\alpha| = r} 2^{\#\alpha} W_{\lambda-\alpha}^{C(\lambda)} = \sum_{|\mu| = r} 2^{m(\lambda/\mu)} W_{\mu}.$$ 

By applying Lemma 1 we see that (10) is true, and this completes the induction. This also finishes the proof of Theorem 1 since the argument shows that (12) holds for every integer $r \geq 0$. $\square$
1.6. We say that boxes \([r, c]\) and \([r', c']\) are \(k''\)-related if \(|c - k| + r = |c' - k| + r'\) (we think of \(k''\) as being equal to \(k - 1\)). For example, the two grey boxes in the diagram of Figure 2 are \(k''\)-related. We call box \([r, c]\) a left box if \(c \leq k\) and a right box if \(c > k\).

If \(\mu \subset \lambda\) are two \(k\)-strict partitions such that \(\lambda/\mu\) is a \(k''\)-horizontal strip, we define \(\lambda_0 = \mu_0 = \infty\) and agree that the diagrams of \(\lambda\) and \(\mu\) include all boxes \([0, c]\) in row zero. We let \(R\) (respectively \(A\)) denote the set of right boxes of \(\mu\) (including boxes in row zero) which are bottom boxes of \(\lambda\) in their column and are (respectively are not) \(k''\)-related to a left box of \(\lambda/\mu\). Let \(N(A)\) denote the number of connected components of \(A\). Moreover, define

\[
\hat{n}(\lambda/\mu) = \ell_k(\mu) - \ell_k(\lambda) + m(\lambda/\mu).
\]

**Lemma 2.** A pair \(\mu \subset \lambda\) of \(k\)-strict partitions forms a \(k''\)-horizontal strip \(\lambda/\mu\) if and only if (i) \(\lambda/\mu\) is contained in the rim of \(\lambda\), and the right boxes of \(\lambda/\mu\) form a horizontal strip; (ii) no two boxes in \(R\) are \(k''\)-related; and (iii) if two boxes of \(\lambda/\mu\) lie in the same column, then they are \(k''\)-related to exactly two boxes of \(R\), which both lie in the same row. We have

\[
\hat{n}(\lambda/\mu) = \begin{cases} 
N(A) & \text{if } \lambda \text{ has positive type and } \mu \text{ does not}, \\
N(A) - 1 & \text{otherwise}.
\end{cases}
\]

**Proof.** We have \(\mu \rightsquigarrow \lambda\) if and only if \(\lambda \rightarrow (p + r, \mu)\) for any \(p \geq |\lambda| + 2k\), where \(r = |\lambda - \mu|\). Observe that a box of \((p + r, \mu) \setminus \lambda\) corresponds to a box of \(\mu\) which is a bottom box of \(\lambda\) in its column. The rest of the proof is a straightforward translation of the definitions in 1.4. \(\square\)

2. Reduction formulas and tableaux

2.1. Let \(x = (x_1, x_2, \ldots)\) and set \(y = (y_1, \ldots, y_k)\) for a fixed integer \(k \geq 1\). Consider the generating series

\[
\prod_{i=1}^{\infty} \frac{1 + x_i t}{1 - x_i t} = \sum_{r=0}^{\infty} q_r(x) t^r \quad \text{and} \quad \prod_{j=1}^{k} (1 + y_j t) = \sum_{r=0}^{\infty} e_r(y) t^r
\]

for the Schur \(Q\)-functions \(q_r(x)\) and elementary symmetric functions \(e_r(y)\). Given any strict partition \(\lambda\) of length \(\ell(\lambda)\), the Schur \(Q\)-function \(Q_\lambda(x)\) is defined by the raising operator expression

\[
Q_\lambda = \prod_{i < j} \frac{1 - R_{ij}}{1 + R_{ij}} q_\lambda
\]
Proof. We compute that $P_{\lambda}(x) = 2^{-\ell(\lambda)} Q_{\lambda}(x)$. In particular $P_0 = 1$ and for each integer $r \geq 1$, we have $P_r = q_r/2$. For each integer $r$, define $\vartheta_r = \vartheta_r(x; y)$ by

$$\vartheta_r = \sum_{i \geq 0} q_{r-i}(x) e_i(y).$$

We let $\Gamma^{(k)} = \mathbb{Z}[\vartheta_1, \vartheta_2, \vartheta_3, \ldots]$ be the ring of theta polynomials. There is a ring isomorphism $\mathcal{C}^{(k)} \to \Gamma^{(k)}$ sending $w_r$ to $\vartheta_r$ for all $r$.

Let $\lambda$ be a $k$-strict partition, and consider the raising operator

$$\tilde{R}^\lambda = \prod_{i<j} (1 - R_{ij}) \prod_{i<j : \lambda_i + \lambda_j > 2k+j-i} (1 + R_{ij})^{-1}.$$

The theta polynomial $\Theta_{\lambda}(x; y)$ is defined by the equation $\Theta_{\lambda} = \tilde{R}^\lambda \vartheta_{\lambda}$. The polynomials $\Theta_{\lambda}$ for all $k$-strict partitions $\lambda$ form a $\mathbb{Z}$-basis of $\Gamma^{(k)}$.

Set

$$\eta_r(x; y) = \begin{cases} e_r(y) + 2 \sum_{i=0}^{r-1} P_{r-i}(x) e_i(y) & \text{if } r < k, \\ \sum_{i=0}^{r-1} P_{r-i}(x) e_i(y) & \text{if } r \geq k \end{cases}$$

and $\eta_k(x; y) = \sum_{i=0}^{k-1} P_{k-i}(x) e_i(y)$. For any $r \geq 0$, we have

$$(14) \quad \vartheta_r = \begin{cases} \eta_r & \text{if } r < k, \\ \eta_k + \eta'_k & \text{if } r = k, \\ 2\eta_r & \text{if } r > k, \end{cases}$$

while $\eta_k - \eta'_k = e_k(y)$. Following [BKT3], we define the ring of eta polynomials

$$B^{(k)} = \mathbb{Z}[\eta_1, \ldots, \eta_k, \eta_{k+1}, \ldots].$$

2.2. For any $k$-strict partition $\lambda$, define $\tilde{\Theta}_{\lambda}(x; y)$ and $\tilde{H}_{\lambda}(x; y)$ by the equations

$$\tilde{\Theta}_{\lambda} = R^\lambda \vartheta_{\lambda} \quad \text{and} \quad \tilde{H}_{\lambda} = 2^{-\ell(\lambda)} \tilde{\Theta}_{\lambda}.$$

**Proposition 1.** For any $k$-strict partition $\lambda$, we have the reduction formula

$$(15) \quad \tilde{H}_\lambda(x; y) = \sum_{p=0}^{\infty} x_1^p \sum_{\mu \leftarrow \lambda - p} 2^{\tilde{\omega}(\mu/\nu)} \tilde{H}_\mu(\tilde{x}; y).$$

**Proof.** We compute that

$$\sum_{r=0}^{\infty} \vartheta_r(x; y)t^r = \frac{1 + x_1 t}{1 - x_1 t} \prod_{i=2}^{\infty} \frac{1 + x_i t}{1 - x_i t} \prod_{j=1}^{k} (1 + y_j t) = \sum_{i=0}^{\infty} x_1^i 2^{\#i} \sum_{s=0}^{s+i} \vartheta_s(\tilde{x}; y) t^{s+i}$$

and therefore, for any integer sequence $\lambda$, we have

$$(16) \quad \tilde{\Theta}_\lambda(x; y) = \sum_{\alpha \geq 0} x_1^{[\alpha]} 2^{\#\alpha} \tilde{\vartheta}_{\lambda - \alpha}(\tilde{x}; y)$$

summed over all compositions $\alpha$. If $R$ denotes any raising operator, then

$$R \vartheta_{\lambda}(x; y) = \vartheta_{R\lambda}(x; y) = \sum_{\alpha \geq 0} x_1^{[\alpha]} 2^{\#\alpha} \vartheta_{R\lambda - \alpha}(\tilde{x}; y) = \sum_{\alpha \geq 0} x_1^{[\alpha]} 2^{\#\alpha} R \vartheta_{\lambda - \alpha}(\tilde{x}; y).$$

Applying the raising operator $R^\lambda$ to both sides of (16), we therefore obtain

$$\tilde{\Theta}_\lambda(x; y) = \sum_{\alpha \geq 0} x_1^{[\alpha]} 2^{\#\alpha} \tilde{\Theta}_{\lambda - \alpha}^{(\lambda)}(\tilde{x}; y) = \sum_{p=0}^{\infty} x_1^p \sum_{\alpha \geq p} 2^{\#\alpha} \tilde{\Theta}_{\lambda - \alpha}^{(\lambda)}(\tilde{x}; y),$$
where \( \Theta_{\lambda}^{\ell(\lambda)} = R^\lambda \partial_{\lambda-\alpha} \) by definition. We now use the mirror identity \( [8] \) and the equation \( \Theta_\lambda = 2^\ell(\lambda) \hat{H}_\lambda \) to complete the proof. \( \square \)

For any \( k \)-strict partition \( \lambda \) of positive type, we define \( \hat{H}_\lambda(x; y) \) by the equation

\[
\hat{H}_\lambda = 2^{-\ell(\lambda)} e_k(y) \Theta_{\lambda-k}
\]

where \( \lambda - k \) means \( \lambda \) with one part equal to \( k \) removed. If \( \lambda \) does not have a part equal to \( k \), we agree that \( \hat{H}_\lambda = 0 \). The following analogue of Proposition \( [1] \) is valid for the polynomials \( \hat{H}(x; y) \).

**Proposition 2.** For any \( k \)-strict partition \( \lambda \) of positive type, we have the reduction formula

\[
\hat{H}_\lambda(x; y) = \sum_{p=0}^{\infty} x_1^p \sum_{\mu \rightarrow \lambda, \mu \rightarrow \lambda-\mu} 2^{\tilde{\eta}(\lambda/\mu)} \hat{H}_\mu(\tilde{x}; \tilde{y})
\]

where the sum is over \( k \)-strict partitions \( \mu \subset \lambda \) of positive type such that \( \lambda/\mu \) is a \( k' \)-horizontal strip.

**Proof.** This follows immediately from \( [17] \) and the reduction formula for the theta polynomial \( \Theta_{\lambda-k}(x; y) \). For the latter, see \([1\) Theorem 4]. \( \square \)

2.3. **Typed \( k \)-strict partitions and eta polynomials.** A typed \( k \)-strict partition \( \lambda \) consists of a \( k \)-strict partition \( (\lambda_1, \ldots, \lambda_\ell) \) together with an integer type(\( \lambda \)) \( \in \{0, 1, 2\} \), such that type(\( \lambda \)) \( > 0 \) if and only if \( \lambda_j = \hat{k} \) for some index \( j \). Suppose that \( \lambda \) is a typed \( k \)-strict partition and \( R \) is any finite monomial in the operators \( R_{ij} \) which appears in the expansion of the power series \( R^\lambda \) in \([8] \). If type(\( \lambda \)) \( = 0 \), then set \( R \star \partial_\lambda = \partial_{R\lambda} \). Suppose that type(\( \lambda \)) \( > 0 \), let \( m = \ell_k(\lambda) + 1 \) be the index such that \( \lambda_m = \hat{k} < \lambda_{m-1} \), and set \( \hat{\alpha} = (\alpha_1, \ldots, \alpha_{m-1}, \alpha_{m+1}, \ldots, \alpha_\ell) \) for any integer sequence \( \alpha \) of length \( \ell \). If \( R \) involves any factors \( R_{ij} \) with \( i = m \) or \( j = m \), then let \( R \star \partial_\lambda = \frac{1}{2} \partial_{R\lambda} \). If \( R \) has no such factors, then let

\[
R \star \partial_\lambda = \begin{cases} 
\eta_k \partial_{R\lambda} & \text{if type(\( \lambda \)) = 1,} \\
\eta_k \partial_{R\lambda} & \text{if type(\( \lambda \)) = 2.}
\end{cases}
\]

The *eta polynomial* \( H_\lambda = H_\lambda(x; y) \) is the element of \( B^{(k)} \) defined by the raising operator formula

\[
H_\lambda = 2^{-\ell(\lambda)} R^\lambda \star \partial_\lambda.
\]

The type of the polynomial \( H_\lambda \) is the same as the type of \( \lambda \). It was shown in \([13\) Thm. 4] that the \( H_\lambda \) for typed \( k \)-strict partitions \( \lambda \) form a \( \mathbb{Z} \)-basis of the ring \( B^{(k)} \).

Recall that \( \text{OG} = \text{OG}(n+1-k, 2n+2) \) denotes the Grassmannian parametrizing subspaces of \( \mathbb{C}^{2n+2} \) of dimension \( n+1-k \) which are isotropic with respect to an orthogonal bilinear form. There is a Schubert class \( \sigma_\lambda \in H^*(\text{OG}, \mathbb{Z}) \) for every typed \( k \)-strict partition \( \lambda \) whose diagram fits inside the rectangle \( R(n, k) \). Moreover, we have a ring epimorphism \( B^{(k)} \rightarrow H^*(\text{OG}, \mathbb{Z}) \) which maps \( H_\lambda \) to \( \sigma_\lambda \), whenever the diagram of \( \lambda \) fits in \( R(n, k) \), and to zero, otherwise.
For any typed $k$-strict partition $\lambda$, we define $\tilde{H}_\lambda$ and $\bar{H}_\lambda$ as above, by ignoring the type of $\lambda$. It follows that the definition (19) of $H_\lambda$ is equivalent to the formula

\[
H_\lambda = \begin{cases} 
\tilde{H}_\lambda & \text{if type}(\lambda) = 0, \\
(\tilde{H}_\lambda + \bar{H}_\lambda)/2 & \text{if type}(\lambda) = 1, \\
(\tilde{H}_\lambda - \bar{H}_\lambda)/2 & \text{if type}(\lambda) = 2.
\end{cases}
\] (20)

**Lemma 3.** Let $\lambda$ be a typed $k$-strict partition and $H_\lambda(0; y)$ be obtained from $H_\lambda(x; y)$ by substituting $x_i = 0$ for all $i \geq 1$. Then we have

\[
H_\lambda(0; y) = \begin{cases} 
0 & \text{if } \lambda_1 > k, \\
s_{\lambda'}(y) & \text{if } \lambda_1 = k \text{ and } \text{type}(\lambda) = 1, \\
0 & \text{if } \lambda_1 = k \text{ and } \text{type}(\lambda) = 2, \\
s_{\lambda'}(y) & \text{if } \lambda_1 < k
\end{cases}
\]

where $\lambda'$ denotes the partition conjugate to $\lambda$.

**Proof.** The raising operator definition of $\tilde{\Theta}_\lambda$ gives

\[
\tilde{\Theta}_\lambda(0; y) = R^\lambda e_\lambda(y)
\] (21)

where $e_\lambda = \prod e_{\lambda_i}(y)$ and $e_r(y)$ denotes the $r$-th elementary symmetric polynomial in $y$. Since $e_r(y) = 0$ for $r > k$, we deduce from (21) that $\tilde{H}_\lambda(0; y) = 0$ unless $\lambda_1 \leq k$. In the latter case we have $\ell_k(\lambda) = 0$, $C(\lambda) = \emptyset$, and

\[
\tilde{\Theta}_\lambda(0; y) = \Theta_\lambda(0; y) = \prod_{i<j} (1 - R_{ij}) e_{\lambda_i}(y) = s_{\lambda'}(y).
\] (22)

Suppose that $\lambda_1 = k$, so that $\text{type}(\lambda) > 0$. If $\text{type}(\lambda) = 1$, then (20) and (22) give

\[
H_\lambda(0; y) = \frac{1}{2} \left( \tilde{\Theta}_\lambda(0; y) + e_k(y) \Theta_{\lambda-k}(0; y) \right)
\]

\[
= \frac{1}{2} \left( s_{\lambda'}(y) + e_k(y) s_{(\lambda-k)'}(0; y) \right) = s_{\lambda'}(y),
\]

since $e_k(y) s_{(\lambda-k)'}(y) = s_{\lambda'}(y)$ by the classical type $A$ Pieri rule. We similarly compute that $H_\lambda(0; y) = 0$ if $\text{type}(\lambda) = 2$. Finally, if $\lambda_1 < k$ then $\text{type}(\lambda) = 0$ and $H_\lambda(0; y) = \tilde{\Theta}_\lambda(0; y) = s_{\lambda'}(y)$, using (22) again. \[\square\]

2.4. **Tableau formulas.** In this subsection we will obtain a tableau description of the eta polynomials $H_\lambda(x; y)$, where the tableaux in question are fillings of the Young diagram of $\lambda$. We first prove a reduction formula for the $x$ variables in $H_\lambda$.

**Definition 2.** If $\lambda$ and $\mu$ are typed $k$-strict partitions with $\mu \subseteq \lambda$, we write $\mu \rightarrow \lambda$ and say that $\lambda/\mu$ is a typed $k'$-horizontal strip if the underlying $k$-strict partitions are such that $\lambda/\mu$ is a $k'$-horizontal strip and in addition $\text{type}(\lambda) + \text{type}(\mu) \neq 3$. In this case we set $\eta(\lambda/\mu) = N(\lambda) - 1$, where the set $\Lambda$ is defined as in (1.6).

**Theorem 2.** For any typed $k$-strict partition $\lambda$, we have the reduction formula

\[
H_\lambda(x; y) = \sum_{p=0}^{\infty} x_1^p \sum_{\mu \rightarrow \lambda \atop \text{type}(\mu) = |\lambda| - p} 2^{n(\lambda/\mu)} H_\mu(\bar{x}; y)
\] (23)

where the inner sum is over typed $k$-strict partitions $\mu$ with $\mu \rightarrow \lambda$ and $|\mu| = |\lambda| - p$. 
Proof. Assume that type(\(\lambda\)) = 1; the proof in the other cases is similar. Using Definition 19 and equations (14) and (15) we obtain

\[
H_\lambda(x; y) = \sum_{p=0}^{\infty} x^p \sum_{|\mu|=|\lambda|-p} 2^{\tilde{n}(\lambda/\mu)} \left( \tilde{H}_\mu(\tilde{x}; y) + \tilde{H}_\mu(\tilde{x}; y) \right)/2
\]

where the inner sum is over all \(k\)-strict partitions \(\mu \subset \lambda\) with \(|\mu| = |\lambda| - p\) such that \(\lambda/\mu\) is a \(k'\)-horizontal strip. The result follows by combining equation (24) with (13) and (20).

Let \(P\) denote the ordered alphabet \(\{1' < 2' < \cdots < k' < 1, 1'' < 2, 2'' < \cdots\}\). The symbols \(1', \ldots, k'\) are said to be marked, while the rest are unmarked. Suppose that \(\lambda\) is any typed \(k\)-strict partition.

Definition 3. a) A typed \(k'\)-tableau \(T\) of shape \(\lambda/\mu\) is a sequence of typed \(k\)-strict partitions

\[
\mu = \lambda^0 \subset \lambda^1 \subset \cdots \subset \lambda^r = \lambda
\]

such that \(\lambda^i/\lambda^{i-1}\) is a typed \(k'\)-horizontal strip for \(1 \leq i \leq r\). We represent \(T\) by a filling of the boxes in \(\lambda/\mu\) with unmarked elements of \(P\) which is weakly increasing along each row and down each column, such that for each \(i\), the boxes in \(T\) with entry \(i\) or \(i''\) form the skew diagram \(\lambda^i/\lambda^{i-1}\), and we use \(i\) (resp. \(i''\)) if and only if type(\(\lambda^i\)) \(\neq 2\) (resp. type(\(\lambda^i\)) = 2), for every \(i \geq 1\). A standard typed \(k'\)-tableau on \(\lambda/\mu\) is a \(k'\)-tableau \(T\) of shape \(\lambda/\mu\) such that the entries 1, 2, \ldots, \(|\lambda - \mu|\), circled or not, each appear exactly once in \(T\). For any typed \(k'\)-tableau \(T\) we define

\[
n(T) = \sum_i n(\lambda^i/\lambda^{i-1}) \quad \text{and} \quad x^T = \prod_i x_i^{m_i}
\]

where \(m_i\) denotes the number of times that \(i\) or \(i''\) appears in \(T\).

b) A typed \(k'\)-bitableau \(U\) of shape \(\lambda\) is a filling of the boxes in \(\lambda\) with elements of \(P\) which is weakly increasing along each row and down each column, such that (i) the unmarked entries form a typed \(k'\)-tableau \(T\) of shape \(\lambda/\mu\) with type(\(\mu\)) \(\neq 2\), and (ii) the marked entries are a filling of \(\mu\) which is strictly increasing along each row. We define

\[
n(U) = n(T) \quad \text{and} \quad (xy)^U = x^T \prod_j y_j^{n_j}
\]

where \(n_j\) denotes the number of times that \(j'\) appears in \(U\).

Theorem 3. For any typed \(k\)-strict partition \(\lambda\), we have

\[
H_\lambda(x; y) = \sum_U 2^{n(U)} (xy)^U
\]

where the sum is over all typed \(k'\)-bitableaux \(U\) of shape \(\lambda\).

Proof. Let \(m\) be a positive integer, \(x^{(m)} = (x_1, \ldots, x_m)\), and let \(H_\lambda(x^{(m)}; y)\) be the result of substituting \(x_i = 0\) for \(i > m\) in \(H_\lambda(x; y)\). It follows from equation (23) that

\[
H_\lambda(x^{(m)}; y) = \sum_{p=0}^{\infty} x_m^p \sum_{|\mu|=\lambda} 2^{n(\lambda/\mu)} H_\mu(x^{(m-1)}; y).
\]
Iterating equation (28) \( m \) times produces
\[
H_\lambda(x^{(m)}; y) = \sum_{\mu, T} 2^{n(T)} x^T H_\mu(0; y)
\]
where the sum is over all typed \( k \)-strict partitions \( \mu \subset \lambda \) and \( k \)-tableau \( T \) of shape \( \lambda/\mu \) with no entries greater than \( m \). We deduce from Lemma 3 that \( H_\mu(0; y) = 0 \) unless \( \mu_1 \leq k \) and \( \text{type}(\mu) \neq 2 \), in which case \( H_\mu(0; y) = s_{\mu'}(y) \). The combinatorial definition of Schur \( S \)-functions \([6, 1(5.12)]\) states that
\[
s_{\mu'}(y) = \sum_S y^S
\]
summed over all semistandard Young tableaux \( S \) of shape \( \mu' \) with entries from 1 to \( k \). We conclude that
\[
H_\lambda(x^{(m)}; y) = \sum_U 2^{n(U)} (xy)^U
\]
summed over all typed \( k' \)-bitableaux \( U \) of shape \( \lambda \) with no entries greater than \( m \). The theorem follows by letting \( m \) tend to infinity.

Example 2. Let \( k = 1 \), \( \lambda = (3, 1) \) with \( \text{type}(\lambda) = 1 \), and consider the alphabet \( P_{1,2} = \{1', 1, 0 < 2, 2, 0\} \). There are twelve 1-bitableaux \( U \) of shape \( \lambda \) with entries in \( P_{1,2} \). The bitableaux
\[
\begin{array}{cccccc}
111 & 112 & 122 & 122 & 1'111 \\
2 & 2 & 1 & 2 & 1 \\
1'11 & 1'12 & 1'22 & 1'11 & 1'22 \\
2 & 2 & 2 & 1' & 1'
\end{array}
\]
satisfy \( n(U) = 0 \). We deduce from Theorem 3 that
\[
H_{3,1}(x_1, x_2; y_1) = (x_1^2 x_2 + 2x_1^2 x_2^2 + x_1 x_2^3) + (x_1^4 + 3x_1^3 x_2 + 3x_1 x_2^2 + x_2^3) y_1
\]
\[
+ (x_1^2 + 2x_1 x_2 + x_2^2) y_1^2
\]
\[
= P_{3,1}(x_1, x_2) + (P_3(x_1, x_2) + P_{2,1}(x_1, x_2)) y_1 + P_2(x_1, x_2) y_1^2.
\]
We similarly find that for \( \lambda = (3, 1) \) of type 2, there are six 1-bitableaux \( U \) of shape \( \lambda \) with entries in \( P_{1,2} \), namely
\[
\begin{array}{cccccc}
111 & 11 2^o & 1^o 2^o 2^o & 1^o 2^o 2^o & 1'11 & 1'1 2^o \\
2^o & 2^o & 1^o & 2^o & 2^o & 2^o
\end{array}
\]
all of which satisfy \( n(U) = 0 \). We deduce from Theorem 3 that
\[
H_{3,1}'(x_1, x_2; y_1) = (x_1^3 x_2 + 2x_1^2 x_2^2 + x_1 x_2^3) + (x_1^2 x_2 + x_1 x_2^2) y_1
\]
\[
= P_{3,1}(x_1, x_2) + P_{2,1}(x_1, x_2) y_1.
\]
Notice that \( H_{3,1} + H_{3,1}' = \frac{1}{2} \Theta_{3,1} \), and compare with [4 Example 7].

Theorem 3 motivates the following definition.

Definition 4. For \( \lambda \) and \( \mu \) any two typed \( k \)-strict partitions with \( \mu \subset \lambda \), let
\[
E_{\lambda/\mu}(x) = \sum_T 2^{n(T)} x^T
\]
where the sum is over all typed \(k'\)-tableaux \(T\) of shape \(\lambda/\mu\).

**Example 3.** (a) There is an involution \(j\) on the set of all typed \(k\)-strict partitions, which is the identity on partitions of type 0 and exchanges type 1 and type 2 partitions of the same shape. If \(T\) is a typed \(k'\)-tableau of shape \(\lambda/\mu\) then we let \(j(T)\) be the typed \(k'\)-tableau of shape \(j(\lambda)/j(\mu)\) obtained by applying \(j\) to each typed partition \(\lambda^i\) which appears in the sequence \(25\) determined by \(T\). Then \(j\) is an involution on the set of all typed \(k'\)-tableaux and it follows from Definition 4 that \(E_{\lambda/\mu}(x) = E_{j(\lambda)/j(\mu)}(x)\).

(b) Suppose that \(\mu_i \geq \min(k, \lambda_i)\) for all \(i\). Then Definition 4 becomes the tableau based definition of skew Schur \(P\)-functions, and therefore \(E_{\lambda/\mu}(x) = P_{\lambda/\mu}(x)\).

(c) Suppose that \(\lambda_1 < k\), so in particular \(C(\lambda) = \emptyset\). Then \(n(\lambda/\mu)\) is equal to the number of edge-connected components of \(\lambda/\mu\), for any partition \(\mu \subset \lambda\). It follows from Worley [W, §2.7] that

\[
\sum_T 2^{n(T)} x^T = S_{\lambda/\mu}(x) := \det(q_{\lambda_i - \mu_j + j - i}(x))_{i,j}.
\]

We therefore have \(E_{\lambda/\mu}(x) = S_{\lambda/\mu}(x)\) and

\[
H_{\lambda}(x; y) = \sum_{\mu \subset \lambda} S_{\lambda/\mu}(x) s_{\mu'}(y),
\]

in agreement with [BKT3] Prop. 5.4(a).

(d) Suppose that there is only one variable \(x\). Then we have

\[
E_{\lambda/\mu}(x) = \begin{cases} 
2^{n(\lambda/\mu)} x^{\lambda - \mu} & \text{if } \lambda/\mu \text{ is a } k'\text{-horizontal strip,} \\
0 & \text{otherwise.}
\end{cases}
\]

**Corollary 1.** (a) Let \(x = (x_1, x_2, \ldots)\) and \(x' = (x'_1, x'_2, \ldots)\) be two sets of variables, and let \(\lambda\) be any typed \(k\)-strict partition. Then we have

\[
H_{\lambda}(x, x'; y) = \sum_{\mu \subset \lambda} E_{\lambda/\mu}(x) H_{\mu}(x'; y),
\]

(30)

\[
H_{\lambda}(x; y) = \sum_{\mu \subset \lambda} E_{\lambda/\mu}(x) s_{\mu'}(y),
\]

(31)

and

\[
E_{\lambda}(x, x') = \sum_{\mu \subset \lambda} E_{\lambda/\mu}(x) E_{\mu}(x'),
\]

(32)

where the sums are over all typed \(k\)-strict partitions \(\mu \subset \lambda\).

(b) For any two typed \(k\)-strict partitions \(\lambda\) and \(\mu\) with \(\mu \subset \lambda\), we have

\[
E_{\lambda/\mu}(x, x') = \sum_{\nu} E_{\lambda/\nu}(x) E_{\nu/\mu}(x'),
\]

(33)

where the sum is over all typed \(k\)-strict partitions \(\nu\) with \(\mu \subset \nu \subset \lambda\).

**Proof.** The proof is the same as that for Corollaries 6 and 7 in [T].

It follows immediately from identity (30) that \(E_{\lambda/\mu}(x)\) is a symmetric function in the variables \(x\). These functions will be studied further in the next section.
3. Stanley symmetric functions and skew elements

3.1. Let $\tilde{W}_{n+1}$ be the Weyl group for the root system of type $D_{n+1}$, and set $\tilde{W}_\infty = \bigcup_{n} \tilde{W}_{n+1}$. The elements of $\tilde{W}_{n+1}$ may be represented as signed permutations of the set $\{1, \ldots, n+1\}$; we will denote a sign change by a bar over the corresponding entry. The group $\tilde{W}_{n+1}$ is generated by the simple transpositions $s_i = (i, i+1)$ for $1 \leq i \leq n$, and an element $\bar{s}_0$ which acts on the right by

$$ (u_1, u_2, \ldots, u_{n+1})\bar{s}_0 = (\bar{u}_2, \bar{u}_1, u_3, \ldots, u_{n+1}). $$

Every element $w \in \tilde{W}_\infty$ can be expressed as a product of $w = s_{a_1} s_{a_2} \cdots s_{a_r}$ of simple reflections $s_i$. If the length $r$ of such an expression is minimal then the sequence of indices $(a_1, \ldots, a_r)$ is called a reduced word for $w$, and $r$ is called the length of $w$, denoted $\ell(w)$. A factorization $w = uv$ in $\tilde{W}_\infty$ is reduced if $\ell(w) = \ell(u) + \ell(v)$.

Following [FS, FK1, FK2, L], we will use the nilCoxeter algebra $W_{n+1}$ of $\tilde{W}_{n+1}$ to define type D Stanley symmetric functions and Schubert polynomials. $W_{n+1}$ is the free associative algebra with unity generated by the elements $u_0, u_1, \ldots, u_n$ modulo the relations

$$
\begin{align*}
u_i^2 &= 0, \\
u_0u_1 &= u_1u_0, \\
u_0u_2u_0 &= u_2u_0u_2, \\
u_iu_{i+1}u_i &= u_{i+1}u_iu_{i+1}, \\
u_iu_j &= u_ju_i, \\
&j > i + 1, \text{ and } (i, j) \neq (0, 2).
\end{align*}
$$

For any $w \in \tilde{W}_{n+1}$, choose a reduced word $a_1 \cdots a_\ell$ for $w$ and define $u_w = u_{a_1} \cdots u_{a_\ell}$. Since the last four relations listed are the Coxeter relations for $D_{n+1}$, it is clear that $u_w$ is well defined, and that the $u_w$ for $w \in \tilde{W}_{n+1}$ form a free $\mathbb{Z}$-basis of $W_{n+1}$.

Let $t$ be an indeterminate and define

$$ A_i(t) = (1 + tu_n)(1 + tu_{n-1}) \cdots (1 + tu_i); $$
$$ D(t) = (1 + tu_n) \cdots (1 + tu_2)(1 + tu_1)(1 + tu_0)(1 + tu_2) \cdots (1 + tu_n). $$

According to [L] Lemma 4.24, for any commuting variables $s, t$ we have $D(s)D(t) = D(t)D(s)$. Set $x = (x_1, x_2, \ldots)$ and consider the product $D(x) := D(x_1)D(x_2) \cdots$. We deduce that the functions $E_w(x)$ in the formal power series expansion

$$ D(x) = \sum_{w \in B_n} E_w(x) u_w $$

are symmetric functions in $x$. The $E_w$ are the type D Stanley symmetric functions, introduced and studied in [BH] [L].

Let $y = (y_1, y_2, \ldots)$. The Billey-Haiman type $D$ Schubert polynomials $\mathcal{D}_w(x; y)$ for $w \in \tilde{W}_{n+1}$ are defined by expanding the formal product

$$ D(x)A_1(y_1)A_2(y_2) \cdots A_n(y_n) = \sum_{w \in \tilde{W}_{n+1}} \mathcal{D}_w(x; y) u_w. $$

The above definition is equivalent to the one in [BH]. One checks that $\mathcal{D}_w$ is stable under the natural inclusion of $\tilde{W}_n$ in $\tilde{W}_{n+1}$, and hence well defined for $w \in \tilde{W}_\infty$. We also deduce the following result from (34) and (35).
Proposition 3. Let \( w \in \tilde{W}_{\infty} \) and \( x' = (x'_1, x'_2, \ldots) \). Then we have

\[
E_w(x, x') = \sum_{uv = w} E_u(x)E_v(x')
\]

and

\[
D_w(x, x'; y) = \sum_{uv = w} E_u(x)D_v(x'; y)
\]

where the sums are over all reduced factorizations \( uv = w \) in \( \tilde{W}_{\infty} \).

3.2. For \( k \neq 1 \), an element \( w \in \tilde{W}_{\infty} \) is \( k \)-Grassmannian if \( \ell(ws_i) = \ell(w) + 1 \) for all \( i \neq k \). We say that \( w \) is 1-Grassmannian if \( \ell(ws_i) = \ell(w) + 1 \) for all \( i \geq 2 \). The elements of \( \tilde{W}_{n+1} \) index the Schubert classes in the cohomology ring of the flag variety \( SO_{2n+2}/B \), which contains \( H^*(OG(n+1-k, 2n+2), \mathbb{Z}) \) as the subring spanned by Schubert classes given by \( k \)-Grassmannian elements. If \( \tilde{P}(k, n) \) denotes the set of typed \( k \)-strict partitions whose Young diagrams fit inside an \( (n+1-k) \times (n+k) \) rectangle, then each typed \( k \)-strict partition \( \lambda \) in \( \tilde{P}(k, n) \) corresponds to a \( k \)-Grassmannian element \( w_\lambda \in \tilde{W}_{n+1} \) which we proceed to describe.

The typed \( k \)-strict partitions \( \lambda \) in \( \tilde{P}(k, n) \) are those whose Young diagram fits inside the non-convex polygon \( \Pi \) obtained by attaching an \( (n+1-k) \times k \) rectangle to the left side of a staircase partition with \( n \) rows. When \( n = 7 \) and \( k = 3 \), the polygon \( \Pi \) looks as follows.

The boxes of the staircase partition that are outside \( \lambda \) lie in south-west to north-east diagonals. Such a diagonal is called related if it is \( k' \)-related to one of the bottom boxes in the first \( k \) columns of \( \lambda \), or to any box \([0, i]\) for which \( \lambda_1 < i \leq k \); the remaining diagonals are called non-related. Let \( u_1 < u_2 < \cdots < u_k \) denote the lengths of the related diagonals. Moreover, let \( \lambda^1 \) be the strict partition obtained by removing the first \( k \) columns of \( \lambda \), let \( \lambda^2 \) be the partition of boxes contained in the first \( k \) columns of \( \lambda \), and set \( r = \ell(\lambda^1) = \ell_k(\lambda) \).

If type(\( \lambda \)) = 1, then the \( k \)-Grassmannian element corresponding to \( \lambda \) is given by

\[
w_\lambda = (u_1 + 1, \ldots, u_k + 1, (\lambda^1)_1 + 1, \ldots, (\lambda^1)_r + 1, \hat{1}, v_1 + 1, \ldots, v_{n-k-r} + 1) \]

while if type(\( \lambda \)) = 2, then

\[
w_\lambda = (u_1 + 1, \ldots, u_k + 1, (\lambda^1)_1 + 1, \ldots, (\lambda^1)_r + 1, \hat{1}, v_1 + 1, \ldots, v_{n-k-r} + 1) \]
where \( v_1 < \cdots < v_{n-k-r} \) are the lengths of the non-related diagonals. Here \( \widehat{\Gamma} = 1 \) or \( \overline{\Gamma} \), so that the total number of barred entries in \( w_\lambda \) is even. Finally, if \( \text{type}(\lambda) = 0 \), then \( u_1 = 0 \), that is, one of the related diagonals has length zero. In this case
\[
w_\lambda = (\widehat{\Gamma}, u_2 + 1, \ldots, u_k + 1, (\lambda^1)_1 + 1, \ldots, (\lambda^1)_r + 1, v_1 + 1, \ldots, v_{n+1-k-r} + 1),
\]
where \( v_1 < \cdots < v_{n+1-k-r} \) are the lengths of the non-related diagonals. For example, the element \( \lambda = (8, 4, 3, 2) \in \mathcal{P}(3, 7) \) of type 1 corresponds to \( w_\lambda = (3, 5, 7, \overline{0}, \overline{2}, 1, 4, 8) \).

\[
\lambda = \begin{array}{cccccccc}
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\end{array}
\]

The element \( w_\lambda \in \widetilde{W}_\infty \) depends on \( \lambda \) and \( k \), but is independent of \( n \). Furthermore, it was proved in [BKT3, Prop. 6.3] that
\[
(H_\lambda(x;y) = \mathfrak{D}_{w_\lambda}(x;y)
\]
for any typed \( k \)-strict partition \( \lambda \).

3.3. The following definition can be formulated for any Coxeter group.

**Definition 5.** An element \( w \in \widetilde{W}_\infty \) is called skew if there exists a typed \( k \)-strict partition \( \lambda \) (for some \( k \)) and a reduced factorization \( w_\lambda = w w' \) in \( \widetilde{W}_\infty \).

Note that if we have a reduced factorization \( w_\lambda = w w' \) in \( \widetilde{W}_{k+1} \) for some typed partition \( \lambda \in \mathcal{P}(k,n) \), then the right factor \( w' \) is \( k \)-Grassmannian, and therefore equal to \( w_\mu \) for some typed \( k \)-strict partition \( \mu \in \mathcal{P}(k,n) \).

**Proposition 4.** Suppose that \( w \) is a skew element of \( \widetilde{W}_\infty \), and let \( \lambda \) and \( \mu \) be typed \( k \)-strict partitions such that the factorization \( w_\lambda = w w_\mu \) is reduced. Then we have \( \mu \subset \lambda \) and \( E_w(x) = E_{\lambda/\mu}(x) \).

**Proof.** By combining (38) with (30) and (37), we see that
\[
\sum_{\mu \subset \lambda} E_{\lambda/\mu}(x) H_\mu(x';y) = H_\lambda(x,x';y) = \sum_{w w_\mu = w_\lambda} E_w(x) \mathfrak{D}_v(x';y)
\]
where the first sum is over all typed \( k \)-strict partitions \( \mu \subset \lambda \) and the second sum is over all reduced factorizations \( w w_\mu = w_\lambda \). The right factor \( v \) in any such reduced factorization of \( w_\mu \) must equal \( w_\nu \) for some typed \( k \)-strict partition \( \nu \), and therefore \( \mathfrak{D}_v(x';y) = H_\nu(x';y) \). Since the \( H_\nu(x';y) \) for \( \nu \) a typed \( k \)-strict partition form a \( \mathbb{Z} \)-basis for the ring \( B^{(k)}(x';y) \) of \( \eta \) polynomials in \( x' \) and \( y \), the desired result follows. \( \square \)

**Definition 6.** Let \( \lambda \) and \( \mu \) be typed \( k \)-strict partitions in \( \mathcal{P}(k,n) \) with \( \mu \subset \lambda \). We say that \( (\lambda, \mu) \) is a compatible pair if there is a reduced word for \( w_\lambda \) whose last \( |\mu| \) entries form a reduced word for \( w_\mu \); equivalently, if we have \( \mathcal{L}(w_\lambda w_\mu^{-1}) = |\lambda - \mu| \).

From Proposition 1, we immediately deduce the next result.
Corollary 2. Let \((\lambda, \mu)\) be a compatible pair of typed \(k\)-strict partitions. Then there is a \(1\)-\(1\) correspondence between reduced factorizations of \(w_\lambda w_\mu^{-1}\) and typed \(k\)-strict partitions \(\nu\) with \(\mu \subset \nu \subset \lambda\) such that \((\lambda, \nu)\) and \((\nu, \mu)\) are compatible pairs.

Theorem 4. Let \(\lambda\) and \(\mu\) be typed \(k\)-strict partitions in \(\mathcal{P}(k,n)\) with \(\mu \subset \lambda\). Then the following conditions are equivalent: (a) \(E_{\lambda/\mu}(x) \neq 0\); (b) \((\lambda, \mu)\) is a compatible pair; (c) there exists a standard typed \(k'\)-tableau on \(\lambda/\mu\). If any of these conditions holds, then \(E_{\lambda/\mu}(x) = E_{\lambda w_\mu^{-1}}(x)\).

Proof. Equation (39) may be rewritten in the form
\[
\sum_{\mu \subset \lambda} E_{\lambda/\mu}(x) H_{\mu}(x'; y) = \sum_{\mu} E_{w_\lambda w_\mu^{-1}}(x) H_{\mu}(x'; y)
\]
where the second sum is over all \(\mu \subset \lambda\) such that \((\lambda, \mu)\) is a compatible pair. It follows that
\[
E_{\lambda/\mu}(x) = \begin{cases} E_{w_\lambda w_\mu^{-1}}(x) & \text{if } (\lambda, \mu) \text{ is a compatible pair,} \\ 0 & \text{otherwise} \end{cases}
\]
and we deduce that (a) and (b) are equivalent. Suppose now that \((\lambda, \mu)\) is a compatible pair with \(|\lambda| = |\mu| + 1\), so that \(w_\lambda = s_i w_\mu\) for some \(i \geq 0\). Observe that if \(x\) is a single variable, then \(E_s(x) = x\), if \(i \leq 1\), and \(E_s(x) = 2x\), if \(i > 1\), and therefore \(E_{\lambda/\mu}(x) \neq 0\). We deduce from Example (3d) that \(\lambda/\mu\) must be a \(k'\)-horizontal strip. Using Corollary (2) it follows that there is a \(1\)-\(1\) correspondence between reduced words for \(w_\lambda w_\mu^{-1}\) and sequences of typed \(k\)-strict partitions
\[
\mu = \lambda^0 \subset \lambda^1 \subset \cdots \subset \lambda^r = \lambda
\]
such that \(|\lambda^i| = |\lambda^{i-1}| + 1\) and \(\lambda^i/\lambda^{i-1}\) is a \(k'\)-horizontal strip for \(1 \leq i \leq r = |\lambda - \mu|\). The latter objects are exactly the standard typed \(k'\)-tableaux on \(\lambda/\mu\). This shows that (b) implies (c), and the converse is also clear.

The previous results show that the non-zero terms in equations (30) and (32) correspond exactly to the terms in equations (37) and (30), respectively, when \(w = w_\lambda\) is a \(k\)-Grassmannian element of \(\tilde{W}_\infty\).

Corollary 3. Let \(w \in \tilde{W}_\infty\) be a skew element and \((\lambda, \mu)\) be a compatible pair such that \(w_\lambda = w w_\mu\). Then the number of reduced words for \(w\) is equal to the number of standard typed \(k'\)-tableaux on \(\lambda/\mu\).

Let \(\iota\) denote the involution on the set of reduced words which interchanges the letters 0 and 1, for example \(\iota(02120) = 12021\). Then the restriction of \(\iota\) to the set of reduced words representing skew elements of \(\tilde{W}_\infty\) corresponds to the involution \(j\) on the set of typed \(k'\)-tableaux defined in Example (3a).

Example 4. Let \(\lambda\) be a typed \(k\)-strict partition with type(\(\lambda\)) \(\in \{0, 1\}\) and let \(\lambda^1\) and \(\lambda^2\) be defined as in (3.2). We can form a standard typed \(k'\)-tableau on \(\lambda\) by filling the boxes of \(\lambda^2\), going down the columns from left to right, and then filling the boxes of \(\lambda^1\), going across the rows from top to bottom. When \(k = 2\) and \(\lambda = (7, 6, 5, 2)\) with type(\(\lambda\)) = 1, the typed \(2'\)-tableau on \(\lambda\) which results is
\[
\begin{array}{cccccccc}
1 & 5 & 9 & 10 & 11 & 12 & 13 \\
2 & 6 & 14 & 15 & 16 & 17 \\
3 & 7 & 18 & 19 & 20 \\
4 & 8
\end{array}
\]
which corresponds to the reduced word
\[ 32043215432043215432 \]
for the 2-Grassmannian element \( w_\lambda = 236541 \in D_4 \). We can form a reduced word for the element \( w'_\lambda = 236541 \) associated \( \lambda = (7, 6, 5, 2) \) with type(\( \lambda \)) = 2 by applying the involution \( \iota \) to obtain
\[ 32143205432143205432. \]
This last word corresponds to the standard typed 2'-tableau
\[
\begin{array}{ccccccccc}
1 & 5 & 9 & 10 & 11 & 12 & 13 \\
2 & 6 & 14 & 15 & 16 & 17 & \\
3 & 7 & 18 & 19 & 20 & \\
4 & 8 & \\
\end{array}
\]

**Example 5.** Consider the 1-Grassmannian element \( w = 3421 \) in \( \widetilde{W}_4 \) associated to the typed 1-strict partition \( \lambda = (4, 2, 1) \) of type 1. The following table lists the nine reduced words for \( w \) and the corresponding standard typed 1-tableaux of shape \( \lambda \).

| word   | tableau |
|--------|---------|
| 1320321 | 1 4 5 6 |
| 2 7     |         |
| 3       |         |
| 1323021 | 1 3 5 6 |
| 2 7     |         |
| 4       |         |
| 1232021 | 1 3 4 5 |
| 2 7     |         |
| 6       |         |
| 1230201 | 1 2 3 5 |
| 4 7     |         |
| 6       |         |
| 1230210 | 1 2 3 5 |
| 4 7     |         |
| 6       |         |
| 1203201 | 1 2 3 4 |
| 5 7     |         |
| 6       |         |
| 1203210 | 1 2 3 4 |
| 5 7     |         |
| 6       |         |
| 3120321 | 1 4 5 7 |
| 2 6     |         |
| 3       |         |
| 3123021 | 1 3 5 7 |
| 2 6     |         |
| 4       |         |

**Corollary 4.** For any two typed \( k \)-strict partitions \( \lambda, \mu \) with \( \mu \subseteq \lambda \), the function \( E_{\lambda/\mu}(x) \) is a nonnegative integer linear combination of Schur \( P \)-functions.

**Proof.** According to [BH] and [L], the type D Stanley symmetric function \( E_{w}(x) \) is a nonnegative integer linear combination of Schur \( P \)-functions. The result follows from this fact together with Theorem [H].

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