Higher-Order Regularity of the Free Boundary in
the Inverse First-Passage Problem

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Abstract

Consider the inverse first-passage problem: Given a diffusion process \( \{X_t\}_{t \geq 0} \) on a probability space \((\Omega, \mathcal{F}, P)\) and a survival probability function \( p \) on \([0, \infty)\), find a boundary, \( x = b(t) \), such that \( p \) is the survival probability that \( X \) does not fall below \( b \), i.e., for each \( t \geq 0 \), \( p(t) = P(\{ \omega \in \Omega \mid X_s(\omega) \geq b(s), \forall s \in (0, t) \}) \). In earlier work, we analyzed viscosity solutions of a related variational inequality, and showed that they provided the only upper semi-continuous (usc) solutions of the inverse problem. We furthermore proved weak regularity (continuity) of the boundary \( b \) under additional assumptions on \( p \). The purpose of this paper is to study higher-order regularity properties of the solution of the inverse first-passage problem. In particular, we show that when \( p \) is smooth and has negative slope, the viscosity solution, and therefore also the unique usc solution of the inverse problem, is smooth. Consequently, the viscosity solution furnishes a unique classical solution to the free boundary problem associated with the inverse first-passage problem.

1 Introduction

1.1 The First-Passage Problem and its Inverse

Let \( X = \{X_t\}_{t \geq 0} \) be the solution to the stochastic differential equation

\[
dX_t = \mu(X_t, t) dt + \sigma(X_t, t) dB_t \quad \forall t > 0,
\]

where \( \{B_t\}_{t \geq 0} \) is a standard Brownian motion defined on a probability space \((\Omega, \mathcal{F}, P)\) and \( \mu, \sigma \) are smooth bounded functions with \( \inf_{R \times [0, \infty)} \sigma > 0 \). The boundary crossing, or first-passage problem for the process \( X \) concerns the following:

1. The Forward Problem: Given a function \( b : (0, \infty) \rightarrow [-\infty, \infty) \), compute the survival probability, \( \mathcal{P}[b] \), that \( X \) does not fall below \( b \), i.e. evaluate

\[
\mathcal{P}[b](t) := P(\{ \omega \in \Omega \mid X_s(\omega) \geq b(s), \forall s \in (0, t) \}) \quad \forall t \geq 0.
\] (1.1)

2. The Inverse Problem: Given a survival probability \( p \), find a barrier \( b \) such that \( \mathcal{P}[b] = p \).

The forward problem is classical and the subject of a large literature. According to Zucca and Sacerdote [2009], the inverse problem was first suggested by A.N. Shiryaev during a Banach Centre meeting, for the case where \( X \) is a Brownian motion, and the first-passage distribution is exponential, i.e.

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the solution of the inverse first-passage problem is the same as the solution of a related optimal stop-
the problem posed by Shiryaev. 

It is shown in Chen et al. [2011] that 
Proposition 1. Suppose that \( \mathcal{X} \) is standard Brownian motion, started at 0 (i.e. \( \mu = 0, \sigma = 1, \) and \( \mathcal{X}_0 = 0 \)), and that \( p \) is continuous with \( p(0) = 1 \). Define:

\[
L(p, T_1, T_2) := \inf_{T_1 \leq t < T_2} \frac{p(s) - p(t)}{t - s}, \quad \forall 0 \leq T_1 < T_2.
\]

1. If \( L(p, T_1, T_2) > 0 \) for some positive \( T_1, T_2 \) with \( T_1 < T_2 \), then \( b \) is continuous on \( (T_1, T_2) \).

2. Assume that \( L(p, 0, T) > 0 \) for every \( T > 0 \). Then \( b \in C([0, \infty)) \).

In particular, if \( p(t) = e^{-\lambda t} \), then \( L(p, 0, T) = e^{\lambda T} > 0 \), yielding continuity of the boundary in the problem posed by Shiryaev.

The purpose of this paper is to study higher-order regularity properties of the boundary \( b \) in the one-sided inverse problem.

Several other papers have also studied the inverse problem. Ekström and Janson [2010] show that the solution of the inverse first-passage problem is the same as the solution of a related optimal stopping problem, and present an analysis of an associated integral equation for the stopping boundary \( b \). Integral equations related to the problem are discussed in Peskir [2002] and Peskir and Shiryaev [2003]. Abundo [2006] studied the small-time behaviour of the boundary \( b \). A rigorous construction of the boundary based on a discretization procedure was recently presented in Potiron [2021].

Remark 1.1. Traditionally, the forward problem is studied for upper-semi-continuous (usc) \( b \) and the conventional survival probability, \( \hat{p} \), is defined in term of the first crossing time, \( \hat{\tau} \), by

\[
\hat{\tau}(\omega) := \inf \{ t > 0 | \mathcal{X}_t(\omega) \leq b(t) \}, \quad \hat{p}(t) := \mathbb{P}(\{ \omega \in \Omega | \hat{\tau}(\omega) > t \}).
\]

Here, the survival probability, \( p = \mathbb{P}[b] \) in (1.1), is defined by

\[
\tau(\omega) := \inf \{ t > 0 | \mathcal{X}_t(\omega) < b(t) \}, \quad p(t) := \mathbb{P}(\{ \omega \in \Omega | \tau(\omega) \geq t \}).
\]

It is shown in Chen et al. [2011] that \( \mathbb{P}[b] \) is well-defined for each \( b \). In addition, define \( b^* \) and \( b^*_- \) by

\[
b^*(t) := \max \left\{ b(t), \lim_{s \to t^-} b(s) \right\}, \quad b^*_-(t) := \lim_{s \to t^+} b(s).
\]

Then (i) \( \mathbb{P}[b] = \mathbb{P}[b^*] \), (ii) when \( b = b^* \), \( \tau = \hat{\tau} \) almost surely, and (iii) when \( b = b^*_- \), \( \mathbb{P}[b] \in C([0, \infty)) \). Since \( \mathbb{P}[b] = \mathbb{P}[b^*] \), it is convenient for the inverse problem to restrict the search to \( b \) in
the class of usc functions, i.e., those \( b \) that satisfy \( b = b^* \). In [Chen et al. 2011] it is shown that for every \( b \in P_0 \), where
\[
P_0 := \{ p \in C([0, \infty)) \mid p(0) = 1 \geq p(s) \geq p(t) > 0 \quad \forall \ t > s > 0 \},
\]
the inverse problem admits a unique usc solution.

1.2 The Free Boundary Problems

We introduce differential operators \( \mathcal{L} \) and \( \mathcal{L}_1 \) defined by
\[
\mathcal{L} \phi := \partial_t \phi - \frac{1}{2} \sigma^2 \partial_x^2 \phi + \mu \partial_x \phi, \quad \mathcal{L}_1 \phi := \partial_t \phi - \frac{1}{2} \sigma^2 \partial_x^2 \phi + \partial_x (\mu \phi).
\]
The survival distribution, \( w \), and survival density, \( u \), are defined by
\[
w(x, t) := \mathbb{P}(\tau \geq t, X_t > x), \quad u(x, t) := -\partial_x w(x, t).
\]
We denote the distribution of \( X_t \) by \( w_0 \) and its density by \( u_0 \):
\[
w_0(x, t) := \mathbb{P}(X_t > x), \quad u_0(x)dx := \mathbb{P}(X_0 \in (x, x + dx)).
\]

When \( b \) is smooth, one can show that \((b, w, p)\) satisfies
\[
\begin{align*}
\mathcal{L} w &= 0 \quad \text{for } x > b(t), t > 0, \\
\partial_x w(x, t) &= 0 \quad \text{for } x \leq b(t), t > 0, \\
w(x, 0) &= w_0(x, 0) \quad \text{for } x \in \mathbb{R}, \ t = 0, \\
p(t) &= w(b(t), t) \quad \text{for } x = b(t), t > 0,
\end{align*}
\]
and \((b, u, p)\) satisfies
\[
\begin{align*}
\mathcal{L}_1 u &= 0 \quad \text{for } x > b(t), t > 0, \\
u(x, t) &= 0 \quad \text{for } x \leq b(t), t > 0, \\
u(x, 0) &= u_0(x) \quad \text{for } x \in \mathbb{R}, \ t = 0, \\
2p(t) &= -\sigma^2 \partial_x u |_{x=b(t)} \quad \text{for } x = b(t), t > 0.
\end{align*}
\]

Note that (1.3) and (1.4) are equivalent in the class of smooth functions via the transformation
\[
u(x, t) = -\partial_x w(x, t), \quad w(x, t) = \int_x^\infty u(y, t) dy \quad \forall x \in \mathbb{R}, t \geq 0.
\]

When \( b \) is given and regular, say, Lipschitz continuous, the forward problem can be easily handled by first solving the initial-boundary value problem consisting of the first three equations in (1.3) and then evaluating \( p \) from the last equation in (1.3). For the inverse problem, both (1.3) and (1.4) are free boundary problems since the domain \( Q_b := \{ (x, t) \mid x > b(t), t > 0 \} \), where the equations, \( \mathcal{L} w = 0 \) and \( \mathcal{L}_1 u = 0 \), are satisfied, is a priori unknown. So far, there is little known concerning the existence, uniqueness, and regularity of classical solutions of the free boundary problems. Here by a classical solution, \((b, w)\), of (1.3) we mean that \( w - w_0 \in C(\mathbb{R} \times [0, \infty)) \), \( \partial_x w \in C(\mathbb{R} \times (0, \infty)) \), \( \partial_t w, \partial_x^2 w \in C(Q_b) \), and each equation in (1.3) is satisfied; similarly, by a classical solution, \((b, u)\), of (1.4), we mean that \( u + w_0 \in C(\mathbb{R} \times [0, \infty)) \), \( \partial_t u, \partial_x^2 u \in C(Q_b) \), and each equation in (1.4) is satisfied. In this paper, we investigate the well-posedness of the free boundary problem and the smoothness of the free boundary.

1.3 The Weak Formulation

In [Cheng et al. 2006], viscosity solutions for the inverse problem, based on the variational inequality
\[
\max \{ \mathcal{L} w, w - p \} = 0 \quad \text{in } \mathbb{R} \times (0, \infty),
\]
are introduced. It is shown that for any given probability distribution \( p \) on \([0, \infty)\), there exists a unique viscosity solution. This was followed up in Chen et al. [2011] in which it was shown that the viscosity solution of the variational inequality gives the solution of the (probabilistic) inverse problem.

For easy reference, we quote the relevant results.

**Definition 1.** Let \( p \in P_0 \) be given where \( P_0 \) is as in (1.2). A viscosity solution of the inverse problem associated with \( p \) is a function \( b \) defined by

\[
b(t) := \inf \{ x \in \mathbb{R} \mid w(x, t) < p(t) \}, \quad \forall t > 0,
\]

provided that \( w \) has the following properties:

1. \( w \in C(\mathbb{R} \times (0, \infty)), \lim_{t \searrow 0} \|w(\cdot, t) - \mathbb{P}(X_t > \cdot)\|_{L^\infty(\mathbb{R})} = 0; \)
2. \( 0 \leq w \leq p \) in \( \mathbb{R} \times (0, \infty) \) and \( Lw = 0 \) in the set \( \{(x, t) \mid t > 0, w(x, t) < p(t)\} \);
3. If for a smooth \( \varphi, x \in \mathbb{R} \) and \( t > \delta > 0 \), the function \( \varphi - w \) attains its local minimum on \([x - \delta, x + \delta] \times [t - \delta, t]\) at \((x, t)\), then \( \mathcal{L}_\varphi(x, t) \leq 0 \).

One can verify that if \((b, w)\) (or \((b, u)\)) is a classical solution of the free boundary problem (1.3) (or (1.4)), then \( b \) is a viscosity solution of the inverse problem associated with \( p \).

**Proposition 2 (Well-posedness of the Inverse Problem).** Let \( p \in P_0 \) be given.

1. Chen et al. [2006]: There exists a unique viscosity solution, \( b \), of the inverse problem associated with \( p \).
2. Chen et al. [2011]: The viscosity solution is a usc solution of the inverse problem, i.e. \( b = b^* \) and \( \mathbb{P}[b] = p \).
3. Chen et al. [2011]: There exists a unique usc solution of the inverse problem associated with \( p \).

It is clear now that the viscosity solution is the right choice for the inverse problem. For convenience, in the sequel, we shall call \( b \), \((b, w)\), \((b, u)\), or \((b, w, u)\), the solution or the viscosity solution of the inverse problem, where \( b \) is the viscosity solution boundary, \( w \) is the viscosity solution for the survival distribution, and \( u = -\partial_x w \) is the viscosity solution for the survival density of the inverse problem associated with \( p \). We shall also call the curve \( x = b(t) \) the free boundary.

### 1.4 The Main Result: Higher Order Regularity

While the work of Chen et al. [2006] and Chen et al. [2011] solves the inverse problem, and presents a basic study of weak regularity, here we make a detailed study of the regularity of the free boundary. The main result of this paper is the following, where \( \lfloor \alpha + \frac{1}{2} \rfloor \) denotes the integer part of \( \alpha + \frac{1}{2} \).

**Theorem 1 (Regularity of the Free Boundary).** Let \( p \in P_0 \) be given and \((b, w, u)\) be the (viscosity) solution of the inverse problem associated with \( p \). Assume that \( p \in C^1((0, \infty)), \dot{p} < 0 \) on \([0, \infty)\), and for some \( \delta > 0 \), either

\[
\text{(i) } u_0 = 0 \text{ on } (-\infty, 0], \quad u_0 \in C^1([0, \delta]), \quad u_0'(0+) > 0, \quad \text{or} \quad \text{(ii) } \dot{p} \in L^1((0, \delta)) . \quad (1.6)
\]

1. Then \((b, w)\) and \((b, u)\) are classical solutions of (1.3) and (1.4) respectively with \( b \in C^{1/2}((0, \infty)) \).
Remark 1.2. To derive the compatibility conditions, we consider, for simplicity, the special case

\[ \sigma_k \]

of variables \( x \) in an interval. Assuming it can be shown that \( \sigma(0) = -\sigma^2(0,0) u_0(0+) \) when \( \alpha \in \left[ \frac{1}{2}, \frac{3}{2} \right] \), then \( b(0) = 0 \) and \( b \in C^\alpha((0, \infty)) \).

3. Finally, if for \( \alpha \geq \frac{1}{2} \) not an integer one has \( p \in C^{\alpha+\frac{1}{2}}((0, \infty)) \), then \( u_0 = 0 \) on \( (-\infty, 0] \), \( u_0 \in C^\alpha([0, \delta]) \), and \( (u_0, p) \) satisfy all compatibility conditions up to order \( [\alpha+\frac{1}{2}] \), including in particular the compatibility condition \( 2\dot{p}(0) = -\sigma^2(0,0) u_0(0+) \) when \( \alpha \in \left( \frac{1}{2}, \frac{3}{2} \right) \), then \( b(0) = 0 \) and \( b \in C^\alpha((0, \infty)) \).

Remark 1.3. To derive the compatibility conditions, we consider, for simplicity, the special case \( \sigma \equiv \sqrt{2} \) and \( \mu \equiv 0 \). Set \( U(x,t) = u(x + b(t), t) \). Then

\[
\begin{align*}
U_t &= U_{xx} + bU_x & \text{in } (0, \infty) \times (0, \infty), \\
U(0, \cdot) &= 0, U_x(0, \cdot) &= -\dot{p}(\cdot) & \text{on } \{0\} \times (0, \infty), \\
U(\cdot, 0) &= u_0(\cdot) & \text{on } [0, \infty) \times \{0\}.
\end{align*}
\]

The \( k \)-th order compatibility condition is the \((k-1)\)-th order derivative of \( \dot{p}(t) = -U_x(0, t) \) at \( t = 0 \) (with differentiation of \( U \) in time being replaced by differentiation in space by \( U_t = U_{xx} + bU_x \)):

\[
\frac{d^k \dot{p}(t)}{dt^k} \bigg|_{t=0} = -\frac{d^{k-1}U_x(0,t)}{dt^{k-1}} \bigg|_{t=0} = -\frac{d^{2k-1}u_0(x)}{dx^{2k-1}} \bigg|_{x=0} + \cdots.
\]

The \( k \)-th order derivative of \( b \) at \( t = 0 \) is obtained by differentiating the equation \( \dot{b}(t) \dot{p}(t) = U_{xx}(0, t) \):

\[
\frac{d^k b(t)}{dt^k} \bigg|_{t=2} = \frac{\partial^{k-1}U_{xx}(0,t)}{\dot{p}(0) \partial^{k-1}} \bigg|_{t=0} + \cdots = \frac{1}{\dot{p}(0)} \frac{d^{2k}u_0(x)}{dx^{2k}} \bigg|_{x=0} + \cdots.
\]

In particular, the first and second order compatibility conditions are

\[
\dot{p}(0) = -u'_0(0), \quad \ddot{p}(0) = -u'''_0(0) - \dot{b}(0)u''_0(0) \bigg|_{b(0)=u'''_0(0)/\ddot{p}(0)}.
\]

Remark 1.3. For \( b \) to be continuous and bounded, it is necessary to assume that \( p \) is strictly decreasing as in Proposition 1. Indeed, if \( p \) is a constant in an open interval, then \( b = -\infty \) in that interval.

The main tool for the proof of Theorem 1 is the hodograph transformation, defined by the change of variables \( x = X(z, t) \), the inverse of \( z = u(x, t) \). Since \( u(b(t), t) = 0 \), we have \( b(t) = X(0, t) \). Taking \( \sigma \equiv \sqrt{2} \) to simplify the exposition, and proceeding formally, we can derive that \( X \) solves quasi-linear pde:

\[
Y_t = Y^{-2}_z Y_{zz} + [z\mu(Y, t)]_z, \quad \dot{p}(t) Y_z(0,t) = -1, \quad u_0(Y(z,0)) = z.
\]

Assuming it can be shown that \( X_{\varepsilon}(\varepsilon, \cdot) \) is positive on \([0, T]\), this system is studied on the set \( \{z(t) \in [0, \varepsilon], \in [0, T]\} \) for any fixed \( T > 0 \) and a small positive \( \varepsilon \) that depends on \( T \). To complete the system, we supply the boundary condition for \( Y \) on \( \{\varepsilon\} \times \{0, T\} \) by \( Y_{\varepsilon}(\varepsilon, t) = X_{\varepsilon}(\varepsilon, t) \).

The classical approach to the hodograph transformation (see, e.g. [Friedman 1982] or [Kinderlehrer and Stampacchia 1980]) employs a bootstrapping strategy, assuming some initial degree of smoothness on \((p, b)\), and then using the regularity theory for (1.8) to strengthen the regularity of \( b \). In particular, standard results for quasi-linear equations ([Ladyzhenskaya et al. 1968, Lieberman 1996]) can be used to derive the existence of a unique classical solution of (1.8), and its regularity (including up to the boundary). The assumptions on \((p, b)\) are sufficient to reverse the hodograph transformation, and transfer the boundary regularity of \( Y \) to \( b(t) = Y(0,t) \) (and to show that indeed \( Y = X \), where \( X \) is
defined through \( x = X(u(x,t),t) \). The results achieved through the classical approach are reviewed in Section 3.

In order to prove Theorem 1, we wish to employ the same strategy, but with weaker assumptions on the initial regularity of \((p,b)\). In doing so, we encounter two main difficulties, the first technical, and the second fundamental. The technical issue is that above the boundary, \( u \) solves \( L_1 u = 0 \), and when \( \mu \neq 0 \) the operator \( L_1 \) has a zeroth order term. The maximum principle arguments employed in analyzing level sets in our proofs require a differential operator with no zeroth order term. We address these related technical hurdles by considering \( v = u/K \) for an appropriate scaling function \( K \), defined as the solution of an auxiliary partial differential equation, such that above the boundary \( L_2 v = 0 \), where \( L_2 = \partial_t - \partial_{xx} + \nu \partial_x \) for some function \( \nu \). The formal hodograph transformation then leads us to consider the partial differential equation:

\[
Y_t = Y_{z}^{-2} Y_{zz} + \nu(Y,t), \quad z \in (0,\varepsilon), t \in (0,T],
\]

(1.9)
together with the boundary condition \( \mathcal{M} Y = 0 \), where

\[
\mathcal{M} Y = \begin{cases} 
Y(z,0) - X_0(z), & z \in [0,\varepsilon], \\
\dot{\rho}(t) Y_z(0,t) + K(Y(0,t),t), & t \in (0,T], \\
Y(z,t) - X_z(\varepsilon,t), & t \in (0,T]. 
\end{cases}
\]

(1.10)

Here, we encounter a fundamental difficulty due to the fact that we do not know a priori that \( X_0 \) is regular up to the boundary, i.e., we do not have the equation \( \dot{\rho}(t) X_z(0,t) + K(X(0,t),t) = 0 \). While we can study the above problem analytically, we have not assumed the requisite regularity to show that \( Y = X \), with \( X(z,t) := \min \{ x \geq b(t) \mid v(x,t) = z \} \). This difficulty is surmounted by defining a family of perturbed equations with boundary operators \( \mathcal{M}^h, h \in \mathbb{R} \), and making comparisons with their solutions \( Y^h \). In particular, for small \( h > 0 \), we show that \( Y^{-h}(0,\cdot) < b < Y^h(0,\cdot) \), \( Y^{-h}(\varepsilon,\cdot) < X(\varepsilon,\cdot) < Y^h(\varepsilon,\cdot) \). Letting \( h \downarrow 0 \), we are then able to obtain that \( X \equiv Y, b = Y(0,\cdot) \), and the required regularity of \( b \).

**Remark 1.4.** For simplicity of exposition, throughout the paper we assume that \( \sigma \equiv \sqrt{2} \). This can be done without loss of generality. Indeed, let

\[
Y(x,t) = \int_0^x \frac{\sqrt{2}}{\sigma(z,t)} dz, \quad \tilde{\mu}(y,t) = \frac{\sqrt{2} \mu(x,t)}{\sigma(x,t)} - \frac{\partial_x \sigma(x,t)}{\sqrt{2}} - \int_0^x \frac{\sqrt{2} \partial_t \sigma(z,t)}{\sigma^2(z,t)} dz \bigg|_{x=X(y,t)}
\]

where \( x = X(y,t) \) is the inverse of \( y = Y(x,t) \). Then by Itô’s lemma, the process \( \{Y_t\} \) defined by \( Y_t := Y(X_t,t) \) is a diffusion process satisfying \( dY_t = \tilde{\mu}(Y_t,t) dt + \sqrt{2} dB_t \). The boundary crossing problem for \( \{X_t\} \) with barrier \( b(t) := Y(b(t),t) \). In terms of the partial differential equations, this is equivalent to the change of variables

\[
y = Y(x,t), \quad \tilde{w}(y,t) = w(X(y,t),t), \quad \tilde{u}(y,t) = -\partial_y \tilde{w}(y,t).
\]

We shall henceforth always assume that \( \sigma \equiv \sqrt{2} \).

The remainder of the paper is structured as follows. In the next section, we recall a few properties of the solution of the inverse problem and prove a smoothing property of the diffusion: under condition (ii) of (1.6), condition (i) is satisfied provided that the initial time \( t = 0 \) is shifted to \( t = s \), for any \( s \) outside a set of measure zero. In Section 3 we provide an interpretation of the free boundary condition \( \sigma^2(b(t),t) u_x(b(t),t) = -2 \dot{\rho}(t) \) for the viscosity solution. In Section 4 we present results that can be derived using the traditional approach to the hodograph transformation \( z = u(X(z,t),t) \). Section 5 presents the proof of Theorem 1 beginning by presenting a required generalization of the Hopf Boundary Point Lemma, then introducing the scaling function \( K \) and the scaled survival density \( v \), and finally analyzing the family \( Y^h \) of solutions to quasi-linear parabolic equations in order to derive our main results.

1In particular, we need that constant functions satisfy the differential equation.

2We can, however, show that \( X \) is well-defined and regular enough inside the domain. It is the regularity for \( X \) up to the boundary (because of the lack of a priori regularity of \( b \)) that is insufficient.
2 Regularity Properties of the Viscosity Solutions of the Inverse Problem

In this section, we collect a few results concerning the regularity of the (viscosity) solution of the inverse problem. Recall that \(w_0(x, t) = P(\mathcal{X}_t > x)\) and \(Q_b = \{(x, t) \mid t > 0, x > b(t)\}\).

Lemma 2.1. Let \((b, w, u)\) be the solution of the inverse problem associated with \(p \in P_0\). Then
\[
\begin{align*}
    u & \in \mathcal{C}^\infty(Q_b), & \mathcal{L}_1 u & = 0 < u \text{ in } Q_b, & \mathcal{L}_1 u & \leq 0 \text{ in } \mathbb{R} \times (0, \infty),
\end{align*}
\]
where the inequalities above hold in the sense of distributions.

In addition, for any \(T > 0\), the following holds:

1. If \(\dot{p} \in L^\infty((0, T))\), then \(u + w_0 \in \bigcap_{\alpha \in (0, 1)} \mathcal{C}^{1, \alpha/2}(\mathbb{R} \times [0, T])\); consequently, \((b, w)\) is a classical solution of the free boundary problem \((1.3)\) on \(\mathbb{R} \times [0, T]\).

2. If \(\inf \{x \mid P(\mathcal{X}_0 \leq x) > 0\} = 0\) and \(\sup_{t \in [0, T]} \dot{p} < 0\), then \(b(0) = 0\) and \(b \in \mathcal{C}([0, T])\).

3. (Smoothing Property) If \(\dot{p} < 0\) on \([0, T]\) and \(\ddot{p} \in L^1((0, T))\), then for a.e. \(t \in (0, T)\),
\[
\begin{align*}
    w_t(t, \cdot) & \in \mathcal{C}^{1/2}(\mathbb{R}), & u(t, \cdot) & \in \mathcal{C}(\mathbb{R}) \cap \mathcal{C}^{3/2}([b(t), \infty)), & \sigma^2(b(t), t) u_x(b(t) + t, \cdot) & = -2\dot{p}(t).
\end{align*}
\]

Proof. For (2.1), see Cheng et al. [2006], Lemma 2.1, and Chen et al. [2011], Proposition 5 (with Theorems 1 and 3 in this reference summarizing the solution of the inverse boundary crossing problem).

To simplify the presentation, we assume that \(\mu \equiv 0\). The general case is analogous\(^3\). The viscosity solution for the survival distribution \(w\) of the inverse problem satisfies the variational inequality \(\max \{\mathcal{L} u, \dot{w} - p\} = 0\), which, according to Cheng et al. [2006] (see also Friedman [1982]), can be approximated, as \(\varepsilon \searrow 0\), by the solution of
\[
\begin{align*}
    w^\varepsilon_t - w^\varepsilon_{xx} & = -\beta(\varepsilon^{-1}([w^\varepsilon - p] \mathbb{R} \times (0, \infty), & w^\varepsilon(\cdot, 0) & = w_0(\cdot, 0) \text{ on } \mathbb{R} \times \{0\}
\end{align*}
\]
where \(\beta(z) = m \cdot \max\{0, z\}\)^3 with \(m = \|\dot{p}\|_{L^\infty((0, T))}\). It is worth mentioning that one can further regularize \((p, w_0, \beta)\) to smooth \((p^\varepsilon, w_0^\varepsilon, \beta^\varepsilon)\) so that the solution is not only smooth, but also monotonically decreasing in \(\varepsilon\); see Cheng et al. [2006] for details. We set \(w^\varepsilon = -\partial_x w^\varepsilon\), and \(\mathcal{G} \varphi = \mathcal{L} \varphi + \beta(\varepsilon^{-1}[\varphi(x, t) - p(t)])\).

1. Since \(\mathcal{G}(p + \varepsilon) = \dot{p} + m \geq 0 = \mathcal{G} w^\varepsilon\) and \(w^\varepsilon(\cdot, 0) \leq p(0)\), the Comparison Principle yields that \(w^\varepsilon \leq p + \varepsilon\). Similarly, \(\mathcal{L}(w_0 - w^\varepsilon) = \beta(\varepsilon^{-1}([w^\varepsilon - p] \mathbb{R} \times (0, \infty)) \geq 0\) gives \(w^\varepsilon \leq w_0\) and \(\mathcal{G} w^\varepsilon \geq 0 = \mathcal{G}(0)\) yields that \(w^\varepsilon \geq 0\). From \(w^\varepsilon \leq p + \varepsilon\), we have:
\[
0 \leq -\mathcal{L}(w^\varepsilon - w_0) = \beta((w^\varepsilon - p) \varepsilon^{-1}) \leq \beta(1) = m \text{ in } \mathbb{R} \times (0, T).
\]
A parabolic estimate (e.g. Krivlov [1999], Lemma 8.7.1, page 122) then implies that
\[
\|w^\varepsilon - w_0\|_{C^{1+\alpha, (1+\alpha)/2}([\mathbb{R} \times [0, T])} \leq mC(\alpha),
\]
for every \(\alpha \in (0, 1)\) where \(C(\alpha)\) is a constant depending only on \(\alpha\).

Let \(\varepsilon_n \searrow 0\), and let \(\beta \in (\alpha, 1)\). Then there exists a subsequence \(\varepsilon_{n_k}\) such that \(w^{\varepsilon_{n_k}} - w_0 \to w - w_0\) in \(C^{1+\alpha,(1+\alpha)/2}([\mathbb{R} \times [0, T])\). As the limit \(w\) is shown in Cheng et al. [2006] to be the unique viscosity

\(^3\)Unlike the assumption that \(\sigma \equiv \sqrt{2}\), there are points in the paper where this is not the case. Hence, we remark on this explicitly when we take the drift to be zero.
solution of the inverse problem associated with $p$, the whole sequence in fact converges. In addition, passing to the limit in the estimate \((2.3)\) we see that $u + w_0x = w_{0x} - w_x \in \cap_{\alpha \in (0,1)} C^{\alpha, \alpha/2}(\mathbb{R} \times [0, T])$.

Note that a classical solution of \((1.3)\) requires that the free boundary condition $w_x(b(t), t) = 0$ be well-defined. Since $w_x = w_0x - [w_{0x} + u]$ is continuous on $\mathbb{R} \times (0, \infty)$, the equation $w_x(b(t), t) = 0$ for $t > 0$ is satisfied in the classical sense. Hence, $(b, w)$ is a classical solution of \((1.3)\). This proves (1).

2. This is shown in Chen et al. [2011], Proposition 6 (see Proposition 1 above).

3. Using energy estimates, we will show that for each $\eta \in (0, T)$, $\int_0^T \int_\mathbb{R} w_{tx}^2(x, t) dx \, dt < \infty$, and therefore in particular $w_{tx}(\cdot, t) \in L^2(\mathbb{R})$ for almost every $t \in (0, T)$, from which Sobolev Embedding yields $w_t(\cdot, t) \in C^{1/2}(\mathbb{R})$. Since $w_t(x, t) = \dot{p}(t)$ for all $x < b(t)$ and $w_t(x, t) = w_{xx} = -u_x$ for all $x > b(t)$, the last assertion of the lemma thus follows.

Suppose $\dot{p} < 0$ on $[0, T)$ and $\ddot{p} \in L^1((0, T))$. Then $\ddot{p} \in C([0, T])$, so $m := \|\ddot{p}\|_{L^\infty((0, T))}$ is finite. As above, we have $0 \leq w^\varepsilon(x, t) \leq w_0$. Using the Comparison Principle as in Cheng et al. [2006], it can be shown that $w^\varepsilon = -w_{xx}^\varepsilon \geq 0$. We then further have $\mathcal{L}(-w_{0x} - w^\varepsilon) = \beta'(\varepsilon^{-1}(w^\varepsilon - p)) \varepsilon^{-1}w^\varepsilon \geq 0$, and comparison yields $w^\varepsilon \leq -w_{0x}$ on $\mathbb{R} \times (0, \infty)$.

Let $\zeta = \zeta(x)$ be a non-negative smooth function satisfying $\zeta(x) = 0$ for $x < -M$, $\zeta(x) = 1$ for $x > 4 - M$, and $0 \leq \zeta'(x) \leq 1$ and $|\zeta''(x)| \leq 1$ for $x \in [-M, 4 - M]$. In the sequel, for convenience of notation, we consider $\beta$ as a function of $(x, t)$, being evaluated at $\varepsilon^{-1}(w^\varepsilon(x, t) - p(t))$. Using the differential equation and integration by parts, one can derive the identity

$$
\frac{d}{dt} \int_\mathbb{R} \left(\frac{1}{2} w_t^\varepsilon \dot{p} + \dot{p} \dot{\beta}\right) \zeta dx + \int_\mathbb{R} \left(\frac{1}{2} w_{xt}^\varepsilon \zeta + |\varepsilon^{-1}(w_t^\varepsilon - \dot{p})| \beta' \zeta - \frac{1}{2} w_{xx}^\varepsilon \zeta dx dt \right)
= \ddot{p} \int_\mathbb{R} \beta \zeta dx,
$$

so that

$$
\int_\mathbb{R} (w_t^\varepsilon)^2 \zeta dx = \ddot{p} \int_\mathbb{R} \beta \zeta dx - \frac{d}{dt} \int_\mathbb{R} \left(\frac{1}{2} w_t^\varepsilon + \dot{p} \dot{\beta}\right) \zeta dx - \int_\mathbb{R} \left(\varepsilon^{-1}(w_t^\varepsilon - \dot{p})^2 \beta' \zeta - \frac{1}{2} w_{xx}^\varepsilon \zeta dx dt \right).
$$

Multiplying both sides of the above equation by $t^2$, and then integrating over $(0, T)$ gives:

$$
\int_0^T \int_\mathbb{R} (w_t^\varepsilon)^2 \zeta dx \, dt = \int_0^T \int_\mathbb{R} \ddot{p}(t) \beta \zeta dx \, dt - \int_0^T t^2 \frac{d}{dt} \int_\mathbb{R} \left(\frac{1}{2} w_t^\varepsilon + \dot{p} \dot{\beta}\right) \zeta dx \, dt
$$

$$
- \int_0^T \int_\mathbb{R} t^2 \left(\varepsilon^{-1}(w_t^\varepsilon - \dot{p})^2 \beta' \zeta - \frac{1}{2} w_{xx}^\varepsilon \zeta dx dt \right)
\leq \int_0^T \int_\mathbb{R} t \ddot{p}(t) \beta \zeta dx \, dt - \int_0^T t^2 \frac{d}{dt} \int_\mathbb{R} \left(\frac{1}{2} w_t^\varepsilon + \dot{p} \dot{\beta}\right) \zeta dx \, dt + \int_0^T \int_\mathbb{R} \beta' w_{xx}^\varepsilon dx \, dt dt
= I_1 + I_2 + I_3.
$$

To control these terms, we make two preliminary estimates. First, since $w_0(\infty, \cdot) = 0$, there exists $N > 0$ such that $w_0(x, t) < p(t)$ for every $(x, t) \in [N, \infty) \times [0, T]$. Consequently, $\beta \equiv 0$ on $[N, \infty) \times [0, T]$, so that, for every $M > 0$,

$$
\int_{-M}^{\infty} \beta \left(\frac{w^\varepsilon(x, t) - p(t)}{\varepsilon}\right) dx \leq (M + N) m \quad \forall t \in [0, T]. \tag{2.4}
$$

Secondly, integrating $(w_t^\varepsilon + \beta)^2 - [w_t^\varepsilon(w_t^\varepsilon + \beta)]_x + w_{xx}^\varepsilon w_t^\varepsilon = -\varepsilon^{-1} \beta' w_{xx}^\varepsilon \leq 0$ over $\mathbb{R}$ we obtain:

$$
\int_\mathbb{R} (w_t^\varepsilon + \beta)^2 dx + \frac{d}{dt} \int_\mathbb{R} w_{xx}^\varepsilon dx \leq 0.
$$

Integrating this inequality multiplied by $2t$ over $[0, s]$, we get, for every $s \in (0, \infty)$,

$$
2 \int_0^s \int_\mathbb{R} (w_t^\varepsilon + \beta)^2 dx \, dt \, ds + \int_\mathbb{R} w_{xx}^\varepsilon(0, s) \, dx \leq \int_0^s \int_\mathbb{R} w_{xx}^\varepsilon(x, t) \, dx \, dt \, ds
$$

$$
\leq \int_0^s \int_\mathbb{R} w_{0x}^2(x, t) \, dx \, dt \leq \int_0^s \max_{\mathbb{R}} |w_{0x}(\cdot, t)| \int_\mathbb{R} |w_{0x}(x, s)| \, dx \, dt \leq \frac{\sqrt{s}}{\sqrt{\pi}}. \tag{2.5}
$$
where we have used the fact that \( \int_R |w_{0x}(x,t)|dx = 1 \) and \( |w_{0x}(x,t)| = \Gamma \ast u_0 \leq \sup_{z \in R} \Gamma(z,t) = (4\pi t)^{-1}e^{-x^2/4t} \).

Returning to the estimation of \( I_1 \), (2.4) immediately gives:

\[
I_1 = \int_0^T \int_R t^2 \ddot{p}(t) \beta \zeta \, dx \, dt \leq (M + N)m \int_0^T t^2 |\ddot{p}(t)| \, dt. \tag{2.6}
\]

For \( I_2 \), we integrate by parts:

\[
I_2 = -\int_0^T t^2 \frac{d}{dt} \left( \int_R \left( \frac{1}{2} w_t^2 + \dot{p} \beta \right) \zeta \, dx \right) dt
= \int_0^T t \int_R \left( \frac{1}{2} w_t^2 + 2 \dot{p} \beta \right) \zeta \, dx \, dt - t^2 \int_R \left( \frac{1}{2} w_t^2 + \dot{p} \beta \right) \zeta \, dx \bigg|_0^T
\leq \int_0^T t \int_R \left( \frac{1}{2} w_t^2 + 2 \dot{p} \beta \right) \zeta \, dx \, dt + T^2 |\dot{p}(T)| \int_R \beta \zeta \, dx. \tag{2.7}
\]

The second term on the right is bounded by (2.4), and the fact that \( \ddot{p} \in L^1([0,T]) \). Now, using that \( (w_t^2)^2 \leq 2(w_t^2 + \beta)^2 + 2\beta^2 \):

\[
\int_0^T t \int_R \left( \frac{1}{2} w_t^2 + 2 \dot{p} \beta \right) \zeta \, dx \, dt \leq 2 \left( \int_0^T \int_R t(w_t^2 + \beta)^2 \zeta \, dx \, dt + \int_0^T \int_R t \beta(\beta + \dot{p}) \zeta \, dx \, dt \right)
\leq \frac{\sqrt{T}}{\pi} + (M + N)m^2 T^2 2, \tag{2.8}
\]

using \( \dot{p} < 0 \), (2.4), and (2.5).

To control \( I_3 \), we again use \( (w_t^2)^2 \leq 2(w_t^2 + \beta)^2 + 2\beta^2 \), so that:

\[
I_3 = \int_0^T \int_R \frac{1}{2} \ddot{w}_{xx}^2 |\zeta_{xx}| \, dx \, dt \leq \int_0^T \int_R t^2((w_t^2 + \beta)^2 + \beta^2) |\zeta_{xx}| \, dx \, dt. \tag{2.9}
\]

As above:

\[
\int_0^T \int_R t^2 \beta^2 |\zeta_{xx}| \, dx \, dt \leq m^2 T^3 \int_R |\zeta_{xx}| \, dx \leq \frac{2}{3} m^2 T^3. \tag{2.10}
\]

Furthermore, from (2.5), we have:

\[
\int_0^T \int_R t^2(w_t^2 + \beta)^2 \, dx \, dt \leq \frac{T \sqrt{T}}{2 \sqrt{\pi}}. \tag{2.11}
\]

Putting things together, we have that:

\[
\int_0^T \int_{4-M}^\infty t^2 w^2_{xx}(x,t) \, dx \, dt \leq C(M,T),
\]

where \( C(M,T) \) is a constant depending on \( M \) and \( T \). Sending \( \varepsilon \to 0 \) we see that the above estimate also holds for \( w \). Finally, since \( \dot{p} < 0 \) on \([0,T]\), we have \( b \in C([0,T]) \). For each \( \eta \in (0,T) \) taking \( M = 4 + \max_{[\eta,T]} |b| \) we obtain

\[
\int_\eta^T \int_R w^2_{tx}(x,t) \, dx \, dt < \infty.
\]

Since \( \eta \) can be arbitrarily small, we conclude that

\[
\int_R w^2_{tx}(x,t) \, dx < \infty \quad \text{for almost every } t \in (0,T).
\]

This completes the proof. \( \square \)
Remark 2.1. We remark that for \((b, u)\) to be a classical solution of the free boundary problem \((1.3)\), we need the existence of the limit of \(u(x, t)\), as \(x \searrow b(t)\), for each \(t \in (0, T]\). Here the conclusion of the third assertion in the previous lemma is not sufficient for \((b, u)\) to be a classical solution. Thus, from an analytical viewpoint, finding classical solutions of the free boundary problem \((1.3)\) is much harder than that of the free boundary problem \((1.3)\).

Remark 2.2. Taking \(X\) to be Brownian motion with \(X_0 \equiv 0\), we have the following:

1. If \(b(\cdot) \equiv a < 0\), then:

\[
p(t) = 1 - 2 \int_{|a|/\sqrt{t}}^{\infty} \varphi(x) \, dx, \quad \dot{p}(t) = 2 \varphi \left( \frac{|a|}{\sqrt{t}} \right) \cdot \frac{|a|}{t^{3/2}},
\]

where \(\varphi\) is the probability density function of a standard normal random variable. Clearly \((1)\) applies \((\dot{p} \in L^\infty((0, T))\), and \((b, w)\) is a classical solution of \((1.3)\), as can be verified by direct computation. However, \(\lim_{t \to 0} \dot{p}(t) = 0\), so that \((2)\) and \((3)\) do not apply (which is not surprising since otherwise we would have \(b(0) = 0\), contradicting the definition of \(b\)).

2. For the exponential survival function \(p(t) = e^{-\lambda t}\) for some \(\lambda > 0\), \(\dot{p}(t) = -\lambda p(t)\) and \(\ddot{p}(t) = \lambda^2 p(t)\), so that both \((2)\) and \((3)\) apply, and in particular \(b(0) = 0\), and the smoothing property in \((3)\) of the lemma holds.

3 The Free Boundary Condition For Viscosity Solutions

We interpret the free-boundary condition \(\partial_x (\sigma^2 u)|_{x=b(t)+} = -2\dot{p}(t)\) for the viscosity solution as follows.

Lemma 3.1. Let \(p \in P_0\) and \((b, w, u)\) be the unique viscosity solution of the inverse problem associated with \(p\). Suppose \(t > 0\), \(b(t) \in \mathbb{R}\) and \(\dot{p}\) is continuous at \(t\). Then for any function \(\ell \in C^1([0, t])\) that satisfies \(\ell(t) = b(t)\), we have:

\[
\lim_{(x,s) \to (b(t), t)} \sigma^2(x,s)u(x,s) \leq -2\dot{p}(t) \leq \lim_{(x,s) \to (b(t), t)} \sigma^2(x,s)u(x,s) \quad \text{whenever} \quad x \searrow \ell(s).
\]

Remark 3.1. If we know that \(b \in C^1\) and \(\partial_x u \in C(Q_0)\), then taking \(\ell = b\) and using L'Hôpital’s rule we see that the two limits in \((3.1)\) are both equal to \(\partial_x (\sigma^2 u)|_{x=b(t)+}\), so \((3.1)\) provides the free boundary condition \(\partial_x (\sigma^2 u)|_{x=b(t)+} = -2\dot{p}(t)\).

Proof. Suppose the first inequality in \((3.1)\) does not hold. Then the first limit in \((3.1)\) is strictly bigger than \(-2\dot{p}(t)\) so there exist small constants \(m > 0\) and \(\delta \in (0, t]\) such that

\[
u(x,s) \geq (2m - \dot{p}(t))(x - \ell(s)) \quad \forall s \in [t - \delta, t], x \in (\ell(s), b(t) + M\delta]
\]

where \(M = \|\dot{\ell}\|_{L^\infty((0, t))}\). Consequently, for each \(s \in [t - \delta, t]\) and \(x \in [\ell(s), b(t) + M\delta]\),

\[
w(x,s) = p(s) - \int_{-\infty}^{x} u(y,s) \, dy \leq p(s) - \int_{\ell(s)}^{x} u(y,s) \, dy \leq p(s) - (m - \frac{\dot{p}(t)}{2})(x - \ell(s))^2.
\]

Now for any sufficiently small positive \(\varepsilon\), consider the smooth function

\[
\phi_{\varepsilon}(x,s) := p(s) - \frac{1}{2}(m - \dot{p}(t))(x - \ell_{\varepsilon}(s))^2, \quad \ell_{\varepsilon}(s) := \ell(s) - (\varepsilon - (t - s)).
\]
Finally, on the right lateral boundary of $Q_\varepsilon$. We compare $\phi_\varepsilon(b(t), t) - w(b(t), t) = -\frac{1}{2}(m - \dot{p}(t))\varepsilon^2 < 0$.

Next we show that $\phi_\varepsilon - w \geq 0$ on the parabolic boundary of $Q_\varepsilon$. 

$\partial_\nu Q_\varepsilon = \{(x, s) \mid s = t - \varepsilon, x \in [\ell(t - \varepsilon), \ell(t - \varepsilon) + \sqrt{\varepsilon}] \} \cup \{(x, s) \mid s \in (t - \varepsilon, t), x \in [\ell_\varepsilon(s), \ell(s) + \sqrt{\varepsilon}] \}.$

When $x = \ell_\varepsilon(s)$, $\phi_\varepsilon(x, s) = p(s)$ and $\phi_\varepsilon - w \geq 0$. For small enough $\varepsilon$, on the remainder of the parabolic boundary of $Q_\varepsilon$, we can verify that $x \in [\ell(s), b(t) + M\delta]$ so that we can use (3.2) to derive

$$\phi_\varepsilon - w \geq \frac{m}{2}[x - \ell(s)]^2 - \frac{1}{2}[m - \dot{p}(t)][x - \ell_\varepsilon(s)]^2$$

$$= \frac{m}{2}[x - \ell(s)]^2 + \frac{1}{2}[m - \dot{p}(t)] \left[2x - \ell(s) - \ell_\varepsilon(s)\right][\ell_\varepsilon(s) - \ell(s)].$$

On the lower part of the parabolic boundary of $Q_\varepsilon$, we have $s = t - \varepsilon$ so $\ell_\varepsilon(s) = \ell(s)$ and $\phi_\varepsilon \geq w$. Finally, on the right lateral boundary of $Q_\varepsilon$, we have $x = \ell(s) + \sqrt{\varepsilon}$ and $0 \leq \ell(s) - \ell_\varepsilon(s) \leq \varepsilon$ so

$$\phi_\varepsilon - w \geq \frac{m}{2}\varepsilon - \frac{1}{2}[m - \dot{p}(t)][2\sqrt{\varepsilon} + \varepsilon]\varepsilon.$$

Thus, $\phi_\varepsilon - w \geq 0$ on the parabolic boundary of $Q_\varepsilon$ provided that $\varepsilon$ is sufficiently small.

Hence, for every small positive $\varepsilon$, there exists $(x_\varepsilon, t_\varepsilon) \in Q_\varepsilon$ such that $\phi_\varepsilon - w$ attains at $(x_\varepsilon, t_\varepsilon)$ the minimum of $\phi_\varepsilon - w$ over $Q_\overline{\varepsilon}$. Now by the definition of $w$ as a viscosity solution (see, e.g. [Cheng et al. 2000]), we have $\mathcal{L}\phi_\varepsilon(x_\varepsilon, t_\varepsilon) \leq 0$. However, we can calculate (with $\mathcal{L} = \partial_\nu - \partial_\varepsilon^2 + \mu\partial_\ell$)

$$\mathcal{L}\phi_\varepsilon(x_\varepsilon, t_\varepsilon) = \dot{p}(t_\varepsilon) + [m - \dot{p}(t)] [(x_\varepsilon - \ell_\varepsilon(t_\varepsilon))\ell(t_\varepsilon) - 1 - \mu(x_\varepsilon, t_\varepsilon)] + 1$$

$$\geq m + [\dot{p}(t_\varepsilon) - \dot{p}(t)] - [m - \dot{p}(t)] \left[\varepsilon + \varepsilon^2\right][M + 1 + |\mu(x_\varepsilon, t_\varepsilon)|].$$

The last quantity is positive if we take $\varepsilon$ sufficiently small. Thus we obtain a contradiction, and that the first inequality in (3.1) holds.

Now we prove the second inequality in (3.1). Since $u \geq 0$, the second inequality in (3.1) is trivially true when $\dot{p}(t) = 0$. Hence, we consider the case $\dot{p}(t) < 0$. Suppose the second inequality in (3.1) does not hold. Then the second limit in (3.1) is strictly less than $-2\dot{p}(t)$ so there exist small constants $m \in (0, -\dot{p}(t)/2)$ and $\delta \in (0, t]$ such that

$$u(x, s) \leq [-\dot{p}(t) - 2m][x - \ell(s)] \quad \forall t \in [t - \delta, t], \ x \in (\ell(s), \ell(t) + M\delta].$$

Since $u \in C^\infty(Q_b)$ and $u > 0$ in $Q_b$, the above inequality implies that $(\ell(s), s) \notin Q_b$. Hence, we must have $\ell(s) \leq b(s)$, for every $s \in [t - \delta, t]$. Consequently, for each $s \in [t - \delta, t]$ and $x \in [\ell(s), b(t) + M\delta]$, 

$$w(x, s) = p(s) - \int_{-\infty}^x u(y, s)dy = p(s) - \int_{\ell(s)}^x u(y, s)ds \geq p(s) + \frac{1}{2}[\dot{p}(t) + 2m][x - \ell(s)]^2. \quad (3.3)$$

Now for any sufficiently small positive $\varepsilon$, consider the smooth function

$$\psi_\varepsilon(x, s) = p(s) + \frac{1}{2}[\dot{p}(t) + m][x - \ell_\varepsilon(s)]^2, \quad \ell_\varepsilon(s) := \ell(s) + [\varepsilon - (t - s)].$$

We compare $w$ and $\psi_\varepsilon$ in the set

$$Q_\varepsilon := \{(x, s) \mid s \in (t - \varepsilon, t], x \in (\ell(s), \ell(s) + \sqrt{\varepsilon}) \}.$$ We claim that the maximum of $\psi_\varepsilon - w$ on $Q_\overline{\varepsilon}$ is positive and is attained at some point $(x_\varepsilon, t_\varepsilon) \in Q_\varepsilon$. First of all, $(\ell_\varepsilon(t), t) \in Q_\varepsilon$ and $\psi_\varepsilon(\ell_\varepsilon(t), t) - w(\ell_\varepsilon(t), t) = p(t) - w(b(t) + \varepsilon, t) > 0$.  


Next we show that $\psi_z-w \leq 0$ on the parabolic boundary of $Q_x$. On the left lateral boundary of $Q_x$, $x = \ell(s) \leq b(s)$, so $w(x,s) = p(s)$ and $\psi_z-w \leq 0$. For the remainder of the parabolic boundary we can use (3.3) to derive

$$\psi_z-w \leq \frac{1}{2}[\dot{p}(t) + m] [x-\ell(s)]^2 - \frac{1}{2}[\dot{p}(t) + 2m] [x-\ell(s)]^2$$

$$= -\frac{m}{2}[x-\ell(s)]^2 - \frac{1}{2}[\dot{p}(t) + m] [2x-\ell(s)-\ell_z(s)] [\ell_z(s)-\ell_z].$$

On the lower side of the parabolic boundary of $Q_x$, $s = t-\varepsilon$ so $\ell(s) = \ell_x(s)$ and $\psi_z-w \leq 0$. Finally, on the right lateral boundary of $Q_x$, we have $x = \ell(s) + \sqrt{\varepsilon}$ and $0 \leq \ell_z(s) - \ell_z \leq \varepsilon$ so

$$\psi_z-w \leq -m \varepsilon + |\dot{p}(t)| + m |\varepsilon|^{3/2}.$$

Thus, $\psi_z-w \leq 0$ on the parabolic boundary of $Q_x$ if $\varepsilon$ is sufficiently small. Consequently, there exists $(x_{\varepsilon}, t_{\varepsilon}) \in Q_x$ such that $\psi_z-w$ attains at $(x_{\varepsilon}, t_{\varepsilon})$ the positive maximum of $\psi_z-w$ over $Q_x$.

We claim that $x_{\varepsilon} > b(t_{\varepsilon})$. Indeed, if $x_{\varepsilon} \leq b(t_{\varepsilon})$, then $\psi_z(x_{\varepsilon}, t_{\varepsilon}) - w(x_{\varepsilon}, t_{\varepsilon}) = \psi_z(x_{\varepsilon}, t_{\varepsilon}) - p(t_{\varepsilon}) \leq 0$ which contradicts the fact that the maximum of $\psi - w$ on $Q_x$ is positive. Thus $x_{\varepsilon} > b(t_{\varepsilon})$. It then implies that $w$ is smooth in a neighborhood of $(x_{\varepsilon}, t_{\varepsilon})$. Consequently, $L\psi_z(x_{\varepsilon}, t_{\varepsilon}) \geq Lw(x_{\varepsilon}, t_{\varepsilon}) = 0$. However,

$$\mathcal{L}\psi_z(x_{\varepsilon}, t_{\varepsilon}) = \dot{p}(t_{\varepsilon}) + |\dot{p}(t) + m| [(x_{\varepsilon} - \ell_z(t_{\varepsilon}))(\mu(x_{\varepsilon}, t_{\varepsilon}) - \ell_z(t_{\varepsilon}) - 1) - 1]$$

$$\leq -m + |\dot{p}(t_{\varepsilon}) - \dot{p}(t)| + |\dot{p}(t) + m| \sqrt{\varepsilon} (|\mu(x_{\varepsilon}, t_{\varepsilon})| + M + \varepsilon).$$

The last quantity is negative if we take $\varepsilon$ sufficiently small. Thus we obtain a contradiction, and the second inequality in (3.1) holds. This completes the proof. \hfill \Box

4 The Traditional Hodograph Transformation

The hodograph transformation considers the inverse, $x = X(z,t)$, of the function $z = u(x,t)$, so that for each fixed $z$, the curve $x = X(z,t)$ is the $z$-level set of $u$. A bootstrapping procedure is applied. By beginning with a weak regularity assumption on the free boundary $b(t) = X(0,t)$, and applying the regularity theory for the partial differential equation satisfied by $X$, one can obtain higher-order regularity of $b$. In this section, we present two results derived using this traditional approach, one for the viscosity solution and the other for the classical solution of the free boundary problem.

Proposition 3. Let $b$ be the solution of the inverse problem associated with $p \in P_0$. Assume that for some interval $I = (t_1, t_2)$, $b \in C^1(I)$, and $p \in C^{\alpha+1/2}(I)$, where $\alpha$ is not an integer. Then $b \in C^\infty(I)$.

Proof. As $b$ is already $C^1$, we need only consider the case $\alpha > 1$, so $p$ is continuously differentiable. Since $b \in C^1(I)$, by working on the function $U(y,t) := u(b(t) + y, t)$ on the fixed domain $[0, \infty) \times I$ with the boundary condition $U(0,t) = 0$ and then translating the regularity of $U$ back to $u(x,t) = U(x-b(t), t)$, one can show that for every $\gamma \in (0,1)$,

$$u \in C^\infty(Q_b) \cap C^{1+\gamma(1+\gamma)/2}(\{(z,t) \mid t \in I, x \in [b(t), \infty)\}).$$

In addition, since $b \in C^1(I)$ and $u \geq 0$, the Hopf Lemma [Protter and Weinberger 1967, Theorem 3.3, pages 170-171] implies that $u_x(b(t), t) > 0$. Consequently, as $u_x \in C^{\alpha+\gamma/2}(\{(x,t) \mid t \in I, x \in [b(t), \infty)\})$, $b \in C^1(I)$, and $p \in C^{\alpha+1/2}(I)$ with $\alpha > 1$, we can use Lemma 3.1 and Remark 3.1 to derive that $2p(t) = -\sigma^2 u_x|_{x=b(t)+} < 0$.

Once we know the continuity of $u_x$ and the positivity of $u_x(b(t), t)$, we can define the inverse $x = X(z,t)$ of $z = u(x,t)$ for $t \in I$ and $x \in [b(t), b_1(t)]$ where $b_1(t) = \min \{x > b(t) \mid u_x(x,t) = 0\}$. Then implicit differentiation gives:

$$X \in C^\infty, \quad X_t = \dot{X}_z - 2 \dot{X}_{zz} + [z\mu(X,t)]_z, \quad \text{in} \quad \{(z,t) \mid t \in I, z \in (0, u(b_1(t), t))\}.$$
Also, \( X_z \in C^{\gamma, \gamma/2}(D) \) where \( D = \{(z, t) \mid t \in I, z \in [0, u(b_1(t), t)]\} \) and \( X_z(0, t) = -1/\hat{p}(t) \) for \( t \in I \). It then follows from the local regularity theory for parabolic equations (see Ladyzhenskaya et al. \[1968\], Theorem IV.5.3, pages 320-322) that when \( p \in C^{\alpha+1/2}(I) \) where \( \alpha > 1 \) is not an integer, since \( X_z(0, \cdot) = -1/\hat{p}(\cdot) \in C^{\alpha-1/2}(I) \) we have \( X \in C^{2\alpha, \alpha}(D) \). Consequently, \( b = X(0, \cdot) \in C^\alpha(I) \). This completes the proof. \( \square \)

**Proposition 4.** Suppose \( \hat{p} < 0 \) on \((0, \infty)\) and \( p \in C^{\alpha+1/2}((0, \infty)) \) where \( \alpha \geq 1/2 \) is not an integer. Assume that \((b, u)\) is a classical solution of the free boundary problem \((1.3)\) satisfying

\[
\lim_{z \to b(\delta, t)} X_z(z, s) = \lim_{z \to b(\delta, t)} u_x(x, s) = -\frac{2\hat{p}(t)}{\sigma^2(b(t), t)} \quad \forall t > 0.
\]

Then \( b \in C^\alpha((0, \infty)) \).

**Proof.** Let \([\delta, T] \subset (0, \infty)\) be arbitrarily fixed. The conditions \((1.3)\) and \( \hat{p} < 0 \) imply that the set \( \{(b(t), t) \mid t \in [\delta, T]\} \) is compact, so that there exists \( \delta_1 > 0 \) such that \( u_x \) is bounded and uniformly positive in \( D = \{(x, t) \mid t \in [\delta, T], x \in (b(t), b(t) + \delta_1)\} \). Consequently, the inverse \( x = X(z, t) \) of \( z = u(x, t) \) is well-defined for \((x, t) \in D\). Setting \( \varepsilon = \min_{t \in [\delta, T]} u(b(t) + \delta_1, t) \) we have that \( X \in C^\infty((0, \varepsilon) \times [\delta, T]) \), \( X_z \) is uniformly positive and bounded in \((0, \varepsilon) \times [\delta, T]\) and

\[
X_z(0, t) := \lim_{z \to b(\delta, t)} X_z(z, s) = \lim_{z \to b(\delta, t)} u_x(x, s) = -\frac{1}{\hat{p}(t)}.
\]

Since \( X \) satisfies \( X_t = X_z^{-2}X_{zz} + [z\mu]_{zz} \) in \((0, \varepsilon) \times [\delta, T]\), as above local regularity then implies that \( X \) defined on \((0, \varepsilon) \times [\delta, T]\) can be extended onto \([0, \varepsilon) \times [\delta, T]\) such that \( X \in C^{2\alpha, \alpha}((0, \varepsilon) \times [\delta, T]) \). Hence \( b = \lim_{z \to 0} X(z, \cdot) = X(0, \cdot) \in C^\alpha([\delta, T]) \). Sending \( \delta \searrow 0 \) and \( T \to \infty \) we conclude that \( b \in C^\alpha((0, \infty)) \). \( \square \)

## 5 Proof of The Main Result

In this section, we prove our main result, Theorem 1 This provides regularity of the viscosity solution of the inverse problem without the a priori regularity assumptions used in the previous section. We begin with a technical result - a generalization of Hopf’s Lemma in the one-dimensional case that is needed in our later arguments. We then introduce the scaling function \( K \), and the new hodograph transformation for the scaled function \( v = u/K \). Finally, we prove Theorem 1 by analyzing a family of perturbed equations related to the PDE satisfied by \( X(z, t) \).

### 5.1 A Generalization of Hopf’s Lemma in the One-Dimensional Case

In order to apply the weak formulation of the free boundary condition, we need the following extension of the classical Hopf Lemma.

**Lemma 5.1 (Generalized Hopf’s Lemma).** Let \( L = \partial_t - a \partial_{xx} + c \partial_x + d \) where \( a, c \), and \( d \) are bounded functions and \( \inf a > 0 \). Assume that \( L \phi \geq 0 \) in \( Q := \{(x, t) \mid 0 < t < T, l(t) < x < r(t)\} \) where \( l \) and \( r \) are Lipschitz continuous functions. Also assume that \( \phi > 0 \) in \( Q \). Then for every \( \delta \in (0, T) \), there exists \( \eta > 0 \) such that

\[
\phi(x, t) \geq \eta[x - l(t)][r(t) - x] \quad \forall x \in [l(t), r(t)], t \in [\delta, T].
\]

Moreover, if in addition \( \phi(l(T), T) = 0 \), then \( \phi_x(l(T), T) \geq \eta[r(T) - l(T)] \); similarly, if \( \phi(r(T), T) = 0 \), then \( \phi_x(r(T), T) \leq -\eta[r(T) - l(T)] \).
Proof. Without loss of generality, we can assume that $\phi(x,0) > 0$ for all $x \in ([l(0), r(0))]$. By modifying $r$ and $l$ in $[0, \delta]$ by $\tilde{l}(t) = l(t) + \varepsilon[\delta - t]$ and $\tilde{r}(t) = r(t) - \varepsilon[\delta - t]$ we can further assume that $\phi(x,0)$ is uniformly positive on $[l(0), r(0)]$; i.e., there exists $\tilde{\eta} > 0$ such that $\phi(x,0) > \tilde{\eta}$ for all $x \in ([l(0), r(0)])$. Furthermore, by approximating $r$ by smooth functions from below and $l$ by smooth functions from above, with the same Lipschitz constants of $r$ and $l$, we can assume that both $r$ and $l$ are smooth functions.

Now for large positive constants $M$ and $L$ to be determined, consider the function

$$
\psi(x,t) = \tilde{\eta}e^{-Mt-L(x-[r(t)+l(t)])/2}/2 \sin \frac{\pi[x-l(t)]}{r(t)-l(t)}.
$$

Direct calculation gives

$$
\mathcal{L}\psi(x,t) = \tilde{\eta}e^{-Mt-L(x-[r(t)+l(t)])/2}/2 \left\{ A \cos \frac{\pi(x-l)}{r-l} + B \sin \frac{\pi(x-l)}{r-l} \right\}
$$

where

$$
A = \frac{\pi(x-r)l' + \pi(l-x)r'}{(r-l)^2} + 2aL\left( x - \frac{r + l}{2} \right) \frac{\pi}{r-l} + c \pi \frac{r}{r-l},
$$

$$
B = \left\{ -M + L\left( x - \frac{r + l}{2} \right) \left( \frac{r'}{2} + \frac{l'}{2} \right) \right\} + a\left\{ L - L^2\left( x - \frac{l + x}{2} \right)^2 + \frac{\pi^2}{(r-l)^2} \right\} - cL\left( x - \frac{r + l}{2} \right) + d.
$$

First taking $L = 2 \max_{[0,T]}((|r'| + |l'| + \|c\|_{L^\infty})/(r-l))/\inf a$ and then taking a suitably large $M$ we see that $\mathcal{L}\psi \leq 0$. Thus, by comparison, we have $\phi(x,t) \geq \psi(x,t)$ in $Q$ and obtain the assertion of the Lemma. 

\hfill \Box

5.2 Scaling the Survival Density

In making comparison arguments in the proof of Theorem II we would like to have that constant functions are solutions of the partial differential equation under consideration. Unfortunately, while this is true for the operator $\mathcal{L} = \partial_t - \partial_{xx}^2 + \mu \partial_x$, it may fail for $\mathcal{L}_1 = \partial_t - \partial_{xx}^2 + \partial_x \mu$. To resolve this technical difficulty, we introduce a suitable scaling of the function $u$.

To this end, let $K = K(x,t)$ be the bounded solution of the initial value problem:

$$
\mathcal{L}_1 K := \partial_t K - \partial_{xx} K + \partial_x (\mu K) = 0 \quad \text{in} \quad \mathbb{R} \times (0, \infty), \quad K(\cdot, 0) = 1 \quad \text{on} \quad \mathbb{R} \times \{0\}. \quad (5.1)
$$

Then $K$ is smooth, uniformly positive, and bounded in $\mathbb{R} \times [0,T]$ for any $T > 0$. Now write

$$
u(x,t) := K(x,t) \varphi(x,t).
$$

It is easy to verify that

$$
\mathcal{L}_1 v = K \mathcal{L}_2 v,
$$

where

$$
\mathcal{L}_2 := \partial_t - \partial_{xx} + \nu \partial_x, \quad \nu(x,t) := \mu(x,t) - \partial_x \log K^2(x,t).
$$

In particular, note that constant functions $c$ satisfy the equation $\mathcal{L}_2 c = 0$.

5.3 The New Hodograph Transformation

As we have seen, the traditional hodograph transformation is defined as the inverse of $z = u(x,t)$. In order to work with the scaled survival density and the operator $\mathcal{L}_2$ introduced above, we instead define $x = X(z,t)$ as the inverse of $z = v(x,t)$:

$$
z = v(X(z,t), t).
$$
If \( z_0 = v(x_0, t_0) > 0 \) and \( v_x(x_0, t_0) \neq 0 \), then by the Implicit Function Theorem, locally the above equation defines a unique smooth \( X \) for \((z, t)\) near \((z_0, t_0)\) that satisfies \( X(z_0, t_0) = x_0 \). In addition, by implicit differentiation,

\[
v_x = \frac{1}{X_z}, \quad v_t = -\frac{X_t}{X_z}, \quad v_{xx} = -\frac{X_{zz}}{X_z^3}.
\]

Thus, \( \mathcal{L}_z v = 0 \) implies that \( X = X(z, t) \) satisfies the following quasi-linear partial differential equation of parabolic type:

\[
X_t = X_z^{-2} X_{zz} + \nu(X, t).
\]

The free boundary is given by \( b(t) = X(0, t) \) on which we can derive the boundary condition as follows. Since \( u(b(t), t) = 0 \) and \( u_x(b(t), t) = -\dot{p}(t) \), we see that \( v_x(b(t), t) = -\dot{p}(t)/K(b(t), t) \).

Hence, we have the non-linear boundary condition

\[
\dot{p}(t) \ X_0(0, t) + K(X(0, t), t) = 0.
\]

This is a standard boundary condition for the quasi-linear parabolic equation for \( X \).

### 5.4 The Basic Assumption

Let \( p \in P_0 \) and \((b, w, u)\) be the unique viscosity solution of the inverse problem associated with \( p \). To prove Theorem 1, we need only establish the assertions of the theorem in a finite time interval \([0, T]\) for any fixed positive \( T \). Hence, in the sequel, we assume, for some \( T > 0 \) and \( x_0 > 0 \), that

\[
p \in C^1([0, T]), \quad \dot{p} < 0 \text{ on } [0, T], \quad u_0 \in C^1([0, x_0]), \quad u_0' > 0 \text{ on } [0, x_0], \quad u_0 = 0 \text{ on } (-\infty, 0].
\]

These conditions are satisfied by the general assumptions made in Theorem 1 if (i) in (1.6) is imposed. If instead of (i), condition (ii) in (1.6) is imposed, we can apply the third assertion of Lemma 2.1 to shift the initial time by considering first the solution \((\hat{b}, \hat{u}) := (b(s + \cdot) - b(s), u(\cdot + b(s), \cdot + s))\) for \( s \) at which \( w_{x_0}(-., s) \in L^2(\mathbb{R}) \), and then sending \( s \searrow 0 \).

Under (5.2), we will show that \( b \in C^{1/2}((0, T]) \) and that \((b, u)\) is a classical solution of the free boundary problem (1.4) on \( \mathbb{R} \times (0, T] \).

First of all, from Lemma 2.1 (1) and (2), we see that

\[
u \in C(\mathbb{R} \times (0, T]) \cup ((-\infty, x_0) \times \{0\}), \quad b \in C([0, T]), \quad b(0) = 0.
\]

### 5.5 The Level Sets.

The hodograph transformation we are going to use is based on the inverse, \( x = X(z, t) \), where \( v = u/K \). That is, for each \( z \), \( X(z, \cdot) \) is the \( z \)-level set of \( v \). For \( X \) to be well-defined, we need to consider \( v \) in the set where \( v_x \) is positive.

We begin by investigating the initial value \( X_0 = X(\cdot, 0) \) specified by \( z = u_0(X_0(z)) \). Assume that \( u_0 \) satisfies (5.2). Then the function \( z = u_0(x), x \in [0, x_0] \), admits a unique inverse, \( x = X_0(z) \), which satisfies:

\[
X_0 \in C^1([0, u_0(x_0))], \quad z = u_0(X_0(z)), \quad X_0'(z) = \frac{1}{u_0'(X_0(z))}, \quad \forall z \in [0, u_0(x_0)].
\]

Next we consider \( X(\cdot, t) \) for small \( t \). Fix \( \varepsilon \in (0, u_0(x_0)) \). There exists \( \delta_1 > 0 \) such that \( X_0(\varepsilon) + 2\delta_1 < x_0 \) and \( b(s) < X_0(\varepsilon) - 2\delta_1 \) for all \( s \in [0, \delta_1] \). Also, since \( u \in C(\mathbb{R} \times (0, \infty) \cup (-\infty, x_0) \times \{0\}) \), there exists \( \delta_2 \in (0, \delta_1] \) such that \( u < \varepsilon \) on \((-\infty, X_0(\varepsilon) - \delta_1] \times [0, \delta_2] \) and \( u > \varepsilon \) on \([X_0(\varepsilon) + \delta_1] \times [0, \delta_2] \).
Finally, from \( L_2v = 0 \) in \([X_0(\varepsilon) - 2\delta_1, X_0(\varepsilon) + 2\delta_1] \times (0, \delta_x)\) we see that \( v_x \) is continuous and uniformly positive and \( v_t = O(t^{-1/2}) \) on \([X_0(\varepsilon) - \delta_1, X_0(\varepsilon) + \delta_1] \times [0, \delta_x]\) for some \( \delta_3 \in (0, \delta_x) \). Hence for each \( t \in [0, \delta_x] \), the equation \( \varepsilon = v(x, t) \). For \( x \in [X_0(\varepsilon) - \delta_1, X_0(\varepsilon) + \delta_1] \), admits a unique solution which we denoted by \( x = X(\varepsilon, t) \). This solution has the property that \( X(\varepsilon, t) = \min\{x > b(t) \mid v(x, t) = \varepsilon\} \). In addition, \( X_t(\varepsilon, t) = -v_t/v_x = O(t^{-1/2}) \). Hence, \( X(\varepsilon, \cdot) \in C^\infty((0, \delta_x)) \cap C^{1/2}([0, \delta_x]) \).

Now we extend the inverse, \( x = X(z, t) \), of \( z = v(x, t) \) to \((z, t) \in (0, z_0) \times [0, T]\) where \( T \) is an arbitrarily fixed positive constant and \( z_0 \) is a small positive constant that depends on \( T \) and on the viscosity solution \( u \). More precisely, we prove the following.

**Lemma 5.2.** Let \( p \in P_0 \) be given and \((b, u)\) be the unique viscosity solution of the inverse problem associated with \( p \). Assume that \([5.2]\) holds. Let \( K \) be defined in \([5.1]\) and \( v := u/K \).

Then there exists \( z_0 > 0 \) such that the function

\[
X(z, t) := \min\{x \geq b(t) \mid v(x, t) = z\} \quad \forall (z, t) \in [0, z_0] \times [0, T]
\]

is well-defined and satisfies

\[
z = v(X(z, t), t), \quad v_x(X(z, t), t) > 0, \quad \forall (z, t) \in (0, z_0) \times [0, T],
\]

\[
X \in C^\infty((0, z_0) \times (0, T]) \cap C^{1,1/2}((0, z_0) \times [0, T]).
\]

**Proof.** We believe that the idea behind this proof may have appeared previously in the literature, but are not aware of a precise reference. For completeness, and possible other applications, we present a full proof. We consider the level sets of \( v \) in \( \mathbb{R} \times (0, \infty) \). We call \( z \in \mathbb{R} \) a critical value of \( v \) if there exists \((x, t) \in \mathbb{R} \times (0, \infty)\) such that \( v(x, t) = z \) and either \( v_t(x, t) = v_x(x, t) = 0 \) or \( v \) is not differentiable at \((x, t)\). Hence, if \( z > 0 \) is not a critical value, then by the Implicit Function Theorem, the level set \( \{(x, t) \in \mathbb{R} \times (0, \infty) \mid v(x, t) = z\} \) consists of smooth curves, each of which either does not have boundary (i.e., lies completely inside \( Q_b \)) or has boundary on \((0, \infty) \times \{0\} \).

Since \( v \in C(\mathbb{R} \times (0, \infty)) \cap C^\infty(Q_b) \) and \( v(x, t) = 0 \) for all \( x \leq b(t) \), by Sard’s Theorem (e.g. \cite{Guillemin1974}, pages 39-45) the set of all critical values of \( v \) has measure zero.

As before, we denote by \( x = X_0(z) \) the inverse of \( z = u_0(x) \), \( x \in (0, x_0) \).

Since \( b(0) = 0 \) and \( b \in C([0, T]) \), we can find a smooth function \( \ell \in C^\infty([0, T]) \) such that \( \ell(0) = x_0 \) and \( \ell(t) > b(t) \) for all \( t \in [0, T] \). We define

\[
\Gamma := \{(\ell(t), t) \mid t \in [0, T]\}, \quad z_0 := \min_{\Gamma} v = \min_{t \in [0, T]} v(\ell(t), t).
\]

Then \( X \) in \([5.3]\) is well-defined and \( X(z, t) \in (b(t), \ell(t)) \) for all \( t \in [0, T] \) and \( z \in (0, z_0) \). In addition, \( X(0, t) = b(t) \) for all \( t \in [0, T] \) and \( X(z, 0) = X_0(z) \) for all \( z \in (0, z_0) \).

Next, let \( \varepsilon \in (0, z_0) \) be a non-critical value of \( v \) and let \( \gamma_\varepsilon \) be the smooth curve in \( \{(x, t) \mid t > 0, v(x, t) = \varepsilon\} \) that connects to \((X_0(\varepsilon), 0)\). We first claim that \( \gamma_\varepsilon \) is a simple curve, i.e., it cannot form a loop. Indeed, if it forms a loop, then the loop is in \( Q_b \) so the differential equation \( v_t = v_{xx} - v v_x \) in \( Q_b \) implies that \( v = \varepsilon \) inside the loop which is impossible, since \( \varepsilon \) is not a critical value. Hence, \( \gamma_\varepsilon \) is a simple curve. We parameterize \( \gamma_\varepsilon \) by its arc-length parameter, \( s \), in the form \( (x, t) = (x(s), t(s)) \), \( s \in (0, l) \), \( (x(0+), t(0+)) = (X_0(\varepsilon), 0) \). It is not difficult to show that \( \lim_{x(+) \to +\infty} v(x(t), t) = 0 \), so \( \gamma_\varepsilon \) stays in a bounded region and we must have \( l < \infty \) and \( t(l-) = 0, x(l-) > x_0 \). In addition, by the earlier discussion of \( X(\varepsilon, t) \) for small positive \( t \) we see that \( t'(s) > 0 \) and \( x(s) = X(\varepsilon, t(s)) \) for all small positive \( s \).

When \( t(s) > 0 \), we can differentiate \( v(x(s), t(s)) = \varepsilon \) to obtain

\[
v_x(x(s), t(s)) x'(s) + v_t(x(s), t(s)) t'(s) = 0, \quad v_{xx} x'^2 + 2v_x x' t' + v_t t'^2 + v_x x'' + v_t t'' |_{x=x(s)} = 0.
\]

Now we define \( l_0 = \sup\{s \in (0, l) \mid t'(s) > 0 \in (0, s)\} \). Since \( t(l) = 0 \), we must have \( l_0 \in (0, l) \) and \( t'(l_0) = 0 \). Consequently, \( x'(l_0)^2 = 1 - t'(l_0)^2 = 1 \). Evaluating \([5.4]\) at \( s = l_0 \), we obtain
\( v_z(x(l_0), t(l_0)) = 0 \). Consequently, evaluating (5.3) at \( s = l_0 \) and using \( v_t = v_{xx} - \nu v_z \) we obtain 
\[ v_z(x(l_0), t(l_0))[1 + t'(l_0)] = 0. \]
Since \( \varepsilon \) is not a critical value of \( v \), we must have \( v_z(x(l_0), t(l_0)) \neq 0 \), so that \( t''(l_0) = -1 \). This implies that \( t'(s) < 0 \) for all \( s \) bigger than and close to \( l_0 \).

Next we define \( l_1 = \sup \{ s \in (l_0, l) \mid t' < 0 \ in \ (l_0, s) \} \). We claim that \( l_1 = l \). Indeed, suppose \( l_1 < l \). Then we must have \( t(l_1) > 0 \) and \( t'(l_1) = 0 \). As above, first evaluating (5.3) at \( s = l_1 \) we obtain \( v_z(x(l_1), t(l_1)) = 0 \) and then evaluating (5.5) at \( s = l_1 \) and using \( v_t = v_{xx} - \nu v_z \) we get that \( t''(l_1) = -1 \). However, this is impossible since \( t'(l_1) = 0 \) and \( t'(s) < 0 \) for all \( s \in (l_0, l_1) \). Thus, we must have \( l_1 = l \). In summary, we have

\[
\begin{align*}
\varepsilon &= 1 \\
\text{by minimizing} & \text{recalling that} \\
\text{to conclude that} & \text{so that} \\
\text{and/or quantitative studies of the free boundary.} & \text{we omit the details.}
\end{align*}
\]

Remark 5.1. Here we have used the Maximum Principle for \( v_z \). The Maximum Principle may not hold for \( u_2 \) since the equation for \( u_z \) is

\[ (u_z)_t = (u_z)_{xx} - \mu u_{xx} - 2\mu u_{xx} - \mu u_{xx} \]

where the non-homogeneous term \( \mu u_{xx} \) may cause difficulties. Of course, we also need to know that \( v \in C^\infty(Q_0) \cap C(\mathbb{R} \times (0, T) \cup ((-\infty, x_0) \times \{0\})) \) and \( \mathcal{L}_2 v \leq 0 \) on \( \mathbb{R} \times (0, \infty) \) so that the Maximum Principle can be applied to \( v - \varepsilon \).

Using the same idea as in the proof, one can derive the following which maybe useful in qualitative and/or quantitative studies of the free boundary.

**Proposition 5.** Let \( (b, u) \) be the solution of the inverse problem associated with \( p \in P_0 \). Assume that \( p \in C^1((0, \infty)) \) and \( \hat{p} < 0 \ on \ [0, \infty) \). Let \( v = u/K \) where \( K \) is defined in (5.1).

1. Denote by \( N(t) \) the number of roots (without counting multiplicity) of \( v_z(\cdot, t) = 0 \) in \( (b(t), \infty) \).

   Then \( N(t) \) is a decreasing function. In particular, if \( N(0) = 1 \), i.e. \( u_0' \) changes sign only once in \( (b(0), \infty) \), then \( N(t) = 1 \ for all \ t > 0 \); that is, \( v_z(\cdot, t) \) changes sign only once in \( (b(t), \infty) \).

2. Suppose \( u_0 \) is the Delta function (i.e. \( x_0 = 0 \ a.s.) \). Then there exists \( X_1 \in C((0, \infty)) \cap C^\infty((0, \infty)) \) such that \( X_1(0) = 0 \ and \ v_z(\cdot, t) > 0 \ in \ (b(t), X_1(t)) \) and \( v_z(\cdot, t) < 0 \ in \ (X_1(t), \infty) \) for every \( t > 0 \).

The first assertion can be proven by a variation of our proof. The second assertion follows from the fact that the Delta function can be approximated by a sequence of the bell-shaped positive functions, each of which has only one local maximum and no local minimum; see [Chen et al. 2008]. We omit the details.
5.6 The Initial Boundary Value Problem

In the sequel, \( X \) is defined by (5.3) and \( \varepsilon \in (0, z_0) \) is a fixed small positive constant. We consider the quasi-linear parabolic initial boundary value problem, for the unknown function \( Y = Y(z, t) \):

\[
\begin{align*}
Y_t &= Y_z^{-2} Y_{zz} + \nu(Y, t), & z \in (0, \varepsilon), t \in (0, T], \\
Y(z, 0) &= X_0(z), & z \in [0, \varepsilon], \\
\dot{p}(t) Y_z(0, t) + K(Y(0, t), t) &= 0, & t \in (0, T], \\
Y_z(\varepsilon, t) &= X_\varepsilon(\varepsilon, t), & t \in (0, T].
\end{align*}
\]

(5.6)

We know from the theory of quasi-linear equations that this problem admits a unique classical solution for \( \varepsilon \) small enough, which is also smooth up to the boundary, so long as \( X_\varepsilon(\varepsilon, t) \) for \( t \in [0, T] \) and \( X_\varepsilon'(z) \) for \( z \in [0, \varepsilon] \) are uniformly positive. The main difficulty is to show that \( X = Y^h \). We know that \( X \) satisfies all of (5.4), except for the third equation (i.e., we don’t know a priori that \( \dot{p}(t) X_z(0, t) + K(X(0, t), t) = 0, t \in (0, T] \), because we don’t have smoothness of \( X \) up to the boundary).

To do this, we consider for each \( h \in \mathbb{R} \), the following initial boundary value problem, for \( Y^h = Y^h(z, t) \):

\[
\begin{align*}
Y^h_t &= (Y^h_z)^{-2} Y^h_{zz} + \nu(Y^h, t), & z \in (0, \varepsilon), t \in (0, T], \\
Y^h(z, 0) &= X_0(z) + h, & z \in [0, \varepsilon], \\
\dot{p}^h(t) Y^h_z(0, t) + K(Y^h(0, t), t) &= 0, & t \in (0, T], \\
Y^h_z(\varepsilon, t) &= X_\varepsilon(\varepsilon, t), & t \in (0, T],
\end{align*}
\]

(5.7)

where \( \{p^h\}_{h \in \mathbb{R}} \) is a family that has the following properties:

\[
p^h \in P_0 \cap C^\infty([0, T]), \quad \dot{p}^h < 0 \text{ on } [0, T], \quad \lim_{h \to 0} \|p^h - p\|_{C^1([0, T])} = 0.
\]

Note that if \( \mu = 0 \) and \( p^h = p \), then \( \nu = 0 \) and \( K \equiv 1 \) so problem (5.7) relates to (5.6) by the simple translation \( Y^h = Y + h \). Here we shall consider the general case.

5.7 Well-Posedness of the Perturbed Problems

We now show that for each \( h \in \mathbb{R} \), (5.7) admits a unique classical solution. Since (5.6) and (5.7) belong to the same type of initial-boundary value problem, we state our result in terms of (5.6). The conclusion for (5.7) is analogous.

**Lemma 5.3.** Let \( K, \nu \) be smooth and bounded functions on \( \mathbb{R} \times [0, T] \). Assume that \( K > 0 \) and

\[
X_0 \in C^1([0, \varepsilon]), X_0' > 0, \quad p \in C^1([0, T]), \quad \dot{p} < 0, \quad X_\varepsilon(\varepsilon, \cdot) \in C([0, T]), X_\varepsilon(\varepsilon, \cdot) > 0.
\]

Then problem (5.6) admits a unique classical solution that satisfies

\[
Y \in C([0, \varepsilon] \times [0, T]) \cap C^\infty((0, \varepsilon) \times (0, T]), \quad Y_\varepsilon \in C([0, \varepsilon] \times (0, T]),
\]

\[
0 < \inf_{[0,\varepsilon] \times (0,T)} Y_\varepsilon \leq \sup_{[0,\varepsilon] \times (0,T)} Y_\varepsilon < \infty.
\]

If in addition \( p \in C^{\alpha+1/2}((t_1, t_2]) \) where \( \alpha \geq 1/2 \) is not an integer and \( (t_1, t_2] \subset (0, T) \), then \( Y \in C^{2\alpha}(0, \varepsilon) \times (t_1, t_2) \) so that \( Y(0, \cdot) \in C^{\alpha}((t_1, t_2]) \).

If \( p \in C^{\alpha+1/2}([0, T]), X_0 \in C^{2\alpha}([0, \varepsilon]), \alpha \geq 1/2 \) is not an integer, and all compatibility conditions at \( (0, 0) \) up to order \( \lfloor \alpha + 1/2 \rfloor \) are satisfied, then \( Y \in C^{2\alpha}([0, \varepsilon] \times [0, T]) \) so \( Y(0, \cdot) \in C^\alpha([0, T]) \).  

\(^5\)Once we have \( X = Y \), and \( b = Y(0, \cdot) \), it follows from classical theory that \( b \) is only 1/2 less differentiable than \( p \).
Proof. According to the general theory of quasi-linear partial differential equations of parabolic type (see [Ladyzhenskaya et al. 1968]), to show that (5.6) admits a unique classical solution, it suffices to establish an a priori estimate for an upper bound and a positive lower bound for $Y_z$. For this purpose, suppose we have a classical solution $Y$ of (5.6). Then using a local analysis we have $Y \in C^\infty((0, \varepsilon) \times (0, T))$. Set $H = Y_z$. Then we can differentiate the first two equations in (5.6) to obtain:

$$
H_t = -(H^{-1})_{zz} + \nu (Y, t) H, \quad z \in (0, \varepsilon), t \in (0, T],
$$

$$
H(z, 0) = X_0'(z), \quad z \in [0, \varepsilon],
$$

$$
\dot{p}(t) H(0, t) + K(Y(0, t), t) = 0, \quad t \in (0, T],
$$

$$
H(\varepsilon, t) = X_z(\varepsilon, t), \quad t \in (0, T].
$$

Now denote

$$
M_1 := \frac{\|K\|_{L^\infty([0, T]*)}}{\inf_{[0, T]} |\dot{p}|}, \quad m_1 := \frac{\max_{[0, T]} K}{\max_{[0, T]} |\dot{p}|},
$$

$$
M_2 := \max_{[0, x_0]} \frac{1}{u_0(x)}, \quad m_2 := \min_{[0, x_0]} \frac{1}{u_0(x)},
$$

$$
M_3 = \max_{t \in [0, T]} X_z(z, t), \quad m_3 = \min_{t \in [0, T]} X_z(z, t),
$$

$$
M = \max\{M_1, M_2, M_3\}, \quad m = \min\{m_1, m_2, m_3\},
$$

$$
k(t) = \|\nu_z(\cdot, t)\|_{L^\infty(\mathbb{R})}.
$$

Then by comparison, we have

$$
m e^{-\int_0^t k(s)ds} \leq H(z, t) = Y_z(z, t) \leq M e^{\int_0^t k(s)ds} \quad \forall t \in [0, T], z \in [0, \varepsilon].
$$

These a priori upper and lower bounds then imply that (5.6) admits a unique classical solution. The remaining assertions follow from the local and global regularity theory of parabolic equations [Ladyzhenskaya et al. 1968]. This completes the proof.

An example, we demonstrate the derivation of the first order compatibility condition:

$$
\frac{1}{\dot{\nu}^2(0)} = \lim_{\varepsilon \to 0} Y_z(0, t) = \lim_{\varepsilon \to 0} Y_z(z, 0) = \frac{1}{u_0(0)}.
$$

We remark that from our definition of $X$, the compatibility of the initial and boundary data at $(\varepsilon, 0)$ for (5.6) is automatically satisfied. For (5.7), since $K(\cdot, 0) \equiv 1$, the first order compatibility condition at $(\varepsilon, 0)$ is also satisfied, so $Y^h \in C^{1,1/2}([0, \varepsilon] \cup [0, T] \setminus \{(0, 0)\})$.

Finally, to demonstrate continuous dependence, we integrate over $(0, \varepsilon) \times \{t\}, t \in (0, T)$, the difference of the differential equations in (5.6) and (5.7) multiplied by $Y - Y^h$ and use integration by parts to obtain

$$
\frac{1}{2} \frac{d}{dt} \int_0^\varepsilon (Y - Y^h)^2 dz + \int_0^\varepsilon \frac{v(Y, t) - v(Y^h, t)}{Y_z Y^h} dz
$$

$$
= \int_0^\varepsilon (Y - Y^h)[\nu(Y, t) - \nu(Y^h, t)] dz + \left[\frac{\dot{p}^h(t)}{K(Y^h(0, t), t)} - \frac{\dot{p}(t)}{K(Y(0, t), t)}\right] [Y(0, t) - Y^h(0, t)].
$$

Upon using the boundedness of $Y_z$ and $Y_z^h$, Cauchy's inequality, and the Sobolev embedding

$$
\|\phi\|_{L^2_\varepsilon([0, \varepsilon])}^2 \leq \frac{1}{2} \int_0^\varepsilon \phi^2(z) dz + \delta \int_0^\varepsilon \phi^2(z) dz \quad \forall \delta > 0,
$$

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and we find that there exists a positive constant $C$ such that

$$\frac{d}{dt} \int_0^{\varepsilon} (Y - Y^h)^2 dz \leq C \int_0^{\varepsilon} (Y - Y^h)^2 + C|\dot{p}^h(t) - \dot{p}(t)|^2 \quad \forall t \in (0, T].$$

Gronwall’s inequality then yields the estimate

$$\max_{t \in [0, T]} \|Y(\cdot, t) - Y^h(\cdot, t)\|_{L^2([0, \varepsilon])}^2 \leq C e^{CT} \left( h^2 + \int_0^T |\dot{p}^h - \dot{p}|^2 dt \right).$$

This estimate in turn implies the $C([0, T]; L^2((0, \varepsilon)))$ convergence of $Y^h$ to $Y$ as $h \to 0$. By Sobolev embedding and the boundedness of $(Y - Y^h)$, this convergence also implies the $L^\infty$ convergence:

$$\lim_{h \to 0} \|Y^h - Y\|_{C([0, \varepsilon] \times [0, T])} = 0.$$

To show that $b = Y(0, \cdot)$, it suffices to show the following:

**Lemma 5.4.** For every $h > 0$, $Y^{-h}(0, \cdot) < b < Y^h(0, \cdot)$ and $Y^{-h}(\varepsilon, \cdot) < X(\varepsilon, \cdot) < Y^h(\varepsilon, \cdot)$ on $[0, T]$.

The proof will be given in the next three subsections.

### 5.8 The Inverse Hodograph Transformation

To show that $Y = X$ and $b = Y(0, \cdot)$, we define $z = v^h(x, t)$ as the inverse function of $x = Y^h(z, t)$:

$$x = Y^h(z, t), z \in [0, \varepsilon], t \in [0, T] \iff z = v^h(x, t), x \in [Y^h(0, t), Y^h(\varepsilon, t)], t \in [0, T].$$

Since $Y^h_z > 0$, the inverse is well-defined. We record the key equation for future reference:

$$x = Y^h(v^h(x, t), t) \quad \forall t \in [0, T], x \in [Y^h(0, t), Y^h(\varepsilon, t)]. \quad (5.8)$$

By implicit differentiation, we find that

$$v^h_x(x, t) = \frac{1}{Y^h_z(z, t)} \bigg|_{z = v^h(x, t)},$$

$$v^h_{xx}(x, t) = -\frac{Y^h_z(z, t)}{Y^h_z(z, t)} \bigg|_{z = v^h(x, t)},$$

$$v^h_i(x, t) = -\frac{Y^h_i(z, t)}{Y^h_z(z, t)} = -\frac{Y^h_{zz}(z, t) - \nu(Y^h(z, t), t) |_{z = v^h(x, t)}}{Y^h_z(z, t)} = v^h_{xx}(x, t) - \nu(x, t)v^h_z(x, t).$$

Finally, setting $u^h(x, t) = K(x, t)v^h(x, t)$ we see that

$$\mathcal{L}_2 u^h = 0, \quad \mathcal{L}_1 u^h(x, t) = 0, \quad \forall t \in (0, T], x \in (Y^h(0, t), Y^h(\varepsilon, t)).$$

When $t = 0$, using $Y^h(z, 0) = X_0(z) + h$, we see from (5.8) that $x = Y^h(v^h(x, 0), 0) = X_0(v^h(x, 0)) + h$. This implies that $x - h = X_0(v^h(x, 0))$, so using $\hat{x} = X_0(u_0(\hat{x}))$ with $\hat{x} = x - h$ we obtain $v^h(x, 0) = u_0(x - h)$ or

$$u^h(x, 0) = v^h(x, 0) = u_0(x - h), \quad \forall x \in [Y^h(0, 0), Y^h(\varepsilon, 0)] = [h, X_0(\varepsilon) + h].$$

Next, substituting $x = Y^h(\varepsilon, t)$ in (5.8) we obtain

$$v^h(Y^h(\varepsilon, t)) = \varepsilon = v(X(\varepsilon, t), t).$$
The boundary condition $Y^h(\varepsilon, t) = X_\varepsilon(\varepsilon, t) = 1/v_x(X(\varepsilon, t), t)$ then implies that
\[ v^h_\varepsilon(Y^h(\varepsilon, t), t) = v_x(X(\varepsilon, t), t). \]
Finally, substituting $x = Y^h(0, t)$ in \( \{5.8\} \) we have
\[ v^h(Y^h(0, t), t) = 0 = v(b(t), t). \]

The boundary condition $\dot{p}^h(t) Y^h(0, t) + K(Y^h(0, t), t) = 0$ then gives
\[ v^h_\varepsilon(Y^h(0, t), t) K(Y^h(0, t), t) = -\dot{p}^h(t). \]

In summary, we see that $v^h = v^h(x, t)$ has the following properties:
\[
\begin{cases}
  \mathcal{L}_2v^h = 0 & \forall t \in (0, T], x \in (Y^h(0, 0), Y^h(\varepsilon, 0)), \\
  v^h(x, 0) = u_0(x - h), & \forall x \in [Y^h(0, 0), Y^h(\varepsilon, 0)], \\
  v^h(Y^h(0, t), t) = 0 = v(b(t), t), & \forall t \in [0, T], \\
  v^h_\varepsilon(Y^h(0, t), t) K(Y^h(0, t), t) = -\dot{p}^h(t), & \forall t \in [0, T], \\
  v^h(Y^h(\varepsilon, t), t) = \varepsilon = v(X(\varepsilon, t), t), & \forall t \in [0, T], \\
  v^h_\varepsilon(Y^h(\varepsilon, t), t) = v_x(X(\varepsilon, t), t), & \forall t \in [0, T].
\end{cases}
\]

Note that $[Y(0, 0), Y^h(\varepsilon, 0)] = [h, X(0) + h]$.

### 5.9 The Proof that $b < Y^h(0, \cdot)$ and $X(\varepsilon, \cdot) < Y^h(\varepsilon, \cdot)$ for $h > 0$

Let $h > 0$ be arbitrarily fixed. We define
\[ T^* := \sup \{t \in [0, T] \mid b < Y^h(0, \cdot) \text{ and } X(\varepsilon, \cdot) < Y^h(\varepsilon, \cdot) \text{ in } [0, t]\}. \]

We know that $X(\varepsilon, \cdot), Y^h(0, \cdot)$ and $Y^h(\varepsilon, \cdot)$ are all continuous and that $b$ is upper-semi-continuous \cite{Chen2011}. Also, $Y^h(0, 0) = h, b(0+0) < 0, \text{ and } Y^h(\varepsilon, 0) = X(0(\varepsilon, 0) + h$. Thus, we have that $T^*$ is well-defined and $T^* \in (0, T]$.

We claim that $b < Y^h(0, \cdot)$ and $X(\varepsilon, \cdot) < Y^h(\varepsilon, \cdot)$ on $[0, T]$. Suppose the claim is not true. Then we must have have $b(t) < Y^h(0, t)$ and $X(\varepsilon, t) < Y^h(\varepsilon, t)$ for all $t \in [0, T^*)$ and

either (i) $X(\varepsilon, T^*) = Y^h(\varepsilon, T^*)$, or (ii) $b(T^*) = Y^h(0, T^*)$.

We shall show that neither of the above can happen. For this we compare $v$ and $v^h$ in the set $Q := \{(x, t) \mid t \in (t_0, T^*], Y^h(0, t) < x < X(\varepsilon, t)\}$, where
\[ t_0 := \inf\{t \in [0, T^*] \mid Y^h(0, t) < X(\varepsilon, t) \text{ in } [t, T^*]\}. \]

Since $b(t) < Y^h(0, t)$ for all $t \in [0, T^*)$, we have $\mathcal{L}_2v = \mathcal{L}_2v^h = 0$ in $Q$. Also, on the left lateral parabolic boundary of $Q$, $x = Y^h(0, t)$, we have $v \geq 0 = v^h$. On the right lateral boundary of $Q$, $x = X(\varepsilon, t)$, we have $v^h \leq \varepsilon = v$. Thus $v \geq v^h$ on the parabolic boundary of $Q$ if $t_0 > 0$. Finally, if $t_0 = 0$, we have $v(x, 0) = u_0(x) > u_0(x - h) = v^h(x, 0)$ for all $x \in [Y^h(0, 0), X(\varepsilon, 0)] = [h, X(0\varepsilon)]$. Thus, $v \geq v^h$ on the parabolic boundary of $Q$. Since $h > 0$, we cannot have $v \equiv v^h$, so the Strong Maximum Principle implies that $v > v^h$ in $Q$.

Now consider case (i): $X(\varepsilon, T^*) = Y^h(\varepsilon, T^*)$. Then as $X(\varepsilon, \cdot)$ is smooth and $v(X(\varepsilon, T^*), T^*) = \varepsilon = v^h(Y^h(\varepsilon, T^*), T^*) = v^h(X(\varepsilon, T^*), T^*)$, the Hopf Lemma implies that $v_x(X(\varepsilon, T^*), t) < v^h_x(X(\varepsilon, T^*), T^*)$. 

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This is impossible since \( v_x^h(X(\varepsilon, T^*), T^*) = v_x^h(Y^h(\varepsilon, T^*), T^*) = v_x(X(\varepsilon, T^*), T^*) \). Thus case (i) cannot happen.

Next, we consider case (ii): \( b(T^*) = Y^h(0, T^*) \). Since \( Y^h(0, \cdot) \) is smooth, the generalized Hopf Lemma implies that there exist \( \eta > 0 \) and \( \delta > 0 \) such that
\[
v(x, s) - v^h(x, s) \geq [x - Y^h(0, s)] \eta, \quad \forall s \in [T^* - \delta, T^*], x \in [Y^h(0, s), b(T^*) + \delta].
\]
However, since \( v_x^h(Y^h(0, s), s)K(Y^h(0, s), t) = -\hat{p}^h(s) \geq -\hat{p}(s) \), the above inequality implies
\[
\lim_{x \to Y^h(0, s), s \to T^*} \frac{u(x, s)}{x - Y^h(0, s)} \geq -\hat{p}(T^*) + \eta K(b(T^*), T^*).
\]
But this contradicts the first inequality in \( \text{(3.1)} \) with \( \ell := Y^h(0, \cdot) \) of Lemma \( \text{(3.1)} \). Hence, case (ii) also cannot happen.

In conclusion, when \( h > 0 \), we must have \( b < Y^h(0, \cdot) \) and \( X(\varepsilon, \cdot) < Y^h(\varepsilon, \cdot) \) on \([0, T]\).

### 5.10 The Proof that \( Y^h(0, \cdot) < b \) and \( Y^h(\varepsilon, \cdot) < X(\varepsilon, \cdot) \) for \( h < 0 \)

Here, we use the facts that \( b(0) = 0 \) and \( b \) is continuous on \([0, T]\), proven in [Chen et al., 2011], and repeated here as Proposition [11]. Also, we need the fact that \( \mathcal{L}_2v^h \leq 0 \) in \( \mathbb{R} \times (0, \infty) \).

Let \( h < 0 \) be arbitrary. We define
\[
T^* := \sup \{ t \in [0, T] \mid Y^h(0, \cdot) < b \text{ and } Y^h(\varepsilon, \cdot) < X(\varepsilon, \cdot) \text{ on } [0, t] \}.
\]
Since \( b, X(\varepsilon, \cdot), Y^h(0, \cdot), Y^h(\varepsilon, \cdot) \) are all continuous and \( Y^h(0, 0) = h < 0 = b(0) \) and \( Y^h(\varepsilon, 0) = X(\varepsilon, 0) + h \), we see that \( T^* \) is well-defined and \( T^* \in (0, T] \).

We claim that \( Y^h(0, \cdot) < b \) and \( Y^h(\varepsilon, \cdot) < X(\varepsilon, \cdot) \) on \([0, T]\). Suppose the claim is not true. Then \( Y(0, t) < b(t) \) and \( Y^h(\varepsilon, t) < X(\varepsilon, t) \) for all \( t \in [0, T^*] \) and
\[
\text{either (i) } X(\varepsilon, T^*) = Y^h(\varepsilon, T^*), \quad \text{or (ii) } b(T^*) = Y^h(0, T^*).
\]
To show that none of the above can happen, we compare \( v \) and \( v^h \) as before in the set
\[
Q := \{ (x, t) \mid t \in (0, T^*], Y^h(0, t) < x < Y^h(\varepsilon, t) \}.
\]
Then, by the definition of \( T^* \), we have \( v(x, t) \leq v^h(x, t) \) on the parabolic boundary of \( Q \) and \( \mathcal{L}_2v^h = 0 \geq \mathcal{L}_2v \) in \( Q \). The Maximum Principle then implies that \( v < v^h \) in \( Q \).

Now consider case (i): \( X(\varepsilon, T^*) = Y^h(\varepsilon, T^*) \). Then as \( Y^h(\varepsilon, \cdot) \) is smooth and \( v(X(\varepsilon, T^*), T^*) = \varepsilon = v^h(X(\varepsilon, T^*), T^*) \), the Hopf Lemma implies that \( v_x(X(\varepsilon, T^*), T^*) > v_x^h(X(\varepsilon, T^*), T^*) \). This is impossible since \( v_x^h(X(\varepsilon, T^*), T^*) = v_x(Y^h(\varepsilon, T^*), T^*) = v_x(X(\varepsilon, T^*), T^*) \). Thus case (i) cannot happen.

Next we consider case (ii): \( b(T^*) = Y^h(0, T^*) \). Again, since \( Y^h(0, \cdot) \) is smooth, the generalized Hopf Lemma implies that there exist \( \eta > 0 \) and \( \delta > 0 \) such that
\[
v^h(x, s) - v(x, s) \geq [x - Y^h(0, s)] \eta, \quad \forall s \in [T^* - \delta, T^*], x \in [Y^h(0, s), b(T^*) + \delta].
\]
However, since \( v_x^h(Y^h(0, s), s)K(Y^h(0, s), s) = -\hat{p}^h(s) \leq -\hat{p}(s) \), the above inequality implies
\[
\lim_{x \to Y^h(0, s), s \to T^*} \frac{u(x, s)}{x - Y^h(0, s)} \leq -\hat{p}(T^*) - \eta K(b(T^*), t).
\]
This contradicts the second inequality in \( \text{(3.1)} \) with \( \ell := Y^h(0, \cdot) \) of Lemma \( \text{(3.1)} \). Hence, case (ii) also cannot happen. Thus, when \( h < 0 \), we have \( Y^h(0, \cdot) < b \) and \( Y^h(\varepsilon, \cdot) < X(\varepsilon, \cdot) \) on \([0, T]\).
5.11 Proof of Theorem [1]

Once we know that $Y^{-h}(0, t) < b(t) < Y^h(0, t)$ for $h > 0$ and $t \in [0, T]$, we can send $h \to 0$ to conclude that $b(t) = Y(0, t)$. Consequently, $b = Y(0, \cdot) \in C^{1/2}([0, T])$.

As we know $\dot{b} \in C([0, T])$, one can show that $Y_z \in C([0, \varepsilon] \times (0, T])$, so

$$\lim_{z \to 0, s \to t} Y_z(z, s) = Y_z(0, t) = -\frac{1}{K(b(t), t)\dot{b}(t)} \quad \forall t \in (0, T].$$

Using $X = Y$ and $u_x(Y(z, t), t) = 1/[K(Y(z, t), t)Y_z(z, t)]$, we then obtain

$$u_x(b(t)+, t) := \lim_{x > b(s), (x, s) \to (b(t), t)} u_x(x, s) = -\dot{b}(t) \quad \forall t \in (0, T].$$

Thus, $(b, u)$ is a classical solution of (1.4) on $\mathbb{R} \times [0, T]$. If, in addition, $p \in C^{\alpha+1/2}((t_1, t_2))$ for some $\alpha > 1/2$ that is not an integer, then by local regularity, $b = Y(0, \cdot) \in C^{\alpha}((t_1, t_2))$. If we further have $u_0 \in C^{2\alpha}([0, x_0])$, $p \in C^{\alpha+1/2}([0, T])$ for some $\alpha \geq 1/2$ that is not an integer, and all compatibility conditions up to the order $[\alpha + 1/2]$ are satisfied, then $Y \in C^{2\alpha, \alpha}([0, \varepsilon] \times [0, T])$ so $b = Y(0, \cdot) \in C^{\alpha}([0, T])$.

Finally, upon noting that $T$ can be arbitrarily large, we also obtain the assertion of Theorem [1] which completes the proof of this theorem.

6 Conclusion

In earlier work, we studied the inverse first-passage problem for a one-dimensional diffusion process by relating it to a variational inequality. We investigated existence and uniqueness, as well as the asymptotic behaviour of the boundary for small times, and weak regularity of the boundary. In this paper, we studied higher-order regularity of the free boundary in the inverse first-passage problem. The main tool used was the hodograph transformation. The traditional approach to the transformation begins with some a priori regularity assumptions, and then uses a bootstrap argument to obtain higher regularity. We presented the results of this approach, but then went further, studying the regularity of the free boundary under weaker assumptions. In order to do so, we needed to perform the hodograph transformation on a carefully chosen scaling of the survival density, and to analyze the behaviour of a related family of quasi-linear parabolic equations. We expect that the method presented here can be applied to other parabolic obstacle problems.

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