Estimates for parameters and characteristics of the confining SU(3)-gluonic field in pions and kaons

Yu. P. Goncharov

Theoretical Group, Experimental Physics Department, State Polytechnical University, Sankt-Petersburg 195251, Russia

Abstract

The estimates of parameters for the classical SU(3)-gluonic field responsible for linear confinement are given for pions and kaons. Also estimates for the characteristics of the mentioned field such as gluon concentrations, electric and magnetic colour field strengths are adduced along with those for quark velocities in mesons under discussion. Possible connection of the obtained results with potential approach and string-like picture of confinement is outlined as well.

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1 Introduction and preliminary remarks

In Ref. [1] for the Dirac-Yang-Mills system derived from QCD-Lagrangian there was found a family of compatible nonperturbative solutions which could pretend to describing confinement of two quarks. Further study of this family as well as its applications to the quarkonia spectra (charmonium and bottomonium) in Refs. [2,3] showed that the above family is in essence unique one [4] and allows one to modify gluon propagator in a nonperturbative way so that the modified propagator could correspond to linear confinement at large distances [5].

Two main physical reasons for linear confinement in the mechanism under discussion are the following ones. The first one is that gluon exchange between quarks is realized with the propagator different from the photon one and existence of such a propagator is direct consequence of the unique confining nonperturbative solutions of the Yang-Mills equations [4]. The second reason
is that, owing to the structure of mentioned propagator, gluon condensate (a classical gluon field) between quarks mainly consists of soft gluons (for more details see [4,5]) but, because of that any gluon also emits gluons (still softer), the corresponding gluon concentrations rapidly become huge and form the linear confining magnetic colour field of enormous strengths which leads to confinement of quarks. Under the circumstances physically nonlinearity of the Yang-Mills equations effectively vanishes so the latter possess the unique nonperturbative confining solutions of the abelian-like form (with the values in Cartan subalgebra of SU(3)-Lie algebra) [4] that describe the gluon condensate under consideration. Moreover, since the overwhelming majority of gluons are soft they cannot leave hadron (meson) until some gluon obtains additional energy (due to an external reason) to rush out. So we deal with confinement of gluons as well.

The approach under discussion equips us with the explicit wave functions that is practically unreachable in other approaches, for example, within framework of lattice theories or potential approach. Namely, for each two quarks (meson) there exists its own set of real constants (for more details see below) $a_j, A_j, b_j, B_j$ parametrizing the mentioned nonperturbative confining gluon field (the gluon condensate) and the corresponding wave functions (nonperturbative modulo square integrable solutions of the Dirac equation in this confining SU(3)-field) while the latter ones also depend on $\mu_0$, the reduced mass of the current masses of quarks forming meson. It is clear that constants $a_j, A_j, b_j, B_j, \mu_0$ should be extracted from experimental data. This circumstance gives possibilities for direct physical modelling of internal structure for any meson and for checking such relativistic models numerically.

So far all applications of the confinement mechanism under consideration have been restricted to the quarkonia [2–5]. The aim of the present paper is to estimate the above parameters for the case of pions and kaons. Of course, when conducting our considerations we should rely on the standard quark model (SQM) based on SU(3)-flavor symmetry (see, e. g., Ref. [6] or the oldies [8]).

Further we shall deal with the metric of the flat Minkowski spacetime $M$ that we write down (using the ordinary set of local spherical coordinates $r, \vartheta, \varphi$ for the spatial part) in the form

$$ds^2 = g_{\mu\nu}dx^\mu \otimes dx^\nu \equiv dt^2 - dr^2 - r^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2). \quad (1)$$

Besides, we have $|\delta| = |\det (g_{\mu\nu})| = (r^2 \sin \vartheta)^2$ and $0 \leq r < \infty$, $0 \leq \vartheta < \pi$, $0 \leq \varphi < 2\pi$.

Throughout the paper we employ the Heaviside-Lorentz system of units with $\hbar = c = 1$, unless explicitly stated otherwise, so the gauge coupling constant $g$ and the strong coupling constant $\alpha_s$ are connected by relation $g^2/(4\pi) = \alpha_s$. 
Further we shall denote $L_2(F)$ the set of the modulo square integrable complex functions on any manifold $F$ furnished with an integration measure, then $L_2^n(F)$ will be the $n$-fold direct product of $L_2(F)$ endowed with the obvious scalar product while $\dagger$ and $\ast$ stand, respectively, for Hermitian and complex conjugation. Our choice of Dirac $\gamma$-matrices conforms to the so-called standard representation and is

$$
\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^b = \begin{pmatrix} 0 & \sigma_b \\ -\sigma_b & 0 \end{pmatrix}, \quad b = 1, 2, 3, \quad \alpha = \gamma^0\gamma = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix},
$$

where $\sigma_b$ denote the ordinary Pauli matrices and $\sigma = \sigma_1i + \sigma_2j + \sigma_3k$. At last $\otimes$ means tensorial product of matrices and $I_3$ is the unit $3 \times 3$ matrix.

When calculating we apply the relations $1 \text{ GeV}^{-1} \approx 0.1973269679 \text{ fm}$, $1 \text{ s}^{-1} \approx 0.658211915 \times 10^{-24} \text{ GeV}$, $1 \text{ V/m} \approx 0.2309956375 \times 10^{-23} \text{ GeV}^2$, $1 \text{ T} \approx 0.6925075988 \times 10^{-15} \text{ GeV}^2$.

Finally, for the necessary estimates we shall employ the $T_{00}$-component (volumetric energy density) of the energy-momentum tensor for a SU(3)-Yang-Mills field which should be written in the chosen system of units in the form

$$
T_{\mu\nu} = -F^a_{\mu\gamma} F^a_{\nu\beta} g^{\alpha\beta} + \frac{1}{4} F^a_{\beta\gamma} F^a_{\alpha\delta} g^{\alpha\beta} g^{\gamma\delta} g_{\mu\nu}.
$$

**2 Main relations**

As was mentioned above, our considerations shall be based on the unique family of compatible nonperturbative solutions for the Dirac-Yang-Mills system (derived from QCD-Lagrangian) studied in details in Refs. [1,4,5]. Referring for more details to those references, let us briefly describe only the relations necessary to us in the present Letter following the notations from [5].

One part of the mentioned family is presented by the unique nonperturbative confining solution of the Yang-Mills equations and looks as follows

$$
A^3_t + \frac{1}{\sqrt{3}} A^8_t = -\frac{a_1}{r} + A_1, -A^3_t + \frac{1}{\sqrt{3}} A^8_t = -\frac{a_2}{r} + A_2, -\frac{2}{\sqrt{3}} A^8_t = \frac{a_1 + a_2}{r} - (A_1 + A_2),
$$

$$
A^3_\varphi + \frac{1}{\sqrt{3}} A^8_\varphi = b_1 r + B_1, -A^3_\varphi + \frac{1}{\sqrt{3}} A^8_\varphi = b_2 r + B_2, -\frac{2}{\sqrt{3}} A^8_\varphi = -(b_1 + b_2) r - (B_1 + B_2)
$$

with the real constants $a_j, A_j, b_j, B_j$ parametrizing the family.

Another part of the family is given by the unique nonperturbative modulo square integrable solutions of the Dirac equation in the confining SU(3)-field of
(4) \( \Psi = (\Psi_1, \Psi_2, \Psi_3) \) with the four-dimensional Dirac spinors \( \Psi_j \) representing the \( j \)th colour component of the meson, which may describe the relativistic bound states of two quarks (mesons) and look as follows (with Pauli matrix \( \sigma_1 \))

\[
\Psi_j = e^{i\omega_j t} r^{-1} \left( \begin{array}{c} F_{j1}(r) \Phi_j(\vartheta, \varphi) \\ F_{j2}(r) \sigma_1 \Phi_j(\vartheta, \varphi) \end{array} \right), \quad j = 1, 2, 3
\]

with the 2D eigenspinor \( \Phi_j = (\Phi_{j1}, \Phi_{j2}) \) of the euclidean Dirac operator \( D_0 \) on the unit sphere \( S^2 \), while the coordinate \( r \) stands for the distance between quarks. The explicit form of \( \Phi_j \) is not needed here and can be found in Ref. [5,7]. For the purpose of the present Letter we shall adduce the necessary spinors below. Spinors \( \Phi_j \) form an orthonormal basis in \( L_2^2(S^2) \).

The energy spectrum of a meson is given by \( \omega = \omega_1 + \omega_2 + \omega_3 \) with

\[
\omega_j = \omega_j(n_j, l_j, \lambda_j) = g A_j +
\]

\[
-\Lambda_j g^2 a_j b_j \pm (n_j + \alpha_j) \sqrt{(n_j^2 + 2n_j \alpha_j + \Lambda_j^2) \mu_0^2 + g^2 b_j^2 (n_j^2 + 2n_j \alpha_j)}
\]

\[
\frac{\mu_0^2}{n_j^2 + 2n_j \alpha_j + \Lambda_j^2}, \quad j = 1, 2, 3,
\]

where \( g \) is the gauge coupling constant, \( a_3 = -(a_1 + a_2), b_3 = -(b_1 + b_2), A_3 = -(A_1 + A_2), B_3 = -(B_1 + B_2), \Lambda_j = \lambda_j - g B_j, \alpha_j = \sqrt{\Lambda_j^2 - g^2 a_j^2}, n_j = 0, 1, 2, ..., \) while \( \lambda_j = \pm (l_j + 1) \) are the eigenvalues of euclidean Dirac operator \( D_0 \) on unit sphere with \( l_j = 0, 1, 2, ..., \mu_0 \) is a mass parameter and one can consider it to be the reduced mass which is equal, e. g. for quarkonia, to half the current mass of quarks forming a quarkonium. As is clear from (6), parameters \( A_{1,2} \) of solution (4) only shift the origin of count for the corresponding energies and we can consider \( A_1 = A_2 = 0 \).

Within the given Letter we need only the radial parts of (5) at \( n_j = 0 \) that are

\[
F_{j1} = C_j P_j r^{\alpha_j} e^{-\beta_j r} \left( 1 - \frac{gb_j}{\beta_j} \right), P_j = gb_j + \beta_j,
\]

\[
F_{j2} = iC_j Q_j r^{\alpha_j} e^{-\beta_j r} \left( 1 + \frac{gb_j}{\beta_j} \right), Q_j = \mu_0 + \omega_j
\]

with \( \beta_j = \sqrt{\mu_0^2 - \omega_j^2 + g^2 b_j^2} \) while \( C_j \) is determined from the normalization condition \( \int_0^\infty (|F_{j1}|^2 + |F_{j2}|^2) dr = \frac{1}{3} \). Consequently, we shall gain that \( \Psi_j \in L_2^2(\mathbb{R}^3) \) at any \( t \in \mathbb{R} \) and, as a result, the solutions of (5) may describe relativistic bound states (mesons) with the energy (mass) spectrum (6).

It is useful to obtain the nonrelativistic limit (when \( c \to \infty \)) for spectrum (6). For that one should replace \( g \to g/\sqrt{\hbar c}, a_j \to a_j/\sqrt{\hbar c}, b_j \to b_j/\sqrt{\hbar c}, \)
$B_j \to B_j/\sqrt{\hbar c}$ and, expanding (6) in $z = 1/c$, we shall get

$$\omega_j(n_j, l_j, \lambda_j) = \pm \mu_0 c^2 \left[ 1 \mp \frac{g^2 a_j^2}{2\hbar^2 (n_j + |\lambda_j|)^2} \right] - \frac{\lambda g^2 a_j b_j}{\hbar (n_j + |\lambda_j|)^2} \pm \mu_0 g^3 B_j a_j^2 f(n_j, \lambda_j) \right] z + O(z^2),$$

where $f(n_j, \lambda_j) = 4\lambda_j n_j (n_j^2 + \lambda_j^2) + \frac{|\lambda_j|}{\lambda_j^2} \left( n_j^4 + 6n_j^2 \lambda_j^2 + \lambda_j^4 \right)$.

As is seen from (8), at $c \to \infty$ the contribution of linear magnetic colour field (parameters $b_j, B_j$) to spectrum really vanishes and spectrum in essence becomes purely Coulomb one (modulo the rest energy). Also it is clear that when $n_j \to \infty$, $\omega_j \to \pm \sqrt{\mu_0^2 + g^2 b_j^2}$.

We may seemingly use (6) with various combinations of signs ($\pm$) before second summand in numerators of (6) but, due to (8), it is reasonable to take all signs equal to + which is our choice within the Letter. Besides, as is not complicated to see, radial parts in nonrelativistic limit have the behaviour of form $F_{j1}, F_{j2} \sim r^{l_j}$, which allows one to call quantum number $l_j$ angular momentum for $j$th colour component though angular momentum is not conserved in the field (4) [1,5]. So for mesons under consideration we should put all $l_j = 0$.

Finally it should be noted that spectrum (6) is degenerated owing to degeneracy of eigenvalues for the euclidean Dirac operator $D_0$ on the unit sphere $S^2$. Namely, each eigenvalue of $D_0 \lambda = \pm (l + 1), l = 0, 1, 2, ...$, has multiplicity $2(l + 1)$ so we has $2(l + 1)$ eigenspinors orthogonal to each other. Ad referendum we need eigenspinors corresponding to $\lambda = \pm 1$ ($l = 0$) so here is their explicit form

$$\lambda = -1 : \Phi = \frac{C}{2} \left( \begin{array}{c} e^{i\varphi} \\ e^{-i\varphi/2} \end{array} \right) e^{i\varphi/2}, \text{ or } \Phi = \frac{C}{2} \left( \begin{array}{c} e^{i\varphi} \\ -e^{-i\varphi/2} \end{array} \right) e^{-i\varphi/2},$$

$$\lambda = 1 : \Phi = \frac{C}{2} \left( \begin{array}{c} e^{-i\varphi} \\ e^{i\varphi/2} \end{array} \right) e^{i\varphi/2}, \text{ or } \Phi = \frac{C}{2} \left( \begin{array}{c} e^{-i\varphi} \\ e^{i\varphi/2} \end{array} \right) e^{-i\varphi/2}$$

with the coefficient $C = 1/\sqrt{2\pi}$ (for more details see Refs. [5,7]).

### 3 Estimates for parameters of SU(3)-gluonic field

Within the present Letter we shall use relations (6) at $n_j = 0 = l_j$ so energy (mass) of mesons under consideration is given by
\[ \mu = \sum_{j=1}^{3} \omega_j(0,0,\lambda_j) = \sum_{j=1}^{3} \left( \frac{-g^2 a_j b_j}{\Lambda_j} + \frac{\alpha_j \mu_0}{|\Lambda_j|} \right) \]  

(10)

and, as a consequence, the corresponding meson wave functions of (5) are represented by (7), (9).

### 3.1 Choice of quark masses and gauge coupling constant

It is evident for employing the above relations we have to assign some values to quark masses and gauge coupling constant \( g \). In accordance with Ref. [6], at present the current quark masses are restricted to intervals \( 1.5 \text{ MeV} \leq m_u \leq 5 \text{ MeV}, 3.0 \text{ MeV} \leq m_d \leq 9 \text{ MeV}, 60 \text{ MeV} \leq m_s \leq 170 \text{ MeV}, \) so we take \( m_u = (1.5 + 5)/2 \text{ MeV} = 3.25 \text{ MeV}, m_d = (3 + 9)/2 \text{ MeV} = 6 \text{ MeV}, m_s = (60 + 170)/2 \text{ MeV} = 115 \text{ MeV}. \) Under the circumstances, the reduced mass \( \mu_0 \) of Table 1 will respectively take values \( m_u/2, m_d/2, m_u m_d/(m_u + m_d), m_u m_s/(m_u + m_s), m_d m_s/(m_d + m_s) \). As to gauge coupling constant \( g = \sqrt{4\pi \alpha_s} \), at present little is known about the values of the strong coupling constant \( \alpha_s = \alpha_s(Q^2) \) at the momentum transfer \( Q \to 0 \). We use the results of recent works [9] on extracting \( \alpha_s \) from the so-called Bjorken sum rule, wherefrom one can conclude that \( \alpha_s \to \pi = 3.1415... \) when \( Q \to 0 \), i.e., \( g \to 2\pi = 6.2831... \). An extrapolation of the results of [9] to the mass scales of mesons under discussion gives rise to the values of \( g \) adduced in Table 1.

### 3.2 Electric formfactor

For each meson with the wave function \( \Psi = (\Psi_j) \) of (5) we can define electromagnetic current \( J^\mu = \Psi(\gamma^\mu \otimes I_3)\Psi = (\Psi^\dagger \Psi, \Psi^\dagger (\alpha \otimes I_3)\Psi) = (\rho, J) \). Electric formfactor \( f(K) \) is the Fourier transform of \( \rho \)

\[
f(K) = \int \Psi^\dagger \Psi e^{-iK r} d^3x = \sum_{j=1}^{3} \int \Psi_j^\dagger \Psi_j e^{-iK r} d^3x = \sum_{j=1}^{3} f_j(K) =
\]

\[
\sum_{j=1}^{3} \int (|F_{j1}|^2 + |F_{j2}|^2) \Phi_j^\dagger \Phi_j \frac{e^{-iK r}}{r^2} d^3x, \quad d^3x = r^2 \sin \vartheta dr d\vartheta d\varphi
\]

(11)

with the momentum transfer \( K \). We can consider vector \( K \) to be directed along z-axis. Then \( Kr = Kr \cos \vartheta \) and at \( n_j = 0 = l_j \), as is easily seen, for any spinor of (9) we have \( \Phi_j^\dagger \Phi_j = 1/(4\pi) \), so with the help of (7) and relations (see Ref. [10]): \( \int_0^\infty r^\alpha e^{-pr} dr = \Gamma(\alpha) p^{-\alpha}, \) Re \( \alpha, p > 0 \), \( \int_0^\infty r^{\alpha-1} e^{-pr} \sin (Kr) dr = \Gamma(\alpha)(K^2 + p^2)^{-\alpha/2} \sin \arctan (K/p), \) Re \( \alpha > -1, \) Re \( p > |\text{Im} K|, \) \( \Gamma(\alpha + 1) =$
\( \alpha \Gamma (\alpha), \ f_0^\pi e^{-iKr \cos \vartheta} \sin \vartheta \, d\vartheta = 2 \sin (Kr)/(Kr) \), we shall obtain

\[
f(K) = \sum_{j=1}^{3} f_j(K) = \sum_{j=1}^{3} \left( \frac{(2\beta_j)^{2\alpha_j+1}}{6\alpha_j} \cdot \frac{\sin \left[ 2\alpha_j \arctan \left( K/(2\beta_j) \right) \right]}{K(2 + 4\beta_j^2)^{\alpha_j}} \right)
\]

\[
= \sum_{j=1}^{3} \left( \frac{1}{3} - \frac{2\alpha_j^2 + 3\alpha_j + 1}{6\beta_j^2} \cdot \frac{K^2}{6} \right) + O(K^4), \tag{12}
\]

wherefrom it is clear that \( f(K) \) is a function of \( K^2 \), as should be, and we can determine the root-mean-square radius of meson in the form

\[
<r> = \sqrt{\sum_{j=1}^{3} \frac{2\alpha_j^2 + 3\alpha_j + 1}{6\beta_j^2}}. \tag{13}
\]

Of course, we can directly calculate \( <r> \) in accordance with the standard quantum mechanics rules as

\[
<r> = \sqrt{\int r^2 \Psi^\dagger \Psi \, d^3x} = \sqrt{\sum_{j=1}^{3} \int r^2 \Psi_j^\dagger \Psi_j \, d^3x}
\]

and the result will be the same as in (13). Therefore, we should not call \( <r> \) of (13) the charge radius of meson – it is just the radius of meson determined by the wave functions of (5) (at \( n_j = 0 = l_j \)) with respect to strong interaction.

### 3.3 Magnetic moment

We can define the volumetric magnetic moment density by \( m = q(r \times J)/2 = q[(yJ_z - zJ_y)i + (zJ_x - xJ_z)j + (xJ_y - yJ_z)k]/2 \) with the meson charge \( q \) and \( J = \Psi^\dagger(\alpha \otimes I_3)\Psi \). Using (5) we have in the explicit form

\[
J_x = \sum_{j=1}^{3} \int \left( F_{j1}^* F_{j2} + F_{j2}^* F_{j1} \right) \frac{\Phi_j^\dagger \Phi_j}{r^2} \, d^3x, \quad J_y = \sum_{j=1}^{3} \int \left( F_{j1}^* F_{j2} - F_{j2}^* F_{j1} \right) \frac{\Phi_j^\dagger \sigma_2 \sigma_1 \Phi_j}{r^2} \, d^3x,
\]

\[
J_z = \sum_{j=1}^{3} \int \left( F_{j1}^* F_{j2} - F_{j2}^* F_{j1} \right) \frac{\Phi_j^\dagger \sigma_3 \sigma_1 \Phi_j}{r^2} \, d^3x. \tag{14}
\]

Magnetic moment of meson is \( M = \int_V m \, d^3x \), where \( V \) is volume of meson. Then at \( l_j = 0 \) we have \( J_y = 0 \) for any spinor of (9), while \( \int_V m_{x,y,z} \, d^3x = 0 \) because of turning to zero either integral over \( \vartheta \) or the one over \( \varphi \), which is easily to check. As a result, magnetic moments of mesons under consideration with the wave functions of (5) (at \( l_j = 0 \)) are equal to zero, as should be according to experimental data [6].
3.4 Numerical results

We employed relations (10) and (13) for obtaining estimates of the confining SU(3)-gluonic field parameters in mesons under discussion and, to impose more restrictions, we considered in (10) each $\omega_j = \mu/3$, $\mu$ is meson mass, though it is not obligatory. Also the experimental estimates, if any, of $< r >$ were used from Refs. [6,11]. The results are adduced in Tables 1, 2. It should be noted that in accordance with SQM $\pi^0 = (\bar{u}u - \bar{d}d)/\sqrt{2}$ is equiprobable superposition of two quarkonia, consequently, we have two set of parameters and, to reach orthogonality for $\bar{u}u$ and $\bar{d}d$ states, we should assign different spinors of (9) at $\lambda = -1$ to the corresponding wave functions of $\pi^0$. Besides, according to SQM the all $\pi$-meson states are orthogonal to each other and, to reach it, we should assign different spinors of (9) at $\lambda = 1$ to the corresponding wave functions of $\pi^\pm$. Analogous situation is for $K$-meson sector.

Table 1
Gauge coupling constant, mass parameter $\mu_0$ and parameters of the confining SU(3)-gluonic field for pions and kaons.

| Particle | $g$ | $\mu_0$ (MeV) | $a_1$ | $a_2$ | $b_1$ (GeV) | $b_2$ (GeV) | $B_1$ | $B_2$ |
|----------|-----|---------------|-------|-------|-------------|-------------|-------|-------|
| $\pi^0 - \bar{u}u$ | 6.2816 | 1.625 | -0.0124758 | -0.00630731 | -0.275416 | 0.211701 | 0.3385 | -0.3526 |
| $\pi^0 - \bar{d}d$ | 6.2816 | 3.00 | 0.0121524 | 0.00614414 | 0.273783 | -0.210436 | 0.3385 | -0.3526 |
| $\pi^+ - \bar{u}d$, $\pi d$ | 6.2745 | 2.1081 | 0.00651124 | 0.0128925 | 0.21007 | -0.273365 | 0.3526 | -0.3385 |
| $K^+ - \bar{u}s$, $\bar{s}u$ | 6.1256 | 3.16068 | 0.0348355 | 0.0523979 | 0.524585 | -0.366167 | 0.5303 | -0.8914 |
| $K_0^+$, $\bar{K}^0 - d\bar{s}$ | 6.11497 | 5.70248 | 0.0494856 | 0.0240874 | 0.104398 | -0.263159 | 0.36052 | -0.784 |

Table 2
Theoretical and experimental meson masses and radii

| Particle | Theoret. (MeV) | Experim. (MeV) | Theoret. $< r >$ (fm) | Experim. $< r >$ (fm) |
|----------|---------------|---------------|----------------------|----------------------|
| $\pi^0 - \bar{u}u$ | $\mu = \omega_1(0,0,-1) + \omega_2(0,0,-1) + \omega_3(0,0,-1) = 134.976$ | 134.976 | 0.602594 | – |
| $\pi^0 - \bar{d}d$ | $\mu = \omega_1(0,0,-1) + \omega_2(0,0,-1) + \omega_3(0,0,-1) = 134.976$ | 134.976 | 0.605026 | – |
| $\pi^+ - \bar{u}d$, $\pi d$ | $\mu = \omega_1(0,0,1) + \omega_2(0,0,1) + \omega_3(0,0,1) = 139.570$ | 139.56995 | 0.607418 | 0.6050 |
| $K^+ - \bar{u}s$, $\bar{s}u$ | $\mu = \omega_1(0,0,-1) + \omega_2(0,0,-1) + \omega_3(0,0,-1) = 493.677$ | 493.677 | 0.564046 | 0.560 |
| $K_0^+$, $\bar{K}^0 - d\bar{s}$ | $\mu = \omega_1(0,0,1) + \omega_2(0,0,1) + \omega_3(0,0,1) = 497.672$ | 497.672 | 0.560964 | – |

4 Estimates of gluon concentrations, electric and magnetic colour field strengths

To obtain further physical characteristics of the confining SU(3)-gluonic field in pions and kaons let us remind that, according to Refs. [3–5], we can confront
the field (4) with the 3-dimensional SU(3)-Lie algebra valued 1-forms of electric $E$ and magnetic $H$ colour fields and also with $T_{00}$-component (volumetric energy density) of the energy-momentum tensor (3)

$$E = \left[\lambda_3(a_1 - a_2) + \lambda_8(a_1 + a_2)\sqrt{3}\right] \frac{dr}{2r^2}, \quad H = -\left[\lambda_3(b_1 - b_2) + \lambda_8(b_1 + b_2)\sqrt{3}\right] \frac{d\vartheta}{2\sin \vartheta},$$

$$T_{00} = T_{tt} = \frac{1}{2} \left( \frac{a_1^2 + a_1 a_2 + a_2^2}{r^4} + \frac{b_1^2 + b_1 b_2 + b_2^2}{r^2 \sin^2 \vartheta} \right) = \frac{A}{r^4} + \frac{B}{r^2 \sin^2 \vartheta},$$

with real $A > 0, B > 0$. When defining the scalar product for the SU(3)-Lie algebra valued 1-forms $A = A^a_\mu dx^\mu, B = B^b_\nu dx^\nu$ by $G(A, B) = \frac{1}{2} g^{\mu\nu} A^a_\mu B^b_\nu \text{Tr}(\lambda_a \lambda_b)$, we shall have electric and magnetic field strengths modulo equal to

$$E = \sqrt{G(E, E)} = \frac{\sqrt{a_1^2 + a_1 a_2 + a_2^2}}{r^2}, \quad H = \sqrt{G(H, H)} = \frac{\sqrt{b_1^2 + b_1 b_2 + b_2^2}}{r \sin \vartheta},$$

so that $T_{00} = [G(E, E) + G(H, H)]/2 = (E^2 + H^2)/2$.

To estimate the gluon concentrations we can employ $T_{00}$-component of (16) and, taking the quantity $\omega = \Gamma$, the whole decay width of a meson, for the characteristic frequency of gluons we obtain the sought characteristic concentration $n$ in the form

$$n = \frac{T_{00}}{\Gamma},$$

so we can rewrite (16) in the form $T_{00} = T_{00}^{\text{coul}} + T_{00}^{\text{lin}}$ conforming to the contributions from the Coulomb and linear parts of the solution (4). The latter gives the corresponding split of $n$ from (18) as $n = n_{\text{coul}} + n_{\text{lin}}$.

The parameters of Tables 1, 2 were employed when computing and for simplicity we put $\sin \vartheta = 1$ in (16)–(17). Also there were used the following present-day whole decay widths of mesons under consideration [6]: $\Gamma = 1/\tau$ with the life times $\tau = 8.4 \times 10^{-17}$ s, $2.6033 \times 10^{-8}$ s, $1.2386 \times 10^{-8}$ s, $0.8953 \times 10^{-10}$ s ($K_0^0$-mode), $5.18 \times 10^{-8}$ s ($K_L^0$-mode), respectively, whereas the Bohr radius $a_0 = 0.529177249 \cdot 10^5$ fm [6]. At last, as has been discussed in Refs. [3,5], we can estimate the quark velocities in the mesons under exploration from the condition

$$v_q = \frac{1}{\sqrt{1 + (\frac{\lambda_B}{\lambda_q})^2}}$$

with the quark Compton wavelength $\lambda_q = 1/m_q$ while we take the quark de Broglie wavelength $\lambda_B = 0.1 r_0$ with $r_0 = < r >$ from Table 2.

Tables 3, 4 contain the numerical results for $n_{\text{coul}}, n_{\text{lin}}, n, E, H, v_q$ for the mesons under discussion.
Table 3
Gluon concentrations, electric and magnetic colour field strengths in pions.

| $\pi^0$—$\bar{u}u$: $r_0 = < r > = 0.602594$ fm, $v_u = 0.999504$ |
|----------------|----------------|----------------|----------------|
| $r$ (fm) | $n_{\text{coul}}$ (m$^{-3}$) | $n_{\text{lim}}$ (m$^{-3}$) | $n$ (m$^{-3}$) | $E$ (V/m) | $H$ (T) |
| 0.1$r_0$ | $0.172647 \times 10^{57}$ | $0.142633 \times 10^{57}$ | $0.315280 \times 10^{57}$ | $0.768576 \times 10^{23}$ | $0.118089 \times 10^{18}$ |
| $r_0$ | $0.172647 \times 10^{53}$ | $0.142633 \times 10^{55}$ | $0.144359 \times 10^{55}$ | $0.768576 \times 10^{21}$ | $0.118089 \times 10^{15}$ |
| 1.0 | $0.217646 \times 10^{52}$ | $0.142633 \times 10^{54}$ | $0.520203 \times 10^{54}$ | $0.279085 \times 10^{21}$ | $0.711597 \times 10^{14}$ |
| 10$r_0$ | $0.172647 \times 10^{49}$ | $0.142633 \times 10^{53}$ | $0.142650 \times 10^{53}$ | $0.768576 \times 10^{19}$ | $0.118089 \times 10^{14}$ |

| $\pi^0$—$\bar{d}d$: $r_0 = < r > = 0.606236$ fm, $v_d = 0.999080$ |
|----------------|----------------|----------------|----------------|
| $r$ (fm) | $n_{\text{coul}}$ (m$^{-3}$) | $n_{\text{lim}}$ (m$^{-3}$) | $n$ (m$^{-3}$) | $E$ (V/m) | $H$ (T) |
| 0.1$r_0$ | $0.159916 \times 10^{57}$ | $0.139255 \times 10^{57}$ | $0.299171 \times 10^{57}$ | $0.739697 \times 10^{23}$ | $0.116682 \times 10^{18}$ |
| $r_0$ | $0.159916 \times 10^{53}$ | $0.139255 \times 10^{55}$ | $0.140854 \times 10^{55}$ | $0.739697 \times 10^{21}$ | $0.116682 \times 10^{15}$ |
| 1.0 | $0.216003 \times 10^{52}$ | $0.139255 \times 10^{54}$ | $0.513951 \times 10^{54}$ | $0.271855 \times 10^{21}$ | $0.707369 \times 10^{14}$ |
| 10$r_0$ | $0.159916 \times 10^{49}$ | $0.139255 \times 10^{53}$ | $0.139270 \times 10^{53}$ | $0.739697 \times 10^{19}$ | $0.116682 \times 10^{14}$ |

| $\pi^\pm$—$\bar{d}$, $\bar{u}$: $r_0 = < r > = 0.607418$ fm, $v_u = 0.999500$, $v_d = 0.999078$ |
|----------------|----------------|----------------|----------------|
| $r$ (fm) | $n_{\text{coul}}$ (m$^{-3}$) | $n_{\text{lim}}$ (m$^{-3}$) | $n$ (m$^{-3}$) | $E$ (V/m) | $H$ (T) |
| 0.1$r_0$ | $0.553136 \times 10^{65}$ | $0.428537 \times 10^{65}$ | $0.981673 \times 10^{65}$ | $0.781444 \times 10^{23}$ | $0.116271 \times 10^{16}$ |
| $r_0$ | $0.553136 \times 10^{61}$ | $0.428537 \times 10^{61}$ | $0.434069 \times 10^{61}$ | $0.781444 \times 10^{21}$ | $0.116271 \times 10^{15}$ |
| 1.0 | $0.752978 \times 10^{60}$ | $0.158412 \times 10^{63}$ | $0.158855 \times 10^{63}$ | $0.288327 \times 10^{21}$ | $0.706251 \times 10^{14}$ |
| 10$r_0$ | $0.553136 \times 10^{57}$ | $0.428537 \times 10^{61}$ | $0.428593 \times 10^{61}$ | $0.781444 \times 10^{19}$ | $0.116271 \times 10^{14}$ |

5 Discussion and concluding remarks

5.1 Discussion

As is seen from Tables 3, 4, at the characteristic scales of each meson the gluon concentrations are large and the corresponding fields (electric and mag-
Table 4
Gluon concentrations, electric and magnetic colour field strengths in kaons.

| $K^{±}, u\pi, \pi$ | $r_0 = < r > = 0.564046$ fm, $v_u = 0.999536$, $v_s = 0.983958$ |
|---------------------|---------------------------------------------------------------|
| $r$ (fm)            | $n_{\text{coul}}$ | $n_{\text{lin}}$ | $n$ | $E$  | $H$  |
| 0.1$r_0$           | $0.699799 \times 10^{66}$ | $0.835931 \times 10^{65}$ | $0.783392 \times 10^{66}$ | $0.402965 \times 10^{24}$ | $0.235429 \times 10^{16}$ |
| $r_0$              | $0.699799 \times 10^{66}$ | $0.835931 \times 10^{65}$ | $0.905911 \times 10^{63}$ | $0.402965 \times 10^{22}$ | $0.235429 \times 10^{15}$ |
| 1.0                | $0.708323 \times 10^{61}$ | $0.265956 \times 10^{63}$ | $0.273033 \times 10^{63}$ | $0.128203 \times 10^{22}$ | $0.132793 \times 10^{15}$ |
| 10$r_0$            | $0.699799 \times 10^{58}$ | $0.835931 \times 10^{61}$ | $0.836631 \times 10^{61}$ | $0.402965 \times 10^{20}$ | $0.235429 \times 10^{14}$ |
| $s_0$              | $0.903288 \times 10^{42}$ | $0.949724 \times 10^{53}$ | $0.457820 \times 10^{12}$ | $0.259942 \times 10^{10}$ |

$K^0, \bar{K}^0, \pi^\pm, \pi^0$ ($K^0$-mode): $r_0 = < r > = 0.560964$ fm, $v_d = 0.999148$, $v_s = 0.984044$

| $r$ (fm)            | $n_{\text{coul}}$ | $n_{\text{lin}}$ | $n$ | $E$  | $H$  |
| 0.1$r_0$           | $0.377302 \times 10^{64}$ | $0.148175 \times 10^{63}$ | $0.392120 \times 10^{64}$ | $0.348022 \times 10^{24}$ | $0.116585 \times 10^{16}$ |
| $r_0$              | $0.377302 \times 10^{60}$ | $0.148175 \times 10^{61}$ | $0.185065 \times 10^{61}$ | $0.348022 \times 10^{22}$ | $0.116585 \times 10^{15}$ |
| 1.0                | $0.376830 \times 10^{59}$ | $0.466279 \times 10^{60}$ | $0.503641 \times 10^{60}$ | $0.109516 \times 10^{22}$ | $0.654000 \times 10^{14}$ |
| 10$r_0$            | $0.377302 \times 10^{56}$ | $0.148175 \times 10^{59}$ | $0.148553 \times 10^{59}$ | $0.348022 \times 10^{20}$ | $0.116585 \times 10^{14}$ |
| $s_0$              | $0.476458 \times 10^{40}$ | $0.166511 \times 10^{51}$ | $0.166511 \times 10^{51}$ | $0.391088 \times 10^{12}$ | $0.123588 \times 10^{10}$ |

$K^0, \bar{K}^0, \pi^\pm, \pi^0$ ($K^0$-mode): $r_0 = < r > = 0.560964$ fm, $v_d = 0.999148$, $v_s = 0.984044$

| $r$ (fm)            | $n_{\text{coul}}$ | $n_{\text{lin}}$ | $n$ | $E$  | $H$  |
| 0.1$r_0$           | $0.218299 \times 10^{67}$ | $0.857308 \times 10^{65}$ | $0.226872 \times 10^{67}$ | $0.348022 \times 10^{24}$ | $0.116585 \times 10^{16}$ |
| $r_0$              | $0.218299 \times 10^{63}$ | $0.857308 \times 10^{63}$ | $0.107561 \times 10^{64}$ | $0.348022 \times 10^{22}$ | $0.116585 \times 10^{15}$ |
| 1.0                | $0.216168 \times 10^{62}$ | $0.269778 \times 10^{63}$ | $0.291395 \times 10^{63}$ | $0.109516 \times 10^{22}$ | $0.654000 \times 10^{14}$ |
| 10$r_0$            | $0.218299 \times 10^{59}$ | $0.857308 \times 10^{61}$ | $0.859491 \times 10^{61}$ | $0.348022 \times 10^{20}$ | $0.116585 \times 10^{14}$ |
| $s_0$              | $0.275668 \times 10^{43}$ | $0.963395 \times 10^{53}$ | $0.963395 \times 10^{53}$ | $0.391088 \times 10^{12}$ | $0.123588 \times 10^{10}$ |

Gluon concentrations (for magnetic colour ones) can be considered to be the classical ones with enormous stregnths. The part $n_{\text{coul}}$ of gluon concentration $n$ connected with the Coulomb electric colour field is decreasing faster than $n_{\text{lin}}$, the part of $n$ related to the linear magnetic colour field, and at large distances $n_{\text{lin}}$ becomes dominant while quarks in mesons under investigation should be considered the ultrarelativistic point-like particles. It should be emphasized that in fact the gluon concentrations are much greater than the estimates given in Tables 3, 4 because the latter are the estimates for maximal possible gluon frequencies, i.e.
for maximal possible gluon impulses (under the concrete situation of pions and kaons). The latter also explains why gluon concentrations are much larger (about 8 orders of magnitude) for charged pions compared to $\pi^0$-meson: just the corresponding life times are different by the same orders so, accordingly, the conforming maximal gluon impulses are in inverse relation.

The given picture is in concordance with the one obtained when considering charmonium in Refs. [2,3,5]. As a result, the confinement mechanism developed in Refs. [1,4,5] is confirmed by the considerations of the present Letter.

It should be noted, however, that our results are of a preliminary character which is readily apparent, for example, from that the current quark masses (as well as the gauge coupling constant $g$) used in computation are known only within the certain limits and we can expect similar limits for the magnitudes discussed in the Letter so it is necessary further specification of the parameters for the confining SU(3)-gluonic field in pions and kaons which can be obtained, for instance, by calculating decay constants and weak formfactors for the given mesons with the help of wave functions discussed above. Also one can obtain the analogous estimates for vector mesons, for instance, by computing the widths of radiative decays for them and so on. We hope to continue analysing other problems of meson spectroscopy elsewhere.

5.2 Connection with potential approach and string-like picture

The results obtained above allow us to shed some light on the following two problems. As is known, during a long time up to now in meson spectroscopy one often uses the so-called potential approach (see, e. g., Refs. [12,13] and references therein). The essence of it is in that the interaction between quarks is modelled on a nonrelativistic confining potential in the form $a/R + kR + c_0$ with some real constants $a, k, c_0$ and the distance between quarks $R$. On the other hand, also for a long time there exists the so-called string-like picture of quark confinement but only at qualitative level (see, e. g., book of Perkins in [8]). Up to now, however, it is unknown as such considerations might be warranted from the point of view of QCD. Let us in short outline as our results (based on and derived from QCD-Lagrangian directly) naturally lead to possible justification of the mentioned directions. Thereto we note that one can calculate energy $E$ of gluon condensate conforming to solution (4) in a volume $V$ through relation $E = \int_V T_{00} r^2 \sin \vartheta dr d\vartheta d\varphi$ with $T_{00}$ of (16)–(17) but one should take into account that classical $T_{00}$ has a singularity along $z$-axis ($\vartheta = 0, \pi$) and we have to introduce some angle $\vartheta_0$ (whose physical meaning is to be clarified a little below) so $\vartheta_0 \leq \vartheta \leq \pi - \vartheta_0$. Then let us choose $V$ as shown in Fig. 1, i. e. the one between two concentric spheres with radii $R_0 < R$ restricted to interior of cone $\vartheta = \vartheta_0$. 

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Without going into details (see also Ref. [4]) we shall obtain

\[ E = E(R) = E_0 + \frac{a}{R} + kR, \]

(20)

where \( E_0 = \frac{4\pi A \cos \vartheta_0}{R_0} - 2\pi BR_0 \ln \frac{1 + \cos \vartheta_0}{1 - \cos \vartheta_0} \), \( a = -4\pi A \cos \vartheta_0 \), \( k = 2\pi B \ln \frac{1 + \cos \vartheta_0}{1 - \cos \vartheta_0} \) with constants \( A, B \) of (16). We recognize the mentioned confining potential in (20) when identifying \( E_0 = c_0 \) and we can see that phenomenological parameters \( a, k, c_0 \) of potential are expressed through more fundamental parameters connected with the unique exact solution (4) of Yang-Mills equations describing confinement. One can notice that the quantity \( k \) (string tension) is usually related to the so-called Regge slope \( \alpha' = 1/(2\pi k) \) and in many if not all of the papers using potential approach it is accepted \( k \approx 0.18 \) GeV\(^2\), \( E_0 = c_0 \approx -0.873 \) GeV (see, e.g., Refs. [12,13]). Also one often uses parametrization \( a = -4\alpha_s(R_0)/3 \), where \( \alpha_s(R_0) \) is the strong coupling constant at \( R_0 \) so when \( R < R_0 \) potential description is not applicable. If using (20) and the results obtained in Table 1 we can in series compute \( \vartheta_0, \alpha_s(R_0), R_0 \) for all mesons under discussion and also for the ground state of charmonium \( \eta_c(1S) \) for that we use the parametrization from Refs. [3–5] with replacing \( a_i \rightarrow a_i/\sqrt{4\pi}, b_i \rightarrow b_i/\sqrt{4\pi} \) since the system of units in those references is different from the one used here. Results of computation are presented in Table 5.

Table 5
Parameters determining the confining potential for pions, kaons and charmonium ground state.

| Particle        | \( \vartheta_0 \) | \( \alpha_s(R_0) \) | \( R_0 \) (fm) |
|-----------------|-------------------|---------------------|---------------|
| \( \pi^0 \rightarrow uu \) | 64.56°          | 0.676 \times 10^{-3} | 0.757         |
| \( \pi^0 \rightarrow dd \) | 64.276°         | 0.532 \times 10^{-3} | 0.957         |
| \( \pi^± \rightarrow ud, \overline{ud} \) | 64.199°         | 0.600 \times 10^{-3} | 0.957         |
| \( K^± \rightarrow u\overline{s}, \overline{u}s \) | 82.464°         | 0.357 \times 10^{-2} | 0.958         |
| \( K^0, \overline{K}^0 \rightarrow d\overline{s}, \overline{d}s \) | 60.272°         | 0.986 \times 10^{-2} | 0.960         |
| \( \eta_c(1S) \rightarrow c\overline{c} \) | 89.934°         | 0.163 \times 10^{-2} | 0.957         |

As well as in Ref. [4], we may consider \( \vartheta_0 \) to be a parameter determining some cone \( \vartheta = \vartheta_0 \) so the quark emits gluons outside of the cone and, generally speaking, the angle \( \vartheta_0 \) is increasing with quark mass. We can see that though a potential could be associated with each meson but the potential is invalid at the meson characteristic scales which follows from Table 2. Moreover, as was shown in Refs. [4,5], potential of form (20) cannot be a solution of the Yang-Mills equations if simultaneously \( a \neq 0, k \neq 0 \). Therefore, it is impossible to obtain compatible solutions of the Yang-Mills-Dirac (Pauli, Schrödinger)
Fig. 2. Formation of string-like picture between quarks

system when inserting potential of form (20) into Dirac (Pauli, Schrödinger) equation. So, we draw the conclusion (mentioned as far back as in Refs. [2]) that the potential approach seems to be inconsistent.

Now if there are two quarks $Q_1, Q_2$ and each of them emits gluons outside of its own cone $\vartheta = \vartheta_{1,2}$ (see Fig. 2) then we have soft gluons (as mentioned in Section 1) in regions I, II and between quarks so a characteristic transverse size $D$ of the gluon condensate is decreasing with increasing quark masses as we just now saw. For heavy quarks the gluon configuration between them practically transforms into a string. As a result, there arises the string-like picture of quark confinement but the latter seems to be warranted enough only for heavy quarks. It should be emphasized that string tension $k$ is determined just by parameters $b_{1,2}$ of linear magnetic colour field from solution (4) [see (20)] which indirectly confirms the dominant role of the mentioned field for confinement.

5.3 Concluding remarks

Considerations of the present Letter as well as ones of Refs. [2,3] show that in meson spectroscopy the approach based on the unique family of compatible nonperturbative solutions for the Dirac-Yang-Mills system derived from QCD-Lagrangian may be employed for both light mesons and heavy quarkonia. Under the circumstances there are no apparent obstacles to apply the approach to any meson. We hope further studies along this direction to confirm the given point of view.

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