Entropy Production in Simple Special Relativistic Fluids

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Abstract

It is well known that, in the absence of external forces, simple non-relativistic fluids involve entropy production only through heat conduction and shear viscosity\textsuperscript{1}. In this work, it is shown that a number density gradient contributes to the local entropy production of a simple relativistic fluid using special relativistic kinetic theory. Also, the presence of an external field may cause strictly relativistic contributions to the entropy production, a fact not widely recognized. The implications of these effects are thoroughly discussed.
I. INTRODUCTION

Force-flux relations in irreversible thermodynamics are phenomenologically proposed as to comply with the second law of thermodynamics. In the usual procedure, one uses the local equilibrium assumption together with the functional dependence \( s(\vec{r}, t) = s(n(\vec{r}, t), \varepsilon(\vec{r}, t)) \) in order to write a balance equation for the entropy density \( s \) [1]. This functional dependence arises from the fact that the most natural choice for the independent scalar state variables is the particle number density \( n \) and the internal energy density \( \varepsilon \). Through them one can establish the space and time dependence of the rest of the thermodynamical variables. In the non-relativistic case the entropy balance equation has the general form

\[
ns + \nabla \cdot \vec{J}_s = \sigma \tag{1}
\]

where \( \vec{J}_s \) is the entropy flux (related with the heat flux). The entropy production \( \sigma \) is then rewritten, after introducing the transport equations for \( n \) and \( \varepsilon \), in the following form

\[
\sigma = \sum \vec{J}_i \cdot \vec{F}_i \tag{2}
\]

where \( \vec{J}_i \) are the still unknown fluxes and \( \vec{F}_i \) the corresponding thermodynamic forces. In this case the thermodynamic forces, as gradients of the state variables, arise naturally as well as the flux-force relations proposed in order to satisfy the extension of the second law of thermodynamics for irreversible processes, \( \sigma \geq 0 \). These are linear relations of the Fourier type, relating the heat flux with the temperature gradient in a simple ideal fluid.

However for high temperature gases, in which relativistic effects in the dynamics of individual molecules are relevant, the transport equations feature new terms [2]. The introduction of such expressions in the entropy balance yields an equation in which the choice of thermodynamic forces is not as direct as in the non-relativistic case. Based on such equation, Ref. [2] proposes the hydrodynamic acceleration as a thermodynamic force corresponding to the heat flux. However, it has been shown that such choice leads to the so-called generic instabilities which in part led to the conclusion that second order type theories were required in order to properly describe the behavior of special relativistic gases [3, 4].

On the other hand, kinetic theory provides a framework for establishing from first principles the relativistic transport equations for the state variables, together with expressions for the corresponding fluxes and production terms. Using these equations and completing the
system with the constitutive equations obtained by solving Boltzmann’s equation to first order in the gradients, one obtains a set of hydrodynamic equations that yields no generic instabilities, and is thus consistent with the Onsager regression of fluctuations hypothesis \[5\] [6]. When this procedure is carried out, one encounters that the force-flux relations feature strictly relativistic effects that are not present in the non-relativistic simple gas and, in some cases, have not been obtained phenomenologically. Such is the case of the coupling of the heat flux with the chemical potential, density or pressure gradient (see Refs. [7], [8] and [9] respectively) which phenomenologically is usually written in terms of the hydrodynamic acceleration. In Ref. [10] it has been shown, by applying the standard techniques of relativistic kinetic theory to the case of a single component charged gas in the presence of an electrostatic field, that the heat flux has an additional driving force given by the gradient of the electrostatic potential. The purpose of this paper is to establish the entropy production for such a case and establish a link to the fluxes-forces formalism. The relation of the production term with the electrostatic field is then identified with Benedicks (thermoelectric) effect [11].

To accomplish this task we have divided this paper as follows. Section II is devoted to establishing the balance equation for the entropy density and the corresponding expression for the entropy production, consistent with the second law. The explicit calculation of the entropy production is shown in Section III. The discussion of the results and final conclusions, as well as possible generalizations, are included in Section IV.

II. THE RELATIVISTIC ENTROPY BALANCE EQUATION

The starting point for establishing an entropy balance equation for a relativistic gas is, as in the non-relativistic case, Boltzmann’s equation. For a high temperature, dilute, non-degenerate gas such equation is given by [9, 12]

\[ v^\alpha \frac{\partial f}{\partial x^\alpha} + \dot{v}^\alpha \frac{\partial f}{\partial v^\alpha} = J (f f') \]  

which corresponds to an evolution equation in phase space for the single-particle distribution function. Here \( v^\alpha \) is the molecular four-velocity, \( x^\alpha \) the space-time position four-vector in a Minkowski metric \( \eta^{\alpha \beta} \) with a \(+ + ++\) signature and \( J (f f') \) is the collision term [12]. The distribution function \( f \) has the same interpretation as in the non-relativistic case, that is a
function such that
\[
f (x^\nu, v^\nu) \, d^3x d^3v \tag{4}
\]
gives the number of points in a cell in the six dimensional phase space with volume \(d^3x d^3v\).

The second term on the left hand side of Eq. (3) contains the four-acceleration of a single particle which, for the case of molecules carrying charge \(q\) and having rest mass \(m\) can be written as
\[
\dot{v}^\mu = \frac{q}{m} v^\nu F^{\mu\nu} \tag{5}
\]
Here \(F^{\mu\nu}\) is the electromagnetic field tensor which for an electrostatic field is given by
\[
F^{\mu\nu} = \begin{pmatrix}
0 & 0 & 0 & -\frac{\phi^1}{c} \\
0 & 0 & 0 & -\frac{\phi^2}{c} \\
0 & 0 & 0 & -\frac{\phi^3}{c} \\
\frac{\phi^1}{c} & \frac{\phi^2}{c} & \frac{\phi^3}{c} & 0
\end{pmatrix} \tag{6}
\]
where \(\phi\) is the electrostatic potential and \(c\) the speed of light.

In order to obtain the entropy balance equation, we follow the standard procedure for obtaining transport equations for the state variables \[9\]. The Boltzmann equation is multiplied by \(k_b \ln f\), where \(k_b\) is Boltzmann’s constant, and integrated over velocity space using the invariant volume element \(d^*v = \frac{\partial v^\mu}{\partial x^\mu} \). In this case the acceleration term vanishes and one is led to (see Appendix A)
\[
- k_b \frac{\partial}{\partial x^\mu} \int v^\mu f (\ln f) \, d^*v = - k_b \int J (ff') \ln f d^*v \tag{7}
\]
which can be written in a more familiar way as
\[
\frac{\partial S^\nu}{\partial x^\nu} = \sigma \tag{8}
\]
where \(S^\nu\) is the entropy four-flux and \(\sigma\) is the entropy production. These quantities are given by
\[
S^\mu = - k_b \int v^\mu f (\ln f) \, d^*v \tag{9}
\]
and
\[
\sigma = - k_b \int J (ff') \ln f d^*v \tag{10}
\]
respectively. From Eq. (8) one can clearly recognize that the extension of the second law of thermodynamics to irreversible processes is given by
\[
\sigma \geq 0 \tag{11}
\]
The next step in order to obtain the entropy production is to introduce Chapman-Enskog’s hypothesis for the distribution function, that is

\[ f = f^{(0)} (1 + \varphi + ...) \quad (12) \]

where \( f^{(0)} \varphi \) is a first order in the gradients correction to the local equilibrium distribution function \( f^{(0)} \), which for a relativistic gas is given by \[14\]

\[ f^{(0)} = \frac{n}{4\pi c^3 zK^2} \exp \left( \frac{2U v}{zc^2} \right) \quad (13) \]

Although the general expression involves an infinite sum, retaining only the first correction corresponds to Navier-Stokes regime which is the main interest of this work. Substituting Eq. (12) in the expression for the entropy production given by Eq. (10) leads to

\[ \sigma = -k_b \int J (ff') \ln (1 + \varphi) d^*v \quad (14) \]

Notice that the term depending on \( f^{(0)} \) vanishes upon integration, being the distribution function a collisional invariant within the Boltzmann equation’s hypothesis. Also, since the correction is assumed to be small we have \( \ln (1 + \varphi) \sim \varphi \) and thus one can write

\[ \sigma = -k_b \int J (ff') \varphi d^*v \quad (15) \]

For the sake of simplicity, we now introduce the relaxation time approximation for the collision term, proposed by Marle \[9\], and write

\[ J (ff') = -\frac{f - f^{(0)}}{\tau} \quad (16) \]

Introducing Eqs. (12) and (16) in Eq. (15) one obtains for the entropy production

\[ \sigma = \frac{k_b}{\tau} \int f^{(0)} \varphi^2 d^*v \quad (17) \]

which is consistent with the extension of the second law (see Eq. (11)). It is worthwhile noticing that the previous expression only holds in the case of a relaxation time approximation. The general expression for the entropy production in the Chapman-Enskog expansion is given by (15).
III. CALCULATION OF THE ENTROPY PRODUCTION

The entropy production can be explicitly calculated from Eq. (15) in the Chapman-Enskog approximation by noticing that the collision kernel can be expressed as

$$J (f f') = \int (f_1' f' - f_1 f) F \sigma d\Omega d\upsilon_1^* = J^{(0)} + J^{(1)} + J^{(2)}$$

(18)

where the ordering corresponds to ascending powers of the Knudsen parameter, roughly the order in the gradients of the thermodynamic variables. The terms in Eq. (18) are given by

$$J^{(0)} = 0$$

$$J^{(1)} = \int f_1^{(0)} f' (\varphi' + \varphi'_1 - \varphi_1 - \varphi) F \sigma d\Omega d\upsilon_1^*$$

$$J^{(2)} = \int f_1^{(0)} f' (\varphi' \varphi'_1 - \varphi_1 \varphi) F \sigma d\Omega d\upsilon_1^*$$

where the order zero approximation vanishes since $f_1^{(0)} f' = f_1^{(0)} f$. Thus, the entropy production corresponding to the first order in the gradients Chapman-Enskog approximation is given by

$$\sigma^{(1)} = -k_b \int f_1^{(0)} f' (\varphi' + \varphi'_1 - \varphi_1 - \varphi) \varphi F \sigma d\Omega d\upsilon_1^* d^* K$$

(19)

Expressing the collision kernel $J^{(1)}$ in terms of the left hand side of Boltzmann’s equation in the corresponding order, one obtains

$$\sigma^{(1)} \simeq -k_b \int \left[ v^\alpha f_\alpha^{(0)} + \frac{q}{m} v_\alpha F^\mu\nu \frac{\partial f^{(0)}}{\partial u^\mu} \right] \varphi d^* K$$

(20)

The term in brackets has already been calculated elsewhere (see equations (7) and (27) in Ref. [10]) and can be written as

$$v^\alpha f_\alpha^{(0)} + \frac{q}{m} v_\alpha F^\mu\nu \frac{\partial f^{(0)}}{\partial u^\mu} = f^{(0)} \gamma(k) h^\mu K^\nu \left\{ \frac{1}{z} \left( \frac{\gamma(k)}{G \left( \frac{1}{z} \right)} - 1 \right), \frac{q}{mc^2} \phi_{\mu}, + \left( 1 - \frac{\gamma(k)}{G \left( \frac{1}{z} \right)} \right) \frac{n_{\mu}}{n} ight\} + \left( 1 - \frac{\gamma(k)}{z} - \frac{\gamma(k)}{G \left( \frac{1}{z} \right)}, \frac{G \left( \frac{1}{z} \right)}{z} \right) \frac{T_{\mu}}{T} \right\}$$

(21)

where $k^\mu$ is the peculiar velocity, defined as the single molecule velocity measured by an observer comoving with the fluid. The speed dependence is included in $\gamma(k)$, given by

$$\gamma(k) = -\frac{U^\mu v_\mu}{c^2}$$

(22)
which is the usual Lorentz factor

$$\gamma(k) = \left(1 - \frac{k^2}{c^2}\right)^{-1/2}$$

(23)

with $k$ being the magnitude of $k^\ell$. Also, the spatial projector $h^{\mu\nu}$ in Eq. (21) is given by

$$h^{\mu\nu} = \eta^{\mu\nu} + \frac{U^\mu U^\nu}{c^2}$$

(24)

Substituting Eq. (21) in Eq. (20) and recalling that the spatial components of the relativistic heat flux are given by

$$J_{[Q]}^\mu = mc^2 h^\mu_\nu \int k^\nu f^{(1)}(1) \gamma^2(k) d^* K$$

(25)

one obtains (see Appendix B)

$$\sigma^{(1)} = -J_{[Q]}^\ell \left\{ \left(1 - \frac{z}{G(\frac{1}{z})} \right) \frac{T,\ell}{T^2} - \frac{z}{G(\frac{1}{z})} \frac{n,\ell}{T} + \frac{z}{G(\frac{1}{z})} \frac{q}{k_b T^2} \phi,\ell \right\}$$

(26)

This is the final result of this work which shows that there are three relativistic contributions to the entropy production in an inhomogeneous charged gas in the presence of an electrostatic field, namely the usual Fourier term, the relativistic contribution due to the particle number density gradient and the Benedicks-like term.

IV. FINAL REMARKS

The phenomenological approach to the establishment of a positive semidefinite local entropy production in a simple relativistic fluid suggests a constitutive equation that couples the heat flux with the hydrodynamic acceleration. Nevertheless, this coupling leads to the so-called generic instabilities first identified by Hiscock and Lindblom back in 1985. In this context, a constitutive equation for the heat flux that solely includes spatial first-order gradients of the local thermodynamic variables was established based on the grounds of relativistic kinetic theory. This constitutive equation solves the generic instabilities problem. Following these ideas, the next logical step corresponds to the analysis of the entropy production based on kinetic theory, so that it can be compared with its phenomenological counterpart.

The main results of this work, basically contained in Eqs. (17) and (26), show that a positive semidefinite expression for the entropy production is obtained using standard kinetic
theory arguments. This expression includes only first order gradients in the local variables and reduces to the classical expression in the limit $z \to 0$. Equation (26) also shows that in the $z \simeq 1$ regime, a Benedicks-type effect arises as a purely relativistic source of entropy [10].

Finally, it is interesting to notice a subtle step involved in the present derivation. The single particle acceleration term present in the left hand side of Eq. (3) is given in covariant form in Eq. (5), which is consistent with special relativity. However, if a gravitational field is considered and general relativistic effects become important, Eq. (5) must be replaced with an expression that takes into account curvature effects in the dynamics of the individual particles (a geodesic for example). In this case it is still unclear if the entropy production would show similar features as those included in Eq. (26) [15-16]. This subject will be addressed in the near future.

Appendix A

In order to obtain the entropy balance equation, Eq. (7), one multiplies Boltzmann’s equation (1) by $k_b \ln f$ and integrates over velocity space. That is

$$\frac{\partial}{\partial x_\alpha} k_b \int v^\alpha f \ln f d^4v + \frac{q}{m} k_b \int v_\alpha F^\mu_\alpha \ln f \frac{\partial f}{\partial v^\alpha} d^4v = k_b \int J (ff') \ln f d^4v$$

(27)

where in the first term on the left hand side use has been made of the fact that the distribution function only depends on space-time coordinates through the state variables. For the acceleration term, one can use

$$\frac{\partial}{\partial v^\alpha} (v_\beta f \ln f) = v_\beta f \frac{\partial}{\partial v^\alpha} (\ln f) + v_\beta \ln f \frac{\partial f}{\partial v^\alpha} + \eta_{\beta\alpha} f \ln f$$

(28)

in order to write

$$\int v_\beta F^{\alpha\beta} \ln f \frac{\partial f}{\partial v^\alpha} d^4v = F^{\alpha\beta} \left[ \int \frac{\partial}{\partial v^\alpha} (v_\beta f \ln f) \, d^4v - \int v_\beta f \frac{\partial}{\partial v^\alpha} (\ln f) \, d^4v - \int \eta_{\beta\alpha} f \ln f \, d^4v \right]$$

(29)

The first term in brackets vanishes since the distribution function tends to zero in the limits of integration. For the second term we have, integrating by parts

$$\int v_\beta \frac{\partial f}{\partial v^\alpha} \, d^4v = -\eta_{\beta\alpha} n$$

(30)
and thus
\[
\int v_\beta F^{\alpha\beta} \ln f \left( v_\beta \right) d^4v = F^{\alpha\beta} \eta_{\beta\alpha} \left( n - \int f \ln f d^4v \right)
\] (31)
which vanishes since \( F^{\alpha\beta} \) is antisymmetric and thus \( F^{\alpha\beta} \eta_{\beta\alpha} = F_\alpha = 0 \) is antisymmetric. Notice also that this is only valid when \( F^{\alpha\beta} \) does not depend on the molecular velocity.

**Appendix B**

In this appendix, the structure of the entropy production given by Eq. (26) is established by substituting the left hand side of the linearized Boltzmann’s equation in the expression for \( \sigma^{(1)} \) given by Eq. (20) which yields

\[
\sigma^{(1)} \simeq k_b \int f^{(0)}(k) h^\ell k^\ell \left\{ \frac{1}{z} \left( \frac{\gamma(k)}{G(\frac{z}{2})} - 1 \right) \frac{q}{m c^2} \phi,\ell + \left( 1 - \frac{\gamma(k)}{G(\frac{z}{2})} \right) \frac{n,\ell}{n} \right\} \varphi d^* K
\]

or, separating terms by the \( \gamma(k) \) dependence

\[
\sigma^{(1)} \simeq k_b \left\{ \left[ \frac{1}{G(\frac{z}{2})} k_b T \Phi,\ell - \frac{1}{G(\frac{z}{2})} \frac{n,\ell}{n} - \left( \frac{1}{z} + \frac{1}{G(\frac{z}{2})} \right) \frac{T,\ell}{T} \right] \int f^{(0)}k^\ell \gamma^2(\frac{z}{2}) \varphi d^* K \right. \\
+ \left. \left[ - \frac{q}{k_b T} \Phi,\ell + \frac{n,\ell}{n} + \left( 1 - \frac{G(\frac{z}{2})}{z} \right) \frac{T,\ell}{T} \right] \int f^{(0)}\gamma(\frac{z}{2})k^\ell \varphi d^* K \right\}
\] (32)

or

\[
\sigma^{(1)} \simeq k_b \left\{ \left[ \frac{1}{G(\frac{z}{2})} k_b T \Phi,\ell - \frac{1}{G(\frac{z}{2})} \frac{n,\ell}{n} - \left( \frac{1}{z} + \frac{1}{G(\frac{z}{2})} \right) \frac{T,\ell}{T} \right] \int f^{(0)}k^\ell \gamma^2(\frac{z}{2}) \varphi d^* K \\
+ \left[ - \frac{q}{k_b T} \Phi,\ell + \frac{n,\ell}{n} + \left( 1 - \frac{G(\frac{z}{2})}{z} \right) \frac{T,\ell}{T} \right] \int f^{(0)}\gamma(\frac{z}{2})k^\ell \varphi d^* K \right\}
\] (33)

Since

\[
\int f^{(0)}k^\ell \gamma^2(\frac{z}{2}) \varphi d^* K = \frac{J^\ell}{mc^2}
\] (34)

and

\[
\int f^{(0)}\gamma(\frac{z}{2})k^\ell \varphi d^* K = 0
\] (35)

one obtains

\[
\sigma^{(1)} \simeq z \left[ \frac{z}{G(\frac{z}{2})} k_b T \Phi,\ell - \frac{z}{G(\frac{z}{2})} \frac{n,\ell}{n} - \left( 1 + \frac{z}{G(\frac{z}{2})} \right) \frac{T,\ell}{T} \right] \frac{J^\ell}{T}
\] (36)

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