ONE-BOX CONDITIONS FOR CARLESON MEASURES FOR THE DIRICHLET SPACE

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Abstract. We give a simple proof of the fact that a finite measure \( \mu \) on the unit disk is a Carleson measure for the Dirichlet space if it satisfies the Carleson one-box condition \( \mu(S(I)) = O(|I|) \), where \( \phi : (0, 2\pi] \to (0, \infty) \) is an increasing function such that \( \int_0^{2\pi} (\phi(x)/x) \, dx < \infty \). We further show that the integral condition on \( \phi \) is sharp.

1. Introduction

Let \( (F, \| \cdot \|_F) \) be a Banach space of measurable functions defined on a measurable space \( X \). A Carleson measure for \( F \) is a positive measure \( \mu \) on \( X \) such that \( F \) embeds continuously into \( L^2(\mu) \). In other words, \( \mu \) is a Carleson measure for \( F \) if there exists a constant \( C \) such that

\[
\int_X |f(x)|^2 \, d\mu(x) \leq C \| f \|_F^2 \quad (f \in F).
\]

Carleson measures were introduced by Carleson [6] in his solution to the corona problem. He considered the case where \( X = \mathbb{D} \), the unit disk, and \( F = H^2 \), the Hardy space. In this case he obtained a rather simple geometric characterization of these measures. Given an arc \( I \) in the unit circle, let us write \( |I| \) for its arclength, and \( S(I) \) for the associated Carleson box, defined by

\[
S(I) := \{ re^{i\theta} : 1 - |I| < r < 1, \ e^{i\theta} \in I \}.
\]

Then a finite measure \( \mu \) on \( \mathbb{D} \) is a Carleson measure for \( H^2 \) if and only if

\[
\mu(S(I)) = O(|I|) \quad (|I| \to 0).
\]

In this article, we are interested in the case where \( X = \mathbb{D} \) and \( F = \mathcal{D} \), the Dirichlet space. By definition, \( \mathcal{D} \) is the space of functions \( f \) holomorphic in \( \mathbb{D} \) whose Dirichlet integral is finite, i.e.,

\[
\mathcal{D}(f) := \frac{1}{\pi} \int_{\mathbb{D}} |f'(z)|^2 \, dA(z) < \infty.
\]

We can make \( \mathcal{D} \) into a Hilbert space by defining

\[
\| f \|_\mathcal{D}^2 := |f(0)|^2 + \mathcal{D}(f) \quad (f \in \mathcal{D}).
\]
Carleson measures for \( \mathcal{D} \) arise in several contexts, notably in characterizing multipliers \([11]\) and interpolation sequences \([4, 5, 9, 10]\). The problem of characterizing Carleson measures themselves has been studied by a number of authors over the years. The analogue of \([11]\) is the condition

\[
\mu(S(I)) = O\left(\left(\frac{1}{|I|}\right)^{-1} \left(\log \log \frac{1}{|I|}\right)^{-\alpha}\right) \quad (|I| \to 0),
\]

which, for a finite measure \( \mu \) on \( \mathbb{D} \), is known to be:

- necessary for \( \mu \) to be Carleson for \( \mathcal{D} \) if \( \alpha = 0 \);
- sufficient for \( \mu \) to be Carleson for \( \mathcal{D} \) if \( \alpha > 1 \);
- neither necessary nor sufficient if \( 0 < \alpha \leq 1 \).

The gap between necessity and sufficiency means that one cannot completely characterize Carleson measures for \( \mathcal{D} \) in terms of a one-box condition like \([1] \) or \([2]\). There do exist complete characterizations, but they are of a more complicated nature; see for example \([1] \) or \([2]\).

We shall discuss the necessity part of \([2]\) briefly at the end of the paper. However, our main interest is the sufficiency part of \([2]\), which is a consequence of the following more precise results.

**Theorem 1.1.** Let \( \mu \) be a finite positive Borel measure on \( \mathbb{D} \) satisfying

\[
\mu(S(I)) = O(\phi(|I|)) \quad (|I| \to 0),
\]

where \( \phi : (0, 2\pi] \to (0, \infty) \) is an increasing function such that

\[
\int_0^{2\pi} \frac{\phi(x)}{x} \, dx < \infty.
\]

Then \( \mu \) is a Carleson measure for \( \mathcal{D} \).

The condition \([4]\) in Theorem 1.1 is sharp in the following sense.

**Theorem 1.2.** Let \( \phi : (0, 2\pi] \to (0, \infty) \) be a continuous increasing function such that \( \phi(x)/x \) is strictly decreasing and

\[
\int_0^{2\pi} \frac{\phi(x)}{x} \, dx = \infty.
\]

Then there exists a finite positive Borel measure \( \mu \) on \( \mathbb{D} \) that satisfies \([3]\) but is not a Carleson measure for \( \mathcal{D} \).

Theorem 1.1 provides a justification of the sufficiency of \([2]\) when \( \alpha > 1 \), and Theorem 1.2 demonstrates its insufficiency when \( \alpha \leq 1 \).

Both theorems were recently obtained by Wynn \([12]\) under additional assumptions on \( \phi \) (see Theorems 1.3 and 1.5 in that paper, as well as the discussion in \( \S 4.2 \)). Wynn’s proofs are rather indirect, since they are a by-product of his work on the so-called discrete Weiss conjecture. Our purpose is to give simpler and more direct proofs. We shall prove Theorems 1.1 and 1.2 in \( \S 2 \) and \( \S 3 \) respectively, and conclude in \( \S 4 \) with some brief remarks about necessity.
2. Proof of Theorem 2.1

Let us write \( \langle \cdot , \cdot \rangle_D \) for the inner product on \( D \). Thus \( \langle f , f \rangle_D = \| f \|_D^2 \) for all \( f \in D \). Also, let

\[
k(z, w) = k_w(z) := 1 + \log \left( \frac{1}{1 - \overline{w}z} \right) \quad (z, w \in \mathbb{D}).
\]

It is easy to verify that \( k \) is a reproducing kernel for \( D \), in the sense that

\[
f(w) = \langle f , k_w \rangle_D \quad (f \in D, \ w \in \mathbb{D}).
\]

We shall need the following dual formulation of the notion of Carleson measure. It is a special case of an abstract result of Arcozzi, Rochberg and Sawyer [2, Lemma 24].

**Theorem 2.1.** Let \( \mu \) be a finite positive Borel measure on \( \mathbb{D} \). Then

\[
\sup_{\| f \|_D \leq 1} \int_D |f(z)|^2 \, d\mu(z) = \sup_{\| g \|_{L^2(\mu)} \leq 1} \left| \int_D \int_D k(w, z)g(z)\overline{g(w)} \, d\mu(z) \, d\mu(w) \right|.
\]

**Corollary 2.2.** A finite positive measure \( \mu \) on \( \mathbb{D} \) is a Carleson measure for \( D \) if and only if

\[
\sup_{\| g \|_{L^2(\mu)} \leq 1} \int_D \log \left( \frac{2}{1 - |wz|} \right) |g(z)||g(w)| \, d\mu(z) \, d\mu(w) < \infty.
\]

**Corollary 2.3.** A finite positive measure \( \mu \) on \( \mathbb{D} \) is a Carleson measure for \( D \) provided that

\[
\sup_{w \in \mathbb{D}} \int_D \log \left( \frac{2}{1 - |wz|} \right) \, d\mu(z) < \infty.
\]

**Proof.** It suffices to check that (8) implies (7). Let \( M \) be the supremum in (8), and write \( L(w, z) := \log(2/|1 - wz|) \). By the Cauchy–Schwarz inequality and the fact that \( L(w, z) = L(z, w) \), we have

\[
\int_D \int_D L(w, z)|g(z)||g(w)| \, d\mu(z) \, d\mu(w) \leq \int_D \int_D L(z, w)|g(w)|^2 \, d\mu(z) \, d\mu(w)
\]

\[
\leq M\| g \|_{L^2(\mu)}^2.
\]

**Proof of Theorem 2.1.** It suffices to show that (8) implies (7). In establishing (8), we can restrict our attention to those \( w \) with \( 1/2 < |w| < 1 \), since the supremum over the remaining \( w \) is clearly finite. For convenience, we extend the domain of definition of \( \phi \) to the whole of \( \mathbb{R}^+ \) by setting \( \phi(t) := \phi(2\pi) \) for \( t > 2\pi \).

Fix \( w \) with \( 1/2 < |w| < 1 \). Using Fubini’s theorem to integrate by parts, we have

\[
\int_D \log \left( \frac{2}{1 - |wz|} \right) \, d\mu(z) = \int_{t=0}^2 \mu \left( \{ z \in \mathbb{D} : |1 - wz| \leq t \} \right) \frac{1}{t} \, dt.
\]

Now \( \{ z \in \mathbb{D} : |1 - wz| \leq t \} = \{ z \in \mathbb{D} : |z - 1/w| \leq t/|w| \} \), which is contained in \( S(I) \) for some arc \( I \) with \( |I| = 8t \). By (3), we therefore have

\[
\mu \left( \{ z \in \mathbb{D} : |1 - wz| \leq t \} \right) \leq C\phi(8t),
\]

where \( C \) is a constant independent of \( w \) and \( t \). We thus obtain

\[
\int_D \log \left( \frac{2}{1 - |wz|} \right) \, d\mu(z) \leq \int_{t=0}^2 C\phi(8t) \frac{1}{t} \, dt = \int_{s=0}^{16} C\frac{\phi(s)}{s} \, ds.
\]

By (4), this last integral is finite. This gives (8).
3. Proof of Theorem 1.2

The proof of Theorem 1.2 is inspired by a construction of Stegenga [11 §4]. Translated into our context, his result is essentially the special case of Theorem 1.2 in which \( \phi(x) := 1 / \log(1/x) \). The construction proceeds via a characterization of Carleson measures for \( \mathcal{D} \), also due to Stegenga. This characterization is expressed in terms of logarithmic capacity, so we take a moment to define this notion and summarize those of its properties that we shall need.

Let \( E \) be compact subset of \( \mathbb{T} \). Its logarithmic capacity \( c(E) \) is defined by the formula
\[
\frac{1}{c(E)} := \inf_{\nu} \int_E \int_E \log \left( \frac{2}{|1 - zw|} \right) d\nu(z) d\nu(w),
\]
where the infimum is taken over all Borel probability measures \( \nu \) on \( E \). It could happen that the infimum is infinite, in which case \( c(E) = 0 \). We shall need the following facts:

- Capacity is upper semicontinuous: if \( E_n \downarrow E \), then \( c(E_n) \downarrow c(E) \).
- For arcs \( I \), we have \( c(I) \asymp 1 / \log(1/|I|) \) as \( |I| \to 0 \).
- Let \( E \) be a generalized Cantor set in the unit circle, formed by taking an arc of length \( l_0 \), removing an arc from its center to leave two arcs of length \( l_1 \), removing an arc from each of their centers to leave four arcs of length \( l_2 \), and so on. Then
\[
(9) \quad c(E) = 0 \iff \sum_{n \geq 0} 2^{-n} \log(1/l_n) = \infty.
\]

For further information about logarithmic capacity, we refer to [7 Chs. III & IV].

We now state Stegenga’s characterization of Carleson measures [11 Theorem 2.3].

**Theorem 3.1.** Let \( \mu \) be a finite positive Borel measure on \( \mathbb{D} \). Then \( \mu \) is a Carleson measure for \( \mathcal{D} \) if and only if there exists a constant \( A \) such that, for every finite set of disjoint closed subarcs \( I_1, \ldots, I_n \) of \( \mathbb{T} \),
\[
(10) \quad \mu \left( \bigcup_{k=1}^n S(I_k) \right) \leq Ac \left( \bigcup_{k=1}^n I_k \right).
\]

With this result under our belt, we can prove Theorem 1.2.

**Proof of Theorem 1.2** We can suppose that \( \lim_{x \to 0} \phi(x) = 0 \), otherwise condition (3) is vacuous. Multiplying \( \phi \) by a constant, we may further suppose that \( \phi(1) = 1 \). Then, for each \( n \geq 0 \), there exists \( l_n \in (0,1] \) such that \( \phi(l_n) = 2^{-n} \). Since \( \phi(x)/x \) is strictly decreasing, \( l_{n+1} < l_n / 2 \) for all \( n \). Let \( E \) be the associated generalized Cantor set, as described above. Let \( \sigma \) the corresponding Cantor–Lebesgue measure (namely the probability measure on \( E \) giving weight \( 2^{-n} \) to each of the \( 2^n \) arcs appearing at the \( n \)-th stage in the construction of \( E \)). Let \( (\delta_n)_{n \geq 0} \) be any decreasing sequence in \((0,1)\). Let \( \mu_n \) be the measure on \( \mathbb{D} \) defined by \( \mu_n((1 - \delta_n)B) := \sigma(B) \) (so \( \mu_n \) is just \( \sigma \), scaled to live on the slightly smaller circle \( |z| = 1 - \delta_n \)). Finally, let \( \mu := \sum_{n \geq 0} 2^{-n} \mu_n \). We claim that \( \mu \) satisfies (3), and that we may choose \( (\delta_n) \) so that \( \mu \) is a Carleson measure for \( \mathcal{D} \).

Let us show that \( \mu \) satisfies (3). Let \( I \) be an arc with \( |I| < l_0 \). Pick \( n \) such that \( l_{n+1} \leq |I| < l_n \). The arcs appearing in the \( n \)-th stage of the construction of \( E \) have
length $l_n$, so $I$ can meet at most two of them, whence $\sigma(I) \leq 2.2^{-n} = 4.2^{-(n+1)} = 4\phi(l_{n+1}) \leq 4\phi(|I|)$. It follows that

$$\mu(S(I)) = \sum_{k \geq 0} 2^{-k} \mu_k(S(I)) \leq \sum_{k \geq 0} 2^{-k} \sigma(I) = 2\sigma(I) \leq 8\phi(|I|).$$

This implies that (3) holds.

Now we show that if $(\delta_n)$ is chosen appropriately, then $\mu$ is not a Carleson measure for $D$. By (5), the integral $\int_0^{l_0} (\phi(x)/x) \, dx$ diverges. On the other hand, it is bounded above by

$$\sum_{n \geq 0} \phi(l_n) \int_{\delta_n}^{l_n} \frac{dx}{x} = \sum_{n \geq 0} 2^{-n} \log(l_n/l_{n+1}) \leq \sum_{n \geq 0} 2^{-n} \log(1/l_{n+1}).$$

Therefore $\sum_n 2^{-n} \log(1/l_n) = \infty$. As mentioned at the beginning of the section, this implies that $c(E) = 0$. Set $E_\delta := \{ \zeta \in \mathbb{T} : d(\zeta, E) \leq \delta \}$. Since capacity is upper semicontinuous, we have $c(E_\delta) \to c(E) = 0$ as $\delta \to 0$. Therefore, we may choose $\delta_n$ so that $c(E_{\delta_n}) < 3^{-n}$ for all $n$. For each $n$, the set $E_{\delta_n}$ is a finite union of closed arcs, $I_1, \ldots, I_k$, each of length at least $\delta_n$. The sets $S(I_j)$ therefore all meet the circle $|z| = 1 - \delta_n$. It follows that

$$\mu \left( \bigcup_{j=1}^k S(I_j) \right) \geq 2^{-n} \mu_n \left( \bigcup_{j=1}^k S(I_j) \right) = 2^{-n} \sigma \left( \bigcup_{j=1}^k I_j \right) \geq 2^{-n} \sigma(E) = 2^{-n}.$$

On the other hand,

$$c \left( \bigcup_{j=1}^k I_j \right) = c(E_{\delta_n}) < 3^{-n}.$$

Thus, for (10) to hold, we must have $2^{-n} < 3A^{-n}$. Obviously, there is no constant $A$ such that this holds for all $n$, so, by Theorem 5.1 the measure $\mu$ is not a Carleson measure for $D$. \hfill \Box

4. Remarks about necessity

(i) As mentioned at the beginning of §3 for arcs $I$ we have $c(I) \propto 1/\log(1/|I|)$ as $|I| \to 0$. Thus, applying Theorem 3.1 with one arc, we see that a necessary condition for $\mu$ to be a Carleson measure for $D$ is that

$$\mu(S(I)) = O \left( \left( \log \frac{1}{|I|} \right)^{-1} \right) \quad (|I| \to 0),$$

as claimed in (2).

(ii) The sufficient condition (5) is not necessary for $\mu$ to be a Carleson measure for $D$. Indeed, consider the measure $\mu$ on $[0,1]$ defined by

$$\mu \left( [0,1] \right) = \left( \log \frac{2}{t} \right)^{-1} \quad (0 \leq t < 1).$$

By Theorem 5.1, $\mu$ is Carleson for $D$. On the other hand, using Fubini’s theorem to integrate by parts, we have

$$\int_{[0,1]} \log \left( \frac{2}{1-wt} \right) \, d\mu(t) = \int_{x=1-w}^2 \frac{dx}{x \log(2w/(w+x-1))} \quad (0 < w < 1),$$

and the right-hand side clearly tends to infinity as $w \to 1^-$, so (5) fails.
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