Tree Level Supergravity and the Matrix Model

Michael Dine, Robert Echols and Joshua P. Gray

Santa Cruz Institute for Particle Physics, Santa Cruz CA 95064

Abstract

It has recently been shown that the Matrix model and supergravity give the same predictions for three graviton scattering. This contradicts an earlier claim in the literature. We explain the error in this earlier work, and go on to show that certain terms in the $n$-graviton scattering amplitude involving $v^{2n}$ are given correctly by the Matrix model. The Matrix model also generates certain $v^6$ terms in four graviton scattering at three loops, which do not seem to have any counterparts in supergravity. The connection of these results with nonrenormalization theorems is discussed.
1 Introduction

One of the important pieces of support for the original Matrix model conjecture was that it successfully reproduced graviton-graviton scattering in supergravity[1, 2]. Subsequently, the model has been shown to reproduce the full helicity-dependent amplitude[7]. As suggested in [1], this agreement can be understood in terms of non-renormalization theorems[8, 9, 10, 11]. That this agreement holds for finite as well as infinite $N$ is an important component of the stronger DLCQ conjecture[12]. Seiberg[13] and Sen[14] have shown that this conjecture follows from well-established duality relations. However, their argument does not necessarily imply that tree level supergravity amplitudes should agree with the leading matrix model computation. Some issues related to these proofs have been discussed in [29, 3]. A puzzle raised by these arguments in the case of propagation in curved backgrounds has been discussed in [15].

In [16], a technique was suggested to extract certain terms in the supergravity and matrix theory $S$-matrices for more than two gravitons. The idea was to consider a hierarchy of distance scales (impact parameters). In the case of three gravitons, for example, one takes one of the gravitons to be far from the other two. In the matrix model, where the graviton separation translates into frequencies of (approximate) harmonic oscillators, one can then analyze the problem by first integrating out the most massive modes, and then study the interactions of the remaining light degrees of freedom. In momentum space, the hierarchy of impact parameters translates into a hierarchy of momentum transfers. This yields an appreciable simplification of the supergravity calculation as well.

Using these methods, it was argued in [16] that there was a discrepancy between the computation of three graviton scattering in the matrix model and in tree level supergravity. Calling the large distance $R$ and the smaller distance $r$, and denoting the velocity of the far-away graviton by $v_3$, the supergravity $S$-matrix contains a term (after Fourier transform):

$$\frac{v_3^3 v_1^2}{r^7 R^7}.$$ (1)

However, it is straightforward to see, by power counting arguments (which we will review in the next section) that no such term can be generated in the matrix model effective action. The authors of [16] then went on to argue that this term could not appear in the Matrix model $S$-matrix.

Subsequently, however, Taylor and Van Raamsdonk[17] pointed out, using simple symmetry considerations, that if one writes an effective action for gravitons in supergravity, this action
cannot contain such terms. (Other criticisms appeared in [18].) Shortly afterwards, Okawa and Yoneya [19] computed the effective action on both the matrix model and supergravity sides, and showed that there is complete agreement. A related computation appeared in [20]. Other calculations have also been reported recently showing impressive agreement between the matrix model and supergravity [21].

It is clear from these remarks that the difficulty in [16] lies in extracting the Matrix model $S$-matrix from the effective action. In the next section of this note, we show how the “missing term” is generated in the $S$-matrix of the matrix model. In order to do this using the effective action approach, it is necessary to resolve certain operator-ordering questions $^1$. To deal with these issues the most efficient approach is the path integral. In section 2, we review first the problem of computing the $S$-matrix from the path integral by studying small fluctuations about classical trajectories. Once this is done, the isolation of the “missing term” is not difficult.

Despite the error in the analysis of [16], the method proposed there yields a considerable simplification in the calculation of the effective action. Indeed, it is possible to calculate certain terms in just a few lines. On the supergravity side, there are also significant simplifications which occur in this limit. One might hope, then, to extract general lessons from this approach. For example, one can easily compare certain tensor structures in $n$-graviton scattering, and perhaps try to understand whether (and why) there is agreement. One can also try to examine, as in [11] the role of non-renormalization theorems.

In section 3, then, we go on to compare certain other terms in three graviton scattering, some of which were not explicitly studied in [19]. These calculations can be performed extremely easily using the methods proposed in [16], on both the matrix model and supergravity sides, and are shown to agree.

Armed with this success, we consider in section 4 scattering of more than three gravitons, and scattering when more dimensions are compactified. Some of the terms in the four graviton scattering amplitude can readily be computed, and compared on both sides. We find agreement of certain terms involving eight powers of velocity. We also find certain terms of order $v^{2n}$ in $n$-graviton scattering, for arbitrary $n$, agree. On the other hand, the matrix model at three loops generates terms of order $v^6$ in four graviton scattering. These do not have the correct scaling with $N$ to generate a Lorentz invariant expression, and it is difficult to see how they can be cancelled by other matrix model contributions to the $S$-matrix. These terms also indicate

$^1$The authors of [16] had convinced themselves that there was no choice of operator ordering which generated the missing term. This was their basic error.
that there are terms at order $v^6$ which are renormalized.

These observations raise a number of questions. In particular, it is not completely clear why the arguments of [13] and [14] imply that the classical supergravity amplitudes should agree with the matrix model result. One might have thought that this should only hold in cases in which there are non-renormalization theorems[3]. These results indicate that already at the level of the four graviton amplitude, there are not non-renormalization theorems, at least in the most naive sense. They also suggest that at order $v^{2n}$, the $n - 1$ loop matrix model diagram reproduces the supergravity amplitude, but that there are discrepancies at three loops and beyond in terms with fewer powers of velocity. We will make some remarks on these issues in the final section, but will not provide a definite resolution.

2 Computing the $S$-Matrix in the Matrix Model

The matrix model is the dimensional reduction of ten dimensional supersymmetric Yang-Mills theory. The action is

$$S = \int dt \left[ \frac{1}{g} \text{tr}(D_t X^i D_t X^i) + \frac{1}{2g} M^6 R_{11}^2 \text{tr}([X^i, X^j][X^i, X^j]) + \frac{1}{g} \text{tr}(i \theta^T D_t \theta + M^3 R_{11} \theta^T \gamma^i [X^i, \theta]) \right]$$

where $R_{11}$ is the eleven dimensional radius, $M$ is the eleven dimensional Planck mass and $g = 2R_{11}$. The $\theta$’s are the fermionic coordinates.

At small transverse velocity and small momentum transfer (with zero $q^+$ exchange) it is a simple matter to compute graviton-graviton scattering in the matrix model. One considers widely separated gravitons, and integrates out the high frequency modes of the matrix model. This yields, at one loop, an effective Lagrangian for the remaining diagonal degrees of freedom which behaves as

$$L_{eff} = \frac{15}{16} \frac{v^4}{r^7} + \text{fermionic terms.}$$

If this effective lagrangian is then treated in Born approximation, one reproduces precisely the supergravity result for the $S$-matrix.

Ref. [16] focused on the problem of multigraviton scattering in the matrix model. For three graviton scattering, it is necessary to compute the terms of order $v^6$ at two loops in the matrix model Hamiltonian. In the three graviton case, there are two relative coordinates
and correspondingly two relative velocities. The basic strategy of [16], which will also be the strategy of this paper, was to consider the case where one of the relative separations, say $x_{13} = x_1 - x_3 = R$, was much larger than $x_{12} = r$. In this limit, oscillators with frequency of order $R$ can be integrated out first, yielding an effective lagrangian for those with mass (frequency) of order $r$ (or zero). This effective lagrangian is restricted by $SU(2)$ symmetry. Finally, one can consider integrating out oscillators with mass of order $r$.

In computing the $S$-matrix for three graviton scattering, as discussed already in [16], it is necessary not only to compute the terms of order $v^6$ in the effective action, but also to consider terms in the scattering amplitude which are of second order in the one loop ($v^4$) effective action. In other words, working with the effective action, it is necessary to go to higher order in the Born series.

In [16], it was observed that terms of the form

$$\frac{v_3^4 v_{12}^2}{R^7 r^7}$$

cannot appear in the effective action of the matrix model. This is seen by simple power counting arguments. The $v_3$ factors can only arise from couplings to heavy fields. Integrating out the fields with mass of order $R$ at one loop, the leading terms involving the light fields $x^a$ are\[11,22\] of the form $v_3^4 x^a x^a / R^9$. Moreover, it was argued that the terms in (4) were not generated by the higher order Born series referred to above. This last point, however, is incorrect, and is the source of the error. In fact, it is easy to find the corresponding term in the matrix model $S$-matrix.

Consider the problem first from a Hamiltonian viewpoint. We wish to compare the supergravity graph of fig. 1 with the contribution of fig. 2 in old fashioned (time-ordered) perturbation theory. The second graph represents the iteration of the one loop effective Hamiltonian to second order. In momentum space, it has the correct $1/q_{15}$ behavior to reproduce the $1/R^7 r^7$ behavior.
of the missing supergravity $S$-matrix term. However, it has also an energy denominator, and various factors of velocity. It is straightforward to check that this energy denominator is proportional to $\frac{1}{2k_2 q_1 + q_1^2}$, the propagator appearing in the covariant diagram of fig. 1. To compare the diagrams in more detail, one also needs matrix elements of the type $\langle \vec{k}_i + q | H' | \vec{k}_i \rangle$ where $H'$ is the one loop Hamiltonian. The leading term in powers of momentum transfer is easily seen to reproduce the corresponding term in the supergravity diagram. In other words, if one ignores the difference in the momenta of the particles in the initial, final and intermediate states, one obtains exact agreement. To see if higher order terms can cancel the energy denominator and reproduce the missing term, it is necessary to keep at least terms linear in the momentum transfer. The problem is that it is not clear how the momentum and $r$ factors are to be ordered in the Hamiltonian. Depending on what one assumes about this ordering, one obtains quite different answers.

Of course, the full model has no such ordering problem. It is only our desire to simplify the calculation using the effective Hamiltonian that leads to this seeming ambiguity. There is an alternative approach, however, which leads to an unambiguous answer, and where one can exploit the simplicity of the one loop effective action. This is to use the path integral. As we will see, the path integral approach permits an unambiguous resolution of the ordering problem.

2.1 Path integral Computation of the S-Matrix

Let us consider the problem of computing the $S$-matrix using the path integral. We will use an approach which is quite close to the eikonal approximation (it is appropriate for small angle scattering) which has been used in most analyses of matrix model scattering. It is helpful, first, to review some aspects of potential scattering. In particular, let us first see how to recover the Born approximation by studying motion near a classical trajectory.
A useful starting point is provided in [7]. In the path integral, it is most natural to compute
the quantity
\[ \langle \vec{x}_f | e^{-iHT} | \vec{x}_i \rangle = \int [dx] e^{iS}. \] (5)

To compute the $S$-matrix, one wants to take the initial and final states to be plane waves, so one multiplies by $e^{i\vec{p}_i \cdot \vec{x}_i} e^{-i\vec{p}_f \cdot \vec{x}_f}$ and integrates over $x_i$ and $x_f$. For small angle scattering in a weak, short-ranged potential, one expects that the dominant trajectories are those for free particles,
\[ \vec{x}_o(t) = \frac{\vec{x}_i + \vec{x}_f}{2} + \vec{v}t \] (6)
where $t$ runs from $-\frac{T}{2}$ to $\frac{T}{2}$, and $\vec{v} = \frac{\vec{x}_f - \vec{x}_i}{T}$. It is convenient to change variables [7] to $\vec{v}$ and $\vec{b}$,
\[ \vec{b} = \frac{\vec{b} + \vec{v}t}{2}. \] (7)
The complete expression for the amplitude is then
\[ \mathcal{A}_{i \rightarrow f} = \int \, d^3\vec{v} \int \, d^3\vec{b} e^{i\vec{b} \cdot (\vec{p}_f - \vec{p}_i)} e^{i\vec{v} \cdot (\vec{b} + \vec{v}t)} \int [dx] e^{iS}. \] (8)

Now if we expand the classical action about this solution, writing
\[ \vec{x} = \vec{x}_o + \delta \vec{x} \] (9)
(note $\delta \vec{x}$ includes both classical corrections to the straight line path and quantum parts) we have a free piece
\[ S_o = \frac{v^2 T}{2}. \] (10)

For large $T$, the $v$ integral can be done by stationary phase, yielding
\[ \vec{v} = \frac{\vec{p}_f + \vec{p}_i}{2}. \] (11)
We will see that this effectively provides the ordering prescription we require for the matrix model problem.

For the case of potential scattering, expand $e^{iS}$ in powers of $V$, and replace the potential by its Fourier transform. The leading semiclassical contribution to the amplitude is then proportional to
\[ \int \, d^3\vec{v} e^{i\vec{v} \cdot (\vec{p}_f - \vec{p}_i)} \int \, d^3q \int dt V(q) e^{i\vec{v} \cdot (\vec{b} + \vec{v}t)}. \] (12)
The \( t \) integral gives a \( \delta \)-function for energy conservation, while the \( b \) integral sets \( \vec{q} = \vec{p}_f - \vec{p}_i \). This is precisely the Born approximation result.

Higher terms in the Born series can be worked out in a similar fashion. Time ordering the terms and replacing the potential by its Fourier transform, the time integrals almost give the expected energy denominators. The terms linear in momentum transfer (involving \( \vec{v} \cdot \vec{q} \)) are given correctly, but the \( \vec{q}^2 \) terms are not. These terms must be generated by the expansion of \( V \) in powers of \( \delta x \), which generates additional powers of \( \vec{q} \). This problem, which is essentially the problem of recoil discussed in [24], will be analyzed in a separate publication [25]. Here we will work to leading order in \( q \), and second order in \( V \).

At second order in \( V \), we need to consider an expression of the form

\[
\int d\vec{v} \int db e^{i\vec{v} \cdot \vec{p}_i} e^{i\vec{b} \cdot (\vec{p}_f - \vec{p}_i)} e^{-\frac{1}{2} \vec{v}^2 T_1} \int_{-T_1}^{T_1} dt_1 \int_{-T_1}^{T_1} dt_2 V(\vec{x}(t_1))V(\vec{x}(t_2)).
\]

Time order the \( t_1, t_2 \) integrals, and Fourier transform each of the factors of \( V \). The integral over \( \vec{v} \) is again done by stationary phase, and the resulting expression has the form:

\[
\frac{1}{2!} \int_{-T_2}^{T_2} dt_1 \int_{-T_2}^{T_2} dt_2 \int db \int dq_1 \int dq_2 V(q_1)V(q_2) e^{i\vec{b} \cdot (\vec{p}_f - \vec{p}_i) + i\vec{q}_1 \cdot (\vec{b} + \vec{v}_1) + i\vec{q}_2 \cdot (\vec{b} + \vec{v}_2)}
\]

(14)

It is now straightforward to do the \( t_1, q_1, \) and \( b \) integrals. The integral over \( t_2 \) yields the energy denominator, \( \frac{1}{\vec{v} q_2} \). This differs from the exact energy denominator by terms of order \( q^2 \). The final integral over \( t_1 \) yields the overall energy conserving \( \delta \)-function. Up to these terms of order \( q^2 \), this is exactly the second order Born approximation expression.

### 2.2 The Ladder Graphs

We are now in a position to compare the supergravity and matrix model ladder graphs (see fig. 1 and 2). On the supergravity side, the calculation is completely standard, and proceeds along the lines of [16]. As there, we take the vertices from [24] and require that the polarizations of the incoming and outgoing gravitons be identical (as is true to leading order in the inverse distance in the matrix model). The \( N_0 N_3 \frac{v^2}{q^2} \) term comes from the second vertex, and is precisely of the same form as in graviton-graviton scattering. The vertex on the first graviton line is

\[
- k_{1\sigma} (k_{1\gamma} - q_{1\gamma}) - (k_{1\sigma} - q_{1\sigma}) k_{1\gamma} + 2k_{1\sigma} k_{1\gamma} + 2(k_{1\sigma} - q_{1\sigma})(k_{1\gamma} - q_{1\gamma}).
\]

(15)

\footnote{To be consistent with the authors [19, 23] who use \( \kappa^2 = 16\pi^5 \), the 3-vertex in [24] needs to be multiplied by 2.}
From the first vertex on the second graviton line, we get a similar expression, replacing $k_1$ with $k_2$ and $q_1$ by $-q_1$. Multiplying these factors together, and including the propagator, gives for the corresponding amplitude:

$$A_1 = (2\kappa)^4(k_2 \cdot k_3)^2(k_1 \cdot k_2)\frac{[(k_1 \cdot k_2) - (q_1 \cdot k_2) + O(q_1^2)]}{q_1^2 q_3^2 (2k_2 \cdot q_1 + q_1^2)}.$$  

(16)

Or

$$A_1 = \frac{\kappa^4 N_1 N_2^2 N_3 v_3^4 v_{12}^2 [(v_{12}^2 - \frac{2}{N_1} \vec{q}_1 \cdot \vec{v}_{12}) + O(q_1^2)]}{q_1^2 q_3^2 (\vec{q}_1 \cdot \vec{v}_{12})},$$  

(17)

expressed in light cone variables with non-relativistic normalization.

Now we want to compare with the matrix model prediction, fig. 2. Recalling the averaging prescription, for the matrix elements of the interaction Hamiltonian we have (dropping terms suppressed by extra powers of $q_3$)

$$\left(\frac{15}{16}\right)^2 N_1 N_2^2 N_3 v_3^4 \left(\frac{\vec{k}_1}{N_1} - \frac{\vec{k}_2}{N_2} - \frac{1}{2} \vec{q}_1 (\frac{1}{N_1} + \frac{1}{N_2})\right)^4$$  

(18)

$$= \left(\frac{15}{16}\right)^2 N_1 N_2^2 N_3 v_3^4 v_{12}^2 [(v_{12}^2 - \frac{2}{N_1} \vec{q}_1 \cdot \vec{v}_{12}) - \frac{2}{N_2} (\vec{q}_1 \cdot \vec{v}_{12}) + O(q_1^2)].$$  

(19)

After Fourier transforming $r$ and $R$ (not shown above), the first term is exactly the term found on the supergravity side. The second term cancels the energy denominator, yielding a contact term,

$$- \kappa^4 N_1 N_2 N_3 \frac{v_3^4 v_{12}^2}{8 q_1^2 q_3^2}.$$  

(20)

Each of the four ladder diagrams yields an identical contribution. The sum is precisely the “missing” term of [16]. At this level, there is no discrepancy between the DLCQ prediction for the scattering amplitude and supergravity.

### 3 Additional contributions to three graviton scattering

The method proposed in [16] should allow us to readily compute certain terms in the matrix model scattering amplitude. Consider, again, the case of three graviton scattering, with $R \gg r$. It is very easy to obtain the terms in the action involving four powers of $v_3$ and two powers of $v_{12}$ (fig. 3). The point, again, is that factors of $v_3$ can be obtained only from loops involving
fields with mass of order $R$, and this costs powers of $R$; to obtain the least suppression, one must attach $v_{12}$ to the light fields.

Integrating out fields with mass of order $R$ yields no terms independent of velocities or quadratic in velocities; at quartic order in velocity, one has:

$$\mathcal{L} = \frac{15}{16} \left( \frac{v_{13}^4}{|\vec{x}_{13}|} + \frac{v_{23}^4}{|\vec{x}_{23}|} \right).$$  \hspace{1cm} (21)

For small $x_{12}$, one can expand in powers of $x_{12}$. The result can be generalized to an $SU(2)$ invariant expression:

$$\delta \mathcal{L} = \frac{15}{64}v_{3}^4\left((\vec{x}_1 + \vec{x}_2) \cdot \nabla R \right)^2 + (x^a \cdot \nabla R)^2 \frac{1}{R^7}. \hspace{1cm} (22)$$

Here $x_1 + x_2$ is the center of mass of the $1-2$ system (combined with the leading term, the expression is translationally invariant). The superscript $a$ is an $SU(2)$ index. Contracting the $x^a$ factors, the leading (infrared divergent and finite) terms cancel. The Euclidean propagator, up to terms quadratic in velocities is given by

$$\langle x^{+i}x^{-j} \rangle = \frac{\delta^{ij}}{\omega^2 + r^2} + \frac{4(v^i v^j) + \text{const} \delta^{ij} v^2}{(\omega^2 + r^2)^3}. \hspace{1cm} (23)$$

Substituting back in our expression above and performing the frequency integral yields

$$N_1 N_2 N_3 \frac{45}{64 R_{11}^4} v_3^4 \left(\vec{v}_{12} \cdot \nabla \right)^2 \frac{1}{R^7}. \hspace{1cm} (24)$$

In deriving this expression it is necessary to keep track of various factors of 2. One comes from the two real massive fields in the loop (or equivalently, written in terms of complex fields, from an extra 2 which appears in the vertex), the other from a factor of $g = 2 R_{11}$ for a 2-loop result. It is easy to show that this is the only contribution with this $r$ dependence and four factors of $v_3$.

Let’s compare this with the supergravity amplitude. There is only one diagram with the tensor structure of (24); this arises from the diagram of fig. 4. There are also several terms...
in individual diagrams of the form $v_4^2 v_2^2 \frac{1}{R^n}$, as well as terms of order $1/R^9 r^5$ with a different tensor structure than the matrix model result. We will shortly explain that, at the level of the $S$-matrix, all of these terms match, just as in the case of the leading $1/R^7 r^7$ term.

Figure 4: Contribution to three graviton amplitude involving three graviton vertex.

Let us first consider the contribution to the supergravity $S$-matrix of the form (24) above. The relevant diagram is shown fig. 4. It is convenient to view $q_1$ and $q_3$ as independent, so $q_2 = -q_1 - q_3$. From (23), the necessary piece of the three graviton vertex is

$$2P_3(k_1\sigma k_2\eta_{\mu\nu}\eta_{\alpha\beta}) + 2P_6(k_1\nu k_2\eta_{\beta\mu}\eta_{\alpha\sigma}) + 4P_6(k_1\nu k_2\eta_{\beta\sigma}\eta_{\gamma\alpha}).$$

(25)

It is then a straightforward exercise to evaluate the diagram. Matters are considerably simplified by using kinematic relations such as $k_1 \cdot q_1 = -\frac{1}{2} q_1^2$, $k_1 \cdot q_2 = \frac{1}{2} q_1^2 - k_1 \cdot q_3$, etc., and dropping terms with the wrong $R$ dependence. After only a few lines of algebra, this yields the covariant form of the amplitude:

$$16\kappa^4[(k_1 \cdot k_3)^2(k_2 \cdot q_3)^2 + (k_2 \cdot k_3)^2(k_1 \cdot q_3)^2 - 2(k_1 \cdot k_3)(k_2 \cdot k_3)(k_1 \cdot q_3)(k_2 \cdot q_3)] \frac{1}{q_1^2 q_3^2}.$$  

(26)

Changing to light cone variables with non-relativistic normalization gives

$$\frac{\kappa^4 N_1 N_2 N_3}{2R_{11}^3} v_3^4 ((\vec{v}_1 - \vec{v}_2) \cdot \vec{q}_3)^2 \frac{1}{q_1^2 q_3^2}.$$  

(27)

Then Fourier transforming gives precisely the matrix model result (24).

There are several other kinematic structures which appear in individual supergravity diagrams which do not arise in the matrix model computation, and thus must be produced by iteration of the one loop action. The cancellation, in fact, is closely related to the cancellation we have studied of the leading term. For example, there are terms from the diagram of fig. 4 which behave as $\frac{v_4^2 v_2^2}{R^3}$. To see how this and other terms cancel, let us return to our earlier discussion. There, we set $q_1 = -q_2$. However, we should be more careful, and write $q_2 = -q_1 - q_3$. Then from fig. 4 we have a contribution

$$-\frac{\kappa^4}{2} N_1 N_2 N_3 v_3^4 v_2^2 \frac{q_1 \cdot q_2 + q_1 \cdot q_3 + q_2 \cdot q_3}{q_1^2 q_2^2 q_3^2}.$$  

(28)
(previously we kept only the first term and set $q_1 = -q_2$). We also have the supergravity term involving the 4-vertex discussed in [16]

$$-\frac{\kappa^4}{2} N_1 N_2 N_3 v_3^4 v_{12}^2 (\frac{1}{q_1^2 q_3^2} + \frac{1}{q_2^2 q_3^2}).$$  \hspace{1cm} (29)$$

On the matrix model side, the higher order Born terms yield

$$-\frac{\kappa^4}{4} N_1 N_2 N_3 v_3^4 v_{12}^2 (\frac{1}{q_1^2 q_3^2} + \frac{1}{q_2^2 q_3^2}).$$  \hspace{1cm} (30)$$

As before, the leading terms match. Expanding in powers of $q_3$, it is not hard to check that the coefficients of $q_1 \cdot q_3$ and $(q_1 \cdot q_3)^2$ match as well.

4 More Gravitons

4.1 n-Graviton Scattering

Certain terms in the four and higher graviton scattering amplitude are easily evaluated by these methods. On the matrix model side, the calculations are particularly simple. One can, for example, consider a generalization of the three graviton calculation above, indicated in fig. 5. At two loops, we saw that we generate in $SU(3)$ an effective coupling,

$$\frac{45}{64} v_3^4 (\vec{v}_1 \cdot \vec{v}_4)^2 \frac{1}{|\vec{x}_4|^7} \frac{1}{|\vec{x}_{31}|^5 + \frac{1}{|\vec{x}_{32}|^5}}.$$  \hspace{1cm} (31)$$

We can generalize this to the case of $SU(4)$, with the hierarchy $x_{4i} \gg x_{3\ell} \gg x_{21}$, where $i = 1, 2, 3$ and $\ell = 1, 2$. In other words, we again suppose that there is a hierarchy of distance scales, with one particle very far from the other three, and one of these three far from the remaining two. Again, we proceed by first integrating out the most massive states, then the next most massive, and so on. After the first two integrations, we generate a term (among others)

$$\frac{45}{256} v_4^4 (\vec{v}_3 \cdot \vec{\nabla}_4)^2 \frac{1}{|\vec{x}_4|^7} (\frac{1}{|\vec{x}_{31}|^5} + \frac{1}{|\vec{x}_{32}|^5}).$$  \hspace{1cm} (32)$$

As before, expand this term in powers of the small distances $x_1, x_2$, and generalize to an $SU(2)$-invariant expression, yielding:

$$\frac{45}{256} v_4^4 (\vec{v}_3 \cdot \vec{\nabla}_4)^2 \frac{1}{|\vec{x}_4|^7} (\vec{x}_a \cdot \vec{\nabla}_3)^2 \frac{1}{|\vec{x}_3|^5}.$$  \hspace{1cm} (33)$$

Finally, the integration of $x_a$ yields various terms. The piece of $\langle x_a x_a \rangle \propto v^i v^j$ yields

$$\frac{135}{256} v_4^4 (\vec{v}_3 \cdot \vec{\nabla}_4)^2 \frac{1}{|\vec{x}_4|^7} (\vec{v}_{12} \cdot \vec{\nabla}_3)^2 \frac{1}{|\vec{x}_3|^5} \frac{1}{|\vec{x}_{12}|^5}.$$  \hspace{1cm} (34)$$
Higher order terms corresponding to $n$-graviton scattering generated in a similar fashion will be discussed below.

![Figure 5: A matrix model diagram contributing to four graviton scattering.](image5)

Another term which is easily obtained is indicated in the diagram in fig. 6. This graph includes the interaction of the light $SU(2)$ fields from integrating out the fields with mass of order $x_4$ at one loop, as well as those obtained by integrating out the fields of mass $x_3$ at one loop. The relevant interactions are

$$
(\frac{15}{64})^2 (v_4^4 v_3^3) [(x^a \cdot \vec{\nabla}^4)^2 \frac{1}{|x_4|^7} (x^a \cdot \vec{\nabla}^3)^2 \frac{1}{|x_3|^7}] \tag{35}
$$

Now contracting the $x^a$ factors as in fig. 6 yields a term:

$$
4(\frac{15}{64})^2 \left[ v_4^4 v_3^3 (\nabla_4^4 \nabla_3^4)^2 \frac{1}{|x_4|^7} (\nabla_3^3 \nabla_3^3)^2 \frac{1}{|x_3|^7} \frac{1}{|x_12|^3} \right] \tag{36}
$$

On the supergravity side, the required computations are somewhat more complicated. The easiest to consider is the first term (34) above. This term is generated by the diagram of fig. 7.
It is not difficult to find the particular tensor structures which give the matrix model expression (34). Focus first on the terms involving $\vec{v}_3 \cdot \vec{\nabla}_4$. These must come from dotting $k_3$ into $q'_2$ or $q_4$. Calling $q'_2 = -q_3 - q_4$, the relevant term in the three graviton vertex is ($\mu, \alpha$ are the polarization indices carried by the graviton with momentum $q'_2$)

$$2 \left[ P_3(q'_2 q_3 \gamma \eta_{\mu \nu} \eta_{\alpha \beta}) + P_6(q'_3 q_3 \gamma \eta_{3 \mu} \eta_{\alpha \beta}) + 2P_6(q'_2 q_3 \mu \eta_{3 \mu} \eta_{\gamma \alpha}) \right] k_4 k_4 k_3 k_3 = (37)$$

Only a few permutations actually contribute, and one obtains simply

$$2(k_3 \cdot q_4)^2 k_4 k_4 k_4$$

So the whole diagram collapses to $\frac{2(k_3 \cdot q_4)^2}{q_3^2}$ times the three graviton term we evaluated earlier. The result agrees completely with the matrix model computation (34).

Indeed, one can now go on to consider similar terms in $n$-graviton scattering. The supergravity graph indicated in fig. 8 can be evaluated by iteration. The coupling of the $n - 1$ graviton is similar to that of the third graviton in the 4-graviton amplitude and can be treated in an identical fashion. The result then reduces to the $n - 1$ graviton computation. So one obtains

$$\frac{15}{16} \left( \frac{3}{4} \right)^{n-2} v_n^4 (\vec{v}_{n-1} \cdot \vec{\nabla}_n)^2 \frac{1}{|\vec{x}_{n-1}|^5} (\vec{v}_{n-2} \cdot \vec{\nabla}_{n-1})^2 \frac{1}{|\vec{x}_{n-1}|^5} \cdots (\vec{v}_3 \cdot \vec{\nabla}_4)^2 \frac{1}{|\vec{x}_4|^5} (\vec{v}_{12} \cdot \vec{\nabla}_3)^2 \frac{1}{|\vec{x}_3|^5} (39)$$

The corresponding term in the matrix model effective action is also obtained by iteration. It is easy to generalize the calculation of fig. 5 to the case above. Repeating our earlier computations gives precisely the result of eqn. (39) above.
The computation of the part of the supergravity amplitude corresponding to eqn. (36) is more complicated. This term is generated by the sum of several diagrams. We will not attempt a detailed comparison here, leaving this, as well as certain other terms, for future work.

Figure 8: Diagram contributing to $n$ graviton scattering.

4.2 Other Dimensions

According to the Matrix model hypothesis, the compactification of $M$-theory to $11-k$ dimensions is described by $k+1$ dimensional super Yang-Mills theory. For graviton-graviton scattering, this has been done in \[30\]. It is a simple matter to extend our analysis to these cases.

As a simple illustration, consider the three graviton case. Working in units where the compact dimensions have $R_k = 1$, then the Fourier transforms needed to convert the supergravity result (37) to an effective potential are

\[
\frac{\kappa^2 v_3^4}{4(2\pi)^{1+k}} \int \frac{d^{9-k} q_3}{(2\pi)^{9-k}} \frac{e^{i\vec{q}_3 \cdot \vec{R}}}{q_3^2} = \frac{v_3^4}{2^{1+k}(2\pi)^{1+k}} \frac{1}{R^{7-k}} \frac{1}{\Gamma\left(\frac{7-k}{2}\right)} \tag{40}
\]

and

\[
\frac{2\kappa^2 v_{i12} v_{j12}^j}{(2\pi)^{1+k}} \int \frac{d^{9-k} q_1}{(2\pi)^{9-k}} \frac{e^{i\vec{q}_1 \cdot \vec{r}}}{q_1^4} = \frac{v_{i12}^i v_{j12}^j}{2^{k}(2\pi)^{1+k}} \frac{1}{r^{5-k}} \frac{1}{\Gamma\left(\frac{5-k}{2}\right)} \tag{41}
\]
On the matrix model side, the loop integrals arising from integrating out the massive states must now be performed in \( k + 1 \) dimensions giving
\[
-6v_3^4 \int \frac{d^{1+k}p}{(2\pi)^{1+k}} \frac{1}{(p^2 + R^2)^4} = \frac{v_3^4}{2^{1+k}(\sqrt{\pi})^{1+k}} \frac{1}{R^{7-k}} \Gamma \left( \frac{7-k}{2} \right)
\] (42)
and
\[
4v_{12}^i v_{12}^j \int \frac{d^{1+k}p}{(2\pi)^{1+k}} \frac{1}{(p^2 + r^2)^3} = \frac{v_{12}^i v_{12}^j}{2^k(\sqrt{\pi})^{1+k}} \frac{1}{r^{5-k}} \Gamma \left( \frac{5-k}{2} \right)
\] (43)
in agreement with the supergravity result above. All of the integrals are convergent for \( k \leq 4 \) and since these same integrals are needed for our \( n \)-graviton result, it is a simple matter to show that the agreement we have found here persists for arbitrary \( n \).

5 Some Puzzles

In the original discussion of [1], as well as in [12], the question was raised: why does the lowest order matrix model calculation reproduce the tree level supergravity result for graviton-graviton scattering. The scattering amplitude is given by a power series in \( gN \), and one ultimately wants to take a limit with \( N \to \infty \), \( g \) fixed. Moreover, one wants to take this limit uniformly in \( r \), i.e. one does not expect to scale distances with \( N \). The answer suggested by these authors was that the explanation lies in a non-renormalization theorem for \( v^4 \) terms, which insures that the one-loop result is exact. The required cancellation was demonstrated at two loops in [8]. Such a theorem for four derivative terms in four dimensional field theory was proven in [9]. The complete proof for the matrix model was finally provided in [10].

The agreement of three graviton scattering in the matrix model with supergravity suggests that there are more non-renormalization theorems governing the various possible terms at order \( v^6 \). Indeed, in the case of \( SU(2) \), a proof was provided in [28]. On the other hand, it is rather easy to see, following reasoning similar to that of [11], that there are operators at order \( v^6 \) which are renormalized in \( SU(N) \), \( N \geq 4 \). In particular, consider the case of four gravitons. In the previous section, we computed the contribution to the amplitude (44) by contracting \( x^a x^a \) in eqn.(23), and took the piece quadratic in \( v^2 \). Taking, instead, the leading, velocity-independent term in this propagator yields a contribution to the effective action,
\[
N_1 N_2 N_3 N_4 = \frac{45}{256} v_4^i \frac{1}{x_4^2} \frac{1}{\nabla_x^2} \frac{1}{x_1 x_3 x_{12}}.
\] (44)
Not only does this represent a renormalization of the \( v^6 \) terms computed at two loops, but the \( N \)-dependence of (44) is not appropriate to a Lorentz-invariant amplitude. One might wonder
if this term can be cancelled by terms generated at higher order in the Born series. However, it is easy to see that this is not the case. One can define an index of an amplitude, $\mathcal{A}$ (written in momentum space), $I_\mathcal{A}$, which is simply the difference of the number of powers of momentum in the numerator and in the denominator. All of the amplitudes we have studied previously have $I_\mathcal{A} = 2$. The iterations of the lower order matrix Hamiltonian also have $I_\mathcal{A} = 2$. However, (44) has $I_\mathcal{A} = -4$. So this can not be the source of the discrepancy. We have checked carefully for other diagrams in the matrix model effective action which might have this structure, and we do not believe there are any.

So we seem to have found, consistent with [19], that the three graviton amplitude is in agreement with the low energy limit of supergravity. We have also shown such agreement for certain terms of order $v^{2n}$ in the $n$-graviton scattering amplitude; further tests of the $v^{2n}$ terms in $n$-graviton scattering will be reported elsewhere. But the contribution in (44) does not seem to correspond to anything in the DLCQ of supergravity. If it is correct that the naive DLCQ of supergravity agrees with the matrix model only in cases where there are non-renormalization theorems, the work reported here suggests that there are non-renormalization theorems for the $v^{2n}$ operators in $SU(n)$, but not for smaller powers of $v$ (in $SU(n)$). It is perhaps possible to show this using reasoning along the lines of [10, 28]. We are currently investigating this possibility. Our results also seem to be compatible with the results of [15], who find that two particle scattering is not described correctly at finite $N$ in curved backgrounds. In the present case, one can think of the two “far away” gravitons as providing a background for the gravitons 1 and 2. The background preserves none of the supersymmetry. We are currently investigating whether this connection can be made precise.

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