Residuated Basic Logic I

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Abstract. We study the residuated basic logic (RBL) of residuated basic algebra in which the basic implication of Visser’s basic propositional logic (BPL) is interpreted as the right residual of a non-associative binary operator · (product). We develop an algebraic system S_{RBL} of residuated basic algebra by which we show that RBL is a conservative extension of BPL. We present the sequent formalization L_{RBL} of S_{RBL} which is an extension of distributive full non-associative Lambek calculus (DFNL), and show that the cut elimination and subformula property hold for it.

1 Introduction and Preliminaries

Basic propositional logic (BPL) was introduced by Visser [?] as a subintuitionistic logic, and recently developed by many other authors ([?], [?], [?], [?], [?]). The aim of this paper is to investigate the relationship between BPL and substructural logics. Inspired from Buszkowski’s idea in [?], we treat the basic implication → in BPL as the right residual of a designated binary operator ·, in addition, admitting some structural rules (weakening and restricted contraction). Thus BPL can be extended conservatively to a substructural logic.

The language of basic propositional logic is extended by adding the binary operator · and its left residual ←. Then we introduce the residuated basic logic (RBL) which is the logic of residuated basic algebras. We

¹Through our communication with Hiroakira Ono (Japan advanced Institute of Science and Technology), we know that Majid Alizadeh (University of Tehran) has the following unpublished result: basic propositional logic with residuation is a conservative extension of basic propositional logic. Ono’s idea is to formalize basic logic with residuation as a sequent system which is an extension of GL in [?]. However, the results in our paper are obtained independently.
Minghui Ma and Zhe Lin present the algebraic system $S_{RBL}$ for residuated basic algebra in terms of which we show that $RBL$ is a conservative extension of $BPL$. We present a Gentzen-style sequent formalization $L_{RBL}$ of $S_{RBL}$ which makes it easy to compare $RBL$ with other substructural logics.

Basic algebra is the algebra for basic propositional logic ([?]). We define residuated basic algebra as bounded distributive lattice order residuated groupoid (cf. [?]) enriched with weakening and restricted contraction $(c_r) a \cdot b \leq (a \cdot b) \cdot b$. It turns out that the reduct of residuated basic algebra restricted to the basic algebra type is in fact basic algebra (cf. theorem [3]). Note that our residuated basic algebra does not contain the unit element of its residuated groupoid reduct. For showing that the residuated basic logic is a conservative extension of $BPL$, we introduce the algebraic system $S_{RBL}$ for residuated basic algebra. The main method for proving the conservative extension is the relational semantics of non-associative Lambek calculus, and some techniques and notations are taken from [?].

The Gentzen-style sequent calculus $L_{RBL}$ is obtained by enriching the sequent calculus of $DFNL$ ($FNL$ with distributive law [?]) with weakening and restricted contraction rules. The logic $FNL$ ([?]) is obtained by adding all lattice operations and their corresponding rules to the logic $NL$ (non-associative Lambek calculus originally introduced by Lambek [?]) which is strongly complete with respect to residuated groupoids ([?]). The associative variant $L$ (Lambek calculus) of $NL$ was also introduced by Lambek [?]. The formalization style of our system $L_{RBL}$ was first independently developed by Dunn [?] and Mints [?] for the positive relevant logic $R^+$, and later used by Kozak [?] to prove the finite model property for $DFL$ (distributive full Lambek calculus). Moreover, we prove that the cut elimination, subformula property and disjunction property hold for $L_{RBL}$. Finally, we discuss the relationships between $BPL$, $RBL$ and other logics. As a consequence, the fragment of $RBL$ restricted to the language of $BPL$ is equivalent to the implicational fragment of $BPL$ in [?].

Now let us give some preliminaries on basic propositional logic, which can be found in [?], [?], [?] and [?]. The language $L_{BPL}$ of basic propositional logic consists of a set $Prop$ of propositional letters and connectives $\land, \lor, \to, \bot$ and $\top$. The set of $L_{BPL}$-formulae is defined recursively by the following rule:

$$A ::= p \mid \bot \mid \top \mid A \lor A \mid A \land A \mid A \to A$$

where $p \in Prop$. Define $\neg A := A \to \bot$, and $A \leftrightarrow B := (A \to B) \land (B \to A)$. 

A BPL-frame is a pair $\mathfrak{F} = (W, R)$ where $W$ is a nonempty set of states, and $R \subseteq W^2$ is a transitive relation, i.e., $\forall x, y, z \in W (xRy \land yRz \rightarrow xRz)$. A BPL-model is a tuple $\mathfrak{M} = (W, R, V)$ where $(W, R)$ is a BPL-frame and $V: \text{Prop} \rightarrow \wp(S)$ is a valuation satisfying the following ‘persistency’ condition: for each propositional letter $p$, if $w \in V(p)$ and $wRu$, then $u \in V(p)$. The definition of satisfaction relation $\mathfrak{M}, w \models A$ is defined as usual ([?], [?]). Especially, for implication, we have the following clause:

$\mathfrak{M}, w \models A \rightarrow B$ iff for all $v \in W$ with $wRu$, $\mathfrak{M}, v \models A$ implies $\mathfrak{M}, v \models B$.

The notion of frame validity $\mathfrak{F} \models A$ is also defined as usual. Note that the truth of every $\mathcal{L}_{\text{BPL}}$-formula is persistent: if $\mathfrak{M}, w \models A$ and $wRu$, then $\mathfrak{M}, v \models A$.

Visser introduced the natural deduction for BPL in [?]. It can be extended to intuitionistic logic ($\text{Int}$) and formal provability logic ($\text{FPL}$). It is already known that, via Gödel’s translation, BPL is embedded into the modal logic K4, FPL into the Gödel-Löb modal logic GL, and Int into the modal logic S4. The Hilbert-style axiomatization of BPL, given by Ono and Suzuki in [?], consists of the following axioms and rules:

1. $A \rightarrow A$
2. $A \rightarrow (B \rightarrow A)$
3. $(A \rightarrow B) \land (B \rightarrow C) \rightarrow (A \rightarrow C)$
4. $(A \rightarrow C) \land (B \rightarrow C) \rightarrow (A \lor B \rightarrow C)$
5. $A \land B \rightarrow A$
6. $A \land B \rightarrow B$
7. $A \rightarrow A \lor B$
8. $B \rightarrow A \lor B$
9. $A \rightarrow (B \rightarrow A \land B)$
10. $(A \rightarrow B) \lor (A \rightarrow C) \rightarrow (A \rightarrow B \land C)$
11. $A \land (B \lor C) \rightarrow (A \land B) \lor (A \land C)$
12. $\bot \rightarrow A$

By $\vdash_{\text{BPL}} A$ we mean that $A$ is provable (or a theorem) in the above Hilbert-style system. The following completeness theorem holds for BPL:

**Theorem 1** ([?], [?]). For all $\mathcal{L}_{\text{BPL}}$-formulas $A$, $\vdash_{\text{BPL}} A$ iff $\mathfrak{F} \models A$ for all BPL-frames $\mathfrak{F}$ iff $\mathfrak{M} \models A$ for all BPL-models $\mathfrak{M}$.

Just as Heyting algebras for intuitionistic logic, there is also a variety of algebras for BPL, which is called the variety of basic algebras.
Definition 1 ([?]). A basic algebra $A = (A, \land, \lor, \top, \bot, \to)$ is an algebra such that $(A, \land, \lor, \top, \bot)$ is a bounded distributive lattice and $\to$ is a binary operation over $A$ satisfying the following conditions: for all $a, b, c \in A$,

1. $a \to (b \land c) = (a \to b) \land (a \to c)$.
2. $(b \lor c) \to a = (b \to a) \land (c \to a)$.
3. $a \to a = \top$.
4. $a \leq \top \to a$.
5. $(a \to b) \land (b \to c) \leq a \to c$.

The operation $\to$ in a basic algebra is said to be basic implication.

Fact 2 ([?]). For any basic algebra $A = (A, \land, \lor, \top, \bot, \to)$ and $a, b, c \in A$,

1. if $a \leq b$, then $c \to a \leq c \to b$, $b \to c \leq a \to c$ and $a \to b = \top$.
2. if $a \land b \leq c$, then $a \leq b \to c$.

Moreover, the relationship between basic algebra and Heyting algebra is clear by the following result from [?]: a basic algebra $A$ is a Heyting algebra iff $a = \top \to a$ for all $a \in A$. Hence every Heyting algebra is a basic algebra.

Given a basic algebra $A$ and an $L_{BPL}$-formula $A$, we say that $A$ is valid in $A$ (notation: $A \models A$), if the equation $A = \top$ holds in $A$, i.e., the formula $A$ denotes the top element in the algebra under any assignment of propositional letters in $A$. Let $BLa$ be the class of all basic algebras. We mean by $BLa \models A$ that $A$ is valid in every basic algebra. It’s already known that $BPL$ is complete with respect to $BLa$ ([?]), i.e., for all $L_{BPL}$-formulae $A$, $\vdash_{BPL} A$ iff $BLa \models A$.

2 Basic Implication and Residuation

It is well-known that the Heyting implication $\to_H$ is the residual of $\land$, i.e., for any $a, b, c$ in a Heyting algebra, $c \leq a \to_H b$ iff $a \land c \leq b$. However, the basic implication is not the residual of $\land$ in basic algebra. Let us introduce an binary operator $\cdot$ the right residual of which is supposed to be the basic implication. Then an algebra $(A, \cdot, \to, \leftrightarrow, \leq)$ is said to be a residuated groupoid, if $(A, \leq)$ is a poset, and $\cdot$, $\to$ and $\leftrightarrow$ are binary operators satisfying the following residuation law:

(RES) $a \cdot b \leq c$ iff $b \leq a \to c$ iff $a \leq c \leftrightarrow b$.

The associativity of the operator $\cdot$ is not assumed in residuated groupoid.
Definition 2. A residuated basic algebra (RBA) is an algebra $A = (A, \wedge, \vee, T, \bot, \rightarrow, \leftarrow, \cdot)$ such that $(A, \wedge, \vee, T, \bot)$ is a bounded distributive lattice and $(A, \rightarrow, \leftarrow, \cdot, \leq)$ is a residuated groupoid satisfying the following axioms: for all $a, b, c \in A$,

$$(w_1) \ a \cdot T \leq a; \quad (w_2) \ T \cdot a \leq a; \quad (c_r) \ a \cdot b \leq (a \cdot b) \cdot b$$

where $\leq$ is the lattice order. Let $\mathbb{RBA}$ be the class of all residuated basic algebras.

For any residuated basic algebra, it is easy to show (i) $a \rightarrow b = \bigvee \{x \in A : a \cdot x \leq b\}$; and (ii) $a \leftarrow b = \bigvee \{x \in A : x \cdot b \leq b\}$ (the least upper bound).

Remark 1. The element $T$ in a residuated basic algebra $A$ is not the unit of $A$’s residuated groupoid reduct. If we enrich residuated basic algebra by constant 1 (the unit of residuated groupoid), then by $(w_1)$ and $(w_2)$, one can easily prove that $1 = T$. It follows that $T \rightarrow a \leq a$, ans so the implication $\rightarrow$ becomes Heyting.

Proposition 1. For any residuated basic algebra $A$ and $a, b, c \in A$,

(i) $\ (b \lor c) \cdot a \leq b \cdot a \lor c \cdot a$.
(ii) $\ a \cdot (b \cdot c) \leq (a \cdot b) \cdot c$.
(iii) $\text{if } a \leq b, \text{ then } c \cdot a \leq c \cdot b \text{ and } a \cdot c \leq b \cdot c$.
(iv) $\text{if } a \leq b, \text{ then } c \rightarrow a \leq c \rightarrow b \text{ and } b \rightarrow c \leq a \rightarrow c$.
(v) $\text{if } a \leq b, \text{ then } a \leftarrow c \leq b \leftarrow a \text{ and } c \leftarrow b \leq c \leftarrow a$.

Proof. It is easy to see that (iii)-(iv) are monotonicity laws which hold in all residuated groupoid. From (iii), we can easily derive that $a \leq b$ and $c \leq d$ imply $a \cdot c \leq b \cdot d$. For (i), since $b \cdot a \leq (b \cdot a) \lor (c \cdot a)$ and $c \cdot a \leq (b \cdot a) \lor (c \cdot a)$, we get $b \leq ((b \cdot a) \lor (c \cdot a)) \leftarrow a$ and $c \leq ((b \cdot a) \lor (c \cdot a)) \leftarrow a$. Hence $(b \lor c) \leq ((b \cdot a) \lor (c \cdot a)) \leftarrow a$, which yields $(b \lor c) \cdot a \leq (b \cdot a) \lor (c \cdot a)$.

For (ii), first by $(w_1)$ and (iii), we have $b \cdot c \leq b$ and $b \cdot c \leq c$. By (iii) again, we get $a \cdot (b \cdot c) \leq a \cdot b$. Since $b \cdot c \leq c$, we get $(a \cdot (b \cdot c)) \cdot (b \cdot c) \leq (a \cdot b) \cdot c$. By $(c_r)$, we get $a \cdot (b \cdot c) \leq (a \cdot b) \cdot c$.

Moreover, for any (residuated) basic algebra $A$ and $a, b \in A$, it is easy to check that $a = b$ iff $\forall x \in A (x \leq a \leftrightarrow x \leq b)$. This gives a way for showing that an equation $a = b$ hold in $A$.

Theorem 3. Let $A = (A, \wedge, \vee, T, \bot, \rightarrow, \leftarrow)$ be a residuated basic algebra. Then $(A, \wedge, \vee, T, \bot, \rightarrow)$ is a basic algebra.
Proof. It suffices to show that all axioms of basic algebra hold in every residuated basic algebra. Obviously, \((A, \land, \lor, \top, \bot)\) is a bounded distributive lattice.

(i) Assume \(x \leq a \to (b \land c)\). By residuation, \(a \cdot x \leq b \land c\). Since \(b \land c \leq b\) and \(b \land c \leq c\), we get \(a \cdot x \leq b\) and \(a \cdot x \leq c\). Thus \(x \leq a \to b\) and \(x \leq a \to c\). Hence \(x \leq (a \to b) \land (a \to c)\). Conversely, assume \(x \leq (a \to b) \land (a \to c)\). Then \(x \leq a \to b\) and \(x \leq a \to c\). By residuation, \(a \cdot x \leq b\) and \(a \cdot x \leq c\). Thus \(a \cdot x \leq b \land c\), and so \(x \leq (a \to b \land c)\). Hence \(a \to (b \land c) = (a \to b) \land (a \to c)\).

(ii) Assume \(x \leq (b \lor c) \to a\). By residuation, we get \((b \lor c) \cdot x \leq a\). Since \(b \leq b \lor c\), we have \(b \cdot x \leq (b \lor c) \cdot x\). Similarly we get \(c \cdot x \leq a\). Thus by residuation, we have \(x \leq b \to a\) and \(x \leq c \to a\). Hence \(x \leq (b \to a) \land (c \to a)\). Conversely, assume \(x \leq (b \to a) \land (c \to a)\). Then \(x \leq b \to a\) and \(x \leq c \to a\). By residuation, \(b \cdot x \leq a\) and \(c \cdot x \leq a\). Then \(b \cdot x \lor c \cdot x \leq a\). By proposition \([1](i)\), we get \((b \lor c) \cdot x \leq b \cdot x \lor c \cdot x\), and so we get \((b \lor c) \cdot x \leq a\). Hence \((b \lor c) \to a = (b \to a) \land (c \to a)\).

(iii) Obviously \(a \to a \leq \top\). Since \(a \cdot \top \leq a\), we get \(\top \leq a \to a\). Hence \(a \to a = \top\). Again, since \(\top \cdot a \leq a\), by residuation we get \(a \leq \top \to a\).

(iv) Assume \(x \leq (a \to b) \land (b \to c)\). Then \(x \leq (a \to b)\) and \(x \leq (b \to c)\), whence \(a \cdot x \leq b\) and \(b \cdot x \leq c\). Then by \([1](iii)\), we get \((a \cdot x) \cdot x \leq b \cdot x\). Hence \((a \cdot x) \cdot x \leq c\). Since \(a \cdot x \leq (a \cdot x) \cdot x\), we have \(a \cdot x \leq c\), which yields \(x \leq a \to c\). Hence \((a \to b) \land (b \to c) \leq (a \to c)\).

Now let us define the residuated basic logic \((RBL)\) of residuated basic algebras. The language \(L_{RBL}\) is the extension of \(L_{BPL}\) by adding binary operators \(\cdot\) and \(\leftarrow\). The set of all \(L_{RBL}\)-formulae is defined recursively by:

\[
A ::= p \mid \bot \mid \top \mid A \land A \mid A \lor A \mid A \cdot A \mid A \to A \mid A \leftarrow A
\]

where \(p \in \text{Prop}\). An assignment \(\sigma\) in a residuated basic algebra \(A\) is a homomorphism from the \(L_{RBL}\)-formula algebra to \(A\). An \(L_{RBL}\)-formula \(A\) is valid in \(A\) (notation: \(A \vDash \sigma\)), if \(\sigma(A) = \top\) for all assignments \(\sigma\) in \(A\). By \(RBA \vDash A\) we mean that \(A \vDash A\) for all residuated basic algebras. Thus we define the residuated basic logic \(RBL = \{A \mid RBA \vDash A\}\) and \(A\) is an \(L_{RBL}\)-formula).

3 Conservative Extension

A logic \(L_2\) is said to be a conservative extension of \(L_1\), if every formula provable in \(L_1\) is provable in \(L_2\). In this section, we introduce an algebraic system \(S_{RBL}\) for residuated basic algebra, and show that \(S_{RBL}\) is a
conservative extension of BPL (cf. theorem 6). It follows that RBL is a conservative extension of BPL.

Simple $S_{RBL}$-sequents are expressions of the form $A \Rightarrow B$ where $A$ and $B$ are $L_{RBL}$-formulae. The algebraic system $S_{RBL}$ for residuated basic algebras consists of the following axioms and rules:

\begin{align*}
(Id) & \quad A \Rightarrow A \\
(D) & \quad A \land (B \lor C) \Rightarrow (A \land B) \lor (A \land C) \\
(W_l) & \quad A \cdot \top \Rightarrow A \\
(W_r) & \quad \top \cdot A \Rightarrow A \\
(RC) & \quad A \cdot B \Rightarrow (A \cdot B) \cdot B \\
(R1) & \quad A \cdot B \Rightarrow C \\
(R2) & \quad B \Rightarrow A \rightarrow C \\
(R3) & \quad A \Rightarrow C \leftarrow B \\
(R4) & \quad A \Rightarrow C \\
(\land L) & \quad A_i \Rightarrow B \\
\frac{A_1 \land A_2 \Rightarrow B, i \in \{1, 2\}}{A_1 \land A_2 \Rightarrow B} \\
(\land R) & \quad C \Rightarrow A \\
\frac{C \Rightarrow A}{C \Rightarrow A \land B} \\
(\lor L) & \quad A \Rightarrow C \\
\frac{A \lor B \Rightarrow C}{A \lor B \Rightarrow C} \\
(\lor R) & \quad C \Rightarrow A_i \\
\frac{C \Rightarrow A_i}{C \Rightarrow A_1 \lor A_2, i \in \{1, 2\}} \\
\end{align*}

By $\vdash_{S_{RBL}} A \Rightarrow B$ we mean that $A \Rightarrow B$ is provable in $S_{RBL}$. We say a $S_{RBL}$-formula $A$ is provable in $S_{RBL}$, if $\vdash_{S_{RBL}} \top \Rightarrow A$.

It is easy to prove that the system $S_{RBL}$ is complete with respect to the class of all bounded distributive lattice order residuated groupoid enriched with weakening and restricted contraction $(c_t)$, which is exactly the class $\mathbb{RBA}$ of all residuated basic algebras. Let us explain some basic notions. An assignment $\sigma$ in a residuated basic algebra $A$ is a homomorphism from the $L_{RBL}$-formula algebra into $A$. A sequent $A \Rightarrow B$ is true under an assignment $\sigma$ in a residuated basic algebra $A$, if $\sigma(A) \leq \sigma(B)$ in $A$. We say that $A \Rightarrow B$ is valid in $A$, if $A \Rightarrow B$ is true under all assignments in $A$.

By $\mathbb{RBA} \models A \Rightarrow B$ we mean that $A \Rightarrow B$ is valid in all residuated basic algebras. The completeness means that $\vdash_{S_{RBL}} A \Rightarrow B$ iff $\mathbb{RBA} \models A \Rightarrow B$.

Let $S_{RBL}^*$ be the system obtained from $S_{RBL}$ by replacing $(W_l), (W_r)$ and $(RC)$ by the following three axioms respectively:

\begin{align*}
(\top^1) & \quad \top \Rightarrow A \rightarrow A \\
(\top^2) & \quad \top \Rightarrow A \rightarrow A \\
(Tr) & \quad (A \rightarrow B) \land (B \rightarrow C) \Rightarrow A \rightarrow C.
\end{align*}

Theorem 4. The system $S_{RBL}^*$ is equivalent to $S_{RBL}$. 
Proof. It suffices to show that \((W_l), (W_r)\) and (RC) are equivalent to \((\top^1), (\top^2)\) and (Tr) respectively. By the proof of theorem 3, we know that \((W_l), (W_r)\) and (RC) imply \((\top^1), (\top^2)\) and (Tr) respectively. Conversely, assume \(\top \Rightarrow A \Rightarrow A\). By (R3), we have \(A \cdot \top \Rightarrow A\). Assume \(A \Rightarrow \top \Rightarrow A\).

By (R3), we have \(\top \cdot A \Rightarrow A\). Finally, assume (Tr) holds. Since \(A \cdot B \Rightarrow A \cdot B\) and \((A \cdot B) \cdot B \Rightarrow (A \cdot B) \cdot B\), by residuation, we obtain \(B \Rightarrow A \Rightarrow (A \cdot B)\) and \(B \Rightarrow (A \cdot B) \Rightarrow ((A \cdot B) \cdot B)\). Hence \(B \Rightarrow (A \Rightarrow (A \cdot B)) \Rightarrow ((A \cdot B) \Rightarrow ((A \cdot B) \cdot B))\). Since \((A \Rightarrow (A \cdot B)) \Rightarrow ((A \cdot B) \Rightarrow ((A \cdot B) \cdot B))\), one gets \(B \Rightarrow A \Rightarrow ((A \cdot B) \cdot B)\). By (R2), we have \(A \cdot B \Rightarrow (A \cdot B) \cdot B\).

Now let us prove that \(S^\ast_{RBL}\) is a conservative extension of BPL (cf. theorem 3). In the whole proof, we use the relational model from [3] (\(S^\ast_{RBL}\)-model in definition 3), in which a ternary relation is used to interpret binary operators. The main technique used in the proof is to construct a \(S^\ast_{RBL}\)-model \(\mathfrak{M}^\ast\) from a BPL-model \(\mathfrak{M}\). By our construction, the model \(\mathfrak{M}^\ast\) is a model for the system \(S^\ast_{RBL}\) (cf. lemma 2). This fact is enough for showing that every \(L_{BPL}\)-formula \(A\) provable in \(S^\ast_{RBL}\) (i.e., \(\vdash_{S^\ast_{RBL}} A\)) is also provable in BPL. Let us introduce the relational semantics first.

**Definition 3.** An residuated basic frame is a pair \(\mathcal{F} = (W, R)\) where \(W\) is a nonempty set of states, and \(R \subseteq W^3\) is a ternary relation. An \(S^\ast_{RBL}\)-model is a triple \(\mathfrak{M} = (W, R, V)\), where \((W, R)\) is a residuated basic frame and \(V : \text{Prop} \rightarrow \wp(W)\) (the powerset of \(W\)) is a valuation function.

**Definition 4.** The satisfiability relation \(\mathfrak{M}, a \models A\) between an \(S^\ast_{RBL}\)-model \(\mathfrak{M}\) with state \(a\) and a \(L_{RBL}\)-formula \(A\) is defined recursively as follows:

\[
- \mathfrak{M}, a \models p, \text{ if } a \in V(p).
- \mathfrak{M}, a \models \top \text{ and } \mathfrak{M}, a \not\models \bot.
- \mathfrak{M}, a \models A \cdot B, \text{ if there exist } a_2, a_3 \in W \text{ such that } R(a, a_2, a_3), \mathfrak{M}, a_2 \models A \text{ and } \mathfrak{M}, a_3 \models B.
- \mathfrak{M}, a \models A \leftarrow B, \text{ if for all } a_1, a_3 \in W \text{ with } R(a_1, a, a_3), \mathfrak{M}, a_3 \models B \text{ implies } \mathfrak{M}, a_1 \models A.
- \mathfrak{M}, a \models A \rightarrow B, \text{ if for all } a_1, a_2 \in W \text{ with } R(a_1, a_2, a), \mathfrak{M}, a_2 \models A \text{ implies } \mathfrak{M}, a_1 \models B.
- \mathfrak{M}, a \models A \land B, \text{ if } \mathfrak{M}, a \models A \text{ and } \mathfrak{M}, a \models B.
- \mathfrak{M}, a \models A \lor B, \text{ if } \mathfrak{M}, a \models A \text{ or } \mathfrak{M}, a \models B.
\]

An \(L_{RBL}\)-formula \(A\) is satisfiable, if there exist a relational model \(\mathfrak{M}\) and a state \(a\) in \(\mathfrak{M}\) such that \(\mathfrak{M}, a \models A\). We say that \(A\) is true in \(\mathfrak{M}\) (notation: \(\mathfrak{M} \models A\)), if \(\mathfrak{M}, a \models A\) for all states \(a\) in \(\mathfrak{M}\). An \(S^\ast_{RBL}\)-sequent \(A \Rightarrow B\) is true at \(a\) in an \(S^\ast_{RBL}\)-model \(\mathfrak{M}\) (notation: \(\mathfrak{M}, a \models A \Rightarrow B\)), if \(\mathfrak{M}, a \models A\) implies
$\mathfrak{J}, a \models B$. A sequent $A \Rightarrow B$ is true in $\mathfrak{J}$ (notation: $\mathfrak{J} \models A \Rightarrow B$), if $\mathfrak{J}, a \models A \Rightarrow B$ for all states $a$ in $\mathfrak{J}$.

Now we construct a tenary $S^*_RBL$-model from a binary BPL-model, which will be shown to be a model for the system $S^*_RBL$.

**Definition 5.** Given a BPL-model $M = (W, R, V)$, define the $S^*_RBL$-model $\mathfrak{J}M = (W', R', V')$ constructed from $M$ as follows:

1. $W' = \{a_1, a_2 | a \in W\}$
2. $R' = \{(b_1, b_2, a_1), (b_2, b_1, a_1), (b_1, b_2, a_2), (b_2, b_1, a_2) | aRb\}$
3. $V'(p) = \{a_i \in W : i \in \{1, 2\} \land a \in V(p)\}$, for each $p \in \text{Prop}$.

Henceforth we show some properties of the induced tenray models.

**Lemma 1.** Let $\mathfrak{M} = (W, R, V)$ be a BPL-model and $\mathfrak{J}^\mathfrak{M} = (W', R', V')$. For any $\mathcal{L}_{BPL}$-formula $A$, $a \in W$ and $a_i \in W'$, $\mathfrak{M}, a \models A$ iff $\mathfrak{J}^\mathfrak{M}, a_i \models A$.

**Proof.** By induction on the complexity of $A$. The cases of propositional letters, disjunction and conjunction are easy. We consider only the case of implication. Let $A = A_1 \rightarrow A_2$. Assume that $\mathfrak{J}^\mathfrak{M}, a_i \not\models A_1 \rightarrow A_2$. Suppose $a_i = a_1$ without loss of generality. Then there exist $a, b \in W$ such that $Rab$ and $R'(b_1, b_2, a_1)$. By the construction of $\mathfrak{J}^\mathfrak{M}$, we have $\mathfrak{J}^\mathfrak{M}, b_2 \models A_1$ and $\mathfrak{J}^\mathfrak{M}, b_1 \not\models A_2$. By induction hypothesis, $\mathfrak{M}, b \models A_1$ and $\mathfrak{M}, b \not\models A_2$. Then $\mathfrak{M}, a \not\models A_1 \rightarrow A_2$. Conversely, assume $\mathfrak{M}, a \not\models A_1 \rightarrow A_2$. Then there exists $b \in W$ such that $Rab$ and $R'(b_1, b_2, a_1)$. By induction hypothesis, $\mathfrak{J}^\mathfrak{M}, b_2 \models A_1$ and $\mathfrak{J}^\mathfrak{M}, b_1 \not\models A_2$. Since $Rab$, we have $R'(b_1, b_2, a_1)$. Hence $\mathfrak{J}^\mathfrak{M}, a \not\models A_1 \rightarrow A_2$.

**Corollary 1.** For any $\mathcal{L}_{BPL}$-formula $A$ and model $\mathfrak{M}$, $\mathfrak{M} \models A$ iff $\mathfrak{J}^\mathfrak{M} \models A$.

**Lemma 2.** For any BPL-model $\mathfrak{M}$ and sequent $A \Rightarrow B$ where $A, B$ are $\mathcal{L}_{BPL}$-formulae, if $\vdash_{S^*_RBL} A \Rightarrow B$, then $\mathfrak{J}^\mathfrak{M} \models A \Rightarrow B$. 
Proof. We need to show that all axioms and rules are admissible in $\mathcal{F}_R$. Axioms except $(\top^1), (\top^2), (\text{Tr})$ are easily shown to be true in $\mathcal{F}_R$. We prove that the three axioms are true in $\mathcal{F}_R$ as follows.

$(\top^1)$ It is obvious that $\mathfrak{M}, a \models A \rightarrow A$ for any $a$ in $\mathfrak{M}$. By lemma [4], we get $\mathfrak{M}, a_i \models A \rightarrow A$. Hence $\mathfrak{M}, a_i \models \top \Rightarrow A \rightarrow A$.

$(\top^2)$ Assume $\mathfrak{M}, a_i \models A$. By lemma [4], we get $\mathfrak{M}, a \models A$. For any state $b$ in $\mathfrak{M}$ such that $Rab$, we have $\mathfrak{M}, b \models A$. Then $\mathfrak{M}, a \models \top \Rightarrow A \rightarrow A$. By lemma [4], we get $\mathfrak{M}, a_i \models \top \rightarrow A$. Hence $\mathfrak{M}, a_i \models A \Rightarrow \top \rightarrow A$.

$(\text{Tr})$ Assume $\mathfrak{M}, a_i \models (A \rightarrow B) \land (B \rightarrow C)$ but $\mathfrak{M}, a_i \not\models A \rightarrow C$. Then there exist $b_1, b_2 \in W'$ such that $R(b_1, b_2, a_i)$, $\mathfrak{M}, b_2 \models A$ and $\mathfrak{M}, b_1 \not\models C$. By $\mathfrak{M}, b_2 \models A$, $\mathfrak{M}, a_i \models A \rightarrow B$ and $R(b_1, b_2, a_i)$, we get $\mathfrak{M}, b_1 \models B$. Since $\mathfrak{M}, b \models B \rightarrow C$, by lemma [4], we get $\mathfrak{M}, a \models B \rightarrow C$, $\mathfrak{M}, b \models B$, $\mathfrak{M}, b \not\models C$. Since $R(b_1, b_2, a_i)$, we get $Rab$. Then by $\mathfrak{M}, a \models B \rightarrow C$ and $\mathfrak{M}, b \models B$, we get $\mathfrak{M}, b \models C$, a contradiction.

Consider the rules $(\text{AL}), (\land R), (\lor L)$ and $(\text{Cut})$. Other rules are shown to be admissible in $\mathcal{F}_R$ similarly. First, for $(\land L)$, assume $\mathfrak{M}, a_1 \models A_1 \Rightarrow B$ and $\mathfrak{M}, a_1 \models A_1 \land A_2$. Then $a_1 \models A_1$, and so $a_1 \models B$. Hence $\mathfrak{M}, a_1 \models A_1 \land A_2 \Rightarrow B$. Second, for $(\text{Cut})$, assume that $a_1 \models A \Rightarrow B$ and $a_1 \models B \Rightarrow C$ and $a_1 \models C$. By definition, we get $a_1 \models C$. Hence $a_1 \models A \Rightarrow C$.

Now it suffices to show the admissibility of residuation rules.

(R1) Assume $\mathfrak{M}, a_1 \models A \cdot B \Rightarrow C$. Suppose $\mathfrak{M}, a_1 \models B$ but $\mathfrak{M}, a_1 \not\models A \rightarrow C$ for some $a_1 \in W'$. Then there are $b_1, b_2$ such that $R'(b_1, b_2, a_1)$, $\mathfrak{M}, b_2 \models A$ and $\mathfrak{M}, b_1 \not\models C$. Thus $\mathfrak{M}, b_1 \models A \cdot B$. By assumption, $\mathfrak{M}, b_1 \models C$, a contradiction.

(R2) Assume $\mathfrak{M}, a_1 \models B \Rightarrow A \rightarrow C$. Suppose $\mathfrak{M}, a_1 \models A \cdot B$ but $\mathfrak{M}, a_1 \not\models C$. Then there exists $b_1 \in W'$ such that $R'(a_1, a_2, b_1)$, $\mathfrak{M}, a_2 \models A$ and $\mathfrak{M}, b_1 \models B$. By assumption, $\mathfrak{M}, b_1 \models A \rightarrow C$. By $\mathfrak{M}, a_2 \models A$, we get $\mathfrak{M}, a_1 \models C$, a contradiction.

(R3) Assume $\mathfrak{M}, a_1 \models A \cdot B \Rightarrow C$. Suppose $\mathfrak{M}, b_2 \models A$ but $\mathfrak{M}, b_2 \not\models C \Leftarrow B$. Then there exists $a_1 \in W'$ such that $R'(a_1, a_2, b_1)$, $\mathfrak{M}, a_1 \models B$ and $\mathfrak{M}, b_1 \not\models C$. Hence $\mathfrak{M}, b_1 \models A \cdot B$. By assumption, we have $\mathfrak{M}, b_1 \models C$, a contradiction.

(R4) Assume $\mathfrak{M}, a_1 \models A \Rightarrow C \Leftarrow B$. Suppose $\mathfrak{M}, a_1 \models A \cdot B$ but $\mathfrak{M}, a_1 \not\models C$. Then there exists $a_1 \in W'$ such that $R'(a_1, a_2, b_1)$, $\mathfrak{M}, a_1 \models B$ and $\mathfrak{M}, a_1 \models C$. By assumption, $\mathfrak{M}, b_2 \models C \Leftarrow B$. Hence $\mathfrak{M}, a_1 \models C$, a contradiction.

Now we can show the following conservative extension theorem.

Theorem 5. For any $\mathcal{L}_\text{BPL}$-formula $A$, $\vdash_{\text{BPL}} A$ iff $\vdash_{\text{SBL}} \top \Rightarrow A$. 

Proof. Assume \( \vdash_{BPL} A \). Then \( \mathcal{BLA} \models A \) since BPL is complete with respect to the class \( \mathcal{BLA} \) of all basic algebras. By theorem 3, \( \mathcal{RBA} \models A \). Hence \( \mathcal{RBA} \models \top \iff A \). By completeness of BPL with respect to the class of all BPL-models, there exists a BPL-model \( M \) such that \( M \not\models A \). Then by corollary 1, \( M \not\models A \). Hence by the definition of truth of a sequent in a \( S_{RBL}^* \)-model, \( M \not\models \top \Rightarrow A \). By lemma 3, we obtain \( \vdash_{S_{RBL}} \top \Rightarrow A \). By theorem 4, we obtain the following theorems immediately.

Theorem 6. For any \( L_{BPL} \)-formula \( A \), \( \vdash_{BPL} A \) iff \( \vdash_{S_{RBL}} \top \iff A \).

Corollary 2. For every \( L_{BPL} \)-formula \( A \), \( \mathcal{BLA} \models A \) iff \( \mathcal{RBA} \models A \).

Theorem 7. The logic RBL is a conservative extension of BPL.

4 A Sequent Calculus for RBL

The algebraic system \( S_{RBL} \) we introduced in section 3 for residuated basic algebra is equivalent to DFN with weakening and the restricted contraction rule (RC). Let us introduce a sequent formalization \( L_{RBL} \) for \( S_{RBL} \), and show the cut elimination which implies the subformula property for \( L_{RBL} \). For general knowledge on sequent calculi for substructural logics, see Ono’s survey paper [\text{?}].

\( L_{RBL} \)-formula structures are defined recursively as follows: (i) every \( L_{RBL} \)-formula is a formula structure; (ii) if \( \Gamma \) and \( \Delta \) are formula structures, then \( \Gamma \circ \Delta \) and \( \Gamma \otimes \Delta \) are formula structures. Each formula structure \( \Gamma \) is associated with a formula \( \mu(\Gamma) \) defined as follows: (i) \( \mu(A) = A \) for every \( L_{RBL} \)-formula \( A \); (ii) \( \mu(\Gamma \circ \Delta) = \mu(\Gamma) \cdot \mu(\Delta) \); (iii) \( \mu(\Gamma \otimes \Delta) = \mu(\Gamma) \land \mu(\Delta) \). Sequents are of the form \( \Gamma \Rightarrow A \) where \( \Gamma \) is an \( L_{RBL} \)-formula structure and \( A \) is an \( L_{RBL} \)-formula.

The sequent calculus \( L_{RBL} \) consists of the following axioms and rules:

\[(\text{Id}) \quad A \Rightarrow A \quad (\top) \quad A \Rightarrow \top \quad (\bot) \quad \bot \Rightarrow A \]

\[(\rightarrow L) \quad \frac{\Delta \Rightarrow A; \quad \Gamma[B] \Rightarrow C}{\Gamma[\Delta \circ (A \rightarrow B)] \Rightarrow C} \quad (\rightarrow R) \quad \frac{A \circ \Gamma \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B} \]

\[(\leftarrow L) \quad \frac{\Gamma[A] \Rightarrow C; \quad \Delta \Rightarrow B}{\Gamma[(A \leftarrow B) \circ \Delta] \Rightarrow C} \quad (\leftarrow R) \quad \frac{\Gamma \circ B \Rightarrow A}{\Gamma \Rightarrow A \leftarrow B} \]

\[(\cdot L) \quad \frac{\Gamma[A \circ B] \Rightarrow C}{\Gamma[A \cdot B] \Rightarrow C} \quad (\cdot R) \quad \frac{\Gamma \Rightarrow A; \quad \Delta \Rightarrow B}{\Gamma \circ \Delta \Rightarrow A \cdot B} \]
The rules (→R) and (←R) are restricted to nonempty sequences Γ. The following semi-associativity ∘-rule is admissible in L_RBL:

\[ (\circ A^1) \frac{\Gamma[(A \circ A_1) \circ A_2] \Rightarrow A}{\Gamma[A \circ (A_1 \circ A_2)] \Rightarrow A} \]

By standard techniques (cf. [?]), it is easy to show that the sequent calculus L_RBL is equivalent to the algebraic system S_RBL, i.e., for any sequent \( \Gamma \Rightarrow A, \vdash_{S_RBL} \mu(\Gamma) \Rightarrow A \) iff \( \vdash_{L_RBL} \Gamma \Rightarrow A \).

For showing the subformula property of L_RBL, we prove that the cut rule except the case that the cut formula is \( \bot \) or \( \top \) can be eliminated. We introduce the generalized mix rule instead of (Cut) and show the ‘mix elimination’, i.e., every sequent derivable in L_RBL has a derivation without using the following mix rule:

\[ (\text{Mix}) \frac{\Delta \Rightarrow A; \; \Gamma[A] \ldots [A] \Rightarrow B}{\Gamma[\Delta] \ldots [\Delta] \Rightarrow B} \]

where \( \Gamma[A] \ldots [A] \) denotes the formula structure containing at least one occurrence of \( A \), and \( \Gamma[\Delta] \ldots [\Delta] \) denotes the formula structure obtained by replacing at least one occurrence of \( A \) in \( \Gamma[A] \ldots [A] \) by the formula structure \( \Delta \). By L_{RBL}^m, we mean the system obtained from L_RBL by replacing the (Cut) by the above (Mix) rule. Obviously, L_RBL is equivalent to L_{RBL}^m.

**Theorem 8.** A sequent is derivable in L_{RBL}^m has a derivation without using (Mix) except the case that the mix formula is \( \bot \) or \( \top \).
Proof. We prove the admissibility of the rule (Mix) by induction: 1) on the complexity of the mixed formula \( A \); 2) on the length of the proof of \( \Gamma[A] \ldots [A] \Rightarrow B \); and 3) on the length of the proof of \( \Delta \Rightarrow A \). Let \( \Delta \Rightarrow A \) be obtained by rule \( R_1 \), and \( \Gamma[A] \ldots [A] \) by rule \( R_2 \). We consider two cases.

Case 1. \( A \) is not produced by \( R_1 \) or \( R_2 \). Let us consider the following subcases.

Case 1.1 \( R_1 = Id \) or \( R_2 = Id \). Assume \( R_1 = Id \). Then \( \Delta = A \) and hence \( \Gamma[\Delta] \ldots [\Delta] = \Gamma[A] \ldots [A] \). Thus we can eliminate this application of (Mix). The case of \( R_2 = Id \) is similar.

Case 1.2 \( R_1 = (\bot) \). Note that the case \( R_2 = (\bot) \) does not apply to the mix rule. When \( R_1 = (\bot) \), the antecedent of the conclusion \( \Gamma[\Delta] \ldots [\Delta] \) contains \( \bot \). Hence by (\( \bot \)) and (W) we get the conclusion sequent.

Case 1.3 \( R_2 = (\top) \). Note that the case \( R_1 = (\top) \) does not occur. Let the premises of (Mix) are \( \Delta \Rightarrow A \) and \( A \Rightarrow \top \), and the conclusion of (Mix) be \( \Delta \Rightarrow \top \). Then \( \Delta \Rightarrow \top \) is obtained by (W\( ^1 \)) and axiom (\( \top \)).

Case 1.4 \( R_1 = (\bot L) \) or \( R_2 = (\bot L) \). Assume \( R_1 = (\bot L) \). Let the premise of \( R_1 \) be \( \Delta[C \odot D] \Rightarrow A \). We use mix rule to \( \Delta[C \odot D] \Rightarrow A \) and \( \Gamma[A] \ldots [A] \Rightarrow B_1 \). We get \( \Gamma[\Delta] \ldots [\Delta] \Rightarrow B_1 \). Then by the rule (\( \bot L \)) we get \( \Gamma[\Delta C \cdot D] \) \ldots \( \Delta[C \cdot D] \Rightarrow B \). The application of mix rule can be eliminated by induction 3). The case of \( R_2 = (\bot L) \) is analogous.

Case 1.5 \( R_1 = (\wedge L) \) or \( R_2 = (\wedge L) \). The proof is quite similar to case 1.4.

Case 1.6 \( R_2 = (\top R) \). Assume that the premises of \( R_2 \) are \( \Gamma_1[A] \ldots [A] \Rightarrow B_1 \) and \( \Gamma_2[A] \ldots [A] \Rightarrow B_2 \). We apply mix rule to \( \Delta \Rightarrow A \) and \( \Gamma[A] \ldots [A] \Rightarrow B_1 \). We get \( \Gamma_1[\Delta] \ldots [\Delta] \Rightarrow B_1 \). Similarly we get \( \Gamma_2[\Delta] \ldots [\Delta] \Rightarrow B_2 \). Then by (\( \top R \)), we get \( \Gamma_1[\Delta] \ldots [\Delta] \odot \Gamma_2[\Delta] \ldots [\Delta] \Rightarrow B_1 \cdot B_2 \). The application of mix rule can be eliminated by induction 2).

Case 1.7 \( R_2 = (\wedge R) \). The proof is quite similar to the case 1.6.

Case 1.8 \( R_2 = (\top R) \). The proof is quite similar to the case \( R_2 = (\bot L) \).

Case 1.9 \( R_1 = (\lor L) \) or \( R_2 = (\lor L) \). For the case \( R_1 = (\lor L) \), assume the premises of \( R_1 \) are \( \Delta[C] \Rightarrow A \) and \( \Delta[D] \Rightarrow A \). We apply mix rule to \( \Delta[C] \Rightarrow A \) and \( \Gamma[A] \ldots [A] \Rightarrow B \), and get \( \Gamma[\Delta[C]] \ldots \Gamma[\Delta[C]] \Rightarrow B \). Similar, we get \( \Gamma[\Delta[D]] \ldots \Gamma[\Delta[D]] \Rightarrow B \). Then by (\( \lor L \)), \( \Gamma[\Delta[C \vee D]] \ldots \Gamma[\Delta[C \vee D]] \Rightarrow B \). The case \( R_2 = (\lor L) \) is rather similar to the case \( R_2 = (\bot R) \).

Case 1.10 \( R_1 = (\rightarrow L) \) or \( R_2 = (\rightarrow L) \). For the case \( R_1 = (\rightarrow L) \), assume the premises of \( R_1 \) are \( \Delta_1 \Rightarrow C \) and \( \Delta_2[D] \Rightarrow A \). First we apply mix rule to \( \Delta_2[D] \Rightarrow A \) and \( \Gamma[A] \ldots [A] \Rightarrow B \) and get \( \Gamma[\Delta_2[D]] \ldots [\Delta_2[D]] \Rightarrow B \). Then we apply (\( \rightarrow L \)) to \( \Delta_1 \Rightarrow C \) and \( \Gamma[\Delta_2[D]] \ldots [\Delta_2[D]] \Rightarrow B \), and
get the sequent \( \Gamma[\Delta_2[\Delta_1 \odot (C \rightarrow D)]] \ldots [\Delta_2[\Delta_1 \odot (C \rightarrow D)]] \Rightarrow B \). By induction 2), the use of mix rule can be eliminated. For the case \( R_2 = (\rightarrow L) \), let the conclusion be \( \Gamma'[\Delta'[A] \ldots [A] \odot (C \rightarrow D)][A] \ldots [A] \Rightarrow B \), and the premises \( \Delta'[A] \ldots [A] \Rightarrow C \) and \( \Gamma'[D][A] \ldots [A] \Rightarrow B \). We apply mix rule first to \( \Delta \Rightarrow A \) and \( \Delta'[A] \ldots [A] \Rightarrow C \), \( \Gamma'[D][A] \ldots [A] \Rightarrow B \) respectively. Again, apply \((\rightarrow L)\) to both resulting sequents. We obtain the sequent \( \Gamma'[\Delta'[\Delta] \odot (C \rightarrow D)][\Delta] \ldots [\Delta] \Rightarrow B \). By induction 2), applications of mix rule can be eliminated.

Case 1.11 \( R_1, R_2 \in \{\oplus A^1, \oplus A^2, \odot C, \odot E, W\} \). Apply the mix rule first to premises, and then use the corresponding rule. It is easy to eliminate applications of mix rule by induction 2).

Case 2. \( A \) is created by \( R_1 \) and \( R_2 \). We consider the following subcases.

Case 2.1 \( A = A_1 \cdot A_2 \). Let the two premises of mix rule are the following: \( \Delta_1 \odot \Delta_2 \Rightarrow A_1 \cdot A_2 \) which is obtained from \( \Delta_1 \Rightarrow A_1 \) and \( \Delta_2 \Rightarrow A_2 \) by \((\cdot R)\), and \( \Gamma[A_1 \cdot A_2][A] \ldots [A] \Rightarrow B \) which is obtained from \( \Gamma[A_1 \odot A_2][A] \ldots [A] \Rightarrow B \) by \((\cdot L)\). Now we apply mix rule to \( \Delta_1 \odot \Delta_2 \Rightarrow A_1 \cdot A_2 \) and \( \Gamma[A_1 \odot A_2][A] \ldots [A] \Rightarrow B \), and get \( \Gamma[A_1 \odot A_2][\Delta_1 \odot \Delta_2] \ldots [\Delta_1 \odot \Delta_2] \Rightarrow B \). Then by applying mix rule to \( \Delta_1 \Rightarrow A \) and the resulting sequent, we get \( \Gamma[\Delta_1 \odot A_2][\Delta_1 \odot \Delta_2] \ldots [\Delta_1 \odot \Delta_2] \Rightarrow B \). Similary, we get \( \Gamma[\Delta_1 \odot A_2][\Delta_1 \odot \Delta_2] \ldots [\Delta_1 \odot \Delta_2] \Rightarrow B \). Thus by induction 2) and 1), mix rule can be eliminated.

Case 2.2 \( A = A_1 \rightarrow A_2 \) or \( A_1 \leftarrow A_2 \). The proof is similar to case 2.1.

Case 2.3 \( A = A_1 \wedge A_2 \). Let the two premises of mix rule are the following: \( \Delta \Rightarrow A_1 \wedge A_2 \) which is obtained from \( \Delta \Rightarrow A_1 \) and \( \Delta \Rightarrow A_2 \) by \((\wedge R)\), and \( \Gamma[A_1 \wedge A_2][A] \ldots [A] \Rightarrow B \) which is obtained from \( \Gamma[A_1 \odot A_2][A] \ldots [A] \Rightarrow B \) by \((\wedge L)\). Now first we apply mix rule to \( \Delta \Rightarrow A_1 \wedge A_2 \) and \( \Gamma[A_1 \odot A_2][A] \ldots [A] \Rightarrow B \), and get \( \Gamma[A_1 \odot A_2][\Delta] \ldots [\Delta] \Rightarrow B \). Then by applying mix rule to \( \Delta \Rightarrow A_1 \) and \( \Delta \Rightarrow A_2 \) and the resulting sequent, we get \( \Gamma[\Delta \odot \Delta][\Delta] \ldots [\Delta] \Rightarrow B \). By \((\odot C)\), we get \( \Gamma[\Delta] \ldots [\Delta] \Rightarrow B \). By induction 2) and 1), mix rule can be eliminated.

Case 2.4 \( A = A_1 \vee A_2 \). The proof is quite similar case 2.3.

**Corollary 3 (Subformula Property).** If \( \Gamma \Rightarrow A \) has a derivation in \( L_{RBL} \), then all formulæ in the derivation are \( \top, \bot \), or subformulæ of \( \Gamma, A \).

**Theorem 9 (Disjunction Property).** For any \( L_{RBL} \)-formulæ \( A \) and \( B \), if \( \top \Rightarrow A \vee B \) is derivable in \( L_{RBL} \), then \( \top \Rightarrow A \) or \( \top \Rightarrow B \) is derivable.

**Proof.** Assume \( \vdash_{L_{RBL}} \top \Rightarrow A \vee B \). Then \( \vdash_{L_{RBL}} \top \Rightarrow A \vee B \). By Theorem \( \top \Rightarrow A \vee B \), the last rule can only be \((\vee R)\).

Since \( L_{RBL} \) is a conservative extension of \( BPL \), we can conclude the following disjunction property of \( BPL \).
Corollary 4. For any $L_{BPL}$-formulae $A$ and $B$, if $\vdash_{BPL} A \lor B$, then $\vdash_{BPL} A$ or $\vdash_{BPL} B$.

5 Relationships with Other Logics

The sequent calculus $LJ$ for $Int$ can be obtained from $LRBL$ by replacing $(\odot C)$ by the following full contraction rule and semi-associative rule:

$$(\odot C^*) \quad \Gamma[\Delta \odot \Delta] \Rightarrow A \quad (\odot A^2) \quad \Gamma[\Delta_1 \odot (\Delta_2 \odot \Delta_3)] \Rightarrow A$$

and dropping the nonempty restriction on $(\to R)$ and $(\leftarrow R)$.

By $(\odot C^*)$, the binary operator $\cdot$ is equivalent to $\land$, and by residuation, the implication becomes Heyting. Thus the following exchange rule in $LJ$ is derivable from $\odot C^*$ and weakening rules:

$$(\odot E) \quad \frac{\Gamma[\Delta \odot A] \Rightarrow A}{\Gamma[A \odot \Delta] \Rightarrow A}$$

but it is not derivable in $LRBL$.

The difference between $LRBL$ and $Int$ can also be shown by the following example. Let us consider the two formulae: $(†) p \land (p \to q) \to (\top \to q)$; $(‡) p \land (p \to q) \to q$. The formula $(†)$ is provable in $Int$ but not in $BPL$. In fact, $Int = BPL + (‡)$ $(?)$. However, the formula $(†)$ is a theorem of $BPL$ and hence provable in $LRBL$. The proof goes as follows.

First we can derive the sequent for any $L_{BPL}$-formulae $A$ and $B$:

$$(\#) \quad \top \cdot (A \land B) \Rightarrow (\top \cdot A) \cdot B$$

in $LRBL$: from $(\odot C)$ and $(\land R)$, we get $\top \cdot (A \land B) \Rightarrow (\top \cdot (A \land B)) \cdot (A \land B)$. By $(\land L)$ and $(\land R)$, we obtain $(\top \cdot (A \land B)) \cdot (A \land B) \Rightarrow (\top \cdot A) \cdot B$. Then by $(\land C)$ we get the required sequent. Now we can derive $\top \Rightarrow p \land (p \to q) \to (\top \to q)$ in $LRBL$: Since $(\top \cdot p) \cdot (p \to q) \Rightarrow q$ is provable, by $(\land C)$ and $(\#)$, we get $\top \odot (p \land (p \to q)) \Rightarrow q$. Then by $(\rightarrow R)$ we get the required sequent.

The contraction rule $(\odot C)$ used in above derivation is significant. It is different from $(\odot C^*)$ in $LJ$ since a nonempty prefix is needed for contraction. This point is also reflected in the difference between the two formulae $(†)$ and $(‡)$. In the consequent of $(†)$, we use $\top$ as a prefix so that $(†)$ is provable in $LRBL$. If we add $\top \to A \Rightarrow A$ as an additional axiom to $LRBL$, we can derive $p \land (p \to q) \Rightarrow q$.

Ishii et.al. $(?)$ introduced the sequent system $LBP$ for $BPL$, and proved that for any $L_{BPL}$-formula $A$, $\vdash_{LBP} A \iff \vdash_{BPL} A$. By the conservative
and, we get the following: for any \( L_{BPL} \)-formula \( A \), \( \vdash_{BPL} A \) iff \( \vdash_{L_{RBL}} \top \iff A \). Now let us show one more result on the sequent calculus \( LBP \) through our conservative extension theorems. The following rule in \( LBP \) is used:

\[
\begin{array}{c}
\Sigma, A \Rightarrow B \\
\hline
\Sigma \Rightarrow A \rightarrow B 
\end{array}
\]

**Proposition 2.** For any \( L_{BPL} \)-formulae \( A \) and \( B \), \( \vdash_{LBP} B \Rightarrow A \) iff \( \vdash_{LBP} \Rightarrow B \rightarrow A \).

**Proof.** We sketch the proof. The left-to-right direction is obvious by the rule \((\rightarrow_{n=0})\) in \( LBP \). Conversely, assume \( \not\vdash_{LBP} B \Rightarrow A \). By completeness of \( LBP \) ([?]), there exists a BPL-model \( M \) such that \( M \not\models B \Rightarrow A \). Thus there exists a state \( x \) in \( M \) such that \( M, x \models B \) but \( M, x \not\models A \). Assume that \( M \) is generated by \( x \) (otherwise, we consider the submodel of \( M \) generated from \( x \), cf.[?]). Let \( M' \) be a new BPL-model obtained from \( M \) by adding a new state \( x' \) which is accessible to \( x \) such that the valuation is the same as that in \( M \). Then by induction on the complexity of \( L_{BPL} \)-formulae, it is easy to show that \((M, x)\) and \((M', x)\) are \( L_{BPL} \)-equivalent. Hence \( M', x' \not\models B \Rightarrow A \). Thus \( 3^{\exists x} x_i \not\models B \Rightarrow A \). By construction of \( 3^{\exists x} \), it is easy to see that \( 3^{\exists x} \not\models B \cdot \top \Rightarrow A \). By lemma 2, \( \forall_{S_{RBL}} B \cdot \top \Rightarrow A \). Therefore, \( \not\vdash_{LBP} \Rightarrow B \rightarrow A \).

**Remark 2.** In [?], Ishii et.al. proved only that the proposition 2 holds in the sequent calculus LFP for Visser’s formal provability logic. It does not hold generally for any sequent \( \Gamma \Rightarrow A \), since the structural operation comma in sequent systems \( LBP \) and \( LFP \) is interpreted by \( \land \), while the structural operation \( \odot \) means \( \cdot \) in \( L_{RBL} \).

Finally, we consider the implicational fragments. By an implicational formula \( A \) we mean a formula constructing from propositional letters using only \( \rightarrow \). The implicational fragment \( BPL^- \) is the set of all implicational formulas provable in \( BPL \). Kikuchi presented in [?] a Hilbert-style axiomatization of \( BPL^- \) which consists of the following axioms and inference rule:

(I) \( A \rightarrow A \)
(K) \( A \rightarrow (B \rightarrow A) \)
(B*) \( (\Gamma \rightarrow (B \rightarrow C)) \rightarrow ((\Gamma \rightarrow (A \rightarrow B)) \rightarrow (\Gamma \rightarrow (A \rightarrow C))) \)
(MP) from $A$ and $A \rightarrow B$ infer $B$.

where $\Gamma$ is a finite sequence of formulas, and $\Gamma \rightarrow A$ is defined as follows:
(i) $\Gamma \rightarrow A = A$ if $\Gamma$ is empty; (ii) $B, \Gamma \rightarrow A = B \rightarrow (\Gamma \rightarrow A)$. The fragment $\text{BPL}^\rightarrow$ is complete with respect to the class of all transitive Kripke models.

The $\rightarrow$-fragment of $\text{RBL}$ is defined as the set $\text{RBL}^\rightarrow$ of all implicational formulae which are valid in $\text{RBA}$. Then by theorem [7] on conservative extension, it is easy to see that $\text{RBL}^\rightarrow$ is a conservative extension of $\text{BPL}^\rightarrow$.

6 Conclusion

The basic implication in Visser’s basic propositional logic $\text{BPL}$ can be formalized in substructural logic as the right residual of the product $\cdot$ (fusion) operation. The resulting logic is the residuated basic logic $\text{RBL}$ of the variety of residuated basic algebras. We develop the algebraic system $\text{SRBL}$ which is a formalization of the equational logic of residuated basic algebras. We show that $\text{SRBL}$ is a conservative extension of $\text{BPL}$. Consequently, $\text{RBL}$ is a conservative extension of $\text{BPL}$. The implicational fragment of $\text{BPL}$ is equal to the $\rightarrow$-fragment of $\text{RBL}$. Moreover, we develop the Gentzen-style sequent formalization $\text{L}_{\text{RBL}}$ for $\text{SRBL}$, and show that the cut elimination and subformula property hold for $\text{L}_{\text{RBL}}$.

Finally, the interpolation lemma, finite model property, and decidability of our sequent calculus $\text{L}_{\text{RBL}}$ can be proved by Buszkowski’s method [?] for showing interpolation and finite embedding property for classes of residuated algebras. The proof will be presented in a forthcoming paper.

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