RATIONAL SPHERES AND DOUBLE DISK BUNDLES

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Abstract. A manifold $M$ is said to be a linear double disk bundle if it has two submanifolds $B_1$ and $B_2$ for which $M$ is diffeomorphic to the normal bundles of the $B_i$ glued together along their common boundary. We show that if $M^n$ is a closed simply connected $n$-manifold with $n$ even which is simultaneously a linear double disk bundle and a rational cohomology sphere, then $M$ must be homeomorphic to a sphere.

1. Introduction

Suppose $B_1, B_2$ are closed manifolds and $D(B_1), D(B_2)$ are linear disk bundles, meaning that they are disk bundles in some Euclidean vector bundles. If the boundaries $\partial D(B_1)$ and $\partial D(B_2)$ are diffeomorphic, say via a diffeomorphism $f$, then we may form a closed smooth manifold $M := D(B_1) \cup_f D(B_2)$ by gluing the two boundaries via $f$. We call any $M$ obtained by this process a linear double disk bundle. We will occasionally use the more general notion of a double disk bundle, where the disk bundles are not necessarily linear.

We caution that the ranks of the two disk bundles $D(B_i)$ are often different. For example, $\mathbb{R}P^2$ is the union of a 2-disk (a rank 2 bundle over a point) and a Möbius band (a rank 1 disk bundle over $S^1$).

While this construction may initially seem to be quite specialized, it is surprisingly ubiquitous in geometry. We just mention a few examples below; a more comprehensive treatment is given in the introduction in [5].

First, every closed cohomogeneity one manifold is a linear double disk bundle, as shown by Mostert [19, Theorem 4 (iv.)] when the orbit space is $[0, 1]$, as a follows from [5, Proposition 3.1d] when the orbit space is $S^1$. In addition, double disk bundles arise in the study of Dupin hypersurfaces and isoparametric submanifolds of spheres [10, 20, 26]. In addition, a linear double disk bundle always admits a Riemannian metric for which the distance spheres about the zero-sections form a singular Riemannian foliation with a codimension 1 regular leaf [23]. Conversely, a singular Riemannian foliation with a codimension 1 regular leaf on a simply connected manifold gives rise to a linear double
disk bundle structure, as a consequence of the slice theorem for singular Riemannian foliations. [18].

In addition to these, we note two main motivations for the study of double disk bundles. The first stems from a question of Grove [11].

**Question 1.1.** Suppose $M$ is a closed manifold which admits a Riemannian metric of non-negative sectional curvature. Must $M$ be a linear double disk bundle?

We refer to a positive answer to Question 1.1 as the Double Soul Conjecture. We observe that some evidence for the Double Soul Conjecture includes Cheeger and Gromoll’s [3] celebrated Soul Theorem, which says that in the non-compact complete case, $M$ must be a vector bundle over a closed totally convex submanifold, called the soul. Further evidence is provided by the fact that, as mentioned above, cohomogeneity one manifolds are all linear double disk bundles. We recall that cohomogeneity one manifolds are one of the two main sources of examples of non-negatively curved Riemannian manifolds (with biquotients being the other main source). Further, all known examples of positively curved closed Riemannian manifolds are linear double disk bundles. We do note, however, that the Double Soul Conjecture is not even known for all compact Lie groups, in particular, for the exceptional Lie groups $E_7$ or $E_8$.

Our second main motivation comes from Fang’s paper on dual submanifolds [7], which, as he mentions, are synonymous with linear double disk bundles. In [7, Theorem 1.4], Fang proves the following theorem.

**Theorem 1.2.** [7, (Fang) Theorem 1.4] Let $\Sigma$ be a homotopy sphere of dimension $n$, and let $M_{\pm} \subseteq \Sigma$ be a pair of dual submanifolds of codimension $m_{\pm} + 1$ with $m_- < m_+$. If $n = 2(m_- + m_+) + 1$, then either $(m_+, m_-) = (5, 4)$ or $m_+ + m_- + 1$ is divisible by a certain function $\delta(m_-)$.

In [7, Remark 1.5], Fang then asks to what extent this result applies to dual submanifolds in rational spheres, providing counterexamples in dimensions of the form $4n - 1$. We use the terminology rational sphere to refer to a closed simply connected smooth manifold whose rational cohomology is isomorphic to that of a sphere.

Given the prevalence of double disk bundles in geometry, any result on their topology will have ramifications in several areas. With this in mind, the main purpose of this article is to prove the following theorem.

**Theorem 1.3.** Suppose $M^n$ is a closed simply connected manifold with the rational cohomology of a sphere with $n \neq 4$, $n$ even. Then $M$ is a linear double disk bundle if and only if $M$ is homeomorphic to a sphere.
If $n = 4$, $M^4$ is a double disk bundle if and only if $M$ is diffeomorphic to a sphere.

Thus, we answer Fang’s question in even dimensions: Theorem 1.2 is true for even dimensional simply connected rational spheres for the simple reason that all such such spheres are already homotopy spheres.

The case $n = 4$ was previously shown by Ge and Radeschi [9] as a consequence of their classification of singular Riemannian foliations in dimension 4. The case $n = 6$ was already shown by the author, Galaz-García, and Kerin [5].

For $n \geq 6$, the backwards implication follows from [24][Theorem 6.3] - every manifold of dimension at least 6 with torsion free cohomology admits a Morse function where the number of critical points matches the total rational Betti number. For a homotopy sphere of dimension at least 6, Morse theory then shows that every homotopy sphere is obtained by gluing two disks (that is, disk bundles over points) along their boundary. For $n \geq 8$, there existence of exotic spheres shows that “homeomorphic” can not be replace with “diffeomorphic” in general.

Lastly, note that, again contingent on the Double Soul Conjecture, Theorem 1.3 also partly explains the classification of even dimensional rational spheres among biquotients [15] and cohomogeneity one manifolds [6]. In both settings, the result is that such a manifold must be diffeomorphic to $S^{2n}$.

More importantly, if the Double Soul Conjecture is proven, it follows immediately from Theorem 1.3 that a non-negatively curved rational sphere of even dimension must be homeomorphic to $S^{2n}$ for some $n$. In contrast, in odd dimensions there are known examples of non-negatively curved rational spheres which are not homotopy spheres. We point out, however, that apart from the 5-dimensional Wu manifold $SU(3)/SO(3)$, all known examples are in dimension 3 (mod 4).

It is natural to wonder to what extent Theorem 1.3 is true when $n$ is odd. When $n$ is congruent to 3 (mod 4), Fang has shown [7, Example 3.3] there are infinitely many homotopy types of rational spheres which are linear double disk bundles. When $n$ is congruent to 1 (mod 4), the problem seems more difficult. We only have partial results.

**Theorem 1.4.** In dimensions 5, 9, and 17, there is a closed simply connected smooth manifold $M$ which is a linear double disk bundle and a rational sphere, but not a homotopy sphere.

Curiously, it follows from [5] that the Wu manifold $SU(3)/SO(3)$ is, up to diffeomorphism, the unique such example in dimension 5. Our
proof of Theorem 1.4 yields spaces whose cohomology rings contain torsion only in degree 3, 5, and 9 respectively, and this torsion is $\mathbb{Z}_2$. We do not know if there are other examples in dimension 9 or 17, nor do we know if any other dimension of the form $4n + 1$ contains an example as in Theorem 1.4. However, we will show the constructions used to prove Theorem 1.4 cannot work in the remaining dimensions, see Proposition 4.1.

The outline of the paper is as follows. In Section 2, we begin with an easy application of the Mayer-Vietoris sequence to establish our recognition principle, Proposition 2.1, for recognizing when a linear double disk bundle is a sphere. We then use rational homotopy theoretic results of Grove and Halperin to understand the possibilities for the rational cohomology groups of the submanifolds $B_i$ and the common boundary $L$ of the normal bundles. The remainder of Section 2 is devoted to proving the simpler cases of Theorem 1.3. By the end of Section 2, we establish that we may assume that both $B_i$ are simply connected rational cohomology spheres and that $L$ has the rational cohomology groups of a product of spheres.

The next section, Section 3, is devoted to showing the hypothesis of the recognition principal are satisfied for any double disk bundle which is rationally an even dimensional sphere. Namely, we show that both $B_i$ are homotopy spheres, and that $L$ has the integral cohomology ring of a product of two spheres. This leads naturally leads to the study of linear sphere bundles of the form $S^{2k} \to E \to B$ where $E$ has the integral cohomology ring of a product of two spheres. Here we prove that a rational sphere $B$ is the base space of such a bundle if and only if $B$ is a homotopy sphere or $k = 1$ and $B = SU(3)/SO(3)$. We establish this by first using Steenrod squares to show that $\dim B = 2k + 1$ (Lemma 3.4). Using a theorem of Quillen [22] which imposes restrictions on the Stiefel-Whitney classes of a spin vector bundle, we then show that $k = 1$ in Proposition 3.9 if $B$ is not a homotopy sphere. Using the known homotopy groups of the Wu manifold, we are then able to deal with this exceptional case, completing the proof of Theorem 1.3.

Finally, in Section 4, we prove Theorem 1.4. The examples are constructed using the space $\mathbb{R}P^2 \# - \mathbb{R}P^2$ with $\mathbb{R} \in \{\mathbb{C}, \mathbb{H}, \mathbb{O}\}$. The fact that this construction is limited to dimension 5, 9, and 17 is then proven in Proposition 4.1.

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2. Basic structure

Throughout this section, \( M^n \) will denote a closed simply connected smooth manifold with the rational cohomology of \( S^n \), \( n \) even, which admits a linear double disk bundle decomposition. We adopt the convention that the phrase “\( M^n \) is a rational sphere” is an abbreviation for “\( M^n \) is a closed simply connected smooth manifold with \( H^\ast(M; \mathbb{Q}) \cong H^\ast(S^n; \mathbb{Q}) \).”

Because \( M \) is a linear double disk bundle, \( M \cong D(B_1) \cup_f D(B_2) \) where both \( B_1 \) and \( B_2 \) are closed manifolds, \( D(B_i) \) are linear disk bundles over \( B_i \), and \( f : \partial D(B_1) \to D(B_2) \) is a diffeomorphism. We write \( L \) for the common diffeomorphism type of \( \partial D(B_i) \) and once and for all choose identifications \( \partial D(B_1) \cong \partial D(B_2) \). Restricting the projection \( D(B_i) \to B_i \) to the boundary, we obtain linear sphere bundles \( S^\ell_i \to L \xrightarrow{\pi_i} B_i \). As shown in [5, Lemma 4.1], we may assume both \( B_i \) are connected.

As mentioned above, any double disk bundle decomposition of a manifold natural leads to a singular Riemannian foliation. We borrow some terminology from that field, referring to \( L \) as the principal leaf and both \( B_i \) as singular leaves.

The double disk bundle decomposition of \( M \) is ideal for applying the Mayer-Vietoris sequence in cohomology. To facilitate its use, we note that for each \( i = 1, 2 \), \( D(B_i) \) deformation retract to \( B_i \) and that the composition \( L \cong D(B_1) \cap D(B_2) \to D(B_i) \to B_i \) is the sphere bundle projection \( \pi_i : L \to B_i \). When using the Mayer-Vietoris sequence, we will always use these identifications.

Our main method for recognizing the homeomorphism type of \( M \) is the following proposition.

**Proposition 2.1** (Recognition Principle). Suppose \( M \) is a rational sphere admitting a double disk bundle decomposition. Suppose, in addition, that the principal leaf \( L \) has the integral cohomology groups of a product of two spheres and that both singular leaves \( B_i \) are homotopy spheres of different dimensions. Then \( M \) is homeomorphic to a sphere.

**Proof.** For both bundles \( S^\ell_i \to L \to B_i \), the Euler class must vanish, or else \( L \) would have the rational cohomology ring of a sphere. Because \( \dim B_1 \neq \dim B_2 \), it follows that the induced map \( H^\ast(B_1) \oplus H^\ast(B_2) \to H^\ast(L) \) is an isomorphism for any \( 0 < * < n - 1 \). Indeed, all the relevant groups vanish unless \( * \in \{ \dim B_1, \dim B_2 \} \).

Now, considering the Mayer-Vietoris sequence associated to the double disk bundle decomposition of \( M \), it immediately follows that \( M \)
has the integral cohomology groups of a sphere. Since $M$ is simply connected by assumption, it is a homotopy sphere. Finally, by the solution of the Poincaré conjecture, $M$ is homeomorphic to a sphere.

\[\square\]

In order to use Proposition 2.1, we need to understand the topology of the $B_i$ and of $L$. We begin with the trivial observation that at least one $\ell_i$ must be odd. To see this, note that since $M$ is a rational cohomology sphere, $\chi(M) = 2$, so the double disk bundle decomposition of $M$ gives the formula

\begin{equation}
2 = \chi(M) = \chi(B_1) + \chi(B_2) - \chi(L).
\end{equation}

If both $\ell_i$ are even, then $B_1, B_2, \text{ and } L$ are all odd dimensional, so have vanishing Euler characteristic. This is a contradiction.

Now, let $F$ denote the homotopy fiber of the inclusion $L \to M$, so we have a fibration $F \to L \to M$. In [12], Grove and Halperin systematically studied the rational homotopy of $F$ in a much broader context where $M$ has the homotopy type of a double mapping cylinder of two spherical fibrations. Adapted to our situation, they proved the following theorem [12, Theorem 1.8].

**Theorem 2.2.** Suppose $M$ is a closed simply connected manifold which has a double disk bundle decomposition with principal leaf $L$ and let $F \to L \to M$ be the fibration associated to the inclusion $L \to M$. Assume at least one of $\ell_\pm$ is odd. Then one of the following happens.

(1) Both singular leaves are non-orientable, both $\ell_i = 1$, and $F$ has the rational homotopy type of the quotient of $S^3 \times S^3 \times \Omega S^7$ by the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$

(2) Exactly one singular leaf $B_i$ is non-orientable, both $\ell_i = 1$, and $F$ has the rational homotopy type of the quotient of $S^1 \times S^3 \times \Omega S^5$ by $\mathbb{Z}_2$

(3) Exactly one singular leaf, say $B_1$ is non-orientable, $\ell_2 = 1$, $\ell_1 \geq 3$ is odd, and $F$ has the rational homotopy type of $S^1 \times S^{2\ell_1+1} \times \Omega S^{2\ell_1+3}$

(4) Both singular leaves are orientable and $F$ has the rational homotopy type of $S^{\ell_1} \times S^{\ell_2} \times \Omega S^{\ell_1+\ell_2+1}$.

In Theorem 2.2, we see for each possibility for $F$, up to cover, has a factor given by a loop space of a sphere $\Omega S^m$. We note that each such loop space has precisely one non-trivial even degree rational homotopy group $\pi_{2s}(F) \otimes \mathbb{Q} \neq 0$, occurring in degree $2s = m - 1$ if $m$ is odd and in degree $2s = 2m - 2$ if $m$ is even. In [6, Proposition 2.7], the author and Kennard prove that when $M$ is a cohomogeneity one manifold
and $L = G/H$ is a principal orbit, then in the long exact sequence in rational homotopy groups associated to the fibration $\mathcal{F} \to L \to M$, the map $\pi_{2s+1}(M) \otimes \mathbb{Q} \to \pi_{2s}(\mathcal{F}) \otimes \mathbb{Q}$ is non-trivial. The proof is done entirely in terms of the minimal models of the involved spaces, and as such, directly generalizes to prove the following proposition.

**Proposition 2.3.** Consider the connecting homomorphism

$$\partial : \pi_{\text{odd}}(M) \otimes \mathbb{Q} \to \pi_{\text{even}}(\mathcal{F}) \otimes \mathbb{Q}$$

coming from the long exact sequence in rational homotopy groups associated to the fibration $\mathcal{F} \to L \to M$. Suppose $2s$ is the unique even degree for which the loop space factor of $\mathcal{F}$ has a non-trivial rational homotopy group. Then $\partial : \pi_{2s+1}(M) \otimes \mathbb{Q} \to \pi_{2s}(\mathcal{F})$ is non-trivial.

In order to make use of Proposition 2.3, we need to know $\pi_{*}(M) \otimes \mathbb{Q}$. To that end, recall that spheres are intrinsically formal in the sense of rational homotopy theory [8]. In particular, the condition $H^{*}(M; \mathbb{Q}) \cong H^{*}(S^{n}; \mathbb{Q})$ implies $\pi_{*}(M) \otimes \mathbb{Q} \cong \pi_{*}(S^{n}) \otimes \mathbb{Q}$. Since $n$ is even, we therefore conclude that $\pi_{2n-1}(M) \otimes \mathbb{Q}$ is non-trivial, with all other odd degree rational homotopy groups trivial.

**Proposition 2.4.** Suppose $M^{n}$ is a rational sphere admitting a linear double disk bundle decomposition. If $n$ is even, then either both $B_{i}$ are non-orientable or both $B_{i}$ are orientable. If both are non-orientable, $M$ is diffeomorphic to $S^{4}$.

**Proof.** First, assume for a contradiction that, say, $B_{2}$ is non-orientable and $B_{1}$ is orientable. From Theorem 2.2, the loop space factor of $F$ has the form $\Omega S^{2\ell_{2}+3}$, where $\ell_{2}$ must be odd. Proposition 2.3 implies $2n - 1 = 2\ell_{2} + 3$. It follows that $\ell_{2} = \frac{2n-4}{2} = n - 2$ is even, giving a contradiction.

Next, assume both $B_{1}$ and $B_{2}$ are non-orientable. From Theorem 2.2, $\pi_{0}(\Omega S^{7}) \otimes \mathbb{Q}$ is the only non-trivial even degree rational homotopy group of the loop space factor. From Proposition 2.3, $\pi_{7}(M) \otimes \mathbb{Q}$ is non-trivial. Hence, $2n - 1 = 7$, so $n = 4$. As mentioned in the introduction, diffeomorphism rigidity follows from Ge and Radeschi’s classification of four-dimensional singular Riemannian foliations [9].

We may henceforth assume that both $B_{1}$ and $B_{2}$ are orientable. From Theorem 2.2, we now observe the loop space factor of $\mathcal{F}$ has the form $\Omega S^{\ell_{1}+\ell_{2}+1}$. It follows from Proposition 2.3 that $\ell_{1} + \ell_{2} = 2n - 2$ if $\ell_{1}$ and $\ell_{2}$ have the same parities, while $\ell_{1} + \ell_{2} = n - 1$ when $\ell_{1}$ and $\ell_{2}$ have different parities. The first possibility is easily handled.
Proposition 2.5. Suppose $M^n$ is a rational sphere admitting a double disk bundle decomposition with both singular leaves $B_i$ orientable. If $\ell_1 + \ell_2 = 2n - 2$, then $M$ is homeomorphic to a sphere.

Proof. Since the codimension of $B_i$ in $M^n$ is $\ell_i + 1$, we find $\ell_i + 1 \leq n$ for both $i$. The equation $\ell_1 + \ell_2 = 2n - 2$ now forces $\ell_1 = \ell_2 = n - 1$. Thus, both $B_i$ are points. As disk bundles over points are just disks, $M$ is a union of two disks, so is a homotopy sphere by the solution the Poincaré conjecture. \hfill $\Box$

We henceforth assume that $\ell_1 + \ell_2 = n - 1$. From the bundles $S^{\ell_i} \to L \to B_i$, we see that $\dim B_i = \ell_{2-i}$. Since $\ell_1$ and $\ell_2$ have different parities, we obviously have $\ell_1 \neq \ell_2$. For the remainder of this paper, we make the following convention.

Convention: We take $\ell_1$ to be odd and $\ell_2$ to be even. We let $\ell_- = \min\{\ell_1, \ell_2\}$ and $\ell_+ = \max\{\ell_1, \ell_2\}$. We use the notation $S^{\ell_i} \to L \to B_i$ to denote the bundles $S^{\ell_i} \to L \to B_i$. That is $B_+$ and $B_-$ are just relabellings of $B_1$ and $B_2$ in some order.

Remark 2.6. The convention was chosen so that $\ell_i$ is congruent to $i$ mod 2. However, since $\ell_1 + \ell_2 = n - 1$ is odd, this has the unfortunate consequence that $\dim B_1 = \ell_2$ is even and $\dim B_2 = \ell_1$ is odd. Similarly, since $\ell_- < \ell_+$, $\dim B_+ < \dim B_-$. Because $M$ is simply connected, we have the following proposition, a proof of which can be found, e.g., in [14, Proposition 1.8]. We remark that Hoelscher’s proof is in the language of cohomogeneity one manifolds, but the proof uses only van Kampen’s theorem and the long exact sequence in homotopy groups associated to the fiber bundles $L \to B_i$ and thus applies in this situation.

Proposition 2.7. Suppose $M$ is a simply connected double disk bundle with principal leaf $L$ and singular leaves $B_i$ for $i = 1, 2$. Consider the two fiber bundles $S^{\ell_i} \to L \to B_i$. Then the images of $\pi_1(S^{\ell_i}) \subseteq \pi_1(L)$ must generate $\pi_1(L)$.

With this in hand, we may now obtain our first restriction on the topology of a singular leaf.

Proposition 2.8. Suppose $M^n$ is a rational sphere which admits the structure of a double disk bundle with both singular leaves orientable and with $\ell_1$ odd and $\ell_2$ even. Then $B_1$ must be a rational sphere.

Proof. Since $\ell_2$ is even, the image of $\pi_1(S^{\ell_2})$ in $\pi_1(L)$ is trivial. From Proposition 2.7, it follows that the map $\pi_1(S^{\ell_1}) \to \pi_1(L)$ must be
surjective. In particular, from the long exact sequence in homotopy groups associated to $S^{\ell_1} \to L \to B_1$, we conclude that $B_1$ is simply connected. Returning to the Euler characteristic equation (2.1), we now observe that $\chi(B_2) = 0$ since $B_2$ is odd-dimensional, and thus $2 = \chi(B_1)$. Since $B_1$ is simply connected, $B_1$ must be a rational sphere.

We also point out that $L$, being a codimension one subspace of the simply connected manifold $M$, is automatically orientable. Since we are assuming both $B_i$ are orientable, both sphere bundles $S^{\ell_i} \to L \to B_i$ must be orientable. Hence, as they are linear by assumption, they each have an Euler class.

We are now ready to deal with the case $\ell_1 = 1$.

**Proposition 2.9.** Suppose $M^n$ is a rational sphere admitting a linear double disk bundle decomposition with both singular leaves $B_i$ orientable, and with $\ell_1$ and $\ell_2$ of different parities. If $\ell_1 = 1$, then $M$ is homeomorphic to a sphere.

**Proof.** First, if $\ell_2 = 2$, then $1 + 2 = n - 1$, so $n = 4$. As mentioned previously, it follows from [9] that $M$ must be diffeomorphic to $S^4$ in this case. Hence, we assume $\ell_2 \geq 4$.

Now, $\dim B_2 = \ell_1 = 1$, so $B_2$ is a circle. The bundle $S^{\ell_2} \to L \to B_2 \cong S^1$ has a trivial Euler class for dimension reasons, so $H^*(L) \cong H^*(S^1 \times S^{\ell_2})$. Because $\ell_2 \geq 4$, the long exact sequences in homotopy groups shows that $\pi_2(L) = 0$.

Now, consider the bundle $S^1 \to L \to B_1$, and recall that $B_1$ is a simply connected rational cohomology sphere. From Proposition 2.7, the induced map $\mathbb{Z} \cong \pi_1(S^1) \to \pi_1(L) \cong \mathbb{Z}$ is surjective, so is an isomorphism. As $\pi_2(L) = 0$, the long exact sequence of homotopy groups now shows that $\pi_3(B_1) = 0$, which then implies $H^2(B_1) = 0$. Thus, the Euler class $e \in H^2(B_1) = 0$ vanishes, so $H^*(B_1)$ injects into the torsion free ring $H^*(L)$. Since we already know $B_1$ is a simply connected rational cohomology sphere, it now follows that $B_1$ is a homotopy sphere.

We now see the hypothesis of Proposition 2.1, the recognition principle, are satisfied, so $M$ must be homeomorphic to a sphere.

We henceforth assume that $\ell_1 \geq 3$. From Proposition 2.7, it now follows that $L$ is simply connected. Further, consideration of the bundles $S^{\ell_i} \to L \to B_i$ now shows that both $B_i$ are simply connected. We now compute the rational cohomology rings of $B_2$ and $L$.
Proposition 2.10. Suppose $M^n$ is a rational sphere admitting a linear double disk bundle decomposition with $\ell_1 \geq 3$ odd and $\ell_2 \geq 2$ even. Then $B_2$ is a rational sphere and $H^*(L; \mathbb{Q}) \cong H^*(S^{\ell_1} \times S^{\ell_2}; \mathbb{Q})$.

Proof. As in the convention, let $\ell_+ = \max\{\ell_1, \ell_2\}$ and $\ell_- = \min\{\ell_1, \ell_2\}$. Consider the fiber bundles $S^{\ell_1} \to L \to B_+$. As $\dim B_+ = \ell_-$, the Euler class vanishes. Since $\ell_-$ and $\ell_+$ have different parities, graded commutativity of the cup product and the Leray-Hirsch theorem implies that $H^*(L; \mathbb{Q})$ requires at least two generators.

Next, consider the fibration $S^{\ell_1} \to L \to B_1$ and recall that $B_1$ has the rational cohomology groups of $S^{\ell_2}$ (Proposition 2.8). Then the rational Euler class $e \in H^{\ell_1+1}(B_1)$ must vanish, for otherwise the Gysin sequence would show that $L$ has the rational cohomology ring of $S^{2\ell_2+1}$, giving a contradiction. It now follows that $L$ has the rational cohomology groups of $S^{\ell_1} \times S^{\ell_2}$.

Finally, to see $B_2$ is a rational sphere, recall from above that $B_2$ is simply connected. Now consider the fiber bundle $S^{\ell_2} \to L \to B_2$. Since $\ell_2$ is even, the integral Euler class is 2-torsion, so the rational Euler class vanishes. Hence, from the Gysin sequence we see that, as groups, $H^*(L; \mathbb{Q}) \cong H^*(S^{\ell_2}; \mathbb{Q}) \otimes H^*(B_2; \mathbb{Q})$, from which it easily follows that $B_2$ has the rational cohomology of $S^{\ell_1}$.

Propositions 2.8 and 2.10 shows that all the hypothesis of Proposition 2.1 are satisfied rationally. The next section is devoted to showing that the hypothesis are satisfied integrally.

3. From rational to integral

We recall that we have already shown that Theorem 1.3 is true except possibly in the case where the $\ell_i$ have different parities, both $B_i$ are simply connected rational spheres, and $L$ has the rational cohomology groups of a product of two spheres. Our goal in this section is to remove the word “rational” so that we may apply Proposition 2.1. We start by showing the $B_i$ of smaller dimension must in fact be a homotopy sphere. Recall our convention that $\ell_- = \min\{\ell_1, \ell_2\}$ and $\ell_+ = \max\{\ell_1, \ell_2\}$, so $\dim B_+ = \ell_- < \ell_+ = \dim B_-$. 

**Proposition 3.1.** Suppose we have two linear bundles $S^{\ell_1} \to L \to B_i$ where $L$ has the rational cohomology groups of $S^{\ell_1} \times S^{\ell_2}$ and both $B_i$ are simply connected rational cohomology spheres with $\dim B_+ < \dim B_-$. Then $B_+$ is a homotopy sphere.
Proof. Since \( \dim B_+ < \ell_+ \), the integral Euler class associated to the bundle \( S^\ell_+ \to L \xrightarrow{\pi_+} B_+ \) vanishes for dimension reasons. From Leray-Hirsch, it follows that \( L \) has the following property.

\((\ast)\) For any non-zero \( x \in H^{\ell_+}(L) \) and for any non-zero \( y \in H^s(L) \) with \( s < \ell_- \), the product \( xy \) is non-zero.

We now assume for a contradiction that there is an \( s \) with \( 0 < s < \ell_- = \dim(B_+) \) for which \( H^s(B_+) \) is non-zero. That is, we assume that \( B_+ \) is not a homotopy sphere. This implies that \( H^s(L) \) is non-zero.

Consider the other bundle \( S^{\ell_-} \to L \xrightarrow{\pi_-} B_- \). From Proposition 2.10 we know the rational Euler class of this bundle must vanish. In particular, the integral Euler class must be torsion. It follows from the Gysin sequence that \( \pi_\ast : H^{\ell_+}(B_-) \cong \mathbb{Z} \to H^{\ell_+}(L) \) is an injection.

Further, because \( s < \ell_- \), the induced map \( \pi_\ast : H^s(B_-) \to H^s(L) \) is an isomorphism, so \( H^s(B_-) \neq 0 \). Let \( y \in H^s(B_-) \) be non-zero and let \( x \in H^{\ell_+}(B_-) \cong \mathbb{Z} \) be non-zero. We note that \( xy \in H^{\ell_+ + s}(B_-) \) must be zero, as its degree is \( \ell_+ + s > \ell_+ = \dim B_- \). So, we see that

\[
0 = \pi_\ast(xy) = \pi_\ast(x)\pi_\ast(y).
\]

Since \( \pi_\ast(x), \pi_\ast(y) \in H^s(L) \) are both non-zero, we have now contradicted \((\ast)\).

As a simple corollary, we now show that \( L \) satisfies the hypothesis of Proposition 2.1.

**Corollary 3.2.** The space \( L \) must have the integral cohomology groups of \( S^{\ell_1} \times S^{\ell_2} \).

**Proof.** Proposition 3.1 shows that \( B_+ \) is a homotopy sphere. Now consider the bundle \( S^{\ell_+} \to L \to B_+ \). Since \( \dim B_+ = \ell_- < \ell_+ \), the integral Euler class vanishes. Now the Gysin sequence breaks into short exact sequences of the form \( 0 \to H^s(B_+) \to H^s(L) \to H^{s-\ell_+}(B_+) \to 0 \). Since \( \mathbb{Z} \) is free, each such short exact sequence splits, so \( H^s(L) \) is torsion free. Since \( L \) has the rational cohomology groups of \( S^{\ell_1} \times S^{\ell_2} \) by Proposition 2.10, the result follows.

Thus, in order to apply Proposition 2.1, we need only show \( B_- \) is a homotopy sphere. One may expect that knowing that \( B_- \) fits into a fiber bundle of the form \( S^{\ell_-} \to L \to B_- \) with \( H^*(L; \mathbb{Z}) \cong H^*(S^{\ell_-} \times S^{\ell_+}) \) is enough to conclude that \( B_- \) is a homotopy sphere.
Unfortunately, this expectation is wrong as the following example shows. Consider the chain of subgroups $S^1 \subseteq SO(3) \subseteq SU(3)$. This gives rise to the homogeneous fibration

$$SO(3)/SO(2) \to SU(3)/SO(2) \to SU(3)/SO(3).$$

Of course, $SO(3)/S^1 = S^2$, so this is a sphere bundle. The $SO(2)$ in $SU(3)$, up to conjugacy, has the form diag$(z, z, 1)$ and thus, from in [1, Lemma 3.3],

$$H^*(SU(3)/SO(2)) \cong H^*(S^2 \times S^5).$$

On the other hand, the Wu manifold $SU(3)/SO(3)$ is well known to be a simply connected 5-dimensional rational sphere with torsion in its cohomology ring: $H^3(SU(3)/SO(3); \mathbb{Z}) \cong \mathbb{Z}_2$.

It turns out that when $k$ is even, this is essentially the only way the expectation is wrong.

**Theorem 3.3.** Suppose $B$ is a rational sphere which is the base of a linear sphere bundle $S^k \to E \to B$ with $H^*(E; \mathbb{Z}) \cong H^*(S^k \times S^m; \mathbb{Z})$. If $k$ is even, then one of the following occurs:

1. $B$ is a homotopy sphere or
2. $B$ is diffeomorphic to the Wu manifold $SU(3)/SO(3)$ and $k = 2$.

The next subsection is devoted to a proof of Theorem 3.3.

**3.1. Proof of Theorem 3.3.** In this section, $B$ will always denote a rational sphere which is the base of a linear sphere bundle $S^k \to E \to B$ with $H^*(E; \mathbb{Z}) \cong H^*(S^k \times S^m; \mathbb{Z})$. First we compute the cohomology ring of such a $B$.

**Lemma 3.4.** Suppose $B$ is a rational sphere which is not a homotopy sphere and assume there is a linear sphere bundle $S^k \to E \to B$ for which $H^*(E; \mathbb{Z}) \cong H^*(S^k \times S^m; \mathbb{Z})$. Then $H^*(B) \cong \mathbb{Z}[e, a]/I$ where $|e| = k + 1, |a| = \dim B$, and $I$ is the ideal

$$I = \langle me, ea^2, e^{s+1} \rangle$$

where $m \geq 2$ and $s = \frac{\dim B - k}{k+1}$. In particular, $\dim B \cong -1 \pmod{k+1}$.

**Proof.** We will compute using the Gysin sequence. Let $0 < t < \dim B$ be the smallest integer with the property that $H^t(B)$ is non-zero torsion. If $t > k + 1$, then the integral Euler class of the bundle $S^k \to E \to B$ vanishes. The Gysin sequence then implies that $H^*(B)$ injects into the torsion free ring $H^*(L)$. This implies $B$ is a homotopy sphere, giving a contradiction. On the other hand, if $t < k + 1$, then we obtain the same contradiction since $H^t(B) \to H^t(E)$ is an injection. Thus, $t = k + 1$. 
Let \( e \in H^{k+1}(B) \) denote the Euler class of the bundle. From the Gysin sequence, \( H^{k+1}(B)/(e) \) injects into the torsion free group \( H^{k+1}(E) \). It follows that \( H^{k+1}(B) \) is cyclic, say of order \( m \), generated by \( e \). Of course, if \( m = 1 \), then \( e = 0 \) so \( B \) is a homotopy sphere. Hence, \( m \geq 2 \).

Continuing, using the fact that \( H^*(L) \) is torsion free, the Gysin sequence also shows that cupping with \( e \), \( \cup e : H^*(B) \to H^{*+k+1}(B) \) is an isomorphism for any * with \( k < * < \dim B - (k+1) \). Together the obvious fact that \( H^{\dim B}(B) \cong \mathbb{Z} \), this completes the determination of the ring \( H^*(B) \), except for the value of \( s \).

Since \( e^{s+1} = 0 \), the non-zero element \( e^s \in H^{s(k+1)}(B) \) is in the kernel of cupping with \( e \). From the Gysin sequence, the map \( H^{s(k+1)+k}(E) \to H^{s(k+1)}(B) \) must be surjective, so that, in particular, \( H^{s(k+1)+k}(E) \neq 0 \). This implies \( s(k+1)+k \in \{0, \dim B, \dim B+k\} \). The first two options obviously cannot occur. If \( s(k+1)+k = \dim B + k \), then

\[
0 \neq e^s \in H^{s(k+1)}(B; \mathbb{Z}) \cong \mathbb{Z},
\]

which is impossible since \( e \) is \( m \)-torsion. Thus, we must conclude that \( s(k+1)+k = \dim B \), so \( s = \frac{\dim B - k}{k+1} \) as claimed.

When \( k \) is even, the Euler class is 2-torsion. Using this fact and Streenrod squares, we can considerably strengthen Lemma 3.4 in this case.

**Lemma 3.5.** Assume the hypothesis of Lemma 3.4 and assume additionally that \( k \) is even. Then \( \dim B = 2k + 1 \) and \( H^*(B) \) has the integral cohomology groups as a sphere, except that \( H^{k+1}(B) \cong \mathbb{Z}_2 \).

**Proof.** We use the notation of the proof of Lemma 3.4.

If \( k \) is even, then the Euler class is 2-torsion, so \( m = 2 \), that is, all non-trivial torsion is 2-torsion. A simple application of the universal coefficients theorem reveals that \( H^*(B; \mathbb{Z}_2) = 0 \) except when * has the form \( t(k+1) - 1 \) or \( t(k+1) \) and \( 0 \leq * \leq \dim B \). Further when \( H^*(B; \mathbb{Z}_2) \neq 0 \), it is isomorphic to \( \mathbb{Z}_2 \).

Suppose \( x \in H^k(B; \mathbb{Z}_2) \) and \( y \in H^{k+1}(B; \mathbb{Z}_2) \) are both non-zero. We clearly have \( y^s \neq 0 \) since \( y \) is the reduction of \( e \) and \( y^{s+1} = 0 \) since \( (s+1)(k+1) = \dim B + 1 \).

Note that \( 2k \) is not of the form \( t(k+1) - 1 \) or \( t(k+1) \), so \( H^{2k}(B; \mathbb{Z}_2) = 0 \). In particular \( x^2 = 0 \).

In order to show that \( \dim B = 2k+1 \), it is sufficient to show \( s = 1 \). To that end, we compute \( Sq^1(x) \), where \( Sq^1 : H^*(B; \mathbb{Z}_2) \to H^{*+1}(B; \mathbb{Z}_2) \) is the Steenrod squaring operation. Recall that \( Sq^1 \) is the same as the Bockstein homomorphism associated to the short exact sequence...
0 → \mathbb{Z}_2 → \mathbb{Z}_4 → \mathbb{Z}_2 → 0, see e.g. \cite[pg. 489]{13}. It again follows from the universal coefficients theorem that the portion of long exact sequence in cohomology groups beginning with \( H^{k-1}(B_2; \mathbb{Z}_2) = 0 \), associated to this exact sequence is

\[ 0 → \mathbb{Z}_2 → \mathbb{Z}_2 → \mathbb{Z}_2 → \mathbb{Z}_2 → 0 = H^{k+2}(B_2; \mathbb{Z}_2). \]

Exactness now ensures that the maps between successive copies of \( \mathbb{Z}_2 \) alternate between isomorphisms and the zero-map, beginning with an isomorphism. In particular, the Bockstein \( \beta \) is an isomorphism, so \( Sq^1(x) = \beta(x) = y \).

Since \( x^2 = 0 \), the Cartan formula gives

\[ 0 = Sq^2(x^2) = Sq^2(x)x + Sq^1(x)^2 + xSq^2(x). \]

Since \( Sq^2(x)x + xSq^2(x) = 0 \) (mod 2), we have \( 0 = Sq^1(x)^2 = y^2 \).

Thus, \( s = 1 \), as claimed.

Suppose \( B \) is a rational sphere and there there is a linear sphere bundle \( S^k → E → B \) with \( H^*(E; \mathbb{Z}) \cong H^*(S^k × S^{\dim B}) \), with \( k = 2 \). Then Lemmas 3.4 and 3.5 tells us that \( \dim B = 5 \) and \( H^*(B; \mathbb{Z}) \cong H^*(SU(3)/SO(3); \mathbb{Z}) \). The Barden-Smale \cite{2, 25} classification of closed simply connected 5-manifolds now shows that \( B \) must be diffeomorphic to \( SU(3)/SO(3) \). Thus, we have the following corollary.

**Corollary 3.6.** Suppose \( B \) is a rational sphere for which there is a linear sphere bundle \( S^2 → E → B \) where \( H^*(E; \mathbb{Z}) \cong H^*(S^2 × S^{\dim B}; \mathbb{Z}) \). Then either \( B \) is a homotopy sphere or \( B = SU(3)/SO(3) \).

Thus, to establish Theorem 3.3, we need only show that if \( k ≥ 4 \) is even, then \( B \) must be a homotopy sphere. The assumption \( k ≥ 4 \) together with Lemma 3.4 implies that \( B \) is at least 3-connected. In particular, any linear sphere bundle over it is automatically spin. This will be the key in what follows.

It is well know that every spin vector bundle also has trivial 3rd Stiefel-Whitney class. However, there are additional restrictions on the Stiefel-Whitney classes, coming from a theorem of Quillen \cite[Theorem 6.5]{22}. To state his theorem, we first set up notation.

We define a function \( h : \mathbb{N} → \mathbb{N} \) as follows. Writing \( s ∈ \mathbb{N} \) as \( s = 8t + u \) with \( 1 ≤ u ≤ 8 \), we set

\[
\begin{align*}
h(s) &= \begin{cases} 
4t & u = 1 \\
4t + 1 & u = 2 \\
4t + 2 & u = 3 \text{ or } 4 \\
4t + 3 & \text{otherwise}
\end{cases}
\end{align*}
\]
Also, we let $BSO(m)$ and $BSpin(n)$ denote the classifying spaces of $SO(m)$ and $Spin(m)$. That is, $BSO(m)$ is the quotient of a free action of $SO(m)$ on contractible space $ESO(m)$, and likewise for $BSpin(n)$. Then, $H^*(BSO(m); \mathbb{Z}_2) \cong \mathbb{Z}_2[w_2, w_3, ..., w_m]$, where $w_i$ is the $i$th Stiefel-Whitney class. Let $\pi : BSpin(m) \to BSO(m)$ be the map induced on the classifying spaces from the two fold cover $Spin(m) \to SO(m)$. Then, Quillen shows:

**Theorem 3.7 (Quillen).** The induced map

$$\pi^* : H^*(BSO(m); \mathbb{Z}_2) \to H^*(BSpin(m); \mathbb{Z}_2)$$

has kernel given by the ideal

$$J = \langle w_2, Sq^1w_2, Sq^2Sq^1w_2, ..., Sq^{q(m)-1}Sq^{q(m)-2}...Sq^4Sq^2Sq^1w_2 \rangle.$$

To aid our understanding of $J$, we need a lemma.

**Lemma 3.8.** Suppose $t$ and $m$ are non-negative integers with $2t+1 \leq m$. Then in $H^*(BSO(m+1); \mathbb{Z}_2)$, there is a polynomial $p(w_2, ..., w_{2t+1})$ such that

$$Sq^{2^t}Sq^{2^t-1}...Sq^2Sq^1w_2 = w_{2^t+1} + p(w_2, ..., w_{2t+1}).$$

**Proof.** An easy induction shows that $Sq^{2^t}Sq^{2^t-1}...Sq^2Sq^1w_2$ is an element of $H^u(BSO(m+1); \mathbb{Z}_2)$ with $u = 2^t+2^{t-1}+...+2+1+2 = 2^{t+1}+1$. Since $2t+1 \leq m$, then , $w_{2^t+1} \in H^*(BSO(m+1); \mathbb{Z}_2)$.

We now prove the lemma using induction. The induction begins by noting that the Wu formula

$$Sq^{i}(w_j) = \sum_{s=0}^{i} \binom{j+s-i-1}{s} w_{i-s}w_{j+s}$$

implies $Sq^1(w_2) = w_3$, so when $t = 0$, we have $p = 0$.

So, assume inductively that $Sq^{2^t-1}...Sq^2Sq^1w_2 = w_{2^t+1} + p(w_2, ..., w_{2t-1})$ for some polynomial $p$. We must compute $Sq^{2^t}(w_{2^t+1} + p)$.

We first note that $Sq^{2^t}p$ is, by the Cartan formula, a polynomial in $Sq^a(w_b)$ for $a \leq 2^t$ and $b \leq 2^t - 1$. Each $Sq^a(w_b)$ can be found using the Wu formula, and all the terms involve Stiefel Whitney classes of degree at most $a + b \leq 2^t + 2^t - 1 = 2^{t+1} - 1$. In short, $Sq^{2^t}p$ is a polynomial $q(w_2, ..., w_{2t+1})$.

Now, the Wu formula gives

$$Sq^{2^t}(w_{2^t+1}) = \sum_{s=0}^{2^t} \binom{2^t + 1 + s - 2^t - 1}{s} w_{2^t-s}w_{2^t+1+s}.$$
The binomial coefficient simplifies to \( \binom{s}{t} = 1 \). Further, when \( s \leq 2^t - 2 \), then these terms only involve Stiefel-Whitney classes of degree at most \( 2^{t+1} - 1 \); these terms can be absorbed into \( q \).

The remaining two terms are when \( s = 2^{t+1} - 1 \), giving \( w_1w_{2^{t+1}} = 0 \) since \( w_1 = 0 \), and \( s = 2^{t+1} \), giving \( w_0w_{2^{t+1} + 1} = w_{2^{t+1} + 1} \). This completes the induction.

\[ \square \]

We are now ready to prove Theorem 3.3 when \( k \geq 4 \).

**Proposition 3.9.** Suppose \( B \) is a rational sphere which is the base of a linear sphere bundle \( S^k \to E \to B \) with \( H^*(E; \mathbb{Z}) \cong H^*(S^k \times S^\text{dim} B; \mathbb{Z}) \) and with \( k \geq 4 \) even. Then \( B \) is a homotopy sphere.

**Proof.** Let \( \xi \) denote the sphere bundle. Assume for a contradiction that \( B \) is not a homotopy sphere. We let \( e = e(\xi) \in H^{k+1}(B; \mathbb{Z}) \) denote the Euler class of \( \xi \). From Lemmas 3.4 and 3.5, we know that \( e \) generates \( H^{k+1}(B; \mathbb{Z}) \cong \mathbb{Z}_2 \) and that \( H^s(B; \mathbb{Z}) = 0 \) for \( 0 < s < k + 1 \). In particular, \( \xi \) is spin. We recall that the mod 2 reduction of the Euler class is the top Stiefel-Whitney class, so \( 0 \neq w_{k+1}(\xi) \in H^{k+1}(B; \mathbb{Z}_2) \cong \mathbb{Z}_2 \).

Since the lowest dimensional non-vanishing Stiefel-Whitney class of any bundle occurs in degree a power of 2, it follows that \( k = 2^{t+1} \) for some \( t \geq 1 \).

We claim that \( t \leq h(k+1) - 1 \). When \( k = 4 \), this inequality becomes \( 1 \leq h(5) - 1 = 2 \), which is true. For \( k = 8 \cdot 2^t - 2 + 1 \), we compute \( h(8 \cdot 2^t - 2 + 1) = 4 \cdot 2^{t-2} = 2^t \), so \( h(\ell + 1) - 1 = 2^t - 1 \). Now the inequality reads \( t \leq 2^t - 1 \), which is obviously true for any \( t \geq 2 \).

From Theorem 3.7, the inequality \( t \leq h(\ell + 1) - 1 \) implies that the element

\[
v := Sq^2 Sq^{2^t - 1} \ldots Sq^2 Sq^1 w_2
\]

is an element of \( \ker \pi^*: H^*(BSO(k+1); \mathbb{Z}_2) \to H^*(BSpin(k+1); \mathbb{Z}_2) \).

From Lemma 3.8 with \( k = 2^{t+1} \), we see that \( v = w_{k+1} + p(w_2, \ldots, w_{k-1}) \) for some polynomial \( p \). Thus, \( \pi^*(w_{k+1}) = \pi^*(p) \).

Now, let \( \phi: B \to BSO(k+1) \) be the classifying map of \( \xi \). Since \( \xi \) is spin, there is a lift \( \overline{\phi}: B \to BSpin(k+1) \), with \( \pi \circ \overline{\phi} = \phi \). Thus,

\[
w_{k+1}(\xi) = \phi^*(w_{k+1}) = \overline{\phi}^* \pi^*(w_{k+1}) = \overline{\phi}^* \pi^*(p) = \phi^*(p) = 0,
\]

where the last equality follows because \( H^*(B; \mathbb{Z}_2) = 0 \) for \( * \leq k - 1 \). This contradicts the fact that \( w_{k+1}(\xi) \neq 0 \).

\[ \square \]

Taken together, Corollary 3.6 and Proposition 3.9 establish Theorem 3.3.
3.2. **Completion of the proof of Theorem 1.3.** We are now ready to complete the proof of Theorem 1.3. Suppose $M^n$ is a rational sphere with $n$ even admitting a linear double disk bundle decomposition. From the work done in Section 2, we may assume we have linear sphere bundles $S^{\ell_i} \to L \to B_i$ where $\ell_1 \geq 3$ is odd, $\ell_2$ is even, both $B_i$ are simply connected rational spheres. In addition, from Proposition 3.1 and Corollary 3.2, the $B_i$ of smaller dimension, $B_+$ must be a homotopy sphere, and $H^*(L) \cong H^*(S^{\ell_1} \times S^{\ell_2})$. Thus, in order to apply Proposition 2.1 and complete the proof of Theorem 1.3, we need only show that $B_-$ is a homotopy sphere. We accomplish this by handling the cases $\ell_1 < \ell_2$ and $\ell_2 < \ell_1$ separately.

**Proposition 3.10.** If $\ell_1 < \ell_2$, then $B_-$ is a homotopy sphere.

**Proof.** If $B_-$ is not a homotopy sphere, then according to Lemma 3.4, we must have $\dim B_- = \ell_- \cong -1 \pmod{\ell_+ + 1}$, so $\ell_- + 1 = t(\ell_+ + 1)$ for some integer $t$. By assumption, $\ell_- = \ell_1$ is odd, so $\ell_+$ is even. Thus, $\ell_- + 1$ is odd, while $t(\ell_- + 1)$ is even. This contradiction establishes that $B_-$ is a homotopy sphere.

**Proposition 3.11.** If $\ell_2 < \ell_1$, then $B_-$ is a homotopy sphere.

**Proof.** Assume for a contradiction that $B_-$ is not a homotopy sphere.

Because $\ell_2 < \ell_1$, the bundle $S^{\ell_-} \to L \to B_-$ has $\ell_- = \ell_2$, so the fiber is even dimensional. Then Theorem 3.3 establishes that $\ell_- = 2$ and $B_\rho \cong SU(3)/SO(3)$. Thus, we have two bundles: $S^5 \to L \to S^2$, and $S^2 \to L \to SU(3)/SO(3)$. We now use known homotopy groups to show there is no such $L$.

First, since $\pi_1(SO(6)) \cong \mathbb{Z}_2$, there are precisely two linear $S^5$ bundles over $S^2$. As shown, e.g., by the author [4, Propositions 4.30 and 4.32], the total space of the non-trivial $S^5$-bundle over $S^2$ is a quotient of $S^3 \times S^5$ by a free $S^1$ action. In particular, the long exact sequence associated to the fiber bundle $S^1 \to S^3 \times S^5 \to (S^3 \times S^5)/S^1$ shows the homotopy groups of the non-trivial bundle are isomorphic to those of the trivial bundle. Thus, $\pi_*(L) \cong \pi_*(S^2 \times S^5)$.

On the other hand, $\pi_4(SU(3)/SO(3))$ since $\pi_4(SU(3)) = 0$ by Bott periodicity. In addition, Puttman and Rigas [21, Theorem 5.6] have shown that

$$\pi_6(SU(3)/SO(3)) \cong \mathbb{Z}_2 \quad \text{and} \quad \pi_5(SU(3)/SO(3)) \cong \mathbb{Z} \oplus \mathbb{Z}_2.$$ 

Thus, the portion of the long exact sequence in homotopy groups associated to the bundle $S^2 \xrightarrow{\rho} L \xrightarrow{\rho} SU(3)/SO(3)$, beginning with $\pi_6(S^2) \cong \mathbb{Z}_{12}$, is
\[ \mathbb{Z}_{12} \xrightarrow{i_1} \mathbb{Z}_{12} \oplus \mathbb{Z}_2 \xrightarrow{\rho_5} \mathbb{Z}_2 \xrightarrow{\partial_5} \mathbb{Z} \oplus \mathbb{Z}_2 \xrightarrow{\rho_5} \mathbb{Z} \oplus \mathbb{Z}_2 \xrightarrow{\partial_5} \mathbb{Z}_2 \xrightarrow{i_6} \mathbb{Z}_2 \xrightarrow{\rho_5} 0, \]

where, for example, \( i_4 : \pi_t(S^2) \to \pi_t(L) \) is the induced map.

Starting on the right side of this sequence, exactness implies \( i_4 \) must be an isomorphism, which implies \( \rho_5 \) is surjective, and hence, an isomorphism. This implies \( \partial_5 \) is an isomorphism, which then implies \( i_6 : \mathbb{Z}_{12} \to \mathbb{Z}_{12} + \mathbb{Z}_2 \) is surjective, which is obviously false. This contradiction shows there is no \( L \) which is simultaneously the total space of a sphere bundle over \( S^2 \) and over \( SU(3)/SO(3) \). Thus, \( B_- \) must be a homotopy sphere in this case as well.

\[ \square \]

4. Rational spheres in dimension \( 4k+1 \)

In this section, we prove Theorem 1.4. In addition, we show the construction which works in dimension 5, 9, and 17, does not generalize to other dimensions.

**Proof.** (Proof of Theorem 1.4)

Let \( \mathbb{K} \in \{\mathbb{C}, \mathbb{H}, \mathbb{O}\} \) and consider \( L = \mathbb{K}P^2_q\mathbb{K}P^2 \). As shown in proof of [27, Theorem 2.1], \( L \) is a linear \( S^k \) bundle over \( B_1 = B_2 = S^k \) where \( k = \dim_{\mathbb{K}} \mathbb{K} \in \{2, 4, 8\} \). Let \( M_f \) be obtained by gluing two copies of the corresponding disk bundles together by a diffeomorphism \( f : L \to L \).

Write \( H^*(L) \cong \mathbb{Z}[u, v]/I \) where \( |u| = |v| = k \) and \( I \) is the ideal \( I = \langle u^2 + v^2, uv \rangle \). We note that for \( \alpha, \beta \in \mathbb{Z}, (\alpha u + \beta v)^2 = 0 \) if and only if \( \alpha = \pm \beta \). In particular, the four elements in \( P = \{ \pm (u + v), \pm (u - v) \} \) are the only primitive elements in \( H^2(L) \) which square to 0. As such, \( f^* \) must preserve \( P \).

Now, for \( i = 1, 2 \), write \( H^*(B_i) \cong \mathbb{Z}[x_i]/x_i^2 \) with \( |x_i| = k \). For \( \pi_i : L \to B_i \) the projection, we wish to determine \( \pi_i^*(x_i) \). A quick glance as the Serre spectral sequence for this bundle reveals that \( \pi_i^*(x_i) \) must be primitive. On the other hand, \( \pi_i^*(x_i)^2 = \pi_i^*(x_i^2) = 0 \), so \( \pi_i^*(x_i) \in P \).

Finally, consider the cohomological Mayer-Vietoris sequence for \( M_f \) associated to the double disk bundle decomposition of \( M_f \). The only non-trivial part of this sequence is

\[ 0 \to H^k(M_f) \to H^k(S^k) \oplus H^k(S^k) \xrightarrow{\psi} H^k(L) \to H^{k+1}(M_f) \to 0. \]

Because for both \( i = 1, 2 \), \( \pi_i^*(x_i) \in P \) and \( f^* \) preserves \( P \), it now follows that with respect to the basis \( \{x_1, x_2\} \) and \( \{u, v\} \), the map \( \psi \) is given by the matrix \( \begin{bmatrix} \pm 1 & \pm 1 \\ \pm 1 & \pm 1 \end{bmatrix} \). In particular, the absolute value of
the determinant is either 0 or 2. Of course, when it is 0, \( M_f \) is not a rational sphere, but when it is 2, \( H^{k+1}(M_f) \simeq \mathbb{Z}_2 \).

Finally, notice that if we set \( \pi_1 = \pi_2 \) and choose \( f \) to be the map which switches the two halves of \( L \), we in fact obtain a rational sphere \( M_f \).

We now show that the above construction cannot be carried out for other values of \( k \). Specifically, we have the following proposition.

**Proposition 4.1.** Suppose \( L \) is a linear \( S^k \)-bundle over \( B^k \) with \( B^k \) a homotopy sphere and with \( k \not\in \{2, 4, 8\} \) and \( k \) even. Let \( D \) denote the corresponding disk bundle, let \( f : L \to L \) be any diffeomorphism, and set \( M_f = D \cup_f D \). If \( M_f \) is a rational sphere, then it is a homotopy sphere.

**Proof.** As shown by Milnor [17], if there is a a principal \( SO(n) \) bundle over \( S^k \) with Stiefel-Whitney class \( w_k \neq 0 \), then \( k = 2, 4, 8 \). Clearly this result extends to replacing \( S^k \) with \( B^k \). Since by assumption, \( k \not\in \{2, 4, 8\} \), we must have \( w_k(L) = 0 \in H^k(B) \).

It now follows from [16] that \( H^*(L) \simeq H^*(S^k \times S^k) \) as rings. Writing \( H^*(L) \simeq \mathbb{Z}[u, v]/u^2 = v^2 = 0 \), one easily sees that \( P = \{\pm u, \pm v\} \) is characterized ring theoretically as consisting of the all the primitive elements which square to 0. Arguing as in the proof of Theorem 1.4, we find that \( \pi^*_i(x_i) \in P \), and thus, the corresponding map \( \psi \) has has rows of the form \( \pm(1, 0) \) or \( \pm(0, 1) \). No matter which possibilities are chosen, clearly the absolute value of the determinant of this matrix is 0 or 1. If follows that either \( M_f \) is not a rational sphere, or it is a homotopy sphere. \qed

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