On the modelling, linear stability, and numerical simulation for advection-diffusion-reaction in poroelastic media

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Abstract

We perform the linear growth analysis for a new PDE-based model for poromechanical processes (formulated in mixed form using the solid deformation, fluid pressure, and total pressure) interacting with diffusing and reacting solutes in the medium. We find parameter regions that lead to interesting behaviour of the coupled system. These mutual dependences between deformation and diffusive patterns are of substantial relevance in the study of morphoelastic changes in biomaterials. We provide a set of computational examples in 2D and 3D that can be used to form a better understanding on how, and up to which extent, the deformations of the porous structure dictate the generation and suppression of spatial patterning dynamics, also related to the onset of mechano-chemical waves.

Keywords: Biot equations, reaction-diffusion, nonlinear coupling, linear stability analysis, mixed finite elements, numerical examples.

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1. Introduction

We propose a new model for the interaction between diffusing species and an underlying poroelastic structure. This work is composed by two main contributions. In [49], we have recently explored the well-posedness of the coupled system and have addressed the stability of a mixed finite element discretisation. On the other hand, in the present companion paper we focus more on the modelling issues, on the spectral linear stability analysis, and we also provide numerical examples dealing with growth and pattern formation, as well as with applications in traumatic brain injury.

As considered here, reaction-diffusion equations are coupled to the balances of mass and linear momentum of the fluid-solid mixture through advection, and though a modification in the reaction

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modulated by changes in volume. In turn, the solutes and the external forces drive the motion of the medium through contractile forces. Even if the present theoretical framework is motivated by examples in cell dynamics, applications sharing the same mathematical and mechano-chemical structure are numerous. These include the formation of inflammatory edema in the context of immune systems [38], oxygen diffusivity in cartilage [25], contaminant transport [4], petroleum extraction [7], drug transport in arteries [8], rock consolidation and fractures [33], tumour localisation and biomass growth [41], or chemically-controlled cell motion [26]. In some of these phenomena, one can observe mechanically-induced transport of the solutes. This effect occurs as the consolidation of the porous media increases the flow of interstitial fluid which in turn contributes to the solute advective transport (see e.g. [40]). On the other hand, the presence of chemical solutes in so-called active poroelastic materials locally modifies morphoelastic properties [36], and these processes can be homogenised to obtain macroscopic models of poroelasticity coupled with advection-reaction-diffusion equations (see e.g. [10, 32]).

Very often, these coupled models are of high-dimensions and very nonlinear, which impedes to obtain exact solutions in closed form. Even if a large variety of numerical methods exist for producing approximate solutions, appropriate methods (in the sense of being robust with respect to model parameters, being convergent, and replicating key properties of the underlying physico-chemical phenomena) have appeared only recently. Modern techniques include [23], where the authors propose mixed finite element schemes and provide a stability analysis for a system of multiple-network poroelasticity, that resembles the model problem we are interested in. As in [23], here we also employ the mixed three-field formulation for poroelasticity from [31, 22], and in [49] we develop the coupling with a primal formulation for the advection-diffusion system.

Beyond the question of the equations’ resolution, enhancement of the complexity of the models leads the stability analysis to become more and more analytically involved. Stability analysis enables to understand and to anticipate partially the essential physical mechanisms of the proposed system with respect to the parameter values. Such tool is particularly useful in the context of patterning systems to locate the parameter space where the model leads to stationary spatially unstable solutions. In the context of the present work, related studies have been performed on particular sub-systems such as reaction-diffusion [13, 43] or decoupled elasticity and diffusion [28, 27]. More recent works tend to integrate further complexity in the model by taking care of multi-layered coupled systems [9], incorporating domain or mechanical growth [11, 19], the coupling between elasticity-diffusion [12], poroelasticity [34, 37, 35], and also porelasticity-diffusion [36], which resembles more the development we advocate in this work.

The remainder of this paper is laid out as follows. The governing equations proposed in [49] are recalled in Section 2. Then, in Section 3 we perform a linear stability analysis around a steady state with zero solid displacement, constant fluid pressure, and constant solute concentrations. To make the analysis as general as possible, we modify the momentum equilibrium that we presented in [49], now including also an acceleration term. We use that to make some model comparisons. We obtain dispersion relations that indicate that the mechano-chemical feedback onsets Turing instabilities (with non-trivial wavenumber) for a range of coupling parameters. We proceed in Section 4 with recalling the locking-free finite element scheme from the companion paper [49]. Then we give some illustrative numerical examples in 2D and 3D collected in Section 5, and we close in Section 6 with a discussion and concluding remarks.

2. Model problem

2.1. Poroelasticity of soft tissue

Let us consider flow of interstitial fluid through a porous medium that is subject to elastic deformations. We will consider that the process occurs in either two- or three-dimensional domains $\Omega \subset \mathbb{R}^d$
with $d \in \{2, 3\}$, and that the fluid does not enter nor leaves the body. As common in the study of flow in porous media, we adopt a description in terms of locally averaged variables. Then, for a given time $t \in (0, t_{\text{final}}]$, poromechanical quantities of interest are in this case the average displacement of the porous structure $u^s(t) : \Omega \rightarrow \mathbb{R}^d$ and the pressure head associated with the fluid flowing through the pores, $p^f(t) : \Omega \rightarrow \mathbb{R}$. We also suppose that gravitational forces have little effect in contributing to the force balances in comparison to other external body forces such as applied loads depending on space and time variables $b(t) : \Omega \rightarrow \mathbb{R}^d$. In the classical theory of consolidation as exposed in the seminal works [6, 47], the system allows to describe physical loading of porous layers and the change of hydraulic equilibrium in a fluid-structure system. There, one assumes as well that the exerted stresses contain shear contributions by the solid phase whereas volumetric contributions appear from both solid and the fluid phases (since the interstitial flow is considered governed by Darcy’s law). This fact motivates the idea from [31, 22] to introduce an auxiliary scalar unknown

$$\psi = \alpha p^f - \lambda \text{div} u^s,$$  

representing the total pressure, or the volumetric part of the total Cauchy stress $\sigma$ (specified in the constitutive equation (2.3), below), where $\alpha$ is the so-called Biot-Willis consolidation (or pressure storage coupling) parameter.

Denoting by $\ell(t) : \Omega \rightarrow \mathbb{R}$ a given volumetric fluid source or sink, the conservation of total pore fluid content can be stated as an equation for the fluid pressure $p^f(t) : \Omega \rightarrow \mathbb{R}$

$$\left( c_0 + \frac{\alpha^2}{\lambda} \right) \partial_t p^f - \frac{\alpha}{\lambda} \partial_t \psi - \frac{1}{\eta} \text{div}(\kappa \nabla p^f) = \ell,$$  

where $\kappa(x)$ is the permeability (or hydraulic conductivity) of the porous medium which can possibly be anisotropic, $\eta$ is the constant viscosity of the pore fluid, and $c_0$ is the constrained specific storage coefficient (which encompasses both the porosity of the solid skeleton and the compressibility of the fluid or of the solid in the meso-scale) [44].

The equations of motion (balance of linear momentum and the constitutive equation relating stress and strains) consist in finding solid displacements $u^s(t) : \Omega \rightarrow \mathbb{R}^d$ such that

$$\sigma = 2\mu \varepsilon(u^s) - \psi I,$$  

$$\rho \partial_{tt} u - \text{div} \sigma = \rho b,$$  

where the total pressure is defined in (2.1), $\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^T)$ is the tensor of infinitesimal strains, $I$ is the identity tensor, $\rho$ denotes is the density of the saturated porous material, and $\mu, \lambda$ are the shear and dilation moduli associated with the constitutive law of the solid structure. These and all other model parameters are assumed positive and bounded, except for the dilation modulus $\lambda$, which approaches infinity for fully incompressible materials. In [49] we did not consider an acceleration term in the equation of balance of linear momentum (2.4) as one typically supposes that solid deformations are much slower than the fluid flow rate. Nevertheless we keep that term here, as we will also explore the influence of inertial effects in the context of linear stability analysis.

Recent applications of such consolidation theory to the poromechanical characterisation of soft living tissues confined to the regime of infinitesimal strains include mainly the formation and development of brain oedema [48, 38] and the importance of including pia mater [45], as well as periodontal layer and tooth interfaces [3]. On the other hand, extensions to large poromechanical deformations and finite growth have been devoted to ventilation processes in the lung parenchyma [5], early stages of feather primordia formation [15], pathological skin expansion [51], or morphoelasticity of arteries [46].

Using poroelasticity to model soft tissues is of high relevance since the permeability of tissue constituents such as collagenous membranes is typically in the orders of $10^{-14}$ to $10^{-12}$ [m²N⁻¹s⁻¹]. If one
considers membranes having a thickness of a few hundred microns, then fluid exchange occurs in the range of seconds and therefore this flow can perfectly affect physiological tissue deformations due to cardiac cycle or breathing [16]. Thus we will work under the assumption that a compound of living cells forms a macroscopic linear poroelastic structure fully saturated with interstitial fluid, and that its motion can be described by (2.1)-(2.4).

2.2. Macroscopic description of two-species motion

Next we turn to the incorporation of two interacting species whose dynamics occurs only by diffusion and reaction. These can represent many different systems, for instance bacteria and neutrophils as in [38], or two different morphogens as in [15]. In any case, the interaction between the diffusive solutes is modelled by a PDE system that includes reaction-diffusion, as well as advection by the velocity of the moving domain. Alternatively, one could also suppose that the species are advected only by the fluid velocity (or by the filtration velocity). Simpler models are able to take advantage of one-dimensional geometries, or of a constant material density of the constituents (e.g. cells) in order to obtain closed-form expressions for the advecting velocity [30]. Here we use the transient form of the equations of motion (2.4),(2.3) to determine such velocity.

We therefore consider the propagation of a generic species with concentration \( w_1 \), reacting with an additional species with concentration \( w_2 \). The problem can be written as follows

\[
\begin{align*}
\partial_t w_1 + \partial_t \mathbf{u} \cdot \nabla w_1 - \text{div} \{ D_1(x) \nabla w_1 \} &= f(w_1, w_2, \mathbf{u}^s), \\
\partial_t w_2 + \partial_t \mathbf{u} \cdot \nabla w_2 - \text{div} \{ D_2(x) \nabla w_2 \} &= g(w_1, w_2, \mathbf{u}^s),
\end{align*}
\]

where \( D_1, D_2 \) are positive definite matrices containing possibly anisotropy of self-diffusion. The net reaction terms depend on parameters that account for the reproduction of species, the removal of species concentration due to reactive interactions, and the intrinsic changes due to local modifications in volume (that is, how the pore microstructure evolves with deformation). For illustrative purposes, and as in [49, 30], we can simply consider hypothetical kinetic specifications, which can also simplify the exposition of the linear stability analysis of Section 3. We choose a modification to the classical Schnakenberg model [42]

\[
\begin{align*}
f(w_1, w_2, \mathbf{u}^s) &= \beta_1 (\beta_2 - w_1 + w_1^2 w_2) + \gamma w_1 \partial_t \text{div} \mathbf{u}^s, \\
g(w_1, w_2, \mathbf{u}^s) &= \beta_1 (\beta_3 - w_2^2 w_1) + \gamma w_2 \partial_t \text{div} \mathbf{u}^s,
\end{align*}
\]

where \( \beta_1, \beta_2, \beta_3, \gamma \) are positive rate constants. As mentioned above, the mechano-chemical feedback operates only by advection and the last two terms defining \( f, g \). These terms are modulated by \( \gamma > 0 \), and therefore they act as a local source for a given species if the solid volume increases, otherwise the additional terms contribute to removal of species concentration [30].

2.3. Active stress

We assume that stresses are exerted by solid, by fluid, and by morphogens. Then the forces are condensed in a macroscopic balance equation for the mixture where we recall that the solid phase is simply considered as an isotropic deformable porous medium and that the fluid phase only contributes volumetrically to the stress through the hydrostatic fluid pressure at the interstitium. Microscopic tension generation is here supposed to occur due to active stresses (2.3) and

\[
\sigma_{\text{total}} = \sigma + \sigma_{\text{act}},
\]

where the active stress operates primarily on a given, constant direction \( \mathbf{k} \), and its intensity depends on a scalar field \( r = r(w_1, w_2) \) and on a positive constant \( \tau \), to be specified later on (see e.g. [20])

\[
\sigma_{\text{act}} = -\tau r \mathbf{k} \otimes \mathbf{k}.
\]
2.4. Initial and boundary conditions under different model configurations

We employ appropriate initial data at rest

\[ w_1(0) = w_{1,0}, \quad w_2(0) = w_{2,0}, \quad u^0(0) = 0, \quad \partial_t u^0(0) = 0, \quad p^f(0) = 0, \quad \psi(0) = 0 \quad \text{in} \quad \Omega \times \{0\}. \]

Regarding boundary conditions, the species concentrations will assume zero diffusive flux boundary conditions on the whole boundary

\[ D_1(x) \nabla w_1 \cdot n = 0 \quad \text{and} \quad D_2(x) \nabla w_2 \cdot n = 0 \quad \text{on} \quad \partial \Omega \times (0, t_{\text{final}}]. \]

For the poromechanics we adopt either Robin conditions for the deformations (mimicking the presence of supporting springs) and zero fluid flux everywhere on the boundary,

\[ [2\mu e(u^0) - \psi I + \sigma_{\text{act}}]n + \zeta u^0 = 0 \quad \text{and} \quad \frac{K}{\eta} \nabla p^f \cdot n = 0 \quad \text{on} \quad \partial \Omega \times (0, t_{\text{final}}], \]

where \( \zeta > 0 \) is the (possibly time-dependent) stiffness of the spring; or as in [49] we can separate the boundary \( \partial \Omega = \Gamma \cup \Sigma \) into two parts \( \Gamma \) and \( \Sigma \) where we prescribe clamped boundaries and zero fluid normal fluxes; and zero (total) traction together with constant fluid pressure, respectively

\[ u^0 = 0 \quad \text{and} \quad \frac{K}{\eta} \nabla p^f \cdot n = 0 \quad \text{on} \quad \Gamma \times (0, t_{\text{final}}], \]

\[ [2\mu e(u^0) - \psi I + \sigma_{\text{act}}]n = 0 \quad \text{and} \quad p^f = 0 \quad \text{on} \quad \Sigma \times (0, t_{\text{final}}]. \]

Each case will be specified in the tests of Sections 3 and 5.

3. Linear stability analysis and dispersion relation

Next we proceed to derive a linear stability analysis similarly as done in [30]. This analysis delivers insight regarding the interaction mechanisms between tissue deformation and diffusing solutes. Our development is however more general and more involved, since we are including acceleration effects in the momentum equilibrium equation. We skip as much as possible the lengthy details of the derivation, and concentrate only in the dispersion relation. Note that since the functions \( f \) and \( g \) are prescribed, we can specify a steady state given by \( w_1 = w_{1,0} = \beta_2 + \beta_3, w_2 = w_{2,0} = \frac{\beta_5}{(\beta_2 + \beta_3)^2}, p = \rho_0, \psi = \psi_0 \) and \( u = 0 \).

Linear stability analysis revolves on the study of the eigenvalue-eigenvector systems evaluated from the linearisation of the coupled PDE around their steady state [43]. This tool enables to understand more deeply the behaviour (stability) of the linearised system with respect to certain types of perturbations around the steady state. The main idea is to postulate a solution of the linearised system, being of the form \( \Phi \exp(\mathbf{i} \mathbf{k} \cdot \mathbf{x} + \phi t) \), where \( \Phi \) is a constant eigenvector of size equal to number of independent variables in our model (e.g., here \( 4 + d \) for \( \mathbb{R}^d \)), while \( \phi \) is the eigenvalue, also called linear growth factor, that is directly linked to the stability of our system. The term \( \mathbf{k} \) (resp. \( k = |\mathbf{k}| \)) is the wave vector (resp. wave number) that is a measure of the spatial structure of our eigen-solution. Depending on the boundary conditions applied to our model, the value of \( k \) is bounded and delimited by the geometry of our domain and the constraint that the eigen-solution must satisfy the boundary conditions. To avoid additional complexity, we centre our analysis on an infinite domain in \( \mathbb{R}^d \), with \( d = \{2, 3\} \).

3.1. General form of the dispersion relation

Following e.g. [50], we can derive a dispersion relation that is eventually defined by the product of two distinct polynomials

\[ P(\phi; k^2) = P_1(\phi; k^2)^{d-1} P_2(\phi; k^2), \]
where $P_1(\phi; k^2) = \rho \phi^2 + \mu k^2$, and where $d = \{2, 3\}$ is the spatial dimension of the infinite domain $\Omega = \mathbb{R}^d$, where the linear stability analysis of the coupled problem (2.2)-(2.4) is performed.

Since $P_2$ is a polynomial with pure imaginary roots, it does not have an influence on the stability of the steady state version of (2.2)-(2.4). Consequently, we can focus our attention on $P_2$ which is a fifth-order polynomial given by

$$P_2(\phi; k^2) = A_5(k^2)\phi^5 + A_4(k^2)\phi^4 + A_3(k^2)\phi^3 + A_2(k^2)\phi^2 + A_1(k^2)\phi + A_0(k^2),$$

where

$$A_5(k^2) = \rho c_0,$$

$$A_4(k^2) = \rho \left( c_0(D_1 + D_2) + \frac{\kappa}{\eta} \right) k^2 - \rho c_0 \frac{\beta_3 - \beta_2 - (\beta_2 + \beta_3)^3}{\beta_2 + \beta_3},$$

$$A_3(k^2) = \rho \left( \frac{\kappa}{\eta} (D_1 + D_2) + c_0 D_1 D_2 \right) k^4 + \left[ c_0 (2\mu + \lambda) + \alpha^2 - \rho c_0 \frac{D_2(\beta_3 - \beta_2) - D_1(\beta_2 + \beta_3)^3}{\beta_2 + \beta_3} - \frac{\kappa \rho}{\eta} \frac{\beta_3 - \beta_2 - (\beta_2 + \beta_3)^3}{\beta_2 + \beta_3} \right] k^2 + \rho c_0 \beta_2 (\beta_2 + \beta_3)^2 - \gamma \left[ \sum_{j=1}^{d} \tilde{Y}_j k_j \right] c_0 (w_{1,0} \theta_1 + w_{2,0} \theta_2),$$

$$A_2(k^2) = \frac{\kappa \rho}{\eta} D_1 D_2 k^6 + \left[ (c_0 (2\mu + \lambda) + \alpha^2) (D_1 + D_2) + \frac{\kappa}{\eta} (2\mu + \lambda) - \frac{\kappa \rho}{\eta} \frac{D_2(\beta_3 - \beta_2) - D_1(\beta_2 + \beta_3)^3}{\beta_2 + \beta_3} \right] k^4 + \left[ \frac{\kappa \rho}{\eta} \beta_1 (\beta_2 + \beta_3)^2 - \left( c_0 (2\mu + \lambda) + \alpha^2 \right) \beta_1 \frac{\beta_3 - \beta_2 - (\beta_2 + \beta_3)^3}{\beta_2 + \beta_3} \right] k^2 - \kappa \gamma \left[ \sum_{j=1}^{d} \tilde{Y}_j k_j \right] \left[ \left( \frac{\kappa}{\eta} \right) (w_{1,0} \theta_1 + w_{2,0} \theta_2) + c_0 (w_{1,0} \theta_1 D_2 + w_{2,0} \theta_2 D_1) \right] k^2 + c_0 \left[ -w_{1,0} \theta_2 \beta_1 \frac{3 \beta_3}{\beta_2 + \beta_3} + w_{2,0} \theta_1 \beta_1 (\beta_2 + \beta_3)^2 + w_{1,0} \theta_1 \beta_1 (\beta_2 + \beta_3)^2 - w_{2,0} \theta_2 \beta_1 (\beta_3 - \beta_2) \right],$$

$$A_1(k^2) = \frac{\kappa}{\eta} (2\mu + \lambda) (D_1 + D_2) + (c_0 (2\mu + \lambda) + \alpha^2) (D_1 D_2) k^6 - \left[ \left( c_0 (2\mu + \lambda) + \alpha^2 \right) \beta_1 \frac{D_2(\beta_3 - \beta_2) - D_1(\beta_2 + \beta_3)^3}{\beta_2 + \beta_3} + \frac{\kappa}{\eta} (2\mu + \lambda) \beta_1 \frac{\beta_3 - \beta_2 - (\beta_2 + \beta_3)^3}{\beta_2 + \beta_3} \right] k^4 + \left( c_0 (2\mu + \lambda) + \alpha^2 \right) \beta_1 (\beta_2 + \beta_3)^2 k^2 - \frac{\kappa \gamma}{\eta} \left[ \sum_{j=1}^{d} \tilde{Y}_j k_j \right] \left[ (w_{1,0} \theta_1 D_2 + w_{2,0} \theta_2 D_1) k^2 - w_{1,0} \theta_1 \beta_1 \frac{2 \beta_3}{\beta_2 + \beta_3} + w_{2,0} \theta_1 \beta_1 (\beta_2 + \beta_3)^2 + w_{1,0} \theta_1 \beta_1 (\beta_2 + \beta_3)^2 - w_{2,0} \theta_2 \beta_1 (\beta_3 - \beta_2) \right],$$

$$A_0(k^2) = \frac{\kappa}{\eta} (2\mu + \lambda) k^4 \left( D_1 D_2 k^4 - \beta_1 \frac{D_2(\beta_3 - \beta_2) - D_1(\beta_2 + \beta_3)^3}{\beta_2 + \beta_3} k^2 + \beta_1^2 (\beta_2 + \beta_3)^2 \right),$$

where the coefficients $w_{j,0}$, for $j = \{1, 2\}$, are the steady state concentrations of generic species $w_j$, and $\theta_j = (\partial \sigma_{act}(w_{0,j}))$ with $\sigma_{act}(\mathbf{w}) = -\tau r(\mathbf{w})$; $\tilde{Y}_j = Y_{-j} + i\tilde{Y}_j$, with $Y_{-j} = \sum \partial_x \mathbf{x}_{k_j}$, $\tilde{Y}_j = \sum \partial_k \mathbf{x}_{k_j}$ and $\mathbf{Y} = \mathbf{k} \otimes \mathbf{k}$.

For the rest of the linear analysis, we impose that $\mathbf{Y} = \mathbf{I}$, with $\mathbf{I}$ the identity matrix, $r^{(1)}(\mathbf{w}) = w_1 + w_2$, $r^{(2)}(\mathbf{w}) = w_1^2$ and $\mathbf{b} = \mathbf{0}$. In such condition, only the coefficients $A_4, A_2, A_1$ are modified and they adopt the following forms:

$$A_4(k^2) = \rho \left( \frac{\kappa}{\eta} (D_1 + D_2) + c_0 D_1 D_2 \right) k^4$$

and
spatial homogeneous case is stable if and only if
satisfy that all real values,
and sufficient set of conditions can be stated so that the roots are in the space of complex non-positive
do not include an acceleration term ($\rho_e$)
eigenvalue systems. Nevertheless, we can observe that for the case that the momentum conservation

3.2. Spatial homogeneous distributions

Due to the high-order of the characteristic polynomial (3.1), it is challenging to determine analytically
the main features of the system. We will therefore resort to numerically solving the corresponding
eigenvalue systems. Nevertheless, we can observe that for the case that the momentum conservation
does not include an acceleration term ($\rho = 0$), the polynomial $P(\phi; k^2)$ is only of order 3. As in [15],
we concentrate on distinguishing some particular scenarios that may include or not the inertial term.
Unless specified otherwise, throughout our analysis we will employ the following parameter values

$$D_1 = 0.05, D_2 = 1.0, \beta_1 = 170, \beta_2 = 0.1305, \beta_3 = 0.7695, E = 3 \cdot 10^4, \nu = 0.495,$$
$$\rho = 1, c_0 = 1 \cdot 10^{-3}, \kappa = 1 \cdot 10^{-4}, \alpha = 0.1, \eta = 1, \gamma = 1 \cdot 10^{-4},$$

which are relevant to the specifications in Tests 1-4 from Section 5. All the computations and graphs in
the remainder of this section have been produced with an in-house MATLAB implementation.

3.2. Spatial homogeneous distributions

For the case $k^2 = 0$, the characteristic polynomial $P_2(\phi; 0) = 0$ reduces to

$$P_2(\phi; 0) = \phi^3 \left[ \rho c_0 \phi^2 - \rho c_0 \beta_1 \frac{\beta_3 - \beta_2 - (\beta_2 + \beta_3)^3}{\beta_2 + \beta_3} \phi + \rho c_0 \beta_1^2 (\beta_2 + \beta_3)^2 \right].$$

Therefore its roots are either equal to zero or defined by the second-order polynomial in brackets. The
so-called Routh-Hurwitz conditions (see e.g. [39]) state that for any polynomial of order 2, a necessary
and sufficient set of conditions can be stated so that the roots are in the space of complex non-positive
real values, $\mathbb{C} = \{ z \in \mathbb{C} : \Re(z) \leq 0 \}$. For a general polynomial $P(\phi) = a_2 \phi^2 + a_1 \phi + a_0$, we need to
satisfy that all $a_i > 0$ (or all $a_i < 0$). In our case, $a_2, a_0$ are positive by definition, and consequently the
spatial homogeneous case is stable if and only if

$$\beta_3 - \beta_2 < (\beta_2 + \beta_3)^3.$$ (3.2)
As $\beta_2, \beta_3$ are positive coefficients, this means that at least the basal source rate $\beta_3$ has to be larger than $\beta_2$ while their difference is lower than $(\beta_2 + \beta_3)^2$. A similar condition is provided in [24]. Additionally, we observe that the system is homogeneously stable irrespective of the parameter values, by simply imposing that $\rho = 0$, i.e., removing the acceleration term in the momentum equilibrium.

3.3. Uncoupled system

This scenario is reached if either $\gamma$ or $\tau$ (or both) are zero. $P_2(\phi; k^2)$ is then a fifth-order polynomial defined as in 3.1 where the terms including $\gamma$ or $\tau$ are dropped from the coefficients $A_i(k^2)$. For a polynomial of order 5, $P(\phi) = \sum_{j=0}^{5} a_j \phi^j$, the Routh-Hurwitz conditions are given by

$$\forall j \quad a_j > 0,$$

$$a_3 a_4 - a_2 a_5 > 0,$$

$$a_2 a_3 a_4 - a_2^2 a_5 - a_1 a_4^2 + a_0 a_2 a_5 > 0,$$

$$a_0 a_2 a_3 a_4 a_5 - a_0 a_2^2 a_5^2 + a_1 a_2 a_3 a_4^2 - a_1 a_2^2 a_4 a_5 - a_1^2 a_4^2 + 2 a_0 a_1 a_2 a_5 - a_2^3 a_4^2 > 0.$$

From (3.2), we see that $\beta_2, \beta_3$ are important parameters of the system. We decide then to perform the analysis with $\beta_2$ as variable; fixing all the other parameters. Based on (3.2) and the constrain on the
parameters, we deduce easily that conditions $a_4$ and $a_5$ are strictly positive, rejecting them to find any patterning space of the system. As the complete analysis of (3.3)-(3.6) is analytically quite involved, we restrict the discussion on the condition that breaks $a_0 > 0$. Beyond its tractable analysis, the choice of this coefficient is legitimised by the fact that $a_1$, $a_2$, $a_3$ can be written as an affine function of $a_0$ with positive coefficients, as long as $\rho \neq 0$. This leads, in the uncoupled system, to the property that if $a_i > 0$ conditions are broken then $a_0 > 0$ is inevitably unsatisfied.

Figure 3.1(A) plots the contour lines of $a_i$, $i = 0, \ldots, 4$. As we can observe for this specific parameter set, $a_2$, $a_4$ are negative only in the region below the magenta dot-dashed curve, corresponding to the limit of the condition (3.2). Consequently, these coefficients are strictly positive while (3.2) is true. Only $a_0$ and $a_1$ present Turing instability, and the latter is just a subset of the $k$-$\beta_2$ space defined by $a_0$. Conditions (3.4)-(3.6) present a similar behaviour with respect to $a_0$ (Figs. 3.1(B)-(D)) and so this enables us to focus our linear analysis on the $a_0$ coefficient only. For the case $\rho = 0$, the resulting polynomial is of order 3 and so the Routh-Hurwitz conditions are defined by

$$\forall j \quad a_j > 0, \quad a_1a_2 - a_0a_3 > 0. \quad (3.7)$$

Note that contrary to the general case, only $a_1$ can be written as an affine function of $a_0$. As the condition (3.2) is no longer needed to obtain homogeneous stability, the coefficients of the affine description of $a_1$ cannot be said strictly positive as before. Nevertheless, as illustrated in Figure 3.2, condition $a_0 > 0$ is the first to be broken with respect to the value of $\beta_2$, and thus we decide to focus the analysis also on that coefficient.

By definition, $a_0$ is a polynomial of even order with respect to $k_2^2$, guaranteeing that there exists at least one local extremum. Therefore, we look for the critical wave number $k_2^c > 0$, obtained by solving the equation $a_0(k_2^2) = 0$, that substituting in $a_0$ will lead to the equation $a_0(k_2^c, \varphi_c) = 0$, with $\varphi_c$ the critical parameter that we want to analyse. In the studied scenario, $a_0^c$ is a cubic polynomial with respect to $k_2^2$, and from the Routh-Hurwitz conditions, the following criteria might be satisfied in order for $a_0$ to be negative for a positive $k_2^2$

$$D_2(\beta_3 - \beta_2) - D_1(\beta_2 + \beta_3)^3 > 0, \quad (3.8)$$

$$36 \left( \beta_1 \frac{D_2(\beta_3 - \beta_2) - D_1(\beta_2 + \beta_3)^3}{\beta_2 + \beta_3} \right)^2 (\beta_2^2(\beta_2 + \beta_3)^2)^2 - 128D_1D_2(\beta_2^2(\beta_2 + \beta_3)^2)^3 > 0. \quad (3.9)$$
Inequality (3.8) enforces that the coefficient has a real positive part and it comes from the analysis of the Routh-Hurwitz conditions. Combined with condition (3.2), it gives an interval for the ratio \((\beta_2 + \beta_3)^2 / (\beta_3 - \beta_2)\) parameter where Turing instabilities are reached. As the discriminant of \(a_0\) is null, we look for the discriminant (3.9) of the derivative \(a_0^{(i)}\) to force \(\kappa^2 \in \mathbb{R}\). Figure 3.3 (panels (A)-(C)) represent the patterning space based on the implicit functions defined in (3.8) and (3.9) for the \((\beta_2, \beta_3)\) space. Figure 3.3 suggests that increasing the value of the production basal rate \(\beta_3\) of the inhibitor \(w_2\) leads to an interval augmentation of possible basal rate \(\beta_2\) of the activator \(w_1\). The condition (3.2) represented by the blue-dot-dashed curve in Figure 3.3(A) is absent for the \(\rho = 0\) scenario, leading to an increase of the patterning space region. Thus, introducing acceleration in the momentum equilibrium equation leads to a restriction of the Turing space.

### 3.4. Coupled system - Null production/degradation rates

In contrast with other reaction-diffusion systems, due to the coupling with the poroelastic deformations we can perfectly have situations where production/degradation rates are missing. For the first case of \(\beta_1 = 0\), corresponding to a pure advection-diffusion chemical system, the characteristic polynomial for \(\rho \neq 0\) is still of order 5 with the new coefficients

\[
\begin{align*}
A_1^{(1)}(k^2) &= A_2(k^2), \quad A_1^{(4)}(k^2) = \rho \left(c_0(D_1 + D_4) + \frac{\kappa}{\eta}\right) k^2, \\
A_1^{(2)}(k^2) &= \rho \left(c_0(D_2 + D_1) + \frac{\kappa}{\eta}\right) k^4 + \left[c_0(2\mu + \lambda) + \alpha^2 + \gamma c_0 \left(\beta_2 + \beta_3\right) \theta_1^{(i)} + \frac{3\beta_3}{(\beta_2 + \beta_3)^2} \theta_2^{(i)}\right] k^2, \\
A_1^{(3)}(k^2) &= \frac{\kappa \rho}{\eta} D_1 D_2 k^6 + \left[(c_0(2\mu + \lambda) + \alpha^2)(D_1 + D_2) + \frac{\kappa}{\eta}(2\mu + \lambda)\right] k^4, \\
A_1^{(4)}(k^2) &= \frac{\kappa}{\eta} (2\mu + \lambda)(D_1 + D_2) + \frac{\gamma \kappa}{\eta} \left(\beta_2 + \beta_3\right) \theta_1^{(i)} D_2 + \frac{3\beta_3}{(\beta_2 + \beta_3)^2} \theta_2^{(i)} D_1, \\
A_1^{(5)}(k^2) &= \frac{\kappa}{\eta}(2\mu + \lambda)D_1 D_2 k^8,
\end{align*}
\]

where each \(\theta^{(i)}\) (for \(i = 1, 2\)) is specified as

\[
\theta^{(1)} = \begin{pmatrix} \theta_1^{(1)} \\ \theta_2^{(1)} \end{pmatrix} = -\tau(1, 1)^T, \quad \theta^{(2)} = -2\tau(1, 0)^T.
\]
Figure 3.4: Contour plots of the Routh-Hurwitz conditions for the characteristic polynomial of system (2.2)-(2.4) for $\beta_1 = 0$ and $\rho = 1$. (A1) Null level set of $P_2$-polynomial coefficients $a_1^{(1)}$ for the coupling term $\theta^{(1)}$. (A2) Level sets (100) of condition (3.4) with null level of the condition (3.4) (red dot-dashed) for the coupling term $\theta^{(1)}$. (A3-A4) Similar analysis for the conditions (3.5) and (3.6) respectively. (B1-B4) Similar analysis for the coupling term $\theta^{(2)}$.

By definition, we observe that the spatial homogeneous situation ($k = 0$) is stable whatever the value of the parameters. For a general $k$, all the coefficients that do not include the coupling parameter $\gamma$ are
Figure 3.5: Contour plots of the Routh-Hurwitz conditions for the characteristic polynomial of system (2.2)-(2.4) for $\beta = 0$ and $\rho = 0$. (A1) Null level set of $P_2$-polynomial coefficients $a_i^{(1)}$ for the coupling term $\theta^{(1)}$. (A2) Level sets (100) of condition (3.7) with associated null level (red dot-dashed) for the coupling term $\theta^{(1)}$. (B1-B2) Similar analysis for the coupling term $\theta^{(2)}$.

strictly positive, and consequently they do not influence the stability of the coupled system. As there is no restriction on the choice of the parameter values for the homogeneous case, any parameter can be chosen as the critical one. In order to analyse the influence of poromechanics on the chemical system, we regard $\tau$ as the parameter of interest.

Figure 3.4 presents contour plots of the Routh-Hurwitz conditions with respect to wave number $k$ and parameter $\tau$ for both $\theta$ defined in (3.10). From Figures 3.4(A1) and 3.4(B1), we observe that $a_3$ is the first coefficient to break the inequality condition with respect to $\tau$, at a low wave number. Along $k$, the parabolic shape of the null levels shows how, depending on the size of the system, any of the three coefficients can break the Routh-Hurwitz inequality. Consequently, the coupled system presents complex instability behaviour and makes difficult to choose only one of them to analyse the full patterning space. Nevertheless, patterns are reachable for large values of $\tau$, irrespective of the wave number (at least in the presented interval). This is contrary to the uncoupled case, where all the $a_i$’s are strictly positive. Figures 3.4(A2)-(A4) and 3.4(B2)-(B4) display the sign of the conditions (3.4)-(3.6) for $\theta^{(1)}$ and $\theta^{(2)}$, respectively. We readily see that the instability region starts already at a value of $\tau$ ($\sim 10^5$) lower than that provided by the $a_i$’s. This emphasises the influence of the mechanical coupling into pattern formation.

The choice of the coupling function $\theta$ has a non-intuitive influence on the Routh-Hurwitz condi-
tions. While it reduces the area of instability region for conditions (3.4) and (3.6) (cf. respectively Figs. 3.4(A2), (B2) and Figs. 3.4(A4), (B4)), it increases it for condition (3.5) (cf. Figs. 3.4(A3), (B3)). Proposing a patterning space based uniquely on the parameters can still be quite difficult due to the complexity of the Routh-Hurwitz conditions and by the evolution of the inequality constrain through the wave number. Nonetheless, we can still show how the poromechanics coupled to a pure advection-diffusion system might generate patterns. In the absence of acceleration, the analysis of the Routh-Hurwitz conditions is quite similar (see Figure 3.5) to the general case. We observe again that for a specific \( \tau \), the conditions can break for any coupling function \( \theta \). Contrary to the case of \( \rho = 1 \), here the defect on the criteria occurs almost at one specific value of \( \tau \) for any wave number \( k \) (see Figs. 3.5(A1), (B1)). The coupling function seems to move closer to each other the levels of the different coefficients \( a_i \), and it also decreases the interval of \( \tau \) that leads to breaking the second condition (3.7).

Secondly, we impose either the basal rate of the activator, \( \beta_2 \), or the inhibitor, \( \beta_3 \), to be equal to zero. Again the characteristic polynomial \( P_2(\rho, k^2) \) for \( \rho \neq 0 \) is still of order 5 with the new coefficients given by

\[
A_5^{(2)}(k^2) = A_5^{(3)}(k^2) = A_3(k^2), \quad A_4^{(2,3)}(k^2) = \rho \left( co(D_1 + D_2) + \frac{\kappa}{\eta} \right) k^2 - \frac{\rho c_0 \beta_1 (1 - \beta_3^2)}{-\rho c_0 \beta_1 (1 + \beta_2^2)},
\]

\[
A_3^{(2,3)}(k^2) = \rho \left( \frac{\kappa}{\eta} (D_1 + D_2) + c_0 D_1 D_2 \right) k^4 + \left\{ \begin{array}{l}
\left[ c_0 (2 \mu + \lambda) + \alpha^2 - \rho c_0 \beta_1 (D_2 - D_1 \beta_2^2) - \frac{\alpha}{\eta} \beta_1 (1 - \beta_3^2) + \Psi(\beta_3; \tau) \right] k^2 + c_0 \rho \beta_1^2 \beta_2^2 \\
\left[ c_0 (2 \mu + \lambda) + \alpha^2 + \rho c_0 \beta_1 (D_2 + D_1 \beta_2^2) + \frac{\alpha}{\eta} \beta_1 (1 + \beta_3^2) + \Psi(\beta_3; \tau) \right] k^2 + c_0 \rho \beta_1^2 \beta_2^2
\end{array} \right.
\]

\[
A_2^{(2,3)} = \frac{\kappa d_0}{\eta} D_1 D_2 k^6 + \left\{ \begin{array}{l}
\left[ c_0 (2 \mu + \lambda) + \alpha^2 \right] (D_1 + D_2) + \frac{\alpha}{\eta} (2 \mu + \lambda) - \frac{\alpha}{\eta} \beta_1 (D_2 - D_1 \beta_2^2) + \Psi(\beta_3; \tau) \right] k^4 \\
\left[ c_0 (2 \mu + \lambda) + \alpha^2 \right] (D_1 + D_2) + \frac{\alpha}{\eta} (2 \mu + \lambda) + \frac{\alpha}{\eta} \beta_1 (D_2 + D_1 \beta_2^2) + \Psi(\beta_3; \tau) \right] k^4 + \left[ \frac{\alpha}{\eta} \beta_1^2 \beta_2^2 - (c_0 (2 \mu + \lambda) + \alpha^2) \beta_1 (1 - \beta_3^2) + \Psi(\beta_3; \tau) \right] k^2 + \frac{\alpha}{\eta} \beta_1^2 \beta_2^2 + (c_0 (2 \mu + \lambda) + \alpha^2) \beta_1 (1 + \beta_3^2) + \Psi(\beta_3; \tau) \right] k^2
\end{array} \right.
\]

\[
A_1^{(2,3)} = \frac{\kappa e_0}{\eta} (D_1 + D_2) + \left\{ \begin{array}{l}
\left[ c_0 (2 \mu + \lambda) + \alpha^2 \right] (D_1 + D_2) + (c_0 (2 \mu + \lambda) + \alpha^2) D_1 D_2 + \Psi(\beta_3; \tau) \right] k^6 \\
\left[ c_0 (2 \mu + \lambda) + \alpha^2 \right] (D_1 + D_2) + (c_0 (2 \mu + \lambda) + \alpha^2) D_1 D_2 + \Psi(\beta_3; \tau) \right] k^6 - \left[ \left[ c_0 (2 \mu + \lambda) + \alpha^2 \right] \beta_1 (D_2 - D_1 \beta_2^2) + \frac{\alpha}{\eta} (2 \mu + \lambda) \beta_1 (1 - \beta_3^2) + \Psi(\beta_3; \tau) \right] k^4 \\
\left[ \left[ c_0 (2 \mu + \lambda) + \alpha^2 \right] \beta_1 (D_2 + D_1 \beta_2^2) + \frac{\alpha}{\eta} (2 \mu + \lambda) \beta_1 (1 + \beta_3^2) - \Psi(\beta_3; \tau) \right] k^4 + \left[ c_0 (2 \mu + \lambda) + \alpha^2 \right] \beta_1 (1 + \beta_3^2) k^4 + \left[ (c_0 (2 \mu + \lambda) + \alpha^2) \beta_1 \beta_2^2 k^2 \right]
\end{array} \right.
\]

where \( \Psi \) is a generic function that summarises the appropriate coupling term defined precisely in (3.1) and the upper (resp. lower) line in braces exhibits the coefficient description for \( \beta_2 = 0 \) (resp. \( \beta_2 = 0 \)). We observe from the new set of coefficients that the stability of the system behaves differently with respect to the basal rate imposed to zero. In an uncoupled scenario, the case \( \beta_3 = 0 \) presents \( a_i \)'s coefficients that are strictly positive whatever the value of the coefficients. This indicates that removing the basal production of the inhibitor prevents any patterning (in the uncoupled scenario), and this occurs for any \( \rho \) and any wave number \( k \). For \( \beta_2 = 0 \), the system can enter an instability region if and
3.5. General System

In a general scenario, we look how the strength of the coupling between the poromechanical and chemical system can affect its linear stability. Figures 3.7 present, for different values of $\gamma$ (which we only if

$$0 < \beta_3 < \sqrt{\min \left( 1, \frac{D_2}{D_1} \right)}.$$  

Coupling the advection-reaction-diffusion system to the poromechanics enables both scenarios to reach instability depending on the parameter values. We analyse again our system focusing on $\tau$. Figure 3.6 presents the null level set of the coefficients for both $\theta$ (cf. (3.10)) and for $\rho = 1$. The case $\beta_2 = 0$ (see Figs. 3.6(A1), (B1)) is significantly influenced by the choice of $\theta$. Going from a linear $\theta^{(1)}$ to a nonlinear $\theta^{(2)}$ coupling function, the instability region is clearly modified, especially for the coefficient $a_2$, leading to an increase of its area. Furthermore, starting from some wave number $k$, the system presents instability whatever the choice of the parameter $\tau$ (e.g., red null level in Fig. 3.6(A1)). In $\beta_3 = 0$, instabilities can be produced using a large value of $\tau$ (see Figs. 3.6(A2), (B2)), implying that coupling the advection-reaction-diffusion system to a mechanical one enables us to bypass the intrinsic stability of the system determined by $\beta_3 = 0$ in a uncoupled scenario, as pointed out above.

Figure 3.6: Null level set of the characteristic polynomial coefficients $a_i^{(2,3)}$ defined in (3.1) for different null basal rate and coupling term $\theta$. (A1) $\beta_2 = 0$ with $\theta^{(1)}$, (A2) $\beta_3 = 0$ with $\theta^{(1)}$. (B1-B2) Similar analysis for $\theta^{(2)}$. In all plots, the density of the solid material is $\rho = 1$. 
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Figure 3.7: Null level set of the characteristic polynomial coefficients $a_i$ with respect to $\tau$ defined in (3.1) for the general coupled system and for different coupling term $\theta$. (A1) $\gamma = 10^{-4}$ with $\theta^{(1)}$. (A2) $\gamma = 10^{-2}$ with $\theta^{(1)}$. (B1-B2) Similar analysis for $\theta^{(2)}$. In all plots, the density of the solid material is imposed to $\rho = 1$.

recall is parameter that couples the poroelasticity to the Schnakenberg model, the null levels of the dispersion relation coefficients. Increasing $\gamma$ leads to a reduction of the critical value of $\tau$ needed to reach instability (compare with Figs. 3.7(A1), (A2)) without affecting the pattern generated by the null level set. Analogous conclusions can be drawn even using a nonlinear coupling function $r^{(2)}(w)$. As in the previous scenario, the different coefficients present a large interval where the Routh-Hurwitz conditions are not satisfied. In summary, the coupled system accommodates the possibility of generating patterns at different length scales.

To conclude this section, we analyse the sign of the $a_i$ coefficients against the value of $\beta_2$, and compare with the results from the uncoupled model studied above. Figures 3.8 display the null level set for an increasing strength of the coupling by augmenting both the value of $\tau$ and $\gamma$. From Figure 3.8(A1) to 3.8(A3), we observe that $a_2$ is particularly affected by the choice of these two parameters. First reducing the region of instability (see Fig. 3.8(A2)), we obtain, in a more strongly coupled system, two instability regions delimited by the red ($a_0$) and green ($a_2$) level sets that overlap with respect to the wave number $k$, but not with respect to $\beta_2$. Consequently, and depending on the strength of the coupling between reaction-diffusion and poroelastic effects, we can discriminate different values of $\beta_2$ that will produce distinct patterns at desired specific scales. This is more clearly seen for the case with $\theta = \theta^{(2)}$, where the interval of $\beta_2$ that leads to instability increases rapidly with the augmentation of $\tau$ and $\gamma$, and tends finally to overlap the region delimited by $a_0$ (see red level in Fig. 3.8(B3)).
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Figure 3.8: Null level set of the characteristic polynomial coefficients of the general coupled system and for different coupling term \( \theta \). (A1) \( \tau = 100, \gamma = 10^{-4} \), with \( \theta^{(1)} \). (A2) \( \tau = 10^{4}, \gamma = 10^{-2} \) with \( \theta^{(1)} \). (A3) \( \tau = 10^{5}, \gamma = 10^{-2} \) with \( \theta^{(2)} \). In all plots, the density of the solid material is imposed to \( \rho = 1 \).

4. Numerical method and implementation

In [49] we propose a discretisation in space using a mixed finite element method. In the lowest-order case, the method consists of piecewise bilinear elements enriched with bubbles for the displacement, piecewise linear and continuous approximations for the fluid pressure and for the solutes, and piecewise constant approximation for total pressure.

The time discretisation is achieved by a backward Euler scheme and an implicit centred difference method for the first and second order time derivatives, respectively. Denoting the fully discrete method reads: From initial data \( u^{s,0}, p_0, \psi_0, w_0, w^0 \) (which will be projections of the continuous initial conditions of each field) and for \( n = 1, \ldots, \) find \( u_h^{s,n+1} \in V_h, p_h^{f,n+1} \in Q_h, \psi_h^{n+1} \in Z_h, w_h^{n+1} \in W_h, w_{2,h}^{n+1} \in W_h \) such that

\[
\begin{align*}
\tilde{a}_1(u_h^{s,n+1}, v_h^{s}) + a_1(u_h^{s,n+1}, v_h^{s}) + b_1(v_h^{s}, \psi_h^{n+1}) &= F_{i_h}^{s,n+1}(v_h^{s}) \quad \forall v_h^{s} \in V_h, & (4.1) \\
\tilde{a}_2(p_h^{f,n+1}, q_h^{f}) + a_2(p_h^{f,n+1}, q_h^{f}) - b_2(q_h^{f}, \delta_h \psi_h^{n+1}) &= G_{p,f}^{n+1}(q_h^{f}) \quad \forall q_h^{f} \in Q_h, & (4.2) \\
b_1(u_h^{s,n+1}, \phi_h) + b_2(p_h^{f,n+1}, \phi_h) - a_3(\psi_h^{n+1}, \phi_h) &= 0 \quad \forall \phi_h \in Z_h, & (4.3) \\
\tilde{a}_4(w_h^{n+1}, s_{1,h}) + a_4(w_h^{n+1}, s_{1,h}) &= J_{f_0}^{n+1}(s_{1,h}) \quad \forall s_{1,h} \in W_h, & (4.4) \\
\tilde{a}_5(w_{2,h}^{n+1}, s_{2,h}) + a_5(w_{2,h}^{n+1}, s_{2,h}) &= J_{g_0}^{n+1}(s_{2,h}) \quad \forall s_{2,h} \in W_h. & (4.5)
\end{align*}
\]

with

\[
\begin{align*}
\tilde{a}_1(u_h^{s,n+1}, v_h^{s}) &:= \int_{\Omega} \delta_t u_h^{s,n+1} \cdot v_h, \quad a_1(u_h^{s,n+1}, v_h^{s}) := 2\mu \int_{\Omega} \varepsilon(u_h^{s,n+1}) : \varepsilon(v_h^{s}), \\
\tilde{a}_2(p_h^{f,n+1}, q_h^{f}) &:= \int_{\Omega} \delta_t p_h^{f,n+1} dX - \int_{\Omega} p_h^{f,n+1} \cdot \nabla \cdot (\varepsilon v_h^{s}) dX, \\
\tilde{a}_4(w_h^{n+1}, s_{1,h}) &:= \int_{\Omega} \delta_t w_h^{n+1} dX - \int_{\Omega} w_h^{n+1} \cdot \nabla \cdot (\varepsilon s_{1,h}) dX.
\end{align*}
\]
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\begin{equation}
\begin{aligned}
  b_1(v_h^n, \phi_h) &:= -\int_\Omega \phi_h \text{div} v_h, \\
  b_2(p_h^{n+1}, \phi_h) &:= \frac{\alpha}{\lambda} \int_\Omega p_h^{n+1} \phi_h, \\
  a_3(\psi_h^{n+1}, \phi_h) &:= \frac{1}{\eta} \int_\Omega \psi_h^{n+1} \phi_h,
  \\
  \tilde{a}_2(p_h^{n+1}, q_h^n) &:= \left( c_0 + \frac{\alpha^2}{\lambda} \right) \int_\Omega \Delta p_h^{n+1} q_h^n, \\
  a_4(u_h^{n+1}, s_1) &:= \delta_t \int_\Omega u_h^{n+1} s_1, \\
  a_5(u_h^{n+1}, s_2) &:= \delta_t \int_\Omega u_h^{n+1} s_2,
\end{aligned}
\end{equation}

and where for the treatment of the ADR problem, we have proceeded as in [29]. The only nonlinearities reside in the terms \( F_{r_h^{n+1}}(v_h^n) \), \( J_{h_1^{n+1}}(s_1) \), and \( J_{h_2^{n+1}}(s_2) \).

With the aim to rewrite the Galerkin scheme (4.1)-(4.5) as a matrix equation, we begin by decomposing the unknowns \( u_h^n, \psi_h, p_h, w_1, \text{and} w_2 \) in terms of the linear basis functions:

\[
\begin{aligned}
  u_h^n &= \sum_{j=1}^{N_1} U_j(\varphi_j), \\
  \psi_h &= \sum_{j=1}^{N_2} P_j(\tilde{\varphi}_j), \\
  p_h &= \sum_{j=1}^{N_3} P_j(\tilde{\varphi}_j), \\
  w_1 &= \sum_{j=1}^{N_4} W_{1,j}(\varphi_{1,j}), \\
  w_2 &= \sum_{j=1}^{N_4} W_{2,j}(\varphi_{2,j}),
\end{aligned}
\]

and substituting in the above equations

\[
\begin{aligned}
  \sum_{j=1}^{N_1} (\varphi_j, \varphi_i) U_j^{n+1} + \Delta t^2 \sum_{j=1}^{N_1} a_1(\varphi_j, \varphi_i) U_j^{n+1} + \Delta t^2 \sum_{j=1}^{N_2} b_1(\varphi_j, \tilde{\varphi}_i) \Phi_j^{n+1} &= \Delta t^2 \sum_{j=1}^{N_4} F_{r_h^{n+1}}(\varphi_j) \\
  + & \sum_{j=1}^{N_1} (\varphi_j, \varphi_i) U_j^{n} - \sum_{j=1}^{N_1} (\varphi_j, \varphi_i) U_j^{n-1} \quad i = 1, ..., N_1, \\
  \sum_{j=1}^{N_2} (\tilde{\varphi}_j, \tilde{\varphi}_i) P_j^{n+1} + \Delta t \sum_{j=1}^{N_2} a_2(\tilde{\varphi}_j, \tilde{\varphi}_i) P_j^{n+1} - \sum_{j=1}^{N_2} b_2(\tilde{\varphi}_i, \tilde{\varphi}_j) \Phi_j^{n+1} &= \Delta t \sum_{j=1}^{N_3} G(\tilde{\varphi}_i) \\
  + & \left( c_0 + \frac{\alpha^2}{\lambda} \right) \sum_{j=1}^{N_2} (\tilde{\varphi}_j, \tilde{\varphi}_i) P_j^{n} - \sum_{j=1}^{N_2} b_2(\tilde{\varphi}_i, \tilde{\varphi}_j) \Phi_j^{n} \quad i = 1, ..., N_2, \\
  \sum_{j=1}^{N_3} b_1(\varphi_j, \tilde{\varphi}_i) U_j^{n+1} + \sum_{j=1}^{N_3} b_2(\tilde{\varphi}_j, \tilde{\varphi}_i) P_j^{n+1} - \sum_{j=1}^{N_3} a_3(\tilde{\varphi}_j, \tilde{\varphi}_i) \Phi_j^{n+1} &= 0 \quad i = 1, ..., N_3, \\
  \sum_{j=1}^{N_4} (\varphi_j, \varphi_i) W_j^{n+1} - \Delta t \sum_{j=1}^{N_4} a_4(\varphi_j, \varphi_i) W_j^{n+1} &= \Delta t \sum_{j=1}^{N_4} J_{h_1^{n+1}}(\varphi_i) + \sum_{j=1}^{N_4} (\varphi_j, \varphi_i) W_j^{n} \quad i = 1, ..., N_4, \\
  \sum_{j=1}^{N_4} (\tilde{\varphi}_j, \tilde{\varphi}_i) W_j^{n+1} - \Delta t \sum_{j=1}^{N_4} a_5(\tilde{\varphi}_j, \tilde{\varphi}_i) W_j^{n+1} &= \Delta t \sum_{j=1}^{N_4} J_{h_2^{n+1}}(\tilde{\varphi}_i) + \sum_{j=1}^{N_4} (\tilde{\varphi}_j, \tilde{\varphi}_i) W_j^{n} \quad i = 1, ..., N_4.
\end{aligned}
\]

Then, we can organise the above system in terms of matrices and vectors:

\[
\begin{aligned}
  A_1 &\in \mathbb{R}^{N_1 \times N_1}, \\
  A_2 &\in \mathbb{R}^{N_1 \times N_1}, \\
  B_1 &\in \mathbb{R}^{N_2 \times N_1}, \\
  B_2 &\in \mathbb{R}^{N_2 \times N_2}, \\
  A_3 &\in \mathbb{R}^{N_3 \times N_2}, \\
  A_4 &\in \mathbb{R}^{N_3 \times N_3}, \\
  A_5 &\in \mathbb{R}^{N_3 \times N_3}, \\
  F &\in \mathbb{R}^{N_1}, \\
  G &\in \mathbb{R}^{N_3}, \\
  J_1 &\in \mathbb{R}^{N_4}, \\
  J_2 &\in \mathbb{R}^{N_4},
\end{aligned}
\]
such that
\[ a_{1,ij} = \Delta t a_1(\varphi_j, \varphi_i), \quad i, j = 1, \ldots, N_1, \quad a_{3,ij} = a_3(\tilde{\varphi}_j, \tilde{\varphi}_i), \quad i, j = 1, \ldots, N_2, \]
\[ a_{2,ij} = (c_0 + \frac{\alpha}{X}) (\tilde{\varphi}_j, \tilde{\varphi}_i), \quad a_{2,ij} = \Delta t a_2(\tilde{\varphi}_j, \tilde{\varphi}_i), \quad i, j = 1, \ldots, N_3, \]
\[ b_{1,ij} = b_1(\varphi_j, \tilde{\varphi}_i), \quad i = 1, \ldots, N_2, \quad j = 1, \ldots, N_1, \quad b_{2,ij} = b_2(\tilde{\varphi}_j, \tilde{\varphi}_i), \quad i = 1, \ldots, N_2, \quad j = 1, \ldots, N_3, \]
\[ \tilde{a}_{4,ij} = (\tilde{\varphi}_j, \tilde{\varphi}_i), \quad a_{4,ij} = \Delta t a_4(\tilde{\varphi}_j, \tilde{\varphi}_i), \quad i, j = 1, \ldots, N_4, \]
\[ G_i = \Delta t G(\tilde{\varphi}_i), \quad i = 1, \ldots, N_3, \quad J_{1,i} = \Delta t J_{f_{n+1}}(\tilde{\varphi}_i), \quad J_{2,i} = \Delta t J_{f_{n+1}}(\tilde{\varphi}_i), \quad i = 1, \ldots, N_4, \]
and then, denoting
\[
\mathbf{A} := \begin{bmatrix}
\tilde{A}_1 + A_1 & \Delta t^2 B_1^T & O & F \\
0 & A_2 + A_2 & -B_2^T & O \\
B_1 & B_2 & -A_3 & O \\
0 & 0 & 0 & \tilde{A}_4 + A_4 + \tilde{A}_5 + A_5 \\
\end{bmatrix}, \quad 
\mathbf{X} := \begin{bmatrix}
U_j \\
\phi_j \\
W_{1,j} \end{bmatrix}, \quad 
\mathbf{H} := \begin{bmatrix}
2\tilde{A}_1 U_j^n - \tilde{A}_1 U_j^{n-1} \\
G + \tilde{A}_2 P_j^n - B_2 \phi_j^n \\
J_1 + \tilde{A}_4 W_{1,j}^n + J_2 + \tilde{A}_5 W_{1,j}^n \\
\end{bmatrix},
\]
we deduce that the fully-discrete matrix problem for (4.6) reads
\[
\mathbf{A} \mathbf{X}^{n+1} = \mathbf{H}^{n},
\]
which will be useful for the development of the numerical tests.

In addition, a Newton method with exact Jacobian is derived for the solution of (4.1)-(4.5) at each time step. Then, regarding both chemical species in a single vector \( \mathbf{W} \), the tangent algebraic systems to be solved at each Newton step (for a given time step) adopt the following form
\[
\tilde{A}_{11} \delta \mathbf{U}_{k+1}^{*} + \tilde{B}_{11} \delta \mathbf{W}_{k+1} = \mathbf{R}_{1,k}, \\
\tilde{A}_{22} \delta \mathbf{P}_{k+1}^{*} - \tilde{B}_{22} \delta \mathbf{W}_{k+1} = \mathbf{R}_{2,k}, \\
\tilde{B}_{13} \delta \mathbf{U}_{k+1}^{*} + \tilde{B}_{23} \delta \mathbf{P}_{k+1}^{*} - \tilde{A}_{33} \delta \mathbf{W}_{k+1} = \mathbf{R}_{3,k}, \\
\tilde{C}_{42} \delta \mathbf{U}_{k+1}^{*} + \tilde{C}_{42} \delta \mathbf{P}_{k+1}^{*} - \tilde{A}_{43} \delta \mathbf{W}_{k+1} = \mathbf{R}_{4,k},
\]
where \( \delta (\cdot)_{k+1} \) represent the vector of nodal values for the incremental unknowns that are premultiplied by the respective elementary matrices constructed with the bilinear forms in (4.6) or their linearisation; that is, the matrix \( \tilde{F}_i \) is induced by the linearisation of \( F_{r_{n+1}}(\cdot), \tilde{A}_{11} \) by \( a_1(\cdot, \cdot) + \tilde{a}_1(\cdot, \cdot), \tilde{B}_{13} \) and \( \tilde{B}_{13}^{*} \) by \( b_1(\cdot, \cdot), \tilde{A}_{22} \) by \( a_2(\cdot, \cdot) + \tilde{a}_2(\cdot, \cdot), \tilde{B}_{23} \) by \( b_2(\cdot, \cdot), \tilde{B}_{32} \) by \( b_3(\cdot, \cdot), \tilde{A}_{33} \) by \( a_3(\cdot, \cdot), \tilde{J}_4 \) by the linearisation of \( J_{f_{n+1}}(\cdot) \) and \( J_{f_{n+1}}(\cdot), \tilde{C}_{42} \) and \( \tilde{C}_{42}^{*} \) by the linearisation of \( c(\cdot, \cdot, \cdot) \) (see its definition in [49, eq. (2.7)]), and \( \tilde{A}_{11} \) by \( \tilde{a}_1(\cdot, \cdot) + a_1(\cdot, \cdot) \) and \( \tilde{a}_3(\cdot, \cdot) \) by \( a_3(\cdot, \cdot) + a_5(\cdot, \cdot) \). The right-hand side vectors \( \mathbf{R}_{i,k} \) account for body forces, mass sources, entities at the previous time step, and residuals from the previous Newton iteration \( k \). The system is solved by the GMRES Krylov solver with incomplete LU factorisation (ILUT) preconditioning. The stopping criterion on the nonlinear iterations is based on a weighted residual norm dropping below the fixed tolerance of \( 1 \cdot 10^{-6} \).

The implementation of all numerical routines has been carried out using the finite element library FEniCS [2].
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5. Numerical tests

5.1. Test 1: Periodic traction preventing stable patterning

In order to investigate the impact that the structural deformation has on the emerging spatial patterns of solutes, we first consider the spatial domain $\Omega = (0, 1) \times (0, 0.6)$, where the clamped boundary is $\Gamma = \{ x : x_1 = 0, x_1 = 1, x_2 = 0 \}$ and the top face constitutes $\Sigma$ where we apply a periodic traction defined by

$$t = \begin{cases} 
(0, -s_0 \sin(\pi t))^T & \text{if } 0.4 \leq x_1 \leq 0.6, \\
0 & \text{otherwise},
\end{cases}$$

with $s_0 = 25000$ (similarly as in the footing problem from e.g. [31]). According to (2.10)-(2.11), on $\Gamma$ we also impose zero fluid pressure fluxes, whereas on $\Sigma$ we set a uniform fluid pressure $p^f = 0$. The parameters that are modified with respect to Test 2 are only the coupling constants of active stress modulation $\tau = 100$ (using again $r = w_1 + w_2$), the direction $k = (1, 0)^T$, the density $\rho = 1$, and the volume-dependent source $\gamma = 0.05$. The resulting patterns (exemplified by transients of the activator chemical $w_1$ and final states of poromechanical variables) are depicted in Figure 5.1. One can readily observe that, apart from altering substantially the distribution of chemical concentrations from the beginning of the simulation, the periodic traction applied on part of the top edge (and which only produces less than a 10% of vertical stretch) prevents the system from reaching a state with stable spatial patterns. For this test we have used a structured mesh.

5.2. Test 2: Small poromechanical effects

Next we take the domain as the disk centred at $(0.5, 0.5)$ with radius 0.5, and assume that the boundary coincides with $\Gamma'$. Then the displacements are set to zero on the whole boundary and we take relatively small values for the coupling constants on the chemical source and on the active stress.
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Figure 5.2: Test 2. Patterns generated with relatively small chemo-mechanical feedback $\gamma = 0.0001$ and clamped boundary conditions, plotted on the undeformed domain at times $t = 0.25$ (top), $t = 0.375$ (middle row), and $t = 1.5$ (bottom panels). From left to right: $w_2$ concentration, solid displacement magnitude, fluid pressure, and total pressure.

$\gamma = 0.0001$, $\tau = 100$, implying in particular that the patterns produced by the coupled poroelastic-advection-diffusion-reaction system are expected to be qualitatively similar to those observed on a fixed domain. In (2.8) we assume a dependence of the form $r = w_1 + w_2$, and the remaining model constants are taken as $D_1 = 0.05$, $D_2 = 1$, $\beta_1 = 170$, $\beta_2 = 0.1305$, $\beta_3 = 0.7695$, $E = 3 \cdot 10^4$, $\nu = 0.495$, $\rho = 1$, $c_0 = 1 \cdot 10^{-3}$, $\kappa = 1 \cdot 10^{-4}$, $\alpha = 0.1$, $\eta = 1$, $k = (x_1 - 0.5, x_2 - 0.5)^T$. The initial condition for the chemicals is a perturbation of the homogeneous steady state $w_1^0 = \beta_2 + \beta_3$, $w_2^0 = \beta_1(\beta_2 + \beta_3)^{-2}$ and for the displacements and fluid pressure we use zero initial conditions. The domain is discretised into an unstructured mesh of 64926 triangles and we employ a fixed time-step $\Delta t = 0.0025$. The system is advanced until $t_{\text{final}} = 1.5$ and plots with patterns of $w_2$, small deformations, as well as fluid and total pressures are shown in Figure 5.2. In the bottom row we can see how the initial perturbation of the steady state evolves into organised dot-shaped spatial structures, seen clearly for the inhibitor chemical $w_2$ and also captured by the total pressure. No deformation occurs along the domain boundary, but the local deformation patterns show also tissue contraction near the zones of high concentration of the activator species $w_1$. 

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Figure 5.3: Test 3. Interplay between linear growth and active stress. Concentrations of $w_1$ and $w_2$ (top and bottom) on the deformed domain, at time $t = 1.25$, for different mild values of the coupling constants $\tau, \tau_2$. The black circle in all plots represents the boundary of the initial domain.

5.3. Test 3: Linear growth and active stress

Maintaining the same domain and discretisation parameters as in Test 1 above, we now fix the mechano-chemical coupling constant $\gamma = 0.01$ and study the competing effect between linear growth with radial traction, and the active stress depending on the concentration of the activator species $w_1$. This is done with an activation of the type $r = \tau_2 t + w_1^2$, and for this 2D case we consider Robin boundary conditions for the solid motion (2.9) with $\zeta = \tau_2$, on the circular boundary (whereas for the 3D case below we impose zero normal displacement on the bottom of the cylinder and a traction $t = \tau_2 n$ on the remainder of the boundary). We set $k$ to be the radial vector, and vary $\tau, \tau_2$. The results are shown in Figure 5.3. From left to right we display snapshots of the chemical patterns produced with the parameter choices $(\tau = 2 \cdot 10^5, \tau_2 = 0.2), (\tau = 10^5, \tau_2 = 2), (\tau = 10^4, \tau_2 = 10)$, and $(\tau = 100, \tau_2 = 20)$.

5.4. Test 4: Linear growth in 3D

We extend the previous test to assess the behaviour of the model and the finite element scheme in a 3D setting. We modify (2.8) to include axial symmetry on another preferential direction for active deformation. The domain is a cylinder of height 0.05 and radius 0.5, and we set

$$\sigma_{\text{act}} = -\tau \left[ \tau_2 t k_{12} \otimes k_{12} + w_1^2 k_3 \otimes k_3 \right],$$

where $k_{12}$ is the radial vector in the plane $x_1 x_2$ and $k_3 = (0, 0, 1)^T$. This implies that the active deformation due to linear growth will occur in the radial direction whereas the stress due to the chemical concentration will act on the $x_3$-direction. On the bottom surface we set zero normal displacements $u \cdot n = 0$ while on the remainder of the boundary we impose zero traction. In Figure 5.4 we show the resulting patterns of $w_1$ concentration for two sets of Lamé and poromechanical-chemical coupling parameters. For the first case we use $E = 1 \cdot 10^4, \nu = 0.499$, and $\tau = 10, \tau_2 = 20, \gamma = 0.05$ and show the patterns on the deformed domain in the left panels; while the plots on the right panels were produced...
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Figure 5.4: Test 4. Interplay between linear growth and active stress for two sets of elasticity and coupling parameters (top and bottom). Concentrations of \( w_1 \) on the deformed domain, at times \( t = 0.25, 0.5, 1.25 \) (left, centre, and right panels, respectively).

with \( E = 1 \cdot 10^3, \nu = 0.3, \) and \( \tau = 50, \tau_2 = 60, \gamma = 0.1 \). We observe stable pattern generation with the first set of model parameters, similar to the expected patterning in the case of pure reaction-diffusion effects, whereas the patterns on the right exhibit large qualitative differences in \( w_1 \) (also in the other species) as well as in the deformation behaviour.

5.5. Test 5: Application to the simulation of brain injuries and calcium propagation

We close this section with an example related to the one-way coupling between poroelastic deformations in the brain (induced by a localised high stress) and the subsequent propagation and reaction of two types of calcium concentration, intra-cellular and extra-cellular, throughout the tissue. This illustrative test is based on the kinetic and 1D models recently advanced in [21]. In there, the authors propose that hydrostatic stress build up due to the brain trauma affect (in an exponentially decreasing manner) the reacting fluxes between the calcium concentrations. We do not include acceleration but we propose to incorporate this in our model using a modification of (2.5)-(2.6) to include a dependence of the reaction terms on the total pressure

\[
\begin{align*}
    f(w_1, w_2, \psi) &= -D_1(w_1 - w_2) + \frac{1}{\chi_1} \left[ -\chi_1 + (1 + \chi_1) \exp(-k|\psi|) \right] \frac{w_2^2}{w_2^2 + k_1^2}, \\
    g(w_1, w_2, \psi) &= -f(w_1, w_2, \psi) + D_2(w_0 - w_2) - \frac{1}{\chi_2} \left[ -\chi_2 + (1 + \chi_2) \exp(-k|\psi|) \right] \frac{w_2}{w_2 + k_2},
\end{align*}
\]

where \( w_1, w_2 \) represent respectively, the extra-cellular and intra-cellular calcium concentrations (in [mM] units) and the model parameters are as in [21]

\[
\begin{align*}
    D_1 &= 2.94 \cdot 10^{-6} \text{ [1/s]}, \quad D_2 = 3.17 \cdot 10^{-5} \text{ [1/s]}, \quad k_1 = 2 \cdot 10^{-4} \text{ [mM]}, \quad k_2 = 5 \cdot 10^{-4} \text{ [mM]}, \\
    \chi_1 &= 2 \cdot 10^3, \quad \chi_2 = 4 \cdot 10^3, \quad k = 4.5 \cdot 10^{-5} \text{ [Pa}^{-1}], \quad w_0 = 0.1 \text{ [mM]}. 
\end{align*}
\]

On the other hand, the fact that calcium activity effects are negligible in producing deformations of the poroelastic structure (at least, when compared to high stress impacts on the skull or with important
kinematic forces building up because of rapid shocks) implies that in our model the total stress (2.7) does not contain an active component modulated by \( w_1, w_2 \). Also, our model is different than the one in [21] in that we do not consider viscoelastic effects but do include poroelasticity of the brain, and we also include diffusion of the calcium concentrations. The remaining constants in the model and the initial conditions adopt the values

\[
E = 3.15 \cdot 10^4 \text{[Pa]}, \quad \nu = 0.45, \quad \rho = 1130 \text{[Kg/m}^3], \quad \frac{\kappa}{\eta} = 10^{-5} \text{[mm}^2\text{Pa}^{-1}\text{s}^{-1}], \quad \alpha = 0.1,
\]
\[
c_0 = 3.9 \cdot 10^{-4} \text{[Pa}^{-1}], \quad u^*(0) = 0, \quad p^f = 0, \quad \psi(0) = 0, \quad w_{1,0} = 1 \text{[mM]}, \quad w_{2,0} = 10^{-4} \text{[mM]}.
\]

The spatial domain consists of a 3D structure of the human brain and the boundaries are split between ventricles and the outer meninginal region of the brain, in contact with the skull. The interstitial flow in this case is by cerebrospinal fluid. The domain consists of an adult human brain atlas [17, 23] and we use a tetrahedral mesh with 29037 vertices. An initial traction of magnitude \( 1.7 \cdot 10^4 [\text{Pa}] \) is applied for \( 5.5 \text{[ms]} \) on a location near the ventricles and on the skull we impose zero normal displacements and zero fluid pressure, whereas on the ventricles we assume zero fluid pressure flux. We employ a timestep of \( \Delta t = 0.1 \text{[ms]} \) and run the simulation until \( t = 180 \text{[s]} \). Transients of the intracellular calcium concentration as well as the total pressure are recorded on two points (one near the ventricles, point A, and another near the meninges, point B), and are displayed in Figure 5.5. One can observe an initial peak of several folds the initial homeostatic value of the intracellular calcium followed by a slowly decaying profile (which however does not goes back to the homeostatic value). We also see that the oscillations in total pressure due to the application of high stresses decrease over time. All this is qualitatively consistent with the model predictions from [21].

6. Concluding remarks

In this paper we have suggested a mathematical model to study the interaction between advecting reaction-diffusion dynamics within a poroelastic material. The set of equations assumes the regime of small strains and the tightly coupling mechanisms are primarily dependent on source functions of change of volume, and active stresses. This work represents an extension with respect to recent three-field models of poroelasticity using total pressure. We have demonstrated the feasibility of the model and of the numerical method to reproduce a variety of coupling scenarios including pattern suppression, linear growth instability, and other morphological changes. We are currently working towards including further details into the model. For example, the effects of chemotaxis and general cross-diffusion, as well as interfacial conditions for two-layered materials as encountered in [3, 14, 44]. Other
aspects to include in future directions are anisotropy of motion and of diffusion, more complicated geometries, and other coupling mechanisms such as stress-assisted diffusion [18].

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