An Improved High-Moment Method for Reliability Analysis Based on Polynomial Chaos Expansion

Jianyu Zhao¹, Jianbin Guo², Jingyan Wang¹ and Hao Xu¹

¹ Beijing Institute of Spacecraft System Engineering, China Academy of Space Technology, Beijing, China
² School of Reliability and Systems Engineering, Beihang University, Beijing, China.

E-mail: guojianbin@buaa.edu.cn

Abstract. For structural reliability analysis, the moment method provides a relatively effective way to estimate the failure probability only using the moment information of a given model. However, it may introduce extra computation costs to obtain the required moments of the model output accurately, especially for some high moment methods like the third-moment method and the fourth-moment method. This study improves the traditional high-moment method based on the polynomial chaos expansion (PCE). The proposed method firstly derives the PCE-based high order moment algorithm by multiplying the orthogonal polynomials of PCE itself. Then, the integration between the PCE and the high-moment method are proposed for failure probability calculation. Besides, different input distributions as well as correlations among input variables are also taken into consideration to expand the scope of suitability. Compared with the existing methods, the proposed method is more efficient and accurate for structural reliability analysis by an engineering example.

1. Introduction
The reliability analysis is one of the most important issues in structural engineering practice. It is usually concerned with the probability of a limit state violation of the structure of interest, which is caused by the uncertainties in design parameters, operating conditions, material properties, applied loadings, etc. In general, it is an intractable problem to solve the failure probability directly. Many reliability methods have been developed to handle this problem. The commonly used methods are presented as follows.

The numerical simulation methods, such as the Monte Carlo simulation (MCS) and its improved versions (i.e., various importance sampling MCS), are widely employed to calculate the failure probability or reliability [1]. But huge computation burden might be introduced when the model in consideration is complex. To mitigate this problem, researchers have developed the semi-analytical methods including the first order reliability method (FORM) and the second order reliability method (SORM) [2]. FORM replaces the original state limit function by its linear approximation at the most probable point (MPP), while SORM makes a second-order approximation of the state limit function at MPP. However, the gradient computation required by both methods is often time-consuming, so does the MPP searching process. An alternative method for structural reliability analysis is to represent the probability density function (PDF) of the limit-state function response explicitly, and then obtain the failure probability by numerical integration of PDF [3]. Some researchers employ the
maximum entropy (ME) principle to approximate the unknown PDF of model output [4]. Its foundation is that the PDF of response is the one having the ME under given moment constraints. But high order moments and iterative matrix inversions are unavailable during the calculation process. Therefore, the efficiency of these methods is weakened. Besides, the moment methods are also developed for failure probability calculation [5]. The basic idea is to represent the output of the limit-state function only by moment information, because a random variable PDF and its moments could determine each other uniquely by the moment generating function. The simplest case is the second-moment method, and those using higher moments are often called as the high-moment methods, like the third-moment method and the fourth-moment method.

It should be noted that these methods above might encounter difficulties when it comes to the implicit limit-state function, because the implicit limit-state function makes it hard to search the MPP, calculate gradients, or obtain moments. Therefore, the response surface methodology is also proposed [6]. It tries to fit the true limit state function by several polynomials, which make the implicit limit-state function evaluation much easier for further computation. Then, various methods mentioned above could execute using the response surface instead of the original limit-state function. However, if input variables are correlated or have different distributions, the calculation burden could further increase.

Compared with other methods for reliability, the moment methodology only requires the statistic characteristics without the MPP searching, the gradient computation, or the iterative matrix inversion. Thus, it is relatively efficient. But the moment estimations may introduce large computational costs by traditional sampling approach. Chakraborty [7] and Lasota [8] used the polynomial chaos expansion (PCE) to obtain low order moments accurately instead of sampling. PCE approximates the limit-state function with orthogonal polynomial basis functions, and it converges in the L_2 sense for any arbitrary stochastic process with finite second moment. Its accuracy could be improved by increasing the PCE degree [9]. Most importantly, due to the orthogonal characteristics, PCE could provide higher order moments by PCE multiplication algorithm without additional samplings. Thus, it could be a powerful tool to extend the moment method.

In this paper, we explore the PCE and multiplication algorithm to obtain high order moments efficiently. Since the correlation among inputs and different distributions could convert into the independent standard normal random variables, and they naturally satisfy the basic form of PCE, it is suitable for PCE to improve the application of traditional moment methods. The presentation of this work is structured as follows. Section 2 presents a review of traditional moment methods. In section 3, the proposed PCE based method is presented. The performance of such method is illustrated by an example in section 4. Discussion and conclusions are summarized in the last section.

2. Review of high moment methods for reliability analysis

We review the classic moment methods in this section, which include the second-moment method, the third-moment method, and the fourth-moment method [5].

For a limit state function \( z = g(x) \), where \( x = (x_1, x_2, \ldots, x_n)^T \in \Omega \) is a \( n \)-dimensional random inputs. If the first two moments are obtained, and assuming that \( z = g(x) \) obeys normal distribution, the reliability index and failure probability based on the second-moment method are expressed as:

\[
P_{f,SM} = Pr\{z < 0\} = \Phi(-\beta_{SM})
\]

\[
\beta_{SM} = \mu_z / \sigma_z
\]

where \( \mu_z \) and \( \sigma_z \) are the mean and standard deviation of \( z = g(x) \) respectively, and \( \Phi(\cdot) \) is the cumulative distribution function of a standard normal random variable.

Again, if the first three moments are obtained, and assuming that the standardized variable \( z_n = (z - \mu_z) / \sigma_z \) obeys the three-parameter lognormal distribution, the skewness, or the third dimensionless central moment, of \( z_n \) is

\[
\alpha_3 = E\left(z_3^n\right) = E\left[(z - \mu_z)^3\right] / \sigma_z^3.
\]
Then, the standard normal random variable \( u \) associated with \( z_u \) could be expressed as:

\[
u = \text{sign}(\alpha_{3c}) \left( \ln \left( 1 + \frac{1}{u^*} \right) \right)^{1/2} \ln \left[ 1 + \frac{1}{u^*} \left( 1 - \frac{z_u}{u^*} \right) \right]
\]

where

\[
u^* = (a + b)^{1/3} + (a - b)^{1/3} - \frac{1}{\alpha_{3c}^c}
\]

\[a = \frac{1}{\alpha_{3c}^c} \left( \frac{1}{\alpha_{3c}^c} + \frac{1}{2} \right)
\]

\[b = -\frac{1}{2\alpha_{3c}^c} (\alpha_{3c}^c + 4)^{1/2}
\]

The reliability index and the failure probability based on the third-moment method are expressed as:

\[
P_{f, TM} = \Phi(-\beta_{TM})
\]

\[
\beta_{TM} = -\text{sign}(\alpha_{3c}) A^{1/2} \ln \left[ 1 + \beta_{3M} / u^* \right]
\]

Furthermore, if the first four moments of \( z = g(x) \) are available, the kurtosis, or the fourth dimensionless central moment, is

\[
\alpha_4 = E \left( z^4 \right) = E \left[ (z - \mu)^4 \right] \sigma_z^4.
\]

Then, it could be proved that the reliability index and the failure probability based on the fourth-moment method are expressed as:

\[
P_{f, FM} = \Phi(-\beta_{FM})
\]

\[
\beta_{FM} = \left( 3(\alpha_4 - 1)\beta_{SM} + \alpha_{3c} \left( \beta_{SM}^2 - 1 \right) \right) \left( 9(1-\alpha_4) + 5\alpha_{3c}^2 (1-\alpha_4) \right)^{1/2}
\]

Besides, consider the case where the elements of inputs are statistically correlated, the Nataf transformation approach is usually employed to convert these inputs into uncorrelated sequence [10]. Specifically, suppose the PDF and the cumulative distribution function (CDF) of inputs \( x_i \in \mathbb{R} \) are \( f_i(x_i) \) and \( F_i(x_i) \) respectively, the correlation coefficient between \( x_i \) and \( x_j \) \((x_i, x_j \in \mathbb{R})\) is \( \rho_{ij} \). Then, the Nataf transformation is given as:

\[
\Phi^{-1} \begin{bmatrix} F_1(x_1) \\ F_2(x_2) \\ \vdots \\ F_n(x_n) \end{bmatrix} = L \begin{bmatrix} \xi_{11} \\ \xi_{21} \\ \vdots \\ \xi_{n1} \end{bmatrix} = \begin{bmatrix} l_{11} & l_{21} & \cdots & l_{11} \\ l_{21} & l_{22} & \cdots & l_{22} \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{bmatrix} \begin{bmatrix} \xi_{1} \\ \xi_{2} \\ \vdots \\ \xi_{n} \end{bmatrix}
\]

where the \( L \) is a lower triangle matrix solved by the Choleskey decomposition of the correlation matrix \( \rho = \rho_{ij} \). Let \( \xi = (\xi_1, \xi_2, \ldots, \xi_n) \) is the standard normal variables, and the elements of \( \xi \) are one to one corresponding to \( x = (x_1, x_2, \ldots, x_n) \). Further more, the reverse transformation above could be given as follows:

\[
x_i = F_i^{-1} \left[ \Phi \left( L_i \xi \right) \right] = F_i^{-1} \left[ \Phi \left( \sum_{j=1}^{n} l_{ij} \xi_j \right) \right], \quad \text{for } i = 1, 2, \ldots, n.
\]

### 3. The proposed improved moment method

#### 3.1 The polynomial chaos expansion algorithm

Consider the limit-state function \( z = g(x) \), where \( x = (x_1, \ldots, x_n) \in \Omega^n \). Suppose that the elements of inputs \( x \) are independent. The output \( z \) is a random variable, and it could be approximated as [9]:

where $N$ is the total number of polynomials, which represents the truncation order of PCE, and $\{c_j\}_{j=0}^{N-1}$ are the coefficients to be determined. $\xi=(\xi_1, \xi_2, \ldots, \xi_N)^T$ is the independent standardized orthogonal random variable associated with inputs $x=(x_1, \ldots, x_n)^T$ by isoprobabilistic transformations. $\{\psi_j\}_{j=0}^{N-1}$ are multi-dimensional hyper-geometric polynomials, and they are defined as the tensor products of the corresponding one-dimensional orthogonal polynomials $\{\phi_i\}_{i=0}^{n-1}$, that is,

$$\psi_j(\xi) = \psi_{\alpha,j}(\xi) = \prod_{k=1}^{N} \phi_{\alpha_{j,k}}(\xi_k)$$

where $\delta_{ij}$ is the Kronecker delta, $\{\phi,\phi\}$ denotes the ensemble average which is the inner product in Hilbert space. The subscript $a_j$ is a tuple defined as $a_j=(a_{j,1},a_{j,2},\ldots,a_{j,n}) \in \mathbb{N}^n$, and it is a multi-indices vector of the set $A$ defined by:

$$A = \left\{ a_j \in \mathbb{N}^n \| a_j \leq p \right\} , \quad N \equiv \binom{n+p}{n} = \frac{(n+p)!}{n!p!}$$

where $p$ is the total degree of the PCE.

Specifically, if the standardized orthogonal random variables belong to normal distribution, the orthogonal polynomials are Hermite polynomials with recursive relation as follows:

$$H_0(x) = 1, \quad H_1(x) = 2x, \quad H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x).$$

The coefficients $\{c_j\}_{j=0}^{N-1}$ could be determined by the probabilistic collocation method (PCM) [11].

Let $\{\xi\}_{i=1}^{M}$ be a set of the random variable samples, and let $\{g(\xi)\}_{i=1}^{M}$ be the corresponding set of model output or response, where $M$ is the number of samples. Denoting $c=(c_0, c_1, \ldots, c_{n-1})^T$, an approximation $\hat{c}$ could be given by the Least Squares (LS) algorithm:

$$\hat{c} = \arg \min_{c} \sum_{i=1}^{M} \left( g(\xi_i) - \sum_{j=0}^{N-1} c_j \psi_j(\xi_i) \right)^2$$

where $M$ is suggested to be chosen as $M=2(N+1)$ [9, 11].

Due to the orthogonality of the basis, the mean value and the variance can be calculated as:

$$\mu = E(z) \approx c_0, \quad \sigma^2 = V(z) \approx \sum_{j=1}^{N} c_j^2 E(\psi_j^2(\xi))$$

Again, when it comes to the case that the elements of the input variable are correlated, e.g., the correlation coefficient between $x_i$ and $x_j$ is $\rho_{ij}$. The Nataf transformation should be employed when calculating the model outputs corresponding to $\{\xi\}_{i=1}^{M}$. Without loss of generality, assume the PDF and CDF of inputs are $F(x_i)$ and $f(x_i)$ ($i=1,2,\ldots,n$) respectively, and then (14) is revised as follows:

$$\hat{c} = \arg \min_{c} \sum_{i=1}^{M} \left( g[F^{-1}(\Phi(L\xi_i))] - \sum_{j=0}^{N} c_j \psi_j(\xi) \right)^2$$

3.2. PCE for high order moment calculation
Consider the fact that PCE consists of orthonormal bases, and the product of two PCE can be expanded as a linear combination of Hermite polynomials, we could make the multiplication of orthogonal polynomials to obtain high order moments. In this paper, we would apply this theory to the truncated PCE, and prove that the multiplication form could provide an accurate high order moment estimation under appropriate PCE degree.

Suppose $u$ and $v$ have PCE formula with the same $n$-dimensional standardized random variables $\xi = (\xi_1, \ldots, \xi_n)^T$ but different degree $p_u$ and $p_v$ respectively. That is, $u = \sum_{|\gamma| = p_u} u_\gamma \psi_\gamma (\xi)$, $v = \sum_{|\nu| = p_v} v_\nu \psi_\nu (\xi)$.

If $E(|uv|^2) < \infty$, then the product of $u$ and $v$ has the PCE formula [12]:

$$uv = \sum_{|\alpha| \leq p_u} \sum_{|\gamma| \leq p_v} \sum_{|\tau| \leq p_v} C(\theta, \gamma, \tau) u_\gamma v_\tau \psi_\alpha (\xi)$$

$$C(\theta, \gamma, \tau) = \left[ \left( \frac{\theta - \gamma + r}{r} \right) \left( \frac{\gamma + r}{r} \right) \left( \frac{\theta - \gamma}{r} \right) \right]^{\frac{1}{2}}$$

(17)

where the subscripts $\alpha$, $\gamma$, $r$, and $\theta$ are tuples associated with PCE terms, e.g., $\alpha = (a_1, a_2, \ldots, a_n)$. We say $\gamma \leq \theta$ if $\gamma_i \leq \theta_i$ for all $i = 1, 2, \ldots, n$. The operation of these subscripts, such as $+$ or $-$, is also defined as component-wise. Especially, the factorial of tuples is defined like $\alpha! = \prod \alpha_i$.

Specifically, the mean of $uv$ is

$$E(uv) = \sum_{|\gamma| \leq p_u} \sum_{|\nu| \leq p_v} C(\theta, \gamma, \tau) u_\gamma v_\tau$$

(18)

In this way, the first four moments of $z^2$ could be computed according to equation (17), that is,

$$z^2 = g^2(x) \approx g_{PCE} (\xi) \cdot g_{PCE} (\xi) = \sum_{|\alpha| \leq 2p} \sum_{|\gamma| \leq p} \sum_{|\tau| \leq p} C(\theta, \gamma, \tau) c_{\theta \gamma \tau} \psi_\alpha (\xi)$$

(19)

For simplicity, let $c^\prime_{\alpha \gamma \tau} = \sum_{\theta \leq \gamma \leq \tau} C(\theta, \gamma, \tau) c_{\theta \gamma \tau}$, then we have $z^2 \approx \sum_{\alpha, \gamma, \tau} c^\prime_{\alpha \gamma \tau} \psi_\alpha (\xi)$.

Secondly, the first four moments could be computed according to equation (15), equation (17), and equation (18), that is,

$$E(z) = \mu_2 \approx E(g_{PCE} (\xi)) = c_0$$

$$E(z^2) \approx E(g_{PCE}^2 (\xi)) = c_0'$$

(20)

Furthermore, the variance, the third dimensionless central moment, and the fourth dimensionless central moment could be computed as equation (21), equation (22), and equation (23) respectively.

$$\sigma^2 = \left( E(z^2) - \mu_2^2 \right)^{1/2} \approx \left( c_0' - c_0^2 \right)^{1/2}$$

$$\alpha_3 = E(z^3) = \sigma_3 E(z_3) \left[ z - \mu_3 \right]$$

$$= \sigma_3 \left( E(z^3) - 3\mu_2 E(z^2) + 2\mu_4 \right) \approx \left( c_0' - c_0^2 \right)^{3/2} \left( \sum_{|\alpha| \leq 2p} c^\prime_{\alpha} - 3c_0 c_0' + 2c_0'' \right)$$

(22)
\[ \alpha_{z} = E(z^4) = \sigma_z^4 E\left((z - \mu_z)^4\right) = \sigma_z^4 \left( E(z^4) - 4E(z^3)\mu_z + 6E(z^2)\mu_z^2 - 3\mu_z^4 \right) \]
\approx \left(c'_0 - c_0^2\right)^2 \left( \sum_{i=1}^{n} c''_i E\left(\psi_d^i(\xi)\right) - 4c_0 \sum_{i=p+1}^{n} c' c'_i + 6c_0^2 c'_0 - 3c_0^4 \right) \quad (23)\]

3.3. The implementation of the proposed method

It should be noted that PCE itself is built in the Hilber space, where the standard normal distribution is preferred. Thus, the Nataf transformation could be seen a necessary step for PCE construction. Therefore, the Nataf transformation and moments solving by PCE are integrated naturally. The details are provided as follows:

For a given limit-state function \( z = g(x) \) with \( x=(x_1,...,x_n)^T \), where the PDF and CDF of each input element are \( f_i(\cdot) \) and \( F_i(\cdot) \) respectively. We assume that the correlation matrix of input random variables are \( \rho=[\rho_{ij}]_{n\times n}. \) If the inputs are independent, \( \rho \) becomes a unit matrix.

With the preparation, the complete algorithm description is introduced as follows:

Step 1: Set the PCE degree \( p \), e.g., \( p=2 \). Then generate \( M=2(n+p)!/[p!1] \) samplings of independent standard normal variables according to PCM.

Step 2: Convert each sampling into the original distribution space with equation (9).

Step 3: Calculate the output of the limit-state function of each sampling, as well as the coefficients of PCE by the LS algorithm with (16), where the orthogonal bases are Hermite bases.

Step 4: Calculate the mean, the variance, the third dimensionless central moment, or the fourth dimensionless central moment with equation (20) - equation (23).

Step 5: Calculate the failure probability \( P_f \) according to equation (1), equation (5), or equation (7).

Step 6: Execute the proposed method with \( p=p+1 \). If the difference of failure probabilities between two computations is less than a threshold value \( \varepsilon \), we terminate the solving process and take the latter failure probability as the final result. Otherwise, another higher PCE degree is used. This procedure is repeated until the convergence condition or threshold value is reached.

The step-by-step procedure of the proposed method is visualized in Figure 1.

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**Figure 1.** Flowchart of the proposed PCE-based approach
4. Application Example
In this example, we provide a satellite assembly, integration, and test (AIT) gantry crane example to demonstrate the proposed method. In the satellite AIT shop, the gantry crane is used to help assemble some large or heavy devices, such as the solar array, the engine, or the communication antenna, in a satellite. The schematic diagram of an AIT gantry crane is shown in Figure 3. The key structure of the gantry crane is that two removable holders are linked by a beam, where the device is lifted by a hook in the middle of the beam.

An important issue of AIT operation is the safety of the gantry crane, that is, the deflection of the gantry crane should be limited. Therefore, the gantry crane could be simplified as a classic beam model in Figure 4. Then, the maximal deflection $\omega_{\text{max}}$ of such drive beam could be computed as:

$$\omega_{\text{max}}(q,E,I,L,F_p) = \frac{L^4}{EI} \left( \frac{F_p}{48} + \frac{5qL}{384} \right),$$

where $q$ is the magnitude of a normally distributed load that represents the dead-weight of the beam, $E$ and $I$ denote Young’s modulus and inertia moment of beam respectively, $L$ is the span length, and $F_p$ is the maximal weight of loads. The detailed distribution parameters of these variables are given in Table 1. Besides, the correlation coefficient between $q$ and $I$ is 0.5, the correlation coefficient between $E$ and $L$ is 0.3, and the correlation coefficient between $E$ and $I$ is 0.3.

**Table 1.** The distribution parameters of the input variables

| Input Variables       | $q$/N/m | $E$/Gpa | $I$/m$^4$ | $L$/m | $F_p$/N |
|-----------------------|---------|---------|-----------|--------|---------|
| Distribution type     | Normal  | Normal  | Normal    | Normal | Normal  |
| Mean value            | 610     | 2.1 × 10$^3$ | 2.4 × 10$^{-4}$ | 18.6   | 4.0 × 10$^3$ |
| coefficient of variation | 0.1   | 0.06    | 0.08      | 0.1    | 0.1     |

**Figure 2.** The schematic diagram of a satellite AIT gantry crane.

**Figure 3.** The equivalent model of a satellite AIT gantry crane.

According to the design requirement, the admissible ratio of deflection to span length is lower than 0.003. Thus, the limit-state function is given as:

$$g(q,E,I,L,F_p) = 0.003 - \frac{\omega_{\text{max}}}{L} = 0.002 - \frac{L^4}{EI} \left( \frac{F_p}{48} + \frac{5qL}{384} \right).$$

We calculated the failure probability of the AIT gantry crane by the proposed method, and the results converge when the PCE degree is 5. The details of the results are presented in Table 4. It shows that the proposed high-moment method could obtain an accurate failure probability, where the relative errors of the third-moment method and the fourth-moment method are less than 0.4%. And the times of original limit-state function evaluations are about 0.05% of the MCS. Therefore, the efficiency of the proposed method is illustrated.
Besides, it should be noted that the times of function evaluations of the second-moment method, the third-moment method, and the fourth-moment method are the same. It is because the original limit state function would be taken only during the process of coefficients computation of PCE. The further moment information, whether it be the first two order moments or high order moments, is obtained based on these coefficients only without any extra samplings.

This example demonstrates that the reliability predicted by the improved method is more efficient than that from the traditional method.

| Methods                        | MCS       | The second-moment method | The third-moment method | The fourth-moment method |
|--------------------------------|-----------|--------------------------|-------------------------|----------------------------|
| Reliability index (RI)         | 2.5703    | 2.1410                   | 2.5807                  | 2.5667                     |
| Failure Probability            | 0.00508   | 0.0084                   | 0.00493                 | 0.00513                    |
| Relative error of RI           | --        | 16.7%                    | 0.40%                   | 0.14%                      |
| Times of Function Evaluations  | 1.0×10^6  | 506                      | 506                     | 506                         |

5. Discussion and Conclusion

This study improves the efficiency of the existing high-moment reliability analysis. Although the traditional moment methods can estimate the reliability only with the moment information of the limit-state function, it may be a time-consuming process to estimate moments by sampling approach. The proposed method could calculate these moments with quite fewer samplings based on PCE technology, because the high moments of the limit-state function of mechanics could be obtained easily by PCE multiplication. And it could be proved that the results of the proposed method are convergent.

Furthermore, the correlation and different distributions of inputs are also taken into consideration. Due to the Nataf transformation, the original distributions are converted into the standard normal distribution, which accords with the probabilistic space associated with PCE. With the statistic characteristics provided by PCE and its multiplication, the high moments are available.

In this way, the main advantage of the proposed method is that:

Firstly, it provides an efficient and convergent way to estimate high order moments of a given system with limited samplings.

Secondly, it improves the traditional moment method for reliability analysis, and it is also suitable in cases where there are correlations among input variables or the input variables have different distributions.

However, it should be noted that the samplings for PCE coefficients might grow quickly with the increase of input variable number according to equation (12). In this way, if there are many parameters in a given system, some necessary analysis should be conducted and the key parameter of interest might be identified for further reliability analysis by the proposed method.

In this paper, the PCE based moment calculation method is presented, which is followed by the integration method between PCE and the moment methods for reliability analysis. It could efficiently and accurately estimate the failure probability with limited samplings. An example indicates the good performance of the proposed method. Further research will focus on the time-vary reliability analysis by the proposed method.

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