APPLICATION OF THE BERNSTEIN POLYNOMIALS FOR SOLVING THE NONLINEAR FRACTIONAL TYPE VOLterra INTEGRO-DIFFERENTIAL EQUATION WITH CAPUTO FRACTIONAL DERIVATIVES

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Abstract. The current work aims at finding the approximate solution to solve the nonlinear fractional type Volterra integro-differential equation

\[ \sum_{k=1}^{m} F_k(x)D^{(k\alpha)}y(x) + \lambda \int_{0}^{x} K(x,t)D^{(\alpha)}y(t)dt = g(x)y^2(x) + h(x)y(x) + P(x). \]

In order to solve the aforementioned equation, the researchers relied on the Bernstein polynomials besides the fractional Caputo derivatives through applying the collocation method. So, the equation becomes nonlinear system of equations. By solving the former nonlinear system equation, we get the approximate solution in form of Bernstein fractional series. Besides, we will present some examples with the estimate of the error.

1. Introduction. The fractional differential and integro-differential equations (FIDEs) had an interesting effect and had been applied in many areas such as in the fields of Mechanics, Physics, applied Mathematics, and Engineering. Most FIDEs don’t have an exact solution, so the approximate and numerical methods used, such as the homotopy perturbation method [9] and the Adomain’s decomposition method [7] are so not effective to find exact solutions. So, the method that should be used is the collocation method [15, 2] that can be used to solve the problem of FIDEs by using the Bernstein polynomials basis [4, 3].

The aim of this paper is to find approximate solution of nonlinear fractional type Volterra integro-differential equations (FVIDEs), as shown bellow

\[ \sum_{k=1}^{m} F_k(x)D^{(k\alpha)}y(x) + \lambda \int_{0}^{x} K(x,t)D^{(\alpha)}y(t)dt = N(x), \quad 0 \leq x \leq 1, \quad (1) \]

where: \( N(x) = g(x)y^2(x) + h(x)y(x) + P(x) \) with the initial condition

\[ D^{(\alpha)}y(r_p) = u_p, \quad (0 \leq r_p \leq R \leq +\infty), \quad p = 0, 1, 2, \ldots, (m - 1). \quad (2) \]

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\(\lambda, r_p\) and \(u_p\) are real constants and \(D^{(\alpha)}y(x)\) is the \(n\)-order derivative of \(y(x)\). Such as \(y(x) \in \zeta([0,1], \mathbb{R})\), \(K(x,t) \in \zeta([0,1]^2, \mathbb{R})\) here \(y(x)\) is the unknown function and \(D^{\alpha}(n-1 \leq \alpha \leq n)\) is Caputo fractional derivative, \(P(x), h(x)\) and \(g(x)\) are the functions defined in \([0,R]\), \(R < \infty\).

We write \(\lambda = 0\) in the equation (1) to obtain a fractional Riccati differential equation [8].

The paper is organized as follows: after this introduction in Section 2, we will introduce the Bernstein polynomials and their properties and some basic definitions about Caputo’s fractional derivatives and their properties. In Section 3 we will define a method for the approximate solution of fractional Problem (1) and (2). Section 4 is devoted to introduce the error analysis method based on the remaining functions developed for the defined method. In Section 5 we will apply the proposed method to some problems and report our numerical results. Finally, the conclusion is given in Section 6.

2. Properties of Bernstein polynomials: basic and definitions.

2.1. Properties of Bernstein polynomials. We give some properties of Bernstein polynomials that we will use later [3, 10, 6]. As we mentioned, \(n\)-degree \(B\)-polynomials are a set of polynomials defined on \([0,1]\) by

\[
B_{i,n}(x) = \binom{n}{i} x^i (1-x)^{n-i}, i = 0, 1, 2, \ldots, n,
\]

where

\[
\binom{n}{i} = \frac{n!}{i!(n-i)!}.
\]

The Bernstein polynomials on \([0,1]\) have the following properties

1. The positive property: For \(i = 0, 1, 2, \cdots, n\) and \(x \in [0,1]\), we have

\[
B_{i,n}(x) \geq 0
\]

2. By using the binomial expansion of \((1-x)^{n-i}\), one can show that:

\[
(1-x)^{n-i} = \sum_{i=1}^{n-i} (-1)^i \binom{n}{i} \binom{n-i}{k} x^i.
\]

3. It has a degree raising property in the sense that any of the lower-degree polynomials (degree < \(n\)) can be expressed as linear combinations of polynomials of degree \(n\). We have

\[
B_{i,n-1}(x) = \left(\frac{n-i}{n}\right) B_{i,n}(x) + \left(\frac{n-i}{n}\right) B_{i-1,n}(x).
\]

Now, we give some basic definitions and properties of fractional calculus.
2.2. Riemann-Liouville fractional derivative.

**Definition 2.1.** Let the real number $q > 0$ and $n \in \mathbb{N}$. The fractional differential operator $D^q$ of order $q$ in the sense of Riemann-Liouville [5] is defined by

$$D^q y(t) = \begin{cases} \frac{1}{\Gamma(n-q)} \frac{d^n}{dx^n} \int_0^t \frac{y(x)}{(t-x)^{q-n+1}} dx, & 0 \leq n - 1 < q < n \\ \frac{d^n}{dx^n} y(t), & q = n, \end{cases}$$

Obviously, when $q = n$, the fractional differential reduces to the ordinary $n$-th derivative of $y(t)$ with respect to $t$.

**Definition 2.2.** $I^q$ denotes the fractional integral operator of order $q$ in the sense of Riemann-Liouville, defined by

$$D^{-q} y(t) = I^q y(t) = \begin{cases} \frac{1}{\Gamma(q)} \int_0^t \frac{y(x)}{(t-x)^{1-q}} dx, & q > 0 \\ y(t), & q = 0. \end{cases}$$

Some basic properties of the fractional operator are listed below for $f \in C_\alpha, \alpha \geq -1, \mu \geq 1, \eta \geq 0, \beta \geq -1$,

1. $I^\mu \in C_0$,
2. $I^\mu I^\beta f(x) = I^\beta I^\mu f(x)$,
3. $I^\mu I^\eta f(x) = I^{\beta+\eta} f(x)$,
4. $D^\beta D^\eta f(x) = D^{\beta+\eta} f(x)$,
5. $D^\beta I^\eta f(x) = f(x)$,
6. $I^\beta D^\eta f(x) = f(x) - \sum_{k=0}^{m-1} \frac{1}{\Gamma(m-k)} f^{(k)}(0^+), \quad m - 1 < \delta < m, \quad m \in \mathbb{N}.$

Furthermore,

$$I^\delta x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\delta+\beta+1)} x^{\delta+\beta}.$$ 

2.3. Caputo fractional derivatives.

**Definition 2.3.** The fractional derivative of $f(x)$ in the Caputo sense is defined as [4, 11, 5]

$$D^\alpha f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t) dt,$$

for $m - 1 < \alpha < m, m \in N, x > 0, f \in C^m_1$ where $D = \frac{d}{dt}$.

For the Caputo derivative we have: $D^\alpha c = 0$. ($c$ is a constant)

$$D^\alpha X^\beta = \begin{cases} 0, & \beta \in N_0 \text{ and } \beta < [\alpha] \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} X^{\beta-\alpha}, & \beta \in N_0 \text{ and } \beta \geq [\alpha] \text{ or } \beta \notin N_0. \end{cases}$$

The linear formula of Caputo fractional derivative is

$$D^\alpha (c_1 f_1(x) + c_2 f_2(x)) = c_1 D^\alpha f_1(x) + c_2 D^\alpha f_2(x),$$

where $c_1$ and $c_2$ are constants.
3. **Method of solution.** In this study, we are developing the approximate Bernstein polynomials by using the collocation method, matrix operations and Caputo fractional derivative to obtain the approximate solution of the Problem (1) and (2). The approximate solution obtained by the following form [5, 11]

\[ Y_{N,\alpha}(x) = \sum_{i=0}^{N} a_{i} B_{iN}^{\alpha}(x - c), \quad x \in [0, 1], \] (3)

where \(0 < \alpha \leq 1\), \(0 \leq c < 1\) and \(a_{i}\) \((i = 0, 1, \ldots, N)\) are the unknown Bernstein coefficients. The equation \(Y_{N,\alpha}(x)\) is defined by

\[ Y_{N,\alpha}(x) = B_{\alpha}(x - c)A. \]

We obtain

\[ Y_{N,\alpha}(x) = (a_{0}B_{0N}^{\alpha}(x - c) + a_{1}B_{1N}^{\alpha}(x - c) + \cdots + a_{N}B_{NN}^{\alpha}(x - c)). \]

In other side we have

\[ B_{N,\alpha}(x - c) = (B_{0n}^{\alpha}(x - c), B_{1n}^{\alpha}, \ldots, B_{NN}^{\alpha}(x - c)), \]

with \(A = (a_{0}, a_{1}, \ldots, a_{n})^{T}\) and we can write \(B_{N,\alpha}(x - c)\) as

\[ B_{N,\alpha}(x - c) = X(x - c)D^{\alpha}. \]

where

\[ X(x - c) = (1, (x - c)^{\alpha}, (x - c)^{2\alpha}, \ldots, (x - c)^{N\alpha}). \]

And

\[ D = \begin{pmatrix}
(-1)^{0}(\frac{\alpha}{0}) & (-1)^{1}(\frac{\alpha}{1}) & (\frac{\alpha}{0})(\frac{n-0}{n}) & \cdots & (\frac{\alpha}{m-0})(\frac{n}{n-m}) \\
0 & (-1)^{0}(\frac{\alpha}{1}) & (\frac{\alpha}{1})(\frac{n-1}{n-0}) & \cdots & (\frac{\alpha}{m-1})(\frac{n-1}{n-0}) \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & (-1)^{0}(\frac{\alpha}{n})
\end{pmatrix}, \]

we obtain

\[ Y_{N,\alpha}(x) = X(x - c)D^{\alpha}A. \]

The \(\alpha\)-order of Caputo fractional derivative of \((x - c)\) is written as [4]

\[ X^{(\alpha)}(x - c) = (0, \frac{\Gamma(\alpha+1)}{\Gamma(1)}, \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}(x - c)^{\alpha}, \ldots, \frac{\Gamma(\alpha\cdot N\alpha+1)}{\Gamma((n-1)\alpha+1)}(x - c)^{(n-1)\alpha}). \]

The relation between \(X^{(\alpha)}(x - c)\) and \(X(x - c)\) is given by

\[ X^{(\alpha)}(x - c) = X(x - c)R, \]

where

\[ R = \begin{pmatrix}
0 & \frac{\Gamma(\alpha+1)}{\Gamma(1)} & \cdots & \cdots & 0 \\
0 & 0 & \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & \frac{\Gamma(\alpha\cdot n\alpha+1)}{\Gamma((n-1)\alpha+1)} \\
0 & \cdots & 0 & \cdots & 0
\end{pmatrix}. \]

The \(2\alpha\)-order of Caputo fractional derivative of \(x - c\) is written as
\[ X^{(2\alpha)}(x-c) = (0, 0, \frac{\Gamma(2\alpha + 1)}{\Gamma(\alpha + 1)} \cdot \frac{\Gamma(3\alpha + 1)}{\Gamma(2\alpha + 1)} (x-c)^\alpha, \ldots, \frac{\Gamma(n\alpha + 1)}{\Gamma((n-1)\alpha + 1)} (x-c)^{(n-2)\alpha}), \]

and the relation between \( X(x-c)^{2\alpha} \) and \( X(x-c) \) is given by

\[ X^{2\alpha}(x-c) = X(x-c)RM, \]

where

\[ M = \begin{bmatrix}
0 & 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & 1 & \ldots & 0 & 0 \\
\vdots & & & & & \\
0 & 0 & \ldots & 0 & 1 \\
0 & 0 & \ldots & 0 & 1 \\
\end{bmatrix}. \]

Similarly, the \((k\alpha)\) order Fractional derivative of \( X(x-c) \) is given by the recurrence relation

\[ X^{(k\alpha)}(x-c) = X(x-c)RM^{k-1} \quad k = 0, 1, \ldots, N. \]  \(4\)

We obtain

\[ Y_N^{(\alpha)}(x) = X(x-c)RM^{(k-1)}D^TA. \]  \(5\)

For \( k = 1 \) the Eq (5) becomes \( Y_N^{(\alpha)}(x) = X(x-c)RM^{(0)}D^TA \)

Note that \( M^0 \) is the matrix unit with dimensions \((N + 1) \times (N + 1)\).

The collocation points defined by [14],

\[ X_i = \frac{i}{N}, i = 0, 1, 2, \ldots, N. \]

We have the matrix equation system:

\[ Y_N^{(k\alpha)} = XRM^{k-1}D^TA, \]  \(6\)

where

\[ X = \begin{bmatrix}
X(x_0-c) \\
X(x_1-c) \\
\vdots \\
X(x_N-c)
\end{bmatrix}, Y_N = \begin{bmatrix}
Y_N(x_0) \\
Y_N(x_1) \\
\vdots \\
Y_N(x_N)
\end{bmatrix}. \]

We have

\[ Y^2 = \begin{bmatrix}
y(x_0) & 0 & \ldots & 0 \\
0 & y(x_1) & \ldots & 0 \\
\vdots & & & \\
0 & \ldots & y(x_N)
\end{bmatrix} \begin{bmatrix}
y(x_0) \\
y(x_1) \\
\vdots \\
y(x_N)
\end{bmatrix} = \tilde{Y}Y, \]

where

\[ \tilde{Y} = \begin{bmatrix}
y(x_0) & 0 & \ldots & 0 \\
0 & y(x_1) & \ldots & 0 \\
\vdots & & & \\
0 & \ldots & y(x_N)
\end{bmatrix}. \]
\[
\begin{bmatrix}
X(x_0 - c)D^T A \\
0 & X(x_1 - c)D^T A & \cdots & 0 \\
\vdots & \cdots & \cdots & \vdots \\
0 & \cdots & X(x_N - c)D^T A
\end{bmatrix}
= 
\begin{bmatrix}
X(x_0 - c) & 0 & \cdots & 0 \\
0 & X(x_1 - c) & \cdots & 0 \\
\vdots & \cdots & \cdots & \vdots \\
0 & \cdots & X(x_N - c)
\end{bmatrix}
\begin{bmatrix}
D^T & 0 & \cdots & 0 \\
0 & D^T & \cdots & 0 \\
\vdots & \cdots & \cdots & \vdots \\
0 & \cdots & D^T
\end{bmatrix}
\cdot
\begin{bmatrix}
A & 0 & \cdots & 0 \\
0 & A & \cdots & 0 \\
\vdots & \cdots & \cdots & \vdots \\
0 & \cdots & A
\end{bmatrix}
\]

We pose \( \hat{Y} = \hat{X}Z\hat{A} \), we have: \( Y^2 = \hat{X}Z\hat{A}D^T A \), where,

\[
\hat{X} = 
\begin{bmatrix}
X(x_0 - c) & 0 & \cdots & 0 \\
0 & X(x_1 - c) & \cdots & 0 \\
\vdots & \cdots & \cdots & \vdots \\
0 & \cdots & X(x_N - c)
\end{bmatrix}
\]

\[
Z = 
\begin{bmatrix}
D^T & 0 & \cdots & 0 \\
0 & D^T & \cdots & 0 \\
\vdots & \cdots & \cdots & \vdots \\
0 & \cdots & D^T
\end{bmatrix}
\]

\[
\hat{A} = 
\begin{bmatrix}
A & 0 & \cdots & 0 \\
0 & A & \cdots & 0 \\
\vdots & \cdots & \cdots & \vdots \\
0 & \cdots & A
\end{bmatrix}
\]

Firstly, let us write Eq (1) under the form

\[
E(x) + I(x) = N(x),
\]

where

\[
E(x) = \sum_{k=1}^{N} F_k(x)D^{(k\alpha)}y(x), \quad I(x) = \lambda \int_0^x K(x, t)D^{(\alpha)}y(t)dt.
\]

And

\[
N(x) = g(x)y^2(x) + h(x)y(x) + P(x).
\]

Let us consider the matrix representation of the \( \lambda \int_0^x K(x, t)D^{(\alpha)}y(t)dt, E(x) \) and \( I(x) \).

\[
I(x) = \lambda \int_0^x K(x, t)D^{\alpha}y(t)dt
\]

\[
= \lambda \int_0^x K(x, t)(1, (t, c)^\alpha, (t - c)^{2\alpha}, \ldots, (t - c)^{N\alpha})RD^T Adt,
\]

and then

\[
I(x) = \lambda V(x)RD^T A,
\]

(7)
where

\[ V(x) = \left( \int_0^x K(x,t)dt, \int_0^x (t-c)^\alpha K(x,t)dt, \ldots, \int_0^x (t,c)^{(N\alpha)} K(x,t)dt \right). \]

To obtain the approximate solution under the form of Eq (3), we substitute the collocation points in Eq (1), we find

\[
\sum_{k=1}^{m} F_k(x_i)D^{(k\alpha)}y_N(x_i) + \lambda \int_0^x K(x_i,t)D^{(\alpha)}y_N(x_i)dt
= g(x_i)y_N^2(x_i) + h(x_i)y_N(x_i) + P(x_i),
\]

for \( i = (0,1,\ldots,N) \).

For \( c = 0 \) the System (8) is written in the following basic matrix form

\[
\sum_{k=1}^{m} F_k D^{(k\alpha)}Y + \lambda \int_0^x K P D^{\alpha}Y = \hat{g} Y^2 + \tilde{h} Y + P, \quad (8)
\]

where

\[
F_k = \begin{bmatrix} F_k(x_0) & 0 & \ldots & 0 \\ 0 & F_k(x_1) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & F_k(x_N) \end{bmatrix}, \quad K_P = \begin{bmatrix} K(x_0,t) \\ K(x_1,t) \\ \vdots \\ K(x_N,t) \end{bmatrix},
\]

\[
P = \begin{bmatrix} P(x_0) \\ P(x_1) \\ \vdots \\ P(x_N) \end{bmatrix}, \quad \hat{h} = \begin{bmatrix} h(x_0) & 0 & \ldots & 0 \\ 0 & h(x_1) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & h(x_N) \end{bmatrix},
\]

\[
\hat{g} = \begin{bmatrix} g(x_0) \\ 0 & g(x_1) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & g(x_N) \end{bmatrix}.
\]

We put the relation (5), (6) and (7) into Eq (8) and then we have the fundamental matrix equation

\[
\left( \sum_{k=1}^{m} F_k X R M^{k-1} D^T + \lambda V_z R D^T - \hat{h} X D^T - \hat{g} \tilde{X} \tilde{Z} \tilde{A} X D^T \right) A = P. \quad (9)
\]

Briefly, Eq (9) can be written in the form \( W(A)A = P \), which corresponds to a system of the \((N+1)\) nonlinear algebraic equation with the unknown coefficients \( a_i \) such \((i = 0,1,2,\ldots,N)\) and

\[
W(A) = \sum_{k=1}^{m} F_k X R M^{k-1} D^T + \lambda V_z R D^T - \hat{h} X D^T - \hat{g} \tilde{X} \tilde{Z} \tilde{A} X D^T. \quad (10)
\]

Then the matrix form of the initial condition (2) becomes

\[
D^{(\alpha)}y(r_p) = u_p, \quad p = 0,1,2,\ldots,(m-1)
\]

\[
D^{\alpha}y(r_p) = (1, (r_p - c)^\alpha, (r_p - c)^{2\alpha}, (r_p - c)^{3\alpha}, \ldots, (r_p - c)^{N\alpha}) R D^T A.
\]
For \( c = 0 \) we have
\[
D^\alpha y(r_p) = (1, r_p^\alpha, r_p^{2\alpha}, r_p^{3\alpha}, \ldots, r_p^{N\alpha})RD^T A = u_p.
\]

Briefly, the matrix form of the initial condition can be expressed as
\[
H_p.A = u_p, \quad p = 0, 1, 2, \ldots, (m - 1),
\]
where
\[
[H_p] = (1, r_p^\alpha, r_p^{2\alpha}, r_p^{3\alpha}, \ldots, r_p^{N\alpha}).RD^T = (u_{p0}, u_{p1}, \ldots, u_{pn}),
\]
to obtain the approximate solution of the problem (1) under condition (2), we replace \( m \) row of the augmented matrix \((\tilde{W}(H), \tilde{P})\) [14], by the new matrix \([H_p]\) thus we have \(\tilde{W}(A).A = \tilde{P}\) where
\[
\tilde{W} = \begin{bmatrix}
\begin{array}{cccccc}
w_{00} & w_{01} & w_{02} & \cdots & w_{0N} & P(x_0) \\
w_{10} & w_{11} & w_{12} & \cdots & w_{1N} & P(x_1) \\
w_{20} & w_{21} & w_{22} & \cdots & w_{2N} & P(x_2) \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
w_{N-m,0} & w_{N-m,1} & w_{N-m,2} & \cdots & w_{N-m,N} & P(x_{N-m-1}) \\
\end{array}
\end{bmatrix},
\]
which is a nonlinear algebraic system, as a result, the coefficients are determined by solving this system. Hence, by substituting the determined coefficients into Eq (3), we obtain the Bernstein polynomial solution (see [1] or [13])
\[
y_{N\alpha}(x) = \sum_{i=0}^{N} a_i B_{iN}^\alpha(x - c).
\]

4. **Error (Analysis-Estimation).** The aim of this section is to determine the error when a function is approximate in terms of \((FVIDEF)\).

To do this, we have the following theorem.

**Theorem 4.1. (Generalized-Taylor’s Formula)** Suppose that
\[
D^{(i\alpha)}y(x) \in C([a, b]) \text{ for } (i = 1, 2, \ldots, (n + 1)),
\]
then we have
\[
y(x) = \sum_{i=0}^{n} \frac{(x - a)^{i\alpha}}{\Gamma(i\alpha + 1)} \left[ D^{(i\alpha)}y(x) \right]_{x=0} + R_n(x), \tag{11}
\]
where
\[
R_n(x) = \frac{(x - a)^{(n+1)\alpha}}{\Gamma(n+1)\alpha + 1} \left[ D^{(n+1)\alpha}y(x) \right]_{x=\xi},
\]
with \( 0 < \varepsilon < x, \text{ for all } x \in [0, 1] \), and \( Y_{n,\alpha}(x) = \sum_{i=0}^{n} \frac{(x-a)^{i\alpha}}{\Gamma(i\alpha+1)} \left[ D^{(i\alpha)}y(x) \right]_{x=0} \).

**Remark 1.** In case of \( \alpha = 1 \) the generalized Taylor’s Formula (11) will be reduced to the classical Taylor’s Formula.
• Error estimation. Suppose that 
\[ D^{(i\alpha)}y(x) \in C([0, 1]) \text{ for } i = 0, 1, 2, \ldots, (n + 1). \]
If \( y(x) \) the exact solution given and \( Y_{n,\alpha}(x) \) is the approximate solution respectively, we name the error function as
\[ R_N(x) = y(x) - Y_{n,\alpha}(x), \]
then the absolute error as:
\[ |y(x) - Y_{n,\alpha}(x)| \leq \frac{M_{\alpha}}{\Gamma((n+1)\alpha+1)}, \]
where
\[ M_{\alpha} = \sup_{\xi \in [0, 1]} |D^{(n+1)\alpha}y(\xi)|. \]
(Therefore the approximations of \( y(x) \) are convergent)

• The maximum absolute error can be estimated approximately
\[ E_{NM} = \max |R_n(x)|, \quad (0 \leq x \leq 1). \]

5. Illustrative examples. In this section, we will present some numerical examples to illustrate and demonstrate the efficiency of the method. We have done all the numerical computations with a computer program Matlab.

Example 1. We consider the following nonlinear fractional type Volterra integro-differential equation (F.VDIEs) defined for \( 0 \leq x \leq 1 \) by
\[ F_1(x)D^{(1/4)}y(x) - h(x)y(x) + \int_0^x x^{\sqrt{t}}D^{(1/4)}y(t)dt = P(x), \quad (12) \]
where
\[ F_1(x) = \sqrt{x}, \quad h(x) = \frac{x}{\Gamma(\frac{7}{4})}, \quad K(xt) = x^{\sqrt{t}}. \]
And
\[ P(x) = -4 \left( \frac{e}{3\Gamma(\frac{11}{4})} \right) x^4 + \left( \frac{e}{2\Gamma(\frac{7}{4})} + 2 \Gamma(\frac{7}{4}) \right) x^3 - \left( \frac{e}{\Gamma(\frac{7}{4})} + \frac{4}{\Gamma(\frac{11}{4})} \right) x^2 + \frac{e}{\Gamma(\frac{7}{4})}, \]
with the initial condition \( D^{(1/4)}y(1) = 0.4702.\)
The exact solution is given by \( y(x) = x(e - 2x) \) knowing that \( e = 2.71828. \) Let us find approximate solution for \( \alpha = 0.25, \ N = 3 \) and \( c = 0, \) we have
\[ Y_{N,\alpha}(x) = \sum_{i=0}^{N-1} a_i B_{i}^{\alpha}(x), \]
and the collocation points are
\[ x_0 = 0, x_1 = \frac{1}{3}, x_3 = \frac{2}{3}, x_4 = 1. \]
From Eq (12) the fundamental matrix is written as
\[ (F_1XRM^0D^T + V_2RD^T - \hat{h}XD^T)A = P, \quad (13) \]
where
\[ F_1 = \begin{bmatrix} 0.5623 & 0 & 0 & 0 \\ 0 & 0.7596 & 0 & 0 \\ 0 & 0 & 0.9035 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \]
\[
D^T = \begin{bmatrix}
1 & 0 & 0 & 0 \\
-3 & 3 & 0 & 0 \\
3 & -6 & 3 & 0 \\
-1 & 3 & -3 & 1 \\
\end{bmatrix}, \quad R = \begin{bmatrix}
0 & \Gamma\left(\frac{5}{4}\right) & 0 & 0 \\
0 & 0 & \Gamma\left(\frac{5}{4}\right) & 0 \\
0 & 0 & 0 & \Gamma\left(\frac{5}{4}\right) \\
0 & 0 & 0 & 0 \\
\end{bmatrix}, \\
M^0 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}, \quad \hat{h} = \begin{bmatrix}
\frac{1}{10\Gamma\left(\frac{5}{4}\right)} & 0 & 0 & 0 \\
0 & \frac{1}{3\Gamma\left(\frac{5}{4}\right)} & 0 & 0 \\
0 & 0 & \frac{2}{3\Gamma\left(\frac{7}{4}\right)} & 0 \\
0 & 0 & 0 & \frac{1}{\Gamma\left(\frac{7}{4}\right)} \\
\end{bmatrix}, \\
\]

\[
Z = \begin{bmatrix}
D^T & 0 & 0 & 0 \\
0 & D^T & 0 & 0 \\
0 & 0 & D^T & 0 \\
0 & 0 & 0 & D^T \\
\end{bmatrix}, \\
P = \begin{bmatrix}
0.2313 \\
0.5511 \\
0.1082 \\
-1.1959 \\
\end{bmatrix}, \quad \hat{A} = \begin{bmatrix}
A & 0 & 0 & 0 \\
0 & A & 0 & 0 \\
0 & 0 & A & 0 \\
0 & 0 & 0 & A \\
\end{bmatrix}, \\
X = \begin{bmatrix}
X(x_0) \\
X(x_1) \\
X(x_2) \\
X(x_3) \\
\end{bmatrix}, \quad \hat{X} = \begin{bmatrix}
X(x_0) & 0 & 0 & 0 \\
0 & X(x_1) & 0 & 0 \\
0 & 0 & X(x_2) & 0 \\
0 & 0 & 0 & X(x_3) \\
\end{bmatrix}, \\
V(x) = \begin{bmatrix}
\frac{4}{3} x^{\frac{3}{2}} \\
\frac{4}{3} x^{\frac{5}{2}} \\
\frac{4}{3} x^{\frac{7}{2}} \\
\frac{4}{3} x^{\frac{9}{2}} \\
\end{bmatrix}, \\
V_x = \begin{bmatrix}
V(0.1) \\
V\left(\frac{1}{3}\right) \\
V\left(\frac{2}{3}\right) \\
V(1) \\
\end{bmatrix} = \begin{bmatrix}
4.2162 \times 10^{-3} & 2.1079 \times 10^{-3} & 1.0161 \times 10^{-3} & 5 \times 10^{-4} \\
\frac{4}{27} \sqrt[3]{\frac{1}{3}} & \frac{2}{27} \sqrt[3]{\frac{1}{3}} & \frac{4}{63} \sqrt[3]{\frac{1}{27}} & \frac{1}{57} \\
\frac{4}{27} \sqrt[3]{\frac{2}{3}} & \frac{8}{27} \sqrt[3]{\frac{2}{3}} & \frac{16}{63} \sqrt[3]{\frac{1}{27}} & \frac{8}{57} \\
\frac{4}{27} \sqrt[3]{\frac{3}{3}} & \frac{8}{27} \sqrt[3]{\frac{3}{3}} & \frac{16}{63} \sqrt[3]{\frac{1}{27}} & \frac{8}{57} \\
\frac{4}{27} \sqrt[3]{\frac{4}{3}} & \frac{8}{27} \sqrt[3]{\frac{4}{3}} & \frac{16}{63} \sqrt[3]{\frac{1}{27}} & \frac{8}{57} \\
\end{bmatrix}. \\
\]

Briefly, Eq (13) can be written in the form \(W(A)A = P\) which corresponds to the system of the \((N + 1)\) nonlinear algebraic equations with the unknown coefficients \(a_i (i = 0, 1, 2, \ldots, N)\) and \(W(A) = (F_1XRM^0D^T - \hat{h}XD^T + V_xRD^T)\).

According to the initial condition and by putting \(n = 3\) and \(u = 0.4702\) we have that \(D^{\frac{1}{2}}y(1) = [H_0]A\) with

\[
[H_0] = X(1)RD^T = (-0.8704, -0.0393, -0.1776, 1.0373). 
\]
From Eq (10), we compute the augmented matrix form, the initial condition as $[H_0]$. From Eq (3) new augmented matrix passed on the condition is found. Hence, by solving (Matlab) this system, the matrix coefficients $A$ are obtained

$$A = \begin{pmatrix} 0 \\ -1.6792 \\ 1.4476 \\ 0.7603 \end{pmatrix}.$$ 

For $(c = 0)$ the determined coefficients are substituted into Eq (3), then we have

$$\tilde{y}_3(x) = -5.0376t^\frac{3}{2} + 14.4356t^\frac{3}{2} - 8.6201t^\frac{3}{2}.$$ 

Also $y(x)$ is chosen and the exact solution is $y(x) = x(e - 2x)$ and the absolute error obtained by the present method are compared in Table 1

| $x_i$ | exact solutions | approximation solutions | errors |
|-------|-----------------|--------------------------|--------|
| 0.0   | 0.0             | 0.0                      | 0.0    |
| 0.1   | 0.2518          | 0.2001                   | 0.042224 $\times 10^{-2}$ |
| 0.2   | 0.4637          | 0.5103                   | 0.0217156 $\times 10^{-1}$ |
| 0.3   | 0.6355          | 0.6858                   | 0.0253009 $\times 10^{-1}$ |
| 0.4   | 0.7673          | 0.7898                   | 0.050625 $\times 10^{-2}$ |
| 0.5   | 0.8591          | 0.8479                   | 0.012544 $\times 10^{-2}$ |
| 0.6   | 0.9109          | 0.8940                   | 0.028561 $\times 10^{-2}$ |
| 0.7   | 0.9227          | 0.8790                   | 0.0190969 $\times 10^{-1}$ |
| 0.8   | 0.8946          | 0.8790                   | 0.024336 $\times 10^{-1}$ |
| 0.9   | 0.8264          | 0.8259                   | 0.025 $\times 10^{-5}$ |
| 1     | 0.7182          | 0.7603                   | 0.0167246 $\times 10^{-1}$ |

Table 1. Exact and approximate solutions and square error for $(n = 3)$ Example 5.1.
Example 2. We consider the following nonlinear fractional type Volterra integro-differential equation (F.VDIEs) is defined for $0 \leq x \leq 1$ by

$$\sqrt{x}(x^2 + 1)D^{(\frac{1}{2})} y(x) + \int_0^x x\sqrt{t} D^{(\frac{1}{2})} y(t) dt = N(x).$$

(14)

where

$$N(x) = g(x)y^2(x) + P(x).$$

And

$$g(x) = \frac{16}{9\Gamma(\frac{3}{2})}, \quad P(x) = -\frac{37}{18\Gamma(\frac{3}{2})} x^3 - \frac{4}{9\Gamma(\frac{3}{2})} x^2 + \frac{1}{\Gamma(\frac{3}{2})} x.$$  

with the initial condition $D^{(\frac{1}{2})} y(1) = 2.6332$.

If we compare the result with Problem (1) we find

$$F_1(x) = \sqrt{x}(x^2 + 1), F_k(x) = 0, \text{ for all } k = 2, 3, \ldots, m,$$

and $k(x, t) = x\sqrt{t}$.

The exact solution is given by

$$y(x) = x(x + 1).$$

Let us find approximate solution for $\alpha = 0.5$, $N = 4$ and $(c = 0)$ we have

$$y_4(x) = \sum_{i=0}^{N=4} a_i B_{i,4}(x).$$

The collocation points are, $x_0 = 0, x_1 = \frac{1}{4}, x_2 = \frac{2}{4}, x_3 = \frac{3}{4}, x_4 = 1$.

From Eq (14), the fundamental matrix is written as

$$(F_1 X R M^0 D^T + V_2 R D^T - \hat{g} \hat{X} Z \hat{A} X D^T) A = P,$$

(15)

where

$$F_1 = \begin{pmatrix}
0.3193 & 0 & 0 & 0 & 0 \\
0 & \frac{17}{32} & 0 & 0 & 0 \\
0 & 0 & \frac{5\sqrt{2}}{8} & 0 & 0 \\
0 & 0 & 0 & \frac{25\sqrt{2}}{32} & 0 \\
0 & 0 & 0 & 0 & 2
\end{pmatrix}, \quad D^T = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
4 & 4 & 0 & 0 & 0 \\
6 & -12 & 6 & 0 & 0 \\
-4 & 12 & -12 & 4 & 0 \\
1 & -4 & 6 & -4 & 1
\end{pmatrix},$$

$$R = \begin{pmatrix}
0 & \Gamma(\frac{3}{2}) & 0 & 0 & 0 \\
0 & 0 & \frac{1}{\Gamma(\frac{3}{2})} & 0 & 0 \\
0 & 0 & 0 & \Gamma(\frac{5}{2}) & 0 \\
0 & 0 & 0 & 0 & \frac{2}{\Gamma(\frac{5}{2})} \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

$$M^0 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}, \quad \hat{g} = \frac{16}{9\Gamma(\frac{3}{2})} \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix},$$

$$Z = \begin{pmatrix}
D^T & 0 & 0 & 0 & 0 \\
0 & D^T & 0 & 0 & 0 \\
0 & 0 & D^T & 0 & 0 \\
0 & 0 & 0 & D^T & 0 \\
0 & 0 & 0 & 0 & D^T
\end{pmatrix}, \quad \hat{A} = \begin{pmatrix}
A & 0 & 0 & 0 & 0 \\
0 & A & 0 & 0 & 0 \\
0 & 0 & A & 0 & 0 \\
0 & 0 & 0 & A & 0 \\
0 & 0 & 0 & 0 & A
\end{pmatrix},$$

$$X = (1, \sqrt{x}, x, x\sqrt{x}, x^2).$$
APPLICATION OF THE BERNSTEIN POLYNOMIALS

\[ X = \begin{pmatrix} X(x_0) \\ X(x_1) \\ X(x_2) \\ X(x_3) \\ X(x_4) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & \frac{1}{3} & \frac{3}{6} & \frac{3}{6} & \frac{3}{6} \\ 1 & \frac{1}{4} & \frac{4}{12} & \frac{4}{12} & \frac{4}{12} \\ 1 & \frac{1}{5} & \frac{5}{20} & \frac{5}{20} & \frac{5}{20} \end{pmatrix}, \]

\[ \hat{X} = \begin{pmatrix} X(x_0) & 0 & 0 & 0 & 0 \\ 0 & X(x_1) & 0 & 0 & 0 \\ 0 & 0 & X(x_2) & 0 & 0 \\ 0 & 0 & 0 & X(x_3) & 0 \\ 0 & 0 & 0 & 0 & X(x_4) \end{pmatrix}, \]

\[ V(x) = \begin{pmatrix} 2 x^2 \frac{2}{3} x^3 \frac{2}{5} x^3 \frac{2}{3} x^4 \frac{2}{7} x^3 \end{pmatrix}, \]

\[ V_x = \begin{pmatrix} V(0,1) \\ V(\frac{1}{2}) \\ V(\frac{1}{2}) \\ V(\frac{1}{2}) \\ V(1) \end{pmatrix} = \begin{pmatrix} 2.1079 \times 10^{-3} \\ 5 \times 10^{-4} \\ 3.1079 \times 10^{-4} \\ 3.33 \times 10^{-3} \\ 2.0346 \times 10^{-5} \\ \frac{2}{5} \\ \frac{1}{10} \\ \frac{1}{10} \\ \frac{1}{10} \\ \frac{2}{5} \end{pmatrix}, \]

\[ P = \begin{pmatrix} 0.1054 \\ -0.2166 \\ -0.5027 \\ -0.0735 \end{pmatrix}. \]

Simply, Eq (15) can be written in the form \( W(A)A = P \), which corresponds to the system of the \((N + 1)\) nonlinear algebraic equations with the unknown coefficients \( a_i (i = 0, 1, 2, \ldots, N) \) and

\[
W(A) = F_1 X R M^0 D^T + V_x R D^T - \hat{g} \hat{X} Z \hat{A} X D^T.
\]

According to the initial condition and by putting \((N = 4)\) and \(u = 2.6332\) we have

\[
D(\frac{1}{2})g(1) = (1, 1, 1, 1) R D^T A = H_0 A,
\]

where \( H_0 = (6.5044, 5.9294, -0.1166, -1.4876, 1.5088) \).

As we did in the previous Example (1) and by using Matlab we obtained

\[
A = \begin{pmatrix} 0 \\ 0.13755 \\ 0.14128 \\ 0.5479 \\ 1.95690 \end{pmatrix}.
\]

For \((c = 0)\) the determined coefficients are substituted into Eq (3), then we have

\[
\tilde{y}_4(x) = 0.5502 x^\frac{1}{2} - 0.8029 x + 2.1469 x^\frac{3}{2} + 0.0627 x^2.
\]
Example 3. We consider the following nonlinear fractional type Volterra integro-differential equation (F DIEs) defined for $0 \leq x \leq 1$ by

$$\sqrt{x}D^{(\frac{1}{2})}y(x) + \sqrt{x} \left( x^2 + \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{5}{2})} \right) D^{(1)}y(x) + \int_0^x \frac{D^{(\frac{1}{2})}y(t)}{\sqrt{x-t}} dt = N(x). \quad (16)$$

where

$$N(x) = \Gamma \left( \frac{5}{2} \right) y^2(x) + \Gamma \left( \frac{5}{2} \right) y(x) + \left( \frac{3}{2} \sqrt{\pi} \right) x.$$  

If we compare this result with Problem (1), we find,

$h(x) = \Gamma(\frac{1}{2})$, $g(x) = \Gamma(\frac{5}{2})$, $P(x) = (\frac{3}{2} \sqrt{\pi})x$, $F_1(x) = \sqrt{x}$, $F_2(x) = \sqrt{x} \left( x^2 + \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2})} \right)$

and $K(x, t) = \frac{1}{\sqrt{x-t}}$, with the initial condition $D^{(\frac{1}{2})}(y(\frac{1}{2})) = 0.6608$, the exact solution is given by $y(x) = x^2$.

Let us find an approximate solution for ($\alpha = 0.5$), $N = 3$ and ($c = 0$).
We have
\[ y_3(x) = \sum_{i=0}^{N=3} a_i \beta^\alpha_i(x), \]
and the collocations points are given by
\[ x_0 = 0, \ x_1 = \frac{1}{3}, \ x_3 = \frac{2}{3}, \ x_4 = 1. \]
From Eq (16) the fundamental matrix is written as
\[ (F_1 X R M^0 D^T + F_2 X R M D^T + V_x R D^T - \hat{h} X D^T - \hat{g} \hat{X} Z \hat{A} X D^T) A = P, \quad (17) \]
where
\[ F_1 = \begin{bmatrix}
0.3162 & 0 & 0 & 0 \\
0 & \sqrt{\frac{1}{3}} & 0 & 0 \\
0 & 0 & \sqrt{\frac{2}{3}} & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}, \quad F_2 = \begin{bmatrix}
0.3123 & 0 & 0 & 0 \\
0 & \frac{7}{9\sqrt{3}} & 0 & 0 \\
0 & 0 & \frac{10}{9\sqrt{3}} & 0 \\
0 & 0 & 0 & \frac{\sqrt{3}}{3} \\
\end{bmatrix}. \]
\[ \hat{g} = \Gamma\left(\frac{5}{2}\right) \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}, \quad \hat{h} = \Gamma\left(\frac{5}{2}\right) \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}, \quad P = \begin{bmatrix}
\frac{3\sqrt{\frac{\pi}{2}}}{\sqrt{\pi}} \\
\frac{3\sqrt{\frac{\pi}{2}}}{\sqrt{\pi}} \\
\frac{3\sqrt{\frac{\pi}{2}}}{\sqrt{\pi}} \\
\\end{bmatrix}. \]
\[ M^0 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}, \quad X(x) = (1, x^\alpha, x^{2\alpha}, x^{3\alpha}), \]
\[ X = \begin{bmatrix}
1 & \cdots & 0 & \cdots & 0 \\
1 & \cdots & \frac{3}{\sqrt{7}} & \cdots & \frac{3}{\sqrt{2}} \\
1 & \cdots & 2\times\frac{3}{\sqrt{7}} & \cdots & 2\times\frac{3}{\sqrt{2}} \\
1 & \cdots & 1 & \cdots & 1 \\
\end{bmatrix}, \quad \hat{X} = \begin{bmatrix}
X(x_0) & 0 & 0 & 0 \\
0 & X(x_1) & 0 & 0 \\
0 & 0 & X(x_2) & 0 \\
0 & 0 & 0 & X(x_3) \\
\end{bmatrix}. \]
\[ D^T = \begin{bmatrix}
1 & 0 & 0 & 0 \\
-3 & 3 & 0 & 0 \\
3 & -6 & 3 & 0 \\
-1 & 3 & -3 & 1 \\
\end{bmatrix}, \quad R = \begin{bmatrix}
0 & \Gamma\left(\frac{3}{2}\right) & 0 & 0 \\
0 & 0 & \frac{1}{\Gamma\left(\frac{1}{2}\right)} & 0 \\
0 & 0 & 0 & \Gamma\left(\frac{5}{2}\right) \\
0 & 0 & 0 & 0 \\
\end{bmatrix}, \]
\[ Z = \begin{bmatrix}
D^T & 0 & 0 & 0 \\
0 & D^T & 0 & 0 \\
0 & 0 & D^T & 0 \\
0 & 0 & 0 & D^T \\
\end{bmatrix}, \quad \hat{A} = \begin{bmatrix}
A & 0 & 0 & 0 \\
0 & A & 0 & 0 \\
0 & 0 & A & 0 \\
0 & 0 & 0 & A \\
\end{bmatrix}, \]
\[ M = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \end{bmatrix}. \]
According to the initial condition and by putting $v = \frac{3}{2}$, $n = 0, 1, 2, \ldots, N $

$$V(x) = \left( \int_0^x \frac{1}{\sqrt{(x-t)}} dt \right)^2 \left( \int_0^x \frac{1}{\sqrt{(x-t)}} dt \right)^2 \left( \int_0^x \frac{4}{\sqrt{(x-t)}} dt \right)^2 \left( \int_0^x \frac{2}{\sqrt{(x-t)}} dt \right)^2.$$

For the part $\int_0^x \frac{t^v}{\sqrt{(x-t)}} dt$ using the formula [12],

$$\int_0^x \frac{t^v}{\sqrt{(x-t)}} dt = \sqrt{\pi x^{v+\frac{1}{2}}} \Gamma(v+1) \Gamma(v+\frac{3}{2}).$$

So that $v = \frac{3}{2}$, $n = 0, 1, 2, \ldots, N$

$$V(x) = \left( 2\sqrt{x}, \frac{\pi}{2}, \frac{4}{3}, \frac{3\pi}{8} x \right).$$

$$V_x = \begin{pmatrix}
V(0.1) \\
V(\frac{1}{3}) \\
V(\frac{7}{10}) \\
V(1)
\end{pmatrix} = \begin{pmatrix}
2 \sqrt{\frac{2}{10}} \\
\frac{2}{\sqrt{3}} \\
2 \sqrt{\frac{2}{3}} \\
2
\end{pmatrix}, \begin{pmatrix}
\frac{\pi}{20} \\
\frac{2}{\sqrt{5}} \\
\frac{2}{\sqrt{3}} \\
\frac{3\pi}{8}
\end{pmatrix}. $$

Briefly, Eq (17) can be written in the form $W(A)A = P$, which corresponds to the system of the $(N+1)$ nonlinear algebraic equations with the unknown coefficients $a_i (i = 0, 1, 2, \ldots, N)$ and

$$W(A) = \left( F_1 XRM^0 + F_2 XRM + V_x R - \hat{h}X - \hat{g}XAZ \right) D^T.$$

According to the initial condition and by putting $n = 3$ and $\mu = 0.6608$ we have $X(\frac{1}{2}) = (1, \sqrt{3}, \frac{1}{2}, \frac{\sqrt{7}}{2})$ and then $D^T y(\frac{1}{2}) = H_0 A$ with $[H_0] = X(\frac{1}{2})RD^T = (0.9261, -0.1309, 0.3992, 0.6645)$ as we did in the previous example from Eq (12) and by using Matlab we obtained

$$A = \begin{pmatrix}
0 \\
0.19 \times 10^{-3} \\
0.418 \times 10^{-3} \\
1.00022
\end{pmatrix}. $$

For ($c = 0$) the determined coefficients are substituted into Eq (3) then we have

$$\tilde{y}_3(x) = 0.57 \times 10^{-3} x^\frac{1}{2} + 0.114 \times 10^{-3} x + x^\frac{3}{2}. $$

| $x$ | exact solutions | approximation solutions | errors |
|-----|-----------------|-------------------------|-------|
| 0.0 | 0               | 0                       | 0     |
| 0.1 | 0.316           | 0.371                   | 0.01 \times 10^{-6} |
| 0.2 | 0.0594          | 0.0897                  | 0.09199 \times 10^{-9} |
| 0.3 | 0.1643          | 0.164465                | 0.01225 \times 10^{-9} |
| 0.4 | 0.2530          | 0.25336                 | 0.01296 \times 10^{-9} |
| 0.5 | 0.3535          | 0.35402                 | 0.02704 \times 10^{-9} |
| 0.6 | 0.4646          | 0.46525                 | 0.04225 \times 10^{-9} |
| 0.7 | 0.5837          | 0.58621                 | 0.02601 \times 10^{-9} |
| 0.8 | 0.7155          | 0.71527                 | 0.00525 \times 10^{-9} |
| 0.9 | 0.8538          | 0.55476                 | 0.09216 \times 10^{-9} |
| 1   | 1               | 1.000684                | 0.00467856 \times 10^{-2} |

Table 3. Exact and approximate solutions and square errors for $(n = 5)$ Example 5.3.
Conclusion. In this paper, we proposed a numerical method to solve the non-linear fractional type Volterra integro-differential equation (FVIDEs) using the collocation method. For the approximate solution, we used the Caputo fractional derivatives and Bernstein polynomials. The fractional derivatives of Caputo and Bernstein’s fractional series transformed the equation into a nonlinear algebraic system by the Matlab software; this gave the solution of the Bernstein coefficients system that allowed us to describe an approximate solution. Some numerical examples were presented to illustrate the theoretical results. As a result, it was observed that this method is effective for Riccati integro-differential equation.

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