Not So SuperDense Coding - Deterministic Dense Coding with Partially Entangled States

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The utilization of a d-level partially entangled state, shared by two parties wishing to communicate classical information without errors over a noiseless quantum channel, is discussed. We analytically construct deterministic dense coding schemes for certain classes of non-maximally entangled states, and numerically obtain schemes in the general case. We study the dependency of the information capacity of such schemes on the partially entangled state shared by the two parties. Surprisingly, for d > 2 it is possible to have deterministic dense coding with less than one ebit. In this case the number of alphabet letters that can be communicated by a single particle, is between d and 2d. In general we show that the alphabet size grows in “steps” with the possible values d, d+1,..., d^2 - 2. We also find that states with less entanglement can have greater communication capacity than other more entangled states.

I. INTRODUCTION

Dense coding, originally introduced by Bennett and Wiesner (1) is the surprising utilization of entanglement to enhance the capacity of a quantum communication channel. Two parties, Alice and Bob, communicate by sending a qubit over a noiseless quantum channel. As no more than two spin states can be perfectly distinguished, Alice can encode only one of two different letters, say '0' or '1', within each particle she sends. This is no better than using a classical communication channel. However, Bennett and Wiesner have shown that if Alice and Bob each hold one particle of a maximally entangled pair, it is possible for the sender, Alice, to transform the two-particle state into 4 orthogonal states by acting locally on her particle. After sending Bob her half of the pair, he will be able to perfectly distinguish the one of four different states by measuring the pair of particles collectively. Surprisingly, this enables the transmission of one of four letters by sending a single qubit, provided that the two parties share initial entanglement.

Numerous aspects of dense coding have been studied. Among these are generalization to pairs of entangled d-level systems (1), to continuous variables (2) and to settings involving more than two parties (3). Other works (4)(5)(6) studied dense coding in the asymptotic limit, where many copies of a partial entangled state are used.

In this paper we consider the case of pure non-maximal entanglement shared between a pair of separated d-level systems. We are not interested in the asymptotic channel capacity, but rather in the deterministic procedure, where the parties wish to perfectly distinguish messages encoded on a single d-level particle. We use exact and numeric methods to study the relation between the given state \( \psi \) whose entanglement is given by its entropy, \( S(\psi) \), to \( N_{max}(\psi) \), the maximal size of alphabet which can be perfectly transmitted. In other words, \( N_{max}(\psi) \) denotes the maximal number of orthogonal states which can be generated by means of a unitary transformation acting locally on one side of the given state.

Our results suggest that for \( d > 2 \), deterministic dense coding processes which utilize non-maximally entangled states are possible for an alphabet size, \( N_{max}(\psi) \), changing in "steps": \( N_{max}(\psi) \in \{d, d+1, \ldots, d^2 - 2\} \) (see Fig. 1). Interestingly, the last step with \( d^2 - 1 \) letters seems to be missing. We have been able to demonstrate these results analytically when \( N_{max} \) is a multiple of \( d \), \( N_{max}(\psi) = kd \), \( k = 2 \ldots d - 1 \), and for the special case \( N_{max}(\psi) = d + 1 \). Using numerical methods, the existence of the other steps has been fully verified for \( d = 3 \) and \( d = 4 \), and partially in higher dimensions.

We have further computed the minimal entanglement \( S_{min}(\psi) \) required to obtain \( N \) letters. For \( N \leq 2d - 1 \), the minimal required entanglement turns out to be less than one ebit (see Fig. 2). Therefore, our method is not equivalent to the trivial approach wherein deterministic concentration brings a non-maximal state to an ebit (1), to be used by utilizing the standard dense coding scheme.

In addition, we find that entanglement, while playing an important role in the communication capacity, does not completely determine \( N_{max}(\psi) \). We show that one can have two states with the property that the less entangled one is in fact better for communication. That is, we can have \( N_{max}(\psi_1) > N_{max}(\psi_2) \) while \( S(\psi_1) < S(\psi_2) \). This is perhaps reminiscent of (7) where it was shown that states with less entanglement can sometimes have a greater probability of being distinguished by separated parties who can only communicate classically. A related situation has been reported in (8) wherein non-maximal states, rather than maximal, were needed to perform certain remote operations.

This paper is organized as follows. We first review deterministic dense coding with maximally entangled d-level systems. Then we proceed to formulate the problem considered in this paper. Section IV treats the two-dimensional case analytically and shows that non-maximal states cannot be used to distinguish more than two letters. In section V we present a geometric ap-
proach, to construct steps with $N_{\text{max}} = kd$. There we show that one can have more communication with less entanglement. In section VII a more general though less intuitive analytic approach is presented, followed by our numerical results. Finally, we summarize our results in section VII.

II. DENSE CODING WITH MAXIMAL ENTANGLEMENT

We consider a bipartite qudit pure state. That is, a system composed of two $d$-level separated subsystems. This system is initially prepared in a maximally entangled state:

$$|\psi(m)\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle_A \otimes |i\rangle_B$$  \hspace{1cm} (1)

where $A$ (B) denotes Alice’s (Bob’s) subsystem. Alice encodes an alphabet of size $d^2$ which we denote as $(m,n)_{m,n=0}^{d-1}$ using a set $\{U_{mn}\}$ of local unitary operators on particle $A$. There are many possible realizations for this set of operators. An elegant, and undoubtedly the most common construction is

$$U_{mn} = (X)^m(Z)^n$$  \hspace{1cm} (2)

where $X$, the shift operator and $Z$, the rotate operator are defined by:

$$X|k\rangle = |(k+1)\text{mod}(d)\rangle$$

$$Z|k\rangle = e^{2\pi ik/d}|k\rangle$$  \hspace{1cm} (3)

It can be easily verified that $|\psi(mn)\rangle = (U_{mn} \otimes \mathbb{I}_B)|\psi_{00}\rangle$ form an orthogonal basis of the two qudits Hilbert space. After encoding the letter $(m,n)$, Alice sends her particle to Bob through the quantum channel. Bob performs a projective measurement of the two-particle state on $\{|\psi(mn)\rangle\}$ to decode the message.

A few remarks are in order here. First, we note that for qudits ($d=2$), this basis is just the well known Bell basis:

$$|\psi_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$|\psi_{10}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

$$|\psi_{11}\rangle = \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle)$$  \hspace{1cm} (4)

Second, trying to intuitively understand the difference between the classical and quantum cases, we note that the shift operators may be regarded as classical, in the sense that they correspond to the possibility of sending $d$ distinct values of a classical $d$-bit. The rotate operators may be regarded as the quantum enhancement, which enables the local realization of $d^2$ orthogonal $d = 2$ qudits states.

III. DETERMINISTIC DENSE CODING WITH NON-MAXIMAL ENTANGLEMENT

We now introduce the main problem this paper addresses. Instead of using a maximally entangled state, we consider an arbitrary pure state. This can be written in the Schmidt representation as:

$$|\psi\rangle = \sum_{i=0}^{d-1} \sqrt{\lambda_i} |i\rangle_A \otimes |i\rangle_B$$  \hspace{1cm} (5)

where $|i\rangle_A$ ($|i\rangle_B$) are the Schmidt basis for system A (B).

We are interested in a maximally sized set of local unitary operators $\{U_i^A\}_{i=1}^{N_{\text{max}}}$ that maintain orthogonality. That is, for all $1 \leq i,j \leq N_{\text{max}}$ we have:

$$\langle \psi (U_i^A \otimes \mathbb{I}_B) (U_j^B \otimes \mathbb{I}_A) | \psi \rangle = \delta_{i,j}$$  \hspace{1cm} (6)

Substituting the state $|\psi\rangle$ into (6) yields:

$$\delta_{i,j} = \sum_{k,l=0}^{d-1} \sqrt{\lambda_k \lambda_l} \langle k | U_i^A | l \rangle_A \otimes \langle k | U_j^B | l \rangle_B$$

$$= \sum_{k=0}^{d-1} \lambda_k \langle k | U_i^A U_j^B | k \rangle$$

$$= \text{Tr}(\Lambda U_i^A U_j^B)$$  \hspace{1cm} (7)

where $\Lambda$ is a $d \times d$ diagonal matrix of the Schmidt coefficients ($\Lambda_{ii} = \lambda_i$). Note that the matrices $U_i$ are unitary in the usual sense $U_i^A U_i^B = \mathbb{I}_d$, but the orthogonality is defined with respect to a non-trivial weights vector (the Schmidt coefficients) rather than the usual trace. For the rest of this paper orthogonality should be understood in this sense.

In this paper our goal is to study the effect of the initial state $|\psi\rangle$ on the maximal size $N_{\text{max}}$ of the set of unitaries satisfying (7). Put in other words, we would like to understand and characterize the relation $N_{\text{max}}$.

We first note that for any choice of $\psi$, there always exists a set of at least $d$ such unitaries. This is the set of shift operators introduced in the previous section. Let us explicitly verify that orthogonality is indeed maintained:

$$\langle \psi | (X^{n_1} \otimes \mathbb{I}_d) (X^m \otimes \mathbb{I}_d) | \psi \rangle$$

$$= \sum_{i,j} \lambda_i \lambda_j \langle (i+n)^{\text{mod}(d)} | (j+m)^{\text{mod}(d)} \rangle \otimes \langle i | j \rangle$$

$$= \delta_{n,m}$$  \hspace{1cm} (8)

That this set is always orthogonal should not surprise us as it corresponds to the classic possibility of encoding $d$ distinct values in a single $d$-bit.

IV. THE TWO-DIMENSIONAL CASE

We first consider the case of non-maximally entangled qubits ($d = 2$). We shall show that for all non-maximally
entangled states, only $N_{\text{max}}(\psi) = 2$ unitaries can be constructed. This means that deterministic dense coding with partial entanglement is not possible in $d < 3$ dimensions; partially entangled qubits have no advantage over pure product states or classical bits.

For convenience, and without loss of generality we assume that $\mathbb{1} \in \{U_1\}$. We parameterize $U = e^{i\theta \cdot \hat{n}} = \cos \theta \mathbb{1} + i \sin \theta (\sigma \cdot \hat{n})$, where $\sigma$ are the Pauli matrices, and $\hat{n}$ is a unit vector. Since $\mathbb{1} \in \{U_1\}$ we must have for all $U \in \{U_1\}$, $\text{Tr}(\Lambda U) = 0$. That is:

$$0 = (\lambda_0 + \lambda_1) \cos \theta + i(\lambda_0 - \lambda_1) \hat{n} \sin \theta \quad (9)$$

which determines $\theta = \frac{\pi}{2}$ and $\hat{n} = 0$. Suppose we want to have a set of three unitaries $\{\mathbb{1}, U_1, U_2\}$. $U_{1(2)}$ must therefore be of the form $U_{1(2)} = i(\sigma_x x_1(2) + \sigma_y y_1(2))$. We must also satisfy:

$$0 = \text{Tr}(\Lambda U_1^\dagger U_2) = (\lambda_0 + \lambda_1)(x_1 x_2 + y_1 y_2) + i(\lambda_0 - \lambda_1)(x_1 y_2 - y_1 x_2) \quad (10)$$

For non-maximal entanglement we have $\lambda_0 - \lambda_1 \neq 0$ and from the normalization we also have $\lambda_0 + \lambda_1 = 1$. In addition we have $x_1^2 + y_1^2 = x_2^2 + y_2^2 = 1$. Combining all these restrictions (10) has no solutions.

V. HIGHER DIMENSIONS, THE GEOMETRIC APPROACH

Regarding the shift operators as “classical”, and the rotate operators as the quantum enhancement made possible by the entanglement of the initial state, one may try to generalize the dense coding scheme by constructing rotations, or phase operators suitable for the given non-maximal entanglement. In analogy to (4), we are looking for a set $\{Z_i\}_{i=1}^3$ defined by:

$$Z_n[k] = e^{i\theta^n_k} |k\rangle$$

where $\theta^n_k$ are real phases whose choice will be discussed shortly. The orthogonality requirement dictates:

$$0 = \langle \psi | Z_1 Z_2 | \psi \rangle = \sum_{i,j} \sqrt{\lambda_i \lambda_j} e^{i(\theta^n_i - \theta^n_j)} \langle ii | jj \rangle$$

$$= \sum_i \lambda_i e^{i(\theta^n_i - \theta^n_i)} \quad (12)$$

We can use a set of $N$ such operators to construct $N d$ orthogonal operators (in the sense of (7)), namely $U_{mn} = (X)^m(Z_n)$ where $0 \leq m < d$ and $0 \leq n < N \leq d$. This construction implies that the total number of operators is a multiple of $d$. In the classical or non-entangled case, it is $1 \cdot d$, and in the maximal case it is $d \cdot d$. To see the effect of the initial state $|\psi\rangle$ on $N_{\text{max}}(\psi)$, let us examine the simple case where we look for $N = 2$ phase operators. Again, we assume that $\mathbb{1} \in \{Z_1\}$, so (12) reduces to $\sum_i \lambda_i e^{i\theta} = 0$. In other words, we are faced with the geometric task of forming a polygon using $d$ vectors of lengths $\{\lambda_0, \lambda_1, \ldots, \lambda_{d-1}\}$. This can always be accomplished if the longest vector is shorter than the sum of the others. Assuming that the $\lambda$s are sorted in descending order this condition is simply $\lambda_0 \leq 1/2$. Not surprisingly, this corresponds to entanglement $S(\psi) \geq 1$. Note that all states with $\lambda_0 \leq 1/2$ are majorized by a Bell state in a $d$-dimensional Hilbert space $(\frac{\sqrt{2}}{2}) (00) + \frac{\sqrt{2}}{2} (11) + \frac{\sqrt{2}}{2} (22) + \frac{\sqrt{2}}{2} (33))$, and thus can be converted to it by LOCC (11, 12). For the Bell state a construction similar to (4) trivially yields $2d$ orthogonal states. Note, however, that in order to deterministically concentrate $|\psi\rangle$ into a Bell state one must use both local operations (LO) and classic communications (CC) (11, 12, 13), whereas in our construction only LO are used. Use of additional communication to convert non-maximally entangled states to maximally entangled ones, would reduce the net gain in communication.

For the case where we want $N > 2$ phase operators satisfying (12), we have not found a simple geometric interpretation. Similar phase factors are also used in the context of deterministic teleportation schemes (14). It can be shown that such phases can be found if, and only if, $\lambda_0 \leq 1/N$. For example, for $d = 4$ and $N = 3$, we can construct $4 \cdot 3 = 12$ operators when $\frac{1}{4} = \lambda_0 = \lambda_1 = \lambda_2 + \lambda_3$ by regarding $\frac{1}{\sqrt{3}} (00) + \frac{\sqrt{3}}{2} (11) + \frac{\sqrt{3}}{2} (22) + \frac{\sqrt{3}}{2} (33))$ as a maximally entangled state in three dimensions, so we can use the powers of the operator $Z$ in (3) with $d = 3$ as $3$ phase operators.

These constructions, as well as the numerical results (Fig. 1) of the following section, lead to the conclusion that in finite dimensional systems, $N_{\text{max}}$. The maximal number of orthogonal unitaries, does not depend directly on the entanglement, but on some other function of the coefficients $\lambda_i$. We can have $S(\psi_1) > S(\psi_2)$, but $N_{\text{max}}(\psi_1) < N_{\text{max}}(\psi_2)$. Naively, one may have expected more entanglement to mean greater communication capacity, yet this is not so.

VI. A GENERAL APPROACH

It turns out that the geometric approach, although guided by the appealing separation into “classical” and quantum operators, does not provide the most general construction. Consider the initial state

$$|\psi_3\rangle = \sqrt{\frac{2}{3}} |00\rangle + \sqrt{\frac{1}{3}} |11\rangle + 0 |22\rangle \quad (13)$$

in $d = 3$ dimensions. Since $\lambda_0 = \frac{2}{3} > \frac{1}{3}$, using the results of the last section, one might be tempted to conclude that the maximal size of a set of orthogonal unitaries is just $d = 3$. But if we abandon the phase and shift operators,
one may consider the set \( \{ \mathbb{I}, X, U_3, U_3^\dagger \} \), where
\[
U_3 = \begin{pmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}
\end{pmatrix}
\]
(14)
is a rotation by \( \frac{\pi}{4} \) within the subspace spanned by \( \{|0\rangle, |2\rangle \} \). This set consists of four orthogonal unitaries (with respect to \( |\psi_0\rangle \)), and as will be discussed in the sequel, \( |\psi_0\rangle \) is the state with minimal entanglement in \( d = 3 \) dimensions admitting more than three orthogonal unitaries. Note that the above construction is by no means unique. It can be generalized to arbitrary dimension \( d \) as follows: The partially entangled state is \( \mathbb{C}^d \otimes \mathbb{C}^d \ni |\psi_d\rangle = \sqrt{\frac{d-1}{d}}|00\rangle + \sqrt{\frac{2}{d}}|11\rangle \), and the set of \( d + 1 \) orthogonal unitaries is \( \{ \mathbb{I}_d, X \} \cap \{ U_d^k \}^{d-2}_{k=0} \), where
\[
U_d^k|0\rangle = -\frac{1}{d-1}|0\rangle + \sqrt{\frac{d}{d-1}} \sum_{j=1}^{d-2} e^{2\pi i j/k} |j+1\rangle
\]
(15)
The effect of \( U_d^k \) on all other basis vectors is restricted only by the unitarity requirement \( U_d^k\dagger U_d^k = \mathbb{I} \). Let us verify explicitly that \( \{ U_d^k \} \) is indeed an orthogonal set (we omit the subscript \( d \)):
\[
\text{trace}(\Lambda U^k U^\dagger) = \frac{d-1}{d} \left( \frac{1}{d} \right) |0\rangle \langle 0| + \frac{1}{d} \langle 1|U^k U^\dagger|1\rangle
\]
\[
= \frac{1}{d(d-1)} \left( 1 + d \sum_{j=1}^{d-2} e^{2\pi i j/(d-k)} \right) + \frac{1}{d}
\]
\[
= \frac{1}{d-1} + \frac{1}{d-1} \sum_{j=0}^{d-2} e^{2\pi i j/(d-k)}
\]
\[
= \frac{1}{d-1} \sum_{j=0}^{d-2} e^{2\pi i j/(d-k)} = \delta_{k,l}
\]
(16)
and trivially
\[
\text{trace}(\Lambda \cdot \mathbb{I} \cdot U^k) = \frac{d-1}{d} \left( \frac{1}{d} \right) |0\rangle \langle 0| + \frac{1}{d} \langle 1|U^k|1\rangle
\]
\[
= -\frac{d-1}{d} \frac{1}{d-1} + \frac{1}{d} = 0
\]
\[
\text{trace}(A X^\dagger U^k) = \frac{d-1}{d} \left( \frac{1}{d} \right) |0\rangle \langle 0| + \frac{1}{d} \langle 1|X^\dagger U^k|1\rangle
\]
\[
= -\frac{d-1}{d} \frac{1}{d-1} + \frac{1}{d} \langle 2|U^k|1\rangle = 0
\]
(17)
This construction can be further generalized to any case where \( \lambda_1 = \frac{d}{N} = \frac{m_1}{m} \) for some integer \( m \). Note that for large \( d \), \( \lambda_1 \approx \frac{d}{d-1} \approx 1 \), which means that the entanglement required for having more than the “classical” \( d \) unitaries approaches zero.

In the general case, we were unable to find a parametrization of eq. (7) which leads to an analytic solution. Numeric results are, however, obtainable. In \( d = 3 \) dimensions, using numeric root finding routines we have mapped the domain of pure states according to the maximal number of orthogonal unitaries \( N_{\text{max}} \) one can construct for a given initial pure state. These results are presented in Figure 1. As we have already seen by example, we find that even when the initial entanglement is less than one ebit, it is possible to construct more than three orthogonal unitaries.

An intriguing observation is that we did not find partially entangled pure states that enable the construction of a maximal set of orthogonal unitaries of size \( d^2 - 1 \) (but we did find all steps \( N_{\text{max}} \leq d^2 - 2 \)). Due to the increasing size of the numeric root finding problems, we have only been able to verify this statement for \( d = 3, 4 \), and, of course, we have proved that this is the case in two dimensions. If true, this is indeed a very peculiar property.

It is also interesting to extract from the numerical results the minimal entanglement necessary to construct \( N \) orthogonal unitaries. One can compare this quantity with the lower bound on the amount of entanglement derived from the channel capacity [6], which when measured in units of dits is given by
\[
C \leq 1 + S(\psi)
\]
(18)
Therefore, the entanglement is bound from below by
$S(\psi) \geq \log_d N - 1$ edits. Figure 2 shows the comparison between the two quantities. It is evident that only when $N$ is a multiple of $d$ do we achieve this bound.

It is instructive to consider the states with minimal entanglement that enable the construction of at least $N$ orthogonal unitaries. For $N < 2d$ this entanglement is less than one ebit. The states with this minimal entanglement have only two non vanishing Schmidt coefficients, so they can be characterized by the value of $\lambda_0$ alone. Table I shows $\lambda_0$ for different values of $N$ and $d$. Although this data has been generated numerically, these values seem to be simple fractions of the two quantities. These results suggest that the state with minimal entanglement admitting at least $d + 1$ orthogonal unitaries in $d$ dimensions is $\sqrt{\frac{d}{d+n}}|00\rangle + \sqrt{\frac{n}{d+n}}|11\rangle$ (the states used (15)), and that the state with minimal entanglement admitting at least $d + n$ ($n = 2 \ldots d$) orthogonal unitaries is $\sqrt{\frac{d}{d+n}}|00\rangle + \sqrt{\frac{n}{d+n}}|11\rangle$.

### VII. CONCLUSIONS

We have shown that deterministic dense coding can be achieved using partially entangled states. While for qubits ($d = 2$), partial entanglement does not help to improve the classical communication capacity, in higher dimensions it is possible to have deterministic dense coding even with less than one ebit (see Fig. 2). When less than one ebit is shared by the parties, the maximal number of alphabet letters that can be communicated by a single particle is $2d-1$. More generally, we show that the alphabet size grows in “steps” and can obtain the values $d, d + 1, \ldots, d^2 - 2$ (Fig. 1). We also find that states with less entanglement can have greater communication capacity than other more entangled states.

Table I summarizes the structure of states with minimal entanglement smaller than one ebit, admitting $d < N < 2d$ unitaries in $d$ dimensions. The resulting structure strongly indicates that geometric constructions, similar to the $N = d + 1$ case (Section VI), can be obtained as well.

A connection between superdense coding and other tasks such as teleportation and distinguishability of operators has been noted in the past. In [10], a one-to-one correspondence between dense coding schemes and quantum teleportation schemes (for maximal entanglement) was established, and we have already pointed out the similarity between the phase operators presented in section [11] and the teleportation protocol with partially entangled states discovered independently in [12]. It would be interesting to understand the correspondence between dense coding and teleportation schemes when partial entanglement is used.

The problem of distinguishing unitary operators and its relation to superdense coding in the maximal case was presented in [13]. The conditions for distinguishing a pair of unitary operations have been specified in [14]. It is interesting that our constructions provide non-trivial sets of unitary operators which can be perfectly distinguished by a single measurement of a specific partially entangled state.

### TABLE I: Values of $\lambda_0$ for states with minimal entanglement such that there exists a construction of $N$ (row index) orthogonal unitary transformations in $d$ (column index) dimensions. Numeric data and conjectured behavior are shown. Note that the entanglement is smaller than one ebit.

| $\lambda_0$ | 2 | 3 | 4 | 5 | 6 | 7 | $\ldots$ | $d$ |
|-------------|---|---|---|---|---|---|---------|----|
| 3           |   |   |   |   |   |   |         |    |
| 4           | 2/4 | 2/3 |   |   |   |   |         |    |
| 5           | 3/5 | 3/4 |   |   |   |   |         |    |
| 6           | 3/6 | 4/6 | 4/5 |   |   |   |         |    |
| 7           | 4/7 | 5/7 | 5/6 |   |   |   |         |    |
| 8           | 4/8 | 5/8 | 6/8 | 6/7 |   |   |         |    |
| 9           | 5/9 | 6/9 | 7/9 |   |   |   |         |    |
| 10          | 5/10| 6/10| 7/10|   |   |   |         |    |
| 11          | 6/11| 7/11|   |   |   |   |         |    |
| 12          | 6/12| 7/12|   |   |   |   |         |    |
| 13          |   |   |   |   |   |   |         |    |
| 14          |   |   |   |   |   |   |         |    |
| $d + 1$     |   |   |   |   |   |   |         |    |
| $d + 2$     |   |   |   |   |   |   |         |    |
| $\vdots$    |   |   |   |   |   |   |         |    |
| $2d$        |   |   |   |   |   |   |         |    |
Finally, it would be interesting to examine whether the construction of a the set of unitaries that satisfy the generalized orthogonality condition (7), sheds light on the recent proposal for probabilistic interpretation of evolutions [17].

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