Abstract

We consider the connection problem of the second nonlinear differential equation

$$\Phi''(x) = (\Phi'^2(x) - 1) \cot \Phi(x) + \frac{1}{x}(1 - \Phi'(x))$$

subject to the boundary condition $\Phi(x) = x - ax^2 + O(x^3)$ ($a \geq 0$) as $x \to 0$. In view of that equation (0.1) is equivalent to the fifth Painlevé (PV) equation after a Möbius transformation, we are able to study the connection problem of equation (0.1) by investigating the corresponding connection problem of PV. Our research technique is based on the method of uniform asymptotics presented by Bassom et al. The monotonically solution on real axis of equation (0.1) is obtained, the explicit relation (connection formula) between the constants in the solution and the real number $a$ is also obtained. This connection formulas have been established earlier by Suleimanov via the isomonodromy deformation theory and the WKB method, and recently are applied for studying level spacing functions.

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1 Introduction and main results

In the present paper we show how the technique of uniform asymptotics introduced by Bassom, Clarkson, Law and McLeod in [1] can be applied to the equation

$$\Phi''(x) = (\Phi'^2(x) - 1) \cot \Phi(x) + \frac{1}{x}(1 - \Phi'(x)),$$ (1.1)
which solutions related to the computation of one particle density matrix of impenetrable bosons at zero temperature\cite{14,16}.

We are focus on the problem of calculating an asymptotics behavior as \( x \to \infty \) of one-parameter class of regular solutions to equation (1.1) defined with the boundary condition

\[
\Phi(x) = x - ax^2 + O(x^3), \quad \text{as} \quad x \to 0,
\]

and on the relevant connection formulae between the different asymptotic parameters which appeared in the above mentioned critical expansions.

Introducing the change of variable

\[
y(s) = \exp(-2i \Phi(x)), \quad s = \frac{x}{2},
\]

in Eq.(1.1), we get for \( y(s) \) the special fifth Painlevé (PV)

\[
\frac{d^2 y}{ds^2} = \left( \frac{1}{2y} + \frac{1}{y-1} \right) \left( \frac{dy}{ds} \right)^2 - \frac{1}{s} \frac{dy}{ds} - 4i \frac{y}{s} + 8 \frac{y(y+1)}{y-1}, \tag{1.4}
\]

which appears in the studying the level spacing functions related to the Fredholm determinant of the sine kernel \( \sin \frac{\pi(x-y)}{\pi(x+y)} \) on the finite interval \((-s, s)\) \cite{15,11}. Let \( s = it \) and \( p(t) = \frac{\sqrt{y+1}}{\sqrt{y-1}} \), then equation (1.4) is equivalent to a special third Painlevé (PIII) equation

\[
\frac{d^2 p}{dt^2} = \frac{1}{p} \left( \frac{dp}{dt} \right)^2 - \frac{1}{t} \frac{dp}{dt} + \frac{1}{t} (p^2 - 1) + p^3 - \frac{1}{p}, \tag{1.5}
\]

which closely related to the studying Bonnet surfaces\cite{2,3}, and the mean curvature and the metric in terms of \( p(t) \) (see (3.115) in \cite{3}). If we set \( w(t) = -p(t) \), then \( w(t) \) satisfies the another special PIII

\[
\frac{d^2 w}{dt^2} = \frac{1}{w} \left( \frac{dw}{dt} \right)^2 - \frac{1}{t} \frac{dw}{dt} + \frac{1}{t} (w^2 - 1) + w^3 - \frac{1}{w}. \tag{1.6}
\]

We mention that this special PIII can be expressed algebraically in terms of a fifth Painlevé transcendent and its first derivative. Consider the following pair of equations

\[
h(\tau) = \frac{w'(t) - w^2(t) - 1}{w'(t) - w^2(t) + 1}, \quad w(t) = \frac{2\tau h(\tau)}{\tau h'(\tau) - h(\tau) + 1}, \tag{1.7}
\]

where \( \tau = \frac{t^2}{2} \). Eliminating \( h \) from (1.7), we get Eq.(1.6) for \( w(t) \); and eliminating \( w \) from (1.7), we get for \( h(\tau) \) the special PV equation

\[
\frac{d^2 h}{d\tau^2} = \left( \frac{1}{2h} + \frac{1}{h-1} \right) \left( \frac{dh}{d\tau} \right)^2 - \frac{1}{\tau} \frac{dh}{d\tau} - \frac{1}{8} \frac{(h-1)^2}{\tau^2 h^2} - \frac{h}{\tau}, \tag{1.8}
\]

which admits the Lax representation \cite{14}. 

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With the help of the preceding derivation we can now see that the Eq. (1.1) is equivalent to Eq. (1.8) after the transform
\[ h(\tau) = 1 + 2 \sin^2 \Phi(x) \frac{\Phi'(x)}{\Phi(x) - 1}, \quad \tau = -\frac{x^2}{8}. \] (1.9)
Hence, we are able to study the connection problem of Eq. (1.1) by using the isomonodromic deformation technique [7] to consider the correspond connection problem for PV (1.8). Basing on the special Lax pair of (1.8), the author of [14] has studied analytical this solution to Eq. (1.1) with the initial condition (1.2) for all real-valued \( a \), and obtained the asymptotic expansion of \( \Phi \) as \( x \to \infty \) and explicit connection formulas by virtue of the isomonodromic deformation technique. Let us summarize the main result of [14] in the theorem as follows:

**Theorem 1.** There exists a unique solution of (1.1) which satisfies (1.2) for any given real number \( a \).

(A) If \( a > \frac{1}{\pi} \), this solution exists for positive real \( x \), and
\[ \Phi(x, a) = -x + \beta \ln x + \gamma + o(1), \quad \text{as } x \to \infty, \] (1.10)
where \( \beta \) and \( \gamma \) are real constants. Furthermore, the relationship between the parameters \( \beta, \gamma \) in (1.10) and the parameters \( a \) in (1.2) are provided by the connection formulas
\[ \beta = -\frac{1}{\pi} \ln(a\pi - 1), \] (1.11)
\[ \gamma = \frac{\pi}{2} + 2 \text{ arg } \Gamma\left(\frac{i\beta}{2} - \frac{1}{2}\right) + \beta \ln 2 + k\pi, \quad k \in \mathbb{Z}. \] (1.12)

(B) If \( a < \frac{1}{\pi} \), this solution exists for all real \( x \), and increases monotonically as \( x \to \infty \),
\[ \Phi(x, a) = x + \beta \ln x + \gamma + o(1), \] (1.13)
where \( \beta \) and \( \gamma \) are real constants. Furthermore, the relationship between the parameters \( \beta, \gamma \) in (1.13) and the parameters \( a \) in (1.2) are provided by the connection formulas
\[ \beta = \frac{1}{\pi} \ln(1 - a\pi), \] (1.14)
\[ \gamma = -2 \text{ arg } \Gamma\left(\frac{i\beta}{2}\right) + \beta \ln 2 - \pi \text{ sign } \beta, \] (1.15)
where \( \beta \neq 0 \) and \( \gamma(0) = 0 \).

(C) If \( a = \frac{1}{\pi} \), this solution exists for all real \( x \) as \( x \) increases monotonically to a finite limit, and, as \( x \to \infty \),
\[ \Phi(x, \frac{1}{\pi}) = \frac{\pi}{2} + o(1). \] (1.16)
On the earlier work of the authors of [4], they have studied numerically this solution with the given asymptotic behavior at the origin (1.2) for the case of \( a > \frac{1}{\pi} \) and proposed (1.10) and (1.11), however, they not obtain the explicit expression (1.12) for \( \gamma \).

Recently, the connection formulas in Theorem 1 are applied for calculations of the Fredholm determinant of the sine kernel \( \sin \pi(x - y)/\pi(x - y) \) on the finite interval \((t, -t)\); see [11].

In this paper, we provided a simpler and more rigorous proof of the Theorem 1, by using the uniform asymptotics method proposed in [1]. For our purposes, we first briefly outline some important properties of the theory of monodromy preserving deformations for the PV transcendents. The reader is referred to [6, 8] for more details.

One of the Lax pairs for the fifth Painlevé equation (1.8) is the system of linear ordinary equations [11]

\[
\frac{\partial \Psi}{\partial \lambda} = \left\{ -i\tau \sigma_3 + \frac{1}{\lambda} \begin{pmatrix} \frac{1}{4} & u \\ v & -\frac{1}{4} \end{pmatrix} + \frac{1}{\lambda^2} \begin{pmatrix} z & q \\ q & -z \end{pmatrix} \right\} \Psi \quad (1.17)
\]

and

\[
\frac{\partial \Psi}{\partial \tau} = \left\{ -i\lambda \sigma_3 + \begin{pmatrix} g & \frac{u}{\tau} \\ \frac{v}{\tau} & -g \end{pmatrix} \right\} \Psi, \quad (1.18)
\]

where \( \tau = -\frac{2^2}{8} \) and

\[
z = -\frac{i}{8} h + 1, \quad q = -\frac{1}{4} \sqrt{h} - 1, \quad u+v = -\frac{i}{2\sqrt{h}}, \quad u-v = \frac{i\tau h \tau}{(1-h)\sqrt{h}}, \quad g = \frac{1}{8\tau}(1+\frac{1}{h}). \quad (1.19)
\]

The compatibility condition \( \Psi_{\lambda x} = \Psi_{x\lambda} \) implies that \( h(\tau) \) satifies the PV equation (1.8).

The equation (1.17) has two irregular points \( \lambda = 0 \) and \( \lambda = \infty \). There exists a canonical solution \( \Psi^{(\infty)} \) defined in a neighborhood of the irregular singular point \( \lambda = \infty \) with the following asymptotics behavior

\[
\Psi^{(\infty)}(\lambda) = E^{(\infty)}(\tau)(I + O(\frac{1}{\lambda})) \lambda^\frac{1}{4}\sigma_3 \exp(-i\tau \lambda \sigma_3), \quad \lambda \to \infty, \quad \arg \lambda = 0, \quad (1.20)
\]

where

\[
E^{(\infty)}(\tau) = \tau^\frac{1}{8}\sigma_3 \exp(\sigma_3 J(\tau)) := d^{\sigma_3} \quad (1.21)
\]

with \( J(\tau) = \frac{1}{\tau} \int_{c}^{\tau} \frac{\text{d}t}{h(t)} \), here \( c \) is a positive constant.

From (1.17) and (1.20) it follows that

\[
\Psi^{(\infty)}(\lambda) = E^{(\infty)}(\tau) \begin{pmatrix} 1 + O(\frac{1}{\lambda}) & a_1 \frac{\lambda}{\sigma_3} \\ a_2 \frac{\lambda}{\sigma_3} & 1 + O(\frac{1}{\lambda}) \end{pmatrix} \lambda^\frac{1}{4}\sigma_3 \exp(-i\tau \lambda \sigma_3) \quad (1.22)
\]

with \( a_1 = \frac{u}{2i\tau d} \) and \( a_2 = -\frac{v}{2i\tau d} \).
On the other hand, (1.17) has another canonical solution $\Psi^{(0)}$ in a neighbourhood of the irregular singular point $\lambda = 0$

$$\Psi^{(0)}(\lambda) = H(\tau)E^{(0)}(\tau)(I + O(\lambda))\lambda^{\frac{i}{4}\sigma_3}\exp\left(\frac{i}{\lambda}\sigma_3\right), \quad \lambda \to 0, \quad \arg \lambda = 0,$$

(1.23)

where the coefficients $H$ and $E^{(0)}$ have the form

$$H(\tau) = \frac{1}{\sqrt{h(\tau)} - 1}(i\sigma_3\sqrt{h} + \sigma_1), \quad E^{(0)}(\tau) = \tau^{\frac{i}{4}\sigma_3}\exp(-\sigma_3 J(\tau)) = \tilde{d}^{\sigma_3}.$$

(1.24)

Since $\Psi^{(\infty)}$ and $\Psi^{(0)}$ are both fundamental solutions, the connection matrix $Q$ can be defined by

$$\Psi^{(\infty)}(\lambda) = \Psi^{(0)}(\lambda)Q,$$

(1.25)

Differentiating both sides of equation (1.25) with respect to $x$, and making use of the fact that both $\Psi^{(\infty)}$ and $\Psi^{(0)}$ satisfy (1.17), it is easily found the isomonodromic condition $\frac{dQ}{dx} = 0$; i.e., $Q$ is a constant matrix.

In the framework of the isomonodromic deformation method, one needs to calculate the monodromy data $Q$ both in terms of the initial condition (1.2) and asymptotics (1.13). Equating then the leading terms of nontrivial monodromy data, one gets connection formulas for the parameters $\beta$, $\gamma$ and $\alpha$. In the limit $x \to 0$ the first term of equation in $\lambda$ (1.17) vanishes, so the $\Psi$-functions can be expressed via the Whittaker functions [14], then the monodromy data as $x \to 0$ is calculated explicitly by use of multiplication formulas of the Whittaker functions and obtain that

$$(Q)_{21} = i2^{-3/4}\sqrt{a\pi}.$$

(1.26)

To estimate the connection matrix in the limit $\tau \to +\infty$. One finds the WKB solution of the $\Psi$-function, which demands a standard procedure of matching near the turning points, involving parabolic cylinder functions (see, [14, p.251]). Eventually, one can obtains the connection matrix for large $x$ as follows by using of the asymptotics behavior of parabolic cylinder functions.

In this paper, we shall provide a hopefully simpler and more rigorous derivation of the asymptotic behavior and the connection formulas in Theorem 11 by using the uniform asymptotics method presented in [1]. Along the same lines we may find the work of Olver [12] and Dunster [3] for coalescing turning points. Initially in [1], the second Painlevé(PII) equation has been taken as an example to illustrate the method. While the difficulty in extending the techniques for PII to other transcendents is also acknowledged by the authors of [1, p.244]. Yet the method has been applied to the connection problems by Wong and Zhang [17, 18], Zeng and Zhao [19], Long, Zeng and Zhou [10]. Recently, Long el at. [9] present a detail asymptotics analysis of the real solutions of the first Painlevé(PI) equation by virtue of the uniform asymptotics method.

The rest of the paper is organized as follows. The proof of Theorem 11 is provided in the Section 2. In Section 3 for the case of $a > \frac{1}{\pi}$ and $a < \frac{1}{\pi}$, we derive uniform
approximations to the solutions of the second-order differential equation obtained from
the Lax pair (1.17) as \( x \to +\infty \) by virtue of the parabolic cylinder functions on the
Stokes curves, respectively. The entry \((2,1)\) of the connection matrix \(Q\) for large \(x\) is
also computed in the section. Some technical details are put in Appendices A and B to
clarify the derivation.

2 Proof of the theorem

To proof of the Theorem \[\dagger\] we need two lemmas as follows.

**Lemma 1.** For \( a > \frac{1}{\pi} \), the asymptotics behavior of the entry \((2,1)\) of the connection
matrix \(Q\) is

\[
(Q)_{21} = \frac{2^{-\frac{1}{4}} \sqrt{\pi e^{-\frac{x^2}{4}}} \Gamma(\frac{1}{2} - i\beta/4)}{\pi} \exp \left( iS + \frac{i}{2} x - \frac{i\beta}{2} \ln x - \frac{i\beta}{2} \ln 2 + \frac{3\pi i}{4} \right).
\]

(2.1)

**Lemma 2.** For \( a < \frac{1}{\pi} \), the asymptotics behavior of the entry \((2,1)\) of the connection
matrix \(Q\) is

\[
(Q)_{21} = \frac{i\sqrt{\beta} 2 \frac{1}{\pi} \sqrt{\pi e^{-\frac{x^2}{4}}} \Gamma(\frac{1}{2} + i\beta/4)}{\pi} \exp \left( -iS + \frac{i}{2} x + \frac{i\beta}{2} \ln x + \frac{i\beta}{2} \ln 2 \right).
\]

(2.2)

The rigorous proofs of those results will be given in the next section. With the help
of the preceding two lemmas we can now prove Theorem \[\dagger\]

**Proof of Theorem \[\dagger\]** We first give the proof when \( a > \frac{1}{\pi} \). Since the connection matrix
\(Q\) must be independent of \(x\), it follows that the right hand sides of (1.26) and (2.1)
are equality. Separating real and imaginary parts and in view of the standard formulae
\( |\Gamma(z)|^2 = \pi \cos h \pi z \) (see,\[13\]) gives the asymptotic behaviors (1.10) and connection
formulas (1.11), (1.12), which completes the proof of statement (A) in Theorem \[\dagger\].

When \( a < \frac{1}{\pi} \), the asymptotic behaviors (1.13) and connection formulas (1.14), (1.15)
are obtained straightforward by equating the expressions (1.26) and (2.2), here has been
made of the formulas \( \Gamma(z + 1) = z \Gamma(z) \) and \( |\Gamma(iy)|^2 = \frac{2\pi e^{2\pi y}}{y(\cos \pi y + e^{-\pi y})} \) and the fact that
\( \Phi(x, 0) = x \) is the solution of the initial problem (1.1) and (1.2). Hence the statement
(B) in Theorem \[\dagger\] is proved.

Using the fact that \( \Phi(x, a) \) is a continuous function of \( a \) \[14, Lemma 1, p.253\], and
taking (1.13) into consideration, according to the definition of \( L \) in (3.55), we obtain

\[
\lim_{x \to \infty} \Phi(x, 1/\pi) = \frac{\pi}{2},
\]

which gives the proof of statement (C) in Theorem \[\dagger\].

The proof of Theorem \[\dagger\] is now complete. \(\square\)
3 Uniform asymptotics and proofs of the lemmas

Making the scaling
\[ \xi = x, \quad \eta = x\lambda, \]
so that (1.17) becomes
\[ \frac{\partial \Psi}{\partial \eta} = \left( \begin{array}{c} \frac{i}{8} \xi + \frac{1}{4} \eta + \frac{\xi}{\eta} z \\ \frac{v}{\eta} + \frac{\xi}{\eta^2} q \end{array} \right) \Psi \]
(3.1)

Let \((\Psi_1, \Psi_2)^T\) be an independent solution of (3.2), and set
\[ \phi = \left( \frac{v}{\eta} + \frac{\xi}{\eta^2} q \right)^{-\frac{1}{2}} \Psi_2, \]
(3.3)
we get from (3.2) the second-order linear differential equation for \(\phi(\eta)\)
\[ \frac{d^2 \phi}{d\eta^2} = \left\{ \xi^2 \frac{i}{8} + \frac{z}{\eta^2} \right\}^2 + \frac{\xi}{\eta} \left( \frac{i}{8} + \frac{z}{\eta^2} \right) + \frac{1}{16\eta^2} + \frac{\xi^2 q^2}{\eta^4} + \frac{1}{\eta^2} \left[ uv + \frac{\xi}{\eta} q(u + v) \right] \]
\[ + \frac{1}{4\eta^2} + \frac{2z}{\eta^3} - \frac{1}{\eta} \left( \frac{i}{8} + \frac{z}{\eta^2} \right) l_1 - \frac{1}{4\eta^2} l_1 - \frac{3}{4\eta^2} l_2 \}
(3.4)
where
\[ l_1 = \frac{v + \frac{2z}{\eta} q}{v + \frac{\xi}{\eta} q}, \quad l_2 = \frac{v + \frac{\xi}{\eta} q}{v + \frac{\xi}{\eta} q}. \]
(3.5)

From (1.19) it is easy to verify that
\[ z^2 + q^2 = -\frac{1}{64}, \quad q(u + v) = \frac{i}{8(h - 1)}. \]
(3.6)
Substituting (3.6) into (3.4) yields
\[ \frac{d^2 \phi}{d\eta^2} = \left\{ -\frac{\xi^2}{64} (1 - \frac{1}{\eta^2})^2 + \frac{\xi^2}{64\eta^2} (16iz - 2) + \frac{\xi}{\eta} \left( \frac{i}{8} + \frac{z}{\eta^2} \right) - \frac{\xi}{\eta} \left( \frac{i}{8} + \frac{z}{\eta^2} \right) l_1 \]
\[ + \frac{uv}{\eta^2} + \frac{2z}{\eta^3} + \frac{1}{8\eta^3 (h - 1)} + \frac{1}{\eta^2} \left[ \frac{5}{16} - \frac{1}{4} l_1 + \frac{3}{4} l_2 - l_2 \right] \}
(3.7)

3.1 Proof of the Lemma [1]

To proof the Lemma [1] we need several lemmas. First, we need to construct the uniform asymptotics solution of equation (3.7) as \(\xi \to \infty\) for \(a > \frac{1}{\pi}\). When \(a > \frac{1}{\pi}\), according to [14] (A.71), the solution of boundary value problem (1.1)-(1.2) has the following asymptotics expansion
\[ \Phi'(\xi, a) = -1 + \frac{\varphi(S)}{\xi} + O(\xi^{-2}), \quad \text{as} \quad \xi \to \infty, \]
(3.8)
where $\varphi(S) = \sin 4S + 2k^2 \sin^2 2S$ with $S = \frac{1}{2}\Phi(\xi, a)$. It follows from the expression of $h$ in (1.9) that
\[
h(\xi) = \cos^2 S \left( 1 - \frac{\varphi(S) \tan^2 S}{2\xi} + O(\xi^{-2}) \right), \quad \text{as } \xi \to \infty. \tag{3.9}
\]
From (3.9) and (1.19), we obtain the following asymptotic behaviors as $\xi \to \infty$
\[
z = \frac{i}{8} \left( 1 + 2 \cot^2 S - \frac{\varphi(S) \csc^2 S}{\xi} + O(\xi^{-2}) \right), \tag{3.10}
\]
\[
 uv = \frac{\xi^2}{16 \sin^2 S} \left( 1 - 2 \tan S + \varphi(S) \left( 1 + \frac{1}{2} \tan^2 S \right) + O(\xi^{-2}) \right), \tag{3.11}
\]
\[
\frac{q}{v} = \frac{i \cot S}{\xi} (1 + O(\xi^{-1})), \tag{3.12}
\]
\[
 qv = -\frac{i \xi \cot S}{16 \sin^2 S} \left( 1 + O(\xi^{-1}) \right). \tag{3.13}
\]
Then, for large $\xi$, substituting (3.10), (3.11) and (3.12) into (3.7), a tedious but straightforward calculation gives
\[
\frac{d^2 \phi}{d\eta^2} = -\xi^2 F(\xi, \eta) \phi, \tag{3.14}
\]
where
\[
F(\xi, \eta) = \frac{1}{64} \left( 1 - \frac{1}{\eta^2} \right)^2 + \frac{F_1(\xi, \eta)}{\xi} + F_2(\eta) O(\frac{1}{\xi^2}), \tag{3.15}
\]
here
\[
F_1(\xi, \eta) = \frac{k^2}{4\eta^2} + \frac{i}{8\eta} \left( 1 - \frac{1}{\eta^2} \right) \left( \frac{1}{2} + \frac{1}{b\eta - 1} \right) \tag{3.16}
\]
with $b = i \tan S$, and
\[
F_2(\eta) = \frac{1}{\eta^2} + \frac{1}{\eta^3}. \tag{3.17}
\]
For large $\xi$, it follows from equation (3.14) that there are two coalescing turning points near $\eta = 1$, and two close to $\eta = -1$. In the present paper, we are only concerned with the two turning points, say $\eta_1$ and $\eta_2$, near $\eta = 1$. When $\eta_j$ approach to 1, it follows from (3.16) that
\[
F_1(\xi, 1) \sim \frac{k^2}{4}. \tag{3.18}
\]
By using (3.14) and (3.18), we get the asymptotic formulas for the two turning points
\[
\eta_j^{-1} = 1 \pm 2\xi^{-1/2}\sqrt{k^2(1 + o(1))}, \quad j = 1, 2, \tag{3.19}
\]
which coalescing to 1 when \( \xi \to \infty \), and the Stokes’ curves defined by
\[
\Im(\xi(\eta + \frac{1}{\eta}))) = 0. \tag{3.20}
\]
Assuming that \( \xi \in \mathbb{R}^+ \), then, it follows from (3.20) that the Stokes lines of the solution \( \phi \) to (3.14) are the positive and the negative real lines in the \( \eta \) plane.

According to the philosophy of uniform asymptotics in [1], we define a number \( \alpha \) by
\[
\frac{1}{2} \pi i \alpha^2 = \int_{-\alpha}^{\alpha} (\tau^2 - \alpha^2)^{1/2} d\tau = \int_{\eta_1}^{\eta_2} F^{1/2}(\xi, s) ds, \tag{3.21}
\]
and a new variable \( \zeta \) by
\[
\int_{\alpha}^{\zeta} (\tau^2 - \alpha^2)^{1/2} d\tau = \int_{\eta_2}^{\eta} F^{1/2}(\xi, s) ds. \tag{3.22}
\]
Here and in (3.21), the cut for the integrand on the left-hand side is the line segment joining \(-\alpha\) and \(\alpha\). The path of integration is taken along the upper edge of the cut. With \( \alpha \) and \( \zeta \) so chosen, then the following lemma is a result from [1, Theorem 1].

**Lemma 3.** Given any solution \( \phi(\eta, \xi) \) of (3.14), there exist constants \( c_1, c_2 \) such that, uniformly for \( \eta \) on the Stokes curves defined by (3.20), as \( \xi \to +\infty \),
\[
\phi(\eta, \xi) = \left( \frac{\zeta^2 - \alpha^2}{F(\xi, \eta)} \right)^{\frac{1}{4}} \left\{ [c_1 + o(1)] D_{\nu}(e^{\pi i/4} \sqrt{2\xi \zeta}) + [c_2 + o(1)] D_{-\nu-1}(e^{-\pi i/4} \sqrt{2\xi \zeta}) \right\}, \tag{3.23}
\]
where \( D_{\nu}(z) \) and \( D_{-\nu-1}(z) \) are solutions of the parabolic cylinder equation and \( \nu \) defined by
\[
\nu = -\frac{1}{2} + \frac{1}{2} i \xi \alpha^2. \tag{3.24}
\]
The next thing to do in calculating the connection matrix \( Q \) as \( \xi \to +\infty \) is to clarify the relation between \( \zeta \) and \( \eta \) in (3.22).

**Lemma 4.** For large \( \xi \) and \( \eta \),
\[
\frac{1}{2} \zeta^2 = \frac{\alpha^2}{2} \ln \zeta + \frac{1}{8} (\eta + \frac{1}{\eta}) - \frac{1}{4} + \frac{i}{4 \xi} \ln \eta - \frac{i}{2 \xi} \ln(1 - b^{-1}) + o(\xi^{-1}), \tag{3.25}
\]
where \( b = i \tan S \), and
\[
\alpha^2 = \frac{k^2}{\xi} + o(\frac{1}{\xi}) \quad \text{as} \quad \xi \to \infty. \tag{3.26}
\]
**Remark 1.** Coupling (3.24) and (3.26) determines the approximate value
\[
\nu = -\frac{ik^2}{2} - \frac{1}{2} + o(1) \quad \text{as} \quad \xi \to +\infty \tag{3.27}
\]
for the order of the parabolic cylinder function \( D_{\nu}(e^{\pi i/4} \sqrt{2\xi \zeta}) \) in (3.23).
Lemma 5. When \( \eta \to 0 \), for large \( \xi \), such \( \xi \eta = o(1) \), then holds

\[
\frac{1}{2} \xi^2 = \frac{\alpha^2}{2} \ln \zeta + \frac{1}{8} (\eta + \frac{1}{\eta}) - \frac{i}{4} \ln \eta - \frac{i}{2 \xi} \ln(1 - b) - \frac{1}{2} \pi i \alpha^2 + o(\xi^{-1}). \tag{3.28}
\]

The proofs of Lemmas 4 and 5 will be given in Appendix A and B, respectively. We are now turning to the proof of Lemma 1.

Proof of Lemma 1. We will concentrate on evaluating the connection matrix \( Q \), for which we need the uniform asymptotic behaviors of \( \phi \). From the definition of \( Q \) it follows that

\[
\Psi^{(\infty)}_{21} = (Q)_{11} \Psi^{(0)}_{21} + (Q)_{21} \Psi^{(0)}_{22}. \tag{3.29}
\]

Hence, the first task is to find the expressions of \( \Psi^{(\infty)}_{21} \), \( \Psi^{(0)}_{21} \) and \( \Psi^{(0)}_{22} \), respectively.

For large \( \xi \), it follows from lemma 3 that two linearly independent asymptotic solutions of equation (3.14) are \( \tilde{\phi}_\nu \) and \( \tilde{\phi}_{-\nu-1} \) which are uniform with respect to \( \eta \) on the Stokes’ curves. Here

\[
\tilde{\phi}_\nu = \left( \frac{\zeta^2 - \alpha^2}{F(\zeta, \eta)} \right)^{\frac{1}{4}} D_\nu(e^{\pi i/4} \sqrt{2 \zeta \xi}) \tag{3.30}
\]

and

\[
\tilde{\phi}_{-\nu-1} = \left( \frac{\zeta^2 - \alpha^2}{F(\zeta, \eta)} \right)^{\frac{1}{4}} D_{-\nu-1}(e^{-\pi i/4} \sqrt{2 \zeta \xi}). \tag{3.31}
\]

By virtue of (3.3), we have

\[
\Psi^{(\infty)}_{21} = (\frac{v}{\eta} + \frac{\xi q}{\eta^2})^{1/2} (f_1 \tilde{\phi}_\nu + f_2 \tilde{\phi}_{-\nu-1}), \tag{3.32}
\]

where \( f_j (j = 1, 2) \) are undetermined constants which can be determined by (1.22).

Similarly, we obtain

\[
\Psi^{(0)}_{21} = (\frac{v}{\eta} + \frac{\xi q}{\eta^2})^{1/2} (\delta_1 \tilde{\phi}_\nu + \delta_2 \tilde{\phi}_{-\nu-1}), \tag{3.33}
\]

\[
\Psi^{(0)}_{22} = (\frac{v}{\eta} + \frac{\xi q}{\eta^2})^{1/2} (\delta_3 \tilde{\phi}_\nu + \delta_4 \tilde{\phi}_{-\nu-1}), \tag{3.34}
\]

where \( \delta_j (j = 1, 2, 3, 4) \) are undetermined constants which can be determined by (1.23).

Substituting (3.32), (3.33) and (3.34) into (3.29), then comparing with the coefficients of \( \tilde{\phi}_\nu \) and \( \tilde{\phi}_{-\nu-1} \), respectively, we get

\[
\begin{aligned}
\delta_1 (Q)_{11} + \delta_3 (Q)_{21} &= f_1 \\
\delta_2 (Q)_{11} + \delta_4 (Q)_{21} &= f_2
\end{aligned}
\]

which gives us that

\[
(Q)_{21} = \frac{\delta_1 f_2 - \delta_2 f_1}{\delta_1 \delta_4 - \delta_2 \delta_3}. \tag{3.35}
\]
To calculate $f_1$, $f_2$ and $\delta_j$ ($j = 1, \ldots, 4$) we proceed as follows. We shall be interested in finding the asymptotic behavior of $\tilde{\phi}_\nu$ and $\tilde{\phi}_{-\nu-1}$ in (3.30), (3.31) for $\eta$ on the Stokes line $\arg \eta = 0$ as $\eta \to \infty$ and $\eta \to 0$, respectively. Then, substituting the obtained results into (3.32), (3.33) and (3.34), we will obtain the asymptotic behavior of $\Psi^{(\infty)}$, $\Psi^{(0)}$ and $\Psi^{(0)}$, respectively, which contain the constants $f_1$, $f_2$ and $\delta_j$ ($j = 1, \ldots, 4$). Combining with the boundary conditions (1.22) and (1.23) for $\Psi^{(\infty)}$ and $\Psi^{(0)}$, one can determine the constants $f_1$, $f_2$ and $\delta_j$ ($j = 1, \ldots, 4$).

From [13], we have the asymptotic behavior of $D_\nu(z)$ for $|z| \to \infty$ as follows:

$$D_\nu(z) \sim \begin{cases} z^{\nu} e^{-\frac{1}{2} z^2}, & \text{arg } z \in (-\frac{3}{4} \pi, \frac{3}{4} \pi), \\ z^{\nu} e^{-\frac{1}{2} z^2} - \frac{\sqrt{2 \pi}}{1-(\nu)} e^{i \pi \nu} z^{-\nu+1} e^{\frac{1}{2} z^2}, & \text{arg } z \in \frac{3}{4} \pi, \\ e^{-2 \pi i (\nu+1)} z^{-\nu} e^{\frac{1}{2} z^2} - \frac{\sqrt{2 \pi}}{1-(\nu)} e^{i \pi \nu} z^{-\nu+1} e^{\frac{1}{2} z^2}, & \text{arg } z \in \frac{5}{2} \pi. \end{cases} \tag{3.36}$$

For $\eta$ on the Stokes line $\arg \eta = 0$ and $\eta \to \infty$, From (3.25) it immediately follows that $\zeta^2 \sim \frac{1}{4} \eta$, then we have $\arg \zeta \sim 0$. Therefore, $\arg(e^{\pi i/4} \sqrt{2 \pi} \zeta) \sim \frac{\pi}{4}$ and $\arg(e^{-\pi i/4} \sqrt{2 \pi} \zeta) \sim -\frac{\pi}{4}$ for $\xi > 0$. From (3.14) we have $F^{-1/4} \sim 2 \frac{1}{4}$ as $\eta \to \infty$ for large $\xi$. Since $(\xi^2 - a^2)^{1/4} \sim \zeta^{1/2}$ as $\eta \to \infty$, by using the appropriate asymptotic formulas of $D_\nu(z)$ in (3.30), we obtain from (3.30) and (3.25) that

$$\tilde{\phi}_\nu \sim A_0 \eta^{\frac{1}{4}} e^{-\frac{1}{4} i \xi \eta}, \text{ as } \eta \to \infty, \tag{3.37}$$

here has been made of (3.24), where

$$A_0 = 2^{i \pi/4} e^{\pi i \nu} e^{\frac{1}{2} \pi \nu} \ln(1 - b^{-1})^{-\frac{1}{4}}. \tag{3.38}$$

Similarly, from (3.31) and (3.25) it follows that

$$\tilde{\phi}_{-\nu-1} \sim B_0 \eta^{-\frac{1}{4}} e^{\frac{1}{4} i \xi \eta}, \text{ as } \eta \to \infty, \tag{3.39}$$

where

$$B_0 = 2^{1-\frac{1}{4} \nu} e^{\pi i \nu} e^{\frac{1}{2} \pi \nu} \ln\left(1 - b^{-1}\right)^{\frac{1}{4}}. \tag{3.40}$$

Substituting (3.37) and (3.39) into (3.32), yields

$$\Psi^{(\infty)}_{21}(\eta) \sim v^{\frac{1}{2}} \eta^{-\frac{1}{2}} \left(f_1 A_0 \eta^{\frac{1}{4}} e^{-\frac{1}{4} i \xi \eta} + f_2 B_0 \eta^{-\frac{1}{4}} e^{\frac{1}{4} i \xi \eta}\right) \text{ as } \eta \to \infty. \tag{3.41}$$

Moreover, it has the asymptotic behavior prescribed in (1.22) when $\eta \to \infty$. Thus, we have

$$v^{\frac{1}{2}} \eta^{-\frac{1}{2}} \left(f_1 A_0 \eta^{\frac{1}{4}} e^{-\frac{1}{4} i \xi \eta} + f_2 B_0 \eta^{-\frac{1}{4}} e^{\frac{1}{4} i \xi \eta}\right) \sim -\frac{d}{2i \tau} v \lambda^{-\frac{3}{2}} \exp(-i \tau \lambda) \tag{3.42}$$

Comparing the coefficients of $e^{\frac{1}{4} i \xi \eta}$ and $e^{-\frac{1}{4} i \xi \eta}$ on both sides of the above asymptotic equation, we get

$$f_1 = 0, \quad f_2 = -i 2^2 \xi^{-\frac{5}{4}} dv^2 B_0^{-1}. \tag{3.43}$$
For \( \eta \) on the Stokes line \( \arg \eta = 0 \) and \( \eta \to 0 \), from (3.28) it immediately follows that
\[ \zeta^2 \sim \frac{1}{4\eta}, \]
then we have \( \arg \zeta \sim \pi \). Therefore, \( \arg(e^{\pi i/4} \sqrt{2\xi} \zeta) \sim \frac{5\pi}{4} \) and \( \arg(e^{-\pi i/4} \sqrt{2\xi} \zeta) \sim \frac{3\pi}{4} \) for \( \xi > 0 \). From (3.14) we have \( F^{-1/4} \sim 2^{\frac{3}{2}} \eta \) as \( \eta \to 0 \) for large \( \xi \). Since \( (\zeta^2 - \alpha^2)^{1/4} \sim \zeta^{1/2} \) as \( \eta \to 0 \), by using the appropriate asymptotic formulas of \( D_\nu(z) \) in (3.30), we obtain from (3.30) and (3.28) that
\[
\tilde{\phi}_\nu \sim \eta \left( C_0 \eta^{-\frac{1}{4}} e^{-\frac{i}{8\eta} \xi} - \frac{\sqrt{2\pi}}{\Gamma(-\nu)} e^{i\pi \nu} D_0 \eta^{\frac{1}{4}} e^{\frac{i}{4\eta} \xi} \right), \quad \text{as } \eta \to 0, \tag{3.44}
\]
where
\[
C_0 = 2^{\frac{3}{4} + \frac{i}{4}} \exp \left( \frac{5\pi i}{4} \nu + \frac{1}{2} \pi i \right) \exp \left( \frac{i}{4} \xi + \frac{\nu}{2} \ln \xi \right) \left( 1 - b \right)^{-\frac{1}{2}}, \tag{3.45}
\]
\[
D_0 = 2^{1 - \frac{i}{2}} \exp \left( \frac{\nu}{4} \xi - \frac{3}{4} \pi i \right) \exp \left( \frac{i}{4} \xi - \frac{\nu + 1}{2} \ln \xi \right) (1 - b)^{-\frac{1}{2}}. \tag{3.46}
\]
Similarly, from (3.31) and (3.28) it follows that
\[
\tilde{\phi}_{-\nu - 1} \sim \eta \left( D_0 e^{\frac{\pi i}{2}(\nu + 1)} \eta^{\frac{1}{4}} e^{\frac{\pi i}{8\eta} \xi} - \frac{\sqrt{2\pi}}{\Gamma(\nu + 1)} e^{\frac{-3\pi i}{2} \nu - \pi i} C_0 \eta^{-\frac{1}{4}} e^{-\frac{i}{8\eta} \xi} \right), \quad \text{as } \eta \to 0, \tag{3.47}
\]
Substituting (3.44) and (3.47) into (3.33) and (3.34), respectively, we obtain
\[
\Psi_{21}^{(0)} \sim \xi^{\frac{1}{2}} q^{\frac{1}{2}} \left[ \left( \delta_1 - \delta_2 \frac{\sqrt{2\pi}}{\Gamma(\nu + 1)} e^{-\frac{3\pi i}{2} \nu - \pi i} \right) C_0 \eta^{-\frac{1}{4}} e^{-\frac{i}{8\eta} \xi} \right.
\]
\[
\left. + \left( \delta_2 e^{\frac{\pi i}{2}(\nu + 1)} - \delta_1 \frac{\sqrt{2\pi}}{\Gamma(\nu)} e^{i\pi \nu} \right) D_0 \eta^{\frac{1}{4}} e^{\frac{i}{8\eta} \xi} \right], \quad \text{as } \eta \to 0, \tag{3.48}
\]
\[
\Psi_{22}^{(0)} \sim \xi^{\frac{1}{2}} q^{\frac{1}{2}} \left[ \left( \delta_3 - \delta_4 \frac{\sqrt{2\pi}}{\Gamma(\nu + 1)} e^{-\frac{3\pi i}{2} \nu - \pi i} \right) C_0 \eta^{-\frac{1}{4}} e^{-\frac{i}{8\eta} \xi} \right.
\]
\[
\left. + \left( \delta_4 e^{\frac{\pi i}{2}(\nu + 1)} - \delta_3 \frac{\sqrt{2\pi}}{\Gamma(\nu)} e^{i\pi \nu} \right) D_0 \eta^{\frac{1}{4}} e^{\frac{i}{8\eta} \xi} \right], \quad \text{as } \eta \to 0. \tag{3.49}
\]
By using the boundary condition (1.23), we have
\[
\frac{d}{d\sqrt{h - 1}} \lambda \xi^{\frac{1}{2}} e^{\frac{i}{8\eta} \xi} \sim \xi^{\frac{1}{2}} q^{\frac{1}{2}} \left[ \left( \delta_1 - \delta_2 \frac{\sqrt{2\pi}}{\Gamma(\nu + 1)} e^{-\frac{3\pi i}{2} \nu - \pi i} \right) C_0 \eta^{-\frac{1}{4}} e^{-\frac{i}{8\eta} \xi} \right.
\]
\[
\left. + \left( \delta_2 e^{\frac{\pi i}{2}(\nu + 1)} - \delta_1 \frac{\sqrt{2\pi}}{\Gamma(\nu)} e^{i\pi \nu} \right) D_0 \eta^{\frac{1}{4}} e^{\frac{i}{8\eta} \xi} \right], \tag{3.50}
\]
\[
\frac{-i\sqrt{h}}{d\sqrt{h - 1}} \lambda \xi^{\frac{1}{2}} e^{-\frac{i}{8\eta} \xi} \sim \xi^{\frac{1}{2}} q^{\frac{1}{2}} \left[ \left( \delta_3 - \delta_4 \frac{\sqrt{2\pi}}{\Gamma(\nu + 1)} e^{-\frac{3\pi i}{2} \nu - \pi i} \right) C_0 \eta^{-\frac{1}{4}} e^{-\frac{i}{8\eta} \xi} \right.
\]
\[
\left. + \left( \delta_4 e^{\frac{\pi i}{2}(\nu + 1)} - \delta_3 \frac{\sqrt{2\pi}}{\Gamma(\nu)} e^{i\pi \nu} \right) D_0 \eta^{\frac{1}{4}} e^{\frac{i}{8\eta} \xi} \right]. \tag{3.51}
\]
From (3.50) and (3.51) it follows that
\[
\frac{\delta_2}{\delta_1} = \frac{e^{\frac{3\pi i}{2} \nu + \pi i} \Gamma(\nu + 1)}{\sqrt{2\pi}}, \quad \frac{\delta_4}{\delta_3} = \frac{\sqrt{2\pi} e^{\frac{3\pi i}{2}(\nu - 1)}}{\Gamma(\nu)}, \quad \frac{\delta_3}{\delta_1} = \frac{-i\sqrt{h}}{d\sqrt{h - 1}} \xi^{\frac{1}{2}} q^{\frac{1}{2}} C_0^{-1} \frac{1 - 2\pi e^{-\pi i \nu - \frac{3\pi i}{4}}}{\Gamma(-\nu) \Gamma(\nu + 1)}. \tag{3.52}
\]
Substituting (3.43) and (3.52) into (3.35), yields
\[
(Q)_{21} = -\frac{\sqrt{2}\pi e^{-\frac{3\pi}{2}}}{\Gamma(\nu + 1)} 2^2 d\xi^{-1} (qv)^{\frac{1}{2}} B_0^{-1} C_0 \left(\sqrt{h - 1}\right) \\
= \frac{2^{-\frac{1}{4}}\sqrt{\pi} e^{-\frac{\pi}{2}}}{\Gamma\left(\frac{1}{2} - i\frac{\pi}{2}\right)} \exp\left(iS + \frac{i}{2}\xi - \frac{i\beta}{2}\ln\xi - \frac{i\beta}{2}\ln 2 - \frac{3\pi i}{4}\right) (1 + O(\xi^{-1})),
\]
where \( \beta = k^2 \). This completes the proof of Lemma 1.

3.2 Proof of the lemma 2

For the case of \( a < \frac{1}{\pi} \), we applying the result \([14, (A.44)]\)
\[
\Phi'(\xi, a) = 1 + 2 \frac{r^2}{\xi} \sin^2(2S) + O(\xi^{-2}), \text{ as } \xi \to \infty,
\]
where \( S = \frac{1}{2}\Phi(\xi, a) \), and \( a \in L \), here \( L \) is defined as follows,
\[
L = \left\{ a \mid \Phi(x) = \Phi(x, a) \text{ is increasing as } x \to \infty, \exists x > 0, \Phi(x, a) > \frac{\pi}{2} \right\}.
\]
By the same argument of approximating equation (3.7) in last subsection, after a carefully calculation, we obtain that
\[
\frac{d^2\phi}{d\eta^2} = -\xi^2 \tilde{F}(\xi, \eta) \phi,
\]
where
\[
\tilde{F}(\xi, \eta) = \frac{1}{64} \left(1 - \frac{1}{\eta^2}\right)^2 + \frac{1}{\xi} \left[ -\frac{1}{4\eta^2} \left( r^2 - \frac{i}{\eta} \right) + \frac{i}{8\eta} \left( 1 - \frac{1}{\eta^2} \right) \left( \frac{1}{2} - \frac{1}{b\eta - 1} \right) \right] + F_2(\eta)O\left(\frac{1}{\xi^2}\right),
\]
with \( b = i \tan S \), and \( F_2(\eta) = 1 + \frac{1}{\eta^2} + \frac{1}{\eta} \). We note that there are two coalescing turning points near 1, and two close to \(-1\). Here we will only concern with the two turning points, say \( \hat{\eta}_1 \) and \( \hat{\eta}_2 \), near 1. When \( \hat{\eta}_j \) approach to 1, by using (3.14), we get the asymptotic formulas for the two turning points
\[
\hat{\eta}_j^{-1} = 1 \pm 2\xi^{-1/2}\sqrt{r^2 - i(1 + o(1))}, \quad j = 1, 2,
\]
which coalescing to 1 when \( \xi \to \infty \), and the Stokes’ curves defined by \( \Im(\xi(\eta + \frac{1}{\eta})) = 0 \). Assuming that \( \xi \in \mathbb{R}^+ \), then, the Stokes lines of the solution \( \phi \) to (3.56) are the positive and the negative real lines in the \( \eta \) plane. Similar to (3.21) and (3.22), if we define \( \hat{\alpha} \) and \( \hat{\theta}(\eta) \) by
\[
\frac{1}{2} \pi i\hat{\alpha}^2 = \int_{-\hat{\alpha}}^{\hat{\alpha}} (\tau^2 - \alpha^2)^{1/2} d\tau = \int_{\hat{\eta}_1}^{\hat{\eta}_2} F^{1/2}(\xi, s) ds,
\]
and
\[
\int_{\hat{\alpha}}^{\hat{\theta}} (\tau^2 - \alpha^2)^{1/2} d\tau = \int_{\eta_2}^{\eta} F^{1/2}(\xi, s) ds,
\]
respectively, then we have the following lemma which is similar to Lemma 3.
Lemma 6. There exist constants \( \hat{c}_1, \hat{c}_2 \) such that

\[
\phi(\eta, \xi) = \left( \frac{\vartheta^2 - \alpha^2}{F(\xi, \eta)} \right)^{\frac{1}{4}} \left\{ [\hat{c}_1 + o(1)] D_{\nu}(e^{\pi i/4} \sqrt{2\xi \vartheta}) + [\hat{c}_2 + o(1)] D_{-\nu-1}(e^{-\pi i/4} \sqrt{2\xi \vartheta}) \right\}, \tag{3.61}
\]

as \( \xi \to +\infty \) uniformly for \( \eta \) on the Stokes curves, where \( D_{\nu}(z) \) and \( D_{-\nu-1}(z) \) are solutions of the parabolic cylinder equation and \( \nu \) defined by \( \nu = -\frac{1}{2} + \frac{1}{2} i \xi \hat{\alpha}^2 \).

Moreover, an argument similar to the one used in Lemma 4 obtain the asymptotic behaviors of \( \vartheta(\eta) \) as \( \eta \to +\infty \) and \( \eta \to 0 \) for large \( \xi \), we state those results as follows without proof.

Lemma 7. For large \( \xi \) and \( \eta \),

\[
\frac{1}{2} \partial^2 = \frac{\alpha^2}{2} \ln \vartheta + \frac{1}{8} (\eta + \frac{1}{\eta}) - \frac{1}{4} + \frac{i}{4\xi} \ln \eta + \frac{i}{2\xi} \ln(1 - b^{-1}) - \frac{i}{\xi} \ln 2 + o(\xi^{-1}), \tag{3.62}
\]

where \( b = i \tan S \), and

\[
\hat{\alpha}^2 = \frac{r^2 - i}{\xi} + o\left(\frac{1}{\xi}\right) \quad \text{as} \quad \xi \to \infty. \tag{3.63}
\]

Remark 2. By using of the definition of \( \nu \) in Lemma 6 and (3.63), we get the approximate value

\[
\nu = \frac{ir^2}{2} + o(1), \quad \text{as} \quad \xi \to +\infty \tag{3.64}
\]

for the order of the parabolic cylinder function \( D_{\nu}(e^{\pi i/4} \sqrt{2\xi \vartheta}) \) in (3.61).

Lemma 8. When \( \eta \to 0 \), for large \( \xi \), such \( \xi \eta = o(1) \), then holds

\[
\frac{1}{2} \partial^2 = \frac{\vartheta^2}{2} \ln \vartheta + \frac{1}{8} (\eta + \frac{1}{\eta}) - \frac{1}{4} - \frac{i}{4\xi} \ln \eta + \frac{i}{2\xi} \ln(1 - b^{-1}) - \frac{i}{\xi} \ln 2 - \frac{1}{2} \pi i \alpha^2 + o(\xi^{-1}). \tag{3.65}
\]

The proofs of those lemmas are analogous to that in Lemma 4 and will not be include here. Now we are in a position to prove the Lemma 2.

Proof of Lemma 2. Based on the Lemma 6, 7 and 8 by suitable modification to the proof of Lemma 1, we can show that the entry \( (2, 1) \) of the connection matrix \( Q \) as \( \xi \to +\infty \) for \( a < \frac{1}{8} \) have the asymptotic behavior

\[
(Q)_{21} = \frac{ir^2 - \frac{4}{3} \sqrt{\pi} e^{-\frac{2\nu}{3}}}{\Gamma(\nu + 1)} \exp \left( -iS + i\frac{1}{2} \xi + \nu \ln \xi + \nu \ln 2 \right) (1 + O(\xi^{-1})). \tag{3.66}
\]

Substituting (3.64) into (3.66) and denoting \( \beta = r^2 \), then we obtain (2.2). This completes the proof of Lemma 2.
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A Appendix A. Proof of Lemma 4

The idea to prove Lemma 4 is to compute the asymptotic behavior of the integrals on the two hand sides of (3.22). Straightforward integration on the left-hand side of (3.22) yields

\[ \int_{-\alpha}^{\alpha} (\tau^2 - \alpha^2)^{1/2} d\tau = \frac{1}{2} \left\{ \zeta (\zeta^2 - \alpha^2)^{1/2} - \alpha^2 \ln(\zeta + (\zeta^2 - \alpha^2)^{1/2}) + \alpha^2 \ln \alpha \right\}, \quad (A.1) \]

Here, the cut for the integrand is again the line segment joining \(-\alpha\) and \(\alpha\), and again we take the integration path along the upper edge of the cut. Because we are going to calculate the higher-order part of the both two sides of (3.22), we will simply ignore the lower-order part in two hand sides, then we obtain that from (A.1) for large \(\zeta\)

\[ \frac{1}{2} \zeta^2 - \frac{1}{2} \alpha^2 \ln(2\zeta) - \frac{1}{4} \alpha^2 + \frac{1}{2} \alpha^2 \ln(\alpha) + O(\alpha^4 \zeta^{-2}) = \int_{\eta_2}^{\eta} F^{1/2}(\xi, s) ds. \quad (A.2) \]

To calculate the right-hand side of (A.2), we split right-hand side into two integrals respectively

\[ \int_{\eta_2}^{\eta} F^{1/2}(\xi, s) ds = \left( \int_{\eta_2}^{\eta^*} + \int_{\eta^*}^{\eta} \right) F^{1/2}(\xi, s) ds := I_1 + I_2 \quad (A.3) \]

where

\[ \eta^* = 1 + 2T \xi^{-1/2}, \quad (A.4) \]

and \(T\) is a large parameter to be specified more precisely later. In \(I_1\) we taking the change

\[ s = 1 + 2t \xi^{-1/2}, \]

and ignore \(F_2\), then \(I_1\) can be evaluated for large \(\xi\) as follows:

\[ I_1 = \frac{T^2}{2\xi} + \frac{k^2}{4\xi} + \frac{k^2}{2\xi} \ln(2T) - \frac{k^2}{4\xi} \ln(-k^2) + o(\xi^{-1}). \quad (A.5) \]

Taking \(T = -\sqrt{-k^2}\) in \(I_1\), we obtain (3.26).
When $|\eta| \to \infty$, from it follows (3.17) that $F_2 = O(\eta^{-2})$, thus, we can ignore $F_2$ in the integral $I_2$ in (A.3), by the binomial expansion, then $I_2$ is given by

$$I_2 \approx \int_\eta^1 \frac{1}{8} (1 - \frac{1}{s^2})ds + \frac{4}{\xi} \int_\eta^1 \frac{F_1(\xi, s)}{1 - \frac{1}{s^2}}ds$$

(A.6)

with error term $o(\xi^{-1})$.

Since

$$\frac{4F_1(\xi, s)}{1 - \frac{1}{s^2}} = \frac{k^2/2}{s - 1} - \frac{k^2/2}{s + 1} - \frac{i/4}{s} + \frac{ib/2}{bs - 1},$$

(A.7)

then the second term in (A.6) is equate to

$$\frac{4}{\xi} \int_\eta^1 \frac{F_1(\xi, s)}{1 - \frac{1}{s^2}}ds = \frac{1}{\xi} \left\{ \frac{i}{4} \ln \eta - \frac{k^2}{2} \ln(2T\xi^{-\frac{3}{2}}) + \frac{k^2}{2} \ln 2 - \frac{i}{2} \ln(1 - b^{-1}) + O(\eta^{-1}) + O(T\xi^{-\frac{3}{2}}) \right\}$$

(A.8)

whilst the first term in $I_2$ is equate to

$$\int_\eta^1 \frac{1}{8} (1 - \frac{1}{s^2})ds = \frac{1}{8}(\eta + \frac{1}{\eta}) - \frac{1}{4} - \frac{T^2}{2\xi} + O(T^3\xi^{-\frac{3}{2}})$$

(A.9)

Combining (A.3), (A.5), (A.6), (A.8) and (A.9) yields

$$\int_\eta^1 F^{1/2}(\xi, s)ds = \frac{1}{8}(\eta + \frac{1}{\eta}) - \frac{k^2}{4\xi} + \frac{i}{4\xi} \ln \eta - \frac{i}{2\xi} \ln(1 - b^{-1}) + \frac{k^2}{2\xi} \ln 2 - \frac{k^2}{2\xi} \ln(\sqrt{-\frac{k^2}{\xi^{1/2}}} + O(T^3\xi^{-\frac{3}{2}}) + O(T\xi^{-2}) + o(\xi^{-1})$$

(A.10)

and so, choosing $T < \xi^{1/2}$ and using (3.26) and (A.2), we obtain (3.25), which completes the proof of Lemma 4.

**B Appendix B. Proof of Lemma 5**

The idea to prove Lemma 5 is to compute the asymptotic behavior of the integral on the right hand side integral in (A.2). When $\eta \to 0$, let $\eta_s = 1 - 2T\xi^{-\frac{3}{2}}$, where $T$ is a large parameter, and split the right hand side integral in (A.2) into two parts

$$\int_\eta^1 F^{1/2}(\xi, s)ds = \left( \int_{\eta_s}^\eta + \int_\eta^1 \right) F^{1/2}(\xi, s)ds = J_1 + J_2.$$ 

(B.1)

The integral $J_1$ can be calculated by the similar manner as in computing $I_1$ (A.5), it follows that

$$J_1 = \frac{T^2}{2\xi} + \frac{k^2}{4\xi} + \frac{k^2}{2\xi} \ln(-2T) - \frac{k^2}{4\xi} \ln(-k^2) + O(\xi^{-1}T^{-2}) + O(T\xi^{-2}).$$

(B.2)
For the second integral $J_2$, according to the expressions of $F_1$ and $F_2$ in (3.16) and (3.17), respectively, we have $F_1 \sim \eta^{-3}$, $F_2(\xi, \eta) \sim \eta^{-3}$ as $\eta \to 0$. Thus we have

$$J_2 = \int_{\eta}^{\eta_2} \left[ \frac{1}{64} \left( 1 - \frac{1}{s^2} \right)^2 + \frac{1}{\xi} F_1(\xi, s) + s^{-3}O(\xi^{-2}) \right]^{\frac{1}{2}} ds$$

$$= \int_{\eta}^{\eta} \left( \frac{1}{8} \left( 1 - \frac{1}{s^2} \right) ds + \frac{4}{\xi} \int_{\eta}^{\eta} \frac{F_1(\xi, s)}{1 - \frac{1}{s^2}} ds + O(\xi^{-2} \ln \eta) + O(\xi^{-1} T^{-2}) + O(\xi^{-3/2} T^{-1}) \right)$$

$$= \frac{1}{8} \left( \frac{1}{\eta} + 1 \right) - \frac{1}{4} - \frac{i}{4\xi} \ln \eta - \frac{i}{2\xi} \ln(1 - b) + \frac{k^2}{2\xi} \ln 2 - \frac{k^2}{2\xi} \ln(2T\xi^{1/2})$$

$$- \frac{T^2}{2\xi} + O(T^3\xi^{-\frac{3}{2}}) + O(T^{-2}) + o(\xi^{-1}) \quad \text{(B.3)}$$

Combining (B.1), (B.2) and (B.3), and choosing $T < \xi^{\frac{1}{8}}$, we have

$$\int_{\eta}^{\eta_2} F_{1/2}(\xi, s) ds = \frac{1}{8} \left( \frac{1}{\eta} + 1 \right) - \frac{1}{4} + \frac{k^2}{4\xi} - \frac{i}{4\xi} \ln \eta - \frac{i}{2\xi} \ln(1 - b) + \frac{k^2}{2\xi} \ln 2$$

$$- \frac{1}{2} \pi i a^2 - \frac{k^2}{4\xi} \ln \left( -\frac{k^2}{\xi} \right) + o(\xi^{-1}). \quad \text{(B.4)}$$

Substituting this into (A.2) yields (3.28). This completes the proof. \[ \square \]

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