Abstract—We propose a design strategy for optimizing antenna positions in linear arrays for far-field Direction of Arrival (DoA) estimation of narrow-band sources in collocated MIMO radar. Our methodology allows to consider any spatial constraints and number of antennas, using as optimization function the Weiss-Weinstein bound formulated for an observation model with random target phase and known SNR, over a pre-determined Field-of-View (FoV). Optimized arrays are calculated for the typical case of a 77GHz MIMO radar of 3Tx and 4Rx channels. Simulations demonstrate a performance improvement compared to the corresponding uniform and minimum redundancy arrays for a wide regime of SNR values.

I. INTRODUCTION

Array design for DoA estimation in automotive MIMO radar demands approaches that constrain the number of antennas and the space available, aimed at improving performance while keeping low complexity and costs. Of particular interest for next generation chip design is the optimization of antenna positions of non-uniform linear arrays and the associated performance of high-resolution DoA estimation algorithms. The array factor motivated pioneering work on non-uniform array design and array current synthesis (beamforming) through inversion techniques and dynamic programming, mainly using metrics such as Mainlobe Width (MLW) and Sidelobe Levels (SSL). The problem of element spacing synthesis has been acknowledged to be a harder optimization problem than current/weight synthesis; indeed, it requires global optimization, like particle swarm optimization, which implies that the SNR follows a Rayleigh distribution and thus an excessive emphasis on low SNR values is made, limiting the design choices regarding the SNR of interest. The work considers target signals distributed as complex Gaussians with zero mean, which implies that the SNR follows a Rayleigh distribution and thus an excessive emphasis on low SNR values is made, limiting the design choices regarding the SNR of interest. The work includes several explicit WWB derivations for DoA estimation, particularly under the WWB and related bounds (called unconditional model) and also for the case where both the SNR and the phase are assumed known. In this work, we consider an alternative model that selects a specific SNR value of interest, and still takes into account the random nature of the target signal phase. Then, we provide an analytical WWB for this model as a function of the so-called test points for the target phase and the DoA for a given FoV. We then validate and compare the performance of these arrays using a standard open-source sparse reconstruction algorithm for angular estimation.

Wichmann arrays as a function of desired aperture, but does not allow to constraint simultaneously the number of elements. Bayesian approaches have been considered more recently using information-theoretic metrics, like the mutual information between measurements and the source angle in terms of array positions, or bounds on the Bayesian Mean-Squared Error (BMSE), which are an indicator of the achievable performance of any estimator and thus quantify the information that can be extracted from the scene for a candidate non-uniform array. The work optimizes the Bayesian and Expected Cramér-Rao bounds (BCRB, ECRB) and the Expected Fisher Information Matrix, which are related to the MLW and are better suited for design at high SNR where large MSE errors due to sidelobe ambiguity are not predominant. Other bounds, like the Weiss-Weinstein bound (WWB) and the Ziv-Zakai bound (ZZB) take into account large estimation errors which occur frequently below a certain SNR due to array sidelobes (“threshold effect”). These large errors are underestimated by the BCRB. On the other hand, a widely used modeling assumption for the construction of the WWB and related bounds (called unconditional model) considers target signals distributed as complex Gaussians with zero mean, which implies that the SNR follows a Rayleigh distribution and thus an excessive emphasis on low SNR values is made, limiting the design choices regarding the SNR of interest. The work includes several explicit WWB derivations for DoA estimation, particularly under the aforementioned unconditional model and also for the case where both the SNR and the phase are assumed known.

In this work, we consider an alternative model that selects a specific SNR value of interest, and still takes into account the random nature of the target signal phase. Then, we provide an analytical WWB for this model as a function of the so-called test points for the target phase and the DoA for a given FoV. We then validate and compare the performance of these arrays using a standard open-source sparse reconstruction algorithm for angular estimation.

II. SPARSE ARRAY DESIGN

This section introduces the observation model for DoA estimation, followed by the optimization metric for array design and the constraints, and finally describes the optimization strategy and presents some examples of optimized 3x4 arrays.
A. Signal model for DoA estimation

Our model for 1-snapshot DoA estimation of a single far-field source for a collocated MIMO linear array is given by

\[ y = se^{ik(d^T + d^R)}u + w. \]  

(1)

Here, the transmitter and receiver locations are denoted by \( d^T := [d^T_1, \ldots, d^T_m] \in \mathbb{R}^m \) and \( d^R := [d^R_1, \ldots, d^R_n] \in \mathbb{R}^n \) (with respect to some reference point), and the positions of the virtual elements \( d^{\text{virt}} \in \mathbb{R}^N, N = mn \), are given by

\[ d^{\text{virt}} = d^T \oplus d^R := [d^T_1 + d^R_1, d^T_1 + d^R_2, \ldots, d^T_m + d^R_n]. \]  

(2)

We also define \( k := \frac{2\pi}{\lambda} \) as the wavenumber; \( u = \sin(\phi) \in [-1, 1] \) as the parameter of interest, where \( \phi \) is the angle (DoA) with respect to boresight, cf. Fig. 1, \( s = |s|e^{i\varphi} \in \mathbb{C} \) represents the target signal; and \( w \sim \mathcal{N}(0, \sigma^2I_N) \) denotes white measurement noise with variance \( \sigma^2 \).

B. Design cost function

We wish to design arrays optimized for angle estimation in a FoV \([u_1, u_2] \subset [-1, 1]\) by identifying transmitter and receiver positions \( d^T \) and \( d^R \) that minimize the Bayesian Mean-Squared Error (BMSE) of any estimator \( \hat{u} \equiv \hat{u}(y) \),

\[ \text{BMSE}(\hat{u}) = \frac{1}{\Delta u} \int_{u \in [u_1, u_2]} \int_{y \in \mathbb{C}} (\hat{u}(y) - u)^2 p(y|u)du, \]

where \( p(y|u) \) is the likelihood of the measurement given \( u \), and we have specified the prior belief of \( u \) as uniformly distributed over the FoV length \( \Delta u := u_2 - u_1 \). The BMSE is in general computationally expensive, so we follow the common practice of replacing it by a tight lower bound whose computation is more convenient. To this end, we use the WWB for model (1) where the target phase \( \varphi \) is uniformly distributed over \([0, 2\pi]\), and \( |s| \) is assumed deterministic and known.\(^{[1]} \) This allows to select the regime of SNR values of interest. The expression of the WWB for this model is derived in the Appendix (see also \((2)\)). Define the SNR as \( c := |s|^2/\sigma^2 \), and the array factor scaled by the number of antennas as \( B(h_u) := \frac{1}{N} \sum_{n=1}^{N} e^{ikd_nh_u} \) for \( d = d^{\text{virt}} \) in (2). Then the family of bounds WWB \( (h_u, h_v) \), parametrized by the so-called test points \((h_u, h_v) \in [0, \Delta u] \times [-2\pi, 2\pi] \) are given in (3). We propose the following cost function for array optimization for a given FoV and SNR,

\[ f_{\Delta u}(d^T, d^R) := \sup_{h_u \in [10^{-4}, \Delta u], h_v \in [-2\pi, 2\pi]} \text{WWB}(h_u, h_v), \]

(4)

which requires, for the evaluation of each candidate array, a global optimization problem of its own to produce the associated tightest bound of the BMSE, \( f_{\Delta u}(d^T, d^R) \leq \text{BMSE}(\hat{u}) \). With this metric, sidelobes that produce large errors, i.e., far from the mainlobe, are penalized more, while in practice, estimated targets outside a window of interest should be considered as false alarms regardless of the error. We compensate this effect by averaging the above cost function over a sequence of FoV lengths, \( S_{\Delta u} = \{\Delta u_1 \leq \cdots \leq \Delta u_l = \Delta u_{\text{desired}}\} \).

C. Design procedure

Given a number of Tx and Rx elements, \( m, n \), that can be placed in the available space \([L^T, U^T]\) and \([L^R, U^R]\), respectively, the optimal locations in our design methodology are given by the solution to the following problem:

\[
\min_{d^T \in \mathbb{R}^m, d^R \in \mathbb{R}^n} \sum_{\Delta u \in S_{\Delta u}} \frac{1}{\Delta u^2} f_{\Delta u}(d^T, d^R) \quad \text{s.t.} \quad d^T_i > d^T_{i+1} > s^T, \quad i \in \{1, \ldots, m-1\} \quad \text{(5a)}
\]
\[
d^R_i > d^R_{i+1} > s^R, \quad i \in \{1, \ldots, n-1\} \quad \text{(5b)}
\]
\[
d^T_i > L^T, \quad d^T_m < U^T \quad \text{(5c)}
\]
\[
d^R_i > L^R, \quad d^R_m < U^R \quad \text{(5d)}
\]

where \( s^T, s^R \) codify the minimum separation between transmitters and receivers to avoid electromagnetic coupling and due to component size that depends on desired antenna power.
the inner optimization, and addition of constraints in the outer optimization. On the SNR.

In this section we compare the DoA estimation performance of the optimized arrays with 3x4 arrays that are commonly considered for the given spatial constraints, cf. Fig. 2. For the comparison, we use a sparse reconstruction algorithm called FOCUSS [27], [28]. Briefly, sparse reconstruction refers to the problem of solving the under-determined system $y = D x$, where $D \in \mathbb{C}^{N \times k}$ has more columns than rows, $k > N$, under the assumption that $x$ is sparse. In the application to DoA estimation, $y \in \mathbb{C}^{N}$ is the 1-snapshot measurement as in [1], and $D$ is called a dictionary, whose columns are the evaluations of the model (1) under the ideal noiseless case and with initial phase $\varphi = 0$, over a grid of hypotheses \{u$_1$, ..., u$_k$\} within a FoV of interest. The solution, or sparse reconstruction $x \in \mathbb{C}^k$, is expected to have nonzeros in the entries corresponding to the true DoA hypotheses. Each entry $i \in \{1, ..., k\}$ of $x$ with a magnitude exceeding a threshold $|x_i| > \gamma$ constitutes a declared target, or declaration, with DoA estimate $u_i$ associated to the corresponding column of the dictionary $D$. We use $k = 300$ in the FoV $\pm 30^\circ$ and a threshold policy optimized for each array using a training set of simulated scenarios in terms of the metrics of interest.

In the array evaluation, we use the following metrics: the Probability of Detection ($P_D$), the False Alarm Rate (FAR), the Probability of Resolution ($P_R$), and the Root MSE (RMSE) of detections. The $P_D$, FAR, and $P_R$ are defined for each Monte Carlo realization, or trial, and are then averaged. In each trial we consider a detection window, $DTW = [−3^\circ, 3^\circ]$, around the realization of the ground-truth targets. If a declaration of the estimation algorithm falls inside a detection window, the corresponding target is said to be detected, otherwise, the declaration is called a false alarm. The $P_D$ is defined, in each trial, as the quotient of the number of detections, divided by the total number of targets. The FAR is consequently defined as the proportion of false alarms, i.e., the quotient of the number of declarations outside of any DTW around true targets, divided by the total number of declarations. In the case

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Based on Matlab code by Héctor Corte, available in MathWorks File Exchange, with some modifications for evaluation of vectorized functions in the inner optimization, and addition of constraints in the outer optimization.

Footnotes:
1. Code available online by Zhilin Zhang and David Wipf.
2. Different than the notion of False Alarm Rate that considers the quotient over the number of grid points.
In Fig. 4 we show that for a 3x4 array and a FoV of Resolution of two targets for a reduced FAR. Future work includes generalizations to two-dimensional arrays, configurations with more antennas, and empirical analysis of resolution limits for two targets with different RCS.

IV. CONCLUSIONS AND FUTURE WORK

This work introduces a strategy to design optimal linear arrays for an arbitrary number of Tx and Rx antennas and desired spatial constraints. The WWB-based objective function includes as design choices the FoV and SNR values of interest, defining a trade-off between MLW and SLL based on a prediction of Bayesian MSE. This is in contrast with metrics that control directly the MLW and SLL. Resulting arrays, optimized for an average of FoVs, make more efficient use of the available space, and show an enhanced performance compared to conventional arrays, particularly the Probability of Resolution of two targets for a reduced FAR. Future work includes generalizations to two-dimensional arrays, configurations with more antennas, and empirical analysis of resolution limits for two targets with different RCS.

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APPENDIX

First we introduce some preliminaries on the WWB, then discuss various DoA observation models, and finally construct the WWB used in this work for array design.

A. Preliminaries on the WWB

Following [15], chapter (4.4.1.4), the family of Weiss-Weinstein-Bounds (WWB) for an observation \( y \in \mathbb{C}^N \) and a vector parameter \( \theta \in \mathbb{R}^q \), with joint probability density \( p(y, \theta) \), is parametrized by the choice of test point \( H := [h_1, ..., h_M] \in \mathbb{R}^{q \times M} \) and obtained as

\[
WWB(H) = HQ^{-1}H^T
\]

with the elements of the matrix \( Q \in \mathbb{R}^{M \times M} \) given by

\[
Q_{ij} := E[f_{y,\theta}(h_i) - f_{y,\theta}(h_i)(h_j - h_j^H)f_{y,\theta}(h_j^H) - f_{y,\theta}(h_j^H)]
\]

where the expectations are computed with the joint density \( p(y, \theta) \), and we define \( f_{y,\theta}(h) \rightarrow \mathbb{R}^q \rightarrow \mathbb{R}_{\geq 0} \) as

\[
f_{y,\theta}(h) = f(y; \theta + h, \theta) := \left( \frac{p(y; \theta + h)}{p(y; \theta)} \right)^{1/2}.
\]

All test points are valid as long as they are selected so that \( Q^{-1} \) exists, which restricts the domain as follows,

\[
\{ H := [h_1, ..., h_M] \in \mathbb{R}^{q \times M} : \Theta \cap (\Theta + h_m) \neq \emptyset, \forall m \},
\]

where \( \Theta := \supp(p_\theta) := \{ \theta \in \mathbb{R}^q : p_\theta(\theta) > 0 \} \) denotes the support of the belief distribution on \( \theta \). The WWB so defined lower bounds the Bayesian Mean Squared Error (BMSE) matrix of any estimator \( \hat{\theta}(y) \),

\[
\Sigma := E[(\hat{\theta}(y) - \theta)(\hat{\theta}(y) - \theta)^H] \geq WWB(H),
\]

in the sense of the Loewner-order (for Hermitian matrices \( A \succeq B \) means that \( A - B \succeq 0 \), which is the notation for \( A - B \) being positive-semidefinite), i.e.,

\[
v^H \Sigma v \geq v^H WWB(H)v
\]

for all vectors \( v \in \mathbb{C}^q \). In particular, if we are interested only in the estimation performance of a certain element of the parameter vector, e.g. \( u = \theta_1 \in \mathbb{R} \), we select \( v = e_1 \) (the first element of the canonical basis in \( \mathbb{R}^q \)), and find a bound for the performance of \( \hat{u}(y) := \hat{\theta}(y)_1 \) in terms of the corresponding entry of the matrix WWB(H),

\[
E[(\hat{u}(y) - u)^2] \geq WWB(H)_{11}.
\]

Next we discuss some observation models for DoA estimation and their consequences, which are captured by the WWB associated to the corresponding joint probability density \( p(y, \theta) \).
B. Conditional and unconditional models for DoA estimation

Here we describe some modeling options of DoA estimation that can leverage the construction of the WWB presented in [21]. We start with the general observation model

\[ y = A(\theta)s + w, \quad (11) \]

where \( A(\theta) \in \mathbb{C}^N \) is a function of an unknown stochastic parameter vector \( \theta \in \mathbb{R}^q \), and \( s \in \mathbb{C} \) can be considered deterministic or stochastic. Here \( w \sim \mathcal{N}_C(0, \sigma^2 I_N) \) denotes white measurement noise with variance \( \sigma^2 \) as in (1). The conditional model in [21] assumes that \( s \in \mathbb{C} \) is deterministic and known, and allows, e.g. to identify \( s \) with the transmitted waveform of the radar, which is of particular relevance for MIMO radar waveform design [29], [30]. Another usage (which we refer to as naive) of the conditional model for DoA estimation assumes that \( s = |s|e^{i\phi} \) is the target signal, in this case deterministic and known, while the steering matrix \( A(\theta) \equiv A(u) = e^{iu} \) (for antenna positions \( d \in \mathbb{R}^N \) in units of \( l/k = \lambda/(2\pi) \)) depends only on the random DoA \( u \). One inconvenience we encountered using this model is the dependence of optimized arrays on the choice of the array reference point used to define \( d \), which we suspect to be caused especially by the modeling choice of known target phase \( \phi \). (In fact, the associated WWB depends in this case on the array factor through the real part \( A \).

In contrast, the unconditional model in [21], employed for DoA estimation, also identifies \( s \in \mathbb{C} \) in (11) with the target signal, but it assumes it obeys a zero-mean complex Gaussian distribution \( s \sim \mathcal{N}_C(0, \sigma_s^2) \). Stated equivalently, the target phase follows a uniform distribution \( \phi \sim U(-\pi, \pi) \), while the target magnitude \( |s| \) is Rayleigh distributed (with scale parameter \( \sigma_s/\sqrt{2} \)). Although this model seems appealing with regard to the phase, the Rayleigh distribution can make an undesired emphasis on target signals with low SNR values. As a consequence, arrays like the uniform have a competitive advantage for values of \( \sigma_s \) that are required to model a moderately high SNR regime in some applications where, in fact, sparse arrays are desirable. This argues against the use of the unconditional model to design arrays for regimes of SNR that are relevant after range-Doppler processing for DoA estimation.

C. Construction of the WWB with random initial phase

As a compromise between the unconditional and (naive) conditional model, we use a measurement model which assumes the target signal magnitude \( |s| \) as deterministic and known, but regards the signal phase \( \phi \) as an uniformly distributed random variable in \((0, 2\pi)\). Denoting \( \theta = [u, \phi]^T \), our measurement model in (1), can be written as

\[ y = A(\theta)|s| + w, \]

with \( A(\theta) := e^{i\phi} e^{iu} \in \mathbb{C}^N \), and prior distribution

\[ p_\theta(\theta) = \frac{1}{2\pi \Delta u} 1_{(0,\Delta u)}(u) 1_{(0,2\pi)}(\phi), \quad (12) \]

where \( 1_{(a,b)} \) is the indicator function, which is 1 in \((a, b)\) and 0 otherwise, and we recall that \( \Delta u = u_2 - u_1 \) quantifies the radar’s FoV. The measurement likelihood function is therefore
given by the Gaussian density\(^7\)
\[
p(y|\theta) = \frac{1}{\pi^N \sigma^{2N}} \exp\left(-\frac{\|y - A(\theta)\|_2^2}{\sigma^2}\right),
\]
and the joint density needed for evaluation of the WWB can be obtained as \(p(y, \theta) = p(y|\theta)p_\theta(\theta)\). Performing the calculations demanded by \(6\) in an analogue fashion to the treatment of the conditional model in \(21\), we obtain\(^8\)
\[
Q_{ij} = \eta(h^i, h^j) + \eta(-h^i, -h^j) - \eta(h^i, -h^j) - \eta(-h^i, h^j)
\]
where \(h^i = [h^i_\text{u}, h^i_\varphi]^\top \in \mathbb{R}^2\) is a column test point, and we employ the following shorthand notations,
\[
\eta(\mu, \rho) := E_{p(y, \theta)}[f_{y, \theta}(\mu)f_{y, \theta}(\rho)] = \hat{\eta}_\theta(\mu, \rho, |\tilde{\Theta}(\mu, \rho)| \frac{\sigma}{2\pi \Delta u}, \eta(\mu, \rho) := \exp(-\frac{c N}{2} [1 - \Re\{e^{i(\rho \varphi - \mu \varphi)} B(\rho - \mu)\}]), \]
\[
B(h_u) = \langle 1, e^{i d h_u} \rangle = \frac{1}{N} \sum_{n=1}^N e^{i d h_u}, \]
where we recall that \(c := |s|^2/\sigma^2 \in \mathbb{R}\) is the SNR, \(d \in \mathbb{R}^N\) codifies the antenna positions in units of \(1/k = \lambda/(2\pi)\), and
\[
|\tilde{\Theta}(\mu, \rho)| := |\Theta \cap (\Theta + \mu) \cap (\Theta + \rho)|
\]
denotes the Lebesgue volume of the parameter-shifted support intersection, with \(\Theta\) defined in \(9\). (Note that requirement \(4\) ensures that \(\eta(0, h_m)\) and \(\eta(h_m, 0)\) are nonzero.)

In this work we use a single-column test point \(H = h = [h_u, h_\varphi]^\top \in \mathbb{R}^{2 \times 1}\) for the WWB formulation in \(6\).\(^9\) WWB\((h) = \frac{1}{\sqrt{2\pi}} hh^\top\). With this choice, the final expression for the DoA component of the WWB as described in \(10\) can be computed as WWB\((h_u, h_\varphi)_{11} = \frac{h_\varphi^2}{\sigma^2}\), yielding the expression \(3\). To obtain a tight bound, we optimize the WWB \(3\) over test-point values\(^{10}\). According to the condition in \(9\), the optimization is performed over
\[
[h_u, h_\varphi] \in (0, \Delta u) \times (-2\pi, 2\pi),
\]
where we have restricted the set according to a symmetry relation that can be derived from \(3\), namely,

WWB\((h_u, h_\varphi)_{11} = \text{WWB}(h_u, -h_\varphi)_{11}\), so that optimization over \((-\Delta u, 0) \times (-2\pi, 2\pi)\) is unnecessary. The optimal test points can be found with a global optimization algorithm like simulated annealing. Global optimization is necessary because this problem is in general nonconvex, as exemplified in Fig. \(6\) where we observe that the number of local minima and the steepness of the function strongly depend on the SNR.

\(^7\)We have included the phase explicitly in the parameter vector \(\theta\) and thus obtained a multidimensional estimation problem instead of the alternative of employing likelihood functions \(p(y|u) = \int p(y|u, \varphi)p(\varphi)\mathrm{d}\varphi\) involving the modified Bessel function of order zero, for which the evaluation of the WWB appears less tractable.

\(^8\) Expand the numerator of \(Q_{ij}\) in \(7\) express the joint density in terms of the likelihood and prior to separate integrations, use some null addition trick to solve the Gaussian integral over observations \(y\), factor out the resulting expression noting that is independent of \(\theta\), and employ the expression for the prior \(13\).

\(^9\) The authors are aware that this choice of test-point matrix conflicts with the condition \(M \geq q\) suggested in \(15\). This condition is unnecessary for the derivation of the covariance inequality, and is only a necessary condition for a maximum-rank bound in \(6\). We make the choice of rank-1 bound for the sake of computational efficiency and because it seems satisfactory for the purpose of having a tight lower bound in \(10\).

\(^{10}\) Other approaches exist: a common alternative chooses a fixed “dense set” of test points related to the peaks of the array factor, as in \(29\) for the Bobrovsky-Zakai bound.