Different canonical formulations of Einstein’s theory of gravity

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Abstract

We describe the four most famous versions of the classical canonical formalism in the Einstein theory of gravity: the Arnowitt-Deser-Misner formalism, the Faddeev-Popov formalism, the tetrad formalism in the usual form, and the tetrad formalism in the form best suited for constructing the loop theory of gravity, which is now being developed. We present the canonical transformations relating these formalisms. The paper is written mainly for pedagogical purposes.

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1. Introduction

The most direct method for constructing a quantum theory is to quantize the corresponding classical theory written in canonical form. Different equivalent canonical formulations of the classical theory may then lead to not completely equivalent versions of the quantum theory. In complicated cases, it is therefore beneficial to use different methods to represent the classical theory in canonical form before quantization. In particular, this concerns the theory of gravity, whose final quantum form has not yet been found. It is not improbable that choosing an appropriate classical canonical formulation, we can here approach a satisfactory solution of the quantization problem. Precisely this approach underlies the so-called loop theory of gravity (see [1] and the references therein), which is currently being developed.

In this paper written mainly for pedagogical purposes, we describe several well-known equivalent classical canonical formulations of the Einstein theory of gravity and relations between these formulations. We first consider the Arnowitt-Deser-Misner (ADM) formalism [2]. We then use the canonical transformation to pass to the Faddeev-Popov (FP) formalism [3]. We next use a change of variables to introduce the frame (tetrad) formalism in the usual form. Finally, we use the canonical transformation to reduce this formalism to the form underlying the loop theory of gravity [1].

We do not consider the problem of quantizing gravity here and restrict ourself to only several remarks on this subject, but the information presented here can be useful in studying this problem.

2. The ADM formalism

First, we consider the classical ADM formalism [2]. Let \( x^\mu \) be coordinates in the Riemannian space-time \( (\mu, \nu, \ldots = 0, 1, 2, 3) \). The coordinate \( x^0 = t \) is called time (we set \( c = 1 \), where \( c \) is the speed of light). We assume that all hypersurfaces \( x^0 = \text{const} \) are spacelike. The space coordinates are denoted by \( x^i (i, k, \ldots = 1, 2, 3) \). We use the metric signature \((-+, +, +, +)\).

We fix a hypersurface \( x^0 = \text{const} \) and let \( \Sigma \) denote it. In the coordinates \( x^i \) the three-dimensional metric induced on \( \Sigma \) coincides with the three-dimensional part of the four-dimensional metric. We let \( \beta_{ik} \) denote this three-dimensional metric and introduce \( \beta^{ik} \) by the condition

\[
\beta_{ik}\beta^{kl} = \delta^l_i.
\]

Then

\[
\beta_{ik} = g_{ik},
\]

\[
\beta^{ik} = g^{ik} - g^{0i}g^{0k} / g^{00},
\]

where \( g_{\mu\nu} \) is the four-dimensional metric and \( g^{\mu\nu} g_{\nu\lambda} = \delta^\mu_\lambda \). We introduce the notation

\[
g = \det g_{\mu\nu}, \quad \beta = \det \beta_{ik}.
\]

As usual,

\[
R^a_{\beta,\gamma\delta} = \partial_\gamma \Gamma^a_{\delta\beta} - \partial_\delta \Gamma^a_{\gamma\beta} + \Gamma^a_{\gamma\rho} \Gamma^\rho_{\delta\beta} - \Gamma^a_{\delta\rho} \Gamma^\rho_{\gamma\beta},
\]

\[
R_{\beta\delta} = R^a_{\beta,\alpha\delta},
\]
\[ R = g^{\beta\delta} R_{\beta\delta}, \tag{7} \]

where \( \Gamma^\alpha_{\beta\gamma} \) are the Christoffel symbols constructed from the metric \( g_{\alpha\mu} \) by a known method.

We determine the quantities \( \Gamma^i_{kl}, R^i_{k,lm}, R_{lm}, R \), formed from \( \beta_{ik}, \partial_l \beta_{ik}, \partial_m \partial_l \beta_{ik} \) precisely as the quantities \( \Gamma^\alpha_{\beta\gamma}, R^\alpha_{\beta\gamma\delta}, R_{\beta\delta}, R \) are constructed from \( g_{\mu\nu}, \partial_\alpha g_{\mu\nu}, \partial_\alpha \partial_\beta g_{\mu\nu} \). We introduce the covariant derivative \( \nabla_i \) acting on \( \Sigma \) by the connection \( \Gamma^i_{kl} \) just as the derivative \( \nabla_\mu \) acts on the entire space-time by the connection \( \Gamma^\alpha_{\beta\gamma} \). We determine the second fundamental tensor \( K_{ik} \) of the hypersurface \( \Sigma \):

\[ K_{ik} = K_{ki} = -\nabla_i n_k \bigg|_\Sigma = n_0 \Gamma^0_{ik} \bigg|_\Sigma = -\frac{1}{\sqrt{-g_{00}}} \Gamma^0_{ik} \bigg|_\Sigma, \tag{8} \]

where the field \( n_\mu(x) \) of unit normals to the surfaces \( x^0 = \text{const} \) is determined by the relations

\[ n_\mu n^\mu = -1, \quad n_\mu = -\delta_\mu^0 \frac{1}{\sqrt{-g_{00}}}, \quad n^\mu = -\frac{g^{0\mu}}{\sqrt{-g_{00}}}. \tag{9} \]

We have the identity

\[ \sqrt{-g} R = \sqrt{-g} \left( R + K^i_i K^l_l - (K^i_i)^2 \right) + 2 \partial_\gamma \left( \sqrt{-g} (n^\gamma \nabla_\delta n^\delta - n^\delta \nabla_\delta n^\gamma) \right), \tag{10} \]

where \( K^i_i = \beta^k_{ik} K^i_{kl} \). The simplest derivation of this identity is based on the well-known Gauss formula relating the curvature tensor of the hypersurface to the curvature tensor of the ambient Riemannian space.

We consider only the gravitational field not interacting with other fields because all specific features of the problem can be clearly seen in this case. We start from the action of the gravitational field

\[ S = \int d^4 x \mathcal{L} \tag{11} \]

where

\[ \mathcal{L} = \frac{1}{2 \kappa} \sqrt{-g} \left( g^{\alpha\beta} \left( \Gamma^\rho_{\alpha\gamma} \Gamma^\gamma_{\rho\beta} - \Gamma^\rho_{\alpha\beta} \Gamma^\gamma_{\rho\gamma} \right) - 2 \Lambda \right). \tag{12} \]

Here \( \kappa = 8 \pi \gamma \), \( \gamma \) is the Newtonian gravitational constant, and \( \Lambda \) is the cosmological constant. Otherwise,

\[ \mathcal{L} = \frac{1}{2 \kappa} \sqrt{-g} (R - 2 \Lambda) + \frac{1}{2 \kappa} \partial_\gamma \left( \sqrt{-g} \left( g^{\alpha\beta} \Gamma^\gamma_{\alpha\beta} - g^{\alpha\beta} \Gamma^\gamma_{\alpha\beta} \right) \right). \tag{13} \]

whence we use identity (10) to obtain

\[ \mathcal{L} = \frac{1}{2 \kappa} \sqrt{-g} \left( R + K^i_i K^l_l - (K^i_i)^2 - 2 \Lambda \right) + \frac{1}{2 \kappa} \partial_\gamma \left( \sqrt{-g} \left( g^{\alpha\beta} \Gamma^\gamma_{\alpha\beta} - g^{\alpha\beta} \Gamma^\gamma_{\alpha\beta} \right) + 2 \sqrt{-g} (n^\gamma \nabla_\delta n^\delta - n^\delta \nabla_\delta n^\gamma) \right). \tag{14} \]

In the case of a closed universe, we can here omit the divergence, and in the case of an island position of masses in an asymptotically three-dimensionally flat space-time, it suffices to only take into account the essential part of the divergence equal to

\[ \frac{1}{2 \kappa} \left( \partial_k \partial_l \beta_{kl} - \partial_k \partial_l \beta_{kl} \right). \tag{15} \]
In the last case \( \Lambda = 0 \). We often omit the divergence and assume that the universe is closed for simplicity; we also often omit the \( \Lambda \) term.

We choose the quantities

\[
\beta_{ik} \equiv g_{ik}, \quad N \equiv \frac{1}{\sqrt{-g^{00}}}, \quad N_i = g_{0i}.
\]

as independent ADM field variables. In what follows, the subscripts \( i, k, \ldots \) are raised and lowered by the three-dimensional tensors \( \beta^{ik} \) and \( \beta_{ik} \). The following relations hold:

\[
g_{ik} = \beta_{ik}, \quad g^{ik} = \beta^{ik} - \frac{N^i N^k}{N^2}, \quad g_{0k} = N_k, \quad g^{0k} = \frac{N^k}{N^2},
\]

\[
g_{00} = -N^2 + N_k N^k, \quad g^{00} = -\frac{1}{N^2}, \quad \sqrt{-g} = N \sqrt{\beta},
\]

\[
n_\mu = -\delta^0_\mu N, \quad n^0 = \frac{1}{N}, \quad n_i = -\frac{N_i}{N},
\]

\[
K_{ik} = \frac{1}{2N} \left( \nabla^3_i N_k + \nabla^3_k N_i - \partial_0 \beta_{ik} \right).
\]

In these variables, Lagrangian density (14) with the divergence omitted becomes

\[
\mathcal{L}^{(ADM)} = N \left\{ 2\tilde{\mathcal{J}}^{ij,kl} K_{ij} K_{kl} + \frac{\sqrt{\beta}}{2\kappa} \left( \nabla^3 R - 2\Lambda \right) \right\},
\]

where

\[
\tilde{\mathcal{J}}^{ij,kl} = \frac{1}{4} \left( \frac{\sqrt{\beta}}{2\kappa} \right) \left( \beta^{ik} \beta^{jl} + \beta^{il} \beta^{jk} - 2\beta^{ij} \beta^{kl} \right).
\]

It is convenient to introduce the symbols

\[
\delta^{ik}_{lm} \equiv \frac{1}{2} \left( \delta^i_l \delta^k_m + \delta^i_m \delta^k_l \right)
\]

and to determine the quantity \( \tilde{\mathcal{J}}^{ij,kl} \) using the condition

\[
\tilde{\mathcal{J}}^{ij,kl} \mathcal{J}_{kl,mn} = \delta^{ij}_{mn}.
\]

In this case, we have

\[
\tilde{\mathcal{J}}^{ij,kl} = \left( \frac{2\kappa}{\sqrt{\beta}} \right) \left( \beta_{ik} \beta_{j} + \beta_{il} \beta_{jk} - \beta_{ij} \beta_{kl} \right), \quad \mathcal{J}_{ij,kl} = \mathcal{J}_{kl,ij} = \mathcal{J}_{ji,kl} = \mathcal{J}_{ij,tk}
\]

and similar relations hold for \( \tilde{\mathcal{J}}^{ij,kl} \). We again omit the inessential part of the divergence arising in the expression for \( \mathcal{L} \).

We set

\[
L = \int_{x^0=\text{const}} d^3 x \mathcal{L}^{(ADM)}
\]

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and determine the conjugate momenta,

\[ P^{(N)}(x) \equiv \frac{\delta L}{\delta (\partial_0 N(x))}, \]  
\[ P^{(N)}(x) \equiv \frac{\delta L}{\delta (\partial_0 N_i(x))}, \]  
\[ P^{ik}(x) \equiv \frac{\delta L}{\delta (\partial_0 \beta_{ik}(x))} = \frac{\partial L^{(ADM)}}{\partial (\partial_0 \beta_{ik}(x))}, \]  

where \( x \equiv (x^1, x^2, x^3) \) and \( \delta/\delta(\cdot) \) is the three-dimensional variational derivative. We immediately obtain the primary constraints

\[ P^{(N)}(x) = 0, \quad P^{(N)}(x) = 0. \]  

We solve these constraints explicitly, i. e., we set \( P^{(N)}(x) \) and \( P^{(N)}(x) \) equal to zero everywhere as they are encountered. Next,

\[ P^{ik}(x) = \frac{\partial L^{(ADM)}}{\partial K_{lm}} \frac{\partial K_{lm}}{\partial (\partial_0 \beta_{ik})} = -2\gamma_{ik,lm} K_{lm}, \]  

hence

\[ K_{ik} = -\frac{1}{2} \gamma_{ik,lm} P_{lm}, \]  

and according to (20), we have

\[ \partial_0 \beta_{ik} = (3) \nabla_i N_k + (3) \nabla_k N_i - 2N K_{ik}. \]  

The density of the generalized Hamiltonian is equal to

\[ H^{(gen)} = P^{ik} \partial_0 \beta_{ik} - L^{(ADM)}. \]  

We again omit the inessential addition to the divergence and obtain

\[ H^{(gen)} = N H_0 + N^i H_i, \]  

where

\[ H_0 = \frac{1}{2} \gamma_{ik,lm} P^{ik} P_{lm} + \left( \frac{\sqrt{\beta}}{2 \kappa} \right) \left( -R + 2\Lambda \right), \]  
\[ H_i = -2\beta_{is} \sqrt{\beta} \nabla_i \left( \frac{P_{ls}}{\sqrt{\beta}} \right), \]  

and we take into account that \( P_{ls}/\sqrt{\beta} \) is a tensor.

The density of the first-order Lagrangian is equal to

\[ L^{(1)} = P^{ik} \partial_0 \beta_{ik} - H^{(gen)} = P^{ik} \partial_0 \beta_{ik} - N H_0 - N^i H_i. \]  

We add the essential part of the divergence and obtain the relation for the island position of masses in asymptotically three-dimensionally flat space:

\[ L^{(ADM)} = P^{ik} \partial_0 \beta_{ik} - H^{(gen)} = P^{ik} \partial_0 \beta_{ik} - N H_0 - N^i H_i - \frac{1}{2 \kappa} \left( \partial_0 \partial_k \beta_{ik} - \partial_k \partial_0 \beta_{ii} \right). \]
We vary $\mathcal{L}^{(ADM)}_{(1)}$ in $N$ and $N^i$ and obtain the secondary constraints

$$\mathcal{H}_0(x) = 0, \quad \mathcal{H}_i(x) = 0.$$  \hfill (40)

In the case of the island position of masses, the total energy reduces to the surface integral

$$H_{\text{total}} = \frac{1}{2\kappa} \int d^3x \left( \partial_i \partial_k \beta_{ik} - \partial_k \partial_k \beta_{ii} \right),$$  \hfill (41)

and we now obtain

$$\mathcal{H}^{(\text{gen})} = N\mathcal{H}_0 + N^i\mathcal{H}_i + \frac{1}{2\kappa} \left( \partial_i \partial_k \beta_{ik} - \partial_k \partial_k \beta_{ii} \right).$$  \hfill (42)

In the case of a closed universe, the total energy is zero.

We introduce the Poisson brackets. If $F_1$ and $F_2$ are two three-dimensional functionals of $\beta_{ik}$ and $P^{ik}$, then

$$\{ F_1, F_2 \} = \int d^3x \left( \frac{\delta F_1}{\delta \beta_{ik}(x)} \delta F_2 - \frac{\delta F_2}{\delta \beta_{ik}(x)} \delta F_1 \right),$$  \hfill (43)

where $\delta/\delta()$ is the three-dimensional variational derivative. Obviously,

$$\{ F_1, F_2 \} = - \{ F_2, F_1 \},$$  \hfill (44)

$$\{ F_1, \{ F_2, F_3 \} \} + \{ F_2, \{ F_3, F_1 \} \} + \{ F_3, \{ F_1, F_2 \} \} = 0,$$  \hfill (45)

$$\{ F_1, F_2 F_3 \} = \{ F_1, F_2 \} F_3 + F_2 \{ F_1, F_3 \}. $$  \hfill (46)

We next use the notation

$$f \equiv f(x), \quad \tilde{f} \equiv f(\tilde{x}).$$  \hfill (47)

In this notation, we obtain

$$\left\{ \beta_{ik}, P^{lm}_{\sim} \right\}_{\sim} = \delta_{ik}^{lm} \delta^3(x - \tilde{x}),$$  \hfill (48)

$$\left\{ \beta_{ik}, \beta_{lm}_{\sim} \right\}_{\sim} = 0, \quad \left\{ P^{ik}, P^{lm}_{\sim} \right\}_{\sim} = 0.$$  \hfill (49)

The following relations hold:

$$\left\{ \mathcal{H}_i, \mathcal{H}_k \right\}_{\sim} = \mathcal{H}_k \partial_i \delta^3(x - \tilde{x}) - \mathcal{H}_i \partial_k \delta^3(x - \tilde{x}),$$  \hfill (50)

$$\left\{ \mathcal{H}_i, \mathcal{H}_0 \right\}_{\sim} = \mathcal{H}_0 \partial_i \delta^3(x - \tilde{x}),$$  \hfill (51)

$$\left\{ \mathcal{H}_0, \mathcal{H}_0 \right\}_{\sim} = \beta^{ik} \mathcal{H}_k \partial_i \delta^3(x - \tilde{x}) - \beta^{ik} \mathcal{H}_k \partial_i \delta^3(x - \tilde{x}),$$  \hfill (52)

where $\partial_i = \frac{\partial}{\partial x^i}$. Clearly, all the constraints in the classical theory are of the first kind. No new constraints arise.
The constraints $H_i$ are generators of three-dimensional transformations of coordinates on the surface $\Sigma$. Indeed, after the change of coordinates

$$x^i \to x'^i + \xi^i(x),$$

where $\xi^i(x)$ are infinitely small, we have

$$\delta \beta_{ik} \equiv \beta'_{ik}(x) - \beta_{ik}(x) = -\nabla_i \xi_k - \nabla_k \xi_i,$$

$$\delta P^{ik} \equiv P'^{ik}(x) - P^{ik}(x) = (\partial \xi^i) P^{ik} + P^{il} \partial_k \xi^l - \partial_l (P^{ik} \xi^l).$$

It can be verified directly that

$$\left\{ \int d^3x \ H_i \xi^i, \beta_{kl} \right\} = \delta \beta_{kl},$$

$$\left\{ \int d^3x \ H_i \xi^i, P^{kl} \right\} = \delta P^{kl}.$$  

Correspondingly, the constraint $H_0$ generates displacements of points of the surface $\Sigma$ along the normal to $\Sigma$. In this case, the variations in $\beta_{ik}$ and $P^{ik}$ correspond to the solutions of the Einstein equations.

We make several remarks about quantizing the described theory. Under quantization, the variables $\beta_{ik}$ and $P^{ik}$ are replaced with operators satisfying the conditions

$$[\beta_{ik}, P^{lm}] = i \delta^{lm}_{ik} \delta^3(x - \tilde{x}),$$

$$[\beta_{ik}, \beta_{lm}] = [P^{ik}, P^{lm}] = 0.$$  

Because constraints (36) and (37) are too complicated to be solved explicitly, these constraints are usually imposed on the state vector. The theory thus obtained is consistent only under the condition that the commutators of the constraints are equal to linear combinations of these constraints with coefficients placed to the left of them. After quantization, the constraints satisfy commutation relations of form (50)-(52) with the bracket $\{ \}$ replaced with $-i[ \ ]$. But the order of the factors $\beta_{ik}$ and $P^{ik}$ chosen in the expressions for the constraints is now important. It may happen that the result of commuting the constraints contains these factors not in the order originally accepted in the constraints and the coefficients of the constraints may arise not only to the left of them. It is easy to see that this does not occur in quantum analogues of relations (50) and (51), and these relations preserve the form after quantization (up to the change $\{ \} \to -i[ \ ]$). In particular, the latter is due to the abovementioned geometric sense of the constraints $H_i$ as generators of transformations of three-dimensional coordinates. This sense is completely preserved under quantization.

The situation with the quantum analogue of relation (52) is quite different. If the operators in the constraints $H_0$ and $H_i$ are located such that these constraints are Hermitian, then the quantity $-i[H_0, H_0] \sim_0$ obtained from the quantity $\{H_0, H_0\}$ is also Hermitian. This means that the non-Hermitian expressions $\beta^{ik} H_k$ cannot appear on the right in an analogue of relation (52) (we take into account that $\beta^{ik}$ and $H_k$ do not commute). The most that can be obtained for the commutator $-i[H_0, H_0] \sim_0$ by choosing the order of the factors in $H_0$ and $H_k$ without violating
the Hermitian property is an expression of the form

\[ \frac{1}{2} (\beta^{ik} \mathcal{H}_k + \mathcal{H}_k \beta^{ki}) \partial_i \delta^3(x - \tilde{x}) - \frac{1}{2} (\beta_{\sim}^{ik} \mathcal{H}_k + \mathcal{H}_k \beta_{\sim}^{ki}) \partial_{\sim} i \delta^3(x - \tilde{x}) = \]

\[ = \beta^{ik} \mathcal{H}_k \partial_i \delta^3(x - \tilde{x}) - \beta_{\sim}^{ik} \mathcal{H}_k \partial_{\sim} i \delta^3(x - \tilde{x}) + \]

\[ + \delta^3(0) (\ldots) \partial_i \delta^3(x - \tilde{x}) - \delta^3(0) (\ldots) \partial_{\sim} i \delta^3(x - \tilde{x}), \tag{59} \]

where (\ldots) and (\ldots) are some nonzero operator-valued functions. The symbols \( \delta^3(0) \) arise from commuting the operators \( \beta^{ik} \) and \( \mathcal{H}_k \) or \( \beta_{\sim}^{ik} \) and \( \mathcal{H}_k \) taken at the same point.

Clearly, an expression of form (59) containing the product \( \delta^3(0) \partial_i \delta^3(x - \tilde{x}) \) does not make sense. A meaningful expression can be obtained from it only by regularization. This raises the question of the possibility of choosing a regularization such that the extra terms in expression (59) become zero and the general covariance of the theory is reestablished after the regularization is removed. But a unique answer to this question has not yet been obtained. In several published works, the problem of regularization and its removal was studied insufficiently rigorously. An explanation for this is that the regularization methods were studied in detail only in the framework of the perturbation theory. But the problem is posed beyond this framework here.

Although there is still a certain ambiguity in this problem, the theory of gravity was quantized by the path-integral method by analogy with quantizing non-Abelian gauge theories (see [3] and the references therein and also [4]). If a satisfactory perturbation theory were thus obtained, then its consistency could be verified directly in the framework of the Feynman diagram formalism, and this would suffice. But it turned out that the constructed perturbation theory is unrenormalizable. Under these circumstances, different approaches for constructing the quantum theory of gravity are now being developed; the most well-known approaches are superstring theory (see, e.g., [5]) and the so-called loop theory of gravity [1].

We also note that the above difficulties in closing the constraint algebra after quantization are also typical of other versions of the canonical formalism in the theory of gravity, which are described below.

### 3. The FP formalism

We now consider the classical canonical FP formalism [4]. We first introduce the quantities

\[ h^{\mu \nu} = \sqrt{-g} g^{\mu \nu}, \tag{60} \]

in terms of which the subsidiary harmonic coordinate condition can be simply written as \( \partial_{\mu} h^{\mu \nu} = 0 \). For the original variables, we take the functions

\[ q^{ik} \equiv h^{0i} h^{k0} - h^{00} h^{ik}, \tag{61} \]

and we write \( q \equiv \det q^{ik} \) in what follows. Moreover, we preserve the functions \( N \) and \( N^i \) contained in the ADM formalism.
The ADM and FP formalisms are related by the canonical transformation

\[
\begin{aligned}
q^{ik} &= \beta \beta^{ik} = \frac{1}{2} \varepsilon^{ilm} \varepsilon^{knp} \beta_{ln} \beta_{mp}, \\
\pi_{ik} &= -\frac{1}{(2\pi)2\sqrt{\beta}} \beta_{ik,lm} P^{lm} = \beta^{-1} \left( \frac{1}{2} \beta_{ik} \beta_{lm} - \beta_{il} \beta_{km} \right) P^{lm}, \\
\beta_{ik} &= \frac{1}{2\sqrt{q}} \varepsilon^{ilm} \varepsilon^{knp} q^{lm} q^{np}, \\
P^{lm} &= -\frac{1}{\sqrt{q}} \left( q^{ik} q^{lm} - q^{il} q^{mk} \right) \pi_{ik},
\end{aligned}
\]  

(62)

where \( \pi_{ik} \) are the momenta conjugate to the generalized coordinates \( q^{ik} \), \( \varepsilon_{ikl} \) is a completely antisymmetric symbol, and \( \varepsilon_{123} = 1 \). In this case

\[
\pi_{ik} \partial_0 q^{ik} = P^{ik} \partial_0 \beta_{ik},
\]

(64)

\[
\left\{ q^{ik}, \pi_{lm} \right\} = \delta^{ik}_{lm} \delta^3(x - \bar{x}),
\]

(65)

\[
\left\{ q^{ik}, q^{lm} \right\} = 0, \quad \left\{ \pi_{ik}, \pi_{lm} \right\} = 0.
\]

(66)

In the FP formalism, the density of the first-order Lagrangian has the form

\[
\mathcal{L}_1^{(FP)} = \pi_{ik} \partial_0 q^{ik} - N \mathcal{H}_0 - N^i \mathcal{H}_i + \frac{1}{2\pi} \partial_0 \partial_k q^{ik},
\]

(67)

where we write the part of the divergence that is essential in the case of the island position of masses in an asymptotically three-dimensionally flat space-time. Now

\[
\mathcal{H}_0 = \frac{2\pi}{q^{1/4}} \left( q^{lp} q^{mq} - q^{lm} q^{pq} \right) \pi_{lm} \pi_{pq} - \frac{q^{1/4}}{2\pi} \left( \frac{3}{R} - 2\Lambda \right),
\]

(68)

\[
\mathcal{H}_i = \frac{2}{q^{1/4}} q^{kl} \left( \nabla_k \left( q^{1/4} \pi_{il} \right) - \nabla_l \left( q^{1/4} \pi_{kl} \right) \right),
\]

(69)

where we must express \( \beta_{ik} \) in \( R \) in terms of \( q^{lm} \) according to (63).

The quantities \( \mathcal{H}_0 \) and \( \mathcal{H}_i \) continue to satisfy relations (50)-(52), and the geometric meaning of these quantities is preserved.

4. The usual frame formalism

We now consider the frame formalism. At each point of space-time, we introduce four mutually pseudo-orthogonal normalized vectors \( e_A^\mu(x) \), where the subscript \( A \) numbers the vectors \( (A = 0, 1, 2, 3) \), the superscript \( \mu \) numbers their components in the coordinate basis \( (\mu = 0, 1, 2, 3) \), and \( x \equiv \{x^0, x^1, x^2, x^3\} \). We assume that

\[
e_A^\mu(x) q_{\mu\nu} e_B^\nu(x) = \eta_{AB},
\]

(70)
where $\eta_{AB} = \text{diag}(-1, 1, 1, 1)$. We also introduce the variables $e^A_\mu(x)$ by the relation
\[
e^A_\mu e^\nu = \delta^\nu_\mu.
\]
(71)

It follows from relations (70) and (71) that
\[
g_{\mu\nu}(x) = e^A_\mu \eta_{AB} e^B_\nu.
\]
(72)

We substitute this expression in expressions (11) and (12) above for the action of the gravitational field and regard $e^A_\mu(x)$ as functions describing this field in what follows. We thus obtain a theory invariant under two groups of transformations: general coordinate transformations and local Lorentz transformations of the variables $e^A_\mu(x)$. The last transformations have the form
\[
e^{\prime A}_\mu(x) = \omega^A_\mu(x) e^B_\mu(x),
\]
(73)
under the condition
\[
\eta_{AB} \omega^A_\mu(x) \omega^B_\nu(x) = \eta_{DE}.
\]
(74)
Relation (74) ensures the invariance of the metric $g_{\mu\nu}(x)$ under such transformations. The variables $e^A_\mu$ and $e^\mu_A$ are called frame parameters.

Each vector referred to the coordinate basis is assigned a vector referred to the frame basis according to the rule
\[
a^A = e^A_\mu a^\mu, \quad a_A = e^A_\mu a^\mu.
\]
(75)

The tensors $T^\mu_\nu,...,A,...,B,...$, which vary in the indices $\mu,...,\nu,...$ as usual under the change of coordinates and which are transformed by the Lorentz matrices in the indices $A,...,B,...$ under the change of parameters $e^A_\mu$ according to (73), can be introduced similarly. The tensor indices $A,...,B,...$ are raised and lowered using the symbols $\eta^{AB}$ and $\eta_{AB}$.

The covariant derivatives are introduced as usual,
\[
\nabla_\mu a^\nu = \partial_\mu a^\nu + \Gamma^\nu_\mu a^\alpha, \quad \nabla_\mu a_\nu = \partial_\mu a_\nu - a_\alpha \Gamma^\alpha_\mu\nu,
\]
(76)
\[
\nabla_\mu a^A = \partial_\mu a^A + A^A_\alpha B a^B, \quad \nabla_\mu a_A = \partial_\mu a_A - a_B A^B_\mu A
\]
(77)
(and similarly in the case of tensors). Under the assumption that
\[
\nabla_\mu e^A_\nu = 0, \quad \nabla_\mu e^\nu_A = 0, \quad \nabla_\mu \eta^{AB} = 0,
\]
(78)
we establish the relation between $\Gamma^\nu_\mu$ and $A^A_\mu$:
\[
\Gamma^\nu_\mu = e^A_\mu A^A_\alpha B e^B_\nu + e^\mu_\alpha \partial_\alpha e^A_\nu,
\]
(79)
\[
A^A_\mu B = e^A_\mu \Gamma^\nu_\mu e^\nu_B + e^\mu_\alpha \partial_\alpha e^A_B,
\]
(80)
where $A^A_\mu B$ is called a frame connection and $\Gamma^\nu_\mu$ is called a coordinate connection.

The frame connection is similar to the gauge field with a Lorentz structure group. Therefore,
\[
A^A_\mu B = A^A_\alpha D \eta_{DB},
\]
(81)
where $A^A_{\alpha D} = -A_{\alpha DA}$. If $A_{\alpha}$ is understood as the matrix $A_{\alpha B}$, then we can construct the analogue of the field strength

$$F_{\mu\nu} = \partial_\mu A_{\nu} - \partial_\nu A_{\mu} + A_{\mu} A_{\nu} - A_{\nu} A_{\mu},$$

(82)

and, moreover,

$$R^\alpha_{\beta,\mu\nu} = e^\alpha_A F_{\mu\nu} A^B_{B \beta}.$$

(83)

It is necessary to use the frame formalism to describe spinors in the Riemannian space-time because the spinor representations of the Lorentz group cannot be extended to the representations of the total linear group. Therefore, the spinors cannot be referred to the local coordinate basis; they can only be referred to the pseudo-orthogonal frame basis. But we use the frame formalism for a different purpose in what follows. As before, we assume that the gravitational field does not interact with other fields.

To remove the gauge arbitrariness completely, we must additionally impose four coordinate and six frame subsidiary conditions. We use this possibility and remove only part of the frame arbitrariness using the three conditions

$$e^\mu_{(0)}(x) = n^\mu,$$

(84)

where $n^\mu(x)$ is the normalized normal to the surface $x^0 = \text{const}$ at the point $x$ (see relations (9)). Hereafter, we enclose the frame indices in parentheses if they are written as numbers and write the coordinate indices without parentheses in this case. Relation (84) contains only three conditions because the vectors $n^\mu$ and $e^\mu_{(0)}$ are normalized. After conditions (84) are introduced, the theory remains invariant under the semidirect product of the group of general coordinate transformations and the group of three-dimensional orthogonal transformations (i.e., the Lorentz transformations under which the vector $e^\mu_{(0)}(x) = n^\mu(x)$ remains unchanged).

On each surface $x^0 = \text{const}$, we introduce the ADM variables $\beta_{ik} \equiv g_{ik}$, $N$ and $N^i$. Next, $i, k, \ldots$ are three-dimensional coordinate indices $(i, k, \ldots = 1, 2, 3)$, and $a, b, \ldots$ are three-dimensional frame indices $(a, b, \ldots = 1, 2, 3)$. By conditions (84), the frame parameters $e^\mu_A$ and $e^A_{\mu}$ can be expressed in terms of $e^i_a$, $e^i_a$, $N$ and $N^i$ as

$$e^0_{(0)} = \frac{1}{N}, \quad e^0_a = 0, \quad e^i_{(0)} = -\frac{N^i}{N}, \quad e^0_{(0)} = N, \quad e^0_{(0)} = 0, \quad e^0_a = e^a N^i.$$

(85)

In this case, not only

$$e^A_{\mu} e^\nu_{A} = \delta^\nu_{\mu},$$

(86)

but also

$$e^a_i e^k_i = \delta^i_k, \quad g_{ik} = \beta_{ik} = e^a_i e^a_k, \quad \beta^{ik} = e^i_a e^k_a.$$

(87)

We set

$$e \equiv \det e^a_i,$$

(88)

and then

$$\beta = e^2.$$

(89)

Having in mind a possible application to the loop theory of gravity, for the main variables, we take the functions

$$Q^i_a \equiv e e^i_a = \sqrt{\beta} e^i_a, \quad N, \quad N^i.$$

(90)
We set
\[ Q \equiv \det Q^i_a, \]  
and
\[ Q = \beta. \]  

We define the quantities \( Q^a_i \) by the relations
\[ Q^a_i Q^k_a = \delta^k_i. \]  

The vectors \( Q^i_a \) (as well as \( e^i_a \)) are tangent to the surface \( x^0 = \text{const} \). It follows from relations (87) that the indices \( a, b, \ldots \) can be raised and lowered using the symbols \( \delta^{ab} \) and \( \delta_{ab} \). Therefore, there is no difference between the superscripts and subscripts \( a, b \), and they can be written for convenience.

We develop the canonical formalism on the hypersurface \( x^0 = \text{const} \) in terms of the three-dimensional variables \( Q^i_a, N, N^i \), preserving the notation \( \Sigma \) for this hypersurface. The simplest way to the goal is to start from the FP formalism (see Sec. 3). According to formulas (62), (87) and (90), we have
\[ q^{ik} = \beta \beta^{ik} = \beta e^i_a e^k_a = Q^i_a Q^k_a. \]  

We substitute this expression in FP Lagrangian (67) and first assume that \( \pi_{ik} \) and \( Q^i_a \) are independent. We see that
\[ \pi_{ik} \partial_0 q^{ik} = 2 \pi_{ik} Q^i_a \partial_0 Q^k_a, \]  

and the variables \( Q^i_a \) are thus assigned the conjugate momenta
\[ \mathcal{P}^a_i = 2 \pi_{ik} Q^k_a. \]  

We hence have
\[ \pi_{ik} = \frac{1}{2} Q^a_k \mathcal{P}^a_i. \]  

But \( \pi_{ik} = \pi_{ki} \), and the new constraints
\[ Q^a_k \mathcal{P}^a_i - Q^a_i \mathcal{P}^a_k = 0 \]  

therefore appear. In view of relations (93), this is equivalent to the three constraints
\[ \Phi^a \equiv \varepsilon^{abc} Q^b \mathcal{P}^c_i = 0, \]  

where \( \varepsilon^{abc} \) is completely antisymmetric and \( \varepsilon^{123} = 1 \).

The action of the frame formalism can now be written in the canonical form
\[ S_{(1)}^{(\text{frame})} = \int d^4 x \mathcal{L}_{(1)}^{(\text{frame})}, \]  

\[ \mathcal{L}_{(1)}^{(\text{frame})} = \mathcal{P}^a_i \partial_0 Q^i_a - N \mathcal{H}_0 - N^i \mathcal{H}_i - \lambda^a \Phi^a, \]  

where we take the new constraints into account using the Lagrange multipliers \( \lambda^a \). We assume that in FP formulas (68) and (69) for \( \mathcal{H}_0 \) and \( \mathcal{H}_i \), the variables \( q^{ik} \) and \( \pi_{ik} \) are expressed in
terms of \( P^a_i \) and \( Q^i_j \) according to (94) and (97). Hereafter, for simplicity, we do not write the divergence in \( \mathcal{L}^{(\text{frame})}_{(1)} \) and consider the case of a closed universe. We have

\[
\mathcal{H}_0 = \frac{1}{4} \left( \frac{2\pi}{\sqrt{Q}} \right) \left( Q^k_Q P_k^i P_i^c - (Q^k_Q P_k^j)^2 \right) - \left( \frac{\sqrt{Q}}{2\pi} \right) \left( (3) \frac{3}{R} - 2\Lambda \right),
\]

\[
\mathcal{H}_i = Q^k_a \left( (3) \nabla^a k P_i^c - (3) \nabla^a c P_k^i \right),
\]

where \((3) \nabla_k^c\) as before is the covariant derivative on the hypersurface \( x^0 = \text{const} \) containing the connection coefficients \((3) \Gamma^i_{kl}\) and \((3) A_{ia}^b\) expressed in terms of \( Q^k_a \) and \( Q^a_k \).

The coefficients \((3) A_{ia}^b\) are determined by analogy with formula (80) by the relations

\[
(3) A_{ia}^b = e^a_i (3) \Gamma^i_{kl} e^l_b - e^a_i \partial_k e^l_b.
\]

We can verify that by condition (84)

\[
(3) A_{ia}^b = (3) A_{ia}^b,
\]

where \((3) A_{ia}^b\) is the three-dimensional part of the four-dimensional connection \( A^a_{\mu} A_B^a \). The quantities \((3) A_{ia}^b\) form an \( SO(3) \) connection. Therefore,

\[
A_{ia}^b = A_{ia}^b = -A_{ia}^b = \varepsilon^{abc} A_{k}^c,
\]

where

\[
A_{k}^c = \frac{1}{2} \varepsilon^{cab} A_{k}^{ab}.
\]

It follows from formulas (90) and (92) that

\[
e^a_i = Q^{-1/2} Q^a_i, \quad e^a_i = Q^{1/2} Q^a_i.
\]

Substituting this in relation (103) and taking expressions (105) and (107) into account, we obtain

\[
A_{k}^c = \frac{1}{2} \varepsilon^{cab} \left( (\partial_k Q^a_i - \partial_i Q^a_k) Q^b_i - Q^a_i (\partial_i Q^b_m) Q^l_m Q^d_i + Q^b_i Q^k_i Q^l_k \partial_k Q^d_i \right) =
\]

\[
= \frac{1}{2} \varepsilon^{cab} \left( Q^a_k \partial_k Q^b_i + Q^d_i Q^k_i Q^l_i \partial_k Q^d_i + Q^d_i \partial_k Q^b_i + Q^a_k Q^b_i Q^d_i \right).
\]

Letting \( A_k \) denote the matrix \((3) A_{ia}^b\), we can determine the three-dimensional field strength

\[
F_{ik} = \partial_i A_k - \partial_k A_i + A_i A_k - A_k A_i.
\]

In this case, the relations

\[
F_{ik}^{ab} = \varepsilon^{abc} F_{ik}^c, \quad R_{k,lm} = e^i_a F_{lm}^{ab} e^b_k
\]
Therefore, the classical algebra of constraints is closed, and all the constraints 
without changing the frames as geometric objects and the quantities \( \Phi \)
(52), but all these terms are proportional to the constraints \( \Phi \). We can verify that the quantities 
frames without changing the coordinates.

The symmetry condition \( \pi_{ik} = \pi_{ki} \) in the framework of the formalism considered in this section 
is satisfied because of constraints (99), while this condition holds in Sec. 3 by definition. The 
Poisson brackets relating \( \pi_{ik} \) and \( \pi_{lm} \) differ from those introduced in Sec. 3. We now have

\[
\left\{ q^{ik}, q^{lm} \right\} = 0, \quad \left\{ q^{ik}, \pi_{lm} \right\} = \delta^{ik}_l \delta^3(x - \bar{x}), \quad (113)
\]

but

\[
\left\{ \pi_{ik}, \pi_{lm} \right\} = \frac{1}{4} Q^b_k Q^b_m Q^d_i Q^c_i \varepsilon^{cda} \Phi^a \delta^3(x - \bar{x}). \quad (114)
\]

New terms therefore appear in the right-hand sides of relations with Poisson brackets (50) -
(52), but all these terms are proportional to the constraints \( \Phi^a \). Moreover,

\[
\left\{ \Phi^a, \Phi^b \right\} = \varepsilon^{abc} \Phi^c \delta^3(x - \bar{x}), \quad \left\{ \Phi^a, \mathcal{H}_0 \right\} = 0, \quad \left\{ \Phi^a, \mathcal{H}_i \right\} = 0. \quad (115)
\]

Therefore, the classical algebra of constraints is closed, and all the constraints \( \mathcal{H}_0, \mathcal{H}_i \) and \( \Phi^a \)
ar constraints of the first kind.

Instead of the constraints \( \mathcal{H}_i \) it is convenient to introduce the constraints

\[
\mathcal{H}^\prime_0 \equiv \mathcal{H}_0 + A_i^c \Phi^c \equiv Q^b_k (\partial_k \Phi^b - \partial_l \Phi^b_k) + (\partial_k Q^b k) \Phi^b = 0. \quad (116)
\]

We can verify that the quantities \( \mathcal{H}^\prime_0 \) generate transformations of three-dimensional coordinates 
without changing the frames as geometric objects and the quantities \( \Phi^a \) generate rotations of 
frames without changing the coordinates.

The algebra of constraints in terms of the quantities \( \mathcal{H}_0, \mathcal{H}^\prime_0 \) and \( \Phi^a \) has the form

\[
\left\{ \mathcal{H}^\prime_0, \mathcal{H}^\prime_k \right\} = \mathcal{H}^\prime_0 \mathcal{H}_k \delta^3(x - \bar{x}) - \mathcal{H}^\prime_k \mathcal{H}_0 \delta^3(x - \bar{x}),
\]

\[
\left\{ \mathcal{H}^\prime_0, \mathcal{H}_0 \right\} = \mathcal{H}_0 \mathcal{H}_0 \delta^3(x - \bar{x}),
\]

\[
\left\{ \mathcal{H}_0, \mathcal{H}_0 \right\} = Q^{a} Q^{b} \mathcal{H}_{ik} \partial_i \delta^3(x - \bar{x}) - Q^{a} Q^{b} \mathcal{H}_{ik} \partial_i \delta^3(x - \bar{x}) + (\ldots)^a \Phi^a - (\ldots)^a \Phi^a,
\]

\[
\left\{ \Phi^a, \Phi^b \right\} = \varepsilon^{abc} \Phi^c \delta^3(x - \bar{x}),
\]

\[
\left\{ \Phi^b, \mathcal{H}^\prime_0 \right\} = -\Phi^b \partial_i \delta^3(x - \bar{x}),
\]

\[
\left\{ \Phi^a, \mathcal{H}_0 \right\} = 0,
\]

where \( (\ldots)^a \) are certain expressions composed of \( Q^i_a \) and \( \Phi^a_i \).
5. The formalism used in the loop theory of gravity

We now turn to the canonical formalism underlying the loop theory of gravity. We can readily verify that three-dimensional frame connection (109) admits the representation

\[
A^a_i = \frac{\delta F}{\delta Q^a_i(x)},
\]

where

\[
F = \frac{1}{2} \int_{x^0 = \text{const}} d^3 x \varepsilon^{abc} Q^b_i \partial_i Q^c_k \partial_k Q^a_i.
\]

Here, \( x \equiv (x^1, x^2, x^3) \), and \( \delta / \delta Q^a_i \) is the three-dimensional variational derivative. Because

\[
\{ p^{\alpha}_i, Q^{\beta}_k \} = -\frac{\delta f[Q^k]}{\delta Q^{\alpha}_i(x)},
\]

where \( f[Q^k] \) is an arbitrary functional of the functions \( Q^k_i(x) \), we have

\[
\{ p^{\alpha}_m, A^c_i \} = \{ p^{\alpha}_c, A^i_m \}.
\]

This permits performing the canonical transformation

\[
Q^a_i \rightarrow Q^a_i, \quad p^a_i \rightarrow p^a_i + b A^a_i,
\]

where \( b \) is a number called the Barbero-Immirzi parameter. This parameter can be assigned any value. By (121), the condition

\[
\{ p^a_i, p^b_k \} = 0.
\]

holds. Because \( A^a_i \) depends only on \( Q^a_i \), the relation

\[
\{ Q^a_i, p^b_k \} = \delta^a_k \delta^b_i \delta^3(x - \bar{x})
\]

is also satisfied.

After (122), we can perform one more canonical change of variables:

\[
Q^a_i \rightarrow B^a_i = \frac{1}{b} p^a_i = A^a_i + \frac{1}{b} p^a_i, \quad p^a_i \rightarrow \Pi^a_i = -b Q^a_i.
\]

Under the change of the three-dimensional coordinates on the hypersurface \( x^0 = \text{const} \) and of the frames satisfying condition (84) all the time, the quantity \( B^a_i \) transforms as the frame connection \( A^a_i \). Therefore, the path integral

\[
\text{tr} W(C) = \text{tr} P \exp \left( - \oint_C dx^i B_i \right),
\]

15
where $B_i$ is the matrix with the entries $B_{ij} = \varepsilon^{abc} B_{i}^{c}$ and the integration is over a closed path on the hypersurface $x^0 = \text{const}$, is invariant under the $SO(3)$ frame transformations generated by the constraints $\Phi^a$. This fact underlies the loop theory of gravity in which the quantities $B_{ij}^a$ and $\Pi_{ij}^a$ are used as canonical variables.

It follows from formulas (125) that

$$Q_i^a = -\frac{1}{b} \Pi^i_a, \quad P_i^a = b (B_i^a - A_i^a),$$

where

$$A_i^a \equiv A_i^a \bigg|_{Q_i^a = -b^{-1} \Pi_i^a}. \quad (128)$$

We substitute this expression in action (101), take (102) and (103) into account, and write the result in the two forms

$$S_1 = \int d^4x L_1, \quad (129)$$

$$L_1 = \Pi^i_a \partial_0 B_i^a - N^i \mathcal{H}_0^i - N^a \mathcal{H}_0^a - \lambda^a \Phi^a = \Pi^i_a \partial_0 B_i^a - N^a \mathcal{H}_0^a - N^i \mathcal{H}_0^i - \lambda^a \Phi^a, \quad (130)$$

where

$$\Phi^a = - \left( \partial_i \Pi^i_a + \varepsilon^{abc} B_i^c \Pi^i_b \right) = - \sqrt{-\Pi} \nabla_i \left( \frac{\Pi^i_a}{\sqrt{-\Pi}} \right) = -D_i \Pi^i_a, \quad (131)$$

$$\mathcal{H}_k' = \mathcal{H}_k + A^a_k \Phi^a = - \Pi^i_c F_i^{ik}(B) + B_k^b \Phi^b, \quad \mathcal{H}_0' = \Pi^i_c \Pi^k_{ib} \varepsilon^{abc} F_i^{kn}(B) + 4b^{-3} \frac{1}{(2\varepsilon)^2} \Pi \left( 1 + \varepsilon^2 b^2 \right) - 2b^{-1} \Lambda \Pi. \quad (132)$$

$$\mathcal{H}_k'' = - \Pi^i_c F_i^{ik}(B) + B_k^b \Phi^b, \quad \mathcal{H}_0'' = \Pi^i_c \Pi^k_{ib} \varepsilon^{abc} F_i^{kn}(B) - \left( 1 + \varepsilon^2 b^2 \right) \left( \Pi^k \Pi^i_{ib} (B_k - A_k^c)(B^c_i - A^c_i) - (\Pi^k_b (B_k - A_k^b))^2 \right) - 2b^{-1} \Lambda \Pi. \quad (133)$$

Here,

$$N' = \frac{N}{4} \left( \frac{2\varepsilon}{\sqrt{Q}} \right), \quad N'' = N_k, \quad N'^{ik} = N^{k},$$

$$\lambda^a = \lambda^a + \frac{N}{4} \left( \frac{2\varepsilon}{\sqrt{Q}} \right) Q_k^d \varepsilon_{dab} ^i - N^k A^b_k + b \varepsilon^a_b \partial_i N, \quad$$

$$N'' = - \frac{N}{(2\varepsilon)b^2 \sqrt{Q}}, \quad N'' = N_k, \quad$$

$$\lambda'' = \lambda^a - N^k A^b_k - N \left( 2\varepsilon b^2 \sqrt{Q} \right)^{-1} Q_k^d \varepsilon_{dab} ^i - 2 \left( 2\varepsilon b \right)^{-1} e^i_a \partial_i N, \quad$$

$$\Pi = \det \Pi^i_a, \quad A^c_k (Q^i_a) \bigg|_{Q_i^a = -b^{-1} \Pi_i^a}, \quad R = \left( 3 \right) R^i_a \bigg|_{Q_i^a = -b^{-1} \Pi_i^a}, \quad F_i^{3k}(B) = \partial_i B_k - \partial_k B_i + B_i B_k - B_k B_i,$$
\( B_i, F_{ik} \) are matrices with the entries
\[
B^a_{\mu i} = \varepsilon^{abc} B^c_{\mu i}, \quad F^a_{\mu i} = \varepsilon^{abc} F^c_{\mu i},
\]
and \( \nabla_i \) is the three-dimensional covariant derivative with the connection \( B^a_{\mu i} \).

We can see that the expressions for \( H'_0 \) and \( H''_0 \) are drastically simplified and become equal to each other if we continue the theory to the complex domain and set
\[
b = \mp i \kappa.
\]

According to (125), we then have
\[
B^c_{\mu i} = A^c_{\mu i} \pm i \kappa P^c_{\mu i}.
\]

But in the frame basis where \( e^{\mu}_{(0)} = n^\mu \), the relations
\[
A^{(0)c}_{i} = N \Gamma^0_{ik} e^k_c = -K_{ik} e^k_c,
\]
hold, and then
\[
\pi_{ik} = \frac{1}{(2 \kappa) \sqrt{\beta}} K_{ik}, \quad \mathcal{P}^a_i = 2 \pi_{ik} Q^k_a = \frac{1}{\kappa} K_{ik} e^k_a = -\frac{1}{\kappa} A^{(0)c}_{i},
\]
which implies that relation (137) becomes
\[
B^c_{\mu i} = A^c_{\mu i} \mp i A^{(0)c}_{i},
\]
i. e.,
\[
B^a_{\mu i} = A^a_{\mu i} \mp i \varepsilon^{abc} A^{(0)c}_{i}.
\]

We can introduce the fields
\[
A^{AB(\pm)}_{\mu} = A^{AB}_{\mu} \mp \frac{i}{2} \eta^{AD} \eta^{BE} \varepsilon_{DEFH} A^{FH}_{\mu},
\]
whence
\[
A^{ab(\pm)}_{i} = A^{ab}_{i} \mp i \varepsilon^{abc} A^{(0)c}_{i}.
\]

The fields \( A^{AB(\pm)}_{\mu} \) satisfy the self-duality (anti-self-duality) conditions
\[
A^{AB(\pm)}_{\mu} = \mp \frac{i}{2} \eta^{AD} \eta^{BE} \varepsilon_{DEFH} A^{FH(\pm)}_{\mu},
\]
and
\[
A^{AB}_{\mu} = A^{AB(+)}_{\mu} + A^{AB(-)}_{\mu}.
\]

Therefore, the expressions for \( \mathcal{K}'_0 \) and \( \mathcal{K}''_0 \) are simplified if
\[
B^a_{\mu i} = A^{ab(+)}_{\mu i},
\]
or
\[
B^a_{\mu i} = A^{ab(-)}_{\mu i}.
\]
In this case, all the constraints depend polynomially on the variables $B_i^a$ and $\Pi_i^a$. But to return to the real domain, the complicated condition

\[ B_i^a + B_i^{a\ast} = 2A(Q)\bigg|_{Q_i^a = -b^{-1}\Pi_i^a} \tag{148} \]

must be satisfied, where $\ast$ denotes complex conjugation in the classical case and Hermitian conjugation after quantization. Because condition (148) is complicated, it is currently preferred to construct the loop theory of gravity for a real value of the parameter $b$ for the case in which the constraint $\mathcal{H}_0'$ (or $\mathcal{H}_0''$) is very complicated.

The above classical canonical formulations of the theory of gravity have found and continue to find application in studying the problem of quantizing this theory.

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