Towards quantification of incompleteness in the pairwise comparisons method

Konrad Kulakowski, IEEE Member, Anna Prusak, Jacek Szybowski

Abstract—Alongside consistency, completeness of information is one of the key factors influencing data quality. The objective of this paper is to define ways of treating missing entries in pairwise comparisons (PC) method with respect to inconsistency and sensitivity. Two important factors related to the incompleteness of PC matrices have been identified, namely the number of missing pairwise comparisons and their arrangements. Accordingly, four incompleteness indices have been developed, simple to calculate, each of them take into account both: the total number of missing data and their distribution in the PC matrix. A numerical study of the properties of these indices has been also conducted using a series of Montecarlo experiments. It demonstrated that both incompleteness and inconsistency of data equally contribute to the sensitivity of the PC matrix. Although incompleteness is only just one of the factors influencing sensitivity, a relative simplicity of the proposed indices may help decision makers to quickly estimate the impact of missing comparisons on the quality of final result.

Index Terms—decision making, pairwise comparisons, incompleteness, data quality, AHP

I. INTRODUCTION

A. On comparing alternatives in pairs

The pairwise comparisons method is referred to as a process of comparing objects in pairs to judge which of them is preferred [34]. In the PC method, the elements in a given set are ranked on a pair-by-pair basis (two at a time), until performing all of the variations. The first evidence of pairwise judgments comes from the XIII-century philosopher Ramon Llull in the context of the election systems and the social choice theory. This system was based on binary comparisons [9]. Specifically, each voting round provides sets of two candidates who should be compared in pairs, and the winner is the one who gathers a majority of voices in the highest number of pairwise comparisons. The PC method proposed by Llull was then reinvented and improved by many other scientists including the XVIII-century French mathematician and philosopher Nicolas de Condorcet [10]. In his election system, the winner (so-called the Condorcet winner) is the one who is always victorious when being compared with any other candidate. However, Condorcet proved that there might be a situation when the winner cannot exist. He provided a three-voters example (the Condorcet triplet) when A is preferred over B, B over C, and C over A, so finally there is no winner [Saari 2009]. The Condorcet method was used in the preference aggregation methods of C. Dodgson in 1876 [22] and A. H. Copeland in 1951, the latter being known as the Copeland’s rule, in which the candidates are ordered by the number of pairwise wins minus the number of pairwise defeats [37], [15].

Another scholar known for his contribution to the PC methodology is an American psychologist and pioneer in psychometric research, Louis L. Thurstone. In 1927 he used Gaussian distribution to analyze pairwise comparisons. His model (also referred to as the law of comparative judgments) was based on three assumptions: 1) whenever a pair of stimuli is presented to a respondent it elicits a continuous preference for each stimulus (which is discriminable); 2) the stimulus with higher value in the comparison is preferred by the respondent; 3) these unobserved preferences are normally distributed [31]. He linked his approach with the psychophysical theory proposed by the XIX-century scholars E. Weber and G. Fechner. The Thurstonian model was reinvented in 1987 by Y. Takane, who added a random error to each paired comparison (so-called Thurstone-Takane model) [31]. In 1952 R. A. Bradley and M. E. Terry proposed an alternate model to the Thurstonian one. They defined the probability that object \( j \) (\( O_j \)) is preferred to object \( k \) (\( O_k \)) in a given comparison \( c_{jk} \). In the psychometric approach, the Bradley-Terry model is often called the BTL model, due to its relation to the choice axiom proposed in 1959 by R. D. Luce [12], [40], [42].

The widely known application of PC method is the Analytic Hierarchy Process (AHP) and the Analytic Network Process (ANP), the multi-criteria decision support techniques developed in the 1970s by the American mathematician, T. L. Saaty [39]. Besides the AHP/ANP methods, other multi-criteria decision techniques based on comparisons of alternatives include ELECTRE, PROMETHEE or MACBETH [16]. However, only the AHP/ANP judgments result in real numbers representing the relative strength of preference [26].

Despite its long history, the PC method (especially with relation to the AHP/ANP and the pairwise comparison matrices) is among the prevalent topics in recent studies, exploring problems such as inconsistency [8], rank reversal [32], [45] and incomplete judgments [34]. These characteristics are important indicators of data quality, which plays a critical role in modern decision-making processes, especially at business and governmental level [1].

B. Quality of data

The literature does not provide a universal set of data quality dimensions. Discrepancies in types and definitions of the
quality characteristics are due to the contextual nature of data quality. According to Batini et al. [1], the six most essential classifications of data quality criteria have been provided by Wand and Wang [43], Wang and Strong [44], Redman [36], Jarke et al. [20], Bovee et al. [3], and Naumann [33]. The analysis of these classifications allowed to distinguish a set of four attributes of data quality most commonly described in the literature. They include accuracy, completeness, consistency, and timeliness.

The first of them, accuracy, is “the extent to which the data is correct, reliable and certified” [44]. Redman [36] defines this term as a measure of the proximity of a data value \(v\) to other values \(v'\). Batini et al. [1] distinguish two types of accuracy, namely syntactic and semantic, specifying that data quality methodologies only consider syntactic accuracy, indicating the closeness of \(v\) to the corresponding definition domain \(D\). In DAMA report [41] data accuracy is defined as “the degree to which data correctly describes the ‘real world’ object or event being described,” and its measure is “the degree to which the data mirrors the characteristics of the real world object or objects it represents.”

Completeness was also defined in multiple ways, for example, “the ability of an information system to represent every meaningful state of a real-world system” [43], or “percentage of real-world information entered in data sources and/or data warehouse” [20]. The authors of [41] defined this criterion as “the proportion of stored data against the potential of 100% complete” measured as “the absence of blank (null or empty string) values or the presence of non-blank values.” In the literature, completeness is often associated with missing values, which exist in the real world but not in the database [1]. This criterion is crucial in pairwise comparison context when some pairs of objects remain with no comparisons, so only partial information is available. This causes other issues such as problems with calculating inconsistency of a partially filled matrix [4]. More information on this criterion is given in Section IV of this paper.

Consistency is one of the fundamental characteristics of data quality but defined in many different ways. Blake and Mangiamelli [2] emphasized that consistency is a multidimensional concept that can be represented by three aspects: representational consistency, integrity, and semantic consistency. Representational consistency refers to the presentation of data in the same format and compatibility with other (e.g., previous) data. Data integrity requires fulfilling four constraints: entity, referential, domain and column. Importantly, violations of entity integrity may lead to redundant or incomplete data. Semantic consistency indicates no contradiction between different data values in a particular set. In the literature, the most commonly discussed are representative consistency in relation to databases and semantic consistency in relation to the introduced values. Concerning the PC method, consistency of data is often regarded in terms of inconsistency indices [6], [27]. The mathematical basis of this measures is provided in Section III.

The last but not least is timeliness. Interpretation of this attribute is different across the literature. Thus, Batini et al. [1] suggested it should be considered in a broader context, as the time-related dimension. According to Wand and Wang [43], timeliness refers to “the delay between a change of a real-world state and the resulting modification of the information system state.” Other time-related dimensions are currency, interpreted as “the degree to which a datum is up-to-date” [36] or “when the information was entered in the sources and/or the data warehouse” [20].

C. Motivation and the organization of the manuscript

Many scientific articles deal with the inconsistency in the pairwise comparisons method. Thus, also many methods for measuring inconsistency have been proposed and thoroughly investigated. As a guide in this rich literature may serve the works [23], [5], [6]. Amazingly the same does not apply to incompleteness. Although some researchers have proposed methods for calculating the ranking for incomplete paired comparisons, the influence of incompleteness to the final result has not been sufficiently studied. One of the exceptions here can be Harker [18], but even this work does not provide us with the methods to measure incompleteness. Therefore, the purpose of this research is to determine the impact of the incompleteness of data on the correctness of the ranking in the PC method. During the work, we have identified two critical factors related to the incompleteness affecting the quality of data. These are the number of missing pairwise comparisons and the arrangement of missing comparisons. For this reason, we propose four incompleteness indices, where each of them depends on both: the total number of missing data and their distribution in the pairwise comparisons matrix. The performed Montecarlo experiments confirmed their usefulness as fast and quick tests of data quality.

The presented paper is composed of V sections including introduction (Section I) and summary (Section V). Section II outlines the theory of the pairwise comparisons method and the PC matrices, explaining phenomena such as incompleteness, inconsistency, and sensitivity. Section III presents four groups of incompleteness indices (\(\alpha\)-index, \(\beta\)-index, tree index, and the compound index) allowing for determining to what extent a given PC matrix based ranking is at risk due to the incompleteness of data. In Section IV, numerical experiments are presented demonstrating relationship between incompleteness, inconsistency and sensitivity.

II. PRELIMINARIES

A. Pairwise comparisons

The pairwise comparisons method very often is used as a way that allows experts to create a ranking based on a series of individual comparisons. The subjects of comparisons are alternatives. Beginning the ranking procedure experts compare alternatives in pairs. Then the results of individual comparisons are used as an input to the appropriate mathematical procedure, which allows computing the final numerical ranking (Fig. 1).

Let \(A = \{a_1, \ldots, a_n\}\) be a finite set of alternatives representing options among which a decision maker can choose. Similarly, let \(C = \{c_{ij} \in \mathbb{R}_+ : i,j = 1, \ldots, n\}\) be a set of expert judgements about each pair \((a_i, a_j) \in A \times A\), so that \(c_{ij}\) is the result of comparisons \(a_i\) against \(a_j\). Assigning
the certain real value \( v \in \mathbb{R}_+ \) represents the expert's opinion that the alternative \( a_i \) is \( v \) times more important than \( a_j \). It is convenient to represent the set of comparisons in the form of a matrix \( C = (c_{ij}) \), hereinafter referred to as the PC (pairwise comparisons) matrix. Since a comparison of a given alternative to itself does not indicate the advantage of any of the two alternatives being compared, the diagonal of \( C \) is composed of ones. Similarly, in most of the cases, it is assumed that if \( a_i \) is \( v \) times more important than \( a_j \) than also \( a_j \) is \( v \) times less important than \( a_i \). The latter observation leads to the equality \( c_{ij} = 1/c_{ji} \). In such a case it is convenient to use the following definition.

**Definition 1.** A matrix \( C = (c_{ij}) \) is said to be reciprocal if for all \( i, j = 1, \ldots, n \) holds \( c_{ij} = 1/c_{ji} \).

The pairwise comparisons method aims to transform the set of paired comparisons (i.e., the PC matrix) into the ranking vector (Fig. 1). Let us define the function that assigns the weight (also called as the importance or the priority) to every vector \( a_i \) for all \( i \). The latter observation leads to the equality \( c_{ij} = 1/c_{ji} \). In such a case it is convenient to use the following definition.

**Definition 2.** Let \( G_C = (V, E, L) \) be a labelled, directed graph with the set of vertices \( V = \{ a_1, \ldots, a_n \} \), the set of edges \( E \subseteq V \times V \setminus \{(a_1, a_1), \ldots, (a_n, a_n)\} \), and the labelling function \( L : E \to \{c_{12}, \ldots, c_{n,n-1}\} \) so that \( L(a_i, a_j) = c_{ij} \). \( G_C \) is said to be induced by the matrix \( C \).

In such a graph vertices correspond to alternatives and edges correspond to the comparisons among the alternatives.

**Definition 3.** Let the output degree of \( a_i \) be denoted by \( \text{outdeg}(a_i) \) and be given as

\[
\text{outdeg}(a_i) = |\{ j : (a_i, a_j) \in E \}|
\]

It is easy to observe that the output degree of vertex \( a_i \) is equal to the number of comparisons of alternative \( a_i \) with others.

**Definition 4.** The ranking function for \( A \) is a function \( w : A \to \mathbb{R}_+ \) that assigns a positive real number to every alternative \( a \in A \).

The role of the ranking computation procedure is to determine the value of \( w \) concerning every alternative. The list of all values \( w(a_1), \ldots, w(a_n) \) we will often write in the form of a transposed vector \( w \):

\[
w = [w(a_1), \ldots, w(a_n)]^T,
\]

Very often \( w \) is called interchangeably as a priority or weight vector. There are several methods of transforming paired comparisons into the ranking. According to the most popular one, referred to in the literature as eigenvalue method (EVM), the ranking is formed as the appropriately rescaled principal eigenvector [38]. Thus, to calculate \( w \) in EVM one have to solve equation

\[
Cw_{\text{max}} = \lambda_{\text{max}}w_{\text{max}},
\]

where \( \lambda_{\text{max}} \) is the spectral radius (principal eigenvalue) of \( C \), then rescale \( w \) so that all its entries sum up to 1.

\[
w = [s \cdot w_{\max}(a_1), \ldots, s \cdot w_{\max}(a_n)]^T,
\]

where

\[
s = \left( \sum_{i=1}^{n} w_{\max}(a_i) \right)^{-1}.
\]

There are a dozen other weighting methods for PC matrices [19], [46], [47], [25], [13]. Among them, the geometric mean method (GMM) deserves particular attention. According to GMM the priority of \( i \)-th alternative is formed as the appropriately rescaled geometric mean of \( i \)-th row of the matrix \( C \). Due to its relative simplicity and theoretical properties in recent times it has gained many supporters.

**Example 5.** Consider a pairwise comparison matrix

\[
C = \begin{pmatrix}
1 & 1 & 2 & 0.5 \\
1 & 1 & 0.25 & 8 \\
0.5 & 4 & 1 & 1 \\
2 & 0.125 & 1 & 1
\end{pmatrix}.
\]

Its principal eigenvalue equals \( \lambda_{\text{max}} \approx 5.8875 \) and its principal eigenvector is given by

\[
w_{\text{max}} = [1.32571, 2.0096, 1.9849, 1]^T.
\]
The sum of its coordinates equals 6.32021, so, after normalization, we obtain a priority vector $[0.20976, 0.31796, 0.31406, 0.15822]^T$. This determines the order of alternatives: $a_2, a_3, a_1, a_4$. Notice that according to EVM, alternative $a_2$ is slightly better than $a_3$. However, since geometric means of the second and third rows of $C$ are equal, GMM assigns the same weights to both alternatives.

**B. Incompleteness**

The priority deriving methods mentioned in the previous section assume that the set of paired comparisons is complete, i.e., every entry $c_{ij}$ of $C$ is known and available. In practice, this condition is not always met. It can happen for many reasons. After taking reciprocity into account, the number of all possible comparisons for $n$ alternatives is $n(n-1)/2$. Thus, when the number of alternatives is large comparing all of them in pairs requires considerable effort. It can not always be possible, for example, because of limited and expensive work time of experts. Harker [17] also points out that an expert, when faced with a comparison between two alternatives $a_i$ and $a_j$, sometimes would rather not compare them directly. This may happen when, e.g., they do not yet have a good understanding of his or her preferences for this particular pair of alternatives. Sometimes experts evade from the answers, especially when taking a position on the given comparisons is morally or ethically tricky, e.g., comparing mortality risk vs. cost. Finally, some data may be lost or damaged.

In response to the above problems, the methods of calculating the ranking based on an incomplete set of pairwise comparisons arose. Probably one of the most popular (and the first one) is the *Harker* method [17]. According to the method based on matrix $C$, a new auxiliary matrix $B = (b_{ij})$ is created where

$$b_{ij} = \begin{cases} 
    c_{ij}, & \text{if } c_{ij} \text{ exists and } i \neq j, \\
    0, & \text{if } c_{ij} \text{ does not exist and } i \neq j, \\
    b_{ii}, & \text{if } i = j,
\end{cases}$$

and $b_{ii}$ means the number of the unanswered questions in the $i$-th row of $C$. Harker has shown that a non-negative quasi-reciprocal matrix $(B + Id)$ can be used for calculation priority ranking as a replacement for an original PC matrix. The natural limitation of the *Harker* method is that in $C$ there must be a series of comparisons between every two alternatives $a_i$ and $a_j$ such that $c_{ik_1}, c_{k_1k_2}, \ldots, c_{k_2j}$ exist. In other words, every two alternatives must be comparable at least indirectly. Every matrix $C$ for which the above condition holds is irreducible and every graph $G = (V, E)$ in which the set of vertices $V = \{v_1, \ldots, v_n\}$ correspond to the set of alternatives $a_1, \ldots, a_n$, and the set of edges $E$ so that there exists the edge $(v_i, v_j)$ in $E$ if $c_{ij}$ is known and defined, is strongly connected [35]. Let us consider the following example.

**Example 6.** Let $C$ be incomplete PC matrix

$$C = \begin{pmatrix}
1 & 3 & ? \\
1/3 & 1 & 3 \\
? & 1/3 & 1
\end{pmatrix},$$

hence the *Harker’s* auxiliary matrix is

$$B + Id = \begin{pmatrix}
1 & 3 & 0 \\
1/3 & 0 & 3 \\
0 & 1/3 & 1
\end{pmatrix} + \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},$$

and

$$B + Id = \begin{pmatrix}
2 & 3 & 0 \\
1/3 & 1 & 3 \\
0 & 1/3 & 2
\end{pmatrix}.$$ 

Thus the rescaled ranking vector obtained by EVM is

$$w = [0.692, 0.23, 0.0769]^T$$

which means that the priority of the first alternative is $w(a_1) = 0.692$, and the second and third: $w(a_2) = 0.23, w(a_3) = 0.0769$, correspondingly.

The corresponding graph has been shown in Fig. 2.

![Fig. 2. The graph of C](image)

**C. Inconsistency**

As $c_{ik}$ represents the results of the comparisons between $i$-th and $k$-th alternative, and $c_{kj}$ expresses the outcome of similitude between $k$-th and $j$-th alternative its natural to expect that $c_{ij} = c_{ik}c_{kj}$. However, the entries of the PC matrix $C$ represent the subjective opinions of experts and due to human imperfection may happen that $c_{ij} \neq c_{ik}c_{kj}$. Whenever it happens, we will call such a situation as inconsistency. If the difference between $c_{ij}$ and $c_{ik}c_{kj}$ is small or happens rarely, it probably will not have much impact on the final result. However, if the difference is large and it happens relatively often, then the results of the pairwise comparisons may be considered unreliable, and hence, the ranking results may not be trustworthy. This observation leads to a question about the degree of inconsistency of PC matrix $C$. A popular way of determining the level of inconsistency in a set of pairwise comparisons is the use of inconsistency indexes. Probably the best-known index is one proposed by *Saaty* in 1977 [38]. It is defined as:

$$CI = \frac{\lambda_{\text{max}} - n}{n - 1},$$

where $\lambda_{\text{max}}$ is the principal eigenvector of $C$, and $n$ is the number of alternatives. It has been proven that $CI$ reaches 0 when the PC matrix $C$ is fully consistent, and it gets the higher values, the more inconsistent $C$ is [38]. Since then,
many different inconsistency indexes have been created. A comprehensive overview of the inconsistency indexes can be found in [6], [5], [29].

D. Sensitivity

Another factor that affects the credibility of the ranking is the sensitivity of the result. By sensitivity we mean here the extent to which the disturbance of input data can change the final result. If a little disorder can significantly modify the result, then the ranking result is unstable and, therefore, not credible. We can be sure if the final result is not accidental.

Reversely, if reasonably small changes in the input data do not cause noticeable modifications of the result, then we can trust that the result obtained is a consequence of decision-making data deliberately introduced into the system. Simply put, it can be assumed that the sensitivity can be used to determine the data quality. The problem is, however, that the sensitivity is hardly measurable. What does it mean "a small change in input"? What change, how to quantify it? What does it mean “noticeable modifications of the result”? A person who wants to deal with the sensitivity analysis must answer all these questions. For the purpose of this article, we have assumed that the inconsistency index determines the input data disturbance. To measure the extent to which the results have been modified, we use two methods: the Manhattan distance \(^1\) and the rescaled Kendall tau distance.

The Manhattan distance between two priority vectors \(w\) and \(u\) is defined as follows:

\[
M_d(w, u) = \sum_{i=1}^{n} |w(a_i) - u(a_i)|.
\]

This metric provides us with information what is the average difference between two different priorities assigned to the same alternatives. As all the entries of priority vectors sum up to 1, the result \(M_d(w, u) \leq 2\).

Very often the ranking results are interpreted only qualitatively. This means that the decision makers are interested in who is the winner, who is in the second and in the third place but not what are the numerical priorities of alternatives. Let \(O: \mathbb{R}_+^n \rightarrow \{1, \ldots, n\}\) be the mapping assigning to every ranking vector \(w\) its ordinal counterpart in such a way that \(i\)-th element of \(O(w)\) indicates the position of \(i\)-th alternative in the ranking (1). For example, if

\[
w = [0.3, 0.5, 0.2]^T
\]

then its ordinal vector is

\[
O(w) = [2, 1, 3]^T.
\]

Qualitative interpretation of ranking vectors leads to a question to what extent both: \(O(w)\) and \(O(u)\) differ from each other. The answer can be the Kendall tau rank distance that counts the number of pairwise disagreements between two ranking lists [21], [14]. Let us define Kendall tau distance formally:

\[
K_d(p, q) = \# \{(i, j) | i < j \text{ and sign}(p(a_i) - p(a_j)) \neq \text{sign}(q(a_i) - q(a_j))\}
\]

where \(p, q\) are ordinal vectors. Since the maximal value of \(K_d(p, q)\) for two \(n\)-element vectors is \(n(n-1)/2\) it is convenient to use the rescaled Kendall tau distance, i.e.

\[
K_{rd}(p, q) = \frac{2K_d(p, q)}{n(n-1)},
\]

so that \(0 \leq K_{rd}(p, q) \leq 1\). The rescaled Kendall tau distance is the second method used in the article for the purpose of measuring discordance between ranking results. Since, vectors produced by EVM, GMM or Harker method are not ordinal before applying \(K_{rd}\) they have to be transformed to their ordinal counterparts using \(O\) mapping.

Sometimes the Kendall tau distance is called a Bubble sort distance. The reason is that when there are no ties their value represents the number of swaps that are done by the bubble sort algorithm [11] when transforming the first list into the second one.

Example 7. Let us consider two ordinal vectors \(p = [1, 2, 4, 3]^T\) and \(q = [3, 4, 1, 2]^T\). It is easy to observe that \(K_{rd}(p, q) = 5\) as the discordant pairs of indices are: \((1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\). Indeed there are five binary swaps needed to transform \(p\) into \(q\). They are:

1) \(p = [1, 2, 4, 3]^T \rightarrow [1, 2, 3, 4]^T\),
2) \([1, 2, 3, 4]^T \rightarrow [1, 3, 2, 4]^T\),
3) \([1, 3, 2, 4]^T \rightarrow [3, 1, 2, 4]^T\),
4) \([3, 1, 2, 4]^T \rightarrow [3, 1, 4, 2]^T\),
5) \([3, 1, 4, 2]^T \rightarrow [3, 4, 1, 2]^T\) = \(q\).

Assuming \(n = 4\), the rescaled value is \(K_{rd}(p, q) = 5/6\).

III. INDICES OF INCOMPLETENESS

A. Incompleteness and sensitivity

According to EVM, the priority vector meets the equation (2). In other words, the weight of every alternative \(w(a_i)\) meets the equation

\[
w(a_i) = \frac{1}{\lambda_{\text{max}}} \sum_{j=1}^{n} c_{ij} w(a_j).
\]

Hence, the priority of one alternative is expressed by the weighted average of all others alternatives. With this regularity, we also deal with the case of GMM [28]. The equation (5) suggests that the disturbance of one single element \(c_{ij}\), assuming that the other elements have not changed, should not affect significantly the value of \(w(a_i)\). However, in the case of an incomplete PC matrix, the relationships between alternatives are weakened. The priorities of individual alternatives are determined by fewer expressions in the form \(c_{ij} w(a_j)\) than normally. It suggests that the susceptibility for disturbances of the rankings calculated based on the incomplete PC matrices is higher than normal. This, of course, should translate to the usually higher sensitivity of such decision models. It means that the completeness of the matrix correlates with the sensitivity of the method. The more comparisons are available, the less vulnerable the model is. One may ask whether the number of missing elements is not enough as an index? To answer this question let us consider the following two PC
matrices with three (six, when the reciprocal elements are taken into account) missing comparisons.

\[
C_1 = \begin{pmatrix}
1 & c_{12} & ? & ? & ? \\
c_{21} & 1 & c_{23} & c_{24} & c_{25} \\
? & c_{32} & 1 & c_{34} & c_{35} \\
? & ? & c_{43} & 1 & c_{45} \\
? & ? & c_{53} & c_{54} & 1
\end{pmatrix}
\quad , \quad (6)
\]

\[
C_2 = \begin{pmatrix}
1 & c_{12} & ? & ? & c_{15} \\
c_{21} & 1 & c_{23} & ? & c_{25} \\
? & c_{32} & 1 & c_{34} & c_{35} \\
? & ? & c_{43} & 1 & c_{45} \\
c_{51} & c_{52} & c_{53} & c_{54} & 1
\end{pmatrix}
\quad . \quad (7)
\]

In the first matrix \(a_1\) is compared only with \(a_2\). Thus, disturbance on \(c_{12}\) completely changes the value \(w(a_1)\). In the second matrix \(a_1\) is compared with \(a_2\) and \(a_5\). Therefore, the same disturbance on \(c_{12}\) will have less impact on the priority \(w(a_1)\). In Section IV this intuition will be confirmed by the Montecarlo experiment. The above consideration leads us to the conclusion that the completeness index, that would be useful in determining the sensitivity of the decision model, should also take into account the arrangement of missing comparisons.

**B. \(\alpha\)-index**

In \(n \times n\) PC matrix a single alternative can be compared with at most \(n - 1\) other alternatives. Therefore, the maximal value of \(\text{outdeg}(a_i)\) for \(i = 1, \ldots, n\) is \(n - 1\) (see Def. 3). Similarly, the number of missing comparisons is given by \(n - 1 - \text{outdeg}(a_i)\). Because the desired behavior is that the newly constructed index should be higher for \(C_1\) than for \(C_2\) the higher value of the expression \(n - 1 - \text{outdeg}(a_i)\) for some particular \(i\) should contribute more to the value of the index than two or more smaller expressions. To achieve this let us raise the expression \((n - 1 - \text{outdeg}(a_i))^\alpha\) to a positive real number \(\alpha > 1\). Thus, the expression

\[
S_\alpha(C) = \sum_{i=1}^{n} (n - 1 - \text{outdeg}(a_i))^\alpha
\]

combines two features together. It raises when the number of missing comparisons increases and providing that there are two matrices of the same size and with the same number of missing comparisons it is higher for this matrix that has larger irregularities in the distribution of missing values. Let us compute the mean of missing values raised to \(\alpha > 1\). As a result, we get the formula:

\[
\frac{1}{n} S_\alpha(C) \quad (8)
\]

which preserves both important features and its value is bounded and varies within the range \([0, (n - 1)^\alpha]\). Hence, in order to get the final form of the index let us divide (8) by \((n - 1)^\alpha\), i.e.

\[
Ild_\alpha(C) = \frac{1}{n} S_\alpha(C) \quad (n - 1)^\alpha
\]

It is clear that \(0 \leq Ild_\alpha \leq 1\). When the PC matrix is fully incomplete, i.e. there are no comparisons between alternatives, \(Ild_\alpha(C) = 0\). Reversely, if \(C\) is complete, i.e. all the alternatives are defined, \(Ild_\alpha(C) = 1\). Providing that the PC matrix is reciprocal, every alternative has to be compared with at least one different alternative. The maximal value of \(Ild_\alpha(C)\) that allows to create the ranking is reached when just one alternative is compared with all the others. Then it is given by \(\frac{n - 1}{n} \cdot \left(\frac{n - 2}{n - 1}\right)^\alpha\). Condition

\[
Ild_\alpha(C) \leq \frac{n - 1}{n} \cdot \left(\frac{n - 2}{n - 1}\right)^\alpha
\]

is necessary but it is not sufficient. Hence, there may exist PC matrices for which \(Ild_\alpha\) is smaller than \(\frac{n - 1}{n} \cdot \left(\frac{n - 2}{n - 1}\right)^\alpha\) but, in spite of this, one can not create the ranking.

**Example 8.** Consider matrices \(C_1\) and \(C_2\) given by (6) and (7). Let us calculate their 2-indices:

\[
Ild_2(C_1) = \frac{1}{8} \sum_{i=1}^{5} (4 - \text{outdeg}(a_i))^2 = \frac{9 + 1 + 1 + 1}{80} = 0.15.
\]

\[
Ild_2(C_2) = \frac{1}{8} \sum_{i=1}^{5} (4 - \text{outdeg}(a_i))^2 = \frac{4 + 1 + 1 + 4}{80} = 0.125.
\]

As we can see the index of the first matrix is greater than the index of the second one, which reflects the fact that the distribution of the missing items in the rows of \(C_2\) is more aligned than in \(C_1\). However, both indices are quite small, as both matrices lack of only 6 elements (out of 20).

**C. \(\beta\)-index**

According the old adage “a chain is only as strong as its weakest link”. Following this common sense observation the second index does not consider the average number of missing comparisons for all alternatives but it focuses on the maximum number of missing comparisons for a single alternative:

\[
M(C, \beta) = \left( \max_{i=1, \ldots, n} (n - 1 - \text{outdeg}(a_i)) \right)^\beta.
\]

Because we can not omit the total number of comparisons the “weakest link” in the form of the above formula has to be combined with the sum:

\[
S(C) = \sum_{i=1}^{n} (n - 1 - \text{outdeg}(a_i)).
\]

Thus, the proposed index gets the form:

\[
Ild_\beta(C) = \frac{M(C, \beta) S(C)}{n(n - 1)^{1+\beta}}
\]

where the multiplier \(1/n(n - 1)^{1+\beta}\) is introduced only for the purpose of fitting the index value to the segment \([0, 1]\).
It is easy to observe that $H_\beta(C)$ is 0 when the matrix $C$ is complete. Reversely, $H_\beta(C) = 1$ if there are no defined values in the matrix except its diagonal.

**Example 9.** Similarly as before let us consider $C_1$ and $C_2$ given by (6) and (7). Their $\beta$ indices (where $\beta = 1$) are

$$H_\beta(C_1) = \max \{3, 1, 1, 1, 0\} \cdot (3 + 1 + 1 + 1 + 0) = \frac{9}{40} = 0.225.$$  

$$H_\beta(C_2) = \max \{2, 1, 1, 2, 0\} \cdot (2 + 1 + 1 + 2 + 0) = \frac{3}{20} = 0.15.$$  

Similarly, as in the case of $\alpha$-index, the matrix $C_1$ gets the higher values of the index than the matrix $C_2$. Both values, however, are quite small as only six elements (out of 20) are missing.

**D. Tree index**

As [24] shows, the existence of a spanning tree in the graph associated with an incomplete pairwise comparison matrix is a necessary condition to generate its missing values, and what follows, to create the ranking. Of course, the more spanning trees we have, the more reliable the data we obtain. The Cayley’s formula [7] states that the number of all spanning trees in a complete graph with $n$ vertices is equal to $n^{n-2}$. If we consider a complete PC matrix, its incompleteness index should be equal to 0. The index should raise with the reduction of the number of spanning trees. However, as we remove the matrix entries one by one, the number of trees decreases exponentially from $n^{n-2}$ to 0. To slow down its drop occurring when we remove the PC matrix elements, it is desirable to divide it over $n^{n-2}$ and apply the $n-2$ root to the ratio. These simple observations lead us to the definition of an alternative incompleteness indicator, which we will call the tree index.

**Definition 10.** The tree-index of a pairwise comparison matrix $C$ is defined by the formula

$$TI(C) = 1 - \frac{NT(C)}{n},$$

where $NT(C)$ denotes the number of the spanning trees in a graph associated with the matrix $C$.

**Remark 11.** Notice that $TI(C) = 0$ if and only if $G_C$ is complete i.e. $C$ has got all elements. On the other hand, $TI(C) = 1$ if and only if $G_C$ is disconnected, which means that we cannot create a priority vector based on the elements of $C$.

According to the Kirchoff’s Theorem [30] the number of spanning trees in a connected graph $G$ with $n$ vertices $v_1, \ldots, v_n$ can be computed as any cofactor of the Laplacian matrix $L(G) = [l_{ij}]$ of $G$, whose elements are given by the formula:

$$l_{ij} = \begin{cases} \deg(v_i), & \text{if } i = j, \\ -1, & \text{if } i \neq j \text{ and } v_i \text{ is connected with } v_j, \\ 0, & \text{otherwise.} \end{cases}$$

**Example 12.** Once more, consider matrices $C_1$ and $C_2$ given by (6) and (7). The corresponding graphs $G_{C_1}$ and $G_{C_2}$ are given in Fig. 3.

![Fig. 3. Graphs $G_{C_1}$ and $G_{C_2}$](image)

Their Laplacian matrices are as follows:

$$L(G_{C_1}) = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 4 & -1 & -1 & -1 \\ 0 & -1 & 3 & -1 & -1 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & -1 & 3 \end{pmatrix},$$

and

$$L(G_{C_2}) = \begin{pmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 3 & -1 & 0 & -1 \\ 0 & -1 & 3 & -1 & -1 \\ 0 & -1 & -1 & 2 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{pmatrix}.$$  

Let us compute the cofactors of the left upper elements of the above matrices:

$$G_{C_{11}} = (-1)^2 \cdot \begin{vmatrix} 1 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{vmatrix} = 16,$$

$$G_{C_{11}} = 2 \cdot (-1)^2 \cdot \begin{vmatrix} 3 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & 4 \end{vmatrix} = 42.$$  

According to the Kirchoff’s Theorem, graphs $G_{C_1}$ and $G_{C_2}$ include 16 and, respectively, 42 spanning trees. Since $n = 5$, the tree incompleteness indices of $C_1$ and $C_2$ are

$$T(C_1) = 1 - \frac{\sqrt{16}}{5} = 0.496,$$

$$T(C_2) = 1 - \frac{\sqrt{42}}{5} = 0.305.$$
Again, the index of $C_1$ is higher than the index of $C_2$, which reflects the fact that removing six elements of a $5 \times 5$ matrix reduces the number of the respective graph’s spanning trees, which lowers the reliability of the resulting priority vector.

It is important to point out that both indices may be useful as measures of incompleteness. The first one measures the location of a matrix on the line between full and (almost) empty (i.e., having only 1s on the main diagonal) matrices. The latter reflects how far a matrix is from matrices which are useless for ordering the alternatives.

### E. Compound indices

As all the indices have the same domain (PC matrices) and codomain $[0,1] \subset \mathbb{R}_+$, then their product will also be a function with the same domain and codomain. It allows us to combine one index with the other to obtain the desirable properties of both. In this context, an interesting proposal seems to be combining $\alpha$ and $\beta$ indices. Thus, let us define a compound $\alpha, \beta$-index as follows:

$$II_{\alpha,\beta}(C) = II_{\alpha}(C) \cdot II_{\beta}(C)$$

As it will turn out in the Section IV, this product allows us to combine together a dynamics of average sensitivity represented by $II_{\alpha}$ together with differences in sensitivity resulting from different arrangements of missing pairwise comparisons. This second feature seems to be better represented by $II_{\beta}$.

### IV. Properties of incompleteness indices - A Numerical Study

#### A. Relationship between incompleteness, inconsistency and sensitivity

An entirely consistent matrix is resistant to reducing the set of paired comparisons. That is because it suffices to compare one alternative with another already ranked to precisely determine the ranking of the former. Hence, as long as it is possible to compute the ranking, i.e., the PC matrix is irreducible, the calculated ranking is the same regardless of which comparisons are missing. However, if a PC matrix is inconsistent, missing comparisons start to matter.

In order to investigate the impact of inconsistency and incompleteness to the sensitivity we randomly prepare 1000 complete and consistent PC matrices $\mathcal{C} = C_1, \ldots, C_{1000}$. Then every matrix from $\mathcal{C}$ was disturbed so that we obtain 41 sets $\mathcal{C}^1, \ldots, \mathcal{C}^{41}$ of matrices with the increasing average incompleteness $CI_{\text{avg}}$ given as

$$CI_{\text{avg}}(\mathcal{C}^i) = \frac{1}{n} \sum_{i=1}^{n} CI(C_i).$$

The average of inconsistencies of those groups starts from $CI_{\text{avg}}(\mathcal{C}^1) = 0.001$, $CI_{\text{avg}}(\mathcal{C}^2) = 0.004$, $CI_{\text{avg}}(\mathcal{C}^3) = 0.008$ and finally they reach $CI_{\text{avg}}(\mathcal{C}^{41}) = 0.385$. Next, we extend every $\mathcal{C}^i$ by adding irreducible incomplete matrices randomly obtained from those originally located there. Let us denote

$^2$As irreducible $n \times n$ matrix must have at least $n-1$ comparisons (we are counting only comparisons over the diagonal) then, for every inconsistent matrix $C \in \mathcal{C}^i$, we generate $n(n-1)/2 - (n-1) = (n^2 - 3n + 2)/2$ incomplete matrices.

the extended $\mathcal{C}^j$ by $\tilde{\mathcal{C}}^j$ and its elements by $C_j^{i,k} \in \tilde{\mathcal{C}}^j$, where $k$ means the number of missing comparisons and $i$ indicates the consistent PC matrix $C_i \in \mathcal{C}$ from which $C_j^{i,k}$ originated. For every $C_j^{i,k}$ we compute incompleteness indices $II_{\alpha}(C_j^{i,k}), II_{\beta}(C_j^{i,k})$ and $TI(C_j^{i,k})$, the measures of sensitivity i.e. Kendall distance $K_{\text{rd}}(w(C_i), w(C_j^{i,k}))$ and the Manhattan distance $M_d(w(C_i), w(C_j^{i,k}))$.

In the Figure 4 we can see the relationship between average value of sensitivity for matrices $C_j^{i,k}$ with the given average inconsistency $CI_{\text{avg}}(\mathcal{C}^j)$ and the average incompleteness given in the form of the three indices $II_{\alpha}(C_j^{i,k}), II_{\beta}(C_j^{i,k})$ and $TI(C_j^{i,k})$. When the considered PC matrices are consistent i.e. $CI_{\text{avg}}(\mathcal{C}) = 0$ then also the resulting rankings do not depend on incompleteness. The distance between rankings obtained from consistent complete and incomplete matrices is 0. However, when inconsistency starts increasing, the impact of incompleteness becomes apparent.

![Fig. 4. Relationship between average consistency level, incompleteness and sensitivity given as the average Manhattan distance between rankings obtained from consistent and inconsistent (and incomplete) 9 × 9 matrices.](image)
translates to the increase of the average Manhattan distance. For very small values of inconsistency ($CI_{\text{avg}} \approx 0.001$) the Manhattan distance is about 0.01 and following the increase of $H_\alpha$ it takes values near 0.04. For the larger values e.g. $CI_{\text{avg}} \approx 0.11$ the value of $M_d$ ranges between 0.1 and 0.4, and similarly for $CI_{\text{avg}} \approx 0.38$ the average values of $M_d$ are between 0.2 and 0.8. This observation indicates that the highly incomplete PC matrices are almost four times more vulnerable to the random disturbances than the complete matrices. As the maximal possible value of the Manhattan distance for vectors whose elements add up to 1 is 2, the value of $M_d = 0.4$ means that this index reaches 20% of its maximal value. The similar behavior can be observed for the other two indices: $H_\beta$ and $TI$ (Figs. 4b and 4c).

The values of Kendall distance reveals the similar properties (Figs. 5a, 5b and 5c). When the inconsistency is small ($CI_{\text{avg}} \approx 0.001$) the average values of Kendall distance are spanned between 0.005 and 0.025 for all indices of incompleteness. Then for moderately inconsistent matrices ($CI_{\text{avg}} \approx 0.11$) they range between 0.05 and 0.15, then for ($CI_{\text{avg}} \approx 0.38$) the values of Kendall index go through 0.09 to 0.25.

It means that for PC matrices with the reasonably high inconsistency we may expect that 25% or more pairs may randomly change their order. Similarly as before the incompleteness may significantly increase (from three to four times) the sensitivity of the PC method.

### B. Impact of the distribution of missing comparisons to the sensitivity

We may suppose that the more missing comparisons to the given alternative the more vulnerable its weight and the position in the ranking. In the extreme case, if the given alternative $a_i$ is compared to only one other alternative $a_j$, i.e., except $c_{ij}$, where $i \neq j$ all other values in the $i$-th row and $j$-th column of $C$ are undefined, the ranking of $a_i$ depends primarily on $c_{ij}$. Any disturbance of $c_{ij}$ can translate into significant changes in the weight of the $i$-th alternative. On the opposite case, the missing comparisons are evenly distributed between alternatives. It ensures the relative safety of each alternative, providing of course, that the number of missing alternatives is not too high. The above observations allow us to indicate an example of the regular and the irregular PC matrix with a fixed number of missing comparisons.

Let us number the selected entries in the $n \times n$ PC matrix in such a way that in the first row $c_{13}$ corresponds to 1, $c_{14} - 2$ and $c_{1,n}$ has assigned number $n - 2$. Similarly, in the second row $c_{24}$ gets the number $n - 1$, $c_{25} - n$ and the last element in the row $c_{2,n}$ gets $2n - 4$. Finally, the last element $c_{n-1,n}$ gets the number $(n^2 - 3n + 2)/2$. Elements directly above the diagonal are not indexed (the above numbering scheme has been shown in the form of a matrix $C_w$).

$$
C_w = \begin{pmatrix}
1 & c_{12} & \ldots & \ldots & \ldots & c_{(n-2)} & c_{(2n-4)} \\
1 & c_{23} & \ldots & \ldots & \ldots & \ldots & \ldots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \vdots & \ddots & c_{n-2,n-1} & \ddots & \ddots & \ddots \\
& & & & & 1 & c_{n-1,n}
\end{pmatrix}
$$

Then, in order to prepare the highly irregular (and highly sensitive) matrix with $x$ missing comparisons it is enough to remove comparisons with assigned numbers from 1 to $x$ and their counterparts below the diagonal. For example, the highly irregular 7 by 7 PC matrix with 9 missing comparisons may look like:

$$
C_w^{[9]} = \begin{pmatrix}
1 & c_{12} & ? & ? & ? & ? & ? \\
c_{21} & 1 & c_{23} & ? & ? & ? & ? \\
? & c_{32} & 1 & c_{34} & c_{35} & c_{36} & c_{37} \\
? & ? & c_{43} & 1 & c_{45} & c_{46} & c_{47} \\
? & ? & c_{53} & c_{54} & 1 & c_{56} & c_{57} \\
? & ? & c_{63} & c_{64} & c_{65} & 1 & c_{67} \\
? & ? & c_{73} & c_{74} & c_{75} & c_{76} & 1
\end{pmatrix}
$$

For the purpose of creating the matrices with the most even distribution of missing values we use another numbering scheme. Let assign number 1 to $c_{13}$, 2 to $c_{24}$, 3 to $c_{35}$, and $n - 2$ to $c_{n-2,n}$. The number $n - 1$ be assigned to $c_{14}$, $2$ to $c_{25}$ and finally $2n - 4$ to $c_{n-3,n}$. The last numbered element is $c_{1n}$ with value of index $(n^2 - 3n + 2)/2$ (the regular numbering scheme is shown as the matrix $C_b$).
we can see plots of the distribution of missing comparisons is.

The number of missing values their increase differs from plots of sensitivity between matrices in the form of sensitivity. For a not very high number of missing values (here 14 which is 50% of all comparisons possible to remove) all the indices seem to mimic the sensitivity charts (Fig. 6). However, for the larger numbers of missing comparisons the differences between PC matrices formed according to $C_b$ and $C_w$ are important. The use of $II\beta$ or $II_{\alpha,\beta}$ can help in this case.

C. Discussion

The first experiment (Section IV-A) clearly shows that both: inconsistency and incompleteness almost equally contribute to the sensitivity of the given PC matrix. This means that when assessing the quality of the matrix its completeness cannot be ignored. On the other hand, Figures 4 and 5 suggest that when the number of missing elements is small, the impact of this deficiency on the final ranking is almost negligible. However, when a lot of comparisons are missing the ranking can be significantly changed due to incompleteness.

Since the proposed indices aim to determine not only a simple number of missing comparisons but also their distributions in way that allows the user to discover potential risks of vulnerability to disturbances. In the second experiment (Section IV-B) we analyze the influence of the distribution of missing elements to the sensitivity of the PC method and the values of incompleteness indices.

The experiments carried out show that all the indices grow (or at least do not decrease) as the number of missing values increases. Similarly all the indices get the greater values when the distribution of missing values is potentially less favorable. However, despite many similarities the values of indices and the values of sensitivity are not identical. Thus, computing

For example, the regular $7 \times 7$ PC matrix with 9 missing comparisons is as follows:

\[
C_b(9) = \begin{pmatrix}
1 & c_{12} & c^{(1)} & \cdots & c^{(n-1)} & \cdots & c^{(2n-2)} \\
1 & c_{23} & c^{(2)} & \cdots & \cdots & \cdots & c^{(n-2)} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \cdots & 1 & \cdots & c^{(n-2)} \\
\vdots & \vdots & \vdots & \cdots & c_{n-1,n} & \cdots & 1
\end{pmatrix}
\]

It is easy to observe that in $C_b(9)$ all alternatives have two missing comparisons (so each of them is compared with the three others), while in $C_w(9)$ alternative $a_1$ is compared only with $a_2$ and $a_3$ is compared only with $a_1$ and $a_3$. In the worst case the disturbances of $c_{12}$ and $c_{23}$ may lead to significant weight changes of $a_1$ and $a_2$.

The question arises to what extent the regular and irregular distribution of missing comparisons translates to the measured sensitivity, and of course to the values of incompleteness indices. In order to answer these questions, we prepared 1000 random inconsistent and incomplete PC matrices $9 \times 9$ with the average inconsistency $CI \approx 0.1$ then we removed their elements according to both: the regular $C_b(i)$ and the irregular $C_w(i)$ pattern subsequently assuming that $i = 0, 1, 2, \ldots, 28$ missing elements. Then we measured the average distance of the ranking vectors obtained from $c_b(i)$ and $c_w(i)$ and complete and not disturbed matrix, and, similarly, we computed the average value of all four indices including the compound $\alpha, \beta$-index.

In the Figures 6a and 6b we can see two plots. The lower plot on both figures represents the average sensitivity of incomplete PC matrices with the missing values distributed according to the $C_b$ scheme. The upper plot corresponds to the average sensitivity of incomplete PC matrices with the missing values distributed according to $C_w$. Both plots look quite similar. They grow as the number of missing comparisons increases, but the plot corresponding to the irregular incompleteness scheme grows faster. It is interesting to note that starting from thirteen missing comparisons the difference in sensitivity between matrices in the form $C_b$ and $C_w$ reaches almost 40%. It shows how important for the distribution of missing comparisons is.

In the similar way we tested all the indices. In the Figure 7 we can see plots of $II_{\alpha}$, $II_{\beta}$, $TI$ and $II_{\alpha,\beta}$ correspondingly.

Although all the indices rise along the increase of the number of missing values their increase differs from plots of sensitivity. For a not very high number of missing values (here 14 which is 50% of all comparisons possible to remove) all the indices seem to mimic the sensitivity charts (Fig. 6). However, for the larger numbers of missing comparisons the differences between PC matrices formed according to $C_b$ and $C_w$ are important. The use of $II_{\beta}$ or $II_{\alpha,\beta}$ can help in this case.

Note that for $n = 9$ we get $\frac{n^2 - 3n + 2}{2} = 28$. 

and analyzing the incompleteness indices can not replace the classical sensitivity analysis. Therefore, incompleteness indices should be treated as kind of a yardstick which allows to quickly detect that incompleteness can be a problem and should be improved. The great advantage of incompleteness indices is the ease of their calculation. As all of them use the number of missing comparisons on their inputs for \( n \times n \) PC matrix we need at most \( O(n^2) \) operations. Performing the sensitivity analysis usually is much more time and resource consuming. Even worse, as the sensitivity analysis tries to answer the questions how the changes in the input data translate to the method outcome, it might happen that the incompleteness as an actual source of problems can be overlooked. The indices of incompleteness eliminate danger. Due to their simplicity they are great for quick and simple test of completeness of the paired decision data.

V. Summary

This paper has developed four incompleteness indices for using with the quantitative pairwise comparisons method with incomplete set of comparisons. These indices can be used as fast and computationally simple data quality tests. The constructed indices have been tested in Montecarlo experiments. Carried trials showed a significant impact of incompleteness expressed by these indices to the sensitivity of the pairwise comparisons based decision model. Although it is clear that the incompleteness only is just one of the factors affecting sensitivity, the defined indices can help the decision makers to discover the risks to sensitivity having their source in the data incompleteness.

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