Spectral Algebras and Non-commutative Hodge-to-de Rham Degeneration

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To the blessed memory of I. R. Shafarevich

Abstract—We revisit the non-commutative Hodge-to-de Rham degeneration theorem of the first author and present its proof in a somewhat streamlined and improved form that explicitly uses spectral algebraic geometry. We also try to explain why topology is essential to the proof.

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INTRODUCTION

For any DG algebra $A$ over a field $K$, one has the Hochschild-to-cyclic, or Hodge-to-de Rham, spectral sequence

$$HH_q(A)((u)) \Rightarrow HP_q(A), \quad \deg u = 2,$$

which relates the Hochschild and periodic cyclic homology of the algebra $A$. It has been conjectured by Kontsevich and Soibelman \cite{7} that if $\text{char } K = 0$ and $A$ is smooth and proper, the spectral sequence degenerates. The conjecture has been proved under some restrictions in \cite{4}, and in full generality in \cite{5}. Recently, a slightly different proof was given by A. Mathew in \cite{9}.

This paper arose as an attempt to generalize these results to other settings of interest for applications (for example, to $\mathbb{Z}/2\mathbb{Z}$-graded DG algebras). As of now, we have not succeeded; however, we think that we can at least streamline and clarify the original proofs of \cite{4, 5}. This is the subject of the present paper.

While the degeneration statement itself is purely homological, all the available proofs use stable homotopy theory. This is quite explicit in \cite{4}, even more explicit in Mathew’s proof, and also implicitly present in \cite{5} (actually, it was deliberately hidden so as to accommodate the readers who do not like topology). The main reason why topology could possibly help can be summarized as follows.

If an algebra $A$ is smooth and proper over $K$, then its Hodge-to-de Rham spectral sequence consists of finite-dimensional $K$-vector spaces, so, by the standard criterion of Deligne, it degenerates if and only if the first page is abstractly isomorphic to the last one. More generally, the Hochschild Homology $HH(A/R)$ exists for an algebra $A$ over any commutative ring spectrum $R$, and we can ask whether there exists an isomorphism

$$HH(A/R) \otimes_R R^{tS^1} \cong HP(A/R), \quad (*)$$

where $R^{tS^1}$ stands for the Tate homotopy fixed points of the spectrum $R$ with respect to the trivial action of the circle $S^1$, and $HP(A/R) = HH(A/R)^{tS^1}$ are the Tate fixed points of $HH(A/R)$ with

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respect to the standard circle action. The homotopy groups \( \pi_*(R^{tS^1}) \) can be computed by the Atiyah–Hirzebruch spectral sequence that starts at \( \pi_*(R((u))) \). If \( R \) is orientable—for example, if \( R \) is a usual commutative ring—then the sequence degenerates, so that \( R^{tS^1} \cong R((u)) \). However, in general, it does not have to, so that \( R^{tS^1} \) can be smaller than \( R((u)) \). Under favorable circumstances, it can become so small that (*) exists for trivial reasons.

In practice, we do not know whether these “favorable circumstances” really occur. However, if one considers a cyclic subgroup \( C_p \subset S^1 \) of some prime order \( p \), then a striking result known as the Segal conjecture shows that for the sphere spectrum \( S \), the Tate fixed point spectrum \( S^{tC_p} \) is simply the \( p \)-completion \( S_p \)—that is, it is as small as it could possibly be (in particular, it is connective). This suggests that one should consider separately all primes and prove the theorem by reducing the statement at each prime \( p \) to a statement about the Tate fixed points \( HH(A/R)^{tC_p} \) that would follow from the Segal conjecture.

If one cuts down to the point, then this is exactly what happens in [4, 5]. Formally, the argument replicates the classical proof of the commutative Hodge-to-de Rham degeneration of Deligne and Illusie [2], and it works by reduction to positive characteristic. The reduction is achieved by a beautiful theorem of B. Toën stating that \( A \cong A_R \otimes_R K \) for some smooth and proper DG algebra \( A_R \) over a finitely generated subring \( R \subset K \) smooth over \( Z \). Then for each residue field \( k \) of some positive characteristic \( p \), one needs to prove degeneration for \( A_k = A_R \otimes_R k \). While in general Hodge-to-de Rham degeneration in positive characteristic is false, it still holds under additional assumptions. In [4, 5], the assumptions are that \( A \) lifts to the second Witt vectors ring \( W_2(k) \) and that the Hochschild cohomology \( HH^i(A) \) vanishes for \( i \geq 2p \). What the second assumption really means though, explicitly in [4] and implicitly in [5], is that \( A \) can be lifted to an algebra over a certain ring spectrum, a topological counterpart of the ring \( W_2(k) \). Degeneration is then due to some very truncated version of the Segal conjecture for the group \( C_p \) proved essentially by hand.

Mathew in [9] has similar assumptions but with Hochschild cohomology replaced by Hochschild homology, and this is because his strategy is different: instead of lifting a \( k \)-algebra \( A \) to a spectrum, he considers it as a spectrum as it is, and then uses deep results about topological Hochschild homology for his proof. It is hard to see how this can be improved, but in retrospect it is clear what can be done with [5]. Instead of first restricting our algebra \( A \) to a ring \( R \subset K \), then localizing \( R \) to ensure that all its residue fields \( k \) are good enough, and then lifting each reduction \( A_k \) to an algebra over a ring spectrum by obstruction theory, one should directly restrict \( A \) to an appropriate ring spectrum \( R \), so that there is no need to lift and no conditions to impose. This is the argument that we sketch in this paper.

One obvious problem with this streamlined argument is that it really has to be done topologically, and one needs an appropriate technology for that. It is more or less clear by now that ideally, one would like to have some model-independent formalism of “enhanced categories,” both stable and unstable, and this formalism should be equipped with a concise and convenient toolkit sufficient for practical applications. At present, the only existing formalism is that of \( \infty \)-categories in the sense of J. Lurie, and that is not model-independent (instead of choosing a category of models, you have to choose a model of your category). What is worse, it does not differentiate cleanly between the model-dependent and model-independent parts, and cannot be used as a black box. There is no convenient toolkit; on the contrary, a rigorous paper written in the \( \infty \)-categorical language has to rely on several thousand pages of Lurie’s foundational work and to give precise references at every second line. In principle, it is possible to do this; a perfect example is the recent paper [10]. However, it seems that the widespread practice these days is not to do this and rely instead on the reader’s conjectural capability to fill in all the missing details.

We emphasize that this is very bad practice that is certain to lead to disaster, and we choose to follow suit. Our justification is that after all, the degeneration theorem has been already proved.
Our goal is to explain the proof and show how it can be improved, not to re-do it with complete rigor. Conversely, having a concrete, detailed, and nontrivial application can show what needs to be a part of any usable future toolkit, and possibly help develop it. To emphasize the provisional nature of our results, we speak of enhanced categories and functors instead of $\infty$-categories, and we state clearly that what we have in the paper is no more than a sketch.

1. PRELIMINARIES

1.1. Enhanced categories. For any enhanced category $\mathcal{C}$, we denote by $\pi_0(\mathcal{C})$ its truncation to an ordinary category. An enhanced functor $\gamma: \mathcal{C} \to \mathcal{C}'$ induces a functor $\pi_0(\gamma): \pi_0(\mathcal{C}) \to \pi_0(\mathcal{C}')$, which we will denote simply by $\gamma$ if there is no danger of confusion. For any enhanced category $\mathcal{C}$ and small category $I$, enhanced functors from $I$ to $\mathcal{C}$ form an enhanced category $\mathcal{C}^I$. We have a natural conservative comparison functor

$$\pi_0: \pi_0(\mathcal{C}^I) \to \pi_0(\mathcal{C})^I,$$  \hfill (1.1)

and if $I = \mathbb{N}$ is the totally ordered set of positive integers considered as a small category in the usual way, then (1.1) is essentially surjective and full. A functor $\gamma: I_0 \to I_1$ induces an enhanced pullback functor $\gamma^*: \mathcal{C}^I_1 \to \mathcal{C}^I_0$. An enhanced category $\mathcal{C}$ is cocomplete if for any small $I$ the pullback functor $\tau^*: \mathcal{C} \to \mathcal{C}^I$ induced by the projection $\tau: I \to \text{pt}$ to the point category $\text{pt}$ admits a left adjoint enhanced functor $\text{hocolim}_I: \mathcal{C}^I \to \mathcal{C}$. An object $c \in \mathcal{C}$ in a cocomplete enhanced category $\mathcal{C}$ is compact if the Yoneda enhanced functor $\text{Hom}(c, -)$ commutes with $\text{hocolim}_I$ for any small filtered $I$. A cocomplete enhanced category $\mathcal{C}$ is compactly generated if the full enhanced subcategory $\mathcal{C}^{\text{pg}} \subset \mathcal{C}$ spanned by compact objects is small and for any object $c \in \mathcal{C}$ we have $c \cong \text{hocolim}_I c$, for an enhanced functor $c: I \to \mathcal{C}^{\text{pg}}$ from a filtered small category $I$. Any small enhanced category $\mathcal{C}$ canonically embeds as a fully faithful enhanced subcategory into its Ind-completion $\text{Ind}(\mathcal{C})$; this is a cocomplete compactly generated enhanced category, and any $c \in \mathcal{C}$ is compact in $\text{Ind}(\mathcal{C}) \supset \mathcal{C}$.

Small enhanced categories themselves form an enhanced category $\text{Cat}$. This category is cocomplete. The full enhanced subcategory $\text{Cat}^{\leq 1} \subset \text{Cat}$ spanned by ordinary small categories is closed under filtered homotopy colimits (but not under all colimits), and truncation defines an enhanced functor $\pi_0: \text{Cat} \to \text{Cat}^{\leq 1}$ left adjoint to the embedding. The functor $\pi_0$ commutes with filtered homotopy colimits, and filtered homotopy colimits in $\text{Cat}^{\leq 1}$ are the classical 2-colimits of ordinary categories.

We will say that an enhanced category $\mathcal{C}$ is Karoubi-closed if so is its truncation $\pi_0(\mathcal{C})$. The following useful lemma is essentially due to B. Toën.

**Lemma 1.1.** Assume given an enhanced functor $\gamma: \mathcal{C} \to \mathcal{C}'$ between cocomplete enhanced categories that preserves filtered homotopy colimits, and assume that $\pi_0(\gamma)$ is conservative and $\mathcal{C}$ is Karoubi-closed. Then $\mathcal{C}'$ is Karoubi-closed.

**Proof.** Assume given an object $c \in \mathcal{C}'$ and an idempotent endomorphism $p: c \to c$ in $\pi_0(\mathcal{C}')$, $p^2 = p$. Let $\mathcal{C}: \mathbb{N} \to \mathcal{C}'$ be the constant enhanced functor with value $c$, and consider the functor $C(p)_n: \mathbb{N} \to \pi_0(\mathcal{C}')$ sending any integer $n \in \mathbb{N}$ to $c$, with transition maps $C(p)_0(n) \to C(p)_0(n + 1)$ equal to $p$. Let $B_0: C(p)_0 \to \pi_0(\mathcal{C})$ be the map equal to $p$ at any $n \in \mathbb{N}$. Since the functor (1.1) is essentially surjective and full for $I = \mathbb{N}$, we can lift $C(p)_0$ to an enhanced functor $C(p): \mathbb{N} \to \mathcal{C}$, $\pi_0(C(p)) \cong C(p)_0$, and $B_0$ lifts to a map $B: C(p) \to C(p)$ of enhanced functors. By adjunction, the isomorphism $c \cong C(p)(0)$ induces a map $A: \mathcal{C} \to C(p)$. Since $\mathcal{C}'$ is cocomplete, $\text{hocolim}_\mathbb{N}$ exists and is functorial, and if we let $c(p) = \text{hocolim}_\mathbb{N} C(p)$, then $A$ and $B$ induce maps

$$a: \ c = \text{hocolim}_\mathbb{N} C \to c(p), \quad b: \ c(p) \to c.$$

Again by adjunction, we have $b \circ a = p$. If the idempotent $p$ does have an image $c'$—that is, we have $c' \in \mathcal{C}'$ and maps $a': c \to c'$ and $b': c' \to c$ such that $b' \circ a' = p$ and $a' \circ b' = \text{id}$ in $\pi_0(\mathcal{C}')$—then
one easily checks that the composition \( a \circ b' : c' \to c(p) \) is an isomorphism, so that \( a \circ b = \text{id} \) by the uniqueness of idempotent images. If not, then since \( \gamma \) commutes with filtered homotopy colimits and \( \mathcal{C} \) is Karoubi-closed, we at least see that \( \gamma(a \circ b) = \gamma(a) \circ \gamma(b) = \text{id} \) in \( \pi_0(\mathcal{C}) \). But since \( \gamma \) is conservative, this implies that \( a \circ b \) is invertible, and then

\[
(a \circ b)^3 = a \circ (b \circ a)^2 \circ b = a \circ p^2 \circ b = a \circ p \circ b = (a \circ b)^2,
\]

so that \( a \circ b = \text{id} \).  

1.2. Spectral algebras. We denote by \( \mathcal{D}(\mathbb{S}) \) the stable enhanced category of spectra. It is cocomplete, compactly generated, and Karoubi-closed (the latter is slightly nontrivial since, for example, the enhanced category of unpointed homotopy types is not). It also carries a natural structure of a symmetric monoidal enhanced category, and the enhanced categories \( \mathcal{D} \text{Alg}(\mathbb{S}) \) and \( \mathcal{D} \text{Comm}(\mathbb{S}) \) of \( E_1 \)- and \( E_\infty \)-algebras in \( \mathcal{D}(\mathbb{S}) \), respectively, are also cocomplete. The stable enhanced category \( \mathcal{D}(\mathbb{S}) \)—or, strictly speaking, its triangulated truncation \( \pi_0(\mathcal{D}(\mathbb{S})) \)—carries a natural \( t \)-structure, with \( \mathcal{D}^{\leq 0}(\mathbb{S}) \subset \mathcal{D}(\mathbb{S}) \) consisting of connective spectra, and a spectrum is discrete if it lies in the heart of this natural \( t \)-structure. Sending \( E \) to \( \pi_0(E) \) identifies the heart with the category of abelian groups. An \( E_\infty \)-algebra in \( \mathcal{D}(\mathbb{S}) \) that is discrete is the same thing as a unital associative commutative ring.

For any positive integer \( N \), we let \( \mathbb{S}(N^{-1}) \) be the localization of the sphere \( \mathbb{S} \) in \( N \). We note that

\[
\mathbb{Q} \cong \text{hocolim}_N \mathbb{S}(N^{-1}),
\]

where the colimit is taken with respect to the divisibility order and \( \mathbb{Q} \) is the field of rationals considered as a discrete \( E_\infty \)-algebra in \( \mathcal{D}(\mathbb{S}) \).

For any \( E_1 \)-algebra \( A \in \mathcal{D} \text{Alg}(\mathbb{S}) \), we have the cocomplete stable enhanced category \( \mathcal{D}(A) \) of left \( A \)-modules, and for any \( E_\infty \)-algebra \( R \) in \( \mathcal{D} \text{Comm}(\mathbb{S}) \), we have the cocomplete stable symmetric monoidal enhanced category \( \mathcal{D}(R) \) of \( R \)-modules and the cocomplete symmetric monoidal enhanced categories \( \mathcal{D} \text{Alg}(R) \) and \( \mathcal{D} \text{Comm}(R) \) of \( E_1 \)- and \( E_\infty \)-algebras in \( \mathcal{D}(R) \), respectively. A map \( R \to R' \) between \( E_1 \)- or \( E_\infty \)-algebras induces an adjoint pair of the tensor product functor \( \mathcal{D}(R) \to \mathcal{D}(R') \), \( M \mapsto R' \otimes_R M \), and the restriction functor \( \mathcal{D}(R') \to \mathcal{D}(R) \). In the \( E_\infty \)-case, the tensor product functor is symmetric monoidal, while the restriction functor is lax symmetric monoidal by adjunction (in the \( \infty \)-categorical setup, this is \([8, \text{Corollary 7.3.2.7}]\)); therefore, they induce adjoint pairs of functors between \( \mathcal{D} \text{Alg}(R) \) and \( \mathcal{D} \text{Alg}(R') \), and between \( \mathcal{D} \text{Comm}(R) \) and \( \mathcal{D} \text{Comm}(R') \). In all these adjoint pairs, the restriction functor commutes with filtered colimits, so that by adjunction the tensor product functor sends compact objects to compact objects.

The enhanced category \( \mathcal{D}(R) \) is compactly generated but there is more. Namely, the forgetful functor \( \mathcal{D}(R) \to \mathcal{D}(\mathbb{S}) \) has a left adjoint free module functor \( F : \mathcal{D}(\mathbb{S}) \to \mathcal{D}(R) \), \( F(V) = V \otimes_S R \), and an object \( M \in \mathcal{D}(R) \) is finitely presented if it is a finite homotopy colimit of objects of the form \( F(E) \), \( E \in \mathcal{D}(\mathbb{S})^{pf} \). Then any object in \( \mathcal{D}(R) \) is a filtered homotopy colimit of finitely presented objects. Since filtered colimits commute with finite limits, any finitely presented object in \( \mathcal{D}(R) \) is compact, and so are its retracts. Conversely, since an isomorphism \( M \cong \hocolim_I M_i \) with filtered \( I \) and compact \( M \) must factor through some \( M_i \), a compact object is a retract of a finitely presented one. Exactly the same holds for \( \mathcal{D} \text{Alg}(R) \), \( \mathcal{D} \text{Comm}(R) \), and \( \mathcal{D}(A) \) for any \( A \in \mathcal{D} \text{Alg}(R) \).

Moreover, the forgetful functor is conservative and commutes with filtered homotopy colimits, so that \( \mathcal{D}(R) \) is Karoubi-closed by Lemma 1.1, and again the same holds for \( \mathcal{D} \text{Alg}(R) \), \( \mathcal{D} \text{Comm}(R) \), and \( \mathcal{D}(A) \). Furthermore, we have the full subcategories \( \mathcal{D}^{\leq 0}(\mathbb{S}) \subset \mathcal{D}(\mathbb{S}) \), \( \mathcal{D} \text{Alg}^{\leq 0}(\mathbb{S}) \subset \mathcal{D} \text{Alg}(\mathbb{S}) \), and \( \mathcal{D} \text{Comm}^{\leq 0}(\mathbb{S}) \subset \mathcal{D} \text{Comm}(\mathbb{S}) \) spanned by connective spectra; these are also compactly generated, and so are \( \mathcal{D}^{\leq 0}(R) \subset \mathcal{D}(R) \), \( \mathcal{D} \text{Alg}^{\leq 0}(R) \subset \mathcal{D} \text{Alg}(R) \), and \( \mathcal{D} \text{Comm}^{\leq 0}(R) \subset \mathcal{D} \text{Comm}(R) \) for any connective \( E_\infty \)-algebra \( R \in \mathcal{D} \text{Comm}^{\leq 0}(\mathbb{S}) \).
Remark 1.2. Compact objects in $\mathcal{D}(R)$ are also known as perfect $R$-modules; this explains our notation (although one usually writes $\mathcal{D}^{\text{pf}}(R)$ instead of $\mathcal{D}(R)^{\text{pf}}$). For algebras, there is no standard terminology. Toën calls compact algebras homotopically finitely presented.

For any $R \in \mathcal{DComm}(S)$ and $A \in \mathcal{DAlg}(R)$, the cocomplete enhanced category $\mathcal{D}(A)$ coincides with the Ind-completion $\text{Ind}(\mathcal{D}(A)^{\text{pf}})$ of its full subcategory of compact objects. Aside from compactness, there is another useful finiteness condition one can impose on $A$-modules: an $A$-module $M \in \mathcal{D}(A)$ is coherent if it is compact as an object in $\mathcal{D}(R)$. We note that unlike compactness, the property of being coherent is preserved by restriction via an algebra map. In fact, any compact object in $\mathcal{D}(R)$ is dualizable, so that we have the endomorphism algebra $\text{End}_R(M) \in \mathcal{DAlg}(R)$, and $M$ is canonically an $\text{End}_R(M)$-module. Then $M$ is tautologically coherent over $\text{End}_R(M)$, and any structure of an $A$-module on $M$ is induced from this canonical $\text{End}_R(M)$-module structure by restriction via an action map $a : A \to \text{End}_R(M)$ in $\mathcal{DAlg}(R)$. We denote by $\mathcal{D}(A)^{\text{coh}} \subset \mathcal{D}(A)$ the full subcategory spanned by coherent modules, and we note that its Ind-completion $\text{Ind}(\mathcal{D}(A)^{\text{coh}})$ is in general different from $\mathcal{D}(R)$.

For any $E_\infty$-algebra $R \in \mathcal{DComm}(S)$, sending an $E_1$- or an $E_\infty$-algebra $R'$ over $R$ to the enhanced category $\mathcal{D}(R')^{\text{pf}}$ of compact $R$-modules gives enhanced functors

$$\mathcal{D}^{\text{pf}} : \mathcal{DAlg}(R), \mathcal{DComm}(R) \to \text{Cat},$$

while sending $R \in \mathcal{DComm}(S)$ to $\mathcal{DAlg}(R)^{\text{pf}}$ or $\mathcal{DComm}(R)^{\text{pf}}$ gives functors

$$\mathcal{DAlg}^{\text{pf}}, \mathcal{DComm}^{\text{pf}} : \mathcal{DComm}(S) \to \text{Cat}.$$  \hspace{1cm} (1.4)

We will need the following fundamental fact.

Proposition 1.3. The enhanced functors (1.3) and (1.4) commute with filtered homotopy colimits.

Outline of a proof. The argument is the same in all cases. For finitely presented objects $M = \text{hocolim}_I F(E_i)$, $I$ finite, the proof is a straightforward induction on the cardinality of $I$. In general, use the characterization of compact objects as retracts of finitely presented ones, and observe that as we have already proved, the necessary retractions must also appear at some finite level. \hspace{1cm} $\Box$

2. FORMAL SMOOTHNESS

For any $E_\infty$-algebra $A \in \mathcal{DComm}(S)$, any $E_\infty$-algebra $R \in \mathcal{DComm}(A)$, and any $R$-module $M \in \mathcal{D}(R)$, we have the split square-zero extension $R \oplus M \in \mathcal{D}(A)$ of $R$ by $M$, and derivations from $R$ to $M$ are splittings $R \to R \oplus M$ of the augmentation map $R \oplus M \to R$. Derivations form an enhanced functor $\text{Der} : \mathcal{D}(R) \to \mathcal{D}(S)$ that is representable by the cotangent module $\Omega(R/A) \in \mathcal{D}(R)$. If $R$ is compact in $\mathcal{DComm}(A)$, then $\Omega(R/A)$ is compact in $\mathcal{D}(R)$. The same module also controls non-split square-zero extensions. In particular, if there are no maps from $\Omega(R/A)$ to the homological shift $M[1]$ of some $M \in \mathcal{D}(R)$, then any square-zero extension

$$M \to R' \to R$$

in $\mathcal{DComm}(A)$ admits a splitting $R \to R'$. The cotangent module $\Omega(-/A)$ is functorial in the appropriate sense and commutes with filtered colimits: for any enhanced functor $R_\ast : I \to \mathcal{DComm}(A)$ with small filtered $I$ and $R = \text{hocolim}_I R_\ast$, we have an enhanced functor from $I$ to $\mathcal{D}(R)$ with values $\Omega(R_i/A) \otimes_{R_i} R_i$ and a natural isomorphism

$$\Omega(R/A) \cong \text{hocolim}_I \Omega(R_\ast, /A) \otimes_{R_\ast} R.$$  \hspace{1cm} (2.1)
Remark 2.1. In the ∞-categorical setting, the sketch above corresponds to [8, Sects. 7.3, 7.4]; however, for some reason, the logic there is reversed: instead of first defining square-zero extensions (for example, by considering the natural symmetric monoidal structure on the filtered version of $\mathcal{D}(S)$), Lurie first defines derivations. The end result is the same.

For any $E_\infty$-algebra $R \in DComm(S)$ and any set $S$, we have the free $R$-module $R[S] \in \mathcal{D}(R)$ generated by $S$. We say that $M \in \mathcal{D}(R)$ is projective if it is a retract of a free $R$-module $R[S]$, and finitely generated projective if $S$ can be chosen to be finite. A finitely generated projective module is compact, and conversely, a compact projective module is finitely generated.

Definition 2.2. For any connective $A \in DComm(S)$, an $E_\infty$-algebra $R \in DComm(A)$ is formally smooth if it is connective and compact in $DComm(A)$ and $\Omega(R/A)$ is a projective $R$-module.

If $A = \mathbb{Q}$ is the field of rationals, then $\mathcal{D}(\mathbb{Q})$ is the derived category of complexes of $\mathbb{Q}$-vector spaces, $DComm(\mathbb{Q})$ is the category of commutative DG algebras over $\mathbb{Q}$, and $A \in DComm(\mathbb{Q})$ is formally smooth if and only if it is a finitely generated smooth $\mathbb{Q}$-algebra placed in the homological degree 0. Over $\mathbb{S}$, formally smooth algebras are not so easy to describe. However, observe that if $R \in DComm(S)^{pl}$ is formally smooth, then $\pi_0(R)$ is at least a finitely generated commutative ring.

Proposition 2.3. For any field $K$ of characteristic 0, there exists an isomorphism $K \cong \text{hocolim}_I R$, for some small filtered $I$ and an enhanced functor $R_I : I \to DComm(S)^{pl}$ whose values $R_i$, $i \in I$, are formally smooth in the sense of Definition 2.2.

Proof. Since $DComm(S)$ is compactly generated, we may assume that $K \cong \text{hocolim}_I R$, for some small filtered $I$ and $R_I : I \to DComm(S)^{pl}$. Moreover, since $K$ is connective and $DComm^{\leq 0}(S)$ is also compactly generated, we may assume that all the $R_i$ are connective. What we need to check is that one can arrange for them to be formally smooth. For this, it suffices to show that any map $r : R \to K$ from a compact connective $R \in D(S)^{pl}$ factors through a formally smooth $E_\infty$-algebra $C$.

Indeed, any finitely generated subring $C_0 \subset K$ lies in a finitely generated smooth $\mathbb{S}$-algebra $C \subset K$. Since $R$ is connective, we have the augmentation map $a : R \to \pi_0(R)$, and $r = b \circ a$ for some map $b : \pi_0(R) \to K$. Then $\pi_0(R)$ is finitely generated, and taking $C_0 = \text{Im} b$, we see that $r$ factors through a finitely generated smooth $\mathbb{Q}$-subalgebra $C \subset K$. But then, by Proposition 1.3, $DComm^{pl}$ commutes with filtered homotopy colimits, and in particular, it commutes with the colimit (1.2).

Thus $C = C_N \otimes_{\mathbb{S}[N^{-1}]} \mathbb{Q}$ for some positive integer $N$. Moreover, since $C$ is formally smooth over $\mathbb{Q}$, we have maps $a : \Omega(C/\mathbb{Q}) \to \mathbb{Q}[S]$ and $b : \mathbb{Q}[S] \to \Omega(C/\mathbb{Q})$, $b \circ a = \text{id}$, for some finite set $S$, and again by Proposition 1.3, we can assume after enlarging $N$ that both are induced by maps

$$a_N : \Omega(C_N/\mathbb{S}(N^{-1})) \to \mathbb{S}(N^{-1})[S] \quad \text{and} \quad b_N : \mathbb{S}(N^{-1})[S] \to \Omega(C_N/\mathbb{S}(N^{-1}))$$

such that $b_N \circ a_N = \text{id}$. Therefore, $C_N$ is formally smooth over $\mathbb{S}(N^{-1})$, and then also over $\mathbb{S}$ since $\mathbb{S} \to \mathbb{S}(N^{-1})$ is a localization. Finally, since $R$ is compact, we can again enlarge $N$ so that the map $R \to C \cong \text{hocolim}_N C_N$ factors through $C_N$, and this finishes the proof. \square

Now, for any prime $p$, denote by $\mathbb{S}_p \in DComm(S)$ the $p$-completion of the sphere $\mathbb{S}$, with its natural map $\mathbb{S}_p \to \mathbb{F}_p$; and for any power $q = p^n$ of $p$, let $\mathbb{S}_q$ be the $n$-fold Galois extension of $\mathbb{S}_p$, with its map $\mathbb{S}_q \to \mathbb{F}_q$ (since $\mathbb{F}_q$ is étale over $\mathbb{F}_p$, the cotangent complex $\Omega(\mathbb{F}_q/\mathbb{F}_p)$ vanishes, so that $\mathbb{S}_q$ exists and is unique).

Lemma 2.4. Assume given an algebra $R \in D(\mathbb{S})$ formally smooth in the sense of Definition 2.2. Then for any finite field $k = \mathbb{F}_q$, any map $a : R \to k$ factors through the canonical map $\mathbb{S}_q \to k$.

Proof. The completed sphere $S = \mathbb{S}_q$ is the homotopy limit of an enhanced functor $S_i : \mathbb{N}^\circ \to DComm(S)$, $n \geq 1$, where $S_1 = k$ and each $S_{n+1}$ is a square-zero extension of $S_n$ by a connective $k$-module $M \in D(k)$. Since (1.1) is full and essentially surjective for $I = \mathbb{N}$, it suffices to extend $a_1 = a : R \to S_1 = k$ to a compatible system of factorizations $a_n : R \to S_n$, $n \geq 2$. This
can be done by induction: at each step, the obstruction to lifting $a_n$ to $a_{n+1}$ lies in the group $\text{Hom}_k(\Omega(R/S), M[1]) \cong \text{Hom}_k(\Omega(R/S) \otimes_S k, M[1])$, and since $\Omega(R/S) \otimes_S k$ is projective and $M$ is connective, this group is trivial. □

3. TOÈN THEOREM

Now fix an $E_\infty$-algebra $R \in \mathcal{D}\text{Comm}(\mathcal{S})$, and assume given some $E_1$-algebra $A \in \mathcal{D}\text{Alg}(R)$. Then $A$ itself can be considered not only as a left $A$-module $A \in \mathcal{D}(A)$ but also as an $R$-module $A \in \mathcal{D}(R)$ and as the diagonal $A$-bimodule $A \in \mathcal{D}(A^\circ \otimes_R A)$, where $A^\circ$ stands for the opposite $E_1$-algebra.

**Definition 3.1.** The algebra $A \in \mathcal{D}\text{Alg}(R)$ is proper (smooth) if $A$ is compact as an object in $\mathcal{D}(R)$ (in $\mathcal{D}(A^\circ \otimes_R A)$, respectively).

Both smoothness and properness are functorial with respect to $R$, so that sending $R$ to the enhanced category $\mathcal{D}\text{Alg}^{\text{sat}}(R)$ of smooth and proper $E_1$-algebras in $\mathcal{D}\text{Alg}(R)$ gives an enhanced functor

$$\mathcal{D}\text{Alg}^{\text{sat}} : \mathcal{D}\text{Comm}(\mathcal{S}) \to \text{Cat}.$$  

(3.1)

The following beautiful theorem has been essentially proved by B. Toën.

**Theorem 3.2.** The functor (3.1) commutes with filtered homotopy colimits.

Strictly speaking, Toën in [12] only considered the situations when $R$ is a commutative ring; let us recall the argument to see that it works for spectral algebras with no changes whatsoever.

**Definition 3.3.** Assume given an $E_\infty$-algebra $R \in \mathcal{D}\text{Comm}(\mathcal{S})$ and two $E_1$-algebras $A, B \in \mathcal{D}\text{Alg}(R)$, and introduce the notation $\mathcal{D}(A, B) = \mathcal{D}(A \otimes_R B)$. Then an object $M \in \mathcal{D}(A, B)$ is coherent if it is compact as a $B$-module.

Toën uses the term “pseudoperfect” instead of “coherent,” but “coherent” is shorter. It is also consistent with earlier terminology: for any $A \in \mathcal{D}\text{Alg}(R)$, we have $\mathcal{D}(A, R) = \mathcal{D}(A \otimes_R R) = \mathcal{D}(A)$, and this identification identifies coherent objects. For any $A, B \in \mathcal{D}\text{Alg}$, we denote by $\mathcal{D}(A, B)^{\text{coh}} \subset \mathcal{D}(A, B)$ the full enhanced subcategory spanned by coherent objects. We observe that for any $A$ the diagonal bimodule $A \in \mathcal{D}(A^\circ, A) = \mathcal{D}(A^\circ \otimes_R A)$ is always coherent.

**Lemma 3.4.** An $E_1$-algebra $A \in \mathcal{D}\text{Alg}(R)$ is smooth (respectively, proper) if and only if for any $B \in \mathcal{D}\text{Alg}(R)$ we have $\mathcal{D}(A^\circ, B)^{\text{coh}} \subset \mathcal{D}(A^\circ, B)^{\text{pf}}$ (respectively, $\mathcal{D}(A^\circ, B)^{\text{pf}} \subset \mathcal{D}(A^\circ, B)^{\text{coh}}$).

**Proof.** For smoothness, note that the free right $A$-module $A \in \mathcal{D}(A^\circ)$ is compact, so that if $\mathcal{D}(A^\circ)^{\text{pf}} \subset \mathcal{D}(A^\circ)^{\text{coh}}$, then $A$ is coherent, that is, compact over $R$. Conversely, being coherent is closed under retracts and finite homotopy colimits, so it suffices to check that if $A$ is proper, then $A^\circ \otimes_R S \otimes_R B$ is coherent for any compact $S \in \mathcal{D}(R)$, which is obvious.

For smoothness, recall that $A \in \mathcal{D}(A^\circ, A)$ lies in $\mathcal{D}(A^\circ \otimes_R A)^{\text{coh}}$, so that if $\mathcal{D}(A^\circ, A)^{\text{coh}} \subset \mathcal{D}(A^\circ, A)^{\text{pf}}$, it is compact. Conversely, note that for any $B$, any coherent $M \in \mathcal{D}(A^\circ, B)$, and any compact $N \in \mathcal{D}(A^\circ, A)$, $N \otimes_A M \in \mathcal{D}(A^\circ, B)$ is compact. Indeed, it again suffices to check this for $N = A^\circ \otimes_R S \otimes_R A$ for some compact $S \in \mathcal{D}(R)$, and then $N \otimes_A M \cong A \otimes_R S \otimes_R M$. But then if $A$ is smooth, any coherent $M$ is isomorphic to $A \otimes_A M$ in $\mathcal{D}(A^\circ, B)$ and is therefore compact. □

**Lemma 3.5.** A smooth and proper $E_1$-algebra $A \in \mathcal{D}\text{Alg}(R)$ is compact.

**Proof.** For any two algebras $A, B \in \mathcal{D}\text{Alg}(R)$, the Hom-space $\text{Hom}(A, B)$ of the enhanced category $\mathcal{D}\text{Alg}(R)$ fits into a functorial homotopy cartesian square

$$\begin{array}{ccc}
\text{Hom}(A, B) & \longrightarrow & \text{Iso}(\mathcal{D}(A^\circ, B)^{\text{coh}}) \\
\downarrow & & \downarrow \\
\text{pt} & \longrightarrow & \text{Iso}(\mathcal{D}(B)^{\text{pf}})
\end{array}$$

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where Iso stands for the enhanced isomorphism groupoid of an enhanced category, the rightmost arrow is the forgetful functor, and the bottom arrow is the embedding onto \( B \in \mathcal{D}(B) \). If \( A \) is smooth and proper, we can replace coherent objects with compact ones by Lemma 3.4 and then recall that \( \mathcal{D}^{\text{pf}} \) commutes with filtered homotopy limits by Proposition 1.3. Since filtered homotopy colimits commute with finite homotopy limits, this proves that \( \text{Hom}(A, -) \) also commutes with filtered homotopy colimits. □

**Lemma 3.6.** A compact algebra \( A \in \mathcal{DA}_{\text{lg}}(R) \) is smooth.

**Proof.** For any bimodule \( M \in \mathcal{D}(A^o, A) \), we have the split square-zero extension \( A \oplus M \in \mathcal{DA}_{\text{lg}}(R) \), and its splittings \( A \to A \oplus M \) correspond to maps \( I_A \to M \) from a non-commutative version \( I_A \in \mathcal{D}(A^o \otimes_R A) \) of the cotangent module. This module \( I_A \) fits into an exact triangle

\[
I_A \to A^o \otimes_R A \to A \to \]

in the triangulated category \( \pi_0(\mathcal{D}(A^o, A)) \); thus it is compact if and only if so is the diagonal bimodule \( A \). □

**Proof of Theorem 3.2.** By Lemma 3.5, we have a full embedding \( \mathcal{DA}_{\text{lg}}^{\text{sat}}(R) \subset \mathcal{DA}_{\text{lg}}(R)^{\text{pf}} \) for any \( R \in \mathcal{D}_{\text{Comm}}(\mathbb{S}) \), so that in particular \( \mathcal{DA}_{\text{lg}}^{\text{sat}}(R) \) is small, and then \( \mathcal{DA}_{\text{lg}}^{\text{pf}} \) commutes with filtered homotopy colimits by Proposition 1.3. Therefore, for any enhanced functor \( R_* : I \to \mathcal{D}_{\text{Comm}}(\mathbb{S}) \) with small filtered \( I \) and \( R \cong \text{hocolim}_I R_* \), the functor

\[
\text{hocolim}_I \mathcal{DA}_{\text{lg}}^{\text{sat}}(R_*) \to \mathcal{DA}_{\text{lg}}^{\text{sat}}(R)
\]

is fully faithful, and we only need to check that it is essentially surjective. In other words, we may assume given an algebra \( A_i \in \mathcal{DA}_{\text{lg}}(R_i)^{\text{pf}} \) such that \( A = A_i \otimes_{R_i} R \) is proper, and we need to show that for some map \( i \to i' \), \( A_{i'} = A_i \otimes_{R_i} R_{i'} \) is already proper (while smoothness is guaranteed by Lemma 3.6).

Since \( A \) is proper and \( \mathcal{D}^{\text{pf}} \) commutes with filtered homotopy colimits, we may assume that \( A \) is isomorphic as an \( R \)-module to \( M_{i'} \otimes_{R_{i'}} R \) for some \( i' \in I \) and \( M_{i'} \in \mathcal{D}(R_{i'}) \). Since \( I \) is filtered, we can choose \( i'' \in I \) with maps \( i \to i'' \) and \( i' \to i'' \), and then replacing \( I \) with \( i'' \setminus I \), we may assume that \( I \) has an initial object \( o \), \( A_o \in \mathcal{DA}_{\text{lg}}(R_o) \) is compact, and we have an isomorphism of \( R \)-modules \( A_o \otimes_{R_o} R \cong M_o \otimes_{R_o} R \) for some \( M_o \in \mathcal{D}(R_o)^{\text{pf}} \). Let \( A_i = A_o \otimes_{R_o} R_i \) and \( M_i = M_o \otimes_{R_o} R_i \), \( i \in I \). Since \( A \) is a coherent \( A \)-module, its \( A \)-module structure is induced by restriction via an action map \( a : A \to \text{End}_R(A) \). By restriction, \( \text{End}_R(A) \) is an \( R_o \)-algebra, and the map \( a \) is adjoint to a map \( a_o : A_o \to \text{End}_R(A) \) in \( \mathcal{DA}_{\text{lg}}(R_o) \). However,

\[
\text{End}_R(A) \cong \text{End}_{R_o}(M_o) \otimes_{R_o} R \cong \text{hocolim}_I \text{End}_{R_i}(M_i),
\]

and since \( A_o \in \mathcal{DA}_{\text{lg}}(R_o) \) is compact, the map \( a_o \) factors through some map \( A_o \to \text{End}_{R_i}(M_i) \), \( i \in I \), adjoint to a map \( a_i : A_i \to \text{End}_{R_i}(M_i) \) in \( \mathcal{DA}_{\text{lg}}(R_i) \). Then by restriction, \( M_i \) becomes a coherent \( A_i \)-module, and since \( A_i \) is compact, \( M_i \in \mathcal{D}(A_i) \) is compact by Lemmas 3.6 and 3.4. Thus we have two compact \( A_i \)-modules, \( M_i \) and \( A_i \) itself, and an isomorphism \( A_i \otimes_{R_i} R \cong M_i \otimes_{R_i} R \) in \( \mathcal{D}(A) \). Since \( \mathcal{D}^{\text{pf}} \) commutes with filtered homotopy colimits by Proposition 1.3, this isomorphism must be induced by an isomorphism \( A_{i'} \cong M_{i'} \) in \( \mathcal{D}(A_{i'}) \) for some \( i' \in I \). But \( M_{i'} \in \mathcal{D}(A_{i'}) \) is not only compact but also coherent, so that \( A_{i'} \) must be proper. □

4. TATE DIAGONAL

Recall that for any \( R \in \mathcal{D}_{\text{Comm}}(\mathbb{S}) \) and any set \( S \), we denote by \( R[S] \) the direct sum of copies of \( R \) numbered by elements \( s \in S \). More generally, for a topological space \( X \), we denote by \( R[X] \in \mathcal{D}(\mathbb{S}) \) the \( R \)-homology spectrum of \( X \). If \( X = G \) is a compact Lie group, then \( R[G] \) is
an \(E_1\)-algebra in \(D\text{Alg}(R)\) with respect to the Pontryagin product, and the projection \(G \to \text{pt}\) induces the augmentation \(E_1\)-map \(R[G] \to R\). Restricting with respect to the augmentation gives a tautological embedding \(a: D(R) \to D(R[G])\) that has adjoints on the left and on the right, \(M \mapsto M_{hG}\) and \(M \mapsto M^{hG}\), respectively, known as the homotopy quotient and the homotopy fixed points functors. If \(R\) is discrete (thus, simply a ring) and the group \(G\) is finite, then \(D(R[G])\) is the derived category of \(R\)-linear representations of the group \(G\), the homotopy quotient is the group homology, and the homotopy fixed points are the group cohomology. In the general situation, the diagonal embedding \(G \to G \times G\) turns \(D(R[G])\) into a symmetric monoidal enhanced category, the tautological embedding \(a\) is symmetric monoidal, and the homotopy fixed points functor is lax symmetric monoidal by adjunction. Thus in particular, \(R^{hG}\) is naturally an \(E_\infty\)-algebra in \(D(R)\), and the homotopy fixed points functor can be refined to a functor

\[
D(R[G]) \to D(R^{hG}), \quad M \mapsto M^{hG}.
\]  

(4.1)

Since \(G\) is assumed to be compact, the algebra \(R[G]\) is proper, so that we have a full embedding

\[
D(R[G])^{\text{pf}} \subset D(R[G])^{\text{coh}}
\]

and the induced embedding

\[
\text{Ind}(D(R[G])^{\text{pf}}) = D(R[G]) \subset D(R[G])^{\text{coh}}.
\]

(4.3)

However, \(R[G]\) is usually not smooth, so that the embeddings (4.2) and (4.3) are not equivalences. We then have a nontrivial enhanced Verdier quotient

\[
D(R[G])^{\text{sing}} = D(R[G])^{\text{coh}} / D(R[G])^{\text{pf}}.
\]

The subcategory \(D(R[G])^{\text{coh}} \subset D(R[G])\) is symmetric monoidal, and the subcategory \(D(R[G])^{\text{pf}} \subset D(R[G])^{\text{coh}}\) is a symmetric monoidal ideal, so that \(D^{\text{sing}}(R[G])\) is also a symmetric monoidal enhanced category in a natural way. On the level of Ind-completions, (4.3) induces a semiorthogonal decomposition

\[
\text{Ind}(D(R[G])^{\text{coh}}) = \langle \text{Ind}(D(R[G])^{\text{sing}}), D(R[G]) \rangle.
\]

(4.4)

The stable enhanced categories \(\text{Ind}(D(R[G])^{\text{coh}})\) and \(\text{Ind}(D(R[G])^{\text{sing}})\) are symmetric monoidal, and so is the projection

\[
l: \text{Ind}(D(R[G])^{\text{coh}}) \to \text{Ind}(D(R[G])^{\text{sing}})
\]

onto the first factor of the decomposition (4.4). The augmentation functor \(D(R) \to D(R[G])\) composed with (4.3) and the projection \(l\) provides a symmetric monoidal functor \(D(R) \to \text{Ind}(D(R[G])^{\text{sing}})\) that has a right adjoint \(Tate\ fixed\ points\) functor

\[
\text{Ind}(D(R[G])^{\text{sing}}) \to D(R), \quad M \mapsto M^{tG}.
\]

(4.6)

By abuse of notation, we will write \(M^{tG} = l(M)^{tG}\) for any \(M\) in the category \(\text{Ind}(D(R[G])^{\text{coh}})\) (and in particular, for any coherent \(M \in D(R[G])\)). The functor (4.6) is lax symmetric monoidal by adjunction, so that \(R^{tG}\) is an \(E_\infty\)-algebra in \(D(R)\), and then as in (4.1), (4.6) can be canonically refined to an enhanced functor

\[
\text{Ind}(D(R[G])^{\text{sing}}) \to D(R^{tG}), \quad M \mapsto M^{tG}.
\]

(4.7)

For any \(M \in D(R[G])^{\text{coh}}\), the decomposition (4.4) induces an exact triangle

\[
M_{hG}[d] \xrightarrow{l} M^{hG} \to M^{tG} \to,
\]

(4.8)
where $d = \dim G$ is the dimension of $G$ and $t$ is a natural trace map induced by the Poincaré duality on $G$ (if the group $G$ is finite, $t$ is just the averaging over the group).

Sometimes Tate fixed points can be computed by localizing the usual homotopy fixed points with respect to certain elements in the homotopy groups of $R^{hG}$. The basic example is $G = S^1$, the unit circle. If (and only if) $R \in \mathcal{D}\text{Comm}(S)$ is orientable as a multiplicative generalized cohomology theory (for example, if $R$ is discrete), we have $\pi_*(R^{hS^1}) \cong \pi_*(R)[u]$, where $u$ is a single generator of cohomological degree 2. In this case, $\pi_*(R^{G}) = \pi_*(R)[u, u^{-1}]$, and for any $M \in \mathcal{D}(R[S^1])^{\text{coh}}$, we have

$$M^{tS^1} \cong M^{hS^1} \otimes_{R^{hS^1}} R^{S^1} \cong \text{hocollim}_n M^{tS^1}[2n],$$

(4.9)

where the colimit is taken with respect to the action $u: M^{hS^1} \rightarrow M^{hS^1}[2]$ of the generator $u \in \pi_2(R^{hS^1})$.

Another example is when $G = C_p \subset S^1$ is the cyclic group of some prime order $p \geq 3$ and $R$ is a ring annihilated by $p$. In this case, $\pi_*(R^{hC_p}) \cong R(\varepsilon, u)$, where $u$ has cohomological degree 2, $\varepsilon$ has cohomological degree 1, and they commute. Tate fixed points $R^{tC_p}$ are again obtained by inverting $u$, and for any coherent $M \in \mathcal{D}(R[C_p])$, we again have

$$M^{tC_p} \cong \text{hocollim}_n M^{hC_p}[2n],$$

(4.10)

with colimit taken with respect to the action of $u$.

If $R$ is not orientable, $\pi_*(R^{S^1})$ is still the abutment of an Atiyah–Hirzebruch spectral sequence whose first page is $\pi_*(R)[u]$, but the spectral sequence does not degenerate, and the periodicity element $u$ does not survive to the last page. We do not know any general method to compute $R^{tS^1}$. The situation for the cyclic group is similar; however, the following striking result holds.

**Lemma 4.1.** Let $R = \mathbb{S}_q$, $q = p^n$, be the $n$-fold étale covering of the $p$-completion $\mathbb{S}_p$ of the sphere, for some $n \geq 1$ and some prime $p$. For any $M \in \mathcal{D}(R)$, consider $M^{\otimes R_p}$ as an object in $\mathcal{D}(R[C_p])$ via the longest cycle permutation action. Then there is a map

$$M \rightarrow (M^{\otimes R_p})^{tC_p},$$

(4.11)

functorial in $M$, and this map is an isomorphism if $M$ is compact. □

This is a version of the Segal conjecture (see [10, Sect. III.1] and references therein). Nikolaus and Scholze call (4.11) the Tate diagonal map. The essential part of the proof is the case $M = R$ (when $M^{\otimes R_p}$ is again $R$ with the trivial $C_p$-action).

Our proof of the Hodge-to-de Rham degeneration relies on an immediate corollary of Lemma 4.1. Observe that for any map $R_0 \rightarrow R_1$ in $\mathcal{D}\text{Comm}(S)$, the augmentation embedding commutes with the tensor product functor $- \otimes_{R_0} R_1$, so that by adjunction we obtain a functorial map

$$M^{tG} \otimes_{R_0^{tG}} R_1^{tG} \rightarrow (M \otimes_{R_0} R_1)^{tG}$$

(4.12)

for any coherent $M$ in $\mathcal{D}(R_0[G])$, where $M^{tG}$ is considered as an $R_0^{tG}$-module via the refinement (4.7).

**Corollary 4.2.** Let $R$ be as in Lemma 4.1, and let $k = \mathbb{F}_q$, $q = p^n$, be the degree-$n$ Galois extension of the prime field $\mathbb{F}_p$, with the natural map $R \rightarrow k$. Then for any compact $M \in \mathcal{D}(R)$ with $M_k = M \otimes_R k$, the map

$$M_k \otimes_k k^{tC_p} \rightarrow (M^{\otimes k}{p})^{tC_p}$$

(4.13)

obtained by composing (4.11) and (4.12) is an isomorphism.

**Proof.** Both sides are functorial in $M$, and the functors are stable enhanced functors; thus, they commute with finite homotopy colimits and with retracts. Therefore, it suffices to consider the case $M = R$, where the statement immediately follows from Lemma 4.1. □
5. HODGE-TO-DE RHAM DEGENERATION

5.1. Cyclic homology. For any $E_\infty$-algebra $R \in D\text{Comm}(S)$ and any $E_1$-algebra $A \in D\text{Alg}(R)$ over $R$, the Hochschild Homology of $A$ over $R$ is defined as the $R$-module $HH(A/R) = A^\otimes A^\otimes R A$. To describe it more explicitly, one uses the bar construction to replace $A$ with a termwise-free simplicial $A$-bimodule; this provides a canonical enhanced functor $(A/R)_\Delta^\sharp : \Delta^\circ \to D(R)$ and an identification $HH(A/R) \cong \text{hocolim}_{\Delta^\circ}(A/R)_\Delta^\sharp$. It is well known that $HH(A/R)$ can be promoted to an object in $D(R[S^1])$. To construct the $S^1$-action, one observes that $(A/R)_\Delta^\sharp$ extends to $A$. Connes’s cyclic category $\Lambda$ of $[1]$; we have an embedding $j : \Delta^\circ \to \Lambda$ and an enhanced functor $(A/R)_\sharp^\Lambda : \Lambda \to D(R)$ such that $j^* (A/R)_\sharp^\Lambda \cong (A/R)_\Delta^\sharp$. For any enhanced functor $E : \Lambda \to D(R)$, one defines $HH(E) = \text{hocolim}_{\Delta^\circ} j^* E, \quad HC(E) = \text{hocolim}_{\Lambda} E,$ and one proves that $HH$ extends to a functor $\tilde{HH} : D(R)^\Lambda \to D(R[S^1])$ (the clearest construction of this extension is given in [3]). In fact, one can say more: the classifying space $|\Lambda|$ of the nerve of the category $\Lambda$ is canonically identified with the classifying space $BS^1$ of the circle, and $D(R[S^1])$ is naturally identified with the full subcategory in $D(R)^\Lambda$ spanned by locally constant enhanced functors. The functor $\tilde{HH}$ is left adjoint to the full embedding $D(R[S^1]) \subset D(R)^\Lambda$. This implies that $HC(E) \cong HH(E)[hS^1]$, and this is known as cyclic homology. Periodic cyclic homology $HP(E)$ is then defined as $HP(E) = HH(E)^{tS^1},$ and one shortens $HP((A/R)_\sharp)$ and $HC((A/R)_\sharp)$ to $HP(A/R)$ and $HC(A/R)$, respectively. If $R$ is discrete (and thus oriented), then $R^{hS^1} \cong R[u], R^{tS^1} \cong R[u, u^{-1}],$ and for any $A \in D\text{Alg}(R)$ we have the spectral sequences 

$$HH(A/R)[u^{-1}] \Rightarrow HC(A/R), \quad HH(A/R)((u)) \Rightarrow HP(A/R).$$  

These are known as the Hodge-to-de Rham spectral sequences.

For any integer $n \geq 1$, we have the cyclic subgroup $C_n \subset S^1$, and its action on $HH$ can be seen directly in terms of the category $\Lambda$. To do this, one defines a category $\Lambda_n$ equipped with an edgewise subdivision functor $i_n : \Lambda_n \to \Lambda$ and a projection $\pi_n : \Lambda_n \to \Lambda$. The projection $\pi_n$ is a bifibration in groupoids whose fiber $pt_n = pt/C_n$ is the connected groupoid with a single object with automorphism group $C_n$. On the level of classifying spaces, $|i_n| : |\Lambda_n| \to |\Lambda|$ is a homotopy equivalence, and the fibration $|\pi_n| : |\Lambda_n| \cong |\Lambda| \to |\Lambda|$ is obtained by delooping once the short exact sequence 

$$1 \to C_n \to S^1 \to S^1 \to 1$$  

of abelian compact Lie groups. The embedding $j : \Delta^\circ \to \Lambda$ fits into a commutative diagram

$$\begin{array}{ccccccc}
\Delta^\circ & \xleftarrow{\pi_n} & \Delta^\circ \times \text{pt}_n & \longrightarrow & \Delta^\circ \\
\downarrow j & & \downarrow j_n & & \downarrow j \\
\Lambda & \xleftarrow{\pi_n} & \Lambda_n & \longrightarrow & \Lambda
\end{array}$$

where the square on the left is cartesian and $\pi_n : \Delta^\circ \times \text{pt}_n \to \Delta^\circ$ is the projection onto the first factor. The classical edgewise subdivision lemma [11] shows that for any $E \in D(R)^\Lambda$ the natural map $\text{hocolim}_{\Delta^\circ} j_n^* j_n^* E \to \text{hocolim}_{\Delta^\circ} j^* E = HH(E)$ is an isomorphism, and its source lies naturally in $D(R)^{\text{pt}_n} \cong D(R[C_n])$. 

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This construction is especially useful if \( n = p \) is an odd prime and \( R \) is a ring annihilated by \( p \). Namely, for any \( R \) and \( M \in \mathcal{D}(R[S^1]) \), the exact sequence (5.2) provides an identification

\[
(M^{hC_p})^{hS^1} \cong M^{hS^1}.
\] (5.4)

If \( R \) is a ring annihilated by \( p \), then the left-hand side carries two periodicity endomorphisms of degree 2: \( u \) coming from \( C_p \), and \( u' \) coming from \( S^1 = S^1/C_p \). The Hochschild–Serre spectral sequence for (5.2) shows that it is the first endomorphism \( u \) that is compatible with the periodicity endomorphism \( u \) on the right-hand side, so that (5.4) coupled with (4.9) and (4.10) provides a map

\[
M^{tS^1} \to (M^{tC_p})^{hS^1}.
\] (5.5)

Moreover, the same Hochschild–Serre spectral sequence shows that \( u' \) actually vanishes, so that \( (M^{tC_p})^{hS^1} = 0 \), and we have \( (M^{tC_p})^{hS^1} \cong (M^{tC_p})_{hS^1}[1] \) by (4.8). Since homotopy quotients commute with homotopy colimits, we conclude that (5.5) is an isomorphism. This allows one to reduce questions about \( M^{tS^1} \) to questions about \( M^{tC_p} \).

5.2. Degeneration theorem. We can now state and prove the Hodge-to-de Rham degeneration theorem. First, assume given a ring \( k \) annihilated by an odd prime \( p \), and an algebra \( A \in \mathcal{D Alg}(k) \). Consider the corresponding enhanced functor \( (A/k)_\sharp : \Lambda \to \mathcal{D}(k) \) and its edgewise subdivision \( j_n^* i_n^*(A/k)_\sharp \) of (5.3). We then have the natural map

\[
\hocolim_{\Delta^o} (j_n^* i_n^*(A/k)_\sharp)^{tC_p} \to (\hocolim_{\Delta^o} (j_n^* i_n^*(A/k)_\sharp))^{tC_p}
\] (5.6)

in \( \mathcal{D}(k[S^1]) \), and its target is identified with \( HH(A/k)^{tC_p} \) by (5.3).

**Lemma 5.1.** Assume that the algebra \( A \) is smooth. Then the map (5.6) is an isomorphism.

**Proof.** Since \( A \) is smooth, the diagonal bimodule \( A \) is compact, and then it is a retract of some piece of the stupid filtration of its bar resolution. Therefore, for some \( n \geq 1 \), the homotopy colimits \( \hocolim_{\Delta^o} \) in (5.6) are retracts of homotopy colimits \( \hocolim_{\Delta^o} \), over the full subcategory \( \Delta^o_{\leq n} \subset \Delta^o \) spanned by finite totally ordered sets with at most \( n \) elements. But the category \( \Delta^o_{\leq n} \) is finite, and the Tate fixed points functor \( (-)^{tC_p} \), being stable, commutes with finite homotopy colimits. \( \square \)

**Remark 5.2.** If \( A \) is not smooth, (5.6) is not an isomorphism, but its source still has an invariant meaning—in fact, \( \hocolim_{\Lambda} (i_n^*(A/k)_\sharp)^{tC_p} \) is the so-called \textit{co-periodic cyclic homology} \( \overline{HP}(A/k) \), a new localizing invariant of DG algebras introduced and studied in [6]. Mathew in [9] has no counterpart of Lemma 5.1, and co-periodic cyclic homology does not appear explicitily. It seems that the real reason for this is that he uses the topological Hochschild homology \( THH(A) \), and one can show that for a DG algebra \( A \) over a finite field \( k \), \( THH(A) \) becomes isomorphic to \( \overline{HP}(A/k) \) after one inverts the Bökstedt periodicity generator. We will return to this elsewhere.

Next, let \( k = \mathbb{F}_q \), \( q = p^n \), be a finite field of odd characteristic \( p \), and let \( R = S_q \) be as in Corollary 4.2.

**Lemma 5.3.** Assume given a smooth and proper algebra \( A \in \mathcal{D}(R) \), with \( A_k = A \otimes_R k \). Then there exists an isomorphism

\[
HP(A/k) \cong HH(A/k) \otimes_k k[u, u^{-1}].
\] (5.7)

**Proof.** Consider the enhanced functor \( (A/R)_\sharp : \Lambda \to \mathcal{D}(R) \), its edgewise subdivision \( i_n^*(A/R)_\sharp \), and its restriction \( j_p^* i_n^*(A/R)_\sharp \) to \( \Delta^o \times \text{pt}_p \subset \Lambda_p \). In this case we have a natural identification.
then it suffices to prove that the Hodge-to-de Rham spectral sequence for $\pi_p^t(A/R)\mathbb{Z}$ degenerates. But this is a spectral sequence of finite-dimensional $k$-vector spaces, one can equip $(M^\otimes k^p)^{C_p}$ with a natural $C_p$-equivariant $Z$-indexed increasing filtration $\beta$, whose associated graded quotients $\text{gr}_n^\beta$ are the shifts $M_/[n]$, and the quotient map $\beta_0(M^\otimes k^p)^{C_p} \to M$ admits a canonical $S$-linear splitting. If $M_\ast$ is of the form $M_\ast = M \otimes k$ for a spectrum $M$, the splitting can be made $k$-linear, and this provides the isomorphisms (4.13) and (5.9). This is the approach taken explicitly in [4] and implicitly in [5] (where the spectrum $M$ is not mentioned by name, and the only thing used is obstructions to its existence). The construction using Lemma 4.1 is obviously much more direct and conceptually clear, but this comes at a price: we have to use the proof of the Segal conjecture as a black box. It would be interesting to see if the technique of [6, 5] can clarify the contents of the black box.

**Remark 5.4.** We note that one does not need the full force of Lemma 4.1 to obtain Corollary 4.2 and Lemma 5.3. In effect, for any complex $A$ of $k$-vector spaces, one can equip $(M\otimes k^p)^{C_p}$ with a natural $C_p$-equivariant $Z$-indexed increasing filtration $\beta$, whose associated graded quotients $\text{gr}_n^\beta$ are the shifts $M_/[n]$, and the quotient map $\beta_0(M\otimes k^p)^{C_p} \to M$ admits a canonical $S$-linear splitting. If $M_\ast$ is of the form $M_\ast = M \otimes k$ for a spectrum $M$, the splitting can be made $k$-linear, and this provides the isomorphisms (4.13) and (5.9). This is the approach taken explicitly in [4] and implicitly in [5] (where the spectrum $M$ is not mentioned by name, and the only thing used is obstructions to its existence). The construction using Lemma 4.1 is obviously much more direct and conceptually clear, but this comes at a price: we have to use the proof of the Segal conjecture as a black box. It would be interesting to see if the technique of [6, 5] can clarify the contents of the black box.

**Theorem 5.5.** Assume given a smooth and proper algebra $A \in \mathcal{DAalg}(K)$ over a field $K$ of characteristic 0. Then the Hodge-to-de Rham spectral sequence for $HP(A/K)$ degenerates.

**Proof.** By Proposition 2.3 and Theorem 3.2, one can choose a formally smooth $E_\infty$-algebra $R \in \mathcal{DComm}(\mathbb{S})$ equipped with a map $a: R \to K$, and a smooth and proper algebra $A_R \in \mathcal{DAalg}(R)$ such that $A_R \otimes_R K \cong A$. Localizing $R$ if necessary, we may assume that it lies in $\mathcal{DComm}(\mathbb{S}(2^{-1}))$. The map $a$ factors through the finitely generated ring $R_0 = \pi_0(R)$, and if we let $A_{R_0} = A_R \otimes_R R_0$, then it suffices to prove that the Hodge-to-de Rham spectral sequence for $HP(A_{R_0}/R_0)$ degenerates. Since $A_R$ is smooth and proper, $A_{R_0}$ is also smooth and proper, so that the Hochschild homology groups $HH_i(A_{R_0}/R_0)$ are finitely generated $R_0$-modules. Then by the Nakayama lemma, to prove that all the differentials in the spectral sequence vanish, it suffices to prove that for any residue field $k$ of the ring $R_0$, with $A_k = A_{R_0} \otimes_{R_0} k$, the Hodge-to-de Rham spectral sequence $HP(A_k/k)$ degenerates. But this is a spectral sequence of finite-dimensional $k$-vector spaces, $k$ is a finite field of odd characteristic, and by Lemmas 5.1 and 2.4 its first and last pages have the same dimensions. 

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REFERENCES

1. A. Connes, “Cohomologie cyclique et foncteurs Extⁿ,” C. R. Acad. Sci. Paris, Sér. I, 296, 953–958 (1983).
2. P. Deligne and L. Illusie, “Relèvements modulo $p^2$ et décomposition du complexe de de Rham,” Invent. Math. 89, 247–270 (1987).
3. V. Drinfeld, “On the notion of geometric realization,” Moscow Math. J. 4 (3), 619–626 (2004).
4. D. Kaledin, “Non-commutative Hodge-to-de Rham degeneration via the method of Deligne–Illusie,” Pure Appl. Math. Q. 4 (3), 785–875 (2008).
5. D. Kaledin, “Spectral sequences for cyclic homology,” in Algebra, Geometry, and Physics in the 21st Century: Kontsevich Festschrift (Birkhäuser, Basel, 2017), Prog. Math. 324, pp. 99–129.
6. D. Kaledin, “Co-periodic cyclic homology,” Adv. Math. 334, 81–150 (2018).
7. M. Kontsevich and Y. Soibelman, “Notes on $A_\infty$-algebras, $A_\infty$-categories and non-commutative geometry. I,” arXiv:math/0606241 [math.RA].
8. J. Lurie, “Higher algebra,” Preprint (Inst. Adv. Stud., Princeton, NJ, 2017), https://www.math.ias.edu/~lurie/papers/HA.pdf
9. A. Mathew, “Kaledin’s degeneration theorem and topological Hochschild homology,” arXiv:1710.09045 [math.KT].
10. T. Nikolaus and P. Scholze, “On topological cyclic homology,” arXiv:1707.01799 [math.AT].
11. G. Segal, “Configuration-spaces and iterated loop-spaces,” Invent. Math. 21, 213–221 (1973).
12. B. Toën, “Anneaux de définition des dg-algèbres propres et lisses,” arXiv:math/0611546 [math.AT].

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