Time-reversal-invariant $C_2$-symmetric higher-order topological superconductors

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We propose a minimal lattice model for two-dimensional class DIII superconductors with $C_2$-protected higher-order topology. While this class of superconductors cannot be topologically characterized by symmetry eigenvalues at high symmetry momenta, we propose a simple Wannier-orbital-based real-space diagnosis to unambiguously capture the corresponding higher-order topology. We further identify and characterize a variety of conventional topological phases in our minimal model, including a weak topological superconductor and a nodal topological superconductor with chiral-symmetry protection. The disorder effect is also systematically studied to demonstrate the robustness of higher-order bulk-boundary correspondence. Our theory lays the groundwork for predicting and diagnosing $C_2$-protected higher-order topology in class DIII superconductors.

Introduction - Anyons are stable exotic quasiparticles with unconventional statistical braiding properties and serve as the cornerstone for topological quantum computation [1,2]. The most well-known anyonic quasiparticle is the zero-dimensional (0D) Majorana zero mode (MZM), which could in principle emerge as a vortex bound state of a two-dimensional (2D) $p+ip$ chiral topological superconductor (TSC) [3,4], or as the end mode of a one-dimensional (1D) $p$-wave TSC [5,6]. Remarkable experimental progress has been made towards realizing Majorana physics in superconducting Rashba nanowires [8,9] and recently in iron-based superconductors [10], where promising signatures of MZMs such as zero-bias tunnel conductance peaks have been observed. However, a “smoking-gun” measurement of confirming MZM’s anyonic nature is missing. Therefore, the experimental existence of MZMs is still not conclusive, which calls for more efforts on topological Majorana physics, perhaps in different systems.

Recently, there has been growing interest in understanding higher-order topology [11]-[14], where anomalous boundary physics could show up on the $(d-n)$-dimensional boundary of a $d$-dimensional topological system with $1 < n \leq d$. In particular, a 2D higher-order TSC is defined to host MZMs on its 0D geometric corners ($d=n=2$ here) [13,20], which is a novel promising Majorana platform. Such corner-localized MZMs are proposed to exist in iron-based superconductors [21,22] and doped WTe$_2$ [23]. However, in most of these proposals, the corner MZMs are relatively “fragile” and can be easily removed upon closing the edge energy gap. The robustness of the corner MZMs, however, can be enhanced against symmetry-preserving edge perturbations if we can introduce additional crystalline-symmetry protections [13,24]. Several groups have proposed various symmetry indicators to classify such symmetry-protected higher-order TSCs based on symmetry eigenvalues at high symmetry momenta [25-29]. In this context, class DIII superconductors with two-fold rotational symmetry $C_2$ are special in the sense that they always host the same $C_2$ symmetry eigenvalues at high symmetry momenta for all possible topologically distinct phases. Consequently, this class of superconductors does not admit an indicator-based classification [30]. While the absence of an indicator does not necessarily imply the absence of topological phases, understanding and characterizing higher-order topology for this symmetry class remains an important open question in spite of considerable recent activities in the topological classification of superconductors.

In this work, we propose a minimal example of a 2D class DIII higher-order topological superconductor with $C_2$ symmetry protection. Despite its simplicity, our minimal model hosts surprisingly rich topological physics, including a higher-order TSC phase with 0D corner-localized Majorana Kramers pairs (MKPs), a weak TSC phase with 1D Majorana edge band, and a chiral-symmetry-protected nodal TSC phase with edge Majorana flat band. The robustness of corner MKPs with respect to disorder effect is systematically studied, unambiguously demonstrating the higher-order bulk-boundary correspondence. Motivated by the recently proposed Majorana counting rule for class D superconductors [31], we develop a real-space Wannier-orbital-based counting rule to predict and diagnose various time-reversal-invariant topological phases emergent in our model. In particular, our real-space counting predicts that the presence of corner MKPs only depends on the information of both lattice geometry and Bogoliubov-de Gennes (BdG) Wannier orbitals, and not the microscopic details of Cooper pairings. We confirm this remarkable prediction through explicit calculations using different types of Cooper pairing.

Majorana counting rule in class DIII - We first define our real-space characterization of higher-order topology for $C_2$-symmetric class DIII superconductors. It is convenient to map the electron and hole degrees of freedom into a pair of Majorana fermions $\alpha_R, s$ and $\beta_R, s$, where $R$ is the real-space position and $s$ is the spin index. The Majorana operators necessarily come in Kramers pairs due to the time-reversal symmetry. This Majorana representation allows us to generalize the concept of “Kitaev limit”
in Ref. [31] to the time-reversal-invariant version. Specifically, a time-reversal-invariant Kitaev limit is achieved for a class DIII superconductor when every bulk MKP formed by electron and hole Kramers pairs is connected to exactly one other MKP via Majorana bonds. However, crucially distinct from class D, the MKP’s spin degrees of freedom enable two different types of Majorana bondings:

- equal-spin bonding: Majorana fermions with the same spin indices are bonded, which effectively leads to \( p \)-wave Cooper pairing;

- opposite-spin bonding: Majorana fermions with opposite spin indices are bonded, which effectively leads to \( s \)-wave/\( d \)-wave Cooper pairing.

For every Kitaev limit, there exists a Kramers pair of maximally localized BdG Wannier orbitals sitting at the center of each bond [32]. Importantly, the position of such Wannier orbital does not rely on the explicit bonding type. We now define a set of counting numbers

\[
\Delta_i = n_i^W - n_i^M,
\]

where \( l = A/B \) is the sub-lattice index, \( a \) and \( b \) are lattice constants along \( x \) and \( y \) directions, and \( s_i \) \((i = 0, x, y, z)\) are the Pauli matrices for spins. The subscript \((\parallel)\) \((\times)\) denotes the equal \((\text{opposite})\)-spin bonding. The Hamiltonian can be transformed to ferromagnetic operators using 

\[
\alpha_{s_i}^A_{R} = (c_{s_i}^A_{R} + c_{s_i}^A_{R}^\dagger) \sqrt{2} \quad \text{and} \quad \beta_{s_i}^R = i(c_{s_i}^A_{R} - c_{s_i}^A_{R}^\dagger) / \sqrt{2}.
\]

Without loss of generality, we choose

\[
\begin{align*}
\tau^1_0 &= \tau^0_0 = \tau^0_0, \\
\tau^A_0 &= -\tau^0_B = \tau^0_0, \\
\tau^A_x &= \tau^B_x = \tau^x_0, \\
\tau^A_y &= -\tau^B_y = -\tau^y_0.
\end{align*}
\]

The bond scheme of Eq. [2] within a unit cell is depicted in Fig. [1a]. In the fermion representation, the momentum-space Bogoliubov-de Genes Hamiltonian is

\[
\mathcal{H}(k) = \begin{bmatrix}
-(t_{00} + t_{xx} \cos k_x) \tau_z + t_{xy} \sin k_x \tau_y \sigma_0 \gamma_0 \\
(t_{0x} + t_{xx} \cos k_x) \tau_y + t_{xy} \sin k_x \tau_x \sigma_0 \gamma_z \\
(1 - \cos k_y) \gamma_0 + \sin k_y \gamma_z / 2 \\
(1 - \cos k_y) \gamma_0 + \sin k_y \gamma_z / 2
\end{bmatrix}
\]

where \( \tau_0 = i \sigma_y K \), particle-hole operator \( \mathcal{P} = \tau_z K \), and rotation \( C_2 = i \tau_z \sigma_z \left[e^{i \theta} (\gamma_z + \gamma_0) + (\gamma_0 - \gamma_z)\right] / 2 \). Here \( \mathcal{K} \) is the complex conjugation.

The gap closing condition of \( \mathcal{H}(k) \) is determined by three independent parameters: \( t_0 = \sqrt{t_{00}^2 + t_{0x}^2} \), \( t_x = \sqrt{t_{xx}^2 + t_{xy}^2} \), \( t_y = \sqrt{t_{yx}^2 + t_{yy}^2} \). Physically, \( t_0 \) characterizes the net bonding strength between onsite MKPs, while \( t_x \) and \( t_y \) characterize the bonding strength for nearest-neighbor MKPs along \( x \) and \( y \) directions, respectively. The Wannier orbital sits at the center of the strongest Majorana bond. For example, if \( t_x > t_{0y} \), we can immediately tell the existence of Wannier orbital pairs sitting at both \( 1c \) and \( 1d \). We note that the position of Wannier orbitals is independent of the spin texture of the bondings.

Meanwhile, we are able to analytically map out the phase diagram in terms of \( t_x/t_0 \) and \( t_y/t_0 \), as shown in Fig. [1b]. Below, we ascertain the topological nature of each phase by smoothly deforming into the corresponding Kitaev limit and applying our Majorana counting rule to perform the topological diagnosis. While such a Kitaev reduction is not necessary for the topological diagnosis, this procedure makes it easier to directly “read out” the position information of Wannier orbitals, greatly facilit-
tating the application of the counting rule.

**Higher-order TSC** - In the regime $t_x > t_0 + t_y$, we deform the Hamiltonian to the Kitaev limit where only $t_x > 0$ and $t_0, t_y \to \epsilon$ with $\epsilon \ll t_x$. As shown in Fig. 1(b), while the two MKPs within one unit cell both originate from maximal Wyckoff positions 1a and 1d, the Wannier orbital pairs sit at 1b and 1c. Therefore, the counting numbers are $\Delta_{x} = 1$ while $\Delta_{b,c} = 0$, indicating the absence of higher-order topology. Indeed, in this limit, the system resembles a set of decoupled 1D time-reversal-invariant TSCs stacking along $x$ direction, manifesting itself as a weak topological superconductor protected by the $x$-directional translational symmetry. As each 1D TSC hosts two end MKPs, the collection of edge MKPs in our weak TSC phase forms two pairs of time-reversal-invariant Majorana flat bands that are individually localized at the upper and lower edges. To see this, we numerically plot the energy spectrum of a $N_x \times N_y$ lattice in Fig. 2(b). Specially, each upper and bottom edge hosts $2N_x$ localized zero-energy states, which confirms the existence of the Majorana flat bands.

If we break the translational symmetry by doubling the unit cell along $x$ direction, the weak topology is destroyed. However, such unit-cell doubling leads to an occupation of one MKP and zero Wannier orbital pair for all four maximal Wyckoff positions. This leads to $\Delta_{b,c,d} = -1$, which satisfies the higher-order topological condition. In the Supplementary Material, we numerically confirm the existence of higher-order topology by adding $x$-directional dimer bonds to the weak phase, which is consistent with the counting rule. This is a demonstration of the versatility of our extended counting rule that can be used as an efficient real-space diagnosis and a powerful tool for model construction.

**Nodal TSC** - For $|t_z - t_0| \leq t_y \leq t_x + t_0$, the system becomes gapless and hosts four 2D Dirac points. The
Energy allows us to define a chiral winding number combination of time-reversal and particle-hole symmetry, which are found to connect Dirac points with opposite ν, where ν ∈ Z. Arising from the non-trivial topology of Dirac points and are protected by the chiral symmetry.

For class DIII superconductors, the chiral symmetry C

... makes it consistent with the time-reversal symmetry requirement ν = 1, while Dirac points II and III have ν = −1, which is consistent with the time-reversal symmetry requirement in the system.

A nontrivial winding number necessarily implies the existence of edge Majorana flat band at zero energy 34

... to characterize the topological nature of the Dirac point 32. Therefore, we numerically calculate the winding number, and find it to be non-zero for all four Dirac points. As shown in Fig. 3(a), Dirac points I and IV share ν = 1, while Dirac points II and III have ν = −1, which is consistent with the time-reversal symmetry requirement in the system.

Disorder Effect - We now study the robustness of the higher-order TSC phase against local disorder. First, we note that there are two important energy scales in our system in the t₀ → 0 limit: (i) the bulk gap 2|tₓ − tᵧ| and (ii) the edge gap 2|tᵧ|. From the higher-order bulk-boundary correspondence, we expect that the corner MKPs could be destroyed only when the disorder strength exceeds the bulk energy gap, if the disorder globally preserves the protecting symmetries.

Numerically, we introduce chemical potential fluctuations by randomizing the strength of on-site bonding t₀ following a Gaussian ensemble, which is centered at zero and with a standard deviation of Δt. For each value of Δt, we average the density of states over 100 random configurations. We now consider two different sets of bonding parameters such that the two corresponding

Figure 2. Energy spectrum and spatial profile of the zero-energy modes in a finite array of 25 × 25 atoms. Nₓ and Nᵧ are unit-cell indices along the x and y–directions. (a) Higher-order TSC: t₀ = 0.2, tₓ = 2, tᵧ = 0.5 and t′ = 0.1. The inset zooms in the four zero-energy states. (b) Weak TSC: t₀ = 0.2, tₓ = 0.5, tᵧ = 1 and t′ = 0.1.

Figure 3. Nodal SC at tₓ = t₀ = 1, tᵧ = 1.5. (a) Bulk-gap across the Brillouin zone having four Dirac points with non-trivial winding numbers. (b) Energy spectrum of the same system in the ribbon configuration. The Majorana flat bands arise from the non-trivial topology of Dirac points and are protected by the chiral symmetry.

Figure 4. Density of states with respect to the disorder strength of a 25 × 25-unit-cell finite system. (a) tₓ = 2, tᵧ = 0.5, t′ = 0.1. (b) tₓ = 1.5, tᵧ = 0.5, t′ = 0.1. The value of the bulk mid-gap (tₓ − tᵧ) is marked by the black dashed lines, which is also the critical disorder strength that destroys the corner MKPs; while the edge mid-gap tᵧ is marked by the white dashed lines.
systems share the same edge gaps but differ in their bulk gaps. As shown in Fig. 4(a) and (b), the critical disorder strength that suppresses corner Majorana modes is close to the value of the bulk gap (black dashed lines), and not the edge gap (white dashed lines). This clearly demonstrates the bulk origin and the consequent stability of higher-order topology in our system.

Conclusion - We propose a minimal model Hamiltonian for higher-order topology in class DIII systems, which contains rather rich topological phenomena such as higher-order TSC phase, weak TSC phase, and nodal TSC phase. While a momentum-space characterization becomes inapplicable, we succeed in deciphering the higher-order topology in real space with our Majorana counting rule, by identifying the spatial information of BdG Wannier orbitals. Our proposed counting rule should serve as an important diagnostic principle in the search for higher-order-topology-based Majorana platforms.

Acknowledgement - R.-X. Z. thanks Sheng-Jie Huang for helpful discussions. This work is supported by the Laboratory for Physical Sciences and Microsoft Corporation. R.-X. Z. is supported by a JQI Postdoctoral Fellowship.

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Majorana Bonding and Wannier orbitals

In this section, we discuss the relation between Majorana bonding and the position of resulting Wannier orbital pair. We assume two Majorana Krammer pairs \((\alpha_s, \gamma_s)\) and \((\beta_{s'}, \gamma_{s'}')\) localized at \(R_s\) and \(R_{s'}\) respectively. The interaction between the two pairs in the basis \((\alpha_s, \gamma_s, \beta_{s'}, \gamma_{s'}')\) is

\[
T = \begin{pmatrix} 0 & A \\ A^\dagger & 0 \end{pmatrix} \tag{6}
\]

where \(A = -A^*\) due to the anti-commutation of Majorana fermions. \(T\) has vanishing diagonal matrix elements because time-reversal invariance forbids coupling between time-reversed partners. The Hamiltonian \(T\) thus has four solutions, corresponding to two Dirac fermions

\[
\nu_1 = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \quad \nu_2 = \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}, \quad \nu_3 = \begin{pmatrix} u_3 \\ v_3 \end{pmatrix}, \quad \nu_4 = \begin{pmatrix} u_4 \\ v_4 \end{pmatrix},
\]

where \(u_1, u_2, v_1, v_2\) are two-component vectors. The position relative to the bond center is given by the matrix \(X = R\sigma_0 \otimes \sigma_z\) where \(R = (R_s - R_{s'})/2\). From the eigenvector equation, we have

\[
\begin{cases}
Au = Ev \\
A^\dagger u = Ev
\end{cases} \Rightarrow |u|^2 = |v|^2 \Rightarrow u = \mathcal{U}v,
\]

with \(\mathcal{U}\) being unitary. As a result, we can define a transformation

\[
M = \begin{pmatrix} 0 & \mathcal{U}^\dagger \\ \mathcal{U} & 0 \end{pmatrix},
\]

so that \(M\nu_1 = \nu_1\) but \(\{M, X\} = 0\). Thus \((\nu_1^T)X\nu_1 = 0\) and similarly, \((\nu_3^T)X\nu_3 = 0\). As a result, we have shown that the Wannier orbital pair is always localized at the middle of the bond, regardless of the bond type.

Majorana counting rule for class DIII systems

We provide a derivation of the Majorana counting rule for class DIII systems presented and utilized in the main text. This derivation is similar to that of the counting rule for class D systems \[31\]. In the Kitaev limit, we are able to “count” \(N\), the number of dangling MKPs on the \(C_2\)-symmetric boundary. In particular, the system is higher-order topological if \(N \equiv 1 \pmod{2}\). \(N\) consists of two parts: (i) a positive contribution from the dangling Wannier orbital pairs \(N^W\); (ii) a negative contribution \(N^M\) from the dangling MKPs that break \(C_2\) symmetry for a finite boundary and need to be removed. Notably, we only consider both \(N^W\) and \(N^M\) contributions from the ones at maximal Wyckoff positions \(q_{1i}\) for \(i = a, b, c, d\). Thus, we define \(n^W_i\) and \(n^M_i\) as the number of MKPs and Wannier orbitals at \(q_{1i}\) within one unit cell, respectively. For a finite size system with \(L_x \times L_y\) lattice sites, we find that

\[
N^W = L_x n^W_a + L_y n^W_b + (L_x + L_y - 1)n^W_d, \\
N^M = -L_x n^M_a - L_y n^M_b - (L_x + L_y - 1)n^M_d,
\]

which leads to

\[
N = -(L_x \Delta_c + L_y \Delta_b + (L_x + L_y - 1)\Delta_d). \tag{11}
\]

Here we have defined the counting number \(\Delta_i = n^W_i - n^M_i\). On the other hand, we want to avoid possible weak topology, which leads to the constraint \(\Delta_b \equiv \Delta_c \equiv \Delta_d \pmod{2}\). Together with Eq. 11 we arrive at the counting rule presented in the main text, that the system is higher-order topological (i.e. \(N\) is odd) only if \(\Delta_{b,c,d} \equiv 1 \pmod{2}\).
Figure 5. The Kitaev limit and the Wannier orbital configuration for various phases in our model. Each dot/diamond represent a Majorana Krammer pair, each line represent a 2 × 2 bonding tensor. (a) Trivial TSC: \( t_0 > t_x + t_y \). (b) Higher-order TSC: \( t_x > t_0 + t_y \). (c) Weak TSC: \( t_y > t_0 + t_z \). (d) Similar to (c) but a dimer bond along \( x \)-direction is added to gap out the flat Majorana band. (e) We calculate the energy spectrum and spatial density of zero-energy modes to demonstrate the higher-order topology of the system in (d). The inset shows a zoom-in plot of the corner MKPs.

Kitaev limit of the gapped phases

The time-reversal-invariant 2D Hamiltonian with \( l = A/B \) as the sub-lattice index is given by

\[
H = i \sum_{R,l} \beta_{R,s}^l \left( t_{0l}^0 \sigma_0 + it_{0x}^0 \sigma_y \right)_{ss'} \alpha_{R,s'}^l + i \sum_{R} \beta_{R,s}^l \left( t_{y}^l \sigma_0 + it_{x}^l \sigma_y \right)_{ss'} \alpha_{R+a,s'}^l
\]

\[
+ i \sum_{R} \beta_{R,s}^A \left( t_{y}^l \sigma_z + t_{y}^l \sigma_x \right)_{ss'} \beta_{R,s'}^B - \alpha_{R,s}^B \left( t_{y}^l \sigma_z - t_{y}^l \sigma_x \right) \alpha_{R+b,s'}^B.
\]

(12)

Without loss of generality, we take \( t_{0l}^0 = t_{0l}^B = t_0 \), \( t_{0x}^A = -t_{0x}^B = t_{0x} \), \( t_{x}^A = t_{x}^B = t_x \), and \( t_{x}^A = -t_{x}^B = -t_x \). We can then define a local transformation to decouple the spinful system into two time-reversal-related copies:

\[
|\alpha_{m,n}^A\rangle = O[\theta_0 - \theta_x]m - \theta_0/2 + \theta_y(2n - 1)] |\alpha_{m,n}^A\rangle,
\]

\[
|\alpha_{m,n}^B\rangle = O[-(\theta_0 - \theta_x)m + \theta_0/2 - 2\theta_yn] |\alpha_{m,n}^B\rangle,
\]

(13)

\[
|\beta_{m,n}^A\rangle = O[(\theta_0 - \theta_x)m + \theta_0/2 + \theta_y(2n - 1)] |\beta_{m,n}^A\rangle,
\]

\[
|\beta_{m,n}^B\rangle = O[-(\theta_0 - \theta_x)m - \theta_0/2 - 2\theta_yn] |\beta_{m,n}^B\rangle.
\]

where \( O[\theta] = \cos \theta \sigma_0 + i \sin \theta \sigma_2 \) is the rotation matrix, \(|\alpha\rangle\) and \(|\beta\rangle\) denote the two-component spinors for brevity, and \( \theta_0 = \text{Arg}(t_x + i t_{xy}) \), \( \theta_x = \text{Arg}(t_{xy} + i t_{xy}) \), \( \theta_y = \text{Arg}(t_{y} + i t_{xy}) \). Under such a transformation, the Hamiltonian becomes a direct sum of two effectively “spinless” Hamiltonians \( H = h \bigoplus h \) where

\[
h = i \sum_{m,n,l} \left( t_{0l} \beta_{m,n}^l \alpha_{m,n}^d + t_x \beta_{m,n}^l \alpha_{m,n+1,n}^d \right) + i \sum_{m,n} \left( t_y \beta_{m,n}^A \beta_{m,n}^B - t_y \alpha_{m,n}^A \alpha_{m,n+1}^A \right),
\]

(14)
and \( t_0 = \sqrt{t_0^2 + t_0^2} \), \( t_x = \sqrt{t_x^2 + t_x^2} \), \( t_y = \sqrt{t_y^2 + t_y^2} \). The bulk gap closing of \( h \) only depends on three parameters \( t_{0,x,y} \), instead of the previous six parameters in the original Hamiltonian. In particular, the system becomes nodal when \( |t_0 - t_x| \leq t_y \leq t_0 + t_x \).

In Fig. 5 we show the real-space Kitaev limit of each gapped phases of and the corresponding Wannier orbital pair configuration. The Kitaev limit is obtained by keeping the strongest bond, and deforming the weaker ones to zero. However, it should be emphasized that the unit cell can be enlarged by adding dimerizing bonds even though these bonds are weak, which could change the values of the counting numbers and could lead to a change of topology. For example, we consider adding an additional dimer bond \( t'_x \) along the \( x \)-direction to gap out the weak topological phase in (c), as shown in Fig. 5 (d), which is defined as

\[
H'_x = i t'_x \sum_{m,n,\sigma} (\beta_{m,n}^{A/B,\sigma} \gamma_{m,n}^{C/D,-\sigma} + \alpha_{m,n}^{C/D,\sigma} \gamma_{m+1,n}^{A/B,-\sigma}).
\]  

(15)

In Fig. 5 (e) with \( t' = 0.1 \), we find two corner-localized MKPs that signals the higher-order topology. This agrees with the prediction of the counting rule that \( \Delta_{b,c,d} = -1 \).

### Nodal superconductor

When \( |t_0 - t_x| < t_y < t_0 + t_x \), the system hosts four nodal points at

\[
\begin{aligned}
k_y &= 2\theta_y, k_x = \pm k_0 + (\theta_0 - \theta_x) \\
k_y &= -2\theta_y, k_x = \pm k_0 + (\theta_x - \theta_0),
\end{aligned}
\]  

(16)

where \( k_0 = \cos^{-1} [(t_y^2 - t_0^2 - t_x^2)/(2t_0 t_x)] \). The \( \mathcal{P} \) and \( \Theta \) naturally define a chiral symmetry \( \mathcal{C} = i\tau_x \otimes \sigma_y \otimes \gamma_0 \). Then

\[
\mathcal{C} = U_C \mathcal{D} U_C^\dagger, \quad U_C^\dagger H(k) U_C = \begin{pmatrix} 0 & \mathcal{N}(k) \\ \mathcal{N}^\dagger(k) & 0 \end{pmatrix}
\]  

(17)

where \( \mathcal{D} \) denotes a diagonal matrix. The winding number around a Dirac point is given by [34]

\[
\nu = \oint \nabla_k \text{Det}[A(k)] dk.
\]  

(18)

When \( \theta_x \neq \theta_0 \) and \( \theta_y \neq 0 \) and \( \pi/2 \), there are four separate Dirac points with non-zero winding numbers shown in the main text. Due to the anti-commutation \( \{\Theta, U_C\} = 0 \), we have

\[
\mathcal{N}^T(-k) = -T^\dagger \mathcal{N}(k) T \Rightarrow \det A(k) = \det A(-k).
\]  

(19)

As a result, the winding numbers around two opposite-momentum Dirac points are the same.