VARIATIONAL AND TOPOLOGICAL METHODS
IN NONLINEAR PROBLEMS

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These lectures are meant as an informal introduction to some of the
techniques used in proving existence of solutions of nonlinear problems of the
form

\[ F(u) = y. \]  \hspace{1cm} (1)

Here \( F \) is a continuous (and usually smooth) mapping from one topological
space \( X \) to another \( Y \), and the spaces are usually infinite dimensional. The
model to keep in mind is one in which these spaces are function spaces
defined in domains on finite-dimensional manifolds, and \( F \) is a system of
nonlinear partial differential operators—or integral operators.

A number of special topics will be presented—in three parts:

I. Global methods: homotopy, in particular topological degree theory, and
extensions. Applications to nonlinear boundary value problems.

II. Variational methods, in which a solution is a stationary point of some
functional. Applications.

III. Local study, perturbation about a solution.

An important analytic aspect of all these problems is that of finding a
priori estimates for the solutions. How one does that varies from problem to
problem and I will barely touch on these technical aspects. I will try, rather,
to avoid technicalities and stress the topological and variational ideas.

The lectures are not addressed to the experts in these fields—for them there
will be little new. They are given with the hope of attracting others to the
subject. Up to now, the topological and abstract ideas used are rather
primitive, and I am confident that there will be enormous further develop­
ment—involving more and more sophisticated topology.

A condensed version of some of this material was presented in [48].

Here is a more specific list of the topics treated.

1.1 Some classical things. Continuity method. Degree theory.
1.2 Some recent extensions of Leray-Schauder degree theory.
1.3 Extension of degree theory to Fredholm maps by Elworthy and
Tromba.

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