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A new modified Kies Fréchet distribution: Applications of mortality rate of Covid-19

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ABSTRACT

The purpose of this paper is to identify an effective statistical distribution for examining COVID-19 mortality rates in Canada and Netherlands in order to model the distribution of COVID-19. The modified Kies Frechet (MKIF) model is an advanced three parameter lifetime distribution that was developed by incorporating the Frechet and modified Kies families. In particular with respect to current distributions, the latest one has very versatile probability functions: increasing, decreasing, and inverted U shapes are observed for the hazard rate functions, indicating that the capability of adaptability of the model. A straight forward linear representation of PDF, moment generating functions, Probability weighted moments and hazard rate functions are among the enticing features of this novel distribution. We used three different estimation methodologies to estimate the pertinent parameters of MKIF model like least squares estimators (LSEs), maximum likelihood estimators (MLEs) and weighted least squares estimators (WLSEs). The efficiency of these estimators is assessed using a thorough Monte Carlo simulation analysis. We evaluated the newest model for a variety of data sets to examine how effectively it handled data modeling. The real implementation demonstrates that the proposed model outperforms competing models and can be selected as a superior model for developing a statistical model for COVID-19 data and other similar data sets.

Introduction

In statistical analysis, extreme value theory (EVT) is very valuable. The EVT was originally related to analyzing the performance of extreme values (EVs). And since EVs have a relatively poor chance of appearing, they may have a very high impact on the observed experiment. Fréchet (F) distribution is the significant model in modelling EVs. The F distribution was originally suggested in [1]. This model is defined in [2] and addressed its broad range of applications in various spheres like accelerated life monitoring, sea waves, horse racing, rainfall, environmental disasters, earthquakes, wind speeds, sea currents, track race records, and so on. More information about the F distribution can be found in the literature; for instance, [3] examined the exponentiated Fréchet model. For relief periods and survival times data, [4] introduced and implemented a new form of the F model. In [5], authors suggested some implementations of the Marshall-Olkin Fréchet distribution.

The CDF and PDF of Fréchet (F) model with scale (α) and shape (β) parameters are

\[ G(y | α, β) = \exp \left[ -\left( \frac{y}{β} \right)^α \right], \]

\[ f(y | α, β) = βαy^{α-1} \exp \left[ -\left( \frac{y}{β} \right)^α \right], \quad α, β > 0, \quad y > 0. \]

The characteristics of the exponentiated Kies distribution were investigated by Kumar and Dharmaja [6]. For product moments of modified Kies (MKI) model through type II progressive censored sample, and also an approximation of model parameters [7]. Focused on the modified Kies (MKI) model family, in [8], authors proposed a novel family of models. If G(γ | Δ) is the reference CDF for a parameter vector

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In current article, we present a detailed comparison of three methods for estimating unknown parameters of MKIF model. Two implementations of real data study have given in Section 4, we analyze estimation methods for MKIF model. Section 5 provides simulation results for the MKIF model. The remaining part of the current study is structured as given: The MKIF model is obtained in Section 2. The statistical characteristics of MKIF model are studied in Section 3. In Section 4, we analyze estimation methods for MKIF model. Section 5 provides simulation results for the MKIF model. Two implementations of real data study have given in Section 6 and conclusion is provided in Section 7.

MKIF model specifications

The CDF and PDF of MKIF model are specified as

\[ F(y | \Theta) = 1 - \exp\left\{ - \left( \frac{-G(y)}{1 - G(y)} \right)^{\eta} \right\}, \quad y > 0, \quad \eta > 0, \quad (5) \]

where the parameter vector \( \Theta = (\eta, \Delta) \). The PDF of (3) is

\[ f(y | \Theta) = \eta \left[ \frac{\tilde{G}(y | \Delta)}{1 - \tilde{G}(y | \Delta)} \right]^{\eta - 1} \exp\left\{ - \left( \frac{-\tilde{G}(y | \Delta)}{1 - \tilde{G}(y | \Delta)} \right)^{\eta} \right\}, \quad y > 0, \quad \eta > 0, \quad (4) \]

In this paper, the three-parameter modified Kies Fréchet (MKIF) distribution, which has several appealing characteristics, is obtained by referring to the distributions earlier. The PDF of the proposed MKIF distribution is extremely versatile, since it can be positive skewed, exponential, or symmetric, allowing for even more tail flexibility. It could model increasing, decreasing, bathtub, and inverted-U hazard rates. Another value of the suggested model is that it has a perfect closed form CDF and is quite simple to handle. Such characteristics make the model strong contender for biomedical life monitoring, actuarial data, and reliability applications.

In the study of any probability distribution, parameter estimation is significant. Because of its appealing properties, MLE is commonly utilized to estimate the parameters of any model. MLEs are unbiased, asymptotically consistent and normally distributed (see [9]). Other estimation techniques developed over time for distributions (see [10–17]) are dependent on various methodologies, like the methods of L-moments estimation (MLE), moment estimation (MOM), least-squares estimation (LSE), probability weighted moment estimation (PWM), weighted least square estimation (WLSE) and maximum product spacing estimation (MPS) and minimum distance estimation. In [18], researchers studied the L-moments and maximum probability approaches for estimating parameters of complementary Beta model. In [19], authors estimated the parameters of generalized power Weibull model employing the MLE and maximum product spacing strategies.

In current article, we present a detailed comparison of three methodologies for estimating unknown parameters of MKIF model, as well as an analysis of the execution of such estimators for different parameter values and sample sizes. We specifically compare MLEs, LSEs, and WLSEs. Theoretically, it is hard to compare the behaviors of various estimation techniques; we conduct detailed simulations study to assess the behaviors of various estimators focused on bias and mean squared error.
$$f(y|\Theta) = \eta \beta^\alpha y^{\beta-1} e^{-\eta^\beta} \left( \frac{e^{-\eta^\beta \gamma}}{1 - e^{-\eta^\beta \gamma}} \right)^{\eta - 1}$$

$$h(y|\Theta) = \frac{f(y|\Theta)}{1 - F(y|\Theta)}$$

$$S(y|\Theta) = \exp \left\{ - \left( \frac{e^{-\eta^\beta \gamma}}{1 - e^{-\eta^\beta \gamma}} \right)^\eta \right\}$$

The CHRF is also called the integrated hrf. It is a measure of risk: higher the $H(t)$ value, higher the risk of failure by $t$ time.

$$H(t) = \int_0^t h(y|\Theta) dy = -\log[S(t)].$$

It is noted that

$$S(t) = e^{-H(0)}$$

$$f(t) = h(t)e^{-H(t)}.$$
have to note that the values for $\Theta$ parameters have indeed been chosen arbitrarily till we captured a broad range of shapes for the parameters concerned. We note that PDF is right and slightly left-skewed or inverted-U shaped, and slightly symmetrical. It is reversed-J shaped for $\alpha = 0.2$ and $\beta = 0.09$ with various values of $\eta$.

Fig. 2 provides flexible hazard rate shapes like increasing, U shaped and decreasing.

Useful expansion of MKIF($\Theta$) model

\[ e^{-s} = \sum_{k=0}^{\infty} (-1)^k \frac{s^k}{k!} \quad \text{and} \quad \exp \left[ -\left( \frac{e^x - 1}{e^x - e^{-1}} \right)^\eta \right] \text{respectively} \]

If $s \in \mathbb{R}^+$, and $|z| < 1$, then power series holds

\[ (1-z)^{-\delta+1} = \sum_{k=0}^{\infty} \binom{\delta+k}{k} z^k \text{, for } |z| < 1 \]

(11)

and

\[ f(y|\Theta) = \eta \beta \alpha y^{\beta-1} \sum_{i,m=0}^{\infty} (-1)^i \binom{m+\eta}{m} \eta \beta \alpha y^{\beta-1} \left( e^{-\left( \frac{y}{\beta} \right)^\eta} \right)^{\eta(x+1)+m} \left( 1 - e^{-\left( \frac{y}{\beta} \right)^\eta} \right)^{\eta} \]

This section covers a valuable expansion of PDF and CDF of MKIF($\Theta$) distribution. The power series and exponential function could be described as

\[ (1-z)^{-\delta+1} = \sum_{k=0}^{\infty} \binom{\delta+k}{k} z^k \text{, for } |z| < 1 \]

(13)

Applying (13) to expand $\left( 1 - e^{\left( \frac{y}{\beta} \right)^\eta} \right)^{-\eta}$, we obtain the PDF of
MKIF(\(\Theta\)) in (6) is expressed as with
\[
\eta_{i,j,m} = \frac{m + \eta}{m} \left( j + m - 1 \right) (-1)^j,
\]
and
\[
f(y|\Theta) = \sum_{i,j,m=0}^{\infty} \eta_{i,j,m} \beta^{1+i+j} \left[ -\left( \frac{\alpha}{\beta} \right)^{1+i+j} \right].
\]
\[
F(y|\Theta) = \sum_{i,j,m=0}^{\infty} \eta_{i,j,m} \beta^{1+i+j} \left[ G(y|\Delta) \right] \left( 1 + \eta \right)^{1+i+j+m-1}.
\]

The MKIF distribution can be presented as a linear mixture of Fréchet (F) models, according to equation (15) As a consequence, the MKIF distribution’s properties can be deduced from those of the Fréchet distribution. In the same way, (11) and (12) indicate that CDF of MKIF in (5) can be written as
\[
F(y|\Theta) = 1 - \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \Lambda_{i,j} G^{i+j}(y|\alpha,\beta) = 1 - \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \Lambda_{i,j} H_{i,j}(y|\alpha,\beta),
\]
where \(\Lambda_{i,j} = \left( j + m - 1 \right) (-1)^j\), and \(H_{i,j}(y|\alpha,\beta) = G^{i+j}(y|\alpha,\beta)\) is CDF of Fréchet model with power \((i+1)\beta\). In addition, the following binomial theorem can be used to extend \(F(y|\Theta)^a\), let \(d\) be positive integer be then
\[
(1-x)^a = \sum_{k=0}^{d} \binom{d}{k} (-1)^k x^k,
\]
(17)
\[
[F(y|\Theta)]^a = \left[ 1 - \exp \left\{ - \left( \frac{e^{-\frac{\alpha}{\beta}}}{1 - e^{-\frac{\alpha}{\beta}}} \right)^a \right\} \right], a \in \mathbb{Z}^+.
\]
Therefore, \(F(y|\Theta)^a\) becomes:
\[
[F(y|\Theta)]^a = \sum_{k=0}^{d} \binom{d}{k} (-1)^k \exp \left\{ - k \left( \frac{e^{-\frac{\alpha}{\beta}}}{1 - e^{-\frac{\alpha}{\beta}}} \right)^a \right\},
\]
(18)
Then, applying (12) and (13) to expand
\[
\exp \left\{ - k \left( \frac{e^{-\frac{\alpha}{\beta}}}{1 - e^{-\frac{\alpha}{\beta}}} \right)^a \right\},
\]
\[ F(y | \Theta) = \sum_{k=0}^{\infty} \sum_{i_1, i_2=0}^{\infty} k^{i_1} \left( \frac{a}{k} \right) \left( i_2 + \eta i_1 - 1 \right) (-1)^{i_1+i_2} e^{-\left( \frac{a}{k \eta} \right)} \left( 1 \right)^{i_1} \left( \eta \right)^{i_1} \left( \frac{\alpha y^\beta}{\beta} \right)^{i_2}. \]  

(19)

Figs. 1 and 2 display maps of PDF and hrf functions of MKIF model for various values of \( \alpha, \beta \) and \( \eta \) respectively. Fig. 2 demonstrates that hrf of MKIF distribution can be rising, decreasing, or inverted U shaped.

Quantile function

The next result can be utilized to simulate values from the MKIF(\( \Theta \)) distribution. The QF of \( Y \) is

\[ Q(q; \Theta) = a \left[ \log \left( 1 \right) \left( 1 + \left[ - \log(0.75) \right] \right) \right] \frac{q^q}{q^{q+1}}, 0 < q < 1. \]  

(20)

As a result, the median, as well as the lower and upper quantiles, are calculated as follows:

\[ \hat{Y} = a \left[ \log \left( 1 \right) \left( 1 + \left[ - \log(0.5) \right] \right) \right] \frac{1}{q}, \]  

(21)

\[ Q(0.25; \Theta) = a \left[ \log \left( 1 \right) \left( 1 + \left[ - \log(0.75) \right] \right) \right] \frac{q^q}{q^{q+1}}, \]  

(22)

\[ Q(0.75; \Theta) = a \left[ \log \left( 1 \right) \left( 1 + \left[ - \log(0.25) \right] \right) \right] \frac{q^q}{q^{q+1}}. \]  

(23)

The accompanying quantile density function is provided by the differentiation of (20)

\[ Q'(q; \Theta) = a \left[ - \log(1-q) \right] \frac{q^q}{q^{q+1}} \left[ \log \left( 1 \right) \left( 1 + \left[ - \log(1-q) \right] \right) \right] \frac{q^q}{q^{q+1}}. \]  

(24)

Skewness and kurtosis based on the quantile function (QF)

Measures identified with moments are a standard procedure to measuring the skewness and kurtosis of a model. These moments, however, do not necessarily occur. Because of this, certain replacements are given by the implementation of the QF. In specific, to measure the skewness, we may utilize the Galton skewness coefficient described as
The following Moors kurtosis coefficient can be used to evaluate the kurtosis.

\[ K_\Omega = \frac{Q(0.875|\Theta) - Q(0.625|\Theta) + Q(0.375|\Theta) - Q(0.125|\Theta)}{IQR}. \]  

At different values of \( \eta \) with \( \alpha = 2 \), and different levels of \( \beta \), Fig. 3 provides maps of the QF and quantile density function. The distribution is found to be U shaped. Fig. 3 display proposed three-dimensional plots of \( \tilde{Y} \), skewness and kurtosis at different levels of \( \beta \). As the higher inputs of the parameters \( \alpha \) and \( \eta \) contribute the higher change in median curve. Also median yields lower values when \( \eta \) approaches to 0.5. On the other hand significant change in the skewness behavior is noticed along \( \alpha \) for smaller values of \( \eta \), but as \( \eta \) increases, it ends up to nearly 0.1. The degree of peakedness of a distribution decreases as \( \eta \) increase. In addition it may also be mesokurtic, platykurtic, or leptokurtic.

Moments and moment generating function (MGF)

The \( r \)-th moment of MKIF model can be evaluated directly by extending the PDF of the MKIF as seen in (14):

\[ \mu_r = E(Y^r) = \int_0^\infty y^r dF(y|\Theta); \quad r = 1, 2, ... \]  

\[ \mu_r = \sum_{i,j,m=0}^{\infty} \sum_{r=1}^{\eta(i+j+1)} \frac{\Gamma(1-r)}{|\eta(1+i)+m+j|^r} \]  

\[ \Gamma(z) \] is the gamma function. The mean of \( Y \) can be obtained by putting \( r = 1 \) in (28). The MGF is widely utilized in characterization of model. The MGF of MKIF model utilizing the Maclaurin series is mentioned as

\[ M(y|\Theta) = E(e^y) = \sum_{r=0}^{\infty} \frac{1}{r!} \sum_{i,j,m=0}^{\infty} \sum_{r=1}^{\eta(i+j+1)} \frac{\Gamma(1-r)}{|\eta(1+i)+m+j|^r} y^r \]  

Probability weighted moments (PWM)

The PWM of MKIF denoted by \( \rho_{r,s} \) is formally defined by

\[ \rho_{r,s} = E[y^r [F(y|\Theta)]] = \int_0^\infty y^r [F(y|\Theta)] f(y|\Theta) dy. \]  

The PWM of MKIF can be determined directly by applying the PDF in (14) and using (19) after replacing \( a \) by \( s \), we have
Fig. 7. Fluctuations of Bias and MSE for parametric Set 3: \((2.0, 2.5, 3.0)\).

\[
\rho_{r,s} = \sum_{k=0}^{\infty} \sum_{l_1, l_2=0}^{\infty} \eta_{i,j,k,l_1,l_2} \frac{k!}{k} \left( \frac{s}{k} \right)^{l_2} \left( l_2 + \eta_{i_1} - 1 \right) \left( \frac{1}{2} \right)^{l_2} \frac{\Gamma \left( 1 - \frac{r}{2} \right)}{\left[ (1+i+l_1+j+l_2) \right]^r} 
\]  

(31)
The certain estimation approaches with simulation

There are many ways to evaluate the parameters of models that each of them has its distinctive attributes and strengths. Three of those strategies are presented momentarily in this section and will be graphically, analyzed in simulation study. Here, $F(.)$ is distribution function of MKIF $(\Theta)$ model. Several statistical characteristics of the MKIF distribution are contributed to this section, considering that $\Theta$ are unknown.

Using three different estimation techniques, we were able to solve the problem of estimating the MKIF distribution parameters. These methods are MLEs, LSEs and WLSEs. From now, $y_1, y_2, ..., y_n$ represent $n$ observed values from $Y$ and their ascending ordering values $y_1 < y_2 < ... < y_n$.

**MLE approach**

There are many techniques for calculating parameters, but the most widely used is the maximum likelihood method. Let $y_1, y_2, ..., y_n$ be a random sample from MKIF model with parameters $\alpha, \beta$ and $\eta$. The likelihood function can be expressed as follows: $L(y|\Theta) = \prod_{i=1}^{n} f(y_i; \Theta)$ or likewise the log-likelihood function for $\alpha, \beta$ and $\eta$ is

$$l(y|\Theta) = \ln \prod_{i=1}^{n} f(y_i; \Theta),$$

$$= n\log(\eta) + n\log(\beta) + n\beta\log(\alpha) - (\beta + 1) \sum_{i=1}^{n} \log y_i - \eta \sum_{i=1}^{n} \left( \frac{\alpha}{y_i} \right)^{\beta}$$

$$- \sum_{i=1}^{n} \left( \frac{e^{-\left( \frac{\alpha}{y_i} \right)^{\beta}}}{1 - e^{-\left( \frac{\alpha}{y_i} \right)^{\beta}}} \right) x - (\eta + 1) \sum_{i=1}^{n} \log \left( 1 - e^{-\left( \frac{\alpha}{y_i} \right)^{\beta}} \right),$$

(32)

**Fig. 8.** Fluctuations of Bias and MSE for parametric Set 4 : (0.9, 0.7, 1.0).
Table 1
MLEs and SEs, of considered distribution for Data Set I.

| Distributions | MLEs | Standard errors |
|---------------|------|-----------------|
| MKIF(\(y; \alpha, \beta, \eta\)) | (2.948513, 1.985900, 1.203498) | (0.2075851, 0.7068093, 0.4833309) |
| MKITL(\(\alpha, \beta\)) | (3.355427, 0.728532) | (0.40463067, 0.02746158) |
| MKIEx(\(\alpha, \beta\)) | (2.2857088, 0.1864468) | (0.27048389, 0.01025465) |
| Fréchet(\(\lambda, \kappa\)) | (2.704450, 3.169386) | (0.1512232, 0.3655088) |

Table 2
MLEs and SEs, of considered distribution for Data Set II.

| Distributions | MLEs | Standard errors |
|---------------|------|-----------------|
| MKIF(\(y; \alpha, \beta, \eta\)) | (4.1595366, 0.7742977, 1.6638938) | (1.2761030, 0.5415642, 1.2870155) |
| MKITL(\(\alpha, \beta\)) | (2.3136894, 0.4875761) | (0.34503557, 0.02886037) |
| MKIEx(\(\alpha, \beta\)) | (1.36564554, 0.09439395) | (0.202082242, 0.00922408) |
| Fréchet(\(\lambda, \kappa\)) | (3.778477, 1.550118) | (0.4732334, 0.2014450) |

Table 3
The goodness-of-fit performance measures for Data I.

| Distribution | AIC | CAIC | BIC | HQIC | K – S | PV |
|--------------|-----|------|-----|------|-------|----|
| MKIF(\(y; \alpha, \beta, \eta\)) | 104.1101 | 104.8601 | 108.8607 | 105.7682 | 0.12053 | 0.6723 |
| MKITL(\(\alpha, \beta\)) | 105.0358 | 105.3994 | 108.2028 | 106.1411 | 0.14536 | 0.4323 |
| MKIEx(\(\alpha, \beta\)) | 111.5294 | 111.893 | 114.6964 | 112.6347 | 0.16947 | 0.2524 |
| Fréchet(\(\lambda, \kappa\)) | 109.8401 | 110.2038 | 113.0072 | 110.9455 | 0.17372 | 0.2274 |
Table 4
The goodness-of-fit performance measures for Data II.

| Distribution      | AIC   | CAIC  | BIC   | HQIC  | K – S | PV   |
|-------------------|-------|-------|-------|-------|-------|------|
| MKIF(\(y, \alpha, \beta, \eta\)) | 159.0168 | 159.9399 | 163.2204 | 160.3616 | 0.07632 | 0.9893 |
| MKITL(\(\alpha, \beta\))     | 157.2194 | 157.6639 | 160.0218 | 158.1159 | 0.07813 | 0.9862 |
| MKIEx(\(\alpha, \beta\))     | 159.8504 | 160.2949 | 162.6528 | 160.7469 | 0.13262 | 0.6196 |
| Fréchet(\(\lambda, \kappa\))  | 166.074  | 166.5185 | 168.8764 | 166.9705 | 0.15126 | 0.4543 |

Fig. 10. Fitted PDFs on histogram of dataset I (left) and Fitted CDFs on empirical CDF of dataset I (right).

Fig. 11. Fitted PDFs on histogram of dataset II (left) and Fitted CDFs on empirical of dataset II (right).

Fig. 12. P-P plots of the MKIF distribution for datasets I and II.
We could calculate the MLEs of the parameters $\alpha, \beta$ and $\eta$ by setting all equations to zero and solving them simultaneously.
Method of ordinary and weighted least squares

The LSEs and WLSEs techniques for estimating unknown parameters are widely recognized [39]. The two techniques for estimating the parameters of MKIF model are discussed here. The LSEs, \( \hat{\alpha}_{\text{LSE}} \), \( \hat{\beta}_{\text{LSE}} \) and \( \hat{\eta}_{\text{LSE}} \), can be achieved by minimizing the following function

\[
\text{LS}(\Theta) = \sum_{i=1}^{n} \left[ F(y_i; \Theta) - \frac{i}{n+1} \right]^2,
\]

with respect to \( \alpha \), \( \beta \) and \( \eta \), where \( \Theta = (\alpha, \beta, \eta) \) These can be extracted equivalently by solving: \( \frac{d\text{LS}(\Theta)}{d\alpha} = 0 \), \( \frac{d\text{LS}(\Theta)}{d\beta} = 0 \), and \( \frac{d\text{LS}(\Theta)}{d\eta} = 0 \) where

\[
\frac{d\text{LS}(\Theta)}{d\alpha} = 2 \sum_{i=1}^{n} \phi_1(\Theta) \left[ 1 - \exp \left\{ - \left( \frac{-e^{-\left( \frac{y_i}{\alpha} \right) \beta}}{1 - e^{-\left( \frac{y_i}{\alpha} \right) \beta}} \right)^\eta \right\} \right] - \frac{i}{n+1},
\]

\( \frac{d\text{LS}(\Theta)}{d\beta} = 2 \sum_{i=1}^{n} \phi_2(\Theta) \left[ 1 - \exp \left\{ - \left( \frac{-e^{-\left( \frac{y_i}{\alpha} \right) \beta}}{1 - e^{-\left( \frac{y_i}{\alpha} \right) \beta}} \right)^\eta \right\} \right] - \frac{i}{n+1},
\]

\( \frac{d\text{LS}(\Theta)}{d\eta} = 2 \sum_{i=1}^{n} \phi_3(\Theta) \left[ 1 - \exp \left\{ - \left( \frac{-e^{-\left( \frac{y_i}{\alpha} \right) \beta}}{1 - e^{-\left( \frac{y_i}{\alpha} \right) \beta}} \right)^\eta \right\} \right] - \frac{i}{n+1},
\]

(37) (38)
\[ \frac{\partial L_S(\Theta)}{\partial \eta} = 2 \sum_{i=1}^{n} \phi_i(\Theta) \left[ \left( 1 - \exp \left( -\frac{e^{\left( \frac{y_i}{\eta} \right)^{\beta}}}{1 - e^{-\left( \frac{y_i}{\eta} \right)^{\beta}}} \right) \right)^\beta \right] - \frac{i}{n+1} \] (39)

Fig. 16. Plotting of the results given in Table 4.

\[ \phi_i(\Theta) = -\beta \sum_{j=1}^{n} \left( \frac{a}{y_j} \right)^{\beta-1} \left( \frac{e^{-\left( \frac{y_j}{\eta} \right)^{\beta}}}{1 - e^{-\left( \frac{y_j}{\eta} \right)^{\beta}}} \right) \] (40)
\[ \varphi_i^2(\Theta) = \sum_{j=1}^{n} \left( \frac{\alpha}{y_0} \right)^j \log \left[ \frac{\alpha}{y_0} \right] \left( \frac{e^{-\frac{1}{\Theta}}}{1-e^{-\frac{1}{\Theta}}} \right)^{y-1} \left( \frac{e^{-\frac{1}{\Theta}}}{1-e^{-\frac{1}{\Theta}}} \right)^{\varphi} \exp \left[ -\left( \frac{e^{-\frac{1}{\Theta}}}{1-e^{-\frac{1}{\Theta}}} \right)^{\varphi} \right], \] (41)

\[ \varphi_i^2(\Theta) = \sum_{j=1}^{n} \left( \frac{e^{-\frac{1}{\Theta}}}{1-e^{-\frac{1}{\Theta}}} \right)^{y} \log \left[ \frac{e^{-\frac{1}{\Theta}}}{1-e^{-\frac{1}{\Theta}}} \right] \exp \left[ -\left( \frac{e^{-\frac{1}{\Theta}}}{1-e^{-\frac{1}{\Theta}}} \right)^{\varphi} \right] + \left( \frac{e^{-\frac{1}{\Theta}}}{1-e^{-\frac{1}{\Theta}}} \right)^{\varphi}, \] (42)

The WLSs, \( \hat{a}_{WLS}, \hat{b}_{WLS} \) and \( \hat{\eta}_{WLS} \), can be determined by minimizing the following function, with respect to \( \alpha, \beta \) and \( \eta \) respectively.

\[ WLS(\Theta) = \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[ F(y_i; \Theta) - \frac{i}{n+1} \right]^2, \] (43)

The estimates can also be obtained by solving: \( dWLS(\Theta)/d\alpha = 0 \), \( dWLS(\Theta)/d\beta = 0 \) and \( dWLS(\Theta)/d\eta = 0 \) where

\[ \frac{dL(\Theta)}{d\alpha} = 2 \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \varphi_i(\Theta) \left[ 1 - \exp \left\{ -\left( \frac{e^{-\frac{1}{\Theta}}}{1-e^{-\frac{1}{\Theta}}} \right)^{\varphi} \right\} - \frac{i}{n+1} \right], \] (44)

\[ \frac{dL(\Theta)}{d\beta} = 2 \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \varphi_i(\Theta) \left[ 1 - \exp \left\{ -\left( \frac{e^{-\frac{1}{\Theta}}}{1-e^{-\frac{1}{\Theta}}} \right)^{\varphi} \right\} - \frac{i}{n+1} \right], \] (45)

where \( \varphi_i(\Theta), j = 1, 2, 3 \) are given in (40–42).

**Graphical analysis**

We can execute simulation experiments graphically to determine finite sample behaviour of the MLEs, LSEs and WLSes. The following algorithm has used to make the decision:

1. Generate thousand samples of size \( n \) from (6). This work is carried out simply by QF and obtained data from uniform distribution.
2. In order, four separate sets of true parameter values \( \alpha, \beta \) and \( \eta \) are used.
   - Set 1: \((0.5, 0.9, 0.8)\)
   - Set 2: \((1, 1.2, 1.5)\)
   - Set 3: \((2.2, 2.5, 3.0)\)
   - Set 4: \((0.9, 0.7, 1.0)\)
3. Calculate the estimates for 1000 samples, say \( \hat{\alpha}, \hat{\beta}, \hat{\eta} \) for \( i = 1, 2, \ldots, 1000 \).
4. Evaluate biases and MSEs. These objectives are obtained with the help of the following formulas:

\[ Bias_{\hat{\theta}}(n) = \frac{1}{1000} \sum_{i=1}^{1000} \left( \hat{\theta}_i - \theta \right), \quad MSE_{\hat{\theta}}(n) = \frac{1}{1000} \sum_{i=1}^{1000} \left( \hat{\theta}_i - \theta \right)^2, \]

where \( \hat{\theta} = (\alpha, \beta, \eta) \).
5. These steps have repeated for \( n = 30, 32, \ldots, 500 \), with the mentioned parameters for MLEs, LSEs and WLSes. The \( Bias_{\hat{\theta}}(n) \) and \( MSE_{\hat{\theta}}(n) \) have been determined. To calculate the value of estimates, we used optim function of R. The outcomes of simulations are indicated in Figs. 5–8. These biases vary with respect to \( n \) in Figs. 5–8 (Left panels) and MSEs differ with respect to \( n \), as seen in Figs. 5–8 (Right panels).

We can deduce that the estimators have the property of asymptotic unbiasedness since the bias tends to zero as \( n \) grows, while the trend in the mean squared error shows consistency since the errors tend to zero as \( n \) grows.

**Final comments on the Simulation Findings**

- Under all estimation methods, the bias of \( \hat{\alpha}, \hat{\beta} \) and \( \hat{\eta} \) reduces as \( n \) increases.
- For the MLEs, the biases of \( \hat{\alpha} \) and \( \hat{\beta} \) are generally positive, but negative biases are also noted for \( \hat{\eta} \) under first three parametric sets.
• The MLEs are overestimated for $\alpha$ and $\beta$ however, it is under-estimated for $\eta$ (see left panel of Figs. 5–8).
• The performance of the LSE and WLSE are quite similar for $\alpha$ and $\beta$ however, it is slightly different for $\eta$ under Set 1–4 (see left panel of Figs. 5–8).
• Under all the approaches, generally, the least MSE of parameters are observed under MLE (see right panel of Figs. 5–8).
• In most cases, the maximum likelihood method of estimation works better than other approaches in terms of MSE, as seen in right panel of Figs. 5–8.
• In most cases, the WLSSE is the next optimum performing estimator, followed by MLE.

The general conclusion from the aforementioned figures is that as the sample size grows, all bias and MSE graphs for all parameters will reach zero. This confirms the accuracy of these estimation methods.

Real data applications

The MKIF model’s potentiality for two real datasets is demonstrated in this section. MKIF$(\theta)$ distribution is collated with other reasonable models, namely: modified Kies Inverted Topp-Leone (MKITL) distribution [25], modified Kies exponential (MKEx) [26], and Fréchet distribution. Some goodness-of-fit measures are used to compare the fitted distributions, including the Akaike Information (AIC), Bayesian Information (BIC), CAIC (consistent Akaike information) and HQIC (Hannan-Quinn information) criterions. In general, lower values of these statistics, the stronger the fit to the data. We also evaluate Kolmogorov-Smirnov($K – S$) statistic along with its P-value (PV) for considered distributions fitted centered on two real datasets. Tables 2 and 3 display the results of these measures. The first dataset includes 36 days of COVID-19 data from Canada, from April 10 to May 15, 2020, as seen at [https://covid19.who.int/]. This data formed of rough mortality rate. In addition, we notice that Almetwally et al. [25] analyzed this dataset to demonstrate the suitability of the new inverted Topp-Leone model. The next set of data is a 30-days COVID-19 data set from the Netherlands, and is obtained between March 31 and April 30, 2020. This data is comprised of an approximate mortality rate and it has also been included by Almonoy et al. [40]. To conclude, for the two datasets, the MKIF$(\theta)$ model shows itself to be the most suitable model, demonstrating its applicability in a realistic environment. The MLE approach has been used to estimate the pertinent parameters of models. The MLEs of the parameters are presented in Tables 1 and 2. As compared to other models used to fit the COVID-19, the MKIF distribution has the largest P-value and the shortest distance of Kolmogorov-Smirnov ($K-S$), seen in Tables 3 and 4. The fit empirical, histogram and PP-graph, profile log-likelihood for the MKIF model for COVID-19 datasets from Canada and Netherlands are seen in Figs. 10–14. The findings of application 1 and 2 are seen in Tables 1–4 and Figs. 10–16.

Final comments on the two applications

1. According to dataset one, MKIF has the largest P-value, as well as the smallest $K-S$ distance.
2. MKIF has been the best model for fitting the dataset I, as seen in Fig. 9.
3. In dataset II, we can notice that MKIF has the largest P-value as well as the smallest $K-S$ distance.
4. The best distribution for fitting the dataset II is MKIF, as seen in Fig. 4.
5. Table 3 shows that the MKITL, MKIEx and Frechet distributions all have poor fit for the first data set.
6. Table 4 shows that MKIF is in great agreement with the MKITL.

Concluding remarks

In current article, we suggest a novel three parameter model, entitled three-parameter modified Kies Fréchet (MKIF) model. The MKIF model is more versatile to study lifetime data than other known models. The SF, hrf, CHRF, linear representation of PDF and CDF, QF and moments of the MKIF model are derived. MLE, LSE, and WLSE approaches are compared. We present two executions based on mortality rate of COVID 19, demonstrating that MKIF model is the best model for fitting this type of data among all its competitors. The parameter estimation of MKIF model is derived by MLE, LSE, and WLSE. The performance of the model under different estimation methods is evaluated using simulation study. Two real-life data suggested, model gives a reasonably preferable fit than MKITL, MKIEx and Fréchet distributions.

CRediT authorship contribution statement

Anum Shafiq: Conceptualization, Methodology, Validation, Formal analysis, Investigation. S.A. Lone: Writing - review & editing, Data curation, Formal analysis, Methodology, Validation, Conceptualization.

Tabassum Naz Sindhu: Writing - review & editing, Data curation, Formal analysis, Methodology, Validation, Conceptualization. Yousef El Khatib: Supervision, Writing – review editing and funding. Qasem M. Al-Mdallal: Supervision, Writing – review editing and funding.

Taseer Muhammad: Writing - review & editing, Data curation, Formal analysis, Methodology, Validation, Conceptualization.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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