Timelike behavior of the pion electromagnetic form factor in the functional formalism.

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The electromagnetic form factor of the pion is calculated within the use of functional formalism. We develop integral representation for the minimal set of Standard Model Green’s functions and derive the dispersion relation for the form factor in the two flavor QCD isospin limit $m_u = m_d$. We use the dressed quark propagator as obtained form the gap equation in Minkowski space and within the Dyson-Schwinger equations formalism to derive the approximate dispersion relation for the form factor for the first time. We evaluate the form factor for the spacelike as well as for the timelike momentum in the presented formalism. A new Nakanishi-like form of integral representation is derived on the basis of the vector Bethe-Salpeter equation for the quark-photon vector with a ladder-rainbow kernel. The Gauge Technique turns out to be a part of the entire structure of the vertex. In the analytic approach presented here, it is shown that the a large amount of the $\rho$-meson peak in the cross section $e^+e^- \rightarrow \pi^+\pi^-$ is governed by the gauge invariance of QED/QCD, i.e. by Gauge Technique constructed quark-photon vertex. This approximation naturally explains the broad shape of the $\rho$-meson peak.

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I. INTRODUCTION

The understanding of QCD as a quantum field theory would not be complete without mastering all domains– the perturbative calculation of high-energy processes, as well as by achieving success in nonperturbative evaluations of low-energy hadronic production processes. In the latter case the timelike character of transferred momenta is an additional challenge. Developing a new nonperturbative technique opens new prospects in this last and not yet well theoretically mastered area. In this respect, the charged meson form factors and the transition meson form factors constitute rather precise data on the electromagnetic structure of the light meson. Simultaneously, both processes are simple enough for theoretical description based on quark and gluon degrees of freedom. The pion form factor $F((\pi))$ in a timelike region of momenta ($s = q^2 > 0$) carries nontrivial information in amplitudes for the production of the two lightest hadrons: a $\pi^+\pi^-$ pair. It has been measured with unprecedented accuracy (0.5% [1]) in the domain of the appearance of a striking resonant structure. Using a phenomenological description based on Breit-Wigner fits, the resonant structure corresponds to a 775 MeV heavy $\rho$ meson, where very nearby at $s = 780$ MeV a small ($\sim 5\%$) admixture of very narrow $\omega$ resonance is also observed [2]. The physical neutral $\rho$, having charged partners, is hence effectively considered as a neutral component of the vector isoriplet, while the physical $\omega$ is, within a good approximation, a strong isosinglet. Moreover, vector mesons could be practically untouched by the QCD non-Abelian anomaly, and both aforementioned light electrically neutral vectors could be practically identical in the isospin limit defined as $m_u = m_d$. However, light vector meson couplings to other mesons are quite different, providing their cross-section shapes differ very dramatically in various processes. We brought some new hints, supported by quantitative results, which offer the fact that various components of QCD/QED vertices contribute very differently in different processes. More concretely, facing the amount of the pion peak obtained, it is very likely that seven out of the eight components of transverse quark-photon vertices could contribute by only a limited amount, suppressed in vicinity of $\rho$-meson peak and its shape is dictated by the gauge invariance more then we expected.

In a spacelike region [$F(t), t = -q^2; t > 0$], the available data [3,11] are represented by a smooth decreasing curve for which the perturbative QCD prediction [8,12] reads

$$F_{\pi}(t) \rightarrow \frac{64\pi^2 f_{\pi}^2}{(11 - 2/3n_f)tL(t)} \left[1 + B_2L_t \frac{20}{384} + B_4L_t \frac{364}{4096} + O(L_t^{-1})\right]^2$$

where $L_t = ln(t/\Lambda_{QCD}^2)$ and the coefficients $B_i$ are related to the nonperturbative part– the pion distribution and the light-cone Bethe-Salpeter wave function. The actual asymptotic predictions within today’s available experimental range $Q^2 \approx 100$ GeV$^2$ moves the validity of perturbative QCD predictions more toward the deep spacelike scale.
New interesting resonant structures, e.g., the deep dip at 1.5 GeV, and other heavier resonances have been found in the shape of $F$ by using the Initial Photon State method in the BABAR 2012 experiment [7]. With increasing energy, a theory like vector meson dominance rapidly becomes a tautology of what is observed in the experiment: the experimental masses of ground state and excited mesons become mass parameters of the theory, while the widths of resonances very much reflect the introduced effective couplings among various meson. Chiral perturbation theory [13] has calculated the electromagnetic form factors near the threshold in various approximations [14–22], while the evaluation at higher energy, $Q > 0.5 GeV$ is out of convergence with the theory and further phenomenological degrees of freedom need to be added in order to continue to higher energy [26]. The functional approach provides good results for spacelike mesonic form factors [27–37], where the approach connects all lengths naturally: it is nonperturbative at low $Q^2$ where QCD is strong and it complies with perturbation theory at spacelike asymptotic. Due to known limitations and obstacles, only a few studies [38–40] based on the quantum field theory functional formalism offer a result for the function $F$ in the entire Minkowski space. A new approach [37] for the evaluation of $F$ based on the integral representation of Bethe-Salpeter functions is employed at level of the constituent quark model (in this approximation, the running of quark masses as well as the momentum dependence of the quark renormalization function is ignored) and the authors restrict themselves to only the spacelike argument of photon momenta. In the presented study, we follow similar lines as the authors in Ref. [37], but we take the momentum dependence in the quark propagator into account. Consequently, for the first time, we calculate the electromagnetic pion form factor in the entire Minkowski space.

The distinct shapes of $\rho$ and $\omega$ resonances appearing in processes where they dominate [say, the former in the function $F(s)$ and the latter in the $3\pi$ production] are a known striking feature. The $\rho$-meson is a broad resonance, while the $\omega$ peak is 20 times more narrow. The 70% contribution of the two pion production cross section $\sigma(ee \rightarrow \pi\pi)$ to the muon anomalous magnetic momentum $a_\mu$ is an integral quantitative expression of the above statement. Single pion or three pions productions dominated by the exchange of $\omega$ (and not $\rho$) mesons in $e^+e^-$ collisions contribute to $a_\mu$ by a remarkably smaller amount. To explain this, new terms with new couplings related with $\rho - \omega - \pi$ mixing are incorporated in the effective theories of QCD [26, 41–44]. These new effective couplings ensure the tree level decay of both mesons: the decay of $\rho \rightarrow \pi\pi$ happens at a point, while the decay of $\omega$ happens through the radiation of pions and the subsequent decay of virtual $\rho \rightarrow \pi\pi$, so $\rho$ participates virtually and its propagation slows the decay of the $\omega$ meson. Although less effective and more demanding in practice, it is also worthwhile to understand this origin from a microscopic explanation based on the quark and gluon degrees of freedom. An understanding of the detailed shape of $\rho$-meson resonance with no more than QCD Lagrangian parameters is certainly not equivalent to an empirical introduction of different phenomenological couplings between light vectors and pseudoscalars.

It is useful [12, 19, 23, 14, 48] to consider the pion form factor as the boundary value of an analytical function which has a cut on the timelike axis of the $q^2$ variable, which starts in the branch point $s_{th} = q_{th}^2 = 4m_\pi^2$, the production threshold. Thus the electromagnetic form factor $F_\tau(t)$ and the production form factor $F(s)$ can be evaluated from the dispersion relation for $F$:

$$F(q^2) = \int_0^\infty d\omega \frac{g(\omega)}{q^2 - \omega + i\epsilon},$$

(1.2)

with the unique spectral function $g$, which represents the imaginary part of $F(s)$ itself, $\Im F(s) = -\pi g(s)$ and which vanishes below $s_{th}$, provided $F(s)$ is the real function there. We will simply write $F_\tau(x)$ for any moment $t$ on the spacelike $x < 0$ or for the timelike argument $x = s > 0$ and for the Euclidean scalar product we have $q_{E}^2 = -t$ in the convention used in this paper.

We present the technique, which within the use of quark and gluon degrees of freedom, leads to the form of the dispersion relation in Eq. (1.2). It does not use predetermined properties of vector mesons; they appear as a solution of Schwinger-Dyson equations for propagators and vertices. Vector meson masses are not an input anymore; furthermore the $\rho$-meson is not taken as a stable hadron-it has no associated real pole in S-matrix and therefore it does not come from the solution of the homogeneous bound state BSE at all. The function $g$ in Eq. (1.2) is then given by a multidimensional integral over the spectral functions of quark propagators and Nakanishi weight functions for the Bethe-Salpeter pion vertex function as well as over the weight functions which appear in the Integral Representation (IR) for the quark-photon vertex.

We will report on an exploratory study of the time-like pion electromagnetic form factor using IRs of QCD Green’s functions, which are derived from nonperturbative truncation of QCD/QED Dyson-Schwinger Equations (DSEs). The IR is introduced in the Sec. IV, and the proof is relegated to the appendices. Proposed IR for vertices instantly offer the analytical continuation of Euclidean solutions for QCD Green’s functions, as well as for hadronic form factors. In order to get the necessary functions for calculation of the pion form factor $F$, we use the solution of a combination of DSEs and the Bethe-Salpeter equation (BSE), which was employed recently for the purpose of calculation of the pion transition form factor [36] and hadron vacuum polarization [38]. Furthermore, we derive a formula for the form factor $F$ in this limit and calculate the integral in Eq. (1.2) numerically.
II. THE ELECTROMAGNETIC PION FORM FACTOR FOR TIMELIKE ARGUMENT AND THE MINIMAL SET OF EQUATIONS OF MOTIONS

The evaluation of the pion form factor is a typical quantum field theory problem which involves bound states as final or initial states. How to calculate such a transition in the BSE approach is generally known \[49\]. Since we deal with gauge theory, which has the additional approximate global symmetries, the Green’s function we used as a building blocks should respect the vectorial as well as axial Ward identities as a constrain. In the case of electromagnetic form factors the working expansion is known \[23, 50\] and here we will consider only the first term, which defines the so called (dressed) Relativistic Impulse Approximation (RIA). This matrix element reads

\[
\mathcal{J}^{\mu}(p, Q) = e F_\pi(Q^2) p^\mu \\
= \frac{2 N_c}{3} \frac{ie}{(2\pi)^4} \text{tr} \left[ G_{EM,u}(k + Q/2, k - Q/2) \Gamma_\pi(k_{\pi-}, p + Q/2) S_d(k + p) \tilde{\Gamma}_\pi(k_{\pi+}, Q/2 - p) \right] + \\
+ \frac{2 N_c}{3} \frac{ie}{(2\pi)^4} \text{tr} \left[ G_{EM,d}(k - Q/2, k + Q/2) \Gamma_\pi(k_{\pi-}, p + Q/2) S_d(k - p) \tilde{\Gamma}_\pi(k_{\pi+}, Q/2 - p) \right] + \\
+ \frac{N_c}{3} \frac{ie}{(2\pi)^4} \text{tr} \left[ G_{EM,u}(k + Q/2, k - Q/2) \Gamma_\pi(k_{\pi-}, Q/2 + p) S_u(k + p) \Gamma_\pi(k_{\pi+}, Q/2 - p) + ... \right], \\
+ \frac{N_c}{3} \frac{ie}{(2\pi)^4} \text{tr} \left[ G_{EM,d}(k + Q/2, k - Q/2) \Gamma_\pi(k_{\pi-}, Q/2 + p) S_u(k + p) \Gamma_\pi(k_{\pi+}, Q/2 - p) + ... \right],
\]

\[(2.1)\]

where the expressions in the first (second) two lines correspond with diagrams where the photon with momentum \( Q \) couples to the up(down)-quark with electric charge \( 2/3(1/3) \). In Eq. \[(2.1)\] \( S_u \) stands for the up quark propagator, \( Q \) is the photon momentum and \( \Gamma_\pi(a, b) \) is the pion vertex function with \( a(b) \) being the relative (total) momentum of quark-antiquark pair. The second line represents the triangle diagram, which has the opposite circulation of momentum (compared to the first one, and we also flip the sign by taking \( k \to k \)). Although we write them explicitly here, it is not difficult to show they contribute equivalently, being individually proportional to the relative momentum of the pionic pair \( p \) and pionic form factor \( F \). Thus, up to the charge prefactor, there are four identical contributions in the isospin limit, for which the propagators of light quarks are equal by definition, \( S_u = S_d \). All propagators and vertices are dressed. For a diagrammatic representation of above see for instance \[37\]. The bare BSE vertices are solution of BSE with the vertex function on the right-hand side of the BSE, being in fact identical to the BSE vertex in the approximation employed here.

The matrix \( G_{EM}^{\mu} \) at each line in the Eq. \[(2.1)\] is the quark-photon semi-amputated vertex defined as

\[
G_{EM,q}^{\mu}(k_{-},k_{+}) = S_q(k_{-}) \Gamma_{EM,q}^{\mu}(k,Q) S_q(k_{+});
\]

\[(2.2)\]

where \( k_{\pm} = k \pm Q/2 \) stands for the momenta of fermionic lines, and where the proper vertex \( \Gamma_{EM,q}^{\mu} \) is determined by its own inhomogeneous BSE, which reads

\[
\Gamma_{EM}^{\mu}(k, P) = \gamma^{\mu} + i \int \frac{d^4l}{(2\pi)^4} S(l_{+}) \Gamma_{EM}^{\mu}(l, P) S(l_{-}) K(l, k, P),
\]

\[(2.3)\]

where we have omitted the quark flavor \( q \). Different flavor combinations enter various form factors of meson, and since flavor is not mixed by our choice of interacting kernels \( K \), we will always mean a single quark flavor quark-photon vertex.

Thus, in order to evaluate the form factor in Eq. \[(2.1)\], one needs to know the quark propagator \( S \), the pion Bethe-Salpeter vertex function \( \Gamma_\pi \) as well as the quark-photon vertex \( \Gamma_{EM,q}^{\mu} \). In the isospin limit the propagators of \( u \) and \( d \) as well as the quark-photon vertices of the \( u \) and \( d \) quarks are identical and by applying charge conjugation one can show that the second line, up to a different prefactor, which turns out to be \(+1/3e\), is equal to the first one. The approximated set of equations for the pion vertex and the quark propagator we used here, were obtained in Refs. \[38, 69\] and will be described in the following section.

III. LIGHT QUARK PROPAGATORS AND THE PION VERTICES

To get the solution for the functions \( S(k) \) and \( \Gamma_\pi(k, P) \) we use the simultaneous solution of DSE for the quark and BSE for the pion, and thus we follow quite a common practice used in Refs. \[38, 51, 61, 62, 63, 69, 70\].
The BSE for the vertex function $\Gamma_\pi$ reads

$$\Gamma_\pi(p, P) = i \int \frac{d^4k}{(2\pi)^4} \gamma_\nu S_q(k+) \Gamma_\pi(k, P) S_q(k-) \gamma_\nu [-g^{\mu\nu} V_\pi(q) - C_\Gamma q^\mu q^\nu (q^2)^2],$$  \hspace{1cm} (3.1)

where the momentum $q = k - p$ and we label $C_\Gamma = 4/3 \xi g^2$, being thus identified with the longitudinal part of the gluon propagator in a class of linear covariant gauges. The first term should not be confused with propagator at all, albeit its momentum dependence of the above kernel was chosen to mimic the so called ladder-rainbow approximation with one gluon exchange and reads

$$V_\pi(q) = \int d\omega \frac{\rho_\pi(\omega)}{q^2 - \omega + i\epsilon} \rho_\pi(\omega) = c_\pi [\delta(\omega - m_\pi^2) - \delta(\omega - m_L^2)] .$$  \hspace{1cm} (3.2)

and it is inspired by solution of DSEs for the gluon DSE in the Landau gauge \[^{[60]}\]. This model was found particularly useful for relatively large region of nontrivial couplings $C_\Gamma$ requiring a certain departure from popular Landau gauge which is our convenient strategy. Obviously, for non-trivial $\xi$, the effective kernel $V_\pi$ is gauge fixing dependent.

We found that the presence of non trivial longitudinal modes improves the convergence of the solution in the form of the IR for the quark propagator. This IR reads

$$S(k) = \int_0^\infty dx \frac{k \rho_c(x) + \rho_s(x)}{k^2 - s + i\epsilon} .$$  \hspace{1cm} (3.3)

where two functions $\rho_c$ and $\rho_s$ fully characterize the quark propagator.

Notably, the longitudinal part of the kernel is the only source of UV divergence in the presented model, which was removed by dimensional renormalization.

In Eq. (3.1) $P$ is the total momentum of the meson satisfying $P^2 = M^2$, $M = 140$ MeV for the ground state, and the arguments in the quark propagator are $k_\perp = k \pm P/2$. The DSE/BSE system provides precise solution for the quark propagator calculated in the gauge $C_\Gamma/(4\pi)^2 = 0.18$ and the kernel couplings \[^{[3.2]}\] $c_\pi/(4\pi)^2 = -1.8$ and $m_\pi^2/m_L^2 = 2/7.5$ where $m_\pi$ in physical units is $m_\pi = 556$ MeV.

The pion BSE vertex function $\Gamma_\pi(P, p)$ is composed from the four scalar functions

$$\Gamma_\pi(P, p) = \gamma_5 (\Gamma_E(P, p) + \delta \Gamma_F(P, p) + PT \Gamma_G(P, p) + \delta \Gamma_H(P, p)) ,$$  \hspace{1cm} (3.4)

where all of them are used to determine the pion mass, and all of them contribute to the electromagnetic form factor. In our exploratory study presented here we simplify and use only the first component formally.

IV. IR DERIVED FROM DSES AND THEIR USE IN CALCULATION OF THE FUNCTION $F_\pi$

A sort of Nakanishi IRs, originally developed for scalar theories \[^{[71]}\] is slowly getting more use in nonperturbative settings of QCD \[^{[37, 39, 72, 73]}\], needless to say a certain controversy on existing actual analytical forms exists \[^{[74]}\]. Independently of the detailed from of IRs for Green’s functions in QCD and the Standard Model, their important property is their great role in performance of analytical integration in momentum space.

IRs for Green’s functions in quantum field theory play an important role since they allow analytical integration. We perform momentum integration in Eq. (2.1) analytically by using the well known formula for the Euclidean space momentum integral. For this purpose we employ IRs for all functions needed, more concretely, we use the IR for the quark propagators in Eq. (3.3) with the solution for $\rho_{c,s}$ as obtained for instance in the work \[^{[38]}\]. Motivated by the following chiral limit Goldberger-Treiman-like identity,

$$\Gamma_E(0, p) = \frac{B(p)}{f_\pi} ,$$  \hspace{1cm} (4.1)

where the scalar function $B$ appears in the inverse of the quark propagator:

$$S(p)^{-1} = \not p A(p) - B(p) ,$$  \hspace{1cm} (4.2)

hence we use a simplified version of the BS vertex, which reads

$$\Gamma_\pi(p, P) = \gamma_5 \frac{1}{N} \int_0^\infty d\omega \frac{\rho_B(\omega)}{p^2 - \omega + i\epsilon} ,$$  \hspace{1cm} (4.3)
and was used with $N$ being the normalization factor satisfying approximately $N = f_\pi$ with its exact value dictated by the canonical normalization of the BSE vertex.

The last missing ingredient is the quark-antiquark-photon vertex $\Gamma_{EM,f}^\mu$ for which we derive its own IR in Appendix A. The version for semiamputated vertex [Eq. (2.2)] reads

$$G_{EM}^\mu(p_-, p_+) = \sum_{i=1}^{8} V_i^\mu T_i(p^2, p.Q, Q^2)$$

$$+ \int_0^\infty d\omega \int_1^{-1} dz \frac{\rho_\omega[\gamma_{\omega}(\omega), \gamma_{\omega}(\omega_+ + \omega \gamma^\mu) + \rho_\omega(\omega)[\gamma_{\omega}(\omega_+ + \gamma^\mu, \gamma_{\omega}(\omega_+) + p.Q + Q^2/4 - \omega + i\epsilon)]^2}{\rho_{\omega}[\gamma_{\omega}(\omega, \alpha, z)]}$$

where, as we show in Appendix, the second line is in fact equivalent to the Gauge Technique Ansatz and the first line completes the entire expression by adding all transverse components independently. The eight transverse components satisfy the condition of transversality $V.Q = 0$ and their concrete form is a matter of convention. Their convenient representation can be chosen in the following way

$$V_1^\mu = \gamma^\mu_T; \quad V_5 = \gamma_5^\mu_T;$$

$$V_2^\mu = p_T^\mu; \quad V_6 = [\gamma^\mu_T, \gamma_5^\mu_T, \gamma^\mu_T];$$

$$V_3^\mu = p_T^\mu; \quad V_7 = [\gamma^\mu_T, \gamma_5^\mu_T];$$

$$V_4^\mu = \gamma^\mu_T(\gamma_5^\mu_T, \gamma^\mu_T); \quad V_8 = \gamma_5^\mu_T \gamma^\mu_T.$$

and the associated scalar functions $T_i$ satisfies 3-dimensional integral representation

$$T_i(p^2, p.Q, Q^2) = \int_0^\infty d\omega \int_1^{-1} dz \frac{\rho_{\omega}[\gamma_{\omega}(\omega, \alpha, z)]}{[p^2 + p.Qz + Q^2/4 - \omega + i\epsilon]^2}.$$  (4.6)

We recall that all functions $T_i$ are for a given gauge uniquely determined by the theory (by the solution of DSEs) through the solution for $\rho_i$. Also, note that somehow arbitrary momentum-dependent prefactors used elsewhere in decomposition [Eq. (4.7)] are not allowed here, unless they fulfill herein proposed IR.

There exist obviously a set of equivalent choices, depending on which part of the transverse components is added to the term which is fixed by gauge covariance. Other definitions of IR are possible and even the single longitudinal component $\gamma_\mu \rightarrow \gamma_{\omega}^\mu$ can be used to express the IR which is fixed by Ward identities. We do not know yet, which choice is more advantageous from other perspectives, e.g. which is more suited for numerical solution. As shown in the Appendix, we have chosen the Gauge Technique inspired form as a tribute to the first nonperturbative solution of DSE for the gauge vertex appearing in the literature [22–28]. Another advantage is that Gauge Technique reduces the proper vertex to the $\gamma$ matrix in the limit of vanishing gauge couplings (all of them in our case).

The value $N = 2$ was chosen to derive the form of IR [Eq. (4.1) from the DSE Eq. (2.3)]. Since the DSE is the equation for the proper vertex, the appropriate IR for this vertex is derived in the first step. Only then, it is shown the derived IR for the proper function $\Gamma_{EM}^\mu$ is equivalent to the proposed IR (4.1) for the semiamputated vertex $G_{EM}^\mu$.

A. Calculation of $F$

In order to get the form factor, we use a quite primitive, albeit not easy approach, and as we use the IRs for all vertices, we match their denominators of them by using Feynman parameterization. This allows the shifting of the loop integration momentum, and we integrate analytically in momentum space. After the momentum integration, the resulting integral involves 9 dimensional integral over the variables of various IRs. The integrand is highly singular for $Q^2 > 0$ thus being not useful in its instant form that we arrive in after momentum integration. Hence, in order to reduce the number of numerical integrations, we use further “gauge technique approximation”, which reduces identical pairs of IR weight functions to the same number of single functions. Thus, for instance

$$\int da d\beta c(a) \beta c(b) \rightarrow \int da \beta c(a),$$

with all $b$’s replaced by $a$’s in the integral kernel. It allows further integration over auxiliary Feynman variables, provided we are left with a 5-dimensional integral at the end. The appropriate derivation is relegated in the Appendix. Furthermore, since weights of BSE vertex function are much less known then the spectral function of the quarks we use the integral reduction for purpose of numerical evaluation here. Numerical results are presented in Fig. 2.
spacelike momentum, where we compare with the experiment. The systematic error is estimated to be around a few percentage at several GeV, however adjusting $F(0) = 1$ is needed as we the proper renormalization does not lead to the correct value automatically.

Using some further approximations we derive the dispersion relation \[ \text{(12)} \] and provide the first estimate for the resulting spectral functions of the pion electromagnetic form factor. The result is valid for low momentum and it consists of the two following terms

$$ F(q^2) = \int_0^\infty d\omega \frac{g_1(\omega)}{q^2 - \omega + i\epsilon} + q^2 \int_0^\infty d\omega \frac{g_2(\omega)}{q^2 - \omega + i\epsilon}, \quad (4.8) $$

where the first term could be responsible for correct normalization $F(0) = 1$ if no approximation (linearization) is made. The functions $g_1$ and $g_2$ are given by an expression involving only a single integration due to the approximation employed. The result is not exact and it suffers from systematic error due to some ignored terms; however it is enough to show that the form factor develops the $\rho$-meson peak. In fact, the Gauge Technique approximated vertex is enough to get almost the entire structure of the $\rho$ meson peak and we ignore all other transverse vertices at this stage. In order to support this statement quantitatively, the dominant contribution to $F_\pi(Q^2)$ has been calculated numerically and its square is compared with world averaged experimental data in Fig. \[ \text{1} \]. The averaged data of the BABAR, BESS, CMD/SND and KLOE experiments \[ \text{1} \] were fitted as described in \[ \text{82} \], noting that there is negligibly small experimental error 0.5% on the $\rho$-meson peak.

Within the used approximation the derivation of the desired dispersion relation \[ \text{(13)} \] is quite straightforward, albeit a bit lengthy and it is delegated to Appendix C of this work.

The phase $\delta$ of the form factor $F = |F|e^{i\delta}$ is shown in Fig \[ \text{3} \]. It overestimates the phase obtained by other methods, but still being satisfactory representative in our initial study. From the obtained phase we estimate that the systematical error can be as much as 30%, which is caused by linearization and other cruel approximations we made. We assume the magnitude possess the same systematics and that it overestimates the experimentally measured magnitude $F$, if the condition $F(0) = 1$ is imposed on the approximated form factor. We lower $F$ by a scale factor $\sqrt{3}$ for purpose of better comparison. Hence there are two lines representing the identical result obtained by our Approximate Dispersion Relations, the dashed corresponds to the standard normalization $F(0) = 1$, the solid line represents the same calculated result, but shifted down due to rescaling. For higher $Q^2$ the result obtained from the dispersion relation becomes untrustworthy due to our pure approximation. Obviously the first line is correct at zero momenta, while the second one reasonably approximates the peak. The difference is systematic error, which is quite large in the present example. This error suppression is an open task and remains a future challenge.

Furthermore, we use the approximated DR and evaluate the form factor in the spacelike domain as well. We recall that this approximation differs with the previous one and we add this result to Fig. \[ \text{2} \] for comparison.

Needless to say, the inclusion of transverse quark-photon form factors could improve the picture, and going beyond isospin approximation could leave some nontrivial imprints on the form factor shape. Some part of the systematical error could be due to this missing contribution, however the missing off-peak contribution is difficult to explain as due only to the absence of transverse quark-photon components. To get rid of this uncertainty, the developed IR in previous section could be used. We expect that the first interesting solution for vertices will be found in the next decade. Beyond our isospin approximation, further integrations (at least two) should appear in practice due to the necessity to use a more sophisticated but unfortunately also a more dimensional IR \[ \text{72, 73} \] in order to evaluate the electromagnetic form factor in nonsymmetric case.

We do not use IRs for the transverse vertices in the part devoted to the numerical study of $F$ due to our simplified approximation. However, the revelation of their entire structure represents important theoretical hint for future studies.

### B. Renormalization within IRs

The use of the proposed IR allows the dimensional regularization to be used to regularize four-dimensional momentum integrals when they show UV divergence.

Regarding the renormalization of the vertex the only allowed UV infinities could be associated with $\gamma$ matrix structure, since transverse form factors have no associated terms in the Lagrangian of the Standard Model. However note, that the second and the eighth transverse components in the list \[ (4.5) \] can, in principle, spoil the renormalization properties for our (till now preferred) choice of power $N = 2$ in the denominator of the IR. Actually, such naive UV divergences appear and it turns out, that they cancel neither mutually nor against the UV term generated by Gauge Technique.
FIG. 1: Calculated magnitude of the pion electromagnetic form factor for $Q^2 > 0$ and comparison with experiments. The error bars are not shown, they are within the visible size of the line and are much smaller then the deviation of presented calculations. The solid line is rescaled by a constant as described in the text.

FIG. 2: Calculated pion electromagnetic form factor for $Q^2 < 0$ and comparison with experiment and asymptotic prediction. The line labeled by ADR stays for evaluation based on further approximations needed to evaluate spectral function in dispersion relation for $F$. For the asymptotic (upper red line) we have chosen the function $F_{\text{asym}}(t) = \frac{64\pi f^2}{9t\Lambda_{\text{QCD}}^2} (1 + 0.1L_0^{-0.1})^2$; $L_0 = \ln(e + t/\Lambda_{\text{QCD}}^2)$; $\Lambda_{\text{QCD}} = 250 \text{MeV}$, which obviously has a correct asymptotic $[\text{Li}]$. Experimental point are: cross $[\text{SS}]$, square $[\text{SS}]$, stars $[\text{SS}]$ and triangle for data from $[\text{SS}]$.

One possibility to get rid of UV divergences from the beginning is that a more general $N$ can be equivalently considered. Assuming different $N$ is allowed and describes the same form factor

$$T_j(k^2,k.Q,Q^2) = \int_0^\infty d\omega \int_1^1 d\alpha \int_{-1}^1 dz \frac{\rho_T\{N\}(\omega,\alpha,z)}{[k^2 + k.Qz + \frac{Q^2}{4}\alpha - \omega + i\epsilon]^{N}} ,$$

Then Nakanishi’s distributions $\rho_N$ with a different integer parameter $N$ are related through the following relation

$$\rho_{T\{N-1\}}(\omega,\alpha,z) = \frac{-1}{N-1} \frac{d\rho_{T\{N\}}(\omega,\alpha,z)}{d\omega} ;$$

$$\rho_{T\{N+1\}}(\omega,\alpha,z) = -N \int_0^\omega d\alpha \rho_{T\{N\}}(\omega,\alpha,z) .$$

(4.10)
FIG. 3: Phase of the pion form factor $F$ as obtained here.

where we assume the Nakanishi weights vanish at boundaries.

Higher values of $N$ are formally allowed; however they would complicate the future evaluation of the hadronic form factor, so we stay with $N=2$ here. To make our calculation meaningful for such a low $N$ we need to prevent the theory from unwanted UV divergences by another way. For this purpose one needs to impose the following sum rules:

$$\int_{\Gamma} d(\omega, \alpha, z) \rho_{2,[2]}(\omega, \alpha, z) = \int_{\Gamma_3} d(\omega, \alpha, z) \rho_{8,[2]}(\omega, \alpha, z) = 0 \quad (4.11)$$

for two weight functions of potentially dangerous transverse form factors.

In the Eq. (4.11) we have introduced the abbreviation for the 3 dimension integration

$$\int_{\Gamma_3} d(\omega, \alpha, z) f \equiv \int_{0}^{\infty} d\omega \int_{1}^{\infty} d\alpha \int_{-1}^{1} dz f \quad (4.12)$$

which will be used for the purpose of brevity.

Actually, the combination of $T_2$ components with the gauge term of the interaction kernel then produce UV divergence, which is proportional to $\gamma_{\mu}$, i.e. to the first component of the proper vertex. Similarly the eighth component, which is quadratic in the relative momentum $p$ of the quark-antiquark pair $\gamma \mu$ $p$ $\not{Q}$ provides linear divergence in the proper vertex. In the dimensional regularization scheme it has the form:

$$\frac{g g^2}{12 \pi^2} (\gamma_{\mu} p - Q_{\mu}) (\epsilon^{-1} + \text{finite}) \int_{\Gamma_3} d(\omega, \alpha, z) \rho_{8,[2]}(\omega, \alpha, z) \quad (4.13)$$

The derivation of entire contributions to quark-photon vertex due to the gauge interaction is delegated shown in the Appendix. Quite generally, within the condition (4.11) one makes our vertex DSE finite within all transverse components properly accounted in.

V. SUMMARY AND DISCUSSION

Our results are a strong hint that there exists a consistent Integral Representation of QCD Green’s functions. If so, it is of a great interest to explore the physical predictions or within their use to calculate physical processes already experimentally measured, but were beyond theoretical capabilities due to timelike character of momenta in nonperturbative low energy strong regime of QCD. We have already derived Integral Representations for the quark-photon gauge vertex showing it contain part, which is identical with the Gauge Technique. Within two approximations we obtained the result for the pion form factor, yet without inclusion of other transverse components of the vertex. The spectral function of the pion electromagnetic form factor has been obtained from the applications of Integral Representation to Dyson-Schwinger equations for the first time. Up to the norm, the form factor agrees with the experimental data at low $Q^2$ in the both spacelike as well as timelike domain of momenta. To this point, let us
mention that the Gauge Technique was used in the so called Spectral Quark Model studies in [79–81], wherein no further transverse vertices were needed to describe the broad shape of $\rho$ meson peak in calculated electromagnetic pion form factors. Although the Spectral Model does not solve the equations of motion for propagators, neither it uses the lattice predictions for this purpose, nevertheless a prognostic feature of Spectral models was that the simple vertex solely dictated by the Abelian gauge invariance could be enough for a gross description. In this paper, we extend the study of [79] in a sense the quark propagators were calculated from the set of QCD DSEs and within a certain ambiguity we confirm the Gauge Technique is enough to provide the gross shape of the pion form factor.

Of course, deficiencies are due to missing $\omega$ meson and due to the absence of isospin symmetry violating contribution. Further shortcomings, e.g. incorrect rate $F_\pi(0)/F_{\rho\pi}(m_\rho)$ appears due to the approximations, e.g. due to the linearization we have used at this stage. Also, the phase follows the Watson theorem very freely. In our case we get $\delta = 250$ at 1GeV which overestimate the values of others $\delta = 150$. Actually, the number of numerical integrations required for the evaluation of any hadronic form factor is the main weakness of proposed method. It is not the nonperturbative evaluation of building blocks: QCD vertices and propagators where the calculations is stuck, but the evaluation of form factor, where large number of entering Green’s functions limit the evaluation. Further improvement of calculation technology, e.g. avoiding a cumbersome number of auxiliary Feynman integrations till now needed for evaluation of hadronic form factor, is a great theoretical challenge for future. Perhaps, a possible generalization of old fashionable Cutkosky rules, seems to be a promising theoretical direction to deal with the problem more efficiently.

We expect an improvement after the smooth and more realistic version of the kernel \(3.2\) will be used. More improvements can be achieved when a correct weight function $\rho_\pi(a)$ or rather its two-dimensional form $\rho_\pi(a, z)$ of the pion BS vertex

$$\Gamma_E(p, P) = \frac{1}{N} \int_0^\infty dz \int_{-1}^1 \frac{\rho_\pi(o, z, m_\pi)}{p^2 + p.P_z + m_\pi^2/4 - o + i\epsilon},$$

will be included.

To get the desired analytical form factor in the Minkowski space, recall that at least quark propagator needs to satisfy a standard two body dispersion- a generalized Källén-Lehmann representation, although the quark spectral function does need to be positive definite function. Most importantly, no other singularities but the single cut is allowed. Nontrivially, here we achieve this goal by our choice of the quark-antiquark interaction kernel. In this respect, for many other DSEs/BSEs studies presented in the literature: [51–59, 61, 62, 64, 65], which are based on the popular version of Maris-Tandy (MT) interaction kernel introduced in [57], the proof of dispersion relation could be more complicated. And at least the derivation of dispersion relation used here would invalidate at very beginning.

Recall, due to the the Gaussian kernel used in MTs, the interaction strength of MT BSE kernel blows up at timelike infinity. This leads to the known behavior: the analytical continuation of quark propagators exhibits infinite number of complex conjugated poles [55, 60, 63, 68]. Such propagators are not analytic in the domain required for the existence of the IR [63] and one can repeat again, the derivation presented here would invalidate from very beginning.

Our modeled DSE/BSE interacting kernel is certainly very primitive, but it includes the important ingredient—purely longitudinal interaction. While there should not be too much interesting physics contained in it, its numerical presence ensures that the Ladder-Rainbow approximation is working in the entire domain of Minkowski momentum space. Identifying a concrete numerical value of gauge parameter requires further knowledge about other QCD vertices, which is out of model reach. However newly, in order to exhibit approximate gauge fixing independence of presented model, we have changed the gauge fixing parameters (the entire coupling $C_T$). Thus we solve the system numerically in a new gauge once again, determine a gauge dependent coupling $c_g$ and in a new gauges we calculate the function $F$. In all cases, only this single parameter was varied in order to meet pionic observable: the pion mass and pionic weak decay constant. The shape of the function $F$ has been reproduced in several different gauges, showing the model is actually the model of quantum gauge theory: the QCD. We plan to perform similar study within improved setup of BSE vertices.

VI. ACKNOWLEDGMENTS

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Appendix A: Integral Representation for quark-photon vertices

The form of the IR for the proper and semi-amputated photon-quark vertex is derived in this Appendix. Both integral representations are related and the appropriate form are derived as a self-consistent solution of the DSE for
the vertex \( \gamma \). The reason to keep both, the IR for proper as well as for semi-amputated vertex is theoretical and practical. While the DSEs are more conveniently solve in terms of proper Greens function, the hadronic form factors are more easily evaluated in terms of semi-amputated vertices.

Using labeling of momentum as described in the main text, the form of Integral Representation we are going to derive for proper vertex reads:

\[
\Gamma^{\mu}_{\text{EM}}(p, Q) = C\gamma_{\mu} + \Gamma^{\mu}_{\text{EM,L}}(p, Q) + \Gamma^{\mu}_{\text{EM,T}}(p, Q)
\]

\[
\Gamma^{\mu}_{\text{EM,L}}(p, Q) = \sum_{i=1}^{4} W_i^{\mu} L_i(p^2, p.Q, Q^2)
\]

\[
\Gamma^{\mu}_{\text{EM,T}}(p, Q) = \sum_{i=1}^{8} V_i^{\mu} T_i(p^2, p.Q, Q^2)
\]

\[
T_i(p^2, p.Q, Q^2) = \int_{0}^{\infty} d\omega \int_{1}^{\infty} d\alpha \int_{-1}^{1} dz \frac{T_i^{[1]}(\omega, \alpha, z)}{F(p, Q; \omega, \alpha, z)}
\]

(A1)

where

\[
F(p, Q; \alpha, \omega, z) = p^2 + p.Qz + \frac{Q^2}{4} \alpha - \omega + i\epsilon
\]

(A2)

for short and the individual quark momentum associated with quark legs are \( p_{\pm} = p \pm Q/2 \), which variable are used to label semi-amputated vertex in the main text. Here \( W_i \) are longitudinal matrices chosen as \( 1Q^\mu, Q^\mu, Q^\mu, Q^\mu \) and \( Q^\mu (\not p, \not Q - notQ/notp) \) respectively, capital letter \( V_i \) stands for the transverse matrix satisfying \( V.Q = 0 \) and \( \gamma^\mu \) has been taken out for calculation convenience. The IR for scalar form factors \( L_i \) satisfies exactly the same IR as the ones for \( T_i \), but with the distribution \( \tau \) replaced by its own Nakanishi weight function, say \( \lambda \).

The bracketed index \([1]\) means that the first power of the denominator appearing in the last line in (A1) has been chosen and if not different the label will be omitted. \( T \) and \( L \) are scalar form factors, while \( V \) are for times four matrices, wherein their Dirac index are not shown for brevity, the unit i.e. \( \delta_{\alpha,\beta} \) in case of the component \( V_5 \) will not be shown as well. Recall, \( \tau, \lambda \) are distributions, they may involve the product of smooth functions with delta functions.

1. **IR based on DSEs and the relation with Nakanishi’s PTIR**

For pedagogical reasons we mention the connection with the PTIR \([71]\) and the IR used herein. First of all, let us recall here that the PTIR has been derived inductively by using perturbation theory from Feynman rules for scalar theories and for various forms of IR we refer to Nakanishi’s original textbook.

Since the form of IR for a Feynman diagram is dictated by the structure of denominators, it is very natural to assume that a sort of PTIR does exist for any renormalizable quantum field theory in 3+1 dimensions. Furthermore, it is useful to assume (at least for a while) that the only difference is that there are as many various Nakanishi weight matrices, wherein their Dirac index are not shown for brevity, the unit i.e. \( \delta_{\alpha,\beta} \) in case of the component \( V_5 \) will not be shown as well. Recall, \( \tau, \lambda \) are distributions, they may involve the product of smooth functions with delta functions.

\[
T, L(p, Q) = \int_{0}^{\infty} d\omega \int dx_1 dx_2 dx_3 \frac{\rho(x_1, x_2, x_3, \omega)}{[p^2 x_1 + p.Q x_2 + Q^2 x_3 - \omega + i\epsilon]}.
\]

(A3)

The polynomial structure and matrices which appear in the numerator of any Feynman diagrams for gauge theory vertex is dictated by Lorentz invariance and are crucial for the number of components, but not for the number of integral variables. Three \( x \)–variables are known to be bounded as the Nakanishi weight function carries the delta function \( \delta(1 - \sum_{i} x_i) \), which is almost entire information we can get from analyzes of Feynman diagrams in general. Obviously, by dividing by \( x_1 \) variable in the kernel and defining a new variables, our proposed IR \([44]\) is included in PTIR inspired from of the gauge vertex; however, this is far from saying it is derivable from PTIR.

In this respect one can only say, that the form of Nakanishi weight functions i.e., the 12 distributions of \( \tau \) and \( \lambda \) can be inspected from the perturbation theory expansion by studying each individual Feynman diagram in separation. There would be a very limited benefit of doing so in strong coupling theory like QCD. Therefore, our proof of the IR \([44]\) does not follow from perturbation theory, but relies on the self-consistent solution of DSE within suggested form of IR implemented. This, when embedded into the r.h.s. of the DSE for the vertex \([23]\), after the integration...
over the momentum in the Euclidean space, reappears on the l.h.s. of DSE again and has exactly identical form that has entered, i.e. the form of IR (A1). The set of weight functions \( \rho_i \) (or equivalently \( \tau_i \) and \( \lambda_i \)) must obey certain conditions: they satisfy a new coupled set integro-differential equations into which the vertex DSE (2.3) is transformed. These equations do not depend on the momenta, but on the three spectral/integral variable \( \omega, \alpha \) and \( z \), with their domain self-consistently determined by the DSEs for vertices and propagators.

For clarity, we should mention here, that the Gauge Technique form of the vertices \( \varepsilon_3, \varepsilon_8 \), which was employed in calculation of meson form factors \( [33, 39, 70] \), represents an approximate subset of IR (1.4). While Gauge Technique vertices are derived Ward identities, however they are not fully self-consistent since they, at any known approximation of DSE, do generate richer structure involving longitudinal as well as transverse vertices. Their entire form is captured by three parametric IR (1.4).

We do not know yet whether the new -integro-differential equations for Nakanishi weights provide a unique solution, however we assume it is the case. We do not even know whether functions \( \rho_i \) exist at all, since the numerical solution is yet out of our reach at the moment. However, when keeping the solution at hand, as has already checked in case of more simple truncation of DSEs system \( [85, 86] \), the consistency with the standard Euclidean formulation can be straightforwardly inspected by the comparison.

In the next subsection of this appendix, we will illustrate the proof on the example of the contribution stemming from the product of the gauge technique vertex and the gauge part of the propagator as well as deriving the IR (1.4) for the particular example of the \( T_5 \) component. In subsequent subsection we write down the relation between IR for proper and semiamputated vertices, which closes the proof. We do not provide the entire list of all contributions, since we do not need them in our approximation. Note especially, the conversion of transverse pieces of \( G_{EM}^{\mu} \) is a quite straightforward task and is illustrated enough in the single component example.

### 2. IR for proper vertices

The IR has two pieces, the first is governed by gauge covariance and the second involves all transverse components independently. Here we show that both terms give rise to an IR for the proper vertex with the same structure of transverse components as well as giving rise longitudinal components in \( \Gamma \). Four later longitudinal terms are in fact independently. Here we show that both terms give rise to an IR for the proper vertex with the same structure of

Contributions to and due to the transverse vertices

Contributions from transverse vertices are exemplified for the most important cases. These are the ones due to the first and the fifth components, the latter is known to be dominant at least in the Landau gauge. Thus in fact, here we show their contributions due to other gauges.

Further, for purpose of discussion of the renormalization we also review the contribution due to the second as well as to the eight component of the transverse part of the vertex.

\( \text{V5:} \) We start with the contribution governed by the fifth component in the Eq. (4.4), i.e. by \( k_T \) where the momentum \( k \) is the relative momentum of produced quark-antiquark pair. Contribution due to metric tensor \( \gamma \times \gamma \) part of the kernel (hence due to the \( \gamma^{\mu}_{ab} \times \gamma^\mu_{\nu,c} / (q^2 - \mu^2) \) matrices and due to the gauge part proceed similarly and we will describe the details for the first example. The first contribution to the proper vertex in our DSE thus reads

\[
-ic_g \int \frac{d^4 k}{(2\pi)^4} \int_{\Gamma_3} d(\omega, \alpha, z) \rho_5(\omega, \alpha, z) \gamma_{\nu} \left( k^{\mu} - \frac{Q^{\mu} Q}{k^2} \right) \gamma^\nu - \ldots, \tag{A4}
\]

where three dots represent the identical integral with the phenomenological parameter \( \mu_g \) replaced as \( \mu_g \rightarrow \Lambda_g \). Using Feynman variable \( x \) to match two denominators, making a standard square completion, shifting the integral variable and integrating over the momentum, we get after some algebra, for the considered contribution:

\[
\frac{4c_g}{(4\pi)^2} \int_{\Gamma_3} d(\omega, \alpha, z) \int_{0}^{1} dx \frac{\rho_5(\omega, \alpha, z) p_{T}^{\mu}}{p^2 + p.Q - x\omega - q^2 (1-x)} - \ldots, \tag{A5}
\]

where we have factorized prefactor \( x(1-x) \) out of the numerator and canceled it against the same factor in the numerator and where the meaning of three dots is just as above in Eq. (A4). We do not write Dirac index and we also omit explicit writing of the Feynman infinitesimal term \( ic \) in most denominators for the purpose of brevity.
In what follows we perform the substitution $\omega \rightarrow \tilde{\omega}$ and then $x \rightarrow \tilde{\alpha}$ such that
\begin{align}
\tilde{\omega} &= \frac{\omega + \mu_g^2}{1 - x}, \quad (A6) \\
\tilde{\alpha} &= \frac{\alpha - z^2 x}{1 - x}, \quad (A7)
\end{align}
which provides the following result for the contribution \[A4\]
\begin{equation}
 p_T^\mu \frac{4 g}{(4 \pi)^2} \int_{-1}^{1} dz \int_1^\infty d\alpha \int_\alpha^{\infty} d\tilde{\alpha} \int_{\tilde{\alpha}^2}^{\infty} d\omega \frac{x(1 - x)^2 \rho_5 [(\tilde{\omega} - \mu_g^2/ x)(1 - x), \alpha, z]}{F(p, Q; \tilde{\omega}, \tilde{\alpha}, z)} - ..., \quad (A8)
\end{equation}
where the ordering of integrals is important. To avoid complicated explicit notation, in case the measure $dx$ is not explicitly written, the letter $x$ will be kept for the following function
\begin{equation}
x = \frac{\tilde{\alpha} - \alpha}{\tilde{\alpha} - z^2}.
\end{equation}
Also as follows from the inverse transformation \[A7\] the variable $\omega_g = \omega$, which reads
\begin{equation}
\omega_g = (\tilde{\omega} - \mu_g^2 / x)(1 - x).
\end{equation}
will be kept for purpose of brevity.

In order to get the desired form of IR for proper vertex we need to change the integration ordering. Also it is convenient to send the information about integration volume into the kernel by using Heaviside step function. Performing this correctly, one can write for the contribution \[A4\] the resulting IR:
\begin{align}
 p_T^\mu \int_{\Gamma_3} d(\tilde{\omega}, \tilde{\alpha}, z) \frac{\tau_{5a}(\tilde{\omega}, \tilde{\alpha}, z)}{F(p, Q; \tilde{\omega}, \tilde{\alpha}, z)}
 &\tau_{5a}(\tilde{\omega}, \tilde{\alpha}, z) = \frac{4 g}{(4 \pi)^2} \int_{-1}^{\tilde{\alpha}} d\alpha \frac{x(1 - x)^2 \theta(\tilde{\omega} - \mu_g^2 / x)}{\tilde{\alpha} - \alpha} \rho_5[\omega_g, \alpha, z] - ... \quad (A11)
\end{align}
where three dots remind us that one should change $\mu_g$ into the parameter $\Lambda$ appropriately, i.e, one should introduce a new variable
\begin{equation}
\omega_L = (\tilde{\omega} - \Lambda^2 / x)(1 - x)
\end{equation}
to define the new variable $\omega_L$ in the function $\rho_5[\omega_L, \alpha, z]$.

Here we could stress the difference from Nakanishi derivation of PTIR. Blindly following a Nakanishi’s derivation would mean the use of the variable $x$ to give rise to our variable $\omega$ (or $\tilde{\omega}$) and we have used slightly different strategy here. In our approach here we avoid numerically inconvenient square roots otherwise presented in the kernel (see toy models without confinement \[S3, S4\]). Recall, the trick we use here, would be impossible without using the fact, that the quark propagator is entirely described by a continuous spectral function. In fact, this is the issue of confinement, which allows us to write the simple equation for IR.

**V5:** We continue with the contribution coming from the transverse vertex $V_5$ matched with the gauge longitudinal interaction part of the kernel $K$. This particularly simple contribution reads
\begin{equation}
 -i C_T \int \frac{d^4 k}{(2 \pi)^4} \int_{\Gamma_3} d(\omega, \alpha, z) \rho_5[\omega, \alpha, z] \frac{(k^\mu - Q^\mu k^\nu Q_{\nu}^\rho)}{[F(k, Q; \omega, \alpha, z)]^2 q^2}. \quad (A13)
\end{equation}
After a few steps sketched in previous cases, this relation can be converted into the following IR:
\begin{align}
 p_T^\mu \int_{\Gamma_3} d(\tilde{\omega}, \tilde{\alpha}, z) \frac{\tau_{5b}(\tilde{\omega}, \tilde{\alpha}, z)}{F(p, Q; \tilde{\omega}, \tilde{\alpha}, z)}
 &\tau_{5b}(\tilde{\omega}, \tilde{\alpha}, z) = \frac{C_T}{(4 \pi)^2} \int_{-1}^{\tilde{\alpha}} d\alpha \frac{\alpha - z^2}{(z^2 - \tilde{\alpha})^2} \rho_5 \left[ \frac{z^2 - \alpha}{z^2 - \tilde{\alpha}}, \omega, \alpha, z \right]. \quad (A14)
\end{align}
Thus, for the resulting total contribution due to the fifth component one just needs to sum up

$$\Delta \tau_5 = \tau_{5a} + \tau_{5b}. \quad (A15)$$

Amazingly, due its simplicity, the IR exhibits self-reproducing property: the contribution to the fifth component $\Delta \tau_5$ is given by the integral over the function $\rho_2$ and no other component is generated. However, the contribution is not complete, the contribution is not entire and other components e.g. $p^\mu \not p Q$ component can contribute as well. Of course, one should keep in mind, the $\rho$ is the Nakanishi weight distribution for the semi-amputated vertex, while $\tau$ is for the proper vertex. Hence the relation between proper and semi-amputated vertices needs to be established. This is the subject of the second part of this Appendix. Before that, we review other important contributions.

The transformation of contributions from terms which involve combinations of momenta is straightforward, albeit quite involved. For the purpose of brevity we write the results in the form of fractions, which include also the second power of $F$, noting the second power of $F$. As the others, it is combined with the gauge part as well with the metric phenomenological interaction in the DSE for the vertex. Explicitly written the first contribution reads

$$-iC_T \int \frac{d^4k}{(2\pi)^4} \int_{\Gamma_3} d(\omega, \alpha, z) \rho_1[\omega, \alpha, z] \frac{\hat{f}(\gamma^\mu - \frac{Q^\mu Q}{Q^2})}{|F(k, Q; \omega, \alpha, z)|^2(q^2)^2}, \quad (A16)$$

which after repeating similar steps as for the $V_5$ component above, leads, after some summations and trivial algebra, into the form:

$$- \gamma_T^\mu \int_{\Gamma_3} d(\bar{\omega}, \hat{\alpha}, z) \int_1^{\tilde{a}} d\alpha \frac{C_T}{(4\pi)^2} \frac{x(1-x)(2-x)\rho_1[\bar{\omega}(1-x), \alpha, z]}{F(p, Q; \omega, \alpha, z)} - p_T^\mu \frac{x^2(1-x)(Q^2 + 2 \not p)}{(\hat{\alpha} - \alpha)}, \quad (A17)$$

noting the second power of $F$ in the second line.

Assuming the boundary condition $\rho_1(0, \alpha, z) = 0$ we use per-parts integration with respect to $\bar{\omega}$, which increases the power of $F$ by a unit. The final, three components IR of desired form then read

$$\int_{\Gamma_3} d(\bar{\omega}, \hat{\alpha}, z) \frac{C_T}{(4\pi)^2 F(p, Q; \omega, \alpha, z)} \int_1^{\tilde{a}} d\alpha \left[ - \gamma_T^\mu x(1-x)(2-x)\rho_1[\bar{\omega}(1-x), \alpha, z] \right] (\hat{\alpha} - \alpha), \quad (A18)$$

V1 g: To calculate the contribution due to the interaction kernel with the metric tensor is more simple. The appropriate contribution reads:

$$-iC_T \int \frac{d^4k}{(2\pi)^4} \int_{\Gamma_3} d(\omega, \alpha, z) \rho_1[\omega, \alpha, z] \frac{\gamma \gamma^\beta (\gamma^\mu - \frac{Q^\mu Q}{Q^2})}{|F(k, Q; \omega, \alpha, z)|^2(q^2)^2} - \cdots, \quad (A19)$$

Repeating basically same steps, which were used to transform the $T_5$ contribution, one gets for the $A19$ the result

$$-\gamma_T^\mu \int_{\Gamma_3} \frac{d(\bar{\omega}, \hat{\alpha}, z)}{F(p, Q; \omega, \alpha, z)} \frac{C_T}{(4\pi)^2} \int_1^{\tilde{a}} d\alpha \frac{2x(1-x)\rho_1[\omega_g, \alpha, z]}{(\hat{\alpha} - \alpha)} \theta \left( \bar{\omega} - \frac{\omega_g^2}{x} \right), \quad (A20)$$

where since the substitution (A7) was made at the end of the transformation $[x$ stands for the fraction (A9) and $\omega_g$ is defined by Eq. (A10)].

V2 For contribution to proper quark-photon vertex due the second transverse component (i.e. $k_T^\mu \not k$) and due to the gauge interaction kernel takes the form

$$-iC_T \int \frac{d^4k}{(2\pi)^4} \int_{\Gamma_3} d(\omega, \alpha, z) \rho_2[\omega, \alpha, z] \frac{\hat{f} \left( k^\mu - \frac{Q^\mu Q}{Q^2} \right)}{|F(k, Q; \omega, \alpha, z)|^2(q^2)^2}, \quad (A21)$$
can be readily transformed into the following form
\[
\gamma^\mu \frac{C_F}{4(4\pi)^2} (1/\epsilon_d - \gamma_E) \int_{\Gamma_3} d(\omega, \alpha, z) \rho_2[\omega, \alpha, z]
\]
\[
+ \gamma^\mu \frac{C_F}{2(4\pi)^2} \int_{\Gamma_3} d(\omega, \alpha, z) \int_0^{\tilde{\alpha}} dx \frac{x^2 (1-x) \left( R_2[\omega(1-x), \alpha, z] - (1-x) (Q_pz + 2p^2) \rho_2[\omega(1-x), \alpha, z] \right)}{F(p, Q; \tilde{\omega}, \tilde{\alpha}, z)}
\]
\[
- p^\mu \frac{C_F}{(4\pi)^2} \int_{\Gamma_3} d(\omega, \alpha, z) \int_0^{\tilde{\alpha}} dx \frac{x^2 (1-x) (1+x) \rho_2[\omega(1-x), \alpha, z]}{F(p, Q; \tilde{\omega}, \tilde{\alpha}, z)}
\]
\[
+ 2 p^\mu \frac{C_F}{(4\pi)^2} \int_{\Gamma_3} d(\omega, \alpha, z) \int_0^{\tilde{\alpha}} dx \frac{x^2 (1-x) \rho_2[\omega(1-x), \alpha, z]}{\tilde{\alpha} - \alpha} \frac{[-Q_pz/2 - Q^2 \tilde{\alpha}/4 + \tilde{\omega}](Qz/2 + \tilde{p})}{\left[ F(p, Q; \tilde{\omega}, \tilde{\alpha}, z) \right]^2}.
\]
(A22)

We plan to publish the details of derivation and further useful relations in a separate Supplemental Material.

The Euler constant \(\gamma_E\), which arises in the equation above, is due to the standard dimensional regularization and should not be confused with projected gamma matrices. As we can see the second component of \(T\) gives rise to nontrivial contributions to the four different components, including the second component itself. The elimination of momentum from the numerator and adjusting the power of \(F\) to the desired value \(N_2\) is matter of standard practice. Furthermore, let us mention that due to the \(z\) dependence, many terms turn out to be zero and the equation above is already in form suited for the first calculation. Note also, there is a single term proportional to \(\gamma_T\), which is divergent in the limit \(d \rightarrow 4; (\epsilon_d \rightarrow 0)\).

**Contributions due to the Gauge Technique vertex**

In the following part we will inspect the proper vertex contribution due to the Gauge Technique IR. As a part of the proof, we will show that the Gauge Technique is equivalent to the proposed IR for semiamputated vertex. Then we will calculate its contribution to proper vertex for its combination with the gauge, i.e. purely longitudinal part of the interaction kernel. As is clear from the previous part devoted to the transverse components, deriving the IR due to the interaction with a metric tensor is a matter of practice, where several well-controlled changes in derivation cannot violate the resulting functional form of the IR.

Considering the DSE with aforementioned inputs on the rhs of the DSE (2.3) means to evaluate the following contribution

\[
-i C_T \int \frac{d^4k}{(2\pi)^4} \frac{d^4l}{(2\pi)^4} G_{GT}^\mu(k_, k_+) \frac{H}{(q^2)^2}
\]

(A23)

where again we label \(C_T = e_q g^2 \xi T_3^2\) and the momentum associated with the internal gluon line is \(q = p - k\).

In the first step, we will show the Gauge Technique vertex, in its conventional form:

\[
G_{GT}^\mu(k_-, k_+) = \int_{-\infty}^{\infty} dx \frac{\rho(x) \gamma^\mu}{|k_- - x + i\epsilon| |k_+ - x + i\epsilon|}
\]

(see some of the papers [75][76][78]) is identical to the second line of IR for semi-amputated vertex (4.4). As we prefer to work with two quark propagator spectral function \(\rho_o\) and \(\rho_s\) we rewrite the above expression into less familiar form

\[
G_{GT}^\mu(k_-, k_+) = \int_0^{\infty} da \rho_o(a)[k_- \gamma^\mu \kappa_- + a \gamma^\mu] + \rho_s(a)[k_- \gamma^\mu + \gamma^\mu k_+]
\]

\[
(k_- - a + i\epsilon)(k_+ - a + i\epsilon)
\]

(A25)

where two functions \(\rho_o\) and \(\rho_s\) are defined on \(R^+\) and they are related with the single function \(\rho\) in the following manner

\[
\rho_o(a) = \frac{\rho(\sqrt{a}) + \rho(-\sqrt{a})}{2\sqrt{a}}
\]

\[
\rho_s(a) = \frac{\rho(\sqrt{a}) - \rho(-\sqrt{a})}{2}.
\]

(A26)

The advantage of our choice is that the function on lhs. takes a nontrivial value at the positive real axis, which simplifies many manipulations we will perform and show in this appendix. In addition we use the following identity

\[
\frac{1}{k_- - a} \frac{1}{k_+ - a} = \int_{-1}^{+1} dz \frac{1}{[k_- + k_+ Qz + Q^2 - a]^2}
\]

(A27)
in order to match the denominators in \((A25)\), getting thus desired form that corresponds to the second line of the entire IR \((4.1)\).

In this way we have proved that the Gauge Technique is a part of proposed IR in addition to converting \(G_{GT}\) into the form suited for evaluation. Substituting Eq. \((A25)\) into the formula \((A23)\) we get at this stage

\[
\sum_{v=a,b,c,d} \Gamma_{v}^{\mu}(k, P) = -i C_{F} \sum_{v=a,b,c,d} \int \frac{d^{4}k}{(2\pi)^{4}} \int_{0}^{\infty} da \int_{-1}^{1} dz \frac{N_{v}^{\mu}(k, p, Q; a)}{[k^{2} + k.Qz + \frac{Q^{2}}{4} - a^{2}(q^{2})^{2}]}, \tag{A28}
\]

\[
N_{a}^{\mu} = \rho_{v}(a) \, \delta(k - Q/2) \gamma^{\mu}(k + Q/2) \, \delta \gamma^{\mu} \, \delta, \quad N_{b}^{\mu} = \rho_{v}(a) \, \delta \gamma^{\mu} \, \delta, \tag{A29}
\]

\[
N_{c}^{\mu} = 2p_{v}(a)k^{\mu}q^{2}; \quad N_{d}^{\mu} = \delta[\gamma^{\mu}, Q] \, \delta. \tag{A30}
\]

From this point, up to the different matrix structure of the numerator, the treatment of the rest is easy as in the case of previous study of transverse contribution. Of course, the IR is two instead of three dimensional, thus aside from a completely continuous part, one can expect the delta function when one uses three dimensional write up \((\delta(\alpha - 1))\). Nevertheless, even so, the IR for the proper function turns to be 3-dimensional.

To derive the IR we will use the variable \(y\) to match the result with \(q^{2}\) in the denominator, which leads to the following result :

\[
\Gamma_{v}^{\mu}(k, P) = -i \int \frac{d^{4}k}{(2\pi)^{4}} \int_{0}^{\infty} dz \int_{-1}^{1} dy \frac{3C_{F} y(1 - y) N_{v}^{\mu}}{\left(\left(k + \frac{Q^{2}}{2} z y - p(1 - y)\right)^{2} - \left(\frac{Q^{2}}{2} z y - p(1 - y)\right)^{2} + p^{2}(1 - y) + \frac{Q^{2}}{4} y - a y\right)^{4}}. \tag{A31}
\]

for each term defined in Eq. \((A28)\).

The rest of transformation is quite universal and we will not repeat it for all terms individually, but we illustrate it for two scalar cases. The first is the integral where we replace the function \(N_{v}^{\mu}\) by \(N_{1} = f(a)\) where \(f(a)\) stands for some continuous functions. Let us label such an auxiliary scalar vertex by the index 1. Integrating over the momentum we directly get

\[
\Gamma_{1}(k, P) = \frac{C_{F}}{(4\pi)^{2}} \int_{-1}^{1} dz \int_{0}^{\infty} da \int_{0}^{\infty} dy \frac{[y(1 - y)]^{-1}}{\left[p^{2} + Q.pz + \frac{Q^{2}}{4} \frac{1 - z y}{1 - y} - \frac{a y}{1 - y}\right]^{2}}. \tag{A32}
\]

For completeness, we repeat the entire transformation here.

Let us perform the substitution \(y \rightarrow \alpha\) such that

\[
\alpha = \frac{1 - z^{2} y}{1 - y}, \quad y = \frac{\alpha - 1}{\alpha - z^{2}}. \tag{A33}
\]

And then the second substitution \(a \rightarrow \omega\) such that

\[
\omega = \frac{a}{1 - y}, \tag{A34}
\]

obtaining thus for \((A32)\) the following expression

\[
\Gamma_{1}(k, P) = \frac{C_{F}}{(4\pi)^{2}} \int_{-1}^{1} dz \int_{0}^{\infty} da \int_{0}^{\infty} d\omega \frac{[\omega - \frac{z^{2} y}{1 - \alpha \frac{z^{2} - \omega^{2}}{1 - \omega}}]}{\left[p^{2} + Q.pz + \frac{Q^{2}}{4} \frac{1 - z y}{1 - y} - \omega + i \epsilon\right]^{2}}. \tag{A35}
\]

As in the previous part, here the variable \(y\), if used without integral measure in any expression will be kept even after the substitutions are performed for the purpose of brevity. Its meaning will be unique throughout this paper and always given by the second Eq. in \((A33)\).

The scalar function \(\Gamma_{1}\) is not yet in desired form and for this purpose we perform per-partes integration with respect to the variable \(\omega\). Doing this we can write

\[
\Gamma_{1}(k, P) = \int_{\Gamma_{3}} d(\omega, \alpha, z) \frac{C_{F}(1 - z^{2})}{(4\pi)^{2}(1 - \alpha)(\alpha - z^{2})} \int_{0}^{\infty} d\omega \frac{[\omega - \frac{z^{2} y}{1 - \alpha \frac{z^{2} - \omega^{2}}{1 - \omega}}]}{\left[p^{2} + Q.pz + \frac{Q^{2}}{4} \frac{1 - z y}{1 - y} - \omega + i \epsilon\right]^{2}}, \tag{A36}
\]

where we have assumed the function \(f\) is vanishing at boundaries. Recall, within a numerical accuracy, this is certainly true for the quark spectral function and we will repeatately exploit the fact that \((\rho_{v,s}(0) = \rho_{v,s}(\infty) = 0)\).
The assiduous reader can note that there is an infrared log divergence involved in the $\alpha$ integral in Eq. (A36). These are standard IF divergences owned to massless gauge boson modes and actually similar divergences appear in $\Gamma^{\mu}$ and make associated form factor procedure dependent. If appears numerically, it could be used to cancel against similar divergences due to the emission of soft real photons in the physical cross section. This fact, however, does not bother us yet, since we are not going to solve the DSEs system in this paper.

Such IF divergence does not appear for a less divergent kernel. Therefore, by replacing for instance $1/q^{4}$ kernel in the Eq. (A32) with $1/q^{2}$ one can get the IR in the following regular form:

$$\int_{\Gamma_{3}} d(\omega, \alpha, z) \frac{C_{\Gamma}}{(4\pi)^{2}} f \left[ \frac{\omega - z^{2}}{\alpha - z^{2}} \right] F(p, Q; \omega, \alpha, z).$$

(A37)

C:
Repeating the game for our vertex (A23) is relatively straightforward. The only complication is that one is faced to a larger number of momentum integrations over various tensors. Here we start with the simplest case, say component $V$ the Eq. (A28). Taking changes into account, one gets two vector contributions, the first contributes to the transverse per-partes is needed to convert this part into the desired IR. Hence we will comment on some steps in more detail.

The conversion of $\Gamma_{s}$ is technically the most demanding, not only this piece involve UV divergence, but a double per-partes is needed to convert this part into the desired IR. Hence we will comment on some steps in more detail.

Here the UV divergent terms, which stem from the first terms of numerator expansions

$$\hat{f}_{1} Q^{\gamma} Q \hat{f}_{1} = \gamma^{\mu} Q^{2} q^{2} + 4q^{\mu} Q.q \hat{f}_{1} - 2Q^{\mu} q^{2} \not{Q} - 2q^{\mu} Q^{2} \not{Q}$$

will be concerned here. We will not list all IRs stemming from other contributions, which are relatively easy to evaluate; we will publish them when an actual numerical solution is available.

In order to see how individual terms arise during the derivation, we will write down a few intermediate steps. Summing the first terms in expansions above, then we get after Feynman paramatrization (i.e. before the transformation (A34)) the following result

$$-iC_{\Gamma} \int d \rho_{\alpha}(a) \int_{0}^{1} d \alpha \int_{-1}^{1} d z \int \frac{d^{4}k}{(2\pi)^{4}} \frac{\gamma^{\mu} \left[ k^{2} - \frac{Q^{2}}{4} \right]}{(k^{2} + p^{2}(1 - x)x + \frac{Q^{2}}{4} x(1 - z^{2}x) + p.Q x(1 - x) - ax)^{3}},$$

(A40)

where $\hat{k} = k + Q Z x/2 - p(1 - x)$ and we omit some prefactors for the purpose of brevity.

We will use the dimensional regularization; thus we label $\epsilon^{-1} = 4 - d$ as the divergent constant in 4 dimensions. After the usual shift the term proportional to $\hat{k}^{2}$ gives

$$\int_{0}^{1} d x \frac{2 \gamma^{\mu} x}{(4\pi)^{2}} \left[ -\frac{2}{\epsilon} - \gamma_{E} + \ln(1 - x) x + \ln F(p, Q; \frac{\alpha}{1 - x}, \frac{1 - z^{2}x}{1 - x}, z) \right],$$

(A41)

where we omit some unimportant prefactors.

After the substitution (A34) (with $x$ instead of $y$) we get the following entire expression:

$$\Gamma^{\mu}_{c}(p, Q) = \gamma^{\mu} \text{Const.} + \gamma^{\mu} \frac{C_{\Gamma}}{(4\pi)^{2}} \int_{\Gamma_{3}} d(\omega, \alpha, z) \frac{2 y(1 - y)(1 - z^{2})}{(z^{2} - \alpha)^{2}} \rho_{\alpha} [\omega(1 - y)] \ln \left[ F(p, Q; \omega, \alpha, z) \right]$$

$$+ \gamma^{\mu} \frac{C_{\Gamma}}{(4\pi)^{2}} \int_{\Gamma_{3}} d(\omega, \alpha, z) \frac{(1 - z^{2})}{(z^{2} - \alpha)^{2}} \rho_{\alpha} [\omega(1 - y)] \frac{Q^{2}}{4} \left( z^{2}y^{2} - 1 \right) + p^{2}(1 - y)y - Q.p z(1 - y)y}{F(p, Q; \omega, \alpha, z)},$$

(A42)

where in order to avoid cluttering notation, we remind here that the letter $y$ is simply (A33) (since $x$ is reserved for a different function in our notational convention). In this process, a constant term (UV divergent) $\gamma^{\mu} \text{Const}$ appears, into which we also sent some constant pieces, which have been generated during the derivation. It should be taken in
mind that the entire i.e. the finite as well infinite part of the vertex could be consistent with the renormalization of the quark DSE due to the WTI.

To transform the first line to the desired IR we use per-partes integration with respect to the variable $\omega$. For this purpose we use the following expression for the primitive function in the numerator

$$R_\omega[\omega, \alpha, z] = \int_0^\omega d\alpha \rho_\omega[\alpha, 1-y]).$$

(A43)

Further, irrespective of the value of boundary term, we set it into the constant term. The remaining of the first line then reads

$$\gamma_\mu \left[ \text{Const.} + \int_{R_\gamma^2} d\omega, \alpha, z \frac{\rho_{\log}(\omega, \alpha, z)}{F(p, Q; \omega, \alpha, z)} \right],$$

(A44)

where the contribution to the vertex weight function is

$$\rho_{\log}(\omega, \alpha, z) = 2C_\gamma \frac{y(1-y)(1-\omega^2)}{(\omega-\alpha)^2} R_\omega[\omega, \alpha, z].$$

(A45)

To convert the second line in Eq. (A42) one can divide term with $p^2$ as the first step. Then the term in the numerator, which is proportional to the variable $Q^2$, can be treated by per-partes integration to cancel it with the prize we get $ln(J)$ instead of $J^{-1}$. In order to get $J$ back in the denominator one can integrate per-partes, but now with respect to the variable $\omega$. The terms involving scalar product $p, Q$ in the numerator can be treated analogously, but instead of the variable $\alpha$ one needs to use the variable $z$. The single resting term has already been derived in this form of IR. The entire result for the second line is then given by (A44) where instead of $\rho_{\log}$ we have the following function

$$C_\gamma \frac{d}{d\alpha} \left[ \frac{(z-\alpha)^2}{(z^2-\alpha)^2} (1-y)(1-\omega)(2y-1)R_\omega[\omega, \alpha, z] \right]$$

$$+ C_\gamma \frac{d}{d\alpha} \left[ \frac{(z-\alpha)^2}{(z^2-\alpha)^2} (1-\omega^2 y^2 + \alpha(1-y)^2) R_\omega[\omega, \alpha, z] \right]$$

$$- C_\gamma \omega \frac{(1-\omega^2)}{(z^2-\alpha)^2}(1-y^2) \rho_v[\omega(1-y)].$$

(A46)

D:

Repeating the game for the last numerator in (A28) gives us

$$\Gamma^\mu_d(p, P) = \int_{-1}^1 dz \int_1^\infty d\omega \int_0^\infty d\alpha \int_0^\infty d\omega \frac{C_\gamma}{(4\pi)^2} \frac{(1-\alpha)(1-\omega^2)}{(z^2-\alpha)^3} M^\mu \rho_s \omega \frac{(1-\omega^2)}{(z^2-\alpha)^2} F(p, Q; \omega, \alpha, z).$$

(A47)

where we do not write Dirac index for brevity.

Using the following identity

$$M^\mu = p^2[\gamma^\mu, Q] + 2p^\mu [\not\! p, Q] + 2Q.p[\not\! p, \gamma^\mu]$$

$$- \frac{z^2}{4} Q^2[\gamma^\mu, Q] + z \left( Q^\mu[\not\! p, Q] + Q^2[\not\! p, \gamma^\mu] \right),$$

(A48)

one can immediately recognize various components of the quark-photon vertex, and thus for instance the last line when implemented in (A47) gives $\sim z(-1)Q^2 V_0$ in the numerator. The last step to get desired IR is the per-partes contribution with respect to the variable $\alpha$, which lower the power of $F$ in the denominator and cancels out unwanted presence of $Q^2$ in the numerator. To convert other terms of $M$ into IR is matter of simple algebra and the repeated use of per-partes integration. The result, together with the numerical solutions, will be published in the future.
3. Integral representation for a semiamputated vertex

Using an accepted form of the semiamputated vertex we have shown the proper vertex $\Gamma^\mu$ satisfies the integral representation, which up to the power of the denominator has the identical form as assumed form of the semiamputated vertex itself. What remains is to show that the IR for SAV is consistent with the obtained IR for proper vertex from DSE solution.

The first power of $F$ in the denominator of the IR is preferable choice for next purpose, which choice simplifies some parts of calculation. However remind, $N = 2$ was a preferred choice in preceding sections. Here we thus should note, that these two weight functions are simply related.

Thus we are going to find relation between the IR of left and right side of the following definition:

$$G_{EM}^\mu(k^+, k^-) = S(k^-)\Gamma_{EM,T}(k, Q) S(k^+).$$  \hspace{1cm} (A49)

with all functions on the rhs. expressed through their own IR. Plugging this IR for the proper vertex, which we recall together with spectral representations for the quark propagators one can write the result

$$G_{EM}^\mu(k^+, k^-) = S(k^-)\Gamma_{EM,T}(k, Q) S(k^+)$$  \hspace{1cm} (A51)

we are prepared to convert the resulting expression

$$\int_0^\infty da db d\omega_{\alpha} \int_1^{z_1} \frac{dz_1 k^2 \rho_\alpha(a) + \rho_\alpha(a)}{(k^2 - a + i\epsilon)(k^2 + k Q z_1 + \frac{Q^2}{2} a - \omega_{\alpha} + i\epsilon)} \frac{\tau^\mu_I(\omega_1, \alpha, z_1) V^\mu_I}{(k^2 - b + i\epsilon)}$$

into the form of suggested IR \[(A51)\] for the lhs of the definition of the SAV.

Let us first briefly describe the core of the proof. As a first step, we commute all $V$’s from the middle position into the front, and use the Feynman rules for denominators to match the propagators $S$ and the proper vertex together. This gives us the IR with some additional presence of the scalar product of momenta in the numerator. If a given term in the numerator belongs to $T_i$, we need only adjust a proper denominator $N = 2$. If there is additional momentum dependence in the prefactor, we use the per-partes integration to remove it with simultaneous change of power of the numerator. At the end, we adjust the power of the denominator to $N = 2$ by per-partes integration with respect to newly defined variable $\omega_{\alpha \nu \nu \nu}$. From all of $V$ we choose the Dirac $\gamma_T$ only, the other terms proceed similarly. In what follows we will not write the Feynman $i\epsilon$; its presence is assumed implicitly.

Using the formula

$$1 \over k^2 - a k^2_+ - b = \int_1^{z_1} dz_G \frac{1}{[k^2 + k Q z_G + \frac{Q^2}{2} - \omega_G]^2}$$  \hspace{1cm} (A53)

and further matching with the denominator of the proper vertex in \[(A52)\]

$$1 \over [k^2 + k Q z_G + \frac{Q^2}{2} - \omega_G]^2 [k^2 + k Q z_T + \frac{Q^2}{2} a - \omega_T]$$

one can write the result

$$G_{EM}^\mu(k, Q) = \int_0^\infty da db \int_0^1 dx \int_1^{z_1} dz_1 dz_G\left[ -2x[\rho_\alpha(a) \rho_\alpha(b) \gamma^\mu(Q^2/4 - k^2) + R^\mu] \over [k^2 + k Q z_G x + z_G(x + (1 - x)) - \omega_G x - \omega_T(1 - x) + i\epsilon]^3 \right]$$

$$R^\mu = (2k^\mu + Q^\mu)(k - Q/2) + \gamma^\mu/2[k, Q] + \rho_\alpha(a) \rho_\alpha(b) \gamma^\mu + \rho_\alpha(a) \rho_\alpha(b)(2k^\mu + Q^\mu)$$

$$- \rho_\alpha(a) \rho_\alpha(b) \gamma^\mu)(k - Q/2 + \rho_\alpha(a) \rho_\alpha(b) \gamma^\mu)(k - Q/2).$$  \hspace{1cm} (A55)
As a next step we perform the following substitutions

\[ \tilde{\alpha} = x + \alpha(1-x) \]
\[ \tilde{\xi} = z_G x + z_T (1-x) \]
\[ \tilde{\omega} = \omega_G x + \omega_T (1-x) \]

(A56)

such that \( \alpha \rightarrow \tilde{\alpha} \), \( z_G \rightarrow \tilde{\xi} \) and \( x \rightarrow \tilde{\omega} \), thus we get the wanted form of the denominator. Doing this explicitly, we can write for the Eq. (A52)

\[
G_{EM}^\mu(k, Q) = \int_0^\infty d\tilde{\omega} d\tilde{\alpha} \int_{-1}^1 d\tilde{\xi} \frac{I}{k^2 + k.Q \tilde{\xi} + \frac{Q^2}{4} \tilde{\alpha} - \tilde{\omega}} \tag{A57}
\]

where \( x \) is the solution of Eqs. (A56), i.e., it is a function \( x(a, b, z_T, \tilde{\xi}, \tilde{\alpha}, \omega_T) \). We label \( \prod_\theta \) the product of step Heaviside functions which define the integration domain and straightforwardly stem from the transformation (A56). They ensure that the numerator is zero in the boundaries of three integrals appearing in (A57). To get the form of IR with desired power of the numerator, one only needs to employ pert partes integrations.

**Appendix B: Evaluating pion form factor within the gauge technique approximation**

The function \( F(Q^2) \) due to the gauge technique vertex in two approximations is derived in this appendix. Both are based on the gauge technique -like linearization, which reduces the number of numerical integrations. Further approximation is made to arrive at the dispersion relation for the form factor \( F(Q) \). Both approximations split at the very end. To begin, we substitute the IR for propagators and vertices, and by changing the ordering of integrations we perform momentum integrations following standard procedures known from perturbation theory.

Thus, after the Feynman paramatization, we can perform integration over the momentum exactly in a way known for evaluation of Feynman integrals in perturbation theory. As we are not in perturbation theory, we remain with number of integrals over the weight functions of all integral representations, as well as all those with three new auxiliary integrals. For the later we use the Feynman variable \( x \) to match the denominators of two IRs for the BSE vertices, then we use the variable \( y \) to match the result with the denominator of the quark propagator, which connects these two vertices, and at the end we will use the variable \( t \) in order to match the result of previous matching with the denominator of the IR for the quark-photon vertex.

The above described steps [for the entire matrix element \( J^\mu(Q|GT) \)] read explicitly

\[
J^\mu(Q) = -i2N_c \int \frac{d^4k}{(2\pi)^4} \int_s \frac{\Gamma(5)yt^2(1-t)U^\mu}{[(k+l)^2 + J]^2}, \tag{B1}
\]

\[
U^\mu = 4\rho_\gamma(\gamma)\rho_\pi(\omega) \left[ (\omega - k^2 + \frac{Q^2}{4}) (p^\mu - k^\nu) + 2k^\mu(k.p - k^2) + \frac{1}{2} Q^\mu k.Q \right]
\]

\[
J = -t^2 + \frac{Q^2}{4}(1-t) - \omega(1-t) + p^2(1-y)t - \gamma(1-y)t + \left[ \frac{p^2}{4} + \frac{Q^2}{16} - ax - b(1-x) \right] yt,
\]

\[
l^2 = \frac{Q^2}{4} o^2 + p^2(-1 + \frac{y}{2} t^2),
\]

\[
o = z(1-t) - \frac{1-2x}{2} yt \tag{B2}
\]

where we have used some shorthand notations: mainly we have also factorized the weight functions \( \rho_\pi \) of the pion vertex functions into the overall measure, for this we use the following abbreviation

\[
\int_s = \int_0^1 dx dy dt \int_0^\infty d\omega \int_{-1}^1 dz \int da \int db \rho_\pi(a)\rho_\pi(b). \tag{B3}
\]

but we omit all trivial terms which were proportional to the product of external momenta \( Q.p = 0 \). However we keep \( p^2 \) variable for the purpose of easier tracking of the presented derivation. We will use the fact that pions are on-shell, i.e. the equation \( p^2 = m_\pi^2 - Q^2/4 \) from the following lines.
For the purpose of integration over the momentum we perform the standard shift \( k \to k - l \) where \( l = \frac{Q}{2} o + p(-1 + y/2)t \), with the polynomial function \( o \) defined by \( \mathrm{B2} \)

\[
o = z(1 - t) + \frac{1 - x}{2}yt,
\]

which after the integration over the momentum provides the nontrivial part of our matrix element in the form

\[
\mathcal{J}^\mu(p, Q) = F(Q^2)p^\mu,
\]

where the pion form factor \( F \) is proportional to the following expression

\[
F(Q^2) = \frac{2N_c}{(4\pi)^2} \int_s t^2(1 - t) \frac{4\rho_s(\gamma_\mu)\rho_s(\omega)}{(4\pi)^2 J^3} \left[ (1 - f)\omega + f\left(-\frac{Q^2}{4}(1 + o^2) + 2p^2 f - p^2 f^2\right)\right]
\[
+ \frac{4\rho_s(\gamma_\mu)\rho_s(\omega)}{(4\pi)^2 J^2} + \frac{4\rho_s(\gamma_\mu)\rho_s(\omega)}{(4\pi)^2 J^3} (1 - 3f),
\]

where we have labeled

\[
f = (1 - y/2)t.
\]

The individual prefactors in Eq. \( \mathrm{B6} \) follow from the standard evaluation performed in Euclidean space, although we come back to the Minkowski metric convention immediately. We just remind the reader with the example

\[
- i \int \frac{d^4k}{(2\pi)^4} \frac{\Gamma(5)k^2}{(k^2 + J)^5} = \frac{2}{(4\pi)^2 J^3}.
\]

Note here that for purpose of consistency, the variable \( p^2 \) was also kept Euclidean for a while, and the on-shell condition \( p^2 = m_\pi^2 - Q^2/4 \) is imposed only afterwards.

For positive timelike \( Q^2 \) the real part of denominator \( J \) pass zero value and albeit not written explicitly, the presence of infinitesimal Feynman imaginary part is assumed.

In what follows it is convenient to split the denominator \( J \), getting

\[
J = \frac{Q^2}{4} \square - \Delta
\]

\[
\Delta = m_\pi^2[(1 - \frac{3}{4}y) t - f^2] - \omega(1 - t) - \gamma(1 - yt) - (ax + b(1 - x))yt
\]

\[
\square = - o^2 + 1 - t - (1 - y)t + f^2.
\]

In Eq. \( \mathrm{B6} \) we do not write trivial terms, including also those, which are proportional linearly to the variable \( o \). These terms are zero as can be inspected by the substitutions \( z \to -z \) and \( x \to 1 - x \) with simultaneous interchange of the pion spectral function arguments \( a \to b \). In this way one gets the identical expression for the appropriate contributions to the form factor, but with opposite sign; hence, it is zero. This, together with on-shell condition \( p.Q = 0 \), causes that term proportional to total momentum \( Q^2 \) to be absent for each diagram individually and the matrix element has an identical Lorentz structure to the charged point like scalar particle.

Before evaluating singular and hence more complicated Minkowski expressions we derive the formula suited for the numerical integrations for spacelike momentum \( Q \). For this purpose we integrate over the variable \( z \) variable analytically. To proceed furthermore, we use "gauge technique" trick again and make linearization in \( \rho \), which allows us to reduce the number of integrations further. As a consequence \( xa + (1 - x)b \to b \) and the integration over the variable \( x \) can be done analytically in closed form. For purpose of numeric, the same is done for the product of the quark spectral function integrals, where after matching by the virtue of gauge technique linearization we make linearization in \( \rho, (\rho_s) \) such that \( \omega(1 - t) - \gamma(1 - yt) \to \gamma(1 - yt) \). In what follows, we will write \( \tilde{\gamma} = \gamma(1 - yt) + byt \)

There are only two necessary integrals for evaluation of the function \( F \) for the spacelike momentum \( Q^2_E = -Q^2 \). The first we show here

\[
\int_{-1}^1 dz \int_0^1 dx J^{-3} = \frac{\theta(a)}{Q^2(1 - t)/4\pi} \frac{D_x}{4a(x^2 + a)} \left[ \frac{1}{4a(x^2 + a)} - \frac{3x \arctan\left[ \frac{\sqrt{x}}{\sqrt{a}} \right]}{4a^{3/2}} \right]
\]

\[
+ \frac{\theta(-a)}{Q^2(1 - t)/4\pi} \frac{D_x}{4a(x^2 + a)} \left[ \frac{1}{4a(x^2 + a)} - \frac{3x \tanh^{-1}\left[ \frac{\sqrt{x}}{\sqrt{-a}} \right]}{4(-a)^{3/2}} \right]
\]

\( \mathrm{B10} \)
with the function $a_d$ defined as
\[
a_d = -\frac{Q_F^2}{4} \left[ (1 - t) - (1 - y)t + f^2 \right] + (1 - 3/4y - f^2)m^2 - \gamma. \tag{B11}
\]

The second required integral reads
\[
\int_{-1}^{1} dz \int_{0}^{1} dx J^{-2} = -\theta(a) \frac{\theta(-a)}{Q_F^2(1 - t)} D_x \left[ \frac{\arctan(x/\sqrt{a})}{a^{3/2}} \right] - \frac{\theta(-a)}{Q_F^2(1 - t)} D_x \left[ \frac{\tanh^{-1}(x/\sqrt{a})}{(-a)^{3/2}} \right] \tag{B12}
\]
where we have introduced abbreviations
\[
D_x[h(x)] = h(x_u) - h(x_d);
\]
\[
x_u = Q_F/2[1 - t - yt/2] ; \quad x_d = Q_F/2[1 - t + yt/2] ; \tag{B13}
\]
for some function $[h(x)]$.

Using the above integrals in Eq. \((B0)\) constitutes the final expression that we have used for the numerical evaluation for the spacelike value of $Q^2$.

1. Derivation of Dispersion Relation

The final expression for $F$ based on formulas derived above is still not yet in a form suited for numerical evaluation in the region of Minkowski momentum $Q^2 > 0$. Recalling the presence of the small Feynman factor $ie$, the log in inverse hyperbolical tangents as well as the function $a$ is badly singular near the real axis of momentum $Q$ and the expression is numerically ill. Since complete analytical integration is still out of our reach, we make further simplification. For this purpose we go back into the expression and ignore the presence of the $a^2$ term in the integrand which allows the conversion of all terms into the desired dispersion relation. We show the derivation for most singular $1/J^3$ term in the Eq. \((B0)\) the conversion of other terms is straightforward within the method used.

Ignoring $a^2$ terms as well as ignoring small terms proportional to $m_\pi$ we can integrate over the variables $x$ and $z$.

The result simply means to replace the integration symbols $\int dx dz$ with a factor 2. The remaining relevant integral we need to evaluate reads
\[
\int_{0}^{1} dt \int_{0}^{1} dy \frac{(1 - y/2)^2yt^2(1 - t)}{J^3}, \tag{B14}
\]
where the denominator reduces as
\[
J = \Box Q^2 \frac{2}{4} - \gamma, \tag{B15}
\]

To proceed further, we perform the last linearization
\[
\int d\gamma \rho_{v,s}(\gamma) \int db \rho_{s}(b) \rightarrow \int d\gamma \tilde{\rho}_{v,s}(\gamma), \tag{B16}
\]
where we assume new functions $\tilde{\rho}$ on the rhs. \((B16)\) are such that resulting form factor $F$ remains unchanged when taking $\gamma \rightarrow \gamma$ in the denominator, i.e. from now
\[
J = \Box Q^2 \frac{2}{4} - \gamma, \tag{B17}
\]
\[\Box = 1 - t - (1 - y)t + (1 - y/2)^2t^2.\]

and we also assume $\tilde{\rho} \approx \rho + \delta_\rho$ with the function $\delta_\rho$ representing corrections.

In addition we introduce the unit in the form
\[
1 = \int_{0}^{\infty} d\alpha \delta(\alpha - \gamma/\Box) \tag{B18}
\]
into the expression \((B14)\) and integrate over the variable $t$. After that we get for \((B14)\) the following expression
\[
\int_{0}^{1} dy \int_{0}^{\infty} d\alpha \frac{\alpha}{\gamma^2} \frac{yt^2(1 - t_\gamma)\theta(t_\gamma)\theta(1 - t_\gamma)}{2(1 - (1 - y/2)t_\gamma)[Q^2 - \alpha + ie]^3} \tag{B19}
\]
where \( t_ - \) is the root of the equation \( \Box \alpha - \gamma = 0 \). Explicitly it reads

\[
t_ - = \frac{1 - \sqrt{\gamma / \alpha}}{1 - y / 2}
\]  

(B20)

noting that since \( \alpha > 0 \), the step function can be equivalently taken as \( \theta(\alpha - \gamma)\theta(1 - \frac{\gamma}{y^2} - \alpha) \). Note that the contribution from the second root \( t_ + = 1 + /.. \) is trivial, since \( t_ + > 1 \), being thus always outside of the interval for the original integral variable \( y \).

Let us change the ordering of the integrations and integrate over the variable \( y \). Theta functions presented in the kernel imply

\[
\int_0^1 dy \int_0^\infty d\alpha \to \int_0^\infty d\alpha \int_0^{2\sqrt{\gamma}} dy
\]  

(B21)

After the integration we get

\[
\int_0^\infty d\alpha \frac{4}{\gamma} \frac{\theta(1 - \sqrt{\gamma / \alpha})\theta(\alpha - \gamma)}{\left[\frac{\gamma^2}{4} - \alpha + i\epsilon\right]^3} \left[-2 - (2\sqrt{\frac{\alpha}{\gamma} - 1)ln(1 - \sqrt{\frac{\alpha}{\gamma}})\right] \left(\sqrt{\frac{\alpha}{\gamma}} - 1\right)^2
\]  

(B22)

After that, we perform double per partes integration with respect to the variable \( \alpha \), such that we get the desired dispersion relation

\[
F(Q^2) = \int_0^\infty d\alpha \frac{g(\alpha)}{\left[\frac{\gamma^2}{4} - \alpha + i\epsilon\right]}
\]

\[
g(\alpha) = \frac{4N_c}{(4\pi^2)} \int_0^\alpha d\gamma 2\hat{\rho}_c(\gamma) K(\alpha, \gamma) + ... 
\]

\[
K(\alpha, \gamma) = \frac{BA^2 + A + B - 1/2}{2\alpha^{3/2}\gamma^{1/2}} - \frac{2 + AB}{\alpha^2} - \frac{B}{\alpha^{1/2}\gamma^{3/2}} + \frac{1}{2\alpha^2}
\]  

(B23)

where

\[
A = 1 - \sqrt{\frac{\alpha}{\gamma}} ; \quad B = ln(1 - \sqrt{\gamma / \alpha})
\]  

(B24)

and where dots represent remaining and not shown contributions (stemming also from the integration over the function \( J^{-2} \), we found these terms can be safely neglected in the approximation employed here).

Our approximation leads to some systematical error: it smoothly overestimates the form factor at medium timelike \( Q^2 \) and the dispersion relation does not provide correct form factor for \( |Q^2| > 2GeV^2 \), hence we call the form factor calculated on the relation (B23) the Approximated Dispersion Relation result. Nevertheless, it offers reasonable comparison with approximation derived in the previous section. Hence, we guess that our ADR does not cripple the function \( F \) below 1GeV too much, keeping the shape of \( \rho \) meson resonance not distorted much.

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