Fuzzy Majority Algorithms for the 1-Median and 2-Median Problems on a Fuzzy Tree

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ABSTRACT
In the classical $p$-median problem, we want to find a set $Y$ containing $p$ points in a given graph $G$ such that the sum of weighted distances from $Y$ to all vertices in $V$ is minimised. In this paper, we consider the 1-median and 2-median problems on a tree with fuzzy weights. We show that the majority property holds for fuzzy 1-median problem on a tree. Then based on a proposed ranking function and the majority property, a fuzzy algorithm is presented to find the median of a fuzzy tree. Finally, the algorithm is extended to solve 2-median problem on fuzzy trees.

1. Introduction
The $p$-median problem is an important issue in the location theory. In this problem, we want to find $p$ best places for locating facility centers to provide the demands of $n$ customers such that the sum of weighted distances from customers to the nearest facility is minimised. The $p$-median problem was first introduced by Hakimi in 1964 [1]. Kariv and Hakimi [2] showed that this problem is $NP$-hard. An $O(pn^2)$ algorithm for the $p$-median and related problems on tree graphs was presented by Tamir [3]. Tragantalerngsak et al. [4] studied single-source capacitated facility location problem and proposed a Lagrangian relaxation-based branch and bound algorithm for this problem. Charikar and Guha [5] presented a constant-factor approximation algorithm for the $k$-median problem. An uncertain model for single-facility location problems on networks was presented by Gao [6].

Since median problem is $NP$-hard, some heuristic methods were presented for finding the solution of this problem (e.g. see [7–10]). Median problem has many applications in the real world such as establishing public services in transportation networks. However, it is seldom known to determine the value of parameters precisely. In most of the cases, according to the opinion of decision-makers the value of parameters is defined approximately by some degree of uncertainty. The fuzzy set theory is one of the best tools for illustrating this uncertain parameter and model the problem in a mathematical form. In 1999, Canos et al. [11] considered the fuzzy $p$-median problem and presented an exact algorithm to solve it. Moreno Perez et al. [12] considered fuzzy location problems on...
networks. Kutangila and Verdegay [13] studied $p$-median problems in a fuzzy environment. Uno et al. [14] considered the facility location problem on a network with fuzzy random weights. In 2013, Mehrjerdi and Nadizadeh [15] presented a greedy clustering method to solve capacitated location-routing problem with fuzzy demands. Then in 2016, Mohammadi et al. [16] considered a bi-objective single allocation $p$-hub center-median problem. Recently, fuzzy location problems have gained more interest and are considered by few authors (e.g. see [12,17–20]).

A special case of the median problem has happened when the graph is a tree. In this paper, a tree network in which the demands of customers are in a fuzzy form is considered (more details about fuzzy trees can be found in [21–24]). First, a new ranking function is proposed. Then, a fuzzy algorithm for finding the median of fuzzy trees is presented. Finally, the algorithm is extended to find 2-median of a fuzzy tree.

This paper is organised into six sections. In the next section, few necessary fuzzy concepts are presented. The new ranking function is introduced in Section 3. Then, in Section 4, fuzzy algorithms for finding one and two medians of fuzzy trees are presented. Some numerical examples are shown in Section 5. Finally in Section 6, the conclusion and few suggestions for future work are given.

2. Preliminaries

In this section, some basic definitions of the fuzzy set theory are presented. These notations can be found in [12,25–27].

**Definition 2.1:** Let $R$ be the real line, then a fuzzy set $\tilde{A}$ in $R$ is a set of ordered pairs $\tilde{A} = \{(x, \mu_A(x)) \mid x \in R\}$, where $\mu_A(x)$ is called the membership function for the fuzzy set. The membership function maps each element of $R$ to a membership value between 0 and 1.

**Definition 2.2:** The support of a fuzzy set $A$ is defined as follows:

$$\text{supp}(A) = \{x \in R \mid \mu_A(x) > 0\}.$$ 

**Definition 2.3:** The core of a fuzzy set $\tilde{A}$ is the set of all points $x$ in $R$ with $\mu_A(x) = 1$.

**Definition 2.4:** A fuzzy set $\tilde{A}$ is a normal set if its core is nonempty. In other words, there is at least one point $x \in R$ with $\mu_A(x) = 1$.

**Definition 2.5:** The $\alpha$-cut or $\alpha$-level set of a fuzzy set $\tilde{A}$ is a crisp set that is defined as follows:

$$A_\alpha = \{x \in R \mid \mu_A(x) > \alpha\}.$$ 

**Definition 2.6:** A fuzzy set $\tilde{A}$ on $R$ is convex, if for any $x, y \in R$ and $\lambda \in [0, 1]$, we have

$$\mu_A(\lambda x + (1 - \lambda)y) \geq \min\{\mu_A(x), \mu_A(y)\}.$$
Definition 2.7: A fuzzy number $\tilde{A}$ is a fuzzy set on the real line that satisfies the condition of normality and convexity.

Definition 2.8: A fuzzy number $\tilde{A}$ on $\mathbb{R}$ is a triangular fuzzy number, if its membership function is defined as follows:

$$
\tilde{A} = \begin{cases} 
\frac{w_\tilde{A} \cdot (x - a_l)}{a_m - a_l} & \text{if } x \in [a_l, a_m] \\
\frac{w_\tilde{A} \cdot (a_u - x)}{a_u - a_m} & \text{if } x \in [a_m, a_u],
\end{cases}
$$

where $0 \leq w_\tilde{A} \leq 1$ is a constant number that is called the height of generalised triangular fuzzy numbers. The parameters $a_l$, $a_m$ and $a_u$ are real numbers. Also fuzzy number is denoted by $\tilde{A} = (a_l, a_m, a_u; w_\tilde{A})$.

Remark 2.1: If $w_\tilde{A} = 1$, then $\tilde{A} = (a_l, a_m, a_u; 1)$ is a normalised triangular fuzzy number and is denoted by $\tilde{A} = (a_l, a_m, a_u)$. Generally, $\tilde{A}$ is called a generalised triangular fuzzy number if $0 \leq w_\tilde{A} \leq 1$ as shown in Figure 1. The fuzzy number is a crisp number if $a_l = a_m = a_u$ and $w_\tilde{A} = 1$.

Remark 2.2: The opposite (or image) of generalised triangular fuzzy number $\tilde{A} = (a_l, a_m, a_u; w_\tilde{A})$ can be defined as $-\tilde{A} = (-a_u, -a_m, -a_l; w_\tilde{A})$.

Remark 2.3: In this paper, the set of all generalised triangular fuzzy numbers on the real line is denoted by $F$ and zero triangular fuzzy number is denoted by $\tilde{0} = (0, 0, 0)$.

Definition 2.9: Function $\psi : F \rightarrow \{-1, 1\}$ is defined as follows:

$$
\forall \tilde{A} \in F : \psi_{\tilde{A}} = \begin{cases} 
1 & \text{if } a_l + a_m + a_u \geq 0 \\
-1 & \text{if } a_l + a_m + a_u < 0.
\end{cases}
$$

Figure 1. Generalised triangular fuzzy number $(a_l, a_m, a_u; w)$. 

A fuzzy graph is the structure

\[ \tilde{G} = (\tilde{V}, \tilde{E}, \tilde{w}, \tilde{l}) \]

Definition 2.10: Let \( \tilde{A} = (a_l, a_m, a_u; w_{\tilde{A}}) \) and \( \tilde{B} = (b_l, b_m, b_u; w_{\tilde{B}}) \) be two generalised triangular fuzzy numbers. Then, arithmetics of fuzzy numbers are defined as follows:

\[
\tilde{A} \oplus \tilde{B} = (a_l + b_l, a_m + b_m, a_u + b_u; \min (w_{\tilde{A}}, w_{\tilde{B}})).
\]

\[ (2) \]

\[
\tilde{A} \ominus \tilde{B} = (a_l - b_u, a_m - b_m, a_u - b_l; \min (w_{\tilde{A}}, w_{\tilde{B}})).
\]

\[ (3) \]

\[
k \cdot \tilde{A} = \begin{cases} (ka_l, ka_m, ka_u; w_{\tilde{A}}) & \text{if } k \geq 0 \\ (ka_u, ka_m, ka_l; w_{\tilde{A}}) & \text{if } k < 0. \end{cases}
\]

\[ (4) \]

Definition 2.11: A generalised triangular fuzzy number \( \tilde{A} = (a_l, a_m, a_u; w_{\tilde{A}}) \) is said to be non-negative if and only if \( a_l \geq 0 \) and is said to be non-positive if and only if \( a_u \leq 0 \).

Definition 2.12: Two generalised triangular fuzzy numbers \( \tilde{A} = (a_l, a_m, a_u; w_{\tilde{A}}) \) and \( \tilde{B} = (b_l, b_m, b_u; w_{\tilde{B}}) \) have the same sign if both of them are non-negative or non-positive.

Now we are able to define the fuzzy graph and fuzzy network. Let \( G = (V, E) \) be a connected undirected graph in which \( V = \{v_1, v_2, \ldots, v_n\} \) is the set of vertices and \( E = \{e_1, e_2, \ldots, e_m\} \) is the set of edges. Each edge \( e \in E \) consists of infinite points that join vertex \( u \) to vertex \( v \) and is represented by \( e = (u, v) \). The vertices \( u, v \in V \) are then named the extremes of \( e \). The function \( l(.) \) is defined on \( E \) and corresponds to each edge \( e \in E \) a positive number \( l(e) \) where denotes the length of edge \( e \). Also a function \( w(.) \) is defined on \( V \), so that for each vertex \( v \in V \), \( w(v) \) is a positive number that indicates the weight of \( v \). According to these concepts, the fuzzy graph and fuzzy network are defined as follows.

Definition 2.13: A fuzzy graph is the structure \( \tilde{G} = (\tilde{V}, \tilde{E}, \rho, \mu) \) where \( \tilde{V} \) is a vertex set, \( \tilde{E} \) is the edge set, and \( \rho : V \rightarrow [0, 1] \) and \( \mu : E \rightarrow [0, 1] \) are the membership functions of vertex set and edge set, respectively, in which

\[ \forall u, v \in V : \max (\mu(e)) \leq \min \{\rho(u), \rho(v)\} ; e = (u, v). \]

A fuzzy graph is denoted by \( \tilde{G} = (\tilde{V}, \tilde{E}) \).

Definition 2.14: Let \( \tilde{G} = (\tilde{V}, \tilde{E}) \) be a fuzzy graph, and \( w(.) \) and \( l(.) \) be the weight and the length functions that are defined on \( \tilde{V} \) and \( \tilde{E} \), respectively. If \( \tilde{w} \) and \( \tilde{l} \) correspond fuzzy positive numbers to all or some elements of \( \tilde{V} \) and \( \tilde{E} \), respectively, then network \( \tilde{N} = (\tilde{V}, \tilde{E}, \tilde{w}, \tilde{l}) \) reduces to a fuzzy network \( \tilde{N} = (\tilde{V}, \tilde{E}, \tilde{w}, \tilde{l}) \). A fuzzy network is classified as follows:

1. (a) The most general form of fuzzy network is \( \tilde{N} = (\tilde{V}, \tilde{E}, \tilde{w}, \tilde{l}) \), where the vertex set and the edge set are fuzzy sets and the weight function and the length function are both fuzzy-valued.
   (b) The fuzzy network with fuzzy vertex and edge sets and either the weight function and the length function or both are crisp-valued. That is \( \tilde{N} = (\tilde{V}, \tilde{E}, \tilde{w}, \tilde{l}) \) or \( \tilde{N} = (\tilde{V}, \tilde{E}, \tilde{w}, \tilde{l}) \) or \( \tilde{N} = (\tilde{V}, \tilde{E}, \tilde{w}, \tilde{l}) \).
(2) (a) The vertex set $V$ is a crisp set, the edge set $\tilde{E}$ is fuzzy set and the weight and the length functions are both fuzzy-valued.

(b) The vertex set $V$ is a crisp set, the edge set $\tilde{E}$ is fuzzy set and either the weight function and the length function or both are crisp-valued. That is $\tilde{N} = (V, \tilde{E}, \tilde{w}, l)$ or $\tilde{N} = (V, \tilde{E}, w, l)$ or $\tilde{N} = (V, E, \tilde{w}, l)$.

(3) (a) The vertex set $V$ and the edge set $E$ are both crisp sets and both the weight function and the length function are fuzzy-valued. That is $\tilde{N} = (V, E, \tilde{w}, l)$.

(b) The vertex set $V$ and the edge set $E$ are both crisp sets and either the weight function and the length function or both are crisp-valued. That is $\tilde{N} = (V, E, \tilde{w}, l)$ or $\tilde{N} = (V, E, w, l)$ or $\tilde{N} = (V, E, l)$.

In this paper, we deal with the fuzzy network $\tilde{N} = (V, E, \tilde{w}, l)$.

3. New ranking function

Ranking of fuzzy numbers is an important procedure in many fuzzy optimisation and decision-making problems. Hence, different methods of ranking fuzzy numbers have been studied by many authors such as [28–31]. However, most of these proposed methods have several disadvantages. For instance, they are not linear, they are unable to distinguish all fuzzy numbers and they cannot rank all generalised fuzzy numbers. Thus in this paper, a new method with less computational process is presented in which there is no integral operator and any radical functions. This new method can rank triangular fuzzy numbers with the best accuracy. Also, it must be mentioned that the ranking results of the new method are the same as famous methods. Also, the new method is a linear ranking method and if two fuzzy numbers have the same ranking according to this method, certainly these two generalised triangular fuzzy numbers are equal. As a result, the presented method can effectively distinguish different generalised triangular fuzzy numbers. Also, the suggested ranking method can rank real numbers as effectively as fuzzy numbers. The simple ranking method is proved to always guarantee the consistency between the ranking of generalised triangular fuzzy numbers and their images. Recently, Jian et al. [32] proposed a triangular approximation operator that preserves the centroid of fuzzy numbers which is an important index for evaluating fuzzy numbers. Thus, it is possible to approximate any arbitrary fuzzy number with a triangular fuzzy number with the same centroid point. Therefore, our method can be used for all fuzzy numbers. In the Appendix, by six comparative examples the superior validity, applicability, efficiency and simplicity of proposed method in comparison with the existing methods are illustrated.

Our new method is based on the centroid point for ranking generalised triangular fuzzy numbers. According to Wang et al. [33], for any generalised triangular fuzzy number $\tilde{A} = (a_i, a_m, a_u, w_\tilde{A})$, its centroid can be determined by

$$G_\tilde{A} (x_\tilde{A}, y_\tilde{A}) = \left( \frac{a_i + a_m + a_u}{3}, \frac{w_\tilde{A}}{3} \right).$$

Clearly, it can be concluded from (5) that the centroid point corresponding to a generalised triangular fuzzy number is obtained based on the parameters $a_i, a_m, a_u$ and $w_\tilde{A}$. Thus, we define a novel ranking $NR$ index of generalised triangular fuzzy number $\tilde{A} = (a_i, a_m, a_u, w_\tilde{A})$
in $\mathbb{R}^4$ as follows:

$$\text{NR}_{\tilde{A}} = \left( \frac{a_l + a_m + a_u}{3}, a_m, \frac{\psi_{\tilde{A}} \cdot (a_u - a_l)}{2}, \frac{\psi_{\tilde{A}} \cdot (w_{\tilde{A}} - 1)}{3} \right) \in \mathbb{R}^4. \quad (6)$$

It can be seen that in some previous existing methods, a real number is assigned to each fuzzy number. Then, the fuzzy numbers are ranked by comparing their corresponding real numbers. However, some important information is lost by this conversion. Hence, in our new method for saving these important information, a unique quartet $\text{NR}$ index in $\mathbb{R}^4$ is assigned to each generalised triangular fuzzy number. The $\text{NR}$ index involves the centroid point, the core, the half-length of support set and the height of generalised triangular fuzzy number that can easily be computed. Actually, the $\text{NR}$ index is easily computed for any triangular fuzzy number with less computation processes and without using operators such as integral or any functions such as radical and so on. In contrast with the existing ranking methods, in the next theorem it is shown that if the $\text{NR}$ indexes of the two generalised triangular fuzzy numbers are equal, then both of these generalised triangular fuzzy numbers are also equal.

**Theorem 3.1:** Let $\tilde{A}, \tilde{B} \in F$. Then $\text{NR}(\tilde{A}) = \text{NR}(\tilde{B})$ if and only if $\tilde{A} = \tilde{B}$.

**Proof:** If two generalised triangular fuzzy numbers $\tilde{A} = (a_l, a_m, a_u, w_{\tilde{A}})$ and $\tilde{B} = (b_l, b_m, b_u, w_{\tilde{B}})$ are equal, then it is clear that $\text{NR}(\tilde{A}) = \text{NR}(\tilde{B})$.

Now assume that we have $\text{NR}(\tilde{A}) = \text{NR}(\tilde{B})$. Thus,

$$\frac{a_l + a_m + a_u}{3} = \frac{b_l + b_m + b_u}{3}.$$

Using the above equation and Definition 2.9, it can be concluded that $\psi_{\tilde{A}} = \psi_{\tilde{B}}$.

Since $\text{NR}(\tilde{A}) = \text{NR}(\tilde{B})$, then

$$\frac{a_l + a_m + a_u}{3} = \frac{b_l + b_m + b_u}{3}, \quad a_m = b_m, \quad \frac{\psi_{\tilde{A}} \cdot (a_u - a_l)}{2} = \frac{\psi_{\tilde{B}} \cdot (b_u - b_l)}{2}$$

and

$$\frac{\psi_{\tilde{A}} \cdot (w_{\tilde{A}} - 1)}{3} = \frac{\psi_{\tilde{B}} \cdot (w_{\tilde{B}} - 1)}{3}.$$

Therefore, $a_l = b_l, a_u = b_u$ and $w_{\tilde{A}} = w_{\tilde{B}}$. Thus, it yields that $\tilde{A} = \tilde{B}$. \(\square\)

Now generalised triangular fuzzy numbers can be ordered using lexicographical order on $\text{NR}$ index. Let $\tilde{A} = (a_l, a_m, a_u; w_{\tilde{A}})$ and $\tilde{B} = (b_l, b_m, b_u; w_{\tilde{B}})$ be two arbitrary generalised triangular fuzzy numbers. According to $\text{NR}$ index, two generalised triangular fuzzy numbers $\tilde{A}$ and $\tilde{B}$ can be compared as follows:

$$\text{NR}(\tilde{A}) <_L \text{NR}(\tilde{B}) \text{ if only if } \tilde{A} \leq \tilde{B},$$

$$\text{NR}(\tilde{A}) =_L \text{NR}(\tilde{B}) \text{ if only if } \tilde{A} \simeq \tilde{B},$$

where the symbol $<_L$ denotes the lexicographical order and defined as follows:

$$(x_1, x_2, x_3, x_4) <_L (y_1, y_2, y_3, y_4) \iff (\exists m = 1, 2, 3, 4) (\forall i < m) (x_i = y_i)$$

and

$$(x_m < y_m).$$
The lexicographical order is a perfect order in $\mathbb{R}^4$ [34], thus the proposed ranking is able to rank all generalised triangular fuzzy numbers. Unlike many existing methods, crisp numbers can be ranked as effectively as triangular fuzzy numbers using the proposed ranking. It is necessary to point out that the proposed method not only is able to rank similar generalised triangular fuzzy numbers but also needs very simple calculations. Some more important properties of the proposed ranking approach are given in the following sections.

**Proposition 3.1:** Let $\tilde{A} = (a_l, a_m, a_u; w_{\tilde{A}})$ be an arbitrary generalised triangular fuzzy number, then

$$NR(-\tilde{A}) = \left( -\frac{a_l + a_m + a_u}{3}, -a_m, -\psi_{\tilde{A}} \cdot \frac{(a_u - a_l)}{2}, -\psi_{\tilde{A}} \cdot \frac{w_{\tilde{A}} - 1}{3} \right) = -NR(\tilde{A}).$$

**Proof:** According to Remark 2.2 and Definition 2.9, we have

$$\psi_{\tilde{A}} = -\psi_{\tilde{A}} \quad \text{and} \quad w_{\tilde{A}} = w_{-\tilde{A}}.$$  

Also, the following relation holds:

$$NR(-\tilde{A}) = \left( \frac{(-a_l) + (-a_m) + (-a_u)}{3}, -a_m, -\psi_{\tilde{A}} \cdot \frac{((-a_l) - (-a_u))}{2}, -\psi_{\tilde{A}} \cdot \frac{w_{\tilde{A}} - 1}{3} \right)$$

$$= \left( -\frac{a_l + a_m + a_u}{3}, -a_m, -\psi_{\tilde{A}} \cdot \frac{(a_u - a_l)}{2}, -\psi_{\tilde{A}} \cdot \frac{w_{\tilde{A}} - 1}{3} \right) = -NR(\tilde{A}). \quad \blacksquare$$

**Proposition 3.2:** For two generalised triangular fuzzy numbers $\tilde{A}, \tilde{B} \in F$, if $\tilde{A} \prec \tilde{B}$, then $-\tilde{B} \prec -\tilde{A}$.

**Proof:** The result can be achieved easily using (6) and Proposition 3.1. From Proposition 3.1, it is obviously concluded that the simple ranking method is proved to always guarantee the consistency between the ranking of triangular fuzzy numbers and their images. \quad \blacksquare

**Proposition 3.3:** Let $\Gamma$ be an arbitrary finite subset of $F$. Then the following properties hold:

1. If $\tilde{A} \in \Gamma$, then $\tilde{A} \leq \tilde{A}$.
2. For $(\tilde{A}, \tilde{B}) \in \Gamma^2$, if $\tilde{A} \leq \tilde{B}$ and $\tilde{A} \geq \tilde{B}$, then $\tilde{A} = \tilde{B}$.
3. For $(\tilde{A}, \tilde{B}, \tilde{C}) \in \Gamma^3$, if $\tilde{A} \leq \tilde{B}$ and $\tilde{B} \leq \tilde{C}$, then $\tilde{A} \leq \tilde{C}$.

**Proof:** According to lexicographical order, the items (1) and (2) can be obtained easily.

To show item (3), since $\tilde{B} \leq \tilde{C}$, according to lexicographical order on $NR$ index, one of the following cases holds:

(i) \hfill \frac{b_l + b_m + b_u}{3} < \frac{c_l + c_m + c_u}{3} \hfill
(ii) \[ \frac{b_l + b_m + b_u}{3} = \frac{c_l + c_m + c_u}{3} \quad \text{and} \quad b_m < c_m, \]

(iii) \[ \frac{b_l + b_m + b_u}{3} = \frac{c_l + c_m + c_u}{3}, \quad b_m = c_m \quad \text{and} \quad \frac{\psi_B \cdot (b_u - b_l)}{2} < \frac{\psi_C \cdot (c_u - c_l)}{2}, \]

(iv) \[ \frac{b_l + b_m + b_u}{3} = \frac{c_l + c_m + c_u}{3}, \quad b_m = c_m, \quad \frac{\psi_B \cdot (b_u - b_l)}{2} = \frac{\psi_C \cdot (c_u - c_l)}{2} \]

\[ \quad \text{and} \quad \frac{\psi_B \cdot (w_B - 1)}{3} < \frac{\psi_C \cdot (w_C - 1)}{3}. \]

Suppose that \[ \frac{b_l + b_m + b_u}{3} < \frac{c_l + c_m + c_u}{3}. \]

Then, since \( \tilde{A} \leq \tilde{B} \), therefore

\[ \frac{a_l + a_m + a_u}{3} < \frac{b_l + b_m + b_u}{3} \quad \text{or} \quad \frac{a_l + a_m + a_u}{3} = \frac{b_l + b_m + b_u}{3}. \]

Thus

\[ \frac{a_l + a_m + a_u}{3} \leq \frac{b_l + b_m + b_u}{3} < \frac{c_l + c_m + c_u}{3}, \]

concluded that \( \tilde{A} < \tilde{C} \).

Similarly, the three other cases for \( \tilde{B} < \tilde{C} \) can be easily proved.

\[ \blacksquare \]

**Proposition 3.4:** Suppose \( \Gamma \) be an arbitrary finite subset of \( F \) and \( (\tilde{A}, \tilde{B}) \in \Gamma^2 \). Then \( \tilde{A} > \tilde{B} \), if \( \text{infsupp}(\tilde{A}) > \text{supsup}(\tilde{B}) \)

**Proof:** According to Definition 2.8, we have

\[ a_u \geq a_m \geq a_l = \text{infsupp}(\tilde{A}) > \text{supsup}(\tilde{B}) = b_u \geq b_m \geq b_l. \]

So

\[ \frac{a_l + a_m + a_u}{3} > \frac{b_l + b_m + b_u}{3}. \]

Now, according to the lexicographical order on \( NR \) index, we conclude \( \tilde{A} > \tilde{B} \).

\[ \blacksquare \]

Also the following property can be easily proved.

**Proposition 3.5:** Let \( \Gamma \) and \( \Gamma' \) be two arbitrary finite subsets of \( F \) in which \( \tilde{A} \) and \( \tilde{B} \) are in \( \Gamma \cap \Gamma' \). Then \( \tilde{A} > \tilde{B} \) on \( \Gamma' \) if and only if \( \tilde{A} > \tilde{B} \) on \( \Gamma \).

**Proposition 3.6:** Let \( \tilde{A}, \tilde{B}, \tilde{C}, \tilde{C} \oplus \tilde{C} \) and \( \tilde{B} \oplus \tilde{C} \) be the elements of \( F \). Suppose that \( \tilde{A}, \tilde{B}, \tilde{C} \) have the same sign and \( \tilde{A} > \tilde{B}. \) Then \( \tilde{A} \oplus \tilde{C} > \tilde{B} \oplus \tilde{C} \).
Proof: Since $\tilde{A} > \tilde{B}$, one of the following cases holds according to lexicographical order on NR index:

(i) \[ \frac{a_l + a_m + a_u}{3} > \frac{b_l + b_m + b_u}{3}, \]

(ii) \[ \frac{a_l + a_m + a_u}{3} = \frac{b_l + b_m + b_u}{3} \quad \text{and} \quad a_m > b_m, \]

(iii) \[ \frac{a_l + a_m + a_u}{3} = \frac{b_l + b_m + b_u}{3}, \quad a_m = b_m \quad \text{and} \quad \frac{\psi_{\tilde{A}} \cdot (a_u - a_l)}{2} > \frac{\psi_{\tilde{B}} \cdot (b_u - b_l)}{2}, \]

(iv) \[ \frac{a_l + a_m + a_u}{3} = \frac{b_l + b_m + b_u}{3}, \quad a_m = b_m \quad \text{and} \quad \frac{\psi_{\tilde{A}} \cdot (a_u - a_l)}{2} = \frac{\psi_{\tilde{B}} \cdot (b_u - b_l)}{2} \]

and

\[ \frac{\psi_{\tilde{A}} \cdot (w_{\tilde{A}} - 1)}{3} > \frac{\psi_{\tilde{B}} \cdot (w_{\tilde{B}} - 1)}{3}. \]

We show that cases (i) and (iii) hold. Other cases can be proved similarly.

To show (i), suppose

\[ \frac{a_l + a_m + a_u}{3} > \frac{b_l + b_m + b_u}{3}. \]

By adding \( (c_l + c_m + c_u)/3 \) to both sides of the above inequality, we obtain

\[ \frac{a_l + c_l + a_m + c_m + a_u + c_u}{3} > \frac{b_l + c_l + b_m + c_m + b_u + c_u}{3}. \]

According to (6) and Definition 10, we conclude, $\tilde{A} \oplus \tilde{C} > \tilde{B} \oplus \tilde{C}$.

Now we show case (iii). Suppose that

\[ \frac{a_l + a_m + a_u}{3} = \frac{b_l + b_m + b_u}{3}, \quad a_m = b_m \quad \text{and} \quad \frac{\psi_{\tilde{A}} \cdot (a_u - a_l)}{2} > \frac{\psi_{\tilde{B}} \cdot (b_u - b_l)}{2}. \]

Since $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ have the same sign, according to Definition 2.10, it is obtained that $\psi_{\tilde{A} \oplus \tilde{C}} = \psi_{\tilde{B} \oplus \tilde{C}} = \psi_{\tilde{A}} = \psi_{\tilde{B}}$. Consequently, we obtain

\[ \frac{\psi_{\tilde{A} \oplus \tilde{C}} \cdot (a_u - a_l)}{2} > \frac{\psi_{\tilde{B} \oplus \tilde{C}} \cdot (b_u - b_l)}{2}. \] (7)

Now, by adding $\psi_{\tilde{B} \oplus \tilde{C}} \cdot (c_u - c_l)/2$ to both sides of the inequality (7), we have

\[ \frac{\psi_{\tilde{A} \oplus \tilde{C}} \cdot (a_u + c_u - a_l - c_l)}{2} > \frac{\psi_{\tilde{B} \oplus \tilde{C}} \cdot (b_u + c_u - b_l - c_l)}{2}. \]

It is concluded from (6) and Definition 2.10 that $\tilde{A} \oplus \tilde{C} > \tilde{B} \oplus \tilde{C}$. □
Proposition 3.7: Let \( \tilde{A} = (a_l, a_m, a_u) \) and \( \tilde{B} = (b_l, b_m, b_u) \) be two triangular fuzzy numbers with the same sign and \( k \in R^+ \). The proposed ranking is linear. That is
\[
NR(k \cdot \tilde{A} \oplus \tilde{B}) = k \cdot NR(\tilde{A}) + NR(\tilde{B}).
\]

Proof: Using Definition 2.10, we have
\[
k \cdot \tilde{A} \oplus \tilde{B} = (ka_l + b_l, ka_m + b_m, ka_u + b_u).
\]
And (6) yields
\[
NR(k \cdot \tilde{A} \oplus \tilde{B}) = \left( \frac{ka_l + b_l + ka_m + b_m + ka_u + b_u}{3}, 0 \right)
\]
\[
\times \left( \frac{k (a_l + a_m + a_u)}{3}, \frac{\psi_{k, \tilde{A} \oplus \tilde{B}} \cdot (ka_u + bu - ka_l - bl)}{2}, 0 \right)
\]
\[
= \left( \frac{k (a_l + a_m + a_u)}{3}, \frac{\psi_{k, \tilde{A} \oplus \tilde{B}} \cdot (ka_u - bl)}{2}, 0 \right)
\]
\[
= k \left( \frac{a_l + a_m + a_u}{3}, a_m, \frac{\psi_{k, \tilde{A} \oplus \tilde{B}} \cdot (a_u - bl)}{2}, 0 \right)
\]
\[
+ \left( \frac{b_l + b_m + b_u}{3}, b_m, \frac{\psi_{k, \tilde{A} \oplus \tilde{B}} \cdot (bu - bl)}{2}, 0 \right).
\]

Since \( \tilde{A} \) and \( \tilde{B} \) have the same sign and \( k \in R^+ \), and according to Definition 2.10, we conclude \( \psi_{k, \tilde{A} \oplus \tilde{B}} = \psi_{\tilde{A}} = \psi_{\tilde{B}} \). Now, by replacing this term in (9), we have
\[
NR(k \cdot \tilde{A} \oplus \tilde{B}) = k \cdot \left( \frac{a_l + a_m + a_u}{3}, a_m, \frac{\psi_{\tilde{A}} \cdot (a_u - bl)}{2}, 0 \right)
\]
\[
+ \left( \frac{b_l + b_m + b_u}{3}, b_m, \frac{\psi_{\tilde{B}} \cdot (bu - bl)}{2}, 0 \right)
\]
\[
= k \cdot NR(\tilde{A}) + NR(\tilde{B}).
\]

Most of the previous existing ranking methods do not satisfy this property. However, it is proved in the recent proposition that the proposed method is a linear ranking method. Generally in the real-world problems, linear ranking requires less computation. Also, in actual applications, the linear ranking can just be used like fuzzy simplex and so on [35,36].

4. Fuzzy median algorithms

In this section, first, a fuzzy algorithm for finding 1-median of a fuzzy tree is given. Then, the algorithm is extended to solve 2-median problem on a fuzzy tree.

4.1. Fuzzy 1-median problem

Let \( X = (V, E) \) be a tree. For any vertex \( v \in V \), the sub-tree \( X_v \) is a tree that is rooted at \( v \) and \( W(X_v) \) is the sum of the weights of all vertices of \( X_v \). Let the median be represented by \( m \).
Suppose that $S$ and $T$ are two sub-trees that are obtained by eliminating the edge $(s, t)$ on $X$ in which $s \in S$ and $t \in T$. For any sub-tree $S$ of $X$, we define

$$f(y, S) = \sum_{x \in S} w(x)d(x, y),$$

$$f(y) = \sum_{x \in X} w(x)d(x, y),$$

$$W(S) = \sum_{x \in S} w(x).$$

For the classical 1-median problem on a tree, Goldman [37] presented a majority algorithm based on the following lemmas.

Lemma 4.1 ([37]): $W(S) \geq W(T)$ if and only if $m \in S$.

Lemma 4.2 ([37]): If $W(S) \geq W(T)$, then the problem of finding the median of $X$ is equivalent to finding the median of sub-tree $N_s$ in which the weight of vertex $s$ is replaced by $w(s) + W(T)$.

Now consider the problem with fuzzy demands. In the following, first we will prove that the above two lemmas hold whenever the demands are in a fuzzy form. Then, the fuzzy majority algorithm are presented.

Lemma 4.3: $\tilde{W}(S) \geq \tilde{W}(T)$ if and only if $m \in S$.

Proof: Let $\tilde{W}(S) \geq \tilde{W}(T)$ and suppose that $y$ is any arbitrary vertex of $T$ and $NR$ is our new presented ranking index. It is sufficient to prove that $\tilde{f}(y) \geq \tilde{f}(s)$.

$$NR(\tilde{f}(y)) = NR(\tilde{f}(y, T) \oplus \tilde{f}(y, S)) = NR(\tilde{f}(y, T) \oplus \sum_{x \in S} \tilde{w}(x)d(x, y))$$

$$= NR(\tilde{f}(y, T) \oplus \sum_{x \in S} \tilde{w}(x)[d(x, s) + d(s, y)])$$

$$= NR(\tilde{f}(y, T) \oplus d(s, y)\tilde{W}(S) \oplus \sum_{x \in S} \tilde{w}(x)d(x, s))$$

$$= NR(\tilde{f}(y, T)) + d(s, y)NR(\tilde{W}(S)) + NR(\sum_{x \in S} \tilde{w}(x)d(x, s)).$$

Since $\tilde{W}(S) \geq \tilde{W}(T)$, then $NR(\tilde{W}(S)) \geq NR(\tilde{W}(T))$. Thus

$$NR(\tilde{f}(y)) \geq NR(\tilde{f}(y, T) + d(s, y)NR(\tilde{W}(T)) + NR(\sum_{x \in S} \tilde{w}(x)d(x, s))$$

$$= NR(\tilde{f}(y, T) \oplus d(s, y)W(T) \oplus \sum_{x \in S} \tilde{w}(x)d(x, s))$$
\[
\begin{align*}
\tilde{f}(y) + \tilde{f}(S) = & \text{NR} \left( \sum_{x \in T} \tilde{w}(x)d(y, x) + \sum_{x \in T} \tilde{w}(x)d(s, y) + \tilde{f}(s, S) \right) \\
\geq & \text{NR} \left( \sum_{x \in T} \tilde{w}(x)d(s, x) + \tilde{f}(s, S) \right) \\
= & \text{NR} \left( \tilde{f}(s, T) + \tilde{f}(S, S) \right) = \text{NR} \left( \tilde{f}(s) \right).
\end{align*}
\]

It means that \( \tilde{f}(y) \geq \tilde{f}(s) \). Thus the median cannot be in sub-tree \( T \).

Now on the contrary, let \( m \in S \) and \( \tilde{W}(S) < \tilde{W}(T) \). Suppose that \( y \) is any arbitrary element of \( S \). Then
\[
\text{NR}(\tilde{f}(y)) = \text{NR}(\tilde{f}(y) + \tilde{f}(y, T)) = \text{NR}(\tilde{f}(y) + \sum_{x \in T} \tilde{w}(x)d(y, x)) \\
= \text{NR}(\tilde{f}(y) + \sum_{x \in T} \tilde{w}(x)[d(y, t) + d(t, x)]) \\
= \text{NR}(\tilde{f}(y) + d(y, t)\tilde{W}(T) + \sum_{x \in T} \tilde{w}(x)d(t, x)) \\
= \text{NR}(\tilde{f}(y, S)) + d(y, t)\text{NR}(\tilde{W}(T)) + \text{NR}(\sum_{x \in T} \tilde{w}(x)d(t, x)).
\]

Since \( \tilde{W}(S) < \tilde{W}(T) \), then \( \text{NR}(\tilde{W}(S)) < \text{NR}(\tilde{W}(T)) \). Therefore,
\[
\text{NR}(\tilde{f}(y)) > \text{NR}(\tilde{f}(y, S)) + d(y, t)\text{NR}(\tilde{W}(S)) + \sum_{x \in T} \tilde{w}(x)d(t, x)
= \text{NR}(\tilde{f}(y, S) + d(y, t)\tilde{W}(S) + \sum_{x \in T} \tilde{w}(x)d(t, x))
= \text{NR}(\sum_{x \in S} \tilde{w}(x)d(x, y) + \sum_{x \in S} \tilde{w}(x)d(y, t) + \sum_{x \in T} \tilde{w}(x)d(t, x))
\geq \text{NR}(\sum_{x \in S} \tilde{w}(x)[d(x, y) + d(y, t)] + \tilde{f}(t, T))
= \text{NR}(\sum_{x \in S} \tilde{w}(x)d(x, t) + \tilde{f}(t, T))
= \text{NR}(\tilde{f}(t, S) + \tilde{f}(t, T)) = \text{NR}(\tilde{f}(t)).
\]

This means that \( m \) is not in the sub-tree \( S \), that is in contradiction with the assumption of lemma. Thus, \( \tilde{W}(S) \geq \tilde{W}(T) \).
**Lemma 4.4:** If \( \tilde{W}(S) \geq \tilde{W}(T) \), then the problem of finding the median of \( X \) is equivalent to finding the median of sub-tree \( N_s \) in which the weight of vertex \( s \) is replaced by \( \tilde{w}(s) \). 

**Proof:** Based on Lemma 4.3, it is concluded that the median is in sub-tree \( T_{r} \). If the median of a crisp tree can be obtained by the arc-deletion algorithm (see e.g. [38]). In the proof, we use Lemma 4.4 to make a new smaller tree and continue the method. 

Now consider the problem of finding two medians of a tree with fuzzy demands. The two medians of a crisp tree can be obtained by the arc-deletion algorithm (see e.g. [38]). In the proof, we use Lemma 4.4 to make a new smaller tree and continue the method.

### 4.2. Fuzzy 2-median problem

Now consider the problem of finding two medians of a tree with fuzzy demands. The two medians of a crisp tree can be obtained by the arc-deletion algorithm (see e.g. [38]). In the proof, we use Lemma 4.4 to make a new smaller tree and continue the method.
same as the classical method of solving 2-median problem, the fuzzy 2-median problem can be solved by the following arc-deletion algorithm.

**Fuzzy 2-median Algorithm**

*Step 1:* For any edge \( e = (s, r) \in E \), delete \( e \). Thus, the tree is partitioned in two sub-trees \( T_s \) and \( T_r \) with the vertex sets \( V_s \) and \( V_r \), respectively. For any of these two sub-trees, find the 1-median and call them \( m_s \) and \( m_r \), respectively.

*Step 2:* For any \( m_s \) and \( m_r \), calculate the following value:

\[
\bar{F}(m_s, m_r) = \sum_{v_j \in V_s} \tilde{w}_{jd}(m_s, v_j) + \sum_{v_j \in V_r} \tilde{w}_{jd}(m_r, v_j).
\]

*Step 3:* Using the proposed ranking method, find the minimum of \( \bar{F}(m_s, m_r) \), \( k = 1, 2, \ldots, m - 1 \). The pair \((m', m'')\) corresponding to this minimum is 2-median of tree, i.e.

\[
NR(\bar{F}(m', m'')) = \min_{k=1,\ldots,m-1} NR(\bar{F}(m_s, m_r)).
\]

*End algorithm.*

Note that the strategy in this algorithm involves in removing one edge of a tree at a time. Removal of an edge decomposes the tree into two parts. Then 1-median of both these trees and their corresponding costs are computed. The edge for which the sum of the cost of 1-medians is the lowest is the optimal split edge, and the two 1-median vertices are the resulting 2-medians of the tree.

5. **Numerical examples**

In this section, the efficiency of fuzzy algorithms is shown using numerical examples.

**Example 5.1:** Consider the tree shown in Figure 2. In this tree, the demands are in normal triangular fuzzy forms as follows: \( \tilde{w}(v_1) = (1.5, 2, 4; 1) \), \( \tilde{w}(v_2) = (3.5, 5, 7; 1) \), \( \tilde{w}(v_3) = (2, 4, 5; 1) \), \( \tilde{w}(v_4) = (0, 1.45, 3; 1) \), \( \tilde{w}(v_5) = (1.75, 4, 5; 1) \), \( \tilde{w}(v_6) = (0, 1, 2; 1) \), \( \tilde{w}(v_7) = (1, 3, 4.5; 1) \).

![Figure 2. A fuzzy tree with 6 vertices.](image-url)
To find 1-median of this tree, the iterations of fuzzy 1-median algorithm are as follows.

**Step 1:** By modifying the tree and rooting at any arbitrary vertex like $v_5$, we have $S = \{v_2, v_6, v_7\}$. Then $\tilde{W}(T), \tilde{W}(T_{v_2}), \tilde{W}(T_{v_6})$ and $\tilde{W}(T_{v_7})$ are calculated as follows:

$$\tilde{W}(T) = \sum_{k=1}^{7} \tilde{w}(v_k) = (9.75, 20.45, 30.5; 1),$$

$$\tilde{W}(T_{v_2}) = \tilde{w}(v_1) + \tilde{w}(v_2) + \tilde{w}(v_3) + \tilde{w}(v_4) = (7, 12.45, 19; 1),$$

$$\tilde{W}(T_{v_6}) = \tilde{w}(v_6) = (0, 1, 2; 1),$$

$$\tilde{W}(T_{v_7}) = \tilde{w}(v_7) = (1, 3, 4.5; 1).$$

**Step 2:** As it is shown below, the sub-tree $T_{v_2}$ has the maximum sum of weights among the three sub-trees ($T_{v_2}$), ($T_{v_6}$) and ($T_{v_7}$).

$$NR(\tilde{W}(T_{v_2})) = (12.81, 12.45, 6, 0),\quad NR(\tilde{W}(T_{v_6})) = (1, 1, 1, 0),$$

$$NR(\tilde{W}(T_{v_7})) = (2.83, 3, 1.75, 0).$$

According to NR index and lexicographical order, these three generalised triangular fuzzy numbers are ordered as $\tilde{W}(T_{v_2}) > \tilde{W}(T_{v_7}) > \tilde{W}(T_{v_6})$. Hence, we have

$$\max \left\{ \tilde{W}(T_{v_2}), \tilde{W}(T_{v_6}), \tilde{W}(T_{v_7}) \right\} = \tilde{W}(T_{v_2}).$$

On the other hand, $NR(\tilde{W}(T)) = (20.23, 20.45, 10.347, 0)$. Therefore, it is clear that $\tilde{W}(T_{v_2}) > \tilde{W}(T)/2$. Hence, let $\tilde{w}(v_2) = \tilde{w}(v_2) + \tilde{W}(T - T_{v_2})$. That is:

$$\tilde{w}(v_2) = \tilde{w}(v_2) + \tilde{w}(v_5) + \tilde{w}(v_6) + \tilde{w}(v_7) = (6.25, 13, 18.5; 1).$$

Let $T = T_{v_2}$ and $r = v_2$ then go to step 1.

**Step 1:** Obviously $S = \{v_1\}$ and $\tilde{W}(T_{v_1}) = (3.5, 7.45, 12; 1)$.

**Step 2:** Since $NR(\tilde{W}(T_{v_1})) = (7.65, 7.45, 4.25, 0)$, it is concluded that $\tilde{W}(T_{v_1}) < \tilde{W}(T)$. Thus $r = v_2$, is the 1-median.

**Example 5.2:** Consider the fuzzy tree of Example 5.1. In this example, we want to find 2-median of the fuzzy tree. Table 1 shows the results of fuzzy 2-median algorithm for this tree. In this table, $m_1$ and $m_2$ are the medians of sub-trees which are obtained by deleting the arc in the first column. In the last column of this table, the objective functions are presented. Based on our presented ranking function, it is concluded that

$$\tilde{F}(v_2, v_5) < \tilde{F}(v_2, v_3) < \tilde{F}(v_1, v_3) < \tilde{F}(v_2, v_4) < \tilde{F}(v_2, v_7) < \tilde{F}(v_2, v_6).$$

| Deleted arc | $m_1$ and $m_2$ | $\tilde{F}(m_1, m_2)$ |
|-------------|-----------------|----------------------|
| $(v_1, v_2)$ | $v_1$ and $v_5$ | (13,29.35,44;1) |
| $(v_1, v_4)$ | $v_2$ and $v_4$ | (15,37,53;1) |
| $(v_1, v_3)$ | $v_2$ and $v_3$ | (7.5,28.8,46;1) |
| $(v_2, v_5)$ | $v_2$ and $v_5$ | (9.5,26.8,42;1) |
| $(v_5, v_6)$ | $v_2$ and $v_6$ | (13.5,37.8,55;1) |
| $(v_5, v_7)$ | $v_2$ and $v_7$ | (14,36.8,54.5;1) |
Thus, the vertices $v_2$ and $v_5$ are two medians of mentioned fuzzy tree that is obtained by deleting the arc $(v_2, v_5)$ and the value of optimal objective function is equal to $(9.5, 26.8, 42; 1)$.

6. Conclusion

Since $p$-median problems are used to model real-world situations, it is necessary to consider the uncertain parameters. On the other hand, for solving this fuzzy problem and most of the other fuzzy decision-making problems, ranking fuzzy numbers is an important tool. In this paper, a new method is presented for ranking all triangular fuzzy numbers. Unlike many existing methods, the proposed method is a linear ranking method and it is a simple implementation that makes use of the algorithm more effectively to solve fuzzy real-world problems. Also, based on this ranking, a unique index is assigned to each fuzzy number. Thus, our new method is an acceptable ranking method for ranking generalised fuzzy numbers and their images. Moreover, crisp numbers can be ranked as effectively as fuzzy numbers using the proposed ranking. Also, it is shown that the new method is more simple, reasonable and consistent with human intuitions than previous methods that are mentioned in the literature. In this paper, we also proposed a fuzzy algorithm to find the 1-median of a tree with fuzzy weights. Then, we extended the algorithm for solving 2-median problem on a tree with fuzzy weights.

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Appendix

Here using six different categories of fundamental examples, the new ranking method is compared with some convenient previous methods presented in the literature.

Example A.1: Let $\tilde{A} = (0.1, 0.3, 0.5)$ and $\tilde{B} = (0.3, 0.5, 0.7)$ be two fuzzy numbers and $-\tilde{A} = (-0.5, -0.3, -0.1)$ and $-\tilde{B} = (-0.7, -0.5, -0.3)$ be the opposite of $\tilde{A}$ and $\tilde{B}$, respectively. Using (6), we obtain

$$NR(\tilde{A}) = (0.3, 0.3, 0.2, 0), \quad NR(\tilde{B}) = (0.5, 0.5, 0.2, 0),$$
$$NR(-\tilde{A}) = (-0.3, -0.3, -0.2, 0), \quad NR(-\tilde{B}) = (-0.5, -0.5, -0.2, 0).$$

It is clear that the first component of $NR(\tilde{B})$ is greater than the first component of $NR(\tilde{A})$. Therefore, $NR(\tilde{A}) < L NR(\tilde{B})$. Thus, $\tilde{A} < \tilde{B}$ (see Figure A1).

![Figure A1. Four triangular fuzzy numbers (0.1,0.3,0.5), (0.2,0.5,0.7), (−0.5, −0.3, −0.1) and (−0.7, −0.5, −0.3).](image)


**Figure A2.** Two sets of fuzzy numbers.

**Table A1.** Comparative results of Example A.2.

| Fuzzy number | New method | Nasseri [42] | Chu-Tsao [39] | Cheng [41] | Chen [40] |
|--------------|------------|--------------|---------------|------------|-----------|
| \(\tilde{A}\) | (6,6,1,0)  | 12           | 3             | 6.021      | 0.5       |
| \(\tilde{B}\) | (6.3,6,0.55,0) | 12.767      | 3.126         | 6.349      | 0.5833    |
| \(\tilde{C}\) | (6.333,6,0,5,0) | 12.854      | 3.985         | 6.3519     | 0.5714    |

In Set 1
\(\tilde{A} < \tilde{B} < \tilde{C}\)\n\(\tilde{A} < \tilde{B} < \tilde{C}\)\n\(\tilde{A} < \tilde{B} < \tilde{C}\)\n\(\tilde{A} < \tilde{B} < \tilde{C}\)

\(\tilde{A} \approx \tilde{B} < \tilde{C}\)\n\(\tilde{A} < \tilde{B} < \tilde{C}\)\n\(\tilde{A} < \tilde{B} < \tilde{C}\)\n\(\tilde{A} < \tilde{B} < \tilde{C}\)

\(\tilde{A} > \tilde{B} > \tilde{C}\)\n\(\tilde{A} > \tilde{B} > \tilde{C}\)\n\(\tilde{A} > \tilde{B} > \tilde{C}\)\n\(\tilde{A} > \tilde{B} > \tilde{C}\)

Also we can see \(\tilde{B} < \tilde{A} < \tilde{A} < \tilde{B}\). This example shows that the proposed method can effectively rank various fuzzy numbers and their images.

**Example A.2:** Consider two sets as follows which are represented in Figure A2:

Set 1: \(\tilde{A} = (5, 6, 7)\), \(\tilde{B} = (5.9, 6, 7)\), \(\tilde{C} = (6, 6, 7)\).

The ranking results of presented ranking function for the two given sets are compared with four other methods in Table A1. As it is seen for the first set of numbers, the results of Chu-Tsao’s method [39] and Chen [40] are unreasonable and incompatible with human intuition. However, for our new method, we obtain the same result as Cheng [41] and Nasseri’s methods [42]. Also, it should be mentioned as an important property of the new approach that it is simpler than Cheng [41] and Nasseri’s methods [42] in computational processes.

**Example A.3:** Consider the following sets:

Set 1: \(\tilde{A} = (0.3, 0.5, 1)\), \(\tilde{B} = (0.1, 0.6, 0.8)\).

Set 2: \(\tilde{A} = (0.1, 0.3, 0.5)\), \(\tilde{B} = (0.2, 0.3, 0.4)\).

Set 3: \(\tilde{A} = (-0.5, -0.3, -0.2)\), \(\tilde{B} = (-0.58, -0.32, -0.17)\), \(\tilde{C} = (-0.7, -0.4, -0.25)\).

The above four sets are shown in Figure A3 and the comparative results are also given in Table A2. It can clearly be seen that

(1) The same reasonable results are obtained using our proposed method and the listed methods for ranking the triangular fuzzy numbers \(\tilde{A}, \tilde{B}\) in Set 1 of Figure A3. Also the new approach is simpler than the listed methods in computational processes.
(2) The fuzzy numbers $\tilde{A}$ and $\tilde{B}$ in Set 2 of Figure A3 cannot be ranked according to Yager [43], Murakami et al. [34], and Cheng’s methods [41]. However, the results of the methods proposed by Lee and Chen [44], Akyar et al. [45], Chen and Chen [46], and the method proposed in this paper are the same. Also, the fuzzy numbers $\tilde{A}$ and $\tilde{B}$ in Set 3 of Figure A3 cannot be ranked using Cheng’s method [41], while the proposed method and all other methods have the same ranking results. This set shows that the fuzzy numbers and their images can be ranked using the proposed method consistently.

(3) Fuzzy numbers $\tilde{A}$ and $\tilde{B}$ in Set 4 of Figure A3 can just be ranked according to the proposed methods by Cheng [41], Akyar et al. [45], and our proposed method.

**Example A.4:** In Figure A4, four fuzzy numbers $\tilde{A} = (0.1, 0.2, 0.3), \tilde{B} = (0.2, 0.5, 0.8), \tilde{C} = (0.3, 0.4, 0.9)$ and $\tilde{D} = (0.6, 0.7, 0.8)$ are presented which were ranked earlier by Yager [43], Fortemps and Roubens [47], and Liou and Wang [48] as shown in Table A3.
Figure A4. Four fuzzy numbers (0.1,0.2,0.3), (0.2,0.5,0.8), (0.3,0.4,0.9) and (0.6,0.7,0.8).

Table A3. Comparative results of Example A.4.

| Fuzzy numbers | \( A \) | \( B \) | \( C \) | \( D \) | Results |
|---------------|---------|---------|---------|---------|---------|
| New method    | (0.2,0.2,0.1,0) | (0.5,0.5,0.3,0) | (0.533,0.4,0.3,0) | (0.7,0.7,0.1,0) | \( A < B < C < D \) |
| Yager [43]    | 0.2     | 0.5     | 0.5     | 0.7     | \( A < B = C < D \) |
| Fortemps and Roubens [47] | 0.2     | 0.5     | 0.5     | 0.7     | \( A < B = C < D \) |
| Liou and Wang [48] | \( \alpha = 1 \) | 0.25 | 0.65 | 0.65 | 0.75 | \( A < B = C < D \) |
|               | \( \alpha = 0.5 \) | 0.2 | 0.5 | 0.5 | 0.7 | \( A < B = C < D \) |
|               | \( \alpha = 0 \) | 0.15 | 0.35 | 0.35 | 0.65 | \( A < B = C < D \) |

Figure A5. Two generalised fuzzy numbers (0.1,0.3,0.5;0.8) and (0.1,0.3,0.5;1).

It is clear from Table A3 that none of the methods are able to rank these fuzzy numbers completely. Yager [43], and Fortemps and Roubens [47] methods are shown to fail to distinguish the fuzzy numbers \( B \) and \( C \). Also, the methods of Liou and Wang [48] cannot distinguish the fuzzy numbers \( B \), \( C \) and \( A \), \( D \), respectively. But, the proposed method gives a complete ranking order.
Table A4. Comparative results of Example A.5.

| Fuzzy numbers | $\tilde{A}$         | $\tilde{B}$        | Results     |
|---------------|----------------------|---------------------|-------------|
| New method    | $(0.3, 0.3, 0.2, -0.067)$ | $(0.3, 0.3, 0.2, 0)$ | $\tilde{A} < \tilde{B}$ |
| Yager [43]    | 0.3                  | 0.3                 | $\tilde{A} \sim \tilde{B}$ |
| Murakami et al. [34] | 0.233         | 0.3                 | $\tilde{A} < \tilde{B}$ |
| Cheng [41]    | 0.461                | 0.5831              | $\tilde{A} < \tilde{B}$ |
| Chen and Chen [46] | 0.2063           | 0.2579              | $\tilde{A} < \tilde{B}$ |
| Lee and Chen [44]  | –                    | –                   | –           |
| Akyar et al. [45]  | –                    | –                   | –           |

Figure A6. Three sets of fuzzy numbers.

Example A.5: In Figure A5, the generalised triangular fuzzy number $\tilde{A} = (0.1, 0.3, 0.5; 0.8)$ and triangular fuzzy number $\tilde{B} = (0.1, 0.3, 0.5; 1)$ are presented. According to the new method, the ranking order is $\tilde{A} < \tilde{B}$. The comparison results of the proposed method with some existing methods are given in Table A4.

According to Table A4, we cannot obtain any reasonable ranking from Yager’s method [43]. Also, Lee’s [44], and Akyar’s methods [45] cannot rank the generalised fuzzy numbers of Figure A5. Obviously, the inconsistency problem of other methods in ranking fuzzy numbers is overcome with the proposed ranking method.

Example A.6: In Figure A6, eight fuzzy numbers $\tilde{A}_1 = (0.7, 0.8, 0.9)$, $\tilde{A}_2 = (0.9, 0.95, 1)$, $\tilde{B}_1 = (0, 0.1, 0.5)$, $\tilde{B}_2 = (0.5, 0.6, 0.7)$, $\tilde{B}_3 = (0.9, 1, 1)$, $\tilde{C}_1 = (0.4, 0.9, 1)$, $\tilde{C}_2 = (0.4, 0.7, 1)$, $\tilde{C}_3 = (0.4, 0.5, 1)$ are adapted from Bortolan and Degani [49]. In Tables A5 and A6, the NR indexes and the comparative results are given, respectively.
Table A5. Calculation result of Example A.6.

| Fuzzy numbers | New method | Cheng [41] | Kerre [30] | Lee and Lee [50] | Liou and Wang [48] | Jain [29] |
|---------------|------------|------------|------------|-----------------|-------------------|-----------|
|               |            | O          | G          | γ = 1           | γ = 0.5           | γ = 0     | k = 1     | k = 2     | k = 0.5 |
| A₁            | (0.8, 0.8, −0.1, 0) | 0.16       | 0.85       | 0               | 0.85              | 0.8       | 0.75       | 0.81       | 0.70    | 0.90    |
| A₂            | (0.95, 0.95, −0.05, 0) | 0.09       | 1          | 1               | 1                 | 0.98      | 0.95       | 0.93       | 0.95    | 0.92    | 0.97    |
| B₁            | (0.033, 0, −0.05, 0) | 0          | 0.89       | 0               | 0.5               | 0.25      | 0          | 0.09       | 0       | 0.26    |
| B₂            | (0.6, 0.6, −0.1, 0) | 0.12       | 0.85       | 0               | 0.5               | 0.75      | 0.7        | 0.65       | 0.63    | 0.42    | 0.78    |
| B₃            | (0.967, 1, −0.05, 0) | 0.1        | 1          | 1               | 1                 | 0.98      | 0.95       | 1          | 1       | 1       |
| C₁            | (0.767, 0.9, −0.3, 0) | 0.46       | 1          | 0.65            | 1                 | 0.95      | 0.8        | 0.65       | 0.90    | 0.84    | 0.95    |
| C₂            | (0.7, 0.7, −0.3, 0) | 0.41       | 0.86       | 0.35            | 0.61              | 0.85      | 0.7        | 0.55       | 0.76    | 0.65    | 0.86    |
| C₃            | (0.633, 0.5, −0.3, 0) | 0.38       | 0.76       | 0.2             | 0.18              | 0.75      | 0.6        | 0.45       | 0.66    | 0.54    | 0.78    |
### Table A6. Comparative results of Example A.6.

| New method | $\tilde{A}_1 < \tilde{A}_2$ | $\tilde{B}_1 < \tilde{B}_2 < \tilde{B}_3$ | $\tilde{C}_3 < \tilde{C}_2 < \tilde{C}_1$ |
|------------|-----------------------------|---------------------------------|---------------------------------|
| Cheng [41] | $\tilde{A}_1 < \tilde{A}_2$ | $\tilde{B}_1 < \tilde{B}_2$ | $\tilde{C}_3 < \tilde{C}_2 < \tilde{C}_1$ |
| Kerre [30] | $\tilde{A}_1 < \tilde{A}_2$ | $\tilde{B}_2 < \tilde{B}_1 < \tilde{B}_2$ | $\tilde{C}_3 < \tilde{C}_2 < \tilde{C}_1$ |
| Lee and Lee [50] | $\tilde{A}_1 < \tilde{A}_2$ | $\tilde{B}_1 = \tilde{B}_2 < \tilde{B}_3$ | $\tilde{C}_3 < \tilde{C}_2 < \tilde{C}_1$ |
| O | $\tilde{A}_1 < \tilde{A}_2$ | $\tilde{B}_1 < \tilde{B}_2 < \tilde{B}_3$ | $\tilde{C}_3 < \tilde{C}_2 < \tilde{C}_1$ |
| G | $\tilde{A}_1 < \tilde{A}_2$ | $\tilde{B}_1 < \tilde{B}_2 < \tilde{B}_3$ | $\tilde{C}_3 < \tilde{C}_2 < \tilde{C}_1$ |
| Liou and Wang [48] | $\gamma = 1$ | $\tilde{A}_1 < \tilde{A}_2$ | $\tilde{B}_1 < \tilde{B}_2 < \tilde{B}_3$ | $\tilde{C}_3 < \tilde{C}_2 < \tilde{C}_1$ |
| | $\gamma = 0.5$ | $\tilde{A}_1 < \tilde{A}_2$ | $\tilde{B}_1 < \tilde{B}_2 < \tilde{B}_3$ | $\tilde{C}_3 < \tilde{C}_2 < \tilde{C}_1$ |
| Jain [29] | $k = 1$ | $\tilde{A}_1 < \tilde{A}_2$ | $\tilde{B}_1 < \tilde{B}_2 < \tilde{B}_3$ | $\tilde{C}_3 < \tilde{C}_2 < \tilde{C}_1$ |
| | $k = 2$ | $\tilde{A}_1 < \tilde{A}_2$ | $\tilde{B}_1 < \tilde{B}_2 < \tilde{B}_3$ | $\tilde{C}_3 < \tilde{C}_2 < \tilde{C}_1$ |
| | $k = 0.5$ | $\tilde{A}_1 < \tilde{A}_2$ | $\tilde{B}_1 < \tilde{B}_2 < \tilde{B}_3$ | $\tilde{C}_3 < \tilde{C}_2 < \tilde{C}_1$ |