Monte Carlo Study of Two-Color QCD with Finite Chemical Potential – Status report of Wilson fermion simulation –

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Using Wilson fermions, we study SU(2) lattice QCD with the chemical potential at $\beta = 1.6$. The ratio of fermion determinants is evaluated at each Metropolis link update step. We calculate the baryon number density, the Polyakov loops and the pseudoscalar and vector masses on $4^4$ and $4^3 \times 8$ lattices. Preliminary data show the pseudoscalar meson becomes massive around $\mu = 0.4$, which indicates the chiral symmetry restoration. The calculation is broken down when approaching to the transition region. We analyze the behavior of the fermion determinant and eigen value distributions of the determinant, which shows a peculiar “Shell-and-Bean” pattern near the transition.

1. INTRODUCTION

Lattice study of QCD has been expected to provide useful informations to understand non-perturbative aspects of quark/gluon physics. Especially at finite temperature, it predicts the confinement/deconfinement transition and is able to describe many features of hadrons and quark gluon plasma (QGP). Lattice QCD offers a sound base of QGP physics, which has become very important issue of physics because of currently going active experiments at CERN SPS and BNL RHIC. See [1].

Contrary to the finite temperature calculation, the progress in lattice QCD study of the finite density has been rather slow. This is because of the well known complex action problem. Indeed after the first QCD dynamical quark simulation with the chemical potential was done for SU(2) color group [2], to our knowledge, no full SU(3) QCD calculations had been tried. A trial to put the phase coming from the determinant into observables suffers from large fluctuation even at $4^4$ size lattice near the phase transition [3, 4]. Stephanov shows that the quench approximation is not the correct $N_f = 0$ limit of full QCD [5]. We still wait for good news concerning Glasgow method (see a good review by Barbour [6] and references therein), and finite density method [7].

Recently the situation has been changed; Due to the progress in analytical investigations [8, 9], we have a hope to obtain informations on concerning QCD by studying QCD-like theories such as SU(2) QCD, models with quarks in the adjoint representation and QCD at finite isospin density; they are expected to have less difficulties in numerical analyses. In these years, there are indeed high activities in Monte Carlo calculations with dynamical quark of these models [10].

In this paper, we report our recent work on SU(2) QCD with Wilson fermions to study finite density states.

2. ALGORITHM

The chemical potential, $\mu$, is introduced in the fermion action, $\bar{\psi}W\psi$, as

\begin{align*}
W(x, x') = & \delta_{x,x'} \\
-\kappa & \sum_{i=1}^{3} \left\{ (1 - \gamma_i)U_i(x)\delta_{x',x+i} + (1 + \gamma_i)U_i^\dagger(x')\delta_{x',x-i} \right\} \\
-\kappa & \left\{ e^{+\mu a}(1 - \gamma_4)U_4(x)\delta_{x',x+4} + e^{-\mu a}(1 - \gamma_4)U_4^\dagger(x')\delta_{x',x-4} \right\}
\end{align*}

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density,

\[ + e^{-\mu a}(1 + \gamma_4) U_{4, x'}(x') \delta_{x', x - 4} \tag{1} \]

by Hasenfratz and Karsch \[17\] to avoid an infinity in the energy density. As the lattice spacing \(a\) tends to zero, Eq.1 gives \(W(\mu) = W(0) + \kappa \mu a \bar{\psi} \gamma_4 \psi + O(a^2)\). The above formula was independently obtained in Ref.[16] by the following naive argument: In the continuum perturbation, the chemical potential is introduced by the substitution \(p_i \to p_i - i \mu\) in fermion propagators. In order to have this continuum limit, the lattice fermion propagator should have the form,

\[
\frac{1}{1 - \kappa} \sum_{\iota=1}^3 \left\{ (1 - \gamma_i) e^{ip_i a} + (1 + \gamma_i) e^{-ip_i a} \right\} - \kappa \left\{ (1 - \gamma_4) e^{i(p_4 - i \mu) a} + (1 + \gamma_4) e^{-i(p_4 - i \mu) a} \right\} \tag{2}
\]

Little is known about the behavior of dynamical fermion simulations when the chemical potential is introduced. We therefore decide to employ an algorithm where the ratio of the determinant,

\[
\frac{\det W(U + \Delta U)}{\det W(U)} = \det(I + W(U)^{-1} \Delta W) \tag{3}
\]

is evaluated explicitly at each Metropolis update process, \(U \to U + \Delta U\), where \(\Delta W \equiv W(U + \Delta U) - W(U)\) \[18, 20\]. An essential ingredient of the algorithm is the following Woodbury formula,

\[
(W + \Delta W)^{-1} = W^{-1} - W^{-1} \Delta W (I + W^{-1} \Delta W)^{-1} W^{-1} \tag{4}
\]

Suppose we update link variables \(U_\mu(x)\)’s only on a subset \(H\) of whole lattice. Then \(\Delta W \neq 0\) only on \(H\). Woodbury formula \(4\) holds even if the matrix space is limited on \(H\). In this case we can get the ratio of the fermion determinant as far as \(U_\mu(x)\)’s are updated inside \(H\). We take a \(2^4\) hypercube as \(H\). When we go to the next hypercube, \((W^{-1})_H\)’s are initialized by CG method.

3. RESULTS

First we calculate the expectation value of the density,

\[
<n> = \frac{1}{\beta V_s} \frac{\partial}{\partial \mu} \log Z \tag{5}
\]

where \(V_s\) is the spatial volume \(N_x N_y N_z\). In Fig.1, we plot \(<n> / T^3\), as a function of \(\mu\), which is dimensionless. The dotted line corresponds to the free quark case obtained by setting \(U_\mu(x) = 1\) and \(\kappa = 1/8\). When \(\kappa\) becomes large, the density reaches to the free case quickly.

The Polyakov line \(<L>\) also increases as a function of \(\mu\) as shown in Fig.2. When the hopping parameter, \(\kappa\), becomes large from 0.158 to 0.185, values of Polyakov line increase. Since the lattice size is small, no sharp increase of \(<L>\) is seen, but large values of \(<L>\) indicate quarks become free from the confinement force at \(\mu > 0.4\).

The behavior of \(<L>\) suggests that we are near the phase transition. But the calculation breaks because of numerical instability at \(\kappa = 0.158, \mu = 0.8\) and \(\kappa = 0.185, \mu = 0.7\) in case of \(4^4\), and we cannot go beyond. To see origins of the instability, we measure the ratio of the fermion determinant \(\det W(U + \Delta U) / \det W(U)\). Figure 3 shows the behavior of the ratio as a function of Monte Carlo sweeps. The ratio changes the value around one, but suddenly it fluctuates very large and the calculation is broken.

In Fig.4, we plot eigen value distributions of \(W\) for \(\mu = 0.0, 0.4, 0.6\) and 0.8 at \(\kappa = 0.156\) on \(4^4\) lattice. As \(\mu\) increases, the distribution of eigen values, \(\lambda_i\), becomes wide in the Real axis, and \(\text{Min Re}(\lambda) < 0\); on the other hand, \(\lambda_i\)'s scatter
Figure 2. Polyakov loop expectation value as a function of $\mu$ for $\kappa$=0.158, 0.180 and 0.185.

Figure 3. The ratio of $\det W$ as a function of Monte Carlo sweeps. It fluctuates around one, but before the simulation is crashed, big fluctuation is observed.

1. Although the data are still very preliminary, we extrapolate them to the chiral limit obtain Fig.\textsuperscript{4}. Error bars are very large, but the pion mass becomes massive around $\mu \sim 0.4$, which means that the chiral symmetry is restored in these regions.

4. CONCLUDING REMARKS

We present numerical study of two-color QCD with the chemical potential with Wilson fermions at $\beta = 1.6$. Although the lattice is very small, most data suggests we are reaching the confinement/deconfinement phase transition.

We employ an algorithm which takes into account the ratio of fermion determinant exactly, and has large Markov step, but we suffer from numerical instability and can not go over the phase transition. Near the phase transition, the distribution of eigen values of $W$ shows a peculiar “Shell-and-Bean” structure. Since the calculation is done at strong coupling region, the strange behavior of $\det W$ might be related with rough configurations far from the continuum \textsuperscript{[21]}. We plan to continue the analysis by using improved gauge actions to clarify the point.
Table 1
Pseudo scalar and vector masses at $\mu=0.0$, 0.2, 0.4 and 0.5 for $\kappa=0.158$ and 0.180.

| $\mu$ | $\kappa = 0.158$ | $\kappa = 0.180$ |
|-------|-----------------|-----------------|
| 0.0   | 1.775(31)       | 1.327(51)       |
| 0.2   | 1.722(09)       | 1.305(37)       |
| 0.4   | 1.738(11)       | 1.428(51)       |
| 0.5   | 1.649(18)       | 1.346(65)       |
| $\rho$ | 1.882(45)       | 1.509(41)       |
| 0.2   | 1.783(13)       | 1.490(41)       |
| 0.4   | 1.788(36)       | 1.660(49)       |
| 0.5   | 1.798(46)       | 1.482(100)      |

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REFERENCES
1. H. Satz, this proceedings, hep-ph/0009099.
2. A. Nakamura, Phys. Lett., 149B (1984) 391.
3. A. Gocksch, Phys. Rev. Lett., 61 (1988) 2054.
4. A. Nakamura, Nucl. Phys. B (Proc. Suppl.) 17 (1990) 395.
5. D. Toussaint, Nucl. Phys. B (Proc. Suppl.) 17 (1990) 539.
6. M. A. Stephanov, Phys. Rev. Lett. 76 (1996) 4472.
7. I. M. Barbour et al., Nucl. Phys. B (Proc. Suppl.) 60A (1998) 220.
8. J. Engels, O. Kaczmarek, F. Karsch and E. Laermann, Nucl. Phys. B558 (1999) 307.
9. J. B. Kogut et al., Nucl. Phys. B582 (2000) 477.
10. D. T. Son and M. A. Stephanov, hep-ph/0005225.
11. M.-P. Lombardo, hep-lat/9907025, hep-lat/9906006.
12. S. Hands, J. B. Kogut, M.-P. Lombardo and S. E. Morrison, Nucl. Phys. B558 (1999) 327.
13. S. Morrison and S. Hands, hep-lat/9902012.
14. S. Hands et al., hep-lat/0006018.
15. S. J. Hands, J. B. Kogut, S. E. Morrison, D. K. Sinclair, hep-lat/0010028.
16. A. Nakamura, Acta. Phys. Pol. B16 (1985) 635.
17. P. Hasenfratz and F. Karsch, Phys. Lett., 125B (1983) 308.
18. I. Barbour et al., J. Comput. Phys. 68 (1987) 227.
19. A. Nakamura et al., Comm. Phys. Comm. 51 (1988) 301.
20. Ph. de Forcrand et al., Phys. Rev. Lett., 58 (1987) 2011.
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