New Generalized Cyclotomic Quaternary Sequences with Large Linear Complexity and a Product of Two Primes Period

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Abstract: Linear complexity is an important criterion to characterize the unpredictability of pseudo-random sequences, and large linear complexity corresponds to high cryptographic strength. Pseudo-Random Sequences with a large linear complexity property are of importance in many domains. In this paper, based on the theory of inverse Gray mapping, two classes of new generalized cyclotomic quaternary sequences with period $pq$ are constructed, where $pq$ is a product of two large distinct primes. In addition, we give the linear complexity over the residue class ring $\mathbb{Z}_4$ via the Hamming weights of their Fourier spectral sequence. The results show that these two kinds of sequences have large linear complexity.

Keywords: stream ciphers; finite field; quaternary sequence; Fourier spectral sequence; linear complexity

1. Introduction

Pseudo-random sequences with large linear complexity and low nontrivial autocorrelation values are widely applied in spread spectrum communication, radar navigation, cryptography, code division multiple access, especially stream cipher. The linear complexity of a sequence is defined as the smallest order of linear feedback shift register that can generate the whole sequence. According to the Berlekamp–Massey algorithm, a large linear complexity should be no less than a half of the period of the sequence [1,2]. Binary sequences with good pseudo-random properties have been studied in depth in recent decades [2]. Compared with binary sequences, quaternary sequences have a higher transmission rate, and a code element can represent more bits of information. Moreover, quaternary sequences have important applications in the four-phase spread spectrum system [3]. Therefore, quaternary sequences are attracting more and more researchers to consider them. Most references have concentrated on the linear complexity of quaternary sequences over $\mathbb{F}_4$ [4–7]. However, there has been less attention to the linear complexity of sequences over $\mathbb{Z}_4$ due to the phenomenon of zero divisors in $\mathbb{Z}_4$ [8].

Inverse Gray mapping is one of the main methods for constructing quaternary sequences [9]. Given two arbitrary binary sequences of equal length, a unique quaternary sequence can be determined by inverse Gray mapping. Kim et al. constructed a class of quaternary sequences with period $2p$ over $\mathbb{Z}_4$ by the use of a Legendre sequence pair. They analyzed the autocorrelation properties and the linear complexity of these sequences [10,11]. Yang et al. defined a class of quaternary sequence on $\mathbb{Z}_4$ by using the Whiteman generalized cyclotomic binary sequence pair and calculated the autocorrelation values [12]. Li et al. analyzed the linear complexity of the sequence which was constructed in [12] by considering the weights of Fourier spectral sequence of the sequence [13,14]. Wang et al. established a class of quaternary sequence on $\mathbb{Z}_4$ based on the balanced Whiteman generalized cyclotomic binary sequence pair and gave the linear complexity of the sequence [15]. Wei et al. introduced the quaternary sequence on $\mathbb{Z}_4$ based on the Ding generalized cyclotomic binary sequence pair and discussed the linear complexity of the sequence [16,17]. The quaternary sequences mentioned above all are constructed by selecting two homogeneous binary sequences.
sequences. It is necessary to confirm whether the quaternary sequences constructed by binary sequences with greater distinction have large linear complexity.

First, this paper proposes a new class of quaternary sequences with period $pq$ based on the Whiteman generalized cyclotomic binary sequence and the Ding generalized cyclotomic binary sequence, which can be denoted by the first class of the generalized cyclotomic quaternary sequence. Second, this paper proposes a new class of quaternary sequences with period $pq$ based on the Ding generalized cyclotomic binary sequence and the new Ding generalized cyclotomic binary sequence [17,18], which can be denoted by the second class of the generalized cyclotomic quaternary sequence. Moreover, the linear complexity of the two quaternary sequences is computed by considering the Hamming weight of their Fourier spectral sequences.

2. Preliminaries

Suppose that $S = \{S_i\}$ is a sequence over $F_r$ with period $N$, where $r$ is an odd prime, $F_r$ is the finite field with $r$ elements, and $N$ divides $r^m - 1$ ($m \geq 1$, $m$ is a positive integer). The linear complexity $LC(S)$ of the sequence $S$ is the smallest positive integer $L$ satisfying

$$s_i + c_1 s_{i-1} + \cdots + c_{L-1} s_{i-L+1} + c_L s_{i-L} = 0, \text{ for } L \leq i \leq N.$$  

where the coefficients $c_1, c_2, \cdots, c_L \in F_r$. The generating polynomial of $S$ is defined by

$$s(x) = \sum_{i=0}^{N-1} s_i x^i \in F_r[x].$$

Definition 1. [1] Let $\theta$ be an element in $F_{r^m}$ of order $N$. Then the discrete Fourier Transform of $S$ is defined as

$$A_k = \sum_{t=0}^{N-1} S(t) \theta^{tk}, \ 0 \leq k \leq N - 1$$

The inverse formula of Equation (1) is given by

$$S(t) = \frac{1}{N} \sum_{k=0}^{N-1} A_k \theta^{-tk}, \ 0 \leq t \leq N - 1$$

where $A_k$ is called a Fourier spectrum of the sequence $S$. Note that $A = \{A_k\}$ is called a Fourier spectrum sequence with period $N$ of $S$.

Lemma 1. [1] $\sum_{i=0}^{N-1} \theta^{di} = \begin{cases} 0, & \text{if } d \equiv 0 \pmod{N} \\ N, & \text{otherwise} \end{cases}$.

Lemma 2. [14] Let $A = \{A_k\}$ be the Fourier spectrum sequence of $S$. Then the linear complexity of $S$ is given by

$$LC(S) = |\{k | A_k \neq 0, \ 0 \leq k \leq N - 1 \}|$$

The linear complexity of $S$ is further derived as

$$LC(S) = N - |\{k | A_k \neq 0, \ 0 \leq k \leq N - 1 \}|$$

Definition 2. Let $a(t)$ and $b(t)$ be a binary sequence with period $N$. Let $\psi[x,y]$ be the inverse Gray mapping defined by

$$\psi[a(t), b(t)] = \begin{cases} 0, & \text{if } (a, b) = (0, 0) \\ 1, & \text{if } (a, b) = (0, 1) \\ 2, & \text{if } (a, b) = (1, 1) \\ 3, & \text{if } (a, b) = (1, 0) \end{cases}.$$
Definition 3. Indicator functions $I_p(t)$ and $I_q(t)$ are defined as

$$I_p(t) = \begin{cases} 1, & t \equiv 0 \pmod{p}, \\ 0, & \text{otherwise.} \end{cases} \quad I_q(t) = \begin{cases} 1, & t \equiv 0 \pmod{q}, \\ 0, & \text{otherwise.} \end{cases}$$

(8)

Definition 4. The quadratic characters $\eta_p(t)$ and $\eta_q(t)$ are defined as

$$\eta_p(t) = \begin{cases} 0, & t \equiv 0 \pmod{p} \\ 1, & t \in \text{QR}_p \\ -1, & t \in \text{NQR}_p \end{cases} \quad \eta_q(t) = \begin{cases} 0, & t \equiv 0 \pmod{q} \\ 1, & t \in \text{QR}_q \\ -1, & t \in \text{NQR}_q \end{cases}$$

(9)

where $\text{QR}_p$ and $\text{NQR}_p$ are the sets of quadratic residues and quadratic non-residues in the set of integers modulo $p$, respectively; By symmetry, $\text{QR}_q$ and $\text{NQR}_q$ are defined similarly.

3. The Linear Complexity of the First Class of Generalized Cyclotomic Quaternary Sequences

Let $p$ and $q$ be two distinct odd primes and set $N = pq$. Define that $P = \{p, 2p, 3p, \ldots, (q-1)p\}$, $Q = \{q, 2q, 3q, \ldots, (p-1)q\}$, then the residue ring $Z_N = \{0\} \cup P \cup Q \cup Z_N^*$, where $Z_N^*$ denotes the set of all invertible elements in $Z_N$. According to the Chinese remainder theorem, we can get $Z_N \cong Z_p \times Z_q$, $t \mapsto (t_1, t_2)$ for $\forall t \in Z_N$, where $t = t_1 \pmod{p}$, $t = t_2 \pmod{q}$.

The two generalized cyclotomic binary sequences are presented as follows.

$$S_1(t) = \begin{cases} 1, & t \in P \\ 0, & t \in \{0\} \cup Q \\ \frac{1-\eta_p(t)\eta_q(t)}{2}, & t \in Z_N^* \end{cases} \quad S_2(t) = \begin{cases} 1, & t \in P \\ 0, & t \in \{0\} \cup Q \\ \frac{1-\eta_p(t)}{2}, & t \in Z_N^* \end{cases}$$

(10)

where $S_1(t)$ is the Whiteman generalized cyclotomic binary sequences of order two with period $pq$ [17], $S_2(t)$ is the Ding generalized cyclotomic binary sequences of order two with period $pq$ [2]. Then, the first class of the generalized cyclotomic quaternary sequence can be expressed by $S'(t) = \psi[S_1(t), S_2(t)]$. Clearly that the sequence $S'(t)$ is different from those in references [12,15,16]. Moreover, when $t$ ranges over $Z_N^*$, every element in $S'(t)$ takes on the same times.

The linear complexity of a periodic sequence can be determined by counting the number of nonzero coefficients of its discrete Fourier transform, which is defined over a finite field [11]. Therefore, a proper field should be found for the linear representation [11].

Let $\theta$ be a primitive $pq$-th root of unity in $F_{r^m}$ where $r \geq 5$ is the odd prime which is not equal to $p$ or $q$ and $F_{r^m}$ is the splitting field of $x^{pq} - 1$. Suppose that $a = \theta^p$, $b = \theta^q$ is the $p$th and $q$th primitive root of unity in the field $F_{r^m}$, respectively.

According to the definitions of indicator function and quadratic character, the sequences $S_1(t)$ and $S_2(t)$ can be expressed as

$$S_1(t) = \frac{1}{2} \left[ 1 - \eta_p(t_1)\eta_q(t_2) + I_p(t_1) - I_q(t_2) - I_p(t_1)I_q(t_2) \right]$$

(11)

$$S_2(t) = \frac{1}{2} \left[ 1 - \left[ 1 - I_p(t_1) \right] \left[ 1 - I_q(t_2) \right] \eta_q(t_2) + I_p(t_1) - I_q(t_2) - I_p(t_1)I_q(t_2) \right]$$

(12)

Then we can derive the representation of $S'(t)$

$$S'(t) = \psi[S_1(t), S_2(t)] = 3S_1(t) + S_2(t) - 2S_1(t)S_2(t) = \frac{1}{2} \left[ 3 - 2\eta_p(t_1)\eta_q(t_2) + I_p(t_1) - 3I_q(t_2) - I_p(t_1)I_q(t_2) - \eta_p(t_1)\eta_p(t_1)\eta_q(t_2) - I_q(t_2)\eta_p(t_1)\eta_q(t_2) \right]$$

(13)
Note that the representation holds for \( r \geq 5 \).
The term of Fourier spectral sequence \( A' = (A'_k) \) of sequence \( S'(t) \) is defined by
\[
A' = \sum_{t=0}^{N-1} S'(t) \theta^{tk} = \frac{1}{2} \left[ 3 \sum_{t=0}^{N-1} \theta^{tk} - 2 \sum_{t=0}^{N-1} \eta_p(t_1) \eta_q(t_2) \theta^{tk} + \sum_{t=0}^{N-1} I_p(t_1) \theta^{tk} \right] - 3 \sum_{t=0}^{N-1} I_q(t_2) \theta^{tk} - \sum_{t=0}^{N-1} I_p(t_1) I_q(t_2) \theta^{tk} + \sum_{t=0}^{N-1} \eta_p(t_1) \eta_q^2(t_2) \theta^{tk} \quad (14)
\]

Lemma 3. [6] Let \( \theta \) be such a primitive \( pq \)th root of unity over \( F_{q^m} \), then
\[
\sum_{t \in Z_N} \theta^t = 0, \quad \sum_{t \in pZ_q} \theta^t = -1 (\text{mod} r), \quad \sum_{t \in qZ_p} \theta^t = -1 (\text{mod} r). \quad (15)
\]

Lemma 4. [13]
\[
\sum_{t=0}^{N-1} \eta_p(t_1) \eta_q(t_2) \theta^{tk} = \left\{ \begin{array}{ll}
0, & k \in \{0\} \cup pZ_p^* \cup qZ_p^* \\
\pm (S_p - S_{N_p}) (S_q - S_{N_q}), & k \in Z_N, \eta_p(k) \eta_q(k) = \pm 1
\end{array} \right. \quad (16)
\]
where \( S_p = \sum_{i \in QR_p} \alpha^i, S_{N_p} = \sum_{i \in NQR_p} \alpha^i; S_q = \sum_{i \in QR_q} \beta^i, S_{N_q} = \sum_{i \in NQR_q} \beta^i. \)

Lemma 5. [13]
\[
\sum_{t=0}^{N-1} I_p(t_1) \theta^{tk} = \left\{ \begin{array}{ll}
q (\text{mod} r), & k \in \{0\} \cup qZ_q^* \\
0, & k \in Z_N^* \cup pZ_q^*
\end{array} \right. \quad (17)
\]
\[
\sum_{t=0}^{N-1} I_q(t_2) \theta^{tk} = \left\{ \begin{array}{ll}
p (\text{mod} r), & k \in \{0\} \cup pZ_p^* \\
0, & k \in Z_N^* \cup qZ_p^*
\end{array} \right. \quad (18)
\]

Lemma 6. [13]
\[
\sum_{t=0}^{N-1} I_p(t_1) I_q(t_2) \theta^{tk} = 1, 0 \leq k \leq N - 1. \quad (19)
\]

Lemma 7.
\[
\sum_{t=0}^{N-1} \eta_p(t_1) \eta_q^2(t_2) \theta^{tk} = \left\{ \begin{array}{ll}
0, & k = 0 \\
\pm (S_p - S_{N_p}), & k \in Z_N^* \\
0, & k \in pZ_q^*
\end{array} \right. \quad (20)
\]
\[
\pm (S_p - S_{N_p}) \ast (q - 1) (\text{mod} r), & k \in qZ_p^*
\end{array} \right.
\]

Proof. By Chinese Remainder Theorem, we know \( t = \frac{q}{p} \cdot t_1 + p \cdot \frac{1}{q} \cdot t_2 \pmod{pq} \), where \( \frac{1}{p} \) represents the inverse element of \( q (\text{mod} p) \), and \( \frac{1}{q} \) represents the inverse element of \( p (\text{mod} q) \). Then
\[
\sum_{t=0}^{N-1} \eta_p(t_1) \eta_q^2(t_2) \theta^{tk} = \sum_{t_1 \in Z_p^*} \eta_p(t_1) \theta^{q \cdot t_1} \sum_{t_2 \in Z_q^*} \eta_q^2(t_2) \theta^{p \cdot \frac{1}{q} \cdot t_2}
\]
\[
= \sum_{t_2 \in Z_q^*} \theta^{q \cdot t_2} \left( \sum_{t_1 \in QR_p} \alpha^{k_1} - \sum_{t_1 \in NQR_p} \alpha^{k_1} \right) \quad (21)
\]
Note that \( p \mid q \cdot p \), then \( \theta^{q \cdot p \cdot \frac{1}{q} \cdot t_2} \) is the \( p \)th primitive root of unity, denoted as \( \alpha \). Similarly, \( \theta^{q \cdot p \cdot \frac{1}{q} \cdot t_2} \) is the \( q \)th primitive root of unity, denoted as \( \beta \).
Lemma 9. The proof is omitted because \( k \) can be easily obtained by the lemmas 3–7. □

Lemma 8. Let \((S_p - S_{Np}) = \delta \) and \((S_p - S_{Np}) = \xi\). Then

\[
A'_k = \begin{cases} 
\frac{(3p+1)(q-1)}{2}, & \text{if } k = 0 \\
\frac{-2\xi - (1+k\delta)}{2}, & \text{if } k \in Z_N, \eta_p(k)\eta_q(k) = 1 \\
\frac{2\xi - (1+k\delta)}{2}, & \text{if } k \in Z_N, \eta_p(k)\eta_q(k) = -1 \\
\frac{-3p+1}{2}, & \text{if } k \in Z_N^* \\
\frac{(q-1)(1+k\delta)}{2}, & \text{if } k \in qZ_p^* 
\end{cases}
\] (22)

Proof. The proof is omitted because \( A' \) can be easily obtained by the lemmas 3–7. □

Lemma 9. [13] \( S_p \in F_r \) if and only if \( r \in QR_p; S_q \in F_r \) if and only if \( r \in QR_q \).

Theorem 1. Suppose that \( r \geq 5 \), the linear complexity of the generalized cyclotomic quaternary sequence \( S'(t) \) with period \( pq \) is calculated as follows.

If \( r \) satisfies one of two cases: \( \eta_p(r)\eta_q(r) = -1; \eta_p(r)\eta_q(r) = 1 \) and \( \pm 2\delta \xi \neq (1 \pm \delta)(\mod r) \),

(1) Then

\[
LC(S') = \begin{cases} 
pq, & \text{if } r \mid (3p+1), r \mid (q-1), r \mid (1 \pm \delta) 
pq - p + 1, & \text{if } r \mid (3p+1), r \mid (q-1), r \mid (1 \pm \delta) 
pq - p, & \text{if } r \mid (3p+1), r \mid (q-1), r \mid (1 \pm \delta) 
pq - q, & \text{if } r \mid (3p+1), r \mid (q-1), r \mid (1 \pm \delta) 
pq - p - q + 1, & \text{otherwise.} 
\end{cases}
\] (23)

If \( r \) satisfies one of two cases: \( \eta_p(r)\eta_q(r) = 1 \) and \( -2\delta \xi = (1 \pm \delta)(\mod r) \), \( \eta_p(r)\eta_q(r) = 1 \) and \( 2\delta \xi = (1 \pm \delta)(\mod r) \), Then

\[
LC(S') = \begin{cases} 
(pq + p + q - 1)/2, & \text{if } r \mid (3p+1), r \mid (q-1), r \mid (1 \pm \delta) 
(pq - p + q + 1)/2, & \text{if } r \mid (3p+1), r \mid (q-1), r \mid (1 \pm \delta) 
(pq - p + q - 1)/2, & \text{if } r \mid (3p+1), r \mid (q-1), r \mid (1 \pm \delta) 
(pq + p - q - 1)/2, & \text{if } r \mid (3p+1), r \mid (q-1), r \mid (1 \pm \delta) 
(pq - p - q + 1)/2, & \text{otherwise.}
\end{cases}
\] (24)

Proof. (1) If \( r \) meets \( \eta_p(r)\eta_q(r) = -1 \), then \( \delta \xi \in F_m \setminus F_r \), when \( k \in Z_N^* \). That is, \( \pm 2\delta \xi - (1 \pm \delta) \neq 0(\mod r) \) for \( k \in Z_N^* \).
If \( r \) meets \( \eta_p(r)\eta_q(r) = 1 \), then \( \delta_\xi \in F_r \). We know \( \pm 2\delta_\xi \neq (1 \pm \delta)(\text{mod} r) \). Easily, we get

\[
A_k^\prime = \begin{cases} 
\frac{(3p+1)(q-1)}{2}, & \text{if } k = 0 \\
\frac{\mp 2\delta_\xi}{2} - (1 \pm \delta), & \text{if } k \in \mathbb{Z}_N^* \\
\frac{-(p+1)}{2}, & \text{if } k \in p\mathbb{Z}_q^* \\
\frac{(q-1)(1 \pm \delta)}{2}, & \text{if } k \in q\mathbb{Z}_p^* 
\end{cases} \tag{25}
\]

The result is clear. (2) Similar proof is omitted. \( \square \)

4. The Linear Complexity of the Second Class of Generalized Cyclotomic Quaternary Sequences

In order to construct cyclic codes, Ding described a new generalized cyclotomy \((V_0, V_1)\), which is a new segmentation of the Ding–Helleseth generalized cyclotomy of order two [2]. By use of this cyclotomic class, Liu et al. constructed a generalized cyclotomic sequence [19]. Let the symbols and the functions be the same as before. It is easy to see that this sequence can be expressed as

\[
S_3(t) = \begin{cases} 
1, & t \in P \\
0, & t \in \{0\} \cup Q \\
\frac{1-\eta_p(t)}{2}, & t \in Z_N^* 
\end{cases} \tag{26}
\]

Define the second class of generalized cyclotomic quaternary sequence with period \( N = pq \) as \( S^\prime(t) = \psi[S_2(t), S_3(t)] \). Clearly, the sequence \( S^\prime(t) \) is different from those in references [12,15,16]. Moreover, when \( t \) ranges over \( \mathbb{Z}_N^* \), every element in \( S^\prime(t) \) takes on the same times.

According to the definitions of indicator function and quadratic character, the sequences \( S_3(t) \) can be expressed as

\[
S_3(t) = \frac{1}{2} \{ 1 - [1 - I_p(t_1)] [1 - I_q(t_2)] \eta_p(t_1) + I_p(t_1) - I_q(t_2) - I_p(t_1)I_q(t_2) \} \tag{27}
\]

Then we can derive the representation of \( S^\prime(t) \)

\[
S^\prime(t) = \psi[S_2(t), S_3(t)] = 3S_2(t) + S_3(t) - 2S_2(t)S_3(t) = \frac{1}{2} \{ 3 + I_p(t_1) - 3I_q(t_2) - 2I_p(t_1) - 2I_q(t_2) - 2I_p(t_1)I_q(t_2) - \eta_p(t_1)\eta_q(t_2) - \eta_q(t_2) + I_q(t_2)\eta_p(t_1)I_q(t_2) + I_p(t_1)\eta_p(t_1) \}
\]

The term of Fourier spectral sequence \( A^\prime = (A^\prime) \) of sequence \( S^\prime(t) \) is defined by

\[
A^\prime = \sum_{t=0}^{N-1} S^\prime(t)\theta^{tk} = \frac{1}{2} \left[ \sum_{t=0}^{N-1} \theta^{tk} + \sum_{t=0}^{N-1} I_p(t_1)\theta^{tk} - 3 \sum_{t=0}^{N-1} I_q(t_2)\theta^{tk} - 2 \sum_{t=0}^{N-1} \eta_q(t_2)\theta^{tk} 
+ 2 \sum_{t=0}^{N-1} I_p(t_1)\eta_q(t_2)\theta^{tk} - \sum_{t=0}^{N-1} I_p(t_1)I_q(t_2)\theta^{tk} - \sum_{t=0}^{N-1} \eta_p(t_1)\eta_q(t_2)\theta^{tk} \right] \tag{29}
\]

Lemma 10. [16]

\[
\sum_{t=0}^{N-1} \eta_q(t_2)\theta^{tk} = \begin{cases} 
0, & k \in \{0\} \cup p\mathbb{Z}_p^* \cup \mathbb{Z}_N^* \\
p^*_q, & k \in p\mathbb{Z}_q^*, k \in \mathbb{Q}\mathbb{R}_q \\
-p^*_q, & k \in p\mathbb{Z}_q^*, k \in \mathbb{N}\mathbb{Q}\mathbb{R}_q 
\end{cases} \tag{30}
\]
Lemma 11. [16]
\[
\sum_{t=0}^{N-1} I_p(t_1)\eta_q(t_2)\theta^t = \begin{cases}
0, & k \in \{0\} \cup q\mathbb{Z}_p^* \\
\xi, & k \in Z_N^* \cup p\mathbb{Z}_q^*, k \in \mathbb{Q}\mathbb{R}_q \\
-\xi, & k \in Z_N^* \cup p\mathbb{Z}_q^*, k \in \mathbb{N}\mathbb{Q}\mathbb{R}_q 
\end{cases}.
\] (31)

Lemma 12. Let \(2\xi (p - 1) + (3p + 1) = \sigma\), then
\[
A'' = \begin{cases}
\frac{(3p+1)(q-1)}{\xi (2 \xi - 1)}, & \text{if } k = 0 \\
\frac{\xi (2 \xi - 1)}{2}, & \text{if } k \in Z_N^*, \eta_p(k) = 1 \\
\frac{-\xi (2 \xi - 1)}{2}, & \text{if } k \in Z_N^*, \eta_p(k) = -1 \\
\frac{\sigma}{p}, & \text{if } k \in p\mathbb{Z}_q^* \\
\xi - \frac{1}{2}, & \text{if } k \in q\mathbb{Z}_p^* 
\end{cases}.
\] (32)

Proof. The proof is omitted because \(A''\) can be easily obtained by the lemmas 3–6, 10,11. \(\square\)

Theorem 2. Suppose that \(r \geq 5\), the linear complexity of generalized cyclotomic quaternary sequence \(S^\sigma(t)\) with period \(pq\) is calculated as follows.

(1) If \(r\) satisfies one of two cases:
\[
\eta_p(r)\eta_q(r) = -1; \eta_p(r)\eta_q(r) = 1 \text{ and } \pm\xi (2 \pm \delta) \neq 1(\text{mod} r).
\]
\[
LC(S^\sigma) = \begin{cases}
pq, & \text{if } r \nmid (3p + 1), r \nmid (q - 1), r \nmid \sigma \\
pq - q + 1, & \text{if } r \nmid (3p + 1), r \nmid (q - 1), r \nmid \sigma \\
pq - p - q + 1, & \text{if } r \mid (q - 1), r \nmid \sigma \\
pq, & \text{if } r \nmid (q - 1), r \nmid \sigma \\
pq - 1, & \text{if } r \mid (3p + 1), r \nmid (q - 1), r \nmid \sigma \\
pq - q, & \text{if } r \mid (3p + 1), r \nmid (q - 1), r \nmid \sigma
\end{cases}
\] (33)

(2) If \(r\) satisfies cases:
\[
\eta_p(r)\eta_q(r) = 1 \text{ and } \pm\xi (2 \pm \delta) = 1(\text{mod} r).
\]
\[
LC(S^\sigma) = \begin{cases}
(pq + p + q - 1)/2, & \text{if } r \mid (3p + 1), r \mid (q - 1), r \mid \sigma \\
(pq + p + q - 1)/2, & \text{if } r \mid (3p + 1), r \nmid (q - 1), r \nmid \sigma \\
(pq - p - q + 1)/2, & \text{if } r \mid (q - 1), r \nmid \sigma \\
(pq - p - q + 1)/2, & \text{if } r \nmid (q - 1), r \nmid \sigma \\
(pq + p + q - 3)/2, & \text{if } r \nmid (3p + 1), r \mid (q - 1), r \nmid \sigma \\
(pq + p + q - 3)/2, & \text{if } r \nmid (3p + 1), r \nmid (q - 1), r \nmid \sigma
\end{cases}
\] (34)

Proof. (1) If \(r\) meets \(\eta_p(r)\eta_q(r) = -1\), then \(\delta\xi \in \mathbb{F}_m \setminus \mathbb{F}_r\), when \(k \in Z_N^*\). That is, \(\pm\xi (2 \pm \delta) \neq 1(\text{mod} r)\) for \(k \in Z_N^*\). If \(r\) meets \(\eta_p(r)\eta_q(r) = 1\), then \(\delta\xi \in \mathbb{F}_r\). So, \(\pm\xi (2 \pm \delta) \neq 1(\text{mod} r)\). Easily, we get
\[
A'' = \begin{cases}
\frac{(3p+1)(q-1)}{2}, & \text{if } k = 0 \\
\frac{\pm\xi (2 \xi - 1)}{2}, & \text{if } k \in p\mathbb{Z}_q^* \\
\frac{\pm\xi}{2}, & \text{if } k \in q\mathbb{Z}_p^* \\
\frac{\xi - 1}{2}, & \text{if } k \in Z_N^*
\end{cases}.
\] (35)

The result is clear.

(2) Similar proof is omitted. \(\square\)
5. Conclusions

Pseudorandom sequences with period $pq$ have been taken seriously, as $pq$ is the RSA modulus, which involves the complex problem of large integer factorization. This paper constructs two classes of new generalized cyclotomic quaternary sequences with period $pq$ over $\mathbb{Z}_4$ by choosing different kinds of generalized cyclotomic binary sequence pairs, and investigates the linear complexity respectively by counting the number of nonzero terms of their Fourier spectral sequence. More quaternary pseudorandom sequences can be constructed according to this idea. We estimate that most of them have large linear complexity, and some of them may have low autocorrection.

In view of symmetry, we suppose that $p < q$. The results show that, the first class of the generalized cyclotomic quaternary sequence has lower linear complexity only if $\eta_p(r)\eta_q(r) = 1, \pm2\delta^2 = (1 \pm \delta)(\text{mod } r)$ and $r|(3p + 1)$; the second one has lower linear complexity only if $\eta_p(r)\eta_q(r) = 1, \pm\delta(2 \pm \delta) = 1(\text{mod } r)$ and $r|\pm\sigma$. In other cases, the linear complexity of the two classes of quaternary sequences is greater than half of the period. Therefore, the two classes of the new sequences in this paper have a large linear complexity in resisting the attack of the Berlekamp–Massey algorithm. Compared with references [12,15,16], the linear complexity of the quaternary sequences constructed in this paper have more values, which make them adapt to more kinds of Linear feedback shift register with different orders. The next step planned is to study the autocorrelation of the two classes of the new quaternary sequences.

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