CONTINUA IN THE GROMOV–HAUSDORFF SPACE

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Abstract. We first prove that for all compact metrizable spaces, there exists a topological embedding of the compact metrizable space into each of the sets of compact metric spaces which are connected, path-connected, geodesic, or CAT(0), in the Gromov–Hausdorff space with finite prescribed values. As its application, we show that the sets prescribed above are path-connected and their non-empty open subsets have infinite topological dimension. By the same method, we also prove that the set of all proper CAT(0) spaces is path-connected and its non-empty open subsets have infinite topological dimension with respect to the pointed Gromov–Hausdorff distance.

1. Introduction

In [8], the author constructed uncountable (cardinality of the continuum) branching geodesics of the Gromov–Hausdorff distance continuously parameterized by a Hilbert cube, passing through or avoiding sets of all spaces satisfying some of the doubling property, the uniform disconnectedness, the uniform perfectness, and passing through sets of all infinite-dimensional spaces and the set of all metric spaces homeomorphic to the Cantor set. This construction implies that the sets described above contain a Hilbert cube as a topological subspace, and hence they have infinite topological dimension with respect to the Gromov–Hausdorff distance.

In [9], by constructing topological embeds of compact metrizable spaces into the Gromov–Hausdorff space, the author proved that the set of all compact metrizable spaces possessing prescribed topological dimension, Hausdorff dimension, packing dimension, upper box dimension, and Assouad dimension, and the set of all compact ultrametric spaces are path-connected and have infinite topological dimension.

In this paper, as a development of the author’s papers [8] and [9], we prove that the sets of all compact metric spaces which are connected, path-connected, geodesic, or CAT(0) have infinite topological dimension as subsets of the Gromov–Hausdorff space, and we also prove that the set of proper CAT(0) spaces has infinite topological dimension with respect to the pointed Gromov–Hausdorff distance. In this paper, the

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topological dimension means the covering dimension. Since we only consider separable metric spaces, the topological dimension in this paper coincides with the large and small inductive dimensions. For the details of dimensions of topological spaces, we refer the readers to [6, 13, 12, and 4].

Let \((X, d)\) be a metric space. In this paper, for \(x, y \in X\), a map \(\gamma : [0, 1] \to X\) is said to be a geodesic from \(x\) to \(y\) if \(\gamma(0) = x\) and \(\gamma(1) = y\), and for all \(s, t \in [0, 1]\) we have \(d(\gamma(s), \gamma(t)) = |s - t| \cdot d(x, y)\). Note that if there exists a curve from \(x\) to \(y\) whose length is \(d(x, y)\), then there exists a geodesic from \(x\) to \(y\) (see [3, Chapter 2]). A metric space is said to be a geodesic space if, for all two points, there exists a geodesic connecting them. A geodesic space \((X, d)\) is said to be a CAT(0) space if, for all geodesic triangles \(\triangle\) in \((X, d)\) and for all \(x, y \in \triangle\), we have \(d(x, y) \leq d_{\mathbb{R}^2}(x, y)\), where \(d_{\mathbb{R}^2}\) is the 2-dimensional Euclidean metric and \(x, y\) are comparison points of \(x, y\) in a comparison triangle \(\overline{\triangle}\) in \(\mathbb{R}^2\) of \(\triangle\). For the details of CAT(0) spaces, we refer the readers to [2]. For a metric space \((Z, h)\), and for subsets \(A, B\) of \(Z\), we define the Hausdorff distance \(\mathcal{HD}(A, B; Z, h)\) of \(A\) and \(B\) in \((Z, h)\) by the infimum of all \(r \in (0, \infty)\) such that for all \(a \in A\) and \(b \in B\) there exist \(u \in A\) and \(v \in B\) with \(h(a, v) \leq r\) and \(h(b, u) \leq r\). For metric spaces \((X, d)\) and \((Y, e)\), the Gromov-Hausdorff distance \(\mathcal{GH}((X, d), (Y, e))\) between \((X, d)\) and \((Y, e)\) is defined as the infimum of all values \(\mathcal{HD}(i(X), j(Y); Z, h)\), where \((Z, h)\) is a metric space, and \(i : X \to Z\) and \(j : Y \to Z\) are isometric embeddings. We denote by \(\mathcal{M}\) the set of all isometry classes of non-empty compact metric spaces. The space \((\mathcal{M}, \mathcal{GH})\) is called the Gromov–Hausdorff space. By abuse of notation, we represent an element of \(\mathcal{M}\) as a pair \((X, d)\) of a set \(X\) and a metric \(d\) rather than its isometry class. We denote by \(\mathcal{C}, \mathcal{P}, \mathcal{G}, \mathcal{Z}\), and \(\mathcal{PZ}\) the subset of \(\mathcal{M}\) consisting of all compact metric spaces which are connected, path-connected, geodesic, and CAT(0), respectively.

Similarly to [9], by constructing a topological embedding of compact metrizable spaces, we prove that \(\mathcal{C}, \mathcal{P}, \mathcal{G}, \mathcal{Z}\), and \(\mathcal{PZ}\) are path-connected and everywhere infinite-dimensional.

Our first main result is the next theorem, which is an analogue of [9 Theorem 1.3]. In the proof of [9, Theorem 1.3], the author used a construction of metrics on the direct sum spaces since that theorem treats properties preserved by the direct sum. We can not apply that method using the direct sum to the class of continua since the direct sum of continua is not connected, especially, it is not a continuum. In contrast to my previous paper [9], to prove Theorem 1.1, we use metrics on the product spaces. We will prove Theorem 1.1 in Section 3.

**Theorem 1.1.** Let \(n \in \mathbb{Z}_{\geq 1}\). Let \(H\) be a compact metrizable space, and \(\{v_i\}_{i=1}^{n+1}\) be \(n + 1\) different points in \(H\). Let \(S\) be any one of \(\mathcal{C}, \mathcal{P}, \mathcal{G}, \mathcal{Z}\). Let \(\{(X_i, d_i)\}_{i=1}^{n+1}\) be a sequence in \(S\) satisfying that

\[
\forall i, j \leq n + 1, \quad d_i(v_k) = d_j(v_k) = d_k(v_k)
\]

and

\[
\forall i, j \leq n + 1, \quad \mathcal{G}(\{(X_i, d_i)\}_{i=1}^{n+1}, (X_j, d_j)) < \infty
\]
\[ \mathcal{GH}((X_i, d_i), (X_j, d_j)) > 0 \text{ for all distinct } i, j. \text{ Then, there exists a topological embedding } \Phi : H \to S \text{ such that } \Phi(v_i) = (X_i, d_i). \]

Applying Theorem 1.1 to \( H = [0, 1]^{\aleph_0} \), by the fact that all non-empty open subsets of \([0, 1]^{\aleph_0}\) have infinite topological dimension, we obtain:

**Corollary 1.2.** The sets \( \mathcal{C}, \mathcal{P}, \mathcal{G}, \) and \( \mathcal{Z} \) are path-connected and all their non-empty open subsets have infinite topological dimension.

**Remark 1.1.** The sets \( \mathcal{C}, \mathcal{G}, \) and \( \mathcal{Z} \) are closed and nowhere dense in \( \mathcal{M} \); however, \( \mathcal{P} \) is not closed. For example, the so-called topologist’s sine curve \( \{(x, y) \in \mathbb{R}^2 \mid x \in (0, 1], y = \sin(1/x)\} \cup \{0\} \times [0, 1] \) is a connected and non-path-connected compact metric space, which is a limit of path-connected spaces \( X_n = \{(x, y) \in \mathbb{R}^2 \mid x \in [2^{-n}, 1], y = \sin(1/x)\} \) with respect to \( \mathcal{GH} \).

Our second result is a non-compact analogue of Theorem 1.1. We construct topological embeddings of compact metrizable spaces into the space of the pointed proper metric space. A metric space is said to be **proper** if all its bounded closed subsets are compact. A pair of a metric space and its point is called a **pointed metric space**. We denote by \( \mathcal{PM} \) the set of pointed isometry classes of pointed proper metric spaces. We represent elements of \( \mathcal{PM} \) in a similar way to \( \mathcal{M} \). Let \( (X, d) \) be a metric space. For \( x \in X \) and for \( r \in (0, \infty) \), we denote by \( B_d(x, r) \) the closed ball centered at \( x \) with radius \( r \). For a subset \( A \) of \( X \) and \( r \in (0, \infty) \), we denote by \( N_d(A, r) \) the set of all \( x \in X \) such that there exists \( a \in A \) with \( d(x, a) < r \). Let \( t \in (0, \infty) \). Let \( (X, d, a) \) and \( (Y, e, b) \) be pointed metric spaces. We denote by \( X \sqcup Y \) the direct sum of \( X \) and \( Y \). A metric \( h \) on \( X \sqcup Y \) is said to be **\((t; a, b)\)-admissible** if \( h|_{X^2} = d \) and \( h|_{Y^2} = e \) and \( h(a, b) < t \) and \( B_h(a, t^{-1}) \subset N_h(Y, t) \) and \( B_h(b, t^{-1}) \subset N_h(X, t) \). We define a quantity \( \mathcal{GH}^*((X, d, a), (Y, e, b)) \) by the infimum of all \( t \in (0, \infty) \) such that there exists a \((t; a, b)\)-admissible metric \( h \) on \( X \sqcup Y \). We also define \( \mathcal{GH}^*((X, d, a), (Y, e, b)) \) by

\[
\mathcal{GH}^*((X, d, a), (Y, e, b)) = \min \left\{ \mathcal{GH}^*((X, d, a), (Y, e, b)), \frac{1}{2} \right\}.
\]

The function \( \mathcal{GH}^* \) is a metric on \( \mathcal{PM} \) (see [5, Corollary 3.14]). Note that the convergence with respect to \( \mathcal{GH}^* \) is equivalent to the ordinary pointed Gromov–Hausdorff convergence (see [5, Proposition 3.5]). The metric \( \mathcal{GH}^* \) is called the **pointed Gromov–Hausdorff distance**.

Our second main result is the next theorem, which will be proven in Section 4. We denote by \( \mathcal{PZ} \) the set of all \( \text{CAT}(0) \) spaces in \( \mathcal{PM} \).

**Theorem 1.3.** Let \( n \in \mathbb{Z}_{>1} \). Let \( H \) be a compact metrizable space, and \( \{v_i\}_{i=1}^{n+1} \) be \( n + 1 \) different points in \( H \). Let \( \{(X_i, d_i, a_i)\}_{i=1}^{n+1} \) be a sequence in \( \mathcal{PZ} \) such that \( \mathcal{GH}^*((X_i, d_i, a_i), (X_j, d_j, a_j)) > 0 \) for all distinct \( i, j \). Then, there exists a topological embedding \( \Phi : H \to \mathcal{PZ} \) such that \( \Phi(v_i) = (X_i, d_i, a_i) \).
Corollary 1.4. The set $\mathcal{P}\mathbb{Z}$ is path-connected and all its non-empty open subsets have infinite topological dimension.

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2. Metric spaces parametrized by a Hilbert space

A topological space is said to be a Hilbert cube if it is homeomorphic to the countable power of the closed unit interval $[0, 1]$ of $\mathbb{R}$. To prove Theorems 1.1 and 1.3 in this section, we construct a family of metric trees injectively and continuously parametrized by a Hilbert cube.

For a set $X$, a map $d : X \times X \rightarrow [0, \infty)$ is said to be a pseudo-metric if $d$ satisfies the triangle inequality and satisfies that $d(x, x) = 0$ and $d(x, y) = d(y, x)$ for all $x, y \in X$. If a pseudo-metric $d$ satisfies that $d(x, y) = 0$ implies $x = y$, then $d$ is a metric. We denote by PMet($X$) the set of all pseudo-metrics on $X$. We define a metric $D_X$ on PMet($X$) by $D_X(d, e) = \sup_{x, y \in X} |d(x, y) - e(x, y)|$. Note that $D_X$ can take the value $\infty$. Let $d \in$ PMet($X$). We denote by $X/d$ the quotient set by the relation $\sim_d$ defined by $x \sim_d y$ if and only if $d(x, y) = 0$. We denote by $[x]_d$ the equivalence class of $x$ by $\sim_d$. We define a metric $[d]$ on $X/d$ by $[d]([x]_d, [y]_d) = d(x, y)$. The metric $[d]$ is well-defined. Remark that if $d$ is a metric, then $(X/d, [d])$ is isometric to $(X, d)$. The following was proven in [9] Corollary 2.13.

Proposition 2.1. Let $T$ be a topological space whose all finite subsets are closed. Let $X$ be a set. If a map $h : T \rightarrow$ PMet($X$) is continuous and $h(t)$ is a metric for all $t \in T$ except finite points, then the map $F : T \rightarrow \mathcal{M}$ defined by $F(t) = (X_{h(t)}, [h(t)])$ is continuous.

A family of metric trees defined in this section is a connected analogue of a family of compact metric spaces homeomorphic to the one-point compactification of the countable discrete space, which is constructed in [8] Definition 4.1 and [9] Definition 4.1.

Definition 2.1. We define $\mathcal{C} = \prod_{i=1}^{\infty} [2^{-2i}, 2^{-2i+1}]$. Note that every $a = \{a_i\}_{i \in \mathbb{Z}_{\geq 1}} \in \mathcal{C}$ satisfies $a_i < 1$ and $a_{i+1} < a_i$ for all $i \in \mathbb{Z}_{\geq 1}$ and $\lim_{i \rightarrow \infty} a_i = 0$. We define a metric $\tau$ on $\mathcal{C}$ by $\tau(x, y) = \sup_{i \in \mathbb{Z}_{\geq 1}} |x_i - y_i|$. Then, $\tau$ generates the topology which makes $\mathcal{C}$ a Hilbert cube.

Definition 2.2. Let $a = \{a_i\}_{i \in \mathbb{Z}_{\geq 1}} \in \mathcal{C}$. We supplementally put $a_0 = 1$. Put $\Upsilon = \{(0, 0)\} \cup (0, 1] \times \mathbb{Z}_{\geq 0}$. To simplify our description, we represent an element $(s, i)$ of $\Upsilon$ as $s_i$. For example, $0_0 = (0, 0)$, $1_n = (1, n)$, and $(1/2)_3 = (1/2, 3)$. We define a metric $R[a]$ on $\Upsilon$ by

$$R[a](s_i, t_j) = \begin{cases} a_i|s - t| & \text{if } i = j; \\ a_is + a_jt & \text{otherwise.} \end{cases}$$
Note that \((\Upsilon, R[a])\) is compact and the space \((\Upsilon, R[a])\) can be considered as a metric subtree of the spider tree which is the plane equipped with the radial metric (see [1, Examples 1.6 and 1.8]). The metric \(R[a]\) is constructed in a similar way to [1, Examples 1.6 and 1.8] with dilated edges, and each \(a_i\) is the scaling factor of the \(i\)-th edge. Since all metric trees are CAT(0) (see [2, (5) in Example 1.15, p.167]), the space \((\Upsilon, R[a])\) is a CAT(0) space. Note that even if \(a \neq b\), the metrics \(R[a]\) and \(R[b]\) generate the same topology on \(\Upsilon\).

**Proposition 2.2.** Let \(a = \{a_i\}_{i \in \mathbb{Z}_{\geq 1}}\) and \(b = \{b_i\}_{i \in \mathbb{Z}_{\geq 1}}\) be in \(C\). Let \(K, L \in (0, \infty)\). If \((\Upsilon, K \cdot R[a])\) and \((\Upsilon, L \cdot R[b])\) are isometric to each other, then \(a = b\).

**Proof.** Let \(f : (\Upsilon, K \cdot R[a]) \rightarrow (\Upsilon, L \cdot R[b])\) be an isometry. The point \(0_0\) of \(\Upsilon\) is the unique point such that \(\Upsilon \setminus \{0_0\}\) has infinitely many connected components. Therefore \(f(0_0) = 0_0\). Let \(A\) be the set of all points \(p\) in \(\Upsilon\) such that \(\Upsilon \setminus \{p\}\) is connected. Then \(A = \{1_i \mid i \in \mathbb{Z}_{\geq 0}\}\), and \(f(A) = A\). We put \(A_0 = A\) and \(A_n = A \setminus \{1_i \mid i = 0, 1, \ldots, n - 1\}\) for each \(n \in \mathbb{Z}_{\geq 1}\). Then, for each \(n \in \mathbb{Z}_{\geq 0}\), the point \(1_n \in \Upsilon\) is characterized as the unique argument of the maximum of the map \(p \in A_n \mapsto R[a](p, 0_0)\), i.e., the point \(1_n\) is the unique point satisfying that \(R[a](1_n, 0_0) = \max_{p \in A_n} R[a](p, 0_0)\). Since \(f\) is an isometry, by induction, \(f(1_n) = 1_n\) for all \(n \in \mathbb{Z}_{\geq 0}\). Thus, for each \(n \in \mathbb{Z}_{\geq 0}\), we have

\[
K \cdot a_n = K \cdot R[a](1_n, 0_0) = L \cdot R[b](1_n, 0_0) = L \cdot b_n,
\]

and hence \(K a_n = L b_n\) for all \(n \in \mathbb{Z}_{\geq 0}\). By \(a_0 = b_0 = 1\), we have \(K = L\). Therefore, we conclude that \(a = b\). \(\square\)

**Proposition 2.3.** For all \(a, b \in C\), we have \(D_\tau(R[a], R[b]) \leq 2\tau(a, b)\).

**Proof.** Let \(s_i, t_j \in \Upsilon\). If \(i = j\), then we have

\[
|R[a](s_i, t_j) - R[b](s_i, t_j)| = |a_i - b_i||s - t| \leq |a_i - b_i| \leq \tau(a, b).
\]

If \(i \neq j\), then we have

\[
|R[a](s_i, t_j) - R[b](s_i, t_j)| \leq |a_i - b_i|s + |a_j - b_j|t \leq 2\tau(a, b).
\]

This finishes the proof. \(\square\)

3. **Topological embeddings**

In this section, we prove Theorem [11]. Before doing that, we prepare and review the basic constructions and properties of metrics.

3.1. **Amalgamation and product of metrics.**

**Proposition 3.1.** Let \(X\) and \(Y\) be sets. Let \(d \in \text{PMet}(X)\) and \(e \in \text{PMet}(Y)\). Assume that there exists a point \(p\) such that \(X \cap Y = \{p\}\).
We define a symmetric function \( h : (X \cup Y)^2 \to [0, \infty) \) by

\[
h(x, y) = \begin{cases} 
  d(x, y) & \text{if } x, y \in X; \\
  e(x, y) & \text{if } x, y \in Y; \\
  d(x, p) + e(p, y) & \text{if } (x, y) \in X \times Y.
\end{cases}
\]

Then, the following statements hold true.

1. The function \( h \) is a pseudo-metric and satisfies \( h|_{X^2} = d \) and \( h|_{Y^2} = e \).
2. If \( d \) and \( e \) are metrics, then so is \( h \).
3. If \((X, d)\) and \((Y, e)\) are geodesic (resp. CAT(0)) metric spaces, then so is \((X \cup Y, h)\).

Proof. The statements (1) and (2) in the proposition are deduced from [7, Proposition 3.2]. The statement on CAT(0) spaces in (3) follows from [2, Theorem 11.1, p.347]. It suffices to show that, if \((X, d)\) and \((Y, e)\) is geodesic spaces, for all \( x \in X \setminus \{p\} \) and \( y \in Y \setminus \{p\} \) there exists a geodesic between \( x \) and \( y \). By the assumption, there exists a geodesic from \( x \) to \( p \) and a geodesic from \( p \) to \( y \). By the definition of \( h \), we can apply [11, Proposition 2.6] to these two geodesics, and we can glue these two geodesics together at the point \( p \). Then, we obtain a geodesic between \( x \) and \( y \) (see also [2, Lemma 5.24, p.67]).

For two metric spaces \((X, d)\) and \((Y, e)\), we denote by \( d \times e \) the \( \ell^2 \)-product metric defined by \((d \times e)((x, y), (u, v)) = \sqrt{d(x, u)^2 + e(y, v)^2}\).

The next lemma can be found in [2, (3) in Example 1.15, p.167].

**Lemma 3.2.** Let \((X, d)\) and \((Y, e)\) be metric spaces. If \((X, d)\) and \((Y, e)\) are geodesic spaces (resp. CAT(0) spaces), then so is \((X \times Y, d \times e)\).

By the definition of CAT(0) spaces, we obtain:

**Lemma 3.3.** Let \( L \in (0, \infty) \). If \((X, d)\) is a geodesic space (resp. CAT(0) space), then so is \((X, L \cdot d)\).

### 3.2. Arcwise-connectedness.

We say that a subset of a topological space is a topological arc if it is homeomorphic to \( \mathbb{R} \). A topological space \( X \) is said to be arcwise-connected if for all two points \( p, q \in X \) with \( p \neq q \), there exists a topological embedding \( f : [0, 1] \to X \) with \( f(0) = p \) and \( f(1) = q \). The following is deduced from [14, Corollary 31.6], and related to the Hahn–Mazurkiewicz theorem.

**Lemma 3.4.** A Hausdorff topological space is path-connected if and only if it is arcwise-connected. In particular, if a Hausdorff space \( X \) has at least two points and \( X \) is a continuous image of \( \mathbb{R} \), then \( X \) contains a topological arc as a subspace.

For a topological space \( X \), we denote by \( \dim_T X \) the topological dimension of \( X \). To construct topological embeddings of compact metrizable spaces into the Gromov–Hausdorff space (especially, to guarantee injectivity), we use the following lemma.
Lemma 3.5. Let $X$ and $Y$ be Hausdorff topological spaces possessing no isolated points. Then any topological arc in the product space $X \times Y$ has no interior points.

Proof. For the sake of contradiction, we suppose that there exists a subset $V$ of $X \times Y$ which is a topological arc with non-empty interiors. Since every point of $\mathbb{R}$ has a neighborhood system consisting of open intervals, by extracting an open subset if necessary, we may assume that $V$ is open in $X \times Y$.

Take a point $(p, q) \in V$. Then, there exists open subsets $A$, $B$ of $X$, $Y$, respectively, with $p \in A$ and $q \in B$ and $A \times B \subset V$. Since $V$ is a topological arc and open in $X \times Y$, there exists an open subset $W$ of $X \times Y$ such that $W$ is a topological arc, and $(p, q) \in W$, and $W \subset A \times B$. Since $X$ and $Y$ have no isolated points, and since the projections $\pi_X$ and $\pi_Y$ are open maps, the sets $\pi_X(W)$ and $\pi_Y(W)$ contain at least two points. Thus, by Lemma 3.4, they contain topological arcs. By this observation, and by $\dim_T \mathbb{R}^2 = 2$, we have $2 \leq \dim_T (\pi_X(W) \times \pi_Y(W))$. By this inequality and $\pi_X(W) \times \pi_Y(W) \subset A \times B$, we have $2 \leq \dim_T V$. This contradicts $\dim_T V = \dim_T \mathbb{R} = 1$. □

3.3. Topological embeddings. For every $n \in \mathbb{Z}_{\geq 1}$, we denote by $\hat{n}$ the set $\{1, \ldots, n\}$. In what follows, we consider that the set $\hat{n}$ is always equipped with the discrete topology.

The next proposition is an analogue of [9, Proposition 4.4]. Unlike the proof of [9, Proposition 4.4], we now use a construction of metrics on product spaces.

Proposition 3.6. Let $S$ be any one of $\mathcal{C}$, $\mathcal{P}$, $\mathcal{G}$, and $\mathcal{Z}$. Let $n \in \mathbb{Z}_{\geq 1}$ and $m \in \mathbb{Z}_{\geq 2}$. Let $H$ be a compact metrizable spaces, and $\{v_i\}_{i=1}^{n+1}$ be $n+1$ many different points in $H$. Put $H^\times = H \setminus \{v_i \mid i = 1, \ldots, n+1\}$. Let $\{(X_i, d_i)\}_{i=1}^{n+1}$ be a sequence of compact metric spaces in $S$ satisfying that $\mathcal{G}(\{(X_i, d_i), (X_j, d_j)\}) > 0$ for all distinct $i, j$. Then there exists a continuous map $F : H \times \hat{m} \to S$ such that

1. for all $i \in \hat{n} + 1$ and $j \in \hat{m}$ we have $F(v_i, j) = (X_i, d_i)$;
2. for all $(s, i), (t, j) \in H^\times \times \hat{m}$ with $(s, i) \neq (t, j)$, we have $F(s, i) \neq F(t, j)$.

Proof. Since the case where $n = 1$ and either $X_1$ or $X_2$ is the one-point metric space follows from cases of $n > 1$, we may assume that $n > 1$, or $n = 1$ and both $X_1$ and $X_2$ contain at least two points.

We denote by $\Lambda(H, m)$ the quotient space of $H \times \hat{m}$ in a such a way that for each $i \in \hat{n} + 1$ we identify $m$ many points $\{(v_i, j) \mid j \in \hat{m}\}$ as a single point. Since $\Lambda(H, m)$ is compact and metrizable, there exists a topological embedding $\rho : \Lambda(H, m) \to \mathcal{C}$ (this is the Urysohn Metrization Theorem, see [10]). Let $\pi_{H, m} : H \times \hat{m} \to \Lambda(H, m)$ be the canonical projection. Since every metrizable space is perfectly normal
(see [13, Proposition 4.18]), for each $i \in \overline{n+1}$, there exists a continuous function $\zeta_i : H \to [0, 1]$ such that $\zeta_i^{-1}(0) = \{v_j \mid j \neq i\}$ and $\zeta_i^{-1}(1) = \{v_i\}$, and there exists a continuous function $\xi : H \to [0, 1]$ with $\xi^{-1}(0) = \{v_i \mid i = 1, \ldots, n + 1\}$. We define $P = \prod_{i=1}^{n+1} X_i$. For each $s \in H$, we define a pseudo-metric $E_s$ on $P$ by

$$E_s(x, y) = \sqrt{\sum_{i=1}^{n+1} (\zeta_i(s) \cdot d_i(x_i, y_i))^2}.$$ 

Note that $E_s$ is a metric if and only if $s \in H^x$, and note that if $s \in H^x$, the metric $E_s$ generates the product topology on $P = \prod_{i=1}^{n+1} X_i$. Since each $X_i$ is compact, the map $W : H \to \text{PMet}(P)$ defined by $W(s) = E_s$ is continuous. By the fact that the connectedness and path-connectedness are invariant under the topological product, and by Lemmas 3.2 and 3.3 if $s \in H^x$, the metric space $(P, E_s)$ is in $\mathcal{S}$. Take $p \in P$. We identify $p$ with $1_0 \in \Upsilon$, and we consider $P \cap \Upsilon = \{p\}$. We put $Z = P \cup \Upsilon$. For each $(s, k) \in H \times \hat{m}$, we define a symmetric function $D_{s,k}$ on $Z \times Z$ by

$$D_{s,k}(x, y) = \begin{cases} E_s(x, y) & \text{if } x, y \in P; \\ \xi(s)R[p \circ \pi_{H,m}(s, k)][x, y] & \text{if } x, y \in \Upsilon; \\ E_s(x, p) + \xi(s)R[p \circ \pi_{H,m}(s, k)](p, y) & \text{if } (x, y) \in P \times \Upsilon. \end{cases}$$

Proposition 3.1 implies that $D_{s,k}$ is a pseudo-metric on $Z$ for all $(s, k) \in H \times \hat{m}$. We see that $D_{s,k}$ is a metric if and only if $s \in H^x$. We also see that for all $i \in \overline{n+1}$ and $k \in \hat{m}$, the quotient metric space $\left(Z_{/D_{s,k}}, [D_{s,k}]\right)$ is isometric to $(X, d_i)$. Note that if $s \neq v_i$ for any $i \in \overline{n+1}$, the quotient metric space $\left(Z_{/D_{s,k}}, [D_{s,k}]\right)$ is isometric to $(Z, D_{s,k})$. By Proposition 3.1 we have $(Z, D_{s,k}) \in \mathcal{S}$ for all $(s, k) \in H^x \times \hat{m}$. We define $F : H \times \hat{m} \to \mathcal{S}$ by

$$F(s, k) = \begin{cases} (X, d_i) & \text{if } s = v_i \text{ for some } i \in \overline{n+1}; \\ (Z, D_{s,k}) & \text{otherwise}. \end{cases}$$

Then the condition (1) is satisfied. Since the map $W : H \to \text{PMet}(P)$ is continuous, by the definition of $D_{s,k}$, and by Proposition 2.3 the map $J : H \times \hat{m} \to \text{PMet}(Z)$ defined by $J(s, k) = D_{s,k}$ is continuous. Therefore, by Proposition 2.1 the map $F$ is continuous.

Next we prove the condition (2). For a metric space $(S, h)$, we denote by $A(S, h)$ the closure of the set of all points possessing neighborhood systems consisting of topological arcs. Note that if metric spaces $(S, h)$ and $(S', h')$ are isometric to each other, then so are $A(S, h)$ and $A(S', h')$. From the assumption that $n > 1$, or $n = 1$ and both $X_1$ and $X_2$ contain at least two points, it follows that $P$ can be represented as the product of two connected spaces possessing at least two points.
By this observation and by Lemma 3.5, we have \( A(P, E_s) = \emptyset \) for all \( s \in H^\times \). Since all points in \( \Upsilon \setminus \{0_0\} \cup \{1_i : i \in \mathbb{Z}_{\geq 0}\} \) have a neighborhood systems consisting of topological arcs, we have \( A(\Upsilon, R(a)) = \Upsilon \) for all \( a \in C \). Thus, by the definition of \( D_{s,k} \), we have \( A(Z, D_{s,k}) = \Upsilon \) for all \( (s, k) \in H^\times \times \hat{m} \). This implies that the metric subspace \( A(Z, D_{s,k}) \) of \( (Z, D_{s,k}) \) is isometric to \( (\Upsilon, \xi(s) \cdot R[\rho \circ \pi_{H,m}(s, k)]) \) for all \( (s, k) \in H^\times \times \hat{m} \). Since \( A \) is isometrically invariant, and since \( \pi_{H,m} \) is injective on \( H^\times \times \hat{m} \), Proposition 2.2 implies that the condition (2) is satisfied. This finishes the proof. 

The proof of Theorem 1.1 is similar to [9, Theorem 1.3].

Proof of Theorem 4.4. Let \( \{(X_i, d_i)\}_{i=1}^{n+1} \) be a sequence of metric spaces in \( S \) such that \( \mathcal{GH}((X_i, d_i), (X_j, d_j)) > 0 \) for all distinct \( i, j \). Put \( m = n + 2 \). Let \( F : H \times \hat{m} \to S \) be a map stated in Proposition 3.6. For all \( i \in [n+1] \) and \( j \in \hat{m} \), we define \( S(i, j) = \{ s \in H^\times : F(s, j) = (X_i, d_i) \} \). By the conditions (1) and (2) in Proposition 3.6 for all \( i \in \hat{m} \), the set \( \bigcup_{i=1}^{m} S(i, j) \) is empty or a singleton. Thus, the number of pairs \( (i, j) \in n+1 \times \hat{m} \) such that \( \text{Card}(S(i, j)) > 0 \) is at most \( n+1 \). By \( m = n + 2 \), we obtain \( \hat{m} \setminus \bigcup_{i=1}^{n+1} \{ j \in \hat{m} : \text{Card}(S(i, j)) > 0 \} \neq \emptyset \), and we can take \( k \) from this set. Then \( \{(X_i, d_i) : i = 1, \ldots, n+1\} \cap F(H^\times \times \{k\}) = \emptyset \). Therefore, the function \( \Phi : H \to S \) defined by \( \Phi(s) = F(s, k) \) is injective, and hence \( \Phi \) is a topological embedding since \( H \) is compact. This completes the proof.

4. The pointed Gromov–Hausdorff space

In this section, we prove Theorem 4.3. In what follows, for a metric space \((X, d)\), we put \( B_d(x, 0) = \{x\} \) and \( B_d(x, \infty) = X \). We first define a rough isometry, which gives a way to estimate \( \mathcal{GH}_{-} \).

Let \((X, d, a)\) and \((Y, e, b)\) be pointed metric spaces. Let \( R \in (0, \infty] \) and \( \varepsilon \in (0, \infty) \) with \( R > \varepsilon \). A map \( f : B_d(a, R) \to Y \) (or \( f : X \to Y \)) is said to be an \((R, \varepsilon)\)-rough isometry from \((X, d, a)\) to \((Y, e, b)\) if the following conditions are satisfied:

1. \( \varepsilon(f(a), b) \leq \varepsilon \).
2. \( B_e(b, R - \varepsilon) \subseteq N_e(f(B_d(a, R)), \varepsilon) \).
3. For all \( x, y \in B_d(a, R) \), we have \( |d(x, y) - \varepsilon(f(x), f(y))| \leq \varepsilon \).

Remark that our definition of a rough isometry is the same as the definition in [5] except that our one is limited to balls and contains the parameter of radius \( R \). The following is deduced from [5, Lemma 3.4].

Lemma 4.1. Let \((X, d, a)\) and \((Y, e, b)\) be pointed metric spaces. Let \( R \in (0, \infty] \) and \( \varepsilon \in (0, \infty) \) with \( R > \varepsilon \). If there exists an \((R, \varepsilon)\)-rough isometry \( f : B_d(a, R) \to Y \) from \((X, d, a)\) to \((Y, e, b)\), then we have

\[
\mathcal{GH}_{-}((X, d, a), (Y, e, b)) < \max \left\{ 2\varepsilon, \frac{1}{R - \varepsilon} \right\}.
\]
The next proposition is an analogue of Proposition 3.6.

**Proposition 4.2.** Let \( n \in \mathbb{Z}_{\geq 1} \) and \( m \in \mathbb{Z}_{\geq 2} \). Let \( H \) be a compact metrizable space, and \( \{v_i\}_{i=1}^{n+1} \) be \( n+1 \) different points in \( H \). Put \( H^\infty = H \setminus \{v_i \mid i = 1, \ldots, n+1\} \). Let \( \{(X_i, d_i, a_i)\}_{i=1}^{n+1} \) be a sequence in \( \mathcal{P}(H) \) satisfying that \( \mathcal{H}^\star((X_i, d_i, a_i), (X_j, d_j, a_j)) > 0 \) for all distinct \( i, j \). Then there exists a continuous map \( F : H \times \hat{\mathcal{P}} \to \mathcal{P}(H) \) such that

1. for all \( i \in \hat{n} + 1 \) and \( j \in \hat{n} \), we have \( F(v_i, j) = (X_i, d_i, a_i) \);
2. for all \( (s, i), (t, j) \in H^\infty \times \hat{n} \) with \( (s, i) \neq (t, j) \), we have \( F(s, i) \neq F(t, j) \).

**Proof.** In what follows, we consider that the set \([0, \infty]\) is equipped with the canonical topology homeomorphic to \([0, 1]\). Since every metrizable space is perfectly normal, and since \([0, \infty]\) is homeomorphic to \([0, 1]\), for each \( i \in \mathbb{N} \) we can take a continuous function \( \sigma_i : H \to [0, \infty] \) such that \( \sigma_i^{-1}(0) = \{v_j \mid j \neq i\} \) and \( \sigma_i^{-1}(\infty) = \{v_i\} \). Put \( P(s) = \prod_{i=1}^{n+1} B_{d_i}(v_i, \sigma_i(s)) \), and \( p = (a_1, \ldots, a_{n+1}) \in P(s) \). We define a metric \( E_s \) on \( P(s) \) by

\[
E_s(x, y) = \sqrt{\sum_{i=1}^{n+1} d_i(x_i, y_i)^2}.
\]

Note that \((P(v_i), E_{v_i})\) is isometric to \((X_i, d_i)\). Since all closed balls of a CAT(0) space are CAT(0) (see [2] (3) in Proposition 1.4, p.160), by Lemma 3.2 we have \((P(s), E_s, p) \in \mathcal{P}(H)\). We define a map \( W : H \to \mathcal{P}(H) \) by

\[
W(s) = \begin{cases} 
(X_i, d_i, a_i) & \text{if } s = v_i \text{ for some } i \in \hat{n} + 1; \\
(P(s), E_s, p) & \text{otherwise}.
\end{cases}
\]

We now prove that the map \( W : H \to \mathcal{P}(H) \) is continuous. Take arbitrary \( s \in H \) and \( \epsilon \in (0, \infty) \).

Case 1. \((s \in H^\infty)\): For a sufficient small neighborhood \( V \) of \( s \), we have \( \max_{i \in \{1, \ldots, n+1\}} |\sigma_i(s) - \sigma_i(t)| \leq \epsilon \) for all \( t \in V \). Since each \((X_i, d_i)\) is a geodesic space, we have

\[
\mathcal{H}(B_{d_i}(a_i, \sigma_i(s)), B_{d_i}(a_i, \sigma_i(t)) ; X_i, d_i) \leq |\sigma_i(t) - \sigma_i(s)|.
\]

Thus, for each \( i \in \hat{n} + 1 \) and for each \( y \in V \), we can take an \((\infty, \epsilon)\)-rough isometry \( f_i : (B_{d_i}(a_i, \sigma_i(s)), d_i, a_i) \to (B_{d_i}(a_i, \sigma_i(t)), d_i, a_i) \). We define a map \( g : P(s) \to P(t) \) by \( g(x) = (f_1(x_1), \ldots, f_{n+1}(x_{n+1})) \). By the triangle inequality for the \( \ell^2 \)-norm, for all \( x, y \in P(s) \), we obtain

\[
|E_s(x, y) - E_t(g(x), g(y))| \leq \left( \sum_{i=1}^{n+1} |d_i(x_i, y_i) - d_i(f_i(x_i), f_i(y_i))|^2 \right)^{1/2} < \sqrt{n + 1} \epsilon.
\]
Thus, $g$ is an $(\infty, \sqrt{n + 1} \epsilon)$-rough isometry. By Lemma 4.1, we have
\[ \mathcal{G}H^*((P(s), E_s, p), (P(t), E_t, p)) < 2\sqrt{n + 1} \epsilon. \]

Case 2. ($s = v_i$ for some $i \in \hat{n} + 1$): Put $R = \epsilon^{-1} + \epsilon$. For a sufficient small neighborhood $V$ of $s(= v_i)$, we have $R < \sigma_i(t)$ and $\max_{j \neq i} \sigma_j(t) \leq \epsilon$ for all $t \in V$. Then, for all $x, y \in P(t)$, we have
\[ |E_t(x, y) - d_i(x, y)| \leq \left( \sum_{j \neq i} d_j(x_j, y_j)^2 \right)^{1/2} \leq 2\sqrt{n} \epsilon. \]

Thus, the $i$-th projection $\pi_i : P(t) \to X_i$ is an $(R, 2\sqrt{n} \epsilon)$-rough isometry. By Lemma 4.1 we have
\[ \mathcal{G}H^*((X_i, d_i, a_i), (P(t), E_t, p)) < 4\sqrt{n} \epsilon. \]

This finishes the proof of the continuity of $W$.

We identify $p \in P(s)$ with $1_0 \in \Upsilon$, and we consider $P(s) \cap \Upsilon = \{ p \}$. We put $Z = P(s) \cup \Upsilon$. By the same way as Proposition 3.6 we define a metric $D_{s,k}$ on $Z$. We also define $F : H \times \hat{m} \to \mathcal{P} Z$ by
\[ F(s, k) = \begin{cases} (X_i, d_i, a_i) & \text{if } s = v_i \text{ for some } i \in \hat{n} + 1; \\ (Z, D_{s,k}, p) & \text{otherwise.} \end{cases} \]

By Proposition 2.3 and the continuity of $W$, the map $F$ is continuous and the condition (1) is satisfied.

By the same way as Proposition 3.6 using the operator $\mathcal{A}$, the map $F$ satisfies the condition (2). This finishes the proof. \hfill \Box

**Proof of Theorem 1.3** By the same method as the proof of Theorem 1.1 using Proposition 4.2 instead of Proposition 3.6, we can prove Theorem 1.3. \hfill \Box

**Remark 4.1.** Theorem 1.1 (Proposition 3.6) for the set $Z$ can be proven by the same method of the proof of Theorem 1.3 (Proposition 4.2); however, this method cannot be used for $\mathcal{C}$, $\mathcal{P}$, or $\mathcal{G}$ since balls of a metric space in $\mathcal{C}$, $\mathcal{P}$, or $\mathcal{G}$ do not necessarily belong to the same class that the ambient space is in. In the proof of Theorem 1.1 (Proposition 3.6) in this paper, we use a method by which we can deal with all $\mathcal{C}$, $\mathcal{P}$, $\mathcal{G}$, and $Z$, simultaneously.

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