Blind Orthogonal Least Squares Based Compressive Spectrum Sensing

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Abstract—As an enabling technique of cognitive radio (CR), compressive spectrum sensing (CSS) based on compressive sensing (CS) can detect the spectrum opportunities from wide frequency bands efficiently and accurately by using sub-Nyquist sampling rate. However, the sensing performance of most existing CSS excessively relies on the prior information such as spectrum sparsity or noise variance. Thus, a key challenge in practical CSS is how to work effectively even in the absence of such information. In this paper, we propose a blind orthogonal least squares based CSS algorithm (B-OLS-CSS), which functions properly without the requirement of prior information. Specifically, we develop a novel blind stopping rule for the OLS algorithm based on its probabilistic recovery condition. This innovative rule gets rid of the need of the spectrum sparsity or noise information, but only requires the computational-feasible mutual incoherence property of the given measurement matrix. Our theoretical analysis indicates that the signal-to-noise ratio required by the proposed B-OLS-CSS for achieving a certain sensing accuracy is relaxed than that by the benchmark CSS using the OMP algorithm, which is verified by extensive simulation results.

Index Terms—Blind stopping rule, compressive spectrum sensing, orthogonal least squares, sparse signal recovery.

I. INTRODUCTION

With the rapid deployment of intelligent transport systems (ITSs), spectrum scarcity in vehicular communications is becoming the bottleneck, since the available bandwidth turns to be insufficient to satisfy the requirement for high-quality wireless services facing the high traffic levels in vehicular applications [1]. To handle such a challenge, cognitive radio (CR) emerges as a key technology by searching for the unused spectrum resources and providing dynamic spectrum access for secondary users (SUs). To detect as much spectrum opportunities as possible from wide frequency bands at sub-Nyquist sampling rate [2], compressive spectrum sensing (CSS) methods have been developed based on compressive sensing (CS) and well acknowledged as a promising wideband spectrum sensing solution [3], [4]. Among various CSS algorithms, the greedy search methods, e.g., orthogonal matching pursuit (OMP) [5] and orthogonal least squares (OLS) [6], exhibit satisfactory sensing performance with fast implementation [7], [8]. The iterative atom selection mechanism of greedy algorithms, however, relies on the spectrum sparsity or noise prior information [9], which is not always available in practice and thus hinders their applications.

Blind greedy (BG) algorithms have been developed to solve the aforementioned dilemma of requiring prior information [10]. In current literature, the blind OMP (B-OMP) algorithm, as a representative BG algorithm, keeps detecting the effective support atomic energy in the residuals blindly [5]. However, the performance of OMP is sensitive to the mutual incoherence property (MIP) [11], [12] of the measurement matrix, that is, MIP should be small enough for effective atom separation, which limits the applicability of B-OMP algorithms in practice.

In contrast, the OLS algorithm enjoys stronger capability for correct atom exploration than OMP, resulting in compelling spectrum recovery performance, even if the measurement matrix exhibits unsatisfactory MIP [13]. Therefore, OLS is capable to guarantee more stable spectrum access of SUs when different measurement matrices are used in practice, which motivates us to investigate blind OLS algorithm for reliable CSS performance without prior information. To the best of our knowledge, there is no study on developing blind stopping rule for OLS. Accordingly, there is no OLS-related blind algorithm design and performance analyses in the current literature of both CS and CSS based CR.

To fill such a technical gap, this paper proposes a blind OLS-based CSS (B-OLS-CSS) algorithm for CR. Specifically, we formulate the bounds of a mapping factor in OLS, which is tighter than the existing ones, by utilizing the probabilistic norm bound and computational-friendly MIP metric. Then, a blind stopping rule for the OLS algorithm is developed via utilizing the MIP-based recovery conditions. To protect primary users’ (PUs’) uninterrupted communications and facilitate SUs’ spectrum access, our stopping rule focuses on the selection of all correct support atoms.

The rest of this paper is organized as follows. In Section II, we introduce notations and system model. In Section III, we present our proposed blind stopping rule, B-OLS-CSS algorithm, and the theoretical analysis. In Section IV, simulation results are given, followed by conclusions in Section V.

II. MATHEMATICAL MODEL

In CR, the received spectrum at a SU is denoted by \( s \in \mathbb{R}^N \), which is sparse based on a certain basis \( \Psi \in \mathbb{R}^{N \times N} \). Let \( s = \Psi x \), where \( x \) is a \( K \)-sparse spectrum that only contains \( K \) nonzero spectrum support entries. Define \( \Phi \in \mathbb{R}^{M \times N} \) as the sampling matrix, where \( M \) and \( N \) are the numbers of sub-Nyquist-rate and Nyquist-rate samples, respectively. Denoting the additive noise as \( \epsilon \sim \mathcal{N}(0, \sigma^2 I_M) \), the compressed measurement vector \( y \in \mathbb{R}^M \) is given by

\[
y = \Phi \Psi x + \epsilon = \Phi \Psi x + \epsilon = Dx + \epsilon, \tag{1}
\]

where \( D = \Phi \Psi \in \mathbb{R}^{M \times N} \) is the measurement matrix. Define \( \text{SNR} = \frac{E(\epsilon_1^2)}{\mathbb{E}[||x||^2]} \), \( \text{SNR}_0 = \frac{\mathbb{E}[||x||^2]}{\mathbb{E}[||\epsilon||^2]} \), and \( \text{SNR}_{\text{min}} \) as the minimum value of \( \text{SNR}_q \quad (q = 1, 2, \ldots, N) \) [5]. The objective of CSS is to recover the spectrum \( x \) from the compressed measurement signal \( y \) given the measurement matrix \( D \).

III. PROPOSED B-OLS-CSS ALGORITHM

In practical vehicular communications, the sparsity value or the noise information is generally not known a priori. In this section, to overcome this issue, we first develop a blind stopping rule for OLS based on our
analytical results of sparse recovery condition which is independent to the signal sparsity value and the noise statistical information. Then, the B-OLS-CSS algorithm is developed accordingly.

A. Blind Stopping Rule and Theoretical Analysis for OLS

In conventional OLS, one atom is selected per iteration using the rule [12]:

\[ i^{t+1} = \arg \max_{j \in \{1, \ldots, N\} \setminus S_i} | \langle P_{S_i}^j D_j \rangle \| \leq \| r^t \| , \]

where \( r^t = P_{S_i}^j (Dx + \epsilon) \) is the residual vector after \( t \) iterations \((1 \leq l \leq K)\). The mapping factor, i.e., \( | \langle P_{S_i}^j D_j \rangle | \), normalizes the corresponding projected atom \( P_{S_i}^j D_j \) during the atom selection procedures, and the residual power decreases the most if the atom is just the one selected by the above rule [6]. If the preset number of iterations is larger than the ground truth sparsity value \( K \) or if the residual is smaller than a predefined threshold according to the noise variance, OLS stops iteration, and the recovered spectrum is calculated by least squares.

The key of BG algorithm is to set the stopping rule. In each iteration, the indices of the nonzero atoms are exploited to calculate the residual vector, which is further used to form the stopping rule. The derived rule stops the algorithm once there is no nonzero component in the residual. Given a \( K \)-sparse spectrum to be recovered, i.e., there exist \( K \) nonzero components, the BG algorithm recovers all its nonzero atoms in the algorithm’s \( K \) iterations. Accordingly, the remaining unoccupied spectrum bands are identified for SUs.

As proved in [5], \( \| D^T r^t \| \leq \| r^t \| \) can be used to detect whether there remains nonzero components in the residual vector. That is, if \( \| D^T r^t \| \leq \| r^t \| \) is smaller than a predefined threshold \( \Theta \), i.e., \( \| D^T r^t \| < \Theta \), the residual vector only contains noise components and OLS stops iteration. In the following, we develop this critical threshold \( \Theta \) that needs to be predefined based on the analytical results for OLS. In doing so, we first present some useful lemmas. In OLS, the tighter the mapping factor bounds, the better the theoretical recovery condition [14]. The following lemmas present a tighter bound for the mapping factor than the existing results [12], [15].

**Lemma 1:** For \( B \in \mathbb{R}^{M \times K} \), whose entries independently and identically satisfy \( \mathcal{N}(0, \frac{1}{M}) \), the smallest singular value \( \gamma_{\text{min}} \) and the largest singular value \( \gamma_{\text{max}} \) with any \( \rho > 0 \) follow: min\( (i_{\gamma_{\text{min}}} \geq 1 \sqrt{\frac{K}{M} - \rho}), \| P_{\gamma_{\text{min}}} \| \leq 1 \leq \frac{1}{\sqrt{\frac{K}{M} + \rho}} \).

The proof can be easily derived from [16, Theorem 2.13].

**Lemma 2:** Suppose \( \mu < \frac{1}{\sqrt{\eta \gamma}} \), then \( \sqrt{\eta \gamma} \leq \| P_{\gamma_{\text{max}}}^j D_j \| \leq 1 \) for \( i \in \{1, 2, \ldots, N\} \setminus S^1 \) with the probability given in Lemma 1, where \( \mu = \max_{j \neq i} \| D^T r^t \| \) is the matrix coherence and \( T = (1 - \frac{K\mu}{1 - K\mu})^{-1} \).

**Proof:** See Appendix A.

The closer the lower bound in Lemma 2 is to 1, the tighter it is. The tightness of this bound depends on the parameter \( \rho \). Next, compared with two existing bounds of the mapping factor, we discuss the bound in our derived Lemma 2 in terms of the range of \( \rho \) which is tighter than the existing ones. In [15] and [12], the authors respectively provide that

\[ \sqrt{1 - \frac{K\mu}{1 - \frac{1}{\mu} \gamma}} \leq \| P_{\gamma_{\text{min}}}^j D_j \| \| \leq 1, \]

\[ \sqrt{1 - \frac{1 - (K - 1)\mu}{1 - (K - 1)\mu}} \gamma \leq \| P_{\gamma_{\text{max}}}^j D_j \| \| \leq 1. \]

**Remark 1:** Letting the left-hand side of the bound in Lemma 2 be larger than those in (2) and (3), we get \( \rho < (K - 1)\mu - \sqrt{K/M} \). With this condition, \( \| P_{\gamma_{\text{max}}}^j D_j \| \)’s lower bound in Lemma 2 is closer to than those in (2) and (3), i.e., our proposed bound is tighter than the existing ones.

To derive the recovery condition for standard OLS, we utilized the upper bound of the reconstructible sparsity, which is the Theorem 2 in our previous work [12]. It indicates that if the real sparsity \( K \) of the spectrum is lower than a specific threshold \( C \), OLS produces reliable recovery. Note that the threshold \( C \) is only related with the matrix coherence of the measurement matrix. Based on the theoretical analysis related to the mapping factor and the reconstructible sparsity, we present the reliable recovery condition for OLS.

**Theorem 1:** Suppose \( K \leq C \) and \( \varepsilon \sim \mathcal{N}(0, \sigma^2 M) \). Define \( \mathcal{P}(\omega) = 1 - \frac{C^2}{\sigma^2} \cdot \frac{\sum_{j \neq \omega} \sigma^2}{\sigma^2} - 2e^{\frac{\omega^2}{2}} - e^{\frac{\omega^2}{4}} \cdot \frac{1}{\sqrt{\pi}} + \frac{1}{\sqrt{\pi}}, \)

where \( \theta = \sqrt{\lambda_1} - \sqrt{\lambda_2} \), \( A_1 = 4(M - C) - 2, A_2 = M + C - 2 \sqrt{(M - \pi)(C - \pi)} > 0 \).

For a given minimum probability of recovery \( P_{\text{min}} \), the OLS algorithm using the stopping rule \( \| D^T r^t \| < \Theta \) can reconstruct the \( K \)-sparse spectrum with the probability \( P > P_{\text{min}} \), if the minimum component \( SNR_{\text{min}} \) satisfies \( SNR_{\text{min}} > \max \{ \varphi_1, \varphi_2 \} \), where \( \mathcal{Q} = \omega \mu, \omega = P^{-1}(P_{\text{min}}) \cdot \varphi_1 = \frac{A_2^2 M^2}{(M - 1)(M - C)^2} - \frac{A_1^2}{2}, \varphi_2 = \frac{A_2^2}{M(1 - \sqrt{\frac{K}{M} - \rho}) - \mu(1 - \sqrt{\frac{K}{M} + \rho}) \pi}, P^{-1}(\cdot) \) represents the inverse function of \( P(\cdot) \).

**Proof:** See Appendix B.

The right-hand-side of the stopping rule \( \sqrt{\frac{K}{M} - \rho} \) contains a constant \( \omega = P^{-1}(P_{\text{min}}) \) and the computable matrix coherence \( \mu \). It indicates that Theorem 1 is operational with an input target probability of recovery \( P_{\text{min}} \) and the calculated \( \mu \). That is, the stopping rule in Theorem 1 can work blindly since it is independent to the sparsity level or the noise prior information. Thus, it is suitable for practical CSS scenario where this information is unavailable.

B. B-OLS-CSS Algorithm

In this subsection, Theorem 1 is utilized to develop the B-OLS-CSS algorithm, which is given in Algorithm 1. Note that the greedy algorithms may not select the exactly correct \( K \) support atoms within \( K \) iterations due to the presence of noise. The B-OLS-CSS algorithm is designed to address this problem by appropriately reducing the right-hand-side of the blind stopping rule in Theorem 1 and hence the algorithm runs more than \( K \) iterations for more opportunities to choose all the \( K \) correct support atoms to the best effort. Based on these arguments, the blind stopping rule in the B-OLS-CSS algorithm is set as \( \| D^T r^t \| < \omega \mu \), where \( \omega = \omega - \rho \). According to Remark 1 and Theorem 1, \( \rho \) satisfies \( 0 < \rho < (C - 1)\mu - \sqrt{C/M} \). A large \( \rho \) induces a loose bound of \( \| P_{\gamma_{\text{max}}}^j D_j \| \), while a small \( \rho \) reduces the expected probability exponentially. Hence, a moderate scale of \( \rho \) is able to effectively balance the required \( SNR_{\text{min}} \) and the expected recovery probability in Theorem 1. The trade-off ensures that B-OLS-CSS iterates slightly more than \( K \) times.

IV. Simulation Results

A. Simulations for Lower Bounds of \( \| P_{\gamma_{\text{max}}}^j D_j \| \) and \( SNR_{\text{min}} \)

In this subsection, we implement simulations to compare Lemma 2 with (2), (3), and Theorem 1 with [5, Theorem 1]. Two \( M \times N \) normalized measurement matrices are generated (where \( M = 1024, N = 8192 \) and \( M = 2048, N = 8192 \)), which are the same as those in [5]. For fair comparison, we fix \( \rho = 0.15 \) in the simulations. Note that \( K \) and \( \mu \) should satisfy \( K \mu < 1 \) and \( \frac{(1 - K - 1)\mu}{(1 - K - 1)\mu} < 1 \) for the existing bounds (2) and (3), respectively. The lower bounds in Lemma 2, (2) and (3) are presented in Fig. 1. It is observed that...
The measurement matrix $\mathbf{P} \in \mathbb{R}^{\rho \times N}$, and recovered index set of nonzero entries $S \subseteq \{1, 2, \ldots, N\}$.

1: **Initialization:** $l = 0$, $\mathbf{r}^0 = \mathbf{y}$, $S^0 = \emptyset$, $\mathbf{x}^0 = \mathbf{0}$.
2: Calculate $\omega = \mathcal{P}^{-1}(\mathcal{P}_{\min})$ and set $\omega' = \omega - \rho$, where $\mathcal{P}^{-1}(\cdot)$ is given in Theorem 1.
3: Calculate the matrix coherence $\mu$ of $\mathbf{D}$.
4: while $|\mathbf{D}^\top \mathbf{r}^l|_\infty > \omega' \mu$ do
5: Set $j^{l+1} = \arg \min_{j \in \{1, \ldots, N\}} \| \mathbf{P}^\perp_{\mathbf{S}^l(j)} \mathbf{y} \|_2$, where $\mathbf{P}^\perp_{\mathbf{S}^l(j)} = \mathbf{I} - \mathbf{D}_{\mathbf{S}^l(j)} \mathbf{D}^\top_{\mathbf{S}^l(j)} \mathbf{D}_{\mathbf{S}^l(j)} \mathbf{D}_{\mathbf{S}^l(j)}^\top$.
6: Augment $\mathbf{S}^{l+1} = \mathbf{S}^l \cup \{j^{l+1}\}$.
7: Estimate $\mathbf{x}^{l+1} = \arg \min_{\mathbf{x} \sup \supp(\mathbf{x}) = \mathcal{S}^{l+1}} \| \mathbf{y} - \mathbf{Dx} \|_2^2$.
8: Update $\mathbf{r}^{l+1} = \mathbf{y} - \mathbf{Dx}^{l+1}$ and set $l = l + 1$.
9: end while
10: return $\mathbf{S} = \mathbf{S}^l$ and $\mathbf{x} = \mathbf{x}^l$.

**Algorithm 1:** B-OLS-CSS Algorithm.

**Input:** The measurement matrix $\mathbf{D}$, compressed measurements $\mathbf{y}$, minimum target probability of recovery $\mathcal{P}_{\min}$, and parameter $\rho$.

**Output:** The recovered spectrum $\mathbf{x} \in \mathbb{R}^N$, and recovered index set of nonzero entries $S \subseteq \{1, 2, \ldots, N\}$.

In this subsection, we present simulations to demonstrate the superiority of our proposed B-OLS-CSS algorithm when compared with the existing benchmarks. The spectrum is regarded to be successfully recovered if the recovered spectrum is within a certain small Euclidean distance of the ground truth. The locations of the nonzero atoms are selected uniformly at random. The nonzero entries in the spectrum are set to be independently and identically distributed as $N(1, 0.01)$ for illustration. We set $\mathcal{P}_{\min} = 0.95$ and $\rho = 0.175$. All CSS algorithms run over 1,000 Monte Carlo trials.

As shown in Fig. 3, for SNR greater than $-5$ dB when sparsity is equal to 4 (or $-3$ dB when sparsity is equal to 6), the performance of the B-OLS-CSS is almost the same as that of the OLS-CSS, where the latter has the perfect information of $K$ and $\sigma$. This reveals that even if the sparsity level or the noise information is unavailable, the sensing performance achieved by B-OLS-CSS algorithm is still competitive.

Moreover, we have implemented simulations to illustrate the sensing performance versus $\omega$ with $K = 4$, $M = 1024$ and $N = 2048$. The results show that in a wide range of $\omega$, i.e., $[1.175, 2.575]$, the performance of B-OLS-CSS can be competitive to that of the conventional OLS-CSS. Based on the aforementioned parameter settings, the value of $\omega$ calculated by Theorem 1 is approximately equal to 1.3, which just falls into the range $[1.175, 2.575]$ that ensures guaranteed performance. This reveals that in practical applications, we can first obtain $\omega$ according to Theorem 1 for B-OLS-CSS, and then perform reliable recovery.

Then, we investigate the sensing performance of the OMP-CSS, CoSaMP-CSS, CSS with the blind OMP in [5] (we call it B-OMP-CSS), OLS-CSS, MOLS-CSS [17] and B-OLS-CSS. We adopt the hybrid measurement matrix $\mathbf{D}$ given in [13]. The $i$-th column of $\mathbf{D}$ satisfies $\mathbf{D}_i = \mathbf{n}_i + \mathbf{c}_i$, where $\mathbf{n}_i \sim N(0, 1)$ and $\mathbf{c}_i$ obeys the uniform distribution on $[0, 1]$. Compared with the Gaussian measurement matrix, the MIP of the hybrid measurement matrix is extremely unsatisfactory. The following Figs. 4 and 5 indicate that our proposed B-OLS-CSS always has desired performance even for the unsatisfactory MIP case. As shown in Fig. 4, when $K = 8$, the performance of B-OLS-CSS is competitive with that of the CoSaMP and MOLS algorithms, and is even better than the OLS-CSS algorithm, which reveals that our proposed blind stopping rule works better than the methods implementing exact $K$ iterations in OLS-CSS. Meanwhile, OMP-CSS behaves the worst, which is consistent with the theoretical statement in [13] that OMP performs poorly in dealing with high coherence measurement matrices.
Furthermore, when $K = 12$, the performance of OMP-CSS, B-OMP-CSS, and OLS-CSS degrades rapidly, and these algorithms end up unable to perform reliable recovery. The performance of our proposed B-OLS-CSS still approaches to that of the CoSaMP and MOLS, which indicates that B-OLS-CSS is more suitable for the practical CR when the prior information is unavailable.

In Fig. 5, the number of measurements $M$ is half of that in Fig. 4. The performance of all the algorithms degrades compared with that in Fig. 4, while our proposed B-OLS-CSS still approaches to CoSaMP-CSS and MOLS-CSS. Fig. 6 shows the probabilities of detection and miss-detection using Gaussian measurement matrix at different SNRs, which are respectively defined by

$P_d = \frac{\sum_{\mathbf{x} \in \mathcal{S}} |\mathbf{D}^T \mathbf{P}(\mathbf{x})\mathbf{s}|}{|\mathcal{S}|}$ and $P_{md} = \frac{|\mathcal{S}| - \sum_{\mathbf{x} \in \mathcal{S}} |\mathbf{D}^T \mathbf{P}(\mathbf{x})\mathbf{s}|}{|\mathcal{S}|}$.

In Fig. 6, $P_d$ of OMP-CSS, CoSaMP-CSS, and MOLS-CSS outperforms the other existing blind CSS techniques in terms of better performance to the one using sparsity and noise information, and it outperforms the other existing blind CSS techniques in terms of better CSS accuracy.

V. CONCLUSION

This work is motivated by the challenge in practical CR that the prior information, e.g., the spectrum sparsity and noise variance, is usually unavailable for CSS techniques. To address this issue, we propose a B-OLS-CSS algorithm, which works properly even without such prior information. The theoretical analysis demonstrates that the SNR required for reliable recovery of our B-OLS-CSS is lower than the existing blind algorithm, leading to theoretical guarantee for the algorithm robustness under the low SNR environments. Simulation results verify that the proposed B-OLS-CSS algorithm can provide comparable performance to the one using sparsity and noise information, and it outperforms the other existing blind CSS techniques in terms of better CSS accuracy.

APPENDIX A

PROOF OF LEMMA 2

Proof: Due to submultiplicativity, we have $\|\mathbf{P}_s^i \mathbf{D}_i\|_2 \leq \rho(\mathbf{D}_s) \rho((\mathbf{D}_s^T \mathbf{D}_s)^{-1}) \|\mathbf{D}_s^T \mathbf{D}_s\|_2$. Based on Lemma 1, we obtain

$\rho(\mathbf{D}_s) = \sqrt{K \mu} \leq \frac{1}{\sqrt{1 - K/M - \rho}}$. Then, similar to the proof in [12], we have $\rho((\mathbf{D}_s^T \mathbf{D}_s)^{-1}) \leq \frac{1}{\sqrt{1 - K/M - \rho}}$. Since $\|\mathbf{D}_s^T \mathbf{D}_s\|_2 \leq \sqrt{K \mu}$, $\|\mathbf{P}_s^i \mathbf{D}_i\|_2 \leq \frac{\sqrt{K \mu^2 (1 + \sqrt{K/M + \rho})}}{1 - \sqrt{K/M - \rho}}$. Finally, the proof is completed because $\|\mathbf{P}_s^i \mathbf{D}_i\|_2 = \sqrt{1 - \|\mathbf{P}_s^i \mathbf{D}_i\|_2^2}$.

APPENDIX B

PROOF OF THEOREM 1

Proof: The proof of Theorem 1 contains three parts: 1) Developing the condition for choosing a correct entry in each iteration; 2) Proving that the OLS algorithm does not stop at the $l$-th iteration ($l < K$); 3) Proving that the OLS algorithm stops after $K$ iterations. Firstly, according to [5, Lemma 4] and [12, Theorem 5], we obtain that if

$\|\mathbf{x}_{s|\hat{S}}\|_2 > \frac{2\sqrt{K - l} (2 - (K - T) \mu)}{(2 - (K - T))^2} \|\mathbf{e}_{s|\hat{S}}\|_\infty$, \quad (4)

OLS selects a correct atom with the probability $P(\|\mathbf{D}^T \mathbf{x}_{s|\hat{S}}\|_\infty \leq \omega_\mu \sigma \theta)$. Since $\frac{1}{\rho} \|\mathbf{x}_{s|\hat{S}}\|_2 = \frac{\sum_{\mathbf{x} \in \hat{S}} M \times \text{SNR}_\psi}{\|\mathbf{x}_{s|\hat{S}}\|_2}$, (4) becomes $\text{SNR}_{\min} > \varphi_1$.}

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Next, based on [18, Lemma 5.1], for $l < K$, we obtain
\[
\frac{\|D^T r^K\|_\infty}{\|r^K\|_2} \geq \frac{\frac{1}{\sqrt{K}}\|D^T P_{SK} x_{0|S}\|_2 - \omega \mu \sigma}{\|P_{SK} D_{0|S} x_{0|S}\|_2 + \sqrt{M + 2\sqrt{M}\log M}\sigma}
\geq \frac{1 - \sqrt{K/M} \rho}{\sqrt{K} - 1} \|x_{0|S}\|_2 - \omega \mu \sigma}
\]
(5)
with the probability $\mathbb{P}(\bigcap_{l=1}^K \{ \|D^T P_{SK} e\|_\infty \leq \omega \mu \sigma, \zeta_{\min} \geq 1 - \sqrt{K/M} - \rho, \zeta_{\max} \leq 1 + \sqrt{K/M} + \rho, \|e\|_2 \leq \sqrt{M + 2\sqrt{M}\log M}\sigma \})$. Then, if $\text{SNR}_{\min} > \varphi_2$, we have $\frac{\|D^T r^K\|_\infty}{\|r^K\|_2} > \omega \mu$. Thus, OLS does not stop at the $l$-th iteration.

Now it remains to prove that OLS stops exactly at the $K$-th iteration. By using [5, Lemma 3], with the probability $\mathbb{P}(\bigcap_{l=1}^K \{ \|D^T P_{SK} e\|_\infty \leq \omega \mu \sigma, \|P_{SK} e\|_2 \geq \theta \sigma \})$, we have
\[
\frac{\|D^T r^K\|_\infty}{\|r^K\|_2} = \frac{\|D^T P_{SK} e\|_\infty}{\|P_{SK} e\|_2} \leq \omega \mu,
\]
which means that OLS stops at the $K$-th iteration. Finally, based on the aforementioned analysis and $K < C$, the proof is completed. $\blacksquare$

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