Relative singularity categories and singular equivalences

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Abstract

Let $R$ be a right noetherian ring. We introduce the concept of relative singularity category $\Delta_{\mathcal{X}}(R)$ of $R$ with respect to a contravariantly finite subcategory $\mathcal{X}$ of mod-$R$. Along with some finiteness conditions on $\mathcal{X}$, we prove that $\Delta_{\mathcal{X}}(R)$ is triangle equivalent to a subcategory of the homotopy category $K_{ac}(\mathcal{X})$ of exact complexes over $\mathcal{X}$. As an application, a new description of the classical singularity category $D_{sg}(R)$ is given. The relative singularity categories are applied to lift a stable equivalence between two suitable subcategories of the module categories of two given right noetherian rings to get a singular equivalence between the rings. In different types of rings, including path rings, triangular matrix rings, trivial extension rings and tensor rings, we provide some consequences for their singularity categories.

Keywords Singularity category · Singular equivalence · Stable category

Mathematics Subject Classification 18E30 · 18G35 · 18G25 · 16G10

1 Introduction

Let $R$ be a right noetherian ring and mod-$R$ the category of finitely generated right $R$-modules. The notion of singularity category of $R$ is defined to be the Verdier quotient $D_{sg}(R) := \frac{D^b(\text{mod-}R)}{K^b(\text{prj-}R)}$, where $D^b(\text{mod-}R)$ denotes the bounded derived category, and $K^b(\text{prj-}R)$ the homotopy category of bounded complexes whose terms are projective. This category was introduced by Buchweitz [11] as a homological invariant of rings. If
$R$ has finite global projective dimension, then every bounded complex admits a finite projective resolution and therefore we have the equality $\mathbb{D}^b(\text{mod-}R) = \mathbb{K}^b(\text{prj-}R)$. In particular, the singularity category of a right noetherian ring of finite global dimension is trivial. Hence, the singularity category provides a homological invariant for rings of infinite global dimension. The singularity category is also used to detect some properties of a singularity category. For example, let $k$ be an algebraically closed field. A commutative complete Gorenstein $k$-algebra $(R, m)$ satisfying $k = R/m$ has an isolated singularity if and only if $\mathbb{D}_{\text{sg}}(R)$ is a Hom-finite category, by work of Auslander [6]. Recently, the singularity category was applied by Orlov to study Landau–Ginzburg modules [37].

Several relative versions of the singularity categories are defined in the literature by the different authors, for example see the papers [13,29,38,39]. What we should mean by relative singularity categories is a special case of the relative ones given in [13,38], and a generalization of the one studied in [29]. The relative concept of the singularity category here is defined with respect to a contravariantly finite subcategory of mod-$R$. To be precise, let $\mathcal{X}$ be a contravariantly finite subcategory of mod-$R$ which contains prj$-R$. The relative singularity category of $R$ with respect to the subcategory $\mathcal{X}$ is defined to be the Verdier quotient

$$\Delta_\mathcal{X}(R) := \frac{\mathbb{D}^b(\text{mod-}\mathcal{X})}{\mathcal{P}},$$

where mod-$\mathcal{X}$ is the category of finitely presented contravariant functors from $\mathcal{X}$ to the category of abelian groups, and $\mathcal{P}$ the thick subcategory of the bounded derived category $\mathbb{D}^b(\text{mod-}\mathcal{X})$ generated by all functors $\text{Hom}_R(-, P) |_{\mathcal{X}}, P \in \text{prj-}R$ (Definition 2.1). In the terminology of [1] or [30], the $\Delta_\mathcal{X}(R)$ is only the silting reduction of $\mathbb{D}^b(\text{mod-}\mathcal{X})$ with respect to presilting subcategory $\mathcal{P}$. It is shown in Theorem 2.2 the classical singularity category $\mathbb{D}_{\text{sg}}(R)$ is a Verdier quotient of the relative singularity categories $\Delta_\mathcal{X}(R)$. Hence due to this connection we try in this paper to use relative singularity categories to study the classical ones. The best cases of the subcategories $\mathcal{X}$ might be more helpful are when the relative global $\mathcal{X}$-dimensions with respect to them are finite. Under such finiteness condition over $\mathcal{X}$, the relative singularity category $\Delta_\mathcal{X}(R)$ will become triangle equivalent to the Verdier quotient $\frac{\mathbb{K}^b(\mathcal{X})}{\mathbb{K}^b(\text{prj-}R)}$. We study this quotient category independently and prove that it is triangle equivalent to the homotopy category $\mathbb{K}^{-,p}(\mathcal{X})$, see Proposition 3.1. Here $\mathbb{K}^{-,p}(\mathcal{X})$ denotes the subcategory of the homotopy category $\mathbb{K}(\mathcal{X})$ of complexes over $\mathcal{X}$ consisting of all complexes which are homotopy-equivalent to a upper bounded exact complex $P$ over $\mathcal{X}$ such that for some $n$, $P^i \in \text{prj-}R$ for all $i \leq n$. In fact, there are only finitely many terms of $P$ to be non-projective. As a consequence we get the following theorem.

**Theorem 1.1** (Theorem 3.3) **Let** prj$-R \subseteq \mathcal{X} \subseteq \text{mod-}R$ **be a contravariantly finite subcategory. Assume the global $\mathcal{X}$-dimension of $R$ is finite. Then, there exists the following equivalence of triangulated categories**

$$\mathbb{D}_{\text{sg}}(R) \simeq \frac{\mathbb{K}^{-,p}(\mathcal{X})}{\mathbb{K}^b(\text{prj-}R)}.$$
As the above theorem says we obtain a description of the singularity category of $R$ such that only exact complexes are involved.

For right noetherian rings $R$ and $R'$, we say that $R$ is singularly equivalent to $R'$ if there exists a triangle equivalence $\mathbb{D}_{\text{sg}}(R) \simeq \mathbb{D}_{\text{sg}}(R')$. Our next purpose is to apply the relative singularity categories to construct singular equivalences. Our attempt of using this approach leads to the following result.

**Theorem 1.2** (Theorem 4.10) Let $\text{prj-}R \subseteq \mathcal{X} \subseteq \text{mod-}R$ and $\text{prj-}R' \subseteq \mathcal{X}' \subseteq \text{mod-}R'$ be contravariantly finite subcategories and to be closed under syzygies. Assume the global $\mathcal{X}$-dimension, resp. $\mathcal{X}'$-dimension, of $R$, resp. $R'$, is finite. Suppose, further, there is a functor $F : \mathcal{X} \to \mathcal{X}'$ such that $F(\text{prj-}R) \subseteq \text{prj-}R'$, the induced functor $F : \mathcal{X} \to \mathcal{X}'$ is an equivalence, and for any $X \in \mathcal{X}$, $F(\Omega_R(X)) \simeq \Omega_{R'}(F(X))$ in $\mathcal{X}'$. If either of the following hold.

1. The subcategories $\mathcal{X}$ and $\mathcal{X}'$ satisfy the condition $(\ast)$ (see Definition 4.1).
2. There are quasi-resolving subcategories $\mathcal{X} \subseteq \mathcal{Y} \subseteq \text{mod-}R$ and $\mathcal{X}' \subseteq \mathcal{Y}' \subseteq \text{mod-}R'$ such that the functor $F$ is a restriction of an exact functor from $\mathcal{Y}$ to $\mathcal{Y}'$. then $R$ and $R'$ are singularly equivalent.

A subcategory of an abelian category is said to be quasi-resolving if it contains all projective objects, closed under direct summands, closed under kernels of epimorphisms. An exact functor from $\mathcal{Y}$ to $\mathcal{Y}'$ means that any short exact sequence in $\text{mod-}R$ with all terms in $\mathcal{Y}$ is mapped by the functor into a short exact sequence in $\text{mod-}R'$.

By the above theorem we observe some sort of relative stable equivalence implies the singular equivalences. It is rather easier to use the stable categories (which have more simple structure) to determine two rings to be singular equivalent. An important example of the subcategories in the theorem is when both are the whole of the module categories. In this special case we are dealing with the stable equivalences which are fundamental equivalences both in the representation theory of algebras and groups and in the theory of triangulated categories. We refer to Corollary 4.10 and Remark 4.12 for a discussion about this special case. Lifting stable equivalences to derived equivalences, and so singular equivalences, are also considered in [4,19], in the setting of self-injective algebras, and for more general algebras in [28]. We will provide some more examples of the subcategories appeared in the above theorems (not necessarily to be the whole of the module categories) to support our main results. Our examples are given by different types of rings, including path rings, triangular matrix rings, trivial extension rings and tensor rings.

The paper is organized as follows. In Sect. 2, we first collect some facts about our functorial approach. Then we introduce the notion of relative singularity categories and describe their connection with the usual singularity categories. In Sect. 3, Theorem 1.1 is proved, and moreover, the Hom-finiteness of the relative singularity categories over Artin algebras is discussed. Section 4 is devoted to the proof of Theorem 1.2 and to applications of this theorem to the problem of lifting a relative stable equivalence to a singular equivalence. In the last section we will give some examples and application of our results.

**Notation and convention** Throughout the paper $R$ denotes a right noetherian ring and with $\text{Mod-}R$, resp. $\text{mod-}R$, the category of, resp. finitely generated, modules
over \( R \). By a module we always mean a right module unless otherwise stated. The subcategory of finitely generated projective modules over \( R \) is denoted by \( \text{prj}-R \).

Let \( \mathcal{A} \) be an additive category. We denote by \( \mathbb{K}(\mathcal{A}) \) the homotopy category of all complexes over \( \mathcal{A} \). Moreover, \( \mathbb{K}^b(\mathcal{A}) \) denotes the full subcategory of \( \mathbb{K}(\mathcal{A}) \) consisting of all bounded above, resp. bounded, complexes. In the case that \( \mathcal{A} \) is abelian, the derived category of \( \mathcal{A} \) will be denoted by \( \mathbb{D}(\mathcal{A}) \), which is the Verdier quotient \( \mathbb{K}(\mathcal{A})/\mathbb{K}_{\text{ac}}(\mathcal{A}) \). Here \( \mathbb{K}_{\text{ac}}(\mathcal{A}) \) is the homotopy category of all exact complexes over \( \mathcal{A} \). By \( \mathbb{D}^b(\mathcal{A}) \), we denote the full subcategory of \( \mathbb{D}(\mathcal{A}) \) consisting of all homologically bounded complexes.

All subcategories are assumed to be full, closed under isomorphisms, direct summands and finite sums. We denote Hom-spaces in the homotopy category by \( \text{Hom}_{\mathbb{K}}(-, -) \). For an \( R \)-module \( M \) in \( \text{mod}-R \), consider a short exact sequence \( 0 \rightarrow \Omega_R(M) \rightarrow P \rightarrow M \rightarrow 0 \) with \( P \) in \( \text{prj}-R \). The module \( \Omega_R(M) \) is then called a syzygy module of \( M \). Note that syzygy modules of \( M \) are not uniquely determined. An \( n \)-th syzygy of \( M \) will be denoted by \( \Omega^n_R(M) \), for \( n \geq 2 \). For a complex \( X \) in \( \mathbb{K}(\mathcal{A}) \), we have in mind the differentials raise the degrees. If \( i > 0 \), resp. \( i < 0 \), we mean by \( X[i] \) the shift of the complex \( X \) to the left, resp. right, \( i \), resp. \(-i\), degrees. A module is considered as a complex concentrated at degree zero when we want to assume it as an object in the category of complexes.

Let \( \mathcal{C} \) be a subcategory of an abelian category \( \mathcal{A} \). Given an object \( M \) in \( \mathcal{A} \), a right \( \mathcal{A} \)-approximation of \( M \) is a map \( g : C \rightarrow M \) with \( C \in \mathcal{C} \) such that for any map \( h : C' \rightarrow M \) with \( C' \in \mathcal{C} \), there is a map \( f : C' \rightarrow C \) such that \( h = g \circ f \). In the case that every object in \( \mathcal{A} \) has a right \( \mathcal{C} \)-approximation, \( \mathcal{C} \) is said to be contravariantly finite in \( \mathcal{A} \). Let \( \text{prj}-R \subseteq \mathcal{A} \) be a subcategory of \( \text{mod}-R \). The stable category of \( \mathcal{A} \) is denoted by \( \mathcal{A}^\vee \). The stable category is defined by setting the objects to be the same as those of \( \mathcal{A} \), and for any \( X \) and \( Y \) in \( \mathcal{A} \), the group of morphisms is given by \( \text{Hom}_R(X, Y) = \text{Hom}_R(X, Y)/\text{PHom}_R(X, Y) \), where \( \text{PHom}_R(X, Y) \) denotes the set of all morphisms \( A \rightarrow B \) which factor through a projective module.

2 Relative singularity categories

Throughout this section unless stated otherwise, let \( \mathcal{X} \) be a contravariantly finite subcategory of \( \text{mod}-R \) containing \( \text{prj}-R \). As a general point in our paper, for avoiding any confusion, at least in our main results we often restate the needed assumptions. In this section, we will introduce the notion of relative singularity category \( \Delta_R(\mathcal{X}) \) with respect to the subcategory \( \mathcal{X} \) as a Verdier Localization of \( \mathbb{D}^b(\text{mod-}\mathcal{X}) \). Then, we intend to make a realization of \( \mathbb{D}_{\text{sg}}(R) \) by a Verdier localization of \( \Delta_R(\mathcal{X}) \). To do this, we need first to recall some functorial construction given in [26].

Let \( F \) be in \( \text{mod-}\mathcal{X} \) and \( \text{Hom}_R(-, X_1) \mid_{\mathcal{X}} \xrightarrow{\text{Hom}_R(-, d) \mid_{\mathcal{X}}} \text{Hom}_R(-, X_0) \mid_{\mathcal{X}} \rightarrow F \rightarrow 0 \) a projective presentation of \( F \). The assumption of \( \mathcal{X} \) being contravariantly finite implies that any morphism in \( \mathcal{X} \) has a weak kernel, consequently \( \text{mod-}\mathcal{X} \) is an abelian category. The functor \( \vartheta : \text{mod-}\mathcal{X} \rightarrow \text{mod-}R \) is defined by sending \( F \) to the cokernel \( \text{Cok}(d) \) of \( d \) in \( \text{mod-}R \). The assignment \( \vartheta \) on morphisms is defined with help of the lifting property, see [26, Remark 2.1] for more details. The functor \( \vartheta \) is an exact functor.
Therefore, it induces a triangle functor $\mathbb{D}_\vartheta^b$ from $\mathbb{D}^b(\text{mod-}\mathcal{X})$ to $\mathbb{D}^b(\text{mod-}\mathcal{R})$. It acts on objects, as well as on roofs, terms by terms. Let $\mathbb{D}_0^b(\text{mod-}\mathcal{X})$ denote the kernel of $\mathbb{D}_\vartheta^b$, by [26, Proposition 3.1], it consists of all complexes $\mathbf{K}$ such that the associated valuated complexes $\mathbf{K}(P)$ on any projective module $P$ in $\text{prj-R}$ are an exact complex of abelian groups. The induced functor from the Verdier localization $\mathbb{D}^b(\text{mod-}\mathcal{X})/\mathbb{D}_0^b(\text{mod-}\mathcal{X})$ to $\mathbb{D}^b(\text{mod-}\mathcal{R})$ will be denoted by $\widetilde{\mathbb{D}}_\vartheta^b$. In [26, Proposition 3.3], it is proved that $\mathbb{D}_\vartheta^b$ is an equivalence of triangulated categories. By $\mathcal{P}$ we will show the thick subcategory of $\mathbb{D}^b(\text{mod-}\mathcal{X})$ generated by all representable functors $\text{Hom}_R(\_ , P) |_{\mathcal{X}}$ in which $P$ runs through modules in $\text{prj-R} \subseteq \mathcal{X}$. As for any $X \in \mathcal{P}$ and $Y \in \mathbb{D}_0^b(\text{mod-}\mathcal{X})$, $\text{Hom}_R(\mathbb{D}^b(\text{mod-}\mathcal{X}))(X, Y) = 0$, then $\mathcal{P}$ can be considered as a thick subcategory of $\mathbb{D}_0^b(\text{mod-}\mathcal{X})$. Moreover, by the same reason, the triangulated category $\mathbb{D}_0^b(\text{mod-}\mathcal{X})$ can be identified as a subcategory of $\mathbb{D}^b(\text{mod-}\mathcal{X})/\mathcal{P}$. For the latter embedding we need to define morphisms in the quotient category with right roofs.

**Definition 2.1** The relative singularity category with respect to a contravariantly finite subcategory $\text{prj-R} \subseteq \mathcal{X} \subseteq \text{mod-}\mathcal{R}$ is the Verdier quotient category

$$\Delta_{\mathcal{X}}(\mathcal{R}) := \frac{\mathbb{D}^b(\text{mod-}\mathcal{X})}{\mathcal{P}}.$$ 

When $\mathcal{X} = \text{prj-R}$, then $\Delta_{\text{prj-R}}(\mathcal{R})$ is not nothing else than the usual singularity category $\mathbb{D}_{\text{sg}}(\mathcal{R})$. For the special case, $\mathcal{X} = \text{Gprj-R}$, the resulting singularity category is called the Gorenstein singularity category, assuming $\text{Gprj-R}$ is contravariantly finite in $\text{mod-}\mathcal{R}$. For instance, over virtually Gorenstein Artin algebras the subcategory of Gorenstein projective modules is always contravariantly finite. This class of algebras which has been introduced in [10].

In the next result a connection between the classic singularity category and the relative one is stated.

**Theorem 2.2** Let $\mathcal{X}$ be a contravariantly finite subcategory of $\text{mod-}\mathcal{R}$ containing $\text{prj-R}$. Then, we have the following equivalences of triangulated categories

$$\mathbb{D}_{\text{sg}}(\mathcal{R}) \simeq \frac{\mathbb{D}^b(\text{mod-}\mathcal{X})/\mathbb{D}_0^b(\text{mod-}\mathcal{X})}{\mathcal{P}} \simeq \frac{\mathbb{D}^b(\text{mod-}\mathcal{X})}{\text{Thick}(\mathbb{D}_0^b(\text{mod-}\mathcal{X}) \cup \mathcal{P})} \simeq \frac{\mathbb{D}^b(\text{mod-}\mathcal{X})/\mathcal{P}}{\mathbb{D}_0^b(\text{mod-}\mathcal{X})}.$$ 

In particular, $\mathbb{D}_{\text{sg}}(\mathcal{R}) \simeq \frac{\Delta_{\mathcal{X}}(\mathcal{R})}{\mathbb{D}_0^b(\text{mod-}\mathcal{X})}$.

**Proof** For the first equivalence, note that by the definition one can see that $\vartheta(\text{Hom}_R(\_ , P) |_{\mathcal{X}}) = P$ for any $P \in \text{prj-R}$, but this fact implies that the functor $\widetilde{\mathbb{D}}_\vartheta^b$ can be restricted to the subcategories $\mathcal{P}$ and $\mathbb{K}^b(\text{prj-R})$. Hence, $\widetilde{\mathbb{D}}_\vartheta^b$ induces a triangle equivalence between the Verdier localization categories $\frac{\mathbb{D}^b(\text{mod-}\mathcal{X})/\mathbb{D}_0^b(\text{mod-}\mathcal{X})}{\mathcal{P}}$ and $\mathbb{D}^b(\text{mod-}\mathcal{R})/\mathbb{K}^b(\text{prj-R})$, as desired. The last two equivalences follow from the universal property of the triangulated quotient categories, see also [36, Proposition 2.3]. \(\square\)
If the ring \( R \) is clear from the context, we often use \( \Delta(\mathcal{X}) \) instead of \( \Delta_{\mathcal{X}}(R) \), e.g., \( \Delta(\text{Gprj}-R) \) instead of the long notation \( \Delta_{(\text{Gprj}-R)}(R) \). Hence, by our theorem, over a virtually Gorenstein Artin algebra \( \Lambda \), we can describe \( \mathcal{D}_{sg}(\Lambda) \) as the Verdier quotient \( \mathcal{D}^b(\text{Gprj-}\Lambda) \cdot \Delta(\text{Gprj}-\Lambda) \).

By putting some finiteness conditions on \( \mathcal{X} \) as in the following, we will obtain more interesting description of \( \mathcal{D}_{sg}(R) \). For a module \( M \) in \( \text{mod-}R \), we say that the \( X \)-dimension of \( M \) is finite, if there an exact sequence
\[
0 \to X_n \to \cdots \to X_1 \to X_0 \to M \to 0
\]
in \( \text{mod-}R \) with all \( X_i \in \mathcal{X} \) and it remains exact by applying \( \text{Hom}_R(X, -) \) for each \( X \in \mathcal{X} \). We say that the global \( \mathcal{X} \)-dimension of \( R \) is finite if any module in \( \text{mod-}R \) has of finite \( \mathcal{X} \)-dimension.

**Lemma 2.3** Assume that the global \( \mathcal{X} \)-dimension of \( R \) is finite. Then, the Yoneda functor induces the following triangle equivalence
\[
\mathcal{D}^b(\text{mod-}\mathcal{X}) \simeq \mathbb{K}^b(\mathcal{X}).
\]

**Proof** Take \( F \) in \( \text{mod-}\mathcal{X} \) with a projective presentation
\[
\text{Hom}_R(-, X_1) |_{\mathcal{X}} \xrightarrow{\text{Hom}_R(-, d) |_{\mathcal{X}}} \text{Hom}_R(-, X_0) |_{\mathcal{X}} \to F \to 0.
\]
By our assumption, there is an exact sequence
\[
0 \to X_n \to \cdots \to X_2 \to \text{Ker}(d) \to 0
\]
with \( X_i \) belong to \( \mathcal{X} \) and remains exact by applying \( \text{Hom}_R(X, -) \) for any \( X \in \mathcal{X} \). By applying the Yoneda functor on the above exact sequence and then gluing the obtained exact sequence in the functor category with \( 0 \to \text{Hom}_R(-, \text{Ker}(d)) |_{\mathcal{X}} \to \text{Hom}_R(-, X_1) |_{\mathcal{X}} \to \text{Hom}_R(-, X_0) |_{\mathcal{X}} \to F \to 0 \), obtained from the projective presentation, we observe the projective dimension of \( F \) in \( \text{mod-}\mathcal{X} \) is finite. On the other hand, it is known that in this case the existence of the triangle equivalence \( \mathcal{D}^b(\text{mod-}\mathcal{X}) \simeq \mathbb{K}^b(\text{prj-}\mathcal{X}) \), where \( \text{prj-}\mathcal{X} \) denotes the category of projective functors in \( \text{mod-}\mathcal{X} \). By our convention since \( \mathcal{X} \) is closed under direct summands then \( \text{prj-}\mathcal{X} \) consists of all representable functors \( \text{Hom}_R(-, X) |_{\mathcal{X}}, \) where \( X \in \mathcal{X} \). Using this fact and the Yoneda functor turns out the triangle equivalence \( \mathbb{K}^b(\text{prj-}\mathcal{X}) \simeq \mathbb{K}^b(\mathcal{X}) \).

The equivalence in the above lemma induces the triangle equivalence
\[
\Delta_{\mathcal{X}}(R) \simeq \mathbb{K}^b(\mathcal{X})/\mathbb{K}^b(\text{prj-}\mathcal{X}),
\]
see the proof of the next theorem for more explanations.

**Theorem 2.4** Let \( \mathcal{X} \) be a contravariantly finite subcategory of \( \text{mod-}R \) containing \( \text{prj-}R \). Assume that the global \( \mathcal{X} \)-dimension of \( R \) is finite. Then, we have the following equivalences of triangulated categories
\[
\mathcal{D}_{sg}(R) \simeq \mathbb{K}^b(\mathcal{X})/\mathbb{K}^b_{ac}(\mathcal{X}) \simeq \mathbb{K}^b(\mathcal{X})/\text{Thick}(\mathbb{K}^b_{ac}(\mathcal{X}) \cup \mathbb{K}^b(\text{prj-}R)) \simeq \mathbb{K}^b(\mathcal{X})/\mathbb{K}^b_{ac}(\mathcal{X}).
\]
The equivalence on objects acts as follows. Let $\mathbf{X}$ be a complex in $\mathcal{D}^b(\text{mod-}\mathcal{X})$ and $\Theta : \mathbf{P} \rightarrow \mathbf{X}$ a $\mathbb{K}$-projective resolution of $\mathbf{X}$. By our assumption the $\mathbb{K}$-projective resolution $\mathbf{P}$ has to be a bounded complex of projective functors. Because of the Yoneda lemma, $\mathbf{P}$ can be presented as a complex obtained by applying the Yoneda functor on some bounded complex of modules in $\mathcal{X}$, say $\mathbf{Q}$. In fact, under the equivalence $\mathbf{X}$ is mapped into a complex in $\mathbb{K}^b(\mathcal{X})$ homotopy equivalent to $\mathbf{Q}$. Therefore, by the construction of the equivalence as already explained, one can see the equivalence can be restricted to $\mathcal{D}^b_0(\text{mod-}\mathcal{X}) \simeq \mathbb{K}^b_{\text{ac}}(\mathcal{X})$ and $\mathcal{P} \simeq \mathbb{K}^b(\text{prj-R})$. For the first restricted equivalence, assume $\mathbf{X}$ belongs to $\mathcal{D}^b_0(\text{mod-}\mathcal{X})$, we consider the triangle $\mathbf{P} \rightarrow \mathbf{X} \rightarrow \text{cone}(\Theta) \rightarrow \mathbf{P}[1]$, as described in the above. The valuation of the triangle on the regular module $R \in \text{prj-R}$ gives us the triangle $\mathbf{P}(R) \rightarrow \mathbf{X}(R) \rightarrow \text{cone}(\Theta)(R) \rightarrow \mathbf{P}(R)[1]$ in the bounded derived category of abelian groups. By the valued triangle we observe that $\mathbf{P}(R)$ is an exact complex of abelian groups. The isomorphism $\mathbf{P}(R) = \text{Hom}_R(R, \mathbf{Q}) \simeq \mathbf{Q}$, as complexes of $R$-modules, yields $\mathbf{Q}$ is an exact complex, so $\mathbf{Q}$ is in $\mathbb{K}^b_{\text{ac}}(\mathcal{X})$, as desired. Now applying Theorem 2.2 and using the above-mentioned observations we can complete the proof. 

Fortunately, over an arbitrary Artin algebra $\Lambda$ there is always a subcategory $\mathcal{X} \subseteq \text{mod-}\Lambda$ which satisfies the assumption we need in Theorem 2.4. For instance, Auslander in his Queen Mary College lecture [5] showed the global dimension of $\text{End}(\bigoplus_{i=0}^n \mathbb{A} / J^i)$, where $J$ is radical and $n$ lowey length of $\Lambda$, respectively, is finite. Consequently, $\mathcal{X} = \mathrm{add-} \bigoplus_{i=0}^n \mathbb{A} / J^i$ holds the needed conditions of Theorem 2.4. Therefore, we can always estimate the singularity categories of Artin algebras via an Artin algebra of finite global dimension.

Recall that a two-sided noetherian ring $R$ is Gorenstein provided that $\text{id}_R R < \infty$ and $\text{id}_{R^{op}} R < \infty$. Let $M$ be in $\text{mod-}R$ and $M^* = \text{Hom}_R(M, R)$. Recall that $M$ is Gorenstein projective provided that there is an exact complex $\mathbf{P}$ of finitely generated projective $R$-modules such that the Hom-complex $\mathbf{P}^* = \text{Hom}_R(\mathbf{P}, R)$ is still exact and $M$ is isomorphic to a certain cocycle $Z^i(\mathbf{P})$ of $\mathbf{P}$. We denote by $\text{Gprj-R}$ the subcategory of $\text{mod-}R$ formed by all Gorenstein projective $R$-modules.

**Example 2.5** let $R$ be a Gorenstein ring. It is known that over Gorenstein rings any module has of finite Gorenstein dimension [22]. So, the subcategory $\text{Gprj-R}$ satisfies the assumption of Theorem 2.4 and so we can apply the theorem to get the equivalences $\mathcal{D}_{\text{sg}}(R) \simeq \frac{\Delta(\text{Gprj-R})}{\mathbb{K}^b_{\text{ac}}(\text{Gprj-R})}$. The subcategory $\text{Gprj-R}$ inherits an exact structure from $\text{mod-}R$, which is Frobenius. Then by the Happel’s work the stable category $\text{Gprj-R}$ becomes naturally a triangulated category. We know by Buchweitz’s theorem [11] the existence of a triangle equivalence $\mathcal{D}_{\text{sg}}(R) \simeq \text{Gprj-R}$. Now by combing the equivalences we get the triangle equivalence $\text{Gprj-R} \simeq \frac{\Delta(\text{Gprj-R})}{\mathbb{K}^b_{\text{ac}}(\text{Gprj-R})}$.

Let us summarize the equivalences of the triangulated categories appeared above in the following diagram. To have the equivalences on the left side we need to assume that the global $\mathcal{X}$-dimension of $R$ is finite.
3 (Relative) Singularity categories via homotopy categories of exact complexes

Throughout this section, let $\text{prj-R} \subseteq \mathcal{X} \subseteq \text{mod-R}$. In this section we will study the Verdier quotient $\mathbb{K}^{b}(\mathcal{X})/\mathbb{K}^{b}(\text{prj-R})$ and show that it can be embedded into the homotopy category of exact complexes over $\mathcal{X}$. Then by applying our results from the preceding section in conjunction with the embedding we give a new description of $\mathbb{D}_{\text{sg}}(R)$ such that only exact complexes are involved.

Let $\mathbb{K}^{-,\text{p}}(\mathcal{X})$ denote the subcategory of the homotopy category $\mathbb{K}(\mathcal{X})$ consisting of all complexes which are homotopy-equivalent to a upper bounded exact complex $P$ over $\mathcal{X}$ such that for some $n$, $P^{i} \in \text{prj-R}$ for all $i \leq n$. In fact, there are only finitely many terms of $P$ to be non-projective. We know that the inclusion $\mathbb{K}^{b}(\text{ac}(\text{Mod-R})) \hookrightarrow \mathbb{K}(\text{Mod-R})$ has a right adjoint, denoted by $\Phi_{1}$. The right adjoint sends a complex $X$ to the complex $Y$ fitting into a triangle $P_{X} \to X \to Y \to P[1]$, where $P_{X}$ is a $\mathbb{K}$-projective complex, see [2, Proposition 1.6]. Since $\mathbb{K}^{b}(\mathcal{X})$ is generated by all complexes concentrated at degree zero with terms in $\mathcal{X}$, then the essential image (in $\mathbb{K}(\mathcal{X})$) of the restricted functor $\Phi_{1}$ over $\mathbb{K}^{b}(\mathcal{X})$ is exactly $\mathbb{K}^{-,\text{p}}(\mathcal{X})$ (see the proof of the next result for more explanations). Since the restricted functor vanishes on $\mathbb{K}^{b}(\text{prj-R})$, hence by the universal property there is a triangle functor $\Psi : \mathbb{K}^{b}(\mathcal{X})/\mathbb{K}^{b}(\text{prj-R}) \to \mathbb{K}^{-,\text{p}}(\mathcal{X})$. To distinguish, we sometimes use $\Psi_{\mathcal{X}}$ when we are dealing with several subcategories, simultaneously.

For any $X$ in $\mathcal{X}$, we fix the following exact complex in $\mathbb{K}^{-,\text{p}}(\mathcal{X})$,

\[ C_{X} : \cdots \to P_{X}^{n} \xrightarrow{d_{X}^{n}} \cdots \to P_{X}^{1} \xrightarrow{d_{X}^{1}} P_{X}^{0} \xrightarrow{d_{X}^{0}} X \to 0 \to \cdots, \]

where $P_{X}^{i}$ are in $\text{prj-R}$, and $X$ at degree 0 and $P_{X}^{i}$ at degree $-i - 1$. Let $P_{X}$ denote the deleted projective resolution induced by $C_{X}$, there is the triangle $P_{X} \to X \to \cdots$.
\[ C_X \to P_X[1]. \] Hence by the triangle, we may assume \( \Phi(X) = C_X \). We will use this convention throughout the paper.

**Proposition 3.1** The functor \( \Psi \), defined in the above, gives the following equivalence of triangulated categories

\[
\frac{\mathbb{K}^b(\mathcal{X})}{\mathbb{K}^b(\text{prj-R})} \simeq \mathbb{K}^{\leq 0}(\mathcal{X}).
\]

**Proof** We first show that the essential image of \( \Psi \), or equivalently \( \Phi \), is equal to \( \mathbb{K}^{\leq 0}(\mathcal{X}) \). Hence \( \Psi \) is dense. Since \( \mathcal{X} \) generates \( \mathbb{K}^b(\mathcal{X}) \), meaning that \( \mathbb{K}^b(\mathcal{X}) \) is the smallest triangulated subcategory containing \( \mathcal{X} \), hence all \( \Psi(X) = C_X \) generates the essential image. Since any \( C_X \) belongs to \( \mathbb{K}^{\leq 0}(\mathcal{X}) \), so the essential image of \( \Psi \) is included in \( \mathbb{K}^{\leq 0}(\mathcal{X}) \).

Conversely, let \( X \) be a complex in \( \mathbb{K}^{\leq 0}(\mathcal{X}) \). We may present it as in the following

\[
\cdots \to P^n \to P^0 \to X^n \to \cdots \to X^1 \to X^0 \to 0
\]

where \( P^i \) in \( \text{prj-R} \). If all \( X^i \), for \( i \geq 1 \) are projective, then \( X \) becomes a projective resolution of \( X^0 \). Hence in this case \( X \simeq \Psi(X^0) \), so it is in the essential image. Take a projective resolution of \( X_0 \)

\[
\cdots \to Q^n \to \cdots \to Q^1 \to Q^0 \to X^0 \to 0.
\]

By the lifting property we get the following chain map

\[
\cdots \to Q^n \to \cdots \to Q^n \to \cdots \to Q^0 \to X^0 \to 0
\]

It is easy to see that the mapping cone of the above chain map is homotopy equivalent to the following complex

\[
\cdots \to Q^n \oplus P^0 \to X^n \to \cdots \to Q^0 \oplus X^2 \to X^1 \to 0.
\]

The obtained complex has non-projective terms in at most \( n - 1 \) degrees. Repeating this argument we can reach to a complex in \( \mathbb{K}^{\leq 0}(\mathcal{X}) \) with at most one non-projective term, which can be considered as a shifting of a projective resolution of some object in \( \mathcal{X} \). It must be homotopy-equivalent to \( C_X[i] \) for some \( i \in \mathbb{Z} \) and \( X \) in \( \mathcal{X} \). Therefore, by such a construction we can see \( X \) must be in the essential image, by the construction it lies in a sequence of the triangles, as described in Remark 3.2. Now we prove the functor \( \Psi \) is full and faithful. Recall that \( \Psi \) acts on objects the same as \( \Phi \), and for a right roof \( X \xleftarrow{f} Z \xrightarrow{g} Y \), with cone(\( g \)) in \( \mathbb{K}^b(\text{prj-R}) \), is defined by \( \Phi(g)^{-1} \circ \Phi(f) \). Assume \( \Phi \) becomes zero on a roof \( X \xleftarrow{f} Z \xrightarrow{g} Y \). Then \( \Phi(f) \) has to be zero. By the
construction of $\Phi$ on morphisms there is the following commutative diagram with the triangles on rows

$$
\begin{array}{ccc}
P_X & \longrightarrow & X \\
& f \downarrow & \Phi(f) \\
P_Z & \longrightarrow & Z
\end{array}
\quad
\begin{array}{ccc}
& \Phi(X) & \longrightarrow P_X[1] \\
\Phi(f) & \downarrow & \\
& \Phi(Z) & \longrightarrow P_Z[1]
\end{array}
$$

Since $\Phi(f)$ is zero then $f$ factors through the morphism $P_Z \to Z$. Since $X$ and $Z$ both are bounded complexes and $P_Z$ a upper bounded complex, we can deduce that the morphism $f$ factors through some brutal truncation $P_Z^{\geq m}$, that is in $\mathbb{K}^b(\text{prj-R})$. Hence $\Psi$ is faithful.

For fullness, since $\mathcal{X}$ generates $\mathbb{K}^b(\mathcal{X})$, it suffices to show that for any $X$ and $Y$ in $\mathcal{X}$, and $i$, $j \in \mathbb{Z}$, the induced group homomorphism

$$
\text{Hom}_{\mathbb{K}^b(\mathcal{X})}(X[i], Y[j]) \to \text{Hom}_{\mathbb{K}^b(\mathcal{X})}(C_X[i], C_Y[j])
$$

is surjective. For simplicity, we only prove for $i = 0$. The exact complexes $C_X$ and $C_Y$, as our notations, are presented as below, respectively,

$$
\begin{align*}
C_X : \cdots & \longrightarrow P^n_X \longrightarrow \cdots \longrightarrow P^1_X \longrightarrow P^0_X \rightarrow X \rightarrow 0 \rightarrow \cdots, \\
C_Y : \cdots & \longrightarrow P^n_Y \longrightarrow \cdots \longrightarrow P^1_Y \longrightarrow P^0_Y \rightarrow Y \rightarrow 0 \rightarrow \cdots,
\end{align*}
$$

where $X$ and $Y$ are settled in degree 0. Let $[f]$ be a homotopy equivalence class in $\text{Hom}_{\mathbb{K}^b(\mathcal{X})}(X, Y[j])$. For $j > 0$, It is easy to see that (only by using the lifting property) $\text{Hom}_{\mathbb{K}^b(\mathcal{X})}(C_X, C_Y[j]) = 0$, so nothing to prove. For $j = 0$, the equivalence class of the right roof $X \xrightarrow{f^0} Y \leftarrow Y$ is mapped into $[f]$ by $\Psi$. It remains for the case $j < 0$. The chain map $f$ is presented as below

$$
\begin{array}{ccc}
\cdots & \longrightarrow & P^0_X \\
& f^1 \downarrow & \leftarrow f^0 \\
\cdots & \longrightarrow & P^{-j}_Y \\
& \longrightarrow & P^{-j-1}_Y \\
& \longrightarrow & \cdots \\
& \longrightarrow & P^0_Y \\
& \longrightarrow & Y \\
& \longrightarrow & 0
\end{array}
$$

Put $Z$ to be the complex $0 \to P^{-j-1}_Y \to \cdots \to P^0_Y \to Y \to 0$, where $P^{-j-1}_Y$ is at degree 0. Let $g : X \to Z$, resp. $s : Y[j] \to Z$, be a chain map to be zero on all degrees except degree 0, resp. $-j$, with $f^0$, resp. $\text{id}_Y$. We claim that the equivalence...
class of the right roof $X \xrightarrow{[g]} Z \xleftarrow{[s]} Y[j]$ is mapped into $[f]$. Consider the following commutative diagram

\[
\begin{array}{cccccccc}
\cdots & P_{Y}^{-j+2} & \rightarrow & P_{Y}^{-j+1} & \rightarrow & P_{Y}^{-j} & \rightarrow & \cdots & 0 & \rightarrow & 0 & \rightarrow & 0 \\
\downarrow & & & \downarrow & & \downarrow & & & \downarrow & & \downarrow & & \\
\cdots & 0 & \rightarrow & 0 & \rightarrow & P_{Y}^{-j-1} & \rightarrow & \cdots & P_{Y}^{0} & \rightarrow & Y & \rightarrow & 0 \\
\downarrow & & & \downarrow & & \downarrow & & & \downarrow & & \downarrow & & \\
\cdots & P_{Y}^{-j+1} & \rightarrow & P_{Y}^{-j} & \rightarrow & P_{Y}^{-j-1} & \rightarrow & \cdots & P_{Y}^{0} & \rightarrow & Y & \rightarrow & 0.
\end{array}
\]

From the above diagram we have the following triangle

$$C_{Y}^{≤j-1}[j-1] \rightarrow Z \rightarrow C_{Y}[j] \rightarrow C_{Y}^{≤j-1}[j].$$

Since $C_{Y}^{≤j-1}[j-1]$ belongs to $\mathcal{K}^{-}(\text{prj-R})$ and $C_{Y}[j]$ an exact complex, then by the triangle we can conclude that $\Phi(Z) \simeq C_{Y}[j]$. In addition, we can see the equalities $\Phi([g]) = [f]$ and $\Phi([s]) = \text{id}_{C_{Y}[j]}$, only by the definition of $\Phi$ on morphisms together with using the triangle. Hence $\Psi$ assigns the morphism $[f]$ to the equivalence class of the right roof $X \xrightarrow{[g]} Z \xleftarrow{[s]} Y[j]$, what we wanted to prove. So the proof is now complete.

Remark 3.2 As a result from the first part of the proof of Proposition 3.1, we can see for any complex $X$ in $\mathcal{K}^{-}(\mathcal{X})$ there is a finite sequences of triangles as the following

$$C_{M_{i}}[r_{i}] \rightarrow X_{i} \rightarrow X_{i+1} \rightarrow C_{M_{i}}[r_{i} + 1],$$

where $0 \leq i \leq n$, $r_{i} \in \mathbb{Z}$ and $M_{i} \in \mathcal{X}$, $X_{0} = X$ and $X_{n+1} = 0$.

As an application of the above proposition in the next result we give a nice description for the usual singularity category $\mathbb{D}_{\text{sg}}(R)$.

Theorem 3.3 Let $\text{prj-R} \subseteq \mathcal{X} \subseteq \text{mod-R}$ be a contravariantly finite subcategory. Assume the global $\mathcal{X}$-dimension of $R$ is finite. Then, there exists the following equivalence of triangulated categories

$$\mathbb{D}_{\text{sg}}(R) \simeq \frac{\mathcal{K}^{-}(\mathcal{X})}{\mathcal{K}_{\text{ac}}^{-}(\mathcal{X})}.$$
Proof By the definition of the equivalence $\Psi$, as defined in the beginning of this section, we have the following commutative diagram with equivalences in the rows

\[
\begin{array}{ccc}
\Delta_{\mathcal{X}}(R) & \xrightarrow{\Psi} & \mathbb{K}^{-,-p}(\mathcal{X}') \\
\mathbb{K}^{-,-p}(\mathcal{X}) & \xrightarrow{\Psi} & \mathbb{K}^{-,-p}(\mathcal{X})
\end{array}
\]

In fact, the restriction of $\Psi$ on $\mathbb{K}^{-,-p}(\mathcal{X})$ in the above diagram is isomorphic to the identity functor. Now in view of the above diagram the functor $\Psi$ induces a triangle equivalence between the Verdier quotients $\Delta_{\mathcal{X}}(R)$ and $\mathbb{K}^{-,-p}(\mathcal{X})$. But we know from Theorem 2.4, the first Verdier quotient is indeed equivalent to $\mathbb{D}_{sg}(R)$. So we are done.

Let us give another application of Proposition 3.1. Let $k$ be a commutative artinian ring. We recall that a $k$-category $\mathcal{C}$ is said to be Hom-finite if for each two objects $X$ and $Y$ in $\mathcal{C}$, $\text{Hom}_\mathcal{C}(X, Y)$ has finite length.

Proposition 3.4 Assume $\Lambda$ is an Artin $k$-algebra. Then the category $\mathbb{K}^{b}_{\text{ac}}(\mathcal{X})$ is Hom-finite. In particular, if $\mathcal{X}$ is a contravariantly finite subcategory in mod-$\Lambda$ and $\Lambda$ has finite global $\mathcal{X}$-dimension, then the relative singularity category $\Delta_{\mathcal{X}}(\Lambda)$ is Hom-finite.

Proof Thanks to the equivalence given in Proposition 3.1 we will show the statement for the triangulated category $\mathbb{K}^{-,-p}(\mathcal{X})$ instead. As Remark 3.2 says any complex in $\mathbb{K}^{-,-p}(\mathcal{X})$ can be constructed via complexes of the form $\mathbb{C}_X$ for some $X \in \mathcal{X}$. Consequently, we only need to show that $\text{Hom}_\mathbb{K}(\mathbb{C}_X[i], \mathbb{C}_Y[j])$ is of finite length (in mod-$k$) for any $X, Y \in \mathcal{X}$ and $i, j \in \mathbb{Z}$. Without of loss generality, we may assume that $i = 0$. If $j > 0$, then $\text{Hom}_\mathbb{K}(\mathbb{C}_X, \mathbb{C}_Y[j]) = 0$, so nothing remains to prove. For $j = 0$, it is clear to see that $\text{Hom}_\mathbb{K}(\mathbb{C}_X, \mathbb{C}_Y) \simeq \text{Hom}_\Lambda(X, Y)$, so we are done as the latter one is of finite length. It remains for $j < 0$. Take $[f]$ in $\text{Hom}_\mathbb{K}(\mathbb{C}_X, \mathbb{C}_Y[j])$. Since $d_{Y}^{-1} \circ f^{0} = 0$, hence $f^{0}$ induces a morphism from $X$ to $\Omega_{\Lambda}^{-j}(Y)$, say $g$. Mapping $[f]$ to $g$ gives an isomorphism of $k$-modules $\text{Hom}_\mathbb{K}(\mathbb{C}_X, \mathbb{C}_Y[j]) \simeq \text{Hom}_\Lambda(X, \Omega_{\Lambda}^{-j}(Y))$. The isomorphism completes the proof.

4 singular equivalences via relative singular and stable equivalences

In this section by our results on relative singularity categories we will give some methods to determine two rings to be singular equivalence. We follow these two following observations. First, assume $\mathcal{X} \subseteq \text{mod-}R$ and $\mathcal{X}' \subseteq \text{mod-}R'$ be the same as in Theorem 3.3. If there is a triangle equivalence between the relative singularity categories $\Delta_{\mathcal{X}}(R) \simeq \Delta_{\mathcal{X}'}(R')$, then it induces a triangle equivalence between the singularity categories $\mathbb{D}_{sg}(R) \simeq \mathbb{D}_{sg}(R')$, by Theorem 3.3, in the case that the equivalence functor can be restricted to the homotopy categories $\mathbb{K}^{b}_{\text{ac}}(\mathcal{X})$ and $\mathbb{K}^{b}_{\text{ac}}(\mathcal{X}')$. Hence, we explore to find some conditions to have the restriction. This idea is exactly applied in [29].
Another way, assume that there is an equivalence between the stable categories $\mathcal{X}$ and $\mathcal{X}'$, say $F$. We know that by [16, Theorem 3.1] one can embed the stable categories $\mathcal{X}$ and $\mathcal{X}'$ in the quotient categories $\mathcal{K}^{b}(\mathcal{X})$ and $\mathcal{K}^{b}(\mathcal{X}')$, respectively. The embeddings are defined in a canonical way, namely, by sending an object $X \in \mathcal{X}$ to the complex concentrated at degree zero with term $X$, similarly for the case $\mathcal{X}'$. What we will follow in this section is to see that how one can extend the equivalence $F$ under certain conditions to a triangle equivalence between the quotient categories, then by using the first observation to get a singular equivalence.

Let us begin with the following definition.

**Definition 4.1** We say that $\text{prj-}R \subseteq \mathcal{X} \subseteq \text{mod-}R$ satisfies condition $(\ast)$: If for any $X \in \mathcal{X}$, $\text{Hom}_{R}(Y, \Omega^{n}_{R}(X)) = 0$ for all $Y$ in $\mathcal{X}$ and all but finitely many $n > 0$, then the projective dimension of $X$ is finite. For the case that $R$ is an Artin algebra $\Lambda$. Then, the condition $(\ast)$ is equivalent to say that the projective dimension of an indecomposable module $X \in \mathcal{X}$ is finite if and only if any indecomposable module $Y$ in $\mathcal{X}'$ appears up to isomorphism as a direct summand of finitely many of syzygies $\Omega_{1}^{\ast}(X)$. Here we need to assume that the syzygies obtained by the minimal projective resolutions.

Following [29, Definition 6.16], we make the following definition.

**Definition 4.2** For a triangulated category $\mathcal{T}$ the triangulated subcategory

$$\mathcal{T}_{r} := \{ X \in \mathcal{T} \mid \text{Hom}_{\mathcal{T}}(Y, X[i]) = 0 \quad \text{for all } Y \text{ and all but finitely many } i \in \mathbb{Z} \}$$

is defined. More explanation, an object $X$ is in $\mathcal{T}_{r}$ if for any $Y$ in $\mathcal{T}$, the Hom-spaces $\text{Hom}_{\mathcal{T}}(Y, X[i])$ is non-zero for only finitely many $i$.

The $\mathcal{T}_{r}$ in [29] is called the **subcategory of right homologically finite objects of $\mathcal{T}$**.

**Proposition 4.3** Assume that $\text{prj-}R \subseteq \mathcal{X} \subseteq \text{mod-}R$ satisfies the condition $(\ast)$ and quasi-resolving. Then, $\mathcal{K}^{-\ast}_{r}(\mathcal{X}) = \mathcal{K}_{ac}^{b}(\mathcal{X})$.

**Proof** Let $X$ be in $\mathcal{K}_{ac}^{b}(\mathcal{X})$. Since $X$ is a bounded complex then it is trivial that $\text{Hom}_{\mathcal{K}}(Y, X[i]) = 0$ for sufficiently small $i$. By the lifting property one can see that $\text{Hom}_{\mathcal{K}}(Y, X[i]) = 0$ for $i > |m| + |n|$, where $m$, resp. $n$, is the least degree such that $Y^{m}$ is non-projective, resp. the biggest degree with $X^{n} \neq 0$. Remember the complexes are assumed that the differentials raise degree. So $\mathcal{K}_{ac}^{b}(\mathcal{X}) \subseteq \mathcal{K}^{-\ast}_{r}(\mathcal{X})$. For the inverse inclusion, let $X \in \mathcal{K}^{-\ast}_{r}(\mathcal{X})$. Denote by $n$ the biggest degree with $X^{n} \neq 0$ and $m$ the least degree such that $X$ is non-projective. For simplicity, we may assume that $n = 0$. For any $Y$ in $\mathcal{X}$ and $i > |m|$, we infer $\text{Hom}_{\mathcal{K}}(C_{Y}[i], X) \cong \text{Hom}_{R}(Y, \Omega_{R}^{\ast}(\ker(d^{m})))$. Note that since $\mathcal{X}$ is closed under kernels of epimorphisms then $\ker(d^{m})$ belongs to $\mathcal{X}$. By the condition $(\ast)$ and in view of the isomorphisms we get the projective dimension of $\ker(d^{m})$ is finite. This implies that $X$ must be homotopy-equivalent to a bounded complex, as desired. $\square$

The above proposition shows that $\mathcal{K}_{ac}^{b}(\mathcal{X})$ has a categorical characterization which is vital for our next result.

Let $\Lambda$ be an Artin algebra. We say that a subcategory of $\text{mod-}\Lambda$ is of **finite representation type** if it has up to isomorphism only finitely many indecomposable modules.
In the case that mod-\(\Lambda\) is of finite representation type, the \(\Lambda\) is called representation-finite.

**Proposition 4.4** Assume \(\Lambda\) is an Artin algebra. Let prj-\(\Lambda\) \(\subseteq\) \(\mathcal{X}\) \(\subseteq\) mod-\(\Lambda\) be closed under syzygies and of finite representation type. Then \(\mathcal{X}\) satisfies the condition \((\ast)\).

**Proof** Let \(X\) in \(\mathcal{X}\) satisfy the property in the condition \((\ast)\), i.e., any indecomposable module \(Y\) in \(\mathcal{X}\) is isomorphic to a direct summand of finitely many syzygies \(\Omega^i_\Lambda(X)\). Since \(\mathcal{X}\) is closed under syzygies then the set \(\{\Omega^i_\Lambda(X)\}_{i \geq 0}\) lies in \(\mathcal{X}\). But since \(\mathcal{X}\) is of finite representation type, then there an indecomposable module \(Y\) in \(\mathcal{X}\) such that it is isomorphic to a direct summand of \(\Omega^i_\Lambda(X)\) for infinitely many \(i\). This gives a contradiction. So we are done. \(\square\)

We remind by our convention any subcategory in the paper is assumed to be closed under direct summands. We used this convention in the proof of the above proposition.

**Proposition 4.5** Assume that prj-\(R\) \(\subseteq\) \(\mathcal{X}\) \(\subseteq\) mod-\(R\) and prj-\(R'\) \(\subseteq\) \(\mathcal{X}'\) \(\subseteq\) mod-\(R'\) are contravariantly finite subcategories in the corresponding ambient categories. Further, \(R\), resp. \(R'\), has finite global \(\mathcal{X}\)-dimension, resp. \(\mathcal{X}'\)-dimension, and both subcategories satisfy the condition \((\ast)\) and being quasi-resolving. Then, if there is a triangle equivalence \(\Delta_\mathcal{X}(R) \simeq \Delta_\mathcal{X}'(R')\) of triangulated categories, then there is a triangle equivalence \(\mathcal{D}_{sg}(R) \simeq \mathcal{D}_{sg}(R')\).

**Proof** The statement is a consequence of Proposition 4.3 and Theorem 3.3. In fact, a triangle equivalence \(\Delta_\mathcal{X}(R) \simeq \Delta_\mathcal{X}'(R')\), because of the intrinsic characterization given in Proposition 4.3, restricts to a triangle equivalence between \(\mathcal{K}_{ac}^b(\mathcal{X}) \simeq \mathcal{K}_{ac}^b(\mathcal{X}')\). Thus it induces a triangle equivalence between the associated Verdier quotient categories \(\Delta_\mathcal{X}(R)/\mathcal{K}_{ac}^b(\mathcal{X})\) and \(\Delta_\mathcal{X}'(R')/\mathcal{K}_{ac}^b(\mathcal{X}')\). By Theorem 3.3, we observe that the Verdier quotient categories are equivalent to the respective singularity categories, so the result. \(\square\)

In the next we give some examples satisfying the assumptions of the above proposition.

**Example 4.6**

1. Let \(\Lambda\) be a CM-finite Gorenstein algebra. Recall that \(\Lambda\) is called CM-finite, if Gprj-\(\Lambda\) has only finitely many isomorphism classes of indecomposable modules. It is known that Gprj-\(\Lambda\) is closed under kernels of epimorphisms, see e.g. [12, Proposition 2.1.7]. Then, in view of Proposition 4.5, Gprj-\(\Lambda\) satisfies all the conditions we need in Proposition 4.5.

2. Let \(\Lambda\) be an Artin algebra with radical square zero, that is \(\text{rad}^2(\Lambda) = 0\). Then the additive closure \(S(\Lambda) = \text{add}(\Lambda \oplus \Lambda/\text{rad}(\Lambda))\) of projective modules and simple modules verifies the conditions. To check, only use this fact that submodules of projective modules are semisimple over Artin algebras with radical square zero.

3. Assume \(\Lambda\) is representation-finite. Then, it is obvious that mod-\(\Lambda\) satisfies the assumptions.

Specializing Proposition 4.5 to the above examples follows:

**Corollary 4.7** Let \(\Lambda\) and \(\Lambda'\) be two Artin algebras. The following statements hold.
1. Assume that \( \Lambda \) and \( \Lambda' \) are CM-finite Gorenstein. Then, if there is a triangulated equivalence \( \Delta(\mathrm{Gprj-}\Lambda) \simeq \Delta(\mathrm{Gprj-}\Lambda') \), then \( \mathbb{D}_{sg}(\Lambda) \simeq \mathbb{D}_{sg}(\Lambda') \).

2. Assume that \( \Lambda \) and \( \Lambda' \) are algebras with radical squarer zero. Then, if there is a triangulated equivalence \( \Delta(S(\Lambda)) \simeq \Delta(S(\Lambda')) \), then \( \mathbb{D}_{sg}(\Lambda) \simeq \mathbb{D}_{sg}(\Lambda') \).

3. Assume that \( \Lambda \) and \( \Lambda' \) are representation-finite. Then, if there is a triangulated equivalence \( \Delta(\mathrm{mod-}\Lambda) \simeq \Delta(\mathrm{mod-}\Lambda') \), then \( \mathbb{D}_{sg}(\Lambda) \simeq \mathbb{D}_{sg}(\Lambda') \).

In concern of the second observation given in the beginning of this section we establish the following construction.

**Construction 4.8** Assume that \( \mathrm{prj-R} \subseteq \mathcal{X} \subseteq \mathrm{mod-R} \) and \( \mathrm{prj-R'} \subseteq \mathcal{X'} \subseteq \mathrm{mod-R'} \) are contravariantly finite subcategories in the corresponding ambient categories and closed under syzygies. Moreover, there is an equivalence \( F : \mathcal{X} \rightarrow \mathcal{X}' \) which is induced from a functor \( F : \mathcal{X} \rightarrow \mathcal{X}' \) preserving projective modules and syzygies, i.e., for any \( X \in \mathcal{X} \), \( F(\Omega R(X)) \simeq \Omega R'(F(X)) \) in \( \mathcal{X}' \). Since the functor \( F \) is additive, then it induces the triangle functor \( \mathbb{K}^b F \) between the bounded homotopy categories \( \mathbb{K}^b(\mathcal{X}) \) and \( \mathbb{K}^b(\mathcal{X}') \). On the other hand, \( F \) preserves projective modules then the functor \( \mathbb{K}^b F \) preserves the bounded homotopy category of projective modules as well. Hence \( \mathbb{K}^b F \) induces a triangle functor \( \mathbb{K}^b F : \mathbb{K}^b(\mathcal{X}) \rightarrow \mathbb{K}^b(\mathcal{X}') \). Denote by \( \mathbb{K}^p F := \Psi_{\mathcal{X}'} \circ \mathbb{K}^b F \circ \Psi_{\mathcal{X}}^{-1} \), see the beginning of Sect. 3 for the definitions of \( \Psi_{\mathcal{X}} \) and \( \Psi_{\mathcal{X}'} \). The introduced functors satisfy the following commutative diagram

\[
\begin{array}{ccc}
\mathcal{X} &=& \mathbb{K}^b(\mathcal{X}) \\
\downarrow F & & \downarrow \mathbb{K}^b F \\
\mathcal{X}' &=& \mathbb{K}^b(\mathcal{X}') \\
\end{array}
\]

Assume, further, \( \mathcal{X} \) and \( \mathcal{X}' \) are quasi-resolving, and the functor \( F \) is exact, meaning that any short exact sequence with terms in \( \mathcal{X} \) is mapped by \( F \) into a short exact sequence in \( \mathcal{X}' \). By applying terms by terms the functor \( F \) and using its exactness, we can extend the functor \( F \) to \( \mathbb{K}^b \mathcal{F} : \mathbb{K}^{-p}(\mathcal{X}) \rightarrow \mathbb{K}^{-p}(\mathcal{X}') \). By the construction, we clearly have the following commutative diagram

\[
\begin{array}{ccc}
\mathbb{K}^b_{ac}(\mathcal{X}) &=& \mathbb{K}^{-p}(\mathcal{X}) \\
\downarrow \mathbb{K}^p \mathcal{F} & & \downarrow \mathbb{K}^p \mathcal{F} \\
\mathbb{K}^b_{ac}(\mathcal{X}') & & \mathbb{K}^{-p}(\mathcal{X}'). \\
\end{array}
\]

Thus because of the above diagram \( \mathbb{K}^p \mathcal{F} \) induces a triangle functor between the corresponding Verdier quotient categories \( \mathbb{K}^{-p}(\mathcal{X}) \) and \( \mathbb{K}^{-p}(\mathcal{X}') \). Now by considering Theorem 3.3, we obtain a triangle functor from \( \mathbb{D}_{sg}(\mathcal{R}) \) to \( \mathbb{D}_{sg}(\mathcal{R}') \), assuming
the global $\mathcal{X}$-dimension, resp. $\mathcal{X}'$-dimension, of $R$, resp. $R'$, is finite. To have the functor $\mathbb{K}^p F$, we can drop the assumption to be closed under kernels of epimorphisms (being quasi-resolving) for the involved subcategories; set another assumption on $F$. That is $F$ to be a restriction of an exact functor from a quasi-resolving subcategory $\mathcal{Y} \subseteq \text{mod-}R$ containing $\mathcal{X}$ to a quasi-resolving subcategory $\mathcal{Y}' \subseteq \text{mod-}R'$ containing $\mathcal{X}'$. As a special case, when $F$ is a restriction of an exact functor from mod-$R$ to mod-$R'$. In this case no need to assume that the functor $F$ to preserve the syzygies, since the exactness of the functor automatically implies it.

**Proposition 4.9** Keep in mind all the notations in the above construction. Then the following statements hold.

1. The functor $\mathbb{K}^p F$ is a triangle equivalence.
2. Assume $\mathcal{X}$ and $\mathcal{X}'$ are quasi-resolving. Then, $\mathbb{K}^p F$ is a triangle equivalence.
3. Assume $F$ is a restriction of an exact functor between quasi-resolving subcategories $\mathcal{X}' \subseteq \mathcal{Y} \subseteq \text{mod-}R$ and $\mathcal{X}' \subseteq \mathcal{Y}' \subseteq \text{mod-}R'$. Then, $\mathbb{K}^p F$ is a triangle equivalence. In this case we still use the notation $\mathbb{K}^p F$, it is indeed the restricted functor $\mathbb{K}^p G \mid \colon \mathbb{K}^{-p}(\mathcal{X}) \to \mathbb{K}^{-p}(\mathcal{X}')$, where $G$ is the exact functor between $\mathcal{Y}$ and $\mathcal{Y}'$.

**Proof** Since the subcategory $\mathcal{C}$ consisting of all $\mathcal{C}_X[i]$ for all $X \in \mathcal{X}$ and $i \in \mathbb{Z}$ generates $\mathbb{K}^{-p}(\mathcal{X})$, see Remark 3.2, then, by using a standard argument, it is enough to show that the restriction of $\mathbb{K}^p F$ on $\mathcal{C}$ is an equivalence. Note that by the definition the image of $\mathcal{C}$ under $\mathbb{K}^p F$ is mapped into the subcategory $\mathcal{C}'$ in $\mathbb{K}^{-p}(\mathcal{X}')$ which is defined similar to $\mathcal{C}$. Let $\mathcal{C}_X[i]$, for some $i \in \mathbb{Z}$ and $X \in \mathcal{X}'$, be in $\mathcal{C}'$. Due to the equivalence $F$ there is a module $X$ in $\mathcal{X}$ with $F(X) \simeq X'$. Then by the uniqueness of projective resolution up to homotopy equivalence we can see that $\mathbb{K}^p F(\mathcal{C}_X[i]) = \mathcal{C}_{F(X)}[i] \simeq \mathcal{C}_X[i]$. So this proves the denseness of the restricted functor. For any pair of modules $A$ and $B$ in $\mathcal{X}$ and $i \in \mathbb{Z}$, we have the following natural isomorphisms

1. If $i = 0$,

$$
\text{Hom}_{\mathbb{K}^p(R)}(\mathcal{C}_A, \mathcal{C}_B) \simeq \text{Hom}_R(A, B)
\simeq \text{Hom}_R(F(A), F(B))
\simeq \text{Hom}_{\mathbb{K}^p(R')}(\mathcal{C}_{F(A)}, \mathcal{C}_{F(B)})
= \text{Hom}_{\mathbb{K}^p(R')}(\mathbb{K}^p F(\mathcal{C}_A), \mathbb{K}^p F(\mathcal{C}_B))
$$

2. If $i > 0$, then

$$
\text{Hom}_{\mathbb{K}^p(R)}(\mathcal{C}_A[i], \mathcal{C}_B) \simeq \text{Hom}_R(A, \Omega^i_R(B))
\simeq \text{Hom}_R(F(A), F(\Omega^i_R(B)))
\simeq \text{Hom}_R(F(A), \Omega^i_R(F(B)))
\simeq \text{Hom}_{\mathbb{K}^p(R')}(\mathcal{C}_{F(A)[i]}, \mathcal{C}_{F(B)})
= \text{Hom}_{\mathbb{K}^p(R')}(\mathbb{K}^p F(\mathcal{C}_A[i]), \mathbb{K}^p F(\mathcal{C}_B))
$$
3. If $i < 0$, then
\[
\text{Hom}_{\text{K}(R)}(C_A[i], C_B) \simeq 0 \simeq \text{Hom}_{\text{K}(R')}(C_{F(A)}[i], C_{F(B)})
\]
\[
= \text{Hom}_{\text{K}(R')}(\mathbb{K}^p F(C_A[i]), \mathbb{K}^p F(C_B)).
\]

The above isomorphisms show that the restriction of $\mathbb{K}^p F$ on $C$ is full and faithful. So we are done. The statements (2) and (3) can be proved by the same argument. So we skip their proofs. \hfill \Box

**Theorem 4.10** Let $\mathcal{X} \subseteq \text{mod-}R$ and $\mathcal{X}' \subseteq \text{mod-}R'$ be as in Construction 4.8, that is, $\text{prj-}R \subseteq \mathcal{X} \subseteq \text{mod-}R$ and $\text{prj-}R' \subseteq \mathcal{X}' \subseteq \text{mod-}R'$ are contravariantly finite and closed under syzygies. Assume, further, the global $\mathcal{X}$-dimension, resp. $\mathcal{X}'$-dimension, of $R$, resp. $R'$, is finite. Suppose there is a functor $F : \mathcal{X} \to \mathcal{X}'$ such that $F(\text{prj-}R) \subseteq \text{prj-}R'$, the induced functor $\tilde{F} : \mathcal{X} \to \mathcal{X}'$ is an equivalence, and for any $X \in \mathcal{X}$, $F(\Omega_R(X)) \simeq \Omega_{R'}(F(X))$ in $\mathcal{X}'$ (as in Construction 4.8). If either of the following hold.

1. The subcategories $\mathcal{X}$ and $\mathcal{X}'$ satisfy the condition $(\ast)$.  
2. The subcategories $\mathcal{X}$ and $\mathcal{X}'$ are quasi-resolving, and the functor $F$ is exact.  
3. The functor $F$ is a restriction of an exact functor between quasi-resolving subcategories $\mathcal{X} \subseteq \mathcal{Y} \subseteq \text{mod-}R$ and $\mathcal{X}' \subseteq \mathcal{Y}' \subseteq \text{mod-}R'$.

then $R$ and $R'$ are singularly equivalent.

**Proof** For (1), we know by Proposition 4.9 the functor $\mathbb{K}^p F$, defined in the construction, is an equivalence between the relative singularity categories $\Delta_{\mathcal{X}}(R) \simeq \Delta_{\mathcal{X}'}(R')$. Thanks to Proposition 4.5 we get the result since the subcategories satisfy the condition $(\ast)$. The proofs of (2) and (3) come from this fact that the triangle equivalence $\mathbb{K}^p F$ is restricted to $\mathbb{K}^p_{\text{sh}}(\mathcal{X}) \simeq \mathbb{K}^p_{\text{sh}}(\mathcal{X}')$, by the construction. Hence $\mathbb{K}^p F$ induces similarly an equivalence between the singularity categories. So we are done these two cases. \hfill \Box

As the above theorem says that some kinds of relative stable equivalences implies the singular equivalences. An interesting cases of $\mathcal{X}$ and $\mathcal{X}'$ is when both equal to the whole of the module categories. In this special case we are dealing with the stable equivalences which have been worked by many researchers.

**Corollary 4.11** Let $R$ and $R'$ be two right noetherian rings. Let $F : \text{mod-}R \to \text{mod-}R'$ be a functor with $F(\text{prj-}R) \subseteq \text{prj-}R'$. Suppose that $F$ induces the stable equivalence $\tilde{F} : \text{mod-}R \to \text{mod-}R'$ with the property $\tilde{F}(\Omega_R(M)) \simeq \Omega_{R'}(F(M))$ for any $R$-module $M$ in $\text{mod-}R$. If one of the two following assumptions is correct.

1. The rings $R$ and $R'$ are two Artin algebras of finite representation type.  
2. The functor $F$ is exact.

then $R$ and $R'$ are singularly equivalent.

**Proof** It is clear that the relative dimension respect to the whole of module category is always finite. Then, the statement is an immediate consequence of Theorem 4.10. \hfill \Box
In the next section we will provide different examples of the subcategories satisfying the assumptions of our main theorem (Theorem 4.10). Let us here only focus on the case that the subcategories to be the whole of the module categories (as the above corollary) and conclude this section by some remarks concerning this special case.

**Remark 4.12** 1. In the case (2) of Corollary 4.11, we are dealing with the stable equivalences induced by exact functors. We refer to [32] for more information related to such stable equivalences. Note that in the contrast of [32] we do not need the quasi-inverse of the functor $F$ (appeared in Corollary 4.11) also induced by an exact functor. An important example of the stable equivalences induced by exact functors is stable equivalences of Morita type. There is a series of works due to Yuming Liu and Changchang Xi [33–35] to give some ways to construct this sort of the stable equivalences and which of properties are preserved. Recall two algebras $\Lambda$ and $\Lambda'$ are said to be stable equivalence of Morita type if there exists a pair of bimodules $\Lambda M_{\Lambda'}$ and $\Lambda' N_{\Lambda}$ such that

1. $M$ and $N$ are projective as left and right modules, respectively;
2. $M \otimes_{\Lambda'} N \simeq A \oplus P$ as $A$-$A$-bimodules for some projective $A$-$A$-bimodule $P$, and $N \otimes_{\Lambda} M \simeq B \oplus Q$ as $B$-$B$-bimodules for some projective $B$-$B$-bimodule $Q$.

By the definition one can prove directly a stable equivalence of Morita type gives also a singular equivalence. The notion of stable equivalences of Morita type is generalized to singular equivalence of Morita type in [18] and also studied in [40]. Corollary 4.11(2) explains how one can get a singular equivalence from a certain given stable equivalence, as happens for the stable equivalences of Morita type. Moreover, the above corollary also can be obtained by using the stabilization theory established by Keller–Vossieck, Beligiannis [9,31]. We also refer the reader to [15] for a nice survey of this theory. For an exact functor $F : \text{mod-} R \to \text{mod-} R'$ as in Corollary 4.11(2), the induced functor $\hat{F}$ is indeed a (left) triangle functor between the left triangulated categories $\text{mod-} R$ and $\text{mod-} R'$. But, by [14, Corollary 2.7] there is a triangle equivalence between the relevant stabilization $S(\text{mod-} R) \simeq S(\text{mod-} R')$. We know from the basic result due to [31] the triangle equivalences $S(\text{mod-} R) \simeq \mathbb{D}_{sg}(R)$ and $S(\text{mod-} R') \simeq \mathbb{D}_{sg}(R')$, so we get the corollary (with assumption of exactnesses of the functor $F$). This observation was mentioned to us by Xiao-Wu Chen. We would like to thank him. In the case that the functor $F$ is not exact it seems that the stabilization theory does not work.

2. It is not true any stable equivalence between two rings gives an singular equivalence between them. For instance, it is known any algebra with radical square zero is stable equivalent to a hereditary algebra [7, Theorem 2.1]. But the singularity categories of hereditary algebras are always trivial, but not to be trivial in general for radical square zero algebras.

3. Let $\Lambda = k[x]/(x^2)$ be the algebra of dual numbers. The assignment $M \mapsto (M, \text{soc}(M))$ gives a functor $F : \text{mod-} \Lambda \to \text{mod-} \Lambda \times \text{mod-} k$. This functor satisfies the required assumption in Corollary 4.11(1), but not necessarily $F$ to be exact.
5 Consequences and examples

We aim in this section to give some application of our results. In different types of rings, including path rings, triangular matrix rings, trivial extension rings and tensor rings, by use of Theorem 2.4, a new description of their singularity categories is given. Then by help of such description we construct some singular equivalences.

5.1 Path rings

Throughout let \( \mathcal{Q} \) be a finite acyclic quiver \( \mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1, s, t) \), where \( \mathcal{Q}_0 \) and \( \mathcal{Q}_1 \) are the set of arrows and vertices of \( \mathcal{Q} \), respectively, and \( s \) and \( t \) are the starting and ending maps from \( \mathcal{Q}_1 \) to \( \mathcal{Q}_0 \), respectively. Assume that a right noetherian ring \( R \) is given, a representation \( X \) of \( \mathcal{Q} \) over \( R \) is obtained by associating to any vertex \( v \) a module \( X_v \) in \( \text{mod-} R \) and to any arrow \( a : v \rightarrow w \) a morphism \( X_a : X_v \rightarrow X_w \) in \( \text{mod-} R \). If \( \mathcal{X} \) and \( \mathcal{Y} \) are two representations of \( \mathcal{Q} \), then a morphism \( f : X \rightarrow Y \) is determined by a family \( \{ f_v \}_{v \in \mathcal{Q}_0} \) so that for any arrow \( a : v \rightarrow w \), the commutativity condition \( Y_a \circ f_v = f_w \circ X_a \) holds. The representations of \( \mathcal{Q} \) over \( \text{mod-} R \) and the morphisms between them, as defined already, form an abelian category which is denoted by \( \text{rep}(\mathcal{Q}, R) \). We can also construct the path ring of \( \mathcal{Q} \) by \( R \) as follows. Let \( \rho \) be the set of all paths in the given quiver \( \mathcal{Q} \) together with the trivial paths associated to the vertices. We write the conjunction of paths from left to right. Now let \( R \mathcal{Q} \) be the free \( R \)-module with basics \( \rho \). An element of \( R \mathcal{Q} \) is written as a finite sum \( \sum_{\rho \in \rho} a_p \rho \), where \( a_p \in R \) and \( a_p = 0 \) for all but finitely many \( \rho \). We can make \( R \mathcal{Q} \) a ring where multiplication is given by concatenation of paths. Then \( R \mathcal{Q} \) is still a right noetherian ring, and as it is a free module of finite rank over the right noetherian ring \( R \). One can prove in the same way for the case \( R \) to be a field \( k \) as in the literature, see for example [8], the category \( \text{mod-} R \mathcal{Q} \) of (right) finitely generated \( \mathcal{Q} \)-modules is equivalent to the category \( \text{rep}(\mathcal{Q}, R) \) of representations of \( \mathcal{Q} \) by \( \mathcal{Q} \)-modules and \( \mathcal{Q} \)-homomorphisms. Hence due to this equivalence we identify \( \text{mod-} R \mathcal{Q} \) with \( \text{rep}(\mathcal{Q}, R) \) which is much easier to work.

For any vertex \( v \) in \( V \) of quiver \( \mathcal{Q} \), let \( e^v : \text{rep}(\mathcal{Q}, R) \rightarrow \text{mod-} R \) be the evaluation functor defined by \( e^v(X) = X_v \), for a representation \( X \). It is shown in [21] that \( e^v \) has a full faithful left adjoint \( e^v_\lambda : \text{mod-} R \rightarrow \text{rep}(\mathcal{Q}, R) \), by sending \( M \) in \( \text{mod-} R \) to the representation \( e^v_\lambda(M) \), that is, for a vertex \( w \in V \), \( e^v_\lambda(M)(w) = \bigoplus_{Q(v,w)}M \), where \( Q(v, w) \) is the set of all paths from \( v \) to \( w \), for an arrow \( a : w_1 \rightarrow w_2 \), \( e^v_\lambda(M)(a) \) is the natural injection from \( \bigoplus_{Q(v,w_1)}M \) to \( \bigoplus_{Q(v,w_2)}M \). As mentioned in [23], subsection 3.1 there is a short exact sequence

\[
\begin{align*}
\hat{\top} 0 & \rightarrow \bigoplus_{a \in E} e^\lambda_{t(a)}(X_{s(a)}) \xrightarrow{\oplus X} \bigoplus_{v \in V} e^v_\lambda(X_v) \xrightarrow{\oplus X} X & \rightarrow 0.
\end{align*}
\]

The above short exact sequence is vital for our next observation.

Let \( \mathcal{M}_R(\mathcal{Q}) \) be the subcategory of \( \text{rep}(\mathcal{Q}, R) \) consisting of all representations being isomorphic to a finite direct sum of representations of the form \( e^v_\lambda(M) \) for some \( v \) in \( V \) and \( M \) in \( \text{mod-} R \). By the classification of projective representations given in [20],
Theorem 3.1] we observe \( \mathcal{M}_R(\mathcal{Q}) \) contains projective representations. For instance, in the case that \( \mathcal{Q} \) is the quiver \( A_2 : 1 \rightarrow 2 \), then the \( \mathcal{M}_R(A_2) \) is formed by all representations which are isomorphic to a finite direct sum of representations of the form either \( 0 \rightarrow M \) or \( M \quad \rightarrow \quad 1 \), for some \( M \) in \( \text{mod-}R \). Equivalently, it contains all representations \( M_1 \xrightarrow{f} M_2 \) such that \( f \) is a split monomorphism.

We usually use \( \text{Hom}_\mathcal{Q}(-, -) \) to show the Hom-space between two representations.

**Lemma 5.1** Use the above notations. The subcategory \( \mathcal{M}_R(\mathcal{Q}) \) is contravariantly finite in \( \text{rep}(\mathcal{Q}, R) \). Moreover, the global \( \mathcal{M}_R(\mathcal{Q}) \)-dimension of \( R\mathcal{Q} \) is at most one.

**Proof** We first consider the first statement. To this end, we show that the morphism \( f_X : \bigoplus_{v \in V} e^v(X) \rightarrow X \) appeared in the short exact sequence (†), introduced in the above, is a right \( \mathcal{M}_R(\mathcal{Q}) \)-approximation. Let us first present \( f_X \) more explicitly as \( (f^v_X)_{v \in V} \), where \( f^v_X : e^v(X_v) \rightarrow X \) is the image of the identity \( \text{id}_{X_v} \) under the isomorphism \( \text{Hom}_\mathcal{Q}(e^v(X_v), X) \simeq \text{Hom}_R(X_v, X_v) \) for any \( v \) in \( V \). To prove, it is enough to show that every morphism as \( h : e^w(\mathcal{M}) \rightarrow X, M \in \text{mod-}R \) and \( w \in V \), factors through \( f_X \). Consider the following natural isomorphisms

\[
\text{Hom}_\mathcal{Q}(e^w(\mathcal{M}), X) \simeq \text{Hom}_R(M, X_w) \simeq \text{Hom}_\mathcal{Q}(e^w(\mathcal{M}), e^w(X_w)).
\]

Denote \( g : e^w(\mathcal{M}) \rightarrow e^w(X_w) \) the image of \( h \) under the composition of the above two isomorphisms, that is indeed, \( e^w(d) \), where \( d : M \rightarrow X_w \) is the image of \( h \) under the first isomorphism. Then by using the adjoint property we observe \( f^w_X \circ g = h \), so \( h \) factors through the \( f_X \). The latter part immediately follows from the short exact sequence (†).

Fortunately, by the above lemma, \( \mathcal{M}_R(\mathcal{Q}) \) satisfies all of the conditions we need to use Theorem 3.3. Hence as an immediate consequence of the theorem we have the following result.

**Theorem 5.2** Let \( \mathcal{Q} \) be an acyclic quiver and \( R \) a right noetherian ring. Then, there is the following triangle equivalence

\[
\mathcal{D}_{\text{sg}}(R\mathcal{Q}) \simeq \frac{\mathcal{K}^{-, \mathcal{b}}(\mathcal{M}_R(\mathcal{Q}))}{\mathcal{K}^{-, \mathcal{b}}_{\text{ac}}(\mathcal{M}_R(\mathcal{Q}))}.
\]

One advantage of the above equivalence is that \( \mathcal{D}_{\text{sg}}(R\mathcal{Q}) \) is completely determined by the representations of the form \( e^v_\lambda(\mathcal{M}) \) which have a more simple structure and easy to handle some difficulties as we shall show in our next result. Let \( F : \text{mod-}R \rightarrow \text{mod-}R' \) be an exact functor the same as in Corollary 4.11(2), that is, \( F(\text{prj-}R) \subseteq \text{prj-}R' \) and the stable induced functor \( F : \text{mod-}R \rightarrow \text{mod-}R' \) is an equivalence. By applying component-wisely \( F \), we get an exact functor \( \tilde{F} : \text{rep}(\mathcal{Q}, R) \rightarrow \text{rep}(\mathcal{Q}, R') \), preserving projective representations. The preservation comes from this fact that for any \( M \in \text{mod-}R \) and \( v \in V \), by the construction, we have \( \tilde{F}(e^v_\lambda(\mathcal{M})) = e^v_\lambda(F(M)) \).

Let us explain our construction for the case \( \mathcal{Q} \) is \( A_2 : 1 \rightarrow 2 \). For each representation

\( \mathcal{E} \) Springer
Lemma 5.3. Let $M$ be in mod-$R$ and $\nu$ in $V$. Then, we have the equality $\mathbb{K}_c^P \tilde{F}(k^v(M)) = k^v(\mathbb{K}_c^P F(M))$. Recall that $C_N$ is the mapping cone of an onto chain map from a projective resolution $P_N$ to $N$, see the beginning of Sect. 3, and also $\mathbb{K}_c^P F$ is defined in Construction 4.8.

Now we prove the promised singular equivalence.

Theorem 5.4. Use the above notation. Then, the induced functor $\mathbb{K}_c^P \tilde{F} : \mathbb{K}^{−p}(M_R(Q)) \rightarrow \mathbb{K}^{−p}(M_{R'}(Q))$ is a triangle equivalence. In particular, $\mathbb{D}_{sg}(RQ) \simeq \mathbb{D}_{sg}(R'Q)$ as triangulated categories.

Proof. As we did before in the proof of Proposition 4.9, it is enough to show that the restricted functor $\mathbb{K}_c^P \tilde{F} : M_R(Q) \rightarrow M_{R'}(Q)$ is an equivalence. Recall that $M_R(Q) \simeq \mathbb{K}^{−p}(M_R(Q))$ consisting of all shiftings of the complexes of the form $\text{C}_{e^v(M)}$ for some $M \in \text{mod-}R$ and $v \in V$, and similarly for $M_{R'}(Q)$. We recall again that $\text{C}_{e^v(M)}$ is an exact complex obtained from a projective resolution of $e^v(M)$. We may assume that $\text{C}_{e^v(M)}$ is obtained by applying $k^v$ on $\text{C}_M$. This makes a sense because $e^v_M$ is an exact functor preserving projective representations. For density, take $\text{C}_{e^v(M)}[i]$ in $M_{R'}(Q)$, where $N \in \text{mod-}R'$. By the equivalence $F$, there is $M$ in mod-$R'$ such that $F(M) \simeq N$. Then one can see by Lemma 5.3, $\mathbb{K}_c^P \tilde{F}(\text{C}_{e^v(M)}[i]) \simeq \text{C}_{e^v(N)}[i]$. Let $\text{C}_{e^v(T)}[j]$ and $\text{C}_{e^v(M)}[i]$ be arbitrary complexes in $M_R(Q)$.
There is the following chain of natural isomorphisms
\[
\text{Hom}_{\mathbb{K}^{\mathbb{Q}(R)}}(C_{e_i^0(M)}[i], C_{e_T^0}[j]) = \text{Hom}_{\mathbb{K}^{\mathbb{Q}(R)}}(k^R_C(M)[i], k^R_C(T)[j]) \\
\simeq \text{Hom}_{\mathbb{K}^{\mathbb{Q}(R)}}(C_M[i], \oplus_{Q(v,w)} CT[j]) \\
\simeq \text{Hom}_{\mathbb{K}^{\mathbb{Q}(R')}}(K_F^P(F(C_M[i])), k^R_C(F(C_T[j]))) \\
\simeq \text{Hom}_{\mathbb{K}^{\mathbb{Q}(R')}}(K_F^P(F(C_M[i])), K_F^P(F(C_T[j]))) \\
\simeq \text{Hom}_{\mathbb{K}^{\mathbb{Q}(R')}}(K_F^P(C_M[i]), K_F^P(CT[j])).
\]

where the second isomorphism obtained from the adjoint pair \((k^R_C, k^R_C)\), for the third \(\mathbb{K}^P_F: \mathbb{K}^{-P}(\text{mod-}R) \to \mathbb{K}^{-P}(\text{mod-}R')\) is the induced functor, introduced in Construction 4.8, which is itself an equivalence by Proposition 4.9(3), for the fourth we again use the adjoint pair \((k^R_C, k^R_C)\) but this turn relative to \(\text{rep}(\mathbb{Q}, R')\), and finally we use Lemma 5.3. \(\square\)

Let \(\mathbb{M}_n(R)\) be the set of all \(n \times n\) square matrices with coefficients in \(R\) for \(n \in \mathbb{N}\). \(\mathbb{M}_n(R)\) is a ring with respect to the usual matrix addition and multiplication. The subset

\[
\mathbb{T}_n(R) = \begin{bmatrix}
R & 0 & \cdots & 0 \\
R & R & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
R & R & \cdots & R
\end{bmatrix}
\]

of \(\mathbb{M}_n(R)\) consisting of all triangular matrices \([a_{ij}]\) in \(\mathbb{M}_n(R)\) with zeros over the main diagonal is a subring of \(\mathbb{M}_n(R)\). It is well known for the quiver

\[
A_n : 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots \longrightarrow n,
\]

there is an isomorphism of rings \(RA_n \cong \mathbb{T}_n(R)\). We specialize our results to the quiver \(A_n\), then:

**Corollary 5.5** Let \(F: \text{mod-}R \to \text{mod-}R'\) be the same as the above and \(n \in \mathbb{N}\). Then, \(T_n(R)\) and \(T_n(R')\) are singularly equivalent.

### 5.2 Triangular matrix rings

Throughout this section let \(R\) and \(S\) be two given right noetherian rings and \(S\)-\(R\)-bimodule \(SM_R\) with \(M_R\) is a finitely generated \(R\)-module. The triangular matrix ring \(T = \begin{bmatrix} R & 0 \\ M & S \end{bmatrix}\) has as its elements matrices \(\begin{bmatrix} r & 0 \\ m & s \end{bmatrix}\) where \(r \in R\), \(s \in S\) and \(m \in M\) with addition defined coordinate wise and multiplication given by \(\begin{bmatrix} r & 0 \\ m & s \end{bmatrix} \begin{bmatrix} r' & 0 \\ m' & s' \end{bmatrix} = \begin{bmatrix} rr' + sm' & 0 \\ mr' + sm' & ss' \end{bmatrix}\). It is not difficult to see that \(T\) is right noetherian as well. It is well-known [25] that \(\text{mod-}T\) is equivalent to the category \(\mathcal{T}\) consisting of all triples \((X, Y)_f\) where \(X \in \text{mod-}R\), \(Y \in \text{mod-}S\) and \(f : Y \otimes_S M_R \to X\) is a \(R\)-linear map. A morphism
(X₁, Y₁) → (X₂, Y₂) is a pair (φ₁, φ₂) where φ₁ : X₁ → X₂, φ₂ : Y₁ → Y₂, such that there is the following commutative diagram

\[
\begin{array}{c}
Y₁ \otimes_S M_R \xrightarrow{f} X₁ \\
\phi₂ \otimes S M_R \downarrow \phi₁ \\
Y₂ \otimes_S M_R \xrightarrow{f'} X₂.
\end{array}
\]

Due to this equivalence we identify mod-T with T. Analogous to representations of quivers, we have two evaluation functors e₁ : mod-T → mod-R and e₂ : mod-T → mod-S in the following way: for every T-module (X, Y)ᵣ, e₁((X, Y)ᵣ) = X and e₂((X, Y)ᵣ) = Y. In this setting, the evaluation functors also have left adjoints. They are explicitly described as follows: For any X ∈ mod-R, e₁(λ(X)) = (X, 0), and for any Y ∈ mod-S, e₂(λ(Y)) = (Y ⊗ S M_R, Y)₁, here “1” denotes the identity morphism Id₁Y⊗SM_R. But in this case the left adjoint are not exact in general. More precisely, e₁ is exact but the latter one might be not in general.

Let \( M(T) \) be the subcategory of mod-T formed by all T-modules which are isomorphic to a direct sum \( e₁(λ(M)) ⊕ e₂(λ(N)) \) for some \( M ∈ \text{mod-R} \) and \( N ∈ \text{mod-S} \). One can see easily \( M(T) \) formed by all objects \( (X, Y)ᵣ \) such that \( f : Y ⊗ S M_R → X \) is a split monomorphism.

**Lemma 5.6** Let \( M(T) \) be defined as in the above. The subcategory \( M(T) \) is contravariantly finite in mod-T, and the global \( M(T) \)-dimension of T is at most one.

**Proof** The proof in general is the same as Lemma 5.1. In this setting we also have a short exact sequence for any \((X, Y)ᵣ\), similar to (†) in Sect. 5.1, as the following

\[
0 → (Y \otimes S M_R, 0)₀ → (X, 0)₀ ⊕ (Y \otimes S M_R, Y)₁ → (X, Y)ᵣ → 0
\]

Analogous to (†) the above short exact sequence is essential to prove the lemma. □

An immediate consequence of lemma above in conjunction with Theorem 3.3 is the following:

**Theorem 5.7** let T be a triangular matrix ring \( T = \begin{bmatrix} R & 0 \\ M & S \end{bmatrix} \) and \( M(T) \) as defined above. Then, there is the following equivalence of triangulated categories

\[
\mathbb{D}_{sg}(T) \simeq \frac{K^{-p}(M(T))}{K^{b}_{ac}(M(T))}.
\]

As an application of the above description of the singularity categories of triangular matrix rings, in the next result, we will show how a certain stable equivalence can be lifted to a singular equivalence.

**Proposition 5.8** let T be a triangular matrix ring \( T = \begin{bmatrix} R & 0 \\ M & S \end{bmatrix} \). Assume the functor \( − \otimes S M_R : \text{mod-S} \to \text{mod-R} \) induces an equivalence between the stable categories
mod-$S \simeq$ mod-$R$. Necessarily, the functor must preserve the projective modules. Then, the following triangle equivalence of the singularity categories occurs:

$$\mathcal{D}_{sg} \left( \left[ \begin{array}{cc} R & 0 \\ M & S \end{array} \right] \right) \simeq \mathcal{D}_{sg} \left( \left[ \begin{array}{cc} R & 0 \\ 0 & R \end{array} \right] \right).$$

**Proof** Setting $T = \left[ \begin{array}{cc} R & 0 \\ M & S \end{array} \right]$ and $T_2(R) = \left[ \begin{array}{cc} R & 0 \\ R & R \end{array} \right]$. Let $\mathcal{M}(T)$ and $\mathcal{M}(T_2(R))$ be defined as in the above for the corresponding triangular matrix rings. Indeed, the subcategory $\mathcal{M}(T_2(R))$ is nothing else than the subcategory $\mathcal{M}_R(A_2)$ of the representations over the quiver $A_2$. For the connivance, we prefer to work with $\mathcal{M}_R(A_2)$ instead of $\mathcal{M}(T_2(R))$. To get the result, we use Theorem 4.10 (3). For this we need to define a functor $F : \mathcal{M}(T) \to \mathcal{M}_R(A_2)$ satisfying the required conditions in the theorem and to be an restriction of an exact functor between some quasi-resolving subcategories. Let $S(T)$ be the subcategory of mod-$T$ formed by all modules $(X, Y)_f$ such that $Y \otimes_S M_R \to X$ is a monomorphism. The subcategory $S(R)$ of mod-$RA_2$ is defined similarly, indeed, it is the category of all (submodules) monomorphisms in mod-$R$, in the sense of Ringel and Schmidmeier. One can see both subcategories are quasi-resolving and contain the subcategories $\mathcal{M}(T)$ and $\mathcal{M}_R(A_2)$, respectively.

Now we define the functor $G : S(T) \to S(R)$ in the following way. Let $(A, B)_f$ in $S(T)$ be given. Then $B \otimes_S M_R \to A$ is a monomorphism. But this means that the monomorphism $f$ lies as an object in $S(R)$. Define $G((A, B)_f) := f$, and on morphisms $G$ is defined in an obvious way. Now we show that the functor $G$ already defined preserves short exact sequences in $S(T)$. Let $0 \to (A^0, B^0)_f^0 \to (A^1, B^1)_f^1 \to (A^2, B^2)_f^2 \to 0$ be a short exact with all terms in $S(T)$. Then it induces the following commutative diagram with exact rows in mod-$R$

$$
\begin{array}{cccccc}
B^0 \otimes_S M_R & \rightarrow & B^1 \otimes_S M_R & \rightarrow & B^2 \otimes_S M_R & \rightarrow & 0 \\
\downarrow f^0 & & \downarrow f^1 & & \downarrow f^2 & & \\
A^0 & \rightarrow & A^1 & \rightarrow & A^2 & \rightarrow & 0.
\end{array}
$$

Since $f^0$ is monomorphism we get the sequence in the top is also a short exact sequence. Hence the above commutative diagram gives us the following short exact sequence in mod-$RA_2$

$$
\begin{array}{cccccc}
B^0 \otimes_S M_R & \rightarrow & B^1 \otimes_S M_S & \rightarrow & B^2 \otimes_R M_S & \rightarrow & 0 \\
\downarrow f^0 & & \downarrow f^1 & & \downarrow f^2 & & \\
A^0 & \rightarrow & A^1 & \rightarrow & A^2 & \rightarrow & 0,
\end{array}
$$

where it is actually the image of the given short exact sequence in $\mathcal{M}(T)$ under the functor $G$. So we are done the claim.

We know by [27, Theorem 3.1] any projective module in mod-$T$ can be written up to isomorphism as $(P, 0)_0 \oplus (Q \otimes_S M_R, Q)_1$ for some $P \in \text{prj-}R$ and $Q \in \text{prj-}S$. Using the fact which the functors $- \otimes_S M_R$ preserves the projective modules, and in view of the characterization of projective modules over triangular matrix rings, we can
verify that the functor \( G \) also preserves the projective modules. Set \( F = G \mid_\mathcal{M}(T) \). The essential image of the restricted functor \( F \) is clearly in \( \mathcal{M}_R(A_2) \). To complete the proof, we must show that the induced functor \( E : \mathcal{M}(T) \to \mathcal{M}_R(A_2) \) is an equivalence.

As any module in \( \mathcal{M}(T) \) (and also similarly in \( \mathcal{M}_R(A_2) \) for the relevant cases) is isomorphic to a finite direct sum of modules of the forms \((X, 0)_0\) or \((Y \otimes S M_R, Y)_1\) where \( X \in \text{mod-}R \) and \( Y \in \text{mod-}S \), hence to prove the claim it is enough to concentrate on such modules.

For density, let \((0 \to Z)\) and \((W \to W)\) in \( \mathcal{M}_R(A_2) \) are given. Since \( - \otimes_S M_R \) induces an equivalence from \( \text{mod-}S \) to \( \text{mod-}R \), there exists projective modules \( P \) and \( Q \) in \( \text{prj-}R \) and \( U \in \text{mod-}S \) with isomorphism \( W \oplus P \simeq (U \otimes_S M_R) \oplus Q \) in \( \text{mod-}R \). With the isomorphism we can get the \( (W \to W) \oplus (P \to P) \simeq F((U \otimes S M_R, U)_1) \oplus (Q \to Q) \). Hence \( F((U \otimes_S M_R, U)_1) \simeq (W \to W) \) in \( \mathcal{M}_R(A_2) \), so the result follows. The case of \((0 \to Z)\) is easy. For fullness and faithfulness, assume that \( D \) and \( E \) are in \( \mathcal{M}(T) \). We must prove the induced group homomorphism \( G : \text{Hom}_T(D, E) \to \text{Hom}_{\mathcal{M}_R}(F(D), F(E)) \) is bijective. To prove, it is enough to consider the following cases.

(1) If \( D = (X, 0)_0 \) and \( E = (Y, 0)_0 \), where \( X, Y \in \text{mod-}R \), then it is clear.
(2) If \( D = (Y \otimes_S M_R, Y)_1 \) and \( E = (X, 0)_0 \), where \( X \in \text{mod-}R \) and \( Y \in \text{mod-}S \), then since both Hom-spaces are zero, then this case is also clear.
(3) If \( D = (X, 0)_0 \) and \( E = (Y \otimes_S M_R, Y)_1 \), then both Hom-spaces are isomorphic to the Hom-space \( \text{Hom}_R(X, Y \otimes S M_R) \) by the relevant adjoint pairs.
(4) If \( D = (Y \otimes_S M_R, Y)_1 \) and \( E = (Z \otimes_S M_R, Z)_1 \), then by using the relevant adjoint pairs and the equivalence induced by \( - \otimes_S M_R \), we have:

\[
\text{Hom}_{\mathcal{M}_R}(F(D), F(E)) \simeq \text{Hom}_R(Y \otimes S M_R, Z \otimes S M_R) \\
\simeq \text{Hom}_S(Y, Z) \\
\simeq \text{Hom}_T(D, E).
\]

Now Theorem 4.10 (3) completes the proof. \( \square \)

Stable equivalences of Morita type are a special case of the above proposition. Hence we have the following corollary.

**Corollary 5.9** Let \( \Lambda \) and \( \Lambda' \) be two Artin algebras of stable equivalence of Morita type. Assume pair of bimodules \((\Lambda' M_{\Lambda}, \Lambda N_{\Lambda'})\) satisfies the required condition, that is, there exists a pair of bimodules \( \Lambda' M_{\Lambda} \) and \( \Lambda N_{\Lambda'} \) such that

1. \( M \) and \( N \) are projective as left and right modules, respectively;
2. \( M \otimes_{\Lambda} N \simeq \Lambda' \oplus P \) as \( \Lambda' - \Lambda' \)-bimodules for some projective \( \Lambda' - \Lambda' \)-bimodule \( P \), and \( N \otimes_{\Lambda} M \simeq \Lambda \oplus Q \) as \( \Lambda - \Lambda \)-bimodule \( Q \).

Then, the following triangle equivalences of the singularity categories occur:

(i) \( \mathbb{D}_{\text{sg}} \left( \left[ \begin{array}{c} \Lambda \\ \Lambda' M_{\Lambda} \end{array} \right] \right) \simeq \mathbb{D}_{\text{sg}} \left( \left[ \begin{array}{c} \Lambda \\ \Lambda \end{array} \right] \right) \).
(ii) \[ \mathbb{D}_{sg} \left( \left[ \begin{array}{cc} \Lambda' & 0 \\ \Lambda & \Lambda' \end{array} \right] \right) \simeq \mathbb{D}_{sg} \left( \left[ \begin{array}{cc} \Lambda' & 0 \\ \Lambda & \Lambda' \end{array} \right] \right). \]

In particular, by Corollary 4.10, we have \[ \mathbb{D}_{sg}(T_2(\Lambda)) \simeq \mathbb{D}_{sg}(T_2(\Lambda')). \]

Hence, \[ \mathbb{D}_{sg} \left( \left[ \begin{array}{cc} \Lambda & 0 \\ \Lambda & \Lambda' \end{array} \right] \right) \simeq \mathbb{D}_{sg} \left( \left[ \begin{array}{cc} \Lambda' & 0 \\ \Lambda & \Lambda' \end{array} \right] \right). \]

5.3 Trivial extension and tensor rings

First we recall some facts on modules over trivial extension rings. Throughout this section, let as usual \( R \) be a right noetherian ring and let \( R M_R \) be a \( R \)-\( R \)-bimodule and \( M_R \) is finitely generated. The trivial extension of \( R \) by \( M \) is a ring, denoted by \( R \ltimes M \), with the underlying abelian group \( R \oplus M \) and the multiplication: \((r, m)(r', m') = (rr', rm' + mr') \) for \( r, r' \) in \( R \) and \( m, m' \) in \( M \). The actions involved in the definition of the multiplication can be viewed in a natural way. Then by our assumption \( R \ltimes M \) is also a right noetherian ring. For further information on \( R \ltimes M \), we refer the reader to [24].

Let us mention here a triangular matrix \( \left[ \begin{array}{c} r \\ M \end{array} \right] \) is a trivial extension of the product ring \( R \times S \) by \( M \). In this section we give a generalization of Theorem 5.7 in terms of trivial extension rings. Similar to the triangular matrix rings and path rings, we have the following short exact sequence in mod-\( R \ltimes M \), for any module \( (X, \sigma) \)

\[ 0 \to (X \ltimes R M_R, \sigma \ltimes_R M) \to (X \ltimes (X \ltimes R M_R), \left[ \begin{array}{c} \sigma \\ 0 \end{array} \right]) \to (X, \sigma) \to 0. \]

See also [14, Lemma 3.1] for the above short exact sequence.

For \( R \)-\( R \)-bimodule \( M \), write \( M \otimes^R R^0 = R \) and \( M \otimes^R R^{j+1} = M \otimes^R R M \otimes^R R^j \) for \( j \geq 0 \). We say that \( M \) is nilpotent, if \( M \otimes^R R^{n+1} = 0 \) for some \( n \geq 0 \). From now on, assume that \( M \) is nilpotent and for \( n \geq 0 \), \( M \otimes^R R^{n+1} = 0 \). For a given \( R \ltimes M \)-module \( (X, \sigma) \), denote by \( \varepsilon_1 \) the above short exact sequence. Replacing \( (X, \sigma) \) with \( (X \otimes_R M_R, \sigma \otimes_R M) \) the starting term of \( \varepsilon_0 \), we get

\[ 0 \to (X \otimes M \otimes^R R^2, \sigma \otimes M \otimes^R R^2) \to ((X \otimes M) \oplus (X \otimes M \otimes^R R^2), \left[ \begin{array}{c} 0 \\ \text{Id}_{X \otimes M \otimes^R R^2} \end{array} \right]) \to (X \otimes M_R, \sigma \otimes_R M) \to 0. \]

Inductively, by replacing with starting term of the short exact sequence \( \varepsilon_i \) we get the short exact sequence \( \varepsilon_{i+1} \)

\[ 0 \to (X \otimes M \otimes^R R^{i+1}, \sigma \otimes M \otimes^R R^{i+1}) \to ((X \otimes M \otimes^R R^i) \oplus (X \otimes M \otimes^R R^{i+1}), \left[ \begin{array}{c} 0 \\ \text{Id}_{X \otimes M \otimes^R R^{i+1}} \end{array} \right]) \to (X \otimes M \otimes^R R^i, \sigma \otimes M \otimes^R R^i) \to 0. \]
For brevity, in the above we drop the subscript “$R$”.

In the rest, we shall also drop in case of need. Denote by $\mathcal{M}(R \ltimes M)$ the subcategory of mod-$R \ltimes M$ consisting of all modules isomorphic to $(X \oplus (X \otimes_R M_R), \begin{bmatrix} 0 & 0 \\ \text{id}_{X \otimes M} & 0 \end{bmatrix})$ for some $X$ in mod-$R$. The middle term of each short exact sequence $\epsilon_i$ is in $\mathcal{M}(R \ltimes M)$. Since $M^\otimes_R(n+1) = 0$ we observe that the starting term in the $\epsilon_n$ has to be in $\mathcal{M}(R \ltimes M)$. Hence by splicing together all the short exact sequences $\epsilon_1, \ldots, \epsilon_n$, we get an exact sequence in mod-$R \ltimes M$ with all terms in $\mathcal{M}(R \ltimes M)$ except possibly the end term

$$0 \rightarrow (X \otimes M^\otimes n, 0) \rightarrow ((X \otimes M^\otimes{n-1}) \oplus (X \otimes M^\otimes{n}), H_n) \rightarrow \cdots \rightarrow (X \oplus (X \otimes_R M_A), H_1) \rightarrow (X, \sigma) \rightarrow 0,$$

where $H_i = \begin{bmatrix} 0 \\ \text{id}_{X \otimes M^\otimes n} \\ 0 \end{bmatrix}$, for $i = 1, \ldots, n$. Note that the the starting term $(X \otimes M^\otimes n, 0)$ is in $\mathcal{M}(R \ltimes M)$ as it is indeed equal to $((X \otimes M^\otimes n) \oplus (X \otimes M^\otimes{n+1}), \begin{bmatrix} 0 \\ \text{id}_{X \otimes M^\otimes n} \\ 0 \end{bmatrix})$.

**Lemma 5.10** The subcategory $\mathcal{M}(R \ltimes M)$ is contravariantly finite.

**Proof** Let $(X, \sigma)$ in mod-$R \ltimes M$ be given. We shall show the the epimorphismem $(X \oplus (X \otimes_R M_A), \begin{bmatrix} 0 \\ \text{id}_{X \otimes M} \\ 0 \end{bmatrix}) \rightarrow (X, \sigma) \rightarrow 0$, appeared in the short exact sequence, introduced in the above, works as a right $\mathcal{M}(R \ltimes M)$-approximation. Assume $[f, g : (Y \oplus (Y \otimes_R M_R)) \rightarrow (X, \sigma)]$, where $f : Y \rightarrow X$ and $g : Y \otimes_R M_R \rightarrow X$, is given. It is a direct verification to check that $\begin{bmatrix} f \\ \text{id}_{Y \otimes M_R} \\ g \end{bmatrix} : Y \oplus (Y \otimes_R M_R) \rightarrow X \oplus (X \otimes_R M_R)$ is a morphism in mod-$R \ltimes M$ and factors $[f, g]$ through $[1, \sigma]$. We are done. \qed

Now the observations provided in the above enables us to apply our results to give a description for $\mathbb{D}_{\text{sg}}(R \ltimes M)$.

**Theorem 5.11** Let $R$ be a right noetherian ring and let $R$-$R$-bimodule $M$ be nilpotent and as right $R$-module is finitely generated. Then, there is the following equivalence of triangulated categories

$$\mathbb{D}_{\text{sg}}(R \ltimes M) \simeq K^{-,b}(\mathcal{M}(R \ltimes M)).$$

**Proof** The theorem is an immediate consequence of Theorem 3.3 and in conjunction with facts provided in the above. Just note that by [24, Corollary 6.1], $\mathcal{M}(R \ltimes M)$ contains the projective modules. \qed

With nilpotent $R$-$R$-bimodule $M$ we can construct another right noetherian ring called tensor ring, that is, $T_R(M) = \bigoplus_{i=0}^\infty M^\otimes_R M$). As mentioned in [17, Section 3], one can identify right modules in mod-$T_R(M)$ with the representations of the endofunctor $- \otimes_R M : \text{mod-}R \rightarrow \text{mod-}R$. By a representation of $- \otimes_R M$, we mean a pair $(X, u)$ with $X$ in mod-$R$ and $u : X \otimes_R M \rightarrow X$. In fact, the modules over $R \ltimes M$ can be considered as the representations of $- \otimes_R M$. A morphism between two representations is the same as one in mod-$R \ltimes M$. Such identifications allow us to consider mod-$R \ltimes M$ as an abelian subcategory of mod-$T_R(M)$. For each module $X$ in
mod-\(R\), following [17, Section2] we define \(\text{Ind}(X) = \bigoplus_{i=0}^{n} X \otimes_{R} M^i\), remember \(n\) is an integer with \(M^i = 0\), and moreover, a morphism \(c_{X} : \text{Ind}(X) \otimes_{R} M_{R} \rightarrow \text{Ind}(X)\) such its restriction to \(X \otimes_{R} M_{R}^i, i > 0\), is the inclusion into \(\text{Ind}(X)\). This defines the \(T_{R}(M)\)-module \((\text{ind}(X), c_{X})\).

The assignment \((\text{Ind}(X), c_{X})\) to each \(X\) gives rise to a functor \(\text{Ind} : \text{mod-}R \rightarrow \text{mod-}T_{R}(M)\). As shown in [17, lemma 2.1] it is a left adjoint of the forgetful functor \(U : \text{mod-}T_{R}(M) \rightarrow \text{mod-}R\) by sending \((X, u)\) to the underlying module \(X\). Due to this adjoint pair one can see that any projective module in \(\text{mod-}T_{R}(M)\) is isomorphic to \((\text{Ind}(P), c_{P})\) for some \(P\) in \(\text{prj-}R\). Furthermore, for any module \((X, u)\) in \(\text{mod-}T_{R}(M)\), there is an exact sequence in \(\text{mod-}T_{R}(M)\) as the following

\[0 \rightarrow (\text{Ind}(X) \otimes_{R} M_{R}) \rightarrow (\text{Ind}(X), c_{X}) \rightarrow (X, u) \rightarrow 0.\]

We refer the reader to [17, Section 2] for more details of the construction of the above sequence. Denote by \(\mathcal{M}(T_{R}(M))\) the subcategory of \(\text{mod-}T_{R}(M)\) formed by all modules which are isomorphic to \((\text{Ind}(X), c_{X})\) for some \(X\) in \(\text{mod-}R\). By using the above sequence one can prove that \(\mathcal{M}(T_{R}(M))\) is a contravariantly finite subcategory in \(\text{mod-}T_{R}(M)\) and the global dimension relative to it is at most one. We leave the proofs to the reader as they can be proved in the similar cases in the preceding sections. Consequently, the subcategory \(\mathcal{M}(T_{R}(M))\) satisfies all the required conditions of Theorem 3.3. Therefore, we have the following result.

**Theorem 5.12** Let \(R\) and \(M\) be the same as in Theorem 5.11. Then, there is the following equivalence of triangulated categories

\[\mathbb{D}_{\text{sg}}(T_{R}(M)) \simeq \mathbb{K}^{-p}(\mathcal{M}(T_{R}(M))) / \mathbb{K}^{b}_{\text{uc}}(\mathcal{M}(T_{R}(M))).\]

One important advantage of the above theorem, and also the similar results in Theorems 5.11, 5.7 and 5.2, is to establish a close connection between the singularity category of \(\mathbb{D}_{\text{sg}}(T_{R}(M))\) and the singularity category \(\mathbb{D}_{\text{sg}}(R)\) of the base ring \(R\).

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