Josephson-Majorana cycle in topological single-electron hybrid transistors

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Charge transport through a small topological superconducting island in contact with a normal and a superconducting electrode occurs through a cycle which involves coherent oscillations of Cooper pairs and tunneling in/out the normal electrode through a Majorana bound state, the Josephson-Majorana cycle. We illustrate this mechanism by studying the current-voltage characteristics of two-terminal superconductor – topological superconductor – normal metal single-electron transistors. At low bias and temperature the Josephson-Majorana cycle is the dominant mechanism for transport.

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FIG. 1. The system consists of a superconducting grain coupled to a quantum wire connected to a normal metal and to a superconductor. Parameters are chosen so that the wire hosts two Majorana bound states at the two ends.

Cooper pair transistors [18, 20] there are important differences which we will discuss in the concluding part of the paper. The observation of transport mediated by the Josephson-Majorana cycle introduced here can be an evidence for the existence of Majorana fermions in systems proposed as topological superconducting wires.

The model — The system we consider is illustrated in Fig. 1. It consists of a topological superconducting island (a nanowire in close proximity to a superconducting island) tunnel-coupled to a normal and a superconducting electrode. The island hosts two Majorana bound states at the ends of the wire associated with the Majorana operators \( \gamma_i \) and \( \gamma_j \), respectively. Fu [15] first pointed out that the parity constraint leads to non-local effects in electron transport. The role of charging effects on the fractional Josephson effect and on the Coulomb blockade through a topological superconducting transistor was studied in Refs. [16] and [17], respectively.

In this work we show how the parity constraint leads to a new charge transport mechanism: The Josephson-Majorana cycle. It takes place when a topological superconducting island is coupled to a superconducting and to a normal lead (see Fig. 1). Charges can flow through the island due to the combined effect of the coherent oscillations of Cooper pairs in the island and the tunneling between the Majorana state and the normal leads. Although the process bears some similarities to the Josephson-quasiparticle mechanism present in

\[
\hat{H} = \hat{H}_M + \hat{H}_{Ch} + \hat{H}_J + \hat{H}_L + \hat{H}_T ,
\]

where the five terms describe coupling of Majorana states, island charging energy, Josephson coupling to the superconducting electrode, metallic normal electrode, and tunneling to the normal lead, respectively. The coupling between the Majorana fermions is given by \( \hat{H}_M = i\lambda \gamma_L \gamma_R \) where the overlap \( \lambda \) between them is exponentially small for distances much larger than the
superconducting coherence length and we will drop this term in the following. Two Majorana states can form a zero-energy fermionic level described by an annihilation operator \( d = (\gamma_L + i\gamma_R)/\sqrt{2} \) which can be either occupied or empty. Therefore not only the number of excess Cooper pairs \( N \) but also the occupation of the \( d \)-level enters the charging energy of the island,

\[
H_{Ch} = E_C (2N + n_d - n_g)^2,
\]

where \( n_d = d^\dagger d \) counts the occupation of the \( d \)-level, \( n_g \) is the gate charge which can be varied continuously by changing the gate voltage, and \( E_C = e^2/(2C) \) is the charging energy expressed in terms of the total capacitance \( C \) of the island. For later convenience we will label the eigenstates of the charging energy with the notation \( |N, n_d\rangle \). In order to describe the Josephson coupling to the superconducting electrode we use the effective Hamiltonian

\[
H_J = -E_J \cos (\varphi_S - \varphi),
\]

where \( E_J \) is the Josephson coupling energy and \( \varphi_S \) and \( \varphi \) are the phase of the superconducting electrode and the island condensate, respectively. For the present purposes any possible coupling to the Majorana state can be safely ignored [21]. The normal lead Hamiltonian is given by the noninteracting model

\[
H_L = \sum_k \epsilon_k c_k^\dagger c_k \text{ with creation (annihilation) operator } c_k^\dagger (c_k) \text{ corresponding to a spinless fermion in the free particle state } k \text{ with energy } \epsilon_k \text{ inside the normal lead. Finally, the tunneling through the Majorana state takes the form [17],}
\]

\[
H_T = \sum_k t_k c_k^\dagger (d + e^{-i\varphi} d^\dagger) + \text{H.c.},
\]

where \( t_k \) is the hopping amplitude to the \( k \)-state in the lead. The advantage of this formulation introduced by Zazunov et al. [17] is the fact that it includes automatically the constraint on the Hilbert space, linking the occupation of the Majorana bound state to the parity of the superconducting condensate. A non-trivial dynamics in the problem arises since \( \varphi \) and \( N \) are canonically conjugated variables, \([N, e^{i\varphi}] = e^{i\varphi} \). There are two terms in the tunneling Hamiltonian, corresponding to regular and anomalous tunneling. The first one describes the transfer of an electron from the \( d \)-level to the normal lead, and the second one the annihilation of a Cooper pair inside the island accompanied by the creation of two electrons, one in the \( d \)-level and one in the normal electrode.

**Josephson-Majorana cycle** — At bias voltage and temperature smaller than the superconducting gap, quasi-particle tunneling is suppressed and coherent Josephson (Cooper pair) tunneling is necessary for transport through the transistor. From Eq. [2] it follows that the resonance condition for coherent Cooper pair oscillations between two charge states that differ by one Cooper pair is fulfilled at integer values of the gate charge \( n_g \). Around \( n_g \approx 1 \), e.g., the lowest energy states are \( |N, n_d\rangle = |0, 0\rangle, |0, 1\rangle \) and \( 1, 0 \). A Josephson-Majorana cycle involving these three states is illustrated in Fig. 2. Starting from \( 0, 1 \), a regular tunneling process releases an electron into the normal lead. Then, the island is recharged with an extra Cooper pair provided by the Josephson coupling. A second (anomalous) tunneling which annihilates a Cooper pair in order to create an electron inside the normal electrode and another one filling the \( d \)-level, completes the cycle. (A cycle with the reverse direction can be obtained by the conjugates of each tunneling process.) Since the energy of the state \( 0, 1 \) is lower than the ones connected by Josephson tunneling, there will be a threshold voltage for the onset of current.

In the remainder of this work we support the simple picture sketched above with more detailed calculations. It is sufficient to compute the current-voltage characteristics to second order in the tunneling amplitudes \( t_k \). The average of the current operator, \( I = i e \sum_k [c_k^\dagger c_k, H] = -2e \text{Im} \sum_k t_k c_k^\dagger (d + e^{-i\varphi} d^\dagger) \), can be conveniently expressed in terms of the reduced density matrix \( \rho \) of the topological superconducting island which is obtained after tracing out the fermionic degrees of freedom of the normal metal. The average steady-state current is

\[
\langle I \rangle = \frac{e \Gamma}{2} \sum_{l,m,n} [D_{n \dagger} D_{ml}(F_{ln} + F_{lm}^\dagger) - D_{ln} D_{ml}^\dagger (2 - F_{nl}^\dagger - F_{ml})] \rho_{nm}
\]

where \( D_{nm} \) and \( \rho_{nm} \) are the matrix elements of the operators \( D = d + e^{-i\varphi} d^\dagger \) and \( \rho \) in the basis defined by the eigenstates \( |\psi_n\rangle \) of the Hamiltonian \( H = H_{Ch} + H_J \) with eigenvalues \( \mathcal{E}_n \). Furthermore, \( F_{nm} = f(\Delta_{nm}) - (i/\pi) \text{Re } \Psi[1/2 + i(\Delta_{nm} - eV)/(2\pi k_B T)] \), where \( \Delta_{nm} = \mathcal{E}_n - \mathcal{E}_m, f(\epsilon) = 1/(1 + e^{(-\epsilon)/k_B T}) \) is the Fermi function of the normal electrode, and \( \Psi \) is the digamma func-
tion. Finally \( \Gamma = 2\pi|\tau|^2N(\epsilon_F) \) is the tunneling rate with \( t_k \sim \tau \) assumed constant close to the Fermi energy \( \epsilon_F \), and \( N(\epsilon_F) \) is the density of states in the normal metal.

A convenient way to represent and compute the reduced density matrix, in particular when higher-order tunneling processes are taken into account, is to use a real-time diagrammatic technique that has been developed to describe transport through a metallic single-electron transistor [22, 23]. The time evolution of the reduced density matrix \( \rho(t) \) can be cast in the form [24]

\[
\hat{\rho}_{nm} + i\Delta_{nm}\rho_{nm} = \sum_{n',m'} \int_{-\infty}^{\infty} dt' W_{nm,n'm'}(t,t')\rho_{n'm'}(t') .
\]

Expanding to second order in the tunneling, this yields for the steady-state limit (\( \hat{\rho} = 0 \)) [25],

\[
i\Delta_{nm}\rho_{nm} = \frac{\Gamma}{2} \sum_{n',m'} [D_{n'n}D_{m'm'}(F_{nm'} + F_{nm'})
+ D_{n'n'}D_{m'm'}(2 - F_{nm'} - F_{nm})] \rho_{n'm'}
- \frac{\Gamma}{2} \sum_{l,n'} [D_{nl}D_{n'l}F_{lm} + D_{nl}^{*}D_{l'm'}(1 - F_{lm})] \rho_{n'm}
- \frac{\Gamma}{2} \sum_{l,n'} [D_{nl}D_{n'l}^{*}F_{lm} + D_{nl}^{*}D_{l'm'}^{*}(1 - F_{nm})] \rho_{nm'} .
\]

In Fig. 3 we show the current as a function of the gate charge \( n_g = CV_g/e \) and the bias charge \( n_V = CV/e \). The current increases stepwise (smeared by temperature) with the bias voltage and shows resonance peaks at integer values of \( n_g \). In the range of the gate charge where only the three charge states \( \{0,0\}, \{0,1\} \) and \( \{1,0\} \) are involved, a simple analytical solution can be obtained for zero temperature when neglecting the imaginary parts of \( F_{nm} \), which are associated with energy renormalization induced by the coupling to the bath. We find the two threshold voltages \( V^{\pm} \)

\[
eV^{\pm} = E_C \left[ 1 \pm 2\sqrt{(1-n_g)^2 + (E_J/4E_C)^2} \right] .
\]

For positive voltage \( V > 0 \) and \( 1/2 < n_g < 3/2 \), the steady-state current is

\[
\langle I \rangle = e\Gamma \begin{cases} 
\frac{2E_J^2}{\Gamma^2 + 4\xi^2 - E_J^2} & V > V^+ \\
\frac{E_J^2}{\Gamma^2 g_c(n_g) + 2\xi^2} & |V^-| < V < V^+ \\
0 & V < |V^-| 
\end{cases}
\]

where \( \xi = \sqrt{4E_C(1-n_g)^2 + E_J^2} \), \( g_c(n_g) = (21 + 13x - x^2 - x^3)/16 \), and \( x = 4E_C(1-n_g)/\xi \). We immediately see that the current vanishes when the Josephson coupling goes to zero. The width of the resonance peaks is \( \delta n_g = E_J/4E_C \) and the peak value for \( V > V^+ \) becomes maximal for \( \Gamma = \sqrt{4\xi^2 - E_J^2} \).

The effect of energy renormalization terms (Lamb shifts) that were neglected in the analytical formulas presented above, is highlighted in Fig. 4 where the resonance condition is slightly displaced from the value \( n_g = 1 \). We remark that the current is antisymmetric under the transformation \( (V,n_g) \to (-V,2-n_g) \). In order to see this we note that the transformation results in the substitutions \( F_{nm} \to 1 - F_{nm} \) and \( D_{nm} \to D_{nm}^{*} \). From Eqs. (5) and (6) we conclude that \( \rho \) remains unchanged while the current changes sign.

Finally, we comment on the abrupt suppression of the current for large bias voltage \( |n_V| > 3/2 \) that is visible in Fig. 4. Beyond the threshold \( n_V = 3/2 \), the new charge state \( \{1,1\} \) becomes available from \( \{1,0\} \) by tunneling. The island is then trapped in this new state since the Josephson coupling to the state \( \{0,1\} \) is suppressed because of the large energy difference of \( |1,1| \) and \( |0,1| \).

As a consequence, transport is blocked.

As already anticipated in the introduction there are crucial differences between the Josephson-Majorana cycle and the Josephson-quasiparticle cycle of Cooper pair...
transistors. The mechanism discussed in the present paper stems from the constraint linking the occupation of the Majorana bound state with the parity of the superconducting condensate. In the Josephson-quasiparticle cycle, it is a process related to the breaking of a Cooper pair into two quasi-particles with energy $\Delta$. Even the presence of an accidental zero energy state close to the normal lead would not give rise to the cycle described in this paper. A transport mechanism involving single particles and Cooper pairs would appear either in higher orders in tunneling or, again, above the superconducting gap. The presence of a zero-energy bound state, without the non-local character of the Majorana states, is not sufficient to realize the cycle introduced in the present paper because it would not display the anomalous tunneling term in the Hamiltonian.

**Conclusions** — In topological hybrid transistors charge transport is dominated at low voltages by a Josephson-Majorana cycle. This is a process in which the Rabi oscillations of Cooper pairs are accompanied by the tunneling of electrons from/to Majorana bound states. We showed the basic principles of this cycle in a superconductor – topological superconductor – normal metal transistor and discussed how the features of the current-voltage characteristics are linked to the presence of Majorana bound states. Because of the two time scales associated to the cycle, $1/E_J$ and $1/\Gamma$, interesting features are expected to appear in the frequency dependence of the current noise. Furthermore, more complex current cycles can be realized if more wires are deposited on top of the central superconducting island. The case of two wires is of particular interest since it is the minimal system to realize a topological qubit even in the presence of interaction as discussed in Ref. [14].

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[24] In the absence of the Josephson coupling, $E_J = 0$, the eigenstates $|\psi_n\rangle$ are defined by the total island charge $|2N + n_\alpha\rangle \equiv |N, n_\alpha\rangle$. Formulated in this basis, the diagrammatic rules for calculating the kernels $W_{nm,m'n'}$ are the same as given in Ref. [23] but with a different rate function, $\alpha^+ (\omega) \rightarrow (I/2\pi) f(\omega)$ and $\alpha^- (\omega) \rightarrow (I/2\pi) [1 - f(\omega)]$ and the extra rule that each crossing of tunneling lines yields a factor $-1$. The transformation of the diagrams into the eigenbasis of $H_{CH} + H_J$ for finite $E_J$ is straightforward.
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