Fragility of spectral clustering for networks with an overlapping structure

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Groups, or communities, commonly overlap in real-world networks. This is a motivation to develop overlapping community detection methods, because methods for non-overlapping communities may not perform well. However, deterioration mechanism of the detection methods used for non-overlapping communities have rarely been investigated theoretically. Here, we analyze the performance of spectral clustering, which does not consider overlapping structures, by using the replica method from statistical physics. Our analysis on an overlapping stochastic block model reveals how the structural information is lost from the leading eigenvector because of the overlapping structure.

I. INTRODUCTION

A graph or a network that represents related data is a common data structure in multivariate statistics, machine learning, and statistical mechanics. Identifying densely connected subgraphs—community detection—is useful for graph analysis. Such subgraphs are referred to as blocks or communities. Spectral clustering is a popular community detection algorithm that is efficient yet highly accurate on random graph models [1–4]. Nevertheless, spectral clustering is not recognized as a state-of-the-art algorithm for real-world networks. This is presumably because of specific features of real-world networks that are missing in simple random graph models. To fill this discrepancy, in this paper, we theoretically investigate how overlapping of communities affects the performance of spectral clustering.

We denote an undirected graph as \( G = (V, E) \), where \( V \) (\( |V| = N \)) is a set of nodes and \( E \) (\( |E| = m \)) is a set of edges. The graph is represented by the \( N \times N \) adjacency matrix \( A \), where \( A_{ij} = 1 \) when a pair of nodes \( i \) and \( j \) is connected by an edge and \( A_{ij} = 0 \) otherwise. The adjacency matrix of graphs with strong (Fig. 1a) and weak (Fig. 1b) non-overlapping community structures are illustrated in Fig. 1.

To identify the community structure, spectral clustering computes the leading eigenvalues and eigenvectors of a regularized adjacency matrix; in this paper, as an example, we focus on the so-called modularity matrix \( Q \) as the regularized adjacency matrix. When the community structure can be clearly identified, the isolated leading eigenvectors have relevant information of the communities, while a bulk of eigenvalues emerges from the randomness of a graph. For example, Fig. 2a shows the spectral density of the modularity matrix corresponding to the adjacency matrix in Fig. 1a. In this case, the largest eigenvalue is clearly separated from the bulk of eigenvalues, and we can extract two communities using the isolated leading eigenvector. On the other hand, Fig. 2b shows the case corresponding to the adjacency matrix in Fig. 1b. The eigenvalue correlated to the community structure is buried in the bulk of eigenvalues, and the spectral density is no longer distinguishable from that of a uniformly random graph. The phase transition point that the eigenvalues do not exhibit community structure at all is referred to as the (algorithmic) detectability limit [1, 7, 8] of spectral clustering.

As a tool for theoretical analysis, we use the replica method that originated from statistical physics. It enables us to calculate the ensemble average over random graph instances. As a result, we obtain a detectability phase diagram that indicates the effect of overlapping on spectral clustering.

Several existing studies have investigated the fragility of spectral clustering. As mentioned above, real-world networks have more complex structures than simple random graphs. Hence, the studies have considered the fragility in case of, e.g., adversarial perturbations [9], noise perturbations [10, 11], tangles and cliques [12], and localization of eigenvectors [7, 8, 13]. In this paper, we analyze the effect of the overlapping structure on the graph spectra. Specifically, we found that, when the size of the community overlap is increased, it is the isolated eigenvalue that is mainly affected. On the other hand, it is the bulk of eigenvalues that is mainly affected when the density of the community overlap is increased.

The rest of the paper is organized as follows. In Sec. II, we introduce the overlapping random graph models that we consider. In Sec. III, we provide the replica analysis for the graph spectra of the random graph model. In Sec. IV, we show the results and their interpretation obtained by the replica analysis. Finally, Sec. V presents a discussion.

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FIG. 1: Adjacency matrices of graphs with a non-overlapping structure. (a) Nodes in the same blocks are more densely connected internally than externally (strong community structure). (b) All nodes are connected with almost the same probability (weak community structure).

FIG. 2: Histograms of the bulk of eigenvalues of the modularity matrix, corresponding to Fig. 1a and Fig. 1b, respectively.

II. OVERLAPPING STOCHASTIC BLOCK MODEL

A. Canonical overlapping SBM

Before considering the overlapping SBM, we first introduce the (canonical) SBM with a general structure. A graph with $K$ groups is generated from the SBM as follows. For each node of the graph, we preassign a planted block label $t = t_i$, $t_i \in \{1, \cdots, K\}$ ($i \in V$). These labels are chosen randomly according to the block size distribution $p_k \left(k \in \{1, \ldots, K\}\right)$, $\sum_k p_k = 1$, $0 < p_k \leq 1$.

Second, each pair of nodes $(i, j)$ is connected by an edge with probability $\rho_{tt}$ independently and randomly; this probability is provided as an element of the $K \times K$ affinity matrix $\rho = [\rho_{kl}]$, $0 \leq \rho_{kl} \leq 1$. Therefore, the probability of a graph instance is expressed as

$$P(A|K, t, \rho) = \prod_{i<j} \rho_{tt}^{A_{ij}} (1 - \rho_{tt})^{1-A_{ij}}. \quad (1)$$
Here, because we consider undirected simple graphs, we assume that $A_{ii} = 0$ and $A_{ij} = A_{ji}$. Moreover, we focus on sparse graphs throughout this paper; i.e., we assume $\rho_{rs} = O(1/N)$ for all $r$ and $s$. When every matrix element of $\rho$ is equal, the model becomes the so-called Erdős–Rényi random graph model.

The overlapping SBM has the following parametrization in the parameter space of the SBM. As illustrated in Fig. 3, the overlapping SBM that we consider consists of three blocks: the first and third blocks correspond to the non-overlapping blocks, and the second block corresponds to the overlapping block. The model parameters of the overlapping SBM are defined as follows.

\begin{equation}
\mathbf{p} = (p_1, p_2, p_3) = (p_1, \alpha p_1, p_1),
\end{equation}

\begin{equation}
\mathbf{\rho} = \begin{pmatrix} p_{in} & p_{in} & p_{out} \\ p_{in} & \sigma p_{in} & p_{in} \\ p_{out} & p_{in} & p_{in} \end{pmatrix} = \begin{pmatrix} 1 & 1 & \epsilon \\ 1 & \sigma & 1 \\ \epsilon & 1 & 1 \end{pmatrix} \rho_{in},
\end{equation}

\begin{equation}
\mathbf{c} = (c_1, c_2, c_3) = (c_1, c_2, c_1).
\end{equation}

We denote the ratio between the community and overlapping blocks as $\alpha = p_2/p_1$. For the parametrization of the affinity matrix, $\rho_{in}$ represents the edge generation probability for pairs of nodes within the same block, while $\rho_{out}$ represents that for pairs of nodes in different blocks. $\epsilon = \rho_{out}/\rho_{in}$ is the parameter that controls the strength of the community structure. Finally, $\sigma$ is the density of the overlapping block. $\mathbf{c}$ is the average degree of each block, where the degree of a node is the number of edges connected to the node. The ratio $c_1/c_2$ can also be expressed as $(1 + \alpha + \epsilon)/(\sigma \alpha + 2)$ using the affinity matrix elements. Therefore, the parameters of the overlapping SBM are constrained as

\begin{equation}
c_1 (\sigma \alpha + 2) = c_2 (1 + \alpha + \epsilon).
\end{equation}

For simplicity, we assume the symmetry between the first and the third blocks, i.e., $p_1 = p_3$ and $c_1 = c_3$. The affinity matrix is a symmetric matrix, because we assume an undirected graph.

The overlapping SBM that we consider consists of three blocks. Nevertheless, in the following sections, we consider the partitioning into two blocks. Therefore, we analyze a model-inconsistent scenario.

### III. REPLICA ANALYSIS

We now calculate the spectrum of the overlapping SBM and show that a phase transition point of the largest eigenvalue exhibits the detectability limit. It should be noted that the same result is obtained in the case of the microcanonical SBM (Appendix B2).

#### A. Spectrum and the detectability limit of the overlapping SBM

As an example of a regularized adjacency matrix, we consider the modularity matrix. Each element of the matrix is defined as

\begin{equation}
M_{ij} = A_{ij} - \frac{d_i d_j}{2m},
\end{equation}

where $d_i (= \sum_{j=1}^{N} A_{ij})$ is the degree of a node $i$ and $m (= |E|)$ is the total number of the edges. Partitioning into two blocks can be identified by the eigenvector of the largest eigenvalue. Thus, our goal is to solve the following maximization problem.

\begin{equation}
\lambda(M) = \frac{1}{N} \max_{\mathbf{x}} \mathbf{x}^T M \mathbf{x}, \quad \text{subj. to } \mathbf{x}^T \mathbf{x} = N,
\end{equation}

where $\lambda(M)$ is the largest eigenvalue of $M$, and $\mathbf{x}^T$ is the transpose of a vector $\mathbf{x}$. This problem can be expressed...
as

\[ f(M, \beta) = -\frac{1}{\beta N} \log Z(M, \beta), \]

\[ \lambda(M) = -2 \lim_{\beta \to \infty} f(M, \beta), \]

\[ Z(M, \beta) = \int dx e^{\frac{\beta}{2} x^\top M x} \delta(x^\top x - N), \]

where \( Z(M, \beta) \) is the partition function. The constraint \( \beta \to \infty \) is imposed by the delta function in (10), and taking we analyze

we are interested in the typical behavior of the graph instances, we analyze

\[ |\lambda(M)|_M = 2 \lim_{\beta \to \infty} \frac{1}{\beta N} |\log Z(M, \beta)|_M, \]

where \([\cdots]_M\) represents the ensemble average over graph instances. Unfortunately, it is difficult to calculate the average \( |\log Z(M, \beta)|_M \) analytically. To overcome this difficulty, we use the replica trick, namely,

\[ |\log Z(M, \beta)|_M = \lim_{n \to 0} \frac{\partial}{\partial n} |\log Z^n(M, \beta)|_M. \]

Here, the exponent \( n \) in \([Z^n]|_M\) is a real value. However, we treat \( n \) as an integer for a moment. In the end, we perform the analytic continuation to the real value.

From Eq. (10), the \( n \)th moment the partition function is obtained as

\[ [Z^n(M, \beta)]_M = \int \left( \prod_{a=1}^{n} dx_a \delta(x_a^\top x_a - N) \right) \left[ \exp \left( \frac{\beta}{2} \sum_a x_a^\top M x_a \right) \right]_M, \]

where \( a \in \{1, \ldots, n\} \) is an index of \( n \) identical copies. For further calculations, we introduce several order parameters and approximations. Detailed calculations are described in Appendix A. As a result, the average largest eigenvalue in the limit of \( N \to \infty \) is obtained by the following saddle-point (extremum) condition of nine auxiliary variables \((\phi, \Omega, \Omega, m_{1k}, m_{2k}, \hat{m}_{1k}, \hat{m}_{2k}, a_k, \hat{a}_k)\).

\[ [\lambda(M)]_M = \text{extr}_{\phi,\Omega,\Omega,m_{1k},m_{2k},\hat{m}_{1k},\hat{m}_{2k},a_k,\hat{a}_k} \left\{ \phi + 2\Omega - \Omega^2 \right. \]

\[ + \frac{1}{2} N \sum_{k,k'} W_{kk'} \left( a_k \left( m_{2k} - \frac{2\Omega}{\sqrt{\sigma}} + \frac{4\Omega^2}{\sigma} \right) + 2m_{1k} \left( m_{1k} - \frac{2\Omega}{\sqrt{\sigma}} \right) + a_k m_{2k} - \frac{m_{2k} - 2\Omega \hat{m}_{1k} + 4\Omega^2}{\sigma} - \frac{m_{2k}}{a_k} \right) \]

\[ - \sum_k p_k \left( \frac{m_{2k} + 2m_{1k} \hat{m}_{1k} + \hat{m}_{2k} - m_{2k}}{a_k - \hat{a}_k} \right) \]

\[ + \left. \frac{1}{N} \sum_k \sum_{i \in V_k} \sum_{d=0}^\infty \frac{P_{\epsilon_k}(d)}{\phi - d \hat{a}_k} \left( d \hat{m}_{2k} + d(d-1) \hat{m}_{1k} \right) \right\}, \]

Here, \( \sum_{i \in V_k} \) is the sum over the node indices that belong to the \( k \)th block. \( W_{kl} \) and \( \hat{c} \) are defined as \( W_{kl} \equiv p_k p_l \) and \( \hat{c} \equiv 2m/N \), respectively. \( P_{\epsilon_k}(d) \) is the Poisson probability mass function of degree \( d \) of each node in block \( k \) that has expectation \( c_k \). \( m_{1k} \) is the mean of the largest eigenvector elements that correspond to the \( k \)th block. Definitions of the other auxiliary variables are described in Appendix A.

The detectability limit is derived by solving the equations of the nine auxiliary variables. In particular, \( m_{11} (= -m_{13}) \) plays an important role for the detectability limit. When \( m_{11}^2 > 0 \), the spectral clustering retains the ability to detect the community structure (detectable condition). On the other hand, when \( m_{11}^2 = 0 \), the result of spectral clustering is uncorrelated to the planted structure (undetectable condition). Accordingly, the phase transition point is derived by the condition \( m_{11}^2 = 0 \). This corresponds to the condition that the largest eigenvalue is buried in the bulk of the eigenvalues, as we mentioned in Introduction.

IV. PERFORMANCE OF THE SPECTRAL CLUSTERING ON THE OVERLAPPING SBM

In this section, using the results obtained by the replica analysis, we show how the size and density of the overlapping block affect the spectrum. We also check the validity of our analytical calculations by comparing them to the results of numerical experiments. We use the microcanonical SBM in the numerical experiments to avoid some problems of the canonical SBM, as mentioned in
The model parameters are set to $c_1 = 10$ and $\sigma = 2$.
Along the line of the results of the numerical experiments determined by constraint (5), degree $c_2$ takes a fixed value. The lines in this figure, from left to right, correspond to the values of $c_2$ from 19 to 11.

Appendix B 4. We used graph-tool [16] to generate graph instances of the microcanonical SBM.

A. Detectability phase diagram and the leading eigenvalue

First, to observe the overall dependency of overlapping structures, we show the detectability phase diagram. Figure 4 shows the detectability phase diagram of the $(\epsilon, \alpha)$ plane, where $\epsilon$ is the strength of the community structure and $\alpha$ is the size of the overlapping block ($\alpha = p_2/p_1$).

The boundary between the blue and orange regions represents the detectability limit of the spectral clustering predicted by the replica analysis. The dots represent the results of the numerical experiments; the color gradient represents the fraction of correctly classified nodes, which is often referred to as the overlap. We can see that both boundaries are in a good agreement. Note that the numerical experiment is possible only on the discrete points in the parameter space because of constraint (5), and $c_2$ can take only natural numbers in the microcanonical SBM. In this experiment, we set $c_1 = 10$ and $\sigma = 2$. Then, the range $c_2$ can take is restricted between 11 and 19 because of the assortative condition $0 \leq \epsilon \leq 1$.

This phase diagram is the result that shows how fragile the spectral clustering is against the overlapping structure.

Figure 5 shows the leading eigenvalue and the edge of the bulk of the eigenvalues, which are predicted by the replica analysis, and the top ten eigenvalues computed in the numerical experiments. We can confirm that the replica analysis accurately describes the behavior of numerical experiments. When $\alpha$ is small, the leading eigenvalue is separated from the bulk of the eigenvalues. As $\alpha$ increases, the leading eigenvalue approaches the bulk of the eigenvalues. As we described in Introduction, when it reaches the bulk of the eigenvalues, the spectral clustering loses ability to detect the community structure, i.e., the detectability limit. Note the value of $\epsilon$ also varies according to (5) as $\alpha$ varies. Thus, the horizontal axis in Fig. 5 corresponds to the line in Fig. 4 with $c_2 = 18$.

B. Effects of the size of the overlapping structure

We now investigate the effect of the overlapping structure on the performance of the spectral clustering when we increase the size of the overlapping block. Because the overlapping block can have denser (or sparser) edge density than the other blocks, the average degree also increases (or decreases) accordingly, as the size of the overlapping block increases. This implies that the width of the bulk of the eigenvalues is trivially influenced, because the bulk is known to depend on the average degree [1].

\[1\] The edge of the bulk of eigenvalue is derived as the largest eigenvalue under the undetectable condition.
FIG. 6: Comparison between the overlapping and bimodal SBMs. This figure shows the eigenvalues of bimodal SBM in addition to those in Fig. 5. The blue dots represent the top ten eigenvalues of the bimodal SBM computed in the numerical experiments. The brown solid and dashed lines represent the leading eigenvalue and the bulk edge of the eigenvalues of the bimodal SBM that are derived by the replica analysis, respectively. The spectrum of the overlapping SBM is plotted as in Fig. 5. These models are identical only when $\alpha = 0$. However, their bulk edges should coincide when $\alpha = 0$ and $\alpha = 1$. For the value of $\epsilon$ of the bimodal SBM, we used the same value as the overlapping SBM, which varies as $\alpha$ increases owing to constraint (5).

However, it is not trivial if it is the only effect. Namely, the overlapping structure may affect the isolated eigenvalue or the bulk in another way. To assess the effect of the overlapping structure rather than the effect of the average degree, we compare the overlapping SBM with the model with no overlapping structures but that has the same degree distribution as the overlapping SBM. In the case of the microcanonical overlapping SBM, the degree distribution is bimodal: all the nodes in the overlapping block have the same degree, while all the other nodes have the other degree. Therefore, we consider the non-overlapping SBM with a bimodal degree distribution (see Appendix C for a detailed definition). We assume that the sizes of the blocks are equal. Hereafter, we refer to this model as the bimodal SBM.

Figure 6 shows the bulks of eigenvalues and the leading eigenvalues of the overlapping and bimodal SBMs. We can confirm that both bulk edges almost coincide. In contrast, the leading eigenvalue of the bimodal SBM is separated from the bulk in the whole space, while that of the overlapping SBM approaches to its bulk as $\alpha$ increases. This indicates that the increase of the size of the overlapping block mainly affects the leading eigenvalue instead of the bulk.

The fact that the bulk is not considerably affected is not very trivial. If we take a closer look, the bulk edges do not exactly coincide in Fig. 6, although the deviation is very small. This is because the models are not identical even when there is no community structure (i.e., $\epsilon = 0$). When $\alpha = 0$, the two models reduce to the $c_1$-regular SBM. Thereby, their bulk edges become equal to $2\sqrt{c_1 - 1}$. When $\alpha = 1$, the overlapping SBM becomes a uniform (one block) model with (average) degree $c_2$, while the bimodal SBM has the community structure with (average) degree $c_2$. However, the bulk edge of the SBM with no overlapping blocks depends only on its average degree. Thus, although the models are not identical, their bulk edges are both $2\sqrt{c_2 - 1}$.

C. Effects of the density of the overlapping structure

Next, we investigate how density $\sigma$ of the overlapping block affects the detectability. As mentioned in the previous subsection, the higher density of the overlapping block trivially makes the width of the bulk of the eigenvalues expand wider.

Figures 7a–7b show the detectability phase diagram derived by the replica analysis and the results of the corresponding numerical experiments for $\sigma = 0.5$ and 2. Notably, the detectable region is wider when $\sigma$ is small. This indicates that the higher density deteriorates the detectability more significantly.

Let us examine $\sigma$ dependency. Figure 8a shows the $\alpha$ dependencies derived by the replica analysis of the canonical SBM. They are the isolated leading largest eigenvalues and the bulk of the eigenvalues for $\sigma = 0, 0.5, 1, 1.5$, and 2. Interestingly, the isolated largest eigenvalue does not depend on $\sigma$ considerably. In contrast, the bulk is highly dependent on $\sigma$. This indicates that the deterioration of the detectability due to $\sigma$ is caused by the expansion of the bulk rather than the shrinkage of the isolated leading eigenvalue. Figure 8b similarly shows the $\epsilon$ dependencies. Again, we can see that the isolated largest eigenvalue does not depend on $\sigma$ considerably while the bulk is highly dependent.

Notably, we cannot test the result of Fig. 8a directly in numerical experiments, because $\alpha$ cannot be varied continuously as $\epsilon$ is fixed. This is due to the constraints of the microcanonical SBM. Similarly, in Fig. 8b $\epsilon$ cannot be varied continuously as $\alpha$ is fixed. Nevertheless, we can draw smooth curves in the replica analysis, because we consider the canonical SBM that is not subject to the constraints of the microcanonical SBM. Importantly, the results of the microcanonical SBM coincide with those of the canonical SBM with the regular approximation at the points where the microcanonical SBM is realizable. We also note that (Appendix B 2) the distinction between the canonical and microcanonical SBMs is invisible in infinite graph size limits.
FIG. 7: Detectability phase diagram of the \((\epsilon, \alpha)\) plane for \(\sigma = 0.5, 2\) and \(c_1 = 10\). A detailed explanation is provided in the caption of Fig. 4.

FIG. 8: (a) Isolated eigenvalues (solid lines) and bulk edges (dashed lines) as a function of \(\alpha\) for \(\sigma = 0, 0.5, 1, 1.5, 2\). Parameters are set to \(c_1 = 10\) and \(\epsilon = 0.3\). The value of degree \(c_2\) varies according to (5). (b) Isolated eigenvalues (solid lines) and bulk edges (dashed lines) as a function of \(\epsilon\). \(\alpha\) is fixed as 0.3. Other experimental conditions are identical to those of Fig. 8a.

V. SUMMARY

We investigated the effect of the size and the density of the overlapping block on the performance of spectral clustering using the replica method. Both larger size and higher density help the isolated eigenvalue to be buried in the bulk of the eigenvalues, i.e., deteriorate the detectability. Importantly, however, their mechanisms are strikingly different. We found that increasing the size of the overlapping block has a prominent effect on making the isolated eigenvalue smaller (Fig. 6). In contrast, increasing the density of the overlapping block makes the bulk width larger, while the isolated eigenvalue remains almost the same (Fig. 8a).

According to our findings, the results of the replica analysis are consistent with those of the numerical experiments. This indicates that the detectability phase transition of the spectral clustering in the present setting is regarded as a phenomenon that can be understood in the scope of the mean-field theory.

Although spectral clustering typically deals with non-overlapping structures, we showed that it is also possible to analyze the model-inconsistent case, such as the overlapping SBM. It is possible, in principle, to investigate even more complex situations using the replica method. However, for example, we would need to deal with saddle-point equations with many variables if we were to analyze
a general three-block SBM. Therefore, we believe that the present model is an extreme case where the analytical calculation is executable and the results are interpretable.

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Appendix A: Derivation of the spectrum and the detectability limit of the canonical SBM

The goal of this appendix is to derive saddle-point expression of the average largest eigenvalue \([\lambda_{\text{largest}}]\). Note that a similar calculation using the replica method can be found in Refs. [2] [8] [17]. We start with the average of \(n\)th moment of the partition function

\[
[Z^n(M, \beta)]_M = \int \left( \prod_{a=1}^{n} dx_a \delta(x_a^\top x_a - N) \right) \left[ \exp \left( \frac{\beta}{2} \sum_a x_a^\top M x_a \right) \right]_M (A1)
\]

\[
= \int \left( \prod_{a=1}^{n} dx_a \delta(x_a^\top x_a - N) \right) \times \left[ \exp \left( -\frac{\beta}{2} \sum_a (\gamma_a^\top x_a)^2 \right) \exp \left( \frac{\beta}{2} \sum_a x_a^\top A x_a \right) \right]_A (A2)
\]

where \(\gamma_i \equiv d_i/\sqrt{2m}\). Introducing order parameters

\[
\Omega_a = \frac{1}{\sqrt{N}} \sum_i \gamma_i x_{ia}, \quad (a \in \{1, \ldots, n\}), (A3)
\]

we can recast exponential factor \(e^{-\frac{\beta}{2} \sum_a (\gamma_a^\top x_a)^2}\) in (A2) as
\[
\exp \left( -\frac{\beta}{2} \sum_a (\chi^T x_a)^2 \right) = \int \prod a \, d\Omega_a \delta \left( \Omega_a - \frac{1}{\sqrt{N}} \sum_i \gamma_i x_{ia} \right) \exp \left( -\frac{\beta N}{2} \sum_a \Omega_a^2 \right) \tag{A4}
\]
\[
= \int \prod a \, \frac{\beta N}{2\pi} d\Omega_a d\hat{\Omega}_a e^{-\frac{\beta N}{2} \sum_a (\Omega_a^2 - 2\hat{\Omega}_a \hat{\Omega}_a)} e^{-\beta \sqrt{\pi} \sum_a \hat{\Omega}_a \gamma_{ia} x_{ia}} \tag{A5}
\]
\[
= \int \prod a \, \frac{\beta N}{2\pi} d\Omega_a d\hat{\Omega}_a e^{-\frac{\beta N}{2} \sum_a (\Omega_a^2 - 2\hat{\Omega}_a \hat{\Omega}_a)} \prod_{ij} e^{-\frac{\beta N}{2} \sum_{\alpha} \hat{\Omega}_{\alpha ij} x_{ia} x_{ja}} \tag{A6}
\]

We have set \( \delta = 2m/N \). Moreover, \( \hat{\Omega}_a \) is the auxiliary variable that is conjugate to \( \Omega_a \). To derive this expression, we transformed the delta function to

\[
\delta \left( \sqrt{\pi} \Omega_a - \sum_i \gamma_i x_{ia} \right) = \int_{-\infty}^{+\infty} \frac{\beta \sqrt{\pi}}{2\pi} d\hat{\Omega}_a e^{-\beta \sqrt{\pi} \Omega_a (\sqrt{\pi} \Omega_a - \sum_i \gamma_i x_{ia})}. \tag{A7}
\]

Inserting Eq. (A6) into the exponential factor in (A2), we obtain

\[
\left[ \exp \left( \frac{\beta}{2} \sum_a x_{ia}^T M x_{ia} \right) \right]_M \approx \int \prod a \, \frac{\beta N}{2\pi} d\Omega_a d\hat{\Omega}_a e^{-\frac{\beta N}{2} (\Omega_a^2 - 2\hat{\Omega}_a \hat{\Omega}_a)} \prod_{ij} \rho_{t_i, t_j} (1 - \rho_{t_i, t_j})^{1 - A_{ij}} e^{\beta \sum_{\alpha} A_{ij} x_{ia} (x_{ja} - \frac{\Omega_a}{\sqrt{\pi}})} \tag{A8}
\]
\[
\approx \int \prod a \, \frac{\beta N}{2\pi} d\Omega_a d\hat{\Omega}_a e^{-\frac{\beta N}{2} (\Omega_a^2 - 2\hat{\Omega}_a \hat{\Omega}_a)} \prod_{ij} \exp \left( \log(1 - \rho_{t_i, t_j}) + \rho_{t_i, t_j} e^{\beta \sum_{\alpha} x_{ia} (x_{ja} - \frac{\Omega_a}{\sqrt{\pi}})} \right). \tag{A9}
\]

Here, we took the configuration average over the canonical SBM 4 and approximated \( \frac{\rho_{t_i, t_j}}{1 - \rho_{t_i, t_j}} \approx \rho_{t_i, t_j} \) by using the fact that \( \rho_{t_i, t_j} = O(N^{-1}) \).

Let us now introduce the order-parameter functions

\[
Q_k(u) = \frac{1}{p_k N} \sum_{i \in V_k} \prod a \, \delta(u_a - x_{ia}), \quad (k \in \{1, \ldots, K\}) \tag{A10}
\]

where \( \sum_{i \in V_k} \) is the sum over the node indices that belong to the \( k \)th block. Then, the last exponential factor in (A9) can be approximated as

\[
\exp \left( \sum_{i<j} \rho_{t_i, t_j} e^{\beta \sum_{\alpha} x_{ia} (x_{ja} - \frac{\Omega_a}{\sqrt{\pi}})} \right) \approx \exp \left( N^2 \int_a du_a dv_a e^{\beta \sum_{\alpha} (u_a - \frac{\Omega_a}{\sqrt{\pi}})} \times \sum_{kk'} Q_k(u) W_{kk'} Q_{k'}(u) \right), \tag{A11}
\]

where we approximated that the contribution from the diagonal elements is negligible, and we defined \( W_{kk'} = p_k \delta_{kk'} \). Inserting Eq. (A11) into (A9), Eq. (A1) is now expressed as

\[
[Z^n(M, \beta)]_M = \int \prod a \, dx_a \int \prod a \, \frac{\beta N}{2\pi} d\Omega_a d\Omega_a \prod_i \delta(x_i x_a^T - N) \times \exp \left( -\frac{\beta N}{2} (\sum_a (\Omega_a^2 - 2\hat{\Omega}_a \hat{\Omega}_a)) + \sum_{i<j} \log(1 - \rho_{t_i, t_j}) \right) + \frac{N^2}{2} \int \prod_a du_a dv_a e^{\beta \sum_{\alpha} (u_a - \frac{\Omega_a}{\sqrt{\pi}})} \sum_{kk'} Q_k(u) W_{kk'} Q_{k'}(u) \right). \tag{A12}
\]

Here, we use the expansion of the delta function

\[
\delta (x_a^T x_a - N) = \int_{-\infty}^{+\infty} \frac{\beta d\phi_a}{4\pi} e^{-\frac{\beta}{4} \phi_a (\sum_i x_{ia}^2 - N)} \tag{A13}
\]

and the identity

\[
1 = \prod_k \frac{p_k N}{\beta} \int \frac{DQ_k}{2\pi} \int dQ_k \times \delta \left( \sum_{i \in V_k} \prod_{a=1}^n \delta(x_{ia} - \mu_a) - p_k N Q_k(\mu) \right) \tag{A14}
\]
\[
= \prod_k \frac{p_k N}{\beta} \int \frac{DQ_k D\hat{Q}_k}{2\pi} \times \delta \left( \sum_{i \in V_k} \prod_{a=1}^n \delta(x_{ia} - \mu_a) - p_k N Q_k(\mu) \right) \tag{A15}
\]

Here, \( \int DQ_k \) is the functional integral with respect to \( Q_k(\mu) \), and \( \hat{Q}_k(\mu) \) was introduced as the conjugate of \( Q_k(\mu) \). To derive Eq. (A15), we used the expansion of the delta function. By inserting the identity, we can focus on a \( Q_k(\mu) \) corresponding to the replacement in (A10). Note that without the insertion of the identity, the replacement of (A10) becomes invalid. From these, we can recast
Eq. (A12) as

\[ Z^n(M, \beta)|_M = \int \prod_a \frac{d\phi_a}{4\pi} \int \prod_a \frac{\beta N}{2\pi} d\Omega_a d\Omega_a \int \prod_k \frac{p_k N}{2\pi} D\hat{Q}_k DQ_k \]

\[
\times \exp \left( -\frac{\beta N}{2} \sum_a (\Omega_a^2 - 2\Omega_a \phi_a) + \sum_{i<j} \log(1 - \rho_{i,j}) \right) \\
- \sum_k p_k N L_k(Q_k, \hat{Q}_k) + K(\{Q_k\}) \\
+ \sum_k \sum_{i \in V_k} \log M_{i,k}(\hat{Q}_k, \{\phi_a\}) \right),
\]

(A16)

where

\[
K(\{Q_k\}) = \frac{N^2}{2} \int \prod_a du_a dv_a e^{\beta u_a \left( v_a - \frac{2\beta}{a} \right)} \\
\times \sum_{kk'} Q_k(u) W_{kk'} Q_{k'}(v),
\]

(A17)

\[
L_k(Q_k, \hat{Q}_k) = \int d\mu Q_k(u) Q_k(u),
\]

(A18)

\[
M_{i,k}(\hat{Q}_k, \{\phi_a\}) = \int \prod_a dx_a e^{Q_k(x_a) - \frac{1}{2} \sum_a \phi_a x_a^2}. \quad (A19)
\]

Here, we assume the functional form of \(Q_k(u)\) and \(\hat{Q}_k(u)\) are restricted to Gaussian mixtures. This indicates that \(Q_k(u)\) and \(\hat{Q}_k(u)\) can be expressed as

\[
Q_k(u) = q^0_k \int dA dH q_k(A, H) \left( \frac{\beta A}{2\pi} \right)^{\frac{A}{2}} \\
\times \exp \left( -\frac{\beta A}{2} \sum_a \left( \mu_a - \frac{H}{A} \right)^2 \right), \quad (A20)
\]

\[
\hat{Q}_k(u) = \hat{q}^0_k \int d\tilde{A} d\tilde{H} \hat{q}_k(A, \tilde{H}) \exp \left( \frac{\beta}{2} \sum_a \left( \tilde{A} \mu^2_a + 2\tilde{H} \mu_a \right) \right), \quad (A21)
\]

where \(q_k(A, H)\) is the weight of a Gaussian distribution with the mean and precision parameter equal to \(H/A\) and \(H\), respectively. \(\tilde{q}_k(A, \tilde{H})\) is defined analogously. \(q^0_k\) and \(\hat{q}^0_k\) are the normalization constants; it can be deduced that \(q^0_k = 1\) and \(\hat{q}^0_k = c_k\) from the saddle-point conditions when \(n = 0\). Inserting Eq. (A20) and (A21) into (A17)–(A19), we have

\[
K(\{Q_k\}) = \frac{N^2}{2} \sum_{kk'} \int dA dH q_k(A, H) \int dA' dH' q_{k'}(A', H') \left( \frac{AA'}{AA' - 1} \right)^{-\frac{1}{2}} \\
\times \exp \left[ \sum_a \frac{\beta}{2} \left( A' \left( H - \frac{2\beta}{V}\right)^2 + 2 \left( H - \frac{2\beta}{V}\right) \frac{H' + AH'^2}{AA' - 1} - \left( H - \frac{2\beta}{V}\right)^2 - \frac{H'^2}{A} \right) \right], \quad (A22)
\]

\[
L_k(Q_k, \hat{Q}_k) = \int dA dH d\tilde{A} \tilde{H} q_k(A, \tilde{H}) \left( \frac{A}{A - \tilde{A}} \right)^{\frac{\beta}{2}} \exp \left[ \frac{n \beta}{2} \left( \frac{(H + \tilde{H})^2 - H^2}{A - A} \right) \right], \quad (A23)
\]

\[
M_{i,k}(\hat{Q}_k, \{\phi_a\}) = \left( \frac{2\pi}{\beta} \right)^{\frac{1}{2}} \sum_{d=0}^{\infty} \frac{1}{d!} d\hat{A}_g d\tilde{H} g k(A_g, \tilde{H}_g) \left( \phi_a - \sum_g \hat{A}_g \right)^{-\frac{1}{2}} \exp \left( \frac{\beta}{2} \left( \sum_g \hat{H}_g \right)^2 \right). \quad (A24)
\]

To derive Eq. (A24), we expanded the exponential as

\[ e^{Q_k(x_i)} = \sum_{d=0}^{\infty} \frac{1}{d!} Q_k^d(x_i). \]

Hereafter, let us assume no distinction among the variables with different replica indices, i.e., \(\phi_a = \phi, \Omega_a = \Omega, \) and \(\hat{\Omega}_a = \hat{\Omega}.\) This is referred to as the replica symmetric assumption. We insert Eq. (A22)–(A24) into (A16) under this assumption. Then, we obtain the following saddle-point equation for the average largest eigenvalue from Eqs. (8), (9), and (12) as
[\lambda(M)]_M = 2 \lim_{\beta \to \infty} \frac{1}{\beta N} \lim_{n \to \infty} \frac{1}{\partial_n} \log[Z^n]_M \\
= \text{extr}_{\phi, \Omega, \hat{\Omega}, \{q_k, \hat{q}_k\}} \left[ \phi + 2\Omega \hat{\Omega} - \Omega^2 + \frac{1}{2} \sum_{kk'} NW_{kk'} \int dA dH q_k(A, H) \int dA' dH' q_{k'}(A', H') \times \left( \frac{A' (H - \frac{2\Omega}{\sqrt{\beta}})^2 + 2 \left( H - \frac{2\Omega}{\sqrt{\beta}} \right) H' + AH'^2}{AA' - 1} - \frac{(H - \frac{2\Omega}{\sqrt{\beta}})^2}{A} - \frac{H'^2}{A'} \right) \right. \\
- \sum_k p_k c_k \int dA dH d\hat{A} d\hat{H} q_k(A, H) \hat{q}_k(\hat{A}, \hat{H}) \left( \frac{(H + \hat{H})^2}{A - \hat{A}} - \frac{H^2}{A} \right) \\
+ \sum_k p_k \sum_{d=0} \mathcal{P}_{c_k}(d) \int \prod_{g=1}^d (d\hat{A}_g d\hat{H}_g \hat{q}_k(\hat{A}_g, \hat{H}_g)) \left( \frac{\sum_g \hat{H}_g}{\phi - \sum_g \hat{A}_g} \right)^2 \right]. \tag{A26} \\

Here, \( \mathcal{P}_{c_k}(d) \) is the probability mass function of degree \( d \) of each node in block \( k \) that has expectation \( c_k \). From the saddle-point condition in Eq. (A26), we obtain the functional equations with respect to \( q_k(A, H) \) and \( \hat{q}_k(\hat{A}, \hat{H}) \) as

\[
q_k(A, H) = \sum_{d=0}^\infty \mathcal{P}_{c_k}(d) \int \prod_{g=1}^{d-1} (d\hat{A}_g d\hat{H}_g \hat{q}_k(\hat{A}_g, \hat{H}_g)) \times \delta \left( H - \sum_{g=1}^{d-1} \hat{H}_g \right) \delta \left( A - \phi + \sum_{g=1}^{d-1} \hat{A}_g \right), \tag{A27}
\]

\[
\hat{q}_k(\hat{A}, \hat{H}) = \frac{1}{c_k} \sum_{k'} N \rho_{kk'} \mathcal{P}_{c_k} \int dA' dH' q_{k'}(A', H') \times \delta \left( \hat{A} - \frac{1}{A'} \right) \delta \left( \hat{H} - \frac{H' - \frac{2\Omega}{\sqrt{\beta}}}{A'} \right). \tag{A28}
\]

To derive Eq. (A27), we used the fact that the expectation of \( H^2 / A \) becomes 0, which is derived by substituting \( \hat{H} = \hat{A} = 0 \). Moreover, the saddle-point condition with respect to \( \phi \) yields

\[
\sum_k p_k \int dA dH Q_k(A, H) \left( \frac{H}{A} \right)^2 = 1, \tag{A29}
\]

where

\[
Q_k(A, H) = \sum_{d=0}^\infty \mathcal{P}_{c_k}(d) \int \prod_{g=1}^d (d\hat{A}_g d\hat{H}_g \hat{q}_k(\hat{A}_g, \hat{H}_g)) \times \delta \left( H - \sum_{g=1}^d \hat{H}_g \right) \delta \left( A - \phi + \sum_{g=1}^d \hat{A}_g \right). \tag{A30}
\]

Unfortunately, solving the functional form of equations is still not analytically tractable. Thus, we introduce further approximations that \( q_k(A) = \delta(A - a_k) \) and \( \hat{q}_k(\hat{A}) = \delta(\hat{A} - \hat{a}_k) \), i.e., we ignore the fluctuation of the precision parameters. This is called the effective medium approximation (EMA) \cite{17, 18}. Performing the EMA for (A26), we arrive at

Equation (A29) corresponds to the normalization constraint in (17). Equations (A27) and (A28) constitute functional equations under constraint (A29), and solving these equations yields the distribution of the largest eigenvector elements. Note that \( q_k(A, H) \) was introduced as the weight in the Gaussian mixture, which approximates the empirical distribution of the largest eigenvector elements in (A10). This indicates that \( q_k(A, H) \) exhibits the probability density of the eigenvector-element distribution.
where \( m_{\ell k} \) and \( \hat{m}_{\ell k} \) stand for the \( \ell \)th moments of \( H \) and \( \hat{H} \), respectively, i.e., \( m_{\ell k} = \int dHH^\ell q_k(H) \) and \( \hat{m}_{\ell k} = \int dHH^\ell \hat{q}_k(\hat{H}) \).

The saddle-point conditions from (A31) lead to the equations for the auxiliary variables \( \phi, \Omega, \hat{\Omega}, m_{\ell k}, \hat{m}_{\ell k}, a_k, \) and \( \hat{a}_k \). Here, we focus on a model with the symmetry between the first and the third blocks: \( p_1 = p_2 \) and \( c_1 = c_3 \). Due to this assumption, we can apply the same assumptions to the physical quantities \( a_k, \hat{a}_k, m_{2k}, \hat{m}_{2k} \), that is, \( a_1 = a_3, \hat{a}_1 = \hat{a}_3, m_{21} = m_{23} \), and \( \hat{m}_{21} = \hat{m}_{23} \). This is because these quantities are the second-order statistics and do not depend on the signs.

Further, we assume \( m_{12} = 0 \). This assumption stems from the fact that the second block corresponds to the overlapping block, which does not contain the two communities. Thus, the corresponding elements of the eigenvector come from a random structure of the graph. Moreover, we classify the solution into the cases of \( m_{11} = 0 \) and \( m_{11} \neq 0 \). For the solution with \( m_{11} = 0 \), we can assume \( m_{13} = 0 \) owing to the symmetry. On the other hand, for the solutions with \( m_{11} \neq 0 \), we can assume \( m_{11} = -m_{13} \) due to the symmetry and the fact that the eigenvector elements of \( \bfx \) tend to have the same signs in the same block. In summary, we have two types of solutions: \( m_{11} = -m_{13} \neq 0, m_{12} = 0 \) and \( m_{11} = m_{12} = m_{13} = 0 \). In fact, the former corresponds to the detectable condition and the latter corresponds to the undetectable condition. The leading eigenvalue is calculated for each of the two conditions, and the detectability limit is derived as the boundary between these two conditions. We further simplify the problem using the regular approximation with respect to the degree, namely the random variables following the Poisson distribution \( d \) in (14) are fixed as their means \( c_k \).

First, under the detectable condition, we can derive the equations for \( a_1, a_2, \hat{a}_1, \) and \( \hat{a}_2 \) from the saddle-point conditions as

\[
\begin{align*}
\lambda(M)|_M &= \phi + 2\hat{\Omega} - \Omega^2 + \frac{1}{2} \sum_{k,k'} \frac{W_{kk'} (a_k m_{2k} - 2\hat{\Omega} \sqrt{\alpha}) + 2m_{1k} \hat{m}_{2k}}{a_k a_{k'} - 1} - \frac{m_{2k} - 2\hat{\Omega} \sqrt{\alpha} m_{1k} + 4\hat{\Omega}}{a_k} - \frac{m_{2k}}{a_{k'}}. \\
\lambda(M)|_M &= \phi + 2\hat{\Omega} - \Omega^2 + \frac{1}{2} \sum_{k,k'} \frac{W_{kk'} (a_k m_{2k} - 2\hat{\Omega} \sqrt{\alpha}) + 2m_{1k} \hat{m}_{2k}}{a_k a_{k'} - 1} - \frac{m_{2k} - 2\hat{\Omega} \sqrt{\alpha} m_{1k} + 4\hat{\Omega}}{a_k} - \frac{m_{2k}}{a_{k'}}. \\
\end{align*}
\]

and the condition of the detectability limit as

\[
D(a_1^{\det}, a_2^{\det}, \hat{a}_1^{\det}, \hat{a}_2^{\det}) = 0, \tag{A37}
\]

where

\[
M_{11} = (1 + \epsilon) \frac{a_1^2 + 1}{(a_1^2 - 1)^2} + \alpha \frac{a_2^2}{(a_1 a_2 - 1)^2} - (1 + \alpha + \epsilon) \frac{1}{(a_1 - a_1^2 c_1 c_1 - 1)}, \tag{A39}
\]

\[
M_{12} = \frac{\alpha}{(a_1 a_2 - 1)^2}, \tag{A40}
\]

\[
M_{21} = \frac{2}{(a_1 a_2 - 1)^2}, \tag{A41}
\]

\[
M_{22} = \frac{2a_1^2}{(a_1 a_2 - 1)^2} + \sigma \alpha \frac{a_2^2 + 1}{(a_2^2 - 1)^2} - (\sigma \alpha + 2) \frac{1}{(a_2 - a_2^2 c_2 c_2 - 1)}. \tag{A42}
\]

The detectability limit (A37) is derived by condition \( \hat{m}_{11}^2 = 0 \), because \( D(a_1, a_2, \hat{a}_1, \hat{a}_2) \) is proportional to \( \hat{m}_{11}^2 \).
Second, under the undetectable condition, we can derive the equations for $a_1, a_2, a_1$, and $a_2$ from the saddle-point conditions as

$$a_1 + (c_1 - 1)a_1 = a_2 + (c_2 - 1)a_2,$$

$$\frac{1}{a_1 - a_1} = \frac{1 + e}{1 + \alpha + e a_1^2 - 1} + \frac{\alpha}{1 + \alpha + \epsilon a_1 a_2 - 1},$$

$$\frac{1}{a_2 - a_2} = \frac{\sigma_1}{\sigma_2 + 2 a_2^2 - 1} + \frac{2}{\sigma_1 + 2 a_1 a_2 - 1},$$

$$D(a_1, a_2, a_1, a_2) = 0.$$  

These equations are analogous to those for the detectable conditions \[\text{A32} \text{A35}\]. A crucial difference is that we have condition $\hat{e}_{11} = 0$ instead of Eq. \[\text{A35}\]. We let the solutions of these equations be $a_1^{\text{und}}, a_2^{\text{und}}, a_1^{\text{und}}$, and $a_2^{\text{und}}$. Using this solution, we obtain the average leading eigenvalue in the undetectable conditions as follows.

$$[\lambda(M)]_M = \phi = a_k^{\text{und}} + (c_k - 1)a_k^{\text{und}}. \quad (k = 1, 2)$$  \[\text{A47}\]

### Appendix B: Microcanonical overlapping SBM

In this appendix, we discuss the microcanonical SBM. In Sec. \[\text{B1}\], we introduce the definition of the microcanonical overlapping SBM. In Sec. \[\text{B2}\], we provide the replica analysis to derive its spectrum and the detectability limit. In Sec. \[\text{B3}\], we derive the saddle-point conditions for normalization constant $N_G$, from which we can derive crucial relations used in Sec. \[\text{B2}\]. Finally, in Sec. \[\text{B4}\], we discuss the distinction between the canonical and microcanonical SBMs and discuss the reason of their use in our numerical experiments.

#### 1. Model definition

Microcanonical SBM is an SBM that is formulated on the basis of different constraints from its canonical model. Although the canonical SBM specifies the expected number of edges within the blocks, the microcanonical SBM specifies the number of edges within the blocks as well as the degree sequence as hard constraints. The microcanonical SBM generates a graph uniformly and randomly from all realizable graphs under these constraints. We denote the sequence of node degrees as $d = [d_i]$. We let $e_{kl}$ be the number of edges between block $k$ and $l$; we denote the corresponding matrix as $e = [e_{kl}]$. Moreover, $t = [t_i] \ t_i \in \{1, \cdots, K\}$ ($i \in V$) are the planted block labels of the nodes. An instance of the microcanonical SBM is generated according to the following probability distribution.

$$P(A | d, e, t) = \frac{1}{\Omega(d, e, t)},$$  \[\text{B1}\]

where $\Omega(d, e, t)$ is the number of all realizable graphs under given $d, e$, and $t$.

We consider a microcanonical SBM with an overlapping structure with the following parametrization.

$$p = (p_1, p_2, p_3) = (p_1, \alpha p_3, p_1),$$  \[\text{B2}\]

$$e = \left( \begin{array}{ccc} 1 & \alpha & \epsilon \\ \alpha & \sigma^2 & \alpha \\ \epsilon & \alpha & 1 \end{array} \right) e_{11},$$  \[\text{B3}\]

$$d_i = c_i.$$  \[\text{B4}\]

Although we can provide an arbitrary degree sequence, for simplicity, we assume the nodes belonging to the same group $k$ have equal degree $c_k$. As in the canonical SBM, the model parameters must satisfy constraint \[5\].

#### 2. Derivation of the spectrum and the detectability limit of the microcanonical SBM

Here, we conduct an analysis analogous to Appendix \[A\] for the microcanonical SBM. As a result of the present analysis, we obtain the same average largest eigenvalues as those of the canonical case in \[\text{A36} \text{A47}\]. However, a different technique is required to impose the microcanonical constraints. The calculations in this appendix are extensions of those in Refs. \[7 \text{8}\]. We start with the $n$th moment of the partition function \[13\].

$$[Z^n (M, \beta)]_M = \left( \prod_{a=1}^{n} dx_a \delta(x_a^\top M x_a - N) \right) \times \left[ \exp \left( \frac{\beta}{2} \sum_a x_a^\top M x_a \right) \right]_M.$$  \[\text{B5}\]

As defined in Appendix \[\text{B1}\], we assume the three blocks model. Then, the exponential factor in \[\text{B5}\] can be recast as

$$x_a^\top M x_a$$

$$= \sum_{ij \in V_1} u_{ij} x_{ia} x_{ja} + \sum_{ij \in V_2} y_{ij} x_{ia} x_{ja} + \sum_{ij \in V_3} z_{ij} x_{ia} x_{ja} + 2 \sum_{i \in V_1 j \in V_2} v_{ij} x_{ia} x_{ja} + 2 \sum_{i \in V_2 j \in V_3} v_{ij} x_{ia} x_{ja} + 2 \sum_{i \in V_1 j \in V_3} w_{ij} x_{ia} x_{ja} - (\gamma^\top x_a)^2,$$  \[\text{B6}\]

where $u_{ij}, y_{ij}, v_{ij},$ and $w_{ij}$ are the adjacency matrix elements. These parameters were introduced to distinguish blocks that obey different statistics. Again, the summation $\sum_{i \in V_k}$ is taken over indices of the nodes that belong to block $k$.

To calculate the ensemble average over the microcanonical SBM, we take the sum over all possible graph configurations as imposing the microcanonical constraints by delta functions. Thus, the configuration average of the exponential factor in \[\text{B5}\] is
\[
\left[ \exp \left( \frac{\beta}{2} \sum_a x_a^\top M x_a \right) \right]_M = \frac{1}{\mathcal{N}_G} \sum_{\{u_{ij}\}, \{w_{ij}\}, \{v_{ij}\}, \{w_{ij}\} \in V_1} \prod_{k \in V_3} \delta \left( \sum_{i \in V_1} u_{i1} + \sum_{m \in V_2} v_{im} + \sum_{n \in V_3} w_{in} - c_1 \right) \prod_{j \in V_2} \delta \left( \sum_{i \in V_1} u_{j1} + \sum_{m \in V_2} v_{jm} + \sum_{n \in V_3} w_{jn} - c_2 \right) \\
\times \prod_{k \in V_3} \delta \left( \sum_{i \in V_1} u_{kl} + \sum_{m \in V_2} v_{km} + \sum_{n \in V_3} w_{kn} - c_3 \right) \delta \left( \sigma p_2 \sum_{i \in V_1} \sum_{j \in V_2} v_{ij} - p_1 \sum_{i,j \in V_2} y_{ij} \right) \delta \left( \sigma p_2 \sum_{i \in V_2} \sum_{j \in V_3} v_{ij} - p_3 \sum_{i,j \in V_2} y_{ij} \right) \\
\times \delta \left( \epsilon \sum_{i,j \in V_1} u_{ij} - \sum_{i \in V_1} \sum_{j \in V_2} v_{ij} \right) \delta \left( \epsilon \sum_{i,j \in V_3} u_{ij} - \sum_{i \in V_1} \sum_{j \in V_3} w_{ij} \right) \exp \left( \frac{\beta}{2} \sum_a x_a^\top M x_a \right). \tag{B7}
\]

Here, \( \mathcal{N}_G \) is the number of all realizable graphs that satisfy the constraints. The first three delta functions in \( \text{(B7)} \) represent Kronecker’s deltas that impose the degree constraints, while the remaining ones represent Dirac’s deltas that impose the constraints with respect to the number of edges between blocks, as specified by matrix \( e \).

We use the integral expression of the delta functions as follows.

\[
\delta(x) = \int \frac{dz}{2\pi} e^{\frac{x}{z^2}}, \tag{B8}
\]

\[
\delta(x) = \int_{-\infty}^{\infty} \frac{d\eta}{2\pi} e^{-\eta x}. \tag{B9}
\]

\[
\frac{1}{\mathcal{N}_G} \int \prod_{k=1,2,3} \prod_{i,j \in V_k} \frac{dz_i}{2\pi} \frac{dz_j}{2\pi} e^{-(1+c_k)} \int \frac{d\zeta}{2\pi} \int \frac{d\xi}{2\pi} \int \frac{d\sigma}{2\pi} \int \frac{d\eta}{2\pi} \int \frac{d\theta}{2\pi} e^{-\frac{\beta}{2} \sum_n (\gamma_n x_n)^2} \\
\times \prod_{i<j} \sum_{u_{ij} \in \{0,1\}} \left( z_i z_j e^{\beta \sum_n x_{ia} x_{ja} - \sigma p_2 - 2\eta p_1 + 2\epsilon p_1} \right) u_{ij} \prod_{v_{ij} \in \{0,1\}} \sum_{r_{ij} \in \{0,1\}} \left( z_i z_j e^{\beta \sum_n x_{ia} x_{ja} - \eta p_2 + \sigma p_1 + 2\epsilon p_1} \right) r_{ij} \\
\times \prod_{i<j} \sum_{u_{ij} \in \{0,1\}} \left( z_i z_j e^{\beta \sum_n x_{ia} x_{ja} - \sigma p_2 - 2\epsilon p_1} \right) u_{ij} \prod_{v_{ij} \in \{0,1\}} \sum_{r_{ij} \in \{0,1\}} \left( z_i z_j e^{\beta \sum_n x_{ia} x_{ja} + \eta + \eta} \right) r_{ij}, \tag{B10}
\]

where parameters \( \zeta, \xi, \tau, \kappa, \eta, \) and \( \theta \) are the auxiliary variables provided by the integral representation of the delta function. Because variables \( u_{ij}, y_{ij}, v_{ij}, \) and \( w_{ij} \) only take binary values, their summations in \( \text{(B10)} \) can be calculated straightforwardly. For example,

\[
\prod_{i<j} \sum_{u_{ij} \in \{0,1\}} \left( z_i z_j e^{\beta \sum_n x_{ia} x_{ja}} \right) u_{ij} \\
= \prod_{i<j} \sum_{u_{ij} \in \{0,1\}} \left( 1 + z_i z_j e^{\beta \sum_n x_{ia} x_{ja}} \right) \approx \prod_{i<j} \exp(z_i z_j e^{\beta \sum_n x_{ia} x_{ja}}). \tag{B11}
\]
To derive the last equation in (B11), we assume that $|z_i|$ and $|z_j|$ are sufficiently small.

Here, we introduce the order-parameter functions

$$Q_k(\mu) = \frac{1}{p_k N} \sum_{i \in V_k} z_i \prod_{a=1}^{n} \delta(x_{ia} - \mu_a), \quad (k = 1, 2, 3)$$  

(B12)

which is similar but not completely equivalent to (A10). Using the order-parameter functions (B12), when $N \gg 1$, Eq. (B11) can be approximated as

$$\prod_{i < j \atop i, j \in V_1} \exp(z_{ij} z_{ji} e^{\beta \sum_{\tau \in \tau_1} x_{\tau, \mu, \nu}})$$

$$\approx \exp \left( \frac{(p_1 N)^2}{2} \int \prod_{a=1}^{n} d\mu_a d\nu_a Q_1(\mu) Q_1(\nu) e^{\beta \sum_{\nu, \mu} \nu_a \mu_a} \right),$$

(B13)

where we approximated that the contribution from the diagonal elements is negligible. Using the similar calculations, (B5) is now written as

$$[Z^n(M, \beta)]_M = e^{N T_n(Q) + N S_n},$$

(B14)

where

$$N T_n(Q)$$

$$= \frac{(p_1 N)^2}{2} \int \prod_{a=1}^{n} d\mu_a d\nu_a Q_1(\mu) Q_1(\nu) e^{\beta \sum_{\nu, \mu} \nu_a \mu_a - 2 T_{p_2} - 2 \eta e}$$

$$+ \frac{(p_2 N)^2}{2} \int \prod_{a=1}^{n} d\mu_a d\nu_a Q_2(\mu) Q_2(\nu) e^{\beta \sum_{\nu, \mu} \nu_a \mu_a + 2 T_{p_3} + 2 \zeta \rho}$$

$$+ \frac{(p_3 N)^2}{2} \int \prod_{a=1}^{n} d\mu_a d\nu_a Q_3(\mu) Q_3(\nu) e^{\beta \sum_{\nu, \mu} \nu_a \mu_a - 2 T_{p_2} - 2 \eta e}$$

$$+ p_1 p_2 N^2 \int \prod_{a=1}^{n} d\mu_a d\nu_a Q_1(\mu) Q_2(\nu) e^{\beta \sum_{\nu, \mu} \nu_a \mu_a - \sigma \zeta \rho + 2 T_{p_1}}$$

$$+ p_2 p_3 N^2 \int \prod_{a=1}^{n} d\mu_a d\nu_a Q_2(\mu) Q_3(\nu) e^{\beta \sum_{\nu, \mu} \nu_a \mu_a - \sigma \zeta \rho + 2 T_{p_3}}$$

$$+ p_1 p_3 N^2 \int \prod_{a=1}^{n} d\mu_a d\nu_a Q_1(\mu) Q_3(\nu) e^{\beta \sum_{\nu, \mu} \nu_a \mu_a + \eta \theta}$$

(B15)

and

$$e^{N S_n} = \int \prod_{i=1}^{N} \prod_{a=1}^{n} dx_{ia} \prod_{a=1}^{n} \delta \left( \sum_{i=1}^{N} x_{ia}^2 - N \right)$$

$$\times \int \sqrt{N} \prod_{a=1}^{n} d\Omega_a \delta \left( \sqrt{N} \Omega_a - \sum_{i} \gamma_i x_{ia} \right) e^{-\frac{1}{4} \Omega_a^2}$$

$$\times \frac{1}{N^G} \int_{k=1,2,3} \prod_{i \in V_k} d\xi_i \left( -2 \pi \right) z_{i}^{-1}$$

$$\times \int \frac{d\xi}{2 \pi} \int \frac{d\xi}{2 \pi} \int \frac{dr}{2 \pi} \int \frac{dr}{2 \pi} \int \frac{d\eta}{2 \pi} \int \frac{d\theta}{2 \pi}.$$  

(B16)

Here, $\Omega_a$ is the order parameter defined in (A3). As in the case of the canonical SBM in (A14), for Eq. (B16), we insert the identity

$$1 = \prod_{k=1,2,3} p_k N \int \frac{DQ_k}{2\pi} \delta \left( \sum_{i \in V_k} \prod_{a=1}^{n} \delta(x_{ia} - \mu_a) - p_k N Q_k(\mu) \right)$$

(B17)

$$= \prod_{k=1,2,3} p_k N \int \frac{DQ_k D\hat{Q}_k}{2\pi} \exp \left( \sum_{k=1,2,3} \int d\mu \hat{Q}_k(\mu) \right)$$

$$\times \left( \sum_{i \in V_k} \prod_{a=1}^{n} \delta(x_{ia} - \mu_a) - p_k N Q_k(\mu) \right) \right),$$

(B18)

In (B17), we perform the functional integration over the space of function $Q_k(\mu)$. It is required to insert identity (B17), because it indicates that we performed the replacement of a function in (B12) by $Q_k(\mu)$. Furthermore, using the integral representation of the delta functions (A7) and (A13), we obtain

$$e^{N S_n}$$

$$= \int \prod_{k=1,2,3} p_k N \frac{DQ_k D\hat{Q}_k}{2\pi} \int \prod_{a=1}^{n} \frac{\beta d\phi_a}{4 \pi} \int \prod_{a=1}^{n} \frac{\beta N d\Omega_a \dot{\Omega}_a}{2 \pi}$$

$$\times \int \frac{d\xi}{2 \pi} \int \frac{d\xi}{2 \pi} \int \frac{dr}{2 \pi} \int \frac{dr}{2 \pi} \int \frac{d\eta}{2 \pi} \int \frac{d\theta}{2 \pi}$$

$$\times \exp \left( - \log N_G - N \sum_{k} K_k(Q_k, \hat{Q}_k) \right)$$

$$+ \frac{\beta N}{2} \sum_{a} \left( 2 \Omega_a \dot{\Omega}_a - \Omega_a^2 + \phi_a \right) - \log \sum_{k}$$

$$\times \log \left( Q_k, \{ \Omega_a \}, \{ \phi_a \} \right),$$

(B19)

where

$$K_k(Q_k, \hat{Q}_k) = p_k \int d\mu Q_k(\mu) \hat{Q}_k(\mu),$$

(B20)

$$L_k \left( Q_k, \{ \Omega_a \}, \{ \phi_a \} \right) = \int \prod_{i \in V_k} \prod_{a} dx_{ia} \prod_{i \in V_k} \left( \hat{Q}_k(x_i) \right)$$

$$\times \exp \left( - \beta \sum_{a} \left( \sqrt{N} \Omega_a \gamma_i x_{ia} + \frac{1}{2} \phi_a x_{ia}^2 \right) \right).$$

(B21)
Here, we used the relation

\[
\int \frac{d z_i}{2 \pi} e^{-z_i (1 + c_k)} = \int \frac{d z_i}{2 \pi} z_i^{-1} \sum_{m=0}^{\infty} \frac{1}{m!} \left( z_i Q_k(x_i) \right)^m \]

(B22)

\[
= \sum_{m=0}^{\infty} \frac{1}{m!} \bar{Q}_k(x_i) \int \frac{d z_i}{2 \pi} z_i^{m-1} \left( 1 + c_k \right) \]

(B23)

\[
= \frac{1}{c_k} \bar{Q}^{c_k}_k(x_i). \quad \text{(B24)}
\]

Now, the variable depending on the node index \( i \) only appears as \( x_i \). Hence, after the integral with respect to \( x_i \) is carried out in \( L_k \left( \hat{Q}_k, \{ \Omega_a \}, \{ \phi_a \} \right) \), Eq. (B19) can be expressed only with integrals over the auxiliary variables \( \phi_a, \Omega_a, \Omega_k, \zeta, \xi, \tau, \kappa, \eta, \theta \) and functional integrals over \( Q_k(\mu) \) and \( \hat{Q}_k(\mu) \).

For further calculations, as in the case of the canonical SBM (Eqs. (A20) and (A21)), we assume the functional form of \( Q_k \) and \( \hat{Q}_k \) are restricted to the Gaussian mixtures as follows.

\[
Q_k(\mu) = T_k \int d A d H q_k(A, H)
\]

\[
\times \frac{1}{2 \pi} \left( \frac{\beta}{2} \right)^{\frac{1}{2}} \exp \left( -\frac{\beta A}{2} \sum_a \left( \mu_a - \frac{H}{A} \right)^2 \right) \],

(B25)

\[
\hat{Q}_k(\mu) = \hat{T}_k \int d A \bar{H} q_k(A, \bar{H}) \exp \left( \frac{\beta}{2} \sum_a \left( \bar{\mu}_a^2 + 2 \bar{H} \mu_a \right) \right),
\]

(B26)

where \( T_k \) and \( \hat{T}_k \) represent the normalization constants. With these functional forms, we can calculate the integrals over \( \mu \) in (B20) and \( x \) in (B21). Then, we obtain the following expressions.

\[
K_k(q_k, \hat{q}_k) = c_k p_h \int d A d H \int d A \bar{H} q_k(A, H) q_k(A, \bar{H}) \]

\[
\times \left( \frac{A}{A - \bar{A}} \right)^{\frac{1}{2}} \exp \left( \frac{\beta}{2} \left( \frac{H + \bar{H}}{A} \right)^2 - \frac{A^2 + A + \bar{H}^2}{A} \right) \],

(B27)

\[
L_k \left( \hat{q}_k, \{ \Omega_a \}, \{ \phi_a \} \right) = T^{c_k}_k \left( \frac{2 \pi}{\beta} \right)^{\frac{1}{2}} \]

\[
\times \int \prod_{g=1}^{n} (d \bar{A}_g d \bar{H}_g \bar{q}_k(\bar{A}_g, \bar{H}_g)) \prod_{a=1}^{n} \left( \phi_a - \sum_{g=1}^{c_k} \bar{A}_g \right)^{-\frac{1}{2}} \times \exp \left( \frac{\beta}{2} \sum_{i \in V_k} \left( \sqrt{N} \Omega_i - \sum_{g=1}^{c_k} \bar{A}_g \right)^2 \right) . \quad \text{(B28)}
\]

In Appendix [B3], we solve for normalization constants \( T_k \) and \( \hat{T}_k \). By using (B54), we can replace \( T_k \hat{T}_k \) with \( c_k \). This is how we eliminated the normalization constants in Eq. (B27). By inserting (B25) and (B26) in (B15), we can calculate the integrals over \( \mu \) and obtain

\[
\mathcal{T}_n = N \int d A d H d A' d H' \left( A A' \right)^{\frac{1}{2}} \exp \left( \frac{\beta}{2} \frac{1}{2} \Xi(A, A', H, H') \right) \]

\[
\times \left( \frac{c_1}{2} \frac{p_1}{p_1 + p_2 + c p_1} q_1(A, H) q_1(A', H') + \frac{c_2}{2} \frac{\sigma p_3}{p_3 + p_2 + c p_3} q_2(A, H) q_2(A', H') \right)
\]

\[
+ \frac{c_3}{2} \frac{p_3}{p_3 + p_2 + c p_3} q_3(A, H) q_3(A', H') + c_2 \frac{p_1 p_2}{p_1 + c p_2 + p_3} q_1(A, H) q_2(A', H')
\]

\[
+ c_2 \frac{p_2 p_3}{p_1 + c p_2 + p_3} q_2(A, H) q_3(A', H') + c_1 \frac{c p_1}{p_1 + p_2 + c p_1} q_1(A, H) q_3(A', H') \right) , \quad \text{(B29)}
\]

where

\[
\Xi(A, A', H, H') = \frac{A' H^2 + AH'^2 + 2 HH'}{AA' - 1} - \frac{H^2}{A} - \frac{H'^2}{A'} .
\]

Here, we used the relations between \( T_1, T_2, \) and \( T_3 \) (B55)–(B61). From the calculations so far, we have performed all the integrals over \( x, x_i, \) and \( \mu \). The functional integrals over \( q_k(\mu) \) and \( \hat{q}_k(\mu) \) in (B19) have been replaced by the integral over the functions \( q_k(A, H) \) and \( \hat{q}_k(A, \hat{H}) \). In summary, the nth moment of the partition function (B14) is now represented by the integrals with respect to auxiliary variables \( \phi_a, \Omega_a, \) and \( \Omega_k \) and the functional integrals over \( q_k(A, H) \) and \( \hat{q}_k(A, \hat{H}) \). Note that the other variables \( \zeta, \xi, \tau, \kappa, \eta, \) and \( \theta \) can be erased when inserting the relations between the normalization constants (B55)–(B61).

Again, as we assumed in the canonical SBM, we impose the replica symmetric assumptions for the parameters \( \phi_a, \Omega_a, \) and \( \Omega_k \), i.e., \( \phi_a = \phi, \Omega_a = \Omega, \) and \( \Omega_k = \Omega \) in Eq. (B27)–(B29). Inserting Eq. (B27)–(B29) under the assumptions into (B14) and taking the limit \( N \to \infty \), the average largest eigenvalue can be expressed as follows.
\[
\lambda(M) = 2 \lim_{\beta \to \infty} \frac{1}{\beta N} \lim_{n \to \infty} \frac{1}{\partial n} \log[Z^n]_M
\]

Moreover, the saddle-point conditions with respect to \(\phi\) and \(\lambda\) from Eq. (B32), we obtain the saddle-point conditions as

\[
\begin{align*}
\text{extr}_{q_k, a_k, \Omega, \Omega, \Omega} & \left\{ \int dA dH \int dA' dH' \zeta(A, A', H, H') 	imes \frac{c_1}{2} \frac{p^2_{\phi}}{p_1 + p_2 + \epsilon_1} q_1(A, H)q_1(A', H') + \frac{c_2}{2} \frac{p^2_{\phi}}{p_1 + \epsilon_2} q_2(A, H)q_2(A', H') + \frac{c_3}{2} \frac{p^2_{\phi}}{p_3 + \epsilon_3} q_3(A, H)q_3(A', H') \right. \\
& \left. + \frac{c_2}{p_1 + \epsilon_2} q_2(A, H)q_2(A', H') + \frac{c_2}{p_1 + \epsilon_2} q_2(A, H)q_2(A', H') + \frac{c_1}{p_1 + \epsilon_1} q_1(A, H)q_1(A', H') \right. \\
& \left. - \sum_{k=1,2,3} c_k p_k \int dA dH \int dA' dH' q_k(A, H)q_k(A', H') \left( \frac{(H + H')^2}{A - A'} - \frac{H^2}{A} \right) \\
& + 2\Omega \hat{\Omega} - \Omega^2 + \phi \\
& \frac{1}{N} \sum_{k=1,2,3} \int \prod_{g=1}^{c_k} \left( dA_g dH_g \hat{q}_k(\hat{A}_g, \hat{H}_g) \right) \sum_{i \in V_k} \left( \frac{\sqrt{N} \hat{\Omega} \gamma_i - \sum_{g=1}^{c_k} \hat{H}_g}{\phi - \sum_{g=1}^{c_k} \hat{A}_g} \right)^2 \Bigg\}.
\end{align*}
\] (B32)

From Eq. (B32), we obtain the saddle-point conditions as

\[
\begin{align*}
\hat{q}_1(\hat{A}, \hat{H}) &= \int dA' dH' \frac{p_1 q_1(A', H') + p_2 q_1(A', H') + \epsilon_1 q_3(A', H')}{p_1 + p_2 + \epsilon_1} \delta \left( \hat{A} - \frac{1}{A'} \right) \delta \left( \hat{H} - \frac{H'}{A'} \right), \\
\hat{q}_2(\hat{A}, \hat{H}) &= \int dA' dH' \frac{p_1 q_1(A', H') + p_2 q_3(A', H')}{p_1 + \epsilon_2} \delta \left( \hat{A} - \frac{1}{A'} \right) \delta \left( \hat{H} - \frac{H'}{A'} \right), \\
\hat{q}_3(\hat{A}, \hat{H}) &= \int dA' dH' \frac{p_3 q_1(A', H') + p_2 q_3(A', H') + \epsilon_3 q_3(A', H')}{p_3 + p_2 + \epsilon_3} \delta \left( \hat{A} - \frac{1}{A'} \right) \delta \left( \hat{H} - \frac{H'}{A'} \right).
\end{align*}
\] (B33)–(B35)

Equations (B33)–(B36) constitute functional equations under the constraint (B37). This constraint corresponds to the normalization constraints in (B3). By solving these equations, we obtain the distribution of the largest eigenvector elements.

Moreover, the saddle-point conditions with respect to \(\phi\) yield

\[
\sum_k p_k \int dA dH Q_k(A, H) \left( \frac{H}{A} \right)^2 = 1,
\] (B37)

where

\[
Q_k(A, H) = \frac{1}{p_k N} \sum_{i \in V_k} \int \prod_{g=1}^{c_k} \left( dA_g dH_g \hat{q}_k(\hat{A}_g, \hat{H}_g) \right) \delta \left( H - \sum_{g=1}^{c_k} \hat{H}_g + \sqrt{N} \hat{\Omega} \gamma_i \right) \delta \left( A - \phi + \sum_{g=1}^{c_k} \hat{A}_g \right).
\] (B38)

As in the canonical case, solving the functional form of equations is still not analytically tractable. Thus, we again introduce the EMA, i.e., the precision parameters of the Gaussian mixtures \(A\) and \(\hat{A}\) are fixed as constants, i.e., \(q_k(A, H) = q(H) \delta(A - a_k)\) and \(\hat{q}_k(\hat{A}, \hat{H}) = \hat{q}_k(\hat{H}) \delta(\hat{A} - \hat{a}_k)\). Performing the EMA for (B32), we have
where \( m_{ik} \) and \( \hat{m}_{ik} \) represent the \( \ell \)th moments of \( H \) and \( \hat{H} \), respectively, i.e., \( m_{ik} = \int dH H^\ell \gamma_k (H) \) and \( \hat{m}_{ik} = \int d\hat{H} \hat{H}^\ell \hat{\gamma}_k (\hat{H}) \).

As in the canonical case, we introduce further assumptions. First, we assume the symmetry between the first and the third blocks, namely \( p_1 = p_2 \) and \( c_1 = c_3 \). Hence, \( a_1 = a_3 = \hat{a}_1 = \hat{a}_3 \), \( m_{21} = m_{23} \), and \( \hat{m}_{21} = \hat{m}_{23} \). Second, we think of two types of solutions: \( m_{11} = -m_{13} \), \( m_{12} = 0 \) and \( m_{11} = m_{12} = m_{13} = 0 \). Under these assumptions, we obtain the same solutions as those of the canonical SBM with the regular approximation. When \( m_{11} = -m_{13} \) and \( m_{12} = 0 \), the average largest eigenvalue is obtained as in Eq. \( \text{(A36)} \). When \( m_{11} = m_{12} = m_{13} = 0 \), the average largest eigenvalue is obtained as in Eq. \( \text{(A47)} \). The detectability limit is given by Eq. \( \text{(A37)} \).

3. Saddle-point conditions for \( \mathcal{N}_G \)

The goal of this subsection is to derive the relations of the normalization constants of the Gaussian mixtures \( T_k \) and \( \hat{T}_k \) in Eqs. \( \text{(B25)} \) and \( \text{(B26)} \). They can be derived using saddle-point conditions for the number of all realizable graphs \( \mathcal{N}_G \). This can be calculated by taking the sum over all possible graph configurations as imposing the microcanonical constraints by delta functions. Thus, we have

\[
\mathcal{N}_G = \sum_{\{u_{i,j}\},\{w_{i,j}\},\{v_{i,j}\}} \prod_{i \in V_1} \delta \left( \sum_{j \in V_2} u_{i,j} + \sum_{m \in V_3} v_{i,m} + \sum_{n \in V_3} w_{i,n} - c_1 \right) \\
\times \prod_{j \in V_2} \delta \left( \sum_{i \in V_1} u_{j,i} + \sum_{m \in V_3} v_{j,m} + \sum_{n \in V_3} w_{j,n} - c_2 \right) \\
\times \prod_{k \in V_3} \delta \left( \sum_{i \in V_1} u_{k,i} + \sum_{m \in V_2} v_{k,m} + \sum_{n \in V_2} w_{k,n} - c_3 \right) \\
\times \delta \left( \sigma p_2 \sum_{i,j \in V_2} v_{i,j} - p_1 \sum_{i,j \in V_2} y_{i,j} \right) \delta \left( \sigma p_2 \sum_{i,j \in V_2} v_{i,j} - p_3 \sum_{i,j \in V_2} y_{i,j} \right) \\
\times\delta \left( p_2 \sum_{i,j \in V_1} u_{i,j} - p_1 \sum_{i,j \in V_1} v_{i,j} \right) \delta \left( p_2 \sum_{i,j \in V_1} u_{i,j} - p_3 \sum_{i,j \in V_1} v_{i,j} \right) \\
\times\delta \left( \epsilon \sum_{i,j \in V_1} u_{i,j} - \sum_{i,j \in V_2} w_{i,j} \right) \delta \left( \epsilon \sum_{i,j \in V_2} u_{i,j} - \sum_{i,j \in V_2} v_{i,j} \right) \delta \left( \epsilon \sum_{i,j \in V_2} u_{i,j} - \sum_{i,j \in V_2} w_{i,j} \right).
\]  \( \text{(B40)} \)
Using the integral representation of the delta function \((B8)\) and \((B9)\), we have

\[
N_G = \sum_{\{u_{ij}\}, \{w_{ij}\}, \{v_{ij}\}, \{u_{ij}\}} \int \prod_{i \in V_1} \frac{dz_i}{2\pi z_i} z_i^{\sum_{j \in V_1} u_{ij} + \sum_{m \in V_2} v_{im} + \sum_{n \in V_3} w_{in} - c_1 - 1} \\
\times \int \prod_{i \in V_2} \frac{dz_i}{2\pi z_i} z_i^{\sum_{j \in V_2} v_{ij} + \sum_{m \in V_3} y_{jm} + \sum_{n \in V_3} w_{in} - c_1 - 1} \\
\times e^{-\frac{1}{2} \sum_{i \in V_1} (\sigma p_i ^2 \sum_{j \in V_2} y_{ij} - p_i \sum_{j \in V_2} y_{ij})} \int \frac{d\xi}{2\pi} e^{-\xi (\sigma p_i \sum_{j \in V_2} y_{ij} - p_i \sum_{j \in V_2} y_{ij})} \\
\times e^{-\frac{1}{2} \sum_{i \in V_1} (\xi \sum_{j \in V_2} v_{ij} - \xi \sum_{j \in V_2} v_{ij})} \int \frac{d\eta}{2\pi} e^{-\eta (\sigma \sum_{j \in V_3} u_{ij} - \sigma \sum_{j \in V_3} u_{ij})} \\
= \int \prod_{k=1,2,3} \prod_{i \in V_k} \frac{dz_i}{2\pi z_i} z_i^{-(1+c_k)} \\
\times \prod_{k=1,2,3} \sum_{u_{ij}} (z_i z_j e^{-2\tau p_2 - 2\eta})^{u_{ij}} \prod_{k=1,2,3} \sum_{v_{ij}} (z_i z_j e^{2\xi p_3 + 2\xi p_1})^{v_{ij}} \prod_{k=1,2,3} \sum_{w_{ij}} (z_i z_j e^{-2\kappa p_2 - 2\theta})^{w_{ij}} \\
\times \prod_{k=1,2,3} \prod_{i \in V_k} \prod_{j \in V_k} \prod_{v_{ij}} (z_i z_j e^{-\sigma \xi p_3 + \kappa p_3})^{v_{ij}} \\
\times \prod_{k=1,2,3} \prod_{i \in V_k} \prod_{j \in V_k} \prod_{w_{ij}} (z_i z_j e^{\eta} \theta)^{w_{ij}}. \tag{B42}
\]

Here, we introduce the order parameters

\[
q_k = \frac{1}{p_k N} \sum_{i \in V_k} z_i, \quad (k = 1, 2, 3) \tag{B43}
\]

Equation \((B42)\) is now written as

\[
N_G = \prod_{k=1,2,3} \left( p_k N \int \frac{dq_k dq_i}{2\pi} \right) \\
\times \int \frac{d\zeta}{2\pi} \int \frac{d\xi}{2\pi} \int \frac{d\eta}{2\pi} \int \frac{d\theta}{2\pi} \\
\times \exp \left( \frac{1}{2} e^{-2\tau p_2 - 2\eta} (p_1 N q_1)^2 + \frac{1}{2} e^{2\xi p_3 + 2\xi p_1} (p_2 N q_2)^2 \\
+ \frac{1}{2} e^{-2\kappa p_2 - 2\theta} (p_3 N q_3)^2 + e^{-\sigma \xi p_3 + \kappa p_3} p_2 p_3 N^2 q_2 q_3 + e^{\eta + \theta} p_1 p_3 N^2 q_1 q_3 \right) \\
+ N \sum_{k=1,2,3} (-\dot{q}_k p_k q_k + p_k c_k \log \dot{q}_k - p_k \log c_k!) \tag{B45}
\]

In the limit \(N \to \infty\), we have the following saddle-point conditions.

\[
e^{2\tau p_2 - 2\eta} p_1 \xi p_3 \eta + \theta = p_3 \xi p_3 \eta + \theta \tag{B46}
\]

\[
e^{-2\xi p_3 - 2\xi p_1} p_2 N q_2 = \frac{1}{2} p_1 \xi p_3 \eta + \theta \tag{B47}
\]

\[
e^{-2\kappa p_2 - 2\theta} p_3 N q_3 = \frac{1}{2} p_2 \xi p_3 \eta + \theta \tag{B48}
\]

\[
e^{-\sigma \xi p_3 + \kappa p_3} p_2 p_3 N^2 q_2 q_3 + e^{\eta + \theta} p_1 p_3 N^2 q_1 q_3 \tag{B49}
\]

\[
e^{-\sigma \xi p_3 + \kappa p_3} p_2 p_3 N^2 q_2 q_3 + e^{\eta + \theta} p_1 p_3 N^2 q_1 q_3 \tag{B50}
\]

Here, we used the same approximation as in \((B11)\). Using
\[ \hat{q}_1 = \frac{p_1 q_1 e^{-2\tau p_2 - 2\nu} + p_2 q_2 e^{-\kappa p_2 + \tau p_1} + p_3 q_3 e^\eta + \theta}{N} \]  
\[ \hat{q}_2 = \frac{p_2 q_2 e^{-2\kappa p_1 + 2\xi p_3} + p_1 q_1 e^{-\kappa p_2 + \tau p_1} + p_3 q_3 e^{-\kappa p_2 + \kappa p_3}}{N} \]  
\[ \hat{q}_3 = \frac{p_3 q_3 e^{-2\kappa p_2 - 2\nu} + p_2 q_2 e^{-\kappa p_2 + \kappa p_3} + p_1 q_1 e^\eta + \theta}{N} \]  
\[ q_k \hat{q}_k = c_k. \quad (k = 1, 2, 3) \]

From Eq. (B46)–(B54), we obtain
\[ q_1^2 = \frac{1}{N} e^{2\tau p_2 + 2\nu} \frac{c_1}{p_1 + p_2 + \epsilon p_1}, \]
\[ q_2^2 = \frac{1}{N} e^{-2\kappa p_1 - 2\xi p_3} \frac{c_2}{p_1 + \sigma p_2 + p_3}, \]
\[ q_3^2 = \frac{1}{N} e^{2\kappa p_2 + 2\nu} \frac{c_3}{p_3 + p_2 + \epsilon p_3}, \]
\[ q_1 q_2 = \frac{1}{N} e^{-\kappa p_2 - \tau p_1} \frac{c_2}{\sigma p_2 + p_1 + p_3}, \]
\[ q_2 q_3 = \frac{1}{N} e^{\kappa p_2 - \kappa p_3} \frac{c_2}{\sigma p_2 + p_1 + p_3}, \]
\[ q_1 q_3 = \frac{1}{N} e^{-(\eta + \theta)} \frac{p_1}{p_3 p_1 + p_2 + \epsilon p_1}, \]
\[ c_1(\sigma p_2 + p_1 + p_3) = c_2(p_1 + p_2 + \epsilon p_3). \]

By substituting (B55)–(B61) into (B45), \( N_{G2} \) is expressed in terms of the model parameters. The order parameters (A10) correspond to the order-parameter functions when \( n = 0 \). This indicates that the normalization constants of the Gaussian mixtures \( T_k \) and \( \hat{T}_k \) in (B25) and (B26) are identical to \( q_k \) and \( \hat{q}_k \), respectively. Accordingly, we obtain the relations between \( T_k \) and \( \hat{T}_k \) as Eqs. (B54)–(B60). Besides, (B61) is identical to the constraint between the model parameters, i.e., the same constraint is derived by both the model definition and the replica analysis.

4. Comparison between the canonical and microcanonical SBMs

In the main text, we used the canonical SBM for deriving the detectability limit, whereas we used the microcanonical SBM for conducting the numerical experiments. This is because the derivation under the canonical SBM is more straightforward and simpler, while the canonical SBM causes a problem when conducting the numerical experiments. The canonical SBM required the regular approximation as an additional approximation to calculate the average largest eigenvalue in the replica analysis. The approximation creates a large difference of the derived solutions from the original ones because of ignoring the fluctuation of the degree distribution. Thus, it becomes difficult to validate the results of the analytical calculation by comparing them to the results of the numerical experiments.

However, the microcanonical SBM does not require the regular approximation because it can be defined with an arbitrary degree sequence, and we can choose one that avoids the effects of the fluctuation. Meanwhile, as mentioned in Sec. IV-A, the microcanonical SBM requires an additional constraint that \( c_1 \) and \( c_2 \) can take only natural numbers. This originates from the fact that it specifies a certain degree for each node as its model parameters.

In short, the canonical SBM is appropriate for explaining the derivation of the detectability limit because of the simplicity. The microcanonical SBM is appropriate for conducting the numerical experiments because it does not require the regular approximation.

Appendix C: Bimodal stochastic block model

In this appendix, we explain the bimodal SBM in detail. This model is a variant of the SBM that has no overlapping structure. The bimodal SBM has a bimodal degree distribution: the nodes take either degree \( c_1 \) or \( c_2 \). We denote the fraction of the nodes that have degree \( c_1 \) as \( b_1 \) and that of \( c_2 \) as \( b_2 \) (\( b_1 + b_2 = 1 \)).

We define the bimodal SBM in the microcanonical formulation. The model is parametrized as follows.
\[ e = \begin{pmatrix} 1 & \epsilon \\ \epsilon & 1 \end{pmatrix} e_{11}, \]
\[ b = (b_1, b_2) = (2p_1, p_2). \]

Here, as defined in Sec. IV, \( e_{kl} \) is the number of edges between blocks \( k \) and \( l \), and \( \epsilon \) is the parameter that controls the strength of community structure. Moreover, \( p_1 \) and \( p_2 \) are the sizes of the first and second blocks of the overlapping SBM, respectively. As mentioned in the main text, the purpose of introducing the bimodal SBM is to compare the overlapping SBM to the SBM with the non-overlapping structure and the same average degree. We can confirm that both models have the same average degree.

Subsequently, we show the average largest eigenvalue of the bimodal SBM under the detectable and undetectable conditions. As in the overlapping SBM, we can calculate it using the replica method. The detailed derivation can be found in Ref. [7].

First, under the detectable condition, we obtain the equation for \( a \) as
\[ \tau \varphi (c_2 A - B)(c_1 A - B) = (a^2 - 1) (\tau \varphi A - B) B, \]
where
\[ A = (\tau \varphi - 1) \Gamma - a, \]
\[ B = \Gamma (\tau \varphi^2 - \tau \varphi) - a \tau \varphi, \]
\[ \Gamma = \frac{1 - \epsilon}{1 + \epsilon}. \]
Here, \( a \) is the precision parameter of the Gaussian mixture, which corresponds to \( a_1 \) and \( a_2 \) in the case of the overlapping SBM. Besides, \( \overline{c_b} \equiv b_1c_1 + b_2c_2 \) and \( \overline{c_b^2} \equiv b_1c_1^2 + b_2c_2^2 \). Note that \( a \) has no indices because of the symmetry between the two blocks. We let the solutions of Eq. (C3) be \( a^{\text{det}} \). Using this solution, we obtain the following expression of the average largest eigenvalue.

\[
[\lambda(M)]_M = \frac{c_1c_2 A}{(a^{\text{det}})^3 B}.
\] (C7)

Second, under the undetectable case, we obtain the equations for \( a \) and \( \phi \) as follows.

\[
\sum_{t=1,2} \frac{b_t c_t^2}{(\phi - c_t/a)^2} = \frac{(a^2 + 1)}{\overline{c_b}} \left( \frac{c_b a}{a^2 - 1} \right)^2.
\] (C8)

When we let the solutions of these equations be \( a^{\text{und}} \) and \( \phi^{\text{und}} \), we obtain the average largest eigenvalue as \( [\lambda(M)]_M = \phi^{\text{und}} \).

**Appendix D: Accuracies of the EMA and the regular approximation**

For the replica analysis, we introduced two approximations: the regular approximation and EMA. Here, we investigate the dependencies of the average degree on the accuracy of each approximation. It is known that when the average degree is sufficiently large, the effect of these approximations can be asymptotically ignored. However, it is not trivial how the approximations affect the results for a graph with a low average degree.

To derive the detectability limit of the canonical SBM, we used both the EMA and the regular approximation. To derive that of the microcanonical SBM, we used the EMA only. Thus, by comparing both results, we can measure how each approximation differs from the original result. Figs. 9a and 9b show the results of the canonical and microcanonical SBMs, respectively. We can see that the results of the replica analysis and the numerical experiments are in agreement for \( c_1 \geq 30 \) in the canonical case. On the other hand, they are in agreement for \( c_1 \geq 6 \) in the microcanonical case. Therefore, we can conclude that the effect of the EMA is smaller than that of the regular approximation. Therefore, for the numerical experiments in Sec. IV, we used the microcanonical SBM and set \( c_1 = 10 \), so that the effect of the approximation can be ignored.
FIG. 9: Largest eigenvalues as a function of $\alpha$. The lines represent the results of the replica analysis and the dots represent those of the numerical experiments. (a) The figure shows the results of the canonical SBM for $c_1 = 10, 14, 18, 22, 26, 30$. (b) The figure shows the results of the microcanonical SBM for $c_1 = 3, 4, 5, 6$. 