The three Comments by Barclay and Maxwell [1], Duncan and Willey [2] and Samuel and Steinfelds [3] all correctly point out that my estimate [4] for \( r_n(1) \), the \( n \)-th order coefficient in the perturbative expansion of the normalized total \( e^+ e^- \) cross-section, does not exhibit the correct flavor or color dependence of the exact calculation [5]. This naturally casts serious doubts upon the validity of the result. I would like to suggest, however, that, in spite of this, my original estimate remains valid as an asymptotic formula for sufficiently large \( n \) and that the leading corrections of \( 0(1/n) \) have a strong flavor and color dependence. Below I give some arguments as to why this might be expected to be so.

It turns out that, if the region of validity of the asymptotic formula is \( n \gg n_0(n_f, N_c) \), then typically \( n_0 \) is expected to be relatively large, i.e. \( > 0(2-3) \). I propose, however, that there is a valley in \( (n_f, N_c) \) parameter space whose bottom is the approximate line \( n_f \approx 2N_c - 1 \) and where \( n_0 \lesssim 0(1) \). This passes through, or near, the point \( (n_f = 5, N_c = 3) \) so, in this sense, the close agreement of my result with the exact calculation for the physical case of interest is, indeed, fortuitous. In the general case of arbitrary \( n_f \) and \( N_c \) one would therefore expect to have to go to much larger values of \( n \) (\( \gtrsim 6 \), say) to obtain reasonable agreement.

The original point of my paper was to derive an asymptotic estimate for \( r_n \) that gave the correct order of magnitude: specifically to answer the question “is \( r_3 \sim 5 \) or 50?” This was in response to the confusion resulting from the incorrect “exact” calculation [3] which originally gave \( r_3 \sim 70 \). Even though corrections to my estimate were discussed in my paper, no serious attempt to evaluate them was made. Since the position of the leading saddle point is at \( k_1 \approx b_1[(n-1) + b'] \), it was natural to retain the combination \( (n + b') \) in the final result. However as I pointed out, this, of course, does not incorporate all \( 0(1/n) \) corrections. There are several other sources of \( 1/n \) contributions such as corrections in going from the \( d_n \) to the \( r_n \) [see my eq. (12)] and from approximating \( \text{Im}D \) by its leading term [see eq. (11)]. The former leads to contributions to \( r_3 \) like \( (8\pi^2/3)(b_2/b_1)r_2 \) and \( (4\pi^2)^2(b_3/3b_1) \) whereas the latter gives an overall modifying factor \( [1+1/3\{b' + r_2/2\pi^2b_1\}] \).
To these must be added corrections to the Gaussian approximation of the saddle-point integration. Although a careful systematic examination of all the corrections has not yet been carried out (it is presently underway) it is clear that they are, in general, large and have a strong dependence on $n_f$ and $N_c$. For example, in $\mathcal{N}S$ with $N_c = 3$ and $n_f = 5$, the factor $r_2/2\pi^2b_1 \approx 1.5$, whereas with $N_c = 5$ and $n_f = 1$, it is almost 3. Furthermore the contribution $(8\pi^2/3)(b_2/b_1)r_2$ which is 1.18 for $N_c = 3, n_f = 5$, is 14 when $N_c = 5, N_f = 1$! Similarly, the term $(4\pi^2)^2b_3/3b_1$ is only 0.5 for $N_c = 3, n_f = 5$ but is 7 at $N_c = 5, n_f = 1$.

A preliminary evaluation of this set of corrections indicates that, in general, they can be uncontrollably large in some cases. However, when $N_c = 3$ and $n_f = 5$ they are relatively small so that use of the asymptotic estimate for $r_3$ can be justified. On the other hand, the remarkable closeness to the exact result is clearly accidental.

Notice, incidentally, that scheme dependence enters via these non-leading contributions. The fact that the leading term is scheme invariant is not an argument against its validity. On the contrary, one can argue on very general grounds that the leading large $n$ behavior of $r_n$ should, in fact, be scheme invariant. The point is that this behavior determines the nature of the divergence of the perturbation series which is itself a reflection of the singularity structure in $g^2$ near $g^2 = 0$. However, the analytic structure in $g^2$ can be determined via the renormalization group (RG) since this requires that $q^2$ and $g^2$ always occur in the combination $q^2e^{K(g)}$[with $K(g) \equiv \int \beta(g)/\beta(g)$] and the analytic properties in $q^2$ are known [see my eq. (9)]. Using the perturbative expansion for $\beta(g)$ around $g^2 = 0$ gives $K(g) \approx 1/b_1g^2 + b'\ln g^2 + 0(g^2)$ when $g^2 \approx 0$. The neglected terms are analytic at $g^2 = 0$. The non-analytic structure at $g^2 = 0$ is therefore completely determined by $b_1$ and $b_2$ both of which are scheme invariant. This therefore shows (i) that the leading large $n$ behavior of the $r_n$ can, in principle, be determined from the RG and $q^2$ analyticity and (ii) that the result will depend only on $b_1$ and $b_2$ and therefore be scheme-invariant.

As a corollary, this also demonstrates the importance of $b_2$ since its presence dramatically changes the analytic structure in $g^2$. From the fact that there are discontinuities only along the positive real axis in $g^2$ one can deduce [see my eqn. (9)] that the $g^2$ (or $k \equiv 1/g^2$) singularities occur when $k/b_1 + b'\ln(k + b_2^2/b_1^2) + \cdots = \ln z \pm 2\pi Ni$ with $N$ an integer and $z$ running from zero to infinity. When $b_2 = 0$ this implies that the $k$-plane [where the integration is to be performed] separates into an infinite number of disconnected sectors parallel to the real axis each separated from the next by $2\pi ib_1$. The appropriate region of integration therefore reduces to $-\infty < \text{Re} k < +\infty$ and $0 \leq \text{Im} k \leq 2\pi b_1$, the boundary being the integration contour. This invalidates a derivation of the null result for $d_n$ claimed
for this case ($b_2 = 0$) in [1]. However, it does emphasize a point that was suppressed in my paper, namely that great care must be taken in defining the contour and domain of integration as determined by the RG before interchanging the $z$ and $k$ integrals. Indeed, when $b' = 0$, this results in no $n!$ behavior from the saddle-point integration. However, if $b'$ is now included, additional singularities are present, for, when $z \approx 0$, one can now have $k \approx -b'$. The structure in the $k$-plane is now quite different and leads, via a saddle-point integration to the $n!$ growth of $r_n$ and the result quoted in my paper. The presence of $b_2$ is crucial; even though it plays a minor role in the final expression, the result is simply not deriveable without it. Thus, conclusions based on its omission, such as in ref. 1 and in the work of Brown and Yaffe [7] are not directly relevant.

The problem of interchanging integrations, which requires care in defining the domain and contour of integration, is nicely illustrated by the example of eq. (2) in ref. 1. To avoid the “zero times infinity” problem that can occur from a cavalier interchange, the $g^2$-contour in this case should first be wrapped around the cut associated with $(-g^2)^{1-s}$ to obtain

$$d(s) = \sin \pi s \int_{L} \frac{dg^2}{\pi} D(g^2)$$

with $L$ lying above the cut on the positive real axis. This is the appropriate coefficient generating function for this case. Now, when the representation eq. (1) or ref. 1 is inserted, no problem arises upon interchange of integrations; thus

$$d(s) = \frac{\sin \pi s}{\pi} \int_0^\infty dz f(z) \int_0^\infty dg^2 (g^2)^{-1-s} e^{-z/g^2}$$

$$= \Gamma(s) \{ \sin \pi s \int_0^\infty \frac{dz}{\pi} f(z) z^{-s} \}.$$

The quantity in curly brackets is just the $s$th coefficient in the perturbative expansion of $f(z)$ as it must be.

Many of these points will be expanded upon in greater detail in forthcoming papers.
References

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[7] L. S. Brown and L. Yaffe, Phys. Rev. 45 R398 (1992). These authors set $b_2 = 0$ and argue that since they cannot derive my result by their method, then analyticity and the RG are not sufficient to determine the large $n$ behavior of $r_n$. However, they do not actually use the full analyticity properties of $D$ in the full complex $q^2$ plane, only the fact that its absorptive part along the positive real axis is determined by $R$.
[8] G. B. West, in “Radiative Corrections” edited by N. Dombey and F. Bondjema (Plenum, N.Y., 1990) p. 487.