DEFORMATION OF EXTREMAL METRICS, COMPLEX MANIFOLDS AND THE RELATIVE FUTAKI INVARIANT

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Abstract. Let $(X, \Omega)$ be a closed polarized complex manifold, $g$ be an extremal metric on $X$ that represents the Kähler class $\Omega$, and $G$ be a compact connected subgroup of the isometry group $\text{Isom}(X, g)$. Assume that the Futaki invariant relative to $G$ is nondegenerate at $g$. Consider a smooth family $(M \to B)$ of polarized complex deformations of $(X, \Omega) \simeq (M_0, \Theta_0)$ provided with a holomorphic action of $G$ which is trivial on $B$. Then for every $t \in B$ sufficiently small, there exists an $h^{1,1}(X)$-dimensional family of extremal Kähler metrics on $M_t$ whose Kähler classes are arbitrarily close to $\Theta_t$. We apply this deformation theory to show that certain complex deformations of the Mukai-Umemura 3-fold admit Kähler-Einstein metrics.

1. Introduction

Every closed Kähler manifold is stable under complex deformations. Indeed, the classical result of Kodaira and Spencer [12] allows us to follow differentiably the Kähler metric under small perturbations of the complex structure. Our goal here is the study of the stability of the extremal condition of a Kähler metric under complex deformations.

Let us recall that a cohomology class $\Omega \in H^2(X, \mathbb{R})$ on a complex manifold $(X, J)$ is called a polarization of $X$ if $\Omega$ can be be represented by the Kähler form $\omega_g$ of a Kähler metric $g$ on $X$. In this case, the pair $(X, \Omega)$ is called a polarized complex manifold. In what follows, we shall sometimes use the metric $g$ and its Kähler form $\omega_g$ interchangeably. We shall denote by $c_1 = c_1(X, J)$ the first Chern class of $(X, J)$.

The set of all Kähler forms on $X$ representing a given polarization $\Omega$ is denoted by $\mathcal{M}_\Omega$. The search for a canonical representative $g$ of $\Omega$ is done by looking for a critical point of the functional

$$
\mathcal{M}_\Omega \to \mathbb{R}
$$

$$
g \mapsto \int_X s_g^2 d\mu_g .
$$

Here, $s_g$ is the scalar curvature of $g$ and $d\mu_g$ its volume form. These critical points are the extremal metrics of Calabi [4]. The condition for $g$ to be such can be stated simply by saying that the gradient of $s_g$ is a real holomorphic vector field, which in itself shows a subtle interplay between extremal metrics and the complex geometry of $X$.

We denote by $\text{Aut}(X)$ the automorphism group of $X$. The space $\mathfrak{h}(X)$ of holomorphic vector fields on $X$ is its Lie algebra. The space $\mathfrak{h}_0(X)$ of holomorphic vector fields with zeroes is an ideal of $\mathfrak{h}(X)$. The identity component $G' = \text{Isom}_0(X, g)$ of the isometry group of the metric metric $g$ is identified with a compact subgroup of
Aut(X), and its Lie algebra is denoted \(\mathfrak{g}'\). If \(g\) is extremal, then \(G'\) is a maximal connected compact subgroup of Aut(X) [5].

1.1. Main result. Let us consider a polarized complex manifold \((X, \Omega)\) and a smooth family of complex deformations \(\mathcal{M} \to B\) of \(X \simeq \mathcal{M}_0\). Here \(B\) is some open neighborhood of the origin in an Euclidean space \(\mathbb{R}^m\) for some \(m \geq 0\) (see [8] for definitions). We let \(E \to B\) be the vector bundle of second fiber cohomology of the fibration \(\mathcal{M} \to B\), whose fibers are \(E_t = H^2(M_t, \mathbb{R})\). A smooth section \(\Theta\) of \(E \to B\) such that \(\Theta_t\) admits a Kähler representative \(\omega_t\) in \(M_t\) for all \(t \in B\) is said to be a polarization of the family \(\mathcal{M} \to B\). We obtain a family \((M_t, \Theta_t)\) of polarized manifolds parametrized by \(t \in B\). A polarized family of complex manifolds \((\mathcal{M} \to B, \Theta)\), and a polarized complex manifold \((X, \Omega)\) together with an isomorphism \(X \simeq \mathcal{M}_0\) that makes \(\Omega\) and \(\Theta_0\) correspond to each other, is said to be a polarized deformation of \((X, \Omega)\).

By shrinking the neighborhood of the origin \(B\) if necessary, the Kodaira-Spencer theory allows us to choose a smooth 2-form \(\beta\) in \(M\) such that \(\omega_t = \beta|M_t\) is a Kähler form in \(M_t\) representing \(\Theta_t\). In this expression, \(\beta|M_t\) denotes the pullback of \(\beta\) by the canonical inclusion \(M_t \to \mathcal{M}\). Such a \(\beta\) is said to represent the polarization \(\Theta\). If \(g\) is a Kähler metric on \(X\) whose Kähler form \(\omega_g\) represents \(\Omega\), \(\beta\) can be constructed so that \(\omega_0\) and \(\omega_g\) agree under the isomorphism \(X \simeq \mathcal{M}_0\).

Let us now assume that the metric \(g\) is extremal. It is then natural to ask if the representative \(\beta\) of the polarization \(\Theta\) can be chosen so that the metric \(g_t\), of Kähler form \(\omega_t\) on \(M_t\), is extremal.

A positive answer to this general statement is not to be expected, and as it has been pointed out in [14], “the answer is an emphatic no” (cf. [3] for some counterexamples). However, if we assume some symmetries for \(\mathcal{M}\) and the nondegeneracy of the relative Futaki invariant, the answer is actually yes provided we allow the polarization \(\Theta\) to be deformed also.

If the metric \(g\) on \(X\) is extremal, then \(G' = \text{Isom}_0(X, g)\) is a maximal connected compact subgroup of Aut(X, g) that acts holomorphically on the central fiber \(\mathcal{M}_0\) of \(\mathcal{M} \to B\). In general though, this action will not extend as a holomorphic action of \(G'\) (cf. [11] for a precise definition of this terminology) to the total space of deformations \(\mathcal{M} \to B\). We shall assume that the action of \(G'\) extends partially, and we have a connected compact subgroup \(G\) of \(G'\) that acts holomorphically on \(\mathcal{M} \to B\) with trivial action on \(B\). We denote by \(C_G^\infty(\mathcal{M})\) the space of \(G\)-invariant smooth functions on \(\mathcal{M}\).

It is then possible to introduce the notion of reduced scalar curvature \(s^G_g\) for any \(G\)-invariant Kähler metric on \(X\) (cf. [2,3], and the Futaki invariant relative to \(G\) (cf. [2,3], [10])

\[
\mathfrak{F}^\infty_{G, \Omega} : \mathfrak{q}/\mathfrak{g} \to \mathbb{R},
\]

where \(\mathfrak{g}\) is the Lie algebra of \(G\) and \(\mathfrak{q}\) is the normalizer of \(\mathfrak{g}\) in \(\mathfrak{h}(X)\). The extremal condition of a metric is encoded in the equation \(s^G_g = 0\), and a \(G\)-invariant extremal metric representing \(\Omega\) have vanishing reduced scalar curvature if, and only if, \(\mathfrak{F}^\infty_{G, \Omega}\) vanishes identically [8, 5]. This characterization of extremal metrics reinterprets in this manner a presentation advocated elsewhere [16, 17, 20].
Let $p$ be the normalizer of $g$ in $g'$. We set $g_0 = g \cap h_0(X)$ and $p_0 = p \cap h_0(X)$, respectively. The differential of the relative Futaki invariant induces a linear map
\[ q/g \to (H^{1,1}(X) \cap H^2(X, \mathbb{R}))^*. \]
We say that $\mathfrak{g}_{G, \Omega}$ is nondegenerate if the restriction of this map to $p_0/g_0$ is injective.

Using the holomorphic action of $G$ on $M \to B$, we refine the construction of the representative $\beta$ of the polarization $\Theta$ so that the Kähler metric $g_t$ it induces on $M_t$ is $G$-invariant (cf. §1.2) for all $t \in B$. We then use the corresponding Kähler form $\omega_t$ as the origin of the affine space of Kähler metrics on $M_t$ that represent the polarization $\Theta_t$. Let $H_t$ be the space of real $g_t$-harmonic $(1,1)$-forms on $M_t$. By the Kodaira-Spencer theory, and perhaps shrinking $B$ to a sufficiently small neighborhood of the origin, the spaces $H_t$ are the fibers of a smooth vector bundle $H \to B$. Given a section $\alpha$ of the bundle $H \to B$ and a sufficiently small function $\phi \in C^\infty(M)$, we consider the family of Kähler metrics $g_{t,\alpha,\phi}$ on $M_t$ defined by the Kähler forms
\[ \omega_{t,\alpha,\phi} = \omega_t + \alpha|_{M_t} + dd^c \phi|_{M_t}. \]
Assuming that $g_0 = g$ is extremal, we seek solutions $g_{t,\alpha,\phi}$ of the equation
\[ s_{g_{t,\alpha,\phi}}^G = 0 \]
for values of $(t, \alpha, \phi)$ in a sufficiently small neighborhood of $(0,0,0)$ where the solution takes on the value $g_{0,0,0} = g$. Introducing suitable Banach spaces, the said equation defines as Fredholm map that is a submersion at the origin if, and only if, the relative Futaki invariant is nondegenerate. The implicit function theorem then yields our main result:

**Theorem A.** Let $(M \to B, \Theta)$ be a polarized family of deformations of a closed polarized complex manifold $(X, \Omega)$. Suppose that $g$ is an extremal metric whose Kähler form $\omega_g$ represents $\Omega$, and that $G$ is a compact connected subgroup of $G' = \text{Isom}_0(X, g)$ such that
- $G$ acts holomorphically on $M \to B$ and trivially on $B$,
- the reduced scalar curvature $s_g^G$ of $g$ is zero,
- the Futaki invariant relative to $G$ is nondegenerate at $g$.

Then, given any $G$-invariant representative $\beta$ of the polarization $\Theta$ such that induced metric $g_0$ on $M_0$ agrees with $g$ via the isomorphism $M_0 \simeq X$, and shrinking $B$ to a sufficiently small neighborhood of the origin if necessary, the space of Kähler metrics on $M_t$ with vanishing reduced scalar curvature lying sufficiently close to the metric $g_t$ induced by $\beta$ is a smooth manifold of dimension $h^{1,1}(X)$. In particular, there are arbitrarily small perturbations $\Theta_t'$ of the polarization $\Theta_t$ such that $\Theta_t'$ is represented by an extremal metric.

**Remark 1.1.1.** Theorem A is a more precise and technical version of Theorem [A]. In particular, it contains more informations about the kind of deformations $\Theta_t'$ of the polarization $\Theta$. This refined version of the above theorem will lead to the applications below.

1.2. Applications. Theorem A generalizes the results of [13, 14]. The case $G = \{1\}$ corresponds to the deformation theory of constant scalar curvature Kähler metrics whereas the case $G = G' = \text{Isom}_0(X, g)$ corresponds to the deformation theory of extremal metrics (cf. [15] for details). The cases corresponding to intermediate
choices for the group $G$ is new. We illustrate the power of Theorem A with the analysis of some new examples of extremal metrics, and some cases of interest. However, we observe that the applicability of our theorem to these follows by elementary reasons, and the nondegeneracy of the relative Futaki invariant is easy to see. It would be of interest to find applications of our result in more demanding situations, which could illustrate the problem at hand in further detail. Some highly nontrivial applications were already given in [14].

1.2.1. Maximal torus symmetry. Extremal Kähler metrics are automatically stable under complex deformations with maximal torus symmetry:

**Corollary B.** Let $(\mathcal{M} \to B, \Theta)$ be a polarized family of deformations of a closed polarized complex manifold $(\mathcal{X}, \Omega)$. Assume that $\Omega$ admits an extremal representative and that $\mathcal{M} \to B$ is endowed with a holomorphic action of a maximal compact torus $G = T \subset \text{Aut}(\mathcal{X})$ acting trivially on $B$. Then for $t \in B$ sufficiently small, $\Theta_t$ is represented by an extremal Kähler metrics on $\mathcal{M}_t$.

Corollary B may look reminiscent of [1, Lemma 4], where a stability result of the extremal condition under complex deformations with symmetries is obtained as an extension of the theory of [13]. This result is carried out under such restrictive assumptions on the deformation of the complex structure and the Kähler class that the scope of its applicability is rather limited.

1.2.2. The Mukai-Umemura 3-fold. We may apply Theorem A to the study of the Mukai-Umemura Fano 3-fold $\mathcal{X}$, with automorphism group $\text{Aut}(\mathcal{X}) = \text{PSL}(2, \mathbb{C})$. Donaldson has proven that this variety admits a Kähler-Einstein metric [7]. Our deformation theorem applies. We obtain the following result:

**Corollary C.** Let $(\mathcal{M} \to B, \Theta)$ be a polarized deformation of the Mukai-Umemura 3-fold with polarization $(\mathcal{X}, c_1(\mathcal{X}))$ and $\Theta_t = c_1(\mathcal{M}_t)$. Assume $\mathcal{M} \to B$ is one of the deformations described at §3.3, endowed with a holomorphic action of a group $G \subset \text{PSL}(2, \mathbb{C})$ isomorphic to a dihedral group of order 8 or a semidirect product $\mathbb{S}^1 \rtimes \mathbb{Z}/2$. Then for every $t \in B$ sufficiently small, $\Theta_t$ is represented by a Kähler-Einstein metric on $\mathcal{M}_t$.

The result above was proved already in [7] using a different approach which was later refined by Székelyhidi [21]. Using this method it is possible to understand general small complex deformations of the Mukai-Umemura 3-fold. It turns out that small deformations corresponding to polystable orbits under the action of $\text{PSL}(2, \mathbb{C})$ are exactly the one which carry Kaehler-Einstein metrics.

1.3. Plan of the paper. In §2 we recall further details and relevant facts about extremal metrics and real holomorphy potentials and relative Futaki invariant that we shall used throughout our work. We proceed in §3 to provide detailed definitions of complex and polarized deformations in a way suitable to our work. The deformation problem that we treat by the implicit function theorem is presented in §4 where we prove an expanded version of our main Theorem A. The applications are given in §5. We begin in §5.1 comparing our result to the LeBrun-Simanca deformation theory, and put the new contributions in proper perspective. We then study deformations with maximal torus symmetry in §5.2 where we prove Corollary B. The Mukai-Umemura manifold and its deformations are discussed in §5.3 where we prove Corollary C.
1.4. Acknowledgments. We would like to thank Professor Simon Donaldson for pointing out an error in the attempt to apply our deformation theory to the Mukai-Umemura 3-fold in an earlier version of this work. We also thank Professor Paul Gauduchon for making the preliminary version of his book [10] available to us.

2. Extremal metrics

Let \((X, \Omega)\) be a closed polarized complex manifold of dimension \(n\). The polarizing assumption on the class \(\Omega\) may be stated by saying that there exist Kähler metrics \(g\) whose Kähler form \(\omega_g\) represents it. We let \(\mathcal{M}_\Omega\) be the set of all such metrics. If \(g \in \mathcal{M}_\Omega\), the projection of the scalar curvature \(s_g\) onto the constants is given by

\[
(2) \quad s_\Omega = 4n\pi \frac{c_1 \cup \Omega^{n-1}}{\Omega^n},
\]

and there exists a vector field \(X_\Omega \in \mathfrak{h}_0 \subset \mathfrak{h}\) [8, 9] such that for every \(g \in \mathcal{M}_\Omega\) we have

\[
(3) \quad \int_X s_g^2 d\mu_g \geq s_\Omega^2 \frac{\Omega^n}{n!} - \mathfrak{F}(X_\Omega, \Omega).
\]

The equality is achieved if, and only if, the metric is extremal. This lower bound was known earlier [11, 18] for metrics in \(\mathcal{M}_\Omega\) that are invariant under a maximal compact subgroup of \(\text{Aut}(X)\). It was proven by Chen [6] to hold in general, for any metric in \(\mathcal{M}_\Omega\). The lower bound varies smoothly as a function of the polarizing class \(\Omega\) [18].

2.1. Holomorphic vector fields. The subset \(\mathfrak{h}_0\) of holomorphic vector fields with zeroes is an ideal of \(\mathfrak{h}\), and the quotient algebra \(\mathfrak{h}/\mathfrak{h}_0\) is Abelian. A smooth complex-valued function \(f\) gives rise to the \((1, 0)\)-vector field \(f \mapsto \partial\# g f \) defined by the expression

\[
g(\partial\# g f, \cdot) = \bar{\partial} f.
\]

This vector field is holomorphic if, and only if, \(\bar{\partial}\partial\# g f = 0\), a condition equivalent to \(f\) being in the kernel of the Lichnerowicz operator

\[
(4) \quad L_g f := (\bar{\partial}\partial\# g)^2 \partial\# g f = \frac{1}{4} \Delta_g^2 f + \frac{1}{2} \partial^{\mu\nu} \nabla_\mu f \nabla_\nu f + \frac{1}{2} (\nabla^7 s_g) \nabla_\tau f.
\]

The ideal \(\mathfrak{h}_0\) consists of vector fields of the form \(\partial\# g f\), for a function \(f\) in the kernel of the Lichnerowicz operator \(L_g\). Or put differently, a holomorphic vector field \(\Xi\) can be written as \(\partial\# g f\) if, and only if, the zero set of \(\Xi\) is nonempty [14]. The kernel \(\mathcal{H}_g\) of \(L_g\) is called the space of holomorphy potentials of \((X, g)\). Since \(L_g\) is elliptic, \(\mathcal{H}_g\) is finite dimensional complex vector space and consists of smooth functions.

Generally speaking, the Lichnerowicz operator \(L_g\) of a metric \(g\) is not a real operator, and so the real and imaginary parts of a function in its kernel do not have to be elements of the kernel also. In studying the geometry of Kähler manifolds, it is often convenient to use real quantities and operators. We introduce the terminology that allows us to do so here. We follow the conventions of [2, 10].

Let us consider the operator \(d^*\) on functions defined by \(d^* = Jd\). Given a real holomorphic vector field \(\Xi\) on \(X\), the Hodge theory provides a unique decomposition \(\xi = \Xi^r = \xi_h + du_\Xi + d^* v_\Xi\), where \(\xi_h\) is a harmonic 1-form, \(u_\Xi, v_\Xi\) are real valued functions, and \(d^* v_\Xi\) is coclosed. The functions \(u_\Xi, v_\Xi\) are uniquely determined up to an additive constant. The dual of this identity gives rise to the decomposition \(\Xi = \Xi_h + \text{grad} u_\Xi + J \text{grad} v_\Xi\), and it follows that \(\Xi^{1,0} = \frac{1}{4} (\Xi - i J\Xi) = \Xi_h^{1,0} + \partial\# g f_\Xi\),
where \( f_\Xi = u_\Xi + iv_\Xi \). It follows that \( \Xi \) is real holomorphic if, and only if, \( \Xi^{1,0} \) is complex holomorphic, that is to say, if, and only if, \( L_g f_\Xi = 0 \). If we extend the Lie bracket operation linearly in each component, the integrability of the complex structure makes of the map \( \Xi \to \Xi^{1,0} \) an isomorphism of Lie algebras.

As indicated above, the condition \( \Xi_h = 0 \) is equivalent to \( \Xi \) being Hamiltonian, and characterizes \( h_0(X) \). If \( \Xi \in h_0(X) \), the potential \( f_\Xi \) of \( \Xi \) is a solution of the equation \( \mathcal{L}_{\Xi^{1,0} \omega_g} = \frac{1}{2} dd^c f \). On the other hand, a holomorphic vector field \( \Xi \) is parallel if, and only if, \( \Xi \) is the dual of a harmonic 1-form.

A vector field \( \Xi \) is a Killing field if, and only if, \( \Xi \) is holomorphic and its potential \( f_\Xi = iv_\Xi \) is, up to a constant, a purely imaginary function \([14]\). In that case, \( v_\Xi \) will be called the Killing potential of \( \Xi \).

If \( \xi \) is a 1-form, we let \( \nabla - \xi \) be the \( J \)-anti-invariant component of the 2-tensor \( \nabla \xi \). Let \( \xi = \Xi^s \) for a real vector field \( \Xi \). The condition \( \nabla - \xi = 0 \) is equivalent to \( \Xi \) being a real holomorphic vector field, and can be expressed as \( \delta \delta \nabla - \xi = 0 \).

We have the identities

\[
\delta \nabla - \xi = \frac{1}{2} \Delta_g \xi - J \xi^s \rho_g \tag{5}
\]

and

\[
\delta \delta \nabla - \xi = \frac{1}{2} \Delta_g \delta \xi - \langle d^c \xi, \rho_g \rangle + \frac{1}{2} \langle \xi, ds_g \rangle , \tag{6}
\]

respectively.

We introduce the operator \( \mathbb{L}_g \) defined by

\[
\mathbb{L}_g f = (\nabla - d)^* (\nabla - d) f = \delta \delta \nabla - df .
\]

This is a real elliptic operator of order four, and if \( f \) is a real valued function, we have that \( \mathbb{L}_g f = 0 \) if, and only if, \( \text{grad} f \) is a real holomorphic vector field; by the observations above, every Hamiltonian Killing field is of the form \( J \text{grad} f \) for a function \( f \) of this type.

By \([8]\) we derive the identities

\[
\delta \delta \nabla - d^c f = -\frac{1}{2} \mathcal{L}_{J \text{grad} s_g} f , \tag{7}
\]

and by \([8]\), we see that

\[
2L_g f = \mathbb{L}_g f + \frac{i}{2} \mathcal{L}_{J \text{grad} s_g} f . \tag{8}
\]

**Lemma 2.1.1.** For any Kähler metric \( g \), the space of real solutions of the equation \( L_g f = 0 \) coincide with the space of real solutions of \( L_g f = 0 \).

**Proof.** By \([8]\), if \( f \) is a real valued function such that \( L_g f = 0 \), then \( \mathbb{L}_g f = 0 \) and \( \mathcal{L}_{J \text{grad} s_g} f = 0 \). Conversely, let us assume that \( \mathbb{L}_g f = 0 \). This implies that \( \Xi = J \text{grad} f \) is a Killing field. Then

\[
\mathcal{L}_{J \text{grad} s_g} f = \langle d^c s_g, df \rangle = -\langle ds_g, d^c f \rangle = -\mathcal{L}_{\Xi s_g} = 0 ,
\]

and by \([8]\), we conclude that \( L_g f = 0 \). \( \square \)

### 2.2. The group of isometries of a Kähler metric.

Let \((\mathcal{X}, \Omega)\) be a polarized complex manifold and \( G, G' \) be connected compact Lie subgroups of \( \text{Aut}(\mathcal{X}) \), with \( G \subset G' \). By taking a \( G' \)-average if necessary, we represent the polarization \( \Omega \) the Kähler form \( \omega_g \) of a \( G' \)-invariant Kähler metric \( g \). We attach to this data a relative Futaki invariant.
2.2.1. **Lie algebras.** The Lie algebras \( g \) and \( g' \) of \( G \) and \( G' \) are naturally identified with subalgebras of the algebra \( h(\mathcal{X}) \) of holomorphic vector fields. We define \( g_0 = g \cap h_0(\mathcal{X}) \) and \( g'_0 = g' \cap h_0(\mathcal{X}) \). We then introduce the Lie algebras

- \( \mathfrak{z} = Z(g) \), the center of \( g \),
- \( \mathfrak{z}' = C_g'(g) \), the centralizer of \( g \) in \( g' \),
- \( \mathfrak{z}'' = C_h(g) \), the centralizer of \( g \) in \( h \),
- \( \mathfrak{p} = N_g'(g) \), the normalizer of \( g \) in \( g' \), and
- \( \mathfrak{q} = N_h(g) \), the normalizer of \( g \) in \( h \).

If \( t \) is any of these Lie algebras, we shall denote by \( t_0 \) the ideal of Hamiltonian vector fields \( t_0 = t \cap h_0(\mathcal{X}) \), and by \( \mathcal{H}_{\mathcal{T}}^{t_0} \) the corresponding spaces of holomorphy potentials.

The space of holomorphy potentials \( \mathcal{H}_{\mathcal{T}}^{t_0} \) (respectively \( \mathcal{H}_{\mathcal{T}}^{t_0} \), \( \mathcal{H}_{\mathcal{T}}^{t_0} \)) is identified to the \( G \)-invariant potentials of \( \mathcal{H}_{\mathcal{T}}^{t_0} \) (respectively \( \mathcal{H}_{\mathcal{T}}^{t_0} \), \( \mathcal{H}_{\mathcal{T}}^{t_0} \)). Notice that \( \mathcal{H}_{\mathcal{T}}^{t_0} \subset \mathcal{H}_{\mathcal{T}}^{t_0} \) and \( \mathcal{H}_{\mathcal{T}}^{t_0} \subset \mathcal{H}_{\mathcal{T}}^{t_0} \) consists of purely imaginary functions whose imaginary parts define Killing potentials.

Now \( \mathfrak{z} \) is an ideal of \( \mathfrak{z}' \) and \( \mathfrak{z}'' \). On the other hand, \( \mathfrak{g} \) is an ideal of \( \mathfrak{p} \) and \( \mathfrak{q} \). We have canonical injections

\[
\mathfrak{z}'/\mathfrak{z} \hookrightarrow \mathfrak{p}/\mathfrak{g}, \quad \mathfrak{z}_0'/\mathfrak{z}_0 \hookrightarrow \mathfrak{p}_0/\mathfrak{g}_0, \quad \mathfrak{z}''/\mathfrak{z} \hookrightarrow \mathfrak{q}/\mathfrak{g}, \quad \mathfrak{z}_0''/\mathfrak{z}_0 \hookrightarrow \mathfrak{q}_0/\mathfrak{g}_0.
\]

**Lemma 2.2.2.** The canonical injections (9) are surjective, and we have canonical isomorphisms of Lie algebras

\[
\mathfrak{z}'/\mathfrak{z} \simeq \mathfrak{p}/\mathfrak{g}, \quad \mathfrak{z}_0'/\mathfrak{z}_0 \simeq \mathfrak{p}_0/\mathfrak{g}_0, \quad \mathfrak{z}''/\mathfrak{z} \simeq \mathfrak{q}/\mathfrak{g}, \quad \mathfrak{z}_0''/\mathfrak{z}_0 \simeq \mathfrak{q}_0/\mathfrak{g}_0.
\]

**Proof.** We prove that \( \mathfrak{p} = \mathfrak{z}' + \mathfrak{g} \). Let \( \Xi \) be a Killing field in \( \mathfrak{g}' \). We have that

\[
\Xi = \Xi_h + J_{\text{grad } \mathfrak{v}_\Xi}
\]

where \( \Xi_h \) is the dual vector field of a harmonic 1-form, and \( \mathfrak{v}_\Xi \) is a real function. The vector field \( \Xi \) belongs to \( \mathfrak{p} \) if, and only if,

\[
[\mathfrak{Y}, \Xi] = \mathcal{L}_{\mathfrak{Y}} \Xi \in \mathfrak{g} \quad \text{for all } \mathfrak{Y} \in \mathfrak{g}.
\]

Since \( \mathcal{L}_{\mathfrak{Y}} \Xi_h = 0 \) because \( \mathfrak{Y} \) is Killing, this condition is equivalent to \( \mathcal{L}_{\mathfrak{Y}} J_{\text{grad } \mathfrak{v}_\Xi} \in \mathfrak{g} \). In turn, this is equivalent to having \( J_{\text{grad }} (\mathfrak{Y} \cdot \mathfrak{v}_\Xi) \in \mathfrak{g} \) since \( \mathfrak{Y} \) preserves \( J \) and the metric, which means that \( \mathfrak{Y} \cdot \mathfrak{v}_\Xi \in i\mathcal{H}_{\mathcal{T}}^{t_0} \) for every \( \mathfrak{Y} \in \mathfrak{g} \). This implies that for every \( \gamma \in G \), we have \( \gamma^* \mathfrak{v}_\Xi = \mathfrak{v}_\Xi \in i\mathcal{H}_{\mathcal{T}}^{t_0} \). We can then average the function \( \mathfrak{v}_\Xi \) under the group action to obtain a \( G \)-invariant function \( \mathfrak{v}_{\Xi} \) on \( X \) such that \( \mathfrak{v} = \mathfrak{v}_\Xi - \mathfrak{v}_{\Xi} \in i\mathcal{H}_{\mathcal{T}}^{t_0} \), and

\[
\Xi = (\Xi_h + J_{\text{grad } \mathfrak{v}_{\Xi}}) + J_{\text{grad } \mathfrak{v}},
\]

where \( J_{\text{grad } \mathfrak{v}} \in \mathfrak{g} \) and \( \Xi_h + J_{\text{grad } \mathfrak{v}_{\Xi}} = \Xi - J_{\text{grad } \mathfrak{v}} \in \mathfrak{g}' \) is \( G \)-invariant and so an element of \( \mathfrak{z}' \).

An analogous argument shows that \( \mathfrak{q} = \mathfrak{z}'' + \mathfrak{g} \), which finishes the proof. \( \square \)

2.2.3. **The reduced scalar curvature.** Let \( L^2_k(\mathcal{X}) \) be the \( k \)th Sobolev space defined by \( g \). The space \( L^2_{k,G}(\mathcal{X}) \) of \( G \)-invariants functions in \( L^2_k(\mathcal{X}) \) can be obtained as metric completion of \( C^\infty_{\mathcal{G}}(\mathcal{X}) \). If \( k > n \), \( L^2_k(\mathcal{X}) \) is a Banach algebra. In fact, the Sobolev embedding theorem says that if \( k > n + l \) we have \( L^2_k(\mathcal{X}) \) continuously contained in \( C^l(\mathcal{X}) \), the space of functions with continuous derivatives of order at most \( l \). We shall work below always imposing this restriction over \( k \).
The $L^2$-Hermitian product on $L^2_{k,G}(\mathcal{X})$ induced by the Riemannian metric $g$ allows us to define the orthogonal $W_{k,g}$ of $i\mathcal{H}^0_g$. Thus we have an orthogonal decomposition

$$L^2_{k,G}(\mathcal{X}) = i\mathcal{H}^0_g \oplus W_{k,g}$$

together with the associated projections

$$\pi^W_g : L^2_{k,G}(\mathcal{X}) \to W_{k,g} \quad \text{and} \quad \pi^G_g : L^2_{k,G}(\mathcal{X}) \to i\mathcal{H}^0_g.$$ 

We introduce the reduced scalar curvature $s^G_g$ defined by

$$s^G_g = \pi^W_g (s_g) = (\mathbb{I} - \pi^G_g)(s_g)$$

By construction, the condition

$$s^G_g = 0$$

is equivalent to $s_g \in i\mathcal{H}^0_g$, and since $\mathcal{H}^0_g \subset \mathcal{H}_g$, this condition implies the extremality of the metric $g$.

2.2.4. The reduced Ricci form. Let $L^2\Lambda^{1,1}_{k,G}(\mathcal{X})$ be the space of real $G$-invariant $(1,1)$-forms on $\mathcal{X}$ in $L^2_k$. We lift the projection $\pi^G_g$ to a projection

$$\Pi^G_g : L^2\Lambda^{1,1}_{k+2,G}(\mathcal{X}) \to L^2\Lambda^{1,1}_{k,G}(\mathcal{X})$$

defined by

$$\Pi^G_g \beta = \beta + dd^c f,$$

where $f = G_g(\pi^W_g(\omega_g, \beta))$ and $G_g$ is the Green operator of $g$ [19, 16, 20]. It follows that $(\omega_g, \Pi^G_g(\beta)) = \pi^G_g(\omega_g, \beta)$.

We apply this projection map to the Ricci form, and define the reduced Ricci form $\rho^G_g$

$$\rho^G := \Pi^G \rho.$$ 

We then obtain the identity

$$\rho_g = \rho^G + \frac{1}{2} dd^c \psi^G_g,$$

where $\psi^G_g = 2G_g(\pi^W_g(\omega_g, \rho)) = G_g \pi^W_g (s_g) = G_g(s^G_g)$. In particular, $\rho^G = \rho_g$ if, and only if, $s^G_g = 0$.

2.2.5. Variational formulas. Given a Kähler metric $g$, we consider infinitesimal deformations of it given by

$$\omega_t = \omega_g + t\alpha + tdd^c \phi$$

of the Kähler class $\omega_g$, where $\phi$ is a smooth function and $\alpha$ is a $g$-harmonic $(1,1)$-form, respectively. These variations have Kähler classes given by $[\omega_g] + t[\alpha]$. In order to avoid confusion, the derivative at $g$ of various geometric quantities will be denoted by $D_g$ below. Sometime later, and when confusion cannot occur, these same derivatives will be denoted by a dot superimposed on the quantities themselves.

The holomorphy potential of a holomorphic vector field depends upon the choice of the metric. Its infinitesimal variation when moving the metric in the direction of $\phi$ or $\alpha$ is described in the following lemma:
Lemma 2.2.6. Let $\Xi$ be a real holomorphic vector field in $h_0(X)$, with holomorphy potential $f_\Xi = u_\Xi + iv_\Xi \in H_g$. The variation $D_gf_\Xi(\phi)$ of $f_\Xi$ at $g$ in the direction of $\phi$ is given by

$$D_gf_\Xi(\phi) = 2\Omega_{\Xi,\alpha}f_\Xi,$$

which is equivalent to the expressions $(D_gu_\Xi)(\phi) = \Omega_\Xi \phi$ and $(D_gv_\Xi)(\phi) = -\mathcal{L}_\Xi \phi$ for the variations of the real and imaginary parts of $f_\Xi$. The variations $D_gu_\Xi(\alpha)$ and $D_gv_\Xi(\alpha)$ at $g$ in the direction of a trace-free form $\alpha$ are given by

$$D_gu_\Xi(\alpha) = -G_g(\delta(J\Xi, J\alpha)),$$
$$D_gv_\Xi(\alpha) = G_g(\delta(E\Xi, J\alpha)).$$

The variation of the reduced scalar curvature $s_g^G$ is described by:

Lemma 2.2.7. Let $g$ be a $G$-invariant Kähler metric on $X$. The variation of the reduced scalar curvature $s_g^G$ when moving the metric in the direction of $\phi$ is given by

$$(D_g s_g^G)(\phi) = -2L_g \phi + \langle d\phi, ds_g^G \rangle.$$

If $s_g^G = 0$, the variation of $s_g^G$ when varying the metric in the direction of a trace-free form $\alpha$ is given by

$$(D_g s_g^G)(\alpha) = \pi^W (G_g(\langle \alpha, dd^c s_g \rangle) - 2 \langle \alpha, \rho_g \rangle).$$

2.3. The relative Futaki invariant. Given any $G$-invariant Kähler metric $g$ on $X$ with Kähler form $\omega_g$ and Ricci-form $\rho_g$, we define the Futaki function of $(G,g)$ by

$$\mathcal{F}^c_{G,\omega_g}(\Xi) = \int_X d\psi^G_g(\Xi) \, d\mu_g = \int_X -\mathcal{L}_\Xi \psi^G_g \, d\mu_g$$

for any real holomorphic vector field $\Xi$ on $X$. Here, $\psi^G_g$ is the Ricci potential $[12]$ of the metric $g$. This function is defined in terms the Kähler metric $g$, but depends only upon the Kähler class $\Omega = [\omega_g]$ $[\Xi \Xi]$, as we briefly recall below. The resulting function $\mathcal{F}^c_{G,\Omega}$ shall be referred to as the Futaki invariant of $\Omega$ relative to $G$, or relative Futaki invariant for short, when the group $G$ and class $\Omega$ are understood. Let us observe that the usual definition of the real Futaki invariant applied to a holomorphic vector field $\Xi$ is given by $\mathcal{F}_{G,\Omega}(\Xi) = \mathcal{F}^c_{G,\Omega}(J\Xi)$. We have introduced this notation for convenience. Put differently, there is a complex valued version of the Futaki invariant such that $\mathcal{F}^c_{G,\Omega}$ is its real part and $\mathcal{F}^c_{G,\Omega}$ the imaginary part.

2.3.1. Properties of the Futaki invariant. The function $\mathcal{F}^c_{G,\omega_g}(\Xi)$ vanishes on parallel holomorphic vector fields. Indeed, let $\Xi_h$ be the dual of a harmonic 1-form $\xi_h$. Then

$$\mathcal{F}^c_{G,\omega_g}(\Xi_h) = \int_X \langle J\xi_h, d\psi^G_g \rangle \, d\mu_g = \int_X \langle \delta J\xi_h, \psi^G_g \rangle \, d\mu_g.$$ Since the space of harmonic 1-forms is $J$-invariant, so $\delta J\xi_h = 0$, and it follows that $\mathcal{F}^c_{G,\omega_g}(\Xi_h) = 0$.

This function $\mathcal{F}^c_{G,\omega_g}$ can be expressed alternatively in terms of the reduced scalar curvature. For if $\Xi$ is a Hamiltonian holomorphic vector field on $(X,g)$, it can be written as $\Xi = \text{grad} u_\Xi + J\text{grad} v_\Xi$ for some real valued functions $u_\Xi, v_\Xi$, and with
\( d \omega \) orthogonal to the space of closed forms. Hence, \( \mathfrak{F}_{G,\omega}(\Xi) = \langle dv, \delta \psi \rangle_{g} = \langle v, \Delta \psi \rangle_{g} \). It follows that

\begin{equation}
\mathfrak{F}_{G,\omega}(\Xi) = \int_{X} v \, s_{g} \, d\mu_{g}.
\end{equation}

The expression above shows that if \( \Xi \) is any Killing field in \( \mathfrak{g} \), then \( \mathfrak{F}_{G,\omega}(\Xi) = 0 \). It follows that \( \mathfrak{F}_{G,\omega} \) induces a \( \mathbb{R} \)-linear map

\begin{equation}
\mathfrak{F}_{G,\omega}: \mathfrak{q}/\mathfrak{g} \to \mathbb{R},
\end{equation}

where \( \mathfrak{q} \) is the normalizer of \( \mathfrak{g} \) in \( \mathfrak{h}(X) \).

The fundamental properties of this function is now stated in the form of a Theorem. This was proven originally by Futaki [8], and generalized a bit later by Calabi [5]. The version given here follows its presentation in [10].

**Theorem 2.3.2.** The relative Futaki function \( \mathfrak{F}_{G,\omega} : \mathfrak{q}/\mathfrak{g} \to \mathbb{R} \) defined in (14) is independent of the particular \( G \)-invariant Kähler representative \( \omega_{g} \) of the class \( \Omega \), and so it induces an invariant function

\[ \mathfrak{F}_{G,\omega}: \mathfrak{q}/\mathfrak{g} \to \mathbb{R} \]

of the class, the relative Futaki invariant of \( \Omega \). The relative Futaki invariant vanishes for a class \( \Omega \) if, and only if, any \( G \)-invariant extremal metric \( g \) that represents \( \Omega \) has vanishing reduced scalar curvature \( s_{g}^{G} \). If \( g \) is any \( G \)-invariant extremal Kähler metric and \( G' = \text{Isom}_{0}(X, g) \), the vanishing of \( \mathfrak{F}_{G,\omega} \) is equivalent to the vanishing of its restriction

\[ \mathfrak{F}_{G,\omega}: \mathfrak{p}_{0}/\mathfrak{g}_{0} \to \mathbb{R}. \]

**Proof.** The invariance is proven in [8], and generalized in [5] Proposition 4.1, p. 110.

The proof of the second statement follows by (13). For if \( s_{g}^{G} = 0 \), then \( \mathfrak{F}_{G,\omega}(\Xi) = 0 \) for every \( \Xi \in \mathfrak{q}_{0} \). Conversely, let us assume that \( \mathfrak{F}_{G,\omega} \) vanishes on \( \mathfrak{q} \) and, therefore, on \( \mathfrak{q}_{0} = \mathfrak{q} \cap \mathfrak{h}_{0}(X) \). Let \( g \) be a \( G \)-invariant extremal Kähler metric that represents the class \( \Omega \), and let \( G' = \text{Isom}_{0}(X, g) \). Then \( \Xi = J \text{grad} \, s_{g} \in \mathfrak{q}_{0}^{0} \), and by [5] Theorem 1, p. 97, \( \Sigma_{\Xi} \Xi = 0 \) for all Killing fields \( \Xi \). Thus, we have that \( \Xi \in \mathfrak{p}_{0} \subset \mathfrak{p} \subset \mathfrak{q} \). But

\[ \mathfrak{F}_{G,\omega}(\Xi) = 0 = \int_{X} s_{g} s_{g}^{G} \, d\mu_{g} = \| s_{g}^{G} \|_{g}^{2}, \]

and so \( s_{g}^{G} = 0 \).

The final statement follows easily by the argument above. \( \square \)

2.3.3. **Nondegeneracy of the Futaki invariant.** Let \( g \) be a metric such that \( [\omega_{g}] = \Omega \). Given any real \( g \)-harmonic \((1,1)\)-form, we may use the metric \( g \) to compute the derivative

\[ \frac{d}{dt} \mathfrak{F}_{G,\omega+ta} \bigg|_{t=0}. \]

We have the following:

**Lemma 2.3.4.** Let \( g \) be a \( G \)-invariant extremal Kähler metric on \( X \) such that \( [\omega_{g}] = \Omega \), and let \( \alpha \) be a real \( g \)-harmonic trace-free \((1,1)\)-form. If \( \Xi = \text{grad} u_{\Xi} + J \text{grad} v_{\Xi} \)
Lemma 2.3.7. The relative Futaki invariant vanishes. In the presence of certain maximal torus symmetries on the underlying manifold, 2.3.6. Nondegeneracy of the Futaki invariant relative to a maximal compact torus. $G$ to $G$, this condition holds for some metric $g$ which finishes the proof.

Definition 2.3.5. This leads to a very important concept in our work:

$q/\mathfrak{g} \to (H^{1,1}(X) \cap H^2(X,\mathbb{R}))^*$. This leads to a very important concept in our work:

**Definition 2.3.5.** Let $G$ and $G'$ be connected compact subgroups of Aut($X$) such that $G \subset G'$. The Futaki invariant $\mathcal{F}_{G,\Omega}$ relative to $G$ is said to be $G'$-nondegenerate at $\Omega$ if the linear map (13) restricted to $\mathfrak{g}_0/\mathfrak{g}_0 \simeq \mathfrak{z}_0/\mathfrak{z}_0 \to (H^{1,1}(X) \cap H^2(X,\mathbb{R}))^*$ is injective. If $g$ is a Kähler metric representing $\Omega$ such that $G' = \text{Isom}_0(X,g)$ and this condition holds for some $G \subset G'$, we say that $g$ is Futaki nondegenerate relative to $G$.

We briefly illustrate this notion in a particular case next.

**2.3.6. Nondegeneracy of the Futaki invariant relative to a maximal compact torus.** In the presence of certain maximal torus symmetries on the underlying manifold, the relative Futaki invariant vanishes.

**Lemma 2.3.7.** Let $X$ be a complex manifold, and let $T$ be a maximal compact torus subgroup of Aut($X$). Then for $G = T$, we have $q/\mathfrak{g} = 0$ and the relative Futaki invariant $\mathcal{F}_{G,\Omega}$ is identically zero for any Kähler class $\Omega$ in $X$. Given any compact connected Lie group $G' \subset$ Aut($X$), then for $G = T \subset G'$ the relative Futaki invariant $\mathcal{F}_{G,\Omega}$ is $G'$-nondegenerate.

**Proof.** The Lie algebra $\mathfrak{g}$ of $G = T$ is Abelian and so equal to its center $\mathfrak{z}$. The centralizer $\mathfrak{z}''$ of $\mathfrak{g}$ in $\mathfrak{h}$ contains $\mathfrak{z}$. If $\mathfrak{z}$ were strictly contained in $\mathfrak{z}''$, we could find an element of $\mathfrak{z}'' \setminus \mathfrak{z}$ that together with $\mathfrak{z}$ would generate an Abelian Lie algebra, and this would contradict the maximality of $\mathfrak{z}$ in $\mathfrak{h}$. Thus, $\mathfrak{z} = \mathfrak{z}''$, and by Lemma 2.2.2, $q/\mathfrak{g} \simeq \mathfrak{z}''/\mathfrak{z} = 0$. The result follows. □
3. Deformations of the complex structure

In this section we recall the theory of smooth deformations of a complex manifold and that of smooth polarized deformations. We illustrate the concept exhibiting a deformation of the Hirzebruch surface $F_1$. This deformation will reappear in one of our applications in §5.

3.1. Complex deformations. A smooth family of complex deformations consists of the following data:

(1) an open connected neighborhood $B$ of the origin in $\mathbb{R}^m$, a smooth manifold $M$ and a smooth proper submersion $\varphi : M \to B$,

(2) an open covering $\{U_j\}_{j \in I}$ of $M$ and smooth complex valued functions $z_j = (z^1_j, \ldots, z^n_j)$ defined on each $U_j$, such that the collection of mappings

$$U_j \cap \varphi^{-1}(t) \to \mathbb{C}^n$$

$$p \mapsto (z^1_j(p), \ldots, z^n_j(p))$$

define a holomorphic atlas on each manifold $\varphi^{-1}(t)$.

Such a family of complex deformations will be denoted simply by $M \to B$, and the fibers $\varphi^{-1}(t)$ together with their canonical complex structure by $M_t$. A complex manifold $X$ and a family of deformations $M \to B$ together with a given isomorphism $X \simeq M_0$ is called a complex deformation of $X$.

Smooth families of complex deformations are smoothly locally trivial. Indeed, let $M$ be the underlying differentiable manifold of the central fiber $M_0$ of a family $M \to B$. At the expense of shrinking $B$ if necessary, we can find a diffeomorphism $M \to B \times M$ such that the diagram

$$\begin{array}{ccc}
M & \to & B \times M \\
\downarrow & & \downarrow \pi_B \\
B & & B \\
\end{array}$$

commutes. Here $\pi_B$ denotes the projection map onto the first factor. We refer to this diffeomorphism as a trivialization of the given family of complex deformations. By way of such a trivialization, we see that all the $M_t$s are diffeomorphic to $M$ for $t$s that are in a sufficiently small neighborhood of the origin in $\mathbb{R}^m$, and that the family of complex manifolds $M_t$ can be seen as a differentiable family $\{J_t\}$ of integrable almost complex structures on $M$. From this point of view, $M_t$ and $(M, J_t)$ are identified as complex manifolds.

3.2. Polarized deformations. Let us now assume that $(X, \Omega)$ is a polarized complex manifold, and that $(M \to B, \Theta)$ is a polarized complex deformation of it. If $\omega_g$ is a Kähler form on $X$ that represents $\Omega$, a deep result of Kodaira and Spencer [12] shows that we can find a smooth family of Kähler metrics on $M_t$ that extends $\omega_g$ and represents $\Theta_t$ for each $t$ (cf. [14]). For in the language of integrable almost complex structures $\{J_t\}$ on $M$ that a trivialization of the deformation allows, the function $t \to h^{p,q}_t = \dim_\mathbb{C}H^q(M, \Omega^p)$ is upper semi-continuous and if $(M, J, \omega_g)$ is Kähler, this function is in fact constant in a sufficiently small neighborhood of the origin. Then it follows that there exists a smooth family $t \to \omega_t$ of 2-forms on $M$ such that $\omega_t$ is Kähler with respect to $J_t$, $[\omega_t] = \Theta_t \in H^2(M; \mathbb{R})$, and $\omega_0$ and $\omega_g$ agree with each other via the identification $X \simeq M_0 \simeq (M, J_0)$. This point of view
is best adapted to our work here. We obtain a 2-form $\beta$ on $\mathcal{M}$ such that $\beta|_{\mathcal{M}_t} = \omega_t$. Such a form $\beta$ is said to represent the polarization $\Theta$.

3.3. Example: the Mukai-Umemura 3-fold. Let $V$ be a 7-dimensional complex vector space, $Gr_3(V)$ be the Grassmanian of complex 3-dimensional subspaces of $V$ and $U \rightarrow Gr_3(V)$ be the tautological rank 3-bundle over $Gr_3(V)$. Notice that $Gr_3(V)$ is 12-dimensional.

Any $\varpi \in \Lambda^2V^*$ defines an section $\sigma_{\varpi}$ of the bundle $\Lambda^2U^* \rightarrow Gr_3(V)$. Let $Z_{\varpi} \subset Gr_3(V)$ be the zero set of $\sigma_{\varpi}$. So $Z_{\varpi}$ is the subset of isotropic 3-planes of $\varpi$, the points $P$ in $Gr_3(V)$ such that $\varpi|_P = 0$. For a generic $\varpi$, $Z_{\varpi}$ is a smooth subvariety of codimension three, and given three linearly independent forms, $\varpi_1, \varpi_2, \varpi_3$, we obtain the 3-fold

$$X_{\Pi} = X_{\varpi_1,\varpi_2,\varpi_3} = Z_{\varpi_1} \cap Z_{\varpi_2} \cap Z_{\varpi_3} \subset Gr_3(V),$$

that depends only on the 3-plane $\Pi$ spanned by the $\varpi_i$ in $\Lambda^2V^*$, and not in the basis chosen to represent it. The action of the group $SL(V)$ on $V$ induces an action on $Gr_3(\Lambda^2V^*)$, and 3-planes $\Pi_1$ and $\Pi_2$ define isomorphic complex varieties $X_{\Pi_1}$ and $X_{\Pi_2}$ if, and only if, the planes $\Pi_1$ and $\Pi_2$ lie in the same $SL(V)$ orbit. We obtain a set of equivalence classes of 3-folds parametrized by the quotient $U/SL(V)$.

There is a Zariski open set $C \subset Gr_3(\Lambda^2V^*)$ of 3-planes $\Pi$ such that $X_{\Pi}$ is a smooth subvariety of dimension 3. It has an obvious family of complex deformations. For if $N = \{(\Pi, x) \in C \times Gr_3(V) | x \in X_{\Pi}\}$, we may consider this smooth complex manifold together with its canonical projection $N \rightarrow C$. The $SL(V)$-action on $C \times Gr_3(V)$ induces an equivariant action on $N \rightarrow C$, and two fibers are isomorphic if, and only if, they are above the same orbit in $C$.

The Mukai-Umemura manifold is a particular smooth 3-fold in this family. It can be described efficiently as follows. The six symmetric power $Sym^2(C^2)$ is the standard irreducible 7-dimensional representation of $SL(2, \mathbb{C})$. We take $V = Sym^6(C^2)$ with its induced $SL(2, \mathbb{C})$-action. The representation $\Lambda^2V^*$ decomposes into irreducible representations as

$$\Lambda^2(Sym^6(C^2)) = Sym^{10}(C^2) \oplus Sym^6(C^2) \oplus Sym^2(C^2),$$

The summand $Sym^2(C^2)$ corresponds to a 3-plane $\Pi_0$ in $\Lambda^2V^*$ that defines the Mukai-Umemura variety $X_{\Pi_0}$. The plane $\Pi_0$ is invariant under $SL(2, \mathbb{C})$, so this group acts naturally on $X_{\Pi_0}$.

Since many of the deformations are equivalent via the $SL(V)$-action, it is important to describe the quotient of $Gr_3(\Lambda^2V^*)$. This is done carefully in [7], where it is proven that the quotient of the tangent space to $Gr_3(\Lambda^2V^*)$ at $\Pi_0$ by the tangent space to the the orbit can be identified with $Sym^3(C^2)$ with its standard $SL(2, \mathbb{C})$-action, the stabilizer of $\Pi_0$ in $SL(V)$. By the theory for equivariant slices of Lie group actions, there is a neighborhood of the origin $B \subset Sym^8(C^2)$ and a $PSL(2, \mathbb{C})$-equivariant embedding $j : B \rightarrow Gr_3(\Lambda^2V^*)$ such that for $t_1, t_2 \in B$, the images $j(t_1)$ and $j(t_2)$ are in the same $SL(V)$-orbit if, and only if, $t_1$ and $t_2$ are in the same $PSL(2, \mathbb{C})$ orbit. If $B$ is taken to be sufficiently small, we have $j(B) \subset C$ and $M \rightarrow B$, defined as the fiber product $M = B \times_C N$, is a smooth family of complex deformations of $\mathcal{M}_0 \simeq X_{\Pi_0}$.

3.3.1. Deformations with symmetries. In particular, the deformations corresponding to polynomials $p = C(u^4 - \alpha u^4)(v^4 - \alpha u^4)$ for $\alpha, C \in \mathbb{C}^\times$ have a stabilizer $G$ in $PSL(2, \mathbb{C})$ isomorphic to a dihedral group of order 8. In the case where $\alpha = 0$, the
stabilizer of \( p \) is the subgroup of \( \text{PSL}(2, \mathbb{C}) \) spanned by the one parameter subgroup 
\[ \lambda \cdot [u : v] = [\lambda u : \lambda^{-1} v] \]
and the rotation \([u : v] \rightarrow [u : v]\). Hence, we have a maximal compact subgroup \( G \) of the stabilizer of \( p \) that is isomorphic to \( \mathbb{Z}/2 \rtimes S_1 \).

We can consider the family of deformation of \( X_{\Pi_0} \) obtained by restricting \( B \) to the subspace of polynomials of the form \( tp \) as above for \( t \in \mathbb{C} \). Thus, we get a family of deformation \( M \rightarrow \mathbb{C} \) endowed with a holomorphic action of \( G \), where \( G \) is a dihedral group of order 8, or the semidirect product of \( \mathbb{Z}/2 \rtimes S_1 \) in the case where \( \alpha = 0 \). In addition, the group \( G \) acts on the central fiber as a subgroup of \( \text{PSL}(2, \mathbb{C}) \), the identity component of the automorphism group of \( X_{\Pi_0} \). Therefore, \( G \) acts trivially on the cohomology of every fiber \( M_t \) of the deformation \( M \rightarrow \mathbb{C} \).

4. Deformations of extremal metrics

In this section we prove a criterion that ensures the stability of the extremal condition of a Kähler metric under complex deformations. We assume some symmetries of the family of deformations and the nondegeneracy of the relative Futaki invariant.

Let \( M \rightarrow B \) be a smooth family of complex deformations of a complex manifold \( X \simeq M_0 \). If we assume that \( X \) is of Kähler type, then it follows that all fibers \( M_t \) are Kähler provided we shrink the set of parameters \( t \in B \) to a sufficiently small neighborhood of the origin \([12]\). In particular, the Lie algebra of holomorphic vector fields \( h(M_t) \) contains the ideal \( h_0(M_t) \) consisting of holomorphic vector fields with zeroes somewhere (cf. [2]). Throughout this section, we shall always assume that \( X \) is of Kähler type and that \( B \) has been so restricted. We shall indicate the occasions where the latter restriction may be necessary.

4.1. Holomorphic group actions. Let \( G \) be a compact connected Lie group acting smoothly on \( M \) such that:

- The fibers \( M_t \) are preserved under the action.
- The induced action on each \( M_t \) is holomorphic,
- \( G \) acts faithfully on \( M_0 \), and it is identified to a subgroup of the connected component of the identity of \( \text{Aut}(M_0) \).

Under this conditions, we say that \( G \) acts holomorphically on \( M \) and trivially on \( B \).

As discussed in [3] we may think of the deformation \( M \rightarrow B \) as a smoothly varying family of integrable almost complex structure \( J_t \) on the underlying manifold \( M \) of the central fiber such that \((M, J_t) \simeq M_t\), and using the smooth trivialisation of the deformation near the central fiber, the holomorphic \( G \)-action on \( M \) can be seen as a smoothly varying family of \( G \)-actions

\[
a : \quad B \times G \times M \rightarrow M \\
(t, g, x) \quad \mapsto \quad a_t(g, x)
\]

where \( a_t \) is a \( J_t \)-holomorphic action that is identified to the \( G \)-action on \( M_t \) modulo the isomorphism \((M, J_t) \simeq M_t\).

By [15], we know that compact Lie group actions are rigid up to conjugation. Hence, there exists a smooth isotopy \( f_t : M \rightarrow M \) that intertwines the \( a_t \) and \( a_0 \) actions,

\[
f_t^{-1} \circ a_t(g, f_t(x)) = a_0(g, x) \quad \text{for all } t \in B, \ g \in G, \ x \in M,
\]
possibly after restricting \( B \) to some smaller neighborhood of the origin. So acting by \( f_t \) on the family of complex structures \( J_t \), we may assume that the action of \( G \) is independent of \( t \), and that it is holomorphic relative to each \( J_t \).

The triviality of the smooth deformation of the action of \( G \) on \( M \) has some strong consequences on the complex geometry. Firstly, there is a canonical morphism of the Lie algebra \( \mathfrak{g} \) of the group \( G \) into the space of smooth vector fields on \( M \),

\[
\xi : \mathfrak{g} \hookrightarrow C^\infty(M, TM).
\]

a map that is injective because the action of \( G \) is assumed to be faithful on the central fiber. We may therefore think of \( \mathfrak{g} \) as a subalgebra of \( C^\infty(M) \). We will do so, and drop \( \xi \) from the notation when no confusion can arise. Notice also that since the \( G \)-action is \( J_t \) holomorphic, \( \mathfrak{g} \) is a subalgebra of \( \mathfrak{h}(M, J_t) \) for all \( t \in B \).

We consider the ideal \( \mathfrak{h}_0(M, J_t) \) of Hamiltonian holomorphic vector fields in \( \mathfrak{h}(M, J_t) \), the space of holomorphic vector fields with a nontrivial zero set \([13]\). Then the ideal of \( \mathfrak{g} \) given by \( \mathfrak{g}_0 = \mathfrak{g} \cap \mathfrak{h}_0(M, J_t) \) consists of the vector fields in \( \mathfrak{g} \subset C^\infty(M, TM) \) that vanish somewhere, and since this properties is independent of the complex structure \( J_t \), \( \mathfrak{g}_0 \) is independent of \( t \).

We summarize the discussion above into the following proposition:

**Proposition 4.1.1.** Let \( \mathcal{M} \to B \) be a family of complex deformations of a manifold \( \mathcal{X} \) of Kähler type. Let \( G \) be a compact connected Lie group acting holomorphically on \( \mathcal{M} \) and trivially on \( B \), and let \( \mathfrak{g} \) be its Lie algebra. If \( B \) is restricted to a sufficiently small neighborhood of the origin, there exists a smooth trivialization \( \mathcal{M}_t \simeq (M, J_t) \) such that \( \mathcal{M}_t \) is Kähler for all \( t \in B \), the action of \( G \) on \( \mathcal{M}_t \) is independent of \( t \), the image of the natural embedding \( \mathfrak{g} \subset C^\infty(M, TM) \) is contained in \( \mathfrak{h}(M, J_t) \) for all \( t \in B \), and \( \mathfrak{g}_0 = \mathfrak{g} \cap \mathfrak{h}_0(M, J_t) \) is an ideal of \( \mathfrak{g} \) that is independent of \( t \).

**Definition 4.1.2.** Let \( \mathcal{M} \to B \) be a family of complex deformations of a manifold \( \mathcal{X} \) of Kähler type, and let \( G \) be a compact connected Lie group acting holomorphically on \( \mathcal{M} \) and trivially on \( B \). A smooth trivialisation \( \mathcal{M}_t \simeq (M, J_t) \) that satisfies the properties of Proposition 4.1.1 is said to be an adapted trivialization for \( \mathcal{M} \to B \) relative to \( G \).

### 4.2. The equivariant deformation problem.

Let \( (\mathcal{M} \to B, \Theta) \) be a polarized family of complex deformations of a polarized manifold \( (\mathcal{X}, \Omega) \), and let \( G \) be a compact connected Lie group acting holomorphically on \( \mathcal{M} \) and trivially on \( B \). Since \( G \) is connected it acts trivially on the cohomology of the fibers \( \mathcal{M}_t \), and in particular on \( \Theta_t \). Let \( (M, J_t) \simeq \mathcal{M}_t \) be an adapted trivialization for \( \mathcal{M} \to B, \Theta \) relative to \( G \). With some abuse of notation, we denote by \( \Theta_t \) the polarization induced on \( M \) via the isomorphism \( \mathcal{M}_t \simeq (M, J_t) \). Then we have the following:

**Lemma 4.2.1.** Let \( (\mathcal{M} \to B, \Theta) \) be a polarized deformation of \( (\mathcal{X}, \Omega) \) \( \simeq (\mathcal{M}_0, \Theta_0) \) provided with a holomorphic action of a compact connected Lie group \( G \) acting trivially on \( B \). Consider an adapted trivialization \( (M, J_t) \simeq \mathcal{M}_t \), and let \( g_t \) be an extremal metric on \( \mathcal{X} \) that represents the Kähler class \( \Omega \). Then, if \( B \) is restricted to a sufficiently small neighborhood of the origin, there exists a smooth family of \( G \)-invariant Kähler metrics \( g_t \) on \( (M, J_t) \) that represent the Kähler classes \( \Theta_t \), and such that \( g_0 \) is identified to the metric \( g \) by the isomorphism \( \mathcal{X} \simeq \mathcal{M}_0 \) up to conjugation by an element of \( \text{Aut}(\mathcal{X}) \).
Proof. The connected component of the identity in the group of isometries denoted \( \text{Isom}_0(X, g) \) is a maximal compact connected subgroup of \( \text{Aut}(X) \) [5]. Thus, we may assume that \( G \subseteq \text{Isom}_0(X, g) \) up to conjugation by an element of \( \text{Aut}(X) \).

Let \( g_0 \) be the \( G \)-invariant metric on \((M, J_0)\) that corresponds to \( g \) by the isomorphism \( X \simeq (M, J_0) \). By the Kodaira-Spencer theory, we can extend \( g_0 \) to a smooth family of \( \text{Kähler} \) metrics \( g_t \) on \((M, J_t)\) that represent \( \Theta_t \) for \( t \) sufficiently small. We can average these metrics if necessary to make of them \( G \)-invariant. Since \( G \) acts isometrically on \( g_0 \), the averaging process leaves \( g_0 \) unchanged. On the other hand, \( G \) acts trivially on the cohomology, and hence, on \( \Theta_t \). So the averaging of the metric \( g_t \) does not change the \( \text{Kähler} \) class that the metric represents. This finishes the proof. \( \Box \)

**Definition 4.2.2.** Let \((M \to B, \Theta)\) be a polarized family of deformations of \((X, \Omega)\) provided with a holomorphic action of a compact connected Lie group \( G \) acting trivially on \( B \). Assume that \( \Omega \) is represented by an extremal metric \( g \). A smooth family of \( G \)-invariant \( \text{Kähler} \) metrics \( g_t \) satisfying the properties given by Lemma 4.2.1 for an adapted trivialization is said to be an adapted smooth family of \( \text{Kähler} \) metrics.

Let us notice that for the adapted family of metrics \( g_t \) of Lemma 4.2.1, the metric \( g_0 \) coincides with the metric \( g \) on \( X \) up to the conjugation by an automorphism. Throughout the rest of the paper, we shall assume that the isomorphism \( X \simeq M_0 \) has been so conjugated so that \( g_0 \) and \( g \) coincide.

In our considerations below, the group \( G' \) of §2.2 will always be the connected component of the identity in the isometry group \( \text{Isom}(X, g) \) of \( g \).

### 4.3. Analytical considerations.

Let \((M \to B, \Theta)\) be a polarized deformation of \((X, \Omega)\), and let \( G \) be a compact Lie group acting holomorphically on \( M \to B \) and trivially on \( B \). We assume that \( G \) is contained in \( G' \). Let \( g \) be an extremal metric on \( X \) that represents the class \( \Omega \). We consider an adapted trivialization \( M_\epsilon \simeq (M, J_1) \) and adapted family of \( \text{Kähler} \) metrics \( g_t \) on \((M, J_t)\) (cf. 4.1 and 4.2) such that \( g_0 = g \).

Let \( L^2_k(M) \) be the \( k \)th Sobolev space defined by \( g \), and let \( L^2_{k,G} \) be space of elements in \( L^2_k \) that are \( G \)-invariant. The latter can be defined as the Banach space completion of \( C^\infty_0(M) \) in the \( L^2_k \) norm.

We denote by \( \omega_t \) the \( \text{Kähler} \) form of the metric \( g_t \) on \((M, J_t)\) of the adapted family. Let \( H_t(M) \) be the space of \( g_t \)-harmonic real \((1,1)\)-forms on \((M, J_t)\). By Hodge theory, we have that

\[
H_t(M) \simeq H^{1,1}(M, J_t) \cap H^2(M, \mathbb{R})
\]

Since \( G \) is connected, it acts trivially on the cohomology of \( M \). By the uniqueness of the harmonic representative of a close form, it therefore acts trivially on \( H_t \) also. Further, the Kodaira-Spencer theory shows that all the spaces \( H_t(M) \) are isomorphic for \( t \) sufficiently small, and form the fibers of a vector bundle \( H(M) \to B \).

For a given function \( \phi \in L^2_{k+4,G}(M) \) and \( \alpha \in H_t(M) \) sufficiently small, one can define a new \( \text{Kähler} \) metric \( g_{t,\alpha,\phi} \) on \((M, J_t)\) with \( \text{Kähler} \) form

\[
\omega_{t,\alpha,\phi} = \omega_t + \alpha + dd^c \phi.
\]
By definition, the deformed metric \( g_{t,\alpha,\phi} \) is automatically invariant under the \( G \)-action and represents the Kähler class \( \Theta_{t,\alpha} = \Theta_t + [\alpha] \).

The real and complex Lichnerowicz operator of \( g_{t,\alpha,\phi} \) will be denoted \( L_{t,\alpha,\phi} \) and \( L_{t,\alpha,\phi} \) respectively. The space of Killing potentials for the metric \( g_{t,\alpha,\phi} \) is given by \( i \ker L_{t,\alpha,\phi} \).

Since \( G \) acts isometrically on \( g_{t,\alpha,\phi} \), the Lie algebra \( \mathfrak{g}_0 \subset \mathfrak{h}_0(M, J_t) \) consists of Hamiltonian Killing fields for the metric \( g_{t,\alpha,\phi} \) as well. Let \( \mathfrak{H}_{t,\alpha,\phi}^{G_0} \) be the space of holomorphiy potentials corresponding to the Killing fields in \( \mathfrak{g}_0 \) for the metric \( g_{t,\alpha,\phi} \). The space \( \mathfrak{H}_{t,\alpha,\phi}^{G_0} \) consists of purely imaginary functions of the form \( iv \) where \( v \) is a Killing potential for some Killing vector field in \( \mathfrak{g}_0 \) relative to the metric \( g_{t,\alpha,\phi} \).

Using the notations of [2.2.1] we introduce \( \mathfrak{H}_{t,\alpha,\phi}^{G_0} \), as the \( G \)-invariant part of \( \mathfrak{H}_{t,\alpha,\phi}^{G_0} \).

An essential feature of \( \mathfrak{H}_{t,\alpha,\phi}^{G_0} \) is that it is identified to \( \mathbb{R} \oplus \mathfrak{g}_0^G \), where \( \mathfrak{g}_0^G \) is the Lie subalgebra of \( \text{Ad}(G) \)-invariant vector fields in \( \mathfrak{g}_0 \) (or equivalently the \( G \)-invariant one when \( \mathfrak{g}_0 \) is considered as a Lie subalgebra of \( \mathfrak{f}(M) \)).

It follows that the spaces \( \mathfrak{H}_{t,\alpha,\phi}^{G_0} \) have constant dimension and that they are the fibers of a vector bundle \( \mathfrak{H}_{t,\alpha,\phi}^{G_0} \) over a neighborhood of the origin in the total space of the bundle \( L_{k+4,G}^2(M) \oplus \mathbf{H}(M) \to B \).

The \( L^2 \)-norm on \( L_{k',G}^2(M) \) induced by the Riemannian metric \( g_{t,\alpha,\phi} \) allows us to define the orthogonal \( W_{k',t,\alpha,\phi} \) of \( i \mathfrak{H}_{t,\alpha,\phi}^{G_0} \) and an orthogonal direct sum

\[
L_{k',G}^2(M) = i \mathfrak{H}_{t,\alpha,\phi}^{G_0} \oplus W_{k',t,\alpha,\phi}
\]

varying smoothly with \((t,\alpha,\phi)\). This construction provides Banach bundles

\[
W_{k'} \to \mathcal{V}
\]

where \( \mathcal{V} \) is a sufficiently small neighborhood of the origin in the total space of \( L_{k+4,G}^2(M) \oplus \mathbf{H}(M) \to B \). We shall denote by

\[
\pi^W_{t,\alpha,\phi} : L_{k',G}^2(M) \to W_{k',t,\alpha,\phi} \quad \text{and} \quad \pi^G_{t,\alpha,\phi} : L_{k',G}^2(M) \to i \mathfrak{H}_{t,\alpha,\phi}^{G_0}
\]

the canonical projection associated to the above splitting. The reduced scalar curvature \( s_{t,\alpha,\phi}^G \) of \( g_{t,\alpha,\phi} \) is given by \( s_{t,\alpha,\phi}^G = \pi^W_{t,\alpha,\phi}(s_{g_{t,\alpha,\phi}}) = (I - \pi^G_{t,\alpha,\phi})(s_{g_{t,\alpha,\phi}}) \). We are looking for particular extremal metrics near \( g \), namely the one with vanishing reduced scalar curvature

\[
(16) \quad s_{t,\alpha,\phi}^G = 0.
\]

The LHS of (16) can be interpreted as a section of the bundle \( W_k \to \mathcal{V} \). Our goal is to seek the zeroes of this section of Banach bundle. Keeping a more prosaic style we can express equation (16) more concretely by using suitable trivialisations of the relevant bundles as follows: the metric \( g_0 \) induces an \( L^2 \)-norm and an associated orthogonal projection \( P = \pi^W_{0} : L_{k+4,G}^2(M) \to W_{k,0} \). For parameters \((t,\alpha,\phi)\) sufficiently small, the restriction

\[
P : W_{k,t,\alpha,\phi} \to W_{k,0},
\]

is automatically an isomorphism.

The bundle of harmonic real \((1,1)\)-forms \( \mathbf{H}(M) \to B \), admits a trivialization in a neighborhood of the central fiber. Thus we have a smooth isomorphism of vector bundle \( h \), up to the cost of shrinking \( B \) to some smaller neighborhood of the origin,
which commutes with the canonical projections

\[
\begin{array}{c}
B \times H_0(M) \\
\downarrow h \\
B
\end{array}
\xrightarrow{h}
\begin{array}{c}
H(M) \\
\downarrow \\
B
\end{array}
\]

and such that \( h \) restricted to the central fiber is the identity. We shall use the notation \( h(t, \alpha) = h_t(\alpha) \in H_t(M) \).

Let \( \mathcal{U} \) be a sufficiently small open neighborhood of the origin in \( B \times H_0(M) \times W_{k+4,0} \) such that the following map is defined

\[
(17) \quad S : \quad \mathcal{U} \rightarrow B \times W_{k,0} \quad \text{with } \quad (t, \alpha, \phi) \mapsto (t, P(s_G^{G}, h_t(\alpha), \phi))
\]

**Lemma 4.3.1.** The map \( S \) is \( C^1 \) and its differential is a Fredholm operator. Assuming that the Kähler metric \( g \) on \( X \) has vanishing reduced scalar curvature \( s_G^g = 0 \), the differential at \( (t, \alpha, \phi) = 0 \) is given by given by a linear operator of the form

\[
\left( \begin{array}{cc}
1 & 0 \\
* & S_{G,g}
\end{array} \right)
\]

where

\[
S_{G,g}(\dot{\alpha}, \dot{\phi}) = -2L_g \dot{\phi} + P(s_G^g(\dot{\alpha}))
\]

is the differential of \( P(s_G^g) \) at \( g \) in the direction of \( (\dot{\alpha}, \dot{\phi}) \). In the case where \( \dot{\alpha} \) is tracefree, we have

\[
S_{G,g}(\dot{\alpha}, 0) = s_G^g(\dot{\alpha}) = P(G_g((\dot{\alpha}, dd^c s_g)) - 2(\dot{\alpha}, \rho)).
\]

**Proof.** The map \( S \) is \( C^1 \) since the reduced scalar curvature depends in a \( C^1 \) manner of the data \( (t, \alpha, \phi) \). The computation of the differential of \( S \) is deduced from Lemma 2.2.7. \qed

At this point, we may compute the index of the differential of \( S \):

**Lemma 4.3.2.** Under the assumption \( s_G^g = 0 \), the index of the differential of \( S \) at the origin is equal to \( h^{1,1}(X) \).

**Proof.** By Lemma 4.3.1 the operator \( S_{G,g} \) is a compact perturbation of the map

\[
H_0(M) \times W_{k+4,0} \rightarrow W_{k,0} \\
(\alpha, \phi) \mapsto -2L_g \phi
\]

The Lichnerowicz operator \( L_g : W_{k+4,0} \rightarrow W_{k,0} \) has index 0. We conclude that the differential of \( S \) has index \( \dim H_0(M) = h^{1,1}(X) \). \qed

### 4.4. Surjectivity

We return to the study of the map \( S \) with the notations of §4.2.

**Proposition 4.4.1.** Under the additional assumption that \( g \) is an extremal metric on \( X \) with \( s_G^g = 0 \), the map \( S \) defined at (17) is a submersion at the origin if and only the relative Futaki invariant \( \mathcal{F}_{G,\Omega}^g \) is non degenerate at \( \Omega \), the Kähler class of \( g \).
Proof. The cokernel of the differential of $\mathcal{S}$ is identified to $\psi \in W_{k,0}$ such that

$$\langle L_g \phi, \psi \rangle = 0, \quad \langle P(s_g^G(\hat{\alpha})), \psi \rangle = 0$$

for all $\phi \in W_{k+4,0}$ and $\hat{\alpha}$ $g$-harmonic $(1,1)$-forms on $(M, J_0) \simeq X$.

The first equation implies that $L_g \psi = 0$. Therefore we have $\psi \in i\mathcal{H}_g^0 \simeq \mathbb{R} \oplus M_0'$. Let $\Xi \in M_0'$ be the Killing field represented by $\psi$. Using the second condition in the particular case where $\hat{\alpha}$ is tracefree, we see that

$$0 = \langle P(s_g^G(\hat{\alpha})), \psi \rangle = \langle s_g^G(\hat{\alpha}), \psi \rangle = \int_X \psi s_g^G(\hat{\alpha}) d\mu_g = \tilde{\mathcal{S}}_{G, \Xi, \Omega}(\hat{\alpha}),$$

where the last equality is given by Lemma 2.3.3. The relative Futaki nondegeneracy condition implies that $\Xi \in M_0$, in other words $\psi \in i\mathcal{H}_g^0$. By definition $\psi$ is orthogonal to $i\mathcal{H}_g$, hence $\psi = 0$ and $\mathcal{S}$ is a submersion.

Conversely, it is easy to check that if the relative Futaki invariant is degenerate, then $\mathcal{S}$ is not a submersion. \hfill \Box

Remark 4.4.2. Under the Futaki nondegeneracy assumption, we may choose any linear subspace $V \subset H_0(M)$ such that the linearized Futaki invariant induces an injective map $p_0/\mathfrak{g}_0 \rightarrow V^*$. Then the corresponding restriction of $\mathcal{S}$ is still a submersive map at the origin.

In conclusion, we obtain the following theorem:

Theorem 4.4.3. Let $(\mathcal{M} \rightarrow B, \Theta)$ be a polarized family of complex deformations of a polarized manifold $(X, \Omega)$. Assume that $\mathcal{M} \rightarrow B$ is endowed with a holomorphic action of a connected compact Lie group $G$ acting trivially on $B$ and that $X$ admits a $G$-invariant extremal metric $g$ with Kähler class $\Xi$ and such that $s_g^G = 0$.

Given an adapted trivialization $\mathcal{M}_t \simeq (M, J_t)$ defined for $t$ sufficiently small, let $g_t$ be any adapted smooth family of $G$-invariant Kähler metrics on $(M, J_t)$ representing $\Theta_t$ (cf. §4.1 and §4.2).

Assume that the relative Futaki invariant $\tilde{\mathcal{S}}_{G, \Xi, \Omega}$ is non degenerate at $g$, then choose a space $V \subset H_0(M)$ such that the linearized relative Futaki invariant restricted to $p_0/\mathfrak{g}_0 \rightarrow V^*$ is injective. Then, the space of solutions

$$S = \{(t, \alpha, \phi) \in \mathcal{U} \mid \alpha \in V \text{ and } s_{g_{t,0}}^G(\alpha, \phi) = 0\}$$

is a smooth manifold of real dimension $\dim V + \dim B$, in a sufficiently small neighborhood of the origin. For any $(t, \alpha, \phi) \in S$, $\alpha$ and $\phi$ are automatically smooth.

The canonical projection $S \rightarrow B$ is a submersion near the origin and the fibers are $\dim V$-dimensional submanifold of $S$ corresponding to families of $G$-invariant Kähler metrics $g_{t,0,\alpha}$ on $\mathcal{M}_t$ with vanishing reduced scalar curvature representing a perturbation of the polarization given by $\Theta_t + [\alpha]$.

Proof. The hypothesis imply that the map $S$ restricted to $\mathcal{U}' = \{(t, \alpha, \phi) \in \mathcal{U} \mid \alpha \in V\}$ is a submersion at the origin. The kernel $K$ of the differential of $S$ has dimension equal to its index $\dim V$. Let $\pi_K : W_{k+4} \rightarrow K$ be the orthogonal projection onto $K$. By definition, the map

$$\mathcal{U} \xrightarrow{\pi_K \times S} K \times B \times W_{k,0}$$

is an isomorphism. The implicit function theorem provides the desired solutions parametrized by $B \times K$. \hfill \Box
Remark 4.4.4. When \( q/g = 0 \), the Futaki invariant is automatically non degenerate and Theorem 4.4.3 applies for any subspace \( V \subset H_0(M) \). Setting \( V = 0 \), the theorem provides for every \( t \) sufficiently small a unique extremal metric on \( M_t \) with Kähler class \( \Theta_t \). In other words, the original polarization \( \Theta \) does not have to be perturbed in this case.

4.5. Generalization to nonconnected groups. Although to this point the group \( G \) has been assumed to be connected, this assumption is not necessary to a large extent. The connectedness was used to ensure that \( G \) acts trivially on the cohomology of the manifold, hence on harmonic forms. Thus, we merely need the assumption that \( G \) acts trivially on the relevant Kähler classes.

For instance, if \( G \) is contained in the connected component of the identity of \( \text{Aut}(X) \), then \( G \) acts trivially on the cohomology of \( X \). We can check that the definition of the Futaki invariant relative to \( G \) still makes sense. The analysis developed at §4.2 extends trivially in this case by working \( G \)-equivariantly. The only difference in this more general framework will appear in Proposition 4.4.1. For in order to make sure that this proposition still holds, we should change slightly the definition of relative Futaki nondegeneracy. Let us recall that the linearized relative Futaki invariant induces a map \( \mathfrak{z}'_{0,G} / \mathfrak{z}_0 \rightarrow (H^{1,1}(X) \cap H^2(X, \mathbb{R}))^* \). If \( G \) is connected, \( \mathfrak{z}_0 \) agrees with the space of \( G \)-invariant Hamiltonian Killing fields on \( (X, g) \). But if \( G \) is not connected, there is a residual action of \( G \) on \( \mathfrak{z}_0 \) by a finite group action. The space of \( G \)-invariant vector fields in \( \mathfrak{z}_0 \) is denoted \( \mathfrak{z}'_{0,G} \). Similarly we denote by \( \mathfrak{z}_{0,G} \) the \( G \)-invariant part of \( \mathfrak{z}_0 \). There is an embedding \( \mathfrak{z}'_{0,G} / \mathfrak{z}_{0,G} \subset \mathfrak{z}'_0 / \mathfrak{z}_0 \) and we shall say that the Futaki invariant \( \mathfrak{f}_{\Theta,G} \) is nondegenerate at \( g \) if the map

\[
\mathfrak{z}'_{0,G} / \mathfrak{z}_{0,G} \rightarrow (H^{1,1}(X) \cap H^2(X, \mathbb{R}))^*
\]

is injective.

Using this new definition of the Futaki nondegeneracy, we can drop the connectivity assumption on \( G \) and just assume that \( G \subset G' = \text{Isom}_0(X, g) \) in Theorem 4.4.3. We can then derive the same conclusions.

5. Applications

In this section we apply Theorem [A] in some particular situations, and especially to produce new examples of Kähler manifolds with extremal metrics.

5.1. Relation to LeBrun-Simanca deformation theory. It is easy to see that Theorem [A] enables us to recover the deformation theory of [14].

5.1.1. Case \( G = \{1\} \). Let \( g \) be a Kähler metric on \( X \) with Kähler class \( \Omega \). Then, we have \( s^G_g = s_g - \bar{s}_g \), where \( \bar{s}_g \) is the average of \( s_g \) on \( X \). So the condition \( s^G_g = 0 \) is equivalent to the property that \( g \) has constant scalar curvature. Furthermore the Futaki invariant relative to \( G = \{1\} \) agrees with the non-relative Futaki invariant. In this case, Theorem [A] is equivalent to the deformation theory for cscK metrics of [14].

5.1.2. Case \( G = G' \). If \( g \) is extremal, then \( G' = \text{Isom}_0(X, g) \) is a maximal connected compact subgroup of \( \text{Aut}(X) \). Then every extremal metric is \( G \)-invariant, up to conjugation, and the condition \( s^G_g = 0 \) for a \( G \)-invariant Kähler metric is equivalent to the metric being extremal.
Theorem A applies. In the case of trivial complex deformations with \( B = \{0\} \), one recovers the openness theorem of [13]. We actually seem to have a more general result for it is possible to allow complex deformations with a holomorphic action of \( G \) acting trivially on \( B \).

5.2. Deformations with maximal torus symmetry. The deformation theory under a maximal compact torus symmetry is particularly well behaved. Let \( X \) be complex manifold and \( g \) an extremal metric on \( X \) with Kähler class \( \Omega \). We shall denote by \( G = T^m \subset \text{Aut}(X) \) a maximal compact torus. Up to conjugation by an automorphism, we may assume that the metric \( g \) is \( G \)-invariant so that \( G \subset G' = \text{Isomo}(X,g) \). The vector field \( \Xi = J \grad s_g \) is in \( z' \subset z'' \). But in this case \( z'' = z \) by maximality of \( z \) (cf. proof of Lemma 2.3.7). Hence \( \Xi \in z \subset g \), which implies \( s^G = 0 \). The Futaki invariant is trivial, hence nondegenerate (cf. Lemmas 2.3.7). So we may apply Theorem A to polarized deformations \( (M \rightarrow B, \Theta) \) of \( (X, \Omega) \) endowed with a holomorphic action of \( G \) acting trivially on \( B \). Thus we have proved Corollary B.

5.3. The Mukai-Umemura 3-fold. Let \( X_{\Pi_0} \) be the Mukai-Umemura 3-fold of §3.3. We use the deformation \( M \rightarrow \mathbb{C} \) of \( X_{\Pi_0} \) that we described there, which is provided with an holomorphic action of a group \( G \) isomorphic to the dihedral group of order 8 or the semi-direct product of \( \mathbb{Z}/2 \) ⋊ \( S_1 \).

Proofs of Corollaries C. We use the fact that \( X_{\Pi_0} \) admits a Kähler-Einstein metric [7]. Notice that although the group \( G \) is disconnected, it acts trivially on the cohomology of \( M_t \) (cf. §4.5). So it suffices to show that the relative Futaki invariant of \( X_{\Pi_0} \) at the Kähler-Einstein metric is nondegenerate in the sense §4.5. This follows trivially from the fact that the space of \( G \)-invariant holomorphic vector fields on \( X_{\Pi_0} \) is reduced to 0 since the action of \( G \) does not fix any point in \( \mathbb{P}^1 \). Thus \( z_{0,G} = 0 \) and the Futaki invariant is nondegenerate necessarily nondegenerate. We may apply Theorem 4.4.3 with the choice of space \( V = 0 \). Thus every class \( \Theta_t \) is represented by a \( G \)-invariant extremal Kähler metric \( g_t \) for \( t \) sufficiently small. Thus the holomorphic vector field \( \grad s_{g_t} \) must be \( G \)-invariant. As the action of \( G \) does not fix any point in \( \mathbb{P}^1 \), it follows that there are no nontrivial \( G \)-invariant holomorphic vector fields on \( M_t \). Thus \( s_{g_t} \) is constant and this finishes the proof of Corollary C.

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