On infinite-dimensional representations of the rotation group and Dirac monopole

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ABSTRACT: The Dirac monopole problem is studied in details within the framework of infinite-dimensional representations of the rotation group, and a consistent pointlike monopole theory with an arbitrary magnetic charge is deduced.

KEYWORDS: monopole, nonassociativity, nonunitary representations, infinite-dimensional representations, indefinite metric Hilbert space
1. Introduction

In 1931 Dirac [1] showed that a proper description of the quantum mechanics of a charged particle of the charge $e$ in the field of the magnetic monopole of the charge $q$ requires the quantization condition $2\mu = n, n \in \mathbb{Z}$ (we set $eq = \mu$ and $\hbar = c = 1$). Well-known group theoretical, topological and geometrical arguments in behalf of Dirac quantization rule are based on employing classical fibre bundle theory or finite dimensional representations of the rotation group [1, 2, 3, 4, 5, 6, 7, 8, 9, 10].

For instance, a realization of the Dirac monopole as U(1) bundle over $S^2$ implies that there exists the division of space into overlapping regions $\{U_i\}$ with nonsingular vector potential being defined in $\{U_i\}$, and yields the correct monopole magnetic field in each region. On the triple overlap $U_i \cap U_j \cap U_k$ it holds

$$\exp(i(q_{ij} + q_{jk} + q_{ki})) = \exp(i4\pi\mu) \quad (1.1)$$

where $q_{ij}$ are the transition functions, and the consistency condition, which is equivalent to the associativity of the group multiplication, requires $q_{ij} + q_{jk} + q_{ki} = 0 \mod 2\pi\mathbb{Z}$. This yields the Dirac selectional rule $2\mu \in \mathbb{Z}$ as a necessary condition to have a consistent U(1)-bundle over $S^2$ [3, 4, 5].

In the presence of the magnetic monopole the operator of the total angular momentum $J$, which includes contribution of the electromagnetic field, obeys the standard commutation relations of the Lie algebra of the rotation group

$$[J_i, J_j] = i\epsilon_{ijk}J_k,$$

and this is true for any value of $\mu$. However, the requirement that $J_i$ generate a finite-dimensional representation of the rotation group yields $2\mu$ being integer and only values $2\mu = 0, \pm 1, \pm 2, \ldots$ are allowed [11, 12, 13, 14, 15, 16].
Thus to avoid the Dirac restrictions on the magnetic charge one needs to consider a nonassociative generalization of U(1) bundle over $S^2$ and give up finite-dimensional representations of the rotation group. Recently we have done the first steps in this direction, developing a consistent pointlike monopole theory with an arbitrary magnetic charge [17, 18, 19]. Here we study in details the Dirac monopole problem within the framework of infinite-dimensional representations of the rotation group.

The paper is organized as follows. In Section II the indefinite metric Hilbert space is introduced. In Section III the properties of infinite-dimensional representations of the rotation group are discussed. In Section IV it is argued that expanding the representations of the rotation group to infinite-dimensional representations allow an arbitrary magnetic charge. In Section V the obtained results and open problems are discussed.

2. Indefinite metric Hilbert space.

Starting from the early 1940s indefinite metric in the Hilbert space has been discussed and used by many authors. Recently a growing interest to this topic has been raised in the context of the so-called PT-symmetric quantum mechanics related with some non-Hermitian Hamiltonians with a real spectrum and pseudo-Hermitian operators [20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30].

Since in conventional quantum mechanics the norm of quantum states given by

$$\int \bar{\psi} \psi dx > 0$$  \hspace{1cm} (2.1)

where $\bar{\psi}$ is the conjugate complex of $\psi$, carries a probabilistic interpretation, the appearance of an indefinite metric in Hilbert space is a severe obstacle. It leads in particular to negative probability of states, that means observables with only positive eigenvalues can get negative expectation values [31].

We treat here the more general situation when the normalization given by

$$\int \bar{\psi} \psi d\mu(x),$$  \hspace{1cm} (2.2)

d$\mu$ being a suitable measure, is not necessary positive. We assume that the integral

$$\int \bar{\psi} \psi' d\mu(x),$$  \hspace{1cm} (2.3)

may be divergent and its value is given by a regularization (for the definition of regularization of an integral see, e.g. [24]).

Following the notations by Pauli [31] we consider an inner product in the indefinite metric Hilbert space $\mathcal{H}_\eta$ defined by the bilinear form of the type

$$(\bar{\psi}, \psi')_\eta = \int \bar{\eta} \psi' d\mu(x),$$  \hspace{1cm} (2.4)

There exist several possibilities of regularizing divergent integral, further (see Sect. 3) we consider the regularization of the integral by analytical continuation in parameter.
in which the operator $\eta$ is only restricted by the condition that it has to be Hermitian and
\[ \int \bar{\psi} \eta \psi d\mu(x) > 0. \tag{2.5} \]

The difference between our construction of the indefinite Hilbert space and suggested in \[31\] arises from the restrictions (2.2) and (2.5). While Pauli requires the positive defined norm (2.2), we don't.

Let functions $\psi_m(x)$ form the basis such that
\[ \int \bar{\psi}_m(x) \psi_{m'}(x) d\mu(x) = \eta_{mm'} \]
where $\eta_{mm'} = (-1)^{\sigma(m)} \delta_{mm'}$ is an indefinite diagonal metric $(-1)^{\sigma(m)} = \pm 1$ depending whether $\sigma(m)$ is even or odd. Defining the action of the operator $\eta$ on $\psi_m$ as
\[ \eta \psi_m = \eta_{mm'} \psi_{m'} \tag{2.6} \]
we find that the set $\{\psi_m\}$ forms the orthonormal basis with respect to the inner product given by
\[ (\psi, \psi_p)_{\eta} = (\psi, \eta \psi_p) = \int \bar{\psi}_m(x) \eta_{pmm'} \psi_{m'}(x) d\mu(x) = \delta_{mp}. \tag{2.7} \]

Since the set $\{\psi_m(x)\}$ forms a basis, an arbitrary function $\psi(x) \in \mathcal{H}$ can be expanded in terms of the $\psi_m(x)$:
\[ \psi(x) = \sum_m c_m^\eta \psi_m, \tag{2.8} \]
where
\[ c_m^\eta = (\psi_m, \psi)_\eta = \eta_{mm'} \int \bar{\psi}_m(x) \psi_{m'}(x) d\mu(x) \tag{2.9} \]

Let $\psi'(x) \in \mathcal{H}$, which can be expanded as follows:
\[ \psi'(x) = \sum_m c_m^{\eta'} \psi_m, \tag{2.10} \]
then the inner product $(\psi, \psi')_\eta$ can be easy calculated that is
\[ (\psi, \psi')_\eta = \int \bar{\psi}(x) \eta \psi'(x) d\mu(x) = \sum_m \bar{c}_m^{\eta'} c_m^\eta. \tag{2.11} \]
In particular:
\[ (\psi, \psi)_\eta = \int \bar{\psi}(x) \eta \psi(x) d\mu(x) = \sum_m |c_m^\eta|^2 > 0. \]

Thus we see that the inner product in the indefinite metric Hilbert space is positive defined scalar product. This provides the standard probabilistic interpretation of the quantum mechanics.

The inner product (2.11) may be written in another form. Let us consider the sum
\[ K(x, x') = \sum_m \psi_m(x) \bar{\psi}_m(x'). \tag{2.12} \]
This yields the following relations:

\begin{align}
\int \psi_m(x')K(x,x')d\mu(x') &= \eta_{mm'}\psi_m(x), \quad (2.13) \\
\eta\psi'(x) &= \int \psi'(x')K(x,x')d\mu(x'), \quad (2.14)
\end{align}

and it is seen that the kernel \( K(x,x') \) plays here a role similar to that of \( \delta \)-function in the standard Hilbert space of quantum mechanics. Now one can rewrite the inner product (2.11) as

\[ (\psi,\psi')_\eta = \int \int \bar{\psi}(x)K(x,x')\psi'(x')d\mu(x)d\mu(x'). \quad (2.15) \]

The expectation value of the observable \( A \) represented by the linear operator acting in \( \mathcal{H} \) is defined by

\[ \langle A \rangle_\eta = \int \bar{\psi}(x)\eta A\psi(x)d\mu(x), \quad (2.16) \]

and the generalization of the Hermitian conjugate operator, being denoted as \( A^\dagger_\eta \), is given by

\[ A^\dagger_\eta = \eta^{-1}A^\dagger \eta \quad (2.17) \]

where \( A^\dagger \) is the Hermitian conjugate operator.

Since the observables are real, we see that for them the related operators have to be self-adjoint in the indefinite metric Hilbert space, that means \( A^\dagger_\eta = A \). In particular, applying this to the Hamiltonian operator \( H \), we have \( H^\dagger_\eta = H \), and assuming that the wave function satisfies the Schrödinger’s equation

\[ i\frac{\partial \psi}{\partial t} = H\psi \]

we obtain

\[ \frac{d}{dt}(\psi|\psi)_\eta = i\bar{\psi}\eta(H^\dagger_\eta - H)\psi = 0, \quad (2.18) \]

that is, the conservation of the wave function normalization.

If we perform a linear transformation

\[ \psi' = S\psi \]

then in order to conserve the normalization of the wave function

\[ (\psi',\psi')_\eta = (\psi,\psi)_\eta \]

we have to demand

\[ \eta' = S^\dagger \eta S. \quad (2.19) \]

In similar manner we find that the observables are invariant,

\[ \langle A' \rangle_\eta = \langle A \rangle_\eta, \quad (2.20) \]
if the operators transform as follows:

$$A' = S^{-1} A S, \quad A^{\dagger}_\eta' = S^{-1} A^{\dagger}_\eta S. \quad (2.21)$$

Assuming that, according to (2.21), the matrix $A$ can be transformed with a suitable $S$ to a normal form such that

$$A\psi_n = a_n \psi_n \quad (2.22)$$

we find

$$(\psi, A\psi)_\eta = \sum_n a_n |c^\eta_n|^2. \quad (2.23)$$

This leads to the conclusion that operator with only positive eigenvalues can not have negative expectation values. In other words, the negative probabilities do not appear in our approach, and the standard interpretation of the wave function normalization is preserved.

### 3. Infinite-dimensional representations of the rotation group

The three dimensional rotation group is locally isomorphic to the group SU(2), and as well known $\text{SO}(3) = \text{SU}(2)/\mathbb{Z}_2$. In what follows the difference between $\text{SO}(3)$ and $\text{SU}(2)$ is not essential and actually we will consider $G=\text{SU}(2)$. The Lie algebra corresponding to the Lie group $\text{SU}(2)$ has three generators and we adopt the basis $J_\pm = J_1 \pm i J_2, \; J_3$. The commutation relations are:

$$\begin{align*}
[J_+, J_-] &= 2J_3, \\
[J_3, J_\pm] &= \pm J_\pm, \\
[J^2, J_\pm] &= 0, \\
[J^2, J_3] &= 0
\end{align*} \quad (3.1)$$

where

$$J^2 = J_3^2 + \frac{1}{2} (J_- J_+ + J_+ J_-) \quad (3.2)$$

is the Casimir operator.

Let $\psi^\lambda_\nu$ be an eigenvector of the operators $J_3$ and $J^2$:

$$\begin{align*}
J_3 \psi^\lambda_\nu &= (\nu_0 + n) \psi^\lambda_\nu, \\
J^2 \psi^\lambda_\nu &= \lambda(\lambda + 1) \psi^\lambda_\nu
\end{align*} \quad (3.3)$$

where $n = 0, \pm 1, \pm 2, \ldots$, and $\nu_0$, just like $\lambda$, is a certain complex number.

There are four distinct classes of representations and each irreducible representation is characterized by an eigenvalue of Casimir operator and the spectrum of the operator $J_3$:

$$33, 34, 35, 36, 37, 38$$

- **Representations unbounded from above and below**, in this case neither $\lambda + \nu_0$ nor $\lambda - \nu_0$ can be integers.

- **Representations bounded below**, with $\lambda + \nu_0$ being an integer, and $\lambda - \nu_0$ not equal to an integer.

- **Representations bounded above**, with $\lambda - \nu_0$ being an integer, and $\lambda + \nu_0$ not equal to an integer.
Representations bounded from above and below, with $\lambda - \nu_0$ and $\lambda + \nu_0$ both being integers, that yields $\lambda = k/2$, $k \in \mathbb{Z}_+$. 

The nonequivalent representations in the each series of irreducible representations are denoted respectively by $D(\lambda, \nu_0)$, $D^+(\lambda)$, $D^- (\lambda)$ and $D(\lambda)$. The representations $D(\lambda, \nu_0)$, $D^+(\lambda)$ and $D^- (\lambda)$ are infinite-dimensional; $D(\lambda)$ is $(2\lambda + 1)$-dimensional representation. 

The irreducible representations $D^+(\lambda)$ and $D(\lambda, \nu_0)$ are discussed in details in [33, 34, 35, 36, 37]. Further we restrict ourselves by the real eigenvalues of the Casimir operator and $J_3$ denoting their by $\ell$ and $m$: $\lambda \to \ell$ and $\nu \to m$.

Representations unbounded from above and below

Let $\psi_m$ be non normalized eigenstates of the operators $J_3$ and $J^2$:

$$J_3 \psi_m = m \psi_m, \quad J^2 \psi_m = \ell (\ell + 1) \psi_m.$$ 

Demand that the commutation relations (3.1) satisfied, yields

$$J_- \psi_m = (\ell + m) \psi_{m-1}, \quad \text{(3.4)}$$
$$J_+ \psi_m = (\ell - m) \psi_{m+1}, \quad \text{(3.5)}$$

Considering the invariance of an inner product $(\psi_m, \psi_{m'})$ with respect to infinitesimal rotations generated by $J_i$ we obtain

$$(\psi_m, (J_+ - J_-) \psi_{m'}) + (\psi_m (J_+ - J_-), \psi_{m'}) = 0,$$

$$(\psi_m, \psi_{m'}) = 0, \quad m \neq m'. \quad \text{(3.7)}$$

Putting $m' = m + 1$ we find

$$(\psi_m, J_- \psi_{m+1}) - (\psi_m J_+ \psi_{m+1}) = 0, \quad \text{(3.8)}$$

that yields the following restriction on the inner product:

$$(\ell + m + 1)(\psi_m, \psi_m) - (\ell - m)(\psi_{m+1}, \psi_{m+1}) = 0. \quad \text{(3.9)}$$

This recursion relationship can be satisfied by writing

$$(\psi_m, \psi_m) = N \Gamma(\ell + m + 1) \Gamma(\ell - m + 1)$$

where $N$ is an arbitrary positive constant, $\Gamma$ is the gamma function, and for $\ell \pm m + 1 < 0$ the value of r.h.s. is given by analytical continuation of the gamma function.

Introducing

$$N_m = (N \mid \Gamma(\ell + m + 1) \Gamma(\ell - m + 1))^{-\frac{1}{2}},$$

we obtain

$$N^2_m (\psi_m, \psi_m) = (-1)^{\sigma(m)}, \quad \text{(3.12)}$$

where

$$(-1)^{\sigma(m)} = \text{sgn}(\Gamma(\ell - m + 1) \Gamma(\ell + m + 1)).$$
$\text{sgn}(x)$ being the signum function.

It follows from Eq. (1.12) that the states $|\ell, m\rangle = N_m \psi_m$ form the orthonormal basis under the inner product given by

$$
\langle m, \ell | \ell, m' \rangle^\eta = N_m^2 \eta_{mm'} \langle \psi_m, \psi_{m'} \rangle = \delta_{mm'},
$$

with the indefinite metric being $\eta_{mm'} = (-1)^{\sigma(m)} \delta_{mm'}$.

We found that the operators $J_\pm$ act on the states $|\ell, n\rangle$ as follows:

- for $-\ell < m < \ell$
  $$
  J_+ |\ell, m\rangle = \sqrt{\ell(\ell + 1) - m(m + 1)} |\ell, m + 1\rangle
  \tag{3.14}
  $$
  $$
  J_- |\ell, m\rangle = \sqrt{\ell(\ell + 1) - m(m - 1)} |\ell, m - 1\rangle
  \tag{3.15}
  $$

- for $|m| > \ell$
  $$
  J_+ |\ell, m\rangle = \sqrt{m(m + 1) - \ell(\ell + 1)} |\ell, m + 1\rangle
  \tag{3.16}
  $$
  $$
  J_- |\ell, m\rangle = -\sqrt{m(m - 1) - \ell(\ell + 1)} |\ell, m - 1\rangle
  \tag{3.17}
  $$

and for the matrix elements one has

$$
(J_+)^{m'm}_{m} = \begin{cases} 
  (J_-)^{m'}_{m}, & \text{if } -\ell < m < \ell \\
  -(J_-)^{m'}_{m}, & \text{if } |m| > \ell
\end{cases}
$$

where ‘bar’ denotes complex conjugation.

One can start with an arbitrary vector $|\ell, m\rangle$ and $\mu$ being an arbitrary number with the fixed value within the given irreducible representation, and apply the operators $J_\pm$ to obtain any state $|\ell, m'\rangle$. Since the eigenvalues of $J_3$ can be changed only by multiples of unity, one has $m = \mu + p$, $p \in \mathbb{Z}$. Thus each irreducible representation $D(\ell, \mu)$ may be characterized by the given values of two invariants $\ell$ and $\mu$. In fact the representations $D(\ell, \mu)$ and $D(-\ell - 1, \mu)$, yielding the same value $Q = \ell(\ell + 1)$ of the Casimir operator, are equivalent and the inequivalent representations may be labeled as $D(Q, \mu)$ [38].

If there exists the number $p_0$ such that $\mu + p_0 = \ell$, we have $J_+ |\ell, \ell\rangle = 0$ and the representation becomes bounded above. In the similar manner if for a number $p_1$ one has $\mu + p_1 = -\ell$, then $J_- |\ell, -\ell\rangle = 0$ and the representation reduces to the bounded below. Finally, finite-dimensional unitary representation arises when there exist possibility of finding $J_+ |\ell, \ell\rangle = 0$ and $J_- |\ell, -\ell\rangle = 0$. It is easy to see that in this case $2\ell$, $2m$ and $2\mu$ all must be integers.

**Representations bounded above**

It is convenient to set $m = \ell - n$ and consider the orthonormal states $|\ell, n\rangle$ instead of $|\ell, m\rangle$. The vectors $|\ell, n\rangle$ form a basis in the space of the representation $D^+(\ell)$, where the operator $J_3$ acts as follows:

$$
J_3 |\ell, n\rangle = (\ell - n) |\ell, n\rangle, \quad n = 0, 1, \ldots, \infty.
$$

\(3.18\)
This representation is characterized by the eigenvalue $\ell$ of the highest-weight state: $|\ell,0\rangle = 0$ and $J_3|\ell,0\rangle = \ell|\ell,0\rangle$. The action of the operators $\{J_{\pm}\}$ on the states is given by

\[
\begin{align*}
J_+|\ell,n\rangle &= \sqrt{n(2\ell - n + 1)}|\ell,n-1\rangle, \\
J_-|\ell,n\rangle &= \sqrt{(n+1)(2\ell - n)}|\ell,n+1\rangle.
\end{align*}
\]

$0 \leq n < 2\ell$

\[
\begin{align*}
J_+|\ell,n\rangle &= \sqrt{n(2\ell - n - 1)}|\ell,n-1\rangle, \\
J_-|\ell,n\rangle &= -\sqrt{(n+1)(2\ell - n)}|\ell,n+1\rangle.
\end{align*}
\]

$n > 2\ell$

We consider a suitable realization of the representation $D^+ (\ell)$ in the space of entire analytical functions $F_\ell = \{f(z) : z \in \mathbb{C}\}$. In this realization the generators $J_\pm$ and $J_3$ act as the first order differential operators:

\[
J_- = -z^2 \partial_z + 2\ell z, \quad J_+ = \partial_z, \quad J_3 = -z \partial_z + \ell,
\] (3.19)

The monomials

\[
\langle z|\ell,n\rangle = N_n z^n,
\]

where $N_n = (\Gamma(n+1)|\Gamma(2\ell - n + 1)|)^{-1/2}$ is the normalization constant, form an orthogonal basis for holomorphic functions analytical in $\mathbb{C}$, and satisfy

\[
(z^n, z^{n'}) = \frac{\Gamma(2\ell + 2)}{2\pi i} \int_D \frac{\bar{z}^n z^{n'} d\bar{z} dz}{(1 + |z|^2)^{\ell+2}} = \Gamma(n+1)\Gamma(2\ell - n + 1)\delta_{np}.
\] (3.20)

For $n > 2\ell$ the value of r.h.s. is given by the analytical continuation of the gamma function [35].

It follows from Eq.(3.20) that the states $|\ell,n\rangle$ form the orthonormal basis under the indefinite metric inner product defined as follows:

\[
\langle n,\ell|\ell,p\rangle = \eta_{pp'}(\langle z|\ell,n\rangle, \langle z|\ell,p'\rangle) = \delta_{np},
\] (3.21)

where $\eta_{np} = (-1)^{\sigma(n)}\delta_{np}$ and

\[
(-1)^{\sigma(n)} = \begin{cases} 
1, & \text{if } 2\ell - n > 0 \\
(-1)^{n+1} \text{sgn} (\sin 2\pi\ell), & \text{if } n - 2\ell > 0
\end{cases}
\]

An arbitrary state of the representation is an entire function of the type

\[
f(z) = \sum_{n=0}^{\infty} f_n \langle z|\ell,n\rangle.
\] (3.22)

The inner product of two entire functions $f(z)$ and $g(z)$ is constructed as follows:

\[
\langle f|g \rangle = \frac{\Gamma(2\ell + 2)}{2\pi i} \int_D \frac{\bar{f} \eta g d\bar{z} dz}{(1 + |z|^2)^{\ell+2}},
\] (3.23)

where the action of the operator $\eta$ is given by

\[
\eta|\ell,n\rangle = \eta_{np}|\ell,p\rangle.
\] (3.24)
Representations bounded below

For the representation bounded below, setting \( m = n - \ell \), we have

\[
J_3|\ell, n\rangle = (n - \ell)|\ell, n\rangle, \quad n = 0, 1, \ldots, \infty.
\]

The action of the operators \( \{J_\pm\} \) on the states is given by

\[
J_-|\ell, n\rangle = \sqrt{n(2\ell - n + 1)}|\ell, n - 1\rangle, \quad 0 \leq n < 2\ell,
\]
\[
J_+|\ell, n\rangle = \sqrt{(n + 1)(2\ell - n)}|\ell, n + 1\rangle, \quad n > 2\ell.
\]

The representation is characterized by the eigenvalue \( \ell \) of the highest-weight state: \(|\ell, 0\rangle\) such that \( J_-|\ell, 0\rangle = 0 \) and \( J_3|\ell, 0\rangle = -\ell|\ell, 0\rangle \).

We consider a realization of the representation \( D^{-}(\ell) \) in the space of analytical functions \( \mathcal{F}^\ell = \{f(z) : z \in \mathbb{C}\} \), such that \( z^{-2\ell}f(z) \) is the meromorphic function. In this realization the generators \( J_\pm \) and \( J_3 \) act as the following differential operators:

\[
J_- = -z^2 \partial_z + 2\ell z, \quad J_+ = \partial_z, \quad J_3 = -z \partial_z + \ell,
\]

The monomials

\[
\langle z|\ell, n\rangle = \mathcal{N}_n z^{2\ell - n},
\]

\( \mathcal{N}_n = (n!\Gamma(2\ell - n + 1))^{-1/2} \) being the same normalization constant as above, form an orthonormal basis such that

\[
\langle n, \ell|p\rangle = \frac{(-1)^{\sigma(n)}\Gamma(2\ell + 2)}{n!\Gamma(2\ell - n + 1)} \frac{1}{2\pi i} \int \frac{z^{2\ell - n}z^{2\ell - p}\tilde{d}\tilde{z}dz}{(1 + |z|^2)^{2\ell + 2}} = \delta_{np}
\]

where

\[
(-1)^{\sigma(n)} = \begin{cases} 
1, & \text{if } 2\ell - n > 0 \\
(-1)^{n+1} \text{sgn}(\sin 2\pi\ell), & \text{if } n - 2\ell > 0
\end{cases}
\]

An arbitrary state of the representation is a function of the type

\[
f(z) = \sum_{n=0}^{\infty} f_n \langle z|\ell, n\rangle. \quad (3.28)
\]

The inner product of the functions \( f(z) \) and \( g(z) \) is constructed as above (see Eq.(3.23)):

\[
\langle f|g\rangle = \frac{\Gamma(2\ell + 2)}{2\pi i} \int \frac{\tilde{f}\eta g\tilde{d}\tilde{z}dz}{(1 + |z|^2)^{2\ell + 2}}. \quad (3.29)
\]
4. Infinite-dimensional representations and Dirac monopole problem

For a non relativistic charged particle in the field of a magnetic monopole the equations of motion

\[ m\ddot{r} = -\frac{\mu}{r^3} \dot{r} \times r \]  

imply that the total angular momentum

\[ J = r \times (p - eA) - \frac{\mu r}{r} \]  

is conserved. The operator of the angular momentum

\[ J = r \times (-i\nabla - eA) - \frac{\mu r}{r}, \]  

having the same properties as a standard angular momentum, obeys the following commutation relations:

\[
\begin{align*}
[H, J^2] &= 0, \\
[H, J_i] &= 0, \\
[J^2, J_i] &= 0, \\
[J_i, J_j] &= i\epsilon_{ijk} J_k
\end{align*}
\]

where \( H \) is the Hamiltonian.

As well known any choice of the vector potential \( A \) being compatible with a magnetic field \( B \) of Dirac monopole must have singularities (the so-called strings), and one can write

\[ B = \text{rot} A_n + h_n \]

where \( h_n \) is the magnetic field of the string.

For instance, Dirac introduced the vector potential as

\[ A_n = q \frac{r \times n}{r(r - n \cdot r)} \]  

where the unit vector \( n \) determines the direction of a string \( S_n \) passing from the origin of coordinates to \( \infty \), and the Schwinger’s choice is

\[ A^{SW} = \frac{1}{2}(A_n + A_{-n}), \]  

with the string being propagated from \( -\infty \) to \( \infty \). Both vector potentials yield the same magnetic monopole field, however the quantization is different. The Dirac condition is \( 2\mu = p \), while the Schwinger one is \( \mu = p, \; p \in \mathbb{Z} \).

These two strings are members of a family \( \{ S^n_\kappa \} \) of weighted strings, which magnetic field is given by

\[ h^n_\kappa = \kappa h_n + (1 - \kappa) h_{-n} \]  

where \( \kappa \) is the weight of a semi-infinite Dirac string. The respective vector potential reads

\[ A^n_\kappa = \kappa A_n + (1 - \kappa) A_{-n}, \]  

and since $A_n^{\kappa} = A_n^{1-\kappa}$, we obtain the following equivalence relation: $S_n^{\kappa} \simeq S_n^{1-\kappa}$.

Two arbitrary strings $S_n^{\kappa}$ and $S_n^{\kappa'}$ are related by the gauge transformation

$$A_n^{\kappa'} = A_n^{\kappa} + d\chi.$$

and vice versa. Then an arbitrary transformation of the strings $S_n^{\kappa} \rightarrow S_n^{\kappa'}$ can be realized as combination $S_n^{\kappa} \rightarrow S_n^{\kappa'}$ and $S_n^{\kappa} \rightarrow S_n^{\kappa'}$, where the first transformation preserving the weight of the string is rotation, and the second one results in changing of the weight string $\kappa \rightarrow \kappa'$ without changing its orientation defined by $n$.

Let denote by $n' = gn, g \in SO(3)$, the left action of the rotation group induced by $S_n^{\kappa} \rightarrow S_n^{\kappa'}$. From rotational symmetry of the theory it follows this gauge transformation can be undone by rotation $r \rightarrow rg$ as follows [4, 5, 17]:

$$A_n^{\kappa'}(r) = A_n^{\kappa}(r') = A_n^{\kappa}(r) + d\alpha((r; g)),$$

$$\alpha(r; g) = e \int_r^{r'} A_n^{\kappa}(\xi) \cdot d\xi, \quad r' = rg$$

where the integration is performed along the geodesic $\tilde{r}r' \subset S^2$.

Now returning to the transformation $S_n^{\kappa} \rightarrow S_n^{\kappa'}$ we obtain

$$A_n^{\kappa'} = A_n^{\kappa} - d\chi_n,$$

$$d\chi_n = 2q(\kappa' - \kappa) \left( \frac{r \times n}{r^2 - (n \cdot r)^2} \right),$$

\(\chi_n\) being polar angle in the plane orthogonal to $n$. It is easy to see that this type of transformations can be undone by combination of the inversion $r \rightarrow -r$ and $\mu \rightarrow -\mu$. In particular, if $\kappa' = 1 - \kappa$ we obtain the mirror string: $S_n^{\kappa} \rightarrow S_{-n}^{\kappa} \simeq S_n^{1-\kappa}$.

Taking into account the spherical symmetry of the system, the vector potential can be considered as living on the two-dimensional sphere of the given radius $r$ and being taken as [3, 4]

$$A_N = q \frac{1 - \cos \theta}{r \sin \theta} \hat{e}_\varphi, \quad A_S = -q \frac{1 + \cos \theta}{r \sin \theta} \hat{e}_\varphi$$

where $(r, \theta, \varphi)$ are the spherical coordinates, and while $A_N$ has singularity on the south pole of the sphere, $A_S$ on the north one. In the overlap of the neighborhoods covering the sphere $S^2$ the potentials $A_N$ and $A_S$ are related by the following gauge transformation:

$$A_S = A_N - 2q d\varphi.$$

This is the particular case of (4.13), (4.13) when $\kappa = 0$ and $\kappa' = 1$.

Choosing the vector potential as $A_N$ we have

$$J_\pm = e^{\pm i\varphi} \left( \pm \frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \varphi} - \mu \frac{\sin \theta}{1 + \cos \theta} \right),$$

$$J_3 = -i \frac{\partial}{\partial \varphi} - \mu,$$

$$J^2 = -\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) - \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} +$$

$$+i \frac{2\mu}{1 + \cos \theta} \frac{\partial}{\partial \varphi} + \mu^2 \frac{1 - \cos \theta}{1 + \cos \theta} + \mu^2$$

(4.18)
where $J_{\pm} = J_x \pm iJ_y$ are the raising and the lowering operators for $J_3 = J_z$.

Writing Schrödinger’s equation

$$\hat{H} \Psi = E \Psi,$$  \hfill (4.19)

in the spherical coordinates, and putting $\Psi = R(r)Y(\theta, \varphi)$ into Eq. (4.19), we get for the angular part the following equation:

$$J^2 Y(\theta, \varphi) = \ell(\ell + 1) Y(\theta, \varphi).$$  \hfill (4.20)

Assuming

$$Y = e^{i(m+\mu)\varphi} z^{p}(1 - z)^{q} F(z),$$

where $z = (1 - \cos \theta)/2$ and $m$ is an eigenvalue of $J_3$, we obtain the resultant equation in the standard form of the hypergeometric equation,

$$z(1 - z) \frac{d^2 F}{dz^2} + (c - (a + b + 1)z) \frac{dF}{dz} - abF = 0$$  \hfill (4.21)

where

$$a = p + q - \ell, \quad b = p + q + \ell + 1, \quad c = 2p + 1,$$  \hfill (4.22)

$$(p + q)(p - q) = m\mu.$$  \hfill (4.23)

As it is known the hypergeometric function $F(a, b; c; z)$ reduces to a polynomial of degree $n$ in $z$ when $a$ or $b$ is equal to $-n$, $(n = 0, 1, 2, \ldots)$, and the respective solution of Eq. (4.21) is of the form \cite{40, 39}

$$F = z^p(1 - z)^q p_n(z)$$  \hfill (4.24)

where $p_n(z)$ is a polynomial in $z$ of degree $n$. Here we are looking for the solutions, like this of the Schrödinger equation (4.20). The requirement of the wave function being single valued force us to take $\alpha = m + \mu$ as an integer and general solution is given by

$$Y^{(\mu,n)}_{\ell} = e^{i\alpha\varphi} Y^{(\delta,\gamma)}_{n}(u),$$  \hfill (4.25)

where $u = \cos \theta$, and

$$Y^{(\delta,\gamma)}_{n}(u) = C_n (1 - u)^{\delta/2} (1 + u)^{\gamma/2} P^{(\delta,\gamma)}_{n}(u),$$  \hfill (4.26)

$P^{(\delta,\gamma)}_{n}(u)$ being the Jacobi polynomials, and the normalization constant $C$ is given by

$$C_n = \left( \frac{2\pi 2^{\delta+\gamma+1} \Gamma(n + \delta + 1) \Gamma(n + \gamma + 1)}{\Gamma(n + 1) \Gamma(n + \delta + \gamma + 1)} \right)^{-1/2} \left( \frac{\Gamma(n + \delta + 1) \Gamma(n + \gamma + 1)}{\Gamma(n + 1) \Gamma(n + \delta + \gamma + 1)} \right)^{1/2}$$

It follows from Eqs. (4.22), (4.23) four different cases (we set $\beta = m - \mu$):

\[
\begin{align*}
Y^{(\mu,n)}_{\ell \pm \mu} &= e^{i\alpha\varphi} Y^{(\alpha,\beta)}_{n}(u) \quad \left\{ \begin{array}{ll}
m = \ell - n, & \ell + \mu \in \mathbb{Z}_+ \\
m = -\ell - n - 1, & \ell - \mu \in \mathbb{Z}_+ 
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
Y^{-(\mu,n)}_{\ell \pm \mu} &= e^{i\alpha\varphi} Y^{(-\alpha,-\beta)}_{n}(u) \quad \left\{ \begin{array}{ll}
m = \ell + n + 1, & \ell + \mu \in \mathbb{Z}_+ \\
m = n - \ell, & \ell - \mu \in \mathbb{Z}_+ 
\end{array} \right.
\end{align*}
\]
The functions \( \{ \pm Y_{\ell \pm \mu}^{(\mu,n)} \} \) form the basis in the indefinite metric Hilbert space for the nonunitary representation \( D^-(\ell \pm \mu, \mu) \) bounded above, and the set of functions \( \{-Y_{\ell \pm \mu}^{(\mu,n)}\} \) for the representation \( D^+(\ell \pm \mu, \mu) \) bounded below. Notice that for \( \mu \) being an arbitrary valued the representations corresponding to \( \ell + \mu \in \mathbb{Z}_+ \) and \( \ell - \mu \in \mathbb{Z}_+ \) are not equivalent.

The wave functions
\[
Y_{\ell}^{(\mu,n)} = \begin{cases} 
+ (\mu,n) & \text{if } \ell = n - \ell - 1, \ell + \mu \in \mathbb{Z}_+ \\
- (\mu,n) & \text{if } \ell = n + \ell + 1, \ell - \mu \in \mathbb{Z}_+ 
\end{cases}
\]
where the upper/lower sign corresponds to the choice of \( (\ell + \mu) \) or \( (\ell - \mu) \), form a complete set of orthonormal solutions with indefinite metric
\[
\eta_{n\mu} = \begin{cases} 
\delta_{n\mu}, \text{ if } 2\ell - n > 0 \\
\delta_{n\mu}(-1)^{n+1} \text{sgn}(\sin 2\pi \ell), \text{ if } n - 2\ell > 0,
\end{cases}
\]
one can set \( m = p \pm \ell \) \((p = 0, \pm 1, \pm 2, \ldots)\), where the upper sign corresponds to the representation defined by \( \ell + \mu \) and the lower one to \( \ell - \mu \).

Similar consideration can be done for the vector potential \( A_5 \). In this case \( \beta = m - \mu \in \mathbb{Z} \) and the corresponding wave functions
\[
\begin{align*}
+ (\mu,n) & \quad Y_{\ell \pm \mu} = e^{i\beta \varphi} Y_n^{(\alpha,\beta)} \quad \begin{cases} m = -\ell - n - 1, \ell + \mu \in \mathbb{Z}_+ \\
m = \ell - n, \ell - \mu \in \mathbb{Z}_+ \end{cases} \\
- (\mu,n) & \quad Y_{\ell \pm \mu} = e^{i\beta \varphi} Y_n^{(-\alpha,-\beta)} \quad \begin{cases} m = n - \ell, \ell + \mu \in \mathbb{Z}_+ \\
m = \ell + n + 1, \ell - \mu \in \mathbb{Z}_+ \end{cases}
\end{align*}
\]
form a complete set of orthonormal basis for the nonunitary representation \( D^\pm(\ell \pm \mu, -\mu) \).

Thus we find the following series of the representations:
\[
\begin{align*}
\ell + \mu \in \mathbb{Z}_+ \Rightarrow & \quad \begin{cases} D^-(\ell + \mu, \mu) : m = \ell - n & \\
D^+(\ell + \mu, \mu) : m = n + \ell + 1 & \\
D^-(\ell + \mu, -\mu) : m = -\ell - n - 1 & \\
D^+(\ell + \mu, -\mu) : m = n - \ell & 
\end{cases} \\
\ell - \mu \in \mathbb{Z}_+ \Rightarrow & \quad \begin{cases} D^-(\ell - \mu, \mu) : m = -\ell - n - 1 & \\
D^+(\ell - \mu, \mu) : m = n - \ell & \\
D^-(\ell - \mu, -\mu) : m = \ell - n & \\
D^+(\ell - \mu, -\mu) : m = n + \ell + 1 & 
\end{cases}
\end{align*}
\]
where \( n = 0, 1, 2, \ldots \). Taking into account the following restriction: \( \ell(\ell + 1) - \mu^2 \geq 0 \), emerging from the Schrödinger equation, the allowed values of \( \ell \) are found to be
\[
\ell + \mu \in \mathbb{Z}_+ \Rightarrow \ell = -\mu + [2\mu] + k, \quad k = 0, 1, 2, \ldots \\
\ell - \mu \in \mathbb{Z}_+ \Rightarrow \ell = \mu + k, \quad k = 0, 1, 2, \ldots
\]
where \([2\mu]\) denotes the integer part of \( 2\mu \).

The function \( Y_{\ell}^{(\mu,n)} \) being a member of the family \( \{Y_{\kappa,\ell}^{(\mu,n)}\} \) of the so-called weighted monopole harmonics such that \( [7] \)
\[
Y_{\kappa,\ell}^{(\mu,n)} = e^{-i2\kappa \mu \varphi} Y_{\ell}^{(\mu,n)}, \quad 2\kappa \mu \in \mathbb{Z}
\] (4.28)
is a solution of the Schrödinger equation corresponding to the choice of the vector potential as

$$A^\kappa = \kappa A_S + (1 - \kappa) A_N,$$

For a given $\mu$ a weight $\kappa$ is quantized parameter in units of $\mu$, and in particular cases $\kappa = 1$ and $\kappa = 1/2$ it yields the Dirac and Schwinger selectional rules respectively.

Since the set of weighted monopole harmonics $\{Y^{(\mu,n)}_{\kappa,\ell}\}$ forms the orthonormal basis in the indefinite metric Hilbert space of the irreducible infinite-dimensional nonunitary representation $D^+ (\ell, \mu) \otimes D^- (\ell, \mu)$, any solution of the Schrödinger’s equation (4.20) can be expanded as

$$\Psi = \sum_{ln} C_{ln} Y^{(\mu,n)}_{\kappa,\ell} \quad (4.29)$$

where $\mu$ is an arbitrary parameter.

When $n + \alpha$, $n + \beta$ and $n + \alpha + \beta$ all are integers $\geq 0$ and $\kappa = 0$ the weighted monopole harmonics are reduced to the monopole harmonics introduced by Wu and Yang [4]. The imposed here restrictions on the values of $n, \alpha$ and $\beta$ yield the finite-dimensional unitary representation of the rotation group and Dirac quantization condition.

5. Discussion and concluding remarks

Involving infinite-dimensional representations of the rotation group, we have deduced a consistent pointlike monopole theory with an arbitrary magnetic charge. It follows from our approach a generalized quantization condition $2\kappa \mu = 0, \pm 1, \pm 2, \ldots$, that can be considered as quantization of the weight string $\kappa$ instead of the monopole charge. In particular cases $\kappa = 1$ and $\kappa = 1/2$ we obtain the Dirac and Schwinger selectional rules respectively.

Using infinite-dimensional representations of the rotation group one has to employ the indefinite metric Hilbert space. What makes difference between our approach and others recently have been developed in the growing number of papers on the subject of $PT$-symmetric quantum mechanics, is absence of “negative probability”. Thus we avoid the problem of the negative probability and preserve the standard probabilistic interpretation of the quantum mechanics.

The other important aspect of the Dirac monopole problem is the gauge-invariant algebra of translations. As is known, the Jacobi identity fails and for the finite translations one has [3, 4]

$$(U_a U_b) U_c = \exp(i\alpha_3 (r; a, b, c)) U_a (U_b U_c) \quad (5.1)$$

where $\alpha_3 = 4\pi \mu \mod 2\pi \mathbb{Z}$, if the monopole is enclosed by the simplex with vertices $(r, r + a, r + a + b, r + a + b + c)$ and zero otherwise [3]. For the Dirac quantization condition being satisfied $\alpha_3 = 0 \mod 2\pi \mathbb{Z}$, and [5.1] provides an associative representation of the translations, in spite of the fact that the Jacobi identity continues to fail. Since a conventional quantum mechanics deals with linear Hilbert space and hence with associative algebra of observables, avoiding of Dirac’s the quantization condition forces us to go beyond the standard quantum mechanical approach and introduce nonassociative algebra of observables [5, 6, 7, 8, 9]. This work is in a progress.
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