Scaling in Local Optimal Paths Cracks

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How local cracks can contribute to the global cracks landscape is a goal of several scientific topics, for example, how bottlenecks can impact the robustness of traffic into a city? In one direction, cracks from cascading failures into networks were generated using a modified Optimal Path-Cracking (OPC) model proposed by Andrade et al [1]. In this model, we broke links of maximum energies from optimal paths between two sites with internal (euclidean) distances \( l \) in networks with linear size \( L \). Each link of this network has an energy value that scales with a power-law that can be controlled using a parameter of the disorder \( \beta \). Using finite-size scaling and the exponents from percolation theory we found that the mass of the cracked links on local optimal paths scales with a power-law \( \beta_0 \) as a separable equation from \( L \) and that can be independent of the disorder parameter.

I. INTRODUCTION

Microcracks can be found in several materials and may diffuse until fracture this material. Modeling the material as a network with several nodes and links and microcracks as nodes or links cracked locally, we can be studying the evolution of the microcracks until fracture the entire network using scaling theory. Here, materials can be a network of a traffic of a city, circuits, an electrical grid, etc. And microcrack can be a local crack as a bottleneck of traffic, a burnt resistor, a burnt transmission line, etc. Some phenomena could be mapped as networks with nodes, links, and properties, that can be studied using scaling theory [2–8].

The link of a network can have a cost, for example, a time travel cost from a start node to an ending node. The set of all links with a time travel cost can be treated as network of time travel of a city. Choosing two points (nodes) of a city (network) to build a path, probably this path is the path with the lowest time travel cost (optimal path). Finding the optimal path between two points in a disordered system is a very important goal of study for science and technology since a long time ago [9–19]. This optimization problem is present in our daily lives when, for example, we make use of the global positioning system (GPS) to trace the best route to arrive at our destination or we choose the optimal path from house to job. The characterization of the optimal path is highly relevant to study fractures [1, 20], polymers in random environments [16], cascading failures [21, 22], transport in porous media [23] and robustness of complex networks [24–27]. When the optimal paths are broken in the network, the robustness is compromised. The robustness of networks has been studied with different algorithm strategies, for example, the geometrical properties of complex networks [28], attacks with the removal of sites [29–31] or the removal of links [32].

This paper aims to study how the process of finding a new optimal path when the current path is broken evolves through time until reaches its final configuration. When optimal paths are broken iteratively under landscapes where energies obey power-laws behavior, namely of disorder, a set of cracks is formed. These cracks grown until the entire network fails. For each type of disorder, the set of cracks obeys a power-law. For strong disorder \( \beta_D > 10^3 \) the fractal dimension of mass of cracks is \( d_f = 1.22 \) and for weak disorder \( \beta_D \simeq 10^{-3} \) the exponent is \( d_f = 2 \) [1, 20]. For strong disorder, this fractal dimension has been reported in several systems such as watershed line [33], perimeter of the percolating cluster at a discontinuous transition [34] and ranked surfaces [35].

In this work, we perform a slightly the optimal path cracking (OPC) model proposed by Andrade et al [1], we define as Local OPC. In this model, the optimal path is determined between two nodes, that belong to the connected component, then the link is broken and a new optimal path is found. This process is repeated until a path cannot be found anymore. In this work we studied the behavior of cracked mass from source node and target node with internal distance \( l \) and its relations with the global scale \( L \). Some studies addressed the critical exponent of the backbone from the giant cluster [36–41] when internal distances are included but there is a lack of information about the behavior on recurrent failures of optimal paths.

This paper is organized as follows. In Sec. II the Local OPC model is presented and its landscape properties. In Sec. III the results obtained from Local OPC is presented for cracked mass and mass of optimal paths and final remarks for Sec. IV.

II. MATERIAL AND METHODS

Our substrate is a square lattice of linear size \( L \), where \( L \) is the number of sites, with periodic boundary conditions from top and left sides. Each link \( i \) has energy value \( \varepsilon_i = e^{\beta_0/(r-1)} \) where \( r \) is a random value between [0, 1] with uniform distribution and \( \beta_D \) is a disorder parameter that has the physical meaning of inverse temperature. The \( \varepsilon \) values obeys a power-law distribution \( P(\varepsilon) \simeq 1/\varepsilon \) subjected to a maximum cutoff given by \( \varepsilon_{\max} = e^{\beta_0} \). The \( \beta_D \) value controls the strength of disorder and broadness of the energy distribution.

The energy of the path is defined as the sum of all energies of its links. The optimal path is shortest path that connects the source site \( (O) \) to target site \( (D) \) with distance \( l \) among all possible paths. We used the Dijkstra algorithm to find the optimal path [42].

The Local OPC can be described as follows. Once the first optimal path between site \( O \) and site \( D \) is determined we
search for the link in the optimal path having the highest energy and its is blocked from landscape. The link cannot longer be part of any path. This is similar to impose an infinite energy to this link. Next, the optimal path is calculated over the remaining accessible links of the landscape, then the link with the highest energy link is again removed and so on until no new paths are available. The blocked links can be viewed as “microcracks” and this process continues recursively until a local failure is formed between site \( O \) and site \( D \) and we can no longer find any path connecting from source site to target site. The cracked links forms a local failure into the system but not necessarily the system is offline. Understanding local failures while maintaining network connectivity is important in understanding many phenomena. This model differs from the standard OPC because it inserts source site and the target site in the landscape.

In Fig. 1 we show the spatial distribution of blocked links in a typical random landscape generated under weak disorder conditions (\( \beta = 0.002 \)) - Fig. 1a - and strong disorder conditions (\( \beta = 700 \)) - Fig. 1b. The broken links from strong disorder condition are contained into domain broken links from weak disorder. As show in Fig. 1b, the amount of no-backbone links reduce significantly then the disorder parameter \( \beta \) increase.

The behavior shown in Fig. 1 is related to the problem of minimum path in disordered landscapes [17, 20]. The passage from weak to strong disorder in the energy distribution reveals a sharp crossover between self-affine and self-similar behaviors of the optimal path. In the strong disorder regime the energy of the minimum path is controlled by the energy of a single site. The parameter \( \beta \) alone determine the limit between weak and strong disorder.

In order to quantify the effect of disorder from local OPC, we perform computer simulations for 5000 realization of square lattice with sizes varying in the range \( 16 \leq L \leq 256 \) and internal distances \( 1 \leq l \leq (L-1) \) with values of weak disorder (\( \beta = 0.002 \)) and strong disorder (\( \beta = 700 \)). The number of sample is 1000 for \( L = 256 \). The generation of random numbers that scales with a power-law distribution results into different landscape of energies. Chose two sites (pair OD) with fixed distance \( l \) we can have different configurations of broken links that suggest not obeying any obvious scale.

III. RESULTS

The cracked mass \( M_{opc} \) shows dependence for a disorder parameter \( \beta \) and the lattice size \( L \) as showed in Fig. 2. The cracked mass grows with disorder parameter decrease [1, 20] Our experiments introduce a third variable that is an internal distance \( l \) between node \( O \) and node \( D \). We see in the Fig 2 the cracked mass enhance smoothly as a function from an internal distance \( l \) until a critical value \( l_c < L \). The critical value \( l_c \) is a boundary effect from a finite-size system. Initially the shape of growing follow an apparent power-law but the grown is quite different from standard OPC and when the internal distance \( l \) is almost the lattice size \( L \) another difference is shown.

The finite size scaling method is applied from cracked mass and is showed in Fig 3. As the site source and node target is contained in the connected component the maximum set of the cracked links not can bigger than the fractal dimension of the standard percolation \( M_{cluster} \sim L^{d_f} \) where \( d_f = 1.89 \) is the fractal dimension of percolating cluster [43–46]. The cracked mass in the thermodynamic limit should be

\[
M_{opc} \sim L^{d_{g}}
\]  

(1)

where \( d_g \) is the global dimension of a percolating cluster that depends from disorder parameter \( \beta \). The size scaling hypothesis is that the cracked mass should be

\[
M_{opc} = L^{d_g}M_0(1/L)
\]  

(2)
Figure 2. Logarithmic dependence on distance $l$ of the mass of all blocked links $M_{opc}$ for $\beta = 700$ in strong disorder conditions (a) and $\beta = 0.002$ in weak disorder conditions (b) for several lattice sizes $L$. The block mass in distance $l$ near lattice size $L$ is reduced because reach boundary conditions.

Figure 3. These results correspond for collapse the cracked mass $M_{opc}$ and internal distance $l$. The collapse of cracked mass occurs when used fractal dimension $d_f = 1.89$ for weak disorder conditions $\beta = 0.002$ (a) and linear dimension $d = 1$ for strong disorder conditions (b). On x-axis the cracked mass collapse occurs when the internal distance is normalized with lattice size $L$. The solid line is a least-square fit to the power-law data excluding the outliers tendency.

where

$$M_0(l/L) \begin{cases} = K & \text{if } l \to L \\ \sim (l/L)^{d_i} & \text{if } l << L \end{cases} \quad (3)$$

and $K$ is a constant value. When $l << L$ we have a new power-law that describe the enhancement of cracked mass

$$M_{opc} \sim L^{d_g-d_i}\ln l \quad (4)$$

where $d_g$ is the global dimension from lattice size $L$ and $d_i$ is an internal dimension from distance $l$. In other cases the internal dimension can be named by co-dimension.

The finite-size scaling of the experiment data is shown in Fig. 3. The cracked mass is enhanced with the internal dimension $d_i = 0.49(1)$ for weak disorder and $d_i = 0.43(1)$ for strong disorder. The data are collapsed when we use the global dimension for weak disorder $d_g = 1.89$ and for strong disorder $d_g = 0.98$. We need to understand the physical meaning the global dimension $d_g$ and its relationship with disorder parameter $\beta$. When $\beta$ is sufficiently low the sampling of the distribution of energy has a significant cutoff near value 1, in other words, the energy variance is shallow and the $\epsilon$ for each link is near value 1. The OPC removes the link with maximum energy for each path found, more precisely, the effect is similar the random cracks for each path in its interactions. In this situation the cracked mass increase as the residual mass from connected components of percolation cluster then the mass scaling obeys $M \sim L^{d_f}$ where $d_f = 1.89$ [37, 43–46] is the fractal dimension of percolation cluster. For other side when $\beta$ is enough strong the sampling of the distribution of energy has full range values $(0, 1]$. The cracked mass collapse for lattice size $L$ as showed in Fig. 3. This behavior is interesting because the percolating cluster is identical into both disorder scenario and the unique
difference is a energy each link. Previous studies showed that the cracked mass in the standard problem using OPC scales with fractal dimension of $d_f = 1.22$ [1, 20, 33–35] and the cracked mass grows linearly with lattice size $L$ with an internal dimension $d_i = 1$ that it is the fractal dimension of the lower backbone [36, 41].

The cracked mass for internal distance $l$ under weak disorder is given by

$$\lim_{\beta \to 0} M_{opc} \sim \begin{cases} L^{1.89} & \text{if } l \to L \\ L^{1.40} & \text{if } l \ll L \end{cases}$$

(5)

and for strong disorder the cracked mass is given by

$$\lim_{\beta \to 1000} M_{opc} \sim \begin{cases} L^{0.98} & \text{if } l \to L \\ L^{0.45} & \text{if } l \ll L \end{cases}$$

(6)

for lattice size $L$.

We studied the shortest path mass using the finite-size scaling. The mass of shortest paths has a similar behavior when the average mass of first paths are collapsed as showed in Fig. 4. We use the equation 4 and the mass of the shortest paths scales $M_{sp} \sim l$ for any lattice size $L$ for weak disorder conditions and the dimension is $d_g = d_i = 1$ that is standard dimension for shortest paths with its energy properties [13–16, 47]. In strong disorder, the average mass of the shortest paths collapses using $d_g = 1.22$ this is the standard fractal dimension [13–15, 17]. The internal dimension found is $d_i = 0.69(1)$ [16, 19] for strong disorder conditions. Thereby the mass of the first path growing to obey the power-law

$$M_{sp} \sim \begin{cases} L^{1.22} & \text{if } l \to L \\ L^{0.53} & \text{if } l \ll L \end{cases}$$

(7)

as previous results from the thermodynamic limit [13–19].

Another set of properties found in the analysis of cracked mass is the sign of cluster after all interactions of the Local OPC. In Fig. 5 we show two extreme scenarios generated from optimal path cracking interactions. The first is completely cracked and the second the landscape is partially cracked but both scenarios do not have a path between the source site and the target site.

A subset network was produced from cracked links and the connected components were determined [48] for both scenarios of energies disorder. The connected components made from cracked links leave a visual sign and the cluster size distribution shows the same inclination that is shown in Fig. 6 for weak disorder conditions. For the network with the same lattice size $L$ the cracked mass for each internal distance $l$ has the same cluster size distributions. In the strong disorder conditions, each site is a cluster thus the distribution is a delta function centered into one that is does not show here.

IV. FINAL REMARKS

In this work we have studied in detail modification of the optimal-path crack (OPC) introduced by Andrade et al. [1] that we call Local Optimal-Path Crack (OPC) for the weak and the strong disorder limits. This modification enables us to find paths between two sites that are contained from the network from an internal distance $l$.

We found interesting exponents for cracked mass links and shortest paths when using a finite-size scaling method. The mass of cracked links obeys a power-laws that scales with the internal distance $l$ and lattice size $L$. Even with values of disorder parameters, the internal dimension $d_i$ has like values such $d_i = 0.49(1)$ for weak disorder conditions and $d_i = 0.43(1)$ for strong disorder conditions. The shortest paths obeys a power-laws that, in the limit $l < L$, the internal dimension $d_i$ has different values $d_i = 1$ for weak disorder conditions and $d_i = 0.69(1)$ for strong disorder conditions.

Concluding, we discover that for all disorders the cracked
Figure 5. The a sample Local Optimal Path Crack (OPC) realization on a 128 X 128 lattice. The blocked links generated under weak disorder conditions (β = 0.002) and strong disorder conditions (β = 7.00) can be categorized in two types, namely, the transversal backbone (red links), the cracked links (gray) and the backbone (black). The backbone is obtained by dual network from transversal backbone. The source site and target site are colored with blue. In (a) and (b) the broken links landscape is shown. This difference is due to the generation of values of the power-law distribution.

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Figure 6. The cluster size distribution $p(s)$ from several samples from lattice size $L = 256$. The cracked links form a subset network that connected components were determined. The system is considered to be in the weak disorder regime for this value and this range of internal distances $2 \leq l \leq 128$. The solid line are the least-squares fits the data of power-laws $p(s) \sim s^{-\tau}$ with $\tau = 2.67(5)$. 

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