One dimensional commutative groups definable in algebraically closed valued fields and in pseudo-local fields

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Abstract

We give a complete list of the one-dimensional groups definable in algebraically closed valued fields and in the pseudo-local fields, up to a finite index subgroup and quotient by a finite subgroup.

1 Introduction

In [1] there is a classification of one dimensional groups definable in the $p$-adic numbers. Here we obtain a similar classification for commutative groups of dimension 1 over other valued fields. Of special importance is the algebraically closed valued fields. This is done in Proposition 56 except in residue characteristic 2 and 3. The only part where this restriction appears is in the study of elliptic curves, it would be good to remove it.

Another interesting field is the one obtained by letting $p$ vary in a $p$-adic field. More precisely if we have $\mathcal{U}$ a non-principal ultra-filter in the prime numbers, then $\Pi_p \mathbb{Q}_p/\mathcal{U}$ is elementary equivalent to $F((t))$ where $F$ is a pseudo-finite field of characteristic 0 (we can also replace $\mathbb{Q}_p$ with any finite extension of it). The classification in this case is obtained in Proposition 57.

The strategy of the proof is similar to the one in [1]. One uses the algebraization result for groups in a very general context obtained in [16], see Section 2.4. A one dimensional connected algebraic group in a perfect field is either the additive group, the multiplicative group, a twisted multiplicative group or an elliptic curve see for example Corollary 16.16 of [14] for the affine case and Theorem 8.27 for the general case. So one has to describe the type-definable subgroups in each of these cases. After this one uses compactness to go from a type-definable...
group morphism to a definable local group morphism and from there to suitably
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The description of type-definable subgroups of algebraic one dimensional
groups proceeds more or less as follows, often there are short exact sequences
$0 \rightarrow G_0 \rightarrow G \rightarrow \Gamma$, and $0 \rightarrow G_0^- \rightarrow G_0 \rightarrow H$ for $H$ an algebraic subgroup

of the residue field. These maps sometimes satisfy that all definable subsets
contain almost all fibers of the image, or at some points some more complicated
weakening of this property. This implies that the classification reduces to the
one in the kernel of the map and in $H$ or $\Gamma$ see Lemma 30. For the kernel $G_0^-$
one has a filtration of the group $w : G_0^- \rightarrow \Gamma$ with intermediate quotients $(k,+)$,
so one proceeds by analogy with $O$ with the filtration $v : O \rightarrow \Gamma$.

We obtain detailed information about the subgroups of algebraic groups that
allows us for instance to identify the maximal stably dominated subgroup in the

case of an elliptic curve as defined in \[11\]. See Proposition 29.

The methods here work for other valued fields when there is good knowl

dge of type-definable subgroups of one-dimensional algebraic groups in the residue
field, and of the definable sets in the value group. So for example the proofs in
Proposition 57 adapt without change to $\mathbb{C}((t))$ and $F((t^q))$ for $F$ pseudo-finite
of characteristic 0.

\section{Preliminaries}

\subsection{Notation}

Throughout the paper for a valued field $K$ we use the notation $O$ for the valuation ring, $\Gamma$ for the valued group and $M$ for the maximal ideal of $O$. We denote $\Gamma_\infty$ the group $\Gamma$ together with one additional distinguished element denoted $\infty$. The valuation $K \rightarrow \Gamma_\infty$ is denoted by $v$. We denote $RV = K^\times / (1 + M)$ and $rv : K^\times \rightarrow RV$ the canonical projection. We also denote $RV_0$ to be $RV$ together with an additional element denoted 0, and also denote by $rv$ the map $rv : K \rightarrow RV_0$ that sends 0 to 0. If $r \in \Gamma$ we denote $B_r = \{x \in K \mid v(x) \geq r\}$ and $B_r^- = \{x \in K \mid v(x) > r\}$. If $r > 0$ denote $U_r = \{x \in K \mid v(1-x) \geq r\}$ and $U_r^- = \{x \in K \mid v(1-x) > r\}$, and also $U_0^- = 1 + M$.

A model of ACVF is a non-trivially valued algebraically closed field.

A model of PL$_0$ is a valued field with residue field pseudo-finite of characteristic 0 and value group a $Z$-group.

We will assume that for every cardinal there is a larger inaccessible cardinal.

We will work in a monster model $K$ which for definiteness we take to be a saturated model of cardinality $\kappa$ for $\kappa$ an inaccessible cardinal. A small set is a set of cardinality smaller than $\kappa$.

This monster model can be replaced by a $\kappa$-saturated strongly $\kappa$-homogeneous model, for sufficiently large $\kappa$ depending on the arguments used, so this assumption on inaccessible cardinals is unnecessary. In fact, for the arguments in the paper $\omega$-saturated is enough.

Definable means definable with parameters, type-definable means a small
intersection of definable sets. If \( A \) is a small set of parameters, \( A \)-definable means definable with parameters in \( A \) and \( A \)-type-definable is a small intersection of \( A \)-definable sets.

A set is \( \lor \)-definable if it is the small disjoint union of definable sets, and a map of \( \lor \)-definable sets is ind-definable if the restriction to a definable set has definable graph.

If \( \Gamma \) is an ordered abelian group and \( a \in \Gamma \), then \( o(a) \) is the set of elements \( x \in \Gamma \) such that \( |nx| < a \) for all \( n \in \mathbb{Z}_{>0} \) and \( O(a) \) is the set of elements such that \( |x| < na \) for a \( n \in \mathbb{Z} \).

If \( K \) is a valued field then for \( a \in K^\times \) we use the shorthand \( O(a) = v^{-1}O(v(a)) \) and \( o(a) = v^{-1}o(v(a)) \).

2.2 Valued fields

For a valued field \( K \) the valued field language consists of a two sorted language, with a sort for \( K \) and a sort for \( \Gamma_{\infty} \), the ring language in \( K \), the ordered group language in \( \Gamma_{\infty} \) with distinguished point \( \infty \) and the function \( v : K \to \Gamma_{\infty} \). We will use the well known fact that if \( K \) is a model of ACVF then \( K \) eliminates quantifiers in this language, and the theory of ACVF together with fixing the characteristic and fixing the characteristic of the residue field is complete. Moreover there is a description of the definable sets \( X \subset K \) due to Holly.

**Proposition 1.** Suppose \( K \) is a model of ACVF. Suppose \( X \subset K \) then \( X \) is a disjoint union of Swiss cheeses. A Swiss cheese is a set of the form \( B \setminus C_1 \cup \cdots \cup C_n \), where \( C_i \) or balls or points and \( B \) is a ball or \( K \) or a point.

See [10].

If \( K \) is a henselian valued field of residue characteristic 0, then the theory of \( K \) is determined by the theory of the residue field \( k \) as a pure field and the theory of \( \Gamma \) as an ordered group, this is the Ax-Kochen-Ershov theorem.

If \( K \) is a henselian valued field of residue characteristic 0 then there is quantifier elimination of the field sort relative to RV. This is a result of Basarav. We will use this in the form below.

**Proposition 2.** If \( K \) is a henselian field of residue characteristic 0 then every definable set \( X \subset K^n \) is a boolean combination of sets quantifier-free definable in the valued field language and sets of of the form \( f^{-1}v^{-1}(Y) \) for \( Y \subset RV^m \) definable and \( f : K^n \to K^m \) given by polynomials. Here \( RV = K^\times / 1 + M \) is given the language that interprets the short exact sequence \( 1 \to k^\times \to RV \to \Gamma \to 0 \), the group language, the field language in \( k^\times \) and the ordered group language in \( \Gamma \).

For a proof see for example Proposition 4.3 of [8].

We also have a description of the definable sets \( X \subset K \) due to Flenner.

**Proposition 3.** If \( K \) is a henselian field of residue characteristic 0 and \( X \subset K \) is a definable set, then there are \( a_1, \ldots, a_n \in K \) and a definable set \( Y \subset RV_0^n \) such that \( X = \{ x \in K^\times \mid (rv(x - a_1), \ldots, rv(x - a_n)) \in Y \} \).
See Proposition 5.1 of [8]

2.3 Definable subsets of RV

Situation 4. Here we consider the following situation. We have $0 \to A \to B \to C \to 0$ a short exact sequence of abelian groups such that $C$ is torsion free and $B/nB$ is finite. The groups $A$ and $C$ come equipped with a language that extends that of groups. In this case we endow $B$ with the language consisting of three sorts for $A$, $B$ and $C$, the group language $(+, 0, -)$ in $B$, the functions between sorts $A \to B$ and $B \to C$ and the extra constants, relations and functions on $A$ and $C$, and one predicate for each $n$, interpreted as $nB$.

The motivating example for the previous situation is the sequence $1 \to k^\times \to RV \to \Gamma \to 0$ in the case $\Gamma/n\Gamma$ and $k^\times/(k^\times)^n$ are finite. In this case $k$ comes equipped with its field language and $\Gamma$ with the ordered group language.

Proposition 5. In Situation 4, we have that every definable set of $B$ is a boolean combination of cosets of $nB$, translates of definable subsets of $A$, and inverse images of definable subsets of $C$.

This is proven in [5] Section 3.

2.4 Definably amenable groups in NTP$_2$ theories

In this section we state the theorem that will allow us to relate a definable abelian group with subgroups of an algebraic group.

Proposition 6. Let $K$ be a field with extra structure which is NTP$_2$ and algebraically bounded. Let $G$ be a definably amenable group in $K$. Then there exists a type-definable subgroup $H \subset G$ of small index, an algebraic group $L$ over $K$ and a type-definable group morphism $H \to L(K)$ of finite kernel.

This is Theorem 2.19 of [16].

See [4] for the definition of NTP$_2$ theories. It is shown there for instance that simple theories and NIP theories are NTP$_2$. [15] has an overview of NIP theories and [20] has an overview of simple theories. One has that a model of ACVF is NIP, and that a pseudo-finite field is simple. Also a henselian valued field of residue characteristic 0 is NTP$_2$ if and only if the residue field as a pure field is, so a model of PL$_0$ is NTP$_2$, for a proof see [4].

A field with extra structure is algebraically bounded if in any model of its theory the algebraic closure in the sense of model theory coincides with the algebraic closure in the sense of fields. A model of ACVF is algebraically bounded and a non-trivially valued henselian valued field of characteristic 0 is algebraically bounded. See [19] for the henselian case.

A definable group $G$ is definably amenable if there is a finitely additive left invariant probability measure in the collection of definable subsets of $G$. A group is amenable if there exists such a measure in the collection of all subsets of $G$, so an amenable group is definably amenable. An abelian group is amenable, see for example Theorem 449C in [9].
2.5 Analytic functions

For algebraically closed valued fields there is an analytic language described in [12]. We briefly review it here. Let $K_0 = \mathbb{F}_p^\text{alg}(t)$ or $\mathbb{C}(t)$ or the maximal unramified extension of $\mathbb{Q}_p$ with the $t$-adic or the $p$-adic valuation. Let $K_1$ be the completion of the algebraic closure of $K_0$, let $O_1$ and $O_0$ be the valuation ring of $K_1$ and $K_0$ respectively. A separated power series is a power series $f \in K_1[[x, y]]$ say $f = \sum_{n, m} a_{n, m} x^n y^m$ which satisfies

1. For $m$ fixed $a_{n, m} \to 0$ as $n \to \infty$.
2. There is $a_{n_0, m_0}$ with $v(a_{n_0, m_0}) = \min_{n, m} v(a_n, a_m)$.
3. For such an $a_{n_0, m_0}$ there are $b_k \in O_1$ with $b_k \to 0$ such that $a_{n, m} a_{n_0, m_0}^{-1} \in O_0(b_k)_k$, for all $n, m$.

Here $O_0(b_k)_k$ is the closure of $O_0[b_k]_k$ in $O_1$.

The set of separated power series is denoted by $K_1[[x, y]]$. Every separated power series $f(x, y)$ produces a map $f : O_1^n \times M_1^m \to K_1$. If we add to the valued field language a function symbol for every separated power series and interpret it as $f$ in $O_1^n \times M_1^m$ and 0 outside, then we obtain a language and a model in this language.

We call a model of the theory of $K_1$ in this language an algebraically closed valued field with analytic functions.

**Proposition 7.** Let $K$ be a model of ACVF with analytic functions, if $X \subset K$ is definable in the analytic language then it is definable in the valued field language.

This property is known as C-minimality. For a proof, see [13].

Analytic languages have been defined in other contexts. We describe another such context we use here. Let $k_1$ be a field of characteristic 0 and take $K_1 = k_1((t))$ with the $t$-adic valuation. We can consider power series $f \in K_1[[x]]$ of the form $f(x) = \sum_{n} a_n x^n$ where $a_n \to 0$ in the $t$-adic topology. If we add to the valued field language a function symbol for each $f$, interpreted as $f(x)$ inside $O_1^n$ and 0 outside, then we have a language and a model in it.

We call a model of the theory of $K_1$ in this language a henselian valued field of residue characteristic 0 and value group a $\mathbb{Z}$-group with analytic functions.

**Proposition 8.** If $K$ is a henselian valued field of residue characteristic 0 and value group a $\mathbb{Z}$-group then a subset $X \subset K$ definable in the analytic language is definable in the valued field language.

**Proof.** This situation fits into the general framework in [7] Section 4.1, where, as in Example 4.4 (2) of [7] one takes $A_{m, n} = A_{m, 0} = K_1(n)$. In [7] a valued field quantifier elimination is proven in certain language we will not describe here, see Theorem 6.3.7 there. From that relative quantifier elimination it follows that if $X \subset RV^n$ is definable in the analytic language, then it is definable in the algebraic language. Finally the framework in [7] satisfies the conditions of
hensel minimality described in \[6\], see Theorem 6.2.1 of \[6\], and so one has that every analytically definable set \(X \subset K\) is the inverse image of \(Y \subset RV_0^n\) under some map of the form \(x \mapsto (rv(x - a_1), \ldots, rv(x - a_n))\) and so it is definable in the valued field language.

When \(K\) is a model of ACVF we will only need to consider \(K_1 = K_0^{alg}\) for \(K_0\) a complete discrete valuation field and the analytic functions taken with all coefficients in a finite extension of \(K_0\). These power series seem algebraically simpler, as only complete discrete valuation rings appear. It is not clear however that for example \(K_1\) is an elementary submodel of its completion in this analytic language, or that \(K_1\) is C-minimal. In residue characteristic 0, \(K_1\) is C-minimal by \[6\].

### 2.6 Elliptic curves

Next we review some properties of elliptic curves. An elliptic curve over a field \(K\) is given by an equation of the form

\[
y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6
\]

Where \(a_i \in K\). This equation is assumed to have discriminant non-zero. The discriminant of the equation is a certain polynomial with integer coefficients in the \(a_i\), see section III.1 of \[18\].

Taking the projective closure one obtains a projective curve \(E \subset P^2_K\). In any field extension of \(K\) there is only one point at infinity \([0 : 1 : 0]\).

This curve is a commutative algebraic group, the identity is the point at infinity, see section III.2 of \[18\] for the equations of the group law.

If \(E(K)\) is the set of \(K\)-points, then this is a definable group in any language that extends the ring language. If \(u \in K^\times\) and \(r, s, t \in K\) then we have an isomorphism of algebraic groups \(E' \rightarrow E\) determined by the equations \(x = u^2x' + r\) and \(y = u^3y' + u^2sx' + t\), the corresponding equation in \(E'\) is given by coefficients \(a'_i\) given in the table 3.1 of \[18\]. With the relation on discriminants being of note, \(u^{12}\Delta' = \Delta\). If \(K\) is a valued field then there is a change of variables such that \(E'\) is defined by an equation with coefficients in \(O\).

Now suppose \(K\) is a valued field, and suppose given a Weierstrass equation with coefficients in \(O\). The homogeneous equation

\[
y^2z + a_1xyz + a_3yz^2 = x^3 + a_2x^2z + a_4xz + a_6z^3
\]

produces a closed subscheme \(E \subset P^2_O\) with generic fiber \(E\). From the properness of \(E\), we have \(E(O) = E(K)\). Reducing the equations mod \(M\) we obtain \(\tilde{E}\) having \(\tilde{E} = E \times_O k \subset P^2_k\), and a map \(E(K) \rightarrow \tilde{E}(k)\). We call \(\tilde{E}_0\) the smooth part of \(\tilde{E}\). If the discriminant of the Weierstrass equation has valuation 0 then \(\tilde{E}\) is an elliptic curve over \(k\), so \(\tilde{E}_0 = \tilde{E}\), and in this case we say the given Weierstrass equation has good reduction. Otherwise \(\tilde{E}\) has only one singular point and it is \(k\)-rational, in this case \(\tilde{E}_0\) is an algebraic group over \(k\) isomorphic to the additive or a possibly twisted multiplicative group, see Proposition III.2.5 of \[18\].
and the remark that follows for a definition of the group law and a proof. If \( \tilde{E}_0 \) is isomorphic to the additive group we say the equation has additive reduction. If it is isomorphic to the multiplicative group we say the equation has split multiplicative reduction. If it is isomorphic to a twisted multiplicative group, we say the equation has a non-split multiplicative reduction.

We denote \( E_0(K) \) to be the inverse image of \( \tilde{E}_0(k) \) under the reduction map \( E(k) \to \tilde{E}(k) \). Then \( E_0(K) \) is a subgroup of \( E(k) \) and the map \( E_0(K) \to \tilde{E}_0(k) \) is a group homomorphism, see Proposition VII.2.1 of \([15]\). Further, if the field \( K \) is henselian then this map is surjective. In any case we will call the kernel \( E_0^\circ(K) \).

Inspecting the definition one gets that \( E_0^\circ(K) \) consists of the point at infinity together with the set of pairs \( (x, y) \in K \) satisfying the Weierstrass equation, such that \( v(x) < 0 \) or equivalently \( v(y) < 0 \). If \( (x, y) \in E_0^\circ(K) \) then from the Weierstrass equation one gets \( 2v(y) = 3v(x) \), so that if we set \( r = -6r = 2v(y) = 3v(x) \), we have \( v(y) = -3r, v(x) = -2r \) and \( r = v(\sqrt[3]{\frac{y}{x}}) \in \Gamma \).

**Lemma 9.** Assume \( K \) is a valued field, and we have a Weierstrass equation over \( O \). Then for the set \( E_0^\circ(K) \) described above the map \( E_0^\circ(K) \to M \) given by \( (x, y) \mapsto -\frac{x}{y} \) and sending the point at infinity to 0 is injective, and satisfies that

1. \( f(a + b) = f(a) + f(b) + s \) for an \( s \) with \( v(s) \geq v(f(a)) + v(f(b)) \).
2. \( f(-a) = -f(a) + r \) for an \( r \) with \( v(r) \geq 2v(f(a)) \).

If \( K \) is henselian the map is surjective.

**Proof.** To see that the map is surjective when \( K \) is henselian one can do the change of variables \( z = \frac{y}{x} \) and \( w = yz^3 = xz^2 \), so that \( x = wz^{-2} \) and \( y = wz^{-3} \). In this case the Weierstrass equation becomes

\[
w^2 + a_1zw^2 + a_3z^3w = w^3 + a_2z^2w^2 + a_4z^4w + a_6z^6
\]

Reducing mod \( M \) we get the equation \( w^2 = w^3 \) which has a unique non-zero solution at \( w = 1 \), so we may apply Hensel’s lemma. This also shows the map is injective.

To see the two displayed equations we may replace \( K \) by its algebraic closure, then because this is first order expressible we may reduce to the case in which \( K \) is the algebraic closure of \( \mathbb{Q}_p \) or \( \mathbb{C}((t)) \) or \( \mathbb{F}_p((t)) \). In this case the Weierstrass equation has coefficients in a finite extension of the fields mentioned and \( a \) and \( b \) also, so we may assume \( K \) is such a finite extension. In particular it is a complete discrete valuation field.

In this case the discussion in section IV.1 of \([15]\) shows that for \( z_1, z_2 \in M \) we have \( f(f^{-1}(z_1) + f^{-1}(z_2)) = z_1 + z_2 + z_1z_2F(z_1, z_2) \) for \( F(T, S) \) a power series in \( O \). Similarly one has \( f(-f^{-1}(z)) = -z + z^2g(z) \) where \( g(S) \) is a power series with coefficients in \( O \). See also Remark IV.2.1 and Example IV.3.1.3 of \([15]\). \( \square \)
If \( K \) is any valued field, and we have a given Weierstrass equation with coefficients in \( O \), then inside \( E \) and for \( a \in \Gamma \) with \( a > 0 \) we have the sets \( E_a \) given by the pairs \((x, y)\) such that \( v(x/y) \geq a \) together with the point at infinity, and \( E_a^- \) the set of pairs \((x, y)\) such that \( v(x/y) > a \) together with the point at infinity.

**Lemma 10.** With the above notation \( E_a \) and \( E_a^- \) are subgroups of \( E(K) \) and the map \( E_a \to B_a/B_a^- \) given by \((x, y) \mapsto -x/y\) and which sends the point at infinity to 0, is a group map. When \( K \) is henselian it is surjective.

**Proof.** This follows easily from Lemma 9. \( \square \)

**Definition 11.** If \( K \) is a valued field of and \( E \) is an elliptic curve then a Weierstrass equation of \( E \) will be called minimal if the valuation of the discriminant is minimal among all possible Weierstrass equations for \( E \).

Such an equation need not exist.

Assume that \( K \) is a valued field elementary equivalent as a valued field to a discrete valuation field. In this case given an elliptic curve \( E \) over \( K \), a minimal Weierstrass equation exists because this is first order expressible and true in a discretely valued field.

Given a minimal Weierstrass equation, another Weierstrass equation related to it by change of variables is minimal if and only if we can obtain one from the other through a change of variables \( x = u^2 x' + r \) and \( y = u^3 y' + u^2 s x' + t \), with \( u \in O^\times \) and \( r, s, t \in O \), see Proposition VII.1.3 of \([18]\).

In case the residue characteristic is not 2 or 3 then there is a change of variables with \( u \in O^\times \) and \( r, s, t \in O \) (coming from completing the square and cube) that produce a minimal Weierstrass equation of the form

\[
y^2 = x^3 + Ax + B
\]

See section III.1 of \([18]\) for details. Two such Weierstrass equations are related through the change of variables \( x = u^2 x' + r \) and \( y = u^3 y' \), and two minimal Weierstrass equation through a \( u \in O^\times \). In this case the coefficients of \( E' \) are given by \( u^4 A' = A \) and \( u^6 B' = B \). It follows that if the value group is a \( \mathbb{Z} \)-group then such an equation is minimal if and only if \( v(A) < 4 \) or \( v(B) < 6 \). If further \( v(A) \) is divisible by 4 and \( v(B) \) is divisible by 6, then one has either \( v(A) = 0 \) or \( v(B) = 0 \) and the equation remains minimal in any finite extension of \( K \).

If \( K \) has residue field of characteristic not 2 or 3 and value group 2 and 3 divisible then the discussion above also shows that a minimal Weierstrass equation exists and can be found of the form \( y^2 = x^3 + Ax + B \), and a Weierstrass equation of that form with \( A, B \in O \) is minimal if and only if \( v(A) = 0 \) or \( v(B) = 0 \).

If we have a valued field \( K \) and an elliptic curve \( E \) over \( K \), we say \( E \) has good, additive, split multiplicative or non-split multiplicative reduction if a minimal Weierstrass equation of \( E \) does, so when we speak of the reduction type of an elliptic curve we assume a minimal Weierstrass equation for it exists.
Similarly the notations $E_0$, $E_0^-$, $E_r$ and $E_r^-$ are understood to be relative to a minimal Weierstrass equation for $E$.

We mention that in case $K$ is a valued field with residue characteristic not 2 or 3 and 2 and 3 divisible value group then an elliptic curve over $K$ has good or multiplicative reduction, by the criterion in Proposition III.2.5 of \cite{18}.

The following two lemmas are a relatively minor point.

**Lemma 12.** Suppose $(R, M, k)$ is a henselian local ring and suppose $f : X \to Y$ is a smooth map of schemes over $R$. Then for every $x \in X(k)$ and $y \in Y(R)$ such that $\bar{y} = f(x)$, there exists $z \in X(R)$ such that $\bar{z} = x$ and $f(z) = y$.

This lemma is well known and we omit the proof.

**Lemma 13.** Let $K$ be a henselian valued field with residue of characteristic 0 and valuation a $\mathbb{Z}$-group. Assume given an elliptic curve $E$ over $K$. Then $nE_0(K) = \pi^{-1}n\tilde{E}_0(k)$, where $\pi : E_0(K) \to \tilde{E}_0(k)$ is the reduction map.

This previous Lemma should be true in more general valued fields, starting with a Weierstrass equation with coefficients in $O$, but I do not know a proof.

**Proof.** This property is first order expressible so without loss of generality $K = k((t))$ is a complete discrete valued field. In this case we get that $\mathcal{E}_0 \subset \mathcal{E}$, the set of smooth points is a group scheme over $O$ with generic fiber $E$ and with $E_0(K) = \mathcal{E}_0(O)$, see Corollaries 9.1 and 9.2 of \cite{17}. Recall that for any group scheme $X$ over a scheme $S$ which is smooth the multiplication map is smooth, and so the multiplication by $n$ map is smooth $\mathcal{E}_0 \to \mathcal{E}_0$. Then this becomes a consequence of Lemma \cite{12}

### 2.7 Tate uniformization

In this section we review the Tate uniformization of an elliptic curve. We will be interested in the case where $K$ is an algebraically closed field of residue characteristic not 2 or 3 and when $K$ is a pseudo-local field of residue characteristic 0 so we treat these two cases separately.

Assume first $K$ is an $\omega$-saturated algebraically closed valued field of residue characteristic not 2 or 3 with analytic functions as in Section 2.5. Define $K_0$ to be $\mathbb{F}_p^a((t))$ or $\mathbb{C}((t))$ or the maximal unramified extension of $\mathbb{Q}_p$ according to the characteristic of $K$ and the residue field of $K$, and let $K_1$ be the completion of algebraic closure of $K_0$. Now there are functions $a_4 : M_1 \to K_1$ and $a_6 : M_1 \to K_1$ given by a separated power series in one variable $K_1[[q]]_a$, defined in Theorem 3.1 of \cite{17}. In fact they are given by elements of $\mathbb{Z}[[q]]$. Then one defines the elliptic curve $E_q$ with equation $y^2 + xy = x^3 + a_4(q)x + a_6(q)$. This equation has discriminant different from 0 for all $q \in M_1$.

One has functions $X(u, q)$ and $Y(u, q)$ defined as

$$X(u, q) = \frac{u}{(1-u)^2} + \sum_{d \geq 1} \left( \sum_{m \mid d} m(u^m + u^{-m} - 2) \right)q^d$$

9
\[ Y(u, q) = \frac{u^2}{(1 - u)^3} + \sum_{d \geq 1} \left( \sum_{m | d} \frac{(m-1)m}{2} u^m - \frac{m(m+1)}{2} u^{-m} + m \right) q^d \]

For \( -v(q) < v(u) < v(q) \) and \( u \neq 1 \). This produces a map \( v^{-1}(-v(q), v(q)) \rightarrow E_q(K_1) \) that takes the \( u = 1 \) to the point at infinity, and \( u \neq 1 \) to \( ((X(u, q), Y(u, q)) \).

This maps extends in a unique way to a surjective group map \( K_1^+ \rightarrow E_q(K_1) \) with kernel \( q^2 \). This is proven in Theorem 3.1 of [17]. This is stated for \( K \) a \( p \)-adic field, but the proof works equally well for a complete rank one valued field. As we use some of the information obtained in the proof we review it next.

The maps \( X \) and \( Y \) are definable in the analytic language. Indeed \( X(u, q) = L_1(u) + L_2(uq, q) + L_3(u^{-1}, q) \) where \( L_1(u) = \frac{u}{1 - u^2} \), and \( L_2, L_3 \) are given by separated power series in two variables \( K[[v, q]] \). Indeed, \( L_2(v, q) = \sum_{k \geq 1, m \geq 1} mv^m q^{m(k - 1)} \), and \( L_3(v, q) = \sum_{k \geq 1, m \geq 1} mv^m q^{m(k - 1)} - 2q^{nk} \) are even given by power series with coefficients in \( \mathbb{Z} \). A similar consideration applies to \( Y \). The maps \((X, Y)\) produce an injective group map \( K_1^+ / q^2 \rightarrow E_q(K_1) \), this follows from formal identities on \( \mathbb{Q}(u)[[q]] \). Now one has that for the Tate map \( t : K_1^+ \rightarrow E_q(K_1) \) one has \( t(O_1^+) \subseteq E_{q,0}(K_1) \) and \( t(1 + M_1) \subseteq E_1^-(K_1) \). Now under the bijections \( M_1 \rightarrow 1 + M_1 \) and \( E_1^-(M_1) \rightarrow M_1 \) described in Lemma 9 we get a map \( \phi : M_1 \rightarrow M_1 \) given by \( t \mapsto t(1 + \sum_{m \geq 1} m! m^{-m} \), and the coefficients belong to the ring \( O_0(q) \) described in Section 2.4. So the argument in [17] shows that \( \phi \) is surjective, the inverse is given as another such power series. Also one gets that \( t(O_1^+) = E_{q,0}(K_1) \) by reducing mod \( 1 + M_1 \) and \( E_1^-(K_1) \) respectively.

Now one decomposes \( E_q(K_1) \setminus E_{q,0}(K_1) \) into some disjoint sets \( U_r, V_r, W \) for \( 0 < r < \frac{1}{2} v(q) \), defined as

\[
U_r = \{(x, y) \in E_q(K_1) \mid v(y) > v(x + y) = r\}
\]

\[
V_r = \{(x, y) \in E_q(K_1) \mid v(x + y) > v(y) = r\}
\]

\[
W = \{(x, y) \in E_q(K_1) \mid v(y) = v(x + y) = \frac{1}{2} v(q)\}
\]

See Lemma 4.1.2 of [17] and its proof. Further Lemma 4.1.4 of [17] shows each of these sets belong to the same \( E_{q,0}(K_1) \) coset, and one sees in its proof that \( V_r = -U_r \). Now I claim that if \( 0 < v(u) = r < \frac{1}{4} v(q) \) then \( t(u) \in U_r \). Indeed in the formula for \( X(u, q) \), one has that if \( a_d = \sum_{m | d} m(u^m + u^{-m} - 2) \) then \( v(a_d) \geq -dr \) and so \( v(a_d q^d) \geq -dr + dv(q) > dr \geq r = v(\frac{u}{1 - u^2}) \), so we conclude \( v(X(u, q)) = r \). A similar computation shows all the summands in \( Y \) have valuation \( > r \) so one gets \( v(Y(u, q)) \geq r \) and \( t(u) \in U_r \) as required. Next I claim that if \( r = v(u) = \frac{1}{2} v(q) \) then \( t(u) \in W \). Indeed, in this case the computation above shows that for the \( d \) term in \( X, v(a_d q^d) \geq dr > r \) if \( d > 1 \) so modulo \( B^-_r, X \) is \( \frac{u}{1 - u^2} + qu^{-1} \). Similarly, modulo \( B^-_r, Y \) becomes \( -qu^{-1} \).

So we obtain \( v(X(u, q) + Y(u, q)) = r = v(Y(u, q)) \) as required. Now because the sets

\[
U'_r = \{ u \in K_1 \mid v(u) = r\}
\]

\[
V'_r = \{ u \in K_1 \mid v(u) = -r\}
\]
\[ W' = \{ u \in K_1 \mid v(u) = \frac{1}{2}v(q) \} \]

for \( 0 < r < \frac{1}{2}v(q) \) are \( O_1^\times \) cosets satisfying \( V_r^{-1} = U_r' \) we conclude that \( U_r, V_r, \) and \( W \) are \( E_{q,0}^\times(K_1) \) cosets satisfying \( t(U_r') = U_r, t(V_r') = V_r, \) and \( t(W') = W, \) and \( t \) is surjective, as required.

Now by elementary equivalence we get that the functions \( a_4 \) and \( a_6 \) produce functions denoted by the same symbol \( a_4, a_6 : M \to K, \) and we obtain for every \( q \in M \) an elliptic curve \( E_q \) and functions \( (X, Y) : v^{-1}(v(q),\{\}) \to E_q \) that extend to a surjective group map \( O(q) \to E_q \) with kernel \( q^{2v}. \) If further \( E \) has multiplicative reduction then there is a unique \( q \) such that \( E \) is isomorphic to \( E_q \) over \( K. \) See Theorem 5.3 of [17].

We collect the previous discussion into a theorem.

**Proposition 14.** Suppose \( K \) is a model of ACVF with analytic functions and residue field of characteristic not 2 or 3 and let \( E \) be an elliptic curve over \( K \) with multiplicative reduction. Then there is a surjective group morphism \( t : O(q) \to E(K) \) with kernel \( q^{2v} \) ind-definable in the analytic language. We have further that:

1. \( v(q) \) equals the valuation of the discriminant of a minimal Weierstrass equation of \( E \).
2. The image of \( O^\times \) is \( E_0^\times \).
3. The image of \( U_0^- \) is \( E_0^- \).
4. For the bijections \( f : U_0^- \to M \) and \( g : E_0^- \to M \) given by \( f(x) = x - 1 \) and \( g(x, y) = -\frac{x}{y} \) we have \( v(gt) = v(f) \).
5. The image of \( U_r \) is \( E_r \) and of \( U_r^- \) is \( E_r^- \) for \( r \in \Gamma_{>0} \).
6. The composition \( E(K) \to O(q)/q^{2v} \to O(v(q))/2v(q) \) is definable in the valued field language.
7. If \( K \) is \( \omega \)-saturated the image of \( o(q) \) in \( E(K) \) is type-definable in the valued field language.

**Proof.** The properties 1-5 of the Tate uniformization map described here are all first order expressible so they follow from the result in \( K_1. \)

For property 6, let \( E \) be an elliptic curve over \( K \) with multiplicative reduction, understood as in the remark following Definition [11] then \( E \) is isomorphic to \( E_q \) for a \( q \in M. \) We conclude that if \( E \) has multiplicative reduction then it has a Weierstrass equation of the form \( y^2 + xy = x^3 + a_4'x + a_6' \) with \( v(a_4') > 0 \) and \( a_6' = a_4' + c \) with \( v(c) \geq 2v(a_4) \). Such an equation is minimal in the sense of Definition [11] Now assume that \( E_q \) is isomorphic to such an elliptic curve. Then \( E \) is obtained from \( E_q \) via a change of variables \( x = u^2x' + r \) and \( y = u^3y' + u^2sx' + t \) with \( u \in O^\times \) and \( r, s, t \in O. \) From the discriminant formula \( \Delta = -a_6 + a_4^2 + 72a_4a_6 - 4a_4^3 - 432a_6^2, \) and \( v^{12}\Delta' = \Delta \) we obtain that \( v(\Delta) = v(a_4') = v(q) = l. \) Now reducing the equations in Table 3.1 of
mod $B_1$ (for $c_4$ and $c_6$) obtain $u^4 = 1$ and $u^6 = 1$ mod $B_1$. From here we get $u^2 = 1$ mod $B_1$. Replacing $u$ by $-u$ if necessary (this has the effect of composing $E_q \to E$ with $- : E \to E$) we conclude $u = 1$ mod $B_1$. Reducing the change of variable equations in Table 3.1 of [18] mod $B_1$ (for $a_1, a_2$ and $a_3$; note that this step uses that the residue field has characteristic not 2 or 3) we obtain $s = r = t = 0$ mod $B_1$. So if $(x', y')$ is the image of $(x, y) \in E_q(K) \setminus E_q,l(K)$ under the map $E_q \to E$ then $x' = x$ and $y' = y$ mod $B_1$.

So if we define $U_r', V_r', W'$ by the same formulas as $U_r, V_r, W$ but using the Weierstrass equation in $E$ then the isomorphism $E_q \to E$ takes $U_r$ to $U_r'$, $V_r$ to $V_r'$ and $W$ to $W'$ (so in particular $U_r', V_r'$ and $W'$ are the $E_0(K)$ cosets).

Further if the image of $(x, y)$ is $(x', y')$ then $v(x + y) = v(x' + y')$ in $U_r$ and $v(y) = v(y')$ in $V_r$. So the map $w : E(K) \to (-\frac{1}{4}v(a_4), \frac{1}{4}v(a_4))$ is given explicitly as $w(P) = 0$ if $P \in E_0(K)$, $w(x, y) = \frac{1}{2}v(a_4)$ if $v(y) = v(x + y)$, $w(x, y) = v(x)$ if $0 < v(y) < v(x + y)$ and $w(x, y) = -v(x)$ if $0 < v(x + y) < v(y)$.

Property 7 follows from 6. □

In our situation we are able to prove using the explicit formulas that $E(K) \to O(q)/q^2 \to O(v(q))/Zv(q)$ is definable in the valued field language.

Related to this we can ask whether for a model of ACVF $K$ with analytic structure and $x \in K^n$, if $y \in \Gamma$ is definable from $x$ in the analytic language then it is definable from $x$ in the valued field language. Also we can ask whether if $a \in K$ is algebraic over $x$ in the valued field language and definable from $x$ in the analytic language then it is definable from $x$ in the valued field language.

The second question together with $C$-minimality is equivalent to saying that for any one-dimensional set $X$ definable in the valued language and any set $Y \subset X$ definable in the analytic language, $Y$ is definable in the valued field language.

The first question together with the second one implies that for any map $f : X \to \Gamma$ definable in the analytic language, where $X$ is one dimensional definable in the valued field language, $f$ is definable in the valued field language.

Using Proposition 2.4 of [1] we obtain the following.

**Proposition 15.** Suppose $K$ is an $\omega$-saturated algebraically closed valued field of residue characteristic not 2 or 3 with analytic functions and let $E$ be an elliptic curve over $K$ with multiplicative reduction. Let $t : O(q) \to E$ be the Tate uniformization map of Proposition 4. Then there is a group $O_E \land$-definable in the valued field language and a group isomorphism $O_E \to O$ ind-definable in the analytic language, such that the composition $O_E \to O \to E$ is ind-definable in the valued field language.

Now suppose $K$ is a henselian valued field of residue characteristic 0 and value group a $Z$-group with analytic functions, as defined in Section 2.4. Then the discussion before goes through and produces the following two theorems.

**Proposition 16.** Suppose $K$ is an $\omega$-saturated henselian valued field of residue characteristic 0 and value group a $Z$-group with analytic functions. Let $E$ be an elliptic curve over $K$ with split multiplicative reduction. Then there is a
surjective group morphism $t : O(q) \to E(K)$ with kernel $q^{\mathbb{Z}}$ ind-definable in the analytic language. We have further that:

1. $v(q)$ equals the valuation of the discriminant of a minimal Weierstrass equation of $E$.
2. the image of $O^\times$ is $E_0$.
3. the image of $U_0^-$ is $E_0^-$.
4. for the bijections $f : U_0^- \to M$ and $g : E_0^- \to M$ given by $f(x) = x - 1$ and $g(x, y) = -\frac{x}{y}$ we have $v(gt) = v(f)$.
5. the image of $U_r$ is $E_r$ and of $U_r^-$ is $E_r^-$ for $r \in \Gamma > 0$.
6. The composition $E(K) \to O(q)/q^{\mathbb{Z}} \to O(v(q))/\mathbb{Z}v(q)$ is definable in the valued field language.
7. the image of $o(q)$ in $E(K)$ is type-definable in the valued field language.

**Proposition 17.** Suppose $K$ is an $\omega$-saturated henselian valued field of residue characteristic 0 and value group a $\mathbb{Z}$-group with analytic functions. Let $E$ be an elliptic curve over $K$ with split multiplicative reduction. Let $t : O(q) \to E$ be the Tate uniformization map of Proposition 16. Then there is a group $O_E \lor$-definable in the valued field language and a group isomorphism $O_E \to O$ ind-definable in the analytic language, such that the composition $O_E \to O \to E$ is ind-definable in the valued field language.

Finally we have in the other reduction cases the following.

**Proposition 18.** Let $K$ be a henselian field of residue characteristic 0 and value group a $\mathbb{Z}$-group. Let $E$ be an elliptic curve over $K$ of good, additive or non-split multiplicative reduction. Then $E(K)/E_0(K)$ is a finite group of order at most 4.

**Proof.** This is first order expressible so without loss of generality $K = k((t))$ is a complete discrete valuation field. In this case the Proposition is Corollary 9.2 (d) of [17].

### 2.8 Almost saturated sets and maps

Given a definable function $f : X \to Y$ and a definable subset $Z \subset X$ then $Z$ is said to be almost saturated with respect to $f$ if there is a finite set $S \subset Y$ such that if $y \in Y \setminus S$ then $f^{-1}(y) \subset Z$ or $f^{-1}(y) \cap Z = \emptyset$. In this case the set $S$ is called an exceptional set of $Z$ with respect to $f$. This is the same as saying that $f(X) \cap f(X \setminus Z) \subset S$, so $Z$ is almost saturated with respect to $f$ if and only if $f(X) \cap f(X \setminus Z)$ is finite.

The function $f : X \to Y$ is said to be always almost saturated or aas for short if every definable set $Z \subset X$ is almost saturated with respect to $f$. 

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Lemma 19. Let $f : X \to Y$ be a definable function. Then $f$ is aas if and only if the map $f_* : S_X(U) \to S_Y(U)$ satisfies that if $f_*(p) = f_*(q)$ is a non-realized type, then $p = q$.

Proof. Suppose $f$ is aas and suppose $p$ and $q$ are types such that $f_*(p) = f_*(q)$ is non-realized. Suppose $Z \subset X$ is definable and $Z \in p$. Then there is $S \subset Y$ finite as in the definition of aas. Then because $f_*(p)$ is not realized we get $f(Z) \setminus S \in f_*(p)$. So we conclude $f^{-1}((f(Z) \setminus S)) \in q$. We have by assumption that $f^{-1}((f(Z) \setminus S)) \subset Z$, so $Z \in q$ as required.

In the other direction, suppose $Z \subset X$ is definable and is not almost saturated relative to $f$. Then $E = f(Z) \cap f(X \setminus Z)$ is a definable set and is infinite. Let $r$ be a complete non-realized global type in $E$. Then there are complete types $p$ and $q$ in $X$ such that $p$ is in $Z$ and $q$ is in $X \setminus Z$, and $f_*(p) = f_*(q) = r$ as required. $\Box$

Now we give some simple properties of this notion.

Lemma 20. Suppose $f : X \to Y$ and $g : Y \to Z$ are definable maps.

1. If $f$ and $g$ are aas then $gf$ is aas.

2. If $gf$ is aas and $g$ has finite fibers then $f$ is aas.

Proof. Suppose $f$ and $g$ are aas and take $W \subset X$ a definable set. Take $S \subset Y$ a finite exceptional set of $W$ with respect to $f$ and $T \subset Z$ a finite exceptional set of $f(W)$ with respect to $g$. Then $f(S) \cup T$ is a finite exceptional set of $W$ with respect to $gf$.

Now suppose $gf$ is aas and $g$ has finite fibers, and suppose $W \subset X$ is a definable set. Take $S \subset Z$ a finite exceptional set of $W$ with respect to $gf$. Then $g^{-1}(S)$ is a finite exceptional set of $W$ with respect to $f$. $\Box$

Next we give some examples of aas maps in henselian fields of characteristic residue characteristic 0 and in algebraically closed valued fields of any characteristic.

Lemma 21. If $K$ is a henselian field of residue characteristic 0 then $K^\times \to RV$ is aas.

Proof. Let $X \subset K^\times$ be definable and let $a_1, \ldots, a_n \in K$ and $Y \subset RV^n_0$ as in Proposition\[8\]

If $x \in K^\times$ and $rv(x) \neq rv(a)$ then $rv(xy-a) = rv(x-a)$ for every $y \in 1+M$.

So we get that $S = \{rv(a) \mid a_i \neq 0\}$ makes $X$ almost saturated relative to $rv$. $\Box$

Lemma 22. If $K$ is a henselian field of residue characteristic 0 or a model of ACVF then $O \to O/M$ is aas.

Proof. Assume $K$ is henselian of residue characteristic 0. If $\pi : O \to k$ is the projection then for $x \in O$ and $a$ such that $\pi(x) \neq \pi(a)$ or for $a \in K \setminus O$, we have $rv(x-a) = rv(x+y-a)$ for all $y \in M$, we conclude by Proposition\[8\].

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Now assume that $K$ is algebraically closed. Being almost saturated is closed under boolean combinations, so by Swiss-cheese decomposition, (Proposition 1), it is enough to show that balls are almost saturated under the map $O \to O/M$. A ball has image a point or it contains all of $O$.

Lemma 23. Suppose $K$ is a henselian field of residue characteristic 0 or an algebraically closed field. Then the map $U_r \to B_r/B_r^-$ given by $1 + x \mapsto \bar{x}$ is aas.

Proof. Under the bijection $U_0^+ \to M$ given by $x \mapsto x - 1$ this becomes the projection $B_r \to B_r/B_r^-$, after rescaling by an element $a \in B_r \setminus B_r^-$ this becomes Lemma 22.

Lemma 24. If $K$ is a henselian field of residue characteristic 0 or an algebraically closed field then $O^\times \to k^\times$ is aas.

Proof. This is equivalent to Lemma 22.

Lemma 25. If $K$ is a model of ACVF or a henselian valued field of characteristic 0 and we are given a Weierstrass equation with coefficients in $O$ then the maps $E_a \to B_a/B_a^-$ are aas.

Proof. Under the bijection of Lemma 9 and after rescaling the result corresponds to Lemma 22.

Lemma 26. Suppose $K$ is a model of ACVF. Then the map $\nu : K \to \Gamma_\infty$ is aas.

Proof. The condition of being almost saturated with respect to $\nu$ is closed under boolean combinations, so by Proposition 1 we see that we only have to prove it for $X = K$, or $X$ equal to a point or a ball. We may assume then that $X$ is a ball. If $0 \in X$ then $X = v^{-1}\nu(X)$ is saturated with respect to $v$. Otherwise $\nu(X) = \{r\}$ so the condition holds too (with $S = \{r\}$).

We mention that the previous result does not hold for most other valued fields, for example $K = k((t))$ where $k$ is a field of characteristic 0 and $k^\times \neq (k^\times)^2$, as then $K^2$ is not almost saturated with respect to $v$.

Lemma 27. Let $K$ be a model of ACVF, and suppose given a Weierstrass equation with coefficients in $O$. Then the map $E(K) \to \hat{E}(k)$ is aas.

Proof. By Lemma 19 and by quantifier elimination we have to prove that if $(x, y) \in E(K')$ and $(x, y) \in E(K')$ are such that $\nu(x, y, r(x', y') \notin \hat{E}(k)$ then $K(x, y) \cong K(x', y')$ as valued fields. This is to say, if $f(T, S), g(T, S) \in K[T, S]$ are polynomials then $\nu(f(x, y)) \leq \nu(g(x, y))$ is equivalent to $\nu(f(x', y')) \leq \nu(g(x', y'))$. Notice that $x, y, x', y' \in O'$ because otherwise $\nu(x, y)$ is the point at infinity. In this case $r(x, y) = (\pi(x), \pi(y))$, for $\pi: O \to k$.

Dividing the polynomials $f, g$ by $h = S^2 + a_1TS + a_3S - a_1T^3 - a_2T^2 - a_6$ in $K[T][S]$ we see that is is enough to consider polynomials with degree in $S$ less than 2. In this case the result will follow from the equation $\nu(f(x, y)) =$
\[ \min\{v(a) \mid a \text{ is a coefficient of } f\}, \] which is in turn equivalent to \( \pi(f(x, y)) \neq 0 \) if \( \pi(f(T, S)) \neq 0 \) for a \( f \in O[T, S] \) with degree in \( S \) less than 2. To see this last point note that \( \pi(x) \notin k \) and so it is transcendental over \( k \). So we have \( \pi(h) \) is an irreducible polynomial in \( k[T, S] \) (it has degree 2 in \( S \), and the degree in \( T \) shows it has no root which is a polynomial in \( K[T] \)), and so \( \pi(h)(\pi(x), S) \) is an irreducible polynomial in \( k(x)[S] \). If \( \pi(f)(x, y) = 0 \) then \( \pi(h)(x, S)|\pi(f)(x, S) \), and so by degree reasons \( \pi(f)(x, S) = 0 \), and this implies \( \pi(f) = 0 \).

Lemma 28. Let \( K \) be a henselian valued field of residue characteristic 0, and assume given a Weierstrass equation with coefficients in \( O \). Then the map \( E(K) \to \tilde{E}(k) \) is aas.

Proof. By Lemma 19 we have to see that for an elementary extension \( K' \) of \( K \) and \( (x, y), (x', y') \in E(K') \) such that \( \pi(x, y), \pi(x', y') \notin \tilde{E}(k) \) and \( \pi(x, y) \equiv_K \pi(x', y') \) then \( (x, y) \equiv_K (x', y') \). We may assume that \( \pi(x) = \pi(x') \).

By elimination of quantifiers relative to RV, we have to see that \( rv(K(x, y)) \equiv_K rv(K(x', y')) \) and that \( K(x, y) \cong K(x', y') \) as valued fields. We have that \( K(x, y) \equiv K(x', y') \) as valued fields as in the previous proof. The previous proof also shows that the image of \( K(x, y) \) and of \( K(x', y') \) under \( v \) is \( \Gamma_{\infty} \), and the kernel of the map \( rv(K(x, y)^\times) \to \Gamma \) and of the map \( rv(K(x', y')^\times) \to \Gamma \) is \( k(\pi(x), \pi(y))^\times \). We conclude that \( rv(K(x, y)) = rv(K(x', y')) \).

The following aside is a consequence of the previous discussion.

Proposition 29. Let \( K \) be a model of ACVF of residue characteristic not 2 or 3. Let \( E \) be an elliptic curve over \( K \). Then \( E_0(K) \subset E(K) \) is the maximal stably dominated connected subgroup of \( E(K) \), obtained in [11] Corollary 6.19.

Proof. From Proposition 13 we get that \( E(K)/E_0(K) \) is \( \Gamma \)-internal and in fact \( E(K)/E_0(K) \cong O(\omega(\Delta))/\mathbb{Z}(\Delta) \) for \( \Delta \) the discriminant of minimal Weierstrass equation of \( E \).

From Lemma 28 we get that the type \( p(x) \) that says \( x \in E_0 \) and \( \pi(x) \in \tilde{E}(k) \) is a complete type. By quantifier elimination in the language of the valued field \( K \) together with the residue map \( O \to k \) we conclude that every definable family \( \{S_a\}_a \) with \( S_a \subset k^n \) is definably definably piecewise in \( a \) of the form \( S_a = T_f(a) \) for a definable function in \( f \) into \( k^m \) and \( T_b \) a definable family in the pure field language. From this and the fact that all types in the stable theory ACVF are definable we obtain that the type \( p \) is definable. The type \( p \) is \( E_0 \)-invariant, as \( \pi_*(gp) = \pi(g)\pi_*(p) = \pi_*(p) \). The second equation because \( \tilde{E}_k \) is an irreducible variety over \( k \).

Finally the type \( p \) is stably dominated via \( \pi \) over some \( A \)-parameters defining \( E \). We recall this means that \( b \vDash p|A \) and \( \pi(b) \vDash \pi_*(p)|B \) then \( f(a) \vDash p|B \) for \( A \subset B \). The first condition implies \( b \in E_0(K) \) and the second that \( \pi(b) \) is generic in \( \tilde{E} \) over \( k_B \) (where \( k_B \) is the residue field of \( acl(B) \)).

In residue characteristic 2 and 3 the equations for a minimal Weierstrass equation become more complicated and it is not clear to me how to prove for example that it stabilizes on finite extensions. This is likely to be the case however, because the construction in [11] does not have this restriction.
Lemma 30. If \( G \to H \) is a definable group morphism which is aas, and \( X \subset G \) is a type-definable subgroup, then either \( f(X) \) is finite or \( X = f^{-1}f(X) \).

Proof. Let \( K \) be the kernel of \( f \). Suppose \( f(X) \) is infinite. There is \( C \) a small set such that if \( h \in H \setminus C \) then \( f^{-1}(h) \cap X = \emptyset \) or \( f^{-1}(h) \subset X \). Indeed if \( X \) is a small intersection of definable sets \( D_i \) and \( S_i \) is a finite exceptional set of \( D_i \) with respect to \( f \), then \( C = \cup_i S_i \) is a small exceptional set of \( X \) with respect to \( f \).

Take \( y \in f(X) \setminus C \). Then \( K \subset f^{-1}(y)f^{-1}(y)^{-1} \subset X \) as required. \( \square \)

The map \( RV \to \Gamma \) is usually not aas, but we have the following weakening.

Lemma 31. Let \( 0 \to A \to B \to C \to 0 \) be an exact sequence of abelian groups as in Situation [4]. If \( X \subset B \) is definable then there is an \( n \in \mathbb{Z}_{>0} \) such that \( X \) is almost saturated with respect to \( B \to B/nA \).

Proof. This condition is closed under boolean combinations in \( X \) so we just need to check the cases described in Proposition [5]. These are clear. \( \square \)

3 Subgroups of one-dimensional algebraic groups

Here we give the type-definable subgroups of an algebraic group of dimension 1 in a model of ACVF or a model of PL₀.

3.1 Subgroups of the additive group

As a first application we have the following. This will be proven again in other characteristics later in Proposition [30], but the proof is illustrative so I include it.

Proposition 32. If \( k \) is an algebraically closed field of residue characteristic 0 and \( X \subset k \) is a type-definable subgroup then \( X \) is a small intersection of definable groups. The definable groups of \( k \) are of the form \( B_r = \{ x \in k \mid v(x) \geq r \} \) or \( B_r^- = \{ x \in k \mid v(x) > r \} \) or \( k \) or 0.

Proof. Suppose \( r \in v(X) \), and take \( \pi : B_r \to k \) a group map with kernel \( B_r^- \) obtained by rescaling. Then \( \pi(X \cap B_r) \) is a non-zero type-definable subgroup of \( k \), and so it is equal to \( k \). By Lemma [30] and [22] we get \( B_r \subset X \). We conclude \( v(X) \subset \Gamma \) is a type-definable upwards closed set, and so it is a small intersection of intervals. And so \( X \) corresponds to the corresponding intersection of balls. \( \square \)

Lemma 33. If \( k \) is a pseudo-finite field of characteristic 0 and \( X \subset k \) is a type-definable subgroup, then \( X = 0 \) or \( X = k \).

Proof. The theory of \( k \) is super-simple, so \( X \) is an intersection of definable groups. See Theorem 5.5.4 of [20]. So we may assume \( X \) is an infinite definable group.
The theory of $k$ is geometric and admits elimination of imaginaries after adding some constants, see [3] and 1.13 in [2], so in this case we have that \( \dim(X) = 1 \), and so \( \dim(k/X) = 0 \). We conclude that $k/X$ is finite. But then $nk = k \subset X$ as required. 

**Proposition 34.** Suppose $K$ is a henselian valued field with value group a $\mathbb{Z}$-group and residue field a pseudo-finite field of characteristic 0. Then every non-trivial type-definable subgroup $X \subset K$ of $(K, +)$ is an intersection of balls around the origin.

**Proof.** This is as in Proposition 32 using Lemma 33. 

Now we reprove Proposition 32 for algebraically closed valued fields in any characteristic.

**Lemma 35.** Suppose $G$ is a commutative group definable in some theory and $v : G \to I$ is a definable function into a totally ordered set, satisfying $v(a + b) \geq \min\{v(a), v(b)\}$, $v(-x) = v(x)$, and $v(0)$ is maximal. Let $B_r = \{a \mid v(a) \geq r\}$ and $B_r^- = \{a \mid v(a) > r\}$. Suppose every definable subset of $G$ is a boolean combination of translates of $B_r$ and $B_r^-$ or the trivial subgroups. Suppose $B_r/B_r^-$ is infinite for every $r \in I$. Then every type-definable subgroup of $G$ is a small intersection of definable subgroups and a definable subgroup is trivial or finite or a finite extension of a $B_r$ or $B_r^-$.

**Proof.** Let us call $B_r$ or $B_r^-$ closed and open balls respectively.

In this situation I claim that if $S$ is a nonempty set of the form $C \setminus \bigcup_i a_i + C_i$ where each $C, C_i$ are trivial groups or balls, then $C = S - S$. Assume first $C$ is of the form $B_r^-$. In this case $S$ contains a set of the form $B_r^- \setminus B_s$ for $s > r$. So without loss of generality $S = B_r^- \setminus B_s$. In this case if we choose $x \in S$ arbitrarily then $r < v(x) < s$. If $y \in B_s$ then $v(y) \geq s > v(x)$ and so $v(x + y) = v(x) \in S$, and $y = (x + y) - x \in S - S$. Similarly if $S$ equals $G \setminus X$ where $X$ is a union of translates of $B_r$, $B_r^-$ or 0. Now suppose $S = B_r \setminus X$ then without loss of generality $X$ is a union of a finite number of translates of $B_r^-$, say $X = F + B_r^-$ with $F$ finite. Denote $\pi : B_r \to B_r/B_r^-$ and suppose $x \in B_r$ is such that $\pi(x) \notin \pi(F)$ and $\pi(x) \notin \pi(F) - \pi(F)$. Then for $f \in F$ and $y \in f + B_r^-$, we get $y = (x + y) - x$ with $x \in S$ and $x + y \in S$, so $y \in S - S$. This finishes the proof of the claim.

Let $X \subset G$ be a type-definable group and let $X \subset D$ be a definable set. We have to find a definable $B$ such that $X \subset B \subset D$ and $B$ is a group of the form claimed. Let $S$ be a symmetric definable set with $X \subset S$ and $3S \subset D$. Then $S$ is a finite union of the form $S = \bigcup_k (a_k + E_k)$ where each $E_k$ is of the form discussed in the first paragraph. So there is $B_i$ which is either a ball or trivial such that $B_i = E_i - E_i$ and $E_i \subset B_i$. If $B = \bigcup_i B_i$ then $B \subset S - S$ and $S \subset F + B$ for $F$ finite.

Now if $H$ is the group generated by $F + B$ then $H/B$ is a finitely generated group, so $H = B_t \oplus \mathbb{Z}^*$ where $B_t/B$ is the torsion part of $H/B$ and so is a finite group. As $X + B$ is a type definable subgroup of $H$ one concludes by compactness that $X + B \subset B_t$. And so being the inverse image of a subgroup

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of the finite group $B/B_t$ it is definable. So $X \subset X + B \subset D$ is the group we wanted.

**Proposition 36.** If $K$ is an algebraically closed field and $X \subset K$ is a type-definable subgroup, then it is an intersection of definable groups, a definable group is a finite extension of a ball around 0.

*Proof.* This is a particular case of Lemma 35.

### 3.2 Subgroups of the multiplicative group

**Lemma 37.** If $\Gamma$ is a divisible ordered abelian group, and $X \subset \Gamma$ is a type-definable subgroup then $X$ is a small intersection of groups of the form $o(a)$ or it is trivial.

The proof in Proposition 3.5 of [1] works.

The following Proposition will be proven again in any characteristic, but I include it as the proof is illustrative.

**Proposition 38.** If $K$ is an algebraically closed field of residue characteristic 0 and $X \subset K^\times$ is a type-definable group, then $X$ is a small intersection of groups of the form $O_{V}^\times$ or $X$ has a finite index subgroup which is a small intersection of groups of the form $U_r$ or of the form $U_{-r}$.

*Proof.* By Lemmas 37, 26 and 30 we are done in the case $v(X) \neq 0$.

Otherwise if $X \subset O^\times$ then by Lemmas 24 and 30 we are done in case $\pi(X) \subset k^\times$ is infinite.

Otherwise we may assume $X \subset U_{0}^\times$. In this case we conclude as in Proposition 32 using Lemma 23.

Next we prove a version of Proposition 38 in case $K$ is a model of $\text{PL}_0$. In this case the residue field is pseudo-finite instead of algebraically closed, so first we describe the subgroups of $k^\times$ for $k$ pseudo-finite. Afterwards we do the modifications necessary to substitute in place of Lemma 26.

**Lemma 39.** If $k$ is a pseudo-finite field and $X \subset k^\times$ is a type-definable subgroup, then $X$ is finite or $X = \bigcap_n X(k^\times)^n$.

*Proof.* As in Lemma 33 we may assume $X$ is infinite definable. In this case we have dim$(k^\times/X) = 0$ and so $(k^\times)^n \subset X$ for some $n$.

**Lemma 40.** Let $0 \to A \to B \to C \to 0$ be an exact sequence of abelian groups as in [4]. Take $n \in \mathbb{Z}_{>0}$. Then the sequence $0 \to A/nA \to B/nA \to C \to 0$ is split exact and any splitting is definable.

For a proof see for example [1] Proposition 8.3.

**Lemma 41.** Let $0 \to A \to B \to C \to 0$ be an exact sequence of abelian groups as in [4]. And let $X \subset B$ be a type-definable group. Then either $X \subset A$ or $X = \bigcap_n (X_n)$ where each $X_n$ is a finite extension of a group of the form $g^{-1}(Y_n) \cap mB$ for $Y_n \subset C$ a type-definable group.
Proof. Because $C$ is torsion-free $g(X)$ is either trivial or infinite. As in Lemma 30 using Lemma 31 we conclude that $\cap_n nA \subset X$ so taking $X_n = X + nA$ we may assume $nA \subset X$ for some $n$. Now if we take $\pi : B \to B/nA \to A/nA$ where $B/nA \to A/nA$ is a projection as in Lemma 40 then for $m = \text{card}(A/nA)$ we get $\pi(mB) = 0$ and also if we take $Y = mB \cap X$ then $X/Y = (X + mB)/mB$ is finite and $Y \subset g^{-1}g(Y) \cap mB \subset X$ as required.

Lemma 42. Let $\Gamma$ be a $Z$-group. If $X \subset \Gamma$ is a type-definable subgroup of $(\Gamma, +)$ then $X$ is a small intersection of groups of the form $o(a)$ and $n\Gamma$ for $n \in \mathbb{Z}_{>0}$.

For a proof see Proposition 3.5 of [1].

Now we put together Lemmas 42, 39 and 41

Lemma 43. Suppose $K$ is a model of $PL_0$. If $X \subset RV$ is a type-definable group, then it is a small intersections of groups which are finite extensions of groups of the form:

1. the trivial group
2. $(k^\times)^n$
3. $RV^n \cap v^{-1}(o(a))$
4. $RV^n$

Proposition 44. Suppose $K$ is a model of $PL_0$. Then every type-definable subgroup $X \subset K^\times$ is a small intersection of groups which are finite extensions of groups of one of the following forms:

1. $(O^\times)^n$
2. $o(a) \cap (K^\times)^n$
3. $(K^\times)^n$
4. $U_a$ for $a > 0$.

Proof. By Lemma 43 and Lemma 30 if $rv(X)$ is infinite then we have cases 1-3. Otherwise we may take $X \subset U_1$ and then we can conclude as in Lemma 34.

Next we prove Proposition 38 in any residue characteristic.

Lemma 45. If $K$ is an algebraically closed valued field and $G$ is a definable commutative group in $K$, and $f : M \to G$ is a bijection satisfying $f^{-1}(f(x) + f(y)) = x + y + r$ for $r$ with $v(r) > \text{Max}\{v(x), v(y)\}$, $f(0) = 0$ and $f^{-1}(-f(x)) = -x + s$ for an $s$ satisfying $v(s) \geq v(x)$, then the function $w : G \to \Gamma_\infty$ given by $w = vf^{-1}$ satisfies the hypothesis of Lemma 32.
**Proof.** Suppose $x, y \in G$ such that $w(x), w(y) \geq t$. We have to see that $w(x + y) \geq t$. There are $a, b \in M$ such that $x = f(a)$ and $y = f(b)$. Then $f^{-1}(x + y) = a + b + r$ with $v(r) > \text{Max}\{v(a), v(b)\} \geq t$ and so $w(x + y) = v(a + b + r) \geq t$.

Now suppose that $w(x) = t$, we will see that $w(-x) \geq t$. If $x = f(a)$ then $f^{-1}(-x) = -a + s$, with $v(s) \geq v(-a) = t$. From here $w(-x) = v(-a + s) \geq t$. Applying this to $-x$ we get $w(x) = w(-x)$.

If we call $G_t$ and $G_t^-$ the sets given by $w(x) \geq t$ and $w(x) > t$ in $G$, then we show that $f(a + B_t) = f(a) + G_t$ and $f(a + B_t^-) = f(a) + G_t^-$. Indeed if $x \in f(a) + E_t$ then there is $b \in B_t$ such that $x = f(a) + f(b)$. Then $f^{-1}(x) = a + b + r$. We have $v(r) > v(b)$ and so $v(b + r) = v(b) \geq t$, so $x \in f(a + B_t)$. In the other direction if $x \in f(a + B_t)$ then there is $b \in M$ such that $x = f(a) + f(b)$. In this case $f^{-1}(x) = a + b + r$ with $v(r) > v(b)$ and so $v(b) = v(b + r) \geq t$.

From this $w(f(b)) = v(b) \geq t$, and we conclude $f(b) \in G_t$ and $x \in f(a) + G_t$. The proof with $B_t^-$ and $G_t^-$ is similar.

Now Swiss cheese decomposition applied to $M$ implies that every definable subset of $G$ is a boolean combination of translate of the trivial group or $G_t, G_t^-$ as required. $\square$

In fact in the applications one has $v(r) \geq v(x) + v(y)$ and $v(s) \geq 2v(x)$.

**Proposition 46.** Suppose $K$ is an algebraically closed field. Suppose $X \subset K^\times$ is a type-definable subgroup. Then $X$ is a small intersection of groups of the form $o(a)$, or it is equal to $K^\times$ or it is equal to $O^\times$, or $X$ is a small intersection of definable groups, which are finite extensions of groups of the form $U_r$ or $U_r^-$.

**Proof.** If $v(X)$ is not trivial then it is infinite because $\Gamma$ is torsion free, so $X = v^{-1}v(X)$ by Lemma 39 and it is either $K^\times$ or a small intersection of groups of the form $o(a)$ by Lemma 37.

Otherwise $X \subset O^\times$. If $\pi : O^\times \to k^\times$ is the residue map then $\pi(X) \subset k^\times$ is a type-definable group. Because $k$ is a pure algebraically closed field we conclude $\pi(X) = k^\times$ or it is finite. In the first case because $\pi$ is aas by Lemma 24 we conclude $X = O^\times$. In the other we may assume $X \subset 1 + M$. For this group we have the function $w : 1 + M \to \Gamma_\infty$ given by $w(1 + x) = v(x)$, this satisfies the properties of Lemma 35, see Lemma 45. $\square$

### 3.3 Subgroups of the elliptic curves

Suppose $K$ is a model of $\text{PL}_0$. We now give the type-definable subgroups of an elliptic curve over $K$.

**Lemma 47.** If $k$ is a pseudo-finite field characteristic 0 and $E$ is an elliptic curve over $k$, then $X \subset E$ type-definable is either finite or $X = \cap_n(nE + X)$.

**Proof.** The proof is the same as that of the Lemma 39. Recall only that $E(k)/nE(k)$ is finite. This is because the $n$-torsion is a subgroup of the $n$-torsion of $E(k^{alg})$ which is finite (and isomorphic to $(\mathbb{Z}/n\mathbb{Z})^2$) and so the multiplication by $n$ map $E(k) \to nE(k)$ has image of dimension 1. $\square$
Proposition 48. Let $K$ be a model of $PL_0$.

Then if $X \subset E_0(K)$ is a type-definable subgroup then $X$ is a small intersection of groups which are finite extensions of groups of the form

1. $nE_0(K)$
2. $E_r$
3. $E^{-r}$

Proof. The proof is the same as in Proposition 44 using Lemma 28 and Lemma 47. We just mention that $nE_0(K) = \pi^{-1}nE_0(k)$ by Lemma 13. □

Proposition 49. Let $K$ be a model of ACVF. Let $E$ be an elliptic curve over $K$ given by a given Weierstrass equation with coefficients in $O$.

If $X \subset E_0$ is a type-definable subgroup then it is equal to $E_0$ or it is a small intersection of groups which are finite extensions of the form $E_r$ or $E^{-r}$.

Proof. The proof is the same as the one in Proposition 46, using the map in Lemma 9. □

3.4 Subgroups of the twisted multiplicative groups

In this section we handle the remaining case for a valued field $K$ pseudo-finite of characteristic 0.

If $K$ is a field and $d \in K^\times \setminus (K^\times)^2$ we call $G(d)$, the subgroup of the multiplicative group $K(\sqrt{d})^\times$ with norm one, a twisted multiplicative group of $K$. $G(d)$ acts on the two-dimensional $K$-vector space $K(\sqrt{d})$ so choosing the basis $\{1, \sqrt{d}\}$ we get an embedding $G(d) \subset \text{GL}_2(K)$, $G(d) = \{ \begin{bmatrix} a & db \\ b & a \end{bmatrix} | a^2 - db^2 = 1 \}$.

Lemma 50. If $k$ is a pseudo-finite field and $d \in k^\times \setminus (k^\times)^2$ and if $X \subset G(d)$ is a type-definable group, then $X$ is either finite or a small intersection of groups that are finite extensions of $G(d)$.

Proof. The proof is the same as in Lemma 47. □

Assume that $K$ is a Henselian valued field of residue characteristic not 2. And assume first that $d \in K^\times \setminus (K^\times)^2$ is such that $v(d) = 0$. Because $K$ is Henselian there exists a unique $K$-invariant extension of the valuation of $K$ to $L = K(\sqrt{d})$ (we will also see this directly). This valuation is then $v : L \to \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_\infty$ given by the equation $v(\alpha) = \frac{1}{2}v(N(\alpha))$. If $\alpha = a + b\sqrt{d}$ and $a, b \in O$ but $a$ and $b$ are not both in $M$, then $v(\alpha) \geq 0$ and if $v(\alpha) \geq r > 0$ then $v(a^2 - b^2d) \geq 2r$ and $v(a^2) = v(b^2d) = 0$. This implies then that the equation $x^2 = d$ has approximate solution $ab^{-1}$ so an application of Hensel’s Lemma implies $d \in (O^\times)^2$. We conclude that $v(a + b\sqrt{d}) = \min\{v(a), v(b)\}$. We have the short exact sequences $1 \to O_L^\times \to L^\times \to \Gamma_L \to 0$, $1 \to U_L^- \to O_L^\times \to k_L^\times \to 1$.
and the bijection $U_L^r \rightarrow M_L$ that gives a filtration $U_{L,r}$ and $U_{L,r}$ satisfying $U_{L,r}/U_{L,r}^- \cong (L,+)$. We now describe the restrictions of this on $G(d)$. To start notice that $G(d) \subset O^\times_L$. Next, we have seen that $k_L = k(\sqrt{d})$, and an application of Hensel's Lemma shows that the image of $G(d)$ in $k_L$ is $G(d)$. Denote the kernel of $G(d) \rightarrow G(d)$ to be $G(d)^-$. These are the elements of the form $a + b\sqrt{d}$ such that $v(a - 1), v(b) > 0$ and $a^2 - b^2d = 1$. In this case $v(a^2 - 1) = v(a - 1) = v(b^2d) = 2v(b)$, so in particular $v(a - 1) = v(b)$.

We conclude now that the map $f : G(d)^- \rightarrow \mathcal{M}$ given by $a + b\sqrt{d} \mapsto b$ is a bijection, (that it is surjective follows from Hensel’s Lemma) and satisfies that $f(\alpha_1\alpha_2) = f(\alpha_1) + f(\alpha_2) + r$ where $r > v(f(\alpha_1) + v(f(\alpha_2))$ and $f(\alpha_1^{-1}) = -f(\alpha_1)$. In particular the groups $G(d)^+ = U_{L,r} \cap G(d)$ and $G(d)^- = U_{L,r}^\prime \cap G(d)$ satisfy $G(d)^+/G(d)^{\prime} = B_r/B_r^{\prime} \cong (K,\mathbb{Z})$ via $f$.

**Lemma 51.** Let $K$ be a Henselian valued field residue characteristic 0. Let $d \in K^\times \setminus (K^\times)^2$ be such that $v(d) = 0$. Then the map $G(d) \rightarrow G(\bar{d})$ is aas.

**Proof.** This proof is the same as in Lemma 25.

**Lemma 52.** Let $K$ be a Henselian valued field residue characteristic 0. Let $d \in K^\times \setminus (K^\times)^2$ be such that $v(d) = 0$. Then the maps $G(d)^+/G(d)^{\prime} \rightarrow k$ are aas.

**Proof.** Under the bijection $G(d)^- \cong M$ and after rescaling this becomes Lemma 25.

Now assume $K$ is Henselian of residue characteristic not 2 and assume the value group is a $\mathbb{Z}$-group. Take $d \in K$ such that $v(d) = 1$. As before we know that the valuation in $K$ extends in a unique way to a $K$-invariant valuation in $K(\sqrt{d})$. In this case we have $v(a + b\sqrt{d}) = \min\{v(a), v(b) + \frac{1}{2}\}$, $O_L = O + O\sqrt{d}$ and $k_L = k$, $\Gamma_L = \Gamma$. As before $G(d) \subset O^\times_L$. Now the residue map $G(d) \rightarrow k$ takes $G(d)$ onto $\{\pm 1\}$. Its kernel is denoted $G(d)^-$ and it consists of the elements $a + b\sqrt{d}$ such that $v(a - 1) > 0$ and $v(b) \geq 0$. In this case we have the groups $U_{r,L}$ for $r \in \Gamma$ given by the elements $a + b\sqrt{d}$ such that $v(a - 1) \geq r$ and $v(b) \geq r$, and the groups $U_{r+\frac{1}{2},L}$ consisting of elements $a + b\sqrt{d}$ such that $v(a - 1) \geq r + 1$ and $v(b) \geq r$. We obtain then the group isomorphisms $U_{r,L}/U_{r+\frac{1}{2},L} \rightarrow B_r/B_{r+1}$ given by $a+b\sqrt{d} \mapsto a-1$ and $U_{r+\frac{1}{2},L}/U_{r+1,L} \rightarrow B_r/B_{r+1}$ given by $a + b\sqrt{d} \mapsto \bar{b}$. If we denote $G(d)_r = U_{r,L} \cap G(d)$ then we have that $G(d)_r = U_{r+\frac{1}{2},L} \cap G(d)$. Indeed if $a^2 - b^2d = 1$ and $v(a - 1) > 0$, then $v(a - 1) = v(a^2 - 1) = v(b^2d) = 2v(b) + 1$ so if $v(b) \geq 0$ then $v(a + b\sqrt{d} - 1) = v(b) + \frac{1}{2}$.

**Lemma 53.** If $K$ is a henselian valued field of residue characteristic 0, and valued group a $\mathbb{Z}$-group, and if $d \in K$ is such that $v(d) = 1$, then the maps $G_r(d) \rightarrow B_r/B_{r+1}$ given by $a + b\sqrt{d} \mapsto b$ is aas.

**Proof.** Indeed the map $G(d)^- \rightarrow O$ given by $a + b\sqrt{d} \mapsto b$, is bijective by Hensel’s Lemma, under this bijection the map becomes as in Lemma 25.
The map \( f : G(d)^{-} \to O \) mentioned in the previous proof satisfied that
\[
v(f(\alpha)) = v(\alpha) - \frac{1}{2} \quad \text{and} \quad f(\alpha^{-1}) = -f(\alpha), \quad f(\alpha_1 + \alpha_2) = f(\alpha_1) + f(\alpha_2) + r \quad \text{with} \quad v(r) > v(f(\alpha_1)) + v(f(\alpha_2)).
\]

**Proposition 54.** Suppose \( K \) is a pseudo-local field of characteristic 0. Suppose \( d \in K^\times \setminus (K^\times)^2 \) and let \( G(d) \) be the corresponding twisted multiplicative group. If \( X \subset G(d) \) is a type-definable group then \( X \) is a small intersection of groups which are finite extensions of groups of the form \( G(d)^n \) or \( G(d)_r \) for \( r \in \Gamma_{>0} \) or \( G(d)^{-} \).

**Proof.** If \( v(d) = 0 \) this is a consequence of Lemmas \( \text{51, 52} \) and \( \text{50} \). If \( v(d) = 1 \) this is consequence of Lemma \( \text{53} \). See the proof of Proposition \( \text{14} \). \( \Box \)

## 4 One dimensional definable groups

Here we give a list of all commutative definable groups of dimension 1 in a model of ACVF of residue characteristic not 2 or 3 and in a model of PL\(_0\).

**Lemma 55.** Let \( 0 \to C \to B \to A \to 0 \) be a short exact sequence of abelian groups definable in some language. Assume that \( C \) is finite, and that for \( n \) equal to the exponent of \( C \), \( A/nA \) is finite and the \( n \)-torsion of \( A \) is finite. Then there is a definable group map \( A \to B \) with finite kernel and cokernel.

**Proof.** Denote \( f : B \to A \) and \( i : C \to B \). Consider \( B' \) the pullback of \( B \to A \) via the multiplication by \( n \) map \( A \to A \). By definition \( B' \) consists of the tuples \((b, a)\) such that \( b \in B, a \in A \) and \( f(b) = na \). Note that the canonical map \( B' \to B \) has finite kernel and cokernel, in fact the kernel is isomorphic to the \( n \)-torsion of \( A \), and the cokernel to \( A/nA \).

We have a short exact sequence \( 0 \to C \to B' \to A \to 0 \). I claim that the exact sequence \( C \to B'/nB' \to A/nA \to 0 \) is a split short exact sequence. Indeed because \( A/nA \) is finite it is enough to see that \( C \to B'/nB' \) is injective with pure image. Because the exponent of \( C \) is \( n \) and that of \( B'/nB' \) divides \( n \) it is enough to see that if \( c \in C \) is such that \( i(c) \in mB' \) then \( c \in mC \) for \( m \) that divides \( n \). So suppose that \((b, a) \in B'\) such that \( m(b, a) = i(c) \). In other words \( ma = 0 \) and \( mb = i(c) \). We have that \( f(b) = na \), but \( na = 0 \) because \( ma = 0 \) and \( m \) divides \( n \), so \( f(b) = 0 \) and \( b = i(c') \) for a \( c' \in C \). We conclude \( c \in mC \) as required.

The composition \( B' \to B'/nB' \) with a projection \( B'/nB' \to C \) is a retraction of \( C \to B' \) so the sequence \( 0 \to C \to B' \to A \to 0 \) is definably split exact. Then the composition of a section \( A \to B' \) with the canonical map \( B' \to B \) is as required. \( \Box \)

**Proposition 56.** Let \( K \) be a model of ACVF of residue characteristic not 2 or 3.

If \( G \) is an abelian one dimensional group definable in \( K \), then there is a finite index subgroup \( H \subset G \) and a finite subgroup \( L \subset H \) such that \( H/L \) is isomorphic to one of the following:
1. \((K, +)\)
2. \((O, +)\)
3. \((M, +)\)
4. \((K^\times, \times)\)
5. \((O^\times, \times)\)
6. \(O(b)/\langle b \rangle\).
7. \((1 + M, \times)\).
8. \((U_r, \times)\).
9. \((U_r^-, \times)\).
10. \(E_0\) for an elliptic curve \(E\).
11. \(E_r\) for \(r > 0\), an elliptic curve \(E\).
12. \(E_r^-\) for \(r > 0\) and an elliptic curve \(E\).
13. \(O_E(b)/\langle b \rangle\) for an elliptic curve \(E\) of multiplicative reduction.

In residue characteristic 0 the finite group \(L\) is not necessary.

Proof. The proof in Proposition 8.2 of [1] goes through.

That the finite kernel is not necessary in residue characteristic 0 is a consequence of Lemma 55. Note that it applies to all the groups mentioned. So now we just need to describe the groups of the form \(B/A\) for \(B\) a group in the list and \(A\) a finite subgroup of \(B\). Cases 1, 2 and 3 are torsion free. In cases 4 and 5 the group \(A\) is finite cyclic and the map \(x \mapsto x^n\) produces an isomorphism \(B/A \cong B\). In case 5 a finite subgroup is of the form \((c, \eta)\) for \(\eta \in O^\times\) an \(m\)th root of unity and \(c^n = b\). In this case \(B/A = O(c)/\langle \eta, c \rangle\). Now the map \(x \mapsto x^m\), \(O(c) \to O(c)\) produces an isomorphism \(O(c)/\langle \eta \eta \rangle \cong O(c)\), so \(B/A \cong O(c)/\langle c^m \rangle\). Cases 6, 7 and 8 are torsion free. In case 9, if \(A \subset E\) is a finite subgroup of an elliptic curve then \(E(K)/A \cong E'(K)\) for an elliptic curve \(E'\). If \(E\) has multiplicative reduction then \(E_0 \cong (O^\times, \times)\) in the analytic language, via the Tate map, so any finite subgroup is cyclic and \(E_0(K)/A \cong E_0(K)\) via the multiplication by \(n\) map. Cases 10, 11, and 12 are torsion free. Case 13 is as in case 5.

We also have that the finite kernel is unnecessary in cases 4, 5, 6, 10, 11 and 14 for any characteristic and additionally in case 1 in the mixed characteristic case. But for example in the mixed characteristic case \(O/pO\ surjects into k\ and so it is no finite and Lemma 55 does not apply.

We mention that Lemma 55 also shows that the finite kernel in Proposition 8.2 in [1] is not necessary.
Proposition 57. Let $K$ a model of PL$_0$.

If $G$ is an abelian one dimensional group definable in $K$, then there is a finite index definable subgroup $H \subset G$ such that $H$ is isomorphic as a definable group to one of the following

1. $(K, +)$
2. $(O, +)$
3. $((K^\times)^n, \times)$
4. $((O^\times)^n, \times)$
5. $O(b^n)/\langle b^n \rangle$.
6. $(U_r, \times)$
7. $G(d)^n$ for $d \in K^\times \setminus (K^\times)^2$ with $v(d) = 0$.
8. $G(d)^-$ for $d \in K^\times$ with $v(d) = 1$.
9. $G(d)_r$ for $d \in K^\times \setminus (K^\times)^2$ with $v(d) = 0$ or $v(d) = 1$ and $r \in \Gamma_{>0}$.
10. $nE(K)/\langle \eta \rangle$ for an elliptic curve $E$ with good reduction and a torsion element $\eta$.
11. $nE_0(K)$ for an elliptic curve $E$ with multiplicative or additive reduction.
12. $E_r$ for $r > 0$ and an elliptic curve $E$.
13. $(O_E(b))^n/\langle b^n \rangle$ for an elliptic curve $E$ of split multiplicative reduction.

Proof. The proof in Proposition 8.2 of [1] goes through.

As before the finite kernel is unnecessary by Lemma 55. We just verify cases 10 and 11. If $E$ has additive reduction then there is a short exact sequence $0 \to E_1(K) \to E_0(K) \to (k, +) \to 0$, so $E_0(K)$ is torsion free. In case $E$ has multiplicative reduction there is a short exact sequence $0 \to E_1(K) \to E_0(K) \to A(k) \to 1$ where $A$ is a possibly twisted multiplicative group over $k$. As $E_1(K)$ is torsion free we conclude that any finite subgroup of $E_0(K)$ is cyclic, because this is the case in $A(k)$, and also the restriction of the reduction map is an isomorphism. We conclude that the multiplication by $n$ map $mE_0(K) \to nmE_0(K)$ produces an isomorphism $mE_0(K)/A \cong nmE_0(K)$ where $A$ is a finite subgroup. In case the curve has good reduction then the $m'$-torsion is included in $\mathbb{Z}/m'/\mathbb{Z}^2$ so a finite group is cyclic or a direct product of two cyclic groups $\langle a \rangle \times \langle b \rangle$ where $\text{ord}(a)\text{ord}(b)$. In the second case the multiplication by $m = \text{ord}(a)$ map produces an isomorphism $nE(K)/A \cong nmE(K)/\langle mb \rangle$. 

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