Persistence of the Thomas-Fermi approximation for ground states supported by the nonlinear confinement

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July 31, 2014

Abstract

We justify the Thomas–Fermi approximation for the elliptic problem with the repulsive nonlinear confinement used in the recent physical literature. The method is based on the resolvent estimates and the fixed-point iterations.

Self-trapping of solitary waves in nonlinear physical media is a commonly known problem of profound significance \cite{1, 2}. An obvious condition is that attractive (alias self-focusing) nonlinearity is necessary for the creation of localized states. Recently, a radically different approach to this problem was proposed in Refs. \cite{3, 4, 5}: repulsive (self-defocusing) nonlinearity that grows at infinity readily gives rise to the self-trapping of localized states, which are stable to weak and strong perturbations alike.

An advantage offered by models with the effective nonlinear confinement is a possibility to find particular solutions in an exact form \cite{6, 7}, and to apply analytical methods to the qualitative approximation of various localized states \cite{8}. The simplest method used for approximating the ground state of energy is the Thomas–Fermi (TF) approximation \cite{9, 10}. Comparison with numerical results has demonstrated that the TF approximation produces quite accurate results for the self-trapped modes with sufficiently large amplitudes \cite{3, 8}. The objective of the present work is to produce a rigorous estimate of the proximity of the TF approximation to true ground states in models with the spatially growing strength of the defocusing cubic nonlinearity.

In a similar context of the elliptic problem with the harmonic confinement and the defocusing cubic nonlinearity, the TF approximation was rigorously justified using calculus of variations \cite{11} and reductions to the Painlevé-II equation \cite{12, 13}. The difficulty that arises in this context is that the Thomas–Fermi approximation is compactly supported and the derivatives of the ground state diverge in a transitional layer near the boundary. Compared to this complication, we show that the justification of the Thomas–Fermi approximation in the elliptic problem with the nonlinear confinement can be obtained from the standard resolvent estimates and fixed-point arguments.
Following to the main model used in Refs. [3]–[8], we consider the elliptic problem with the repulsive nonlinear confinement,
\[- \epsilon^2 \Delta u + V(x)u^3 - u = 0, \quad x \in \mathbb{R}^d, \quad d = 1, 2, 3, \tag{1}\]
where \(\epsilon\) is a small parameter corresponding to the TF approximation, \(\Delta\) is the \(d\)-dimensional Laplacian, \(u\) is a positive stationary state to be found, and, in accordance with what is said above, the strength of the nonlinear confinement \(V\) is supposed to satisfy the following properties: (i) \(V(x) \geq V_0 > 0\) for all \(x \in \mathbb{R}^d\), and (ii) \(V(x) \to \infty\) as \(|x| \to \infty\). Further constraints on the smoothness of \(V\) and its growth at infinity will be needed for the main result. Note, however, that no symmetry assumptions are needed on \(V\).

The formal TF solution of the elliptic problem is found for \(\epsilon = 0\) and corresponds to the spatially decaying positive eigenfunction:
\[u_0(x) = \frac{1}{\sqrt{V(x)}}, \quad x \in \mathbb{R}^d. \tag{2}\]
If we require \(u_0 \in L^2(\mathbb{R}^d)\) so that the stationary state can be normalized in the \(L^2(\mathbb{R}^d)\) norm, then \(1/V\) needs to be integrable. However, this requirement is not needed for the main persistence result formulated as follows.

**Theorem 1** Assume that \(\nabla \log(V) \in H^2(\mathbb{R}^d)\) for \(d = 1\) or \(\nabla \log(V) \in H^3(\mathbb{R}^d)\) for \(d = 2, 3\). There exist positive constants \(\epsilon_0\) and \(C_0\) such that for every \(\epsilon \in (0, \epsilon_0)\), there exists a unique solution \(u = u_0 + U\) of the nonlinear elliptic problem (1) with \(U \in H^1(\mathbb{R}^d)\) satisfying
\[\|U\|_{H^1} \leq C_0 \epsilon^2. \tag{3}\]

To study the persistence of the TF approximation, we set \(u := w/\sqrt{V}\) and decompose \(w := 1 + r\). In this way, the nonlinear elliptic problem (1) can be rewritten for the perturbation function \(r\):
\[L_\epsilon r = \epsilon^2 F + N(r), \tag{4}\]
where \(N(r) = -3r^2 - r^3\) is the nonlinear term,
\[F = \sqrt{V} \Delta \frac{1}{\sqrt{V}} = -\frac{\Delta V}{2V} + \frac{3|\nabla V|^2}{4V^2}, \tag{5}\]
is the source term, and
\[L_\epsilon = 2 - \epsilon^2 \Delta + \epsilon^2 \frac{1}{V} \nabla V \cdot \nabla - \epsilon^2 F \tag{6}\]
is the linearized operator at the TF approximation. Further, we write \(L_\epsilon\) as a sum of two operators,
\[\bar{L}_\epsilon := 2 - \epsilon^2 \Delta - \frac{\epsilon^2 |\nabla V|^2}{4V^2} \tag{7}\]
and
\[L_\epsilon - \bar{L}_\epsilon := \epsilon^2 \frac{\nabla V \cdot \nabla}{V} + \frac{\epsilon^2}{2} \nabla \left( \frac{\nabla V}{V} \right). \tag{8}\]
Note that the quadratic form associated with \(L_\epsilon - \bar{L}_\epsilon\) is zero after the integration by parts. We establish invertibility of \(\bar{L}_\epsilon\) on any element of \(L^2(\mathbb{R}^d)\) in the following lemma.
Lemma 1 Assume that $\nabla \log(V) \in L^\infty(\mathbb{R}^d)$. There exists a positive constant $\epsilon_0$ such that for every $\epsilon \in (0, \epsilon_0)$ and for every $f \in L^2(\mathbb{R}^d)$, the following is true:

$$\|\tilde{L}_\epsilon^{-1} f\|_{L^2} + \epsilon \|\nabla \tilde{L}_\epsilon^{-1} f\|_{L^2} \leq \|f\|_{L^2}.$$  

(9)

Additionally, if $\Delta \log(V) \in L^\infty(\mathbb{R}^d)$, then for every $f \in H^1(\mathbb{R}^d)$, the following is true as well:

$$\|\tilde{L}_\epsilon^{-1} f\|_{H^1} \leq \|f\|_{H^1}.$$  

(10)

Proof. Under the condition of $\nabla \log(V) \in L^\infty(\mathbb{R}^d)$, the last term in $\tilde{L}_\epsilon$ is a small bounded negative perturbation to the first positive term, whereas the second term, $-\epsilon^2 \Delta$, is a nonnegative operator. The bilinear form

$$a(u, w) := \int_{\mathbb{R}^d} \left(2 \bar{w} u + \epsilon^2 \nabla \bar{w} \cdot \nabla u - \frac{\epsilon^2 |\nabla V|^2}{4V^2} \bar{w} u \right) dx$$  

(11)

satisfies the boundedness and coercivity assumptions in the $H^1(\mathbb{R}^d)$ space:

$$|a(u, w)| \leq C(V) \|u\|_{H^1} \|w\|_{H^1}, \quad a(u, u) \geq \epsilon^2 \|u\|_{H^1}^2,$$  

(12)

where the constant $C(V) > 2$ depends on $\|\nabla \log(V)\|_{L^\infty}$. By the Lax-Milgram Theorem, for every $f \in L^2(\mathbb{R}^d)$, there is a unique $u \in H^1(\mathbb{R})$ such that

$$\|u\|_{L^2}^2 + \epsilon^2 \|\nabla u\|_{L^2}^2 \leq a(u, u) = \int_{\mathbb{R}^d} \bar{u} f dx.$$  

(13)

By the Cauchy–Schwarz inequality, we obtain the bounds (9). Under the additional condition of $\Delta \log(V) \in L^\infty(\mathbb{R}^d)$, we apply operator $\nabla$ to $\tilde{L}_\epsilon u = f$ and write the corresponding equation in the weak form,

$$a(\nabla u, \nabla u) = \int_{\mathbb{R}^d} \nabla \bar{u} \cdot \nabla f dx + \frac{\epsilon^2}{4} \int_{\mathbb{R}^d} \bar{u} \nabla u \cdot \nabla \left(\frac{|\nabla V|^2}{V^2}\right) dx.$$  

(14)

Again applying the Cauchy–Schwarz inequality and using the smallness of $\epsilon^2$, we obtain the bound (10).

Using Lemma 1, the persistence problem (4) can be rewritten as the fixed-point equation:

$$r = \Phi_\epsilon(r) := \epsilon^2 \tilde{L}_\epsilon^{-1} F + \tilde{L}_\epsilon^{-1}(\tilde{L}_\epsilon - L_\epsilon)r + \tilde{L}_\epsilon^{-1}N(r).$$  

(15)

Using the contraction mapping principle, we prove the following result.

Lemma 2 Assume that $\nabla \log(V), \Delta \log(V) \in L^\infty(\mathbb{R}^d)$ and $\nabla \log(V) \in H^2(\mathbb{R}^d)$. There exist positive constants $\epsilon_0$ and $C_0$ such that for every $\epsilon \in (0, \epsilon_0)$, there exists a unique solution $r \in H^1(\mathbb{R}^d)$ of the fixed-point equation (15) satisfying

$$\|r\|_{H^1} \leq C_0 \epsilon^2.$$  

(16)
Proof. We will prove that under the assumptions of the theorem, operator $\Phi_\varepsilon$ is a contraction on the ball $B_\delta(H^1(\mathbb{R}^d))$ of radius $\delta$ if $\delta = O(\varepsilon^2)$ as $\varepsilon \to 0$.

From the assumption of $\nabla \log(V) \in L^\infty(\mathbb{R}^d) \cap H^2(\mathbb{R}^d)$, we realize that $F \in H^1(\mathbb{R}^d)$. Applying the bound (10), we obtain that, for $\varepsilon > 0$ sufficiently small, there is $C_1 > 0$ such that
\[ \| \varepsilon^2 \tilde{L}_\varepsilon^{-1} F \|_{H^1} \leq \varepsilon^2 C_1. \] (17)
By Sobolev’s embedding of $H^1(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$ for any $p \geq 2$ if $d = 1$, for $2 \leq p \leq \infty$ if $d = 2$, and $2 \leq p \leq 6$ if $d = 3$, and by the estimate (9), we obtain that, for $\varepsilon > 0$ sufficiently small, there is $C_2 > 0$ such that
\[ \| \tilde{L}_\varepsilon^{-1} N(r) \|_{H^1} \leq \varepsilon^{-1} \| N(r) \|_{L^2} \leq \varepsilon^{-1} (3 \| r \|^2_{L^2} + \| r \|^3_{L^6}) \leq C_2 \varepsilon^{-1} (\| r \|^2_{H^1} + \| r \|^3_{H^1}). \] (18)
Finally, under the conditions of $\nabla \log(V), \Delta \log(V) \in L^\infty(\mathbb{R}^d)$, we have the bounds
\[ \| (\tilde{L}_\varepsilon - L_\varepsilon) u \|_{L^2} \leq \varepsilon^2 \| \nabla \log(V) \|_{L^\infty} \| \nabla u \|_{L^2} + \frac{1}{2} \varepsilon^2 \| \Delta \log(V) \|_{L^\infty} \| u \|_2, \] (19)
hence, using estimate (9), we obtain that, for $\varepsilon > 0$ sufficiently small, there is $C_3 > 0$ such that
\[ \| \tilde{L}_\varepsilon^{-1}(\tilde{L}_\varepsilon - L_\varepsilon) u \|_{H^1} \leq \varepsilon^{-1} \| (\tilde{L}_\varepsilon - L_\varepsilon) u \|_{L^2} \leq \varepsilon C_3 \| u \|_{H^1}. \] (20)
From these three estimates, it is clear that $\Phi_\varepsilon$ maps a ball $B_\delta(H^1(\mathbb{R}^d))$ of radius $\delta = C_0 \varepsilon^2$ to itself, where $C_0 > C_1$ independently of $\varepsilon$. Similar estimates on the Lipschitz continuous nonlinear term $N(r)$ and perturbation operator $\tilde{L}_\varepsilon^{-1}(\tilde{L}_\varepsilon - L_\varepsilon)$ show that $\Phi_\varepsilon$ is a contraction on the ball $B_\delta(H^1(\mathbb{R}^d))$ of radius $\delta = C_0 \varepsilon^2$. Hence, the assertion of the theorem follows from the Banach fixed-point theorem.

Remark 1 Sobolev’s embedding of $H^s(\mathbb{R}^d)$ to $L^\infty(\mathbb{R}^d)$ for $s > \frac{d}{2}$ allows us to replace the three conditions of the theorem by only one condition: $\nabla \log(V) \in H^2(\mathbb{R})$ if $d = 1$ and $\nabla \log(V) \in H^3(\mathbb{R}^d)$ if $d = 2$ or $d = 3$. With this refinement, Theorem 1 follows from Lemma 2 after the decomposition $u = u_0(1 + r)$ is used.

Remark 2 If the condition of the theorem are replaced by weaker conditions
\[ \nabla \log(V), \Delta \log(V) \in L^\infty(\mathbb{R}^d) \cap L^2(\mathbb{R}^d), \] (21)
then, applying bound (9), we obtain
\[ \| \varepsilon^2 \tilde{L}_\varepsilon^{-1} F \|_{H^1} \leq \varepsilon \| F \|_{L^2}, \] (22)
but radius $\delta = C_0 \varepsilon^2$ is critical for the contraction of mapping $\Phi_\varepsilon$ because of the quadratic term in $N(r)$. Hence, fixed-point arguments can only be used if $\| F \|_{L^2}$ is sufficiently small.

In the end of this article, we discuss several examples.
• If $V$ grows algebraically at infinity with any rate $\alpha > 0$, that is, if
\[ V(x) \sim |x|^\alpha \quad \text{as} \quad |x| \to \infty \]
(the nonlinear confinement of this kind was adopted in Ref. [3]), then
\[ |\nabla \log(V)| \sim |x|^{-1} \quad \text{and} \quad \Delta \log(V) \sim |x|^{-2}. \]
Assuming smoothness of $V$, these conditions provide $F \in H^1(\mathbb{R}^d)$ if $d = 1, 2, 3$, hence Theorem 1 holds for such potentials for any $\alpha > 0$. (Of course, $u \in L^2(\mathbb{R}^d)$ if and only if $\alpha > d$). Some exact expressions are available for particular $V$ and $\varepsilon$ [7].

• If $V$ grows like the exponential or Gaussian function (such as in the models introduced in Refs. [4, 8]), then the assumption $F \in H^1(\mathbb{R}^d)$ fails for any $d = 1, 2, 3$. Nevertheless, if $V = (1 + \beta |x|^2)e^{\alpha|x|^2}$ with $\alpha, \beta > 0$, then the analytic expression is available [4] for a particular value of $\varepsilon = \varepsilon_0$:
\[
\begin{aligned}
u &= \frac{\varepsilon \alpha}{\sqrt{\beta}} e^{-\frac{\varepsilon}{2|\varepsilon|}}, \\
\varepsilon_0 &= \frac{\sqrt{\beta}}{\sqrt{\alpha^2 + d\alpha \beta}}.
\end{aligned}
\] (23)

However, because $F \notin H^1(\mathbb{R}^d)$ ($F$ is not even bounded at infinity), it is not clear if there exists a family of stationary states for any $\varepsilon \in (0, \varepsilon_0)$ that connects the TF approximation (2) and the exact solution (23).

• If $V$ is a symmetric double-well potential, then Theorem 1 justifies the construction of a symmetric stationary state $u$. Symmetry-breaking bifurcation may happen in double-well potentials, but it cannot happen to the symmetric state due to uniqueness arguments. Therefore, such a bifurcation may only happen to an anti-symmetric stationary state.

In conclusion, we have produced a rigorous proof of the proximity of the self-trapped states, produced by the TF (Thomas-Fermi) approximation in the recently developed models with the spatially growing local strength of the defocusing cubic nonlinearity, to the true ground state, in the space of any dimension, $d = 1, 2, 3$. As an extension of the analysis, it may be interesting to justify the empiric use of the TF approximation for the description of self-trapped modes with intrinsic vorticity (by themselves, they are not ground states, but may play such a role in the respective reduced radial models [6, 8, 11]). Another relevant extension can be developed for two-component models with the nonlinear confinement of the same type [5].

Acknowledgments: The work of D.P. is supported by the Ministry of Education and Science of Russian Federation (the base part of the state task No. 2014/133).
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