Critical equations with Hardy terms in the Heisenberg group

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Abstract
In this paper, we are concerned with the study of a critical $(p, q)$ equation with Hardy terms on the Heisenberg group. Existence of entire solutions is obtained via an application of some concentration–compactness type results and the mountain pass theorem. Our results are presented in the model case of the $(p, q)$ horizontal Laplacian equations, but the method can be extended to deal with a more general class of problems with operators of $(p, q)$ growth.

Keywords Heisenberg group · Variation methods · critical exponents · Hardy terms

Mathematics subject classification MSC 35R03 · MSC 35B33 · MSC 35J92 · MSC 35A15 · MSC 35H20 · MSC 35J75

1 Introduction
In this paper, we complete the study started in [33] by giving some applications of the results contained therein. More precisely, we consider the critical equation with Hardy terms in $H^p$.


\[-\Delta_{H,p}u - \Delta_{H,q}u + |u|^{p-2}u + |u|^{q-2}u - \sigma \psi q \frac{|u|^{q-2}u}{r^q} = \lambda f(\xi, u) + |u|^\gamma - 2u, \quad (E)\]

where \( \sigma \) and \( \lambda > 0 \) are real parameters and \( Q = 2n + 2 \) is the homogeneous dimension of \( \mathbb{H}^n \). The exponents \( p \) and \( q \) are such that \( 1 < p < q < Q \), where \( q^* = qQ/(Q - q) \) is the critical exponent related to \( q \). Moreover, the operator \( \Delta_{H,q} \) with \( \phi \in \{p,q\} \), appearing in equation \((E)\), is the well known horizontal \( \phi \)-Laplacian on the Heisenberg group, which is defined as

\[\Delta_{H,\phi}\phi = \text{div}_H(|D_H\phi|^{p-2}D_H\phi) \quad \text{for all } \phi \in C^2(\mathbb{H}^n).\]

The function \( r \) in \((E)\) denotes the Korányi norm of the Heisenberg group \( \mathbb{H}^n \), given by

\[r(\xi) = r(z, t) = (|z|^4 + t^2)^{1/4},\]

where \( \xi = (z, t) \) and \( z = (x, y) \in \mathbb{R}^n \times \mathbb{R}^n, t \in \mathbb{R}, \) and \( |z| \) is the Euclidean norm in \( \mathbb{R}^{2n} \) of \( z \).

Finally, the weight function \( \psi \) in \((1.3)\) is related to the horizontal Hilbertian norm of the horizontal gradient of \( r \), in short \( \psi = |D_H r|_H \) in \( \mathbb{H}^n \setminus \{O\} \). Its presence in equation \((E)\) will be explained later on, by means of the Hardy inequality in \( \mathbb{H}^n \). For further details we refer to Sect. 2.

The study of critical equations in the context of the stratified Lie groups is a fast growing and fascinating topic. The main reason behind this interest is the strong connection between this subject and the Yamabe problem on CR manifolds. We refer to \([8, 20, 24, 25]\) and to the references therein for many details in the special case \( \phi = 2 \) on Carnot groups. We also cite \([3, 34]\) for the general case when \( 1 < \phi < 2 \) in the Heisenberg setting.

On the other hand, only few recent works deal with \((p, q)\) horizontal Laplacian problems in the whole \( \mathbb{H}^n \). The so-called \((p, q)\) operators are introduced by Marcellini in \([22, 23]\) in the Euclidean context. Marcellini considers functions with different growth near the origin and at infinity (unbalanced growth). Since then, the topic has been extensively studied and we just mention few recent contributions which are very relevant to the present paper. More precisely, when \( p = 2 \) and \( 2 < q < n \), the \((p, q)\) equations were studied in \([28–30]\). In particular, in \([28]\) the authors prove existence and multiplicity of solutions of a parametric nonlinear non-homogeneous Dirichlet problem in a bounded domain \( \Omega \subset \mathbb{H}^n \), with \( \partial \Omega \) of class \( C^2 \). In \([27]\), the authors give existence and multiplicity of solutions for a parametric \((p, q)\)-equations with sign-changing reaction and Robin boundary conditions. These contributions are also related to the works of Zhikov \([40, 41]\), where the so-called double phase operators are studied in connection with phenomena arising in nonlinear elasticity. We mention the paper \([1]\) for further details.

In order to state the main results of the present paper, it is crucial to introduce the best constant in the Folland–Stein inequality. By \([12]\), we know that for all \( \phi \), with \( 1 < \phi < Q \), there exists a positive constant \( C_{\phi^*} = C_{\phi^*}(\phi, Q) \), related to the associated critical exponent \( \phi^* = \phi Q/(Q - \phi) \), such that for all \( \phi \in C^{\infty}_c(\mathbb{H}^n) \)

\[\|\phi\|_{\phi^*} \leq C_{\phi^*}\|D_H\phi\|_{\phi^*}, \quad (1.1)\]

where the horizontal gradient of \( \phi \) is the vector

\[D_H\phi = (X_1\phi, \ldots, X_n\phi, Y_1\phi, \ldots, Y_n\phi),\]
and \(\{X_j, Y_j\}_{j=1}^n\) is the basis of horizontal left invariant vector fields on \(\mathbb{H}^n\), that is
\[
X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad j = 1, \ldots, n.
\]

By [16], we know that the best constant of the Folland–Stein inequality is achieved in the so called Folland–Stein space \(S^{1,\varphi} (\mathbb{H}^n)\), \(1 < \varphi < Q\), which is defined as the completion of \(C_c^\infty (\mathbb{H}^n)\) with respect to the norm
\[
\|D_H \varphi\|_{\varphi} = \left( \int_{\mathbb{H}^n} |D_H \varphi|_H^{\varphi} d\xi \right)^{1/\varphi}.
\]
Hence, we can write the best constant \(C_{\varphi^*}\) of the Folland–Stein inequality as
\[
C_{\varphi^*} = \inf_{u \in S^{1,\varphi} (\mathbb{H}^n) \setminus \{0\}} \frac{\|D_H u\|_{\varphi}^{\varphi}}{\|u\|_{\varphi^*}^{\varphi}},
\]
and clearly \(C_{\varphi^*} > 0\).

For our purposes, it is also crucial to introduce the best Hardy–Sobolev constant \(\mathcal{H}_\varphi = \mathcal{H}_{\varphi} (\varphi, Q)\), which is given for \(1 < \varphi < Q\) by
\[
\mathcal{H}_\varphi = \inf_{u \in S^{1,\varphi} (\mathbb{H}^n) \setminus \{0\}} \frac{\|D_H u\|_{\varphi}^{\varphi}}{\|u\|_{H_\varphi}^{\varphi}}, \quad \|u\|_{H_\varphi} = \int_{\mathbb{H}^n} |u|^{\varphi} \psi_\varphi^{-\varphi} d\xi.
\]
The weight function \(\psi\) appearing in (1.3) is defined as \(\psi = |D_H r|_H\). For further details we refer to Sect. 2.

The main difficulty of working with Hardy terms is that the Hardy embedding is continuous, but not compact, that is \(S^{1,\varphi} (\mathbb{H}^n) \hookrightarrow L^\varphi (\mathbb{H}^n, \psi^{\varphi} r^{-\varphi} d\xi)\).

On \(f\) in (\(\mathcal{E}\)) we assume the following condition:

\((\mathcal{F})\) \(f\) is a Carathéodory function, with \(f(\cdot, u) = 0\) for all \(u \leq 0\) and \(f(\cdot, u) > 0\) for all \(u > 0\), satisfying the two properties

\[(f_1) \quad \text{there exist } m \text{ and } m, \text{ with } p < m < m < q^*, \text{ such that for every } \varepsilon > 0 \text{ there exists } C_\varepsilon > 0 \text{ for which the inequality}
\[
|f(\xi, u)| \leq m\varepsilon |u|^{m-1} + mC_\varepsilon |u|^{m-1} \quad \text{for any } u \in \mathbb{R}
\]
holds for a.e. \(\xi \in \mathbb{H}^n\);

\[(f_2) \quad \text{there exists } \theta, \text{ with } q < \theta < q^*, \text{ such that the inequality}
\[
0 \leq \theta F(\xi, u) \leq f(\xi, u)u \quad \text{for all } u \in \mathbb{R}
\]
holds for a.e. \(\xi \in \mathbb{H}^n\), where \(F(\xi, u) = \int_0^u f(\xi, v) dv\) for a.e. \(\xi \in \mathbb{H}^n\) and all \(u \in \mathbb{R}\).

Clearly, the natural space where finding solutions of (\(\mathcal{E}\)) is
\[
W = HW^{1,\varphi} (\mathbb{H}^n) \cap HW^{1,q} (\mathbb{H}^n),
\]
endowed with the norm
\[\|u\| = \|u\|_{HW^{1,p}} + \|u\|_{HW^{1,q}}\]

for all \(u \in W\), where \(HW^{1,p}(\mathbb{H}^n)\) is the horizontal Sobolev space defined in Sect. 2.

**Theorem 1.1** Suppose that \((\mathcal{F})\) holds. Then, for any \(\sigma \in (-\infty, \mathcal{H}_q)\), there exists \(\lambda_\sigma = \lambda_\sigma(\sigma, Q, q, \theta) > 0\) such that equation \((\mathcal{E})\) admits at least one nontrivial solution \(u\) in \(W\) for all \(\lambda \geq \lambda_\sigma\).

Theorem 1.1 extends and complements in several directions previous results, such as the theorems contained in \([10, 11]\) in \(\mathbb{R}^n\) and \([3, 31–34]\) in \(\mathbb{H}^n\). Existence is obtained via the mountain pass lemma of Ambrosetti and Rabinowit and follows somehow the ideas of \([10, 11]\). A very delicate step is the tricky compactness Theorem 3.1, which extends to the Heisenberg group analogous results obtained in the Euclidean setting in Lemma 2.3 of \([5]\), see also Lemma 3.3 of \([7]\), Lemma 2.2 of \([6]\) and Theorem 2.8 of \([11]\). Moreover, the "triple loss of compactness" in \((\mathcal{E})\), caused by the simultaneous presence of the Hardy and critical terms in the whole Heisenberg group \(\mathbb{H}^n\), forces to study the exact behavior of the Palais–Smale sequences, in the spirit of Lions. This analysis is deeply connected with the concentration phenomena taking place and strongly relies on the results in \([33]\). More precisely, existence is based on Theorems 1.1 and 1.2 of \([33]\), where the concentration–compactness principle of Lions \([18, 19]\), or CC principle, and its variant, that is, the CC principle at infinity of Chabrowski \([4]\), both are proved involving the Hardy–Sobolev embedding in the Folland–Stein space. Actually, applying the same arguments of Theorem 1.1, see Sect. 3, we could obtain existence of solutions for a much more general class of the \((p, q)\) operators, of the type considered in \([10, 11, 31]\). More precisely, we could replace the sum of the operators in equation \((\mathcal{E})\) by a more general divergence type operator of the form

\[- \text{div}_H (A(|D_H u|_H) D_H u),\]

where \(A : \mathbb{R}^+ \to \mathbb{R}^+, \mathbb{R}^+ = (0, \infty)\), is a strictly positive and strictly increasing function of class \(C^1(\mathbb{R}^+)\) satisfying the assumptions described in \([31]\) and \(\text{div}_H\) denotes the horizontal divergence, which is defined along any horizontal vector field \(X = \{v^i X^i_j + \bar{v}^i Y^i_j\}_{j=1}^n\), of class \(C^1(\mathbb{H}^n, \mathbb{R}^{2n})\) as

\[\text{div}_H X = \sum_{j=1}^n [X^i_j(v^i) + Y^i_j(\bar{v}^i)].\]

Furthermore, \(A\) is assumed to be such that \(tA(t) \to 0\) as \(t \to 0^+\). The function \(A\) denotes the potential of \(t \mapsto tA(t)\), which is 0 at 0. Without entering into further details, we just give some examples covered by \(A\), when \(1 < p < q < Q\). If \(A(t) = t^p/p + t^q/q\), we recover the \((p, q)\) horizontal Laplacian operator. Moreover, if \(A(t) = t^p/p + 1/2(1 + t^q)^{2/q}\), we get

\[-\Delta_{H,p} u - \text{div}_H \left( \frac{|D_H u|^{q-2}_H D_H u}{(1 + |D_H u|^{q}_H)^{1-2/q}} \right)\]

while if \(A(t) = \sqrt{1 + t^2} - 1 + t^4/4\), so that \(1 < p = 2 < q = 4 < Q\), we obtain,
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The range of examples is actually very wide and for more details we refer to [11, 31]. However, for notational simplicity, here we stick to the model case of the $(p, q)$ horizontal Laplacian operator.

The paper is organized as follows. Section 2 is dedicated to a short recap of the main definitions and properties of the Heisenberg group. In Sect. 3, we introduce the functional setting of equation $(E)$ and we present some useful tools which will be crucial in the paper, such as Theorem 3.1. Finally, Sect. 4 is devoted to the proof of the main Theorem 1.1, via a combination of standard variation methods and a more delicate analysis of the exact behavior of the Palais–Smale sequences, in the spirit of Lions.

\section{The Heisenberg group $\mathbb{H}^n$}

In this section we present the basic properties of the Heisenberg group $\mathbb{H}^n$. For a complete and exhaustive treatment we refer, e.g., to [13, 14, 16, 38].

Let $\mathbb{H}^n$ be the Heisenberg group of topological dimension $2n + 1$, that is the Lie group which has $\mathbb{R}^{2n+1}$ as a background manifold and whose group structure is given by the non–Abelian law

$$\xi \circ \xi' = (z + z', t + t' + 2 \sum_{i=1}^{n} (y_i x'_i - x_i y'_i))$$

for all $\xi, \xi' \in \mathbb{H}^n$, with

$$\xi = (z, t) = (x_1, \ldots, x_n, y_1, \ldots, y_n, t) \quad \text{and} \quad \xi' = (z', t') = (x'_1, \ldots, x'_n, y'_1, \ldots, y'_n, t').$$

The $2n + 1$ left–invariant vector fields on $\mathbb{H}^n$

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t},$$

for $j = 1, \ldots, n$, form a basis for the real Lie algebra of $\mathbb{H}^n$ of left–invariant vector fields. This basis satisfies the Heisenberg canonical commutation relations

$$[X_j, Y_k] = -4\delta_{jk} T, \quad [Y_j, Y_k] = [X_j, X_k] = [Y_j, T] = [X_j, T] = 0.$$

Moreover, all the commutators of length greater than two vanish, and so $\mathbb{H}^n$ is a nilpotent graded stratified group of step two. A left invariant vector field $X$ that belongs to the span of $\{X_j, Y_j\}_{j=1}^{n}$, is called horizontal.

For each real number $R > 0$, we consider the dilation $\delta_R : \mathbb{H}^n \to \mathbb{H}^n$ naturally associated with the Heisenberg group structure, which is defined by

$$\delta_R(\xi) = (Rz, R^2 t) \quad \text{for all} \quad \xi = (z, t) \in \mathbb{H}^n. \quad (2.1)$$

It is easy to verify that the Jacobian determinant of dilatations $\delta_R$ is constant and equal to $R^{2n+2}$, where the natural number $Q = 2n + 2$ is the homogeneous dimension of $\mathbb{H}^n$. 

\[ Springer \]
The Korányi norm on $\mathbb{H}^n$ is given by

$$r(\xi) = r(z, t) = (|z|^4 + t^2)^{1/4} \quad \text{for all } \xi = (z, t) \in \mathbb{H}^n.$$  

Consequently, the Korányi norm is homogeneous of degree 1, with respect to the dilations $\delta_R$, $R > 0$, that is

$$r(\delta_R(\xi)) = r(Rz, R^2t) = (|Rz|^4 + R^4t^2)^{1/4} = R r(\xi) \quad \text{for all } \xi = (z, t) \in \mathbb{H}^n.$$  

The corresponding distance, the so called Korányi distance, is

$$d_K(\xi, \xi') = r(\xi^{-1} \xi') \quad \text{for all } (\xi, \xi') \in \mathbb{H}^n \times \mathbb{H}^n.$$

Throughout the paper, we denote by $B_R(\xi_0) = \{ \xi \in \mathbb{H}^n : d_K(\xi, \xi_0) < R \}$ the Korányi open ball of radius $R$ centered at $\xi_0$. For simplicity we put $B_R = B_R(O)$, where $O = (0, 0)$ is the natural origin of $\mathbb{H}^n$.

The Lebesgue measure on $\mathbb{R}^{2n+1}$ is invariant under the left translations of the Heisenberg group. Thus, since the Haar measures on Lie groups are unique up to constant multipliers, we denote by $d\ell$ the Haar measure on $\mathbb{H}^n$ that coincides with the $(2n + 1)$–Lebesgue measure and by $|U|$ the $(2n + 1)$–dimensional Lebesgue measure of any measurable set $U \subseteq \mathbb{H}^n$. Furthermore, the Haar measure on $\mathbb{H}^n$ is $Q$–homogeneous with respect to dilations $\delta_R$. Consequently,

$$|\delta_R(U)| = R^Q |U|, \quad d(\delta_R \xi) = R^Q d\xi.$$

In particular, $|B_R| = |B_1| R^Q$.

We define the horizontal gradient of a $C^1$ function $u : \mathbb{H}^n \to \mathbb{R}$ by

$$D_H u = \sum_{j=1}^n [(X_j u) X_j + (Y_j u) Y_j].$$

Clearly, $D_H u \in \text{span} \{X_j, Y_j\}_{j=1}^n$. In span $\{X_j, Y_j\}_{j=1}^n \cong \mathbb{R}^{2n}$ we consider the natural inner product given by

$$(X, Y)_H = \sum_{j=1}^n (x^j y^j + \overline{x}^j \overline{y}^j)$$

for $X = \{x^j X_j + \overline{x}^j Y_j\}_{j=1}^n$ and $Y = \{y^j X_j + \overline{y}^j Y_j\}_{j=1}^n$. The inner product $(\cdot, \cdot)_H$ produces the Hilbertian norm

$$|X|_H = \sqrt{(X, X)_H}$$

for the horizontal vector field $X$.

For any horizontal vector field function $X = X(\xi), X = \{x^j X_j + \overline{x}^j Y_j\}_{j=1}^n$, of class $C^1(\mathbb{H}^n, \mathbb{R}^{2n})$, we define the horizontal divergence of $X$ by

$$\text{div}_H X = \sum_{j=1}^n [X_j(\overline{x}^j) + Y_j(\overline{X}^j)].$$

Similarly, if $u \in C^2(\mathbb{H}^n)$, then the Kohn–Spencer Laplacian, or equivalently the horizontal Laplacian, or the sub–Laplacian, in $\mathbb{H}^n$, of $u$ is
$$\Delta_H u = \sum_{j=1}^{n} (X_j^2 + Y_j^2) u$$

$$= \sum_{j=1}^{n} \left( \frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} + 4y_j \frac{\partial^2}{\partial x_j \partial t} - 4x_j \frac{\partial^2}{\partial y_j \partial t} \right) u + 4|\zeta|^2 \frac{\partial^2 u}{\partial t^2}. $$

According to the celebrated Theorem 1.1 due to Hörmander in [15], the operator $\Delta_H$ is hypoelliptic. In particular, $\Delta_H u = \text{div}_H D_H^2 u$ for each $u \in C^2(\mathbb{H}^n)$. A well known generalization of the Kohn–Spencer Laplacian is the horizontal $\varphi$–Laplacian on the Heisenberg group, $\varphi \in (1, \infty)$, defined by

$$\Delta_{H, \varphi} \varphi = \text{div}_H (|D_H \varphi|_H^{\varphi-2} D_H \varphi) \text{ for all } \varphi \in C_c^{\infty}(\mathbb{H}^n).$$

Let us now review some classical facts about the first–order Sobolev spaces on the Heisenberg group $\mathbb{H}^n$. Denote by $HW^{1,\varphi}(\Omega)$ the horizontal Sobolev space consisting of the functions $u \in L^p(\Omega)$ such that $D_H u$ exists in the sense of distributions and $|D_H u|_H \in L^p(\Omega)$, endowed with the natural norm

$$\|u\|_{HW^{1,\varphi}(\Omega)} = \left( \|u\|_{L^p(\Omega)}^{\varphi} + \|D_H u\|_{L^p(\Omega)}^{\varphi} \right)^{1/\varphi},$$

where $\|u\|_{L^p(\Omega)} = \left( \int_{\Omega} |u|^{\varphi} \, d\xi \right)^{1/\varphi}$ and $\|D_H u\|_{L^p(\Omega)} = \left( \int_{\Omega} |D_H u|_H^{\varphi} \, d\xi \right)^{1/\varphi}$.

Thanks to [12] we know that if $1 \leq \varphi < Q$, then the embedding

$$HW^{1,\varphi}(\Omega) \hookrightarrow L^s(\Omega) \text{ for all } s \in [\varphi, \varphi^*], \quad \varphi^* = \frac{\varphi Q}{Q - \varphi},$$

is continuous.

Let us also briefly recall a version of the Rellich theorem in the Heisenberg group. We refer to [13, 14, 16, 21], where this topic is extensively treated and we just recall that, if $1 \leq \varphi < Q$ and $B_R(\xi_0)$ is any Korányi ball, then the embedding

$$HW^{1,\varphi}(B_R(\xi_0)) \hookrightarrow L^s(B_R(\xi_0))$$

is compact, provided that $1 \leq s < \varphi^*$.

Finally, following [13], we denote by

$$\psi(\xi) = |D_H r(\xi)|_H = \frac{|\zeta|}{r(\xi)}, \quad \text{for } \xi = (z, t) \neq (0, 0).$$

Note that the density function $\psi^2$ is homogeneous of degree zero with respect to the family of dilations $\delta_R$, introduced in (2.1). Moreover, $0 \leq \psi \leq 1$, $\psi(0, t) \equiv 0$, $\psi(z, 0) \equiv 1$. In the Euclidean space the presence of the density $\psi$ is outshone by the flat geometry of $\mathbb{R}^n$, which yields $\psi \equiv 1$.

The Hardy inequality in $\mathbb{H}^n$ was obtained in [13] when $\varphi = 2$ and the extended in [26] for all $\varphi > 1$, see also [9]. We report here this second version.

**Theorem 2.1** (Hardy inequality in $\mathbb{H}^n$, [26]) Let $1 < \varphi < Q$ and let $\varphi \in C_c^{\infty}(\mathbb{H}^n \setminus \{O\})$. Then, the following inequality holds.
Clearly, by a density argument inequality (2.4) is still valid for in $S^1$ and $\mathbb{R}^n$. Moreover, (2.4) implies that $\mathcal{S}_1, \mathbb{R}^n$ continuously. However, as already noted, the embedding (2.5) is not compact, even locally in any neighborhood of $O$.

For computational simplicity, we prefer to use the Korányi norm and distance instead of the Carnot–Carathéodory distance, though the Korányi distance does not reflect the sub–Riemannian structure of the Heisenberg group. However, the two metrics are closely related. Interestingly, in the setting of the Heisenberg group it was shown by Yang in [39] that the $L$–gauge $d(x) –$ sometimes also called the Korányi–Folland or Kaplan gauge – can be replaced by the Carnot–Carathéodory distance, and the Hardy inequality in the Heisenberg group remains valid with the same best constant $\frac{\varrho}{Q-\varrho}$. For further comments and related results we refer to [35].

Let us conclude the section introducing a suitable sequence of mollifiers in the Heisenberg group. To this aim, we first define the group convolution as follows. If $u \in L^1(\mathbb{H}^n)$ and $v \in L^\varrho(\mathbb{H}^n)$, with $1 \leq \varrho < \infty$, then for a.e. $x \in \mathbb{H}^n$ the function

$$
\eta \mapsto u(x) \eta(x) v(\eta)
$$

is in $L^1(\mathbb{H}^n)$. Moreover, $u \ast v$, defined a.e. on $\mathbb{H}^n$ by

$$(u \ast v)(\xi) = \int_{\mathbb{H}^n} u(\eta) \eta^{-1}(v(\eta)) d\eta,$$

is called convolution of $u$ and $v$. By the analogue of the Young theorem $u \ast v$ belongs to $L^\varrho(\mathbb{H}^n)$ and

$$
\|u \ast v\|_\varrho \leq \|u\|_1 \|v\|_\varrho.
$$

For further details we refer to [12, 17, 37]. Using the convolution, it is possible to generate a sequence of mollifiers $(\rho_k)_k$ on $\mathbb{H}^n$, with the properties that $\rho_k \in C_\infty(\mathbb{H}^n)$, $\rho_k \geq 0$ in $\mathbb{H}^n$ and $\int_{\mathbb{H}^n} \rho_k d\xi = 1$ for any $k$. Furthermore, if $\varrho \in [1, \infty)$ then as $k \to \infty$

$$
\rho_k \ast u \to u \text{ in } L^\varrho(\mathbb{H}^n), \quad D_H(\rho_k \ast u) \to D_H u \text{ in } L^\varrho(\mathbb{H}^n, \mathbb{R}^{2n})
$$

for all $u \in HW^{1,\varrho}(\mathbb{H}^n)$.

3 Functional setting and preliminary results

In this section we present some useful results and comments and from now on we assume that the structural assumptions required in Theorem 1.1 hold. Clearly, $(\mathcal{E})$ has a variational structure and the Euler–Lagrange functional $I : W \to \mathbb{R}$ associated to $(\mathcal{E})$ is given by
Indeed, for all $\varphi \in \mathcal{F}$, the functional $I(u)$ is well defined and of class $C^1(W)$ by the assumption $(\mathcal{F})$, and

$$I(u) = \int_{\mathbb{H}^n} \left( \left| D_1 u \right|^p + \left| D_2 u \right|^q - \frac{\sigma}{q} \int_{\mathbb{H}^n} |u|^q \frac{\psi^q}{r^q} d\xi \right) - \lambda \int_{\mathbb{H}^n} F(\xi, u) d\xi - \frac{1}{q^*} \int_{\mathbb{H}^n} |u|^{q^*} d\xi$$

for all $u \in W$. Hence, the (weak) solutions of $(\mathcal{E})$ are exactly the critical points of $I$.

From the main properties summarised in Sect. 2 we easily get the next result.

**Lemma 3.1** The embedding $W \hookrightarrow L^\varphi(\mathbb{H}_n)$ is continuous for all $\varphi \in [p, q^*]$, and

$$\|u\|_{\varphi'} \leq C_\varphi \|u\| \quad \text{for all } u \in W, \quad (3.1)$$

where $C_\varphi$ depends on $\varphi$, $Q$, $p$ and $q$. Moreover, if $\varphi \in [1, q^*)$, then for all $R > 0$ the embedding

$$W \hookrightarrow L^\varphi(B_R)$$

is compact.

Before studying equation $(\mathcal{E})$, let us recall the crucial inequality, originally proved by Simon in [36]. For all $s \in (1, \infty)$ there exists $\kappa > 0$, depending only on $s$, such that

$$|X - Y|_H^s \leq \kappa \left\{ \begin{array}{ll}
\mathcal{P}_s(X, Y), & s \geq 2, \\
\mathcal{P}_s(X, Y)^{s/2}(|X|^s + |Y|^s)^{(2-s)/2}, & 1 < s < 2,
\end{array} \right. \quad (3.2)$$

where $\mathcal{P}_s(X, Y) = (|X|^s H - |Y|^s H, X - Y)_H$ for all $X$ and $Y$ in the span of $\{X_j, Y_j\}_{j=1}^m$. The structural assumptions of Theorem 1.1 lead to geometry of the mountain pass theorem of Ambrosetti and Rabinowitz for the functional $I$ at special levels.

**Lemma 3.2** (Geometry of mountain pass) For all $\sigma \in (-\infty, \mathcal{H}_q)$ and $\lambda > 0$ (i) There exists a nontrivial function $e \in C_\infty^\infty(\mathbb{H}^n)$, independent of $\lambda$ and $\sigma^\gamma$, with $e \geq 0$ in $\mathbb{H}^n$ and $\int_{\mathbb{H}^n} |e|^{q_0} d\xi > 0$ such that

$$I(e) < 0, \quad \|e\| \geq 2;$$

(ii) There exist numbers $j = j(\sigma, \lambda) > 0$ and $\rho = \rho(\sigma, \lambda) \in (0, 1]$ such that

$$I(u) \geq j \quad \text{for any } u \in W \text{ with } \|u\| = \rho.$$

The proof of Lemma 3.2 is standard and again very similar to the demonstration which first appears in Lemma 2.4 of [11] and Lemma 4.1 of [32] and so there is no reason to produce it here.
Now, for fixed $\sigma \in (-\infty, \mathcal{H}_q)$ and $\lambda > 0$, thanks to the geometry given in Lemma 3.2, we introduce the special levels of $I$

$$c_{\sigma, \lambda} = \inf_{\gamma \in \Gamma} \max_{r \in [0,1]} I(\gamma(r)),$$  

(3.3)

where $\Gamma = \{ \gamma \in C([0,1], W) : \gamma(0) = 0, I(\gamma(1)) < 0 \}$. Obviously, $c_{\sigma, \lambda} > 0$ thanks to Lemma 3.2, since $\|v\| > \rho$. The next asymptotic property of the levels $c_{\sigma, \lambda}$ as $\lambda \to \infty$ is crucial to overcome the difficulties due to the presence of the Hardy terms and the critical nonlinearities. This result was observed in the Euclidean space in [10] and in [11] for the scalar and vectorial case, respectively. We also refer to [31] for a similar feature in the Heisenberg context.

**Lemma 3.3** For any $\sigma \in (-\infty, \mathcal{H}_q)$ it results

$$\lim_{\lambda \to \infty} c_{\sigma, \lambda} = 0.$$

Again the proof of Lemma 3.3 follows directly from that of Lemma 2.5 of [11] and it is not reported here.

Let us now prove a tricky compactness theorem of independent interest. The argument we use first appears in the Euclidean context in the proof of Lemma 2.3 in [5], see also Lemma 3.3 of [7] and Lemma 2.2 of [6] and Theorem 2.7 of [11].

**Theorem 3.1** Let $\varphi \in (p, q^*)$ and let $(u_k)_k$ be a bounded sequence in $W$. Then there exists $u \in W$ such that, up to a subsequence, $u_k \to u$ strongly in $L^\varphi(B_R^p)$ as $k \to \infty$.

**Proof** Fix $\varphi$ and $(u_k)_k$ as in the statement. Then, there exists $u \in W$ such that, up to a subsequence,

$$\begin{align*}
  u_k &\to u & \text{weakly in } W, \\
  u_k &\to u & \text{a.e. in } H^p, \\
  u_k &\to u & \text{in } L^p(B_R)
\end{align*}$$

(3.4)

for any $p \in [1, q^*)$ and radius $R$.

We first claim that

$$\lim_{k \to \infty} \|u_k\|_{\varphi} = \|u\|_{\varphi}.$$  

(3.5)

To this aim fix $\epsilon > 0$. There exist positive numbers $s_0$, $S_0$ such that $0 < s_0 < S_0$ and $|s|^\varphi \leq \epsilon |s|^p$ for all $s \in \mathbb{R}$, with $|s| \leq s_0$, while $|s|^\varphi \leq \epsilon |s|^{q^*}$ for all $s \in \mathbb{R}$, with $|s| \geq S_0$, since $p < \varphi < q^*$. Hence, for any $s \in \mathbb{R}$

$$|s|^\varphi \leq \epsilon (|s|^p + |s|^{q^*}) + \mathbb{1}_{[s_0, S_0]}(|s|)|s|^\varphi.$$

Now fix $R > 0$. For all $k \in \mathbb{N}$

$$\int_{B_R} |u_k|^\varphi d\xi \leq \epsilon \int_{B_R} (|u_k|^p + |u_k|^{q^*}) d\xi + S_0^\varphi \int_{A_k \cap B_R} \mathbb{1}_{[s_0, S_0]}(|u_k|) d\xi,$$

(3.6)

where $A_k = \{ x \in H^p : s_0 \leq |u_k| \leq S_0 \}$. Furthermore, by (2.2) there exists $C > 0$ such that
\[
\int_{\mathbb{H}^n} \left( |u_k|^p + |u_k|^q \right) d\xi \leq C \quad \text{for all } k \in \mathbb{N}. \tag{3.7}
\]

Therefore
\[
s_0^p |A_k| \leq \int_{\mathbb{H}^n} \left( |u_k|^p + |u_k|^q \right) d\xi \leq C \quad \text{for all } k \in \mathbb{N}.
\]

Hence \( \sup_{n \in \mathbb{N}} |A_k| \leq C s_0^{-p} \).

We claim that \( \lim_{R \to \infty} |A_k \cap B_{R_j}^c| = 0 \) uniformly in \( k \in \mathbb{N} \). Let us first show that
\[
\lim_{R \to \infty} |A_k \cap B_{R_j}^c| = 0 \quad \text{for all } k \in \mathbb{N}. \tag{3.8}
\]

Otherwise, there exist \( k_0 \in \mathbb{N}, \delta > 0 \) and radii \( R_j \uparrow \infty \) as \( j \to \infty \) such that
\[
|A_{k_0} \cap B_{R_j}^c| \geq \delta \quad \text{for all } j \in \mathbb{N}.
\]

Clearly, \( |A_{k_0} \cap B_{R_j}^c| \leq |A_{k_0}| \leq C s_0^{-p} \) for all \( j \in \mathbb{N} \). Put \( \Omega_j = B_{R_j}^c \setminus B_{R_{j+1}}^c \) for all \( j \in \mathbb{N} \). Then \( \Omega_j \cap \Omega_{\ell} = \emptyset \) if \( j \neq \ell \), and
\[
B_{R_j}^c = \bigcup_{\ell = j}^{\infty} \Omega_{\ell}, \quad |A_{k_0} \cap B_{R_j}^c| = \sum_{\ell = j}^{\infty} |A_{k_0} \cap \Omega_{\ell}| \geq \delta \quad \text{for all } j \in \mathbb{N}.
\]

Therefore, \( \infty > |A_{k_0} \cap B_{R_j}^c| = \sum_{\ell = j}^{\infty} |A_{k_0} \cap \Omega_{\ell}| = \infty \), which is the desired contradiction. Hence, (3.8) holds.

On the other hand, (3.4) gives that \( u \in HW^{1,p}(\mathbb{H}^n) \cap HW^{1,\overline{p}}(\mathbb{H}^n) \) and \( u_k \to u \) a.e. in \( \mathbb{H}^n \). Hence, there exists \( R_0 = R_0(\varepsilon) \geq 1 \) such that for all \( R \geq R_0 \)
\[
\int_{B_R} |u|^p d\xi \leq \varepsilon.
\]

Put \( t_1 = R_0 \) and construct \( t_\ell \uparrow \infty \) such that \( D_\ell = B_{t_\ell}^c \setminus B_{t_{\ell+1}}^c, B_{R_j}^c = \bigsqcup_{\ell = 1}^{\infty} D_\ell \) and
\[
\int_{D_\ell} |u|^p d\xi \leq \frac{\varepsilon}{2^\ell} \quad \text{for all } \ell \in \mathbb{N}.
\]

Clearly, \( D_\ell \) is a bounded domain for all \( \ell \in \mathbb{N} \), and \( D_\ell \cap D_m = \emptyset \) if \( \ell \neq m \). It is possible to apply the reverse Fatou lemma to the sequence \( k \mapsto \Psi_k = |u_k|^p \mathbb{1}_{A_k} \) in each \( D_\ell \), since \( \Psi_k \leq S_0^p \in L^1(D_\ell) \). Consequently,
\[
\limsup_{k \to \infty} \int_{D_\ell \cap A_k} |u_k|^p d\xi = \limsup_{k \to \infty} \int_{D_\ell} |u_k|^p \mathbb{1}_{A_k} d\xi \leq \int_{D_\ell} \limsup_{k \to \infty} |u_k|^p \mathbb{1}_{A_k} d\xi
\]
\[
\leq \int_{D_\ell} \limsup_{k \to \infty} |u_k|^p d\xi = \int_{D_\ell} |u|^p d\xi \leq \frac{\varepsilon}{2^\ell}.
\]

Then,
\[
\limsup_{k \to \infty} \int_{A_k \cap B_{R_0}^c} |u_k|^{p} d\xi = \limsup_{k \to \infty} \sum_{\varepsilon = 1}^{\infty} \int_{D_{\varepsilon} \cap A_k} |u_k|^{p} d\xi \leq \sum_{\varepsilon = 1}^{\infty} \limsup_{k \to \infty} \int_{D_{\varepsilon} \cap A_k} |u_k|^{p} d\xi \leq \sum_{\varepsilon = 1}^{\infty} \int_{D_{\varepsilon}} |u|^{p} d\xi \leq \varepsilon.
\]

Therefore,
\[
s_0^p \limsup_{k \to \infty} |A_k \cap B_{R_0}^c| \leq \limsup_{k \to \infty} \int_{A_k \cap B_{R_0}^c} |u_k|^{p} d\xi \leq \varepsilon.
\]

(3.9)

Clearly, \((A_k \cap B_{R}^c) \subseteq (A_k \cap B_{R_0}^c)\) for all \(R \geq R_0\) and \(k \in \mathbb{N}\). Hence, the claim \(\lim_{R \to \infty} |A_k \cap B_{R}^c| = 0\) uniformly in \(k \in \mathbb{N}\) is proved by (3.8) and (3.9).

In particular, for any \(\varepsilon > 0\) there exist \(R_0 \geq 1\) and \(\delta_0 \in (0, \varepsilon C S_0^{\varphi})\) such that \(|A_k \cap B_{R_0}^c| \leq \delta_0\) for all \(k \in \mathbb{N}\), and
\[
\int_{A_k \cap B_{R_0}^c} |u_k| d\xi \leq |A_k \cap B_{R_0}^c| \leq \delta_0 \leq \frac{\varepsilon C}{S_0^{\varphi}} \quad \text{for all } k \in \mathbb{N}.
\]

Hence, (3.6) and (3.7) yield
\[
\limsup_{k \to \infty} \|u_k\|_{L^p(B_{R_0}^c)}^p \leq 2\varepsilon C.
\]

(3.10)

Now, (3.4) implies that \(u_k \to u\) in \(L^p(B_{R_0})\). Therefore, for all \(R \geq R_0 = R_0(\varepsilon)\) by (3.10)
\[
\limsup_{k \to \infty} \int_{B_{R_0}} |u_k|^{p} d\xi = \limsup_{k \to \infty} \int_{B_{R_0}} |u_k|^{p} d\xi + \limsup_{k \to \infty} \int_{B_{R_0}^c} |u_k|^{p} d\xi
\]
\[
= \int_{B_{R_0}} |u|^{p} d\xi + \limsup_{k \to \infty} \int_{B_{R_0}^c} |u_k|^{p} d\xi
\]
\[
\leq \int_{\mathbb{R}^n} |u|^{p} d\xi - \int_{B_{R_0}} |u|^{p} d\xi + 2\varepsilon C
\]
\[
\leq \int_{\mathbb{R}^n} |u|^{p} d\xi - \int_{B_{R}} |u|^{p} d\xi + 2\varepsilon C.
\]

Letting \(R \to \infty\) we have
\[
\limsup_{k \to \infty} \int_{\mathbb{R}^n} |u_k|^{p} d\xi \leq \|u\|_{L^p}^p + 2\varepsilon C
\]

and so as \(\varepsilon \to 0\) we obtain
\[
\limsup_{k \to \infty} \|u_k\|_{L^p} \leq \|u\|_{L^p}.
\]

This, together with (3.4), gives at once the validity of (3.5). Finally, applying the Clarkson and Radon theorems we obtain \(u_k \to u\) in \(L^p(\mathbb{R}^n)\). \(\square\)
4 Proof of Theorem 1.1

In this section we prove the existence of nontrivial solutions for (E), and so all the structural assumptions required in Theorem 1.1 are supposed to hold. Let us now prove the following crucial lemma.

Lemma 4.1 Let \((u_k)_k \subset W\) be a Palais–Smale sequence of \(I\) at the level \(c_{\sigma, \lambda}\) for all \(\sigma \in (-\infty, H_q)\) and \(\lambda > 0\). Then,

(i) up a subsequence, \(u_k \rightharpoonup u\) in \(W\) as \(k \to \infty\),

(ii) there exists \(\lambda^* = \lambda^*(\sigma)\) such that the weak limit \(u\) is a critical point of \(I\) for any \(\lambda \geq \lambda^*\).

Proof (i) Let \((u_k)_k \subset W\) be a Palais–Smale sequence of \(I\) at level \(c_{\sigma, \lambda}\) for any \(\lambda > 0\), that is

\[
I(u_k) \to c_{\sigma, \lambda} \quad \text{and} \quad I'(u_k) \to 0 \quad \text{in } W' \quad \text{as } k \to \infty.
\]

In order to prove our claim, we start showing that \((u_k)_k\) is bounded in \(W\). Thanks to (4.1), there exists \(d_{\sigma, \lambda} > 0\) such that as \(k \to \infty\)

\[
c_{\sigma, \lambda} + d_{\sigma, \lambda} \|u_k\| + o(1) \geq I(u_k) - \frac{1}{\theta} \langle I'(u_k), u_k \rangle
\]

\[
= \left( \frac{1}{p} - \frac{1}{\theta} \right) \|u_k\|_{H^{1,p}}^p + \left( \frac{1}{q} - \frac{1}{\theta} \right) \|u_k\|_{H^{1,q}}^q
\]

\[- \sigma \left( \frac{1}{q} - \frac{1}{\theta} \right) \|u_k\|_{H_q}^q
\]

\[+ \frac{\lambda}{\theta} \int_{\mathbb{H}^n} \left( f(\xi, u_k) - \theta F(\xi, u_k) \right) d\xi + \left( \frac{1}{q} - \frac{1}{q^*} \right) \|u_k\|_{H_q}^{q^*}
\]

\[\geq \left( \frac{1}{q} - \frac{1}{\theta} \right) \left( \|u_k\|_{H^{1,p}}^p + \|u_k\|_{H^{1,q}}^q \right) - \frac{\sigma^+}{H_q} \left( \frac{1}{q} - \frac{1}{\theta} \right) \|u_k\|_{H^{1,q}}^q
\]

\[\geq \ell \left( \|u_k\|_{H^{1,p}}^p + \|u_k\|_{H^{1,q}}^q \right)
\]

by (f2), where \(d_{\sigma, \lambda} = o(1)\) for every \(\lambda\) and \(\sigma\) and

\[
\ell = \left( 1 - \frac{\sigma^+}{H_q} \right) \left( \frac{1}{q} - \frac{1}{\theta} \right) > 0.
\]

since \(q < \theta < q^*\).

Assume by contradiction that \(\|u_k\| \to \infty\) as \(k \to \infty\). Now we have three possibilities as \(k \to \infty\)

1. \(\|u_k\|_{H^{1,p}} \to \infty\) and \(\|u_k\|_{H^{1,q}} \to \infty\);
2. \(\|u_k\|_{H^{1,p}} \to \infty\) and \(\|u_k\|_{H^{1,q}}\) is bounded;
3. \(\|u_k\|_{H^{1,p}}\) is bounded and \(\|u_k\|_{H^{1,q}} \to \infty\).
In the first case, for \( k \) sufficiently large obviously \( \|u_k\|^q_{\mathcal{H}^{1,q}} \geq \|u_k\|^p_{\mathcal{H}^{1,p}} \), being \( p < q \). Therefore, we get
\[
c_{\sigma,\lambda} + d_{\sigma,\lambda} \|u_k\| + o(1) \geq \epsilon \left( \|u_k\|^p_{\mathcal{H}^{1,p}} + \|u_k\|^q_{\mathcal{H}^{1,q}} \right)
\geq \epsilon \left( \|u_k\|^p_{\mathcal{H}^{1,p}} + \|u_k\|^p_{\mathcal{H}^{1,p}} \right)
\geq 2^{1-p}\epsilon \left( \|u_k\|^p_{\mathcal{H}^{1,p}} + \|u_k\|^p_{\mathcal{H}^{1,p}} \right) = 2^{1-p}\epsilon \|u_k\|^p
\]
which implies, as \( k \to \infty \), that \( 0 \geq 2^{1-p}\epsilon > 0 \). This is obviously impossible.

In the second case, we know that
\[
c_{\sigma,\lambda} + d_{\sigma,\lambda} \|u_k\| + o(1) \geq \epsilon \left( \|u_k\|^p_{\mathcal{H}^{1,p}} + \|u_k\|^q_{\mathcal{H}^{1,q}} \right) \geq \epsilon \|u_k\|^p_{\mathcal{H}^{1,p}}
\]
and so dividing both sides by \( \|u_k\|^p_{\mathcal{H}^{1,p}} \), we obtain
\[
d_{\sigma,\lambda} \|u_k\|^p_{\mathcal{H}^{1,p}} + \|u_k\|^q_{\mathcal{H}^{1,q}} + o(1) \geq \epsilon.
\]
This in turn implies \( 0 \geq \epsilon > 0 \) as \( k \to \infty \), and gives the required contradiction.

Finally, the third case is analogous to the second one.

Therefore, since we ruled out all the three possibilities, we conclude that \((u_k)_k\) is bounded in \( W \). Thus, since \( W \) is a reflexive Banach space, there exists \( u \in W \) such that \( u_k \rightharpoonup u \) in \( W \). For simplicity, in what follows we denote by \((u_k)_k\) every subsequence extracted from the original sequence.

\( (ii) \) From part (i), there exist nonnegative numbers \( t_{\sigma,\lambda}, J_{\sigma,\lambda}, \delta_{\sigma,\lambda} \) such that
\[
\begin{cases}
    u_k \to u \text{ in } W, & \|u_k\|^p_{\mathcal{H}^{1,p}} + \|u_k\|^q_{\mathcal{H}^{1,q}} \to t_{\sigma,\lambda}, \\
    u_k \to u \text{ in } L^q(\mathbb{H}^n, \nu^q/\varrho^q), & \|u_k - u\|_{\mathcal{H}^s} \to J_{\sigma,\lambda}, \\
    u_k \to u \text{ in } L^p(\mathbb{B}_R), & u_k \to u \text{ a.e. in } \mathbb{H}^n, \\
    |\delta| \leq g_R \text{ a.e. in } \mathbb{H}^n, & \text{for some } g_R \in L^p(\mathbb{B}_R) \text{ and all } R > 0,
\end{cases}
\]
(4.3)

with \( p \in [1, q^*) \), by (1.3), (2.2) and Lemma 3.1.

Moreover, since in particular \( u_k \to u \) in \( S^{1,q}(\mathbb{H}^n) \), by Theorem 1.1 of [33], there exist three nonnegative Radon measures \( \mu, \nu \) and \( \omega \on \mathbb{H}^n \), an at most countable set \( J \), a family of points \( \{\xi_j\}_{j \in J} \subset \mathbb{H}^n \), two families of nonnegative numbers \( \{\mu_j\}_{j \in J} \) and \( \{\nu_j\}_{j \in J} \) and three nonnegative numbers \( v_0, \mu_0, \alpha_0 \), such that
\[
|u_k|^q d\xi^* \rightharpoonup \nu = |u|^q d\xi + v_0\delta_O + \sum_{j \in J} v_j \delta_{\xi_j} \quad \text{in } \mathcal{M}(\mathbb{H}^n),
\]
\[
|D_H u_k|^q d\xi^* \rightharpoonup \mu \geq |D_H u|^q d\xi + \mu_0\delta_O + \sum_{j \in J} \mu_j \delta_{\xi_j} \quad \text{in } \mathcal{M}(\mathbb{H}^n),
\]
(4.4)
\[
|u_k|^q |\nu|^q |\varphi|^q d\xi \rightharpoonup \omega = |u|^q |\nu|^q |\varphi|^q d\xi + \omega_0\delta_O \quad \text{in } \mathcal{M}(\mathbb{H}^n),
\]
\[
\frac{\nu_j^{q/q'}}{C} \leq \frac{H_j}{C q'} \quad \text{for all } j \in J, \quad \frac{\nu_0^{q/q'}}{\alpha_0} \leq \frac{\mu_0 - \sigma\alpha_0}{T_\sigma},
\]
where
\[ \mathcal{I}_\sigma = \inf_{u \in S^1_q(\mathbb{H}^n)} \frac{\|D_H u\|_q^q - \sigma \|u\|_{H_q}^q}{\|u\|_q^q}, \quad (4.5) \]

\( C_q \) is defined in (1.2) when \( \varphi = q \), while \( \delta_O, \delta_j \) are the Dirac functions at the points \( O \) and \( \xi_j \) of \( \mathbb{H}^n \), respectively. From (4.1)–(4.3) we derive

\[ c_{\sigma, \lambda} + o(1) \geq \mathcal{C}(\|u_k\|_{W^{1, p}} + \|u_k\|_q^q) + \left( \frac{1}{\theta} - \frac{1}{q^*} \right) \int_{\mathbb{H}^n} |u_k|^{q^*} d\xi \quad (4.6) \]

as \( k \to \infty \).

First we assert that

\[ \lim_{\lambda \to \infty} t_{\sigma, \lambda} = 0. \quad (4.7) \]

Otherwise, \( \limsup_{\lambda \to \infty} t_{\sigma, \lambda} = t_\sigma > 0 \). Hence there is a sequence \( j \mapsto \lambda_j \uparrow \infty \) such that \( t_{\sigma, \lambda_j} \to t_\sigma \) as \( j \to \infty \). Then, letting \( j \to \infty \) we get from (4.6) and Lemma 3.3 that

\[ 0 \geq \mathcal{C} t_\sigma > 0. \]

This contradiction proves the assertion (4.7). Moreover, since \( u_k \to u \) in \( W \), we know by the weak lower semicontinuity of the norms that

\[ \|u\|_{W^{1, p}} + \|u\|_{W_{q}}^q \leq t_{\sigma, \lambda}. \]

Thus, (1.3), (2.2) and (4.7) imply that

\[ \|u\|_{H_q}^q \leq C_1 t_{\sigma, \lambda} \to 0, \quad \|u\|_{q^*}^q \leq C_2 t_{\sigma, \lambda} \to 0 \quad (4.8) \]

as \( \lambda \to \infty \).

Fix now a test function \( \varphi \in C^\infty_0(\mathbb{H}^n) \), such that \( 0 \leq \varphi \leq 1 \), \( \varphi \equiv 1 \) in \( B_1 \), while \( \varphi \equiv 0 \) in \( B_2^c \), and \( |D_H \varphi|_{\infty} \leq 2 \). Take \( \varepsilon > 0 \) and put \( \varphi_{\varepsilon, j}(\xi) = \varphi(\delta_{1/\varepsilon}(\xi \ominus \varepsilon^{-1})), \xi \in \mathbb{H}^n \), for any fixed \( j \in J \), where \( \{\xi_j\}_j \) is introduced in (4.4). Fix \( j \in J \). Then \( \varphi_{\varepsilon, j} u_k \in W \) and so \( \langle I'(u_k), \varphi_{\varepsilon, j} u_k \rangle = o(1) \) as \( k \to \infty \). Therefore, as \( k \to \infty \)

\[ o(1) = A_{p, q}(u_k, \varphi_{\varepsilon, j} u_k) - \sigma \int_{\mathbb{H}^n} |u|^q \varphi_{\varepsilon, j} \frac{\psi_j}{r^q} d\xi \]

\[ - \lambda \int_{\mathbb{H}^n} f(\xi, u_k) \varphi_{\varepsilon, j} u_k d\xi - \int_{\mathbb{H}^n} |u|^q \varphi_{\varepsilon, j} d\xi \]

\[ = \int_{\mathbb{H}^n} \left\{ |D_H u_k|_{H_q}^{p-2} + |D_H u_k|_{H_q}^{q-2} \right\} (D_H u_k, D_H \varphi_{\varepsilon, j}) d\xi \]

\[ + \int_{\mathbb{H}^n} \left\{ |D_H u_k|_{H_q}^p + |D_H u_k|_{H_q}^q \right\} \varphi_{\varepsilon, j} d\xi + \int_{\mathbb{H}^n} (|u_k|^p + |u_k|^q) \varphi_{\varepsilon, j} d\xi \]

\[ - \sigma \int_{\mathbb{H}^n} |u|^q \varphi_{\varepsilon, j} \frac{\psi_j}{r^q} d\xi - \lambda \int_{\mathbb{H}^n} f(\xi, u_k) \varphi_{\varepsilon, j} u_k d\xi - \int_{\mathbb{H}^n} |u_k|^q \varphi_{\varepsilon, j} d\xi \quad (4.9) \]

Now, by the Cauchy–Schwarz inequality
\[
\limsup_{k \to \infty} \left| \int_{\mathbb{H}^n} \left( \left| D_H u_k \right|^{p-2}_H + \left| D_H u_k \right|^{q-2}_H \right) (D_H u_k, D_H \varphi_{e,j})_H u_k d\xi \right| \\
\leq \limsup_{k \to \infty} \int_{B(\xi,2e)} \left( \left| D_H u_k \right|^{p-1}_H \right) \left( \left| u_k D_H \varphi_{e,j}(\xi) \right|_H^p d\xi \right)^{1/p} \\
\leq \limsup_{k \to \infty} \int_{B(\xi,2e)} \left( \left| D_H u_k \right|^{q-1}_H \right) \left( \left| u_k D_H \varphi_{e,j}(\xi) \right|_H^q d\xi \right)^{1/q} \\
\leq \limsup_{k \to \infty} \int_{B(\xi,2e)} \left( \left| u D_H \varphi_{e,j}(\xi) \right|_H^p d\xi \right)^{1/p} + c_1 \left( \int_{B(\xi,2e)} \left| u D_H \varphi_{e,j}(\xi) \right|_H^q d\xi \right)^{1/q} \\
\leq c_0 \left( \int_{B(\xi,2e)} \left| u D_H \varphi_{e,j}(\xi) \right|_H^p d\xi \right)^{1/p} + c_1 \left( \int_{B(\xi,2e)} \left| u \right|_H^q d\xi \right)^{1/q}.
\]

where \( c_0 = \sup_{k \in \mathbb{N}} \left\| D_H u_k \right\|^{p-1}_H \) and \( c_1 = \sup_{k \in \mathbb{N}} \left\| D_H u_k \right\|^{q-1}_H \). Moreover, we put \( c_\varphi = \left( \int_{B_1} \left| D_H \varphi(\eta) \right|_H^p d\eta \right)^{1/Q} \), being

\[
\int_{B_1(\xi)} \left| D_H \varphi_{e,j}(\xi) \right|_H^p d\eta = \int_{B_1(\xi)} \frac{1}{\epsilon^Q} \left| D_H \varphi(\delta_{1/\epsilon}(\xi \circ \epsilon^{-1})) \right|_H^p d\eta \\
= \int_{B_1} \left| D_H \varphi(\eta) \right|_H^p d\eta.
\]

Here \( \eta = \delta_{1/\epsilon}(\xi \circ \epsilon^{-1}) \) is the change of variable, with \( d\eta = \epsilon^{-Q} d\xi \). Consequently,

\[
\lim \limsup_{k \to \infty} \int_{\mathbb{H}^n} \left( \left| D_H u_k \right|^{p-2}_H + \left| D_H u_k \right|^{q-2}_H \right) (D_H u_k, D_H \varphi_{e,j})_H u_k d\xi = 0. \tag{4.10}
\]

Moreover, by the properties of \( \varphi \) and (4.3), as \( k \to \infty \)

\[
0 \leq \int_{\mathbb{H}^n} \left( \left| u_k \right|^p + \left| u_k \right|^q \right) \varphi_{e,j} d\xi \leq \int_{B(\xi,2e)} \left( \left| u_k \right|^p + \left| u_k \right|^q \right) d\xi \\
\to \int_{B(\xi,2e)} \left( \left| u \right|^p + \left| u \right|^q \right) d\xi,
\]

since \( 1 < p < q < q^* \). In conclusion,

\[
\lim \limsup_{k \to \infty} \int_{\mathbb{H}^n} \left( \left| u_k \right|^p + \left| u_k \right|^q \right) \varphi_{e,j} d\xi = 0. \tag{4.11}
\]

Similarly, by \((f_1)\) and (4.3), as \( k \to \infty \)
since $1 < p < m < m < q^*$. Therefore,

$$\lim_{\varepsilon \to 0^+} \lim_{k \to \infty} \int_{\mathbb{B}^q} f(\xi, u_k) \varphi_{\varepsilon, j} u_k d\xi = 0. \quad (4.12)$$

In conclusion, passing to the limit in (4.9), using (4.4) and (4.10)–(4.12), we obtain the crucial formula for all $j \in J \cup \{0\}$

$$\int_{\mathbb{B}^q} \varphi_{\varepsilon, j} d\mu + o(1) \leq \sigma \int_{\mathbb{B}^q} \varphi_{\varepsilon, j} d\omega + \int_{\mathbb{B}^q} \varphi_{\varepsilon, j} d\nu \quad (4.13)$$

as $\varepsilon \to 0^+$. Now, by Lemma 3.3 there exists $\lambda^* = \lambda^*(\sigma, Q, q, \theta) = \lambda^*(\sigma) > 0$ such that

$$c_{\sigma, \lambda} < \left(\frac{1}{\theta} - \frac{1}{q^*}\right) \min\{C_{q^*}^{Q/q}, I_{\sigma}^{Q/q}\} \quad \text{for all } \lambda \geq \lambda^*. \quad (4.14)$$

We divide the proof in two steps.

**Step 1.** Let us first show that $\nu_j = 0$ for all $j \in J \cup \{0\}$. Assume by contradiction that $\nu_j > 0$ for some $j \in J \cup \{0\}$, that is $\xi_j$ is a singular point of the measure $\nu$ (note that we put $\xi_0 = O$).

Thus, (4.4) and (4.13) imply $C_{q^*} \nu_j^{q/q^*} \leq \mu_j \leq \nu_j$ and, since by contradiction $\nu_j > 0$, this yields $\nu_j \geq C_{q^*}^{Q/q^*}$. Moreover, by (4.4) and (4.6) we know that

$$c_{\sigma, \lambda} + o(1) \geq \left(\frac{1}{\theta} - \frac{1}{q^*}\right) \int_{\mathbb{B}^q} \varphi_{\varepsilon, j} d\nu$$

as $k \to \infty$ since $\varphi_{\varepsilon, j} \leq 1$ and so, sending $k \to \infty$ and $\varepsilon \to 0^+$, we have for all $\lambda \geq \lambda^*$

$$c_{\sigma, \lambda} \geq \left(\frac{1}{\theta} - \frac{1}{q^*}\right) \nu_j \geq \left(\frac{1}{\theta} - \frac{1}{q^*}\right) C_{q^*}^{Q/q^*} > 0.$$

This is an obvious contradiction by (4.14). Hence, $\nu_j = \mu_j = 0$ for all $j \in J$ and for all $\lambda \geq \lambda^*$, as desired. Similarly, when the center of the ball is $O$, then (4.13) gives $I_{\sigma} \nu_0^{q/q^*} + \sigma \omega_0 \leq \mu_0 \leq \sigma \omega_0 + \nu_0$. Assume by contradiction that $\nu_0 \neq 0$. Then, $\nu_0 \geq I_{\sigma}^{Q/q}$. As above, by (4.6) we obtain as $k \to \infty$ and $\varepsilon \to 0^+$

$$c_{\sigma, \lambda} \geq \left(\frac{1}{\theta} - \frac{1}{q^*}\right) \nu_0 \geq \left(\frac{1}{\theta} - \frac{1}{q^*}\right) I_{\sigma}^{Q/q} > 0,$$

which is again a contradiction by (4.14). Therefore, $\nu_0 = 0$ and so $\mu_0 = \sigma \omega_0$ for all $\lambda \geq \lambda^*$.

**Step 2.** Let us now prove that $\omega_0 = 0$. Assume again by contradiction that $\omega_0 > 0$. By (1.3) and differentiation, we get for all $\tau > 0$ the existence of a constant $C_\tau$ such that
\[ \mathcal{H}_q \int_{\mathbb{H}^n} |u_k \phi_{x,0}|^q \psi^q \frac{d\xi}{r(\xi)^q} \leq \int_{\mathbb{H}^n} |D_H(u_k \phi_{x,0})|_H^q \frac{d\xi}{r(\xi)^q}, \]  

(4.15)

where by (4.3), the Hölder inequality and a change of variable, we can estimate the last term as follows

\[ \lim_{k \to \infty} \int_{B_{2\varepsilon}} |D_H \phi_{x,0}|_H^q |u_k|^q d\xi = \int_{B_{2\varepsilon}} |D_H \phi_{x,0}|_H^q |u|^q d\xi \leq c_\phi \left( \int_{B_{2\varepsilon}} |u|^q d\xi \right)^{q/q^*}, \]

where \( c_\phi \) was defined above. Consequently,

\[ \lim_{\varepsilon \to 0^+} \lim_{k \to \infty} \int_{\mathbb{H}^n} |D_H \phi_{x,0}|^q |u_k|^q d\xi = 0, \]

from which, using (4.3) and (4.15), we get

\[ \mathcal{H}_q \int_{\mathbb{H}^n} |\phi_{x,0}|^q d\omega + o(1) \leq (1 + \tau) \int_{\mathbb{H}^n} |\phi_{x,0}|^q d\mu, \]

for all \( \tau > 0 \) which yields, sending \( \tau \to 0^+ \), that \( \mathcal{H}_q \omega_0 \leq \mu_0 \). Thus, we obtain from the previous step

\[ \mathcal{H}_q \omega_0 \leq \mu_0 = \sigma \omega_0 < \mathcal{H}_q \omega_0. \]

This is impossible. Hence \( \omega_0 = 0 \) for all \( \lambda \geq \lambda^* \), as desired.

In summary, we have shown that there exists \( \lambda^* = \lambda^*(\sigma) \) such that for all \( \lambda \geq \lambda^* \)

\[ |u_k|^q d\xi \overset{\ast}{\rightharpoonup} |u|^q d\xi, \quad |u_k|^q \frac{\psi^q}{r^q} d\xi \overset{\ast}{\rightharpoonup} |u|^q \frac{\psi^q}{r^q} d\xi \quad \text{in} \quad \mathcal{M}(\mathbb{H}^n), \]

as \( k \to \infty \), that is for all \( \phi \in C_0^\infty(\mathbb{H}^n) \) it results

\[ \lim_{k \to \infty} \int_{\mathbb{H}^n} \phi |u_k|^q d\xi = \int_{\mathbb{H}^n} \phi |u|^q d\xi, \]

\[ \lim_{k \to \infty} \int_{\mathbb{H}^n} \phi |u_k|^q \frac{\psi^q}{r^q} d\xi = \int_{\mathbb{H}^n} \phi |u|^q \frac{\psi^q}{r^q} d\xi. \]

(4.16)

From now on in the proof we assume that \( \lambda \) is fixed, with \( \lambda \geq \lambda^* \). Our goal is now to prove that for all \( R > 0 \) we have

\[ \int_{B_R} |D_H u_k - D_H u|^q d\xi = o(1) \quad \text{as} \quad k \to \infty. \]

(4.17)

To this aim, take \( R > 0 \) and \( \phi \in C_0^\infty(\mathbb{H}^n) \) such that \( 0 \leq \phi \leq 1 \text{ in } \mathbb{H}^n, \phi \equiv 1 \text{ in } B_R, \phi \equiv 0 \text{ in } B^c_{2R} \text{ and } \|D_H \phi\|_\infty \leq 2 \). Clearly,
\[ o(1) = \langle I'(u_k), \varphi u_k \rangle - \langle I'(u_k), \varphi u \rangle = \langle I'(u_k), \varphi (u_k - u) \rangle \]
\[ = A_{p,q}(u_k, \varphi (u_k - u)) - \sigma \int_{\mathbb{H}^n} |u_k|^{q-2} u_k \varphi (u_k - u) \frac{\psi q}{r^q} d\xi \]
\[ - \lambda \int_{\mathbb{H}^n} f(\xi, u_k) \varphi (u_k - u) d\xi - \int_{\mathbb{H}^n} |u_k|^{q-2} u_k \varphi (u_k - u) d\xi \]
\[ = \int_{\mathbb{H}^n} (|D_H u_k|_{H}^{p-2} + |D_H u_k|_{H}^{q-2}) (D_H u_k, D_H \varphi) (u_k - u) d\xi \]
\[ + \int_{\mathbb{H}^n} (|D_H u_k|_{H}^{p-2} + |D_H u_k|_{H}^{q-2}) (|D_H u_k|^2 - (D_H u_k, D_H u)_{H}) \varphi d\xi \]
\[ + \int_{\mathbb{H}^n} (|u_k|^{p-2} + |u_k|^{q-2}) u_k \varphi (u_k - u) d\xi \]
\[ - \sigma \int_{\mathbb{H}^n} |u_k|^{q-2} u_k \varphi (u_k - u) \frac{\psi q}{r^q} d\xi - \lambda \int_{\mathbb{H}^n} f(\xi, u_k) \varphi (u_k - u) d\xi \]
\[ - \int_{\mathbb{H}^n} |u_k|^{q-2} u_k \varphi (u_k - u) d\xi, \]
so that
\[ \int_{\mathbb{H}^n} (|D_H u_k|_{H}^{p-2} + |D_H u_k|_{H}^{q-2}) (|D_H u_k|^2 - (D_H u_k, D_H u)_{H}) \varphi d\xi \]
\[ = \langle I'(u_k), \varphi u_k \rangle - \langle I'(u_k), \varphi u \rangle - \int_{\mathbb{H}^n} (|D_H u_k|_{H}^{p-2} + |D_H u_k|_{H}^{q-2}) (D_H u_k, D_H \varphi) (u_k - u) d\xi \]
\[ - \int_{\mathbb{H}^n} (|u_k|^{p-2} + |u_k|^{q-2}) u_k \varphi (u_k - u) d\xi \]
\[ + \sigma \int_{\mathbb{H}^n} |u_k|^{q-2} u_k \varphi (u_k - u) \frac{\psi q}{r^q} d\xi + \lambda \int_{\mathbb{H}^n} f(\xi, u_k) \varphi (u_k - u) d\xi \]
\[ + \int_{\mathbb{H}^n} |u_k|^{q-2} u_k \varphi (u_k - u) d\xi \]
\[ = \langle I'(u_k), \varphi u_k \rangle - \langle I'(u_k), \varphi u \rangle \]
\[ - \int_{\mathbb{H}^n} (|D_H u_k|_{H}^{p-2} + |D_H u_k|_{H}^{q-2}) (D_H u_k, D_H \varphi) (u_k - u) d\xi \]
\[ - \int_{\mathbb{H}^n} (|u_k|^{p-2} + |u_k|^{q-2}) u_k \varphi (u_k - u) d\xi \]
\[ + \sigma \int_{\mathbb{H}^n} |u_k|^{q-2} u_k \varphi (u_k - u) \frac{\psi q}{r^q} d\xi + \lambda \int_{\mathbb{H}^n} f(\xi, u_k) \varphi (u_k - u) d\xi \]
\[ + \int_{\mathbb{H}^n} |u_k|^{q-2} u_k \varphi (u_k - u) d\xi \]
By the Cauchy–Schwarz and the Hölder inequalities
\[ \left| \int_{\mathbb{H}^n} (|D_H u_k|_{H}^{p-2} + |D_H u_k|_{H}^{q-2}) (D_H u_k, D_H \varphi) (u_k - u) d\xi \right| \]
\[ \leq 2 \int_{B_{2R}} \left( |D_H u_k|_{H}^{p-1} |u_k - u| d\xi + |D_H u_k|_{H}^{q-1} |u_k - u| \right) d\xi \]
\[ \leq 2 \|D_H u_k\|_{p}^{p-1} \left( \int_{B_{2R}} |u_k - u|^p d\xi \right)^{1/p} \]
\[ + 2 \|D_H u_k\|_{q}^{q-1} \left( \int_{B_{2R}} |u_k - u|^q d\xi \right)^{1/q}, \]
which in turns gives, by (4.3),
\[ \lim_{k \to \infty} \int_{\mathbb{H}^n} (|D_H u_k|^{p-2}_H + |D_H u_k|^{q-2}_H)(D_H u_k, D_H \varphi)(u_k - u) d\xi = 0. \quad (4.19) \]

Similarly,
\[
\int_{\mathbb{H}^n} (|u_k|^{p-2} + |u_k|^{q-2}) u_k\varphi(u_k - u) d\xi 
\leq \|u_k\|_p^{p-1} \left( \int_{B_{2R}} |u_k - u|^p d\xi \right)^{1/p} + \|u_k\|_q^{q-1} \left( \int_{B_{2R}} |u_k - u|^q d\xi \right)^{1/q},
\]
which yields, again by (4.3),
\[ \lim_{k \to \infty} \int_{\mathbb{H}^n} (|u_k|^{p-2} + |u_k|^{q-2}) u_k\varphi(u_k - u) d\xi = 0. \quad (4.20) \]

Moreover, by \((f_1)\), the Hölder inequality and (4.3), as \( k \to \infty \)
\[
0 \leq \int_{\mathbb{H}^n} f(\xi, u_k)\varphi(u_k - u) d\xi 
\leq \int_{B_{2R}} \varphi(m|u_k|^m - |u_k - u| + mC_1|u_k|^{m-1}|u_k - u|) d\xi 
\leq C \left\{ \left( \int_{B_{2R}} |u_k - u|^m d\xi \right)^{1/m} + \left( \int_{B_{2R}} |u_k - u|^m d\xi \right)^{1/m} \right\} \to 0, \quad (4.21) \]
since \( 1 < p < m < q^* \), where \( C = m \sup_{k \in \mathbb{N}} \|u_k\|^{m-1} + mC_1 \sup_{k \in \mathbb{N}} \|u_k\|^{m-1} \) and \( C < \infty \) by Lemma 3.1. Finally, as \( k \to \infty \)
\[ \int_{\mathbb{H}^n} |u_k|^{q-2} u_k\varphi(u_k - u) \frac{\psi^q}{r^q} d\xi \to 0, \quad (4.22) \]
and
\[ \int_{\mathbb{H}^n} |u_k|^{q-2} u_k\varphi(u_k - u) d\xi \to 0, \quad (4.23) \]
by (2.2), (4.3) and (4.16), since \( |u_k|^{q-2} u_k \to |u|^{q-2} u \) in \( L^{q'}(\mathbb{H}^n) \) and, similarly, \( |u_k|^{q-2} u_k \to |u|^{q-2} u \) in \( L^{q'}(\mathbb{H}^n, \psi^{q'r-q}) \) in virtue of Proposition A.8 of [2], which can be applied in this contest, since the weight function \( \psi^{q'r-q} \) is of class \( L^1_{\text{loc}}(\mathbb{H}^n) \), being \( \psi = |\psi| \leq 1 \) and \( 1 < q < Q \).

Thus, combining (4.18)–(4.23), we have as \( k \to \infty \)
\[ \int_{\mathbb{H}^n} (|D_H u_k|^{p-2}_H + |D_H u_k|^{q-2}_H)(|D_H u_k|^2 - (D_H u_k, D_H \varphi)(u_k - u) d\xi = o(1). \quad (4.24) \]

Now, using the notations of (3.2), by convexity
\[ \mathcal{P}_p(D_H u_k, D_H u) = \left( |D_H u_k|^{p-2}_H D_H u_k - |D_H u|^{p-2}_H D_H u, D_H u_k - D_H u \right)_H \geq 0 \]
a.e. in \( \mathbb{H}^n \) and for all \( k \). Let us now distinguish two different cases.

Case \( q \geq 2 \). By (3.2), there exists \( \kappa > 0 \), depending only on \( q \), such that
as $k \to \infty$ by (4.24). Thus, (4.17) is proved when $q \geq 2$.

Case $1 < q < 2$. By (3.2) with $s = q$ and the Hölder inequality with $p = 2/q$ and $p' = 2/(2 - q)$, there exists $\kappa > 0$, depending only on $q$, such that

$$
\frac{1}{\kappa} \int_{B_R} |D_H u_k - D_H u|^q_d \xi \leq \int_{B_R} |D_H u_k - D_H u|^q_d \xi \leq \int_{B_R} \|D_H u_k\|^q_d + \|D_H u\|^q_d (2-q)/2 \quad (4.25)
$$

$$
\leq M^{(2-q)/2} \left( \int_{B_R} \|D_H u_k\|^q_d + \|D_H u\|^q_d \phi \right)^{q/2} \quad (4.26)
$$

as $k \to \infty$ by (4.24), since $\|D_H u_k\|^q_d + \|D_H u\|^q_d \leq M$ for all $k$ and some positive constant $M$, as stated.

Therefore (4.17) is proved for all $q$, with $1 < q < Q$. Hence, $D_H u_k \to D_H u$ in $L^q(B_R; \mathbb{R}^{2n})$ for all $R > 0$. Consequently, up to subsequences, not relabelled, we get that

$$
D_H u_k \to D_H u \quad \text{a.e. in } \mathbb{H}^n, \quad (4.27)
$$

and for all $R > 0$ there exists a function $h_R \in L^q(B_R)$ such that $|D_H u_k| \leq h_R$ a.e. in $B_R$ for all $k$.

Fix $\phi$ in $C_c^\infty(\mathbb{H}^n)$ and let $R > 0$ so large that $\text{supp } \phi \subset B_R$. Since by assumption $(I^n(u_k), \phi) = o(1)$ as $k \to \infty$, we have
\[ A_{p,q}(u_k, \phi) = \sigma \int_{\mathbb{R}^n} |u_k|^{q-2} u_k \phi \frac{\partial^q}{\partial \xi^q} + \lambda \int_{\mathbb{R}^n} f(\xi, u_k) \phi d\xi + \int_{\mathbb{R}^n} |u_k|^{q-2} u \phi d\xi + o(1). \]  

(4.28)

We want to pass to the limit in (4.28). By the above construction we have a.e. in \( B_R \)
\[ |D_H u_k|^p + |D_H u_k|^q \leq (h_R^p + h_R^q) |D_H \phi| \quad \text{is} \quad \mathbf{q} \in L^1(B_R). \]

Therefore, the dominated convergence theorem gives as \( k \to \infty \)
\[ \int_{\mathbb{R}^n} (|D_H u_k|^p + |D_H u_k|^q) (D_H u_k, D_H \phi) d\xi \]
\[ = \int_{B_R} (|D_H u_k|^p + |D_H u_k|^q) (D_H u_k, D_H \phi) d\xi \]
\[ \to \int_{\mathbb{R}^n} (|D_H u|^p + |D_H u|^q) (D_H u, D_H \phi) d\xi . \]

Similarly, using (4.3), we get a.e. in \( B_R \)
\[ |D_H u_k|^p + |D_H u_k|^q \leq (g_R^p + g_R^q) |D_H \phi| \quad \text{is} \quad \mathbf{g} \in L^1(B_R), \]

and so the dominated convergence theorem gives as \( k \to \infty \)
\[ \int_{\mathbb{R}^n} (|u_k|^p + |u_k|^q) u_k \phi d\xi \to \int_{\mathbb{R}^n} (|u|^p + |u|^q) u \phi d\xi . \]

Moreover, (f_i) implies
\[ |f(\xi, u_k) \cdot \phi| \leq m |u_k|^{m-1} |\phi| + m C_1 |u_k|^{m-1} |\phi| \leq \mathbf{0} \in L^1(B_R), \]

so by again the dominated convergence theorem as \( k \to \infty \)
\[ \int_{\mathbb{R}^n} f(\xi, u_k) \phi d\xi \to \int_{\mathbb{R}^n} f(\xi, u) \phi d\xi . \]

Finally, letting \( k \to \infty \) in (4.28), using the above arguments and (4.16), we get at once that
\[ A_{p,q}(u, \phi) = \sigma \int_{\mathbb{R}^n} |u|^{q-2} u \phi \frac{\partial^q}{\partial \xi^q} + \lambda \int_{\mathbb{R}^n} f(\xi, u) \phi d\xi + \int_{\mathbb{R}^n} |u|^{q-2} u \phi d\xi . \]  

(4.29)

along any \( \phi \) in \( C_0^\infty(\mathbb{R}^n) \). This implies, by a density argument, that \( u \) is a solution of (4) and (4.30).

Indeed, define the sequence of cut–off functions
\[ \zeta_k(\xi) = \zeta(\delta_{1/k}(\xi)), \quad \xi \in \mathbb{R}^n, \]  

where \( \delta_{1/k} \) is the dilation of parameter \( 1/k \), as introduced in (2.1). Let \( \varphi \in W \) and put \( \phi_k = \zeta_k(\rho_k \ast \varphi) \), where \( (\rho_k)_k \) is the sequence of mollifiers introduced in Sect. 2 and \( (\zeta_k)_k \) is a sequence of cut–off functions defined in (4.30).

From Theorem 2.2 in \[31\], it is evident that the sequence \( (\phi_k)_k \) is in \( C_c^\infty(\mathbb{R}^n) \) and has the properties that \( \phi_k \to \varphi \) in \( W \) and, up to subsequences, \( \phi_k \to \varphi, D_H \phi_k \to D_H \varphi \) a.e. in \( \mathbb{R}^n \) as \( k \to \infty \), and there exist functions \( \Phi \in L^p(\mathbb{R}^n) \) and \( \Psi \in L^q(\mathbb{R}^n) \) such that \( |\phi_k| \leq \Phi \) and \( |\Psi| \leq \Psi \).
$|D_H\phi_k|_H \leq \Phi$ and $|\phi_k| \leq \Psi$, $|D_H\phi_k|_H \leq \Psi$ a.e. in $\mathbb{H}^n$ and for all $k$. Of course, (4.29) holds true along $(\phi_k)_k$ for all $k$. Passing to the limit as $k \to \infty$ under the sign of integrals by the dominated convergence theorem, we obtain the validity of (4.29) for all $\varphi \in W$. In conclusion,

$$\langle I'(u), \varphi \rangle = 0 \quad \text{for all } \varphi \in W,$$

that is $u$ is a solution of $(\mathcal{E})$ for all $\lambda \geq \lambda^*$. This completes the proof of (ii).

\begin{lemma}
(The (PS)$_{c_{\sigma, \lambda}}$ condition)] Let $(u_k)_k \subset W$ be a Palais–Smale sequence of $I$ at the level $c_{\sigma, \lambda}$. Then, for any $\sigma \in (-\infty, \mathcal{H}_q)$ and any $\lambda \geq \lambda^*$, up to a subsequence, $u_k \to u$ in $W$.

\end{lemma}

\begin{proof}
Let $(u_k)_k \subset W$ be a Palais–Smale sequence of $I$ at the level $c_{\sigma, \lambda}$ and fix $\lambda \geq \lambda^*$ such that (4.14) holds. Define the following quantities

$$\nu_\infty = \lim_{R \to \infty} \limsup_{k \to \infty} \int_{B_R} |u_k|^{q^*} d\xi, \quad \mu_\infty = \lim_{R \to \infty} \limsup_{k \to \infty} \int_{B_R} |D_H u_k|_H^q d\xi,$$

$$\omega_\infty = \lim_{R \to \infty} \limsup_{k \to \infty} \int_{B_R} |u_k|^{\Psi q^*} \frac{d\xi}{r^q}.$$

From Theorem 1.2 of [33], we know that

$$\limsup_{k \to \infty} \int_{\mathbb{H}^n} |u_k|^{q^*} d\xi = \nu(\mathbb{H}^n) + \nu_\infty,$$

$$\limsup_{k \to \infty} \int_{\mathbb{H}^n} |D_H u_k|_H^q d\xi = \mu(\mathbb{H}^n) + \mu_\infty,$$

$$\limsup_{k \to \infty} \int_{\mathbb{H}^n} |u_k|^{\Psi q^*} \frac{d\xi}{r^q} = \omega(\mathbb{H}^n) + \omega_\infty, \quad \nu q^{q^*} \leq \frac{\mu_\infty - \sigma \omega_\infty}{I_\sigma},$$

where $\mu, \nu, \omega$ are the measures introduced in (4.4).

We first prove that $\nu_\infty = \omega_\infty = 0$. To this aim, consider the function $\chi \in C^\infty(\mathbb{H}^n)$ such that $0 \leq \chi \leq 1$, $\chi = 0$ in $B_1$ and $\chi = 1$ in $B_2$. Take $R > 0$ and put

$$\chi_R(\xi) = \chi(\delta_{1/R}(\xi)), \quad \xi \in \mathbb{H}^n.$$

Note that supp $\chi_R \subset B_R^c$ and supp $D_H \chi_R \subset B_{2R} \setminus B_R$. Lemma 4.1 implies at once that $\chi_R u_k \in W$ for any $k$ and so $\langle I'(u_k), \chi_R u_k \rangle = o(1)$ as $k \to \infty$. Therefore, denoting by simplicity $A_{\delta_{1/R}}(\phi) = |D_H \phi|_H^\delta + |D_H \phi|_H^\gamma$ for all $\phi \in W$, we obtain as $k \to \infty$
\[ o(1) = \int_{\mathbb{H}^n} \left\{ A_{p-2,q-2}(u_k) \langle D_H u_k, D_H (\chi_R u_k) \rangle_H + (|u_k|^p + |u_k|^q) \chi_R \right\} d\xi \]
\[ - \sigma \int_{\mathbb{H}^n} |u|^q \chi_R \frac{p}{r^q} d\xi - \lambda \int_{\mathbb{H}^n} f(\xi, u_k) \chi_R u_k d\xi - \int_{\mathbb{H}^n} |u_k|^q \chi_R d\xi \]
\[ = \int_{\mathbb{H}^n} \left\{ A_{p-2,q-2}(u_k) \langle D_H u_k, D_H \chi_R \rangle_H u_k d\xi \right\} \]
\[ + \int_{\mathbb{H}^n} \left\{ A_{p,q}(u_k) \chi_R d\xi + \int_{\mathbb{H}^n} (|u_k|^p + |u_k|^q) \chi_R d\xi \right\} \]
\[ - \sigma \int_{\mathbb{H}^n} |u|^q \chi_R \frac{p}{r^q} d\xi - \lambda \int_{\mathbb{H}^n} f(\xi, u_k) \chi_R u_k d\xi - \int_{\mathbb{H}^n} |u_k|^q \chi_R d\xi, \] (4.33)

so that
\[
\int_{\mathbb{H}^n} \{ A_{p,q}(u_k) \chi_R d\xi = - \int_{\mathbb{H}^n} \{ A_{p-2,q-2}(u_k) \langle D_H u_k, D_H \chi_R \rangle_H u_k d\xi \}
\[ - \int_{\mathbb{H}^n} (|u_k|^p + |u_k|^q) \chi_R d\xi + \sigma \int_{\mathbb{H}^n} |u|^q \chi_R \frac{p}{r^q} d\xi \]
\[ + \lambda \int_{\mathbb{H}^n} f(\xi, u_k) \chi_R u_k d\xi + \int_{\mathbb{H}^n} |u_k|^q \chi_R d\xi + o(1). \] (4.34)

Now, by the Cauchy–Schwarz inequality and the properties of \( \chi_R \)

\[
\limsup_{k \to \infty} \left( \int_{\mathbb{H}^n} \{ A_{p-2,q-2}(u_k) \langle D_H u_k, D_H \chi_R \rangle_H u_k d\xi \right) \]
\[ \leq \limsup_{k \to \infty} \int_{B_{2r} \setminus B_r} A_{p-1,q-1}(u_k) |u_k| \cdot |D_H \chi_R|_H d\xi \]
\[ \leq \limsup_{k \to \infty} \|D_H u_k\|_{p-1}^p \left( \int_{B_{2r} \setminus B_r} |u_k|^p D_H \chi_R(\xi)_{H}^p d\xi \right)^{1/p} \]
\[ + \limsup_{k \to \infty} \|D_H u_k\|_{q-1}^q \left( \int_{B_{2r} \setminus B_r} |u_k|^q D_H \chi_R(\xi)_{H}^q d\xi \right)^{1/q} \]
\[ \leq c_0 \left( \int_{B_{2r} \setminus B_r} |u|^p d\xi \right)^{1/p} + c_1 \left( \int_{B_{2r} \setminus B_r} |u|^q d\xi \right)^{1/q} \]
\[ \leq c \chi \left\{ c_0 \left( \int_{B_{2r}} |u|^p d\xi \right)^{1/p} + c_1 \left( \int_{B_{2r}} |u|^q d\xi \right)^{1/q} \right\}. \]

where \( c_0 = \sup_{k \in \mathbb{N}} \|D_H u_k\|_{p-1}^p \) and \( c_1 = \sup_{k \in \mathbb{N}} \|D_H u_k\|_{q-1}^q \), while the change of variable \( \eta = \delta_{1/\epsilon}(\xi) \) gives \( c_\delta = \left( \int_{B_{2r} \setminus B_r} |D_H \chi(\eta)|_{H}^{q \epsilon} d\eta \right)^{1/q \epsilon} \). Consequently,

\[
\lim \limsup_{k \to \infty} \int_{\mathbb{H}^n} \{ A_{p-2,q-2}(u_k) \langle D_H u_k, D_H \chi_R \rangle_H u_k d\xi \} = 0. \] (4.35)

By Theorem 3.1, up to a subsequence, as \( k \to \infty \)
\[ u_k \to u \text{ strongly in } L^p(\mathbb{H}^n) \text{ for any } p \in (p^*, q^*). \] (4.36)
Therefore, \((f)\) with \(\epsilon = 1\), together with the Hölder inequality give as \(k \to \infty\)

\[
\limsup_{k \to \infty} \int_{\mathbb{R}^n} f(\xi, u_k) \chi_R u_k d\xi \leq \limsup_{k \to \infty} \int_{\mathbb{R}^n} \chi_R \left( |u_k|^m + m C_1 |u_k|^m \right) d\xi
\]

\[
\leq m \limsup_{k \to \infty} \int_{B_R} |u_k|^m d\xi + m C_1 \limsup_{k \to \infty} \int_{B_R} |u_k|^m d\xi
\]

\[
= m \int_{B_R} |u|^m d\xi + m C_1 \int_{B_R} |u|^m d\xi.
\]

Consequently, by (4.36) we have

\[
\lim_{k \to \infty} \limsup_{R \to \infty} \int_{\mathbb{R}^n} f(\xi, u_k) \chi_R u_k d\xi = 0. \tag{4.37}
\]

Now, again by [33], we know that

\[
\mu_\infty = \lim_{k \to \infty} \limsup_{R \to \infty} \int_{\mathbb{R}^n} |D Micha |^q u_k|^q d\xi,
\]

\[
\nu_\infty = \lim_{k \to \infty} \limsup_{R \to \infty} \int_{\mathbb{R}^n} |\Psi |^q |u_k|^q d\xi,
\]

\[
\omega_\infty = \lim_{k \to \infty} \limsup_{R \to \infty} \int_{\mathbb{R}^n} |\Psi |^q |u_k|^q \Psi d\xi. \tag{4.38}
\]

Hence, (4.34)–(4.38) imply that \(\mu_\infty \leq \sigma \omega_\infty + \nu_\infty\), and so (4.32) gives

\[
\mathcal{I}_\sigma^{\nu_\infty/q} + \sigma \omega_\infty \leq \mu_\infty \leq \sigma \omega_\infty + \nu_\infty.
\]

Assume by contradiction that \(\nu_\infty \neq 0\). Then, \(\nu_\infty \geq \mathcal{I}_\sigma^{\nu_\infty/q}\). Thanks to (4.6), and (4.16), this yields

\[
c_{\sigma, k} = \lim_{k \to \infty} \left\{ f(u_k) - \frac{1}{\theta} (f(u_k), u_k) \right\} \geq \limsup_{k \to \infty} \left( \frac{1}{\theta} - \frac{1}{q^*} \right) \int_{\mathbb{R}^n} |u_k|^q d\xi
\]

\[
= \left( \frac{1}{\theta} - \frac{1}{q^*} \right) \left\{ \int_{\mathbb{R}^n} |u|^q d\xi + \nu_\infty \right\} \geq \left( \frac{1}{\theta} - \frac{1}{q^*} \right) \mathcal{I}_\sigma^{\nu_\infty/q},
\]

which contradicts (4.14). Thus \(\nu_\infty = 0\) and so

\[
\mu_\infty = \sigma \omega_\infty.
\]

It remains to prove that \(\omega_\infty = 0\), which in turn implies \(\mu_\infty = 0\). But this is analogous to the proof of \(\omega_0 = 0\), since (1.3) applied to \(\chi_R u_k\) yields that for all \(\tau > 0\) there exists \(C_\tau > 0\) such that

\[
\mathcal{H}_\tau \int_{\mathbb{R}^n} |u_k|^q \Psi d\xi \leq \|D Micha |^q u_k\|^q_q \leq (1 + \tau) \int_{\mathbb{R}^n} |\chi_R|^q |D Micha |^q u_k| d\xi
\]

\[
+ C_\tau \int_{\mathbb{R}^n} |D Micha |^q u_k|^q d\xi. \tag{4.39}
\]

where, arguing as before,
Then, (4.41), together with (4.40) and (4.44), yields
\[ \lim_{R \to \infty} \limsup_{k \to \infty} \int_{B_{Rk} \setminus B_R} |D_H X_R|^q |u_k|^q d\xi = \lim_{R \to \infty} \limsup_{k \to \infty} \int_{B_{Rk} \setminus B_R} |D_H X_R|^q |u_k|^q d\xi = 0. \]

Now, assume by contradiction that \( \omega_\infty > 0 \). Hence, as \( \tau \to 0^+ \)
\[ \mathcal{H}_q \omega_\infty \leq \mu_\infty = \sigma \omega_\infty < \mathcal{H}_q \omega_\infty, \]
which is impossible.

Therefore, by (4.16) and (4.32), since \( \nu_\infty = \omega_\infty = 0 \), up to a subsequence we have
\[ \lim_{k \to \infty} \|u_k\|_{q^*}^q = \nu(\mathbb{R}^N) + \nu_\infty = \|u\|_{q^*}^q, \]
\[ \lim_{k \to \infty} \|u_k\|_{H^q}^q = \nu(\mathbb{R}^N) + \nu_\infty = \|u\|_{H^q}^q. \] (4.40)

Now, Lemma 4.1, part (ii), yields as \( k \to \infty \)
\[ o(1) = \langle f'(u_k), u_k \rangle, \quad o(1) = \langle f'(u), u \rangle. \]

Subtracting the two formulas we obtain
\[ o(1) = \|u_k\|_{H^{1,p}}^p + \|u_k\|_{H^{1,q}}^q - \|u\|_{H^{1,p}}^p - \|u\|_{H^{1,q}}^q - \sigma (\|u_k\|_{H^q}^q - \|u\|_{H^q}^q) - \lambda \int_{\mathbb{R}^n} \{f(\xi, u_k)u_k - f(\xi, u)u\} d\xi \]
\[ - \|u_k\|_{q^*}^q + \|u\|_{q^*}^q. \] (4.41)

Now, adding and subtracting the term \( f(\xi, u_k)u \) we have
\[ \int_{\mathbb{R}^n} \{f(\xi, u_k)u_k - f(\xi, u)u\} d\xi = \int_{\mathbb{R}^n} f(\xi, u_k)(u_k - u)d\xi \]
\[ + \int_{\mathbb{R}^n} \{f(\xi, u_k) - f(\xi, u)\} ud\xi, \] (4.42)
where by (4.36), (f1) with \( \varepsilon = 1 \) and the Hölder inequality, it is easy to see that
\[ \int_{\mathbb{R}^n} f(\xi, u_k)(u_k - u)d\xi \leq \int_{\mathbb{R}^n} \left\{ m|u_k|^{m-1}|u_k - u| + mC_1|u_k|^{m-1}|u_k - u| \right\} d\xi \]
\[ \leq C(\|u_k - u\|_m + \|u_k - u\|_m) \to 0, \]
since \( p < m < m < q^* \). On the other hand, the dominated convergence theorem implies, since (f1) holds, that
\[ \int_{\mathbb{R}^n} \{f(\xi, u_k) - f(\xi, u)\} ud\xi \to 0 \] (4.43)
as \( k \to \infty \). Therefore, (4.42)–(4.43) give as \( k \to \infty \)
\[ \int_{\mathbb{R}^n} \{f(\xi, u_k)u_k - f(\xi, u)u\} d\xi \to 0. \] (4.44)

Then, (4.41), together with (4.40) and (4.44), yields
Finally, from an application of the Brézis and Lieb lemma we conclude, thanks to (4.45), that
\[ \|u_k\|^p_{H^{\phi, p} \cap H^{\psi, q}} \to \|u\|^p_{H^{\phi, p} \cap H^{\psi, q}}, \] as \( k \to \infty \). Consequently, \( u_k \to u \) in \( W \) as \( k \to \infty \).

\[ \|u_k - u\|^p_{H^{\phi, p} \cap H^{\psi, q}} + \|u_k - u\|^q_{H^{\phi, p} \cap H^{\psi, q}} \to 0 \]
as \( k \to \infty \). Consequently, \( u_k \to u \) in \( W \) as \( k \to \infty \).

\[ \square \]

**Proof of Theorem 1.1** First, Lemma 3.2 guarantees that for any \( \sigma \in (-\infty, \mathcal{H}_q) \) and any \( \lambda \geq \lambda_* \), the functional \( I \) has the geometry of the mountain pass lemma. Thus, \( I \) admits a Palais–Smale sequence \( (u_k) \) at level \( c_{\sigma, \lambda} \), which, in virtue of Lemma 4.1 part (i), up to a subsequence, weakly converges to some limit \( u \in W \). Moreover, by Lemma 4.1 part (ii), the weak limit \( u \in W \) is a critical point of \( I \). Now, for all \( \sigma \in (-\infty, \mathcal{H}_q) \) and all \( \lambda \geq \lambda_* \), the functional \( I \) satisfies the \((PS)_{c_{\sigma, \lambda}}\) condition, as asserted in Lemma 4.2. Therefore, up to a subsequence, \( u_k \to u \) in \( W \) as \( k \to \infty \) and so \( u \) is a nontrivial solution of equation (\( \mathcal{E} \)).

\[ \square \]

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