ON THE RIGOROUS MATHEMATICAL DERIVATION
FOR THE VISCOSOUS PRIMITIVE EQUATIONS WITH
DENSITY STRATIFICATION∗

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Abstract   In this paper, we rigorously derive the governing equations describing the motion
of a stable stratified fluid, from the mathematical point of view. In particular, we prove that
the scaled Boussinesq equations strongly converge to the viscous primitive equations with
density stratification as the aspect ratio goes to zero, and the rate of convergence is of the
same order as the aspect ratio. Moreover, in order to obtain this convergence result, we also
establish the global well-posedness of strong solutions to the viscous primitive equations with
density stratification.

Key words   Boussinesq equations; primitive equations; density stratification; hydrostatic
approximation; strong convergence

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1 Introduction

The primitive equations are considered to be a fundamental model in geophysical flows
([32, 33, 37, 40, 41]). For large-scale oceanic dynamics, an important feature is that the vertical
scale of the ocean is much smaller than the horizontal scale, which means that we can use
hydrostatic approximation to simulate the motion of the ocean in the vertical direction. Owing
to this fact, and the high accuracy of hydrostatic approximation, the primitive equations for
oceanic dynamics can be formally derived from the Boussinesq equations (see [10, 26]).

The small aspect ratio limit from the Navier-Stokes equations to the primitive equations
was first studied by Azérad-Guillén [1] in a weak sense, then by Li-Titi [28] in a strong sense
with error estimates, and finally by Furukawa et al. [14] in a strong sense but under a relaxed
regularity on the initial condition. Subsequently, the strong convergence of solutions of the
scaled Navier-Stokes equations to the corresponding ones of the primitive equations with only
horizontal viscosity was obtained by Li-Titi-Yuan [30]. In addition, the rigorous justification

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Let us consider the anisotropic Boussinesq equations defined on \( \Omega \times (0, 1) \times (0, 1) \), where \( \tau = H/L \) is given by

\[
\tau = \frac{H}{L}
\]

with density stratification, from the mathematical point of view. These two facts show that the density stratification term is of great significance both physically and mathematically. Therefore, the aim of this paper is to derive, rigorously, the governing equations describing the motion of a stable stratified fluid, i.e., the viscous primitive equations with density stratification, from the mathematical point of view.

Let \( \Omega_\tau = M \times (-\tau, \tau) \) be a \( \tau \)-dependent domain, where \( M = (0, 1) \times (0, 1) \). Here, \( \tau = H/L \) is called the aspect ratio, which measures the ratio of the vertical scale \( H \) to the horizontal scale \( L \) of the ocean. For large-scale ocean circulation, the aspect ratio \( \tau \) is close to \( 10^{-3} \), so it is much less than 1.

Denote by \( \nabla_h = (\partial_x, \partial_y) \) the horizontal gradient operator. Then the horizontal Laplacian operator \( \Delta_h \) is given by

\[
\Delta_h = \nabla_h \cdot \nabla_h = \partial_{xx} + \partial_{yy}.
\]

Let us consider the anisotropic Boussinesq equations defined on \( \Omega_\tau \),

\[
\begin{aligned}
\partial_t u + (u \cdot \nabla) u + \nabla \pi + \frac{g \hat{k}}{\rho_b} \hat{k} &= \mu_h \Delta_h u + \mu_z \partial_{zz} u, \\
\partial_t \rho + u \cdot \nabla \rho &= \kappa_h \Delta_h \rho + \kappa_z \partial_{zz} \rho, \\
\nabla \cdot u &= 0,
\end{aligned}
\]

(1.1)

where the three dimensional velocity field \( u = (v, w) = (v_1, v_2, w) \), the pressure \( \pi \) and the density \( \rho \) are the unknowns. \( g \) is the gravitational acceleration and \( \rho_b \) is the reference constant density. \( \hat{k} = (0, 0, 1) \) is unit vector pointing to the \( z \)-direction. \( \mu_h \) and \( \mu_z \) represent the horizontal and vertical viscosity coefficients, respectively, while \( \kappa_h \) and \( \kappa_z \) represent the horizontal and vertical heat conduction coefficients, respectively.

For simplicity, the reference constant density \( \rho_b \) is set to be \( \rho_b = 1 \). In fact, the anisotropic Boussinesq equations (1.1) have an elementary exact solution \( (u, \pi, \rho) = (0, \bar{\rho}(z), \bar{\rho}(z)) \) satisfying the hydrostatic approximation

\[
\frac{d\bar{\rho}(z)}{dz} + g\bar{\rho}(z) = 0.
\]

Assume that

\[
p(x, y, z, t) = \pi(x, y, z, t) - \bar{\rho}(z), \quad \rho(x, y, z, t) = \rho(x, y, z, t) - \bar{\rho}(z).
\]

Then the anisotropic Boussinesq equations (1.1) become

\[
\begin{aligned}
\partial_t v + (v \cdot \nabla_h) v + w \partial_z v + \nabla_h p &= \mu_h \Delta_h v + \mu_z \partial_{zz} v, \\
\partial_t w + v \cdot \nabla_h w + w \partial_z w + \partial_z p + g\rho &= \mu_h \Delta_h w + \mu_z \partial_{zz} w, \\
\partial_t \rho + v \cdot \nabla_h \rho + w \partial_z \rho + \left( \frac{d\bar{\rho}(z)}{dz} \right) = \kappa_h \Delta_h \rho + \kappa_z \partial_{zz} \rho + \kappa_z \frac{d^2\bar{\rho}(z)}{dz^2}, \\
\nabla_h \cdot v + \partial_z w &= 0.
\end{aligned}
\]

(1.2)
Let $N = \left(-g \frac{d\bar{\rho}(z)}{dz}\right)^{1/2}$. If $N > 0$, then $N$ is called the buoyancy or Brunt-Viisilä frequency. When $\frac{d\bar{\rho}(z)}{dz} < 0$, the density decreases with height, and the lighter fluid is above the heavier fluid, which is a situation referred to as stable stratification.

First, we transform the anisotropic Boussinesq equations (1.2), defined on the $\tau$-dependent domain $\Omega_\tau$, to the scaled Boussinesq equations defined on a fixed domain. To this end, we introduce new unknowns with the subscript $\tau$,

$$u_\tau = (v_\tau, w_\tau), v_\tau(x, y, z, t) = v(x, y, \tau z, t),$$

$$w_\tau(x, y, z, t) = \frac{1}{\tau}w(x, y, \tau z, t), \quad p_\tau(x, y, z, t) = p(x, y, \tau z, t),$$

$$\rho_\tau(x, y, z, t) = (g\tau)\rho(x, y, \tau z, t), \quad \bar{\rho}_\tau(z) = \bar{\rho}(\tau z),$$

for any $(x, y, z) \in \Omega =: M \times (-1, 1)$ and for any $t \in (0, \infty)$. Then the last two scalings allow us to write the pressure and the density non-dimensionally as

$$\bar{\rho}_\tau(z) + p_\tau(x, y, z, t) = \bar{\rho}(\tau z) + p(x, y, \tau z, t) = \pi(x, y, \tau z, t)$$

and

$$\bar{\rho}_\tau(z) + \rho_\tau(x, y, z, t) = (g\tau)(\bar{\rho}(\tau z) + \rho(x, y, \tau z, t)) = (g\tau)\rho(x, y, \tau z, t),$$

respectively.

Suppose that $\mu_h = \kappa_h = 1$ and that $\mu_z = \kappa_z = \tau^2$. Under these scalings, the anisotropic Boussinesq equations (1.2) defined on $\Omega_\tau$ can be written as the scaled Boussinesq equations

$$\begin{cases}
\partial_t v_\tau + (v_\tau \cdot \nabla_h) v_\tau + w_\tau \partial_z v_\tau + \nabla_h p_\tau = \Delta_h v_\tau + \partial_z v_\tau, \\
\tau (\partial_t w_\tau + v_\tau \cdot \nabla_h w_\tau + w_\tau \partial_z w_\tau) + \frac{1}{\tau} (\partial_z p_\tau + \rho_\tau) = \tau \Delta_h w_\tau + \tau \partial_z w_\tau, \\
\partial_t \rho_\tau + v_\tau \cdot \nabla_h \rho_\tau + w_\tau \partial_z \rho_\tau + w_\tau \frac{d\bar{\rho}_\tau}{dz} = \Delta_h \rho_\tau + \partial_z \rho_\tau + \frac{d^2 \bar{\rho}_\tau}{dz^2}, \\
\nabla_h \cdot v_\tau + \partial_z w_\tau = 0,
\end{cases}
(1.3)$$

defined on the fixed domain $\Omega$.

When the fluid is steadily stratified, we can assume, for simplicity, that $\bar{\rho}(z) = 1 - (1/g)N^2z$ for some positive constant $N^2$, where $N$ represents the strength of the stable stratification. This assumption leads to $\bar{\rho}_\tau(z) = (g\tau)\bar{\rho}(\tau z) = g\tau - \tau^2 N^2 z$, and hence the third equation of the scaled Boussinesq equations (1.3) becomes

$$\partial_t \rho_\tau + v_\tau \cdot \nabla_h \rho_\tau + w_\tau \partial_z \rho_\tau - \tau^2 N^2 w_\tau = \Delta_h \rho_\tau + \partial_z \rho_\tau.$$

Set $\tau^2 N^2 = 1$, i.e., $N \sim 1/\tau$, which means that the stratification effect is very strong. In such a case, the scaled Boussinesq equations (1.3) can be rewritten as

$$\begin{cases}
\partial_t v_\tau - \Delta v_\tau + (v_\tau \cdot \nabla_h) v_\tau + w_\tau \partial_z v_\tau + \nabla_h p_\tau = 0, \\
\tau^2 (\partial_t w_\tau - \Delta w_\tau + v_\tau \cdot \nabla_h w_\tau + w_\tau \partial_z w_\tau) + \partial_z p_\tau + \rho_\tau = 0, \\
\partial_t \rho_\tau - \Delta \rho_\tau + v_\tau \cdot \nabla_h \rho_\tau + w_\tau \partial_z \rho_\tau - w_\tau = 0, \\
\nabla_h \cdot v_\tau + \partial_z w_\tau = 0,
\end{cases}
(1.4)$$

Next, we supply the scaled Boussinesq equations (1.4) with the following boundary and initial conditions:

$$v_\tau, w_\tau, p_\tau \text{ and } \rho_\tau \text{ are periodic in } x, y, z,$$

$$v_\tau(0, x, y, \tau z, t), v_\tau(\Omega, x, y, \tau z, t), w_\tau(x, y, z, t), w_\tau(x, y, \tau z, t),$$

$$p_\tau(0, x, y, \tau z, t), p_\tau(\Omega, x, y, \tau z, t), \rho_\tau(x, y, z, t), \rho_\tau(x, y, \tau z, t)$$

are periodic in $x, y, \tau z, t$.
\[ (v_\tau, w_\tau, \rho_\tau)_{t=0} = (v_0, w_0, \rho_0). \]  

(1.6)

Here \((v_0, w_0, \rho_0)\) is given. Moreover, we also equip system (1.4) with the following symmetry condition:

\[ v_\tau, w_\tau, p_\tau \text{ and } \rho_\tau \text{ are even, odd, even and odd with respect to } z, \text{ respectively.} \]  

(1.7)

Note that the above symmetry condition is preserved by the scaled Boussinesq equations (1.4), i.e., that it holds provided that the initial data satisfy this symmetry condition. Due to this fact, throughout this paper, we always suppose that the initial data satisfy that

\[ v_0, w_0 \text{ and } \rho_0 \text{ are periodic in } x, y, z, \text{ and are even, odd and odd in } z, \text{ respectively.} \]  

(1.8)

In this paper, we will use the same notation, \(L^p(\Omega)\) or \(H^m(\Omega)\), to denote both a space itself and its finite product spaces. For convenience, we denote by notations \(|\cdot|_p\) and \(|\cdot|_{p,M}\) the \(L^p(\Omega)\) norm and the \(L^p(M)\) norm, respectively. Moreover, since the scaled Boussinesq equations (1.4) satisfy the symmetry condition (1.7), it follows from the divergence-free condition that \(w_0\) is uniquely determined as

\[ w_0(x, y, z) = -\int_0^z \nabla h \cdot v_0(x, y, \xi) d\xi \]  

(1.9)

for any \((x, y) \in M\) and \(z \in (-1, 1)\). Hence only the initial condition of \((v_\tau, \rho_\tau)\) is given throughout the paper.

For the proof of the global existence of weak solutions to the scaled Boussinesq equations (1.4), we refer to the work of Lions-Temam-Wang [26, Part IV]. Specifically, for any initial data \((u_0, \rho_0) = (v_0, w_0, \rho_0) \in L^2(\Omega)\), with \(\nabla \cdot u_0 = 0\), we can prove that there exists a global weak solution \((v_\tau, w_\tau, p_\tau)\) of the scaled Boussinesq equations (1.4), subject to the boundary and initial conditions (1.5)–(1.6) and the symmetry condition (1.7). Moreover, by an argument similar to that of Lions-Temam-Wang [26, Part IV], we can also show that we have a unique local strong solution \((v_\tau, w_\tau, \rho_\tau)\) for initial data \((u_0, \rho_0) = (v_0, w_0, \rho_0) \in H^1(\Omega)\), with \(\nabla \cdot u_0 = 0\). The weak solutions of the scaled Boussinesq equations (1.4) are defined as follows:

**Definition 1.1** Given \((u_0, \rho_0) = (v_0, w_0, \rho_0) \in L^2(\Omega)\), with \(\nabla \cdot u_0 = 0\), we say that a space periodic function \((v_\tau, w_\tau, \rho_\tau)\) is a weak solution of system (1.4), subject to the boundary and initial conditions (1.5)–(1.6) and the symmetry condition (1.7), if

(i) \((v_\tau, w_\tau, \rho_\tau) \in C_w([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))\) for any \(T > 0\), where \(C_w\) is the space of weakly continuous function in time;

(ii) \((v_\tau, w_\tau, \rho_\tau)\) satisfies the integral equality

\[
\int_0^\infty \int_\Omega \left\{ (-v_\tau \cdot \partial_t \varphi_h - \tau^2 w_\tau \partial_t \varphi_3 - \rho_\tau \partial_t \psi + \rho_\tau \varphi_3 - w_\tau \psi) \\
+ \nabla v_\tau \cdot \nabla \varphi_h + \tau^2 \nabla w_\tau \cdot \nabla \varphi_3 + \nabla \rho_\tau \cdot \nabla \psi \\
+ [(u_\tau \cdot \nabla) v_\tau \cdot \varphi_h + \tau^2 (u_\tau \cdot \nabla w_\tau) \varphi_3 + (u_\tau \cdot \nabla \rho_\tau) \psi] \right\} dxdydzdt
\]

\[
= \int_\Omega (v_0 \cdot \varphi_h(0) + \tau^2 w_0 \varphi_3(0) + \rho_0 \psi(0)) dxdydz
\]

for any spatially periodic function \((\varphi, \psi) = (\varphi_h, \varphi_3, \psi)\), with \(\varphi_h = (\varphi_1, \varphi_2)\), such that \(\nabla \cdot \varphi = 0\) and \((\varphi, \psi) \in C_w^\infty(\Omega \times [0, \infty))\).
Remark 1.2 Similar to the theory of three-dimensional Navier-Stokes equations (see Temam [39, Ch.III, Remark 4.1] and Robinson et al. [36, Theorem 4.6]), we can prove that \((v_\tau, w_\tau, \rho_\tau)\) satisfies the energy inequality

\[
\frac{1}{2} \left( \|v_\tau(t)\|_2^2 + \tau^2 \|w_\tau(t)\|_2^2 + \|\rho_\tau(t)\|_2^2 \right) + \int_0^t \left( \|\nabla v_\tau\|_2^2 + \tau^2 \|\nabla w_\tau\|_2^2 + \|\nabla \rho_\tau\|_2^2 \right) \, ds
\]

\[
\leq \frac{1}{2} \left( \|v_0\|_2^2 + \tau^2 \|w_0\|_2^2 + \|\rho_0\|_2^2 \right) + \int_0^t \left( \|\nabla v_\tau\|_2^2 + \tau^2 \|\nabla w_\tau\|_2^2 + \|\nabla \rho_\tau\|_2^2 \right) \, ds
\]

(1.10)

for a.e. \(t \in [0, \infty)\), as long as the weak solution \((v_\tau, w_\tau, \rho_\tau)\) is obtained by the Galerkin method.

In consequence, this paper aims to study the small aspect ratio limit for system (1.4). In other words, when the aspect ratio \(\tau\) goes to zero, we are going to prove that the scaled Boussinesq equations (1.4) converge to the viscous primitive equations with density stratification

\[
\begin{align*}
\partial_t v - \Delta v + (v \cdot \nabla h)v + w \partial_z v + \nabla h p &= 0, \\
\partial_z p + \rho &= 0, \\
\partial_t \rho - \Delta \rho + v \cdot \nabla h \rho + w \partial_z \rho - w &= 0, \\
\nabla h \cdot v + \partial_z w &= 0,
\end{align*}
\]

(1.11)
in a suitable sense, where the density stratification term \(w\) in the third equation of system (1.11) provides additional dissipation for this system. Moreover, the resulting system (1.11) satisfies the same boundary and initial conditions (1.5)–(1.6) and the symmetry condition (1.7) as system (1.4).

Next, we want to recall some results concerning the primitive equations. The global existence of weak solutions of the full primitive equations was first given by Lions-Temam-Wang [25–27], but the question of uniqueness to this mathematical model is still unknown, except for some special cases [3, 20, 22, 29, 38]. The existence and uniqueness of strong solutions to the primitive equations with full dissipation in different settings is due to the work of Cao-Titi [10], Kobelkov [21], Kukavica-Ziane [23, 24], Hieber-Kashiwabara [18], and Hieber et al. [17], as well as Giga et al. [15]. The study of the global strong solutions to the primitive equations is naturally carried out in the cases of partial dissipation. More details on these cases can be found in the work of Cao-Titi [11], Fang-Han [13], Li-Yuan [31], and Cao-Li-Titi [5–9]. However, the inviscid primitive equations with or without rotation are known to be ill-posed in Sobolev spaces, and the smooth solutions may develop singularity in finite time, see Renardy [35], Han-Kwan and Nguyen [16], Ibrahim-Lin-Titi [19], Wong [42], and Cao et al. [4].

The rest of this paper is organized as follows: some auxiliary lemmas frequently used in the proof are collected in Section 2; the main results of the paper are stated in Section 3; the global well-posedness of strong solutions to the viscous primitive equations with density stratification (1.11) is established in Section 4; the proofs of Theorems 3.2 and 3.3 are presented in Sections 5 and 6, respectively.

2 Preliminaries

In this section, we present some Ladyzhenskaya-type inequalities in three dimensions for a class of integrals which are frequently used throughout the paper.

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Denote by $\phi$ to the viscous primitive equations with density stratification (1.11) for initial data $(v_0, \rho_0)$ depending continuously on $x$ and $y$ in $(0, T)$, with $\partial_0 v = 0$, $\partial_0 \rho = 0$, and $(v_0, w_0, \rho_0) \in L^2(0, T; H^1(\Omega))$ and $(\partial_0 v, \partial_0 \rho) \in L^2(0, T; L^2(\Omega))$.

The global existence of weak solutions to the scaled Boussinesq equations (1.4) basically follows the proof in Lions-Temam-Wang [26, Part IV]. For initial data $(v_0, \rho_0) \in H^1(\Omega)$, it can be deduced from (1.9) that $(v_0, w_0, \rho_0) \in L^2(\Omega)$, which implies that system (1.4) has a global weak solution $(v, w, \rho)$. For this case, we have the following strong convergence theorem.

**Theorem 3.2** Take a periodic function pair $(v_0, \rho_0) \in H^1(\Omega)$ such that

$$\int_{-1}^{1} \nabla_h \cdot v_0(x, y, z)dz = 0, \int_{\Omega} v_0(x, y, z)dx dy dz = 0, \text{ and } \int_{\Omega} \rho_0(x, y, z)dx dy dz = 0.$$

Suppose that $(v, w, \rho)$ is a global weak solution of system (1.4), satisfying the energy inequality (1.10), and that $(v, \rho)$ is the unique global strong solution of system (1.11), with the same boundary and initial conditions (1.5)–(1.6) and the symmetry condition (1.7). Let $(V, W, \Gamma) = (v - v, w - w, \rho - \rho)$. 

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Then, for any $T > 0$, it holds that
\[
\sup_{0 \leq t \leq T} \left(\| (V_\tau, \tau W_\tau, \Gamma_\tau) \|^2_{H^1} \right) (t) + \int_0^T \| \nabla (V_\tau, \tau W_\tau, \Gamma_\tau) \|^2_{H^1} dt \leq \tau^2 \widehat{K}_1(T),
\]
where $\widehat{K}_1(t)$ is a nonnegative continuously increasing function that does not depend on $\tau$. As a result, we have the strong convergences
\[
(v_\tau, \tau w_\tau, \rho_\tau) \to (v, 0, \rho) \quad \text{in} \quad L^\infty (0, T; L^2(\Omega)),
\]
\[
(\nabla v_\tau, \tau \nabla w_\tau, \nabla \rho_\tau, w_\tau) \to (\nabla v, 0, \nabla \rho, w) \quad \text{in} \quad L^2 (0, T; L^2(\Omega)),
\]
and the rate of convergence is of the order $O(\tau)$.

Next we assume that initial data $(v_0, \rho_0)$ belongs to $H^2(\Omega)$. Then, from (1.9), it follows that $(v_0, w_0, \rho_0)$ belongs to $H^1(\Omega)$. By an argument similar to that of Lions-Temam-Wang [26, Part IV], there exists a unique local strong solution $(v_\tau, w_\tau, \rho_\tau)$ to system (1.4), subject to the boundary and initial conditions (1.5)–(1.6) and the symmetry condition (1.7). In this case, we also have the following strong convergence theorem.

**Theorem 3.3** Take a periodic function pair $(v_0, \rho_0) \in H^2(\Omega)$ such that
\[
\int_{-1}^1 \nabla_h \cdot v_0(x, y, z) dz = 0, \quad \int \Omega v_0(x, y, z) dz dx dy = 0, \quad \text{and} \quad \int \Omega \rho_0(x, y, z) dz dx dy = 0.
\]
Suppose that $(v_\tau, w_\tau, \rho_\tau)$ is the unique local strong solution of system (1.4), and that $(v, \rho)$ is the unique global strong solution of system (1.11), with the same boundary and initial conditions (1.5)–(1.6) and the symmetry condition (1.7). Let
\[
(V_\tau, W_\tau, \Gamma_\tau) = (v_\tau - v, w_\tau - w, \rho_\tau - \rho).
\]
Then, for any $T > 0$, there is a small positive constant $\tau(T) = \frac{3\beta_0}{4\sqrt{K_2(T)}}$ such that system (1.4) exists a unique strong solution $(v_\tau, w_\tau, \rho_\tau)$ on the time interval $[0, T]$, and that system (6.1)–(6.4) (see Section 6, below) has the estimate
\[
\sup_{0 \leq t \leq T} \left(\| (V_\tau, \tau W_\tau, \Gamma_\tau) \|^2_{H^1} \right) (t) + \int_0^T \| \nabla (V_\tau, \tau W_\tau, \Gamma_\tau) \|^2_{H^1} dt \leq \tau^2 \widehat{K}_3(T),
\]
provided that $\tau \in (0, \tau(T))$, where both $\widehat{K}_2(t)$ and $\widehat{K}_3(t)$ are the nonnegative continuously increasing functions that do not depend on $\tau$. As a result, we have the strong convergences
\[
(v_\tau, \tau w_\tau, \rho_\tau) \to (v, 0, \rho) \quad \text{in} \quad L^\infty (0, T; H^1(\Omega)),
\]
\[
(\nabla v_\tau, \tau \nabla w_\tau, \nabla \rho_\tau, w_\tau) \to (\nabla v, 0, \nabla \rho, w) \quad \text{in} \quad L^2 (0, T; H^1(\Omega)),
\]
\[
w_\tau \to w \quad \text{in} \quad L^\infty (0, T; L^2(\Omega)),
\]
and the rate of convergence is of the order $O(\tau)$.

**Remark 3.4** It should be pointed out that the case where $\tilde{g}(z) = \text{Constant}$ has been studied by the authors (see [34]). Compared with [34], the third equation of the resulting limit system here contains the density stratification term $w$. In order to establish the global $H^1$ theory for system (1.11), we carry out the a priori estimates on $v$ and $\rho$, simultaneously. In this way, the global well-posedness of strong solutions to the viscous primitive equations with density stratification (1.11) is obtained, and the first order energy estimate on strong solutions will be used in the proof of Theorem 3.2. Moreover, Theorem 3.3 is proven by establishing the second order energy estimate on strong solutions of system (1.11).
4 Global Well-Posedness of the Primitive Equations

In this section, we establish the global well-posedness of strong solutions to the viscous primitive equations with density stratification (1.11), subject to the boundary and initial conditions (1.5)–(1.6) and the symmetry condition (1.7).

Before this, we use the symmetry condition (1.7) to reformulate system (1.11). This symmetry condition indicates that \( w|_{z=0} = 0 \). Integrating the last equation of system (1.11) with respect to \( z \) yields

\[
w(x, y, z, t) = -\int_0^z \nabla_h \cdot v(x, y, \xi, t)\,d\xi.
\]

We integrate the second equation of system (1.11) with respect to \( z \) to obtain that

\[
p(x, y, z, t) = p_\gamma(x, y, t) - \int_0^z \rho(x, y, \xi, t)\,d\xi,
\]

in which \( p_\gamma(x, y, t) \) represents the unknown surface pressure as \( z = 0 \). Based on the above relations, we can recast system (1.11) as

\[
\begin{align*}
\partial_t v - \Delta v + (v \cdot \nabla_h) v - \left( \int_0^z \nabla_h \cdot v(x, y, \xi, t)\,d\xi \right) \partial_z v + \nabla_h p_\gamma(x, y, t) \\
- \int_0^z \nabla_h \rho(x, y, \xi, t)\,d\xi = 0,
\end{align*}
\]

(4.1)

\[
\begin{align*}
\partial_t \rho - \Delta \rho + v \cdot \nabla_h \rho - \left( \int_0^z \nabla_h \cdot v(x, y, \xi, t)\,d\xi \right) \partial_z \rho + \int_0^z \nabla_h \cdot v(x, y, \xi, t)\,d\xi = 0,
\end{align*}
\]

(4.2)

satisfying the boundary and initial conditions

\[
v and \rho are periodic in x, y, z,
\]

\[
(v, \rho)|_{t=0} = (v_0, \rho_0),
\]

and the symmetry condition that

\[
v and \rho are even and odd with respect to z, respectively.
\]

4.1 \( L^2 \) Estimates on \( v \) and \( \rho \)

Taking the \( L^2(\Omega) \) inner product of (4.1) and (4.2) with \( v \) and \( \rho \), respectively, and integrating by parts, we obtain

\[
\frac{1}{2} \frac{d}{dt} \left( \|v\|_2^2 + \|\rho\|_2^2 \right) + \|\nabla v\|_2^2 + \|\nabla \rho\|_2^2
\]

\[
= \int_\Omega \left( \int_0^z \nabla_h \rho(x, y, \xi, t)\,d\xi \right) \cdot v\,dx\,dy\,dz - \int_\Omega \left( \int_0^z \nabla_h \cdot v(x, y, \xi, t)\,d\xi \right) \rho\,dx\,dy\,dz = 0,
\]

where we have used the facts that

\[
\int_\Omega \left( \nabla_h \cdot v - \left( \int_0^z \nabla_h \cdot v(x, y, \xi, t)\,d\xi \right) \partial_z v \right) \cdot v\,dx\,dy\,dz = 0,
\]

\[
\int_\Omega \left( v \cdot \nabla_h \rho - \left( \int_0^z \nabla_h \cdot v(x, y, \xi, t)\,d\xi \right) \partial_z \rho \right) \rho\,dx\,dy\,dz = 0
\]

and

\[
\int_\Omega \nabla_h p_\gamma(x, y, t) \cdot v\,dx\,dy\,dz = 0.
\]
Integrating the differential equation above in time between 0 to \( t \), we have that
\[
\left( \|v\|_2^2 + \|\rho\|_2^2 \right) (t) + \int_0^t \left( \|\nabla v\|_2^2 + \|\nabla \rho\|_2^2 \right) ds \leq \eta_1, \tag{4.3}
\]
where
\[
\eta_1 = C \left( \|v_0\|_{H^1}^2 + \|\rho_0\|_{H^1}^2 \right).
\]

### 4.2 \( L^4 \) Estimates on \( v \) and \( \rho \)

Multiplying (4.1) and (4.2) by \( |v|^2 v \) and \( |\rho|^2 \rho \), respectively, and integrating over \( \Omega \), then it follows from integration by parts that
\[
\begin{align*}
\frac{1}{4} \frac{d}{dt} \|v\|_4^4 &+ \int_\Omega \|v\|^2 \left( |\nabla v|^2 + 2 |\nabla |v||^2 \right) dx dy dz \\
&+ \int_\Omega \frac{1}{4} \|\rho\|_4^4 + \int_\Omega \|\rho\|^2 \left( |\nabla \rho|^2 + 2 |\nabla |\rho||^2 \right) dx dy dz \\
&= \int_\Omega \left( \int_0^z v(x,y,\xi,t) d\xi \right) \left( \nabla_h \left( |\rho|^2 \rho \right) \right) dx dy dz \\
&- \int_\Omega \left( \int_0^z \rho(x,y,\xi,t) d\xi \right) (\nabla_h \cdot |v|^2 v) dx dy dz - \int_\Omega \nabla_h p_\gamma(x,y,t) \cdot |v|^2 v dx dy dz \\
&= : D_1 + D_2 + D_3, \tag{4.4}
\end{align*}
\]
in which we have used the facts that
\[
\begin{align*}
\int_\Omega \left( v \cdot \nabla_h \right) v - \left( \int_0^z \nabla_h \cdot v(x,y,\xi,t) d\xi \right) \partial_x v \right] \cdot |v|^2 v dx dy dz &= 0, \\
\int_\Omega \left[ v \cdot \nabla_h \rho - \left( \int_0^z \nabla_h \cdot v(x,y,\xi,t) d\xi \right) \partial_x \rho \right] |\rho|^2 \rho dx dy dz &= 0.
\end{align*}
\]

We now estimate the first integral term \( D_1 \) on the right-hand side of (4.4). Using Hölder’s inequality yields that
\[
D_1 : = \int_\Omega \left( \int_0^z v(x,y,\xi,t) d\xi \right) \nabla_h \left( |\rho|^2 \rho \right) dx dy dz \\
\leq C \int_M \left( \int_{-1}^1 |v| dz \right) \left( \int_{-1}^1 |\rho|^2 |\nabla_h \rho| dz \right) dx dy \\
\leq C \int_M \left( \int_{-1}^1 |v| dz \right) \left( \int_{-1}^1 |\rho|^2 dz \right)^{3/4} \left( \int_{-1}^1 \left| \nabla_h \rho \right|^2 dz \right)^{1/4} dx dy \\
\leq C \left( \int |v|^4 dx dy dz \right)^{1/4} \left( \int_\Omega |\rho|^4 dx dy dz \right)^{1/4} \left( \int_\Omega \left| \nabla_h \rho \right|^2 dx dy dz \right)^{1/4} \\
\leq C \|v\|_4 \|\rho\|_4 \left( \int_\Omega \left| \nabla \rho \right|^2 \rho dx dy dz \right)^{1/2}.
\]

Due to Young’s inequality, we have that
\[
D_1 \leq C \|v\|_4^2 \|\rho\|_4^2 + \frac{3}{8} \int_\Omega |\rho|^2 \left| \nabla \rho \right|^2 dx dy dz \\
\leq C \left( \|v\|_4^4 + \|\rho\|_4^4 \right) + \frac{3}{8} \int_\Omega |\rho|^2 \left| \nabla \rho \right|^2 dx dy dz. \tag{4.5}
\]

A similar argument as that for the integral term \( D_1 \) gives that
\[
D_2 : = \int_\Omega \left( - \int_0^z \rho(x,y,\xi,t) d\xi \right) (\nabla_h \cdot |v|^2 v) dx dy dz
\]
Substituting (4.8) into (4.7) and then using Young’s inequality yields that
\[
D_3 = -\int_\Omega \nabla_h p_\gamma(x, y, t) \cdot |v|^2 \, dx \, dy \, dz
\leq \int_M |\nabla_h p_\gamma(x, y, t)| \left( \int_{-1}^1 |v|^2 \, dz \right) \, dx \, dy
\leq C \|v\|^{1/2} \|\nabla v\|^{1/2} \left( \|v\|_2^2 + \|v\|_4 \|\nabla v\|_2^{1/2} \right) \|\nabla_h p_\gamma\|_{2,M}.
\]
(4.7)

Applying the operator $\text{div}_h$ to equation (4.1) and integrating the resulting equation with respect to $z$ from $-1$ to $1$, we can see that $p_\gamma(x, y, t)$ satisfies the system
\[
\begin{cases}
-\Delta_h p_\gamma = \frac{1}{2} \int_{-1}^1 \nabla_h \cdot \left( \nabla_h \cdot (v \otimes v) - \int_0^z \nabla_h \rho \, dz \right) \, dx \, dy, \\
\int_M p_\gamma(x, y, t) \, dx \, dy = 0, \quad p_\gamma \text{ is periodic in } x, y,
\end{cases}
\]
where the condition $\int_M p_\gamma(x, y, t) \, dx \, dy = 0$ is imposed to guarantee the uniqueness of $p_\gamma(x, y, t)$.

By virtue of the elliptic estimates and Poincaré’s inequality, we obtain that
\[
\|\nabla_h p_\gamma\|_{2,M} \leq C \left\| \int_{-1}^1 \left( \nabla_h \cdot (v \otimes v) - \int_0^z \nabla_h \rho \, dz \right) \, dz \right\|_{2,M}
\leq C \left( \|v\|_2 \|\nabla v\|_2 + \|\nabla \rho\|_2 \right)
\leq C \left( \|v\|_2 \|\nabla v\|_2 + \|\nabla \rho\|_2 \right).
\]
(4.8)

Substituting (4.8) into (4.7) and then using Young’s inequality yields that
\[
D_3 \leq C \|v\|^{1/2} \|\nabla v\|^{1/2} \left( \|v\|_4^2 + \|v\|_4 \|\nabla v\|_2^{1/2} \right) \left( \|\nabla \rho\|_2 + \|v\|\nabla v\|_2 \right)
\leq C \|v\|^{1/2} \|\nabla v\|^{1/2} \left( \|\nabla \rho\|_2 \|v\|_2^2 + \|\nabla \rho\|_2 \|v\|_4 \|\nabla v\|_2^{1/2} \right)
+ C \|v\|^{1/2} \|\nabla v\|^{1/2} \left( \|v\|_2 \|\nabla v\|_2 + \|\nabla \rho\|_2 \|v\|_4 \|\nabla v\|_2^{3/2} \right)
\leq C \left( \|v\|_2 \|\nabla v\|_2 + \|v\|_2 \|\nabla v\|_2^2 + \|\nabla \rho\|_2^2 \right) \left( \|v\|_4^2 + \|\nabla \rho\|_2^4 \right)
+ \frac{1}{4} \left( \|v\|_2 \|\nabla v\|_2 + \|\nabla \rho\|_2^2 \right) + \frac{1}{4} \|v\|\nabla v\|_2^2.
\]
(4.9)

Adding (4.5), (4.6) and (4.9) leads to
\[
\frac{d}{dt} \left( \|v\|_4^4 + \|\nabla \rho\|_4^4 \right) + \int_\Omega \left( |v|^2 |\nabla v|^2 + \|\nabla \rho\|_2^2 \right) \, dx \, dy \, dz
\leq C \left( \|v\|_2 \|\nabla v\|_2 + \|v\|_2 \|\nabla \rho\|_2 \|\nabla v\|_2^2 + \|\nabla \rho\|_2^2 + 1 \right) \left( \|v\|_4^4 + \|\nabla \rho\|_4^4 \right)
+ \left( \|v\|_2 \|\nabla v\|_2 + \|\nabla \rho\|_2^2 \right).
\]

Owing to Grönwall’s inequality, we get from (4.3) that
\[
\left( \|v\|_4^4 + \|\nabla \rho\|_4^4 \right) (t) + \int_0^t \int_\Omega \left( |v|^2 |\nabla v|^2 + \|\nabla \rho\|_2^2 \right) \, dx \, dy \, dz \, ds
\leq \exp \left\{ C \int_0^t \left( \|v\|_2 \|\nabla v\|_2 + \|v\|_2 \|\nabla v\|_2 + \|\nabla \rho\|_2^2 + 1 \right) \, ds \right\}
\]
For the first integral term
\[ D(4.11), \]
the last integral term
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Next, we estimate the second integral term
\[ D \]
\[ \text{obtain that} \]
\[ L \]
\[ \text{Taking the} \]
\[ =: \]
\[ = \]
\[ \leq \]
\[ \int \]
\[ \int \]
\[ \int \]
\[ =: D_1 + D_2 + D_3. \] (4.11)

For the first integral term \( D_1 \) on the right-hand side of (4.11), we use integration by parts, Hölder’s inequality and Young’s inequality to get that
\[ D_1 := \int_\Omega \left[ \left( \int_0^z \nabla_h \cdot v(x,y,\xi,t)d\xi \right) \partial_{zz} \rho - \left( \int_0^z \nabla_h \rho dy \right) \partial_{zz} v \right] dx dy dz \]
\[ = \int_\Omega \left[ \partial_{zz} \rho \left( \int_0^z \nabla_h \cdot v(x,y,\xi,t)d\xi \right) \partial_{zz} v \right] dx dy dz \]
\[ \leq \left\| v \right\|_2 \left\| \nabla_h \partial_{zz} \rho \right\|_2 + \left\| \rho \right\|_2 \left\| \nabla_h \partial_{zz} v \right\|_2 \]
\[ \leq C \left( \left\| v \right\|_2^2 + \left\| \rho \right\|_2^2 \right) + \frac{1}{6} \left( \left\| \nabla \partial_{zz} v \right\|_2^2 + \left\| \nabla \partial_{zz} \rho \right\|_2^2 \right). \]

Next, we estimate the second integral term \( D_2 \) on the right-hand side of (4.11). Using integration by parts, Hölder’s inequality, the Lebesgue interpolation inequality, the Sobolev embedding theorem, and Poincaré’s inequality gives that
\[ D_2 := \int_\Omega \left[ v \cdot \nabla_h \rho - \left( \int_0^z \nabla_h \cdot v(x,y,\xi,t)d\xi \right) \partial_{zz} \rho \right] dx dy dz \]
\[ = \int_\Omega \left[ \left( \nabla_h \cdot \partial_{zz} \rho \right) \partial_{zz} \rho + (\partial_{zz} v \cdot \nabla_h \partial_{zz} \rho) \partial_{zz} \rho - 2 \left( v \cdot \nabla_h \partial_{zz} \rho \right) \partial_{zz} \rho \right] dx dy dz \]
\[ \leq \left\| \rho \right\|_4 \left\| \nabla \partial_{zz} \rho \right\|_4 \left\| \nabla \partial_{zz} \rho \right\|_2 \left( \left\| \rho \right\|_4 \left\| \partial_{zz} \rho \right\|_4 \right) \left\| \nabla h \partial_{zz} \rho \right\|_2 \]
\[ \leq C \left\| \rho \right\|_4 \left\| \nabla \partial_{zz} \rho \right\|_2^{1/4} \left\| \nabla \partial_{zz} \rho \right\|_2^{3/4} \left\| \nabla \partial_{zz} \rho \right\|_2 + C \left\| v \right\|_4 \left\| \partial_{zz} \rho \right\|_2^{1/4} \left\| \nabla \partial_{zz} \rho \right\|_2^{7/4} \]
\[ + \left\| \nabla \partial_{zz} \rho \right\|_2^{3/4} \left\| \nabla \partial_{zz} \rho \right\|_2 \]
\[ \text{By virtue of Young’s inequality, we have that} \]
\[ D_2 \leq C \left( \left\| v \right\|_4^4 + \left\| \rho \right\|_4^4 \right) \left( \left\| \nabla \partial_{zz} \rho \right\|_2^2 + \left\| \partial_{zz} \rho \right\|_2^2 \right) + \frac{1}{6} \left( \left\| \nabla \partial_{zz} v \right\|_2^2 + \left\| \nabla \partial_{zz} \rho \right\|_2^2 \right). \]
With an argument similar to that for the second integral term \( D_2 \) on the right-hand side of (4.11), the last integral term \( D_3 \) can be estimated as
\[ D_3 := \int_\Omega \left[ v \cdot \nabla h \right] v - \left( \int_0^z \nabla_h \cdot v(x,y,\xi,t)d\xi \right) \partial_{zz} \rho \]
Due to Hölder’s inequality and Young’s inequality, the first integral term \( D_1 \) on the right-hand side of (4.13) can be bounded as

\[
D_1 = \int_\Omega \left[ \left( \int_0^z \nabla h \cdot v(x, y, \xi, t) d\xi \right) (\Delta \rho - \partial_t \rho) - \left( \int_0^z \nabla h \rho(x, y, \xi, t) d\xi \right) \cdot (\Delta v - \partial_t v) \right] dxdydz
\]

\[
\leq C \| \nabla h \rho \|_2 (\| \partial_t \rho \|_2 + \| \Delta \rho \|_2) + C \| \nabla v \|_2 (\| \partial_t v \|_2 + \| \Delta v \|_2)
\]

\[
\leq C \left( \| \nabla v \|^2_2 + \| \nabla \rho \|^2_2 + \| \partial_t v \|^2_2 + \| \partial_t \rho \|^2_2 + \| \Delta v \|^2_2 + \| \Delta \rho \|^2_2 \right). \tag{14.14}
\]
For the second integral term $D_2$ on the right-hand side of (4.13), we use Lemma 2.1, Poincaré’s inequality and Young’s inequality to obtain that

$$D_2 := \int_{\Omega} [v \cdot \nabla_h \rho (\Delta \rho - \partial_t \rho) + (v \cdot \nabla_h) v \cdot (\Delta v - \partial_t v)] dxdydz$$

$$\leq \int_{M} \left( \int_{-1}^{1} \left| (v + \partial_z v) dz \right| \left( \int_{-1}^{1} \left| \nabla_h \rho \right| \left( \left| \partial_t \rho \right| + \left| \Delta \rho \right| \right) dz \right) dxdy \right.$$ 

$$+ \int_{M} \left( \int_{-1}^{1} \left| (v + \partial_z v) dz \right| \left( \int_{-1}^{1} \left| \nabla_h \rho \right| \left( \left| \partial_t \rho \right| + \left| \Delta \rho \right| \right) dz \right) dxdy \right.$$ 

$$\leq C \left( \left\| v \right\|^2_2 \left\| \nabla v \right\|^2_2 + \left\| \partial_z v \right\|^2_2 \left\| \nabla \partial_z v \right\|^2_2 \right) \left( \left\| \nabla \rho \right\|^2_2 + \left\| \Delta \rho \right\|^2_2 \right)$$

$$+ C \left( \left\| v \right\|^2_2 \left\| \nabla v \right\|^2_2 + \left\| \partial_z v \right\|^2_2 \left\| \nabla \partial_z v \right\|^2_2 \right) \left( \left\| \nabla \rho \right\|^2_2 + \left\| \Delta \rho \right\|^2_2 \right)$$

$$\leq C \left( \left\| v \right\|^2_2 \left\| \nabla v \right\|^2_2 + \left\| \partial_z v \right\|^2_2 \left\| \nabla \partial_z v \right\|^2_2 \right) \left( \left\| \nabla \rho \right\|^2_2 + \left\| \Delta \rho \right\|^2_2 \right)$$

$$+ \frac{1}{6} \left( \left\| \partial_t v \right\|^2_2 + \left\| \Delta v \right\|^2_2 + \left\| \partial_t \rho \right\|^2_2 + \left\| \Delta \rho \right\|^2_2 \right). \quad (4.15)$$

Finally, it remains to estimate the last integral term $D_3$ on the right-hand side of (4.13). A similar argument as that for $D_2$ yields that

$$D_3 := \int_{\Omega} \left( \int_{0}^{s} \nabla_h \cdot v (x, y, \xi, t) dx \right) \left[ \partial_x \rho (\Delta \rho - \partial_t \rho) + \partial_z v \cdot (\partial_t v - \Delta v) \right] dxdydz$$

$$\leq C \left( \left\| \partial_x v \right\|^2_2 \left\| \nabla \partial_x v \right\|^2_2 + \left\| \partial_z v \right\|^2_2 \left\| \nabla \partial_z v \right\|^2_2 \right) \left( \left\| \nabla \rho \right\|^2_2 + \left\| \Delta \rho \right\|^2_2 \right)$$

$$+ \frac{1}{6} \left( \left\| \partial_t v \right\|^2_2 + \left\| \Delta v \right\|^2_2 + \left\| \partial_t \rho \right\|^2_2 + \left\| \Delta \rho \right\|^2_2 \right). \quad (4.16)$$

Summing (4.14), (4.15) and (4.16) gives that

$$\frac{d}{dt} \left( \left\| \nabla v \right\|^2_2 + \left\| \rho \right\|^2_2 \right) \left( t \right) + \int_{0}^{t} \left( \left\| \partial_t v \right\|^2_2 + \left\| \Delta v \right\|^2_2 + \left\| \partial_t \rho \right\|^2_2 + \left\| \Delta \rho \right\|^2_2 \right) ds$$

$$\leq C \left( \left\| v \right\|^2_2 \left\| \nabla v \right\|^2_2 + \left\| \partial_z v \right\|^2_2 \left\| \nabla \partial_z v \right\|^2_2 + \left\| \partial_x v \right\|^2_2 \left\| \nabla \partial_x v \right\|^2_2 + \left\| \partial_z v \right\|^2_2 \left\| \nabla \partial_z v \right\|^2_2 + \left\| \partial_x v \right\|^2_2 \right) \left( \left\| \nabla \rho \right\|^2_2 + \left\| \Delta \rho \right\|^2_2 \right)$$

$$\leq \eta_4(t), \quad (4.18)$$

where

$$\eta_4(t) := e^{C(t+2)(\eta_1^2 + \eta_2^2(t)+1)} \left( \left\| v \right\|^2_{H^1} + \left\| \rho \right\|^2_{H^1} \right).$$

Based on the above energy estimates, we now give the proof of Theorem 3.1.

**Proof of Theorem 3.1** Adding (4.3) and (4.18), we have that

$$\sup_{0 \leq s \leq t} \left( \left\| v \right\|^2_{H^1} + \left\| \rho \right\|^2_{H^1} \right) (s) + \int_{0}^{t} \left( \left\| \partial_t v \right\|^2_2 + \left\| \nabla v \right\|^2_{H^1} + \left\| \partial_t \rho \right\|^2_2 + \left\| \nabla \rho \right\|^2_{H^1} \right) ds \leq \eta_1 + \eta_4(t),$$

where $\eta_4(t)$ is a nonnegative continuously increasing function defined on $[0, \infty)$. For any $T > 0$, the following estimate holds

$$\sup_{0 \leq t \leq T} \left( \left\| v \right\|^2_{H^1} + \left\| \rho \right\|^2_{H^1} \right) (t) + \int_{0}^{T} \left( \left\| \partial_t v \right\|^2_2 + \left\| \nabla v \right\|^2_{H^1} + \left\| \partial_t \rho \right\|^2_2 + \left\| \nabla \rho \right\|^2_{H^1} \right) dt \leq \eta_1 + \eta_4(T).$$

\[ \square \] Springer
5 Strong Convergence for $H^1$ Initial Data

In this section, assume that initial data $(v_0, \rho_0)$ belongs to $H^1(\Omega)$ with

$$\int_{t_1}^{t_2} \nabla_h \cdot v_0 \, dz = 0, \quad \text{for all } (x,y) \in M.$$ 

We prove that the scaled Boussinesq equations (1.4) strongly converge to the viscous primitive equations with density stratification (1.11) as the aspect ratio $\tau$ goes to zero.

The following proposition is formally obtained by testing the scaled Boussinesq equations (1.4) with $(v, w, \rho)$. As for the rigorous justification for this proposition, we refer to the work of Li-Titi [28] and Bardos et al. [2].

**Proposition 5.1** Given a periodic function pair $(v_0, \rho_0) \in H^1(\Omega)$ with

$$\int_{t_1}^{t_2} \nabla_h \cdot v_0 \, dz = 0 \quad \text{and} \quad w_0(x, y, z) = -\int_{0}^{z} \nabla_h \cdot v_0(x, y, \xi) \, d\xi,$$

suppose that $(v_\tau, w_\tau, \rho_\tau)$ is a global weak solution of system (1.4), satisfying the energy inequality (1.10), and that $(v, \rho)$ is the unique global strong solution of system (1.11). Then it holds that

$$\begin{align*}
&\left(\int_{\Omega} (v_\tau \cdot v + \tau^2 w_\tau w + \rho_\tau \rho) \, dx \, dy \, dz\right) (r) - \frac{\tau^2}{2} \|w(r)\|^2_2 \\
&+ \int_{0}^{t} \int_{\Omega} \left(\nabla v_\tau \cdot \nabla v + \tau^2 \nabla w_\tau \cdot \nabla w + \nabla \rho_\tau \cdot \nabla \rho\right) \, dx \, dy \, dz \, dt \\
&= \|v_0\|^2_2 + \int_{0}^{t} \int_{\Omega} \left[-(u_\tau \cdot \nabla) v_\tau \cdot v - \tau^2 (u_\tau \cdot \nabla w_\tau) w - (u_\tau \cdot \nabla \rho_\tau) \rho\right] \, dx \, dy \, dz \, dt \\
&+ \frac{\tau^2}{2} \|w_0\|^2_2 + \tau^2 \int_{0}^{t} \int_{\Omega} \left(\int_{0}^{z} \partial_t v(x, y, \xi, t) \, d\xi\right) \cdot \nabla_h W_\tau \, dx \, dy \, dz \, dt \\
&+ \|\rho_0\|^2_2 + \int_{0}^{t} \int_{\Omega} (v_\tau \cdot \partial_t v + \rho_\tau \partial_t \rho + w_\tau \rho - \rho_\tau w) \, dx \, dy \, dz \, dt \tag{5.1}
\end{align*}$$

for any $r \in [0, \infty)$.

With the help of this proposition, we can now estimate the difference function $(V_\tau, W_\tau, \Gamma_\tau)$.

**Proposition 5.2** Let $(V_\tau, W_\tau, \Gamma_\tau) = (v_\tau - v, w_\tau - w, \rho_\tau - \rho)$. Under the same assumptions as in Proposition 5.1, it holds that

$$\sup_{0 \leq s \leq t} \left(\|V_\tau(s, \tau W_\tau(s, \Gamma_\tau))\|^2_2\right) + \int_{0}^{t} \|\nabla(V_\tau(s, \tau W_\tau(s, \Gamma_\tau)))\|^2_2 \, ds \leq \tau^2 \mathcal{K}_1(t)$$

for any $t \in [0, \infty)$, where

$$\mathcal{K}_1(t) = C e^{C \eta^2(t)} \left[q_4(t) + \eta_4^2(t) + \left(\|v_0\|^2_2 + \tau^2 \|w_0\|^2_2 + \|\rho_0\|^2_2\right)^2\right].$$

Here $C$ is a positive constant that does not depend on $\tau$. \(\square\)
Proof Multiplying the first three equations in system (1.11) by \(v_r, w_r\) and \(\rho_r\), respectively, and integrating over \(\Omega \times (0, T)\), it follows from integration by parts that

\[
\int_0^T \int_\Omega (v_r \cdot \partial_t v + \rho_r \partial_t \rho + \nabla v_r : \nabla v + \nabla \rho_r \cdot \nabla \rho) \, dx dy dz \, dt
= \int_0^T \int_\Omega [\rho w_r - \rho w_r - (u \cdot \nabla)v \cdot v_r - (u \cdot \nabla \rho) \rho_r] \, dx dy dz \, dt.
\]

Replacing \((v_r, w_r, \rho_r)\) with \((v, w, \rho)\), a similar argument gives that

\[
\frac{1}{2} \left( \|v_r(0)\|^2 + \|\rho(0)\|^2 \right) + \int_0^T \left( \|\nabla v_r\|^2 + \|\nabla \rho\|^2 \right) \, dt = \frac{1}{2} \left( \|v_0\|^2 + \|\rho_0\|^2 \right).
\]

Thanks to Remark 1.2, the weak solution \((v_r, w_r, \rho_r)\) of system (1.4) satisfies the energy inequality

\[
\frac{1}{2} \left( \|v_r(t)\|^2 + \tau^2 \|w_r(t)\|^2 + \|\rho_r(t)\|^2 \right) + \int_0^T \left( \|\nabla v_r\|^2 + \tau^2 \|\nabla w_r\|^2 + \|\nabla \rho_r\|^2 \right) \, dt
\leq \frac{1}{2} \left( \|v_0\|^2 + \tau^2 \|w_0\|^2 + \|\rho_0\|^2 \right).
\]

Subtracting the sum of (5.1) and (5.2) from the sum of (5.3) and (5.4), we have that

\[
\frac{1}{2} \left( \|V_r(t)\|^2 + \tau^2 \|W_r(t)\|^2 + \|\Gamma_r(t)\|^2 \right) + \int_0^T \left( \|\nabla V_r\|^2 + \tau^2 \|\nabla W_r\|^2 + \|\nabla \Gamma_r\|^2 \right) \, dt
\leq \int_0^T \int_\Omega [(u_r \cdot \nabla \rho_r) + (u \cdot \nabla \rho) \rho_r] \, dx dy dz \, dt
+ \tau^2 \int_0^T \int_\Omega \left[ \left( \int_0^z \partial_t v(x, y, \xi, t) \right) \cdot \nabla_h W_r - \nabla w \cdot \nabla W_r \right] \, dx dy dz \, dt
+ \int_0^T \int_\Omega [(u_r \cdot \nabla) v_r \cdot v + (u \cdot \nabla) v \cdot v_r] \, dx dy dz \, dt + \tau^2 \int_0^T \int_\Omega (u_r \cdot \nabla w_r) w \, dx dy dz \, dt
= : J_1 + J_2 + J_3 + J_4.
\]

First, we estimate the integral term \(J_1\) on the right-hand side of (5.5). Using Hölder’s inequality, Lemma 2.1 and Young’s inequality gives that

\[
J_1 = \int_0^T \int_\Omega [(u_r \cdot \nabla \rho_r) + (u \cdot \nabla \rho) \rho_r] \, dx dy dz \, dt
= \int_0^T \int_\Omega [(V_r \cdot \nabla_h \Gamma_r) - (\partial_z W_r) \Gamma_r \rho - W_r \Gamma_r \partial_z \rho] \, dx dy dz \, dt
= \int_0^T \int_\Omega [(V_r \cdot \nabla_h \Gamma_r) + (\partial_z W_r) \Gamma_r \rho] \, dx dy dz \, dt
+ \int_\Gamma \int_\Omega [\Gamma_r (\partial_z \rho)] \left( \int_0^2 \nabla_h V_r \right) \, dx dy dz \, dt
\leq \int_0^T \int_\Omega [(|V_r||\nabla_h \Gamma_r||\rho| + |\nabla_h V_r||\Gamma_r||\rho|) \, dx dy dz \, dt
+ \int_M \left( \int_{-1}^1 |\nabla_h V_r| \, dz \right) \left( \int_{-1}^1 |\Gamma_r||\partial_z \rho| \, dz \right) \, dx dy
\leq C \left( \int_0^T \|\nabla \rho\|^2 \, dt + \|\Delta \rho\|^2 \right) \left( \int_0^T \|V_r\|^2 + \|\Gamma_r\|^2 \right) \, dt + \frac{1}{8} \int_0^T \left( \|\nabla V_r\|^2 + \|\nabla \Gamma_r\|^2 \right) \, dt.
\]
we obtain that
\[
J_2 := \tau^2 \int_0^T \int_0^\infty \left[ - \left( \int_0^2 \frac{\partial_t v(x, y, \xi, t) \cdot \nabla_h W_\tau - \nabla w \cdot \nabla W_\tau}{d} \right) dz d\tau dt
\]
\[
\leq C \tau^2 \int_0^T \left( \|\partial_t v\|_2^2 + \|\nabla_h w\|_2^2 + 2\|\partial_x w\|_2^2 \right) dt + \frac{1}{8} \int_0^T \tau^2 \|\nabla W_\tau\|_2^2 dt
\]
\[
\leq C \tau^2 \int_0^T \left[ \|\partial_t v\|_2^2 + \int_\Omega \left( \int_0^2 \nabla_h (\nabla \cdot v) d\xi \right)^2 \right] dt
\]
\[
+ C \tau^2 \int_0^T \|\nabla_h v\|_2^2 dt + \frac{1}{8} \int_0^T \tau^2 \|\nabla W_\tau\|_2^2 dt
\]
\[
\leq C \tau^2 \int_0^T \left( \|\partial_t v\|_2^2 + \|\Delta v\|_2^2 \right) dt + \frac{1}{8} \int_0^T \tau^2 \|\nabla W_\tau\|_2^2 dt. \tag{5.7}
\]

The bounds for $J_3$ and $J_4$ on the right-hand side of (5.5) can be found in Li-Titi [28, Proposition 4.2]. These two integral terms can be estimated as
\[
J_3 := \int_0^T \int_\Omega [(u_\tau \cdot \nabla) v_\tau \cdot v + (u \cdot \nabla) v \cdot v_\tau] dxdydzdt
\]
\[
\leq C \int_0^T \|\nabla v\|_2^2 \|\Delta v\|_2^2 \|V_\tau\|_2^2 dt + \frac{1}{8} \int_0^T \|\nabla V_\tau\|_2^2 dt \tag{5.8}
\]
and
\[
J_4 := \tau^2 \int_0^T \int_\Omega (u_\tau \cdot \nabla w_\tau) w dxdydzdt
\]
\[
\leq C \tau^2 \int_0^T \left( \|u_\tau\|_2^2 \|\nabla v_\tau\|_2^2 + \|\nabla v\|_2^2 \|\Delta v\|_2^2 + \tau^2 \|w_\tau\|_2^2 \|\nabla w_\tau\|_2^2 \right) dt
\]
\[
+ \frac{1}{8} \int_0^T \left( \|\nabla V_\tau\|_2^2 + \tau^2 \|\nabla W_\tau\|_2^2 \right) dt, \tag{5.9}
\]
respectively.

Adding (5.6), (5.7), (5.8) and (5.9) yields that
\[
h(t) := \left( \|V_\tau, \Gamma_\tau\|_2^2 + \tau^2 \|W_\tau\|_2^2 \right) (t) + \int_0^t \left( \|\nabla(V_\tau, \Gamma_\tau)\|_2^2 + \|\nabla W_\tau\|_2^2 \right) ds
\]
\[
\leq C \int_0^t \|\nabla v\|_2^2 \|\Delta v\|_2^2 \|V_\tau\|_2^2 ds + C \tau^2 \int_0^t \left( \|\partial_t v\|_2^2 + \|\Delta v\|_2^2 \right) ds
\]
\[
+ C \tau^2 \int_0^t \left( \|u_\tau\|_2^2 \|\nabla v_\tau\|_2^2 + \|\nabla v\|_2^2 \|\Delta v\|_2^2 + \tau^2 \|w_\tau\|_2^2 \|\nabla w_\tau\|_2^2 \right) ds
\]
\[
+ C \int_0^t \|\nabla \rho\|_2^2 \|\Delta \rho\|_2^2 \left( \|V_\tau\|_2^2 + \|\Gamma_\tau\|_2^2 \right) ds =: H(t)
\]
for a.e. $t \in [0, \infty)$. Taking the derivative of $H(t)$ with respect to $t$, we obtain that
\[
H'(t) \leq C \tau^2 \left( \|v_\tau\|_2^2 \|\nabla v_\tau\|_2^2 + \|\nabla v\|_2^2 \|\Delta v\|_2^2 + \tau^2 \|w_\tau\|_2^2 \|\nabla w_\tau\|_2^2 \right)
\]
\[
+ C \left( \|\nabla v\|_2^2 \|\Delta v\|_2^2 + \|\nabla \rho\|_2^2 \|\Delta \rho\|_2^2 \right) \left( \|V_\tau\|_2^2 + \|\Gamma_\tau\|_2^2 \right) + C \tau^2 \left( \|\partial_t v\|_2^2 + \|\Delta v\|_2^2 \right)
\]
\[
\leq C \left( \|\nabla v\|_2^2 \|\Delta v\|_2^2 + \|\nabla \rho\|_2^2 \|\Delta \rho\|_2^2 \right) H(t) + C \tau^2 \left( \|\partial_t v\|_2^2 + \|\Delta v\|_2^2 \right)
\]
\[
+ C \tau^2 \left( \|v_\tau\|_2^2 \|\nabla v_\tau\|_2^2 + \|\nabla v\|_2^2 \|\Delta v\|_2^2 + \tau^2 \|w_\tau\|_2^2 \|\nabla w_\tau\|_2^2 \right).
\]
We subtract system (1.11) from system (1.4) to obtain system (4.18) and (5.4) that
\[ H(t) = \left( \| (V_t, \Gamma_t) \|^2_2 + \tau^2 \| W_t \|^2_2 \right) (t) + \int_0^t \left( \| \nabla (V_t, \Gamma_t) \|^2_2 + \tau^2 \| \nabla W_t \|^2_2 \right) ds \]
\[ \leq C \tau^2 \exp \left\{ C \int_0^t \left( \| \nabla v \|^2_2 \| \Delta v \|^2_2 + \| \nabla \rho \|^2_2 \| \Delta \rho \|^2_2 \right) ds \right\} \]
\[ \times \int_0^t \left( \| \partial_t v \|^2_2 + \| \Delta v \|^2_2 + \| v \|^2_2 \| \nabla v \|^2_2 + \| \nabla \rho \|^2_2 \| \Delta \rho \|^2_2 + \tau^4 \| \omega \|^2_2 \right) \] \[ \leq C \tau^2 e^{C\eta_0^2(t)} \left[ \eta_1(t) + \eta_2(t) + \left( \| v_0 \|^2_2 + \tau^2 \| w_0 \|^2_2 + \| \rho_0 \|^2_2 \right)^2 \right]. \]
This completes the proof. \( \square \)

Based on Proposition 5.2, the proof of Theorem 3.2 is as follows:

**Proof of Theorem 3.2** For any \( T > 0 \), according to Proposition 5.2, the following estimate holds:
\[ \sup_{0 \leq t \leq T} \left( \| (V_t, \tau W_t, \Gamma_t) \|^2_2 \right) (t) + \int_0^T \| \nabla (V_t, \tau W_t, \Gamma_t) \|^2_2 dt \leq \tau^2 \tilde{K}_1(T). \]
Here
\[ \tilde{K}_1(T) = C e^{C\eta_0^2(T)} \left[ \eta_1(T) + \eta_2(T) + \left( \| v_0 \|^2_2 + \| w_0 \|^2_2 + \| \rho_0 \|^2_2 \right)^2 \right], \]
and \( C \) is a positive constant that does not depend on \( \tau \). It can be deduced from the above estimate that
\( (v_t, \tau w_t, \rho_t) \rightarrow (v, 0, \rho) \) in \( L^\infty (0, T; L^2(\Omega)) \),
and \( (\nabla v_t, \tau \nabla w_t, \nabla \rho_t, \nabla \rho_t, \nabla w_t) \rightarrow (\nabla v, 0, \nabla \rho, w) \) in \( L^2 \left( 0, T; L^2(\Omega) \right) \).

Obviously, the rate of convergence is of the order \( O(\tau) \). The theorem is proven. \( \square \)

### 6 Strong Convergence for \( H^2 \) Initial Data

In this section, assume that initial data \( (v_0, \rho_0) \) lies in \( H^2(\Omega) \), where initial velocity \( v_0 \) satisfies that
\[ \int_{-1}^1 \nabla_h \cdot v_0(x, y, z) dz = 0, \quad \text{for all } (x, y) \in M. \]
We prove that the scaled Boussinesq equations (1.4) strongly converge to the viscous primitive equations with density stratification (1.11) as the aspect ratio \( \tau \) goes to zero. In this case, there is a unique local strong solution \( (v_t, w_t, \rho_t) \) to system (1.4), subject to the boundary and initial conditions (1.5)–(1.6) and the symmetry condition (1.7). Denote by \( T^* \) the maximal existence time of this local strong solution.

Let
\[ (U_t, \Gamma_t, P_t) = (V_t, W_t, \Gamma_t, P_t), \]
\[ (V_t, W_t, \Gamma_t, P_t) = (v_t - v, w_t - w, \rho_t - \rho, p_t - p). \]
We subtract system (1.11) from system (1.4) to obtain system
\[ \partial_t V_t - \Delta V_t + (U_t \cdot \nabla) V_t + (u \cdot \nabla) V_t + (U_t \cdot \nabla) v + \nabla_h P_t = 0, \] (6.1)
\( \square \) Springer
\[ \tau^2(\partial_t W_T - \Delta W_T + U_T \cdot \nabla W_T + U_T \cdot \nabla w + u \cdot \nabla W_T) + \partial_x P_T \\
+ \Gamma_T + \tau^2(\partial_t w - \Delta w + u \cdot \nabla w) = 0, \quad (6.2) \]
\[ \partial_t \Gamma_T - \Delta \Gamma_T + U_T \cdot \nabla \Gamma_T + U_T \cdot \nabla \rho + u \cdot \nabla \Gamma_T - W_T = 0, \quad (6.3) \]
\[ \nabla_h \cdot V_T + \partial_2 W_T = 0, \quad (6.4) \]
defined on \( \Omega \times (0, T^*_T). \)

**Proposition 6.1** Suppose that \((v_0, \rho_0) \in H^2(\Omega)\) with \(\int_{-1}^{1} \nabla_h \cdot v_0\, dz = 0\). Then system (6.1)–(6.4) has the basic energy estimate
\[
\sup_{0 \leq s \leq t} \left( \left\| (V_T, \tau W_T, \Gamma_T) \right\|^2 \right) (s) + \int_0^t \left\| \nabla (V_T, \tau W_T, \Gamma_T) \right\|^2 \, ds \leq \tau^2 K_1(t)
\]
for any \( t \in [0, T^*_T) \), where
\[
K_1(t) = C e^{C \eta_1^2(t)} \left[ \eta_1(t) + \eta_2(t) + \left( \left\| v_0 \right\|^2 + \tau^2 \left\| v_0 \right\|^2 + \left\| \rho_0 \right\|^2 \right)^2 \right].
\]
Here \( C \) is a positive constant that does not depend on \( \tau \).

It is important to note that the Proposition 6.1 is a direct consequence of Proposition 5.2. Moreover, the basic energy estimate on system (6.1)–(6.4) can also be obtained by the energy method. The strong solution \((v_T, w_T, \rho_T)\) to system (1.4) is local, so is the basic energy estimate. In order to obtain the first order energy estimate for system (6.1)–(6.4), we need to perform the second order energy estimate on system (1.11).

**Proposition 6.2** Suppose that \((v_0, \rho_0) \in H^2(\Omega)\) with \(\int_{-1}^{1} \nabla_h \cdot v_0\, dz = 0\). Then system (1.11) has the second order energy estimate
\[
\sup_{0 \leq s \leq t} \left( \left\| \Delta v \right\|^2 + \left\| \Delta \rho \right\|^2 \right) (s) + \int_0^t \left( \left\| \nabla \partial_t v \right\|^2 + \left\| \nabla \Delta v \right\|^2 + \left\| \nabla \partial_t \rho \right\|^2 + \left\| \nabla \Delta \rho \right\|^2 \right) \, ds \leq \eta_3(t)
\]
for any \( t \in [0, \infty) \), where
\[
\eta_3(t) = e^{C(t(t/2) \eta_1^2(t))} \left[ \left\| v_0 \right\|^2 + \left\| \rho_0 \right\|^2 \right].
\]

**Proof** Taking the \( L^2(\Omega) \) inner product of (4.1) and (4.2) with \( \Delta (\Delta v - \partial_t v) \) and \( \Delta (\Delta \rho - \partial_t \rho) \), respectively, we deduce from integration by parts that
\[
\frac{d}{dt} \left( \left\| \Delta v \right\|^2 + \left\| \Delta \rho \right\|^2 \right) + \left\| \nabla \partial_t v \right\|^2 + \left\| \nabla \Delta v \right\|^2 + \left\| \nabla \partial_t \rho \right\|^2 + \left\| \nabla \Delta \rho \right\|^2
\]
\[
= \int_\Omega \nabla \left( \int_0^z \nabla_h \cdot v(x, y, \xi, t)\, d\xi \right) \cdot \nabla (\Delta \rho - \partial_t \rho) \, dz\, dy\, dz
\]
\[
+ \int_\Omega \nabla \left( \int_0^z \nabla_h \rho(x, y, \xi, t)\, d\xi \right) \cdot \nabla (\partial_t v - \Delta v) \, dz\, dy\, dz
\]
\[
+ \int_\Omega \nabla \left( \nabla_h \rho \right) - \left( \int_0^z \nabla_h \cdot v(x, y, \xi, t)\, d\xi \right) \partial_x \rho \cdot \nabla (\Delta \rho - \partial_t \rho) \, dz\, dy\, dz
\]
\[
+ \int_\Omega \nabla \left( \nabla_h \rho \right) v - \left( \int_0^z \nabla_h \cdot v(x, y, \xi, t)\, d\xi \right) \partial_x v \cdot \nabla (\Delta v - \partial_t v) \, dz\, dy\, dz
\]
\[
= : G_1 + G_2 + G_3 + G_4. \quad (6.5)
\]

For the first integral term \( G_1 \) on the right-hand side of (6.5), we use Hölder’s inequality and Young’s inequality to get that
\[
G_1 : = \int_\Omega \nabla \left( \int_0^z \nabla_h \cdot v(x, y, \xi, t)\, d\xi \right) \cdot \nabla (\Delta \rho - \partial_t \rho) \, dz\, dy\, dz
\]
\[
\begin{align*}
&= \int_\Omega \nabla_h \left( \int_0^z \nabla_h \cdot v(x, y, \xi, t) d\xi \right) \cdot \nabla_h (\Delta \rho - \partial_t \rho) dx dy dz \\
&\quad + \int_\Omega (\nabla_h \cdot v) (\partial_z \Delta \rho - \partial_z \partial_t \rho) dx dy dz \\
&= \int_\Omega \left[ \int_0^z \nabla_h \cdot v(x, y, \xi, t) d\xi \right] (\partial_t \Delta \rho - \partial_t \partial_t \rho) dx dy dz \\
&\quad + \int_\Omega (\nabla_h \cdot v) (\partial_z \Delta \rho - \partial_z \partial_t \rho) dx dy dz \\
&\leq \int_M \left( \int_{-1}^1 (|\partial_t \nabla_h v| + |\partial_t \Delta \rho|) d\xi \right) dx dy dz \\
&\quad + \int_\Omega |\nabla_h v| (|\partial_z \partial_t \rho| + |\partial_t \Delta \rho|) dx dy dz \\
&\leq C \left( \|\nabla v\|^2_2 + \|\Delta v\|^2_2 \right) + \frac{1}{8} \left( \|\nabla \partial_t \rho\|^2_2 + \|\nabla \Delta \rho\|^2_2 \right) . \tag{6.6}
\end{align*}
\]

Next, we estimate the second integral term \(G_2\) on the right-hand side of (6.5). Using the same method as for the integral term \(G_1\) gives that
\[
\begin{align*}
G_2 &:= \int_\Omega \nabla \left( \int_0^z \nabla_h \rho(x, y, \xi, t) d\xi \right) \cdot \nabla (\partial_t v - \Delta v) dx dy dz \\
&\leq C \left( \|\nabla \rho\|^2_2 + \|\Delta \rho\|^2_2 \right) + \frac{1}{8} \left( \|\nabla \partial_t \rho\|^2_2 + \|\nabla \Delta \rho\|^2_2 \right) . \tag{6.7}
\end{align*}
\]
Due to Lemma 2.2 and Young’s inequality, the third integral term \(G_3\) on the right-hand side of (6.5) can be estimated as
\[
\begin{align*}
G_3 &:= \int_\Omega \nabla \left[ v \cdot \nabla_h \rho - \left( \int_0^z \nabla_h \cdot v(x, y, \xi, t) d\xi \right) \partial_z \rho \right] \cdot \nabla (\Delta \rho - \partial_t \rho) dx dy dz \\
&= \int_\Omega (\partial_t u \cdot \nabla \rho + u \cdot \partial_t \nabla \rho) \cdot \nabla (\Delta \rho - \partial_t \rho) dx dy dz \\
&\leq C \|\partial_t \nabla v\|^{1/2}_2 \|\partial_t \Delta v\|^{1/2}_2 \|\nabla \rho\|^{1/2}_2 \|\Delta \rho\|^{1/2}_2 \left( \|\partial_t \partial_t \rho\|_2 + \|\partial_t \Delta \rho\|_2 \right) \\
&\quad + C \|\nabla v\|^{1/2}_2 \|\Delta v\|^{1/2}_2 \|\partial_t \nabla \rho\|^{1/2}_2 \|\Delta \rho\|^{1/2}_2 \left( \|\partial_t \partial_t \rho\|_2 + \|\partial_t \Delta \rho\|_2 \right) \\
&\leq C \left( \|\nabla v\|^2_2 \|\Delta v\|^2_2 + \|\nabla \rho\|^2_2 \|\Delta \rho\|^2_2 \right) \left( \|\Delta v\|^2_2 + \|\Delta \rho\|^2_2 \right) \\
&\quad + \frac{1}{8} \left( \|\nabla \Delta v\|^2_2 + \|\nabla \partial_t \rho\|^2_2 + \|\nabla \Delta \rho\|^2_2 \right) . \tag{6.8}
\end{align*}
\]
For the last integral term \(G_4\) on the right-hand side of (6.5), a similar argument as to that for the integral term \(G_3\) gives that
\[
\begin{align*}
G_4 &:= \int_\Omega \nabla \left[ (v \cdot \nabla_h) \nu - \left( \int_0^z \nabla_h \cdot v(x, y, \xi, t) d\xi \right) \partial_z \nu \right] \cdot \nabla (\Delta \nu - \partial_t \nu) dx dy dz \\
&= \int_\Omega (\partial_t u \cdot \nabla \nu + u \cdot \partial_t \nabla \nu) \cdot \nabla (\Delta \nu - \partial_t \nu) dx dy dz \\
&\leq C \|\nabla v\|^2_2 \|\Delta v\|^2_2 \left( \|\nabla \nu\|^2_2 + \|\nabla \Delta \nu\|^2_2 \right) + \frac{1}{8} \left( \|\nabla \partial_t \nu\|^2_2 + \|\nabla \Delta \nu\|^2_2 \right) . \tag{6.9}
\end{align*}
\]
Substituting (6.6)–(6.9) into (6.5), we have that
\[
\frac{d}{dt} \left( \|\Delta v\|^2_2 + \|\Delta \rho\|^2_2 \right) + \frac{1}{2} \left( \|\nabla \partial_t \nu\|^2_2 + \|\nabla \Delta \nu\|^2_2 \right) .
\]
\[
\leq C \left( \|\nabla v\|_2^2 \|\Delta v\|_2^2 + \|\nabla \rho\|_2^2 \|\Delta \rho\|_2^2 + 1 \right) \left( \|\Delta v\|_2^2 + \|\Delta \rho\|_2^2 \right).
\]

Thanks to Grönwall’s inequality, it follows from (4.18) that
\[
\left( \|\Delta v\|_2^2 + \|\Delta \rho\|_2^2 \right) (t) + \int_0^t \left( \|\nabla \partial_t v\|_2^2 + \|\nabla \Delta v\|_2^2 + \|\nabla \partial_t \rho\|_2^2 + \|\nabla \Delta \rho\|_2^2 \right) ds \\
\leq \exp \left\{ C \int_0^t \left( \|\nabla v_0\|_2^2 + \|\nabla \rho_0\|_2^2 + 1 \right) \right\} \left( \|\Delta v_0\|_2^2 + \|\Delta \rho_0\|_2^2 \right).
\]

The proof is completed. \(\square\)

With the help of Proposition 6.2, we can perform the first order energy estimate on system (6.1)–(6.4).

**Proposition 6.3** Suppose that \((v_0, \rho_0) \in H^2(\Omega)\) with \(\int_{\Omega} \nabla_y v_0 dz = 0\). Then there exists a small positive constant \(\beta_0\) such that system (6.1)–(6.4) has the first order energy estimate
\[
\sup_{0 \leq s \leq t} \left( \|\nabla (V_r, \tau W_r, \Gamma_r)\|_2^2 \right) (s) + \int_0^t \|\Delta (V_r, \tau W_r, \Gamma_r)\|_2^2 ds \leq \tau^2 K_2(t)
\]
for any \(t \in [0, T^*_r)\), provided that
\[
\sup_{0 \leq s \leq t} \left( \|\nabla (V_r, \Gamma_r)\|_2^2 + \tau^2 \|\nabla W_r\|_2^2 \right) (s) \leq \beta_0^2,
\]
where
\[
K_2(t) = C e^{C(1+\tau^4)} \eta_3(t) \left[ \eta_5(t) + \eta_6(t) \right].
\]
Here \(C\) is a positive constant that does not depend on \(\tau\).

**Proof** Multiplying the first three equations in system (6.1)–(6.4) by \(-\Delta V_r, -\Delta W_r\) and \(-\Delta \Gamma_r\) respectively, then integrating over \(\Omega\) and finally integrating by parts gives
\[
\frac{1}{2} \frac{d}{dt} \left( \|\nabla (V_r, \Gamma_r)\|_2^2 + \tau^2 \|\nabla W_r\|_2^2 \right) + \|\Delta (V_r, \Gamma_r)\|_2^2 + \tau^2 \|\Delta W_r\|_2^2
\]
\[
= \int_{\Omega} (U_r \cdot \nabla \Gamma_r + u \cdot \nabla \Gamma_r + U_r \cdot \nabla \rho) \Delta \Gamma_r dx dy dz
\]
\[
+ \tau^2 \int_{\Omega} (\partial_t w - \Delta w + u \cdot \nabla w) \Delta W_r dx dy dz
\]
\[
+ \tau^2 \int_{\Omega} (U_r \cdot \nabla W_r + u \cdot \nabla W_r + U_r \cdot \nabla w) \Delta W_r dx dy dz
\]
\[
+ \int_{\Omega} \left[ (U_r \cdot \nabla) V_r + (u \cdot \nabla) V_r + (U_r \cdot \nabla) v \right] \cdot \Delta V_r dx dy dz
\]
\[
= : I_1 + I_2 + I_3 + I_4. \tag{6.10}
\]

For the first integral term \(I_1\) on the right-hand side of (6.10), we apply Lemma 2.2 and Poincaré’s inequality and Young’s inequality to obtain that
\[
I_1 = : \int_{\Omega} (U_r \cdot \nabla \Gamma_r + u \cdot \nabla \Gamma_r + U_r \cdot \nabla \rho) \Delta \Gamma_r dx dy dz
\]
\[
\leq C \|\nabla V_r\|_2^{1/2} \|\Delta V_r\|_2^{1/2} \|\nabla \Gamma_r\|_2^{1/2} \|\Delta \Gamma_r\|_2^{1/2} \|\Delta \Gamma_r\|_2
\]
\[
+ C \|\nabla v\|_2^{1/2} \|\Delta v\|_2^{1/2} \|\nabla \Gamma_r\|_2^{1/2} \|\Delta \Gamma_r\|_2^{1/2}
\]
\[
+ C \|\nabla V_r\|_2^{1/2} \|\Delta V_r\|_2^{1/2} \|\nabla \rho\|_2^{1/2} \|\Delta \rho\|_2^{1/2} \|\Delta \Gamma_r\|_2.
\]

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\[ \leq \frac{5}{64} \left( \| \Delta V_r \|_2^2 + \| \Delta \Gamma_r \|_2^2 \right) + C \left( \| \nabla V_r \|_2^2 + \| \nabla \Gamma_r \|_2^2 \right) \]
\[ \times \left( \| \Delta V_r \|_2^2 + \| \Delta \Gamma_r \|_2^2 \right) + \| \Delta \rho \|_2^2 \| \nabla \Delta \rho \|_2^2 + \| \Delta v \|_2^2 \| \nabla \Delta v \|_2^2 \right). \tag{6.11} \]

The integral terms \( I_2, I_3 \) and \( I_4 \) on the right-hand side of (6.10) can be bounded as

\[ I_2 := \tau^2 \int_\Omega (\partial_t w - \Delta w + u \cdot \nabla w) \Delta W_r \, dx \, dy \, dz \]
\[ \leq \tau^2 C \left( \| \Delta v \|_2^2 \| \nabla \Delta v \|_2^2 + \| \nabla \partial_t v \|_2^2 + \| \nabla \Delta v \|_2^2 \right) + \frac{5}{64} \tau^2 \| \Delta W_r \|_2^2, \tag{6.12} \]

\[ I_3 := \tau^2 \int_\Omega (U_r \cdot \nabla W_r + u \cdot \nabla W_r + U_r \cdot \nabla w) \Delta W_r \, dx \, dy \, dz \]
\[ \leq \frac{5}{64} \left( \| \Delta V_r \|_2^2 + \tau^2 \| \Delta W_r \|_2^2 \right) + C \left( \| \nabla V_r \|_2^2 + \tau^2 \| \nabla W_r \|_2^2 \right) \]
\[ \times \left( \| \Delta V_r \|_2^2 + \| \Delta W_r \|_2^2 \right) + (1 + \tau^4) \| \Delta v \|_2^2 \| \nabla \Delta v \|_2^2 \right) \tag{6.13} \]

and

\[ I_4 := \int_\Omega [(U_r \cdot \nabla) V_r + (u \cdot \nabla) V_r + (U_r \cdot \nabla) v] \cdot \Delta V_r \, dx \, dy \, dz \]
\[ \leq C \left( \| \nabla V_r \|_2^2 \| \Delta V_r \|_2^2 + \| \Delta v \|_2^2 \| \nabla \Delta v \|_2^2 \right) + \frac{5}{64} \| \Delta V_r \|_2^2 \]
\[ \leq C \| \nabla V_r \|_2^2 \left( \| \Delta V_r \|_2^2 + \| \Delta v \|_2^2 \| \nabla \Delta v \|_2^2 \right) + \frac{5}{64} \| \Delta V_r \|_2^2, \tag{6.14} \]

respectively. The details of the above calculations can be found in [28, Proposition 5.2].

Combining the estimates for (6.11), (6.12), (6.13) and (6.14), we obtain that

\[ \frac{1}{2} \frac{d}{dt} \left( \| \nabla (V_r, \Gamma_r) \|_2^2 + \tau^2 \| \nabla W_r \|_2^2 \right) + \frac{11}{16} \left( \| \Delta (V_r, \Gamma_r) \|_2^2 + \tau^2 \| \Delta W_r \|_2^2 \right) \]
\[ \leq C_\delta \left( \| \nabla (V_r, \Gamma_r) \|_2^2 + \tau^2 \| \nabla W_r \|_2^2 \right) \left[ \left( \| \Delta (V_r, \Gamma_r) \|_2^2 + \tau^2 \| \Delta W_r \|_2^2 \right) \right. \]
\[ + \| \Delta \rho \|_2^2 \| \nabla \Delta \rho \|_2^2 + (1 + \tau^4) \| \Delta v \|_2^2 \| \nabla \Delta v \|_2^2 \] \[ \left. + \tau^2 C_\delta \left( \| \Delta v \|_2^2 \| \nabla \Delta v \|_2^2 + \| \nabla \partial_t v \|_2^2 \right) \right) \| \nabla \Delta v \|_2^2 \right). \tag{6.15} \]

Using the assumption given by the proposition,

\[ \sup_{0 \leq s \leq t} \left( \| \nabla (V_r, \Gamma_r) \|_2^2 + \tau^2 \| \nabla W_r \|_2^2 \right) (s) \leq \beta_0^2, \]

and choosing \( \beta_0 = \sqrt{\frac{3}{16c_\delta}} \), it can be deduced from inequality (6.15) that

\[ \frac{d}{dt} \left( \| \nabla (V_r, \Gamma_r) \|_2^2 + \tau^2 \| \nabla W_r \|_2^2 \right) + \left( \| \Delta (V_r, \Gamma_r) \|_2^2 + \tau^2 \| \Delta W_r \|_2^2 \right) \]
\[ \leq C_\delta \left[ \| \Delta \rho \|_2^2 \| \nabla \Delta \rho \|_2^2 + (1 + \tau^4) \| \Delta v \|_2^2 \| \nabla \Delta v \|_2^2 \right] \left( \| \nabla (V_r, \Gamma_r) \|_2^2 + \tau^2 \| \nabla W_r \|_2^2 \right) \]
\[ + \tau^2 C_\delta \left( \| \Delta v \|_2^2 \| \nabla \Delta v \|_2^2 + \| \nabla \partial_t v \|_2^2 \right) \| \nabla \Delta v \|_2^2 \right). \]

Note that the fact that \((V_r, W_r, \Gamma_r)|_{t=0} = 0\). Applying Grönwall’s inequality to the above inequality, it follows from Proposition 6.2 that

\[ \left( \| \nabla (V_r, \Gamma_r) \|_2^2 + \tau^2 \| \nabla W_r \|_2^2 \right) (t) + \int_0^t \left( \| \Delta (V_r, \Gamma_r) \|_2^2 + \tau^2 \| \Delta W_r \|_2^2 \right) \, ds \]
\[ \leq \tau^2 C_\delta \exp \left\{ C_\delta \int_0^t \left[ \| \Delta \rho \|_2^2 \| \nabla \Delta \rho \|_2^2 + (1 + \tau^4) \| \Delta v \|_2^2 \| \nabla \Delta v \|_2^2 \right] \, ds \right\} \]
so is omitted here. Based on Proposition 6.4, we give the proof of Theorem 3.3.

Here both $C$ and $\tilde{\tau}$ are positive constants that do not depend on $\tau$.

The proof of Proposition 6.4 is similar to that given by Pu-Zhou [34, Proposition 4.1], and so is omitted here. Based on Proposition 6.4, we give the proof of Theorem 3.3.

**Proof of Theorem 3.3** For any $T > 0$, thanks to Proposition 6.4, there exists a small positive constant $\tau(T) = \frac{3\delta_0}{4\sqrt{\mathcal{K}_2(T)}}$ such that $T^*_\tau > T$, provided that $\tau \in (0, \tau(T))$, which implies that system (1.4), corresponding to the boundary and initial conditions (1.5)–(1.6) and the symmetry condition (1.7), has a unique strong solution $(v_\tau, w_\tau, \rho_\tau)$ on the time interval $[0, T]$ for all $\tau \in (0, \tau(T))$. Moreover, the following estimate holds

$$\sup_{0 \leq t \leq T} \left( \|v_\tau(t)\|_{H^1}^2 \right) + \int_0^T \|\nabla v_\tau(t)\|_{H^1}^2 dt \leq \tau^2 \left( \tilde{K}_1(T) + \tilde{K}_2(T) \right),$$

where

$$\tilde{K}_1(T) = C e^{C \tilde{\tau}^2(T)} \left[ \eta_4(T) + \eta_3^2(T) \right] + \left( \|v_0\|_2^2 + \|w_0\|_2^2 + \|\rho_0\|_2^2 \right)^2,$$

and

$$\tilde{K}_2(T) = C' e^{C \tilde{\tau}^2(T)} \left[ \eta_5(T) + \eta_3^2(T) \right].$$

Here both $C$ and $C'$ are positive constants that do not depend on $\tau$.

The proof of Proposition 6.4 is similar to that given by Pu-Zhou [34, Proposition 4.1], and so is omitted here. Based on Proposition 6.4, we give the proof of Theorem 3.3.

**Proof of Theorem 3.3** For any $T > 0$, thanks to Proposition 6.4, there exists a small positive constant $\tau(T) = \frac{3\delta_0}{4\sqrt{\mathcal{K}_2(T)}}$ such that $T^*_\tau > T$, provided that $\tau \in (0, \tau(T))$, which implies that system (1.4), corresponding to the boundary and initial conditions (1.5)–(1.6) and the symmetry condition (1.7), has a unique strong solution $(v_\tau, w_\tau, \rho_\tau)$ on the time interval $[0, T]$ for all $\tau \in (0, \tau(T))$. Moreover, the following estimate holds

$$\sup_{0 \leq t \leq T} \left( \|v_\tau(t)\|_{H^1}^2 \right) + \int_0^T \|\nabla v_\tau(t)\|_{H^1}^2 dt \leq \tau^2 \left( \tilde{K}_1(T) + \tilde{K}_2(T) \right) =: \tau^2 \tilde{K}_3(T).$$

Here $\tilde{K}_3(t)$ is a nonnegative continuously increasing function that does not depend on $\tau$. Finally, it is clear that the strong convergences stated in Theorem 3.3 are the direct consequences of the above estimate. The theorem is thus proven. \(\square\)

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