Lower bound on the radii of circular orbits in the extremal Kerr black-hole spacetime

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Abstract

It is often stated in the physics literature that maximally-spinning Kerr black-hole spacetimes are characterized by near-horizon co-rotating circular geodesics of radius $r_{\text{circular}}$ with the property $r_{\text{circular}} \rightarrow r_H^{+}$, where $r_H$ is the horizon radius of the extremal black hole. Based on the famous Thorne hoop conjecture, in the present compact paper we provide evidence for the existence of a non-trivial lower bound $\frac{r_{\text{circular}} - r_H}{r_H} \gtrsim (\mu/M)^{1/2}$ on the radii of circular orbits in the extremal Kerr black-hole spacetime, where $\mu/M$ is the dimensionless mass ratio which characterizes the composed black-hole-orbiting-particle system.
I. INTRODUCTION

The circular geodesic motions of test particles in rotating Kerr black-hole spacetimes have attracted the attention of physicists and mathematicians during the last five decades (see [1–4] and references therein). The characteristic radii of these astrophysically important orbits are bounded from below by the radius of the equatorial null circular geodesic which, for a given value of the black-hole angular momentum, is characterized by the smallest possible circumference.

Intriguingly, as discussed by many authors [1–7], in the maximally-spinning (extremal) Kerr black-hole spacetime, there exist circular orbits of radius $r_{\text{circular}}$ which are characterized by the limiting near-horizon behavior $r_{\text{circular}}/r_H \to 1^+$ [8, 9].

The main goal of the present compact paper is to provide evidence, which is based on the famous Thorne hoop conjecture [10], for the existence of a non-trivial lower bound $r_{\text{circular}} > r_{\text{min}} > r_H$ on the radii of circular orbits in the maximally-spinning Kerr black-hole spacetime. In particular, as we shall show below, according to the hoop conjecture the smallest possible circular radius $r_{\text{min}} = r_{\text{min}}(\mu/M)$ is determined by the dimensionless mass ratio $\mu/M$ [11] which characterizes the composed extremal-black-hole-orbiting-particle system.

II. CIRCULAR ORBITS IN THE MAXIMALLY-SPINNING KERR BLACK-HOLE SPACETIME

We consider a particle of proper mass $\mu$ which orbits around a maximally-spinning (extremal) Kerr black-hole of mass $M$ (with $\mu/M \ll 1$) and angular momentum $J = M^2$. The extremal curved black-hole spacetime is characterized by the line element [1]

$$ds^2 = -(1-2Mr/\Sigma)dt^2 - (4M^2r\sin^2\theta/\Sigma)dt d\phi + (\Sigma/\Delta)dr^2 + \Sigma d\theta^2 + (r^2 + M^2 + 2M^3r\sin^2\theta/\Sigma)\sin^2\theta d\phi^2,$$  

(1)

where $(t, r, \theta, \phi)$ are the familiar Boyer-Lindquist spacetime coordinates and the metric functions in (1) are defined by the relations

$$\Delta \equiv (r - M)^2 \quad ; \quad \Sigma \equiv r^2 + M^2 \cos^2\theta.$$  

(2)
The degenerate horizon of the extremal black-hole spacetime is defined by the radial functional relation \( \Delta(r = r_H) = 0 \), which yields the simple relation

\[
r_H = M . \tag{3}
\]

The dimensionless energy ratio \( E(r)/\mu \) (energy per unit mass as measured by asymptotic observers) associated with a circular orbit of radius \( r \) in the equatorial plane of the extremal (maximally-spinning) Kerr black-hole spacetime is given by the functional expression

\[
\frac{E(r)}{\mu} = \frac{r \pm M^{1/2}r^{1/2} - M}{r^{3/4}(r^{1/2} \pm 2 M^{1/2})^{1/2}} . \tag{4}
\]

The upper/lower signs in the dimensionless energy expression (4) correspond to co-rotating/counter-rotating circular orbits in the black-hole spacetime, respectively. From Eq. (4) one finds the near-horizon co-rotating energy ratio

\[
\frac{E(x)}{\mu} = \frac{1}{\sqrt{3}} + \frac{2}{3\sqrt{3}} \cdot x + O(x^2) , \tag{5}
\]

where the dimensionless parameter \( x \) (with the near-horizon property \( x \ll 1 \)) is defined by

\[
r \equiv M(1 + x) . \tag{6}
\]

III. THE THORNE HOOP CONJECTURE AND A LOWER BOUND ON THE RADIi OF CIRCULAR ORBITS IN THE EXTREMAL BLACK-HOLE SPACE-TIME

The total energy (as measured by asymptotic observers) of the composed extremal-Kerr-black-hole-orbiting-particle system is given by

\[
E_{\text{total}} = M + E_{\text{ISCO}} + O(E_{\text{ISCO}}^2/M) . \tag{7}
\]

According to the Thorne hoop conjecture, the composed system will form an engulfing horizon if it can be placed inside a ring whose circumference \( C \) is equal to (or smaller than) \( 4\pi E_{\text{total}} \). That is, Thorne’s famous hoop conjecture asserts that

\[
C(E_{\text{total}}) \leq 4\pi E_{\text{total}} \implies \text{Black-hole horizon exists .} \tag{8}
\]
As we shall now show, the hoop relation (8) provides compelling evidence for the existence of a lower bound (with \( r_{\text{circular}} > r_{\text{min}} > r_H \)) on the radii of circular orbits in the maximally-spinning (extremal) Kerr black-hole spacetime. In particular, using Eqs. (1) and (2) one obtains the functional expression \[ C(r) = 2\pi \sqrt{r^2 + M^2} + 2M^3/r \],

(9)

for the circumference \( C = C(r) \) of an equatorial circular orbit in the extremal Kerr black-hole spacetime. Substituting (6) into (9), one finds the simple near-horizon relation

\[
C(x \ll 1) = 4\pi M \cdot \left[ 1 + \frac{3}{8} \cdot x^2 + O(x^3) \right].
\]

(10)

Taking cognizance of Eqs. (5), (7), (8), and (10), one obtains the dimensionless lower bound

\[
x_{\text{circular}} > x_{\text{min}} = \left( \frac{8}{3\sqrt{3}} \cdot \frac{\mu}{M} \right)^{1/2}.
\]

(11)

on the scaled radii of circular orbits in the composed extremal-Kerr-black-hole-orbiting-particle system. In particular, according to the Thorne hoop conjecture \[10\], composed black-hole-particle configurations whose circular orbits are characterized by the relation \( x_{\text{circular}} \leq x_{\text{min}} \) are expected to be engulfed by a larger horizon with \( r_{\text{horizon}} \geq r_{\text{circular}} \) [see Eqs. (8) and (11)].

IV. SUMMARY

A remarkable feature of the maximally-spinning (extremal) Kerr black-hole spacetime, which has been discussed in the physics literature by many authors (see [1–7] and references therein), is the existence of co-rotating circular orbits which are characterized by the limiting radial behavior \( r_{\text{circular}}/r_H \rightarrow 1^+ \).

In the present compact paper we have used the famous Thorne hoop conjecture \[10\] in order to provide evidence for the possible existence of a larger horizon (with \( r_{\text{horizon}} \geq r_{\text{circular}} \geq r_H \)) that engulfs composed extremal-Kerr-black-hole-orbiting-particle configurations which violate the dimensionless relation (11). In particular, our analysis has revealed the intriguing fact that circular orbits in the extremal Kerr black-hole spacetime are restricted to the radial region [see Eqs. (8), (6), and (11)]

\[
r_{\text{circular}} > r_H \cdot \left[ 1 + \left( \frac{8}{3\sqrt{3}} \cdot \frac{\mu}{M} \right)^{1/2} + O\left( \frac{\mu}{M} \right) \right].
\]

(12)
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[8] Note that maximally spinning Kerr black-hole spacetimes are characterized by the relations $M = a = r_H$, where $\{M, a, r_H\}$ are respectively the mass, angular momentum per unit mass, and the degenerate horizon radius of the extremal black hole.
[9] We shall use natural units in which $G = c = 1$.
[10] K. S. Thorne, in Magic without Magic: John Archibald Wheeler, edited by J. Klauder (Freeman, San Francisco, 1972).
[11] Here $\mu$ is the proper mass (with $\mu \ll M$) of the orbiting particle.
[12] We shall henceforth focus our attention on the co-rotating circular orbits of the Kerr black-hole spacetime. These are the orbits which approach the black-hole horizon ($r_{\text{circular}}^{\text{co-rotating}} / r_H \to 1^+$) in the extremal $a/M \to 1^-$ limit.
[13] Here we have substituted the characteristic relations $dt = dr = d\theta = 0, \theta = \pi/2$, and $\Delta \phi = 2\pi$ for equatorial circular orbits in the line element of the extremal Kerr black-hole spacetime.
[14] It is worth noting that the circumference of an engulfing hoop which is perpendicular to the equatorial plane (that is, with the properties $dt = dr = d\phi = 0$ and $\Delta \theta = 2\pi$) of the maximally-spinning (extremal) black hole is smaller than the calculated circumference which characterizes the equatorial ($\theta = \pi/2$) engulfing hoop [this simple fact stems from Eqs.
and (2) with the characteristic property \( \Sigma < r^2 + M^2 + 2M^3/r \). One therefore concludes that if the equatorial hoop (9) is characterized by the inequality (8), then the perpendicular engulfing hoop will also be characterized by the same inequality.