Asymptotic Bethe-Ansatz Results for a Hubbard Chain with 
1/sinh-Hopping

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Abstract

We investigate spin-1/2 electrons with local Hubbard interaction and variable range hopping amplitudes which decay like \(\sinh(\kappa)/\sinh(\kappa r)\). Assuming integrability the Asymptotic Bethe Ansatz approach allows us to derive the generalized Lieb-Wu integral equations from the two-particle phase shift. Due to the nesting property there is a metal-to-insulator transition at 
\(U_c(\kappa > 0) = 0^+\). The charge gap in the singular limit \(\kappa = 0\) opens when the interaction strength equals the bandwidth, 
\(U_c(\kappa = 0) = W > 0\).

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Exact solutions play a crucial role in our understanding of strongly correlated systems. The algebraic and the coordinate Bethe Ansatz constitute the main line of approach to integrable systems \[1\]. Nevertheless, there are exactly solvable Hamiltonians whose eigenstates cannot be cast into the standard Bethe Ansatz form. A particular example is the popular Calogero-Sutherland-Moser Hamiltonian \[2,3\] that has attracted recently a great deal of attention in connection with the universal properties of disordered systems and random matrix theory \[4,5\].

Recently, Ruckenstein and one of us (F.G.) introduced a $1/r$-Hubbard model that incorporates long-range hopping and on-site repulsion, and that includes the lattice version of the Calogero-Sutherland-Moser model in the strong-coupling limit at half band-filling \[6\]. The integrability as well as the structure of the eigenfunctions of this model are still poorly understood. The analysis of the two-body problem in the $1/r$-Hubbard model indicates that the eigenstates are neither of the Bethe Ansatz form as in the Hubbard model, nor of the Jastrow-type as in the Calogero-Sutherland model \[7\]. Since the wave functions do not become plane waves even at large distances, Sutherland’s Asymptotic Bethe Ansatz \[1,3\] cannot be directly applied to the $1/r$-Hubbard model.

In this article we introduce a model with variable range hopping that interpolates between the standard and the $1/r$-Hubbard model. For half-filled bands and in the strong coupling limit it reduces to the antiferromagnetic $1/\sinh^2(\kappa r)$-Heisenberg or Inozemtsev model \[8\]. This model is a special case of an exchange interaction model \[9\] most recently explicitly solved in Ref. \[10\]. The $1/\sinh(\kappa r)$-Hubbard model is a straightforward but non-trivial generalization of the Inozemtsev model to an itinerant electron system \[11\].

We consider a one-dimensional Hubbard model \[12\] for $N$ spin-1/2 electrons with hopping amplitudes $t(l - m)$ and on-site repulsion $U$,

$$\hat{H} = \sum_{l \neq m, \sigma} t(l - m)\hat{c}_{l, \sigma}^+\hat{c}_{m, \sigma} + U \sum_{l} \hat{n}_{l, \uparrow}\hat{n}_{l, \downarrow},$$

(1)

where the lattice sums on the ring run from $-L/2$ to $L/2 - 1$. The Hubbard model (1) is known to be exactly solvable in two cases: (i) $t(l - m) = -t\delta_{l-m, \pm 1}$ for the standard Hubbard
model [13] and (ii) \( t(l - m) = -it(-1)^{l-m}[d(l-m)]^{-1} \) for the 1/r-Hubbard model [1] where \( d(l-m) = (L/\pi)\sin[\pi(l-m)/L] \) denotes the chord distance between the sites \( l \) and \( m \).

Henceforth we will use \( t \) as our energy unit. Here we introduce a model on the infinite chain \((L \to \infty)\) with hopping amplitudes \( t(l - m) = -i\sinh(\kappa)(-1)^{l-m} / \sinh[\kappa(l-m)] \) where \( \kappa^{-1} \) controls the effective range of the hopping.

The dispersion relation \( \epsilon(k) \) of our model (1) is given by \( \epsilon(k) = (-2i) \sum_{n=1}^{\infty} t(n) \sin(kn) \).

It is odd under parity, and can be expressed in terms of a logarithmic derivative of theta-functions. For \( \kappa = 0 \) one has \( \epsilon(k) = k \) with a discontinuity at the Brillouin zone boundary, while for \( \kappa \to \infty \) one finds \( \epsilon(k) = 2 \sin k \). \( \epsilon(k) \) is continuous for all \( \kappa > 0 \) with zeros at \( k = \pi, k = 0 \). The maximum (minimum) at \( k = \pi/2 \) (\( k = -\pi/2 \)) for \( \kappa = \infty \) is gradually shifted as function of \( \kappa \) to higher (lower) momenta until it reaches \( k = \pi \) (\( k = -\pi \)) for \( \kappa = 0 \).

The corresponding low-energy (g-ology [14]) Hamiltonian involves left- and right-movers with different velocities. The limit \( \kappa \to 0 \) is singular as the dispersion relation involves suddenly only right-movers, while the left-movers' velocity diverges like \( 1/\kappa \), i.e., a discontinuity occurs in \( \epsilon(k) \) at the Brillouin zone boundary. Hence the physics of the metal-to-insulator transition for \( \kappa > 0 \) and for \( \kappa = 0 \) (1/r-Hubbard model) is completely different. At half-filling the distance \( 2k_F = \pi n \) between the two Fermi points becomes half a reciprocal lattice vector ("perfect nesting"). The bands for right- and left-movers effectively cross each other, and their degeneracy is lifted for all \( U > 0 \). The critical value for the transition is thus \( U_c(\kappa > 0) = 0^+ \). In the chiral 1/r-Hubbard model where left-movers are absent one obtains a finite critical value, \( U_c(\kappa = 0) = 2\pi \), see Ref. [13].

We discuss the symmetric orbital part \( \psi(x_1, x_2) \) of the two-particle wave function in the limit of a large system with open boundary conditions. In first quantization the Schrödinger equation for the wave function \( \psi(x_1, x_2) \) reads for \( x_1 \neq x_2 \)

\[
E\psi(x_1, x_2) = F(x_1, x_2) + F(x_2, x_1) + t(x_1 - x_2) \left( \psi(x_2, x_2) - \psi(x_1, x_1) \right),
\]

while for \( x_1 = x_2 \) we find \((E - U)\psi(x_1, x_1) = 2F(x_1, x_1)\). Here we defined \( F(x_1, x_2) = \sum_{x \neq x_1, x_2} t(x_1 - x)\psi(x, x_2) \).
We seek symmetric, spin singlet scattering solutions \( \psi(x_1 \ll x_2) = e^{i(k_1 x_1 + k_2 x_2)} - e^{i\theta(k_1, k_2)}e^{i(k_2 x_1 + k_1 x_2)} \), where \( \theta(k_1, k_2) \) is the phase shift, and \( k_1, k_2 \) are the quasi-momenta. Furthermore, we should choose \( \psi(x_1, x_2) \) such that we can employ the trigonometric identity

\[
\sinh(z_1)\sinh(z_2) = \frac{\sinh(z_1 - z_2)}{\cosh(z_2) - \cosh(z_1)}
\]

This naturally leads to

\[
\psi(x_1, x_2) = \frac{1 - \delta_{x_1, x_2}}{2\sinh[\kappa(x_2 - x_1)]} \left( Ae^{i(k_1 x_1 + k_2 x_2)} - Be^{i(k_2 x_1 + k_1 x_2)} \right) - e^{-\kappa(x_2 - x_1)} \left( Ae^{i(k_1 x_2 + k_2 x_1)} - Be^{i(k_1 x_1 + k_2 x_2)} \right) + \delta_{x_1, x_2} \lambda e^{i(k_1 + k_2)x_1}
\]

as choice for the wave function \((B/A = e^{i\theta(k_1, k_2)})\). It has precisely the form of Inozemtsev’s two-magnon state \[8\].

The calculation of \( F(x_1, x_2) \) has to be done with care because of the infinite lattice sums. For \( x_1 \neq x_2 \) the Schrödinger equation gives \( E = \epsilon(k_1) + \epsilon(k_2) \), and

\[
\lambda = (A - B) + \frac{(A + B)(\epsilon(k_1) - \epsilon(k_2))}{2i\sinh(\kappa)},
\]

while the equation for \( x_1 = x_2 \) becomes \( (A - B)E - i(A + B)(\eta(k_1) - \eta(k_2)) = \lambda(E - U) \)

with the abbreviation \( \eta(k) = 2\sinh(\kappa) \sum_{n=1}^{\infty} (-1)^n \cos(kn) \cosh(\kappa n) / \sinh^2(\kappa n) = \eta(-k) \).

The phase shift is then found as

\[
\theta(k_1, k_2) = -2\tan^{-1}\left[ \frac{H(k_2) - H(k_1)}{U/2} \right]
\]

with \( H(k) = [-\eta(k) + \epsilon(k)(\epsilon(k) - U)/(2\sinh(\kappa))] / 2 \).

Sutherland \[13\] observed that the two-particle phase shift is sufficient to obtain the (scattering) spectrum of a model, even if the wave functions cannot explicitly be constructed as long as the model is integrable. It is far from clear that the model \([1]\) is integrable. Nonetheless we investigate the consequences of this conjecture.

We employ periodic boundary conditions to quantize the pseudo momenta \( k_i \). The error introduced here is exponentially small, of order \( \exp(-\kappa L) \). It is clear how to set up the Asymptotic Bethe Ansatz equations for \( N - M \) up-spins and \( M \) down-spins in a straightforward generalization of the Lieb-Wu equations \[13\] because \( H(k) \) plays the role of \( \sin k \):
\[ Lk_j = 2\pi I_j + \sum_{\beta=1}^{M} \Theta \left( 2H(k_j) - 2\Lambda_\beta \right) \]  
\[ - \sum_{j=1}^{N} \Theta \left( 2\Lambda_\alpha - 2H(k_j) \right) = 2\pi J_\alpha - \sum_{\beta=1}^{M} \Theta \left( \Lambda_\alpha - \Lambda_\beta \right) \]

for \( j = 1, \ldots, N; \alpha = 1, \ldots, M \), where \( \Theta(x) = -2 \tan^{-1}(2x/U) \). \( I_j \) are integers (half-odd integers) for \( M \) even (odd), \( J_\alpha \) are integers (half-odd integers) for \( N - M \) even (odd).

We are interested in the ground state energy per site, \( E/L \), of the model in the thermodynamic limit. In this case the equations (3) can be transformed into integral equations for the densities \( \rho(k) = 1/[L(k_{j+1} - k_j)], \sigma(\Lambda) = 1/[(\Lambda_{\alpha+1} - \Lambda_\alpha)] \) in the usual way:

\[
2\pi\rho(k) = 1 + H'(k) \int_{B_1}^{B_2} d\lambda \sigma(\lambda) K [H(k) - \lambda; U/4] \]

(7a)

\[
2\pi\sigma(\Lambda) = \int_{Q_1}^{Q_2} dk \rho(k) K [H(k) - \Lambda; U/4] - \int_{B_1}^{B_2} d\lambda \sigma(\lambda) K [\Lambda - \lambda'; U/2] \]

(7b)

with \( K [x; y] = 2y/(x^2 + y^2) \), and \( H'(k) = \partial H(k)/\partial k \). The ground state energy density is calculated as \( E/L = \int_{Q_1}^{Q_2} dk \rho(k) \epsilon(k) \).

The integration limits \( Q_1, Q_2, B_1, \) and \( B_2 \) are no longer symmetric around zero. They are determined by the particle numbers, \( \int_{Q_1}^{Q_2} dk \rho(k) = N/L, \int_{B_1}^{B_2} d\lambda \sigma(\lambda) = M/L \), and the condition that a charge (spin) excitation at \( Q_1 (B_1) \) has the same energy as the corresponding excitation at \( Q_2 (B_2) \). The latter conditions are most conveniently described in the pseudo particle picture [15]. The pseudo particle dispersions for charge and spin for zero external magnetic field follow from

\[
\epsilon_c(k) = \epsilon(k) + \int_{B_1}^{B_2} d\lambda \epsilon_s(\lambda) K [H(k) - \lambda; U/4] \]

(8a)

\[
2\pi\epsilon_s(\Lambda) = \int_{Q_1}^{Q_2} dk H'(k) \epsilon_c(k) K [H(k) - \Lambda; U/4] - \int_{B_1}^{B_2} d\lambda \epsilon_s(\lambda) K [\Lambda - \lambda'; U/2] . \]

(8b)

The integration bounds have to fulfill \( \epsilon_c(Q_1) = \epsilon_c(Q_2), \epsilon_s(B_1) = \epsilon_s(B_2) \). The ground state energy and chemical potential can then be expressed in terms of the pseudo particle energies as \( E/L = \int_{Q_1}^{Q_2} dk \epsilon_c(k)/(2\pi) \), and \( \mu = \text{Max}_{Q_1 \leq k \leq Q_2} \epsilon_c(k) \).

For zero external magnetic field, \( M = L/2 \), one finds \( B_2 = \infty = -B_1 \). For half-filling, \( N = L \), one further obtains \( Q_2 = \pi = -Q_1 \). Therefore we can solve the integral equations analytically using Fourier transformation. With the help of the four functions
\[
\tilde{J}_0^{(s)}(\omega) = \left\langle \text{Re} \right\rangle \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{i\omega H(k)} \\
\tilde{J}_1^{(s)}(\omega) = \left\langle \text{Im} \right\rangle \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{i\omega H(k)} \epsilon(k) H'(k) 
\]

one may express the densities and the ground state energy as

\[
\sigma(\lambda) = \int_0^{\infty} \frac{d\omega}{2\pi \cosh(\omega U/4)} \left[ \cos(\omega \lambda) \tilde{\lambda}_0^{(s)}(\omega) + \sin(\omega \lambda) \tilde{\lambda}_0^{(s)}(\omega) \right] 
\]

\[
\rho(k) = \frac{1}{2\pi} + H'(k) \int_0^{\infty} \frac{d\omega}{\pi(1 + e^{\omega U/2})} \left[ \cos(\omega H(k)) \tilde{\lambda}_0^{(s)}(\omega) + \sin(\omega H(k)) \tilde{\lambda}_0^{(s)}(\omega) \right] 
\]

\[
E/L = \int_0^{\infty} \frac{2d\omega}{1 + e^{\omega U/2}} \left[ \tilde{\lambda}_0^{(s)}(\omega) \tilde{\lambda}_1^{(s)}(\omega) + \tilde{\lambda}_0^{(s)}(\omega) \tilde{\lambda}_1^{(s)}(\omega) \right] 
\]

For \( \kappa \to \infty \) we have \( \tilde{\lambda}_0^{(s)}(\omega) = 0 \), \( \tilde{\lambda}_1^{(s)}(\omega) = J_0(\omega) \), \( \tilde{\lambda}_1^{(s)}(\omega) = -2J_1(\omega)/\omega \), where \( J_0, J_1 \) are Bessel functions, and eq. (9c) becomes the Lieb-Wu result for the ground state energy [13].

For \( \kappa \to 0 \) one may rescale \( \omega = z/\kappa \) which allows to perform the calculation analytically. The final result is \( E/L(U \leq W) = (-W + U)/4 - U^2/(12W) \), \( E/L(U \geq W) = -W^2/(12U) \), with the band-width \( W = 2\pi \), in complete agreement with Ref. [8]. \( E/L \) is monotonically increasing as function of \( U/t \) and shows no singular behavior as function of \( \kappa \).

The pseudo particle dispersions become

\[
\epsilon_s(\lambda) = \int_0^{\infty} \frac{d\omega}{\cosh(\omega U/4)} \left[ \cos(\omega \lambda) \tilde{\lambda}_1^{(s)}(\omega) + \sin(\omega \lambda) \tilde{\lambda}_1^{(s)}(\omega) \right] 
\]

\[
\epsilon_c(k) = \epsilon(k) + 2 \int_0^{\infty} \frac{d\omega}{1 + e^{\omega U/2}} \left[ \cos[\omega H(k)] \tilde{\lambda}_1^{(s)}(\omega) + \sin[\omega H(k)] \tilde{\lambda}_1^{(s)}(\omega) \right] 
\]

The gap \( \Delta \mu = \mu(n \to 1^+) - \mu(n \to 1^-) = U - 2\mu(n \to 1^-) \) is given by \( \Delta \mu = U - 2\text{Max}(\epsilon_c(k)) \).

For large \( U/W \) one finds \( \Delta \mu(U \gg W) = U - W + O(W^2/U) \). The analytical structure of \( \Delta \mu \) is very similar for all \( \kappa > 0 \) such that the physics cannot be different from \( \kappa = \infty \) where \( U_c = 0^+ \). Consequently, \( U_c(\kappa > 0) = 0^+ \), and the gap is exponentially small for \( U \ll W \). The situation changes for \( \kappa = 0 \). Explicitly, \( \epsilon_c(k, \kappa = 0) = \left[ 2k + U - \sqrt{U^2 + W^2 - 2WUk/\pi} \right]/4 \) such that \( \Delta \mu = 0 \) for \( U \leq W \), and \( \Delta \mu = U - W \) for \( U \geq W \), i.e., \( U_c(\kappa = 0) = W \), in agreement with Ref. [8] and the discussion above.

The ground state energy for strong coupling and half-filling can be cast into the form

\( (2\sinh(k)A = W/2; \psi(x) \) is the digamma function)
\[ \frac{E}{L} = -\frac{1}{U} \int_{-A}^{A} \frac{dx}{2\pi} e(x) \int_{-A}^{A} dx' \rho(x) \text{Re} \left[ \psi \left( \frac{1}{2} + i \frac{x - x'}{2} \right) - \psi \left( 1 + i \frac{x - x'}{2} \right) \right] \]  

\[ e(x) = 2 \sinh(\kappa) \left[ \eta \left( \epsilon_+^{-1}(2x \sinh(\kappa)) \right) - \eta \left( \epsilon_-^{-1}(2x \sinh(\kappa)) \right) \right] \]

\[ \rho(x) = \frac{\sinh(\kappa)}{\pi} \left[ \frac{1}{\epsilon' \left( \epsilon_+^{-1}(2x \sinh(\kappa)) \right)} - \frac{1}{\epsilon' \left( \epsilon_-^{-1}(2x \sinh(\kappa)) \right)} \right]. \]

Here, \( k_\pm = \epsilon_\pm^{-1}(y) \) are the two solutions of \( \epsilon(k) = y \) with \( \epsilon'(k_+) > 0, \epsilon'(k_-) < 0 \). Numerical results for \( \kappa = \infty; 1; 0.1; 0 \) are \( \lim_{U \to \infty} (UE/L) = -4 \ln 2 \approx -2.773; -2.826; -3.197; -\pi^2/3 \approx -3.290 \), respectively. The expressions (11) agree with the result obtained by Sutherland et al. [10] which supports our assumptions about integrability of the model (1). They also provide an explicit solution for the integral equations in Ref. [10] for the Inozemtsev model.

We have calculated the critical exponents that control the asymptotic behavior of correlation functions by finite size scaling in conformal theory [15]. The final formulae generalize those of Ref. [15] to the case of different velocities for right- and left-moving electrons [16].

In sum, we are confident that the model (1) is integrable and can thus be solved using the Asymptotic Bethe Ansatz. A complete construction of scattering states at finite densities is still lacking even for the \( 1/\sinh^2(\kappa r) \)-Heisenberg model [8]. Models with variable-range exchange thus remain an interesting problem in mathematical physics.

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