Stable and unstable regimes in Bose-Fermi mixture with attraction between components

A.M. Belemuk and V.N. Ryzhov
Institute for High Pressure Physics, Russian Academy of Sciences, Troitsk 142190, Moscow Region, Russia

S.-T. Chui
Bartol Research Institute, University of Delaware, Newark, DE 19716

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A collapse of the trapped boson-fermion mixture with the attraction between bosons and fermions is investigated in the framework of the effective Hamiltonian for the Bose system. The properties of the $^{87}$Rb and $^{40}$K mixture are analyzed quantitatively at $T = 0$. We find numerically solutions of modified Gross-Pitaevskii equation which continuously go from stable to unstable branch. We discuss the relation of the onset of collapse with macroscopic properties of the system. A comparison with the case of a Bose condensate of atomic $^7$Li system is given.

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I. INTRODUCTION

Bose-Einstein condensation (BEC) in ultracold atomic gas clouds with repulsive and attractive interatomic interactions [1][2][3] have been the subject of intense theoretical and experimental interest in recent years. Besides the studies using the bosonic atoms, growing interest is focused on the cooling of fermionic atoms [4]. Cooling of trapped fermionic atoms to a temperature regime where a Fermi gas can be considered as degenerate has been possible by sympathetic cooling in the presence of a second boson or fermion component. Quantum degeneracy was first reached with mixtures of bosonic $^7$Li and fermionic $^6$Li atoms [5][6]. Later, experiments to cool mixtures of $^{23}$Na and $^6$Li [7], as well as $^{87}$Rb and $^{40}$K [8][9], to ultralow temperatures succeeded. The Bose gas, which can be cooled evaporatively, is used as a coolant, the fermionic system being in thermal equilibrium with the cold Bose gas through boson-fermion interaction in the region of overlapping of the systems.

Collisional interaction between bosons and fermions greatly affect the properties of the degenerate mixture. Theoretical considerations predict the phenomenon of component separation for systems with a positive coupling constant [10], instabilities and significant modification of the properties of individual component in a case of boson-fermion attraction [11][12]. Effect of density fluctuations in a Bose condensate on the fermion-fermion interaction with relevant implications for the achievement of fermionic superfluidity has been investigated in [13]. The presence of a sufficiently attractive boson-fermion interaction can bring about the formation of stable fermionic bright solitons [14].

The simultaneous collapse of the two species has been observed in a $^{40}$K$-^{87}$Rb mixture by Modugno and co-workers [8]. K and Rb atoms were prepared in doubly polarized states $|F = 9/2, m_F = 9/2>$ and $|2, 2>$, respectively. As was shown, as the number of bosons is increased there is an instability value $N_{Bc}$ at which a discontinues leakage of the bosons and fermions occurs, and collapse of boson and fermion clouds is observed. The collapse was found for the following critical numbers of bosons and fermions: $N_{Bc} \approx 10^5$; $N_K \approx 2 \times 10^4$. For investigating the stability diagram of the K-Rb system the mean-field model of Gross-Pitaevskii (GP) type, coupled to the Thomas-Fermi equation for fermions has been used [11][15][16]. The ground state and the stability of the system against collapse was considered with an imaginary-time evolution scheme. The signature of the instability is the failure of the convergence procedure toward the ground state of the system, characterizing by an indefinite growth of the central density [16].

In this paper we study the instability and collapses of the trapped boson-fermion mixture due to the boson-fermion attractive interaction, using the effective Hamiltonian for the Bose system [17][18]. The effective Hamiltonian incorporates the three-particle elastic collisions induced by the boson-fermion interaction. We analyze quantitatively properties of the $^{87}$Rb and $^{40}$K mixture with an attractive interaction between bosons and fermions at $T = 0$. The stability of this system on the basis variational condensate wave function was studied in [18], and good agreement with experiment [8] was found. We estimate the instability boson number $N_{Bc}$ for the collapse transition by numerical calculation of the modified GP equation and give a comparison with the similar picture in a single Bose-condensate with attractive interaction. Our instability analysis will involve dependences of the chemical potential $\mu$ and the number of boson particles $N$ on the value of central density $n_c$ of the Bose-condensate wave function $\phi(r)$. Considerations based on $n_c$-dependence were introduced earlier in Ref. [19] for studying the stability of a Bose condensate of atomic $^7$Li in a magnetic trap at nonzero temperature. The calculations [19] confirmed that the gas becomes mechanically unstable when the free energy of the system as a function of the central density of the gas approaches a maximum value. In our case we present arguments based on the second variation of the energy functional. We exhibit ex-
and consists of an integration over a complex field \( \phi(r) \) that reveal an instability point of energy functional and relate their to a broad class of variations which does not preserve the normalization.

**II. EFFECTIVE BOSE HAMILTONIAN**

First of all we briefly discuss the effective boson Hamiltonian \[17, 18\]. Our starting point is the functional-integral representation of the grand-canonical partition function of the Bose-Fermi mixture. It has the form \[20, 21\]:

\[
Z = \int D[\phi^*]D[\phi]D[\psi^*]D[\psi] \exp \left\{ -\frac{1}{\hbar} (S_B(\phi^*, \phi) + S_F(\psi^*, \psi) + S_{\text{int}}(\phi^*, \phi, \psi^*, \psi)) \right\}
\]

and consists of an integration over a complex field \( \phi(\tau, r) \), which is periodic on the imaginary-time interval \( [0, \hbar \beta] \), and over the Grassmann field \( \psi(\tau, r) \), which is antiperiodic on this interval. Therefore, \( \phi(\tau, r) \) describes the Bose component of the mixture, whereas \( \psi(\tau, r) \) corresponds to the Fermi component. The term describing the Bose gas has the form:

\[
S_B(\phi^*, \phi) = \int_0^{\hbar \beta} d\tau \int dr \left\{ \phi^*(\tau, r) \left( \frac{\partial}{\partial \tau} - \frac{\hbar^2 \nabla^2}{2m_B} + V_B(r) - i\mu_B \right) \phi(\tau, r) + \frac{g_B}{2} |\phi(\tau, r)|^4 \right\}
\]

Because the Pauli principle forbids s-wave scattering between fermionic atoms in the same hyperfine state, the Fermi-gas term can be written in the form:

\[
S_F(\psi^*, \psi) = \int_0^{\hbar \beta} d\tau \int dr \left\{ \psi^*(\tau, r) \left( \frac{\partial}{\partial \tau} - \frac{\hbar^2 \nabla^2}{2m_F} \right) \psi(\tau, r) \right\}
\]

The term describing the interaction between the two components of the Fermi-Bose mixture is:

\[
S_{\text{int}}(\phi^*, \phi, \psi^*, \psi) = g_{BF} \int_0^{\hbar \beta} d\tau \int dr |\psi(\tau, r)|^2 |\phi(\tau, r)|^2,
\]

where \( g_B = 4\pi \hbar^2 a_B/m_B \) and \( g_{BF} = 2\pi \hbar^2 a_{BF}/m_1 \), \( m_1 = m_B m_F/(m_B + m_F) \), \( m_B \) and \( m_F \) are the masses of bosonic and fermionic atoms respectively, \( a_B \) and \( a_{BF} \) are the s-wave scattering lengths of boson-boson and boson-fermion interactions. \( K \) and \( Rb \) atoms in the trap are in the potentials with an elongated symmetry (\( \lambda \)-trap asymmetry parameter)

\[
V_B(r) = \frac{m_B \omega_B^2}{2} (\rho^2 + \lambda z^2), \quad V_{BF}(r) = \frac{m_F \omega_F^2}{2} (\rho^2 + \lambda z^2)
\]

The trap parameters \( \omega_B \) and \( \omega_F \) are chosen in such a way that \( m_B \omega_B^2/2 = m_F \omega_F^2/2 \), so \( \omega_F = \sqrt{m_B/m_F} \omega_B \). Parameters \( \mu_B \) and \( \mu_F \) are the chemical potentials for the Bose and fermi systems respectively. The chemical potential of an ideal fermi gas in a trap is \( \mu_F = \hbar \omega_F (6 \lambda N_F)^{1/3} \).

The integral over Fermi fields

\[
Z_{\text{F}} = \int D[\psi^*]D[\psi] \exp \left\{ -\frac{1}{\hbar} (S_F(\psi^*, \psi) + S_{\text{int}}(\phi^*, \phi, \psi^*, \psi)) \right\}
\]

is Gaussian, we can calculate this integral and obtain the partition function of the Fermi system as a functional of Bose field \( \phi(\tau, r) \)

\[
Z_{\text{F}} = \exp \left\{ -\frac{1}{\hbar} S_{\text{eff}} \right\}, \quad S_{\text{eff}} = \int_0^{\hbar \beta} d\tau \int dr \left( \frac{\hbar^2}{2m_B} V_B(r) - i\mu_B \right) |\phi(\tau, r)|^2 + \frac{g_{eff}}{2} |\phi(\tau, r)|^4 + \frac{g_{BF}}{3} |\phi(\tau, r)|^6 \right\},
\]

where

\[
V_{\text{eff}} = k_0 \frac{m_B \omega_B^2}{2} r^2, \quad k_0 = (1 - \frac{3}{2} \kappa \mu_F^{1/2} g_{BF}), \quad \kappa = \frac{\sqrt{2m_B^{3/2}}}{3\pi^2 \hbar^3}
\]

\[
g_{\text{eff}} = g_B - \frac{3}{2} \kappa \mu_F^{1/2} 2 g_{BF}, \quad g_{BF} = \frac{3 \kappa}{8 \mu_F^{1/2} g_{BF}}
\]

The first three terms in \[11\] have the conventional Gross-Pitaevskii \[23\] form, and the last term is a result of boson-fermion interaction. The interaction with Fermi gas leads to modification of the trapping potential. For the attractive fermion-boson interaction the system should behave as if it was confined in a magnetic trapping potential with larger frequencies than the actual ones, in agreement with experiment \[5\]. Boson-fermion interaction also induces the additional attraction between Bose atoms which does not depend on the sign of \( g_{BF} \). The last term in \( H_{\text{eff}} \) \[11\] corresponds to the three-particle elastic collisions induced by the boson-fermion interaction. In contrast with inelastic 3-body collisions which result in the recombination and removing particles from the system \[24, 25\], this term for \( g_{BF} < 0 \) leads to increase of the gas density in the center of the trap in order to lower the total energy.

**III. NUMERICAL PROCEDURE**

To simplify the formalism we introduce dimensionless variables for the spatial coordinate, the energy, and the
wave function as
\[ r = a_\perp r', \quad E = \hbar \omega_\perp E', \quad \phi(r) = \frac{1}{\sqrt{a_\perp^2}} \phi'(r) \]

where the typical length and energy of the harmonic external potential are \( a_\perp = \sqrt{\hbar/m_B \omega_\perp}, \quad \hbar \omega_\perp = \hbar \omega_B. \)

Then the effective Hamiltonian takes the form (the primes omitted)
\[
H_{\text{eff}} = \int \left\{ \frac{1}{2} \nabla \phi^2 + \left( k_0 \phi^2 + \frac{\lambda z^2}{2} - \mu_B \right) |\phi|^2 + \frac{u}{4} |\phi|^4 + \frac{v}{6} |\phi|^6 \right\} d^3r \tag{2}
\]

where we introduced dimensionless parameters \( u = 2 g_{\text{eff}}/a_\perp^3 \hbar \omega_\perp \) and \( v = 2 g_{\text{eff}}^B/a_\perp^6 \hbar \omega_\perp. \) The wave function \( \phi \) is normalized to the number of atoms in the condensate \( \int d^3r |\phi(r)|^2 = N. \) In the \( T \to 0 \) limit considered, \( N \) coincides with the total number of bosonic atoms in the trap. The explicit form of the ground-state wave function is obtained by minimizing the energy functional. The first order variation of the energy functional gives the modified Gross- Pitaevskii equation
\[
\left( -\frac{\nabla^2}{2} + k_0 \phi^2 + \frac{z^2}{2} - \mu_B + \frac{u}{2} |\phi|^2 + \frac{v}{2} |\phi|^4 \right) \phi = 0 \tag{3}
\]

The parameters of the \(^{87}\text{Rb}\) and \(^{40}\text{K}\) mixture with an attractive interaction between the bosons and the fermions are the following \(^8\): \( a_B = 5.25 \text{ nm}, \ a_{\text{BF}} = -21.7^{+4.3}_{-3.8} \text{ nm}. \) The magnetic potential had an elongated symmetry, with harmonic oscillation frequencies for Rb atoms \( \omega_\perp = \omega_B = 2\pi \times 215 \text{ Hz} \) and \( \omega_{Bz} = \lambda \omega_B = 2\pi \times 163 \text{ Hz}. \) At these parameter values characteristic length \( a_\perp = 735 \text{ nm}, \) chemical potential for fermions \( \mu_F \approx 31 \hbar \omega_B, \ \omega_F \approx 1.47 \omega_B, \ k_0 = 1.07, \ u = 0.11, \ v = -0.0003. \) Because of a small deviation \( k_0 \) from unity we from now on put \( k_0 = 1. \) Note also that we look for the ground state of Eq. (3), i.e. the function \( \phi(r) \) can be treated as real one.

To clarify the main features of instabilities of the system we consider isotropic picture when a problem can be considered effectively as a one-dimensional. The case of non-spherical symmetry of the trap is recovered at final stage by multiplying the critical number of bosons \( N_c \) by the reverse trap asymmetry ratio \( 1/\lambda. \)

So we get the equation \( (\mu = \mu_B)\):
\[
\Delta \phi = (r^2 - 2\mu + u \phi^2 + v \phi^4) \phi, \tag{4}
\]

Solutions of Eq. (4) will be compared to those for the single component Bose condensate with attractive interactions. As an example of Bose system with attractive interaction we choose \(^7\text{Li} \(^{26}\). The s-wave scattering length is \( a = -27.3 a_0, \) where \( a_0 \) is the Bohr radius. The transverse frequency is \( \omega_\perp/2\pi = 163 \text{ Hz}, \) so the corresponding characteristic length is \( a_\perp = 2.97 \times 10^{-3} \text{ cm} \) and \( u = 8\pi a/a_\perp = -0.012. \)

\[
\begin{align*}
\text{FIG. 1: Evolution of the profile of the bosonic condensate wave function } \phi(r) \text{ with increasing the central density } n_c'.
\end{align*}
\]

It is convenient to look for the numeric solutions of Eq. (4) introducing the new parameter: central density \( n_c = \phi^2(0). \) Numerical integration of Eq. (4) with boundary conditions
\[
\phi'(0) = 0, \quad \phi(r) \to 0, \quad r \to \infty
\]
defines the family of solutions \( \phi(r, n_c) \) depending on central density, the chemical potential \( \mu(n_c) \) being also a function of \( n_c. \) Such an approach when one considers \( n_c \) as an input parameter except for \( \mu \) enables to find solutions in the region of instability and to go continuously from stable to unstable branch in the parameter space. This approach differs from an imaginary-time scheme \(^{27}\), where the stability is indicated by requiring the convergence procedure to the final value. Solving Eq. (4), one can easily estimate the effective energy \( E_{\text{eff}} \) corresponding to the functional \( (2) \) and the ground state energy \( E = E_{\text{eff}} + \mu N, \) both as functions of central density \( n_c. \)

Derivative \( d\phi(r, n_c)/dn_c \) is of especial interest, because it determines a change in the number of particle through the variation in the central density \( n_c \)
\[
\frac{dN}{dn_c} = 2 \int d^3r \phi(r, n_c) \frac{d\phi(r, n_c)}{dn_c}
\]

If at some \( n_c \) there comes about \( dN/dn_c = 0 \) it results in the appearance of zero mode in density fluctuations and the onset of instability \(^{19}\).

It is convenient to consider \( \phi(r, n_c) \) and \( d\phi(r, n_c)/dn_c \) as functions of rescaled central density parameter \( n'_c \) = \( |u| n_c. \) The results are plotted in Fig. 1 and 2 where \( \phi(r, n_c) \) is supposed to be normalized to 1, and distance \( r \) is given in units of \( a_\perp. \) Figure 1 shows the evolution of the profile of the condensate wave function with increasing the central density. For comparison, the solution for the isotropic harmonic oscillator, \( \phi_{ho} = \pi^{-3/4} \exp(-r^2/2), \)
which corresponds to the ground state of the ideal Bose gas ($n = 0, \nu = 0$), is also shown. For $n'_c \lesssim 20$, one sees the behavior characteristic of the Bose gas with repulsion, namely the cloud density becomes more flat at the trap center, with increasing radius of the boson cloud. For $n'_c \gtrsim 20$, the solution changes qualitatively: the central density begins to increase. Figure 2 shows the evolution of the derivative $d\phi(r, n_c)/dn_c$ for $^7$Li-system with attractive interaction ($u < 0$) (the left panel) and for BF mixture (the right panel). The behavior of BF mixture at relatively high densities ($n'_c \gtrsim 20$) has similar features with $^7$Li-system. Then $n'_c$ increases there is a continuous change of the shape of function $d\phi/dn_c$. It acquires a negative minimum at $r \lesssim a_\perp$, which results in a saturation and a maximum in $N(n_c)$ dependence.

To relate conditions for stability of a system towards small changes in its density profile with thermodynamic functions let us consider the total energy $E$ as a functional of the condensate wave function $\phi(r)$ and its gradient $\nabla \phi(r)$

$$E = \int d^3r \mathcal{E}(\phi(r), \nabla \phi(r))$$

The first order variation $\delta E$ should be considered with the constraint

$$\delta N = 0, \quad N = \int d^3r |\phi(r)|^2$$

As usual we broaden the class of allowable variation using the Lagrange procedure with multiplier $\mu$

$$E_{\text{eff}} = E - \mu N, \quad \delta E_{\text{eff}} = 0$$

Effective energy density contains one more variable $\mu$ $E_{\text{eff}} = \mathcal{E}_{\text{eff}}(\phi(r), \nabla \phi(r), \mu)$ and functional $E_{\text{eff}}$ coincides with the effective Hamiltonian (2). At $T = 0$ functional $E$ is nothing but the free energy of the system, and $E_{\text{eff}}$ is the thermodynamic potential $\Omega = F - \mu N$.

Consider functionals $E$ and $E_{\text{eff}}$ taking it on the solution of Eq. (1) and on particular class of variations

$$\phi(r) = \phi(r, n_c), \quad \delta \phi = \frac{d\phi(r, n_c)}{dn_c}$$

Then they become a function of central density $E = E(n_c), E_{\text{eff}} = E_{\text{eff}}(n_c)$. The first order variations $\delta E(\phi, \delta \phi)$ and $\delta E_{\text{eff}}(\phi, \delta \phi, \mu)$ considering on functions (1) are nothing but the first derivatives of functions $E(n_c)$ and $E_{\text{eff}}(n_c)$. Due to equalities

$$\frac{\delta E}{\delta \phi(r)} = 2\mu \phi, \quad \frac{\delta E_{\text{eff}}}{\delta \phi(r)} = 0, \quad \frac{\partial E_{\text{eff}}}{\partial \mu} = -\phi^2$$

there are simple relations (8)

$$\frac{dE}{dn_c} = \int d^3r \frac{\delta E}{\delta \phi(r)} \frac{d\phi}{dn_c} = \mu(n_c) \frac{dN}{dn_c}$$

$$\frac{dE_{\text{eff}}}{dn_c} = \int d^3r \left\{ \frac{\delta E_{\text{eff}}}{\delta \phi(r)} \frac{d\phi}{dn_c} + \frac{\partial E_{\text{eff}}}{\partial \mu} \frac{d\mu}{dn_c} \right\} = -\frac{d\mu}{dn_c} N(n_c)$$

which relates extremum points of $E(n_c)$ and $E_{\text{eff}}(n_c)$ with an extremum of $\mu(n_c)$ and $N(n_c)$. Note that variations (8) do not satisfy constraint (9), which holds only when $dN/dn_c = 0$. It means that (8) forms a broader class of variations and include those of which do not conserve the number of particle.

Now we relate the behavior of $\mu(n_c)$ and $N(n_c)$ with the second order variation $\delta^2 E_{\text{eff}}(\phi, \delta \phi, \nabla \delta \phi, \mu)$ taking it on functions (1). It is related with $d^2 E_{\text{eff}}/dn_c^2$ through the equality

$$\frac{d^2 E_{\text{eff}}}{dn_c^2} = \delta^2 E_{\text{eff}}(n_c) +$$

$$+ \int d^3r \left( 2 \frac{\partial^2 \mathcal{E}_{\text{eff}}}{\partial \phi \partial \mu} \frac{d\phi(r, n_c)}{dn_c} \frac{d\mu}{dn_c} + \frac{\partial \mathcal{E}_{\text{eff}}}{\partial \mu} \frac{d^2 \mu}{dn_c^2} \right) d^3r$$

Taking into account that $\partial^2 \mathcal{E}_{\text{eff}}/\partial \phi \partial \mu = -2\phi$ we obtain a simple relation

$$\delta^2 E_{\text{eff}}(n_c) = \frac{d\mu(n_c)}{dn_c} \frac{dN}{dn_c}$$

Eq. (9) shows that there is a simple connection between $\delta^2 E_{\text{eff}}(n_c)$ (taken on a particular class of variation) and the behavior of $\mu(n_c)$ and $N(n_c)$. At the point of instability of the system, where $dN/dn_c = 0$, the second variation $\delta^2 E_{\text{eff}}(n_c)$ is equal to zero. As for the second variation of functional $E$, one can write an equality

$$\delta^2 E(n_c) = \frac{d\mu(n_c)}{dn_c} \frac{dN}{dn_c} + 2\mu(n_c) \int \left( \frac{d\phi(r, n_c)}{dn_c} \right)^2 d^3r$$

which involves an additional term.
IV. RESULTS AND DISCUSSION

To compare the qualitative behavior and properties near collapse transition of $^7$Li- system with BF mixture, we have calculated functions $\mu(n'_c)$, $N(n'_c)$ and $E(n'_c)$ shown at left panels in Fig. 3 and Fig. 4, respectively. Right panels show the dependencies $\mu$, $E$ and $E_{\text{eff}}$ on the number of particle. Function $\mu(n'_c)$ for $^7$Li system (Fig. 3) gives no sign of singularities near collapse transition. The same can be said about $E_{\text{eff}}(n'_c)$ due to Eq. 11. However, there is a common feature in behavior $^7$Li and BF mixture, namely extremums of $N(n'_c)$ and $E(n'_c)$. For system with attraction such a point is at $n'_c \approx 12$, and for BF mixture at $n'_c \approx 23$. This value of central density corresponds to the onset of instability of the system towards collapse. This feature was recognized in Ref. 19 and connected with the presence of zero mode fluctuation of density at this point.

Note, that the extremum for $^7$Li is very wide in $n'_c$. Points of extremum in $N(n_c)$ and $\mu(n'_c)$ is connected due to Eqs. 11 and 13 with extremum of $E(n'_c)$ and $E_{\text{eff}}(n'_c)$. Functions $\mu(N)$, $E(N)$, and $E_{\text{eff}}(N)$ have forms characteristic of multi-value behavior. Curves with asterisks are those parts of $\mu$, $E$ and $E_{\text{eff}}$ which lie at $n'_c < n'_c(0)$ and with triangles are those which lie at $n'_c > n'_c(0)$.

At $n'_c \lesssim 23$ behavior of BF mixture is similar to those of Bose gas with repulsion. Numerical results show $28, 29$ that the density profile $n(r)$ can be accurately described in the framework of Thomas-Fermi (TF) approximation up to $n'_c \sim 20$. In this region the positive zero point energy and boson-boson repulsion energy (the first two terms in Eq. 4) stabilize the system. However, if the central density grows too much, the kinetic energy and boson-boson repulsion are no longer able to prevent the collapse of the gas. We see similar behavior of the system with attraction and BF mixture at $n'_c > 23$. Likewise the case of Bose condensate with attraction (see, for example, $23, 24, 25$), the collapse is expected to occur when the number of particles in the condensate exceeds the critical value $N_{BC}$. For BF mixture curves $\mu(n)$, $E(N)$ and $E_{\text{eff}}(N)$ have a point of termination which corresponds to the maximum number.
of particle \( N_{cr} \sim 6000 \). Taking into account the asymmetry parameter \( \lambda \approx 0.076 \) we obtain \( N_{cr} \sim 10^5 \) which is in good agreement with the experiment \[26\]. A small difference in \( E_{eff} \) for stable and unstable branches arises solely from a very small difference in chemical potentials of these states and not connected with computational accuracy. The small difference in chemical potentials of this branches (\( \mu(N) \)- curves with asterisks and triangles in Fig. 3) is due to a small value of the three-body interaction term \( \nu \).

To find the disappearance of the local minimum of functional \( E_{eff} \) which points to the instability of the system, we should explore the second order variation \( \delta^2 E_{eff} \). \( \delta^2 E_{eff} \) changes a sign from positive to negative one at the point of instability. In terms of the steepest descent method absence of local minimum implies that the convergence towards the local minimum falls down. The second order variation \( \delta^2 E_{eff} \) is given by the quadratic form on \( \delta \phi \) and \( \nabla \delta \phi \) and for the functional \[29\] has the form (for our purposes it is sufficient to consider only real \( \delta \phi \))

\[
\delta^2 E_{eff} = \int \{ (r^2 - 2\mu + 3u\phi^2 + 5v(\phi^2)) (\delta \phi)^2 + (\nabla \delta \phi)^2 \} \, d^3r
\]

Numerical calculations show that \( \delta^2 E_{eff} > 0 \) on solution \( \phi = \phi(r, n_c) \) if we take \( \delta \phi \) as a gaussian, satisfying condition \[30\]. This implies that an extremum of the Hamiltonian is a local minimum. In the case of Bose condensate with attraction the existence of the barrier around the metastable state was confirmed in Ref. \[30\] by extensive variational studies of the nearby wave-function.

At the stable branch \( (n'_c < n'^c_{at}) \) the value of \( d\mu/dN \) is negative for \(^7\)Li system and is positive for BF mixture. In a homogeneous one-component system \( (N/V = const) \)

\[
d\mu/dN = \partial \mu/\partial \rho = 1/\rho^2 \chi_T (\rho = m |\phi|^2 \text{ is the mass density}, \chi_T \text{ is the isothermal compressibility of the system})
\]

and the criterion of thermodynamic stability \( \chi_T > 0 \) reduces to the requirement that \( d\mu/dN \) should be positive. It is easily generalized for an inhomogeneous system which can be treated in the framework of local density approximation. In the local density approximation the density profile \( n(r) = |\phi|^2 \) depends on \( N \) as a parameter and monotonically expands with increasing of particle number. So the density \( n(r, N) \) undergoes a steady increase: \( dn(N, N)/dN > 0 \) at any point \( r \) within a radius of external potential \( V_{ext}(r) \). A density profile is determined from the equation

\[
\mu_{loc}(n, N)) + V_{ext}(r) = \mu(N)
\]

So, in the local density approximation criterium of local stability \( \partial \mu_{loc}/\partial n > 0 \) through the relation \( d\mu/dN = (\partial \mu_{loc}/\partial n) (dn/dN) \) gives the stability condition \( d\mu/dN > 0 \) for an inhomogeneous system in external potential.

BF mixture at \( 7 \lesssim n'_c \lesssim 20 \) safely can be considered in the TF approximation (Fig. 5) \[29\]. In this case the second variation \( \delta^2 E_{eff} \) changes

\[
\phi^2_{TF}(r) = n_{cr} \left[ 1 - \sqrt{1 - \frac{R^2 - r^2}{R_{cr}^2}} \right], \quad R^2 = 2\mu \quad (10)
\]

where \( R_{cr} = u^2/(4|v|), \ n_{cr} = u/(2|v|), \ R \leq R_{cr} \) and is considered at interval \( 0 \leq r \leq R \). The evolution of the profile corresponds to a monotonic expanding of the boson cloud with increasing number of bosons. That is why the stable branch of BF mixture corresponds to the positive value of \( d\mu/dN \).

In conclusion, our analysis of the stability of K-RB Fermi-Bose mixture on the basis of effective Bose Hamiltonian shows the clear resemblance to the behavior of \(^7\)Li system. There is a value of central density at which small variations of density profile conserve the number of particle \( \delta N = 0 \) and the second variation \( \delta^2 E_{eff} \) changes...
the sign. The value we determine for \( N_c \) is in very good accordance with experiment. Points of extremum of functions \( \mu(n_c) \) and \( N(n_c) \) is related with the first and the second derivatives of functions \( E(n_c) \) and \( E_{\text{eff}}(n_c) \).

We note that the investigation of the actual dynamics after the system has been driven into the unstable region would require a description that go beyond the stationary scenario of Eq. 11, in similar fashion to what happens during the collapse of a single Bose-Einstein condensate with attractive interaction [24, 25]. Here we will not discussed these aspects, since we are concerned with the determination of the critical values for the onset of instability. Another interesting issue concerns the relevance of finite temperature effects, which are not included in the present treatment.

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