On Symmetric Sets of Projectors for Reconstruction of a Density Matrix

Alexander Yu. Vlasov

Abstract
In this work are presented sets of projectors for reconstruction of a density matrix for an arbitrary mixed state of a quantum system with the finite-dimensional Hilbert space. It was discussed earlier [1] a construction with $(2^n - 1)n$ projectors for the dimension $n$. For $n = 2$ it is a set with six projectors associated with eigenvectors of three Pauli matrices, but for $n > 2$ the construction produces not such a ‘regular’ set. In this paper are revisited some results of previous work [1] and discussed another, more symmetric construction with the Weyl matrix pair (as the generalization of Pauli matrices). In the particular case of prime $n$ it is the mutually unbiased set with $(n+1)n$ projectors. In appendix is shown an example of application of complete sets for discussions about separability and random robustness.

1 Introduction
Let us consider [1] the $n$-dimensional Hilbert space $\mathcal{H}_n$ and a density matrix $\hat{\rho} \in \mathcal{H}_n^* \otimes \mathcal{H}_n$ of a quantum system. Let we have also a set of $N$ elements $|v_\alpha\rangle \in \mathcal{H}_n$ and the set $S_N(\mathcal{H}_n)$ of projectors $\hat{P}_\alpha = P(v_\alpha) = |v_\alpha\rangle\langle v_\alpha|$. Each projector here may describe a probability of an outcome of some measurement

$$p_\alpha = \text{Tr}(\hat{P}_\alpha \hat{\rho}), \quad \alpha = 1, \ldots, N$$

(1.1)

and so $N$ projectors via Eq. (1.1) produce some formal linear map

$$\hat{L}_S : \mathcal{F}(n) \to \mathbb{R}^N$$

(1.2)

from $n^2$-dimensional real space $\mathcal{F}(n)$ of $n \times n$ Hermitian matrices $\hat{\rho}$ to the formal vector of probabilities $p_\alpha$ defined by Eq. (1.1).

It shold be mentioned, that a physical density matrix also must have the trace one and to be non-negative definite, but because the space of such matrices is not linear, it is useful to introduce some constructions for the linear space of Hermitian matrices, e.g., it is clear, that existence of an inverse map to $\hat{L}_S$ is the sufficient condition for reconstruction of the density matrix.

2 Preliminaries
2.1 Classification
Let us recollect a classification of sets $S_N(\mathcal{H}_n)$ suggested in [1]:

- **Representative**: exists right inverse of $\hat{L}_S$, i.e., it is possible to reconstruct the Hermitian matrix using the formal vector of probabilities $\mathbb{R}^N$.

- **Minimal representative**: exists ‘usual’ inverse of $\hat{L}_S$, i.e., the representative set with $N = n^2$ elements and an isomorphism $\hat{L}_S^{-1} : \mathbb{R}^{n^2} \to \mathcal{F}(n)$.

- **Affine minimal**: it is possible to use the set with $N = n^2 - 1$ and the condition $\text{Tr}(\hat{\rho}) = 1$ to reconstruct a density matrix, e.g., it is the minimal representative set without one vector.

---

1 A Hermitian matrix $H$ may be described by $n^2$ real parameters: $n$ real elements $H_{kk}$ and $n(n-1)/2$ pairs of elements $\{\text{Re}(H_{kl}), \text{Im}(H_{kl}); k > l\}$.
• **Complete:** $S_N(H_n)$ is representative and may be constructed by using a union of few orthogonal bases in $H_n$.

• **Almost perfect:** $S_{m,n}(H_n)$ is the complete set constructed by using a disjoint union of $m$ orthogonal bases in $H_n$, i.e., $N = mn$.

• **Perfect:** the set is almost perfect and each element is orthogonal only to $n-1$ vectors from its own basis.

It is useful to introduce in the present paper two new kinds of complete sets:

• **Mutually unbiased:** the perfect set constructed as a union of mutually unbiased bases (MUB) $\mathcal{E}$, viz, for any two elements of different bases we have $|\langle v|w \rangle|^2 = 1/n$.

• **Symmetric:** the almost perfect set with the possibility of a transformation between any two bases using symmetries of $S_{m,n}(H_n)$, i.e., the unitary automorphisms of the corresponding set of vectors $|v\rangle$, $\alpha = 1, \ldots, mn$.

2.2 Some previous results

Let us mention few other facts proved in [1]:

**Theorem 2.1** Three properties are equivalent:

1. A set of projectors $S_N(H_n)$ is representative.

2. Any complex $n \times n$ matrix may be represented as a linear combination of the projectors $\hat{P}(v\alpha)$, $\alpha = 1, \ldots, N$ with complex coefficients.

3. Any Hermitian $n \times n$ matrix may be represented as a linear combination of the projectors $\hat{P}(v\alpha)$, $\alpha = 1, \ldots, N$ with real coefficients.

One consequence of the **Theorem 2.1** is [1]:

**Lemma 2.1** An (almost) perfect set may not be minimal, becasue it must have at least $N = n^2 + n$ elements.

**Proof:** Due to the **Theorem 2.1** we must have the dimension of the linear span of $\hat{P}_\alpha$ not less than $n^2$. For the (almost) perfect set with $m$ different bases the dimension is also not bigger than $mn - m + 1$ due to $m$ different presentation of same element, the unit matrix, as sum of all $n$ projectors for a basis. So

$m \geq n + 1$ (i.e., $mn - m + 1 \geq n^2$) and $N = n^2 + n$ is the lower limit for a perfect set. □

**Note:** This limit is satisfied for mutually unbiased sets discussed below.

The complex decomposition mentioned in the **Theorem 2.1** let us use a convenient non-Hermitian basis like the set of $n^2$ matrices $\hat{E}(kl)$ with one unit in a cell $(k,l)$, i.e., $\langle \hat{E}(kl) \rangle_{ij} = \delta_{ki}\delta_{lj}$ or, less formally, $\hat{E}(kl) = |k\rangle \langle l|$. Decomposition in such a basis is simpler, but due to the **Theorem 2.1** may be used in proofs of representativity as well. E.g., the coefficients of the decomposition may be simply found, if to introduce some scalar product on the linear space of the matrices, say

$$\langle \hat{A}, \hat{B} \rangle_* = \text{Tr}(\hat{A}^* \hat{B}).$$

(2.1)

This product is also in agreement with Eq. (1.1), $p_\alpha = \langle \hat{P}_\alpha, \hat{\rho} \rangle_*$. Of course, it is always possible to produce a Hermitian basis with the property $\langle H_\alpha, H_\beta \rangle_* = \delta_{\alpha\beta}$ by application of the standard Gram-Schmidt procedure, but the simpler non-Hermitian basis $\hat{E}(kl)$ already has the property of orthogonality$^2$ in the matrix norm Eq. (2.1)

$$\langle \hat{E}(kl), \hat{E}(ij) \rangle_* = \text{Tr}(|k\rangle \langle l| |j\rangle \langle i|) = (|l\rangle \langle i|)(|i\rangle \langle k|) = \delta_{ki}\delta_{lj}. $$

The $\hat{E}(kl)$ basis has also other useful property: tensor products of such matrices from bases in dimensions $n$ and $l$ produce a basis in dimension $nl$ with $(nl)^2$ matrices of same kind (viz, with only one nonzero element, the unit in some cell). Such a property is usefull for other important theorem [1]:

**Theorem 2.2** (Composition theorem)

Let $S_N(H_n)$ and $S_P(H_p)$ are representative (complete, almost perfect) sets with $N$ and $P$ vectors for two Hilbert spaces with dimensions $n$ and $p$. Then a set

$^2$Here $(kl)$ may be considered as a multi-index, or ‘linearized’ as $\alpha = (k-1)n + l.$
$S_{NP}(\mathcal{H}_n \otimes \mathcal{H}_p)$ based on $NP$ tensor products of elements from both sets is, respectively, representative (complete, almost perfect) set for the composite system, but such a tensor product of perfect sets is only almost perfect.

Note: It should be added, that a tensor product of symmetric sets is also symmetric, because a tensor product of two symmetries, i.e., linear transformations between two bases, is again the symmetry, but a tensor product of mutually unbiased sets is not mutually unbiased, because there are orthogonal vectors in different bases.

### 2.3 Examples of the sets

Examples of representative and complete sets were presented in \([1]\). Let us choose a basis $|k\rangle$, $k = 1, \ldots, n$ of the Hilbert space, then a representative set with $n^2$ projectors may be constructed using $n + n(n - 1)/2 + n(n - 1)/2 = n^2$ vectors presented below \([1] [2]\):

| $n$ vectors of the basis: | $|k\rangle$, | $\frac{1}{\sqrt{2}}(|k\rangle + |l\rangle)$, $k < l$, | $\frac{1}{\sqrt{2}}(|k\rangle + i|l\rangle)$, $k < l$. |
| --- | --- | --- | --- |
| $\frac{n(n-1)}{2}$ vectors: | $|k\rangle$, | $\frac{1}{\sqrt{2}}(|k\rangle - |l\rangle)$, $k < l$, | $\frac{1}{\sqrt{2}}(|k\rangle - i|l\rangle)$, $k < l$. |
| $\frac{n(n-1)}{2}$ vectors: | $|k\rangle$, | $\frac{1}{\sqrt{2}}(|k\rangle - |l\rangle)$, $k < l$, | $\frac{1}{\sqrt{2}}(|k\rangle - i|l\rangle)$, $k < l$. |

The representative set is not complete, but it is possible to use yet another $n(n - 1)$ vectors: to produce a complete set with $(2n - 1)n$ projectors

The set is complete, because it is the union of $n^2 - n + 1$ bases:

- An initial basis: $|k\rangle$, $k = 1, \ldots, n$.
- $n(n - 1)/2$ bases produced by substitution of two elements in initial basis: $\frac{1}{\sqrt{2}}(|k\rangle + |l\rangle)$ and $\frac{1}{\sqrt{2}}(|k\rangle - |l\rangle)$ instead of $|k\rangle$ and $|l\rangle$.
- $n(n - 1)/2$ bases produced by substitution: $\frac{1}{\sqrt{2}}(|k\rangle + i|l\rangle)$ and $\frac{1}{\sqrt{2}}(|k\rangle - i|l\rangle)$ instead of $|k\rangle$ and $|l\rangle$.

Certainly, this complete set is not a disjoint union for $n > 2$, because here exist different bases with $n - 2$ common elements.

### 3 Symmetric complete sets of projectors

The example above contains $(2n - 1)n$ elements, but only for $n = 2$ it is a perfect and symmetric set. In such a case there are three bases with two vectors corresponding eigenvectors of three Pauli matrices

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. $$

Similar approach may be used in a higher dimension due to auxiliary lemmas suggested below.

#### 3.1 Auxiliary lemmas

Let us consider an unitary matrix $\hat{M}$ and the orthonormal basis produced by eigenvectors $|\mu_k\rangle$, $k = 1, \ldots, n$

$$\hat{M}|\mu_k\rangle = \lambda_k|\mu_k\rangle, \quad |\mu_k\rangle|\mu_j\rangle = \delta_{ij}. $$

Orthogonal projectors associated with such a matrix are defined as

$$\hat{P}_{\hat{M},k} \equiv |\mu_k\rangle\langle\mu_k|. $$

**Lemma 3.1** Any power $\hat{M}^p$ of the matrix Eq. \((3.1)\) may be expressed as a linear combination of associated projectors $\hat{P}_{\hat{M},k}$ Eq. \((3.2)\).

**Proof:**

$$\hat{M}^p = \hat{M}^p \sum_{k=1}^{n} |\mu_k\rangle\langle\mu_k| = \sum_{k=1}^{n} \lambda_k^p|\mu_k\rangle\langle\mu_k| = \sum_{k=1}^{n} \lambda_k^p \hat{P}_{\hat{M},k} \tag{3.2} $$

**Lemma 3.2** Let we have some set of unitary matrices $\hat{M}_{(i)}$ and any complex matrix may be expressed as a sum of powers $\hat{M}_{(i)}^p$ with complex coefficients\(^3\), then it is possible to use orthonormal bases of eigenvectors of $\hat{M}_{(i)}$ for construction of a complete set of projectors.

\(^3\)The power zero, $\hat{M}_{(i)}^0 = 1$ is also taken into account here.
Proof: Due to the Lemma 3.1 we have the decomposition of any complex matrix using eigenvectors, due to the Theorem 2.1 the set is representative and it is the union of orthogonal bases, i.e., complete. □

E.g., it is true for Pauli matrices because together with the unit they are basis of complex 2×2 matrices.

3.2 Weyl pair

An analogue of such unitary basis in a higher dimension \( n \) is \( n^2 \) matrices \( \hat{U}^k \hat{V}^l, k,l = 0,\ldots,n-1 \), where \( \hat{U}, \hat{V} \) is the Weyl pair 5.

\[
\hat{U} = \begin{pmatrix}
0 & 1 & \cdots & 0 \\
0 & 0 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{pmatrix}, \quad \hat{V} = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & \zeta & \cdots & 0 \\
0 & 0 & \cdots & \zeta^2 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \zeta^{n-1}
\end{pmatrix}.
\]

(where \( \zeta = \exp \frac{2\pi i}{n} \)), but the Lemma 3.2 does not use products of powers. It is necessary to choose a minimal set of \( \hat{U}^a \hat{V}^b, 0 \leq a, b < n \) with same eigenvectors as the whole set.

Lemma 3.3 (Discrete version of von Neumann uniqueness theorem) Let \( \hat{A}, \hat{B} \in SU(n) \),

\[
\hat{A}\hat{B} = \zeta^j \hat{B}\hat{A}, \quad \gcd(j,n) = 1,
\]

(viz. \( j \) and \( n \) are coprime) then exists a unitary transformation \( \hat{S} \):

\[
\hat{S}\hat{A}\hat{S}^{-1} = \hat{U}^j, \quad \hat{S}\hat{B}\hat{S}^{-1} = \hat{V}.
\]

Proof: Let us rewrite Eq. 3.4 as

\[
\hat{A}\hat{B}\hat{A}^{-1} = \zeta^j \hat{B},
\]

but \( \hat{A}\hat{B}\hat{A}^{-1} \) and \( \hat{B} \) have same eigenvalues and so due to Eq. 3.4 \( \hat{B} \) and \( \zeta^j \hat{B} \) have same eigenvalues, but it is possible only for a set of \( n \) numbers \( \zeta^j, k = 0,\ldots,n-1 \). If \( \gcd(j,n) = 1 \), this set after some permutation corresponds to the set \( \zeta^k \), \( k = 0,\ldots,n-1 \). It is precisely eigenvalues of \( \hat{V} \) and so diagonalization of \( \hat{B} \) by transition \( \hat{S} \) to the basis of eigenvectors of \( \hat{B} \) is just \( \hat{V} = \hat{S}\hat{B}\hat{S}^{-1} \). In this new basis another matrix is \( \hat{A}' = \hat{S}\hat{A}\hat{S}^{-1} \) and it is clear from Eq. 3.4, that \( \hat{A}' \) is the cyclic shift of eigenvectors of \( \hat{V} \) with ‘step’ \( j \), i.e., \( \hat{A}' = \hat{U}^j \) and so Eq. 3.4 holds. □

Two particular examples of \( \gcd(j,n) = 1 \) are \( \{j = 1, \forall n\} \) (see 6 for more details with this particular example) and \( \{\forall j, n \text{ is prime}\} \) (it is a main application in this article).

It should be mentioned, that if \( \gcd(j,n) = k, n = kl \), then matrices are reducible, with \( k \)-dimensional subspaces corresponding to equal eigenvalues, e.g., it is tensor products like \( \hat{V}_l \otimes \mathbf{1}_k \).

3.3 Constructions of symmetric sets

3.3.1 Prime dimension

Theorem 3.1 Let \( n \) is a prime number, then eigenvectors of \( n+1 \) matrices: \( \hat{U}, \hat{U}^m \hat{V}, m = 0,\ldots,n-1 \) produce a complete set of projectors.

Proof: It can be written for \( m = 1,\ldots,n-1: (\hat{U}^m \hat{V})^l = \alpha(\hat{U}^m \hat{V}^l) \) with complex \( \alpha \) due to properties of the Weyl pair: \( \hat{U}\hat{V} = \zeta\hat{V}\hat{U}, \hat{U}^n = \hat{V}^n = 1 \). The equation \( ml \pmod{n} = k \) for any \( k,l \) always has some solution \( m \), if \( n \) prime, because in such a case arithmetic modulo \( n \) is field. So any matrix \( \hat{U}^k \hat{V}^l, k,l = 1,\ldots,n-1 \) may be presented as a power of \( \hat{U}^m \hat{V} \) and together with powers of \( \hat{U} \) and \( \hat{V} \) it is any matrix \( \hat{U}^k \hat{V}^l \), \( k,l = 0,\ldots,n-1 \), but it is the basis. So any complex matrix may be expressed as a sum of powers of \( n+1 \) matrices suggested above with complex coefficients and due to the Lemma 3.2 it is possible to use eigenvectors of the matrices for construction of a complete set of projectors. □

Theorem 3.2 The complete set of projectors described above in the Theorem 3.1 is also perfect, symmetric and mutually unbiased.

Proof: Any mutually unbiased set is perfect by definition. Let us prove, that our set is mutually unbiased 6 and symmetric. Any two different matrices \( \hat{A}, \hat{B} \) considered in the Theorem 3.1 has property \( \hat{A}\hat{B} = \zeta^j \hat{B}\hat{A} \) for some \( j = 1,\ldots,n-1 \) and the Lemma 3.3 let us consider a new basis, there \( \hat{A} \) and \( \hat{B} \) may be rewritten as \( \hat{U}^j \) and \( \hat{V} \). Elements of this new basis are eigenvectors \( |b_k\rangle \) of \( \hat{B} \). In the basis \( \hat{A} \) is cyclic \( j \)-shift, and eigenvectors of \( \hat{A} \) may be written as \( |a_k\rangle = \sum_{l=0}^{n-1} \zeta^{jl} |b_l\rangle / \sqrt{n} \), i.e., \( |\langle a_k | b_m \rangle| = 1/\sqrt{n} \).
\(\forall k, m\). It was considered an arbitrary pair \(\hat{A}, \hat{B}\) between \(n + 1\) matrices, viz, all bases are mutually unbiased (see also [6]).

Let us prove now, that the set is symmetric. If to show, that exists a symmetry between any basis and eigenvectors of \(V\) (i.e., the initial basis of the Hilbert space)\(^4\) then a symmetry of any two bases may be expressed via two such transformations as \(T_1 T_2^{-1}\). Due to the Lemma 3.3 (with \(A = \hat{V}, B = \hat{U}\)) exists transformation \(\hat{U} \mapsto \hat{V}, \hat{V} \mapsto \hat{U}^{-1}\) (it is discrete Fourier transform). It is the symmetry, because maps \(\hat{U} k \hat{V}^l \mapsto \hat{V} k \hat{U}^{-l} = \hat{V} k \hat{U}^{n-l}\), and so it is an automorphism for the set of operators \(\hat{U}k\hat{V}^l\) and they eigenvectors. Due to the Lemma 3.3 (with \(A = \hat{U}, B = \hat{V}^k\)) exists a transformation \(\hat{V} \hat{U} k \mapsto \hat{V} k \hat{U}^{-k}\) and it is also the symmetry. Here are \(n\) transformations \(T_j\) to the canonical basis of \(\hat{V}\), and \(T_j T_k^{-1}\) is the symmetry between two arbitrary bases. \(\square\)

### 3.3.2 Non-prime dimension

If the dimension is not prime, it is possible to use a tensor product of few symmetric sets to construct a symmetric set in the composite dimension due to the Note after the Theorem 2.2. It was also mentioned, that a tensor product of unbiased sets is not unbiased, but it is always almost perfect, as a product of perfect sets. Really mutually unbiased bases exist not only in prime dimensions, but for any power \(p^l\) of prime \(p\). The research of application of such bases for construction of a complete set is an interesting problem, but it is outside of the scope of present work.

It should be mentioned, that \(\hat{U} k \hat{V}^l\) are an unitary basis in an arbitrary dimension, and so bases of eigenvectors may be always used for construction of a complete set due to the Lemma 3.2 (applied to the ‘power’ \(p = 1\)). But only for prime dimension \(n + 1\) operators introduced in the Theorem 3.1 have all properties necessary for constructions used above.

If the dimension is not prime, there are following problems:

1. The whole set \(\hat{U} k \hat{V}^l\) may not be represented as powers of the \(n + 1\) matrices from the Theorem 3.1.

2. The operators \(\hat{U}, \hat{V}\) have subspaces with equal eigenvalues and so eigenvectors for such subspaces may be presented using arbitrary combinations, viz, not in an unique way.

3. The Lemma 3.3 used for the proof of symmetry does not work if gcd\((j, n) > 1\).

So it is possible to construct a complete set of projectors using products of the Weyl pair in any dimension, but it have less ‘regular’ structure if the dimension is not prime.

Using the tensor product structure with sets of prime dimensions \(p_k\) for \(n = \prod_k p_k\), it is possible to construct almost perfect, symmetric sets with \(m = \prod_k (p_k + 1) > n + 1\), \(N = mn\) elements, but at least for the power of prime, exist mutually unbiased, i.e., perfect sets with \(m = n + 1\) and the smallest size \(N\). Even if for products of different primes, mutually unbiased sets do not exist, may be it is possible using eigenvectors of \(\hat{U} k \hat{V}^l\) at least construct complete, or (almost) perfect, or symmetric set with dimension smaller than \(\prod_k (p_k + 1)\)? Possibly the question devotes further research.

### References

[1] A. Yu. Vlasov, “Probabilities, tensors and qubits,” [arXiv:quant-ph/0104126](arXiv:quant-ph/0104126) (2001).

[2] C. M. Caves, C. A. Fuchs, and R. Schack, “Unknown quantum states: The quantum de Finetti representation,” [arXiv:quant-ph/0104088](arXiv:quant-ph/0104088) (2001).

[3] H. Barnum, “Information-disturbance tradeoff in quantum measurement on the uniform ensemble (and on the mutually unbiased bases),” [arXiv:quant-ph/0205155](arXiv:quant-ph/0205155) (2000).

\(^4\)Of course unitary transformation between two basis always exists, even \(n!\) such transformations, but here is necessary to find a symmetry of the set, e.g., it must maps the set of all vectors to itself.
APPENDIX

A Separability and robustness

To show application of a complete set of projectors, let us demonstrate a proof for analogue of some theorem from \[A1, A2\] for an arbitrary number of finite-dimensional quantum systems.

**Theorem A.1 (ZHSL-BCJLPS')** A density matrix $\hat{\rho}$ of any composite system may be represented as

$$\hat{\rho} = \alpha \hat{\rho}_s - \beta \mathbf{1}, \quad \alpha, \beta \in \mathbb{R}, \quad \alpha > 0, \quad (A.1)$$

where $\hat{\rho}_s$ is a density matrix of a separable state, i.e.,

$$\hat{\rho}_s = \sum_I \alpha_I \hat{\rho}^{(1)}_I \otimes \cdots \otimes \hat{\rho}^{(k)}_I, \quad \alpha_I > 0 \quad (A.2)$$

and each $\hat{\rho}^{(i)}_I$ is a valid density matrix of $i$-th subsystem.

**Proof:** Let us consider a complete set of projectors for each subsystem, then due to the Theorem 2.2 it is possible to construct a set of projectors for the whole system as the tensor product. Using this representative set, due to the Theorem 2.1 it is possible to write any density matrix as

$$\hat{\rho} = \sum_{\alpha} k_\alpha \hat{P}_\alpha, \quad k_\alpha \in \mathbb{R}, \quad (A.3)$$

where not all $k_\alpha$ are necessary positive, but instead of negative terms it is possible to write $k_\alpha \hat{P}_\alpha = k_\alpha \mathbf{1} + (-k_\alpha)(1 - \hat{P}_\alpha)$, and for the complete set $1 - \hat{P}_\alpha$ always may be represented as the sum of other projectors\(^5\) and so Eq. (A.3) may be rewritten as

$$\hat{\rho} = \sum_{\alpha} k'_\alpha \hat{P}_\alpha - k_1 \mathbf{1}, \quad k'_\alpha > 0, \quad (A.4)$$

and because we use construction of the complete set as the tensor product $\hat{P}_\alpha = \hat{P}_{\alpha_1} \otimes \cdots \otimes \hat{P}_{\alpha_k}$, Eq. (A.4) coincides with Eq. (A.1) after substitution Eq. (A.2), and $\hat{P}_{\alpha_i} = |v_{\alpha_i}\rangle \langle v_{\alpha_i}|$ is the valid density matrix (of a pure state). $\square$

This proof for composition of the arbitrary number of systems with the arbitrary (possibly different) finite dimensions — is a generalized analogue of \[A2\] for two-dimensional systems and Pauli matrices. It should be mentioned also, that a minimum of $\beta$ for different $\hat{\rho}_s$ in Eq. (A.1) characterises a measure of separability of quantum systems known as the random robustness \[A3\]. The proof of the Theorem \[A1\] above is constructive, but not necessary provides this minimal value of $\beta$. Does it possible to suggest some optimization strategy using complete sets of projectors? It is yet another interesting problem.

References

[A1] K. Žyczkowski, P. Horodecki, A. Sanpera, and M. Lewenstein, “On the volume of the set of mixed entangled states,” Phys. Rev. A 58, 883 (1998); [arXiv:quant-ph/9804024]

[A2] S. L. Braunstein, C. M. Caves, R. Jozsa, N. Linden, S. Popescu, and R. Schack, “Separability of very noisy mixed states and implications for NMR quantum computing,” Phys. Rev. Lett. 83, 1054 (1999); [arXiv:quant-ph/9811018]

[A3] G. Vidal and R. Tarrach, “Robustness of entanglement,” Phys. Rev. A 59, 141 (1999); [arXiv:quant-ph/9806094] (1998).

\(^5\)Sum of all projectors for the given basis is the unit and for a complete set each elements belongs to some basis.