An Upper Bound on the Minimum Weight of Type II \( \mathbb{Z}_{2k} \)-Codes

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Abstract

In this paper, we give a new upper bound on the minimum Euclidean weight of Type II \( \mathbb{Z}_{2k} \)-codes and the concept of extremality for the Euclidean weights when \( k = 3, 4, 5, 6 \). Together with the known result, we demonstrate that there is an extremal Type II \( \mathbb{Z}_{2k} \)-code of length \( 8m \) (\( m \leq 8 \)) when \( k = 3, 4, 5, 6 \).

Key Words: Type II code, Euclidean weight, extremal code, theta series

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1 Introduction

Let \( \mathbb{Z}_{2k} \) be the ring of integers modulo \( 2k \), where \( k \) is a positive integer. In this paper, we take the set of elements of \( \mathbb{Z}_{2k} \) to be either \( \{0, 1, \ldots, 2k-1\} \) or \( \{0, \pm 1, \ldots, \pm(k-1), k\} \). A \( \mathbb{Z}_{2k} \)-code \( C \) of length \( n \) (or a code \( C \) of length \( n \) over \( \mathbb{Z}_{2k} \)) is a \( \mathbb{Z}_{2k} \)-submodule of \( \mathbb{Z}_{2k}^n \). The Euclidean weight of a codeword \( x = (x_1, x_2, \ldots, x_n) \) is \( \sum_{i=1}^{n} \min\{x_i^2, (2k - x_i)^2\} \). The minimum Euclidean weight \( d_E(C) \) of \( C \) is the smallest Euclidean weight among all nonzero codewords of \( C \).

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A $\mathbb{Z}_{2k}$-code $C$ is self-dual if $C = C^\perp$ where the dual code $C^\perp$ of $C$ is defined as $C^\perp = \{ x \in \mathbb{Z}_{2k}^n \mid x \cdot y = 0 \text{ for all } y \in C \}$ under the standard inner product $x \cdot y$. As described in [14], self-dual codes are an important class of linear codes for both theoretical and practical reasons. It is a fundamental problem to classify self-dual codes of modest lengths and determine the largest minimum weight among self-dual codes of that length.

A binary doubly even self-dual code is often called Type II. For $\mathbb{Z}_4$-codes, Type II codes were first defined in [2] as self-dual codes containing a $(\pm 1)$-vector and with the property that all Euclidean weights are divisible by eight. Then it was shown in [11] that, more generally, the condition of containing a $(\pm 1)$-vector is redundant. For general $k$, Type II $\mathbb{Z}_{2k}$-codes were defined in [1] as a self-dual code with the property that all Euclidean weights are divisible by $4k$. It is known that a Type II $\mathbb{Z}_{2k}$-code of length $n$ exists if and only if $n$ is divisible by eight [1].

The aim of this paper is to show the following theorem.

**Theorem 1.** Let $C$ be a Type II $\mathbb{Z}_{2k}$-code of length $n$. If $k \leq 6$ then the minimum Euclidean weight $d_E(C)$ of $C$ is bounded by

$$d_E(C) \leq 4k \left\lfloor \frac{n}{24} \right\rfloor + 4k.$$  \hspace{1cm} (1)

**Remark 2.** The upper bound (1) is known for the cases $k = 1$ [13] and $k = 2$ [2]. For $k \geq 3$, the bound (1) is known under the assumption that $\lfloor n/24 \rfloor \leq k - 2$ [1].

We say that a Type II $\mathbb{Z}_{2k}$-code meeting the bound (1) with equality is extremal for $k \leq 6$. For the following cases

$$(k, m) = (4, 7), (4, 8), (5, 6), (5, 7), (5, 8), (6, 4), (6, 5), (6, 6), (6, 7) \text{ and } (6, 8),$$

an extremal Type II $\mathbb{Z}_{2k}$-code of length $8m$ is constructed for the first time in Section 4. Together with the known results on the existences of extremal Type II codes, we have the following theorem.

**Theorem 3.** If $k \leq 6$ then there is an extremal Type II $\mathbb{Z}_{2k}$-code of length $8m$ for $m \leq 8$.

The existences of a binary extremal Type II code of length 72 and a 72-dimensional extremal even unimodular (Type II) lattice are long-standing open questions. In this paper, we propose the following new question.

**Question.** Is there an extremal Type II $\mathbb{Z}_{2k}$-code of length 72 for $k \leq 6$?

We remark that if there is an Type II $\mathbb{Z}_{2k}$-code of length 72 ($k = 4, 5, 6$) then a 72-dimensional extremal even unimodular lattice can be obtained by Construction A.

All computer calculations in this paper were done by Magma [3].
2 Preliminaries

An \( n \)-dimensional (Euclidean) lattice \( \Lambda \) is a subset of \( \mathbb{R}^n \) with the property that there exists a basis \( \{e_1, e_2, \ldots, e_n\} \) of \( \mathbb{R}^n \) such that \( \Lambda = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \cdots \oplus \mathbb{Z}e_n \), i.e., \( \Lambda \) consists of all integral linear combinations of the vectors \( e_1, e_2, \ldots, e_n \). The dual lattice \( \Lambda^* \) of \( \Lambda \) is the lattice \( \{x \in \mathbb{R}^n \mid \langle x, y \rangle \in \mathbb{Z} \text{ for all } y \in \Lambda \} \) where \( \langle x, y \rangle \) is the standard inner product. A lattice with \( \Lambda = \Lambda^* \) is called unimodular. The norm of a vector \( x \) is \( \langle x, x \rangle \). A unimodular lattice with even norms is said to be even. A unimodular lattice containing a vector of odd norm is said to be odd. An \( n \)-dimensional even unimodular lattice exists if and only if \( n \equiv 0 \pmod{8} \) while an odd unimodular lattice exists for every dimension. The minimum norm \( \min(\Lambda) \) of \( \Lambda \) is the smallest norm among all nonzero vectors of \( \Lambda \). For \( \Lambda \) and a positive integer \( m \), the shell \( \Lambda_m \) of norm \( m \) is defined as \( \{x \in \Lambda \mid \langle x, x \rangle = m \} \). Two lattices \( \Lambda \) and \( \Lambda' \) are isomorphic, denoted \( \Lambda \cong \Lambda' \), if there exists an orthogonal matrix \( A \) with \( \Lambda' = \Lambda \cdot A = \{xA \mid x \in \Lambda \} \). Two lattices \( \Lambda \) and \( \Lambda' \) are neighbors if both lattices contain a sublattice of index 2 in common.

The theta series \( \Theta_\Lambda(q) \) of \( \Lambda \) is the following formal power series

\[
\Theta_\Lambda(q) = \sum_{x \in \Lambda} q^{\langle x, x \rangle} = \sum_{m=0}^{\infty} |\Lambda_m| q^m.
\]

For example, when \( \Lambda \) is the \( E_8 \)-lattice

\[
\Theta_\Lambda(q) = E_4(q) = 1 + 240 \sum_{m=1}^{\infty} \sigma_3(m) q^{2m} = 1 + 240q^2 + 2160q^4 + 6720q^6 + \cdots,
\]

where \( \sigma_3(m) \) is a divisor function \( \sigma_3(m) = \sum_{0 < d \mid m} d^3 \). Moreover the following theorem is known (see [6, Chap. 7]).

**Theorem 4.** If \( \Lambda \) is an even unimodular lattice then

\[
\Theta_\Lambda(q) \in \mathbb{C}[E_4(q), \Delta_{24}(q)],
\]

where \( \Delta_{24}(q) = q^2 \prod_{m=1}^{\infty} (1 - q^{2m})^{24} \).

We now give a method to construct even unimodular lattices from Type II codes, which is called Construction A [1]. Let \( \rho \) be a map from \( \mathbb{Z}_{2k} \) to \( \mathbb{Z} \) sending \( 0, 1, \ldots, k \) to \( 0, 1, \ldots, k \) and \( k + 1, \ldots, 2k - 1 \) to \( 1 - k, \ldots, -1 \), respectively. If \( C \) is a self-dual \( \mathbb{Z}_{2k} \)-code of length \( n \), then the lattice

\[
A_{2k}(C) = \frac{1}{\sqrt{2k}} \{\rho(C) + 2k\mathbb{Z}^n\}
\]
is an \( n \)-dimensional unimodular lattice, where
\[
\rho(C) = \{(\rho(c_1), \ldots, \rho(c_n)) \mid (c_1, \ldots, c_n) \in C\}.
\]
The minimum norm of \( A_{2k}(C) \) is \( \min\{2k, d_E(C)/2k\} \). Moreover, if \( C \) is Type II then the lattice \( A_{2k}(C) \) is an even unimodular lattice \( \mathbb{I} \).

The symmetrized weight enumerator of a \( \mathbb{Z}_{2k} \)-code \( C \) is
\[
\text{swe}_C(x_0, x_1, \ldots, x_k) = \sum_{c \in C} x_0^{n_0(c)} x_1^{n_1(c)} \cdots x_{k-1}^{n_{k-1}(c)} x_k^{n_k(c)},
\]
where \( n_0(c), n_1(c), \ldots, n_{k-1}(c), n_k(c) \) are the numbers of 0, \( \pm 1, \ldots, \pm (k-1), k \) components of \( c \), respectively \( \mathbb{I} \). Then the theta series of \( A_{2k}(C) \) can be found by replacing \( x_1, x_2, \ldots, x_k \) by
\[
f_0 = \sum_{x \in 2k\mathbb{Z}} q^{x^2/2k}, f_1 = \sum_{x \in 2k\mathbb{Z}+1} q^{x^2/2k}, \ldots, f_k = \sum_{x \in 2k\mathbb{Z}+k} q^{x^2/2k},
\]
respectively.

### 3 Proof of Theorem \( \mathbb{I} \)

In this section, we give a proof of Theorem \( \mathbb{I} \). Our proof is an analogue of that of \( [2, \text{Corollary 13}] \) (see also \( [12] \)). We remark that in the proof of \( [2, \text{Corollary 13}] \) (\( \Delta/E_3^8 \)4) (p. 973, right, l. -7) should be \( (tE_3^4/\Delta) \) and \( (4\mathbb{Z})^8/2 \) (p. 973, right, l. -5) should be \( 2\mathbb{Z}^8 \).

**Proof.** Let \( C \) be a Type II \( \mathbb{Z}_{2k} \)-code of length \( n \). Then the even unimodular lattice \( A_{2k}(C) \) contains the sublattice \( \Lambda_0 = \sqrt{2k}\mathbb{Z}^n \) which has minimum norm \( 2k \). We set \( \Theta_{\Lambda_0}(q) = \theta_0, n = 8j \) and \( j = 3\mu + \nu \) (\( \nu = 0, 1, 2 \)), that is, \( \mu = \lfloor n/24 \rfloor \). In this proof, we denote \( E_4(q) \) and \( \Delta_{24}(q) \) by \( E_4 \) and \( \Delta \), respectively. By Theorem \( \mathbb{I} \) the theta series of \( A_{2k}(C) \) can be written as
\[
\Theta_{A_{2k}(C)}(q) = \sum_{s=0}^{\mu} a_s E_4^j \Delta^s = \sum_{r \geq 0} |A_{2k}(C)_r| q^r = \theta_0 + \sum_{r \geq 1} \beta_r q^r.
\]

Suppose that \( d_E(C) \geq 4k(\mu+1) \). We remark that a codeword of Euclidean weight \( 4km \) gives a vector of norm \( 2m \) in \( A_{2k}(C) \). Then we choose the \( a_0, a_1, \ldots, a_\mu \) so that
\[
\Theta_{A_{2k}(C)}(q) = \theta_0 + \sum_{r \geq 2(\mu+1)} \beta_r q^r.
\]
Here, we set \( b_{2s} \) as \( E_4^{-j} \theta_0 = \sum_{s=0}^{\infty} b_{2s}(\Delta/E_4^3)^s \). That is, \( \theta_0 = \sum_{s=0}^{\infty} b_{2s} E_4^{-j-3s} \Delta^s \).

Then
\[
\sum_{s=0}^{\mu} a_s E_4^{-j-3s} \Delta^s = \Theta_{A_{2k}(C)}(q) = \sum_{s=0}^{\infty} b_{2s}(\Delta/E_4^3)^s + \sum_{r \geq 2(\mu+1)} \beta_r^* q^r.
\]

Comparing the coefficients of \( q^i \) (\( 0 \leq i \leq 2\mu \)), we get \( a_s = b_{2s} \) (\( 0 \leq s \leq \mu \)). Hence we have
\[
- \sum_{r \geq (\mu+1)} b_{2s} E_4^{-j-3r} \Delta^r = \sum_{r \geq 2(\mu+1)} \beta_r^* q^r.
\]

In (2), comparing the coefficient of \( q^{2(\mu+1)} \), we have
\[
\beta_{2(\mu+1)}^* = -b_{2(\mu+1)}.
\]

All the series are in \( q^2 = t \), and Bürman’s formula shows that
\[
b_{2s} = \frac{1}{s!} \frac{d^{s-1}}{dt^{s-1}} \left( \frac{d}{dt} (E_4^{-j} \theta_0) \right) (tE_4^3/\Delta)^s \right|_{\{t=0\}}.
\]

Using the fact that \( \theta_0 = \theta_1^* \) where \( \theta_1 \) is the theta series of the lattice \( \sqrt{2k} \mathbb{Z}^8 \),
\[
b_{2s} = -j \frac{d^{s-1}}{dt^{s-1}} \left( E_4^{-3s-j-1} \theta_1^{-1} (\theta_1 E'_4 - \theta'_1 E_4) (t/\Delta)^s \right) \right|_{\{t=0\}},
\]

where \( f' \) is the derivation of \( f \) with respect to \( t = q^2 \).

The condition that there is a codeword of Euclidean weight \( 4k(\mu + 1) \) is equivalent to the condition \( \beta_{2(\mu+1)}^* > 0 \). It is sufficient to show that the coefficients of \( \theta_1^{-1} (\theta_1 E'_4 - \theta'_1 E_4) \) are positive up to the exponent \( \mu \) since \( E_4 \) and \( 1/\Delta \) have positive coefficients.

By Proposition 3.4 in [1], there exists a Type II \( \mathbb{Z}_{2k} \)-code of length 8 for every \( k \). Hence let \( C_8 \) be a Type II \( \mathbb{Z}_{2k} \)-code of length 8. Then \( A_{2k}(C_8) \) is the \( E_8 \)-lattice. In addition, we can write
\[
E_4 = \text{swe}_C (f_0, f_1, \ldots, f_k) \text{ and } \theta_1 = f_0^8.
\]

Deriving
\[
E_4/\theta_1 = \text{swe}_C (1, f_1/f_0, \ldots, f_k/f_0),
\]
we find
\[
\theta_1^{-1} (\theta_1 E'_4 - \theta'_1 E_4) = \frac{\partial \text{swe}_C (f_0, f_1, \ldots, f_k)}{\partial x_1} f_0^{8j-1} (f_0 f'_1 - f'_0 f_1) + \cdots + \frac{\partial \text{swe}_C (f_0, f_1, \ldots, f_k)}{\partial x_k} f_0^{8j-1} (f_0 f'_k - f'_0 f_k).
\]
Hence it is sufficient to show that \( f_0^{8j-1}(f_0 f'_1 - f'_0 f_1), \ldots, f_0^{8j-1}(f_0 f'_k - f'_0 f_k) \) have positive coefficients up to \( \mu \). We only consider the case \( f_0^{8j-1}(f_0 f'_1 - f'_0 f_1) \) and the other cases are similar. We have that
\[
t(f_0 f'_1 - f'_0 f_1) = \sum_{x,y \in \mathbb{Z}} \frac{(1 + 2ky)^2 - (2kx)^2}{4k} t^{((1+2ky)^2 + (2kx)^2)/4k},
\]
then
\[
f_0^s(f_0 f'_1 - f'_0 f_1) = \sum_{x,y,x_1,\ldots,x_s \in \mathbb{Z}} \frac{(1 + 2ky)^2 - (2kx)^2}{4k} t^{((1+2ky)^2 + (2kx)^2 + (2kx_1)^2 + \cdots + (2kx_s)^2)/4k}. \tag{3}
\]

Fix one of the choices \( y, x, x_1, \ldots x_s \in \mathbb{Z} \) and define \( l \) as follows:
\[
l = (1 + 2ky)^2 + (2kx)^2 + (2kx_1)^2 + \cdots + (2kx_s)^2. \tag{4}
\]
Consider all permutations on the set \( \{x, x_1, \ldots, x_s\} \). As the sum of coefficients of \( t^{l/4k} \) in the right hand side of (3) under these cases, we have that some positive constant multiple by
\[
\frac{(s + 1)(1 + 2ky)^2 - (2kx)^2 - (2kx_1)^2 - \cdots - (2kx_s)^2}{4k} = \frac{(s + 2)(1 + 2ky)^2 - l}{4k}. \tag{5}
\]
If \( l < s + 2 \) then (5) is positive. Since we consider the case \( s = 8j - 1, l < n + 1 \). Hence if the exponent \( l/4k \) of \( t \) is less than \( (n + 1)/4k \) then (5) is positive. This means that if \( \mu < (n + 1)/4k \) then (5) is positive. This condition \( \mu < (n + 1)/4k \) is satisfied since \( k \leq 6 \). Thus for any choice \( y, x, x_1, \ldots x_s \), (5) is positive. The coefficient of \( t^{l/4k} \) in the right hand side of (3) is the sum of those coefficients (5), that is, positive. This completes the proof of Theorem 1.

\[ \square \]

4 Extremal Type II \( \mathbb{Z}_{2k} \)-codes

An extremal Type II \( \mathbb{Z}_{2k} \)-code of length \( 8m \) is currently known for the cases \( (k, m) \) listed in the second column in Table I. In this section, an extremal Type II \( \mathbb{Z}_{2k} \)-code of length \( 8m \) is constructed for the first time for the cases \( (k, m) \) listed in the last column in Table I.
Table 1: Existence of extremal Type II $\mathbb{Z}_{2k}$-codes of length $8m$

| $k$ | $m$ (known cases) | $m$ (new cases) |
|-----|------------------|-----------------|
| 1   | 1, 2, . . . , 8, 10, 11, 13, 14, 17 | 6, p. 194, [8]  |
| 2   | 1, 2, . . . , 8 | 7, 8 (Proposition 5, C_{10,56}, C_{10,64}) |
| 3   | 1, 2, . . . , 8 | 6, 7, 8 (Proposition 8, C_{10,56}, C_{10,64}) |
| 4   | 1, 2, . . . , 6 | 7, 8 (C_{8,56}, C_{8,64}) |
| 5   | 1, 2, . . . , 5 | 6, 7, 8 (Proposition 8, C_{10,56}, C_{10,64}) |
| 6   | 1, 2, 3 | 4, 5, 6, 7, 8 (Proposition 7) |

Let $A$ and $B$ be $n \times n$ negacirculant matrices, that is, $A$ and $B$ have the following form

$$
\begin{pmatrix}
  r_0 & r_1 & r_2 & \cdots & r_{n-1} \\
  -r_{n-1} & r_0 & r_1 & \cdots & r_{n-2} \\
  -r_{n-2} & -r_{n-1} & r_0 & \cdots & r_{n-3} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  -r_1 & -r_2 & -r_3 & \cdots & r_0
\end{pmatrix}.
$$

If $AA^T + BB^T = -I_n$, then it is trivial that

$$
\begin{pmatrix}
  I_{2n} & A & B \\
  -B^T & A^T
\end{pmatrix}
$$

generates a self-dual code where $I_n$ denotes the identity matrix of order $n$ and $A^T$ is the transpose of $A$.

Table 2: New extremal Type II $\mathbb{Z}_{2k}$-codes

| Codes | $r_A$ | $r_B$ |
|-------|-------|-------|
| C_{8,56} | (0, 0, 4, 3, 4, 1, 6, 3, 1, 1, 1, 1, 1, 2) | (1, 1, 0, 0, 0, 0, 0, 3, 2, 0, 0, 0, 0, 4) |
| C_{8,64} | (0, 0, 0, 2, 0, 7, 3, 2, 0, 0, 5, 3, 1, 4, 0, 2) | (0, 0, 1, 0, 0, 0, 0, 1, 7, 1, 3, 0, 1, 2, 2, 0) |
| C_{10,56} | (0, 0, 0, 2, 5, 1, 4, 1, 2, 0, 5, 0, 4, 1) | (0, 0, 0, 0, 0, 1, 0, 5, 0, 3, 9, 1, 0) |
| C_{10,64} | (0, 0, 4, 3, 2, 0, 0, 1, 9, 0, 0, 9, 1, 2, 0) | (1, 3, 0, 1, 0, 6, 9, 4, 6, 2, 0, 5, 0, 0, 2, 3) |
| C_{12,32} | (0, 0, 7, 6, 0, 1, 7, 10) | (0, 1, 0, 4, 1, 0, 3, 11) |
| C_{12,40} | (0, 0, 0, 2, 1, 0, 5, 9, 2, 10) | (0, 1, 0, 1, 0, 0, 11, 1, 0, 4) |
| C_{12,56} | (2, 1, 1, 2, 4, 11, 0, 5, 0, 0, 6, 1, 5, 7) | (1, 0, 5, 3, 0, 8, 0, 2, 0, 7, 7, 0, 0, 4) |

Using the above construction method, we have found extremal Type II $\mathbb{Z}_8$-codes $C_{8,56}$ and $C_{8,64}$ of lengths 56 and 64, respectively, and extremal Type II $\mathbb{Z}_{10}$-codes $C_{10,56}$ and $C_{10,64}$ of lengths 56 and 64, respectively. The
first rows $r_A$ and $r_B$ of the matrices $A$ and $B$ in their generator matrices (6) are listed in Table 2. Hence we have the following:

**Proposition 5.** For lengths 56 and 64, there is an extremal Type II $\mathbb{Z}_{2k}$-code when $k = 4$ and 5.

An $n$-dimensional even unimodular lattice is called *extremal* if it has minimum norm $2\lfloor n/24 \rfloor + 2$. The existence of an extremal Type II $\mathbb{Z}_{10}$-code of length 48 is established by considering the existence of a 10-frame in some extremal even unimodular lattice. Recall that a set $\{f_1, \ldots, f_n\}$ of $n$ vectors $f_1, \ldots, f_n$ in an $n$-dimensional unimodular lattice $L$ with $\langle f_i, f_j \rangle = \ell \delta_{i,j}$ is called an $\ell$-frame of $L$, where $\delta_{i,j}$ is the Kronecker delta. It is known that an even unimodular lattice $L$ contains a $2k$-frame if and only if there is a Type II $\mathbb{Z}_{2k}$-code $C$ such that $A_{2k}(C) \simeq L$.

**Proposition 6.** There is an extremal Type II $\mathbb{Z}_{10}$-code of length 48.

**Proof.** Let $C_{5,48}$ be the $\mathbb{F}_5$-code with generator matrix (6) where the first rows $r_A$ and $r_B$ of the matrices $A$ and $B$ are

$$r_A = (2, 3, 0, 2, 2, 3, 2, 3, 2, 2, 0) \quad \text{and} \quad r_B = (3, 0, 4, 4, 0, 1, 0, 0, 4, 0, 0, 1),$$

respectively. Then this code $C_{5,48}$ is a self-dual code and the lattice $A_5(C_{5,48}) = \frac{1}{\sqrt{5}} \{ x \in \mathbb{Z}^{48} \mid x \ (\text{mod 5}) \in C_{5,48} \}$ is an odd unimodular lattice. The lattice has theta series $1 + 393216q^5 + 26201600q^6 + \cdots$. We have verified that $A_5(C_{5,48})$ has an even unimodular neighbor $L_{48}$ which is extremal. Clearly the lattice $A_5(C_{5,48})$ contains the 5-frame $\{\sqrt{5}e_1, \sqrt{5}e_2, \ldots, \sqrt{5}e_{48}\}$ where $e_i$ ($i = 1, 2, \ldots, 48$) denotes the $i$-th unit vector $(\delta_{i,1}, \delta_{i,2}, \ldots, \delta_{i,48})$ of length 48. Then the set $F = \{\sqrt{5}(e_{2i-1} \pm e_{2i}) \mid i = 1, 2, \ldots, 24\}$ is a 10-frame of the even sublattice of $A_5(C_{5,48})$. Hence $F$ is also a 10-frame of the extremal even unimodular neighbor $L_{48}$. Therefore there is a Type II $\mathbb{Z}_{10}$-code $C_{10,48}$ of length 48 such that $A_{10}(C_{10,48}) \simeq L_{48}$. Moreover, the code $C_{10,48}$ must be extremal since the lattice $L_{48}$ is extremal.

Similar to the above proposition, the existence of 12-frames in extremal even unimodular lattices yields that of some extremal Type II $\mathbb{Z}_{12}$-codes.

**Proposition 7.** There is an extremal Type II $\mathbb{Z}_{12}$-code of length $8m$ for $m = 4, 5, 6, 7$ and 8.

**Proof.** It is known that there is an extremal Type II $\mathbb{Z}_6$-code of length $8m$ for $m = 4, 5, 6, 7$ and 8 (see Table 1). We denote these codes by $C_{6,8m}$ ($m = 4, 5, 6, 7$ and 8), respectively. Since $C_{6,8m}$ is an extremal Type II code, the lattice $A_6(C_{6,8m})$ is an extremal even unimodular lattice for
\( m = 4, 5, 6, 7 \) and 8. Moreover, clearly the lattice \( A_6(C_{6,8m}) \) contains the 6-frame \( \{ \sqrt{6}e_1, \sqrt{6}e_2, \ldots, \sqrt{6}e_{8m} \} \) where \( e_i \) denotes the \( i \)-th unit vector of length \( 8m \). Then the set \( \{ \sqrt{6}(e_{2i-1} \pm e_{2i}) \mid i = 1, 2, \ldots, 4m \} \) is a 12-frame of \( A_6(C_{6,8m}) \). Hence there is a Type II \( \mathbb{Z}_{12} \)-code \( N_{8m} \) of length \( 8m \) such that \( A_{12}(N_{8m}) \cong A_6(C_{6,8m}) \). Moreover, the code \( N_{8m} \) must be extremal since the lattice \( A_6(C_{6,8m}) \) is extremal. \( \square \)

Remark 8. Similar to Proposition 5, by considering generator matrices (6), we have found extremal Type II \( \mathbb{Z}_{12} \)-codes \( C_{12,32}, C_{12,40} \) and \( C_{12,56} \) of lengths 32, 40 and 56, respectively where the first rows \( r_A \) and \( r_B \) of the matrices \( A \) and \( B \) in (6) are listed in Table 2.

Together with the known results on the existences of extremal Type II codes (see Table 1), Propositions 5, 6 and 7 give Theorem 3.

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