Study on General Riemann Hypothesis

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Abstract It is well known, General Riemann Hypothesis have many equivalent propositions. In this paper, we will prove an equivalent proposition that is related to the prime theorem for arithmetic series. However, the prime theorem for arithmetic series can be represented as the sieve functions. In order to improve the sieve method, we introduced the transformation of sieve function. For this, we obtain some useful lemmas. The difference of similar sieve functions will be determined within a certain range. Furthermore, we proved the equivalent proposition of General Riemann Hypothesis. So we proved General Riemann Hypothesis and Riemann Hypothesis to be true.

Keywords Sieve method · Riemann Hypothesis · General Riemann Hypothesis

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1 Introduction

In 1859, Riemann published a famous paper ‘on the number of primes less than a given magnitude’. [1] In this article, he introduced the zeta function that be defined as a complex series for $Re(s) > 1$,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

(1)
The zeta function has analytic continuation to the whole complex plane, except a simple pole at $s = 1$. In the region $0 < \sigma < 1$, the zeta function has infinite complex zeros, that is called as non-trivial zeros of zeta function.

**Riemann Hypothesis:** all non-trivial zeros of zeta function lie on the line $\text{Re}(s) = \frac{1}{2}$.

Afterwards, Dirichlet introduced the L-function that be defined for $\text{Re}(s) > 1$,

$$L(s, \chi) = \sum_{n=1}^{\infty} \chi(n)n^{-s}$$

where $\chi(n)$ is Dirichlet characteristics. The function $L(s, \chi)$ also has analytic continuation to the whole complex plane, in the region $0 < \sigma < 1$, the function $L(s, \chi)$ has infinite complex zeros, that is called as non-trivial zeros of the function $L(s, \chi)$.

**General Riemann Hypothesis:** all non-trivial zeros of Dirichlet L-function lie on the line $\text{Re}(s) = \frac{1}{2}$.

For $\chi(n) = 1$, $L(s, \chi) = \zeta(s)$, therefore, let General Riemann Hypothesis be true, then Riemann Hypothesis be also true.

It is difficult to directly prove General Riemann Hypothesis. However, General Riemann Hypothesis has many the equivalent propositions. For example, Proposition A

$$\pi(x) = \frac{x}{\log x} + O(\sqrt{x \log x})$$

which proposition is equivalent to Riemann Hypothesis.

Proposition B

$$\pi(x; q, l) = \frac{\pi(x)}{\phi(q)} + O(\sqrt{x \log x}), (q, l) = 1$$

which proposition is equivalent to General Riemann Hypothesis, let the Proposition B be true, then General Riemann Hypothesis be true.

In this paper, we will prove the Proposition B, and then prove General Riemann Hypothesis. We will mainly referred to H.Halberstam, H.E. Richert, ‘Sieve Method, [2], C.D. Pan, C.B. Pan, ‘Basic Analytic Number Theory’, [3] and E. C. Tichmarch’s book ‘ The Theory of the Riemann Zeta-function’ ,[4] and another articles. [5-12]

First, we will introduce the transformation of sieve function to improve sieve method, and then use this method to prove the Proposition B. Many notations will be used in this paper.

**Notations**

- $B = O(A)$, or $B << A$: there be positive constant $c$ such that $|B| \leq cA$, for $A > 0$
- $B \gg A$: there be positive constant $c$ such that $|B| \geq cA$, for $A > 0$
- $\mu(d)$: Möbius function
- $\chi(n)$: Dirichlet characteristics
- $\nu(d)$: the number of different prime divisors of $d$
\( \phi(q) \): Euler function
\( \forall \lambda_i \): all of \( \lambda_i \)

2 The Transformation of Sieve Function

In number theory, let there exist an integer set \( A = \{ a : a \leq N \} \), \( A_d = \{ a : a \mid d, a \in A \} \), then the sieve function be general represented as

\[ S(A; P(z), z) = \sum_{a \in A} \sum_{d \mid (a, P(z))} \mu(d) = \sum_{d \mid (P(z))} \mu(d) |A_d| \] (5)

where

\[ P(z) = \prod_{p \leq z} p \] (6)

\( \mu(d) \) is M"obius function, \( |A_d| \) is size of set \( A_d \).

However, it is difficult to simple calculate the value of sieve function \( S(A; P(z), z) \) in the general case. In order to improve the sieve method, we will introduce transformation of sieve function.

The sieve function can be written as another form

\[ S(A; P(z), z) = |A| + \sum_{1 < d \mid (P(z))} \mu(d) |A_d| \] (7)

let us introduce an integer \( \lambda \), that be called as transformation factor, such that

\[ |A = |A'| \] (8)

and

\[ |A_d | \rightarrow |A'_d | = |A_d | + \lambda, 1 < d \mid q \] (9)

then it have

\[ \sum_{1 < d \mid q} \mu(d)(|A'_d |) = \sum_{1 < d \mid q} \mu(d)(|A_d | + \lambda) = \sum_{1 < d \mid q} \mu(d)|A_d | + \sum_{1 < d \mid q} \mu(d)\lambda \] (10)

because

\[ \sum_{d \mid q} \mu(d)\lambda = 0 \] (11)

\[ \sum_{1 < d \mid q} \mu(d)\lambda = -\lambda \] (12)

so it have

\[ S(A'; P(z), z) = S(A; P(z), z) + \sum_{1 < d \mid q} \mu(d)\lambda = S(A; P(z), z) - \lambda \] (13)

that be called as transformation of sieve function in this paper, and written as

\[ S(A; P(z), z) \rightarrow S(A'; P(z), z) \] (14)
However, for \( S(A; P(z), z) \rightarrow S(A'; P(z), z) \), it have
\[
|A| = |A'|
\]

The sieve function can be transformed as many times, that will form a group of transformations. There are some useful concepts about sieve transformation in this paper.

1. Standard transformation of sieve function: let there be transformation \( S(A; P(z), z) \rightarrow S(A'; P(z), z) \)
\[
S(A'; P(z), z) = S(A; P(z), z) + \sum_{i=1}^{n} \sum_{d | q_i} \mu(d) \lambda_i = S(A; P(z), z) - \sum_{i=1}^{n} \lambda_i
\]
such that the signs of transformation factors \( \lambda_i \) be all same, namely
\[
\forall \lambda_i \leq 0
\]
or
\[
\forall \lambda_i \leq 0
\]
then which transformation be called as standard transformation of sieve function.

2. Identical transformation of sieve function: let there be transformation \( S(A; P(z), z) \rightarrow S(A'; P(z), z) \),
\[
S(A'; P(z), z) = S(A; P(z), z) + \sum_{i=1}^{n} \sum_{d | q_i} \mu(d) \lambda_i = S(A; P(z), z) - \sum_{i=1}^{n} \lambda_i
\]
such that
\[
\sum_{i=1}^{n} \lambda_i = 0
\]
so it have
\[
S(A'; P(z), z) = S(A; P(z), z)
\]
the value of sieve function \( S(A; P(z), z) \) be constant, which be called as identical transformation of sieve function, and be written as \( S(A'; P(z), z) = S(A; P(z), z) \).

3. Special transformation of sieve function: let there be transformation \( S(A; P(z), z) \rightarrow S(A'; P(z), z) \), such that satisfying two conditions
\[
S(A; P(z), z) = S(A'; P(z), z)
\]
\[
\sum_{p \leq z} |A_p| = \sum_{p \leq z} |A'_p|
\]
then transformation \( S(A; P(z), z) \rightarrow S(A'; P(z), z) \) be called as special transformation of sieve function. There are two basic ways of special transformations of sieve functions as follows.
The first way: let us make transformation $S(A; P(z), z) \to S(A'; P(z), z)$, such that
\begin{align*}
|A_d| &\to |A'_d| = |A_d| + \lambda, 1 < d | q_1 \\
|A_d| &\to |A'_d| = |A_d| - \lambda, 1 < d | q_2
\end{align*}
for this, it have
\begin{equation}
S(A; P(z), z) = S(A'; P(z), z)
\end{equation}
\begin{equation}
\sum_{p \leq z} |A'_p| = \sum_{p \leq z} |A_p| + \lambda(\nu(q_1) - \nu(q_2))
\end{equation}
where $\nu(q)$ is the number of different prime factors of integer $q$, let $\nu(q_1)$ and $\nu(q_2)$ satisfy equation
\begin{equation}
\lambda(\nu(q_1) - \nu(q_2)) = 0
\end{equation}
and, $\nu(q_1) = \nu(q_2)$, then it have
\begin{equation}
S(A'; P(z), z) = S(A; P(z), z)
\end{equation}
the transformation $S(A; P(z), z) = S(A'; P(z), z)$ be special transformation of sieve function.

The second way: let us make transformation $S(A; P(z), z) \to S(A''; P(z), z)$, and $S(A'; P(z), z) \to S(A''; P(z), z)$, such that
\begin{align*}
|A_d| &\to |A'_d| = |A_d| + \lambda_1, 1 < d | q_1 \\
|A_d| &\to |A'_d| = |A_d| - \lambda_1, 1 < d | q_3 \\
|A'_d| &\to |A''_d| = |A'_d| + \lambda_2, 1 < d | q_2 \\
|A'_d| &\to |A''_d| = |A'_d| - \lambda_2, 1 < d | q_3
\end{align*}
for this, it have
\begin{equation}
S(A; P(z), z) = S(A'; P(z), z) = S(A''; P(z), z)
\end{equation}
\begin{equation}
\sum_{p \leq z} |A'_p| = \sum_{p \leq z} |A_p| + \lambda_1(\nu(q_1) - \nu(q_3))
\end{equation}
\begin{equation}
\sum_{p \leq z} |A''_p| = \sum_{p \leq z} |A'_p| + \lambda_2(\nu(q_2) - \nu(q_3))
\end{equation}
and
\begin{equation}
\sum_{p \leq z} |A''_p| = \sum_{p \leq z} |A_p| + \lambda_1(\nu(q_1) - \nu(q_3)) + \lambda_2(\nu(q_2) - \nu(q_3))
\end{equation}
let $\nu(q_1)$, $\nu(q_2)$, $\nu(q_3)$ satisfy the equation
\begin{equation}
\lambda_1(\nu(q_1) - \nu(q_3)) + \lambda_2(\nu(q_2) - \nu(q_3)) = 0
\end{equation}
then it have
\[
\sum_{p \leq z} |A_p| = \sum_{p \leq z} |A_p''|
\] (40)

For this, the transformation \( S(A; P(z), z) = S(A''; P(z), z) \) be also special transformation of sieve function.

Of curse, there will be many other ways of special transformation of sieve functions.

3 Lemmas

**Lemma 1:** Let there be transformation \( S(A; P(z), z) \rightarrow S(B; P(z), z) \), then it must have
\[
|A| = |B|
\] (41)

As mentioned above, this is the fundamental property of sieve function transformation.

**Lemma 2:** Let \(|A| = |B|\), then there must be transformation
\[
S(A; P(z), z) \rightarrow S(B; P(z), z)
\] (42)

**Proof** It is clearly that there must be transformations
\[
S(A; P(z), z) \rightarrow |A|
\] (43)
\[
S(B; P(z), z) \rightarrow |B|
\] (44)

and inverse transformations
\[
|A| \rightarrow S(A; P(z), z)
\] (45)
\[
|B| \rightarrow S(B; P(z), z)
\] (46)
in addition \(|A| = |B|\), it must have transformations,
\[
S(A; P(z), z) \rightarrow |A| \rightarrow |B| \rightarrow S(B; P(z), z)
\] (47)
namely, there must be transformation
\[
S(A; P(z), z) \rightarrow S(B; P(z), z)
\] (48)

Of curse, for \(|A| = |B|\), there may be many other ways of transformation \( S(A; P(z), z) \rightarrow S(B; P(z), z) \).

**Lemma 2** be proved.

**Lemma 3:** Let there be transformations \( S(A; P(z), z) \rightarrow S(A''; P(z), z), S(A; P(z), z) \rightarrow S(B; P(z), z) \), then there must be transformation
\[
S(A''; P(z), z) \rightarrow S(B; P(z), z)
\] (49)
Proof Let there be transformations

\[ S(A; P(z), z) \to S(A'; P(z), z) \]  (50)

\[ S(A; P(z), z) \to S(B; P(z), z) \]  (51)

according to Lemma 1, then it have

\[ |A| = |A'| = |B| \]  (52)

for this, according to Lemma 2, there must be transformation

\[ S(A'; P(z), z) \to S(B; P(z), z) \]  (53)

Lemma 3 be proved.

**Lemma 4:** Let there be \( S(A; P(z), z) \to S(B; P(z), z) \) to be standard transformation, then it have

\[ S(A; P(z), z) - S(B; P(z), z) \ll \sum_{p \leq z} ||A_p| - |B_p|| \]  (54)

Proof let \( S(A; P(z), z) \to S(B; P(z), z) \) be standard transformation of sieve function

\[ S(B; P(z), z) = S(A; P(z), z) + \sum_{i=1}^{n} \sum_{1 \leq d | q_i} \mu(d) \lambda_i = S(A; P(z), z) - \sum_{i=1}^{n} \lambda_i \]  (55)

which the signs of transformation factors \( \lambda_i \) be all same,

\[ \forall \lambda_i \leq 0 \]  (56)

or

\[ \forall \lambda_i \geq 0 \]  (57)

then it have

\[ S(A; P(z), z) - S(B; P(z), z) \ll \sum |\lambda_i| \]  (58)

it is clearly, for \( \forall \lambda_i \geq 0 \), or \( \forall \lambda_i \leq 0 \), it have

\[ \sum |\lambda_i| \leq \sum_{p \leq z} ||A_p| - |B_p|| \]  (59)

therefore, for \( \forall \lambda_i \geq 0 \), or \( \forall \lambda_i \leq 0 \), it have

\[ S(A; P(z), z) - S(B; P(z), z) \ll \sum_{p \leq z} ||A_p| - |B_p|| \]  (60)

Lemma 4 be proved.
Lemma 4: Let $|A| = |B|$, then we have
\[ S(A; P(z), z) - S(B; P(z), z) \ll \sum_{p \leq z} ||A_p| - |B_p|| \] (61)

**Proof** According to Lemma 2, let $|A| = |B|$, then there must be transformation $S(A; P(z), z) \rightarrow S(B; P(z), z)$.

Let $S(A; P(z), z) \rightarrow S(B; P(z), z)$ be standard transformation, then by Lemma 4, it have
\[ S(A; P(z), z) - S(B; P(z), z) \ll \sum_{p \leq z} ||A_p| - |B_p|| \] (62)

Let $S(A; P(z), z) \rightarrow S(B; P(z), z)$ be not standard transformation, we can make a series of special transformation
\[ S(A; P(z), z) = S(A_1; P(z), z) = \cdots = S(A^n; P(z), z) \] (63)
such that $S(A^n; P(z), z) \rightarrow S(B; P(z), z)$ be standard transformation.

For example, let $S(A; P(z), z) \rightarrow S(B; P(z), z)$ be not standard transformation, there be two transformation factors,
\[ |B_d| = |A_d| + \lambda_1, 1 < d|q_1|, \] (64)
\[ |B_d| = |A_d| + \lambda_2, 1 < d|q_2|, \] (65)
where $\lambda_1 > 0, \lambda_2 < 0, |\lambda_1| \leq |\lambda_2|$. For $\nu(q_1) = \nu(q_2)$, we can make special transformations $S(A; P(z), z) = S(A'; P(z), z)$ by the first way, such that
\[ |A_d'| = |A_d| + \lambda_1, 1 < d|q_1|, \] (66)
\[ |A_d'| = |A_d| - \lambda_1, 1 < d|q_2|, \] (67)
\[ \sum_{p \leq z} |A_p| = \sum_{p \leq z} |A'_p| \] (68)
however, according to Lemma 3, there will also be transformation $S(A'; P(z), z) \rightarrow S(B; P(z), z)$, and have
\[ |B_d| = |A_d'| + \eta_1 = |A_d'| + \lambda_1, 1 < d|q_1| \] (69)
and
\[ |B_d| = |A_d'| + \eta_2 = |A_d'| + \lambda_1 + \lambda_2 = |A_d| + \lambda_2, 1 < d|q_2| \] (70)
for this, it have
\[ \eta_1 = 0 \] (71)
\[ \eta_2 = \lambda_1 + \lambda_2 \leq 0 \] (72)
namely, \( \eta_1 = 0, \eta_2 \leq 0 \), which become same signs.

For \( \lambda_1 > 0, \lambda_2 < 0, |\lambda_1| \leq |\lambda_2|, \nu(q_1) \neq \nu(q_2) \), we can make special transformations \( S(A; P(z), z) = S(A'; P(z), z) \), and \( S(A'; P(z), z) \) by the second way, such that

\[
|A_d'| = |A_d| + \lambda_1, 1 < d|q_1 \tag{73}
\]

\[
|A_d''| = |A_d| - \lambda_1, 1 < d|q_3 \tag{74}
\]

and

\[
|A_d'''| = |A_d'| + \lambda_2, 1 < d|q_2 \tag{75}
\]

\[
|A_d''''| = |A_d''| - \lambda_2, 1 < d|q_3 \tag{76}
\]

for this, it have

\[
S(A; P(z), z) = S(A'; P(z), z) = S(A'''; P(z), z) \tag{77}
\]

\[
\sum_{p \leq z} |A_p'| = \sum_{p \leq z} |A_p| + \lambda_1 (\nu(q_1) - \nu(q_3)) \tag{78}
\]

\[
\sum_{p \leq z} |A_p''| = \sum_{p \leq z} |A_p| + \lambda_2 (\nu(q_2) - \nu(q_3)) \tag{79}
\]

let \( \nu(q_1), \nu(q_2), \nu(q_3) \) satisfy the equation

\[
\lambda_1 (\nu(q_1) - \nu(q_3)) + \lambda_2 (\nu(q_2) - \nu(q_3)) = 0 \tag{80}
\]

then it have

\[
S(A; P(z), z) = S(A'''; P(z), z) \tag{81}
\]

\[
\sum_{p \leq z} |A_p| = \sum_{p \leq z} |A_p''| \tag{82}
\]

similarly, it also have transformation \( S(A'''; P(z), z) \to S(B; P(z), z) \), that have

\[
|B_d| = |A_d''''| + \eta_1 = |A_d''''|, 1 < d|q_1 \tag{83}
\]

\[
|B_d| = |A_d''''| + \eta_2 = |A_d''''|, 1 < d|q_2 \tag{84}
\]

\[
|B_d| = |A_d''''| + \eta_3 = |A_d''''| + \lambda_1 + \lambda_2, 1 < d|q_3 \tag{85}
\]

for this, it have

\[
\eta_1 = 0 \tag{86}
\]

\[
\eta_2 = 0 \tag{87}
\]

\[
\eta_3 = \lambda_1 + \lambda_2 \leq 0 \tag{88}
\]

namely, \( \eta_1 = 0, \eta_2 = 0, \eta_3 \leq 0 \), which become same signs.

By applying the above ways, we can make a series of special transformation of sieve functions

\[
S(A; P(z), z) = S(A^1; P(z), z) = \cdots = S(A^n; P(z), z) \tag{89}
\]
such that $S(A^n; P(z), z) \rightarrow S(B; P(z), z)$ be standard transformation of sieve function, namely, $\forall \eta_i \leq 0$ or $\forall \eta_i \geq 0$. For this, according to Lemma 4, we have

$$S(B; P(z), z) - S(A^n; P(z), z) < \sum_{p \leq z} |A_n^p| - |B_p| = \sum_{p \leq z} |A_n^p| - \sum_{p \leq z} |B_p|$$

(90)

since $S(A; P(z), z) = S(A^n; P(z), z)$ be special transformation,

$$S(A; P(z), z) = S(A^n; P(z), z)$$

(91)

then it have

$$S(A; P(z), z) - S(B; P(z), z) < \sum_{p \leq z} |A_p| - \sum_{p \leq z} |B_p| = \sum_{p \leq z} |A_p| - |B_p|$$

(93)

Lemma 5 be proved.

**Lemma 6**: Let $|A| - |B| \neq 0$, then we have

$$S(A; P(z), z) - S(B; P(z), z) < |A| - |B| + \sum_{p \leq z} |A_p| - |B_p|$$

(94)

Proof For $|A| - |B| \neq 0$, we can introduce a sieve function $S(A'; P(z), z)$, such that

$$|A'| = |B|$$

(95)

$$|A_d'| = |A_d|, d > 1$$

(96)

then it have

$$S(A; P(z), z) - S(A'; P(z), z) < |A| - |B|$$

(97)

because $|A'| = |B|$, according to Lemma 2, there must be transformation

$$S(A'; P(z), z) \rightarrow S(B; P(z), z)$$

(98)

furthermore, according to Lemma 5, we have

$$S(A'; P(z), z) - S(B; P(z), z) < \sum_{p \leq z} |A'_p| - |B_p| = \sum_{p \leq z} |A_p| - |B_p|$$

(99)

to sum up, we have

$$S(A; P(z), z) - S(B; P(z), z) < |A| - |B| + \sum_{p \leq z} |A_p| - |B_p|$$

(100)

Lemma 6 be proved.
4 Proof of General Riemann Hypothesis

Now, we will prove the Proposition B that be equivalent to General Riemann Hypothesis and can be represented as the following theorem.

**Theorem 1:** Let \((q, l) = 1\), and \(l < q\), then it have

\[
\pi(x; q, l) = \frac{\pi(x)}{\phi(q)} + O(\sqrt{x} \log x) 
\]

\((101)\)

**Proof** In the number theory, the function \(\pi(x; q, l)\) be defined as

\[
\pi(x; q, l) = \sum_{p \equiv l \pmod{q}, p \leq l} 1 
\]

\((102)\)

because \(\pi(x; 2, 1) = \pi(x)\), so we will focus on studying functions \(\pi(x; q, l)\), for \(q > 2\).

However, the primes are distributed among the irreducible residue classes of \(\text{mod}(q)\). The number of irreducible residue classes of \(\text{mod}(q)\) is equal to Euler function

\[
\phi(q) = q \prod_{p | q} \left(1 - \frac{1}{p}\right) 
\]

\((103)\)

According to the prime number theorem of arithmetic series, for \((q, l_i) = 1\), and \(l_i < q\), it have

\[
\pi(x; q, l_i) = \frac{\pi(x)}{\phi(q)} + r_i 
\]

\((104)\)

where \(r_i\) is error term, let the primes be uniformly distributed in the irreducible residue classes of \(\text{mod}(q)\), then it have

\[
\pi(x; q, l_i) = \frac{\pi(x)}{\phi(q)} 
\]

\((105)\)

namely, the mean value of \(\pi(x; q, l_i)\) is equal to \(\frac{\pi(x)}{\phi(q)}\).

The function \(\pi(x; q, l_i)\) can also be represented as a sieve functions. For example, let us set \(A = \{a : n = kq + l, n \leq x\}\), \(A_d = \{a : a \in A, a|P(z)\}\), \((q, l) = 1\), and \(l < q\), then it have

\[
S(A; P(z), \sqrt{x}) = \sum_{d \in \{P(z)\}} \mu(d) |A_d| 
\]

\((106)\)

\[
|A| = \left[\frac{x - l}{q}\right] 
\]

\((107)\)

\[
|A_d| = \left[\frac{x - l}{dq}\right] 
\]

\((108)\)

for this, it have

\[
\pi(x; q, l) = S(A; P(z), \sqrt{x}) + O(\sqrt{x} \log x) 
\]

\((109)\)
Let us set $B = \{b : n = kq + m, n \leq x\}$, $B_d = \{b : b \in B, b|P(z)\}$, $(q, m) = 1$, and $m < q$, then it have

$$S(B; P(z), \sqrt{x}) = \sum_{d|(P(z))} \mu(d)|B_d|$$

(110)

$$|B| = \left\lfloor \frac{x - m}{q} \right\rfloor$$

(111)

$$|B_d| = \left\lfloor \frac{x - m}{dq} \right\rfloor$$

(112)

for this, it have

$$\pi(x; q, m) = S(B; P(z), \sqrt{x}) + O(\sqrt{x \log x})$$

(113)

According to Lemma 6, we have

$$S(A; P(z), \sqrt{x}) - S(B; P(z), \sqrt{x}) << ||A| - |B|| + \sum_{p \leq \sqrt{x}} ||A_p| - |B_p||$$

(114)

for $(q, l) = 1$, $(q, m) = 1$ and $l, m < q$,

$$||A| - |B|| = \left\lfloor \frac{x - l}{q} \right\rfloor - \left\lfloor \frac{x - m}{q} \right\rfloor << 2$$

(115)

$$\sum_{p \leq \sqrt{x}} ||B_p| - |A_p|| << \sum_{p \leq \sqrt{x}} \left| \left\lfloor \frac{x - l}{dq} \right\rfloor - \left\lfloor \frac{x - m}{dq} \right\rfloor \right| << \sqrt{x \log x}$$

(116)

therefore, we have

$$S(A; P(z), \sqrt{x}) - S(B; P(z), \sqrt{x}) << \sqrt{x \log x}$$

(117)

for this, we have

$$\pi(x; q, l) - \pi(x; q, m) << \sqrt{x \log x}$$

(118)

However, in general case, for $(q, l_i) = 1$, $(q, l_j) = 1$, and $l, l_j < q$, $q > 2$, we can also prove that

$$\pi(x; q, l_i) - \pi(x; q, l_j) << \sqrt{x \log x}$$

(119)

add condition, the mean value of $\pi(x; q, l_i)$ is equal to $\frac{\pi(x)}{\phi(q)}$, we have

$$\pi(x; q, l_i) = \frac{\pi(x)}{\phi(q)} + O(\sqrt{x \log x})$$

(120)

Otherwise, let

$$\pi(x; q, l_i) > \frac{\pi(x)}{\phi(q)} + O(\sqrt{x \log x})$$

(121)

and

$$\pi(x; q, l_j) < \frac{\pi(x)}{\phi(q)} + O(\sqrt{x \log x})$$

(122)
then it have
\[ \pi(x; q, l_i) - \pi(x; q, l_j) \gg \sqrt{x} \log x \] (123)
that contradicts the previous results
\[ \pi(x; q, l_i) - \pi(x; q, l_j) \ll \sqrt{x} \log x \] (124)
To sum up, it must have
\[ \pi(x; q, l_i) = \frac{\pi(x)}{\phi(q)} + O(\sqrt{x} \log x) \] (125)
namely, for \((q, l) = 1, l < q\), it have
\[ \pi(x; q, l) = \frac{\pi(x)}{\phi(q)} + O(\sqrt{x} \log x) \] (126)

Theorem 1 be proved.

For this, we proved the proposition B to be true, therefore, General Riemann Hypothesis is true, and Riemann Hypothesis is also true.

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