ANALYSIS OF THE SPECTRUM OF A 2 × 2 OPERATOR MATRIX.
DISCRETE SPECTRUM ASYMPTOTICS

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Abstract. We consider a 2 × 2 operator matrix $A_\mu$, $\mu > 0$ related with the lattice systems describing two identical bosons and one particle, another nature in interactions, without conservation of the number of particles. We obtain an analogue of the Faddeev equation and its symmetric version for the eigenfunctions of $A_\mu$. We describe the new branches of the essential spectrum of $A_\mu$ via the spectrum of a family of generalized Friedrichs models. It is established that the essential spectrum of $A_\mu$ consists of at most three bounded closed intervals and their location is studied. For the critical value $\mu_0$ of the coupling constant $\mu$ we establish the existence of infinitely many eigenvalues, which are located in the both sides of the essential spectrum of $A_\mu$. In this case, an asymptotic formula for the discrete spectrum of $A_\mu$ is found.

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1. Introduction and statement of the problem

It is well-known that [25], if $H$ is a bounded linear operator in a Hilbert space $\mathcal{H}$ and a decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ into two Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$ is given, then $H$ always admits a block operator matrix representation

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}$$

with bounded linear operators $H_{ij} : \mathcal{H}_j \to \mathcal{H}_i$, $i, j = 1, 2$. In addition, $H = H^*$ if and only if $H_{ii} = H_{ii}^*$, $i = 1, 2$ and $H_{21} = H_{12}^*$. Such operator matrices often arise in mathematical physics, e.g. in quantum field theory, condensed matter physics, fluid mechanics, magnetohydrodynamics and quantum mechanics. One of the special class of $2 \times 2$ block operator matrices is the Hamiltonians acting in the one- and two-particle subspaces of a Fock space. It is related with a system describing three-particles in interaction without conservation of the number of particles in Fock space. Here off-diagonal entries of such block operator matrices are annihilation and creation operators.

Operator matrices of this form play a key role for the study of the energy operator of the spin-boson Hamiltonian with two bosons on the torus. In fact, the latter is a 6 × 6 operator matrix which is unitarily equivalent to a 2 × 2 block diagonal operator with two copies of a particular case of $H$ on the diagonal, see e.g. [16]. Consequently, the location of the essential spectrum and finiteness of discrete eigenvalues of the spin-boson Hamiltonian are determined by the corresponding spectral information on the operator matrix $H$. We
recall that the spin-boson model is a well-known quantum-mechanical model which describes
the interaction between a two-level atom and a photon field. We refer to [12] and [8] for
excellent reviews from physical and mathematical perspectives, respectively. Independently
of whether the underlying domain is a torus \( \mathbb{T}^d \) or the whole space \( \mathbb{R}^d \), the full spin-boson
Hamiltonian is an infinite operator matrix in Fock space for which rigorous results are very
hard to obtain. One line of attack is to consider the compression to the truncated Fock space
with a finite number \( N \) of bosons, and in fact most of the existing literature concentrates on
the case \( N \leq 2 \). For the case of \( \mathbb{R}^d \) there are some exceptions, e.g. [6], [7] for arbitrary finite
\( N \) and [26] for \( N = 3 \), where a rigorous scattering theory was developed for small coupling
constants. In [9] it is shown that the discrete spectrum of the spin-boson model with two
photons in \( \mathbb{R}^d \) is finite and the essential spectrum consists of a half-line, the bottom of which
is a unique zero of a simple Nevanlinna function.

For the case when the underlying domain is a torus, the spectral properties of some versions
of \( H \) were investigated in [3], [4], [10], [17], [22]. An important problem of the spectral theory
of such matrix operators is the infiniteness of the number of eigenvalues located outside the
essential spectrum. We mention that, the infiniteness of the discrete eigenvalues below the
bottom of the essential spectrum of the Hamiltonian in Fock space, which has a block operator
matrix representation, and corresponding discrete spectrum asymptotics were discussed in
[3], [19]. These results were obtained using the machinery developed in [24] by Sobolev.

In the present paper we consider a \( 2 \times 2 \) operator matrix \( A_\mu \), \((\mu > 0 \) is a coupling constant
related with the lattice systems describing two identical bosons and one particle, another
nature in interactions, without conservation of the number of particles. This operator acts in
the direct sum of one- and two-particle subspaces of the bosonic Fock space and it is related
with the lattice spin-boson Hamiltonian [16], [18]. We find the critical value \( \mu_0 \) of the coupling
constant \( \mu \), to establish the existence of infinitely many eigenvalues lying in both sides of
essential spectrum of \( A_{\mu_0} \) and to obtain an asymptotics for the number of these eigenvalues.

We point out that the latter assertion seems to be quite new for the discrete models and
similar result have not been obtained yet for the three-particle discrete Schrödinger operators
and operator matrices in Fock space. In all papers devoted to the infiniteness of the number
of eigenvalues (Efimov’s effects) the situation on the neighborhood of the left edge of essential
spectrum are discussed, see for example [12], [3], [4], [11], [17]. Since the essential spectrum of the
three-particle continuous Schrödinger operators [15], [23], [24] and standard spin-boson model
with at most two photons [13], [14] coincides with half-axis \([\kappa; +\infty)\), the main results of the
present paper are typical only for lattice case, and they do not have analogues in the continues
case.

Now we formulate the problem. Let \( \mathbb{T}^3 \) be the three-dimensional torus, the cube \((−\pi, \pi]^3
with appropriately identified sides equipped with its Haar measure. Let \( L_2(\mathbb{T}^3) \) be the Hilbert
space of square integrable (complex) functions defined on \( \mathbb{T}^3 \) and \( L_2^s((\mathbb{T}^3)^2) \) be the Hilbert
space of square integrable (complex) symmetric functions defined on \((\mathbb{T}^3)^2\). Denote by \( \mathcal{H} \)
the direct sum of spaces \( \mathcal{H}_1 := L_2(\mathbb{T}^3) \) and \( \mathcal{H}_2 := L_2^s((\mathbb{T}^3)^2) \), that is, \( \mathcal{H} := \mathcal{H}_1 \oplus \mathcal{H}_2 \). The spaces
\( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) are called one- and two-particle subspaces of a bosonic Fock space \( \mathcal{F}_s(L_2(\mathbb{T}^3))\)
over \( L_2(\mathbb{T}^3) \), respectively.

Let us consider a \( 2 \times 2 \) operator matrix \( A_\mu \) acting in the Hilbert space \( \mathcal{H} \) as

\[
A_\mu := \begin{pmatrix}
A_{11} & \mu A_{12} \\
\mu A_{12}^* & A_{22}
\end{pmatrix}
\]
with the entries
\[(A_{11}f_1)(k) = w_1(k)f_1(k), \quad (A_{12}f_2)(k) = \int_{T^3} f_2(k, s)ds,\]
\[(A_{22}f_2)(k, p) = w_2(k, p)f_2(k, p), \quad f_i \in \mathcal{H}_i, \quad i = 1, 2.\]

Here \(\mu > 0\) is a coupling constant, the functions \(w_1(\cdot)\) and \(w_2(\cdot, \cdot)\) have the form
\[w_1(k) := \varepsilon(k) + \gamma, \quad w_2(k, p) := \varepsilon(k) + \varepsilon(\frac{1}{2}(k + p)) + \varepsilon(p)\]
with \(\gamma \in \mathbb{R}\) and the dispersion function \(\varepsilon(\cdot)\) is defined by
\[(1.1) \quad \varepsilon(k) := \sum_{i=1}^{3} (1 - \cos k_i), \quad k = (k_1, k_2, k_3) \in \mathbb{T}^3,\]

\(A_{12}^*\) denotes the adjoint operator to \(A_{12}\) and
\[(A_{12}^*f_1)(k, p) = \frac{1}{2}(f_1(k) + f_1(p)), \quad f_1 \in \mathcal{H}_1.\]

Under these assumptions the operator \(\mathcal{A}_\mu\) is bounded and self-adjoint.

We remark that the operators \(A_{12}\) and \(A_{12}^*\) are called annihilation and creation operators \([5]\), respectively. In physics, an annihilation operator is an operator that lowers the number of particles in a given state by one, and it is the adjoint of the annihilation operator.

2. Faddeev’s equation and essential spectrum of \(\mathcal{A}_\mu\)

In this section we obtain an analogue of the Faddeev type integral equation for eigenvectors of \(\mathcal{A}_\mu\) and investigate the location and structure of the essential spectrum of \(\mathcal{A}_\mu\).

Throughout the present paper we adopt the following conventions: Denote by \(\sigma(\cdot)\), \(\sigma_{\text{ess}}(\cdot)\) and \(\sigma_{\text{disc}}(\cdot)\), respectively, the spectrum, the essential spectrum, and the discrete spectrum of a bounded self-adjoint operator.

Let \(\mathcal{H}_0 := \mathbb{C}\). To study the spectral properties of the operator \(\mathcal{A}_\mu\) we introduce a family of bounded self-adjoint operators (generalized Friedrichs models) \(\mathcal{A}_\mu(k), k \in \mathbb{T}^3\) which acts in \(\mathcal{H}_0 \oplus \mathcal{H}_1\) as \(2 \times 2\) operator matrices
\[\mathcal{A}_\mu(k) := \begin{pmatrix} A_{00}(k) & \frac{\mu}{\sqrt{2}} A_{01} \\ \frac{\mu}{\sqrt{2}} A_{01}^* & A_{11}(k) \end{pmatrix},\]
with matrix elements
\[A_{00}(k)f_0 = w_1(k)f_0, \quad (A_{01}f_1) = \int_{T^3} f_1(t)dt,\]
\[(A_{11}(k)f_2)(p) = w_2(k, p)f_1(p), \quad f_i \in \mathcal{H}_i, \quad i = 1, 2.\]

From the simple discussions it follows that \(\sigma_{\text{ess}}(\mathcal{A}_\mu(k)) = [m(k), M(k)]\), where the numbers \(m(k)\) and \(M(k)\) are defined by
\[(2.1) \quad m(k) := \min_{p \in \mathbb{T}^3} w_2(k, p), \quad M(k) := \max_{p \in \mathbb{T}^3} w_2(k, p).\]

For any \(k \in \mathbb{T}^3\) we define an analytic function \(I(k; \cdot)\) in \(\mathbb{C} \setminus \sigma_{\text{ess}}(\mathcal{A}_\mu(k))\) by
\[I(k; z) := \int_{T^3} \frac{dt}{w_2(k, t) - z}.\]
Then the Fredholm determinant associated to the operator $A_\mu(k)$ is defined by

$$\Delta_\mu(k; z) := w_1(k) - z - \frac{\mu^2}{2} I(k; z), \quad z \in \mathbb{C} \setminus \sigma_{\text{ess}}(A_\mu(k)).$$

A simple consequence of the Birman-Schwinger principle and the Fredholm theorem implies that for the discrete spectrum of $A_\mu(k)$ the equality

$$\sigma_{\text{disc}}(A_\mu(k)) = \{ z \in \mathbb{C} \setminus [m(k); M(k)] : \Delta_\mu(k; z) = 0 \}$$

holds.

Set

$$m := \min_{k, p \in T^3} w_2(k, p), \quad M := \max_{k, p \in T^3} w_2(k, p),$$

$$\Lambda_\mu := \bigcup_{k \in T^3} \sigma_{\text{disc}}(A_\mu(k)), \quad \Sigma_\mu := [m; M] \cup \Lambda_\mu.$$

For each $\mu > 0$ and $z \in \mathbb{C} \setminus \Sigma_\mu$ we define the integral operator $T_\mu(z)$ acting in the Hilbert spaces $L_2(T^3)$ by

$$(T_\mu(z)g)(p) = \frac{\mu^2}{2\Delta_\mu(p; z)} \int_{T^3} \frac{g(t)dt}{w_2(p, t) - z}.$$

The following theorem is an analog of the well-known Faddeev’s result for the operator $A_\mu$ and establishes a connection between eigenvalues of $A_\mu$ and $T_\mu(z)$.

**Theorem 2.1.** The number $z \in \mathbb{C} \setminus \Sigma_\mu$ is an eigenvalue of the operator $A_\mu$ if and only if the number $\lambda = 1$ is an eigenvalue of the operator $T_\mu(z)$. Moreover the eigenvalues $z$ and 1 have the same multiplicities.

We point out that the integral equation $g = T_\mu(z)g$ is an analogue of the Faddeev type system of integral equations for eigenfunctions of the operator $A_\mu$ and it is played crucial role in the analysis of the spectrum of $A_\mu$. For the proof of Theorem 2.1 we show the equivalence of the eigenvalue problem $A_\mu f = zf$ to the equation $g = T_\mu(z)g$.

The following theorem describes the location of the essential spectrum of the operator $A_\mu$ by the spectrum of the family of generalized Friedrichs models $A_\mu(k)$.

**Theorem 2.2.** For the essential spectrum of $A_\mu$ the equality $\sigma_{\text{ess}}(A_\mu) = \Sigma_\mu$ holds. Moreover the set $\Sigma_\mu$ consists of no more than three bounded closed intervals.

The inclusion $\Sigma_\mu \subset \sigma_{\text{ess}}(A_\mu)$ in the proof of Theorem 2.1 is established with the use of a well-known Weyl criterion, see for example [22]. An application of Theorem 2.1 and analytic Fredholm theorem (see, e.g., Theorem VI.14 in [23]) proves inclusion $\sigma_{\text{ess}}(A_\mu) \subset \Sigma_\mu$.

In the following we introduce the new subsets of the essential spectrum of $A_\mu$.

**Definition 2.3.** The sets $\Lambda_\mu$ and $[m; M]$ are called two- and three-particle branches of the essential spectrum of $A_\mu$, respectively.

The definition of the set $\Lambda_\mu$ and the equality

$$\bigcup_{k \in T^3} [m(k); M(k)] = [m; M]$$

together with Theorem 2.2 give the equality

$$\sigma_{\text{ess}}(A_\mu) = \bigcup_{k \in T^3} \sigma(A_\mu(k)).$$
Here the family of operators $A_\mu(k)$ have a simpler structure than the operator $A_\mu$. Hence, in many instance, (2.2) provides an effective tool for the description of the essential spectrum.

Using the extremal properties of the function $w_2(\cdot, \cdot)$, and the Lebesgue dominated convergence theorem one can show that the integral $I(0; 0)$ is finite, where $\bar{0} := (0, 0, 0) \in T^3$, see [20, 21].

For the next investigations we introduce the following quantities

$$\mu_0^0(\gamma) := \sqrt{2\gamma \left(I(0, 0)\right)^{-1/2}} \quad \text{for } \gamma > 0;$$
$$\mu_1^0(\gamma) := \sqrt{24 - 2\gamma \left(I(0, 0)\right)^{-1/2}} \quad \text{for } \gamma < 12.$$

Since $T^3$ is compact, and the functions $\Delta_\mu(\cdot; 0)$ and $\Delta_\mu(\cdot; 18)$ are continuous on $T^3$, there exist points $k_0, k_1 \in T^3$ such that the equalities

$$\max_{k \in T^3} \Delta_\mu(k; 0) = \Delta_\mu(k_0; 0), \quad \min_{k \in T^3} \Delta_\mu(k; 18) = \Delta_\mu(k_1; 18)$$

hold.

Let us define the following notations:

$$\gamma_0 := \left(12 \frac{I(k_0; 0)}{I(0; 0)} - \varepsilon(k_0)\right) \left(1 + \frac{I(k_0; 0)}{I(0; 0)}\right)^{-1};$$
$$\gamma_1 := (18 - \varepsilon(k_1)) \left(1 - \frac{I(k_1; 18)}{I(0; 0)}\right).$$

Denote

$$E_{\mu}^{(1)} := \min \left\{ \Lambda_\mu \cap (-\infty; 0] \right\}; E_{\mu}^{(2)} := \max \left\{ \Lambda_\mu \cap (-\infty; 0] \right\};$$
$$E_{\mu}^{(3)} := \min \left\{ \Lambda_\mu \cap [18; \infty) \right\}; E_{\mu}^{(4)} := \max \left\{ \Lambda_\mu \cap [18; \infty) \right\}.$$

We formulate the results, which are precisely describe the structure of the essential spectrum of $A_\mu$. The structure of the essential spectrum depends on the location of the parameters $\mu > 0$ and $\gamma \in \mathbb{R}$.

**Theorem 2.4.** Let $\mu = \mu_0^0(\gamma)$, with $\gamma < 12$. The following equality holds

$$\sigma_{\text{ess}}(A_\mu) = \begin{cases} [E_1; E_2] \cup [0; 18], & \text{if } \gamma < \gamma_0; \\ [E_1; 18], & \text{if } \gamma_0 \leq \gamma < 6; \\ [0; 18], & \text{if } 6 \leq \gamma < 12. \end{cases}$$

**Theorem 2.5.** Let $\mu = \mu_1^0(\gamma)$, with $\gamma > 0$. The following equality holds

$$\sigma_{\text{ess}}(A_\mu) = \begin{cases} [0; 18], & \text{if } 0 < \gamma \leq 6; \\ [0; E_{\mu}^{(4)}], & \text{if } 6 < \gamma \leq \gamma_1; \\ [0; 18] \cup [E_{\mu}^{(3)}; E_{\mu}^{(4)}], & \text{if } \gamma > \gamma_1. \end{cases}$$

The proof of these two theorems are based on the existence conditions of the eigenvalue $z_\mu(k)$ of the operator $A_\mu(\cdot)$ and the continuity of $z_\mu(\cdot)$ on its domain.
3. Birman-Schwinger principle and discrete spectrum asymptotics of the operator $A_\mu$

Let us denote by $\tau_{\min}(A_\mu)$ and $\tau_{\max}(A_\mu)$ the lower and upper bounds of the essential spectrum $\sigma_{\text{ess}}(A_\mu)$ of the operator $A_\mu$, respectively, that is,

$$
\tau_{\min}(A_\mu) \equiv \min \sigma_{\text{ess}}(A_\mu), \quad \tau_{\max}(A_\mu) \equiv \max \sigma_{\text{ess}}(A_\mu).
$$

For an interval $\Delta \subset \mathbb{R}$, $E_\Delta(A_\mu)$ stands for the spectral subspace of $A_\mu$ corresponding to $\Delta$. Let us denote by $\sharp\{\cdot\}$ the cardinality of a set and by $N_{(a,b)}(A_\mu)$ the number of eigenvalues of the operator $A_\mu$, including multiplicities, lying in $(a, b) \subset \mathbb{R} \setminus \sigma_{\text{ess}}(A_\mu)$, that is,

$$
N_{(a,b)}(A_\mu) := \dim E_{(a,b)}(A_\mu).
$$

For a $\lambda \in \mathbb{R}$, we define the number $n(\lambda, A_\mu)$ as follows

$$
n(\lambda, A_\mu) := \sup\{\dim F : (A_\mu u, u) > \lambda, u \in F \subset H, ||u|| = 1\}.
$$

The number $n(\lambda, A_\mu)$ is equal to the infinity if $\lambda < \max \sigma_{\text{ess}}(A_\mu)$; if $n(\lambda, A_\mu)$ is finite, then it is equal to the number of the eigenvalues of $A_\mu$ bigger than $\lambda$.

By the definition of $N_{(a,b)}(A_\mu)$, we have

$$
N_{(-\infty;z]}(A_\mu) = n(-z, -A_\mu), -z > -\tau_{\min}(A_\mu),
$$

$$
N_{(z;+\infty)}(A_\mu) = n(z, A_\mu), z > \tau_{\max}(A_\mu).
$$

In our analysis of the discrete spectrum of $A_\mu$ the crucial role is played by the compact operator $\hat{T}_\mu(z)$, $z \in \mathbb{R} \setminus [\tau_{\min}(A_\mu); \tau_{\max}(A_\mu)]$ in the space $L_2(\mathbb{T}^3)$ as integral operator

$$
(\hat{T}_\mu(z)g)(p) = \frac{\mu^2}{2\sqrt{\Delta_\mu(p; z)}} \int_{\mathbb{T}^3} \frac{g(t)dt}{\sqrt{\Delta_\mu(t; z)(w_2(p, t) - z)}}, \quad \text{for} \quad z < \tau_{\min}(A_\mu),
$$

$$
(\hat{T}_\mu(z)g)(p) = -\frac{\mu^2}{2\sqrt{-\Delta_\mu(p; z)}} \int_{\mathbb{T}^3} \frac{g(t)dt}{\sqrt{-\Delta_\mu(t; z)(w_2(p, t) - z)}}, \quad \text{for} \quad z > \tau_{\max}(A_\mu).
$$

The following lemma is a realization of the well-known Birman-Schwinger principle for the operator $A_\mu$ (see [3]).

**Lemma 3.1.** For $z \in \mathbb{R} \setminus [\tau_{\min}(A_\mu); \tau_{\max}(A_\mu)]$ the operator $\hat{T}_\mu(z)$ is compact and continuous in $z$ and

$$
N_{(-\infty;z]}(A_\mu) = n(1, \hat{T}_\mu(z)) \quad \text{for} \quad z < \tau_{\min}(A_\mu),
$$

$$
N_{(z;+\infty)}(A_\mu) = n(1, \hat{T}_\mu(z)) \quad \text{for} \quad z > \tau_{\max}(A_\mu).
$$

This lemma can be proven quite similarly to the corresponding result of [3]. Let $\mathbb{S}^2$ being the unit sphere in $\mathbb{R}^3$ and

$$
S_r : L_2((0, r), \sigma_0) \to L_2((0, r), \sigma_0), \quad r > 0, \quad \sigma_0 = L_2(\mathbb{S}^2)
$$

be the integral operator with the kernel

$$
S(t; y) = \frac{25}{8\pi^2\sqrt{6}} \frac{1}{5\cos(hy) + t},
$$

$$
y = x - x', \quad x, x' \in (0, r), \quad t = (\xi, \eta), \quad \xi, \eta \in \mathbb{S}^2.
$$

For $\lambda > 0$, define

$$
U(\lambda) = \frac{1}{2} \lim_{r \to \infty} r^{-1} n(\lambda, S_r).
$$

The existence of the latter limit and the fact $U(1) > 0$ shown in [24].
From the definitions of the quantities $\mu^0_l(\gamma)$ and $\mu^0_r(\gamma)$ it is easy to see that $\mu^0_l(6) = \mu^0_r(6)$.

We set $\mu^0 : = \mu^0_l(6)$.

We can now formulate our last main result.

**Theorem 3.2.** The following relations hold:

$$\#(\sigma_{\text{disc}}(A_{\mu^0}) \cap (-\infty, 0)) = \#(\sigma_{\text{disc}}(A_{\mu^0}) \cap (18, \infty)) = \infty;$$

$$(3.1) \quad \lim_{z \to 0} \frac{N(-\infty, z)(A_{\mu^0})}{|\log |z||} = \lim_{z \to 18} \frac{N(z, \infty)(A_{\mu^0})}{|\log |z - 18||} = U(1).$$

Clearly, by equality (3.1) the infinite cardinality of the parts of discrete spectrum of $A_{\mu^0}$ in $(-\infty; 0)$ and $(18; +\infty)$ follows automatically from the positivity of $U(1)$.

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