Finite Difference Method on Flat Surfaces with a Flat Unitary Vector Bundle

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Abstract
We establish an asymptotic relation between the spectrum of the discrete Laplacian associated to discretizations of a tileable surface with a flat unitary vector bundle and the spectrum of the Friedrichs extension of the Laplacian with Neumann boundary conditions. Our proof is based on the precise study of the singularities of the eigenvectors near the conical points and corners of a half-translation surface, and on establishing Harnack-type estimates on “almost harmonic” discrete functions, defined on the graphs, which approximate the given surface.

Keywords
Graph Laplacian · Spectrum · Flat surfaces · Discretizations

Mathematics Subject Classification 53C99

1 Introduction
Many natural combinatorial invariants of a finite graph $G$ can be expressed through the spectrum $\text{Spec}(\Delta_G)$ of the combinatorial Laplacian $\Delta_G$, defined as the difference between the degree and the adjacency operators. For example, the number of connected components corresponds to the multiplicity of 0 in $\text{Spec}(\Delta_G)$. The largest eigenvalue of $\Delta_G$ in a regular graph $G$ is twice the degree of the vertices if and only if the graph is bipartite. The smallest positive eigenvalue of $\text{Spec}(\Delta_G)$ is related to the Cheeger isoperimetric constant, which detects “bottlenecks”, see Cheeger [8].

According to a famous Matrix-tree theorem of Kirchoff, the product of non-zero eigenvalues of $\Delta_G$ corresponds to the number of marked spanning trees on $G$. This theorem has been generalized by Forman in [18, Theorem 1] to the setting of a unitary line bundle on a graph and by Kenyon in [21, Theorems 8,9] to vector bundles of rank 2 endowed with $\text{SL}_2(\mathbb{C})$-connections (we give a precise meaning for those notions in
Sect. 2.1). They showed that a sum of cycle-rooted spanning forests (cf. [21, §4] for related definitions), weighted by some function depending on the monodromy of the connection, calculated over the cycle, can be expressed through the determinant of the Laplacian twisted by a connection on a vector bundle (see (1.1)).

Now, if instead of considering a single graph \( G \) with a vector bundle over it, we consider a family \( \Sigma_n, n \in \mathbb{N}^* \) of graphs endowed with vector bundles \( F_n \), constructed as approximations of a certain surface \( \Sigma \) with a flat unitary vector bundle \( F \), we could ask ourselves how the spectral invariants of \( (\Sigma_n, F_n) \) would behave, as \( n \to \infty \).

We consider in this article tileable surfaces \( (\Sigma, g^{T\Sigma}) \) with conical singularities. This means that \( \Sigma \) can be tiled completely and without overlaps over subsets of positive Lebesgue measure by euclidean squares of area 1. In particular, the conical singularities \( \text{Con}(\Sigma) \) have angles \( \frac{k\pi}{2}, k \in \mathbb{N}^* \setminus \{4\} \), the boundary \( \partial \Sigma \) can be tiled by the boundaries of the tiles, and the angles of the boundary corners \( \text{Ang}(\Sigma) \) are of the form \( \frac{k\pi}{2}, k \in \mathbb{N}^* \setminus \{2\} \).

For example, if \( \Sigma \) is a torus, then it can be tiled by euclidean squares of the same size if and only if the ratio of its periods is rational. If \( \Sigma \) is a rectangular domain in \( \mathbb{C} \) with one boundary component, then it can be tiled by euclidean squares of the same size if and only if the ratios between the lengths of the sides of \( \Sigma \) are rational. Our surfaces generalize also the so-called pillowcase covers, which can be characterized as ramified coverings of \( \mathbb{C}P^1 \), branched over four points, cf. Zorich [27].

The main result of this article shows that for a given tileable surface \( (\Sigma, g^{T\Sigma}) \) with a flat unitary vector bundle \( (F, h^F, \nabla^F) \), and a suitably chosen discretization \( \Sigma_n \) of \( \Sigma \), the spectral theory of the graph Laplacian \( \Delta_{\Sigma_n}^F \) on \( \Sigma_n \), associated with the discretization \( (F_n, h^{F_n}, \nabla^{F_n}) \) of \( (F, h^F, \nabla^F) \), is an approximation, up to a renormalization, of the spectral theory of the Friedrichs extension of the Laplacian \( \Delta_{\Sigma}^F \) with Neumann boundary conditions on \( \partial \Sigma \).

More precisely, the Laplacian \( \Delta_G^V \), associated to a graph \( G \) and a vector bundle with connection \( (V, h^V) \) over \( G \) (see Sect. 2.1 for definitions), is the linear operator on sections \( f \) of \( V \), given by

\[
\Delta_G^V f(v) = \sum_{(v,v') \in E(G)} (f(v) - \phi_{v'v} f(v')), \quad v \in V(G),
\]

(1.1)

where \( \phi_{v'v} \) are the parallel transports along the edges. For trivial vector bundle, \( \Delta_G^V \) is equal to \( \Delta_G \).

We fix a tileable surface \( (\Sigma, g^{T\Sigma}) \) with conical singularities. We also fix a flat unitary vector bundle \( (F, h^F, \nabla^F) \) on the compactification

\[
\overline{\Sigma} := \Sigma \cup \text{Con}(\Sigma),
\]

(1.2)

of \( \Sigma \) (i.e. we require \( (\nabla^F)^2 = 0 \) and we suppose that the connection \( \nabla^F \) preserves the metric \( h^F \)). In particular, the monodromies of \( \nabla^F \) around conical points are trivial.

We fix a tiling of \( \Sigma \). We construct a graph \( \Sigma_1 = (V(\Sigma_1), E(\Sigma_1)) \) by taking vertices \( V(\Sigma_1) \) as the centers of tiles and edges \( E(\Sigma_1) \) in such a way that the resulting graph \( \Sigma_1 \) is the nearest-neighbor graph with respect to the flat metric \( g^{T\Sigma} \) on \( \Sigma \). This means...
that an edge connects two vertices if and only if they are the closest neighbors with respect to the metric $g^T\Sigma$. One can see that in certain cases our graph $\Sigma_1$ can have double edges and even loops, see Remark 2.1. Thus, in general, $\Sigma_1$ is a multigraph. We omit this subtlety and call $\Sigma_1$, by an abuse of notation, a graph.

The vector bundle $F_1$ over $\Sigma_1$ and the Hermitian metric $h^{F_1}$ on $F_1$ are constructed by the restriction from $F$ and $h^F$. The connection $\nabla^{F_1}$ is constructed using the parallel transport of $\nabla^F$ with respect to the straight path between the vertices. It is a matter of a simple verification to see that as $(F, h^F, \nabla^F)$ is unitary, the vector bundle $(F_1, h^{F_1}, \nabla^{F_1})$ is unitary as well. By considering regular subdivisions of tiles into $n^2$ squares, $n \in \mathbb{N}^*$, and repeating the same procedure, we construct a family of graphs $\Sigma_n = (V(\Sigma_n), E(\Sigma_n))$ with unitary vector bundles $(F_n, h^{F_n}, \nabla^{F_n})$ over $\Sigma_n$, for $n \in \mathbb{N}^*$. Since the monodromy of the vector bundle is trivial around conical singularities, in the Laplacian of $(\Sigma_n, F_n, \nabla^{F_n})$ one can simply ignore the loops and replace each multiedge by an edge with multiplicity. Note that we also have a natural injection

$$V(\Sigma_n) \hookrightarrow \Sigma. \quad (1.3)$$

Let us consider an example of a rectangular domain $\Sigma$ in $\mathbb{C}$ with integer vertices. Then the graphs $\Sigma_n$ coincide with subgraphs of $\frac{1+i\sqrt{-1}}{2n} + \frac{1}{n}\mathbb{Z}^2$, which stay inside of $\Sigma$. See Fig. 1.

We denote by $(\nabla^F)^*$ the formal adjoint of $\nabla^F$ with respect to the $L^2$-metric induced by $g^T\Sigma$ and $h^F$. We denote by $\Delta^F_\Sigma$ the Laplacian on $\Sigma$ associated with $(F, h^F, \nabla^F)$. It is a differential operator acting on the smooth sections of $F$ by

$$\Delta^F_\Sigma := (\nabla^F)^* \nabla^F. \quad (1.4)$$

If $(F, h^F, \nabla^F)$ is trivial, $\Delta^F_\Sigma$ coincides with the usual Laplacian, given by the formula $-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$.

In this paper, we always consider $\Delta^F_\Sigma$ with Neumann boundary conditions on $\partial \Sigma$. In other words, for a normal $n$ to the boundary, the sections $f$ from the domain of our Laplacian satisfy

$$\nabla^F_n f = 0 \quad \text{over } \partial \Sigma. \quad (1.5)$$
It is well-known that unlike for smooth manifolds, the Laplacian \( \Delta^F_{\Sigma} \) is not necessarily *essentially self-adjoint*. Thus, to define the spectrum of \( \Delta^F_{\Sigma} \), we will be obliged to specify the self-adjoint extension of \( \Delta^F_{\Sigma} \) we are working with. We choose the Friedrichs extension and, by an abuse of notation, we denote it by the same symbol \( \Delta^F_{\Sigma} \). See Sect. 2.2 for some properties and definitions on Friedrichs extension and Sect. 2.3 for an explicit description of its domain.

As in the case of smooth domains, the spectrum of \( \Delta^F_{\Sigma} \) is discrete, cf. Proposition 2.3. We order it non-decreasingly as follows

\[
\text{Spec}(\Delta^F_{\Sigma}) = \{\lambda_1, \lambda_2, \ldots \}.
\]  

(1.6)

The main goal of this article is to study the relationship between \( \text{Spec}(\Delta^F_{\Sigma_n}) \) and \( \text{Spec}(\Delta^F_{\Sigma}) \). For this, let \( \lambda^n_i, i \in \mathbb{N}^* \) be a non-decreasing sequence, such that for any \( n \in \mathbb{N}^* \), we have

\[
\text{Spec}(n^2 \cdot \Delta^F_{\Sigma_n}) = \{\lambda^n_1, \lambda^n_2, \ldots \}.
\]  

(1.7)

**Theorem 1.1** *In the notations of (1.6) and (1.7), for any \( i \in \mathbb{N} \), as \( n \to \infty \), we have*

\[
\lambda^n_i \to \lambda_i.
\]  

(1.8)

**Remark 1.2** *(a) In particular, from Theorem 1.1, we see that any spectral invariant depending on a finite number of eigenvalues with fixed indices is related in the limit with the spectrum of \( \Delta^F_{\Sigma} \).

(b) Using the results of this article, one can estimate the difference \(|\lambda^n_i - \lambda_i|\) for any \( i \in \mathbb{N} \), as \( n \to \infty \). As this would involve some additional work, which is not related to our main application [17], we decided to omit those estimates and leave their derivation to an interested reader.

The second result is a similar statement in realms of eigenvectors. Morally, it says that the eigenvectors of \( n^2 \cdot \Delta^F_{\Sigma_n} \) converge in \( L^2(\Sigma, F) \) to the eigenvectors of \( \Delta^F_{\Sigma} \). However, as the eigenvalues might have some multiplicity, and the finitely dimensional space \( \text{Map}(V(\Sigma_n), F_n) \) doesn’t inject in \( L^2(\Sigma, F) \) canonically, we need to introduce some further notation for a rigorous statement.

Assume that the eigenvalue \( \lambda_i, i \in \mathbb{N}^* \), of \( \Delta^F_{\Sigma} \) has multiplicity \( m_i \). Let \( f_{i,j} \in L^2(\Sigma, F), j = 1, \ldots, m_i, \) be some orthonormal basis of eigenvectors of \( \Delta^F_{\Sigma} \) corresponding to the eigenvalue \( \lambda_i \). By Theorem 1.1, there is a series of eigenvalues \( \lambda^n_{i,j}, j = 1, \ldots, m_i, \) of \( n^2 \cdot \Delta^F_{\Sigma_n} \), converging to \( \lambda_i \), as \( n \to \infty \). Moreover, no other eigenvalue of \( n^2 \cdot \Delta^F_{\Sigma_n}, n \in \mathbb{N}^* \), comes close to \( \lambda_i \) asymptotically.

In the beginning of Sect. 3.2, we define a “linearization” functional

\[
L_n : \text{Map}(V(\Sigma_n), F_n) \to L^2(\Sigma, F).
\]  

(1.9)
One should think of it as a sort of linear interpolation, (1.3), which “blurs” the function near the set Con(Σ) of conical points of Σ and the set Ang(Σ) of non-smooth points of the boundary ∂Σ.

We denote by ∥·∥_{L^2(Σ_n, F_n)} the $L^2$-norm on Map($V(Σ_n), F_n$), given by

$$\|f\|_{L^2(Σ_n, F_n)}^2 = \sum_{v \in V(Σ_n)} h^{F_n}(f(v), f(v)).$$ (1.10)

**Theorem 1.3** We use the same notation as in Theorem 1.1. For any $i \in \mathbb{N}$, there is $N$ such that for any $n \geq N$, there are $f^n_{i,j} \in Map(\Sigma_n, F_n), 1 \leq j \leq m_i$, which are pairwise orthogonal, satisfy $\|f^n_{i,j}\|_{L^2(Σ_n, F_n)}^2 = n^2$, and which are in the span of the eigenvectors of $n^2 \cdot Δ^{F_n}_{Σ_n}$, corresponding to the eigenvalues $λ^n_{i,j}, j = 1, \ldots, m_i$, such that, as $n \to \infty$, in $L^2(Σ, F)$, the following limit holds

$$L_n(f^n_{i,j}) \to f_{i,j}.$$ (1.11)

**Remark 1.4** We emphasize that $f^n_{i,j}$ are not in general eigenvectors. Nevertheless, if one considers a sequence $f^n_i$ of eigenvectors of $n^2 \cdot Δ^{F_n}_{Σ_n}$, such that $\|f^n_i\|_{L^2(Σ_n, F_n)}^2 = n^2$, and the corresponding sequence of eigenvalues is bounded by a fixed constant, then, up to extracting a subsequence, $L_n(f^n_i)$ will converge to an eigenvector of $Δ^{F}_{Σ}$. This follows directly from the discreteness of the spectrum of $Δ^{F}_{Σ}$ and Theorems 1.1, 1.3. See also Remark 3.14 for a related statement.

In the proofs of Theorems 1.1 and 1.3, we were inspired by the approximation theory of Dodziuk, [12], and Dodziuk–Patodi, [13]. Note, however, that there are two big differences between their theory and ours. They work with simplicial complexes, and the Laplacian from [12, 13] depends largely on the geometry of the space and has no relation to the combinatorial Laplacian (1.1) we are considering in this article (which depends only on the combinatorics of the approximation graph). Second, the boundary of their manifold is smooth. Non-smoothness of the boundary in our case raises some technical problems which did not appear in the articles [12, 13].

To obtain our results, we apply a well-known strategy that relies, on one side, on the variational characterisation of eigenvalues and, on the other side, on two operators that allow to transport functions from $Σ$ to functions on $Σ_n$ and the other way round. This naturally leads to the study of the behaviour of the corresponding Rayleigh quotients with respect to those operators.

On the continuous side, to estimate Rayleigh quotients, we use weak elliptic regularity results on polygons due to Grisvard, [19]. This enables us to overcome the lack of elliptic estimates near conical singularities and corners, which entail the non-differentiability of the eigenvectors.

To estimate the Rayleigh quotients on the discrete side, which is a bit more involved, we develop Harnack-type estimates in Theorem 3.11 for the eigenvectors corresponding to small eigenvalues of $Δ^{F_n}_{Σ_n}$ to prove that the discrete eigenvectors are...
“asymptotically continuous”. We rely on the potential theory on lattices, developed by Duffin [14] and Kenyon [20].

This article is organized as follows. In Sect. 2 we introduce the main notions related to flat surfaces and their discretizations. We define the Friedrichs extension of the Laplacian and give an explicit description of its domain. In Sect. 3, we prove Theorems 1.1, 1.3, modulo Harnack-type estimates, which are proved in Sect. 4.

This paper goes in line of a general idea that operators on discretizations in fine mesh limit are related to the associated continuous operators, see Dodziuk [12], Dodziuk–Patodi [13], Müller [25], Colin de Verdière [10, 11], Chelkak–Smirnov [9], Burago–Ivanov–Kurylev [7].

Our results will be used in [17] to relate the asymptotic expansion of the number of spanning trees and weighted cycle-rooted spanning forests on the discretizations of flat surfaces to the corresponding zeta-regularized determinants. The results of those papers are announced in [16]. The general philosophy of our analysis is very much influenced by global analysis framework developed by Bismut–Gillet–Soulé [2, 3], Bismut–Lebeau [5], Ma–Marinescu [22].

Notation. For a graph $G$, we denote by $V(G)$, $E(G)$ the sets of vertices and edges of $G$, respectively. For an (oriented) edge $e$, we denote by $h(e)$ the head of $e$, and by $t(e)$ the tail of $e$. For a Hermitian vector bundle $(V, hV)$ on a finite graph $G$ (resp. over the edges of $G$), we denote by $\langle \cdot, \cdot \rangle_{L^2(G,V)}$ the $L^2$-scalar product on the set $\text{Map}(V(G), V)$ (resp. $\text{Map}(E(G), V)$), defined by the following formula

$$\langle f, g \rangle_{L^2(G,V)} = \sum_{v \in V(G)} hV(f(v), g(v)), \quad f, g \in \text{Map}(V(G), V),$$

$$\langle f, g \rangle_{L^2(G,E)} = \sum_{e \in E(G)} hV(f(e), g(e)), \quad f, g \in \text{Map}(E(G), V).$$

Recall that the divergence operator $d^*_G : \text{Map}(E(G), \mathbb{C}) \to \text{Map}(V(G), \mathbb{C})$ is defined by

$$(d^*_G g)(P) := \sum_{e \in E(G), t(e)=P} g(e), \quad g \in \text{Map}(E(G), \mathbb{C}).$$

The following identities can be verified directly

$$\Delta_G = d^*_G d_G, \quad \langle \Delta_G f, g \rangle_{L^2(G)} = \langle d_G f, d_G g \rangle_{L^2(G)}.$$

For $k \in \mathbb{N}$, we denote by

$$\mathcal{C}^k(\Sigma, F) := \left\{ f \in \mathcal{C}^k(\Sigma, F) : \nabla^l f \in L^\infty(\Sigma, (T^*\Sigma)^{\otimes l} \otimes F), \text{ for } l \leq k \right\},$$

where $T^*F$ is endowed with the induced metric from $g^{T\Sigma}$. We warn the reader that in some references, for $\Sigma \subset \mathbb{C}$, the notation $\mathcal{C}^k(\Sigma, F)$ is used for the set of functions that
are restrictions to $\Sigma$ of $k$ times continuously differentiable functions that are defined on a neighbourhood of $\Sigma$. Such a definition would not make too much sense here at a conical point, and, hence, we avoid it.

Finally, for $P \in \Sigma, r > 0$, we denote by $B_\Sigma(r, P)$ the geodesic ball around $P$ in $\Sigma$ of radius $r$. Similarly, for $P \in \Sigma_n, r \in \mathbb{N}$, we denote by $B_{\Sigma_n}(r, P)$ the geodesic ball (in graph distance) around $P$ in $\Sigma_n$ of radius $r$.

2 Analysis on Flat Surfaces

Here we recall some functional analysis on flat surfaces and introduce the main objects of this article. More precisely, in Sect. 2.1, we recall the basics of flat surfaces and the main notions related to vector bundles over graphs. In Sect. 2.2, we define the Friedrichs extension of the Laplacian and study some of its spectral properties. Finally, in Sect. 2.3, we describe the domain of the Friedrichs extension through the singularities of the functions from it.

2.1 Tileable Surfaces and Their Discretizations

Here we recall the definition of flat surfaces, explain some properties of the discretizations of tileable surfaces and give a short introduction to vector bundles over graphs.

By Gauss-Bonnet theorem, the only closed Riemann surface admitting a flat metric has the topology of the torus. However, any Riemann surface can be endowed with a flat metric having a finite number of cone-type singularities. Let us explain this point more precisely.

A cone-type singularity is a Riemannian metric

$$ds^2 = dr^2 + r^2 dt^2,$$

(2.1)

on the manifold

$$C_\theta := \{(r, t) : r > 0; t \in \mathbb{R}/\theta \mathbb{Z}\},$$

(2.2)

where $\theta > 0$. In what follows, when we speak of cones, we assume $\theta \neq 2\pi$ implicitly.

By a flat metric with a finite number of cone-type singularities we mean a metric defined away from a finite set of points such that there is an atlas for which the metric looks either like the standard metric on $\mathbb{R}^2$, or like the conical metric (2.1) on an open subset of (2.2).

For a flat surface $(\Sigma, g^{T\Sigma})$ with a finite number of cone-type singularities, we denote by Con$(\Sigma)$ the conical points of the surface $\Sigma$, and by Ang$(\Sigma)$ the points where two different smooth components of the boundary meet (corners). We denote by $\angle : \text{Con}(\Sigma) \to \mathbb{R}$ the function which associates to a conical point its angle and by $\angle : \text{Ang}(\Sigma) \to \mathbb{R}$ the function which associates the interior angle between the smooth components of the boundary.

For the most part of this paper, we will be interested in tileable surfaces $\Sigma$. Those are flat surfaces with a finite number of cone-type singularities with a fixed tiling by
equal euclidean squares. We normalize the metric \( g^{\Sigma} \) on \( \Sigma \) so that the squares of the fixed tiling have area 1. Let us now set up some notations associated with discretization \( \Sigma_n \) of \( \Sigma \), constructed in Introduction. For \( P \in \text{Con}(\Sigma) \cup \Delta(\Sigma) \), we define the set \( V_n(P) \) as the set of the nearest neighbors of \( P \) from the vertex set \( V(\Sigma_n) \) with respect to the flat metric on \( \Sigma \). It’s easy to verify that for \( n \geq 2 \), \( \|V_n(P) = \frac{\angle(P)}{\pi} \). We define \( U_n(P) \) as the smallest star-like set centered at \( P \), containing \( V_n(P) \) with the edges connecting them and (in case if \( P \in \Delta(\Sigma) \)) the geodesics from \( V_n(P) \) to \( \partial \Sigma \).

**Remark 2.1** For \( n \geq 2 \), the edges of \( \Sigma_n \) have at most double multiplicity, see Fig. 2. Moreover, the number of edges with double multiplicity is equal to \( \#(P \in \text{Con}(\Sigma) : \angle(P) = \pi) \). Also, there might be loops, and the number of them is equal to the number of conical singularities of angle \( \frac{\pi}{2} \). We are, thus, working with multigraphs, but by an abuse of notation, we call them graphs.

Now, by a vector bundle \( V \) on a graph \( G = (V(G), E(G)) \), we mean the choice of a vector space \( V_v \) for any \( v \in V(G) \) so that for any \( v, v' \in V(G) \), the vector spaces \( V_v \) and \( V_{v'} \) are isomorphic. The set of sections \( \text{Map}(V(G), V) \) of \( V \) is defined by

\[
\text{Map}(V(G), V) = \oplus_{v \in V(G)} V_v.
\] (2.3)

A connection \( \nabla V \) on a vector bundle \( V \) is the choice for each (oriented) edge \( e = (v, v') \in E(G) \) of an isomorphism \( \phi_{vv'} : V_v \to V_{v'} \) between the corresponding vector spaces, such that \( \phi_{vv'} = \phi_{v'v}^{-1} \). This isomorphism is called the parallel transport of vectors in \( V_v \) to vectors in \( V_{v'} \).

A Hermitian metric \( h^V \) on the vector bundle \( V \) is a choice of a positive-definite Hermitian metric \( h_v \) on \( V_v \) for each \( v \in V(G) \). We say that a connection \( \nabla V \) is unitary with respect to \( h^V \) if its parallel transports preserves \( h^V \).

The Laplacian \( \Delta_G^V \) on \((V, \nabla V)\) is the linear operator \( \Delta_G^V : \text{Map}(V(G), V) \to \text{Map}(V(G), V) \), defined for \( f \in \text{Map}(V(G), V) \) by (1.1). Remark that unlike Laplace–Beltrami operator on a smooth manifold, we don’t use the metric to define the Laplacian (1.1).

Thus, in general, the operator \( \Delta_G^V \) is not self-adjoint, see for example [21, §3.2]. However, if the connection is unitary with respect to \( h^V \), then it becomes self-adjoint (cf. Kenyon [21, §3.3]).

Let’s represent the Laplacian \( \Delta_G^V \) in the form (1.4). One can extend the definition of a vector bundle to the edges of \( G \). A vector bundle \( V' \) over \( V(G) \oplus E(G) \) is by definition a collection of vector spaces \( V_e \) for each edge \( e \in E(G) \) as well as \( V_v \) for each vertex \( v \in V(G) \). Similarly, we extend the definition of Hermitian metric. A connection \( \nabla V' \) on \( V' \) is a choice of a connection \( \nabla V \) on \( V \) as well as connection...
isomorphisms $\phi_{ve} : V_v \rightarrow V_e, \phi_{ev} : V_e \rightarrow V_v$ for each $v \in V(G)$ and $e \in E(G)$, and satisfying $\phi_{ve} = \phi_{ev}^{-1}$ and $\phi_{v'e} = \phi_{ev'} \circ \phi_{ve}$. Similarly to (2.3), we define

$$\text{Map}(E(G), V) := \bigoplus_{e \in E(G)} V_e. \quad (2.4)$$

Quite easily, for any vector bundle $V$ and a connection $\nabla^V$ on $V(G)$, we may extend it to a vector bundle $V'$ and a connection $\nabla^{V'}$ on $E(G) \oplus V(G)$. For $V$, endowed with a Hermitian metric $h^V$, for which the connection $\nabla^V$ is unitary, the vector bundles $V_e, e \in E(G)$ can be endowed with metrics and connections $\phi_{ev}, e \in E(G), v \in V(G)$ so that $\nabla^{V'}$ is unitary as well.

There is a natural map $\nabla^V_G : \text{Map}(V(G), V) \rightarrow \text{Map}(E(G), V)$, defined by

$$(\nabla^V_G f)(e) = \phi_{t(e)} f(t(e)) - \phi_{h(e)} f(h(e)), \quad f \in \text{Map}(V(G), V), \quad (2.5)$$

where $t(e)$ and $h(e)$ are tail and head respectively of an oriented edge $e$. We define the operator $(\nabla^V_G)^* : \text{Map}(E(G), V) \rightarrow \text{Map}(V(G), V)$ by the formula

$$((\nabla^V_G)^* f)(v) = \sum_{e \in E(G), t(e)=v} \phi_{ev} f(e). \quad (2.6)$$

It is an easy verification (cf. Kenyon [21, §3.3]) that for the Laplacian, defined by (1.1), we have

$$\Delta^V_G = (\nabla^V_G)^* \nabla^V_G. \quad (2.7)$$

In general $(\nabla^V_G)^*$ is not the adjoint of $\nabla^V_G$ with respect to the appropriate $L^2$-metrics (see (1.10)). But if the connection $\nabla^V$ is unitary, it is indeed the case, cf. [21, §3.3]. In this article, all connections are unitary, and thus, by (2.7), the associated discrete Laplacians are self-adjoint and positive.

### 2.2 Properties of Friedrichs Extension of the Laplacian

In this section, we study the Laplacian $\Delta^F_\Sigma$ of a flat surface $\Sigma$ with conical singularities and piecewise geodesic boundary endowed with a flat unitary vector bundle $(F, h^F, \nabla^F)$ over $\Sigma$, (1.2). The main statements of this section are the discreteness of the spectrum of the Friedrichs extension of $\Delta^F_\Sigma$, see Proposition 2.3, and the Green identity for the elements from the domain of $\Delta^F_\Sigma$, see Proposition 2.5. The content of this section is most certainly not new, but we weren’t able to find a complete reference for all the results contained here.

We consider $\Delta^F_\Sigma$ as an operator acting on the functional space $\mathcal{C}^\infty_{0, N}(\Sigma, F)$, where

$$\mathcal{C}^\infty_{0, N}(\Sigma, F) := \left\{ f \in \mathcal{C}^\infty_0(\Sigma \setminus \text{Ang}(\Sigma), F) : \nabla_n f = 0 \text{ over } \partial \Sigma \right\}. \quad (2.8)$$
where \( n \) is the normal to \( \partial \Sigma \). Note that \( \mathcal{E}_{0,N}^\infty (\Sigma, F) \) is dense in \( L^2(\Sigma, F) \). However, unlike in the case of a manifold with smooth boundary, the operator \( \Delta^F_\Sigma \) is in general not essentially self-adjoint.

Let \( \text{Dom}_{\text{max}} (\Delta^F_\Sigma) \) denote the maximal domain of \( \Delta^F_\Sigma \). In other words, for \( u \in L^2(\Sigma, F) \), we have \( u \in \text{Dom}_{\text{max}} (\Delta^F_\Sigma) \) if and only if \( \Delta^F_\Sigma u \in L^2(\Sigma, F) \), where \( \Delta^F_\Sigma u \) is viewed as a distribution.

Let’s denote by \( W^k_p (\Sigma, F) \) the Sobolev space on \( \Sigma \), defined as

\[
W^k_p (\Sigma, F) = \left\{ u \in L^p (\Sigma, F) : \nabla^l u \in L^p (\Sigma, (T^* \Sigma)^{\otimes l} \otimes F) \quad \text{for} \quad 0 \leq l \leq k \right\}.
\]

We denote by \( \| \cdot \|_{W^k_p (\Sigma, F)} \) the norm on \( W^k_p (\Sigma, F) \), given for \( u \in W^k_p (\Sigma, F) \) by

\[
\| u \|_{W^k_p (\Sigma, F)} = \sum_{0 \leq l \leq k} \| \nabla^l u \|_{L^p (\Sigma, (T^* \Sigma)^{\otimes l} \otimes F)}.
\]

**Theorem 2.2** (Rellich–Kondrachov) The inclusion \( W^1_2 (\Sigma, F) \hookrightarrow L^2(\Sigma, F) \) is compact.

**Proof** Let \( u_j, j \in J \) be some bounded sequence in \( W^1_2 (\Sigma, F) \). Choose a finite number of functions \( f_i : \Sigma \to \{0, 1\}, i \in I \) such that for any \( x \in \Sigma \) away from a negligible set, there exists exactly one \( i \in I \) such that \( f_i(x) = 1 \), and, for any \( i \in I \), \( U_i := \text{supp}(f_i) \) is isomorphic to a subdomain of \( \mathbb{C} \) with piecewise smooth boundary. By [1, Theorem 6.2, p. 144], Rellich–Kondrachov theorem holds for \( U_i \). We conclude by extracting a converging subsequence of each summand on the right-hand side of the following expression \( u_k = \sum_{i \in I} f_i u_k \).

For any densely defined positive symmetric operator, one can construct in a canonical way a self-adjoint extension, called *Friedrichs extension*, by the completion of the associated quadratic form, cf. [26, Theorem X.23]. Once the definition is unraveled, the domain \( \text{Dom}_{Fr} (\Delta^F_\Sigma) \) of the Friedrichs extension of the Laplacian \( \Delta^F_\Sigma \) on \( \Sigma \) with Neumann boundary conditions on \( \partial \Sigma \) satisfies

\[
\text{Dom}_{Fr} (\Delta^F_\Sigma) \subset \text{Dom}_{\text{max}} (\Delta^F_\Sigma) \cap W^1_2 (\Sigma, F).
\]

The value of the Friedrichs extensions of the Laplacian \( \Delta^F_\Sigma \) on \( f \in \text{Dom}_{Fr} (\Delta^F_\Sigma) \) is defined in the distributional sense. By the definition of \( \text{Dom}_{\text{max}} (\Delta^F_\Sigma) \), it lies in \( L^2(\Sigma, F) \).

**Proposition 2.3** The spectrum of \( \Delta^F_\Sigma \) is discrete.

**Proof** Since the Friedrichs extension is non-negative by construction (cf. [26, Theorem X.23]), the kernel of \( 1 + \Delta^F_\Sigma \) is empty. Thus, the inverse \( (1 + \Delta^F_\Sigma)^{-1} \) is well defined. By (2.11), the image of \( (1 + \Delta^F_\Sigma)^{-1} \) lies in \( W^1_2 (\Sigma, F) \). By this and Theorem 2.2, \( (1 + \Delta^F_\Sigma)^{-1} \) is a compact operator. In particular, it has a discrete spectrum with only one possible accumulation point at 0. \( \square \)
Now, recall that for a smooth codimension 1 submanifold \( \Gamma \subset \Sigma \), transversal to the boundary \( \partial \Sigma \), \( p \geq 1 \) and \( k \geq 1 \), the trace (or restriction) operator

\[
W_p^k(\Sigma, F) \rightarrow W_p^{k-1}(\Gamma, F), \quad f \mapsto f|_{\Gamma}, \tag{2.12}
\]

is well defined, cf. [19, Theorem 1.5.1.1]. We apply (2.12) implicitly when we mention an integration over \( \Gamma \) of a function from \( W_p^k(\Sigma, F) \), \( k \geq 1 \).

**Proposition 2.4** The subset \( C^\infty_{0,N}(\Sigma, F) \) is dense inside of \( W_2^1(\Sigma, F) \).

**Proof** A standard density result says that \( C^\infty(\Sigma \setminus \text{Ang}(\Sigma), F) \) is dense inside of \( W_2^1(\Sigma, F) \). Thus, it is enough to show that for any \( f \in C^\infty(\Sigma \setminus \text{Ang}(\Sigma), F) \), one can find \( g \in C^\infty_{0,N}(\Sigma, F) \), such that \( f - g \) lies in the closure of \( C^\infty(\Sigma \setminus \partial \Sigma, F) \) inside of \( W_2^1(\Sigma, F) \). To do so, it is enough to take \( g \in C^\infty_{0,N}(\Sigma, F) \), verifying \( g|_{\partial \Sigma} = f|_{\partial \Sigma}, \Omega \) to be a domain with smooth boundary such that \( \supp f \cup \supp g \subset \Omega \), and to apply [19, Corollary 1.5.1.6], which says that for a domain \( \Omega \) with smooth boundary, any \( h \in W_2^1(\Omega) \), verifying \( h|_{\partial \Omega} = 0 \), lies in the closure of \( C^\infty(\Omega \setminus \partial \Omega) \). \( \square \)

**Proposition 2.5** (Green’s identity) For any open subset \( U \subset \Sigma \) with piecewise smooth boundary \( \partial U \) not passing through \( \text{Con}(\Sigma) \) and \( \text{Ang}(\Sigma) \), and any \( u, v \in \text{Dom}_F(\Delta_S^F) \), we have

\[
\langle \Delta_S^F u, v \rangle_{L^2(U,F)} = \langle \nabla^F u, \nabla^F v \rangle_{L^2(U,T^*\Sigma \otimes F)} - \int_{\partial U} \nabla^F_n u \cdot vd\nu_{\partial U}, \tag{2.13}
\]

where \( n \) is the outward normal to the boundary \( \partial U \) (to simplify the notations, we omit the pointwise scalar product induced by \( h^F \) in the last integral).

**Proof** First of all, elliptic estimates, the fact that \( u, \Delta_S^F u \in L^2(\Sigma, F) \), (2.11), (2.11) and (2.12) imply that all the terms in (2.13) are well defined. Now, let’s prove (2.13) for \( u \in \text{Dom}_F(\Delta_S^F) \) and \( v \in C^\infty_{0,N}(\Sigma, F) \). Let \( W \subset \Sigma \), \( W \cap (\text{Con}(\Sigma) \cup \text{Ang}(\Sigma)) = \emptyset \), be an open set with smooth boundary \( \partial W \), such that

\[
\text{supp}(v) \subset W. \tag{2.14}
\]

By elliptic estimates, the restriction \( u|_W \) of \( u \) to \( W \) lies inside of \( W_2^2(W, F) \). From the classical Green’s identity, cf. Grisvard [19, Lemma 1.5.3.8], we see that

\[
\langle \Delta_S^F u, v \rangle_{L^2(W,F)} = \langle \nabla^F u, \nabla^F v \rangle_{L^2(W,T^*\Sigma \otimes F)} - \int_{\partial W} \nabla^F_n u \cdot vd\nu_{\partial W}. \tag{2.15}
\]

Now, by (2.14) and the fact that \( \Delta_S^F u \in L^2(\Sigma, F) \), we deduce \( \langle \Delta_S^F u, v \rangle_{L^2(W,F)} = \langle \Delta_S^F u, v \rangle_{L^2(\Sigma,F)} \). By (2.14) and the fact that \( u \in W_2^1(\Sigma, F) \), we deduce \( \langle \nabla^F u, \nabla^F v \rangle_{L^2(W,F)} = \langle \nabla^F u, \nabla^F v \rangle_{L^2(\Sigma,F)} \). By (2.14), the fact that \( u \in W_2^1(W, F) \), thus, \( \nabla^F_n u \in L^2(\partial W, F) \), and the identity \( \nabla^F_n u = 0 \) over \( \partial \Sigma \), we deduce that

\[
\int_{\partial W} \nabla^F_n u \cdot vd\nu_{\partial W} = \int_{\partial U} \nabla^F_n u \cdot vd\nu_{\partial U}. \tag{2.16}
\]
From (2.15)-(2.16), we conclude that (2.13) holds for \( u \in \text{Dom}_{Fr}(\Delta^F_{\Sigma}) \) and \( v \in \mathcal{C}_{0,N}^\infty(\Sigma, F) \).

Now, let’s argue why (2.13) also holds for \( u \in \text{Dom}_{Fr}(\Delta^F_{\Sigma}) \) and \( v \in W^1_2(\Sigma, F) \). Indeed, by Proposition 2.4, there is a sequence \( u_m \in \mathcal{C}_{0,N}^\infty(\Sigma, F) \) such that \( u_m \to v \) in \( W^1_2(\Sigma, F) \). Then, since \( \nabla^F u \in L^2(\partial U, F) \), we see by the continuity of the trace operator that the last term of (2.13) associated to \( v_m \), converges to the respective term associated to \( v \). Similarly, we get the convergence for the first and the second terms of (2.13). In the limit we get (2.13) for \( u \in \text{Dom}_{Fr}(\Delta^F_{\Sigma}) \) and \( v \in W^1_2(\Sigma, F) \). By (2.11), we get (2.13) for \( u, v \in \text{Dom}_{Fr}(\Delta^F_{\Sigma}) \).

**Corollary 2.6** The kernel of \( \Delta^F_{\Sigma} \) consists of flat sections of \( F \). In other words,

\[
\ker \Delta^F_{\Sigma} \simeq H^0(\Sigma, F).
\]

**Proof** First of all, it’s easy to see that the flat sections are from \( \text{Dom}_{Fr}(\Delta^F_{\Sigma}) \) (for example, it is a trivial consequence of Proposition 2.8), and they are trivially from the kernel of \( \Delta^F_{\Sigma} \). Now, let \( u \in \ker(\Delta^F_{\Sigma}) \). Then from Proposition 2.5, we deduce \( \langle \Delta^F_{\Sigma} u, u \rangle_{L^2(\Sigma, F)} = \langle \nabla^F u, \nabla^F u \rangle_{L^2(\Sigma, T^*\Sigma \otimes F)} \). Thus, \( \nabla^F u = 0 \), hence \( u \) is a flat section.

\[\square\]

### 2.3 A Description of the Domain of Friedrichs Extension

In this section, we give a more explicit description of the domain of the Friedrichs extension of the Laplacian. This description will later play an important role in the proof of Theorem 1.1. We don’t claim originality on this section, but again we weren’t able to find a complete reference for it.

We describe \( \text{Dom}_{Fr}(\Delta^F_{\Sigma}) \) by prescribing some asymptotical behavior near \( \text{Con}(\Sigma) \cup \text{Ang}(\Sigma) \) to the functions from it. To begin, we suppose that \( \partial \Sigma = \emptyset \) and that \( \langle F, h^F, \nabla^F \rangle \) is a trivial line bundle. As all our considerations are local, it is sufficient to consider a surface \( \Sigma \) with only one conical point \( P \) of the conical angle \( \theta = \angle(P) \).

For \( k \in \mathbb{N} \), we introduce the functions \( C^{k}_{\pm, \theta} \) on the model cone \( C_\theta \), (2.2), by

\[
C^{k}_{\pm, \theta}(r, \phi) = r^{\pm \frac{2\pi k}{\theta}} \exp\left(\sqrt{-1} \frac{2\pi k \phi}{\theta}\right), \quad \text{for } k > 0,
\]

\[
C^{0}_{\pm, \theta}(r, \phi) = 1, \quad C^{0}_{0, \theta}(r, \phi) = \log(r).
\]

Those functions are formal solutions to \( \Delta_{C_\theta} u = 0 \) on the cone \( C_\theta \), (2.2). The functions \( C^{k}_{\pm, \theta}, k < \frac{\theta}{2\pi} \) belong to the \( L^2 \)-space \( L^2(C_\theta) \) with respect to the conical metric (2.1).

Let \( \chi_P \) be a smooth function on \( \Sigma \) which is equal to 1 near \( P \) and such that in a vicinity of the support of \( \chi_P \), the manifold \( \Sigma \) is isometric to the neighborhood of the tip of \( C_\theta \). We use this isometry to view the function \( \chi_P C^{k}_{\pm, \theta} \) as a function on \( \Sigma \).

Let \( \text{Dom}_{\min}(\Delta_{\Sigma}) \) denote the minimal closure of \( \Delta_{\Sigma} \), viewed as an operator on \( \mathcal{C}_{0,N}^\infty(\Sigma) \). In other words, for \( u \in L^2(\Sigma) \), we have \( u \in \text{Dom}_{\min}(\Delta_{\Sigma}) \) if and only if there exists a sequence of functions \( u_m \in \mathcal{C}_{0,N}^\infty(\Sigma) \) such that \( u_m \to u \) in \( L^2(\Sigma) \) and
ΔΣu_m → w for some w ∈ L^2(Σ). Clearly, in this case, we have ΔΣu = w on the level of distributions.

Recall that the maximal domain Dom_{max}(ΔΣ) was defined in the beginning of Sect. 2.2.

**Proposition 2.7** (Mooers [24, Proposition 2.3]) The following identity holds

\[
\text{Dom}_{\text{max}}(ΔΣ) = \text{Dom}_{\text{min}}(ΔΣ) + \sum_{0 \leq k < \frac{\theta}{2\pi}} \langle χ P C^k_{+\beta} \rangle + \sum_{0 \leq k < \frac{\theta}{2\pi}} \langle χ P C^k_{-\beta} \rangle, \tag{2.19}
\]

where by \(\langle v \rangle\) we mean a vector space spanned by the vectors \(v \cdot e_i\), where \(e_i, i = 1, \ldots, \text{rk}(F)\) is a flat local frame in the neighborhood of a fixed point over which the summation is done.

Now, the description of the set of all self-adjoint extensions of ΔΣ looks as follows, cf. Mooers [24, Theorem 2.1]. Denote by \(M\) the linear subspace of \(L^2(Σ, F)\) spanned by the functions \(χ P C^k_{±\beta}\) with \(0 \leq k < \frac{\theta}{2\pi}\). The dimension, 2d, of \(M\) is even. To get a self-adjoint extension of ΔΣ, one chooses a subspace \(N\) of \(M\) of dimension \(d\) such that for any \(u, v \in N\), we have

\[
\langle ΔΣu, v \rangle_{L^2(Σ)} - \langle u, ΔΣv \rangle_{L^2(Σ)} = 0. \tag{2.20}
\]

To any such \(N\) there corresponds a self-adjoint extension of ΔΣ with domain \(\text{Dom}_{\text{min}}(ΔΣ) + N\).

The domain of the Friedrichs extension \(\text{Dom}_{Fr}(ΔΣ)\) corresponds to the choice of functions \(χ P C^k_{±\beta}\) with \(0 \leq k < \frac{\theta}{2\pi}\). To verify this, by Proposition 2.7, (2.11) and the description above, it is enough to see that the above functions span the only subspace of dimension \(\frac{1}{2} \dim(\text{Dom}_{\text{max}}(ΔΣ)/\text{Dom}_{\text{min}}(ΔΣ))\) inside of \(M\), which lies in \(W^1_2(Σ, F)\). This is very easy to see directly, see also Proposition 2.9 below.

To make further notation easier, for \(P ∈ \text{Con}(Σ)\), we denote

\[
C^k_P := χ P C^k_{+\beta(P)}. \tag{2.21}
\]

Now let’s come back to the original surface Σ with piecewise geodesic boundary \(\partial Σ\) and a flat unitary vector bundle \((F, h^F, ∇^F)\) over \(\overline{Σ}\). Consider a double manifold \(\widetilde{Σ} := Σ ∪ Σ^*,\) where \(Σ^*\) is isomorphic to Σ but with the opposite orientation, and Σ and \(Σ^*\) are glued along the boundary \(\partial Σ\) in an obvious way. Then its easy to see that \(\widetilde{Σ}\) has a structure of a flat surface with conical angles \(α\), where \(α ∈ \angle(\text{Con}(Σ))\) is of double multiplicity coming from the conical angles of Σ and \(Σ^*,\) and \(2\beta\), where \(β ∈ \angle(\text{Ang}(Σ))\), coming from corners of the boundary.

The manifold \(\widetilde{Σ}\) has a natural involution \(i\), interchanging Σ and \(Σ^*\). The fixed point set of \(i\) is equal to \(\partial Σ ↪ \widetilde{Σ}\). We denote by \(π : \widetilde{Σ} → Σ\) the obvious projection, for which we have \(π ∘ i = π\). The pullback \(π^*(F, h^F, ∇^F)\) gives a flat unitary vector bundle \((\widetilde{F}, h^{\widetilde{F}}, ∇^{\widetilde{F}})\) over \(\widetilde{Σ}\).

Define the functions \(A^k_Q, Q ∈ \text{Ang}(Σ), k ∈ \mathbb{N}\), by

\[
A^k_Q(r, φ) := χ_Q(C^k_{+2β}(r, φ) + C^k_{+2β}(r, -φ)), \tag{2.22}
\]

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where $\chi_Q$ is a smooth function on $\Sigma$ which is equal to 1 near $Q$ and such that in a vicinity of the support of $\chi_Q$, the manifold $\Sigma$ is isometric to a neighborhood of a standard angle

$$A_\theta := \{(r, t) : r \geq 0; 0 \leq t \leq \theta\}/(0,t)\sim(0,t'),$$

(2.23)

endowed with the metric (2.1). Remark that $A^k_Q(r, \phi)$ satisfy Neumann conditions on $\partial A_\theta$.

Directly from the definition of the Friedrichs extension, we see that the domain $\text{Dom}_{Fr}(\Delta^F_\Sigma F)$ of the Friedrichs extensions of the Laplacian $\Delta^F_\Sigma F$ on $\Sigma$ with Neumann boundary conditions on $\partial \Sigma$ is related to the domain $\text{Dom}_{Fr}(\Delta^E_\Sigma \tilde{F})$ of the Friedrichs extensions of the Laplacian $\Delta^E_\Sigma \tilde{F}$ as follows: $u \in \text{Dom}_{Fr}(\Delta^F_\Sigma F)$ if and only if $\pi^*u \in \text{Dom}_{Fr}(\Delta^E_\Sigma \tilde{F})$. Moreover, the following identity holds

$$\pi^*(\Delta^F_\Sigma F u) = \Delta^E_\Sigma \tilde{F}(\pi^*u).$$

(2.24)

From the above description of the case with empty boundary and (2.24), we deduce

**Proposition 2.8** The domain $\text{Dom}_{Fr}(\Delta^F_\Sigma F)$ of Friedrichs extension can be described as

$$\text{Dom}_{Fr}(\Delta^F_\Sigma F) = \text{Dom}_{\text{min}}(\Delta^E_\Sigma \tilde{F}) + \sum_{P \in \text{Con}(\Sigma)} \sum_{0 \leq k < \angle(P)/2\pi} \langle C^k_P \rangle + \sum_{Q \in \text{Ang}(\Sigma)} \sum_{0 \leq k < \angle(Q)/\pi} \langle A^k_Q \rangle,$$

(2.25)

where we used the same notation as in Proposition 2.7 for $\langle v \rangle$.

**Proposition 2.9** For $P \in \text{Con}(\Sigma)$ and $Q \in \text{Ang}(\Sigma)$, we have

$$C^k_P \in W^l_p(\Sigma) \text{ if and only if } k = 0 \text{ or } \frac{2\pi k}{\angle(P)} > l - \frac{2}{p},$$

$$A^k_Q \in W^l_p(\Sigma) \text{ if and only if } k = 0 \text{ or } \frac{\pi k}{\angle(Q)} > l - \frac{2}{p}.$$

(2.26)

In addition, for any $k \in \mathbb{N}$, $P \in \text{Con}(\Sigma)$ and $Q \in \text{Ang}(\Sigma)$, we have

$$C^k_P, A^k_Q \in \text{Dom}_{Fr}(\Delta_\Sigma F).$$

(2.27)

**Proof** The statement (2.26) follows by an explicit calculation, cf. Grisvard [19, Preface and Lemma 4.4.3.5]. The statement (2.27) is proved by using the same argument as the one we used before (2.21). □
Proposition 2.10 For any \((\Sigma, g^T \Sigma)\) and \((F, h^F, \nabla F)\) as before, the following inclusion holds

\[
\text{Dom}_{Fr}(\Delta^F_{\Sigma}) \subset L^\infty(\Sigma, F).
\] (2.28)

In addition, there exists \(p_0 > 1\), which depends only on the sets \(\angle(\text{Con}(\Sigma))\) and \(\angle(\text{Ang}(\Sigma))\), such that

\[
\text{Dom}_{Fr}(\Delta^F_{\Sigma}) \subset W^{2}_{p_0}(\Sigma, F).
\] (2.29)

Remark 2.11 In fact, the Friedrichs extension is the only extension satisfying (2.28). This can be easily seen from the general description of self-adjoint extensions, given after Proposition 2.7.

Proof In [19, Theorem 4.3.1.4], Grisvard proved the following a priori estimate: there exists \(C > 0\) such that for any \(u \in W^2_2(\Sigma, F)\), the following elliptic estimate holds

\[
\|u\|_{W^2_2(\Sigma, F)} \leq C(\|u\|_{L^2(\Sigma, F)} + \|\Delta_{\Sigma} F u\|_{L^2(\Sigma, F)}).
\] (2.30)

We note that Grisvard proved this estimate only for polygons with no vector bundles, but the proof remains obviously valid for general flat surfaces with piecewise geodesic boundary and flat unitary vector bundle. From (2.30), we deduce that

\[
\text{Dom}_{\min}(\Delta^F_{\Sigma}) \subset W^2_2(\Sigma, F).
\] (2.31)

From Proposition 2.9, we see that one can choose \(p > 1\) very close to 1 so that \(C^k_P \in W^2_2(\Sigma)\) for \(P \in \text{Con}(\Sigma)\) and \(0 \leq k < \frac{\angle(P)}{2\pi}\) and \(A^k_Q \in W^2_2(\Sigma)\) for \(Q \in \text{Ang}(\Sigma)\) and \(0 \leq k < \frac{\angle(Q)}{\pi}\). By this, (2.25) and (2.31), we deduce (2.29).

Now, by Sobolev estimates, cf. Adams [1, Theorem 5.4 Part II, Case C”], the following holds

\[
W^2_2(\Sigma, F) \subset L^\infty(\Sigma, F).
\] (2.32)

We note that Adams proves (2.32) only for planar domains satisfying cone property (cf. Adams [1, p. 66] for a definition of the cone property), but since a flat surface can be decomposed into sectors, which obviously satisfy the cone property, the inclusion (2.32) continues to hold for all \(\Sigma\). Now, (2.28) follows from (2.32) and the trivial fact that \(C^k_P, A^k_Q \in L^\infty(\Sigma)\).

3 Finite Difference Method, Proofs of Theorems 1.1, 1.3

In this section, we investigate the extension of finite difference method from lattices \(\frac{1}{n} \mathbb{Z}^2, n \in \mathbb{N}^*\) in \(\mathbb{R}^2\) to general graphs \(\Sigma_n\) “approximating” tileable surfaces \(\Sigma\) (see Sect. 2.1). In particular, we prove Theorems 1.1, 1.3. We conserve the notation from Theorems 1.1.
This section is organized as follows. The main result of Sect. 3.1 is

**Theorem 3.1** *For any* \( i \in \mathbb{N} \), *the following bound holds*

\[
\lambda_i \geq \limsup_{n \to \infty} \lambda_i^n.
\]  
(3.1)

In Sect. 3.2, modulo some Harnack-type inequality, Theorem 3.11 (which we prove in Sect. 4), we prove the following statement.

**Theorem 3.2** *For any* \( i \in \mathbb{N} \), *the following bound holds*

\[
\liminf_{n \to \infty} \lambda_i^n \geq \lambda_i.
\]  
(3.2)

Clearly, Theorems 3.1, 3.2 and Theorem 1.1 are equivalent. However, we prefer to state them separately, since the techniques we use in the proofs of Theorems 3.1 and 3.2 are rather different.

Then, in Sect. 3.3, we show that Theorem 1.3 follows almost formally from the arguments, developed in the course of the proof of Theorem 1.1.

### 3.1 Regularity of Eigenvectors on a Flat Surface, a Proof of Theorem 3.1

The main goal of this section is to prove Theorem 3.1. The main idea is to start from an eigenvector \( f_i \) corresponding to an eigenvalue \( \lambda_i \) of the Friedrichs extensions of the Laplacian \( \Delta^F_\Sigma \), i.e.

\[
\Delta^F_\Sigma f_i = \lambda_i f_i,
\]  
(3.3)

and to construct a section \( R_n f_i \in \text{Map}(V(\Sigma_n), F_n) \) by restriction, (1.3). We prove that the Rayleigh quotients associated to \( f_i \) and \( R_n f_i \) are close enough. Then Theorem 3.1 would follow from a simple application of the min-max theorem. The main difficulty is that due to the lack of elliptic regularity near singularities, \( f_i \) does not extend smoothly up to \( \text{Con} (\Sigma) \cup \text{Ang} (\Sigma) \).

More precisely, we define the restriction operator through the injection (1.3) as follows

\[
R_n : \mathcal{C}^0 (\Sigma \setminus \partial \Sigma, F) \to \text{Map}(V(\Sigma_n), F_n), \quad (R_n f)(v) := f(v).
\]  
(3.4)

Let’s first show that the rescaled discrete Laplacians \( n^2 \cdot \Delta^F_{\Sigma_n} \) approximate weakly the smooth Laplacian \( \Delta^F_\Sigma \) with Neumann boundary conditions. More precisely, the following holds.

**Proposition 3.3** *There is a constant* \( C > 0 \) *such that for any* \( f \in \mathcal{C}^3 (\Sigma, F) \), *satisfying (1.5), any* \( n \in \mathbb{N}^* \) *and any* \( P \in V(\Sigma_n) \), *the following bound holds*

\[
\left| n^2 \cdot \Delta^F_{\Sigma_n} (R_n f)(P) - (\Delta^F_\Sigma f)(P) \right| \leq \frac{C}{n} \| f \|_{\mathcal{C}^3 (B_{\Sigma_n}(\frac{3}{n}, P))},
\]  
(3.5)
Proof As the geometry of discretization is prescribed by (1.3), (3.5) follows from
Taylor expansion, applied in three different situations: when the vertex $P \in V(\Sigma_n)$
has degree 2, 3 or 4.

Now, we consider the eigenspace $V_k$, $k \in \mathbb{N}$, associated with the first $k$ eigenvalues
of $\Delta^F_\Sigma$. In the end of this section, by studying explicitly the structure of the singularities
of $f \in V_k$ near $\text{Con}(\Sigma)$ and $\text{Ang}(\Sigma)$, we show the following

**Theorem 3.4** For any $f, g \in V_k$, as $n \to \infty$, we have

$$\langle \Delta^F_{\Sigma_n} (R_n f), R_n g \rangle_{L^2(\Sigma_n, F_n)} \to \langle \Delta^E f, g \rangle_{L^2(\Sigma, F)}.$$  

(3.6)

Let’s see how Theorem 3.1 follows from Theorem 3.4.

**Proof of Theorem 3.1** By elliptic regularity, we know that the eigenvectors $f_i, i \in \mathbb{N}$ of
$\Delta^F_\Sigma$ are smooth in the interior of $\Sigma$. By this and (2.28), we conclude $f_i \in C^\infty(\Sigma, F) \cap L^\infty(\Sigma, F)$. From this and the fact that the tiles have area 1, we see that for any
$f, g \in V_k$, as $n \to \infty$:

$$\frac{1}{n^2} \langle R_n f, R_n g \rangle_{L^2(\Sigma_n, F_n)} \to \langle f, g \rangle_{L^2(\Sigma, F)}.$$  

(3.7)

Construct a vector space $V^*_k := R_n(V_k)$. From Theorem 3.4 and (3.7), we get

$$\lim_{n \to \infty} \sup_{f \in V^*_k} \left\{ \frac{n^2 \cdot (\Delta^F_{\Sigma_n} f, f)}{\langle f, f \rangle_{L^2(\Sigma_n, F_n)}} \right\} = \lambda_k.$$  

(3.8)

Now, by the characterization of the eigenvalues through Rayleigh quotient, we have

$$\lambda^n_k = \inf_{V \subset \text{Map}(V(\Sigma_n), F_n)} \sup_{f \in V} \left\{ \frac{n^2 \cdot (\Delta^F_{\Sigma_n} f, f)}{\langle f, f \rangle_{L^2(\Sigma_n, F_n)}} : \dim V = k \right\}.$$  

(3.9)

Clearly, for $n$ big enough, we have $\dim V^*_k = k$. By this, from (3.9), we see that

$$\lambda^n_k \leq \sup_{f \in V^*_k} \left\{ \frac{n^2 \cdot (\Delta^F_{\Sigma_n} f, f)}{\langle f, f \rangle_{L^2(\Sigma_n, F_n)}} \right\}$$  

(3.10)

By (3.8) and (3.10), we deduce Theorem 3.1. ⊓⊔

To finish the proof of Theorem 3.1, we only need to prove Theorem 3.4. The main
difficulty lies in the fact that in general, the finite differences $n^2 \cdot \Delta^F_{\Sigma_n} R_n(f_i)$ might
“explode” near $\text{Con}(\Sigma) \cup \text{Ang}(\Sigma)$. We use the theory of elliptic regularity for polygons
developed by Grisvard, [19], to prove that this divergence poses no problem once we are
concerned with the $L^2$-product in (3.6). Alternatively, one can use the weighted elliptic
estimates from the book of Maz’ya-Nazarov-Plamenevskii [23, §1.3.6] to deduce those
results.
Recall that the functions $C^k_P$ and $A^k_Q$ for $P \in \text{Con}(\Sigma)$, $Q \in \text{Ang}(\Sigma)$ and $k \in \mathbb{N}$ were defined in (2.21) and (2.22). The main result of this section goes as follows.

**Theorem 3.5** There are coefficients $\gamma^k_{P,i} \in \mathbb{R}$ for $P \in \text{Con}(\Sigma)$, $k \in \mathbb{N}, 0 < \frac{2\pi k}{z(P)} \leq 2, 1 \leq i \leq \text{rk}(F)$ and $\alpha^k_{Q,i} \in \mathbb{R}$ for $Q \in \text{Ang}(\Sigma)$, $k \in \mathbb{N}, 0 < \frac{\pi k}{z(Q)} \leq 2, 1 \leq i \leq \text{rk}(F)$ such that the section $g_i$, defined as

$$g_i := f_i - \sum_{P \in \text{Con}(\Sigma)} \sum_{0 < \frac{2\pi k}{z(P)} \leq 2} \gamma^k_{P,i} \cdot e^i \cdot C^k_P - \sum_{Q \in \text{Ang}(\Sigma)} \sum_{0 < \frac{\pi k}{z(Q)} \leq 2} \alpha^k_{Q,i} \cdot e^i \cdot A^k_Q,$$

(3.11)

satisfies the following regularity properties (see (1.15))

$$g_i \in C^2(\Sigma, F).$$

(3.12)

In (3.11), we used the Einstein summation convention for local flat frames $e_j$ of $F$ based at respective points.

Let’s now see how Theorem 3.4 follows from Theorem 3.5. After that, we prove Theorem 3.5.

**Proof of Theorem 3.4** By elliptic regularity, the functions $f_i, g_i$ are smooth away from $\text{Con}(\Sigma) \cup \text{Ang}(\Sigma)$. So, by Proposition 3.3, it’s clear that to prove (3.6), it is enough to prove that for any $\epsilon > 0$, there is $c > 0$, such that for any $P \in \text{Con}(\Sigma) \cup \text{Ang}(\Sigma)$, we have

$$\langle \Delta^F_{\Sigma} f_i, f_j \rangle_{L^2(B_{\Sigma}(c,P))} \leq \epsilon,$$

(3.13)

$$\langle \Delta^F_{\Sigma} (R_n f_i), R_n f_j \rangle_{L^2(B_{\Sigma_n}(cn,P))} \leq \epsilon.$$  

(3.14)

The bound (3.13) follows from $\Delta^F_{\Sigma} f_i, f_j \in L^2(\Sigma, F)$. Let’s prove (3.14). By (2.18), (3.5) and the identity $\Delta^F_{\Sigma} C^k_P = 0$, which holds near $P$, there is $C > 0$ such that for any $n \in \mathbb{N}^*$, we have

$$\left| \Delta^F_{\Sigma} (R_n C^k_P)(Q) \right| \leq \frac{C}{n^3} \left( \frac{\text{dist}_{\Sigma_n}(P, Q)}{n} \right)^{\frac{2\pi k}{z(P)} - 3}.$$

(3.15)

Thus, we conclude that for any $k, l \in \mathbb{N}$, and $\alpha := \frac{2\pi k}{z(P)} + \frac{2\pi l}{z(P)}$, we have

$$\langle \Delta^F_{\Sigma} (R_n C^k_P), R_n C^l_P \rangle_{L^2(B_{\Sigma_n}(cn,P))} \leq \frac{1}{n^\alpha} \sum_{Q \in B_{\Sigma_n}(cn,P)} \text{dist}_{\Sigma_n}(P, Q)^{\alpha - 3}. \quad (3.16)$$

However, trivially, for any $\epsilon > 0$, there is $c > 0$ such that for $n$ big enough, we have

$$\sum_{Q \in B_{\Sigma_n}(cn,P)} \text{dist}_{\Sigma_n}(P, Q)^{\alpha - 3} \leq 4 \frac{\epsilon(P)}{2\pi} \sum_{i=0}^{cn} i^{\alpha - 2} \leq \epsilon n^\alpha.$$

(3.17)
Now, by (3.12), we conclude that there is $C > 0$ such that
\[
\| n^2 \cdot \Delta_{\Sigma}^F (R_n g_i) \|_{L^\infty(\Sigma, F)} \leq C. \tag{3.18}
\]
By (3.16), (3.17) and the analogous estimates for $A^k_Q$, $Q \in \text{Ang}(\Sigma)$, $k \in \mathbb{N}$, (3.18), we get (3.14), which finishes the proof. \hfill \Box

**Proof of Theorem 3.5** By Proposition 2.10, there is $p > 1$, such that we have
\[
f_i \in \text{Dom}_{Fr}(\Sigma_1^F) \subset W_p^2(\Sigma, F). \tag{3.19}\]
Then, by Grisvard [19, Theorem 5.1.3.1], we know that there exists a function $u \in L^2(\Sigma, F)$ satisfying Neumann boundary conditions on $\partial \Sigma$, solving the equation
\[
\Delta_{\Sigma}^F u = \lambda_i f_i, \tag{3.20}
\]
and for which there are $\gamma_{P}^{k,j} \in \mathbb{R}$ for $P \in \text{Con}(\Sigma)$, $k \in \mathbb{N}$, $0 < \frac{2\pi k}{\angle(P)} \leq 4 - \frac{2}{p}$, $1 \leq j \leq \text{rank}(F)$ and $\alpha_{Q}^{k,j} \in \mathbb{R}$ for $Q \in \text{Ang}(\Sigma)$, $k \in \mathbb{N}$, $0 < \frac{\pi k}{\angle(Q)} \leq 4 - \frac{2}{p}$, $1 \leq j \leq \text{rank}(F)$ such that
\[
u = \sum_{P \in \text{Con}(\Sigma)} \sum_{0 < \frac{2\pi k}{\angle(P)} \leq 4 - \frac{2}{p}} \gamma_{P}^{k,j} \cdot e_j \cdot C^k_{P} - \sum_{Q \in \text{Ang}(\Sigma)} \sum_{0 < \frac{\pi k}{\angle(Q)} \leq 4 - \frac{2}{p}} \alpha_{Q}^{k,j} \cdot e_j \cdot A^k_{Q} \in W_p^4(\Sigma, F), \tag{3.21}
\]
where we used Einstein summation convention for local flat frames $e_j$ based at respective points.

We remark that Grisvard’s result was proved for polygons. For flat surfaces the proof remains identical, as one could obtain a cone by gluing a corner with Dirichlet boundary conditions and another corner with Neumann boundary conditions, and the result of Grisvard was proved by using local techniques (see in particular [19, p. 252, 253]) for corners with arbitrary boundary conditions and arbitrary angles (see [19, Remarks 4.3.2.7, 5.1.3.7 and §4.2]).

Recall that by Sobolev embedding theorem (cf. Adams [1, Part II, Theorem 6.2]), we have
\[
W_p^4(\Sigma, F) \subset W_2^1(\Sigma, F). \tag{3.22}\]
Remark that (3.22) was proved in [1] only for domains satisfying the cone property (see [1, §4] for a definition) and without vector bundles. However, similarly to the proof of Theorem 2.2, by decomposing $\Sigma$ into sectors, which obviously satisfy the cone property, we see that (3.22) holds.

By Proposition 2.9, (3.21) and (3.22), we see that $u \in W_2^1(\Sigma, F)$. However, by Grisvard [19, Lemma 4.4.3.1, Remark 5.1.3.6], the solutions of $\Delta_{\Sigma}^F u = \lambda_i f_i$, $u \in$
$W_2^1(\Sigma, F)$, satisfying Neumann boundary conditions are unique up an element from $\ker \Delta^F_\Sigma$. By (2.11), we see that $f_i \in W_2^1(\Sigma, F)$, thus, by the argument above, the fact that $\Delta^F_\Sigma f_i = \lambda_i f_i$ and (3.20), there is $k \in \ker \Delta^F_\Sigma$, such that

$$u = f_i + k.$$  

(3.23)

Now, by Sobolev embedding theorem, cf. [1, p.97, (8)], for $p > 1$, the following inclusion holds

$$W_p^4(\Sigma, F) \hookrightarrow C^2(\Sigma, F).$$  

(3.24)

We conclude by (3.21), (3.23) and (3.24).

Remark 3.6 To get only Theorem 3.1, without a more precise statement, Theorem 3.4, it is possible to avoid the technicalities near the cone points by working at the level of quadratic forms, and using the fact that $C^\infty_{0, N}(\Sigma, F)$ is dense in $W_2^1(\Sigma, F)$. We, however, decided to avoid this path for several reasons. First, we feel that Theorem 3.4 and its proof adds more to our understanding of the smooth Laplacian, and how the discrete Laplacians converge to it. Second, in our proof of Theorem 1.3 in Sect. 3.3, we use Theorem 3.4.

3.2 Linearization Functional, a Proof of Theorem 3.2

The main goal of this section is to prove Theorem 3.2. The main idea is similar to the one from Sect. 3.1, but the methods are crucially different. We start from an eigenvector $f_i^n$ corresponding to an eigenvalue $\lambda_i^n$ of the discrete model $n^2 \cdot \Delta^F_{\Sigma_n}$ and construct by “linearization” an element $L_n(f_i^n) \in L^2(\Sigma, F)$. We prove that the Rayleigh quotients associated to $f_i^n$ and $L_n(f_i^n)$ are close enough. Then Theorem 3.2 would follow from a simple application of the min-max theorem.

We note that all the statements of this section are local in nature. To simplify largely our notation, we will fix a local flat frame of $(F, \nabla F)$ around some point, and constantly write all the statements of this section in this local frame. Of course, the final result will never depend on the initial choice of the frame.

Let’s define the “linearization” functional $L_n : \text{Map}(V(\Sigma_n), F_n) \to \mathcal{C}^0(\Sigma, F)$. Recall that the sets $V_n(P)$ and $U_n(P)$ were defined in Sect. 2.1. First, for $f \in \text{Map}(V(\Sigma_n), F_n)$, let’s define $f^{avg} \in \text{Map}(V(\Sigma_n), F_n)$ by averaging the function $f$ on $V_n(P)$, i.e.

$$f^{avg}(v) := \begin{cases} \frac{1}{\# V_n(P)} \sum_{Q \in V_n(P)} f(Q), & \text{if there is } P \in \text{Con}(\Sigma) \cup \text{Ang}(\Sigma), \\ v \in V_n(P), & \text{otherwise.} \end{cases}$$  

(3.25)

By definition, the functional $L_n$ satisfies $L_n(f) = L_n(f^{avg})$. Now, we define $L_n$ by describing explicitly the value of $(L_n f)(z)$ for any $z \in \Sigma$ and any $f \in \text{Map}(V(\Sigma_n), F_n)$, satisfying $f = f^{avg}$.
Suppose \( z \in U_n(P) \) for some \( P \in \text{Con}(\Sigma) \cup \text{Ang}(\Sigma) \). We define
\[
(L_n f)(z) := f(Q), \quad \text{where} \quad Q \in V_n(P).
\]
By the assumption, \( f = f^{avg} \). So \((L_n f)(z)\) doesn’t depend on the choice of \( Q \).

Next, suppose that \( \text{dist}(z, \partial \Sigma) < \frac{1}{2n} \) and that for any \( P \in \text{Con}(\Sigma) \cup \text{Ang}(\Sigma) \), we have \( z \notin U_n(P) \). Then let \( P, Q \in V(\Sigma_n) \) be the two closest points to \( z \) in \( V(\Sigma_n) \). In case if there are several choices, take any. The geometrical place of points \( z \in \Sigma \), satisfying \( \text{dist}(z, \partial \Sigma) < \frac{1}{2n} \), and having \( P \) and \( Q \) as their closest points in \( V(\Sigma_n) \), is a rectangle. We denote this rectangle by \( R \). The points \( P \) and \( Q \) have either the same \( x \) or \( y \) coordinates, where \( x \) and \( y \) are linear coordinates having axes parallel to the boundaries of the tiles of \( \Sigma \). Suppose that they share the same \( y \) coordinate, the other case is treated similarly. We renormalize \( x \) coordinate so that it satisfies \( x(P) = 1 \) and \( x(Q) = 0 \). Then we define
\[
(L_n f)(z) := f(P)x(z) + f(Q)(1 - x(z)).
\]
Finally, suppose that \( z \) is in none of the cases considered above. Consider some triangulation of \( \Sigma \setminus (B_\Sigma(\frac{1}{2n}, \partial \Sigma) \cup \bigcup_{P \in \text{Con}(\Sigma) \cup \text{Ang}(\Sigma)} U_n(P)) \) with vertices at \( V(\Sigma_n) \). Let \( P, Q, R \in V(\Sigma_n) \) be the vertices of the triangle \( T \) containing \( z \). If there are several choices, take any. We define \((L_n f)(z)\) as the value of the unique affine function \( L \) at \( z \), satisfying \( L(P) = f(P), L(Q) = f(Q) \) and \( L(R) = f(R) \) (with respect to the coordinates \( x, y \) as in the previous step). This procedure describes \( L_n \) completely. For a schematic description of the functional \( L_n \), see Fig. 3.

The “linearization” functional satisfies a number of important properties.

**Proposition 3.7** For any \( f \in \text{Map}(V(\Sigma_n), F_n) \), we have
\[
L_n(f) \in W^1_2(\Sigma, F).
\]

**Proof** Recall first that for any domain \( \Omega \), the set \( W^1_2(\Omega) \) is closed under taking maximums, cf. [15, p. 292]. This means that if \( f, g \in W^1_2(\Omega) \), then \( \max\{f, g\} \in W^1_2(\Omega) \). Similarly, we have \( \min\{f, g\} \in W^1_2(\Omega) \). Proposition 3.7 follows from this and the fact
that, any continuous piece-wise linear function can be represented through a compositions of max, min, applied to a number of linear functions.  

**Proposition 3.8** Suppose that \( f, g \in \text{Map}(V(\Sigma_n), F_n) \) satisfy \( f = f^{\text{avg}}, g = g^{\text{avg}} \). Then the following identity holds

\[
\langle \nabla^F(L_n(f)), \nabla^F(L_n(g)) \rangle_{L^2(\Sigma, T^*\Sigma \otimes F)} = \langle \nabla_{\Sigma_n}^F f, \nabla_{\Sigma_n}^F g \rangle_{L^2(\Sigma_n, F_n)}.
\]  

(3.29)

The proof of Proposition 3.8 is a direct calculation, and it is given in the end of this section.

Now, for an eigenvector \( f_i^n \in \text{Map}(V(\Sigma_n), F_n), i \in \mathbb{N}, n \in \mathbb{N}^* \), corresponding to the eigenvalue \( \lambda_i^n \) of \( n^2 \cdot \Delta_{\Sigma_n}^F \), the functional \( L_n \) satisfies the following proposition, the proof of which is an easy application of Theorem 3.1 and will be given in the end of this section.

**Proposition 3.9** For any \( \phi \in \mathcal{C}^1(\Sigma), i, j \in \mathbb{N} \) fixed, as \( n \to \infty \), the following estimation holds

\[
\langle \phi \cdot L_n(f^n_i), L_n(f^n_j) \rangle_{L^2(\Sigma, F)} = \frac{1}{n^2} \langle \phi \cdot f^n_i, f^n_j \rangle_{L^2(\Sigma_n, F_n)} + o(1).
\]  

(3.30)

Now we can state the most important result of this section.

**Theorem 3.10** For any \( i, j \in \mathbb{N} \) fixed, as \( n \to \infty \), the following estimation holds

\[
\langle \nabla^F(L_n(f^n_i)), \nabla^F(L_n(f^n_j)) \rangle_{L^2(\Sigma, T^*\Sigma \otimes F)} = \langle \nabla_{\Sigma_n}^F f^n_i, \nabla_{\Sigma_n}^F f^n_j \rangle_{L^2(\Sigma_n, F_n)} + o(1).
\]  

(3.31)

Our proof of Theorem 3.10 relies on the following technical statement, to the proof of which we devote a separate Sect. 4.

**Theorem 3.11** (Harnack-type inequality) We fix \( \lambda > 0 \). Suppose that a sequence \( f_n \in \text{Map}(V(\Sigma_n), F_n), n \in \mathbb{N}^*, \| f_n \|^2_{L^2(\Sigma_n, F_n)} = n^2 \), satisfies

\[
\| \Delta_{\Sigma_n}^F f_n \| \leq \frac{\lambda}{n^2} |f_n|,
\]  

(3.32)

Then, as \( n \to \infty \), the following limit holds

\[
\max_{(P, Q) \in E(\Sigma_n)} \left| f_n(P) - f_n(Q) \right| \to 0.
\]  

(3.33)

Let’s see how Theorem 3.11 and Proposition 3.8 imply Theorem 3.10.

**Proof of Theorem 3.10** Recall that for \( f \in \text{Map}(V(\Sigma_n), F_n) \), we defined \( f^{\text{avg}} \) in (3.25). By Proposition 3.8, we conclude
\[ \langle \nabla^F (L_n(f_i^{n, \text{avg}})), \nabla^F (L_n(f_j^{n, \text{avg}})) \rangle \mid_{L^2(\Sigma, F)} = \langle \nabla^F_{\Sigma_n} (f_i^{n, \text{avg}}), \nabla^F_{\Sigma_n} (f_j^{n, \text{avg}}) \rangle \mid_{L^2(\Sigma_n, F_n)}. \] (3.34)

However, by the construction of \( f_i^{n, \text{avg}} \), there is \( C > 0 \), which depends purely on the sets \( \angle(\text{Con}(\Sigma)) \) and \( \angle(\text{Ang}(\Sigma)) \), such that the difference between left and right-hand sides of (3.31) and (3.34) are bounded by

\[
C \sum_{P \in \text{Con}(\Sigma) \cup \text{Ang}(\Sigma)} \max_{R, Q \in V(\Sigma_n), \text{dist}_{\Sigma_n}((R, Q), V_n(P)) \leq 2} \left| f_i^n(R) - f_i^n(Q) \right| \cdot \left| f_j^n(R) - f_j^n(Q) \right|. \quad (3.35)
\]

By Theorem 3.11 and (3.34), (3.35), we conclude. \( \square \)

**Proof of Theorem 3.2** By Theorem 3.10, Proposition 3.9, applied for \( \phi = 1 \), and (2.7), we see that for any \( k \in \mathbb{N} \) fixed, as \( n \to \infty \), we have

\[
\lambda_k^n = \frac{\langle n^2 \cdot \Delta^F_{\Sigma_n} (f_i^n), f_i^n \rangle \mid_{L^2(\Sigma_n, F_n)}}{\langle f_i^n, f_i^n \rangle \mid_{L^2(\Sigma_n, F_n)}} = \frac{\langle \nabla^F (L_n(f_i^n)), \nabla^F (L_n(f_i^n)) \rangle \mid_{L^2(\Sigma, T^* \Sigma \otimes F)}}{\langle L_n(f_i^n), L_n(f_i^n) \rangle \mid_{L^2(\Sigma, F)}} + o(1). \quad (3.36)
\]

By Propositions 2.4, 3.7, we conclude that for any \( i \in \mathbb{N}, n \in \mathbb{N}^* \), there is a sequence of functions \( g_{i,k}^n \in C_0^\infty(\Sigma, F), k \in \mathbb{N} \) such that in \( L^2(\Sigma, F) \) and \( L^2(\Sigma, T^* \Sigma \otimes F) \), as \( k \to \infty \), we have

\[
g_{i,k}^n \to L_n(f_i^n), \quad \nabla^F (g_{i,k}^n) \to \nabla^F (L_n(f_i^n)). \quad (3.37)
\]

Moreover, since the functions \( L_n(f_i^n) \) are constant in the neighborhood of \( \text{Con}(\Sigma) \cup \text{Ang}(\Sigma) \), we see that the functions \( g_{i,k}^n \) can be chosen to be constant as well. In particular, we have

\[
g_{i,k}^n \in \text{Dom}_{F^*} (\Delta^F_{\Sigma}). \quad (3.38)
\]

Then for any \( n \in \mathbb{N}^* \), we can choose \( k_{n,i} \) such that, as \( n \to \infty \), the function \( g_i^n := g_{i,k_{n,i}}^n \) satisfies

\[
\begin{align*}
\langle \nabla^F (L_n(f_i^n)), \nabla^F (L_n(f_j^n)) \rangle \mid_{L^2(\Sigma, T^* \Sigma \otimes F)} &= \langle \nabla^F g_i^n, \nabla^F g_j^n \rangle \mid_{L^2(\Sigma, T^* \Sigma \otimes F)} + o(1), \\
\langle L_n(f_i^n), L_n(f_j^n) \rangle \mid_{L^2(\Sigma, F)} &= \langle g_i^n, g_j^n \rangle \mid_{L^2(\Sigma, F)} + o(1). \quad (3.39)
\end{align*}
\]

Note, however, that by Proposition 2.5 and (3.38), we have

\[
\langle \nabla^F g_i^n, \nabla^F g_j^n \rangle \mid_{L^2(\Sigma, T^* \Sigma \otimes F)} = \langle \Delta^F_{\Sigma} g_i^n, g_j^n \rangle \mid_{L^2(\Sigma, F)}. \quad (3.40)
\]
Now, let’s consider a vector space $V^n_k \subset C^\infty_0(\Sigma, F)$, spanned by $g^n_1, \ldots, g^n_k$. By Proposition 3.9, applied for $\phi = 1$, we see that for $n$ big enough, we might choose $k_{n,i}$ big enough so that we have $\dim V^n_k = k$. We use the characterization of the eigenvalues of $\Delta^F_\Sigma$ through Rayleigh quotient

$$\lambda_k = \inf_{V \subset \text{Dom}_F(\Delta^F_\Sigma)} \sup_{f \in V} \left\{ \frac{\langle \Delta^F_\Sigma f, f \rangle_{L^2(\Sigma,F)}}{\langle f, f \rangle_{L^2(\Sigma,F)}} : \dim V = k \right\}. \quad (3.41)$$

In particular, by (3.38), we conclude

$$\lambda_k \leq \sup_{f \in V^n_k} \frac{\langle \Delta^F_\Sigma f, f \rangle_{L^2(\Sigma,F)}}{\langle f, f \rangle_{L^2(\Sigma,F)}}. \quad (3.42)$$

However, by (3.36), (3.39), (3.40), we deduce that for any $k \in \mathbb{N}$, we have

$$\liminf_{n \to \infty} \sup_{f \in V^n_k} \frac{\langle \Delta^F_\Sigma f, f \rangle_{L^2(\Sigma,F)}}{\langle f, f \rangle_{L^2(\Sigma,F)}} = \liminf_{n \to \infty} \lambda^n_k. \quad (3.43)$$

From (3.42) and (3.43), we deduce Theorem 3.2. □

**Proof of Proposition 3.8** The proof is a rather boring verification. We decompose

$$\begin{align*}
\langle \nabla^F(L_n(f)), \nabla^F(L_n(g)) \rangle_{L^2(\Sigma, T^* \Sigma \otimes F)} &= \sum_{P \in \text{Con}(\Sigma) \cup \text{Ang}(\Sigma)} \int_{U_n(P)} \langle \nabla^F(L_n(f)), \nabla^F(L_n(g)) \rangle(z)dv_\Sigma(z) \\
&+ \sum_R \int_R \langle \nabla^F(L_n(f)), \nabla^F(L_n(g)) \rangle(z)dv_\Sigma(z) \\
&+ \sum_T \int_T \langle \nabla^F(L_n(f)), \nabla^F(L_n(g)) \rangle(z)dv_\Sigma(z),
\end{align*} \quad (3.44)$$

where $\sum_R$ means the sum over the rectangles near $\partial \Sigma$ from the definition of $L_n$ and $\sum_T$ means a sum over the triangles of the triangulation used in the definition of $L_n$, see Fig. 3. As $L_n(f)$ is constant near $U_n(P)$ for $P \in \text{Con}(\Sigma) \cup \text{Ang}(\Sigma)$, the first term in (3.44) vanishes.

Now, let $P, Q \in V(\Sigma_n)$ correspond to a rectangle $R$. An easy calculation shows

$$\int_R \langle \nabla^F(L_n(f)), \nabla^F(L_n(g)) \rangle(z)dv_\Sigma(z) = \frac{1}{2} (f(P) - f(Q)) \cdot (g(P) - g(Q)). \quad (3.45)$$
Finally, let $P, Q, R \in V(\Sigma_n)$ be the vertices of a triangle $T$. Then, similarly to (3.45), we have

$$\int_T \langle \nabla^F(L_n(f)), \nabla^F(L_n(g)) \rangle(z) d\Sigma(z)$$

$$= \frac{1}{2} (f(P) - f(Q)) \cdot (g(P) - g(Q)) + \frac{1}{2} (f(P) - f(R)) \cdot (g(P) - g(R)).$$

(3.46)

By (1.14), (3.44), (3.45) and (3.46), we conclude. □

**Proof of Proposition 3.9** Note that by Theorem 3.1, for any $i \in \mathbb{N}$, there is a constant $C_i > 0$ such that for any $n \in \mathbb{N}^*$, we have

$$\lambda^n_i < C_i. \quad (3.47)$$

By the construction of the linearization functional $L_n$, there is $C$, which depends only on the set $\angle(\text{Con}(\Sigma) \cup \text{Ang}(\Sigma))$ and on $\|\phi\|_{\theta^1(\Sigma, F)}$, such that for $P \in V(\Sigma_n)$ and $x \in \Sigma$, satisfying $\text{dist}_\Sigma(P, x) < \frac{1}{n}$, the following holds

$$\left| (\phi \cdot L_n(f^n_i))(x) - (\phi \cdot f^n_i)(P) \right| \leq C \max_{\text{dist}_\Sigma(P, Q) \leq 2} \left| f^n_i(P) - f^n_i(Q) \right| + \frac{C}{n} |f^n_i(P)|.$$

(3.48)

From (3.48), and the fact that our normalization is chosen so that the area of the initial tiles is 1, we see that there is $C > 0$ such that for any $i, j \in \mathbb{N}$, $n \in \mathbb{N}^*$, we have

$$\left| \phi L_n(f^n_i), L_n(f^n_j) \right|_{L^2(\Sigma, \mathcal{F})} - \frac{1}{n^2} \left| \phi f^n_i, f^n_j \right|_{L^2(\Sigma)} \leq \frac{C}{n^2} \left| \phi f^n_i, f^n_j \right|_{L^2(\Sigma)}$$

$$+ \frac{C}{n^2} \sum_{P \in V(\Sigma_n)} |f^n_j(P)| \cdot \max_{\text{dist}_\Sigma(P, Q) \leq 2} |f^n_i(P) - f^n_i(Q)|$$

$$+ \frac{C}{n^2} \sum_{P \in V(\Sigma_n)} |f^n_j(P)| \cdot \max_{\text{dist}_\Sigma(P, Q) \leq 2} |f^n_j(P) - f^n_j(Q)|.$$  

(3.49)

Now, by Cauchy inequality, there is a constant $C$, which depends only on the set $\angle(\text{Con}(\Sigma) \cup \text{Ang}(\Sigma))$ and on $\|\phi\|_{\theta^1(\Sigma, F)}$, such that we have

$$\sum_{P \in V(\Sigma_n)} |f^n_j(P)| \cdot \max_{\text{dist}_\Sigma(P, Q) \leq 2} |f^n_j(P) - f^n_j(Q)| \leq C \cdot \left\| f^n_j \right\|_{L^2(\Sigma)} \left\| \nabla^F f^n_j \right\|_{L^2(\Sigma)}.$$

(3.50)

However by (3.47) and the bound on the norm of $f^n_i$, we see that

$$\left\| \nabla^F f^n_i \right\|_{L^2(\Sigma)} = \left\| \Delta^F_{\Sigma_n} f^n_i \right\|_{L^2(\Sigma)}^{1/2} \leq C_i. \quad \Box$$

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Also, from the bound on the norm of $f^n_i$, $f^n_j$ and Cauchy inequality, we have
\[
\frac{C}{n^3} \langle f^n_i, f^n_j \rangle_{L^2(\Sigma_n)} \leq \frac{C}{n} \|\phi\|_{L^\infty(\Sigma, F)}.
\] (3.52)

From (3.49), (3.50), (3.51) and (3.52), we conclude. \qed

### 3.3 Convergence of the Eigenvectors, a Proof of Theorem 1.3

The main goal of this section is to prove Theorem 1.3. We will see here that Theorem 1.3 follows almost formally from the approximation theory we developed in Sects. 3.1 and 3.2.

Assume that the eigenvalue $\lambda_i$, $i \in \mathbb{N}^*$ of $\Delta^F_{\Sigma_n}$ has multiplicity $m_i$. By Theorem 1.1, we see that there is a series of eigenvalues $\lambda^n_{i,j}$, $j = 1, \ldots, m_i$ of $n^2 \cdot \Delta^F_{\Sigma_n}$ (possibly equal, but in general not), converging to $\lambda_i$, as $n \to \infty$. Moreover, no other eigenvalue of $n^2 \cdot \Delta^F_{\Sigma_n}$ comes close to $\lambda_i$ asymptotically. We denote by $V^n_i$ the vector space spanned by the eigenvectors $f^n_{i,j}$ of $n^2 \cdot \Delta^F_{\Sigma_n}$, corresponding to the eigenvalues $\lambda^n_{i,j}$, $j = 1, \ldots, m_i$, and let
\[
\pi_{V^n_i} : \text{Map}(V(\Sigma_n), F_n) \to V^n_i
\] (3.53)
be the orthogonal projection onto this space. Recall that the “linearization” $L_n : \text{Map}(V(\Sigma_n), F_n) \to L^2(\Sigma, F)$ functional was defined in the beginning of the Sect. 3.2, and the restriction functional $R_n : \mathcal{C}^0(\Sigma, F) \to \text{Map}(V(\Sigma_n), F_n)$ was defined in (3.4). The main result of this section is the following

**Theorem 3.12** For fixed $i, j \in \mathbb{N}^*$, $j \leq m_i$, as $n \to \infty$, in $L^2(\Sigma, F)$, we have
\[
L_n(\pi_{V^n_i}(R_n f_{i,j})) \to f_{i,j},
\] (3.54)
where $f_{i,j} \in \mathcal{C}^0(\Sigma, F)$, $j = 1, \ldots, m_i$ are as in Theorem 1.3.

**Remark 3.13** Clearly Theorem 3.12 implies Theorem 1.3 by Proposition 3.9 and (3.7).

**Proof** For simplicity of the presentation, we suppose that the spectra of $n^2 \cdot \Delta^F_{\Sigma_n}$ and $\Delta^F_{\Sigma}$ are simple, i.e. the eigenvalues have multiplicity 1. The proof of the general case remains verbatim, but the notation becomes more difficult. We denote by $f_i$, $\|f_i\|_{L^2(\Sigma, F)} = 1$, the eigenvectors of $\Delta^F_{\Sigma}$ corresponding to the eigenvectors $\lambda_i$, and by $f^n_i$, $\|f^n_i\|_{L^2(\Sigma_n, F_n)} = n^2$, the eigenvectors of $n^2 \cdot \Delta^F_{\Sigma_n}$ corresponding to the eigenvectors $\lambda^n_i$.

We decompose
\[
R_n f_i = \alpha^n_i f^n_i + \sum_{l \neq i} \beta^i_l n f^n_l,
\]
\[
L_n f^n_i = \gamma^n_i f_i + \sum_{l \neq i} \delta^i_l f_l,
\] (3.55)
for some $\alpha^n_i, \gamma^n_i$ and $\beta^{i,n}_l, \delta^{i,n}_l$, for $i, l, n \in \mathbb{N}$. Then Theorem 3.12 would follow if we prove that, as $n \to \infty$, the following convergence holds

$$\alpha^n_i \to 1, \quad \sum_{l \neq i} (\beta^{i,n}_l)^2 \to 0.$$  \hfill (3.56)

Let’s show that (3.56) holds by induction on $i$. It clearly holds for $i = 0$ by Corollary 2.6, because the restriction of a flat section is obviously flat for the discrete connection, and, hence, lies in the kernel of the discrete Laplacian. Suppose we proved it for $i - 1$, let’s show that it also holds for $i$.

Indeed, we know by (3.7) that for any $j \leq i - 1$, as $n \to \infty$, the following convergence holds

$$\frac{1}{n^2} \langle R_n f_i, R_n f_j \rangle_{L^2(\Sigma_n, F_n)} \to \langle f_i, f_j \rangle_{L^2(\Sigma, F)} = 0.$$  \hfill (3.57)

However, by (3.55), we have

$$\frac{1}{n^2} \langle R_n f_i, R_n f_j \rangle_{L^2(\Sigma_n, F_n)} = \alpha^n_i \beta^{j,n}_i + \alpha^n_j \beta^{i,n}_j + \sum_{l \neq i, j} (\beta^{i,n}_l)^2 \lambda^n_l.$$  \hfill (3.58)

From the induction hypothesis (3.56), (3.57) and (3.58), we conclude that for any $j \leq i - 1$, as $n \to \infty$, the following limit holds

$$\beta^{i,n}_j \to 0.$$  \hfill (3.59)

Now, by Theorem 3.4, as $n \to \infty$, the following convergence holds

$$\langle \Delta_{\Sigma_n}^F (R_n f_i), R_n f_i \rangle_{L^2(\Sigma_n, F_n)} \to \lambda_i.$$  \hfill (3.60)

However, by (3.55), we have

$$\langle \Delta_{\Sigma_n}^F (R_n f_i), R_n f_i \rangle_{L^2(\Sigma_n, F_n)} = (\alpha^n_i)^2 \cdot \lambda^n_i + \sum_{l \neq i} (\beta^{i,n}_l)^2 \lambda^n_l.$$  \hfill (3.61)

Since $\lambda^n_l, l \geq j + 1$ is asymptotically strictly bigger than $\lambda^n_i$, as $n \to \infty$, by (3.59), (3.60), (3.61), we deduce the first statement of (3.56). The second statement is shown analogically, where one has to use Theorem 3.10 instead of Theorem 3.4 and Proposition 3.9 instead of (3.7).

\[\square\]

**Remark 3.14** In an analogous way, one can prove that in the notations of Theorem 1.3, for any $i, j \in \mathbb{N}, 1 \leq j \leq m_i$, and for the orthogonal projection $\pi_i$ onto the eigenspace
of $\Delta_{\Sigma}^F$, corresponding to the eigenvalue $\lambda_i$, as $n \to \infty$, in $L^2(\Sigma_n, F_n)$, we have

$$R_n(\pi_i(L_n f^n_{i,j})) \to f^n_{i,j},$$

where $f^n_{i,j} \in \text{Map}(\Sigma_n, F_n)$, are the eigenvectors of $\Delta_{\Sigma_n}^F$ as in Theorem 1.3.

4 Harnack-Type Inequality for Discrete Laplacian

The theory of harmonic functions on lattices has received considerable attention in the past; for example, see Duffin [14], Kenyon [20], Bücking [6]. Much less attention has been drawn to functions which are almost harmonic in the sense that their Laplacian is not zero, but evaluated at some point, it is close to zero in comparison with the value of the function itself at this point. The main goal of this section is to give a proof of Theorem 3.11, which is a statement in this direction.

4.1 Asymptotic Continuity of Discrete Eigenvectors, a Proof of Theorem 3.11

This section is devoted to the proof of Theorem 3.11. The following result is central here

**Theorem 4.1** We fix $\lambda > 0$. Suppose that a sequence $f_n \in \text{Map}(\Sigma_n, F_n)$ satisfies the same assumptions as in Theorem 3.11. Then, as $n \to \infty$, the following limit holds

$$\frac{1}{\sqrt{\log n}} \max_{P \in V(\Sigma_n)} |f_n(P)| \to 0.$$  

(4.1)

We prove Theorem 4.1 in Sect. 4.2. Let's see how Theorem 4.1 implies Theorem 3.11.

**Proof of Theorem 3.11** First, by (3.32), we see that

$$\langle \Delta_{\Sigma_n}^F f_n, f_n \rangle_{L^2(\Sigma_n, F_n)} \leq \lambda.$$  

(4.2)

By (1.14) and (4.2), we deduce that

$$\sum_{e \in E(\Sigma_n)} (\nabla_{\Sigma_n}^F f)^2 \leq \lambda.$$  

(4.3)

As a consequence, we obtain

$$\max_{e \in E(\Sigma_n)} |(\nabla_{\Sigma_n}^F f)(e)| \leq \sqrt{\lambda}.$$  

(4.4)

The main idea of the proof is to show that if there is an edge $e$, so that $|(\nabla_{\Sigma_n}^F f)(e)|$ is asymptotically bounded from below by a nonzero constant, then the neighboring edges of $e$ will contribute a lot to the sum in the left-hand side of (4.3) up to the point so that the bound (4.3) doesn’t hold.
More rigorously, suppose that the statement of Theorem 3.11 is false. Then we can find $c > 0$ and $e_n \in E(\Sigma_n)$, $n \in \mathbb{N}$, such that for any $n \in \mathbb{N}^*$, up to choosing a subsequence, we have

$$|\langle \nabla \Sigma_n^F_n f_n (e_n) \rangle | \geq c.$$  \hfill (4.5)

Denote by $\partial B_{\Sigma_n}(r, P) \subset E(\Sigma_n)$, $P \in V(\Sigma_n)$ the subset defined as follows

$$\partial B_{\Sigma_n}(r, P) = \left\{ e \in E(\Sigma_n) : h(e) \in B_{\Sigma_n}(r, P), t(e) \notin B_{\Sigma_n}(r, P) \right\}.$$  \hfill (4.6)

Let $P_n = h(e_n)$. For $e \in E(\Sigma_n)$, we say that $e \in B_{\Sigma_n}(r, P_n)$ if $h(e), t(e) \in B_{\Sigma_n}(r, P_n)$. Then by (1.14), the following identity holds

$$\langle \Delta^{F_n}_{\Sigma_n} f_n, f_n \rangle_{L^2(B_{\Sigma_n}(r, P_n), F_n)} = \sum_{e \in B_{\Sigma_n}(r, P_n)} |\langle \nabla \Sigma_n^F_n f_n (e) \rangle |^2$$

$$+ \sum_{e \in \partial B_{\Sigma_n}(r, P_n)} (\nabla \Sigma_n^F_n f_n (e) \cdot f_n (h(e)).$$  \hfill (4.7)

However, by Theorem 4.1 and (3.32), there is $N \geq 0$ such that for $n \geq N$, $r \leq \sqrt{n}$, we have

$$\langle \Delta^{F_n}_{\Sigma_n} f_n, f_n \rangle_{L^2(B_{\Sigma_n}(r, P_n), F_n)} \leq \frac{\lambda}{n^2} \left\| f_n \right\|_{L^2(B_{\Sigma_n}(r, P_n), F_n)} \leq \lambda n^{-1/2}. \hfill (4.8)$$

From (4.5), (4.7) and (4.8), we deduce that for any $n \geq N$ and $r \leq \sqrt{n}$ we have

$$\sum_{e \in \partial B_{\Sigma_n}(r, P_n)} |\langle \nabla \Sigma_n^F_n f_n (e) \rangle | \cdot |f_n (h(e))| \geq c^2 - \frac{\lambda}{\sqrt{n}}. \hfill (4.9)$$

However, by Theorem 4.1, for any $\epsilon > 0$, there is $N$ such that for all $n \geq N$, we have

$$\sum_{e \in \partial B_{\Sigma_n}(r, P_n)} |\langle \nabla \Sigma_n^F_n f_n (e) \rangle | \cdot |f_n (h(e))| \leq \epsilon \sqrt{\log(n)} \sum_{e \in B_{\Sigma_n}(r, P_n)} |\langle \nabla \Sigma_n^F_n f_n (e) \rangle |. \hfill (4.10)$$

Now, from trivial geometric considerations, there exists $C > 0$, which depends only on the sets $\angle(\text{Con}(\Sigma))$ and $\angle(\text{Ang}(\Sigma))$, such that for any $r \leq n$, we have $\# \partial B_{\Sigma_n}(r, P) \leq C r$. By this and arithmetic mean inequality, we have

$$\left( \sum_{e \in \partial B_{\Sigma_n}(r, P_n)} |\langle \nabla \Sigma_n^F_n f_n (e) \rangle | \right)^2 \leq C r \sum_{e \in \partial B_{\Sigma_n}(r, P_n)} |\langle \nabla \Sigma_n^F_n f_n (e) \rangle |^2. \hfill (4.11)$$
From (4.9), (4.10) and (4.11), for any $\epsilon > 0$, there is $N$ such that for all $n \geq N$ and $r \leq \sqrt{n}$:

$$\sum_{e \in \partial B_{\Sigma_n}(r, P_n)} |(\nabla_{\Sigma_n}^F f_n)(e)|^2 \geq \frac{c^4}{2 \log(n)} \frac{1}{\epsilon^2 Cr}. \tag{4.12}$$

From (4.3) and (4.12), we see that for any $\epsilon > 0$, there is $N \geq 0$ such that for any $n \geq N$, we have

$$\sqrt{n} \sum_{r=1}^{\sqrt{n}} \frac{1}{r} \leq \epsilon \log(n), \tag{4.13}$$

which is nonsense. Thus, the initial assumption (4.5) is false and Theorem 3.11 holds. $\square$

### 4.2 Uniform Bound on Discrete Eigenvectors, a Proof of Theorem 4.1

The main goal of this section is to give a uniform bound on the values of discrete eigenvectors, i.e. to prove Theorem 4.1. As all the statements of this section are local in nature, we may assume without loosing the generality that $(F, h^F, \nabla F)$ is trivial of rank 1.

For $c > 0$, we define

$$V_c(\Sigma_n) := \{ P \in V(\Sigma_n) : \text{dist}_\Sigma(P, \text{Con}(\Sigma) \cup \text{Ang}(\Sigma)) > c \}. \tag{4.14}$$

The following theorem says that in the interior of $\Sigma$, the bound (4.1) can be significantly improved. The proof of it uses the potential theory on $\mathbb{Z}^2$, and it is given in Sect. 4.3.

**Theorem 4.2** We fix $\lambda > 0$. For any $c > 0$, there is a constant $C(\lambda) > 0$, such that for any sequence $f_n \in \text{Map}(V(\Sigma_n), F_n)$ as in Theorem 3.11, we have

$$\max_{P \in V_c(\Sigma_n)} |f_n(P)| \leq C(\lambda). \tag{4.15}$$

Our proof of Theorem 4.1 relies on Theorem 4.2, maximum principle and the construction of the discrete flow $E_{\mathbb{Z}^2}^n$ on the $n \times n$ subgraph inside of $\mathbb{Z}^2$, as follows: for $a, b \in \mathbb{N}, a + b \leq n$, let

$$E_{\mathbb{Z}^2}^n((a, b), (a + 1, b)) = (a + 1) \left( \frac{1}{a + b + 1} - \frac{1}{a + b + 2} \right),$$

$$E_{\mathbb{Z}^2}^n((a, b), (a, b + 1)) = \frac{1}{a + b + 2} - a \left( \frac{1}{a + b + 1} - \frac{1}{a + b + 2} \right). \tag{4.16}$$

And for other edges, $E_{\mathbb{Z}^2}^n$ is defined to be zero. Then it is a matter of a tedious calculation, to see that for $Q \in V(\Sigma_n)$, the following holds (see (1.13) for the definition of
\[(d^n_{\mathbb{Z}^2} E^n_{\mathbb{Z}^2})(Q) = \begin{cases} -1, & \text{for } Q = (0, 0), \\ (n + 1)^{-1}, & \text{for } Q = (a, b) \in \mathbb{N}^2 \text{ with } a + b = n, \\ 0, & \text{otherwise.} \end{cases} \] (4.17)

Remark that by (4.16), for any edge \( e \) in \( \mathbb{Z}^2 \), we have
\[
|E^n_{\mathbb{Z}^2}(e)| \leq \frac{4}{\text{dist}_{\mathbb{Z}^2}(0, h(e)) + 1}. \] (4.18)

Hence, there is \( C > 0 \) such that for any \( n \in \mathbb{N}^* \), we have
\[
\|E^n_{\mathbb{Z}^2}\|_{L^2(\Sigma_n)} \leq \left(32 \sum_{i=1}^{n} \frac{1}{i + 1}\right)^{1/2} \leq C \sqrt{\log(n)}. \] (4.19)

**Proof of Theorem 4.1** The proof has two parts. In the first part, by using the flow \( E^n_{\mathbb{Z}^2} \), considered above, we prove that there is a constant \( C_0(\lambda) \) such that for any \( n \in \mathbb{N}^* \), the following bound holds
\[
\frac{1}{\sqrt{\log n}} \max_{P \in V(\Sigma_n)} \left| f_n(P) \right| \leq C_0(\lambda). \] (4.20)

In the second step, we “bootstrap” the estimate (4.20) to get Theorem 4.1.

We start by giving a proof of (4.20). The main idea is to estimate the change of \( f_n \) along a flow \( E^n_P \), \( P \in V(\Sigma_n) \), constructed similarly to \( E^n_{\mathbb{Z}^2} \), and the values of \( f_n \) near the conical points and angular singularities with the values of \( f_n \) at \( V_1(\Sigma_n) \) to conclude by Theorem 4.2.

More precisely, let \( P \in V(\Sigma_n) \). Choose a horizontal and vertical rays from \( P \) such that the quadrant with side 1 between them is isomorphic to a standard quadrant in \( \mathbb{R}^2 \) and all the points at the distance bigger than 1 from \( P \) in this quadrant lie in \( V_1(\Sigma_n) \). This is always possible up to a multiplication of the initial metric by 4. Then the part of the graph \( \Sigma_n \), which gets trapped inside of the quadrant of size 1 is isomorphic to the part of \( \mathbb{Z}^2 \). Our flow \( E^n_P \in \text{Map}(E(\Sigma_n), \mathbb{C}) \) has support inside this quadrant and it coincides with \( E^n_{\mathbb{Z}^2} \), normalized so that \( P \) corresponds to \((0, 0)\).

As a consequence of (4.17), we see that
\[
\langle d^n_{\Sigma} f_n, E^n_P \rangle_{L^2(\Sigma_n)} = \langle f_n, d^n_{\Sigma} E^n_P \rangle_{L^2(\Sigma_n)} = -f_n(P) + \frac{1}{n + 1} \sum_{Q \in \text{supp}(d^n_{\Sigma} E^n_P) \setminus P} f_n(Q). \] (4.21)

By Cauchy inequality, (4.2) and (4.19), we deduce
\[
\langle d^n_{\Sigma} f_n, E^n_P \rangle_{L^2(\Sigma_n)} \leq C \sqrt{\lambda} \sqrt{\log(n)}. \] (4.22)
By (4.21), (4.22), we see that
\[
|f_n(P)| \leq \frac{1}{n + 1} \sum_{Q \in \text{supp}(d_{\Sigma_n}^* E_P^n) \setminus P} |f_n(Q)| + C \sqrt{\lambda} \sqrt{\log(n)}. \tag{4.23}
\]

By (4.23), \(\text{supp}(d_{\Sigma_n}^* E_P^n) \setminus P \in V_1(\Sigma, F)\), \#supp\(d_{\Sigma_n}^* E_P^n) = n + 2\) and Theorem 4.2, we see that the bound (4.20) holds for \(C_0(\lambda) = C \sqrt{\lambda} + C(\lambda)\), where \(C(\lambda)\) is from Theorem 4.2 for \(c = 1\).

Now, let's show that the bound (4.20) can be “bootstrapped” to Theorem 4.1. In fact, by Theorem 4.2, we only need to prove Theorem 4.1 in the neighborhood of \(\text{Con}(\Sigma) \cup \text{Ang}(\Sigma)\).

Let's fix \(P \in \text{Con}(\Sigma) \cup \text{Ang}(\Sigma)\). Recall that the set \(V_n(P)\) was defined in Sect. 2.1.

Consider a function \(h_P\), given by
\[
h_P(Q) = \text{dist}_{\Sigma_n}(Q, V_n(P))^2, \quad Q \in V(\Sigma_n). \tag{4.24}
\]

By trivial verification, we see that inside of \(B_{\Sigma_n} \cap V_n(P)\), the following bound holds
\[
\Delta_{\Sigma_n} h_P \leq -1. \tag{4.25}
\]

Let \(C_0(\lambda)\) be from (4.20). Consider a function
\[
g_n = f_n + 2C_0(\lambda) \sqrt{\log(n)} \frac{\lambda}{n^2} h_P. \tag{4.26}
\]

Then by (3.32), (4.20) and (4.25), we see that over \(B_{\Sigma_n}(n, V_n(P))\), we have
\[
\Delta_{\Sigma_n} g_n < 0. \tag{4.27}
\]

Thus, by the maximum principle, the maximum of \(g_n\) on \(B_{\Sigma_n}(r, V_n(P))\), \(r \leq n\), is attained on \(S_{\Sigma_n}(r, V_n(P))\). Thus, for any \(\epsilon\), satisfying \(0 < \epsilon < 1\), we have
\[
\max_{Q \in B_{\Sigma_n}(\epsilon n, V_n(P))} \left( f_n(Q) + 2C_0(\lambda) \sqrt{\log(n)} \frac{\lambda}{n^2} h_P(Q) \right) 
\leq \max_{R \in S_{\Sigma_n}(\epsilon n, V_\epsilon(P))} f_n(R) + 2C_0(\lambda) \epsilon^2 \lambda \sqrt{\log(n)}. \tag{4.28}
\]

From Theorem 4.2, we see that there is \(C_0(\lambda, \epsilon) > 0\) such that
\[
\max_{R \in V_\epsilon(\Sigma_n)} |f_n(R)| \leq C_0(\lambda, \epsilon). \tag{4.29}
\]

From (4.24), (4.28), (4.29) and the fact that \(S_{\Sigma_n}(\epsilon n, V_n(P)) \subset V_\epsilon(\Sigma_n)\), we conclude that for any \(0 < \epsilon < 1\) and \(n \in \mathbb{N}^*\), we have
\[
\max_{Q \in B_{\Sigma_n}(\epsilon n, V_n(P))} f_n(Q) \leq 2C_0(\lambda) \epsilon^2 \sqrt{\log(n)} + C_0(\lambda, \epsilon). \tag{4.30}
\]
Similarly, we can bound $f_n$ from below. By this, (4.29) and (4.30), we conclude. □

### 4.3 Interior Bounds on Almost Harmonic Functions, a Proof of Theorem 4.2

The main goal of this section is to get interior bounds on almost harmonic functions and to prove Theorem 4.2. The proof of this theorem relies heavily on the discrete potential theory introduced by Duffin, [14], in dimension 3 and developed by Kenyon, [20], in dimension 2.

Since Theorem 4.2 is a statement about the value of $f_n$ at points far away from conical and angle singularities, it would follow from the following two theorems. The first one corresponds to the bound on the interior points of $\Sigma_1$.

**Theorem 4.3** We fix $\lambda > 0$. There is a constant $C(\lambda) > 0$, such that for any sequence $f_n \in \text{Map}(V(\mathbb{Z}^2), \mathbb{C})$ satisfying $\|f_n\|^2_{L^2(B_{\mathbb{Z}^2}(n,0))} \leq n^2$, and

$$\left| \Delta_{\mathbb{Z}^2} f_n \right| \leq \frac{\lambda}{n^2} |f_n|, \quad \text{over} \quad B_{\mathbb{Z}^2}(n,0), \quad (4.31)$$

we have the following bound

$$|f_n(0)| \leq C(\lambda). \quad (4.32)$$

And the second one corresponds to the bound on the points near the boundary $\partial \Sigma$.

**Theorem 4.4** We fix $\lambda > 0$. There is a constant $C(\lambda) > 0$, such that for any sequences $P_n \in \mathbb{N} \times \mathbb{Z}$, $\text{dist}_{\mathbb{N} \times \mathbb{Z}}(P_n, (0,0)) \leq \frac{n}{2}$ and $f_n \in \text{Map}(V(\mathbb{N} \times \mathbb{Z}), \mathbb{C})$ satisfying $\|f_n\|^2_{L^2(B_{\mathbb{N} \times \mathbb{Z}}(n,P_n))} \leq n^2$ and

$$\left| \Delta_{\mathbb{N} \times \mathbb{Z}} f_n \right| \leq \frac{\lambda}{n^2} |f_n|, \quad \text{over} \quad B_{\mathbb{N} \times \mathbb{Z}}(n, P_n), \quad (4.33)$$

we have the following bound

$$|f_n(P_n)| \leq C(\lambda). \quad (4.34)$$

Before embarking on a proof of Theorem 4.3, let’s see how it would imply Theorem 4.4.

**Proof of Theorem 4.4** Let $i : \mathbb{Z}^2 \to \mathbb{Z}^2$ be an involution, given by $(x, y) \mapsto (1-x, y)$. Clearly, we may identify the orbits of $i$ with $\mathbb{N} \times \mathbb{Z}$. Thus, we have the $i$-invariant mapping $\pi : \mathbb{Z}^2 \to \mathbb{N} \times \mathbb{Z}$. We construct $\tilde{f}_n$ by

$$\tilde{f}_n = f_n \circ \pi. \quad (4.35)$$

Trivially, the following identity holds

$$\Delta_{\mathbb{Z}^2} \tilde{f}_n = (\Delta_{\mathbb{N} \times \mathbb{Z}} f_n) \circ \pi. \quad (4.36)$$
From this, we see that $\tilde{f}_n$ satisfies the assumptions of Theorem 4.3, from which we conclude. \hfill \Box

Let’s now prove Theorem 4.3. For simplicity, for $n \in \mathbb{N}$, we denote

$$B_{\mathbb{Z}^2}^0(n, 0) := \{z \in \mathbb{Z}^2 : |z| \leq n\},$$

where $|z|$ is the modulus of $z \in \mathbb{C}$ (and not the distance between 0 and $z$ in $\mathbb{Z}^2$).

Let’s now recall some well-known facts from the potential theory on lattices.

**Proposition 4.5** There exists a unique function $G_{\mathbb{Z}^2}^n : \mathbb{Z}^2 \to \mathbb{R}$ satisfying

$$\text{supp}(G_{\mathbb{Z}^2}^n) \subset B_{\mathbb{Z}^2}^0(n, 0),$$

$$(\Delta_{\mathbb{Z}^2} G_{\mathbb{Z}^2}^n)(z) = \begin{cases} 1, & \text{for } z = 0, \\ 0, & \text{for } z \in B_{\mathbb{Z}^2}^0(n, 0) \setminus \{0\}. \end{cases}$$

**Proof** The uniqueness follows from the maximum principle. The existence is proved by an explicit iterative procedure, see Duffin [14, p. 242]. \hfill \Box

**Proposition 4.6** (Duffin [14, Theorem 2, §3] in dimension 3, Kenyon [20, Theorem 7.3] in dimension 2, cf. also Bücking [6, Theorem A.2]) There is a unique function $G_{\mathbb{Z}^2} : \mathbb{Z}^2 \to \mathbb{R}$ satisfying

$$G_{\mathbb{Z}^2}(z, w) = \begin{cases} 1, & \text{for } z = w, \\ 0, & \text{otherwise,} \end{cases}$$

$$G_{\mathbb{Z}^2}(z, z) = 0,$$

$$G_{\mathbb{Z}^2}(z, w) = O(\log |z - w|), \text{ as } |z - w| \to \infty,$$

where by $\Delta_{\mathbb{Z}^2}$ we mean a Laplacian evaluated with respect to the $z$-coordinate. Moreover, as $|z - w| \to \infty$, such a function satisfies the following asymptotic expansion

$$G_{\mathbb{Z}^2}(z, w) = -\frac{1}{2\pi} \log(2|z - w|) - \frac{\gamma_{EM}}{2\pi} + O\left(\frac{1}{|z - w|}\right),$$

where $\gamma_{EM}$ is the Euler-Mascheroni constant.

For $n \in \mathbb{N}$, we denote

$$S_{\mathbb{Z}^2}^0(n, 0) := B_{\mathbb{Z}^2}^0(n, 0) \setminus B_{\mathbb{Z}^2}^0(n - 1, 0).$$

**Proposition 4.7** Let $G_{\mathbb{Z}^2}^n$ be a function from Proposition 4.5. There is a constant $C > 0$ such that for any $n \in \mathbb{N}^*$, the following bound holds

$$\max_{z \in S_{\mathbb{Z}^2}^0(n, 0)} G_{\mathbb{Z}^2}^n(z) \leq \frac{C}{n}.$$
Proof The proof relies on Proposition 4.6 and the maximum principle, see Duffin [14, Lemma 2] for a proof of the corresponding result in dimension 3 and Bücking [6, Proposition A.3] for an analogical proof in dimension 2.

As a consequence of Proposition 4.7, we get the following

**Proposition 4.8** There is a constant $C > 0$, such that for any $n \in \mathbb{N}^*$, $z \in B_{C}^{\mathbb{Z}^2} (n, 0)$, we have

$$G_n^{\mathbb{Z}^2}(z) \leq C \log \left( \frac{n + 2}{|z| + 1} \right),$$

$$\| G_n^{\mathbb{Z}^2} \|_{L^2(\mathbb{Z}^2)} \leq Cn.$$  \hspace{1cm} (4.43)

**Proof** We begin with a proof of the first bound of (4.43). Consider $f_n : \mathbb{Z}^2 \to \mathbb{R}$, defined by

$$f_n = G_{n+1}^{\mathbb{Z}^2} - G_n^{\mathbb{Z}^2}. \hspace{1cm} (4.44)$$

By (4.38), the function $f_n$ is harmonic in $B_{C}^{\mathbb{Z}^2} (n, 0)$. Thus, by the maximal principle, it attains its maximum on $S_{C}^{\mathbb{Z}^2} (n + 1, 0)$. By this and Proposition 4.7, we see that there is $C > 0$ such that for any $n \in \mathbb{N}^*$, we have

$$|f_n| \leq \frac{C}{n}. \hspace{1cm} (4.45)$$

From this, we conclude by induction that

$$G_n^{\mathbb{Z}^2}(z) \leq \sum_{i=|z|}^{n} \frac{C}{i}. \hspace{1cm} (4.46)$$

By standard bounds on the partial harmonic sums, we obtain from (4.46) the first estimate in (4.43). It is now an easy exercise to deduce the second estimate in (4.43) from the first.

Finally we are ready to give

**Proof of Theorem 4.3** Similarly to (4.6), we define $\partial B_{C}^{\mathbb{Z}^2} (r, 0) \subset E(\mathbb{Z}^2)$. By (1.14), we see that for any $r \in \mathbb{N}^*$, $r \leq n$, the following discrete analogue of Green identity holds

$$\langle \Delta^{\mathbb{Z}^2} f_n, G_r^{\mathbb{Z}^2} \rangle_{L^2(B_{C}^{\mathbb{Z}^2} (r, 0))} - \langle f_n, \Delta^{\mathbb{Z}^2} G_r^{\mathbb{Z}^2} \rangle_{L^2(B_{C}^{\mathbb{Z}^2} (r, 0))} = \sum_{e \in \partial B_{C}^{\mathbb{Z}^2} (r, 0)} \left( f_n(h(e)) G_r^{\mathbb{Z}^2}(t(e)) - f_n(h(e)) G_r^{\mathbb{Z}^2}(t(e)) \right). \hspace{1cm} (4.47)$$
However, by Proposition 4.8 and (4.31), we deduce that there is a constant $C > 0$ such that for any $n \in \mathbb{N}^*$, $r \in \mathbb{N}^*$, $r \leq n$, we have
\[
\langle \Delta_Z f_n, G_r \rangle_{L^2(B^2_C(r,0))} \leq C\lambda. \tag{4.48}
\]
By (4.38), we have
\[
\langle f_n, \Delta_Z G_r \rangle_{L^2(B^2_C(r,0))} = f_n(0). \tag{4.49}
\]
From Proposition 4.7 and (4.38), we see that there is a constant $C > 0$ such that for any $n \in \mathbb{N}^*$, $r \in \mathbb{N}^*$, $n/2 \leq r \leq n$, we have
\[
\left| \sum_{e \in \partial B^2_C(r,0)} \left( f_n(h(e))G_r^2(t(e)) - f_n(h(e))G_r^2(t(e)) \right) \right| \leq C \sum_{e \in \partial B^2_C(r,0)} |f_n(h(e))| \tag{4.50}
\]
By combining (4.47)-(4.50), we see that for any $n \in \mathbb{N}^*$, $r \in \mathbb{N}^*$, $n/2 \leq r \leq n$, we have
\[
|f_n(0)| \leq C\lambda + C \sum_{e \in \partial B^2_C(r,0)} |f_n(h(e))|. \tag{4.51}
\]
By taking average of (4.51) for $r \in \mathbb{N}^*$, $n/2 \leq r \leq n$, we have
\[
|f_n(0)| \leq C\lambda + \frac{2C}{n^2} \sum_{P \in B^2_C(n,0)} |f_n(P)|. \tag{4.52}
\]
By mean inequality and the assumptions on the norm of $f_n$, for some constant $C > 0$, we have
\[
\left( \frac{1}{n^2} \sum_{P \in B^2_C(n,0)} |f_n(P)| \right)^2 \leq \frac{C}{n^2} \sum_{P \in B^2_C(n,0)} |f_n(P)|^2 \leq C. \tag{4.53}
\]
By (4.52) and (4.53), we conclude. $\square$

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