Finite-dimensional pointed Hopf algebras of type $A_n$ related to the Faddeev-Reshetikhin-Takhtajan $U(R)$ construction

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Abstract

Two "quantum enveloping algebras", here denoted by $U(R)$ and $U^\sim(R)$, are associated in \cite{FRTa} and \cite{FRTb} to any Yang-Baxter operator $R$. The latter is only a bialgebra, in general; the former is a Hopf algebra.

In this paper, we study the pointed Hopf algebras $U(R_Q)$, where $R_Q$ is the Yang-Baxter operator associated with the multi-parameter deformation of $GL_n$ supplied in \cite{AST}; cf also \cite{S, Re}.

Some earlier results concerning these Hopf algebras $U(R_Q)$ were obtained in \cite{To, CLMT, CM}; a related (but different) Hopf algebra was studied in \cite{DP}.

The main new results obtained here concerning these quantum enveloping algebras are: 1) We list, in an extremely explicit form, those

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quantum enveloping algebras \( U(R_Q) \) which are finite-dimensional—let \( \mathcal{U} \) denote the collection of these. 2) We verify that the pointed Hopf algebras in \( \mathcal{U} \) are quasitriangular and of Cartan type \( A_n \) in the sense of Andruskiewich-Schneider. 3) We show that every \( U(R_Q) \) is a Hopf quotient of a double cross-product (hence, as asserted in 2), is quasitriangular if finite-dimensional.) 4) CAUTION: These Hopf algebras are NOT always cocycle twists of the standard 1-parameter deformation. This somewhat surprising fact is an immediate consequence of the data furnished here—clearly a cocycle twist will not convert an infinite-dimensional Hopf algebra to a finite-dimensional one! Furthermore, these Hopf algebras in \( \mathcal{U} \) are (it is proved) not all cocycle twists of each other. 5) We discuss also the case when the quantum determinant is central in \( A(R_Q) \), so it makes sense to speak of a \( Q \)-deformation of the special linear group.

**Introduction**

Throughout the remainder of this paper, \( k \) will denote an algebraically closed ground-field of characteristic 0.

Let \( G \) be an affine algebraic group over \( k \); we then associate to \( G \) in the usual way two \( k \)-Hopf algebras: \( A(G) \), whose elements are representative functions on \( G \), and \( U(G) \), whose underlying \( k \)-algebra is the enveloping algebra of the Lie algebra of \( G \). Those Hopf algebras certified by workers in the field as being “quantum groups”, fall into two main classes: those ‘deforming’ the type \( U(G) \), the “quantum enveloping algebras”, and those ‘deforming’ the type \( A(G) \), which will here be called “quantum groups”.

Perhaps the earliest systematic construction of infinite families of these two types of Hopf algebras, was furnished by the seminal work of Faddeev, Reshetikhin and Takhtejan (\[FRTa\], \[FRTb\]). The starting point of their marvellous construction may be taken to be a \( K \)-linear transformation

\[
R : V \otimes V \mapsto V \otimes V
\]

where \( V \) is a finite-dimensional vector-space over \( K \), and \( R \) satisfies the Yang-Baxter condition, say in the form

\[
(R \otimes I_V) \circ (I_V \otimes R) \circ (R \otimes I_V) = (I_V \otimes R) \circ (R \otimes I_V)(I_V \otimes R) \quad (1)
\]

Given such an \( R \), there is associated in \[FRTa\] a Hopf algebra \( A(R) \), first proved in \[LT\] to have the property (which seems characteristic for
quantum groups") that its finite-dimensional comodules form in a natural way a braided monoidal category (as defined in [JS]). Moreover, the paper [FRTa] goes on to construct inside \((A(R))^\circ\) a bialgebra, which here will be denoted by \(U^\sim(R)\); this is in general only a bialgebra, but not a Hopf algebra. This result was improved by Faddeev, Reshetikhin and Takhtajan in a later paper [FRTb], where they construct inside \((A(R))^\circ\) a bialgebra—which will here be denoted by \(U^\sim(R)\)—properly containing the earlier construction \(U^\sim(R)\), and where they show that this larger bialgebra \(U(R)\) indeed has in a natural way the structure of a Hopf algebra.

The most usual applications of quantum groups, have involved the 1-parameter deformations of quantum enveloping algebras, given by the Drin’feld-Jimbo-Lusztig construction. Thus, there was some interest aroused in the early '90s, by the construction of \(\binom{N}{2} + 1\)-parameter deformations of \(A(GL_n)\) and \(U(gl_n)\), as given in [AST, Re, S]. This work involved the construction of a solution \(R = R_Q\) to (1), where the multiparameter

\[ Q = \{r, q_{i,j} : 1 \leq i < j \leq N\} \]

is made up of \(\binom{N}{2} + 1\) non-zero elements \(r, q_{i,j}\) in \(k\).

We are thus led to study the quantum enveloping algebra \(U(R_Q)\) (furnished by applying to \(R_Q\) the constructions of Faddeev,Reshitikhin and Takhtajian discussed above.) This was first done in complete generality in the paper [To], while in the later-appearing papers [CLMT] and [CM], the proofs in [To] were substantially simplified, at the expense of requiring the parameter \(r\) in \(Q\), not to be a root of 1—this additional assumption is however never satisfied for the cases to be considered in the present paper, as will be explained below.

Actually, the Hopf algebra studied in [To], only coincides with the [FRTb] construction \(U(Q)\) when \(r \neq 1\); when \(r = 1\) it represents a ‘closure’ as \(r \to 1\) whose further study we reserve for a later paper. For the sake of completeness, let us mention yet another incarnation of \(U(R_Q)\), constructed (in work appearing prior to [To] ) by Dobrev and Parashar, in [DP]. Their Hopf algebra (whose definition utilizes \(R_Q\) in an interesting way which is outside the scope of the present paper) is never finite-dimensional, hence is not directly relevant to our present purposes. In the addendum to [DP] they give a multi-parameter \(Q\)-deformation of \(U(sl_n)\)—which also is never finite-dimensional.

There has recently been an interest in the study of pointed finite-dimensional Hopf algebras, stimulated by the remarkable results obtained in this direction.
by Andruskiewich and Schneider (AS1, AS2). The purpose of the present paper is to study in this light, the rather ancient work on $U(R_Q)$ discussed above. Since we are thus here interested only in finite-dimensional Hopf algebras, only the construction presented in [To] will be relevant. It will be proved below that, for $r \neq 1$, $U(R_Q)$ is finite-dimensional, if and only if: (*) $r$ and each $q_{i,j}$ is a root of unity.

(From this it is easily deduced that, as asserted above, none of the versions of $U(R_Q)$ constructed in [Re, S, DP, CLMT, CM] is ever finite-dimensional.)

Also, it will be proved below that, when (*) holds, the pointed Hopf algebra $U(R_Q)$ is quasi-triangular, and has Andruskiewich-Schneider Cartan matrix of type (as it ‘should’ be!) $A_n$. These Hopf algebras are not all twistings of each other though. Indeed, we exhibit in Theorem 2.2 families of finite-dimensional Hopf algebras arising from $U(R_Q)$ with non-isomorphic groups of group-like elements, and thus these Hopf algebras can not be obtained from each other by twists. Although not all pointed Hopf algebras of type $A_n$ can be realized as $U(R_Q)$, the particular new family described in [AS1, Ex.7.27] can be so realized.

The structure of the group of group-like elements of $U(R_Q)$ strongly depends on the determinant element of $A(R_Q)$. Recall that the bialgebra $O_q(M_n(C))$ can be obtained as $A(R_Q)$ for a special choice of the parameters in $Q$ and that its determinant group-like element is central. The Hopf algebra $U_q(gl_n)$ and Lusztig’s finite-dimensional Hopf algebra $u_q(gl_n)$ are both obtained as the corresponding $U(R_Q)$, depending on whether or not $q$ is a root of unity, while $U_q(sl_n)$ and $u_q(sl_n)$ are Hopf subalgebras respectively. We show in Theorem 2.2 how this situation is generalized for a Hopf algebra of the form $U(R_Q)$ where $A(R_Q)$ admits a central determinant. Example 2.5 is then a special examples of a (new) Hopf algebra of this type. We construct a Hopf algebras of type $A_3$ with a group of group-like elements generated by one element (while the group of group-like elements of $u_q^{\geq 0}(sl_3)$ is generated by two elements).

In the second part we prove in Theorem 3.2 that $U(R_Q)$ is always a Hopf quotient of a double crossproduct of its ”$\geq 0$“ and ”$\leq 0$“ parts. This implies in particular that $U(R_Q)$ is quasitriangular when it is finite dimensional.
1 Preliminaries

Throughout we assume that the base field \( k \) is algebraically closed of characteristic 0.
\( \text{gr} H \) as a biproduct:

In what follows we give is a brief overview of this subject based on one of the many possible references (see for example \([\text{AS1, AS2}]\)).

Let \( H \) be a pointed Hopf algebra over an algebraically closed field of characteristic 0, let \( H_n, n \geq 0 \) denote the coradical filtration of \( H \) and set \( H_{-1} = k \). Let \( G = G(H) = H_0 \) and let

\[
\text{gr} H = \bigoplus_{n \geq 0} \text{gr} H(n)
\]

where \( \text{gr} H(n) = H_n/H_{n-1} \) for all \( n \geq 0 \). Then \( \text{gr} H \) is a graded Hopf algebra. There is a Hopf algebra projection \( \pi : \text{gr} H \to \text{gr} H(0) = kG \) and a Hopf algebra injection \( i : kG \to \text{gr} H \). By \([\text{R1}]\) this implies that we have a biproduct

\[
\text{gr} H \cong R \# kG
\]

where \( R = \{ x \in \text{gr} H \mid (id \otimes \pi) \circ \Delta(x) = x \otimes 1 \} \) is the algebra of the coinvariants of the induced \( H \)-coaction.

It is known that \( R \) is a graded braided Hopf algebra in the category of left Yetter-Drinfeld modules over \( kG \). The action of \( kG \) on \( R \) is given by the adjoint action of the group and the coaction is given by \( \rho = (\pi \otimes id) \circ \Delta \). The original Hopf algebra is then a lifting of \( R \# kG \).

By \([\text{R1}]\) there is a coalgebra projection \( \Pi : R \# kG \to R \) given by

\[
\Pi = id \ast (i \circ S \circ \pi)
\]

The following lemma follows directly from the definition of \( \Pi \). We include it here for completeness.

**Lemma 1.1.** All group-like elements of \( R \# kG \) are mapped by \( \Pi \) to 1. A skew primitive element \( x \) such that \( \Delta(x) = g' \otimes x + x \otimes g \), \( g, g' \in G \) is mapped to \( xg^{-1} \) which is a primitive element of \( R \).

The vector space \( V = P(R) \) of primitive elements of \( R \) is a Yetter-Drinfeld submodule of \( R \) with a braiding (called the infinitesimal braiding)

\[
c : V \otimes V \to V \otimes V \text{ given by } c(v \otimes w) = \sum (v_{-1} \cdot w) \otimes v
\]
where $\rho(v) = \sum v_{-1} \otimes v_{0} \in H \otimes V$.

The **Nichols algebra** of $V$, $B(V)$, is in this case the subalgebra of $R$ generated by $V$.

If the group $G$ is abelian and $V$ is finite-dimensional then the braiding is given by a family of scalars $l_{ij} \in k, 1 \leq i, j \leq n$ so that
\[
c(x_i \otimes x_j) = l_{ij}(x_j \otimes x_i)
\]
where $\{x_1, \ldots, x_n\}$ is a basis of $V$. We say that the braiding is of **Cartan-FL-type** if there exist $q \neq 1$ so that for $1 \leq i, j \leq n$
\[
l_{ij}l_{ji} = q^{d_{ij}}
\]
where $(a_{ij})$ is a generalized symmetrizable Cartan matrix with positive integers $\{d_1, \ldots, d_n\}$ so that $d_{ij}a_{ij} = d_{ji}a_{ji}$.

The Cartan matrix is invariant under twisting (in the sense of [AS2, §2] which is a variation of Reshetikhin [Re]). More precisely, a twist for a Hopf algebra $H$ is an invertible element $F \in H \otimes H$ which satisfies
\[
(\Delta \otimes \text{Id})(F)(F \otimes 1) = (\text{Id} \otimes \Delta)(F)(1 \otimes F)
\]
and
\[
(\varepsilon \otimes \text{Id})(F) = (\text{Id} \otimes \varepsilon)(F) = 1.
\]

Given a twist $F$ for $H$, one can define a new Hopf algebra $H^F$ where $H^F = H$ as an algebras and the coproduct is determined by
\[
\Delta^F(a) = F^{-1}\Delta(a)F, \quad S^F(a) = Q^{-1}S(a)Q
\]
for every $a \in H$, where $Q := m \circ (S \otimes \text{Id})(F)$.

In this context we are interested in the particular case when $F \in kG \otimes kG$ and the group $G = G(H)$ is commutative.

Let $\hat{G}$ denote the group of characters of the abelian group $G$ and let $\sigma$ be a (convolution)-invertible 2-cocycle on $\hat{G}$. Then $\sigma$ gives rise to a twist $F \in kG \otimes kG$. It is proved that the infinitesimal braiding of $H^F$ is of the same Cartan matrix as that of $H$.

The Hopf algebras $A(R_Q)$ and $U(R_Q)$:

The reader is referred to [To] for the full details; we follow the notations there.
Let $Q = \{ r \neq 1, p_{i,j}\}_{1 \leq i < j \leq n}$ be $\binom{n}{2} + 1$ non-zero elements of $k$ and

$$q_{i,j} := r/p_{i,j}.$$ 

Set

$$\kappa_j^i = \begin{cases} p_{ij} & i < j, \\ r & i = j, \\ q_{ji} = r/p_{ji} & i > j \end{cases}$$

(2)

Let $V$ be a vector space with a basis $\{v_1, \ldots, v_n\}$ and let $R_Q : V \otimes V \to V \otimes V$ be the following Yang-Baxter operator:

$$R_Q(v_i \otimes v_j) = \sum_{k,l=1}^{n} R_{ij}^{kl} v_k \otimes v_l$$

where

$$R_{ij}^{kl} = \begin{cases} \kappa_j^i & i = l, j = k \\ r - 1 & i = k, j = l, i > j \\ 0 & \text{otherwise} \end{cases}$$

(3)

Let $A(R_Q)$ be the associated bialgebra constructed by the FRT-construction and described in [AST]. Recall that $A(R_Q)$ is generated as an algebra by $\{T_j^i\}_{1 \leq i,j \leq n}$ so that

$$\Delta(T_j^i) = \sum_k T_k^i \otimes T_j^k.$$  

(4)

Let $U_Q = U(R_Q) \subset (A(R_Q))^0$ be the FRT-construction of the $U$-Hopf algebra described in [T]o.

We recall that for $1 \leq i \leq n$ the group-like elements $K_i, L_i \in U_Q$ are the algebra automorphisms defined on the generators $T_j^i$ of $A(R_Q)$ by

$$K_i(T_j^i) = \delta_{ji} \kappa_j^i \quad L_i(T_j^i) = \delta_{ji} (\kappa_j^i)^{-1}$$

(5)

and $G(U_Q)$ is an abelian group generated by $\{K_i^{\pm 1}, L_i^{\pm 1}\}$.

There are skew-primitive elements $E_{i+1}^i \in U_Q, \; i = 1, \ldots, n - 1$ defined on the generators $T_j^i$ of $A(R_Q)$ by

$$E_{i+1}^i(T_k^i) = \delta_{ki} \delta_{i+1}$$

so that

$$\Delta(E_{i+1}^i) = K_{i+1} \otimes E_{i+1}^i + E_{i+1}^i \otimes K_i.$$  

(6)
There are also skew-primitive elements $F_{i+1}^{i+1} \in U_Q$ defined by

$$F_{i+1}^{i+1} (T^l_k) = \delta_{k,i+1} \delta_{l,i}$$

so that

$$\Delta(F_{i+1}^{i+1}) = L_i \otimes F_{i+1}^{i+1} + F_{i+1}^{i+1} \otimes L_{i+1} \quad (7)$$

The commuting relations for the $K_i$'s and the $E^j_{j+1}$'s are given by:

$$\kappa_{j+1}^j K_i E^j_{j+1} = \kappa_{j+1}^j E^j_{j+1} K_i \quad (8)$$

The algebra $U_Q$ is generated by \{ $K_t$, $L_t$, $E^{i}_{i+1}$, $F_{i+1}^{i+1}$ $|$ $1 \leq t \leq n$, $1 \leq i \leq n-1$. \}

It was proved [10 Th. 8.5] that $U_Q$ has a PBW basis given by products of powers of elements \{ $E^i_j$, $F^i_j$, $g$ $|$ $1 \leq i < j \leq n$, $1 \leq k < l \leq n$ \} where $g \in G(U_Q)$. The elements \{ $E^i_j$ \} can be defined successively starting from the elements \{ $K_i$, $E_{i+1}^{i+1}$ \} via the identities in [10 (5.14) - (5.18)]. Similarly, the elements \{ $F^i_k$ \} can be defined successively starting from the elements \{ $L_t$, $F_{i+1}^{i+1}$ \} via the identities in [10 (5.14) - (5.18)]. If $r$ is a root of unity then the additional identities

$$(E^i_j)^{e(Q)} = (F^i_j)^{e(Q)} = 0$$

hold, where $e(Q)$ is defined in [10 (8.1)]. Any other relation among the $E^i_j$'s or among the $F^i_k$'s can be derived from these sets of identities.

## 2 Hopf algebras of type $A_n$ arising from $U_Q$

In this section we discuss gr $U_Q$ and show that it provides new examples of Hopf algebras of type $A_n$.

Consider the (Hopf)-subalgebra $B^l \subset U_Q$ generated by:

$$B^l = \langle K_j^{i+1}, E^i_{i+1}, j = 1, \ldots n, i = 1, \ldots, n-1 \rangle$$

Let gr $B^l = R \# kG(B^l)$. By Lemma 1.1 we have that

$$V = Sp_k \{ x_i := E^i_{i+1} K^{-1} \quad i = 1, \ldots n-1 \} \quad (9)$$

is a subspace of primitive elements contained in $R$. Observe that for each $i$,

$$\rho(x_i) = K^{-1}_i K_{i+1} \otimes x_i.$$
Let $SLG$ be the group generated by the $n - 1$ group-like elements

$$SLG = \langle K_i = K_i^{-1}K_{i+1}, \ i = 1, \ldots, n - 1 \rangle$$

Then $V$ is a Yetter-Drinfeld module over $kSLG$. We consider the Hopf algebra $B(V)\#kSLG \subset \text{gr} \ B^l$.

Since the Yang-Baxter operator $R_Q$ is related to $GL_n$ one would expect the following proposition.

**Proposition 2.1.** Let $V$ and $SLG$ be as above, then $B(V)\#kSLG$ is of type $A_n$.

**Proof.** We need to compute the coefficients $l_{ij}$ of the braiding. Since $\rho(x_i) = K_i \otimes x_i$ it follows that

$$c(x_i \otimes x_j) = K_i \cdot x_j \otimes x_i =$$

By (8) $K_i \cdot x_j = \kappa_i^j(\kappa_{i+1}^j)^{-1}x_j$ which is given explicitly by:

$$\kappa_j^i(\kappa_{j+1}^i)^{-1} = \begin{cases} q_{j,i}q_{j+1,i}^{-1} & j < i - 1, \\ q_{i-1,i}r^{-1} & j = i - 1, \\ r^{-1}p_{i,i+1}^{-1} & j = i, \\ q_{i,j}p_{i,j}^{-1} & j > i \end{cases}$$

It follows that $K_i \cdot x_j = K_i^{-1}K_{i+1} \cdot x_j = l_{ij}x_j$ where

$$l_{ij} = \begin{cases} q_{j,i+1}q_{j+1,i+1}^{-1}q_{j+1,i} & j < i - 1, \\ q_{i-1,i+1}q_{i,i+1}^{-1}q_{i-1,i}r & j = i - 1, \\ q_{i,i+1}r^{-1}p_{i,i+1}^{-1} & j = i, \\ r^{-1}p_{i+1,i+2}^{-1}p_{i+1,i+1}^{-1}p_{i,i+2} & j = i + 1, \\ p_{i+1,i+1}p_{i+1,i+1}^{-1}p_{i,i+1}^{-1}p_{i,j} & j > i + 1 \end{cases}$$

Hence

$$l_{ij}l_{ji} = \begin{cases} 1 = l_{ii}^0 & |i - j| > 2, \\ r^{-2} = l_{ii}^2 & i = j, \\ r = l_{ii}^r & |i - j| = 1 \end{cases}$$

Thus the braiding is of Cartan type $A_n$ as claimed. \(\square\)
The structure of $B(V)$ depends only on $r$. It is finite-dimensional when $r$ is a root of unity by [10] or by [AS1]. In fact $B(V)$ is isomorphic to the ”positive” part of either $u_q(sl_n)$ or $U_q(sl_n)$, (where $u_q(g)$ is Lusztig’s finite-dimensional Hopf algebra derived from $U_q(g)$ when $q$ is a root of unity).

The other parameters determine the group $G(U_Q)$. The group will be infinite if any of the parameters $p_{i,j}, r$ is not a root of unity. This follows from the definition of the generators $K_i$ given in [9]. Thus $B(V)$ may be finite-dimensional while $SLG$ is infinite. Furthermore, some of the $K_i$’s may be of finite order while the others are of infinite order.

It is proved in [CM, Lemma 4.2] that if $r$ is not a root of unity then the group $G(B_l)$ ($G(B_r)$) is freely generated by the $n$ elements $K_i$.

For the finite-dimensional case we show how different choices of the parameters $\{p_{i,j}\}$ may provide non-isomorphic groups $G(B_l)$ and $SLG$.

**Theorem 2.2.** Let $Q = \{r \neq 1, p_{i,j} \in k\}_{1 \leq i < j \leq n}$ be $(\begin{smallmatrix} n \\ 2 \end{smallmatrix}) + 1$ non-zero elements in $k$, let $A(R_Q), U_Q$ be the FRT constructions associated with the Yang-Baxter operator $R_Q$ and let $V$ and $SLG$ be defined as in [7] and [11]. Assume $r$ is a root of unity of order $N$ and each $p_{i,j}$ is a root of unity of order $N_{ij}$. Then

1. The groups $G(B_l)$ and thus $SLG$ are finite.

2. If $\{N, N_{ij}\}_{1 \leq i < j \leq n}$ are relatively prime then each $\overline{K}_i$ is of order $Nm_i$ where $m_i \neq m_l$ for $i \neq l$. In this case $B(V)\#kSLG$ is the family defined in [AS], Example 7.27

**Proof.**

1. By [4] and [5] we have that $(K_i)^l(T)^k = \delta_{jk}(\kappa^i_j)^l$. Hence the order of $K_i$ depends on the order of $\{\kappa^i_j, j = 1, \ldots, n\}$. Now, $p_{i,j}q_{ij} = r$ for all $i < j$ hence if the order of each $p_{i,j}$ is $N_{ij}$ then the order of each $q_{ij}$ is lcm$\{N, N_{ij}\}$ and thus the order of each $K_i = \text{lcm}\{N, N_{ij}, 1 \leq j \leq n\}$.

2. For any $1 \leq i \leq n - 1$ set $M_i = \Pi_{j=i+1}^n N_{ij}, M_n = 1$ and $m_i = M_{i+1} M_i$. It follows by part 1 the order of each $K_i$ is $NM_i$ and hence the order of each $\overline{K}_i$ is $Nm_i$.

Set $p_{i,i} = 1$ and $p_{j,i} = p_{i,j}^{-1}$ for $i < j$. Recall [AST] Th. 3] that $A(R_Q)$ has a normal group-like element which is central if and only if $P_l = P_k$ for all $l, k$ where

$$P_l = r^l \prod_{j=1}^n p_{i,j}$$
Since for all $i < j$, $q_{i,j} = r/p_{i,j}$ it is not hard to check that

$$P_l = r \prod_{j=1}^{l-1} q_{jl} \prod_{j=l+1}^{n} p_{i,j} = \prod_{j=1}^{n} \kappa_j^l. \tag{11}$$

Set

$$\sigma = K_1 \cdots K_n \tag{12}$$

We have:

**Proposition 2.3.** If the determinant element of $A(R_Q)$ is central then $\sigma$ is a central group-like element of $U_Q$.

**Proof.** By (5) we have that for any $i$, $K_i(T^k_i)$ is nonzero if and only if $l = k$. Thus (4) implies that for any $u \in U_Q$, $g \in G(U_Q)$,

$$gu(T^k_i) = g(T^l_i)u(T^k_i) \quad \text{and} \quad ug(T^k_i) = u(T^k_i)g(T^k_i).$$

Observe that the element $\sigma = K_1 \cdots K_n$ satisfies $\sigma(T^l_i) = P_l$ for all $l$. Thus if the determinant is central then $\sigma(T^l_i) = \sigma(T^k_i)$ for all $l, k$. This implies that for all $u \in U_Q$, $k, l$

$$\sigma u(T^k_i) = \sigma(T^l_i)u(T^k_i) = u(T^k_i)\sigma(T^k_i) = u\sigma(T^k_i)$$

Now, for any $a, b \in A(R_Q)$, $u \in U_Q$ we have $\sigma u(ab) = \sum \sigma u_1(a)\sigma u_2(b)$ and $u\sigma(ab) = \sum u_1\sigma(a)u_2\sigma(b)$. So we can prove by induction on the length of monoms in $A(R_Q)$ that $u\sigma(a) = \sigma u(a)$ for all $a \in A(R_Q)$ which proves our claim. \(\square\)

We consider now some properties of the finite-dimensional Hopf algebras which are obtained when the bialgebra $A(R_Q)$ admits a central determinant.

**Theorem 2.4.** Let $Q = \{r \neq 1, p_{i,j} \in k\} \subseteq i < j < n$ be roots of unity, let $A(R_Q)$, $U_Q$ be the FRT constructions associated with the Yang-Baxter operator $R_Q$ and let $V$ and $SLG$ be defined as in (9) and (10). Assume that the determinant element of $A(R)$ is central, then:

1. All the group-like elements $K_i$ have the same order, hence all the $K_i$’s have the same order.

2. If the common order of the $K_i$ is relatively prime to $n$ then $G(B^1) = \sigma \times SLG$ where $\sigma$ is the central group-like defined in (12).
Proof. 1. By (11), the order of each \( P_i = \text{lcm}\{N, N_{ij} \mid 1 \leq j \leq n\} \) which is also the order of \( K_i \). Thus if the \( P_i \)'s are all equal then the order of the \( K_i \)'s are all equal.

2. Note that \( K_i K_j^{-1} \in \text{SLG} \) for all \( i \neq j \), hence \( K_i^{n-1} \prod_{j \neq i} K_j^{-1} \in \text{SLG} \) and so \( K_i^n = \sigma K_i^{n-1} \prod_{j \neq i} K_j^{-1} \in \sigma \text{SLG} \) for all \( i \). If the order of each \( K_i \) is \( m \) and \((m, n) = 1\) then this implies that \( K_i \in \sigma \text{SLG} \).

As it is known, by letting \( r = q^2 \) and \( p_{ij} = q \) for \( i < j \) the corresponding \( U_Q \) is \( U_q(gl_n) \) if \( q \) is not a root of unity and \( u_q(gl_n) \) otherwise.

The following is an example of a finite-dimensional \( U_Q \) so that the determinant of \( A(R_Q) \) is central. But unlike \( u_q(sl_n) \) and \( u_q(gl_n) \) the generators \( \overline{K_i} \) and \( K_i \) are not free. This implies that this Hopf algebra can not be obtained by twisting a known one.

Example 2.5. Let \( n = 3 \) and let \( q \) be a 7\(^{th}\) root of unity. Let

\[
\begin{align*}
    r &= q \\
    p_{12} &= p_{23} = q_{13} = q^2 \\
    q_{23} &= q_{12} = p_{13} = q^{-1}
\end{align*}
\]

Then the determinant element of \( A(R_Q) \) is central by (11). A direct computation using (5) shows that

\[
K_3 = K_1^{-2} K_2^3 \quad \text{hence} \quad \overline{K}_1^2 = \overline{K}_2.
\]

In the next example the \( n - 1 \) generators \( \overline{K}_i \) are free, but not the \( K_i \).

Example 2.6. Given \( n \), let \( q \) be a root of unity of order \( n + 1 \). As for the 1-parameter deformation \( u_q(gl_n) \), let

\[
\begin{align*}
    r &= q^2 \\
    p_{ij} &= q_{ij} = q.
\end{align*}
\]

A direct computation yields that \( \sigma = \prod_{i=1}^{n} K_i = 1 \). Thus \( \text{SLG} = G(B^l) \) by Theorem 2.2.4.

3 \( U_Q \) as a double crossproduct

In this section we show that for any choice of \{\( r \neq 1, p_{ij} \}_{1 \leq i < j \leq n}\) the Hopf algebra \( U_Q \) can be considered as a quotient of a double crossproduct which is quasitriangular in the finite dimensional case.
We recall first the definition of the double crossproduct \[ R_2 \]. Let \( B \) and \( H \) be bialgebras so that \( B \) is a left \( H \)-module coalgebra, \( H \) is a left \( B \)-module coalgebra and certain comparability conditions are satisfied. The double crossproduct \( B \rtimes H \) \( \text{Mj, 6.43} \) is is the tensor product \( B \otimes H \) with a coalgebra structure given by the \( \Delta_B \otimes \Delta_H \). The multiplication is defined with respect to the two given module structures. We omit here the general definition of the product; instead we will consider the following:

Let \( H \) be a Hopf algebra with a bijective antipode and \( B \) a sub Hopf algebra of \( (H^0) \text{cop} \). Then

\[
\begin{align*}
    h \mapsto p = \langle p_2, h \rangle p_1 & \quad p \mapsto h = \langle p_1, h \rangle p_2 & \quad p \mapsto h = \langle p, h_1 \rangle h_2 & \quad h \mapsto p = \langle p, h_1 \rangle h_2 \\
\end{align*}
\]

are defined for all \( h \in H, p \in B \). One can define an \( H \)-module structure on \( B \) and a \( B \)-module structure on \( H \) that satisfy all the necessary conditions for the double crossproduct. In this case the product in \( B \rtimes H \) is given explicitly by

\[
(p \rtimes h)(p' \rtimes h') = \sum p p_2 \rtimes (S^{-1} p'_1 \mapsto h \mapsto p'_3) h' \tag{13}
\]

If \( H \) is finite dimensional and \( B = (H^*) \text{cop} \) the double crossproduct is the Drinfeld double \( D(H) \).

We recall also more definitions and results from \( \text{[R5]} \). The bialgebra \( A(R_Q) \) is endowed with an invertible braiding \( < | >_R \). The braiding is given on generators by

\[
<T_i^j|T^k_j> = R_{ij}^{kl}
\]

where \( R_{ij}^{kl} \) are given in \( \text{[3]} \).

Let \( \lambda^+, \lambda^-, \rho^+, \rho^- : A(R_Q) \to A(R_Q)^* \) be the following maps:

\[
\begin{align*}
    \lambda^+(a) = &\ < a \mid >_R & \rho^+(a) = &\ < - \mid a >_R \\
    \lambda^-(a) = &\ < a \mid >_{R^{-1}} & \rho^-(a) = &\ < - \mid a >_{R^{-1}} \\
\end{align*}
\]

for all \( a \in A(R_Q) \). Recall that \( \lambda^+ \) is an anti-algebra and a coagebra map given explicitly by:

\[
\lambda^+(T^j_i) = \begin{cases} 
(r - 1) E^{j}_{i} & i > j, \\
K_{i} & i = j, \\
0 & \text{otherwise}
\end{cases} \tag{14}
\]
and \( \rho^+ \) is an algebra and an anti-coalgebra map given by [10, (6.14)] (after a slight modification) by

\[
\rho^+(T^j_i) = \begin{cases} 
    r^{-2}(r - 1)S^{-1}F^j_i & i < j, \\
    L^{-1}_i & i = j, \\
    0 & \text{otherwise}.
\end{cases}
\]  

(15)

Note that

\[ B^l = \langle K_i, K^{-1}_i, E^i_{i+1} \rangle \]

is the algebra generated by \( \{ \text{im}\lambda^+, K^{-1}_i \} \)

Set

\[ B^r = \langle L_i, L^{-1}_i, F^j_{i+1} \rangle \]

is the algebra generated by \( \{ \text{im}\rho^+, L_i \} \)

Observe that (6) and (7) imply that \( B^l \) and \( B^r \) are Hopf algebras. Moreover, \( U_Q = B^lB^r \).

**Lemma 3.1.** There exists a Hopf algebra injection

\[ \theta : (B^r) \rightarrow (B^l)^{0\text{cop}}. \]

**Proof.** Define a map \( \hat{\theta} : \text{Im}(\rho^+) \rightarrow (\text{Im}(\lambda^+))^* \) as follows: For \( v = \rho^+(b) \), \( w = \lambda^+(a) \), let

\[ \hat{\theta}(v)(w) := \langle a|b \rangle \]  

(16)

From the definition of \( \lambda^+ \) and \( \rho^+ \) it follows that \( \hat{\theta} \) is well defined, injective, algebra and anti-coalgebra map. We extend \( \hat{\theta} \) step by step:

**Step 1:** For any \( 1 \leq j \leq n \) we wish to extend \( \hat{\theta}(L^{-1}_j) \) to an element of \( \text{hom}(B^l, k) = (B^l)^* \). In order to have it we need only to define \( \hat{\theta}(L^{-1}_j)(K^{-1}_i) \) for all \( 1 \leq i \leq n \). Note that (15) and the definition of \( \hat{\theta} \) imply that

\[ \hat{\theta}(L^{-1}_j)(K^{-1}_i) = \rho^+(T^j_i)(\lambda^+(T^i_j)) = \langle T^i_j|T^j_i \rangle = \kappa^j_i. \]

Thus define:

\[ \hat{\theta}(L^{-1}_j)(K^{-1}_i) = (\hat{\theta}(L^{-1}_j)(K^{-1}_i))^{-1} = (\kappa^j_i)^{-1}. \]

Observe that \( \hat{\theta}(L^{-1}_j) \) is multiplicative as an element of \( (\text{Im}(\lambda^+))^* \), therefore the extension of \( \hat{\theta}(L^{-1}_j) \) defines an element in \( G((B^l)^*) \).

**Step 2:** We extend the domain of \( \hat{\theta} \) by letting

\[ \hat{\theta}(L_j) = (\hat{\theta}(L^{-1}_j))^{-1} \in G((B^l)^*). \]
for all $1 \leq j \leq n$.

Step 3: In order to extend $\hat{\theta}$ to a map $\theta : B^r \to (B^l)^*$. The only undefined values are those involving terms of the form $\theta(F^{j+1}_i)(K^{-1}_i)$. Since By (11) $S^{-1}F^{j+1}_i = -L^{-1}_{j+1}F^{j+1}_i L^{-1}_j$ it follows that

$$\theta(F^{j+1}_i)(K^{-1}_i) = \theta(S^{-1}F^{j+1}_i)(K_i) = -(\theta(L^{-1}_{j+1})(K_i))(\theta(F^{j+1}_i)(K_i)),$$

where the right hand side has been defined in the previous steps. Thus define:

$$\theta(v)(w) = \begin{cases} \hat{\theta}(v)(w) & w \in Im(\lambda^+) \\ \hat{\theta}(S^{-1}v)(K_i) & w = K^{-1}_i \end{cases}$$

and extend $\theta(v)$ to monoms containing $K^{-1}_i$ in $B^l$ with respect to the co-product in $(B^r)^{op}$. Then $\theta$ is the desired injection.

Before proving the main theorem of this section we wish to precede with some calculations. By abuse of notations denote $\theta(L_j)$ by $L_j$. Set

$$e_i = (r - 1)E^i_{i+1}, \quad f_i = r^{-2}(r - 1)F^{i+1}_i \quad (17)$$

Observe that $e_i = \lambda^+(T^i_{i+1})$ by (14) and $S^{-1}f_j = \rho^+(T^{j+1}_i)$ by (15). By (3) and (14) we have

$$L^{-1}_j(K_i) = \langle T^i_i | T^j_j \rangle = \kappa^i_j \quad (18)$$

$$L^{-1}_j(e_i) = \langle T^i_i | T^j_j \rangle = 0$$

Since $\theta$ is an anti-coalgebra map it follows that $S \circ \theta = \theta \circ S^{-1}$, and thus

$$S\theta(f_j) = \theta(S^{-1}f_j) = \theta(\rho^+(T^{j+1}_i)).$$

Hence (3) and (16) imply that

$$\begin{align*}
S\theta(f_j)(K_i) &= \theta(\rho^+(T^{j+1}_i))(\lambda^+(T^i_i)) = \langle T^i_i | T^{j+1}_j \rangle = R^{j+1,i}_{i,j} = 0 \\
S\theta(f_j)(e_i) &= \theta(\rho^+(T^{j+1}_i))(\lambda^+(T^i_{i+1})) = \langle T^i_{i+1} | T^{j+1}_j \rangle = R^{j+1,i}_{i+1,j} = \delta_{ij}(r-1).
\end{align*} \quad (19)$$

Now, by (3) we have $S^{-1}(e_i) = -K^{-1}_i e_i K^{-1}_{i+1}$ hence (8) implies that $S^{-2}(e_i) = K_{i+1}K^{-1}_i e_i K^{-1}_{i+1} K_i = r^{-1}e_i$. Therefore,

$$S^{-1}\theta(f_j)(e_i) = S\theta(f_j)(S^{-2}e_i) = r^{-1}S\theta(f_j)(e_i) = \delta_{ij}r^{-1}(r-1) \quad (20)$$
It follows that
\[
\theta(f_j)(e_i) = S\theta(f_j)(S^{-1}e_i) = \theta(f_j)(K_i^{-1}e_iK_{i+1}^{-1}) \\
= -S\theta(f_j)(e_i)L_{j+1}^{-1}(K_{i+1}^{-1}) \\
= -(\kappa_i^j\kappa_{i+1}^{j+1})^{-1}\delta_{ij}(r-1) \\
= -\delta_{ij}r^{-2}(r-1) \\
\text{(by (2))}
\]

We identify \(B^r\) with its image \(\theta(B^r) \subset (B^l)^{\text{co}p}\). Then \(B^r \bowtie B^l\) is defined and we use the above calculations to prove:

**Theorem 3.2.** \(U_Q\) is a Hopf-algebra quotient of \(B^r \bowtie B^l\).

**Proof.** Define \(\phi : B^r \bowtie B^l \to U_Q\) on generators by:
\[
\phi(u \bowtie w) = uw, \\
u \in B^r, \ w \in B^l.\]

Since \(U_Q = B^lB^r\) it follows that \(\phi\) is surjective. Since the coproduct in the double is the tensor coproduct it follows that \(\phi\) is a coalgebra map. Thus we need only to check that \(\phi\) is an algebra map.

Since \(B^r\) and \(B^l\) are contained as Hopf algebras in the double it follows from the definition of \(\phi\) that it is enough to check multiplicity on generators of the form \((\varepsilon \bowtie w)(u \bowtie 1)\).

We will start from the relations among the group-like elements: Observe that (13) implies that:
\[
(\varepsilon \bowtie K_i)(L_j \bowtie 1) = L_j \bowtie (L_j^{-1} \leftarrow K_i \leftarrow L_j) = L_j \bowtie K_i.\]

Thus all group-like elements in the double commute which is preserved in \(U_Q\) and so
\[
\phi((\varepsilon \bowtie K_i)(L_j \bowtie 1)) = \phi(L_j \bowtie K_i) = \phi(\varepsilon \bowtie K_i)\phi(L_j \bowtie 1).\]

Next we check relations between \(K_i\) and \(f_j\). Observe that
\[
(\varepsilon \bowtie K_i)(f_j \bowtie 1) = \\
= f_{j,2} \bowtie (S^{-1}f_{j,1} \leftarrow K_i \leftarrow f_{j,3}) \quad (\text{by (13)}) \\
= L_j^{-1}(K_i)L_{j+1}(K_i)(f_j \bowtie K_i) \quad (\text{by applying } \Delta^2(f_j) \text{ and by (19)}) \\
= \kappa_i^j(\kappa_{j+1}^{i+1})^{-1}(f_j \bowtie K_i) \quad (\text{by (18)})
\]
Hence by the definition of \( \phi \),

\[
\phi((\varepsilon \bowtie K_i)(f_j \bowtie 1)) = \\
= (\kappa_{j+1}^i)^{-1}\kappa_j^i \phi(f_j \bowtie K_i) \quad \text{(by above)} \\
= (\kappa_{j+1}^i)^{-1}\kappa_j^i f_j K_i \\
= K_i f_j \quad \text{(by (5))} \\
= \phi(\varepsilon \bowtie K_i) \phi(f_j \bowtie 1).
\]

Similarly,

\[
(\varepsilon \bowtie e_i)(L_j \bowtie 1) = L_j \bowtie (L_j^{-1} \rightarrow e_i \leftarrow L_j) = \kappa_i^j(\kappa_{i+1}^j)^{-1}(L_j \bowtie e_i)
\]

which by [10, 5.21] are the same identities as in \( U_Q \).

Consider now the relations between the \( \{ e_i \} \)'s and the \( \{ f_j \} \)'s. For the sake of convenience let \( L_i K_j \) denote the element \( L_i \bowtie K_j = (\varepsilon \bowtie K_j)(L_i \bowtie 1) \) in the double. We show first that:

\[
\kappa_i^j(\varepsilon \bowtie e_i)(f_j \bowtie 1) - \kappa_i^j (f_j \bowtie e_i) = \delta_{ij} r^{-1}(r - 1)(L_i K_{i+1} - L_{i+1} K_i) \quad (22)
\]

Indeed,

\[
(\varepsilon \bowtie e_i)(f_j \bowtie 1) = \\
= \sum f_{j+2} \bowtie (S^{-1} f_{j+1} \rightarrow e_i \leftarrow f_{j+3}) \quad \text{(by (13))} \\
= \sum S^{-1} f_{j+1}(e_{i+3}) f_{j+3}(e_{i+1}) (f_{j+2} \bowtie e_{i+2} \\
= L_{j+1}^i(K_i) f_j(e_i)(L_{j+1} \bowtie K_i) + L_{j+1}^i(K_i) L_j(K_{i+1}) (f_j \bowtie e_i) \\
+ S^{-1}(f_j)(e_i) L_j(K_{i+1}) (L_j \bowtie K_{i+1}) \quad \text{(by applying } \Delta^2 \text{ to } e_i \text{ and } f_j \text{ and by (18) and (19))} \\
= -\delta_{ij} r^{-2}(r - 1)\kappa_{i+1}^i(L_{i+1} \bowtie K_i) + \kappa_{j+1}^j(\kappa_{j+1}^i)^{-1}(f_j \bowtie e_i) \\
+ r^{-1}(r - 1)(\kappa_{i+1}^i)^{-1}(L_i \bowtie K_{i+1}) \quad \text{(by (18), (20) and (21))} \\
= \delta_{ij} \kappa_{i+1}^i r^{-2}(r - 1)(L_i \bowtie K_{i+1} - L_{i+1} \bowtie K_i) + \kappa_{j+1}^j(\kappa_{j+1}^i)^{-1}(f_j \bowtie e_i) \quad \text{(by (2))}
\]

Since by (2) \( \kappa_i^j \kappa_{i+1}^i = r \) (22) follows. Now, by [10, 5.23a] the following hold in \( U_Q \):

\[
\kappa_j^i E_{i+1}^j F_{j+1}^i - \kappa_{j+1}^i F_{j}^j E_{i+1}^i = \delta_{ij} r^{-1}(r - 1)(L_i K_{i+1} - L_{i+1} K_i) \quad (23)
\]
Hence:

$$\kappa_{i+1}^j \phi(\varepsilon \triangleright e_i) \phi(f_j \triangleright 1) - \kappa_i^j \phi(f_j \triangleright e_i) =$$

$$= (1 - r)^2 r^{-2} (\kappa_{i+1}^j E_i^j F_{i+1}^j + \kappa_i^j F_i^j E_i^j) - \kappa_{i+1}^j F_j^j E_{i+1}^j$$  (by (17))

$$= \delta_{ij} r (r - 1)^{-1} (1 - r)^2 r^{-2} (L_i K_{i+1} - L_{i+1} K_i)$$  (by (23))

$$= r^{-1} (r - 1) \delta_{ij} (L_i K_{i+1} - L_{i+1} K_i)$$

$$= \phi(\kappa_{i+1}^j (\varepsilon \triangleright e_i) (f_j \triangleright 1) - \kappa_i^j (f_j \triangleright e_i))$$  (by (22))

Therefore,

$$\phi((\varepsilon \triangleright e_i)(f_j \triangleright 1)) = \phi(\varepsilon \triangleright e_i) \phi(f_j \triangleright 1)).$$

This conclude the proof that \( \phi \) is multiplicative.

We have shown that \( U_Q \) is a homomorphic image of the double crossproduct. Note that the relations among the \( \{E_i^j, F_i^j\}_{i<j} \) are the same in \( U_Q \) and in the double. Thus the kernel of \( \phi \) may contain relations only among the group-like elements \( \{K_i, L_j\} \).

**Corollary 3.3.** If \( U_Q \) is finite dimensional then \( B^l \cong (B^r)^{\text{cop}} \). Thus the double is isomorphic to \( D(B^l) \) and so \( U_Q \) is quasitriangular.

**Proof.** If \( U_Q \) is finite dimensional then \( B^l = \text{im} \lambda^+ \) and \( B^r = \text{im} \rho^+ \). By Lemma 3.1 \( B^l \subset (B^l)^* \) as vector spaces. But one can prove similarly that \( B^l \subset (B^r)^* \) via \( \phi^* \). Hence they all have the same dimension and the double crossproduct is indeed a Drinfeld double.

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