Cusps in $K \rightarrow 3\pi$ decays

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Abstract

The pion mass difference generates a pronounced cusp in $K \rightarrow 3\pi$ decays. As has recently been pointed out by Cabibbo and Isidori, an accurate measurement of the cusp may allow one to pin down the S-wave $\pi\pi$ scattering lengths to high precision. Here, we present and illustrate an effective field theory framework that allows one to determine the structure of this cusp in a straightforward manner. The strictures imposed by analyticity and unitarity are respected automatically.

Key words: Chiral symmetries, Analytic properties of the S-matrix, Decays of K-mesons, Meson-meson interactions

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1. The S-wave $\pi\pi$ scattering lengths $a_0, a_2$ have been predicted with percent level accuracy some time ago [1, 2], and first steps for an experimental verification of this prediction have been performed in Ref. [3, 4]. Recently, it has been pointed out by Cabibbo and Isidori [5, 6] that isospin violating effects generate a pronounced cusp in $K \rightarrow 3\pi$ decays whose experimental investigation may allow one to determine the combination $a_0 - a_2$ with high precision. A first analysis of data based on this proposal has appeared [7]. (The strong impact of the unitarity cusp on $\pi^0\pi^0$ scattering close to threshold was already mentioned in [8].) In order for this program to be carried through successfully, one needs to determine the structure of the cusp with a precision that matches the experimental accuracy. In view of the large amount of data available [7], this is a considerable task. In the present letter, we present a method which – we believe – has the potential to achieve this goal.

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In [5, 6], the structure of the singularity at the cusp is investigated using unitarity, analyticity and cluster decomposition properties of the $S$-matrix. In addition, an approximation scheme is used, which consists in expanding the decay amplitude in powers of $\pi\pi$ scattering lengths. The latest work [6] retains effects up to order (scattering lengths)$^2$ and omits explicit electromagnetic effects. Here, we present a Lagrangian framework, which automatically satisfies unitarity and analyticity constraints and, in addition, allows one to include electromagnetic contributions in a standard manner. Specifically, we use a non-relativistic framework that has already proven to be useful in the description of bound states [9–25]. In this framework – in contrast to relativistic field theory – an expansion in powers of scattering lengths emerges automatically from the loop expansion. Moreover, it is a scheme that provides a proper power counting. [The low-energy expansion proposed here is closely related to early work performed in the sixties by many authors (for a review see [26]), who used $S$-matrix methods to investigate the production of particles – in particular also in $K \rightarrow 3\pi$ decays – in the threshold region. The method presented here may be considered an effective field theory realisation of these approaches.]

The strategy that we follow in this letter is the following. First we write down the most general non-relativistic Lagrangian relevant for this decay and determine all four-pion couplings therein through a matching procedure in terms of the threshold parameters of $\pi\pi$ scattering. In the next step, we evaluate the $K \rightarrow 3\pi$ decay amplitudes to two loops. This results in an explicit representation of the $S$-matrix elements which is valid in the whole decay region and slightly beyond. We propose to analyse the experimental data with the use of this representation. We plan to include real and virtual photon corrections at a later stage. On the other hand, we do keep the pion and kaon masses at their physical values, which is a fully consistent procedure in this framework. This guarantees that the various branch points and cusps occur at the proper place in the Mandelstam plane.

We display the results without a detailed derivation, which will be provided in a forthcoming publication [27].

2. We consider the neutral and charged decay modes $K^+(p_K) \rightarrow \pi^0(p_1)\pi^0(p_2)\pi^+(p_3)$ and $K^+(p_K) \rightarrow \pi^+(p_1)\pi^+(p_2)\pi^-(p_3)$. The kinematical variables are defined as usual: $s_i = (P_K - p_i)^2$ with $p_i^2 = M_i^2$, $i = 1, 2, 3$, where $M_{\pi^+} = M_\pi$ and $M_{\pi^0}$ denote the masses of the charged and neutral pions, respectively, and $\Delta_\pi = M_\pi^2 - M_{\pi^0}^2 \neq 0$. In the centre-of-mass (CM) frame $P_K = (M_K, 0)$, with $M_K$ the charged kaon mass,

$$p_i^0 = \frac{M_K^2 + M_i^2 - s_i}{2M_K}, \quad p_i^2 = \frac{\lambda(M_K^2, M_i^2, s_i)}{4M_K^2},$$

where $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$ is the triangle function.
Below we also use the velocities $v_{jk}$ and kinetic energies $T_i$,

$$v_{jk}^2(s_i) = \frac{\lambda(s_i, M_{j}^2, M_{k}^2)}{s_i^2}, \quad T_i = p_i^0 - M_i. \quad (2)$$

3. A non-relativistic approach to describe decays $K \to 3\pi$ can be justified, if the typical kinetic energies of the decay products are much smaller than the masses. This can be achieved by considering a world where the strange quark mass is taken to be smaller than its actual value. Then, a consistent counting scheme arises, if one introduces a formal parameter $\varepsilon$ and counts $T_i$ as a term of order $\varepsilon^2$, the pion momenta as order $\varepsilon$, whereas the pion and kaon masses are counted as $O(1)$. From $\sum_i T_i = M_K - \sum_i M_i$, one concludes that the difference $M_K - \sum_i M_i$ is then a quantity of order $\varepsilon^2$ as well. The pion mass difference $\Delta_\pi$ is also counted as $O(\varepsilon^2)$. The effective field theory framework, which we construct below, enables one to obtain a systematic expansion of the amplitudes in $\varepsilon$. For sufficiently small $m_s$, the expansion in $\varepsilon$ is expected to work very well.

Together with $\varepsilon$, our theory has another expansion parameter, namely a characteristic size of the $\pi\pi$ threshold parameters, which we denote generically as $a$. In particular, the amplitudes in the non-relativistic framework are given in form of an expansion in several low-energy couplings $C_i, D_i$, which can be expressed in terms of the threshold parameters of the relativistic $\pi\pi$ scattering amplitude. We expect the expansion in $a$ to converge rapidly because of the smallness of the scattering lengths. These two expansions are correlated [27]: because one-loop integrals are of order $\varepsilon$, adding a pion loop generated by a four-pion vertex increases both the order in $a$ and in $\varepsilon$ by one. A consistent power counting is achieved: to a given order in $a$ and in $\varepsilon$, a well-defined finite number of diagrams contribute.

Increasing now $m_s$ to its physical value again, convergence in the $\varepsilon$-expansion is not a priori evident, because $T_i/M_i$ can become as large as 0.4, and the corresponding maximal momentum $|p|$ is then not much smaller than the pion mass. However, let us note the non-relativistic framework is only used to correctly reproduce the non-analytic behaviour of the decay amplitudes in the kinematical variables $s_1, s_2, s_3$, and to thus provide a parametrisation consistent with unitarity and analyticity – a trivial polynomial part in the amplitudes can be removed by a redefinition of the couplings in the Lagrangian. In addition, from the analysis of the experimental data one knows [6] that in the whole physical region the real part of the decay amplitude can be well approximated by a polynomial in $s_1, s_2, s_3$ with a maximum degree 2. We interpret this fact as an experimental indication for a good convergence of the $\varepsilon$-expansion for the quantities one is interested in.

We now proceed with the construction of the non-relativistic Lagrangian framework. In the decay amplitudes, we shall restrict ourselves to terms up to
and including $O(\epsilon^2, a \epsilon^3, a^2 \epsilon^2)$.

4. It is convenient to formulate the non-relativistic approach in a manner that describes the two-particle subsystems in a manifestly covariant way [27]. We start with the $\pi\pi$ interaction and consider the following five channels in $\pi^a \pi^b \to \pi^c \pi^d$: (1) (00; 00), (2) (+0; +0), (3) (+−; 00), (4) (−−; −−), (5) (++; ++). The Lagrangian takes the form

$$\mathcal{L}_{\pi\pi} = 2 \sum_{\pm} \Phi_\pm W_\pm (i \partial_t - W_\pm) \Phi_\pm + 2 \Phi_0 W_0 (i \partial_t - W_0) \Phi_0 + \sum_{i=1}^{5} \mathcal{L}_i, \quad (3)$$

where $\Phi_i$ is the non-relativistic pion field operator, $W_\pm = \sqrt{M_\pi^2 - \Delta}$, $W_0 = \sqrt{M_\pi^2 + \triangle}$, with $\Delta$ the Laplacian, and $\mathcal{L}_i = x_i C_i \left( \Phi_c^\dagger \Phi_d^\dagger \Phi_a \Phi_b + h.c. \right) + x_i D_i \left\{ (W_c \Phi_c^\dagger W_d^\dagger \Phi_a \Phi_b + \Phi_c^\dagger \Phi_d^\dagger W_a \Phi_b \Phi_b + \nabla \Phi_c^\dagger \nabla \Phi_d^\dagger \Phi_a \Phi_b - h_i \Phi_c^\dagger \Phi_d^\dagger \Phi_a \Phi_b) + h.c. \right\} + \ldots, \quad (4)$$

with $h_1 = 2M_\pi^2$, $h_2 = 2M_\pi M_{\pi^0}$, $h_3 = 3M_\pi^2 - M_{\pi^0}^2$, $h_4 = h_5 = 2M_\pi^2$. The ellipsis stands for terms of order $\epsilon^4$ as well as for P-wave contributions, which occur at order $\epsilon^2$ in the $\pi\pi$ amplitude. They do not enter the $K \to 3\pi$ matrix elements at the order of accuracy considered, so we omit them here. The low-energy constants $C_i, D_i$ are matched to the physical scattering lengths below. To simplify the resulting expressions, we have furthermore introduced the scaling $x_1 = x_5 = 1/4$, $x_2 = x_3 = x_4 = 1$. Finally, note that we omit local 6-pion couplings as well. Their contribution to the $K \to 3\pi$ amplitude is purely imaginary in the non-relativistic framework, and of order $\epsilon^4$, see also [6, 27].

The pion propagator is given by

\[ i\langle 0\mid T\Phi_a(x)\Phi_b^\dagger(y)\mid 0 \rangle = \delta_{ab} \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip(x-y)}}{2w_a(p)(w_a(p) - p_0 - i0)}, \quad a, b = \pm, 0, \quad (5) \]

and $w_\pm(p) = \sqrt{M_\pi^2 + p_\pm^2}$, $w_0(p) = \sqrt{M_{\pi^0}^2 + p^2}$. The loops are evaluated by using the following prescription: one expands the square root in a series, calculates the emerging integrals using dimensional regularisation, and sums up the series. In this manner one ensures that the power counting in the non-relativistic theory is not destroyed by the loop corrections.

All loops in $\pi\pi$ scattering can be expressed through the basic integral

\[ J_{ab}(P^2) = \int \frac{d^Dl}{i(2\pi)^D} \frac{1}{2w_a(l)2w_b(P - l)} \frac{1}{(w_a(l) - l_0)(w_b(P - l) - P_0 + l_0)}, \quad (6) \]
with $P^2 = P_0^2 - P^2$. In the limit $D \to 4$,

$$J_{ab}(P^2) = \frac{i}{16\pi} v_{ab}(P^2),$$

which is a quantity of order $\epsilon$.

5. The couplings $C_i, D_i$ can be expressed in terms of the threshold parameters of the underlying relativistic theory. In the isospin symmetry limit, the expansion of the relativistic amplitude reads

$$\text{Re} \bar{T}_i(s, t) = \bar{A}_i \left\{ 1 + \frac{\bar{r}_i}{4M^2_\pi} (s - 4M^2_\pi) \right\} + \ldots.$$  

(8)

The ellipsis stands for higher orders in $\epsilon$, as well as for P-wave contributions. The bar indicates the isospin symmetric limit, at $M_\pi = 139.57$ MeV. In terms of the standard dimensionless scattering lengths $a_0$ and $a_2$, one has

$$3\bar{A}_1 = N(a_0 + 2a_2), \quad 2\bar{A}_2 = Na_2, \quad 3\bar{A}_3 = N(a_2 - a_0),$$

$$6\bar{A}_4 = N(2a_0 + a_2), \quad \bar{A}_5 = Na_2; \quad N = 32\pi,$$

(9)

with $a_0 - a_2 = 0.265 \pm 0.004$ [1, 2]. The products $\bar{A}_i\bar{r}_i$ denote effective ranges. Still in the isospin symmetry limit, the couplings $C_i, D_i$ are related to these threshold parameters in the following manner,

$$2\bar{C}_i = \bar{A}_i, \quad 8M^2_\pi \bar{D}_i = \bar{A}_i\bar{r}_i.$$  

(10)

Taking isospin breaking into account, one finds at leading order in chiral perturbation theory [28]

$$2C_{1,2,5} = \bar{A}_{1,2,5}(1 - \eta), \quad 2C_3 = \bar{A}_3(1 + \eta/3), \quad 2C_4 = \bar{A}_4(1 + \eta),$$

(11)

where $\eta = \Delta_\pi/M^2_\pi = 6.5 \times 10^{-2}$. Isospin breaking corrections in $D_i$ do not contribute at the order considered here.

6. It remains to display the $K \to 3\pi$ Lagrangian,

$$\mathcal{L}_K = 2K^\dagger W_K \left( i\partial_t - W_K \right) K$$

$$+ \frac{1}{2} G_0 \left( K^\dagger \Phi_+ \Phi^2_0 + h.c. \right) + \frac{1}{2} G_1 \left( K^\dagger (W_+ - M_\pi) \Phi_+ \Phi^2_0 + h.c. \right)$$

$$+ \frac{1}{2} H_0 \left( K^\dagger \Phi_- \Phi^2_+ + h.c. \right) + \frac{1}{2} H_1 \left( K^\dagger (W_- - M_\pi) \Phi_- \Phi^2_+ + h.c. \right) + \ldots,$$

(12)

where $K$ denotes the non-relativistic field for the $K^+$ meson, $W_K = \sqrt{M^2_K - \Delta}$, and the ellipsis stands for the higher-order terms in $\epsilon$. Note that all couplings $G_i, H_i$ are assumed to be real. Their contribution to the decay matrix elements at tree level is provided below, in the amplitudes $A_{N,C}$. 

5
The complete Lagrangian of the theory is \( \mathcal{L}_K + \mathcal{L}_{\pi \pi} \). The tree-level expressions for the amplitudes, generated by \( \mathcal{L}_K \), are modified by final state interactions of the pions, generated by loops evaluated with \( \mathcal{L}_{\pi \pi} \). We use the notation

\[
\mathcal{M}_{00+} = M_{N}^{\text{tree}} + M_{N}^{1\text{-loop}} + M_{N}^{2\text{-loops}} + \ldots \quad [K^+ \to \pi^0 \pi^0 \pi^+] ,
\]

\[
\mathcal{M}_{++-} = M_{C}^{\text{tree}} + M_{C}^{1\text{-loop}} + M_{C}^{2\text{-loops}} + \ldots \quad [K^+ \to \pi^+ \pi^+ \pi^-] \quad (13)
\]

for the decay amplitudes and the Condon-Shortley phase convention for the pions. Our amplitudes are normalised such that the decay rates are given by

\[
d\Gamma = \frac{1}{2M_K} \frac{(2\pi)^4 \delta^{(4)}(P_f - P_i)}{2(2\pi)^3 p_i^0} |M|^2 \prod_{i=1}^3 \frac{d^3 p_i}{2(2\pi)^3 p_i^0} . \quad (14)
\]

7. We now display the tree and one-loop results and modify the notation for the couplings \( C_i, D_i \) in order to make the formulae more transparent:

\[
(C_1, C_2, C_3, C_4, C_5) = (C_{00}, C_{+0}, C_x, C_+, C++) , \quad (15)
\]

and analogously for the \( D_i \). We find

\[
\mathcal{M}_N^{\text{tree}} = A_N(s_3) ,
\]

\[
\mathcal{M}_N^{1\text{-loop}} = B_{N1}(s_3) J_{+-}(s_3) + B_{N2}(s_3) J_{00}(s_3)
+ \{B_{N3}(s_1) J_{+0}(s_1) + (s_1 \leftrightarrow s_2)\} ,
\]

\[
\mathcal{M}_C^{\text{tree}} = A_C(s_3) ,
\]

\[
\mathcal{M}_C^{1\text{-loop}} = B_{C1}(s_3) J_{++}(s_3)
+ \{B_{C2}(s_1) J_{+-}(s_1) + B_{C3}(s_1) J_{00}(s_1) + (s_1 \leftrightarrow s_2)\} , \quad (16)
\]

where

\[
A_N(s_3) = G_0 + G_1\left(\frac{p_3^0}{M_\pi} - M_\pi \right) ,
\]

\[
B_{N1}(s_3) = 2\left(C_x + D_x(s_3 - \bar{s}_x)\right) \left\{H_0 + H_1\left(\frac{p_1^0 + p_2^0}{2} - M_\pi \right)\right\} ,
\]

\[
B_{N2}(s_3) = \left(C_{00} + D_{00}(s_3 - \bar{s}_{00})\right) \left\{G_0 + G_1\left(\frac{p_3^0}{M_\pi} - M_\pi \right)\right\} , \quad (17)
\]

\[
B_{N3}(s_1) = 2\left(C_{+0} + D_{+0}(s_1 - \bar{s}_{+0})\right) \left\{G_0 + G_1\left(\frac{p_2^0 + p_3^0}{2}\left(1 + \frac{\Delta_\pi}{s_1}\right) - M_\pi \right)\right\} ,
\]

\[
2 \text{ To render the formulae more compact, we keep some terms in Eqs. (17), (18) that contribute at order } a\epsilon^5. \]
In the above expressions, $\bar{s}_i$ denotes the physical threshold in the $i$ channel and $p_0^i$ are given by Eq. (1). Note that, according to this equation, the masses in the relation of $p_0^i$ to $s_i$ differ in the neutral and charged channels.

8. There are two topologically distinct two-loop graphs that describe pion-pion rescattering in the final state, see Fig. 1. At the order of accuracy we are working, it is sufficient to consider the case of non-derivative couplings. In this case, the contributions of both diagrams depend only on the variable $s$, where

$$Q^\mu = (q_1 + q_2)^\mu, \quad Q^2 = s.$$  

The diagram in Fig. 1B, apart from a factor containing coupling constants, is given by a product of two one-loop diagrams which were already calculated in Eq. (6). The non-trivial contribution from Fig. 1A is proportional to

$$\mathcal{M}(s) = \int \frac{d^Dl}{i(2\pi)^D} \frac{d^Dk}{i(2\pi)^D}$$

$$\times \frac{1}{2 w_a(l + k)} \frac{1}{w_a(l + k) - M_K + l^0 + k^0} \frac{1}{2 w_b(l)} \frac{1}{w_b(l) - l^0}$$

$$\times \frac{1}{2 w_c(k)} \frac{1}{w_c(k) - k^0} \frac{1}{2 w_d(Q - k)} \frac{1}{w_d(Q - k) - Q^0 + k^0}.$$  

(20)
A detailed discussion of this integral will be provided in [27]. Here we simply note that the most general representation for diagram Fig. 1A can be written in the form

$$\mathcal{M}(s) = F(M_a, M_b, M_c, M_d; s) + P(s) J_{cd}(s) + P'(s), \quad (21)$$

where $F$ is ultraviolet-finite and contains the full non-analytic behaviour of the two-loop diagram in the low-energy domain. Further, $J_{cd}$ is the one-loop function Eq. (7), and $P$, $P'$ are real polynomials. We have suppressed the dependence of $F$ on the mass $M_i$ generated by $Q^2$, see below.

In the following, we use a simplified form of $F$, where part of its imaginary part is dropped – this omission affects the decay width at order $\alpha^3$ only and is therefore of no relevance here. We use the integral representation [27]

$$F(M_a, M_b, M_c, M_d; s) = \frac{N_0}{64\pi^3\sqrt{s}} \int_0^1 dy \frac{dg(y, s)}{\sqrt{y}} \left( \ln g(y, s) - \ln g(y, \bar{s}) \right) + O(\epsilon^4), \quad (22)$$

where

$$N_0 = \frac{M_K}{2\sqrt{s_0}} \left( 1 - \frac{(M_a - M_b)^2}{s_0} \right)^{1/2} \frac{1}{(2(M_K^2 + M_c^2) - (M_a + M_b)^2 - s_0)^{1/2}},$$

$$s_0 = M_K^2 + M_c^2 - 2M_K \left( M_c^2 + \frac{Q^2(1 + \delta)^2}{4} \right)^{1/2},$$

$$g(y, s) = -\epsilon(1-y)q_0^2 - y\Delta^2 + \frac{y(1-y)Q^2(1 + \delta)^2}{4(1 + yQ^2/s)} - i0,$$

$$q_0^2 = \frac{\lambda(s, M_c^2, M_d^2)}{4s}, \quad \bar{s} = (M_c + M_d)^2,$$

$$\Delta^2 = \frac{\lambda(M_K^2, M_c^2, (M_a + M_b)^2)}{4M_K^2}, \quad \delta = \frac{M_c^2 - M_d^2}{s}. \quad (23)$$

Approaching threshold from above, we find

$$F(M_a, M_b, M_c, M_d; s) = -\frac{q_0}{128\pi^2(M_c + M_d)} \frac{\lambda^{1/2}(\bar{s}_0, M_a^2, M_b^2)}{\bar{s}_0} + O(q_0^2), \quad (24)$$

where $\bar{s}_0$ denotes $s_0$ at $q_0^2 = 0$. This last relation shows that $F$ is of order $\epsilon^2$.

9. Our prescription for the representation of the decay amplitudes at $O(\alpha^2)$ is as follows: we evaluate the contributions from all the graphs displayed in Fig. 2 and Fig. 3. Further, in the graphs of the type Fig. 1A, we retain only
Fig. 2. Two-loop graphs contributing to the decay $K^+ \to \pi^0\pi^0\pi^+$ in the non-relativistic effective theory. The graphs obtained by a permutation of identical particles in the final state are not shown.

the non-analytic piece $F$, whereas the polynomials $P, P'$ are included in the tree-level couplings $G_i, H_i$. This choice of a particular representation of $F$ is equivalent to a renormalisation prescription.

With this convention, we find for the amplitudes at order $a^2\epsilon^2$

$$\mathcal{M}_I^{2\text{-loops}} = \mathcal{M}_I^A(s_1, s_2, s_3) + \mathcal{M}_I^B(s_1, s_2, s_3) ; \ I = N, C \ ,$$

(25)

where

$$\mathcal{M}_N^A = 4H_0C_{++}C_xF_+(M_\pi, M_\pi, M_\pi, M_\pi; s_3)$$

$$+ 2G_0C_{++}^2F_+(M_\pi^0, M_\pi^0, M_\pi, M_\pi; s_3)$$

$$+ 2H_0C_{++}C_xF_+(M_\pi, M_\pi, M_\pi, M_\pi; s_3)$$

$$+ 4G_0C_{00}C_{++}F_+(M_\pi, M_\pi^0, M_\pi; s_3)$$

$$+ \left\{ 4H_0C_xC_{++}F_0(M_\pi, M_\pi, M_\pi, M_\pi^0; s_1) + 2G_0C_{++}^2F_0(M_\pi^0, M_\pi, M_\pi^0, M_\pi; s_1) + (s_1 \leftrightarrow s_2) \right\} ,$$

$$\mathcal{M}_N^B = 4H_0C_xC_{++}J_{++}^2(s_3) + G_0C_{00}^2J_{00}^2(s_3)$$

$$+ 2\left[ G_0C_x^2 + H_0C_xC_{00} \right] J_{+-}(s_3)J_{00}(s_3)$$

$$+ \left\{ 4G_0C_{++}^2J_{++}^2(s_1) + (s_1 \leftrightarrow s_2) \right\} ,$$

(26)
Fig. 3. Two-loop graphs contributing to the decay $K^+ \to \pi^+\pi^+\pi^-$ in the non-relativistic effective theory. The graphs obtained by a permutation of identical particles in the final state are not shown.

$$
\mathcal{M}_C^A = 2G_0C_2C_{+++}F_-(M_{\pi^0}, M_{\pi^0}, M_\pi, M_\pi; s_3), \\
+ 4H_0C_{---}C_{+++}F_-(M_\pi, M_\pi, M_\pi, M_\pi; s_3) \\
+ \left\{ 4H_0C^2_{---}F_+(M_\pi, M_\pi, M_\pi, M_\pi; s_1) \\
+ 2G_0C_2C_{---}F_+(M_{\pi^0}, M_{\pi^0}, M_\pi, M_\pi; s_1) \\
+ 2H_0C_{+++}C_{---}F_+(M_\pi, M_\pi, M_\pi, M_\pi; s_1) \\
+ 4G_0C_{++0}C_2F_+(M_\pi, M_\pi, M_{\pi^0}, M_{\pi^0}; s_1) + (s_1 \leftrightarrow s_2) \right\}, \\
$$

$$
\mathcal{M}_C^B = H_0C^2_{+++}J^2_{+++}(s_3) \\
+ \left\{ 4H_0C^2_{---}J^2_{---}(s_1) + G_0C_2C_{00}J^2_{00}(s_1) \\
+ 2[H_0C^2_2 + G_0C_2C_{++-}]J_{+-}(s_1)J_{00}(s_1) + (s_1 \leftrightarrow s_2) \right\}. 
$$

Here, $F_i(\ldots; s)$ stands for the integral $F(\ldots; s)$, evaluated at $Q^2 = \lambda(M_K^2, M_{\pi^0}^2, s)/4M_K^2$, with $i = \pm, 0$.

10. The decay amplitudes depend on the four real $K \to 3\pi$ coupling constants $H_i, G_i$ and on the threshold parameters for $\pi\pi$ scattering. Combining the tree and one-loop result Eqs. (16)–(18) with the two-loop contributions Eqs. (25)–(27), we obtain the neutral and charged decay amplitudes up to and including terms of order $\epsilon^2, a\epsilon^3$ and $a^2\epsilon^2$, expressed in terms of the one- and two-loop integrals $J$ and $F$ displayed in Eqs. (7) and (22), respectively. This representation is valid in the whole decay region, and is the main result of this article.
The decay amplitude $K^+ \rightarrow \pi^0\pi^0\pi^+$ obeys what we refer to as the threshold theorem: the coefficient of the leading non-analytic piece, which is proportional to $v_+(s_3)$, is given by a product of two factors, the decay amplitude $K^+ \rightarrow \pi^+\pi^+\pi^-$ and the scattering amplitude $\pi^+\pi^- \rightarrow \pi^0\pi^0$, both evaluated at threshold. Thus, the heuristic argumentation of Ref. [5], which serves as a cornerstone of the whole method, is confirmed in the effective field theory framework. This threshold theorem has its analogue in hadronic atoms, viz., in the modification of energy levels and decay widths through hadronic interactions [15–25]. Of course, aside from the determination of the leading term in $v_+$, our approach also allows a systematic evaluation of higher-order contributions $v_+^3, v_+^5, \ldots$.

11. We now compare the content of this letter with the recent work of Cabibbo and Isidori [6] (CI), who have proposed an alternative representation for the $K \rightarrow 3\pi$ decay amplitudes. Assuming certain analytic properties of the decay amplitudes, CI derive a representation of the amplitudes up to and including terms of order $a^2$, using analyticity, unitarity, and cluster decomposition properties of the $S$ matrix.

Conceptual aspects of our methods have already been compared in the introduction. Here we add that, firstly, we do provide an explicit representation of the decay amplitudes that is valid in the whole decay region, including all powers of velocities generated by the graphs considered. Secondly, we note that our explicit two-loop calculation confirms the expected [6] analytic properties of the amplitudes in the vicinity of the $\pi^+\pi^-$ threshold. On the other hand, away from this threshold, but still in the physical decay region, branch points develop [27], contrary to the expectations spelled out in [6].

After these general remarks we compare the amplitudes in more detail at one and two loops. In the actual calculations in Ref. [6], an approximation has been used: the angular integrals have been replaced by averages, where the integrand is evaluated at a certain value of $\cos \theta$. As CI note, this approximation is exact, if the integrand is at most a linear function in $\cos \theta$. Since this is true at $O(a\epsilon^2)$, one expects that our results at this order algebraically agree with CI. We have checked that this is indeed the case (up to a few typos [29]).

At two loops, our results are algebraically different from those of Ref. [6]. The reason can be traced back to the following. At two loops, the discontinuity cannot, in general, be obtained from an integration over $\cos \theta$ without further ado: the path of integration has to be deformed into the complex plane – it does not simply run from $\cos \theta = -1$ to $\cos \theta = 1$ along the real axis [26]. In fact, near the pseudothreshold $s_3 = (M_K - M_\pi)^2$, the deformed path runs to infinity, thus generating an infinity in the discontinuity there. In the case where all internal masses are equal to the charged pion mass, integrating $\cos \theta$ along the real axis does generate the correct discontinuity up to $s_3 = (M_K^2 - M_\pi^2)/2$. 

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In the case where two neutral pions are running in the inner loop, the situation is more complicated, because an anomalous threshold develops in the lower half plane. Still, integrating along the path mentioned generates the correct discontinuity near threshold. [These difficulties do not arise in our approach – the function $F$ in Eq. (22) is smooth on the upper rim of the cut, while its discontinuity develops the singularities mentioned, at the positions predicted by the Landau equations [27, 30, 31].]

A replacement of the angular integrals by an average [6], where the integrand is evaluated at a certain value of $\cos \theta$, can therefore be a reasonable approximation only in the vicinity of the threshold $s_3 = 4M^2_\pi$. Indeed, our result agrees (up to a typo [29]) with Ref. [6] to the order considered here, at threshold. Away from threshold, the expressions differ.

Finally, we shortly comment on the (revised) article by Gamiz et al. [32], which appeared only very recently. It is the aim of that article to provide an error analysis of the procedure proposed in [5, 6]. The authors investigate the process $K \rightarrow 3\pi$ in the framework of chiral perturbation theory, and approximate two-loop graphs by retaining their discontinuity only – an approach which generates fake singularities in the transition amplitude, as just mentioned. An ad hoc prescription is invoked in [32] to avoid these singularities when investigating the cusp, see e.g. their comment after Eq. (4.20).

12. In summary, we have investigated $K \rightarrow 3\pi$ decays within a non-relativistic effective Lagrangian framework. The amplitudes are calculated in a systematic double expansion in the kinetic energies of the decay products (which we count as terms of order $\epsilon^2$), and in the threshold parameters of elastic $\pi\pi$ scattering (which are generically denoted by $a$). We provide an explicit representation of the amplitudes at order $\epsilon^2, a\epsilon^3, a^2\epsilon^2$ – valid in the whole decay region – in terms of the (real) $K \rightarrow 3\pi$ coupling constants $G_i, H_i$ and of the threshold parameters $a$.

Our amplitudes agree with the ones of Cabibbo and Isidori [6], up to and including terms of order $a\epsilon^3$. On the other hand, at order $a^2$, they differ away from threshold. We propose to repeat the analysis of the experimental data of the NA48/2 collaboration [7] with our representation of the amplitudes, for several reasons: i) In view of the aimed precision, one ought to examine the importance of the mentioned differences in the determination of $\pi\pi$ scattering lengths. ii) It would be useful to extend the fit to the full decay region, and to the charged decay modes $K^+ \rightarrow \pi^+\pi^+\pi^-$ as well, in order to determine a maximal set of $\pi\pi$ threshold parameters. iii) As was already pointed out in [6], cusps also occur at the border of the Dalitz plot. Investigating data in those regions may allow one to determine different combinations of scattering lengths.
It remains to investigate the importance of higher orders in the low-energy expansion, and to apply radiative corrections, which can be evaluated in the field-theoretical framework used here in a standard manner. The effects generated by the \( \pi^+\pi^- \) bound state at the \( \pi^+\pi^- \) threshold can also be investigated within the same approach (see, e.g., [15–23]). We plan to include these effects in forthcoming publications.

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