Low energy transition in spectral statistics of 2D interacting fermions

Pil Hun Song\textsuperscript{(a)} and Dima L. Shepelyansky\textsuperscript{(b)}

\textsuperscript{(a)}Max-Planck-Institut für Kernphysik, Postfach 103980, 69029 Heidelberg, Germany
\textsuperscript{(b)}Laboratoire de Physique Quantique, UMR 5626 du CNRS, Université Paul Sabatier, F-31062 Toulouse Cedex 4, France

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We study the level spacing statistics $P(s)$ and eigenstate properties of spinless fermions with Coulomb interaction on a two dimensional lattice at constant filling factor and various disorder strength. In the limit of large lattice size, $P(s)$ undergoes a transition from the Poisson to the Wigner-Dyson distribution at a critical total energy independent of the number of fermions. This implies the emergence of quantum ergodicity induced by interaction and delocalization in the Hilbert space at zero temperature.

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The experimental observation of the metal-insulator transition in two dimensions (2D) by Kravchenko et al. \cite{1} has attracted a great interest to interacting fermions in a disordered potential. Indeed, according to the well-established result \cite{2}, all states are localized for noninteracting particles in 2D. Therefore, in the view of the experimental result \cite{1}, a new theory should be developed to understand the interaction effects between the localized fermionic states. However, in spite of various theoretical attempts, a coherent theory for such systems is still not available. While for highly excited states, it has been shown that the repulsive/attractive interaction can induce a delocalization of two interacting particles \cite{2,3}, the properties of low energy states are not understood yet. Recently, in addition to experimental and theoretical investigations, a number of attempts have been made to study these many fermionic systems through numerical simulations \cite{6,7,8}. Even though several interesting features have been reported, the systems studied there are not large enough to observe interaction induced delocalization.

In this paper we use another numerical approach based on the analysis of spectral properties of multi-particle fermionic systems. Indeed, Shklovskii et al. have shown that the level spacing statistics is a powerful tool to analyze the Anderson transition in disordered systems \cite{9}. When the states are localized, the levels are not correlated and the statistics is given by the Poisson distribution, $P(s) = P_P(s) = \exp(-s)$, while in the metallic phase, the states are ergodic and the statistics is close to the Wigner surmise, $P_W(s) = (\pi s/2) \exp(-\pi s^2/4)$. The critical transition point is characterized by an intermediate statistics which depends on the boundary conditions \cite{10} and the spatial dimension of the system \cite{11}. This approach has also been used to determine the quantum chaos border and the interaction induced thermalization in finite fermionic systems \cite{12} and to detect Anderson transition for two electrons with the Coulomb interaction on 2D disordered lattice \cite{13,14}. All these results demonstrate that the approach developed in \cite{10} allows to investigate efficiently the transition from nonergodic (localized) to ergodic eigenstates.

Here we use the above method to study the change of the spectral statistics, $P(s)$, with excitation energy $E$ in a model of spinless fermions with Coulomb interaction on 2D disordered lattice. The Hamiltonian reads

$$H = V \sum_{<ij>} a_i^\dagger a_j + \sum_i w_i n_i + U \sum_{i,j} \frac{n_i n_j}{r_{ij}},$$

where $a_i^\dagger (a_i)$ is the fermion creation (annihilation) operator at site $i$, $V$ is the hopping between the nearest neighbors, the diagonal energies $w_i$ are randomly distributed within the interval $[-W/2, W/2]$, $n_i = a_i^\dagger a_i$ is the occupation number at site $i$ and $U$ is the strength of the Coulomb interaction with $r_{ij}$ the interparticle distance. The particles move in a 2D cell of size $L \times L$ with the periodic boundary conditions applied. The Coulomb interaction is taken between electrons in one cell of size $L$ and with 8 charge images in nearby 8 cells as in \cite{3}. The number of particles $N_p$ and the cell size were varied within the intervals $2 \leq N_p \leq 20$ and $8 \leq L \leq 28$. With the notations in the Eq. (1), the parameter $r_s$, which measures the ratio of the Coulomb energy to the Fermi energy, $\epsilon_F$, is given by $r_s = U/(2V\sqrt{\pi \nu})$, where $\nu = N_p/L^2$ is the filling factor and $\epsilon_F = 4\pi \nu V$. The majority of our data have been obtained for $\nu \approx 1/32$ (nearest rational value) and for $U/V = 2$, which corresponds to $r_s = 3.22$.

To study the level spacing statistics $P(s)$, we generalize the approach used in \cite{10} for many particles. First, one particle eigenstates (orbitals) at $U = 0$ are obtained and the Hamiltonian (1) is rewritten in this basis using the two-body interaction matrix elements. We consider the first $M$ orbitals from the lowest energy and the Hamiltonian multi-particle matrix, constructed only from these orbitals, is diagonalized at the final stage. To the maximum, $N_p = 20$ particles with up to $M = 42$ orbitals has been considered and the resulting matrix size is $N_m \approx 5000$. This size is significantly smaller than $M C_{N_p}$ since the condition $\sum_{i=1}^{N_p} m_i \leq \sum_{i=1}^{N_p-1} i + M$ applies for
the one-particle orbital index $m_i$. Such a pyramid rule allows one to use efficiently only low energy multi-particle states and to make a striking reduction of the resulting total matrix size without any serious modification of the low energy states. We checked that our level statistics results at low energy are not sensitive to the variation of $M$ and $N_m$ (see for example insert in Fig.1). To compute the spectral statistics $P(s)$ at a given total excitation energy $E$, which is counted from the ground state energy, disorder average has been performed over $N_D = 5000$ (for low energy) and $N_D = 1000$ (for higher energy) configurations. In this way, the total statistics for $P(s)$ for small energy interval varies from $N_S = 10^4$ at low $E$ to $N_S = 3 \times 10^5$ at high $E$.

A change of $P(s)$ with $W$, at a given $E$ and all other parameters fixed, is shown in Fig. 1 for $N_p = 10$. As $W$ decreases, $P(s)$ evolves from the Poissonian to the Wigner-Dyson distribution. To measure the proximity of $P(s)$ to one of these two limiting cases, it is convenient to use the parameter $\eta$ which is defined as $\eta = \int_0^s (P(s) - P_W(s)) ds / \int_0^s (P_p(s) - P_W(s)) ds$, where $s_0 = 0.4729\ldots$ is the smaller intersection point of $P_p(s)$ and $P_W(s)$. In this way $\eta = 1$ corresponds to $P_p(s)$ and $\eta = 0$ to $P_W(s)$. According to Fig. 1, $\eta$ changes almost by an order of magnitude when $W$ decreases by factor of two. This shows that for strong disorder, $W/V = 15$, the multi-particle states at given $E$ are not ergodic (localized) while for weaker disorder, $W/V = 7$, they become ergodic and characterized by the random matrix spectral statistics.

We note that for given values of disorder the one-particle inverse participation ratio (IPR), $\xi_1$ (the number of sites contributing a one-particle eigenstate) is much smaller than the total number of sites $L^2$ for the case of Fig. 1:

![Graph of level statistics $P(s)$ for 10 particles at $L = 18$, $r_s = 3.22$ in the energy interval $0.25 < E/B < 0.3$ (with $B = 4V$): $W/V = 15$, $\eta = 0.93$ (○); $W/V = 10$, $\eta = 0.51$ (●); $W/V = 7$, $\eta = 0.10$ (•). Total statistics is $N_S = 10^5$; $1.4 \times 10^7$; $1.8 \times 10^5$, respectively. Full lines show the Poisson distribution and the Wigner surmise. Insert, made for $N_p = 6$ case of Fig.2b, illustrates that at low $E/B$ the statistics $P(s)$, characterized by $\eta$, is independent of matrix size variation $N_m = 797(○), 1231 (□), 2235 (full diamond)$.

![Graph of dependence of $\eta$ on the rescaled total energy $E/B$ for various number of particles $N_p$: 2(△), 3 (full triangle up), 4 (□), 6 (full diamond), 8 (○), 10 (●), 14 (○) and 20 (full triangle down); (a) $W/V = 10$ and (b) $W/V = 7$; filling factor $\nu \approx 1/32$ and $r_s = 3.22$, $8 \leq L \leq 25$. Inserts show the dependence of $\xi_s/N_p$ on $E/B$ by same symbols; arrows mark the critical energy $E_c$ from the main figure.

$\xi_1 = 3.4$ and $4.2$ ($W/V = 15$); 5.2 and $11.6$ ($W/V = 10$); 8.2 and $36.7$ ($W/V = 7$), where the formers are for the ground state and the latters for the center of the band. This means that the transition to ergodicity is induced by the interaction which becomes effectively more strong when the one-particle localization length increases (similar to the two interacting particle case [4,5]). At the
same time, the comparison with the data for $N_p = 2$ [14] at the same $E$ ($\eta \approx 0.92, 0.82$ and $0.51$ for $W/V = 15$, 10 and 7, respectively) shows that the delocalization effect due to multi-particle interaction is stronger for weak disorder while for strong disorder it does not affect localization.

![Graph](image)

**FIG. 3.** Level statistics $P(s)$ at the crossing points of Fig. 2 (with the same symbols). The upper data are for $W/V = 10$, $0.225 < E/B < 0.275$, $\eta_c \approx 0.56$, $N_S = 10^4, 4 \times 10^4$ and $10^5$ for $N_p = 3, 6$ and 10, respectively. The lower data (shifted down by 0.4) are for $W/V = 7$, $0.125 < E/B < 0.175$, $\eta_c \approx 0.33$, $N_S = 10^4, 1.4 \times 10^4$ and $5 \times 10^4$ for $N_p = 3, 6$ and 14, respectively. The full lines are the Poisson distribution and the Wigner surmise.

In Fig. 2 we present the variation of $\eta$ with the total energy $E$ for various number of particles at fixed filling factor and two values of disorder. For $N_p > 2$, all curves have approximately the same crossing point: $E_c \approx 0.25B$ ($W/V = 10$), $E_c \approx 0.15B$ ($W/V = 7$) and $E_c \approx 0.1B$ ($\eta_c \approx 0.19, W/V = 5$, not shown). As $N_p$ increases, the statistics approaches the Poissonian case for $E < E_c$, while it goes to the Wigner case for $E > E_c$. The transition point, $E_c$, is characterized by an intermediate statistics, which is independent of the system size as shown in Fig. 3. The energy $E_c$ grows with the increase of $W$ so that for $W/V = 15$ the crossing point is not found clearly within our computations (in this case $\eta \approx 0.8 - 1.0$ for $0 < E/B \leq 1$ and $N_p > 2$). The above data give a strong evidence that the ground state and the excited states with $E < E_c$ are localized in the limit of the infinite system size. However, for excitations with $E > E_c$ the statistics is close to $P_W(s)$ that implies the ergodicity of eigenstates and their space delocalization.

To show that the eigenstate properties are qualitatively different below and above $E_c$ we determined the variation of IPR $\xi_S$ with $N_p$ and $E$ as presented in the inserts of Fig. 2. This IPR $\xi_S$ is computed in the basis of noninteracting Slater determinants and determines how many of such states are required to construct an eigenstate. The results are obtained by averaging over 100 disorder realizations so that the statistical error is less than 5%. For $E < E_c$ the ratio $\xi_S/N_p$, which gives the number of noninteracting states per particle, remains constant while for $E > E_c$ it grows rapidly with $N_p$. For example, the ground state average value is $\xi_S/N_p = 0.35; 0.40; 0.46$ for $W/V = 10; 7; 5$ (the latter is not shown) with only $\pm 21; 13; 18\%$ variation when $N_p$ changes from 3 to 10; 14; 20, respectively for each $W/V$. On the contrary, for $E = 0.4B > E_c$ the ratio $\xi_S/N_p$ grows in $2.7; 9; 20$ times when $N_p$ changes from 3 to 8 for $W/V = 10; 7; 5$ respectively. These results show that the delocalization in the Hilbert space of Slater determinants takes place for $E > E_c$ with $E_c$ independent of number of particles. A crossover from localized to extended states in the Hilbert space was extensively discussed recently for the metallic phase with extended one-particle states [16,13]. Our data in Fig. 2 represent the first evidence for a delocalization transition in the Hilbert space for the insulating phase with localized one-particle states.

![Graph](image)

**FIG. 4.** Dependence of $\epsilon_\eta/B$ on the number of particles $N_p$, obtained from Fig. 2: $W/V = 10$ with $\eta$ ($E_\eta$) = 0.4 (full diamond) and $W/V = 7$ with $\eta$ ($E_\eta$) = 0.2 (●), where $\epsilon_\eta = E_\eta/N_p$. The straight line shows the slope when $E_\eta = \text{const}$. Insert gives the dependence of maximal $\eta$ on $r_s$ for $W/V = 7$ and $N_p = 6$: $U/V = 2$, $8 \leq L \leq 28$ (full diamond) and $L = 14$, $0.25 \leq U/V \leq 2$ (∞).

An unusual property of the above transition is that it takes place at the finite total energy. This means that the energy per particle $\epsilon = E/N_p$, or the temperature, is equal to zero at the transition point as $N_p \to \infty$. In this
sense this transition can be considered as a quantum zero temperature transition. To ensure that in our numerical computations $\epsilon$ was sufficiently low, we present variation of $\epsilon_{\eta}$, defined at the fixed $\eta$ level, with the number of particles in Fig. 4. According to these data $\epsilon_{\eta}$ drops more than an order of magnitude and becomes approximately by an order of magnitude smaller than the Fermi energy $\epsilon_F \approx 0.1B$.

Dependence of $\eta_m$ on $r_s$ is found in the insert of Fig. 4. The value of $r_s$ is varied in two different ways: i) by changing the cell size $L$ or ii) by changing $U/V$. The first case i) allows to obtain large $r_s$ value and shows that the statistics approaches the Poissonian case. This result is in a qualitative agreement with the experimental data \[1\], where the localized (insulating) phase appears for large $r_s$, and with the theoretical argument given in \[2\], according to which the two-body interaction drops as the filling factor decreases. The same arguments explain why $\eta$ goes to 1 when the interaction strength $U$ decreases as in the second case ii). One can expect that the variation of $\eta_c$ with $r_s$ at large $N_p$ and fixed $\nu$ will be qualitatively similar to that one of $\eta_m$ in Fig. 4.

The comparison of the energy $E_c$, at which the transition at fixed $\nu$ takes place (see Fig. 2), with the transition energy for two particles ($E_{2c}/B = 1.2$ for $W/V = 10$ and $E_{2c}/B = 0.56$ for $W/V = 7$ \[3\]) shows that $E_c$ is significantly smaller than $E_{2c}$ ($E_c/E_{2c} \approx 0.2$). This signifies that the multi-particle interaction is more efficient for delocalization than the two-body interaction for only two particles. Our qualitative understanding for the transition to the quantum ergodicity at fixed $\nu$ and fixed total energy $E_c$ is based on the following scenario. In the analogy with the approach developed in \[3\] for $N_p = 2$, the dynamics of three or four particles at finite energy $E$ can be assumed to be equivalent to one particle dynamics in a disordered system with effective spatial dimension $3 < d_{eff} < 2N_p$. With the growth of $E$, the rate of this one-particle hopping increases \[3\] and the delocalization transition finally takes place. As a result, in the original system, the energy can be transferred by this small group of particles from one place to another that allows to establish the ergodicity in the system of large size at fixed filling factor. The values of $\eta_c$ found here (Fig. 2) are larger than that for 3D Anderson model ($\eta_c \approx 0.2$), that makes the above scenario consistent with the result of \[12\] according to which $\eta_c$ grows with the increase of the spatial dimension $d$. We also note that the approach of $\eta$ to 1 near the ground state is not so surprising. Indeed, even in the metallic quantum dots one has $\eta \approx 1$ near the ground state since the interaction between particles is effectively weak \[13\] and the transition to ergodicity with $\eta = 0$ takes place only at higher energy \[17\].

In conclusion, our results show that 2D fermions, which are strongly localized by disorder without interaction, become ergodic due to Coulomb interaction. The transition to ergodicity, in lattice and Hilbert spaces, and the random matrix statistics takes place at the finite total energy or zero temperature. These results are in favor of the experimentally observed metal-insulator transition \[8\], even if our approach does not allow to investigate the conductance dependence on temperature which is the main experimental method to detect the transition. Furthermore, they show that the variable range hopping transport can be induced by electron-electron interaction in agreement with recent experiments \[13\].

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\[13\] We can try to adopt the approach \[1\] to the present case. According to \[1\] the two-body matrix element between particles at a distance $L$ is $U_c \sim U/L^3$ and the number of such effectively interacting particles at temperature $T$ is
$n_{\text{eff}} \sim \nu L^2 T/\epsilon_F$. Then the spacing between directly coupled multi-particle states is $\Delta_c \sim \Delta_2/n_{\text{eff}}^2$ where at low energy the two-particle spacing is $\Delta_2 \sim V/\xi_1$, since each particle can jump only inside $\xi_1$ sites \[4,14\]. The transition to $\eta = 0$ takes place for $U_s > \Delta_c$ \[13\] that gives the critical energy $E_c \sim T n_{\text{eff}} \sim (N_p/\nu)^{1/2} V^2/U \xi_1$. However, this border is higher than the numerically found one. This indicates that the delocalization of small group of particles is dominating.

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