Dark matter distribution function from non-extensive statistical mechanics

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Abstract

We present an analytical and numerical study of the velocity distribution function of self gravitating collisionless particles, which include dark matter and star clusters. We show that the velocity distribution derived through the Eddington’s formula is identical to the analytical one derived directly from the generalized entropy of non-extensive statistical mechanics. This implies that self gravitating collisionless structures are to be described by non-extensive thermo-statistics. We identify a connection between the density slope of dark matter structures, $\gamma$, from $\rho \sim r^{-\gamma}$, and the entropic index, $q$, from the generalized entropy, $S_q$. Our numerical result confirms the analytical findings of earlier studies and clarifies which is the correct connection between the density slope and the entropic index. We use this result to conclude that from a fundamental statistical mechanics point of view the central density slope of self gravitating collisionless dark matter structures is not constrained, and even cored dark matter structures are allowed with $\gamma = 0$. We find that the outer density slope is bounded by $\gamma = 10/3$.

Key words: cosmology: dark matter — gravitation — galaxies: kinematics and dynamics — cosmology: theory — methods: analytical

1 Introduction

The existence of dark matter (DM) has been established beyond any doubt, and the formation and evolution of cosmological structures through gravitational attraction is reasonably well understood. The velocity distribution of the resulting structures, however, remains unknown. This is in stark contrast to the well established velocity distribution of ideal gases, $f(v) \sim \exp(-mv^2/2kT)$, which is derived from the extensive Boltzmann-Gibbs entropy. Dark matter, however, experiences long-range gravitational interaction, and may therefore not obey the rule of extensivity.
At the same time numerical simulations provide predictions of steep central density cusps with power law slopes, $\rho \sim r^{-\gamma}$, with $\gamma$ from 1 to 1.5 within a few percent of the virial radius of the halo (Navarro, Frenk & White 1996, Moore et al. 1998). However, recent careful studies (Diemand, Moore & Stadel 2004, Reed et al. 2003, Stoehr 2004, Navarro et al. 2004) indicate that the resolved region has still not converged on a central density slope. Analytical analyses of the Jeans equations seem to indicate that the most shallow allowed slope is $\gamma = 1$ (Hansen 2004). Such steep inner numerically resolved slopes are, however, not supported by observations. By measuring the rotation curve of a galaxy one can in principle determine the density profile of its DM halo. Low surface brightness galaxies and spirals, where the observed dynamics should be DM dominated, seem to show slowly rising rotation curves (Rubin et al. 1985, Courteau 1997, Palunas & Williams 2000, de Blok et al. 2001, de Blok, Bosma & McGaugh 2003, Salucci 2001, Swaters et al. 2002, Corbelli 2003) indicating that these DM halos have constant density cores. Galaxy clusters, where baryons can play even less of a role, may show a similar discrepancy. Arcs (Sand, Treu & Ellis 2002) and strong lensing fits of multiple image configurations and brightnesses (Tyson, Kochanski & dell’Antonio 1998) also indicate shallow cores in clusters.

Facing this apparent disagreement between observations and numerical simulations it is very important to understand if the pure dark matter central density slopes are constrained from a fundamental statistical mechanics point of view. We will here combine an analytical and a numerical approach. First we show that the dark matter distribution functions are exactly the ones derived from non-extensive statistical mechanics. We then find a connection between the density slope, $\gamma$, and the entropic index, $q$. A given theoretical bound on $q$ will then lead to a bound on $\gamma$. We show that the bounds we can imagine still allow any density slope, including a core with $\gamma = 0$. The outer slope is bounded by $10/3$, which is close to the findings of N-body codes. Our findings can be very useful and directly applicable if one can envisage stronger theoretical bounds on $q$.

2 Eddington’s formula

Let us first derive the actual velocity distribution function of self gravitating collisionless structures numerically.

We will be considering the simplest possible dark matter structures (actually any collisionless system, including star clusters), which are spherical and isotropic. According to recent numerical N-body results this is a reasonably good approximation in the central equilibrated part of dark matter structures. We will argue later that statistical mechanics in a natural way forces a system
towards isotropy. Any given density profile, \( \rho(r) \), can be inverted (Eddington 1916) to give the particle velocity distribution function, \( f(E) \), through

\[
f(E) = \frac{1}{\sqrt{8\pi^2}} \int_0^E \frac{d^2\rho}{d\Psi^2} \frac{d\Psi}{\sqrt{E-\Psi}},
\]

(1)

where \( \Psi(r) \) is the relative potential as a function of radius, and \( E \) is the relative energy, \( E = \Psi - \frac{mv^2}{2} \). For recent applications and technical details see (Binney 1982, Kazantzidis et al. 2004). For the numerical inversion we typically consider structures with 10 orders of magnitude in radius, and always confirm that the resulting velocity distributions have converged. We consider only isotropic structures and leave non-isotropy for a future analysis. For simplicity we construct very simple structures which depend only on one free parameter, namely the density slope \( \gamma \), from \( \rho(r) \sim r^{-\gamma} \). The resulting velocity distribution functions are presented as (coloured) symbols in figure 1 for the slopes \( \gamma = 1.0, 1.5, 2 \) and 2.3.

It is important to note that for \( \gamma > 2 \) there is a maximal velocity. This is a real feature, purely related to the density profile, and is not simply the result of a finite structure. This shows directly that the velocity distribution function cannot be an exponential or any sum of exponentials, since exponential functions have tails going to infinite energy. We have thus shown explicitly that the often discussed sum of exponentials (Lynden-Bell 1967, Kull, Treumann

Fig. 1. The velocity distribution functions. The (coloured) symbols are from inverting a given density distribution according to Eddington’s formula, for the density slopes of \( \gamma = 1.0, 1.5, 2.0 \) and 2.3. The solid lines are the theoretical formulae from Tsallis statistics, using entropic indices of \( q = 1.39, 1.21, 1.0 \) and 0.86. The curves are arbitrarily normalized, and use different velocity dispersion, \( \sigma^2 \), in order to make the figure more readable and the difference in shapes more evident. The labels describe the lines from left to right.
& Böhringer 1997, Iguchi et al. 2004) cannot be correct, and instead a more general functional form for the velocity distribution function must be sought. We will now turn to non-extensive thermo-statistics and show that the correct functional form for the velocity distribution function is easy to derive.

3 Tsallis statistics

Statistical mechanics for classical gases can be derived from the Boltzmann-Gibbs assumption for the entropy, \( S_{BG} = -k \sum p_i \ln p_i \), where \( p_i \) is the probability for a given particle to be in the state \( i \), and the sum is over all states. For normal gases the probability, \( p(v) \), coincides with the velocity distribution function, \( f(v) \). This classical statistics can be generalized to Tsallis (or non-extensive) statistics (Tsallis 1988), which depends on the entropic index \( q \)

\[
S_q = -k \sum_i p_i^q \ln_q p_i ,
\]

where the \( q \)-logarithm is defined by, \( \ln_q p = (p^{1-q} - 1)/(1-q) \), and for \( q = 1 \) the normal Boltzmann-Gibbs entropy is recovered, \( S_{BG} = S_1 \). The probabilities still obey, \( \sum p_i = 1 \), while the particle distribution function is now given by \( f(v) = p^q(v) \). Thus, for \( q < 1 \) one privileges rare events, whereas \( q > 1 \) privileges common events. One also sees directly that entropy is maximized for isotropic distributions, which supports our original isotropy assumption. For a summary of applications see (Tsallis 1999), and for up to date list of references see http://tsallis.cat.cbpf.br/biblio.htm.

Average values are calculated through the particle distribution function, and one e.g. has the mean energy (Tsallis, Mendes & Plastino 1998)

\[
U_q = \frac{\sum p_i^q E_i}{c_p} ,
\]

where \( c_p = \sum p_i^q \), and \( E_i \) are the energy eigenvalues. Optimization of the entropy in eq. (2) under the constraints leads to the probability (Silva, Plastino & Lima 1998, Tsallis, Prato & Plastino 2003)

\[
p_i = \frac{[1 - (1-q)\beta_q (E_i - U_q)]^{1/(1-q)}}{Z_q} ,
\]

where \( Z_q \) normalizes the probabilities, \( \beta_q = \beta/c_p \), and \( \beta \) is the optimization Lagrange multiplier associated with the average energy. Adding a constant
energy, \( \epsilon_0 \), to all the energy eigenvalues leads to \( U_q \rightarrow U_q + \epsilon_0 \), which leaves all the probabilities, \( p_i \), invariant (Tsallis, Mendes & Plastino 1998). When \( \alpha \) is positive, where

\[
\alpha = 1 + (1 - q)\beta_q U_q ,
\]

then eq. (4) can be written as

\[
p_i = \frac{(1 - (1 - q)(\beta_q/\alpha) E_i)^{1/(1-q)}}{Z_q^\prime}.
\]

On figure 1 we plot 4 solid (black) lines, corresponding to \( q = 1.39, 1.21, 1.0 \) and 0.86, and well chosen \( \beta \)'s.

It is clear from figure 1 that the velocity distribution function, \( f(v) \), inverted from a density distribution function with slope \( \gamma = 2.3 \), where \( \rho \sim r^{-\gamma} \), is extremely well fitted with the non-extensive form in eq. (6) using \( q = 0.86 \). Similarly, the Maxwell distribution (the exponential, \( q = 1 \)) is recovered with an isothermal sphere (with \( \gamma = 2 \)), and finally, the \( \gamma = 1.5 \) is well fitted with \( q = 1.21 \). One should keep in mind that the probabilities, \( p(v) \), are related to the physical distributions through \( f(v) = p^\alpha(v) \).

We consider this excellent agreement in figure 1 as a strong indicator that equilibrium self gravitating structures indeed are to be described by non-extensive statistical mechanics. We are thus giving theoretical support to the use of Tsallis entropy (and the resulting velocity distribution function) when analysing gravitational structures. Such analysis was first conducted in (Lavagno et al. 1998), where an entropic index of \( q \approx 0.23 \) was found to describe accurately the peculiar velocity of galaxy clusters. It is worth emphasizing that the maximal velocity for \( q < 1 \), \( v_{\text{max}} = \sqrt{2\alpha/m\beta_q(1 - q)} \) is an inherent property of the non-extensive thermo-statistics (for \( q < 1 \)) and is not related to any finite size of the system. We also note that the \( \alpha \) in eq. (5) may be important for cluster temperature observations (Hansen 2004).

The structures we are considering are polytropes, which are described by the polytropic index, \( n \), making the connection between pressure and density, \( p \sim \rho^{1+1/n} \). It was first shown analytically by (Plastino & Plastino 1993) that these structures are exactly described by non-extensive statistical mechanics, and a connection was found between \( n \) and the entropic index, \( q \). Thus, the finding of this section is a numerical confirmation of the original work of (Plastino & Plastino 1993), which says that the distribution function for polytropes is infact given by eq. (6). As we will discuss below, the actual connection between \( n \) and \( q \) is not clear from the literature, and our independent results are useful in distinguishing which of the analytical results is correct.
Naturally, for a more general density profile where $\gamma$ is not constant, the actual velocity distribution function will be a sum over terms of the shape given by eq. (6), with $f(v) = \rho^\alpha(v)$. One can only approximate the velocity distribution function with one term of the form given in eq. (6) when $\gamma(r)$ varies sufficiently slowly with radius.

4 A connection between the density slope and the entropic index?

We can now proceed as in the previous section with various density profiles, and find the corresponding $q$. The result is presented in figure 2. The fitted $q$'s are accurate at the few percent level. This figure is presented only in the range $0 < \gamma < 2.5$, since steeper slopes lead to numerical problems in our simple inversion of Eddington’s formula. This is not a problem since we are interested in exactly this range in density profiles. As pointed out in section 3, our numerical finding that the self-gravitating structures are described by non-extensive statistical mechanics was first shown analytically by (Plastino & Plastino 1993). There seems, however, to be some confusion in the literature, as to what the connection between $n$ and $q$ really is. Polytropes follow the Lane-Emden equation, and for power-law density profiles one thus has, $n = \gamma/(\gamma - 2)$. Now, there are 3 different connections between $n$ and $q$ frequently used in the literature (see e.g. (Taruya & Sakagami 2003, Chavanis 2003, Tsallis, Prato & Plastino 2003) for different expressions) and we can therefore use our independent numerical method to distinguish which is the correct one. We find that the formula derived in (Tsallis, Prato & Plastino 2003, Lima & de Souza 2004, Silva & Alcaniz 2003), $n = (5 - 3q)/(2(1 - q))$ fits our results very well. We can therefore conclude that the correct connection between the density slope, $\gamma$, and the entropic index, $q$, is given by

$$q = \frac{10 - 3\gamma}{6 - \gamma}.$$  \hspace{1cm} (7)

This is shown as the thin (green) solid line in figure 2. We note that this equation may play an important role in explaining the observed correlation between the density slope and the anisotropy of dark matter structures (Hansen & Moore 2004).

It is interesting to ask the question: which are the theoretical constraints on the entropic index $q$? We saw above that eq. (6) is only valid for positive $\alpha$. Actually the expression in eq. (6) is also valid for negative $\alpha$, however, a negative effective temperature (a negative velocity dispersion in our case), $\beta_q' = \beta_q/\alpha$ is difficult to interpret for cosmological structures. Since we find numerically that $U_q\beta_q = 3/2$, one sees that $\alpha = 0$ for $q = 5/3$. The same conclusion, namely $q < 5/3$, was reached by (Boghosian 1999) by demanding a
positive proportion between temperature and internal energy; and also reached by (Silva & Alcaniz 2003) through an analysis of the velocity dispersion.

We see in figure 2, that \( q = 5/3 \) corresponds to a cored dark matter distribution with \( \gamma = 0 \). Thus all dark matter density slopes are allowed from a fundamental statistical mechanics point of view. Our findings are directly applicable if one can imagine a stronger theoretical bound on \( q \). E.g. (Boghosian 1999) considered the ideal gas case, and found that positiveness of the thermal conductivity leads to \( q < 7/5 \). Such a bound would according to our figure 2 imply a bound on the central density profile of \( \gamma > 1 \), in agreement with numerical and analytical suggestions (Diemand, Moore & Stadel 2004, Reed et al. 2003, Navarro et al. 2004, Hansen 2004). It has indeed been suggested (Xiao, Sun & Hao 2004) that the numerical methods induce an artificial conductivity, which would then imply a central slope of \(-1\).

The lower bound on \( q \) is given by \( q > 0 \) (Lima, Silva & Plastino 2001), which implies a bound on the dark matter profile of \( \gamma < 10/3 \), when using eq. (7). This is very close to the findings of numerical simulations, which is \( \gamma \leq 3 \). Further, if one would have a bound of \( q \geq 1/3 \) (as suggested by (Lima, Silva & Santos 2001)), then one would have a bound on the outer density profile of \( \gamma \leq 3 \).

\[ \text{Fig. 2. A given density profile, } \rho \sim r^{-\gamma}, \text{ gives a specific distribution function, } f(v), \text{ which can be fitted with two free parameters, namely the entropic index, } q, \text{ and the velocity dispersion, } \sigma^2. \text{ We present the resulting } q \text{ as a function of the density slope, } \gamma. \text{ The theoretical limit of } q < 5/3 \text{ is presented with a dashed line. The thin (green) solid line is the curve given in eq. (7).} \]
5 Conclusions

We compare the velocity distribution function for self gravitating collisionless structures (e.g. dark matter or star clusters) with the analytical expectation from non-extensive statistical mechanics, and find excellent agreement between the two. This is a strong indication that self gravitating systems of collisionless particles must be described by non-extensive statistical mechanics.

We identify a connection between the density power slope, $\gamma$, from $\rho \sim r^{-\gamma}$, and the entropic index, $q$, from the generalized entropy, $S_q$. We show that $q$ is only bounded from above by $q < 5/3$, which implies that all dark matter central slopes are allowed from this fundamental statistical mechanics point of view, including central cores with $\gamma = 0$. We show that the outer density slope is bounded by $\gamma \leq 10/3$. If one can imagine stronger theoretical bounds on $q$ then our findings are directly applicable to give stronger bounds on the allowed density slope.

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