Cosmological Born-Infeld gravity coupled to Born-Infeld electrodynamics

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Abstract. We present a detailed study about static spherically symmetric and electrically charged black hole solutions with non-zero cosmological constant \( \Lambda \). We study the Eddington-inspired Born-Infeld (known as EiBI) gravity coupled to the Born-Infeld electrodynamics. We analyze metric functions and employ Jana-Kar method to solve it. This black hole solution is parameterized by \( M, q, \kappa, \lambda \) and \( \Lambda \) which allows us to distinguish black hole, extremal black hole and naked singularity.

1. Introduction

General relativity has long been studied in numerous environment. Historically, it was Einstein who built the theory known as Einstein field equation from evaluating Einstein-Hilbert action. Several main feature of Einstein field equation is a black hole which has singularity enclosed by horizon(s). As far as we know, this singularity is like tiny region where the surface form shrink into a point. Therefore, tidal gravitational force are very strong, so matter gets stretched and squeezed out. According to this phenomenon, many researchers has been studied on geometrical structure of black holes in different matter sector.

Born-Infeld electrodynamics in 1930 [1] found that the electric field on electron self-energy has a finite value. It has attracted huge interest in recent times and also has good avenue when we approach it into gravitational field. It was Demianski who first studied static electromagnetic geon in asymptotically-flat space-time [2]. In 2003, Fernando and Krug [3] finds a solution in the presence of cosmological constant \( \Lambda \). From this point, we note that Einstein field equation was successful as a classical theory of gravity.

Nevertheless, it is always possible to study black hole in modified theory. Among the theories of gravity, a new kind theory called Eddington inspired Born-Infeld (EiBI) has been proposed by Banados and Ferreira [4] in 2010. The EiBI theory is a modification of the standard GR action. EiBI theory also has distinctive feature from GR such as it avoids the appearance of singularities in the early universe. When the EiBI is in vacuum \((\kappa \to 0)\), the action will be reduced to Einstein-Hilbert action. Although still fairly new, many aspects have been studied using this EiBI theory. It is reported that black hole in EiBI coupled to Maxwell electrodynamics has been studied in [5, 6] and tells us such horizon is Schwarzschild-like. Later, Jana-Karr finds a solution from EiBI-Born-infeld in 4 dimensions in asymptotically-flat space-time [7]. Also, EiBI theory has been widely studied in higher dimension [8, 9] and other compact object [10, 11, 12].

In this paper, we extend the Jana-Karr’s work by adding cosmological constant \( \Lambda \). We restrict ourselves to finding the exact solution in the electrostatic ansatz. This paper is structured as follows. Section 2 discusses Eddington inspired Born-Infeld theory as modified gravity and introduces new metric known as auxiliary metric \( q_{\mu \nu} \). Section 3 discusses the structure of
cosmological Born-Infeld gravity coupled to Born-Infeld electrodynamics. In that section, we evaluating three EiBI equation whose obtained in the previous section and transform it into Schwarzschild gauge. In section 4, we will discuss the result in order to distinguish black hole, extremal black hole and naked singularity. Finally, some concluding remarks are given in Section 5.

2. Eddington inspired Born-Infeld Theory

We start with EiBI action \[ S(g, \Gamma, \Phi) = \frac{1}{8\pi\kappa} \int d^4x \left( \sqrt{-g_{\mu\nu} + \kappa R_{\mu\nu}}(\Gamma) - \lambda \sqrt{-g} \right) + S_M(g, \Phi) \] (1)

Where \( \kappa \) is so called EiBI parameter. Cosmological constant \( \Lambda \) which the main of this work is embedded in \( \lambda \) by \( \lambda = 1 + \kappa \Lambda \). We also note that \( q_{\mu\nu} \) and \( g_{\mu\nu} \) is auxiliary and physical metric. Here, we set \( G = 1 \). In order to evaluate (1), we follow Vollick proposal \[ ; \] that is, use Palatini formalism and treat \( \Gamma \) and \( g \) as two independent fields. So the auxiliary metric defined as

\[ q_{\mu\nu} = g_{\mu\nu} + \kappa R_{\mu\nu} \] (2)

Varying EiBI action with respect to \( \Gamma \), leads us to the metric compatible expression. After some algebra, we get

\[ \Gamma^\sigma_{\alpha\beta} = \frac{1}{2} q^{\sigma\alpha} \left( \partial_\alpha q_{\beta\rho} + \partial_\rho q_{\alpha\beta} - \partial_\beta q_{\alpha\rho} \right) \] (3)

As we can see, \( \Gamma \) is now consist of auxiliary metric \( q \), not \( g \). If we varying the action (1) with respect to metric tensor \( g_{\mu\nu} \), we get the EiBI fields equation

\[ \sqrt{-qq^{\mu\nu}} = \lambda \sqrt{-g} \sqrt{-g} \sqrt{-8\pi\kappa g^{\mu\nu}} - 8\pi\kappa \sqrt{-g} T^{\mu\nu} \] (4)

with \( q \) and \( g \) is determinant of \( q_{\mu\nu} \) and \( g_{\mu\nu} \) and \( T^{\mu\nu} \) is energy-momentum tensor. Several nontrivial \( T^{\mu\nu} \) in EiBI theory has been studied in \[ 5, 6, 7 \]. As we mention before, we extend Jana-Karr works by adding cosmological constant \( \Lambda \) on its gravity sector. In the next section, we discuss this work in detail.

3. Cosmological Born-Infeld gravity coupled to Born-Infeld electrodynamics

We start with Born-Infeld lagrangian for matter fields,

\[ \mathcal{L}_{BI} = \frac{b^2}{4\pi} \left[ 1 - \sqrt{1 + \frac{F}{b^2}} \right] \] (5)

With \( F = \frac{1}{2} F_{\mu\nu} F^{\mu\nu} \) and \( b \) is Born-Infeld parameter. If we takes \( b \to \infty \), Born-Infeld lagrangian (5) can be reduced into Maxwell lagrangian with higher-order corrections,

\[ \mathcal{L}_M = -F_{\mu\nu} F^{\mu\nu} + O(F^4) \] (6)

Hence, the resulting of tensor energy-momentum which is derived from the variation of action (1) with respect to metric tensor is
In this work, we use electrostatic case \((A_\mu=\phi,0,0,0)\). Varying equation (1) with respect to \(A_\mu\) yields

\[
\nabla_\mu \left( \frac{F^{\mu\nu}}{\sqrt{1 + b^2 F^{\alpha\beta} F_{\alpha\beta}}} \right) = 0.
\]

Ansatz we use in this paper is the same as Jana-Kar

\[
g_{\mu\nu} dx^\mu dx^\nu = -U(\bar{r}) e^{2\phi(\bar{r})} dt^2 + U(\bar{r}) e^{2v(\bar{r})} d\bar{r}^2 + V(\bar{r}) \bar{r}^2 d\Omega_2^2,
\]

And

\[
q_{\mu\nu} dx^\mu dx^\nu = -e^{2\phi(\bar{r})} dt^2 + e^{2v(\bar{r})} d\bar{r}^2 + \bar{r}^2 d\Omega_2^2,
\]

Where \(d\Omega_2^2\) is the metric of an unit 2-sphere. In this metric, \(\bar{r}\) is radial coordinate, but the physical radial distance is \(r^2 = \bar{r}^2\). This physical radial distance is interpreted as area of 2-sphere. The advantage of this metric is absence of differential expression in field equation (4). Before evaluate these field equation, it is important to solve the equations of electromagnetic motion. In electrostatics, it is easy to see that the electric field is given by

\[
E(\bar{r}) = -\frac{d\phi}{d\bar{r}} = \frac{q U e^{\nu+\phi}}{\sqrt{\nu^2 \bar{r}^4 + q^2}}
\]

with \(q\) an integration constant which can usually be identified as the charge. Hence, the non-zero of tensor energy-momentum is

\[
\tau^{00} = \frac{e^{-2\nu b^2}}{U^4 \pi} \left( \frac{\sqrt{\nu^2 \bar{r}^4 + q^2}}{\nu \bar{r}} - 1 \right),
\]

\[
\tau^{11} = -\frac{e^{-2\nu b^2}}{U^4 \pi} \left( \frac{\sqrt{\nu^2 \bar{r}^4 + q^2}}{\nu \bar{r}} - 1 \right),
\]

\[
\tau^{22} = \frac{b^2}{4\pi \nu \bar{r}^2} \left( 1 - \frac{\nu \bar{r}^2}{\sqrt{\nu^2 \bar{r}^4 + q^2}} \right),
\]

and

\[
\tau^{33} = \frac{\tau^{22}}{\sin^2 \theta}
\]

Inserting this into field equation (4). After some algebra, we get the expression for \(V(\bar{r})\) and \(U(\bar{r})\) from polynomial equation
\[(\lambda^2 - 4b^2\kappa) V^2 - 2(\lambda - 2b^2\kappa) V + 1 - \frac{4\kappa^2 q^2 b^2}{\rho^4} = 0. \quad (16)\]

Of course, there are two possible solutions for \(V(\bar{\rho})\) and \(U(\bar{\rho})\). Therefore, we have

\[V(\bar{\rho}) = \frac{1 - \frac{2\kappa b^2}{\lambda} \left(1 + \sqrt{1 + \frac{\lambda^2 q^2 - 4\kappa^2 \lambda}{b^2 \bar{\rho}^4}} \right)}{\lambda - 4\kappa b^2 \kappa} \quad (17)\]

And

\[U(\bar{\rho}) = \frac{(\lambda - 4\kappa b^2 \kappa) \left(1 + \sqrt{1 + \frac{\lambda^2 q^2 - 4\kappa^2 \lambda}{b^2 \bar{\rho}^4}} \right)}{b^2 \bar{\rho}^4} \quad (18)\]

Equations (17) and (18) are metric solutions which can be obtained from polynomial equations. Also, we can conclude that the advantages of this metric is removing differential expression in field equation (4). Next, we evaluate equation (2) to obtain the metric \(e^{2\psi(\bar{\rho})}\)

\[\left(\frac{1 - U}{\kappa}\right) e^{2\psi} = e^{-2\nu + 2\psi \left(\frac{-2\psi'}{\bar{\rho}} + \nu' \psi'' - \psi'^2 - \psi''\right)}, \quad (19)\]

\[\left(\frac{1 - U}{\kappa}\right) e^{2\nu} = \left(\frac{2\psi'}{\bar{\rho}} + \nu' \psi'' - \psi'^2 - \psi''\right), \quad (20)\]

\[\left(\frac{1 - \nu}{\bar{\rho}}\right) \bar{\rho}^2 = 1 - \frac{1}{2} \left(\bar{\rho}' e^{2\psi}\right)'. \quad (21)\]

The first two equations are solved by \(\mu = -\nu\). Inserting it into (21) yields

\[e^{2\psi(\bar{\rho})} = 1 - \frac{2M}{\bar{\rho}} - \frac{\bar{\rho}^2}{3\kappa} + \frac{1}{\kappa \bar{\rho}} \int V(\bar{\rho}) \bar{\rho}^2 d\bar{\rho}. \quad (22)\]

Inserting metric \(V(\bar{\rho})\) into equation (21), we get

\[e^{2\psi(\bar{\rho})} = \left(1 - \frac{2M}{\bar{\rho}} - \frac{\bar{\rho}^2}{3\kappa} + \frac{1}{\kappa \bar{\rho}} \int V(\bar{\rho}) \bar{\rho}^2 d\bar{\rho} \right) \cdot \frac{\bar{\rho}^2}{3\kappa(4b^2\kappa - \lambda^2)} \left(2b^2\kappa \left(\sqrt{\frac{q^2(\lambda^2 - 4b^2\kappa)}{b^2 \bar{\rho}^4}} + 1 + \frac{4\kappa(q^2(4b^2\kappa - \lambda^2) - b^2 \bar{\rho}^4)}{q(4b^2\kappa - \lambda^2) - b^2 \bar{\rho}^4} \bar{\rho} \right) - 1 \right) \quad (23)\]

Where \(F[*, *]\) denotes Elliptic function [14]. When \(\Lambda = 0\), the solution in equation (22) reduces to the standard Schwarzschild form. Equations (17), (18) and (23) are the exact solution of cosmological Born-Infeld gravity coupled to Born-Infeld electrodynamics in radial coordinate \(\bar{\rho}\). This exact physical metric solution comes from gauge (9) that has \(\bar{\rho}\) on the metric. Fortunately, it is possible to bring it into another gauge that has \(r\) on its metric functions, called Schwarzschild gauge. In order to transform it, we follow Jana-Kar
works by making $\alpha \equiv 4k\beta^2 = 1$ in equation (16). Hence, the metric functions $V$, $U$ and $e^{2\psi(r)}$ becomes

$$V(\mathcal{F}) = \frac{\sqrt{\frac{4k(\lambda-1)\lambda q^2}{\mathcal{F}^2}+1+2\lambda-1}}{2(\lambda-1)\lambda},$$

and

$$U(\mathcal{F}) = \frac{(1-2\lambda)}{(-2\lambda^2+3\lambda-\frac{1}{2})\sqrt{1+\frac{4kq^2\beta(\lambda-1)}{\mathcal{F}^2}+\lambda-\frac{3}{2}}}.$$

Now we want to transform this metric from gauge (9) into Schwarzschild gauge by making

$$W = r^2 \equiv \frac{\sqrt{\frac{4k(\lambda-1)\lambda q^2}{\mathcal{F}^2}+1+2\lambda-1}}{2(\lambda-1)\lambda}$$

we obtain the line element

$$ds^2 = -U(r)e^{2\psi(r)}dt^2 + U(r)e^{-2\psi(r)} \frac{r^2(2\lambda+\frac{r^2}{4kq^2+r^2})}{2(\sqrt{4kq^2+r^4}(2\lambda-1)r^2)} dr^2 + r^2 d\Omega^2 \quad (27)$$

Where

$$U(r) = \frac{(1-2\lambda)}{(-2\lambda^2+3\lambda-\frac{1}{2})\sqrt{1+\frac{4kq^2\beta(\lambda-1)}{w(r)^2}+\lambda-\frac{3}{2}}}$$

and

$$e^{2\psi(r)} = 1 - \frac{2M}{w(r)} - \frac{w(r)^2}{3\kappa} + \frac{w(r)^2}{3\kappa} \left[ 1+\frac{4kq^2\beta(\lambda-1)}{w(r)^2}+1+2\lambda-1 \right]$$

$$\frac{1}{w(r)} \frac{w(r)^2}{\sqrt{1+\frac{4kq^2\beta(\lambda-1)}{w(r)^2}+\lambda-\frac{3}{2}}}$$

$$2kq^2 \left[ \frac{w(r)^2}{\sqrt{1+\frac{4kq^2\beta(\lambda-1)}{w(r)^2}+\lambda-\frac{3}{2}}} \right] + 4F(\sinh^{-1}\left(\frac{\sqrt{\frac{4k(\lambda-1)\lambda q^2}{w(r)^2}}}{\sqrt{2}}\right),-1,1) \right] \right)}{w(r)^4+4k(\lambda-1)\lambda q^2}$$

And
Equation (27)-(30) are exact solutions which we want to analyze its behaviour in the next section

4. Result and Discussion

From the previous section, we study static spherically-symmetric solutions of EiBI-Born-Infeld with the presence of cosmological constant $\Lambda$. Equation (27)-(30) are exact solutions in this paper. Since black holes solution is parameterized by $M$, $q$, $\kappa$, $\lambda$ and $\Lambda$, it is tempting to distinguish whether those parameter obeys black hole solution, extremal black hole or naked singularity. Metric solution in cosmological Einstein-Born-Infeld has been studied by Sharmanthie Fernando and Don Krug [3] and found that these metric behaviour causing at the most two horizons. While in EiBI (Eddington inspired Born-Infeld), has the same amount of horizons. Next, we will present the metric behaviour of cosmological EiBI-Born-Infeld black hole. In Figure 1, we show a typical plot of metric $g_{tt}$ and $g_{rr}$ as a function of $r$, where $r$ is physical radial distance as we mention before. We plot them for some particular values of $\kappa$ and $q$ in de-Sitter space ($\Lambda > 0.9$). We varying the value of $M$ and take $M = 1$ as a maximum parameter since we want to analyze in planck mass orde. Important to note that this metric is plotted by constraint $\alpha \equiv 4\kappa b^2 = 1$. Based on this 4d metric plot, it turns out that the behaviour of this metric in 4d is also the same as [3]; that is, can have at the most two positive roots. We note some generic feature that for $\kappa$ and $M$ positive, the horizon increases with decreasing $M$ parameter. Another point we want to analyse is when we consider at $r \to 0$. We can see in 4d plot, the metric function indicating a point charge [7]. In large $r$, these metric can be seen as asymptotically-de-Sitter space-time.

![Figure 1](image-url)
Table 1. Table for numerical value of $r_+$ in different value of $M$ with $\kappa = 5, q = 0.3, \Lambda = 0.9$. We note that $r_+$ is horizon radius and $r_e$ is extremal radius as we can see, for small $M$ can have at the most two positive roots.

| M    | $r_{1+}$ | $r_e$  | $r_{2+}$ |
|------|----------|--------|----------|
| 1    | ~        | ~      | ~        |
| 0.782| ~        | 0.96   | ~        |
| 0.5  | 0.3      |        | 1.43     |

While in $r_e$, we only have one positive root. Nevertheless, it is tempting to investigate the metric $g_{rr} = \frac{u(r)}{e^{2\Phi(r)}}$. The exact solution already appear in equation (28) and (29). For detailed study, we also present a typical profile of metric $g_{rr}$ in figure 1. We varying the parameter $M$ as a function of $r$ with chosen parameter ($\kappa = 5, q = 0.3, \Lambda = 0.9$). As we can see, metric $g_{rr}$ is inverse of metric $g_{tt}$. For $M = 1$, suffers naked singularity in metric $g_{tt}$. Consequently, the $g_{rr}$ metrics are regular everywhere. For extremal $M_e$, suffers singularity at $r_e = 0.96$ whereas for $M = 0.5$ suffers two singularities. In this discussion, we also note a generic feature that for positive $\kappa$ and $q$, the singularity point increase with decreasing $M$ parameter.

5. Conclusion

In this work, we study static spherically symmetric electrically charged black hole solution in the presence of cosmological constant $\Lambda$. For gravity sector, we use Born-Infeld lagrangian and employ Palatini formalism to evaluate the metric. After evaluating EiBI equation, we get the solution in gauge (9) and transform it into Schwartzschild gauge. Our solution reduces to Jana-Kar when $\Lambda = 0$ ($\lambda = 1$) and also reduces to Reissner-Nordstrom when $\kappa \to 0$ which caused by natural fraction $\alpha = 1 \to b^2 = \frac{1}{4\kappa}$. All in all, we conclude that the horizon increases with decreasing $M$ parameter and allows us to distinguish black hole, extremal black hole and naked singularity.

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