NON-EXPANSIVE BIJECTIONS, UNIFORMITIES AND POLYHEDRAL FACES

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Dedicated to the memory of Bernardo Cascales

Abstract. We extend the result of B. Cascales at al. about expand-contract plasticity of the unit ball of strictly convex Banach space to those spaces whose unit ball is the union of all its finite-dimensional polyhedral extreme subsets. We also extend the definition of expand-contract plasticity to uniform spaces and generalize the theorem on expand-contract plasticity of totally bounded metric spaces to this new setting.

1. Introduction

Let $E_1, E_2$ be metric spaces. A map $F: E_1 \to E_2$ is called non-expansive (resp. non-contractive) if $d(F(x), F(y)) \leq d(x, y)$ (resp. $d(F(x), F(y)) \geq d(x, y)$) for all $x, y \in E_1$. A metric space $E$ is called expand-contract plastic (or simply, an EC-space) if every non-expansive bijection from $E$ onto itself is an isometry.

[5, Satz IV] or [10, Theorem 1.1] imply that every totally bounded metric space is an EC-space, but there are examples of EC-spaces that are not totally bounded (and even unbounded).

In general bounded closed subsets of infinite-dimensional Banach spaces are not EC-spaces, see [2, Example 2.7]. It is not known whether it is true that for every Banach space $X$ its unit ball $B_X$ is an EC-space. There is no known counterexample and there are some known partial positive results: finite-dimesional spaces (the unit ball is compact), strictly convex Banach spaces (see [2, Theorem 2.6]) or $\ell_1$-sum of strictly convex Banach spaces (see [7, Theorem 3.1]). A more general problem is studied in [12]:

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Problem 1.1. Let $X$ and $Y$ be Banach spaces and $F : B_X \to B_Y$ be a bijective non-expansive map. Is $F$ an isometry?

There are positive answers when $Y$ is $\ell_1$, a finite-dimensional Banach space or a strictly convex Banach space (see [12]).

The unit sphere of a strictly convex space consists of its extreme points. The main result of our paper is Theorem 4.11, in which we substitute extreme points by finite-dimensional polyhedral extreme subsets. Namely, we demonstrate that if $X$, $Y$ are Banach spaces, $F : B_X \to B_Y$ is a bijective non-expansive map and $S_Y$ is the union of all its finite-dimensional polyhedral extreme subsets, then $F$ is an isometry.

Let us briefly explain the structure of the paper. In Section 2 we extend the results about EC-spaces in totally bounded metric spaces to totally bounded uniform spaces, see Theorem 2.3 and Lemma 2.4.

In Section 3 we recollect some known results about bijective non-expansive maps between unit balls that we will need in the sequel.

The goal of Section 4 is to demonstrate the main result. On the way we collect as much as possible information about preimages under a bijective non-expansive map of finite-dimensional faces of the unit ball. Using this information we obtain positive answers for the Problem 1.1 for the case when $X$ is strictly convex (Theorem 4.2) and for the case when $S_Y$ is the union of all its finite-dimensional polyhedral extreme subsets (Theorem 4.11).

We dedicate this paper to the memory of Bernardo Cascales, who passed away in April, 2018. From the very beginning our activity related to EC-spaces was motivated by Bernardo’s interest to the subject. It was his idea to search for a definition of EC-spaces that could be applicable to topological vector spaces and to uniform spaces. It was Bernardo who communicated to us the Ellis’ result [4] and the way how this result was used by Isaac Namioka [11] in his elegant demonstration of EC-plasticity of compact metric spaces. We were planning to start a joint project, but...
The uniform space \((E, U)\) is called totally bounded if for every \(U \in \mathcal{B}\) there is a finite subset \(\tilde{E} \subset E\) such that \(E = U[\tilde{E}]\).

Let us recall the following definitions that were introduced in [3] and extend the concepts of non-expansive, non-contractive and isometric maps to uniform spaces.

**Definition 2.1.** Let \((E, U)\) be a uniform space, \(B\) a basis of the uniformity and \(F : E \to E\) a map. We say that \(F\) is non-contractive for the basis \(B\) if for every \(V \in B\)

\[(F(x), F(y)) \in V \Rightarrow (x, y) \in V\]

We say that \(F\) is non-expansive for the basis \(B\) if for every \(V \in B\)

\[(x, y) \in V \Rightarrow (F(x), F(y)) \in V.\]

We say that \(F\) is an isobasism for the basis \(B\) if for every \(V \in B\)

\[(F(x), F(y)) \in V \iff (x, y) \in V.\]

For unexplained standard definitions and terminology we refer to [8, Chapter 6].

The next proposition will be used in the proof of Theorem 2.3.

**Proposition 2.2** (Ellis [4]). Let \(K\) be a compact space, let \(S \subset C(K, K)\) be a semigroup for the composition, and let \(\Sigma := \overline{S} \subset K^K\). The following are equivalent:

1. each member of \(\Sigma\) is onto,
2. each member of \(\Sigma\) is one to one,
3. \(\Sigma\) is a group and \(id : K \to K\) is the identity element of the group.

**Theorem 2.3.** Let \(K\) be a compact Hausdorff uniform space, \(B\) a basis for the uniformity made of open sets in \(K \times K\). If \(F : K \to K\) is a non-contractive bijection for the basis \(B\), then \(F\) is an isobasism for the basis \(B\).

**Proof.** The demonstration follows the idea of Namioka’s unpublished proof of EC-plasticity of compact metric spaces [11], and is presented here with his kind permission.

Observe that since \(F\) is non-contractive, then \(F^{-1}\) is non-expansive and then

\[(x, y) \in V \Rightarrow (F^{-1}(x), F^{-1}(y)) \in V;\]

so \(F^{-1}\) is a continuous bijection between compact spaces and then \(F\) is a continuous function. Consider the semigroup

\[S = \{F^n : n \in \mathbb{N}\} \subset C(K, K)\]

and let \(G \in \Sigma = \overline{S}\) be the pointwise closure of \(S\). Choose a net \((G_i)_{i \in I}\) in \(S\) that converges to \(G\). Let \(x, y \in K\) and \(V \in \mathcal{B}\) be such that
\[(G(x), G(y)) \in V. \] There is \(j \in I\) such that \((G_j(x), G_j(y)) \in V\). Let \(n \in \mathbb{N}\) be such that \(G_j = F^n\). Then since \(F\) is non-contractive we have that
\[(F^n(x), F^n(y)) \in V \Rightarrow (F^{n-1}(x), F^{n-1}(y)) \in V \Rightarrow \cdots \Rightarrow (x, y) \in V.\]

We have proved that
\[(2.2) \quad (G(x), G(y)) \in V \Rightarrow (x, y) \in V\]
for every \(G \in \Sigma\), \(x, y \in K\) and \(V \in \mathcal{B}\). Then we have that \(G\) is one to one. By Proposition 2.2, \(\Sigma\) is a group so \(F^{-1} \in \Sigma\) and then by (2.2) we have that
\[(F^{-1}(x), F^{-1}(y)) \in V \Rightarrow (x, y) \in V\]
for every \(x, y \in K\) and \(V \in \mathcal{B}\), so \(F^{-1}\) is non-contractive and then \(F\) is an isobasism for the basis \(\mathcal{B}\). \(\square\)

We know that every totally bounded metric space is an EC-space. The above theorem generalizes this result for uniformities when the space is compact. We can use some ideas of [10] to get the following results for uniformities in totally bounded spaces.

**Lemma 2.4.** Let \((E, \mathcal{U})\) be a totally bounded uniform space, \(\mathcal{B}\) a basis for the uniformity in \(E \times E\) and \(F : E \to E\) a non-contractive bijection for \(\mathcal{B}\). Then \(F\) satisfies that for every \(V \in \mathcal{B}\)
\[(2.3) \quad (x, y) \in V \Rightarrow \text{for each } W \in \mathcal{U} \text{ there is } k \in \mathbb{N} \text{ such that } (F^k(x), F^k(y)) \in W \circ V \circ W.\]

**Proof.** Choose \(x, y \in E\) and \(V \in \mathcal{B}\) with \((x, y) \in V\). Choose \(W \in \mathcal{U}\), \(W' \in \mathcal{B}\) a subset of \(W\), \(Z \in \mathcal{B}\) such that \(Z \circ Z \subset W'\) and \(U \in \mathcal{B}\) such that \(U \subset Z \cap Z^{-1}\). Since \(E\) is totally bounded there is a finite set \(\tilde{E} \subset E\) such that \(E = U[\tilde{E}]\). Then there is a infinite set \(M \subset \mathbb{N}\) and \(z_1, z_2 \in \tilde{E}\) such that \(\{F^n(x) : n \in M\} \subset U[z_1]\) and \(\{F^n(y) : n \in M\} \subset U[z_2]\).

Pick \(n, m \in M\) with \(m > n\) and let \(k = m - n\). Then
\[(F^m(x), F^n(x)) = (F^m(x), z_1) \circ (z_1, F^n(x)) \in U \circ U^{-1} \subset Z \circ Z \subset W'.\]
Then by (2.1) we have that \((F^k(x), x) \in W' \subset W\). Analogously \((y, F^k(y)) \in W\). Then
\[(F^k(x), F^k(y)) = (F^k(x), x) \circ (x, y) \circ (y, F^k(y)) \in W \circ V \circ W.\]

\(\square\)

**Corollary 2.5.** Let \((E, \mathcal{U})\) be a totally bounded uniform space, \(\mathcal{B}\) a basis for the uniformity and \(F : E \to E\) a non-contractive bijection for \(\mathcal{B}\). Then \(F\) satisfies that
\[(2.4) \quad (x, y) \in V \Rightarrow (F(x), F(y)) \in \overline{V}\]
for every \(V \in \mathcal{B}\).
Proof. Choose \( x, y \in E \) and \( V \in \mathcal{B} \) with \((x, y) \in V\). By Lemma 2.4 we have that for each \( W \in \mathcal{B} \) there is \( k \in \mathbb{N} \) such that \((F^k(x), F^k(y)) \in W \circ V \circ W\). Then since \( F^k \) is a bijection, we can choose \( w, z \in E \) such that \((F^k(x), F^k(w)) \in W\), \((F^k(w), F^k(z)) \in V\) and \((F^k(z), F^k(y)) \in W\). Since \( F \) is a non-contractive map we have that \((F(x), F(w)) \in W\), \((F(w), F(z)) \in V\) and \((F(z), F(y)) \in W\) so \((F(x), F(y)) \in W \circ V \circ W\) and then
\[
(F(x), F(y)) \in \bigcap_{W \in \mathcal{B}} W \circ V \circ W = V.
\]

Let \((E, d)\) be a metric space, if we denote \( U_\varepsilon = \{(x, y) : d(x, y) < \varepsilon\}\) then \( \mathcal{B} = \{U_\varepsilon : \varepsilon > 0\} \) is a basis of the uniformity and \( F : E \to E \) is non-expansive, non-contractive or an isometry for the metric \( d \) if and only if \( F \) is non-expansive, non-contractive or an isobasism for the basis of the uniformity \( \mathcal{B} \). Then Corollary 2.5 implies the following result:

**Corollary 2.6 ([5, Satz IV]).** Let \((E, d)\) be a totally bounded metric space and \( F : E \to E \) a bijective non-contractive (or non-expansive) map. Then \( F \) is an isometry.

**Corollary 2.7.** Let \( X \) be a topological vector space, \( A \subset X \) a totally bounded set and \( \mathcal{B} \) a basis of closed neighborhoods of 0. Let \( F : A \to A \) be a bijection such that for every \( x, y \in A \) and \( V \in \mathcal{B} \)
\[
F(x) - F(y) \in V \Rightarrow x - y \in V.
\]
Then \( f \) satisfies that for every \( x, y \in A \) and \( V \in \mathcal{B} \)
\[
x - y \in V \Rightarrow F(x) - F(y) \in V.
\]

**Proof.** This result follows from Corollary 2.5 applied to the set \( A \) and the basis for a uniformity \( \{U_V : V \in \mathcal{B}\} \) where \( U_V = \{(x, y) : x - y \in V\} \) for each \( V \in \mathcal{B} \).

The following result is a reformulation of the last corollary:

**Corollary 2.8.** Let \( X \) be a topological vector space, \( A \subset X \) a totally bounded set and \( \mathcal{B} \) a basis of closed neighborhoods of 0. Let \( F : A \to A \) be a bijection. If there is \( x, y \in A \) and \( V \in \mathcal{B} \) such that
\[
x - y \in V \text{ and } F(x) - F(y) \notin V,
\]
then there is \( z, w \in A \) and \( W \in \mathcal{B} \) such that
\[
F(z) - F(w) \in W \text{ and } z - w \notin W.
\]

### 3. Notation and auxiliary statements for Banach spaces

In this short section we fix the necessary notation and recollect some known results that we will need in the sequel. Below the letters \( X \) and \( Y \) always stand for real Banach spaces. We denote by \( S_X \) and \( B_X \) the unit sphere and the closed unit ball of \( X \) respectively. For a convex
set \( A \subset X \) denote by \( \text{ext}(A) \) the set of extreme points of \( A \); that is, \( x \in \text{ext}(A) \) if \( x \in A \) and for every \( y \in X \setminus \{0\} \) either \( x + y \not\in A \) or \( x - y \not\in A \). Recall that \( X \) is called strictly convex if all elements of \( S_X \) are extreme points of \( B_X \), or in other words, \( S_X \) does not contain non-trivial line segments. Strict convexity of \( X \) is equivalent to the strict triangle inequality \( \|x + y\| < \|x\| + \|y\| \) holding for all pairs of vectors \( x, y \in X \) that do not have the same direction. For subsets \( A, B \subset X \) we use the standard notation \( A + B = \{ x + y : x \in A, y \in B \} \) and \( aA = \{ ax : x \in A \} \).

Proposition 3.1 (P. Mankiewicz’s [9]). If \( A \subset X \) and \( B \subset Y \) are convex subsets with non-empty interior, then every bijective isometry \( F : A \to B \) can be extended to a bijective affine isometry \( \bar{F} : X \to Y \).

Taking into account that in the case of \( A, B \) being the unit balls every isometry maps 0 to 0, this result implies that every bijective isometry \( F : B_X \to B_Y \) is the restriction of a linear isometry from \( X \) onto \( Y \).

Proposition 3.2 (Brower’s invariance of domain principle [1]). Let \( U \) be an open subset of \( \mathbb{R}^n \) and \( f : U \to \mathbb{R}^n \) be an injective continuous map, then \( f(U) \) is open in \( \mathbb{R}^n \).

Proposition 3.3 ([6, Proposition 4]). Let \( X \) be a finite-dimensional normed space and \( V \) be a subset of \( B_X \) with the following two properties: \( V \) is homeomorphic to \( B_X \) and \( V \supset S_X \). Then \( V = B_X \).

The remaining results of this section listed below appeared first in [2] for the particular case of \( X = Y \). The generalizations to the case of two different spaces were made in [12] and [7].

The following theorem appears in [12, Theorem 2.1] and it can be demonstrated repeating the proof of [2, Theorem 2.3] almost word to word (see [13, Theorem 2.3] for details).

Theorem 3.4. Let \( F : B_X \to B_Y \) be a bijective non-expansive (briefly, a BnE) map. In the above notations the following hold.

1. \( F(0) = 0 \).
2. \( F^{-1}(S_Y) \subset S_X \).
3. If \( F(x) \) is an extreme point of \( B_Y \), then \( x \) is also an extreme point of \( B_X \), \( F(ax) = aF(x) \) for all \( a \in [-1, 1] \).

Moreover, if \( Y \) is strictly convex, then

(i) \( F \) maps \( S_X \) bijectively onto \( S_Y \);
(ii) \( F(ax) = aF(x) \) for all \( x \in S_X \) and \( a \in [-1, 1] \).

Lemma 3.5 ([12, Lemma 2.3]). Let \( F : B_X \to B_Y \) be a BnE map such that \( F(S_X) = S_Y \). Let \( V \subset S_X \) be the subset of all those \( v \in S_X \) that \( F(av) = aF(v) \) for all \( a \in [-1, 1] \). Denote \( A = \{ tx : x \in V, t \in [-1, 1] \} \), then \( F|_A \) is a bijective isometry between \( A \) and \( F(A) \).
Lemma 3.6 ([7, Lemma 2.9]). Let $F : B_X \to B_Y$ be a BnE map such that for every $v \in F^{-1}(S_Y)$ and every $t \in [-1,1]$ the condition $F(tv) = tF(v)$ holds true. Then $F$ is an isometry.

**Proposition 3.7 ([12, Theorem 3.1]).** Let $F : B_X \to B_Y$ be a BnE map. If $Y$ is strictly convex, then $F$ is an isometry.

Let us list some more definitions.

- An **extreme subset** of a set $B \subset X$ is a subset $C \subset B$ with the property
  \[ \forall y_1, y_2 \in B \ \forall \alpha \in (0,1) \ (\alpha y_1 + (1-\alpha)y_2 \in C) \implies (y_1, y_2 \in C). \]
- The **generating subspace** of a convex set $C$ is $\text{span}(C - C)$.
- The **dimension** of a convex set $C$ is the dimension of its generating subspace.
- For a convex set $B \subset X$ we will say that a point $x \in B$ is **$n$-extreme** if for any $(n+1)$-dimensional subspace $E \subset X$ and any $\varepsilon > 0$ there is an element $e \in S_E$, such that $x + \varepsilon e \notin B$.
- For $n \in \mathbb{N}$ a point $x$ of the convex set $B$ is called **sharp $n$-extreme in** $B$ if it is $n$-extreme and is not $(n-1)$-extreme.

Remark, that in the definition we do not demand the convexity of extreme subsets. This is done in order to enjoy the following easy to verify property: the union of any collection of extreme subsets of $B$ is an extreme subset of $B$. Nevertheless, we mostly deal with convex sets and convex extreme subsets. Observe also that being 0-extreme point and being extreme point of $B$ in the usual sense are equivalent. Every $n$-extreme point of $B$ is also $(n+1)$-extreme point of $B$. Every $n$-dimensional convex extreme subset $C$ of a convex set $B$ consists of $n$-extreme points of $B$ and contains a sharp $n$-extreme point. If $E$ is the generating subspace of the $n$-dimensional convex extreme subset $C \subset B$, then $x \in C$ is a sharp $n$-extreme point of $B$ if and only if $x$ belongs to the relative interior of $C$ in the affine subspace $x+E = C+E$. For a convex set $C \subset X$ with generating subspace $E$ by $\partial C$ we denote the relative boundary of $C$ in $C+E$.

Evidently, in a normed space for collinear vectors $x,y$ looking in the same direction (codirected vectors) we have

\[ \|x + y\| = \|x\| + \|y\|. \]

In spaces that are not strictly convex the converse statement is not true, which motivates the following definition.

**Definition 3.8.** Elements $x,y \in X$ are said to be **quasi-codirected**, if they satisfy (3.1).

By the triangle inequality, in order to verify (3.1) it is sufficient to check $\|x + y\| \geq \|x\| + \|y\|$. The next lemma is well-known, but this is the example of a fact which is much easier to demonstrate than to find out when and who observed it first ☺
Lemma 3.9. If \( x, y \in X \) are quasi-codirected, then for every \( a, b > 0 \) the elements \( ax \) and \( by \) are quasi-codirected as well.

Proof. By symmetry we may assume \( a \geq b \). Then, \( \|ax + by\| = \|(a+b)y\| = a\|x\| + b\|y\| \). □

Geometrically speaking \( x, y \in S_X \) are quasi-codirected, if the whole segment \([x, y] := \{tx + (1-t)y : t \in [0, 1]\}\) lies on the unit sphere. If \( C \subset S_X \) is convex, then every two elements of \( C \) are quasi-codirected.

4. Non-expansive maps and finite-dimensional faces

The aim of this section is, in the setting of Section 3 and using some similar ideas, to obtain as much as possible information about preimages of finite-dimensional faces of the unit ball. The main result is Theorem 4.11 that gives a positive answer for the Problem 1.1 when \( S_Y \) is the union of all its finite-dimensional polyhedral extreme subsets.

Let us start with a very simple observation.

Lemma 4.1. Let \( X, Y \) be Banach spaces, \( F : B_X \to B_Y \) be a BnE map, and \( y_1, y_2 \in S_Y \) be quasi-codirected. Then,

1. \( F^{-1}(y_1) \) is quasi-codirected with \( -F^{-1}(-y_2) \), so
2. if \( F^{-1}(-y_2) = -F^{-1}(y_2) \), then \( F^{-1}(y_1) \) is quasi-codirected with \( F^{-1}(y_2) \).
3. In particular if \( y_2 \) is an extreme point of \( B_Y \), then \( F^{-1}(y_1) \) is quasi-codirected with \( F^{-1}(y_2) \).

Proof.
\[
\|F^{-1}(y_1) + (-F^{-1}(-y_2))\| = \|F^{-1}(y_1) - F^{-1}(y_2)\| \\
\geq \|y_1 + (-y_2)\| = \|y_1 + y_2\| = 2.
\]

The above lemma readily implies the following natural counterpart to Proposition 3.7.

Theorem 4.2. Let \( X, Y \) be Banach spaces, \( X \) be strictly convex and \( F : B_X \to B_Y \) be a BnE map. Then \( F \) is an isometry.

Proof. According to Proposition 3.7 it is sufficient to demonstrate that \( Y \) is strictly convex. Assume to the contrary that \( S_Y \) contains a non-void segment \([y_0, y_1] := \{ty_1 + (1-t)y_0 : t \in [0, 1]\}\). Since \( X \) is strictly convex, the only element of \( S_X \) quasi-codirected with \( F^{-1}(y_1) \) is \( F^{-1}(y_1) \) itself. But, according to (1) of Lemma 4.1 all elements \(-F^{-1}(-y_1)\), where \( y_1 := ty_1 + (1-t)y_0, t \in [0, 1]\), are quasi-codirected with \( F^{-1}(y_1) \). This contradiction completes the proof. □
Let $Y$ be a Banach space, $y_1, y_2 \in S_Y$ be quasi-codirected. Denote

$$D_1(y_1, y_2) := (y_1 + B_Y) \cap (-y_2 + B_Y)$$

(4.1)

$$= \{ y \in Y : \|y_1 - y\| \leq 1 \text{ and } \|y_2 + y\| \leq 1 \} = \{ y \in Y : \|y_1 - y\| = \|y_2 + y\| = 1 \}.$$

Some evident properties of $D_1(y_1, y_2)$ are listed below without proof.

**Lemma 4.3.** Let $Y$ be a Banach space, $y_1, y_2 \in S_Y$ be quasi-codirected. Then

- $D_1(y_1, y_2)$ is a convex closed subset of $Y$.
- $0 \in D_1(y_1, y_2)$.
- $tD_1(y_1, y_2) \subset D_1(y_1, y_2)$ for every $t \in [0, 1]$.  
- $D_1(y_1, y_2) \subset 2B_Y$, consequently
- $\frac{1}{2}D_1(y_1, y_2) \subset D_1(y_1, y_2) \cap B_Y$.

**Lemma 4.4.** Let $Y$ be a Banach space, $y_1, y_2 \in S_Y$ be quasi-codirected, and $h \in Y$ be such that $y_1 \pm h \in S_Y$. Then

(4.2)

$$\left\{ \frac{1}{2}(y_1 - y_2) \pm \frac{1}{2}h \right\} \subset D_1(y_1, y_2).$$

In particular, substituting $y_2 = y_1$ we obtain

$$\pm \frac{1}{2}h \in D_1(y_1, y_1).$$

Substituting $h = 0$ we obtain

$$\frac{1}{2}(y_1 - y_2) \in D_1(y_1, y_2),$$

which implies that for all $t \in [0, 1/2]$

(4.3)

$$t(y_1 - y_2) \in D_1(y_1, y_2).$$

**Proof.** We have to verify two inequalities:

$$\left\| \frac{1}{2}(y_1 - y_2) \pm \frac{1}{2}h + y \right\| \leq 1 \text{ and } \left\| \frac{1}{2}(y_1 - y_2) \pm \frac{1}{2}h - y_1 \right\| \leq 1.$$

Each of them reduces to the same inequality

$$\left\| \frac{1}{2}(y_1 + y_2) \pm \frac{1}{2}h \right\| \leq 1.$$

Let us demonstrate this: $\|(y_1 + y_2) \pm h\| = \|y_2 + (y_1 \pm h)\| \leq \|y_2\| + \|y_1 \pm h\| = 2$. □

**Lemma 4.5.** Let $Y$ be a Banach space, $C \subset S_Y$ be a convex extreme subset, and $E$ be the generating subspace of $C$. Then $D_1(y_1, y_2) \subset E$ for every $y_1, y_2 \in C$. 

Proof. Let $y \in D_1(y_1, y_2)$. Then, $y_1 - y, y_2 + y \in B_Y$ and
\[
\frac{1}{2}((y_1 - y) + (y_2 + y)) = \frac{1}{2}(y_1 + y_2) \in C.
\]
Consequently, by the definition of extreme subset, $y_2 + y \in C$, so $y = (y_2 + y) - y_2 \in C - C \subset E$. \hfill \Box

Lemma 4.6. Let $X, Y$ be Banach spaces, $F : B_X \rightarrow B_Y$ be a BnE map, $y_1, y_2 \in S_Y$ be quasi-codirected, $x_1 = F^{-1}(y_1) \in S_X$, $x_2 = -F^{-1}(-y_2) \in S_X$. Then
\[
F(D_1(x_1, x_2) \cap B_X) \subset D_1(y_1, y_2) \cap B_Y.
\]
In particular, $F\left(\frac{1}{2}D_1(x_1, x_2)\right) \subset D_1(y_1, y_2) \cap B_Y$.

Proof. According to (1) of Lemma 4.1, $x_1$ and $x_2$ are quasi-codirected, so the set $D_1(x_1, x_2)$ is well-defined. Consider arbitrary $x \in D_1(x_1, x_2) \cap B_X$. We have $\|x_1 - x\| \leq 1$ and $\|(x_2 - x) - x\| = \|x_2 + x\| \leq 1$, so $\|F(x_1) - F(x)\| \leq 1$ and $\|F(-x_2) - F(x)\| \leq 1$. In other words, $\|y_1 - F(x)\| \leq 1$ and $\|(-y_2) - F(x)\| = \|y_2 + F(x)\| \leq 1$, which means that $F(x) \in D_1(y_1, y_2)$. \hfill \Box

Lemma 4.7. Let $X, Y$ be Banach spaces, $F : B_X \rightarrow B_Y$ be a BnE map, $n \in \mathbb{N}$, and $C \subset S_Y$ be an $n$-dimensional convex extreme subset. Then for every $y_1 \in C$ its preimage $x_1 = F^{-1}(y_1) \in S_X$ is an $n$-extreme point of $B_X$.

Proof. Denote $x_2 = -F^{-1}(-y_1) \in S_X$. Assume that $x_1$ is not $n$-extreme point of $B_X$. Then, according to the definition, there exist an $(n + 1)$-dimensional subspace $E \subset X$ and an $\varepsilon > 0$ such that $x_1 + \varepsilon B_E \subset S_X$. According to Lemma 4.4
\[
\frac{1}{2}(x_1 - x_2) + \varepsilon B_E \subset D_1(x_1, x_2).
\]
The above inclusion implies that $\frac{1}{2} D_1(x_1, x_2)$ contains an $(n + 1)$-dimensional ball. Then Lemma 4.6 implies that $D_1(y_1, y_1)$ contains a homeomorphic copy of $(n + 1)$-dimensional ball, which is impossible by Lemma 4.5. \hfill \Box

Note, that under conditions of the previous lemma $x$ may be also $m$-extreme point for some $m < n$. Now we are coming to the most important and at the same time most difficult result of the paper.

Theorem 4.8. Let $X, Y$ be Banach spaces, $F : B_X \rightarrow B_Y$ be a BnE map, then for every $n \in \mathbb{N}$ the preimage of any $n$-dimensional convex polyhedral extreme subset $C \subset S_Y$ is an $n$-dimensional convex polyhedral extreme subset of $S_X$. Moreover, the equality $-F^{-1}(-C) = F^{-1}(C)$ holds true.

Proof. We will use the induction in $n$. The initial case of $n = 0$ (i.e., of extreme points) is covered by the assertion (3) of Theorem 3.4. Let us
assume that the theorem is demonstrated for extreme subsets of dimension smaller than \( n \), and let us demonstrate it for a given \( n \)-dimensional polyhedral extreme subset \( C \subset S_Y \). Denote \( E \) the generating subspace of \( C \), \( \dim E = n \). The boundary \( \partial C \) of polyhedron \( C \) consists of finite union of its convex \( (n-1) \)-dimensional polyhedral extreme subsets, so, by the inductive hypothesis, \( A := F^{-1}(\partial C) \) also consists of finite union of some convex \( (n-1) \)-dimensional polyhedral extreme subsets of \( S_X \). Consequently, \( A \) is an extreme subset of \( B_X \). Also, \( A \) is compact, and \( F|_A \) performs a homeomorphism between \( A \) and \( F(A) = \partial C \). Let \( y_1 \in C \setminus \partial C \) be an arbitrary point. Denote \( x_1 = F^{-1}(y_1) \). Since \( y_1 \) is quasi-codirected with every point \( y_2 \in \partial C \), \( x_1 \) is quasi-codirected with the corresponding \( x_2 = -F^{-1}(-y_2) \in S_X \). By the inclusion (4.3) and Lemmas 4.6 and 4.5

\[
F\left(t(x_1 - x_2)\right) \in F\left(\frac{1}{2}D_1(x_1, x_2)\right) \subset D_1(y_1, y_2) \subset E
\]

for all \( t \in [0, \frac{1}{2}] \). By the inductive hypothesis, when \( y_2 \) runs through \( \partial C \) the corresponding \( x_2 \) runs through the whole \( A \). So, denoting \( \tilde{A} = [0, \frac{1}{2}] (x_1 - A) = \{t(x_1 - x_2) : t \in [0, \frac{1}{2}], x_2 \in A\} \) we obtain

(4.4)

\[
F(\tilde{A}) \subset E.
\]

Let us demonstrate that the segments \((0, \frac{1}{2}] (x_1 - x_2)\) with different \( x_2 \in A \) are pairwise disjoint. We will argue by contradiction. Let two segments of the form \((0, \frac{1}{2}] (x_1 - \tilde{x}_2), (0, \frac{1}{2}] (x_1 - \tilde{x}_2)\) with \( \tilde{x}_2, \tilde{x}_2 \in A, \tilde{x}_2 \neq \tilde{x}_2 \) intersect at some point \( y \). Then the corresponding closed segments \([0, \frac{1}{2}] (x_1 - \tilde{x}_2), [0, \frac{1}{2}] (x_1 - \tilde{x}_2)\) intersect in two points \((0 \text{ and } y)\), so either they coincide or one segment contains the other one. That is, \((x_1 - \tilde{x}_2)\) and \((x_1 - \tilde{x}_2)\) are codirected. There are two cases:

\[
(x_1 - \tilde{x}_2) = \lambda(x_1 - \tilde{x}_2) \quad \text{or} \quad (x_1 - \tilde{x}_2) = \lambda(x_1 - \tilde{x}_2) \quad \text{with some } 0 < \lambda < 1.
\]

We will discuss the first one, the second one is analogous. We get \( \tilde{x}_2 = \lambda\tilde{x}_2 + (1 - \lambda)x_1 \), so these three points are on the same segment and \( \tilde{x}_2 \) lies between \( x_1 \) and \( \tilde{x}_2 \). Since \( A \) is extreme subset, we get \( x_1 \in \tilde{A} \), which contradicts the fact \( y_1 \notin \partial C \).

The set \((x_1 - A)\) is homeomorphic to the unit sphere of \( \mathbb{R}^n \). Let us show, that \( \tilde{A} \) is homeomorphic to the unit ball of \( \mathbb{R}^n \), with 0 mapped to 0. Let \( S^n \) and \( B^n \) denote the unit sphere and the unit ball of \( \mathbb{R}^n \) respectively, and \( h : S^n \to (x_1 - A) \) be a homeomorphism. One may define the mapping \( H : B^n \to \tilde{A} \) as

\[
H(x) = \begin{cases} 
0, & \text{when } x = 0 \\
\frac{1}{2}||x||h\left(\frac{x}{||x||}\right), & \text{when } x \in B^n \setminus \{0\}.
\end{cases}
\]

Obviously, this mapping is bijective and continuous at 0. We are going to show that \( H \) is continuous at all points. Let us consider some
sequence \( \{x_n\}_{n=1}^{\infty} \) in \( B^n \) converging to an \( x \in B^n \setminus \{0\} \), that is
\[
\lim_{n \to \infty} x_n = x \neq 0.
\]
Then
\[
\lim_{n \to \infty} H(x_n) = \lim_{n \to \infty} \frac{1}{4} \|x_n\| h \left( \frac{x_n}{\|x_n\|} \right) = \frac{1}{4} \lim_{n \to \infty} \|x_n\| \lim_{n \to \infty} h \left( \frac{x_n}{\|x_n\|} \right) = \frac{1}{4} \|x\| h \left( \frac{x}{\|x\|} \right) = H(x).
\]
So, \( H \) is a bijective continuous map from compact \( B^n \) to Hausdorff space, thus \( H \) is a homeomorphism.

Consequently, \( F(\tilde{A}) \subset E \) is homeomorphic to the unit ball of \( \mathbb{R}^n \), with 0 being a relative (in \( E \)) interior point of \( F(\tilde{A}) \).

Consider now any point \( \tilde{y}_2 \in C \setminus \partial C, \tilde{y}_2 \neq y_1 \), such that the corresponding \( \tilde{x}_2 = -F^{-1}(-\tilde{y}_2) \) is not equal to \( x_1 \). By the same reason as before, the segment \( F \left( [0, \frac{1}{\alpha}] (x_1 - \tilde{x}_2) \right) \subset D_1(y_1, \tilde{y}_2) \subset E \). The set \( F \left( [0, \frac{1}{\alpha}] (x_1 - \tilde{x}_2) \right) \) is a continuous curve in \( E \) connecting \( F(\frac{1}{\alpha}(x_1 - \tilde{x}_2)) \) with 0, which is an interior point of \( F(\tilde{A}) \). So there is a \( t_0 \in (0, \frac{1}{\alpha}] \) such that \( F(t_0(x_1 - \tilde{x}_2)) \in F(\tilde{A}) \), that is \( t_0(x_1 - \tilde{x}_2) \in \tilde{A} \). This means that for some \( t_1 \in (0, \frac{1}{\alpha}] \) and some \( x_2 \in A \) we have \( t_0(x_1 - \tilde{x}_2) = t_1(x_1 - x_2) \).

In other words, there is an \( \alpha > 0 \) such that
\[
(4.5) \quad x_1 - \tilde{x}_2 = \alpha(x_1 - x_2).
\]
Let us demonstrate that \( \alpha < 1 \). Indeed, if \( \alpha \geq 1 \), the above formula would give the representation
\[
x_2 = \left(1 - \frac{1}{\alpha}\right) x_1 + \frac{1}{\alpha} \tilde{x}_2
\]
of \( x_2 \in A \) as a convex combination of \( x_1, \tilde{x}_2 \in S_X \setminus A \), which contradicts the fact that \( A \) is extreme in \( S_X \).

Since \( \alpha < 1 \), the formula (4.5) gives the representation
\[
\tilde{x}_2 = (1 - \alpha)x_1 + \alpha x_2
\]
of \( \tilde{x}_2 \) as a convex combination of \( x_1 \) and some \( x_2 \in A \).

If we consider the BnE mapping \( G : B_X \rightarrow B_Y \) defined as \( G(x) = -F(-x) \), all the above reasoning is applicable for \( G \) as well, because by the inductive hypothesis \( G^{-1}(\partial C) = F^{-1}(\partial C) = A \). Since \( \tilde{x}_2 = G^{-1}(\tilde{y}_2) \) and \( x_1 = -G^{-1}(-y_1) \) the roles of these elements for \( G \) interchange, and we deduce that also \( x_1 \) is a convex combination of \( \tilde{x}_2 \) and some \( x_3 \in A \). So, we obtain the following properties of sets \( F^{-1}(C \setminus \partial C) \) and \( G^{-1}(C \setminus \partial C) \):

Properties.
(i) For every \( u \in F^{-1}(C \setminus \partial C) \)
\[
G^{-1}(C \setminus \partial C) \subset \{ tx + (1-t)u : t \in [0, 1], x \in A \}.
\]
(ii) For every $v \in G^{-1}(C \setminus \partial C)$
\[ F^{-1}(C \setminus \partial C) \subset \{ tx + (1 - t)v: t \in [0,1], x \in A \}. \]

(iii) For every $u \in F^{-1}(C \setminus \partial C)$ and every $v \in G^{-1}(C \setminus \partial C)$, $u \neq v$ there are (unique) elements $w, z \in A$ such that $[u, v] \subset [w, z]$.

Properties (i) and (ii) imply that $F^{-1}(C)$ and $G^{-1}(C)$ lie in some finite-dimensional subspace of $X$. Since both these sets are bounded and closed, they are compacts. Continuous mappings $F$ and $G$ map corresponding compacts $F^{-1}(C)$ and $G^{-1}(C)$ to $C$ bijectively, so both $F^{-1}(C)$ and $G^{-1}(C)$ are homeomorphic to $C$, i.e. homeomorphic to the unit ball of $\mathbb{R}^n$. Since the set $\{ tx + (1 - t)u: t \in [0,1], x \in A \}$ for a fixed $u$ is also homeomorphic to the unit ball of $\mathbb{R}^n$ and $A$ corresponds to the unit sphere and belongs to both $\{ tx + (1 - t)u: t \in [0,1], x \in A \}$ and $G^{-1}(C)$, the inclusion (i) and Proposition 3.3 imply that

(i)' for every $u \in F^{-1}(C \setminus \partial C)$
\[ G^{-1}(C) = \{ tx + (1 - t)u: t \in [0,1], x \in A \}, \]

and by the same reason

(ii)' For every $v \in G^{-1}(C \setminus \partial C)$
\[ F^{-1}(C) = \{ tx + (1 - t)v: t \in [0,1], x \in A \}. \]

In particular, from (i)' it follows that every $u \in F^{-1}(C \setminus \partial C)$ belongs to $G^{-1}(C)$, so $F^{-1}(C) \subset G^{-1}(C)$, and (ii)' implies the inverse inclusion $G^{-1}(C) \subset F^{-1}(C)$, so
\[ G^{-1}(C) = F^{-1}(C). \]

Coming back to the already used inclusion (4.3) and Lemmas 4.6 and 4.5 we obtain that for all $x_1, x_2 \in F^{-1}(C)$
\[ F \left( \frac{1}{4}(x_1 - x_2) \right) \in E, \]
in other words
(4.6) \[ F \left( \frac{1}{4}(F^{-1}(C) - F^{-1}(C)) \right) \subset E. \]

Recall, that by the inductive hypothesis, $A = F^{-1}(\partial C)$ consists of finite union of some convex $(n - 1)$-dimensional polyhedral extreme subsets $\hat{W}_i$, $i = 1, \ldots, N$ which are preimages of corresponding parts of $\partial C$. Let us fix some $v \in F^{-1}(C \setminus \partial C)$. Denote
\[ W_i = \left\{ tx + (1 - t)v: t \in [0,1], x \in \hat{W}_i \right\}. \]

These $W_i$ are $n$-dimensional convex polyhedrons, and, according to (ii)',
\[ F^{-1}(C) = \bigcup_{i=1}^{N} W_i. \]
We state that all polyhedrons $W_i$ (and also their union $F^{-1}(C)$) are situated in one and the same $n$-dimensional affine subspace $\tilde{E}$.

To this end, consider the generating subspaces $Z_i = \text{span}(W_i - W_i)$ of $W_i$ and let us demonstrate that all of $Z_i$ are equal one to another, i.e. all of them are equal to some $n$-dimensional linear subspace $Z$. Then $\tilde{E} = v + Z$ will be the $n$-dimensional affine subspace $\tilde{E}$ we are looking for.

Let us argue “ad absurdum”. Assume that $Z_i \neq Z_j$ for some $i \neq j$. Then $Z_i + Z_j$ has dimension strictly greater than $n$, and
\[
\dim(W_i - W_j) = \dim(\text{span}((W_i - W_j) - (W_i - W_j))) = \dim(Z_i + Z_j) > n.
\]
Taking into account that $W_i - W_j \subset (F^{-1}(C) - F^{-1}(C))$ the dimension of $F^{-1}(C) - F^{-1}(C)$ is strictly greater than $n$, which makes the inclusion (4.6) impossible.

It remains to demonstrate that $F^{-1}(C)$ is convex and is an extreme subset. For the convexity let us show that $F^{-1}(C) = B_X \cap \tilde{E}$. We have already known, that $F^{-1}(C) \subset B_X \cap \tilde{E}$. Let us show the inverse inclusion. Again we will argue by contradiction. Suppose there is a point $z \in (B_X \cap \tilde{E}) \setminus F^{-1}(C)$. We may fix some $v \in F^{-1}(C \setminus \partial C)$ and consider the segment $[z, v]$. As we already remarked, $F^{-1}(C)$ is homeomorphic to $C$ and hence to $B^n$, that is, $v$ lies in the relative interior of $F^{-1}(C)$ in $\tilde{E}$. So, the segment $[z, v]$ must intersect $A = F^{-1}(\partial C)$ in some point. In other words, there is $\lambda \in (0, 1)$ such that $\lambda z + (1 - \lambda)v \in A$, which contradicts the fact, that $A$ is an extreme subset in $B_X$.

\[\square\]

**Theorem 4.9.** Let $X$, $Y$ be Banach spaces, $F : B_X \to B_Y$ be a $\text{BnE}$ map, then for every $n$-dimensional convex polyhedral extreme subset $C \subset S_Y$ the following equality holds true: $F(\text{conv}(0, F^{-1}(C))) = \text{conv}(0, C)$.

**Proof.** We will carry out the proof by induction in $n$. For $n = 0$ (i.e., when $C$ is extreme point) the required equality may be obtained from the assertion (3) of Theorem 3.4. Suppose our theorem is proved for all extreme subsets of dimension smaller than $n$, and let us show the same for a given $n$-dimensional polyhedral extreme subset $C \subset S_Y$. Consider $x \in F^{-1}(C \setminus \partial C)$ and $\alpha \in (0, 1)$. Since $F$ is non-expansive we have
\[
\|F(\alpha x)\| \leq \|\alpha x\|, \text{ and } \|F(x) - F(\alpha x)\| \leq \|x - \alpha x\|.
\]
Also
\[
1 = \|F(x)\| \leq \|F(\alpha x)\| + \|F(x) - F(\alpha x)\| \leq \|\alpha x\| + \|x - \alpha x\| = 1.
\]
That is why  
\[ \|F(\alpha x)\| + \|F(x) - F(\alpha x)\| = 1. \]
So one may write \( F(x) \) as a convex combination  
\[ F(x) = \frac{\|F(\alpha x)\|}{\|F(\alpha x)\| + \|F(x) - F(\alpha x)\|} \cdot F(\alpha x) + \frac{\|F(x) - F(\alpha x)\|}{\|F(\alpha x)\| + \|F(x) - F(\alpha x)\|} \cdot (F(x) - F(\alpha x)). \]
Since \( F(x) \in C \) and \( C \) is extreme subset in \( B_X \) we get  
\[ \frac{F(\alpha x)}{\|F(\alpha x)\|} \in C \]  
and  
\[ F(x) - F(\alpha x) \]  
\[ \in \partial \text{conv}(0, C). \]  
By the inductive hypothesis \( F(\text{conv}(0, A)) = \text{conv}(0, \partial C) \) and  
\[ \partial \text{conv}(0, C) \subset F(\text{conv}(0, F^{-1}(C))). \]
Besides, \( \text{conv}(0, F^{-1}(C)) \) is homeomorphic to \( B^{n+1} \) and \( \partial \text{conv}(0, C) \) is homeomorphic to \( S^{n+1} \). In this way Proposition 3.3 implies the statement of the theorem. □

**Lemma 4.10.** Let \( X, Y \) be Banach spaces, \( F : B_X \to B_Y \) be a BnE map, then  
\[ \|F(\alpha x)\| = \|\alpha x\| = \alpha \]  
for all \( x \in F^{-1}(S_Y), \alpha \in [0,1]. \)

**Proof.** Since \( F \) is non-expansive, we may use inequalities (4.7) and (4.8). The inequality (4.8) implies  
\[ \|F(\alpha x)\| + \|F(x) - F(\alpha x)\| = \|\alpha x\| + \|x - \alpha x\|, \]
and application of (4.7) concludes the proof. □

**Theorem 4.11.** Let \( X, Y \) be Banach spaces, \( F : B_X \to B_Y \) be a BnE map and \( S_Y \) be the union of all its finite-dimensional polyhedral extreme subsets. Then \( F \) is an isometry.

**Proof.** Let us first show, that \( F(S_X) = S_Y \). Since  
\[ S_Y = \bigcup_{i \in I} C_i, \]
where \( C_i \) are finite-dimensional polyhedral extreme subsets of \( S_Y \) and \( I \) is some index set, one may deduce  
\[ B_Y = \bigcup_{i \in I} \text{conv}(0, C_i). \]
Due to bijectivity of \( F \), theorem 4.9 implies  
\[ B_X = \bigcup_{i \in I} \text{conv}(0, F^{-1}(C_i)). \]
Consequently, there is no other norm-one points in \( B_X \) except for points from \( F^{-1}(C_i) \), and we get  
\[ S_X = \bigcup_{i \in I} F^{-1}(C_i) = F^{-1}(S_Y). \]
To prove that \( F \) is an isometry we will use lemmas 3.6 and 3.5. We are going to show for the set \( V \) from lemma 3.5 that  
\[ F^{-1}(C) \subset V \]
for every \( n \)-dimensional polyhedral extreme subset \( C \) of \( S_Y \). To do that, we will use induction by dimension. For 0-dimensional sets, i.e. extreme points, the statement we need follows from item (3) of theorem 3.4. Now suppose that the inclusion is proved for all \((n-1)\)-dimensional polyhedral extreme subsets and let us prove it for dimension \( n \). Consider some \( n \)-dimensional extreme subset \( C \) in \( S_Y \). For every pair \( x,y \in F^{-1}(C) \) there are \( u,v \in F^{-1}(\partial C) \) such that \( x = \lambda u + (1 - \lambda)v \) and \( y = \mu u + (1 - \mu)v, \lambda, \mu \in (0,1) \). Without loss of generality one may account \( \lambda > \mu \). Since \( \partial C \) consists of \((n-1)\)-dimensional polyhedral extreme subsets, the inductive hypothesis and lemma 3.5 give that \( \|u - v\| = \|F(u) - F(v)\| \). Since \( F \) is non-expansive,

\[
\|u - v\| = \|F(u) - F(v)\| \leq \|F(u) - F(x)\| + \|F(x) - F(y)\|
\]

\[
+ \|F(y) - F(v)\| \leq \|u - x\| + \|x - y\| + \|y - v\|
\]

\[
= (1 - \lambda)\|u - v\| + (\lambda - \mu)\|u - v\| + \mu\|u - v\| = \|u - v\|.
\]

So we get \( \|F(u) - F(x)\| = \|u - x\|, \|F(y) - F(v)\| = \|y - v\|, \|F(x) - F(y)\| = \|x - y\| \). Thus, \( F \) is bijective isometry between \( F^{-1}(C) \) and \( C \) and Proposition 3.1 implies that \( F \) is affine on \( F^{-1}(C) \). Lemma 4.10 together with Theorem 4.9 give the equality \( F(\alpha F^{-1}(C)) = \alpha C \) for \( \alpha \in [0,1] \), and application of the “moreover” part of theorem 4.8 extends this to \( \alpha \in [-1,1] \). The same way as before, the inductive hypothesis and lemma 3.5 imply that \( F \) is bijective isometry between \( \alpha F^{-1}(C) \) and \( \alpha C \), so \( F \) is affine on \( \alpha F^{-1}(C) \). We are going to show that \( F(\alpha x) = \alpha F(x) \) for all \( x \in F^{-1}(C), \alpha \in [-1,1] \). Every \( x \in F^{-1}(C) \) is of the form \( x = \lambda u + (1 - \lambda)v \), where \( u,v \in F^{-1}(\partial C) \) and \( \lambda \in (0,1) \).

We obtain

\[
F(\alpha x) = F(\lambda u + (1 - \lambda)v) = \lambda F(u) + (1 - \lambda)F(v),
\]

because \( F \) is affine on \( \alpha F^{-1}(C) \). By the inductive hypothesis \( F(\alpha u) = \alpha F(u), F(\alpha v) = \alpha F(v) \), so

\[
F(\alpha x) = \lambda \alpha F(u) + (1 - \lambda)\alpha F(v) = \alpha(\lambda F(u) + (1 - \lambda)F(v)).
\]

It remains to use the fact that \( F \) is affine on \( F^{-1}(C) \) to conclude that

\[
F(\alpha x) = \alpha F(\lambda u + (1 - \lambda)v) = \alpha F(x).
\]

So, the required inclusion (4.10) is demonstrated. At last, (4.9) and the written above imply that for every \( v \in F^{-1}(S_Y) \) and every \( t \in [-1,1] \)

\[
F(tv) = tF(v).
\]

So, the application of lemma 3.6 completes the proof of the theorem.

\[\square\]

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