Necessary conditions for variational regularization schemes

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Abstract

We study variational regularization methods in a general framework, more precisely those methods that use a discrepancy and a regularization functional. While several sets of sufficient conditions are known to obtain a regularization method, we start with an investigation of the converse question: how could necessary conditions for a variational method to provide a regularization method look? To this end, we formalize the notion of a variational scheme and start with a comparison of three different instances of variational methods. Then we focus on the data space model and investigate the role and interplay of the topological structure, the convergence notion and the discrepancy functional. Especially, we deduce necessary conditions for the discrepancy functional to fulfil usual continuity assumptions. The results are applied to discrepancy functionals given by Bregman distances and especially to the Kullback-Leibler divergence.

1. Introduction

By ‘variational regularization’ we mean every method that is used to approximate an ill-posed problem by well-posed minimization problems. We start with a mapping \( F : X \rightarrow Y \) between two sets \( X \) and \( Y \) and equations

\[ F(x) = y. \]

A common problem with inverse problems is that of instability, i.e. that arbitrary small disturbances in the right-hand side \( y \) (e.g. by replacing a ‘correct’ \( y \) in the range of \( F \) with one in an arbitrarily small neighborhood) may lead to unwanted effects such as that no solution exists anymore or that solutions with a perturbed right-hand side differ arbitrarily from the true solutions. In topological spaces \( X \) and \( Y \) we can formulate the problem of instability more precisely. The equation \( F(x^{\text{exact}}) = y^{\text{exact}} \) is unstable, if there exists a neighborhood \( \mathcal{U} \) of \( x^{\text{exact}} \) such that for all neighborhoods \( \mathcal{V} \) of \( y^{\text{exact}} \) there exists \( y^\delta \in \mathcal{V} \) such that \( F^{-1}(y^\delta) \cap \mathcal{U} = \emptyset \) (cf [26, 28]).
Variational regularization methods replace the equation \( F(x) = y \) by a minimization problem for an (extended) real valued functional such that the minimizers are suitable approximate solutions of the equation. The most widely used variational method is Tikhonov regularization [42], but other methods are used as well. Starting from a detailed analysis of this method in Hilbert spaces, there are several recent studies on Tikhonov regularization in the context of more general spaces like Banach spaces [23, 36, 38, 39] or even topological spaces [15, 16, 20, 35, 45]. Especially, the discrepancy functional, which measures the distance between the measured data and the reconstructed data, have come into the focus of recent research. A Poisson noise model motivates the use of a Kullback–Leibler divergence and is applied in fluorescence microscopy and optical/infrared astronomy [7], for inverse scattering problems and phase retrieval problems [24], and for STED- and 4Pi-microscopy and positron emission tomography [10]. Moreover, a kind of Burg entropy is used for multiplicative noise which has applications to remove speckle noise in synthetic aperture radar imaging [2]. By now, a quite general set of sufficient assumptions is available under which Tikhonov regularization has the desired regularizing properties, i.e. stable solvability of the minimization problems and suitable approximation of the true solution if the noise vanishes. These sufficient assumptions are helpful to check if a chosen setting for variational regularization is indeed suited. On the other hand, when designing a regularization method it would be helpful to know in advance which setting works and which is not going to work. Hence, in this paper we begin with a study of the converse analysis and aim at providing necessary conditions on variational methods such that regularization is achieved. Such conditions would also be helpful in designing new variational methods as they rule out several options. Moreover, necessary conditions are a further step toward the understanding of the nature of variational regularization.

We remark that we are aware that necessary conditions cannot be expected to be very strong (as an example, a minimization problem can be changed quite arbitrary without changing the minimizer itself). However, there are already a few results of this flavor known in specialized contexts which we list here.

**Theorem 1.1** (No uniform bounded linear regularization, [14, remark 3.5]). If the linear and bounded operator \( F : X \to Y \) between Hilbert spaces \( X \) and \( Y \) does not have closed range and \((L_\alpha)_{\alpha > 0}\) is a family of linear and bounded operators from \( Y \) to \( X \) such that for all \( x \in X \) it holds that \( L_\alpha Fx \) converges to \( x \) for \( \alpha \to 0 \), then \( (\|L_\alpha\|) \) is unbounded.

In other words, linear regularization methods are necessarily not uniformly bounded.

The next example of a necessary condition deals with the problem of parameter choice. We need the Moore–Penrose pseudo-inverse \( F^\dagger \) of a bounded linear mapping between Hilbert spaces, cf [5].

**Theorem 1.2** (Bakushinskii Veto, [3]). Let \( F : X \to Y \) be a bounded linear operator between Hilbert spaces and \((L_\alpha)_{\alpha > 0}\) be a family of continuous mappings from \( Y \) to \( X \). If there is a mapping \( \alpha : Y \to [0, \infty[ \) such that

\[
\limsup_{\delta \to 0} \{\|L_\alpha(y^\delta) - F^\dagger y\| : y^\delta \in Y, \|y - y^\delta\| \leq \delta\} = 0
\]

then \( F^\dagger \) is bounded.

In other words, parameter choice rules which are regularizing (in the sense of [14, definition 3.1]) for ill-posed problems (i.e. unbounded \( F^\dagger \)) necessarily need to use the noise level.

An example for \textit{a priori} parameter choice rules was proven by Engl.
Theorem 1.3 (Decay conditions for a priori parameter choice rules for linear methods. [14, proposition 3.7] and [13]). Let \( F \) and \((L_\alpha)\) be as in theorem 1.1, and \( \alpha : [0, \infty[ \rightarrow [0, \infty[ \) be an a priori parameter choice rule. Then it holds that

\[
\lim_{\delta \to 0} \sup \{\|L_\alpha(\delta)y^\delta - F^\dagger y\| : y^\delta \in Y, \|y - y^\delta\| \leq \delta\} = 0
\]

if and only if

\[
\lim_{\delta \to 0} \alpha(\delta) = 0 \quad \text{and} \quad \lim_{\delta \to 0} \delta \|L_\alpha(\delta)\| = 0.
\]

In other words, a priori parameter choice rules necessarily need to fulfil certain decay conditions.

Finally we mention the ‘converse results’ from [33] which say that for Tikhonov regularization in Hilbert spaces certain convergence rates imply that certain source conditions are fulfilled (see [17] for generalization to other regularization methods).

Before we start our investigation of necessary conditions for variational regularization in section 3, we start with a section in which we formalize the notion of a ‘variational scheme’ and investigate a few different variational methods.

2. Variational schemes: Tikhonov, Morozov, and Ivanov

In this section we formalize the notion of a variational scheme which can be used to build variational regularization methods. We start by fixing the ingredients of an inverse problem. In this paper we take the point of view that an inverse problem consists of a mapping \( F : (X, \tau_X) \to (Y, \tau_Y) \) between two topological spaces, usually called forward operator. The space \( X \) is the solution space and \( Y \) is called data space. We further assume that \( F \) is continuous, i.e. the forward problem (calculating the data for some given solution) is well posed. In contrast, the solution of an equation \( F(x) = y \) for some given \( y \) does not need to be well posed. As in [26] and the more recent references [22, 20] we use topological spaces since the functionals we consider do not take any linear structure into account which would justify the use of linear or normed spaces.

A variational scheme consists of all ingredients which are needed to classify and analyze the associated minimization problems and their minimizers under perturbations of the data \( y \). Hence, it should encode information about the notions of convergence, ‘proximity’, and the objective functional to be minimized. However, we do not allow for totally arbitrary objective functionals but we rather use the intuition that a variational scheme involves two functionals: a ‘similarity measure’ or ‘discrepancy functional’ \( \rho \) and a ‘regularization functional’ \( R \). The functional \( \rho \) is used to measure ‘similarity’ in the data space in the sense that \( \rho(F(x), y) \) is small if \( x \) explains the data \( y \) well. The functional \( R \) on the solution space is used to measure how well \( x \) fits prior knowledge in the sense that \( R(x) \) is small for an \( x \) which fulfills the prior knowledge well.

Definition 2.1 (Variational scheme). By a variational scheme for a given inverse problem \( F : (X, \tau_X) \to (Y, \tau_Y) \) we understand a tuple \( \mathcal{M} = (\rho, R, S) \), consisting of

- the discrepancy functional \( \rho : Y \times Y \to [0, \infty[ \), for which we assume that \( \rho(y, y) = 0 \) for all \( y \in Y \),
- the regularization functional \( R : X \to [0, \infty[ \), and
- a sequential convergence structure \( S \) on \( Y \).

That is, \( S \) is a mapping which maps any element in \( Y \) to a set of sequences in \( Y \) such that the constant sequence \( (y) \) is an element of \( S(y) \) and that if a sequence is in \( S(y) \) then so
does any of its subsequences. Usually, we denote \((y_n) \in S(y)\) by \(y_n \xrightarrow{S} y\) and say that \((y_n)\) converges to \(y\) (with respect to \(S\)); see also [4, section 1.7].

While most ingredients of a variational scheme are standard, we remark on the sequential convergence structure \(S\). Often decaying noise is described in terms of norm-convergence, a notion which is not available here and sometimes may not even be appropriate (see, e.g. [16] and [35]). Therefore, the sequential convergence structure will be used to describe ‘vanishing noise’ in \(Y\), i.e. the vanishing of noise is modeled by convergence of a sequence \((y_n)\) to noise free data \(y\) w.r.t. \(S\). Of course, a topology induces a sequential convergence structure but not all sequential convergence structures are topological (e.g. pointwise almost everywhere convergence is not induced by a topology [34]). Moreover, a sequential convergence structure induced by a topology may not encode all information of the topology (consider the case of convergence w.r.t. \(Y\)). Note that we do not assume that convergence w.r.t. \(S\) is topological since this is not used in standard proofs for regularizing properties (e.g. [23]). Moreover, the topology \(\tau_Y\) may induce a different convergence structure which is more tied to the mapping properties of \(F\). Of course, there will be further relations between \(\tau_Y, S\) and \(\rho\) in the following, and indeed, section 3 mainly deals with these relations, but for the general variational scheme we keep them mostly unrelated.

We mention that we included the value \(\infty\) in the range of the discrepancy functional \(\rho\) and the regularization functional \(R\) to model that certain data may be considered ‘incomparable’ or that certain solutions may be deemed to be impossible. As usual, the value \(\infty\) is excluded for minimizers by definition and we use the notation \(\text{dom}R = \{x : R(x) < \infty\}\) (similarly for \(\rho\)).

Variational regularization methods can be build from variational schemes as follows. Instead of solving \(F(x) = y\) we aim at two goals. Find an \(x \in X\) such that \(x\) explains the data \(y\) well, in the sense that \(\rho(F(x), y)\) is small, and \(x\) fits to our prior knowledge in the sense that \(R(x)\) is small. In other words: we have two objective functionals \(x \mapsto \rho(F(x), y)\) and \(x \mapsto R(x)\) which we would like to ‘jointly minimize’ and such problems go under the name of ‘vector optimization’. A core notion there is that of ‘Pareto-optimal solutions’, i.e., solutions \(x^*\) such that there does not exist an \(x\) such that \(R(x) \leq R(x^*)\) and \(\rho(F(x), y) \leq \rho(F(x^*), y)\) and one of both inequalities is strict [9, section 4.7]. Note that for ‘exact data’, i.e. \(y^{\text{exact}}\) in the range of \(F\), the notion of Pareto optimality induces a notion of generalized solutions of the equation \(F(x) = y\) (see [16] for a slightly different notion).

**Definition 2.2.** Let \((\rho, R, S)\) be a variational scheme for \(F : (X, \tau_X) \to (Y, \tau_Y)\) and \(y^{\text{exact}}\) be in the range of \(Y\). We say that \(\hat{x}\) is a \(\rho\)-generalized R-minimal solution of \(F(x) = y^{\text{exact}}\) if \(\rho(F(\hat{x}), y^{\text{exact}}) = 0\) and \(R(\hat{x}) = \min\{R(x) : \rho(F(x), y^{\text{exact}}) = 0\}\).

Using the two objective functionals \(\rho(F(\cdot), y)\) and \(R\) we can build at least three different minimization problems which aim at finding Pareto optimal solutions. These three problems are well known in the inverse problems community and in fact can be traced back to the pioneering works in the Russian school: Tikhonov regularization [42] sets \(T_{\alpha,y}(x) := \rho(F(x), y) + \alpha R(x)\) for some \(\alpha > 0\) and considers

\[
T_{\alpha,y}(x) \to \min_{x \in X}.
\]

In other words, choose a weighting between ‘good data fit’ and ‘good fit to prior knowledge’ and minimize the weighted objective functional. Ivanov regularization [25] uses \(\tau > 0\) and considers

\[
\rho(F(x), y) \to \min_{x \in X} \text{ s.t. } R(x) \leq \tau.
\]
In other words, choose the solution with the best data-fit which also fits the prior knowledge up to a predefined amount. Finally, Morozov regularization [32] uses δ > 0 and considers
\[ R(x) \rightarrow \min_{x \in X} \quad \text{s.t.} \quad \rho(F(x), y) \leq \delta. \] (3)
In other words, choose the solution which fits best the prior knowledge among the ones which explain the data up to a predefined amount.

These methods are treated and compared, e.g., in [27, chapter 3.5] in the case of Banach spaces and \( \rho(F(x), y) = \|F(x) - y\|^p \) and \( R(x) = \|Lx\|^q \) with a (possibly unbounded) linear operator \( L \) (where (2) is called ‘method of quasi-solutions’ and (3) goes under the name ‘method of the residual’). We state a result on the relation of the minimizers of these methods in our abstract framework of a variational scheme without any convexity assumptions on \( R \) or \( \rho \).

**Theorem 2.3.** Let \((\rho, R, S)\) be a variational scheme for \( F : (X, \tau_X) \rightarrow (Y, \tau_Y) \).

(i) If there exists a unique solution \( x_\tau \) of (2), then it is the unique solution of (3) with \( \delta = \rho(F(x_\tau), y) \).

(ii) If there exists a unique solution \( x_\delta \) of (3), then it is the unique solution of (2) with \( \tau = R(x_\delta) \).

(iii) If there exists a unique solution \( x_\alpha \) of (1) then it solves (2) with \( \tau = R(x_\alpha) \) and (3) with \( \delta = \rho(F(x_\alpha), y) \).

**Proof.** (i) With \( \delta = \rho(F(x_\tau), y) \), it is clear that \( x_\tau \) is feasible for the optimization problem (3) and the objective value is \( R(x_\tau) \). Assume that there is a solution \( \tilde{x} \neq x_\tau \) of (3) with \( R(\tilde{x}) \leq R(x_\tau) \leq \tau \). Then, \( \tilde{x} \) would be feasible for (2) with objective \( \rho(F(\tilde{x}), y) \leq \delta = \rho(F(x_\tau), y) \) which is a contradiction to the uniqueness of the solution \( x_\tau \). The proof of (ii) mimics the proof of (i). For (iii) again, assume that there exists a solution \( \tilde{x} \neq x_\alpha \) of (2). Then, one sees that \( T_{\alpha, \delta}(\tilde{x}) \leq T_{\alpha, \delta}(x_\alpha) \) contradicting the uniqueness of \( x_\alpha \). The proof is similar for the last claim. □

We remark that the missing implications in theorem 2.3 are not true without additional assumptions.

**Example 2.4** (Unique Ivanov and Morozov minimizers need not be Tikhonov minimizers). We illustrate this by a simple one-dimensional example. Let \( X = Y = \mathbb{R} \), \( F = \text{id} \) and consider the regularization functional \( R(x) = |x + 1| \) (saying that the solution should be close to \(-1\)) and as discrepancy functional the so-called Bregman distance with respect to the strictly convex function \( x \mapsto x^4 \), i.e., \( \rho(x, y) = y^4 - x^4 - 4x^3(y - x) \). We choose \( \tau = 1 \) and \( y = 1 \) and obtain \( x_\tau = 0 \) as the unique solution of (2) (which is also the unique solution of (3) with \( \delta = 1 \)). But there is no \( \alpha > 0 \) such that \( x = 0 \) minimizes \( T_{\alpha, \delta} \) (cf figure 1).

In the above examples it holds that \( x_\tau \) is a stationary point of the mapping \( x \mapsto \rho(F(x), y) \).

Note that the precise form of \( R \) is not important in this example, several other \( R \) with \( R'(0) > 0 \) would also work. Indeed, we can deduce from the next proposition that it is necessary for \( x_\tau \) to be also a (local) Tikhonov minimizer that not both of these properties are fulfilled.

**Proposition 2.5** ([35, Theorem 4.13]). Let \((\rho, R, S)\) be a variational scheme for \( F : (X, \tau_X) \rightarrow (Y, \tau_Y) \), \( X \) be a normed space and let \( y \in Y \). Furthermore, assume that the mappings \( f(x) = \rho(F(x), y) \) and \( R \) obey directional derivatives \( R'(x^\alpha; v) \) and \( f'(x^\alpha; v) \) for all directions \( v \in X \).

If \( x^* \) is a local minimizer of \( T_{\alpha, \delta} \) for some \( \alpha > 0 \) then for every \( v \) it holds that
\[ -\alpha R'(x^*; v) \leq f'(x^*; v). \]

Moreover, if the directional derivatives of \( f \) and \( R \) at \( x^* \) are linear in \( v \) and \( R'(x^*; \cdot) \neq 0 \), then \( f'(x^*; \cdot) \neq 0 \).
Figure 1. Illustration of example 2.4.

In other words: if we have a solution $x^*$ of (2) with $R'(x^*; \cdot) \neq 0$ which is also a local minimizer of $T_{\alpha, y}$ then it is not stationary for $x \mapsto \rho(F(x), y)$.

**Remark 2.6.** Under convexity assumptions on $f(x) = \rho(F(x), y)$ and $R$ one can show that Ivanov minimizers (or Morozov minimizers) are indeed also Tikhonov minimizers for some parameter $\alpha > 0$ if they are not minimizers of the constraint. This is related to the fact that the subgradients of convex functions describe the normal vectors to the sublevel sets of the respective function; see e.g. [40].

Although the variational problems (1)–(3) share their solutions under the circumstances presented above, they often differ with respect to their practical application.

It has been remarked already in early works (see, e.g., [41]) that Ivanov and Morozov regularizations are related to different types of prior knowledge on the exact equation $F(x^{\text{exact}}) = y^{\text{exact}}$. Morozov regularization is related to prior knowledge about the exact data or the noise level, i.e., upper estimates on the quantity $\rho(y^{\text{exact}}, y)$. Ivanov regularization is related to prior knowledge about the exact solution, i.e., about upper estimates about the quantity $R(x^{\text{exact}})$. Hence, the choice between Morozov and Ivanov regularization should be based upon the available prior knowledge at hand. However, there are more factors, which should be taken into account when choosing the variational method. Since the three optimization problems (1)–(3) may belong to different ‘subclasses’ of optimization problems their solution may have different computational complexity.

**Example 2.7** (Linear problems in Hilbert space). In this classical setting, $X$ and $Y$ are Hilbert spaces, $F$ is bounded and linear and we use $\rho(Fx, y) = \|Fx - y\|_Y^2$ and $R(x) = \|x\|_X^2$. In this case, the Tikhonov problem has an explicit solution $x_\alpha = (F^*F + \alpha I)^{-1}F^*y$ which can be treated numerically in several convenient ways (since the operator which has to be inverted is self-adjoint and positive definite).

However, for both Ivanov and Morozov regularization no closed solution exists in general and one usually resorts to solving a series of Tikhonov problems, adjusting the parameter $\alpha$ such that the Ivanov or Morozov constraint is fulfilled [19].

**Example 2.8** (Sparse regularization). We consider regularization of a linear operator equation $Ku = g$ with an operator $K : \ell^2 \to Y$ with a Hilbert space $Y$ by means of a sparsity constraint [11, 30, 21]. In this setting one works with the discrepancy functional $\rho(Ku, g) = \frac{1}{2}\|Ku - g\|_Y^2$ and the regularization functional $R(x) = \|u\|_1$ (extended by $\infty$ if the $1$-norm does not exist). In this case Tikhonov regularization consists of solving a convex, non-smooth, and unconstrained optimization problem (it is a non-smooth convex program, however, with additional structure).
Morozov regularization consists of solving a non-smooth and convex optimization problem with a (smooth) convex constraint (and it can be cast as a second-order cone-program), and Ivanov regularization requires solving a smooth and convex optimization problem with a non-smooth convex constraint (it is a quadratic program).

Looking a little bit closer at this classification and the properties of $\rho$ and $R$, we observe that Ivanov regularization gives in fact the 'easiest' problem since it obeys a smooth objective function and a constraint with a fairly easy structure (e.g., it is easy to calculate projections onto the constraint). On the other hand, the Morozov problem is 'difficult' since it involves a non-smooth objective over a fairly complicated convex set (in the sense that projections onto the set $\|Ku - g\| \leq \delta$ are costly to calculate). Indeed, this rationale is behind the SPGL1 method [43, 44]. It replaces the Morozov problem with a sequence of Ivanov problems, solving each by a spectral projected gradient method, resulting in one of the fastest methods available for Morozov regularization with $\ell^1$ regularization functional.

In conclusion, the choice between the three variational methods should be based on the available prior knowledge and also on the tractability and the complexity of the corresponding optimization problem (often leading to a combination of two methods).

3. Necessary conditions for Tikhonov schemes

In this section we analyze regularization properties of the Tikhonov method. First we formalize our requirements for a scheme to be regularizing in the Tikhonov case. As usual we formulate conditions on existence, stability and convergence of the minimizers, cf [39].

**Definition 3.1** (Tikhonov regularization scheme). A variational scheme $(\rho, R, S)$ for $F : (X, \tau_X) \rightarrow (Y, \tau_Y)$ is called Tikhonov regularization scheme, if the following conditions are fulfilled.

(R1) **Existence.** For all $\alpha > 0$ and all $y \in Y$ it holds that $\arg\min_{x \in X} T_{\alpha, y}(x) \neq \emptyset$.

(R2) **Stability.** Let $\alpha > 0$ be fixed, $y_n S \rightarrow y$ and $x_n \in \arg\min_{x \in X} T_{\alpha, y_n}(x)$. Then $(x_n)$ converges subsequentially in $\tau_X$ and for each subsequential limit $\bar{x}$ of $(x_n)$ it holds that $\bar{x} \in \arg\min_{x \in X} T_{\alpha, y}(x)$.

(R3) **Convergence.** Let $F(x) = y$ have an exact solution $x^{\text{exact}}$ such that $R(x^{\text{exact}}) < \infty$ and $y_n S \rightarrow y$. Then there exists a sequence $(\alpha_n)_{n}$ of positive real numbers such that $x_n \in \arg\min_{x \in X} T_{\alpha_n, y_n}(x)$ converges subsequentially in $\tau_X$ and every subsequential limit $\bar{x}$ is a $\rho$-generalized $R$-minimal solution of $F(x) = y$.

3.1. Trivial necessary conditions

First we list fairly obvious necessary conditions to be regularizing in the Tikhonov sense. To that end, we introduce the solution operator

$$A : Y \times ]0, \infty[ \rightarrow 2^X$$

$$(y, \alpha) \mapsto \arg\min_{x \in X} T_{\alpha, y}(x)$$

for the Tikhonov problem (1). For fixed $\alpha > 0$ we denote $A_{\alpha}(y) = A(y, \alpha)$. We consider $A$ and $A_{\alpha}$ as set valued mappings and use the respective notation (see, e.g., [40]), especially the notion of the domain $\text{dom} A_{\alpha} = \{ y \in Y : A_{\alpha}(y) \neq \emptyset \}$ and the graph $\text{gr}(A_{\alpha}) = \{(y, x) \in Y \times X : x \in A_{\alpha}(y)\}$.
Moreover, we recall that a topology is called sequential if it can be described by sequences, i.e., every sequentially closed set is closed.

**Remark 3.2.** Let \((\rho, R, S)\) be a variational scheme for \(F : (X, \tau_X) \to (Y, \tau_Y)\). Then obviously (R1) is fulfilled if and only if \(\text{dom}A = Y \times \{0, \infty\}\), which implies that \(\text{dom}R \cap F^{-1}(\text{dom}\rho(\cdot, y)) \neq \emptyset\) and hence range \(F \cap \text{dom}\rho(\cdot, y) \neq \emptyset\) does hold for all \(y \in Y\).

**Theorem 3.3.** Let \((\rho, R, S)\) be a variational scheme for \(F : (X, \tau_X) \to (Y, \tau_Y)\) that fulfills (R2), \(\alpha > 0\) and \(y \in Y\). Then

(i) \(A_\alpha(y)\) is sequentially compact and so is \((\bigcup_{n \in \mathbb{N}} A_\alpha(y_n)) \cup A_\alpha(y)\) for every sequence \((y_n)\) in \(Y\) such that \(y_n \xrightarrow{S} y\).

(ii) The implication

\[
\begin{align*}
&y_n \xrightarrow{S} y \\
x_n \xrightarrow{\tau_x} x \\
x_n \in A(y_n, \alpha)
\end{align*}
\]

does hold, i.e. the mapping \(A_\alpha\) is sequentially closed.

If \(S\) is induced by a topology \(\tau\) and \(\tau \times \tau_X\) is sequential, then \(\text{gr}(A_\alpha)\) is closed for every \(\alpha > 0\).

If furthermore \(A_\alpha\) is single valued, then (R2) does hold if and only if \(A_\alpha\) is continuous w.r.t. \(S\) and the sequential convergence structure of \(\tau_X\).

**Proof.**

(i) Let \((x_n)\) be a sequence in \(A_\alpha(y)\) and consider the constant sequence \(y_n := y\). Then \(y_n \rightarrow y\) and \(x_n \in A_\alpha(y_n)\) do hold. Therefore (R2) implies the existence of a convergent subsequence of \((x_n)\) converging to an element of \(A_\alpha(y)\).

To prove the second assertion, let \((x_k)_{k \in \mathbb{N}}\) be a sequence in \((\bigcup_{n \in \mathbb{N}} A_\alpha(y_n)) \cup A_\alpha(y)\). We distinguish two cases.

(a) There exists a \(\tilde{y} \in \{y_n : n \in \mathbb{N}\} \cup \{y\}\) such that \(x_k \in A_\alpha(\tilde{y})\) for infinitely many \(k \in \mathbb{N}\).

Then \((x_k)\) has a subsequence in \(A_\alpha(\tilde{y})\) and the assertion is covered by the first part of the proof.

(b) For every \(\tilde{y} \in \{y_n : n \in \mathbb{N}\} \cup \{y\}\) there are at most finitely many \(k \in \mathbb{N}\) such that \(x_k \in A_\alpha(\tilde{y})\). Without loss of generality we can assume that \(x_k \not\in A(y)\) for all \(k \in \mathbb{N}\), that the \(x_k\) are pairwise distinct and that there is at most one \(x_k \in A(y_n)\) for all \(n \in \mathbb{N}\) (otherwise we could choose an appropriate subsequence).

Then, the sequence given by \(\tilde{y}_k := y_n\) if \(x_k \in A(y_n)\) is well defined and \(\{\tilde{y}_k : k \in \mathbb{N}\}\) is an infinite subset of \{\(y_n : n \in \mathbb{N}\}\). Hence \((y_n)\) and \((\tilde{y}_k)\) have a subsequence \((\tilde{y}_{k_n})\) in common.

Now \((\tilde{y}_{k_n})\) being a subsequence of \((y_n)\) implies \(\tilde{y}_{k_n} \xrightarrow{S} y\) and due to construction \(x_{k_n} \in A(\tilde{y}_{k_n}) = \arg\min_{x \in X} T_{\rho, \tilde{y}_{k_n}}(x)\) does hold. Applying (R2) yields the existence of a convergent subsequence of \((x_{k_n})\) with limit in \(A_\alpha(y)\), which completes the proof.

(ii) The first assertion is just a reformulation of (R2). As to the second assertion, in the case of topological convergence the sequential closedness of \(A_\alpha\) is equivalent to sequential closedness of \(\text{gr}(A_\alpha)\) w.r.t. \(\tau \times \tau_X\). Since the latter topology is sequential, the assertion follows.
3.2. A closer look on the data space

There exists a vast amount of settings that provide sufficient conditions for a Tikhonov scheme with non-metric discrepancy term to be regularizing. Here we start from a theorem which is extracted from [35, 15, 16].

**Theorem 3.4.** Let \( \mathcal{M} = (\rho, R, S) \) be a variational scheme for a continuous mapping \( F : (X, \tau_X) \to (Y, \tau_Y) \) that fulfills the following list of assumptions.

(A1) The sublevelsets \( \{ x \in X : R(x) \leq M \} \) are sequentially compact w.r.t \( \tau_X \) for all \( M > 0 \), so in particular \( R \) is sequentially lower semicontinuous.

(A2) \( \text{dom} \ T_{a_n} \neq \emptyset \) for all \( y \in Y \).

(A3) \( (x, y) \mapsto \rho(F(x), y) \) is sequentially \( \tau_X \times \tau_Y \) lower semi continuous.

(A4) The sequential convergence structure \( S \) is given by

\[
y_n \overset{S}{\to} y \text{ if and only if } \rho(y, y_n) \to 0 \tag{CONV}
\]

and furthermore it fulfills

\[
y_n \overset{S}{\to} y \text{ implies } \rho(z, y_n) \to \rho(z, y) \text{ for all } z \in \text{dom} \rho(\cdot, y). \tag{CONT}
\]

(A5) \( y_n \overset{S}{\to} y \) implies \( y_n \overset{T}{\to} y \)

Then \( \mathcal{M} \) is a Tikhonov regularization scheme.

**Proof.** We will only give a sketch of the proof, for details we refer to [23, 35, 16].

Since \( \rho \) and \( R \) are non-negative (A1)–(A3) imply (R1) (existence of minimizers).

Let \( (x_n) \) be a sequence of minimizers as in (R2). Then \( (R(x_n)) \) is bounded due to (A2) and [CONT]. Hence, (A1) delivers a convergent subsequence. Let \( \tilde{x} \) be the limit of such a subsequence. Then, (A5), (A3) and [CONT] yield \( T_{a_n} \tilde{x} \leq T_{a_n} x \) for all \( x \in X \). Consequently, (R2) is fulfilled (stability).

Let \( F(x^*) = y, R(x^*) < \infty \) and \( (y_n) \) be a sequence such that \( y_n \overset{S}{\to} y \). Then, due to [CONV], there exists \( a_n \) such that

\[
a_n \to 0 \text{ and } \frac{\rho(y, y_n)}{a_n} \to 0 \text{ as } n \to \infty \tag{4}
\]

does hold (e.g. \( a_n = \sqrt{\rho(y, y_n)} \)).

Therefore \( R(x_n) \leq \frac{1}{a_n} T_{a_n} (x^*) \) for \( x_n \in \text{argmin}_{x \in X} T_{a_n} x \) and together with (A1) this yields subsequential convergence and \( R(\tilde{x}) \leq R(x^*) \) for every subsequential limit \( \tilde{x} \). Using [CONV] we get \( \rho(F(x_n), y_n) \to 0 \), which yields \( \rho(F(\tilde{x}), y) = 0 \) due to (A5) and (A3). \( \square \)

**Remark 3.5.** In [35] it is additionally assumed that \( \rho(z, y) = 0 \) implies \( y = z \). This allows us to formulate (R3) with \( R \)-minimal solutions in the strict sense (i.e. with \( F(x) = y \)) instead of \( \rho \)-generalized \( R \)-minimal solutions.

In item (A4) it would be sufficient if [CONT] only holds for \( z \in \text{dom} \rho(\cdot, y) \cap F(X) \).

As remarked earlier, it is hard to obtain necessary conditions for a general Tikhonov scheme to be regularizing. Hence, we have chosen to start with the analysis of the data space \( Y \). This is motivated by the fact that there are three different objects that pose additional structure on \( Y \), namely the topology \( \tau_Y \), the sequential convergence structure \( S \) and the discrepancy functional \( \rho \). Obviously, not every combination of these three objects will lead to a regularization scheme. We start from theorem 3.4 and the conditions [CONV], [CONT] and (A5) and investigate the interplay of \( \tau_Y, S \) and \( \rho \) and deduce necessary conditions on
their relations. We are aware that the conditions [CONV], [CONT] and (A5) are not necessary for a scheme to be regularizing, but they appear as natural conditions in the context of regularization. However, we will get two different topologies whose convergent sequences, under appropriate circumstances, come naturally to fulfill one of the conditions [CONV] and [CONT], respectively, both given in a constructive way. Moreover, they will provide means to analyze other topologies having the desired convergent sequences (see remark 3.8 for details). Applied to specific classes of discrepancy functionals this could allow a deeper structural insight on what [CONT] does really mean and may tackle a subclass for which theorem 3.4 is eligible without further adaptions.

Remark 3.6. Every topology $\tau$ induces a sequential convergence structure $S(\tau)$ via $y_n \overset{S(\tau)}{\to} y$ if and only if $y_n \overset{\tau}{\to} y$. In the further course of the paper we will say, that a topology $\tau$ satisfies [CONV] respectively [CONT] if and only if the sequential convergence structure induced by the topology has the respective property.

Now we define the two topologies mentioned above, the first one designed to satisfy [CONV], the second to satisfy [CONT].

Definition 3.7. Let $Y$ be a set and $\rho : Y \times Y \rightarrow [0, \infty]$ such that $\rho(y, y) = 0$ for all $y \in Y$.

(i) We call $B^\rho(\varepsilon)(z) := \{y \in Y : \rho(z, y) < \varepsilon\}$ the $\varepsilon$-ball w.r.t $\rho$ centered at $z$ and set $\tau_\rho := \{U \subseteq Y : \forall z \in U \exists \varepsilon > 0 \text{ such that } B^\rho(\varepsilon)(z) \subseteq U\}$. 

(ii) Let $Z \subseteq Y$ and $\tilde{Y} \subseteq Y$ and let $[0, \infty]$ be equipped with the one-point compactification of the standard topology on $[0, \infty]$. For $z \in Z$ we define $f_z : \tilde{Y} \rightarrow [0, \infty]$ by $\tilde{y} \mapsto \rho(z, \tilde{y})$.

By $\tau_{IN}$ we denote the initial topology on $\tilde{Y}$ w.r.t the family $(f_z)_{z \in Z}$, i.e., the coarsest topology on $\tilde{Y}$ for which all the $f_z$ are continuous.

Note that the notation $\tau_{IN}$ does not reflect the dependence on $\tilde{Y}$ and $Z$. Hence, throughout the paper we will always mention explicitly the $\tilde{Y}$ and $Z$ involved.

Remark 3.8. The two additional sets $Z$ and $\tilde{Y}$ are introduced to allow us to model a broader class of discrepancy functionals and to construct a larger variety of topologies. First, note that there are non-symmetric discrepancy functionals and even ones in which the domains of $\rho(\cdot, y)$ and $\rho(z, \cdot)$ differ. Especially, both arguments of $\rho$ have different meanings. The first argument takes images of solutions $x$ under $F$ which can have additional structure (e.g. due to discretization), while the second argument takes measured data which may also have additional characteristics. Moreover, a smaller $Z$ will allow for a coarser topology (and this will be helpful if the range of $F$ is a ‘small’ set) and a smaller $\tilde{Y}$ can model only a restrictive set of possible data (e.g. strictly non-negative one). This purpose could also be met by restricting ourselves to $\rho : Z \times \tilde{Y} \rightarrow [0, \infty]$ and $F : X \rightarrow Z$ for appropriately chosen $Z$ and $\tilde{Y}$ in the first place.

The reason for not doing so is that the topology $\tau_{IN}$ is not merely designed to be itself a possible member of a regularization scheme but also as a tool to analyze other topologies that induce a convergence structure as in theorem 3.4 (A4).

The original aim of constructing the two topologies was to derive conditions on topologies on the whole of $Y$ whose sequential convergence structures fulfill both conditions demanded
in theorem 3.4 (A4) by sandwiching them between \( \tau_\rho \) and \( \tau_{IN} \). In favor of this purpose we want \( \tau_\rho \) to be some sort of maximal topology fulfilling [CONV] and \( \tau_{IN} \) to be minimal with the property [CONT].

As we will see in theorem 3.12 the first request is achieved easily if there is any topology w.r.t the family \( \{ \rho(z, \cdot) \mid z \in Y \} \) we certainly would satisfy [CONT], but we would lose minimality as soon as there is a \( y \in Y \) with \( \text{dom}\rho(\cdot, y) \subseteq Y \). Therefore we have to choose an index set \( Z \subseteq Y \) in this case to avoid more continuous \( \rho(z, \cdot) \) than required by [CONT] and hence \( \tau_{IN} \) too fine. As to \( \tilde{Y} \): if there exist \( y_1, y_2 \in Y \) such that \( \text{dom}\rho(\cdot, y_1) \neq \text{dom}\rho(\cdot, y_2) \) there will not exist a \( Z \subseteq Y \) such that the convergence condition from [CONT] is only fulfilled for \( z \) which also satisfy the finiteness condition. We can cure this by choosing \( Z \) smaller as long as the intersection of all these domains is nonempty, otherwise choosing \( \tilde{Y} \subseteq Y \) may allow us to apply the analysis at least to topological subspaces.

Also this approach is not carried out to its full extent; we would like to leave the way open to do so.

**Example 3.9** (Metrics and powers of norms). If \( \rho \) is a metric on \( Y, Z = \tilde{Y} = Y \) and \( S \) defined as in [CONV]. Then \( S \) and \( \rho \) satisfy [CONT]: Let \( \rho(y, y_n) \to 0 \) and \( z \in Y \). Then

\[
\rho(z, y_n) \leq \rho(z, y) + \rho(y, y_n) \to \rho(z, y).
\]

Clearly, the triangle inequality could be replaced by a quasi-triangle inequality and hence, for the popular case of \( \rho(z, y) = \|z - y\|^p_Y \) of the \( p \)th power of a norm (with \( p > 0 \)) a similar claim is valid: [CONV] says that \( S \) is the norm-convergence (independent of the value of \( p \)), and hence, [CONT] is fulfilled.

Moreover, in the cases of (quasi-)metrics and positive powers of norms the topologies \( \tau_\rho \) and \( \tau_{IN} \) coincide and are the metric or norm topology, respectively.

We may caution the reader that in general the two topologies \( \tau_\rho \) and \( \tau_{IN} \) may be different, as is shown by the following (somewhat pathological) example.

**Example 3.10.** In \( Y = \tilde{Y} = Z = \mathbb{R}^2 \) consider

\[
\rho(z, y) = \begin{cases} 0 & \#\{i \mid y_i \neq z_i\} \leq 1 \\ 1 & \text{else.} \end{cases}
\]

In other words, two elements are considered equal if they differ only in one coordinate. In this case one can show that the topology \( \tau_\rho \) is the indiscrete topology (i.e., the only open sets are \( Y \) and the empty set). However, \( \tau_{IN} \) has the subbasis

\[
B_{1/2}(z) = \{ y \in \mathbb{R}^2 : \#\{i \mid y_i \neq z_i\} \leq 1 \}
\]

and hence \( \tau_{IN} \) is finer than \( \tau_\rho \). Moreover, \( \tau_{IN} \) has fewer convergent sequences than \( \tau_\rho \) (in which every sequence converges to every point).

For the reader’s convenience we recall some properties of the topologies \( \tau_\rho \) and \( \tau_{IN} \) that will be used in the further course of the paper.

**Lemma 3.11.** The following properties hold for \( \tau_\rho \).

(i) \( \tau_\rho \) is a sequential topology.

(ii) A mapping from \( Y \) to an arbitrary topological space is \( \tau_\rho \)-continuous if and only if it is sequentially continuous w.r.t \( \tau_\rho \).

(iii) \( \rho(y, y_n) \to 0 \) implies \( y_n \xrightarrow{\tau_\rho} y \).
The following holds for $\tau_N$.

(iv) For arbitrary $Z, \tilde{Y} \subseteq Y$ sequential convergence w.r.t $\tau_{IN}$ can be characterized as follows.
Let $(y_n)_{n \in \mathbb{N}}$ be a sequence in $\tilde{Y}$ and $y \in \tilde{Y}$. Then $y_n \overset{\tau_{IN}}{\rightarrow} y$ if and only if $\rho(z, y_n) \rightarrow \rho(z, y)$ for all $z \in Z$.

(v) If additionally $\tilde{Y} \subseteq Z$ does hold, $y_n \overset{\tau_{IN}}{\rightarrow} y$ implies $\rho(y, y_n) \rightarrow 0$.

Proof. For (i) see [1, section 2.4]. Item (ii) is a direct consequence of $\tau_{\rho}$ being sequential and (iii) is clear from the definition of open sets w.r.t $\tau_{\rho}$. Then, the first implication of (iv) is due to the sequential continuity of continuous maps and the converse holds because the set $\{f_1^{-1}(V) : z \in \tilde{Z}, V \subseteq [0, \infty) \text{ open}\}$ is a subbase for $\tau_N$. Finally, (v) is the continuity of $f_\tau$ at $y$. □

Now we investigate the relation of $\tau_{\rho}$ to the property [CONV].

**Theorem 3.12.** Let $\tau$ be a topology on $Y$. Then the following does hold.

(i) The property
$$\rho(y, y_n) \rightarrow 0 \implies y_n \overset{\tau}{\rightarrow} y$$
does hold if and only if $\tau$ is coarser than $\tau_{\rho}$.

(ii) If $\tau$ has property [CONV], then so does $\tau_{\rho}$. In particular $\tau_{\rho}$ is the finest topology with that property.

Proof.

(i) Let $\tau$ be coarser than $\tau_{\rho}$, then every $\tau_{\rho}$-convergent sequence is also $\tau$-convergent, and therefore (5) does hold.

Now let $\tau$ be a topology where (5) does hold. Suppose there exists $U \in \tau$ and $U \not\in \tau_{\rho}$. Then there is an $u \in U$ such that for all $n \in \mathbb{N}$ there exists a $y_n \in B^\rho_1(u) \setminus U$.

Evidently $\rho(u, y_n) \rightarrow 0$ does hold and since (5) does hold w.r.t $\tau$, this implies $y_n \overset{\tau}{\rightarrow} u$ in contradiction to $y_n \not\in U$ for all $n \in \mathbb{N}$.

(ii) Let $\tau_{\tilde{U}}$ be a topology that fulfils [CONV] and $(y_n)$ a $\tau_{\rho}$ convergent sequence with limit $y$.

Due to (i) $\tau$ is coarser than $\tau_{\rho}$, therefore $y_n \overset{\tau}{\rightarrow} y$ and consequently $\rho(y, y_n) \rightarrow 0$. □

So, if $S$ is induced by a topology at all, this is also done by the relatively well-behaved (i.e. sequential) topology $\tau_{\rho}$.

One case in which this applies is marked out in the following remark.

**Remark 3.13.** If $S$ provides unique limits, it is induced by a topology. Because a sequence $S$-converges given all its subsequences have a subsequence tending to the same limit, this is guaranteed, e.g., by [4, proposition 1.7.15], [29].

We have two further immediate consequences of theorem 3.12.

**Corollary 3.14.** Let (A4) of theorem 3.4 hold. Then (A5) of theorem 3.4 does hold for a topology $\tau_Y$ on $Y$ if and only if $\tau_Y$ is coarser than $\tau_{\rho}$.

In the case of $\rho(z, y) = \|z - y\|_p$ of Banach space norm, this means that $\tau_Y$ is coarser than the norm topology, i.e. this condition which has been required previously (cf [23]) is somehow necessitated.

**Corollary 3.15.** If there is a topology $\tau$ where $\rho(y, y_n) \rightarrow 0$ implies $y_n \overset{\tau}{\rightarrow} y$ such that [CONT] is fulfilled, then $\tau_{\rho}$ also fulfils [CONT].
Since we are only interested in sequential convergence, this allows us to take $\tau_\rho$ as a sort of model topology.

**Remark 3.16.** In general, the set $\tau_S$ of all sequentially open sets w.r.t to a sequential convergence structure $S$ on $Y$ is a topology on $Y$. As has been shown in [16, proposition 2.10], in the case that $S$ is given by [CONV], it is sufficient for [CONV] to hold for the topology $\tau_S$ as well, that $S$ fulfils [CONT].

Therefore assumption (A4) implies that $\tau_\rho$ also has [CONV] and this again implies that $\tau_S = \tau_\rho$, since $\tau_\rho$ is sequential. Moreover, in this case the sets $B_\rho^\varepsilon(y)$ are open for all $\varepsilon > 0$, $y \in Y$ (see also [16]) and therefore constitute a base for $\tau_\rho$.

The next theorem deals with the question what consequences it has if [CONT] does hold in $\tau_\rho$.

**Theorem 3.17.** Let $Z \subseteq \bigcap_{y \in Y} \text{dom} \rho(\cdot, y)$ be nonempty and $\bar{\tilde{Y}} = Y$.

(i) $\tau_{IN}$ is coarser than $\tau_\rho$.

(ii) If $Z = Y$ then $\tau_\rho$ and $\tau_{IN}$ both satisfy [CONV]. In particular they have the same convergent sequences.

**Proof.**

(i) Since $\rho(z, \cdot)$ is sequentially continuous for all $z \in Z$, it is also continuous and therefore $\tau_{IN}$ is coarser than $\tau_\rho$.

(ii) Due to (i) convergence w.r.t. $\tau_\rho$ yields convergence w.r.t. $\tau_{IN}$ and hence $\rho(y, y_n) \to 0$ implies $y_n \xrightarrow{\text{IN}} y$. Since $Y \subseteq Z$ the converse is also true and therefore $\tau_{IN}$ satisfies [CONV], and so does $\tau_\rho$.

**Remark 3.18.** If $\bar{\tilde{Y}} \subseteq Y$ and $(\tau_\rho|_{\bar{\tilde{Y}}}) = (\tau_\rho)_{|_{\bar{\tilde{Y}}}}$ is sequential (e.g. if $\bar{\tilde{Y}}$ open or closed w.r.t $\tau_\rho$, see [18]), then $(\tau_\rho|_{\bar{\tilde{Y}}})$ does hold. In a setting where $\bar{\tilde{Y}} \subseteq Z \subseteq Y$, this together with theorem 3.17 would still guarantee that $\tau_{IN}$ and the subspace topology of $\tau_\rho$ on $\bar{\tilde{Y}}$ provide the same convergent sequences.

If $\rho(z, \cdot)$ is $\tau_\rho$-continuous at every $y \in Y$ for all $z \in Z$ regardless of the finiteness condition in [CONT], then we can drop the assumption $Z \subseteq \bigcap_{y \in Y} \text{dom} \rho(\cdot, y)$ in theorem 3.17.

So, in the setting of theorem 3.17 sequential convergence in $\tau_\rho$ and $\tau_{IN}$ coincides. In general the sequential convergence structures of these topologies can be different from each other, cf example 3.10.

### 3.3. Application to Bregman discrepancies

We conclude section 3 by an application to a special class of discrepancy functionals, namely ones that stem from Bregman distances which appear, e.g., in the case of Poisson noise or multiplicative noise [6, 31, 36]. Especially, this gives an example that illustrates how theorem 3.17 can be used to gain necessary conditions on the discrepancy functional for theorem 3.4 to apply.

Also, we treat the question, when $\rho(y_1, y_2) = 0$ implies $y_1 = y_2$ in this case.

In the following let $V$ be a Banach space and $J : V \to [0, \infty)$ proper, convex, $Z = Y = \text{dom} J$ and $\bar{\tilde{Y}} \subseteq \{ y \in Y : J \text{ has a single valued subdifferential at } y \}$. The mapping
which maps to the unique subgradient of $J$ is denoted by $\nabla J$. As distance functional $\rho$ we consider the Bregman distance w.r.t. $J$, i.e., for $(z, y) \in Y \times \tilde{Y}$ the functional

$$D_J(z, y) = J(z) - J(y) - \langle \nabla J(y), z - y \rangle.$$ 

**Lemma 3.19.** Let $y_1, y_2 \in \tilde{Y}$. Then $D_J(y_1, y_2) = 0$ if and only $\nabla J(y_1) = \nabla J(y_2)$. In the case $\tilde{Y} = V$ the property

$$D_J(y_1, y_2) = 0 \Leftrightarrow y_1 = y_2$$

for all $y_1, y_2 \in V$ holds if and only if $J$ is strictly convex.

**Proof.** First let $\rho(y_1, y_2) = 0$. Then $J(y_1) = J(y_2) + \langle \nabla J(y_2), y_1 - y_2 \rangle$ and hence linearity of $\nabla J(y_2)$ and non-negativity of $\rho$ imply $J(v) - J(y_1) - \langle \nabla J(y_2), v - y_1 \rangle = \rho(v, y_2) \geq 0$ for all $v \in V$. Therefore $\nabla J(y_2)$ is a subgradient of $J$ in $y_1$. Since the subgradient of $J$ is single valued at $y_1$ this yields $\nabla J(y_2) = \nabla J(y_1)$.

Now let $\nabla J(y_2) = \nabla J(y_1)$. Then $0 \geq -\rho(y_1, y_2) = \rho(y_2, y_1) \geq 0$.

For the second statement, note that we only need to show that a function is strictly convex if and only if the mapping $\nabla J$ is injective. Assume that $J$ is strictly convex but that there is $\xi = \nabla J(y) = \nabla J(z)$ for $y \neq z$. Plugging $y$ and $z$ in the respective (strict) subgradient inequalities gives

$$J(y) - J(z) > \langle \xi, y - z \rangle$$

and adding both inequalities we arrive at the contradiction $0 > 0$. Moreover, assuming that $J$ is not strictly convex, there are $y \neq z$ such that $J(y) - J(z) = \langle \nabla J(z), y - z \rangle$. But then it holds for all $z'$ that $J(z') - J(y) - \langle \nabla J(z), z' - y \rangle = J(z') - J(z) - \langle \nabla J(z), z' - z \rangle \geq 0$ which shows that $\nabla J(y) = \nabla J(z)$, i.e. that $\nabla J$ cannot be injective.

The Bregman distance $D_J$ is in general not symmetric and the behavior in both coordinates can be quite different (e.g., $D_J$ is always convex in the first coordinate but not necessarily so for the second). In the literature, both the discrepancies

$$\rho_1(F(x), y) = D_J(F(x), y)$$

and

$$\rho_2(F(x), y) = D_J(y, F(x))$$

are used (see [31] for the first variant and [36, 6] for the second).

First, we analyze the variant $\rho_1(z, y) = D_J(z, y)$ which corresponds to the Tikhonov function $T_{\rho_1}(x) = D_J(F(x), y) + oR(x)$. The following lemma explores what convergence w.r.t $\tau_{TN}$ actually looks like.

**Lemma 3.20.** For all sequences $(y_n)$ in $\tilde{Y}$, $y \in \tilde{Y}$ the following does hold: $y_n \overset{\tau_{TN}}{\longrightarrow} y$ if and only if $\rho_1(y_n, y) \rightarrow 0$ and $\langle \nabla J(y_n) - \nabla J(y), y - z \rangle \rightarrow 0$ for all $z \in Z$. Moreover, $y_n \overset{\tau_{TN}}{\rightarrow} y$ if and only if $\rho_1(y_n, y) \rightarrow 0$ and $\nabla J(y_n) \overset{\star}{\rightharpoonup} \nabla J(y)$ in span$(Z)^\ast$. In particular $(\nabla J)_\beta : \tilde{Y} \rightarrow \text{span}(Z)^\ast$ is sequentially $\tau_{TN}$-weak* continuous.

**Proof.** The identity $\rho_1(z, y_n) - \rho_1(z, y) = \rho_1(y_n, y) + \langle \nabla J(y_n) - \nabla J(y), y - z \rangle$ holds for all $z \in Z$.

So clearly $\rho_1(y_n, y) \rightarrow 0$ and $\langle \nabla J(y_n) - \nabla J(y), y - z \rangle \rightarrow 0$ imply $y_n \overset{\tau_{TN}}{\rightarrow} y$.

Conversely, let $y_n \overset{\tau_{TN}}{\rightarrow} y$ hold. Then $\rho_1(y_n, y) \rightarrow 0$ and hence $0 = \lim_{n \rightarrow \infty}(\rho_1(z, y_n) - \rho_1(z, y) - \rho_1(y_n, y)) = \lim_{n \rightarrow \infty}(\nabla J(y_n) - \nabla J(y), y - z)$. \(\square\)

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Corollary 3.21. Let \( \text{dom } J = \tilde{Y} = V \).

(i) If \( \tau_{\rho_1} \) satisfies [CONT], then \( \nabla J \) is \( \tau_{\rho_1} \)-weak* continuous.

(ii) \( \tau_{\text{TN}} \) provides unique sequential limits if and only if \( J \) is strictly convex.

Proof. (i) is a direct consequence of the previous lemma and (ii) is a direct consequence of lemma 3.19 and the definition of \( \tau_{\text{TN}} \). \( \square \)

So, if \( J \) is strictly convex, in the setting of corollary 3.21 it is necessary for theorem 3.4 to apply to Bregman discrepancies that \( J \) has \( \tau_{\rho_1} \)-weak* continuous derivative, since in this case the sequential convergence structure is given by \( \tau_{\rho_1} \) (due to remark 3.13 and theorem 3.12).

Moreover, in this case, a Tikhonov regularization scheme with discrepancy \( \rho_1 \) guarantees convergence to an exact solution given \( J \) is strictly convex.

To complement lemma 3.20, we now analyze the variant \( \rho_2(z, y) = D_J(y, z) \) which corresponds to the Tikhonov functional \( T_{\alpha, \gamma}(x) = D_J(y, F(x)) + \alpha R(x) \). Similarly to lemma 3.20 we state the following characterization of convergence with respect to \( \tau_{\text{TN}} \).

Lemma 3.22. For all sequences \( (y_n) \) in \( \tilde{Y} \), \( y \in \tilde{Y} \) the following does hold: \( y_n \xrightarrow{\tau_{\text{TN}}} y \) if and only if \( \rho_2(y, y_n) \to 0 \) and \( \langle \nabla J(y) - \nabla J(z), y - y_n \rangle \to 0 \) for all \( z \in \tilde{Y} \).

Proof. In this case the identity \( \rho_2(z, y_n) - \rho_2(z, y) = \rho_2(y, y_n) + \langle \nabla J(y) - \nabla J(z), y - y_n \rangle \) does hold for all \( z \in Z \). So clearly \( \rho_2(y, y_n) \to 0 \) and \( \langle \nabla J(y) - \nabla J(z), y - y_n \rangle \to 0 \) imply \( y_n \xrightarrow{\tau_{\text{TN}}} y \).

Conversely, let \( y_n \xrightarrow{\tau_{\text{TN}}} y \) hold. Then \( \rho_2(y, y_n) \to 0 \) and hence \( 0 = \lim_{n\to\infty} (\rho_2(z, y_n) - \rho_2(z, y) - \rho_2(y, y_n)) = \lim_{n\to\infty} \langle \nabla J(y) - \nabla J(z), y - y_n \rangle \). \( \square \)

3.4. The Kullback–Leibler divergence

If the noise is modeled by a Poisson process, the appropriate discrepancy functional is the so-called Kullback–Leibler divergence [6, 37]. This model fits into the context of Bregman distances and we provide the setup as in [37]. Consider a bounded set \( \Omega \) in \( \mathbb{R}^n \) equipped with the Lebesgue measure \( \mu \) and set \( V = L^1(\Omega) \). We define

\[
J(y) = \int_\Omega y \log(y) - y \, d\mu \quad y \geq 0 \text{ a.e., } y \log(y) \in L^1(\Omega) \quad \text{else}.
\]

From [36] we take the following facts. The functional \( J \) is strictly convex, we have \( Z = \text{dom } J = \{ y \in L^1(\Omega) : y \geq 0 \text{ a.e., } y \log(y) \in L^1(\Omega) \} \) and it holds that

\[
\partial J(y) = \begin{cases} \{ \log(y) \} & y \geq \epsilon \text{ a.e. for some } \epsilon > 0, y \in L^\infty(\Omega) \\ \emptyset & \text{else.} \end{cases}
\]

Hence, we denote \( \nabla J(y) = \log(y) \) and we have

\[
\tilde{Y} = \{ y \in L^1(\Omega) \cap L^\infty(\Omega) : y \geq \epsilon \text{ for some } \epsilon > 0 \}.
\]

The associated Bregman distance is also known as Kullback–Leibler divergence

\[
D_{\text{KL}}(z, y) = \int_\Omega z \log \left( \frac{z}{y} \right) - z + y \, d\mu.
\]

From [8] it is known that

\[
\|z - y\|_1^2 \leq \left( \frac{2}{3} \|y\|_1 + \frac{4}{3} \|z\|_1 \right) D_{\text{KL}}(z, y).
\]
Lemma 3.23. It holds that \( \text{span}(Z) = L^1(\Omega) \) and consequently \( \text{span}(Z)^* = L^\infty(\Omega) \).

**Proof.** Consider \( y \in L^1(\Omega) \) and \( \epsilon > 0 \) and define
\[
y^+_\epsilon(x) = \begin{cases} \frac{1}{\epsilon} y(x) & \frac{1}{\epsilon} < y(x) < \frac{1}{\epsilon} \\ y(x) & y(x) \leq \frac{1}{\epsilon} \end{cases}, \quad y^-_\epsilon(x) = \begin{cases} -\frac{1}{\epsilon} y(x) & -\frac{1}{\epsilon} < y(x) < -\epsilon \\ -\epsilon & y(x) \leq -\epsilon \end{cases}.
\]
Then it holds that \( y^+_\epsilon, y^-_\epsilon \in Z \) and \( (y^+_\epsilon + y^-_\epsilon) \to y \) in \( L^1(\Omega) \). Since every continuous linear functional on \( \text{span}(Z) \) can be extended continuously to \( \text{span}(Z) = L^1(\Omega) \), we conclude that \( \text{span}(Z)^* = L^\infty(\Omega) \). \( \Box \)

First we look at \( \rho_1 \) as in lemma 3.20. Here \( \rho_1(z, y) = D_{KL}(z, y) \), i.e., the measured data is in the second argument of the Kullback–Leibler divergence as, e.g., in [6, 37]. We deduce directly from lemma 3.20 that a sequence \( (y_n) \) converges in \( \tau_{KL} \) to \( y \) if and only if
\[
\rho_1(y_n, y) = D_{KL}(y_n, y) \to 0 \quad \text{and} \quad \log(y_n) \overset{*}{\to} \log(y) \quad \text{in} \quad L^\infty(\Omega).
\]
We can describe this notion of convergence in more familiar terms.

**Theorem 3.24.** In the case of \( \rho_1(z, y) = D_{KL}(z, y) \) and \( \tau_{KL} \) defined by definition 3.7, it holds that
\[
y_n \tau_{KL} y \iff \left\{ \begin{array}{l}
y_n \to y \quad \text{in} \ L^1(\Omega) \\
\log(y_n) \overset{*}{\to} \log(y) \quad \text{in} \ L^\infty(\Omega).
\end{array} \right.
\]

**Proof.** The implication ‘\( \Rightarrow \)’ follows from lemma 3.20 and the fact that \( D_{KL}(y_n, y) \to 0 \Rightarrow \|y_n - y\|_1 \to 0 \) (see [36]). For the converse direction, observe that
\[
D_{KL}(y_n, y) = \int_\Omega y(\log(y) - \log(y_n)) \, d\mu + \int_\Omega y - y_n \, d\mu
\]
and that both terms on the right-hand side converge due to the assumptions (and the implicit non-negativity assumption). \( \Box \)

Loosely speaking, one can interpret the assumption that \( \log(y_n) \) converges weakly* in \( L^\infty(\Omega) \) as a condition that \( y_n \) is not allowed to converge to zero on a non-null set which seems to be a natural condition in this context.

In view of theorem 3.4 we can conclude the following. If one aims at Tikhonov regularizing schemes with Kullback–Leibler divergence \( \rho_1(F(x), y) \) and wants to apply theorem 3.4, then the appropriate model for ‘data \( y \) converging to noiseless data \( y' \)’ is given by ‘strong convergence in \( L^1 \) plus weak* convergence in \( L^\infty \).

Second, we look at the case of \( \rho_2 \) as in lemma 3.22. Here \( \rho_2(z, y) = D_{KL}(y, z) \), i.e., the data is in the first argument of the Kullback–Leibler divergence as, e.g., in [31]. We conclude directly from lemma 3.22 that a sequence \( (y_n) \) converges in \( \tau_{KL} \) to \( y \) if and only if
\[
D_{KL}(y_n, y) \to 0
\]
and
\[
\int_\Omega (\log(y) - \log(z))(y - y_n) \, d\mu \to 0 \quad \text{for all} \ z \in \bar{Y}.
\]
In fact the condition in the second line is precisely weak convergence in \( L^1(\Omega) \). Indeed for any \( w \in L^\infty(\Omega) \) and \( y \in \bar{Y} \) we can define \( z \in \bar{Y} \) via \( z = \exp(w - \log(y)) \) and see that
\[
\int_\Omega (\log(y) - \log(z))(y - y_n) \, d\mu = \int_\Omega w(y - y_n) \, d\mu
\]
and hence, the condition is indeed weak convergence in \( L^1(\Omega) \). Since, as already noticed, \( D_{KL}(y_n, y) \to 0 \) implies that \( y_n \to y \) in \( L^1(\Omega) \) strongly, we see that (6) implies (7) and conclude:
Theorem 3.25. In the case \( \rho_2(z, y) = D_{\text{KL}}(y, z) \) and \( \tau_{\text{IN}} \) defined by definition 3.7 it holds that
\[ y_n \xrightarrow{\tau_{\text{IN}}} y \iff D_{\text{KL}}(y_n, y) \to 0. \]

Remark 3.26. Note that the convergence in \( \tau_{\text{IN}} \) (i.e. \( D_{\text{KL}}(y_n, y) \to 0 \)) is stronger than strong convergence in \( L^1(\Omega) \), even if all \( y_n \) and \( y \) are uniformly bounded away from zero. To see this consider the following counterexample. Let \( \Omega = [0, 1] \), \( \epsilon > 0 \) and define \( y \equiv \epsilon \). Now define
\[ y_n(x) = \begin{cases} n & \text{if } 0 \leq x \leq \left( n \log(n) \right)^{-1} \\ \epsilon & \text{if } x > \left( n \log(n) \right)^{-1} \end{cases} \]
(note that \( y, y_n \in \overline{Y} \)). Then it holds that \( \|y_n - y\|_1 \to 0 \) but
\[
D_{\text{KL}}(y_n, y) = \int_0^1 y_n(x) \log \left( \frac{y_n(x)}{y(x)} \right) - y_n(x) + y(x) \, dx \\
= \int_0^{\left( n \log(n) \right)^{-1}} n \log(n/\epsilon) - n + \epsilon \, dx \\
= \frac{n \log(n/\epsilon) - n + \epsilon}{n \log(n)} \\
\to 1.
\]

As a final remark on the Kullback–Leibler divergence, we note that the above discussion on topologies deduced from the Kullback–Leibler divergence when introduced as Bregman distance and considered in \( L^1(\Omega) \), gives another motivation for the use of a positive ‘baseline’ when working with Poisson noise, i.e. the measured data (and hence, also the regularized quantities) are shifted away from zero by adding a small positive constant as, e.g., in [45].

4. Conclusion

We examined variational regularization in a quite general setting and started a study on necessary conditions for variational schemes to be regularizing. Although it seems that little can be said about necessary conditions in general we obtained several results in this direction. Especially, we tried to clarify the relations between the different players in the data space, e.g. the convergence structure, the topology and the discrepancy functional. Here we started from a list of conditions which is known to guarantee regularizing properties and deduced necessary conditions for the topologies and the discrepancy functional. For Bregman discrepancies we illustrated that our results imply necessary conditions for the continuity of the derivative of the functional which induces the Bregman distance and pointed out structural difference when the measured data is in the first or second argument of the Bregman distance, respectively. In the particular case of the Kullback–Leibler divergence, we also characterized convergence in the natural topology \( \tau_{\text{IN}} \) for both cases.

Although our results are fairly abstract, they are first steps towards the analysis of necessary conditions which can be used to figure out essential limitations of variational schemes. Next steps could be to analyze the other ingredients of a variational scheme, namely the solution space \( X \), its topology, the regularization functional and of course, the operator. Other directions for future research are to consider special classes of discrepancy functionals with additional structure and to extend the analysis to Morozov and Ivanov regularization.

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