Dirac cones and chiral waves in soft matter

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(Dated: June 17, 2020)

Abstract

We study the propagation of in-plane elastic waves in a soft thin strip; a specific geometrical and mechanical hybrid framework which we expect to exhibit Dirac-like cone. We separate the low frequencies guided modes (typically 100 Hz for a centimetric strip) and obtain experimentally the full dispersion diagram. Dirac-like cones are evidenced and go along with remarkable wave features such as negative phase velocities or zero group velocity (ZGV). Our measurements are convincingly supported by a model (and numerical simulation) for both Neumann and Dirichlet boundary conditions. Finally, we perform selective generation of one-way chiral modes by carefully setting the source position and polarization. All these results reveal that soft materials are good candidates for observing atypical wave-based phenomena, an interesting feature since they make most of the biological tissues.
Graphene has probably become the most studied material in the last decades. It displays unique electronic properties resulting from the existence of the so-called Dirac cones \[1, 2\]. At these degeneracy points, the motion of electrons is described in quantum mechanics by the Dirac equation: the dispersion relation becomes linear and electrons behave like massless fermions \[3\]. As a result, interesting transport phenomena such as the Klein tunneling or the Zitterbewegung effect have been reported \[4, 5\]. But Dirac cones are not specific to graphene. They correspond to transition points between different topological phases of matter \[6\]. This discovery has enabled the understanding of topologically protected transport phenomena, such as the quantum Hall effect \[7–9\].

Dirac cones are the consequence of a specific spatial patterning rather than a purely quantum phenomenon. Inspired by these tremendous findings from condensed matter physics, the wave community thus started to search for classical analogs in photonic crystals \[10, 11\] (or phononic crystals when dealing with acoustic waves \[12\]) and more recently in metamaterials \[13\]. Abnormal transport properties similar to the Zitterbewegung effect were highlighted \[14–17\]. In recent years, the quest for photonic (and phononic) topological insulators \[18\] has become a leading topic. This specific state of matter results from the opening of a band gap at the Dirac frequency and is praised for its application to robust one-way wave-guiding \[19–23\]. Surprisingly, similar degeneracies have been observed for unexpected photonic lattices as the consequence of an accidental adequate combination of parameters \[24\]. Such Dirac-like cones have a fundamentally different nature as they occur in the \(k \rightarrow 0\) limit \[25\] but still offer interesting features: wave-packets propagate with a non-zero group velocity while exhibiting no phase variation, just like in a zero-index material \[26–29\].

A similar accidental \(k \rightarrow 0\) Dirac-like cone can be observed in the dispersion relation of elastic waves propagating in a simple plate. In this context, the cone results from the coincidence of two cut-off frequencies occurring when the Poisson ratio is exactly of \(\nu = 1/3\) \[30, 32\]. This condition drastically restricts the amount of potential materials to nearly the single Duraluminum alloy. However, a most recent investigation showed that, for in-plane waves propagating in a thin strip, the degeneracy is expected for \(\nu = 1/2\) \[33\], that is to say for soft materials. This supposes that the strip configuration is the key to observe Dirac cones and decline the associated transport effects in the world of soft matter.

In this article, we study the in-plane elastic waves propagating in a soft thin strip and pro-
pose an experimental platform to map the field displacements. We provide full experimental and analytical description of these in-plane waves both for free and rigid edge conditions. We notably extract the low-frequency part of the dispersion diagram for both the configurations. Not only we clearly evidence the existence of Dirac-like cones for such a simple geometry, but we also highlight some other remarkable wave phenomena such as backward modes or zero group velocity (ZGV) modes. Eventually, we exploit these features to perform selective excitation resulting in a one-way propagation of chiral modes, and in the separation of the two contributions of a ZGV wave.

We realize soft thin strips with a silicone elastomer (Smooth-On Ecoflex® 00-30). The monomer and cross-linking agent are mixed in a 1:1 ratio and left for curing during approximately half a day. Meanwhile, the blend is seeded with some dark pigments (smaller than 500 µm) for tracking purposes. Once cured, the strip dimensions are of $L \times w \times h = 600 \text{ mm} \times 39 \text{ mm} \times 3 \text{ mm}$ and the polymer is found to have a density of $\rho = 1010 \text{ kg.m}^{-3}$. The strip is then suspended with its lower end immersed in a glycerol container to avoid spurious reflections and out-of plane motions (Fig. 1). Indeed, all along the article, we will focus on the in-plane vibrations (i.e. displacements $u_1$ and $u_2$ corresponding respectively to directions $x_1$ and $x_2$) resulting in a guided propagation along the vertical direction $x_1$. A 3D-printed clamp tightens the strip at a specific location and is designed with conical termination in order to ensure point-like actuation. The clamp is mounted on a shaker (Tira Vib 51120) driven monochromatically by an external signal generator and amplifier (Tira Analog Amplifier BAA 500) with frequencies ranging from 1 to 200 Hz.

The in-plane displacements are monitored via a digital camera (Basler acA4112-20um) positioned approximately 2 meters away from the strip. Stroboscopic imaging (60 images per period) is performed by slightly detuning the frame rate with respect to the excitation frequency. The video data is then processed with a Digital Image Correlation (DIC) algorithm [34] in order to track the motion of the dark seeds. Complex monochromatic displacement fields are hence mapped on $620 \times 4112$ pixels. Typical displacement fields $u_1$ and $u_2$ measured when shaking at 110 Hz are reported on Fig. 2(a). This method is sensitive to displacement magnitudes in the micrometer range and enables the probing of the displacement within the whole scanned area in spite of the significant viscous damping expected for such silicone polymers.
FIG. 1. **Experimental setup:** a soft elastic strip (of dimensions $L = 600$ mm, $w = 39$ mm, $h = 3$ mm) seeded with dark pigments (for motion tracking purposes) is suspended. A shaker connected to a clamp induces in-plane displacement propagating along the strip.

The interpretation of the displacement maps is not straightforward. As for any wave-guiding process the field gathers contributions from several modes. Given the system geometry, we project the data on their symmetrical (resp. anti-symmetrical) component with respect to the vertical central axis (dotted line in Fig. 2). For improved extraction per-
FIG. 2. Free edges field maps and dispersion. Here $w = 39$ mm. (a) Real part of the raw displacements at 110 Hz and (b) the three corresponding singular vectors (see text). (c) Experimental (symbols) and analytical (solid lines) dispersion curves. Transparency renders the ratio $\text{Im}(k)/\text{Abs}(k)$. Filled grey and blue symbols are extracted symmetrical and anti-symmetrical modes. Empty ones are just built by symmetry.

Performances, a single value decomposition is then performed on the projected field and only the main components are kept. As depicted in Fig.2 at 110 Hz, the raw data decomposes onto three main contributions: two anti-symmetrical modes (denoted $A_0$ and $A_1$) and one symmetrical mode (denoted $S_0$). Each of them clearly exhibits its own spatial frequency $k$ in the $x_1$ direction; a value extracted by performing a simple Fourier analysis of the right pseudo-eigenvector after the singular value decomposition. The procedure is repeated from 1 to 200 Hz to obtain the full dispersion diagram displayed in Fig.2(c) (filled symbols are the extracted ones, empty ones are obtained by symmetry with respect to the $k = 0$ axis). Several branches with different symmetries and behaviours can be distinguished. They are numbered with increasing cut-off frequencies. Note that the Fourier analysis only provides the real part of the wave number $k$. In fact, due to viscous dissipation, this quantity is complex. As a matter of fact, this is well pictured by the decaying character of the field maps.

Those experimental results are supported by theoretical predictions (solid line) obtained with a simplified model (and with 3-D simulations). Indeed, one can show that the in-plane modes of a given strip are analogous to the Lamb waves propagating in a virtual 2-D plate of appropriate effective mechanical properties \[33\]. When the strip is made of a soft material, the analogy holds for a plate with a shear wave velocity of $v_T$, a longitudinal velocity of
FIG. 3. **Fixed edges dispersion.** Experimental (symbols) and theoretical (solid lines) dispersion curves for a strip of width \( w = 50.6 \text{ mm} \) with fixed edges. Symmetrical modes (resp. anti-symmetrical) are labelled in grey (resp. blue).

exactly \( 2\nu_T \) and a thickness of \( w \). Strikingly, this amounts to acknowledging that, for a thin strip of soft material, the low frequency in-plane guided waves are independent of the bulk modulus (or equivalently of the longitudinal wave velocity) and of the strip thickness \( h \). One can then retrieve the full dispersion solely from the knowledge of the strip’s shear modulus \( G \), width \( w \) and density \( \rho \). Of course, the intrinsic dispersive properties of the soft material as well as its lossy character must be taken into account. A simple and commonly accepted model for describing the low frequency rheology of silicone polymers is the fractional Kelvin-Voigt model \([35,37]\), for which the complex shear modulus writes \( G = G_0[1 + (i\omega\tau)^n] \). This formalism being injected in the 2-D formalism, our measurements are convincingly adjusted (solid lines in Fig.2) when the following set of parameters is input: \( G_0 = 26 \text{ kPa}, \tau = 260 \mu\text{s} \) and \( n = 0.33 \). Note that this choice of parameters turns out to match relatively well the measurements obtained with a traditional rheometer. The transparency of the theoretical line represents the weight of the imaginary part of the wave-number \( k \). When \( k \) becomes essentially imaginary, the solution is evanescent which explains why it cannot be extracted from the experiment.
FIG. 4. **Selective generation.** Chronophotographic sequences (12 snapshots) over a full oscillation cycle. (a) The source is positioned at the strip centre and shaken vertically at 136 Hz: symmetric diverging waves are observed on both parts. (b) Two sources facing each other are rotated counter-wisely at 136 Hz: only the symmetric wave propagating in the $x_1 > 0$ direction is excited. (c) Two sources are shaken horizontally at 102 Hz: a stationary wave associated to an anti-symmetric ZGV mode is observed. (d) Two sources are rotated at 102 Hz: the anti-symmetric waves are separated and a negative phase velocity is observed on the top part. For all sequences, the red dashed lines are visual guides highlighting the zero displacements and the sketches show the source shape and motion.

Let us now comment on a few interesting features of this dispersion diagram. First, at low frequencies, the single symmetrical branch (labelled $S_0$) presents a linear slope, thus defining a non-dispersive propagation or equivalently a propagation at a constant wave velocity. The latter corresponds experimentally to a value of exactly $\sqrt{3}v_T$ which confirms the prediction from [33]. This is somehow counter-intuitive: the displacement of $S_0$ is quasi-exclusively polarized along the $x_1$ direction, giving it the aspect of a pseudo-longitudinal wave, but it propagates at a speed independent of the longitudinal velocity. At 150 Hz, two branches cross linearly in the $k \to 0$ limit. This is the signature of a Dirac-like cone [24, 30, 38]. Note how well it is defined in spite of the significant damping present in the system. Here, the cone is the result of the incompressible nature of the soft elastomer. Indeed, when $v_L \gg v_T$ (or $\nu \approx 0.5$), the second and third cut-off frequencies automatically coincide [33]. In other words, any thin soft strip would display such a Dirac-like cone. Because the cone is located in $k = 0$, the lower frequency part of the $S_2$ branch features negative wavenumbers (solid symbols). This means that the phase and group velocities are anti-parallel [39, 40] and
results in backward propagation of the wave-fronts. It is interesting to note that the wave number extraction from experimental maps provides negative values (filled symbols) which evidences a negative phase velocity, a feature that has been the scope of many developments in the metamaterial field [41, 42], and that occurs spontaneously here.

From now on, we implement Dirichlet boundary conditions on a \( w = 50.6 \) mm strip by clamping its edges in a stiff aluminium frame. The previous experimental steps are repeated and the dispersion curves extracted (Fig. 3). Again, a Dirac-like cone is observed for this configuration. However, the crossing branches are now the anti-symmetrical ones. Additionally, due to the fixed boundaries, the lower order branches \( A_0 \) and \( S_0 \) (from Fig. 2(b)) have been suppressed, and the first cut-off frequency is now associated with a symmetrical mode. It is also worth noticing how the slope of the \( A_1 \) branch switches sign for a non-zero wave number. As a consequence, the group velocity vanishes for a given frequency. This is the signature of a ZGV point; a phenomenon which has been previously observed in rigid plates [43–45]. Here again, the prediction obtained with a 2-D model assuming rigid boundaries convincingly matches the experiment.

Let us now illustrate the rich physics associated to this dispersion diagram by selectively exciting a few modes. To begin with, the source is placed in the middle of the strip and shaken vertically at 136 Hz. Such excitation is intrinsically symmetrical and only \( S_1 \) can be excited at this frequency. The chronophotographic sequence displayed on Fig. 4(a) reports twelve successive snapshots of the displacement \( u_1 \) taken over a full period of vibration at 136 Hz. As expected, the field pattern respects the \( S_1 \) symmetry. Also, the zeroes of the field (red dashed lines) move away from the source, which corresponds to diverging waves.

On either side of the strip, there are two solutions with identical profiles but opposite phase velocities: in other words they are time-reversed partners. Thus, the bottom part of the strip hosts the solution \( S_1 \) while its top part supports \( S_1^* \). Furthermore, the transverse field \( u_2 \) (not shown here, see video in Supp. Mat.) is in phase quadrature compared to \( u_1 \) at this frequency. This essentially suggests that the in-plane displacement is circularly polarized; an interesting feature since such a polarization is known to flip under a time-reversal operation. One can easily take advantage of this feature by imposing a chiral excitation. To this end, we use a source made of two counter-rotating clamps at equal distances from the centre of the strip (this is realised by driving two distinct clamps with 4 different speakers connected to a Presonus AudioBox 44VSL soundboard). As depicted
in Fig. 4(b), the wave propagating towards \( x_1 > 0 \) (namely \( S_1 \)) is unaffected, however, the solution propagating in the opposite direction \( (S_1^*) \) vanishes. By controlling the source chirality, we performed selective feeding and one-way wave transport, a feature which has recently stimulated the research of classical analogues to the quantum hall effect \([46-49]\).

One can also try to capture the strip behaviour near the ZGV point. As it is associated with an anti-symmetrical motion, the system is shaken horizontally by two clamps driven simultaneously at 102 Hz, and the field displacement \( u_2 \) over a full cycle is represented in Fig. 4(c). It exhibits a very unique property: still standing zeroes denote the stationary character of the solution. This feature is a classical signature of ZGV points. To understand it, let us take a look back at Fig. 3. Causality imposes that \( A_1 \) and \( A_2 \) (filled symbols, solid lines) propagate in the bottom part of the strip while their time partners \( A_1^* \) and \( A_2^* \) (empty symbols, dashed lines) go to the top part. Interestingly at the ZGV frequency (102 Hz), \( A_1 \) and \( A_2 \) (resp. \( A_1^* \) and \( A_2^* \)) have opposite wave numbers and interfere in a standing wave. The principle is comparable to that of a vibrating string with fixed terminations for example. Yet, in the present case, the backward wave does not result from edge’s reflections but rather from the dispersion itself.

Again, introducing some chirality will result in breaking the time-reversal symmetry. The sources are now rotated in an anti-symmetrical manner (see inset) resulting in the measurements reported on Fig. 4(d). The propagative nature of the field is retrieved on both sides: the zeroes of the field are travelling. Note that, on the upper part, the wavefronts are anti-causal, \textit{i.e.} they seem to move towards the source which is typical of a negative phase velocity. Strictly speaking, only \( A_1 \) (resp \( A_2^* \)) remains in the lower part (resp. upper part) of the strip. Thanks to the chiral excitation, we have separated the two contributions of a ZGV point, and highlight their unique nature as a superposition of two modes propagating in opposite directions.

In this article, we report the observation of Dirac-like cones in a soft material in spite of a significant dissipation due to viscous effects. The associated dispersion is also found to induce atypical wave phenomena such as a negative phase velocity and a zero group velocity. For both the Dirichlet and Neumann boundaries, a convincing agreement is found between experiments, our theoretical simplified model and numerical analysis. Additionally, we perform the selective feeding of chiral waves into the strip. These experimental results show how the physics of semi-conductors can echo in a drastically different context. Beyond
the purely wave physics phenomena, the problem of the soft strip may stimulate interest in different domains in a near future. From a material point of view, we show how a very simple platform can provide comprehensive information about the visco-elastic properties of a soft solid leading to new technologies to probe the rheology. From a biological point of view, understanding the complex physics associated with a geometry that is ubiquitous in the human tissues and organs, is a major challenge. Imaging and therapeutic methods based on elastography could benefit from an in-depth understanding of the specific dynamic response of tendons, myocardium or vocal cords among others. Some physiological mechanisms could also be unveiled by accounting for the atypical vibrations of a soft strip. In the inner ear, for instance, the sound transduction is essentially driven by a combination of two soft strips namely the basilar and tectorial membranes. Overall, we might discover that physiology had already tamed the features expected for graphene-based applications.

We thank Sander Wildeman for sharing his DIC algorithm, Gatien Clément for early experimental developments, Jérôme Laurent for sharing his Lamb modes dispersion code and Pascale Arnaud for video edition. This work has been supported by LABEX WIFI (Laboratory of Excellence within the French Program Investments for the Future) under references ANR-10-LABX-24 and ANR-10-IDEX-0001-02 PSL*, and by Agence Nationale de la Recherche under reference ANR-16-CE31-0015.

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