A LOWER BOUND FOR $\chi(O_S)$

VINCENZO DI GENNARO

Abstract. Let $(S, \mathcal{L})$ be a smooth, irreducible, projective, complex surface, polarized by a very ample line bundle $\mathcal{L}$ of degree $d > 25$. In this paper we prove that $\chi(O_S) \geq -\frac{1}{8}d(d - 6)$. The bound is sharp, and $\chi(O_S) = -\frac{1}{8}d(d - 6)$ if and only if $d$ is even, the linear system $|H^0(S, \mathcal{L})|$ embeds $S$ in a smooth rational normal scroll $T \subset \mathbb{P}^5$ of dimension 3, and here, as a divisor, $S$ is linearly equivalent to $\frac{d}{2}Q$, where $Q$ is a quadric on $T$. Moreover, this is equivalent to the fact that the general hyperplane section $H \in |H^0(S, \mathcal{L})|$ of $S$ is the projection of a curve $C$ contained in the Veronese surface $V \subset \mathbb{P}^5$, from a point $x \in V \setminus C$.

Keywords: Projective surface, Castelnuovo-Halphen’s Theory, Rational normal scroll, Veronese surface.

MSC2010: Primary 14J99; Secondary 14M20, 14N15, 51N35.

1. Introduction

In [6], one proves a sharp lower bound for the self-intersection $K_S^2$ of the canonical bundle of a smooth, projective, complex surface $S$, polarized by a very ample line bundle $\mathcal{L}$, in terms of its degree $d = \deg \mathcal{L}$, assuming $d > 35$. Refining the line of the proof in [6], in the present paper we deduce a similar result for the Euler characteristic $\chi(O_S)$ of $S$ [1, p. 2], in the range $d > 25$. More precisely, we prove the following:

Theorem 1.1. Let $(S, \mathcal{L})$ be a smooth, irreducible, projective, complex surface, polarized by a very ample line bundle $\mathcal{L}$ of degree $d > 25$. Then:

$$\chi(O_S) \geq -\frac{1}{8}d(d - 6).$$

The bound is sharp, and the following properties are equivalent.

(i) $\chi(O_S) = -\frac{1}{8}d(d - 6)$;

(ii) $h^0(S, \mathcal{L}) = 6$, and the linear system $|H^0(S, \mathcal{L})|$ embeds $S$ in $\mathbb{P}^5$ as a scroll with sectional genus $g = \frac{1}{8}d(d - 6) + 1$;

(iii) $h^0(S, \mathcal{L}) = 6$, $d$ is even, and the linear system $|H^0(S, \mathcal{L})|$ embeds $S$ in a smooth rational normal scroll $T \subset \mathbb{P}^5$ of dimension 3, and here $S$ is linearly equivalent to $\frac{d}{2}(H_T - W_T)$, where $H_T$ is the hyperplane class of $T$, and $W_T$ the ruling (i.e. $S$ is linearly equivalent to an integer multiple of a smooth quadric $Q \subset T$).
By Enriques’ classification, one knows that if \( S \) is unruled or rational, then \( \chi(O_S) \geq 0 \). Hence, Theorem 1.1 essentially concerns irrational ruled surfaces.

In the range \( d > 35 \), the family of extremal surfaces for \( \chi(O_S) \) is exactly the same for \( K_S^2 \). We point out there is a relationship between this family and the Veronese surface. In fact one has the following:

**Corollary 1.2.** Let \( S \subseteq \mathbb{P}^r \) be a nondegenerate, smooth, irreducible, projective, complex surface, of degree \( d > 25 \). Let \( L \subseteq \mathbb{P}^r \) be a general hyperplane. Then \( \chi(O_S) = -\frac{1}{8}d(d - 6) \) if and only if \( r = 5 \), and there is a curve \( C \) in the Veronese surface \( V \subseteq \mathbb{P}^5 \) and a point \( x \in V \setminus C \) such that the general hyperplane section \( S \cap L \) of \( S \) is the projection \( p_x(C) \subseteq L \) of \( C \) in \( L \sim \mathbb{P}^4 \), from the point \( x \).

In particular, \( S \cap L \) is not linearly normal, instead \( S \) is.

2. Proof of Theorem 1.1

**Remark 2.1.** (i) We say that \( S \subseteq \mathbb{P}^r \) is a *scroll* if \( S \) is a \( \mathbb{P}^1 \)-bundle over a smooth curve, and the restriction of \( O_S(1) \) to a fibre is \( O_{\mathbb{P}^1}(1) \). In particular, \( S \) is a geometrically ruled surface, and therefore \( \chi(O_S) = \frac{1}{8}K_S^2 [1, \text{Proposition III.21}] \).

(ii) By Enriques’ classification \([1, \text{Theorem X.4 and Proposition III.21}]\), one knows that if \( S \) is unruled or rational, then \( \chi(O_S) \geq 0 \), and if \( S \) is ruled with irregularity \( > 0 \), then \( \chi(O_S) \geq \frac{1}{8}K_S^2 \). Therefore, taking into account previous remark, when \( d > 35 \), Theorem 1.1 follows from \([6, \text{Theorem 1.1}]\). In order to examine the range \( 25 < d \leq 35 \), we are going to refine the line of the argument in the proof of \([6, \text{Theorem 1.1}]\).

(iii) When \( d = 2\delta \) is even, then \( \frac{1}{8}d(d - 6) + 1 \) is the genus of a plane curve of degree \( \delta \), and the genus of a curve of degree \( d \) lying on the Veronese surface.

Put \( r + 1 := h^0(S, L) \). Therefore, \( |H^0(S, L)| \) embeds \( S \) in \( \mathbb{P}^r \). Let \( H \subseteq \mathbb{P}^{r-1} \) be the general hyperplane section of \( S \), so that \( L \cong O_S(H) \). We denote by \( g \) the genus of \( H \). If \( 2 \leq r \leq 3 \), then \( \chi(O_S) \geq 1 \). Therefore, we may assume \( r \geq 4 \).

**The case \( r = 4 \).**

We first examine the case \( r = 4 \). In this case we only have to prove that, for \( d > 25 \), one has \( \chi(O_S) > -\frac{1}{4}d(d - 6) \). We may assume that \( S \) is an irrational ruled surface, so \( K_S^2 \leq 8\chi(O_S) \) (compare with previous Remark 2.1 (ii)). We argue by contradiction, and assume also that

\[
(1) \quad \chi(O_S) \leq -\frac{1}{8}d(d - 6).
\]

We are going to prove that this assumption implies \( d \leq 25 \), in contrast with our hypothesis \( d > 25 \).

By the double point formula:

\[
d(d - 5) - 10(g - 1) + 12\chi(O_S) = 2K_S^2,
\]

and \( K_S^2 \leq 8\chi(O_S) \), we get:

\[
d(d - 5) - 10(g - 1) \leq 4\chi(O_S).
\]
And from $\chi(O_S) \leq -\frac{1}{8}d(d-6)$ we obtain
\begin{equation}
10g \geq \frac{3}{2}d^2 - 8d + 10.
\end{equation}

Now we distinguish two cases, according that $S$ is not contained in a hypersurface of degree $< 5$ or not.

First suppose that $S$ is not contained in a hypersurface of $\mathbb{P}^4$ of degree $< 5$. Since $d > 16$, by Roth’s Theorem ([12, p. 152], [8, p. 2, (C)]), $H$ is not contained in a surface of $\mathbb{P}^3$ of degree $< 5$. Using Halphen’s bound [9], we deduce that
\[ g \leq \frac{d^2}{10} + \frac{d}{2} + 1 - \frac{2}{5}(\epsilon + 1)(4 - \epsilon), \]
where $d - 1 = 5m + \epsilon$, $0 \leq \epsilon < 5$. It follows that
\[ \frac{3}{2}d^2 - 8d + 10 \leq 10g \leq \frac{d^2}{4} + 5d + 10 \left( 1 - \frac{2}{5}(\epsilon + 1)(4 - \epsilon) \right). \]
This implies that $d \leq 25$, in contrast with our hypothesis $d > 25$.

In the second case, assume that $S$ is contained in an irreducible and reduced hypersurface of degree $s \leq 4$. When $s \in \{2, 3\}$, one knows that, for $d > 12$, $S$ is of general type [2, p. 213]. Therefore, we only have to examine the case $s = 4$. In this case $H$ is contained in a surface of $\mathbb{P}^3$ of degree 4. Since $d > 12$, by Bezout’s Theorem, $H$ is not contained in a surface of $\mathbb{P}^3$ of degree $< 4$. Using Halphen’s bound [9], and [8, Lemme 1], we get:
\[ g \leq \frac{d^2}{8} + \frac{d}{2} + 1 \leq \frac{d^2}{8} + 1. \]
Hence, there exists a rational number $0 \leq x \leq 9$ such that
\[ g = \frac{d^2}{8} + d \left( \frac{x - 9}{8} \right) + 1. \]
If $0 \leq x \leq \frac{15}{2}$, then $g \leq \frac{d^2}{8} - \frac{3}{16}d + 1$, and from (2) we get
\[ \frac{3}{20}d^2 - \frac{4}{5}d + 1 \leq g \leq \frac{d^2}{8} - \frac{3}{16}d + 1. \]
It follows $d \leq 24$, in contrast with our hypothesis $d > 25$.

Assume $\frac{15}{2} < x \leq 9$. Hence,
\[ \left( \frac{d^2}{8} + 1 \right) - g = -d \left( \frac{x - 9}{8} \right) < \frac{3}{16}d. \]
By [9] proof of Proposition 2, and formula (2.2)], we have
\[ \chi(O_S) \geq 1 + \frac{d^3}{96} - \frac{d^2}{16} - \frac{5d}{3} - \frac{349}{16} - (d - 3) \left[ \left( \frac{d^2}{8} + 1 \right) - g \right] \]
\[ > 1 + \frac{d^3}{96} - \frac{d^2}{16} - \frac{5d}{3} - \frac{349}{16} - (d - 3) \frac{3}{16}d = \frac{d^3}{96} - \frac{d^2}{4} - \frac{53}{48}d - \frac{333}{16}. \]
Combining with (1), we get
\[ \frac{d^3}{96} - \frac{d^2}{4} - \frac{53}{48}d - \frac{333}{16} + \frac{1}{8}d(d - 6) < 0, \]
i.e.
\[ d^3 - 12d^2 - 178d - 1998 < 0. \]
It follows \( d \leq 23 \), in contrast with our hypothesis \( d > 25 \).

This concludes the analysis of the case \( r = 4 \).

**The case \( r \geq 5 \).**

When \( r \geq 5 \), by [3] Remark 2.1, we know that, for \( d > 5 \), one has \( K^2_S > -d(d-6) \), except when \( r = 5 \), and the surface \( S \) is a scroll, \( K^2_S = 8\chi(O_S) = 8(1-g) \), and

\[
g = \frac{1}{8}d^2 - \frac{3}{4}d + \frac{(5-\epsilon)(\epsilon + 1)}{8},
\]

with \( d - 1 = 4m + \epsilon \), \( 0 < \epsilon \leq 3 \). In this case, by [6] pp. 73-76, we know that, for \( d > 25 \), \( S \) is contained in a smooth rational normal scroll of \( \mathbb{P}^5 \) of dimension 3. Taking into account that we may assume \( K^2_S \leq 8\chi(O_S) \) (compare with Remark 2.1 (i) and (ii)), at this point Theorem 1.1 follows from [6] Proposition 2.2, when \( d > 30 \).

In order to examine the remaining cases \( 26 \leq d \leq 30 \), we refine the analysis appearing in [6]. In fact, we are going to prove that, assuming \( r = 5 \), \( S \) is a scroll, and (3), it follows that \( S \) is contained in a smooth rational normal scroll of \( \mathbb{P}^5 \) of dimension 3 also when \( 26 \leq d \leq 30 \). Then we may conclude as before, because [6] Proposition 2.2 holds true for \( d \geq 18 \).

First, observe that if \( S \) is contained in a threefold \( T \subset \mathbb{P}^5 \) of dimension 3 and minimal degree 3, then \( T \) is necessarily a smooth rational normal scroll [6] p. 76]. Moreover, observe that we may apply the same argument as in [3] p. 75-76] in order to exclude the case \( S \) is contained in a threefold of degree 4. In fact the argument works for \( d > 24 \) [6] p. 76, first line after formula (13)].

In conclusion, assuming \( r = 5 \), \( S \) is a scroll, and (3), it remains to exclude that \( S \) is not contained in a threefold of degree \( < 5 \), when \( 26 \leq d \leq 30 \).

Assume \( S \) is not contained in a threefold of degree \( < 5 \). Denote by \( \Gamma \subset \mathbb{P}^3 \) the general hyperplane section of \( H \). Recall that \( 26 \leq d \leq 30 \).

- **Case I:** \( h^0(\mathbb{P}^3, \mathcal{I}_T(2)) \geq 2 \).

It is impossible. In fact, if \( d > 4 \), by monodromy [3] Proposition 2.1, \( \Gamma \) should be contained in a reduced and irreducible space curve of degree \( \leq 4 \), and so, for \( d > 20 \), \( S \) should be contained in a threefold of degree \( \leq 4 \) [3 Theorem (0.2)].

- **Case II:** \( h^0(\mathbb{P}^3, \mathcal{I}_T(2)) = 1 \) and \( h^0(\mathbb{P}^3, \mathcal{I}_T(3)) > 4 \).

As before, if \( d > 6 \), by monodromy, \( \Gamma \) is contained in a reduced and irreducible space curve \( X \) of degree \( \deg(X) \leq 6 \). Again as before, if \( \deg(X) \leq 4 \), then \( S \) is contained in a threefold of degree \( \leq 4 \). So we may assume \( 5 \leq \deg(X) \leq 6 \).

Since \( d \geq 26 \), by Bezout’s Theorem we have \( h_T(i) = h_X(i) \) for all \( i \leq 4 \). Let \( X' \) be the general plane section of \( X \). Since \( h_X(i) \geq \sum_{j=0}^{i} h_X(j) \), we have \( h_X(3) \geq 14 \) and \( h_X(4) \geq 19 \) [7] pp. 81-87]. Therefore, when \( d \geq 25 \), taking into account [7] Corollary (3.5)], we get:

\[
\begin{align*}
    h_T(1) &= 4, \\
    h_T(2) &= 9, \\
    h_T(3) &\geq 14, \\
    h_T(4) &\geq 19, \\
    h_T(5) &\geq 22, \\
    h_T(6) &\geq \min\{d, 27\}, \\
    h_T(7) &= d.
\end{align*}
\]
It follows that:
\[ p_a(C) \leq \sum_{i=1}^{+\infty} d - h_\Gamma(i) \leq (d - 4) + (d - 9) + (d - 14) + (d - 19) + (d - 22) + 3 = 5d - 65, \]
which is \( \frac{1}{8}d(d - 6) + 1 \) for \( d \geq 26 \). This is in contrast with (3).

- **Case III:** \( h^0(\mathbb{P}^3, \mathcal{I}_\Gamma(2)) = 1 \) and \( h^0(\mathbb{P}^3, \mathcal{I}_\Gamma(3)) = 4 \).

We have:
\[ h_\Gamma(1) = 4, \ h_\Gamma(2) = 9, \ h_\Gamma(3) = 16, \ h_\Gamma(4) \geq 19, \ h_\Gamma(5) \geq 24, \ h_\Gamma(6) = d. \]

It follows that:
\[ p_a(C) \leq \sum_{i=1}^{+\infty} d - h_\Gamma(i) \leq (d - 4) + (d - 9) + (d - 16) + (d - 19) + (d - 24) = 5d - 72, \]
which is \( \frac{1}{8}d(d - 6) + 1 \) for \( d \geq 26 \). This is in contrast with (3).

- **Case IV:** \( h^0(\mathbb{P}^3, \mathcal{I}_\Gamma(2)) = 0 \).

We have:
\[ h_\Gamma(1) = 4, \ h_\Gamma(2) = 10, \ h_\Gamma(3) \geq 13, \ h_\Gamma(4) \geq 19, \ h_\Gamma(5) \geq 22, \ h_\Gamma(6) \geq \min\{d, 28\}, \ h_\Gamma(7) = d. \]

It follows that:
\[ p_a(C) \leq \sum_{i=1}^{+\infty} d - h_\Gamma(i) \leq (d - 4) + (d - 10) + (d - 13) + (d - 19) + (d - 22) + 2 = 5d - 66, \]
which is \( \frac{1}{8}d(d - 6) + 1 \) for \( d \geq 26 \). This is in contrast with (3).

This concludes the proof of Theorem 1.1.

**Remark 2.2.** (i) Let \( Q \subseteq \mathbb{P}^3 \) be a smooth quadric, and \( H \in |\mathcal{O}_Q(1, d - 1)| \) be a smooth rational curve of degree \( d \) [11, p. 231, Exercise 5.6]. Let \( S \subseteq \mathbb{P}^4 \) be the projective cone over \( H \). A computation, which we omit, proves that
\[ \chi(\mathcal{O}_S) = 1 - \left( \frac{d - 1}{3} \right). \]

Therefore, if \( S \) is singular, it may happen that \( \chi(\mathcal{O}_S) < -\frac{1}{3}(d - 6) \). One may ask whether \( 1 - \left( \frac{d - 1}{3} \right) \) is a lower bound for \( \chi(\mathcal{O}_S) \) for every integral surface.

(ii) Let \( (S, \mathcal{L}) \) be a smooth surface, polarized by a very ample line bundle \( \mathcal{L} \) of degree \( d \). By Harris’ bound for the geometric genus \( p_g(S) \) of \( S \) [10], we see that \( p_g(S) \leq \left( \frac{d - 1}{3} \right) \). Taking into account that for a smooth surface one has \( \chi(\mathcal{O}_S) = h^0(S, \mathcal{O}_S) - h^1(S, \mathcal{O}_S) + h^2(S, \mathcal{O}_S) \leq 1 + h^2(S, \mathcal{O}_S) = 1 + p_g(S) \), from Theorem 1.1 we deduce (the first inequality only when \( d > 25 \)):
\[ -\left( \frac{d^2 - 1}{2} \right) \leq \chi(\mathcal{O}_S) \leq 1 + \left( \frac{d - 1}{3} \right). \]
3. Proof of Corollary 1.2

- First, assume that $\chi(\mathcal{O}_S) = -\frac{1}{6}d(d-6)$.

By Theorem 1.1, we know that $r = 5$. Moreover, $S$ is contained in a nonsingular threefold $T \subseteq \mathbb{P}^5$ of minimal degree 3. Therefore, the general hyperplane section $H = S \cap L$ of $S$ ($L \cong \mathbb{P}^4$ denotes the general hyperplane of $\mathbb{P}^5$) is contained in a smooth surface $\Sigma = T \cap L$ of $L \cong \mathbb{P}^4$, of minimal degree 3.

This surface $\Sigma$ is isomorphic to the blowing-up of $\mathbb{P}^2$ at a point, and, for a suitable point $x \in V \setminus L$, the projection of $\mathbb{P}^5 \setminus \{x\}$ on $L \cong \mathbb{P}^4$ from $x$ restricts to an isomorphism

$$p_x : V \setminus \{x\} \to \Sigma \setminus E,$$

where $E$ denotes the exceptional line of $\Sigma$ [11 p. 58].

Since $S$ is linearly equivalent on $T$ to $\frac{d}{2}(H_T - W_T)$ ($H_T$ denotes the hyperplane section of $T$, and $W_T$ the ruling), it follows that $H$ is linearly equivalent on $\Sigma$ to $\frac{d}{2}(H_\Sigma - W_\Sigma)$ (now $H_\Sigma$ denotes the hyperplane section of $\Sigma$, and $W_\Sigma$ the ruling of $\Sigma$). Therefore, $H$ does not meet the exceptional line $E = H_\Sigma - 2W_\Sigma$. In fact, since $H_\Sigma^2 = 3$, $H_\Sigma \cdot W_\Sigma = 1$, and $W_\Sigma^2 = 0$, one has:

$$(H_\Sigma - W_\Sigma) \cdot (H_\Sigma - 2W_\Sigma) = H_\Sigma^2 - 3H_\Sigma \cdot W_\Sigma + 2W_\Sigma^2 = 0.$$

This implies that $H$ is contained in $\Sigma \setminus E$, and the assertion of Corollary 1.2 follows.

- Conversely, assume there exists a curve $C$ on the Veronese surface $V \subseteq \mathbb{P}^5$, and a point $x \in V \setminus C$, such that $H$ is the projection $p_x(C)$ of $C$ from the point $x$.

In particular, $d$ is an even number, and $H$ is contained in a smooth surface $\Sigma \subseteq L \cong \mathbb{P}^4$ of minimal degree, and is disjoint from the exceptional line $E \subseteq \Sigma$. By [8 Theorem (0.2)], $S$ is contained in a threefold $T \subseteq \mathbb{P}^5$ of minimal degree. $T$ is nonsingular. In fact, otherwise, $H$ should be a Castelnuovo’s curve in $\mathbb{P}^4$ [6 p. 76]. On the other hand, by our assumption, $H$ is isomorphic to a plane curve of degree $\frac{d}{7}$. Hence, we should have:

$$g = \frac{d^2}{6} - \frac{2}{3}d + 1 = \frac{d^2}{8} - \frac{3}{4}d + 1$$

(the first equality because $H$ is Castelnuovo’s, the latter because $H$ is isomorphic to a plane curve of degree $\frac{d}{7}$). This is impossible when $d > 0$.

Therefore, $S$ is contained in a smooth threefold $T$ of minimal degree in $\mathbb{P}^5$.

Now observe that in $\Sigma$ there are only two families of curves of degree even $d$ and genus $g = \frac{d^2}{7} - \frac{2}{7}d + 1$. These are the curves linearly equivalent on $\Sigma$ to $\frac{d}{2}(H_\Sigma - W_\Sigma)$, and the curves equivalent to $\frac{d+1}{6}H_\Sigma + \frac{d-2}{2}W_\Sigma$. But only in the first family the curves do not meet $E$. Hence, $H$ is linearly equivalent on $\Sigma$ to $\frac{d}{2}(H_\Sigma - W_\Sigma)$. Since the restriction $\text{Pic}(T) \to \text{Pic}(\Sigma)$ is bijective, it follows that $S$ is linearly equivalent on $T$ to $\frac{d}{2}(H_T - W_T)$. By Theorem 1.1 $S$ is a fortiori linearly normal, and of minimal Euler characteristic $\chi(\mathcal{O}_S) = -\frac{1}{6}d(d-6)$. 

A LOWER BOUND FOR $\chi(\mathcal{O}_S)$

REFERENCES

[1] Beauville, A.: Surfaces algébriques complexes, Astérisque 54, société mathématiques de france, 1978.
[2] Braun, R. - Floystad, G.: A bound for the degree of smooth surfaces in $\mathbb{P}^4$ not of general type, Compositio Math., 93(2), 211-229 (1994).
[3] Chiantini, L. - Ciliberto, C.: A few remarks on the lifting problem, Astérisque, 218, 95-109 (1993).
[4] Chiantini, L. - Ciliberto, C. - Di Gennaro, V.: The genus of projective curves, Duke Math. J., 70(2), 229-245 (1993).
[5] Di Gennaro, V.: A note on smooth surfaces in $\mathbb{P}^4$, Geometriae Dedicata, 71, 91-96 (1998).
[6] Di Gennaro, V. - Franco, D.: A lower bound for $K_S^2$, Rendiconti del Circolo Matemático di Palermo, H. Ser (2017) 66:69-81.
[7] Eisenbud, D. - Harris, J.: Curves in Projective Space, Sém. Math. Sup. 85 Les Presses de l’Université de Montréal, 1982.
[8] Ellingsrud, G. - Peskine, Ch.: Sur les surfaces lisses de $\mathbb{P}_4$, Invent. Math., 95, 1-11 (1989).
[9] Gruson, L. - Peskine, Ch. Genre des courbes dans l’espace projectif, Algebraic Geometry: Proceedings, Norway, 1977, Lecture Notes in Math., Springer-Verlag, New York 687, 31-59 (1978).
[10] Harris, J.: A Bound on the Geometric Genus of Projective Varieties, Ann. Scuola Norm. Sup. Pisa Cl. Sci., 8(4), 35-68 (1981).
[11] Hartshorne, R.: Algebraic Geometry, GTM, 52, Springer-Verlag, 1983.
[12] Roth, L.: On the projective classification of surfaces, Proc. London Math. Soc., 42, 142-170 (1937).

Università di Roma "Tor Vergata", Dipartimento di Matematica, Via della Ricerca Scientifica, 00133 Roma, Italy.

Email address: digennar@axp.mat.uniroma2.it