INJECTING FINITENESS TO PROVE
COMPLETENESS FOR FINITE LINEAR TEMPORAL LOGIC

ERIC CAMPBELL AND MICHAEL GREENBERG

Abstract. Temporal logics over finite traces are not the same as temporal logics over potentially infinite traces [2, 6, 5]. Roşu first proved completeness for linear temporal logic on finite traces ($LTL_f$) with a novel coinductive axiom [14]. We offer a different proof, with fewer, more conventional axioms. Our proof is a direct adaptation of Kröger and Merz’s Henkin-Hasenjaeger-style proof [11]. The essence of our adaption is that we “inject” finiteness: that is, we alter the proof structure to ensure that models are finite. We aim to present a thorough, accessible proof.

§1. Introduction. Temporal logics have proven useful in a remarkable number of applications, in particular reasoning about reactive systems. To accommodate the nonterminating nature of such systems, temporal logics have used a possibly infinite model of time. For nearly thirty years after Pnueli’s seminal work [13], the prevailing wisdom held that proofs about infinite-trace temporal logics were sound for finite models of time. Researchers have recently overturned that conventional wisdom: some formulae are valid only in finite models (and vice versa) [2, 6, 5].

Having realized that finite temporal logics differ from (possibly) infinite ones, we may wonder: how do these finite temporal logics behave? What are their model and proof theories like? Can we adapt existing metatheoretical techniques from infinite settings, or must we come up with new ones? Reworking the model theory of temporal logics for finite time is an uncomplicated exercise: the standard model is a (possibly infinite) sequence of valuations on primitive propositions; to consider only finite models, simply restrict the model to finite sequences of valuations. The proof theory is more challenging. In practice, it is sufficient to (a) add an axiom indicating that the end of time eventually comes, (b) add an axiom to say what happens when the end of time arrives, and (c) to relax (or strengthen) axioms from the infinite logic that may not hold in finite settings. For an example of (c), consider $LTL_f$. It normally holds that the next modality commutes with implication, i.e., $\circ(\phi \Rightarrow \psi) \Leftrightarrow (\circ \phi \Rightarrow \circ \psi)$, i.e., in the next moment $\phi$ implies $\psi$ iff $\phi$ in the next moment implies $\psi$ in the next moment; in a finite setting, we must relax the if-and-only-if to merely the left-to-right direction.
Once we settle on a set of axioms, what does a proof of deductive completeness look like? We believe that it is possible to adapt existing techniques for infinite temporal logics to finite ones. As evidence, we offer a proof of completeness for linear temporal logic over finite traces ($LTL_f$) with a conventional structure: we define a graph of positive-negative pairs of formulae (PNPs), following Kröger and Merz’s presentation [11]. The only change we make to their construction is that when we prove our satisfiability lemma—the core property relating the PNP graph to provability—we “inject” finiteness, adding a formula that guarantees a finite model.

We claim the following contributions:

- Evidence for the claim that the metatheory for infinite temporal logics readily adapts to finite temporal logics by means of injecting finiteness (Section 2 situates our work; Section 3 explains our model of finite time).
- A proof of deductive completeness for linear temporal logic on finite traces ($LTL_f$; Section 4) with fewer axioms than any prior proof [14].

§2. Related work. Pnueli [13] proved his temporal logic programs to be sound and complete over traces of “discrete systems” which may or may not be finite; Lichtenstein et al. [12] extended LTL with past-time operators and allowed more explicitly for the possibility of finite or infinite traces.

Baier and McIlraith were the first to observe that some formulae are only valid in infinite models, and so $LTL_f$ and other ‘truncated’ finite temporal logics differ from their infinite originals [2]. Roșu [14] offers a translation from $LTL_f$ to LTL that preserves satisfiability of formulae, but makes no claims about the inverse translation. De Giacomo and Vardi showed that satisfiability and validity were PSPACE-complete for these finite logics, relating $LTL_f$ and linear dynamic logic ($LDL_f$) to other logics (potentially infinite LTL, FO[$<$, star-free regular expressions, MSO on finite traces) [6]; later, de Giacomò et al. were able to directly characterize when $LTL_f$ and $LDL_f$ formulae are sensitive to infiniteness [5]. De Giacomo and Vardi have also studied the synthesis problem for our logic of interest [7, 9]. Most recently, D’Antoni and Veanes offered a decision procedure for MSO on finite sequences, but without a deductive completeness result [4].

Roșu [14] was the first to show a deductive completeness result for a finite temporal logic: he showed $LTL_f$ is deductively complete by replacing the induction axiom with a coinduction axiom $\text{COIND}$: if $\vdash \neg \phi \Rightarrow \phi$ then $\vdash \square \phi$. He shows that $\text{COIND}$ is equivalent to the combination of

---

1In Roșu’s paper, empty circles mean “weak next” and filled ones mean “next”, while we follow Kröger and Merz and do the reverse—even when quoting Roșu [11].
the conventional induction axiom Ind (if \( \vdash \phi \Rightarrow \Diamond \phi \) then \( \vdash \phi \Rightarrow \Box \phi \)) axiom and a finiteness axiom Fin, \( \vdash \Diamond \top \). Roșu proves consistency using “maximally consistent” worlds, i.e., in a greatest fixed-point style.

Our goal is to show that existing, conventional methods for infinite temporal logics suffice for proving that finite temporal logics are deductively complete. For LTL\(_f\), we take the conventional inductive framing, extending Kröger and Merz’s axioms with the axiom \( \vdash \Diamond \top \), i.e., \( \Diamond \text{end} \) (we call this axiom Finite). Surprisingly, we are able to prove completeness with only six temporal axioms—one fewer than Roșu’s seven, though he conjectures his set is minimal. It turns out that some of his axioms are consequences of others—his necessitation axiom \( \Box \phi \) can be proved from \( \Box \top \) and coInd (we use Induction and WkNextStep, our equivalent of \( \Box \top \), in Lemma 13). Our results for LTL\(_f\) show that a smaller axiom set exists. In fact, we could go still smaller: using Roșu’s proofs, we can replace Finite and Induction with coInd, for only five axioms! Our proof offers a separate contribution, beyond shrinking the number of axioms needed and giving a thorough, accessible presentation: we follow Kröger and Merz’s least fixed-point construction quite closely, adapting their proof from LTL to LTL\(_f\) by injecting finiteness. The key idea is that existing techniques for infinite systems readily adapt to finite ones: we can reuse model theory which uses potentially infinite models so long as we can force the theory to work exclusively with finite models.

2.1. Applications. For de Giacomo and Vardi, LTL\(_f\) is useful for AI planning applications [6, 5, 7, 9]. The second author first encountered finite temporal logics when designing Temporal NetKAT [3]. NetKAT is a specification language for network configurations [1] based on Kleene algebra with tests [10]. Temporal NetKAT extends NetKAT with the ability to write and analyze policies using past-time finite linear temporal logic, e.g., a packet may not arrive at the server unless it has previously been at the firewall. Our interest in the deductive completeness of LTL\(_f\) comes directly from the Temporal NetKAT work: the completeness result for Temporal NetKAT’s equivalence relation relies on deductive completeness for LTL\(_f\).

§3. Modeling finite time. Our logic, LTL\(_f\), uses a finite model of time: traces. A trace over a fixed set of propositional variables is a (possibly infinite) sequence \((\eta_1, \eta_2, \ldots)\) where \(\eta_i\) is a valuation, i.e., a function establishing the truth value (t or f) for each propositional variable. We refer to each valuation as a ‘state’, with the intuition that each valuation represents a discrete moment in time.

Definition 1 (Valuations and Kripke structures). Given a set of variables \(\text{Var}\), a valuation is a function \(\eta : \text{Var} \to \{t, f\}\). A Kripke structure or
a trace is a finite, non-empty sequence of valuations; we write $K^n \in \text{Model}_n$ to refer to a model with $n \in \mathbb{N}^+$ valuations, i.e., $K^n = (\eta_1, \ldots, \eta_n)$.

We particularly emphasize the finiteness of our Kripke structures, writing $K^n$ and explicitly stating the number of valuations as a superscript each time. The number $n$ is not directly accessible in our logic, though the size of models is observable (e.g., the LTL$_f$ formula $\circ \circ \circ \top$ is satisfiable only in models with four or more steps). Our traces are not only finite, but they are necessarily non-empty—all LTL$_f$ formulae would trivially hold in empty models.

Suppose we have $\text{Var} = \{x, y, z\}$. As a first example, the smallest possible model is one with only one time step, $K^1 = \eta_1$, where $\eta_1$ is a function from $\text{Var}$ to the booleans, i.e., a subset of $\text{Var}$. As a more complex example, consider the following model $K^4$ with four time steps:

$$K^4 = \begin{pmatrix}
\{x\} & \{x, y\} & \emptyset & \{x, y, z\} \\
\eta_1 & \eta_2 & \eta_3 & \eta_4
\end{pmatrix}$$

In the first state, $x$ holds but $y$ and $z$ do not (i.e. $\eta_1(x) = t$, but $\eta_1(y) = \eta_1(z) = f$); then $x$ and $y$ hold; then no propositions hold; and then all primitive propositions hold.

Our logic uses Kripke structures to interpret formulae, defining a function $K^n_i : \text{Formula} \to \{t, f\}$. (Put another way: we define a function $\text{interp} : \text{Model}_n \times \{1, \ldots, n\} \times \text{Formula} \to \{t, f\}$, writing $K^n_i(\phi)$ for $\text{interp}(K^n, i, \phi)$.) We lift this interpretation function to define validity and satisfiability.

**Definition 2** (Semantic satisfiability and validity). For an interpretation function $K^n_i : \text{Formula} \to \{t, f\}$, we say for $\phi \in \text{Formula}$:

- $K^n$ models $\phi$ iff $K^n_i(\phi) = t$;
- $\phi$ is satisfiable iff $\exists K^n$ such that $K^n$ models $\phi$;
- $K^n \models \phi$ (pronounced "$K^n$ satisfies $\phi$") iff $\forall 1 \leq i \leq n$, $K^n_i(\phi) = t$;
- $\models \phi$ (pronounced "$\phi$ is valid") iff $\forall K^n$, $K^n \models \phi$; and
- $\mathcal{F} \models \phi$ for $\mathcal{F} \subseteq \text{Formula}$ (pronounced "$\phi$ is valid under $\mathcal{F}$") iff $\forall K^n$, if $\forall \psi \in \mathcal{F}$, $K^n \models \psi$ then $K^n \models \phi$.

§4. LTL$_f$: linear temporal logic on finite traces. Linear temporal logic is a classical logic for reasoning on potentially infinite traces. The syntax of linear temporal logic on finite traces (LTL$_f$) is identical to that of its (potentially) infinite counterpart. We define LTL$_f$ as a propositional logic with two temporal operators (Figure[1]). The propositional fragment comprises: variables $v$ from some fixed set of propositional variables $\text{Var}$; the false proposition, $\bot$; and implication, $\phi \Rightarrow \psi$. The temporal fragment comprises two operators: the next modality, written $\circ \phi$, which means that $\phi$ holds in the next moment of time; and, weak until, written $\phi \mathcal{W} \psi$, which means that $\phi$ holds until either (a) the end of time, or (b) $\psi$ holds.
Injecting Finiteness for LTLF Completeness

Syntax

Variables \( v \in \text{Var} \)

LTL\(_f\) formulae \( \phi, \psi \in \text{LTL}_f \) ::= \( v \mid \bot \mid \phi \Rightarrow \psi \mid \circ \phi \mid \phi \mathcal{W} \psi \)

Encodings

\(-\phi = \phi \Rightarrow \bot \)

\(\phi \land \psi = \neg(\neg \phi \lor \neg \psi)\)

\(\text{end} = \neg \circ \top \)

\(\Box \phi = \phi \mathcal{W} \bot \)

\(\Diamond \phi = \neg \Box \neg \phi \)

\(\phi \mathcal{U} \psi = \phi \mathcal{W} \psi \land \Diamond \psi\)

Semantics

\(K^n_i(v) = \eta_i(v)\) (1)

\(K^n_i(\bot) = \top\) (2)

\(K^n_i(\phi \Rightarrow \psi) = \begin{cases} \top & K^n_i(\phi) = \top \text{ or } K^n_i(\psi) = \top \\ \bot & \text{otherwise} \end{cases}\) (3)

\(K^n_i(\circ \phi) = \begin{cases} K^n_{i+1}(\phi) & i < n \\ \bot & i = n \end{cases}\) (4)

\(K^n_i(\phi \mathcal{W} \psi) = \begin{cases} \top & \forall i \leq j \leq n, \ K^n_j(\phi) = \top \text{ or } \\ \exists i \leq k \leq n, \ K^n_k(\psi) = \top \text{ and } \\ \forall i \leq j < k, \ K^n_j(\phi) = \top \\ \bot & \text{otherwise} \end{cases}\) (5)

Figure 1. LTL\(_f\) syntax and semantics

These core logical operators encode a more conventional looking logic (Figure 1), with the usual logical operators and an enriched set of temporal operators. Of these standard encodings, we remark on two in particular: end, the end of time, and \(\circ \phi\), the weak next modality. In the usual (potentially infinite) semantics, it is generally the case that \(\circ \top\) holds, i.e., that the true proposition holds in the next state, i.e., that there is a next state. But at the end of time, there is no next state—and so \(\circ \top\) ought not adhere. In every state but the last, we have \(\circ \top\) as usual. We can therefore define end = \(\neg \circ \top\)—if end holds, then we must be at the end of time. It’s worth noting here that negation does not generally commute with the next modality\(^2\) observe that \(\neg \circ \top\) holds only at the end of time, but \(\circ \neg \top\) holds nowhere. In fact, \(\neg \circ \neg \phi\) holds at the end of time for every possible \(\phi\), and everywhere else, \(\neg\) and \(\circ\) commute, i.e. \(\neg \circ \neg \phi\) holds if and only if \(\circ \neg \phi\) holds. Bearing these facts in mind, we define the weak

\(^2\)This is not true in the possibly-infinite semantics, where \(\models \neg \circ \phi \Leftrightarrow \circ \neg \phi\)
Axioms

- all propositional tautologies

Taut

\[ \vdash \phi \Rightarrow \psi \iff (\bullet \phi \Rightarrow \bullet \psi) \]  

WkNextDistr

\[ \vdash \text{end} \Rightarrow \neg \circ \phi \]  

EndNextContra

\[ \vdash \Diamond \text{end} \]  

Finite

\[ \vdash \phi \mathcal{W} \psi \iff (\phi \land \bullet (\phi \mathcal{W} \psi)) \]  

WkUntilUnroll

\[ \vdash \phi \Rightarrow \psi \; \vdash \phi \Rightarrow \bullet \phi \]  

WkNextStep

\[ \vdash \phi \Rightarrow \square \psi \]  

Induction

Consequences

\[ \vdash \neg (\circ \top \land \circ \bot) \]  

Lemma 6

\[ \vdash \neg \circ \phi \iff \text{end} \lor \circ \neg \phi \]  

Lemma 7

\[ \vdash \bullet \phi \iff \circ \phi \lor \text{end} \]  

Lemma 8

\[ \vdash \neg \text{end} \land \bullet \neg \phi \Rightarrow \neg \bullet \phi \]  

Lemma 9

\[ \vdash (\bullet \phi \land \psi) \iff \bullet \phi \land \bullet \psi \]  

Lemma 10

\[ \vdash \neg \bullet \phi \Rightarrow \bullet \neg \phi \]  

Lemma 11

\[ \vdash \square \phi \iff \phi \land \square \phi \]  

Lemma 12

\[ \vdash \phi \]  

Lemma 13

\[ \mathcal{F} \vdash \phi \text{ iff assuming } \vdash \psi \text{ for each } \psi \in \mathcal{F} \text{ we have } \vdash \phi \]

Figure 2. LTL$_f$ proof theory

next modality as $\bullet \phi = \neg \circ \neg \phi$. Weak next ($\bullet \phi$) is insensitive to the end of time and strong next ($\circ \phi$) is sensitive to the end of time. To realize these intuitions, we must define our model.

The simple, standard model for LTL is a possibly-infinite trace; we restrict ourselves to finite traces (Definition 1). Given a Kripke structure $K^n$, we assign a truth value to a proposition $\phi$ at time step $1 \leq i \leq n$ with the function $K^n_i(\phi)$, defined as a fixpoint on formulae. The definitions for $K^n_i$ in the propositional fragment are straightforward implementations of the conventional operations. The definitions for $K^n_i$ in the temporal fragment also assign the usual meanings, being mindful of the end of time. When there is no next state, the formula $\circ \phi$ is necessarily false; when there is no next state, the formula $\phi \mathcal{W} \psi$ degenerates into $\phi \lor \psi$.

Why? Suppose we are at the end of time; one of two cases adheres. Either we have $\phi$ until the end of time (which is now!), or we have $\psi$ and have satisfied the until. We can verify our earlier intuitions about end and $\bullet \phi$. Observe that $K^n_i(\text{end}) = t$ exactly when $i = n$; similarly, $K^n_i(\bullet \top) = t$ for all $1 \leq i \leq n$. 
By way of example, consider \( K^4 \) from Section 3. We have \( K^4 \models y \Rightarrow x \), because \( K^4_i (y \Rightarrow x) = t \) for all \( i \), i.e., whenever \( y \) holds, so does \( x \).

Similarly, \( K^4 \) models \( x \lor y \) with the \( k \) in the existential equal to 2; we have \( K^4 \) models \( z \lor x \), too, but trivially with \( k = 1 \). The formula \( \Box z \) doesn’t hold in any state of \( K^4 \), but \( K^4 \models \Diamond z \). We prove a semantic deduction theorem appropriate to our setting: rather than producing a bare implication, deduction produces an implication whose premise is under an ‘always’ modality.

**Theorem 3 (Semantic deduction).** \( F \cup \{ \phi \} \models \psi \) iff \( F \vdash \Box \phi \Rightarrow \psi \).

**Proof.** We prove each direction separately. From left to right, suppose \( F \cup \{ \phi \} \models \psi \). Let \( K^n \) be given such that \( K^n \models \chi \) for all \( \chi \in F \). We show that \( K^n (\Box \phi \Rightarrow \psi) \) for all \( i \).

Let an \( i \) be given. If \( K^n_i (\Box \phi) = t \), we are done immediately—so suppose \( K^n_j (\Box \phi) = t \) for all \( j \geq i \). It remains to be seen that \( K^n (\psi) = t \) for all \( j \geq i \).

Let \( j \) be given. We can extract a smaller Kripke structure from \( K^n \); call it \( K^{m-i} = (\eta_i, \ldots, \eta_{m-i}) \), noting that \( K^{m-i}_k = K^n_{k-1+i} \) for all \( 1 \leq k \leq m - i \).

Then, our assumption that \( K^n \models \chi \) for all \( \chi \in F \cup \{ \phi \} \) implies \( K^{m-i} \models \chi \) for all \( \chi \in F \cup \phi \). We already assumed that \( F \vdash \{ \phi \} \models \psi \), so we can conclude that \( K^{m-i} \models \psi \). Hence, \( K^{m-i} \) assigns \( t \) to \( \Box \phi \Rightarrow \psi \), and so \( K^n (\Box \phi \Rightarrow \psi) = t \).

From right to left, suppose \( F \models \Box \phi \Rightarrow \psi \). Let \( K^n \) be given such that \( K^n \models \chi \) for all \( \chi \in F \cup \{ \phi \} \). We must show that \( K^n (\psi) \) for all \( i \). Since \( K^n \models \chi \) for all \( \chi \in F \), then \( K^n_i (\Box \phi \Rightarrow \psi) = t \) by assumption. Furthermore, we know that \( K^n \models \phi \), i.e., \( K^n_i (\phi) = t \) for all \( i \leq j \leq n \).

Then \( K^n (\Box \phi) = t \) by definition, so the implication in the assumption cannot hold vacuously: so \( K^n (\psi) = t \) as desired.

For our axioms (Figure 2), we adapt Kröger and Merz’s presentation [11]. Two axioms are new: **FINITE** says that time will eventually end; **EndNextContra** says that at the end of time, there is no next state. Other axioms are lightly adapted: wherever one would ordinarily use the (strong) next modality, \( \circ \phi \), we instead use weak next, \( \bullet \phi \). Using strong next would be unsound in finite models. We can, however, characterize the relationship between the next modality, negation, and the end of time (“Consequences” in Figure 2 and Section 4.2).

Roșu proves completeness with a slightly different set of axioms, replacing **FINITE** and **INDUCTION** with a single coinduction axiom he calls **COIND**:

\[
\vdash \bullet \phi \Rightarrow \phi \\
\vdash \Box \phi
\]

He proves that **COIND** is equivalent to the conjunction of **FINITE** and **INDUCTION**, so it does not particularly matter which axioms we choose.
In order to emphasize how little must change to make our logic finite, we keep our presentation as close to Kröger and Merz’s as possible.

4.1. Soundness. Proving that our axioms are sound is, as usual, relatively straightforward: we verify each axiom in turn.

Theorem 4 (LTLₚ soundness). If ⊢ φ then ⊨ φ.
Proof. By induction on the derivation of ⊢ φ. Our proof refers to the various cases in the definition of the model (Figure 1).

(TAUT) As for propositional logic.
(WKNextDistr) We have ⊢ •(φ ⇒ ψ) ⇔ (•φ ⇒ •ψ). To show validity in the model, let Kⁿ be given. We show that Kⁿ assigns true to the left-hand side iff it assigns true to the right-hand side.

\[ K^n \models •(φ ⇒ ψ) \]
iff \[ ∀1 \leq i \leq n, \ K^i(•(φ ⇒ ψ)) = t \]
iff \[ ∀1 \leq i \leq n − 1, \ K^{i+1}(φ ⇒ ψ) = t \]
iff \[ ∀1 \leq i \leq n − 1, \ K^{i+1}(φ) = f \text{ or } K^{i+1}(ψ) = t \]
iff \[ ∀1 \leq i \leq n, \ K^i(•φ) = f \text{ or } K^i(•ψ) = t \]
where K^i(•ψ) = t trivially
iff \[ ∀1 \leq i \leq n, \ K^i(φ ⇒ ψ) \]
iff \[ K^n \models •φ ⇒ •ψ \]

By unfolding the encodings of logical operators, we can derive that
\[ K^n \models •(φ ⇒ ψ) ⇔ (•φ ⇒ •ψ). \]

(ENDNextContra) We have ⊢ end ⇒ ¬ ◦ φ; let Kⁿ be given to show Kⁿ |− f end ⇒ ¬ ◦ φ, i.e., that Kⁿ assigns t to the given formula at each 1 ≤ i ≤ n. Let i be given.

We have Kⁿ end ⇒ ¬ ◦ φ. There are two cases: i < n and i = n. When i < n, we have Kⁿ end = Kⁿ (¬ ◦ T). Since i < n, then Kⁿ (◦ T) = Kⁿ+1 T = Kⁿ+1 (¬ ⊥) = t, and so Kⁿ end = f. The implication is thus vacuous: Kⁿ (end ⇒ ¬ ◦ φ) = t, by the first clause of case [3].

When i = n, we have Kⁿ (◦ φ) = f, so Kⁿ (¬ ◦ φ) = t. Therefore Kⁿ end ⇒ ¬ ◦ φ = t, by the second clause of case [3].

(Finite) We have ⊢ ◦ end; let Kⁿ be given to show Kⁿ |− ◦ end, i.e., that for all 1 ≤ i ≤ n, we have Kⁿ( ◦ end) = t. Let i be given.

Unfolding our encodings, we must show that:
\[ K^n( ◦ end) \]
= Kⁿ (¬ ◦ end) = Kⁿ (¬ (¬ end W ⊥)) = t

By cases [2] and [3], it suffices to show that Kⁿ (¬ end W ⊥) = f. By case [3], there are two ways for the weak-until to be assigned true; we show that neither adheres. First, observe that Kⁿ ⊥ = f for all 1 ≤ k ≤ n, so there is no k to satisfy the second clause of case [3].

---

[3] They use a slightly less-expressive logic, omitting W and U. We extend their methodology to include these operators.
Next, observe that when \( j = n \), where \( K^n_{\text{end}} = \mathsf{f} \), so the first clause of case (5) cannot be satisfied. Since neither case holds, we find \( K^n_{\text{end}} W \bot = \mathsf{f} \).

(WkUntilUnroll) We have \( \vdash \phi W \psi \iff \psi \lor (\phi \land (\mathsf{f} W \psi)) \); let \( K^n \) be given to show that \( K^n \models \phi W \psi \iff \psi \lor (\phi \land (\mathsf{f} W \psi)) \), i.e., that for all \( 1 \leq i \leq n \), the left-hand side of our formula is assigned true by \( K^n_i \) iff the right-hand side is. Let an \( i \) be given; we prove each side independently.

From left to right, we have \( K^n_i (\phi W \psi) = \mathsf{t} \) iff \( \forall i \leq j \leq n \), \( K^n_j (\phi) = \mathsf{t} \) or \( \exists i < k \leq n \), \( K^n_k (\phi) = \mathsf{t} \) and \( \forall i < j < k \), \( K^n_j (\phi) = \mathsf{t} \). We go by cases. If \( \phi \) always holds, then \( K^n_i (\phi) = \mathsf{t} \) and \( K^n_{i+1} (\phi W \psi) = \mathsf{t} \), so \( K^n_i (\psi \lor (\phi \lor (\mathsf{f} W \psi))) = \mathsf{t} \). If, on the other hand, \( \phi \) holds until \( \psi \) eventually holds, we ask: \( i = k \)? If so, then \( K^n_i (\psi) = \mathsf{t} \) implies \( K^n_i (\psi \lor (\phi \lor (\mathsf{f} W \psi))) = \mathsf{t} \). If not, then \( K^n_{i+1} (\phi W \psi) \) holds by the second clause of (5), using our same \( k \). Therefore, \( K^n_i (\phi W \psi) = \mathsf{t} \) along with \( K^n_i (\phi) = \mathsf{t} \) (since \( j \) can be \( i \)), so \( K^n_i (\psi \lor (\phi \lor (\mathsf{f} W \psi))) = \mathsf{t} \).

From right to left, we have \( K^n_i (\psi \lor (\phi \lor (\mathsf{f} W \psi))) = \mathsf{t} \); we must show \( K^n_i (\phi W \psi) = \mathsf{t} \). We go by cases on which side of the disjunction holds. If \( K^n_i (\psi) = \mathsf{t} \), then \( k = i \) witnesses the second clause of case (5) with \( k = i \). If \( K^n_i (\phi) = \mathsf{t} \), \( K^n_{i+1} (\phi W \psi) = \mathsf{t} \). Then we ask: \( i = n \)? If so, we are done by the first clause of case (5). If \( i < n \), then \( K^n_{i+1} (\phi W \psi) = \mathsf{t} \). If it holds because \( \forall i + 1 \leq j \leq n \), \( K^n_j (\phi) = \mathsf{t} \), then \( K^n_i (\phi) = \mathsf{t} \) completes first clause of case (5). Otherwise, \( K^n_{i+1} (\phi W \psi) = \mathsf{t} \), because there is some \( i + 1 \leq k \leq n \) such that \( K^n_k (\psi) = \mathsf{t} \) and \( \forall i + 1 \leq j < k \), \( K^n_j (\phi) = \mathsf{t} \). Since we also have \( K^n_i (\phi) = \mathsf{t} \), \( k \) witnesses the second clause of case (5).

(WkNextStep) We have \( \vdash \mathsf{f} \); as our IH on \( \vdash \phi \), we have \( \models \phi \), i.e., \( \forall K^n \forall 1 \leq i \leq n \), \( K^n_i (\phi) = \mathsf{t} \). Let a \( K^n \) be given to show \( K^n \models \mathsf{f} \), i.e., \( \forall 1 \leq i \leq n \), \( K^n_i (\phi) = \mathsf{t} \). Let \( i \) be given. If \( i = n \), then \( K^n_{i+1} (\mathsf{f} \phi) = \mathsf{f} \), so \( K^n_i (\mathsf{f} \phi) = K^n_i (\neg \neg \neg \phi) = \mathsf{t} \). If \( i < n \), then by definition (4), \( K^n_i (\neg \neg \phi) = K^n_{i+1} (\neg \neg \phi) \). By the IH, we know that \( K^n_{i+1} (\phi) = \mathsf{t} \), so \( K^n_i (\neg \neg \phi) = \mathsf{f} \), and \( K^n_i (\neg \neg \phi) = \mathsf{f} \). Therefore \( K^n_i (\phi) = K^n_i (\neg \neg \neg \phi) = \mathsf{t} \).

(Induction) We have \( \vdash \phi \Rightarrow \Box \psi \); as our IHs, we have (1) \( \models \phi \Rightarrow \psi \) and (2) \( \models \phi \Rightarrow \mathsf{f} \), i.e., every Kripke structure \( K^n \) assigns \( \mathsf{t} \) to those formulae at every index. Let \( K^n \) be given to show that \( \forall 1 \leq i \leq n \), \( K^n_i (\phi \Rightarrow \Box \psi) = \mathsf{t} \). If \( K^n_i (\phi) = \mathsf{f} \), the implication vacuously holds; instead consider the case where \( K^n_i (\phi) = \mathsf{t} \). Let \( k = n - i \); we go by induction on \( k \) to show \( K^n_k (\Box \psi) = \mathsf{t} \). When \( k = 0 \), it must be the case that \( n = i \). We have \( K^n_i (\phi) = \mathsf{t} \); by outer IH (1), it must be that \( K^n_i (\psi) = \mathsf{t} \), which means that \( K^n_k (\psi) = \mathsf{t} \) by the first clause of case (5).
When $k = k' + 1$, we have here $i < n$. We must show that $K^n_i(\square \psi) = t$. Since $K^n_i(\phi) = t$, the outer IHs immediately give $K^n_i(\psi) \models_{IH} (1)$ $t$ (outer IH on $\vdash \phi \Rightarrow \psi$) and $K^n_i(\bullet \phi) = t$ (outer IH on $\vdash \phi \Rightarrow \bullet \phi$). Furthermore, we have $i < n$; unfolding $\bullet \phi$ gives $K^n_{i+1}(\neg \phi) = f$, or equivalently $K^n_{i+1}(\phi) = t$. So $K^n_{i+1}(\square \psi) = K^n_{i+1}(\psi W \perp) = t$ by the inner IH. Since $K^n_{i+1} \models \neg \perp$, the above conclusion that $K^n_{i+1}(\psi W \perp) = t$ must hold because $\forall i + 1 \leq j \leq n$, $K^n_j(\psi) = t$ (as per case (5)). Since $K^n_i(\psi) = t$ as well, we have $\forall i \leq j \leq n K^n_j(\psi) = t$; therefore $K^n_i(\square \psi) = K^n_i(\psi W \perp) = t$. \qed

We can also prove a deduction theorem for our proof theory analogous to Theorem 4.

**Theorem 5** (Deduction). $\mathcal{F} \cup \{ \phi \} \vdash \psi$ iff $\mathcal{F} \vdash \square \phi \Rightarrow \psi$.

**Proof.** From left to right, by induction on the derivation:

- $(\psi$ an axiom or $\psi \in \mathcal{F}$) We have $\mathcal{F} \vdash \square \phi \Rightarrow \psi$ by Taut.
- $(\text{WkNextStep})$ We have $\psi = \bullet \chi$. WkNextStep concludes $\bullet \chi$ from $\mathcal{F} \cup \{ \phi \} \vdash \chi$, which gives the IH of $\mathcal{F} \vdash \square \phi \Rightarrow \chi$. Applying WkNextStep to the IH, we have $\mathcal{F} \vdash \bullet(\square \phi \Rightarrow \chi)$; by WkNextDistr, we have $\mathcal{F} \vdash \bullet \square \phi \Rightarrow \bullet \chi$. By Lemma 12, we know that $\vdash \square \phi \Rightarrow \phi \land \bullet \square \phi$, so by Taut we have $\mathcal{F} \vdash \square \phi \Rightarrow \psi$.
- $(\text{Induction})$ We have $\psi = \chi \Rightarrow \square \rho$, with $\mathcal{F} \cup \{ \phi \} \vdash \chi \Rightarrow \rho$ and $\mathcal{F} \cup \{ \phi \} \vdash \chi \Rightarrow \bullet \chi$. By the IH, we know that $\mathcal{F} \vdash \square \phi \Rightarrow \chi \Rightarrow \rho$ and $\mathcal{F} \vdash \square \phi \Rightarrow \chi \Rightarrow \bullet \chi$. By Taut, we have $\mathcal{F} \vdash \square \phi \land \chi \Rightarrow \rho$ and $\mathcal{F} \vdash \square \phi \land \chi \Rightarrow \bullet \chi$, so by Induction we have $\mathcal{F} \vdash \square \phi \land \chi \Rightarrow \square \rho$. By Taut, we find $\mathcal{F} \vdash \square \phi \Rightarrow \chi \Rightarrow \square \rho$.

From right to left, we have $\mathcal{F} \vdash \square \phi \Rightarrow \psi$ and must show $\mathcal{F} \cup \{ \phi \} \vdash \psi$. By Taut, we have $\mathcal{F} \cup \{ \phi \} \vdash \square \phi \Rightarrow \psi$. We must prove that $\mathcal{F} \cup \{ \phi \} \vdash \square \phi$, which gives $\mathcal{F} \cup \{ \phi \} \vdash \psi$ via Taut.

By WkNextStep and Taut, we know that $\mathcal{F} \cup \{ \phi \} \vdash \phi \Rightarrow \bullet \phi$. We therefore have by induction that $\mathcal{F} \cup \{ \phi \} \vdash \phi \Rightarrow \square \phi$. By Taut, we can conclude $\mathcal{F} \cup \{ \phi \} \vdash \phi$ and subsequently $\mathcal{F} \cup \{ \phi \} \vdash \square \phi$. \qed

### 4.2. Consequences.

Before proceeding to the proof of completeness, we prove a variety of properties in LTL$_f$ necessary for the proof: characterizations of the modality (Lemmas 6, 8, and 12) and distributivity over connectives (Lemmas 9, 13). We also derive Rosu’s necessitation axiom, $N_{\square}$ (Lemma 14).

Together Lemma 9 and Lemma 14 completely characterize the relationship between the weak next modality and negation: we can pull a negation out of a weak next modality when not at the end; we can push a negation in whether or not the end has arrived.

**Lemma 6** (Modal consistency). $\vdash \neg(\circ T \land \circ \perp)$
Proof. Suppose for a contradiction that \( \vdash \circ \top \land \circ \bot \). We have \( \vdash \top \) by Taut, so \( \vdash \bullet \top \) by WkNextStep. But \( \bullet \top \) desugars to \( \neg \circ \neg \top \), i.e., \( \neg \circ \bot \) -- a contradiction.

Lemma 7 (Negation of next). \( \vdash \neg \circ \phi \iff \text{end} \lor \circ \neg \phi \)

Proof. From left to right, we have \( \vdash \text{end} \lor \neg \text{end} \) by the law of the excluded middle (Taut). If \( \text{end} \) holds, we are done. Otherwise, suppose \( \vdash \neg \circ \phi \land \neg \text{end} \); we must show \( \vdash \circ \neg \phi \). By resugaring and Taut, we have \( \vdash \bullet \neg \phi \); by WkNextDistr, we have \( \vdash \neg \circ \phi \). By desugaring and Taut, we have \( \vdash \circ \neg \phi \).

From right to left, the law of the excluded middle yields \( \vdash \text{end} \lor \neg \text{end} \) (Taut). If \( \text{end} \) holds, then we have \( \vdash \neg \circ \phi \) by EndNextContra immediately. So we have \( \vdash \neg \text{end} \land \circ \neg \phi \) and we must show \( \vdash \circ \neg \phi \). By resugaring and Taut, we have \( \vdash \bullet \neg \phi \). By desugaring and Taut, we have \( \vdash \circ \neg \phi \) as desired.

Lemma 8 (Weak next/nex equivalence). \( \vdash \bullet \phi \iff \circ \phi \lor \text{end} \)

Proof. From left to right, suppose \( \vdash \bullet \phi \). \( \bullet \phi \) desugars to \( \neg \circ \neg \phi \). By Lemma 7, we have \( \text{end} \lor \circ \neg \phi \). If \( \text{end} \) holds, we are done immediately by Taut. Otherwise, we have \( \circ \neg \phi \), which gives us \( \circ \phi \) by Taut, as well.

From right to left, suppose \( \vdash \circ \phi \lor \text{end} \). By the law of the excluded middle, we have \( \text{end} \lor \neg \text{end} \) (Taut). If \( \text{end} \) holds, then we are done immediately by EndNextContra. If \( \circ \phi \land \neg \text{end} \) holds, then we can show \( \vdash \bullet \phi \) by desugaring to \( \circ \neg \phi \Rightarrow \bot \). Suppose for a contradiction that \( \circ \neg \phi \). We have \( \circ \phi \) and \( \circ \neg \phi \), so \( \circ \bot \). But from \( \neg \text{end} \), we have \( \circ \top \) -- and by Lemma 6 we have a contradiction.

Lemma 9 (Weak next negation before the end).

\[ \vdash \neg \text{end} \land \bullet \neg \phi \Rightarrow \neg \bullet \phi \]

Proof. We have \( \neg \text{end} \land \bullet \neg \phi \). By unrolling syntax, we have \( \neg \text{end} \land \neg \circ \neg \phi \). By Taut, we have \( \neg \text{end} \land \neg \circ \phi \). By Lemma 8, \( \circ \phi \Rightarrow \bullet \phi \), so we have \( \neg \bullet \phi \).

Lemma 10 (Weak next distributes over conjunction).

\[ \vdash \bullet (\phi \land \psi) \iff \bullet \phi \land \bullet \psi. \]

Proof. From \( \bullet (\phi \Rightarrow \psi) \), we have \( \bullet (\neg (\phi \Rightarrow \neg \psi)) \) by Taut. By Taut again, we have \( \text{end} \lor \neg \text{end} \); by the definition of \( \bullet \), we can refactor our formula into \( \bullet (\neg (\phi \Rightarrow \neg \psi)) \land \text{end} \lor \neg \circ (\phi \Rightarrow \neg \psi) \land \neg \text{end} \), also by Taut. By EndNextContra, we have \( \text{end} \Rightarrow \neg \circ (\phi \Rightarrow \neg \psi) \land \neg \text{end} \), so we can eliminate the left-hand disjunct by Taut, to find \( \text{end} \lor \neg \circ (\phi \Rightarrow \neg \psi) \land \neg \text{end} \). By Lemma 8, we can weaken our next modality to find \( \text{end} \lor \neg (\bullet (\phi \Rightarrow \neg \psi)) \land \neg \text{end} \). Using Lemma 8, we can change \( \bullet \neg \psi \) to \( \neg \bullet \psi \), because we have \( \neg \text{end} \) in that disjunct; we now have
end ∨ ¬(● φ ⇒ ¬ ◦ ψ) ∧ ¬end. By Taut, we can rearrange the implication to find end ∨ (● φ ∧ ◦ ψ) ∧ ¬end. By EndNextContra, we can introduce ● φ and ◦ ψ on the left-hand disjunct, to find ● φ ∧ ◦ ψ ∧ end ∨ ● φ ∧ ● ψ ∧ ¬end. Finally, Taut allows us to rearrange our term to ● φ ∧ ● ψ ∧ (end ∨ ¬end), where the rightmost conjunct falls out and we find ● φ ∧ ● ψ. ⊣

**Lemma 11 (Weak next negation).** ⊢ ¬ ● φ ⇒ ● ¬ φ

**Proof.** By definition, ¬ ● φ is equivalent to ¬ ◦ ¬ φ. By Taut, we eliminate the double negation to find ◦ ¬ φ. By Lemma 7 from right to left, we have ¬ ◦ φ. By Taut, we reintroduce an inner double negation, to find ¬ ◦ ¬¬ φ. By definition, we have • ¬ φ. ⊣

**Lemma 12 (Always unrolling).** ⊢ □ φ ⇔ φ ∧ • □ φ

**Proof.** Desugaring, we must show ⊢ (φ W ⊥) ⇔ φ ∧ • (φ W ⊥). By WkUntilUnroll, ⊢ (φ W ⊥) ⇔ ⊥ ∨ (φ ∧ • (φ W ⊥)). By Taut we can eliminate the ⊥ case of the disjunction on the right. ⊣

**Lemma 13 (Necessitation).** If ⊢ φ then ⊢ □ φ.

**Proof.** We apply Induction with φ = ⊤ and ψ = φ to show that ⊢ T ⇒ □ φ, i.e., ⊢ □ φ by Taut. We must prove both premises: ⊢ T ⇒ φ and ⊢ T ⇒ • T. Since we’ve assumed ⊢ φ, we have the first premise by Taut. By Taut and WkNextStep, we have ⊢ • T, and so ⊢ T ⇒ • T by Taut. ⊣

4.3. Completeness. To show deductive completeness for LTL$_f$, we must find that if |= φ then ⊢ φ. To do so we construct a graph that does two things at once: first, paths from the root of the graph to a terminal state correspond to Kripke structures which φ satisfies; second, consistency properties in the graph relate to the provability of the underlying formula φ.

Our construction follows the standard Henkin-Hasenjaeger-style least-fixed point approach found in Kröger and Merz’s book [11]: we construct a graph whose nodes assign truth values to each subformula of our formula of interest, φ, by putting each subformula in either the true, “positive” set or in the false, “negative” set. To show that |= φ implies ⊢ φ, we build a graph for ¬ φ that guarantees that ¬ φ is inconsistent, i.e., ⊢ ¬¬ φ, and so ⊢ φ via double-negation elimination (since LTL$_f$’s propositional core is classical). Our proof itself is classical, using the law of the excluded middle to define the proof graph (comps) and prove some of its properties (Lemma 27).

What about the ‘f’ in LTL$_f$? Nothing described so far differs in any way from the Henkin-Hasenjaeger graph approach used by Kröger and Merz [11]. Kröger and Merz’s graphs were always finite, but their notion of satisfying paths forces paths to be infinite. We restrict our attention to terminating paths: paths where not only is our formula of interest
satisfied, but so is $\Diamond \text{end}$. To ensure such paths exist, we inject $\Diamond \text{end}$ into the root node of the graph.

The proof follows the following structure (Figure 3): we define the nodes of the graph (Definition 14); we define the edge relation on the graph (Figure 4) and show that it maps appropriately to time steps in the proof theory (Lemma 21 finds a consistent successor; Lemma 16 shows the successor is a state in our graph); we show that the graph structure results
in a finite structure with appropriate consistency properties (Lemma 25); we define which paths in the graph represent our Kripke structure of interest (Lemma 27 shows that our graph’s transitions correspond to the semantics; Lemma 29 guarantees that we have appropriate finite models). The final proof comes in two parts: we show that consistent graphs correspond to satisfiable formulae (Theorem 30), which we then use to show completeness (Theorem 31).

**Definition 14 (PNP).** A positive-negative pair (PNP) $\mathcal{P}$ is a pair of finite sets of formulae $(\text{pos}(\mathcal{P}), \text{neg}(\mathcal{P}))$. We refer to the collected formulas of $\mathcal{P}$ as $F_{\mathcal{P}} = \text{pos}(\mathcal{P}) \cup \text{neg}(\mathcal{P})$; we call the set of all PNPs PNP.

We write the literal interpretation of a PNP $\mathcal{P}$ as:

$$\hat{\mathcal{P}} = \bigwedge_{\phi \in \text{pos}(\mathcal{P})} \phi \land \bigwedge_{\psi \in \text{neg}(\mathcal{P})} \neg \psi.$$ 

We say $\mathcal{P}$ is inconsistent if $\vdash \neg \hat{\mathcal{P}}$; conversely, $\mathcal{P}$ is consistent when it is not the case that $\vdash \neg \hat{\mathcal{P}}$, i.e., $\nvdash \neg \hat{\mathcal{P}}$.

Positive-negative pairs are the nodes of our proof graph—each node is a collection of formulae that hold (or not) in a given moment in time. Before we can even begin constructing the graph, we show that they adequately characterize a moment in time: that is, they are without contradiction, can be ‘saturated’ with all of the formulae of interest, and respect the general rules of our logic. Readers may be familiar with ‘atoms’, but PNPs are themselves not atoms; complete PNPs are more or less atoms (Figure 4).

**Lemma 15 (PNP properties).** For all consistent PNPs $\mathcal{P}$:

1. $\text{pos}(\mathcal{P}) \cap \text{neg}(\mathcal{P}) = \emptyset$;
2. For all $\phi$, either $(\text{pos}(\mathcal{P}) \cup \{\phi\}, \text{neg}(\mathcal{P}))$ or $(\text{pos}(\mathcal{P}), \text{neg}(\mathcal{P}) \cup \{\phi\})$ is consistent;
3. $\bot \notin \text{pos}(\mathcal{P})$;
4. if $\{\phi, \psi, \phi \Rightarrow \psi\} \subseteq F_{\mathcal{P}}$, then $\phi \Rightarrow \psi \in \text{pos}(\mathcal{P})$ iff $\phi \in \text{neg}(\mathcal{P})$ or $\psi \in \text{pos}(\mathcal{P})$;
5. if $\vdash \phi \Rightarrow \psi$ and $\phi \in \text{pos}(\mathcal{P})$ and $\psi \in F_{\mathcal{P}}$, then $\psi \in \text{pos}(\mathcal{P})$.

**Proof.** Let a given PNP $\mathcal{P}$ be consistent. We show each case by reasoning based on whether each formula is assigned to the positive or the negative set of $\mathcal{P}$, deriving contradictions as appropriate.

1. Suppose for a contradiction that $\phi \in \text{pos}(\mathcal{P}) \cap \text{neg}(\mathcal{P})$. We have $\vdash \neg(\phi \land \neg \phi)$ by Taut, but $\vdash \hat{\mathcal{P}} \Rightarrow \phi \land \neg \phi$, and so $\vdash \neg \hat{\mathcal{P}}$ by Taut—making $\mathcal{P}$ inconsistent, a contradiction.
2. If $\phi \in \text{pos}(\mathcal{P})$ or $\phi \in \text{neg}(\mathcal{P})$ already, we are done; we already know by (1) that $\phi \notin \text{pos}(\mathcal{P}) \cap \text{neg}(\mathcal{P})$. So $\phi$ does not already occur in $\mathcal{P}$. Suppose for a contradiction that adding $\phi$ to either set is
injecting finiteness for ltlf completeness

15

inconsistent, i.e. both \( \vdash \neg (\widehat{P} \land \phi) \) and \( \vdash \neg (\widehat{P} \land \neg \phi) \). By Taut, that would imply that \( \vdash \neg \widehat{P} \land (\phi \lor \neg \phi) \), which is the same as simply \( \vdash \neg \widehat{P} \)—a contradiction.

3. Suppose for a contradiction that \( \bot \in \text{pos}(P) \); by Taut we have \( \vdash \widehat{P} \Rightarrow \bot \), which is syntactic sugar for \( \vdash \neg \widehat{P} \)—a contradiction.

4. Suppose \( \{ \phi, \psi, \phi \Rightarrow \psi \} \subseteq F_P \). If \( \phi \Rightarrow \psi \in \text{pos}(P) \), we must show that \( \phi \in \text{pos}(P) \) or that \( \psi \in \text{neg}(P) \). Suppose for a contradiction that neither is in the appropriate set; we then have \( \vdash \widehat{P} \Rightarrow (\phi \Rightarrow \psi) \land \phi \land \neg \psi \); by Taut, we can then conclude \( \vdash \neg \widehat{P} \)—a contradiction.

If, on the other hand, \( \phi \in \text{neg}(P) \) or \( \psi \in \text{pos}(P) \), we must show that \( \phi \Rightarrow \psi \in \text{pos}(P) \). Suppose for a contradiction that its not the case that \( \phi \Rightarrow \psi \in \text{pos}(P) \). Since \( \phi \Rightarrow \psi \in F_P \), then \( \phi \Rightarrow \psi \in \text{neg}(P) \). We have either \( \vdash \widehat{P} \Rightarrow \neg(\phi \Rightarrow \psi) \land \neg \phi \) or \( \vdash \widehat{P} \Rightarrow \neg(\phi \Rightarrow \psi) \land \psi \). By Taut, we can convert \( \neg(\phi \Rightarrow \psi) \) into \( \phi \land \neg \psi \—and either way we can find by Taut that \( \vdash \neg \widehat{P} \), a contradiction.

5. Suppose \( \vdash \phi \Rightarrow \psi \) and \( \psi \in \text{pos}(P) \) with \( \phi \in F_P \). We must show that \( \phi \in \text{pos}(P) \). Suppose for a contradiction that \( \phi \in \text{neg}(P) \). We then have \( \vdash \widehat{P} \Rightarrow (\phi \Rightarrow \psi) \land \psi \land \neg \phi \); by Taut, we can then find \( \vdash \neg \widehat{P} \), which is a contradiction.

\( \vdash \)

Our goal is to generate successors states to build a graph of PNPs; to do so, we define two functions: a step function \( \sigma \) and a closure function \( \tau \) (Figure 4). The closure function \( \tau \) takes a PNP and produces all of its subterms that are relevant for the current state, i.e., it doesn’t go under the next modality. Our closure \( \tau \) is slightly smaller than the commonly seen Fischer-Ladner closure [8]: we don’t include every possible negation and we stop when we reach a next modality. We write \( P \preceq Q \) (read “\( P \) is extended by \( Q \)” or “\( Q \) extends \( P \)” ) when \( Q \)’s positive and negative sets subsume \( P \)’s (Figure 1). We say \( P \) is complete when \( F_P = \tau(P) \). We say a complete PNP \( Q \) is a completion of \( P \) when \( P \preceq Q \) and \( Q \) is consistent and complete. We define the set of all consistent completions of a given PNP \( P \) as \( \text{comps}(P) \). The step function \( \sigma \) takes a PNP and generates those formulae which must hold in the next step, thereby characterizing the transitions in our graph. The set of completions, \( \text{comps} \), is not a constructive set, since we have (as yet) no way to determine whether a given PNP is consistent or not.

First, we show that each PNP implies its successor (Lemma 16); next, consistent PNPs produce consistent successors (Lemma 17).

Lemma 16 (Transitions are provable). For all \( P \in \text{PNP} \), we have \( \vdash \widehat{P} \Rightarrow \bullet \sigma(P) \).
Transition functions

\[ \sigma_i^+ (P) = \{ \phi \mid \circ \phi \in \text{pos}(P) \} \]
\[ \sigma_2^+ (P) = \{ \phi W \psi \mid \phi W \psi \in \text{pos}(P), \psi \in \text{neg}(P) \} \]
\[ \sigma_3^- (P) = \{ \phi \mid \circ \phi \in \text{neg}(P) \} \]
\[ \sigma_4^- (P) = \{ \phi W \psi \mid \phi W \psi \in \text{neg}(P), \phi \in \text{pos}(P) \} \]
\[ \sigma(P) = (\sigma_1^+ (P) \cup \sigma_2^+ (P), \sigma_3^- (P) \cup \sigma_4^- (P)) \]
\[ \tau(v) = \{ v \} \]
\[ \tau(\phi \Rightarrow \psi) = \{ \phi \Rightarrow \psi \} \]
\[ \tau(Wp) = \{ \phi W \psi \} \]
\[ \tau(F) = \bigcup_{\phi \in F} \tau(\phi) \]

Extensions, completions, and possible assignments

\[ \leq \subseteq \text{PNP} \times \text{PNP} \quad \text{comps} : \text{PNP} \rightarrow 2^{\text{PNP}} \quad \text{assigns} : 2^{\text{LTL}_f} \rightarrow 2^{\text{PNP}} \]

\[ \mathcal{P} \leq \mathcal{Q} \text{ iff } \text{pos}(P) \subseteq \text{pos}(Q) \text{ and } \text{neg}(P) \subseteq \text{neg}(Q) \]
\[ \text{comps}(F) = \{ P \mid F_P = \tau(F) \} \]
\[ \text{assigns}(F) = \{ \mathcal{Q} \mid F_{\mathcal{Q}} = \tau(F), \mathcal{P} \leq \mathcal{Q}, \mathcal{Q} \text{ consistent} \} \]

**Figure 4.** Step and closure functions; extensions and completions

**Proof.** Unfolding the definition of \( \widehat{P} \), we must show

\[ \vdash \left[ \bigwedge_{\phi \in \text{pos}(P)} \phi \land \bigwedge_{\psi \in \text{neg}(P)} \psi \right] \Rightarrow \left[ \bigwedge_{\phi \in \text{pos}(\sigma(P))} \phi \land \bigwedge_{\psi \in \text{neg}(\sigma(P))} \psi \right] . \]

By cases on the clauses of \( \sigma \), we show that \( \widehat{P} \) implies each of the parts of \( \widehat{\sigma(P)} \), tying the cases together by Taut and Lemma 10:

\( \sigma_1^+ \) Suppose \( \phi \in \sigma_1^+ (P) \) because \( \circ \phi \in P \). We have \( \vdash \widehat{P} \Rightarrow \bullet \phi \) because \( \circ \phi \Rightarrow \bullet \phi \) by Lemma 8.

\( \sigma_2^+ \) Suppose \( \phi W \psi \in \sigma_2^+ (P) \) because \( \vdash \phi W \psi \in \text{pos}(P) \) and \( \psi \in \text{neg}(P) \). Then, \( \vdash \widehat{P} \Rightarrow \neg \psi \land \phi W \psi \). We have \( \widehat{P} \Rightarrow \bullet \phi W \psi \) by WkUntilUnroll and Taut.

\( \sigma_3^- \) Suppose \( \phi \in \sigma_3^- (P) \) because \( \circ \phi \in \text{neg}(P) \). We have \( \vdash \widehat{P} \Rightarrow \bullet \neg \phi \) because \( \neg \circ \phi \Rightarrow \bullet \neg \phi \) by Lemmas 7 and 8.
Suppose $\phi \models W \psi \in \sigma_4^-(P)$ because $\phi \models W \psi \in \neg(\sigma(P))$ and $\phi \in \pos(\sigma(P))$. We have:

\[
\vdash \widehat{\sigma} \Rightarrow \neg(\phi \models W \psi) \land \phi
\]

iff

\[
\vdash \widehat{\sigma} \Rightarrow \neg(\psi \lor (\phi \land \bullet(\phi \models W \psi))) \land \phi \quad \text{WKUNTILUNROLL}
\]

iff

\[
\vdash \widehat{\sigma} \Rightarrow \neg(\psi \land (\neg\phi \lor \bullet(\phi \models W \psi))) \land \phi \quad \text{TAUT}
\]

iff

\[
\vdash \widehat{\sigma} \Rightarrow \neg(\psi \land \bullet(\phi \models W \psi)) \land \phi \quad \text{TAUT}
\]

Lemma 11

implies

\[
\vdash \widehat{\sigma} \Rightarrow \psi \land (\neg(\phi \land \bullet(\phi \models W \psi))) \land \phi
\]

iff

\[
\vdash \widehat{\sigma} \Rightarrow \bullet(\neg(\phi \models W \psi)) \quad \text{TAUT}
\]

Lemma 17 (Transitions are consistent). For all consistent PNP states $P$, if $\vdash \widehat{\sigma} \Rightarrow \neg\end$, then $\sigma(P)$ is consistent.

Proof. Let $P$ be a consistent PNP such that $\vdash \widehat{\sigma} \Rightarrow \neg\end$. Assume for the sake of contradiction, that $\neg \sigma(P)$. Then we can write, by WKNEXT, $\vdash \bullet(\neg\sigma(P))$, which is equivalent to $\vdash \neg\sigma(\sigma(P))$. By Lemma 16, $\vdash \widehat{\sigma} \Rightarrow \bullet(\sigma(P)) \land \neg\end$, which by Lemma 8 and Taut gives $\vdash \widehat{\sigma} \Rightarrow \sigma(\sigma(P))$. Now we can derive $\vdash \widehat{\sigma} \Rightarrow \bot$, or equivalently $\neg\widehat{\sigma}$—a contradiction. Therefore $\sigma(P)$ is consistent.

Lemma 18 (Inconsistent PNP's have no completions). If a PNP $P$ is inconsistent, then $\text{comps}(P) = \emptyset$.

Proof. Let $P$ be given; suppose for a contradiction that there exists $Q \in \text{comps}(P)$, i.e., $F_Q = \tau(P)$ and $P \preceq Q$ and $Q$ is consistent. We have $\vdash \widehat{\sigma} \Rightarrow \widehat{\sigma}(P) \land \neg\end$, which by Lemma 8 and Taut gives $\vdash \widehat{\sigma} \Rightarrow \sigma(\sigma(P))$. Now we can derive $\vdash \widehat{\sigma} \Rightarrow \bot$, or equivalently $\neg\widehat{\sigma}$—a contradiction.

In order to fully define our graph, we must show that not only are successors of PNP's provable, so are their completions. We do so in two steps: first, we show that there is always some provable assignment of propositions in each set of formulas; next, conditionally provable assignments are in fact completions.

Lemma 19 (Assignments are provable). $\vdash \bigvee_{P \in \text{assigns}(\sigma)} \widehat{P}$

Proof. By induction on the size of $\sigma$. When $|\sigma| = 0$, We have $\vdash \top$ by Taut. When $|\sigma| = n + 1$, let $\phi \in \sigma$ be a maximal formula, i.e., $\phi \notin \tau(\sigma - \{\phi\})$. We have $\text{assigns}(\sigma) = \{P \mid P' \in \text{assigns}(\sigma'), \sigma_P =$
\[ \mathcal{P} = \{ \phi \} \cup \tau(\phi) \}, \text{i.e., each formula in } \tau(\phi) \text{ not already assigned in } \mathcal{P}' \text{ is put in either the positive or negative set of } \mathcal{P}. \text{ That is, we take each formula in } \mathcal{P}' \text{ and conjoin } \psi \lor \neg \psi \text{ for each } \psi \in \tau(\phi). \text{ We know by the IH that } \vdash \bigvee_{\mathcal{P}' \in \text{assigns}(\mathcal{F})} \mathcal{P}', \text{ so by Taut we have } \vdash \bigvee_{\mathcal{P} \in \text{assigns}(\mathcal{F})} \mathcal{P}. \]

**Lemma 20 (Consistent assignments are completions).**

*For all consistent PNPs \( \mathcal{P} \) and for all \( Q \in \text{assigns}(\mathcal{P}) \), if \( \vdash \mathcal{P} \Rightarrow \mathcal{Q} \) then \( Q \in \text{comps}(\mathcal{P}) \).*

**Proof.** Let \( \mathcal{P} \) and \( Q \in \text{assigns}(\mathcal{P}) \) be given such that \( \vdash \mathcal{P} \Rightarrow \mathcal{Q} \).

Suppose for a contradiction that \( Q \notin \text{comps}(\mathcal{P}) \). It must be the case that either \( Q \) does not extend \( \mathcal{P} \) or \( Q \) is inconsistent—we show that both cases are contradictory.

If \( \mathcal{P} \not\preceq Q \), then there exists some formula \( \phi \) such that \( \phi \in \text{pos}(\mathcal{P}) \) and \( \phi \notin \text{neg}(Q) \) or vice versa (since \( Q \in \text{assigns}(\mathcal{P}) \), every formula must be accounted for). Then, \( \vdash \mathcal{P} \Rightarrow \phi \) and \( \vdash \mathcal{Q} \Rightarrow \neg \phi \). Then \( \vdash \mathcal{P} \land \mathcal{Q} \Rightarrow \phi \land \neg \phi \), which by Taut means \( \vdash \neg (\mathcal{P} \land \mathcal{Q}) \), or equivalently, that \( \vdash \mathcal{P} \Rightarrow \neg \mathcal{Q} \).

When combined with the assumption that \( \vdash \mathcal{P} \Rightarrow \mathcal{Q} \), via Taut, we can derive \( \vdash \neg \mathcal{P} \)—a contradiction with \( \mathcal{P} \)'s consistency.

If, on the other hand \( Q \) is inconsistent, then we can see from \( \vdash \mathcal{P} \Rightarrow \mathcal{Q} \) that \( \vdash \neg \mathcal{Q} \)—and by Taut, it must be that \( \vdash \neg \mathcal{P} \), which contradicts \( \mathcal{P} \)'s consistency.

Combining the last two proofs we find that consistent completions are provable.

**Lemma 21 (Consistent completions are provable).**

*For all consistent PNPs \( \mathcal{P} \), we have \( \vdash \mathcal{P} \Rightarrow \bigvee_{Q \in \text{comps}(\mathcal{P})} \mathcal{Q} \).*

**Proof.** By Lemma 19 we have \( \vdash \bigvee_{Q \in \text{assigns}(\mathcal{P})} \mathcal{Q} \). By Taut, we have \( \vdash \mathcal{P} \Rightarrow \bigvee_{Q \in \text{assigns}(\mathcal{P})} \mathcal{Q} \). By Lemma 20 we know that we only need to keep those \( Q \in \text{assigns}(\mathcal{P}) \) which are also in \( \text{comps}(\mathcal{P}) \), and so we have \( \vdash \mathcal{P} \Rightarrow \bigvee_{Q \in \text{comps}(\mathcal{P})} \mathcal{Q} \) as desired.

Having established the fundamental properties of consistent completions, we set about defining proof graphs, the structure on which we build our proof. Starting from a PNP formed from a given formula, we can construct a graph where nodes are PNPs and a node \( \mathcal{P} \)'s successors are consistent completions of \( \sigma(\mathcal{P}) \).

**Definition 22 (Proof graphs).** For a consistent and complete PNP \( \mathcal{P} \) (i.e., where \( \mathcal{F}_\mathcal{P} = \tau(\mathcal{P}) \) and it is not the case that \( \vdash \neg \mathcal{P} \)), we define a *proof graph* \( \mathcal{G}_\mathcal{P} \) as follows: (a) \( \mathcal{P} \) is the root of \( \mathcal{G}_\mathcal{P} \); (b) \( \mathcal{P} \) has an edge to the root of \( \mathcal{G}_Q \) for each \( Q \in \text{comps}(\sigma(\mathcal{P})) \).
Since $\mathcal{P}$ is composed of a finite number of formulae, the set of all subsets of $\tau(\mathcal{P})$ is finite, as are any assignments of those subsets to PNPs. Hence the number of nodes in the proof graph must be finite\[^4\]

Our innovation in adapting the completeness proof to finite time is \textit{finiteness injection}, where we make sure that $\Diamond \text{end}$ is in the positive set of the root of the proof graph. After injecting finiteness, every node of the proof graph either has $\text{end}$ in its positive set (and no successors) or all of its successors have $\Diamond \text{end}$ in their positive set.

Every lemma we prove, from here to the final completeness result, has some premise concerning the end of time: by only working with PNPs with $\Diamond \text{end}$ in the positive set, we guarantee that time eventually ends.

\textbf{Lemma 23 (end injection is invariant).} If $\mathcal{P}$ is a consistent and complete PNP with $\Diamond \text{end} \in \text{pos}(\mathcal{P})$, then either:

- $\text{end} \in \text{pos}(\mathcal{P})$ and $\mathcal{P}$ has no successors (i.e., $\text{comps}(\sigma(\mathcal{P})) = \emptyset$), or
- $\text{end} \in \text{neg}(\mathcal{P})$ and for all $\mathcal{Q} \in \text{comps}(\sigma(\mathcal{P}))$, we have $\Diamond \text{end} \in \text{pos}(\mathcal{Q})$.

\textbf{Proof.} Recall that $\Diamond \text{end}$ desugars to $\neg (\neg \circ \top \ W \bot)$. Since $\mathcal{P}$ is complete, we know that $\text{end} \in \mathcal{F}_P$. If $\text{end}$ (i.e., $\neg \circ \top$) is in $\text{pos}(\mathcal{P})$ and $\mathcal{P}$ is consistent, it must be the case that $\circ \top \in \text{neg}(\mathcal{P})$ by Lemma 15. We therefore have that $\top \in \sigma^-_3(\mathcal{P})$, so $\vdash \sigma(\mathcal{P}) \Rightarrow \neg \top$, i.e., $\sigma(\mathcal{P})$ is inconsistent—and therefore $\text{comps}(\sigma(\mathcal{P})) = \emptyset$, because there are no consistent completions of an inconsistent PNP (Lemma 18). If, on the other hand, $\text{end}$ (i.e., $\neg \circ \top$) is in $\text{neg}(\mathcal{P})$ and $\mathcal{P}$ is consistent, then it must be the case that $\circ \top \in \text{pos}(\mathcal{P})$. We must have have $\Diamond \text{end} \in \sigma^+_3(\mathcal{P})$, which means $\Diamond \text{end} \in \text{pos}(\sigma(\mathcal{P}))$. It must be therefore be the case that $\Diamond \text{end} \in \text{pos}(\mathcal{Q})$ for all $\mathcal{Q} \in \text{comps}(\sigma(\mathcal{P}))$, since each such $\mathcal{Q}$ must be an extension of $\sigma(\mathcal{P})$. \hfill \dashv

We can go further, showing that $\Diamond \text{end}$ is in fact in every node’s positive set, and every node is consistent and complete.

\textbf{Lemma 24 (Proof graphs are consistent).} For all consistent and complete PNPs $\mathcal{P}$, every node $\mathcal{Q} \in \mathcal{G}_P$ is consistent and complete. If $\Diamond \text{end} \in \text{pos}(\mathcal{P})$, then $\Diamond \text{end} \in \text{pos}(\mathcal{Q})$.

\textbf{Proof.} By induction on the length of the shortest path from $\mathcal{P}$ to $\mathcal{Q}$ in $\mathcal{G}_P$. When $n = 0$, we have $\mathcal{Q} = \mathcal{P}$, so we have $\mathcal{P}$’s completeness and consistency by assumption; the second implication is immediate.

When $n = n' + 1$, we have some path $\mathcal{P}, \mathcal{P}_2, \mathcal{P}_3, \ldots, \mathcal{P}_{n'}, \mathcal{Q}$. We know that $\mathcal{P}_{n'}$ is complete and consistent; we must show that $\mathcal{Q}$ is complete and consistent. By construction, we know that $\mathcal{Q} \in \text{comps}(\sigma(\mathcal{P}_{n'}))$, so $\mathcal{Q}$

\[^4\text{Confusingly, Kröger and Merz [11] call this graph an “infinite tree” in their proof of completeness for potentially infinite LTL, even though it turns out to be finite in that setting, as well.}\]
must be consistent and complete by definition. By the III, we know that \( \Diamond \text{end} \in \text{pos}(P_{n'}) \), so by Lemma 23 we find the same for \( \mathcal{Q} \).

Each node has the potential for successors: for each node \( \mathcal{Q} \in \mathcal{G}_P \), we can prove that \( \mathcal{Q} \) implies that the disjunction of every other node’s literal interpretation holds in the next moment of time.

**Lemma 25 (Step implication).** For all consistent and complete PNPs \( \mathcal{P} \) where \( \Diamond \text{end} \in \text{pos}(\mathcal{P}) \) then \( \vdash \lor_{\mathcal{Q} \in \mathcal{G}_P} \mathcal{Q} \Rightarrow \lor_{\mathcal{Q} \in \mathcal{G}_P} \mathcal{Q} \).

**Proof.** Let \( \mathcal{Q} \in \mathcal{G}_P \) be given. We show that each \( \mathcal{Q} \) implies the right-hand side. By Lemma 16, we know that \( \vdash \mathcal{Q} \Rightarrow \mathcal{Q} \). By Lemma 24 we know that \( \mathcal{Q} \) is consistent and complete and \( \Diamond \text{end} \in \text{pos}(\mathcal{Q}) \). Since \( \mathcal{Q} \) is complete, we know that \( \text{end} \in \mathcal{F}_\mathcal{Q} \). We now show that \( \vdash \mathcal{Q} \Rightarrow \lor_{\mathcal{Q}' \in \text{comps}(\sigma(\mathcal{Q}))} \mathcal{Q}' \), by cases on where \( \text{end} \) occurs in \( \mathcal{F}_\mathcal{Q} \).

If \( \text{end} \in \text{pos}(\mathcal{Q}) \), then \( \text{comps}(\sigma(\mathcal{Q})) = \emptyset \) by Lemma 23 we must find \( \vdash \mathcal{Q} \Rightarrow \bot \). Since \( \vdash \mathcal{Q} \Rightarrow \text{end} \), we are done by Lemma 8 with \( \phi = \bot \). If, on the other hand, \( \text{end} \in \text{neg}(\mathcal{Q}) \), we have \( \vdash \mathcal{Q} \Rightarrow \neg \text{end} \), so by Lemma 17 we know that \( \sigma(\mathcal{Q}) \) is consistent. We therefore have \( \vdash \sigma(\mathcal{Q}) \Rightarrow \lor_{\mathcal{Q}' \in \text{comps}(\sigma(\mathcal{Q}))} \mathcal{Q}' \) by Lemma 21.

We have \( \vdash \mathcal{Q} \Rightarrow \lor_{\mathcal{Q}' \in \mathcal{G}_P} \mathcal{Q}' \), because \( \text{comps}(\sigma(\mathcal{Q})) \subseteq \mathcal{G}_P \) by definition.

Since we find this for each \( \mathcal{Q} \), we conclude \( \vdash \lor_{\mathcal{Q} \in \mathcal{G}_P} \mathcal{Q} \Rightarrow \lor_{\mathcal{Q} \in \mathcal{G}_P} \mathcal{Q} \).

We have so far established that the proof graph \( \mathcal{G}_P \) is rooted at \( \mathcal{P} \), preserves any finiteness we may inject, and each node has provable successors. We are nearly done: we show that our proof graph corresponds to a Kripke structure which models \( \mathcal{P} \).

**Definition 26 (Terminal nodes and paths).** A node \( \mathcal{Z} \in \mathcal{G}_P \) is terminal when \( \circ \top \in \text{neg}(\mathcal{Z}) \). A path \( \mathcal{P}_1, \ldots, \mathcal{P}_n \) is terminal when \( \mathcal{P}_n \) is terminal.

**Lemma 27 (Proof graphs are models).** For all consistent and complete PNPs \( \mathcal{P} \), if \( \mathcal{P}_1, \mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n \) is a terminal path in \( \mathcal{G}_P \), then for all \( i \):

1. For all formulae \( \phi \), if \( \circ \phi \in \mathcal{F}_{\mathcal{P}_i} \), then \( \circ \phi \in \text{pos}(\mathcal{P}_i) \) iff \( \phi \in \text{pos}(\mathcal{P}_{i+1}) \).
2. For all formulae \( \phi \) and \( \psi \), if \( \phi \psi \in \mathcal{F}_{\mathcal{P}_i} \), then \( \phi \psi \in \text{pos}(\mathcal{P}_j) \) iff either \( \phi \in \text{pos}(\mathcal{P}_i) \) or for all \( j \geq i \) there is some \( k \geq i \) such that \( \psi \in \text{pos}(\mathcal{P}_k) \) and \( \forall i \leq j < k, \phi \in \text{pos}(\mathcal{P}_j) \).

**Proof.**

1. We have \( \mathcal{P}_{i+1} \in \text{comps}(\sigma(\mathcal{P}_i)) \) by definition. One the one hand, if \( \circ \phi \in \text{pos}(\mathcal{P}_i) \), we have \( \phi \in \text{pos}(\mathcal{P}_i) \), and so all consistent completions have \( \phi \) in the positive set—in particular, \( \mathcal{P}_{i+1} \).

2. On the other hand, if we have \( \phi \in \text{pos}(\mathcal{P}_{i+1}) \), it must be the case that \( \circ \phi \) is in one of \( \text{pos}(\mathcal{P}_i) \) or \( \text{neg}(\mathcal{P}_i) \) because \( \circ \phi \in \mathcal{F}_{\mathcal{P}_i} \). In the former case, we are done immediately. Suppose for a contradiction
We have $P \circ \phi \in \neg(\sigma(P))$. Since $P_i+1$ is a completion of $\sigma(P_i)$, it must be that $\neg(\sigma(P)) \subseteq \neg(\sigma(P))$. Since $\neg(\sigma(P))$, we must have $\phi \in \neg(\sigma(P))$, so $\phi \in \neg(\sigma(P))$. But we have $\phi \in \text{pos}(\sigma(P))$ by assumption—and we have contradicted the consistency of $\sigma(P)$ (Lemma 24).

2. We have $P_j+1 \in \text{comps(}\sigma(P_j))$ for all $j$, by definition. Further, we know that $\phi \land \psi \in \text{pos}(\sigma(P))$ implies that $\{\phi, \psi\} \subseteq \mathcal{F}_P$. We go by cases on where $\phi \land \psi$ occurs in $\mathcal{F}_P$, going from left-to-right both times and proving the contrapositive for the right-to-left implication.

$(\phi \land \psi \in \text{pos}(\sigma(P)))$ We must show that either $\phi \in \text{pos}(\sigma(P))$ for all $j \geq i$ or there is some $k \geq i$ such that $\psi \in \text{pos}(\sigma(P))$ and $\forall i \leq j < k, \phi \in \text{pos}(\sigma(P))$. We go by induction on $n$. When $n = 0$, we either have $\psi \in \text{pos}(\sigma(P))$ (and so $k = i$) or we have $\psi \in \text{neg}(\sigma(P))$ (and then the path ends). When $n = n'+1$, we know the path from $P_i$ to $P_i+1$ has either $\phi$ in every positive set or eventually $\psi$ occurs after $\delta$s. In the latter case, we can simply reuse the $k$ from the inductive hypothesis. In the former case, we know $\{\phi, \psi \in \text{pos}(\sigma(P))\}$, so by the above we can find that either $\psi \in \text{pos}(\sigma(P))$ or $\phi \in \text{neg}(\sigma(P))$ has $\phi \land \psi \in \text{pos}(\sigma(P))$. By the above again, we can find that either $\psi \in \text{pos}(\sigma(P))$ (and so $k = i$) or $\phi \in \text{pos}(\sigma(P))$ (and we have $\phi \in \text{pos}(\sigma(P))$ for all $j \geq i$).

$(\phi \land \psi \in \text{pos}(\sigma(P)))$ We have $\phi \land \psi \in \text{neg}(\sigma(P))$, so we must show that it is not the case that either $\phi \in \text{pos}(\sigma(P))$ for all $j \geq i$ or there is some $k \geq i$ such that $\psi \in \text{pos}(\sigma(P))$ and $\forall i \leq j < k, \phi \in \text{pos}(\sigma(P))$. We show that all paths out of $P_i$ have $\phi$ in the positive set for zero or more transitions, but eventually neither $\phi$ nor $\psi$ holds.

First, we show that if $\phi \land \psi \in \text{neg}(\sigma(P))$, then $(a) \psi \in \text{neg}(\sigma(P))$ and $(b)$ either $\phi \in \text{neg}(\sigma(P))$ or $\psi \in \text{neg}(\sigma(P))$ and $\forall \mathcal{Q} \in \text{comps}(\sigma(P))$, we have $\phi \land \psi \in \text{neg}(\mathcal{Q})$. Since $\phi \land \psi \in \text{neg}(\sigma(P))$, we have
\[ \vdash \hat{P} \Rightarrow -(\psi \lor \phi \land \bullet(\phi \ W \psi)) \] by \textsc{WkUntilUnroll}. By \textsc{Taut} we have \[ \vdash \hat{P} \Rightarrow -\psi \land (-\phi \lor -\bullet(\phi \ W \psi)) \] by desugaring and \textsc{Taut} we have \[ \vdash \hat{P} \Rightarrow -\psi \land (-\phi \lor -\circ(\phi \ W \psi)) \]. To have \( P \) consistent, it must be that \( \psi \in \text{neg}(P) \). If \( \phi \in \text{neg}(P) \), we are done—we have satisfied (a) and (b). Suppose \( \phi \in \text{pos}(P) \). By the definition \( \sigma_4 \), we now have \( \phi \ W \psi \in \text{neg}(\sigma(P)) \), so it must be the case that for any completion \( Q \), we have \( \phi \ W \psi \in \text{neg}(Q) \). Now, finally, suppose \( \phi \ W \psi \in \text{neg}(P_i) \). For \( P_i \) to be consistent, it must be that \[ \vdash \hat{P_i} \Rightarrow \hat{P_{i+1}} \Rightarrow \ldots \Rightarrow \hat{P_n} \] (since no node is terminal until \( P_n \)). One such node must have \( \phi \in \text{neg}(P_i) \): apply the reasoning above to see that no node can have \( \psi \in \text{pos}(P_i) \) and, furthermore, if \( \phi \in \text{pos}(P_i) \) then \( \phi \ W \psi \in \text{neg}(P_{i+1}) \). If it does not happen before the terminal node, the last one has no successor, so \textsc{WkUntilUnroll} shows that necessarily \( \phi \in \text{neg}(P_n) \). 

Here we slightly depart from Kröger and Merz's presentation: since their models can be infinite, they must make sure that their paths are able to in some sense ‘fulfill’ temporal predicates. We, on the other hand, know that all of our paths are finite, so our reasoning is simpler. First, there must exist some terminal node.

**Lemma 28 (Injected finiteness guarantees terminal nodes).** For all consistent and complete PNP's \( P \), if \( \Diamond \text{end} \in \text{pos}(P) \) then there is a terminal node \( Z \in G_P \).

**Proof.** Suppose for a contradiction that \( \Diamond T \in \text{pos}(Q) \) for all \( Q \in G_P \). We have assumed \( \Diamond \text{end} \in \text{pos}(P) \); by desugaring, \( \Diamond \text{end} \) amounts to \( -\Box \Box \text{end} \), i.e., \( -(-\text{end} \ W \bot) \), i.e., \( -(\Diamond T \ W \bot) \).

We have \( \vdash Q \Rightarrow \Diamond T \) for each \( Q \in G_P \) by assumption, so by \textsc{Taut}, we have \( \vdash \bigvee_{Q \in G_P} Q \Rightarrow \Diamond T \). We have \( \vdash \bigvee_{Q \in G_P} \Diamond \Rightarrow \bullet \bigvee_{Q \in G_P} \) by Lemma 25. So by \textsc{Induction}, \( \vdash \bigvee_{Q \in G_P} \Rightarrow \Box \Diamond T \), i.e., \( \Diamond T \ W \bot \). Since \( P \in G_P \), we know \( \vdash \hat{P} \Rightarrow \Diamond T \ W \bot \) by Lemma 25 again. But \( \Diamond \text{end} \in \text{pos}(P) \) means that \( \vdash \hat{P} \Rightarrow \Diamond \text{end} \), so \( \vdash \hat{P} \Rightarrow \neg(\Diamond T \ W \bot) \), as well! It must then be the case that \( \vdash \neg \hat{P} \), which contradicts our assumption that \( P \) is consistent.

We therefore conclude that there must exist some node \( Z \in G_P \) such that \( \Diamond T \in \text{neg}(Z) \). 

Since our proof graph is constructed connectedly from the root on out, the existence of a terminal node implies the existence of a terminal path from the root to that node. 

**Corollary 29 (Injected finiteness guarantees terminal paths).** For all consistent and complete PNP's \( P \), if \( \Diamond \text{end} \in \text{pos}(P) \) then there is a terminal path \( \hat{P}, \hat{P}_2, \ldots, \hat{P}_{n-1}, Z \in G_P \).
We must show that $K$ the variables in $P$ so is to construct a Kripke structure. Suppose our terminal path is of the form $P$ appropriately finite model. into the positive set—we inject finiteness to make sure we’re building an $\vdash$ we have $P$ virtually considers a version of $P$ with $\Diamond \text{end}$ (the $\text{FINITE}$ axiom) injected into the positive set—we inject finiteness to make sure we’re building an appropriately finite model.

**Theorem 30 (LTL$_f$ satisfiability)**. If $P$ is a consistent PNP, then $\hat{P}$ is satisfiable.

**Proof**. Let $P' = (\{\Diamond \text{end}\} \cup \text{pos}(P), \text{neg}(P))$. If $P$ is consistent, then so is $P'$. (If not, it must be because $\vdash \hat{P} \Rightarrow \neg \Diamond \text{end}$; by Taut and Finite, we have $\vdash \neg \hat{P}$ and $P$ is not consistent.)

To show that $\hat{P}$ is satisfiable, we use the terminal path from Corollary to construct a Kripke structure. Suppose our terminal path is of the form $P, P_2, \ldots, P_n$; let $K^n = (\eta_1, \ldots, \eta_n)$ where we define:

$$\eta_i(v) = \begin{cases} t & v \in \text{pos}(P_i) \\ f & \text{otherwise} \end{cases}$$

We must show that $K^n_i(\hat{P}) = t$; it suffices to show that $K^n_i(\hat{P}') = t$, since the variables in $P$ and $P'$ are identical. We prove that for all $\phi \in F_{P'}$, we have $K^n_i(\phi) = t$ iff $\phi \in \text{pos}(P_i)$. We go by induction on $\phi$; throughout, we rely on the fact that every node is consistent and complete (Lemma 24).

$(\phi = v) \ K^n_i(v) = t$ iff $\eta_i(v) = t$ iff $v \in \text{pos}(P_i)$.

$(\phi = \bot) \ K^n_i(\bot) = f$ by definition and $\bot \notin \text{pos}(P_i)$ by Lemma 15.

$(\phi = \psi \Rightarrow \chi)$ Let an $i$ be given. We know $P_i$ is a consistent and complete PNP, so $\{\psi, \chi\} \in F_{P_i}$. By the IH, we have $K^n_i(\psi) = t$ iff $\psi \in \text{pos}(P_i)$ and similarly for $\chi$. We have $K^n_i(\psi \Rightarrow \chi) = t$ iff $K^n_i(\psi) = f$ or $K^n_i(\chi) = t$ iff $\psi \in \text{neg}(P_i)$ or $\chi \in \text{pos}(P_i)$ (by the IHs) iff $\psi \Rightarrow \chi \in \text{pos}(P_i)$ (again by Lemma 15).

$(\phi = \circ \psi)$ Let an $i$ be given. We have $K^n_i(\circ \psi) = t$ iff $i > n$ and $K^{n+1}_i(\psi) t$ iff in $K^n_i(\psi) = t$ iff $\psi \in \text{pos}(P_{i+1})$ (by the IH) iff $\circ \psi \in \text{pos}(P_i)$ (since $\circ \psi \in F_{P'}$, by Lemma 27).

$(\phi = \psi \mathcal{W} \chi)$ We have $K^n_i(\psi \mathcal{W} \chi) = t$ iff either for all $i \leq j \leq n$, $K^n_j(\psi) = t$ or there exists an $i \leq k \leq n$ such that $K^n_k(\chi) = t$ and for all $i \leq j < k$ we have $K^n_j(\psi) = t$. By the IH, those hold iff formulae are in appropriate positive sets; by Lemma 27 those formulae are in appropriate positive sets iff $\psi \mathcal{W} \chi$ is in the appropriate positive set. At this point, $K^n_i(\hat{P}) = t$ is a special case where $i = 1$. \qed
Finally, we can show completeness. The proof is the usual one, where we to find a proof of \( \vdash \phi \) we use \( \models \phi \) to see that \( \neg \phi \) is unsatisfiable—and then the PNP for \( \neg \phi \) is inconsistent, and so \( \vdash \neg \neg \phi \), which yields \( \vdash \phi \).

**Theorem 31 (LTL\(_f\) completeness).** If \( \models \phi \) then \( \vdash \phi \).

**Proof.** If \( \models \phi \), then for all Kripke structures \( K^n \), we have \( K^n_i(\phi) = t \) for all \( i \). Conversely, it must also be the case that \( K^n_i(\neg \phi) = f \) for all \( i \), and so \( \neg \phi \) is unsatisfiable. In other words, the PNP \((\emptyset, \{\phi\})\) is unsatisfiable. By the contrapositive of Theorem 30 it must be the case that \((\emptyset, \{\phi\})\) is inconsistent, i.e., \( \vdash \neg \neg \phi \). By Taut, we can conclude that \( \vdash \phi \). \( \dashv \)

We extend the proof of completeness to allow for assumptions in the usual way.

**Corollary 32 (LTL\(_f\) completeness, with contexts).** If \( \mathcal{F} \models \phi \) then \( \mathcal{F} \vdash \phi \).

**Proof.** By induction on the size of \( \mathcal{F} \). If \( |\mathcal{F}| = 0 \), then by Theorem 31. If \( |\mathcal{F}| = n + 1 \), we have \( \{\phi_1, \ldots, \phi_{n+1}\} \models \psi \). By Theorem 3 we have \( \{\phi_1, \ldots, \phi_n\} \vdash \Box \phi_{n+1} \Rightarrow \psi \). By the IH, we have \( \{\phi_1, \ldots, \phi_{n+1}\} \vdash \psi \). \( \dashv \)

§5. Decision procedure. We have implemented a satisfiability decision procedure for LTL\(_f\)\(^5\). Our method is based Kröger and Merz’s tableau-based decision procedure [11]. Kroger and Merz generate tableaux where the states are PNPs; they proceed to unfold propositional and then temporal formulae while checking for closedness. If a certain kind of path exists in the resulting graph, then the formula is satisfiable—we can use that path to generate a Kripke structure.

The closed nodes of their tableaux are inductively defined as those which are manifestly contradictory (e.g., \( \bot \in \text{pos}(P) \) or \( \text{pos}(P) \cap \text{neg}(P) \neq \emptyset \)), those where all of their successors are contradictory (e.g. \( \bot \lor \bot \in \text{pos}(P) \) isn’t obviously contradictory, but both of its temporal successors are), and those where a negated temporal formula is never actually falsified (e.g., if \( \Box \phi \in \text{neg}(P) \) and we are generating an infinite Kripke structure, we had better falsify \( \phi \) at some point). The third criterion is a critical one: Kröger and Merz, by default, generate infinite paths in their tableaux, which correspond to infinite Kripke structures. If they were to drop their third criterion, they would find infinite paths where, say, \( \neg \Box \phi \) is meant to hold but \( \phi \) is never falsified. Such “dishonest” infinite paths must be carefully avoided.

Our decision procedure diverges slightly from theirs. First, we generalize their approach from just having always (\( \Box \)) to include weak until (\( \mathcal{W} \)). Next, we simplify their approach to exclude the third condition on paths.

\(^5\)https://github.com/ericthewry/ltlf-decide
Since we deal with finite models of time, we’ll never consider infinite paths—and so we avoid the issue of dishonest infinite paths wholesale.

Our simplified notion of closedness means we can implement a more efficient algorithm. While Kröger and Merz need to keep the tableau around in order to identify the “honest” strongly connected components of the tableau, we need not do so. We can perform a perfectly ordinary graph search without having to keep the whole tableau in memory. (We do have to keep the states of the tableau in memory, though.) To be clear: we claim no asymptotic advantage, and our algorithm remains exponential; rather, our implementation is simpler. We don’t report on the efficiency of implementation at all—rather, the code is written in Literate Haskell and is meant to be expository and tutorial.

§6. Discussion. We have studied a finite temporal logic for linear time: LTL$_f$. We were able to adapt techniques for infinite temporal logics to show deductive completeness in a finite setting. We are by no means the first to prove completeness for LTL$_f$, but we do so (a) in direct analogy to existing methods and (b) improving on Roșu’s axioms [14]. The proof of deductive completeness calls for only minor changes to the proof with potentially infinite time: we inject finiteness by inserting $\Diamond$ end into our proof graphs, allowing us to directly adapt methods from an infinite logic; injecting finiteness simplifies the selection of the path used to generate the Kripke structure in the satisfiability proof (Lemma 28 and Corollary 29). We believe that the technique is general, and will adapt to other temporal logics; we offer this proof as evidence.

To be clear, we claim that the proof of completeness for a ‘finitized’ logic is relatively straightforward once you find the right axioms. We can offer only limited guidance on finding the right axioms. Finite temporal logics should have an axiom saying that time is, indeed, finite; some sort of axiom will be needed to establish the meaning of temporal modalities at the end of time (e.g., Finite); when porting axioms from the infinite logic, one must be careful to check that the axioms are sound at the end of time (e.g., EndNextContra), when temporal modalities may change in meaning (e.g., changing distribution over implication to use the weak next modality, as in WkNextDistr).

Acknowledgments. The comments of anonymous FoSSaCS reviewers helped improve this work.

REFERENCES

[1] Carolyn Jane Anderson, Nate Foster, Arjun Guha, Jean-Baptiste Jeannin, Dexter Kozen, Cole Schlesinger, and David Walker, Netkat: Semantic foundations for networks, Symposium on Principles of Programming
Languages (New York, NY, USA), POPL ’14, ACM, 2014, pp. 113–126.

[2] Jorge A. Baier and Sheila A. McIlraith, Planning with first-order temporally extended goals using heuristic search, National Conference on Artificial Intelligence, AAAI’06, AAAI Press, 2006, pp. 788–795.

[3] Ryan Beckett, Michael Greenberg, and David Walker, Temporal netkat, Programming Language Design and Implementation (New York, NY, USA), PLDI ’16, ACM, 2016, pp. 386–401.

[4] Loris D’Antoni and Margus Veanes, Monadic second-order logic on finite sequences, Symposium on Principles of Programming Languages (New York, NY, USA), POPL 2017, ACM, 2017, pp. 232–245.

[5] Giuseppe De Giacomo, Riccardo De Masellis, and Marco Montali, Reasoning on ltl on finite traces: Insensitivity to infiniteness, National Conference on Artificial Intelligence, AAAI’14, AAAI Press, 2014, pp. 1027–1033.

[6] Giuseppe De Giacomo and Moshe Y Vardi, Linear temporal logic and linear dynamic logic on finite traces, International Joint Conference on Artificial Intelligence, Association for Computing Machinery, 2013, pp. 854–860.

[7] Giuseppe De Giacomo and Moshe Y. Vardi, Synthesis for ltl and ldl on finite traces, International Joint Conference on Artificial Intelligence, IJCAI’15, AAAI Press, 2015, pp. 1558–1564.

[8] Michael J Fischer and Richard E Ladner, Propositional dynamic logic of regular programs, Journal of Computer and System Sciences, vol. 18 (1979), no. 2, pp. 194–211.

[9] Giuseppe De Giacomo and Moshe Y. Vardi, LTLf and LDLf synthesis under partial observability, International Joint Conference on Artificial Intelligence, IJCAI’16, AAAI Press, 2016, pp. 1044–1050.

[10] Dexter Kozen, Kleene algebra with tests, ACM Trans. Program. Lang. Syst., vol. 19 (1997), no. 3, pp. 427–443.

[11] Fred Kröger and Stephan Merz, Temporal logic and state systems, Springer, 2008.

[12] Orna Lichtenstein, Amir Pnueli, and Lenore Zuck, The glory of the past, Workshop on Logic of Programs, Springer, 1985, pp. 196–218.

[13] A. Pnueli, The temporal logic of programs, Foundations of Computer Science, Oct 1977, pp. 46–57.

[14] Grigore Roșu, Finite-trace linear temporal logic: Coinductive completeness, International Conference on Runtime Verification, RV ’16, Springer, 2016, pp. 333–350.

CORNELL UNIVERSITY
ITHACA, NY, USA
E-mail: ehc86@cornell.edu

STEVENS INSTITUTE OF TECHNOLOGY
HOBOKEN, NJ, USA
E-mail: michael.greenberg@stevens.edu