1. Introduction and Main Results

In 1984, Björner [4] showed that every interval in the Bruhat order of a Coxeter group $W$ is the “face poset” of some stratified space in which each closed stratum (resp., the boundary of each stratum) has the homology of a ball (resp., of a sphere). Passing to the Euler characteristic, this result implies Verma’s formula [19; 20] for the Möbius function of the Bruhat order—namely, $\mu(u, v) = (-1)^{\ell(v) - \ell(u)} (u \leq v)$, where $\ell$ denotes the length function. (This is, in turn, equivalent to saying that each Bruhat interval contains equally many elements of even and odd length.)

In fact, Björner proved a stronger result: Every interval in the Bruhat order is the face poset of a regular cell complex (i.e., closed strata actually are balls). However, the construction of such complex in [4] was entirely “synthetic” (essentially, a succession of cell attachments; cf. [5, 4.7.23]). Furthermore, it was based on the existence of a combinatorial shelling, which by itself easily implies Verma’s formula, bypassing all geometry. A question posed in [4] asked for a natural geometric construction of a stratified space with the desired properties.

In this paper, we propose such a construction for the case where $W$ is the Weyl group of a semisimple group $G$. In the type-$A$ case, where $W$ is the symmetric group and $G$ the special linear group, we prove that our stratified spaces indeed have the required homological properties. The spaces we construct are links of cells in the Bruhat decomposition of the totally nonnegative part of the unipotent radical of $G$.

In the remainder of Section 1, we present the details of this construction and state our main results and conjectures. The rest of the paper is devoted to proofs.

Let $G$ be a semisimple, simply connected algebraic group defined and split over $\mathbb{R}$. Let $B$ and $B_-$ be two opposite Borel subgroups of $G$, so that $H = B_- \cap B$ is an $\mathbb{R}$-split maximal torus in $G$; we denote by $N$ and $N_-$ the unipotent radicals of $B$ and $B_-$, respectively.

For the type $A_{n-1}$: the group $G$ is the real special linear group $\text{SL}(n, \mathbb{R})$; $H$, $B$, and $B_-$ are the subgroups of diagonal, upper triangular, and lower triangular matrices, respectively; $N$ and $N_-$ are the subgroups of $B$ and $B_-$ that consist of matrices whose diagonal entries are equal to 1.

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We denote by $Y$ the set of all totally nonnegative elements in $N$. In the case of the special linear group, $Y$ consists of the upper-triangular unipotent matrices all of whose minors are nonnegative. The general definition was first suggested by Lusztig (see [17] and references therein). In our current notation, Lusztig defined $Y$ as the multiplicative submonoid of $N$ generated by the elements $\exp(te_i)$, $t \geq 0$, where the $e_i$ are the Chevalley generators of the Lie algebra of $N$. An alternative description in terms of nonnegativity of certain “generalized minors” was given in [10] (cf. our Proposition 2.9).

Let $W$ be the Weyl group of $G$. The length of an element $w \in W$ is denoted by $\ell(w)$. The group $W$ is partially ordered by the Bruhat order, defined geometrically by

$$u \leq v \iff B_+ uB_- \subseteq B_+ vB_-.$$ 

The Bruhat decomposition $G = \bigcup_{w \in W} B_+ w B_-$ induces the partition of $Y$ into mutually disjoint totally positive varieties $Y_w = Y \cap B_+ w B_-$, $w \in W$ (this terminology is borrowed from [9]).

We denote $Y_w = Y_w^\circ$. The varieties $Y_w^\circ$ were first studied by Lusztig in [16], where, in particular, the following basic properties were obtained.

**Proposition 1.1** [16]. *Each totally positive variety $Y_w^\circ$ is a cell; more precisely, $Y_w^\circ$ is homeomorphic to $\mathbb{R}^{\ell(w)}$. Furthermore, $Y_w = \bigcup_{u \leq w} Y_u^\circ$.***

**Example:** $G = \mathrm{SL}(3, \mathbb{R})$. In this case,

$$Y = \left\{ x = \begin{bmatrix} 1 & x_{12} & x_{13} \\ 0 & 1 & x_{23} \\ 0 & 0 & 1 \end{bmatrix} : x_{12} \geq 0, x_{23} \geq 0, x_{13} \geq 0, \begin{vmatrix} x_{12} & x_{13} \\ 1 & x_{23} \end{vmatrix} \geq 0 \right\}.$$ 

Thus the set $Y$ is described in the coordinates $(x_{12}, x_{23}, x_{13})$ as the closure of one of the pieces into which the plane $x_{13} = 0$ and the hyperbolic paraboloid $x_{12}x_{23} = x_{13}$ partition the 3-space—namely, the piece containing the point $(1, 1, \frac{1}{2})$. The semialgebraic set $Y$ decomposes naturally into six algebraic strata: the origin, two rays (the positive semi-axes for $x_{12}$ and $x_{23}$), two 2-dimensional pieces connecting them, and the 3-dimensional interior. These are the six Bruhat strata $Y_w^\circ$ for $w \in W = S_3$ (the symmetric group). Figure 1 shows a planar cross-section of this stratification (or, equivalently, the link of the 0-dimensional cell). The adjacency of the strata $Y_w^\circ$ is indeed described by the Bruhat order on $S_3$, in agreement with Proposition 1.1.

![Figure 1](image-url)  
**Figure 1** Totally nonnegative varieties $Y_w^\circ$ in the special case $G = \mathrm{SL}(3, \mathbb{R})$
For \( u, v \in W \) \((u \leq v)\), the Bruhat interval \([u, v]\) is defined by \([u, v] = \{ u \leq w \leq v \}\), with the partial order inherited from \( W \). Similarly, \((u, v) \overset{\text{def}}{=} \{ u < w \leq v \}\).

In view of Proposition 1.1, it is natural to suggest that the geometric model for a Bruhat interval \([u, v]\) (or \((u, v)\)) is provided by the link

\[
\text{Lk}(u, v) = \text{lk}(Y^u_v, Y_v)
\]
of the cell \(Y^u_v\) inside the subcomplex \(Y_v \subset Y\).

The following are our main results.

**Theorem 1.2.** For any \( u \leq v \), the link \( \text{Lk}(u, v) \) is well-defined as a stratified space. The strata \( S_{u,v,w} = \text{Lk}(u, v) \cap Y^u_w \) are labeled by the elements \( w \in (u, v] \), and each stratum \( S_{u,v,w} \) is an open smooth manifold of dimension \( \ell(w) - \ell(u) - 1 \).

The closures and boundaries of the strata \( S_{u,v,w} \) are given by

\[
\bar{S}_{u,v,w} = \bigcup_{u < w' \leq w} S_{u,v,w'}, \quad \partial S_{u,v,w} = \bigcup_{u < w' < w} S_{u,v,w'}.
\]

Here, by a stratified space we mean a decomposition of a semialgebraic set into disjoint smooth submanifolds labeled by the elements of a partially ordered set, as described for example in [11, Sec. 1.1]. Although the stratifications we consider seem to satisfy Whitney’s regularity conditions (cf. [11, Sec. 1.2]), we will not need to verify these conditions to justify our constructions.

**Theorem 1.3 (Type A only).** All the strata \( S_{u,v,w} \) are orientable.

**Theorem 1.4 (Type A only).** Each closed stratum \( \bar{S}_{u,v,w} \) is contractible. Moreover, the contraction can be chosen so that it restricts to a contraction of the open stratum \( S_{u,v,w} \).

These theorems ensure that the stratified spaces \( \text{Lk}(u, v) \) have the desired homological properties, as we will now explain. Let \( H_i(X) \) (resp., \( H_i(X, X') \)) denote, as usual, the ordinary \( i \)th homology group CW complex \( X \) (resp., pair of CW complexes \( X' \subset X \)). The corresponding Euler characteristics are denoted by \( \chi(X) \) and \( \chi(X, X') \), respectively.

**Corollary 1.5 (Type A only).** For any \( u < w \leq v \), we have

\[
H_i(\bar{S}_{u,v,w}, \partial \bar{S}_{u,v,w}) = \begin{cases} 
\mathbb{Z} & \text{if } i = \ell(w) - \ell(u) - 1, \\
0 & \text{otherwise.}
\end{cases}
\]

Consequently, \( \chi(\bar{S}_{u,v,w}, \partial \bar{S}_{u,v,w}) = (-1)^{\ell(w)-\ell(u)-1} \).

**Proof.** We will need the Lefschetz duality isomorphism [8, Ex. 18.3]:

\[
H_i(X, A) \simeq H^{n-i}(X \setminus A; \mathbb{Z}), \quad i > 0,
\]

where \( X \) is a compact topological space and \( A \) its closed subset such that \( X \setminus A \) is a smooth orientable \( n \)-dimensional manifold. Take \( X = \bar{S}_{u,v,w} \) and \( A = \partial \bar{S}_{u,v,w} \).

Then Theorems 1.2 and 1.3 ensure that the above conditions are satisfied, with
Corollary 1.6 (Verma’s formula for the type $A$). \[ \sum_{u \leq w \leq v} (-1)^{\ell(w)} = 0. \]

Thus every Bruhat interval is an Eulerian poset [18].

**Proof.** The additivity of the Euler characteristic [7, V.5.7], which applies in view of Theorem 1.2, gives

\[ \chi(Lk(u,v)) = \sum_{u < w \leq v} \chi(S_{u,v,w} \cap \partial S_{u,v,w}). \]

In the last identity, the left-hand side is equal to 1 by Theorem 1.4, while the right-hand side is equal to \( \sum_{u < w \leq v} (-1)^{\ell(w)-\ell(u)-1} \) by Corollary 1.5. Simplifying, we obtain the desired formula.

For the type $A$, we prove the following refinement of Theorem 1.2. Let us define the stratified space $Y_{[u,v]}$ by $Y_{[u,v]} = \bigcup_{w \in [u,v]} Y_{w}^\circ$. Note that $Lk(u,v) = \lim\{Y_{w}^\circ, Y_{w}\} = \lim\{Y_{w}^\circ, Y_{[u,v]}\}$.

**Theorem 1.7 (Type $A$ only).** The stratified space $Y_{[u,v]}$ has the structure of the direct product of the cell $Y_{u}^\circ$ and the cone over the link $Lk(u,v)$. More precisely, there exists an isomorphism of stratified spaces $Y_{[u,v]}$ and $Y_{u}^\circ \times \text{Cone}(Lk(u,v))$ whose restriction to each stratum is a diffeomorphism.

**Remark 1.8.** Theorem 1.3 can be deduced from Theorem 1.7 as follows. By (1.1), the stratum $S_{u,v,w}$ coincides with the interior of $Lk(u,w)$. Thus Theorem 1.7 asserts, in particular, that the cell $Y_{w}^\circ$ is a direct product of the cell $Y_{u}^\circ$ and the interior of the cone over $Lk(u,w)$. Both cells $Y_{w}^\circ$ and $Y_{u}^\circ$ are evidently orientable. Therefore (see e.g. [12, Ex. 3.2.24]), the interior of the cone over $Lk(u,w)$ is orientable and so is the interior of $Lk(u,w)$.

**Conjecture 1.9.** Theorems 1.4 and 1.7 (hence Theorem 1.3 and Corollary 1.5) hold for any semisimple algebraic group $G$.

We believe that Conjecture 1.9 can be strengthened as follows.

**Conjecture 1.10.** Each stratum $S_{u,v,w}$ (resp., its closure and its boundary) is homeomorphic to an affine space (resp., a closed ball and a sphere) of dimension $\ell(v) - \ell(u) - 1$ (resp., $\ell(v) - \ell(u) - 1$ and $\ell(v) - \ell(u) - 2$). Thus $Lk(u,v)$ is a regular cell complex.

Assuming Conjecture 1.10 holds, each stratified link $Lk(u,v)$ provides a geometric realization of the “generalized synthetic Schubert variety” whose existence was hypothesized by Björner [4].

We hope to extend the construction of the spaces $Lk(u,v)$ to an arbitrary simply laced Coxeter group, and possibly further, so that the analogs of all statements.
formulated above would still hold. (Note that Björner’s original result applies to intervals in any Coxeter group.)

It should be mentioned that one of our ‘‘hidden motivations’’ has been the desire to better understand the combinatorics of Kazhdan–Lusztig polynomials. It was already pointed out in their original paper [14] that Verma’s formula is equivalent to the assertion that the constant term of any Kazhdan–Lusztig polynomial is 1.

The remainder of this paper is organized as follows. Sections 2–3 introduce some useful Lie-theoretic machinery; in particular, we define a projection onto a cell $Y_u$ that plays a crucial role in subsequent proofs. In Section 4, we prove Theorem 1.2. Section 5 contains the proofs of Theorems 1.4 and 1.7. These proofs are based on a technical lemma (Lemma 5.4), which is proved in Section 6 for the special case of $G = SL(n)$; this is the only ‘‘type-specific’’ ingredient of our proofs.

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2. Preliminaries

This section introduces necessary technical background; throughout it, we do not claim any originality. The notation used here is consistent with [9]. In particular, we denote by

$$G_0 = B_- B = N_+ H N$$

the set of elements of $x \in G$ that have a Gaussian decomposition; for the latter, we use the notation $x = [x_-] [x_0] [x_+]$.

We think of the Weyl group $W$ as the quotient of the normalizer of $H$ modulo $H$, and we identify each element $w \in W$ with a fixed representative in $G$.

**Lemma 2.1.** For $w \in W$, we have $w^{-1} B_- w \subset G_0$ and $w^{-1} B w \subset G_0$. Moreover, $w^{-1} N_- w \subset N_- N$ and $w^{-1} N w \subset N_- N$.

**Proof.** Since $B = H N$, $B_- = H N_-$, and $w$ normalizes $H$, it suffices to prove the last statement. It is well known (cf. [9, Prop. 2.12] or [2, (5.3)]) that any $x \in N$ is uniquely factored as $x = x_1 x_2$ with $x_1 \in N \cap w N_- w^{-1}$ and $x_2 \in N \cap w N w^{-1}$. Hence $N \subset w N_- w^{-1} \cdot w N w^{-1} = w N_- N w^{-1}$, as desired.

**Lemma 2.2.** If $z \in w^{-1} B_- w$ then $[z]_-, [z]_+ \in w^{-1} N_- w$. Analogously, if $z \in w^{-1} B w$ then $[z]_- , [z]_+ \in w^{-1} N w$.

**Proof.** It is enough to show that $z \in w^{-1} B w$ implies $[z]_+ \in w^{-1} N w$. Just as in the proof of Lemma 2.1, we can write $z = h w^{-1} x_1 x_2 w$, where $h \in H$, $x_1 \in N \cap w N_- w^{-1}$, and $x_2 \in N \cap w N w^{-1}$. Then $z = (h w^{-1} x_1 w) (w^{-1} x_2 w)$, where the factors belong to $B_-$ and $N$, respectively. Thus $[z]_+ = w^{-1} x_2 w \in w^{-1} N w$, as desired.
We define the subgroups
\[ N_-(w) = w^{-1}Bw \cap N_-, \]
\[ N(w) = w^{-1}Bw \cap N = w^{-1}Nw \cap N, \]
and the set
\[ N^w = B_- wB_- \cap N. \]

**Lemma 2.3.** For any \( w \in W \) and \( x_w \in N^w \), there exists a unique \( y \in N_-(w) \) satisfying \( x_w = [wy]_+ \). Specifically, \( y = w^{-1}[x_w w^{-1}]_+ w \).

**Proof.** Immediate from [9, Prop. 2.10 and 2.17]. \( \square \)

**Lemma 2.4.** Let \( x_w \in N^w \), \( b_- \in B_- \), and \( x_w b_- \in G_0 \). Then \( [x_w b_-]_+ \in N^w \).

**Proof.** \( [x_w b_-]_+ \in B_- x_w b_- \subseteq B_- B_- wB_- b_- = B_- wB_- \). \( \square \)

The following statement, though obvious, is quite useful.

**Lemma 2.5.** If \( x \in G_0 \) and \( y \in G \), then \( [[x]_+ y]_+ = [xy]_+ \), provided one of the two sides is well-defined.

**Lemma 2.6.** For any \( x_w \), \( \tilde{x}_w \in N^w \), there exists a unique \( n_1 \in N_-(w) \) satisfying \( x_w = [\tilde{x}_w n_1]_+ \). Specifically, \( n_1 = w^{-1}([\tilde{x}_w w^{-1}]_+^{-1}[x_w w^{-1}]_+ w \).

**Proof.** Uniqueness follows from the uniqueness part of Lemma 2.3, together with Lemma 2.5 and the fact that \( N_-(w) \) is a group. In more detail: Assume that \( x_w = [\tilde{x}_w n_1]_+ = [\tilde{x}_w n'_1]_+ \), where \( n_1 \neq n'_1 \) and \( n_1, n'_1 \in N_-(w) \). Let \( y \) be as in Lemma 2.3. Then \( x_w = [\tilde{x}_w n_1 \cdot n_1^{-1} n'_1]_+ = [x_w n_1^{-1} n'_1]_+ = [w y n_1^{-1} n'_1]_+ \), where \( y \neq y n_1^{-1} n'_1 \in N_-(w) \), a contradiction.

With the notation \( y = w^{-1}[x_w w^{-1}]_+ w \) and \( \tilde{y} = w^{-1}[\tilde{x}_w w^{-1}]_+ w \), it remains to check that \( n_1 = \tilde{y}^{-1} y \) satisfies \( x_w = [\tilde{x}_w n_1]_+ \). Indeed, \( x_w = [wy]_+ = [w \tilde{y} n_1]_+ = [x_w n_1]_+ \).

We now turn to total nonnegativity. Let us first recall Lusztig’s original definition [16], whereby the set \( Y \) of totally nonnegative elements in \( N \) is defined as the multiplicative monoid generated by the elements
\[ x_t(t) = \exp(t e_i), \]
where \( t \geq 0 \) and the \( e_i \) are the Chevalley generators of the Lie algebra of \( N \). One of the first results in [16] is the following description of the Bruhat stratum \( Y^w = Y \cap B_- wB_- \).

**Proposition 2.7** [16]. Let \( (a_1, \ldots, a_l) \) be a reduced word for \( w \in W \). Then the map
\[ (t_1, \ldots, t_l) \mapsto x_{a_1}(t_1) \cdots x_{a_l}(t_l) \]
is a bijection between \( \mathbb{R}^l_{>0} \) and \( Y^w \).
(It is clear from this description that $Y^\circ_w$ is indeed a cell of dimension $l = \ell(w)$; cf. Proposition 1.1.) One is tempted to use the parameterizations (2.2) to prove our main theorems. Unfortunately, this approach encounters substantial difficulties, due chiefly to the fact that the relationship between parameterizations of adjacent cells is generally quite complicated. In what follows, we make very little use of Proposition 2.7.

**Proposition 2.8** [17, 6.3]. The cell $Y^\circ_w$ is a connected component in $N^w$ (in the ordinary topology).

For $u \in W$, we denote $Y_{\geq u} = \bigcup_{v \geq u} Y^\circ_v$.

To state the next result, we will need the notion of a generalized minor of an element $x \in G$ (see [9, Sec. 14]). Generalized minors are certain regular functions on $G$ that can be defined as suitably normalized matrix coefficients corresponding to pairs of extremal weights in some fundamental representation of $G$. In the case of type $A$, this notion coincides with the ordinary notion of a minor of a square matrix.

**Proposition 2.9** [10, Thm. 3.1]. An element $x \in G$ is totally nonnegative, in the sense of Lusztig [16], if and only if all its generalized minors are nonnegative.

**Lemma 2.10.** If a generalized minor does not vanish at some point $x \in Y^\circ_u$, then it vanishes nowhere in $Y^\circ_u$ and, moreover, nowhere in $Y_{\geq u}$.

**Proof.** For the type $A$, this is an immediate corollary of [1, Prop. 5.2.2]. The general case can be deduced from (highly nontrivial) [3, Prop. 7.4]. According to the latter, for any generalized minor $\Delta$ and any sequence of indices $a = (a_1, \ldots, a_m)$, the function $P_a(t_1, \ldots, t_m) = \Delta(x_{a_1}(t_1) \cdots x_{a_m}(t_m))$ (cf. (2.2)) is either identically zero or a polynomial with positive integer coefficients. (The type-$A$ version of this statement is well known; see e.g. [1, Thm. 2.4.4].) Since $\Delta$ does not vanish at some point in $Y^\circ_u$, we know that $P_a$ is a nonzero polynomial for any reduced word $a$ for $u$. For $v \geq u$, any reduced word $b$ for $v$ contains some reduced word $a$ for $u$ as a subword (see [13, 5.10]). Hence $P_a$ is a specialization of $P_b$ obtained by setting some of the variables equal to zero. Then $P_a \neq 0$ implies $P_b \neq 0$. On the other hand, $P_b$ is a polynomial with positive coefficients, so $P_b(t_1, t_2, \ldots) \neq 0$ for any $t_1, t_2, \ldots > 0$ or, equivalently, $\Delta(x) \neq 0$ for any $x \in Y^\circ_v$.

**Lemma 2.11.** For any $u \in W$, we have $B_- u B_- \subseteq G_0 u$. In particular, $N^u \subseteq G_0 u$.

**Proof.** This follows from Lemma 2.1.

**Corollary 2.12.** $Y_{\geq u} \subseteq G_0 u$.

**Proof.** By [9, Cor. 2.5], the set $G_0 u$ is defined by several inequalities of the form $\Delta \neq 0$, where $\Delta$ is a generalized minor. Since $Y^\circ_u \subseteq G_0 u$ (by Lemma 2.11), none of these minors vanishes on $Y^\circ_u$ and hence none vanishes anywhere on $Y_{\geq u}$, by Lemma 2.10.
Theorem 2.13 ([6, Cor. 1.2]; cf. [15, Sec. 1.2]). For \( u, v \in W \), the intersection \( B_-vB_- \cap BuB_- \) is non-empty if and only if \( u \leq v \).

Corollary 2.14. \( G_0u \subset \bigcup_{v \geq u} B_-vB_- \).

Proof. Let \( x \in G_0u \) and \( x \in B_-vB_- \), and let \( v \in W \). Then, by Theorem 2.13, \( B_-vB_- \cap BuB_- \neq \emptyset \implies B_-vB_- \cap Bu \neq \emptyset \implies u \leq v \), as desired.

Corollary 2.15. \( Y \geq u \subset Y \cap G_0u \).

Proof. The inclusion \( Y \geq u \subset Y \cap G_0u \) is Corollary 2.12. The opposite inclusion is immediate from Corollary 2.14.

Lemma 2.16. \( Y \geq u \subset Y \geq u \).

Proof. By [16, Lemma 2.14], for any \( w_1, w_2 \in W \) we have \( Y^{w_1}w_2 = Y^{w_1}w_2 \) for some \( w_3 \in W \). Moreover, it is clear from the proof of this statement in [16] that \( w_3 \geq w_1 \), and the lemma follows.

Example: \( G = SL(3, \mathbb{R}) \). Let \( u = s_1 \), the transposition of 1 and 2 in the symmetric group \( W = S_3 \). Then, using the notation \( x = \begin{bmatrix} 1 & x_{12} & x_{13} \\ 0 & 1 & x_{23} \\ 0 & 0 & 1 \end{bmatrix} \) for the elements \( x \in N \), we have

\[
N(u) = \{ x_{12} = 0 \}, \quad N^u = \left\{ x_{12} \neq 0, \quad x_{13} = 0, \quad x_{23} = 0 \right\}, \quad N \cap G_0u = \{ x_{12} \neq 0 \}
\]

and

\[
Y^u_\geq = \left\{ x_{12} > 0, \quad x_{13} = 0, \quad x_{23} = 0 \right\}, \quad Y^u_\geq = \left\{ x_{12} > 0, \quad x_{13} \geq 0, \quad x_{23} \geq 0 \right\}.
\]

3. Projecting on a Cell

In this section, we introduce a projection \( \pi_u \): \( Y \geq u \rightarrow Y^u_\geq \) that will later be used to construct and study the links \( Lk(u, v) = Lk(Y^u_\geq, Y^v_\geq) \). This projection can be viewed as the totally positive version of the projection of an affine open neighborhood of a Schubert cell onto the cell itself, which arises from the direct product decomposition described by Kazhdan and Lusztig in [15, Secs. 1.3–1.4].

Let us fix an element \( u \in W \).

Lemma 3.1. If \( x \in G_0u \cap G_0 \) (in particular, if \( x \in Y \geq u \)—cf. Corollary 2.12), then \( u^{-1}[xu^{-1}]_+u \in G_0 \) and \( u[u^{-1}[xu^{-1}]_+]_- \in G_0 \).

Proof. The first statement follows from Lemma 2.1. Proof of the second: For some \( b_- \in B_- \) and \( b \in B \), we have \( u[u^{-1}[xu^{-1}]_+]_- = uu^{-1}b_-xu^{-1}ub = b_-xb \in G_0 \).

Lemma 3.2. The map \( (x_u, x^u) \mapsto x = x_u.x^u \) is a bijection

\[ N^u \times N(u) \rightarrow N \cap G_0u. \]
The inverse map $x \mapsto (x_u, x^u)$ is given by

$$x_u = [u[u^{-1}[xu^{-1}] + u] - ]_+$$  (3.1)

and

$$x^u = [u^{-1}[xu^{-1}] + u]_+.$$  (3.2)

Furthermore, if $x \in N \cap G_0u$ is totally nonnegative (i.e., $x \in Y_{\geq u}$—cf. Corollary 2.15), then $x_u$ is totally nonnegative (i.e., $x_u \in Y_u^\circ$).

**Proof.** Assume that $x_u \in N^u$, $x^u \in N(u)$, and $x = x_u x^u$. Then $x = x_u x^u \in G_0 u \cdot u^{-1} N u = G_0 u$ (by Lemma 2.11 and the definition of $N(u)$), as claimed.

Let us prove that the map in question is a surjection. Let $x \in N \cap G_0 u$, and let $x_u$ and $x^u$ be given by (3.1) and (3.2); note that the right-hand sides of these formulas are well-defined (by Lemma 3.1). Thus $x_u = [u[y] - ]_+$ and $x^u = [y]_+$, where $y = u^{-1}[xu^{-1}] + u$. Then $x_u \in B_u[y]_+ \subset B_u B_u$ and $x^u \in N(u)$ (by Lemma 2.2). Furthermore, $x_u x^u = [u[y] - ][u[y] + ]_+ = [u y]_+$ (since $y \in N \cap N$ by Lemma 2.1) and therefore $x_u x^u = [u y]_+ = [[xu^{-1}] + u]_+ = x$ (by Lemma 2.5).

Let us now prove injectivity. Again, suppose $x_u \in N^u$, $x^u \in N(u)$, and $x = x_u x^u$. We will show that $x_u$ and $x^u$ can be recovered from $x$ via (3.1)–(3.2). Since $x \in N \cap G_0 u$, the right-hand sides of (3.1)–(3.2) are well-defined (by Lemma 3.1). Then

$$[u[u^{-1}[xu^{-1}] + u] - ]_+ = [u[u^{-1}[xu^{-1}] + u] - u]_+$$

$$= [u[u^{-1}[xu^{-1}] + u] - u]_+$$

$$= [u^{-1}[xu^{-1}] + u]_+$$

$$= x_u$$

(proving (3.1).

Let us prove (3.2). Denote $A = u^{-1}[xu^{-1}] + u$. We have

$$x = [x]_+ = [u[u^{-1}[xu^{-1}] + u] + ]_+ = [u A]_+$$

(by Lemma 2.1). On the other hand, we already proved that $x_u = [u[A] - ]_+$. Thus $x = x_u[A]_+$ (i.e., $x_u = [A]_+$), as desired.

It remains to prove that $x_u$ is totally nonnegative whenever $x$ is. Assume that $x \in Y_u^\circ \subset Y_{\geq u}$. Consider a path that connects $x$ with a point $x_0 \in Y_u^\circ$ and stays inside $Y_u^\circ$ (such a path exists because $Y_u^\circ$ is connected and its boundary contains $Y_u^\circ$; see Proposition 1.1). The image of this path under the projection $N \cap G_0 u \rightarrow N^u$ connects $x_u$ with $x_0$. Since $x_0 \in Y_u^\circ$, Proposition 2.8 implies that $x_u \in Y_u^\circ$.

In view of Lemma 3.2, the formula

$$\pi_u(x) = [u[u^{-1}[xu^{-1}] + u] - ]_+$$  (3.3)

defines a continuous projection $\pi_u$: $Y_{\geq u} \rightarrow Y_u^\circ$. (The map $\pi_u$ is a projection since $x = x \cdot 1$ gives the factorization in question for $x \in Y_u^\circ$.)

**Example:** $G = SL(3, \mathbb{R})$, $u = s_1$. For $x \in Y_{\geq u}$ (or, more generally, $x \in N \cap G_0 u$), the factorization $x = x_u x^u$ is given by
The fiber of the projection \( \pi_u : x \mapsto x_u \) over a point \( x_u = \begin{bmatrix} 1 & a \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \in Y_u^\circ \) \((a > 0)\) is therefore
\[
\pi_u^{-1}(x_u) = Y \cap \{ x_{12} = a \} = \left\{ \begin{bmatrix} 1 & a & x_{13} \\ 0 & 1 & x_{23} \\ 0 & 0 & 1 \end{bmatrix} : ax_{23} \geq x_{13} \geq 0 \right\}.
\]

4. Transversals and Links. Proof of Theorem 1.2

Our next goal is to prove that the restriction of the projection \( \pi_u \) onto \( Y_{[u, v]} \) is globally trivialized along \( Y_u^\circ \).

**Lemma 4.1.** For any \( \tilde{x} \in N \cap G_{0u} \) and any \( x_u \in N^u \), there exists unique \( n_\geq \in N_{\geq}(u) \) such that the element \( x' = [\tilde{x} n_\geq]_+ \) is well-defined and belongs to \( x_u N(u) \).

If, moreover, \( \tilde{x} \) and \( x_u \) are totally nonnegative, then \( x' \) is also totally nonnegative. We thus obtain a cell-preserving projection
\[
\rho_{x_u} : Y_{\geq u} \to \pi_u^{-1}(x_u)
\]
\[
\tilde{x} \mapsto x'
\] (see Figure 2).

![Figure 2](image)

**Figure 2** The projection \( \rho_{x_u} \)

*Proof.* Let \( \tilde{x} = \tilde{x}_u \tilde{x}^u \), where \( \tilde{x}_u \in N^u \) and \( \tilde{x}^u \in N(u) \), as in Lemma 3.2. Let \( n_\geq \in N_{\geq}(u) \) be such that \( x_u = [\tilde{x}_u n_\geq]_+ \) (such \( n_\geq \) exists and is unique by Lemma 2.6).

Set
\[
n_\leq = [\tilde{x}^u]^{-1} n_\geq.
\] (4.2)

Since both \( \tilde{x}^u \) and \( n_\geq \) belong to \( u^{-1} Nu \), the element \( n_\leq \) is well defined in view of Lemma 2.1, and belongs to \( N_-(u) \) by Lemma 2.2. Let us prove that the element \( n_\leq \) defined by (4.2) has the desired properties; that is, \( x' = [\tilde{x} n_\leq]_+ \) is well-defined and belongs to \( x_u N(u) \), as shown in the following diagram:
We denote by $\text{Total nonnegativity of}$ Theorem 4.2. (1)

$\pi_n \downarrow \downarrow \downarrow \downarrow$

$\tilde{x}_n \leadsto x' = [\tilde{x}_n]_+$ (4.3)

Denote $z = \tilde{x}u$. Once again, $z \in \overline{\mathcal{U}u} \subset N - N$, and (4.2) implies $[z]_- = [\tilde{x}u]_{\mathcal{N}} = [\tilde{x}u[(\tilde{x}u)^{-1}n_1]_{\mathcal{N}}] = n_1$. Then $\tilde{x}_n - \tilde{x}_u z = \tilde{x}_n n_1[z]_+ \in G_0$ (because $[\tilde{x}_n]_+=x_u$), so $x'$ is indeed well-defined. Furthermore, $x' = x_u[z]_+ \in \mathcal{U}N(u)$ (by Lemma 2.3), as desired.

Uniqueness is proved by a similar argument. Suppose that $n_\mathcal{N} \subset N_\mathcal{N}(u)$ is such that $x' = [\tilde{x}_n]_+ \in \mathcal{U}N(u)$. As before, denote $z = \tilde{x}u$. Then $z = [z]_-$ and $n_\mathcal{N} = [\tilde{x}_n]_+ = [\tilde{x}_n[z]_+ \cdot [z]_+$. Since $[\tilde{x}_n[z]_+]_+ \in N_\mathcal{N}$ (by Lemma 2.4) and $[z]_+ \in N(u)$, it follows from Lemma 3.2 that $[\tilde{x}_n[z]_+]_+ = x_u = [\tilde{x}_n]_+$. Hence, by Lemmas 2.2 and 2.6, $n_1 = [z]_- = [\tilde{x}u]_{\mathcal{N}}$, implying (4.2).

It remains to prove the second part of the lemma. In view of Lemma 2.6, a path connecting $\tilde{x}_n$ and $x_u$ within $\mathcal{U}^\mathcal{N}$ gives a continuous deformation of the identity $1 \in G$ into $n_1 \in N_\mathcal{N}(u)$, which gives rise (via (4.2)) to a continuous deformation of $1$ into $n_\mathcal{N}$ and, finally, to a path connecting $\tilde{x}$ and $x' = [\tilde{x}_n]_+$ within the Bruhat cell containing $x$. Hence $x'$ is totally nonnegative by Proposition 2.8.

**Example:** $G = \text{SL}(3, \mathbb{R})$, $u = s_1$. For

$$
\tilde{x} = \begin{bmatrix}
1 & \tilde{x}_{12} & \tilde{x}_{13} \\
0 & 1 & \tilde{x}_{23} \\
0 & 0 & 1
\end{bmatrix} \in \mathcal{U}^\mathcal{N} \quad \text{and} \quad x_u = \begin{bmatrix}
1 & a & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \in \mathcal{U}^\mathcal{N},
$$

computations give

$$
n_{-1} = \begin{bmatrix}
1 & 0 & 0 \\
a^{-1} - \tilde{x}_{12}^{-1} & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
$$

and

$$
x' = \rho_{x_u}(\tilde{x}) = [\tilde{x}_n]_+ = \begin{bmatrix}
1 & a \\
\frac{a\tilde{x}_{13}}{\tilde{x}_{12}} & \tilde{x}_{12} \tilde{x}_{23} - \tilde{x}_{13} \\
0 & 1 \\
0 & 0 & 1
\end{bmatrix}.
$$

Total nonnegativity of $x'$ does indeed follow from total nonnegativity of $\tilde{x}$ and $x_u$.

We denote by $w_u$ the element of maximal length in $W$.

**Theorem 4.2.** (1) For $x_u \in \mathcal{U}^\mathcal{N}$, the set $x_uN(u)$ is a smooth submanifold in $N \cap G \mathcal{U}u$ diffeomorphic to the affine space $\mathbb{R}^{(w_u) - (u)}$. Furthermore, $x_uN(u)$ is transversal to every Bruhat stratum $N_{\mathcal{W}}$, $w \geq u$ (and hence to every stratum $\mathcal{U}u \subset \mathcal{U}^\mathcal{N}$).

(2) For $x_u, \tilde{x}_n \in \mathcal{U}^\mathcal{N}$, the map $\rho_{x_u}$ described in Lemma 4.1 establishes a diffeomorphism between $\tilde{x}uN(u)$ and $x_uN(u)$. This diffeomorphism respects total nonnegativity and the Bruhat stratification; more precisely, it restricts to a stratified diffeomorphism between the fibers $\pi^{\mathcal{U}}(\tilde{x}_u)$ and $\pi^{\mathcal{U}}(x_u)$. 


Proof. The map \( n_+ \mapsto x_+ n_+ \) establishes a diffeomorphism between \( N(u) \cong \mathbb{R}^{(\ell(u)-\ell(w))} \) and \( x_+ N(u) \). To prove transversality, consider a point \( x \in x_+ N(u) \cap N^w \).

It will be enough to show that \( x^w N(u) \) is transversal to the smooth submanifold \( [x N_{-}(u)]_+ \) of dimension \( \ell(u) \) in \( N^w \). Assume the contrary—in other words, that there exists a common tangent vector \( v \) to \( [x N_{-}(u)]_+ \) and \( x^w N(u) \) at the point \( x \).

Let us evaluate the differential \( D \) of the projection \( N \cap G_0 u \to N^w \) at the vector \( v \):

On the one hand, the projection is constant on \( x^w N(u) \) and hence \( D(v) = 0 \):

On the other hand, in view of (4.2), the restriction of the projection onto \( [x N_{-}(u)]_+ \) is a diffeomorphism and hence \( D(v) \neq 0 \)—a contradiction.

Let us prove the second part of the theorem. From (4.2) and (4.3), we have

\[
\xi_{i}(t) = \sum_{j} x_{i,j} + (\pi_0^{-1}(x_{i,j}) \cap Y_{[u,w]}).
\]

In particular, \( Y_{[u,w]} \leq u \cong Y^w \leq \pi_0^{-1}(x_{i,j}) \).

Proof of Theorem 1.2. Corollary 4.3 shows that the link of \( Y^w_{[u,w]} \) in \( Y_{[u,w]} \) is well-defined (up to a stratified diffeomorphism); it is explicitly given by

\[
\text{Lk}(u,v) = (\pi_0^{-1}(x_{i,j}) \cap S_{x_{i,j}}(x_{i,j})),
\]

where \( x_{i,j} \) is an arbitrary point on \( Y^w_{[u,w]} \) and \( S_{x_{i,j}}(x_{i,j}) \) is a small sphere centered at \( x_{i,j} \).

The first two statements of Theorem 1.2 follow right away. The equalities (1.1) follow from the analogous property for the Bruhat stratification of \( Y \) (cf. Proposition 1.1), combined with Corollary 4.3.

5. Proofs of Theorems 1.4 and 1.7

Recall that the elements \( x_i(t) \) are defined by (2.1). For the type \( A_{n-1} \), \( x_i(t) \) is the \( n \times n \) matrix that differs from the identity matrix in a single entry (equal to \( t \)) located in row \( i \) and column \( i+1 \).

Definition 5.1. We define the regular map \( \text{str} : N \to C \) by the conditions \( \text{str}(x_i(t)) = t \) and \( \text{str}(xy) = \text{str}(x) + \text{str}(y) \). In particular, in the case of type \( A \), we have \( \text{str}(x) = x_{i,i+1} \), the sum of the matrix elements immediately above the main diagonal.

Definition 5.2. For \( \tau > 0 \), let \( d(\tau) \in H \) be uniquely defined by the conditions \( d(\tau) \circ a_i = \tau \) for all simple roots \( a_i \). Then \( d(\tau) x_i(a) d(\tau)^{-1} = x_i(\tau a) \) for any \( i \). For the type \( A_{n-1} \),
and the automorphism \( x \mapsto d(\tau)xd(\tau)^{-1} \) of the group \( N \) multiplies each matrix entry \( x_{ij} \) of \( x \) by \( \tau^{i-j} \).

Note that the automorphism \( x \mapsto d(\tau)xd(\tau)^{-1} \) preserves the cells \( N^w \) and the subgroups \( N(w) \); it also preserves total nonnegativity.

**Definition 5.3.** For \( u \in W \) and \( x_u \in Y_u^\circ \), we define the vector field \( \psi \) on \( \pi_u^{-1}(x_u) \) by

\[
\psi(x) = \frac{d}{d\tau}(\rho_{x_u}(d(\tau)xd(\tau)^{-1})) \bigg|_{\tau=1}
\]

(recall that \( \rho_{x_u} \) is defined by (4.1)).

**Lemma 5.4 (Type A only).** The vector field \( \psi \) vanishes nowhere on \( \pi_u^{-1}(x_u) \) except at the point \( x_u \). The directional derivative \( \nabla_0 \str(x) \) is positive at every point \( x \neq x_u \).

The proof of this lemma is given in Section 6.

**Proof of Theorem 1.7.** In view of Corollary 4.3, it remains to show that the fiber \( \pi_u^{-1}(x_u) \cap Y_{[u,v]} \) has the structure of the cone over the link \( \text{Lk}(u,v) \).

The vector field \( \psi \) can be extended (by the same formula (5.2), with \( x_u = \pi_u(x) \)) to the open subset \( N \cap G_0 u \) of \( N \). Furthermore, \( \psi(x) \) is given by rational functions in the affine coordinates of \( x \); therefore, the theorem of uniqueness and existence of solutions applies to this extension of \( \psi \) (and hence to \( \psi \) itself). Since \( \psi \) is tangent to each stratum of the preimage \( \pi_u^{-1}(x_u) \) (all these strata are smooth by part (1) of Theorem 4.2), it follows that every trajectory of \( \psi \) is contained in a single stratum.

The intersection \( Y \cap \{ \str \leq c \} \) is compact for any \( c > 0 \). Lemma 5.4 then implies that, for every \( x_0 \in \pi^{-1}(x_u) \), the solution \( x(t) \) of the Cauchy problem \( \dot{x} = \psi(x) \) and \( x(0) = x_0 \) exists for \( t < 0 \) exists for \( t \in (-\infty, 0] \). (Otherwise, the trajectory \( T_\tau = \{ x(t) : t \leq 0 \} \) would hit the boundary of the stratum containing \( x_0 \).)

The trajectory \( T_\tau \) must have limit points; let \( x_{\text{lim}} \) be one of them. The function \( s : t \mapsto \str(x(t)), t < 0 \), is increasing (by Lemma 5.4) and bounded from below.

Therefore \( \lim_{t \to -\infty} s(t) = 0 \), implying that \( \nabla_0 \str(x_{\text{lim}}) = 0 \). By Lemma 5.4, this means that \( x_{\text{lim}} = x_u \). Thus every trajectory of \( \psi \) originates at the point \( x_u \) (at \( t = -\infty \)). A similar argument shows that \( \lim_{t \to t^+} \str(x(t)) = +\infty \), where \( t^+ \) denotes the upper limit of the maximal domain of definition of \( x(t) \) (so \( t^+ \in [0, \infty) \)). We conclude that the function \( \str \) increases from \( \str(x_u) \) to \( \infty \) along each trajectory of \( \psi \), except for the trajectory \( x(t) = x_u \). Thus every nontrivial trajectory \( T \subset Y_{[u,v]} \) intersects the set

\[
L_\varepsilon(u,v) = L_{\varepsilon,x_u}(u,v) = \pi_u^{-1}(x_u) \cap Y_{[u,v]} \cap \{ x : \str(x) = \str(x_u) + \varepsilon \}
\]
at exactly one point; see Figure 3. Therefore, \( \pi^{-1}(x_u) \cap Y_{[u,v]} \) is diffeomorphic to the cone \( \text{Cone}(L_c(u,v)) \). (In particular, \( \pi^{-1}(x_u) \cong \text{Cone}(L_c(u, w_u)) \).) This implies that \( L_c(u,v) \) is isomorphic (as a stratified space) to the link \( \text{Lk}(u,v) \). Theorem 1.7 is proved.

**Proof of Theorem 1.4.** For \( x \in \pi^{-1}(x_u) \cap Y_{[u,v]} \), let \( \lambda_{u,v}(x) \) denote the unique point of intersection of the link \( L_c(u,v) \) with the trajectory of \( \psi \) that passes through \( x \) (see the sentence containing (5.3)).

Fix \( z \in Y_{[u,v]} \), and define \( R_{u,v} : L_c(u,v) \times [0,1] \rightarrow L_c(u,v) \) by

\[
R_{u,v}(x,\tau) = \lambda_{u,v}(\rho_\tau(\pi(x)d(\tau)^{-1}d(1-\tau))dx(1-\tau)^{-1})).
\]

(Note that \( d(\tau)d(\tau)^{-1} \in Y_{[u,v]} \), and therefore \( d(\tau)d(\tau)^{-1}d(1-\tau)dx(1-\tau)^{-1} \in Y_{[u,v]} \), by Lemma 2.16; hence the right-hand side of (5.4) is well-defined.)

Let us show that the map \( R_{u,v} \) is a deformation retraction of \( L_c(u,v) \) into a point. First, \( R_{u,v} \) is continuous with respect to \( \tau \). Second, \( \lim_{\tau \to 0} d(\tau)dx(\tau)^{-1} = 1 \) and \( \lim_{\tau \to 1} d(\tau)dx(\tau)^{-1} = z \). Therefore \( R_{u,v}(x,0) = x \). On the other hand, \( R_{u,v}(x,1) = \lambda_{u,v}(\rho_{\pi(z)}(\pi(x))) \), a point independent of \( x \).

It remains only to observe that \( S_{u,v,w} \cong S_{u,w,w} \cong L_c(u,w) \).}

### 6. Proof of Lemma 5.4

Our first goal is an explicit formula for the vector field \( \psi \).

Throughout this section, we use the following notation. For \( x \in Y_{[u,v]} \), denote

\[
\begin{align*}
    x_u &= \pi_u(x) = [u[u^{-1}[xu^{-1}]_+u]_-]_+, \\
    A &= u^{-1}[xu^{-1}]_+u, \\
    y &= [A]_-, \\
    x^u &= [A]_+ = x_u^{-1}x.
\end{align*}
\]

Thus \( x = x_u x^u, x_u = [uy]_+ \) (agreeing with Lemma 2.3), and \( A = xy^u \).

Let us fix an arbitrary totally nonnegative element \( d \in H \). We will need some basic properties of the cell-preserving automorphism \( x \mapsto dxd^{-1} \).
Lemma 6.1. The automorphism \( x \mapsto dxd^{-1} \) of \( Y_{\geq n} \) commutes with the maps \( x \mapsto x_u \) and \( x \mapsto x^n; \)

\[
(dx^{-1})_u = dx_ud^{-1}, \quad (dx^{-1})^u = dx^u d^{-1}.
\]

Proof. In view of Lemma 3.2, the statement follows from the factorization

\[
dxd^{-1} = dx_ud^{-1} \cdot dx^u d^{-1},
\]

where the two factors on the right belong to \( Y_{\geq n} \) and \( N(u) \), respectively.

Lemma 6.2. The automorphism \( x_u \mapsto dx_ud^{-1} \) of \( Y_{\geq n} \) commutes with the map \( x_u \mapsto y \) (cf. Lemma 2.6). In other words,

\[
dx_ud^{-1} = [u \cdot dyd^{-1}]_+.
\]

The unique element \( n_1 \in N_- (u) \) such that \( x_u = [dx_ud^{-1} n_1]_+ \) (cf. Lemma 2.6) is given by \( n_1 = dyd^{-1} y \).

Proof. First part: \( dx_ud^{-1} = d[u_y]_+ d^{-1} = [dudy^{-1}]_+ = [udyd^{-1}]_+ \). The second part is then a special case of Lemma 2.6.

Lemma 6.3. \( \rho_{x_u}(dxd^{-1}) = x ([dA^{-1}d^{-1}A]_+)^{-1} \).

Proof. Applying (4.1) to \( \tilde{x} = dxd^{-1} \) and using Lemmas 6.1 and 6.2 along with (6.1), we obtain

\[
\rho_{x_u}(dxd^{-1}) = [dx^{-1} d (x^n)^{-1} d (dyd^{-1} y^{-1})]_+ = [dxd^{-1} ([dA^{-1}d^{-1}A]_+)]_+ \\
= [dxd^{-1} \cdot dA^{-1} d^{-1} A]_+ ([dA^{-1} d^{-1} A]_+)^{-1} \\
= [xA^{-1} d^{-1} A]_+ ([dA^{-1} d^{-1} A]_+)^{-1}.
\]

It remains to show that \( [xA^{-1} d^{-1} A]_+ = x \). This is done as follows:

\[
[xA^{-1} d^{-1} A]_+ = [x_u y^{-1} d A]_+ = [uy^{-1} d A]_+ = [uyy^{-1} d A]_+ \\
= [udy^{-1} y]_+ x^n = [uy]_+ x^n = x_u x^n = x.
\]

Let \( g, n, \) and \( b_- \) denote the Lie algebras of groups \( G, N, \) and \( B_- \), respectively. Let \( \pi_n \) denote the projection \( g \to n \) along \( b_- \). (In the case \( g = sl_n, \) \( \pi_n \) replaces all lower triangular entries of a traceless matrix by zeros.)

Lemma 6.4. Let \( f(\tau) = f_-(\tau) f_+(\tau) \), where \( f_-(\tau) \in B_- \) and \( f_+(\tau) \in N \) for all \( \tau > 0 \). Assume that \( f(1) = 1 \). Then \( f_{\tau}'(1) = \pi_n f'(1) \).

Proof. The equality \( f(1) = 1 \) implies \( f_-(1) = f_+(1) = 1 \). Then \( f'(1) = f'_-(1) f'_+(1) + f_-(1) f'_+(1) = f'_-(1) + f'_+(1) \). Since \( f'_-(1) \in b_- \) and \( f'_+(1) \in n \), we are done.

Proposition 6.5. \( \psi(x) = x \pi_n (A^{-1} d'A) \).

Proof. By Lemma 6.3, \( \rho_{x_u}(d(\tau) x d(\tau)^{-1}) = x ([d(\tau) A^{-1} d(\tau)^{-1} A]_+)^{-1} \). Hence

\[
\psi(x) = \frac{d}{d\tau} \left( \rho_{x_u}(d(\tau) x d(\tau)^{-1}) \right) \bigg|_{\tau=1} = x \frac{d}{d\tau} \left( [d(\tau) A^{-1} d(\tau)^{-1} A]_+ \right)^{-1} \bigg|_{\tau=1}.
\]
Applying Lemma 6.4 and observing that \(d(1) = 1\), we obtain:

\[
\frac{d}{dt} \left( [d(\tau)A^{-1}d(\tau)^{-1}A_{+}]^{-1} \right)_{\tau=1} = -\frac{d}{dt} \left( [d(\tau)A^{-1}d(\tau)^{-1}A_{+}] \right)_{\tau=1} \\
= -\pi_{n} \frac{d}{dt} (d(\tau)A^{-1}d(\tau)^{-1}A) \\
= -\pi_{n} (d'(1) - A^{-1}d'(1)A) = \pi_{n}(A^{-1}d'(1)A),
\]

implying the claim. \(\square\)

In the rest of this section, we consider only the case of the type \(A_{n-1}\). Thus \(W\) is the symmetric group \(S_n\). We treat the elements of \(W\) as bijective maps \(\{1, \ldots, n\} \rightarrow \{1, \ldots, n\}\) and choose permutation matrices as their representatives in \(G = GL(n)\).

**Lemma 6.6.**

1. If \(1 \leq i \leq j \leq n\) and \(u(j) \leq u(i) \leq u(j+1)\), then \((A^{-1})_{j,i} \cdot A_{i,j+1} \geq 0\).
2. If \(1 \leq j < i \leq n\) and \(u(j) \leq u(i) \leq u(j+1)\), then \((A^{-1})_{j,i} \cdot A_{i,j+1} \leq 0\).
3. Otherwise, \((A^{-1})_{j,i} \cdot A_{i,j+1} = 0\).

**Proof.** For a matrix \(a \in u^{-1}Nu\), the matrix element \(a_{ij}\) vanishes unless \(u(i) \leq u(j)\). It follows that \((A^{-1})_{j,i} : A_{i,j+1} = 0\) unless \(u(j) \leq u(i) \leq u(j+1)\), proving part (3) of the lemma. To prove parts (1) and (2), set \(s = u(i)\), \(p = u(j)\), and \(q = u(j+1)\); thus \(p \leq s \leq q\).

In what follows, we denote by \(z^{i_1, \ldots, i_r}\) the determinant of the submatrix of a matrix \(z\) formed by the rows \(i_1, \ldots, i_r\) and the columns \(j_1, \ldots, j_r\) (in that order). Using the definition of \(A\) and that \(x \in N\), we obtain

\[
A_{i,j+1} = ([xu^{-1}]_{+})_{s,q} = ([xu^{-1}]_{+})_{1\ldots s-1, q} \\
= \frac{(xu^{-1})_{1\ldots s-1, q}}{(xu^{-1})_{1\ldots s, 1\ldots s}} = \frac{x_{u^{-1}(l), u^{-1}(s-1), u^{-1}(q)}}{x_{u^{-1}(l), u^{-1}(s)}}.
\]

Because \(x\) is totally nonnegative, the sign of \(A_{i,j+1}\) is either zero or \((-1)^{n_1+n_2}\), where

\[
n_1 = \text{card}\{l : l \leq s-1, u^{-1}(l) > i\}, \\
n_2 = \text{card}\{l : l \leq s-1, u^{-1}(l) > j+1\}.
\]

The sign of \((A^{-1})_{j,i}\) can be determined in a similar fashion. Using the notation \(\hat{i}\) to indicate that the index \(i\) is being removed, we obtain:

\[
(A^{-1})_{j,i} = ([xu^{-1}]_{+})_{p,q} = (-1)^{(s-p)}([xu^{-1}]_{+})_{1\ldots s-1, \hat{p}, \ldots, \hat{n}} \\
= (-1)^{(s-p)} \frac{(xu^{-1})_{1\ldots s-1, \hat{p}, \ldots, \hat{n}}}{(xu^{-1})_{1\ldots s-1, \hat{s}}}. \\
= (-1)^{(s-p)} \frac{x_{u^{-1}(l), u^{-1}(p), u^{-1}(s)}}{x_{u^{-1}(l), u^{-1}(s-1)}}.
\]
Hence the sign of \((A^{-1})_{j,i}\) is either zero or \((-1)^{s-p+n_1+n_2+n_3+n_4}\), where \(n_1\) is the same as before and

\[
\begin{align*}
n_3 &= \text{card}\{l : l \leq p - 1, \ u^{-1}(l) > j\}, \\
n_4 &= \text{card}\{l : p < s, \ u^{-1}(l) < j\}.
\end{align*}
\]

Since

\[
n_3 + s - p - n_4 = \text{card}\{l : l \leq s, \ u^{-1}(l) > j\},
\]

we conclude that the sign of \(A_{i,j+1}(A^{-1})_{j,i}\) is either zero or \((-1)^{\text{card}\{l : l=s, u^{-1}(l)>j\}}\). matching the claim of the lemma.

Let us extend the notation \(\text{str}(x) = \sum_i x_{i,i+1}\) to arbitrary matrices \(x\).

**Lemma 6.7.** Let \(v = \text{diag}(n, n-1, \ldots, 1)\). Then \(\text{str}(A^{-1}vA) \geq 0\).

**Proof.** For \(k = 1, \ldots, n\), let \(v_k = \text{diag}(1, \ldots, 1, 0, \ldots, 0)\) denote the diagonal matrix whose first \(k\) diagonal entries are equal to 1, and all other entries vanish. The equality \(A^{-1}A = 1\) implies

\[
\sum_{i=1}^{j} (A^{-1})_{j,j} \cdot A_{i,j+1} + \sum_{i=1}^{n} (A^{-1})_{j,j} \cdot A_{i,j+1} = 0,
\]

where (by Lemma 6.6) all terms in the first sum are nonnegative while all terms in the second sum are nonpositive. Then \((A^{-1}v_kA)_{i,j+1} = \sum_{i=1}^{k} (A^{-1})_{j,j} \cdot A_{i,j+1} \geq 0\), implying \(\text{str}(A^{-1}v_kA) \geq 0\). Since \(v = \sum_{k=1}^{n} v_k\), the lemma follows.

We are now prepared to complete the proof of Lemma 5.4. First, let us note that \(d(\tau)x_{u}d(\tau)^{-1} \in Y_u^\circ\); hence \(\rho_{x_{u}}d(\tau)x_{u}d(\tau)^{-1} \in Y_u^\circ \cap \pi_{u}^{-1}(x_{u}) = \{x_{u}\}\). Thus \(\psi(x_{u}) = 0\).

Let \(x \in \pi^{-1}(x_{u})\), \(x \neq x_{u}\). Since \(d'(1) = v + \lambda\) for some scalar \(\lambda\), Proposition 6.5 yields \(\psi(x) = x\pi_n(A^{-1}vA)\). Therefore,

\[
\text{str}(\psi(x)) = \text{str}(x) + \text{str}(\pi_n(A^{-1}vA)) = \text{str}(x) + \text{str}(A^{-1}vA)
\]

(here we identify the tangent vector \(\psi(x)\) with the corresponding traceless matrix). Since \(\text{str}(x) > 0\), Lemma 6.7 implies that \(\text{str}(\psi(x)) > 0\); in particular, \(\psi(x) \neq 0\). Then \(\text{str}(x + \psi(x)d\tau) = \text{str}(x) + \text{str}(\psi(x))d\tau\); hence \(\nabla_{\psi} \text{str}(x) = \text{str}(\psi(x)) > 0\), as desired.

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