Generic Base Change, Artin’s Comparison Theorem, and the Decomposition Theorem for Complex Artin stacks

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Abstract

We prove the generic base change theorem for stacks, and give an exposition on the lisse-analytic topos of complex analytic stacks, proving some comparison theorems between various derived categories of complex analytic stacks. This enables us to deduce the decomposition theorem for perverse sheaves on complex Artin stacks with affine stabilizers from the case over finite fields.

1 Introduction

In the study of the topology of complex algebraic varieties, the notion of intersection (co)homology and the decomposition theorem have played an important role. They are also quite useful in some other fields, such as representation theory. See [6] for a detailed introduction.

In [4] the notion of perverse sheaves was generalized to spaces with group actions (the so-called equivariant perverse sheaves), and the decomposition theorem was proved in this case. The notion of (middle) perverse sheaves has also been generalized to algebraic stacks [16], and the decomposition theorem has been proved for algebraic stacks of finite type with affine stabilizers over a finite field [24]. The result for algebraic stacks over the complex numbers was also announced in [24], 3.15, and we publish the proof in this article. For the necessity of the assumption on the stabilizers, we direct the reader to [24], Section 1) for a counter-example of Drinfeld.

1.1. Let $k$ be a field and let $\mathcal{X}$ be a $k$-algebraic stack. We say that $\mathcal{X}$ has affine stabilizers if for every $x \in \mathcal{X}(k)$, the group scheme $\text{Aut}_x$ is affine. Note that, since being affine is fpqc local on the base, for any finite field extension $k'/k$ and any $x \in \mathcal{X}(k')$, the $k'$-group scheme $\text{Aut}_x$ is affine.

Here is the main result, in a simplified and global form.

Theorem 1.2. Let $f : X \rightarrow Y$ be a proper morphism of finite diagonal between complex Artin stacks of finite type, with affine stabilizers, and let $f^{\text{an}} : X^{\text{an}} \rightarrow Y^{\text{an}}$ be the associated morphism of complex analytic stacks. Then there exist locally closed irreducible smooth substacks $Y_{\alpha} \subset Y$, irreducible $\mathbb{C}$-local systems $L_{\alpha\beta}$ on $Y_{\alpha}$, and integers $d_{\alpha\beta} \geq 0$, the index set for $(\alpha, \beta)$ being finite, such that we have a decomposition

$$IH^n(X^{\text{an}}, \mathbb{C}) \cong \bigoplus_{\alpha, \beta} IH^{n-d_{\alpha\beta}}(Y^{\text{an}}_{\alpha}, L_{\alpha\beta})$$

for each $n \in \mathbb{Z}$.

See [12, 24] for the general and local version of the theorem.

We briefly mention some technical issues. One would like to deduce the decomposition theorem over $\mathbb{C}$ from that over finite fields, as in [3], Section 6). In order for the argument
to work, one must generalize the generic base change theorem to stacks. Also, in order
to obtain a topological statement, one has to prove some comparison theorems between
different topologies. Roughly speaking, the generic base change theorem relates lisse-étale
sheaves over \( \mathbb{C} \) with lisse-étale sheaves over \( \mathbb{F} \) (algebraic closure of a finite field), and the
comparison theorems relate lisse-étale sheaves over \( \mathbb{C} \) with lisse-analytic sheaves over \( \mathbb{C} \).

**Organization.** In \( \S 2 \) we prove the generic base change for \( Rf_* \) and \( R\mathcal{H}om \), and in
\( \S 3 \) we develop the theory of constructible sheaves and their derived categories on complex
analytic stacks that are algebraic; in particular, we give the comparison between the lisse-étale
topos and the lisse-analytic topos, and the comparison between the adic version and
the topological version of the derived category of the lisse-analytic topos. In \( \S 4 \) after giving
a comparison between bounded derived categories with prescribed stratification over the
complex numbers and over an algebraic closure of a finite field, we finish the proof of the
decomposition theorem for stacks over \( \mathbb{C} \).

**Notations and Conventions 1.3.**

1.3.1. Let \( (\Lambda, m) \) be a complete DVR of mixed characteristic, with finite residue field \( \Lambda_0 \)
of characteristic \( \ell \) and uniformizer \( \lambda \). Let \( \Lambda_n = \Lambda/m^{n+1} \), for \( n \in \mathbb{N} \).

1.3.2. By an Artin stack, or an algebraic stack, we mean an algebraic stack in the sense of
M. Artin (\[20\], 1.2.22) of finite type over the base. We will use \( \mathcal{X}, \mathcal{Y}, \cdots \) to denote algebraic
stacks over a general base \( S \) in \( \S 2 \) In \( \S 3 \) we use them to denote complex algebraic stacks,
and \( \mathcal{X}, \mathcal{Y}, \cdots \) for complex analytic stacks. By a presentation of an algebraic stack \( \mathcal{X} \), we
mean a smooth surjection \( \pi : X \rightarrow \mathcal{X} \) where \( X \) is a scheme.

1.3.3. By a variety over \( k \) we mean a separated reduced \( k \)-scheme of finite type. For a
\( k \)-algebraic stack \( \mathcal{X} \), we say that it is essentially smooth if \( (\mathcal{X}^a)_k \) is smooth over \( \overline{k} \).

1.3.4. For a map \( f : X \rightarrow Y \) and a complex of sheaves \( K \) on \( Y \), we sometimes write
\( H^n(X, K) \) for \( H^n(X, f^*K) \).

1.3.5. We will denote \( Rf_*, Rf_!, Lf^* \) and \( Rf^! \) by \( f_*, f_!, f^* \) and \( f^! \) respectively in most part
of this paper, except in \( \S 3.4, 3.5 \), where we use symbols like \( R\gamma_* \) and \( R\pi_* \) to emphasize
that we are considering the derived functors.

1.3.6. We will only consider the middle perversity. We use \( p_{\mathcal{H}^i} \) and \( p_{\tau \leq i} \) to denote coho-
logy and truncations with respect to this perverse \( t \)-structure.

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## 2 Generic base change

As mentioned in \( \S 1 \) an important step in deducing the decomposition theorem over \( \mathbb{C} \) from
that over \( \mathbb{F}_q \) (as in \( \S 3 \), Section 6) will be to compare the derived categories of the fiber over
and the fiber over \( \mathcal{P} \) of some stack over a local ring with mixed characteristics. For doing that, we prove the generic base change theorem (as in \([8]\), Th. finitude) for stacks in this section.

2.1. Let \( S \) be a scheme satisfying the following condition denoted \((LO)\): it is a noetherian affine excellent finite-dimensional scheme in which \( \ell \) is invertible, and all \( S \)-schemes of finite type have finite \( \ell \)-cohomological dimension. The theory of derived categories and the six operations in \([14, 15]\) then applies to algebraic stacks over \( S \) locally of finite type. As mentioned in \([13, 2]\), we will only consider those of finite type over \( S \).

We refer to \([23, \S 3]\) for the definition and basic properties of stratifiable complexes in detail; see also \([15, \text{Section 3}]\) or \([23, \text{Definition 2.2}]\) for more discussion on the \( \lambda \)-adic derived category of an algebraic stack. Here we only give a quick review of the definitions.

Let \( \mathcal{A} = \mathcal{A}(\mathcal{X}) \) be the abelian category \( \text{Mod}(\lambda_{\text{lis-\'et}}, \Lambda) \) of \( \Lambda \)-modules on the simplicial lisse-\'etale topos of \( \mathcal{X} \), and let \( \mathcal{D}(\mathcal{A}) \) be the ordinary derived category of \( \mathcal{A} \). An object \( M \in \mathcal{D}(\mathcal{A}) \) is called a \( \lambda \)-complex (resp. an \( AR \)-null complex) if all cohomology systems \( \mathcal{H}^i(M) \) are \( AR \)-adic (resp. \( AR \)-null). Let \( \mathcal{D}_c(\mathcal{A}) \) be the full subcategory of \( \lambda \)-complexes, and let \( D_c(\mathcal{X}, \Lambda) \), the \( \lambda \)-adic derived category of \( \mathcal{X} \), to be the quotient of \( \mathcal{D}_c(\mathcal{A}) \) by the full subcategory of \( AR \)-null complexes.

For a pair \( (\mathcal{S}, \mathcal{L}) \), where \( \mathcal{S} \) is a stratification of the algebraic stack \( \mathcal{X} \), and \( \mathcal{L} \) assigns to every stratum \( U \in \mathcal{S} \) a finite set \( \mathcal{L}(U) \) of isomorphism classes of simple locally constant \( \Lambda_0 \)-sheaves on \( U \), we define \( \mathcal{D}_{\mathcal{S}, \mathcal{L}}(\mathcal{A}) \) to be the full subcategory of \( \mathcal{D}_c(\mathcal{A}) \) consisting of complexes of projective systems \( K = (K_n)_n \) such that, for all \( i, n \in \mathbb{Z} \) and for every \( U \in \mathcal{S} \), the restrictions \( \mathcal{H}^i(K)|_U \) are lcc with Jordan-Hölder components contained in \( \mathcal{L}(U) \). Define \( D_{\mathcal{S}, \mathcal{L}}(\mathcal{X}, \Lambda) \) to be its essential image under the localization \( \mathcal{D}_c(\mathcal{A}) \to D_c(\mathcal{X}, \Lambda) \); in other words, it is the quotient of \( \mathcal{D}_{\mathcal{S}, \mathcal{L}}(\mathcal{A}) \) by the thick subcategory of \( AR \)-null complexes. It is a triangulated category. Similarly, one can define, for each \( n \geq 0 \), a triangulated full subcategory \( D_{\mathcal{S}, \mathcal{L}}(\mathcal{X}, \Lambda_n) \) of \( D_c(\mathcal{X}, \Lambda_n) \): it consists of those complexes \( K \) such that \( \mathcal{H}^i(K)|_U \) are lcc with Jordan-Hölder components contained in \( \mathcal{L}(U) \), for each integer \( i \) and stratum \( U \in \mathcal{S} \).

Finally we define \( D_{\mathcal{S}}^{\text{stra}}(\mathcal{X}, \Lambda) \) to be the 2-direct limit of all the \( D_{\mathcal{S}, \mathcal{L}}(\mathcal{X}, \Lambda_n) \)’s; similarly for \( D_{\mathcal{S}}^{\text{stra}}(\mathcal{X}, \Lambda_n) \).

2.2. For a morphism \( f : \mathcal{X} \to \mathcal{Y} \) of \( S \)-algebraic stacks and \( K \in D_c^+(\mathcal{X}, \Lambda_n) \) (resp. \( D_c^+(\mathcal{X}, \Lambda) \)), we say that the formation of \( f_! K \) commutes with generic base change, if there exists an open dense subscheme \( U \subset S \) such that for any morphism \( g : S' \to U \subset S \) with \( S' \) satisfying \((LO)\), the base change morphism \( g^*f_*K \to f_{S'}^*g'^*K \) is an isomorphism. Recall that the base change morphism is defined as follows: one applies \( f_* \) to the adjunction map \( K \to g_*^*g'^*K \), and then uses the adjunction \((g^*, g_*^)\) to obtain the base change morphism, as shown in the following 2-Cartesian diagram

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{g'} & \mathcal{X}_{S'} \\
\downarrow f & & \downarrow f_{S'} \\
\mathcal{Y} & \xrightarrow{g'} & \mathcal{Y}_{S'} \\
\downarrow S & \xrightarrow{g} & \downarrow S' \\
S & & S'.
\end{array}
\]

Lemma 2.3. (i) Let \( P : \mathcal{Y} \to \mathcal{Y} \) be a presentation, and let the following diagram be
2-Cartesian:

\[ X \xleftarrow{P'} X_Y \]

\[ f \]

\[ Y \xleftarrow{P} Y. \]

Then for \( K \in D_c^+(\mathcal{X}, \Lambda) \), the formation of \( f_*K \) commutes with generic base change if and only if the formation of \( f'_*(P'^*K) \) commutes with generic base change.

(ii) Let \( K' \rightarrow K \rightarrow K'' \rightarrow K'[1] \) be an exact triangle in \( D_c^+(\mathcal{X}, \Lambda) \), and let \( f : \mathcal{X} \rightarrow \mathcal{Y} \) be an \( S \)-morphism. If the formations of \( f_*K' \) and \( f_*K'' \) commute with generic base change, then so does the formation of \( f_*K \).

(iii) Let \( f : \mathcal{X} \rightarrow \mathcal{Y} \) be a schematic morphism, and let \( K \in D_c^+(\mathcal{X}, \Lambda) \) for some finite set \( L \) of isomorphism classes of simple lcc \( \Lambda_0 \)-sheaves on \( \mathcal{X} \). Then the formation of \( f_*K \) commutes with generic base change.

(iv) Let \( K \in D_c^+(\mathcal{X}, \Lambda) \), and let \( j : \mathcal{U} \rightarrow \mathcal{X} \) be an open immersion with complement \( i : Z \rightarrow \mathcal{X} \). For \( g : S' \rightarrow S \), consider the following diagram obtained by base change:

\[ \begin{array}{ccc}
U & \xleftarrow{j} & \mathcal{X} \\
\downarrow{g_U} & & \downarrow{g} \\
\mathcal{U} & \xleftarrow{j_U} & \mathcal{X} \\
\downarrow{f} & & \downarrow{i} \\
\mathcal{Y} & \xleftarrow{i} & Z \\
\end{array} \]

Suppose that the base change morphisms

\[
g'^*(fj)_*(j^*K) \rightarrow (f_{S'}j_{S'})_*(g_{U})_{*}(j^*K),
\]

\[
g'^*(fi)_*(i^*K) \rightarrow (f_{S'}i_{S'})_*(g_{Z})_{*}(i^*K) \quad \text{and}
\]

\[
g'^*(j_*)K \rightarrow j_{S'}g'^*(i^*K)
\]

are isomorphisms, then the base change morphism \( g'^*f_*K \rightarrow f_{S'}g'^*K \) is also an isomorphism.

(v) Let \( f : \mathcal{X} \rightarrow \mathcal{Y} \) be a schematic morphism of \( S \)-algebraic stacks, and let \( K \in D_c^{+, \text{stra}}(\mathcal{X}, \Lambda) \). Then the formation of \( f_*K \) commutes with generic base change on \( S \).

(vi) Let \( f : \mathcal{X} \rightarrow \mathcal{Y} \) be a morphism of \( S \)-algebraic stacks, and let \( j : \mathcal{U} \rightarrow \mathcal{Y} \) be an open immersion with complement \( i : Z \rightarrow \mathcal{Y} \). Let \( K \in D_c^{+, \text{stra}}(\mathcal{X}, \Lambda) \). Then there exists an open dense subscheme \( S^0 \subset S \), such that for any map \( g : S' \rightarrow S \), with associated diagram in which the squares are 2-Cartesian:

if the base change morphisms

\[
g_{U} \cdot f_{U*}(j^*K) \rightarrow f_{U*}g_{U*}(j^*K) \quad \text{and} \quad g_{Z} \cdot f_{Z*}(i^*K) \rightarrow f_{Z*}g_{Z*}(i^*K)
\]

\[ \begin{array}{ccc}
\mathcal{X}_{U, S'} & \xleftarrow{i_{S'}} & \mathcal{X}_{S'} \\
\downarrow{f_{S'}} & & \downarrow{f} \\
\mathcal{X} & \xleftarrow{i} & Z \\
\downarrow{f} & & \downarrow{g} \\
\mathcal{U} & \xleftarrow{j} & \mathcal{Y} \\
\downarrow{g_U} & & \downarrow{g} \\
\mathcal{U} & \xleftarrow{j} & \mathcal{Y} \\
\downarrow{f} & & \downarrow{g} \\
\mathcal{U} & \xleftarrow{j} & \mathcal{Y} \\
\end{array} \]
are isomorphisms, then the base change morphism \( g^* f_* K \to f_{S'} g'^* K \) is an isomorphism over \( Y_{S' \times S S^0} \).

Similar results hold with \( \Lambda \) replaced by \( \Lambda_n \) \((n \geq 0)\), and the proof is the same.

**Proof.** (i) Given a map \( g : S' \to S \), consider the following diagram

\[
\begin{array}{ccc}
X_Y & \xrightarrow{g'_Y} & X_{Y,S'} \\
\downarrow{f'} & & \downarrow{f'_S} \\
Y & \xrightarrow{g'} & Y_{S'} \\
\end{array}
\]

where all squares are 2-Cartesian. For the base change morphism \( g'^* f_* K \to f_{S'} g'^* K \) to be an isomorphism on \( Y_{S'} \), it suffices for it to be an isomorphism locally on \( Y_{S'} \). In the following commutative diagram

\[
\begin{array}{ccc}
P_{S'} g'^* f_* K & \xrightarrow{0} & P_{S'} f_{S'} g'^* K \\
\downarrow{(1)} & & \downarrow{(2)} \\
g'^* P^* f_* K & \xrightarrow{(3)} & f_{S'} P_{S'} g'^* K \\
\downarrow{(4)} & & \downarrow{(5)} \\
g'^* f'_i P'^* K & \xrightarrow{(5)} & f_{S'} g'^* P'^* K \\
\end{array}
\]

(1) and (4) are canonical isomorphisms given by \( "P^* g^* \simeq g^* P^*" \), and (2) and (3) are canonical isomorphisms given by \( "P^* f_* = f_* P^*" \), which follows from the definition of \( f_* \) on the lisse-étale site. Therefore, (0) is an isomorphism if and only if (5) is an isomorphism.

(ii) This follows easily from the axioms of a triangulated category (or 5-lemma):

\[
\begin{array}{ccc}
g'^* f'_K & \xrightarrow{\sim} & g'^* f_* K & \xrightarrow{\sim} & g'^* f'_K \\
\downarrow{f_{S'} g'^* K} & & \downarrow{f_{S'} g'^* K} & & \downarrow{f_{S'} g'^* K} \\
f_{S'} g'^* K & \xrightarrow{\sim} & f_{S'} g'^* K & \xrightarrow{\sim} & f_{S'} g'^* K \\
\end{array}
\]

(iii) By (i) we may assume that \( f : X \to Y \) is a morphism of \( S \)-schemes. Note that the property of being trivialized by a pair of the form \((\{X\}, \mathcal{L})\) is preserved when passing to a presentation. By definition \( f_* K \) is the class of the system \( (f_* \mathcal{K}_n)_n \), so it suffices to show that there exists an open dense subscheme of \( S \) over which the formation of \( f_* \mathcal{K}_n \) commutes with base change, for every \( n \). By the spectral sequence

\[R^n f_* \mathcal{H}^q(\mathcal{K}_n) \implies R^{n+q} f_* \mathcal{K}_n\]

and (ii), it suffices to show the existence of an open dense subscheme of \( S \), over which the formations of \( f_* L \) commute with base change, for all \( L \in \mathcal{L} \). This follows from \([\mathbb{S}, \text{Th. finitude}]\).

(iv) Consider the commutative diagram

\[
\begin{array}{ccc}
g'^* f_i i^! K & \xrightarrow{(1)} & f_{S'} g'^* i^! K & \xrightarrow{(2)} & f_{S'} i_S^* g_i^* i^! K \\
\downarrow{(3)} & & \downarrow{(3)} & & \downarrow{(4)} \\
g'^*(f_i)_i^! K & \xrightarrow{(5)} & (f_{S'} i_{S'})_i^! K \\
\end{array}
\]
(1) and (4) are canonical isomorphisms, (5) is an isomorphism by assumption, and (3) is the base change morphism for $i_*$, which is an isomorphism by (15, 12.5.3), since $i_*=i_!$. Therefore, (2) is an isomorphism. Similarly, consider the commutative diagram

$$
g^* f_* j_*^* K \xrightarrow{(1)} f_{S'} g^! j_*^* K \xrightarrow{(2)} f_{S'} g_{U*} j_*^* K \xrightarrow{(3)} f_{S'} j_{S'}^* g_{U*} j_*^* K \xrightarrow{(4)} g'^* (fj)_* j_*^* K \xrightarrow{(5)} (f_{S'} j_{S'})_* g_{U*} j_*^* K.
$$

(1) and (4) are canonical isomorphisms, and (3) and (5) are isomorphisms by assumption, so (2) is an isomorphism. Then apply (ii) to the exact triangle $i_* i^! K \to K \to j_* j^* K \to$. 

(v) By (i), we may assume that $f : X \to Y$ is a morphism of $S$-schemes. Assume that $K$ is trivialized by $(\mathcal{S}, \mathcal{L})$, and let $j : U \to X$ be the immersion of an open stratum in $\mathcal{S}$ with complement $i : Z \to X$. Then $j^* K \in D^+(U, \mathcal{L}(U), \Lambda)$, so by (iii), the formation of $j_*(K|_U)$ commutes with generic base change. This is the third base change isomorphism in the assumption of (iv). By noetherian induction and (iv), we may replace $X$ by $U$ and assume that $\mathcal{S} = \{X\}$. The result follows from (iii).

(vi) In the commutative diagrams

$$
g^* j_* f_{U*} i_*^! K \xrightarrow{(1)} j_{S'}^! g_{U*} i_*^! K \xrightarrow{(2)} j_{S'}^! g_{U*} j_*^! K \xrightarrow{(3)} j_{S'}^! g_{U*} j_*^! K \xrightarrow{(4)} g^! (fj)_* j_*^! K \xrightarrow{(5)} j_{S'}^! g_{U*} j_*^! K
$$

and

$$
g^* i_* f_{Z*} i^! K \xrightarrow{(6)} i_{S'}^! g_{Z*} i^! K \xrightarrow{(7)} i_{S'}^! g_{Z*} i^! K \xrightarrow{(8)} g^* (fi)_* i^! K \xrightarrow{(9)} (f_{S'} i_{S'}^! )_* g_{Z*} i^! K \xrightarrow{(10)} i_{S'}^! g_{Z*} i^! K,
$$

(2), (5), (7) and (10) are canonical isomorphisms, (3) and (8) are isomorphisms by assumption, (6) is an isomorphism by proper base change, and (1) is an isomorphism after shrinking $S$ by (v). Therefore, (4) and (9) are isomorphisms. Also by (v), the base change morphism $g'^* j'_*(j'^* K) \to j_{S'}^! g_{U*} j_*^! (j^* K)$ becomes an isomorphism after shrinking $S$. Hence by (iv), the base change morphism $g^* f_* K \to f_{S'} g'^* K$ is an isomorphism after shrinking $S$.

2.4. For $K \in D_c^-(\mathcal{X}, \Lambda_n)$ and $L \in D_c^+(\mathcal{X}, \Lambda_n)$, and for a morphism $g : \mathcal{Y} \to \mathcal{X}$, the base change morphism $g^* R\shom_{\mathcal{X}}(K, L) \to R\shom_{\mathcal{Y}}(g^* K, g^* L)$ is defined as follows. By adjunction $(g^*, g_*)$, it corresponds to the morphism

$$R\shom_{\mathcal{X}}(K, L) \to g_* R\shom_{\mathcal{Y}}(g^* K, g^* L) \simeq R\shom_{\mathcal{X}}(K, g_* g^* L)$$

obtained by applying $R\shom_{\mathcal{X}}(K, -)$ to the adjunction morphism $L \to g_* g^* L$. One can define the base change morphism for $\Lambda$-coefficients in the same way.

Note that if $K' \to K \to K'' \to K'[1]$ is an exact triangle, and the base change morphisms for $R\shom(K', L)$ and $R\shom(K'', L)$ are isomorphisms, then so is the base change morphism for $R\shom(K, L)$; similarly for the position of $L$.

We say that the formation of $R\shom_{\mathcal{X}}(K, L)$ commutes with generic base change on $S$, if there exists an open dense subscheme $U \subset S$ such that for any morphism $g : S' \to U \subset S$ with $S'$ satisfying (LO), the base change morphism

$$g'^* R\shom_{\mathcal{X}}(K, L) \to R\shom_{\mathcal{X}|_{S'}}(g'^* K, g'^* L)$$

is an isomorphism. Here $g' : \mathcal{X}|_{S'} \to \mathcal{X}$ is the natural projection.
The following is the main result of this section.

**Theorem 2.5.** (i) Let \( f : \mathcal{X} \rightarrow \mathcal{Y} \) be a morphism of \( S \)-algebraic stacks. For every \( K \in D_+^{\text{str}}(\mathcal{X}, \Lambda_n) \) (resp. \( D_c^{+\text{str}}(\mathcal{X}, \Lambda) \)), the formation of \( f_*K \) commutes with generic base change on \( S \).

(ii) For all \( K, L \in D_c^K(\mathcal{X}, \Lambda_n) \), the formation of \( R\mathcal{H}om_X(K, L) \) commutes with generic base change on \( S \).

**Proof.** (i) We can always replace a stack by its maximal reduced closed substack, so we will assume that all stacks in the proof are reduced.

Suppose that \( K \) is \((\mathcal{S}, \mathcal{L})\)-stratifiable for some pair \((\mathcal{S}, \mathcal{L})\). By (2.3 i, iii, iv), we can replace \( \mathcal{Y} \) by a presentation and replace \( \mathcal{X} \) by an open stratum in \( \mathcal{S} \), to assume that \( \mathcal{Y} = Y \) is a scheme, that \( \mathcal{S} = \{ X \} \), that the relative inertia \( \mathcal{I}_f \) is flat over \( X \) and has components over \( X \) \((2, 5.1.14)\); let

\[
\mathcal{X} \xrightarrow{\pi} X \xrightarrow{b} Y
\]

be the rigidification with respect to \( \mathcal{I}_f \). Replacing \( \mathcal{X} \) by the inverse image of an open dense subscheme of the \( S \)-algebraic space \( X \), we may assume that \( X \) is a scheme. Let \( \mathcal{F} = \pi_*K \), which is stratifiable \((23, 3.9)\). By (2.3 v), the formation of \( b_*\mathcal{F} \) commutes with generic base change. To finish the proof, we shall show that the formation of \( \pi_*K \) commutes with generic base change. As in the proof of (2.3 iii), it suffices to show that there exists an open dense subscheme \( U \) of \( S \), over which the formations of \( \pi_*L \) commute with any base change \( g : S' \rightarrow U \), for all \( L \in \mathcal{L} \).

By \((2, 5.1.5)\), \( \pi \) is smooth, so étale locally it has a section. By (2.3 i) we may assume that \( \pi : BG \rightarrow X \) is a neutral gerbe, associated to a flat group space \( G/X \). By (2.3 vi) we can use dévissage and shrink \( X \) to an open subscheme. Using the same technique as the proof of \((23, 3.9)\), we can reduce to the case where \( G/X \) is smooth. For the reader’s convenience, we briefly recall this reduction. Shrinking \( X \), we may assume that \( X \) is an integral scheme with function field \( k(X) \), and that \( G/X \) is a group scheme. There exists a finite field extension \( k''/k(X) \) such that \( G_{\text{red}} \) is smooth over \( \text{Spec} \ k'' \). Let \( k' \) be the separable closure of \( k(X) \) in \( k'' \). Purely inseparable morphisms are universal homeomorphisms. By taking the normalization of \( X \) in these field extensions, we get a finite generically étale surjection \( X' \rightarrow X \), such that \( G_{\text{red}} \) is generically smooth over \( X' \). Shrinking \( X \) and \( X' \) we may assume that \( X' \rightarrow X \) is an étale surjection, and replacing \( X \) by \( X' \) (2.3 i) we may assume that \( G_{\text{red}} \) is generically smooth over \( X \), and shrinking \( X \) further we may assume that \( G_{\text{red}} \) is smooth over \( X \). Finally we may replace \( G \) by \( G_{\text{red}} \), since the morphism \( BG_{\text{red}} \rightarrow BG \) is representable and radicial.

Now \( P : X \rightarrow BG \) is a presentation, and we consider the associated smooth hypercover. Let \( f_p : G^p \rightarrow X \) \((p \geq 1)\) be the structural maps, and let the following squares be \( 2 \)-Cartesian:

\[
\begin{array}{cccc}
G^p & \xrightarrow{f_p} & X^p & \xrightarrow{P^p} \quad (BG)^p & \xrightarrow{\pi^p} & X^p & \xrightarrow{\pi^p} & S^p \\
\downarrow g_p & & \downarrow g^p & & \downarrow g^p & & \downarrow g & & \downarrow g \\
G & \xrightarrow{f_p} & X & \xrightarrow{P} & BG & \xrightarrow{\pi} & X & \xrightarrow{\pi} & S.
\end{array}
\]

We have the spectral sequence \((15, 10.0.9)\)

\[R^q f_{p*} P^* L \Rightarrow R^{p+q} \pi_* L\]

and similarly for the base change to \( S' \). We can regard the map \( f_p \) as a product \( \prod_p f_1 \) and apply the Künneth formula (shrinking \( X \) we can assume that \( X \) satisfies the condition (LO), and we can apply \((15, 11.0.14)\))

\[f_{p*} P^* L = f_{1*} (f_1^* P^* L) \otimes_{\Lambda_0} f_{1*} \Lambda_0 \otimes_{\Lambda_0} \cdots \otimes_{\Lambda_0} f_{1*} \Lambda_0.
\]
Shrink $S$ so that the formations of $f_1, f^*_i P^* L$ and $f_1, \Lambda_0$ commute with any base change on $S$. From the base change morphism of the spectral sequences

$$g^* R^q f_p f^* P^* L \to g^* R^{p+q} \pi_* L$$

we see that the base change morphism (1) is an isomorphism.

(ii) Let $g : S' \to S$ be any morphism, $P : X \to X'$ be a presentation, and consider the 2-Cartesian diagrams

For the base change morphism

$$g^* R \mathcal{H}om_X(K, L) \to R \mathcal{H}om_{X_{S'}}(g^* K, g^* L)$$

to be an isomorphism, we can check it locally on $X_{S'}$. Consider the commutative diagram

$$P^* g^* R \mathcal{H}om_X(K, L) \to P^* R \mathcal{H}om_{X_{S'}}(g^* K, g^* L)$$

where (2) and (5) are canonical isomorphisms, (3) and (4) are isomorphisms by ([14], 4.2.3), and (6) is an isomorphism after shrinking $S$ ([8], Th. finitude, 2.10). Therefore (1) is an isomorphism after shrinking $S$.

**Remark 2.5.1.** (i) This result strengthens ([21], 9.10 ii), in that the open subscheme in $S$ can be chosen to be independent of the index $i$ as in $R^i f_* F$.

(ii) As we only used $f_*$, not $f_!$, in the proof of the generic base change theorem, it may seem that the hypothesis (LO) on the base $S$ (cf. [21]) is unnecessary. However, in the proof of ([23], 3.9), when proving that $f_*$ preserves stratifiability, which is needed in (2.5), we worked with the case for $f_!$ first, in order to do noetherian induction. Possibly this hypothesis on cohomological dimension can be removed in the future.
3 Complex analytic stacks

In this section, we give some fundamental results on constructible sheaves and derived categories on the lisse-analytic topos of the analytification of a complex algebraic stack. The two main results in this section are: the comparison between the adic derived categories of the lisse-étale topos and the lisse-analytic topos (3.4.11), and the comparison between the adic derived category and the topological derived category of the lisse-analytic topos (3.5.9).

3.1 Lisse-analytic topos

Stacks over topological categories have already been discussed, for instance in [18, 25]. Strictly speaking, Toën only discussed analytic Deligne-Mumford stacks in [25], and Noohi only discussed topological stacks in [18] (and mentioned analytic stacks briefly).

Since we are mainly interested in analytifications of complex algebraic stacks, and will not study analytic spaces and analytic stacks in full generality in this paper, we will make a global assumption on analytic spaces: we only consider analytic spaces of finite dimension. This rules out infinite disjoint unions of spaces of increasing dimensions, and is consistent with our assumption that algebraic stacks are of finite type (13.2).

A morphism $f : X \to Y$ of complex analytic spaces is smooth if for every point $x \in X$, there exist open neighborhoods $x \in U \subset X$ and $f(x) \in V \subset Y$, with $f(U) = V$, such that $f|_U : U \to V$ is isomorphic to the projection $pr_1 : V \times Z \to V$ for some complex manifold $Z$ (one can certainly take $Z$ to be a polydisk). In topology, this is usually called a submersion, but we will use the algebro-geometric terminology of smoothness in the paper, if there is no confusion.

Definition 3.1.1. Let $\mathrm{Ana-Sp}$ be the site of complex analytic spaces with the analytic topology. A stack $\mathcal{X}$ over this site is called an analytic stack, if the following hold:

(i) the diagonal $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable (by analytic spaces) and, letting the inertia $\mathcal{I}_\mathcal{X}$ of $\mathcal{X}$ be the fiber product $\mathcal{X} \times_{\Delta, \mathcal{X} \times \mathcal{X}, \Delta} \mathcal{X}$ with $p_1 : \mathcal{I}_\mathcal{X} \to \mathcal{X}$ the first projection, the complex Lie group $p_1^{-1}(x)$ has finitely many connected components, for every $x \in \mathcal{X}(\mathbb{C})$, and

(ii) there exists a smooth surjection $P : X \to \mathcal{X}$, where $X$ is an analytic space.

We will call $P : X \to \mathcal{X}$ in (ii) an analytic presentation of $\mathcal{X}$.

3.1.2. Similar to the lisse-étale topos of an algebraic stack, one can define the lisse-analytic topos $\mathcal{X}_{\text{lis-an}}$ of an analytic stack $\mathcal{X}$ to be the topos associated to the lisse-analytic site $\mathrm{Lis-an}(\mathcal{X})$ defined as follows:

- **Objects**: pairs $(U, u : U \to \mathcal{X})$, where $U$ is an complex analytic space and $u$ is a smooth morphism;
- **Morphisms**: a morphism $(U, u \in \mathcal{X}(U)) \to (V, v \in \mathcal{X}(V))$ is given by a pair $(f, \alpha)$, where $f : U \to V$ is a morphism of analytic spaces and $\alpha : u \cong vf$ is an isomorphism in $\mathcal{X}(U)$; the composition law is evident;
- **Open coverings**: $\{(j_i, \alpha_i) : (U_i, u_i \in \mathcal{X}(U_i)) \to (U, u \in \mathcal{X}(U))\}_{i \in I}$ is an open covering if the maps $j_i : U_i \to U$ are open immersions and their images cover $U$.

As in ([13], 12.2.1), one can show that, to give a sheaf $F \in \mathcal{X}_{\text{lis-an}}$ is equivalent to giving a sheaf $F_{U,u}$ in the analytic topos $U_{\text{an}}$ of $U$ for every $(U, u) \in \mathrm{Lis-an}(\mathcal{X})$, and a morphism $\theta_{f,\alpha} : f^{-1}F_{V,v} \to F_{U,u}$ for every morphism $(f, \alpha) : ((U, u) \to (V, v))$ in $\mathrm{Lis-an}(\mathcal{X})$, such that

- $\theta_{f,\alpha}$ is an isomorphism if $f$ is an open immersion, and
- for every composition $(U, u) \xrightarrow{(f,\alpha)} (V, v) \xrightarrow{(g,\beta)} (W, w)$
we have $\theta_{f,\alpha} \circ f^{-1}(\theta_{g,\beta}) = \theta_{g,\beta}(f) \circ \alpha$.

The sheaf $F$ is Cartesian if $f_{\alpha}$ is an isomorphism, for every $(f, \alpha)$. By abuse of notation, we will also denote “$F_{U, u}$” and “$f_{\alpha}$” by “$F_U$” and “$f$” respectively, if there is no confusion about the reference to $u$ and $\alpha$.

This topos is equivalent to the “lisse-étale” topos $\mathcal{X}_{\text{lisse-ét}}$ associated to the site Lis-ét($\mathcal{X}$) with the same underlying category as that of Lis-an($\mathcal{X}$), but the open coverings are surjective families of local isomorphisms. This is because the two topologies are cofinal: for a local isomorphism $V \to U$ of analytic spaces, there exists an open covering $\{V_i \subset V\}_i$ of $V$ by analytic subspaces, such that for each $i$, the composition $V_i \subset V \to U$ is an open immersion.

3.1.3. Let $\mathcal{C}^\bullet$ be a complex of sheaves of abelian groups in $\mathcal{X}_{\text{lisse-an}}$. For a morphism $f : U \to V$ in Lis-an($\mathcal{X}$), we have $\theta_f^n : f^* C^n_V \to C^n_U$ for each component $C^n$, and these maps commute with the differentials in $\mathcal{C}^\bullet$ (by definition of morphisms of sheaves), hence they give a chain map $\theta_f^\bullet : f^* \mathcal{C}^\bullet \to \mathcal{C}^\bullet$. If the cohomology sheaves $\mathcal{H}^n(\mathcal{C}^\bullet)$ are all Cartesian, then $\theta_f^\bullet$ is an quasi-isomorphism, for every $f$.

3.2 Locally constant sheaves and constructible sheaves

Let $R$ be a commutative ring with identity. For a sheaf of sets (resp. a sheaf of $R$-modules) on the analytic site of an analytic space, we say that the sheaf is locally constant constructible, abbreviated as lcc, if it is locally constant with respect to the analytic topology, and stalks are finite sets (resp. finitely generated $R$-modules).

Let $\mathcal{X}$ be an analytic stack. For a Cartesian sheaf $F \in \mathcal{X}_{\text{lisse-an}}$, we say that $F$ is locally constant (resp. lcc) if the conditions in the following (3.2.1) hold. The following lemma is an analytic version of ([21], 9.1).

**Lemma 3.2.1.** Let $F \in \mathcal{X}_{\text{lisse-an}}$ be a Cartesian sheaf. Then the following are equivalent.

(i) For every $(U, u) \in \text{Lis-an}(\mathcal{X})$, the sheaf $F_U$ is locally constant (resp. lcc).

(ii) There exists an analytic presentation $P : X \to \mathcal{X}$ such that $F_X$ is locally constant (resp. lcc).

The same statement holds for a Cartesian sheaf $F$ of $R$-modules.

**Proof.** We only need to show that (ii) implies (i). There exists an open covering $U = \cup U_i$, such that over each $U_i$, the smooth surjection $X \times_{P, X, u} U \to U$ has a section $s_i :

\begin{align*}
X \times_X U & \longrightarrow X \\
\downarrow s_i & \\
U_i \longrightarrow U & \quad \text{and} \quad \downarrow u \\
\quad \downarrow P & \\
& \longrightarrow \mathcal{X}.
\end{align*}

Therefore $F_{U_i} \simeq s_i^{-1} F_{X \times_X U}$, which is locally constant (resp. lcc).

3.2.2. Let $\mathcal{X}$ be a complex algebraic stack. Following ([15], 20), one can define its associated analytic stack $\mathcal{X}^{\text{an}}$ as follows. If $X_1 \Rightarrow X_0 \to \mathcal{X}$ is a smooth groupoid presentation, then $\mathcal{X}^{\text{an}}$ is defined to be the analytic stack given by the presentation $X_1^{\text{an}} \Rightarrow X_0^{\text{an}}$, and it can be proved that this is independent of the choice of the presentation, up to an isomorphism that is unique up to 2-isomorphism. Similarly, for a morphism $f : \mathcal{X} \to \mathcal{Y}$ of complex algebraic stacks, one can choose their presentations so that $f$ lifts to a morphism of groupoids, hence induces a morphism of their analytifications, denoted $f^{\text{an}} : \mathcal{X}^{\text{an}} \to \mathcal{Y}^{\text{an}}$. The analytification functor preserves finite 2-fiber products.

Sometimes we write $\mathcal{X}(\mathbb{C})$ for the analytification $\mathcal{X}^{\text{an}}$ or the associated lisse-analytic topos. For a $\mathbb{C}$-algebraic space $X$, we denote by $X(\mathbb{C})$ the analytification or the associated
analytic topos. There is a possible confusion which will not occur in the sequel: for an analytic space $X$, these two topos are not the same. The restriction functor defines an equivalence from Cartesian sheaves $X_{\text{lis-an, cart}}$ to $X_{\text{an}}$.

### 3.2.3
Let $X = \mathcal{X}_{\text{an}}$ for a complex algebraic stack $\mathcal{X}$, and let $P : X \to \mathcal{X}$ be a presentation. Let $R$ be a commutative ring with identity. For a sheaf $F$ of sets (resp. $R$-modules) on $X_{\text{lis-an}}$, we say that $F$ is \emph{algebraically constructible} (or just \emph{constructible}), if it is Cartesian, and that for every $(U, u) \in \text{Lis-ét}(\mathcal{X})$, the sheaf $F_U$ is constructible, i.e. lcc on each stratum in an \emph{algebraic} stratification of the analytic space $U$. In the following, when there seems to be a confusion about the coefficient ring $R$, we will mention it explicitly.

One could also define a notion of \emph{analytic constructibility}, using analytic stratifications rather than algebraic ones, but this notion will not give us a comparison between the constructible derived categories of the lisse-étale topos and of the lisse-analytic topos.

The notion of constructible sheaves (and some variants) on complex analytic spaces are defined in \cite{[10], 4.1}.

#### Lemma 3.2.4
Let $F$ be a Cartesian sheaf of sets (resp. $R$-modules) on $X_{\text{lis-an}}$, and let $P : X \to \mathcal{X}$ be a presentation as above. Then the following are equivalent.

(i) $F$ is constructible.

(ii) $F_X$ is constructible on $X(\mathbb{C})$ (in the algebraic sense above).

(iii) There exists an algebraic stratification $\mathcal{S}$ on $\mathcal{X}$, such that for each stratum $V(\mathbb{C})$, the sheaf $F_V$ is lcc.

#### Proof.
(i)$\Rightarrow$(ii) is clear.

(ii)$\Rightarrow$(iii). Let $\mathcal{S}_X$ be a stratification of the scheme $X$, such that for each $U \in \mathcal{S}_X$, the sheaf $F_U$ is lcc. Let $U$ be an open stratum in $\mathcal{S}_X$, and let $V$ be the image of $U$ under the map $P$; then $V$ is an open substack of $\mathcal{X}$, and $P_U : U \to V$ is a presentation. By \cite[3.2.1]{[21]} we see that $F_{V_{\text{an}}}$ is lcc. Since $X - P^{-1}(V) \to \mathcal{X} - V$ gives an algebraic presentation of $(\mathcal{X} - V)_{\text{an}} = X - V_{\text{an}}$, and

$$F|_{X - V_{\text{an}}}|_{(X - P^{-1}(V))_{\text{an}}} \simeq F_X|_{(X - P^{-1}(V))_{\text{an}}}$$

is still constructible, by noetherian induction we are done.

(iii)$\Rightarrow$(i). Let $(U, u) \in \text{Lis-ét}(\mathcal{X})$. Then $a_{\text{an}} : \mathcal{S}$ is an algebraic stratification of $U(\mathbb{C})$, and it is clear that $F_U$ is lcc on each stratum of this stratification. \hfill $\square$

### 3.2.5
Assume the ring $R$ is noetherian. Then the constructible $R$-modules on $X_{\text{lis-an}}$ form a full subcategory Mod$_c(X, R)$ of Mod$(\mathcal{X}, R)$ that is closed under taking kernels, cokernels and extensions (i.e. it is a Serre subcategory). To see this, we first show that Cartesian sheaves form a Serre subcategory.

Let $(f, \alpha) : (U, u) \to (V, v)$ be a morphism in Lis-an($\mathcal{X}$). The functor $f^* : \text{Mod}(V_{\text{an}}, R) \to \text{Mod}(U_{\text{an}}, R)$ is exact, because $f^*F = R_U \otimes_{f^{-1}R_V} f^{-1}F$. Let $a : F \to G$ be a morphism of Cartesian sheaves. Then $\text{Ker}(f^*a) : f^*F \to f^*G$ is an isomorphism:

$$f^*\text{Ker}(a) \longrightarrow f^*F \overset{f^*a}{\longrightarrow} f^*G \quad \text{isomorphism}$$

The proof for cokernels and extensions (using 5-lemma) is similar. One can also mimic the proof in \cite[3.8, 3.9]{[21]} to prove a similar statement for analytic stacks, in the more general
setting where the coefficient ring is a flat sheaf. In this paper, we will only need the case of a constant coefficient ring.

Then by Lemma 3.2.4 iii, it suffices to show that lcc \( R \)-modules form a Serre subcategory. This follows from ([1], IX, 2.1).

### 3.3 Derived categories

#### 3.3.1. Again let \( X = \mathcal{X}^{an} \). We follow [15] and define the derived category \( D_c(X_{lis-an}, \Lambda) \) of constructible \( \Lambda \)-adic sheaves (by abuse of language, as usual) as follows.

A complex of projective systems \( M \) in the ordinary derived category \( \mathcal{D}(X_{lis-an}^n, \Lambda) \) of the simplicial topos \( X_{lis-an}^n \), ringed by \( \Lambda = (\Lambda_n)_n \), is called a \( \lambda \)-complex if for every \( i \) and \( n \), the sheaf \( \mathcal{H}^i(M_n) \) is constructible and the cohomology system \( \mathcal{H}^i(M) \) is AR-adic. A \( \lambda \)-module is a \( \lambda \)-complex concentrated in degree 0, i.e. the \( \mathcal{H}^i \)'s are AR-null for \( i \neq 0 \). Then we define \( D_c(X_{lis-an}, \Lambda) \) to be the quotient of the full subcategory \( \mathcal{D}_c(X_{lis-an}^n, \Lambda) \) of \( \lambda \)-complexes by the full subcategory of AR-null complexes (i.e. those with AR-null cohomology systems).

This quotient inherits a standard \( t \)-structure, and we define the category \( \Lambda-Sh_c(X) \) of constructible \( \Lambda \)-adic sheaves on \( X_{lis-an} \) to be its core, namely the quotient of the category of \( \lambda \)-modules by the thick full subcategory of AR-null systems. By ([12], p.234), this is equivalent to the category of adic systems, i.e. those projective systems \( F = (F_n)_n \), such that for each \( n \), \( F_n \) is a constructible \( \Lambda \)-module on \( X_{lis-an} \), and the induced morphism \( F_n \otimes_{\Lambda_n} \Lambda_{n-1} \to F_{n-1} \) is an isomorphism.

Passing to localizations and 2-colimits, one can also define the categories \( D_c(X_{lis-an}, E_\lambda) \) and \( D_c(X_{lis-an}, \overline{Q}_\ell) \), as well as their cores with respect to the standard \( t \)-structures: the categories of constructible \( E_\lambda \)- and \( \overline{Q}_\ell \)-sheaves on \( X_{lis-an} \).

#### 3.3.2. Let \( X_\bullet \) be a strictly simplicial analytic space. Then we have the subcategory \( \text{Ab}_{cart}(X_\bullet) \) of Cartesian abelian sheaves on \( X_\bullet \) in \( \text{Ab}(X_\bullet) \), as defined in ([13], 12.4.2). It can be proved in the same way as in (3.2.5) that this is a Serre subcategory, which enables us to define the triangulated subcategory \( \mathcal{D}_{cart}(X_\bullet, \mathbb{Z}) \) of the ordinary derived category \( \mathcal{D}(X_\bullet, \mathbb{Z}) \), consisting of complexes with Cartesian cohomology sheaves.

If \( \mathcal{X} \) is an analytic stack and \( X_\bullet \to \mathcal{X} \) is a strictly simplicial hypercover of \( \mathcal{X} \) by analytic spaces, one can also consider the local topoi \( \mathcal{X}_{lis-an}|_{X_\bullet} \) (cf. ([1], IV, 5)) and have the notion of Cartesian sheaves on it. Then \( \text{Ab}_{cart}(\mathcal{X}|_{X_\bullet}) \subset \text{Ab}(\mathcal{X}|_{X_\bullet}) \) is a Serre subcategory, and we may define the triangulated subcategory \( \mathcal{D}_{cart}(\mathcal{X}|_{X_\bullet}, \mathbb{Z}) \subset \mathcal{D}(\mathcal{X}|_{X_\bullet}, \mathbb{Z}) \). These constructions apply as well to a general coefficient ring \( R \) in place of \( \mathbb{Z} \).

Now let \( X_\bullet \) be a strictly simplicial \( \mathbb{C} \)-algebraic space and \( R \) be a noetherian ring. A sheaf \( F \) of sets (resp. \( R \)-modules) on \( X_\bullet(\mathbb{C}) \) is said to be constructible if it is Cartesian and all components \( F_n \) on \( X_n(\mathbb{C}) \) are constructible. Constructible \( R \)-modules on \( X_\bullet(\mathbb{C}) \) form a Serre subcategory, and one can define the triangulated subcategory \( \mathcal{D}_{c}(X_\bullet(\mathbb{C}), R) \) consisting of complex with constructible cohomology sheaves. When \( X_\bullet \to \mathcal{X} \) is a strictly simplicial hypercover of a complex algebraic stack \( \mathcal{X} \) and \( \mathcal{X} = \mathcal{X}^{an} \), we also have \( \mathcal{D}_{c}(\mathcal{X}|_{X_\bullet(\mathbb{C})}, R) \).

Following ([15], 10.0.6), we define the adic derived category \( D_c(X_\bullet(\mathbb{C}), \Lambda) \) as follows.

A sheaf \( F \in \text{Mod}(X_\bullet(\mathbb{C})^N, \Lambda) \) is AR-adic if it is Cartesian (i.e. each \( F_n \in \text{Mod}(X_n(\mathbb{C}), \Lambda_n) \) is Cartesian), and \( F|_{X_\bullet(\mathbb{C})} \) is AR-adic for every \( i \). A complex \( C \in \mathcal{D}(X_\bullet(\mathbb{C})^N, \Lambda) \) is a \( \lambda \)-complex (resp. an AR-null complex) if its cohomology sheaf \( \mathcal{H}^i(C) \) is AR-adic and \( \mathcal{H}^i(C_m)|_{X_n(\mathbb{C})} \in \text{Mod}(X_n(\mathbb{C}), \Lambda_m) \) is constructible, for every \( i, m, n \) (resp. \( C|_{X_n(\mathbb{C})} \) is AR-null, for every \( n \)). Finally we define \( D_c(X_\bullet(\mathbb{C}), \Lambda) \) to be the quotient of the full subcategory \( \mathcal{D}_c(X_\bullet(\mathbb{C})^N, \Lambda) \subset \mathcal{D}(X_\bullet(\mathbb{C})^N, \Lambda) \) consisting of all \( \lambda \)-complexes by the full subcategory of AR-null complexes.
3.3.3. Let \( R \) be a noetherian ring as before. Then we may also define a topological constructible derived category as follows. Let \( \mathcal{D}(\mathcal{X}_{\text{lis-an}}, R) \) be the ordinary derived category of sheaves of \( R \)-modules on \( \mathcal{X}_{\text{lis-an}} \). Then \([3.2.5]\) allows us to define its triangulated subcategories

\[
\mathcal{D}_c(\mathcal{X}_{\text{lis-an}}, R) \subset \mathcal{D}_{\text{cart}}(\mathcal{X}_{\text{lis-an}}, R),
\]

consisting of complexes with constructible cohomology sheaves and Cartesian cohomology sheaves, respectively. The cores of the standard \( t \)-structures on them are \( \text{Mod}_c(\mathcal{X}, R) \) and \( \text{Mod}_{\text{cart}}(\mathcal{X}, R) \) respectively. The main examples we have in mind of the ring \( R \) for this topological setting are \( \Lambda, \mathbb{Q}, \mathbb{C} \) and \( \mathbb{Q}_\ell \).

In particular, when \( R = \Lambda \), to emphasize the difference from the category \( \Lambda\text{-}\text{Sh}_c(\mathcal{X}) \) of \( \Lambda \)-adic sheaves on \( \mathcal{X} \), we will often denote by \( \text{Mod}_0(\mathcal{A}_X) \) the category of constructible \( \Lambda_X \)-modules. In \([3.5.9]\), we will show that the two categories \( D_c(\mathcal{X}_{\text{lis-an}}, \Lambda) \) and \( \mathcal{D}_c(\mathcal{X}_{\text{lis-an}}, \Lambda) \) are equivalent.

For simplicity, we will drop “lis-an” in \( \mathcal{D}_c(\mathcal{X}_{\text{lis-an}}, R) \), if there is no confusion. Also we will drop “lis-ét” in \( D_c(\mathcal{X}_{\text{lis-ét}}, R) \).

3.4 Comparison between the derived categories of lisse-étale and lisse-analytic topoi

Given an algebraic stack \( \mathcal{X}/\mathbb{C} \), let \( \mathcal{X} = \mathcal{X}^{\text{an}} \), and let \( P : X \to \mathcal{X} \) be a presentation, with analytification \( P^{\text{an}} : X(\mathbb{C}) \to \mathcal{X} \). Let \( \epsilon : X_\bullet \to \mathcal{X} \) be the associated strictly simplicial smooth hypercover, and let \( e^{\text{an}} : X^{\text{an}}(\mathbb{C}) \to \mathcal{X} \) be the analytification. They induce morphisms of topoi, denoted by the same symbol. Consider the following morphisms of topoi:

\[
\begin{array}{ccc}
\mathcal{X}_{\text{lis-an}} & \xrightarrow{\epsilon^{\text{an}}} & \mathcal{X}_{\text{lis-an}}|_{X^{\text{an}}(\mathbb{C})} \\
\downarrow{\epsilon} & & \downarrow{\epsilon} \\
\mathcal{X}_{\text{lis-ét}} & \xrightarrow{\gamma} & X^{\text{an}}(\mathbb{C}) \\
\end{array}
\]

We will show that \( R\epsilon_* \circ R\xi_* \circ e^{\text{an}}_* \) gives an equivalence between \( D_c(\mathcal{X}, \Lambda) \) and \( D_c(\mathcal{X}, \Lambda) \), and that it is compatible with pushforwards. It is proved in \([14], 2.2.6\) that, \( (\epsilon^*, R\epsilon_*) \) induce an equivalence between the triangulated categories \( \mathcal{D}_c(\mathcal{X}, \Lambda_n) \) and \( \mathcal{D}_c(X_\bullet, \Lambda_n) \). As in \([15], 10.0.8\), this gives an equivalence between \( D_c(\mathcal{X}, \Lambda) \) and \( D_c(X^{\text{an}}_\bullet, \Lambda) \). We mimic the proof there and prove the analytic analogue.

**Proposition 3.4.1.** (i) Let \( R \) be a noetherian commutative ring with identity. Then the pairs of functors \( (e^{\text{an}}^*, R\epsilon_*) \), \( (\delta^{\text{an}}^*, R\delta_*^*) \) and \( (\gamma^{\text{an}}^*, R\gamma_*^{\text{an}}) \) induce equivalences of triangulated categories

\[
\begin{array}{ccc}
\mathcal{D}_c(\mathcal{X}, \Lambda) & \xrightarrow{R\epsilon_*} & \mathcal{D}_c(X^{\text{an}}_\bullet, \Lambda) \\
\end{array}
\]

which is commutative, as well as an equivalence

\[
D_c(\mathcal{X}, \Lambda) \leftarrow D_c(X^{\text{an}}_\bullet, \Lambda).
\]

(ii) Let \( X \) be a \( \mathbb{C} \)-algebraic space, and let \( \xi = \xi_X : X(\mathbb{C}) \to X^{\text{ét}}_\Lambda \) be the natural morphism of topoi. Then \( R\xi_* \) is defined on the unbounded constructible derived category, and the functors \( (\xi^*, R\xi_*) \) induce an equivalence between \( D_c(X(\mathbb{C}), \Lambda) \) and \( D_c(X, \Lambda) \).
(iii) Let $f : X \to Y$ be a morphism of $\mathbb{C}$-algebraic spaces, and let $\xi_X, \xi_Y$ be as in (ii). Then for every $F \in D^+_c(X, \Lambda)$, the natural morphism

$$\xi_Y f_* F \to f^\text{an}_*(\xi_X F)$$

is an isomorphism. Recall (1.3.5) that $f_*$ and $f^\text{an}_*$ here are derived functors $Rf_*$ and $Rf^\text{an}_*$, respectively.

**Proof.** (i) Firstly, note that the restriction functor $\delta^\text{an}_*: \text{Ab}(\mathfrak{X}_{\text{lis-an}}|X_\bullet(\mathbb{C})) \to \text{Ab}(X_\bullet(\mathbb{C}))$ is exact so that $R\delta^\text{an}_* = \delta^\text{an}_*$, since the topologies are the same. We have $\epsilon^\text{an,*} \simeq \delta^\text{an,*} \circ \gamma^\text{an,*}$, since they are all restrictions. Therefore, it suffices to prove that $(\delta^\text{an,*}, R\delta^\text{an}_*)$ and $(\gamma^\text{an,*}, R\gamma^\text{an}_*)$ induce equivalences of triangulated categories.

For an abelian sheaf $F$ on $X_\bullet(\mathbb{C})$, given by $F_n \in \text{Ab}(X_n(\mathbb{C}))$ for each $n$ and the transition map $\theta_a : a^*F_n \to F_m$ for each morphism $a : m \to n$ in $\Delta^{+\text{op}}$ (i.e. for each order-preserving injection $a : \{0, \cdots, n\} \to \{0, \cdots, m\}$), the sheaf $\delta^\text{an,*}F$ assigns to the object

$$U \xrightarrow{u} X_n(\mathbb{C}) \xrightarrow{\mathcal{X}}$$

the sheaf $u^*F_n$ on $U_{\text{an}}$, and to each morphism

$$U' \xrightarrow{u'} X_m(\mathbb{C}) \xrightarrow{\varphi} U \xrightarrow{u} X_n(\mathbb{C}) \xrightarrow{\mathcal{X}}$$

the transition map $\theta_{a,\varphi} : \varphi^*u^*F_n \xrightarrow{\theta_{a,\varphi}} u'^*F_m$.

It is then clear that $(\delta^\text{an,*}, \delta^\text{an}_*)$ induce equivalences of categories

$$\text{Ab}_{\text{cart}}(\mathfrak{X}|X_\bullet(\mathbb{C})) \leftrightarrow \text{Ab}_{\text{cart}}(X_\bullet(\mathbb{C})), \quad \text{Ab}_c(\mathfrak{X}|X_\bullet(\mathbb{C})) \leftrightarrow \text{Ab}_c(X_\bullet(\mathbb{C})).$$

For $F \in \mathcal{D}_{\text{cart}}(X_\bullet(\mathbb{C}), \mathbb{Z})$ and $G \in \mathcal{D}_{\text{cart}}(\mathfrak{X}|X_\bullet(\mathbb{C}), \mathbb{Z})$, we see that the adjunction and coadjunction morphisms

$$F \to \delta^\text{an}_*\delta^\text{an,*}F, \quad \delta^\text{an,*}\delta^\text{an}_*G \to G$$

are isomorphisms by applying $\mathcal{H}^i$. Hence $(\delta^\text{an,*}, \delta^\text{an}_*)$ induce equivalences

$$\mathcal{D}_{\text{cart}}(\mathfrak{X}|X_\bullet(\mathbb{C}), \mathbb{Z}) \leftrightarrow \mathcal{D}_{\text{cart}}(X_\bullet(\mathbb{C}), \mathbb{Z}), \quad \mathcal{D}_c(\mathfrak{X}|X_\bullet(\mathbb{C}), \mathbb{Z}) \leftrightarrow \mathcal{D}_c(X_\bullet(\mathbb{C}), \mathbb{Z}),$$

and also with $\mathbb{Z}$ replaced by any noetherian ring $R$.

To show that $\gamma^\text{an,*}$ induces an equivalence with coefficient $R$, we will apply (14), 2.2.3, All the transition morphisms of topoi in the strictly simplicial ringed topos $(\mathfrak{X}_{\text{lis-an}}|X_\bullet(\mathbb{C}), R)$ as well as $\gamma^\text{an} : (\mathfrak{X}_{\text{lis-an}}|X_\bullet(\mathbb{C}), R) \to (\mathfrak{X}_{\text{lis-an}}, R)$ are flat. Let $\mathcal{C} = \text{Mod}_c(\mathfrak{X}, R)$, which is a Serre subcategory of $\text{Mod}(\mathfrak{X}, R)$ by (3.2.5), and let $\mathcal{C}_\bullet$ be the essential image of $\mathcal{C}$ under $\gamma^\text{an,*} : \text{Mod}(\mathfrak{X}, R) \to \text{Mod}(\mathfrak{X}|X_\bullet(\mathbb{C}), R)$. We will see shortly that $\mathcal{C}_\bullet = \text{Mod}_c(\mathfrak{X}|X_\bullet(\mathbb{C}), R)$. To apply (14), 2.2.3 we need to verify the assumption (14), 2.2.1, which has two parts:
• ([14], 2.1.7) for the ringed sites \((\text{Lis-an}(\mathcal{X}))_{X_i(\mathbb{C})}, R)\) with \(\mathcal{C}_i\) the essential image of \(\mathcal{C}\) under the restriction \(\text{Mod}(\mathcal{X}, R) \to \text{Mod}(\mathcal{X}|_{X_i(\mathbb{C})}, R)\). This means that, for every object \(U\) in this site, there exists an analytic open covering \(U = \bigcup U_\alpha\) and an integer \(n_0\), such that for every \(F \in \mathcal{C}_i\) and \(n \geq n_0\), we have \(H^n(U_\alpha, F) = 0\). This follows from ([10], 3.1.7, 3.4.1).

• \(\gamma^{\text{an}*} : \mathcal{C} \to \mathcal{C}_\bullet\) is an equivalence with quasi-inverse \(R\gamma^{\text{an}}_\bullet\). For \(F \in \text{Mod}(\mathcal{X}, R)\), its image \(\gamma^{\text{an}*}F\) is the sheaf that assigns to the object

\[
\begin{array}{ccc}
U & \xrightarrow{u} & X_n(\mathbb{C}) \\
\downarrow & & \downarrow \\
\mathcal{X}
\end{array}
\]

the sheaf \(u^*F_{X_n(\mathbb{C})}\), and to each morphism

\[
\begin{array}{ccc}
U' & \xrightarrow{u'} & X_m(\mathbb{C}) \\
\downarrow & & \downarrow \\
U & \xrightarrow{u} & X_n(\mathbb{C}) \\
\downarrow & & \downarrow \\
\mathcal{X}
\end{array}
\]

the transition map \(\theta_{a,\varphi} : \phi^* u^* F_{X_n(\mathbb{C})} \xrightarrow{\theta_{a,\varphi}} u^* F_{X_m(\mathbb{C})}\).

So it is clear that \(\gamma^{\text{an}*}\) sends Cartesian (resp. constructible) sheaves to Cartesian (resp. constructible) sheaves. To verify this assumption, we need the analytic version of ([21], 4.4, 4.5), which we state in the following for the reader’s convenience.

Let \(\text{Des}(X(\mathbb{C})/\mathcal{X}, R)\) be the category of pairs \((F, \alpha)\), where \(F \in \text{Mod}(X(\mathbb{C}), R)\), and \(\alpha : p_1^* F \to p_2^* F\) is an isomorphism on \(X_1(\mathbb{C})\) (where \(p_1\) and \(p_2\) are the natural projections \(X_1(\mathbb{C}) \rightrightarrows X_0(\mathbb{C}) = X(\mathbb{C})\), such that \(p_{13}(\alpha) = p_{23}(\alpha) \circ p_{12}(\alpha) : p_1^* F \to p_3^* F\) on \(X_2(\mathbb{C})\)). Here \(p_i : X_2 \to X_0\) are the natural projections. There is a natural functor \(\alpha : \text{Mod}_{\text{cart}}(\mathcal{X}, R) \to \text{Des}(X(\mathbb{C})/\mathcal{X}, R)\), sending \(M\) to \((F, \alpha)\), where \(F = M_{X(\mathbb{C})}\) and \(\alpha\) is the composite

\[
p_1^* F \xrightarrow{p_1^*} M_{X_1(\mathbb{C})} \xrightarrow{(p_2^*)^{-1}} p_2^* F.
\]

There is also a natural functor \(\alpha : \text{Mod}_{\text{cart}}(X_\bullet(\mathbb{C}), R) \to \text{Des}(X(\mathbb{C})/\mathcal{X}, R)\) sending \(F = (F_i)_{i \in I}\) to \((F_0, \alpha)\), where \(\alpha\) is the composite

\[
p_1^* F_0 \xrightarrow{\text{can}} F_1 \xrightarrow{\text{can}} p_2^* F_0,
\]

and the cocycle condition is verified as in ([21], 4.5.4).

**Lemma 3.4.2.** The natural functors in the diagram

\[
\begin{array}{ccc}
\text{Mod}_{\text{cart}}(X_\bullet(\mathbb{C}), R) & \xrightarrow{\text{res}} & \text{Mod}_{\text{cart}}(\mathcal{X}, R) \\
\downarrow & & \downarrow \\
\text{Mod}_{\text{cart}}(\mathcal{X}, R) & \xrightarrow{A} & \text{Des}(X(\mathbb{C})/\mathcal{X}, R) \\
\downarrow & & \downarrow \\
\text{Mod}_{\text{cart}}(\mathcal{X}, R) & \xrightarrow{B} & \text{Des}(X(\mathbb{C})/\mathcal{X}, R)
\end{array}
\]

are all equivalences, and the diagram is commutative up to natural isomorphism.
The proof in ([21], 4.4, 4.5) carries verbatim to analytic stacks. In particular, by (3.2.4) the restriction
\[ \text{Mod}_c(X, R) \to \text{Mod}_c(X_\bullet(C), R) \]
is an equivalence.

Note that \( \mathcal{C}_\bullet = \text{Mod}_c(X|_{X_\bullet(C)}, R) \). Clearly every object in \( \mathcal{C}_\bullet \) is constructible. Conversely, for any constructible \( R \)-module \( G_\bullet \) on \( X|_{X_\bullet(C)} \), we have \( G_\bullet \cong \delta^{an}_\bullet F_\bullet \) where \( F_\bullet \) is the restriction \( \delta^{an}_\bullet G_\bullet \in \text{Mod}_c(X_\bullet(C), R) \) of \( G_\bullet \). By (3.4.2) we see that \( F_\bullet \) is the restriction of \( F \in \text{Mod}_c(X, R) \) for a unique (up to isomorphism) constructible \( R_X \)-module \( F \), therefore \( G_\bullet \cong \gamma^{an,*}F \) from that \( \delta^{an}_\bullet G_\bullet \cong F_\bullet \cong \text{res} F = \delta^{an}_\bullet (\gamma^{an,*}F) \), and hence \( G_\bullet \in \mathcal{C}_\bullet \).

From the 2-commutative diagram
\[
\begin{array}{ccc}
\text{Mod}_c(X|_{X_\bullet(C)}, R) & \xrightarrow{\gamma^{an,*}} & \text{Mod}_c(X_\bullet(C), R) \\
\text{Mod}_c(X, R) & \xrightarrow{\text{res}} & \text{Mod}_c(X_\bullet(C), R) \\
\end{array}
\]
we see that \( \gamma^{an,*} : \mathcal{C} \to \mathcal{C}_\bullet \) is an equivalence.

By ([14], 2.2.3), the functors \( (\gamma^{an,*}, R\gamma^{an}_*) \) induce an equivalence
\[
\mathcal{D}_c(X, R) \leftrightarrow \mathcal{D}_c(X|_{X_\bullet(C)}, R).
\]

Note that for \( M \in \mathcal{D}_c(X^N, \Lambda_\bullet) \) (resp. \( \mathcal{D}_c(X_\bullet(C)^N, \Lambda_\bullet) \)), each level \( M_n \) is in \( \mathcal{D}_c(X, \Lambda_n) \) (resp. \( \mathcal{D}_c(X_\bullet(C), \Lambda_n) \)), and the property of \( M \) being AR-adic (resp. AR-null) is intrinsic ([12], V, 3.2.3). So AR-adic (resp. AR-null) complexes on the two sides correspond under this equivalence, and we have equivalences
\[
\mathcal{D}_c(X^N, \Lambda_\bullet) \leftrightarrow \mathcal{D}_c(X_\bullet(C)^N, \Lambda_\bullet), \quad D_c(X, \Lambda) \leftrightarrow D_c(X_\bullet(C), \Lambda).
\]

(ii) We prove it for torsion coefficients first, and then pass to adic coefficients. For torsion coefficients, we prove it for schemes first, and then apply descent ([14], 2.2.3) to deduce it for algebraic spaces.

Let \( X/\mathbb{C} \) be a scheme and let \( G \) be a sheaf of abelian groups on \( X(\mathbb{C}) \). Then the sheaf \( R^i\xi_*G \) on \( X_{\text{ét}} \) is the sheafification of the presheaf
\[
(U \to X) \mapsto H^i(U(\mathbb{C}), G).
\]

By ([10], 3.1.7, 3.4.1), \( R^i\xi_*G = 0 \) for all sheaves \( G \) and all \( i > 1 + 2 \dim X \), so \( R\xi_* \) has finite cohomological dimension, and the functor
\[
R\xi_* : \mathcal{D}(X(\mathbb{C}), \Lambda_n) \to \mathcal{D}(X, \Lambda_n)
\]
takes the full subcategory \( \mathcal{D}_c(X(\mathbb{C}), \Lambda_n) \) into \( \mathcal{D}_c(X, \Lambda_n) \).

Given \( F \in \mathcal{D}_c(X, \Lambda_n) \) and \( G \in \mathcal{D}_c(X(\mathbb{C}), \Lambda_n) \), we want to show that the adjunction and coadjunction morphisms
\[
F \to R\xi_*\xi^*F, \quad \xi^*R\xi_*G \to G
\]
are isomorphisms. The analytification functor \( \xi^* \) is exact and \( \text{cd}(R\xi_*) < \infty \), so by applying \( \mathcal{H}^i \) on both sides, we may assume that \( F \) and \( G \) are bounded, or even constructible sheaves. By the comparison ([3], 6.1.2 (A’)), \( G \) is algebraic (i.e. \( G = \xi^*\tilde{G} \) for some \( \Lambda_n \)-sheaf \( \tilde{G} \) on \( X_{\text{ét}} \)).

The sheaves \( R^i\xi_*\xi^*F \) and \( R^i\xi_*G \) are sheafifications of the functors on \( \text{Ét}(X) \)
\[
(U \to X) \mapsto H^i(U(\mathbb{C}), F^{an}), \quad (U \to X) \mapsto H^i(U(\mathbb{C}), G)
\]
respectively. By the comparison theorem of Artin ([1], XVI, 4.1), we have
\[ H^i(U(\mathbb{C}), F^{an}) = H^i(U, F), \quad H^i(U(\mathbb{C}), G) = H^i(U, G), \]
so they both sheafify to zero if \( i > 0 \) ([17], 10.4), and to \( F \) and \( \tilde{G} \) respectively if \( i = 0 \). It follows that the adjunction and coadjunction morphisms are both isomorphisms, and we have an equivalence
\[ (\xi^*, R\xi_*) : \mathcal{D}_c(X(\mathbb{C}), \Lambda_n) \leftrightarrow \mathcal{D}_c(X, \Lambda_n) \]
for each \( n \).

Now let \( X \) be a \( \mathbb{C} \)-algebraic space, and take a simplicial étale hypercover \( \epsilon : X_\bullet \to X \) of \( X \) by schemes. As in (i), we can apply ([14], 2.2.3) to show that the morphisms of topoi
\[ X_\bullet,\text{ét} \to X,\text{ét} \]
induce equivalences
\[ \mathcal{D}_c(X_\bullet, \Lambda_n) \leftrightarrow \mathcal{D}_c(X, \Lambda_n) \]
for each \( n \). Similarly, \( \epsilon^{an} : X_\bullet(\mathbb{C}) \to X(\mathbb{C}) \) induces an equivalence
\[ \mathcal{D}_c(X_\bullet(\mathbb{C}), \Lambda_n) \to \mathcal{D}_c(X(\mathbb{C}), \Lambda_n). \]
Therefore, the commutative diagram of topoi
\[ X_\bullet(\mathbb{C}) \xrightarrow{\epsilon^{an}} X(\mathbb{C}) \]
\[ \xi_\bullet \downarrow \quad \downarrow \xi \]
\[ X_\bullet,\text{ét} \xrightarrow{\epsilon} X,\text{ét} \]
leads to a commutative diagram
\[ \mathcal{D}_c(X_\bullet(\mathbb{C}), \Lambda_n) \xrightarrow{\sim} \mathcal{D}_c(X(\mathbb{C}), \Lambda_n) \]
\[ \sim \downarrow \quad \downarrow^{(1)} \]
\[ \mathcal{D}_c(X_\bullet,\text{ét}, \Lambda_n) \xrightarrow{\sim} \mathcal{D}_c(X,\text{ét}, \Lambda_n), \]
and (1) is an equivalence.

Since AR-adic (resp. AR-null) complexes in \( \mathcal{D}(X(\mathbb{C})^N, \Lambda_\bullet) \) and in \( \mathcal{D}(X^N, \Lambda_\bullet) \) correspond, \( (\xi^*, R\xi_*) \) induce equivalences
\[ \mathcal{D}_c(X(\mathbb{C})^N, \Lambda_\bullet) \leftrightarrow \mathcal{D}_c(X^N, \Lambda_\bullet), \quad \mathcal{D}_c(X(\mathbb{C}), \Lambda) \leftrightarrow \mathcal{D}_c(X, \Lambda). \]

(iii) Applying \( \mathcal{H}^i \) on both sides, we need to show that
\[ \xi^*_Y R^if_*F \to R^if^{an}_*(\xi^*_X F) \]
is an isomorphism. The normalization functor \( F \to \tilde{F} \) has finite cohomological dimension, so \( \tilde{F} \) is essentially bounded below. Replacing \( F \) by various levels \( \tilde{F}_n \) of its normalization, we reduce to the case where \( F \in \mathcal{D}_+(X, \Lambda_n) \). Then one can replace \( F \) by \( \tau_{\leq i} F \) and reduce to the case where \( F \) is bounded, or even a constructible \( \Lambda_n \)-sheaf. For schemes, this follows from Artin’s comparison theorem ([1], XVI, 4.1). For algebraic spaces, this follows from the case of schemes by descent.

Explicitly, to prove that the base change morphism is an isomorphism, we may pass to an étale presentation of \( Y \) and apply smooth base change theorem, hence reduce to the
case where $Y$ is a scheme. Then let $X \rightarrow X$ be a simplicial étale cover by schemes, and let $f_p : X_p \rightarrow Y$ ($p \geq 1$) be the composition $X_p \rightarrow X \xrightarrow{f} Y$. The commutative diagram of topoi

\[
\begin{array}{ccc}
X_p(C) & \xrightarrow{f_{an}} & Y(C) \\
\downarrow \xi_{X_p} & & \downarrow \xi_Y \\
X_{\cdot, et} & \xrightarrow{f} & Y_{\cdot, et}
\end{array}
\]

leads to a morphism of spectral sequences

\[
\begin{array}{ccc}
\xi_Y^* R^q f_{an}(F|_{X_p}) & \xrightarrow{(1)} & \xi_Y^* R^{p+q} f_* F \\
\sim \xrightarrow{(2)} & \xrightarrow{\sim} & \sim \\
R^q f_{an}^*(\xi_{X_p}^*(F)|_{X_p(C)}) & \xrightarrow{\sim} & R^{p+q} f_{an}^*(\xi_X^* F),
\end{array}
\]

where (1) is an isomorphism, therefore (2) is an isomorphism.

**Corollary 3.4.3.** There is a natural equivalence between the triangulated categories $D_c(X, \Lambda)$ and $D_c(X, \Lambda)$, compatible with pushforwards by morphisms of complex algebraic stacks.

**Proof.** As mentioned before, combining ([14], 2.2.6) and (3.4.1) we see that $R\epsilon_* \circ R\xi_{\cdot, \cdot, \cdot}^{an,*}$ gives an equivalence between $D_c(X, \Lambda)$ and $D_c(X, \Lambda)$. Let $f : X \rightarrow Y$ be a morphism of $\mathbb{C}$-algebraic stacks, and we choose a commutative diagram

\[
\begin{array}{ccc}
X_{\cdot} & \xrightarrow{\tilde{f}} & Y_{\cdot} \\
\downarrow \epsilon_X & & \downarrow \epsilon_Y \\
X & \xrightarrow{f} & Y
\end{array}
\]

Then we have the following diagram

\[
\begin{array}{ccc}
D_c^+(X, \Lambda) & \xrightarrow{\epsilon_{X, \cdot, \cdot}^{an,*}} & D_c^+(X_{\cdot, \cdot, \cdot}, \Lambda) \\
\downarrow f_{an}^* & & \downarrow \tilde{f}_* \\
D_c^+(\mathbb{C}, \Lambda) & \xrightarrow{\epsilon_{\cdot, \cdot, \cdot}^{an,*}} & D_c^+(Y_{\cdot, \cdot, \cdot}, \Lambda)
\end{array}
\]

\[
\begin{array}{ccc}
D_c^+(X, \Lambda) & \xrightarrow{\epsilon_{X, \cdot}^{an,*}} & D_c^+(X_{\cdot, \cdot, \cdot}, \Lambda) \\
\downarrow f_{an}^* & & \downarrow \tilde{f}_* \\
D_c^+(\mathbb{C}, \Lambda) & \xrightarrow{\epsilon_{\cdot, \cdot}^{an,*}} & D_c^+(Y_{\cdot, \cdot, \cdot}, \Lambda)
\end{array}
\]

where the horizontal arrows are all equivalences of triangulated categories. The square on the left commutes by construction, and the commutativity of the squares in the middle and on the right follows from (3.4.1 iii) and ([15], p.202) respectively.

It remains to show that the equivalence is “natural” in the sense that, if $P' : X' \rightarrow X$ is another presentation, the induced equivalence is naturally isomorphic to the one induced by $X$. The usual argument of taking 2-fiber product reduces us to assume that one presentation dominates the other, and the claim is clear in this case.

**3.5 Comparison between the two derived categories on the lisse-analytic topos**

In (3.3.1) and (3.3.3), we defined two derived categories, denoted by $D_c(X, \Lambda)$ and $D_{\cdot, \cdot, \cdot}(X, \Lambda)$ respectively. Before proving that they are equivalent, we give some preparation on the analytic analogues of some concepts and results in [15].
3.5.1. For now let \( \mathfrak{X} \) be any complex analytic stack, not necessarily algebraic. As in [11], let \( \pi : (\mathfrak{X}_{\text{lis-an}}^N, \Lambda_n) \to (\mathfrak{X}_{\text{lis-an}}, \Lambda) \) be the morphism of ringed topoi, with \( \pi_* = \lim \) and \( \pi^* = (- \otimes_{\Lambda} \Lambda_n)_n \). We then have derived functors \( R\pi_* \) and \( L\pi^* \) between \( \mathcal{D}(\mathfrak{X}^N, \Lambda_n) \) and \( \mathcal{D}(\mathfrak{X}, \Lambda) \), and \( L\pi^* \) has (co)homological dimension 1. Denote \( \text{Mod}(\mathfrak{X}^N, \Lambda_n) \) by \( \mathcal{A}(\mathfrak{X}) \) or just \( \mathcal{A} \).

Variant: more generally, one can consider other coefficients for the topoi \( \mathfrak{X}_{\text{lis-an}}^N \) and \( \mathfrak{X}_{\text{lis-an}} \), for instance the constant sheaf \( \mathbb{Z} \), namely one is considering all sheaves of abelian groups. In this case we have derived functors \( R\pi_* \) and \( L\pi^* = L\pi^{-1} = \pi^{-1} \) between \( \mathcal{D}(\mathfrak{X}^N, \mathbb{Z}) \) and \( \mathcal{D}(\mathfrak{X}, \mathbb{Z}) \).

When \( \mathfrak{X} = X \) is an analytic space (always assumed to be finite dimensional), we often consider the morphism \( \pi_X : \mathfrak{X}_{\text{an}}^N \to X_{\text{an}} \) between analytic topoi. The derived functor \( R\pi_{X*} : \mathcal{D}(\mathfrak{X}^N, \mathbb{Z}) \to \mathcal{D}(X, \mathbb{Z}) \) has finite cohomological dimension: this is a consequence of ([15], 2.1.1) and ([10], 3.1.7, 3.4.1). In this case, by \( \pi_X \) we always mean this morphism between analytic topoi, rather than the lisse-analytic topos of \( X \), unless otherwise stated.

It follows from (3.5.3 i) below that, for an analytic stack \( \mathfrak{X} \) with a presentation \( X \), we have \( \text{cd}(R\pi_*) \leq \text{cd}(R\pi_{X*}) < \infty \).

Lemma 3.5.2. Let \( M \) be an AR-null complex in \( \mathcal{D}(\mathcal{A}(\mathfrak{X})) \). Then \( R\pi_! M = 0 \).

Proof. The case when \( M \) is essentially bounded below follows from ([11], 1.1). In particular, since each of \( \mathcal{A}_!^\epsilon(M) \) and \( \tau_{>i} M \) is AR-null, we have \( R\pi_* \mathcal{A}_!^\epsilon(M) \cong R\pi_* \tau_{>i} M = 0 \). For the general case, we apply ([14], 2.1.10), with \( \epsilon = \pi \) and \( \mathcal{A}_! = \) the entire category of \( \Lambda_n \)-modules on \( \mathfrak{X}_{\text{lis-an}}^N \); the conditions in loc. cit. are trivially satisfied, and the assumption ([14], 2.1.7) for the ringed topos \( (\mathfrak{X}_{\text{lis-an}}, \Lambda_n) \) is verified by ([10], 3.1.7, 3.4.1).

For \( M \in \mathcal{D}(\mathcal{A}) \), let \( \tilde{M} \) be the normalization of \( M : \tilde{M} = L\pi^! R\pi_* M \). We say that \( M \) is normalized if the coadjunction morphism \( \tilde{M} \to M \) is an isomorphism. As mentioned before, if \( M \in \mathcal{D}(\mathcal{A}(X_{\text{an}})) \) for an analytic space \( X \), one defines the normalization \( \tilde{M} \) similarly, using the morphism \( \pi_X \) of analytic topoi. In this case, the normalization functor has finite cohomological dimension.

The analytic versions of ([15], 2.2.1, 3.0.11, 3.0.10) hold and can be proved verbatim, as we state in the following.

Proposition 3.5.3. (i) For \( (U, u) \) in Lis-an(\( \mathfrak{X} \)) and \( M \in \mathcal{D}(\mathfrak{X}^N, \mathbb{Z}) \), we have \( R\pi_{U*}(M_U) = (R\pi_* M)_U \) in \( \mathcal{D}(\mathcal{U}_{an}, \mathbb{Z}) \).

(ii) For \( (U, u) \) in Lis-an(\( \mathfrak{X} \)) and \( M \in \mathcal{D}(\mathfrak{X}, \Lambda) \), we have \( L\pi^*_{U*}(M_U) = (L\pi^* M)_U \) in \( \mathcal{D}(\mathcal{A}(\mathcal{U}_{an})) \).

(iii) Let \( M \in \mathcal{D}(\mathcal{A}) \). Then it is normalized if and only if the natural morphism

\[
\Lambda_{n-1} \otimes_{\Lambda} M_n \to M_{n-1}
\]

is an isomorphism for each \( n \).

3.5.4. Let \( f : X \to Y \) be a morphism of complex analytic spaces. Then we have a natural isomorphism

\[
Rf_* \circ R\pi_X^* \cong R\pi_Y^* \circ Rf^N_* : \mathcal{D}(X^N, \mathbb{Z}) \to \mathcal{D}(Y, \mathbb{Z}).
\]

In fact, \( f \) defines a morphism of their analytic topoi \( f : X_{\text{an}} \to Y_{\text{an}} \), and we have a commutative diagram of topoi:

\[
\begin{array}{ccc}
X_{\text{an}} & : & Y_{\text{an}} \\
\pi_X & \downarrow & \pi_Y \\
X_{\text{an}} & \stackrel{f}{\longrightarrow} & Y_{\text{an}}.
\end{array}
\]
To see this, one verifies either \( f^{\mathbb{N},-1} \circ \pi^{-1}_Y \equiv \pi^{-1}_X \circ f^{-1} \) (which is clear) or \( \pi_{Y*} f^{\mathbb{N}}_* \equiv f_* \circ \pi_{X*} \) (namely, \( f_* \) preserves limits).

One may generalize it to algebraic morphisms between algebraic analytic stacks and their adic derived categories, using descent. As we will not use it in the sequel, we do not give the proof in detail here.

**Lemma 3.5.5.** (i) Let \( X \) be a \( \mathbb{C} \)-scheme and \( F \in \mathcal{M}od_c(\Lambda_{X(\mathbb{C})}) \). Then there is an integer \( N \geq 0 \) such that \( F/\ker(\lambda^N) \) is a flat \( \Lambda_{X(\mathbb{C})} \)-sheaf. Also, for each \( n \geq 0 \), the sheaf \( F \otimes \Lambda_n \) is constructible.

(ii) Let \( X \) be an analytic space and let \( F \) be an analytically constructible \( \Lambda_X \)-module. Then there is an integer \( N \geq 0 \) with the same property as above.

**Proof.** (i) This is a consequence of \((\text{[3]}, \, 6.1.2, \, (A''))\) and \((\text{[3]}, \, \text{Rapport}, \, 2.8)\). But since the sketchy proof of \((\text{[3]}, \, 6.1.2, \, (A''))\) is not very clear to me, we prove (i) directly.

By definition there is a stratification \( \mathcal{S} \) of \( X \) such that for each stratum \( i_V : V \hookrightarrow X \), the sheaf \( i'_V F \) on \( V(\mathbb{C}) \) is a lcc \( \Lambda_{V(\mathbb{C})} \)-module. Thus there exists an integer \( N \geq 0 \) such that \( i'_V F/\ker(\lambda^N \circ i'_V F) \) is flat. To conclude, we take \( N = \max_V \{ n_V \} \) and use the fact that \( i'_V \) is exact (hence preserves \( \text{"Ker}(\lambda^N)\)”).

The second statement follows from \((3.2.3)\), as \( F \otimes \Lambda_n = \text{Coker}(\lambda^{n+1} \circ F) \).

(ii) The same proof applies; \( \mathcal{S} \) is now an analytic stratification, with finitely many strata in it.

Now we assume that our analytic stack \( \mathcal{X} \) is the analytification of an algebraic one \( \mathcal{X} \) (except in \((3.5.8)\), which apply to non-algebraic ones too). Next we show that \( R\pi_* \) and \( L\pi^* \) preserve constructibility. This depends in an essential way on the lisse-analytic topology, as the corresponding statement in the algebraic category is false (cf. \((15), \, 3.0.16)\)).

**Proposition 3.5.6.** The functors \( R\pi_* \) and \( L\pi^* \) restrict to functors between \( \mathcal{D}_c(\mathcal{A}(\mathcal{X})) \) and \( \mathcal{D}_c(\mathcal{X}, \Lambda) \).

**Proof.** 1) We prove that if \( M = (M_n)_n \in \mathcal{D}_c(\mathcal{A}(\mathcal{X})) \), then \( R^i \pi_* M \) is a constructible \( \Lambda_X \)-module for each \( i \). Let us start with the following lemma, a little stronger than needed.

**Lemma 3.5.7.** If \( M \in \mathcal{D}_{cart}(\mathcal{X}^N, \mathbb{Z}) \), then \( R\pi_* M \in \mathcal{D}_{cart}(\mathcal{X}, \mathbb{Z}) \).

**Proof.** Let \( f : U \to V \) be a morphism in Lis-an(\( \mathcal{X} \)). It induces a commutative diagram of topoi

\[
\begin{array}{ccc}
U^\text{an} & \xrightarrow{f^\text{an}} & V^\text{an} \\
\pi_U \downarrow & & \downarrow \pi_V \\
U^\text{an} & \xrightarrow{f} & V^\text{an}.
\end{array}
\]

In particular, we have the base change morphism

\[
bc : f^* R\pi_{V*} M_V \to R\pi_{U*} f^\text{an,*} M_V.
\]

The transition morphism \( \Theta_f : f^* (R\pi_* M) \to (R\pi_* M)_U \) is the composition (via \((3.5.3)\))

\[
f^* R\pi_{V*} M_V \xrightarrow{bc} R\pi_{U*} f^\text{an,*} M_V \xrightarrow{R\pi_{U*}(\theta_f)} R\pi_{U*} M_U,
\]

where \( \theta_f \) is the transition morphism for \((M_n)_n \) and is an isomorphism \((3.1.3)\). We need to show that the base change morphism is an isomorphism.

Since \( R\pi_{U*} \) and \( R\pi_{V*} \) have finite cohomological dimensions, by taking \( \mathcal{H}^i \) of both sides of the base change morphism, we may assume that \( M \) is essentially bounded (i.e. the
projective system $\mathcal{H}^i(M)$ is AR-null for $|i| \gg 0$, or even a projective system of Cartesian sheaves \((3.2.5)\).

A standard fiber-product argument reduces us to assume that $f$ is smooth (cf. \((13), \text{12.3.1})\). Since the problem is local on $U$, we may assume that $f$ is isomorphic to the projection $\text{pr}_1 : V \times Z \to V$ from a product of $V$ with a complex manifold $Z$.

Let us denote by $f_{ps}^*$ the presheaf inverse image functor, and by $\mathcal{R}^i\pi_{U*}M_U$ the presheaf that assigns to an open $U' \subset U$ the group $H^i(\pi_U^*U', M_U)$, and similarly for $\mathcal{R}^i\pi_{V*}M_V$.

Then we have morphisms of presheaves on $U$:

$$f_{ps}^*\mathcal{R}^i\pi_{V*}M_V \to \mathcal{R}^i\pi_{U*}f_{ps}^*\mathcal{R}^i\pi_{V*}M_V \to \mathcal{R}^i\pi_{U*}f^*\mathcal{R}^i\pi_{V*}M_V$$

from which the base change morphism for cohomology sheaves $R^i\pi_{V*}M_V$ is deduced by sheafification. Explicitly, to an open $U' \subset U$, the left-hand side assigns the colimit of the direct system $H^i(\pi_U^*V', M_V)$ parametrized by open sets $V' \subset V$ containing $f(U')$, and the right-hand side assigns $H^i(\pi_U^*U', f^*M_V)$, and the morphism of presheaves is induced from the natural maps

$$H^i(V', M_{n,V}) \to H^i(U', f^*M_{n,V})$$

given by the mapping $f|_{U'} : U' \to V'$. To show that this morphism of presheaves sheafifies to an isomorphism, it suffices (by checking stalks) to show that it is an isomorphism on a topological basis on $U$, which can be taken to be open sets of the form $U' = V' \times Z'$, with $V' \subset V$ and $Z' \subset Z$ open, and the $Z''$'s are polydisks. Then $V' = f(U')$ is open, and by the Künneth formula, the morphism

$$H^i(V', M_{n,V}) \to H^i(U', M_{n,V} \boxtimes \mathbb{Z})$$

is an isomorphism, since $R\Gamma(Z', \mathbb{Z}) = \mathbb{Z}$. \hfill \Box

Now let $M \in \mathcal{D}_c(\mathcal{A}(X))$. Since $R\pi_*M \in \mathcal{D}_{\text{cart}}(X, \Lambda)$, to show that it is constructible, by \((3.2.3) \text{3.5.3(i)}\) we may pass to an algebraic presentation $X(\mathbb{C}) \to X$, so we may assume that $X = X(\mathbb{C})$ for some $\mathbb{C}$-scheme $X$. Since $R\pi_*$ has finite cohomological dimension, we may assume that $M$ is an AR-adic projective system with constructible components. By \((12), \text{V, 3.2.3})$ we may assume that $M$ is an adic system.

Next we will reduce to the case where $M$ has lcc components. Applying \((3), \text{6.1.2 (A')})$ to the components $M_\alpha$ and the transition maps $\rho_\alpha : M_\alpha \to M_{\alpha-1}$, we see that $\widetilde{M}$ is algebraic, i.e. it is the analytification of an adic system $\widetilde{M} = (M_n, \tilde{\rho}_n)$ with constructible components on $X_{ad}$. By \((8), \text{Rapport, 2.5})$, there exists a stratification of $X$ over each stratum of which $\widetilde{M}$ is lisse. Let $j : U \hookrightarrow X$ be an open stratum, with complement $i : Z \to X$. Then $M_{n,U(\mathbb{C})}$, being the analytification of the lcc sheaf $\widetilde{M}_{n,U(\mathbb{C})}$, is lcc; it corresponds to the representation

$$\pi_1(U(\mathbb{C})) \longrightarrow \pi_1^q(U) \longrightarrow \text{GL}(\widetilde{M}_{n,U(\mathbb{C})}),$$

and $U(\mathbb{C})$ is locally contractible \((5), \text{4.4})$. We apply $R\pi_*$ to the exact triangle

$$i_{\mathbb{C}}^n R^i\pi_{U(\mathbb{C})}^n M \longrightarrow M \longrightarrow R^i j_{\mathbb{C}}^n M_{U(\mathbb{C})}$$

to obtain (by \((3.5.4)\))

$$i_* R\pi_{U(\mathbb{C})}^n M \longrightarrow R\pi_* M \longrightarrow R^i j_* R\pi_{U(\mathbb{C})}^n M_{U(\mathbb{C})}.$$
Lemma 3.5.8. Under the assumption that the $M_n$'s are lcc on $\mathfrak{X}$, we have $R^i\pi_*M = 0$ for $i \neq 0$.

Proof. By (5.3.i) we may pass to an analytic presentation $X \to \mathfrak{X}$, so assume that $\mathfrak{X} = X$ is an analytic space. Then $R^i\pi_*M$ is the sheaf on $X_{an}$ associated to the presheaf that assigns to each open set $U \subset X$ the group $H^i(\pi^*U, M)$. We only need to consider those open sets $U$ which are contractible, since they generate a basis for the topology of $X$, by (5, 4.4). Then $M_n|_U$, a priori locally constant, are constant sheaves defined by finite sets, hence $H^i(U, M_n) = 0$ for $i \neq 0$, and $H^0(U, M_n)$ are finite so that $\varprojlim_n H^0(U, M_n) = 0$. The result then follows from (15, 2.1.i).

It remains to show that $\pi_*M$ is constructible; in fact it is a lcc $\Lambda_X(\mathbb{C})$-module. To see this, we may replace $X(\mathbb{C})$ by a contractible open subset by (3.2.1), hence assume that each $M_n$ is constant. Then $\pi_*M = \varprojlim_n M_n$ is also constant, because this sheaf limit is just the presheaf limit.

2) Now we prove that if $F \in \mathcal{D}_c(\mathfrak{X}, \Lambda)$, then $L\pi^*F \in \mathcal{D}_c(\mathfrak{X})$.

First let $F \in \mathcal{D}_{cart}(\mathfrak{X}, \Lambda)$, and let us show that $L\pi^*F$ is Cartesian, i.e. for all $i$, each component of the projective system $L^i\pi^*F$ is Cartesian. Since $L\pi^*$ has finite cohomological dimension we may assume that $F$ is a Cartesian sheaf, by (3.2.5). Then $(L\pi^*F)_n = F \otimes \Lambda^n_{\Lambda}$ is represented by the complex $F \to \Lambda^{n+1}$, both the kernel and cokernel of which are Cartesian sheaves, by (3.2.5).

Now let $F \in \mathcal{D}_c(\mathfrak{X}, \Lambda)$, and let us show that $L\pi^*F$ is a $\Lambda$-complex, i.e. each cohomology $L^i\pi^*F$ is an AR-adic system of constructible sheaves. By (3.2.4), this can be checked on an algebraic presentation $X(\mathbb{C})$, so we assume that $\mathfrak{X} = X(\mathbb{C})$ and $F \in \mathfrak{M}_c(\Lambda_X(\mathbb{C}))$. By (3.5.5) we reduce to two cases: $F$ is flat, or $F$ is annihilated by $\Lambda$.

If $F$ is flat, then $L\pi^*F = \pi^*F$ is the adic sheaf $(F \otimes \Lambda^n)_n$, with constructible components (3.5.5) (even with respect to the same algebraic stratification for $F$).

If $\lambda F = 0$, then using the following projective system of $\Lambda$-flat resolutions of the $\Lambda_n$'s

$$
\begin{array}{cccccccc}
0 & \to & \Lambda & \to & \Lambda^{n+1} & \to & \Lambda_n & \to & 0 \\
\downarrow{\lambda} & & \downarrow{1} & & \downarrow{1} & & \downarrow{1} & & \\
0 & \to & \Lambda & \to & \Lambda & \to & \Lambda_{n-1} & \to & 0 
\end{array}
$$

we see that $L\pi^*F$ is represented by the following complex of systems:

$$
\begin{array}{cccccccc}
0 & \to & F & \to & F & \to & 0 \\
\downarrow{1} & & \downarrow{1} & & \downarrow{1} & & \\
0 & \to & F & \to & F & \to & 0. 
\end{array}
$$

Therefore, $\mathcal{H}^0(L\pi^*F)$ is

$$
\cdots \to F \to F \to \cdots
$$

which is adic with constructible components, and $\mathcal{H}^{-1}(L\pi^*F)$ is

$$
\cdots \to F \to F \to \cdots
$$

which is AR-null (hence AR-adic) with constructible components.

By (3.5.2) the functor $R\pi_* : \mathcal{D}_c(\mathfrak{X}) \to \mathcal{D}_c(\mathfrak{X}, \Lambda)$ factors through the quotient category $D_c(\mathfrak{X}, \Lambda) :$

$$
\mathcal{D}_c(\mathfrak{X}) \xrightarrow{Q} D_c(\mathfrak{X}, \Lambda) \xrightarrow{R\pi_*} \mathcal{D}_c(\mathfrak{X}, \Lambda).
$$
Thus the normalization functor, when restricted to $\mathcal{D}_c(\mathcal{A})$, factors through $D_c(X, \Lambda)$, and for $K \in D_c(X, \Lambda)$ we still denote $L\pi^*R\pi_*K$ by $\hat{K}$.

**Proposition 3.5.9.** (i) The functors $(Q \circ L\pi^*, R\pi_*)$ induce an equivalence $D_c(X, \Lambda) \longleftrightarrow \mathcal{D}_c(X, \Lambda)$.

(ii) Let $f : X \rightarrow Y$ be a morphism of complex algebraic stacks, and let $f^{an} : X \rightarrow \mathfrak{Y}$ be its analytification. Then the following diagram commutes:

$$
\begin{array}{ccc}
D_c^+(X, \Lambda) & \xrightarrow{R\pi_{X,*}} & D_c^+(X, \Lambda) \\
\downarrow f^{an} & & \downarrow f^{an} \\
D_c^+(\mathfrak{Y}, \Lambda) & \xrightarrow{R\pi_{\mathfrak{Y},*}} & D_c^+(\mathfrak{Y}, \Lambda).
\end{array}
$$

**Proof.** (i) We will show that the adjunction and coadjunction maps are isomorphisms. For coadjunction maps, this is the analytic version of ([15], 3.0.14).

**Lemma 3.5.10.** Let $M \in \mathcal{D}_c(\mathcal{A}(X))$. Then the coadjunction map $\hat{M} \rightarrow M$ has an AR-null cone.

**Proof.** It can be proved in the same way as ([15], 3.0.14). We go over the proof briefly. First note that, if

$$M' \rightarrow M \rightarrow M'' \rightarrow M'[1]$$

is an exact triangle in $\mathcal{D}_c(\mathcal{A})$ and the coadjunction map is an isomorphism for two vertices, then it is so for the third. In particular, by (3.5.2), if $M$ and $M'$ in $\mathcal{D}_c(\mathcal{A})$ are AR-isomorphic (that is, their images in $D_c(X, \Lambda)$ are isomorphic), then the coadjunction map is an isomorphism for $M$ if and only if it is so for $M'$.

By (3.5.3), we may pass to an algebraic presentation $P : X(\mathbb{C}) \rightarrow \mathfrak{X}$, so assume that $\mathfrak{X} = X(\mathbb{C})$. The normalization functor has finite cohomological dimension, so one can assume that $M$ is a $\lambda$-module. By ([12], V, 3.2.3) we may assume that $M$ is an adic system with constructible components. Therefore, $M$ is algebraic by ([13], 6.1.2 (A')), and by ([8], Rapport, 2.8) one reduces to two cases: $M$ is flat (i.e. each component $M_n$ is a flat $\Lambda_n$-sheaf), or $\Lambda M = 0$, i.e. $M$ is AR-isomorphic to (hence we may assume that it is) the constant system $(M_0)_n$.

If $M$ is flat, then the natural map

$$M_n \otimes_{\Lambda_n} \Lambda_{n-1} \simeq M_n \otimes_{\Lambda_n} \Lambda_{n-1} \xrightarrow{\sim} M_{n-1}$$

is an isomorphism, so by (3.5.3) iii) $M$ is normalized, hence the cone of $\hat{M} \rightarrow M$ is zero.

If $M$ is the constant adic system $(M_0)_n$, then $R\pi_*M = M_0$ by ([15], 2.2.3). We saw in the proof of (3.5.6) that $\mathcal{H}^0(L\pi^*M_0)$ is $(M_0)_n$ and that $\mathcal{H}^{-1}(L\pi^*M_0)$ is AR-null. Therefore, the natural map $L\pi^*M_0 \rightarrow M$ is an AR-isomorphism.

Then we prove the following, slightly general than needed.

**Lemma 3.5.11.** Let $X$ be an analytic stack (not necessarily algebraic), and let $F \in \mathcal{D}_c(X, \Lambda)$ be a complex with analytically constructible cohomology sheaves. Then the adjunction map $F \rightarrow R\pi_*L\pi^*F$ is an isomorphism.

**Proof.** Let us denote $R\pi_*L\pi^*F$ by $\hat{F}$. Note that if $F' \rightarrow F \rightarrow F'' \rightarrow F'[1]$ is an exact triangle, and the adjunction map is an isomorphism for two vertices, then it is so for the third.
That the map $F \to \tilde{F}$ is an isomorphism is a local property, since it is equivalent to the vanishing of all the cohomology sheaves of the cone, which can be checked locally. So by (3.5.3 ii), we may replace $X$ by an analytic presentation $X$. Since the functor $F \mapsto \tilde{F}$ has finite cohomological dimension, we may assume that $F$ is a sheaf. By (3.5.5 ii) we reduce to two cases: $F$ is flat, or $F$ is annihilated by $\lambda$. The second case follows from (10), 2.2.3, so we assume that $F$ is flat.

We want to reduce to the case when $F$ is locally constant. Let $j : U \hookrightarrow X$ be the open immersion of a subspace over which $F$ is locally constant, and let $i : Z \hookrightarrow X$ be the complement. Consider the exact triangle

$$i_*N \xrightarrow{1} F \xrightarrow{Rj_*F_U} ,$$

where $N = Ri^!F$ is in $\mathcal{D}^b_c(Z, \Lambda)$ by (10), 4.1.5 i). It suffices to show that the adjunction maps for $i_*N$ and $Rj_*F_U$ are isomorphisms.

By (3.5.1) we have $R\pi_X \circ i_*N \simeq i_* \circ R\pi_{Z,*}$. Also we have $L\pi^*_X \circ i_* \simeq i_*^N \circ L\pi^*_Z$, since $i_*$ is extension by zero (that $i_*^n(N \otimes_{\Lambda} \Lambda_n) \simeq i_n^N \otimes_{\Lambda} \Lambda_n$ also follows from the projection formula in topology (10), 2.3.29)). Therefore, the adjunction map for $i_*N$ on $X$ is obtained by applying $i_*$ to the adjunction map for $N$ on $Z$:

$$i_*N \xrightarrow{} R\pi_{X,*}L\pi_X^*i_*N \simeq i_*R\pi_{Z,*}L\pi_Z^*N,$$

which is an isomorphism by noetherian hypothesis.

Again by (3.5.1) we have $R\pi_{X,*} \circ i_*^N \simeq i_* \circ R\pi_{U,*}$. We will show that $Rj_*^N \circ L\pi_U^* \simeq L\pi_X^* \circ Rj_*$ on $\mathcal{D}_c(U, \Lambda)$, even though we only need this isomorphism for bounded complexes. Let $F \in \mathcal{D}_c(U, \Lambda)$. For each $n$ we have a natural morphism $\Lambda_n \otimes^L \Lambda j_*^n \rightarrow \Lambda_n \otimes^L \Lambda F$.

Consider the short exact sequence

$$0 \rightarrow \Lambda \xrightarrow{\lambda^{n+1}} \Lambda \xrightarrow{\lambda} \Lambda_n \rightarrow 0.$$

Let $F \rightarrow I$ be a $K$-injective resolution of $F$ (cf. 22). Then $\Lambda_n \otimes^L \Lambda I$ is also a $K$-injective complex (one sees this by applying (22), 1.3) to the exact triangle

$$I \xrightarrow{\lambda^{n+1}} \Lambda_n \otimes^L I \xrightarrow{} I$$

obtained from the short exact sequence above tensored with $I$), and $j_*(\Lambda_n \otimes^L I) = \Lambda_n \otimes^L j_* I$, since by applying $Rj_*$ to the exact triangle of $K$-injective complexes

$$I \xrightarrow{\lambda^{n+1}} \Lambda_n \otimes^L I \xrightarrow{} I$$

we get

$$j_*I \xrightarrow{\lambda^{n+1}} j_* I \xrightarrow{} j_*(\Lambda_n \otimes^L I) \xrightarrow{} ,$$

and by applying $- \otimes^L j_* I$ to the short exact sequence above we get

$$j_*I \xrightarrow{\lambda^{n+1}} j_* I \xrightarrow{} \Lambda_n \otimes^L j_* I \xrightarrow{} .$$

Thus by (22), 5.12, 6.7) we have

$$Rj_*(\Lambda_n \otimes^L F) = j_*(\Lambda_n \otimes^L I) = \Lambda_n \otimes^L Rj_* F.$$

Therefore, the adjunction map for $Rj_*F_U$ on $X$ is obtained by applying $Rj_*$ to the adjunction map for $F_U$ on $U$. Hence we may assume that $F$ is a locally constant sheaf on $X$. Replacing $X$ by an open cover, we assume that $F$ is constant, defined by a free module $\Lambda^r$. By additivity we may assume that $r = 1$. Then $L\pi^* \Lambda = \Lambda$, and $\pi_* \Lambda = \lim \Lambda = \Lambda$.

We conclude by applying (3.5.3) to deduce that $R^i \pi_* \Lambda = 0$ for $i \neq 0$. □
Therefore, \((Q \circ L\pi^*, \overline{R\pi_*})\) induce an equivalence between \(D_c(\mathcal{X}, \Lambda)\) and \(\mathbb{D}_c(\mathcal{X}, \Lambda)\).

(ii) If \(X_0 \to X\) is a strictly simplicial algebraic smooth hypercover, we have \(D_c(\mathcal{X}, \Lambda) \simeq D_c(X_0, \Lambda)\) and \(\mathbb{D}_c(\mathcal{X}, \Lambda) \simeq \mathbb{D}_c(X_0, \Lambda)\) by (3.4.1 i). So we may assume that \(\mathcal{X} = X\) and \(\mathcal{Y} = Y\) are analytifications of algebraic schemes. By definition of \(\overline{R\pi_*}\), it suffices to show that the following diagram commutes

\[
\begin{array}{ccc}
\mathbb{D}_c^+(\mathcal{A}(X)) & \xrightarrow{R\pi^*} & \mathbb{D}_c^+(X, \Lambda) \\
\downarrow f^N \circ \pi^* & & \downarrow f^N \\
\mathbb{D}_c^+(\mathcal{A}(Y)) & \xrightarrow{R\pi^*} & \mathbb{D}_c^+(Y, \Lambda),
\end{array}
\]

and this follows from the commutativity of the diagram of topoi

\[
\begin{array}{ccc}
X^N_{\text{an}} & \xrightarrow{\pi^*} & X_{\text{an}} \\
\downarrow f^N \circ \pi^* & & \downarrow f^N \\
Y^N_{\text{an}} & \xrightarrow{\pi^*} & Y_{\text{an}},
\end{array}
\]

Note that the corresponding diagram for \(f^N : X \to Y\) does not even make sense, since if \(f\) is not smooth, it does not necessarily induce a morphism of their lisse-analytic topoi. \(\square\)

**Remark 3.5.12.** Similarly, \(\overline{R\pi_*}\) induces a fully faithful functor \(D_c(\mathcal{X}, \overline{\mathbb{Q}_\ell}) \to \mathbb{D}_c(\mathcal{X}, \overline{\mathbb{Q}_\ell})\), which is compatible with \(f^N\) when restricted to \(D_c^+\).

**Remark 3.5.13.** This result (3.5.9), together with (3.4.3) and (15, 3.1.6), generalizes (3, 6.1.2, (B’)). Taking their cores, we obtain

\[\Lambda\text{-Sh}_c(\mathcal{X}) \simeq \Lambda\text{-Sh}_c(\mathcal{X}) \simeq \mathbb{M}_{\text{disc}}(\Lambda\chi),\]

generalizing (loc. cit., (A’)).

4 Decomposition Theorem over \(\mathbb{C}\)

Let \((\Lambda, m)\) be a complete DVR as before, with residue characteristic \(\ell \neq 2\). Let \(\mathcal{X}\) be an algebraic stack over \(\text{Spec} \mathbb{C}\). We first prove a comparison theorem between the lisse-étale topos over \(\mathbb{C}\) and over \(\mathbb{F}\), and then use this together with (3.4.3, 3.5.9) to deduce the decomposition theorem for \(\mathbb{C}\)-algebraic stacks with affine stabilizers.

4.1 Comparison between the lisse-étale topos over \(\mathbb{C}\) and over \(\mathbb{F}\)

Let \((\mathcal{F}, \mathcal{L})\) be a pair on \(\mathcal{X}\) with \(\Lambda_0\)-coefficients. By refining we may assume that all strata in \(\mathcal{F}\) are essentially smooth and connected. Let \(A \subset \mathbb{C}\) be a subring of finite type over \(\mathbb{Z}\), large enough so that there exists a triple \((\mathcal{X}_S, \mathcal{F}_S, \mathcal{L}_S)\) over \(S := \text{Spec} A\) giving rise to \((\mathcal{X}, \mathcal{F}, \mathcal{L})\) by base change, that \(\mathcal{X}_S\) is flat over \(S\), and that \(1/\ell \in A\). Then \(S\) satisfies the condition (LO): the hypothesis on \(\ell\)-cohomological dimension follows from (Π, X, 6.2). We may shrink \(S\) to assume that strata in \(\mathcal{F}_S\) are smooth over \(S\) with geometrically connected fibers, which is possible because one can take a presentation \(P : \mathcal{X}_S \to \mathcal{X}_S\) and shrink \(S\) so that the strata in \(P^* \mathcal{F}_S\) are smooth over \(S\) with geometrically connected fibers. Let \(a : \mathcal{X}_S \to S\) be the structural map.

Let \(A \subset V \subset \mathbb{C}\), where \(V\) is a strict henselian discrete valuation ring whose residue field \(s\) is an algebraic closure of a finite residue field of \(A\). Let \((\mathcal{X}_V, \mathcal{F}_V, \mathcal{L}_V)\) be the triple over \(V\) obtained by base change, and let \((\mathcal{X}_s, \mathcal{F}_s, \mathcal{L}_s)\) be its special fiber. Then we have morphisms

\[\mathcal{X} \xrightarrow{u} \mathcal{X}_V \xleftarrow{i} \mathcal{X}_s.\]
Proposition 4.1.1. (stack version of ([3], 6.1.9)) For $S$ small enough, the functors

$$D_{\mathcal{X}, L}^b(\mathcal{X}', \Lambda_n) \xrightarrow{u_n^*} D_{\mathcal{X}', L'}(\mathcal{X}', \Lambda_n) \xrightarrow{i_n^*} D_{\mathcal{X}, L}(\mathcal{X}, \Lambda)$$

and

$$D_{\mathcal{X}, L}^b(\mathcal{X}', \Lambda) \xrightarrow{u^*} D_{\mathcal{X}', L'}(\mathcal{X}', \Lambda) \xrightarrow{i^*} D_{\mathcal{X}, L}(\mathcal{X}, \Lambda)$$

are equivalences of triangulated categories with standard $t$-structures.

Proof. These restriction functors are clearly triangulated functors preserving the standard $t$-structures.

By (2.5), we can shrink $S = \text{Spec } A$ so that for any $F$ and $G$ of the form $j_!L$, where $j : \mathcal{U}_S \to \mathcal{X}_S$ in $\mathcal{S}$ and $L \in L_S(\mathcal{U}_S)$, the formations of $R\mathcal{H}om_{\mathcal{X}_S}(F, G)$ commute with base change on $S$, and that the complexes $a_*\mathcal{E}xt^q_{\mathcal{X}_S}(F, G)$ on $S$ are lcc (see Remark 4.1.3 below for explanation) and of formation compatible with base change, i.e. the cohomology sheaves are lcc, and for any $g : \mathcal{S}' \to S$, the base change morphism for $a_*$:

$$g^*a_*\mathcal{E}xt^q_{\mathcal{X}_S}(F, G) \to a_{\mathcal{S}'*}g^*a_*\mathcal{E}xt^q_{\mathcal{X}_S}(F, G)$$

is an isomorphism. Then using the same argument as in [3], the claim for $u_n^*$ and $i_n^*$ follows. For the reader’s convenience, we explain the proof in [3] in more detail.

Note that the spectra of $V$, $\mathbb{C}$ and $s$ have no non-trivial étale surjections mapping to them, so their small étale topoi are equivalent to the topos of sets. In particular, $Ra_{\mathcal{S}*}$ (resp. $Ra_{\mathcal{S}'*}$ and $Ro_{\mathcal{S}*}$) is just $Rj$. Let us show the full faithfulness of $u_n^*$ and $i_n^*$ first. For $K, L \in D_{\mathcal{X}, L}^b(\mathcal{X}_V, \Lambda_n)$, let $K_{\mathbb{C}}$ and $L_{\mathbb{C}}$ (resp. $K_s$ and $L_s$) be their images under $u_n^*$ (resp. $i_n^*$). Then the full faithfulness follows from the more general claim that, the maps

$$\mathcal{E}xt^i_{\mathcal{X}}(K_{\mathbb{C}}, L_{\mathbb{C}}) \xrightarrow{u_n^*} \mathcal{E}xt^i_{\mathcal{X}_V}(K, L) \xrightarrow{i_n^*} \mathcal{E}xt^i_{\mathcal{X}_s}(K_s, L_s)$$

are bijective for all $i$.

Since $Hom_{D_\mathcal{S}, (\mathcal{X}, \Lambda_n)}(K, -)$ and $Hom_{D_\mathcal{S}, (\mathcal{X}, \Lambda_n)}(-, L)$ are cohomological functors, by 5-lemma we may assume that $K = F$ and $L = G$ are $\Lambda_n$-sheaves. Let $j : \mathcal{U}_S \to \mathcal{X}_S$ be the immersion of an open stratum in $\mathcal{S}$, with complement $i : \mathcal{Z}_S \to \mathcal{X}_S$. Using the short exact sequence

$$0 \to j_{\mathcal{V}!}j_V^*F \to F \to i_{\mathcal{V}*}i_V^*F \to 0$$

and noetherian induction on the support of $F$ and $G$, we may assume that they take the form $j_{\mathcal{V}!}L$, where $j$ is the immersion of some stratum in $\mathcal{S}$, and $L$ is a sheaf in $\mathcal{L}_V$. The spectral sequence

$$R^p a_{\mathbb{C}*}\mathcal{E}xt^q_{\mathcal{X}_{\mathbb{C}}}(F_{\mathbb{C}}, G_{\mathbb{C}}) \Rightarrow Ext^{p+q}_{\mathcal{X}_{\mathbb{C}}}(F_{\mathbb{C}}, G_{\mathbb{C}})$$

is natural in the base $\mathbb{C}$, which can be $V$, $\mathbb{C}$ or $s$. The assumption on $S$ made before implies that the composite base change morphism

$$g^*a_*\mathcal{E}xt^q_{\mathcal{X}_S}(F, G) \to a_{\mathcal{S}'*}g^*a_*\mathcal{E}xt^q_{\mathcal{X}_S}(F, G) \to a_{\mathcal{S}'*}\mathcal{E}xt^q_{\mathcal{X}_{\mathcal{S}'}}(g^*F, g^*G)$$

is an isomorphism, for all $g : \mathcal{S}' \to S$. Therefore, the maps

$$\mathcal{E}xt^i_{\mathcal{X}}(F_{\mathbb{C}}, G_{\mathbb{C}}) \xrightarrow{u_n^*} \mathcal{E}xt^i_{\mathcal{X}_V}(F, G) \xrightarrow{i_n^*} \mathcal{E}xt^i_{\mathcal{X}_s}(F_s, G_s)$$

are bijective for all $i$. The claim (hence the full faithfulness of $u_n^*$ and $i_n^*$) follows.

This claim also implies their essential surjectivity. To see this, let us give a lemma first.

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Lemma 4.1.2. Let $F : \mathcal{C} \to \mathcal{D}$ be a triangulated functor between triangulated categories. Let $A, B \in \text{Obj } \mathcal{C}$, and let $F(A) \xrightarrow{u} F(B) \to C' \to F(A)[1]$ be an exact triangle in $\mathcal{D}$. If the map

$$F : \text{Hom}_\mathcal{C}(A, B) \to \text{Hom}_\mathcal{D}(F(A), F(B))$$

is surjective, then $C'$ is in the essential image of $F$.

Proof. Let $u : A \to B$ be a morphism such that $F(u) = v$. Let $C$ be the mapping cone of $u$, i.e. let the triangle $A \xrightarrow{u} B \to C \to A[1]$ be exact. Then its image

$$F(A) \xrightarrow{v} F(B) \xrightarrow{v} F(C) \xrightarrow{v} F(A)[1]$$

is also an exact triangle. This implies that $C' \simeq F(C)$. \hfill $\square$

Now we can show the essential surjectivity of $u^*_n$ and $i^*_n$. For $K \in D^b_{\mathcal{X}, \mathcal{L}}(\mathcal{X}, \Lambda_n)$, to show that $K$ lies in the essential image of $u^*_n$, using the truncation exact triangles and (4.1.2), we reduce to the case where $K$ is a sheaf. Using noetherian induction on the support of $K$, we reduce to the case where $K = j_! L$, where $j : \mathcal{U} \to \mathcal{X}$ is the immersion of a stratum in $\mathcal{X}$, and $L \in \mathcal{L}(\mathcal{U})$. This is in the essential image of $u^*_n$. Similarly, $i^*_n$ is also essentially surjective.

Next, we prove that $u^*$ and $i^*$ are equivalences.

We claim that for $K, L \in D^b_{\mathcal{C}}(\mathcal{X}_V, \Lambda)$, if the morphisms

$$\text{Hom}_{D_{\mathcal{C}}(\mathcal{X}_V, \Lambda)}(\widehat{K}_n, \widehat{L}_n) \xrightarrow{u^*_n} \text{Hom}_{D_{\mathcal{C}}(\mathcal{X}_V, \Lambda)}(\widehat{K}_n, \widehat{L}_n, \zeta)$$

are bijective for all $n$, then the morphisms

$$\text{Hom}_{D_{\mathcal{C}}(\mathcal{X}_V, \Lambda)}(K, L) \xrightarrow{u^*} \text{Hom}_{D_{\mathcal{C}}(\mathcal{X}_V, \Lambda)}(K, L, \zeta)$$

are bijective. Let $\Box$ be one of the bases $V, \mathcal{C}$ or $s$. Since $K$ and $L$ are bounded, we see from the spectral sequence

$$R^0a_{\Box, s} \text{Ext}^q_{\mathcal{A}_\mathcal{C}}(\widehat{K}_n, \widehat{L}_n, \zeta) \Rightarrow Ext_{\mathcal{A}_\mathcal{C}}^{p+q}(\widehat{K}_n, \widehat{L}_n, \zeta)$$

and the finiteness of $\text{RHom}$ and $Ra_{\Box, s}$ (14, 4.2.2, 4.1) that, the groups $Ext^{-1}(\widehat{K}_n, \widehat{L}_n, \zeta)$ are finite for all $n$, hence they form a projective system satisfying the condition (ML) (cf. EGA 0III, 13.1.2). By (15, 3.1.3), we have an isomorphism

$$\text{Hom}_{D_{\mathcal{C}}(\mathcal{X}_V, \Lambda)}(K, L) \xrightarrow{u^*} \lim_{\rightarrow n} \text{Hom}_{D_{\mathcal{C}}(\mathcal{X}_V, \Lambda)}(\widehat{K}_n, \widehat{L}_n, \zeta),$$

natural in the base $\Box$, and the claim follows.

Since when restricted to $D^b_{\mathcal{X}_V, \mathcal{L}}$, the functors $u^*_n$ and $i^*_n$ are fully faithful for all $n$, we deduce that $u^*$ and $i^*$ are also fully faithful.

Finally we prove the essential surjectivity of $u^*$ and $i^*$. Let $K \in D^b_{\mathcal{X}, \mathcal{L}}(\mathcal{X}, \Lambda)$. By the full faithfulness of $u^*$ and (4.1.2), we may assume that $K$ is in the core $(D^b_{\mathcal{X}, \mathcal{L}})^{\vee}$ of $D^b_{\mathcal{X}, \mathcal{L}}$.
with respect to the standard $t$-structure. Then there exists an AR-adic representative $M = \{M_n, \rho_n : M_n \to M_{n-1}\}$ of $K$ in $\mathcal{A}(\mathcal{X})$ that is trivialized by $(\mathcal{S}, \mathcal{L})$; for instance $M = \mathcal{H}(\mathcal{K})$ by (23, 3.5). By (12, V, 3.2.3), since $M$ is AR-adic, it satisfies the condition (MLAR) (see (12, V, 2.1.1) for definition) and, if we denote by $N = (N_n)_n$ the projective system of the universal images of $M$, there exists an integer $k \geq 0$ such that $l_k(N) := (N_{n+k} \otimes \Lambda_n)_n$ is an adic system. By construction, the system $l_k(N)$ is trivialized by $(\mathcal{S}, \mathcal{L})$, and is AR-isomorphic to $M$, so we may assume that $M$ is adic.

The functor $u^*_n$ induces an equivalence on the cores with respect to the standard $t$-structures:

$$(u^*_n)^\circ : D^{b}_{\mathcal{S}, \mathcal{L}}(\mathcal{X}_n, \Lambda_n) \to D^{b}_{\mathcal{S}, \mathcal{L}}(\mathcal{X}, \Lambda_n)^\circ.$$ 

Let $M_V = \{M_{n,V}, \rho_{n,V} : M_{n,V} \to M_{n-1,V}\}$ be the unique (up to isomorphism) extension of $M$ to $\mathcal{X}_V$, where $M_{n,V}$ (resp. $\rho_{n,V}$) is an object (resp. a morphism) in $D^{b}_{\mathcal{S}, \mathcal{L}}(\mathcal{X}_V, \Lambda_n)^\circ$. The induced morphism $\overline{\rho}_{n,V} : M_{n,V} \otimes \Lambda_{n-1} \to M_{n-1,V}$ is an isomorphism because it is sent to the isomorphism $\overline{\rho}_n : M_n \otimes \Lambda_{n-1} \to M_{n-1}$ via the equivalence $(u_{n-1}^*)_\circ$. This shows that $M_V$ is an adic system of sheaves on $\mathcal{X}_V$, each level being trivialized by $(\mathcal{S}, \mathcal{L})$, and it gives an object in $D^{b}_{\mathcal{S}, \mathcal{L}}(\mathcal{X}_V)$ whose image in $D^{b}_{\mathcal{S}, \mathcal{L}}(\mathcal{X}, \Lambda)$ is sent to $K$ under $u^*$:

$$\begin{array}{ccc}
\mathcal{D}^{b}_{\mathcal{S}, \mathcal{L}}(\mathcal{X}_V) & \xrightarrow{Q_V} & D^{b}_{\mathcal{S}, \mathcal{L}}(\mathcal{X}_V, \Lambda) \\
\downarrow u^* & & \downarrow u^* \\
\mathcal{D}^{b}_{\mathcal{S}, \mathcal{L}}(\mathcal{X}) & \xrightarrow{Q} & D^{b}_{\mathcal{S}, \mathcal{L}}(\mathcal{X}, \Lambda),
\end{array}$$

the functors $Q$ and $Q_V$ in the diagram being the localization functors. This shows that $u^*$ (and similarly, $i^*$) is essentially surjective.

**Remark 4.1.3.** Note that, in contrast to the case of schemes, $a \in \text{ext}^q_{\mathcal{X}_S}(F, G)$ in the proof is in general an unbounded complex, so we need to explain why one can shrink $S$ such that all cohomology sheaves are lcc (for all $q$, too). In the proof of (23, Th. 3.9), that stratifiable complexes are stable under the six operations, we actually proved more, namely, given a pair $(\mathcal{S}, \mathcal{L})$ on $\mathcal{X}$ and a morphism $f : \mathcal{X} \to \mathcal{Y}$ of $S$-algebraic stacks, there exists a pair $(\mathcal{S}', \mathcal{L}')$ on $\mathcal{Y}$ such that $f_*$ takes $D^{b}_{\mathcal{S}, \mathcal{L}}(\mathcal{X}, \Lambda)$ into $D^{b}_{\mathcal{S}', \mathcal{L}'}(\mathcal{Y}, \Lambda)$; similar results hold for the other operations, as well as for $\Lambda_n$-coefficients. So in our case, since the $\Lambda_0$-sheaves of the form $j_!L$ are finite in number, we see that the sheaves $R^pa_*\text{ext}^q_{\mathcal{X}_S}(F, G)$ (for all $p, q \in \mathbb{Z}$ and $F, G$ of the form $j_!L$) are trivialized by a pair $(\mathcal{S}', \mathcal{L}')$ on $S$, and consequently we may replace $S$ by an affine open subscheme in an open stratum in $\mathcal{S}'$.

**4.1.4.** We will show that if the $\mathbb{C}$-algebraic stack $\mathcal{X}$ has affine stabilizers (1.1), then $\mathcal{X}_s$ obtained as above also has affine stabilizers.

It is not difficult to see that the formation of the inertia stack is compatible with base change in the following sense. For a 2-Cartesian diagram

$$\begin{array}{ccc}
\mathcal{X}' & \xrightarrow{g'} & \mathcal{X} \\
\downarrow f' & & \downarrow f \\
\mathcal{Y}' & \xrightarrow{g} & \mathcal{Y}
\end{array}$$

of algebraic stacks over any base $S$ (not necessarily locally of finite type), let $I_{f}$ and $I_{f'}$ be the relative inertia stacks for $f$ and $f'$ respectively, then $I_{f'} \simeq I_{f} \times_X Y \simeq I_{f} \times_X \mathcal{X}'$. Also, if $i : \mathcal{V} \to \mathcal{X}$ is an immersion, then the restriction of $I_{X/S}$ to $\mathcal{V}$ is $I_{Y/S}$, i.e. $I_{Y/S} \simeq I_{X/S} \times_X \mathcal{V}$. 28
Let $P : X_V \rightarrow \mathcal{X}_V$ be a presentation, and let the following squares be 2-Cartesian:

$$
\begin{array}{ccc}
I & \rightarrow & I_V \\
\downarrow & & \downarrow \\
X & \rightarrow & X_V
\end{array}
\quad \begin{array}{ccc}
I & \leftarrow & I_s \\
\downarrow & & \downarrow \\
X & \leftarrow & X_s
\end{array}
$$

Since any $\mathbb{C}$- or $s$-point of $\mathcal{X}_V$ can be lifted to $X_V$, we may replace $\mathcal{X}_V$ by $X_V$. Since $X_V$ is flat over $\text{Spec} \, V$, and the generic point $\eta \in \text{Spec} \, V$ is an open subset, by (EGA IV, 2.3.10) we see that the generic fiber $X_\eta$ is dense in $X_V$. This is also true with $X_V$ replaced by any stratum in $P^* \mathcal{S}$ (by our assumption that any stratum in $\mathcal{S}$ is $S$-smooth, a fortiori, $S$-flat).

We may assume that $I_V$ is flat over $X_V$, by stratifying $X_V$. Replacing $X_V$ by its maximal reduced subschemes if necessary, we may assume that $X_V$ is integral. Let $x$ be its generic point, which is a field of characteristic 0. Therefore $I_{V,x}$ is smooth over $x$, hence $I_V$ is smooth over a dense open subset of $X_V$. By noetherian induction, we may assume that $I_V$ is smooth over $X_V$. Then we apply the lower semi-continuity of abelian ranks for smooth group schemes ([16], X, 8.7), to deduce that all fibers of $I_V \rightarrow X_V$ are affine (note that all fibers of $I_\eta \rightarrow X_\eta$ are affine).

4.1.5. Also, if $f : \mathcal{X} \rightarrow \mathcal{Y}$ is a proper morphism of finite diagonal between $\mathbb{C}$-algebraic stacks, then one can choose $S$ and $V$ such that $f$ extends to a proper morphism of finite diagonal $f_V : \mathcal{X}_V \rightarrow \mathcal{Y}_V$ between $V$-algebraic stacks. Clearly one has a proper extension $f_V$. The base change of the diagonal morphism

$$
\Delta_{f_V} : \mathcal{X}_V \rightarrow \mathcal{X}_V \times_{\mathcal{Y}_V} \mathcal{X}_V
$$

to the geometric generic point $\text{Spec} \, \mathbb{C} \rightarrow \text{Spec} \, V$ is the diagonal morphism of $f$

$$
\Delta_f : \mathcal{X} \rightarrow \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}.
$$

As $f_V$ is separated, $\Delta_{f_V}$ is representable and proper, so it suffices to show that $\Delta_{f_V}$ is quasi-finite (EGA IV, 8.11.1), which is equivalent to $\mathcal{I}_{f_V} \rightarrow \mathcal{X}_V$ being quasi-finite. As in [4.1.4], we may replace $\mathcal{X}_V$ by a presentation $X_V$ (and replace $\mathcal{I}_{f_V}$ by $I_{f_V} := I_{f_V} \times_{X_V} X_V$ as well), and stratify $X_V$ to assume that $I_{f_V}$ is a flat group scheme over $X_V$ (assumed integral). Now $I_{f_\eta}$ is finite over $X_\eta$ (as $\eta \rightarrow \mathbb{C}$ is faithfully flat), and $X_\eta$ is dense in $X_V$ as before, by ([2], VIb, Cor. 4.3) we see that $I_{f_V}$ is quasi-finite over $X_V$.

In particular, the special fiber $f_s : \mathcal{X}_s \rightarrow \mathcal{Y}_s$ is also proper and of finite diagonal.

4.2 The proof

Let $\mathcal{X}$ be a $\mathbb{C}$-algebraic stack, with analytification $\mathfrak{X}$. Let $\Omega$ be a field of characteristic 0; the examples that we have in mind are $\Omega = \mathbb{Q}$, $\mathbb{C}$, $E_\lambda$ or $\overline{\mathbb{Q}}_\ell$.

4.2.1. Following the idea of [16], one can define $\Omega$-perverse sheaves on $\mathfrak{X}_{\text{lis-an}}$ as follows. Let $P : X \rightarrow \mathcal{X}$ be a presentation of relative dimension $d$, and let $P_{\text{an}} : X(\mathbb{C}) \rightarrow \mathfrak{X}$ be its analytification. Let $p = p_{1/2}$ be the middle perversity on $X(\mathbb{C})$. Define $P \mathcal{D}_c^{\leq 0}(\mathfrak{X}, \Omega)$ (resp. $P \mathcal{D}_c^{= 0}(\mathfrak{X}, \Omega)$) to be the full subcategory of objects $K \in \mathcal{D}_c(\mathfrak{X}, \Omega)$ such that $P_{\text{an}}^* K[d]$ is in $P \mathcal{D}_c^{\leq 0}(X(\mathbb{C}), \Omega)$ (resp. $P \mathcal{D}_c^{= 0}(X(\mathbb{C}), \Omega)$). As in [16], 4.1, 4.2, one can show that these subcategories do not depend on the choice of the presentation $P$, and they define a t-structure, called the (middle) perverse t-structure on $\mathfrak{X}$. 

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4.2.2. Following ([3], 6.2.4), one can define complexes of sheaves of geometric origin as follows. Let \( \mathcal{F} \) be a \( \Omega \)-perverse sheaf on \( X_{\text{lis-an}} \) (resp. a \( \Omega_{\ell} \)-perverse sheaf on \( X_{\text{lis-\acute{e}t}} \)). We say that \( \mathcal{F} \) is semi-simple of geometric origin if it is a semi-simple perverse sheaf, and every irreducible constituent belongs to the smallest family of simple perverse sheaves on complex analytic stacks (resp. lisse-\acute{e}tale sites of \( \mathbb{C} \)-algebraic stacks) that

(a) contains the constant sheaf \( \Omega \) over a point, and is stable under the following operations:

(b) taking the constituents of \( \mathcal{F} \), for \( T = f_\ast, f_!, f^\ast, f^!, R\mathcal{H}om(-, -) \) and \( - \otimes - \),

where \( f \) is an arbitrary algebraic morphism between stacks.

A complex \( K \in D_c^b(\mathcal{X}, \Omega) \) (resp. \( K \in D_c^b(\mathcal{X}, \Omega_{\ell}) \)) is said to be semi-simple of geometric origin if it is isomorphic to the direct sum of the \( (p\mathcal{H}^i K)[−i] \)'s, and each \( p\mathcal{H}^i K \) is semi-simple of geometric origin. Notice that this property is not local for the smooth topology, as the example in ([24], Section 1) shows.

One can replace the constant sheaf \( E_\lambda \) by its ring of integers \( \mathcal{O}_\lambda \), and deduce that every complex \( K \in D_c^b(\mathcal{X}, \Omega) \) that is semi-simple of geometric origin has an integral structure, hence belongs to the essential image of \( D^b_c(\mathcal{X}, \Omega_{\ell}) \to D^b_c(\mathcal{X}, \Omega_{\ell}) \). Therefore, we can apply (3.5.12).

**Lemma 4.2.3.** (stack version of ([3], 6.2.6)) Let \( \mathcal{F} \) be a simple \( \Omega_{\ell} \)-perverse sheaf of geometric origin on \( \mathcal{X} \). For \( A \subset \mathbb{C} \) large enough, the equivalence (4.1.1)

\[
D^b_{\mathcal{X}, \mathcal{L}}(\mathcal{X}, \Omega_{\ell}) \leftrightarrow D^b_{\mathcal{Y}, \mathcal{L}}(X, \Omega_{\ell})
\]

takes \( \mathcal{F} \) to a simple perverse sheaf \( \mathcal{F}_s \) on \( X_s \), such that \( (X_s, \mathcal{F}_s) \) is deduced by base extension from a pair \( (X_0, \mathcal{F}_0) \) defined over a finite field \( \mathbb{F}_q \), and \( \mathcal{F}_0 \) is \( \iota \)-mixed by Lafforgue’s result. Then apply ([24], 3.4).  

Proof. Being of geometric origin, \( \mathcal{F}_s \) is obtained by base extension from some simple perverse sheaf \( \mathcal{F}_0 \) on \( X_0 \), which is \( \iota \)-mixed by Lafforgue’s result. Then apply ([24], 3.4).  

Finally, we are ready to prove the stack version of the decomposition theorem over \( \mathbb{C} \).

**Theorem 4.2.4.** (stack version of ([3], 6.2.5)) Let \( f : \mathcal{X} \to \mathcal{Y} \) be a proper morphism of finite diagonal between \( \mathbb{C} \)-algebraic stacks with affine stabilizers. If \( K \in D^b_c(\mathcal{X}, \Omega) \) is semi-simple of geometric origin, then \( f_\ast K \) is also bounded, and is semi-simple of geometric origin on \( \mathcal{Y} \).

**Proof.** We can replace \( D^b_c(\mathcal{X}, \Omega) \) by \( D^b_c(\mathcal{X}, \Omega_{\ell}) \), then by \( D^b_c(\mathcal{X}, \Omega_{\ell}) \) (using (3.5.9 ii, 3.5.12)), and finally by \( D^b_c(\mathcal{X}, \Omega_{\ell}) \) (using (3.4.3)).

From ([19], 5.17) we know that there is a canonical isomorphism \( f_! \simeq f_\ast \) on \( D^c_\mathcal{X}(\mathcal{X}, \Omega_{\ell}) \). For \( K \in D^b_c \), we have \( f_! K \in D^c_\mathcal{X} \) and \( f_\ast K \in D^b_\mathcal{X} \), hence \( f_\! K \in D^b_c \).

**Lemma 4.2.5.** We can reduce to the case where \( K \) is a simple perverse sheaf \( \mathcal{F} \).

**Proof.** First, we show that the statement for simple perverse sheaves of geometric origin implies the statement for semi-simple perverse sheaves of geometric origin. This is clear:

\[
f_\ast(\bigoplus_i \mathcal{F}_i) = \bigoplus_i f_\ast \mathcal{F}_i = \bigoplus_i \bigoplus_j p\mathcal{H}^j(f_\ast \mathcal{F}_i)[−j] = \bigoplus_j p\mathcal{H}^j(f_\ast(\bigoplus_i \mathcal{F}_i))[−j].
\]

Then we show that the statement for semi-simple perverse sheaves implies the general statement. If \( K \) is semi-simple of geometric origin, we have

\[
f_\ast K = \bigoplus_i f_\ast p\mathcal{H}^j(K)[−i] = \bigoplus_i \bigoplus_j p\mathcal{H}^j f_\ast p\mathcal{H}^j(K)[−i − j].
\]
Taking $^p\mathcal{H}^n$ on both sides, we get
\[ ^p\mathcal{H}^n(f_*K) = \bigoplus_{i+j=n} ^p\mathcal{H}^{i}f_* ^p\mathcal{H}^{j}(K), \]
therefore $f_*K = \bigoplus_n ^p\mathcal{H}^n(f_*K)[-n]$ and each summand is semi-simple of geometric origin.

Now assume that $K$ is a simple perverse sheaf $\mathcal{F}$, which is stratifiable ([23], 3.4 v). By (1.2.3), $\mathcal{F}$ corresponds to a simple perverse sheaf $\mathcal{F}_s$ which is induced from an $\iota$-pure perverse sheaf $\mathcal{F}_0$ by base change. By (2.5), the formation of $f_*$ over $\mathbb{C}$ is the same as the formation of $f_{s,*}$ over $\mathbb{F}$ and of $f_{0,*}$ over a finite field. By (1.1.4 1.1.5) and (24), 3.9 iii), $f_{0,*}\mathcal{F}_0$ is also $\iota$-pure. By (24), 3.11, 3.12), we have
\[ f_{s,*}\mathcal{F}_s \simeq \bigoplus_{i \in \mathbb{Z}} ^p\mathcal{H}^i(f_{s,*}\mathcal{F}_s)[-i], \]
and each $^p\mathcal{H}^i(f_{s,*}\mathcal{F}_s)$ is semi-simple of geometric origin. Therefore $f_*\mathcal{F}$ (and hence $f_{an}^\star\mathcal{F}^{an}$) is semi-simple of geometric origin.

We hope to work out Saito’s theory of mixed Hodge modules for complex analytic stacks in the future, which may lead to an alternative proof of the Decomposition theorem.

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