METHOD OF VARIATIONS OF POTENTIAL OF QUASI-PERIODIC SCHRÖDINGER EQUATION

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ABSTRACT. We study the one-dimensional discrete quasi-periodic Schrödinger equation

\[-\varphi(n + 1) - \varphi(n - 1) + \lambda V(x + n\omega)\varphi(n) = E\varphi(n), \quad n \in \mathbb{Z}\]

We introduce the notion of variations of potential and use it to define “typical” potential. We show that for “typical” $C^3$ potential $V$, if the coupling constant $\lambda$ is large, then for most frequencies $\omega$, the Lyapunov exponent is positive for all energies $E$.

1. Introduction

Given any function $V : \mathbb{T} \to \mathbb{R}$, we have a family of quasi-periodic discrete Schrödinger equations

\[(1.1) \quad -\varphi(n + 1) - \varphi(n - 1) + \lambda V(x + n\omega)\varphi(n) = E\varphi(n), \quad n \in \mathbb{Z}\]

where $(x, \omega) \in \mathbb{T} \times \mathbb{T}$ are parameters.

Equation (1.1) can be rewritten as a first order difference equation:

\[
\begin{pmatrix}
\varphi(n + 1) \\
\varphi(n)
\end{pmatrix} = 
\begin{pmatrix}
\lambda V(x + n\omega) - E & -1 \\
1 & 0
\end{pmatrix} 
\begin{pmatrix}
\varphi(n) \\
\varphi(n - 1)
\end{pmatrix}.
\]

The monodromy matrix of this equation is

\[M_{[a, b]}(x, \omega, E) = \begin{pmatrix}
\lambda V(x + b\omega) - E & -1 \\
1 & 0
\end{pmatrix} \cdots \begin{pmatrix}
\lambda V(x + a\omega) - E & -1 \\
1 & 0
\end{pmatrix}.
\]

To study the properties of the spectrum and eigenfunctions of equation (1.1) for generic $(x, \omega)$, we make use of the Lyapunov exponent. It is defined by

\[L(\omega, E) = \lim_{n \to \infty} \frac{1}{n} \int_{\mathbb{T}} \|M_{[1, n]}(x, \omega, E)\| \, dx\]

By Kingman’s subadditive ergodic theorem, if the shift $x \mapsto x + \omega$ is ergodic, then the limit exists and

\[L(\omega, E) = \lim_{n \to \infty} \frac{1}{n} \|M_{[1, n]}(x, \omega, E)\|\]

for almost all $x \in \mathbb{T}$.

For analytic potential $V$, Bourgain, Goldstein[BG] and Goldstein, Schlag[GS1] showed that the Lyapunov exponent is positive for large $|\lambda|$. Furthermore, if $L(\omega_0, E_0) > 0$, then for most $\omega$ close to $\omega_0$, there exists $\delta > 0$ such that

the spectrum of (1.1) in $(E_0 - \delta, E_0 + \delta)$ is pure point and the corresponding eigenfunctions decay exponentially.

This property is called Anderson localization. (See the recent monograph by Bourgain[Bo] for more details.)

Generalization of the method of [BG] and [GS] for potential in a Gevrey-class was established by Klein[Kl].

In this paper, we use Goldstein and Schlag’s methods introduced in a recent work[GS2] to study the spectrum and eigenfunctions of equation (1.1). We develop these methods for smooth potentials in the perturbative regime.
It is well known that the monodromy matrix is of the form
\[
M_{[a,b]}(x,\omega, E) = \begin{pmatrix}
    f_{[a,b]}(x,\omega, E) & -f_{[a+1,b]}(x,\omega, E) \\
    f_{[a,b-1]}(x,\omega, E) & -f_{[a+1,b-1]}(x,\omega, E)
\end{pmatrix}
\]
where
\[
f_{[p,q]}(x,\omega, E) = \det[H_{[p,q]}(x,\omega) - E]
\]

The spectrum of the Schrödinger equation
\[
-\varphi(n+1) - \varphi(n-1) + \lambda V(x + n\omega)\varphi(n) = E\varphi(n),
\]
with zero boundary conditions \( \varphi(a-1) = 0 \) and \( \varphi(b+1) = 0 \), consists of the eigenvalues of \( H_{[a,b]}(x,\omega) \).

If \( V(x + m\omega) \approx V(x + n\omega) \) for some \( m, n \in [a,b] \), \( m \neq n \), one says that the equation exhibits “resonance”. Difficulties in studying the spectrum of \( H_{[a,b]}(x,\omega) \) arise when resonances occur; two eigenvalues of \( H_{[a,b]}(x,\omega) \) can be very close and the corresponding eigenfunctions need not be “localized”.

We develop methods to deal with resonances by defining some functions \( E^{(s)}(x,\omega) \) which take their values in \( \text{sp} H_{[-N_s,N_s]}(x,\omega) \), for appropriate \( N_{s+1} \approx e^{N_s^s}, s = 1,2,\ldots \). In Part I, for \( C^1 \) potential, we eliminate a small part of \( \mathbb{T} \times \mathbb{T} \) from the domain of \( E^{(s)}(x,\omega) \) at each step. This allows us to exclude the case when two eigenvalues of \( H_{[a,b]}(x,\omega) \), \( b - a \approx N_{s+1} \), become too close. We are then able to define exponentially decaying eigenfunctions. The result is the following theorem.

**Theorem 1.1.** Given \( V \in C^3(\mathbb{T}) \), let \( \mathcal{J} = V(\mathbb{T}) \). There exists \( \lambda_0 = \lambda_0(V) \) such that, for any \( |\lambda| > \lambda_0 \), the following holds:

There is \( \Omega = \Omega(V,\lambda) \in [0,1], \text{mes}([0,1] \setminus \Omega) \lesssim \lambda^{-1/2} \), such that for any \( \omega \in \Omega \) there exists \( E_\omega \subset \lambda \mathcal{J} \), \( \text{mes}\left(\mathcal{J} \setminus \lambda^{-1}E_\omega\right) \lesssim \lambda^{-1/2} \), with \( L(\omega, E) \gtrsim \log \lambda \) for all \( E \in E_\omega \). Also, for any \( E \in E_\omega \), there is \( x \in \mathbb{T} \) and \( \{\varphi(n)\} \), \( |\varphi(n)| \lesssim e^{-c|n|} \) such that
\[
-\varphi(n+1) - \varphi(n-1) + V(x + n\omega)\varphi(n) = E\varphi(n).
\]

In a recent paper, Bjerklöv [Bj] has obtained similar results.

The major drawback of the method in Part I is that part of the spectrum is also eliminated. To include all spectral values, we use variations of potential in Part II. Given a \( C^3 \) potential \( V \), any \( C^3 \) function \( \tilde{V} \) satisfying the conditions
\[
\max_{x \in \mathbb{T}} |V(x) - \tilde{V}(x)| < \delta
\]
\[
\max_{x \in \mathbb{T}} |V'(x) - \tilde{V}'(x)| < \delta
\]
\[
\max_{x \in \mathbb{T}} |V''(x) - \tilde{V}''(x)| < \delta
\]
can be written, near \( x = 0 \), in the form
\[
\tilde{V}(x) = V(x) + \eta + \xi x + \frac{1}{2} \theta x^2 + x^3 R(x)
\]
where \(|\eta|, |\xi|, |\theta| < \delta, R \in C^3(\mathbb{T} \setminus \{0\})\), \(|\partial^\alpha R| \lesssim 1\) for any index \(|\alpha| \leq 2\). More generally, since \(\mathbb{T}\) is compact, we can find large integer \(T\) so that

\[
\tilde{V}(x) = V(x) + \sum_{m=1}^{T} \left[ \eta_m + \xi_m (x - \frac{m}{T}) + \frac{1}{2} \theta_m (x - \frac{m}{T})^2 + (x - \frac{m}{T})^3 R_m (x - \frac{m}{T}) \right]
\]

for all \(x \in \mathbb{T}\), where \(\eta = (\eta_1, \ldots, \eta_T), \xi = (\xi_1, \ldots, \xi_T), \theta = (\theta_1, \ldots, \theta_T) \in \prod_{1}^{T} [-\delta, \delta]\), and \(R_m \in C^3(\mathbb{T} \setminus \{0\})\), \(|\partial^\alpha R_m| \lesssim 1\) for any index \(|\alpha| \leq 2\). This motivates the following definition.

**Definition.** Let \(T\) be a large integer, \(0 < \delta \ll \frac{1}{T}\). Suppose \(R_m(\eta, \xi, \theta; x)\) are \(C^3\) functions, \(m = 1, 2, \ldots, T, (\eta, \xi, \theta) \in \prod_{1}^{T} [-\delta, \delta], x \in \mathbb{T}\), satisfying the following conditions:

\[
\begin{align*}
|\partial^\alpha R_m(\eta, \xi, \theta; x)| &\lesssim \frac{1}{T} & \text{for any index } |\alpha| \leq 3 \\
R_m(0, 0, 0; x) &\equiv 0 \\
R_m(\eta, \xi, \theta; x) &\equiv 0 \quad \text{for } |x| \geq \frac{1}{2T}.
\end{align*}
\]

Define a \((T, \delta)\)-variation of potential by

\[
W(\eta, \xi, \theta; \{R_m\}; x) = \sum_{m=1}^{T} v_m \left( \eta_m + \xi_m (x - \frac{m}{T}) \right)
\]

where

\[
v_m(\eta_m, \xi_m, \theta_m; x) = \eta_m + \xi_m x + \frac{1}{2} \theta_m x^2 + x^3 R_m(\eta_m, \xi_m, \theta_m; x)
\]

By (1.4) and (1.5),

\[
v_m(0, 0, 0; x) \equiv 0
\]

and

\[
v_m(\eta_m, \xi_m, \theta_m; x) = 0 \quad \text{for } |x| \geq \frac{1}{2T}.
\]

Denote the collection of \((T, \delta)\)-variations of potential by \(\mathcal{S}(T, \delta)\). The set of parameters \((\eta, \xi, \theta)\) has measure \((2\delta)^{3T}\). We want to define a notion of “typical” potential by using the normalized measure on this set of parameters. Hence, a set \(S \subset \mathcal{S}(T, \delta)\) is called \((1 - \varepsilon)\)-typical if

\[
|S| := \min_{\{R_m\}} \frac{1}{(2\delta)^{3T}} \text{mes}\left\{(\eta, \xi, \theta) \in [-\delta, \delta]^{3T} : W(\eta, \xi, \theta; \{R_m\}; \cdot) \in S\right\} \geq 1 - \varepsilon
\]

The main result in this paper, summarized in the following theorem, is that for typical \(C^3\) potential \(\tilde{V}\), we have positive Lyapunov exponent for all energies \(E\).

**Theorem 1.2.** Given any \(V \in C^3(\mathbb{T}), |V'(x)| + |V''(x)| \geq c > 0\), there is \(\lambda_0 = \lambda_0(V)\) such that for \(|\lambda| > \lambda_0\), one has a collection of variations \(\{S_\ell = S_\ell(V, \Lambda)\}_{\ell=1}^{\infty}, S_\ell \subset \mathcal{S}(T^{(\ell)}, \delta^{(\ell)}), \log T^{(\ell+1)} \simeq (T^{(\ell)})^{\alpha}, 0 < \alpha \ll 1, \sum_{\ell=1}^{\infty} (1 - |S_\ell|) \leq \lambda^{-\beta}\), so that for any potential

\[
\tilde{V}(x) = V(x) + \sum_{\ell=1}^{\infty} W^{(\ell)}(\eta^{(\ell)}, \xi^{(\ell)}, \theta^{(\ell)}; \{R_m^{(\ell)}\}; x)
\]

where \(W^{(\ell)} \in S_\ell\), there exists \(\Omega = \Omega(\lambda, \tilde{V}), \text{mes}(\mathbb{T} \setminus \Omega) \leq \lambda^{-\beta}\), so that the Lyapunov exponent \(L(\omega, E) \geq \frac{1}{T} \log \lambda\) for any \(\omega \in \Omega, E \in \mathbb{R}\).
There are two central technical problems which one has to deal with in order to establish Theorem 1.2. The first one consists of the splitting of eigenvalues of equation (1.1) on a finite interval \([-N, N]\). The technology for this splitting was developed in the recent work by Goldstein and Schlag[GS2] in the case of analytic potential. It is based on avalanche principle[GS1], elimination of resonances and localized eigenfunctions on a finite interval. We modify this method for smooth potential and prove the following result:

**Proposition 1.3.** Using the notation of Theorem 1.2, there exists integers \(T'_s, \log T'_s \sim \log T^{(s)}\), such that for any nested sequence of intervals \(\mathcal{F}_{s,k_s} = (\frac{k_s}{T'_s}, \frac{k_s+1}{T'_s})\), and \(x \in \mathbb{T}, \omega \in \Omega\), there is a sequence integers \(N_s = N_s(x, \omega)\), with \(\log N_s \sim \log T'_s\), so that

\[
|E_1 - E_2| > \exp(-N'_s)
\]

for distinct eigenvalues \(E_1, E_2 \in (\text{sp} \mathcal{H}_{[-N_s,N_s]}(x, \omega)) \cap \mathcal{F}_{s,k_s}\).

The second problem is as follows. The eigenvalues of the problem (1.1) on a finite interval \([1,N]\) have a smooth parameterization as \(E_1(x) < E_2(x) < \ldots < E_N(x), x \in \mathbb{T}\). This general result is due to the self-adjointness of the problem (1.1) and non-degeneracy of the the eigenvalues of (1.1) restricted on a finite interval. The problem is how to eliminate multiple resonance, i.e. \(|E_i(x) - E_j(x + m\omega)| < \delta\) and \(|E_i(x) - E_k(x + n\omega)| < \delta\) for \(m \neq n\). To deal with this problem, we need to evaluate the quantity

\[
|\partial_x E_j| + |\partial_{xx} E_j|
\]

from below.

This problem was also studied in [GS2]; for analytic potential, the problem was solved using discriminant of polynomials and Sard-type arguments. This method has no modification for smooth potentials. To solve this problem, we need to introduce variations of the potential. The most basic idea of our method is as follows.

“Typical” smooth functions \(F(x)\) are Morse functions, i.e. the quantity

\[
|\partial_x F| + |\partial_{xx} F|
\]

has a “good” lower bound, gauged according to the “size” of \(F\). On the other hand, there is a basic relation between \(\partial_x E_j\) and the potential \(V(x)\):

\[
\partial_x E_j = \sum_{k=1}^{N} V'(x + k\omega)|\varphi_j(x)(k)|^2
\]

where \(\varphi_j(x)(.)\) is a normalized eigenfunction of (1.1) on the interval \([1,N]\) corresponding to \(E_j(x)\). Relation (1.9) enables one to express the “genericity” of the potential \(V\) in terms of the lower bound for (1.7), provided \(\varphi_j(x)(.)\) is exponentially localized. We use Sard-type arguments to show that the total mass of those \(\omega\) for which there is no “response” in (1.9) under the variations of \(V\), is extremely small.

We will use a KAM-type approach to prove Theorem 1.2. At each scale, we need to establish the following inductive hypothesis:

h1. Localization in finite interval. This means that at each scale, we can find \(N_s\) such that the eigenfunctions of \(H_{[-N_s,N_s]}\) decay exponentially for \(|n| > \sqrt{N_s}\).

h2. Estimate the number of eigenvalues and separation of eigenvalues.

h3. \(\partial_x E + \partial_{xx} E > \exp(-N'_s)\)

h4. Elimination of multiple resonances.

In section 6, we first use a Diophantine condition to obtain an upper bound of the number of entries that can lead to resonance. Using this bound, we can find \(N_1\) such that the entries where resonance occur is far away from the edge of \([-N_1,N_1]\). This will give us exponentially decaying eigenfunctions, hence (h1) in the first scale. We show how (h1) implies (h2) in the first scale in Section 7; this will be the base case for Proposition 1.3.

In section 8, we introduce variations of potential. Using the results from Appendix F, we show that for typical potential, if the eigenvalues are separated, then (1.7) has a “good” lower bound. Due to its technical
nature, we defer the proof of (h4) in Appendix E. There, we show that if (h3) is satisfied, then multiple resonances do not occur for most frequencies $\omega$. This forms a major part of the inductive hypothesis used in Section 9. With no multiple resonances, we can define inductively $C^3$ functions $E_k(x, \omega)$ which take values in $\text{sp} H_{[-N_s, N_s]}(x, \omega)$, with exponentially decaying eigenfunctions.

In Section 10, we take $\tilde{V}$ as a limit of the varied potential. We show that lower estimate for (1.7), which we have established for $E^{(s)}(x, \omega)$, still holds for the eigenvalues of $H_{[-N_s, N_s]}(x, \omega)$ with the limiting potential $\tilde{V}$. Then, we will be able to prove Theorem 1.2.

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Part 1.

2. Elimination of flat slope

In Part I, we consider a $C^1$ potential $V(x)$. It is well known that problems in studying the eigenvalues and eigenfunctions for quasi-periodic potential, to large extent, arises from the small denominator in the Green's function. The KAM type approach to this problem developed by Goldstein, Schlag[GS1, GS2] consists of so-called multi-scale analysis which relate the eigenvalues of a relatively short interval with those of a longer interval.

At initial scale, the small denominator is of the form $V(x + m\omega) - V(x + n\omega)$, $-N_1 \leq m, n \leq N_1$. Clearly, when $|V'(x)|$ is small, this brings additional difficulties to the problem, since $|V(x) - V(x + n\omega)|$ will be small for most frequencies $\omega$. In this section, we eliminate the part of the torus where the slope of $V$ is flat, i.e. $|V'(x)|$ is small. We then eliminate a set of frequencies that lead to resonance.

Throughout Part I, let $V \in C^1(T)$, $\max_{x \in T} V(x) - \min_{x \in T} V(x) \leq \max_{x \in T} |V'(x)| \leq C_0$; furthermore, assume $V$ has $m_0$ monotonicity intervals. Set $A = \{x \in T : |V'(x)| < \varepsilon\}$. Next two lemmas give an upper bound of the measure of the spectral values that we have to exclude due to the elimination of flat slope.

**Lemma 2.1.** $\text{mes} V(A) \leq \varepsilon$

*Proof.* $\varepsilon \geq \int_A |V'(x)| dx = \int_{\mathbb{R}} \#\{x \in A : V(x) = E\} dE \geq \text{mes} V(A)$. \hfill \Box

Take $J = \{\min_{x \in \mathbb{T}} V(x), \max_{x \in \mathbb{T}} V(x)\}$. Let $J \setminus V(A) = \bigcup_i [\alpha_i, \beta_i]$, $h_i = \beta_i - \alpha_i$. Define $h : [0, \infty) \to [0, \infty)$ by $h(y) = \sum_i h_i \leq y$. $h$ is increasing, continuous from the right, and $h(0) = 0$. Hence, there is $\delta > 0$ such that $h(\delta) < \varepsilon$. Fix $0 < \delta < \varepsilon^{10} \ll \frac{1}{\varepsilon}$ such that $h(\delta) < \varepsilon$. Set $R = \{i : h_i > \delta\}$. Let

$$E^{(0)} = \bigcup_{i \in R} [\alpha_i + 2\delta^2, \beta_i - 2\delta^2]$$

**Lemma 2.2.** $\# R \leq C_0 \delta^{-1}$, $\text{mes}(J \setminus E^{(0)}) \leq 2\varepsilon + 4C_0\delta$.

*Proof.*

$$(\# R)\delta \leq \sum_{i \in R} h_i \leq C_0$$

$$\text{mes}(J \setminus E^{(0)}) = \text{mes} V(A) + h(\delta) + (\# R)(4\delta^2) \leq 2\varepsilon + 4C_0\delta$$

Next, we eliminate some frequencies to avoid resonances. Having eliminate flat slope, we can obtain an upper bound of the measure of the set of frequencies we eliminate.

For any $i \in R$, let $\alpha_i = y_{i0} < y_{i1} < \cdots < y_{in}$, $\beta_i = \delta^2 \leq y_{ik} - y_{i, k-1} < 2\delta^2$ for $2 < k < n_i - 1$ and $y_{i1} - y_{i0} = y_{i2} - y_{i1} = \delta^2 = y_{i, n_i - 1} - y_{i, n_i - 2} = y_{i, n_i} - y_{i, n_i - 1}$. For $1 \leq k \leq n_i$, consider $T_{ik} = V^{-1}([y_{i, k-1}, y_{ik}])$. 

Since $V$ has $m_0$ monotonicity intervals, $T_{ik}$ consists of at most $m_0$ intervals, each with length less than $(2\delta^2)^{-1} < 2\delta$.

Say $T_{ik} = \bigcup_{j=1}^{J_k} [a_{ikj}, b_{ikj}]$, $J_k \leq m_0$. Set \( \bar{B}_{ikj} = \bigcup_{|\ell-k| \leq 2} \bigcup_{1 \leq m \leq J_\ell} \{ [a_{i\ell m}, b_{i\ell m}] - [a_{ikj}, b_{ikj}] \} \),

\[
\bar{B}_{ikj}(n) = \{ \omega \in [0, 1] : \{ n\omega \} \in \bar{B}_{ikj} \},
\]

\[
B_{ikj} = \bigcup_{0 < |n| \leq N_1^2} \bar{B}_{ikj}(n) \quad \text{where } N_1 = \lfloor \delta^{-1/4} \rfloor
\]

\[
D_i^{(1)} = \bigcup_{1 \leq k < n, j=1} J_k \bigcup_{a_{ikj}, b_{ikj}} \times (\{0,1\} \setminus B_{ikj})
\]

**Lemma 2.3.** Let $E \in [\alpha_i + 2\delta^2, \beta_i - 2\delta^2]$, $i \in R$. Suppose $(x, \omega) \in D_i^{(1)}$ such that $|V(x) - E| \leq \delta^2$. Then $|V(x + j\omega) - E| > \delta^2$ for $0 < |j| \leq N_1^2$.

**Proof.** $E \in [y_{ik-1}, y_{ik}]$ for some $k$, $2 < k < n_i - 1$. $|V(x) - E| \leq \delta^2 \Rightarrow x \in T_{i\ell}$, $|\ell - k| \leq 1$. Say $x \in [a_{i\ell m}, b_{i\ell m}]$. If $|V(x + j\omega) - E| \leq \delta^2$ then $x + j\omega \in T_{i\tilde{\ell}}$, $|\tilde{\ell} - k| \leq 1$. Thus, we have $|\ell - \tilde{\ell}| \leq 2 \Rightarrow j\omega \in B_{i\ell m} \Rightarrow \omega \in B_{i\tilde{\ell} m}(j)$. This contradicts the construction of $D_i^{(1)}$ if $0 < |j| \leq N_1^2$. Hence $|V(x + j\omega) - E| > \delta^2$ for $0 < |j| \leq N_1^2$.

Define

\[
D^{(1)} = \bigcup_{i \in R} D_i^{(1)}
\]

**Corollary 2.4.** If $E \in \mathcal{E}^{(0)}$, $(x, \omega) \in D^{(1)}$, $|V(x) - E| \leq \delta^2$. Then $|V(x + j\omega) - E| > \delta^2$ for $0 < |j| \leq N_1^2$.

For later sections, we also need an upper bound of the number of components in $D^{(1)}$.

**Definition.** A set $U_1 \subset \mathbb{R}$ (or $U_2 \subset \mathbb{R}^2$) is said to be $K$–simple if $U_1 = \bigcup_{k=1}^K J_k$ where $J_k$ are intervals $(U_2 = \bigcup_{k=1}^K D_k$ where $D_k$ are rectangles). If $U_1$ is $K$–simple, write compl$U_1 \leq K$.

**Lemma 2.5.** For any $i \in R$, compl$B_{ikj} \leq 5m_0\delta^{-1}$, mes$B_{ikj} < 40m_0\delta^{1/2}$.

**Proof.** $\{ [a_{i\ell m}, b_{i\ell m}] - [a_{ikj}, b_{ikj}] \}$ is an interval. Hence compl$\bar{B}_{ikj} \leq \sum_{|\ell-k| \leq 2} J_\ell \leq 5m_0$

\[
\text{compl } \bar{B}_{ikj}(n) \leq n(5m_0)
\]

\[
\text{compl } B_{ikj} \leq 5m_0 \sum_{0 < |n| \leq N_1^2} n \leq 5m_0(N_1^2)^2 \leq 5m_0\delta^{-1}
\]

\[
\text{mes } \{ [a_{i\ell m}, b_{i\ell m}] - [a_{ikj}, b_{ikj}] \} \leq (b_{i\ell m} - a_{i\ell m}) + (b_{ikj} - a_{ikj}) < 4\delta
\]

\[
\text{mes } \bar{B}_{ikj}(n) = \text{mes } \bar{B}_{ikj} < (5m_0)(4\delta) = 20m_0\delta
\]

\[
\text{mes } B_{ikj} < (2N_1^2)(20m_0\delta) \leq 40m_0\delta^{1/2}
\]

\[
\square
\]

**Corollary 2.6.** compl$D^{(1)} \leq 5m_0^2C_0\delta^{-3}$
Proof.

\[
\text{compl } D_i^{(1)} \leq n_i m_0 (5m_0 \delta^{-1})
\]

\[
\text{compl } D^{(1)} \leq 5m_0^2 \delta^{-1} \sum_{i \in R} n_i \leq 5m_0^2 \delta^{-1} (C_0 \delta^{-2})
\]

\[\square\]

Corollary 2.7. Let \( L = \sum_{i \in R} \sum_{1 < k < n_i} \sum_j (b_{ikj} - a_{ikj}) \). Then

\[
\text{mes} \left( \bigcup_{i \in R} \bigcup_{1 < k < n_i} \bigcup_j (a_{ikj}, b_{ikj}) \times [0, 1] \right) \leq 40 m_0 \delta^{1/2} L
\]

To simplify notation, henceforth we will write \( D^{(1)} = \bigcup_k \bigcup_{e_n} (a^{(1)}_k, b^{(1)}_k) \times (\omega^{(1)}_k, \omega^{(1)}_k) \).

3. Eigenvalues and Eigenfunctions at first scale

We now construct a function \( E^{(1)}(x, w) \), which takes values in \( \text{sp } H_{[-N_1, N_1]}(x, \omega) \), with corresponding eigenfunction \( \varphi^{(1)}(x, \omega) \), which is exponentially decaying. We will also derive some additional properties of \( E^{(1)} \).

Take \( \lambda = \delta^{-1} \),

\[
H_{[-N_1, N_1]}(x, \omega) = \begin{pmatrix}
\lambda V(x - N_1 \omega) & 1 & \cdots & 1 \\
-1 & \lambda V(x - (N_1 - 1)\omega) & \cdots & -1 \\
\cdots & \cdots & \cdots & \cdots \\
-1 & -1 & \cdots & \lambda V(x + N_1 \omega)
\end{pmatrix}
\]

Lemma 3.1. Let \( (x, \omega) \in D^{(1)} \). Then \( H_{[-N_1, N_1]}(x, \omega) \) has a unique eigenvalue in \( [\lambda V(x - 2, \lambda V(x + 2)] \).

Proof. \( (x, \omega) \in D^{(1)} \implies V(x) \in \mathcal{E}^{(0)} \), \( |V(x + j\omega) - V(x)| \geq \delta^2 = \lambda^{-1/2} \) for \( 0 < |j| \leq N_1 \). By Lemma B.4, \( H_{[-N_1, N_1]}(x, \omega) \) has a unique eigenvalue in \( [\lambda V(x - 2, \lambda V(x + 2)] \). \[\square\]

Denote the eigenvalue of \( H_{[-N_1, N_1]}(x, \omega) \) in Lemma 3.1 by \( E^{(1)}(x, \omega) \).

Lemma 3.2. \( E^{(1)}(x, \omega) \) is \( C^1 \) in \( D^{(1)} \). There are \( C^1 \) functions \( \{ \varphi^{(1)}(x, \omega)(n) \}_{|n| \leq N_1} \) defined in \( D^{(1)} \) such that

\[
\sum_{n=-N_1}^{N_1} |\varphi^{(1)}(x, \omega)(n)|^2 = 1,
\]

\[
H_{[-N_1, N_1]}(x, \omega) \varphi^{(1)}(x, \omega) = E^{(1)}(x, \omega) \varphi^{(1)}(x, \omega).
\]

Also,

\[
\partial_x E^{(1)}(x, \omega) = \sum_{n=-N_1}^{N_1} V'(x + n \omega) |\varphi^{(1)}(x, \omega)(n)|^2
\]

\[
\partial_{\omega} E^{(1)}(x, \omega) = \sum_{n=-N_1}^{N_1} n V'(x + n \omega) |\varphi^{(1)}(x, \omega)(n)|^2
\]

Proof. These follow from general results in Appendix A. \[\square\]

Lemma 3.3. \( |\varphi^{(1)}(x, \omega)(n)| < \lambda^{-\frac{1}{2}|n|} |\varphi^{(1)}(x, \omega)(0)| \).
Proof. We prove the case for \( n > 0 \); the case for \( n < 0 \) is similar.

By Poisson’s formula,
\[
\varphi^{(1)}(x, \omega)(n) = \left( H_{[1, N_1]} E^{(1)}(x, \omega) \right)^{-1}(n, 1) \varphi^{(1)}(x, \omega)(0)
\]
Since
\[
|\lambda V(x + m\omega) - E^{(1)}(x, \omega)| \geq |\lambda V(x + m\omega) - \lambda V(x)| - |\lambda V(x) - E^{(1)}(x, \omega)|
\]
\[
> \lambda^{1/2} - 2 \quad \text{for } 1 \leq m \leq N_1 .
\]
and \( N_1 \leq \delta^{-1/4} = \lambda^{1/16} \), by Corollary C.8, one has
\[
|\varphi^{(1)}(x, \omega)(n)| < \lambda^{-\frac{1}{16}}|\varphi^{(1)}(x, \omega)(0)| .
\]

\[ \square \]

Corollary 3.4.
\[
\frac{1}{2} \lambda^{39/40} < |\partial_x E^{(1)}(x, \omega)| \leq C_0 \lambda \quad |\partial_\omega E^{(1)}(x, \omega)| \leq C_0 N_1 \lambda
\]

Proof.
\[
\sum_{n \neq 0} |\varphi^{(1)}(x, \omega)(n)|^2 \leq 2 \sum_{n=1}^\infty \lambda^{-\frac{1}{4}n} \leq 4 \lambda^{-\frac{1}{4}}
\]

\( (x, \omega) \in D^{(1)} \implies |V'(x)| \geq \varepsilon > \delta^{1/4} = \lambda^{-\frac{1}{4}} \)
\[
|\partial_x E^{(1)}(x, \omega)| \geq \lambda |V'(x)| |\varphi^{(1)}(x, \omega)(0)|^2
\]
\[
- \lambda \sum_{n \neq 0} |V'(x + n\omega)| |\varphi^{(1)}(x, \omega)(n)|^2
\]
\[
\geq \lambda \epsilon (1 - 4 \lambda^{-\frac{1}{4}}) - \lambda C_0 (4 \lambda^{-\frac{1}{4}}) > \frac{1}{2} \lambda^{\frac{39}{40}}
\]

\[ \square \]

We now construct a set of frequencies such that the image of \( E^{(1)}(x, \omega) \) contains most of the spectral values. We will need this in Section 4 when we eliminate the situation when \( \text{sp}(H_{[-N_1, N_1]}(x, \omega)) \) and \( \text{sp}(H_{[-N_1, N_1]}(x + j\omega, \omega)) \) are close to each other. Recall that \( L = \sum_k (b_k^{(1)} - a_k^{(1)}) \).

Lemma 3.5. There exists \( \Omega^{(1)} \subset [0, 1] \), \( \text{mes}(\{0, 1 \} \setminus \Omega^{(1)}) \leq \delta^{1/4} \) such that for any \( w \in \Omega^{(1)} \), one has
\[
\text{mes}(V^{-1}(E^{(0)}) \setminus D_w^{(1)}) \leq 40m_0 \delta^{1/4} L
\]
\[
\text{compl}(V^{-1}(E^{(0)}) \setminus D_w^{(1)}) \leq 40m_0 C_0 \delta^{-7/4} L .
\]

Proof. By Corollary 2.7,
\[
\int_0^1 \text{mes}(V^{-1}(E^{(0)}) \setminus D_w^{(1)})d\omega = \text{mes}\left( \bigcup_k (a_k^{(1)}, b_k^{(1)}) \times [0, 1] \setminus D^{(1)} \right) \leq 40m_0 \delta^{1/2} L .
\]

Let \( B^{(1)} = \left\{ \omega \in [0, 1] : \text{mes}(V^{-1}(E^{(0)}) \setminus D_w^{(1)}) \geq 40m_0 \delta^{1/4} L \right\} \). Then \( \text{mes} B^{(1)} \leq \delta^{1/4} \).
Take $\Omega^{(1)} = [0, 1] \setminus B^{(1)}$. If $\omega \in \Omega^{(1)}$, then $\text{mes}(V^{-1}(E^{(0)}) \setminus D^{(1)}_\omega) < 40m_0 \delta^{1/4}/L$. Each interval $(a_k^{(1)}, b_k^{(1)})$ is contained in either $D^{(1)}_\omega$ or $V^{-1}(E^{(0)}) \setminus D^{(1)}_\omega$. Since $b_k^{(1)} - a_k^{(1)} = \delta^2 C_0^{-1}$,

$$\# \{(a_k^{(1)}, b_k^{(1)}) \subset (V^{-1}(E^{(0)}) \setminus D^{(1)}_\omega)\} \leq (40m_0 \delta^{1/4}/L)(\delta^2 C_0^{-1})^{-1}.$$  

□

For $\omega \in \Omega^{(1)}$, let $V(V^{-1}(E^{(0)}) \setminus D^{(1)}_\omega) = \bigcup_{r=1}^{R_\omega} (E_{\omega r}, \overline{E}_{\omega r})$, $R_\omega \leq 40m_0C_0\delta^{-7/4}/L$. Define

$$E^{(1)}_\omega = E^{(0)} \setminus \bigcup_r (E_{\omega r} - \delta^2, E_{\omega r} + \delta^2).$$

**Lemma 3.6.** For any $\omega \in \Omega^{(1)}$, one has $\text{mes}(E^{(0)} \setminus E^{(1)}_\omega) \leq 120m_0C_0\delta^{1/4}$. Suppose $|V(x) - E| \leq \delta^2$, $E \in E^{(1)}_\omega$. Then $x \in D^{(1)}_\omega$.

Proof.

$$\text{mes}(E^{(0)} \setminus E^{(1)}_\omega) \leq (40m_0 \delta^{1/4})C_0 + (40m_0C_0\delta^{-7/4})(2\delta^2) = 120m_0C_0\delta^{1/4}.$$

The second assertion follows from the construction of $E^{(1)}_\omega$. □

Next two lemmas provide us some flexibility concerning the size of the monodromy matrices we can use. We will use them in Section 4 to obtain the hypothesis of the Avalanche Principle.

**Lemma 3.7.** Let $\omega \in \Omega^{(1)}$, $\lambda^{-1} E \in E^{(1)}_\omega$. Suppose $N_1 \leq b - a \leq 5N_1$. Then

$$\log |f_{[a,b]}(x, \omega, E)| \geq (b - a) \log(\lambda^{1/2} - 2) + \log \text{dist}(E, \text{sp} H_{[a,b]}(x, \omega))$$

Proof. Let $\text{sp} H_{[a,b]}(x, \omega) = \{\mu_n\}_{n=a}^b$, $|\lambda V(x + n\omega) - \mu_n| \leq 2$. Say

$$|E - \mu_{n_0}| = \text{dist}(E, \text{sp} H_{[a,b]}(x, \omega)).$$

If $|E - \mu_{n_0}| > \lambda^{1/2} - 2$ then $|E - \mu_n| > \lambda^{1/2} - 2$ for all $n$. Hence

$$\log |f_{[a,b]}(x, \omega, E)| = \sum_{n \neq n_0} \log |E - \mu_n| + |E - \mu_{n_0}| \\
\geq (b - a) \log(\lambda^{1/2} - 2) + \log \text{dist}(E, \text{sp} H_{[a,b]}(x, \omega)).$$

If $|E - \mu_{n_0}| \leq \lambda^{1/2} - 2$ then $|E - \lambda V(x + n_0\omega)| \leq \lambda^{1/2} \implies x + n_0\omega \in D^{(1)}_\omega$. Therefore, $|E - \lambda V(x + n\omega)| > \lambda^{1/2}$ for $n \neq n_0 \implies |E - \mu_n| > \lambda^{1/2} - 2$ for $n \neq n_0$. Hence

$$\log |f_{[a,b]}(x, \omega, E)| \geq (b - a) \log(\lambda^{1/2} - 2) + \log \text{dist}(E, \text{sp} H_{[a,b]}(x, \omega)).$$

□

**Lemma 3.8.** Let $E \in \text{sp} H_{[a,b]}(x, \omega)$, $N_1 \leq b - a \leq 5N_1$, $w \in \Omega^{(1)}$, $\lambda^{-1} E \in E^{(1)}_w$. If

$$|\lambda^{-1} E - V(x + j\omega)| < \lambda^{-1/2}$$

for some $j$, $a + N_1^\delta < j < b - N_1^\delta$ then $|E - E^{(1)}(x + j\omega, \omega)| < 10\lambda^{-\frac{3}{2}}N_1^\delta$.

Proof. $|\lambda^{-1} E - V(x + j\omega)| < \lambda^{-1/2} = \delta^2 \implies x + j\omega \in D^{(1)} \implies E^{(1)}(x + j\omega, \omega)$ is defined. Also, similar to the construction of $E^{(1)}$, $E$ is the only eigenvalue of $H_{[a,b]}(x, \omega)$ in $[V(x + j\omega) - 2, V(x + j\omega) + 2]$. 

Let
\[ p = \max\{a, j - N_1\}, \quad q = \min\{b, j + N_1\}, \]
\[ Y = \left[ \sum_{\ell = p}^{q} |\varphi^{(1)}(x + j\omega, \omega)(\ell - j)|^2 \right]^{1/2} \]
\[ \tilde{\varphi}(\ell) = \begin{cases} \frac{1}{Y} \varphi^{(1)}(x + j\omega, \omega)(\ell - j) & \text{if } \ell \in [p, q] \\ 0 & \ell \in [a, b] \setminus [p, q] \end{cases} \]

Then \( \|\tilde{\varphi}\| = 1 \).

\[ p + N_1^d < j < q - N_1^d \implies 1 - Y^2 \leq 2 \sum_{n = N_1^d}^{\infty} |\lambda^{-\frac{1}{2}}n|^2 \leq \frac{1}{2} \]
\[ \| [H_{[a,b]}(x, \omega) - E^{(1)}(x + j\omega, \omega)] \tilde{\varphi} \| \leq Y^{-1}[|\varphi(x + j\omega, \omega)(p - 1)| + |\varphi(x + j\omega, \omega)(q + 1)|] \leq 10^{-\frac{1}{2}}N_1^d \]

Using Lemma A.5, since \( E \) is the only eigenvalue of \( H_{[a,b]}(x, \omega) \) in \([V(x + j\omega) - 2, V(x + j\omega) + 2]\), we have \( |E - E^{(1)}(x + j\omega, \omega)| < 10^{-\frac{1}{2}}N_1^d \). \( \square \)

4. Elimination of Resonances

In this section, we use the result from Appendix E to avoid the situation when \( \text{sp}(H_{[-N_1,N_1]}(x, \omega)) \) and \( \text{sp}(H_{[-N_1,N_1]}(x + j\omega, \omega)) \) are close to each other, by eliminate part of the domain in \( D^{(1)} \). Then we construct a function \( E^{(2)}(x, \omega) \) which takes value in \( \text{sp}(H_{[-N_2,N_2]}(x, \omega)) \), \( N_2 \ll N_1^d \), and the corresponding eigenfunctions \( \varphi^{(2)}(x, \omega) \). We will obtain some basic properties of \( E^{(2)} \) and \( \varphi^{(2)} \) in this section; additional properties, including hypothesis for Avalanche Principle, will be derived in section 5, where the general inductive construction is done.

Recall that a set \( U \subset \mathbb{R}^2 \) is called \( K \)-simple, denoted compl \( U \leq K \), if \( U = \bigcup_{k=1}^{K} D_k \) where \( D_k \) are rectangles.

**Lemma 4.1.** There is \( m \)-simple set \( \bar{D}^{(2)} \subset D^{(1)}, m \leq e^{2N_1^d} \), which satisfies the following conditions:

1. \( \text{mes}(D^{(1)} \setminus \bar{D}^{(2)}) \leq e^{-\frac{1}{2}}N_1^d L \)
2. If \((x, \omega) \in \bar{D}^{(2)} \) and \((x + j\omega, \omega) \in D^{(1)} \) for some \( j, 0 < |j| \leq N_2^d \), where \( N_2 = [e^{N_1^d}] \), \( \tau < \vartheta \), then \( |E^{(1)}(x, \omega) - E^{(1)}(x + j\omega, \omega)| > 4e^{-N_1^d} \).

**Proof.** For \( 0 < |j| \leq N_2^d \), by construction of \( D^{(1)} \), \( x \in D_\omega^{(1)} \implies |V(x) - V(x + j\omega)| > \lambda^{-\frac{1}{2}} \). If \( x + j\omega \in D_\omega^{(1)} \), then
\[ |E^{(1)}(x, \omega) - E^{(1)}(x + j\omega, \omega)| \geq |\lambda V(x) - \lambda V(x + j\omega)| > 4e^{-N_1^d} \].

Consider \( D_k := \left( (a^{(1)}_k, b^{(1)}_k) \times [0, 1] \right) \cap D^{(1)} \). Let \( a^{(1)}_k = x_0 < x_1 < \cdots < x_n = b^{(1)}_k \), \( x_i - x_{i-1} < e^{-N_1^d} \), \( n \leq 2e^{N_1^d} (b^{(1)}_k - a^{(1)}_k) \). For any \( |j| > N_2^d \), let \( T_k(j) = \{ (x, \omega) \in D_k : x + j\omega \in D_\omega^{(1)} \} \), \( T_k(j) \) consists of at most \( |j|C_0\delta^{-2} \) components. (See Corollary 2.6) On each of these components, the function
\[ F_j(x, \omega) = E^{(1)}(x, \omega) - E^{(1)}(x + j\omega, \omega) \]
dist
rectangles, the boundary consists of two vertical lines and two lines with slope $C$.

Suppose \( \frac{1}{j} \leq \| \cdot \| \), \( \frac{1}{j} \ll (2C_0 \lambda) \left( \frac{1}{4} |j\lambda^{39/40}| \right)^{-1} \).

We get rectangles \( [x_{i-1}, x_i] \times [\omega_{ik}, \bar{\omega}_{ik}] \) such that
\[
\{ (x, \omega) \in T_k(j) : |F_j(x, \omega)| \leq 2e^{-N_\delta^p} \} \subseteq \bigcup_i \bigcup_{\ell} [x_{i-1}, x_i] \times [\omega_{ik}, \bar{\omega}_{ik}]:= D_k(j)
\]
\[
\text{mes } D_k(j) \leq |j|C_0\delta^{-2}(b_k^{(1)} - a_k^{(1)}) \left[ \frac{2(4e^{-N_1^p})}{\lambda^{39/40}} + 2e^{-N_2^p} \frac{2C_0\lambda}{\lambda^{39/40}} \right] < (b_k^{(1)} - a_k^{(1)})e^{-\frac{3}{2}N_1^p}
\]
\[
\text{compl } D_k(j) \leq 2e^{N_1^p}(b_k^{(1)} - a_k^{(1)})|j|C_0\delta^{-2}
\]

Set
\[
\tilde{D}^{(2)} = D^{(1)} \setminus \left( \bigcup_k \left( \bigcup_{N_1^p < |j| \leq N_2^p} D_k(j) \right) \right)
\]
\[
\text{mes } (D^{(1)} \setminus \tilde{D}^{(2)}) \leq \sum_k \sum_{N_1^p < |j| \leq N_2^p} (b_k^{(1)} - a_k^{(1)})e^{-\frac{3}{2}N_1^p} < e^{-\frac{1}{2}N_1^p}L.
\]

\[
m = \text{compl } \tilde{D}^{(2)} \leq \sum_k \sum_{N_1^p < |j| \leq N_2^p} 2e^{N_1^p}(b_k^{(1)} - a_k^{(1)})|j|C_0\delta^{-2} \leq e^{2N_1^p}
\]

By taking away a set of measure zero in \( \tilde{D}^{(2)} \), we obtain
\[
D^{(2)} = \bigcup_k \bigcup_{\ell} (a_k^{(2)}, b_k^{(2)}) \times (\omega_{k\ell}^{(2)}, \bar{\omega}_{k\ell}^{(2)}) \subseteq \tilde{D}^{(2)}
\]
where \( b_k^{(2)} - a_k^{(2)} < e^{-N_1^p}, \omega_{k\ell}^{(2)} - \bar{\omega}_{k\ell}^{(2)} < e^{-N_1^p}, \text{compl } D^{(2)} \leq e^{2N_1^p} \).

We now show that \( E^{(1)}(x, \omega) \) and \( \text{sp } H_{[a,b]}(x, \omega), b-a \approx N_1 \), are separated, provide that the entries near the edge of the interval \( [a, b] \) are separated from \( V(x) \), i.e. \( |V(x) - V(x + j\omega)| > \lambda^{1/2} \) for all \( j \in [a, a + N_1^p] \cup [b - N_1^p, b] \). The hypothesis of the Avalanche Principle will follow from the following lemma.

**Lemma 4.2.** Let \( (x, \omega) \in D^{(2)}, \omega \in \Omega^{(1)}, \lambda^{-1}E \in \mathcal{E}_\omega^{(1)} \), \( |E - E^{(1)}(x, \omega)| < 2e^{-N_1^p} \). Suppose \( |V(x + j\omega) - \lambda^{-1}E| > \lambda^{-1/2} \) whenever \( a \leq j \leq a + N_1^p \) or \( b - N_1^p \leq j \leq b \) where \( 1 \leq a \leq a + N_1 \leq b < a + 5N \). Then dist(\( \text{sp } H_{[a,b]}(x, \omega), E \) ≥ \( e^{-N_1^p} \).

**Proof.** Let \( \text{sp } H_{[a,b]} = \{ \mu_j \}_{j=a}^b \), \( |\mu_j - \lambda V(x + j\omega)| \leq 2 \). If \( |E - \lambda V(x + j\omega)| > \lambda^{1/2} \) for all \( j \in [a, b] \) then \( |E - \mu_j| > \lambda^{1/2} \) \( \Rightarrow \) dist(\( \text{sp } H_{[a,b]}(x, \omega), E \) > \( \lambda^{1/2} - 2 \).

Suppose \( |E - \lambda V(x + j_0\omega)| \leq \lambda^{1/2} \), \( a + N_1^p < j_0 < b - N_1^p \). Since \( E \in \mathcal{E}_\omega^{(1)} \), \( x + j_0\omega \in \mathcal{D}_\omega^{(1)} \). So \( |E - \lambda V(x + j\omega)| > \lambda^{1/2} \) for all \( j \neq j_0, a \leq j \leq b \), dist(\( \text{sp } H_{[a,b]}(x, \omega), E \) = \( |E - \mu_{j_0}| \). By Lemma 3.8, \( |\mu_{j_0} - E^{(N_1^p)}(x + j_0\omega)| < 10\lambda^{-\frac{1}{2}}N_1^p < e^{-N_1^p} \). Since \( x \in \mathcal{D}_\omega^{(2)} \), one has \( |E^{(1)}(x, \omega) - E^{(1)}(x + j_0\omega)| > 4e^{-N_1^p} \).

Therefore,
\[
|E - \mu_{j_0}| \geq |E^{(1)}(x, \omega) - E^{(1)}(x + j_0\omega)| - |E^{(1)}(x, \omega) - E| - |E^{(1)}(x + j_0\omega) - \mu_{j_0}| \geq e^{-N_1^p}
\]
\[ \Box \]
Corollary 4.3. Let \( (x, \omega) \in D(2), \omega \in \Omega(1), \lambda^{-1}E \in \mathcal{E}(1), |E - E(1)(x, \omega)| < e^{-N_1^\vartheta} \). Suppose \(|V(x + j\omega) - \lambda^{-1}E| \geq \lambda^{-1/2}\) whenever
\[
a \leq j \leq a + N_1^\vartheta \quad \text{or} \quad b - N_1^\vartheta \leq j \leq b + N_1^\vartheta \quad \text{or} \quad c - N_1^\vartheta \leq j \leq c
\]
where
\[
1 \leq a \leq a + N_1 \leq b \leq b + 1 + N_1 \leq c \leq a + 5N_1 \leq N_2^2
\]
Then
\[
\log ||M_{[a,b]}(x, \omega, E)|| \geq (b - a) \log(\lambda^{1/2} - 2) - N_1^\vartheta
\]
\[
\log ||M_{(b,c]}(x, \omega, E)|| \geq (c - b - 1) \log(\lambda^{1/2} - 2) - N_1^\vartheta
\]
and
\[
\log ||M_{[a,b]}(x, \omega, E)|| + \log ||M_{(b,c]}(x, \omega, E)|| - \log ||M_{[a,c]}(x, \omega, E)|| \leq 20 \log(\lambda C_0) + 20N_1^\vartheta.
\]

Proof. This follows from Lemma 3.7, Lemma C.10 and Lemma 4.2. □

Let \( x \in D(\omega), \omega \in \Omega(1), \lambda^{-1}E \in \mathcal{E}(1), |E - E(1)(x, \omega)| < 2e^{-N_1^\vartheta} \). Suppose \( 0 < a < b \leq N_2 \) such that \(|V(x + j\omega) - \lambda^{-1}E| \geq \lambda^{-1/2}\) for \( a \leq j \leq a + N_1^\vartheta \) and \( b - N_1^\vartheta < j \leq b \). Let \( a < j_1 < j_2 < \cdots < j_m < b \) be such that \(|V(x + j\omega) - \lambda^{-1}E| \geq \lambda^{-1/2}\). By construction of \( D(1), j_1 - j_{i-1} > N_1^\vartheta \). So choose \( a = k_0 < k_1 < \cdots < k_n = b \) such that \( \min_{i \leq \ell} |j_i - k_\ell| > N_1^\vartheta, N_1 < k_\ell - k_{\ell-1} \leq 2N_1 \).

Let
\[
A_1 = M_{[a,k_1]}(x, \omega, E) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},
\]
\[
A_\ell = M_{[k_{\ell-1},k_\ell]}(x, \omega, E), \quad \ell = 2, 3, \ldots, n - 1
\]
\[
A_n = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} M_{[k_{n-1},k_n]}(x, \omega, E)
\]

Lemma 4.4. With notation as above,
\[
\log |f_{[a,b]}(x, \omega, E)| = \sum_{\ell=2}^{n} \log ||A_\ell A_{\ell-1}|| - \sum_{\ell=2}^{n-1} \log ||A_\ell|| + O(\lambda^{-cN_1})
\]

Proof. If \([k_{\ell-1}, k_\ell]\) does not contain any \( j_i \), then \(|\lambda V(x + j\omega) - E| \geq \lambda^{1/2}\) for all \( j \in [k_{\ell-1}, k_\ell] \). Hence
\[
\log ||A_\ell|| \geq N_1 \log(\lambda^{1/2} - 2).
\]
If \( j_i \in [k_{\ell-1}, k_\ell] \), since \( k_{\ell-1} + N_1^\vartheta < j < k_\ell - N_1^\vartheta \), dist(\( \text{sp} H_{[k_{\ell-1}, k_\ell]}(x, \omega, E) \geq e^{-N_1^\vartheta} \) (Lemma 4.2). By Lemma 3.7, \( \log ||A_\ell|| \geq N_1 \log(\lambda^{1/2} - 2) - N_1^\vartheta \).

Also, applying Lemma C.10, we get
\[
\log ||A_\ell|| + \log ||A_{\ell-1}|| - \log ||A_\ell A_{\ell-1}|| \leq 20(\log(\lambda C_0) + N_1^\vartheta).
\]

The assertion follows from the Avalanche Principle. □

Next, we show that there is \( N \approx N_2 \) such that the Green’s function \( (H_{[1,N]}(x, \omega) - E)^{-1}(k, 1) \) decay exponentially. In fact, the Green’s function decay exponentially if resonance does not occur near the edge.

Lemma 4.5. Let \( (x, \omega) \in D(2), \omega \in \Omega(1), \lambda^{-1}E \in \mathcal{E}(1), |E - E(1)(x, \omega)| < e^{-N_1^\vartheta} \). Suppose \(|V(x + j\omega) - \lambda^{-1}E| \geq \lambda^{-1/2}\) for \( j \in [N - N_1^\vartheta, N] \) where \( N_2 \leq N < N_2^2 \). Then \( E \notin \text{sp} H_{[1,N]}(x, \omega) \) and
\[
\log |[H_{[1,N]}(x, \omega) - E]^{-1}(n, 1)| \leq -\frac{1}{5} n \log \lambda
\]
for \( n > N_1 - N_1^\vartheta \).
Proof. We can choose $p_k, q_k$ such that

$$N_1 \leq q_k - p_k \leq 2N_1, \quad p_k < q_k - 1 < p_k + 1 < q_k$$

$$q_k - 1 - p_k > N_1^\theta,$$

$$|V(x + m\omega) - \lambda^{-1}E| > \lambda^{-1/2} \quad \text{for} \quad |m - p_k| \leq N_1^\theta \text{ or } |m - q_k| \leq N_1^\theta.$$

The first assertion follows from Lemma C.4 and C.9 if we can show that $\text{dist}(H_{p_k, q_k}(x, \omega), E) \geq e^{-N_1^\theta}.$ But this follows from Lemma 4.2.

Applying Lemma 4.4, and note that we can choose $k_i$ such that $\min_i |n - k_i| = |n - k_{i_0}| \leq 2N_1^\theta,$ we get

$$\log \left| [H_{[1, N]}(x, \omega) - E]^{-1}(k_{i_0}, 1) \right| = \log |f(k_{i_0}, N)| - \log |f_{[1, N]}|$$

$$= - \sum_{\ell = 1}^{i_0+1} (\log \|A_{\ell}A_{\ell-1}\| - \|A_{\ell}\|) + O(\lambda^{-cN_1})$$

$$= - \sum_{\ell = 1}^{i_0} \log \|A_{\ell}\| + \sum_{\ell = 2}^{i_0+1} (\log \|A_{\ell}\| + \log \|A_{\ell-1}\| -$$

$$\log \|A_{\ell}A_{\ell-1}\|) + O(\lambda^{-cN_2})$$

$$\leq -k_{i_0}\log(\lambda^{1/2} - 2) + i_0N_1^\theta + i_0(20\log(\lambda C_0) + N_1^\theta)$$

$$\leq -\frac{1}{4}k_{i_0}\log \lambda$$

and

$$\log \left| [H_{[1, N]}(x, \omega) - E]^{-1}(n, 1) \right| \leq -\frac{1}{4}k_{i_0}\log \lambda + 2N_1^\theta \log(\lambda C_0 + 2)$$

$$\leq -\frac{1}{5}n \log \lambda$$

□

In the next two lemmas, we find $N \approx N_2$ such that the entries near the edge of $[1, N]$ are separated from $V(x)$.

**Lemma 4.6.** Suppose $x \in \mathbb{T}$, $\omega \in \Omega(1)$, $\lambda^{-1}E \in \mathcal{E}_1$, $N \gg N_1$. Then either

(a) $|\lambda V(x + j\omega) - E| > \frac{1}{2}\lambda^{1/2}$ for $N - N_1 < j \leq N$; or

(b) $|\lambda V(x + j\omega) - E| > \frac{1}{2}\lambda^{1/2}$ for $N < j \leq N + N_1$.

**Proof.** If $|\lambda V(x + j_0\omega) - E| < \frac{1}{2}\lambda^{1/2}$ for some $N - N_1 < j_0 \leq N$ then $(x + j_0\omega, \omega) \in \mathcal{D}(1)$ since $E \in \mathcal{E}_1$. Hence $|\lambda V(x + j\omega) - \lambda V(x + j_0\omega)| > \lambda^{1/2}$ for $|j - j_0| \leq N_2^2$. In particular, this is true for all $j, N < j \leq N + N_1$. Thus $|\lambda V(x + j\omega) - E| > \frac{1}{2}\lambda^{1/2}$ for $N < j \leq N + N_1$. □

Let $(x_1, \omega_1), (x_2, \omega_2) \in (a_k^{(2)}, b_k^{(2)}) \times (\omega_{kl}^{(2)}, \omega_{kl}^{(2)}) \subset \mathcal{D}(2)$. For $N_2 - N_1 < j \leq N_2 + N_1$,

$$|(x_1 + j\omega_1) - (x_2 + j\omega_2)| \leq |x_1 - x_2| + |j| |\omega_1 - \omega_2|$$

$$\leq (b_k^{(2)} - a_k^{(2)}) + (N_2 + N_1)(\omega_{kl}^{(2)} - \omega_{kl}^{(2)})$$

$$\leq e^{-N_1^\theta}.$$

Suppose $|\lambda V(x_1 + j\omega_1) - E_1| > \frac{1}{2}\lambda^{1/2}$ for $N_2 - N_1 < j \leq N_2$. The for $|E_2 - E_1| < \frac{1}{2}\lambda^{1/2}$,

$$|\lambda V(x_2 + j\omega_2) - E_2| > \frac{\lambda^{1/2}}{3}.$$
for $N_2 - N_3 < j \leq N_2$. Hence, we have the following lemma.

**Lemma 4.7.** For each $(a_k^{(2)}, b_k^{(2)}) \times (\omega_k^{(2)}, \omega_{k\ell}^{(2)})$, there exist $N_{\ell k}^{(2)}$, $|N_{\ell k}^{(2)} - N_2| \leq N_1$, such that for any $(x, \omega) \in (a_k^{(2)}, b_k^{(2)}) \times (\omega_k^{(2)}, \omega_{k\ell}^{(2)}) \cap \Omega^{(2)}$, $\lambda^{-1} E \in E_{\omega}^{(1)}$, $|E - E(x, \omega)| < e^{-N_2^0}$, one has

$$|\lambda V(x + j\omega) - E| > \frac{\lambda^{1/2}}{3}$$

for $N_{\ell k}^{(2)} - N_1 < j \leq N_{\ell k}^{(2)}$. Furthermore, the Green's function $[H_{[1, N_{\ell k}^{(2)}]}(x, \omega) - E]^{-1}(n, 1)$ decay exponentially.

**Proof.** With no resonances near the edge, the second assertion follows from Lemma 4.5. \[\square\]

For each component $(a_k^{(2)}, b_k^{(2)}) \times (\omega_k^{(2)}, \omega_{k\ell}^{(2)})$, fix $N_{\ell k}^{(2)}, N_{\ell k}^{(2)}$ such that the Green’s functions $(H_{[1, N_{\ell k}^{(2)}]} - E)^{-1}(j, 1)$ and $(H_{[-N_{\ell k}^{(2)}, -1]} - E)^{-1}(-j, -1)$ decay exponentially for all $(x, \omega) \in (a_k^{(2)}, b_k^{(2)}) \times (\omega_k^{(2)}, \omega_{k\ell}^{(2)})$. We can now construct $C^1$ function $E^{(2)}(x, \omega)$ which take values in $sp H_{[-N_{\ell k}^{(2)}, N_{\ell k}^{(2)}]}(x, \omega)$.

**Lemma 4.8.** If $(x, \omega) \in (a_k^{(2)}, b_k^{(2)}) \times (\omega_k^{(2)}, \omega_{k\ell}^{(2)})$, then

$$sp H_{[-N_{\ell k}^{(2)}, N_{\ell k}^{(2)}]}(x, \omega) \cap \left( E^{(1)}(x, \omega) - e^{-N_1}, E^{(1)}(x, \omega) + e^{-N_1} \right) \neq \emptyset .$$

**Proof.** Let

$$\psi(j) = \begin{cases} \varphi^{(1)}(x, \omega)(j) & |j| \leq N_1 \\ 0 & |j| > N_1 \end{cases}$$

$$\| [H_{[-N_{\ell k}^{(2)}, N_{\ell k}^{(2)}]}(x, \omega) - E^{(1)}(x, \omega)] \psi \| \leq |\varphi^{(1)}(x, \omega)(1)| + |\varphi^{(1)}(x, \omega)(-N_1)|$$

$$\leq 2(\lambda^{-\frac{1}{2}N_1}) < e^{-N_1} .$$

The assertion now follows from Lemma A.4. \[\square\]

**Lemma 4.9.** Suppose $E_1, E_2 \in sp H_{[-N_{\ell k}^{(2)}, N_{\ell k}^{(2)}]}(x, \omega) \cap \left( E^{(1)}(x, \omega) - e^{-N_1^0}, E^{(1)}(x, \omega) + e^{-N_1^0} \right)$, $H_{[-N_{\ell k}^{(2)}, N_{\ell k}^{(2)}]}(x, \omega) \psi_i = E_i \psi_i$, $\| \psi_i \| = 1$. If $|\psi_i(k)| \leq e^{-c|k|}$ for $|k| \geq N_1$ then $E_1 = E_2$.

**Proof.** Assume $E_1 \neq E_2$. Let

$$\varphi_i(k) = \begin{cases} \psi_i(k) \left( \sum_{|j| \leq N_1} |\psi_i(j)|^2 \right)^{-1/2} & |k| \leq N_1 \\ 0 & |k| > N_1 \end{cases}$$

$$\| \varphi_i \| = 1, \left( [H_{[-N_1, N_1]}(x, w) - E_i] \varphi_i \right) \leq 2(2e^{-N_1}).$$

Since

$$sp H_{[-N_1, N_1]}(x, \omega) \cap \left( E(N_1) - \frac{1}{2} \lambda^{1/2}, E(N_1) + \frac{1}{2} \lambda^{1/2} \right) = \left\{ E(N_1)(x, \omega) \right\} ,$$

by Lemma A.5

$$\min_{|c| = 1} \| \varphi^{(N_1)}(x, \omega) - c \varphi_i \| < \sqrt{2}(4e^{-N_1}) \left( \frac{1}{2} \lambda^{1/2} \right)^{-1} .$$

Hence, there is $|c| = 1$ such that

$$400 \lambda^{-1} e^{-2N_1} > \| \varphi_1 - c \varphi_2 \|^2 = 2 + 2 \Re \overline{c} \varphi_1, \varphi_2$$

$$\Longrightarrow |\langle \varphi_1, \varphi_2 \rangle| > 1 - 200 \lambda^{-1} e^{-2N_1} .$$
Since $H_{[-\tilde{N}_k, -N_{k\varepsilon}]}(x, \omega)$ is hermitian,

\[
0 = |\langle \psi_1, \psi_2 \rangle| \geq \left| \sum_{|k| \leq N_1} \sum_{\epsilon} \psi_1(k)\psi_2(k)\right| - \sum_{|k| > N_1} |\psi_1(k)\psi_2(k)| \\
\geq |\langle \varphi_1, \varphi_2 \rangle| - \sum_{|k| > N_1} e^{-2|k|} > 0
\]

Lemma 4.8 and Lemma 4.9 show that $H_{[-\tilde{N}_k, -N_{k\varepsilon}]}(x, \omega)$ has unique eigenvalue $E^{(2)}(x, \omega) \in (E^{(1)}(x, \omega) - \frac{1}{2} e^{-N_1^g}, E^{(1)}(x, \omega) + \frac{1}{2} e^{-N_1^g})$ for $\omega \in \Omega^{(1)}$, $x \in D^{(2)}$, $|E^{(2)}(x, \omega) - E^{(1)}(x, \omega)| < e^{-N_1}$, the eigenfunction $\varphi^{(2)}(x, \omega)$ corresponding to $E^{(2)}(x, \omega)$ decay exponentially, $||\varphi^{(2)}(x, \omega)|| = 1$. Using Lemma A.5, by multiplying $c$ to $\varphi^{(2)}$, $|c| = 1$, we can choose

\[
\| \varphi^{(1)}(x, \omega) - \varphi^{(2)}(x, \omega) \| \leq e^{-N_1}
\]

where

\[
\varphi^{(1)}(x, \omega)(k) = \begin{cases} 
\varphi^{(1)}(x, \omega; k) & |k| \leq N_1 \\
0 & |k| > N_1 
\end{cases}
\]

Together with Lemma 4.7, this shows that $|\varphi^{(2)}(x, \omega)(k)| \leq e^{-\frac{k}{2}}$ for all $k$, since $\varphi^{(1)}(x, \omega)$ decay exponentially.

5. Inductive Construction of Eigenvalue $E^{(s+1)}$ and Eigenfunction $\varphi^{(s+1)}$

In this section, we will derive Theorem 1.1. First, to construct $E^{(s+1)}$, we assume $E^{(1)}, \ldots, E^{(s)}, \varphi^{(1)}, \ldots, \varphi^{(s)}$ have been defined with the following properties:

1. The domain of $E^{(k)}$ and $\varphi^{(k)}(\cdot)(j)$ is $D^{(k)}$, $D^{(1)} \supset D^{(2)} \supset \cdots \subset D^{(s)}$: compl $D^{(k)} < e^{4N_{k-1}^g}$, mes $(D^{(k)} - D^{(k-1)}) < e^{-N_{k-1}^g}$ for $1 \leq k \leq s$ where $N_k = e^{N_{k-1}^g}$.
2. $D^{(k)} = \bigcup_{i, j} \{ (a^{(k)}_{ij}, b^{(k)}_{ij}) \times (\omega^{(k)}_{ij}, \omega^{(k)}_{ij}) \}$, $|a^{(k)}_{ij} - a^{(k)}_{ij}| < e^{-N_{k-1}^g}$, $\omega^{(k)}_{ij} - \omega^{(k)}_{ij} < e^{-N_{k-1}^g}$. For each $i, j$, there is $\tilde{N}^{(k)}_{ij}, N^{(k)}_{ij} = N_k$ such that $H_{[-\tilde{N}^{(k)}_{ij}, -N^{(k)}_{ij}]}(x, \omega)\varphi^{(k)}(x, \omega) = E^{(k)}(x, \omega)\varphi^{(k)}(x, \omega)$ for all $(x, \omega) \in (a^{(k)}_{ij}, b^{(k)}_{ij}) \times (\omega^{(k)}_{ij}, \omega^{(k)}_{ij})$.
3. $\sum |\varphi^{(k)}(x,\omega)(j)|^2 = 1$: $|\varphi^{(k)}(x,\omega)(j)| \leq \lambda^{-\frac{j}{2}}|j|$, $\frac{1}{2} \lambda^{39/40} < |\partial_x E^{(s+1)}(x,\omega)| \leq C_0 \lambda$: $|\partial_x E^{(s+1)}(x,\omega)| \leq C_0 N_{s+1} \lambda$.
4. $|E^{(k)}(x,\omega) - E^{(k-1)}(x,\omega)| < e^{-N_{k-1}}$, $\|\varphi^{(k)}(x,\omega) - \varphi^{(k-1)}(x,\omega)\| < e^{-N_{k-1}}$ where

\[
\varphi^{(k-1)}(x,\omega)(n) = \begin{cases} 
\varphi^{(k-1)}(x,\omega)(n) & \text{if } -N^{(k-1)}_{ij} \leq n \leq N^{(k-1)}_{ij} \\
0 & \text{otherwise} 
\end{cases}
\]

\[
|E^{(k-1)}(x,\omega) - E^{(k-1)}(x + j\omega,\omega)| \geq 2e^{-N_{k-1}^g} \text{ if } (x, \omega) \in D^{(k)} \text{ and } (x + j\omega, \omega) \in D^{(k-1)}, |j| \leq N_{k-1}^2 
\]

6. $\text{sp } H_{[-\tilde{N}^{(k)}_{ij}, N^{(k)}_{ij}]}(x, \omega) \cap (E^{(k-1)}(x, \omega) - e^{-N_{k-1}^g}, E^{(k)}(x, \omega) + E^{-N_{k-1}^g}) = \{ E^{(k)}(x, \omega) \}$.

7. There is $\Omega^{(1)} \supset \Omega^{(2)} \supset \cdots \Omega^{(s-1)}$, mes $\Omega^{(k-1)} \cap \Omega^{(k)} < e^{-\frac{k}{2}N_{k-1}^g}$, such that $\text{mes } (D^{(k)} - D^{(k-1)}) < e^{-\frac{k}{2}N_{k-1}^g}$ for all $\omega \in \Omega^{(k)}$.

8. For $\omega \in \Omega^{(k)}$, there is $E_\omega \subset \tilde{E}^{(k-1)} \subset \cdots \subset E_\omega^{(1)}$, mes $E_\omega^{(k-1)} \cap E^{(k)} < e^{-\frac{k}{2}N_{k-1}^g}$, such that if $x \in D^{(k)}$, $E \in E_\omega^{(1)}$, $|E - E^{(k-1)}(x, \omega)| < e^{-N_{k-1}^g}$ then $x \in D_\omega^{(k)}$. 

Lemma 5.1. Let \((x, \omega) \in \mathcal{D}^{(k)}, \omega \in \Omega^{(k)}, \lambda^{-1}E \in \mathcal{E}^{(k)}, e^{-N_{k+1}^\theta} < |E^{(k)}(x + j\omega, \omega) - E| < e^{-N_k^\theta}.\) Suppose \(|E^{(k)}(x + j\omega, \omega) - E| > e^{-N_k^\theta}\) whenever
\[-a \leq j \leq -a + N_k^\theta \text{ or } b - N_k^\theta \leq j \leq b, \quad x + j\omega \in \mathcal{D}^{(k)}\]
where \(a, b \approx N_{k+1}.\) Then
\[
\log |f_{[a,b]}(x, \omega, E)| \geq \frac{1}{4}(a + b) \log \lambda
\]
For the first scale, these conditions were proven in Section 3. Furthermore, we assume that for \((x, \omega) \in \mathcal{D}^{(k+1)}, \omega \in \Omega^{(k)}, \lambda^{-1}E \in \mathcal{E}^{(k)}, |E - E^{(k)}(x, \omega)| < 2e^{-N_k^\theta},\) the following conditions (and the corresponding conditions in \([-N_{k+1}])\) hold.

10. Suppose \(|E^{(k-1)}(x + j\omega, \omega) - E| > e^{-N_{k-1}^\theta}\) whenever
\[a \leq j \leq a + N_k^\theta \text{ or } b - N_k^\theta \leq j \leq b, \quad x + j\omega \in \mathcal{D}^{(k-1)}\]
where \(1 \leq a \leq a + N_k \leq b < a + 5N.\) Then \(\text{dist}(\text{sp} H_{[a,b]}(x, \omega), E) \geq e^{-N_k^\theta}.\)

11. Suppose \(|E^{(k-1)}(x + j\omega, \omega) - E| > e^{-N_{k-1}^\theta}\) whenever
\[a \leq j \leq a + N_k^\theta \text{ or } b - N_k^\theta \leq j \leq b + N_k^\theta \text{ or } c - N_k^\theta \leq j \leq c, \quad x + j\omega \in \mathcal{D}^{(k-1)}\]
where
\[1 \leq a \leq a + N_k \leq b \leq b + N_k \leq c \leq a + 5N \leq N_{k+1}^2\]
Then
\[
\log ||M_{[a,b]}(x, \omega, E)|| \geq (b - a) \log(\lambda^{1/2} - 2) - N_k^\theta
\]
\[
\log ||M_{(b,c]}(x, \omega, E)|| \geq (c - b - 1) \log(\lambda^{1/2} - 2) - N_k^\theta
\]
and
\[
\log ||M_{[a,b]}(x, \omega, E)|| + \log ||M_{(b,c]}(x, \omega, E)|| - \log ||M_{[a,c]}(x, \omega, E)|| \leq 20 \log(\lambda C_0) + 20N_k^\theta
\]

12. Suppose \(|E^{(k-1)}(x + j\omega, \omega) - E| > e^{-N_{k-1}^\theta}\) for \(j \in [N - N_1^\theta, N]\) where \(N_{k+1} \leq N < N_{k+1}^2.\) Then \(E \notin \text{sp} H_{[1,N]}(x, \omega)\) and
\[
\log \left| \left[ H_{[1,N]}(x, \omega) - E \right]^{-1}(n, 1) \right| \leq -\frac{1}{5}n \log \lambda
\]
for \(n > N_k - N_1^\theta.\)

With some abuse of notation, we use \(E^{(0)}(\omega)\) to denote \(\lambda\Omega\) and \(e^{-N_0^\theta}\) means \(\lambda^{1/2}.\) Then the last three conditions were proven, for \(k=1,\) in Section 4. We now begin to prove these conditions inductively for higher scale. As in section 4, we need to eliminate the case when \(\text{sp} H_{[-N_{k+1},N]}(x, \omega)\) and \(\text{sp} H_{[-N_{k+1},N]}(x + j\omega, \omega)\) are close.

Lemma 5.1. There is \(D^{(s+1)} \subset D^{(s)}, \mathcal{D}^{(s+1)} = \bigcup_{j=1}^M D_j^{(s+1)}\) where \(D_j^{(s+1)} = \bigcup_{t} \left( a_j^{(s+1), t}, b_j^{(s+1), t} \right) \times \left( \omega_j^{(s+1), t}, \bar{\omega}_j^{(s+1), t} \right),\)
\[b_j^{(s+1), t} - a_j^{(s+1), t} < e^{-N_{s+1}^\theta}, \omega_j^{(s+1), t} - \bar{\omega}_j^{(s+1), t} < e^{-N_{s+1}^\theta}, M \leq 2N_{s+1}^\theta, \text{ which satisfies the following conditions:}\]

(1) \(\text{mes}(D^{(s)} \setminus D^{(s+1)}) \leq e^{-\frac{1}{4}N_{s+1}^\theta} L\)

(2) If \((x, \omega) \in D^{(s+1)}\) and \((x + j\omega, \omega) \in D^{(s)}\) for some \(0 < |j| \leq N_{s+1}^2\) where \(N_{s+1} = \lfloor e^{N_{s+1}} \rfloor, \tau < \vartheta, \text{ then}\)
\[|E^{(s)}(x, \omega) - E^{(s)}(x + j\omega, \omega)| > 4e^{-N_{s+1}^\theta} .\]

Proof. For \(|n| \leq N_{s+1}^2,\) this follows from conditions (4) and (5). For \(|n| > N_{s+1}^2,\) the proof is essentially the same as proof of Lemma 4.1, given the estimate on derivatives of \(E^{(s)}\) from condition (3). (Also see remark after Lemma 4.1.) \(\square\)

We will first prove conditions (10)–(12) for \(k=s.\)
Lemma 5.2. Let \((x, \omega) \in D^{(s+1)}\), \(\omega \in \Omega^{(s)}\), \(\lambda^{-1} E \in E^{(s)}\), \(|E - E^{(s)}(x, \omega)| < 2e^{-N_s^g}\). Suppose \(|E^{(s-1)}(x + j\omega, \omega) - E| > e^{-N_s^{g-1}}\) whenever \(a \leq j \leq a + N_s^g\) or \(b - N_s^g \leq j \leq b\) where \(1 \leq a \leq a + N_s \leq b < a + 5N\). Then dist\((\text{sp} H_{[a,b]}(x, \omega), E) \geq e^{-N_s^g}\).

Proof. Let \(a < j_1 < j_2 < \cdots < j_m \leq b\) be such that \(x + j_1\omega \in D^{(s)}\) and \(|E - E^{(s)}(x + j_1\omega, \omega)| < \frac{1}{4}e^{-N_s^g}\). Then

\[
|E^{(s-1)}(x + j_1\omega, \omega) - E^{(s-1)}(x + j_{i+1}\omega, \omega)| \leq |E^{(s-1)}(x + j_1\omega, \omega) - E^{(s)}(x + j_1\omega, \omega)| \\
+ |E^{(s)}(x + j_1\omega, \omega) - E^{(s)}(x + j_i+1\omega, \omega)| \\
+ |E^{(s)}(x + j_i+1\omega, \omega) - E^{(s-1)}(x + j_i+1\omega, \omega)| \\
< e^{-N_s^g} + \frac{1}{2}e^{-N_s^g} + \frac{1}{2}e^{-N_s^g} + e^{-N_s^g} < 2e^{-N_s^g}
\]

By condition (5), \(|j_i - j_{i+1}| > N_s^2\). Also, \(j_1 > a + N_s^g\) and \(j_m < b - N_s^g\). The assertion follows from condition (12) and Lemma C.9.

Corollary 5.3. Let \((x, \omega) \in D^{(s+1)}\), \(\omega \in \Omega^{(s)}\), \(\lambda^{-1} E \in E^{(s)}\), \(|E - E^{(s)}(x, \omega)| < e^{-N_s^g}\). Suppose \(|E^{(s)}(x + j\omega, \omega) - E| > e^{-N_s^{g-1}}\) whenever

\[
a \leq j \leq a + N_s^g \quad \text{or} \quad b - N_s^g \leq j \leq b + N_s^g \quad \text{or} \quad c - N_s^g \leq j \leq c, \quad x + j\omega \in D^{(s)}
\]

where

\[
1 \leq a \leq a + N_s \leq b \leq b + 1 + N_1 \leq c \leq a + 5N_s \leq N_{s+1}^2
\]

Then

\[
\log \|M_{[a,b]}(x, \omega, E)\| \geq (b - a) \log(\lambda^{1/2} - 2) - N_s^g
\]

\[
\log \|M_{[b,c]}(x, \omega, E)\| \geq (c - b - 1) \log(\lambda^{1/2} - 2) - N_s^g
\]

and

\[
\log \|M_{[a,b]}(x, \omega, E)\| + \log \|M_{[b,c]}(x, \omega, E)\| - \log \|M_{[a,c]}(x, \omega, E)\| \leq 20 \log(\lambda C_0) + 20N_s^g.
\]

Proof. This follows form Lemma 5.2, condition (9) and Lemma C.10.

Lemma 5.4. Let \((x, \omega) \in D^{(s+1)}\), \(\omega \in \Omega^{(s)}\), \(\lambda^{-1} E \in E^{(s)}\), \(|E - E^{(s)}(x, \omega)| < 2e^{-N_s^g}\). Suppose \(|E^{(s-1)}(x + j\omega, \omega) - E| > e^{-N_s^{g-1}}\) for \(j \in [N - N_s^g, N]\) where \(N_{s+1} \leq N < N_{s+1}^2\). Then \(E \notin \text{sp} H_{[1,N]}(x, \omega)\) and

\[
\log \|H_{[1,N]}(x, \omega) - E\|^{-1}(n, 1) \leq -\frac{1}{2}n \log \lambda
\]

for \(n > N_s - N_s^g\).

Proof. Let \(1 < j_1 < j_2 < \cdots < j_m \leq N\) be such that \(x + j_1\omega \in D^{(s)}\) and \(|E - E^{(s)}(x + j_1\omega, \omega)| < \frac{1}{4}e^{-N_s^g}\). By condition (5), \(|j_i - j_{i+1}| > N_s^2\). Also, \(j_1 > N_s^2\) and \(j_m < N - N_s^g\). (See the proof of Lemma 5.2) By condition (11), we can partition \([1, N]\) so that the hypothesis of Avalanche Principle is satisfied, and the assertion follows. (See Lemma 4.4 and Lemma 4.5)

Next, we show that there is \(N \approx N_{s+1}\) such that the Green’s function \((H_{[1,N]}(x, \omega) - E)^{-1}(k, 1)\) decays exponentially and then we will be able to define \(E^{(s+1)}\) and \(\phi^{(s+1)}\).

Let \((x, \omega) \in D^{(s+1)}\), \(\omega \in \Omega^{(s)}\), \(\lambda^{-1} E \in E^{(s)}\), \(|E - E^{(s)}(x, \omega)| < 2e^{-N_s^g}\). By condition (5), if there is \(n_0 \in [N_{s+1} - N_s, N_{s+1}]\) such that \(x + n_0\omega \in D^{(s)}\) and \(|E - E^{(s)}(x + n_0\omega, \omega)| < 2e^{-N_s^g}\) then \(|E - E^{(s)}(x + n, \omega)| \geq 2e^{-N_s^g}\) for all \(n \in (N_{s+1}, N_s + 1 + N_s)\).

For each \(i, j\), we can find \(N_{ij}^{(s+1)}\), \(|N_{ij}^{(s+1)} - N_{s+1}| \leq N_s^g\) such that \(|E - E^{(s)}(x + n, \omega)| \geq e^{-N_s^g}\) for all \(n \in [N_{ij}^{(s+1)} - N_s, N_{ij}^{(s+1)} + 1], (x, \omega) \in (\alpha_{ij}^{(s+1)}, \beta_{ij}^{(s+1)}), \omega_{ij}^{(s+1)}\). Hence by Lemma 5.4, the Green’s function \([H_{[1,N]}^{(s+1)}(x, \omega) - E]^{-1}(n, 1)\) decay exponentially.
Fix $N_{ij}^{(s+1)}$, $N_{ij}^{(s+1)}$, such that $H_{[-N_{ij}^{(s+1)}, N_{ij}^{(s+1)}]}(x, \omega)$ has a unique eigenvalue $E^{(s+1)}(x, \omega)$ in $(E^{(s)}(x, \omega) - e^{-N_{ji}^{(s)}}$, $E^{(s)}(x, \omega) + e^{-N_{ji}^{(s)}}$ for all $(x, \omega) \in \left(\alpha_j^{(s+1)}, \beta_j^{(s+1)}\right) \times \left(\omega_j^{(s+1)}, \bar{\omega}_j^{(s+1)}\right)$. $\|E^{(s+1)}(x, \omega) - E^{(s)}(x, \omega)\| < e^{-N_{ji}^{(s)}}$. (See Lemma 4.8 and Lemma 4.9) We can define $\varphi^{(s+1)}(x, \omega)(n)$ such that

$$H_{[-N, N]}(x, \omega)\varphi^{(s+1)}(x, \omega) = E^{(s+1)}(x, \omega)\varphi^{(s+1)}(x, \omega)$$

$$\sum_n |\varphi^{(s+1)}(x, \omega)(n)|^2 = 1$$

$$\|\varphi^{(s)}(x, \omega) - \varphi^{(s+1)}(x, \omega)\| \leq e^{-N_{s}^{(s)}} e^{-N_{s}^{(s)}} < e^{-\frac{1}{2}N_{s}}$$

where $\varphi^{(s)}$ is as defined in condition (4).

**Lemma 5.5.**

$$|\varphi^{(s+1)}(x, \omega)(n)| < \lambda^{-\frac{1}{4}}$$

$$\frac{1}{2} \lambda^{39/40} < |\partial_x E^{(s+1)}(x, \omega)| \leq C_0 \lambda$$

$$|\partial_\omega E^{(s+1)}(x, \omega)| \leq C_0 N_{s+1} \lambda$$

**Proof.** The first assertion follows from Lemma 5.4 and the fact that $\varphi^{(s)}(x, \omega)(n)$ decay exponentially. This also gives the estimate of the derivatives of $E^{(s)}$. (See Corollary 3.4) □

Lemma 5.1 and Lemma 5.5 establish conditions (1)–(6) for the $s+1$ scale.

Let $B^{(s+1)} = \left\{ \omega \in \Omega^{(s)}: \text{mes}\left(\left(E^{(s)}(x, \omega) - e^{-N_{s}^{(s)}}\right) \setminus D^{(s+1)}_{\omega}\right) \geq e^{-\frac{1}{4}N_{s}^{(s)}}\right\}$, $\Omega^{(s+1)} = \Omega^{(s)} \setminus B^{(s+1)}$. As in Lemma 3.5, $\text{mes}(\Omega^{(s)} \setminus \Omega) \leq e^{-\frac{1}{4}N_{s}^{(s)}}$. Also, for any $\omega \in \Omega^{(s+1)}$, one has

$$\text{mes}\left(E^{(s)}(x, \omega)^{-1}(E^{(s)}(x, \omega) \setminus D^{(s+1)}_{\omega})\right) \leq e^{-N_{s}^{(s)/2}}.$$ 

For $\omega \in \Omega^{(s+1)}$, suppose $D^{(s+1)}_{\omega} = \bigcup_r (a_r^{(s)}(\omega), b_r^{(s)}(\omega))$. Let

$$\tilde{E}_{\omega,r}^{(s+1)} = \text{Ran}(E^{(s)}(x, \omega))_{D^{(s+1)}(a_r^{(s)}(\omega), b_r^{(s)}(\omega))} = \bigcup_k (E_{r,k}, E_{r,k})$$

$$\tilde{E}_{\omega}^{(s+1)} = \bigcup_r \bigcup_k (E_{r,k} + e^{-N_{s}^{(s)}}, E_{r,k} - e^{-N_{s}^{(s)}})$$

$$\tilde{E}_{\omega}^{(s+1)} = \bigcup_r \bigcup_k (E_{r,k} + 2e^{-N_{s}^{(s)}}, E_{r,k} - 2e^{-N_{s}^{(s)}})$$

Then $\text{mes}(E^{(s)} \setminus E^{(s+1)}) < e^{-N_{s}^{(s)/4}}$. It is clear that for $\omega \in \Omega^{(s+1)}$, if $x \in D^{(s)}_{\omega}$, $E \in E^{(s+1)}_{\omega}$, $|E - E^{(s)}(x, \omega)| < e^{-N_{s}^{(s)}}$ then $x \in D^{(s+1)}_{\omega}$. This establishes conditions (7) and (8).

The next two lemma give us condition (9) and complete the inductive procedure.

**Lemma 5.6.** Let $\omega \in \Omega^{(s+1)}$, $E \in E^{(s+1)}_{\omega}$. Suppose $|E - E^{(s+1)}(x + j \omega, \omega)| \geq e^{-2N_{s}^{(s)}}$ whenever $(x + j \omega, \omega) \in D^{(s+1)}$ for $|j| \leq 4N_{s+1}$. Then

$$\log |f_{[-N, N]}(x, \omega, E)| \geq \frac{N_{ij}^{(s+1)} + \tilde{N}_{ij}^{(s+1)}}{4} \log \lambda$$

**Proof.** For $x = x + j \omega, |j| \leq 3N_{s+1}$, if $(x + j \omega, \omega) \in D^{(s)}$, $|j| \leq 4N_{s}$, then either

1. $(x + j \omega, \omega) \notin D^{(s+1)}$, or
2. $(x + j \omega, \omega) \in D^{(s+1)}, |E - E^{(s+1)}(x + j \omega, \omega)| > e^{-2N_{s}^{(s)}}$
In (1), $E \in \mathcal{E}^{(s+1)} \Rightarrow |E^{(s)}(\hat{x}, j\omega, -) - E| > \frac{1}{2} e^{-N_s^g}$. In (2),
\[ |E - E^{(s)}(x, \omega))| \geq |E - E^{(s+1)}(x, \omega)) - |E^{(s+1)}(x, \omega)) - E^{(s)}(x, \omega))| > \frac{1}{2} e^{-2N_s^g}. \]

By condition (11), we can apply Avalanche Principle to obtain the assertion. \hfill \Box

**Lemma 5.7.** Let $(x, \omega) \in \mathcal{D}^{(s+1)}$, $\omega \in \Omega^{(s+1)}$, $\lambda^{-1} E \in \mathcal{E}^{(s+1)}$, $e^{-N_{s+1}^g} < |E^{(s+1)}(x + j\omega, \omega)) - E| < e^{-N_s^g}$.

Suppose $|E^{(s)}(x + j\omega, \omega)) - E| > e^{-N_s^g}$ whenever
\[-a \leq j \leq -a + N_k^g \text{ or } b - N_s^g \leq j \leq b, \quad x + j\omega \in \mathcal{D}^{(k)}\]

where $a, b \approx N_{k+1}$. Then
\[ \log |f_{[-a, b]}(x, \omega, E)| \geq \frac{1}{4} (a + b) \log \lambda \]

**Proof.** By construction
\[ \text{sp} \mathcal{H}_{[-N, N]}(x, \omega) \cap \left( E^{(s)}(x, \omega) - e^{-N_s^g}, E^{(s)}(x, \omega) + e^{-N_s^g} \right) = \left\{ E^{(s+1)}(x, \omega) \right\} \]
\[ |E^{(s)}(x, \omega) - E^{(s+1)}(x, \omega)| < e^{-N_s} \]
\[ \Rightarrow |E - E^{(s)}(x + j\omega, \omega)| \geq e^{-N_s^g} \text{ for } (x + j\omega, \omega) \in \mathcal{D}^{(s)}, 0 < |j| \leq N_{s+1}^g \]
\[ \Rightarrow \text{there exist } |\bar{E} - E| \leq e^{-2N_s^g} \text{ such that } |\bar{E} - E^{(s)}(x + j\omega, \omega)| \geq e^{-2N_s^g}, |j| \leq N_{s+1} \]

By Lemma 5.6, \[ \log |f_{[-N, N]}(x, \omega, E)| > \frac{N_{s+1} N^{(s+1)}}{4} \log \lambda. \] Applying Lemma B.5 gives
\[ \log |f_{[-N, N]}(x, \omega, E)| > \frac{N_{s+1}^{(s+1)}}{4} \log \lambda - 2N_{s+1}. \]

\hfill \Box

By taking the limit as $s$ goes to infinity, we can prove Theorem 1.1:

**Theorem 5.8.** There is $\Omega = \Omega(V, \lambda) \in [0, 1]$, $\text{mes}(\{0, 1\} \setminus \Omega) \lesssim \lambda^{-\frac{1}{2}}$, such that for any $\omega \in \Omega$ there exists $\mathcal{E}_\omega \subset \mathcal{J}$, $\text{mes}(\mathcal{J} \setminus \lambda^{-1} \mathcal{E}_\omega) \lesssim \lambda^{-\frac{1}{2}}$, with $\lambda(\omega, E) \gtrsim \log \lambda$ for all $E \in \mathcal{E}_\omega$. Also, there is $x \in \mathcal{T}$ and $\{\varphi(n)\}$, $|\varphi(n)| \lesssim e^{-c|n|}$ such that
\[-\varphi(n+1) + \varphi(n-1) + V(x + n\omega)\varphi(n) = E\varphi(n). \]

**Proof.** Take $\Omega = \bigcap \Omega^{(k)}$, $\mathcal{E}_\omega = \bigcap \mathcal{E}_\omega^{(k)}$. For any $\omega \in \Omega$ and $E \in \mathcal{E}_\omega$, one has
\[ \frac{1}{N_{ij}^{(k)} + N_{ij}^{(k)}} \log \| M_{[-N_{ij}^{(k)}, N_{ij}^{(k)}}(x_k, \omega, E) \| \gtrsim \log \lambda, \]

$k = 1, \ldots$ where $e^{-N_k} < |E - E^{(k)}(x_k, \omega)| < e^{-N_{k-1}}$, $x_k \in \mathcal{D}_\omega^{(k)}$.

A subsequence of $\{x_k\}$ converges. Without loss of generality, we may assume $x = \lim x_k$. Take $\varphi(n) = \lim \varphi^{(k)}(x_k, \omega)(n)$. \hfill \Box

**Part 2.**

6. Resonance at first scale

In Part II, we introduce variations of potential (Section 8), and show that for typical variation, the resulting potential leads to positive Lyapunov exponent for all spectral value. In this section, we show that by choosing appropriate length for the interval in the first scale, we can get an upper estimate of the number of terms where resonances can occur.

Let $V \in C^2(\mathbb{T})$ be such that $|V'(x)| + |V''(x)| \geq 2c > 0$ for all $x \in \mathbb{T}$. Note that the assumption implies $V$ has finitely many critical points. Let $x_1, \ldots, x_n$ be the critical points of $V$. There exists $\delta > 0$ such that
\[ |V'(y)| \geq c \ |y - x_i| \quad \text{whenever } |y - x_i| < \delta \]
Lemma 6.1. Take any $E \in \mathbb{R}$, $0 < \varepsilon < \min\{\frac{1}{6} \varepsilon^2, C_0^2\}$. Then

$$V^{-1}(E - \varepsilon, E + \varepsilon) = \bigcup_{i=1}^{m} (a_i, b_i), \quad m \leq m_0, \quad b_i - a_i \lesssim \varepsilon^{1/2}$$

Proof. Each monotonicity interval of $V$ can intersect with at most one component $(a_i, b_i)$ of $V^{-1}(E - \varepsilon, E + \varepsilon)$ so $m \leq m_0$.

$$\int_{x_j}^{x_{j+\delta}} |V'(y)| dy \geq \int_0^{\delta} c dt = \frac{c}{2} \delta^2$$

so $(a_i, b_i)$ does not contain any $(x_j, x_{j+\delta})$. Similarly, $(a_i, b_i)$ does not contain any $(x_j - \delta, x_j)$. Therefore, $(a_i, b_i)$ intersects with at most one $(x_{j_1}, x_{j_1} + \delta)$ and one $(x_{j_2} - \delta, x_{j_2})$. Suppose $(a_i, b_i) \cap (x_{j_1}, x_{j_1} + \delta) = (y_1, y_2)$

$$2\varepsilon \geq \int_{y_1}^{y_2} |V'(y)| dy \geq c \int_{a_i}^{b_i} (y - x_{j_1}) dy = \frac{c}{2} \left( (y_2 - x_{j_1})^2 - (y_1 - x_{j_1})^2 \right) \geq \frac{c}{2} (y_2 - y_1)^2$$

Similarly, if $(a_i, b_i) \cap (x_{j_2} - \delta, x_{j_2}) = (y_3, y_4)$ then $2\varepsilon \geq \frac{c}{2} (y_4 - y_3)^2$.

$$b_i - a_i = (y_2 - y_1) + (y_4 - y_3) + \text{mes} \left[ (a_i, b_i) \setminus \bigcup_{j} (x_j - \delta, x_j + \delta) \right] \leq 2 \left( \frac{4\varepsilon}{c} \right)^{1/2} + \frac{2\varepsilon}{C_0}$$

Fix $N \gg 1$, $\lambda = N^{8\cdot 2^{m_0}}$, such that

$$\lambda^{-1/2} < \min\{\frac{1}{6} \varepsilon^2, C_0^2\}.$$ 

Lemma 6.2. For any $x \in \mathbb{T}$, $E \in \mathbb{R}$, if $\omega$ satisfies the Diophantine condition

$$\|\ell\omega\| \gtrsim |\ell|^{-\beta} \quad 1 < \beta < 2$$

then

$$\# \left\{ |\ell| \leq N^{2^{m_0}} : x + \ell \omega \in V^{-1}(E - \lambda^{-\frac{1}{2}}, E + \lambda^{-\frac{1}{2}}) \right\} \leq m_0$$

provided $\delta < \delta_0(m_0, \beta)$.

Proof. Suppose, for contradiction, that

$$x + \ell_1 \omega, x + \ell_2 \omega \in (a_i, b_i) \subset V^{-1}(E - \lambda^{-1/2}, E + \lambda^{-1/2}).$$
Then, by Lemma 6.1,
\[ N^{-\beta \tau_m} < |\ell_1 - \ell_2|^{-\beta} \lesssim \| (\ell_1 - \ell_2) \omega \| < b_i - a_i \lesssim \left( \lambda^{-1/2} \right)^{1/2} = N^{-2 \cdot 2^{m_0}} \]
This is false for sufficiently large \( N \) (or equivalently, for sufficiently small \( \delta \)). \( \square \)

It is well known that almost all \( \omega \in \mathbb{T} \) satisfied condition (6.1). In particular, given any interval \((\omega, \tilde{\omega})\), there is \( \omega \in (\omega, \tilde{\omega}) \) such that condition (6.1) is satisfied.

With the upper bound provided by Lemma 6.2, we can now find \( N^{(1)} \) such that all the resonance term is far away from the edge of \([- (N^{(1)})^2, (N^{(1)})^2] \). In this case, the Green’s function \( H_{[1, (N^{(1)})^2]} (x, \omega)(k, N^{(1)} + 1) \) will decay exponentially and we get exponentially decaying eigenfunctions.

**Lemma 6.3.** For any \( x \in \mathbb{T} \) and any \( \omega \) satisfying condition (6.1), there is \( N^{(1)}(x, \omega) \leq N^{2m_0-1} \) such that
\[ |V(x + j\omega) - V(x)| \geq \lambda^{-1/2} \]
for all
\[ j \in \left[ - \left( N^{(1)}(x, \omega) \right)^2, -N^{(1)}(x, \omega) \right] \cup \left[ N^{(1)}(x, \omega), \left( N^{(1)}(x, \omega) \right)^2 \right]. \]

**Proof.** If \( |V(x + j\omega) - V(x)| \geq \lambda^{-1/2} \) for all \( j \in [-N^2, -N) \cup (N, N^2) \) then we can take \( N^{(1)}(x, \omega) = N \).

So assume there is \( j_1 \in [-N^2, -N) \cup (N, N^2) \) such that \( |V(x + j_1\omega) - V(x)| < \lambda^{-1/2} \). If \( |V(x + j\omega) - V(x)| \geq \lambda^{-1/2} \) for all \( j \in [-N^2, -N) \cup (N, N^2) \) then we can choose \( N^{(1)}(x, \omega) = N^2 \).

So assume there is \( j_2 \in [-N^2, -N^2) \cup (N^2, N^4) \) such that \( |V(x + j_2\omega) - V(x)| < \lambda^{-1/2} \). Continue this process if there is \( j_3 \in [-N^2, -N^{2^{j_2}}) \cup (N^{2^{j_2}}, N^{2^{j_2+1}}) \) such that \( |V(x + j_3\omega) - V(x)| < \lambda^{-1/2} \). Since
\[ \# \left\{ 0 < |j| \leq N^{2m_0} : |V(x + j\omega) - V(x)| < \lambda^{-1/2} \right\} \leq m_0 - 1 \]
we can find \( N^{(1)}(x, \omega) \leq N^{2m_0-1} \) such that
\[ |V(x + j\omega) - V(x)| \geq \lambda^{-1/2} \]
for all
\[ j \in \left[ - \left( N^{(1)}(x, \omega) \right)^2, -N^{(1)}(x, \omega) \right] \cup \left[ N^{(1)}(x, \omega), \left( N^{(1)}(x, \omega) \right)^2 \right]. \] \( \square \)

Take \( K, L \in \mathbb{N} \) such that \( 20C_1 \lambda^{1/2} < K < 30C_1 \lambda^{1/2}, 20C_1 \lambda^{5/8} < L < 30C_1 \lambda^{5/8} \). Let \( \tilde{x}_i = \frac{i}{K}, i = 1, 2, \ldots, K \) and \( \tilde{\omega}_j = \frac{j}{L}, j = 1, 2, \ldots, L \). Define
\[ U_i = (\tilde{x}_i - \frac{1}{32C_1} \lambda^{-1/2}, \tilde{x}_i + \frac{1}{32C_1} \lambda^{-1/2}), \quad V_j = (\tilde{\omega}_j - \frac{1}{32C_1} \lambda^{-5/8}, \tilde{\omega}_j + \frac{1}{32C_1} \lambda^{-5/8}) \]
Then \( \mathbb{T} \times \mathbb{T} = \bigcup_{i,j} U_i \times V_j \). Moreover, there is \( \tilde{\omega}_j \in V_j \) satisying condition (6.1) for each \( j \).

**Lemma 6.4.** For each \( i, j \), there is \( N^{(1)}_{ij} \leq N^{2m_0-1} \) such that \( |V(x + k\omega) - V(x)| \geq \frac{7}{8} \lambda^{-1/2} \) for all \( (x, \omega) \in U_i \times V_j \), \( k \in \left[ - \left( N^{(1)}_{ij} \right)^2, -N^{(1)}_{ij} \right] \cup \left( N^{(1)}_{ij}, \left( N^{(1)}_{ij} \right)^2 \right) \).

**Proof.** Let \( N^{(1)}_{ij} = N^{(1)}(\tilde{x}_i, \tilde{\omega}_j) \). For any \( (x, \omega) \in U_i \times V_j \), \( k \in \left[ - \left( N^{(1)}_{ij} \right)^2, -N^{(1)}_{ij} \right] \cup \left( N^{(1)}_{ij}, \left( N^{(1)}_{ij} \right)^2 \right) \), we have
\[ |V(x + k\omega) - V(x)| \geq \left| V(\tilde{x}_i + k\tilde{\omega}_j) - V(\tilde{x}_i) \right| - \left| V(\tilde{x}_i + k\tilde{\omega}_j) - V(\tilde{x}_i + k\omega) \right| - \left| V(\tilde{x}_i + k\omega) - V(x + k\omega) \right| - |V(x) - V(\tilde{x}_i)| \geq \lambda^{-1/2} - C_1 |k(\tilde{\omega}_j - \omega)| - 2C_1 |\tilde{x}_i - x| \geq \frac{7}{8} \lambda^{-1/2} \] \( \square \)
Notation: \( \Lambda_{ij}^{(1)} = \left[-(N_{ij}^{(1)})^2, (N_{ij}^{(1)})^2\right] \).

Let
\[
J_{ij}^{(1)} := \# \left\{ \ell \in \Lambda_{ij}^{(1)} : |V(x_i + \ell \hat{\omega}_j) - V(x_i)| < \frac{1}{2} \lambda^{-1/2} \right\},
\]
\[
\leq \# \left\{ |\ell| \leq N^{2m_0} : |V(x_i + \ell \hat{\omega}_j) - V(x_i)| < \frac{1}{2} \lambda^{-1/2} \right\} \leq m_0.
\]

We have distinct eigenvalues
\[
E_{ijk}^{(1)}(x_i, \hat{\omega}_j) \in \text{sp} \ H_{\Lambda_{ij}^{(1)}}^{(1)}(x_i, \hat{\omega}_j), \quad 1 \leq k \leq J_{ij}^{(1)},
\]
such that there is \( \ell_k \in [-N_{ij}^{(1)}, N_{ij}^{(1)}] \), \( |E_{ijk}^{(1)}(x_i, \hat{\omega}_j) - \lambda V(x_i + \ell_k \hat{\omega}_j)| \leq 2, |V(x_i + \ell_k \hat{\omega}_j) - V(x_i)| < \frac{1}{2} \lambda^{-1/2} \).

By perturbation theory and Lemma A.1, we can define \( C^3 \) functions \( E_{ijk}^{(1)}(x, \omega), \varphi_{ijk}^{(1)}(x, \omega)(\ell), \ell \in \Lambda_{ij}^{(1)} \), for \((x, \omega) \in U_i \times V_j\) such that
\[
H_{\Lambda_{ij}^{(1)}}^{(1)}(x, \omega) \varphi_{ijk}^{(1)}(x, \omega) = E_{ijk}^{(1)}(x, \omega) \varphi_{ijk}^{(1)}(x, \omega)
\]
\[
\partial_x E_{ijk}^{(1)}(x, \omega) = \lambda \sum_{\ell \in \Lambda_{ij}^{(1)}} V'(x + \ell \omega)|\varphi_{ijk}^{(1)}(x, \omega)(\ell)|^2
\]
\[
\partial_\omega E_{ijk}^{(1)}(x, \omega) = \lambda \sum_{\ell \in \Lambda_{ij}^{(1)}} \ell V'(x + \ell \omega)|\varphi_{ijk}^{(1)}(x, \omega)(\ell)|^2
\]
\[
\sum_{\ell \in \Lambda_{ij}^{(1)}} |\varphi_{ijk}^{(1)}(x, \omega)(\ell)|^2 = 1
\]
Furthermore,
\[
|E_{ijk}^{(1)}(x, \omega) - \lambda V(x)| \leq |E_{ijk}^{(1)}(x, \omega) - E_{ijk}^{(1)}(x, \hat{\omega})| + |E_{ijk}^{(1)}(\hat{x}_i, \omega) - E_{ijk}^{(1)}(\hat{x}_i, \hat{\omega})|
\]
\[
+ |E_{ijk}^{(1)}(\hat{x}_i, \hat{\omega}) - \lambda V(x + \ell_k \hat{\omega})| + |\lambda V(x + \ell_k \hat{\omega}) - \lambda V(x)|
\]
\[
\leq (\lambda C_1)|x - \hat{x}_i| + (\lambda N^{2m_0} C_1)|\omega - \hat{\omega}| + 2 + \frac{1}{2} \lambda^{1/2}
\]
\[
\leq \frac{3}{4} \lambda^{3/4}
\]

Lemma 6.5. For any \((x, \omega) \in U_i \times V_j, N_{ij}^{(1)} < \ell \leq (N_{ij}^{(1)})^2\), one has
\[
\left(H \left( N_{ij}^{(1)}, (N_{ij}^{(1)})^2 \right)/(x, \omega) - E_{ijk}^{(1)}(x, \omega) \right)^{-1} (N_{ij}^{(1)} + 1, \ell) \lesssim \lambda^{-\frac{3}{4}(\ell - N_{ij}^{(1)})}.
\]

Proof.
\[
|\lambda V(x + m \omega) - E_{ijk}^{(1)}(x, \omega)| \geq |\lambda V(x + m \omega) - \lambda V(x)| - |\lambda V(x) - E_{ijk}^{(1)}(x, \omega)|
\]
\[
\geq \frac{7}{8} \lambda^{1/2} - \left( \frac{3}{4} \lambda^{1/2} \right) \geq \frac{1}{8} \lambda^{1/2}
\]
for any \( m \in (N_{ij}^{(1)}, (N_{ij}^{(1)})^2]\). Also \((N_{ij}^{(1)})^2 \leq N^{2m_0} \leq \lambda^{1/2}\). The assertion follows from Corollary C.8. \(\square\)

Corollary 6.6. \(|\varphi_{ijk}^{(1)}(x, \omega)(\ell)| \leq \lambda^{-\frac{3}{4}(\ell - N_{ij}^{(1)})}, N_{ij}^{(1)} < |\ell| \leq (N_{ij}^{(1)})^2\).

In the next few sections, we will derive some properties of \( E_{ijk}^{(1)} \). But we need to know that these are the only eigenvalues of \( H_{\Lambda_{ij}^{(1)}}(x, \omega) \) near \( V(x) \).
Lemma 6.7. For any \((x, \omega) \in U_i \times V_j\),
\[
\text{sp } H_{\Lambda_i}^1(x, \omega) \cap \left( \lambda V(x) - \frac{1}{4} \lambda^{1/2}, \lambda V(x) + \frac{1}{4} \lambda^{1/2} \right) \subset \bigcup_k \left\{ E_{ijk}^{(1)}(x, \omega) \right\}.
\]

Proof. By construction, \(\text{sp } H_{\Lambda_i}^1(\bar{x}_i, \omega_j) \cap \left( \lambda V(\bar{x}_i) - \frac{1}{2} \lambda^{1/2}, \lambda V(\bar{x}_i) + \frac{1}{2} \lambda^{1/2} - 2 \right) \subset \bigcup_k \left\{ E_{ijk}^{(1)}(\bar{x}_i, \omega_j) \right\}.
\]
Suppose
\[
\mu(x_0, \omega_0) \in \text{sp } H_{\Lambda_i}^1(x_0, \omega_0) \cap \left( \lambda V(x_0, \omega_0) - \frac{1}{4} \lambda^{1/2}, \lambda V(x_0, \omega_0) + \frac{1}{4} \lambda^{1/2} \right)
\]
for some \((x_0, \omega_0) \in U_i \times V_j\). Then we can define \(C^1\) function
\[
\mu(x, \omega) \in \text{sp } H_{\Lambda_i}^1(x, \omega), \quad |\partial_x \mu(x, \omega)| \leq \lambda C_1, \quad |\partial_{\omega} \mu(x, \omega)| \leq \lambda N^{2m_0} C_1
\]
on \(U_i \times V_j\). But
\[
|\mu(\bar{x}_i, \omega_j) - \lambda V(\bar{x}_i)| \leq \left| \mu(\bar{x}_i, \omega_j) - \mu(x_0, \omega_j) \right| + \left| \mu(x_0, \omega_0) - \mu(x_0, \omega_j) \right| + \left| \mu(x_0, \omega_0) - \lambda V(x_0) \right| + \left| \lambda V(x_0) - \lambda V(\bar{x}_i) \right|
\]
\[
\leq \lambda C_1 |\bar{x}_i - x_0| + \lambda N^{2m_0} C_1 |\omega_j - \omega_0| + \frac{1}{4} \lambda^{1/2} + \lambda C_1 |x_0 - \bar{x}_i|
\]
\[
\leq \frac{1}{2} \lambda^{1/2} - 2
\]
Hence \(\mu(\bar{x}_i, \omega_j) = E_{ijk}^{(1)}(\bar{x}_i, \omega_j)\) for some \(k\). Therefore \(\mu(x, \omega) = E_{ijk}^{(1)}(x, \omega)\).
\(\square\)

7. Separation of eigenvalues at first scale

We develop a method to find a lower estimate for the separation between eigenvalues of \(H_{\Lambda_i}(x, \omega)\). This method, first introduced in [GS2] for analytic potential, is based on the orthogonality of eigenfunctions corresponding to different eigenvalues of the self-adjoint matrix \(H_{\Lambda_i}(x, \omega)\).

Lemma 7.1. Let \((x, \omega) \in U_i \times V_j\), \(E_1, E_2 \in (\lambda V(x) - \frac{3}{4} \lambda^{1/2}, \lambda V(x) + \frac{3}{4} \lambda^{1/2})\). Then for any \(n \in \left(-\left(N_i^{(1)}\right)^2, -N_i^{(1)}\right)\), one has
\[
|\log |f_{\left(-\left(N_i^{(1)}\right)^2, n\right]}(x, \omega, E_1)| - \log |f_{\left(-\left(N_i^{(1)}\right)^2, n\right]}(x, \omega, E_2)|| \leq |E_1 - E_2|.
\]

Proof. Let \(\left\{ \mu_m \right\}_{m=-\left(N_i^{(1)}\right)^2}^n = \text{sp } H_{\left(-\left(N_i^{(1)}\right)^2, n\right]}(x, \omega), \quad |\mu_m - \lambda V(x + m\omega)| \leq 2.\) Then for any \(E \in (\lambda V(x) - \frac{3}{4} \lambda^{1/2}, \lambda V(x) + \frac{3}{4} \lambda^{1/2})\)
\[
|\mu_m - E| \geq |\lambda V(x + m\omega) - \lambda V(x)| - |\lambda V(x + m\omega) - \mu_m| - |\lambda V(x) - E|
\]
\[
\geq \frac{7}{8} \lambda^{1/2} - 2 - \frac{3}{4} \lambda^{1/2}
\]
\[
\geq \frac{1}{16} \lambda^{1/2}.
\]
For each \(m\), there exists \(E \in (E_1, E_2) \subset (\lambda V(x) - \frac{1}{4} \lambda^{1/2}, \lambda V(x) + \frac{1}{4} \lambda^{1/2})\) such that
\[
|\log |\mu_m - E_1| - \log |\mu_m - E_2|| = \frac{|E_1 - E_2|}{|\mu_m - E|} \leq 2 \lambda^{-1/2} |E_1 - E_2|
\]
\[
|\log |f_{\left(-\left(N_i^{(1)}\right)^2, n\right]}(x, \omega, E_1)| - \log |f_{\left(-\left(N_i^{(1)}\right)^2, n\right]}(x, \omega, E_2)|| \leq \sum_m |\log |\mu_m - E_1| - \log |\mu_m - E_2||
\]
\[
\leq \left(N_i^{(1)}\right)^2 |16 \lambda^{-1/2} |E_1 - E_2|
\]
\[
\leq \lambda^{1/8} |16 \lambda^{-1/2} |E_1 - E_2|
\]
\[
\leq |E_1 - E_2|.
Corollary 7.2. Let \((x, \omega), E_1, E_2, n\) be as in Lemma 7.1. Then

\[
|f_{[-(N_{ij}^{(1)})^2, n]}(x, \omega, E_1) - f_{[-(N_{ij}^{(1)})^2, n]}(x, \omega, E_2)| \leq |E_1 - E_2|e^{E_1 - E_2} |f_{[-(N_{ij}^{(1)})^2, n]}(x, \omega, E_1)|.
\]

Proof. With \(\{\mu_m\}\) as in proof of Lemma 7.1, note that \(\text{sgn}(\mu_j - E_1) = \text{sgn}(\mu_j - E_2)\). So

\[
\text{sgn} f_{[-(N_{ij}^{(1)})^2, n]}(x, \omega, E_1) = \text{sgn} f_{[-(N_{ij}^{(1)})^2, n]}(x, \omega, E_2).
\]

Hence, there is \(\Gamma\) between \(|f_{[-(N_{ij}^{(1)})^2, n]}(x, \omega, E_1)|\) and \(|f_{[-(N_{ij}^{(1)})^2, n]}(x, \omega, E_2)|\) such that

\[
|f_{[-(N_{ij}^{(1)})^2, n]}(x, \omega, E_1) - f_{[-(N_{ij}^{(1)})^2, n]}(x, \omega, E_2)| = \log |f_{[-(N_{ij}^{(1)})^2, n]}(x, \omega, E_1)| - \log |f_{[-(N_{ij}^{(1)})^2, n]}(x, \omega, E_2)| \leq |E_1 - E_2| \cdot \Gamma.
\]

But

\[
\Gamma \leq \max \left\{ |f_{[-(N_{ij}^{(1)})^2, n]}(x, \omega, E_1)|, |f_{[-(N_{ij}^{(1)})^2, n]}(x, \omega, E_2)| \right\} \leq e^{E_1 - E_2} |f_{[-(N_{ij}^{(1)})^2, n]}(x, \omega, E_1)|.
\]

Lemma 7.3. For any \((x, \omega) \in U_1 \times V_j, 1 \leq k \leq J_{ij}^{(1)},\)

\[
\sum_{N_{ij}^{(1)} < |n| \leq (N_{ij}^{(1)})^2} |f_{[-(N_{ij}^{(1)})^2, n]}(x, \omega, E_{ijk}^{(1)}(x, \omega))|^2 < 4\lambda^{-1} \sum_{n \in A_{ij}^{(1)}} |f_{[-(N_{ij}^{(1)})^2, n]}(x, \omega, E_{ijk}^{(1)}(x, \omega))|^2.
\]

Proof. \(\{f_{[-(N_{ij}^{(1)})^2, n]}(x, \omega, E_{ijk}^{(1)}(x, \omega))\}_{n \in A_{ij}^{(1)}}\) satisfies the Schrödinger equation with zero boundary condition.

Hence, there is \(\mu \in \mathbb{R}\) such that

\[
f_{[-(N_{ij}^{(1)})^2, n]}(x, \omega, E_{ijk}^{(1)}(x, \omega)) = \mu \varphi_{ijk}^{(1)}(x, \omega)(n).
\]

By Corollary 6.6,

\[
\sum_{N_{ij}^{(1)} < |n| \leq (N_{ij}^{(1)})^2} |f_{[-(N_{ij}^{(1)})^2, n]}(x, \omega, E_{ijk}^{(1)}(x, \omega))|^2 = \mu^2 \sum_{N_{ij}^{(1)} < |n| \leq (N_{ij}^{(1)})^2} |\varphi_{ijk}^{(1)}(x, \omega)(n)|^2
\]

\[
\leq 2\mu^2 \sum_{n > N_{ij}^{(1)}} \lambda^{-1}|n - N_{ij}^{(1)}|
\]

\[
\leq 2\mu^2 \frac{\lambda^{-1}}{1 - \lambda^{-1}}
\]

\[
< 4\lambda^{-1} \mu^2
\]

\[
= 4\lambda^{-1} \sum_{n \in A_{ij}^{(1)}} |f_{[-(N_{ij}^{(1)})^2, n]}(x, \omega, E_{ijk}^{(1)}(x, \omega))|^2.
\]

Lemma 7.4. For any \((x, \omega) \in U_1 \times V_j, |E_{ijk}^{(1)}(x, \omega) - E_{ij\ell}^{(1)}(x, \omega)| > e^{-|A_{ij}^{(1)}|^{3/4}}\) if \(k \neq \ell\).
Proof. For any $n \in [-N_{ij}^{(1)}, N_{ij}^{(1)}]$, 

$$
|f_{[-(N_{ij}^{(1)})^2,n]}(x,\omega, E_{ijk}^{(1)}(x)) - f_{[-(N_{ij}^{(1)})^2,n]}(x,\omega, E_{i\ell j}^{(1)}(x,\omega))| 
\leq \left\| \left( f_{[-(N_{ij}^{(1)})^2,n]}(x,\omega, E_{ijk}^{(1)}(x,\omega)) \right) - \left( f_{[-(N_{ij}^{(1)})^2,n]}(x,\omega, E_{i\ell j}^{(1)}(x,\omega)) \right) \right\| 
= \left\| M_{[-N_{ij}^{(1)}-1,n-1]}(x,\omega, E_{ijk}^{(1)}(x,\omega)) \left( f_{[-(N_{ij}^{(1)})^2,-N_{ij}^{(1)}-1]}(x,\omega, E_{i\ell j}^{(1)}(x,\omega)) \right) - M_{[-N_{ij}^{(1)}-1,n-1]}(x,\omega, E_{i\ell j}^{(1)}(x,\omega)) \left( f_{[-(N_{ij}^{(1)})^2,-N_{ij}^{(1)}-2]}(x,\omega, E_{i\ell j}^{(1)}(x,\omega)) \right) \right\| 
\leq \left\| M_{[-N_{ij}^{(1)}-1,n-1]}(x,\omega, E_{ijk}^{(1)}(x,\omega)) \right\| + \left\| M_{[-N_{ij}^{(1)}-1,n-1]}(x,\omega, E_{i\ell j}^{(1)}(x,\omega)) \right\| 
= \left\| f_{[-(N_{ij}^{(1)})^2,N_{ij}^{(1)}]}(x,\omega, E_{ijk}^{(1)}(x,\omega)) \right\| + \left\| f_{[-(N_{ij}^{(1)})^2,-N_{ij}^{(1)}-2]}(x,\omega, E_{i\ell j}^{(1)}(x,\omega)) \right\| 
\leq |E_{ijk}^{(1)}(x,\omega) - E_{i\ell j}^{(1)}(x,\omega)| (\lambda C_2)^{n+N_{ij}^{(1)}} |\mu| 
+ (\lambda C_2)^{n+N_{ij}^{(1)}+1} \left| E_{ijk}^{(1)}(x,\omega) - E_{i\ell j}^{(1)}(x,\omega) \right| e^{|E_{ijk}^{(1)}(x,\omega) - E_{i\ell j}^{(1)}(x,\omega)| |\mu|}
$$

where $C_2 = \log(2\max_{y \in T} |V(y)| + 1)$, $|\mu|^2 = \sum_{m \in \Lambda_{ij}^{(1)}} |f_{[-(N_{ij}^{(1)})^2,n]}(x,\omega, E_{ijk}^{(1)}(x,\omega))|^2$. ($\mu$ is the same as in the proof of Lemma 7.3.)

If $|E_{ijk}^{(1)}(x,\omega) - E_{i\ell j}^{(1)}(x,\omega)| \leq e^{-|\Lambda_{ij}^{(1)}|^{3/4}} = e^{-|\Lambda_{ij}^{(1)}|^{3/4}}$ then

$$
|f_{[-(N_{ij}^{(1)})^2,n]}(x,\omega, E_{ijk}^{(1)}(x,\omega)) - f_{[-(N_{ij}^{(1)})^2,n]}(x,\omega, E_{i\ell j}^{(1)}(x,\omega))| \leq e^{-\frac{2}{3}|\Lambda_{ij}^{(1)}|^{-3/4}} |\mu|
$$

for all $n \in [-N_{ij}^{(1)}, N_{ij}^{(1)}]$. 


Therefore,
\[
\sum_{n \in \Lambda_{ij}^{(1)}} \left| f_{[-(N_{ij}^{(1)})^2,n]}(x,\omega, E_{ijk}^{(1)}(x,\omega)) - f_{[-(N_{ij}^{(1)})^2,n]}(x,\omega, E_{ij\ell}^{(1)}(x,\omega)) \right|^2 \\
= \sum_{N_{ij}^{(1)} < |n| \leq (N_{ij}^{(1)})^2} \left| f_{[-(N_{ij}^{(1)})^2,n]}(x,\omega, E_{ijk}^{(1)}(x,\omega)) - f_{[-(N_{ij}^{(1)})^2,n]}(x,\omega, E_{ij\ell}^{(1)}(x,\omega)) \right|^2 \\
+ \sum_{|n| \leq (N_{ij}^{(1)})^2} \left| f_{[-(N_{ij}^{(1)})^2,n]}(x,\omega, E_{ijk}^{(1)}(x,\omega)) - f_{[-(N_{ij}^{(1)})^2,n]}(x,\omega, E_{ij\ell}^{(1)}(x,\omega)) \right|^2 \\
\leq \sum_{N_{ij}^{(1)} < |n| \leq (N_{ij}^{(1)})^2} \left[ |f_{[-(N_{ij}^{(1)})^2,n]}(x,\omega, E_{ijk}^{(1)}(x,\omega))|^2 + |f_{[-(N_{ij}^{(1)})^2,n]}(x,\omega, E_{ij\ell}^{(1)}(x,\omega))|^2 \right] \\
+ (2N_{ij}^{(1)} + 1)e^{-|\Lambda_{ij}^{(1)}|/|\mu|^2} \\
\leq 4\lambda^{-1} \sum_{n \in \Lambda_{ij}^{(1)}} \left[ |f_{[-(N_{ij}^{(1)})^2,n]}(x,\omega, E_{ijk}^{(1)}(x,\omega))|^2 + |f_{[-(N_{ij}^{(1)})^2,n]}(x,\omega, E_{ij\ell}^{(1)}(x,\omega))|^2 \right] \\
+ (2N_{ij}^{(1)} + 1)e^{-|\Lambda_{ij}^{(1)}|/|\mu|^2} \\
\leq \lambda^{-1/2} \sum_{n \in \Lambda_{ij}^{(1)}} \left[ |f_{[-(N_{ij}^{(1)})^2,n]}(x,\omega, E_{ijk}^{(1)}(x,\omega))|^2 + |f_{[-(N_{ij}^{(1)})^2,n]}(x,\omega, E_{ij\ell}^{(1)}(x,\omega))|^2 \right].
\]

But \( E_{ijk}^{(1)}(x,\omega) \neq E_{ij\ell}^{(1)}(x,\omega) \implies \left\{ f_{[-(N_{ij}^{(1)})^2,n]}(x,\omega, E_{ijk}^{(1)}(x,\omega)) \right\}_{n \in \Lambda_{ij}^{(1)}} \) and \( \left\{ f_{[-(N_{ij}^{(1)})^2,n]}(x,\omega, E_{ij\ell}^{(1)}(x,\omega)) \right\}_{n \in \Lambda_{ij}^{(1)}} \) are orthogonal. Hence
\[
\sum_{n \in \Lambda_{ij}^{(1)}} \left| f_{[-(N_{ij}^{(1)})^2,n]}(x,\omega, E_{ijk}^{(1)}(x,\omega)) - f_{[-(N_{ij}^{(1)})^2,n]}(x,\omega, E_{ij\ell}^{(1)}(x,\omega)) \right|^2 \\
= \sum_{n \in \Lambda_{ij}^{(1)}} \left[ |f_{[-(N_{ij}^{(1)})^2,n]}(x,\omega, E_{ijk}^{(1)}(x,\omega))|^2 + |f_{[-(N_{ij}^{(1)})^2,n]}(x,\omega, E_{ij\ell}^{(1)}(x,\omega))|^2 \right].
\]

This contradiction shows that
\[
|E_{ijk}^{(1)}(x,\omega) - E_{ij\ell}^{(1)}(x,\omega)| > e^{-|\Lambda_{ij}^{(1)}|/|\mu|^2}.
\]

8. Variations of Potential

In this section we introduce variation of potential and the notion of “typical” variations. We then derive some properties of \( E_{ijk}^{(1)} \) for typical variations.

**Definition.** Let \( T \) be a large integer, \( 0 < \delta \ll \frac{1}{T} \). Suppose \( R_m(\eta_m, \xi_m, \theta_m; x) \) are \( C^3 \) functions, \( m = 1, 2, \ldots, T \), \( (\eta, \xi, \theta) \in \prod_{1}^{3T}[\delta, \delta] \), \( x \in T \), satisfying the following conditions:

\[
\begin{align*}
(8.1) \quad & |\partial_\alpha R_m(\eta_m, \xi_m, \theta_m; x)| \leq \frac{1}{T} \quad \text{for any index } |\alpha| \leq 3 \\
(8.2) \quad & R_m(0,0,0;x) = 0 \\
(8.3) \quad & R_m(\eta_m, \xi_m, \theta_m; x) = -x^{-3}(\eta_m + \xi_m x + \frac{1}{2}\theta_m x^2) \quad \text{for } |x| \geq \frac{1}{2T}.
\end{align*}
\]
Define a \((T, \delta)\)-variation of potential by
\[
W(\eta, \xi, \theta; \{R_m\}; x) = \sum_{m=1}^{T} v_m \left( \eta_m, \xi_m, \theta_m; x - \frac{m}{T} \right)
\]
where
\[
v_m(\eta_m, \xi_m, \theta_m; x) = \eta_m + \xi_m x + \frac{1}{2} \theta_m x^2 + x^3 R_m(\eta_m, \xi_m, \theta_m; x)
\]

By (8.2) and (8.3),
\[
v_m(0, 0, \{R_m\}; x) \equiv 0
\]
and
\[
v_m(\eta_m, \xi_m, \theta_m, \{R_m\}; x) = 0 \quad \text{for } |x| \geq \frac{1}{2T}.
\]

Denote the collection of \((T, \delta)\)-variations of potential by \(S(T, \delta)\). A set \(S \subset S(T, \delta)\) is called \((1-\varepsilon)\)-typical if
\[
|S| := \min_{\{R_m\}} \frac{1}{(2\delta)^{3T}} \text{mes}\left\{ (\eta, \xi, \theta) \in [-\delta, \delta]^{3T} : v(\eta, \xi, \theta, \{R_m\};.) \in S \right\} \geq 1 - \varepsilon
\]

\textbf{Remark 8.1.} We assume \(\delta \ll \lambda^{-1/2}\). Since \(\max_{y \in \mathbb{Z}} |\tilde{V}(y) - V(y)| \lesssim \delta\), one has
\[
|\tilde{V}(x + j\omega) - \tilde{V}(x)| \geq \frac{7}{8} \lambda^{-1/2} \quad \text{for } j \in \Lambda_{ij}^{(1)} \setminus [-N_{ij}^{(1)}, N_{ij}^{(1)}]
\]

for any \((x, \omega) \in U_i \times V_j\). Hence, we can define \(\tilde{E}_{ijk}^{(1)}(x, \omega)\) and \(\tilde{\varphi}_{ijk}^{(1)}(x, \omega)\) on \(U_i \times V_j\), such that
\[
\tilde{H}_{\tilde{\Lambda}_{ij}^{(1)}}(x, \omega) \tilde{\varphi}_{ijk}^{(1)}(x, \omega) = \tilde{E}_{ijk}^{(1)}(x, \omega) \tilde{\varphi}_{ijk}^{(1)}(x, \omega) , \quad \|	ilde{\varphi}_{ijk}^{(1)}(x, \omega)\| = 1
\]
\(\tilde{E}_{ijk}^{(1)}(x, \omega)\) and \(\tilde{\varphi}_{ijk}^{(1)}(x, \omega)\) have the same properties of \(E_{ijk}^{(1)}(x, \omega)\) and \(\varphi_{ijk}^{(1)}(x, \omega)\), shown in Section 6 and Section 7.

\textbf{Remark 8.2.} These functions depend on \(\{\eta_m\}, \{\xi_m\}, \{\theta_m\}\). It is for the simplicity of notations that the dependence is not explicitly written.

For the remainder of this section, we fix \((\tilde{x}, \tilde{\omega}) \in U_i \times V_j\), \(\tilde{\omega}\) satisfies condition (6.1).

\textbf{Remark 8.3.} We can define an one-to-one correspondence between \(\Lambda_{ij}^{(1)}\) and a subset of \([1, T]\) as follow: For any \(\ell \in \Lambda_{ij}^{(1)}\), there is unique \(\ell' \in [1, T] \cap \mathbb{N}\) such that \(-\frac{1}{2T} \leq \{\tilde{x} + \ell \tilde{\omega}\} - \frac{|\tilde{x}|}{T} \leq \frac{1}{2T}\). If \(m, \ell \in \Lambda_{ij}^{(1)}, m \neq \ell,\) then the Diophantine condition implies \(m' \neq l'\).

\textbf{Remark 8.4.} If \(p \neq q'\) for all \(q \in \Lambda_{ij}^{(1)}\), then \(\partial_{\tilde{x}} \tilde{H}_{\tilde{\Lambda}_{ij}^{(1)}}(\tilde{x}, \tilde{\omega}) = 0\).

Let \(g = \frac{1}{2} \text{dist}(\tilde{E}_{ijk}^{(1)}(\tilde{x}, \tilde{\omega}), \text{sp} \tilde{H}_{\tilde{\Lambda}_{ij}^{(1)}}(\tilde{x}, \tilde{\omega}) \setminus \{E_{ijk}^{(1)}(\tilde{x}, \tilde{\omega})\}) > \frac{1}{2} e^{-|\Lambda_{ij}^{(1)}|^{3/4}}\). We now show that if the eigenvalues are separated, then we have control how much the eigenfunctions change when we perturbed the potential.

\textbf{Lemma 8.5.}
\[
\|\partial_{\tilde{x}} \tilde{\varphi}_{ijk}^{(1)}(\tilde{x}, \tilde{\omega})\| \leq \begin{cases} \frac{\lambda}{2T} & \text{if } \exists q \in \Lambda_{ij}^{(1)} \text{ such that } q' = p \\ 0 & \text{otherwise} \end{cases}
\]
\[
\|\partial_{\tilde{\omega}} \tilde{\varphi}_{ijk}^{(1)}(\tilde{x}, \tilde{\omega})\| \leq \begin{cases} \frac{\lambda}{2T} & \text{if } \exists q \in \Lambda_{ij}^{(1)} \text{ such that } q' = p \\ 0 & \text{otherwise} \end{cases}
\]
\[
\|\partial_{\tilde{x}} \partial_{\tilde{\omega}} \tilde{\varphi}_{ijk}^{(1)}(\tilde{x}, \tilde{\omega})\| \leq \begin{cases} \frac{\lambda}{2T} & \text{if } \exists q \in \Lambda_{ij}^{(1)} \text{ such that } q' = p \\ 0 & \text{otherwise} \end{cases}
\]
Proposition 8.6. \[ \begin{align*}
\eta & = (\eta_1, \ldots, \eta_T) \subset (-\delta, \delta)^T. \text{ The next lemma shows that for “typical” } \xi, \theta, \text{ the eigenvalues are Morse functions.} \\
\end{align*}\]

Proof. By Lemma A.2,
\[ \left\| \partial_{x^p} \tilde{\varphi}_{ijk}^{(1)}(\tilde{x}, \tilde{\omega}) \right\| \leq \left\| \frac{-1}{2\pi i} \oint_{|z - \tilde{\varphi}_{ijk}^{(1)}(\tilde{x}, \tilde{\omega})| = \rho} \left[ z - \tilde{H}_\Lambda^{(1)}(\tilde{x}, \tilde{\omega}) \right]^{-1} \partial_{\xi^p} \tilde{H}_\Lambda^{(1)}(\tilde{x}, \tilde{\omega}) \left[ z - \tilde{H}_\Lambda^{(1)}(\tilde{x}, \tilde{\omega}) \right]^{-1} dz \right\| \]
\[ \leq \begin{cases} 
\frac{1}{\rho} (2\pi \rho) \frac{\Lambda^{(1)} T}{\theta^2} & \text{if } \exists \, \rho \in \Lambda^{(1)}_i \text{ such that } \rho' = \rho \\
0 & \text{otherwise}
\end{cases} \]
The proofs for the other two inequalities are similar. \[ \square \]

Fix \( \eta = (\eta_1, \ldots, \eta_T) \subset (-\delta, \delta)^T. \) The next lemma shows that for “typical” \( \xi, \theta, \) the eigenvalues are Morse functions.

Proposition 8.6. For \( \lambda^2 \frac{\Lambda^{(1)}_i}{\theta} \ll \varepsilon \ll 1, \) one has
\[ \text{mes}\left\{ (\xi, \theta) \in (-\delta, \delta)^2 : \left| \partial_{x^p} \tilde{E}_{ijk}^{(1)}(\tilde{x}, \tilde{\omega}) \right| \leq \varepsilon, \left| \partial_{\xi^p} \tilde{E}_{ijk}^{(1)}(\tilde{x}, \tilde{\omega}) \right| \leq \varepsilon \right\} \leq \left( \Lambda^{(1)}_i \lambda \right)^2 \]

Proof. Let \( \tilde{\varphi}_{00}(x, \omega) = \tilde{\varphi}_{ijk}^{(1)}(x, \omega) \bigg|_{(\xi, \phi) = (0, 0)}, \tilde{E}_{00}(x, \omega) = \tilde{E}_{ijk}^{(1)}(x, \omega) \bigg|_{(\xi, \phi) = (0, 0)} \)
\[ \lambda^{-1} \partial_{x^p} \tilde{E}_{00}(\tilde{x}, \tilde{\omega}) = \sum_{\ell \in \Lambda^{(1)}_i} \partial_{x^p} \tilde{V}(\tilde{x} + \ell \tilde{\omega}) \tilde{\varphi}_{ijk}^{(1)}(\tilde{x}, \tilde{\omega})(\ell) \left| \tilde{\varphi}_{ijk}^{(1)}(\tilde{x}, \tilde{\omega})(\ell) \right|^2 \]
\[ = \sum_{\ell \in \Lambda^{(1)}_i} \left[ V'(\tilde{x} + \ell \tilde{\omega}) + \xi_\ell + O\left(\frac{\delta}{T}\right) \right] \left[ \tilde{\varphi}_{00}(\tilde{x}, \tilde{\omega})(\ell) \right] + O(\frac{\Lambda^{(1)}_i \delta}{\theta T}) \]
\[ = \lambda^{-1} \partial_{x^p} \tilde{E}_{00}(\tilde{x}, \tilde{\omega}) + \sum_{\ell \in \Lambda^{(1)}_i} \xi_\ell \left| \tilde{\varphi}_{00}(\tilde{x}, \tilde{\omega})(\ell) \right|^2 + O(\frac{\Lambda^{(1)}_i \delta}{\theta T}) \]

By Lemma F.1,
\[ (2\delta)^{-T} \text{mes}\left\{ \xi : \left| \lambda^{-1} \partial_{x^p} \tilde{E}_{00}(\tilde{x}, \tilde{\omega}) + \sum_{\ell \in \Lambda^{(1)}_i} \xi_\ell \left| \tilde{\varphi}_{00}(\tilde{x}, \tilde{\omega})(\ell) \right|^2 \right| \leq \lambda^{-1} \varepsilon \right\} \]
\[ = (2\delta)^{-\Lambda^{(1)}_i} \text{mes}\left\{ \xi_\ell \in \Lambda^{(1)}_i : \left| \lambda^{-1} \partial_{x^p} \tilde{E}_{00}(\tilde{x}, \tilde{\omega}) + \sum_{\ell \in \Lambda^{(1)}_i} \xi_\ell \left| \tilde{\varphi}_{00}(\tilde{x}, \tilde{\omega})(\ell) \right|^2 \right| \leq \lambda^{-1} \varepsilon \right\} \]
\[ \leq \Lambda^{(1)}_i \lambda^{-1} \varepsilon \delta. \]

This expression does not depend on \( \theta. \) So
\[ \left\{ (\xi, \theta) : \left| \partial_{x^p} \tilde{E}_{ijk}^{(1)}(\tilde{x}, \tilde{\omega}) \right| \leq \varepsilon \right\} \subset \Xi \times (-\delta, \delta)^{\Lambda^{(1)}_i} \]
where \( (2\delta)^{-T} \text{mes} \Xi \leq \Lambda^{(1)}_i \lambda^{-1} \varepsilon \delta. \)
Fix $\xi_0 \in \Xi$. Let $\tilde{\varphi}_0(x, \omega) = \tilde{\varphi}^{(1)}_{ijk}(x, \omega)|_{\theta = 0}$, $\tilde{E}_0(x, \omega) = \tilde{E}^{(1)}_{ijk}(x, \omega)|_{\theta = 0}$

\[
\lambda^{-1}\partial_{xx}\tilde{E}^{(1)}_{ijk}(\tilde{x}, \tilde{\omega}) = \sum_{\ell \in \Lambda^{(1)}_{ij}} \partial_{xx} \tilde{V}(\tilde{x} + \ell \tilde{\omega})|_{\varphi^{(1)}_{ijkl}(\tilde{x}, \tilde{\omega})(\ell)|^2 + 2 \sum_{\ell \in \Lambda^{(1)}_{ij}} \partial_x \tilde{V}(\tilde{x} + \ell \tilde{\omega})|_{\varphi^{(1)}_{ijkl}(\tilde{x}, \tilde{\omega})(\ell)| \partial_x \varphi_{ijkl}(\tilde{x}, \tilde{\omega})(\ell)
\]

\[
= \sum_{\ell \in \Lambda^{(1)}_{ij}} \left[ V''(\tilde{x} + \ell \tilde{\omega}) + O\left(\frac{1}{T}\right) \right] \left[ |\tilde{\varphi}_0(\tilde{x}, \tilde{\omega})(\ell)|^2 + O\left(\frac{|A^{(1)}_{ij}|}{\delta \lambda} \right) \right] + 2 \sum_{\ell \in \Lambda^{(1)}_{ij}} \left[ V'(\tilde{x} + \ell \tilde{\omega}) + O\left(\frac{\delta}{T}\right) \right] \left[ |\tilde{\varphi}_0(\tilde{x}, \tilde{\omega})(\ell)| + O\left(\frac{|A^{(1)}_{ij}|}{\delta \lambda} \right) \right] \partial_x \tilde{\varphi}_0(\tilde{x}, \tilde{\omega})(\ell)
\]

\[
= \lambda^{-1}\partial_{xx}\tilde{E}_0(\tilde{x}, \tilde{\omega}) + \sum_{\ell \in \Lambda^{(1)}_{ij}} \theta_{\ell} |\tilde{\varphi}_0(\tilde{x}, \tilde{\omega})(\ell)|^2 + O\left(\frac{|A^{(1)}_{ij}|^2}{\delta \lambda} \right)
\]

Similar calculation shows that

\[
(2\delta)^{-T} \operatorname{mes}\left\{ \theta : |\partial_{xx}\tilde{E}^{(1)}_{ijk}(\tilde{x}, \tilde{\omega})| \leq \varepsilon \text{ for fix } \xi_0 \in \Xi \right\} \leq |A^{(1)}_{ij}| \frac{\lambda^{-1} - \varepsilon}{\delta}
\]

\]
there is a set $\Omega_s = \Omega_s(\tilde{V}) \subset \Omega_{s-1} \subset \mathbb{T}$, $\text{mes}(\Omega_{s-1}\setminus\Omega_s) \leq e^{-N^s/2}$, satisfying the following inductive hypothesis:

For each $i, j$ there is $\Lambda_{ij}^{(s)} = \left[-\left(N_{ij}^{(s)}\right)^2, (N_{ij}^{(s)})^2\right]$ and $C^3$ functions $\tilde{E}_{ijk}^{(s)}$ and $\varphi_{ijk}^{(s)}$ defined on $U_i \times V_j$,

$$\tilde{H}_{\Lambda_{ij}^{(s)}}(x, \omega)\varphi_{ijk}^{(s)}(x, \omega) = \tilde{E}_{ijk}^{(s)}(x, \omega)\varphi_{ijk}^{(s)}(x, \omega)$$

such that, for any $(x, \omega) \in U_i \times \Omega_s \cap V_j$,

1. $\tilde{H}_{\Lambda_{ij}^{(s)}}(x, \omega) \cap \left(\lambda V(x) - \frac{1}{2}\lambda^{1/2}, \lambda + \frac{1}{2}\lambda^{1/2}\right) = \{\tilde{E}_{ijk}^{(s)}(x, \omega)\}_{k=1}^{K_{ij}^{(s)}}, \quad K_{ij}^{(s)} \leq N_{ij}^{(s)}$;

2. If $E \in (\lambda V(x) - \frac{1}{2}\lambda^{1/2}, \lambda V(x) + \frac{1}{2}\lambda^{1/2})$, dist $(E, \text{sp} H_{\Lambda_{ij}^{(s)}}(x, \omega)) < e^{-N^s}$ then for any $N \approx (N_{ij}^{(s)})^2$ one has
   $$\log \left|\left[H_{[1, N]}(x, \omega) - E\right]^{-1}(n, 1)\right| \leq -\frac{1}{4}n \log \lambda$$

for $N_{ij}^{(s)} < n \leq N$. In particular, $|\tilde{E}_{ijk}^{(s)}(x, \omega)(n)| \leq \lambda^{-\frac{1}{2}|\{k\}|^{-N_{ij}^{(s)}}}$ if $N_{ij}^{(s)} < |n| \leq (N_{ij}^{(s)})^2$;

3. $|\tilde{H}_{\Lambda_{ij}^{(s)}}^{(s)}(x, \omega) - \tilde{E}_{ijk}^{(s)}(x, \omega)| > e^{-1/4}\left|N_{ij}^{(s)}\right|^2$ if $k \neq \ell$;

4. $|\partial_x\tilde{E}_{ijk}^{(s)}(x, \omega)| + |\partial_x\tilde{E}_{ijk}^{(s)}(x, \omega)| \geq e^{-N^s/2}$.

The above conditions, for $s=1$, are proven in Section 6–8.

Fix $V^{(s)} := \tilde{V}$ in the form of (9.1). To simplify notations, we will write $E_{ijk}(x, \omega)$ for $\tilde{E}_{ijk}(x, \omega)$ and $\varphi_{ijk}(x, \omega)$ for $\tilde{\varphi}_{ijk}(x, \omega)$. Let $N_{s+1} \approx e^{N^s}$.

To begin the inductive procedure, we use results from Appendix E to eliminate a set of frequencies to avoid the situation where the distance between $\text{sp} H_{\Lambda_{ij}^{(s)}}(x, \omega)$ and $\text{sp} H_{N_{s+1}}(x + n\omega, \omega)$ is small. After that, we will be able to apply the Avalanche Principle.

**Lemma 9.1.** There exists $B^{(s+1)} \subset O_s$, $\text{mes} B^{(s+1)} \leq e^{-\frac{1}{2}N^s}$, such that for each $U_i \times V_j$, there is $N_{ij}^{(s+1)}$, $\log N_{ij}^{(s+1)} \approx \log N_{s+1}$, so that dist $(E, \text{sp} H_{\Lambda_{ij}^{(s)}}(x + n\omega, \omega)) > \frac{1}{2}e^{-N_{ij}^{(s+1)}}$ whenever $|E - E_{ijk}^{(s)}(x, \omega)| < \frac{\epsilon}{2}$ for any $x \in U_i$, $x + n\omega \in U_p$, $\omega \in V_j \cap \Omega^{(s)} \setminus B^{(s+1)}$, for all $n \in \left[-(N_{ij}^{(s+1)})^2, -(N_{ij}^{(s+1)})^2\right]$.

**Proof.** If $|E_{ijk}^{(s)}(x, \omega) - E_{ijk}^{(s)}(x + n\omega, \omega)| > \epsilon$ for all $p, r$, $N_{s+1} < |n| \leq (N_{s+1})^2$, $(x, w) \in U_i \times V_j \cap \Omega^{(s)}$ then choose $N_{ij}^{(s+1)} = N_{s+1}$. Otherwise, let

$$F_1(x, \omega) = E_{ijk}^{(s)}(x, \omega) - E_{ijk}^{(s)}(x + n_1\omega, \omega)$$

$$F_2(x, \omega) = E_{ijk}^{(s)}(x + n_1\omega, \omega) - E_{ijk}^{(s)}(x + n_2\omega, \omega)$$

where $N_{s+1} \leq |n_1| \leq (N_{s+1})^2$, $(N_{s+1})^4 \leq |n_2| \leq (N_{s+1})^8$.

Since $|\partial_x E_{ijk}^{(s)}(x, \omega)| + |\partial_x E_{ijk}^{(s)}(x, \omega)| \geq e^{-N^s/2}$ by Theorem E.6,

$$\text{mes} B_j^{(s+1)} := \text{mes} \left\{\omega \in V_j \cap \Omega_s : \exists x \in U_i, x + n_1\omega \in U_k \text{ such that } |F_1(x, \omega)| \leq \epsilon/2, |F_2(x + n_1\omega, \omega)| \leq \epsilon/2 \right\} \leq \epsilon^2 |U| |V_j|$$

Let $B^{(s+1)} = \sum_j B_j^{(s+1)}$,

$$\text{mes}(O_s \setminus B^{(s+1)}) \leq \sum_{i, j} \epsilon^3 |U_i| |V_j|$$

$$\leq \lambda^2 \epsilon \leq \epsilon^{\theta/2}.$$ 

□
Lemma 9.2. Let \( (x, \omega) \in U \times V \cap \Omega_s \setminus B^{(s+1)} \), \( E \in (\lambda V(x) - \frac{1}{4} \lambda^{1/2}, \lambda V(x) + \frac{1}{4} \lambda^{1/2}) \). Suppose \( \text{dist} \left( E, \text{sp} \, H_{\Lambda_{ij}}(x, \omega) \right) < \frac{1}{2} e^{-N^s} \). Then \( \text{dist} \left( E, \text{sp} \, H_{[-a,b]}(x + n\omega, \omega) \right) > \frac{1}{2} e^{-N^s} \) where \( N^{(s+1)}_{ij} \lesssim |n| \leq (N^{(s+1)}_{ij})^2 \), \( a, b > N_s \).

Proof. The assertion follows from condition (2) and Lemma C.9. \( \square \)

Corollary 9.3. Let \( x, \omega, E \) as in Lemma 9.2, \( -(N^{(s+1)}_{ij})^2 \leq c \leq d \leq N^{(s+1)}_{ij} \), \( d - c > 10N_s \) then we can choose \( c = a_0 < a_1 < \cdots < a_K = d \), \( a_k - a_{k-1} > N_s \),

\[
A_k = M_{(a_k-1, a_k]}(x, \omega, E) \quad k = 2, \ldots, K - 1
\]

\[
A_1 = M_{[a_0, a_1]}(x, \omega, E) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
\]

\[
a_K = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} M_{(a_{K-1}, a_K]}(x, \omega, E)
\]

such that

\[
\log |f_{[c,d]}(x, \omega, E)| = \sum_{k=1}^{K} \log \| A_k A_{k-1} \| - \sum_{k=2}^{K-1} \log \| A_k \| + O \left( N_s + a \right).
\]

Proof. The hypothesis of Avalanche principle is verified by Lemmas 9.2 and C.10. \( \square \)

Similar to Section 7, we need now to show that the spectrum of \( H_{\Lambda_{ij}^{(s+1)}}(x, \omega) \) near \( E_{ij_k}^{(s)}(x, \omega) \) is separated.

Lemma 9.4. If \( E_1, E_2 \in \left( E_{ij_k}^{(s)}(x, \omega) - \frac{1}{2} e^{-\langle N^{(s)} \rangle^{a}}, E_{ij_k}^{(s)}(x, \omega) + \frac{1}{2} e^{-\langle N^{(s)} \rangle^{a}} \right) \) then \( -(N^{(s+1)}_{ij})^2 \leq a < b < -N^{(s+1)}_{ij}, \ |\Lambda_{pq}^{(s)}| < b - a \leq 5 |\Lambda_{pq}^{(s)}| \), one has

\[
\left| \log |f_{[a,b]}(x, \omega, E_1)| - \log |f_{[a,b]}(x, \omega, E_2)| \right| \leq e^{\frac{1}{2} N^s} \left| E_1 - E_2 \right|.
\]

Proof. Let \( J = \{ j \in [a, b] : | E_{ij_k}^{(s)}(x + j\omega, \omega) - E_{ij_k}^{(s)}(x, \omega) | < e^{-N^s} \} \). Then \# \( J \leq N^{(s+1)}_{ij} \).

Let \( \mu_j \) \( j = a \) = \( \text{sp} \, H_{[a,b]}(x, \omega) \), \( |\mu_j - \lambda V(x + j\omega) | \leq 2 \)

\[
\left| \log |f_{[a,b]}(x, \omega, E_1)| - \log |f_{[a,b]}(x, \omega, E_2)| \right| \leq \sum_{j \in J} | \log |\mu_j - E_1| - \log |\mu_j - E_2| | + \sum_{j \notin J} | \log |\mu_j - E_1| - \log |\mu_j - E_2| |
\]

\[
\leq N^{(s+1)}_{ij} \left( \frac{1}{2} e^{-\langle N^{(s)} \rangle^{a}} \right)^{-1} \left| E_1 - E_2 \right| + (b - a) \left( \frac{1}{2} \lambda^{1/2} \right)^{-1} \left| E_1 - E_2 \right|.
\]

\( \square \)

Corollary 9.5. \( E_1, E_2 \) as in Lemma 9.4, then

\[
\left| \log |f_{[-(N^{(s+1)}_{ij})^2, a]}(x, \omega, E_1)| - \log |f_{[-(N^{(s+1)}_{ij})^2, a]}(x, \omega, E_2)| \right| \leq e^{\frac{1}{2} N^s} \left| E_1 - E_2 \right|.
\]

Corollary 9.6. If \( E_1, E_2 \in \left( E_{ij_k}^{(s)}(x, \omega) - \frac{1}{2} e^{-N^s}, E_{ij_k}^{(s)}(x, \omega) + \frac{1}{2} e^{-N^s} \right) \) \( \cap \) \( \text{sp} \, H_{\Lambda_{ij}^{(s+1)}}(x, \omega) \), \( E_1 \neq E_2 \), then

\[
\left| E_1 - E_2 \right| > e^{-\langle N^{(s+1)} \rangle^{a}}.
\]
Lemma 9.7. There is a set of variations $S_{s+1}$, $1 - |S_{s+1}| \leq e^{-N_{s+1}^{A}/2}$, such that for any $W \in S_{s+1}$, the potential $V = V^{(s)} + W$ has the following property: There is a set $\Omega_{s+1} = \Omega_{s+1}(\tilde{V}) \subset \Omega_{s}$, mes$(\Omega_{s} \setminus \Omega_{s+1}) \lesssim e^{-N_{s+1}^{A}/2}$ so that for any $i, j, k$ one has
\[
|\partial_{x_{i}} E^{(s+1)}(x, \omega)| + |\partial_{x_{i}} E^{(s+1)}(x, \omega)| > e^{-N_{s+1}^{A}}
\]
for all $x \in U_{i}$, $\omega \in \Omega_{s+1} \cap V_{j}$.


For the first scale, we know that the number of eigenvalues close to $V(x)$ is bounded by $m_{0} \ll N_{ij}^{(i)}$ since there are at most $m_{0}$ resonance entries. For the general inductive step, we will count the number of eigenvalues of $H_{\Lambda_{ij}^{(i+1)}}(x, \omega)$ near $E_{ij}^{(s)}(x, \omega)$ in the following lemma.

Lemma 9.8. If $\omega \in \Omega_{s+1}$, then
\[
\# \left\{ E \in \sp H_{\Lambda_{ij}^{(i+1)}}(x, \omega) : \text{dist}(E, H_{\Lambda_{ij}^{(i)}}(x, \omega)) < \frac{1}{2} e^{-N_{s}^{\beta}} \right\} \lesssim |\Lambda_{ij}^{(i+1)}|^{1/2}
\]

Proof. By Corollary 9.3, if $H_{\Lambda_{ij}^{(i+1)}} \varphi = E\varphi, \|\varphi\| = 1, \text{dist}(E, H_{\Lambda_{ij}^{(i)}}(x, \omega)) < \frac{1}{2} e^{-N_{s}^{\beta}}$ then $|\varphi(n)| \lesssim e^{-|n|-N_{ij}^{(s)}}$ if $|n| > N_{ij}^{(i+1)}$.

Let $L = \bigoplus_{\text{dist}(E, H_{\Lambda_{ij}^{(i)}}(x, \omega)) < \frac{1}{2} e^{-N_{s}^{\beta}}} \ker (H_{\Lambda_{ij}^{(i+1)}} - EI)$. Then $|\varphi(n)| \lesssim e^{-|n|-N_{ij}^{(i+1)}}$ if $|n| > N_{ij}^{(2)}$ for all $\varphi \in L$.

But by Lemma A.6, $\dim L \lesssim N_{ij}^{(i+1)} \lesssim |\Lambda_{ij}^{(i+1)}|^{1/2}$.


Lemma 9.9. If $w \in \Omega_{s+1}$, $e^{-N_{s}^{\beta}} < \text{dist}(E, \sp H_{\Lambda_{ij}^{(i+1)}}(x, \omega))$ then $\log |f_{\Lambda_{ij}^{(i+1)}}(x, \omega)| > \frac{|\Lambda_{ij}^{(i+1)}|^{1/2}}{4} \log \lambda$.

Proof. If $|E - E_{pjk}^{(s)}(x+n_{0}, \omega)| \geq \frac{1}{2} e^{-N_{s}^{\beta}}$ for $n, p, k$ then the assertion follows immediately from the Avalanche Principle.

So suppose $|E - E_{pjk}^{(s)}(x+n_{0}, \omega)| < \frac{1}{2} e^{-N_{s}^{\beta}}$, $\omega \in \Omega_{s+1}$. Then $|E - E_{pjk}^{(s)}(x+n_{0}+n, \omega)| \geq \frac{1}{2} e^{-N_{s}^{(1)}}$,

$N_{pjk}^{(i+1)} < |n| \leq (N_{pjk}^{(i+1)})^{2}$. Take $\tilde{E} = |E - E_{pjk}^{(s)}(x+n_{0}, \omega)| < \frac{1}{4} e^{-N_{s}^{\beta}}$ such that $|\tilde{E} - E_{pjk}^{(s)}(x+n_{0}, \omega)| \geq \frac{1}{4} e^{-N_{s}^{\beta}}$. By Lemma 9.8 and Lemma B.5,

$\log |f_{\Lambda_{ij}^{(i+1)}}(E)| > \log |f_{\Lambda_{ij}^{(i+1)}}(\tilde{E})| - |\Lambda_{ij}^{(i+1)}|^{1/2}(N_{ij}^{(i+1)})^{\gamma}$.


10. Proof of Main Theorem

We have defined $C^{3}$ functions $V^{(1)}, V^{(2)}, \ldots, \|V^{(\ell+1)} - V^{(\ell)}\| \lesssim e^{-N_{i}^{(\ell)}},$ and $T \supset \Omega^{(2)} \supset \Omega^{(3)} \supset \ldots, \text{mes}(\Omega^{(\ell+1)} \setminus \Omega^{(\ell)}) \lesssim e^{-N_{i}^{(\ell)}}$. Let $\tilde{V} = \lim V^{(\ell)} \in C^{3}(T), \Omega = \bigcap \Omega^{(\ell)}$,

$\widehat{H}_{\Lambda_{ij}^{(\ell)}}(x, \omega) = \begin{pmatrix}
\tilde{V}(x - |\Lambda_{ij}^{(\ell)}| \omega) & -1 \\
-1 & \tilde{V}(x - (|\Lambda_{ij}^{(\ell)}| - 1) \omega) & -1 \\
& \ddots & \ddots & \ddots \\
& -1 & \tilde{V}(x + |\Lambda_{ij}^{(\ell)}| \omega)
\end{pmatrix}$
Then $\|\hat{H}_{\Lambda^{(t)}}(x,\omega) - H_{\Lambda_{ij}^{(t)}}(x,\omega)\| \lesssim e^{-(N^{(t)})^{\alpha}}$.

To establish Theorem 1.2, we need to show that the eigenvalues and eigenfunctions of $\hat{H}_{\Lambda_{ij}^{(t)}}(x,\omega)$ have the same properties shared by the eigenvalues and eigenfunctions of the potential $V^{(t)}$. We do this by comparing the corresponding eigenvalues and eigenfunctions.

By Lemma B.4, we can define

$$\hat{E}_{ijk}^{(t)}(x,\omega) \in \text{sp} \hat{H}_{\Lambda_{ij}^{(t)}}(x,\omega), \quad \omega \in \Omega, \quad x \in U_{ij}$$

such that $|\hat{E}_{ijk}^{(t)}(x,\omega) - E_{ijk}^{(t)}(x,\omega)| \lesssim e^{-(N^{(t)})^{\alpha}}$.

**Lemma 10.1.** dist $\left(\hat{E}_{ijk}^{(t)}(x,\omega), \text{sp} \hat{H}_{\Lambda_{ij}^{(t)}}(x,\omega) \setminus \{\hat{E}_{ijk}^{(t)}(x,\omega)\}\right) > e^{-(N^{(t)})^{\alpha}}$.

**Proof.** This follows from Corollary 9.4, Lemma B.4 and the definition of $\hat{E}_{ijk}^{(t)}(x,\omega)$ above. □

**Lemma 10.2.** There is $\|\hat{\phi}_{ijk}^{(t)}(x,\omega)\| = 1$ such that

$$\hat{H}_{\Lambda_{ij}^{(t)}}(x,\omega)\hat{\phi}_{ijk}^{(t)}(x,\omega) = \hat{E}_{ijk}^{(t)}(x,\omega)\hat{\phi}_{ijk}^{(t)}(x,\omega),$$

$$||\hat{\phi}_{ijk}^{(t)}(x,\omega) - \hat{\phi}_{ijk}^{(t)}(x,\omega)|| \lesssim e^{-(N^{(t)})^{\alpha}}$$

**Proof.**

$$\|\hat{H}_{\Lambda_{ij}^{(t)}}(x,\omega)\hat{\phi}_{ijk}^{(t)}(x,\omega) - \hat{E}_{ijk}^{(t)}(x,\omega)\hat{\phi}_{ijk}^{(t)}(x,\omega)\| \leq \|\hat{H}_{\Lambda_{ij}^{(t)}}(x,\omega) - H_{\Lambda_{ij}^{(t)}}(x,\omega)\| + \|H_{\Lambda_{ij}^{(t)}}(x,\omega) - E_{ijk}^{(t)}(x,\omega)\| \|\hat{\phi}_{ijk}^{(t)}(x,\omega)\|$$

$$+ |E_{ijk}^{(t)}(x,\omega) - \hat{E}_{ijk}^{(t)}(x,\omega)|$$

$$\lesssim e^{-(N^{(t)})^{\alpha}}$$

The assertions follows from Lemma A.5 and Lemma 10.1. □

**Corollary 10.3.** $w \in \Omega$. If $|E - \hat{E}_{ijk}^{(t)}(x,\omega)| < \frac{1}{2} e^{-(N^{(t)})^{\alpha}}$ then

$$\text{dist} \left( E, \text{sp} \hat{H}_{\Lambda_{ij}^{(t)}}(x+n\omega,\omega) \right) > \frac{1}{2} e^{-(N^{(t)})^{\alpha}}$$

for all $N^{(t+1)} < |n| \leq (N^{(t+1)})^{2}$. Also,

$$\# \left( E \in \text{sp} \hat{H}_{\Lambda_{ij}^{(t+1)}}(x,\omega) : \text{dist} \left( E, \text{sp} \hat{H}_{\Lambda_{ij}^{(t)}}(x,\omega) \right) < \frac{1}{2} e^{-(N^{(t)})^{\alpha}} \right) \lesssim |\Lambda_{ij}^{(t+1)}|^{1/2}$$

**Proof.** Same as in Lemma 9.1 and Lemma 9.6. □

Let $f_{\Lambda_{ij}^{(t)}}(x,\omega,\lambda) = \det(\hat{E}_{\Lambda_{ij}^{(t)}}(x,\omega) - \lambda)$.

**Lemma 10.4.** If $\omega \in \Omega \cap V^{(t)}_{j}, x \in U_{i}$

$$\text{dist} \left( E, \text{sp} H_{\Lambda_{ij}^{(t)}}(x,\omega) \right) > \frac{1}{2} e^{-(N^{(t)})^{\alpha}}$$

Then $\log |\hat{f}_{\Lambda_{ij}^{(t)}}(x,\omega,\lambda)| > \frac{|\Lambda_{ij}^{(t)}|}{4} \lambda^{1/2} \log \lambda$.

**Proof.** For $\ell = 1$, let $\mu_{k} = \text{sp} H_{\Lambda_{ij}^{(t)}}(x,\omega)$, $|\lambda \hat{V}(x+k\omega) - \mu_{k}| \leq 2$.

Let $K = \{k : |E - \mu_{k}| \leq \lambda^{1/2}\}$. If $k \in K$, then

$$|V(x+k\omega) - \lambda^{-1}E| \leq |V(x+k\omega) - \hat{V}(x+k\omega)| + |\hat{V}(x+k\omega) - \lambda^{-1}E|$$

$$\lesssim \lambda^{-1/2}.$$
By Lemma 6.2, \( \# K \leq m_0 \). Hence
\[
\log |f_{A_{ij}^{(1)}}(x, \omega, E)| > |A_{ij}^{(1)}(x, \omega) - m_0| \log \lambda^{1/2} - m_0 (N^{(1)})^3 
\]
\[
> \frac{|A_{ij}^{(1)}|}{4} \log \lambda 
\]
For \( \ell > 1 \), there is \( |E - E'| \leq e^{-N^{(\ell-1)}} \) such that \( \text{dist}(E, \text{sp} H_{A_{ij}^{(1)}}(x + n \omega)) > e^{-N^{(\ell-1)}} \). By the Avalanche Principle
\[
\log |f_{A_{ij}^{(\ell)}}(x, \omega, E)| > \frac{|A_{ij}^{(\ell)}|}{4} \log \lambda 
\]
In view of Corollary 10.3, we get
\[
\log |f_{A_{ij}^{(\ell)}}(x, \omega, E)| > \frac{|A_{ij}^{(\ell)}|}{4} \log \lambda 
\]
(see Lemma 9.7).

\[ \square \]

**Lemma 10.5.** \( |\partial_x E_{ijk}^{(\ell)}(x, \omega) - \partial_x E_{ijk}^{(\ell)}(x, \omega)| \leq e^{-N^{(\ell)}} \).

**Proof.**
\[
|\partial_x E_{ijk}^{(\ell)}(x, \omega) - \partial_x E_{ijk}^{(\ell)}(x, \omega)| \leq \lambda \sum_{n \in \Lambda_{ij}^{(\ell)}} |\tilde{V}'(x + n \omega)| \left| \tilde{f}_{ij}^{(\ell)}(x, \omega) \right|^2
\]
\[
\leq \lambda \sum_{n} \left( |\tilde{V}'(x + n \omega) - V'(x + n \omega)| \right) + \lambda \sum_{n} |V'(x + n \omega)| \left| \tilde{f}_{ij}^{(\ell)}(x, \omega) \right|^2
\]
\[
\leq e^{-N^{(\ell)}}
\]

\[ \square \]

**Lemma 10.6.** \( \|\partial_x \phi_{ijk}^{(\ell)}(x, \omega) - \partial_x \phi_{ijk}^{(\ell)}(x, \omega)\| \leq e^{-\frac{1}{2}N^{\ell}} \).

**Proof.**
\[
\partial_x \phi_{ijk}^{(\ell)}(x, \omega) = \frac{\int_{|z - E_{ijk}^{(\ell)}| = \varrho} (H_{A_{ij}^{(\ell)}}(x, \omega) - z)^{-1} \left( \partial_x H_{A_{ij}^{(\ell)}}(x, \omega) \right)(H_{A_{ij}^{(\ell)}} - z)^{-1} dz}{(H_{A_{ij}^{(\ell)}}(x, \omega))}(H_{A_{ij}^{(\ell)}} - z)^{-1} \phi_{ijk}^{(\ell)}(x, \omega)
\]
where \( \varrho = \frac{1}{2}e^{-N^{(\ell)}} \). Hence
\[
\|\partial_x \phi_{ijk}^{(\ell)}(x, \omega) - \partial_x \phi_{ijk}^{(\ell)}(x, \omega)\| \leq \int \left( |(\tilde{H} - z)|^{-1} \left( \partial_x \tilde{H} \right)(\tilde{H} - z)^{-1} - (H - z)^{-1} \left( \partial_x H \right)(H - z)^{-1} \right)
\]
\[
+ \left( \int |(H - z)|^{-1} \left( \partial_x H \right)(H - z)^{-1} \right) \phi_{ijk}^{(\ell)}(x, \omega) - \phi_{ijk}^{(\ell)}(x, \omega)\|
\]
\[
\leq \|\tilde{H} - z\|^{-1} \|\partial_x H\| \|\tilde{H} - z\|^{-1} \|\tilde{H} - z\|^{-1} \|
\]
\[
\leq e^{-\frac{1}{2}N^{(\ell)}}
\]
since \( \|\tilde{H} - z\|^{-1} = (H - z)^{-1} \left( (H - z) - (\tilde{H} - z) \right)(H - z)^{-1} \) and \( \|\tilde{H} - z\|^{-1} \leq e^{N^{\ell}} \), \( \|\tilde{H} - z\|^{-1} \leq e^{N^{\ell}} \).

\[ \square \]
Lemma 10.7. \(|\partial_{xx}\tilde{E}^{(\ell)}_{ijk}(x,\omega) - \partial_{xx}E^{(\ell)}_{ijk}(x,\omega)| \lesssim e^{-\frac{1}{4}(N^{(\ell)})^\nu}\).

Proof. 

\[
|\partial_{xx}\tilde{E}^{(\ell)}_{ijk}(x,\omega) - \partial_{xx}E^{(\ell)}_{ijk}(x,\omega)| \leq \lambda \sum_{n \in \Lambda^{(\ell)}_{ij}} \left| \tilde{V}''(x+n\omega)|\tilde{\varphi}^{(\ell)}_{ijk}(x,\omega)(n)|^2 - V''(x+n\omega)|\varphi^{(\ell)}_{ijk}(x,\omega)(n)|^2 \right|
+ 2\lambda \sum_{n \in \Lambda^{(\ell)}_{ij}} \left| \tilde{V}'(x+n\omega)\tilde{\varphi}^{(\ell)}_{ijk}(x,\omega)\partial_x \tilde{\varphi}^{(\ell)}_{ijk} - V'(x+n\omega)\varphi^{(\ell)}_{ijk}(x,\omega)\partial_x \varphi^{(\ell)}_{ijk} \right|
\lesssim e^{-\frac{1}{4}(N^{(\ell)})^\nu}.
\]

Corollary 10.8. \(|\partial_{xx}\tilde{E}^{(\ell)}_{ijk}(x,\omega)| + |\partial_{xx}E^{(\ell)}_{ijk}(x,\omega)| \geq e^{-\frac{1}{4}N^\nu}\).

Proof. Since \(|\partial_{xx}\tilde{E}^{(\ell)}_{ijk}(x,\omega)| + |\partial_{xx}E^{(\ell)}_{ijk}(x,\omega)| \geq e^{-\frac{1}{4}(N^{(\ell)})^\nu}\), the assertion follows from Lemma 10.5 and Lemma 10.7. \(\square\)

Theorem 10.9. Given any \(V \in C^{3}(\mathbb{T})\), \(|V'(x)| + |V''(x)\| \geq c > 0\), there is \(\lambda_0 = \lambda_0(V)\) such that for \(|\lambda| > \lambda_0\), one has a collection of perturbed potentials \(\{S_\ell = S_\ell(V, \lambda)\}_{\ell=1}^\infty\), \(S_\ell \subset \mathcal{S}(T^{(\ell)}, \delta_\ell)\), \(\log T^{(\ell+1)} \asymp (T^{(\ell)})^\alpha\), \(0 < \alpha \ll 1\), \(\sum_{\ell=1}^\infty (1 - |S_\ell|) \leq \lambda^{-\beta}\), so that for any potential

\[
\tilde{V}(x) = V(x) + \sum_{\ell=1}^\infty W^{(\ell)}(\eta^{(\ell)}, \xi^{(\ell)}, \theta^{(\ell)}, \{R^{(\ell)}_m\}; x)
\]

where \(W^{(\ell)} \in S_\ell\), there exists \(\Omega = \Omega(\lambda, \tilde{V})\), \(\text{mes } \Omega \leq \lambda^{-\beta}\), so that the Lyapunov exponent \(L(\omega, E) \geq \frac{1}{4}\log \lambda\) for any \(\omega \in \Omega\), \(E \in \mathbb{R}\).

Proof. Fix \(E, \omega\). For each \(\ell\), from Corollary 11.8, we have

\[
\text{mes } \left\{ x \in \mathbb{T} : \text{dist} \left( E, \text{sp } H_{\lambda^{(\ell)}}(x,\omega) \right) \leq e^{-\frac{1}{4}(N^{(\ell)})^\nu} \right\} \leq e^{-\frac{1}{4}(N^{(\ell)})^\nu}.
\]

Let \(\mathcal{J} = \bigcup_{\ell=1}^\infty \left\{ x \in \mathbb{T} : \text{dist} \left( E, \text{sp } H_{\lambda^{(\ell)}}(x,\omega) \right) \leq e^{-\frac{1}{4}(N^{(\ell)})^\nu} \right\}\). If \(x \in \mathbb{T} \setminus \mathcal{J}\) then

\[
\frac{1}{|\lambda^{(\ell)}|} \log \| M_{\lambda^{(\ell)}}(x,\omega,E) \| \geq \frac{1}{|\lambda^{(\ell)}|} \log \| f_{\lambda^{(\ell)}}(x,\omega,E) \|
\geq \frac{1}{4} \log \lambda.
\]

From above, \(\text{mes } \mathbb{T} \setminus \mathcal{J} > 0\). But \(\frac{1}{|\lambda^{(\ell)}|} \log \| M_{\lambda^{(\ell)}}(x,\omega,E) \| \to L(\omega, E)\) for almost all \(x\) since \(\omega \in \mathbb{R} \setminus \mathbb{Q}\). Therefore \(L(\omega, E) \geq \frac{1}{4} \log \lambda\). \(\square\)

Appendix A. Matrix Functions

Let \(A : (a,b) \to M_{m \times m}(\mathbb{C})\) be \(C^1\), \(A(x)\) self-adjoint for every \(x \in (a,b)\). Suppose \(E : (a,b) \to \mathbb{R}\) is such that \(E(x)\) is a simple eigenvalue of \(A(x)\), and

\[
\text{sp } A(x) \cap \left[ E_0 - \frac{3\delta}{2}, \ E_0 + \frac{3\delta}{2} \right] = \{E(x)\} \subset \left( E_0 - \frac{\delta}{2}, \ E_0 + \frac{\delta}{2} \right), \ \delta > 0.
\]
Let $P(x)$ be the orthogonal projection on the eigenspace of $A(x)$ with eigenvalue $E(x)$. By Riesz formula for orthogonal projection of self-adjoint matrix

$$P(x) = \frac{1}{2\pi i} \oint_{|z - E_0| = \delta} (zI - A(x))^{-1} \, dz$$

$$P'(x) = \frac{1}{2\pi i} \oint_{|z - E_0| = \delta} (zI - A(x))^{-1} A'(x) (zI - A(x))^{-1} \, dz$$

Hence $P : (a, b) \to M_{m \times m}(\mathbb{C})$ is $C^1$

$$\|P'(x)\| \leq \frac{1}{2\pi} \left( \frac{\delta}{2} \right)^{-1} \|A'(x)\| \left( \frac{\delta}{2} \right)^{-1} (2\pi \delta) = \frac{1}{4\delta} \|A'(x)\| \, .$$

**Lemma A.1.** $A : (a, b) \to M_{m \times m}(\mathbb{C}), E : (a, b) \to \mathbb{R}$ as above. Then $E$ is $C^1$ and $E'(x) = (A'(x) \varphi(x), \varphi(x))$ where $\varphi(x)$ is a normalized eigenvector of $A(x)$ with eigenvalue $E(x)$.

**Proof.** For any $x_0 \in (a, b)$, choose $\varphi_0 \in \text{Ran} P(x_0), \|\varphi_0\| = 1$. Then $(P(x_0)\varphi_0, \varphi_0) = 1$. There exists neighborhood $U$ of $x_0$ such that $(P(x)\varphi_0, \varphi_0) > 0$ for every $x \in U$.

$$E(x) = \frac{(A(x)P(x)\varphi_0, \varphi_0)}{(P(x)\varphi_0, \varphi_0)} \text{ for } x \in U$$

$E$ is $C^1$ since $A$ and $P$ are $C^1$.

$$E'(x_0) = \frac{(A'(x_0)P(x_0)\varphi_0, \varphi_0) + (A(x_0)P'(x_0)\varphi_0, \varphi_0)}{(P(x_0)\varphi_0, \varphi_0)} - \frac{(A(x_0)P(x_0)\varphi_0, \varphi_0)}{(P(x_0)\varphi_0, \varphi_0)^2}$$

$$= (A'(x_0)\varphi_0, \varphi_0) + (P'(x_0)\varphi_0, A(x_0)\varphi_0) - (A(x_0)\varphi_0, \varphi_0)(P'(x_0)\varphi_0, \varphi_0)$$

$$= (A'(x_0)\varphi_0, \varphi_0) + E(x_0)(P'(x_0)\varphi_0, \varphi_0) - E(x_0)(P'(x_0)\varphi_0, \varphi_0)$$

$$= (A'(x_0)\varphi_0, \varphi_0) \, .$$

**Lemma A.2.** Let $x_0, \varphi_0, U$ be as in proof of Lemma A.1. Define $\varphi(x) = (P(x)\varphi_0, \varphi_0)^{-1/2} (P(x)\varphi_0)$ for $x \in U$. Then $\varphi(x)$ is a normalized eigenvector of $A(x)$ with eigenvalue $E(x)$. $\varphi : U \to \mathbb{R}^n$ is $C^1$ and $\varphi'(x_0) = P'(x_0)\varphi_0$.

**Proof.** Since $P(x)$ is an orthogonal projection,

$$(P(x)\varphi_0, \varphi_0) = (P(x)^2\varphi_0, \varphi_0) = (P(x)\varphi_0, P(x)\varphi_0) = \|P(x)\varphi_0\|^2 \, .$$

Therefore, $\|\varphi(x)\| = 1$.

Since $P$ is $C^1$, $\varphi$ is $C^1$,

$$\varphi'(x_0) = -\frac{1}{2}(P(x_0)\varphi_0, \varphi_0)^{-\frac{3}{2}} (P'(x_0)\varphi_0, \varphi_0)(P(x_0)\varphi_0) + (P(x_0)\varphi_0, \varphi_0)^{-\frac{1}{2}} (P'(x_0)\varphi_0)$$

$$= -\frac{1}{2}(P'(x_0)\varphi_0, \varphi_0)\varphi_0 = P'(x_0)\varphi_0 \, .$$

It remains to show $(P'(x_0)\varphi_0, \varphi_0) = 0$.

$$P(x) = P(x)^2$$

$$P'(x_0) = P'(x_0)P(x_0) + P(x_0)P'(x_0)$$

$$(P'(x_0)\varphi_0, \varphi_0) = (P'(x_0)P(x_0)\varphi_0, \varphi_0) + (P(x_0)P'(x_0)\varphi_0, \varphi_0)$$

$$(P'(x_0)\varphi_0, \varphi_0) = 2(P'(x_0)\varphi_0, \varphi_0) \, .$$

Hence $(P'(x_0)\varphi_0, \varphi_0) = 0$.  

□
Lemma A.3. Let $A : (a, b) \to M_{m \times m}(\mathbb{C})$, $E : (a, b) \to \mathbb{R}$ as above. If $A$ is $C^2$, then $E$ is $C^2$ and
\[ |E''(\xi)| \leq \|A''(\xi)\| + \frac{1}{2\delta}\|A'(\xi)\|^2. \]

Proof. For any $x_0 \in (a, b)$, define $\varphi(x)$ in a neighborhood $U$ of $x_0$ as in Lemma A.2, $\|\varphi(x)\| = 1$,
\[ A(x)\varphi(x) = E(x)\varphi(x) \quad \text{for every} x \in U \]
\[ E'(x) = \big( A'(x)\varphi(x), \varphi(x) \big) \quad \text{for every} x \in U. \]

Since $\varphi(x)$ is $C^1$, if $A$ is $C^2$ then $E'$ is $C^1$. So $E$ is $C^2$.
\[ E''(x_0) = \big( A''(x_0)\varphi(x_0), \varphi(x_0) \big) + \big( A'(x_0)P'(x_0)\varphi(x_0), \varphi(x_0) \big) + \big( A'(x_0)\varphi(x_0), P'(x_0)\varphi(x_0) \big) \]
\[ = \big( A''(x_0)\varphi(x_0), \varphi(x_0) \big) + 2\big( A'(x_0)P'(x_0)\varphi(x_0), \varphi(x_0) \big) \]
\[ \leq \|A''(x_0)\| + 2\|A'(x_0)\|\|P'(x_0)\| \]
\[ \leq \|A''(x_0)\| + \frac{1}{2\delta}\|A'(x_0)\|^2. \]

Let $V : \mathbb{R} \to \mathbb{R}$ be continuous, piecewise smooth, 1–periodic. Define
\[ H_{[m,n]}(x, w) = \begin{pmatrix} V(x + mw) & -1 & & & \\ -1 & V(x + (m + 1)w) & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & -1 & \\ & & & & V(x + nw) \end{pmatrix} \]

Then $H_{[m,n]}(x, w)$ is continuous; $H_{[m,n]}(., w)$ and $H_{[m,n]}(x, .)$ are piecewise smooth. It is well-known that the eigenvalues of $H_{[m,n]}(x, w)$ are simple. So the eigenvalues $E_{[m,n]}^{(1)}(x, w) < \cdots < E_{[m,n]}^{(n)}(x, w)$ of $H_{[m,n]}(x, w)$ are continuous.

Let $\{ \varphi_{[m,n]}^{(j)}(x, \omega)(j) \}_{m \leq j \leq n}$ be a normalized eigenvector of $H_{[m,n]}(x, \omega)$ with eigenvalue $E_{[m,n]}^{(j)}(x, \omega)$.

Suppose $\partial_x H_{[m,n]}(x_0, \omega_0)$ exists. For fixed $k$, let
\[ E_0 = E_{[m,n]}^{(k)}(x_0, \omega_0), \]
\[ \delta = \frac{1}{2} \text{dist} \left( E_{[m,n]}^{(k)}(x_0, \omega_0), \{ E_{[m,n]}^{(j)}(x_0, \omega_0) \}_{j \neq k} \right) > 0. \]

Since $E_{[m,n]}^{(j)}$ are continuous, there exists neighborhood $U$ of $x_0$ such that for every $x \in U$
\[ \text{sp} \ H_{[m,n]}(x_0, \omega_0) \cap \left[ E_0 - \frac{3\delta}{2}, E_0 + \frac{3\delta}{2} \right] = \{ E_{[m,n]}^{(k)}(x_0, \omega_0) \} \subset \left( E_0 - \frac{\delta}{2}, E_0 + \frac{\delta}{2} \right). \]

By Lemma A.1,
\[ \partial_x E_{[m,n]}^{(k)}(x_0, \omega_0) = \sum_{j = m}^{n} V'(x_0 + j\omega_0) |\varphi_{[m,n]}^{(k)}(x_0, \omega_0)(j)|^2. \]

Similarly,
\[ \partial_\omega E_{[m,n]}^{(k)}(x_0, \omega_0) = \sum_{j = m}^{n} jV'(x_0 + j\omega_0) |\varphi_{[m,n]}^{(k)}(x_0, \omega_0)(j)|^2. \]

\[ \square \]

Lemma A.4. If $A$ is $n \times n$, self-adjoint, and $\|A\varphi\| < \varepsilon$ for some $\varphi$, $\|\varphi\| = 1$. Then there exists $\lambda \in \text{sp} A \cap (-\varepsilon, \varepsilon)$. 

Proof. If $A$ is not invertible, then take $\lambda = 0$. If $A^{-1}$ exists, then $1 = \|\varphi\| = \|A^{-1}A\varphi\| < \|A^{-1}\varepsilon\|$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $A$,

$$\frac{1}{\varepsilon} < \|A^{-1}\| = \frac{1}{\min|\lambda_i|} \Rightarrow \min|\lambda_i| < \varepsilon.$$

\[\square\]

**Lemma A.5.** Let $A$ be $n \times n$, self-adjoint. Suppose $\|A\varphi\| < \varepsilon$, $A\psi = 0$, $\|\varphi\| = \|\psi\| = 1$. If $|\mu| > \delta$ for every $\mu \in \text{sp} A \setminus \{0\}$ then $\min_{|c|=1} \|\varphi - c\psi\| \lesssim \varepsilon/\delta$ provided 0 is simple eigenvalue.

**Proof.** Let $\psi_\mu$ be normalized eigenvectors of $A$ with eigenvalues $\mu \neq 0$,

$$\varepsilon^2 > \|A\varphi\|^2 = \lambda^2((\varphi, \psi))^2 + \sum_{\mu \neq 0} \mu^2((\varphi, \psi_\mu))^2 > \delta^2 \sum_{\mu \neq 0} |(\varphi, \psi_\mu)|^2$$

$$\|\varphi - c\psi\|^2 = \|\varphi\|^2 - \overline{c}(\varphi, \psi) - c(\psi, \varphi) + |c|^2 \|\psi\|^2$$

$$= 2 \begin{cases} 1 - \text{Re}(\overline{c}(\varphi, \psi)) & |c| = 1 \\ 1 - |(\varphi, \psi)|^2 & |(\psi, \varphi)| \leq 1 \\ 1 - |(\varphi, \psi)|^2 & |(\psi, \varphi)| > 1 \end{cases}$$

$$= 2 \sum_{\mu \neq 0} |(\varphi, \psi_\mu)|^2$$

$$< 2\varepsilon^2 / \delta^2$$

\[\square\]

**Lemma A.6.** Let $\mathcal{L}, \mathcal{M}$ be subspaces of $\mathbb{R}^n$. If $\dim \mathcal{L} > \dim \mathcal{M}$, then there is $\psi \in \mathcal{L}$, $\|\psi\| = 1$, such that $(\psi, \varphi) = 0$ for all $\varphi \in \mathcal{M}$.

**Proof.** Suppose, for contradiction, that $\mathcal{L} \cap \mathcal{M}^\perp = \{0\}$. Since $\mathbb{R} = \mathcal{M} \oplus \mathcal{M}^\perp$, we have $\mathcal{L} \subset \mathcal{M}$. But this is impossible because $\dim \mathcal{L} > \dim \mathcal{M}$. Hence, there is $\psi \in \mathcal{L} \cap \mathcal{M}^\perp$, $\|\psi\| = 1$.

\[\square\]

**Appendix B. Mini-max principle**

Mini-max Principle: Let $A$ be $n \times n$ hermitian matrix, with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Then

$$\lambda_j = \min_{\dim M = j} \max_{x \in M, \|x\| = 1} (Ax, x)$$

$$= \max_{\dim M = n - j + 1} \min_{x \in M, \|x\| = 1} (Ax, x).$$

**Lemma B.1.** Let $A, B$ be $n \times n$ hermitian matrices, with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$ and $\mu_1 \leq \cdots \leq \mu_n$ respectively. Suppose there is $\alpha > 0$, $y \in \mathbb{R}^n$ such that $Ax = Bx + \alpha(x, y)y$ for all $x \in \mathbb{R}^n$. Then the eigenvalues of $A$ and $B$ interlace, i.e. $\mu_1 \leq \lambda_1 \leq \mu_2 \leq \lambda_2 \leq \cdots \leq \mu_n \leq \lambda_n$. 
Proof.

\[
\mu_j = \min_{x \in M_j, \|x\|=1} (Bx, x) \quad \text{for some } M_j, \dim M_j = n - j + 1
\]

\[
\lambda_j = \max_{\dim M = n - j + 1} \min_{x \in M, \|x\|=1} (Ax, x) 
\geq \min_{x \in M_j, \|x\|=1} (Bx, x) = \min_{x \in M_j, \|x\|=1} [(Bx, x) + \alpha |(x, y)|^2] \geq \mu_j
\]

On the other hand,

\[
\lambda_j = \min_{x \in \tilde{M}_j, \|x\|=1} (Ax, x) \quad \text{for some } \tilde{M}_j, \dim \tilde{M}_j = n - j + 1
\]

Let \( U_j \subset \tilde{M}_j \cap (\mathbb{R}y) \perp \), \( \dim U_j = n - j \) for \( j < n \).

\[
\mu_{j+1} = \max_{\dim M = n - j} \min_{x \in M, \|x\|=1} (Bx, x) 
\geq \min_{x \in U_j, \|x\|=1} (Bx, x) = \min_{x \in U_j, \|x\|=1} (Ax, x) \geq \lambda_j.
\]

Corollary B.2. Let \( A, B \) be hermitian matrices, with eigenvalues \( \lambda_1 \leq \cdots \leq \lambda_n \) and \( \mu_1 \leq \cdots \leq \mu_n \) respectively. Suppose \( \text{rank}(A - B) = k \). Then

\[
\mu_j \leq \lambda_j + k \\
\lambda_j \leq \mu_{j+k}.
\]

Lemma B.3. Let \( A, B \) be \( n \times n \) hermitian matrices, \( \text{rank}(A - B) = k > 0, B \neq 0 \). Then

\[
\log |\det B| - \log |\det A| \leq 2k \log \|B\| - 2k \log \text{dist}(sp A, 0).
\]

Proof. \( \det A = 0 \iff \text{dist}(sp A, 0) = 0 \). In this case, the inequality is trivial. So assume \( \det A \neq 0 \).

Let \( \lambda_1 \leq \cdots \leq \lambda_n \) and \( \mu_1 \leq \cdots \leq \mu_n \) be the eigenvalues of \( A \) and \( B \) respectively. Suppose \( \mu \leq 0 < \mu_{i+1} \).

By Lemma B.2, \( \lambda_{j-k} \leq \mu_j < 0 \) for \( k < j \leq i \); \( 0 < \mu_\ell \leq \lambda_{\ell+k} \) for \( i < \ell \leq n - k \).

\[
\log |\det B| - \log |\det A| = \sum_{k \leq j \leq i} (\log |\mu_j| - \log |\lambda_{j-k}|) + \sum_{i < j \leq n-k} (\log |\mu_j| - \log |\lambda_{j-k}|) 
+ \sum_j \log |\mu_j| + \sum_{j > n-k} \log |\mu_j| - \sum_{i-k < j \leq i+k} \log |\lambda_j| 
\leq 2k \log \|B\| - 2k \log \text{dist}(sp A, 0).
\]

Lemma B.4. Let \( A, B \) be \( n \times n \) hermitian matrices, with eigenvalues \( \lambda_1 \leq \cdots \leq \lambda_n \) and \( \mu_1 \leq \cdots \leq \mu_n \) respectively. Then \( |\lambda_j - \mu_j| \leq \|A - B\|, j = 1, \ldots, n \).
Proof.

\[ \mu_j = \max_{x \in M_j} (Bx, x) \quad \text{for some } M_j, \dim M_j = j \]

\[ \lambda_j = \min_{\dim M = j} \max_{x \in M} (Ax, x) \leq \max_{x \in M_j} (Ax, x) \]

\[ \leq \max_{x \in M_j} (Bx, x) + \max_{x \in M_j} ((A - B)x, x) \]

\[ \leq \mu_j + \|A - B\|. \]

Similarly, \( \mu_j \leq \lambda_j + \|A - B\|. \)

\[ \square \]

**Lemma B.5.** Let \( A \) be an \( n \times n \) hermitian matrix. Given an interval \((E', E'')\), let \( E = E' - n(E'' - E') \), \( E = E'' + n(E' - E') \). If \( m = \#(sp A \cap (E, E')) \), then for any \( E_1, E_2 \in (E', E'') \) one has

\[ \log |\det(A - E_2)| - \log |\det(A - E_1)| \leq 1 + m \log \|A - E_2\| - m \log \text{dist}(sp A, E_1) \]

**Proof.** We prove the case for \( E_1 < E_2 \). (The case for \( E_2 < E_1 \) is similar.) Let \( \lambda_1 \leq \cdots \leq \lambda_n \) be the eigenvalues of \( A \), \( \lambda_r < E < \lambda_{r+1}, \lambda_s < T \leq \lambda_{s+1} \).

For \( j > s \),

\[ |E_2 - \lambda_j| < |E_1 - \lambda_j| \]

For \( j \leq r \),

\[ \frac{|E_2 - \lambda_j|}{|E_1 - \lambda_j|} \leq \frac{|E_2 - E_1| + |E_1 - \lambda_j|}{|E_1 - \lambda_j|} \leq \frac{E'' - E'}{n(E'' - E')} + 1 = 1 + \frac{1}{n} \]

For \( r < j \leq s \),

\[ \frac{|E_2 - \lambda_j|}{|E_1 - \lambda_j|} \leq \frac{\|A - E_2\|}{\text{dist}(sp A, E_1)} \]

\[ \log |\det(A - E_2)| = \log \prod_{j=1}^{n} |E_2 - \lambda_j| \]

\[ = \log \prod_{j=1}^{n} |E_1 - \lambda_j| + \sum_{j \leq r} \log |E_2 - \lambda_j| + \sum_{r < j \leq s} \log |E_1 - \lambda_j| + \sum_{j > s} \log |E_2 - \lambda_j| \]

\[ \leq \log |\det(A - E_1)| + n \log(1 + \frac{1}{n}) + m \left( \log \|A - E_2\| - \log \text{dist}(sp A, E_1) \right) \]

The assertion now follows since \( n \log(1 + \frac{1}{n}) \leq 1 \).

\[ \square \]

**APPENDIX C. 1 DIMENSIONAL DIFFERENCE SCHRODINGER EQUATION**

Consider the 1 dimensional difference Schrödinger equation:

\[ -\varphi(k - 1) - \varphi(k + 1) + \lambda v(k) \varphi(k) = E \varphi(k), \quad k \in \mathbb{Z} \]

where \( v(k) \in \mathbb{R}, |v(k)| \leq C \).

The eigenvalues of this equation with Dirichlet boundary conditions, \( \varphi(a - 1) = \varphi(b + 1) = 0 \), are the same as the eigenvalues of

\[ H_{[a, b]} = \begin{pmatrix} \lambda v(a) & -1 & \cdots & -1 \\ -1 & \lambda v(a + 1) & \cdots & -1 \\ \vdots & \ddots & \ddots & \vdots \\ -1 & \cdots & -1 & \lambda v(b) \end{pmatrix} \]
Given initial condition \( \varphi(a - 1), \varphi(a) \), the solution of this equation, for \( b \geq a \), can be written as

\[
\begin{pmatrix}
\varphi(b + 1) \\
\varphi(b)
\end{pmatrix} = M_{[a,b]}(E) \begin{pmatrix}
\varphi(a) \\
\varphi(a - 1)
\end{pmatrix}
\]

where \( M_{[a,b]}(E) \) is the monodromy matrix

\[
M_{[a,b]}(E) = \prod_{k=a}^{b} A_k, \quad A_k = \begin{pmatrix}
\lambda v(k) - E & -1 \\
1 & 0
\end{pmatrix}
\]

The entries of the monodromy \( M_n \) are

\[
M_{[a,b]}(E) = \begin{pmatrix}
f_{[a,b]}(E) & -f_{[a+1,b]}(E) \\
f_{[a,b-1]}(E) & -f_{[a+1,b-1]}(E)
\end{pmatrix}
\]

where \( f_{[a,b]}(E) = \det(H_{[a,b]} - E) \).

If \( E \notin \text{sp} H_{[a,b]} \), then by Cramer’s rule,

\[
(H_{[a,b]} - E)^{-1}(k, \ell) = (H_{[a,b]} - E)^{-1}(\ell, k) \quad [a \leq k \leq \ell \leq b]
\]

\[
= \frac{f_{[a,k-1]}(E)f_{[\ell+1,b]}(E)}{f_{[a,b]}(E)}.
\]

By convention, \( f_{[a,a-1]}(E) = f_{[b+1,b]}(E) = 1 \).

Poisson’s Formula: If \( -\varphi(n - 1) - \varphi(n + 1) + v(n) \varphi(n) = E \varphi(n), \ E \notin \text{sp} H_{[a,b]} \). Then for \( a \leq m \leq b \)

\[
\varphi(m) = (H_{[a,b]} - E)^{-1}(m, a) \varphi(a - 1) + (H_{[a,b]} - E)^{-1}(m, b) \varphi(b + 1).
\]

Notation:

\[
T_{[a,b]} = H_{[a,b]} - \text{diag}(\lambda v(a), \ldots, \lambda v(b))
\]

\[
= \begin{pmatrix}
0 & -1 & & & \\
-1 & 0 & -1 & & \\
& -1 & 0 & \ddots & \\
& & \ddots & -1 & \\
& & & -1 & 0
\end{pmatrix}
\]

Note that \( \|T_{[a,b]}\| \leq 2 \).

Fix \( a, b \). Let \( n = b - a + 1 \).

Lemma C.1. If \( \mu = \text{min} \{\lambda v(j) - E\} > 2 \), then \( \text{dist}(\text{sp} H_{[a,b]}, E) > \mu \) and \( \log |f_{[a,b]}(E)| > n \log \mu \).

Proof. Let \( \mu_1 \leq \cdots \leq \mu_n \) be the eigenvalues of \( (H_{[a,b]} - E), \widehat{\mu}_j = \lambda v(i_j) - E, \ \widehat{\mu}_1 \leq \cdots \leq \widehat{\mu}_n \). By Lemma B.4, \( |\mu_j - \widehat{\mu}_j| \leq \|T_{[a,b]}\| \leq 2 \). Hence \( |\mu_j| \geq 2\mu - 2 > \mu \). Therefore

\[
\log |f_{[a,b]}(E)| = \Sigma \log |\mu_j| > n \log \mu.
\]

Lemma C.2. If \( \mu = \text{min} \{\lambda v(j) - E\} > 2 \), then \( \#(\text{sp}(H_{[a,b]} - E) \cap [-\mu, \mu]) \leq 1 \) and

\[
\log |f_{[a,b]}(E)| > (n - 1) \log \mu + \log \text{dist}(\text{sp} H_{[a,b]}, E).
\]
Lemma C.3. If \( \mu = \frac{1}{2} \min_{j \neq j_0} |\lambda v(j) - E| > 2, |E| \leq \max |\lambda v(j)| = \lambda C \) then
\[
\log |(H_{[a,b]} - E)^{-1}(k, \ell)| < -\frac{1}{2}(\ell - k) \log \mu
\]
provided \( |\ell - k| > \lambda^0, \lambda > \lambda_0(\vartheta, C) \).

**Proof.** Without loss of generality, assume \( \ell > k \). Let
\[
H_{[a,b,k,\ell]} = \begin{pmatrix} H_{[a,k-1]} & H_{[k,\ell]} \\ H_{[\ell+1,b]} \end{pmatrix}
\]
\[
f_{[a,b,k,\ell]}(E) = \det(H_{[a,b,k,\ell]} - E)
\]
\[
\text{rank}(H_{[a,b]} - H_{[a,b,k,\ell]}) \leq 4. \text{ Since } \log \text{dist}(\text{sp} H_{[a,b]}, E) > \log \mu > 0, \text{ by Lemma B.3}
\]
\[
\log |f_{[a,b,k,\ell]}(E)| - \log |f_{[a,b]}(E)| \leq 8 \log \|H_{[a,b,k,\ell]} - E\|
\]
\[
\log |(H_{[a,b]} - E)^{-1}(k, \ell)| = \log |f_{[a,k-1]}(E)| + \log |f_{[\ell+1,b]}(E)| - \log |f_{[a,b]}(E)|
\]
\[
= \log |f_{[a,b,k,\ell]}(E)| - \log |f_{[a,b]}(E)| - \log |f_{[k,\ell]}(E)|
\]
\[
\leq 8 \log \|H_{[a,b,k,\ell]} - E\| - \log |f_{[k,\ell]}(E)|
\]
\[
< -\frac{1}{2}(\ell - k) \log \mu \quad \text{provided } (\ell - k) > \lambda^0
\]

**Lemma C.4.** If \( \mu = \frac{1}{2} \min_{j \neq j_0} |\lambda v(j) - E| > 2, |E| \leq \max |\lambda v(j)| = \lambda C, E \notin \text{sp} H_{[a,b]} \). Then
\[
\log |(H_{[a,b]} - E)^{-1}(k, \ell)| < -\frac{1}{2}|k - \ell| \log \mu + 8 \log \text{dist}(\text{sp} H_{[a,b]}, E)|
\]
provided \( |k - \ell| > \lambda^0, \lambda > \lambda_0(\vartheta, C) \).

**Proof.** Without loss of generality, assume \( k < \ell \). If \( j_0 < k < \ell \) or \( k < \ell < j_0 \), then
\[
\log |(H_{[a,b]} - E)^{-1}(k, \ell)| = \log |f_{[a,k-1]}(E)| + \log |f_{[\ell+1,b]}(E)| - \log |f_{[a,b]}(E)|
\]
\[
= \log |f_{[a,b,k,\ell]}(E)| - \log |f_{[a,b]}(E)| - \log |f_{[k,\ell]}(E)|
\]
\[
\leq 8 \log \|H_{[a,b,k,\ell]} - E\| + 8 \log \text{dist}(\text{sp} H_{[a,b]}, E)| - (\ell - k + 1) \log \mu
\]
\[
< -\frac{1}{2}|k - \ell| \log \mu + 8 \log \text{dist}(\text{sp} H_{[a,b]}, E)|
\]
Suppose \( k \leq j_0 \leq \ell \). Let
\[
\tilde{H}_{[a,b]}(E) = \text{diag}(\lambda v(a), \ldots, \lambda v(j_0 - 1), 2\mu + E, \lambda v(j_0 + 1), \ldots, \lambda v(b)) + T_{[a,b]}
\]
\[
\tilde{f}_{[a,b]}(E) = \det(\tilde{H}_{[a,b]}(E) - E).
\]
By Lemma C.3,
\[
\log |(\widetilde{H}_{[a,b]}(E) - E)^{-1}(k, \ell)| \leq 8 \log(2\lambda C + 2) - (\ell - k + 1) \log \mu
\]
\[
\text{rank}(\widetilde{H}_{[a,b]}(E) - H_{[a,b]}) = 1. \text{ By Lemma B.3}
\]
\[
\log |\tilde{f}_{[a,b]}(E)| - \log |f_{[a,b]}(E)| \leq 2 \log \|\widetilde{H}_{[a,b]}(E) - E\| - 2 \log \text{dist}(\text{sp} H_{[a,b]}, E)
\]
\[
\leq 2 \log(2\lambda C + 2) + 2 \log \text{dist}(\text{sp} H_{[a,b]}, E)
\]
\[
\log |(H_{[a,b]} - E)^{-1}(k, \ell)| = \log |f_{[a,k-1]}(E)| + \log |f_{[\ell+1,b]}(E)| - \log |f_{[a,b]}(E)|
\]
\[
= \log \|H_{[a,b]}(E) - E\|^{-1}(k, \ell) + \log |\tilde{f}_{[a,b]}(E)| - \log |f_{[a,b]}(E)|
\]
\[
\leq 10 \log(2\lambda C + 2) - (\ell - k + 1) \log \mu + 2 \log \text{dist}(\text{sp} H_{[a,b]}, E)
\]
\[
< -\frac{1}{2} |k - \ell| \log \mu + 8 \log \text{dist}(\text{sp} H_{[a,b]}, E)
\]

\[\square\]

**Lemma C.5.** Let $K \in M_{n\times n}(\mathbb{C})$. Then
\[
|\log |\det(I + K)|| \lesssim n\|K\| \text{ provided } \|K\| < \frac{1}{2}.
\]

**Proof.** \(\text{sp}(I + K) \subset \{z \in \mathbb{C} : |z - 1| \leq \|K\|\} \), \(\det(I + K) = \Pi \lambda\) where the product runs over the eigenvalues of \(I + K\) with corresponding algebraic multiplicity. So
\[
(\frac{1}{2})^n < (1 - \|K\|)^n \leq |\det(I + K)| \leq (1 + \|K\|)^n
\]
if \(\|K\| < \frac{1}{2}\). Therefore
\[
|\log |\det(I + K)|| \lesssim n\|K\|.
\]
\[\square\]

**Lemma C.6.**
\[
|\log |\det \begin{pmatrix} a_1 & 1 & 1 \\ 1 & a_2 & 1 \\ 1 & \cdots & 1 \\ 1 & a_n \end{pmatrix} | - \sum_{j=1}^{n} \log |a_j| \lesssim \frac{n}{\min |a_j|}
\]
provided \(\min |a_j| > 2\).

**Proof.**
\[
\det \begin{pmatrix} a_1 & 1 & 1 \\ 1 & a_2 & 1 \\ 1 & \cdots & 1 \\ 1 & a_n \end{pmatrix} = \left( \prod_{j=1}^{n} a_j \right)^n \begin{pmatrix} 1 & 1/a_1 & 1/a_2 \\ 1/a_2 & 1 & 1/a_3 \\ 1/a_3 & \cdots & \ddots \\ 1/a_n & \cdots & \ddots \end{pmatrix}
\]
\[
|\log |\det \begin{pmatrix} a_1 & 1 & 1 \\ 1 & a_2 & 1 \\ 1 & \cdots & 1 \\ 1 & a_n \end{pmatrix} | - \sum_{j=1}^{n} \log |a_j| = |\log |\det \left( I + \begin{pmatrix} 0 & 1/a_1 & 1/a_2 \\ 1/a_2 & 0 & 1/a_3 \\ 1/a_3 & \cdots & \ddots \end{pmatrix} \right) | \lesssim \frac{n}{\min |a_j|}
\]
provided \(\min |a_j| > 2\).
\[\square\]
Lemma C.7. If \( \min_{a \leq k \leq b} |\lambda v(k) - E| \geq \lambda^A \), \( a \leq i < j < b \). Then

\[
\left| \frac{f_{[a,i]}(E)f_{[j,b]}(E)}{f_{[a,b]}(E)} \right| \leq \lambda^{-A|j-i-1|}
\]

provided \( n = b - a + 1 \lesssim \lambda^A \).

Proof.

\[
\log \left| f_{[a,i]}(E) \right| \leq \sum_{k=a}^{i} \log |\lambda v(k) - E| + n\lambda^{-A}
\]
\[
\log \left| f_{[j,b]}(E) \right| \leq \sum_{k=j}^{b} \log |\lambda v(k) - E| + n\lambda^{-A}
\]
\[
\log \left| f_{[a,b]}(E) \right| \geq \sum_{k=a}^{b} \log |\lambda v(k) - E| - n\lambda^{-A}
\]
\[
\log \left| \frac{f_{[a,i]}(E)f_{[j,b]}(E)}{f_{[a,b]}(E)} \right| \leq -\sum_{i<k<j} \log |\lambda v(k) - E| + 3n\lambda^{-A}
\]
\[
\leq -(j-i-1)\log \lambda^A + 3n\lambda^{-A}
\]

\( \square \)

Corollary C.8. \( |(H_{[a,b]} - E)^{-1}(i,j)| \leq \lambda^{-A(i-j)} \) if \( \min_{a \leq k \leq b} |\lambda v(k) - E| \geq \lambda^A \), \( n \lesssim \lambda^A \).

Lemma C.9. Suppose \( [a, b] = \bigcup_{k=1}^{K} [a'_k, b'_k] \), \( a'_1 = a \), \( b'_K = b \), \( a'_k < b'_{k-1} < a'_{k+1} < b'_k \) for \( k = 2, 3, \ldots, K-1 \), \( b'_{k-1} - a'_k > n^\tau \) for \( k = 2, 3, \ldots, K \). Assume that

\[
|(H_{[a',b']} - E)^{-1}(i,j)| \leq \exp(-\gamma|i-j|)
\]
when \( |i-j| \geq \frac{1}{2}n^\tau \) for all \( k \). Then \( E \notin \text{sp} \ H_{[a,b]} \) provided \( n > n_0(\tau, \gamma) \).

Proof. Suppose, for contradiction, that \( H_{[a,b]} \varphi = E \varphi \), \( \| \varphi \| = 1 \). Let

\[
x_k = \left[ \frac{1}{2}(a'_k + b'_{k-1}) \right], \quad k = 2, \ldots, K.
\]

Then \( a'_k < x_k < x_{k+1} < b_k \). For any \( j \in [x_k, x_{k+1}] \), by Poisson’s formula

\[
\varphi(j) = (H_{[a'_k,b'_k]} - E)^{-1}(j,a'_k)\varphi(a'_k - 1) + (H_{[a'_k,b'_k]} - E)^{-1}(j,b'_k)\varphi(b'_k + 1)
\]
\[
|\varphi(j)| \leq 2\exp\left[-\frac{\gamma}{2}n^{\tau}\right]
\]
since \( |j-a'_k| \geq \frac{1}{2}n^\tau \), \( |j-b'_k| \geq \frac{1}{2}n^\tau \).

Similarly, for \( j \in [a, x_2] \cup [x_K, b] \),

\[
|\varphi(j)| \leq \exp\left[-\frac{\gamma}{2}n^{\tau}\right]
\]

Since

\[
[a, b] = \left( \bigcup_{k=2}^{K-1} [x_k, x_{k+1}] \right) \cup [a, x_2] \cup [x_K, b]
\]
\[
1 = \sum_{j=a}^{b} |\varphi(j)|^2 \leq n \left[ 4\exp(-\gamma n^{\tau}) \right]
\]
Therefore

\( \log \| M_{[a,b]}(E) \| + \log \| M_{[b+1,c]}(E) \| - \log \| M_{[a,c]}(E) \| \leq 20 [\log(\lambda C_0) - \log \kappa] \)

provided \( \lambda C_0 \gg 1 \).

**Proof.**

\[
\| M_{[a,c]}(E) \| \geq |f_{[a,c]}(E)|
\]

\[
\| M_{[a,b]}(E) \| \leq |f_{[a,b]}(E)| + |f_{[a,b-1]}(E)| + |f_{[a+1,b-1]}(E)| + |f_{[a+1,b]}(E)|
\]

\[
\| M_{[b+1,c]}(E) \| \leq |f_{[b+1,c]}(E)| + |f_{[b+1,c-1]}(E)| + |f_{[b+2,c]}(E)| + |f_{[b+1,c-1]}(E)|
\]

(C.1)

\[
\frac{\| M_{[a,b]}(E) \| \| M_{[b+1,c]}(E) \|}{\| M_{[a,c]}(E) \|} \leq \left( \frac{|f_{[a,b]}(E)| + \cdots + |f_{[b+1,c]}(E)| + \cdots}{|f_{[a,c]}(E)|} \right).
\]

Expand the numerator, we get 16 terms. One of the terms is

\[
|f_{[a+1,b-1]}(E)| |f_{[b+2,c-1]}(E)| = \left| \det \begin{pmatrix} H_{[a+1,b-1]} & I_2 \\ I_1 & H_{[b+2,c-1]} \end{pmatrix} \right| = |\det \tilde{H}|
\]

\( \text{rank}(H_{[a,c]} - \tilde{H}) = 8 \). By Lemma B.3

\[
\log |f_{[a+1,b-1]}(E) f_{[b+1,c-1]}(E)| - \log |f_{[a,c]}(E)| \leq 16 \log \| \tilde{H} \| - 16 \log \text{dist}(\text{sp} H_{[a,c]}, E)
\]

Therefore

\[
\frac{|f_{[a+1,b-1]}(E)| |f_{[b+1,c-1]}(E)|}{|f_{[a,c]}(E)|} \leq \left( \frac{2 \lambda C_0 + 2}{\kappa} \right)^{16}
\]

Same estimate holds for other terms in (C.1). So

\[
\frac{\| M_{[a,b]}(E) \| \| M_{[b+1,c]}(E) \|}{\| M_{[a,c]}(E) \|} \leq 16 \left( \frac{2 \lambda C_0 + 2}{\kappa} \right)^{16}
\]

Hence

\[
\log \| M_{[a,b]}(E) \| + \log \| M_{[b+1,c]}(E) \| - \log \| M_{[a,c]}(E) \| \leq \log 16 + 16 \log \left( \frac{2 \lambda C_0 + 2}{\kappa} \right)
\]

**Appendix D. Avalanche Principle**

**Proposition D.1.** *(Goldstein, Schlag)* [GS1] Let \( A_1, \ldots, A_n \) be a sequence of 2 \( \times \) 2 matrices whose determinants satisfy

\[
\max_{1 \leq j \leq n} |\det A_j| \leq 1.
\]

Suppose that

\[
\min_{1 \leq j \leq n} \|A_j\| \geq \mu > n
\]

and

\[
\max_{1 \leq j < n} \left[ \log \|A_{j+1}\| + \log \|A_j\| - \log \|A_{j+1}A_j\| \right] < \frac{1}{2} \log \mu.
\]

Then

\[
\left| \log \|A_n \ldots A_1\| + \sum_{j=2}^{n-1} \log \|A_j\| - \sum_{j=1}^{n-1} \log \|A_{j+1}A_j\| \right| \leq \frac{n}{\mu}.
\]
Proposition D.2. \textbf{(Goldstein, Schlag)}[GS2] For \(a_0 < a_1 < \cdots < a_n\), if the monodromy matrices \(M_{(a_{j-1}, a_j)} =: A_j\) satisfy the conditions of Proposition 1, then
\[
\left| \log \|M_{(a_0, a_n)}\| + \sum_{j=2}^{n-1} \log \|M_{(a_{j-1}, a_j)}\| - \sum_{j=3}^{n-1} \log \|M_{(a_{j-2}, a_j)}\| \right| \lesssim \frac{n}{\mu}.
\]
Moreover, if \(A_j := M_{(a_{j-1}, a_j)}\) for \(1 < j < n\),
\[
A_1 := M_{(a_0, a_1)} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad A_n := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} M_{(a_{n-1}, a_n)}
\]
satisfy the conditions of the proposition, then
\[
\left| \log f_{[a_0, a_n]}(E) + \sum_{j=2}^{n-1} \log \|M_{(a_{j-1}, a_j)}\| - \sum_{j=3}^{n-1} \log \|M_{(a_{j-2}, a_j)}\| - \log \|M_{(a_0, a_2)} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \log \| \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} M_{(a_{n-2}, a_n)} \right| \lesssim \frac{n}{\mu}
\]

Proposition E.1. Let \(f(x, y)\) be \(C^1\) on \((a, b) \times (c, d) =: D\). Assume \(\partial_y f(x, y) \geq \mu > 0\), \(|\partial_x f(x, y)| \leq \kappa\) for all \((x, y) \in D\). Given \(\rho > 0\), there exists \(n\)-simple set \(D_0 \subset D\), \(n = 2m\), \(|D \setminus D_0| \leq (b-a)\left(\frac{2n}{\mu} + \frac{\kappa}{\rho} \frac{\mu}{\kappa} \right)\) such that \(|f(x, y)| \geq \rho\) on \(D_0\).

Proof. For \(x \in (a, b)\), define \(s^+(x) = \{y \in (c, d) : f(x, y) \geq \rho\}\), \(s^-(x) = \{y \in (c, d) : f(x, y) \leq -\rho\}\) and
\[
y^+(x) = \begin{cases} d & \text{if } s^+ = \emptyset \\ \inf s^+ & \text{if } s^+ \neq \emptyset \end{cases}
\]
\[
y^-(x) = \begin{cases} c & \text{if } s^- = \emptyset \\ \sup s^- & \text{if } s^- \neq \emptyset \end{cases}
\]
Let
\[
J_k = \left(a + \frac{k-1}{m}(b-a), a + \frac{k}{m}(b-a)\right), \quad x_k = a + \frac{2k-1}{2m}(b-a), \quad k = 1, 2, \ldots, m,
\]
\[
y_{k,1} = \min \left\{ y^+(x_k) + \frac{1}{2m}, d \right\}
\]
\[
y_{k,2} = \max \left\{ y^-(x_k) - \frac{1}{2m}, c \right\}
\]
Take
\[
D_{k,1} = J_k \times (y_{k,1}, d)
\]
\[
D_{k,2} = J_k \times (c, y_{k,2})
\]
\[
D_0 = \bigcup_k (D_{k,1} \cup D_{k,2})
\]
Then $|f(x, y)| \geq \rho$ on $D_0$ and

$$0 < y_{k, 1} - y_{k, 2} \leq \frac{2\rho}{\mu} + \frac{\kappa}{m \mu} \implies |D \setminus D_0| \leq (b - a)\left(\frac{2\rho}{\mu} + \frac{\kappa}{m \mu}\right)$$

\[ \square \]

**Lemma E.2.** Suppose $f : [a, b] \to \mathbb{R}$ is $C^2$, monotone. If $|f''(x)| \geq C$ for all $x \in (a, b)$, then

$$\text{mes}\{x \in [a, b] : |f(x)| \leq \varepsilon\} \leq \left(\frac{\varepsilon}{C}\right)^{1/2}.$$ 

**Proof.** Without loss of generality, assume $f$ is increasing. (Otherwise, consider $-f$.)

Let $(\tilde{a}, \tilde{b}) \subset [\bar{a}, \bar{b}] = \{x \in [a, b] : |f(x)| \leq \varepsilon\}$. If $f''(x) \geq C$ then there is $\tilde{c} \in (\tilde{a}, \tilde{b})$ such that

$$f(\tilde{b}) - f(\tilde{a}) = f' (\tilde{a})(\tilde{b} - \tilde{a}) + \frac{1}{2} f''(\tilde{c})(\tilde{b} - \tilde{a})^2$$

$$2\varepsilon \geq \frac{1}{2} C(\tilde{b} - \tilde{a})^2$$

$$\tilde{b} - \tilde{a} \lesssim \left(\frac{\varepsilon}{C}\right)^{1/2}.$$ If $f''(x) \leq -C$ then

$$f(\tilde{a}) - f(\tilde{b}) = f' (\tilde{b})(\tilde{a} - \tilde{b}) + \frac{1}{2} f''(\tilde{c})(\tilde{b} - \tilde{a})^2$$

$$f(\tilde{b}) - f(\tilde{a}) = f' (\tilde{b})(\tilde{b} - \tilde{a}) + \frac{1}{2} [-f''(\tilde{c})](\tilde{b} - \tilde{a})^2$$

$$2\varepsilon \geq \frac{1}{2} C(\tilde{b} - \tilde{a})^2$$

$$\tilde{b} - \tilde{a} \lesssim \left(\frac{\varepsilon}{C}\right)^{1/2}.$$ 

$$\text{mes}\{x \in [a, b] : |f(x)| \leq \varepsilon\} = \sup \tilde{b} - \tilde{a} \lesssim \left(\frac{\varepsilon}{C}\right)^{1/2}$$

\[ \square \]

**Corollary E.3.** Suppose $f : [a, b] \to \mathbb{R}$ is $C^3$, such that $|f'(x)| + |f''(x)| \geq C, |f''(x)| + |f'''(x)| \leq \kappa$. Then for $\varepsilon \lesssim C \lesssim \kappa(b - a)$, one has

$$\text{mes}\{x \in [a, b] : |f(x)| \leq \varepsilon\} \lesssim \left(\frac{\varepsilon}{C}\right)^{1/2}(b - a).$$

**Proof.** Let $x_i = a + \frac{k}{N}a, i = 0, 1, 2, \ldots, N$ where $\frac{C}{\varepsilon} \leq \frac{b}{N} \leq \frac{C}{\varepsilon}$. On each $x_i$, either $|f'(x_i)| \geq \frac{C}{2}$ or $|f'''(x_i)| \geq \frac{C}{2}$.

If $|f'''(x_i)| \geq \frac{C}{2}$ then $|f''(x)| \geq \frac{C}{2}$ for all $x \in [x_{i-1}, x_i]$ since $\text{sgn} f''$ does not change in $[x_{i-1}, x_i]$, $f$ has at most two monotonicity intervals in $[x_{i-1}, x_i]$. Hence $\text{mes}\{x \in [x_{i-1}, x_i] : |f(x)| \leq \varepsilon\} \lesssim \left(\frac{\varepsilon}{C}\right)^{1/2}$.

If $|f'(x_i)| \geq \frac{C}{2}$ then $|f'(x)| \geq \frac{C}{2}$ for all $x \in [x_{i-1}, x_i]$. So $\text{mes}\{x \in [x_{i-1}, x_i] : |f(x)| \leq \varepsilon\} \leq \frac{C}{2\varepsilon} \lesssim \left(\frac{C}{\varepsilon}\right)^{1/2}$. Therefore

$$\text{mes}\{x \in [a, b] : |f(x)| \leq \varepsilon\} \lesssim \left(\frac{C}{\varepsilon}\right)^{1/2} \lesssim \kappa \left(\frac{C}{\varepsilon}\right)^{1/2}(b - a).$$

\[ \square \]

Let $F_\ell(x, y)$ be $C^3$ on $(a_\ell, b_\ell) \times (c, d)$, where $(a_\ell, b_\ell), (c, d) \subset \mathbb{T}, \ell \in \{1, 2\}$. Suppose

$$0 < \mu \leq |\partial_y F_1(x, y)|$$

$$|\partial_y F_1|, |\partial_x F_1|, |\partial_{xx} F_1|, |\partial_{yy} F_1|, |\partial_{xy} F_1| \leq \kappa$$
Since $\partial_y F_1 \neq 0$, we can define $y(. , t)$ such that $F_1(x, y(x, t)) = t$ for $x \in \pi_1(F_1^{-1}(t))$. Furthermore,

$$|\partial_x y(x, t)| \leq \frac{\kappa}{\mu}, \quad |\partial_{xx} y(x, t)| \leq \frac{4\kappa^3}{\mu^2}$$

Let $x_i = a_1 + \frac{i}{N_1}, i = 0, 1, 2, \ldots, |N(b_1 - a_1)| =: N_1$. Define

$$\xi_i = \min \{ |\partial_y y(x, t)| : (x, t) \in [x_{i-1}, x_i] \times [-\varepsilon, \varepsilon] \} .$$

Suppose $\xi_i \leq \rho$. Then there is $t \in [-\varepsilon, \varepsilon]$ such that

$$\max \{ y([x_{i-1}, x_i], t) \} \leq \frac{\rho}{N} + \frac{4\kappa^3}{\mu^2}.$$ 

Since $|y(x, s) - y(x, t)| \leq \frac{|x - t|}{N}$,

$$\bigcup_{t \in [-\varepsilon, \varepsilon]} \{ y([x_{i-1}, x_i], t) \} \subset [\tilde{e}_i, \tilde{d}_i]$$

$\tilde{d}_i - \tilde{e}_i \leq \frac{\rho}{N} + \frac{4\kappa^3}{\mu^2} + \frac{2\kappa}{\mu}$. Write $(c, d) \backslash \bigcup_{\xi_i \leq \rho} [\tilde{e}_i, \tilde{d}_i] = \bigcup_{j = 1}^{J} (c_j, d_j), J \leq N_1 + 1$. On each $(c_j, d_j)$, define the inverse functions of $y$, i.e. $F_1(\phi_{jk}(y(t), y)) = t$ for $y \in (c_j, d_j) \cap \pi_2(F_1^{-1}(t)), k = 1, 2, \ldots, K, K_j \leq N_1 + \frac{b_1 - a_1}{d - e} \leq 2N_1$ provided $N \geq \frac{1}{d - e} \frac{\kappa}{\mu}$.

**Lemma E.4.** Let $y_0 \in (c_j + \frac{2\kappa}{\mu}, d_j - \frac{2\kappa}{\mu})$. Then $|\phi_{jk}(y_0, s) - \phi_{jk}(y_0, t)| \leq \frac{2\kappa}{\mu^2}$ for any $s, t \in [-\varepsilon, \varepsilon]$ such that $\phi_{jk}(y_0, s)$ and $\phi_{jk}(y_0, t)$ are defined.

**Proof.** Let $a = \phi_{jk}(y_0, s), b = \phi_{jk}(y_0, t)$. Then $F_1(a, y_0) = s$. Since $|\partial_y F_1(x, y)| > \mu$ and $|s - t| \leq 2\varepsilon$, there is $y_1 \in [y_0 - \frac{2\kappa}{\mu}, y_0 + \frac{2\kappa}{\mu}] \subset (c_j, d_j)$ such that $F_1(a, y_1) = t$. Hence, $y(x, t)$ is defined for all $x$ between $a$ and $b$. Therefore, there exists $\bar{x}$ between $a$ and $b$ such that

$$\frac{2\varepsilon}{\mu} \geq |\phi_{jk}(y_0, s) - \phi_{jk}(y_0, t)| = \frac{|y(a, s) - y(a, t)|}{|\partial_y F_1(\phi_{jk}(y_0, s)) - \partial_y F_1(\phi_{jk}(y_0, t))|} = |\partial_x y(\bar{x}, t)| > \rho$$

$\square$

**Lemma E.5.** Suppose $b_1 - a_1 > \varepsilon^{-1/100}$ and $d - c > \varepsilon^{-1/100}$. If

$$|\partial_x F_2(x, y)| + |\partial_{xx} F_2(x, y)| \geq C > \varepsilon^{-1/100}$$

for all $(x, y) \in (a_2, b_2) \times (c, d)$, then for $\kappa \leq \varepsilon^{-1/100}, \mu \geq \varepsilon^{-1/100}$, one has

$$\max \{ y \in (c, d) : \exists x \in (a_1, b_1), n > \varepsilon^{-1/10} \text{ such that } |F_1(x, y)| \leq \varepsilon, |F_2(x + ny, y)| \leq \varepsilon \} \lesssim \varepsilon^\theta(d - c)(b_1 - a_1).$$

**Proof.** Choose $\varepsilon = \varepsilon^{1/25}, N \approx \varepsilon^{-1/10}$. Fix $j, k, t, n$. Let $f(y) = F_2(\phi_{jk}(y, t) + ny, y)$. Then

$$f'(y) = (\partial_x F_2)(\partial_y \phi_{jk} + n) + \partial_y F_2$$

$$f''(y) = (\partial_{xx} F_2)(\partial_y \phi_{jk} + n)^2 + (\partial_x F_2)(\partial_{yy} \phi_{jk} + 2\partial_{xy} F_2)(\partial_y \phi_{jk} + n) + \partial_{yy} F_2$$

for all $y$ where $f$ is defined.

$$|f'(y)| + |f''(y)| \geq \begin{cases} \frac{C}{2}(n - \varepsilon^{-1}) - \kappa & \text{if } |\partial_x F_2| \geq \frac{C}{2} \\ \frac{C}{2}(n - \varepsilon^{-1})^2 - \kappa \frac{4\kappa^3}{\mu^2} - 2\kappa(n + \kappa) - \kappa & \text{if } |\partial_{xx} F_2| \geq \frac{C}{2} \\ nC/4 & \end{cases}$$
By Corollary E.3,
\[ \operatorname{mes}\left\{ y \in (c_j + \frac{2\varepsilon}{\mu}, d_j - \frac{2\varepsilon}{\mu}) : |f(y)| \leq \varepsilon^{1/2} \right\} \lesssim \frac{\kappa}{nC} \left( \frac{1}{nC} \right)^{1/2} (d_j - c_j) \]

Also, by Lemma E.4,
\[ |F_2(\phi_{jk}(y, s) + ny, y) - F_2(\phi_{jk}(y, t) + ny, y)| \leq \kappa |\phi_{jk}(y, s) - \phi_{jk}(y, t)| \leq \frac{2K\varepsilon}{\mu \rho} \]

for all s such that \( \phi_{jk}(y, s) \) is defined. If \( |f(y)| > \varepsilon^{1/2} \) then
\[ |F_2(\phi_{jk}(y, s) + ny, y)| > |f(y)| - \frac{2K\varepsilon}{\mu \rho} > \varepsilon \]

Hence,
\[ \operatorname{mes}\left\{ y \in (c, d) : \exists x \in (a_1, b_1), n > \frac{1}{\varepsilon^{1/10}} \text{ such that } |F_1(x, y)| \leq \varepsilon, |F_2(x + ny, y)| \leq \varepsilon \right\} \lesssim (N_1 + 1)(2N_1) \frac{K}{C} \left( \frac{1}{nC} \right)^{1/2} (d - c) \sum_{n > \varepsilon^{-1/10}} \frac{1}{n^{1/2}} + N \left( \frac{\rho}{N} + \frac{4K^3}{\mu^2N^2} + \frac{6\varepsilon}{\mu} \right) \lesssim \varepsilon^9 (d - c)(b_1 - a_1) \]

Fix \( n_1 \ll n_2 \).

Let \( F_1(x, \omega) = E_1(x, \omega) - E_2(x + n_1\omega, \omega) \) for \((x, \omega) \in (a_1, b_1) \times (c, d)\)

\[ F_2(\tilde{x}, \omega) = E_2(\tilde{x}, \omega) - E_3(\tilde{x} + n_2\omega, \omega) \] for \((\tilde{x}, \omega) \in (a_2, b_2) \times (c, d)\)

where \(|\partial_x E_\ell(x, \omega)| + |\partial_{xx} E_\ell(x, \omega)| > \delta\). Also, assume
\[ |\partial_\omega E_\ell|, |\partial_{x\omega} E_\ell|, |\partial_{xx\omega} E_\ell| \leq \kappa \]

**Theorem E.6.** For \( \varepsilon \ll n_1 \delta \),
\[ \operatorname{mes}\left\{ \omega \in (c, d) : \exists x \in (a_1, b_1), x + n_1\omega \in (a_2, b_2) \text{ such that } |F_1(x, \omega)| \leq \varepsilon, |F_2(x + n_1\omega, \omega)| \leq \varepsilon \right\} \lesssim \varepsilon^9 (d - c)(b_1 - a_1) \]

**Proof.** Let \( \tilde{x}_i = a_1 + \frac{n_1 \delta}{\mu} i \), \( \tilde{\omega}_i = \omega + \frac{n_1 \delta}{\mu} j \), \( D_{ij} = [\tilde{x}_{i-1}, \tilde{x}_i] \times [\tilde{\omega}_{j-1}, \tilde{\omega}_j] \). Either

1. \(|\partial_\omega F_1| > \frac{n_1 \delta}{\mu}\) for all \((x, \omega) \in D_{ij}\); or
2. \(|\partial_{x\omega} F_1| > \frac{n_1 \delta}{\mu}\) for all \((x, \omega) \in D_{ij}\).

Choose \( \mu \) so that \( \left( \frac{n_1 \delta}{\mu} \right)^{100} > \mu \gg \varepsilon^{1/100} \). In case (1),
\[ \operatorname{mes}\left\{ \omega \in (\tilde{\omega}_{j-1}, \tilde{\omega}_j) : \exists x \in (\tilde{x}_{i-1}, \tilde{x}_i) \text{ such that } |F_1(x, \omega)| \leq \varepsilon, |F_2(x + n_1\omega, \omega)| \leq \varepsilon \right\} \lesssim \varepsilon^9 (\tilde{\omega}_j - \tilde{\omega}_{j-1})(\tilde{x}_i - \tilde{x}_{i-1}) \]

In case (2), consider \( G_1 = \partial_\omega F_1, G_2 = F_2 \). Applying Lemma E.5 to \( G_1, G_2 \) gives
\[ \operatorname{mes} \Omega_{ij} := \operatorname{mes}\left\{ \omega \in (\tilde{\omega}_{j-1}, \tilde{\omega}_j) : \exists x \in (\tilde{x}_{i-1}, \tilde{x}_i) \text{ such that } |G_1(x, \omega)| \leq \mu, |G_2(x + n_1\omega, \omega)| \leq \mu \right\} \lesssim \mu^9 (\tilde{\omega}_j - \tilde{\omega}_{j-1})(\tilde{x}_i - \tilde{x}_{i-1}) \]

Also,
\[ \operatorname{mes}\left\{ \omega \in (\tilde{\omega}_{j-1}, \tilde{\omega}_j) \setminus \Omega_{ij} : \exists x \in (\tilde{x}_{i-1}, \tilde{x}_i) \text{ such that } |F_1(x, \omega)| \leq \varepsilon, |F_2(x + n_1\omega, \omega)| \leq \varepsilon \right\} \lesssim \varepsilon^9 (\tilde{\omega}_j - \tilde{\omega}_{j-1})(\tilde{x}_i - \tilde{x}_{i-1}) \]

Hence,
\[ \operatorname{mes}\left\{ \omega \in (c, d) : \exists x \in (a_1, b_1), |F_1(x, \omega)| \leq \varepsilon, |F_2(x + n_1\omega, \omega)| \leq \varepsilon \right\} \lesssim \varepsilon^9 (d - c)(b_1 - a_1) \]
Appendix F. Volume of Hyperplane

Given $0 \neq a = (a_1, \ldots, a_N) \in \mathbb{R}^N$, $C \in \mathbb{R}$. Let $P(a, C) = \{x \in \mathbb{R}^N, a \cdot x = C\}$. Suppose $a_i \neq 0$. For any $\delta > 0$ define

$$P_{i, \delta}(a, C) = \{x \in P(a, C) : |x_j| \leq \delta \text{ for } j \neq i\}.$$ 

Then $\text{Vol}_{N-1}(P_{i, \delta}(a, C)) = \frac{|a|}{|a_i|}(2\delta)^{N-1}.$

Lemma F.1. Suppose $a_j \geq 0$ for all $j = 1, \ldots, N$, $\Sigma a_j = 1$. For any $C \in \mathbb{R}$, $\delta, \varepsilon > 0$, let

$$P_\varepsilon(a, C, \varepsilon) = \{\xi \in (\xi_1, \ldots, \xi_N) : |\xi_j| \leq \delta \text{ for all } j, |a \cdot \xi - C| \leq \varepsilon\}$$

Then $(2\delta)^{-N} \text{mes } P_\varepsilon(a, C, \varepsilon) \leq N \varepsilon \delta$.

Proof. Write $\xi = t \frac{a_i}{\|a_i\|} + \eta$ where $\eta \cdot a = 0$. Then $\xi \cdot a = t \|a\|$. From the hypothesis, there is $|a_i| \geq \frac{1}{N}$

$$\xi \in P_\varepsilon(a, C, \varepsilon) \implies t \in \left[ \frac{C}{\|a\|} - \frac{\varepsilon}{\|a\|}, \frac{C}{\|a\|} + \frac{\varepsilon}{\|a\|} \right] =: J$$

$$\text{mes } P_\varepsilon(a, C, \varepsilon) \leq \int_J P_{i, \delta}(a, y) dy = \frac{2\varepsilon}{|a_i|}(2\delta)^{N-1}$$

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