CURRENT SUPERALGEBRAS AND UNITARY REPRESENTATIONS

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Abstract. In this paper we determine the projective unitary representations of finite dimensional Lie supergroups whose underlying Lie superalgebra is \( g = A \otimes \mathfrak{t} \), where \( \mathfrak{t} \) is a compact simple Lie superalgebra and \( A \) is a supercommutative associative (super)algebra; the crucial case is when \( A = \Lambda_\ast(\mathbb{R}) \) is a Grassmann algebra. Since we are interested in projective representations, the first step consists in determining the cocycles defining the corresponding central extensions. Our second main result asserts that, if \( \mathfrak{t} \) is a simple compact Lie superalgebra with \( \mathfrak{t}_1 \neq \{0\} \), then each (projective) unitary representation of \( \Lambda_\ast(\mathbb{R}) \otimes \mathfrak{t} \) factors through a (projective) unitary representation of \( \mathfrak{t} \) itself, and these are known by Jakobsen’s classification. If \( \mathfrak{t}_1 = \{0\} \), then we likewise reduce the classification problem to semidirect products of compact Lie groups \( K \) with a Clifford–Lie supergroup which has been studied by Carmeli, Cassinelli, Toigo and Varadarajan.

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1. Introduction

In a similar fashion as projective unitary representations $\pi: G \to \text{PU}(\mathcal{H})$ of a Lie group $G$ implement symmetries of quantum systems modelled on a Hilbert space $\mathcal{H}$, projective unitary representations of Lie supergroups implement symmetries of super-symmetric quantum systems [2]. Here the Hilbert space is replaced by a super Hilbert space $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, i.e., a direct sum of two subspaces corresponding to a $\mathbb{Z}_2$-grading of $\mathcal{H}$. We deal with Lie supergroups as Harish–Chandra pairs $\mathcal{G} = (G, \mathfrak{g})$, where $G$ is a Lie group and $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a Lie superalgebra, where $\mathfrak{g}_0$ is the Lie algebra of $G$, and we have an adjoint action of $G$ by automorphisms of the Lie superalgebra $\mathfrak{g}$ extending the adjoint action of $G$ on its Lie algebra $\mathfrak{g}_0$.

The concept of a unitary representation of a Lie supergroup $\mathcal{G}$ consists of a unitary representation $\pi$ of $G$ by grading preserving unitary operators and a representation $\chi_\pi$ of the Lie superalgebra $\mathfrak{g}$ on the dense subspace $\mathcal{H}^\infty$ of smooth vectors for $G$ such that natural compatibility conditions are satisfied (see Definition [2] for details). To accomodate the fact that the primary interest lies in projective unitary representations, one observes that projective representations lift to unitary representations of central extensions by the circle group $\mathbb{T}$ acting on $\mathcal{H}$ by scalar multiplication. Having this in mind, one can directly study unitary representations of central extensions (see [6] for more details on this passage).

The corresponding classification problem splits into two layers. One is to determine the even central extensions of a given Lie supergroup $\mathcal{G}$ and the second consists of determining for these central extensions the corresponding unitary representations.

The existence of an invariant measure implies that for any finite dimensional Lie group $G$, unitary representations exist in abundance, in particular the natural representation on $L^2(G)$ is injective. This is drastically different for Lie supergroups, for which all unitary representations may be trivial. The reason for this is that, for every unitary representation $\chi: \mathfrak{g} \to \text{End}(\mathcal{H}^\infty)$ and every odd element $x_1 \in \mathfrak{g}_1$, the operator $-i\chi([x_1, x_1])$ is non-negative. This imposes serious positivity restrictions on the representations on the even part $\mathfrak{g}_0$, namely that $-i\chi(x) \geq 0$ for all elements in the closed convex cone $\mathcal{C}(\mathfrak{g}) \subseteq \mathfrak{g}_0$ generated by all brackets $[x_1, x_1]$ of odd elements. Accordingly, $\mathfrak{g}$ has no faithful unitary representation if the cone $\mathcal{C}(\mathfrak{g})$ is not pointed (cf. [9]). Put differently, the kernel of any unitary representation contains the ideal $\text{rad}(\mathfrak{g})$ of the Lie superalgebra $\mathfrak{g}$ generated by the linear subspace $\mathcal{C} := \mathcal{C}(\mathfrak{g}) \cap -\mathcal{C}(\mathfrak{g})$ of $\mathfrak{g}_0$ and all those elements $x \in \mathfrak{g}_1$ with $[x, x] \in \mathcal{C}$.

A particularly simple but nevertheless important class of Lie superalgebras are the Clifford–Lie superalgebras $\mathfrak{g}$ for which $[\mathfrak{g}_0, \mathfrak{g}] = \{0\}$ ($\mathfrak{g}_0$ is central), so that the Lie bracket of $\mathfrak{g}$ is determined by a symmetric bilinear map $\mu: \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_0$. If $\mathfrak{g}_0 = \mathbb{R}$ and the symmetric bilinear form $\mu$ is indefinite, then $\mathfrak{g}$ has no non-zero unitary representations.

In [1] the authors have determined the structure of finite dimensional Lie superalgebras $\mathfrak{g}$ for which finite dimensional unitary representations exist. This property implies in particular that $\mathfrak{g}$ is compact in the sense that $e^{\text{ad}\mathfrak{g}_0} \subseteq \text{Aut}(\mathfrak{g})$ is a compact subgroup, but, unlike the purely even case, this condition is not sufficient for the existence of finite dimensional projective unitary representations. In particular, it is shown in [1] that only four families of simple compact Lie superalgebras have finite dimensional projective unitary representations: $\mathfrak{su}(n|m; \mathbb{C}), \mathfrak{psu}(n|m; \mathbb{C}), \mathfrak{c}(n)$ and $\mathfrak{pq}(n)$ (see Subsection [2] for details).
In this paper we take the next step by considering current Lie superalgebras \( g = A \otimes \mathfrak{k} \), where \( \mathfrak{k} \) is a compact simple Lie superalgebra and \( A \) is a supercommutative associative (super)algebra and study the projective unitary representations, resp., the unitary representations of central extensions of these Lie superalgebras. Since we are interested in the phenomena caused by the superstructure, the main interest lies in algebras \( A \) generated by their odd part \( A_1 \). As the supercommutativity implies that the squares of odd elements in \( A \) vanish, any such \( A \) is a quotient of a Graßmann algebra. Therefore the main point is to understand current superalgebras of the form \( A_s(\mathbb{R}) \otimes \mathfrak{k} \), where \( \mathfrak{k} \) is a compact simple Lie superalgebra and \( A_s(\mathbb{R}) \) is the Graßmann algebra with \( s \) generators.

Our main result are the following. As explained below, we first have to understand the structure of the central extensions, resp., of the even 2-cocycles. This is described in Section 3 and works as follows. Suppose that \( \kappa \) is a non-degenerate invariant symmetric bilinear form on \( \mathfrak{k} \) which is invariant under all derivations of \( \mathfrak{k} \). Then any \( D \in \text{der}(\mathfrak{k}) \) and any linear map \( f : A \to \mathbb{R} \) leads to a 2-cocycle
\[
\eta_{f,D}(a \otimes x, b \otimes y) := (-1)^{|b||x|} f(ab) \kappa(Dx, y).
\]
There is a second class of natural cocycles on \( A \otimes \mathfrak{k} \). To describe it, we call a bilinear map \( F : A \times A \to \mathbb{R} \) a Hochschild map if
\[
F(a, b) = -(-1)^{|a||b|} F(b, a) \quad \text{and} \quad F(ab, c) = F(a, bc) + (-1)^{|b||a|} F(b, ac)
\]
hold for \( a, b, c \in A \). If \( S : \mathfrak{k} \to \mathfrak{k} \) is \( \kappa \)-symmetric and contained in the centroid \( \text{cent}(\mathfrak{k}) \), i.e., it commutes with all right brackets, then
\[
\xi_{F,S}(a \otimes x, b \otimes y) := (-1)^{|b||x|} F(a, b) \kappa(Sx, y)
\]
also defines a 2-cocycle. Our first main result Theorem 3.10 asserts that each 2-cocycle on \( A \otimes \mathfrak{k} \) is equivalent to a finite sum of cocycles of the form \( \eta_{f,D} \) and \( \xi_{F,S} \).

Our second main result is Theorem 4.9 which asserts that, if \( \mathfrak{k} \) is a simple compact Lie superalgebra with \( \mathfrak{k}_1 \neq \{0\} \), then each unitary representation of \( g = A_s(\mathbb{R}) \otimes \mathfrak{k} \) factors through the quotient map \( \varepsilon \otimes \text{id}_\mathfrak{k} : g \to \mathfrak{k} \) corresponding to the augmentation homomorphism \( \varepsilon : A_s(\mathbb{R}) \to \mathbb{R} \). This result shows that, if \( \mathfrak{k} \) is not purely even, the passage to the current superalgebra does not lead to more unitary representations than what we have seen in [1] for simple compact Lie superalgebras. Note that their irreducible representations have been determined in terms of highest weights by Jakobsen [5]. For an argument that all irreducible unitary representations are of this form, see [9].

This leaves us with the case where \( \mathfrak{k} = \mathfrak{k}_0 \) is a (purely even) compact Lie algebra. If \( A_s^+(\mathbb{R}) = \ker \varepsilon \), then
\[
g \cong (A_s^+(\mathbb{R}) \otimes \mathfrak{k}) \rtimes \mathfrak{k}
\]
is a semidirect sum of the compact Lie algebra \( \mathfrak{k} \) and the ideal \( g^+ := A_s^+(\mathbb{R}) \otimes \mathfrak{k} \). In Theorem 4.9 we show that every unitary representation of any central extension \( \hat{g} \) of \( g \) annihilates the ideal \( I := A_s^{>3}(\mathbb{R}) \otimes \mathfrak{k} \), resp., its central extension \( I \). As the quotient \( \hat{g}/I \) is a semidirect product \( \hat{\mathfrak{h}} \rtimes \mathfrak{h} \), where \( \hat{\mathfrak{h}} \) is a Clifford–Lie superalgebra, the classification of the unitary representations of the corresponding Lie supergroup \( \hat{N} \rtimes K \) can be determined with the methods developed in details in [2]. We provide a detailed description of these results in Appendix A. Theorem A.9 contains the classification of irreducible unitary representations of any semidirect product supergroup of the form \( \mathcal{G} = \mathcal{N} \rtimes K \), where \( K \) is a compact Lie group. As we have seen above, this combined with the other results provides a complete description of
the irreducible unitary representations of current superalgebras of the form $A \otimes \mathfrak{g}$, where $\mathfrak{g}$ is a compact simple Lie superalgebra.

The structure of the paper is as follows. We collect preliminaries in Section 2. In Section 3 we then turn to the central extensions, resp., the 2-cocycles of current superalgebras $\mathfrak{g} = A \otimes \mathfrak{k}$, culminating in Theorem 3.16. In Section 4 our description of the cocycles is used to determine the unitary representations of central extensions of the current superalgebras $A \otimes \mathfrak{k}$. Here we first turn to the case where $\mathfrak{g}$ is compact with $\mathfrak{k} = \{0\}$ and then to the case where $\mathfrak{g}$ is a simple compact Lie superalgebra with $\mathfrak{k} \neq \{0\}$. In both cases we reduce the classification problem to situations for which the solutions are known. In the first case we end up with semidirect products $\mathfrak{n} \rtimes \mathfrak{g}$ covered by [2] and in the second case with central extensions of $\mathfrak{g}$ itself. Throughout the paper, we always assume that Lie superalgebras under consideration are finite dimensional.

2. Preliminaries

In this section we provide precise definitions required for unitary representations of Lie supergroups.

A pre-Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ is called a pre-Hilbert superspace if $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ is a superspace such that $(\mathcal{H}_0, \mathcal{H}_0) = \{0\}$. A pre-Hilbert superspace $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ is called a Hilbert superspace if $\mathcal{H}$ is a complete space with respect to the metric induced by $\langle \cdot, \cdot \rangle$. For a pre-Hilbert superspace $(\mathcal{H}, \langle \cdot, \cdot \rangle)$, an endomorphism $T \in \text{End}(\mathcal{H})$ is said to be symmetric with respect to the inner product $\langle \cdot, \cdot \rangle$ if $\langle Tx, y \rangle = \langle x, Ty \rangle$ for all $x, y \in \mathcal{H}$; it is called nonnegative, denoted by $T \geq 0$, if $\langle Tx, x \rangle \geq 0$ for all $x \in \mathcal{H}$. A linear homogeneous isomorphism between two pre-Hilbert superspaces, preserving the corresponding inner products, is called unitary. We write $\text{Aut}(\mathcal{H})$ (resp. $\text{Aut}(\mathcal{H})_{\text{even}}$) for the group of all unitary (resp. even) unitary automorphisms of $\mathcal{H}$.

**Definition 2.1.** (i) Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert superspace and $G$ be a finite dimensional Lie group. A unitary representation of $G$ on $\mathcal{H}$ is a pair $(\pi, \mathcal{H})$, where $\pi : G \to \text{Aut}(\mathcal{H})$ is a group homomorphism such that, for each $\zeta \in \mathcal{H}$, the orbit map

$$\pi^\zeta : G \to \mathcal{H}, \quad g \mapsto \pi(g)\zeta$$

is continuous. An element $\zeta \in \mathcal{H}$ is called a smooth vector if $\pi^\zeta$ is a smooth function [7] [III.3]. We denote by $\mathcal{H}^\infty$ the set of all smooth vectors of $(\pi, \mathcal{H})$ and recall that it is a dense subset of $\mathcal{H}$ as $G$ is finite dimensional [3]. As $G$ acts by homogeneous operators, $\mathcal{H}^\infty$ is a sub-superspace of $\mathcal{H}$.

(ii) A unitary representation of a real Lie superalgebra $\mathfrak{g}$ in a pre-Hilbert superspace $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ is a real Lie superalgebra homomorphism $\chi : \mathfrak{g} \to \text{End}(\mathcal{H})$ satisfying

$$\langle \chi(X)(u), v \rangle = \langle u, -i[X]\chi(X)(v) \rangle$$

for $X \in \mathfrak{g}$, $u, v \in \mathcal{H}$.

We then refer to $\mathcal{H}$ as a unitary $\mathfrak{g}$-module.

(iii) Two unitary representations are said to be equivalent (or isomorphic) if their actions intertwine with an even unitary operator.

**Definition 2.2.** (i) A Lie supergroup is a pair $\mathcal{G} := (G, \mathfrak{g})$, in which $G$ is a (finite dimensional) Lie group and $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a finite dimensional real Lie superalgebra such that
• $g_0$ is the Lie algebra of $G$,
• there is a smooth action $\sigma : G \rightarrow \text{Aut}(g)$ of $G$ on $g$,
• the differential of $\sigma$ is the adjoint action of $g_0$ on $g$.

We denote the Lie supergroup $(G, g)$ by $(G, g, \sigma)$ if we want to emphasise $\sigma$.

(ii) A Lie subsupergroup of a Lie supergroup $G = (G, g)$ is a pair $H = (H, h)$ in which $H$ is a Lie subgroup of $G$, $h = h_0 \oplus h_1$ is a Lie sub-superalgebra of $g$ and the action of $H$ on $h$ is the restriction of the action of $G$ on $g$. A Lie subsupergroup is called special if $g_1 = h_1$.

(iii) If $(G, g)$ is a Lie supergroup, then an automorphism of $(G, g)$ is a pair $(\gamma, \beta) \in \text{Aut}(G) \times \text{Aut}(g)$ such that $\beta|_{g_0}$ coincides with the differential $d\gamma$ and $\beta g \beta^{-1} = \sigma(\gamma(g))$ for $g \in G$.

(iv) If $N = (N, n)$ is a Lie supergroup, $K$ a Lie group and 

$$\alpha = (\alpha_N, \alpha_n) : K \rightarrow \text{Aut}(N, n)$$

is a smooth group homomorphism, then we can form the semidirect product Lie supergroup $N \rtimes_\alpha K := (N \times_{\alpha_N} K, n \rtimes_{\beta_n} \mathfrak{k})$, where $\beta_n: \mathfrak{k} \rightarrow \text{der}(n)$ is the derived action of $\mathfrak{k}$ (via $\alpha_n$) by even derivations on the Lie superalgebra $\mathfrak{k}$.

**Definition 2.3.** A unitary pre-representation of a Lie supergroup $(G, g, \sigma)$ in a Hilbert superspace $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ is a triple $(\pi, \chi, \mathcal{B})$, where

• $\pi : G \rightarrow \text{Aut}(\mathcal{H})_{\text{even}}$ is a unitary representation of $G$,
• $\mathcal{B} \subseteq \mathcal{H}$ is a $\pi(G)$-invariant dense subspace contained in the space $\mathcal{H}^\infty$ of smooth vectors of $(\pi, \mathcal{H})$,

(a) $\chi : g \rightarrow \text{End}(\mathcal{B})$ is a unitary representation of $g$ in $\mathcal{B}$,
(b) $\chi(X) = d\pi(X)|_{\mathcal{H}^\infty}$ for $X \in g_0$.

A unitary representation is a unitary pre-representation for which $\mathcal{B} = \mathcal{H}^\infty$ is the full space of smooth vectors of $(\pi, \mathcal{H})$. According to [2] (see also [8, Lemma 4.4]), every unitary pre-representation extends uniquely to a unitary representation. We simply denote a unitary representation $(\pi, \chi, \mathcal{H}^\infty)$ by $(\pi, \chi)$.

(iv) Suppose that $(G, g)$ is a Lie supergroup and $(\pi, \chi_\pi)$ is a unitary representation of $(G, g)$ in a Hilbert superspace $\mathcal{H}$. A closed sub-superspace $\mathcal{K}$ of $\mathcal{H}$ for which $\pi(g)\mathcal{K} \subseteq \mathcal{K}$ and $\chi_\pi(X)\mathcal{K}\subseteq \mathcal{K}$, for all $g \in G$ and $X \in g$, is called a submodule of $\mathcal{H}$. The unitary representation $\pi$ (and correspondingly the unitary module $\mathcal{H}$) is called irreducible if $\mathcal{H}$ has no nontrivial submodule.

As we shall need it below, we recall the construction of unitarily induced representations in the context of Lie supergroups (see [2] §3 for more details).

**Definition 2.4** (Induced Representation). Suppose that $G = (G, g)$ is a Lie supergroup and $H = (H, h)$ is a special Lie subsupergroup of $(G, g)$ (Definition 2.2). Suppose $(\rho, \chi_\rho, \mathcal{H}^\infty)$ is a unitary representation of $H$ and that the (purely even) homogeneous space $H \setminus G$ carries a $G$-invariant measure $\mu$. Define $\mathcal{H}$ as the space of (equivalence classes of) measurable functions $f : G \rightarrow \mathcal{H}$ such that

(a) for any $g \in G$ and $h \in H$, we have $f(hg) = \rho(h)f(g)$,
(b) $\int_{H \setminus G} ||f(g)||^2 d\mu(Hg) < \infty$.

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3 For a Lie superalgebra $g$, a linear map $D : g \rightarrow g$ is called a derivation of $g$ if, for $x, y \in g$, $D[x, y] = [Dx, y] - (-1)^{|x||y|}[Dy, x]$. The set of derivations of $g$ is denoted by $\text{der}(g)$.\[\]

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In other words, this is the Hilbert space of \( L^2 \)-sections of the Hilbert bundle \( \mathcal{H} \times_H \mathcal{G} \) over \( \mathcal{H} \setminus \mathcal{G} \) associated to the \( \mathcal{H} \)-principal bundle \( \mathcal{G} \).

We define \( \pi : \mathcal{G} \to \text{Aut}(\mathcal{H}) \) by

\[
(\pi(g)f)(g_0) = f(g_0g) \quad \text{for} \quad f \in \mathcal{H}, g, g_0 \in \mathcal{G}.
\]

Let \( \mathcal{B} \subseteq \mathcal{H} \) be the subspace of \( \mathcal{H}^\infty \) consisting of all smooth functions \( f : \mathcal{G} \to \mathcal{H} \) with compact support modulo \( \mathcal{G} \). Then \( \mathcal{B} \) is a dense \( \mathcal{G} \)-invariant subspace of \( \mathcal{H}^\infty \) and we define \( \chi_\pi : \mathcal{G} \to \text{End}(\mathcal{B}) \) by

\[
(\chi_\pi(X)f)(g) = \chi_\pi(g \cdot X)f(g) \quad \text{for} \quad g \in \mathcal{G}, X \in \mathfrak{g}, f \in \mathcal{B}.
\]

Now \( (\pi, \chi_\pi) \) defines a unitary pre-representation of \( (\mathcal{G}, \mathfrak{g}) \) in \( \mathcal{H} \) and its canonical extension to a unitary representation is called the induced representation and denoted by \( (\pi, \chi_\pi) := \text{Ind}_{\mathcal{G}}^{\mathcal{H}}(\rho, \chi_\rho) \).

**Definition 2.5.** Suppose \( \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) is a finite dimensional real Lie superalgebra.

We write

\[
\mathcal{C}(\mathfrak{g}) \subseteq \mathfrak{g}_0
\]

for the closed convex cone generated by the brackets \([x, x], x \in \mathfrak{g}_1 \) (cf. [9]). The ideal \( \text{urad}(\mathfrak{g}) \) of \( \mathfrak{g} \) generated by \( \mathcal{C} := \mathcal{C}(\mathfrak{g}) \cap -\mathcal{C}(\mathfrak{g}) \) and all those elements \( x \in \mathfrak{g}_1 \) with \([x, x], x \in \mathcal{C} \) is called the unitary radical of \( \mathfrak{g} \). We say that the convex cone \( \mathcal{C}(\mathfrak{g}) \) of \( \mathfrak{g} \) is pointed if \( \mathcal{C}(\mathfrak{g}) \cap -\mathcal{C}(\mathfrak{g}) = \{0\} \).

**Lemma 2.6.** If there exists a linear functional \( \lambda \in \mathfrak{g}_0^* \) with

\[
\lambda([x_1, x_1]) > 0 \quad \text{for} \quad 0 \neq x_1 \in \mathfrak{g}_1,
\]

then the cone \( \mathcal{C}(\mathfrak{g}) \) is pointed.

**Proof.** Let \( C := \{x_1 \in \mathfrak{g}_1 : \lambda([x_1, x_1]) = 1\} \). As the level set of a positive definite form on \( \mathfrak{g}_1 \), the set \( C \) is compact. Hence \( K := \{[x_1, x_1] : x_1 \in C\} \) is a compact subset of \( \mathfrak{g}_0 \) contained in the affine hyperplane \( \lambda^{-1}(1) \). Therefore the closed convex cone \( \mathbb{R}^+ \text{conv}(K) = \mathcal{C}(\mathfrak{g}) \) is pointed.

The following lemma shows that \( \text{urad}(\mathfrak{g}) \) is contained in the kernel of every unitary representation of \( \mathfrak{g} \), hence the name.

**Lemma 2.7.** If \( (\chi, \mathcal{H}) \) is a unitary representation of the real Lie superalgebra \( \mathfrak{g} \) in a Hilbert superspace, then the following assertions hold:

(i) For \( x \in \mathfrak{g}_1 \), \( \chi(x) = 0 \) if and only if \( \chi([x, x]) = 0 \).

(ii) \( -i\chi(x) \geq 0 \) for \( x \in \mathcal{C}(\mathfrak{g}) \).

(iii) \( \text{urad}(\mathfrak{g}) \subseteq \ker \chi \).

**Proof.** (i) For \( x \in \mathfrak{g}_1 \) and \( u \in \mathcal{H}^\infty \), we have \( \chi([x, x]) = 2\chi(x)^2 \) and

\[
-2i\chi([x, x])u, u = -2i(\chi(x)^2u, u) = 2(-i\chi(x)u, -i\chi(x)u) \geq 0.
\]

(ii) follows from (2.1).

(iii) First (i) and (ii) imply that \( \mathcal{C}(\mathfrak{g}) \cap -\mathcal{C}(\mathfrak{g}) \subseteq \ker \chi \), and as \( \ker \chi \) is an ideal of \( \mathfrak{g} \), the assertion follows.

**Remark 2.8.** (The relation to super-hermitian forms)

(i) Suppose that \( \mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \) is a superspace equipped with an even super-hermitian form \( (\cdot, \cdot) : \mathcal{H} \times \mathcal{H} \to \mathbb{C} \), that is,

- \( (\mathcal{H}_0, \mathcal{H}_1) = \{0\} \)
- \( (\cdot, \cdot) \) is linear in the first component,
\[ (u, u) > 0 \text{ and } -i(v, v) > 0 \text{ for } 0 \neq u \in \mathcal{H}_0 \text{ and } 0 \neq v \in \mathcal{H}_1, \]
\[ (u, v) = (-1)^{|u||v|}(v, u). \]

Then
\[ \langle u, v \rangle := i^{-|u||v|}(u, v) \quad (u, v \in \mathcal{H}) \]
defines a hermitian form for which \( \mathcal{H}_0 \) and \( \mathcal{H}_1 \) are orthogonal subspaces of \( \mathcal{H} \), i.e., (\( \mathcal{H}, \langle \cdot, \cdot \rangle \)) is a Hilbert superspace. A homogeneous linear endomorphism \( T : \mathcal{H} \to \mathcal{H} \) is called \textit{supersymmetric} with respect to the super-hermitian form \( \langle \cdot, \cdot \rangle \) if
\[ (Tu, v) = (-1)^{|T||u|}(u, Tv) \quad (u, v \in \mathcal{H}). \]

A linear endomorphism \( T = T_0 + T_1 \in \text{End}(\mathcal{H}) \) is called \textit{supersymmetric} with respect to \( \langle \cdot, \cdot \rangle \) if \( T_0 \) and \( T_1 \) are supersymmetric with respect to \( \langle \cdot, \cdot \rangle \).

The mapping \( T \mapsto e^{[T|\hat{T}]T} \) defines a bijection from the set of supersymmetric linear endomorphisms of \( \mathcal{H} \) with respect to \( \langle \cdot, \cdot \rangle \) onto symmetric linear endomorphisms of \( \mathcal{H} \) with respect to \( \langle \cdot, \cdot \rangle \). Moreover, if \( \mathfrak{g} \) is a real Lie superalgebra and \( \chi : \mathfrak{g} \to \text{End}(\mathcal{H}) \) is a Lie superalgebra homomorphism, then \( \chi \) is a unitary representation of \( \mathfrak{g} \) in the Hilbert superspace \( (\mathcal{H}, \langle \cdot, \cdot \rangle) \) if and only if, for each \( X \in \mathfrak{g} \), \( \chi(X) \) is a skew-supersymmetric with respect to \( \langle \cdot, \cdot \rangle \), i.e.,
\[ (\chi(X)u, v) = -(-1)^{|\chi(X)||u|}(u, \chi(X)v) \quad (u, v \in \mathcal{H}). \]

3. Current superalgebras

In this section, we assume \( \mathbb{F} \) is a field of characteristic zero and unless otherwise mentioned, we consider all vector spaces and tensor products over \( \mathbb{F} \). The main result of this section is Theorem 3.16 describing the structure of the 2-cocycle of current superalgebras of the form \( \mathfrak{A} \otimes \mathfrak{t} \).

3.1. Invariant forms and 2-cocycles. For a Lie superalgebra \( \mathfrak{g} \) and a superspace \( M \), a bilinear map \( \omega : \mathfrak{g} \times \mathfrak{g} \to M \) is called a 2-cocycle with coefficients in \( M \) if
\[ \omega(x, y) = -(-1)^{|x||y|}\omega(y, x) \]
\[ \omega([x, y], z) = \omega(x, [y, z]) - (-1)^{|x||y|}\omega(y, [x, z]) \]
for all \( x, y, z \in \mathfrak{g} \). The set of all 2-cocycles with coefficients in \( M \) is denoted by \( Z^2(\mathfrak{g}, M) \). A 2-cocycle \( \omega \) is called a 2-coboundary if there is a linear map \( f : \mathfrak{g} \to M \) with \( \omega(x, y) = f([x, y]) \) for all \( x, y \in \mathfrak{g} \). The set of 2-coboundaries is denoted by \( B^2(\mathfrak{g}, M) \) and the quotient space
\[ H^2(\mathfrak{g}, M) := Z^2(\mathfrak{g}, M)/B^2(\mathfrak{g}, M) \]
is called the second cohomology of \( \mathfrak{g} \) with coefficients in \( M \). Two 2-cocycles are called cohomologous if their difference is a 2-coboundary.

2-cocycles of a Lie superalgebra \( \mathfrak{g} \) are in correspondence with its central extensions: If \( \mathfrak{g} \) is a Lie superalgebra and \( \omega \) is a 2-cocycle of \( \mathfrak{g} \) with coefficients in a superspace \( M \), taking \( \hat{\mathfrak{g}} \) to be the superspace \( \mathfrak{g} \oplus M \) and defining
\[ [\cdot, \cdot]_\omega : \hat{\mathfrak{g}} \times \hat{\mathfrak{g}} \to \hat{\mathfrak{g}}, \quad (x + m, x' + m') \mapsto [x, x'] + \omega(x, x') \]
for \( x, x' \in \mathfrak{g} \) and \( m, m' \in M \), \( \hat{\mathfrak{g}} \) together with \([\cdot, \cdot]_\omega \) is a Lie superalgebra and the canonical projection map \( \pi : \hat{\mathfrak{g}} \to \mathfrak{g} \) is a central extension.
Remark 3.2. We note that if $(M, \cdot)$ is a right $g$-module, then $M$, together with the action
\begin{equation}
  x \cdot a := (-1)^{|x||a|} a \cdot x 
  \quad \text{for } x \in g, a \in M
\end{equation}
is a left $g$-module. Conversely, if $(M, \cdot)$ is a left $g$-module, then $M$ together with the action $a \cdot x := (-1)^{|x||a|} x \cdot a$ $(x \in g, a \in M)$ is a right $g$-module.

Although, if we have a left $g$-module, we automatically have a right $g$-module and vice versa, our preference is to use right actions as they simplify working with degrees.

A linear map $\varphi$ from a $g$-module $M$ to a $g$-module $N$ is called a $g$-$\text{module homomorphism}$ if $\varphi(mx) = \varphi(m)x$ for all $m \in M$ and $x \in g$.

For a $g$-module $M$, a $\text{derivation}$ of $g$ in $M$ is a linear map $d: g \rightarrow M$ satisfying
\[ d[x, y] = d(x)y - (-1)^{|x||y|} d(y)x \]
for all $x, y \in g$. We denote the set of all derivations of $g$ in $M$ by $\text{der}(g, M)$. A derivation $d \in \text{der}(g, M)$ is called $\text{inner}$ if there is $m \in M$ with $d(x) = mx$ for all $x \in g$. The $\text{first cohomology}$ $H^1(g, M)$ of $g$ with coefficients in $M$ is the quotient space $\text{der}(g, M)/\text{ider}(g, M)$, where $\text{ider}(g, M)$ is the set of inner derivations of $g$ in $M$. A derivation of $g$ in $M$ is called $\text{outer}$ if it is not inner.

Definition 3.3. Suppose that $\mathfrak{t}$ is a Lie superalgebra.

(i) For a superspace $M$, a bilinear map $\alpha: \mathfrak{t} \times \mathfrak{t} \rightarrow M$ is called $\text{supersymmetric}$ (resp. $\text{skew-supersymmetric}$) if for $x, y \in \mathfrak{t}$, $\alpha(x, y)$ equals $(-1)^{|x||y|}\alpha(y, x)$ (resp. $-(-1)^{|x||y|}\alpha(y, x)$) and it is called $\text{invariant}$ if
\[ \alpha([x, y], z) = \alpha(x, [y, z]) \quad \text{for } x, y, z \in \mathfrak{t}. \]
The set of all bilinear maps from $\mathfrak{t} \times \mathfrak{t}$ to $M$ is denoted by $\text{Bil}(\mathfrak{t}, M)$ and the set of all supersymmetric invariant bilinear maps from $\mathfrak{t} \times \mathfrak{t}$ to $M$ is denoted by $\text{Sym}(\mathfrak{t}, M)^\mathfrak{t}$.

(ii) The subsuperspace
\[ \text{cent}_{\mathfrak{t}}(\mathfrak{t}) := \{ \gamma \in \text{End}(\mathfrak{t}) | (\forall a, b \in \mathfrak{t}) \gamma [a, b] = [\gamma(a), b] \} \]
of the superspace $\text{End}(\mathfrak{t})$ is called the $\text{centroid}$ of $\mathfrak{t}$. The Lie superalgebra $\mathfrak{t}$ is called $\text{absolutely simple}$ if $\text{cent}_{\mathfrak{t}}(\mathfrak{t}) = \mathfrak{t}\text{id}$.

Suppose that $\mathfrak{t}$ is a finite dimensional Lie superalgebra and $\kappa$ is an invariant nondegenerate supersymmetric bilinear form. Then the map
\begin{equation}
  \varphi: \mathfrak{t} \rightarrow \mathfrak{t}^*, \quad \varphi(x) = \kappa(x, \cdot)
\end{equation}
is a linear bijection and so, for $S \in \text{End}(\mathfrak{t})$, there is a unique endomorphism $S^*$ of $\mathfrak{t}$ satisfying
\[ \kappa(Sx, y) = (-1)^{|x||y|}\kappa(S^*y, x) \quad \text{for } x, y \in \mathfrak{t}. \]
Lemma 3.4. For $T \in \text{End}(\mathfrak{t})$, define the bilinear map

$$\kappa_T : \mathfrak{t} \times \mathfrak{t} \rightarrow \mathbb{F}, \quad (x, y) \mapsto \kappa(T(x), y).$$

This assignment has the following properties:

(i) $\kappa_T$ is supersymmetric (resp. skew-supersymmetric) if and only if $T^* = T$ (resp. $T^* = -T$).
(ii) $\kappa_T$ is invariant if and only if $T \in \text{cent}_F(\mathfrak{t})$.
(iii) $\kappa_T$ satisfies $\kappa_T([x, y], z) = \kappa_T(x, [y, z]) = (-1)^{|x||y|}\kappa_T(y, [x, z])$, for all $x, y, z \in \mathfrak{t}$, if and only if $T$ is a derivation.

Proposition 3.5. Suppose that $\kappa$ is homogeneous and define

$$\theta : \text{End}(\mathfrak{t}) \rightarrow \text{Bil}(\mathfrak{t}, \mathbb{F}), \quad T \mapsto \kappa_T.$$

This assignment has the following properties:

(a) The restriction of $\theta$ to $\text{der}_-(\mathfrak{t}) := \{D \in \text{der}(\mathfrak{t}) | D^* = -D\}$ is a linear isomorphism onto the superspace $Z^2(\mathfrak{t}) := Z^2(\mathfrak{t}, \mathbb{F})$.
(b) The restriction of $\theta$ to the superspace $\text{cent}_+(\mathfrak{t}) := \{S \in \text{cent}_\mathbb{F}(\mathfrak{t}) | S^* = S\}$ is a linear isomorphism from $\text{cent}_+(\mathfrak{t})$ onto the superspace $\text{Sym}(\mathfrak{t})^\mathbb{F} := \text{Sym}(\mathfrak{t}, \mathbb{F})^\mathbb{F}$.

Definition 3.6. The bilinear form $\kappa$ is called derivation invariant if $\text{der}(\mathfrak{t}) = \text{der}_-(\mathfrak{t})$.

Example 3.7. If $\mathfrak{t}$ is a sub-superalgebra of a Lie superalgebra $\mathfrak{g}$ such that $\kappa$ is a restriction of an invariant supersymmetric bilinear form on $\mathfrak{g}$ and each derivation of $\mathfrak{t}$ is of the form $\text{ad}_x$ for some $x \in \mathfrak{g}$, then $\kappa$ is derivation invariant.

Remark 3.8. For $x, y \in \mathfrak{t}$ and $f \in \mathfrak{t}^*$, one can define $(f \cdot x)(y) := f([x, y])$. Now $\mathfrak{t}^*$ together with this action is a $\mathfrak{t}$-module and $\varphi$ defined in \((3.2)\) is a $\mathfrak{t}$-module isomorphism. Also for $\alpha \in Z^2(\mathfrak{t})$, the linear map $\zeta_\alpha : \mathfrak{t} \rightarrow \mathfrak{t}^*$ defined by $\zeta_\alpha(x)(y) := \alpha(x, y)$, for $x, y \in \mathfrak{t}$, is an element of $\text{der}(\mathfrak{t}, \mathfrak{t}^*)$. Moreover, $\zeta_\alpha$ is an inner derivation if and only if $\alpha$ is a 2-coboundary. Identifying $\mathfrak{t}$ and $\mathfrak{t}^*$ via $\varphi$ and using Proposition 3.5, we can embed $H^2(\mathfrak{t}) := H^2(\mathfrak{t}, \mathbb{F})$ in $H^1(\mathfrak{t}, \mathfrak{t})$ by considering $H^2(\mathfrak{t})$ as those outer derivations of $\mathfrak{t}$ belonging to $\text{der}_-(\mathfrak{t})$.

Assumption: From now on to the end of this section, we assume that $\mathfrak{t}$ is a finite dimensional perfect Lie superalgebra equipped with a nondegenerate homogeneous invariant supersymmetric bilinear form $\kappa$.

We set $I$ to be the subsuperspace of the exterior algebra\(^4\) $\Lambda \mathfrak{t}$ spanned by

$$[x, y] \wedge z - x \wedge [y, z] + (-1)^{|x||y|} y \wedge [x, z] \quad \text{for } x, y, z \in \mathfrak{t}.$$ 

Then the dual space of the quotient space $\Lambda_d(\mathfrak{t}) := \Lambda^2 \mathfrak{t}/I$ is nothing but the superspace $Z^2(\mathfrak{t})$ of 2-cocycles of $\mathfrak{t}$ with trivial coefficients. Throughout this section, we fix $\{D_1, \ldots, D_n\} \subseteq \text{der}_-(\mathfrak{t})$ such that $\{\kappa_{D_i} | 1 \leq i \leq n\}$ is a basis for $Z^2(\mathfrak{t})$.

\(^4\) For a vector superspace $V$, by $\Lambda V = \oplus_{n=0}^\infty \Lambda^n V$ and $S V = \oplus_{n=0}^\infty S^n V$, we denote respectively the exterior superalgebra as well as the symmetric superalgebra of the vector superspace $V$ and "$\wedge$" and "$\vee$" denote the multiplication maps on $\Lambda V$ and $S V$ respectively.
Since \( \mathfrak{k} \) is finite dimensional, it follows that there is a unique basis \( \{ \lambda_1, \ldots, \lambda_n \} \) for \( \Lambda_d(\mathfrak{k}) \) such that

\[
\tilde{\alpha} : \Lambda_d(\mathfrak{k}) \longrightarrow M, \quad \tilde{\alpha}(x, y) = \sum_{i=1}^n \kappa(D_i, x, y)\lambda_i
\]

where \( \tilde{\alpha} \) stands for the equivalence classes in \( \Lambda_d(\mathfrak{k}) \). The degree 2-subspace \( S^2(\mathfrak{k}) \) has a natural \( \mathfrak{k} \)-module action and the dual space of the quotient space \( S^2(\mathfrak{k})/S^2(\mathfrak{k})\mathfrak{k} \) is isomorphic to \( \text{Sym}(\mathfrak{k})^\mathfrak{k} \). Now we fix a subset \( \{ S_1, \ldots, S_m \} \) of \( \text{cent}_+(\mathfrak{k}) \) such that \( \{ \kappa_{S_i} \mid 1 \leq i \leq m \} \) is a basis for \( \text{Sym}(\mathfrak{k})^\mathfrak{k} \). So there is a unique basis \( \{ \mu_1, \ldots, \mu_m \} \), which we fix throughout this section, for \( S^2(\mathfrak{k})/S^2(\mathfrak{k})\mathfrak{k} \) such that

\[
\tilde{\alpha} : \Lambda_d(\mathfrak{k}) \longrightarrow M, \quad \tilde{\alpha}(x, y) = \sum_{i=1}^m \kappa(S_i, x, y)\mu_i
\]

in which by the abuse of notations, we again use \( \tilde{\alpha} \) for the equivalence classes in \( S^2(\mathfrak{k})/S^2(\mathfrak{k})\mathfrak{k} \).

Suppose \( M \) is a superspace. If \( \alpha \) is a 2-cocycle of \( \mathfrak{k} \) with coefficients in \( M \), \( \alpha \) induces the linear map

\[
\tilde{\alpha} : \Lambda_d(\mathfrak{k}) \longrightarrow M, \quad \tilde{\alpha}(x, y) = \alpha(x, y).
\]

Next suppose \( \alpha : \mathfrak{k} \times \mathfrak{k} \longrightarrow M \) is a supersymmetric invariant bilinear map, then \( \alpha \) induces the linear map

\[
\tilde{\alpha} : S^2(\mathfrak{k})/S^2(\mathfrak{k})\mathfrak{k} \longrightarrow M, \quad \tilde{\alpha}(x, y) = \alpha(x, y).
\]

**Proposition 3.9.** Let \( M \) be a superspace. For \( T \in \text{End}(\mathfrak{k}) \) and \( m \in M \), define

\[
\nu_{T,m} : \mathfrak{k} \times \mathfrak{k} \longrightarrow M, \quad (x, y) \mapsto \kappa_T(x, y)m = \kappa(T(x), y)m.
\]

(i) If \( m \in M \) and \( S \in \text{cent}_+(\mathfrak{k}) \), then \( \nu_{S,m} \) is a supersymmetric invariant bilinear map. If \( D \in \text{der}_-(\mathfrak{k}) \), then \( \nu_{D,m} \) is a 2-cocycle of \( \mathfrak{k} \) and if \( m \neq 0 \), then \( \nu_{D,m} \) is a 2-coboundary if and only if \( D \) is an inner derivation.

(ii) If \( \alpha \in Z^2(\mathfrak{k}, M) \), then \( \alpha = \sum_{i=1}^n \nu_{D_i, \tilde{\alpha}(\mu_i)} \).

(iii) If \( \alpha : \mathfrak{k} \times \mathfrak{k} \longrightarrow M \) is a supersymmetric invariant bilinear map, then

\[
\alpha = \sum_{i=1}^m \nu_{S_i, \tilde{\alpha}(\mu_i)}.
\]

**Proof.** (i) By Lemma 3.3, for \( m \in M \), \( D \in \text{der}_-(\mathfrak{k}) \) and \( S \in \text{cent}_+(\mathfrak{k}) \), \( \nu_{D,m} \) is a 2-cocycle and \( \nu_{S,m} \) is a supersymmetric invariant bilinear map. For the last statement, suppose that \( \nu_{D,m} \) is a 2-coboundary. Then there is a linear map \( \ell : \mathfrak{k} \longrightarrow M \) such that for \( x, y \in \mathfrak{k} \), \( \kappa(Dx, y)m = \ell([x, y]) \). This gives a linear map \( f \in \mathfrak{k}^* \) and a unique \( t_f \in \mathfrak{k} \) such that for \( x, y \in \mathfrak{k} \),

\[
\kappa(Dx, y) = f[x, y] = \kappa(t_f, [x, y]) = \kappa([t_f, x], y).
\]

This in turn implies that \( D = \text{ad}(t_f) \), in other words \( D \) is an inner derivation. Conversely if \( D \) is an inner derivation, it is immediate that \( \kappa_D(\cdot, \cdot)m \) is a coboundary as \( \kappa \) is invariant.

(ii) Considering (3.3) and (3.5), for \( x, y \in \mathfrak{k} \), we have

\[
\alpha(x, y) = \tilde{\alpha}(\tilde{x} \wedge y) = \tilde{\alpha}\left( \sum_{i=1}^n \kappa(D_i x, y)\lambda_i \right) = \sum_{i=1}^n \kappa(D_i x, y)\tilde{\alpha}(\lambda_i) = \sum_{i=1}^n \nu_{D_i, \tilde{\alpha}(\lambda_i)}(x, y).
\]
This completes the proof.

(iii) Use the same argument as in part (ii). □

3.2. 2-cocycles of current superalgebras. Throughout this subsection, $A$ denotes a unital supercommutative associative superalgebra. For a superspace $M$, we refer to a bilinear map $F: A \times A \to M$ satisfying

1. $F(a, b) = -(-1)^{|a||b|}F(b, a)$,
2. $F(ab, c) = F(a, bc) + (-1)^{|b||a|}F(b, ac)$

for $a, b, c \in A$, a Hochschild map. For a Hochschild map $F: A \times A \to M$ and $a \in A$, using (2), we have

$$F(1, a) = F(1, a1) = F(a, 1) - F(a, 1) = 0.$$

Definition 3.10. We set $g := A \otimes \mathfrak{k}$ and, for the sake of simplicity, for $a \in A$ and $x \in \mathfrak{k}$, we denote $a \otimes x$ by $ax$. We recall that $g$ together with

$$[ax, by] := (-1)^{|x||b|}(ab)[x, y]$$

for $a, b \in A$ and $x, y \in \mathfrak{k}$, is a Lie superalgebra.

Definition 3.11. Suppose $\omega: g \times g \to M$ is a 2-cocycle of $g$ with coefficients in a superspace $M$. For an element $a \in A$ and a homogeneous element $b \in A$, define

$$\omega_{a, b}: \mathfrak{k} \times \mathfrak{k} \to M, \quad (x, y) \mapsto (-1)^{|x||b|}\omega(ax, by).$$

We say $\omega$ is $\mathfrak{k}$-cocyclic if, for all homogeneous elements $a, b \in A$ the form $\omega_{a, b}$ is a 2-cocycle. We also say $\omega$ is $\mathfrak{k}$-invariant if, for all $a, b \in A$, $\omega_{a, b}$ is an invariant bilinear map.

Example 3.12. Suppose that $M$ is a superspace. For a linear map $f: A \to M$ and $D \in \text{der}_-(\mathfrak{k})$,

$$\eta_{f, D}: g \times g \to M, \quad \eta_{f, D}(ax, by) := (-1)^{|b||x|}f(ab)\kappa(Dx, y), \quad (a, b \in A, x, y \in \mathfrak{k})$$

is a $\mathfrak{k}$-cocyclic 2-cocycle of $g$. Also for a Hochschild map $F: A \times A \to M$ and an element $S \in \text{cent}_+(\mathfrak{k})$,

$$\xi_{F, S}: g \times g \to M, \quad \xi_{F, S}(ax, by) := (-1)^{|b||x|}F(a, b)\kappa(Sx, y), \quad (a, b \in A, x, y \in \mathfrak{k})$$

is a $\mathfrak{k}$-invariant 2-cocycle of $g$.

Lemma 3.13. (i) If $\omega$ is a $\mathfrak{k}$-cocyclic 2-cocycle, for homogeneous elements $a, b \in A$, we have

$$\omega_{a, b} = (-1)^{|a||b|}\omega_{b, a} \quad \text{and} \quad \omega_{ab, 1} = \omega_{a, b} = \omega_{1, ab}.$$

(ii) If $\omega$ is a $\mathfrak{k}$-invariant 2-cocycle, then for homogeneous elements $a, b, c \in A$, we have

$$\omega_{a, b} = -(-1)^{|a||b|}\omega_{b, a} \quad \text{and} \quad \omega_{ab, c} = \omega_{a, bc} + (-1)^{|a||b|}\omega_{b, ac}.$$

Proof. (i) Suppose that $a, b \in A$ and $x, y, z \in \mathfrak{k}$ are homogeneous elements. Then we have

$$\omega_{a, b}(x, y) = -(-1)^{|x||y|}\omega_{a, b}(y, x) = (-1)^{|x||y|+|b||y|}\omega(by, bx) = (-1)^{|a||b|+|a||x|}\omega(bx, ay)$$

$$= (-1)^{|a||b|}\omega_{b, a}(x, y).$$
We also have
\[ \omega_{ab,1}(y, z, x) = \omega(ab[y, z], x) = (-1)^{|y||b|} \omega([ay, bz], x) \]
\[ = (-1)^{|y||b|} \omega(ay, b[z, x]) = (-1)^{|a||b|+|a||z|+|y||z|} \omega(bz, a[y, x]) \]
\[ = \omega_{ab}(y, z, x) = (-1)^{|a||b|+|y||z|} \omega_{ab}(z, [y, x]) \]
\[ = \omega_{ab}(y, z, x). \]

This completes the proof as \( \mathfrak{k} \) is perfect.

(ii) Suppose that \( \omega \) is a \( \mathfrak{k} \)-invariant 2-cocycle and \( a, b \in A \) are homogeneous. As \( \omega_{ab} \) is invariant, for \( x, y, z \in \mathfrak{k} \), we have
\[ \omega_{ab}(x, y, z) = \omega_{ab}(y, x, z) = (-1)^{|y||z|} \omega_{ab}(x, [y, z]) \]
\[ = (-1)^{|a||b|+|a||y|+|a||z|} \omega_{ab}(y, ax) = (-1)^{|a||b|+|y||z|} \omega_{ab}(x, ay) \]
\[ = (-1)^{|a||b|+|y||z|} \omega_{ab}(x, ay) = (-1)^{|a||b|+|y||z|} \omega_{ab}(x, ay) \]
\[ = (-1)^{|a||b|+|y||z|} \omega_{ab}(x, ay). \]

This shows that \( \omega_{ab} \) is supersymmetric as \( \mathfrak{k} \) is perfect. So for \( x, y, z \in \mathfrak{k} \), we have
\[ \omega_{ab}(x, y, z) = (-1)^{|x||y|} \omega_{ab}(y, x) = (-1)^{|b||x|+|y||z|} \omega(x, ay) \]
\[ = (-1)^{|a||b|+|z||y|} \omega_{ab}(x, ay) = (-1)^{|a||b|+|z||y|} \omega_{ab}(x, ay) \]
\[ = (-1)^{|a||b|+|z||y|} \omega_{ab}(x, ay). \]

Proposition 3.14. Suppose that \( \mathfrak{k} \) is a finite dimensional perfect Lie superalgebra equipped with a nondegenerate invariant supersymmetric bilinear form \( \kappa \). If \( \kappa \) is derivation invariant, then each 2-cocycle of \( \mathfrak{g} = A \otimes \mathfrak{k} \) is of the form \( \omega_1 + \omega_2 \), where \( \omega_1 \) is a \( \mathfrak{k} \)-invariant 2-cocycle and \( \omega_2 \) is a \( \mathfrak{k} \)-cocyclic 2-cocycle.

Proof. Suppose that \( \omega : \mathfrak{g} \times \mathfrak{g} \rightarrow M \) is a 2-cocycle. Then \( \omega \) induces a linear map \( f : \Lambda^2 \mathfrak{g} \rightarrow M \). The linear maps
\[ \sigma_+ : \Lambda^2 A \otimes S^2 \mathfrak{k} \rightarrow \Lambda^2 \mathfrak{g} \]
\[ a \wedge b \otimes x \wedge y \mapsto \frac{1}{2}((-1)^{|x||b|}(ax \wedge by) + (-1)^{|a||b|+|a||x|}(ay \wedge bx)) \]
for \( a, b \in A, x, y \in \mathfrak{k} \)

and
\[ \sigma_- : S^2 A \otimes \Lambda^2 \mathfrak{k} \rightarrow \Lambda^2 \mathfrak{g} \]
\[ a \otimes b \otimes x \wedge y \mapsto \frac{1}{2}((-1)^{|x||b|}(ax \wedge by) - (-1)^{|a||b|+|a||x|}(ay \wedge bx)) \]
for \( a, b \in A, x, y \in \mathfrak{k} \).
are embeddings and
\[ \Lambda^2 \mathfrak{g} \simeq (\Lambda^2 A \otimes S^2 \mathfrak{k}) \oplus (S^2 A \otimes \Lambda^2 \mathfrak{k}). \]

On the other hand, the linear map
\[ e_1 : A \rightarrow S^2 A, \quad a \mapsto a \vee 1, \]

is also an embedding. The linear map
\[ \mu : S^2 A \rightarrow A, \quad a \vee b \mapsto ab, \]
satisfies \( S^2 A = \text{im}(e_1) \oplus \ker(\mu). \) So
\[ \Lambda^2 \mathfrak{g} \simeq (\Lambda^2 A \otimes S^2 \mathfrak{k}) \oplus (A \otimes \Lambda^2 \mathfrak{k}) \oplus (\ker(\mu) \otimes \Lambda^2 \mathfrak{k}). \]

We next define \( \omega \in \{ i \mid f \in \Lambda^2 \mathfrak{k} \} \) such that\( \omega \) is also an embedding. The linear map
\[ \bar{f} : \Lambda^2 A \rightarrow \Lambda^2 \mathfrak{k}, \quad f \mapsto f_1 + f_2 + f_3 \]

where
\[ f_1 : \Lambda^2 A \otimes S^2 \mathfrak{k} \rightarrow M, \quad f_2 : \Lambda \otimes \Lambda^2 \mathfrak{k} \rightarrow M, \quad f_3 : \ker(\mu) \otimes \Lambda^2 \mathfrak{k} \rightarrow M. \]

In view of Remark 3.8 and the fact that \( \mathfrak{k} \) is perfect and \( \kappa \) is derivation invariant, the super versions of Corollary 3.3, Theorem 3.7 and (13) of \(^{10}\), imply that \( f_3 = 0 \) and that \( f_1 \) and \( f_2 \) respectively induce maps \( \bar{f}_1 : A \times A \rightarrow \text{Sym}(\mathfrak{k}, M) \) and \( \bar{f}_2 : A \rightarrow Z^2(\mathfrak{k}, M) \) with
\[ \bar{f}_1(a, b)(x, y) = f_1((a \wedge b) \otimes (x \vee y)) \quad \text{and} \quad \bar{f}_2(a)(x, y) = f_2(a \otimes (x \wedge y)). \]

We next define \( \omega_j : \mathfrak{g} \times \mathfrak{g} \rightarrow M, j = 1, 2, \) by
\[ \omega_1(ax, by) := (-1)^{|x||b|} f_1(a \wedge b \otimes x \vee y) \quad \text{for} \ a, b \in A \ \text{and} \ x, y \in \mathfrak{k} \]

and
\[ \omega_2(ax, by) := (-1)^{|x||b|} f_2(ab \vee 1, x \wedge y) \quad \text{for} \ a, b \in A \ \text{and} \ x, y \in \mathfrak{k}. \]

Then \( \omega_1 \) and \( \omega_2 \) are 2-cocycles (see \(^{10}\) Thm. 3.7)). Also \( f = f_1 + f_2 \) implies \( \omega = \omega_1 + \omega_2 \). As \( \text{im}(\bar{f}_2) \subseteq \text{Sym}(\mathfrak{k}, M) \), \( \omega_1 \) is a \( \mathfrak{k} \)-invariant 2-cocycle and as \( \text{im}(\bar{f}_2) \subseteq Z^2(\mathfrak{k}, M) \), \( \omega_2 \) is a \( \mathfrak{k} \)-cocyclic 2-cocycle. \( \square \)

**Proposition 3.15.** For \( D_i, \lambda_i \) and \( S_j, \mu_j \) from \(^{35}\) and \(^{36}\), the following assertions hold for any \( \omega \in Z^2(\mathfrak{g}, M) \):

(i) If \( \omega \) is \( \mathfrak{k} \)-cocyclic, then there are linear maps \( f_1, \ldots, f_n \in \text{Hom}_\mathfrak{k}(A, M) \) such that\( \omega = \sum_{i=1}^n \eta_{f_i, D_i} \).

(ii) If \( \omega \) is \( \mathfrak{k} \)-invariant, then there are Hochschild maps \( F_i : A \times A \rightarrow M, i = 1, \ldots, m, \) such that \( \omega = \sum_{i=1}^m \xi_{F_i, S_i} \).

**Proof.** (i) For \( a, b \in A \), \( \omega_{a,b} \) is a 2-cocycle. Consider \( \varnothing_{a,b} \) as in \(^{35}\) and for \( i \in \{1, \ldots, n\} \), take
\[ f_i : A \rightarrow M, \quad f_i(a) := \varnothing_{1,a}(\lambda_i). \]

Then \( f_i \) is a linear map and by Proposition \(^{35}\)(iii) and Lemma \(^{33}\)(i), we have
\[ \omega_{a,b} = \sum_{i=1}^n \nu_{D_i, \varnothing_{a,b} (\lambda_i)} = \sum_{i=1}^n \nu_{D_i, \varnothing_{1,a} (\lambda_i)} = \sum_{i=1}^n \kappa_{D_i, f_i(ab)}. \]

(ii) For \( a, b \in A \), consider \( \varnothing_{a,b} \) as in \(^{36}\) and for \( i \in \{1, \ldots, m\} \), take\( F_i : A \times A \rightarrow M, \) \( (a, b) \mapsto \varnothing_{a,b}(\mu_i). \)
Then by Lemma 3.13, $F_i$ is a Hochschild map and by Proposition 3.9, we have

$$\omega_{a,b} = \sum_{i=1}^{m} \nu_{S_i, \omega_{a,b}(\mu_i)} = \sum_{i=1}^{m} F_i(a, b) \kappa_{S_i}.$$ 

This completes the proof. □

Recalling Example 3.12 and using Propositions 3.14 and 3.15, we arrive at the following structure theorem for 2-cocycles:

**Theorem 3.16.** Suppose that $\kappa$ is derivation invariant and $D_1, \ldots, D_t, S_1, \ldots, S_m \in \text{cent}_+(\mathfrak{g})$ are such that the cohomology classes $[\kappa_{D_i}]$ form a basis of $H^2(\mathfrak{g})$ and $S_1, \ldots, S_m \in R_{\text{id}}$ form a basis of $\text{Sym}(R)$. Then each 2-cocycle of $g = A \otimes \mathfrak{g}$ with values in a superspace $M$, is cohomologous to a sum

$$\tilde{\omega} = \sum_{i=1}^{t} \eta_{f_i, D_i} + \sum_{j=1}^{m} \xi_{F_j, S_j},$$

i.e.,

$$\tilde{\omega}(ax, by) = (-1)^{|b||x|} \sum_{i=1}^{t} f_i(ab) \kappa(D_i x, y) + (-1)^{|b||x|} \sum_{j=1}^{m} F(a, b) \kappa(S_j x, y),$$

where the $f_i: A \to M$ are linear maps and the $F_j: A \times A \to M$ are Hochschild maps.

4. Unitary representations of current superalgebras

A finite dimensional real Lie algebra $g$ is called compact if it is the Lie algebra of a compact Lie group $G$. Then the subgroup of Aut($g$) generated by $e^{\text{ad}(g)}$ is compact [4, Pro. 12.1.4]. Further, compactness of $g$ is equivalent to the existence of a faithful finite dimensional unitary representation [4, Thm. 12.3.9, Lem. 12.1.2]. This is different for Lie superalgebras. We call a finite dimensional real Lie superalgebra $g = g_0 \oplus g_1$ compact if the subgroup of Aut($g$) generated by $e^{\text{ad}(g_0)}$ has compact closure. As we have already seen in the introduction, this does not imply the existence of a faithful finite dimensional unitary representation. For a classification of compact Lie superalgebras with faithful unitary representations we refer to [4]. Our aim in this section is to investigate the existence of (projective) unitary representations for current superalgebras of the form $A \otimes \mathfrak{f}$, where $\mathfrak{f}$ is a simple compact Lie superalgebra and $A$ is graded commutative.

4.1. Compact Lie algebras. We start with the case where $\mathfrak{f}$ is a simple compact Lie algebra. Fix a simple compact Lie algebra $\mathfrak{f}$ with the Killing form $\kappa$ and a compact Lie group $K$ with Lie algebra $\mathfrak{f}$.

**Remark 4.1.** Suppose that $A$ is a unital supercommutative associative superalgebra. Since $\mathfrak{f}$ is a compact simple Lie algebra, $H^2(\mathfrak{f}) = \{0\}$ and $\text{cent}_+(\mathfrak{f}) = R_{\text{id}}$. By Theorem 3.16, a 2-cocycle $\omega$ of $A \otimes \mathfrak{f}$ is equivalent to one of the form

(4.1) $\omega(ax, by) := \omega_f(ax, by) := F(a, b) \kappa(x, y)$ for $a, b \in A$, $x, y \in \mathfrak{f}$,

where $F$ is a Hochschild map. Here we use that $|x| = 0$ for every $x \in \mathfrak{f} = \mathfrak{f}_0$.

In the following proposition we use Theorem A.9 from the appendix.
Proposition 4.2. Let $M$ be a superspace and let $A = A_0 \oplus A_1$ be a finite dimensional unital supercommutative associative superalgebra equipped with a consistent $\mathbb{Z}$-grading $A = A_0 \oplus A_1 \oplus A^2$ satisfying $A^0 = \mathbb{R}$ and $A_1 A_1 = A^2$. Consider the central extension $\hat{A}$ of the Lie superalgebra $\mathfrak{g} := A \otimes \mathfrak{k}$ defined by a 2-cocycle of the form $\omega = \omega_F$, where $F : A \times A \rightarrow M$ is an even Hochschild map. Then $n := ((A^1 \oplus A^2) \otimes \mathfrak{k}) \oplus M$ is a Clifford–Lie superalgebra and $\mathfrak{g} \simeq n \rtimes \mathfrak{k}$. If $K$ is a simply connected Lie group with Lie algebra $\mathfrak{k}$, then the equivalence classes of irreducible unitary representations of the Lie supergroup $N \rtimes K$ are determined by the $K$-orbits $O_{\lambda}$ in $\mathcal{E}(\mathfrak{n})^*$ and, for any fixed $\lambda$, they are in one-to-one correspondence with odd irreducible representations of $K^\mathfrak{m}$; see (A.7).

Proof. We identify $A^0 \otimes \mathfrak{k}$ with $\mathfrak{k}$, so that $n$ is an ideal of $\hat{A}$ and $\hat{A} \simeq n \rtimes \mathfrak{k}$. As $A^2 = A^1 A^1$ and $A^3 = \{0\}$, we get $F(A^2, A^2) = \{0\}$. We next note that for $X \in A^2 \otimes \mathfrak{k}$ and $Y \in (A^1 \otimes \mathfrak{k}) \oplus (A^2 \otimes \mathfrak{k})$, we have $\omega(X, Y) = 0$ as $F$ is even and $F(A^2, A^2) = \{0\}$ and so

$$[X, Y]_\omega = [X, Y] + \omega(X, Y) = 0.$$  

This implies that $n_0 = (A^2 \otimes \mathfrak{k}) \oplus M_0 \subseteq Z(\mathfrak{n})$, i.e., $n$ is an ideal of $\hat{A}$ which is a Clifford–Lie superalgebra. Now the assertion follows from Theorem A.9. \qed

Proposition 4.3. Assume $A = A_0 \oplus A_1$ is a unital supercommutative associative superalgebra and $\omega$ is a 2-cocycle for $\mathfrak{g} := A \otimes \mathfrak{k}$ with corresponding central extension $(\hat{\mathfrak{g}}, [\cdot, \cdot]_\omega)$. Suppose $x \in \mathfrak{k}$ and $a_1, \ldots, a_n \in A$, $n \geq 2$, are homogeneous elements with $a_1^2 = \cdots = a_n^2 = 0$. Then $X := a_1 \cdots a_n \otimes x$ satisfies $[X, X]_\omega = 0$. In particular, if $n \geq 3$ is odd and $a_1, \ldots, a_n$ are odd elements, $X \in \text{urad}(\hat{\mathfrak{g}})$.

Proof. Recall Remark 4.4 and suppose $F$ is a Hochschild map on $A \times A$ such that for $a, b \in A$ and $x, y \in \mathfrak{k}$, $\omega(ax, by) = F(a, b)\kappa(x, y)$. For $a := a_1 \cdots a_t$, we have

$$F(a, a) = F(a_1, a_2 \cdots a_1 a_1 \cdots a_t) + (-1)^{|a_1|(|a_2| + \cdots + |a_t|)}F(a_2 \cdots a_t, a_1 a_1 \cdots a_t) = 0$$

For $x \in \mathfrak{k}$, this leads to

$$[a_1 \cdots a_t \otimes x, a_1 \cdots a_t \otimes x]_\omega = \kappa(x, x)F(a_1 \cdots a_t, a_1 \cdots a_t) = \kappa(x, x)F(a, a) = 0.$$

If $t \geq 3$ is odd and $a_1, \ldots, a_t$ are odd elements, $X := a_1 \cdots a_t \otimes x$ is an odd element of $\hat{\mathfrak{g}}$ so that the remaining assertion follows from Lemma A.9. \qed

Example 4.4. A typical (and universal) example of a unital supercommutative associative superalgebra is the real unital supercommutative associative superalgebra $\Lambda_s(\mathbb{R})$ generated by odd elements $\epsilon_1, \ldots, \epsilon_s$ subject to the relations

$$\epsilon_i \epsilon_j + \epsilon_j \epsilon_i = 0 \quad (1 \leq i, j \leq s).$$

It is called the (real) Graßmann superalgebra in $s$ generators $\epsilon_1, \ldots, \epsilon_s$. The Graßmann superalgebra $\Lambda_s(\mathbb{R})$ has a natural consistent $\mathbb{Z}$-grading

$$\Lambda_s(\mathbb{R}) = \bigoplus_{m \in \mathbb{Z}} \Lambda^m_s(\mathbb{R})$$

with

$$\Lambda^0_s(\mathbb{R}) = \mathbb{R}, \quad \Lambda^m_s(\mathbb{R}) = \text{span}_\mathbb{R}\{\epsilon_{i_1} \cdots \epsilon_{i_m} \mid 1 \leq i_1 < \cdots < i_m \leq s\} \quad \text{and} \quad \Lambda^s_s(\mathbb{R}) = \{0\}.$$  

\footnote{This means that the $\mathbb{Z}_2$-grading induced from the $\mathbb{Z}$-grading on $A$ coincides with the original $\mathbb{Z}_2$-grading on $A$.}
for $1 \leq m \leq s$ and $n \in \mathbb{Z} \setminus \{0, \ldots, m\}$. We set

$$
\Lambda^s_{\text{odd}}(R) := \bigoplus_{m=0}^{\infty} \Lambda_s^{2m+1}(R), \quad \Lambda^s_{\text{even}}(R) := \bigoplus_{m=1}^{\infty} \Lambda_s^{2m}(R) \quad \text{and} \quad \Lambda^s_{\text{+}}(R) := \bigoplus_{m=1}^{\infty} \Lambda_s^m(R).
$$

The following theorem reduces the problem of classifying projective irreducible unitary representation of $g = \Lambda_s(R) \otimes \mathfrak{t}$ to the case of semidirect products of $\mathfrak{t}$ with Clifford–Lie superalgebras discussed in detail in Appendix A.

**Theorem 4.5.** Consider a current superalgebra $g = \Lambda_s(R) \otimes \mathfrak{t}$ and a central extension $g = g \oplus M$ by a superspace $M$, defined by a 2-cocycle $\omega = \omega_F$, where $F: \Lambda_s(R) \times \Lambda_s(R) \rightarrow M$ is a Hochschild map. We set

$$
R := \text{span}_R \{ F(a, b) \mid a \in \Lambda_s(R), b \in \oplus_{m=3}^{s} \Lambda_s^m(R) \} \subseteq M.
$$

(i) The ideal

$$
I := (\oplus_{m=3}^{s} \Lambda_s^m(R) \otimes \mathfrak{t}) \oplus R
$$

of $\hat{g}$ lies in the unitary radical $\text{urad}(\hat{g})$ of $\hat{g}$; in particular, for each unitary representation $\pi$ of $\hat{g}$, we have $I \subseteq \ker(\pi)$.

(ii) Suppose that $\omega$ is even, i.e., that $F$ is even. Set

$$
\hat{n} := (\Lambda_s^\infty(R) \otimes \mathfrak{t}) \otimes M \quad \text{and} \quad n := \hat{n}/I.
$$

Then $\hat{g} \simeq \hat{n} \times \mathfrak{t}$ and $\hat{g}/I \simeq n \times \mathfrak{t}$. Moreover, $n$ is a Clifford–Lie superalgebra and if $K$ is a simply connected Lie group with Lie algebra $\mathfrak{t}$, then the equivalence classes of irreducible unitary representations of the Lie supergroup $N \times K$ are determined by the $K$-orbits $O_{\lambda}$ in $\mathfrak{c}'(n)^*$ and, for any fixed $\lambda$, they are in one-to-one correspondence with odd irreducible representations of $K_s^\infty$ as defined in (A7).

**Proof.** (i) Assume $r, t$ are positive integers and $j_1, \ldots, j_{2t+1}, i_1, \ldots, i_r \in \{1, \ldots, s\}$. We know from Proposition 4.3 that

$$
e_{j_1} \ldots e_{j_{2t+1}} \otimes \mathfrak{t} \subseteq \text{urad}(\hat{g}).
$$

For $x \in \mathfrak{t}$, we have

$$
F(e_{j_1} \ldots e_{j_{2t+1}}, e_{i_1} \ldots e_{i_r}) \kappa(x, x) = [e_{j_1} \ldots e_{j_{2t+1}} \otimes x, e_{i_1} \ldots e_{i_r} \otimes x]_\omega \in [\text{urad}(\hat{g}), \hat{g}]_\omega \subseteq \text{urad}(\hat{g}).
$$

Now choosing $x \in \mathfrak{t}$ with $\kappa(x, x) \neq 0$, we get that

$$
(4.2) \quad F(e_{j_1} \ldots e_{j_{2t+1}}, e_{i_1} \ldots e_{i_r}) \in \text{urad}(\hat{g})
$$

This implies that, for $x, y \in \mathfrak{t}$,

$$
e_{j_1} \ldots e_{j_{2t+1}} e_{i_1} \ldots e_{i_r} \otimes [x, y] = [e_{j_1} \ldots e_{j_{2t+1}} \otimes x, e_{i_1} \ldots e_{i_r} \otimes y]_\omega - F(e_{j_1} \ldots e_{j_{2t+1}}, e_{i_1} \ldots e_{i_r}) \kappa(x, y) \in [\text{urad}(\hat{g}), \hat{g}]_\omega + \text{urad}(\hat{g}) \subseteq \text{urad}(\hat{g}).
$$

As $\mathfrak{t}$ is perfect, this implies that

$$
(4.3) \quad \oplus_{m=3}^{s} \Lambda_s^m(R) \otimes \mathfrak{t} \subseteq \text{urad}(\hat{g}).
$$

This in turn shows that, for $x \in \mathfrak{t}$,

$$
F(e_{j_1} \ldots e_{j_{2t+2}}, e_{i_1} \ldots e_{i_r}) \kappa(x, x) = [e_{j_1} \ldots e_{j_{2t+2}} \otimes x, e_{i_1} \ldots e_{i_r} \otimes x]_\omega \in \text{urad}(\hat{g}).
$$
Choosing $x \in \mathfrak{k}$ with $\kappa(x, x) \neq 0$, one gets that $F(\epsilon_{j_1} \cdots \epsilon_{j_{2m+2}}, \epsilon_i \cdots \epsilon_{i_m})$ lies in urad($\hat{g}$). This together with \((1.2)\) and the fact that $F(1, \Lambda_s(R)) = \{0\}$ gives that $R \subseteq \text{urad}(\hat{g})$. So by \((4.3)\), we get that
\[ I = (\oplus_{m=3}^{\infty} \Lambda_s^m(R) \otimes \mathfrak{k}) \oplus R \subseteq \text{urad}(\hat{g}). \]
This completes the proof as urad($\hat{g}$) lies in the kernel of each unitary representation of $\hat{g}$ by Lemma \[2.7\]
(ii) Set
\[ A := \Lambda_s(R)/(\oplus_{m=3}^{\infty} \Lambda_s^m(R)) \quad \text{and} \quad T := M/R \]
and consider the induced Hochschild map $\widehat{F} : A \times A \rightarrow T$. Then
\[ \hat{g}/I \simeq (A \otimes \mathfrak{k}) \oplus T \]
is a central extension of $A \otimes \mathfrak{k}$ corresponding to the 2-cocycle $\xi_{F, \text{id}}$. Therefore, irreducible unitary representations of $\hat{g}$ are exactly those of $\hat{g}/I \simeq (A \otimes \mathfrak{k}) \oplus T \cong n \times \mathfrak{k}$ and so the assertion follows from Proposition \[4.2\]

**Proposition 4.6.** The Lie superalgebra $\hat{g} := \Lambda_s(R) \otimes \mathfrak{k}$ has a central extension with a faithful unitary representation if and only if $s \leq 2$.

**Proof.** Suppose first that $s \geq 3$. For all $x \in \mathfrak{k}$ and distinct indices $i_1, \ldots, i_{2m+1}$ ($m \geq 1$), Proposition \[4.3\] implies that $\epsilon_{i_1} \cdots \epsilon_{i_{2m+1}} \otimes x$ has square zero in each central extension $\hat{g}$ of $g$. In particular, $\hat{g}$ has no faithful unitary representation.

Now we assume that $s \in \{1, 2\}$. We consider the even bilinear map
\[ F : \Lambda_s(R) \times \Lambda_s(R) \rightarrow \mathbb{R} \]
\[ (\epsilon_i, \epsilon_j) \mapsto \delta_{i,j} \quad (1 \leq i, j \leq s) \]
\[ (b, a), (a, b) \mapsto 0 \quad (a \in \Lambda_s(R), \ b \in \Lambda_s(R)_0). \]
We shall show that $F$ is a Hochschild map: Suppose that $a, b, c$ are homogeneous elements of $\Lambda_s(R)$ with respect to the $\mathbb{Z}$-grading on $\Lambda_s(R)$. If at least one of $a, b$ is even, then we have $F(a, b) = F(b, a) = 0$. If both $a$ and $b$ are odd, then we may assume $a = \epsilon_i$ and $b = \epsilon_j$ for some $1 \leq i, j \leq s$ which in turn implies that $F(a, b) = F(b, a)$. Therefore in both cases
\[ F(a, b) = (-1)^{|a||b|} F(b, a). \]
Moreover, we know that $F$ is even, $F(\Lambda_s(R)_0, \Lambda_s(R)) = F(\Lambda_s(R), \Lambda_s(R)_0) = \{0\}$ and $\Lambda_s^3(R) = \{0\}$, so we get that $F(ab, c) = F(a, bc) = F(b, ac) = 0$. In fact, the only critical case for $F(ab, c)$ is that, say $a \in \Lambda_s^0(R)$ and $b, c \in \Lambda_s^1(R)$. Then $F(a, bc) = 0$ and by \[4.4\], we have $F(ab, c) = F(b, ac)$. Therefore we always have
\[ F(ab, c) = F(a, bc) + (-1)^{|a||b|} F(b, ac) \]
and thus $F$ is a Hochschild map. So
\[ \omega : g \times g \rightarrow \mathbb{R}, \quad (ax, by) \mapsto F(a, b)\kappa(x, y) \]
is a 2-cocycle whose restriction to $g_1 \times g_1$ is definite as the Killing form $\kappa$ of $\mathfrak{k}$ is negative definite ($\mathfrak{k}$ is compact and simple). Finally Lemma \[2.6\] implies that the cone $\mathcal{C}(\hat{g})$ of the central extension $\hat{g}$ is pointed. As $\hat{g} \cong n \rtimes \mathfrak{k}$ is a semidirect product of the compact Lie algebra $\mathfrak{k}$ with a Clifford–Lie superalgebra, Theorem \[A.2\] now implies that the unitary representations of $\hat{g}$ separate the points. As we may form arbitrary direct sums, a faithful unitary representation exists. \[\square\]
4.2. Compact simple Lie superalgebras. By Theorem 2.3 of [1], one knows the classification of finite dimensional compact Lie superalgebras; to state that classification, we first need to recall the structure of the Lie superalgebras involved.

Denote by $M_{m \times n}(\mathbb{C})$ the set of all $m \times n$-matrices with complex entries and for a complex matrix $A$, denote by $A^t$ and $A^*$ the transposition and the conjugation of the matrix $A$ respectively. For a square matrix (resp. block matrix) $A$, by $\text{tr}(A)$ (resp. $\text{str}(A)$), we mean the trace (resp. supertrace) of $A$. We set

$$u(n; \mathbb{C}) := \{ A \in M_{n \times n}(\mathbb{C}) \mid A^* = -A \} \text{ and } su(n; \mathbb{C}) := \{ A \in u(n; \mathbb{C}) \mid \text{str}(A) = 0 \}.$$  

Also for two positive integers $p, q$, we denote by $\mathfrak{gl}(p, q)$ the Lie superalgebra of all block matrices of dimension $(p, q)$ with entries in $\mathbb{C}$ and denote by $\mathfrak{s}(p, q)$ the sub-superalgebra of $\mathfrak{gl}(p, q)$ containing all elements with zero supertrace. For $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{gl}(p, q)$, we put the superconjugation of $X$ to be

$$X^\# := \begin{pmatrix} A^* & -iC^* \\ -iB^* & D^* \end{pmatrix} \in \mathfrak{gl}(p, q)$$

and set

$$u(p; q; \mathbb{C}) := \{ X \in \mathfrak{gl}(p, q) \mid X^\# = -X \} = \left\{ \begin{pmatrix} A & B \\ iB^* & D \end{pmatrix} \mid A \in u(p; \mathbb{C}), D \in u(q; \mathbb{C}) \right\}$$

which is a compact real form of $\mathfrak{gl}(p, q)$.

I. $su(p|q; \mathbb{C})$ and $\mathfrak{psu}(p|p; \mathbb{C})$ Suppose $p, q$ are two positive integers with $p \geq q$ and let $1_p$ (resp. $1_q$) be the identity matrix of order $p$ (resp. $q$). We set

$$su(p|q; \mathbb{C}) := \{ X \in u(p|q; \mathbb{C}) \mid \text{str}(X) = 0 \}$$

and

$$\mathfrak{psu}(p|p; \mathbb{C}) := su(p|p; \mathbb{C})/\mathbb{R}1.$$  

$\mathfrak{su}(p|q; \mathbb{C})_0 \simeq \mathfrak{su}(p; \mathbb{C}) \oplus \mathfrak{su}(q; \mathbb{C}) \oplus i\mathbb{I}$ with $\mathbb{I} := \frac{1}{p}1_p + \frac{1}{q}1_q$ and $\mathfrak{su}(p|q; \mathbb{C})_1 \simeq \mathbb{C}^p \otimes \mathbb{C}^q$. Since $\mathbb{R}1$ has a trivial intersection with $\mathfrak{su}(p|p; \mathbb{C})_1$, to simplify our notations, we take $\mathfrak{psu}(p|p; \mathbb{C})_1 = \mathfrak{su}(p|p; \mathbb{C})_1$. The supertrace form $\kappa$, that is the bilinear form mapping $(A, B)$ to $\text{str}(AB)$, is an even supersymmetric bilinear form on $\mathfrak{su}(p|q; \mathbb{C})$ whose restriction to $\mathfrak{su}(p; \mathbb{C})$ is negative definite while its restriction to $\mathfrak{su}(q; \mathbb{C})$ is positive definite. Moreover, if $p > q$, the restriction of $\kappa$ to $i\mathbb{I}$ is positive definite while the radical of $\kappa$ is $\mathbb{R}1$ if $p = q$; in the latter case, $\kappa$ induces a nondegenerate even supersymmetric bilinear form on $\mathfrak{psu}(p|p; \mathbb{C})$ denoted again by $\kappa$. These all together imply that

$$\text{(4.5) } \text{in both cases } \mathfrak{su}(p|q; \mathbb{C}) (p > q) \text{ and } \mathfrak{psu}(p|p; \mathbb{C}), \text{ there exist nonzero even elements } x, y \text{ with } 0 \neq \kappa(x, x) = -\kappa(y, y) \text{ and } \kappa(x, y) = 0.$$  

We now suppose that $p > q \geq 1$ and note that $\mathfrak{su}(p|q; \mathbb{C})_1$ is an irreducible $\mathfrak{su}(p|q; \mathbb{C})_0$-module on which $i\mathbb{I}$ acts as $(\frac{1}{p} - \frac{1}{q})\text{id}$. Also $\mathfrak{psu}(p|p; \mathbb{C})_0$ acts irreducibly on $\mathfrak{psu}(p|p; \mathbb{C})_1 \simeq \mathbb{C}^p \otimes \overline{\mathbb{C}^q}$.

The Lie superalgebra $\mathfrak{su}(p|q; \mathbb{C})$ is a compact real form of $\mathfrak{sl}(p, q)$ and $\mathfrak{psu}(p; \mathbb{C})$ is a compact real form of $\mathfrak{sl}(p, p)/\mathbb{C}1$. On the other hand, the Cartan–Killing form of $\mathfrak{su}(p|q; \mathbb{C})$ is nondegenerate which in turn implies that all derivations of $\mathfrak{su}(p|q; \mathbb{C})$ are inner and so

$$H^2(\mathfrak{su}(p|q; \mathbb{C})) = \{ 0 \}.$$
while the restriction of $D := \text{ad} \left( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right)$ to $\mathfrak{su}(p|p; \mathbb{C})$ induces an even outer derivation of $\mathfrak{psu}(p|p; \mathbb{C})$ vanishing on the even part; this in fact forms a basis for $H^2(\mathfrak{psu}(p|p; \mathbb{C})) \simeq \mathbb{R}$ (see Remark 3.5).

For our future use, we note that for
derivation of $\kappa$ we have

\[
\kappa(x_*, y_*) = \kappa(Dx_*, y_*) = 0 \quad \text{and} \quad [x_*, y_*] = u + v,
\]

for some nonzero elements $u$ and $v$ of irreducible components of $\mathfrak{psu}(p|p; \mathbb{C})_0$. Also for the elementary matrix $e_{r,s}$ with 1 in $(r, s)$-entry and 0 elsewhere and

\[
z_* := \begin{pmatrix} 0 & e_{r,s} \\ i e_{s,r} & 0 \end{pmatrix} \in \mathfrak{su}(p|q; \mathbb{C})_1,
\]

we have

\[
\kappa(z_*, z_*) = 0 \quad \text{and} \quad [z_*, z_*] = u + v + z
\]

where $u, v$ are nonzero elements of irreducible components of $\mathfrak{su}(p|q; \mathbb{C})_0$ and $z$ is a central element of $\mathfrak{su}(p|q; \mathbb{C})$.

II. $\mathfrak{c}(n)$ ($n \geq 2$) : Suppose $n \geq 2$ and let $\mathfrak{c}(n)$ be the compact real form of the orthosymplectic Lie superalgebra $\mathfrak{osp}(2|2n - 2)$ with $\mathfrak{c}(n)_0 \simeq \mathbb{R} \oplus \mathfrak{sp}(n - 1)$, where $\mathfrak{sp}(n - 1)$ is the compact real form of the symplectic Lie algebra of rank $n - 1$ and $\mathfrak{c}(n)_1$ is an irreducible $\mathfrak{c}(n)_0$-module isomorphic to $\mathbb{H}^{n-1}$ where $\mathbb{H}$ is the quaternion algebra; more precisely, $\mathfrak{c}(n)$ is the set of all matrices

\[
\begin{pmatrix}
\alpha & 0 & M & N \\
0 & -\alpha & iN & -iM \\
-iM^t & N^t & A & B \\
-iN^t & -M^t & -A^* & -A^t
\end{pmatrix}
\]

where $\alpha \in i\mathbb{R}$, $M, N \in M_{1 \times n}(\mathbb{C})$, $A, B \in M_{n \times n}(\mathbb{C})$, $B^t = B$, and $A^* = -A$. The Cartan–Killing form $\kappa$ of $\mathfrak{c}(n)$ is a real nonzero scalar multiple of its supertrace form. All derivations of $\mathfrak{c}(n)$ are inner because $\kappa$ is nondegenerate, in particular $H^2(\mathfrak{c}(n)) = \{0\}$. The compact Lie algebra $\mathfrak{sp}(n - 1)$ has a negative definite Killing form and the trace form on $\mathfrak{sp}(n - 1)$ is a positive real scalar multiple of the Killing form. Therefore $\kappa$ restricted to $\mathbb{R} \subseteq \mathfrak{c}(n)_0$ is negative definite while the restriction of $\kappa$ to $\mathfrak{sp}(n - 1) \subseteq \mathfrak{c}(n)_0$ is positive definite. So it is easy to find $x_1 \in \mathbb{R}$ and $x_2 \in \mathfrak{sp}(n - 1)$ such that

\[
\kappa(x_1, x_2) = 0 \quad \text{and} \quad \kappa(x_1, x_1) = -\kappa(x_2, x_2) \neq 0;
\]

in particular, $\kappa(x_1 + x_2, x_1 + x_2) = 0$.

III. $\mathfrak{pq}(n)$ ($n > 2$) : Suppose $n > 2$. Set

\[
\mathfrak{q}(n) = \left\{ \begin{pmatrix} a & (1 - i)b \\ (1 - i)b & a \end{pmatrix} \bigg| a, b \in \mathfrak{u}(n; \mathbb{C}), \text{tr}(b) = 0 \right\}
\]

and take

\[
\mathfrak{pq}(n) := \mathfrak{q}(n)/\mathbb{R} \mathfrak{1} \simeq \left\{ \begin{pmatrix} a & (1 - i)b \\ (1 - i)b & a \end{pmatrix} \bigg| a, b \in \mathfrak{su}(n; \mathbb{C}) \right\}.
\]
The real Lie superalgebra $\mathfrak{q}(n)$ is a real form of the Lie superalgebra
$$\tilde{Q}(n) := \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \in \mathfrak{sl}(n+1, n+1) \mid \text{tr}(b) = 0 \right\}$$
and $\mathfrak{pq}(n)$ is a real form of $Q(n) := \tilde{Q}(n)/\mathbb{C}1$. The even part $\mathfrak{pq}(n)_0 \simeq \mathfrak{su}(n, \mathbb{C})$ is a simple Lie algebra acting irreducibly on $\mathfrak{pq}(n)_1 \simeq \mathfrak{su}(n, \mathbb{C})$, by the adjoint representation. The restriction of $D := \text{ad} \left( \begin{array}{ccc} 0 & 1 \\ i1_n & 0 \end{array} \right)$ to $\mathfrak{pq}(n)$, which is an outer derivation of $\mathfrak{pq}(n)$ vanishing on $\mathfrak{pq}(n)_0$, forms a basis for $H^2(\mathfrak{pq}(n))$ regarding Remark 3.8. Also the bilinear form
$$\kappa: \mathfrak{pq}(n) \times \mathfrak{pq}(n) \rightarrow \mathbb{R}$$
$$\left( \begin{array}{ccc} a \\ (1-i)b \\ a \end{array} \right), \left( \begin{array}{ccc} a' \\ (1-i)b' \\ a' \end{array} \right) \mapsto \text{tr}(ab' + a'b)$$
is an odd invariant nondegenerate supersymmetric bilinear form on $\mathfrak{pq}(n)$. It is easily verified that the bilinear form $\kappa_D: \mathfrak{k} \times \mathfrak{k} \rightarrow \mathbb{R}, (x, y) \mapsto \kappa(D(x), y)$, is definite on $\mathfrak{k}_1 \times \mathfrak{k}_1$.

**Theorem 4.7** ([1] Thm. 2.3]). Each simple compact Lie superalgebra $\mathfrak{k}$ is either a simple compact Lie algebra or isomorphic to one of the Lie superalgebras
\begin{equation}
\mathfrak{su}(n|m; \mathbb{C}), n > m, \quad \text{and} \quad \mathfrak{psu}(n|n; \mathbb{C}), \quad \mathfrak{pq}(n), \quad \mathfrak{c}(n), n \geq 2.
\end{equation}

Note that we have seen in (I)-(III) above that any of these Lie superalgebras $\mathfrak{k}$ carries a nondegenerate supersymmetric invariant homogeneous bilinear form.

**Remark 4.8.** (i) It is easily verified that, if a real vector space $V$ equipped with a bilinear form $(\cdot, \cdot)$ has an orthogonal decomposition $V = V_1 \oplus V_2$ of nonzero subspaces such that $(\cdot, \cdot)$ is negative definite on $V_1$ and positive definite on $V_2$, then $\{ x \in V \mid (x, x) = 0 \}$ spans the vector space $V$. Now if $\mathfrak{k}$ is one of the real Lie superalgebras $\mathfrak{su}(n|m; \mathbb{C})$ ($n > m$), $\mathfrak{psu}(n|n; \mathbb{C})$ ($n \geq 2$) or $\mathfrak{c}(n)$ ($n \geq 2$) and $\kappa$ is the bilinear form introduced in (I) and (II), then
$$\{ x \in \mathfrak{k}_0 \mid \kappa(x, x) = 0 \}$$
spans $\mathfrak{k}_0$.

(ii) Let $\mathfrak{k}$ be one of the simple compact Lie superalgebras in (4.11) and $\kappa$ be the nondegenerate supersymmetric invariant bilinear form on $\mathfrak{k}$ introduced in (I)-(III). Since the real Lie superalgebra $\mathfrak{k}$ is a real form of a simple Lie superalgebra, it is absolutely simple and so $\mathfrak{cen}_{\mathfrak{k}}(\mathfrak{k}) = \mathbb{R}\text{id}$. In particular, by Proposition 3.5 up to scalar multiple, $\kappa$ is the unique nonzero supersymmetric invariant bilinear form on $\mathfrak{k}$.

(iii) Suppose that $\mathfrak{k}$ is one of the simple compact Lie superalgebras $\mathfrak{su}(n|m; \mathbb{C})$ with $n > m$ or $\mathfrak{c}(n)$ with $n \geq 2$. We know from (I) and (II) that $\mathfrak{k}_0$ is a reductive Lie algebra and the center $Z(\mathfrak{k}_0)$ of $\mathfrak{k}_0$ is isomorphic to $\mathbb{R}$. A direct calculation shows that the bilinear form $b_j: \mathfrak{k}_1 \times \mathfrak{k}_1 \rightarrow \mathbb{R}$ mapping $(x, y)$ to the projection of $[x, y]$ on the center component of $[x, y] \in \mathfrak{k}_0$, with respect to the decomposition of $\mathfrak{k}_0$ stated in (I) and (II), is a definite form. This in particular implies that $\mathfrak{sp}(\mathfrak{k})$ is pointed, (Lemma 2.3; see also [1] Thm. 5.4 & Lem. 3.4)).

(iv) Suppose that $n \geq 2$ is a positive integer. Then for
$$B_j := \text{diag}(0, \ldots, 0, 1_{j \text{th}}, 0, \ldots, 0) \quad \text{for } 1 \leq j \leq n$$
and
\[ X_j := \begin{pmatrix} 0 & B_j \\ iB_j & 0 \end{pmatrix} \in \text{su}(n; \mathbb{C})_1 \quad \text{for } 1 \leq j \leq n, \]
we have \( \sum_{j=1}^{n} [X_j, X_j] \in iR1 \). Also for \( b_j := i(B_j - B_{j+1}) \) \((1 \leq j \leq n - 1)\) and \( b_n := i(B_n - B_1) \), we have
\[ Y_j := \begin{pmatrix} 0 & (1 - i)b_j \\ (1 - i)b_j & 0 \end{pmatrix} \in \text{pq}(n; \mathbb{C})_1 \quad (1 \leq j \leq n) \]
and \( \sum_{j=1}^{n} [Y_j, Y_j] \in iR1 \). This implies that, for \( \mathfrak{t} = \text{psu}(n; \mathbb{C}) \) \((n \geq 2)\) or \( \mathfrak{t} = \text{pq}(n) \)(\(n > 2\)), \( \mathcal{C}(\mathfrak{t}) \) is not pointed.

**Theorem 4.9.** Suppose that \( \mathfrak{t} \) is a simple compact Lie superalgebra with \( \mathfrak{t}_1 \neq \{0\} \) and \( s \) is a positive integer. Then \( \Lambda^+_s(\mathbb{R}) \otimes \mathfrak{t} \) lies in the kernel of each unitary representations of \( \mathfrak{g} = \Lambda_s(\mathbb{R}) \otimes \mathfrak{t} \); in particular unitary representations of \( \mathfrak{g} \) are in one-to-one correspondence with unitary representations of \( \mathfrak{t} \).

**Proof.** We first note that \( \mathfrak{g} = (\Lambda^+_s(\mathbb{R}) \otimes \mathfrak{t}) \oplus \mathfrak{t} \) and that \( \Lambda^+_s(\mathbb{R}) \otimes \mathfrak{t} \) is an ideal of \( \mathfrak{g} \). So each unitary representation \( \pi \) of \( \mathfrak{t} \) can be extended to a unitary representation of \( \mathfrak{g} \) with \( \Lambda^+_s(\mathbb{R}) \otimes \mathfrak{t} \subseteq \ker(\pi) \).

Next recall that \( \mathfrak{t}_0 \) is a reductive Lie algebra and set \( \mathfrak{h} := [\mathfrak{t}_0, \mathfrak{t}_0] \). Suppose that \( \pi \) is a unitary representation of \( \mathfrak{g} \). Then for a nonnegative integer \( t \) and elements \( i_1, \ldots, i_{2t+1} \in \{1, \ldots, s\} \), thanks to Lemma 2.7(i), we have
\[ \epsilon_{i_1} \cdots \epsilon_{i_{2t+1}} \otimes \mathfrak{t}_0 \subseteq \ker(\pi). \]
Therefore, for \( x, y \in \mathfrak{h} \) and \( i_1, \ldots, i_{2t+2} \in \{1, \ldots, s\} \),
\[ \epsilon_{i_1} \cdots \epsilon_{i_{2t+2}} \otimes [x, y] = [\epsilon_{i_1} \epsilon_{i_2} \cdots \epsilon_{i_{2t+1}} \otimes x, \epsilon_{i_{2t+2}} \otimes y] \in \ker(\pi). \]
So as \( \mathfrak{h} \) is perfect, we get using (4.12) that
\[ \Lambda^+_s(\mathbb{R}) \otimes \mathfrak{h} \subseteq \ker(\pi). \]
Fixing \( x \in \mathfrak{h} \) and \( y \in \mathfrak{t}_1 \) with \( [x, y] \neq 0 \), we have for each \( i_1, \ldots, i_t \in \{1, \ldots, s\} \),
\[ 0 \neq \epsilon_{i_1} \cdots \epsilon_{i_t} \otimes [y, x] = [1 \otimes y, \epsilon_{i_1} \cdots \epsilon_{i_t} \otimes x] \in \ker(\pi) \cap (\Lambda^+_s(\mathbb{R}) \otimes \mathfrak{t}_1). \]
So it follows that
\[ \Lambda^+_s(\mathbb{R}) \otimes \mathfrak{t}_1 \subseteq \ker(\pi) \]
because \( \mathfrak{t}_1 \) is an irreducible \( \mathfrak{t}_0 \)-module. We finally note that, as \( \mathfrak{t} \) is simple, \( \mathfrak{t}_0 = [\mathfrak{t}_1, \mathfrak{t}_1] \) and so
\[ \Lambda^+_s(\mathbb{R}) \otimes \mathfrak{t}_0 = \Lambda^+_s(\mathbb{R}) \otimes [\mathfrak{t}_1, \mathfrak{t}_1] = [1 \otimes \mathfrak{t}_1, \Lambda^+_s(\mathbb{R}) \otimes \mathfrak{t}_1] \subseteq [\mathfrak{g}, \ker(\pi)] \subseteq \ker(\pi). \]
This completes the proof. \( \square \)

One knows from 1 Lem.'s 3.2 & 3.4(v)] that faithful finite dimensional unitary representations do not exist for \( \text{psu}(n; \mathbb{C}) \) and \( \text{pq}(n) \) \((n > 2)\) while \( \text{su}(n; \mathbb{C}) \) and \( \text{q}(n) \) \((n > 2)\) which are respectively universal central extensions of \( \text{psu}(n; \mathbb{C}) \) and \( \text{pq}(n) \) \((n > 2)\) have finite dimensional faithful unitary representations. We also know form Theorem 4.9 that there is no faithful unitary representation for \( \mathfrak{g} = \Lambda_s(\mathbb{R}) \otimes \mathfrak{t} \) but we are interested in faithful unitary representations of central extensions of \( \mathfrak{g} \).

In what follows recalling (4.5) and (4.10), we will see that for each central extension \( \mathfrak{g} \) of \( \mathfrak{g} \), the ideal \( \text{urad}(\mathfrak{g}) \) is non-zero. In particular, \( \mathfrak{g} \) does not have faithful unitary representations (Lemma 2.7).
Theorem 4.10. Let $s$ be a positive integer and $\mathfrak{k}$ be a compact simple Lie superalgebra with $\mathfrak{k}_1 \neq \{0\}$. Suppose $\widehat{\mathfrak{g}} = \mathfrak{g} \oplus M$ is a perfect central extension of $\mathfrak{g} = \Lambda_s(\mathfrak{R}) \otimes \mathfrak{k}$ such that $\text{urad}(\widehat{\mathfrak{g}})$ is a proper ideal, then

$$\text{urad}(\widehat{\mathfrak{g}}) = (\Lambda_s^+(\mathfrak{R}) \otimes \mathfrak{k}) \oplus (M \cap \text{urad}(\widehat{\mathfrak{g}})).$$

In particular, either all unitary representations of $\widehat{\mathfrak{g}}$ are trivial, or they factor through a central extension of $\mathfrak{k}$.

Proof. Suppose that $\omega$ is the 2-cocycle corresponding to $\widehat{\mathfrak{g}}$. We split the proof into three steps:

**Step 1.** $\Lambda_s^+(\mathfrak{R}) \otimes \mathfrak{k} \subseteq \text{urad}(\widehat{\mathfrak{g}})$: To show this, we use a type–by–type approach:

- **Remark 4.8(ii) and Theorem 4.19** there exist a linear map $f$ on $\Lambda_s(\mathfrak{R})$ and a Hochschild map $F$ on $\Lambda_s(\mathfrak{R}) \times \Lambda_s(\mathfrak{R})$ such that for $a, b \in \Lambda_s(\mathfrak{R})$ and $x, y \in \mathfrak{k}$,

$$\omega(a \otimes x, b \otimes y) = (-1)^{|x| |y|} \left( F(a, b) \kappa(x, y) + f(ab) \kappa(Dx, y) \right).$$

- **Stage 1.** $\Lambda_{s,0}^{\text{odd}}(\mathfrak{R}) \otimes \mathfrak{k}_0 \subseteq \text{urad}(\widehat{\mathfrak{g}})$: As $\kappa$ is odd and $D |_{\mathfrak{k}_0} = 0$, we have

$$(4.13) \quad [a \otimes x, b \otimes y]_\omega = ab \otimes [x, y] \quad \text{for} \quad x, y \in \mathfrak{k}_0, \ a, b \in \Lambda_s(\mathfrak{R}).$$

This shows that, for $a \in \Lambda_s(\mathfrak{R})$ and $x \in \mathfrak{k}_0$, $a \otimes x$ is an odd element of $\widehat{\mathfrak{g}}$ with square zero and so by Lemma 4.7(i),

$$\Lambda_{s,0}^{\text{odd}}(\mathfrak{R}) \otimes \mathfrak{k}_0 \subseteq \text{urad}(\widehat{\mathfrak{g}}).$$

- **Stage 2.** $\Lambda_s^+(\mathfrak{R}) \otimes \mathfrak{k}_0 \subseteq \text{urad}(\widehat{\mathfrak{g}})$: Since $\mathfrak{k}_0$ is simple, Stage 1 together with (4.13) implies that

$$\Lambda_s^{\text{even}}(\mathfrak{R}) \otimes \mathfrak{k}_0 = \sum_{m=1}^{\infty} \Lambda_s^{2m-1}(\mathfrak{R}) \Lambda_s^1(\mathfrak{R}) \otimes [\mathfrak{k}_0, \mathfrak{k}_0]$$

$$= \sum_{m=1}^{\infty} [\Lambda_s^{2m-1}(\mathfrak{R}) \otimes \mathfrak{k}_0, \Lambda_s^1(\mathfrak{R}) \otimes \mathfrak{k}_0]_\omega \subseteq \text{urad}(\widehat{\mathfrak{g}}).$$

Therefore, we get the result using Stage 1.

- **Stage 3.** $\Lambda_s^+(\mathfrak{R}) \otimes \mathfrak{k} \subseteq \text{urad}(\widehat{\mathfrak{g}})$: For $x \in \mathfrak{k}_0$, $y \in \mathfrak{k}_1$ and $a \in \Lambda_s^+(\mathfrak{R})$, by (3.7) and Stage 2, we have

$$a \otimes [x, y] = a \otimes [x, y] + F(a, 1) \gamma(x, y) + f(a) \kappa(Dx, y) = [a \otimes x, 1 \otimes y]_\omega \in [\text{urad}(\widehat{\mathfrak{g}}), \widehat{\mathfrak{g}}]_\omega \subseteq \text{urad}(\widehat{\mathfrak{g}}).$$

So choosing $x \in \mathfrak{k}_0$ and $y \in \mathfrak{k}_1$ with $u := [x, y] \neq 0$ ($\mathfrak{k}_1$ is an irreducible $\mathfrak{k}_0$-module), we have $\Lambda_s^+(\mathfrak{R}) \otimes u \subseteq \text{urad}(\widehat{\mathfrak{g}})$. Now for $x_1, \ldots, x_k \in \mathfrak{k}_0$, we have

$$\Lambda^+(\mathfrak{R}) \otimes \{x_k, \ldots, [x_1, u] \ldots\} = [1 \otimes x_k, \ldots, 1 \otimes x_1, \Lambda^+(\mathfrak{R}) \otimes u]_\omega \subseteq [\widehat{\mathfrak{g}}, \text{urad}(\widehat{\mathfrak{g}})]_\omega \subseteq \text{urad}(\widehat{\mathfrak{g}}).$$

As $\mathfrak{k}_1$ is irreducible, this implies that $\Lambda_s^+(\mathfrak{R}) \otimes \mathfrak{k}_1 \subseteq \text{urad}(\widehat{\mathfrak{g}})$ and so Stage 2 completes the proof.

$\mathfrak{f} = \mathfrak{psu}(n|m; \mathbb{C})$: Consider the outer derivation $D$ of $\mathfrak{f}$ introduced in (I). By Remark 4.8(ii) and Theorem 4.19 there are a linear map $f$ on $\Lambda_s(\mathfrak{R})$ and a Hochschild map $F$ on $\Lambda_s(\mathfrak{R}) \times \Lambda_s(\mathfrak{R})$ such that for $a, b \in \Lambda_s(\mathfrak{R})$ and $x, y \in \mathfrak{k}$,

$$\omega(a \otimes x, b \otimes y) = (-1)^{|x| |y|} \left( F(a, b) \kappa(x, y) + f(ab) \kappa(Dx, y) \right).$$
• Stage 1. \( \Lambda_+^{\text{odd}}(R) \otimes t_0 \subseteq \text{urad}(\mathfrak{g}) \): Suppose \( a \in \Lambda_+^{\text{odd}}(R) \) and \( x \in t_0 \) with \( \kappa(x, x) = 0 \).

Then, as \( Dx = 0 \), we have

\[
[a \otimes x, a \otimes x]_\omega = a^2 \otimes [x, x] + \sum_{a} \frac{F(a, a) \kappa(x, x) + f(a^2) \kappa(Dx, x)}{2} = 0.
\]

So by Lemma 2.7(i), we get

\[
\{a \otimes x \mid a \in \Lambda_+^{\text{odd}}(R), x \in t_0, \kappa(x, x) = 0 \} \subseteq \text{urad}(\mathfrak{g}).
\]

Therefore, by Remark 4.8(i), we have

\[
\Lambda_+^{\text{odd}}(R) \otimes t_0 \subseteq \text{urad}(\mathfrak{g}).
\]

• Stage 2. \( \Lambda_+^{\text{even}}(R) \otimes t_1 \subseteq \text{urad}(\mathfrak{g}) \): Since \( \kappa \) is even and \( D \mid t_0 = 0 \), we have

\[
[a \otimes x, b \otimes y]_\omega = ab \otimes [x, y] \quad \text{for } x \in t_0, y \in t_1, a, b \in \Lambda_+^{\text{even}}(R).
\]

So Stage 1 implies that for \( x, x_1, \ldots, x_k \in t_0 \) and \( y \in t_1 \),

\[
\Lambda_+^{\text{even}}(R) \otimes [x_k, \ldots, [x_1, [x, y]]] \subseteq \text{urad}(\mathfrak{g}).
\]

Now we are done as \( t_1 \) is an irreducible \( t_0 \)-module.

• Stage 3. \( \Lambda_+^{\text{even}}(R) \otimes t \subseteq \text{urad}(\mathfrak{g}) \): Recalling \( x_+ \) and \( y_+ \) from (4.6) and using (4.7), we have

\[
\Lambda_+^{\text{even}}(R) \otimes [x_+, y_+] = [\Lambda_+^{\text{odd}} \otimes x_+, \Lambda_+^{\text{odd}} \otimes y_+]_\omega \subseteq \text{urad}(\mathfrak{g}).
\]

We recall that \( \mathfrak{psu}(n|n; \mathbb{C}) = t_0^1 \oplus t_0^2 \) with \( t_0^1, t_0^2 \cong \mathfrak{su}(n; \mathbb{C}) \) and that \( [x_+, y_+] = u + v, 0 \neq u \in t_0^1 \) and \( 0 \neq v \in t_0^2 \). Now for \( x_1, \ldots, x_k \in t_0 \), using (4.15), we have

\[
\Lambda_+^{\text{even}}(R) \otimes [x_k, \ldots, [x_1, [x, u]]] \subseteq \text{urad}(\mathfrak{g}).
\]

Also, for \( y_1, \ldots, y_k, y \in t_0^2 \), we have

\[
\Lambda_+^{\text{even}}(R) \otimes [y_k, \ldots, [y_1, [y, v]]] \subseteq \text{urad}(\mathfrak{g}).
\]

Since \( t_0^1 \) and \( t_0^2 \) are simple, these imply that \( \Lambda_+^{\text{even}}(R) \otimes t_0 \subseteq \text{urad}(\mathfrak{g}) \). This together with Stages 1–2 gives that \( \Lambda_+^{\text{even}}(R) \otimes t \subseteq \text{urad}(\mathfrak{g}) \).

\[ \mathfrak{f} = \mathfrak{su}(m|n; \mathbb{C}), m \neq n \]: In this case, \( t_0 = t_0^1 \oplus t_0^2 \oplus \mathfrak{f} \mathfrak{l} \mathfrak{z} \) where \( i \mathfrak{z} \) is a central element and \( t_0^1, t_0^2 \) as well as \( t_0^2 \) are simple ideals of \( t_0 \). Since \( H^2(\mathfrak{f}) = \{0\} \) by (I), using Remark 4.8(ii) and Theorem 8.10, there exists a Hochschild map \( F \) on \( \Lambda_+^{\text{even}}(R) \times \Lambda_+^{\text{even}}(R) \) such that for \( a, b \in \Lambda_+^{\text{even}}(R) \) and \( x, y \in \mathfrak{f} \),

\[
\omega(ax, by) = (-1)^{|x||y|} F(a, b) \kappa(x, y).
\]

• Stage 1. \( \Lambda_+^{\text{odd}}(R) \otimes t_0 \otimes \Lambda_+^{\text{even}}(R) \otimes t_1 \subseteq \text{urad}(\mathfrak{g}) \): For \( a \in \Lambda_+^{\text{odd}}(R) \) and \( x \in t_0 \) with \( \kappa(x, x) = 0 \), we have

\[
[a \otimes x, a \otimes x]_\omega = a^2 \otimes [x, x] + F(a, a) \kappa(x, x) = 0.
\]

A modification of the argument in Stages 1–2 of the previous case implies that

\[
(\Lambda_+^{\text{odd}}(R) \otimes t_0) \otimes (\Lambda_+^{\text{even}}(R) \otimes t_1) \subseteq \text{urad}(\mathfrak{g}).
\]
• Stage 2. $\Lambda^\text{even}(R) \otimes (t_{0}^1 + t_{0}^2) \subseteq \text{urad}(\widehat{g})$ : Choose $z, u, v$ and $z$ as in (4.13) and (4.19) and use a slight modification of the argument in Stage 3 of the previous case.

• Stage 3. $\Lambda^+_s(R) \otimes \mathfrak{t} \subseteq \text{urad}(\widehat{g})$ : Choose $x \in t_{0}^1$ with $\kappa(x, x) \neq 0$, then for $b \in \Lambda^+_s(R)$ and $a \in \Lambda^+_s(R)$, by Stages 1–2, we have

$$\kappa(x, x) F(a, b) = [a \otimes x, b \otimes x]_\omega \in [\widehat{g}, \text{urad}(\widehat{g})]_\omega \subseteq \text{urad}(\widehat{g}).$$

It means that

$$F(a, b) \in \text{urad}(\widehat{g}) \quad \text{for } a \in \Lambda^+_s(R) \text{ and } b \in \Lambda^+_s(R).$$

This together with Stage 1 implies that for $y \in \mathfrak{t}_1$, we have

$$\Lambda^\text{even}(R) \otimes [y, y] \subseteq [\Lambda^\text{odd}(R) \otimes y, \Lambda^\text{odd}(R) \otimes y]_\omega + F(\Lambda^\text{odd}(R), \Lambda^\text{odd}(R)) \kappa(y, y)$$

(4.16) \subseteq \text{urad}(\widehat{g}).

Fix a nonzero $y \in \mathfrak{t}_1$. As we mentioned in Remark 4.3 iii), $[y, y]$ has a nonzero component in $\mathbb{R} \mathfrak{d} \mathfrak{l}$. So by (4.16) and Stage 2, we get that $\Lambda^\text{even}(R) \otimes \mathfrak{r} \mathfrak{l} \subseteq \text{urad}(\widehat{g})$. This together with Stages 1–2, completes the proof.

**Step 2. urad(\widehat{g}) \cap ((R \otimes \mathfrak{t}) \otimes M) \subseteq M :** We first suppose $\mathfrak{t} = \mathfrak{psu}(n)$ and consider $f$ as above. If $(1 \otimes x_0) + m_0 \in \text{urad}(\widehat{g})$ for some $m_0 \in M$ and nonzero $x_0 \in \mathfrak{t}$, then we have for $y \in \mathfrak{t}$

(4.17) $1 \otimes [x_0, y] + f(1) \kappa(Dx_0, y) = [1 \otimes x_0, 1 \otimes y]_\omega = [1 \otimes x_0 + m_0, 1 \otimes y]_\omega \in \text{urad}(\widehat{g})$.

But $\mathfrak{t}$ is simple, so this implies that, for each $x \in \mathfrak{t}$, there is $r_x \in M$ with

(4.18) $(1 \otimes x) + r_x \in \text{urad}(\widehat{g})$.

Since $\widehat{g}$ is perfect, for $x \in \mathfrak{t}$ and $m \in M$, there are $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathfrak{t}$ and homogeneous elements $a_1, \ldots, a_n, b_1, \ldots, b_n \in \Lambda^+_s(R)$ with $a_1 = 1$ if $a_i \in \Lambda^+_s(R)$, such that $x + m = \sum_{i=1}^{n} [a_i \otimes x_i, b_i \otimes y_i]_\omega$. So as $\Lambda^+_s(R) \otimes \mathfrak{t} \subseteq \text{urad}(\widehat{g})$, (4.18) implies that

$$x + m = \sum_{i=1}^{n} [a_i \otimes x_i, b_i \otimes y_i]_\omega + \sum_{i=1}^{n} [a_i \otimes x_i, b_i \otimes y_i]_\omega$$

$$= \sum_{i=1}^{n} [a_i \otimes x_i, b_i \otimes y_i]_\omega + \sum_{i=1}^{n} [(a_i \otimes x_i) + r_{x_i}, b_i \otimes y_i]_\omega \in \text{urad}(\widehat{g})$$

which is a contradiction as $\text{urad}(\widehat{g})$ is proper. So we are done in this case. Repeating this argument, one gets the result for $\mathfrak{t} = \mathfrak{psu}(n; \mathbb{C})$ as well.

Next assume $\mathfrak{t}$ is one of the remaining types. If $(1 \otimes x_0) + m_0 \in \text{urad}(\widehat{g})$ for some $m_0 \in M$ and nonzero $x_0 \in \mathfrak{t}$, then for $y \in \mathfrak{t}$, we have

$$1 \otimes [x_0, y] = [1 \otimes x_0, 1 \otimes y]_\omega = [1 \otimes x_0 + m_0, 1 \otimes y]_\omega \in \text{urad}(\widehat{g}).$$

But $\mathfrak{t}$ is simple, so it follows that $1 \otimes \mathfrak{t} \subseteq \text{urad}(\widehat{g})$. This implies that $\Lambda^+_s(R) \otimes \mathfrak{t} \subseteq \text{urad}(\widehat{g})$ because $\Lambda^+_s(R) \otimes \mathfrak{t} \subseteq \text{urad}(\widehat{g})$. Since $\widehat{g}$ is perfect, we get that

$$\widehat{g} = [\Lambda^+_s(R) \otimes \mathfrak{t}, \Lambda^+_s(R) \otimes \mathfrak{t}]_\omega \subseteq \text{urad}(\widehat{g})$$

This is a contradiction and so we are done.
Step 3. urad(\mathfrak{g}) = (\Lambda^+ (\mathbb{R}) \otimes \mathfrak{k}) \oplus (M \cap \text{urad}(\mathfrak{g})) : By Step 1, \Lambda^+ (\mathbb{R}) \otimes \mathfrak{k} \subseteq \text{urad}(\mathfrak{g})$. Therefore, we have
\text{urad}(\mathfrak{g}) = (\Lambda^+ (\mathbb{R}) \otimes \mathfrak{k}) \oplus (\text{urad}(\mathfrak{g}) \cap (\mathbb{R} \otimes \mathfrak{k} \oplus M))
and so we are done by Step 2. \qed

Appendix A. Unitary Representations of Semidirect Products

In this section, we describe the classification of irreducible unitary representations of Lie supergroups which are semidirect products \(N \rtimes K\) of a finite dimensional Clifford–Lie supergroup \(N\) and a compact Lie group \(K\) (cf. Definition 2.2). This classification has been obtained in [2], but since it is also central in our context, we recall this result in some detail, and this requires a number of ingredients, such as representations of Clifford algebras and Mackey’s little group theory for Lie supergroups.

A.1. Clifford algebras. Let \(V\) be a finite dimensional real vector space and \(\mu\) a symmetric bilinear form on \(V\). We write \(C = C(V, \mu)\) for the corresponding Clifford algebra and \(\iota : V \to C\) for the structure map satisfying
\[\iota(v)^2 = \mu(v, v)1\quad \text{for} \quad v \in V.\]
As \(\iota\) is injective, we shall identify \(V\) with the subset \(\iota(V) \subseteq C\). We consider the \(\mathbb{Z}_2\)-grading \(C = C_0 \oplus C_1\) on \(C\) induced from the natural \(\mathbb{Z}\)-grading on \(C\) and recall the parity operator
\[\Pi_C : C \to C, \quad x_0 + x_1 \mapsto x_0 - x_1.\]
The unit group \(C^\times\) acts on \(C\) by the automorphisms
\[\alpha_g(c) := \Pi_C(g)cg^{-1}, \quad x \in C^\times, c \in C.\]
The stabilizer of \(V\) is the Clifford group
\[\Gamma(V) := \{g \in C^\times : \alpha_g(V) = V\}.\]
The map
\[\alpha : \Gamma(V) \to O(V), \quad x \mapsto \alpha_x|V\]
defines a group homomorphism. Let \(x \mapsto x^T\) denote the unique anti-automorphism of \(C\) that coincides with the identity on \(V\). Then \(x^T x = N(x) 1\) for \(x \in \Gamma(V)\) and some \(N(x) \in \mathbb{R}^\times\), which defines a group homomorphism \(N : \Gamma(V) \to \mathbb{R}^\times\). Its kernel is the Pin group \(\text{Pin}(V) = \text{Pin}(V, \mu) := \ker(N)\). If \(\mu\) is positive definite, then each element of \(\text{Pin}(V)\) is a product \(v_1 \cdots v_n\) of unit length vectors \(v_j \in V\) and we obtain the following short exact sequence
\[1 \to \{\pm 1\} \to \text{Pin}(V) \xrightarrow{\alpha} O(V) \to 1.\]
The universal property of Clifford algebras implies that each element \(T \in O(V)\) induces an automorphism \(\nu_T\) of \(C\) with \(\nu_T|V = T\). This defines a natural action \(\nu : O(V) \to \text{Aut}(C)\).

Definition A.1. A selfadjoint representation \((\pi, \mathcal{H})\) of a Clifford algebra \(C = C(V, \mu)\) on a Hilbert superspace \(\mathcal{H}\) is an algebra homomorphism \(\pi : C \to \text{End}(\mathcal{H})\) for which all operators \(\pi(v), v \in V,\) are odd and symmetric. Therefore selfadjoint representations are in one-to-one correspondence with linear maps \(\pi : V \to \text{Herm}_1(\mathcal{H})_{\text{odd}}\) satisfying
\[\pi(v)^2 = \mu(v, v)1\quad \text{for} \quad v \in V.\]
By [2] Lem. 11, we have:

**Proposition A.2.** If $\mu$ is positive definite, then there exists a finite dimensional irreducible selfadjoint representation $(\tau, \mathcal{N})$ of $\mathcal{C}(V, \mu)$ which is unique if $\dim V$ is odd and unique up to parity reversal if $\dim V$ is even. Any other selfadjoint representation $(\pi, \mathcal{H})$ of $\mathcal{C}(V, \mu)$ can be chosen purely even.

**Definition A.3.** If $G$ is a group and $\mathcal{H}$ a Hilbert superspace, then a homomorphism $\pi : G \to \text{Aut}(\mathcal{H})$ is called a graded unitary representation with respect to the subgroup $G_1 := \pi^{-1}(\text{Aut}(\mathcal{H})_{\text{even}})$ if $G_1$ is a proper subgroup. It is called even if $\pi(G) \subseteq \text{Aut}(\mathcal{H})_{\text{even}}$, i.e., if $G_1 = G$.

**Remark A.4.** If $G_1 \subseteq G$ is a subgroup of index two (hence normal), any unitary representation $(\pi, \mathcal{H}_0)$ of $G_1$ on a (purely even) Hilbert superspace $\mathcal{H}_0$ admits, up to equivalence, a unique extension to a graded representation $(\hat{\pi}, \mathcal{H})$ of $G$ on a Hilbert superspace $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$. It is equivalent to $\text{Ind}_{G_1}^G(\pi)$ with its natural 2-grading.

In the following proposition we formulate the outcome of Lemmas 13/14 in [2] in terms of a central extension of the orthogonal group.

**Proposition A.5.** Suppose that $\mu$ is positive definite and that $(\tau, \mathcal{N})$ is an irreducible selfadjoint representation of $\mathcal{C}(V, \mu)$. Then there exists a central extension $q_0 : \tilde{O}(V, \mu) \to O(V, \mu)$ such that:

(i) The subgroup $q_0^{-1}(\text{SO}(V, \mu))$ is equivalent, as a central extension, to $\text{Spin}(V) := \alpha^{-1}(\text{SO}(V)) \subseteq \text{Pin}(V)$.

(ii) There exists a graded unitary representation $\hat{\tau} : \tilde{O}(V, \mu) \to \text{Aut}(\mathcal{N})$ extending the even representation $\tau|_{\text{Spin}(V)}$ of $\text{Spin}(V, \mu)$ such that

$$\hat{\tau}(g)\tau(c)\hat{\tau}(g)^{-1} = \tau(\nu_{\rho}(c)) \quad \text{for} \quad g \in \tilde{O}(V, \mu), c \in \mathcal{C}(V, \mu).$$

If $\dim V$ is odd, then there exists an even representation $\hat{\tau}$ with this property.

**A.2. Clifford–Lie superalgebras.** We now explain how selfadjoint representations of Clifford algebras relate to unitary representations of Clifford–Lie supergroups.

A finite dimensional (real) Lie superalgebra $\mathfrak{n} = n_0 \oplus n_1$ is called a Clifford–Lie superalgebra if $n_0$ is a subset of the center of $\mathfrak{n}$. Then $N := (n_0, +)$ is a Lie group with Lie algebra $n_0$ and $\mathcal{N} := (N, \mathfrak{n})$ is a Lie supergroup for which $N$ acts trivially on $\mathfrak{n}$; in [2] these supergroups are called super translation groups.

**Definition A.6.** We say that a unitary representation $(\pi, \chi, \mathcal{H})$ of $\mathcal{N}$ is $\lambda$-admissible for $\lambda \in n_0^*$ if

$$\pi(x) = e^{i\lambda(x)}1 \quad \text{for} \quad x \in n_0.$$

If a $\lambda$-admissible unitary representation exists, then Lemma 2.7(ii) implies that $\lambda \in \mathcal{C}(\mathfrak{n})^*$. By Schur’s Lemma, every irreducible unitary representation of $\mathcal{N}$ is $\lambda$-admissible for some $\lambda \in \mathcal{C}(\mathfrak{n})^*$.

Fix $\lambda \in \mathcal{C}(\mathfrak{n})^*$. Then

$$\mu_{\lambda} : n_1 \times n_1 \longrightarrow \mathbb{R}, \quad \mu_{\lambda}(x, y) := \frac{1}{2}\lambda([x, y]).$$
is a positive semidefinite symmetric bilinear form on $n_1$, hence defines on the quotient space

$$n_{1, \lambda} := n_1 / \{ x \in n_1 : \mu_\lambda(x, x) = 0 \}$$

a positive definite form $\overline{\mu}_\lambda$. We write

$$(A.5) \quad C_\lambda := C(n_{1, \lambda}, \overline{\mu}_\lambda)$$

for the corresponding Clifford algebra and $C_{\lambda, C}$ for its complexification. This is a $C^*$-algebra whose representations are precisely the complex linear extensions of selfadjoint representations of $C_\lambda$. The corresponding structure map lifts to a linear map

$$\iota : n_1 \to C_\lambda, \quad x \mapsto \overline{x}$$

satisfying

$$\iota(x)^2 = \mu_\lambda(x, x)1 = \frac{1}{2} \lambda([x, x]) \quad \text{for} \quad x \in n_1.$$ 

Therefore the map

$$(A.6) \quad \iota_\lambda : n \to C_{\lambda, C}, \quad x \mapsto \begin{cases} \iota\lambda(x)1 & \text{for} \quad x \in n_0 \\ \iota\lambda(x) & \text{for} \quad x \in n_1 \end{cases}$$

is a homomorphism of Lie superalgebras if $C_{\lambda, C}$ in endowed with the natural Lie superbracket defined on homogeneous elements by the super-commutator

$$[a, b] := ab - (-1)^{|a||b|} ba, \quad a, b \in C_{\lambda, C}.$$ 

If $\rho$ is a selfadjoint representation of $C_\lambda$, then $\rho_C \circ \iota_\lambda$ is a $\lambda$-admissible unitary representation of $n$, so that $\pi := e^{i\lambda}$ on $n_0$ leads to a unitary $\lambda$-admissible representation of the Lie supergroup $N$. If, conversely, $(\pi, \chi_\pi)$ is a $\lambda$-admissible representation of $N$, then the universal property of $C_\lambda$ implies the existence of a selfadjoint representation $\rho$ of $C_\lambda$ with $\rho_C \circ \iota_\lambda = \chi_\pi$. This establishes a one-to-one correspondence between $\lambda$-admissible unitary representations of $N$ and selfadjoint representations of $C_\lambda$, which are completely described in Proposition [A.2]. We thus obtain:

**Proposition A.7.** ($\lambda$-admissible representations) Suppose that $\lambda \in \mathcal{C}(n)^*$. Then there exists a finite dimensional irreducible unitary representation $(\pi_\lambda, \chi_\pi, \mathcal{H}_\lambda)$ of $N = (N, n)$ which is unique if $\dim n_{1, \lambda}$ is odd and unique up to parity reversal if $\dim n_{1, \lambda}$ is even. Any other $\lambda$-admissible unitary representation $(\pi, \mathcal{H})$ is of the form $\pi = 1 \otimes \pi_\lambda$, where $\mathcal{H} = \mathcal{M} \otimes \mathcal{N}$ is a tensor product of Hilbert superspaces. If $\dim n_{1, \lambda}$ is odd, then the multiplicity space $\mathcal{M}$ can be chosen purely even.

**Corollary A.8.** (Irreducible representations) Each irreducible unitary representation of $\mathcal{N} = (N, n)$ is finite dimensional and equivalent to some $(\pi_\lambda, \lambda)$ with $\lambda \in \mathcal{C}(n)^*$ or to $\pi_{\lambda, j}$, where $\Pi$ denotes parity reversal, i.e., $\pi_{\lambda, 0} = \pi_\lambda$ but $\mathcal{H}_{1, j} := \mathcal{H}_{1-j}$ for $j = 0, 1$. If $\dim n_{1, \lambda}$ is odd, then both are equivalent.

### A.3. Semidirect products

Now we assume that $K$ is a compact Lie group acting on $n$ by a group homomorphism $\rho : K \to \text{Aut}(n) \cong \text{Aut}(N)$. The action $\rho$ induces the action $\rho^*$ of $K$ on the dual cone

$$\mathcal{C}(n)^* = \{ \lambda \in n_0^* : (\forall x \in n_1) \lambda([x, x]) \geq 0 \} \quad \text{by} \quad \rho_k^*(\lambda) := \lambda \circ \rho_k^{-1},$$

for $k \in K$ and $\lambda \in \mathcal{C}(n)^*$. We write $\mathcal{O}_\lambda$ for the corresponding orbit of $\lambda$.

We now explain how the irreducible unitary representations of the Lie supergroup $G := \mathcal{N} \rtimes K = (N \rtimes K, n)$ can be classified. This classification has been derived in [2] (Theorems 4 and 5) by generalizing Mackey’s Imprimitivity to the super context.
and by using the corresponding technique of unitary induction (Definition 2.4). We now give a precise formulation of this result.

Fix \( \lambda \in \mathcal{C}(n)^{\ast} \) and consider its stabilizer group \( K_{\lambda} \subseteq K \). In the Mackey context, \( \mathcal{G}_{\lambda} := \mathcal{N} \rtimes K_{\lambda} \) plays the role of the little group from which we want to induce representations to \( \mathcal{G} \). Therefore we first have to classify the unitary representations of \( \mathcal{G}_{\lambda} \) which are \( \lambda \)-admissible in the sense that \( \pi(x) = e^{i\lambda(x)}1 \) for \( x \in n_{0} \). This can be done with the tools developed in the preceding two subsections.

As \( K_{\lambda} \) preserves \( \lambda \), its action on \( n_{1} \) factors through an orthogonal representation

\[
\rho_{\lambda} : K_{\lambda} \to O(n_{1,\lambda}).
\]

Note that \( K_{\lambda} \) need not be connected, so that the range of \( \rho_{\lambda} \) need not be contained in the identity component \( SO(n_{1,\lambda}) \). This causes some trouble in the constructions because it leads to graded unitary representations. Therefore a key point in the construction in [2] is to consider the following subgroup of \( K_{\lambda} \):

\[
(A.7) \quad K_{\lambda}^{\circ} := \begin{cases} \rho_{\lambda}^{-1}(SO(n_{1,\lambda})) & \text{if } \rho_{\lambda}(K_{\lambda}) \not\subseteq SO(n_{1,\lambda}) \text{ and } \dim(n_{1,\lambda}) \text{ even} \\ K_{\lambda} & \text{otherwise.} \end{cases}
\]

So either \( K_{\lambda}^{\circ} \) equals \( K_{\lambda} \) or it is a subgroup of index 2. From the central extension \( \hat{\mathcal{O}}(n_{1,\lambda}) \) of \( O(n_{1,\lambda}) \) by \( \{ \pm 1 \} \) (Proposition A.5), we obtain a central extension

\[
\hat{K}_{\lambda} = \rho_{\lambda}^{\ast}\hat{\mathcal{O}}(n_{1,\lambda}) = \{(k, g) \in K_{\lambda} \times \hat{\mathcal{O}}(n_{1,\lambda}) : \rho_{\lambda}(k) = \overline{g}\}.
\]

Recall the Clifford algebra \( C_{\lambda} := \mathcal{C}(n_{1,\lambda}, \pi_{\lambda}) \) from (A.5) and its irreducible representation \((\tau, \mathcal{S}_{\lambda})\). We then obtain with Proposition A.5 a graded unitary representation \( \kappa_{\lambda} \) of \( \hat{K}_{\lambda} \) on \( \mathcal{N} \) by

\[
\kappa_{\lambda} : \hat{K}_{\lambda} \to \text{Aut}(\mathcal{S}_{\lambda}), \quad \kappa_{\lambda}(k, g) := \kappa(g)
\]

satisfying

\[
\kappa_{\lambda}(k, g)\tau(\iota(x))\kappa(k, g)^{-1} = \tau(\iota(\rho_{\lambda}(k)x)) \quad \text{for} \quad (k, g) \in \hat{K}_{\lambda}, x \in n_{1}.
\]

For the corresponding unitary representation \( \chi_{\lambda} \) of \( n \) on \( \mathcal{S}_{\lambda} \), this leads to

\[
\kappa_{\lambda}(k, g)\chi_{\lambda}(x)\kappa_{\lambda}(k, g)^{-1} = \chi_{\lambda}(\rho_{\lambda}(k)x) \quad \text{for} \quad (k, g) \in \hat{K}_{\lambda}, x \in n,
\]

so that it combines with \( \kappa_{\lambda} \) to a unitary representation \((\hat{\pi}_{\lambda}, \hat{\chi}_{\lambda})\) of \( \hat{\mathcal{G}}_{\lambda} = \mathcal{N} \rtimes \hat{K}_{\lambda} \) defined by

\[
\hat{\pi}_{\lambda}(x, k) = e^{i\lambda(x)}\kappa(k), \quad \hat{\chi}_{\lambda}(x, y) = \chi_{\lambda}(x) + d\kappa_{\lambda}(y), \quad x \in n, y \in t_{\lambda}.
\]

We call a unitary representation \((\pi, \mathcal{H})\) of \( \hat{K}_{\lambda}^{\circ} \) odd if \( \pi(-1) = -1 \). Note that \( \pi(-1) \) is always a unitary involution and that \( \pi(-1) \in \{ \pm 1 \} \) holds if \( \pi \) is irreducible by Schur’s Lemma.

Theorem 4 of [2] now asserts the existence of a functor \( r \mapsto \theta_{r}^{\lambda} \) which assigns to an odd unitary representation \((r, \mathcal{H})\) of \( \hat{K}_{\lambda}^{\circ} \) a unitary representation of the little supergroup \( \mathcal{N} \rtimes K_{\lambda} \) which is \( \lambda \)-admissible. This establishes in particular a bijection of the equivalence classes of irreducible odd unitary representation of \( \hat{K}_{\lambda}^{\circ} \) on (purely even) Hilbert spaces with the irreducible unitary representation of the Lie supergroup \( \mathcal{N} \rtimes K_{\lambda} \).
More concretely, starting from an odd unitary representation \( r \) of \( \hat{K}_\lambda \), we first
construct the induced representation \((\hat{r}, \hat{H})\) (Remark A.4) which is a graded representation of the full group \( \hat{K}_\lambda \) on a Hilbert superspace. Now
\[
\theta_\lambda^r(x, k) := e^{i\lambda(x)\hat{r}(k)} \otimes \kappa_\lambda(k)
\]
defines the corresponding unitary representation of \( N \rtimes K_\lambda \) (here we use that \( r \) and \( \kappa_\lambda \) are both odd and graded with respect to the subgroup \( \hat{K}_\lambda \)) and the associated representation of the Lie superalgebra is determined by
\[
\chi(x) = \tau_\lambda(\iota_\lambda(x)) \quad \text{for} \quad x \in n_1.
\]
As \( \theta_\lambda^r \) is a unitary representation of the little supergroup \( N \rtimes K_\lambda \), unitary induction now leads to a unitary representation
\[
\Theta_\lambda^r := \text{Ind}_{N \rtimes K_\lambda}(\theta_\lambda^r)
\]
of the Lie super \( N \rtimes K \).

**Theorem A.9.** Fix \( \lambda \in \mathcal{C}(n)^\ast \). If \((r, \mathcal{H})\) is an odd unitary representation of \( \hat{K}_\lambda \), then \( \theta_\lambda^r \) is a unitary representation of the Lie supergroup \( N \rtimes K \). If \( r \) is irreducible, then so is \( \Theta_\lambda^r \) and all irreducible unitary representations of \( N \rtimes K \) are obtained in this way. If \( \Theta_\lambda^r \) and \( \Theta_\lambda^{r'} \) are equivalent, then \( \lambda \) and \( \lambda' \) lie in the same \( K \)-orbit in \( \mathcal{C}(n)^\ast \) and \( \Theta_\lambda^r \cong \Theta_\lambda^{r'} \) if and only if \( r \cong r' \). Hence the equivalence classes of irreducible unitary representations of \( N \rtimes K \) are determined by the \( K \)-orbits \( O_\lambda \) in \( \mathcal{C}(n)^\ast \) and, for any fixed \( \lambda \), they are in one-to-one correspondence with odd irreducible representations of \( \hat{K}_\lambda \).

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