A Sobolev Space Inroad to Riemann Integrability

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Abstract  A conditioned equivalence is proved for a certain weighted Sobolev space to the space of Riemann integrable functions. An equivalence representing a new result that not only asserts the sufficiency (but non-necessity) nature of bounded variation of functions for their Riemann integrability, but also reveals a potential for some novel computational findings.

Keywords: bounded variation, Sobolev space, Stieltjes integrals, continuity of functions, Riemann integrability

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1. Introduction

The transparent classical definition of the Riemann sum type of integral, see e.g. [1], was shown by Graves in [2], to admit an easy extension to a following integration of Banach space-valued functions, [3]. Indeed, let \( f: I \to X \) be given, where \( X \) is a Banach space with the norm \( \| \cdot \|_X \) and the compact interval \( I = [a, b] \subset \mathbb{R} = (-\infty, \infty) \) is endowed with a length measure \( \lambda \). \( f(x) \) is said to be Riemann integrable and \( \gamma \in X \) its Riemann integral if for every \( \varepsilon > 0 \), \( \exists \) a constant gauge \( \delta \in \mathbb{R}^+ = [0, \infty) \), such that for every \( \delta \)-fine \( K \)-partition \( \{x_i, I_i \in I \} \), see e.g. [3], the inequality

\[
\sum_{i=1}^{n} f(x_i) \lambda(I_i) - \gamma < \varepsilon \quad \text{for all} \quad \bigcap_{i=1}^{n} I_i = I,
\]

holds.

Moreover, if in the previous statement \( \delta \in \mathbb{R}^+ \) is replaced by \( \delta: I \to \mathbb{R}^+ \), \( f \) would be Henstock-Kurzweil, [4], integrable, whereas if both \( \delta \in \mathbb{R}^+ \) and \( K \)- partition, [3], \( f \) becomes MacShane, [5], integrable. The same approach was followed by Bochner, in [6], for Lebesgue integrability in Banach spaces, i.e. when the above \( \lambda \) is a Lebesgue measure, to introduce the Bochner integral.

Lebesgue integrability happens also, however, to be a defining operation for all Sobolev spaces, see e.g. [7], and Riemann integrability in Sobolev spaces may sound rather unusual. The goal of this note, is to demonstrate that a linearly weighted Sobolev space can serve, nonetheless, as a natural space for Riemann integrable functions. The obtained result not only constitutes an addition to the existing literature, e.g. [3], on vector-valued integration, but also reveals a potential for some novel computational findings.

2. Main Result

The space of \( \rho \)-weighted Riemann integrable functions over a closed interval \( I \),

\[
\mathcal{R}(I; \rho) = \{ f : \int_I |f| \rho \, dx < \infty \} \subset \{ f : \int_I |f| \rho \, dx < \infty \}, \tag{1}
\]

where both integrals are conceived in the Riemann sense, is well known to be semi-normed, see e.g. [1,7]. A semi-norm on a functional vector space \( X \), with \( f, g \in X \) and \( c \in \mathbb{C} \), is a function \( \zeta : X \to \mathbb{R} \) with the properties:

i) \( \zeta(cf) = |c| \zeta(f) \), absolute homogeneity,

ii) \( \zeta(f+g) \leq \zeta(f) + \zeta(g) \), subadditivity.

It is clear that \( \text{Ker}(\zeta) = \{ f \in X : \zeta(f) = 0 \} \) is a subspace of \( X \). Moreover, a semi-norm \( \zeta \) becomes a norm \( \| \cdot \| \) if \( \text{Ker}(\zeta) = \{ 0 \} \). Hence a norm \( \| \cdot \| : X \to \mathbb{R}^+ \) is a semi-norm with the additional property:

iii) \( \|f\| \geq 0 \), \( \forall f \in X \), with the equality if \( f = 0 \), point separability.

Both integrals in (1), namely

\[
\zeta_{\mathcal{R};\rho}(f) = \|f\|_{\mathcal{R};\rho} = \int_I |f| \rho \, dx,
\]

or

\[
\zeta_{\mathcal{R};\rho}(f) = \left| \int_I f \rho \, dx \right|,
\]

can be semi-norms for \( \mathcal{R}(I; \rho) \), because they happen to satisfy the previous properties (i) and (ii) but may violate (iii). Indeed \( \zeta_{\mathcal{R};\rho}(f) = 0 \) if \( f = 0 \) only a.e., that is when \( f \approx 0 \), while \( \zeta_{\mathcal{R};\rho}(f) = 0 \), when e.g.

\[
f(x) = \begin{cases} 
0 \leq x < \frac{1}{2} & \rho(x) = 1, I = [0,1]. \\
\frac{1}{4} \leq x \leq 1 & 0 < \frac{1}{2}
\end{cases}
\]
This makes \( \mathcal{R}(I;\rho) \) a vector space of equivalence classes of \( \rho \)-weighted Riemann integrable functions on \( I \), where \( f \) can be equivalent to \( g \) iff \( f = g \) a.e. Nonetheless, by application of the Riesz-Fischer theorem, \([1,7]\), it can readily be proved that \( \mathcal{R}(I;\rho) \) is complete under the seminorm \( \| f \|_{\mathcal{R},\rho} \), i.e. \( \mathcal{R}(I;\rho) \) is a Banach space.

Since \( \mathcal{R}(I;\rho) \subset L^1(I;\rho) \), one should not confuse \( \mathcal{R}(I;\rho) \) with the \( L^1 \) seminorm
\[
\| f \|_{\mathcal{R},\rho} = \int \| f \|_\rho \, dx ,
\]
of the \( p = 1 \) case of \( L^p \) spaces, in which the integral is conceived in the Lebesgue sense. Note moreover that \( L^1 \) happens to be unusual among the \( L^p \) spaces in that it is not reflexive. Incidentally, conventional \( \rho \)-weighted Sobolev spaces \( W^{p,k}(I;\rho) \) are related to these \( L^p \) spaces and have additional structure viz.
\[
W^{p,k}(I;\rho) = \{ f : D^k f \in L^p(I;\rho), \forall |\alpha| = 0,1,2,3, \ldots, k \} ,
\]
as they are endowed with the norm
\[
\| f \|_{W^{p,k},\rho} = \left\{ \sum_{0 \leq |\alpha| \leq k} \left( \int \| D^\alpha f \|_\rho \right)^p \right\}^{1/p} .
\]

Here, the \( D^\alpha f \) derivatives, up to a given \( k \)-order, are understood in a weak sense, to make the \( W^{p,k}(I;\rho) \) space complete, thus a Banach space.

**Definition 1.** (Functions of bounded variation, \([8,9]\)). Let us subdivide the \( I=[a,b] \) interval arbitrarily into \( n \) subintervals to form a finite partition
\[
P = \left\{ a = x_0 < x_1 < x_2 < \ldots < x_{k-1} < \ldots < x_k < \ldots < x_n = b \right\} ,
\]
with the sum
\[
\nu = \sum_{k=1}^{n+1} \left| f(x_k) - f(x_{k-1}) \right| .
\]

A function \( f : I \to \mathbb{R} \) is of bounded variation (BV), or \( f \in BV(I) \), is characterized by the norm
\[
\| f \|_{BV} = \sup_n \nu ,
\]
and is defined when
\[
\frac{b}{a} f(x) < \infty .
\]

**Definition 2.** (Hölder continuous functions), \( f : I \to \mathbb{R} \) is said to be Hölder continuous (HC) if \( \exists \alpha > 0 \) such that
\[
| f(\bar{x}) - f(\hat{x}) | \leq L | \bar{x} - \hat{x} | ^\alpha , \ a \leq \bar{x} < \hat{x} \leq b ,
\]
\( \forall \bar{x}, \hat{x} \in I , \ 0 \leq \alpha \leq 1 , \) and \( L \) is some \( > 0 \) constant.

The special case of \( \alpha = 1 \) corresponds to the case of Lipschitz continuous (LC) functions; and it is well known that Lipschitzianity is weaker than continuous differentiability (CD), and stronger than uniform continuity (UC) or continuity (C).

As for the \( \mathcal{R}(I;\rho) \) space of interest to this work, its related \( L^1(I;\rho) \) space naturally invokes \( W^{1,1}(I;\rho) \), which is known, \([5,7]\), to be isomorphic to the space of functions of BV, endowed with the norm \( \| f \|_{BV} \), defined by (9-10).

Hence \( W^{1,1}(I;\rho) \) is a space of absolutely continuous functions on \( I \) or equivalence classes of \( \rho \)-weighted Lebesgue integrable functions that are a.e. absolutely continuous. It is a Banach space w.r.t. the norm
\[
\| f \|_{W^{1,1},\rho} = \sum_{0 \leq |\alpha| \leq 1} \int \| D^\alpha f \|_\rho \, dx
\]
and
\[
\| f \|_{W^{1,1},\rho} = \| f \|_{BV} + \| D f \|_{BV} ,
\]
where the \( \mathcal{R} \) index refers to integration in the Riemann sense of functions essentially in BV(\( I \)), isomorphic with \( W^{1,1}(I;\rho) \).

To demonstrate then that \( \| f \|_{W^{1,1},\rho} \) is a norm, we examine it against the previous three related properties. Although the second term \( \| D f \|_{BV} \) in (13) is always a seminorm since it annihilates constants, i.e. violates the norm property (iii), the first term \( \| f \|_{BV} \) does not. This guarantees, incidentally, satisfaction of (iii) by the entire \( \| f \|_{W^{1,1},\rho} \). Property (i) is clearly satisfied by both components of \( \| f \|_{W^{1,1},\rho} \), then by \( \| f \|_{W^{1,1},\rho} \) itself. It is also straightforward to prove that both \( \| D f \|_{BV} \) and \( \| f \|_{BV} \) satisfy property (ii). To illustrate this, let \( I=[0,1] \), \( \rho(x)=1 \) on \( I \) and let \( f, g \in W^{1,1}(I;\rho) \) be \( f(x) = x \) and \( g(x) = -x. \)

Clearly, \( \| D f \|_{\mathcal{R}} , \| D g \|_{\mathcal{R}} , \| f \|_{\mathcal{R}} , \| g \|_{\mathcal{R}} = 1 \), while \( |D(f+g)|_{\mathcal{R}} = 0 \), whence \( \| D f \|_{\mathcal{R}} , \| D g \|_{\mathcal{R}} , \| f \|_{\mathcal{R}} , \| g \|_{\mathcal{R}} \) is a satisfaction of the triangle inequality by \( \| f \|_{\mathcal{R}} \). The inequality \( \| f \|_{\mathcal{R}} + \| g \|_{\mathcal{R}} \leq \| f + g \|_{\mathcal{R}} \) can similarly be established to confirm satisfaction of property (ii) by the sum, \( \| f \|_{\mathcal{R}} \), of these two terms.

**Theorem 1.** Let \( I = [a,b] \subset \mathbb{R} \), then
\[
W^{1,1}(I;\rho) \cap \{ LC, HC, UC, BV, C \} (I) = \mathcal{R}(I) ,
\]
i.e. \( \forall f \in W^{1,1}(I;\rho) \cap \{ LC, HC, UC, BV, C \} (I) \).

Proof. Consider integration by parts of the Stieltjes integral \( \int_a^b f(x) \, d\rho(x) \), see e.g. \([8,10]\), via
\[
\int_a^b f(x) \, d\rho(x) = \rho(x) f(x) \bigg|_a^b - \int_a^b \rho(x) \, df(x) .
\]
On one hand, existence of either one of the Stieltjes integrals implies existence of the other, [11], with a satisfaction of
\[ \rho(x) f(x) \bigg|_{a}^{b} < \infty, \quad (16) \]
only with both (15) and (16) valid, if
i) \( f(x) \in C(I) \) and \( \rho(x) \in BV(I) \),
ii) \( f(x) \in R(I) \) and \( \rho(x) \in LC(I) \),
iii) \( f(x) \in _{x} \alpha - HC(I) \) and \( \rho(x) \in _{x} \beta - HC(I) \); \( \alpha + \beta > 1 \), and
the three statements are reflexive by replacement of \( f \) with \( \rho \). Results that cohere with the following classical inclusion over \( I \), [13],
\[ CD \subseteq LC \subseteq HC \subseteq UC \subseteq BV \subseteq C \subseteq R. \quad (17) \]
Next assume in (15) that \( \rho(x) = x \) to reduce it to
\[ \int_{a}^{b} f(x) dx = xf(x) \bigg|_{a}^{b} - \int_{a}^{b} x(Df) dx. \quad (18) \]
Relation (18) happens remarkably to operate as a "functional quadruple" with two inputs, \( \rho \) and \( f \), and two outputs, \( R(I) \) and \( W^{R}(I;x) \). In this quadruple, the possible \( CD, LC, HC, UC, BV, C, R(I) \) spaces for \( \rho \) and \( f \) combine, according to the results (i)-(iii) and their reflections, in order to maintain the outputs fixed. In particular, (18) guarantees existence of \( f \in W^{R}(I;x) \) simultaneously in any space of the set \( LC, HC, UC, BV, C \) and not only in the isomorphic BV(I), when \( \rho(x) = x \in CD(I) = \{ CD \cap LC \cap HC \cap UC \cap BV \cap C \} R(I) \).

Clearly therefore if \( f \in W^{R}(I;x) \), then \( f \in R(I) \), which is the required result.

We should underline here that BV of \( f(x) \) over \( I \) is not necessary, though sufficient for its Riemann integrability. The same is true if the BV condition is replaced by the stronger LC condition, [3] or [9]. Moreover, LC(I) implies CD(I) a.e., and many functions can \( \in LC(I) \) while \( \not \in CD(I) \). So LC is weaker than CD, while being stronger than BV or C. Now it is time to state the important definition that follows, on Riemann integrability in a linearly-weighted Riemann-Sobolev space.

**Definition 3.** \( f : I \rightarrow W^{R}(I;x) \) is Riemann integrable in \( W^{R}(I;x) \), whenever
\[ \int_{a}^{b} x(Df) dx \quad \text{exists}. \]

**Remark 1.** It should be noted that satisfaction of \( x f(x) \bigg|_{a}^{b} \) alone in (18) does not necessarily mean existence of the \( V f(x) \), i.e. that \( f(x) \) is of BV. But with added integrability of \( Df(x) \) over \( I \), w.r.t. the \( x \) weight, \( f(x) \) becomes, by Theorem 1, Riemann integrable over \( I \), even if it is not of BV.

**Remark 2.** Lipschitzianity happens to guarantee that \( f(x) \) is of BV, or even is absolutely continuous over \( I \).

### 3. Demonstrative Examples

The first example is meant to illustrate the sufficiency nature of BV of \( f(x) \) over \( I \) for its Riemann integrability over the same \( I \).

**Example 1.** Consider the unique infinitely discontinuous and wildly oscillatory function :
\[ f(x) = \sum_{k=1}^{\infty} (-1)^{k-1} \left[ u \left(x - \frac{1}{k+1}\right) - u \left(x - \frac{1}{k}\right) \right] x, \]
which is neither differentiable nor Lipschitzian over \( I = [0,1] \). Indeed, this function, being continuous up to a discrete set of points, i.e. is continuous a.e., should obviously be Riemann integrable. Our purpose here, however, is to arrive at the same conclusion via theorem 1. Hence it can be graphically demonstrated that
\[ \frac{b}{a} \int_{a}^{b} f(x) = \sup_{k=1}^{\infty} \sum_{k=1}^{\infty} \left[f(x_{k}) - f(x_{k-1}) \right] = \sup_{k=1}^{\infty} \sum_{k=1}^{\infty} \left[ \frac{1}{k+1} - \frac{1}{k} \right] = 1, \]
i.e. it is of BV over \([0,1]\). Obviously,
\[ Df(x) = \sum_{k=1}^{\infty} (-1)^{k-1} \left[ \delta \left(x - \frac{1}{k+1}\right) - \delta \left(x - \frac{1}{k}\right) \right] \]
and
\[ xDf(x) = f(x) + \sum_{k=1}^{\infty} (-1)^{k-1} \left[ \delta \left(x - \frac{1}{k+1}\right) - \delta \left(x - \frac{1}{k}\right) \right] x^{2}. \]

Then by (18) we may write
\[ \int_{0}^{1} f(x) dx = xf(x) \bigg|_{0}^{1} = \int_{0}^{1} f(x) + \sum_{k=1}^{\infty} (-1)^{k-1} \left[ \delta \left(x - \frac{1}{k+1}\right) - \delta \left(x - \frac{1}{k}\right) \right] x^{2} dx \]
Now since \( x f(x) \bigg|_{0}^{1} = 0 \), the previous relation becomes
\[ \int_{0}^{1} f(x) dx = -\frac{1}{2} \sum_{k=1}^{\infty} (-1)^{k-1} \left[ \frac{\delta \left(x - \frac{1}{k+1}\right)}{k+1} - \frac{\delta \left(x - \frac{1}{k}\right)}{k} \right] x^{2} dx \]
\[ = -\frac{1}{2} \sum_{k=1}^{\infty} (-1)^{k-1} \left[ \frac{1}{k+1} - \frac{1}{k} \right] x^{2}. \]
Substitution of the identity
\[ \sum_{k=1}^{\infty} \left(\frac{-1}{k+1}\right)^{k-1} x^{k} = 1 - \sum_{k=1}^{\infty} \left(\frac{-1}{k+1}\right)^{k-1} x^{k}, \]
in the previous relation reduces it to
\[ \int_{0}^{1} f(x) dx = \sum_{k=1}^{\infty} \left(\frac{-1}{k+1}\right)^{k-1} x^{k} - \frac{1}{2} = \mu(2) - \frac{1}{2}. \]
The above Dirichlet’s Eta function \( \eta(s) \), of argument 2, is related to the associated Riemann Zeta function \( \zeta(2) \), via 
\[ \eta(2) = (1 - \frac{1}{2^2}) \zeta(2) . \]
So as \( \zeta(2) = \pi^2/6 \), we may rewrite the last relation as 
\[ \int_0^1 f(x) \, dx = (\pi^2/12) - (1/2), \]
which is Riemann integrable.

### 3.1. Novel Rational Expansion

One of the rather unexpected applications of theorem 1 is the possibility of generating expansions of some fractions in certain unusual partial fractions. Such a possibility is highlighted in the example that follows.

**Example 2.** According to Theorem 1, localization of nondifferentiability is not a required restriction. For an illustration, consider a function defined by 
\[ f(x) = \sum_{k=1}^{\infty} \frac{1}{k^2} \]
with linear chords, of distinct slopes, between consecutive vertices. This \( f(x) \) is not of BV on \( I = [0,1] \). It can be graphically verified that 
\[ \int_0^1 f(x) \, dx = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \]
and 
\[ \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \]
Furthermore, the behavior of its chord’s slopes as \( k \to \infty \) is not of BV on \( I \). Moreover, the behavior of its chord’s slopes as \( k \to \infty \) is not of BV on \( I \). Nevertheless, it is not difficult to verify the Riemann integrability of this function over \( I \), either directly graphically, or via the application of Theorem 1.

**Graphical integration.** When integrating the posing saw-like oscillation \( f(x) \) on \( [0,1] \), we encounter different directions of flow in the oscillation numbers, \( k \), and in values of the \( x \)-axis. It turns out to be more convenient to use the map \( z = 1/x \) and integrate \( f(z) \) over \( z \in [0,1] \). In fact \( x = 1/k \) becomes \( z = k-1 \). Then for odd \( k \) only, i.e.

\[ \int_0^1 f(x) \, dx = \sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{\pi^2}{6} \]

and \( y_m^+ = y_m^- \) are straight lines with opposite slopes. Direct evaluation of the sum, \( A_m \), of the areas of the hanging on the \( z \)-axis three triangles, which is the integral of the \( m \)-th oscillation, can, by some lengthy calculations, be shown to be

\[ A_m = \frac{1}{m(2m-1)^2} \]

Hence

\[ \int_0^1 f(x) \, dx = \int_0^1 f(z) \, dz = \sum_{m=1}^{\infty} A_m. \]

Integration via Theorem 1. Here \( y_m^- \) and \( y_m^+ \) need to be represented analytically as

\[ y_m^-(z) = -(4m-1)z + (4m-3), \]
\[ y_m^+(z) = (4m+1)z - (4m-1), \]

to allow for expressing the associated function \( f(z) \) as

\[ f(z) = \sum_{m=1}^{\infty} y_m^-(z) \left[ u \left( z - \frac{2m-2}{2m-1} \right) - u \left( z - \frac{2m-1}{2m} \right) \right] \]
\[ + y_m^+(z) \left[ u \left( z - \frac{2m-1}{2m} \right) - u \left( z - \frac{2m-1}{2m+1} \right) \right]. \]

Since \( f(z) \big|_0^1 = 0 \), then application of theorem 1 results with

\[ \int_0^1 f(z) \, dz = -\frac{1}{2} \sum_{m=1}^{\infty} \int_0^1 \left[ \begin{array}{c}
\frac{-(4m-3)}{2m} \\
\frac{4m-1}{2m+1}
\end{array} \right] \, dz 
\]

which can be shown, after some lengthy details, to lead to

\[
\int_0^1 f(z) \, dz = \int_0^1 f(x) \, dx = \sum_{m=1}^\infty B_m, \quad (25)
\]

with

\[
B_m = \frac{(m-1)}{m(2m-1)(2m+1)} + \frac{m}{(2m+1)^2} - \frac{(m-1)}{(2m-1)^2}. \quad (26)
\]

It is straightforward to verify that \(B_m \equiv A_m\) and that (25) and (27) are identical. Interestingly, however, \(B_m\) is not a rational expansion of the rational \(A_m\) in simple partial fractions viz

\[
A_m = \frac{1}{m(2m-1)^2 (2m+1)^2}
\]

\[
= \frac{a}{m} + \frac{b m+c}{(2m+1)^2} + \frac{d m+e}{(2m-1)^2} = B_m, \quad (27)
\]
in which \(a = 1, b = -2, c = -3/2, d = -2,\) and \(e = 3/2,\) as a must.

Clearly the first term in (26) is not a simple partial fraction for our \(A_m\). Hence comparison of (26) and (27) illustrates that \(B_m\), in (26), is remarkably a novel rational expansion of \(A_m\), of (23), in unusual partial fractions. This fact can perhaps motivate other "recreational" or unforeseen applications for Theorem 1, along the arguments used in [15], for example.

4. Conclusion

The result of this note is meant to complement (and not to replace) the theory of Riemann integration of a.e. continuous functions. It implements for the first time, however, Sobolev space arguments to assert the well known result that BV is not necessary (though sufficient) for Riemann integrability. The distinguishing feature of the function of example 2, from that of example 1, is its non BV on the respective \(I). \) It is demonstrated that BV is not necessary for the Riemann integrability of example 2 over \([0.1],\) indicating the surplus in its sufficiency (non-necessity) implied by Theorem 1. The computational potential of Theorem 1, as a technique of integration, raises, according to example 2, an issue that calls for further exploration.

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