JACOBI IDENTITY FOR VERTEX ALGEBRAS IN HIGHER DIMENSIONS

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Abstract. Vertex algebras in higher dimensions provide an algebraic framework for investigating axiomatic quantum field theory with global conformal invariance. We develop further the theory of such vertex algebras by introducing formal calculus techniques and investigating the notion of polylocal fields. We derive a Jacobi identity which together with the vacuum axiom can be taken as an equivalent definition of vertex algebra.

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1. Introduction

Two–dimensional conformal field theory is important in physics as providing models of quantum field theory (QFT). It also plays a role in other areas of mathematical physics as well as in statistical physics and condensed matter physics. A vertex algebra is essentially the same as a chiral algebra in conformal field theory (see [7, 20] and the book [12]). In more details, the field content in two–dimensional conformal field theories splits into two chiral parts consisting of fields that depend separately on one of the two light–cone variables. Observable chiral fields have commutators supported on the diagonal, i.e., vanishing for non-coinciding arguments. It turns out that chiral fields form a purely algebraic structure under their operator product expansion, which is called a vertex algebra. Axiomatistically, the notion of vertex algebra was first introduced by R. E. Borcherds [8]. Vertex algebras arose naturally in the representation theory of infinite–dimensional Lie algebras and in the construction of the “moonshine module” for the Monster finite simple group [8, 19]. Now the theory of vertex algebras is a rapidly developing area of mathematics (see the books [19, 21, 17, 24]). “Multi–dimensional” generalizations of vertex algebras were considered in [9, 23, 26, 28].

The vertex algebras introduced in [28] for higher space–time dimension arose naturally within a one–to–one correspondence with axiomatic QFT models satisfying the additional symmetry condition of global conformal invariance (GCI). The incorporation of GCI within the framework of axiomatic QFT, together with the problem of finding (nonperturbatively) such models in higher dimensions, has been studied previously in e.g. [31, 30, 32, 29] (see also the groundbreaking early work [34]). In this way constructing models of higher–dimensional QFT with GCI becomes a purely algebraic problem. Let us point out that even for general QFT (without GCI) there are not any known models that satisfy the Wightman axioms in space–time dimension greater or equal to four, which cannot be realized by free or generalized free (Heisenberg) fields. In fact, this problem has remained open for more than fifty years.

On the other hand, even in dimension one (i.e., in chiral conformal field theory), vertex algebras are far from full classification and are quite intricate in general. A different algebraic structure known as a vertex Lie algebra has been introduced by V. G. Kac [21] (see also [33, 17, 14]) (it is also called a “conformal algebra” but we will not use this terminology here since it can be confused with the usual conformal Lie algebra in higher dimensions). This is the structure formed by the commutators of fields, i.e., by the singular part of their operator product expansion. Thus the relationship between vertex Lie algebras and vertex algebras is somewhat similar to the relationship between Lie algebras and associative algebras. It turns out that this new algebraic structure is more tractable and, in particular, classification results for vertex Lie (super)algebras can be obtained [11, 15] (see also [21, 22]). The theory of vertex Lie algebras was further developed in e.g. [10, 4], and a “multi–dimensional” generalization was considered in [1].

In the present paper we initiate an investigation of the notion of vertex Lie algebra for the vertex algebras in higher dimensions of [28]. In dimension one, the main axiom for vertex Lie algebras is the so-called Jacobi identity, which is related to the Jacobi identity of [19] for vertex algebras (and the Borcherds identity of [21]). Recall that in dimension one vertex algebras can be defined in terms of the Jacobi identity (see [19, 18, 21, 24]). The main result of the present paper is a
generalization of this Jacobi identity to higher dimensions. In addition, we show that together with the vacuum axiom this identity can be taken as a definition of vertex algebra equivalent to the definition of [28].

As in dimension one, we derive our Jacobi identity from certain “commutativity” and “associativity” identities (cf. [18, 25, 2, 24]). However, in the one–dimensional case the Jacobi identity can be simplified so that it does not involve external sufficiently large parameters. This is no longer the case in higher dimensions, and our Jacobi identity entails the same degree of difficulty as the “associativity” identity. The main difference with the one–dimensional case is that now the singular part of the operator product expansion contains infinitely many terms. Nevertheless, it follows from our Jacobi identity that the singular modes close an algebraic structure under the commutator, which would be the higher–dimensional analog of vertex Lie algebra.

The paper is organized as follows. The next two sections are devoted to an important technical preparation, which can be useful not only for this work but also for future investigations of vertex algebras in higher dimensions. This includes an introduction of several spaces of formal series in Sect. 2 and a higher–dimensional residue functional in Sect. 3 (additional material is contained in Appendix A). In Sect. 4 we recall the notions of fields, locality and operator product expansion in higher dimensions, mainly following [28], but extending our considerations also to polylocal fields. Our main result, the Jacobi identity, is contained in Sect. 5, together with several integral versions and a commutator formula. Concluding remarks are presented in Sect. 6.

2. Spaces of Formal Series

In this section we introduce various spaces of formal series, which will be used throughout the paper. In particular, we define the notion of a formal distribution, and we discuss formal series expansions.

2.1. Notation. In this subsection we fix some notation to be used throughout the paper, mostly following the notation of [28]. We fix a positive integer $D$, which will play the role of space–time dimension, and we denote by $z, z_i, w, \text{etc.}, D$-component variables:

$$z = (z^1, \ldots, z^D), \quad z_i = (z_i^1, \ldots, z_i^D), \quad w = (w^1, \ldots, w^D).$$

We will denote by $z_{ij}$ the difference

$$z_{ij} := z_i - z_j = (z_i^1 - z_j^1, \ldots, z_i^D - z_j^D),$$

and not a new variable. We introduce the standard scalar product:

$$z_1 \cdot z_2 := \sum_{\alpha=1}^D z_1^\alpha z_2^\alpha, \quad z^2 := z \cdot z.$$  

Note that $z^2$ stands for the Euclidean square of the vector $z$, while $z^2$ is its second component.

All vector spaces considered will be over the field $\mathbb{C}$ of complex numbers. For a vector space $V$, let $V[z]$ (respectively, $V[z]$) be the space of polynomials (respectively, formal power series) in $z$ with coefficients in $V$. For a 1-component variable $\varrho$, we denote by $V[\varrho, \varrho^{-1}]$ the space of formal power series in $\varrho$ and $\varrho^{-1}$ with coefficients in $V$, and by $V[\varrho]_{\varrho} := V[\varrho][\varrho^{-1}]$ the space of Laurent series.
Note that $V[z]$ is a $\mathbb{C}[z]$-module and $V[z]$ is a $\mathbb{C}[z]$-module. We denote by $\mathbb{C}[z]_x$ (respectively, $\mathbb{C}[z]_x$) the localization of $\mathbb{C}[z]$ (respectively, $\mathbb{C}[z]$) with respect to the multiplicative system $\{z^k\}_{k=1,2,...}$. Let $V[z]_x$ and $V[z]_x$ be the localizations of the corresponding modules. Then $V[z]_x$ (respectively, $V[z]_x$) is a module over $\mathbb{C}[z]_x$ (respectively, $\mathbb{C}[z]_x$).

We introduce the formal derivatives on $V[z]$ and $V[z]$:

$$
\partial_z := (\partial_{z_1}, \ldots, \partial_{z_D}), \quad \partial_{z^\alpha} := \frac{\partial}{\partial z^\alpha},
$$

and the Euler and Laplace operators:

$$
z \cdot \partial_z = \sum_{\alpha=1}^D z^\alpha \partial_{z^\alpha}, \quad \partial_z^2 = \sum_{\alpha=1}^D (\partial_{z^\alpha})^2.
$$

Then a polynomial $f(z) \in V[z]$ is **homogeneous** of degree $m$ iff $(z \cdot \partial_z - m)f(z) = 0$; it is **harmonic** iff $\partial_z^2 f(z) = 0$. We denote by $V[z]_{\text{har}}$ (respectively, $V[z]_{\text{har}}$) the spaces of harmonic polynomials (respectively, formal power series). Note that a formal power series is harmonic if and only if each of its homogeneous components is a harmonic polynomial.

Finally, we denote by $\mathbb{Z}_+$ the set of non-negative integers, and by $\mathbb{N}$ the set of positive integers. The notation $N \gg 0$ means that $N > 0$ is sufficiently large.

### 2.2. Harmonic Decomposition

The classical harmonic decomposition is the fact that every polynomial of $z \in \mathbb{C}^D$ can be divided by $z^2$ with a unique harmonic remainder. One can view this in a more abstract way as follows. Let $V$ be an arbitrary vector space. It is easy to see that the linear operators $\partial_z^2$, $z^2$ and $z \cdot \partial_z + D/2$ generate a representation of $\mathfrak{sl}_2$ on $V[z]$, namely

$$
[\partial_z^2, z^2] = 2 z \cdot \partial_z + D, \quad [z \cdot \partial_z, \partial_z^2] = -2 \partial_z^2, \quad [z \cdot \partial_z, z^2] = 2 z^2.
$$

In particular, we have the following useful formula

$$
[\partial_z^2, (z^2)^n] = 4n(z^2)^{n-1}(n-1 + z \cdot \partial_z + D/2).
$$

It follows from the representation theory of $\mathfrak{sl}_2$ that every homogeneous polynomial $\phi(z) \in V[z]$ of degree $k$ can be written uniquely in the form

$$
\phi(z) = \sum_{2n+m=k} (z^2)^n h_m(z),
$$

where $h_m(z)$ are harmonic homogeneous polynomials of degree $m$ (see e.g. [28, Lemma 1.1] for a direct proof). There is a similar harmonic decomposition for elements $\phi(z)$ of the localized space $V[z]_x$: the only difference is that we allow $k$ and $n$ to be negative (and the sum is still finite).

If we allow infinite sums, the largest space that we get is the space $V[[z, 1/z^2]]$ of formal series (see [28])

$$
\phi(z) = \sum_{n \in \mathbb{Z}} \sum_{m=0}^\infty \sum_{\sigma=1} h_m \phi_{l(n,m,\sigma)} (z^2)^n h_{m,\sigma}(z), \quad \phi_{l(n,m,\sigma)} \in V.
$$

Here $\{h_{m,\sigma}(z)\}_{\sigma=1,\ldots,n}$ is a basis of the space of harmonic homogeneous polynomials of degree $m$ and

$$
h_m = \binom{m + D - 1}{D - 1} - \binom{m + D - 3}{D - 1}.
$$
Note that the localized space $V[z]_{z^2}$ introduced in Sect. 2.1 can be naturally embedded in $V[z, 1/z^2]$ as the set of elements (2.9) for which the sum over $n$ is bounded from below.

Elements (2.9) are called $V$-valued formal distributions, and the coefficients $\phi_{(n, m, \sigma)}$ in expansion (2.9) are called modes of $\phi(z)$. The space $V[z, 1/z^2]$ has a natural structure of a differential module over the algebra $\mathbb{C}[z]_{z^2}$ (see [28, Sect. 1]), i.e., a $\mathbb{C}[\partial_z]$-module equipped with a compatible action of $\mathbb{C}[\partial_z]$ so that the Leibniz rule for differentiation is satisfied. We will review and generalize this in the next subsection.

2.3. Differential Module Structure. In this subsection we will describe the algebraic structure of the space $V[z, 1/z^2]$ in a way that can be generalized to more general vector spaces of “formal functions.”

As a consequence of harmonic decomposition (2.8), the vector space $V[z]$ is naturally isomorphic to $V[\varrho] \llbracket \varrho \rrbracket^{\text{har}}$, the vector spaces of harmonic polynomials with coefficients polynomials in a 1-component variable $\varrho \equiv z^2$. Similarly, we have $V[z]_{z^2} \cong V[\varrho, \varrho^{-1}] \llbracket \varrho \rrbracket^{\text{har}}$ and

$$\begin{align*}
V[z]_{z^2} &\cong V[\varrho] \llbracket \varrho \rrbracket^{\text{har}}, \\
V[z, 1/z^2] &\cong V[\varrho, \varrho^{-1}] \llbracket \varrho \rrbracket^{\text{har}},
\end{align*}$$

using the notation of Sect. 2.1 and 2.2.

We will now describe the differential module structures of the spaces $V[z]$ and $V[z]_{z^2}$ (over the algebras $\mathbb{C}[z]$ and $\mathbb{C}[\partial_z]$, respectively) in a way that is applicable to the space $V[z, 1/z^2]$ and is suitable for generalization. Set $R = V[\varrho]$ or $V[\varrho, \varrho^{-1}]$, respectively. Let $f(\varrho) \in R$ and let $h(z) \in R[z]^{\text{har}}$ be a harmonic homogeneous polynomial of degree $m$. Then for each $\alpha = 1, \ldots, D$, the polynomials $\partial_{z^\alpha} h(z)$ and

$$A_\alpha h(z) := z^\alpha h(z) - (D + 2m - 2)^{-1} z^2 \partial_{z^\alpha} h(z)$$

are harmonic as well. Indeed, for $(A_\alpha h)(z)$ this follows from (2.7). Note that the right–hand side of (2.12) is well defined for $m = 0$ since in this case $\partial_{z^\alpha} h(z) = 0$.

We deduce that the action of $z^\alpha$ and $\partial_{z^\alpha}$ on $R[z]^{\text{har}}$ is given by the formulas:

$$\begin{align*}
z^\alpha f(\varrho) h(z) &\equiv f(\varrho) (A_\alpha h)(z) + (D + 2m - 2)^{-1} \varrho f(\varrho) (\partial_{z^\alpha} h)(z), \\
\partial_{z^\alpha} f(\varrho) h(z) &\equiv f(\varrho) (\partial_{z^\alpha} h)(z) + 2 z^\alpha \left(f'(\varrho) h(z)\right),
\end{align*}$$

where $\varrho \equiv z^2$ and $f'(\varrho)$ denotes the derivative $df(\varrho)/d\varrho$, and in the right–hand side of (2.14) one has to apply (2.13) in order to get a result in $R[z]^{\text{har}}$.

Now we observe that by linearity Eqs. (2.13), (2.14) give rise to a well-defined action of $z^\alpha$ and $\partial_{z^\alpha}$ on the space of harmonic formal power series $R[z]^{\text{har}}$. This follows from the fact that the linear operators $A_\alpha$ and $\partial_{z^\alpha}$ on $R[z]^{\text{har}}$ are graded (of degree $+1$ and $-1$, respectively) with respect to the polynomial degree in $z$. We also notice that the right–hand sides of Eqs. (2.13), (2.14) involve only the differential $\mathbb{C}[\varrho]$–module structure of $R$. These observations are summarized in the following statement.

**Proposition 2.1.** Let $R$ be a differential module over $\mathbb{C}[\varrho]$ with a derivation $f \mapsto f'$. Then Eqs. (2.13), (2.14) define on $R[z]^{\text{har}}$ and $R[z]^{\text{har}}$ structures of differential modules over $\mathbb{C}[z]$ with derivations $\partial_{z^\alpha}$. If $R$ is a differential $\mathbb{C}[\varrho, \varrho^{-1}]$–module, then $R[z]^{\text{har}}$ and $R[z]^{\text{har}}$ are differential $\mathbb{C}[z]_{z^2}$–modules.
Proof. We have to check that for every $\phi(z) \in R[z]^{\text{har}}$ or $R[z]^{\text{har}}$, the following relations are satisfied:

\[
z^\alpha z^\beta \phi(z) = z^\beta z^\alpha \phi(z), \quad \sum_{\alpha=1}^{D} (z^\alpha)^2 \phi(z) = z^2 \phi(z), \quad \partial_z z^\alpha \phi(z) = \partial_z z^\alpha \phi(z), \quad \partial_z^2 (z^\beta \phi(z)) = z^\beta \partial_z^2 \phi(z) + \delta^\beta_\alpha \phi(z), \quad \partial_z ((z^2)^{-1} \phi(z)) = (z^2)^{-1} \partial_z \phi(z) - 2(z^2)^{-2} z^\alpha \phi(z).
\]

This can be verified by a straightforward computation, or can be deduced from the fact that these relations hold for $R = V[\varrho]$ and $V[\varrho, \varrho^{-1}]$.

In particular, using isomorphism (2.11) and taking $R = V[\varrho, \varrho^{-1}]$ in the above proposition, we obtain a structure of a differential $\mathbb{C}[z, z^2]$-module on the space $V[z, 1/z^2]$ of formal distributions (cf. [28]). Note that $V[z]$ is also a $\mathbb{C}[z]$-module, and $V[z, z^2]$ is a $\mathbb{C}[z, z^2]$-module. However, $V[z, 1/z^2]$ is not a $\mathbb{C}[z]$-module because $V[\varrho, \varrho^{-1}]$ is not a $\mathbb{C}[z]$-module. Although obvious, the next remark plays an important role in the theory.

Remark 2.1. The action of $\mathbb{C}[z, z^2]$ on $V[z, z^2]$ does not have zero divisors. In other words, if $p(z) \phi(z) = 0$ for $p(z) \in \mathbb{C}[z, z^2]$, $\phi(z) \in V[z, z^2]$, then either $p(z) = 0$ or $\phi(z) = 0$. Note that this is not the case for the $\mathbb{C}[z]$-module $V[z, 1/z^2]$ (see [28, Example 1.1]).

2.4. Generalized Formal Distributions. As another application of Proposition 2.1, we will define spaces of formal distributions that involve non-integral powers of $z^2$. We will take $R$ to be a space of formal series in real powers of $\varrho$,

\[
R = V[\varrho^\Gamma] := \left\{ f(\varrho) = \sum_{\gamma \in \Gamma} f_{\gamma} \varrho^\gamma \bigg| f_{\gamma} \in V \right\},
\]

where $\Gamma$ is a $\mathbb{Z}$-invariant subset of $\mathbb{R}$, i.e., such that

\[
\Gamma + \mathbb{Z} := \{ \gamma + m \mid \gamma \in \Gamma, m \in \mathbb{Z} \} \subseteq \Gamma.
\]

Then $R$ is a differential $\mathbb{C}[\varrho, \varrho^{-1}]$-module (with $(\varrho^\gamma)' = \gamma \varrho^{\gamma-1}$), and $R[z]^{\text{har}}$ is a differential $\mathbb{C}[z, z^2]$-module, which we denote as $V[z, (z^2)^{\Gamma}]$. We can view the elements of $V[z, (z^2)^{\Gamma}]$ as infinite series (cf. (2.9), (2.11))

\[
\phi(z) = \sum_{\gamma \in \Gamma} \sum_{m=0}^{\infty} \sum_{\sigma=1}^{b_{m,\sigma}} \phi_{\gamma, m, \sigma} (z^2)^\gamma h_{m,\sigma}(z), \quad \phi_{n,m,\sigma} \in V.
\]

For $\Gamma = \mathbb{Z}$, the above-defined module $V[z, (z^2)^{\mathbb{Z}}]$ coincides with $V[z, 1/z^2]$. We will call elements (2.17) (generalized) formal distributions. Note that this construction works also for subsets $\Gamma \subseteq \mathbb{C}$ but we will restrict our considerations to $\mathbb{R}$.

We define inductively the vector spaces

\[
V[z_1, (z_1^2)^{\Gamma_1}; \ldots; z_n, (z_n^2)^{\Gamma_n}]
\]

\[
:= \left( V[z_1, (z_1^2)^{\Gamma_1}; \ldots; z_{n-1}, (z_{n-1}^2)^{\Gamma_{n-1}}] \right)[z_n, (z_n^2)^{\Gamma_n}]
\]

of formal distributions in several $D$-dimensional vector variables $z_1, \ldots, z_n$. When all $\Gamma_i = \mathbb{Z}$, we will denote the space (2.18) as $V[z_1, 1/z^2_1; \ldots; z_n, 1/z^2_n]$. 
Remark 2.2. Choosing instead $R = V[\partial^\Gamma]$ in the above construction, we obtain a differential $\mathbb{C}[z]_{zz^2}$-module $R[\bar{z}]^{\bar{\text{bar}}}$ denoted as $V[z, (z^2)^\Gamma]$; it consists of all finite sums of the form (2.17).

Remark 2.3. Note that Eq. (2.7) is valid for $n \in \Gamma$, and it implies

$$\partial^2_z ((z^2)\gamma h_{m,\sigma}(z)) = 4\gamma \left(\gamma + m + \frac{D}{2} - 1\right)(z^2)^{\gamma - 1} h_{m,\sigma}(z).$$

In particular, $(z^2)^\gamma h_{m,\sigma}(z)$ is harmonic if and only if $\gamma = 0$ or $\gamma = -\frac{D}{2} + 1 - m$.

Example 2.1. Let us consider the case $D = 1$. Then $z = z$ is a 1-component variable and the harmonic polynomials are just the affine polynomials $a + bz$. Thus, every element of $V[z, (z^2)^\Gamma]$ has the form

$$\phi(z) = \sum_{\gamma \in \Gamma} (z^2)^\gamma (a_\gamma + b_\gamma z), \quad a_\gamma, b_\gamma \in V.$$ 

It is easy to see that when $\Gamma$ is an additive subgroup of $\mathbb{R}$ containing $\frac{1}{2}\mathbb{Z}$, we have the following direct sum of differential $\mathbb{C}[z]_{zz^2}$-modules:

$$V[z, (z^2)^\Gamma] = (1 + (z^2)^{-1/2} z)V[z, (z^2)^\Gamma] \oplus (1 - (z^2)^{-1/2} z)V[z, (z^2)^\Gamma].$$

Note that here $(z^2)^{-1/2} z$ is viewed as an element of $V[z, (z^2)^\Gamma]$, and is not equal to 1.

2.5. $\Gamma$–Localization. The vector space of localized formal power series $V[\bar{z}]_{zz^2}$ can be embedded in $V[z, 1/z^2]$ as the subspace of all elements (2.9) for which the sum over $n$ is bounded from below. This space is a (differential) module over the localized algebra $\mathbb{C}[\bar{z}]_{zz^2}$. We will later need the following generalizations of these spaces.

Let $\Gamma_1, \ldots, \Gamma_s$ be additive subgroups of $\mathbb{R}$ containing $\mathbb{Z}$; then each $\Gamma_i$ is automatically $\mathbb{Z}$-invariant (see (2.16)). Let $A$ be a commutative associative algebra, and let $f_1, \ldots, f_s$ be some fixed elements of $A$. Then we define

$$A_{f_1^{r_1} \cdots f_s^{r_s}} := (\mathbb{C}[\Gamma_1] \otimes_C \cdots \otimes_C \mathbb{C}[\Gamma_s] \otimes_C A)/J,$$

where $\mathbb{C}[\Gamma_i]$ is the group algebra of $\Gamma_i$ (with elements denoted as $e^\gamma$) and $J$ is the ideal generated by elements of the form

$$e^{\gamma_1} \otimes \cdots \otimes e^{\gamma_s} \otimes g - e^\gamma \otimes \cdots \otimes e^\gamma \otimes g$$

for which there exist $\gamma_i \in \Gamma_i$ such that $\gamma_i + \gamma'_i, \gamma_i + \gamma''_i \in \mathbb{Z}_+$ ($i = 1, \ldots, s$) and

$$f_1^{\gamma_1 + \gamma'_1} \cdots f_s^{\gamma_s + \gamma''_s} g = f_1^{\gamma_1 + \gamma''_1} \cdots f_s^{\gamma_s + \gamma'_s} g$$

in the algebra $A$. The so-defined commutative associative algebra $A_{f_1^{r_1} \cdots f_s^{r_s}}$ will be called the $\Gamma_1, \ldots, \Gamma_s$–localization of $A$ with respect to $f_1, \ldots, f_s$. When all groups $\Gamma_i$ are equal to $\Gamma$, we will just call it $\Gamma$–localization. The image in $A_{f_1^{r_1} \cdots f_s^{r_s}}$ of an element $e^\gamma \otimes \cdots \otimes e^\gamma \otimes g$ will be denoted as $f_1^{\gamma_1} \cdots f_s^{\gamma_s} g$.

Obviously, the $\mathbb{Z}$–localization of $A$ with respect to $f_1, \ldots, f_s$ coincides with the localized algebra $A_{f_1 \cdots f_s}$. In this case the localization with respect to a set $\{f_1, \ldots, f_s\}$ is naturally isomorphic to the localization with respect to the product $f_1 \cdots f_s$. For general $\Gamma$ this is not true.
If $M$ is an $A$–module, then in the same way one defines the localization $M_{f_1, \ldots, f_s}$ as an $A_{f_1, \ldots, f_s}$–module. In addition, if $M$ is a differential $A$–module, then the localization $M_{f_1, \ldots, f_s}$ is a differential $A_{f_1, \ldots, f_s}$–module with the same set of derivations. Indeed, every derivation of $A$ extends by the Leibniz rule to $A_{f_1, \ldots, f_s}$, since $J$ is an invariant subspace. We note also that if $M$ has no zero divisors the same is true for $M_{f_1, \ldots, f_s}$ (cf. Remark 2.1).

As a special case of the above construction, we get a differential $\mathbb{C}[z]((z^2))$–module $V[[z]]((z^2))$. The latter can be identified with the $\mathbb{C}[z]z^2$–submodule of $V[[z]]((z^2))$, consisting of all elements (2.17) for which the sum over $\gamma$ is over the union of finitely many sets of the form $\{\gamma_i + \mathbb{Z}_+\}$.

2.6. Formal Expansions. Let $\Gamma$ be an additive subgroup of $\mathbb{R}$ containing $\mathbb{Z}$, and let $V$ be a vector space. Recall the $\Gamma$–localizations defined in Sect. 2.5 and the spaces of formal series defined by (2.17), (2.18). There are obvious embeddings

$$\mathbb{C}[z, w]((z^2)), \mathbb{C}[z]((z^2)) \subseteq \mathbb{C}[z]((z^2)) \subseteq \mathbb{C}[z, (z^2)^\Gamma; w, (w^2)^\Gamma].$$

We emphasize that the spaces $\mathbb{C}[z]z^2[w, w^2]$ and $\mathbb{C}[w, w^2]z^2$ are different, because elements of the former space have only finitely many negative powers of $w$, but possibly infinitely many negative powers of $z^2$. Note also that the first two spaces in (2.25) are rings, and hence the second one is a module over the first.

For $\gamma \in \Gamma$, we define $t_{z,w}((z-w)^2)^\gamma$ as the Taylor expansion of $((z-w)^2)^\gamma$ in $w$ around 0, to wit

$$(z-w)^2)^\gamma := \sum_{k=0}^{\infty} \frac{(-w \cdot \partial_k)^k}{k!} (z^2)^\gamma \in \mathbb{C}[z]((z^2)) [w].$$

One defines $t_{z,w}((z+w)^2)^\gamma$ in the same way, while $t_{w,z}((z-w)^2)^\gamma$ is defined using the identifications

$$(z-w)^2)^\gamma = ((w-z)^2)^\gamma, \quad ((-z)^2)^\gamma = ((z)^2)^\gamma.$$

More generally, for $\psi(z, w) \in V[[z, w]]((z^2))$ we set

$$t_{z,w} (\psi(z, w) ((z-w)^2)^\gamma) := \psi(z, w) t_{z,w} ((z-w)^2)^\gamma,$$

thus obtaining a $\mathbb{C}[z, w]((z^2))$–linear map

$$t_{z,w} : \mathbb{C}[z, w]((z^2)) \rightarrow \mathbb{C}[z]((z^2)) [w].$$

Note that the map $t_{z,w}$ commutes with all partial derivatives $\partial_{z^\alpha}$ and $\partial_{w^\alpha}$.

We define another version of $t_{z,w}$ as follows:

$$t_{z,w} : \mathbb{C}[z-w]((z-w)^2)^\gamma \rightarrow \mathbb{C}[z]((z^2)) [w]((w^2)^\gamma),$$

$$\phi(z-w, w) \mapsto e^{-w \cdot \partial_k} \phi(z, w)|_{w'=w},$$

where in the left–hand side $z-w$ is viewed as an independent variable. Equation (2.30) is well defined because $e^{-w \cdot \partial_k} \phi(z, w) \in \mathbb{C}[z]((z^2)) [w, w']((w^2)^\gamma)$ (while an analogous formal expansion $t_{z,w}$ on $V[w]((w^2)) (z-w)^2)^\gamma$ does not make sense). Obviously, maps (2.29) and (2.30) agree with each other on the intersection of their domains. The two versions of $t_{z,w}$ can be related to each other by the following statement.
We define the spaces of successively localized formal series

\[
V[z_1]_{(z_1^2)^r} \cdots [z_1]_{(z_1^2)^r} := (V[z_1]_{(z_1^2)^r} \cdots [z_{n-1}]_{(z_{n-1}^2)^r})[z_n]_{(z_n^2)^r},
\]

which will be used in the sequel. The space (3.2) is a module over the algebra \( \mathbb{C}[z_1]_{(z_1^2)^r} \cdots [z_1]_{(z_1^2)^r} \). Again, one should keep in mind that in (3.2) one would get a different space if the variables are put in different order.

3. Residue Functional

In this section we introduce an important linear functional on the spaces of formal distributions, which we call the residue functional. We discuss its fundamental properties and we prove an analog of the Cauchy formula. A geometric interpretation of the residue is given in Appendix A.

3.1. Definition and Main Properties. In this subsection we introduce, for an arbitrary vector space \( V \), an important linear map \( V[z, (z^2)^R] \rightarrow V \), which will be denoted as \( \phi(z) \mapsto \text{Res}_z \phi(z) \) and will be called the \textbf{residue}. Observing that every element of \( V[z, (z^2)^R] \) can be uniquely represented as a formal series of the form

\[
\phi(z) = \sum_{\gamma \in \mathbb{R}} (z^2)^\gamma \phi_\gamma(z) \quad \text{with} \quad \phi_\gamma(z) \in V[z]^\text{har},
\]

we define

\[
\text{Res}_z \sum_{\gamma \in \mathbb{R}} (z^2)^\gamma \phi_\gamma(z) := \phi_{-\frac{D}{2}}(0).
\]

Theorem 3.1. (a) The linear map (3.2) is \( \partial_z \)-invariant in the sense that

\[
\text{Res}_z \partial_z^\alpha \phi(z) = 0
\]

for all \( \phi(z) \in V[z, (z^2)^R] \) and \( \alpha = 1, \ldots, D \).

(b) The \textbf{bilinear form}

\[
(f, g) := \text{Res}_z f(z)g(z), \quad f, g \in \mathbb{C}[z, (z^2)^R],
\]

is nondegenerate.

(c) Let \( h_m(z) \) and \( h'_{m'}(z) \) be harmonic homogeneous polynomials of degrees \( m \) and \( m' \), respectively. Then

\[
\langle (z^2)^\gamma h_m(z), h'_{m'}(z) \rangle = 0 \quad \text{if} \quad m \neq m' \quad \text{or} \quad 2\gamma + m + m' \neq -D,
\]
and in the opposite case this coincides with the unique, up to a multiplicative constant, $O(D)$–invariant scalar product on the vector space of harmonic homogeneous polynomials of degree $m (= m')$ given by

\[
\langle (z^2)^{-m - \frac{D}{2}} h_m(z), h'_{m'}(z) \rangle.
\]

**Proof.** (a) Let $h(z)$ be a harmonic homogeneous polynomial of degree $m$. From Eqs. (2.12)–(2.14) we deduce the harmonic decomposition

\[
\partial_\gamma \left( (z^2)^\gamma h(z) \right) = 2\gamma (z^2)^{\gamma-1} (A_\gamma h)(z) + (1 + 2\gamma(D + 2m - 2)^{-1}) (z^2)^\gamma \partial_\gamma h(z).
\]

Now let us apply $\text{Res}_z$ to the right–hand side of this equation. The first term will give zero, because $(A_\gamma h)(0) = 0$. Similarly, the second term can give a nonzero result only if $m = 1$ and $\gamma = -D/2$ but then the coefficient vanishes.

(b, c) Property (3.5) follows from the harmonic decomposition

\[
h_m(z) h'_{m'}(z) = \sum_{n=0}^{\min(m,m')} (z^2)^n h''_{m+m'-2n}(z),
\]

where $h''_{m'}(z)$ are uniquely determined harmonic homogeneous polynomials of degree $m'' = |m-m'|, \ldots, m+m'$. It is known that for $m = m'$, the constant polynomial $h''_0 \in \mathbb{C}$ defines an $O(D)$–invariant nondegenerate scalar product on the space of harmonic polynomials of degree $m$. This proves the remaining statements. $\square$

From now on, we will assume that the bases \{h_{m,\sigma}(z)\}_{\sigma=1,\ldots,b_m}$ of harmonic homogeneous polynomials of degree $m$ are orthonormal, so that

\[
\text{Res}_z \left( (z^2)^\gamma h_{m,\sigma}(z) h'_{m',\sigma'}(z) \right) = \delta_{\gamma,-\frac{D}{2}-m} \delta_{m,m'} \delta_{\sigma,\sigma'},
\]

in accord with Theorem 3.1(c). Consequently, the modes of a formal series $\phi(z)$ given by (2.17) can be recovered as residues:

\[
\phi_{\{\gamma,m,\sigma\}} = \text{Res}_z \phi(z) (z^2)^{-\frac{D}{2}-\gamma-m} h_{m,\sigma}(z).
\]

This justifies the name “formal distributions.”

**Corollary 3.2.** Let $\Gamma$ be a $\mathbb{Z}$–invariant subset of $\mathbb{R}$ (i.e., $\Gamma + \mathbb{Z} \subseteq \Gamma$), and set

\[
\Gamma' = -\Gamma + \frac{D}{2} := \left\{ -\gamma + \frac{D}{2} \mid \gamma \in \Gamma \right\}.
\]

Then

\[
V[z,(z^2)^\Gamma'] \cong \text{Hom}_\mathbb{C} \left( \mathbb{C}[z,(z^2)^\Gamma], V \right)
\]

as differential $\mathbb{C}[z]_{z^2}$–modules. In particular, $\mathbb{C}[z,(z^2)^\Gamma]$ is the dual $\mathbb{C}[z]_{z^2}$–module of $\mathbb{C}[z,(z^2)^\Gamma']$.

Note that by the recursive definition (2.18) the residue functional is defined also on formal distributions in several $D$–dimensional variables. For instance, we have

\[
V[z,(z^2)^\mathbb{R}], w, (w^2)^\mathbb{R}] \xrightarrow{\text{Res}_w} V[z,(z^2)^\mathbb{R}] \xrightarrow{\text{Res}_z} V.
\]

Then under the natural isomorphism

\[
V[z,(z^2)^\mathbb{R}], w, (w^2)^\mathbb{R}] \cong V[w,(w^2)^\mathbb{R}], z,(z^2)^\mathbb{R}]
\]
the “Fubini theorem” is satisfied, namely,

\[
(3.13) \quad \text{Res}_z \text{Res}_w \phi(z, w) = \text{Res}_w \text{Res}_z \phi(z, w)
\]

for \( \phi(z, w) \in V[z, (z^2)^{\Gamma}] \).

The following proposition describes all \( \partial_z \)-invariant linear functionals \( V[z, (z^2)^\Gamma] \to V \) (cf. (3.3)).

**Proposition 3.3.** Let \( \Gamma \) be a \( \mathbb{Z} \)-invariant subset of \( \mathbb{R} \), and let \( \Omega: V[z, (z^2)^\Gamma] \to V \) be a linear map that is \( \partial_z \)-invariant, i.e., such that \( \Omega(\partial_z \phi) = 0 \) for all \( \phi \in V[z, (z^2)^\Gamma] \) and \( \alpha = 1, \ldots, D \).

(a) If \( D \geq 2 \), then there exists a complex constant \( C \) such that \( \Omega(\phi) = C \text{Res}_z \phi \) for all \( \phi \in V[z, (z^2)^\Gamma] \). In particular, if \( \frac{D}{2} \notin \Gamma \) then \( \Omega = 0 \).

(b) If \( D = 1 \), then there exist complex constants \( C \) and \( C' \) such that \( \Omega(\phi) = C \text{Res}_z \phi + C' \text{Res}'_z \phi \) for all \( \phi \in V[z, (z^2)^\Gamma] \) (\( z \) is now a 1-component variable), where \( \text{Res}'_z \) is defined by

\[
(3.14) \quad \text{Res}'_z \sum_{\gamma \in \mathbb{R}} (z^2)^\gamma (a_\gamma + b_\gamma z) := b_{-1}
\]

(see Example 2.1).

**Proof.** The space of all \( \partial_z \)-invariant linear maps \( \Omega: V[z, (z^2)^\Gamma] \to V \) is isomorphic to the vector space

\[
\text{Hom}_\mathbb{C}(V[z, (z^2)^\Gamma] / \partial_z V[z, (z^2)^\Gamma], V),
\]

where

\[
\partial_z V[z, (z^2)^\Gamma] := \partial_z V[z, (z^2)^\Gamma] + \cdots + \partial_z^n V[z, (z^2)^\Gamma].
\]

Fix \( \gamma \in \Gamma \) and a harmonic homogeneous polynomials \( h(z) \) of degree \( m \). We will prove that for \( D \geq 2 \) one has

\[
(3.15) \quad (z^2)^\gamma h(z) \in \partial_z V[z, (z^2)^\Gamma] \quad \text{if} \quad (\gamma, m) \neq \left(-\frac{D}{2}, 0\right),
\]

while for \( D = 1 \) one has

\[
(3.16) \quad (z^2)^\gamma h(z) \in \partial_z V[z, (z^2)^\Gamma] \quad \text{if} \quad (\gamma, m) \neq \left(-\frac{1}{2}, 0\right) \quad \text{or} \quad (\gamma, m) \neq (-1, 1).
\]

Indeed, using the equalities (see (2.19))

\[
\partial_x^2 ((z^2)^{\gamma+1} h(z)) = 4(\gamma + 1) \left(\gamma + m + \frac{D}{2}\right) (z^2)^\gamma h(z)
\]

and

\[
\sum_{\alpha = 1}^D \partial_{z^\alpha} ((z^2)^{\gamma+1} \partial_{z^\alpha} h(z)) = 2m(\gamma + 1)(z^2)^\gamma h(z),
\]

we conclude that \( (z^2)^\gamma h(z) \in \partial_z V[z, (z^2)^\Gamma] \) if \( \gamma \neq -1 \) and \( (\gamma, m) \neq \left(-\frac{D}{2}, 0\right) \). Finally, in the case \( \gamma = -1 \), we have

\[
\sum_{\alpha = 1}^D \partial_{z^\alpha} ((z^2)^{-1} \partial_{z^\alpha} h(z)) = (D + m - 2)(z^2)^{-1} h(z).
\]

This proves (3.15) and (3.16).
Now observing that by Theorem 3.1(a) we have \((z^2)^{-D/2} \not\in \partial_z V[z,(z^2)^\Gamma]\), we complete the proof of part (a). To prove part (b), it remains to check that \(\operatorname{Res}'_z\) is \(\partial_z\)-invariant, which is straightforward. □

**Example 3.1.** Let \(D = 1\); then elements of \(V[z,(z^2)^\Gamma]\) have the form (2.20). In particular, for \(\Gamma = \mathbb{Z}\), we can write every element \(\phi(z) \in V[z,1/z^2]\) uniquely as

\[
\phi(z) = \sum_{n \in \mathbb{Z}} c_n z^n, \quad c_n \in V.
\]

In other words, \(V[z,1/z^2]\) can be identified with the space of formal series \(V[z,z^{-1}]\).

The functional \(\operatorname{Res}_z\) vanishes on a series (3.17), while the functional \(\operatorname{Res}'_z\) coincides with the usual residue: \(\operatorname{Res}'_z \phi(z) = c_{-1}\).

### 3.2. Translation Invariance and Cauchy Formula

Let \(V\) be a vector space, as before. One of the most important properties of the residue map (3.2) is its translation invariance.

**Proposition 3.4.** (Formal translation invariance.)

\[
\operatorname{Res}_z \tau_{z,w} \phi(z + w) = \operatorname{Res}_z \phi(z), \quad \phi(z) \in V[z,(z^2)^\Gamma].
\]

This equation is also valid for elements of \(V[z,(z^2)^\mathbb{R}]\).

**Proof.** Recall that, by definition (cf. (2.26)), \(\tau_{z,w} \phi(z + w) = e^{w \cdot \partial_z} \phi(z)\). Then Eq. (3.18) follows from Theorem 3.1(a). □

We proceed to finding an analog of the Cauchy kernel for our residue functional.

**Proposition 3.5.** For \(\psi(z) \in V[z], \gamma \in \mathbb{R}, n \in \mathbb{N}\), we have:

\[
\operatorname{Res}_z (z^2)^\gamma \psi(z) = 0 \quad \text{if} \quad \gamma + \frac{D}{2} > 0 \quad \text{or} \quad \gamma + \frac{D}{2} \not\in \mathbb{Z},
\]

(3.19)

\[
\operatorname{Res}_z (z^2)^{-\frac{D}{2}} \psi(z) = \psi(0),
\]

(3.20)

and

\[
\operatorname{Res}_z (z^2)^{-\frac{D}{2} - n} \psi(z) = K_n^{-1}((\partial_z^2)^n \psi)(0),
\]

(3.21)

where

\[
K_n := 2^{2n} \prod_{k=1}^{n} k \left( k + \frac{D}{2} - 1 \right).
\]

(3.22)

**Proof.** Eqs. (3.19) and (3.20) are straightforward from the definition of the residue functional. To prove (3.21), it is enough to assume that \(\psi(z)\) is a homogeneous polynomial of degree \(2n\). Then we apply induction on \(n\), starting with (3.20), and using the \(\partial_z\)-invariance (3.3) and the relation

\[
\sum_{\alpha=1}^{D} \partial_{z^\alpha} ((z^2)^{-\frac{D}{2} - n + 1}(\partial_{z^\alpha} \psi)(z))
\]

\[
= -4n \left(n + \frac{D}{2} - 1\right)(z^2)^{-\frac{D}{2} - n} \psi(z) + (z^2)^{-\frac{D}{2} - n + 1}(\partial_z^2 \psi)(z).
\]
This completes the proof. □

As a corollary of Proposition 3.5, for even $D$ we have a local formula for the residue of an element $\phi(z) \in V \llbracket z \rrbracket$:

\begin{equation}
\text{Res}_z \phi(z) = K_N^{-1}(\partial_z^N \phi(z))|_{z=0}, \quad N \gg 0.
\end{equation}

**Proposition 3.6.** (Higher-dimensional “Cauchy formula.”)

\begin{equation}
\text{Res}_z \iota_{z,w}((z-w)^2)^{-\frac{D}{2}} \psi(z) = \psi(w) \quad \text{for} \quad \psi(z) \in V \llbracket z \rrbracket.
\end{equation}

**Proof.** Consider the formal series

\[ \phi(z, w) := \iota_{z,w}((z-w)^2)^{-D/2} \psi(z) \in V \llbracket z \rrbracket \llbracket w \rrbracket. \]

By (3.18), we have

\[ \text{Res}_z \phi(z, w) = \text{Res}_z \iota_{z,w} \phi(z + w', w). \]

We can put $w' = w$ in the right-hand side and obtain

\[ \text{Res}_z \iota_{z,w} \phi(z + w, w) = \text{Res}_z \iota_{z,w} (z^2)^{-D/2} \psi(z + w) = \psi(w), \]

using (3.20). □

### 3.3. Harmonic Decomposition of $\iota_{z,w}((z-w)^2)^\gamma$

As before, let \( \{ h_{m,\sigma}(z) \} \) be an orthonormal basis of the space of harmonic homogeneous polynomials of degree $m$ (see (3.7)). Introduce the polynomials

\begin{equation}
H_m(z, w) := \sum_{\sigma=1}^{h_m} h_{m,\sigma}(z) h_{m,\sigma}(w).
\end{equation}

Note that $H_m(z, w)$ is the unique, up to a multiplicative constant, $O(D)$–invariant polynomial that is separately harmonic and homogeneous in $z$ and $w$ of degree $m$. Combining Eq. (3.7) with the Cauchy formula (3.24), we obtain that

\begin{equation}
\iota_{z,w}((z-w)^2)^{-\frac{D}{2}} \psi(z) = \sum_{m,n=0}^{\infty} (z^2)^{-\frac{D}{2}-m-n} (w^2)^n H_m(z, w).
\end{equation}

Recall that for every $\gamma \in \mathbb{R}$ and $n \in \mathbb{Z}_+$ the binomial coefficient $\binom{\gamma}{n}$ is defined as $\gamma(\gamma-1)\cdots(\gamma-n+1)/n!$ and is a polynomial of $\gamma$ of degree $n$.

**Proposition 3.7.** For every $\gamma \in \mathbb{R}$, we have

\begin{equation}
\iota_{z,w}((z-w)^2)^\gamma = \sum_{m,n=0}^{\infty} K_{m,n}(\gamma) (z^2)^{\gamma-m-n} (w^2)^n H_m(z, w),
\end{equation}

where

\begin{equation}
K_{m,n}(\gamma) := \frac{(-1)^n}{\binom{\frac{D}{2}-1+\gamma}{n}} \binom{\gamma}{m+n}. \quad (3.28)
\end{equation}

**Proof.** It follows from definition (2.26) and the $O(D)$–invariance that $\iota_{z,w}((z-w)^2)^\gamma$ has the form (3.27). Moreover, it is clear from (2.26) that for each fixed $m,n \in \mathbb{Z}_+$, the coefficient $K_{m,n}(\gamma)$ is a polynomial of $\gamma$. Then to establish (3.28) it will suffice to prove it for infinitely many values of $\gamma$. 

We will prove by induction that formula (3.28) holds for all $\gamma$ such that $-\frac{D}{2} - \gamma \in \mathbb{Z}_+$. For $\gamma = -\frac{D}{2}$ it gives $K_{m,n}(\frac{D}{2}) = 1$, which agrees with (3.26). Next, assume that (3.27) and (3.28) hold for some $\gamma$. Apply the Laplace operator $\partial_z^2$ to both sides of (3.27) and use (2.19) to find:

$$\gamma (\gamma + \frac{D}{2} - 1) t_{z,w} ((z-w)^2)^{\gamma-1}$$

$$= \sum_{m,n=0}^{\infty} K_{m,n}(\gamma) (\gamma - m - n)(\gamma - n + \frac{D}{2} - 1) (z^2)^{\gamma-1-m-n} (w^2)^n H_m(z,w).$$

Comparing this to (3.27) with $\gamma = 1$ instead of $\gamma$, we obtain that (3.28) holds for $\gamma = 1$. This completes the proof. \(\square\)

For $\gamma = -\frac{D}{2} + 1$, expansion (3.27) takes the particularly simple form

$$t_{z,w} ((z-w)^2)^{\frac{D}{2}+1} = \sum_{m=0}^{\infty} \frac{\frac{D}{2} - 1}{\frac{D}{2} - 1 + m} (z^2)^{\frac{D}{2}+1-m} H_m(z,w).$$

Note that both sides of this equation are harmonic with respect to both $z$ and $w$ (see Remark 2.3). Let us also point out that for fixed $\gamma \in \mathbb{Z}_+$, the coefficient $K_{m,n}(\gamma)$ vanishes whenever $m + n > \gamma$. Using (3.27) for $\gamma = 0, 1, 2, \ldots$, one can find the polynomials $H_m(z,w)$ recursively; for example,

$$H_0(z,w) = 1, \quad H_1(z,w) = D z \cdot w,$$

$$H_2(z,w) = \left(\frac{D}{2} + 1\right)(D(z \cdot w)^2 - z^2 w^2).$$

### 3.4. Formal Delta–Function

In this subsection we define a formal distribution in two variables, which plays the role of the delta–distribution. Let us consider the $\mathbb{Z}$–invariant set $\mathbb{Z}' := \frac{D}{2} + \mathbb{Z}$ (cf. (3.9)), which coincides with $\mathbb{Z}$ when $D$ is even. We define the following formal distribution in two variables

$$\delta(z,w) := \sum_{n \in \mathbb{Z}'} \sum_{m=0}^{\infty} (z^2)^{-\frac{D}{2} - m - n} (w^2)^n H_m(z,w) \in \mathbb{C}[z,(z^2)^{\mathbb{Z}'};w,(w^2)^{\mathbb{Z}}].$$

**Proposition 3.8.** The above-defined $\delta(z,w)$ is the unique element of $\mathbb{C}[z,(z^2)^{\mathbb{Z}'};w,(w^2)^{\mathbb{Z}}]$ with the property that

$$\text{Res}_z \phi(z) \delta(z,w) = \phi(w) \quad \text{for all} \quad \phi(z) \in \mathbb{C}[z]_{z^2}.$$ 

*In addition, it satisfies*

$$\phi(z) \delta(z,w) = \phi(w) \delta(z,w), \quad \phi(z) \in \mathbb{C}[z]_{z^2}$$

*and*

$$\partial_z^{\alpha} \delta(z,w) = -\partial_w^{\alpha} \delta(z,w), \quad \alpha = 1, \ldots, D.$$  

**Proof.** Property (3.32) and the uniqueness of $\delta(z,w)$ follow from the orthogonality relation (3.7) (cf. Corollary 3.2). Then Eq. (3.32) and the $\partial_z$–invariance of the residue (3.3) imply that, for every $\phi(z) \in \mathbb{C}[z]_{z^2}$,

$$\text{Res}_z \phi(z) (\partial_z^{\alpha} + \partial_w^{\alpha}) \delta(z,w)$$

$$= - \text{Res}_z \left( \partial_z^{\alpha} \phi(z) \right) \delta(z,w) + \partial_w^{\alpha} \text{Res}_z \phi(z) \delta(z,w) = 0.$$
This proves (3.34). Similarly, (3.33) follows from the equalities

\[ \text{Res}_z \psi(z) \phi(z) \delta(z, w) = \psi(w) \phi(w) = \text{Res}_z \psi(z) \phi(w) \delta(z, w), \]

for all \( \psi(z), \phi(w) \in \mathbb{C}[z,w]. \) □

**Remark 3.1.** Let \( \Gamma \) be any \( \mathbb{Z} \)-invariant subset of \( \mathbb{R} \), and let \( \Gamma' = -\Gamma + \frac{d}{2} \) (see (3.9)). Corollary 3.2 implies that there exists a unique element \( \delta_\Gamma(z,w) \in \mathbb{C}[z,(z^2)^{\Gamma'}; w, (w^2)^\Gamma] \) such that Eq. (3.32) holds for \( \delta_\Gamma(z,w) \) and all \( \phi(z) \in \mathbb{C}[z(z^2)^\Gamma]. \) Then (3.34) is satisfied as well, while the analog of (3.33) holds only when \( \Gamma \) is a subgroup of \( \mathbb{R} \) (this is needed for \( \mathbb{C}[z(z^2)^\Gamma] \) to be a ring). Finally, note that \( \delta_\Gamma(z,w) = \delta_\Gamma(w,z) \) when \( \Gamma = \Gamma'. \)

Observe that \( \delta(z,w) \) is symmetric, i.e., \( \delta(z,w) = \delta(w,z) \), if and only if \( D \) is even. Switching \( z \) and \( w \) in (3.26) and using (2.27), we obtain that for even \( D \) we have:

\[
\begin{align*}
\delta(z,w) &= \iota_{z,w} \left( (z-w)^2 \right)^{-\Phi} + \iota_{w,z} \left( (z-w)^2 \right)^{-\Phi} \\
&\quad + \sum_{m=0}^{\infty} \sum_{n=1}^{\frac{D}{2}-1+m} \left( z^2 \right)^{-\Phi-m+n} (w^2)^{-n} H_m(z,w).
\end{align*}
\]

(3.35)

This splitting of \( \delta(z,w) \) as a sum of three terms suggests the introduction of a natural partition of the space of formal distributions (without assuming that \( D \) is even). For a formal distribution \( \phi(z) \in \mathcal{V}[z,z^2] \), written as in (2.9), we define its parts \( \phi(z)_+, \phi(z)_- \), and \( \phi(z)_- \), as follows. We let \( \phi(z)_+ \) be given by (2.9) with the sum over \( n \in \mathbb{Z} \) restricted to \( n \geq 0, n \in \mathbb{Z} \). For \( \phi(z)_- \) we restrict the sum to \( 0 \leq \frac{D}{2} - m, n \in \mathbb{Z} \) (we first sum over \( m \) and then over \( n \)). For \( \phi(z)_\sim \) we restrict the sum to \( -\frac{D}{2} + 1 - m \leq n \leq -1, n \in \mathbb{Z} \). We call \( \phi(z)_+ \) the **regular part** of \( \phi(z) \), and we define the **singular part** as

\[
\phi(z)_{\text{s.p.}} := \phi(z)_- + \phi(z)_\sim.
\]

(3.36)

Then, obviously, \( \phi(z)_+ \in \mathcal{V}[z] \) and

\[
\phi(z) = \phi(z)_+ + \phi(z)_- + \phi(z)_{\sim} = \phi(z)_+ + \phi(z)_{\text{s.p.}}.
\]

(3.37)

It is important that the above partition of the space of formal distributions is \( \mathbb{C}[\partial_w]-\)invariant, i.e.,

\[
(\partial_w - \phi(z))_{\sim} = \partial_w - \phi(z)_+, \quad \text{for } * = +, -, \sim, \text{s.p.} \quad \text{and } \alpha = 1, \ldots, D.
\]

(3.38)

Let us point out that the product \( \phi(z) \iota_{z,w} ((z-w)^2)^{-\Phi} \) is well defined and belongs to the space \( \mathcal{V}[z,1/z^2][w] \). Then we can generalize Cauchy formula (3.24), using Eq. (3.26), to obtain

\[
\text{Res}_w \phi(z) \iota_{z,w} ((z-w)^2)^{-\Phi} = \phi(w)_+, \quad \text{for } \phi(z) \in \mathcal{V}[z,1/z^2].
\]

(3.39)

Thus, \( \iota_{z,w} ((z-w)^2)^{-\Phi} \) may be called \( \delta^+(z,w) \), and one can also introduce formal distributions \( \delta^-(z,w) \) and \( \delta^\sim(z,w) \) that give the \( - \) and \( \sim \) parts of \( \phi(w) \), respectively, and such that \( \delta(z,w) = \delta^+(z,w) + \delta^-(z,w) + \delta^\sim(z,w) \). In the case when \( D \) is even, this splitting of \( \delta(z,w) \) coincides with the one in Eq. (3.35).

**Remark 3.2.** Introduce the formal distribution (cf. (3.26))

\[
\delta_{\text{bar}}(z,w) := (z^2 - w^2) \iota_{z,w} ((z-w)^2)^{-\Phi} = \sum_{m=0}^{\infty} \left( z^2 \right)^{-\Phi+1-m} H_m(z,w).
\]
It is harmonic with respect to both \( z \) and \( w \), and has the property that 
\[
\text{Res}_z \ h(z) \, \delta^\dagger_{\text{har}}(z, w) = h(w) \quad \text{for all} \quad h(z) \in \mathbb{C}[z]\text{har}.
\]

It follows from (3.31) that \( \delta(z, w) = \delta_1(z^2 - w^2) \, \delta^\dagger_{\text{har}}(z, w) \), where 
\[
\delta_1(x - y) := \sum_{n \in \mathbb{Z}} x^{-1-n} \, y^n \in \mathbb{C}[x, x^{-1}; y, y^{-1}]
\]
is the formal delta–distribution in the usual \( D = 1 \) theory of vertex algebras (see e.g. [19, 21, 24]). Notice that, even though \( (z^2 - w^2) \, \delta_1(z^2 - w^2) = 0 \), one can not conclude from here that \( \delta(z, w) = 0 \), because the product \( \delta_1(z^2 - w^2) \, \iota_{z,w}((z-w)^2)^{-\frac{D}{2}} \) is not well defined (see the discussion in [24, Sect. 2.1] and in particular Eq. (2.1.17)).

3.5. Transformation Properties. For completeness, in this subsection we will investigate the transformation properties of the residue functional and the \( \iota \)-operation under the conformal inversion \( z \mapsto z/z^2 \).

We observe that the substitution \( \phi(z) \mapsto \phi(z/z^2) \) is a well-defined involution of \( V[z, (z^2)^\mathbb{R}] \) if we set \( (z/z^2)^\gamma := (z^2)^{-\gamma} \). Explicitly, if \( \phi(z) \) is given by (2.17), then
\[
\phi\left(\frac{z}{z^2}\right) := \sum_{\gamma \in \Gamma} \sum_{m=0}^{\infty} \sum_{\sigma=1}^{h_m} \phi_{\gamma,m,\sigma}(z^2)^{-\gamma-m} h_m,\sigma(z),
\]
because \( h_m,\sigma(z/z^2) = (z^2)^{-m} h_m,\sigma(z) \). Clearly, under this isomorphism, the \( \mathbb{C}[z];z^2 \)-module \( V[z, (z^2)^\mathbb{R}] \) is mapped onto \( V[z, (z^2)^{-1}] \).

Now let \( \Gamma \) be an additive subgroup of \( \mathbb{R} \) containing \( \mathbb{Z} \). For \( \gamma \in \Gamma \), we define
\[
\left(\frac{z}{z^2} - \frac{w}{w^2}\right)^\gamma := (z^2)^{-\gamma} (w^2)^{-\gamma} ((z-w)^2)^\gamma,
\]
which agrees with the usual formula for \( \gamma = 1 \). Then the substitution \( \psi(z, w) \mapsto \psi(z/z^2, w/w^2) \) defines an automorphism of \( V[z, w];(z^2)^\mathbb{R}((w^2)^\mathbb{R}) \).

Proposition 3.9. Let \( \phi(z) \in V[z, (z^2)^\mathbb{R}] \) and \( \psi(z, w) \in V[z, w];(z^2)^\mathbb{R}((w^2)^\mathbb{R}) \). Then we have:
\[
\text{Res}_z \phi\left(\frac{z}{z^2}\right) = \text{Res}_z \left(\frac{z^2}{z^2}\right)^{-D} \phi(z)
\]
and
\[
\iota_{z,w} \left(\frac{z}{z^2}, \frac{w}{w^2}\right) = \left(\iota_{w',z'} \psi(z', w') \right) \bigg|_{z'=z/z^2, w'=w/w^2}.
\]

Proof. Eq. (3.42) is immediate from the definition of the residue (cf. (2.17), (3.40)). To prove (3.43), it suffices to check it for \( \psi(z, w) = ((z - w)^2)^\gamma \), in which case the statement follows from (2.27), (2.27) and (3.41). □

One can define involutive automorphisms
\[
\phi(z) \mapsto \phi(-z), \quad \psi(z, w) \mapsto \psi(-z, -w)
\]
of \( V[z, (z^2)^\mathbb{R}] \) and \( V[z, w];(z^2)^\mathbb{R}((w^2)^\mathbb{R}) \) by setting (cf. (2.27))
\[
(z^2)^{\gamma_1} (w^2)^{\gamma_2} ((z-w)^2)^{\gamma_3} f(z, w) \mapsto (z^2)^{\gamma_1} (w^2)^{\gamma_2} ((z-w)^2)^{\gamma_3} f(-z, -w)
\]
for \( f(z, w) \in V[z, w] \). By the definition of \( \iota_{z,w} \) and \( \iota_{w,z} \) (see Sect. 2.6), the so-defined operation commutes with both of them. It is also clear that it anti-commutes with all partial derivatives \( \partial_{z^{\alpha}}, \partial_{w^{\alpha}} \), and satisfies

\[
(3.46) \quad \text{Res}_z \phi(-z) = \text{Res}_z \phi(z), \quad \phi(z) \in V[z,(z^2)^R].
\]

**Remark 3.3.** In analogy with the above automorphism \((3.45)\), one can define an automorphism \( \Theta \) by setting

\[
\Theta: (z^2)^{\gamma_1} (w^2)^{\gamma_2} ((z - w)^2)^{\gamma_3} f(z, w) \mapsto e^{2\pi i(\gamma_1 + \gamma_2 + \gamma_3)} (z^2)^{\gamma_1} (w^2)^{\gamma_2} ((z - w)^2)^{\gamma_3} f(z, w)
\]
for \( f(z, w) \in V[z, w] \). Then \( \Theta \) commutes with the \( \iota \)–operations and with the partial derivatives, and instead of \((3.46)\) one has: \( \text{Res}_z (\Theta \phi)(z) = (-1)^D \text{Res}_z \phi(z) \).

### 4. Fields and Locality

In this section we investigate the notions of fields, locality and operator product expansion in higher dimensions, mainly following \[28\]. We give the definition of vertex algebra and we provide two examples of vertex algebras.

#### 4.1. Polylocal Fields

In this subsection we introduce the notion of a field of several variables, and we generalize the results of \[28, \text{Sect. 2}\] about the existence of operator product expansion.

Let \( V = V_0 \oplus V_1 \) be a \( \mathbb{Z}_2 \)-graded vector space (i.e., a *superspace*). Then \( \text{End} V = (\text{End} V)_0 \oplus (\text{End} V)_1 \) is a \( \mathbb{Z}_2 \)-graded associative algebra, and we will denote its Lie super bracket by

\[
(4.1) \quad [A, B] := AB - (-1)^{pq}BA, \quad \text{for } A \in (\text{End} V)_p, \ B \in (\text{End} V)_q.
\]

We will suppose that \( V \) is endowed with an action of mutually commuting even endomorphisms \( T_1, \ldots, T_D \) (called *translation endomorphisms*) and with an even vector \( |0\rangle \) (called *vacuum*), such that \( T_1 |0\rangle = \cdots = T_D |0\rangle = 0 \).

Let \( A(z_1, \ldots, z_m) \) be an \( (\text{End} V) \)-valued formal distribution; in other words, let

\[
(4.2) \quad A(z_1, \ldots, z_m) \in (\text{End} V)[z_1, 1/z_1^2; \ldots; z_m, 1/z_m^2].
\]

It is called a **field** in \( z_1, \ldots, z_m \) (or just an *\( m \)-field*) iff for every \( v \in V \) one has

\[
(4.3) \quad A(z_1, \ldots, z_m) v \in V[z_1, \ldots, z_m]_{z_1^2 \cdots z_m^2}.
\]

This means that for every \( v \in V \) there is a non-negative integer \( N_{A,v} \) such that

\[
(4.4) \quad A(z_1, \ldots, z_m) v = (z_1^2 \cdots z_m^2)^{-N_{A,v}} \psi_{A,v}(z_1, \ldots, z_m)
\]
for some \( \psi_{A,v}(z_1, \ldots, z_m) \in V[z_1, \ldots, z_m] \).

If \( A \) is an \( m \)-field, then for every partition \( \{1, \ldots, m\} = J_1 \sqcup \cdots \sqcup J_r \) (disjoint union), the restriction

\[
(4.5) \quad \tilde{A}(u_1, \ldots, u_r) v := (z_1^2 \cdots z_m^2)^{-N_{A,v}} \psi_{A,v}(z_1, \ldots, z_m) \bigg|_{z_j := u_j \text{ for } j \in J_s}
\]

makes sense and defines again a field.
An $m$–field (or, more generally, an $(\text{End} V)$–valued formal distribution) $A$ is called translation invariant iff

\[ [T_\alpha, A(z_1, \ldots, z_m)] = \sum_{k=1}^m \partial_{z_k^\alpha} A(z_1, \ldots, z_m) \]

for every $\alpha = 1, \ldots, D$.

Let us point out that a product $A(z_1, \ldots, z_m) B(w_1, \ldots, w_n)$ of two fields is not a field in general. Indeed, by the above definition, for every $v \in V$ we have

\[ A(z_1, \ldots, z_m) B(w_1, \ldots, w_n) v \in V[z_1, \ldots, z_m w_1^2 \cdots w_n^2], \]

and it may contain infinitely many negative powers of $z_1^2, \ldots, z_m^2$. As a consequence, the restriction of the product (4.8) for coinciding arguments is not well defined in general. We will show below that one can “regularize” this product to make a field if the following definition is satisfied.

Two fields (or, more generally, $(\text{End} V)$–valued formal distributions) $A$ and $B$ are called mutually local iff there exists a non-negative integer $N_{A,B}$ such that

\[ \left( \prod_{j=1}^m \prod_{k=1}^n (z_j - w_k)^2 \right)^{N_{A,B}} [A(z_1, \ldots, z_m), B(w_1, \ldots, w_n)] = 0. \]

A 1–field that is local with respect to itself is usually called a local field; a 2–field that is local with respect to itself is called a bilocal field. An $m$–field, for general $m$, which is local with respect to itself, is called a polylocal field.

In the following statement we sum up some consequences of the above definitions.

**Theorem 4.1.** Let $A(z_1, \ldots, z_m)$ and $B(z_1, \ldots, z_n)$ be an $m$–field and an $n$–field, respectively, which are mutually local as above.

(a) Every restriction (4.6) of $A$ is also a field and is mutually local with respect to $B$.

(b) If the field $A$ is translation invariant, then its restrictions are also translation invariant fields.

(c) If $A$ is translation invariant, then $A(z_1, \ldots, z_m)[0] \in V[z_1, \ldots, z_m]$.

(d) Every partial derivative $\partial_{z_k^\alpha} A$ is a field and is mutually local with respect to $B$. If the field $A$ is translation invariant, then $\partial_{z_k^\alpha} A$ is also translation invariant.

(e) The formal distribution

\[ F_{A,B}(z_1, \ldots, z_m, w_1, \ldots, w_n) \]

\[ := \left( \prod_{j=1}^m \prod_{k=1}^n (z_j - w_k)^2 \right)^{N_{A,B}} A(z_1, \ldots, z_m) B(w_1, \ldots, w_n) \]

is an $(m+n)$–field. If the fields $A$ and $B$ are local with respect to a $p$–field $C(z_1, \ldots, z_p)$, then $F_{A,B}$ is also local with respect to $C$. If both fields $A$ and $B$ are translation invariant, then $F_{A,B}$ is also translation invariant.

**Proof.** Statements (a) and (b) follow easily from definitions.

Statement (c) for $m = 1$ is proved in [32, Proposition 3.2(a)], and that proof can be straightforwardly generalized for general $m$ (note that one can take $h_2 = h_1$ there).
To prove (d), one “commutes” the derivative $\partial_{z^i}$ through the polynomial $(z_k - w_j)^N$, as it is done in a more general case in [28, Lemma 2.3].

(e) Note that, by (4.8) and (4.9), for every $v \in V$ the series $F_{A,B}(z_1, \ldots, z_m, w_1, \ldots, w_n) v$ belongs to the intersection

$$V \llbracket z_1, \ldots, z_m \rrbracket [z_1^2, \ldots, z_m^2] \cap V \llbracket w_1, \ldots, w_n \rrbracket [w_1^2, \ldots, w_n^2] \llbracket z_1, \ldots, z_m \rrbracket \llbracket w_1, \ldots, w_n \rrbracket,$$

which is exactly $V \llbracket z_1, \ldots, z_m, w_1, \ldots, w_n \rrbracket [z_1^2, \ldots, z_m^2, w_1^2, \ldots, w_n^2]$. But this means, by definition, that $F_{A,B}$ is an $(m + n)$–field. The remaining part of the statement is straightforward. □

As a corollary of Theorem 4.1, every $m$–field $A(z_1, \ldots, z_m)$ can be expanded in 1–fields as follows. Consider for $v \in V$ the formal expansion

$$t_{z,w_1} \cdots t_{z,w_{m-1}} A(z + w_1, \ldots, z + w_{m-1}, z) v$$

(4.11)

$$:= \exp(w_1 \cdot \partial_{z_1} + \cdots + w_{m-1} \cdot \partial_{z_{m-1}}) A(z_1, \ldots, z_m) v \bigg|_{z_1=\ldots=zm=z} \in V \llbracket z \rrbracket [z_1^2, \ldots, w_{m-1}^2].$$

This is a formal power series in $w_1, \ldots, w_{m-1}$ with coefficients of the form $\psi_i(z) v \in V \llbracket z \rrbracket [z_1^2, \ldots, w_{m-1}^2]$ for some uniquely defined fields $\psi_i(z)$ ($i$ running over some index set). All $\psi_i(z)$ are fields because they are obtained from $A(z_1, \ldots, z_m)$ by the operations of differentiation and restriction. If, in addition, $A$ is translation invariant and is local with respect to some other fields $B$, $C$, etc., then all the fields $\psi_i(z)$ are also translation invariant and local with respect to $B$, $C$, etc.

The formal expansion (4.11) is called the operator expansion of $A(z_1, \ldots, z_m)$. Applying this expansion to the field $F_{A,B}$ (4.10), we get what is called the operator product expansion (OPE) of two mutually local fields $A$ and $B$.

Example 4.1. Let us consider, for comparison, the $D = 1$ case of OPE. Recall from Example 3.1 that now $z = z$ is a 1–component variable and the space of $(\text{End } V)$–valued formal distributions is identified with $(\text{End } V) [z, z^{-1}]$. Then our notions of fields and locality coincide with the ones used in vertex algebra theory (see [20, 13, 25, 21, 24]). For two mutually local fields $a(z)$ and $b(z)$ with parities $p_a$ and $p_b$, respectively, one introduces their $n$–th product for $n \in \mathbb{Z}$ by

$$\left( a(w)(n)b(w) \right) c := \text{res}_z a(z) b(w) c t_{z,w} (z - w)^n - (-1)^{p_a p_b} \text{res}_z b(w) a(z) c t_{w,z} (z - w)^n,$$

(4.12)

where $\text{res}_z z^k := \delta_{k,-1}$ is the usual residue functional (it corresponds to our $\text{Res}_z$; see Example 3.1). By the Cauchy theorem for $\text{res}_z$ (see [19, 21, 24]), one gets an equivalent definition

$$\left( a(w)(n)b(w) \right) c = \frac{1}{N!} \partial_{z}^{N} \left( (z - w)^{N+n+1} a(z) b(w) c \right) \bigg|_{z = w}, \quad N \gg 0,$$

(4.13)

where the right–hand side is independent of $N$. Our approach to the OPE of $a(z)$ and $b(z)$ corresponds precisely to definition (4.13). Then Dong’s Lemma, the fact that the field $a(z)(n)b(z)$ is local with respect to every field $c(z)$ local with respect to $a$ and $b$, is a simple corollary of Theorem 4.1 (e), (d), (a).
4.2. Completeness and State–Field Correspondence. Definition of Vertex Algebra. The translation invariance and locality properties allow us to introduce a state–field correspondence for a vertex algebra in higher dimensions, as in Sect. 3 and 4 of [28]. Here we will reproduce these results in a more concise way.

As in the previous subsection, let $V$ be a superspace endowed with an action of mutually commuting even endomorphisms $T_\alpha$ ($\alpha = 1, \ldots, D$) and a vacuum vector $|0\rangle$. A system of fields $\{\phi_i(z)\}_{i \in I}$ is called local iff $\phi_i(z)$ and $\phi_j(z)$ are mutually local for every $i, j \in I$. The system $\{\phi_i(z)\}$ is called translation invariant iff every $\phi_i(z)$ is translation invariant. Finally, the system $\{\phi_i(z)\}$ is called complete (with respect to the vacuum $|0\rangle$) iff the coefficients of all formal series $\phi_{i_1}(z_1) \cdots \phi_{i_n}(z_n) |0\rangle$ ($n \in \mathbb{N}$) together with $|0\rangle$ span the whole vector space $V$. In other words, the system $\{\phi_i(z)\}$ is complete iff the vacuum $|0\rangle$ is a cyclic vector for the associative subalgebra of $\operatorname{End} V$ generated by the modes of all fields $\phi_i(z)$.

**Theorem 4.2.** Let $\{\phi_i(z)\}_{i \in I}$ be a translation invariant, local and complete system of fields. Then for every $a \in V$ there exists a unique field, denoted as $Y(a, z)$, which is translation invariant, local with respect to all $\phi_i(z)$, and such that

$$Y(a, z) |0\rangle_{z=0} = a.$$ 

**Proof.** Let us consider the vector space $\mathcal{F}$ of all translation invariant 1–fields that are local with respect to $\phi_i(z)$ for all $i \in I$. By Theorem 4.1(c), there is a well-defined linear map

$$\mathcal{F} \to V, \quad \chi(z) \mapsto \chi(z) |0\rangle_{z=0}. \quad \quad \quad (4.15)$$

It follows from translation invariance that

$$\chi(z) |0\rangle = e^{z^T} (\chi(w) |0\rangle)_{w=0}, \quad z \cdot T := z^1 T_1 + \cdots + z^D T_D.$$ 

Then Theorem 3.1 from [28] implies that map (4.15) is injective. The theorem will be proved as soon as we show that map (4.15) is surjective.

Consider for every fixed $m = 1, 2, \ldots$ and $i_1, \ldots, i_m \in I$ the $m$–field

$$A(z_1, \ldots, z_m) := \left( \prod_{1 \leq k < \ell \leq m} (z_k^2)^{N_{k\ell}} \right) \phi_{i_1}(z_1) \cdots \phi_{i_m}(z_m),$$

where $z_{kl} = z_k - z_l$ and $N_{k\ell}$ are the integers fulfilling the locality condition (4.9) for $\phi_{i_k}$ and $\phi_{i_\ell}$. We then claim that all coefficients of $A(z_1, \ldots, z_m) |0\rangle$ belong to the image of (4.15). To prove this, first note that by Theorem 4.1(e) $A$ is a translation invariant $m$–field that is local with respect to $\phi_i(z)$ for all $i \in I$. Then all coefficients $\psi_i(z)$ in the operator expansion of $A$ are contained in $\mathcal{F}$ (see (4.11)). It follows from Theorem 4.1(c) that for $v = |0\rangle$ the right–hand side of (4.11) is simply the Taylor expansion of

$$A(z + w_1, \ldots, z + w_{m-1}, z) |0\rangle \in V[z, w_1, \ldots, w_{m-1}].$$

Then it is clear that all coefficients of $A(z_1, \ldots, z_m) |0\rangle$ belong to the image of (4.15).

On the other hand, iterating (4.8) we obtain that (cf. (2.32)):

$$\phi_{i_1}(z_1) \cdots \phi_{i_m}(z_m) |0\rangle \in V[z_1]_{z_1^2} \cdots [z_m]_{z_m^2}.$$ 

The right–hand side of (4.16) is a module over the algebra $\mathbb{C}[z_1]_{z_1^2} \cdots [z_m]_{z_m^2}$, in which the polynomial $\prod_{k < \ell} (z_k^2)^{N_{k\ell}}$ is invertible: its inverse is given by applying the
expansion \( \prod_{k<l} \tau_{z_k,z_l} \) (see Sect. 2.6). Therefore,

\[
\phi_{i_1}(z_1) \cdots \phi_{i_m}(z_m) |0\rangle = \left( \prod_{1 \leq k < l \leq m} \tau_{z_k,z_l} (z_k^2)^{-N_{k,l}} \right) A(z_1, \ldots, z_m) |0\rangle.
\]

This implies that every coefficient of (4.16) can be expressed as a linear combination of coefficients of \( A(z_1, \ldots, z_m) |0\rangle \), and hence belongs to the image of map (4.15). But by completeness the coefficients of (4.16) span \( V \); therefore, (4.15) is surjective. □

Corollary 4.3. Let \( \chi(z) \) be a translation invariant field, which is local with respect to a translation invariant, local and complete system of fields \( \{ \phi_i(z) \} \). Then \( \chi(z) \) is a local field and \( \chi(z) = Y(a,z) \) for \( a = \chi(z) |0\rangle |_{z=0} \).

Theorem 4.2 leads naturally to the following definition [28].

Definition 4.1. A vertex algebra \( V \) over \( \mathbb{C}^D \) is a superspace \( V \) endowed with:

(a) an action of mutually commuting, even endomorphisms \( T_1, \ldots, T_D \) (translation endomorphisms),

(b) an even vector \( |0\rangle \) (vacuum) such that \( T_1 |0\rangle = \cdots = T_D |0\rangle = 0 \),

(c) a parity preserving linear map (state–field correspondence)

\[ V \rightarrow (\text{End} \ V)[z, 1/z^2], \quad a \mapsto Y(a,z), \]

such that \( \{ Y(a,z) \}_{a \in V} \) is a translation invariant, local system of fields and \( a = Y(a,z) |0\rangle |_{z=0} \) for all \( a \in V \).

Corollary 4.4. Every translation invariant, local and complete system of fields \( \{ \phi_i(z) \} \) generates on \( V \) a unique structure of a vertex algebra.

Note that when \( D = 1 \), Definition 4.1 is equivalent to the definition of a usual (chiral) vertex algebra (see e.g. [21, 24]). We will use the notation

\[
(4.17) \quad a(z) \equiv Y(a,z)
\]
as it is customary in the usual \( D = 1 \) theory of vertex algebras. For a vertex algebra \( V \) and elements \( a, b \in V \), we denote by \( N(a,b) \) the smallest non-negative integer fulfilling the locality condition for \( a(z) \) and \( b(z) \), i.e.,

\[
(4.18) \quad N(a,b) := \min \left\{ N \in \mathbb{Z}^+ \mid ((z-w)^2)^N [a(z), b(w)] = 0 \right\}.
\]

Remark 4.1. One can introduce a more general notion of vertex algebra that involves non-integral powers of \( z^2 \) in the definition of a field and in the notion of locality. For \( D = 1 \) this would correspond to the “generalized vertex algebras” of [16, 13, 27], as explained in [3]. For \( D = 2 \), a related notion was introduced in [23].

4.3. Examples of Vertex Algebras. For completeness, in this subsection we present two simple examples of vertex algebras. We refer to [28, 32, 5, 6] for additional examples.

Our first example shows that the notion of a vertex algebra includes as a special case that of a (super)commutative associative algebra (cf. [8]). We call a vertex algebra \( V \) over \( \mathbb{C}^D \) holomorphic if \( Y(a,z) \in (\text{End} \ V)[z] \) for all \( a \in V \). The following statement is a straightforward generalization of the corresponding one in the case \( D = 1 \) (see [8, 21, 24]).
Proposition 4.5. (a) If $V$ is a holomorphic vertex algebra over $\mathbb{C}^D$, then $a \ast b := Y(a, z) b|_{z=0}$ defines on $V$ the structure of a super-commutative associative algebra with a unit $|0\rangle$ and even derivations $T_1, \ldots, T_D$.

(b) Conversely, given a super-commutative associative algebra $V$ with a product $\ast$, a unit $|0\rangle$, and with mutually commuting even derivations $T_1, \ldots, T_D$, then $Y(a, z) b := (e^{z^2 a}) \ast b$ defines the structure of a holomorphic vertex algebra on $V$.

Our second example is the vertex algebra generated by a harmonic scalar free field $\varphi(z)$ in even space-time dimensions $D > 2$ (see [28, Sect. 5] for a more general construction). Consider a new formal variable $p := (p_1, \ldots, p_D)$ and introduce the vector spaces

$$P := \mathbb{C}[p]/p^2 \mathbb{C}[p] \cong \mathbb{C}[p]^{\text{har}}, \quad V := \mathbb{C}[P].$$

Here $V$ is defined as the algebra of all polynomials of the elements of $P$, i.e., it is the symmetric algebra of $P$. We will denote by $[f(p)]$ the equivalence class of a polynomial $f(p) \in \mathbb{C}[p]$ in $P$. Then $V$ is the linear span over $\mathbb{C}$ of all monomials of the form $[f_1(p)] \cdots [f_n(p)]$ (note that $[f_1(p)][f_2(p)] \neq [f_1(p)f_2(p)]$ in $V$).

We define an $(\text{End} V)$–valued formal distribution $\varphi(z)$ by the formula

$$\varphi(z) [f_1(p)] \cdots [f_n(p)] := \sum_{k=0}^{\infty} \frac{1}{k!} (z \cdot p)^k [f_1(p)] \cdots [f_n(p)]$$

(4.20)

$$+ \sum_{l=1}^{n} f_l(-\partial_z) (z^2)^{-\frac{D}{2}+1} [f_1(p)] \cdots [\hat{f_l}(p)] \cdots [f_n(p)],$$

where a hat over a term means that it is omitted in the product. This does not depend on the choice of representatives $f_l(p)$ because $\partial_z^2 (z^2)^{-\frac{D}{2}+1} = 0$. Since the right–hand side of (4.20) contains only finitely many negative powers of $z^2$, it follows that $\varphi(z)$ is a field. After a straightforward computation, one finds

$$\varphi(z_1) \varphi(z_2) [f_1(p)] \cdots [f_n(p)] = e^{-z_2 \cdot \partial_{z_1}} (z_1^2)^{-\frac{D}{2}+1} [f_1(p)] \cdots [f_n(p)] + \cdots$$

(4.21)

where the remaining terms are symmetric under the exchange of $z_1$ and $z_2$. Hence,

$$[\varphi(z_1), \varphi(z_2)] = (t_{z_1,z_2} (z_{12}^2)^{-\frac{D}{2}+1} - t_{z_2,z_1} (z_{12}^2)^{-\frac{D}{2}+1}) \text{id}_V,$$

(4.22)

where $z_{12} = z_1 - z_2$. Therefore, $\varphi(z)$ is a local field, since

$$t_{z_1,z_2} (z_{12}^2)^{-\frac{D}{2}-1} [\varphi(z_1), \varphi(z_2)] = 0.$$

(4.23)

We define endomorphisms $T_1, \ldots, T_D$ of $V$ by the formula

$$T_\alpha [f_1(p)] \cdots [f_n(p)] := \sum_{l=1}^{n} [p_\alpha f_l(p)] [f_1(p)] \cdots [\hat{f_l}(p)] \cdots [f_n(p)]$$

(4.24)

for $\alpha = 1, \ldots, D$. In particular, $T_\alpha |0\rangle = 0$, where $|0\rangle$ is the constant polynomial $1 \in V$. One can easily verify that the endomorphisms $T_1, \ldots, T_D$ commute with each other, and the field $\varphi(z)$ is translation invariant. Let us point out that $\varphi(z)|0\rangle = e^{z^2} [\varphi(z)]|0\rangle = e^{z^2} 1$, and $\varphi(z)|0\rangle|_{z=0} = 1 \neq 1 = |0\rangle$.

Using (4.20), one can prove by induction on $n$ that every monomial $[f_1(p)] \cdots [f_n(p)]$ is a linear combination of the coefficients of $\varphi(z_1) \cdots \varphi(z_n)|0\rangle$. Therefore, the field $\varphi(z)$ is complete, and one can apply Corollary 4.4 to generate on $V$ the structure of a vertex algebra over $\mathbb{C}^D$. Finally, we note that the field $\varphi(z)$ is
harmonic, i.e., it satisfies the Laplace equation $\partial^2_\phi \psi(z) = 0$. This follows from (4.19), (4.20) and the fact that the function $(z^2)^{-\frac{D+1}{2}}$ is harmonic.

5. Jacobi Identity

This section contains the main results of the paper. We first prove certain “formal commutativity and associativity” relations, and then use them to derive our Jacobi identity. Integral versions of the Jacobi identity are also presented. We show that together with a partial vacuum axiom it can be taken as an equivalent definition of the notion of vertex algebra over $\mathbb{C}[D]$. We derive a formula for the commutator of two fields, and we prove that the singular parts of fields close a substructure under the commutator.

5.1. Commutativity and Associativity. In this subsection we will extend the “associativity” of [28, Theorem 4.3] by giving a connection between the degrees of the poles in a product $a(z)b(w)$ of two fields and the integers $N(a,b)$ introduced in (4.18). In the case of usual $D = 1$ vertex algebras our results agree with the “formal commutativity and associativity” of [17, 24].

Theorem 5.1. (“Formal commutativity and associativity.”) Let $V$ be a vertex algebra, and let $a, b, c \in V$, where $a$ and $b$ have parities $p_a$ and $p_b$, respectively. Then there exists a localized formal series

\begin{equation}
\psi_{a,b,c}(z_1, z_2) = \frac{\psi_{a,b,c}(z_1, z_2)}{(z_1^2)^{N(a,c)}(z_2^2)^{N(b,c)}(z_1^2 z_2^2)^{N(a,b)}},
\end{equation}

where $\psi_{a,b,c}(z_1, z_2) \in V[[z_1, z_2]]$, $z_{12} = z_1 - z_2$,

such that

\begin{equation}
\mathcal{H}_{a,b,c}(z_1, z_2) = (-1)^{p_a p_b} \mathcal{H}_{b,a,c}(z_2, z_1)
\end{equation}

and

\begin{align}
a(z_1) b(z_2) c &= \epsilon_{z_1, z_2} \mathcal{H}_{a,b,c}(z_1, z_2), \\
b(z_2) a(z_1) c &= (-1)^{p_a p_b} \epsilon_{z_2, z_1} \mathcal{H}_{a,b,c}(z_1, z_2), \\
(a(z_3) b(z_2) c &= \epsilon_{z_2, z_3} \mathcal{H}_{a,b,c}(z_2 + z_3, z_2).
\end{align}

Proof. Eqs. (5.2)–(5.5) follow from Proposition 4.2 and Theorem 4.3 of [28]. The explicit form (5.1) of $\mathcal{H}_{a,b,c}(z_1, z_2) \in V[z_1, z_2]\, z_1^2 z_2^2 z_{12}^2$ follows from Eq. (4.18) and the following lemma, which provides another description of the numbers $N(a, b)$. □

Lemma 5.2. For any two elements $a$ and $b$ in a vertex algebra $V$, we have

\begin{equation}
N(a, b) = \min \left\{ N \in \mathbb{Z}_+ \left| (z^2)^N a(z) b \in V[z] \right. \right\}.
\end{equation}

Proof. Denote the right-hand side of (5.6) by $N'(a, b)$, and consider the formal distribution

\begin{equation}
F_{a,b}(z_1, z_2) := (z_1^2)^{N(a,b)} a(z_1) b(z_2).
\end{equation}

Due to Theorem 4.1(e), (c), $F_{a,b}$ is a translation invariant, bilocal field and $F_{a,b}(z_1, z_2) |0\rangle \in V[z_1, z_2]$. Therefore, setting $z_2 = 0$ we find from $b(z_2)|0\rangle_{z_2=0} = b$ that $N'(a, b) \leq N(a, b)$.
Consider now the formal distribution
\[
F'_{a,b}(z_1, z_2) := (z_1^2)_{12}^{N'(a,b)} a(z_1) b(z_2).
\]
As in the proof of Theorem 4.1(e), \(F'_{a,b}\) is translation invariant and local (as a formal distribution) with respect to all fields \(c(z)\) for \(c \in V\). It follows from translation invariance that
\[
(F'_{a,b}(z_1, z_2)|0) = e^{xz-T} \left( (z_1^2)_{12}^{N'(a,b)} a(z_1) b \right) \in V[z_1, z_2] = V[z_1, z_2].
\]
On the other hand, by locality (assuming that \(c \in V\) has a fixed parity \(p_c\)) we get
\[
\left( (z_1 - w)^2 (z_2 - w)^2 \right)^N F'_{a,b}(z_1, z_2) (c(w)|0) = \left( (z_1 - w)^2 (z_2 - w)^2 \right)^N (-1)^{p_a+p_b} F'_{a,b}(z_1, z_2) (c(w)|0)
\]
for \(N \gg 0\). It follows from Eq. (5.7) and Theorem 4.1 that both sides of the above equation belong to the intersection of \(V[z_1, 1/z_1^2; z_2, 1/z_2^2][w] \) and \(V[w]_{w^2} [z_1, z_2] \). But the latter space is exactly \(V[z_1, z_2, w]\). Therefore,
\[
\left( (z_1 - w)^2 (z_2 - w)^2 \right)^N F'_{a,b}(z_1, z_2) (c(w)|0) \in V[z_1, z_2, w],
\]
and setting \(w = 0\) we find that \(F'_{a,b}\) is a field.

In the same way one proves that
\[
F'_{b,a}(z_1, z_2) := (z_1^2)_{12}^{N'(a,b)} b(z_2) a(z_1)
\]
is a field; note that \(N'(a,b) = N'(b,a)\) because of the “quasisymmetry” relation
\[
a(z) b = (-1)^{p_a p_b} e^{x-T} (b(-z) a), \quad a, b \in V
\]
(see [28, Proposition 4.4]). Locality for \(a(z)\) and \(b(z)\) implies that
\[
(z_1^2)_{12}^N \left( F'_{a,b}(z_1, z_2) - (-1)^{p_a p_b} F'_{b,a}(z_1, z_2) \right) c = 0, \quad N \gg 0.
\]
On the other hand, since \(F'_{a,b}\) and \(F'_{b,a}\) are fields,
\[
(F'_{a,b}(z_1, z_2) - (-1)^{p_a p_b} F'_{b,a}(z_1, z_2)) c \in V[z_1, z_2]_{z_1^2 z_2^2},
\]
which is a \(\mathbb{C}[z_1, z_2]_{z_1^2 z_2^2}\)-module with no zero divisors. Hence, \(F'_{a,b} = (-1)^{p_a p_b} F'_{b,a}\) and \(N(a,b) \vDash N'(a,b)\). \(\square\)

From the proof of Lemma 5.2 and from Theorem 4.1, we deduce the following corollary.

**Corollary 5.3.** Let \(V\) be a vertex algebra, and let \(A(z_1, \ldots, z_m)\) be an \((\text{End } V)\)-valued formal distribution, which is translation invariant and local with respect to all fields \(Y(c, z)\) (\(c \in V\)). Then \(A\) is an \(m\)-field if and only if \(A(z_1, \ldots, z_m)|0) \in V[z_1, \ldots, z_n]\).

Next, we will derive from Theorem 5.1 an “associativity” property, which generalizes Eqs. (4.2) and (7.3) from [2] (see also [13, 24]).

**Proposition 5.4.** (“Associativity.”) For every three elements \(a, b, c\) in a vertex algebra \(V\) and for \(L \geq N(a, c)\), we have:
\[
((z + w)^2)^L a(z) b(w) c = ((z + w)^2)^L t_{z,w} a(z + w) b(w) c,
\]
\[
(z^2)^L a(z) b(w) c = \left( (u + z - w)^2 \right)^L t_{z,w} (a(z - w) b)(u) c |_{u = w},
\]
\[
(z^2)^L a(z) b(w) c = \left( (u + z - w)^2 \right)^L t_{z,w} (a(z - w) b)(u) c |_{u = w},
\]
\[
(z^2)^L a(z) b(w) c = \left( (u + z - w)^2 \right)^L t_{z,w} (a(z - w) b)(u) c |_{u = w},
\]
where the expression under the substitution in the right-hand side of (5.10) belongs to \(\ell_{z,w} V[z,w,u]_{(z-w)^2 u^2}\) and setting \(u = w\) makes sense.

Proof. We can assume without loss of generality that \(L = N(a,c)\). Then, by Theorem 5.1, the left-hand side of (5.9) is equal to
\[
\frac{\psi_{a,b,c}(z+w,w)}{(z^2)^{N(a,b)}(w^2)^{N(b,c)}} \in V[z,w]_{z^2 w^2},
\]
while the right-hand side is
\[
\ell_{z,w} \ell_{z+w,w} \frac{\psi_{a,b,c}(z+w,w)}{((z+w)-w)^2} \frac{\psi_{a,b,c}(z,w)}{(z-w)^2} \frac{\psi_{a,b,c}(u+z-w,u)}{(u-w)^2} \in V[z,w]_{z^2 w^2},
\]
respectively, and obviously they become equal after the substitution \(u = w\).

5.2. Jacobi Identity. In this subsection, for any three elements in a vertex algebra over \(\mathbb{C}^D\), we derive an identity that generalizes the Jacobi identity of [19] (and the Borcherds identity of [21]) for usual \(D = 1\) vertex algebras.

Theorem 5.5. (“Jacobi identity.”) Let \(V\) be a vertex algebra, and let \(a, b, c \in V\), where \(a\) and \(b\) have fixed parities \(p_a\) and \(p_b\), respectively. Then for \(L \geq N(a,c)\) and for every \(F(z,w) \in \mathbb{C}[z,w]_{(z^2)^{p_a p_b} (w^2)^{p_a p_b} (z-w)^2}\), we have
\[
\begin{align*}
&\quad a(z) b(w) c \ell_{z,w} F(z,w) - (-1)^{p_a p_b} b(w) a(z) c \ell_{w,z} F(z,w) \\
&= (z^2)^{-L} \left( [(u+z-w)^2]^L \ell_{z,w} \ell_{z+w,z} (a(z-w)b)(u) c F(z,w) \right)_{u=w},
\end{align*}
\]
where the expression under the substitution in the right-hand side belongs to \((\ell_{z,w} - \ell_{w,z}) V[z,w,u]_{(z^2)^{p_a p_b} (w^2)^{p_a p_b} (z-w)^2 u^2}\) and setting \(u = w\) makes sense.

Proof. By the same argument as in the proof of Eq. (5.10) above, one finds separately
\[
\begin{align*}
&\quad a(z) b(w) c \ell_{z,w} F(z,w) \\
&= (z^2)^{-L} \left( [(u+z-w)^2]^L \ell_{z,w} (a(z-w)b)(u) c F(z,w) \right)_{u=w},
\end{align*}
\]
and
\[
\begin{align*}
&\quad (-1)^{p_a p_b} b(w) a(z) c \ell_{w,z} F(z,w) \\
&= (z^2)^{-L} \left( [(u+z-w)^2]^L \ell_{w,z} (a(z-w)b)(u) c F(z,w) \right)_{u=w},
\end{align*}
\]
for \(L \geq N(a,c)\). Taking the difference we obtain (5.11).

The main subtlety of Eq. (5.11) is that in the right-hand side one cannot make the substitution \(u = w\) in each of the factors separately. It is only after we multiply them that this substitution makes sense. The reason is that, in contrast to the case \(D = 1\), the expression \((\ell_{z,w} - \ell_{w,z}) a(z-w)b\) involves an infinite sum, and hence in general \((\ell_{z,w} - \ell_{w,z}) a(z-w)b(w)c\) is not well defined. We refer to Sect. 5.4 and 5.5 below for additional discussion.
It is clear from the proof of Eq. (5.10) that if we multiply the right–hand sides of Eqs. (5.12) and (5.13) by \((u^2)^M\) for \(M \geq N(b, c)\), they will become regular in \(u\) (i.e., not containing negative powers of \(u^2\)). Then we will be able to represent the substitution \(u = w\) by Cauchy formula (3.24). Thus we obtain the following equivalent integral form of Jacobi identity (5.11).

**Corollary 5.6.** For every elements \(a, b, c\) in a vertex algebra \(V\), \(a\) and \(b\) having fixed parities \(p_a\) and \(p_b\), respectively, and for every \(L \geq N(a, c), M \geq N(b, c)\), we have:

\[
a(z) b(w) c t_{z,w} F(z, w) - (-1)^{p_a p_b} b(w) a(z) c t_{w,z} F(z, w)
\]

\[
= \text{Res}_{u} (z^2)^{-L} (w^2)^{-M} ((u + z - w)^2)^L (u^2)^M t_{u,w} ((u - w)^2)^{-2w} \times (t_{z,w} - t_{w,z}) \left( (a(z - w) b)(u) c F(z, w) \right)
\]

for \(F(z, w) \in \mathbb{C}[z, w]_{(z^2)^2(w^2)^2((z-w)^2)^2}\).

**Proof.** It remains to note that the right–hand side of (5.14) makes sense. Indeed, the product of the Cauchy kernel \(t_{u,w} ((u - w)^2)^{-2w}\) and the third line in (5.14) is well defined in the space \((t_{z,w} - t_{w,z}) V[u]_{llbracket z, w \rrbracket (z^2)^2(w^2)^2((z-w)^2)^2}^\mathbb{R}\). \(\Box\)

**Remark 5.1.** One can give an alternative proof of Theorem 5.5 by using “associativity” relation (5.9), “quasisymmetry” relation (5.8), and generalizing the arguments of [2, Sect. 7] to the case of arbitrary \(D\) (see also [18, 25, 24]). With obvious modifications, Eq. (5.11) remains valid for generalized vertex algebras (see Remark 4.1 and [16, 13, 27, 3]).

We will show in Sect. 5.4 below that Jacobi identity (5.11), together with a partial vacuum axiom, can be taken as an equivalent definition of vertex algebra over \(\mathbb{C}^D\).

### 5.3. Integral Borcherds Formula

In this subsection we will derive an integral version of Jacobi identity (5.11), which in particular gives a formula for the commutator of modes that generalizes the Borcherds commutator formula from [8] (see also [19, 18, 21, 24]).

Let us introduce the following additive subgroup of \(\mathbb{R}\),

\[
\mathbb{Z} := \mathbb{Z} + \frac{D}{2} \mathbb{Z} = \begin{cases} 
\mathbb{Z}, & \text{if } D \text{ is even;} \\
\frac{1}{2} \mathbb{Z}, & \text{if } D \text{ is odd.}
\end{cases}
\]

As a consequence of Corollary 3.2, every \((\text{End } V)\text{–valued formal distribution } \phi(z) \in (\text{End } V)[z, 1/z^2] \) can be considered as a linear map

\[
\mathbb{C}[z]_{(z^2)^2} \rightarrow \text{End } V, \quad f(z) \mapsto \text{Res}_z \phi(z) f(z).
\]

Thus \(\mathbb{C}[z]_{(z^2)^2}\) plays the role of a vector space of test functions, and for even space–time dimension \(D\) it is exactly \(\mathbb{C}[z]_{z^D} \equiv \mathbb{C}[z]_{(z^2)^D}\). According to Eq. (3.8), the modes of \(\phi(z)\) can be obtained by integrating \(\phi(z)\) (with \(\text{Res}_z\)) against appropriate test functions.

Now let us take \(\text{Res}_u \text{ Res}_w\) of both sides of Eqs. (5.12) and (5.13) for \(F(z, w) \in \mathbb{C}[z, w]_{(z^2)^2(w^2)^2((z-w)^2)^2}\), and represent the substitution \(u = w\) as \(\text{Res}_u\) as done in
Eq. (5.14). We are going to rewrite the resulting identities in the form
\[
\text{Res}_z \text{ Res}_w a(z) b(w) c \; t_{z,w} F(z, w)
\]
(5.17) \hspace{1cm}
= \text{Res}_z \text{ Res}_w K^+_{L,M}(z, w; F) (a(z) b(w) c),
\]
\[
\text{Res}_z \text{ Res}_w b(w) a(z) c \; t_{w,z} F(z, w)
\]
(5.18) \hspace{1cm}
= (-1)^{p_a p_b} \text{Res}_z \text{ Res}_w K^-_{L,M}(z, w; F) (a(-z) b)(w) c,
\]
where $K^\pm_{L,M}$ are to be determined. To arrive at the above formulas, we will use translation invariance of the residue to replace $z$ with $z + w$ in (5.12) and $w$ with $w + z$ in (5.13). More precisely, we have the following lemma.

**Lemma 5.7.** For every $G(z, w) \in V\llbracket z, w \rrbracket_{(z^2)(w^2)}^{(z-w)^2}$ we have
\[
\text{Res}_z \text{ Res}_w \; t_{z,w} \; G(z, w) = \text{Res}_z \text{ Res}_w \; t_{z,w} \; G(z+w, w).
\]

**Proof.** Translation invariance of the residue (see (3.18)) implies the identity
\[
\text{Res}_z \text{ Res}_w \; t_{z,w} \; G(z, w) = \text{Res}_z \text{ Res}_w \; t_{z,u} \; t_{z+u,w} \; G(z+u, w).
\]
Then since the expression under the residue in the right-hand side belongs to the space $V\llbracket z \rrbracket_{(z^2)}^{(z-w)^2}$, we can set there $u = w$. But
\[
t_{z,w} \; t_{z+u,w} \; G(z+w, w) = t_{z,w} \; G(z+w, w)
\]
by “Taylor formula” (2.31). □

Applying Lemma 5.7 to the right-hand side of (5.12), we obtain (5.17) with
\[
K^+_{L,M}(z, w; F) := \text{Res}_u \; t_{z,u} \; F(u+z, u) \; ((u+z)^2)^{-L} \; (u^2)^{-M}
\]
\[
\times \; t_{w,u} \; ((w-u)^2)^{-\frac{M}{2}} \; ((z+w)^2)^L \; (w^2)^M.
\]

Similarly, after a renaming of the variables, (5.13) leads to (5.18) with
\[
K^-_{L,M}(z, w; F) := \text{Res}_u \; t_{u,z} \; F(u, u+z) \; ((u+z)^2)^{-M} \; (u^2)^{-L}
\]
\[
\times \; t_{w,u+z} \; ((w-u+z)^2)^{-\frac{M}{2}} \; ((w-z)^2)^L \; (w^2)^M.
\]
Notice that the expressions after $\text{Res}_u$ in the right-hand sides of (5.20) and (5.21) are well-defined elements of $\mathbb{C}\llbracket w \rrbracket_{(w^2)^2}^{(w^2)^2} \llbracket z \rrbracket_{(z^2)^2}^{(z^2)^2} \llbracket u \rrbracket_{(u^2)^2}$, and in fact the former belongs to $\mathbb{C}\llbracket z, w \rrbracket_{(z^2)^2}^{(z^2)^2} \llbracket u \rrbracket_{(u^2)^2}^{(u^2)^2}$. Then (5.20) and the formula
\[
K^+_{L,M}(z, w; F) = t_{w,z} \; K^+_{L,M}(z, w-z; F_{op}), \quad F_{op}(z, w) := F(w, z)
\]

imply that
\[
K^+_{L,M}(z, w; F) \in \mathbb{C}\llbracket z, w \rrbracket_{(z^2)^2}^{(z^2)^2} \llbracket w \rrbracket_{(w^2)^2}^{(w^2)^2}, \quad K^-_{L,M}(z, w; F) \in \mathbb{C}\llbracket w \rrbracket_{(w^2)^2}^{(w^2)^2} \llbracket z \rrbracket_{(z^2)^2}^{(z^2)^2}.
\]

Taking the difference of Eqs. (5.17) and (5.18), and using (3.46), we obtain the following result.

**Theorem 5.8.** With the above notation, in any vertex algebra, we have:
\[
\text{Res}_z \; \text{Res}_w \; a(z) \; b(w) \; c \; t_{z,w} \; F(z, w)
\]
(5.24) \hspace{1cm}
= (-1)^{p_a p_b} \text{Res}_z \; \text{Res}_w \; b(w) \; a(z) \; c \; t_{w,z} \; F(z, w)
\]
\[
= \text{Res}_z \; \text{Res}_w \; K_{L,M}(z, w; F) \; (a(z) b)(w) c,
\]
where

$$K_{L,M}(z, w; F) := K_{L,M}^+(z, w; F) - K_{L,M}^-(z, w; F).$$

In particular, when $F(z, w) = f(z) g(w)$ is a product of two test functions, Eq. (5.24) gives a formula for the commutator of modes, generalizing the Borcherds formula.

### 5.4. The Jacobi Identity As Alternative Axiom. The Case $D = 1$.

In this subsection we derive some consequences of our Jacobi identity (5.11). We prove that together with a partial vacuum axiom it can be taken as an equivalent definition of vertex algebra over $\mathbb{C}^D$. We also show that for $D = 1$ it reduces to (an equivalent form of) the Jacobi identity of [19].

First of all, it is clear from the definitions that if $(z^2)^N a(z)b \in V[\![z]\!]$, then $(\tau_{z,w} - \tau_{w,z}) a(z - w) b ((z - w)^2)^N = 0$. Therefore, Jacobi identity (5.11) implies locality. Our next step is to show that it also implies “associativity.”

**Lemma 5.9.** Let $V$ be a vector space, let $c$ be an element of $V$, and let $a(z), b(w)$ be two fields on $V$. Assume that Eq. (5.11) holds for some fixed $p_a, p_b, L$ and for all $F(z, w) \in C[z, w]^{z^2, (z - w)^2 \geq}$ (see (5.15)). Then Eq. (5.9) holds for some $L' \geq L$.

**Proof.** Let $L' \geq L$ be large enough so that $(z^2)^L' a(z) c \in V[\![z]\!]$. Obviously, if Eq. (5.11) holds for some $L$ then it holds for all $L' \geq L$, so let us just assume $L' = L$. Applying $\text{Res}_z$ to both sides of (5.11) with $F(z, w) = (z^2)^L f(z - w)$, where $f(z) \in C[\![z]\!]^{z^2 \geq}$ is an arbitrary test function, we obtain:

$$\text{Res}_z (z^2)^L a(z) b(w) c \tau_{z,w} f(z - w) = \text{Res}_z \left[ \left( (u + z - w)^2 \right)^L \tau_{z,w} \left( a(z - w) b(u) c f(z - w) \right) \right]_{u=w}.$$ 

Now using the translation invariance of the residue (see (3.18)), we get:

$$\text{Res}_z \left[ \left( (z + w)^2 \right)^L \tau_{z,w} a(z + w) b(w) c f(z) \right] = \left[ \text{Res}_z \left[ \left( (u + z)^2 \right)^L \left( a(z) b(u) c f(z) \right) \right] \right]_{u=w}.$$ 

After the substitution $u = w$, this gives exactly Eq. (5.9) (cf. (5.16)). □

Now we can prove the following statement, which shows that a vertex algebra can be defined in terms of Jacobi identity as in [19] for the $D = 1$ case (see also [21, 24]).

**Theorem 5.10.** Let $V$ be a vector superspace endowed with an even vector $|0\rangle$ and with a parity preserving linear map $Y: V \leftrightarrow Y(a, z) \equiv a(z)$ to the space of fields on $V$. Assume that Jacobi identity (5.11) holds for every fixed $a, b, c \in V$ with parities $p_a, p_b$ of $a$ and $b$, respectively, for some $L \geq 0$ and for all $F(z, w) \in C[z, w]^{z^2, (z - w)^2 \geq}$. Finally, let the following “partial vacuum axiom” be satisfied:

$$Y(|0\rangle, z) a = a, \quad \text{Res}_z (z^2) \frac{\partial}{\partial z} Y(a, z) |0\rangle = a \quad \text{for all} \quad a \in V.$$ 

Then there exist uniquely determined mutually commuting even endomorphisms $T_1, \ldots, T_D$ of $V$, which make $V$ a vertex algebra over $\mathbb{C}^D$. 


Proof. We have already pointed out that locality of \(a(z)\) and \(b(w)\) follows from Jacobi identity for \(F(z, w) = ((z-w)^2)^N\) with \(N \gg 0\). We will derive the rest of the axioms of vertex algebra (Definition 4.1) from Eqs. (5.9) and (5.26) (cf. Lemma 5.9).

Putting in Eq. (5.9) \(b = |0\rangle\) and \(L \gg 0\) such that \((z^2)^L a(z) c \in V[z]\), we obtain that

\[
\left((z + w)^2\right)^L (a(z) |0\rangle)(w) c \in V[z, w].
\]

Since \((a(z) |0\rangle)(w) c \in V[w, z]^2[z][z]^2\), it makes sense to multiply the above equation by \(\iota_{w,z}((z + w)^2)^{-L}\) and get

\[
(a(z) |0\rangle)(w) c \in \iota_{w,z} V[z, w][z][z]^2 \subset V[w, z][z].
\]

Then letting \(c = |0\rangle\) and using the second equality in (5.26) (with respect to \(w\)), we deduce from here that \((a(z) |0\rangle \in V[z] \) for all \(a \in V\). Thus the second equality in (5.26) can be restated as \((a(z) |0\rangle)_{|z=0} = a\).

We define the translation endomorphisms \(T_1, \ldots, T_D\) of \(V\) by the formula

\[
T_\alpha a := \partial_{z^{\alpha}} a(z) |0\rangle \big|_{z=0}, \quad \alpha = 1, \ldots, D,
\]

which should hold if \(V\) is a vertex algebra. Then putting \(c = |0\rangle\) in Eq. (5.9), we deduce that

\[
T_\alpha (a(z) b) - a(z) (T_\alpha b) = \partial_{z^{\alpha}} a(z) b,
\]

while the substitution \(b = |0\rangle\) in Eq. (5.9) implies \((T_\alpha a)(z) = \partial_{z^{\alpha}} a(z)\). The remaining axioms (Definition 4.1(a), (b)) are then immediate. \(\Box\)

Remark 5.2. With obvious modifications, Theorem 5.5 and Theorem 5.10 hold also for modules over vertex algebras (see [28, Sect. 6] for the definition of module).

The main subtlety of Jacobi identity (5.11) is that one can not make the substitution \(u = w\) in each of the factors separately. However, in the next proposition we will show that this can be done if the field \(a(z)\) has a special form. Recall that the regular and singular parts of a formal distribution were defined in Sect. 3.4.

Proposition 5.11. Let \(a, b\) be elements in a vertex algebra \(V\), with parities \(p_a\) and \(p_b\), respectively. Assume that for some \(n \in \mathbb{Z}\) the singular part of \((z^2)^n a(z) b\) belongs to \(V[z]^2\). Then we have:

\[
\begin{align*}
& a(z) b(w) \iota_{w,z} ((z-w)^2)^n - (-1)^{p_a + p_b} b(w) a(z) \iota_{w,z} ((z-w)^2)^n \\
& = \left(\iota_{w,z} - \iota_{z,w}\right) \left(\left((a(z) b)(w) ((z-w)^2)^n\right)\right),
\end{align*}
\]

(5.27)

and the right-hand side is well defined.

Proof. For an arbitrary fixed \(c \in V\) and \(L \gg 0\), set \(F(z, w) = ((z-w)^2)^n\) in Eq. (5.11). Because the regular part of \(((z-w)^2)^n a(z) - w\) is killed by \(\iota_{z,w} - \iota_{w,z}\), we can replace \(((z-w)^2)^n a(z) - w\) by its singular part in the right-hand side of (5.11). But by assumption the coefficients of the singular part of \((z^2)^n a(z) b\) span a finite-dimensional subspace of \(V\). Therefore, in (5.11) one can substitute \(u = w\) in each factor separately, and the right-hand side of (5.27) makes sense. After putting \(u = w\) in the other factor in (5.11) it cancels with \((z^2)^L\). \(\Box\)

Remark 5.3. When \(D = 1\), the assumption of Proposition 5.11 is satisfied for every pair of elements \(a, b \in V\) and every \(n \in \mathbb{Z}\). In this case, the collection of identities (5.27) is equivalent to the Jacobi identity of [19].
Let us note that for $D > 1$ the assumption of Proposition 5.11 is in fact quite restrictive. For $n = 0$ it holds for the scalar free field $\varphi(z)$ discussed in Sect. 4.3 (because $\varphi(z)$ is harmonic) but it does not hold for the Wick square $\varphi(z)^2$. Furthermore, it does not hold for $\varphi(z)$ itself when $n < 0$. On the other hand, the assumption is satisfied for $n = 0$ and any “generalized free field” (see [28, Sect. 5]), thus providing a version of the Wick Theorem (note that the element $b$ is arbitrary).

5.5. Degree Cutoffs and Commutator Formula. In this subsection, we derive a commutator formula, which shows in particular that the singular modes of fields close an algebraic structure under the commutator.

We have remarked at the end of the previous subsection that the main difficulty for $D > 1$ as opposed to $D = 1$ is that the singular part of $a(z)b$ involves an infinite sum in general. To circumvent this problem we introduce “degree cutoffs” as follows. For a formal distribution $\phi(x)$, written as in (2.17), and for any $N \in \mathbb{R}$, we define the cutoff $\phi(z)^{\leq N}$ by restricting the sums over $m$ and $\gamma$ in (2.17) to indices with $m + 2\gamma \leq N$. In other words, we restrict the sum to terms with degrees in $z$ less than or equal to $N$. We denote the remaining part of $\phi(z)$ by $\phi(z)^{> N} := \phi(z) - \phi(z)^{\leq N}$. In the same way, we define cutoffs $\phi(z)^{s,p,N}$ of the singular part of $\phi(z)$ (see (3.36)). Note that all these operations are commuting projections on the space of formal distributions.

Even though the singular part $a(z)^{s,p,N}b$ may involve infinitely many terms with arbitrarily high degrees in $z$, it is important that for fixed $N \in \mathbb{Z}$ the cutoff of the singular part $a(z)^{s,p,N}b$ is finite, i.e., it belongs to $V[z]^{s,z}$. Then the same argument as in the proof of Proposition 5.11 gives the following result.

Lemma 5.12. Let $a, b, c$ be elements in a vertex algebra $V$, where $a$ and $b$ have parties $p_a$ and $p_b$, respectively. Then for every $n, N \in \mathbb{Z}$ and every $L \geq N(a,c)$, we have:

\[
\begin{align*}
& a(z) b(w) \tau_{z,w}((z-w)^2)^n - (-1)^{p_a p_b} b(w) a(z) \tau_{w,z}((z-w)^2)^n \\
& = (\tau_{z,w} - \tau_{w,z}) \left( (a(z-w)^{\leq N} b)(w) ((z-w)^2)^n \right) \\
& + (z^2)^{-L} \left( (u+z-w)^2 \right)^L (\tau_{z,w} - \tau_{w,z}) (a(z-w)^{> N} b)(u) c ((z-w)^2)^n \bigg|_{u=w}.
\end{align*}
\]

We will now apply this lemma in the case $n = 0$ when it reduces to a formula for the commutator of the fields $a(z), b(w)$. Then, because of the presence of $\tau_{z,w} - \tau_{w,z}$, in the right–hand side of the above equation one can replace $a(z-w)^{\leq N}$ and $a(z-w)^{> N}$ by their singular parts and obtain:

\[
\begin{align*}
& [a(z), b(w)] c = (\tau_{z,w} - \tau_{w,z}) (a(z-w)^{s,p,N} b)(w) \\
& + (z^2)^{-L} ((u+z-w)^2)^L (\tau_{z,w} - \tau_{w,z}) (a(z-w)^{s,p,N} b)(u) c \bigg|_{u=w}.
\end{align*}
\]

The next result shows that the singular parts of fields themselves close a structure with respect to the commutator.

Proposition 5.13. For every three elements $a, b, c$ in a vertex algebra $V$ and for every $L \geq N(a,c)$, one has:

\[
\begin{align*}
& [a(z)^{s,p,N} b(w)^{s,p,N}] c \\
& = \left( (z^2)^{-L} ((u+z-w)^2)^L (\tau_{z,w} - \tau_{w,z}) (a(z-w)^{s,p,N} b)(u)^{s,p,N} c \bigg|_{u=w} \right)^{s,p,N}.
\end{align*}
\]
where the outer s.p. in the right-hand side designates taking the singular part with respect to both $z$ and $w$.

Note that for $D > 1$ a product of two singular terms may contain a regular part; that is why in (5.29) we must include the outer projection onto the singular parts.

**Proof of Proposition 5.13.** We will prove that (5.29) holds for all terms with total degree in $z$ and $w$ up to $N$, for every fixed $N \in \mathbb{Z}$. For this purpose, we consider all terms of total degree $\leq N$ in (5.28), and take the singular parts with respect to both $z$ and $w$. We will consider separately the resulting two terms in the right-hand side.

In the first term, the expansion $\iota_w(z - w)$ will not contribute because it produces terms regular in $z$. Since $\iota_z(w - \iota_w(z - w))$ is regular in $w$, it will not contribute either, and we will obtain

$$
\left( \iota_{z,w}(a(z - w)_{s.p.} b)(w)_{s.p.} c \right)_{s.p.}
$$

Reversing the above reasoning, this expression can be rewritten as

$$
\left( \iota_{z,w} - \iota_{w,z}(a(z - w)_{s.p.} b)(w)_{s.p.} c \right)_{s.p.}
$$

Then, as in the derivation of Lemma 5.12, it is equal to

$$
\left( \left[ (z^2)^{-L} \left( (u + z - w)^2 \right)^L \iota_{z,w} - \iota_{w,z}(a(z - w)_{s.p.} b)(u)_{s.p.} c \right]_{u=w} \right)_{s.p.}
$$

It remains to prove that the second term resulting from the right-hand side of (5.28) is equal to

$$
\left( \left[ (z^2)^{-L} \left( (u + z - w)^2 \right)^L \iota_{z,w} - \iota_{w,z}(a(z - w)_{s.p.} b)(u)_{s.p.} c \right]_{u=w} \right)_{s.p.}
$$

This follows from the fact that

$$
\left[ (z^2)^{-L} \left( (u + z - w)^2 \right)^L \iota_{z,w} - \iota_{w,z}(a(z - w)_{s.p.} b)(u)_{s.p.} c \right]_{u=w}
$$

contains only terms with total degree in $z$ and $w$ strictly greater than $N$. □

### 6. Concluding Remarks

In this paper we develop further the theory of vertex algebras in higher dimensions. We start by introducing useful formal calculus techniques including various spaces of formal series and a formal residue functional. This residue functional is uniquely determined (up to a multiplicative constant) by the property that it is translation invariant (Theorem 3.1), and so it plays the role of the integral. In addition, it satisfies an analog of the Cauchy formula (Eq. (3.24)). The modes of fields can be obtained by integrating the fields (with respect to our residue functional) against certain test functions.

Our main goal was to understand the algebraic structure obeyed by the modes of local fields with respect to the commutator. For this purpose we derived an analog of the Jacobi identity for vertex algebras in higher dimensions (Theorem 5.5). Since the commutator of two local fields is expressed in terms of the singular part of their operator product expansion, a natural question arises whether the singular parts of fields close a structure under the commutator. Utilizing a certain degree cutoff technique we proved that this is indeed the case (Proposition 5.13).
Thus, if we denote by \( a[z] b \) the singular part \( a(z)_{s.p.} b \), we find that it closes the following structure. The map \( a, b \mapsto a[z] b \in (V[z]_{a2})_{s.p.} \) is parity preserving and bilinear on a superspace \( V \) endowed with an action of mutually commuting even endomorphisms \( T_1, \ldots, T_D \), and the following axioms are satisfied:

(a) (translation invariance) \( [T_\alpha, a[z]] b = (T_\alpha a)[z] b = \partial_z^\alpha a[z] b; \)

(b) (skew–symmetry) \( a[z] b = (-1)^{p_a p_b} \left( e^{x T} (b[z] a) \right)_{s.p.}; \)

(c) (Jacobi identity)

\[
[a[z], b[w]] c = \left( (z^2)^{-L} \left[ ((u + z - w)^2)^L (t_{z,w} - t_{w,z}) (a[z] b) |u| c \right]_{u=w} \right)_{s.p.}
\]

for \( L \gg 0 \), and the expression under the substitution in the right–hand side belongs to the space \( (t_{z,w} - t_{w,z}) V[z, w, u]_{(z-w)\in\mathbb{A}_2} \) where setting \( u = w \) makes sense. It is expected that the obtained algebraic structure will play in higher dimensions the same role as vertex Lie algebras do in dimension one.

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**Appendix A. Geometric Realization of the Residue**

In this appendix we will provide a geometric definition of the residue functional introduced in Sect. 3.1. We will suppose that the space–time dimension \( D \) is even.

Let us introduce the following 1–parameter family \( \{\overline{M}_r\}_{r>0} \) of \( D \)–dimensional real submanifolds of \( \mathbb{C}^D \):

\[
\overline{M}_r := \{ z \in \mathbb{C}^D \mid z = \lambda u, \ \lambda \in \mathbb{C}, \ |\lambda| = r, \ u \in S^{D-1} \subset \mathbb{R}^D \},
\]

where \( S^{D-1} \) denotes the unit sphere in \( \mathbb{R}^D \). Note that \( \overline{M}_1 \) is exactly the *conformally compactified Minkowski space* (see [31, 28]).

We introduce a parameterization of \( \overline{M}_r \),

\[
\overline{M}_r \ni z = r e^{i \zeta} u \quad \text{ for } \ \zeta \in [0, \pi), \ u \in S^{D-1},
\]

which shows that \( \overline{M}_r \) is diffeomorphic to \( (S^1 \times S^{D-1})/\mathbb{Z}_2 \), with the points \( (e^{i \zeta}, u) \) and \( (-e^{i \zeta}, -u) \) being identified. In particular, all \( \overline{M}_r \) are orientable for even \( D \).

Thus, the volume form \( dz^1 \wedge \cdots \wedge dz^D \) on \( \mathbb{C}^D \) can be restricted to \( \overline{M}_r \) and gives rise to a complex measure there. In parameterization (A.2), we have

\[
dz^1 \wedge \cdots \wedge dz^D |_{\overline{M}_r} = i r^D e^{i D \zeta} d\zeta \wedge ds(u),
\]
where \( d\sigma(u) \) is the \( O(D) \)-invariant volume form \( dz^1 \wedge \cdots \wedge dz^D |_{S^{D-1}} \) on the unit sphere \( S^{D-1} \).

An important property of the family \( \{\mathcal{M}_r\} \) is that if \( z \in \mathcal{M}_r \) and \( w \in \mathcal{M}_{r'} \) for \( r \neq r' \) then \( (z - w)^2 \neq 0 \). Indeed, writing
\[
\begin{align*}
\text{(A.4)} & \quad z = r e^{i\zeta} u, \quad w = r' e^{i\zeta'} u', \quad u \cdot u' = \cos \alpha = (e^{i\alpha} + e^{-i\alpha})/2, \\
\end{align*}
\]
we find that
\[
\begin{align*}
\text{(A.5)} & \quad (z - w)^2 = (r e^{i\zeta} - r' e^{i(\zeta + \alpha)})(r e^{i\zeta} - r' e^{i(\zeta - \alpha)}).
\end{align*}
\]
This shows that for \( n \in \mathbb{Z} \) the formal Taylor expansion \( \iota_{z,w}(\langle z - w \rangle^2) \) in the above parameterization corresponds to a geometric series expansion for \( r > r' \).

Note also that the conformal inversion \( z \mapsto z/z^2 \) maps \( M_r \) onto \( M_{r'} \).

Proposition A.1. For \( f(z) \in \mathbb{C}[z^2] \) and \( g(z, w) \in \mathbb{C}[z, w]_{z^2 w^2(z-w)^2} \), we have:
\[
\begin{align*}
\text{(A.6)} & \quad \text{Res}_z f(z) = (i\pi \mathcal{V}_{D-1})^{-1} \int_{\mathcal{M}_r} f(z) \, dz^1 \wedge \cdots \wedge dz^D, \\
\text{(A.7)} & \quad \text{Res}_z \iota_{z,w} g(z, w) \\
& = (i\pi \mathcal{V}_{D-1})^{-1} \int_{\mathcal{M}_r} g(z, w) \, dz^1 \wedge \cdots \wedge dz^D \quad \text{for} \quad w \in \mathcal{M}_{r'}, \ r' < r, \\
\text{(A.8)} & \quad \text{Res}_z \iota_{w,z} g(z, w) \\
& = (i\pi \mathcal{V}_{D-1})^{-1} \int_{\mathcal{M}_r} g(z, w) \, dz^1 \wedge \cdots \wedge dz^D \quad \text{for} \quad w \in \mathcal{M}_{r'}, \ r' > r,
\end{align*}
\]
where \( \mathcal{V}_{D-1} = \int_{S^{D-1}} d\sigma(u) \).

Proof. It is enough to check (A.6) for the functions \( (z^2)^n h(z) \), where \( n \in \mathbb{Z} \) and \( h(z) \) is a harmonic homogeneous polynomial of degree \( m \). Then (A.6) follows from (A.3) and the formulas
\[
\int_{S^{D-1}} h(u) \, d\sigma(u) = \delta_{m,0} h(0) \mathcal{V}_{D-1}, \quad \int_0^\pi e^{iD\zeta + 2i\zeta} d\zeta = \pi \delta_{r, -\frac{\pi}{2}}.
\]
Equation (A.7) follows from (A.6) because the expansion \( \iota_{z,w} g(z, w) \) converges uniformly to \( g(z, w) \) for \( z \in \mathcal{M}_r, w \in \mathcal{M}_{r'} \) and fixed \( r > r' \) (see (A.5)). Finally, (A.8) is proved in the same way as (A.7) but for \( r < r' \). \( \square \)

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