The Cangemi-Jackiw manifold in high dimensions and symplectic structure

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Abstract

The notion of Poincaré gauge manifold (\( \mathcal{G} \)), proposed in the context of an (1 + 1) gravitational theory by Cangemi and Jackiw (D. Cangemi and R. Jackiw, Ann. Phys. (N.Y.) 225 (1993) 229), is explored from a geometrical point of view. First \( \mathcal{G} \) is defined for arbitrary dimensions and in the sequence a symplectic structure is attached to \( T^*\mathcal{G} \). Treating then the case of 5-dimensions, a (4,1) de-Sitter space, applications are presented studying representations of the Poincaré group in association with kinetic theory and the Weyl operators in phase space. The central extension in the Aghassi-Roman-Santilli group (Aghassi, Roman and Santilli, Phys. Rev. D 1 (1970) 2753) is derived as a subgroup of linear transformations in \( \mathcal{G} \) with 6 dimensions.

I. INTRODUCTION

This paper is dedicated to an investigation of the manifold (to be denoted by \( \mathcal{G} \)) proposed by Cangemi and Jackiw (CJ) in the study of an (1+1)-dimensional gravity as a gauge

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theory. In the CJ formalism, the gauge group is given by the generators of the Lie algebra
\[ [x_1, x_2] = x_4, \quad [x_3, x_2] = x_1, \quad [x_3, x_1] = x_2, \]
which is a central extension, specified by \( x_4 \), of the (1 + 1)-Poincaré group. This Lie algebra is in correspondence to a symmetric bilinear invariant form given by
\[
\eta = \begin{pmatrix}
g^{\mu\nu} & 0 & 0 \\
0 & 0 & -1 \\
0 & -1 & 0
\end{pmatrix},
\]
where \( g^{\mu\nu} \) is the (1 + 1)-Minkowski metric, with \( \text{diag}(g^{\mu\nu}) = (-1, 1) \). If, in particular, \( g^{\mu\nu} \) is the Euclidean metric, in which \( \text{diag}(g^{\mu\nu}) = (1, 1) \), then \( \eta \) is the light-cone metric \([4,5]\).

Among different applications of this approach \([6-9]\), an analysis of \( \eta \)-like metrics and associated Lie algebras \([10]\) has shown that this correspondence is not unique, and that in five dimensions \( \eta \), with \( g^{\mu\nu} \) being the Euclidean metric in three dimensions, can be used to write
the relativistic and the non-relativistic physics in a geometric unified way. One interesting result in this analysis is that the Galilean physics can be derived as a manifestly covariant theory \([11-18]\). Such results point to the richness of \( \mathcal{G} \) and for the need of other additional studies, as for instance by exploring higher dimensions and the symplectic structure attached to \( T^*\mathcal{G} \). These aspects are addressed here.

To begin with, in Section 2, the CJ manifold is defined in \( N \) dimensions and then
the linear group of transformations in this \( N \) dimensional space is studied. The notion of
symplectic 2-form is introduced in Section 3. Therefore (canonical) representations of Lie
algebras in phase space, including representations with generalized van Hove operators \([19]\),
are analyzed. In Section 4, considering \( \mathcal{G} \) in 5-dimensions and using the notion of embedding
(following along the lines established in Ref. \([10]\)), a relativistic kinetic equation is derived
and studied. This equation, which is but an example of the classical thermofield dynamics
formalism \([20]\), has a local collision term obtained from symmetry elements via a Lagrangian
written in phase space. A particular solution leads then to a wave-travelling distribution
function. The symplectic structure is also used with the Weyl product (first introduced in
the context of the Wigner-function picture of quantum mechanics \([21]\)), and applied to the
analysis of a scheme proposed by McDonald and Kaufman [22] to derive kinetic equations governing the behavior of the electromagnetic waves in a dispersive medium. The main result here is to show that the symplectic structure associated to $T^*\mathcal{G}$ provides a geometric interpretation for the McDonald and Kaufman method.

In Section 5 $\mathcal{G}$ is studied in 6-dimensions. In this case we show that a subgroup of the linear transformations in $\mathcal{G}$ is a general set of affine transformations in the $(3+1)$-Minkowski space with a central charge. This kind of group was studied by Aghassi, Roman and Santilli [23–25], motivated by the following problem. In the contraction of the Poincaré group, taking $c \to \infty$, the extended Galilei group is recovered but with mass interpreted as a central charge and not as the value of a Casimir invariant, as it is in the case of the Poincaré Lie algebra. Analyzing this kind of difficulty (representing a problem at least at the level of rigor), Aghassi, Roman and Santilli considered a general set of linear transformations in the $(3+1)$-Minkowski space including, beyond the Lorentz rotations and the translations, a boost given by $b^\mu u$, where $b^\mu$ is a 4-velocity and $u$ is a proper-time scalar. In this scenario, mass arises via a central extension of this group, which is a kind of Galilean relativistic symmetry. In our analysis, using the CJ manifold, without the use of central extension, mass emerges as a Casimir invariant in the Lie algebra of the linear group of transformations in the 6-dimensional $\mathcal{G}$ space. The connection with the usual field theory is discussed via the scalar representation. Final concluding remarks are presented in Section 6.

II. THE CJ MANIFOLD AND TENSOR FIELDS

Let $(\mathcal{G}, \eta)$ be a $N$-dimensional pseudo-Riemannian manifold, where the metric $\eta$ is given by

$$\eta = \eta_{\mu\nu} e^\mu \otimes e^\nu,$$

(2)

with $\{e^\mu\}$ being basis of cotangent space of $\mathcal{G}$ at arbitrary point $p \in \mathcal{G}$; the components of $\eta$, say $\eta_{\mu\nu}$, are represented by the matrix
\[
(\eta_{\mu\nu}) = \begin{pmatrix}
I_{ij} & 0 & 0 & \ldots & 0 \\
0 & B_1 & 0 & \ldots & 0 \\
0 & 0 & B_2 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & B_{N-k}
\end{pmatrix}_{N,N},
\]

where \( I_{ij} \) is the \( k \)-dimensional Euclidean metric, \( \text{diag}(I_{ij}) = (1, 1, \ldots, 1) \) (with \( i, j \leq k \)), and

\[
B_i = \begin{pmatrix}
0 & -1 \\
-1 & 0
\end{pmatrix}.
\]

The restriction is that \( N - k \) is an even number.

Consider two arbitrary vectors \( X \) and \( Y \) in \( \mathcal{G} \), which are written in terms of the basis \( \{e_\mu\} \) of \( T^*_p \mathcal{G} \) in the form \( X = X^\mu e_\mu \) and \( Y = Y^\nu e_\nu \). Using Eq. (3), the inner product is

\[
(X \mid Y) = \eta(X, Y) = \sum_{i=1}^{k} X^i Y^i - \sum_{i=1}^{N-k} (X^{k+2i}Y^{k+2i-1} + X^{k+2i-1}Y^{k+2i}).
\]

The rules raising and lowering indices, determined by \( X^\mu = \eta^{\mu\nu}X_\nu \) and \( X_\mu = \eta_{\mu\nu}X^\nu \) with \( \eta_{\mu\nu} = \eta^{\mu\nu} \), give rise to the following relations,

\[
X^i = X_i, \text{ if } 1 \leq i \leq k;
\]

\[
X^\mu = X^{k+j} = -X_{\mu+a}, \text{ if } k + 1 \leq \mu \leq N; a = \pm 1,
\]

where the parameter \( a \) is +1 if \( j \) is odd or −1 if \( j \) is even.

Since that in Eq. (2) the idea of a \( (q = 0, r = 2) \)-tensor at \( p \in \mathcal{G} \) is introduced, we can now generalize it for an arbitrary tensor at \( p \), say \( J_{r,p}^q(\mathcal{G}) \), by the form

\[
T: T^*_p \mathcal{G}^{(1)} \otimes \ldots \otimes T^*_p \mathcal{G}^{(q)} \otimes T_p \mathcal{G}^{(1)} \otimes \ldots \otimes T_p \mathcal{G}^{(r)} \mapsto \mathcal{R},
\]

such that \( T \) is written in terms of the bases as

\[
T = T^{\mu_1 \ldots \mu_q}_{\mu_1 \ldots \mu_q} e_{\mu_1} \otimes \ldots \otimes e_{\mu_q} \otimes e^{\nu_1} \otimes \ldots \otimes e^{\nu_r},
\]
where \( T^{\mu_1\ldots\mu_q}_{\nu_1\ldots\nu_r} = T(e^{\mu_1}, \ldots, e^{\mu_q}, e_{\nu_1}, \ldots, e_{\nu_r}) \).

It is interesting at this point to analyze linear transformations in \( \mathcal{G} \) of type

\[
\mathfrak{g}^\mu = C^\mu_\nu x^\nu + a^\mu, \tag{9}
\]

with \( |\mathcal{G}| = 1 \). Considering infinitesimal transformations, such that \( C^\mu_\nu = \delta^\mu_\nu + \epsilon^\mu_\nu \), we can see that the generators of these transformations are given by the operators

\[
M_{\mu \nu} := -i (x_\mu \partial_\nu - x_\nu \partial_\mu), \tag{10}
\]

\[
P_\mu := -i \partial_\mu, \tag{11}
\]

which are defined in the space of functions at \( p \in \mathcal{G} \). These generators satisfy the Lie algebra,

\[
[M_{\mu \nu}, M_{\rho \sigma}] = -i (\eta_{\nu \rho} M_{\mu \sigma} - \eta_{\mu \rho} M_{\nu \sigma} + \eta_{\mu \sigma} M_{\nu \rho} - \eta_{\nu \sigma} M_{\mu \rho}), \tag{12}
\]

\[
[P_\mu, M_{\rho \sigma}] = -i (\eta_{\mu \rho} P_\sigma - \eta_{\mu \sigma} P_\rho), \tag{13}
\]

\[
[P_\mu, P_\nu] = 0. \tag{14}
\]

Eqs. (12)-(14) will be called a \( g \)-Lie algebra.

For the case of 5-dimensions, in order to rederive the results of Ref. [10], we introduce the notation,

\[
J_{lm} = M_{lm} = -i (x^l \partial_m - x^m \partial_l), \tag{15}
\]

being \( 1 \leq l, m \leq k \). We also have

\[
G_{lm} = M_{k+l,m} = i (x^{k+l+a} \partial_m + x^m \partial_{k+l}), \tag{16}
\]

if \( 1 \leq l \leq N - k \) and \( 1 \leq m \leq k \) \( (a = \pm 1 \text{ if } l \text{ is odd or even, respectively}) \), and finally

\[
D_{lm} = M_{k+l,k+m} = i (x^{k+l+a_l} \partial_{k+m} - x^{k+m+a_m} \partial_{k+l}), \tag{17}
\]

if \( 1 \leq l, m \leq N - k \) \( (a_l, a_m = \pm 1 \text{ if } l, m \text{ are odd or even, respectively}) \).

For the sake of applications in next sections, it is interesting to observe the nature of different embeddings from the Euclidean space \( \mathcal{E} \) into \( \mathcal{G} \). Let us then exemplify two particular embeddings.
The first case of embedding is defined by

$$\mathcal{I}_1 : A \mapsto A = \left( A, \frac{A^{k+1}}{\sqrt{2}}, \frac{A^{k+1}}{\sqrt{2}}, \ldots, \frac{A^{k+N-k}}{\sqrt{2}}, \frac{A^{k+N-k}}{\sqrt{2}} \right),$$

(18)

where $A$ is a vector in the $k$-dimensional Euclidean space ($E^k$). Therefore, from the Eq. (5), the norm of this class of vectors in $\mathcal{G}$ is

$$\|A\|^2 \overset{\mathcal{I}_1}{=} A^2 - \sum_{i=1}^{N-k} (A^{k+i})^2.$$  

(19)

We can have vectors of type $\|A\|^2 = 0$ or $\|A\|^2 \geq 0$. Hence this embedding gives rise to a Minkowski space in $(N-1)$-dimensions, which is a hyperplane in $\mathcal{G}$. In other words

$$A^\mu A_\mu = \eta_{\mu\nu} A^\mu A^\nu \overset{\mathcal{I}_1}{=} A_\mu A^\mu = g_{\mu\nu} A^\mu A^\nu,$$

(20)

where $g_{\mu\nu}$ is just the Minkowski metric, given by $\text{diag} (g_{\mu\nu}) = (-1, 1, \ldots, 1)$ which has $N-1$ entries and $A^{k+1} \equiv A^0$. Then the 5-dim vectors become $A = (A^0, A)$. We will use the embedding $\mathcal{I}_1$ to study representations of the Poincaré Group in phase space.

The second case of embedding to be considered is the following

$$\mathcal{I}_2 : A \mapsto A = \left( A, 0, A^{k+2}, 0, \ldots, A^{k+N-k}, 0 \right).$$

(21)

which results in vectors with norm $(A|A) = \eta_{\mu\nu} A^\mu A^\nu \overset{\mathcal{I}_2}{=} A^2.$

### III. SYMPLECTIC STRUCTURES IN $T^*G$

Let us define a $2N$-dimensional phase space as a manifold $\Gamma$ defined via the following symplectic 2-form $w$,

$$w = dq^\mu \wedge dp_\mu; \quad \mu = 1, 2, ..N,$$

(22)

and by a vector field

$$X_f = \frac{\partial f}{\partial p_\mu} \frac{\partial}{\partial q^\mu} - \frac{\partial f}{\partial q^\mu} \frac{\partial}{\partial p_\mu},$$

(23)
where \( q = (q^1, \ldots, q^N) = (\mathbf{q}, q^{k+1}, \ldots, q^N) \) is a point of \( \mathcal{G} \) (hereafter we use the notation: \( \mathcal{G}_{N,k} \) to indicate the dimension \( N \) of the space and \( k \) standing for the dimension of \( \mathbf{I}_{ij} \) in the metric) characterizing the configuration space in generalized coordinates, and \( f \) is a \( C^\infty \) function in \( \Gamma \), such that a Poisson bracket \( \{ f, g \} \) is introduced by

\[
w(X_f, X_h) = \{ f, h \} = dq^\mu(X_f)dp_\mu(X_h) - dp_\mu(X_f)dq^\mu(X_h),
\]

\[
= \eta^{\mu\nu}\left( \frac{\partial f}{\partial q^\mu} \frac{\partial h}{\partial p^\nu} - \frac{\partial f}{\partial p^\nu} \frac{\partial h}{\partial q^\mu} \right). \tag{24}
\]

On the other hand we have

\[
\{ f, h \} = df(X_h) = \langle df, X_h \rangle. \tag{25}
\]

and

\[
\langle df, X_{\{h,g\}} \rangle + \langle dh, X_{\{g,f\}} \rangle + \langle dg, X_{\{f,h\}} \rangle = 0,
\]

such that

\[
[X_h, X_f] = X_{\{f,h\}}, \tag{26}
\]

inducing then a representation of Lie groups. Considering a Lie algebra characterized by the structure constants \( C^k_{ij} \), the canonical representation can be introduced by following usual methods, that is,

\[
\{ l_i, l_j \} = C^k_{ij}l_k, \tag{27}
\]

where \( l_j \) are functions in \( \Gamma \) corresponding to generators of the Lie symmetries. A canonical representation for such Lie algebra, given by Eqs.(12)-(14), but with Lie product defined by (24) as in (27), is provided by the generators,

\[
M_{\mu\nu} = q_\mu p_\nu - p_\mu q_\nu, \quad \text{and} \quad P_\mu = p_\mu. \tag{28}
\]

On the other hand, the vector fields \( X_f \) and the functions \( f \) induce a unitary representation in the Hilbert space \( \mathcal{H}(\Gamma) \) built up from the complex functions in \( \Gamma \) of type \( L^2(\text{Lebesgue}) \).
integral). Actually, from an arbitrary function \( f \) we can construct unitary operators from the vector field, that is \( \hat{f} = -iX_f \). It is worth to note that \( f \) also induces multiplicative operators of \( \mathcal{H}(\Gamma) \) given by \( \mathcal{T} = 1f \). As a consequence we have

\[
[\hat{M}_{\mu\nu}, \hat{M}_{\rho\sigma}] = -i \left( \eta_{\nu\rho}\hat{M}_{\mu\sigma} - \eta_{\mu\rho}\hat{M}_{\nu\sigma} + \eta_{\mu\sigma}\hat{M}_{\nu\rho} - \eta_{\nu\sigma}\hat{M}_{\mu\rho} \right), \tag{29}
\]

\[
[\hat{P}_\mu, \hat{M}_{\rho\sigma}] = -i \left( \eta_{\mu\rho}\hat{P}_\sigma - \eta_{\mu\sigma}\hat{P}_\rho \right), \tag{30}
\]

\[
[\hat{P}_\mu, \hat{P}_\rho] = 0, \tag{31}
\]

with the auxiliary extra conditions

\[
[\hat{M}_{\mu\nu}, M_{\rho\sigma}] = -i \left( \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\rho}M_{\nu\sigma} + \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\sigma}M_{\mu\rho} \right), \tag{32}
\]

\[
[\hat{P}_\mu, M_{\rho\sigma}] = -i \left( \eta_{\mu\rho}P_\sigma - \eta_{\mu\sigma}P_\rho \right), \tag{33}
\]

\[
[\hat{P}_\mu, P_\rho] = 0, \tag{34}
\]

where \( M_{\rho\sigma} \) and \( P_\sigma \) are given in Eq. (28) and \( \hat{M}_{\mu\nu} \) is

\[
\hat{M}_{\mu\nu} = ip_\nu \frac{\partial}{\partial p^\mu} - ip_\mu \frac{\partial}{\partial p^\nu} + iq_\nu \frac{\partial}{\partial q^\mu} - iq_\mu \frac{\partial}{\partial q^\nu}; \quad \hat{P}_\mu = -i \frac{\partial}{\partial q^\mu}. \tag{35}
\]

We remark introducing unitary operators defined by

\[
\hat{f} = f - \frac{1}{2} \left( q^\mu \frac{\partial f}{\partial q^\mu} + p^\mu \frac{\partial f}{\partial p^\mu} \right) \frac{1}{2} X_f,
\]

which is a generalization of the van Hove bracket \([19]\), Eqs. (29)-(34) are derived with \( \hat{M}_{\rho\sigma} \) given again by (35) and

\[
\hat{P}_\mu = \frac{1}{2} p_\mu - i \frac{\partial}{\partial q^\mu}. \tag{36}
\]

In the next section we use the two embeddings treated at the end of Section 2 in association with this phase space approach.

IV. APPLICATIONS IN KINETIC THEORY
A. Poincaré Group in phase space

For the embedding described by Eq.(18), we have for \( N = 5 \), \( \Im_1 : q \mapsto q = (q, \frac{q_4}{\sqrt{2}}, \frac{q_4}{\sqrt{2}}) \), and \( \Im_1 : p \mapsto p = (p, \frac{p_4}{\sqrt{2}}, \frac{p_4}{\sqrt{2}}) \), with \((q, p) \in \Gamma\). So doing, taking into account Eq.(20), we get

\[
q^\mu q_\mu = \eta^{\mu\nu} q_\mu q_\nu \xrightarrow{\Im_1} q^\mu q_\mu = g^{\mu\nu} q_\mu q_\nu
\]

being now \( \mu, \nu = 0, 1, 2, 3 \) in \( g^{\mu\nu} \). We have with this embedding, \( M_{\mu\nu}, \widehat{M}_{\mu\nu} \) and \( \widehat{P}_\mu (\mu, \nu = 0, 1, 2, 3) \) given formally by Eqs. (28), (35) and (36), and \( \widehat{P}_\mu = p_\mu \xrightarrow{\Im_1} (p_0, p_1, p_2, p_3) \). Eqs.(29)-(34) are then replaced by

\[
\left[ \widehat{M}_{\mu\nu}, \widehat{M}_{\rho\sigma} \right] = -i \left( g_{\nu\rho} \widehat{M}_{\mu\sigma} - g_{\mu\rho} \widehat{M}_{\nu\sigma} + g_{\mu\sigma} \widehat{M}_{\nu\rho} - g_{\nu\sigma} \widehat{M}_{\mu\rho} \right), \\
\left[ \widehat{P}_\mu, \widehat{M}_{\rho\sigma} \right] = -i \left( g_{\mu\rho} \widehat{P}_\sigma - g_{\mu\sigma} \widehat{P}_\rho \right), \\
\left[ \widehat{P}_\mu, \widehat{P}_\rho \right] = 0,
\]

and

\[
\left[ \widehat{M}_{\mu\nu}, M_{\rho\sigma} \right] = -i \left( g_{\nu\rho} M_{\mu\sigma} - g_{\mu\rho} M_{\nu\sigma} + g_{\mu\sigma} M_{\nu\rho} - g_{\nu\sigma} M_{\mu\rho} \right), \\
\left[ \widehat{P}_\mu, M_{\rho\sigma} \right] = -i \left( g_{\mu\rho} P_\sigma - g_{\mu\sigma} P_\rho \right), \\
\left[ \widehat{P}_\mu, P_\rho \right] = 0.
\]

This algebra is a representation of the Poincaré-Lie algebra in the relativistic phase space derived in the context of the classical thermofield dynamics formalism [20]. Some Casimir invariants are given by

\[
I_1 = P^\mu P_\mu, \\
I_2 = \widehat{P}^\mu \widehat{P}_\mu, \\
I_3 = P^\mu \widehat{P}_\mu,
\]

resulting immediately, from \( I_3 \), in the relativistic kinetic equation

\[
p^\mu \partial_\mu \phi(q, p) = 0
\]
without the collision term, where $\phi(q, p)$ is a classical amplitude of probability in phase space $[20]$. The density of probability is given by $f(q, p) = |\phi(q, p)|^2$, still satisfying the same Eq. (46). In Ref. [20], using the notion of propagator, a phenomenological collision term was derived for Eq. (46). Here we would like to emphasize that Eq. (46) can be derived from a variational principle so that interaction terms can be introduced by imposing symmetry as a physical criterious, as we proceed in usual field theory. For instance, from the Lagrangian density

$$L = \frac{1}{2} \phi^*(q, p) p^\mu \partial_\mu \phi(q, p) + \frac{1}{2} \phi(q, p) p^\mu \partial_\mu \phi^*(q, p) - \lambda V(\phi, \phi^*),$$

we obtain the Euler-Lagrange equation

$$p^\mu \partial_\mu \phi(q, p) = \lambda \frac{\partial V(\phi, \phi^*)}{\partial \phi^*}.$$  

A model for $V(\phi)$ preserving the phase invariance $\phi \rightarrow -\phi$ is given by $V(\phi) = \frac{1}{4} \phi^4$, resulting in

$$p^\mu \partial_\mu \phi(q, p) = \lambda \phi^3.$$  

To evaluate the interest of this method, let us consider Eq. (48) in $(1 + 1)$ dimensions, that is

$$(\partial_t + \frac{p^1}{p^0} \partial_x) \phi(x, p) = \frac{\lambda}{p^0} \phi^3.$$  

Assuming a wave-travelling solution,

$$\phi(x, p) \equiv \psi(vt + x).$$

and defining $y = vt + x$, Eq. (48) reads

$$\partial_y \psi = \frac{\lambda}{(vp^0 + p^1)^3} \psi^3.$$  

A solution of this equations is given by

$$\phi(x, p) = \psi(x, p) = \sqrt{\frac{vp^0 + p^1}{\lambda(vt + x)}}.$$
Observe that the probability density function in the phase space is interpreted here by
\( f(q, p) = |\phi(q, p)|^2 \). Therefore we find

\[
  f(x, p) = \frac{vp^0 + p^1}{(vt + x)^\lambda}.
\]

It is worthy noticing the fact that this simple distribution function can not be derived so
trivially, as it was here, by using the usual formalism based on the Boltzmann-like equations
for \( f(x, p) \), since the collision term is often given by cumbersome arguments. In the sequence,
as another application, we use the embedding given in Eq. (21) and the Weyl operators.

**B. Weyl representation for the Electromagnetic waves in dispersive medium**

McDonald and Kaufman [22] have derived a wave kinetic equation governing the behavior
of electromagnetic waves in dispersive medium in the following way. The electric field \( E(q, t) \)
propagating in an inhomogeneous, time-varying medium can be described by the integral
equation

\[
  \int d^3q' dt' D(q, t; q', t') E(q, t) = j(x, t).
\]

Eq. (50) can be seen as an operator equation given by \( D \cdot E = j \). Using its adjoint counter-
part, we can write

\[
  D \cdot (EE^\dagger) = (jj^\dagger)(D^\dagger)^{-1},
\]

which is an equation for the spectral tensor of the electric field \((EE^\dagger)\) written in terms
of the dispersion operator \( D \) and the spectral current-source tensor \((jj^\dagger)\). A phase space
representation for Eq. (51) is then derived by using the Weyl representation for the product
of operators, resulting in [22]

\[
  D\Delta(q, t; k, w)(EE^\dagger) = (jj^\dagger)\Delta(q, t; k, w)(D^\dagger)^{-1},
\]

where
\[ \Delta(q, t; k, w) = \exp \frac{i}{2} \left( \frac{\partial}{\partial q^i} \frac{\partial}{\partial k_i} - \frac{\partial}{\partial k_i} \frac{\partial}{\partial q^i} + \frac{\partial}{\partial w} \frac{\partial}{\partial t} - \frac{\partial}{\partial t} \frac{\partial}{\partial w} \right). \] (53)

The main point here is that we can find a geometric interpretation for the McDonald and Kaufman theory [22]. Let us consider the embedding given in Eq.(21) in 5-dimensions, such that \( \mathcal{F}_2 : q \mapsto q = (q, t, 0) \). The conjugate variable of \( q \), say \( k \), is of the form \( k \mapsto k = (k, 0, w) \). Then, from Eq.(22) we have

\[ w = dq^\mu \wedge dk_\mu \]
\[ = dq^i \wedge dk_i - dt \wedge dw, \] (54)

and

\[ X_f = \frac{\partial f}{\partial k_i} \frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial}{\partial k_i} + \frac{\partial f}{\partial w} \frac{\partial}{\partial t} - \frac{\partial f}{\partial t} \frac{\partial}{\partial w}. \]

such that

\[ w(X_f, X_h) = \{f, h\} \]
\[ = dq^\mu(X_f)dp_\mu(X_h) - dp_\mu(X_f)dq^\mu(X_h) \]
\[ = \eta^{\mu\nu} \left( \frac{\partial f}{\partial q^\mu} \frac{\partial h}{\partial p^\nu} - \frac{\partial f}{\partial p^\nu} \frac{\partial h}{\partial q^\mu} \right) \]
\[ = \frac{\partial f}{\partial q^i} \frac{\partial h}{\partial k_i} - \frac{\partial f}{\partial k_i} \frac{\partial h}{\partial q^i} + \frac{\partial f}{\partial w} \frac{\partial h}{\partial t} - \frac{\partial f}{\partial t} \frac{\partial h}{\partial w}. \] (55)

With this Poisson bracket, we construct a Weyl product of functions in phase space \( f(q, p) \) and \( g(q, p) \) (interpreted as a phase space representation of a product of two corresponding operators \( F \cdot G \)) by

\[ f \ast g = f(q, p) \Delta(q, t; k, w) g(q, p). \] (56)

where \( \Delta(q, t; k, w) \) is given, from Eq.(53), in Eq.(53).

V. THE CONNECTION BETWEEN THE CJ MANIFOLD AND THE ARS-GROUP

Let us consider the \( \mathcal{G} \) manifold in six dimensions, with \( N = 6 \) and \( k = 2 \) (\( \mathcal{G}_{6,2} \)). The metric \( \eta \) is then
\[
(\eta_{\mu\nu}) = \begin{pmatrix}
I_2 & 0 & 0 \\
0 & \mathbb{B}_1 & 0 \\
0 & 0 & \mathbb{B}_2
\end{pmatrix},
\tag{57}
\]

where \( \text{diag}(I_2) = (1,1) \) and \( \mathbb{B}_i \) is given by the Eq. (4). Let us denote a vector in \( \mathcal{G}_{6,2} \) by \( \xi = (\xi^1, ..., \xi^6) \). Through a mapping, \( \eta \) can made partially diagonal, that is introduce the transformation \( U : \mathcal{G}_{6,2} \rightarrow \mathcal{G}_{6,2} \),

\[
\begin{align*}
\xi^i & \mapsto \xi^i, \\
U : \xi^3 & \mapsto \frac{1}{\sqrt{2}} (\xi^4 - \xi^3), \\
U : \xi^4 & \mapsto \frac{1}{\sqrt{2}} (\xi^4 + \xi^3),
\end{align*}
\tag{58}
\]

\[
U : \xi^5 \mapsto \xi^5,
\]

\[
U : \xi^6 \mapsto \xi^6.
\]

These relations can be rewritten as

\[
x^\mu = U^\mu_\nu \xi^\nu
\tag{59}
\]

where \( \mu, \nu = 1, ..., 6 \), the \( x^\mu \) are the new variables, and \( U^\mu_\nu \) are the transformation coefficients given by

\[
(U^\mu_\nu) = \begin{pmatrix}
I_2 & 0 & 0 & 0 \\
0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 0 & I_2
\end{pmatrix},
\tag{60}
\]

with \( U^{-1} = U \).

Considering vectors \( \xi, \zeta \in \mathcal{G}_{6,2} \), and \( x = U\xi \) and \( y = U\zeta \), the inner product between them is

\[
(\xi \mid \zeta) = \eta_{\mu\nu} \xi^\mu \zeta^\nu
= (U^\mu_\alpha \eta_{\mu\nu} U^\nu_\beta) x^\alpha y^\beta
= \omega_{\alpha\beta} x^\alpha y^\beta.
\tag{61}
\]
where \( \omega_{\alpha\beta} = U_\alpha^\mu \eta_{\mu\nu} U_\beta^\nu \), or explicitly,
\[
(\omega_{\alpha\beta}) = \begin{pmatrix}
g_{\mu\nu} & 0 & 0 \\
0 & 0 & -1 \\
0 & -1 & 0
\end{pmatrix},
\]
with \( \text{diag}(g_{\mu\nu}) = (1, 1, 1, -1) \) being the \((3 + 1)\) Minkowski metric; Then, we can write Eq.(61) as
\[
(\xi | \zeta) = (x | y) = x \cdot y - x^4 y^4 - x^5 y^5 - x^6 y^6.
\]
Analyzing linear transformations of type \( \bar{\xi}^\mu = G_\nu^\mu \xi^\nu + \chi^\mu \), such that \(|G| = 1\), with the use of Eq.(59) we obtain
\[
\bar{\xi}^\mu = \bar{G}_\nu^\mu x^\nu + a^\mu,
\]
where \( \bar{G}_\nu^\mu = U_\alpha^\mu G_\alpha^\beta U_\beta^\nu \). Then the generators of these transformations described by Eq.(64) are
\[
\bar{M}_{\mu\nu} = U_\mu^\alpha M_{\alpha\beta} U_\beta^\nu, \quad \bar{A}_\mu = U_\mu^\alpha A_\alpha,
\]
with \( \bar{M} \) standing for the rotations and \( \bar{A} \) for translations. Hence, we get the following Lie algebra
\[
\begin{align*}
[\bar{M}_{\mu\nu}, \bar{M}_{\rho\sigma}] &= -i \left( \omega_{\nu\rho} \bar{M}_{\mu\sigma} - \omega_{\nu\mu} \bar{M}_{\rho\sigma} + \omega_{\mu\sigma} \bar{M}_{\nu\rho} - \omega_{\mu\rho} \bar{M}_{\nu\sigma} \right), \\
[\bar{A}_\mu, \bar{M}_{\rho\sigma}] &= -i \left( \omega_{\mu\rho} \bar{A}_\sigma - \omega_{\mu\sigma} \bar{A}_\rho \right), \\
[\bar{A}_\mu, \bar{A}_\nu] &= 0.
\end{align*}
\]
Let us introduce the following notation for the generators in Eq.(63),
\[
\begin{align*}
\bar{M}_{\mu\nu} &= -\bar{M}_{\nu\mu} := J_{\mu\nu}, \quad \bar{M}_{5\mu} = -\bar{M}_{\mu5} := Q_\mu, \\
\bar{M}_{6\mu} &= -\bar{M}_{\mu6} := N_\mu, \quad \bar{M}_{65} = -\bar{M}_{56} := D, \\
\bar{A}_\mu &= P_\mu, \quad \bar{A}_5 := P_5, \quad \bar{A}_6 := S,
\end{align*}
\]
where now the indices $\mu, \nu$ run as $\mu, \nu = 1, 2, 3, 4$. Using this indices, Eqs.(66)–(68) can then be rewritten in the $(3 + 1)$ Minkowski space, that is

\begin{align}
\left[ J_{\mu\nu}, J_{\rho\sigma} \right] &= -i \left( g_{\nu\rho} J_{\mu\sigma} - g_{\mu\rho} J_{\nu\sigma} + g_{\mu\sigma} J_{\nu\rho} - g_{\nu\sigma} J_{\mu\rho} \right), \\
\left[ P_{\mu}, J_{\rho\sigma} \right] &= -i \left( g_{\mu\rho} P_{\sigma} - g_{\mu\sigma} P_{\rho} \right), \\
\left[ P_{\mu}, P_{\nu} \right] &= \left[ Q_{\mu}, Q_{\nu} \right] = \left[ J_{\mu\nu}, S \right] = \left[ P_{\mu}, S \right] = 0, \\
\left[ Q_{\mu}, J_{\rho\sigma} \right] &= -i \left( g_{\mu\rho} Q_{\sigma} - g_{\mu\sigma} Q_{\rho} \right), \\
\left[ P_{\mu}, Q_{\nu} \right] &= i g_{\mu\nu} P_{5}, \quad \left[ S, Q_{\mu} \right] = i P_{\mu}, \\
\left[ P_{5}, J_{\rho\sigma} \right] &= \left[ P_{5}, Q_{\mu} \right] = \left[ P_{5}, P_{\mu} \right] = \left[ P_{5}, S \right] = 0,
\end{align}

(72)–(77)

We see by means of Eqs.(72)–(77) that the generators $J, P, Q$ and $S$ form a closed algebra, which is nothing but the Lie algebra of the called extended relativistic Galilei group, or $ARS$–group \cite{23–25}. The operators $J_{\mu\nu}$ and $P_{\mu}$ stand for the generators of the Poincaré group, $Q_{\mu}$ are the generators of the “relativistic Galilean boost”, $S$ is the generator of translations in the $x^{6}$ coordinate, interpreted as the proper time, and $P_{5}$ is associated with $l^{-1} 1$, where $l$ is a constant with dimension of length. Following this approach, the particular transformations generated by $J, P, Q$ and $S$ are, from Eq.(64), of the form

\begin{align}
\bar{x}^{\mu} &= \Lambda^{\mu}_{\nu} x^{\nu} + b^{\mu} x^{6} + a^{\mu}, \\
\bar{x}^{5} &= x^{5} + \left( \Lambda^{\mu}_{\nu} x^{\nu} \right) b_{\mu} + \frac{1}{2} b^{\mu} b_{\mu} x^{6} + a^{5}, \\
\bar{x}^{6} &= x^{6} + a^{6}.
\end{align}

(78)–(80)

These Eqs.(78)–(80) are identified as the transformations of the ARS-group, with $\Lambda^{\mu}_{\nu}$ representing the $SO(3,1)$, $b^{\mu}$ standing for the relativistic Galilean boost, and $a^{\mu}$ for the translations. The name “relativistic Galilei group” is justified by the likeness between it and the Galilei group, although it describes relativistic physics. Eq.(79) arises as a compatibility condition, because it gives the isometry of the transformations in $G_{6,2}$. The generator $P_{5}$, which is here an invariant in the subalgebra generated by $J, P, Q$ and $S$, is considered in the $ARS$–approach as a central extension of the group given in Eqs.(78) and Eq.(80).
In order to see the compatibility of this $ARS$–formalism with the usual one, let us derive the scalar representation but from the CJ manifold context, that is using unitary representation for the linear transformations in $G_{6,2}$. From Eqs.(72)-(77) two of the Casimir invariants are given by

\begin{align}
I_1 &= P_\mu P^\mu, \\
I_2 &= P_5.
\end{align}

These invariants $I_1$ and $I_2$ can be used to get physical representations of the $ARS$–group. Considering the simplest case, i.e. a faithful unitary representation in the Hilbert space $\mathcal{H}(G_{6,2})$ of scalar functions with $I_1 = -\partial_\mu \partial^\mu$ and $I_2 = -i\partial_5$, we have from the Schur’s Lemma,

\begin{align}
-\partial_\mu \partial^\mu \Psi &= k \Psi, \\
-i\partial_5 \Psi &= l^{-1} \Psi,
\end{align}

where $\Psi = \Psi(x^1,\ldots,x^6)$ is a scalar function, and $k$ and $l$ are arbitrary constants. So, substituting Eq.(84) in Eq.(83) and setting $k = 0$, we get

\[(\nabla^2 - \partial_4^2 - 2il^{-1}\partial_6) \Psi = 0.\]  

(85)

Fixing on functions of type $\Psi(x) = \Phi(x^1,x^2,x^3,x^4) \chi(x^5,x^6)$ and $\frac{1}{2}im^2l$ as the separation constant, we obtain

\[(\nabla^2 - \partial_4^2 + m^2) \Phi = 0.\]  

(86)

Hence, identifying $\partial_4 \equiv \partial_t$, $m$ being the mass and $\Phi$ the wave function of the particle, Eq.(86) is just the Klein-Gordon equation.

VI. CONCLUDING REMARKS

In this work we have explored the Cangemi-Jackiw space ($\mathcal{G}$), a generalized light-cone manifold. Tensor fields and symplectic structures have been introduced for arbitrary dimensions. In particular, canonical representations of Lie symmetry are defined. Through the
notion of embeddings of the Euclidean space into $\mathcal{G}$ and exploring methods of field theory, we have derived a representation for the Poincaré group in phase space which is closely associated with the relativistic kinetic theory. Using then arguments based on symmetry and covariance, a Lagrangian with an interaction term is written in phase-space. The Euler-Lagrange equations give rise to a model-dependent kinetic equation, which is a counterpart in phase space of the usual $\lambda\phi^4$ theory. A solution for this equation provides a travelling-wave distribution function. The simplicity in the reasoning leading to such a result lies on two ingredients (which are not usual in phase-space theories): the notion of amplitude and a covariant Lagrangian. In this context, different possibilities for the potential $V(\phi)$ in Eq. (47), as for instance $V(\phi) = k \cos(\beta \phi)$, remain to be explored in future works.

The formalism has also been used to provide geometrical basis for the McDonald and Kaufman approach [22], which describes the behavior of the electromagnetic field in dispersive medium. This interpretative result opens a possibility for using their method, under this new geometrical understanding, to consider other complex situations, such as the derivation of kinetic equations for non-abelian fields. Finally, we have employed $\mathcal{G}$ in 6-dimensions to derive the Aghassi-Roman-Santilli group [23–25], which is a kind of Galilei group defined in the $(3 + 1)$-Minkowski space. With these applications the remarkable quiescence of the CJ manifold is made more evident.

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REFERENCES

[1] D. Cangemi and R. Jackiw, Phys. Rev. Lett. 69 (1992) 233.
[2] D. Cangemi and R. Jackiw, Phys. Lett. B 299 (1993) 24.
[3] D. Cangemi and R. Jackiw, Ann. Phys. (N.Y.) 225 (1993) 229.
[4] D. Bazeia and R. Jackiw, Ann. Phys. (N.Y.) 270 (1998) 246.
[5] L. Susskind, Phys. Rev. 165 (1968) 1535.
[6] C. R. Nappi and E. Witten, Phys. Rev. Lett. 71 (1993) 3751; W. Witten, Comm. Math. Phys. 92 (1984) 455.
[7] D.I. Olive, E. Rabinovici, and A Schwimmer, Phys. Lett.B 321 (1994) 361;
[8] K. Sfetsos, Phys. Lett.B 324 (1994) 335.
[9] J. Lukierski, P.C. Stichel, and W.J. Zakrzewski, Ann. Phys. 260 (1997) 224.
[10] M. de Montigny, F. C. Khanna, A. E. Santana, E. S. Santos, J. D. M. Vianna, Ann. Phys (N.Y.) 277 (1999) 144.
[11] C. Duval, G. Burdet, H.P. Künzle, and M. Perrin, Phys. Rev. D 31 (1985) 1841.
[12] H. P. Künzle, Canad. J. Phys. 64 (1986) 185.
[13] H. P. Künzle and C. Duval, Clas. Quant. Grav. 3 (1986) 957.
[14] H. P. Künzle and C. Duval, *Semantical aspects of spacetime theories*, U. Majer and H.-J. Schmidt, eds. (BI-Wissenschaftsverlag, Mannheim, 1994) p. 113.
[15] Y. Takahashi, Fortschr. Phys. 36 (1988) 63.
[16] Y. Takahashi, Fortschr. Phys. 36 (1988) 83.
[17] M. Omote, S. Kamefuchi, Y. Takahashi and Ohnuki, Fortschr. Phys. 37 (1989) 933.
[18] A.E. Santana, F.C. Khanna, and Y. Takahashi, Prog. Theor. Phys. 99 (1998) 327.
[19] L. Van Hove, Proc. R. Acad. Sci. 26 (1951) 1.

[20] M. C. B. Andrade, A. E. Santana, J. D. M. Vianna, J. Phys. A: Math. Gen. 33 (2000) 4015.

[21] E. P. Wigner, Phys. Rev. 40 (1932) 749.

[22] S. W. McDonald and A. N. Kaufman, Phys. Rev. A 32 (1985) 1708.

[23] J. J. Aghassi, P. Roman and R. M. Santilli, Phys. Rev. D 1 (1970) 2753.

[24] J. J. Aghassi, P. Roman and R. M. Santilli, J. Math. Phys. 11 (1970) 2297.

[25] J. J. Aghassi, P. Roman and R. M. Santilli, N. Cimento 5 (1971) 551.