THE DIAMETRAL STRONG DIAMETER 2 PROPERTY OF
BANACH SPACES IS THE SAME AS THE DAUGAVET
PROPERTY

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Abstract. We demonstrate the result stated in the title, thus answering an open question asked by Julio Becerra Guerrero, Ginés López-Pérez and Abraham Rueda Zoca in J. Conv. Anal. 25, no. 3 (2018).

1. Introduction

According to the famous theorem [3] demonstrated by Daugavet in 1963, the norm identity
\[ \|\text{Id} - T\| = 1 + \|T\|, \]
which has become known as the Daugavet equation, holds for compact operators on \( C[0,1] \).

Around 20 years ago the following general concept was introduced [8]: a Banach space \( X \) has the Daugavet property if the equation (1.1) is fulfilled for every rank one operator \( T \in L(X) \) (here and below \( L(X) \) denotes the space of all bounded linear operators \( G: X \to X \)).

Surprisingly, the Daugavet property of \( X \) implies the validity of (1.1) for much wider classes of operators than operators of rank 1: for example, for compact operators, weakly compact operators [8], operators that do not fix a copy of \( \ell_1 \) [13], narrow operators [9], SCD-operators [1], etc.

Although the Daugavet property is of isometric nature, it has a number of linear-topological consequences. For example, a space with the Daugavet property cannot be reflexive, it cannot possess an unconditional basis, and so on. We refer to [10, Section 1.4] for more motivation and the history of the subject.

Let \( B_X \) be the unit ball of a Banach space \( X \). A slice of \( B_X \) is a non-empty part of \( B_X \) that is cut out by a real hyperplane. Given \( x^* \in X^* \) with \( \|x^*\| = 1 \) and \( \alpha > 0 \), denote the corresponding slice as
\[ \text{Slice}(x^*, \alpha) := \{x \in B_X : \text{Re}x^*(x) > 1 - \alpha\}. \]
The Daugavet property of $X$ can be reformulated in terms of slices of the unit ball:

(i) the property holds if and only if for every slice $S$ of $B_X$, every $\varepsilon > 0$ and every $x \in S_X$ there is $s \in S$ with $\|x - s\| > 2 - \varepsilon$.

There are two characterizations more [8, 13], relevant to the subject of this paper, where slices are substituted by relatively weakly open subsets or convex combinations of slices, respectively:

(ii) $X$ has the Daugavet property if and only if for every non-empty relatively weakly open subset $U$ of $B_X$, every $\varepsilon > 0$ and every $x \in S_X$ there is $y \in U$ with $\|x - y\| > 2 - \varepsilon$.

(iii) $X$ has the Daugavet property if and only if for every finite collection $S_1, \ldots, S_n$ of slices of $B_X$ and every collection of positive scalars $\lambda_1, \ldots, \lambda_n$ with $\lambda_1 + \ldots + \lambda_n = 1$ the corresponding convex combination $C = \lambda_1 S_1 + \ldots + \lambda_n S_n$ of slices has the property that for every $\varepsilon > 0$ and every $x \in S_X$ there is $y \in C$ with $\|x - y\| > 2 - \varepsilon$.

In [6] it was asked whether the condition $x \in S_X$ in (i) can be changed to $x \in S \cap S_X$. In order to formalize this question the following definition was introduced:

**Definition 1.1.** A Banach space $X$ is said to be a space with bad projections if and only if for every slice $S$ of $B_X$, every $\varepsilon > 0$ and every $x \in S_X$ there is $s \in S$ with $\|x - s\| > 2 - \varepsilon$. We denote this condition $X \in SBP$, or “$X$ is an SBP space”.

The name is motivated by the fact that $X \in SBP$ if and only if $\|\text{Id} - P\| \geq 2$ for every projection $P \in L(X)$ of rank 1.

It was demonstrated [6] that every absolute sum of two SBP spaces is an SBP space. On the other hand, the Daugavet property is inherited only by $\ell_1$ and $\ell_\infty$ sums, consequently there are SBP spaces that do not have the Daugavet property.

Motivated by this result, Becerra Guerrero, López-Pérez and Rueda Zoca [2] introduced the following concepts.

**Definition 1.2 ([2, Definition 2.1]).** A Banach space $X$ is said to have the diametral diameter two property (DD2P) if for every non-empty relatively weakly open subset $U$ of of $B_X$, every $\varepsilon > 0$ and every $x \in U \cap S_X$ there is $y \in U$ with $\|x - y\| > 2 - \varepsilon$.

Comparing this with the characterization (ii), one easily sees that the Daugavet property implies the DD2P. On the other hand, DD2P is stable under all $\ell_p$ sums [2, Theorem 2.11], so the inverse implication does not hold.

Finally, we are prepared for the main subject of interest of this paper.

**Definition 1.3 ([2, Definition 3.1]).** A Banach space $X$ is said to have the diametral strong diameter two property (DSD2P) if for every convex combination $C$ of slices of $B_X$, every $\delta > 0$ and every $x \in C$ there is $y \in C$ with $\|x - y\| > 1 + \|x\| - \delta$. 
Again, from the characterization (iii) of the Daugavet property, one can deduce [2, Example 3.3] that the Daugavet property implies the DSD2P.

The aim of this article is to demonstrate that the inverse implication is also true, so the diametral strong diameter 2 property of Banach spaces is the same as the Daugavet property. The validity of this inverse implication was suggested in [2, Question 4.1]. This question remained open since 2015, when the arXive preprint of [2] was published, and was mentioned as open problem in [4, 5, 11, 12]. Our result, combined with the known results about the Daugavet property, gives also the positive answers to Questions 4.3 and 4.4 of [2] (the last one was already solved directly in [4]).

2. The main result

Let us start with a useful geometrical concept.

Definition 2.1 ([7, Definition 1.2.9]). Let $X$ be a normed space, $\varepsilon > 0$ and $x, y \in X$. The elements $x, y$ are said to be $\varepsilon$-quasi-codirected if $\|x + y\| \geq \|x\| + \|y\| - \varepsilon$.

Lemma 2.2 ([7, Lemma 1.2.10]). Let $x, y \in X$ be $\varepsilon$-quasi-codirected. Then for every $a, b > 0$ the elements $ax, by$ are $(\varepsilon \max\{a, b\})$-quasi-codirected.

Proof. Without loss of generality we may assume $a \geq b$. Then $a = \max\{a, b\}$ and

$$
\|ax + by\| = \|a(x + y) - (a - b)y\| \geq a\|x + y\| - (a - b)\|y\|
\geq a(\|x\| + \|y\| - \varepsilon) - (a - b)\|y\| = a\|x\| + b\|y\| - a\varepsilon.
$$

□

Theorem 2.3. Let $X$ be a Banach space with the diametral strong diameter two property. Then $X$ has the Daugavet property.

Proof. We are going to demonstrate that $X$ satisfies the characterization (i). Let $\varepsilon > 0$, $x \in S_X$ and a slice $S = \text{Slice}(x^*, \alpha)$ of $B_X$ be given, where $x^* \in X^*$, $\|x^*\| = 1$ and $\alpha > 0$. We need to demonstrate the existence of $s \in S$ with $\|x - s\| > 2 - \varepsilon$. To this end, take $\beta = \min\{\alpha, \varepsilon / 2\}$ and consider the slices

$$
S_1 = \text{Slice}(x^*, \beta) \subset S, \quad S_2 = \text{Slice}(-x^*, \beta) = -S_1
$$

and the convex combination $C$ of slices defined as

$$
C = \frac{1}{2} S_1 + \frac{1}{2} S_2 = \left\{ \frac{1}{2} y_1 - \frac{1}{2} y_2 : y_1, y_2 \in S_1 \right\}.
$$

(2.1)

Remark that every element $z \in S_1$ has $\|z\| \geq \Re x^*(z) > 1 - \beta \geq 1 - \varepsilon / 2$ and $S_1$ has not empty norm-interior

$$
W = \{ z \in X : \|z\| < 1, \Re x^*(z) > 1 - \beta \}.
$$

Then $0 \in \frac{1}{4} W - \frac{1}{4} W \subset C$ is a norm-interior point of $C$, so there is such an $r \in (0, \frac{1}{2})$ that $r B_X \subset C$. Consider $rx \in C$ and $\delta \in (0, \varepsilon / (2r))$. According
to Definition 1.3, there is \( y \in C \) with \( \|rx - y\| > 1 + r\|x\| - \delta \). By (2.1), we may represent \( y \) as \( y = \frac{1}{2}s - \frac{1}{2}h \) with \( s, h \in S_1 \). Then

\[
\left\| rx - \frac{1}{2}s \right\| = \left\| rx - y + \frac{1}{2}h \right\| \geq \|rx - y\| - \frac{1}{2} > r\|x\| + \frac{1}{2} - \delta.
\]

This means that the elements \( rx \) and \(-\frac{1}{2}s\) are \( \delta\)-quasi-codirected. Applying Lemma 2.2, we obtain that the elements \( x \) and \(-s\) are \( \frac{\delta}{r}\)-quasi-codirected, that is

\[
\|x - s\| \geq \|x\| + \|s\| - \frac{\delta}{r} \geq 2 - \epsilon - \frac{\delta}{r} > 2 - \epsilon.
\]

Since \( s \in S_1 \subset S \), this completes the proof. \( \square \)

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