Hypergraph Limits: A Regularity Approach

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ABSTRACT: A sequence of k-uniform hypergraphs $H_1, H_2, \ldots$ is convergent if the sequence of homomorphism densities $t(F, H_1), t(F, H_2), \ldots$ converges for every k-uniform hypergraph $F$. For graphs, Lovász and Szegedy showed that every convergent sequence has a limit in the form of a symmetric measurable function $W : [0, 1]^2 \rightarrow [0, 1]$. For hypergraphs, analogous limits $W : [0, 1]^{2k-2} \rightarrow [0, 1]$ were constructed by Elek and Szegedy using ultraproducts. These limits had also been studied earlier by Hoover, Aldous, and Kallenberg in the setting of exchangeable random arrays. In this paper, we give a new proof and construction of hypergraph limits. Our approach is inspired by the original approach of Lovász and Szegedy, with the key ingredient being a weak Frieze-Kannan type regularity lemma. © 2014 Wiley Periodicals, Inc. Random Struct. Alg., 47, 205–226, 2015

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1. INTRODUCTION

One of the starting points in the theory of dense graph limits is the seminal paper by Lovász and Szegedy [13] where they constructed limit objects for convergent sequences of dense graphs. The subject has grown enormously since then with many exciting developments (see Lovász’s recent monograph [12]).

For any two graphs $F$ and $G$, let $\text{hom}(F, G)$ denote the number of homomorphism from $F$ to $G$, i.e., maps $V(F) \rightarrow V(G)$ that carry every edge of $F$ to an edge of $G$. The homomorphism density $t(F, G)$ is defined to be the probability that a random map $V(F) \rightarrow V(G)$ is a homomorphism, i.e.,

$$t(F, G) := \frac{\text{hom}(F, G)}{|V(G)|^{\binom{|V(F)|}{2}}}.$$
A sequence of graphs $G_1, G_2, \ldots$ is called *convergent* if the sequence $t(F, G_1), t(F, G_2), \ldots$ converges for every graph $F$. Convergent graph sequences were defined and studied in [4, 5]. The main result of Lovász and Szegedy [13] is that for every convergent graph sequence there is a limit object in the form of a *graphon*, which is a symmetric measurable function $W : [0, 1]^2 \to [0, 1]$ (here symmetric means that $W(x, y) = W(y, x)$) such that $t(F, G_n) \to t(F, W)$ as $n \to \infty$ for all graphs $F$. Here $t(F, W)$ is defined by

$$t(F, W) := \int_{[0,1]^{V(F)}} \prod_{i \in E(F)} W(x_i, x_j)dx_1dx_2 \cdots dx_{|V(F)|}$$

The natural extension of these limits to hypergraphs was considered by Elek and Szegedy [7]. They constructed using ultraproducts an “ultralimit hypergraph” for any sequence of hypergraphs, and established a correspondence principle which enabled them to convert statements about finite hypergraphs, such as hypergraph regularity and removal lemmas [9, 15, 16], to measure-theoretic claims about ultralimit spaces. One of the consequences of their work is the existence of a limit object in the form of a measurable function $W : [0, 1]^{k^2-2} \to [0, 1]$ for any convergent sequence of $k$-uniform hypergraphs.

These limit objects had actually appeared earlier in a different form, in the study of exchangeable random arrays, initiated by Hoover [10], Aldous [1], and Kallenberg [11] during the 1980s, building on the classic de Finetti’s theorem on exchangeable random variables. This connection is explained in the survey [3] by Austin, where he credits Tao [17] for initiating the link between exchangeable random variables and hypergraphs. These connections for graphs are also explained in the survey by Diaconis and Janson [6] as well as Aldous’ ICM talk [2].

The purpose of this paper is to provide a new proof of the existence of hypergraph limits. Our approach is based on weak Frieze-Kannan [8] type regularity partitions, in line with mainstream perspectives on dense graph limits. The proof does not use any exchangeable random variables or ultraproducts, and the construction of the limit is subjectively more concrete than earlier proofs. Our proof is inspired by the original approach of Lovász and Szegedy [13], and the paper is self-contained other than an application of the Martingale Convergence Theorem.

### 1.1. Convergence and Limit Object

For any $k$-uniform hypergraphs $F$ and $H$, let $\text{hom}(F, H)$ denote the number of homomorphisms from $F$ to $H$, i.e., maps $V(F) \to V(H)$ that carry every edge of $F$ to an edge of $H$. Define $t(F, H) := \text{hom}(F, H)/|V(H)|^{|V(F)|}$. This is the probability that a random map $V(F) \to V(H)$ is a homomorphism.

**Definition 1.1** (Convergence). A sequence of $k$-uniform hypergraphs $H_1, H_2, \ldots$ is called *convergent* if the sequence $t(F, H_1), t(F, H_2), \ldots$ converges for every $k$-uniform hypergraph $F$.

For any positive integer $n$, define $[n] := \{1, 2, \ldots, n\}$. For any set $A$, define $r(A)$ to be the collection of all nonempty subsets of $A$, and $r_+(A)$ to be collection of all nonempty proper subsets of $A$. More generally, let $r(A, m)$ denote the collection of all nonempty subsets of $A$ of size at most $m$. So for instance, $r_+(\{k\}) = r(\{k\}, k - 1)$. We will also use the shorthand $r[k]$ and $r_+(k)$ to mean $r(\{k\})$ and $r_+(\{k\})$ respectively.
Any permutation $\sigma$ of a set $A$ induces a permutation on $r(A,m)$. We say that a function $W : [0, 1]^{r([k],m)} \to [0, 1]$ is symmetric if it remains invariant under any permutation of the coordinates induced by any permutation of $[k]$. For example, $W : [0, 1]^{r([3],0)} \to [0, 1]$ being symmetric means that

$$W(x_1, x_2, x_3, x_{12}, x_{13}, x_{23}) = W(x_{\sigma_1}, x_{\sigma_2}, x_{\sigma_3}, x_{\sigma_1 \sigma_2}, x_{\sigma_1 \sigma_3}, x_{\sigma_2 \sigma_3})$$

(1)

for any permutation $\sigma$ of $\{1, 2, 3\}$. Here we write $x_i$ for $x_{[i]}$ and $x_{ij}$ for $x_{[i,j]}$.

**Definition 1.2.** A $k$-uniform hypergraphon is a symmetric measurable function $W : [0, 1]^{r([k],0)} \to [0, 1]$.

**Example 1.3.** A 3-uniform hypergraphon is a measurable function $W : [0, 1]^6 \to [0, 1]$ satisfying the symmetry condition (1).

For any $k$-uniform hypergraph $F$ and hypergraphon $W$, define the homomorphism density by

$$t(F, W) := \int_{[0, 1]^{r(V(F),k-1)}} \prod_{A \in E(F)} W(x_{r(A)})dx$$

Our convention throughout the paper is that if $x = (x_A : A \subseteq A) \in [0, 1]^A$ is a vector whose coordinates are indexed by some set system $A$, and $B \subseteq A$ is a subcollection, then we write $x_B = (x_B : B \in B) \in [0, 1]^B$ to mean the restriction of the vector to the coordinates indexed by $B$.

**Example 1.4.** If $K_4^{(3)} = \{123, 124, 134, 234\}$ is the complete 3-uniform hypergraph on 4 vertices and $W$ is a 3-uniform hypergraphon, then

$$t(K_4^{(3)}, W) = \int_{[0,1]^{10}} W(x_1, x_2, x_3, x_{12}, x_{13}, x_{23})W(x_1, x_2, x_4, x_{12}, x_{14}, x_{24})W(x_1, x_3, x_4, x_{13}, x_{14}, x_{34}) \cdot W(x_2, x_3, x_4, x_{23}, x_{24}, x_{34})dx_1dx_2dx_3dx_4dx_{12}dx_{13}dx_{14}dx_{23}dx_{24}dx_{34}.$$
1.2. Why are There so Many Coordinates?

It may initially seem somewhat strange that we need 6 coordinates to describe the limit of 3-uniform hypergraphs, whereas every 3-uniform hypergraph can be described in terms of a 3-dimensional adjacency array. These extra dimensions do not arise for limits of graphs, but they are essential for hypergraphs. Here is a standard example illustrating why functions of 3-dimensional adjacency array. These extra dimensions do not arise for limits of graphs, but for 3-uniform hypergraphs, whereas every 3-uniform hypergraph can be described in terms of a 3-coordinate for every proper subset of \([0, 1]\). Then with probability one, \(t(F, H_n) \to 2^{-|\partial F|}\) for every 3-uniform hypergraph \(F\), where \(\partial F\) is the collection of unordered pairs of vertices of \(F\) that are contained in some edge of \(F\). The limit of \(H_n\) is different from, say, the constant hypergraphon 1/2, which is the limit of a sequence of 3-uniform hypergraphs where every triple of vertices is taken to be an edge independently with probability 1/2. To describe the limit of \(H_n\), we need to incorporate the limit of \(G_n\) into the data, and this is achieved by the three extra coordinates.

We know that the graph sequence \(G_n\) converges to the constant graphon with value 1/2. To build the limit of \(H_n\), we partition each of the last three coordinates, \(x_{12}, x_{13}, x_{23}\) into two intervals \([0, 1/2] \cup (1/2, 1]\), corresponding to the limit of \(G_n\) and the limit of its complement. The limiting hypergraphon has constant value 1 on \([0, 1]^3 \times [0, 1/2]^3\) (as the edges of \(H_n\) are supported on \(G_n\)) and 0 elsewhere. Intuitively, the first three coordinates encode the vertex types, the last three coordinates encode the vertex-pair types. This hypergraphon is \([0, 1]\)-valued since it is deterministic once the vertex and vertex-pairs types are set. If we modify the sequence \(H_n\) so that each triangle of \(G_n\) is included as an edge of \(H_n\) with some probability \(p\) independently, then the limiting hypergraphon would be constant \(p\) on \([0, 1]^3 \times [0, 1/2]^3\) and 0 elsewhere.

For \(k\)-uniform hypergraphs, we can similarly impose some structure at each level, corresponding to \(j\)-element subsets of vertices, for every \(1 \leq j \leq k\). This is why we need a coordinate for every proper subset of \([k]\) to describe hypergraph limits.

1.3. Random Hypergraph Model

To further illustrate the involvement of the \(2^k - 2\) coordinates in a hypergraphon, let us review the associated random hypergraph model.

Recall that if \(W : [0, 1]^2 \to [0, 1]\) is a graphon, then we have the following natural random graph model \(G(n, W)\) on \(n\) vertices: choose i.i.d. uniform \(x_1, x_2, \ldots, x_n \in [0, 1]\), and let there be an edge between vertices \(i\) and \(j\) with probability \(W(x_i, x_j)\) independently. It was shown \([13, \text{Cor. 2.6}]\) using Azuma’s inequality that \(G(n, W)\) converges to the limit \(W\) almost surely.

Similarly, a \(k\)-uniform hypergraphon \(W\) gives a natural model \(G(n, W)\) of a random \(k\)-uniform hypergraph on \(n\) vertices: choose a uniformly random \(x \in [0, 1]^{\binom{n}{k}} - 1\) and add the edge \(B = \{i_1, \ldots, i_k\} \subseteq [n]\) with probability \(W(x_{i_1}, \ldots, x_{i_k})\) independently. Essentially the same proof for graphs extend over to show \([7, \text{Thm. 11}]\) that \(G(n, W)\) converges to \(W\) in the sense of Theorem 1.5, as \(n \to \infty\) with probability one. Observe that the random hypergraphs \(H_n\) of triangles in \(G(n, 1/2)\) discussed earlier is a special case of this model.

1.4. Analytic Version and Compactness

It will be convenient to prove an analytic version of Theorem 1.5. We say that a sequence of \(k\)-uniform hypergraphons \(W_1, W_2, \ldots\) is convergent if the sequence \(t(F, W_1), t(F, W_2), \ldots\) converges for every \(k\)-uniform hypergraph \(F\).
Theorem 1.6. If $W_1, W_2, \ldots$ is a convergent sequence of $k$-uniform hypergraphons, then there exists a $k$-uniform hypergraphon $\tilde{W}$ so that $t(F, W_n) \to t(F, \tilde{W})$ as $n \to \infty$ for every $k$-uniform hypergraph $F$.

In this case we say that $W_n$ converges to $\tilde{W}$. Here is an equivalent formulation of the theorem.

Theorem 1.7. Every sequence $W_1, W_2, \ldots$ of $k$-uniform hypergraphons contains a subsequence that converges to some $k$-uniform hypergraphon $\tilde{W}$.

Theorem 1.7 implies Theorem 1.6 trivially since we can just take the limit $\tilde{W}$ produced by Theorem 1.7. The converse is true because $[0, 1]^N$ is sequentially compact, so we can restrict $(W_n)$ to some subsequence $(W_{n_i})$ so that $t(F, W_{n_i})$ converges as $i \to \infty$ for every $F$.

We shall prove Theorem 1.7 with respect to another notion of convergence based on regular partitions, which implies the convergence of homomorphism densities. The partition-based convergence gives some structural insight into the convergence of hypergraphs.

There is a neat interpretation of Theorem 1.7 in terms of compactness, discovered by Lovász and Szegedy [14] in the case of graphons. Let $\mathcal{W}_0^{(k)}$ denote the set of $k$-uniform hypergraphons. Give $\mathcal{W}_0^{(k)}$ the weakest topology for which the functions $t(F, \cdot)$ are continuous for every $k$-uniform hypergraph $F$. Identify $W$ with $W'$ if $t(F, W) = t(F, W')$ for every $k$-uniform hypergraph $F$. Call this topology the *left-convergence topology* of $\mathcal{W}_0^{(k)}$.

Corollary 1.8. The space $\mathcal{W}_0^{(k)}$ with the left-convergence topology is compact.

Proof. The space is metrizable with the metric $\delta(W, W') = \sum_{i \geq 1} 2^{-i} |t(F_i, W) - t(F_i, W')|$ where $(F_i)$ is some enumeration of all isomorphism classes of $k$-uniform hypergraphs. We know that compactness is equivalent to sequential compactness in metric spaces, and Theorem 1.7 shows that the space is sequentially compact.

When $k = 2$, Lovász and Szegedy [14] showed that $\mathcal{W}_0^{(2)}$ is compact under the cut metric topology, and Borgs, Chayes, Lovász, Sós, and Vesztergombi [4] showed that the cut metric topology is equivalent to the left-convergence topology. Lovász and Szegedy interpreted the compactness with respect to the cut metric as an analytic form of the regularity lemma, and they showed that the compactness of the space of graphons implies strong versions of the regularity lemma. Unfortunately, for $k \geq 3$, we do not know of a useful extension of the cut metric to hypergraphs (and there may be some reasons to believe that such a natural metric might be too much to ask for). This is one of the main obstacles in working with convergence of hypergraphs. It would be nice to have a simple and useful description of distance between hypergraphs which agrees with the topology induced by homomorphism densities.

1.5. Organization

In §2 we review the Lovász-Szegedy construction of graph limits. In §3 we give an informal sketch of the proof of the existence of $3$-uniform hypergraph limits. Most of the ideas, minus the technical hairiness, are contained in §3. The proof of the main result is contained in §4–6. §4 collects some of the notation used in the proof. §5 contains the regularity...
2. LIMITS OF GRAPHONS

For any symmetric measurable function \( W : [0, 1]^2 \to \mathbb{R} \), the cut norm is defined by

\[
\|W\|_\square := \sup_{S, T \subseteq [0, 1]} \left| \int_{S \times T} W(x, y) dx dy \right|,
\]  

(2)

where \( S \) and \( T \) range over all measurable subsets of \([1]\). We have the identity

\[
\|W\|_\square = \sup_{u, v : [0, 1] \to [0, 1]} \left| \int (u(x) - W(x, y)) v(y) dx dy \right|
\]  

(3)

where \( u \) and \( v \) range over all measurable functions \([0, 1] \to [0, 1]\). Indeed, since the integral in (3) is linear in both \( u \) and \( v \), one can restrict to \([0, 1]\)-valued \( u \) and \( v \), thereby reducing to (2).

Recall that a graphon is a symmetric measurable function \( W : [0, 1]^2 \to [0, 1] \). For any measure preserving bijection \( \phi : [0, 1] \to [0, 1] \) and any graphon \( W \), define \( W^\phi \) by \( W^\phi(x, y) = W(\phi(x), \phi(y)) \). We define the cut distance between graphons by

\[
\delta_\square(U, W) = \inf_{\phi} \|U^\phi - W\|_\square,
\]

where the infimum is taken over all measure preserving bijections \( \phi : [0, 1] \to [0, 1] \). The cut distance can be defined for pairs of graphs by considering their associated graphons. Graphs that are close in cut distance are also close in homomorphism densities, by the following counting lemma.

**Lemma 2.1 (Counting lemma).** For any graphons \( U \) and \( W \) and any graph \( F \), we have

\[
|t(F, U) - t(F, W)| \leq e(F) \|U - W\|_\square
\]

where \( e(F) \) is the number of edges of \( F \).

We illustrate the proof through the example \( F = K_3 \).

\[
t(K_3, U) - t(K_3, W) = \int_{[0, 1]^3} (U(x, y)U(x, z)U(y, z) - W(x, y)W(x, z)W(y, z)) dx dy dz
\]

\[
= \int_{[0, 1]^3} (U(x, y) - W(x, y))W(x, z)W(y, z) dx dy dz
\]

\[
+ \int_{[0, 1]^3} U(x, y)(U(x, z) - W(x, z))W(y, z) dx dy dz
\]

\[
+ \int_{[0, 1]^3} U(x, y)U(x, z)(U(y, z) - W(y, z)) dx dy dz
\]
Each of the three terms in the final sum is bounded in absolute value by \( \|U - W\|_\square \). For example, for the first term, for every fixed value of \( z \), the integral has the form (3), and so it is bounded in absolute value by \( \|U - W\|_\square \), and the same bound holds after integrating \( z \) by the triangle inequality. It follows that 
\[
|t(K_3, U) - t(K_3, W)| \leq 3\|U - W\|_\square.
\]

For any graphon \( W \) and any partition \( Q \) of the interval \([0, 1]\) into a finite collection of measurable subsets, let \( W_Q \) be a graphon which is the step function obtained from \( W \) by replacing its value at \((x, y) \in Q_i \times Q_j\) by the average value of \( W \) on \( Q_i \times Q_j \), for any \( Q_i, Q_j \in Q \). (if either \( Q_i \) or \( Q_j \) has measure zero, then assign value 0 on \( Q_i \times Q_j \)). For graphs, think of \( Q \) as a partition of the vertex set, and \( W_Q \) as recording the edge densities between pairs of vertex subsets.

A key tool in the construction of graph limits is the following weak regularity lemma due to Frieze and Kannan [8] (see also [14, Lem 3.1]). It can be proved by an \( L^2 \)-energy increment argument.

**Lemma 2.2.** (Weak regularity lemma). For every \( \varepsilon > 0 \) and every symmetric measurable function \( W : [0, 1]^2 \to [0, 1] \), there is some partition \( Q \) of \([1]\) into at most \( 2^{2/\varepsilon^2} \) parts such that \( \|W - W_Q\|_\square \leq \varepsilon \).

Lovász and Szegedy [14] showed that with respect to the cut metric, after identifying graphons with cut distance zero, the space of all graphons is compact. Equivalently:

**Theorem 2.3.** (Lovász and Szegedy [14]). Every sequence \( W_1, W_2, \ldots \) of graphons contains a subsequence converging to some graphon \( \tilde{W} \) in cut distance.

Let us recall the idea of the proof of Theorem 2.3. Let \( \varepsilon > 0 \). We apply the weak regularity lemma to approximate every \( W_n \) by some \((W_n)_{Q_n}\). By replacing each \( W_n \) by some \( W_n^{\phi_n} \) for some measure preserving bijection \( \phi_n \), we may assume that the partition \( Q_n \) divides \([0, 1]\) into intervals. Take a subsequence so that the lengths of the intervals converge, and the values of \((W_n)_{Q_n}\) inside the boxes induced by the partition also converge, i.e., the value inside the \((i, j)\)-th box of \((W_n)_{Q_n}\) converges to some value as \( n \to \infty \) (may be different limits for different \((i, j)\)). Then in this subsequence, \((W_n)_{Q_n}\) converges pointwise almost everywhere to some limit \( \tilde{U}_1 \), which is also a step function.

Now repeat the same procedure with a smaller \( \varepsilon' < \varepsilon \). We obtain new partitions \( Q_n' \) which are refinements of previous partitions. Call the resulting limit \( \tilde{U}_2 \). Note that steps of \((W_n)_{Q_n'}\) are refinements of the steps of \((W_n)_{Q_n}\), and the values of the latter can be obtained from the former by averaging over each step. Thus a similar relation holds for \( \tilde{U}_2 \) and \( \tilde{U}_1 \).

Now, we repeat this procedure for a sequence of \( \varepsilon_k \) tending to zero. We obtain a sequence \( \tilde{U}_1, \tilde{U}_2, \ldots \) of step functions so that each \( \tilde{U}_i \) can be obtained from \( \tilde{U}_{i+1} \) by average over each step. It follows that if \((X, Y)\) is a uniform random point in \([0, 1]^2\), then the sequence \((\tilde{U}_1(X, Y), \tilde{U}_2(X, Y), \ldots)\) is a martingale. Since every \( \tilde{U}_i \) is bounded, the Martingale Convergence Theorem\(^1\) implies that the martingale converges with probability 1, and hence there is some \( \tilde{W} : [0, 1]^2 \to [0, 1] \) which is the pointwise almost everywhere limit of \( \tilde{U}_i \)’s. One then checks that \( \tilde{W} \) is the desired limit.

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\(^1\) The Martingale Convergence Theorem (see [18, Thm. 11.5]) says that every \( L^1 \)-bounded martingale converges almost surely. Our martingales are actually bounded uniformly within \([0, 1]\).

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In summary, the above proof consists of two main steps:

1. For each error tolerance $\epsilon$, apply a weak regularity lemma to get a finite-dimensional step function approximation of each graphon. Take a subsequence so that the step functions converge.
2. Take a decreasing sequence of $\epsilon$ tending to zero, we obtain refining chains of regularity partitions, and the corresponding subsequential limits $\hat{U}_s$ form a martingale. The existence of the final limit graphon follows by the Martingale Convergence Theorem.

3. LIMITS OF 3-UNIFORM HYPERGRAPHS

In this section we sketch the idea for 3-uniform hypergraph limits. To keep things simple, consider a sequence $H_1, H_2, \ldots$ of 3-uniform hypergraphs (as opposed to hypergraphons).

We begin with an initial attempt that does not quite work. For a 3-variable function $W : [0,1]^3 \to \mathbb{R}$, we might extend the cut norm (5) as follows (assume everything is measurable from now on):

\[
\|W\|_0 = \sup_{R,S,T \subseteq [0,1]} \left| \int_{R \times S \times T} W(x,y,z)dx dy dz \right|.
\]  

(4)

For each hypergraph $H$, one can easily extend the weak regularity lemma, Lemma 2.2, to obtain a partition $Q$ of the vertex set of $H$ into at most $2^{3/2}$ parts so that $\|W^H - W^H_Q\|_0 \leq \epsilon$ (regard $W^H$ as a 3-variable function for now, and $W^H_Q$ is derived from $W$ by averaging over each cells induced by $Q$). Theorem 2.3 also extends with virtually no change in the proof. That is, allowing permutations of vertices, some subsequence of $H_n$ converges with respect to the vertex-cut norm (4) to a 3-variable symmetric function $\bar{W} : [0,1]^3 \to [0,1]$.

Unfortunately, the vertex-cut norm (4) is not strong enough to guarantee a counting lemma. We want to say that if $H_1$ and $H_2$ are close with respect to some cut norm, then $t(F,H_1)$ and $t(F,H_2)$ are close. If we carry through the proof of Lemma 2.1, we find that $|t(F,H_1) - t(F,H_2)| \leq \epsilon(F)\|W^{H_1} - W^{H_2}\|_0$ holds when $F$ is a linear hypergraph, i.e., where every two edges of $F$ intersect in at most one vertex. However, when $F$ is not linear, say $F = K_4^{(3)}$, then this claim is completely false, as $t(F,H_1)$ and $t(F,H_2)$ can be separated even when $\|W^{H_1} - W^{H_2}\|_0$ is small. A counterexample for 3-uniform hypergraphs can be built by taking triangles of the random graph $G(n,p)$, and then keeping each triangle as a 3-uniform edge with some probability $q$. With parameters $(p,q) = (1/2, 1)$ and $(1,1/8)$, we obtain 3-uniform hypergraphs that are close with respect to the vertex-cut norm, and yet they have very different $K_4^{(3)}$ densities.

Now let us scrap the vertex-cut norm (4). The proof of the counting lemma, Lemma 2.1, extends with respect to the following modified cut norm (again we use a 3-variable $W$ for now):

\[
\|W\|_2 = \sup_{u,v,w : [0,1]^3 \to [0,1]} \left| \int_{[0,1]^3} W(x,y,z)u(x,y)v(x,z)w(y,z)dx dy dz \right|.
\]  

(5)

For this cut norm, the counting lemma $|t(F,H_1) - t(F,H_2)| \leq \epsilon(F)\|W^{H_1} - W^{H_2}\|_2$ holds. However, like trying to fit a large rug in a small room, we quickly run into another
issue: this norm is too strong and we do not have the compactness result corresponding to Theorem 2.3. Indeed, taking the sequence $H_n$ of triangles of $\mathbb{G}(n, 1/2)$ from §1.2, the two hypergraphs $H_n$ and $H_m$ are typically not close with respect to $\| \cdot \|_{WH}$, although they are close in homomorphism densities.

Even though we do not have compactness with respect to $\| \cdot \|_{WH}$, we can still hope for a slightly weaker topology that gives convergence of homomorphism densities. We can extend the weak regularity lemma, Lemma 2.2, to $\| \cdot \|_{WH}$, where now instead of partitioning the vertex set $V = V(H)$, we partition the edges of the underlying complete graph $K_V = (\mathbb{Y})$, i.e., the collection of unordered pairs of $V$. So now $\mathbb{Q}$ is a partition $K_V = G_1 \cup \cdots \cup G_m$ of the edges of $K_V$ into $m$ graphs. The partition $\mathbb{Q}$ of $K_V$ induces a partition $\mathbb{Q}^*$ on triplets of vertices:

$$(x, y, z) \sim_{\mathbb{Q}^*} (x', y', z') \iff (x, y) \sim_{\mathbb{Q}} (x', y'), (x, z) \sim_{\mathbb{Q}} (x', z'), \text{ and } (y, z) \sim_{\mathbb{Q}} (y', z').$$

Being somewhat sloppy with notation for the time being, we can form $W^H_{n}$ by averaging $W^H$ inside each cell of $\mathbb{Q}^*$. Then the weak regularity lemma guarantees us a partition $\mathbb{Q}$ of $K_V$ into at most $2^{1/\epsilon^2}$ parts so that $\| W^H - W^H_{n} \|_{WH} \leq \epsilon$, and $| t(F, W^H) - t(F, W^H_{n}) | \leq e(F) \epsilon$ by the counting lemma.

For each hypergraph in the sequence $H_1, H_2, \ldots,$ apply the weak regularity lemma (for a uniform $\epsilon$) to obtain a partition $\mathbb{Q}_n$ of the complete graph on $V (H_n)$ into $m$ graphs:

$$K_{V(H_n)} = G_{n,1} \cup \cdots \cup G_{n,m},$$

where $m$ depends on $\epsilon$ but not on $n$.

By applying Theorem 2.3 on the graph sequence $(G_{n,1})_{n \geq 1}$, we can find a graphon $\tilde{Y}_1 : [0, 1]^2 \to [0, 1]$ so that $G_{n,1}$ converges to $\tilde{Y}_1$ as $n \to \infty$ along some subsequence. By further restricting to subsequences, we can find a $\tilde{Y}_j$ for each $1 \leq j \leq m$ so that $G_{n,j}$ converges to $\tilde{Y}_j$ as $n \to \infty$ along a subsequence.

For each $n$, $(G_{n,1}, \ldots, G_{n,m})$ is a partition of $K_{V(H_n)}$, so the same holds for the resulting limit $\mathbb{Q}$ in the sense that $\tilde{Y}_1 + \cdots + \tilde{Y}_m = 1$ almost everywhere as functions $[0, 1]^2 \to [0, 1]$. Next we build a partition $\hat{\mathbb{Q}}$ of the cube $[0, 1]^3 = [0, 1]^{\mathbb{Y}_2}$ (coordinates indexed by $x_1, x_2, x_{12}$) by stacking together subsets whose heights are given by $\tilde{Y}_j$. More precisely, $\hat{\mathbb{Q}} = \{\hat{\mathbb{Q}}_1, \ldots, \hat{\mathbb{Q}}_m\}$ where

$$\hat{\mathbb{Q}}_j = \{(x_1, x_2, x_{12}) \in [0, 1]^3 : (\tilde{Y}_1 + \cdots + \tilde{Y}_{j-1})(x_1, x_2) \leq x_{12} < (\tilde{Y}_1 + \cdots + \tilde{Y}_j)(x_1, x_2)\}.$$

This is the first place where the “extra” coordinates such as $x_{12}$ arise even though we started with hypergraphs not requiring these extra coordinates. They arise because the limit graphon $\tilde{Y}_1$ of a sequence of graphs $G_{n,1}$ is not always a $\{0, 1\}$-valued function.

The partition $\hat{\mathbb{Q}}$ of $[0, 1]^{\mathbb{Y}_2}$ induces a partition $\hat{\mathbb{Q}}^*$ of $[0, 1]^6 = [0, 1]^3_{\mathbb{Y}_2}$:

$$(x_1, x_2, x_3, x_{12}, x_{13}, x_{23}) \sim_{\hat{\mathbb{Q}}^*} (x'_1, x'_2, x'_3, x'_{12}, x'_{13}, x'_{23}) \iff (x_i, x_j, x_{ij}) \sim_{\hat{\mathbb{Q}}}(x'_i, x'_j, x'_{ij}) \forall 1 \leq i < j \leq 3.$$

The partition $\hat{\mathbb{Q}}^*$ should not be viewed as a regularization partition for any $H_n$ (indeed, the extra coordinates do not even appear in $H_n$). Instead, the partitions $\mathbb{Q}_n$ themselves become increasing close to $\hat{\mathbb{Q}}$. There is a correspondence of cells of $\mathbb{Q}_n$ with those of $\hat{\mathbb{Q}}$, and this induces a correspondence between cells of $\mathbb{Q}_n^*$ with those of $\hat{\mathbb{Q}}^*$.

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2 Provided that the limits of the various graph sequences are taken in a compatible way. This is a source of technical/notational annoyance later on, and it is the reason for introducing branching partitions in §6.

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Now we construct the first limiting hypergraphon $\tilde{U}_1$ as a step function $[0, 1]^6 \to [0, 1]$ that is constant on each part of $\tilde{Q}^*$. On each part of $\tilde{Q}^*$, we assign to $\tilde{U}_1$ the limiting value of the average of $W_n$ on the corresponding cell of $Q_n^*$, limit taken as $n \to \infty$ along a further restricted subsequence. We have constructed $\tilde{U}_1$, which plays a similar role as $\tilde{U}_1$ near the end of §2.

However, unlike §2, $\tilde{U}_1$ is not close in $\| \cdot \|_{\mathbb{F}^2}$ to $H_n$ for large $n$. It is a limit in the following sense: we first $\epsilon$-regularized $H_n$, and then took the graph limit of the partitions, created a new partition of $[0,1]^6$ using these lower order limits, and then constructed a step-function $U_1$ using this limiting partition and the limiting values on the steps. We knew from the earlier counting lemma (referred to later on as Counting Lemma I) that

$$|t(F, H) - t(F, W_{Q_n}^H)| \leq e(F)\epsilon. \quad (6)$$

By what we will call Counting Lemma II, we have (here $n \to \infty$ along a subsequence)

$$\lim_{n \to \infty} t(F, W_{Q_n}^H) = t(F, \tilde{U}_1). \quad (7)$$

Here is some intuition why (7) holds. Both $W_{Q_n}^H$ and $\tilde{U}_1$ are step functions. We can split them up into weighted sums of indicator functions, on which the claim reduces to checking homomorphism densities for the graphons corresponding to parts of the partitions $Q_n$ and $\tilde{Q}$. We know that the graphs which are the parts of $Q_n$ converge to the graphons from which $\tilde{Q}$ is built. So the graph homomorphism densities converge.

This shows that $\tilde{U}_1$ is a $O(e(F)\epsilon)$-approximation to a subsequence of $H_n$ in terms of $F$-densities. Now, take a smaller $\epsilon' < \epsilon$, and build another $\tilde{U}_2$, where the new partitions $Q_n$ are refinements of the previous ones. Continuing this process, we obtain a sequence $\tilde{U}_1, \tilde{U}_2, \ldots$ which is a martingale as before. The Martingale Convergence Theorem gives a pointwise almost everywhere limit $\tilde{W}$ of $\tilde{U}_s, s \to \infty$, and $\tilde{W}$ is the desired limit.

In proving 3-uniform hypergraph limits, we used the existence of graph limits. In general, we prove the existence of $k$-uniform hypergraph limits by induction on $k$. There are a few further technical difficulties. For example, we need to make sure that the limit of a sequence of partitions remains a partition, so the limit needs to be taken in a compatible way. Since we are working with multiple partitions, we will need to deal with homomorphisms from $F$ to a vector of hypergraphons, where the edges of $F$ individually land in different hypergraphons. The details are addressed in the rest of this paper.

4. NOTATION

One (not so trivial) source of difficulty in working with hypergraphs is the complexity of notation. This section collects some of the notation and conventions used in the rest of this paper. Some notations were already introduced in §1.

We shall omit the word “measurable” as everything we consider is assumed to be measurable.

4.1. Hypergraphs

A $k$-uniform hypergraph $F$ is some finite collection of $k$-element subsets of some ground set, which we denote by $V(F)$. So when we talk about an element of $F$, we mean an edge of $F$, and $|F|$ means the number of edges of $F.$
4.2. Subsets, Partitions, and Hypergraphons

**Definition 4.1** (Symmetric sets and partitions). A *symmetric (measurable)* subset of $[0,1]^r$ is one which is invariant under the action of all permutations of $[k]$. A symmetric (measurable) partition of $[0,1]^r$ is a partition of $[0,1]^r$ into a finite collection of symmetric subsets.

A symmetric subset $P \subseteq [0,1]^r$ is associated to a $k$-hypergraphon $W^P : [0,1]^{r-[k]} \rightarrow [0,1]$ by integrating out the top coordinate:

$$W^P(x_{r-[k]}) := \int_0^1 1_P(x_{r-[k]}) dx_{[k]}.$$  \hfill (8)

For example, for $k = 3$, we have $P \subseteq [0,1]^3$, with coordinates indexed by $r[2] = \{1, 2, 12\}$, and

$$W^P(x_1, x_2) = \int_0^1 1_P(x_1, x_2, x_{12}) dx_{12}.$$  

This operation collapses the final coordinate in $P$. It will be helpful to think of $P$ and $W^P$ as representing the same object. For example, when $k = 2$ this means we do not care how $P$ is placed along the $x_{12}$ coordinate, as we only care about how much $P$ intersects line segments of the form $\{x_1\} \times \{x_2\} \times [0,1]$. And conversely, for given a $W : [0,1]^2 \rightarrow [0,1]$, there are many $P \subseteq [0,1]^2$ satisfying $W^P = W$, e.g., any set of the form $P = \{(x,y,z) : a(x,y) \leq z \leq b(x,y)\}$ where $b(x,y) - a(x,y) = W(x,y)$.

4.3. Homomorphism Densities

For any tuple of $k$-uniform hypergraphons $W = (W_1, \ldots, W_m)$, any $k$-uniform hypergraph $F$, and any map $\alpha : F \rightarrow [m]$, define the homomorphism density

$$t_\alpha(F,W) := \int_{[0,1]^{r(F)+k-1}} \prod_{e \in F} W_{\alpha(e)}(x_{r(e)}) dx.$$  

**Example 4.2.** If $k = 2, F = K_3 = \{12, 13, 23\}, \alpha = (12 \mapsto 1, 13 \mapsto 2, 23 \mapsto 3)$, then

$$t_\alpha(F,W) = \int_{[0,1]^3} W_1(x_1,x_2)W_2(x_1,x_3)W_3(x_2,x_3) dx_1 dx_2 dx_3$$

For any symmetric partition $P = (P_1, \ldots, P_m)$ of $[0,1]^r$, define

$$W^P := (W^{P_1}, \ldots, W^{P_m}) \quad \text{and} \quad t_\alpha(F,P) := t_\alpha(F,W^P).$$  \hfill (9)

4.4. Quotient and Stepping Operators

Let $W : [0,1]^{r-[k]} \rightarrow [0,1]$ be a $k$-uniform hypergraphon and $Q$ a symmetric partition of $[0,1]^{r-[k-1]}$ into $q$ parts $Q_1, Q_2, \ldots, Q_q \subseteq [0,1]^{r-[k-1]}$. The *quotient* $W/Q$ is a $2q^k$-tuple of numbers in $[0,1]$ defined by assigning to each $k$-tuple $f = (f_1, \ldots, f_k) \in [q]^k$ a pair $(v_f, w_f)$, referred to as (volume, average), as follows:

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• Volume: \( v_f \) equals the integral

\[
v_f := \int_{x \in [0,1]^{r<k}} 1_{\Omega_f} (x_{r([k],1)}) 1_{\Omega_2} (x_{r([k],2)}) \cdots 1_{\Omega_k} (x_{r([k],k)}) \, dx. \tag{10}
\]

• Average: If \( v_f = 0 \), then we set \( w_f = 0 \). Otherwise, \( w_f \) is defined to be

\[
w_f := \frac{1}{v_f} \int_{x \in [0,1]^{r<k}} W(x_{r([k],1)}) 1_{\Omega_f} (x_{r([k],1)}) 1_{\Omega_2} (x_{r([k],2)}) \cdots 1_{\Omega_k} (x_{r([k],k)}) \, dx. \tag{11}
\]

Intuitively, the partition \( Q \) induces a partition \( Q' \) of \([0,1]^{[k]}\) into parts enumerated by \( f \in [q]^k \). Each cell of \( Q' \) has a volume \( v_f \) and an average value \( w_f \) of \( W \) on the cell.

If we have another \( k \)-uniform hypergraphon \( W' \), and a symmetric partition \( Q' \) of \([0,1]^{[k−1]}\) into \( q \) parts (\( Q \) and \( Q' \) have the same number of parts) with volumes and weights \((v'_f, w'_f)\), we define

\[
d_f(W/Q, W'/Q') := \sum_{f \in [q]^k} (|v_f − v'_f| + |w_f − w'_f|). \tag{12}
\]

For any symmetric subset \( P \subseteq [0,1]^{[k]} \), we write

\[ P/Q := W^P/Q. \]

A \( Q \)-step function \( U : [0,1]^{r<k} \rightarrow \mathbb{R} \) is a function of the form

\[
U(x) = \sum_{f = (f_1, \ldots, f_k) \in [q]^k} u_f 1_{\Omega_f} (x_{r([k],1)}) 1_{\Omega_2} (x_{r([k],2)}) \cdots 1_{\Omega_k} (x_{r([k],k)}), \tag{13}
\]

for some real values \( u_f \). Since \( Q \) is a partition, the indicator functions in (13) all have disjoint support, which together partition the domain \([0,1]^{r<k}\). Usually \( U \) is a symmetric function, which is equivalent to having an additional symmetry constraint on \( u_f \), namely that \( u_f = u_{f'} \) whenever \( f' \) is obtained from \( f \) by a permutation of the coordinates.

The \( Q \)-stepping operator, denoted by a subscript \( Q \), turns a \( k \)-uniform hypergraphon \( W \) into a symmetric \( Q \)-step function \( W^Q \) by averaging over each induced cell of \( Q' \). More precisely, we define \( W_Q : [0,1]^{r<k} \rightarrow [0,1] \) to be (using \( v_f \) and \( w_f \) from \( W/Q \) defined earlier)

\[
W_Q(x) := \sum_{f = (f_1, \ldots, f_k) \in [q]^k} w_f 1_{\Omega_f} (x_{r([k],1)}) 1_{\Omega_2} (x_{r([k],2)}) \cdots 1_{\Omega_k} (x_{r([k],k)}).
\]

We can also apply the stepping operator to a tuple of hypergraphons. If \( W = (W_1, \ldots, W_m) \), then

\[ W_Q := ((W_1)_Q, \ldots, (W_m)_Q). \]

In particular, if \( P = \{P_1, \ldots, P_m\} \) is a partition of \([0,1]^{[k]}\), then we write

\[ W^P_Q := ((W_1)^{P_1}_Q, \ldots, (W_1)^{P_m}_Q) = (W^P_{Q_1}, \ldots, W^P_{Q_m}). \]
4.5. Cut Norm

**Definition 4.3.** For any symmetric function \( W : [0, 1]^r \rightarrow \mathbb{R} \), define

\[
\| W \|_{\square} := \sup_{u_1, \ldots, u_k : [0, 1]^r \rightarrow [0, 1]} \left| \frac{1}{k} \sum_{i=1}^k W(\mathbf{x}_r(\mathbf{i})) \mathbf{x}_r(\mathbf{i}) \right|. \tag{14}
\]

Note that by linearity of the expression inside the absolute value in (14), it suffices to consider functions \( u_i \)'s which are indicator functions \( 1_{B_i} \) of symmetric subsets \( B_i \subseteq [0, 1]^{[k-1]} \). The usual cut norm corresponds to the case \( k = 2 \). The following example shows \( k = 3 \).

**Example 4.4.** For any symmetric function \( W : [0, 1]^r \rightarrow \mathbb{R} \), \( W \) \( \| \|_{\square} \) equals to

\[
\sup_{u_1, u_2, u_3} \left| \frac{1}{6} \int_{[0,1]^6} W(x_1, x_2, x_3, x_{12}, x_{13}, x_{23}) u_1(x_2, x_3, x_{23}) u_2(x_1, x_3, x_{13}) u_3(x_1, x_2, x_{12}) dx_1 dx_2 dx_3 dx_{12} dx_{13} dx_{23} \right|
\]

where \( u_1, u_2, u_3 \) vary over all symmetric functions \( [0, 1]^{[2]} \rightarrow [0, 1] \).

5. REGULARITY AND COUNTING LEMMAS

**Definition 5.1.** Let \( W \) be a \( k \)-uniform hypergraphon and \( \mathcal{Q} \) a symmetric partition of \( [0, 1]^{[k-1]} \). We say that \( (W, \mathcal{Q}) \) is weakly \( \epsilon \)-regular if \( \| W - W_{\mathcal{Q}} \|_{\square} \leq \epsilon \).

For a symmetric subset \( P \subseteq [0, 1]^{[k]} \), we say that \( (P, \mathcal{Q}) \) is weakly \( \epsilon \)-regular if \( (W_P, \mathcal{Q}) \) is.

**Lemma 5.2 (Weak regularity lemma).** Let \( k \geq 2 \) and \( \epsilon > 0 \). Let \( W = (W_1, \ldots, W_m) \) be a tuple of \( k \)-uniform hypergraphons. Let \( \mathcal{Q} \) be a symmetric partition of \( [0, 1]^{[k-1]} \). Then there exists a partition \( \mathcal{Q}' \) refining \( \mathcal{Q} \) so that every part of \( \mathcal{Q} \) is refined into exactly \( 2^{\ln m / \epsilon^2} \) parts (allowing empty parts) so that \( (W_i, \mathcal{Q}') \) is weakly \( \epsilon \)-regular for every \( 1 \leq i \leq m \).

**Proof.** We build the partition incrementally, starting with \( \mathcal{Q} \). At a given stage, suppose the partition is \( \mathcal{R} \). If \( (W_i, \mathcal{R}) \) is weakly \( \epsilon \)-regular for every \( i \) then we stop. Otherwise there is some \( i \) with \( \| W_i - (W_i)_\mathcal{R} \|_{\square} > \epsilon \), so there exists symmetric subsets \( B_1, \ldots, B_k \subseteq [0, 1]^{[k-1]} \) such that

\[
\left| \int_{[0,1]^{[k]}} (W_i - (W_i)_\mathcal{R})(\mathbf{x}_r(\mathbf{k})) \prod_{i=1}^k 1_{B_i}(\mathbf{x}_r(\mathbf{k})_{[\mathbf{i}]}) d\mathbf{x} \right| > \epsilon. \tag{15}
\]

Let \( B : [0, 1]^{[k]} \rightarrow [0, 1] \) be the function (not necessarily symmetric)

\[
B(\mathbf{x}) := \prod_{i=1}^k 1_{B_i}(\mathbf{x}_r(\mathbf{k})_{[\mathbf{i}]}) d\mathbf{x}.
\]
For two functions $U, U' : [0, 1]^{c[k]} \to [0, 1]$, define the inner product

$$
\langle U, U' \rangle = \int_{[0,1]^{c[k]}} U(x)U'(x)dx.
$$

We will use the following easy fact: if $U'$ is a $Q$-step function, then $\langle U, U' \rangle = \langle U_Q, U' \rangle$.

Now let $\mathcal{R}'$ be the minimal partition refining $\mathcal{R}$ and $B_1, \ldots, B_k$. Since $((W_i)_{\mathcal{R}'})_{\mathcal{R}} = (W_i)_{\mathcal{R}}$, applying the fact above, we obtain

$$
((W_i)_{\mathcal{R}'}, (W_i)_{\mathcal{R}}) = ((W_i)_{\mathcal{R}'}, (W_i)_{\mathcal{R}}).
$$

Since $B$ is an $\mathcal{R}'$-step function, we have $\langle (W_i)_{\mathcal{R}'}, B \rangle = \langle W_i, B \rangle$. So by (15)

$$
|\langle (W_i)_{\mathcal{R}'}, (W_i)_{\mathcal{R}} \rangle - |\langle W_i, (W_i)_{\mathcal{R}} \rangle| > \epsilon.
$$

Since $\|U\|^2_2 = \langle U, U \rangle$ for any $U$, we obtain by (16), the Cauchy-Schwarz inequality, and (17)

$$
\| (W_i)_{\mathcal{R}'} \|^2_2 - \| (W_i)_{\mathcal{R}} \|^2_2 = \| (W_i)_{\mathcal{R}'} - (W_i)_{\mathcal{R}} \|^2_2 \geq |\langle (W_i)_{\mathcal{R}'}, (W_i)_{\mathcal{R}} \rangle - |\langle W_i, (W_i)_{\mathcal{R}} \rangle| |^2 > \epsilon^2.
$$

Furthermore, for every $1 \leq j \leq m$, $\| (W_i)_{\mathcal{R}'})_i \|^2_2 \geq \| (W_i)_{\mathcal{R}} \|^2_2$ by convexity since $((W_i)_{\mathcal{R}'})_i = (W_i)_{\mathcal{R}}$.

The quantity $\| (W_i)_{\mathcal{R}} \|^2_2 + \cdots + \| (W_m)_{\mathcal{R}} \|^2_2$ is at most $m$, and each iteration above increases the sum by at least $\epsilon^2$. So there can be at most $m/\epsilon^2$ iterations. At the end we obtain a partition $Q'$ so that $(W_i, Q')$ is weakly $\epsilon$-regular for every $1 \leq i \leq m$. Each time we introduced at most $k$ new sets to refine the partition, so $\mathcal{R}'$ refines each part of $\mathcal{R}$ into at most $2^k$ subparts. After at most $m/\epsilon^2$ iterations, each part of the original partition $Q$ is refined into at most $2^{k/m^2}$ parts. We can throw in some empty parts so that each part of $Q$ is refined into exactly $\lceil 2^{km^2} \rceil$ parts.

**Lemma 5.3** (Counting lemma I). Let $U = (U_1, \ldots, U_m)$ and $W = (W_1, \ldots, W_m)$ be two $m$-tuple of $k$-uniform hypergraphons and $Q$ a symmetric partition of $[0, 1]^{c(k-1)}$. Suppose that $\| W_i - U_i \|_{l^{k-1}} \leq \epsilon$ for each $i$. Then for any $k$-uniform hypergraph $F$ and any map $\alpha : F \to [m]$, we have

$$
|t_\alpha(F, U) - t_\alpha(F, W)| \leq |F|\epsilon.
$$

**Proof.** Let $V = V(F)$ and $F = \{e_1, \ldots, e_{|F|}\}$. Write as a telescoping sum

$$
t_\alpha(F, U) - t_\alpha(F, W) = \sum_{j=1}^{|F|} \int_{[0,1]^{c(k-1)}} \left( \prod_{i=1}^{|F|} U_{\alpha(e_i)}(x_{r\subset(e_i)}) - \prod_{i=1}^{|F|} W_{\alpha(e_i)}(x_{r\subset(e_i)}) \right) dx_{r\subset(V,k-1)}.
$$

The $j$-th term in the final sum is bounded by $\| U_{\alpha(e_j)} - W_{\alpha(e_j)} \|_{l^{k-1}} \leq \epsilon$. Indeed, if we fix all variables other than $x_{r\subset(e_j)}$, then all the factors except for $(U_{\alpha(e_j)} - W_{\alpha(e_j)})(x_{r\subset(e_j)})$ have

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the form \( u(x_{(f)}) \) for some \( f \subseteq e_j \), where \( f \) is the intersection of \( e_j \) with another edge \( e'_j \). So the integral can be bounded by the \((k - 1)\)-cut norm, as claimed.

\[ R_{\alpha}^{(k)}(x) = \sum_{f \subseteq e_j} \sum_{j \in [k]} x_{(f)}^{(j)}. \]

Lemma 5.4 (Counting lemma II). Let \( U = (U_1, \ldots, U_m) \) and \( W = (W_1, \ldots, W_m) \) be two \( m \)-tuples of \( k \)-uniform hypergraphons. Let \( Q = [Q_1, \ldots, Q_q] \) and \( R = [R_1, \ldots, R_q] \) be symmetric partitions of \([0, 1]^{k-1}\). Suppose that \( d_1(U_i/Q, W_i/R) \leq \delta \) for each \( i \). Then for any \( k \)-uniform hypergraph \( F \) and any map \( \alpha : F \rightarrow [m] \),

\[
|t_\alpha(F, U_Q) - t_\alpha(F, W_R)| \leq |F|\delta + \sum_{\beta : \partial F \rightarrow [q]} |t_\beta(\partial F, Q) - t_\beta(\partial F, R)|,
\]

where the sum is taken over all maps \( \beta : \partial F \rightarrow [q] \), and \( \partial F \) is the \((k - 1)\)-uniform hypergraph on \( V(F) \) consisting of \((k - 1)\)-element subsets of \( V(F) \) that are contained in some edge of \( F \).

Proof. We can replace each \( U_i \) by \((U_i)_Q\) as this does not change \( U_i/Q \) or \( t_\alpha(F, U_Q) \). So we may assume that every \( U_i \) is a symmetric \( Q \)-step function, i.e., \( U_{\mathcal{R}} = U \). Similarly, assume that every \( W_i \) is a symmetric \( R \)-step function.

For each \( f \in [q]^k \), let \((v_{i,f}, w_{i,f})\) denote the volume and average corresponding to \( f \) in \( U_i/Q \), and let \((v'_{i,f}, w'_{i,f})\) denote the same for \( W_i/R \).

For each \( 1 \leq i \leq m \), construct a symmetric \( Q \)-step function \( U'_i \) from \( U_i \) by changing its value on the step corresponding to \( f \) from \( w_{i,f} \) to \( w'_{i,f} \). So \( U'_i/Q \) has \((v_{i,f}, w'_{i,f})\) as its volumes and averages. In other words,

\[
U_i(x_{r<k}) = \sum_{f=(f_1, \ldots, f_q)} w_{i,f} 1_{Q_{f_1}}(x_{r(k)\setminus[1]}) \cdots 1_{Q_{f_k}}(x_{r(k)\setminus[k]}); \quad (19)
\]

\[
U'_i(x_{r<k}) = \sum_{f=(f_1, \ldots, f_q)} w'_{i,f} 1_{Q_{f_1}}(x_{r(k)\setminus[1]}) \cdots 1_{Q_{f_k}}(x_{r(k)\setminus[k]}); \quad (20)
\]

\[
W_i(x_{r<k}) = \sum_{f=(f_1, \ldots, f_q)} w'_{i,f} 1_{R_{f_1}}(x_{r(k)\setminus[1]}) \cdots 1_{R_{f_k}}(x_{r(k)\setminus[k]}). \quad (21)
\]

Write \( U' = (U'_1, \ldots, U'_m) \). We have

\[
\|U_i - U'_i\|_1 = \sum_{f \in [q]^k} v_{i,f} |w_{i,f} - w'_{i,f}| = \sum_{f \in [q]^k} (|v_{i,f} w_{i,f} - v'_{i,f} w'_{i,f}| + |w_{i,f} v_{i,f} - v_{i,f}|)
\]

\[
\leq \sum_{f \in [q]^k} (|v_{i,f} w_{i,f} - v'_{i,f} w'_{i,f}| + |v_{i,f} - v_{i,f}|) = d_1(U_i/Q, W_i/R) \leq \delta.
\]

So \( \|U_i - U'_i\|_1 \leq \delta \) for each \( i \). It follows that

\[
|t_\alpha(F, U) - t_\alpha(F, U')| \leq |F|\delta. \quad (22)
\]

(This follows from Counting Lemma I, but it is in fact even easier.) From (20) we have

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Now we are almost ready to build the limiting object. We will proceed by induction on $k$. BRANCHING PARTITIONS

**Definition 6.1.** A degree $p = (p_1, p_2, \ldots) \in \mathbb{N}^k$ (symmetric) branching partition $\mathcal{P}$ of $[0, 1]^{[k]}$ is a collection of symmetric subsets $P_i$ of $[0, 1]^{[k]}$, collected into levels, where each level $P_i$ is a symmetric partition of $[0, 1]^{[k]}$:

- **Level 0**: $\mathcal{P}_0 = \{[0, 1]^{[k]}\}$
- **Level 1**: $\mathcal{P}_1 = \{P_1, P_2, \ldots, P_{p_1}\}$ is a symmetric partition of $[0, 1]^{[k]}$.
- **Level $l$ ($l \geq 2$)**: $\mathcal{P}_l$ is a refinement of $\mathcal{P}_{l-1}$, where each part of $\mathcal{P}_{l-1}$ gets further refined into exactly $p_l$ parts.

The lemma follows from combining (22) and (25) using the triangle inequality. □

6. BRANCHING PARTITIONS

Now we are almost ready to build the limiting object. We will proceed by induction on $k$ (for $k$-uniform hypergraphons). The situation is very simple when $k = 1$, since in this case a hypergraphon is simply a number between 0 and 1. To build the limiting hypergraphon in general, we will need to repeatedly apply the weak regularity lemma to obtain a refining chain of partitions. Since we need to apply induction on $k$, we need to have a stronger induction hypothesis that involves a sequence of not just single hypergraphons, but refining chains of partitions. This motivates the following definition of a branching partition, which is a special case a filtration, in the language of probability. See Figure 1.

**Definition 6.1.** A degree $p = (p_1, p_2, \ldots) \in \mathbb{N}^k$ (symmetric) branching partition $\mathcal{P}$ of $[0, 1]^{[k]}$ is a collection of symmetric subsets $P_i$ of $[0, 1]^{[k]}$, collected into levels, where each level $P_i$ is a symmetric partition of $[0, 1]^{[k]}$:

- **Level 0**: $\mathcal{P}_0 = \{[0, 1]^{[k]}\}$
- **Level 1**: $\mathcal{P}_1 = \{P_1, P_2, \ldots, P_{p_1}\}$ is a symmetric partition of $[0, 1]^{[k]}$.
- **Level $l$ ($l \geq 2$)**: $\mathcal{P}_l$ is a refinement of $\mathcal{P}_{l-1}$, where each part of $\mathcal{P}_{l-1}$ gets further refined into exactly $p_l$ parts.

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An index at level \( l \) is a tuple \( i = (i_1, i_2, \ldots, i_l) \in [p_1] \times [p_2] \times \cdots \times [p_l] \), which points to the symmetric subset \( P_i = P_{i_1 \ldots i_l} \in \mathcal{P}_l \) at level \( l \), where \( P_i \) is the \( i \)-th part in the refinement of the part \( P_{i_1 \ldots i_{l-1}} \) at level \( l - 1 \), whenever \( l \geq 2 \) (all partitions are ordered).

Font correction. \( \mathcal{P} \) is a branching partition, \( \mathcal{P} \) is a partition, and \( P \) is a subset of \([0,1]^{[k]}\).

**Example 6.2.** A symmetric subset \( P \subseteq [0,1]^{[k]} \) or a \( k \)-uniform hypergraphon \( W \) (related by (8)) can be thought of as a degree \( (2,1,1,1,\ldots) \) branching partition: level \( l \) is \( P \) and \( P^c \) (the complement of \( P \) in \([0,1]^{[k]}\)) and all subsequent levels are trivial refinements.

We can generalize the notion of regularity from Definition 5.1 to branching partitions as follows.

**Definition 6.3.** Let \( P \) be a branching partition of \([0,1]^{[k]} \) and \( \mathcal{Q} \) a branching partition of \([0,1]^{[k-1]} \). We say that \((\mathcal{P},\mathcal{Q})\) is weakly \((\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_r)\)-regular if for every \( s \geq 1 \), whenever \( P \subseteq [0,1]^{[k]} \) is a member of \( P \) of level at most \( s \), and \( \mathcal{Q}_s \) is the level \( s \) partition of \([0,1]^{[k-1]} \) in \( \mathcal{Q} \), the pair \((P,\mathcal{Q}_s)\) is weakly \( \varepsilon_s \)-regular.

**Lemma 6.4** (Weak regularity lemma for branching partitions). For every \( k \geq 2, p = (p_1,p_2,\ldots) \in \mathbb{N}^\infty \) and \( \varepsilon = (\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_r) \in \mathbb{N}^\infty \), we can find a \( q = (q_1,q_2,\ldots) \in \mathbb{N}^\infty \) so that the following holds: for every degree \( p \) branching partition \( P \) of \([0,1]^{[k]} \), there exists a degree \( q \) branching partition \( \mathcal{Q} \) of \([0,1]^{[k]} \) so that \((\mathcal{P},\mathcal{Q})\) is weakly \( \varepsilon \)-regular.

**Proof.** Take \( q_s = \left\lfloor 2^{k(p_1+p_2+\cdots+p_{s-1})/\varepsilon_s} \right\rfloor \). We build \( \mathcal{Q} \) successively by level. To obtain the level \( s \) partition in \( \mathcal{Q} \), applying Lemma 5.2 with \( \varepsilon = \varepsilon_s, W \) the collection of hypergraphons corresponding to all members of \( P \) of level at most \( s \), and \( \mathcal{Q} \) the level \( s - 1 \) partition in \( \mathcal{Q} \).

Now we introduce two notions of convergence for branching partitions. The first notion, called left-convergence, is based on convergence of homomorphism densities. The second notion, called partitionable convergence, is based on convergence of regularity partitions. We will show, using our counting lemmas, that partitionable convergence implies left-convergence.

Notation. Given degree \( p = (p_1,p_2,\ldots) \) branching partitions \( \mathcal{P}_1,\mathcal{P}_2,\ldots \) and \( \mathcal{Q} \) of \([0,1]^{[k]} \) and degree \( q = (q_1,q_2,\ldots) \) branching partitions \( \mathcal{Q}_1,\mathcal{Q}_2,\ldots \) and \( \mathcal{Q} \) of \([0,1]^{[k-1]} \), we use the following notation to refer to the partitions and parts in these branching partitions.

- For each \( l \geq 1 \), \( P_{n,l} \) is the level \( l \) partition in \( \mathcal{P}_n \), and \( \mathcal{P}_l \) is the level \( l \) partition in \( \mathcal{P} \).
- For each \( s \geq 1 \), \( Q_{n,s} \) is the level \( s \) partition in \( \mathcal{Q}_n \), and \( \mathcal{Q}_s \) is the level \( s \) partition in \( \mathcal{Q} \).
- For each index \( i = (i_1,i_2,\ldots,i_l) \in [p_1] \times \cdots \times [p_l] \), \( P_{n,i} \) is the index \( i \) element of \( \mathcal{P}_n \) and \( \mathcal{P}_i \) is the index \( i \) element of \( \mathcal{P} \).

**Definition 6.5.** (Left-convergence: \( \mathcal{P}_n \rightarrow \mathcal{P} \)). We say that a sequence \( \mathcal{P}_1,\mathcal{P}_2,\ldots \) of degree \( p \) branching partitions of \([0,1]^{[k]} \) left-converges to another degree \( p \) branching partition \( \mathcal{Q} \) of \([0,1]^{[k]} \), written \( \mathcal{P}_n \rightarrow \mathcal{P} \), if

\[
\lim_{n \to \infty} t_{\alpha}(F,\mathcal{P}_{n,l}) = t_{\alpha}(F,\mathcal{P}_l) \quad \text{for all } F, l, \alpha
\]
where $F$ ranges over all k-uniform hypergraphs, $l$ ranges over all positive integers, and $\alpha$ ranges over all maps $F \rightarrow [p_1 \cdots p_l]$. Recall from (9) that $t_\alpha(F, \mathcal{P}) := t_\alpha(F, W^p)$ for a partition $\mathcal{P}$.

**Definition 6.6** (Partitionable convergence: $\mathcal{P}_n \rightarrow \mathcal{P}$). We say that a sequence $\mathcal{P}_1, \mathcal{P}_2, \ldots$ of degree $p = (p_1, p_2, \ldots)$ branching partitions of $[0, 1]^{[k]}$ partitionably converges to another degree $p$ branching partition $\mathcal{P}$ of $[0, 1]^{[k]}$, written $\mathcal{P}_n \rightarrow \mathcal{P}$, if the following is satisfied (the definition is inductive on $k$).

- When $k = 1$, for every index $i = (i_1, \ldots, i_t) \in [p_1] \times \cdots \times [p_l]$, we have $\lim_{n \rightarrow \infty} \lambda(P_{n,i}) = \lambda(P_i)$, where $\lambda$ is the Lebesgue measure on $[0, 1]$.

- When $k \geq 2$, there exists some $q \in \mathbb{N}^k$ and degree $q$ branching partitions $\mathcal{Q}_1, \mathcal{Q}_2, \ldots$ and $\mathcal{Q}$ of $[0, 1]^{[k-1]}$ satisfying:
  
  a. $(\mathcal{Q}_n, \mathcal{Q}_n)$ is weakly $(1, 1/2, 1/3, \ldots)$-regular for every $n$;
  b. $\mathcal{Q}_n \rightarrow \mathcal{Q}$ as $n \rightarrow \infty$ (defined inductively);
  c. For every $s \geq 1$ and every index $i \in [p_1] \times \cdots \times [p_l]$, one has $\lim_{n \rightarrow \infty} d_i(P_{n,i}/\mathcal{Q}, \tilde{P}_i/\mathcal{Q}) = 0$;
  d. For every member $\tilde{P} \subseteq [0, 1]^{[k]}$ of $\mathcal{Q}$, one has $(W^\tilde{P})_{\mathcal{Q}_s} \rightarrow W^\tilde{P}$ pointwise almost everywhere as $s \rightarrow \infty$.

**Lemma 6.7.** (Partitionable convergence implies left-convergence). If $\mathcal{P}_n \rightarrow \mathcal{P}$, then $\mathcal{P}_n \rightarrow \mathcal{P}$.

**Proof.** We use induction on $k$. When $k = 1$, the claim is trivial. Now assume $k \geq 2$.

We need to show that (26) holds. Fix $F, l, \alpha$. Let $m = p_1 \cdots p_l$. Let $\mathcal{Q}_n$ and $\mathcal{Q}$ be as in Definition 6.6, and let $q = (q_1, q_2, \ldots)$ be the degree of $\mathcal{Q}$.

Let $\epsilon > 0$. By Definition 6.6(d), $W^\tilde{P}_{\mathcal{Q}_s}$ converges pointwise almost everywhere in each coordinate to $W^\tilde{P}_s$ as $s \rightarrow \infty$, so $\lim_{s \rightarrow \infty} t_\alpha(F, W^\tilde{P}_{\mathcal{Q}_s}) = t_\alpha(F, W^\tilde{P}_s)$. We can find an $s \geq \max\{l, |F|/\epsilon\}$ so that $|t_\alpha(F, W^\tilde{P}_{\mathcal{Q}_s}) - t_\alpha(F, \tilde{P}_s)| \leq \epsilon$. Fix this value of $s$.

By Definition 6.6(b) we have $\mathcal{Q}_n \rightarrow \mathcal{Q}$, so $\mathcal{Q}_n \rightarrow \mathcal{Q}$ by the induction hypothesis. Thus

$$\lim_{n \rightarrow \infty} t_\beta(\partial F, Q_{n,s}) = t_\beta(\partial F, \tilde{Q}_s)$$

(27)

for all $\beta : \partial F \rightarrow [q_1 q_2 \cdots q_l]$. See Lemma 5.4 for the definition of $\partial F$. We have

$$|t_\alpha(F, \mathcal{P}_{n,l}) - t_\alpha(F, \tilde{P}_s)| \leq |t_\alpha(F, \mathcal{P}_{n,l}) - t_\alpha(F, W^\mathcal{P}_{n,i})| + |t_\alpha(F, W^\mathcal{P}_{n,i}) - t_\alpha(F, \tilde{P}_s)|$$

$$+ |t_\alpha(F, W^\tilde{P}_{\mathcal{Q}_s}) - t_\alpha(F, \tilde{P}_s)|$$

(28)

As $n \rightarrow \infty$, the first term on the right hand side of (28) has a limsup of at most $|F|/s \leq \epsilon$ by Counting Lemma I (Lemma 5.3) since $(P, Q_{n,i})$ is $1/s$-regular for every $P \in \mathcal{P}_{n,l}$ by Definition 6.6(a). The second term on the RHS of (28) goes to zero by Counting Lemma II (Lemma 5.4), Definition 6.6(c), and (27). The third term on the RHS of (28) is at most $\epsilon$ using our choice of $s$. It follows that $\limsup_{n \rightarrow \infty} |t_\alpha(F, \mathcal{P}_{n,l}) - t_\alpha(F, \tilde{P}_s)| \leq 2\epsilon$. Since $\epsilon$ can be made arbitrarily small, we obtain $\lim_{n \rightarrow \infty} t_\alpha(F, \mathcal{P}_{n,l}) = t_\alpha(F, \tilde{P}_s)$ as desired.
Proposition 6.8. Let \( p \in \mathbb{N}^+ \). Let \( \mathcal{P}_1, \mathcal{P}_2, \ldots \) be a sequence of degree \( p \) branching partitions of \([0, 1]^{[k]}\). Then there exists another degree \( p \) branching partition \( \tilde{\mathcal{P}} \) of \([0, 1]^{[k]}\) so that \( \mathcal{P}_n \longrightarrow \tilde{\mathcal{P}} \) as \( n \rightarrow \infty \) along some infinite subsequence.

Proof. We use induction on \( k \). The claim is easy when \( k = 1 \), since we can pick a subsequence so that for each index \( i \), the measure \( \lambda(P_{n,i}) \) converges to some value \( a_i \) as \( n \rightarrow \infty \), and we can take the limit \( \tilde{\mathcal{P}} \) to be a branching partition where \( \tilde{P}_i \) is an interval with length \( a_i \).

Now assume \( k \geq 2 \). By Lemma 6.4, there exists a \( q \in \mathbb{N}^+ \) so that for every \( n \) we can find a degree \( q \) branching partition \( \mathcal{Q}_n \) of \([0, 1]^{[k-1]}\) so that \((\mathcal{P}_n, \mathcal{Q}_n)\) is weakly \((1, 1/2, 1/3, \ldots)\)-regular, thereby satisfying (a) in Definition 6.6. Applying the induction hypothesis, we can restrict to a subsequence so that \( \mathcal{Q}_n \longrightarrow \tilde{\mathcal{Q}} \) for some branching partition \( \tilde{\mathcal{Q}} \) of \([0, 1]^{[k-1]}\) (here and onwards in this proof we abuse notation by only considering convergence as \( n \rightarrow \infty \) along some subsequence. We will be repeatedly taking subsequences, and the conclusion will follow by a standard diagonalization argument). So (b) is satisfied.

By further restricting to a subsequence, we may assume that for each \( s \geq 1 \) and each index \( i \), the quotient \( P_{n,i}/Q_{n,s} \) converges coordinate-wise as \( n \rightarrow \infty \). Let \( W_{n,i} := W_{P_{n,i}} \) be the hypergraphon associated to \( P_{n,i} \). Let \( \tilde{W}_{i,s} : [0, 1]^{<k} \rightarrow [0, 1] \) be a symmetric \( \tilde{Q}_s \)-step function, with values assigned so that \( d_1(W_{n,i}/Q_{n,s}, \tilde{W}_{i,s}/\tilde{Q}_s) \rightarrow 0 \) as \( n \rightarrow \infty \). This is possible since we previously assumed that \( P_{n,i}/Q_{n,s} \) converges coordinatewise as \( n \rightarrow \infty \), so that are now simply putting in the limiting values of the “average” coordinates into a template for a symmetric \( \tilde{Q}_s \)-step function in order to construct \( \tilde{W}_{i,s} \). To see that the “volume” coordinates (10) of \( Q_{n,s} \) converge to those of \( \tilde{Q}_s \), note that this amount to the claim that \( \lim_{n \rightarrow \infty} t_\beta(K_{k-1}^{[k-1]}, Q_{n,s}) = t_\beta(K_{k-1}^{[k-1]}, \tilde{Q}_s) \) for every \( \beta : K_{k-1}^{[k-1]} \rightarrow [q] \), where \( K_{k-1}^{[k-1]} \) is the \((k-1)\)-uniform simplex, i.e., the collection of all \((k-1)\)-element subsets of \([k]\). The convergence of these homomorphism densities follows from \( \mathcal{Q}_n \longrightarrow \tilde{\mathcal{Q}} \) which in turn follows from \( \mathcal{Q}_n \longrightarrow \tilde{\mathcal{Q}} \) and Lemma 6.7.

Claim 1. \((\tilde{W}_{i,s+1})_{\tilde{Q}_s} = \tilde{W}_{i,s}\).

Proof of Claim 1. We have

\[
\lim_{n \rightarrow \infty} d_1(W_{n,i}/Q_{n,s}, \tilde{W}_{i,s}/\tilde{Q}_s) = 0 \quad (29)
\]

and

\[
\lim_{n \rightarrow \infty} d_1(W_{n,i}/Q_{n,s}, \tilde{W}_{i,s+1}/\tilde{Q}_{s+1}) = 0 \quad (30)
\]

Since \( \tilde{Q}_{s+1} \) is a refinement of \( \tilde{Q}_s \), by merging together parts in \( W_{n,i}/Q_{n,s+1} \) and \( \tilde{W}_{i,s+1}/\tilde{Q}_{s+1} \), we deduce from (30)

\[
\lim_{n \rightarrow \infty} d_1(W_{n,i}/Q_{n,s}, \tilde{W}_{i,s+1}/\tilde{Q}_s) = 0 \quad (31)
\]

From (29) and (31) we obtain \( \tilde{W}_{i,s+1}/\tilde{Q}_s = \tilde{W}_{i,s}/\tilde{Q}_s \), which implies \((\tilde{W}_{i,s+1})_{\tilde{Q}_s} = \tilde{W}_{i,s}\), since both sides are \( \tilde{Q}_s \)-step functions.

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Fig. 1. A branching partition.

It follows that $\tilde{W}_{i,1}, \tilde{W}_{i,2}, \tilde{W}_{i,3}, \ldots$ is a martingale with respect to the filtration\(^3\) induced by $\tilde{Q}_1, \tilde{Q}_2, \ldots$. By the Martingale Convergence Theorem, there exists some $\tilde{W}_i$, so that $\tilde{W}_{i,s} \to \tilde{W}_i$ pointwise almost everywhere as $s \to \infty$. Furthermore $(\tilde{W}_i)_{\tilde{Q}_s} = \tilde{W}_{i,s}$.

Claim 2. Let $l \geq 1$, and let $i = (i_1, \ldots, i_{l-1}) \in [p_1] \times \cdots \times [p_{l-1}]$ an index at level $l - 1$, which points to a part in $\mathcal{P}_l$ that splits into indices $\{j_1, \ldots, j_{p_l}\} = i \times [p_l]$ at level $l$. Then

$$\tilde{W}_{i_1} + \cdots + \tilde{W}_{i_{p_l}} = \tilde{W}_i \text{ almost everywhere.}$$

Proof of Claim 2. Since $P_{n,j_1}, \ldots, P_{n,j_{p_l}}$ is a partition of $P_{n,i}$, we have

$$W_{n,j_1} + \cdots + W_{n,j_{p_l}} = W_{n,i}$$

Taking the $Q_{n,s}$ quotient of both sides and then take the limit as $n \to \infty$, we find the following equality for these $\tilde{Q}_s$-step functions.

$$\tilde{W}_{j_1,s} + \cdots + \tilde{W}_{j_{p_l},s} = \tilde{W}_{i,s}$$

Taking $s \to \infty$ and using the pointwise almost everywhere convergence of $\tilde{W}_{j,s} \to \tilde{W}_j$ as $s \to \infty$ for every index $j$, we obtain Claim 2.

Claim 2 tells us that we can find a branching partition $\tilde{\mathcal{P}}$ of $[0, 1]^{[k]}$ so that the part $\tilde{P}_l$ satisfies $W_{\tilde{P}_l} = \tilde{W}_l$. Visually we can build the level $s$ of $\tilde{\mathcal{P}}$ by stacking together subsets of $[0, 1]^{[k]}$ that correspond to $\tilde{W}_{j,s}$, ranged over all indices $j$ at level $s$. Then $P_{n,i}/Q_{n,s} = W_{n,i}/Q_{n,s} \to d_i W_{i,s}/\tilde{Q}_s = \tilde{W}_i/\tilde{Q}_s = \tilde{P}_i/\tilde{Q}_s$, so (c) is satisfied. Also (d) is satisfied since $W_{\tilde{P}_l} = \tilde{W}_{i,s} \to W_i = W_{\tilde{P}_l}$ pointwise almost everywhere as $s \to \infty$ (from our application of the Martingale Convergence Theorem).

Proof of Theorem 1.6. Let $\mathcal{P}_n$ be the degree $(2, 1, 1, 1, \ldots)$ branching partition built from $W_n$ as in Example 6.2. Proposition 6.8 implies that there exists a branching partition $\tilde{\mathcal{P}}$ so that $\mathcal{P}_n \to \tilde{\mathcal{P}}$ along a subsequence, and hence $\mathcal{P}_n \to \tilde{\mathcal{P}}$ along a subsequence by Lemma 6.7. Let $\tilde{P}$ be the index (1) element of $\tilde{\mathcal{P}}$. The associated hypergraphon $\tilde{W} = \tilde{W}^{\tilde{P}}$ is the

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\(^3\)To be more precise, let $[0, 1]^{[k]}$ be the probability space equipped with the uniform Lebesgue measure. For each $s \geq 1$ let $\mathcal{B}_s$ be the minimal $\sigma$-algebra on $[0, 1]^{[k]}$ generated by functions of the form $1_Q(S_{i,(j_1,\ldots,j_k)})$ ranged over $Q \in Q_s$ and $j \in [k]$. Then $\tilde{W}_{i,s}$ is a $\mathcal{B}_s$-measurable random variable, and Claim 1 implies that $\tilde{W}_{i,1}, \tilde{W}_{i,2}, \ldots$ is a martingale adapted to the filtration.
desired limit of $W_n$. By applying (26) with $l = 1$ and $\alpha \equiv 1$, we see that $t(F, W_n) \to t(F, \tilde{W}_n)$ along the subsequence.

We conclude the paper with a conjecture that partitionable convergence is equivalent to left-convergence, thereby proposing a converse to Lemma 6.7.

**Conjecture 6.9.** \( \mathcal{P}_n \to \tilde{\mathcal{P}} \text{ if and only if } \mathcal{P}_n \rightharpoonup \tilde{\mathcal{P}}. \)

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