Self-consistent system of equations for a kinetic description of the low-pressure discharges accounting for the nonlocal and collisionless electron dynamics

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Abstract

For low-pressure discharges, when the electron mean free path is larger or comparable with the discharge length, the electron dynamics is essentially nonlocal. Moreover, the electron energy distribution function (EEDF) deviates considerably from a Maxwellian. Therefore, an accurate kinetic description of the low-pressure discharges requires knowledge of the nonlocal conductivity operator and calculation of the nonMaxwellian EEDF. The previous treatments made use of simplifying assumptions: a uniform density profile and a Maxwellian EEDF. In the present study a self-consistent system of equations for the kinetic description of nonlocal, nonuniform, nearly collisionless plasmas of low-pressure discharges is derived. It consists of the nonlocal conductivity operator and the averaged kinetic equation for calculation of the nonMaxwellian EEDF. The importance of accounting for the nonuniform plasma density profile on both the current density profile and the EEDF is demonstrated.
List of variables:
- $T_b$ is the bounce period,
- $T$ is half of the bounce period, $T(\varepsilon_x) = T_b/2 = \int_{x_-}^{x_+} dx/|v_x(x, \varepsilon_x)|$,
- $\Omega_b$ is the bounce frequency, $\Omega_b = 2\pi/T_b$,
- $L$ is the gap width,
- $R$ is half of the gap width, which is used to model a cylinder geometry, $R = L/2$,
- $w$ is the electron kinetic energy, $w = w_x + w_y + w_z = m(v_x^2 + v_y^2 + v_z^2)/2$,
- $\varphi(x)$ is the electron potential energy, $\varphi(x) = -e\phi(x)$, and
- $\phi$ is the electrostatic potential,
- $\varepsilon$ is the total electron energy, $\varepsilon = w + \varphi$,
- $\omega$ is the frequency of the rf electric field,
- $\nu$ is the electron elastic collision frequency,
- $\lambda$ is the electron elastic mean free path,
- $\nu_k^e$ is the electron inelastic mean free path for the process number $k$,
- $x_-(\varepsilon)$, $x_+(\varepsilon)$ are the left and right turning points $[\varepsilon = \varphi(x\pm)]$,
- $\tau$ is the time of flight from the left turning point $x_-(\varepsilon_x)$ to $x$: $\tau(x, \varepsilon_x) = \int_{x_-}^{x_+} dx/|v_x(x, \varepsilon_x)|$,
- $\theta$ is the variable angle for bounce motion, defined as $\theta(x) = \pi\text{sgn}(v_x)\tau(x)$, $/T(\varepsilon_x)$
- $\Phi(x, \varepsilon_x)$ is the generalized phase of the rf electric field, $\Phi(x, \varepsilon_x) = \int_{x_-}^{x_+} (-i\omega + \nu)dx/|v_x(x, \varepsilon_x)|$.
- $\delta$ is the width of the skin layer,
- $f$ is the electron velocity distribution function (EVDF), $f = f_0(\varepsilon) + f_1$, where $f_0(\varepsilon)$ is the main part of the electron velocity distribution function averaged over velocity directions and over available space for electrons with a given total energy $\varepsilon$, which is referred in the following as the electron energy probability function (EEPF). Notwithstanding the fact that $f$ is defined in the velocity space, we shall look for $f(\varepsilon)$ as a function of the energy. The EEPF is normalized as $n = \int f d^3v = 4\pi\sqrt{2}/m^{3/2} \int_0^\infty f_0(\varepsilon)\sqrt{\varepsilon - \varphi(x)}d\varepsilon$, where $n$ is the electron density, and the factor $4\pi\sqrt{2}/m^{3/2}$ is included in the definition of $f_0$ for convenience. $f_1$ is the rapidly varying, anisotropic part of the EVDF,
- $E_{sc}(x)$ is the space charge stationary electric field,
- $E_{rf}(x, t)$ is the rf nonstationary electric field,
- $St(f)$ is the collision integral,
- $V^{rf}$ is the oscillatory electron velocity driven by rf electric field.

I. INTRODUCTION

Low pressure radio-frequency discharges are extensively utilized for plasma processing and lighting [1]. Simulation of discharge properties is a common tool for optimization of the plasma density profiles and ion and electron fluxes. Recent plasma technology tends to decrease the gas pressures down to the millitorr range. For these low pressures it is easier to maintain uniform plasmas with well controlled parameters. Due to the large value of the electron mean free path ($\lambda$) the electron current is determined not by the local rf electric field (Ohm’s law), but rather is a function of the whole profile of the rf electric field on distances of order $\lambda$ (anomalous skin effect). Therefore, a rather complicated non-local conductivity operator has to be determined for the calculation of the rf electric field penetration into the plasma. Moreover, the electron energy distribution function (EEDF) is typically non-Maxwellian in these discharges [2]. Hence, for accurate calculation of the
discharge characteristics at low pressures, the EEDF needs to be computed self-consistently. Self-consistency is an important and difficult issue for the kinetic simulations of a plasma. The EEDF, nonlocal conductivity and plasma density profiles are all nonlinear and nonlocally coupled. That is why, the self-consistency aspect of the model is the main concern of this study. The so-called “nonlocal” approach relies on the direct semi-analytic solution of the Boltzmann equation in the limiting regime where the electron relaxation length is much large than the discharge gap, but the electron mean free path is small compared with the discharge dimension \( \lambda \). The nonlocal approach has been successfully applied to the self-consistent kinetic modelling of various low-pressure discharges, where the electron mean free path is small: the capacitively coupled plasmas [3], [4], the inductively coupled plasmas [5], [6], [7]; the dc discharges [11], [12]; the afterglow [13], and the surface-wave discharges [14]. The additional references can be found in reviews: [15], [16], and [17].

If gas pressure lowered even farther, the electron mean free path becomes comparable or even larger than the discharge dimension and numerous collisionless phenomena dominate the discharge characteristics [18]. In the present paper the nonlocal approach is generalized for the low-pressure discharges to account for the collisionless heating and transit-time (electron inertia) effects in the discharge description.

Present analysis considers only an inductively coupled plasma. But the approach has been designed in the most generalized way, so that derivations can be readily performed for other discharges. For example, in Ref. [19] the capacitive discharge; in Ref. [20] the electron-cyclotron-resonance discharge and in Ref. [21] surface-wave discharge were considered with self-consistent account for collisionless heating.

Most previously reported theoretical studies assume a uniform plasma, in a semi-infinite [22] or a slab geometry [23]. In this case the analytical treatment considerably simplifies, because electron trajectories are straight. In the semi-infinite case, electrons traverse the region of the rf electric field (skin layer) and are reflected back into the plasma at the discharge walls. An acquired velocity kick then dissipates in the plasma on distances of order the electron mean free path. If the plasma dimension is small or comparable with \( \lambda \), the subsequent kicks are correlated. The resonance between the wave frequency and the bounce frequency of the electron motion between walls may result in modification of the nonlocal conductivity [24], [25] and may yield an enhanced electron heating [26]. The anomalous skin effect has been studied experimentally in cylindrical [24] and planar discharges [27]. Additional references can be found in the reviews of classical and recent works on the anomalous skin effect in gas discharge plasmas [28], [29]. The theoretical studies in cylindrical geometry are much more cumbersome, and has been done for uniform plasma in Refs. [30], [31], [32] and for a parabolic potential well in Ref. [33]. Qualitative results in the cylindrical geometry are similar to the results in the plane geometry, therefore, in the present study only one-dimensional slab geometry is considered.

For the case of a bounded uniform plasma, the electrostatic potential well is flat in the plasma and infinite at the wall (to simulate the existence of sheaths). In this square potential well, electrons are reflected back into the plasma only at the discharge walls. In a realistic non-uniform plasma, however, the position of the turning points will depend on the electron total (kinetic plus potential) energy and the actual shape of the potential well, i.e., low total energy electrons bounce back at locations within the plasma and may not reach regions of high electric field at all. As a result the current density profiles in a nonuniform plasma may considerably differ from the profiles in a uniform plasma. The theory of the anomalous skin effect for an arbitrary profile of the electrostatic potential and a Maxwellian EVDF was
developed by Meierovich et al. in Refs. [34], [35], and [36] for the slab geometry. Although rigorous analytical results of non-uniform plasmas have been reported, the detailed self-consistent simulations related to such plasmas and comparison with experimental data are lacking. Such simulations were completed recently and presented in our separate publications [37] and [38] and will be additionally reported elsewhere.

The kinetic description of the anomalous skin effect is based on a well-known mechanism of collisionless power dissipation—the Landau damping [39]. In the infinite plasma, the resonance particles moving with a velocity (v) close to the wave phase velocity, so that \( \omega = \mathbf{v} \cdot \mathbf{k} \), intensively interact with wave fields. Therefore, the collisionless electron heating (and the real part of the surface impedance) depends on the magnitude of a Fourier harmonic of the electric field \( [E(k)] \) and the number of the resonant particles \( [f(v_x = \omega/k), x = k/k] \). That is why, the momentum acquired in the skin layer of width \( \delta \) is maximal if the projection of velocity perpendicular to the plasma boundary (x-axis direction) is of order \( \omega/\delta \). If the interaction with the skin layer are repeated in a resonance manner the momentum changes are mounted up. Therefore, the main contribution to the electron heating and the resistive part of the surface impedance comes from these resonant electrons. In a bounded plasma, the resonance condition requires the bounce period \( (T_b) \) be equal to one or several rf electric field periods: \( T_b = 2\pi n/\omega \), where \( n \) is an integer number. The maximum interaction occurs for \( n = 1 \) (see below). For a slab of width \( L \), \( T_b = 2L/v_x \). The maximum electron heating occurs if both aforementioned conditions are satisfied simultaneously, which gives \( \omega/\delta = v_x \) and \( 2L/v_x = 2\pi/\omega \) or \( L = \delta \pi/2 \). Hence, the optimum conditions for the power transfer to the plasma corresponds to the plasma of size comparable with the 3 times of the skin depth. Because the bounce frequency depends on the electrostatic potential, accounting for the plasma nonuniformity is important for a correct calculation of the efficient power coupling.

As discussed before, the collisionless heating is determined by the number of resonant particles, and, hence, is dependent on the EEDF. The EEDF, in its turn, is controlled by the collisionless heating. The only particles, which are in resonance with a wave, are heated by the collisionless heating. It means that in the regime of the collisionless dissipation, the form of the electron energy distribution function is sensitive to the wave spectrum. Therefore, the plateau in the EEDF can be formed in the regions of intensive collisionless heating, if the wave phase velocities are confined in some interval [40]. The evidences of a plateau formation for the capacitive discharge plasma were obtained in Ref. [41]. The cold electrons, which are trapped in the discharge center, do not reach periphery plasma regions where an intensive rf electric field is located, and as a result, these electrons are not heated by the rf electric field. The coupling between the EEDF shape and collisionless heating may result in new nonlinear phenomena: an explosive generation of the cold electrons [12]. The experimental evidences of the collisionless heating on the EEDF were obtained in Ref. [27], [13], [44] and [45].

In the linear approximation the collisionless dissipation does not depend explicitly on the collision frequency. However, as shown in Ref. [16], if the electron elastic collision frequency is too small, heating can actually decrease due to nonlinear effects akin to the nonlinear Landau damping.

The present article presents a self-consistent system of equations describing the non-local electron kinetics in a 1-D slab (bounded) non-uniform plasma. The system consists of a nonlocal conductivity operator, and an averaged over fast electron motions kinetic equation for the EEDF. Transit time (non-local) effects on the current density profile and collisionless heating are of particular interest. Rigorous derivations for the nonlocal conductivity
operator have been performed. The analytic results of Ref. [34] for the Maxwellian EEDF were generalized for the nonMaxwellian EEDF. The spectral method was developed to find the rf electric field profile. A quasilinear approach was used for calculating the collisionless heating. The quasilinear theory developed in Ref. [47] was generalized for an arbitrary value of the collision frequency. As a result, the simulations can be done in a wide range of the background gas pressures ranging from the collisional case ($\lambda \ll \delta$) to the fully collisionless case ($\lambda > L$). Self-consistency of the nonlocal conductivity operator and the energy diffusion coefficient has been verified: both yield the same expression for the power deposition. The robust time-averaging procedure was designed for the kinetic equation in a most general way. As a result, the procedure can be readily repeated for other discharges, see for example [19] and [20].

II. CALCULATION OF ANISOTROPIC PART OF THE ELECTRON VELOCITY DISTRIBUTION FUNCTION

In low-pressure discharges, where the energy relaxation length is large compared with the plasma width, the main of the electron velocity distribution function (EVDF) is a function of the total energy only [15], [16] and [17]. Therefore, we look for $f = f_0 + f_1$, where $f_0(\varepsilon)$ is a function of the total energy $\varepsilon$, $\varepsilon = w + \varphi(x)$, $w = m(v_x^2 + v_y^2 + v_z^2)/2$ is the kinetic energy, $\varphi = -e\phi$ is the electron electrostatic potential energy, and $\phi$ is the electrostatic potential. $f_1$ does not contribute to the electron density (the integral $\int f_1 d^3v = 0$), but $f_1$ contributes to the electron current (the integral $\int f_0 v d^3v = 0$). Typically the mean electron flow velocity ($V_{rf} = \int v f_1 d^3v/\int f_0 d^3v$) is small compared with the thermal velocity ($V_T \equiv \sqrt{2T/m}$). Therefore, the isotropic part of the EVDF is larger than the anisotropic part $f_1 \sim (V_{rf}/V_T)f_0 << f_0$ [15], [16], and [17].

Vlasov’s equation reads:

$$\frac{\partial f_1}{\partial t} + v_x \frac{\partial f_1}{\partial x} + \frac{eE_{sc}(x)}{m} \frac{\partial f_1}{\partial v_x} + \frac{eE_y(x,t)}{m} \frac{\partial (f_0 + f_1)}{\partial v_y} = St(f_1 + f_0), \tag{2.1}$$

where $E_{sc}(x)$ is the space-charge stationary electric field, and $E_y(x,t)$ is the rf nonstationary electric field, $St(f)$ is the collision integral. In the Eq.(2.1), we used the fact that

$$v_x \frac{\partial f_0(\varepsilon)}{\partial x} + \frac{eE_{sc}(x)}{m} \frac{\partial f_0(\varepsilon)}{\partial v_x} = v_x \frac{\partial f_0(\varepsilon)}{\partial x} |_{\varepsilon_x = 0}, \tag{2.2}$$

because $\varepsilon_x$ is constant along a trajectory. After applying the standard quasilinear theory, Eq.(2.1) splits into two equations [17]: a linear equation for $f_1$

$$\frac{\partial f_1}{\partial t} + v_x \frac{\partial f_1}{\partial x} + \frac{eE_{sc}(x)}{m} \frac{\partial f_1}{\partial v_x} + \frac{eE_y(x,t)}{m} \frac{\partial f_0}{\partial v_y} = St(f_1), \tag{2.3}$$

and a quasilinear equation for $f_0$

$$\frac{eE_y(x,t)}{m} \frac{df_1}{dv_y} = St(f_0), \tag{2.4}$$

where upper bar denotes space-time averaging over the phase space available for the electron with the total energy $\varepsilon$ [18], [19], [20].
The rf electric field $E_y(x, t) = E_{y0}(x) \exp(-i\omega t)$ and the anisotropic part of the EVDF $f_1 = f_{10} \exp(-i\omega t)$ are harmonic functions, where $\omega$ is the discharge frequency. In what follows the subscript 0 is omitted. Eq.(2.3) becomes

$$-i\omega f_1 + v_x \frac{\partial f_1}{\partial x}|_{x=x} + e v_y E_y(x) \frac{df_0}{d\varepsilon} = -\nu f_1. \tag{2.5}$$

In transformation from Eq.(2.3) to Eq.(2.5) the BGK approximation was used $St(f_1) = -\nu f_1$, where $\nu$ is the transport collision frequency and introduced the new variable: the total energy along $x$-axis $\varepsilon_x = m v_x^2/2 + \varphi(x)$. There have been a number of studies, which explored the effects of the exact collision integral on collisionless phenomena [51], [52]. These treatments use expansion in series of spherical functions in velocity spaces. The exact calculation are important only if the collision frequency is a strong function of the poloidal scattering angle. If the differential cross section does not depend on the poloidal scattering angle, the BGK approximation is correct exactly [52]. For partially ionized plasma electron-neutral collisions are the most frequent scattering mechanism. At typical electron energies in the low-pressure discharges 1-5 eV [2], the differential cross section weakly depends on the poloidal scattering angle, and, therefore, the BGK approximation has a good accuracy.

Equation (2.5) can be solved by a number of different methods. First, let us consider a direct solution. Alternative derivation using Fourier series is performed in Appendix C. After some straightforward algebra described in Appendix A, the symmetric part of the EVDF $f_{1s} \equiv 1/2(f_{1+} + f_{1-})$ is given by

$$f_{1s}(v, x) = -mv_y V_{y}^{rf}(x, \varepsilon_x) \frac{df_0}{d\varepsilon}, \tag{2.6}$$

where $V_{y}^{rf}(x, \varepsilon_x) = 1/2(V_{y+}^{rf} + V_{y-}^{rf})$, $V_{y}^{rf}$ are the oscillatory velocities of an electron with a given $\varepsilon_x$, $\pm$ signs denote $v_x > 0$ and $v_x < 0$, respectively:

$$V_{y}^{rf}(x, \varepsilon_x, v_{\perp}) = \frac{e}{m \sinh \Phi_+} \left[ \cosh \Phi \int_{x_+}^{x} E_y(x') \cosh(\Phi_+ - \Phi') d\tau', \cosh(\Phi_+ - \Phi) \int_{x_-}^{x} E_y(x') \cosh \Phi' d\tau' \right], \tag{2.7}$$

$$\tau \equiv \int_{x_-}^{x} \frac{dx}{|v_x(x, \varepsilon_x)|}, \tag{2.8}$$

$$\Phi(x, \varepsilon_x, v_{\perp}) \equiv \int_{x_-}^{x} (-i\omega + \nu) d\tau, \tag{2.9}$$

$$\Phi_+(\varepsilon_x, v_{\perp}) \equiv \Phi(x_+, \varepsilon_x, v_{\perp}), \tag{2.10}$$

where $x_-(\varepsilon_x), x_+(\varepsilon_x)$ are the left and right turning points, respectively, for the electron with energy $\varepsilon_x$ [corresponding to zero velocity $v_x$ or $\varepsilon_x = e\varphi(x_-)$], $\tau$ is the time of flight from the left turning point $x_-(\varepsilon_x)$ to $x$, $v_{\perp} = \sqrt{v_y^2 + v_x^2}$. The functions $V_{y}^{rf}$ and $\Phi$ depend on the electron speed via the collision frequency $\nu(v)$.

In the local limit the electron mean free path is large $\lambda >> \delta$ and phase $Im(\Phi) >> 1$. Therefore, $\cosh \Phi \approx \sinh \Phi \approx 1/2 \exp(\Phi)$. The main contribution in the both integrals in Eq.(7.10) are near the point $x' = x$ and hence $d\Phi = (-i\omega + \nu) d\tau$

$$V_{y}^{rf} \approx \frac{e}{m} \frac{E_y(x)}{-i\omega + \nu}, \tag{2.11}$$

as it should be in the local limit.


III. CALCULATION OF NONLOCAL CONDUCTIVITY

Knowing the EVDF $f_{1s}$, one can calculate the current density

$$j = \frac{em^{3/2}}{4\pi \sqrt{2}} \int f_{1s} v_y d^3v. \quad (3.1)$$

Substituting $f_{1s}$ from Eq.(2.6) into Eq.(3.1) and making the transformation to the spherical coordinates in the velocity space $dv_x dv_y dv_z = v^2 dv \sin \vartheta d\vartheta d\psi$ in ($\cos \vartheta = v_x/v$; $\tan \psi = v_y/v_z$ ) Eq.(3.1) becomes

$$j(x) = -e\sqrt{2m^{3/2}} \int_{0}^{\infty} w \left\langle v_y^2 V_r f \right\rangle \frac{df_0(\varepsilon)}{d\varepsilon} dv,$$  \quad (3.2)

where the averaged over velocity direction factor $\left\langle v_y^2 V_r f \right\rangle$ is

$$\left\langle v_y^2 V_r f \right\rangle \equiv \frac{v^2}{4\pi} \int_{0}^{\pi} \int_{0}^{2\pi} V_r f(x, \varepsilon_x, v)[\sin \vartheta]^3[\cos \psi]^2 d\psi d\vartheta. \quad (3.3)$$

Because $V_r f$ does not depend on $\psi$, the integration over $\psi$-angle can be completed. Changing integral from $\vartheta$ to $v_x = v \cos \vartheta$ gives

$$\left\langle v_y^2 V_r f \right\rangle \equiv \frac{1}{4v} \int_{-v}^{v} V_r f(x, \varepsilon_x, v_{\perp})(v^2 - v_x^2) dv_x,$$  \quad (3.4)

or

$$\left\langle v_y V_r f \right\rangle \equiv \frac{1}{2m\sqrt{w}} \int_{\varphi(x)}^{\varepsilon} V_r f(x, \varepsilon_x, v_{\perp}) \frac{\varepsilon - \varepsilon_x}{\sqrt{\varepsilon_x - \varphi(x)}} d\varepsilon_x. \quad (3.5)$$

Substituting Eq.(3.5) into Eq.(3.2) and changing integration from $v$ to $\varepsilon$ yields

$$j(x) = -\frac{e}{2} \int_{0}^{\infty} \left[ \int_{\varphi(x)}^{\varepsilon} V_r f(x, \varepsilon_x) \frac{\varepsilon - \varepsilon_x}{\sqrt{\varepsilon_x - \varphi(x)}} d\varepsilon_x \right] \frac{df_0(\varepsilon)}{d\varepsilon} d\varepsilon. \quad (3.6)$$

Further simplifications are possible if the collision frequency $\nu$ is small ($\nu << \omega$) or $\nu$ does not depend on electron velocity. In this case $V_r f(x, \varepsilon_x, v_{\perp})$ is the only function of $(x, \varepsilon_x)$. Integrating Eq.(3.3) in parts yields

$$j(x) = \frac{e}{2} \int_{\varphi(x)}^{\infty} \left[ \left. \int_{\varphi(x)}^{\varepsilon} \frac{V_r f(x, \varepsilon_x) d\varepsilon_x}{\sqrt{\varepsilon_x - \varphi(x)}} \right] f_0(\varepsilon) d\varepsilon \right] f_0(\varepsilon) d\varepsilon. \quad (3.7)$$

If $V_r f$ is a constant Eq.(3.7) gives trivial result: $j = evV_r f$.

Introducing a new function $\Gamma(\varepsilon)$

$$\Gamma(\varepsilon) \equiv \int_{0}^{\infty} f_0(\varepsilon) d\varepsilon, \quad (3.8)$$

and integrating Eq.(3.7) in parts one more time gives

$$j(x) = \frac{e}{2} \int_{\varphi(x)}^{\infty} \frac{V_r f(x, \varepsilon_x) \Gamma(\varepsilon)}{\sqrt{\varepsilon - \varphi(x)}} d\varepsilon \quad (3.9)$$
For the Maxwellian EVDF $f_0$, Eq. (3.9) is equivalent to Lieberman’s et al. result [34]. Substituting Eq. (2.7) into Eq. (3.9) yields the nonlocal conductivity operator

$$j_y(x) = \int_0^x G(x, x') E_y(x') dx' + \int_x^L G(x', x) E_y(x') dx'$$

(3.10)

where

$$G(x, x') = \frac{\varepsilon^2}{2\sqrt{2m}} \int_{\text{max}(\varphi, \varphi')}^{\infty} \frac{\cosh \Phi \cosh(\Phi_+ - \Phi')}{\sinh \Phi_+} \frac{\Gamma(\varepsilon)}{\sqrt{\varepsilon - \varphi(x)} \sqrt{\varepsilon - \varphi(x')}} d\varepsilon.$$  

(3.11)

Note that $G(x, x')$ has a logarithmic singularity at $x = x'$ [34], but because calculation of the electron current in Eq. (3.10) requires additional integration, there is no singularity in the current.

In the limit of large gap, where $\delta < \lambda < L$, $\text{Re}(\Phi) \gg 1$ and $\cosh \Phi \cosh(\Phi_+ - \Phi') / \sinh \Phi_+ \to 1/2[\exp(\Phi - \Phi') + \exp(-\Phi - \Phi')]$. And the region of integration in Eq. (3.10) beyond the skin layer can be omitted. In the local limit, where $\lambda << \delta$, Eq. (3.10) gives the standard local conductivity, see Eq. (2.11).

IV. CALCULATION OF THE TRANSVERSE RF ELECTRIC FIELD PROFILE

Maxwell’s equations can be reduced to a single scalar equation for the transverse electric field [28]

$$\frac{d^2 E_y}{dx^2} + \frac{\omega^2}{c^2} E_y = -\frac{4\pi i\omega}{c^2} [j(x) + I\delta(x) - \delta_{\text{anti}}I\delta(x - L)],$$

(4.1)

where the electron current $j$ is given by Eq. (3.10), $I$ is the current in the coil, $\delta_{\text{anti}} = 0$, if the there is no any coil with the current located at $x = L$, and $\delta_{\text{anti},k} = 1$, if there is a coil with the current $-I$ at $x = L$. The 1D slab system of two currents flowing in opposite directions describes very well a cylindrical configuration, where a coil produces rf currents at both plasma boundaries $x = -R$ and $x = R$, $R = L/2$ [23], [24]. The Eq. (4.1) and Eq. (3.10) can be solved numerically using a finite difference scheme. There is major difficulty in such approach. Straightforward computing the complex Green’s function in Eq. (3.11) is slow and time consuming [37]. The better approach is to solve the integro-differential Eq. (4.1) making use of a spectral method. In 1D geometry the electric field can be represented as a sum of harmonic functions. The method is described in the Appendix D.

V. AVERAGING OF KINETIC EQUATION FOR THE MAIN PART OF THE EVDF

Kinetic equation for $f_0$ averaged over the discharge period is

$$v_x \frac{\partial f_0}{\partial x} + \frac{e}{m} E_{se}(x) \frac{df_0}{dv_x} + \frac{e}{2m} \text{Re} \left[ E_{y}^*(x) \frac{df_1}{dv_y} \right] = St(f_0),$$

(5.1)

$$St(f) = St_{el}^v(f) + St_{el}^e(f) + St_{ee}(f) + St_{inel}(f),$$

(5.2)
\[ \text{St}_{el}^v(f) = \int (f' - f) d\sigma, \quad (5.3) \]

\[ \text{St}_{el}^e(f) = \frac{\partial}{v \partial w} (v V_{el} f), \quad (5.4) \]

\[ \text{St}_{ee}(f) = \frac{\partial}{v \partial w} \left( v D_{ee} \frac{\partial}{w} f \right) + \frac{\partial}{v \partial w} (v V_{ee} f), \quad (5.5) \]

\[ \text{St}_{inel}(f_0) = \sum_k \left[ \sqrt{\frac{w + \varepsilon_k^*}{w}} \nu_k^* f_0 (w + \varepsilon_k^*) - \nu_k^* f_0 \right], \quad (5.6) \]

where \( w = \frac{mv^2}{2} \) is the kinetic energy, \( \text{St}_{el}^v(f) \) is the part of the elastic scattering collision integral with differential cross section \( d\sigma \), which changes the electron momentum but does not alter the kinetic energy, \( \text{St}_{el}^e(f) \) accounts for energy change in elastic collisions, \( \text{St}_{ee}(f_0) \) is the electron-electron collision integral, and \( \text{St}_{inel}(f_0) \) is the sum over all inelastic collisions with the electron energy loss \( \varepsilon_k^* \) and inelastic collision frequency \( \nu_k^* \) (see details for ionization and wall losses in [5], [20]). Here, the coefficients \( D_{ee}, V_{ee}, V_{el} \) are given by [54], [49], [5].

\[ V_{el} = \frac{2m}{M} w \nu, \quad (5.7) \]

\[ V_{ee} = 2w \nu_{ee} \int_0^w dw \sqrt{w f}, \quad (5.8) \]

\[ D_{ee} = \frac{4}{3} w \nu_{ee} \int_0^w dw w^{3/2} f + w^{3/2} \int_w^\infty dw f, \quad (5.9) \]

\[ \nu_{ee} = \frac{4\pi \Lambda_{ee} n}{m^2 v^3}, \quad (5.10) \]

where \( \nu_{ee} \) is the Coulomb collision frequency and \( \Lambda_{ee} \) is the Coulomb logarithm. Note that at large electron energies \( V_{ee} = 2w \nu_{ee} \) and \( D_{ee} = 2w T_e \nu_{ee} \), where \( T_e = 2/3 \int_0^w dw w^{3/2} f/n \).

If the electron mean free path is large compared with gap \( (\lambda >> L) \), the first two terms on the left hand side are dominant. Therefore, \( f_0 \) is approximately a function of \( \varepsilon_x \) only, not a function of both variables \( x, v_x \) separately. Similarly, \( \text{St}_{el}^v(f) \) is the largest term from the remaining terms in the equation. Therefore, \( f_0 \) is approximately isotropic, and is a function of \( \varepsilon \) only, so that \( \text{St}(f_0) = 0 \) [50]. This assumption was verified by comparison with particle-in-cell simulations in Ref. [3] for a capacitive coupled plasma, in Ref. [5], [32] for an inductive coupled plasma, and in Ref. [20] for an ECR discharge and in Ref. [53] experimentally.

To find \( f_0 \), it is necessary to average over fast electron bouncing and over all velocity angles. First, let us average over fast electron bouncing. In order to do so, we integrate all terms of Eq.(5.1) over the full period of electron bouncing

\[ \oint dt \text{Term}(x, v_x) \equiv \int_{x_+}^{x_+} \frac{dx}{v_x} \text{Term}(x, v_x > 0) + \int_{x_+}^{x_+} \frac{dx}{|v_x|} \text{Term}(x, v_x < 0), \]

where \( \text{Term}(x, v_x) \) is a term in Eq.(5.1). Because the first two terms represent the full time derivative \( df/dt \) along trajectory, they disappear after integration, and Eq.(5.1) becomes

\[ \oint dt \frac{e}{2m} \text{Re} \left[ E_y^v(x) \frac{df_1}{dv_y} \right] = \oint dt \text{St}(f_0). \quad (5.11) \]
Second, we integrate Eq. (5.11) over all possible perpendicular velocities \(dv_y dv_z\) with a given total energy \(m(v_y^2 + v_z^2) < 2\varepsilon\). Total averaging is a triple integral
\[
\overline{\text{Term}(x, v)} = \frac{1}{4\pi} \int \int dv_y dv_z \int dt \text{Term}(x, v),
\]
where the factor \(1/4\pi\) is introduced for the normalization purposes. Note, that integral Eq. (5.12) is simply averaging over all phase space available for the electron with the total energy \(\varepsilon\).

If the \(\text{Term}(x, v)\) depends on velocity only via speed \(v\), like, e.g., the inelastic collision integral Eq. (5.6), then integration of Eq. (5.13) in spherical coordinates gives
\[
\overline{\text{Term}(x, v)}(\varepsilon) = \int_{x-}^{x+} dx v(x, \varepsilon) \text{Term}[x, v(x, \varepsilon)],
\]
where
\[
\overline{v(x, \varepsilon)} = \sqrt{2[\varepsilon - \varphi(x)]/m}.
\]

Thus, averaging of the terms, which are functions of the position and the kinetic energy only, reduces to the integrating over the entire available discharge volume weighted with the velocity, which is the standard procedure that also appears in the averaged kinetic equations for the local (collisional) case \[17\].

**A. Calculation of the nonlocal energy diffusion coefficient**

The averaged energy diffusion term originates from the averaged left hand side of Eq. (5.11), which gives
\[
\frac{eE_y(x, t)}{m} \frac{df_1}{dv_y} = \frac{e}{8\pi} \int \int dx d^3v \delta[\varepsilon - w - \varphi(x)] Re \left[ \frac{E_y^*(x)}{dv_y} \frac{df_1}{d\varepsilon} \right].
\]

Using chain rule for the integration in \(dv_y\) and the fact that \(d\delta(\varepsilon - w - \varphi)/d\varepsilon = mv_y d\delta(\varepsilon - w - \varphi)/d\varepsilon\), Eq. (5.16) becomes
\[
\frac{eE_y(x, t)}{m} \frac{df_1}{dv_y} = \frac{em}{8\pi} d\varepsilon \Re \int dx d^3v \delta[\varepsilon - w - \varphi(x)] v_y E_y^*(x) f_1.
\]

Substituting \(f_1\) from Eq. (2.6) and integrating in the velocities \(v_y\) and \(v_z\) yields
\[
\frac{eE_y(x, t)}{m} \frac{df_1}{dv_y} = -\frac{d}{d\varepsilon} D_\varepsilon \frac{df_0}{d\varepsilon},
\]
where \(D_\varepsilon\) is the energy diffusion coefficient
\[
D_\varepsilon = \frac{e}{4m} Re \int_0^\varepsilon d\varepsilon_x (\varepsilon - \varepsilon_x) \int_{x-(\varepsilon_x)}^{x+(\varepsilon_x)} dx \frac{v_y}{v_x} E_y^*(x) V_x f(x, \varepsilon_x).
\]

Equation (5.19) is the general expression for the energy diffusion coefficient: in the limiting regime of the small mean free path (\(\lambda << \delta\)) it tends to the local limit, in the intermediate
pressure range ($\delta \ll \lambda \ll L$) Eq.(5.19) corresponds to the hybrid heating: electron motion in the skin layer is collisionless, but the randomization of the velocity kick acquired during a single pass through the skin layer occurs due to collisions in the plasma bulk, and in the opposite limit of the large mean free path ($\lambda > L$) Eq.(5.19) describes collisionless heating (see Appendix B,C for details). If the collision frequency does not depend on the kinetic energy the direct substitution of Eq.(2.7) for $V_{rf}$ gives

$$D_\varepsilon(\varepsilon) = \left( \frac{\pi}{4} \sum_{n=-\infty}^{\infty} \int_0^\infty d\varepsilon_x \right) \left| E_{yn}(\varepsilon_x) \right|^2 \frac{\Omega_b(\varepsilon_x)}{\Omega_b(\varepsilon_x)n - \omega} \frac{\nu}{\nu^2 + \nu_0^2},$$  (5.20)

where

$$E_{yn}(\varepsilon_x) = \frac{1}{\pi} \int_0^\pi E_y(\theta) \cos(n\theta) d\theta.$$  (5.21)

Note that $D_\varepsilon(\varepsilon)$ in the last equation accounts for the bounce resonance $\Omega_b(\varepsilon_x)n = \omega$ and the transit time resonance $\omega = v/\delta$, which corresponds to the maxima of $E_{yn}(\varepsilon_x)$.

VI. SELF-CONSISTENT SYSTEM OF EQUATIONS

In summary, the self-consistent system of equations for the kinetic description of low-pressure discharges accounting for nonlocal and collisionless electron dynamics contains:

1. The averaged kinetic equation for $f_0$ reads

$$-\frac{d}{d\varepsilon} \left( D_\varepsilon + D_{ee} \right) \frac{df_0}{d\varepsilon} - \frac{d}{d\varepsilon} \left[ V_{ee} + V_{el} \right] f_0 = \sum_k \left[ \nu_k \sqrt{\left( w + \varepsilon_k \right)} \sqrt{\varepsilon - \varphi(x)} f_0(\varepsilon + \varepsilon_k) - \nu_k f_0 \right],$$  (6.1)

where the upper bar denotes averaging according to Eq.(5.14) and $D_{ee}$ is given by Eq.(5.9), $V_{ee}$ by Eq.(5.8), $V_{el}$ by Eq.(5.7), and $D_\varepsilon$ by Eq.(5.19) or by Eq.(5.20).

2. The rf electric field is determined from the Maxwell Eq.(4.1), where the electron current is given by Eq.(3.10).

3. The electrostatic potential is obtained using the quasineutrality condition

$$n_i(x) = \int_{\varphi(x)}^\infty f_0(\varepsilon) \sqrt{\varepsilon - \varphi(x)} d\varepsilon,$$  (6.2)

where $n_i(x)$ is the ion density profile given by a set of number particles and ion momentum equations [37]. Eq.(6.2) is solved in the form of a differential equation [3]

$$\frac{d\varphi}{dx} = -T_e^{scr}(x) \frac{d\ln[n_i(x)]}{dx},$$  (6.3)

where $T_e^{scr}(x)$ is the electron screening temperature

$$T_e^{scr}(x) = \left[ \frac{1}{2n(x)} \int_{\varphi(x)}^\infty f_0(\varepsilon) \frac{d\varepsilon}{\sqrt{\varepsilon - \varphi(x)}} \right]^{-1}.$$  (6.4)

4. The power deposition can be computed as

$$P(x) = \frac{1}{2} Re \left[ E_y(x) j(x) \right].$$  (6.5)
Substituting Eq. (3.6) and changing the integration order, Eq. (6.5) becomes

\[
P = -\sqrt{2m} \int_{0}^{\infty} D_{\varepsilon}(\varepsilon) \frac{df_{0}(\varepsilon)}{d\varepsilon} d\varepsilon .
\]  

(6.6)

Equation (6.6) can be used as a consistency check.

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VII. APPENDIXES

A. Derivation of \( f_1 \)

Direct integration of Eq. (2.5) yields

\[
f_{1+}(x, v) = -e v_y \frac{df_{0}}{d\varepsilon} \left[ \int_{x_{-}}^{x_{+}} e^{-(\Phi(x)-\Phi(x'))} E' d\tau' + f_{1+}(x_{-}) e^{-\Phi(x)} \right].
\]  

(7.1)

\[
f_{1-}(x, v) = -e v_y \frac{df_{0}}{d\varepsilon} \left[ -\int_{x_{-}}^{x_{+}} e^{\Phi(x)-\Phi(x')} E_y' d\tau' + f_{1-}(x_{-}) e^{\Phi(x)} \right].
\]  

(7.2)

where \( \pm \) signs denote \( v_x > 0 \) and \( v_x < 0 \), respectively, and for brevity we introduced \( E_y' \equiv E_y(x') \) and \( d\tau = dx/|v_x| \). The two constants \( f_{1+}(x_{-}), f_{1-}(x_{-}) \) are to be determined from the boundary condition at the turning points, and

\[
\Phi(x) \equiv \int_{x_{-}}^{x} (-i \omega + \nu) d\tau .
\]  

(7.3)

Due to continuity of the EVDF

\[
f_{1-}(x_{-}) = f_{1+}(x_{-}), \quad f_{1-}(x_{+}) = f_{1+}(x_{-}).
\]  

(7.4)

Substituting the boundary condition at the turning points Eqs. (7.4) into Eqs. (7.1) and (7.2) yields

\[
- \int_{x_{-}}^{x_{+}} e^{-\Phi'} E_y' d\tau' + f_{1-}(x_{-}) e^{\Phi} = \int_{x_{-}}^{x_{+}} e^{-(\Phi_+ - \Phi')} E_y' d\tau' + f_{1+}(x_{-}) e^{-\Phi},
\]

or

\[
f_{1+}(x_{-}) = \frac{e}{\sinh \Phi_+} \int_{x_{-}}^{x_{+}} \cosh(\Phi_+ - \Phi') E_y' d\tau',
\]  

(7.5)

where \( \Phi \equiv \Phi(x), \Phi' \equiv \Phi(x'), \) and \( \Phi_+ \equiv \Phi(x_+) \). \( f_1 \) enters into the current calculation only as a sum \( f_{1+} + f_{1-} \). Therefore, we compute \( f_{1s} \equiv 1/2(f_{1+} + f_{1-}) \) from Eqs. (7.1) and (7.2)

\[
f_{1s} = -e v_y \frac{df_{0}}{d\varepsilon} \left\{ f_{1+}(x_{-}) \cosh \Phi - \int_{x_{-}}^{x} \sinh(\Phi - \Phi') E_y(\theta') d\tau' \right\},
\]  

(7.6)

substituting \( f_{1+}(x_{-}) \) from Eq. (7.5) gives

\[
f_{1s} = -m v_y V_{y} \frac{df_{0}}{d\varepsilon},
\]  

(7.7)
where
\[ V_{rf}^y = \frac{e}{m \sinh \Phi_+} \left[ \cosh \Phi \int_{x_-}^{x_+} \cosh(\Phi_+ - \Phi') E_y' d\tau' - \sinh \Phi_+ \int_{x_-}^{x_+} \sinh(\Phi - \Phi') E_y' d\tau' \right]. \tag{7.8} \]

Splitting the first term into two integrals \( \int_{x_-}^{x_+} = \int_{x_-}^{x} + \int_{x}^{x_+} \), and accounting for the fact that
\[ \cosh \Phi \cosh(\Phi_+ - \Phi) - \sinh \Phi_+ \sinh(\Phi - \Phi') = \cosh \Phi \cosh(\Phi_+ - \Phi') \tag{7.9} \]
gives
\[ V_{rf}^y = \frac{e}{m \sinh \Phi_+} \left[ \cosh \Phi \int_{x_-}^{x} E_y' \cosh(\Phi_+ - \Phi') d\tau' + \cosh(\Phi_+ - \Phi) \int_{x}^{x_+} E_y' \cosh \Phi' d\tau' \right]. \tag{7.10} \]

### B. Diffusion coefficient in the energy space

The equation for the energy diffusion coefficient
\[ D_\varepsilon = \frac{e^2}{4m^2} \text{Re} \int_0^\varepsilon d\varepsilon_x \left( \varepsilon - \varepsilon_x \right) \int_{x_-}^{x_+} \frac{dx}{v_x} E_y^*(x) V_{rf}^y(x, \varepsilon_x) \tag{7.11} \]
has correct limits in local and nonlocal cases.

1. **local limit** \( \lambda << L \)

In the local limit of the small mean free path, substituting \( V_{rf}^y \) from Eq.(2.11) into Eq.(7.11) gives
\[ D_\varepsilon = \frac{e^2}{4m^2} \text{Re} \int_0^\varepsilon d\varepsilon_x \left( \varepsilon - \varepsilon_x \right) \int_{x_-}^{x_+} \frac{dx}{\sqrt{2(\varepsilon_x - \varphi)/m}} \frac{E_y^*(x) E_y(x)}{-i\omega + \nu}. \tag{7.12} \]

Changing the order of the integration and accounting for the fact that
\[ \frac{1}{m^2} \int_0^\varepsilon d\varepsilon_x \frac{\varepsilon - \varepsilon_x}{\sqrt{2(\varepsilon_x - \varphi)/m}} = \frac{2}{3} v^3 \]
\[ D_\varepsilon = \frac{e^2}{6} \text{Re} \int_{x_-}^{x_+} dx |E_y|^2 \frac{\nu v^3}{(\omega^2 + \nu^2)}, \tag{7.13} \]
which corresponds to the local limit \[16, 17\].

2. **nonlocal limit** \( \delta << \lambda << L \)

In the nonlocal limit, collisions during the electron motion in the skin layer are rare. Therefore, \( V_{rf}^y \) is simply velocity ”kick” due the rf electric field. Recalling that \( V_{rf}^y(x, \varepsilon) = (V_{rf}^y + V_{rf}^y)/2 \), the last factor in Eq.(7.11) can be written as
\[ \frac{1}{2m} \int_{x_-}^{x_+} \frac{dx}{v_x} E_y^*(x) V_{rf}^y(x, \varepsilon_x) = \left\langle \frac{1}{2} \int d\tau \frac{d\Delta V_y(\tau)}{d\tau} \Delta V_y(\tau) \right\rangle = \frac{1}{4} \left\langle \Delta V_{y\infty}^2 \right\rangle, \tag{7.14} \]
where $\int dt$ is an integral along the electron trajectory entering and leaving the skin layer, and $\Delta V_{y\infty}$ is the total velocity kick after a single path through the skin layer, and the angular brackets denote averaging over phases of the rf field. Eq. (7.15) simplifies to

$$D_\varepsilon = \frac{1}{8} \int_0^\varepsilon d\varepsilon_x (\varepsilon - \varepsilon_x) \langle \Delta V_{y\infty}^2 \rangle.$$  \hspace{1cm} (7.15)

In the limit of a uniform plasma Eq. (7.15) was proposed in Ref. [26], [32].

3. collisionless limit $\lambda \gg L$

The energy diffusion coefficient Eq. (7.11) with substitution of $V_{yrf}$ from Eq. (7.10) is determined by the following integral:

$$\text{Int} = \int_x^{x^+} E_y(x') V_{yrf} d\tau = \frac{1}{\sinh \Phi_+} \int_x^{x^+} E_y(x) d\tau \left[ \cosh \Phi \int_x^{x^+} E_y(x') \cosh(\Phi_+ - \Phi') d\tau' + \cosh(\Phi_+ - \Phi) \int_x^{x^+} E_y(x') \cosh \Phi' d\tau' \right].$$

The term in brackets is $\cosh \Phi \int_x^{x^+} E_y(x') \cosh(\Phi_+ - \Phi') d\tau' + \cosh(\Phi_+ - \Phi) \int_x^{x^+} E_y(x') \cosh \Phi' d\tau'$. Therefore,

$$\text{Int} = \frac{\cosh \Phi_+}{\sinh \Phi_+} \int_x^{x^+} E_y(x) \cosh \Phi' d\tau' \int_x^{x^+} E_y(x') \cosh \Phi' d\tau' + \text{Int}1,$$

where

$$\text{Int}1 = \int_x^{x^+} E_y(x') d\tau' \left[ \cosh \Phi \int_x^{x^+} E_y(x') \sinh \Phi' d\tau' + \sinh \Phi \int_x^{x^+} E_y(x') \cosh \Phi' d\tau' \right].$$

Integrating in parts gives $\text{Int}1 = \int_x^{x^+} \sinh \Phi \cosh \Phi' d\tau' \int_x^{x^+} E_y(x') (E_y' + E_y E_y') \cosh \Phi' d\tau'$. Therefore,

$$D_\varepsilon = \frac{1}{4} \text{Re} \int_0^\varepsilon d\varepsilon_x (\varepsilon - \varepsilon_x) \coth \Phi_+ \int_x^{x^+} E_y(x) \cosh \omega T' d\tau' \int_x^{x^+} E_y(x') \cosh \omega T'' d\tau'',$$  \hspace{1cm} (7.16)

where $\sinh \Phi_+ = \imath \sin \omega T + \nu T \cos \omega T$. Main contribution comes from the points, where $\omega T = \pi n$ and $\coth \Phi_+ = \pi \sum_{n=1}^{\infty} \delta(\omega T - \pi n)$.

$$D_\varepsilon(\varepsilon) = \frac{\pi e^2}{4} \sum_{n=-\infty}^{\infty} \int_{-\infty}^\varepsilon d\varepsilon_x \left| Ef \right|^2 (\varepsilon - \varepsilon_x) \delta \left[ \omega T(\varepsilon_x) - \pi n \right],$$  \hspace{1cm} (7.17)

$$Ef(\varepsilon_x) = \int_{x_-(\varepsilon_x)}^{x_+(\varepsilon_x)} E_y(x') \cos \omega T' d\tau'$$  \hspace{1cm} (7.18)

This corresponds to the previous results [17].

C. Alternative derivations in Fourier space.

The direct calculation described in the previous sections are rather cumbersome. The alternative derivation can be done easier using Fourier series.

It is convenient to introduce the variable angle of the bounce motion

$$\theta(x, \varepsilon_x) = \frac{\pi \text{sgn}(v_x)}{T(\varepsilon_x)} \int_{x_-(\varepsilon_x)}^{x_+(\varepsilon_x)} dx \frac{dx}{|v_x(\varepsilon_x)|},$$  \hspace{1cm} (7.19)
where \( T \) is the half of the bounce period for the electron motion in the potential well \( \varphi(x) \), which is given by

\[
T(\varepsilon_x) = \int_{x_-}^{x_+} \frac{dx}{|v_x(\varepsilon_x)|}. \tag{7.20}
\]

The bounce frequency for the electron in the potential well is \( \Omega_b(\varepsilon_x) = \pi/T(\varepsilon_x) \). Utilizing angle variable, Eq.\((2.5)\) simplifies to become

\[
-i\omega f_1 + \Omega_b(\varepsilon_x) \frac{\partial f_1}{\partial \theta} \bigg|_{\varepsilon_x} + v_y e E_y(\theta) \frac{df_0}{d\varepsilon} = -\nu f_1, \tag{7.21}
\]

where \( \pm \) signs denote \( v_x > 0 \) and \( v_x < 0 \), respectively and \( \Omega_b(\varepsilon_x) = \pi/T(\varepsilon_x) \) is the bounce frequency in the potential well.

We shall use Fourier series in variable \( \theta \):

\[
g(x,\varepsilon_x) = \sum_{n=-\infty}^{\infty} g_n \exp(in\theta) \tag{7.22}
\]

\[
g_n = \frac{1}{2\pi} \left[ \int_{-\pi}^{\pi} g(\theta,\varepsilon_x) \exp(-in\pi\theta) \, d\theta \right]. \tag{7.23}
\]

Note that in the last integral, the region \( 0 < \theta < \pi \) corresponds to \( v_x > 0 \), and the region \( -\pi < \theta < 0 \) corresponds to \( v_x < 0 \). Utilizing the Fourier series Eq.\((7.23)\), the Vlasov equation becomes

\[
(i\Omega_b - i\omega + \nu) f_{1n} = -E_{yn} v_y \frac{df_0}{d\varepsilon}, \tag{7.24}
\]

where

\[
E_{yn}(\varepsilon_x) = \frac{1}{\pi} \left[ \int_{0}^{\pi} E_y(\theta) \cos(n\theta) \, d\theta \right]. \tag{7.25}
\]

Making use of Fourier series Eq.\((7.22)\), Eq.\((7.24)\) gives

\[
f_{1s}(x,\varepsilon_x) = -mv_y V^{rf}_{y}(x,\varepsilon_x) \frac{df_0}{d\varepsilon}, \tag{7.26}
\]

\[
V^{rf}_{y}(x,\varepsilon_x) = \frac{e}{m} \sum_{n=-\infty}^{\infty} \frac{E_{yn} \cos[n\theta(x)]}{in\Omega_b - i\omega + \nu}. \tag{7.27}
\]

Eq.\((7.26)\) is the alternative form of Eq.\((7.10)\).

Substituting the function \( V^{rf}_{y}(x,\varepsilon_x) \) from Eq.\((7.26)\) into Eq.\((3.9)\) gives the current density

\[
j(x) = \frac{e^2}{2m} \sum_{n=-\infty}^{\infty} \int_{\varphi(x)}^{\infty} \frac{\Gamma(\varepsilon)}{\sqrt{\varepsilon - \varphi(x)}} \frac{E_{yn} \cos[n\theta(x)]}{in\Omega_b - i\omega + \nu} \, d\varepsilon. \tag{7.27}
\]

The averaged energy coefficient is given by Eq.\((7.11)\). Substituting the function \( V^{rf}_{y}(x,\varepsilon_x) \) from Eq.\((7.26)\) into Eq.\((7.11)\) gives

\[
D_\varepsilon = \frac{e^2}{4} \operatorname{Re} \int_{0}^{\varepsilon} d\varepsilon_x (\varepsilon - \varepsilon_x) \int_{x_-}^{x_+} \frac{dx}{v_x} E_y^*(x) \sum_{n=-\infty}^{\infty} \frac{E_{yn} \cos[n\theta(x)]}{in\Omega_b - i\omega + \nu}, \tag{7.28}
\]

or

\[
D_\varepsilon(\varepsilon) = \frac{\pi e^2}{4} \sum_{n=-\infty}^{\infty} \int_{0}^{\varepsilon} d\varepsilon_x |E_{yn}(\varepsilon_x)|^2 (\varepsilon - \varepsilon_x) \frac{\nu}{\Omega_b(\varepsilon_x) [\Omega_b(\varepsilon_x) n - \omega]^2 + \nu^2}. \tag{7.28}
\]

Note that Eq.\((7.28)\) is valid for any collision frequency, and Eq.\((7.14)\) is valid only for \( \nu \ll \omega \).
D. Solving the Maxwell equations for the rf electric field using Fourier Series

System that has an antenna at \( x = 0 \) and a grounded electrode at \( x = L \) in the uniform plasma were studied in Ref. \[25\]. The papers \[23\] and \[31\] considered a cylindrical-like system in the uniform plasma. Both papers used Fourier series to solve Maxwell’s equations. Here, we shall generalize the procedure for a case of a nonuniform plasma.

Similarly to the previous analysis, it is convenient to continue the rf electric field symmetrically \( E_y(x) = E_y(-x) \) outside of the slab. Then, the electric field is given by Fourier series \[25\]:

\[
E_y(x) = \sum_{s=0}^{\infty} \Xi_s \cos(k_s x),
\]

where \( s \) is an integer, \( k_s = (2s+1)\pi/(2L) \), for the case of the grounded electrode, and \( k_s = (2s+1)\pi/L \), for the case of the cylindrical-like system. Substituting (7.29) into Eq.(4.1) and integrating with the weight \( \cos(k_s x)/L \) over the region \([-L, L]\) yields

\[
\left(-k_s^2 + \frac{\omega^2}{c^2}\right)\Xi_s = -\frac{4\pi i\omega}{c^2} \left[j_s + I_\text{anti,k}\right],
\]

where

\[
j_s = \frac{2}{L} \int_{0}^{L} j(x) \cos(k_s x) dx.
\]

Substituting the equation for the current density Eq.(7.27) gives

\[
j_s = \frac{n e^2}{m} \frac{1}{s\Omega_{bT}} \sum_{l=0}^{\infty} \Xi_l Z_{s,l}^{\text{gen}} \left(\frac{\omega + i\nu}{s\Omega_{bT}}\right),
\]

where \( \Omega_{bT} = v_T\pi/L \), and we introduced the generalized plasma dielectric function

\[
Z_{s,l}^{\text{gen}} \left[\xi = \frac{\omega + i\nu}{s\Omega_{bT}}\right] = \sqrt{\frac{2}{m}} \frac{s\Omega_{bT}}{L} \sum_{n=-\infty}^{\infty} \int_{0}^{\infty} \frac{\Gamma(\xi)}{in\Omega_b(\xi) - is\Omega_{bT}\xi} \frac{\pi}{\Omega_b(\xi)} G_{s,n}(\xi) G_{l,n}(\xi) d\xi,
\]

where the coefficients \( G_{l,n}(\xi) \) are the Fourier transform of \( \cos(k_l x) \) in the bounce motion of the electron in potential well:

\[
G_{l,n}(\xi) = \frac{1}{T} \left[ \int_{0}^{T} \cos[k_l x(\tau)] \cos\left(\frac{\pi n \tau}{T}\right) d\tau \right].
\]

In the limit of a uniform plasma \( \tau = x/v_x \), \( \tau = L/v_x \) which gives

\[
G_{l,n}(\xi) = \frac{1}{L} \left[ \int_{0}^{L} \cos(k_l x) \cos\left(\frac{n\pi x}{L}\right) dx \right].
\]

For a cylindrical-like system coefficients \( G_{l,n}(\xi) \) are particular simple

\[
G_{l,n}(\xi) = \frac{1}{2} \delta_{|l|,|n|}.
\]
and the generalized plasma dielectric function is

\[ Z_{g_{s,l}}^\text{gen}(\xi) = \delta_{s,l} Z(\xi), \]  

(7.37)

where \( Z(\xi) \) is the "standard" plasma dielectric function

\[ Z(\xi) = \pi^{-1/2} \int_{-\infty}^{\infty} dt \frac{\exp(-t^2)}{t - \xi}. \]  

(7.38)

Eq. (7.32) is identical to the results of Ref. [23] for cylindrical-like configuration uniform plasma with a Maxwellian EVDF.

Coefficients \( G_{l,n}(\varepsilon) \) can be effectively computed using the fast Fourier transform [55]. The off-diagonal coefficients are generally very small, that is why using this spectral method makes computing much faster than the straight forward finite difference method used in Ref. [37].

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