THE TAMELY RAMIFIED FUNDAMENTAL LOCAL EQUIVALENCE
AT INTEGRAL LEVEL

JUSTIN CAMPBELL AND GURBIR DHILLON

ABSTRACT. Let $G$ be an almost simple algebraic group with Langlands dual $\hat{G}$, and fix a noncritical integral level $\kappa$ for $G$, with dual level $\hat{\kappa}$ for $\hat{G}$. We prove an equivalence between $\kappa$-twisted Whittaker $D$-modules on the affine flag variety of $G$ and affine Category $O$ for $\hat{G}$ at level $-\hat{\kappa}$, as conjectured by Gaitsgory. To do so, we prove an affine version of Milicic-Soergel’s Whittaker localization of blocks of Category $O$.

1. INTRODUCTION

Let $G$ be an almost simple complex algebraic group with Langlands dual $\hat{G}$. The Fundamental Local Equivalence is a conjectural identification, proposed by Gaitsgory and Lurie, between the category of twisted Whittaker $D$-modules on the affine Grassmannian for $G$ and the category of representations at the dual level of the affine Lie algebra of $\hat{G}$, integrable for its arc group [14]. This conjecture provides a deformation of the Geometric Satake equivalence to all Kac-Moody levels, and is expected to play a similarly fundamental role in the quantum geometric Langlands program as the Satake equivalence in the usual Langlands correspondence.

The Fundamental Local Equivalence is a conjecture of unramified nature, in that the $D$-modules live on the quotient of loop group of $G$ by its arc group, and the representations of the affine Lie algebra of $\hat{G}$ are integrable for its arc group. Gaitsgory has conjectured a tamely ramified Fundamental Local Equivalence, wherein one replaces arc subgroups with Iwahori subgroups. In this paper we prove the latter conjecture, under an integrality assumption on the level.

2. STATEMENT OF RESULTS

To formulate the Fundamental Local Equivalence precisely, we introduce some notation. We begin with the twisted Whittaker $D$-modules on the affine flag variety. Write $\mathfrak{g}$ for the Lie algebra of $G$, $\mathcal{L}G$ for its algebraic loop group, and $\text{Fl}_G$ for its affine flag variety. There is a canonical embedding

$$((\mathfrak{g}^* \otimes \mathfrak{g}^*)^G \longrightarrow \text{Pic}(\text{Fl}_G) \otimes_{\mathbb{Z}} \mathbb{C},$$

which sends the Killing form to the line bundle coming from the Tate extension of $\mathcal{L}G$. Accordingly, to an invariant bilinear form $\kappa$ one may associate the corresponding cocomplete dg-category of twisted $D$-modules $D_\kappa(\text{Fl}_G)$. This category, like many others which
appear below, is not the derived category of its heart, but rather a renormalization thereof introduced by Frenkel-Gaitsgory. It carries a \( t \)-structure such that its bounded below category agrees with the naïve one, and an action of \( D_\kappa(\mathfrak{L} G) \), the renormalized dg-category of \( \kappa \)-twisted \( D \)-modules on the loop group, via left convolution.

To form its Whittaker subcategory, for a Borel \( B \) of \( G \) write \( N \) for its unipotent radical. For a nondegenerate character \( \psi \) of \( \mathfrak{L} N \) of conductor zero, we will take the associated co-complete dg-category of \( (\mathfrak{L} N, \psi) \)-equivariant twisted \( D \)-modules. Writing \( I \) for the Iwahori subgroup of \( \mathfrak{L} G \) corresponding to \( B \), we will write this category as

\[
D_\kappa(I \setminus \mathfrak{L} G / \mathfrak{L} N, \psi) \simeq D_{-\kappa} (\mathfrak{Fl}_G \setminus \mathfrak{L} N, \psi).
\]

We refer the reader to Section 4 of [6] for the definition of the category of invariants for a group ind-scheme such as \( \mathfrak{L} N \).

We next describe the Langlands dual side, i.e. Iwahori-integrable modules for the affinization of the dual Lie algebra. Write \( \hat{\mathfrak{g}} \) for the Lie algebra of \( \hat{G} \), and \( \mathfrak{L} \hat{G} \) for its loop group. For an invariant bilinear form \( \hat{\kappa} \) on \( \hat{\mathfrak{g}} \), consider the affine Lie algebra \( \hat{\mathfrak{g}}_{\hat{\kappa}} \), and its renormalized dg-category of smooth representations \( \hat{\mathfrak{g}}_{\hat{\kappa}} \text{-mod} \), as introduced by Frenkel–Gaitsgory [12]. Let \( I \) be the Iwahori subgroup of \( \mathfrak{L} \hat{G} \) corresponding to \( I \), and consider the corresponding category of \( I \)-equivariant objects

\[
\hat{\mathfrak{g}}_{\hat{\kappa}} \text{-mod}^I.
\]

Finally, recall that one may identify Kac-Moody levels, i.e. the lines of invariant bilinear forms, for \( G \) and \( \hat{G} \). Namely, writing \( W_f \) for their Weyl group, we have:

\[
(\mathfrak{g}^* \otimes \mathfrak{g}^*)^G \simeq (\mathfrak{h}^* \otimes \mathfrak{h}^*)^{W_f} \simeq (\mathfrak{h} \otimes \mathfrak{h})^{W_f} \simeq (\hat{\mathfrak{g}}^* \otimes \hat{\mathfrak{g}}^*)^{\hat{G}}.
\]

We will follow the convenient practice in the subject of incorporating into the above equivalence a shift by the critical levels \( \kappa_c \) for \( G \) and \( \hat{\kappa}_c \) for \( \hat{G} \), as we spell out more carefully in Section 7. For a level \( \kappa \) for \( G \), write \( \bar{\kappa} \) for the corresponding level of \( \hat{G} \). We may now state the tamely ramified Fundamental Local Equivalence, as conjectured by Gaitsgory in Sections 0.3.2–0.3.3 of [16].

**Conjecture 2.1.** For any nonzero \( \kappa \), there is an equivalence:

\[
D_\kappa(I \setminus \mathfrak{L} G / \mathfrak{L} N, \psi) \simeq \hat{\mathfrak{g}}_{\hat{\kappa}} \text{-mod}^I.
\]

In this paper we prove Conjecture 2.1 under an integrality assumption on the level. Recall the basic level \( \kappa_b \) for \( \mathfrak{g} \), which gives the short coroots of \( \mathfrak{g} \) squared length two.

**Theorem 2.3.** Suppose that \( \kappa \) is a nonzero integral multiple of \( \kappa_b \). Then Conjecture 2.1 is true for \( \kappa \).

To our knowledge, Theorem 2.3 gives the first known cases of Conjecture 2.1. We now briefly indicate the structure of the argument. A basic pattern in proving equivalences in the Langlands program is that, given the combinatorial nature of Langlands duality, one often shows that both sides of the proposed equivalence admit the same combinatorial
description. The Category $\mathcal{O}$ of an affine Lie algebra is known to admit such a Coxeter-theoretic description, via Kazhdan-Lusztig theory and Soergel modules, and we will use this to provide the desired combinatorial control of both sides of (2.2).

It is quite plausible that the Kac-Moody side of (2.2) should be controlled by Category $\mathcal{O}$, provided one handles issues of cocompletion, compactness, and renormalization appropriately. Accordingly, we show in Theorem 5.15 that for negative $\kappa$ it is roughly the ind-completion of the bounded derived category of $\mathcal{O}$. While this would be false for positive $\kappa$, we may instead reduce to the negative level case via Kac-Moody duality, i.e. the pairing between representations at positive and negative levels afforded by semi-infinite cohomology.

The Whittaker side of (2.2) looks superficially more distant from Category $\mathcal{O}$ than the Kac-Moody side. However, for a semisimple Lie algebra, it was known that integral blocks of its Category $\mathcal{O}$ may be realized as holonomic partial Whittaker sheaves on its flag variety [17], [21], [22], [29]. Nonetheless, a generalization to affine type is not straightforward, due to the unavailability of a center for noncritical affine algebras. Indeed, even the special case of identifying a regular block of Category $\mathcal{O}$ at a negative integral level with Schubert-constructible perverse sheaves on the affine flag variety, which has long been anticipated by experts, has not previously appeared in the literature.

Despite this, based on ongoing work of the second named author on the geometric representation theory of affine $W$-algebras, we were led to suspect that the relation between integral blocks of $\mathcal{O}$ and partial Whittaker sheaves should persist in affine type, and indeed it does.

To state this precisely, suppose $G$ is simply connected. Write $\tilde{I}$ for the prounipotent radical of $I$, and let $\chi$ be an additive character of $\tilde{I}$. One may form the category of partial Whittaker sheaves on the affine flag variety $D(I \setminus \mathfrak{L}G/\tilde{I}, \chi)$.

Next, we introduce the relevant Kac-Moody representations. Let $\mu$ be an integral antidominant weight for the affinization of $\mathfrak{g}$, and consider the block $\mathcal{O}_\mu$ of Category $\mathcal{O}$ containing the simple module with highest weight $\mu$. Suppose that $\chi$ and $\mu$ are compatible, in that the affine simple roots on which $\chi$ is nonzero coincide with the affine simple roots whose associated reflections stabilize $\mu$. Then we have

**Theorem 2.4.** There is a canonical up to scalars equivalence between the compact objects in the heart of the Whittaker category and the corresponding block of affine Category $\mathcal{O}$

$$D(I \setminus \mathfrak{L}G/\tilde{I}, \chi)^{\mathcal{O}_\mu} \simeq \mathcal{O}_\mu. \quad (2.5)$$

When $\chi$ vanishes, Theorem 2.4 yields the identification between a regular block of affine Category $\mathcal{O}$ and Schubert constructible perverse sheaves alluded to above. To our knowledge this has not appeared in the literature. However, the relation between objects on both side, e.g. simple and Verma modules, via a global sections functor, was achieved by Kashiwara–Tanisaki [19], and a monodromic variant of this equivalence was obtained by Frenkel–Gaitsgory [13]. Our proof makes essential use of their results.
Let us mention in passing that Theorem 2.4 has nice applications to (i) relations between translation functors and nearby cycles conjectured by the first named author, new already in finite type, (ii) localization of singular blocks of affine Category $\mathfrak{O}$ on affine partial flag varieties, and (iii) a relation between Drinfeld-Sokolov reduction and translation functors conjectured by the second named author. These will be explained in the forthcoming paper [8].

It remains to indicate how to prove Theorem 2.3 using Theorem 2.4. Our assumption on the level $\kappa$ is precisely that the twist on the Whittaker side is trivial. Moreover, by a theorem of Raskin, one may identify the $(\mathfrak{L}N, \psi)$- and $(\mathfrak{I}, \text{Ad}_{t-\rho} \psi)$-invariants of $\text{Fl}_G$, after which we may use Theorem 2.4 to identify the Whittaker sheaves coming from each connected component of $\text{Fl}_G$ with Kac-Moody representations. Having reduced the problem to identifying blocks of affine Category $\mathfrak{O}$ for $\mathfrak{g}$ and $\hat{\mathfrak{g}}$, we finish by using a wonderful theorem of Fiebig, which extends Soergel’s reconstruction of blocks from their integral Weyl groups to infinite type. A more involved variant of this argument, which is in progress by the second named author, appears to apply to the general case of Conjecture 2.1.

We conclude with two final remarks. First, as we explain in Conjecture 7.14, the Coxeter-theoretic combinatorics we utilize may be understood as relatively lossless projections of a local geometric Shimura correspondence. Second, ongoing work of Brubaker–Buciumas–Bump–Gustafsson produces non-metaplectic Iwahori Whittaker functions via the partition functions of solvable lattice models, and explains a relation between these functions and non-symmetric Macdonald polynomials [7]. Via the correspondence between Iwahori Whittaker functions and affine Hecke eigensheaves in $D(\mathfrak{L}G/\mathfrak{L}N, \psi)$, there should be interesting connections between their work and the present, and their metaplectic generalizations.

**Organization of the paper.** Sections 3-6 are devoted to a proof of Theorem 2.4. Namely, in Section 3, we give an argument for the case of nondegenerate Whittaker characters in finite type which makes the Hecke equivariance manifest. In Section 4, we discuss compact generators for the Whittaker category, and analyze the action of the affine Hecke category on the vacuum object. In Section 5, we provide a similar analysis for the Iwahori-equivariant category of $\hat{\mathfrak{g}}_{\kappa}$-mod. In Section 6, we use the results of Sections 4 and 5 to bootstrap the result of Section 3 to a proof of Theorem 2.4. Finally, in Section 7, we prove Theorem 2.3.

**Conventions regarding dg-categories.** In this paper we understand “dg-category” to mean “$\mathbb{C}$-linear stable $(\infty,1)$-category.” We denote by DGCat the $(\infty,1)$-category of idempotent-complete dg-categories with exact $\mathbb{C}$-linear functors, and write DGCat_{cont} for the $(\infty,1)$-category of cocomplete (presentable) dg-categories with colimit-preserving functors.

We occasionally make use of the Lurie tensor product on DGCat_{cont}. Recall that for two objects $\mathcal{E}$ and $\mathcal{D}$, the dg-category $\mathcal{E} \otimes \mathcal{D}$ is characterized by the universal property that a $\mathbb{C}$-linear colimit-preserving functor $\mathcal{E} \otimes \mathcal{D} \to \mathcal{E}$ is equivalent to a functor $\mathcal{E} \times \mathcal{D} \to \mathcal{E}$ which is $\mathbb{C}$-linear and preserves colimits in each variable separately. The unit of this tensor structure
is Vect, the dg-category of complexes of \( \mathbb{C} \)-vector spaces. We write \( \text{Fun}(\mathcal{C}, \mathcal{D}) \) for the dg-category of continuous functors \( \mathcal{C} \to \mathcal{D} \), i.e. the inner Hom in \( \text{DGCat}_{\text{cont}} \). More generally, for a monoidal dg-category \( \mathcal{A} \) acting on \( \mathcal{C} \) and \( \mathcal{D} \), we have the dg-category \( \text{Fun}_\mathcal{A}(\mathcal{C}, \mathcal{D}) \).

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3. **The Categorical Sign Representation**

Let \( L \) be a finite-dimensional reductive group with Lie algebra \( \mathfrak{l} \), and fix a triangular decomposition \( \mathfrak{l} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n} \). Write \( B \) for the Borel of \( L \) with Lie algebra \( \mathfrak{h} \oplus \mathfrak{n} \), and \( N \) for its unipotent radical. Let \( I \) be the set of vertices of the Dynkin diagram of \( L \), and for \( i \in I \) write \( \alpha_i, \check{\alpha}_i \), for the corresponding simple roots and coroots of \( \mathfrak{l} \). Denote by \( W_f \) the Weyl group of \( \mathfrak{l} \).

Recall that the Hecke algebra associated to \( L \) is a \( \mathbb{Z}[q^{\pm 1/2}] \)-algebra with standard generators \( T_i \), for \( i \in I \), satisfying a braid relation and the quadratic relation

\[
(T_i + 1)(T_i - q) = 0, \quad \text{for all } i \in I.
\]

Accordingly, it has a sign representation \( \mathbb{C}_{\text{sgn}} \), on which each \( T_i \) acts as -1. In this section, we explain that there are two equivalent incarnations of the categorical sign representation \( \text{Vect}_{\text{sgn}} \) of the categorical Hecke algebra \( \text{D}(B \backslash L/B) \). Namely, we will see it roughly as a maximally singular block of Category \( \mathcal{O} \) or as nondegenerate Whittaker modules on the flag variety.

Let us introduce the Whittaker side. Let \( N^- \) denote the connected subgroup of \( L \) with Lie algebra \( \mathfrak{n}^- \), and recall its one-parameter subgroups \( N_i \), for \( i \in I \), whose Lie algebras are the negative simple root spaces. Fix a nondegenerate additive character \( \psi : N^- \to \mathbb{G}_a \), i.e. one whose restriction to each \( N_i \) is nonzero. We may accordingly form the category of Whittaker \( \mathcal{D} \)-modules on the flag variety of \( L \), i.e.

\[
\text{D}(B \backslash L/N^-, \psi) := \text{D}(B \backslash L)^{N^- \cdot \psi},
\]

which is a full subcategory of \( \text{D}(B \backslash L) \) because \( N^- \) is unipotent. This carries an action of \( \text{D}(B \backslash L/B) \) via left convolution. As a category, it is equivalent to Vect, with generator \( \mathcal{W}_\psi \), its unique indecomposable object which lies in the heart \( \text{D}(B \backslash L)^\circ \).

Let us describe the singular side. Write \( Z \) for the center of the universal enveloping algebra of \( \mathfrak{l} \), and for a central character \( \chi : Z \to \mathbb{C} \) write \( \mathfrak{l} \text{-mod}_\chi \) for the category of \( \mathfrak{l} \)-modules with generalized central character \( \chi \). I.e., this is the full subcategory of \( \mathfrak{l} \text{-mod} \) consisting of objects whose cohomology groups are all set-theoretically supported on \( \chi \) when restricted from \( U(\mathfrak{l}) \) to \( Z \). For any \( \chi \), the category \( \mathfrak{l} \text{-mod}_\chi \) carries a canonical action of \( \text{D}(L) \), and hence its \( B \)-equivariant category \( \mathfrak{l} \text{-mod}_\chi^B \) carries a canonical action...
of \( D(B\backslash L/B) \). Recall the Harish-Chandra isomorphism \( Z \simeq \Sym h^{W_f} \), and the resulting identification of central characters with dot orbits of \( W_f \) on \( h^* \).

Suppose \( \chi \) is maximally singular, i.e. the corresponding \( W_f \)-orbit in \( h^* \) has a single element, which we continue to denote by \( \chi \). Writing \( \rho \) for the half sum of the positive roots, we may take \( \chi = -\rho \), and if \( l \) is semisimple this is the unique choice. Then \( l\text{-mod}^B \) as a category is canonically equivalent to \( \Vect \), with generator the unique Verma module \( M_\chi := \text{ind}_l^B C_\chi \) in the block.

**Theorem 3.1.** For \( \chi \) maximally singular, there is a \( t \)-exact equivalence of \( D(B\backslash L/B) \)-modules:

\[
D(B\backslash L/N^-, \psi) \simeq l\text{-mod}^B \chi.
\]

This theorem is originally due to Milicic-Soergel [22] via a different argument, where they use the language of Harish-Chandra modules rather than the Hecke category.

**Proof.** We will produce a \( D(B\backslash L/B) \)-equivariant map from \( D(B\backslash L/N^-, \psi) \) to \( l\text{-mod}^B \chi \), and check it sends our generators to one another. To produce the desired functor, we will provide a \( D(L) \)-equivariant functor

\[
F : D(L/N^-, \psi) \longrightarrow l\text{-mod} \chi
\]

and then take \( B \)-invariants. By Corollary 2.4.6 in [22], the category \( D(L/N^-, \psi) \) can also be realized as Whittaker coinvariants, i.e. for any \( D(L) \)-module \( C \) we have a canonical equivalence

\[
\text{Fun}_{D(L)}(D(L/N^-, \psi), C) \simeq C^{N^-, \psi}. \tag{3.2}
\]

To define \( F \), it therefore suffices to specify an object of \( l\text{-mod}^{N^-, \psi} \chi \), which we construct as follows. Consider the universal Whittaker module \( M(\psi) := \text{Ind}_n^L C_\psi \), which lies in \( l\text{-mod}^{N^-, \psi} \). Recall that, as with any central character, the forgetful functor from \( l\text{-mod} \chi \) to \( l\text{-mod} \) admits a continuous right adjoint:

\[
\text{Oblv}_\chi : l\text{-mod} \chi \rightleftarrows l\text{-mod} : \text{Ri}^1_{\chi}.
\]

These functors restrict to an adjunction between the Whittaker subcategories, and we set \( M(\psi, \chi) := \text{Ri}^1_{\chi} M(\psi) \).

It remains to argue that the resulting functor \( F^B : D(B\backslash L/N^-, \psi) \to l\text{-mod}^B \) is an equivalence. It suffices to show that \( F^B \) sends \( W_\psi \) to \( M_\chi \), up to a cohomological shift. Writing \( \delta_e \) for the delta \( D \)-module on the identity of \( L \), by the construction of (3.2) we have \( F(Av_{N^-, \psi}, \delta_e) \simeq M(\psi, \chi) \). It follows that

\[
F^B(Av_B, Av_{N^-, \psi}, \delta_e) \simeq Av_{B, \ast} F(Av_{N^-, \psi}, \delta_e) \simeq Av_{B, \ast} M(\psi, \chi).
\]
To calculate the latter, note that
\[
\text{Hom}_{l}\text{-mod}^{\hat{\chi}}(M_{\chi}, Av_{B,\ast} M(\psi, \hat{\chi})) \simeq \text{Hom}_{l}\text{-mod}(M_{\chi}, M(\psi)) \\
\simeq \text{Hom}_{b}\text{-mod}(C_{\chi}, \text{Res}_{\hat{\chi}} M(\psi)) \simeq \mathbb{C}[\dim B],
\]
where in the last step we use that \(M(\psi)\) is a free rank one \(U(b)\)-module. Noting that \(Av_{B,\ast} Av_{\ast,\psi,\ast} \delta_{c} \simeq W_{\psi}[\dim N]\), it follows that the functor associated to \(M(\psi, \hat{\chi})[\dim h]\) produces the desired \(t\)-exact equivalence. \(\square\)

4. Partial Whittaker sheaves

In this section, we study partial Whittaker sheaves on the affine flag variety. We would like to argue that (i) these form a highest weight category, and (ii) all the (co)standard objects can be reconstructed from the one with the smallest support via the action of the affine Hecke algebra.

However, since for (i) we are not working with an abelian category of holonomic D-modules, but rather a renormalized unbounded derived dg-category of arbitrary D-modules, what we will actually prove is that the latter can be reconstructed from the compact objects in the heart of the \(t\)-structure, which indeed form a highest weight category.

4.1. Whittaker sheaves on ind-schemes. The argument for (i) holds rather generally. So for the moment, let \(X\) be any finite type algebraic variety equipped with an action of a prounipotent group scheme \(U\) with only finitely many orbits. Accordingly, \(X\) has a stratification
\[
X = \bigsqcup_{\lambda \in \Lambda} C_{\lambda}
\]
by the orbits \(C_{\lambda}\), and by prounipotence of \(U\) each orbit is an affine space. For any additive character \(\chi : U \to \mathbb{G}_{a}\), consider the Whittaker category \(D(X)^{U,\chi}\), which is a full subcategory of \(D(X)^{U}\) by prounipotence of \(U\).

**Proposition 4.1.** Fix a stratum \(C_{\lambda}\). Then \(D(C_{\lambda})^{U,\chi}\) is nonzero if and only if \(\chi\) vanishes on the stabilizer of one (equivalently, any) point \(c \in C_{\lambda}\). If \(D(C_{\lambda})^{U,\chi}\) is nonzero, then it admits a \(t\)-exact equivalence with \(\text{Vect}\), and compact objects in \(D(C_{\lambda})^{U,\chi}\) are bounded with lisse cohomology sheaves.

**Proof.** This statement is well-known, but due to its importance for us we remind the reader of the proof. Choose \(c \in C_{\lambda}\) and denote its stabilizer in \(U\) by \(U_{c}\). Taking the fiber at \(c\) induces an equivalence
\[
D(C_{\lambda})^{U,\chi} \simeq \text{Vect}^{U_{c},\chi},
\]
where the action of \(U_{c}\) on \(\text{Vect}\) is the trivial one. Since \(U_{c}\) is connected, if \(\chi|_{U_{c}}\) is nonzero then \(\text{Vect}^{U_{c},\chi} \neq 0\). On the other hand, since \(U_{c}\) is prounipotent, if \(\chi|_{U_{c}} = 0\) then we have an equivalence \(\text{Vect}^{U_{c},\chi} \simeq \text{Vect}\). In the latter case, the resulting equivalence \(\text{Vect} \simeq D(C_{\lambda})^{U,\chi}\) can be normalized to send \(\mathbb{C}\) to \(\tilde{\chi}_{!} \exp[1 - \dim C_{\lambda}]\), where \(\tilde{\chi} : C_{\lambda} \to \mathbb{G}_{a}\) is the smooth map.
induced by $\chi$ and $\exp$ denotes the exponential $D$-module on $G_a$, placed in cohomological degree zero. □

Let $\Lambda \subset \tilde{\Lambda}$ index the strata $C_\lambda$ with $D(C_\lambda)^{U,\chi}$ nonzero. For each $\lambda \in \Lambda$, write $j_{\lambda,!, j_{\lambda,*}}$ for the $!$- and $*$- extensions, respectively, of the indecomposable object in $D(C_\lambda)^{U,\chi, \mathcal{O}}$. Note that $\Lambda$ carries a natural partial order coming from orbit closures, i.e. for $\lambda, \nu \in \Lambda$, we have $\lambda \leq \nu$ if and only if $C_\lambda$ lies in the closure of $C_\nu$.

Theorem 4.2. Write $A \subset D(X)^{U,\chi, \mathcal{O}}$ for the full subcategory consisting of coherent $D$-modules. Then

1. $A$ is a highest weight category with standard objects $j_{\lambda,!}$ and costandard objects $j_{\lambda,*}$ for $\lambda \in \Lambda$.
2. The tautological map $A \to D(X)^{U,\chi}$ induces a fully faithful embedding $D^b(A) \to D(X)^{U,\chi}$.
3. The above embedding exhibits $D(X)^{U,\chi}$ as the ind-completion of $D^b(A)$. I.e., $D(X)^{U,\chi}$ is compactly generated with compact objects given by the essential image of $D^b(A)$.

Proof. For (1), we observe that since $U$ acts on $X$ with finitely many orbits, any object of $A$ is automatically holonomic. Since we do not know a reference for the twisted equivariant case, we include an argument.

For an object $M$ of $A$, consider a stratum $C_\lambda$ maximal in its support. By coherence, Proposition 4.1 implies the restriction of $M$ to $C_\lambda$ is of the form $e^\chi \otimes V$, for a finite dimensional vector space $V$. We obtain a canonical map $j_{\lambda,!} \otimes V \to M$. Write $M'$ for its image, which is automatically finite length since $j_{\lambda,!}$ is holonomic. By induction on the number of cells in the support of a module, we deduce that $M/M'$ is of finite length too, whence so is $M$. A similar argument shows that the simple objects of $A$ are the intermediate extensions $j_{\lambda,!*}$ for $\lambda \in \Lambda$.

Checking the remaining conditions of $A$ being highest weight, as enumerated in Section 3.2 of [5], are straightforward, except for possibly the vanishing of $\text{Ext}^2_A(j_{\lambda,!}, j_{\nu,*})$ for $\lambda, \nu \in \Lambda$. However, this injects into $H^2 \text{RHom}_{D(X)^{U,\chi}}(j_{\lambda,!}, j_{\nu,*})$, and it is straightforward to see that

\[ \text{RHom}_{D(X)^{U,\chi}}(j_{\lambda,!}, j_{\nu,*}) \simeq \text{RHom}_{D(X)}(j_{\lambda,!}, j_{\nu,*}) \simeq \begin{cases} \mathbb{C} & \lambda = \nu, \\ 0 & \lambda \neq \nu. \end{cases} \tag{4.3} \]

For (2), one can use the presentation of $D^b(A)$ as the homotopy category of bounded complexes of tilting objects, cf. Proposition 1.3 of [3], and Equation (4.3). Equivalently, one can use that the $j_{\lambda,!}$ generate $D^b(A)$, as do the $j_{\lambda,*}$.

For (3), it suffices to check that the $j_{\lambda,!}$ for $\lambda \in \Lambda$, generate $D(X)^{U,\chi, \mathcal{O}}$. Suppose $M$ is an object of $D(X)^{U,\chi, \mathcal{O}}$ such that $\text{RHom}(j_{\lambda,!, M}) \simeq 0$ for all $\lambda \in \Lambda$. Taking a $C_\lambda$ open in $X$, it follows by adjunction from Proposition 4.1 that the restriction of $M$ to $C_\lambda$ vanishes. Hence $M$ is pushed forward from the complement of $C_\lambda$, and we finish by induction on the number of strata. □
Suppose now $X$ is an ind-scheme presented as a filtered colimit $\lim_{\alpha} X_\alpha$ of finite type schemes under closed embeddings. In this case, we have

$$D(X) = \lim_{\alpha} D(X_\alpha),$$

where the colimit is taken in $\text{DGCat}_{\text{cont}}$ with the transition functors given by direct image.

Suppose that $X$ carries an action of a prounipotent group $U$ such that each $X_\alpha$ is $U$-stable. Recall from Section 4 of [6] that for a category $\mathcal{C}$ acted on by $D(U)$, there is a canonical equivalence between invariants and coinvariants $\mathcal{C}^U \simeq \mathcal{C}_U$, and similarly for twisted (co)invariants. In particular, taking $U$-invariants commutes with colimits, and hence

$$D(X)^{U,\chi} = \left(\lim_{\alpha} D(X_\alpha)^{U,\chi}\right).$$

Suppose that $U$ acts on each $X_\alpha$ with only finitely many orbits. In this case, let $\Lambda_\alpha$ index the orbits on $X_\alpha$ supporting Whittaker sheaves, and set $\Lambda := \lim_{\alpha} \Lambda_\alpha$. Note that $\Lambda$ carries a partial order induced from those of the $\Lambda_\alpha$, i.e. the closure relation on strata, and for any $\lambda \in \Lambda$ the collection $\{\nu \in \Lambda : \nu \leq \lambda\}$ is finite. For each $\lambda \in \Lambda$, consider as before the $!$- and $*$- extension $j_{\lambda,!}, j_{\lambda,*}$, and further consider the indecomposable tilting object $T_\lambda$.

**Proposition 4.4.** For each $\alpha$, write $A_\alpha$ for the subcategory of $D(X_\alpha)^{U,\chi,c}$ defined in Theorem 4.2. Then for each $\alpha \rightarrow \beta$, the map $D^b(A_\alpha) \rightarrow D^b(A_\beta)$ is fully faithful, and the tautological functor

$$\lim_{\alpha} D^b(A_\alpha) \rightarrow D(X)^{U,\chi}$$

exhibits the latter as the ind-completion of the former, where the colimit is taken in $\text{DGCat}$. In particular, $D(X)^{U,\chi,c}$ is equivalent to the bounded homotopy category of the tilting objects $T_\lambda$, for $\lambda \in \Lambda$.

**Proof.** That $D^b(A_\alpha) \rightarrow D^b(A_\beta)$ is fully faithful can be seen by (i) considering $A_\alpha$ as a ‘closed’ highest weight subcategory of $A_\beta$, or (ii) via the Kashiwara lemma and Theorem 4.2(2).

To see the compact generation claim, given a diagram of compactly generated dg-categories $\mathcal{C}_\alpha$ such that the transition maps $i_{\alpha,\beta}$ preserve compactness, its colimit taken in $\text{DGCat}_{\text{cont}}$ is canonically equivalent to the ind-completion of the colimit of the compact objects $\lim_{\alpha} \mathcal{C}_\alpha$ taken in $\text{DGCat}$. The claim about tilting objects follows from the explicit form of filtered colimits in $\text{DGCat}$, cf. [27], and the analogous claim for each $A_\alpha$. □

4.2. Whittaker sheaves on the affine flag variety. In this subsection, $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$ will denote a finite dimensional simple Lie algebra, with associated simply connected form $G$. Recall that $\mathfrak{L}G$ denotes the corresponding loop group and $I$ its standard Iwahori subgroup. Denote the simple roots by $\alpha_i$, where $i \in \hat{I}$ ranges over the vertices of the affine Dynkin diagram.

For a subset of $\hat{I}$ of finite type, write $L$ for the corresponding Levi subalgebra, $I$ for the corresponding Levi subgroup, and $B_L$ for its standard Borel. Write $W$ for the Weyl group of $G$, $W_L$ for the Weyl group of $L$, and denote the longest element of the latter by $w_0^L$. 
As before, write $\tilde{I}$ for the pronipotent radical of $I$, and consider its conjugate

$$\tilde{I}_L^- := w_0^L \tilde{I} w_0^L,$$

An additive character $\psi : \tilde{I}_L^- \to \mathbb{G}_a$ is at most nonzero on the simple root one-parameter subgroups $w_0^LN_{\alpha_i}w_0^L$ for $i \in \tilde{J}$. Fix a $\psi$ which is nonzero precisely on the negative simple roots of $L$. We will be concerned with the partial Whittaker category

$$D(I \setminus LG/\tilde{I}_L^-, \psi) := D(I \setminus LG)_{\tilde{I}_L^-, \psi}.$$

Recall that the orbits of $\tilde{I}_L^-$ on $I \setminus LG$ are given by the finite-dimensional Schubert cells

$$C_w := Iw\tilde{I}_L^- \quad \text{for } w \in W.$$

From Proposition 4.1, it follows that $C_w$ supports a Whittaker sheaf if and only if $w$ is of minimal length in its right $W_L$-coset. Let us write $W_L$ for the set of such minimal length coset representatives, and for each $w \in W_L$ write $j_w^\psi$, $j_w^\psi_!$ for the corresponding standard and costandard objects. Note that $j_e^\psi \simeq j_e^\psi_!$, where $e$ denotes the identity element of $W$, and write $\delta^\psi$ for this simple object.

The Hecke algebra $D(I \setminus LG/I)$ acts on the partial Whittaker category by left convolution

$$D(I \setminus LG/I) \otimes D(I \setminus LG/\tilde{I}_L^-, \psi) \longrightarrow D(I \setminus LG/\tilde{I}_L^-, \psi)$$

$$M \otimes N \mapsto M \ast N.$$

For $w \in W$, write $j_{w,!}, j_{w,*}$ for the respective extensions in $D(I \setminus LG/I)$ of the constant D-module from $IwI$.

**Proposition 4.5.** For any $w \in W_L$ we have:

1. $j_{w,!*} \ast \delta^\psi \simeq j_{w,!}^\psi$.
2. $j_{w,*} \ast \delta^\psi \simeq j_{w,*}^\psi$.
3. Under the equivalences of (1) and (2), applying $- \ast \delta^\psi$ to the canonical map

$$j_{w,!} \longrightarrow j_{w,*}$$

yields a nonzero map $j_{w,!}^\psi \rightarrow j_{w,*}^\psi$.

**Proof.** By the assumption that $w \in W_L$, the relevant convolution morphism between Schubert cells is an isomorphism, cf. Section 2 of [2] for more details in an ostensibly special case. \[\square\]

### 5. Affine Category $\mathcal{O}$ and Harish-Chandra modules

In this section, first we perform the comparison between Category $\mathcal{O}$ and Harish-Chandra modules for the Iwahori. Second, we show the linkage principle is a decomposition as affine Hecke algebra modules. Note this assertion is non-trivial in affine type due to the absence of central characters.
5.1. Compact generators for Harish-Chandra modules. Let $\kappa$ be any non-critical level, and $\hat{g}_\kappa$ the corresponding affine Lie algebra. Write $\hat{g}_\kappa$-mod for its renormalized cocomplete dg-category of representations, as introduced by Frenkel–Gaitsgory in Sections 22-23 of [12], which carries an action of $D_\kappa(\mathfrak{L}G)$. For a compact open subgroup $H$ of $\mathfrak{L}G$, we may accordingly form the invariants category $\hat{g}_\kappa$-mod$^H$. For our application, we are interested in $\hat{g}_\kappa$-mod$^I$, for $I$ the Iwahori subgroup of $\mathfrak{L}G$. In the next subsections we will use the following assertion.

**Proposition 5.1.** Write $\Lambda$ for the character lattice of $I$. Then the Verma modules $M_\lambda$, for all $\lambda \in \Lambda$, are compact generators of $\hat{g}_\kappa$-mod$^I$.

The reader happy to take Proposition 5.1 on faith may wish to skip the remainder of this subsection. We will deduce Proposition 5.1 from a more general presentation of $\hat{g}_\kappa$-mod$^H$, stated in unpublished notes of Gaitsgory [15]. Consider the abelian category of Harish-Chandra modules. Within its unbounded derived dg-category, take the pretriangulated envelope of the representations induced from finite-dimensional representations of $H$. Denote its ind-completion by $(\hat{g}_\kappa, H)$-mod.

**Theorem 5.2.** There is a canonical equivalence

$$(\hat{g}_\kappa, H)$-mod \simeq \hat{g}_\kappa$-mod$^H.$

In the remainder of this subsection, we give a proof of Theorem 5.2, largely following the argument sketched in loc. cit. Write $H$ as an inverse limit

$$H = \lim_{\leftarrow} H_i, \quad \text{for } i \in \mathbb{Z}^{\geq 0},$$

where the $H_i$ are finite dimensional and each morphism is surjective with unipotent kernel. Its renormalized category of representations is the colimit in DGCat$^\text{cont}$ of the categories of representations of the $H_i$ under inflation:

$$\text{Rep}(H) = \lim_{\leftarrow} \text{Rep}(H_i).$$

Denote the unbounded derived category of its abelian category of representations by $\text{Rep}(H)^\text{na"{i}ve}$, and note that there is a tautological map

$$\text{Rep}(H) \longrightarrow \text{Rep}(H)^\text{na"{i}ve}.$$

**Lemma 5.4.** Consider the pretriangulated envelope of the finite-dimensional representations of $H$ taken in $\text{Rep}(H)^\text{na"{i}ve}$. Then the morphism (5.3) identifies its ind-completion with $\text{Rep}(H)$.

Proof. Note that for each $i$, the category $\text{Rep}(H_i)$ is compactly generated by its finite-dimensional representations. Since the inflation maps preserve compactness, it follows that $\text{Rep}(H)$ is compactly generated by the finite dimensional representations of $H$. It remains
to check that \((5.3)\) is fully faithful on such objects, for which we give two arguments. First,
this follows from using the cobar complex and the identification \(\mathcal{O}_H \simeq \varprojlim \mathcal{O}_{H_i}\).

Second, we can use the following result of Raskin, which appears as Lemma 5.4.3 of [26].

**Theorem 5.5.** Let \(\alpha \mapsto \mathcal{C}_\alpha\) be a filtered diagram of cocomplete dg-categories equipped with
\(t\)-structures. Assume for each \(\alpha \to \beta\) that the corresponding morphism
\[\mathcal{C}_\alpha \to \mathcal{C}_\beta\]
is \(t\)-exact and admits a continuous right adjoint, and set
\[\mathcal{C} := \varprojlim \mathcal{C}_\alpha.\]

Then there is a unique \(t\)-structure on \(\mathcal{C}\) which is compatible with filtered colimits and such
that each insertion \(\mathcal{C}_\alpha \to \mathcal{C}\) is \(t\)-exact. Moreover suppose that for each \(\alpha\) the canonical map
of bounded below categories
\[D(\mathcal{C}_\alpha^+) \to \mathcal{C}_\alpha^+\]
is an equivalence. Then the same is true for \(\mathcal{C}^+\), provided that the diagram \(\alpha \mapsto \mathcal{C}_\alpha\) is
countable.

In our situation, it is straightforward to see that \((5.3)\) is \(t\)-exact and restricts to an
equivalence
\[\text{Rep}(H)^\wedge \to \text{Rep}(H)_{\text{na"ıve}}^\wedge.\]
By Raskin’s theorem, it follows that \((5.3)\) is an equivalence on bounded below objects, and
in particular fully faithful on the compact generators of \(\text{Rep}(H)\). \(\square\)

Writing \(h\) for the Lie algebra of \(H\), and writing \(h_i\) for the Lie algebra of \(H_i\), we have
\[h = \varprojlim h_i.\]
For each \(i\), write \(h_i\)-mod for the cocomplete dg-category of representations of \(h_i\). The
renormalized category of smooth representations of \(h\) is the colimit of those of the \(h_i\) under inflation:
\[h\text{-mod} = \varinjlim h_i\text{-mod}.\]
Denote the unbounded derived category of the the abelian category of smooth representa-
tions of \(h\) by \(h\text{-mod}^\text{na"ıve}\). As before, there is a tautological functor
\[h\text{-mod} \to h\text{-mod}^\text{na"ıve}. \quad (5.6)\]

**Lemma 5.7.** Consider the pretriangulated envelope of the finitely generated smooth repre-
sentations of \(h\) in \(h\text{-mod}^\text{na"ıve}\). Then the morphism \((5.6)\) identififies its ind-completion with
\(h\text{-mod}.\)

**Proof.** Similar to the second argument of Lemma 5.4. \(\square\)

Note that each \(h_i\)-mod carries an action of \(D(H)\), and hence so does \(h\text{-mod}\). In partic-
ular, we can form the invariant category
\[h\text{-mod}^H.\]
Lemma 5.8. There is a canonical equivalence

\[ \mathfrak{h} \text{-mod}^H \simeq \text{Rep}(H). \]

Proof. Since taking invariants for a group scheme commutes with colimits, we have

\[ \mathfrak{h} \text{-mod}^H \simeq \varinjlim (\mathfrak{h}_i \text{-mod}^{H_i}). \]

Fix an \( i \), and write \( U_i \) for the kernel of the projection \( H \to H_i \). Since \( U_i \) acts trivially on \( \mathfrak{h}_i \text{-mod} \) and is prounipotent, we have

\[ \mathfrak{h}_i \text{-mod}^{H_i} \simeq (\mathfrak{h}_i \text{-mod}^{U_i})^{H_i} \simeq \mathfrak{h}_i \text{-mod}^{H_i} \simeq \text{Rep}(H_i). \]

Under the above equivalences, the transition maps identify with inflation, hence

\[ \mathfrak{h} \text{-mod}^H \simeq \varinjlim (\mathfrak{h}_i \text{-mod}^{H_i}) \simeq \varinjlim \text{Rep}(H_i) \simeq \text{Rep}(H), \]

as desired. \( \square \)

Recall that \( \mathfrak{h} \text{-mod} \) and \( \hat{\mathfrak{g}}_\kappa \text{-mod} \) carry \( t \)-structures such that their bounded below parts coincide with those of the unrenormalized categories. In particular, there is an adjunction

\[ \text{Ind} : \mathfrak{h} \text{-mod}^+ \rightleftarrows \hat{\mathfrak{g}}_\kappa \text{-mod}^+ : \text{Oblv}. \] (5.9)

Lemma 5.10. The adjunction (5.9) induces an adjunction

\[ \text{Ind} : \mathfrak{h} \text{-mod} \rightleftarrows \hat{\mathfrak{g}}_\kappa \text{-mod} : \text{Oblv} \] (5.11)

such that \( \text{Oblv} \) is conservative.

Proof. To obtain the full adjunction, one may consider the composite

\[ \mathfrak{h} \text{-mod}^c \longrightarrow \mathfrak{h} \text{-mod}^+ \longrightarrow \hat{\mathfrak{g}}_\kappa \text{-mod}^+ \longrightarrow \hat{\mathfrak{g}}_\kappa \text{-mod} \]

and ind-extend. To see that \( \text{Oblv} \) is conservative, note that \( \mathfrak{h} \) contains a basis \( u_j \) of open subalgebras about 0 in \( \hat{\mathfrak{g}}_\kappa \). Applying \( \text{Ind} \) to \( \text{Ind}^\mathfrak{h}_u \mathbb{C} \) therefore yields a collection of compact generators for \( \hat{\mathfrak{g}}_\kappa \text{-mod} \), which is equivalent to the conservativity of its right adjoint. \( \square \)

We will show in the appendix that

Lemma 5.12. Both functors in (5.11) carry a canonical datum of \( D(H) \)-equivariance.

By the lemma, (5.11) induces an adjunction

\[ \text{Ind} : \mathfrak{h} \text{-mod}^H \rightleftarrows \hat{\mathfrak{g}}_\kappa \text{-mod}^H : \text{Oblv}. \]

Via Lemma 5.8 we may rewrite this as

\[ \text{Ind} : \text{Rep}(H) \rightleftarrows \hat{\mathfrak{g}}_\kappa \text{-mod}^H : \text{Oblv}. \] (5.13)

Finally, let us turn to Harish-Chandra modules. Write \( (\hat{\mathfrak{g}}_\kappa, H) \text{-mod}^{\text{naive}} \) for the unbounded derived category of the abelian category \( (\hat{\mathfrak{g}}_\kappa, H) \text{-mod}^{\text{\dagger}} \). There is an adjunction

\[ \text{Ind} : \text{Rep}(H)^{\text{naive}} \rightleftarrows (\hat{\mathfrak{g}}_\kappa, H) \text{-mod}^{\text{naive}} : \text{Oblv}. \]
By the definition of the renormalization of Harish-Chandra modules, \((\hat{g}_\kappa, H)\)-mod\(^c\) is the pretriangulated envelope of the image of \(\text{Rep}(H)^c\). As Oblv is \(t\)-exact, it follows that \((\hat{g}_\kappa, H)\)-mod\(^c\) consists of almost perfect objects in \((\hat{g}_\kappa, H)\)-mod\(^c\), i.e. if an object of \((\hat{g}_\kappa, H)\)-mod\(^c\) belongs to \((\hat{g}_\kappa, H)\)-mod\(\geq -n\) for some \(n \geq 0\), then it is compact in \((\hat{g}_\kappa, H)\)-mod\(\geq -n\). In particular, we have a canonical identification 
\[
(\hat{g}_\kappa, H)\text{-mod}^+ \simeq (\hat{g}_\kappa, H)\text{-mod}^{\text{naive}, +}.
\]

By an argument similar to Lemma 5.10, we obtain an adjunction
\[
\text{Ind} : \text{Rep}(H) \rightleftarrows (\hat{g}_\kappa, H)\text{-mod} : \text{Oblv} \quad (5.14)
\]
wherein Oblv is conservative. Now observe that the monads on \(\text{Rep}(H)\) coming from (5.11), (5.14) canonically identify. Hence Theorem 5.2 follows by Barr-Beck-Lurie.

5.2. Category \(\mathcal{O}\) and Harish-Chandra modules. Fix a negative level \(\kappa\), i.e.
\[
\kappa \notin \kappa_c + \mathbb{Q}_{\geq 0} \kappa_b.
\]
Consider the corresponding affine Lie algebra \(\hat{g}_\kappa\) and its usual abelian category \(\mathcal{O}\). Recall that \(\Lambda\) denotes the character lattice of the Iwahori, and within \(\mathcal{O}\) consider the full subcategory \(\mathcal{O}_\Lambda\) of objects whose weights lie in \(\Lambda\). Note that there is a tautological functor
\[
\mathcal{O}_\Lambda \longrightarrow \hat{g}_\kappa\text{-mod}^{I, \triangledown}.
\]

We will now explain how to reconstruct the entire renormalized derived category of Harish-Chandra modules from this embedding.

For \(\lambda \in \Lambda\), recall the Verma module \(M_\lambda\), its simple quotient \(L_\lambda\), the dual Verma \(A_\lambda\), and the indecomposable tilting module \(T_\lambda\). This induces a partial order on \(\Lambda\), where \(\lambda \leq \nu\) if \(L_\lambda\) is a subquotient of \(M_\nu\). For each \(\nu \in \Lambda\), introduce the basic closed set
\[
\Lambda^{\leq \nu} = \{ \lambda \in \Lambda : \lambda \leq \nu \}.
\]
A subset \(\Lambda' \subset \Lambda\) is said to be closed if it is a union of basic closed sets. Finally, by our assumption on the level, each \(\Lambda^{\leq \nu}\) is finite, hence we can write \(\Lambda\) as an ascending union
\[
\Lambda = \bigcup_{i \geq 0} \Lambda_i,
\]
where each \(\Lambda_i\) is finite and closed.

**Theorem 5.15.** For each \(i\), write \(\mathcal{O}_{\Lambda_i}\) for the Serre subcategory of \(\mathcal{O}_\Lambda\) consisting of finite successive extensions of the \(L_\lambda\) with \(\lambda \in \Lambda_i\). Then

1. \(\mathcal{O}_{\Lambda_i}\) is a highest weight category with standard objects \(M_\lambda\) and costandard objects \(A_\lambda\) for \(\lambda \in \Lambda_i\).
2. For each \(i\), the tautological map \(D^b\mathcal{O}_{\Lambda_i} \to \hat{g}_\kappa\text{-mod}^I\) is fully faithful.
3. The induced map
\[
\lim_{\to} D^b\mathcal{O}_{\Lambda_i} \longrightarrow \hat{g}_\kappa\text{-mod}^I,
\]
where the colimit is taken in DGCat, exhibits the latter as the ind-completion of the former.
(4) \( \hat{\mathfrak{g}}_\kappa - \text{mod}^I \) is canonically equivalent to bounded homotopy category of tilting modules \( T_\lambda \) for \( \lambda \in \Lambda \).

**Proof.** Assertion (1) is well known. For assertion (2), it suffices to check that for \( \mu, \nu \in \Lambda \) we have

\[
\text{RHom}_{(\hat{\mathfrak{g}}_\kappa, I)}(M_\mu, A_\nu) \cong \text{RHom}_\Lambda(\mathbb{C}_\mu, A_\nu)
\]

This is also well known, but we recall the calculation. If we write \( U \) for the prounipotent radical of the Iwahori, and for \( \lambda \in \Lambda \) we write \( C_\lambda \) for the corresponding character representation of the torus \( T \), we have

\[
\text{RHom}_{(\hat{\mathfrak{g}}_\kappa, I)}(M_\mu, A_\nu) \cong \text{RHom}_{\text{Rep}(T)}(C_\mu, A_\nu) \cong \text{RHom}_{\text{Rep}(T)}(C_\mu, C_{\nu^U}) \cong \text{RHom}_{\text{Rep}(T)}(C_\mu, C_{\nu}).
\]

For (3), note that by point (2) the map out of the colimit is fully faithful. Moreover, note that \( D^b(O_{\Lambda_i}) \) coincides with the pretriangulated envelope of the \( M_\lambda \) for \( \lambda \in \Lambda_i \). By Proposition 5.1 it follows that the map has essential image the compact objects of \( \hat{\mathfrak{g}}_\kappa - \text{mod}^I \), as desired. Finally, (4) follows from (3). \( \square \)

### 5.3. The Linkage Principle

In this section, \( \kappa \) continues to be any negative level. The highest weights of Category \( O \) are parametrized by the dual Cartan \( \mathfrak{h}^* \).

We will need the linkage principle, i.e. the decomposition of \( O \) into blocks, which we now briefly review. Recall the loop algebra

\[
\mathfrak{L}g = \mathfrak{g} \otimes \mathbb{C}((z)).
\]

There is a unique invariant bilinear form \( \kappa_b \) on \( \mathfrak{g} \) for which the short coroots have squared length two. Associated to \( \kappa_b \) is the central extension of \( \mathfrak{L}g \)

\[
0 \rightarrow \mathbb{C}c \rightarrow \hat{\mathfrak{g}}_{\kappa_b} \rightarrow \mathfrak{L}g \rightarrow 0,
\]

with Lie bracket given by the formula

\[
[X \otimes f, Y \otimes g] = [X, Y] \otimes fg + \kappa_b(X, Y) \text{Res } fg c
\]

for all \( X, Y \in \mathfrak{g} \) and \( f, g, c \in \mathbb{C}((z)) \). The affine coroots determine linear functionals on the extended dual Cartan \( \mathfrak{h}^* \otimes \mathbb{C}c^\vee \). We will only need the real affine coroots, which we denote by \( \hat{\Phi} \). Associated to this is a linear action of the affine Weyl group \( W \) on the extended dual Cartan \( \mathfrak{h}^* \otimes \mathbb{C}c^\vee \). Label the simple affine coroots \( \hat{\alpha}_i \) for \( i \in \hat{I} \), and note there is a unique element \( \hat{\rho} \) in \( \mathfrak{h}^* \otimes \mathbb{C}c^\vee \) satisfying \( \langle \hat{\rho}, \hat{\alpha}_i \rangle = 1 \) for all \( i \in \hat{I} \). Associated to this is the \( \hat{\rho} \)-shifted dot action of \( W \). Writing \( \kappa = k\kappa_b \) for some \( k \in \mathbb{C} \), the highest weights for \( \hat{\mathfrak{g}}_{\kappa} \) identify with the affine hyperplane

\[
\mathfrak{h}^*_\kappa := \mathfrak{h}^* + kc^\vee.
\]

The linear and dot actions of \( W \) preserve \( \mathfrak{h}^*_\kappa \). For \( \lambda \in \mathfrak{h}^*_\kappa \), there is an associated collection of real coroots

\[
\hat{\Phi}_\lambda := \{ \hat{\alpha} \in \hat{\Phi} : \langle \lambda + \hat{\rho}, \hat{\alpha} \rangle \in \mathbb{Z} \}.
\]
Write $W_{\lambda}$ for the corresponding integral Weyl group, i.e. the subgroup of $W$ generated by the reflections $s_{\tilde{\alpha}}$ for $\tilde{\alpha} \in \tilde{\Phi}_{\lambda}$. With this, we can state the block decomposition of $\mathcal{O}$, due to Deodhar-Gabber-Kac [9].

**Theorem 5.16. (Linkage principle)** For $\lambda \in h^*_k$, the block $\mathcal{O}_\lambda$ of $\mathcal{O}$ for $\hat{g}_\kappa$ containing $L_\lambda$ has highest weights $W_{\lambda} \cdot \lambda$.

Let us apply this to $\mathcal{O}_\Lambda$. Note that the integral weights have a common integral Weyl group, i.e.

**Proposition 5.17.** For any $\lambda, \lambda' \in \Lambda$, we have $\tilde{\Phi}_{\lambda} = \tilde{\Phi}_{\lambda'}$.

**Proof.** Write $\tilde{\Phi}_f$ for the coroots of $g$. There is a standard enumeration of the affine coroots as $\tilde{\alpha}_{(n)}$, for a choice of finite coroot $\tilde{\alpha} \in \tilde{\Phi}_f$ and integer $n \in \mathbb{Z}$. One may calculate that

$$\langle \lambda + \rho, \tilde{\alpha}_{(n)} \rangle = \langle \lambda + \rho, \tilde{\alpha} \rangle + \frac{n}{2} (\kappa - \kappa_c)(\tilde{\alpha}, \tilde{\alpha}). \quad (5.18)$$

Now note that if $\lambda \in \Lambda$, the first summand on the right hand side of (5.18) is an integer. □

Let us write $W_\kappa$ for the integral Weyl group arising from Proposition 5.17. Write $\mathcal{O}^{f,1}_\Lambda$ for the full subcategory of $\mathcal{O}_\Lambda$ consisting of finite length objects, and similarly $\mathcal{O}^{f,1}_\lambda$ for a block therein. Theorem 5.16 yields a decomposition of abelian categories

$$\mathcal{O}^{f,1}_\Lambda = \bigoplus_{\lambda \in W_\kappa \setminus \Lambda} \mathcal{O}^{f,1}_\lambda. \quad (5.19)$$

We would like to bootstrap this to an analogous decomposition for Harish-Chandra modules. Accordingly, for $\lambda \in W_\kappa \setminus \Lambda$, write $\hat{g}_\kappa$-mod$^f_\Lambda$ for the full cocomplete subcategory of $\hat{g}_\kappa$-mod$^f$ with compact generators $M_\mu$ for $\mu \in W_\kappa \cdot \lambda$.

**Proposition 5.20.** There is a decomposition in DGCat$_{\text{cont}}$:

$$\hat{g}_\kappa$-mod$^f = \bigoplus_{\lambda \in W_\kappa \setminus \Lambda} \hat{g}_\kappa$-mod$^f_\lambda. \quad (5.21)$$

**Proof.** Recall the $\Lambda_i, i \geq 0$, from Theorem 5.15. For each $i$ and $\lambda \in W_\kappa \setminus \Lambda$, set

$$\mathcal{O}_{\Lambda,i} := \mathcal{O}_\Lambda \cap \mathcal{O}_{\Lambda_i}.$$  

For fixed $i$, it follows from Equation 5.19 that we have a decomposition in DGCat

$$D^b \mathcal{O}_{\Lambda,i} = \bigoplus_{\lambda \in W_\kappa \setminus \Lambda} D^b \mathcal{O}_{\lambda,i}.$$  

Taking a colimit over $i$ in DGCat and ind-completing, we are therefore done by Theorem 5.15(3). □

**Corollary 5.22.** For arbitrary noncritical $\kappa'$, there is a decomposition in DGCat$_{\text{cont}}$:

$$\hat{g}_{\kappa'}$-mod$^f = \bigoplus_{\lambda \in W_{\kappa'} \setminus \Lambda} \hat{g}_{\kappa'}$-mod$^f_\lambda.$$
Proof. The decomposition at positive rational levels follows from the decomposition at negative rational levels by Kac-Moody duality. More carefully, Arkhipov–Gaitsgory showed in Section 2.2 of [1] that semi-infinite cohomology gives a perfect pairing
\[
C_{\hat{X}^+}(\mathfrak{g}_{2\kappa_+}, \mathfrak{g}_- \otimes \mathfrak{g}_{-\kappa'+2\kappa_+}) : \hat{g}_{\kappa'} \otimes \hat{g}_{-\kappa'+2\kappa_+} \rightarrow \text{Vect}
\]
which was shown to be $\mathfrak{L}G$-equivariant by Raskin [25]. This therefore induces a perfect pairing on $I$-equivariant categories. The associated contravariant equivalence on compact objects interchanges $\mathcal{M}_\lambda$ at level $\kappa'$ with $M_{-\lambda-2\rho}[\dim N]$ at level $-\kappa'+2\kappa_c$, cf. Lemma 9.8 of [10]. □

5.4. Action of the affine Hecke algebra. In this section, we take $G$ to be simply connected and $\kappa$ to be negative integral, i.e.
\[
\kappa \in \kappa_c + \mathbb{Z}_{<0}K_b.
\]
In this situation, there is a canonical monoidal equivalence $D_\kappa(\mathfrak{L}G) \simeq D(\mathfrak{L}G)$, and hence an action of the affine Hecke algebra $\mathcal{H} := D(I \setminus \mathfrak{L}G/I)$ on $\hat{g}_\kappa$-mod$^I$. Recall that $\lambda \in \Lambda$ is antidominant if
\[
\langle \lambda + \check{\rho}, \check{\alpha}_i \rangle \neq 1, 2, 3, \ldots
\]
for all $i \in \hat{I}$. Each orbit of $W_\kappa$ on $\Lambda$ contains a unique antidominant weight, hence we may identify $W_\kappa \setminus \Lambda$ with such weights. For an antidominant weight $\lambda$, its stabilizer $W_\lambda$ in $W$ is a parabolic subgroup, and we denote by $W^\lambda$ the set of minimal length coset representatives for $W/W_\lambda$. The goal of this subsection is to prove the following two theorems.

Theorem 5.23. The direct sum decomposition (5.21) is one of affine Hecke algebra modules. I.e., for each $\lambda \in \Lambda$ antidominant, the action of $\mathcal{H}$ preserves $\hat{g}_\kappa$-mod$^I$.

To state the second theorem, for $\lambda$ antidominant, write $\delta_\lambda$ for $M_\lambda \simeq L_\lambda$. For each $w \in W$ write $j_{w,(!)}, j_{w,*}$ for the corresponding standard and costandard objects in $D(I \setminus \mathfrak{L}G/I)$, cf. the discussion preceding Proposition 4.5.

Theorem 5.24. Fix an antidominant weight $\lambda \in \Lambda$ and $w \in W^\lambda$.

1. $j_{w,(!)} \ast \delta_\lambda \simeq M_{w,\lambda}$.
2. $j_{w,*} \ast \delta_\lambda \simeq A_{w,\lambda}$.
3. Under the equivalences of (1) and (2), applying $- \ast \delta_\lambda$ to the canonical map
\[
\tilde{j}_{w,(!)} \rightarrow \tilde{j}_{w,*}
\]
yields a nonzero map $M_{w,\lambda} \rightarrow A_{w,\lambda}$.

5.4.1. The regular case. We first handle those $\lambda$ which are regular, i.e. have trivial stabilizer in $W$, following the affine localization theorem due to Kashiwara-Tanisaki [19], Beilinson-Drinfeld [1], and Frenkel-Gaitsgory [13].

Write $\tilde{I}$ for the promipotent radical of $I$, and consider the enhanced flag variety $\mathfrak{L}G/\tilde{I}$. This carries a right action of $T$, and hence we can consider the monodromic category $D_\kappa(\mathfrak{L}G/\tilde{I})^{T,w,\lambda}$. We have a global sections functor
\[
D_\kappa(\mathfrak{L}G/\tilde{I})^{T,w,\lambda} \xrightarrow{\text{ObvlR}} \text{IndCoh}(\mathfrak{L}G/\tilde{I})^{T,w,\lambda} \xrightarrow{\Gamma_{\text{IndCoh}}} \text{Vect}^{T,w} \xrightarrow{(-)^hT} \text{Vect}.
\]
Here IndCoh(\(\mathfrak{L}G/\mathfrak{I}\)) denotes the dg-category of ind-coherent sheaves on \(\mathfrak{L}G/\mathfrak{I}\), and \(\Gamma^{IndCoh}\) is the functor of ind-coherent global sections. We remark that IndCoh(\(\mathfrak{L}G/\mathfrak{I}\)) is canonically equivalent to QCoh(\(\mathfrak{L}G/\mathfrak{I}\)), but the usual global sections functor \(\Gamma\) on the latter is not continuous. However, \(\Gamma^{IndCoh}\) can be characterized as the unique continuous functor which agrees with \(\Gamma\) on complexes supported on quasicompact closed subschemes of \(\mathfrak{L}G/\mathfrak{I}\).

The composition (5.25) canonically lifts to a strongly \(\mathfrak{L}G\)-equivariant functor

\[ \Gamma^T : D_\kappa(\mathfrak{L}G/\mathfrak{I})^{T,w,\lambda} \to \hat{\mathfrak{g}}_\kappa \text{-mod}, \]

(5.26)

cf. Proposition A.4 from the appendix.

**Theorem 5.27.** After taking \(I\)-invariants, \(\Gamma^T\) gives an equivalence

\[ \Gamma^T : D_\kappa(I\backslash\mathfrak{L}G/I)^{T,w,\lambda} \simeq \hat{\mathfrak{g}}_\kappa \text{-mod}_I. \]

(5.28)

**Proof.** For the Schubert cell

\[ j^0_w : Iw\mathfrak{I} \to \mathfrak{L}G \]

corresponding to \(w \in W\), consider the standard and costandard objects \(j^0_{w!,j^0_{w,*}}\). As shown in Theorem 4.2 and Proposition 4.4, the \(j^0_{w!,}\) are compact generators, as are the \(j^0_{w,*}\). We will use the following result of Kashiwara–Tanisaki, which appears as Theorem 3.4.1 in [19].

**Theorem 5.29.** *(Kashiwara-Tanisaki)* For any \(w \in W\), we have

1. \(\Gamma^T j^0_{w,!} \simeq M_{w,\lambda}\),
2. \(\Gamma^T j^0_{w,*} \simeq A_{w,\lambda}\),
3. \(\Gamma^T\) sends a nonzero map \(j^0_{w,!} \to j^0_{w,*}\) to a nonzero map \(M_{w,\lambda} \to A_{w,\lambda}\).

We are therefore done by Theorems 5.15 and 5.16, as both categories are identified with the ind-completion of the same bounded homotopy category of tiltings.

We may now deduce Theorems 5.23 and 5.24 in the regular case.

**Proof of Theorem 5.23 for regular \(\lambda\).** Consider any \(D(\mathfrak{L}G)\)-representations \(\mathcal{C}, \mathcal{D}\) equipped with an equivariant map \(\mathcal{C} \to \mathcal{D}\). After taking \(I\)-invariants, we obtain a map of \(\mathcal{H}\) modules \(\mathcal{C}^I \to \mathcal{D}^I\). In particular, the essential image of \(\mathcal{C}^I\) is preserved by the action of the affine Hecke algebra. Taking \(\mathcal{C}, \mathcal{D}\) as in Theorem 5.27 gives the claim.

**Proof of Theorem 5.24 for regular \(\lambda\).** By Theorem 5.27, it suffices to address the analogous claim in \(D(I\backslash\mathfrak{L}G/I)\), where it is clear.

5.5. **The singular case.**

**Proof of Theorem 5.23 for singular \(\lambda\).** For any \(\lambda \in \Lambda\), not necessarily antidominant, consider the global sections functor as in (5.28). Note that it sends \(j_{w,*}\) to an object in degree zero with the same Jordan-Hölder content as \(A_{w,\lambda}\). It follows that the subcategory of \(\hat{\mathfrak{g}}_\kappa \text{-mod}_I\) generated under colimits by essential image of \(\Gamma^T\) coincides with \(\hat{\mathfrak{g}}_\kappa \text{-mod}_I^\lambda\).
For \( \lambda \) singular antidominant, one can show that Theorem 5.29 holds for \( w \in W^\lambda \), which implies Theorem 5.24. However, we will instead deduce Theorem 5.24 from the regular case via translation functors.

For any two levels \( \kappa_1, \kappa_2 \), the usual tensor product of representations gives a map

\[
\hat{g}_{\kappa_1} \text{-mod}^{\text{naive}} \otimes \hat{g}_{\kappa_2} \text{-mod}^{\text{naive}} \longrightarrow \hat{g}_{\kappa_1 + \kappa_2} \text{-mod}^{\text{naive}}.
\]

Restricting to compact objects of the renormalized derived categories, we obtain a map

\[
\hat{g}_{\kappa_1} \text{-mod}^{c} \otimes \hat{g}_{\kappa_2} \text{-mod}^{c} \longrightarrow \hat{g}_{\kappa_1 + \kappa_2} \text{-mod}^{+},
\]

where \( \otimes^{c} \) denotes the tensor product on pre-triangulated idempotent-complete dg-categories. Ind-extending, we obtain a map

\[
\hat{g}_{\kappa_1} \text{-mod} \otimes \hat{g}_{\kappa_2} \text{-mod} \longrightarrow \hat{g}_{\kappa_1 + \kappa_2} \text{-mod}.
\]

Notice that the left hand side of (5.30) carries a diagonal action of \( D_{\kappa_1 + \kappa_2} (\mathcal{L}G) \). In the appendix, we show in Corollary A.2 the following

**Lemma 5.31.** The functor (5.30) carries a canonical \( D_{\kappa_1 + \kappa_2} (\mathcal{L}G) \)-equivariant structure.

Suppose that \( \kappa_1 \) is integral. By passing to integrable objects, we obtain an equivariant functor

\[
\hat{g}_{\kappa_1} \text{-mod}^{\mathcal{L}G} \otimes \hat{g}_{\kappa_2} \text{-mod} \longrightarrow \hat{g}_{\kappa_1 + \kappa_2} \text{-mod}.
\]

If we further pick an object \( M \) of \( \hat{g}_{\kappa_2} \text{-mod}^{\mathcal{L}G} \), this induces an equivariant functor

\[
M \otimes - : \quad \hat{g}_{\kappa_2} \text{-mod} \longrightarrow \hat{g}_{\kappa_1 + \kappa_2} \text{-mod}.
\]

We may further take \( I \)-invariants to obtain a Hecke-equivariant functor

\[
M \otimes - : \quad \hat{g}_{\kappa_2} \text{-mod}^{I} \longrightarrow \hat{g}_{\kappa_1 + \kappa_2} \text{-mod}^{I}.
\]

Suppose that \( \kappa_2 \) and \( \kappa_1 + \kappa_2 \) are negative integral. If we pick \( \lambda, \mu \in \Lambda \), we may include and project on the corresponding blocks to form the composition

\[
\lambda M_{\nu} : \quad \hat{g}_{\kappa_2} \text{-mod}^{\nu} \longrightarrow \hat{g}_{\kappa_2} \text{-mod}^{I} \longrightarrow \hat{g}_{\kappa_1 + \kappa_2} \text{-mod}^{I} \longrightarrow \hat{g}_{\kappa_1 + \kappa_2} \text{-mod}^{I}.
\]

By Theorem 5.23 the composite is a Hecke equivariant functor.

Now take \( \kappa \) negative integral, and \( \lambda \) a singular antidominant weight. Recall that a weight \( \pi \) is called dominant if

\[
\langle \pi, \bar{\alpha}_i \rangle \in \{0, 1, 2, \ldots\}
\]

for all \( i \in \hat{I} \). We may write \( \lambda = \nu + \pi \), where \( \pi \) is dominant and \( \nu \) is regular antidominant. We will take \( \lambda M_{\nu} \) to be the usual translation functor \( \mathcal{T} \), for which we need to check the following.

**Lemma 5.32.** For \( \pi \) dominant, \( L_{\pi} \) carries a canonical \( \mathcal{L}G \)-equivariant structure.

*Proof.* This follows from affine Borel-Weil-Bott and the equivariance of the global sections functor, cf. Propositions A.4 A.5 in the appendix. \( \square \)
Proof of Theorem [5.24] for singular \( \lambda \). Having already proved the regular case, it suffices to know for \( w \in W^\lambda \) that

\[
\mathcal{T}M_{w \cdot \nu} \simeq M_{w \cdot \lambda} \quad \mathcal{T}A_{w \cdot \nu} \simeq A_{w \cdot \lambda} \quad \mathcal{T}L_{w \cdot \nu} \simeq L_{w \cdot \lambda}.
\]

The first and third are explicitly stated by Kashiwara-Tanisaki in Section 3 of [20], and the middle follows from recalling that translation commutes with the standard duality \( \mathbb{D} \) on \( \mathcal{O} \).

\[\square\]

6. Affine Milicic-Soergel equivalence

Let \( G \) be simply connected, and fix a standard proper parahoric \( P = LU \) in \( \mathcal{L}G \). Let \( \kappa \) be a negative integral level, and \( \lambda \) an antidominant weight at level \( \kappa \) whose stabilizer in \( W \) is \( W^L \).

Theorem 6.1. There is a canonical \( t \)-exact and \( \mathcal{H} \)-equivariant equivalence

\[
D(I\mathcal{L}G/I^\circ_L, \psi) \simeq \hat{g}_\kappa - \text{mod}^I_\lambda.
\] (6.2)

Proof. Recall the objects \( \delta^\psi \) and \( \delta^\lambda \). We claim it suffices to produce an \( \mathcal{H} \)-equivariant functor in either direction which interchanges \( \delta^\psi \) and \( \delta^\lambda \). Suppose we have such a functor \( F \). By assumption, \( W^L = W^\lambda \), and hence by Proposition 4.5 and Theorem 5.24 we have that \( F \) interchanges \( M_{w \cdot \lambda} \) and \( j^\psi_{w, !} \) and \( A_{w \cdot \lambda} \) and \( j^\psi_{w, *} \) for any \( w \in W^L \). Moreover, for any \( y, w \in W^L \) it induces an isomorphism

\[
\text{RHom}(M_{y \cdot \lambda}, A_{w \cdot \lambda}) \simeq \text{RHom}(j^\psi_{y, !}, j^\psi_{w, *}),
\] (6.3)

since these vanish unless \( y = w \), in which case they are one dimensional and identified by Proposition 4.5(3) and Theorem 5.24(3). It then follows that \( F \) is an equivalence, e.g. by Proposition 4.4 and Theorem 5.15(4).

As in Theorem 3.1 to construct such an \( F \) we will produce an \( \mathcal{L}G \)-equivariant functor

\[
D(\mathcal{L}G/I^\circ_L, \psi) \rightarrow \hat{g}_\kappa - \text{mod}^I.
\]

This is equivalent to specifying an object of \( \hat{g}_\kappa - \text{mod}^I_{\mathcal{L}G, \psi} \), which will be the parabolic induction of the corresponding object for \( L \).

More precisely, write \( B_L \) for the Borel of \( L \) contained in \( I \), and write \( M^L_\lambda \) for the corresponding Verma module for \( I \) of highest weight \( \lambda \). Recall \( N^-_L \) denotes the unipotent radical of the opposite Borel of \( L \) whose Lie algebra contains \( h \), and note that \( \psi \) restricts to a nondegenerate character of \( N^-_L \). In the proof of Theorem 3.1 we produced an object \( M(\psi, \chi) \) of \( I - \text{mod}^\circ N^-_L, \psi \) such that, up to a cohomological shift, we had

\[
\text{Av}_{B_L, *} M(\psi, \chi) \simeq M^L_\lambda.
\]

Write \( p \) for the Lie algebra of \( P \), and take the parabolic induction

\[
M := \text{Ind}^\hat{g}_p^I \text{Infl}^p I M(\psi, \chi).
\]
As in the proof of Theorem 3.1, it remains to calculate the $I$-average of $M$, and check that it agrees with $\delta^\lambda$ up to a cohomological shift. The short exact sequence of group schemes

$$1 \rightarrow U_P \rightarrow I \rightarrow B_L \rightarrow 1$$

yields a canonical isomorphism of functors

$$\text{Av}_{I,*} \cong \text{Av}_{B_L,*} \circ \text{Av}_{U_P,*}.$$

Since $M$ is by construction $U_P$ equivariant, by the prounipotence of $U_P$ we have

$$\text{Av}_{U_P,*} M \cong M.$$

As inflation and induction are both canonically $B_L$ equivariant functors, cf. the appendix, we have

$$\text{Av}_{B_L,*} \text{Ind}_{p}^g \text{Inf}_{p}^\lambda \text{M}(\psi, \hat{\chi}) \cong \text{Ind}_{p}^g \text{Inf}_{p}^\lambda M_{\lambda} \cong M_{\lambda},$$

as desired. □

Finally, let us note that one can switch between $\hat{I}^{-}$ and $\hat{I}$-invariants. Namely, if $\mathcal{C}$ is a strong representation of $\mathcal{L}G$, $g$ an element of $\mathcal{L}G(\mathcal{C})$, and $H$ an open subgroup of $\mathcal{L}G$, then there is a canonical identification

$$\delta_{g} * : \mathcal{C}^H \cong \mathcal{C}^{Hg^{-1}}.$$ (6.4)

In particular, if we write $\psi_L$ for an additive character of $\hat{I}$ which is nonzero precisely on the simple positive roots of $L$, and we choose a lift of $w_{0}^{L}$ to $L$, we obtain

**Corollary 6.5.** There is a canonical $t$-exact and $\mathcal{H}$-equivariant equivalence

$$D(I \setminus \mathcal{L}G/\hat{I}, \psi_L) \simeq \hat{\mathfrak{g}}_{\kappa} \text{-mod}_{\hat{\lambda}}^I.$$

**Remark 6.6.** We expect that the argument for Theorem 6.1 should be adaptable, mutatis mutandis, to $\mathcal{L}G$ replaced by a Kac–Moody group and $L$ a standard Levi of finite type.

### 7. The fundamental local equivalence

Let $\mathfrak{g}$ be a simple Lie algebra, and $\hat{\mathfrak{g}}$ its Langlands dual. Recall the notion of dual levels

$$\kappa \in \text{Sym}^2(\mathfrak{g}^*)^\mathfrak{g}, \quad \hat{\kappa} \in \text{Sym}^2(\hat{\mathfrak{g}}^*)^\hat{\mathfrak{g}}.$$

It will be helpful for us to be more concrete. Write $\kappa_{b}(\mathfrak{g})$ for the basic level of $\mathfrak{g}$, and $\kappa_{b}(\hat{\mathfrak{g}})$ for the basic level of $\hat{\mathfrak{g}}$. There is a corresponding trivialization of the lines of invariant forms

$$\mathbb{C} \cong \text{Sym}^2(\mathfrak{g}^*)^\mathfrak{g}, \quad \mathbb{C} \cong \text{Sym}^2(\hat{\mathfrak{g}}^*)^\hat{\mathfrak{g}},$$

interchanging $1 \in \mathbb{C}$ and the basic levels. Denote the common lacing number of $\mathfrak{g}$ and $\hat{\mathfrak{g}}$ by $r$. Finally, write $h^\vee$ for the dual Coxeter number of $\mathfrak{g}$, and $Lh^\vee$ for the dual Coxeter number of $\hat{\mathfrak{g}}$. Then explicitly the level dual to $-h^\vee + k$, for $k \in \mathbb{C}^\times$, is

$$-Lh^\vee + \frac{1}{rk}.$$ (7.1)
7.1. The extended affine Weyl group and Dynkin diagram automorphisms. Let $G$ be an almost simple group with Lie algebra $\mathfrak{g}$. Then it is well known that its fundamental group acts via Dynkin diagram automorphisms on $\hat{\mathfrak{g}}_{\kappa}$, cf. Section 1 of [18]. It will be convenient to use the following construction of that action, which is probably known.

Associated to the fixed triangular decomposition of $\mathfrak{g}$ is a maximal torus $T$ of $G$. Writing $\hat{\Lambda}_G$ for its coweight lattice, one forms the extended affine Weyl group

$$W^{\text{ext}} := W_f \ltimes \hat{\Lambda}_G.$$ 

The inclusion of the coroot lattice into $\hat{\Lambda}_G$ gives a short exact sequence

$$1 \rightarrow W \rightarrow W^{\text{ext}} \rightarrow \pi_1(G) \rightarrow 1.$$ 

Let us write $\text{Dy}(\hat{\mathfrak{g}}_{\kappa})$ for the automorphism group of the Dynkin diagram of $\hat{\mathfrak{g}}_{\kappa}$, which we identify with the corresponding automorphisms of $\hat{\mathfrak{g}}_{\kappa}$. The desired homomorphism $\text{Dy}$ fits into an exact sequence

$$1 \rightarrow W \rightarrow W^{\text{ext}} \rightarrow \text{Dy}(\hat{\mathfrak{g}}_{\kappa}).$$ 

The construction is as follows. Consider the normalizer $N_G(T)$, and form its loop group $\mathfrak{L}N_G(T)$. Its component group identifies with $W^{\text{ext}}$, and its neutral component with that of $\mathfrak{L}T$. For any $\mathbb{C}$-point $\dot{w}$ of $\mathfrak{L}N_G(T)$, we may restrict its adjoint action to the standard Iwahori

$$I \xrightarrow{\dot{w}} \text{Ad}_{\dot{w}} I.$$ 

The latter group is another Iwahori subgroup containing $T$, and hence may be conjugated by some $y$ in $\mathfrak{L}N_{G_s}(T)$ back to $I$, where $G_s$ denotes the simply connected cover of $G$. The composition

$$I \xrightarrow{\dot{w}} \text{Ad}_{\dot{w}} I \xrightarrow{y} I$$ 

when restricted to the abelian quotient of the prounipotent radical yields the desired element of $\text{Dy}(\hat{\mathfrak{g}}_{\kappa})$. It is straightforward to check that this is independent of the choice of $y$, and that this action factors through the component group $W^{\text{ext}}$, provided that one remembers that the $\mathbb{C}$-points of the arc group $\mathfrak{L}^+T$ and of the neutral component of $\mathfrak{L}T$ coincide.

In the following subsection we will use the following observation and accompanying notation. For $w$ an element of $W^{\text{ext}}$, write $\text{Dy}_w$ for the image of $w$ under $\text{Dy}$. Then for any standard Levi $L$ of $\mathfrak{L}G$, its image $\text{Dy}_w L$ is again a standard Levi.

7.2. The Whittaker side of the FLE. We will now rewrite the Whittaker category on the affine flag variety in terms of Kac-Moody representations. As in Corollary 6.5, for a standard Levi $L$ of $\mathfrak{L}G_s$, write $\psi_L$ for any additive character of $I$ which is nonzero precisely on the positive simple roots of $L$. Note that all such are interchanged under the action of $T$ as in Equation (6.4). By a slight abuse of notation, we will often write $G$ instead of the standard Levi $G_s$.

Fix an auxiliary negative integral level $\kappa'$ such that there exists a regular antidominant weight at level $\kappa'$. At such a level, for each proper standard Levi $L$ we may choose an antidominant weight $\mu_L$ whose stabilizer in $W$ is $W_L$. Finally, write $I_s$ for the Iwahori of $\mathfrak{L}G_s$. 
Proposition 7.2. There is an $\mathcal{H}$-equivariant isomorphism
\[
D(I\backslash LG/\mathcal{L}N, \psi) \simeq \bigoplus_{\lambda \in \pi_1(G)} \hat{g}_{\lambda'} \text{-mod}_{D_{\mathfrak{l}}(-\lambda-\rho)G} (7.3)
\]

Proof. As shown by Raskin in Section 7 of [23], the Whittaker category coincides with the baby Whittaker category, i.e.
\[
D(I\backslash LG/\mathcal{L}N, \psi) \simeq D(I\backslash LG/ Ad_{t-\rho} \hat{I}, Ad_{t-\rho} \psi_G).
\]
We will now decompose the latter as a $D(Fl_{G})$-module over the connected components of $Fl_{G}$:
\[
D(I\backslash LG/ Ad_{t-\rho} \hat{I}, Ad_{t-\rho} \psi_G) \simeq \bigoplus_{\lambda \in \pi_1(G)} D(Ad_{t-\rho} I_s \backslash LG_s/ Ad_{t-\rho} \hat{I}, Ad_{t-\rho} \psi_G) \simeq \bigoplus_{\lambda \in \pi_1(G)} D(I_s\backslash LG_s/ Ad_{t-\rho} I, Ad_{t-\rho} \hat{I}, Ad_{t-\rho} \psi_G)
\]
Applying Corollary 6.5 to the summands in the last expression yields the proposition. □

7.3. The Kac-Moody side of the FLE. We are interested in studying affine Category $O$ for $\hat{g}$ at a level
\[
\kappa = -Lh^\vee + \frac{1}{r_k} (7.4)
\]
where $k \in \mathbb{Z}^{<0}$. We will write $\hat{W}$ for the affine Weyl group of $\hat{g}$, and continue to write $W$ for the affine Weyl group of $g$. Recall that the weight lattice for the Iwahori $\hat{I}$ of $\mathfrak{L}G$ identifies with $\Lambda_G$, and that all such weights have a common integral Weyl group $\hat{W}_\kappa$, cf. Proposition 5.17.

Proposition 7.5. For a level of the form (7.4), there is a canonical isomorphism of Coxeter groups
\[
\hat{W}_\kappa \simeq W.
\]

Proof. After composing with the automorphism of $h_\hat{g}$ given by translation by $\hat{\rho}$, the action of $\hat{W}$ on $h_\hat{g}$ identifies it with $W_f \times \frac{1}{rk} Q^l$, where $W_f$ is the finite Weyl group acting linearly, and $Q^l$ is the long coroot lattice acting by translations. The form $\kappa_b$ for $\hat{g}$ induces a $W_f$ invariant inner product on $h$ for which the long coroots have squared length two. Accordingly, for a coroot $\alpha$, write $\hat{\alpha}_l$ for its long multiple, i.e.
\[
\hat{\alpha}_l = \begin{cases} \hat{\alpha}, & \kappa_b(\hat{\alpha}, \hat{\alpha}) = 2, \\ r\hat{\alpha}, & \kappa_b(\hat{\alpha}, \hat{\alpha}) = \frac{2}{r} \end{cases}
\]
For any \( \lambda \in \mathfrak{h} \), let us write \( t^\lambda \) for translation by \( \lambda \). Finally, recall the enumeration of real affine roots
\[
\Phi \simeq \Phi_f \times \mathbb{Z}.
\]
With this notation, the affine reflection for \( \alpha_{(n)} \), where \( \alpha \in \Phi_f \) and \( n \in \mathbb{Z} \), is given by
\[
s_{\alpha_{(n)}} = t^{-\frac{n}{2r_k} \hat{\alpha}_i} s_\alpha t^{\frac{n}{2r_k} \hat{\alpha}_i}.
\]  
(7.6)
As discussed in Proposition 5.17, for \( \alpha_{(n)} \) to lie in \( \check{W} \) is equivalent to
\[
- n \frac{\kappa_b(\alpha, \alpha)}{2r_k} \in \mathbb{Z}.
\]  
(7.7)
Recall that \( \kappa_b(\alpha, \alpha) \) is 2 or \( 2r \), depending on whether \( \alpha \) is short or long. If we write \( \Phi_s \) for the short coroots, and \( \Phi_l \) for the long coroots, then it follows from (7.7) that \( \check{W} \) is generated by the reflections corresponding to
\[
\Phi_s \times \mathbb{Z} \quad \sqcup \quad \Phi_l \times k\mathbb{Z}.
\]  
(7.8)
Combining this with Equation (7.6), it follows that \( \check{W} \) is identified with \( W_f \rtimes \check{Q} \), where \( \check{Q} \) denotes the coroot lattice. Applying \( \kappa_b \) for \( g \) identifies this with \( W_f \rtimes Q_l \), the semidirect product of \( W_f \) and the long root lattice, acting on \( \mathfrak{h}^* \), i.e. \( W \). Moreover, it is straightforward to see these identifications intertwine the set of simple reflections corresponding to the walls of the negative alcove.

Recall that the simple generators of \( \check{W} \) correspond to the positive elements of (7.6) which cannot be written as a nontrivial sum of other positive elements of (7.6), cf. Section 2 of [20]. If we consider the finite simple roots \( \alpha_i \), for \( i \in I \), along with the highest root \( \theta \), we obtain the following.

**Corollary 7.9.** The simple reflections of \( \check{W} \) are given by
\[
s_{\alpha_i(0)}, \text{ for } i \in I, \text{ and } s_{-\theta, (-k)} = t^{-\frac{\theta}{2}} s_\theta t^{\frac{\theta}{2}}.
\]

It follows from Proposition 7.5 that the block decomposition of \( \mathcal{O} \) is given by the dot action of \( W_f \rtimes \check{Q} \) on \( \check{\Lambda}_G \). We now determine the antidominant weights under this action. As notation, let us call a coweight \( \check{\mu} \in \check{\Lambda}_G \) **negative minuscule** if for every positive root \( \alpha \) of the finite root system, we have
\[
-1 \leq \langle \alpha, \check{\mu} \rangle \leq 0.
\]

**Lemma 7.10.** A coweight \( \check{\lambda} \in \check{\Lambda}_G \) is antidominant with respect to \( \check{W} \) if and only if \( \check{\lambda} + \check{\rho} \) is negative minuscule.

**Proof.** By Corollary 7.9, it follows that \( \check{\lambda} \) is antidominant if and only if
\[
\langle \alpha_i, \check{\lambda} + \check{\rho} \rangle \leq 0, \text{ for all } i \in I, \text{ and } -1 \leq \langle \theta, \check{\lambda} + \check{\rho} \rangle.
\]
Since \( \theta \) is the highest root, the claim follows. \( \square \)
Let us index the simple reflections of \( \hat{W}_k \) by \( \hat{J} \), the affine simple roots of \( \hat{g} \), as in Corollary 7.9. If \( \hat{\lambda} + \hat{\rho} \) is the negative minuscule coweight \( -\hat{\omega}_i \) for some \( i \in I \), i.e. for any \( j \in I \) we have

\[
\langle \hat{\omega}_i, \alpha_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}
\]

it follows that the stabilizer of \( \hat{\lambda} \) is the parabolic subgroup of \( \hat{W}_k \) generated by the simple reflections corresponding to \( \hat{I} \setminus i \).

7.4. **Combining the two sides.** To compare the two sides, it remains to examine the summands arising in the Whittaker category as in Proposition 7.2, which were indexed by \( \pi_1(G) \). To account for the \( \hat{\rho} \)-shift in Lemma 7.11, write \( G_{\text{ad}} \) for the adjoint form \( G \), and \( \hat{\Lambda}_{G_{\text{ad}}} \) for the coweight lattice. Recall that under the identification \( \pi_1(G_{\text{ad}}) \cong \hat{\Lambda}_{G_{\text{ad}}}/\hat{Q} \), we may associate to each element \( \hat{\lambda} \) in \( \pi_1(G_{\text{ad}}) \) a unique coset representative in \( \hat{\Lambda} \) which is minuscule.

**Lemma 7.11.** For \( \hat{\lambda} \in \pi_1(G_{\text{ad}}) \), write \( \hat{\omega}_i \) for its unique minuscule representative. Then \( \text{Dy}_{-\hat{\lambda}} \hat{G} \) is the standard Levi of \( \mathcal{L}G_s \) corresponding to \( \hat{J} \setminus i \).

This lemma essentially appears with slightly different language and conventions in [18], but we include a somewhat different proof for the convenience of the reader.

**Proof.** We must show that \( \text{Dy}_{-\hat{\omega}_i} \) sends the affine node 0 to \( i \). This is equivalent to \( \text{Dy}_{\hat{\omega}_i} \) sending \( i \) to 0.

Write \( P = LU_P \) for the standard Levi of \( G \) corresponding to \( \hat{J} \setminus i \). Since \( \hat{\omega}_i \) is minuscule, if we expand \( \theta \) as a sum of simple roots, the coefficient of \( \alpha_i \) is one. It follows that the Lie algebra of \( U_P \) is simple as an \( L \)-module, and in particular that the longest element \( w_0^L \) of the Weyl group of \( L \) exchanges \( \theta \) and \( \alpha_i \). From this observation, it is straightforward to see that \( \text{Dy}_{\hat{\omega}_i} \) may be given, after picking lifts of \( w_0 \) and \( w_0^L \) to \( G \), as the composition

\[
\mathcal{L}G_s \xrightarrow{\text{Ad}_{\hat{\omega}_i}} \mathcal{L}G_s \xrightarrow{\text{Ad}_{w_0^L}} \mathcal{L}G_s \xrightarrow{\text{Ad}_{w_0}} \mathcal{L}G_s.
\]

In particular, this sends the finite simple root \( \alpha_i \) to the affine root \( \alpha_0 \). \( \square \)

Having assembled all the necessary ingredients, we are ready to prove our main theorem.

**Proof of Theorem 2.3.** Recall that \( \kappa \) is negative. On the Whittaker side, Proposition 7.2 yields the decomposition

\[
D(T\backslash \mathcal{L}G/\mathcal{L}N, \psi) \cong \bigoplus_{\hat{\lambda} \in \pi_1(G)} \hat{g}_{\kappa'} -\text{mod}^{I_S}_{\text{Dy}(\hat{-\lambda-\hat{\rho}})} \label{7.12}
\]

On the Kac-Moody side, Subsection 7.3 and Proposition 5.20 yields

\[
\hat{g}_{\kappa} -\text{mod}^I \cong \bigoplus_{\hat{\lambda} \in \pi_1(G)} \hat{g}_{\kappa} -\text{mod}^I_{\hat{-\lambda-\hat{\rho}}} \label{7.13}
\]
For fixed $\lambda$, by Theorem 5.15 the compact objects in the heart of the corresponding summands of (7.12), (7.13) are blocks of affine Category $\mathcal{O}$ for $\mathfrak{g}$ and $\mathfrak{g}'$, respectively. Under the identification of their integral Weyl groups given in Proposition 7.5, the stabilizers of their antidominant representatives coincide by Lemma 7.11. Soergel’s theory, as proved for Kac-Moody algebras by Fiebig [11], yields a $t$-exact equivalence between the corresponding blocks. Applying Theorem 5.15, this induces an equivalence between the full blocks of Harish-Chandra modules, as desired.

Finally, the case of positive $\kappa$ follows from the negative case by dualizing. Namely, the standard self-duality of $D(\text{Fl}_G)$ carries a canonical datum of $\mathcal{L}G$-equivariance. By the identification of Whittaker invariants and coinvariants, proved by Raskin in Section 2 of [26], this induces a duality of the Whittaker categories, as explained e.g. in Section 4 of [10]. On the Kac-Moody side, one may similarly use Kac-Moody duality, and the canonical identification of $\mathfrak{I}$-invariants and coinvariants.

As a final remark, let us give an alternative perspective on the proof of Theorem 2.3, based on a categorification of the local Shimura correspondence [28]. A general expectation of local quantum geometric Langlands is the exchanging of $I$-invariants and $I'$-invariants. Applying this to the endomorphisms of the corepresenting objects, namely $D_\kappa(\text{Fl}_G)$ and $D_{\kappa'}(\text{Fl}_{\mathfrak{g}'}_G)$, we obtain the following conjecture, due to Gaitsgory.

**Conjecture 7.14.** There is a canonical monoidal equivalence

$$D_\kappa(\mathfrak{I} \mathcal{L}G/\mathfrak{I}) \simeq D_{\kappa'}(\mathfrak{I}' \mathcal{L}\mathfrak{g}'/\mathfrak{I}).$$

(7.15)

Under this identification, the equivalence (2.2) intertwines the affine Hecke actions on each side.

Conjecture 7.14 allows for a repackaging of the above argument which replaces Soergel modules by modules for the metaplectic affine Hecke categories identified by (7.15). That being said, Conjecture 7.14 itself should follow from a Soergel bimodule argument, which is work in progress by the second named author.

**Appendix A. Equivariance data for some standard functors in Geometric Representation Theory**

A.1. **Weak invariants for groups of infinite type.** The material in this subsection is due to Raskin. It is outlined in the note [24], and will be developed in detail in the forthcoming paper [25]. We summarize the relevant definitions here for the reader’s convenience.

In what follows, by a group we mean either (i) an algebraic group of finite type, (ii) an affine group scheme with a prounipotent radical of finite codimension, or (iii) an ind-affine group ind-scheme $G$ containing a subgroup $K$ of type (ii) for which $G/K$ is ind-proper, and in particular of ind-finite type. Such a group has a Lie algebra $\mathfrak{g}$ which is (i) discrete, (ii) linearly compact, or (iii) a Tate vector space, respectively.

In these cases one can make sense of weak invariants for a category $\mathcal{C}$ acted on strongly by $G$. This is standard in the case (i), so suppose $G$ is of type (ii) and write $G = \lim_i G_i$
with $G_i = G/G^i$ of finite type for all $i$. Then we put

$$
C^{G,w} := \lim_{i} (C^{G^i})_{G^i,w},
$$

with the colimit taken in DGCat$_{\text{cont}}$. For instance, the category

$$
\text{Rep}(G) := \text{Vect}^{G,w}
$$

is the renormalized dg-category of $G$-representations. Indeed, instead of the naïve category of $\mathcal{O}_G$-comodules, we obtain

$$
\text{Rep}(G) \simeq \lim_{i} \text{Rep}(G_i).
$$

In case (iii), we can consider the “weak Hecke category”

$$
\mathcal{H}^{w}_{G,K} := \text{IndCoh}(G/K)^{K,w}
$$

equipped with its natural convolution monoidal structure.

Now for a category $\mathcal{C}$ acted on strongly by $G$, the category $\mathcal{C}^{K,w}$ as defined above acquires an $\mathcal{H}^{w}_{G,K}$-action. In these terms we can define

$$
\mathcal{C}^{G,w} := \text{Fun}_{\mathcal{H}^{w}_{G,K}}(\text{Rep}(K), \mathcal{C}^{K,w}).
$$

In fact $\mathcal{C}^{G,w}$ does not depend on the choice of subgroup $K$, but we will not need this.

As in [6], one defines $D^*(G)$ and $D^!(G)$. These are algebra and coalgebra objects in DGCat$_{\text{cont}}$ respectively, mutually dual, each with two commuting strong $G$-actions. By definition, a strong action of $G$ on a category is just a $D^*(G)$-action. Further, one can consider the renormalized category of Lie algebra representations $\mathfrak{g}$-mod, and the canonical identification

$$
\mathfrak{g}\text{-mod} \simeq D^!(G)^{G,w},
$$

where one takes weak invariants with respect to the right action of $G$ and retains a strong action via left multiplication. Moreover, for a multiplicative twisting $\mathcal{T}$ on $G$, consider the corresponding extension $\mathfrak{g}_\kappa$ of $\mathfrak{g}$. One can form the algebra object $D^*_\kappa(G)$ of twisted $D$-modules, its dual coalgebra object $D^!\kappa(G)$, the renormalized category of Lie algebra representations $\mathfrak{g}_\kappa\text{-mod}$, and the identification

$$
\mathfrak{g}_\kappa\text{-mod} \simeq D^!\kappa(G)^{G,w}
$$

of strong $G$-categories at level $\kappa$.

### A.2. Restriction and induction.

**Proposition A.1.** Let $\pi : H \to G$ be a group morphism, $\mathcal{T}$ a multiplicative twisting on $G$, and $\mathfrak{h}_\kappa \to \mathfrak{g}_\kappa$ the corresponding morphism of centrally extended Lie algebras. Then the natural restriction map

$$
\text{Res} : \mathfrak{g}_\kappa\text{-mod} \to \mathfrak{h}_\kappa\text{-mod}
$$

carries a canonical datum of $H$-equivariance.
Proof. The restriction functor identifies with the composite
\[ D^1_k(G)^{G,w} \xrightarrow{\text{Obly}} D^1_k(G)^{H,w} \xrightarrow{\pi^*} D^1_k(H)^{H,w} \]
and the two appearing functors carry natural \( H \)-equivariant structures. \( \square \)

Corollary A.2. Let \( \mathcal{T}, \mathcal{T}' \) be two multiplicative twistings, and \( \mathfrak{g}_\kappa, \mathfrak{g}_{\kappa'} \) the corresponding central extensions of \( \mathfrak{g} \). The tensor product of representations
\[ \mathfrak{g}_\kappa \text{-mod} \otimes \mathfrak{g}_{\kappa'} \text{-mod} \to \mathfrak{g}_{\kappa+\kappa'} \text{-mod} \]
carries a canonical datum of \( D^*_{\kappa+\kappa'}(G) \)-equivariance.

Proof. One has a natural identification of renormalized derived categories
\[ \mathfrak{g}_\kappa \text{-mod} \otimes \mathfrak{g}_{\kappa'} \text{-mod} \simeq (\mathfrak{g} \oplus \mathfrak{g})_{\kappa+\kappa'} \text{-mod}, \]
where as usual \( (\mathfrak{g} \oplus \mathfrak{g})_{\kappa+\kappa'} \) denotes the quotient of \( \mathfrak{g}_\kappa \oplus \mathfrak{g}_{\kappa'} \) identifying the two canonical central elements. Under this identification, the above functor corresponds to restriction along the diagonal embedding \( G \to G \times G \). \( \square \)

Proposition A.3. Let \( \iota : H \to G \) be the embedding of a sub-group scheme such that \( G/H \) is of ind-finite type. Then the induction functor
\[ \text{Ind} : h_\kappa \text{-mod} \to g_\kappa \text{-mod}, \]
left adjoint to \( \text{Res} \), carries a canonical datum of \( H \)-equivariance.

Proof. We decompose \( \text{Ind} \) as follows. By the assumption on \( \iota \), the morphism \( H \to G \) is of finite presentation. In particular, \( \iota^! \) admits a left adjoint
\[ \iota^! \text{dR,}_*: D^1_k(H) \to D^1_k(G). \]
Writing \( p : G/H \to \text{Spec} \, k \), we also consider the weakly \( H \)-equivariant functor
\[ p^! : \text{Vect} \to \text{IndCoh}(G/H), \]
which after passing to weak \( H \)-invariants yields an \( \mathcal{H}^w_{G,H} \)-linear functor \( \text{Rep}(H) \to \mathcal{H}^w_{G,H} \).

Now the induction functor identifies with the composition
\[ D^1_k(H)^{H,w} \xrightarrow{\text{dR,}_*} D^1_k(G)^{H,w} \simeq \text{Fun}_{\mathcal{H}^w_{G,H}}(\mathcal{H}^w_{G,H}, D^1_k(G)^{H,w}) \]
\[ \xrightarrow{\text{op}^!} \text{Fun}_{\mathcal{H}^w_{G,H}}(\text{Rep}(H), D^1_k(G)^{H,w}) = D^1_k(G)^{G,w}, \]
and the two appearing functors carry strongly \( H \)-equivariant structures. \( \square \)

One should be able to show that the \( H \)-equivariant structures on \( \text{Res} \) and \( \text{Ind} \) are compatible under their adjunction. We will only use this compatibility in the case that \( G/H \) is ind-proper, where it is clear. Namely, the functor
\[ D^1_k(G)^{G,w} \xrightarrow{\text{Obly}} D^1_k(G)^{H,w} \]
is given by precomposition with \( p^!_{\text{IndCoh}} \), which in the ind-proper case is left adjoint to \( p^! \).
A.3. **Global sections on the enhanced flag variety.** Let us write \( \tilde{\mathfrak{Fl}} \) for the enhanced affine flag variety. Following Section 7 of Beilinson-Drinfeld [4], by its formal smoothness, for any level \( \kappa \) one has the global algebra of differential operators \( \Gamma(\tilde{\mathfrak{Fl}}, D_\kappa) \) which contains a copy of \( \hat{\mathfrak{g}}_{-\kappa} \). Moreover, ind-coherent global sections defines a functor
\[
\Gamma^{\text{IndCoh}} : D_\kappa(\tilde{\mathfrak{Fl}}) \to \text{Vect}.
\]
These sections carry the structure of discrete right modules for \( \Gamma(\tilde{\mathfrak{Fl}}, D_\kappa) \), and in particular left modules for \( \hat{\mathfrak{g}}_\kappa \). Let us denote this enhanced functor by
\[
\Gamma_{BD} : D_\kappa(\tilde{\mathfrak{Fl}}) \to \hat{\mathfrak{g}}_\kappa\text{-mod}.
\]

It is expected that \( \Gamma_{BD} \) carries a canonical datum of strong equivariance.

We now show enough consequences of this to suffice for the purposes of this paper.

**Proposition A.4.** Suppose \( \mathcal{C} \) carries an action of \( D_\kappa(\mathfrak{L}G) \), and one is given a weakly equivariant functor \( F : \mathcal{C} \to \text{Vect} \). Then \( F \) canonically factorizes through a strongly equivariant functor \( F : \mathcal{C} \to \hat{\mathfrak{g}}_\kappa\text{-mod}_\text{Oblv} \to \text{Vect} \).

**Proof.** Raskin in [25] shows that one has a right adjoint to
\[
\text{Oblv} : D_\kappa(\mathfrak{L}G)\text{-mod} \to \mathfrak{L}G\text{-mod}_{\text{weak}}.
\]
It therefore remains to apply the construction of *loc. cit.* to \( \text{ Vect} \). Namely, letting \( H \) run over compact open subgroups of \( \mathfrak{L}G \), we have
\[
\text{Oblv}^R(\text{ Vect}) \simeq \lim_{\longrightarrow} \text{ Vect}^{\mathfrak{L}G, w}_{\mathfrak{H}} \simeq \hat{\mathfrak{g}}_\kappa\text{-mod},
\]
as desired. \( \square \)

From the Proposition, we obtain an *a priori* different lift of \( \Gamma^{\text{IndCoh}} \) to an equivariant functor
\[
\Gamma : D_\kappa(\tilde{\mathfrak{Fl}}) \to \hat{\mathfrak{g}}_\kappa\text{-mod}.
\]
While we expect that \( \Gamma \simeq \Gamma_{BD} \), we will only show

**Proposition A.5.** The functors \( \Gamma, \Gamma_{BD} \) coincide after restriction to hearts, i.e. one has a natural isomorphism of functors
\[
\Gamma, \Gamma_{BD} : D_\kappa(\tilde{\mathfrak{Fl}}) \to \hat{\mathfrak{g}}_\kappa\text{-mod}^igcirc
\]
which induces the identity natural transformation between
\[
\text{Oblv} \circ \Gamma, \text{Oblv} \circ \Gamma_{BD} : D_\kappa(\tilde{\mathfrak{Fl}}) \to \text{Vect}^igcirc.
\]

**Proof.** Because we are working with the abelian category of Lie algebra representations, we may reduce to the analogous assertion for each \( \mathfrak{sl}_2 \) corresponding to an affine simple root \( \alpha_i \) for some \( i \in \mathfrak{j} \).

For fixed \( \alpha_i \), let \( B \) be any Borel subgroup of the corresponding copy of \( SL_2 \). Since \( B \) was arbitrary, it suffices to show the claim for \( B \). Let \( I \) be an Iwahori subgroup of \( LG \) containing \( B \). We may present \( \tilde{\mathfrak{Fl}} \) as an ind-scheme via the colimit of the Schubert varieties...
$X_w$ for $w \in W$, where $X_w$ is the closure of $Iw^-I$. Recall that $X_w$ has rational singularities, and its Bott-Samelson resolution $\tilde{X}_w$ carries a $B$-action for which the projection $\tilde{X}_w \to X_w$ is $B$-equivariant. We are therefore reduced to the analogous assertion for a linear algebraic group acting on a smooth algebraic variety, where it is straightforward. □

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