Numerical solutions of the modified Lane–Emden equation in $f(R)$-gravity

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ABSTRACT

The modified Lane–Emden equation for stellar hydrostatic equilibrium in $f(R)$-gravity is numerically solved by using an iterative procedure. Such an integro–differential equation can be obtained in the weak field limit approximation of $f(R)$-gravity by considering a suitable polytropic equation of state in the modified Poisson equation. The approach allows, in principle, to deal with still unexplored self-gravitating systems that could account for exotic stellar structures that escape standard stellar theory.

Key words: equation of state – gravitation – hydrodynamics – methods: numerical.

1 INTRODUCTION

Extended theories of gravity (ETGs) are acquiring more and more interest in modern astrophysics and cosmology due to several in-escapable issues and shortcomings that, in principle, could be addressed by revising or extending general relativity (GR; Bogdanos et al. 2010; Capozziello, De Laurentis & Stabile 2010b; Capozziello et al. 2011; De Laurentis & Capozziello 2011, 2012; De Laurentis & De Martino 2013b). In particular, observational and theoretical issues related to dark matter (DM) and dark energy could be encompassed in a new picture where gravitational interaction is depending on the scale of the self-gravitating systems without requiring further material ingredients (Capozziello & De Laurentis 2012) that, up to now, seem extremely elusive (see the recent results of the LUX Collaboration 2013). In particular, one of the most famous observational issues which needs DM, in addition to the visible one, concerns dynamics of stellar objects in galaxies, a problem arising since the beginning (Oort 1932). The disagreement between the observed and expected mass needed to explain dynamics soon revealed to be present even at cosmological scales (Zwicky 1933). Besides, the observed accelerating behaviour of the Hubble flow (Riess et al. 2004) needs to be addressed at fundamental level either by revising the standard cosmological model or finding out cosmic fluids capable of giving rise to cosmic acceleration.

It is however worth pointing out that the possibility to consider modifications of the Newtonian dynamics should be taken into account not only when considering the coupling of matter and gravity field of each other influencing distinct objects, but also in the case of self-gravitating systems. These can be of different types and sizes: collapsing molecular clouds, instability strip and main-sequence stars, hypermassive stars, and post-evolved very compact objects such as magnetars (Capozziello et al. 2012a; De Laurentis & De Martino 2013a). In particular, exotic compact objects are not only matter of speculative theories but are gaining observational evidence of their presence (Muno et al. 2006) and very often escape interpretation in the framework of standard Newtonian theory.

Specifically, the structure and evolution of very massive stars is a key field of investigation for interpreting peculiar objects like Gamma-Ray Burst (GRB). The nature of progenitors could play a crucial role in determining the energetics and hydrodynamical processes related to the central engine in order to provide the observed enormous amount of $10^{51}–54$ erg of electromagnetic energies. Suitable candidates for GRB progenitors are thought to be Wolf–Rayet stars with hydrogen-stripped envelopes and masses of the helium core $M \approx 15–30 M_\odot$.

Additionally, numerical simulations claim for the possibility of pair-instability supernovae with $M \gtrsim 130 M_\odot$ as possible progenitor, if not for all, at least for a part of GRBs (Chardonnet, Chechetkin & Titarchuk 2010). It is thus evident that addressing the problem of stellar stability over a wide range of masses and radii has important implications for several aspects of their pre- and post-evolutionary history. These theoretical issues can be considered valid both at descriptive and predictive level: on the one hand, there is the need to explain the existence of objects which seem to have masses higher than their expected value (i.e. Volkoff magnetars); on the other hand, providing theoretically stable stellar models (not yet observed) can have important implications both for stellar evolution and for testing

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In particular, modified gravity theories may lead to extra branch of solutions for neutron stars as recently pointed out in Astashenok, Capozziello & Odintsov (2013, hereafter A13). In general, the stability of stellar structures is strictly related to the hydrostatic equilibrium conditions. Equations governing such equilibrium contain the gradient of the gravitational potentials and are, ultimately, related to the field equations governing the coupling between matter and gravitational field. In the Newtonian theory, if the thermodynamical relations of stellar structures are determined by a polytropic equation of state (EOS) between pressure and density, the gravitational potential \( \Phi \) is determined through the well-known Lane–Emden equation (LEE). The weak-field approximation in GR case leads to a generally non-linear second-order differential equation governing the behaviour of the gravitational potential over the polytropic stellar structure (Kippenhahn & Weigert 1990).

However, when the Hilbert–Einstein action is modified considering a function \( f(R) \) of the Ricci scalar, in particular, if we expand this function up to \( R^2 \), the weak field equations lead to a system of coupled modified Poisson equations between \( \Phi \) and the scalar curvature invariant \( R \) (Capozziello et al. 2011). This means, equivalently, that two gravitational potentials have to be considered being the field equations of fourth order (Capozziello & De Laurentis 2011). This system gives rise to a modified Lane–Emden equation (MLEE) for \( \Phi \) which contains an additional convolutional term related to the Yukawa-like correction in the gravitational potential (Capozziello & De Laurentis 2012). The importance of this term depends on the interaction length-scale over which it acts.

It is worth quoting previous studies in this respect where several results have been achieved for stellar structures in ETGs. In Kobayashi & Maeda (2008), the authors did not achieve non-relativistic and relativistic stellar configurations in \( f(R) \) pointing out that the dynamics of the effective scalar degree of freedom in the strong gravity regime could prevent stable systems. In the strong gravity regime for \( f(R) \)-gravity, numerical solutions were found by Babichev & Langlois (2009) corresponding to static star configurations. In their paper, the authors, despite of the results of Kobayashi & Maeda (2008), pointed out that the choice of the EOS for the star is crucial for the existence of solutions and realistic neutron stars are studied.

The claim that stars with relativistically deep potentials cannot exist in \( f(R) \)-gravity was confuted in Upadhye & Hu (2009). Numerical examples of stable stars in \( f(R) \)-gravity are studied. The main result in this paper is that as soon as a star is larger in mass, non-linear chameleon effects screen much of mass, stabilizing gravity at the stellar centre.

In Arapoglu, Deliduman & Yavuz Eksi (2011), the structure of neutron stars in perturbative \( f(R) = R + R^2 \) gravity models with realistic equations of state was studied. Observational constraints on the mass–radius relation were discussed, in particular coming from Gravity Probe B, pointing out that deviations from GR could be small in this picture.

In this paper, we consider numerical solutions of MLEE coming from \( f(R) \)-gravity. We want to show that, beside standard stellar structures, coming from the Newtonian limit of GR, other structures could be addressed. The eventual detection of such structures could constitute a formidable test for alternative theories of gravity, in particular for ETGs.

The paper is organized as follows: in Section 2, we review briefly how the MLEE comes out in the context of \( f(R) \)-gravity. In Section 3, a description of the algorithm used to numerically integrate the MLEE is given. In Section 4, we show the numerical results for some values of polytropic index comparing them with GR results.

In particular, we discuss the role of the Yukawa-like correction coming out in the Newtonian potential. The general expressions for computing stellar masses and gravitational binding energies are reported in Section 5, where we also show the expressions of their relative values for \( f(R) \) and GR stellar models. In Section 6, we discuss our results and further developments which will be needed to build-up physical stellar models, and finally in Section 7, we draw our conclusions.

2 THE MODIFIED LANE–EMDEN EQUATION

The MLEE has been derived in the context of \( f(R) \)-gravity, for self-gravitating systems in the linear approximation of weak field (Capozziello et al. 2011). We refer the reader to that paper for the mathematical details and its derivation. Here, we briefly report the most relevant steps.

In the metric approach, the \( f(R) \) field equations are obtained by varying the action with respect to \( g_{\mu \nu} \) (Capozziello & De Laurentis 2011; Nojiri & Odintsov 2011; Capozziello & Francaviglia 2008; Capozziello, de Laurentis & Faraoni 2010a). We get

\[
\begin{align*}
\left\{ \begin{array}{l}
\mathcal{L}_f(R_{\mu \nu} - \frac{f(R)}{2} g_{\mu \nu} - f'(R) g_{\mu \nu} \Box f'(R) = \mathcal{X} T^{(m)}_{\mu \nu}, \\
3 \Box f'(R) + f'(R) R - 2 f(R) = \mathcal{X} T^{(m)}(R) 
\end{array} \right. 
\end{align*}
\]

where the second equation is the trace. Here, \( T^{(m)}_{\mu \nu} = \frac{1}{2} \hat{g}^{\sigma \tau} \mathcal{L}_f(\Box_{\mu \nu}) \) is the energy-momentum tensor of matter, while \( r = T^\sigma_{\sigma} \) is the trace, \( f' = \frac{df(R)}{dR} \), \( \Box = \Box_\mu \nu \) and \( \mathcal{X} = 8 \pi G \). The conventions for the Ricci tensor is \( R_{\mu \nu} = R^{\rho}_{\mu \nu \rho} \) while for the Riemann tensor is \( R^\alpha_{\mu \nu \rho} = T^\alpha_{\beta \mu \nu} = \ldots \). The affinities are the standard Christoffel symbols \( \Gamma^\alpha_{\beta \gamma} = \frac{1}{2} \hat{g}^{\mu \lambda} (\partial_{[\beta} g_{\lambda \gamma]} + g_{[\beta \gamma]} g_{\alpha \lambda} - g_{\alpha \beta} g_{\gamma \lambda}) \). The adopted signature is \((+ - - -)\).

The field equations (1) can be rewritten as

\[
G_{\mu \nu} = \mathcal{X} (T^{(m)}_{\mu \nu} + T^{(c)}_{\mu \nu}),
\]

where \( G_{\mu \nu} \) is the Einstein tensor and \( T^{(c)}_{\mu \nu} \) is the curvature energy-momentum tensor that comes out when we introduce curvature higher order terms in the Lagrangian (Capozziello & Fang 2002). It is defined as

\[
\mathcal{X} T^{(c)}_{\mu \nu} = \left[ 1 - f'(R) \right] R_{\mu \nu} + \frac{1}{2} \hat{g}_{\mu \nu} f(R) - R_{\mu \nu} \nabla_x \nabla_y f'(R) - \frac{1}{2} \hat{g}_{\mu \nu} \Box f'(R),
\]

and vanishes as soon as \( f(R) = R \). As we can see, in the right-hand side (r.h.s) of equation (2), there are two fluids: a standard matter fluid and a curvature fluid. In this way, we can treat even the ETGs as the GR in the presence of two sources (see Capozziello, De Laurentis & Lambiase 2012b, for details). Note that these quantities satisfy the Bianchi identities, in fact, using the following properties

\[
[V_{\mu}, V_\nu] V^\sigma = - R^{\sigma}_{\mu \rho \nu} V^\rho ,
\]

where \( V^\sigma \) is a generic vector, and

\[
[V_{\mu}, V_\nu] f'(R) = 0,
\]

it is straightforward to show that

\[
V_\alpha \left( \mathcal{X} T^{(c)}_{\mu \nu} \right) = 0,
\]

so that the total energy-momentum tensor can be written as

\[
T_{\mu \nu} = T^{(m)}_{\mu \nu} + T^{(c)}_{\mu \nu},
\]

1 Here, we use the convention \( c = 1 \).
and then \( \nabla^\mu T_{\mu \nu} = 0 \). Let us now perturb the metric tensor up to \( c^{-2} \) so that the Ricci scalar becomes

\[
R \sim R^{(2)} + O(4),
\]

and the \( n \)th derivative of the Ricci function can be developed as

\[
f^n(R) \sim f^n(R^{(2)} + O(4))
\]

\[
\sim f^n(0) + f^n(0)R^{(2)} + O(4),
\]

where \( R^{(n)} \) indicates a quantity of order \( O(n) \). From lowest order of field equations (1), we have \( f(0) = 0 \) which trivially follows from the assumption that the space–time is asymptotically Minkowskian. Equations (1) at \( O(2) \) order (Newtonian level) become

\[
\begin{align*}
R^{(2)} - \frac{\Delta}{z} - f''(0)\Delta R^{(2)} &= X T^{(0)}_t, \\
-3f''(0)\Delta R^{(2)} - R^{(2)} &= X T^{(0)}_t,
\end{align*}
\]

where \( \Delta \) is the Laplacian in the flat space, \( R^{(2)} = -\Delta \Phi \) and for simplicity we set \( f(0) = 1 \). We recall that the energy momentum for a perfect fluid considered is the following

\[
T_{\mu \nu} = (\epsilon + p)u_\mu u_\nu - pg_{\mu \nu},
\]

where \( p \) is the pressure and \( \epsilon \) is the energy density. Then, we have

\[
\begin{align*}
\Delta \Phi + \frac{\epsilon}{z} + f''(0)\Delta R^{(2)} &= -X \rho, \\
3f''(0)\Delta R^{(2)} + R^{(2)} &= -X \rho,
\end{align*}
\]

where \( \rho \) is the mass density. We note that for \( f(0) = 0 \) we have the Newtonian mechanics with \( \Delta \Phi = -4\pi G\rho \). Thus, equation (12) can be considered as the modified Poisson equation for \( f(R) \)-gravity.

From the Bianchi identity (satisfied for first line of equation 1), we find

\[
\frac{\partial \rho}{\partial x_i} = \frac{1}{2} (p + \epsilon) \frac{\partial \ln g_{tt}}{\partial x_i},
\]

Let us suppose now that matter satisfies a polytropic EOS \( \rho = K \rho^n \) with \( K \) free independent parameter or a constant with fixed value \( n \) and the polytropic exponent. Since \( \epsilon = \rho c^2 \), equation (13) satisfies this relation

\[
\frac{\gamma}{\gamma - 1} \rho^{\gamma - 1} = \Phi \rightarrow \rho = \left[ \frac{\gamma - 1}{\gamma K} \right]^{\frac{1}{\gamma - 1}} \Phi^{\frac{1}{\gamma - 1}} = A_\Phi \Phi^n,
\]

where \( n \) is the polytropic index, which is defined by \( n = 1/\gamma \). We remind that in the non-relativistic limit \( \gamma = 5/3 \) and \( n = 3/2 \), while for the relativistic case \( \gamma = 4/3 \) and \( n = 3 \). Note that for these cases the polytropic constant \( K \) is fixed and can be derived from a combination of natural constants (Kippenhahn & Weigert 1990).

Here, using equation (12) and after some calculations, we get the general form of the MLEE which reads as

\[
\frac{d^2 w(z)}{dz^2} + \frac{2 dw(z)}{dz} + w(z) = \frac{m_0}{8 z} \int_0^{\xi_0} dz' \left[ e^{-m_0(z+z')} - e^{-m_0(z'+z)} \right] w(z')^n,
\]

where \( \rho_c \) is the central density, while \( K \) and \( n \) are the constant and the index of the polytropic EOS

\[
P = K \rho^n, \quad \gamma = 1 + \frac{1}{n},
\]

The quantity \( m_0 \xi_0 \) in the right-hand side of equation (15) is related to the effective length-scale of the Yukawa-like correction term to the Newtonian potential. Defining \( L_N = 1/m \), we may write \( m_0 \xi_0 = \xi_0/L_N \); when \( L_N/\xi_0 \gg 1 \), \( m_0 \xi_0 \to 0 \), the gravitational potential asymptotically becomes Newtonian, and the right-hand side term of equation (15) goes to zero, in turn recovering the classical case.

It is well known that analytical solutions for the classical LEE exist only for three values of the polytropic index \( n \), equal to 0, 1 and 5, respectively. Of these, only \( n = 0 \) and \( n = 1 \) give finite-radius solutions. The situation of course gets even more complicated in the case of MLEE, for its full integro–differential nature – in this case analytical solutions are available only for \( n = 0 \) (see Capozziello et al. 2011). A systematic investigation of polytropic stellar structures thus requires the development of numerical methods aimed at obtaining general solution of equation (15). This is actually necessary for both \( f(R) \) and GR models, given that, e.g., degenerate non-relativistic (\( n = 3/2 \)) and relativistic (\( n = 3 \)) polytropic equations of state are needed to build up stellar models.

An important question has to be discussed at this point. We have developed our calculation in the so-called Jordan frame where further gravitational degrees of freedom, coming from \( f(R) \)-gravity, are non-linearly mixed to the standard GR degrees of freedom in the l.h.s. of the field equations. Considering conformal transformations, it is possible to disentangle these further degrees of freedom and then reducing to the Einstein frame by bringing them on the r.h.s. of field equations. However, as it is shown in detail in Stabile, Stabile & Capozziello (2013), the conformal transformations have the effect of shifting the non-minimal coupling (in our case \( f(R)^{-1} \)) from the gravitational to the matter sector and then the meaning of quantities like gravitational potentials, pressure, matter density and mass result more complicated and have to be accurately discussed. Furthermore, the weak field limit breaks the gauge invariance and the conformal invariance and then the analysis in the Einstein frame could lead to ambiguous results. In other words, the gain of simplifying the gravitational sector in the Einstein frame is lost considering the matter sector. In particular, the polytropic EOS, \( p = K \rho^n \), defined by minimally coupled pressure and matter density in the Jordan frame, assumes a completely different meaning in the Einstein frame leading to difficult physical interpretations. Due to these difficulties and ambiguities, we preferred to perform the whole analysis in the Jordan frame where standard GR can be easily recovered in the limit \( f(R) \to R \).

### 3 DESCRIPTION OF THE ALGORITHM

We numerically solved equation (15) using an iterative procedure described as follows. If we define \( w_i(z) \) the function at the \( i \)-th iteration and define the integral term of the r.h.s. as \( N[w_i - 1] \), the MLEE can be written as

\[
\frac{d^2 w_i}{dz^2} + \frac{2 dw_i}{dz} + w_i = \frac{m_0}{8 z} N[w_i - 1].
\]

In the framework of a numerical procedure, the term on the r.h.s. of equation (19) can be viewed as a source term of a non-linear second-order differential equation. We can now rewrite the equation...
The robustness of the solving algorithm described in Section 3 allowed us the possibility to explore the solutions of the MLEE for any value of the polytropic index \( n < 5 \) and the Yukawa length-scale \( L_{Y}/\xi_{0} \) and compare them with the case of GR. The latter case simply corresponds to setting \( m\xi_{0} = 0 \) in equation (15).

In Fig. 1, we report the function \( w(z) \) for \( L_{Y}/\xi_{0} = 1 \) and four different peculiar values of the polytropic index \( n \). For each value of \( n \), we also reported the solution of the classical LEE (\( m\xi_{0} = 0 \)); we remind the reader that when \( n = 0 \) and \( n = 1 \) the classical solution admits the analytical expressions \( w^{0}(z) = 1 - \xi^{2}/6 \) and \( w^{1}(z) = \sin(z)/z \), respectively.

The other two function \( w(z) \) in Fig. 1 correspond to the polytropic EOS of a degenerate non-relativistic (\( n = 3/2 \)) and relativistic (\( n = 3 \)) gas, and can be of relevant astrophysical interest for instance in the study stellar structure of compact objects. As already known, for any fixed value of \( n \), the radius is higher for GR-models, and the higher the polytropic index, the higher is the stellar radius \( z_{\max} = R/\xi_{0} \).

To investigate how more generally \( z_{\max} \) depends on the Yukawa length-scale value, we subsequently performed a set of runs with a grid of values of \( L_{Y}/\xi_{0} \) spanning several order of magnitude, with results reported in Fig. 2. For clarity plotting purposes, we show results for values of \( n \) from 1 to 3 with step size of 0.5. From the figure, it can be seen that the deviations from the GR-values occur in a range \( 10^{-3} \lesssim L_{Y}/\xi_{0} \lesssim 1 \), which does not approximatively depend on the polytropic index \( n \). The region characterizing the deviations between GR and \( f(R) \) solutions has a sharp increase around \( L_{Y}/\xi_{0} \sim 10^{-4} \), followed by a plateau and a subsequent fast transition to the classical solution at \( L_{Y}/\xi_{0} \sim 1 \). For \( n \geq 2 \) the right-hand part of the plateau also shows a secondary superposed increase of \( z_{\max} \) before the steep decay to the classical solution, while in the region around \( L_{Y}/\xi_{0} \sim 10^{-4} \) the value of \( z_{\max} \) is slightly lower than its GR-solution.

It is also worth noting that in all cases \( L_{Y}/\xi_{0} < z_{\max} \), or \( L_{Y} < R^{*} \), thus the Yukawa correction term, no matter which is its true value in the considered region, acts on a characteristic length-scale lower...
than the stellar radius. We consider this point of key importance in the framework of the determination of the stellar structure.

In Fig. 3, we report the behaviour of the stellar radius as a function of the polytropic index $n$ between 0 and 7/2 and for $10^{-4} \leq L_1/\xi_0 \leq 10$, which corresponds to the range of Yukawa length-scales where deviations occur from the classical solutions (see Fig. 2). The data can be well fitted by a function $z_{\text{max}}(n) = a \exp(-b n) + c$ with $a$ in the range $\sim 0.16-0.27$, $b \sim 1$ and $c$ in the range $\sim 2.4-2.9$.

5 STELLAR MASS AND GRAVITATIONAL ENERGY BINDING

The solution of the LEE allows, for given values of the central density, the polytropic index $n$ and the constant $K$ of the polytropic relation $p = K \rho^{1+1/n}$, to find the mass and the gravitational binding energy of a polytropic star. In the classical solution of the equation, the values are

$$M = 4\pi \rho_c R^3 \left( -\frac{1}{z} \frac{dw}{dz} \right)_{z = 0}^{z = z_{\text{max}}}, \quad (23)$$

and

$$E_g = -\frac{3}{5-n} \frac{GM^2}{R}. \quad (24)$$

In the more general case of MLEE, $E_g$ and $M$ can be written as

$$M = \int_0^{R_{\text{max}}} \rho(x)dx, \quad (25)$$

and

$$E_g = \int_0^{R_{\text{max}}} \Phi(x)\rho(x)dx. \quad (26)$$

While keeping in mind the definitions $\Phi = w\Phi_e$, $\Phi_e = \rho^{1/n}(n + 1)K$, and $\rho = \rho_c w^n$, equations (25) and (26) can be rewritten as

$$M = 4\pi \rho_c R^3 \int_0^{z_{\text{max}}} w^n z^2 dz, \quad (27)$$

and

$$E_g = -4\pi \rho_c^3 (n + 1)K \rho_c^{1/n+1} \int_0^{z_{\text{max}}} w^{n+1} z^2 dz, \quad (28)$$

where $w(z)$ is a solution of the LEE (15), and depends on $n$ and $m \xi_0$. When $m \xi_0 \to 0$, equations (25) and (26) can be solved analytically and reduce to equations (23) and (24), respectively, while in the general case of $f(R)$-gravity they need to be solved numerically.

Let us now consider for any given polytropic index $n$, the solutions of equation (15) for the case of $f(R)$ and GR gravity, which we label as $w_{GR}(R)$ and $w_{GR}$, respectively. Using equations (27) and (28), we can now write

$$M_{f(R)}/M_{GR} = \frac{\int_0^{z_{\text{max}}} w_{f(R)}^n z^2 dz}{\int_0^{z_{\text{max}}} w_{GR}^n z^2 dz}, \quad (29)$$

and

$$E_{f(R)}^g/E_{GR}^g = \frac{\int_0^{z_{\text{max}}} w_{f(R)}^{n+1} z^2 dz}{\int_0^{z_{\text{max}}} w_{GR}^{n+1} z^2 dz}. \quad (30)$$

It is evident that equations (29) and (30) are very useful for finding relative masses and gravitational binding energies between $f(R)$ and GR stellar models, for any desired choice of the parameters $K$ and $\rho_c$, which indeed do not appear. The results here reported are obtained for a polytropic index value $n = 3$ but actually they are the same for any value of $n$ (see text).

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and

$$E_{f(R)}^g/E_{GR}^g = \frac{\int_0^{z_{\text{max}}} w_{f(R)}^{n+1} z^2 dz}{\int_0^{z_{\text{max}}} w_{GR}^{n+1} z^2 dz}. \quad (30)$$

It is evident that equations (29) and (30) are very useful for finding relative masses and gravitational binding energies between $f(R)$ and GR stellar models, for any desired choice of the parameters $K$ and $\rho_c$, which indeed do not appear. The results here reported are obtained for a polytropic index value $n = 3$ but actually they are the same for any value of $n$ (see text).

Looking at Fig. 4, it is worth noting that the value of the ratios $M_{f(R)}/M_{GR}$ and $E_{f(R)}^g/E_{GR}^g$ are very closed each other. This is not surprising, given that in equations (29) and (30), the only difference is the change in the power of the $w(z)$-function from $n$ to $n + 1$.

The most relevant result is that in $f(R)$-gravity masses and associated gravitational energy binding values can exceed up to 50 per cent of their classical value for $10^{-3} \lesssim L_1/\xi_0 \lesssim 10^{-1}$. Furthermore, we have two characteristic gravitational radii, the standard Schwarzschild one

$$R_s = \frac{2GM}{c^2}, \quad (31)$$

where $M$ is defined by equation (27) and, as we have seen, a length-scale related to the Yukawa correction. Keeping in mind the definition of $\xi_0$, equation (16), and defining $L_1 = a\xi_0$, where $a = 1/m\xi_0$ and $m^2 = -1/3f''(0)$, we obtain

$$\frac{L_1}{R_s} = \frac{2c^2}{3K\rho_c^{1/n}(1 + n)F(w, n)}. \quad (32)$$
where

$$F(w, n) = \int_0^{r_{\text{max}}} w^n z^2 dz.$$  \hspace{1cm} (33)

For values of $n$ in the range $1-4$, the ratio defined in equation (32) ranges from $\sim 10^4 K_{14}^{-1}(\rho_{100})^{-1}$ to $\sim 10^4 K_{14}^{-1}(\rho_{10})^{-1/4}$, where $\rho_{100} = 10^{-2} \rho$, and $K_{14} = 10^{-14} K$. Looking at Fig. 4, the range of values of $\alpha = L_{\nu}/\xi_0$ for which significant deviations from standard Newtonian theory occur is $10^{-4} \leq \alpha \leq 0.1$. This means that a second gravitational radius, acting on scales larger than the Schwarzschild radius, can be defined. From a dynamical point of view, in principle, this fact can be interpreted as the possibility to stabilize more massive self-gravitating systems that cannot be achieved in standard GR. In other words, very massive exotic objects that could not be obtained as solutions of the standard theory of stellar structures could be achieved in the framework of $f(R)$-gravity.

### 6 DISCUSSION

The development of an efficient algorithm aimed to solve the LEE for stellar structures in $f(R)$-gravity allows us to open unexplored possibilities for self-gravitating systems and stellar structures. In particular, we have seen that higher order gravitational corrections give the possibility to extend the range of stable stellar structures by inducing a further characteristic gravitational radius that, if compared with the Schwarzschild one, can rule these new structures. In general, one can get more massive structures that remain stable.

The results reported in Figs 2 and 4 give, for a wide range of Yukawa length-scale values, the star radius in units of polytropic radius $\xi_0$ and the ratio of mass and gravitational energy binding between $f(R)$ and GR models, respectively. The deviation from classical results occur in the range $10^{-4} \leq L_{\nu}/\xi_0 \leq 10$. As $L_{\nu} = \sqrt{-3f''(0)}$, these results are as general as possible, in the sense that they do not depend on the explicit form of $f(R)$. This kind of approach allows actually to explore all the possible ranges of masses and radii of polytropic stellar structures. Specific values are instead given once the functional form of $f(R)$ with its parameters is provided (see A13).

In order to construct fully self-consistent star models one needs to take into account physical EOS for matter.

A case of particular interest can be for instance that of neutron stars. Considering the analytical description of the EOS for high-density matter based on results of the Brussels–Montreal group and reported in A13, it is possible to see that for $10^{14} \leq \rho \leq 10^{16} \text{ g cm}^{-3}$, the relation $P(\rho)$ cannot be described by a single polytropic EOS. To work in a polytropic regime, one should divide the given interval of $\rho$ in different sub-intervals within which the function $P(\rho)$ is approximatively polytropic, with its own constants $K_i$ and $\gamma_i$.

Thus, the correct way of building up a stellar model would be the following: once the initial central density $\rho_0$ is given, from table 2 and equation 3 of A13, one derives the corresponding polytropic EOS with $P_1 = K_1 \rho^{\gamma_1}$ and then starts to integrate equation (15) until the density remains in the interval where $K_i$ and $\gamma_i$ are the same than those at the star centre. Once $\rho$, while decreasing, moves to a different interval corresponding to new values $K_2$ and $\gamma_2$, the integration of equation (15) stops, and a new run must be performed starting with initial conditions $P_2(0) = P_1(z_2)$ and $P_2(0) = P_1(z_2)$, where $z_2$ is the transition radius between two neighbour polytropic zones. The procedure should be repeated until the stellar radius is reached ($w = 0$).

The use of a single polytropic EOS for a whole star should be indeed considered as no more than a toy model to gain some idea of the order of magnitude of the physical quantities involved, such as stellar radius and mass. However, when comparing the theoretical predictions with the observational data, the required precision of e.g., derived mass–radius relations makes the above described procedure unavoidable.

Specifically, in Astashenok, Capozziello & Odintsov (2014), the so-called hyperon puzzle in the theory of neutron stars is addressed considering $f(R) = R + \alpha R^2 + \beta R^3$ models. For simple hyperon EOS, it is possible to obtain the maximal neutron star mass which satisfies the recent observational data for PSR J1614-2230 (Demorest et al. 2010). The soft hyperon EOS, considered there, is usually treated as non-realistic in the standard GR. On the other hand, the numerical analysis of mass–radius relation for massive neutron stars with hyperon EOS in $f(R)$-gravity turns out to be consistent with observations. In other words, the same $f(R)$-gravity model can solve at once three problems: consistent description of the maximal mass of neutron star, realistic mass–radius relation and account for hyperons in EOS.

A further remark concerns the Chameleon mechanism (Khoury & Weltman 2004) and the Solar system tests that make the models phenomenologically viable. Such constraints are evade depending on the strength of the Yukawa-like correction that we are considering into the gravitational potential. According to the MLEE (15), in the regime $L_{\nu}/\xi_0 \gg 1$ the gravitational potential becomes asymptotically Newtonian. This means that equation (15) becomes the standard LEE derived when GR is restored. This regime is achieved as soon as one is well far from the stellar structures that we are considering here.

At this point is worth stressing the main differences of our approach with respect to the results already achieved on this topics by other authors, in particular by Kobayashi & Maeda (2008), Babichev & Langlois (2009) and Upadhye & Hu (2009). In all these studies, the authors used the conformal transformations in order to disentangle the further gravitational degrees of freedom coming from $f(R)$ and then take into account spherical symmetry in the Einstein frame. The existence and stability of relativistic stars strictly depend on the consistency of EOS with the stellar dynamical system. As already stressed in the Introduction, Kobayashi & Maeda (2008) found non-relativistic star configurations while Babichev & Langlois (2009) and Upadhye & Hu (2009) derive both relativistic and non-relativistic star configurations. Furthermore, specific $f(R)$ models are chosen there allowing the existence of both relativistic and non-relativistic stars.

In our case, as discussed in Section 2, we develop the analysis in the Jordan frame and request that $f(R)$ function is analytical. The consequence of this choice is twofold: From one side, standard matter remains minimally coupled to the system and then, by choosing a suitable EOS, one can easily discriminate between relativistic and non-relativistic regimes (see Section 2). Besides, we achieve the MLEE (15) that allows us to control, on the stellar structure, how modifications deriving from $f(R)$ affect the model. In particular, the boundary conditions result modified being the MLEE an integro–differential equation, different from the standard LEE that is a differential equation. Such an approach allows, in principle, to deal with relativistic and non-relativistic star configurations under the same standard, and GR results are immediately recovered as soon as $f(R) \rightarrow R$. As pointed out in Astashenok et al. (2014), peculiar objects like PSR J1614-2230 can be described in such a general scheme.

Finally, we conclude that more realistic numerical star solutions will be considered as soon as different regimes for EOS are implemented for the same $f(R)$-gravity model. This will be the argument of a forthcoming paper.
7 CONCLUSIONS

The algorithm that we developed provides intriguing prospects for the study of self-gravity systems. This would be particularly desired for the study of main-sequence models with a central nuclear burning core surrounded by a convective envelope. Besides, the algorithm could work also to describe structures out of the main sequence, peculiar objects and systems in the instability strip since boundary conditions on the MLEE are less restrictive with respect to those requested for standard LEE. Finally, as discussed in A13, in the case of observation of self-gravitating systems that escape standard description of GR, this could constitute a straightforward test for alternative theories of gravity.

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