A Survey of the Development of Geometry up to 1870*

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September 4, 2014

Abstract

This is an expository treatise on the development of the classical geometries, starting from the origins of Euclidean geometry a few centuries BC up to around 1870. At this time classical differential geometry came to an end, and the Riemannian geometric approach started to be developed. Moreover, the discovery of non-Euclidean geometry, about 40 years earlier, had just been demonstrated to be a "true" geometry on the same footing as Euclidean geometry. These were radically new ideas, but henceforth the importance of the topic became gradually realized. As a consequence, the conventional attitude to the basic geometric questions, including the possible geometric structure of the physical space, was challenged, and foundational problems became an important issue during the following decades.

Such a basic understanding of the status of geometry around 1870 enables one to study the geometric works of Sophus Lie and Felix Klein at the beginning of their career in the appropriate historical perspective.

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*This monograph was written up in 2008-2009, as a preparation to the further study of the early geometrical works of Sophus Lie and Felix Klein at the beginning of their career around 1870. The author apologizes for possible historiographic shortcomings, errors, and perhaps lack of updated information on certain topics from the history of mathematics. Comments or corrections from the reader are most welcome, and may contribute to an improved later edition.
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1 Euclidean geometry, the source of all geometries

At the end of the 18th century, the notion of geometry was largely synonymous with Euclidean geometry, namely the classical Greek geometry which had prevailed for more than 2000 years. In the 17th century, Kepler, Galileo and Newton were leading figures in the Copernican revolution which had paved the way for the birth of modern science, and moreover, which finally abandoned the long lasting doctrines and the supreme authority of Aristotle among the scholars. Certainly, the 17th century was also a century of great advances in mathematics, such as the rise of calculus, but still the basic role of Euclidean geometry was not challenged. Indeed, in his famous work *Principia Mathematica* (1687) Newton was careful to recast his demonstrations in geometric terms, and analytical calculations are almost completely missing. In particular, Newton formulated his basic laws for the universe in the framework of Euclidean geometry.

However, during the 18th century another dominating authority had established himself in the intellectual world, namely the German philosopher Immanuel Kant (1724–1804). He maintained that Euclidean geometry was the only absolute geometry, known to be true *a priori* in our mind as an inevitable necessity of thought, and no other geometry was thinkable. Moreover, Kant regarded the Newtonian universe as the true model of the physical space, supported by our endowed intuition about space and time, and independent of experience. In reality, the Kantian doctrine on the nature of space and geometry hampered the development of science, until the outburst of the inevitable geometric "revolution" in the 19th century led to the discovery of non-Euclidean geometries and radically changed geometry as a mathematical science.
1.1 Early geometry and the role of the real numbers

Geometry is in fact encountered in the first written records of mankind. But what are the origins of the Euclidean geometry? In his truly remarkable work, organized into 13 books usually referred to as *Euclid’s Elements*, Euclid (325–265 BC) presented a large part of the geometric and algebraic knowledge of Babylonian, Egyptian and Greek scholars at his time. He did this in a deductive style, which has become known as the axiomatic method of mathematics. Geometry as developed in the *Elements* is usually referred to as *synthetic* geometry. The basic undefined geometric objects are points, lines, planes, whereas notions such as line segments, angles and circles can be defined, and moreover, the undefined notion of congruence expresses "equality" among them. Starting from five basic postulates and five common notions (or rules of logic), a logical chain consisting of 465 propositions were deduced and presented in the *Elements*. For the convenience of the reader, let us be more specific and state the postulates — somewhat modernized— as follows:

\[ E_1 : \text{There is a unique line passing through any two distinct points.} \]

\[ E_2 : \text{Any segment on a line may be extended by any given segment.} \]

\[ E_3 : \text{For every point } O \text{ and every other point } P, \text{ there is circle with center } O \text{ and radius } OP. \]

\[ E_4 : \text{All right angles are congruent to each other.} \]

\[ E_5 : \text{If a line intersects two other lines so that the two interior angles on one side of it are together less than two right angles, then the two lines will meet at a point somewhere on that side.} \]

The ancient geometry flourished for about 1000 years, primarily in Greece and Alexandria in Egypt. Besides Euclid some of the prominent men from this epoch are Thales of Miletus (ca. 600 BC), Pythagoras (585–501 BC), Plato (429–348 BC), Eudoxus of Cnidus (408–355? BC), Aristotle (384–322 BC), Archimedes of Syracuse (287–212 BC), Appolonius of Perga (262–190 BC), and Pappus of Alexandria (290–350 AD). The Pythagorean school played a crucial role, developing number theory which they also linked to geometry, number mystics and music theory. In fact, Pythagoras himself is said to have coined the words "philosophy" and "mathematics". The Platonic Academy in Athens became the mathematical center of the world and, for example, Aristotle and Eudoxus had been students at this academy.

Many of the ancient mathematical texts have been lost, including advanced works by Euclid and Appolonius. For example, Appolonius introduced the names *ellipse, parabola* and *hyperbola* in his famous treatise Conic Sections, but some of the volumes are irretrievably lost. Moreover, Euclid’s three-volume work called the *Porisms*, apparently on advanced geometry, is also lost. However, the *Elements* of Euclid have survived through the centuries by manuscript copies which have transmitted the geometric truths to new generations, and with minor
changes. Through Euclid's authority, the *Elements* teach geometry with a definite approach, but with no motivation even of the most sophisticated terms. Moreover, previous or alternative approaches, which could have filled some of the gaps, are ignored. The first printed edition was in 1482, and over thousand editions have appeared since then. In effect, the *Elements* have remained in use as a textbook practically unchanged for more than 2000 years, and up to modern times the first six books have served as the student's usual introduction to geometry.

It is somewhat surprising that Euclid did not use numbers in his geometry. Namely, the geometric objects such as line segments, angles, areas etc. are not measured by numbers, but they are related to each other in terms of congruence, similarity and proportion (or ratio). The idea behind this is, loosely speaking, that congruent objects have the same "shape and size", that is, they are *similar* and have the same *magnitude*. In the simplest case of line segments, denoted by letters $a, b, c, \ldots$, congruence means "same magnitude". The magnitude of $a$ relative to $b$ is represented by the proportion $a : b$, but let us write it as a ratio $a/b$. The idea of proportion is the clue to proving many theorems, since by subdivision of the geometric figure similitude can be reduced to congruence. The ultimate procedure for defining and handling the ratios $a/b$ in Greek mathematics, as presented in books V –VI of the *Elements*, is generally attributed to Eudoxus. Previously, the Pythagoreans had failed at this point, having stumbled into the existence of incommensurable ratios and hence the discovery of irrationality, which influenced the further development of Greek geometry in a fundamental way.

In more detail, the Pythagorean approach was to associate a (rational) number $m/n$ to each ratio $a/b$ of magnitudes, based on their belief that $a$ and $b$ are always commensurable, that is, they are both integral multiples of some suitably small $c$, say $a = mc, b = nc$. However, taking $a$ and $b$ to be the side and diagonal of a square, and assuming $n/m$ is reduced so that $n$ and $m$ are relatively prime, this example yields the contradictory identity $2n^2 = m^2$. Thus they had, in fact, encountered a pair of incommensurable line segments. Hippasus (ca. 500 BC) is credited with the discovery, but most likely he first discovered the incommensurability of the side and diagonal of a regular pentagon.

Later, Aristotle pointed out how rational numbers can approximate any ratio by a "take away from each other" procedure, which is closely related to the continued fraction construction in modern mathematics. Namely, if $a_0$ and $a_1$ are the magnitudes to be compared and $a_0 > a_1$ say, one finds successively unique positive integers $m_1, m_2, \ldots$, so that $a_{p-1} = m_p a_p + a_{p+1}$ with $a_p > a_{p+1}$, for $p = 1, 2$ etc. Let us write

$$a_0/a_1 \longleftrightarrow [m_1, m_2, m_3, \ldots] = m_1 + \frac{1}{m_2 + \frac{1}{m_3 + \ldots}} \quad (1)$$

which also indicates that the procedure may never stop, in which case $a_0/a_1$ is incommensurable. Otherwise, $a_{p+1} = 0$ for some $p$, and the procedure is known as the *Euclidean algorithm*. Then $a_0 = ma_p, a_1 = na_p$ for some $m$ and $n$, and
the above finite continued fraction ending with \( m_p \) is just the rational number \( m/n \) measuring the commensurable ratio \( a_0/a_1 \).

In a more elegant way, however, Eudoxus’s approach to ratios is the principal source to the modern view of real numbers, irrational or not. Regarding ratio as an undefined relation between magnitudes, he declares that \( a/b = c/d \), if for any two positive integers \( n \) and \( m \), one of the following three relations holds for the pair \((a, b)\), namely

\[
i) \ ma = nb \quad \text{or} \quad ii) \ ma > nb \quad \text{or} \quad iii) \ ma < nb, \tag{2}
\]

if and only if the corresponding relation holds for the pair \((c, d)\). Furthermore, he declares that \( a/b \) is less than \( c/d \) if for some \( m \) and \( n \)

\[
na < mb \quad \text{and} \quad nc > md
\]

For real numbers \( a, b, c, d \) in the modern sense, the definition (2) of equality between ratios certainly yields \( a/b = c/d \) as numbers. Furthermore, Aristotle demonstrates the density of the rationals, locating an incommensurable ratio \( a/b \) by comparing it with a special sequence of approximating commensurable ratios, similar to the modern definition of real numbers by decimal fractions. On the other hand, Eudoxus compares \( a/b \) with all commensurable ratios and thus anticipates the modern view that a real number is determined by its order relations with respect to all rationals.

However, in the Elements the existence of the ratio \( a/b \) of two given magnitudes \( a, b \) is not questioned. What is needed is the existence of integers \( m, n \) so that \( ma > b \) and \( nb > a \), which was taken for granted. In the 19th century it became clear that this property, referred to as the Archimedian axiom, has to be postulated, directly or indirectly. This is one of many examples illustrating the Greek philosophy of mathematics which, despite the use of the deductive method, is very different from the modern ideal of an axiomatic system, where all conclusions are strictly deduced from a few fundamental axioms. For Plato or Aristotle, geometric objects are real and knowable and their basic properties are, anyhow, settled and may not be explicitly announced.

Another major contribution of Eudoxus is his method of exhaustion, which is elaborated in Book XII of the Elements. Namely, by using a concept close to the idea of integration in modern calculus, he shows how to subdivide a known magnitude into decreasingly small pieces whose totality approaches that of an unknown magnitude. He uses the method to show, for example, that the volume of a pyramid is one third of the volume of the prism with the same base and height. The method can be used to compute areas and volumes bounded by specific curves and surfaces. In 1906 an unknown treatise of Archimedes called The method was discovered, in which he did not repudiate ”infinitesimal” methods. It is likely that he made progress beyond Eudoxus in this direction, but the improved tools must have failed to meet the rigour of the Elements and hence eliminated by Euclid. The ideas of Eudoxus and Archimedes were, indeed, anticipating the integral calculus initiated by Newton and Leibniz in the 17th century.
Skillful techniques were developed to apply the theory of proportions, and the ancient use of the ratios came very close to the segment arithmetic introduced by Hilbert (1899). Namely, by choosing a "unit" segment $e$, so that $a$ (or rather its magnitude) can be identified with the ratio $a/e$, the algebraic operations addition and multiplication could be effectuated by compass and ruler constructions. A well known theorem, due to Menelaus of Alexandria (ca. 70–130 AD), is the following formula involving the product of three ratios:

$$\frac{AP}{PB} \frac{BO}{QC} \frac{QC}{RA} = -1 \quad (3)$$

where $P, Q, R$ are points on the (possibly extended) edges $AB, BC, CA$ of a triangle $ABC$, respectively. The theorem says that the three points are collinear if and only if the identity $(3)$ holds.

### 1.1.1 Geometric algebra, constructivism, and the real numbers

The ancient scholars certainly developed some mathematical rigor and logical analysis through the problem of incommensurability. The area formula for a rectangle proved by the Pythagoreans is valid only when the sides are commensurable, but following Eudoxus the formula holds in general. The traditional opinion, say up to the beginning of the 20th century, was that the Greeks created geometric algebra by translating algebraic relations into geometry. For example, the Pythagoreans solved quadratic equations by a geometric procedure involving the notion of area. Namely, the equations

$$A = (a + x)x \quad \text{and} \quad A = (a - x)x \quad (4)$$

express the area of two rectangles arising from a rectangle with sides $a$ and $x$, by adding the excess (hyperbole) or subtracting the defect (ellipse) represented by the square of side $x$. However, for the last 50 years no historian would phrase the origins of geometric algebra in this way.

On the other hand, quadratic equations correspond to the geometric problems which the Greeks could study by means of the ruler and compass constructions, still an indispensable tool in the modern education of plane geometry. Today we can say that the ratios $a/b$ obtained in this way, starting from a unit element $e$, will only yield the so-called constructible numbers, that is, obtained from the rationals by the successive addition of square roots. Some modern writers (e.g. Hartshorne [2000: 42]) claim the above geometric approach to algebra prevented Euclid and other ancient scholars from conceiving of real numbers beyond the constructible ones, for example, $\sqrt{2}$ or transcendental numbers such as $\pi$.

The numbers $\sqrt{2}$ and $\pi$ are the solutions of two famous geometric problems of antiquity which remained unsolved, namely the "dublication of the cube", and the "quadrature of the circle" which Anaxagoras (499-428 BC) first attempted to solve. Eudoxus would have approximated such irrational "numbers" by rationals, but we must remind ourselves that in Greek mathematics the irrational
numbers did, in fact, not have the status of being "numbers". After all, the irrationality of $\pi$ was first proved by Lambert in the 18th century.

The problem Eudoxus solved with his theory of proportions was not really understood until the late 19th century, more than 2000 years after the *Elements* were written down. During this long period a clear conception of the nature of the real numbers seemed to be missing, and even doubts about the soundness of irrationals were expressed by some scholars. For example, the German algebraist Michael Stifel (1487–1567), who discovered logarithms and was the first to use the term "exponent", argued in 1544 that "just as an infinite number is not a number, so an irrational number is not a true number". Finally, in 1871 Richard Dedekind reexamined the ancient problems on incommensurables, and with his epoch-making essay *Continuity and irrational numbers* (1872), he established the theory of real numbers on a logical foundation and without the extraneous influence of geometry. It should be mentioned, however, that the Dedekind cut construction is essentially the same idea as Eudoxus used.

It is interesting to observe that the ancient Euclidean constructivism has survived up to modern times, manifesting itself in the belief that mathematics should deal only with constructible numbers and with a finite number of operations. The famous Berlin professor Kronecker, who Klein and Lie met during the fall 1869, was the first to doubt the significance of non-constructive existence proofs. He is well known for his remark that "God created the integers, all else is the work of man", and he opposed the use of irrational numbers. As late as in 1886, when Klein’s previous student Lindemann lectured on his proof from 1882 that $\pi$ is transcendental, Kronecker complimented Lindemann on a beautiful proof, but as he added, it proved nothing since transcendental numbers did not exist.

In their study of equations like (4) the ancient scholars also came close to the development of coordinate (or analytic) geometry. For example, by putting $A = y^2$ in (4) the equations describe a hyperbola and an ellipse in the coordinates $x$ and $y$. However, with the Greek passion for geometry, Menaechmus (390–320 BC), who was a student of Eudoxus, was led to the observation that these curves are plane sections of a cone. His study was continued by Archimedes, and Appolonius finished the project with his celebrated treatise *Conic Sections*, where he also set forth the principal properties of *conjugate* diameters. In a way coordinates were used, but always in an rather awkward language dictated by the geometry. Therefore, today we regard analytic geometry as originating from Descartes and Fermat in the early 17th century.

### 1.1.2 The downfall of the ancient geometry

The Second Alexandrian School (around 300 AD), with the mathematicians Diophantus and Pappus, continued the tradition dating back to Pythagoras and brought again fame to Alexandria which lasted for another century or two. Pappus was actually on the track of a new type of geometric truths, naturally belonging to projective geometry. These are statements involving only points, lines and their incidence relations. According to the celebrated *Pappus’s theo-
rem, if $A, B, C$ and $A', B', C'$ are points on different lines $l$ and $l'$ respectively, and $AB'$ denotes the line joining $A$ and $B'$ etc., then the three points of intersection

$$P = AB' \cap A'B, \quad Q = BC' \cap B'C, \quad R = CA' \cap C'A$$

lie on the same line. Pappus was the last of the great geometers from classical Greek mathematics, and he stated this result as an exercise in one of his books. Most likely Pappus knew about Menelaus’s theorem (cf. (3)), which can be used to solve his exercise.

But the downfall of Greek geometry was unrelenting; it came with the decadence of the classical Greek culture around the 5th century AD. At this time the flow of written records and oral traditions carrying the unofficial mathematical knowledge was suddenly broken, probably due to political events or pressure of Roman culture. In addition, the surviving literature, influenced by the selective role played by Euclid’s *Elements*, was unable to inspire further mathematical creativity.

Stepping forward to the early Middle Ages and the Islamic golden age, one finds that progress in geometry beyond Euclid’s *Elements* and the Alexandrian School was still rather modest. Certainly, there were some extensions and refinements of the Euclidean postulates, as well as attempts to prove the parallel postulate and further studies of conic sections. Thus, there seemed to be no remaining challenges from the ancient geometry. Stepping forward another century or two, the works of Kepler, Desargues and Pascal, in the spirit of Pappus, also remind us that the ancient geometry was not capable of much further extension.

### 1.1.3 The ancient geometry: Its failures and its final algebraization

The idea of transformations is absent in Euclid’s *Elements*, and therefore the *Elements* study the properties of individual triangles from a static viewpoint only. This must have limited the scope of geometric thought and its interaction with related sciences such as kinematics and mechanics. This is all the more astounding since transformations were, in fact, known and used long before Euclid and, for example, the modern notion of symmetry played an important role in the geometry of Thales in the 6th century BC. Even stone age decorations witness that it was known very early to mankind. But geometric transformations amount to changing of figures according to specific rules, so they were eliminated from Euclid’s work as they seemed to belong to mechanics rather than geometry. This is certainly far from the modern viewpoint where motions and deformations of triangles are ideal study object of kinematic geometry.

Despite its fame as the outstanding example of a deductive theory, the claim that all propositions are deduced logically from the definitions, axioms or postulates, is unfounded and modern criticism have shown essential gaps, as pointed out earlier. Without the basic idea of a transformation the *Elements* use artificial methods, which cannot be justified from the basic postulates. For example, to circumvent or demonstrate congruency between triangles, Euclid applies a
superposition method — an unstated congruence axiom — which allows him to move a triangle from one place to another. Tacitly assuming this he establishes the properties usually referred to as the congruence propositions SSS (side-side-side) and SAS (side-angle-side). Unfortunately, this "magic" method is still found in school geometry textbooks nowadays.

So, for many reasons, during the 19th century there was a growing conviction that the classical geometry needed a thorough upgrading of its logical fundament. Based on the joint efforts of many previous geometers, David Hilbert (1862-1943) finally presented in his Foundations of geometry (1899) and subsequent works the modern axiomatization of the classical geometries (Euclidean, hyperbolic, or projective). In particular, the Euclidean space was given a sound basis using about 15 axioms, in the spirit of Euclid, with congruence via its SAS property postulated as an axiom.

In analytic geometry (see below), the real Cartesian plane \( \mathbb{R}^2 \) is used as a model of the Euclidean plane. Thus the geometry study in the plane is reduced to algebra, and geometric properties depend on properties of the real numbers, justified by the Cantor–Dedekind axiom dating back to 1872. In fact, in their study of axiomatic systems and dependency relations, Hilbert and others introduced the Cartesian plane \( \mathbb{F}^2 \) over various (ordered) fields \( \mathbb{F} \), even skew fields. Points and lines are introduced as in \( \mathbb{R}^2 \), but the geometric properties of these planes reflect the algebraic properties of \( \mathbb{F} \). For example, Pappus's theorem (see 5) holds in \( \mathbb{R}^2 \), and it is an interesting result due to Hilbert that the field \( \mathbb{F} \) is commutative if and only if the theorem holds in \( \mathbb{F}^2 \).

Finally, it should be noted that critical questions with regard to Euclid’s Elements are associated with two major events in the development of geometry in the 19th century, in which Félix Klein was largely involved around 1870. The first event is concerned with the fifth postulate \( E_5 \), the so-called parallel postulate, and the discovery of non-Euclidean geometry. Whereas the first four postulates are local in nature and easy to accept, the parallel postulate is less intuitive since one cannot “see” what happens indefinitely far out in the plane. It was precisely the doubts about the nature of this postulate which spurred the discovery, and early works of Klein in 1871–72 contributed to the understanding of the new geometry which he called hyperbolic geometry (see Chapter 4).

The other event was the appearance of the notion of symmetry group in geometry, which heavily uses transformations and thus provides an alternative approach to the classical geometry. This new viewpoint is exemplified in Klein’s Erlanger Programm (1872). For example, the congruences in Euclidean geometry are the rigid motions and they constitute the symmetry group of the geometry. Conversely, in Klein’s approach a geometry is largely characterized by its group. Namely, by focusing attention on the group itself, geometric properties are precisely those which are invariant under the group. But certainly the idea of transformation groups and the study of their possible structures extend far beyond the known classical geometries at that time. Sophus Lie’s general study of continuous groups arose naturally from his geometric experiences in the early 1870’s, in those years when Klein and Lie were in close contact and mutually stimulated each other.
1.2 The decline of pure geometry and rise of analytic geometry

Recall that since antiquity the language of algebra was largely provided by geometry, and maybe pure algebra made little progress since the ancients believed the theory of ratios could not be granted algebraically. So, it seemed that only persons unaware of the Greek scruples would be in the position to resume and develop genuine algebra. But a mathematical symbolic language was still missing, and this may have hampered the progress during the Middle Ages. The creator of such a language, and usually regarded as the father of modern algebraic notation, is the French lawyer and mathematician François Viète (1540–1603). Some of the symbols he introduced are still in use today.

Following Viète and greatly influenced by Pappus, there was a revival of mathematics in France in the 17th century, with leading figures such as Girard Desargues (1591–1661), René Descartes (1596–1650), Pierre Fermat (1601–1665), and Blaise Pascal (1623–1662). In the hands of Descartes and Fermat, the symbolic analysis led to an entirely new way of investigating mathematical problems, namely the analytic or coordinate approach to geometry. Actually, Menaechmus (390–320 BC) and Appolonius came close to such an approach in their study of conic sections. But, whereas Appolonius had failed by attaching the coordinates to the conic itself, Descartes and Fermat initiated the new approach which attaches a coordinate system to the underlying plane, thus enabling them to study the different figures in their mutual relations. Therefore, the beginning of the analytic geometric approach is generally attributed to Descartes and Fermat.

Actually, both Euclid and Descartes unified algebra and geometry, but their approaches were converse to each other. With his basic mathematical work La Geometrie (1637), Descartes initiated the algebraization of geometry by associating to each point in the Euclidean plane a pair \((x, y)\) of real numbers called coordinates, namely the (signed) distances from the point to two fixed perpendicular axes. In fact, Descartes considered skew coordinate systems as well, but we shall not do so. The pair \((x, y)\) became known as Cartesian (or rectangular) coordinates, and the totality of pairs as the Cartesian plane, usually denoted by \(\mathbb{R}^2\) for set theoretic reasons. Henceforth, geometric figures and their properties could be expressed and studied in terms of coordinates; for example, Descartes and Fermat represented a curve by an equation \(f(x, y) = 0\) in the variables \(x, y\). Thus, various geometric locii could now be described in the Cartesian plane in terms of algebraic or transcendental functions and equations. Notice, however, that the Cartesian distinction between geometrical and mechanical curves corresponds to the terms algebraic and transcendental in modern terminology. To begin with, orthogonality of the coordinate axes was not necessarily assumed. Originally, Descartes the coordinate axes could be more general, were not necessarily perpendicular, but we remark that perpendicularity of the coordinate axes was not originally assumed.

This was the birth of coordinate geometry, usually referred to as analytic geometry after Lacroix introduced the term for the first time in his famous two-
volume textbook *Traité de calcul ..* (1797–98). But it was also the birth of algebraic geometry, in the sense that the equations involve the coordinates in a purely algebraic way. Equations of type \( ax + by + c = 0 \) represent the straight lines, whereas those of degree 2

\[
ax^2 + bxy + cy^2 + dx + ey + f = 0
\]  

(6)

represent curves which were found to be the familiar conic sections. Descartes himself initiated a classification of algebraic curves according to the degree of the equation. In the Cartesian plane, problems of higher degree, even transcendental or with 3 variables, could also be handled in a similar way.

As a consequence of all this, however, the ancient pure geometry fell into oblivion, at least temporarily, because the new approach turned out to be a much more powerful method of proof and discovery. Therefore, the algebraic and analytic methods continued to dominate geometry almost to the exclusion of synthetic methods. In effect, the Euclidean plane or space was virtually replaced by the Cartesian plane \( \mathbb{R}^2 \) or 3-space \( \mathbb{R}^3 \), respectively. In reality, however, this identification rests on a kind of “continuity” postulate, tacitly accepted since the days of Descartes, saying that to each point of the line there corresponds a real number, and conversely. This kind of subtlety, however, was not fully understood until late in the 19th century.

In the 18th century, Euler was a leading mathematician, writing on essentially everything, so he combined algebra, analysis and geometry to solve many types of problems. In solid analytic geometry, his formulae for translation and rotation of the axes, in terms of the so-called Euler angles, are still well known and used today. About 100 years after Descartes and Fermat attempted a unified treatment of binary quadratics (6) and conics, Euler gave for the first time a unified treatment of the general quadratic equations

\[
a x^2 + bxy + cz^2 + 2dxy + 2eyz + 2fxz + gx + hy + iz + k = 0
\]  

(7)

which involve three variables and have up to ten terms. His work indicated that the equation can be reduced by transformation to the five canonical forms of quadrics, namely an ellipsoid, two types of hyperboloids and two types of paraboloids, but he did not list all the degenerate types.

Another leading 18th century mathematician, born three decades after Euler, was Lagrange, whose work resembles Euler’s in its elegance and generality, but neither of the two were typical geometers. At some occasions the analyst Lagrange even boasted of his omission of diagrams or figures. In 1773 Lagrange turned his attention to the basic problem in solid geometry, namely the geometry of four points. They span a tetrahedron, and with one point chosen to be the origin Lagrange was seeking analytic formulae for its various geometric invariants, such as area, volume, center of gravity, and centers and radii of the inscribed and circumscribed spheres. His “tetranometry” presented maybe one of the earliest associations of linear algebra with analytic geometry, but as Lagrange himself put it, the importance of the work lay more in the point of view than in the substance (cf. Boyer[1956], Chap. III).
On the other hand, the invention of infinitesimal calculus by Newton and Leibniz in the later half of the 17th century did not only divert attention from pure geometry, but even geometry as a whole. Instead, by exploring the consequences of the fundamental theorem of calculus, a new type of equations arose, namely differential and integral equations, which have ever since played an important role in the modern developments of mathematics, analytic mechanics, and the natural sciences. In reality, however, one important aspect of geometry profited largely from the new tools in analysis, namely the metric study of Euclidean geometry, with all its embedded curves and curved surfaces. So, here we are witnessing the birth of differential geometry, to which Euler and Monge gave many basic contributions.

But Lagrange’s interests tended towards physics, and with his famous *Mechanique analytique* (1788) he developed the whole subject of mechanics, from the time of Newton, into a branch of mathematical analysis, starting from a few basic principles and using the theory of differential equations. Mechanics may be regarded as the geometry of a (3+1)-dimensional space, Lagrange remarked, but he did not initiate such a multidimensional coordinate geometry.

Among synthesists as well as analysts, the greatest geometer of the 18th century was Monge. We shall encounter him many times later, notably for his basic contributions to differential geometry and projective geometry. Lagrange must have envied him for his fertility of imagination and geometrical innovation, but they both seemed to have realized, more than anyone before, that analysis and geometry can be combined into a very useful alliance. So, geometry was rapidly approaching a new stage, towards the turn to the 19th century, when Paris became the major scene for many historical events, political as well as scientific.

### 1.3 The advent of the new geometries

During the 19th century geometry developed roughly in three major directions:

1. differential geometry,
2. projective geometry,
3. non-Euclidean geometry,

and in the following chapters we shall discuss them separately in some detail, up to around 1870, say. It is natural to use this year as a limit for our review, and for many reasons. At this time projective geometry had established itself as the central topic of geometry, in fact, as the new geometry of the century. Since the beginning of the century, the methodology of analytic geometry had been successfully developed during the decades prior to 1870, to the extent of being the "golden age" of analytic geometry (cf. also Boyer[1956]). Classical differential geometry also came to an end around 1870, as the Riemannian geometric approach became known among geometers and was facing a rapid development. Moreover, the discovery of non-Euclidean geometry, about 40 years earlier, had just been demonstrated to be a "true" geometry on the same footing as Euclidean geometry. These were radically new ideas, but henceforth the importance of the topic became gradually realized. This was the status
of geometry anno 1870, when Felix Klein og Sophus Lie came to Paris in the
spring, at the beginning of their mathematical career.

On the other hand, with the 1870's a new era started for mathematics as
a whole, where the distinction between the various fields such as algebra and
geometry became increasingly more difficult. Geometry was clearly dominated
by the analytic approach, but there were also geometers who opposed the idea
of reducing geometric thinking to analytic geometry and perhaps relying too
much on physical intuition or experience. The discovery of non-Euclidean geom-
etry, which became known as hyperbolic geometry, also challenged the conven-
tional attitude to the basic geometric questions, including the possible geometric
structure of the physical space. Therefore, from the synthesist’s viewpoint, the
foundations of projective geometry as well as Euclidean and hyperbolic geom-
etry, maybe in contrast to analysis, still needed a thorough revision of its basic
concepts and postulates. Consequently, the foundational problems became an
important issue during these decades, and for the sake of completeness we have
also included a brief account on these topics in Chapter 5.

Now, let us return to the year 1787, when Lagrange came to Paris and joined
the group of leading mathematicians, such as Monge, Laplace, and Legendre.
A great flowering of French mathematics was about to start, in the spirit of the
French revolution and the new Republic (1789–99), followed by the interlude
of the emperor Napoleon. The École Normale and École Polytechnique were
founded in 1794–95, and these schools were given a leading role with regard to
the higher education.

Among the faculty of the new schools we find for example Lagrange, Monge,
Carnot and Lacroix. In fact, as a favorite of Napoleon, Monge also served as
the director of École Polytechnique, responsible for the education of military
engineers and officers. Geometry played an important role in the mathematics
curriculum, and with his lectures during the years 1795–1809, Monge’s enthu-
siasm inspired pupils like Brianchon, Dupin, Servois, Biot, Gergonne, Poncelet
and others. They contributed to the further development of Monge’s ideas in
differential geometry, but also to the rediscovery and revival of projective geom-
etry, which had somehow fallen into oblivion since the days of Desargues and
Pascal 150 years earlier.

The new generation of mathematicians also included prominent figures like
Fourier, Cauchy, Poisson and Chasles, but perhaps the innovations in geometry
were, after all, less than in some other areas. An important step was taken
by the establishment of a periodical at École Polytechnique, which certainly
made publication easier than before, since the traditional publication channels
were only the journals at the science academies. Even better, in 1810 an earlier
student of Monge, J. D. Gergonne (1771–1859), had just retired as an artillery
officer, and turning to mathematics he founded and edited his own Gergonne’s
Annales, the first periodical devoted entirely to mathematics. The journal ter-
minated when he finally retired in 1832, but fortunately, in 1826 the first purely
mathematical journal in Germany, Crelle’s Journal, had been established in
Berlin.
Truly, Paris was the center from which the new spirit of analytic geometry spread to the rest of the world. Former students of Monge, S.F. Lacroix (1765–1843) and J.B. Biot (1774–1862), are well known for their popular textbooks on the topic, which also influenced writers in other countries. Lacroix was said to be the most prolific writer in "modern times", and when Plücker reported that his introduction to analytic geometry took place in 1825, he referred to the 6th edition of Biot's textbook (cf. Boyer [2004: 245]). But numerous textbooks similar to these appeared in many countries, see for example Salmon’s treatise [1865a] which first appeared in 1862, and its popular German translation [1865b] which Klein used.

Differential geometry in the tradition of Euler, Monge, Dupin and others continued to flourish in France in the 19th century. But after Monge, the next great step was actually made by Gauss in Germany, a master of many mathematical disciplines. His major work [1828] on curved surfaces also established him in the forefront of geometry. On the French side, however, they did not keep an eye on what the Germans were doing until around 1850, but during the remaining two decades of the pre-Riemannian era differential geometry progressed steadily with participants also from England, Italy and other countries. (For a review, see Reich [1973].)

On the other hand, the French geometers also missed the discovery of non-Euclidean geometry in the 1820’s. This landmark in the history of geometry is generally attributed to Bolyai in Hungary, Lobachevsky in Russia, as well as Gauss. We refer to Chapter 4 for more information, including the long prehistory of the topic.

Finally, turning to projective geometry, let us start with the remark that the history of geometry is full of discoveries and rediscoveries, as well as rivalry, priority conflicts, and controversies. This is amply exemplified by the revival of projective geometry at the turn of the century, initiated by Monge’s pupil L.N.M. Carnot (1753–1823) with his major work Géométrie de position (1803). Although he includes a brief section on coordinate geometry, in fact the most general view of coordinate systems since Newton’s time, his main purpose is to "free geometry from the hieroglyphics of analysis”. Monge himself abided by the joint use of analysis and pure geometry, but gradually several outstanding mathematicians held the opinion that synthetic geometry had been unfairly and unwisely neglected in the past, and now they would make an effort to revive and extend that approach. The champion of the synthetic method was Poncelet, a previous officer of Napoleon. With his classic Traité des propriétés projectives des figures (1822) he is said to have introduced projective geometry as a new discipline. But he had strong opponents like Gergonne and Chasles, who headed the analytic trend and its use of algebra, and they were also joined by the foremost analysts in Germany, namely Möbius and Plücker.

Although the reception of Poncelet’s work was rather poor in France, his ideas were followed up in Germany. In fact, his strict synthetic approach was taken over by Steiner, the first of a German school who favored strict geometric methods to the extent of even detesting analysis. Steiner’s noble goal was to develop projective geometry as a unification of the classical geometry, whereas
his compatriot von Staudt wanted to establish projective geometry independent of Euclidean geometry and its metric concepts. Von Staudt almost succeeded in the 1860’s, but around 1870 a flaw was discovered and traced back to the implicit usage of the Euclidean parallel postulate. However, this was later remedied by others, and remaining fundamental questions about projective spaces were settled during the following decades by German and Italian geometers.

The controversy between the proponents of the two geometric approaches lasted for many decades. In retrospect, there were in fact good reasons for this since the methods of analysis were incomplete and even logically unsound. The pure geometer rightly questioned the validity of the analytic proofs and would credit them merely with suggesting the results. The analyst, on the other hand, could retort only that the geometric proofs were clumsy and not so elegant. (cf. Kline [1972: Chap.35, §3]).

In reality, analytic geometry is based upon the Cartesian geometry, and the subtle distinction between the Euclidean plane (appropriately defined) and the real Cartesian plane \( \mathbb{R}^2 \) critically depends on the properties of the real numbers \( \mathbb{R} \). However, the Cantor–Dedekind axiom (1872) finally placed the arithmetization of analytic geometry upon a solid logical foundation. Therefore, we may well regard 1872 as the terminal year of classical analytic geometry, 200 years after Descartes and Fermat (cf. also Boyer[1956], Ch.IX).

2 Classical differential geometry

Classical differential geometry is a term used since 1920 for pre-Riemannian differential geometry in the tradition of Euler, Clairaut, Monge, Gauss, Dupin, say, to the end of the 1860’s. Curves and surfaces in 3-space can be rather complicated geometric object, but the development of calculus has provided infinitesimal techniques to analyze these objects and to distinguish between and classify the abundance of various geometric forms. Some of the simplest ones are lines, circles, planes, and spheres, and these are also used for comparison reason or for approximations of the more general ones. The basic notion of curvature, essentially a measure of the deviation from linearity, was introduced for these purposes. For modern references, cf. e.g. Spivak[1979], Vol. II, III, and Rosenfeld[1988].

2.1 Curves and their geometric invariants

Let us start with a brief review of curves and their curvature theory. The ancients investigated many aspects of the conic sections, which are curves of the second degree. In his study of plane curves Euler continued with a classification of cubic curves, and moreover, he also listed 146 different types of quartics. But Clairaut (1831) was the first who published a treatise on space curves; for him a space curve was the intersection of two surfaces.

In the older literature we encounter many well known geometric curves, often constructed as the solution of a mechanical problem, and usually they
are expressed by transcendental functions. One of the constructions yields a roulette, namely a curve generated by a curve rolling on another curve. For example, the focus of a parabola rolling on a straight line traces out a catenary, and a fixed point on a circle rolling on a straight line traces out a cycloid. The catenary has the shape of a flexible chain suspended by its ends and acted on by gravity. Its equation is

\[ y = a \cosh \left( \frac{x}{a} \right) \]  

(8)

and was obtained by Leibniz. Let us also recall the Newton–Leibniz controversy which provoked a great rivalry between the British and the continentals. Johann Bernoulli (1696) proposed to Leibniz, l’Hôpital, and others, the celebrated “brachistocroni” problem, to find the path between two fixed points which minimizes the time of fall of a point mass acted on by gravity. The problem reached Wallis and Gregory in Oxford, who could not solve the problem, but when the challenge reached Newton, he found out, in a few hours, that the brachistocroni is a cycloid.

The tractrix is another distinguished type of curve in the history of mathematics. It is characterized by the property that the length of its tangent between a line (y-axis) and the point of tangency is the constant a, which leads to the equation

\[ \pm y = a \text{arcsech} \left( \frac{x}{a} \right) - \sqrt{a^2 - x^2}. \]  

(9)

The problem was posed by Leibniz, but it was first studied by Huygens in 1692, who also gave the curve its name.

Various other curves are associated with a given curve. For example, when light rays are reflected off a fixed curve, the curve arising as the envelope of the reflected rays is known as a caustic curve. Such curves were also studied by Huygens (1678), and the Bernoulli, l’Hôpital, and Lagrange also gave their contributions. Recall that a family of curves \( F(x, y, a) = 0 \) depending on a parameter a has an enveloping curve which at each of its points is tangent to some member of the family. Its equation is found by elimination of a from the equations

\[ \frac{\partial F}{\partial a}(x, y, a) = F(x, y, a) = 0. \]  

(10)

The evolute of a curve can be constructed in this way, namely as the curve enveloped by the family of normal lines of the given curve. Huygens (1673) is credited for being the first to study the evolute, but the geometric idea dates back, in fact, to the work of Appollonius on conics. The reverse of the evolute is an involute, which is not unique, since different curves may have the same evolute. We mention that the evolute of the tractrix is the catenary, but the cycloid is its own evolute (with a shift). Similarly, the logarithmic spiral, described in polar coordinates \((r, \theta)\) by the equation \( r = ae^{b\theta} \), is congruent to its own evolute. The study of envelope in calculus dates back to Leibniz, and it is implicit in the early works on the evolute. As Lagrange also pointed out, a singular solution of a differential equation is generally an envelope of the integral curves.
The only intrinsic geometric invariant of a curve is its arc-length, and the arc-length $s$ measured from a starting point provides a natural parametrization of the curve. However, what is interesting with curves, as exemplified by the curves described above, is their extrinsic geometry, namely their metric properties and position relative to the ambient Euclidean plane or space. For a plane curve parametrized by $s$, its curvature measures the rate of change of its tangent direction at a given point, that is,

$$\kappa = \frac{d\alpha}{ds}$$

where $\alpha$ is the oriented angle (in radians) between a fixed direction in the plane and the tangent direction at $p$.

Note that $\alpha = \alpha(s)$ is also the arc-length traced out on a circle by the unit tangent vector. The sign of $\kappa$ depends on the direction of the curve and the orientation of the plane. The radius of curvature, $\rho = \pm 1/\kappa$, is the radius of the osculating circle, which is the limiting circle passing through three points on the curve and tending to $p$. In fact, these concepts were already known to Leibniz, who also had similar thoughts about the osculating sphere and the curvature of a surface (cf. Rosenfeld [1988: 280]).

For a curve in the plane, the center of the osculating circle is a focal point, situated at the distance $\rho$ from $p$ on the normal line, and as the point $p$ varies the focal points trace out the focal curve. The latter curve is, in fact, the evolute of the given curve, but now arising from a different viewpoint.

Euler took up the subject of space curves in 1775. He extended the definition (11) of curvature by taking $\alpha \geq 0$ as the length of the spherical curve traced out by the unit tangent vector. Let $x, y, z$ be rectangular coordinates in $\mathbb{R}^3$ so that $\sqrt{x^2 + y^2 + z^2}$ measures the distance from the origin. In terms of these coordinates the arc-length element $ds$ is subject to the constraint

$$ds^2 = dx^2 + dy^2 + dz^2$$

and the curvature expresses as

$$\kappa = \frac{d\alpha}{ds} = \sqrt{\frac{d^2x}{ds^2} + \frac{d^2y}{ds^2} + \frac{d^2z}{ds^2}}$$

Before 1850 the geometric properties of space curves were laboriously investigated, and many types of curves have been described in the literature. They were also called curves of double curvature, because the curve is characterized by two scalar functions $\kappa_i(s)$, $i = 1, 2$, namely the above curvature $\kappa(s) > 0$ and the torsion $\tau(s)$, classically known as the first and second curvature. A huge step forward was taken around 1850, with the advent of the Frenet–Serret formulae, which in modern vector algebra notation read

$$\frac{dt}{ds} = \kappa n, \quad \frac{dn}{ds} = -\kappa t + \tau b, \quad \frac{db}{ds} = -\tau n$$

(13)

Here $(t, n, b)$ is an orthonormal frame along the given curve $\gamma(s)$, actually explained by the above equations, where the velocity vector $t = d\gamma/ds$ is the unit
tangent, \( \mathbf{n} \) is called the *principal normal*, and \( \mathbf{b} = \mathbf{t} \times \mathbf{n} \) is the *binormal*. The equations were obtained independently by J.A. Serret in 1851 and J.F. Frenet (1816–1900) in 1852, but partly also in Frenet’s thesis (1847). Camille Jordan discovered in 1874 their generalization for curves with curvatures \( \kappa_1, \ldots, \kappa_{n-1} \) in \( n \)-space.

Geometrically, the space curve is completely determined by the system (13). Thus, for given functions \( \kappa(s) > 0 \) and \( \tau(s) \) on the interval \([0, L] \) and a given initial position and direction, the system yields by integration a unique space curve of length \( L \). The plane spanned by \( \mathbf{t} \) and \( \mathbf{n} \) is the *osculating plane*, and hence the torsion is measuring the rate at which the plane is changing. The *osculating circle*, lying in this plane, has radius equal to the radius of curvature \( \rho = 1/\kappa \), as before. The *normal plane* is the plane spanned by \( \mathbf{n} \) and \( \mathbf{b} \), and hence the principal normal \( \mathbf{n} \) spans the line common to the osculating plane and the normal plane.

### 2.2 Surfaces and their curvature invariants

Henceforth, let us turn to surfaces in 3-space and those special curves confined to them. The curvature theory of surfaces is considerably more complicated than that of curves, mainly because of the interlocking relationship between their intrinsic and extrinsic geometry. Surfaces are typically presented and studied via parametrizations, or as level surfaces of functions. Euler also introduced differential equations to define surfaces, such as solutions of the equation

\[
Pdx + Qdy + Rdz = 0. \tag{14}
\]

But a systematic study of surfaces appeared for the first time in Monge’s book [1807], *Applications of analysis to geometry*. Monge introduced notions like “families of surfaces” and he used his theory of surfaces to elucidate the solutions of partial differential equations. His theory of surfaces and his geometric approach to the study of differential equations inspired many geometers in the 19th century.

Euler worked out the foundations of the analytic theory of curvature of surfaces. In 1767 he characterized the curvature of a surface at a point \( p \) by describing all normal curvatures \( \kappa_\theta \), where the number \( \kappa_\theta \) is the signed curvature (see (11)) of the curve, called the *normal section*, cut out by a plane \( P_\theta \) perpendicular to the surface. The plane \( P_\theta \) contains the line \( l \) perpendicular to the surface at \( p \), so it is determined by the angle \( \theta \) which specifies the tangential direction of the curve at \( p \).

What did Euler find out about the numbers \( \kappa_\theta \)? If \( \kappa_\theta \) is not a constant for all \( \theta \), then \( \kappa_\theta \) takes its minimum (resp. maximum) value \( \kappa_1 \) (resp. \( \kappa_2 \)) for directions \( \theta_1 \) and \( \theta_2 \) which differ by 90°, and by choosing the zero angle \( \theta = 0 \) so that \( \kappa_0 \) is smallest, Euler’s formula reads

\[
\kappa_\theta = \kappa_1 \cos^2 \theta + \kappa_2 \sin^2 \theta. \tag{15}
\]

Actually, Euler expressed his formula in terms of curvature radii, and the above formula is a modification by Dupin (1837). The numbers \( \kappa_1 \) and \( \kappa_2 \) became
known as the principal curvatures, and the corresponding directions $\theta_i$ are the principal directions. Dupin was a student of Monge at Ecole Polytechnique in Paris. His geometric ideas were influential for many decades, and below we shall occasionally return to some of his achievements.

More generally, in 1776 J.B. Meusnier (1754–1793) extended Euler’s result to finding the curvature of the curve $C$ cut out by any plane $P$ through $p$. In this case, let $C$ have tangent direction $\theta$ at $p$, let $P_\theta$ be the corresponding normal section, and let $\varphi < \pi/2$ be the (dihedral) angle between the planes $P_\theta$ and $P$. Then the curvature $\kappa = \tilde{\kappa}_\varphi$ of the curve $C$ at $p$ is determined by Meusnier’s identity (for a proof, see (34))

$$\tilde{\kappa}_\varphi \cos \varphi = \kappa_\theta. \quad (16)$$

2.2.1 The Gaussian approach

We shall deduce the formulae (15) and (16), but in the more general setting due to Gauss. With his paper [1828] on general investigations of curved surfaces, Gauss unified the surface theories of Euler and Monge, but he also went much further. This paper, briefly called the Disquisitiones, is maybe the single most important work in the history of differential geometry. Here he also proved his celebrated results on the intrinsic geometry of surfaces. Gauss related the curvature of a surface $S$ with the variation of the tangent planes, or equivalently, the normal directions. So, as an astronomer himself he borrowed from astronomy the notion of spherical representation and introduced the so-called Gauss map

$$\eta : S \rightarrow S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\} \quad (17)$$

which, for an orientable surface, specifies the “outward” normal direction by a continuously varying unit vector $\eta_p$ perpendicular to $S$ at each $p$.

In his early investigations of curvature, Gauss defined the total curvature of a subset $R$ of $S$ as the area of the image $\eta(R)$, with negative sign if $\eta$ reverses orientation. Then, by comparison with the area of $R$ itself and letting $R$ decrease to a point $p$, he defined the curvature of $S$ at $p$ as the limit (if it exists)

$$K = \lim_{R \rightarrow p} \frac{\text{Area}(\eta(R))}{\text{Area}(R)}.$$ 

This definition, perhaps not so rigorous, is still useful in special cases; for example, $K$ must vanish on a region $R$ whose image $\eta(R)$ has zero area. Obvious examples are cones or cylinders, where the image of $\eta$ lies on a curve of finite length.

Both before and after Gauss various definitions of curvature have been proposed, say by Euler, Meusnier, Monge and Dupin, but they were never established and fell into oblivion. Instead, the adopted definition well known today is due to the more rigorous Gaussian approach leading to the definition of $K$ as the product of Euler’s principal curvatures $\kappa_i$, as follows. Firstly, observe that the tangent plane $T_p$ of $S$ at $p$, viewed as a linear subspace of $\mathbb{R}^3$, is a
Euclidean plane with the inner product $v \cdot w$ inherited from $\mathbb{R}^3$. Secondly, by noticing that $T_p$ naturally identifies with the tangent plane of the sphere $S^2$ at $\eta_p$, the differential of $\eta$ at $p$ becomes a linear operator today known as the shape operator (or Weingarten map)

$$d\eta : T_p \to T_p$$

(18)

Therefore, associated with this operator, or rather its negative $-d\eta$, is the bilinear form

$$\Pi_p(v, w) = -d\eta(v) \cdot w$$

(19)

called the second fundamental form. Below we shall see why this form is actually symmetric, and consequently it has eigenvalues $\kappa_1 \leq \kappa_2$ and orthonormal eigenvectors $t_1, t_2$ in $T_p$ so that

$$\Pi_p(t_1, t_1) = \kappa_1, \quad \Pi_p(t_2, t_2) = \kappa_2, \quad \Pi_p(t_1, t_2) = 0$$

(20)

Moreover, the numbers $\kappa_i$ are just Euler’s principal curvatures, and hence the vectors $t_i$ point in the principal directions.

The Gaussian curvature $K$ and the mean curvature $H$ are defined to be

$$K = \kappa_1 \kappa_2, \quad H = \frac{1}{2}(\kappa_1 + \kappa_2).$$

(21)

With his “Theorema Egregium” Gauss showed that $K$ is an intrinsic invariant, that is, it depends only on the geometry of the surface itself, so that $K$ is unchanged when the surface is bent or isometrically deformed in any way. Contrary to this, however, $H$ reflects the way the surface is embedded in the ambient space, so it is an extrinsic invariant.

The surface is nowadays called flat (resp. minimal) if $K = 0$ (resp. $H = 0$) holds at all points, and the reason for these terms will become clear later. More generally, if there is a functional relation $W(\kappa_1, \kappa_2) = 0$, for example when $K$ or $H$ is constant, the surface is referred to as a Weingarten surface (or W-surface), after J. Weingarten (1836–1910) who made important contributions, in the 1860’s and onward, to the theory of surfaces in the spirit of Gauss. Note, however, Klein and Lie used the term “W-surface” with a different meaning (see letter 3.3.1870). At a single point $p$ the surface is said to be hyperbolic, elliptic, parabolic, or planar if (i) $K < 0$ or (ii) $K > 0$, or (iii) $K = 0 \neq H$ or (iv) $K = H = 0$, respectively, at the point $p$. The terms hyperbolic, elliptic, parabolic, with reference to the sign of the curvature, is due to Klein (1871).

Let $(x, y)$ be the coordinate system of the tangent plane $T_p$ relative to the principal frame $(t_1, t_2)$. The Dupin indicatrix at $p$ is the following curve

$$\mathcal{D}_p : \kappa_1 x^2 + \kappa_2 y^2 = \pm 1.$$  

(22)

It consists of either two hyperbolas, an ellipse, two parabolas, two parallel lines, or is empty, according to whether $p$ is hyperbolic, elliptic, parabolic, or planar, respectively. The indicatrix describes the local geometry around $p$, as follows.
The two planes parallel to \( T_p \) and at a small distance \( \delta \) cuts the surface in a set which projects orthogonally to a set \( C_\delta \) in \( T_p \). By scaling \( C_\delta \) with the factor \((2\delta)^{-1/2}\), the limiting set as \( \delta \to 0 \) is the curve \( \mathcal{D}_p \). (For a proof, see Spivak [1975], Vol. 3, p. 68.) At a hyperbolic point \( p \) the two hyperbolas have asymptotic lines given by

\[
y = \pm (\sqrt{-\kappa_1/\kappa_2}) x.
\]

Therefore, the vectors \( t_\theta = \cos \theta \, t_1 + \sin \theta \, t_2 \), with \( \tan \theta = \pm \sqrt{-\kappa_1/\kappa_2} \), represent the directions of the asymptotes, and they yield vanishing normal curvature

\[
\Pi_p(t_\theta, t_\theta) = \kappa_1 \cos^2 \theta + \kappa_2 \sin^2 \theta = \kappa_\theta = 0.
\] (23)

In the literature these directions are called the asymptotic directions at the point \( p \), and we note that the directions are perpendicular if \( \kappa_1 = -\kappa_2 \), that is, when \( H = 0 \). On the other hand, at a parabolic point there is only one asymptotic direction, namely the principal direction \( t_i \) corresponding to \( \kappa_i = 0 \). Let us also say that all directions are both principal and asymptotic at a planar point.

The indicatrix is useful in the geometric study of two of the most interesting families of curves on surfaces, namely the lines of curvature and the asymptotic lines. These are the curves which at each point are heading in a principal or asymptotic direction, respectively. Euler (1760) was the first who investigated the lines of curvature, and his work inspired Monge to develop his general theory of curvature which he applied in 1795 to the central quadrics

\[
\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 = c.
\]

However, asymptotic lines were introduced by Dupin. In 1813 he published his *Developpements de geometrie*, with many contributions to differential geometry such as the idea of asymptotic lines. The indicatrix was not invented by him, but he showed how to make more effective use of this suggestive conic. For example, a pair of conjugate diameters of this conic are referred to as conjugate tangents in Dupin’s theory.

The classical approach to surfaces is, of course, simplified by the usage of vector calculus. So, let \( S \) be a given surface in Euclidean 3-space parametrized by coordinates \( u, v \), assumed (for simplicity) valid on all \( S \). Thus, the parametrization

\[
\Phi : (u, v) \to \Phi(u, v) = (x(u, v), y(u, v), z(u, v))
\] (24)

is a one-to-one smooth map from a region \( D \) in the \( uv \)-plane onto \( S \), and the coordinate vectors \( \frac{\partial \Phi}{\partial u}, \frac{\partial \Phi}{\partial v} \) at the point \( p \) span the tangent plane of \( S \) at \( p \). Let us define its positive orientation and associated normal field \( \eta \) on \( S \), in other words the Gauss map \( (17) \), by taking

\[
\eta = \left| \frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} \right|^{-1} \left( \frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} \right)
\] (25)

By expressing \( dx, dy, dz \) in terms of \( du \) and \( dv \) and substituting into \( (12) \), we obtain the squared line element restricted to the surface, namely the quadratic
form
\[ d\mathbf{s}^2|_S = Edu^2 + 2Fdu\,dv + Gdv^2 \]  
whose coefficients are the following inner products
\[ E = \left| \frac{\partial \Phi}{\partial u} \right|^2, \quad F = \frac{\partial \Phi}{\partial u} \cdot \frac{\partial \Phi}{\partial v}, \quad G = \left| \frac{\partial \Phi}{\partial v} \right|^2 \]  
Euler considered, for example, the case of a surface which can be developed on the plane with rectangular coordinates \( u, v \). Then a small triangle on \( S \) must be mapped to an isometric triangle on the plane, from which he deduced, in effect, that the coordinate vectors \( \frac{\partial \Phi}{\partial u}, \frac{\partial \Phi}{\partial v} \) are orthonormal, so the quadratic form \( (26) \) becomes the following simple form
\[ d\mathbf{s}^2 = du^2 + dv^2 \]
which characterizes the geometry of the Euclidean plane.

The surface theory of Gauss involves in fact two quadratic forms in \( du, dv \), referred to as the first and second fundamental form,

\[ I = Edu^2 + 2Fdu\,dv + Gdv^2 \]
\[ II = Ldu^2 + 2Mdu\,dv + Ndv^2 \]
where \( I \) is the metric form \( (26) \), and the coefficients of \( II \) are the normal components of the second order derivatives of the map \( (27) \),

\[ L = \eta \cdot \frac{\partial^2 \Phi}{\partial u^2}, \quad M = \eta \cdot \frac{\partial^2 \Phi}{\partial u \partial v}, \quad N = \eta \cdot \frac{\partial^2 \Phi}{\partial v^2}, \]

Indeed, the extrinsic geometry of the surface is encoded in the second form \( (30) \). This approach is motivated by the study of parametrized curves \( t \to \gamma(t) = \Phi(u(t), v(t)) \) on the surface, whose acceleration is the vector

\[ \gamma'' = \frac{d^2 \Phi}{dt^2} = u'^2 \frac{\partial^2 \Phi}{\partial u^2} + 2u'v' \frac{\partial^2 \Phi}{\partial u \partial v} + v'^2 \frac{\partial^2 \Phi}{\partial v^2} + \left( u'' \frac{\partial \Phi}{\partial u} + v'' \frac{\partial \Phi}{\partial v} \right) \]
and consequently its normal component times \( dt^2 \), namely the expression \( (\gamma'' \cdot \eta)dt^2 \), is just the above quadratic form \( II \). In particular, using the natural parameter \( t = s \), the Frenet-Serret formulas \( (13) \) yield

\[ II = (\gamma'' \cdot \eta)ds^2 = (\kappa \cos \varphi)ds^2 \]
where \( \kappa \) is the curvature of the curve \( \gamma \) and \( \varphi \) is the angle between its principal normal \( \mathbf{n} \) and the surface normal \( \eta \).

Finally, by combining \( (26) \) with \( (33) \) there is the general formula

\[ \kappa \cos \varphi = \frac{Lu'^2 + 2Mu'v' + Nv'^2}{Eu'^2 + 2Fv'v' + Gv'^2} = \Pi_p(t_{\theta}, t_{\theta}) \]
where $t_\theta$ is the unit tangent of the curve at $p$. In particular, this explains why the form (14) is symmetric. For the curve cut out by the normal section $P_\theta$ we have $\kappa = \kappa_\theta$ and $\cos \varphi = \pm 1$, so the above formula (34) yields, in fact, that Euler’s “normal” curvature $\kappa_\theta$ equals $\Pi_p(t_\theta, t_\theta)$. Consequently, we also find that Meusnier’s formula (13) is just the special case of (34) for curves which are planar.

Recall from (21) the definition of the curvature $K$ as the product of the principal curvatures

$$\kappa_1 = \Pi_p(t_1, t_1), \quad \kappa_2 = \Pi_p(t_2, t_2)$$

which are the minimum and maximum values of $\Pi_p(t, t)$ as $t$ runs over all tangent vectors of unit length, that is, $1 = |t|^2 = \Pi_p(t, t)$. The metric form $\Pi_p$ has matrix coefficients $E,F,G$ with respect to the basis \{\($\partial \Phi / \partial u$, $\partial \Phi / \partial v$\)}), so let us apply the Gram–Schmidt algorithm and replace it by an orthonormal basis and calculate the associated symmetric matrix $\Omega$ of the form $\Pi_p$. Then it will be clear that the eigenvalues of $\Omega$ are just the numbers $\kappa_i$, and a simple calculation yields the following useful formulas for the Gaussian and the mean curvature

$$K = \kappa_1 \kappa_2 = \frac{LN - M^2}{EG - F^2}, \quad H = \frac{1}{2} (\kappa_1 + \kappa_2) = \frac{1}{2} \frac{EN - 2FM + GL}{EG - F^2}. \quad (35)$$

For later usage, let us calculate these quantities for the graph of a function $z = f(x, y)$, the most familiar case in elementary vector calculus. The function in (24) becomes $\Phi(x,y) = (x, y, f(x,y))$, and writing $f_x, f_y, f_{xx}$ etc. for the partial derivatives of $f$, the expressions in (35) read

$$K = \frac{f_{xx}f_{yy} - f_x^2}{(1 + f_x^2 + f_y^2)^2}, \quad H = \frac{(1 + f_x^2)f_{yy} + (1 + f_y^2)f_{xx} - 2f_x f_y f_{xy}}{(1 + f_x^2 + f_y^2)^{3/2}}. \quad (36)$$

Meusnier derived this formula for $H$ in 1776. To prove his “Theorema Egregium” Gauss expressed $K$ in terms of the three functions $E,F,G$ (27). But this also implies there is a general formula for $LN - M^2$, purely in terms of $E,F,$ and $G$ without invoking any parametrization function. Liouville and Brioschi also derived such explicit formulas, see below.

Gauss himself introduced the kind of coordinates $(u, v)$ called *isothermal* coordinates, a term due to Lamé (1833), so that the metric $ds^2$ and its Gaussian curvature takes the simple form

$$ds^2 = e^\varphi (du^2 + dv^2), \quad K = -\frac{1}{2} e^{-\varphi} \left( \frac{\partial^2 \varphi}{\partial u^2} + \frac{\partial^2 \varphi}{\partial v^2} \right). \quad (37)$$

He proved only the existence of isothermal coordinates when the given functions $E,F,$ and $G$ are analytic, whereas in the differentiable case their existence was not proved until the 20th century. In the special case of constant curvature, the expression (37) for $K$ is often referred to as Liouville’s equation. Liouville is well-known, for example, for his studies of conformal geometry, where metrics differing only by a function multiple, say $ds^2 = e^\varphi ds'^2$, are regarded as “identical” and are said to be *conformally* equivalent.
2.2.2 French and Italian response to the Gaussian approach

Actually, the geometric works of Gauss, which culminated with his *Disquisitiones* (1828), were largely unnoticed or neglected for many years to come, say up to around 1850 or so. Ten years after the publication of the *Disquisitiones*, E.F.A Minding (1806–1885) was the first who continued the work of Gauss, with his special study of surfaces of constant curvature (see below). In France, the tradition after Monge and his pupils still dominated the preferences and geometrical way of thinking. S.D. Poisson (1784–1840) published in 1832 a memoir on the curvature of surfaces, including a historical review, but without mentioning Gauss (see Reich [1973]). During the following decade, essentially only the versatile engineer and applied mathematician G. Lamé, who in 1832 had accepted a chair of physics at the École Polytechnique, wrote papers on differential geometry. Among his interests we find triply orthogonal systems of surfaces, and through his studies of heat conduction he was also led to a general theory of curvilinear coordinates. Up to the 1860’s, the “best” non-trivial example of a triply orthogonal system was the family of confocal quadrics, and both Lamé [1839] and Jacobi [1839] introduced elliptic (or ellipsoidal) coordinates using this system. Lamé used them to separate and solve the Laplace equation $\Delta(f) = 0$, whereas Jacobi calculated the geodesics on the ellipsoid.

But in the early 1840’s some younger French geometers, such as Bertrand and Bonnet in their early twenties, made reference to Gauss and his *Disquisitiones* in their first papers, and at the end of the decade an ebullient interest in the geometric works of Gauss 20 years earlier burst out in France and Italy. In particular, as Monge’s celebrated paper [1807] was republished by Liouville in 1850, he also gave a new proof of the Theorema Egregium which appeared in the appendix, together with the original latin version of the *Disquisitiones*.

Liouville had an explicit formula for $K$ in 1851 which, in fact, the Italian Beltrami made clever use of in his papers in the mid 1860’s. In Italy, F. Brioschi (1824–97) had greatly influenced the direction of higher education and research in mathematics, and among his doctoral students we find Cremona (1853) and Beltrami (1856). In 1863 he founded the Technical University in Milan, where he served both as a director and professor of mathematics and hydraulics. Maybe Beltrami’s teacher had not yet worked out his formula by 1865, but it is Brioschi’s formula for $K$ rather than Liouville’s which is most easily found in textbooks and online pages on differential geometry. In orthogonal coordinates $u, v$, namely when $F = 0$, the formula simplifies to

$$K = \frac{-1}{2\sqrt{EG}} \left( \frac{\partial}{\partial u} \left( \frac{G_u}{\sqrt{EG}} \right) + \frac{\partial}{\partial v} \left( \frac{E_v}{\sqrt{EG}} \right) \right).$$ (38)

Next, let us also recall some geometric invariants naturally arising in the study of curves $\gamma(s)$ on a given oriented surface $S$, using their arc-length $s$ as the natural parameter. Along the curve there is the velocity vector $\gamma'(s) = t$, the acceleration $\gamma''(s) = \kappa n$, and the resulting Frenet-Serret frame $(t, n, \bf{b})$, see [13]. But there is also the orthonormal Darboux frame $(t, u, \eta)$, with $u$ in the tangent plane, chosen so that $t \times u = \eta$ is the positively directed normal, as
before. Again, in analogy with (13), differentiation with respect to \( s \) yields a dynamical system with a skew symmetric matrix, namely for suitable coefficients \( \kappa_g, \kappa_\eta, \tau_g \) we can write
\[
\begin{align*}
\gamma' &= \kappa_g u + \kappa_\eta \eta, \\
\gamma'' &= -\kappa_g t + \tau_g \eta.
\end{align*}
\]
The leftmost equation expresses the decomposition of the acceleration (32) into its tangential and normal component
\[
\gamma'' = \gamma' = \kappa_g u + \kappa_\eta \eta
\]
called the geodesic and normal curvature vector, respectively. It follows that \( \kappa_\eta \) is just the normal curvature introduced by Euler, see (15), namely \( \kappa_\eta = \kappa_\theta \) where the angle \( \theta \) gives the direction of \( t \), so that
\[
\kappa^2 = \kappa_g^2 + \kappa_\eta^2, \quad \Pi_{\gamma(s)}(t, t) = \kappa_\eta.
\]
Gauss referred to \( \kappa_g \) as the “Seitenkrümmung”, and the term tangential curvature was also used. However, the term used today is geodesic curvature, which dates back to Bonnet (1848).

2.2.3 Curves on a surface and the Gauss-Bonnet theorem

The terms introduced above serve to characterize the following three classical main types of curves \( \gamma(s) \) confined to a surface \( S \), as follows:

- **Geodesic curves**: Their geodesic curvature \( \kappa_g \) vanishes. The curves are locally characterized by the geodesic equation \( \kappa_g = 0 \), which in local coordinates is a 2nd order nonlinear differential equation. Therefore, a geodesic starting from a point \( p_0 \) is uniquely determined by its initial direction \( t(0) \). Their crucial geometric property is the “shortest length” property, which generally holds for small segments of the curve, but not necessarily for longer segments. But surely, a shortest curve between two given points must be a geodesic. Geodesics are the natural generalization of the “straight lines” in Euclidean or hyperbolic geometry. Being a geodesic curve is an intrinsic property, that is, it remains a geodesic under isometric deformations.

- **Asymptotic lines**: Their normal curvature \( \kappa_\eta \) vanishes. Other equivalent conditions are (i) \( \kappa_g = \pm \kappa \), or (ii) the osculating plane (if defined) equals the tangent plane, or (iii) the Frenet–Serret frame and the Darboux frame differ at most by signs, namely \((t, n, b) = (t, \pm u, \pm \eta)\). Moreover, the identity \( \tau_g = \tau \) (if defined) also holds. The equation \( \kappa_\eta = 0 \) means the vanishing of the second fundamental form (30) along the curve, namely
\[
Ldu^2 + 2Mdu dv + Ndv^2 = 0
\]
By regarding this relation as a quadratic equation this yields (in general) two vector fields on the surface, whose integral curves are the asymptotic lines.
• **Lines of curvature:** Their geodesic torsion \( \tau_g \) vanishes. Equivalently, their velocity vector \( \gamma'(s) = t(s) \) is a principal direction \( t_i(s) \), for all \( s \). It is easy to express the principal directions as the zero directions of the following quadratic form

\[
(EM - FL)d\mu^2 + (EN - GL)d\mu dv + (FN - GM)d\nu^2 = 0, \quad (42)
\]

and in analogy with the case this also yields two vector fields on the surface, whose integral curves are the lines of curvature.

Euler (1732) was the first to work out a differential equation for the shortest curve between two points on a surface which solves equation (14) (see Rosenfeld[1988: 282]). Dupin’s theorem, which we shall return to below, provides in specific cases a geometric construction of the lines of curvature curves. Obvious examples of asymptotic curves, which are geodesics as well, are straight lines lying on the surface. Their unit speed motion along the line has a constant velocity vector \( t \), so \( t' = 0 \) and consequently \( \kappa_g = \kappa_\eta = 0 \) by (39). But \( \kappa_\eta \) lies between \( \kappa_1 \) and \( \kappa_2 \), so \( K = \kappa_1 \kappa_2 \leq 0 \) must hold along the line. Moreover, strict equality \( K = 0 \) holds if and only if our line is also a line of curvature, or equivalently, the tangent plane as the same normal direction and hence is constant along the line. The reader may verify these statements, say using (39) and (19).

In his differential geometric studies Monge had found envelopes of families of surfaces. For a one-parameter family \( F(x, y, z, a) = 0 \) enveloping a surface, the procedure is similar to that of curves, see the equations (10). For each value of \( a \) the equations yield a curve, called the characteristic, on the surface labelled by \( a \). Then by varying \( a \) the characteristics sweep out the enveloping surface. For example, the Dupin cyclides are surfaces which can be enveloped by a one-parameter family of spheres, in fact, in two different ways.

A 2-parametric system of lines is usually called a ray systems or a line congruence. They play an important role in line geometry, where their envelope is referred to as the focal set, or the focal surface (Brennfläche), which may degenerate to a curve called the evolute. A typical focal surface \( F \) has two components \( F_1 \) and \( F_2 \), and the ray system consists of their common tangent lines. This also applies to the study of surfaces, by considering the ray system of lines normal to a given surface \( S \).

On the other hand, using the notion of curvature there is another construction of the focal sets \( F_i \) of a given surface. Namely, by pointwise pushing the surface in the normal direction a distance equal to its curvature radii, one obtains the sets \( F_i \) as the locus of points

\[
p + \frac{1}{\kappa_i(p)} n(p), \quad p \in S, \quad i = 1, 2.
\]

Letting \( S \) be a torus obtained by rotating a circle, for example, the two sets will be a circle and a straight line (axis if symmetry).
In his paper [1828] Gauss considered geodesic triangles $\Delta$ on a surface $S$, and denoting the angles by $\alpha, \beta,$ and $\gamma$ he derived the simple formula

$$\int \int KdS = \alpha + \beta + \gamma - \pi$$

which expresses the “angular excess” as the total curvature of the triangle. Twenty years later, Bonnet 1848 considered the general case of a triangle with piecewise smooth boundary edges $C_i$, and he extended the formula by adding correction terms on the left side, namely the line integral of the geodesic curvature $\kappa_g$ along the three edges. This is the essence of the general Gauss–Bonnet formula (44).

The remaining part amounts to some combinatorial bookkeeping which arises when $n$ triangles fill up a region $S_n$ of an oriented surface $S$, and we apply and add together the modified formula (43) for each triangle. It is assumed that any two triangles are either disjoint or have just one common vertex or edge. Then the number of vertices ($V$), edges ($E$) and triangles ($T$) are related by the Euler characteristic $\chi(S_n) = V - E + F$ of the region, and this is the only global topological invariant needed here. Clearly, for a closed disk and a sphere the number is 1 and 2, respectively. The boundary $\partial S_n$ of the region consists of those edges belonging to a single triangle.

Next, let us make use of the induced orientation of the triangles and their edges, noting that adjacent triangles induce opposite directions on their common edge. Therefore, the line integrals away from the boundary cancel each other. Now, let us replace the angles of a triangle by its oriented outer angles $\alpha' = \pi - \alpha, \beta' = \pi - \beta$ etc., which measure the “jumps” of the tangential direction at the vertices. In particular, at each corner of the boundary there is a net angular “jump”, and let us move them to the left side of our equation. Then the remaining “jumps” on the right side, in fact, add up to a multiple of $2\pi$, namely

$$\int \int KdS + \int_{\partial S_n} \kappa_g ds + \sum_{\partial S_n} (\text{jumps}) = 2\pi \chi(S_n).$$

This is the Gauss–Bonnet formula, valid for a compact oriented surface $S (= S_n)$ with a piecewise smooth boundary, where the ”jumps" disappear if the boundary is smooth. It is very likely that Gauss had ideas about how to generalize his formula (43), but he did not publish more on the topic. Bonnet proved special cases of equation (44) in 1848.

### 2.3 Surfaces in classical analytic geometry

#### 2.3.1 Ruled and developable surfaces

Let us have a closer look at some types of surfaces frequently encountered in the classical literature, starting with the ruled surfaces. A surface is ruled if for each point there is a straight line on the surface passing through the point. Obvious
examples are cylinders and cones. It is doubly ruled if there are two distinct lines passing through each point, and familiar examples are the hyperbolic paraboloid and the hyperboloid of one sheet:

\[ z = \frac{x^2}{a^2} - \frac{y^2}{b^2}, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1. \]  

(45)

In fact, these are the only doubly ruled surfaces of degree 2, and the only triply ruled surface is the plane.

A ruling of the surface is a 1-parameter family of straight lines on the surface, which yields a parametrization of the surface of type

\[ (s, t) \rightarrow \gamma(s) + t\delta(s) \]  

(46)

where \( \gamma(s) \) and \( \delta(s) \) are given space curves. Each of the lines, parametrized by \( t \), is called a generatrix, whereas the curve \( \gamma(s) \) cutting every line is the directrix, and \( \delta(s) \) is the director. Monge and his school gave these surfaces some thought, but later they played an important role in Plücker’s study of line geometry. Recall that any surface must have curvature \( K \leq 0 \) along a straight line, consequently a ruled surface satisfies \( K \leq 0 \) everywhere.

Monge’s work was sometimes preceded by Euler’s, but he worked independently and with his own originality. For example, developable surfaces were introduced by Euler and Monge in 1772 and 1785, respectively, using their own definition. To begin with, let us define a developable surface in 3-space to be a ruled surface with vanishing Gaussian curvature, \( K = 0 \). Note that the above hyperboloid \((45)\) has negative curvature, so it is an example of a ruled surface which is not developable. For the simpler and final definition of “developable”, see \((49)\) below.

Ruled and developable surfaces appear naturally in surface theory, as in the following construction. For any chosen curve \( \gamma \) on the surface \( S \), the family of normal lines to \( S \) along \( \gamma \) spans a ruled surface \( \bar{S} \) with \( \gamma \) as its directrix. Then, by a theorem of Bonnet, the curve \( \gamma \) is a line of curvature of \( S \) if and only if the associated ruled surface \( \bar{S} \) is developable.

Classically, the developable surfaces were found to belong to four types:

\( (i) \) planes, \( (ii) \) cylinders, \( (iii) \) cones, \( (iv) \) tangent developables.  

(47)

However, this list is not complete, not even among the analytic developable ones. As an example, there is an analytic developable, homeomorphic to the Möbius strip (see Spivak[1975], vol. 3, p. 355). Ruled surfaces different from the types \((47)\) became also known as scrolls (Schraubenfläche).

Let us give parametrizations \((46)\) of the developable surfaces of type \((ii)\) - \((iv)\) in \((47)\). They are actually generated by a single space curve \( \gamma(s) \) with velocity \( \gamma'(s) \neq 0 \), on some interval \( s \in (a, b) \), in the following way:

- Generalized cylinder: Take \( \gamma(s) \) to be a curve in the \( xy \)-plane and let \( \delta \) be the constant vector \((0, 0, 1)\). The surface \((46)\) has principal curvature \( \kappa_1 = 0 \) in the vertical direction, and \( \kappa_2 = \kappa(s) \) is the curvature of \( \gamma(s) \), with principal direction along \( \gamma'(s) \). Consequently \( K = 0 \) and \( H = \frac{1}{2} \kappa(s) \).
• Generalized cone: Choose a vector $v_0$ as the vertex of the cone, and let $\delta(s) = \gamma(s) - v_0$. The vertex $v_0$, corresponding to $t = -1$ in (46), is a singular point of the surface. Away from the vertex, $\gamma(s) - v_0$ points in the principal direction with $\kappa_1 = 0$, so $K = \kappa_1\kappa_2 = 0$.

• Tangent developable: Assume $\gamma(s)$ is an arc-length parametrization, $\gamma(s)$ has curvature $\kappa(s) \neq 0$, and take $\delta(s) = \gamma'(s)$. The surface (46) has two sheets which for $t = 0$ meet along the curve $\gamma(s)$ as a cuspical edge. Away from this curve, the surface has the principal curvature $\kappa_1 = 0$ in the direction of $\gamma'(s)$, so again the total curvature vanishes.

Already in 1825 Gauss published a paper on conformal transformations, where he compared two surfaces whose metric expressions $ds^2$ (see (26)) differ by a scalar function, say

$$Edu^2 + 2Fdudv + Gdv^2 = \mu^2(E'du^2 + 2F'dudv + G'dv^2).$$

He observed that for $\mu = 1$ a complete equality (Gleichheit) holds between the surfaces, so that one surface can be developed onto the other. Thus he extended the notion of “developable”, which in this context also became known as applicable, and thus a major classical problem has been to determine those surfaces applicable to a given surface, not just the plane.

Briefly, applicable surfaces have the same metric expression $ds^2$ (see (26)), and they were regarded as deformations of each other (locally). In modern terminology, they are (locally) isometric, but not necessarily congruent, that is, transformable to each other by a rigid motion of the ambient space. For example, the catenoid and the helicoid are applicable, one can in fact “stretch and twist” the catenoid continuously to the form of a helicoid without changing the intrinsic geometry. They are examples of minimal surfaces discussed below, and certainly they look very different.

Following the footsteps of Gauss, Minding’s theorem (1839) says that surfaces in 3-space with the same constant curvature $K$ are mutually applicable, that is, locally isometric. In particular, surfaces with vanishing curvature $K = 0$ are applicable with the plane and hence they can be constructed by suitably bending of plane regions. Therefore they are also ruled, namely we have the following three equivalent conditions for surfaces, briefly said to be flat:

(i) developable, (ii) $K = 0$, (iii) locally isometric to a plane.

In France, Bonnet and Bour investigated the curvature of special surfaces such as ruled surfaces or surfaces of rotation, as well as the developability for ruled surfaces. For example, a theorem of Bour says that a scroll is applicable to some surface of rotation. But first of all, when it comes to the classical problem concerning applicable surfaces, Weingarten in Germany was the first who made a major step forward when he described in 1863 a class of surfaces applicable to a given surface of rotation.
At this point, let us illustrate with some examples why the study of asymptotic curves played such an important role in the classical surface theory. According to a theorem of Bonnet, an isometry between non-ruled surfaces which maps a family of asymptotic curves into the asymptotic curves of the other surface must be a rigid motion (cf. Chern [1991]). The problem of finding another surface applicable to a given surface $S$ amounts analytically to solving a certain Monge–Ampère equation, whose characteristics are the asymptotic curves of $S$. This system of equations may well become over-determined if additional constraints are posed on the surface. There is the following result concerning a given space curve $\gamma$ lying on $S$, namely if $\gamma$ is asymptotic then one can construct infinitely many surfaces $S'$ through $\gamma$ which are applicable to $S$. On the other hand, if $\gamma$ is not asymptotic and is kept fixed, then there no such surface except $S$ itself.

### 2.3.2 Minimal surfaces

Next, let us turn to minimal surfaces, whose well known physical models are the soap bubbles suspended between strings and always tend to minimize their area. Ever since the first discoveries in the 18th century the determination of minimal surfaces has posed a challenge to geometers up to present time. It started with Euler’s discovery in 1744 that a catenary (8) rotated around the $x$-axis yields a minimal surface of revolution. This is the well known catenoid, which for any finite segment of the catenary has the smallest area among all surfaces suspended between the two boundary circles. By a simple variational analysis on the area of a surface of rotation, Euler derived a differential equation for the rotated curve having the catenary as its solution. He may not have proved it rigorously, but with the original definition the catenoid is, in fact, the only minimal surface of rotation different from the plane. This was also verified by Bonnet in the 1850’s.

Using a more general variational analysis of area, Lagrange (1860) studied minimal surfaces on the explicit form $z = f(x, y)$. He considered the area

$$A(t) = \int_U (1 + \tilde{f}_x^2 + \tilde{f}_y^2)^2 \, dx \, dy$$

of a variation $\tilde{f}(x, y; t)$ of a minimal surface ($t = 0$) over an open region $U$ in the $xy$-plane, with boundary fixed for all $t$. By demanding $A'(0) = 0$ for all variations he derived the associated Euler-Lagrange equation

$$\left(1 + f_y^2\right)f_{xx} + \left(1 + f_x^2\right)f_{yy} - 2f_xf_yf_{xy} = 0.$$  

(50)

It is a rewarding exercise at an undergraduate level to deduce Euler’s catenary from this equation by expressing it in polar coordinates $(r, \theta)$ and setting $\theta = 0$. The solutions of the resulting reduced differential equation, $rf'' + f'(1 + f'^2) = 0$, are those curves in the $(r, z)$-plane which generate the rotationally symmetric minimal surfaces $z = f(r)$. The reduced equation is easily solved by quadrature and yields the catenary family of curves, $a \cosh(\frac{\theta}{a} + b) - r = 0$, as expected.
This simple calculation applied to an example from the early 18th century, indeed, illustrates the basic idea of “equivariant” differential geometry, a modern reduction technique based on the interaction between symmetry and the least action principle. This lies at the heart of Sophus Lie’s symmetry approach to differential equations, but for many reasons the method was not properly developed until the late 20th century.

Meusnier discovered in 1776 the helicoid, which is closely related to the catenoid. In cylindrical coordinates it has the simple equation \( z = c\theta \). Until about 1830 these two surfaces besides the plane were the only known minimal surfaces, but at this time H.F. Scherk (1798–1885) discovered his first surface, implicitly defined by \( e^{az}\cos ay - \cos ax = 0 \). His discovery was regarded as sensational, and in 1831 he was awarded a prize at the Jablonowski Society in Leipzig. The reader can find the Scherk surface by seeking the solutions of equation (50) of the splitting type \( z = g(x) + h(y) \), by applying the method of separation of variables. However, the helicoid is the only ruled minimal surface in 3-space other than the plane, as was shown by E.C. Catalan (1814–1894) in 1842.

On the other hand, Meusnier derived in 1776 also his formula (36) for the mean curvature \( H \), and by comparison with formula (50) he thus observed that \( H = 0 \) is a necessary condition for minimality. In fact, in the 19th century the simple and local condition \( H = 0 \) became the new and modern definition of a minimal surface. In effect, the original meaning of a minimal surface, being infinitely extensible and with no boundary curve, was abandoned.

Henceforth minimal surfaces became the natural surface analogue of geodesic curves, which are locally of shortest length, but perhaps not the shortest curve between any two of its points. The mathematical problem of existence of a minimal surface with a given boundary became known as the Plateau problem, after the Belgian physicist Joseph Plateau (1801–1883), who conducted extensive studies of soap films. We mention briefly that the existence of a solution, for a given boundary curve, was not proved until 1931, but little could be said about the geometric properties of the solution.

During the 19th century complex analysis gradually became an important direction of mathematics. Then it also turned out that complex analytic functions have a close connection with minimal surfaces, and in the 1860’s Weierstrass, Riemann and Enneper found representation formulas which parametrize a minimal surfaces for each pair \((f, g)\) of functions. Thus Enneper and Weierstrass created a whole class of new parametrizations, roughly by taking two functions on a domain \( D \) say, with \( fg^2 \) holomorphic, and define the surface as the set of points

\[
(x, y, z) = \text{Re} \left( \int f(1 - g^2) d\zeta, \int if(1 + g^2) d\zeta, \int 2fg d\zeta \right). \quad (51)
\]

The difficulty lies in controlling the global behavior of the surface, which may have singularities such as self-intersections. We also mention that H.A. Schwarz and his collaborators solved the Plateau problem in 1865 for special boundary
curves, by finding appropriate functions to be inserted into the above formulae (51). As the calculus of variations and topological methods were developed, the study of minimal surfaces took new directions in the 20th century, and the methods of Weierstrass and Schwarz came more in the background.

2.3.3 Dupin’s cyclides and some related topics

Dupin’s cyclides are among those surfaces whose remarkable properties attracted considerable attention in the early 19th century and, in fact, since the 1980’s they still do. They constitute a 3-parameter family, and originally the cyclide is defined as the envelope of the 1-parameter family of spheres tangent to three fixed spheres. In fact, Dupin discovered them still as an undergraduate student, but they appeared first in his *Developpements* in 1813. Later, in his book *”Applications de Geometrie“* (1822), he called them *cyclides*. Algebraically, a cyclide is of order three or four, and having circular lines of curvature is one of their essential properties. Moreover, their two focal surfaces $F_i$ degenerate to curves of second order.

But there are still many other ways of characterizing them, alternative definitions are due to Liouville, J.C. Maxwell (1868), Casey (1871) and Cayley (1873). In recent years there are, in fact, many indepth studies of their algebraic and geometric properties. For a survey of all this, including modern applications to geometric design, we refer to Chandru et al. [1989]. On the other hand, in the mid 1860’s Dupin’s cyclides became somewhat subordinate to another family of surfaces also named cyclides, namely the generalized cyclides discovered by Darboux and Moutard. As a consequence, the previous cyclides were rather overshadowed by the new family of surfaces, having different properties which for the purpose of the Darboux school were regarded more important. We shall return to them below.

An interesting aspect of Dupin’s cyclides is their symmetry properties; they can, in fact, be generated and described neatly in terms of conformal transformations. To explain this, first recall that these are the angle preserving transformations, a property which is also evident from their action on the squared line element (or metric) of the space, namely

$$\text{d}s^2 \rightarrow \mu^2 \text{d}s^2 \quad (\text{or } (\text{d}s^2 = 0) \rightarrow (\text{d}s^2 = 0))$$

(52)

where the function $\mu^2 > 0$ may depend on the transformation. As indicated, in Klein’s letters “conformal” is also expressed by the invariance of the equation $\text{d}s^2 = 0$.

In our case, the space is the Euclidean space $\mathbb{R}^3$ (or a subregion) with the metric (12), and therefore the celebrated Liouville’s theorem (1846) on conformal mappings describes them as the composition of similarities (that is, Euclidean motions and homotheties) and inversions. The latter type consists of the classical geometric transformations called inversion with respect to a sphere, and Liouville called them transformation by reciprocal radii. The transformation interchanges the inside and outside of a fixed sphere and inverts the radial
distance. For a sphere of radius $\rho$ centered at $v_0$, the vector algebra expression of the inversion is

$$v \rightarrow v_0 + \rho^2 \frac{v - v_0}{|v - v_0|^2}.$$  (53)

Liouville showed that a cyclide can be obtained from a torus of revolution, or a circular cylinder or a circular conic, by a suitable choice of inversion. The cyclide is also anallagmatic, in the sense of being its own inverse with respect to at least one inversion.

The corresponding transformation of the plane is the inversion with respect to a circle, defined similarly. If straight lines are regarded as circles of infinite radius, one can say briefly that the transformation transforms circles into circles with one circle fixed. The method of inversion seems to be attributed to Steiner (1824), but it was used in special cases by Poncelet (1822), Plücker and others, and it was studied most extensively by Möbius (1855). It was also discovered through physical considerations; for example, it appeared as “the method of images” in the work on electrostatics by W. Thomson (1845). Inversion was one of the first non-linear transformations to be deeply studied in geometry.

As a natural generalization, Cremona introduced in 1854 the general birational transformation on the plane, which became known as Cremona transformations and developed by him in the 1860’s. They were found to have many applications such as the reduction of singularities of curves, and the study of elliptic integrals and Riemann surfaces.

Remarks on groups

In retrospect, it is tempting to interpret the above geometric objects in the light of modern group theory. By adding to $\mathbb{R}^3$ a point at infinity, our space becomes (via stereographic projection) conformally the same as the 3-sphere $S^3$, with its transformation group $CG(3)$ consisting of the tenfold infinity of conformal transformations. Namely, it is a 10-dimensional Lie group, containing the orthogonal group $SO(4)$ as the group of isometries of the sphere. Now, the latter clearly contains the 2-dimensional torus group $T$, which together with all its conjugates $T'$ in $CG(3)$ act on the sphere with tori as their orbits. The images of these tori in $\mathbb{R}^3 \cup \{\infty\}$ are the Dupin cyclides, and they can be permuted among themselves by transformations from $CG(3)$.

The above description of surfaces using groups has a striking similarity with Klein and Lie’s approach to W-surfaces, as part of their study of W-configurations in 1870–71. We touch this topic only briefly and refer to Hawkins [2000] §1.2, for supporting evidence that it was during the collaboration on this project, based upon Lie’s paper [1870a], that Klein and Lie developed the basic general principles leading to the idea of continuous groups of transformations, in analogy with the customary definition of a group of substitutions (or permutations) in algebra. In this study they were working with specific projective transformations on complex projective 3-space. Namely, in the mentioned paper on tetrahedral line complexes, Lie had focused attention on the totality $G$ of
projective transformations fixing the vertices of a given tetrahedron— in modern
terms $G$ is a (complex) 3-dimensional torus. At this time Klein and Lie used
terms like “cycle” or “closed system” for families of transformations closed un-
der composition, tacitly assuming the family would also be closed under taking
inverses as in the algebraic case. Since the surfaces they were seeking are just
the orbits of the various 2-dimensional subtori $T$ of $G$, the classification of these
surfaces would necessarily involve a certain group classification problem. The
problem was rather intractable and hence postponed, but really they did not
return to it (cf. note to letter of 12.9.71).

2.3.4 Pseudospherical surfaces
Surfaces of constant curvature $K$ in Euclidean 3-space are clearly natural geo-
metric objects, and for $K = 0$, resp. $K > 0$, the prototype examples are the
plane and the round sphere respectively. Minding (1839) also posed the ques-
tion about the uniqueness of the sphere among closed surfaces with constant
$K > 0$. The affirmative answer was not given until Liebmann’s theorem (1900),
valid for non-singular and $C^2$-differentiable surfaces. However, in the case of
$K < 0$ little was known until Minding made an explicit study of such surfaces
in 1838. He discovered, in fact, three types of surfaces of rotation, and some
non-rotational surfaces as well. The geometrically simplest one is the tractoid,
namely the surface obtained by rotating the tractrix (9), and it became known
as the pseudosphere.

By using the expression (30) for $K$ one is led to the following differential
equation
\begin{equation}
    f_{xx}f_{yy} - f_{xy}^2 + (1 + f_x^2 + f_y^2)^2 = 0
\end{equation}
valid for surfaces on the explicit form $z = f(x, y)$ and with $K = -1$. Then, an
application of the same reduction to (54) as was applied to equation (50) when
we found the catenoid, will also yield a differential equation for the tractrix
which is solvable by quadrature. The pseudosphere was, in fact, already known
to Gauss, who referred to it as the “opposite” of the sphere in a note written
in the 1820’s. For many years, the term pseudosphere was used confusingly in
the literature for any surface of constant negative curvature. After all, Minding
had concluded that all these surfaces are isometric, or more precisely, appli-
cable to each other. It was Beltrami who finally in 1868 referred to them as
pseudospherical surfaces, in order to “avoid circumlocation” as he puts it, 30
years after Minding’s results had appeared. We refer to Coddington [1905] for
an interesting account of the historical development of pseudospherical surfaces
during the years 1837–1887.

We also remark that in terms of asymptotic coordinates $(u, v)$, that is, the
coordinate lines are the asymptotic lines, equation (54) takes the original form
of the sine-Gordon equation
\begin{equation}
    \psi_{uv} = \sin \psi
\end{equation}
with $\psi$ as the angle between the asymptotic curves. As a consequence, the
study of pseudospherical surfaces is equivalent to that of the above equation, and
the equation was much used for that purpose in the 19th century. In modern mathematical theories the equation is also found to be interesting because it has soliton solutions.

2.3.5 Coordinate geometry, triply orthogonal systems, and generalized cyclides

The classical study of the differential geometry of 3-space led, in fact, to several kinds of differential equations with modern applications nowadays. For us, it is appropriate to recall some of the early developments on triply orthogonal systems. Although applications of particular examples occurred already in the works of Leibniz and Euler, most of the early contributions are due to French geometers such as Lamé, Dupin, Liouville, Bonnet and Darboux.

A coordinate system in 3-space (or a subregion) amounts to a triple \( S^{(\alpha)}, S^{(\beta)}, S^{(\gamma)} \) of 1-parameter families of surfaces, called coordinate surfaces, such that every point \( p \) lies on a unique surface from each family, which yields a bijective correspondence \( p \leftrightarrow (\alpha, \beta, \gamma) \). The triple is said to be orthogonal if the coordinate surfaces intersect each other perpendicularly. Then, according to the celebrated Dupin’s theorem, two orthogonally intersecting surfaces must intersect along lines of curvature. Dupin published his proof in his previously mentioned paper Développements (1813); for a modern proof, see Spivak[1875], Vol. 3. The orthogonality property of the coordinates is also reflected by the corresponding expression for the arc-length element (12), namely

\[
\begin{align*}
\text{ds}^2 &= P\text{d}\alpha^2 + O\text{d}\beta^2 + R\text{d}\gamma^2 \\
&= (56)
\end{align*}
\]

where the coefficients \( P, Q, \) and \( R \) are functions depending on the geometry of the surfaces.

During the first half of the 19th century French geometry developed roughly in two directions, namely with focus on differential geometry or projective geometry, in the tradition of Monge and Poncelet, respectively. According to Hawkins[2000: 28], the “French metrical geometry” — a term often used by Klein and Lie — referred to those geometers who combined concepts from both methodologies. In fact, their basic approach is within the framework of conformal geometry rather than projective geometry, partly inspired by Liouville’s theorem (1846) which gives a precise description of the totality of conformal transformations of 3-dimensional space (or higher). During the 1860’s in Paris, a bright aspiring student arose from the elite schools for mathematical training and gradually developed his ideas which placed him centrally among the “metrical” geometers around 1870. His name was Gaston Darboux, born the same year as Sophus Lie.

Darboux’s first two papers, which appeared in the Nouvelles Annales in 1864, are concerned with his construction of a family of 4th degree curves which he referred to as cyclic. He starts with the planar sections of tori and the intersection of a sphere with other quadratic surfaces, and he also includes their inverses with respect to spheres. Among the cyclic curves one finds many of the classically well known curves, such as the Descartes ovals, the scissoid, and the
lemniscate. But, as a student at École Normale, Darboux became familiar with the works of Lamé, Dupin and Bonnet on triply orthogonal systems of surfaces, and this topic became, in fact, his major interest for a long time.

At this time the best example of such surfaces was still the confocal surfaces of degree 2 (see note to letter 13.12.70), and the surfaces were all defined by a single equation. Years before, Kummer had studied analogous families of plane curves, \( f(x, y, s) = 0 \), \( s \) a parameter, namely for each point there are two curves passing through it, and they meet orthogonally. He also found that the curves had to be confocal, that is, they have the same foci. In fact, the orthogonality and the confocal property amount to the same thing. For example, the orthogonal family of Descartes ovals have three common foci. Here we shall simply remark that the definition of foci can also be extended to algebraic curves of degree \( n > 2 \). On the other hand, the situation is different in 3-space, and Darboux was able to construct a triply orthogonal family which is not confocal.

On August 1, 1864, Darboux presented to the Academy of Science his discovery of the following triply orthogonal and confocal system of surfaces, expressed by one algebraic equation of degree 4

\[
\mu (x^2 + y^2 + z^2)^2 + \frac{a\lambda - 4h}{\alpha - \lambda} x^2 + \frac{b\lambda - 4h}{\beta - \lambda} y^2 + \frac{c\lambda - 4h}{\gamma - \lambda} z^2 - h = 0 \quad (57)
\]

where \( \alpha, \beta, \gamma, h, \) and \( \mu \) are constants and \( \lambda \) is the parameter. For example, the case \( \mu = 0 \) yields a 1-parameter family of confocal quadrics. The above surfaces were subsequently called (generalized) cyclides, but their properties are quite different from those of Dupin’s cyclides. Clearly, the intersection of a cyclide with a sphere is a cyclic curve.

However, on the same day as Darboux, the 15 year older Moutard announced to the Academy that he had discovered the same system of surfaces. Moutard was an expert on anallagmatic surfaces, namely surfaces invariant under an inversion, and for surfaces of degree 4 he found that they were invariant under 5 different inversions if they contain the imaginary circle at infinity as a double curve. The last statement simply means the 4th order terms have the same form as in (57). But it is a geometric statement in our space extended to the complex projective 3-space. Here, the portion of our extended complex surface which lies in the plane at infinity is expressed by the equation derived from (57) by ignoring all terms of degree lower than 4, namely the equation

\[
x^2 + y^2 + z^2 = 0 \quad (58)
\]

with multiplicity 2. This equation describes the imaginary circle at infinity, as it was referred to in classical projective geometry. By seeking the lines of curvature of his surfaces Moutard found the family of cyclides. We refer to their papers Darboux [1864], [1865] and Moutard [1864a, b].

From his triply orthogonal system of cyclides Darboux derived the new coordinate system \( (\lambda_1, \lambda_2, \lambda_3) \) of space, a kind of generalized elliptic coordinates, and expressed the metric \( ds^2 \) on the form (56). As in the well known case of the confocal quadrics, Darboux discovered that the intersection curves on the
surfaces, which by Dupin’s famous theorem are lines of curvature, also form an isotermal system of curves on each surface. But more importantly, he also came across the following extension of Dupin’s theorem, namely when two orthogonal one-parameter families of surfaces intersect along lines of curvature, there is always a third family of surfaces intersecting the other two orthogonally. He used this result to find the condition a given family of surfaces, say \( f(x, y, z) = \lambda \), has to satisfy in order to belong to some triply orthogonal system. His answer to this was a certain partial differential equation of order 3 in two independent variables, but it was not calculated explicitly since it was too complicated. But he did, for example, determine those orthogonal system for which the lines of curvature are in a plane. All these results were published in his classic memoir [1866], actually his first memoir on orthogonal systems, and it was subsequently presented as his doctoral thesis.

Certainly, orthogonal systems of surfaces is a field with which Darboux’s name will always be associated. But it was rather Cayley (1872) who first succeeded in finding a tractable differential equation, which in a simple way determines triply orthogonal systems, maybe of a special kind. However, Darboux quickly analyzed and realized the essence of Cayley’s work, which he extended to the case of \( n \) variables. Now he was able to prove various results on orthogonal system in higher dimensions, for example, the determination of orthogonal systems consisting of surfaces of degree two, or orthogonal systems containing a given surface. The topic continued to play a major role in Darboux’s geometric works for many decades, as can be seen from his voluminous treatises on surfaces (1878 and 1898).

However, as it often happens in mathematics, new techniques and viewpoints may suddenly lead to substantial simplifications of previous hard work. Thus, it is appropriate to mention the work of G.M. Green (1891–1919) in USA, who was partly inspired by Darboux in France and E.J. Wilczynski (1876–1932) in USA. The young Green wrote a 27 page paper in 1913 which largely overrides Darboux’s many pages devoted to triply orthogonal systems. He used a pair of simultaneous partial differential equations of order two, but the new idea comes from the projective geometric setting which he learned from Wilczynski. In the late 1870’s the French G.H. Halphen (1844–89) examined differential equations invariant under projective transformations, and the topic of his doctoral dissertation in 1878 was differential invariants. Apart from the early investigations of Halphen, Wilczynski is largely regarded as the founder of projective differential geometry, a geometric setting where he was the first to demonstrate the utility of completely integrable systems of homogeneous linear differential equations.

### 2.4 Riemann and the birth of modern differential geometry

Riemann’s Habilitation lecture at Göttingen in 1854 is generally regarded as the birth of modern differential geometry, but it became generally known only after its first publication in 1867. It is regarded as a classic of mathematics, which is even more remarkable, since the audience of Riemann’s lecture was the
Philosophical Faculty of Göttingen, and its purpose was to demonstrate lecturing ability. Certainly, the essay [1867] is almost devoid of explicit mathematical content, but Gregory [1989], for example, argues convincingly that the paper is better understood if we see Riemann speaking primarily as a philosopher, but with a rather powerful mathematical methodology.

In fact, as a student Riemann had taken philosophy classes, and he was well acquainted with, but also disagreed with, the Kantian view of space. As his influences Riemann names only two persons, namely his supervisor Gauss, who was in the audience in 1854, and the philosopher Johann F. Herbart (1776-1841), who is also known as the founder of pedagogy as an academic discipline. Herbart held the chair after Kant in Königsberg, and became a professor of philosophy at Göttingen University in 1833.

Gauss was probably the only one who realized the depth of Riemann’s ideas. According to Freudenthal [1970–1990], the lecture was too far ahead of its time to be appreciated in those days, and it was not fully understood until 60 years later, when the mathematical apparatus developed from Riemann’s lecture provided the frame for the physical ideas in Einstein’s general theory of relativity. In the meantime, however, Riemann’s ideas were a major source of inspiration for many upcoming geometers, such as Beltrami, Helmholtz, Clifford, Cristoffel, Lipschitz, Lie, Klein, Killing (1847–1923), Poincaré, Ricci-Curbastro, and Levi-Civita. But Riemann himself did, in fact, not publish any work on differential geometry in his lifetime. An English translation by Clifford of Riemann’s lecture appeared first in the journal Nature (1873), but Spivak [1979], Vol. 2, also has an English translation, supplemented with comments.

2.4.1 Riemann’s lecture and his metric approach to geometry

The mathematical contents of Riemann’s essay is often summarized roughly as a generalization to higher dimensions of Gauss’s results on the intrinsic geometry of surfaces. But we shall have a closer look at the essay, which is divided into three parts. First of all, he states that he is aiming at a continuous space model, whereas the measurable properties of a discrete space are simply determined by counting. Next, he observes the confusing status of non-Euclidean geometry, which was not generally accepted at that time. He attributes this problem to the fact that geometers do not distinguish clearly between the topological and metric properties of space, which he discusses in Part 1 and Part 2 of his address, respectively.

Here we use the modern term “topology” for the 20th century discipline which grew out from geometry and “analysis situs” during the decades after Riemann. Many fundamental ideas in topology date, in fact, back to the works of Riemann, his Italian friend Enrico Betti, and later also Poincaré, see for example Betti [1871], Poincaré [1895]. Riemann refers to the underlying space as a multiply or n-fold extended quantity, anticipating the modern concept of a (differentiable) n-dimensional manifold, and for simplicity we shall also use this term in the sequel. In particular, the surfaces studied by Gauss are 2-dimensional manifolds. In contrast to this, it should be noted that the notion
of space is undefined in the axiomatic development of geometry, although its properties are implicitly described by the axioms.

Now, Riemann argues that topological considerations alone would not be sufficient, say, to deduce Euclid’s parallel postulate, and he points out that experimental data are needed to determine the actual metric properties of Physical Space, which he takes up in the final Part 3 of his address. Riemann had a strong background in theoretical physics, influenced by the physics professors W.E. Weber (1804–91) and J.B. Listing (1808–82). In fact, he was Weber’s assistant for 18 months, and besides, Listing must also be counted among the early pioneers of topology.

According to Riemann, the measurable properties of Space are, after all, the subject of geometry. The distance between two points is measured by physical instruments, for example, one uses a rod or some optical instrument, and in those days the measurable properties of space were found to agree completely with Euclidean geometry. But the instruments and the notion of a rigid body lose their validity when it comes to infinitely small distances where, in fact, the metric may even disagree with the ordinary assumptions of geometry. This is a possible scenario the physicists should be prepared to meet, and as a preparation he presents a vastly general vision of geometry in Part 2.

In Part 2 Riemann displays a hypothesis on the metric structure of the space which is as general as could be imagined at that time. Almost all mathematical results appear here, but at the same time he is deliberately vague on many points and avoids technical definitions and calculations, since many in the audience had little knowledge of mathematics. According to Riemann, a given manifold can be endowed with many different metric relations, each characterized by the form of the infinitesimal distance $d_s$, which Riemann postulates to be expressible as a positive quadratic form

$$ds^2 = \sum g_{ij}dx_idx_j$$

(59)

in the differentials $dx_i$ of the local coordinates, where the coefficient functions $g_{ij}(x)$ may vary from point to point. If they are constant, then the metric is Euclidean, and a linear change of coordinates will transform the metric to the standard form

$$ds^2 = dy_1^2 + dy_2^2 + \ldots + dy_n^2$$

(60)

which may also be regarded as the infinitesimal Pythagorean law. Therefore, by continuity the geometry of (59) agrees optimally with the Euclidean geometry in the vicinity of each point. Expressions of the type (59) have become known as a Riemannian metric, and by the term “space” we shall mean a manifold with a given metric of this kind. The global geometric properties, such as a distance function measuring the distance between two points, will follow from the metric by integrating $ds$ along curves. For Riemann, the physical space is merely an example of a 3-dimensional space and, contrary to the Kantian viewpoint, he argues that the actual determination of its metric (59) is a matter of physical measurements.
Next, Riemann turns to the metric properties behind the basic relation involving the $n(n + 1)/2$ functions $g_{ij} = g_{ji}$. He argues that the degrees of freedom of the functions are found by subtracting $n$ of them, due to the freedom in the arbitrary choice of $n$ coordinate functions. Thus Riemann concludes there is some set of $n(n − 1)/2$ functions which determine the metric completely, and he proposes to choose the functions

$$K_{1,2}, K_{1,3}, K_{1,n}, K_{2,3}, \ldots, K_{n−1,n}\quad(61)$$

which at each point $p$ give the Gaussian curvature in $n(n − 1)/2$ independent surface directions.

The above “sectional” curvature $K_{ij}$ is calculated as the Gaussian curvature of a surface $S_{ij}$ with the specified surface direction at $p$, which the modern reader will interpret as the 2-dimension tangent plane at $p$. Implicit in Riemann’s argument is the fact that the number $K_{ij}$ only depends on the surface direction and not on the actual choice of $S_{ij}$. On the other hand, it is a major point of Riemann that the “sectional” curvatures (61), in turn, determine the metric relations (59).

Riemann pays most attention to spaces of constant curvature, meaning that all the numbers $K_{ij}$ are equal to the same constant $K$, valid for any point. Here he gives the following “standard” expression for the metric in appropriate homogeneous coordinates, namely

$$ds^2 = \frac{1}{(1 + \frac{1}{4} \sum x_i^2)^2} \sum dx_i^2\quad(62)$$

In particular, when $K = 0$, as in the Euclidean plane or space, $ds^2$ is the sum of squares of complete differentials, and Riemann proposes to refer to these spaces as flat. The round sphere is the familiar example of a space with constant $K > 0$, whereas Minding’s pseudosphere has constant $K < 0$, but its connection with hyperbolic geometry was not known to Riemann.

In Part 3 Riemann focuses on the possible models for the physical space. He points out that every determination from experience remains inexact, and this circumstance becomes important when the empirical determinations are extended beyond the observational limits into the immeasurably large or the immeasurably small. Moreover, in the first case Riemann also distinguishes between unboundedness and infinitude, noting that an unbounded universe may possibly have finite size (volume).

Riemann argues that Space must be some unbounded 3-dimensional manifold, and he does not exclude the possibility of having a variable curvature. This was a radical idea which even many decades later was deemed too speculative by most “experts”. But first of all, he points to further empirical evidence which put the spaces of constant curvature into the forefront. Loosely speaking, these are the spaces which look the same at each point and in every direction, and in terms of well known physical terms Riemann describes this spatial property as “free movability of rigid bodies”, in the sense that they can be “freely
shifted and rotated”. Thus, he asserts the following two properties of a space are equivalent:

\( (i) \) the space has constant curvature; \( (ii) \) rigid bodies can be freely moved.

But notions such as “rigid body” and “free movability” are physical rather than geometrical, and Riemann makes no effort to explain statement (ii) in geometric terms, as Helmholtz tried with his axiom \( H3 \), see Section 5.4.

### 2.4.2 The beginning of modern differential geometry

The publication of Riemann’s lecture in 1867 was met with widespread acclaim, and many were influenced by his ideas. Shortly afterwards, Helmholtz in Germany and Clifford in England published their own interpretations and extensions, which also helped bring the attention to a wider community. As a physicist, Helmholtz preferred to link the foundations of geometry with the physical statement (ii) in (63), whereas others were challenged by the necessity of having statement (ii) reexpressed purely in terms of derived geometric concepts. This is closely related to the Riemann–Helmholtz space problem, which we shall return to in Section 5.4.

Christoffel, Lipschitz and Schering were the first who started to elaborate the Riemannian approach to geometry, based upon the postulated infinitesimal structure of \( ds^2 \). As a consequence, the transformation theory of quadratic differential forms like (59) became a central topic. Whereas Riemann described in his lecture—but with no calculations—the conditions for \( ds^2 \) to be transformable to the flat metric, Christoffel took the next step in 1869 by determining the necessary and sufficient conditions for a quadratic differential form to be transformable into another one, by a suitable change of coordinates. Lipschitz treated the same problem in 1870, but the solution of Christoffel turned out to be more useful. His analysis led him to the invention of a process which enabled him to derive a sequence of tensors from a given one. This process was named covariant differentiation by Ricci in 1887. Riemannian geometry was finally supplemented in the early 20th century by the important notion of parallel transport, which brings tangent vectors along closed curves and expresses their total change of direction using the curvature tensor. This notion was developed by Levi-Civita in 1917 and independently by J.A. Schouten (1883–1971) in 1918. (See also note to letter 22.1.73).

The modern geometer uses the Riemann curvature tensor \( \{R_{ijkl}\} \) expressed in the language of tensor calculus to investigate the curvature properties of a space. But it is natural to inquire whether Riemann himself was ever in possession of this tensor. In fact, some of the mathematical analysis underlying Riemann’s address in 1854 can be found in the second part of his Pariser Arbeit, which is an essay he submitted in 1861 to the Academy of Science in Paris in competition for a prize, announced in 1858 and relating to a question on heat conduction. Again there is a quadratic differential expression like (59), but
not interpreted as a metric this time, and Riemann is seeking the integrability conditions under which it can be transformed to the simple form (60). He introduces expressions which are essentially the components of the curvature tensor \( R_{ijkl} \), and he shows the integrability conditions are the vanishing of the components. In particular, in the two variable case the condition is the vanishing of the Gauss curvature, \( K = 0 \).

But Riemann was not awarded the prize, perhaps because the way he obtained his results was not satisfactorily explained. Neither was the prize awarded to anyone else, and it was finally withdrawn in 1868. An English translation of an extract from the prize essay’s second part can be found in Spivak [1979], Vol. 2, pp.179–182. In 1872 Riemann’s mathematical papers came into the hands of Clebsch, who was Riemann’s successor in Göttingen. But he died the same year, and they were temporarily passed over to Dedekind and Weber. His Collected Mathematical Works were edited and published by H. Weber in 1876, but the prize essay appeared first in the 2nd edition in 1892.

3 Projective geometry

3.1 The origins of projective geometry

Projective geometry is the first of the “new” geometries of the 19th century, naturally arising from the classical Euclidean geometry. It is also the simplest and most fundamental of these geometries. In a way, it is concerned with the aspect of figures that remain unaltered when the observer changes his position, and even goes to infinity. The geometry originates from the idea of perspective, namely the study of the geometric rules of perspective drawing, by which spatial objects and relations in 3-space are projected onto a 2-dimensional plane. The principles behind all this have been explored since antiquity.

The earliest and most basic projective invariant is the cross-ratio of four collinear points, see (67); here we are using the modern term introduced by Clifford (1878). This ratio has been shown, more recently, to be the unique key invariant of projective geometry. In the surviving book after Menelaus, called *Sphaerica*, there is, in fact, a theorem in spherical geometry which corresponds to the invariance of the cross-ratio. Appolonius and Pappus were cognisant of the simpler theorem valid in the plane, but the origins of that discovery is an open question. The invariance of the cross-ratio under perspective projection can, in fact, be deduced from the ancient theorem, already known to Thales (600 BC), saying that a line drawn parallel with one side of a triangle cuts the other two sides proportionally. Pappus wrote about Euclid’s lost books, the Porisms, he gave 38 different porisms and also suggested that Euclid knew about the invariance of the cross-ratio.

It has been quite a favorite sport among geometers to reconstruct, with varying success, the lost ancient works. The work of Chasles, *Les trois livres de porismes d’Euclid* (1860) is at least recognized as an elaborate and ingenious “restoration” of the porisms. H. Zeuthen, who studied with Chasles in the
1860's, shared his interest in ancient Greek mathematics and he investigated in detail the work of Apollonius on conic sections, such as his method of determining the foci of a central conic. Moreover, in his treatise *Die Lehre von den Kegelschnitten im Altertum* (1886), Zeuthen suggests as a possibility that the Porisms were just a by-product of a fully developed projective geometry on conics.

### 3.1.1 Developments in the 16th and 17th century

Let us mention two important medieval scholars, both from Nuremberg in Bavaria, namely Johann Werner (1468-1522) and Albrecht Dürer (1471-1528), the latter best known as an artist painter. Werner worked on spherical trigonometry and was maybe the last writer in the medieval tradition of conic sections with some original contribution. But it does not seem that he knew about the cross-ratio. Dürer was also one of the most important Renaissance mathematicians; his remarkable achievements were through the applications of mathematics to art, but he also developed new and important ideas within mathematics itself. His masterpiece was the descriptive geometric treatise on the human proportions, finished in 1523 but published posthumously. The reason for his delay is largely because he felt it was necessary first to write an educational elementary mathematical treatise, which he published in 1525 as four books through his own publishing company. These were, in fact, the first mathematical books published in German. In the last book, for example, he wrote on regular and semi-regular solids, his own theory of shadows, and an introduction to the theory of perspective. In 1527 he also published a work on (military) fortification, maybe as a response to the threat of an invasion by the Turks felt by the people of Germany at that time.

It is remarkable that the celebrated Pappus’s theorem (cf. Section 1.1.2) naturally belongs to projective geometry, although the subject was not developed until 1500 years later. But during this long time span there were also a few other momentous discoveries of the same kind, notably the theorems of Desargues (1639) and Pascal (1640). Inspired by the works of Appolonius and Pappus, Desargues studied geometric objects such as conics from a new viewpoint by focusing on perspective or central projections, and properties invariant under these.

To describe the basic theorem of Desargues, take two triangles $ABC$ and $A'B'C'$ in perspective from a point $O$, that is, the three points $O, A, A'$, resp. $O, B, B'$, resp. $O, C, C'$ are collinear. Then the three intersection points of the corresponding sides of the triangles, $AB$ and $A'B'$, $BC$ and $B'C'$, $AC$ and $A'C'$, when suitably extended, are lying on the same line.

Although the work of Desargues did not receive much acclaim by his contemporaries, there were important exceptions and his lectures in Paris influenced French geometers such as Descartes and Pascal. The latter published an essay, at the age of 17, with several projective geometric theorems, such as the famous Pascal’s *hexagon theorem* saying that for any hexagon inscribed in a nondegenerate conic, the three points of intersection of the opposite sides are collinear.
Pappus's theorem is actually the other case where the conic degenerates into two lines.

A central projection may well map parallel lines to intersecting lines, and therefore, in the new geometry parallel lines are no longer special. This suggests that parallel lines behave as if they intersect at some common ideal point "at infinity", an idea already suggested by work of Kepler (1571–1630) and Desargues. The introduction of these ideal points, one for each "direction" of lines, yields the projective plane as an extension of the Euclidean plane. The basic geometric objects are still the points and lines, but there is one new line, called the line at infinity, consisting of all those new points. The theorems of Pappus, Menelaus, Desargues and Pascal may as well be regarded as statements valid in the extended plane. In fact, here they become simpler than in the Euclidean plane, with no exceptional cases since two lines always intersect.

Desargues and Pascal also knew about the invariance of the cross-ratio. Desargues introduced notions such as harmonic sets and involution, and he carried the ancient polar theory about poles and polars, much further than Appollonius. In plane geometry, this is a construction which uses a given conic (ellipse, parabola, or hyperbola) to associate a line (polar) to each point (pole) and vice versa. Namely, the conic has two tangents, say at \( p_1 \) and \( p_2 \), passing through a given point \( p_0 \), and the line through \( p_1 \) and \( p_2 \) is the polar of \( p_0 \). This correspondence is involutive, and moreover, it has the property that the polars of the points on a line \( \lambda \) constitute the lines passing through the pole of \( \lambda \). Desargues started with a circle as the given conic, and he treated a diameter as the polar of a point at infinity. Here we mention that if the circle is mapped onto a conic by a central projection, the image of a pair of mutually perpendicular diameters will be a pair of conjugate diameters in the sense of Appolonius. Desargues also introduced self-polar or self-conjugate triangles, namely each side of the triangle lies on the polar of the opposite vertex.

However, the rather weird terminology used by Desargues has not survived, whereas the terms pole and polar, still used today, were first introduced by his later compatriots F. J. Servois (1768–1847) and Gergonne, in 1811 and 1813 respectively. Before that, however, Euler, Legendre, Monge, and Brianchon had also used the pole-polar construction. Monge and his students Servois and Brianchon were in the forefront of projective geometry at the beginning of the 19th century, but Servois is perhaps better known for initiating the algebraic theory of operators, and he came close to discovering the quaternions before Hamilton.

The late 17th century scholar in Paris, Philippe de La Hire (1640–1718), was originally an artist (painter), and his interests in geometry arose from his study of perspective in art. Perhaps he deserves to be considered, after Pascal, a direct disciple of Desargues in projective geometry. His famous treatise *Nouvelle méthode* (1673), which clearly displays the influence of Desargues, is a broad projective approach to the study of conic sections. Utilizing his own method of projection, he also reproved all 364 theorems of Appolonius. Somewhat strangely, however, he did not mention Desargues, claiming he was not aware of the latter’s work until after the publication of his own.
In Italy, the geometer Giovanni Ceva (1647–1734) was interested in the synthetic geometry of triangles, and he is known for the rediscovery of the ancient Menelaus’s theorem as well as his own celebrated Ceva’s theorem from 1678. The latter theorem states that three lines from the vertices $A,B,C$ of a triangle to points $P,Q,R$ on the opposite sides, respectively, are concurrent precisely when the product of the ratios in which the sides are divided is equal to 1:

$$\frac{AR}{RB} \cdot \frac{BP}{PC} \cdot \frac{CQ}{QA} = 1.$$  (64)

In fact, Ceva’s theorem was known to early Arab mathematicians, and it dates back to an 11th century king of Saragossa.

3.1.2 Euler’s affine geometry and Monge’s descriptive geometry

The subject of projective geometry dragged along and fell into oblivion, say for the next 100–150 years after La Hire and Ceva. However, as a prelude to the resurrection of projective geometry in the early 19th century, it is timely to recall first the important step made by Euler in the 18th century, when he investigated those properties of Euclidean plane figures which remain invariant under parallel projection from one plane to another. He coined the term affine for this purpose. Thus he initiated affine geometry as a kind of geometry featuring the parallelism in Euclidean geometry. The affine and the Euclidean plane are identical as sets, having the same points and lines. But the notion of angle is undefined, and comparison of lengths in different directions is also meaningless. The generality of affine motions, generated by all possible parallel projections, is simply illustrated in a fixed Cartesian plane model, where the affine motions are coordinate transformations of type

$$(x, y) \rightarrow (ax + by + e, cx + dy + f), \quad ad - bc \neq 0,$$  (65)

whereas suitable (orthogonality) conditions on $a, b, c,$ and $d$ are needed for a Euclidean motion. So, affine geometry is to be regarded as a “relaxation” of Euclidean geometry.

Next, let us return to descriptive geometry, which originated with the medieval works of Dürer, to overcome the problems of projection and to describe the movement of bodies in space. But his ideas were not put on a sound mathematical basis until the work of Monge. As a young man in the early 1760’s, Monge studied orthogonal projections in 3-space and represented a figure by its ”shadows” in mutually perpendicular planes. Then he devised a method to reconstruct the original figure from the ”shadows”. Thus orthographic projection, the graphic method used in modern mechanical drawing, evolved from Monge’s simple scheme, and the discipline became known as descriptive geometry.

The story goes that Monge’s ideas originated from a problem on fortifications, which he solved for the French military. This reminds us of Dürer’s treatise in 1527, which also dealt with fortifications but probably outdated by Monge 250 years later? Monge’s solutions were so successful that the military kept
the method secret for 30 years and forbade Monge to publish them. His own account on the subject appeared around 1795, when École Polytechnique was established. In France, several outstanding geometers were, in fact, educated at military (artillery) schools and were involved in the Revolution or the ensuing military activities. In particular, quite many students at École Polytechnique became soldiers or officers in Napoleon’s army and later played important roles in the political and academic life.

Monge gave lectures on descriptive geometry at École Polytechnique, where this topic became a permanent part of the curriculum, and he emphasized geometric visualization of mathematical and physical problems. The French style soon became a model for other schools and military academies in other countries, including the United States, and descriptive geometry became a major topic. However, with regard to visualization of the geometry, T. Olivier (1793-1853) went even beyond Monge by building a geometric model collection for pedagogical purposes. Some of the models were ruled surfaces, even with moving parts to illustrate how the surfaces are generated, others were designed to illustrate how curves arise as the intersection of certain surfaces. Selling models, particularly in the United States, became an "industry" which gave Olivier quite a good income. In Germany this kind of enterprise was promoted, for example, by Plücker, who had studied a year in Paris around 1823 and was Klein’s influential teacher in Bonn in 1867–68. At Clebsch’s suggestion, in 1869 L.C. Wiener (1826–96) constructed plaster models of cubics surfaces and others, which were later exhibited in London and Chicago. Klein also became a proponent for plaster and string models in Germany, and in the early 1870’s his interest in models was expressed in letters to Lie.

Carnot was Monge’s early student, who also joined him in establishing the École Polytechnique. He was teaching there, with a strong engineering background, but is still best known as a geometer. His desire was to overcome the increase of generality due to the algebraic methods of Descartes, so he tried to simplify pure geometry and give it a universal setting. Of particular interest are his ideas about correlative figures, obtained by continuous deformation of a given figure. By first establishing geometric relations in the simplest case, where the involved quantities are positive numbers, with no further restrictions he assumed the relations are identities which still hold when the figure is replaced by a correlative figure. The convenience of this principle or rather “axiom” was demonstrated by deducing generalizations of the theorems of Menelaus and Ceva, and by establishing several theorems of Euclid’s Elements from one single theorem. He published these results in 1801–1803, and the principle is, in fact, a forerunner of Poncelet’s continuity principle, see below. Carnot’s military masterpiece, like that of Dürer and Monge, was also on fortification, published in 1809.

3.2 The rise of projective geometry in the 19th century

At the end of the 18th century Euclidean geometry was still the basic frame for geometric thought, but with the turn of the century the situation changed
dramatically. After all, our visual world has the geometry of a projective rather than a Euclidean space, so many geometers believed in the points at infinity and perhaps also regarded the fundamental concepts of geometry to be projective. Gradually, the conspicuous beauty and elegance of projective geometry made it a favorite study among many geometers, who swarmed into the new “gold field” and quickly uncovered the most accessible treasures. Thus, the rise of projective geometry made it synonymous with modern geometry of the 19th century, and during the first decades geometers in France and Germany played a leading role. Paris continued to be at the center of the scene, and the priority of French mathematicians in the creation of projective geometry cannot be denied. Monge’s role was very influential, both as a director and teacher at École Polytechnique. He can be said to be the first modern specialist in geometry as a whole. Other prominent geometers were Carnot, Poncelet, Gergonne and Chasles in France, whereas Möbius, Steiner, von Staudt, and Plücker were at the forefront in Germany.

3.2.1 Poncelet and the creation of projective geometry

In Paris around 1800, unexpected geometric ideas dating back to the 17th century and even back to ancient times, were rediscovered and further investigated. C.J. Brianchon (1783-1864) discovered the long forgotten Pascal’s hexagon theorem shortly after he entered École Polytechnique. This led him to another hexagon theorem as well, by a skillful application of the ancient polarity associated with a conic (cf. Lie-Scheffers [1896], Kap.1, §3). Brianchon’s theorem (1806) says that for any hexagon circumscribed about a conic, the three diagonals meet at a common point. In fact, the theorems of Pascal and Brianchon are the first clear-cut significant example of a pair of dual theorems. Brianchon’s theorem was later generalized to the case of a $(4n + 2)$-gon, by Möbius (1847). But from the long list of geometers who originated from the school of Monge, Poncelet ranks first and foremost (cf. e.g. Darboux[1904: 101]).

Poncelet’s early mathematical career is particularly interesting. He had learned about the works of Monge, Carnot and Brianchon at École Polytechnique, from which he graduated in 1810, at the age of 22. But he was older than usual, due to health problems, and now he chose a military career. Being trained as an engineer he took part in Napoleon’s ill-fated invasion of Russia in 1812, where he participated in the terrible battle at Krasnoy and barely survived. As a prisoner, kept at Saratov until the defeat of Napoleon in 1814, Poncelet discovered and wrote down the basic principles of projective geometry, which were kept in his notebook when he finally returned to France in the fall of 1814. During the following years he developed his new ideas in a systematic way, being employed as a Captain of Engineers and a teacher of mechanics at Metz, until 1825 when he accepted the position as professor of mechanics. Poncelet published many articles on geometry and mechanics, but he also continued his military career, and he was highly regarded for his mechanical inventions.

During his early studies as well as in the Saratov notes, Poncelet applied analytic geometry. However, for some reason, after his return to Paris he changed
his taste and became a staunch advocate of synthetic geometry. His famous _Traité_ (1822) ignited a tremendous surge forward in the geometrical developments, which also took place at major universities abroad. With his studies of the relationship between a figure and its image by central projections, Poncelet took the final step towards a precise mathematical description of the ancient geometric conception of perspectivity. In fact, Desargues had initiated this subject in 1639, but his forgotten work did not come to light until 1845.

For example, when a plane figure is illuminated by light rays emanating from an outside point, its shadow in any other plane is a projective image of the figure. As we have already observed, the parallel projections studied by Euler are the special case of central projections from ideal points infinitely far away. By collectively referring to properties of figures preserved by all these induced transformations as _projective_, Poncelet actually introduced a completely new discipline called projective geometry. We mention that Möbius and Chasles, who developed the theory in a different way, used the alternative terms _collineation_ and _homography_ for a projective transformation, respectively. Other, such as von Staudt, also used the term collineation.

Whereas Möbius and Chasles used analytic methods without hesitation, Poncelet was strongly against the use of coordinates. But to achieve the generality of analysis he found it necessary to introduce into synthetic geometry imaginary points as well as ideal points. Thus he made a bold attack on imaginary points, with a courage and thoroughness far ahead of his predecessors. For this purpose he built upon Carnot’s idea about correlative figures, and in his _Traité_ he introduced the _Principle of Continuity_, a term coined by himself. Another important principle, which also came in the limelight in the 1820's, is the Principle of duality, and we shall return to both of them below.

Poncelet wrote about imaginary points and lines without having a general definition, although occasionally he gave a rather complicated geometric definition. Anyhow, the principle of continuity paved the way for the introduction of imaginaries in geometry, whose geometrical interpretation sometimes had a quasi-mystical status, exemplified by Steiner’s reference to the "ghosts in the shadowy kingdom of geometry" (cf. Rowe [1989: 212]).

Poncelet announced for the first time one of the basic principles of modern geometry, namely that every circle in the plane passes through two immovable imaginary points, known as the _absolute_ points or _circular_ points at infinity. They are common to all circles in the Euclidean plane. In the same vein he also introduced the _spherical_ circle at infinity, which all spheres in Euclidean 3-space have in common. But it is, indeed, more of a triumph for the analyst, such as Plücker, that he can easily "calculate" these points using complex numbers. Namely, in terms of homogeneous coordinates $x_i$ of the projectively extended plane or space, the two loci of points are typically given by (see (66))

\begin{align}
(i) \quad x_3 &= 0, \quad x_1^2 + x_2^2 = 0, \\
(ii) \quad x_4 &= 0, \quad x_1^2 + x_2^2 + x_3^2 = 0
\end{align}

(66)

Here and in the sequel we shall adhere to the usual meaning of homogeneous coordinates, namely they are only determined up to a common non-zero factor.
Moreover, according to Poncelet, two conics intersect in four points, real or complex, and two real conics with no real common point have two common imaginary chords. As a curiosity, let us also mention the Poncelet–Steiner theorem, postulated by Poncelet and verified by Steiner, that all Euclidean constructions (with ruler and compass) can be carried out with ruler alone plus a single circle and its center.

3.2.2 The Principle of Continuity and the Principle of Duality

The Principle of Continuity has, indeed, a long history, and it was observed or enunciated by various scholars before Poncelet, in one form or another. It dates at least back to Kepler in the early 16th century, and Leibniz stated it as a general law applicable in a broad philosophical sense, see Kleiner [2006]. For example, Boscovich (1711-1787) enunciated and used the principle, and it was used by Monge and Carnot before Poncelet applied it in 1813. Among geometers, however, it was not generally accepted until 1822, when it was formulated by Poncelet in his Traité.

Kline [1972: 844]) contends that the principle of continuity was, in fact, accepted during the 19th century as intuitively clear and therefore had the status of an axiom. It was freely used by geometers and they never deemed that it required a proof. A modern formulation amounts to the statement that if an analytic identity involving a finite number of variables holds for all real values, then it also holds by analytic continuation for all complex values (cf. Bell [1945: 340]). However, although Poncelet could not justify his use of the principle, he refused to present it as a simple consequence of analysis. As a result, he became involved in lengthy controversies with other mathematicians such as A.L. Cauchy (1789-1867), the pioneer of real and complex analysis.

Let us return to the beginning of the 19th century, when Brianchon discovered two closely related theorems. At that time neither Brianchon nor his contemporaries realized the general principle underlying his discovery, namely the Principle of duality. This is one of the cornerstones in projective geometry, and the use of it virtually doubles the geometric “harvest”, at one stroke and without extra labour. In the projective plane the principle relies on the fact that two lines have a unique intersection point, and conversely, two points determine a unique line. In effect, by formally replacing ”point” by ”line” and vice versa, in any theorem involving only points, lines and the incidence relation, one obtains an equally valid statement called the dual theorem, as exemplified by the theorems of Pascal and Brianchon.

The duality principle was first questioned by Brianchon, but as a new discovery the principle was claimed by both Poncelet and Gergonne. The latter had discovered the duality principle by observing the symmetry of the incidence relations between points and lines, thus anticipating the self-duality of projective geometry which is so evident from the modern axiomatic viewpoint. In the plane and the 3-space, the principle will apply to all statements which do not involve metric properties, and the term duality was introduced by Gergonne to denote the relationship between the original and the dual theorem. He was so
obsessed by duality that he modified submitted papers to his Annales, worked out the dual versions of theorems and presented the mutually dual theorems side by side in two columns. It is possible that Gergonne, after all, saw a deep independent geometric principle, but he did nothing to establish a logical basis for it.

In Monge’s Géométrie descriptive (1799), one of the topics is the pole and polar construction in plane geometry, and Monge also gives elegant proofs bringing in the third dimension. On the other hand, Poncelet (1826) was the first to establish the duality principle using this theory, and he exploited the principle to its limit. Depending on a given conic, the polarity construction amounts to a geometric transformation, namely the polar transformation which in a certain way maps points to lines and lines to points and thus realizes the duality principle. This is, indeed, the first example of a geometric transformation taking geometric objects of one type to objects of another type. The correspondence between points and lines was called a correlation by Chasles, a term still used today, but more generally, this is a projective mapping of the projective n-space onto its ”dual” projective n-space.

However, Gergonne had discovered the same principle, but in a different way, and for him the conics behind Poncelet’s geometric resiprocation were of subordinate importance. Now both claimed priority for the discovery of the principle. But the underlying reason for it became a controversial topic and there was a debate going on for many years, involving Poncelet, Gergonne, Plücker, Möbius, Chasles and others, before the principle was finally clarified.

In fact, duality in the projective plane also extends to a duality between plane curves, because the points along a given curve yield by polarity a family of lines which envelop another curve, called the dual curve. It should be noted that the dual of a conic is also a conic, because the conic has two tangents passing through any point off the curve, which by polarity are mapped to two collinear points on the dual curve.

We mention that Plücker used the term reciprocity when he referred to duality. In the second volume of [1828-31] he discussed this topic, based on his new idea of taking lines as the fundamental geometric Elements. He displayed the reciprocity at work in the geometry of conics, treated as envelopes of lines and expressed in terms of homogeneous line coordinates. It is also noteworthy that decades later, Plücker and Lie constructed many types of reciprocities or geometric transformations which generalize the above duality principle. The simple idea, related to the same principle, is that families of points and lines may be intimately related and reciprocally associated with respect to a curve. The latter may be swept out in two ways, (i) either as generated by the motion of a point, or (ii) enveloped by the turning motion of a straight line (the tangent). In particular, a curve will be dual to another curve, in a way preserving tangency, and this leads us to contact transformations, contact geometry, and the geometrical works of Lie in the early 1870’s. In his more general setting the old principle appears as a simple special case, namely a ”linear” version which Lie always referred to as the Poncelet-Gergonne’s reciprocity principle.
3.2.3 The cross-ratio and harmonic sets

As Appolonius must have observed, in Poncelet’s geometry ellipses, parabolas and hyperbolas are congruent since they are obtained by central projection from a circle. In such a generality, one may wonder what properties of figures are invariant under all projective mappings, and hence are independent of magnitude, distance and angle. To this end, consider the following two types of ratios, involving three (resp. four) different collinear points:

\[(P, Q; R) = \frac{PR}{QR}, \quad (P, Q; R, S) = \frac{(P, Q; R)}{(P, Q; S)} = \frac{PR QS}{QR PS}\]

First of all, the 3-point ratio \((P, Q; R)\) is easily seen to be invariant under affine, but not projective transformations. On the other hand, the ancients (e.g. Appolonius, Pappus) knew about the double ratio \((P, Q; R, S)\), namely the cross-ratio, and its invariance under projections. In France, this ratio was also used by Desargues and Pascal in the 17th century, but with the exception of Brianchon, leading French pioneers of projective geometry such as Poncelet and Chasles were probably unaware of this invariant at the time when it appeared in the works of their German counterparts Steiner and Möbius.

Chasles got the idea of cross-ratio through his attempts to understand Euclid’s lost works, and he knew that Pappus also had the idea. Concerning the 17th century scholars, he had seen what La Hire had written, but the works of Desargues had escaped him first. Independent proofs of the invariance of the cross-ratio were given by Möbius (1827), Steiner (1832), and Chasles (1837). Möbius and Chasles referred to the ratio as the *Doppelverhältniss* or *unharmonic ratio* (rapport anharmonique), respectively.

Möbius, Steiner and Chasles also introduced the cross-ratio of four lines of a pencil, that is, lines in a plane meeting at a point \(O\). In fact, this approach is seen to be more basic and it also yields a simple proof of the projective invariance of the cross-ratio in both cases. Suppose a line \(l\) intersects the lines \(p, q, r, s\) from the pencil at the points \(P, Q, R, S\), and let \(pq\) denote the angle between \(p\) and \(q\) at \(O\). Then by simple trigonometry

\[(P, Q; R, S) = \frac{PR QS}{QR PS} = \frac{\sin pq \sin rs}{\sin pr \sin qs}\]

and the right hand side is defined to be the cross-ratio of the lines. On the other hand, the invariance of the cross-ratio under projection also follows from the expression \((68)\). Steiner did not consider negative quantities in his geometry, so for him the angles in \((68)\) are positive, whereas Möbius and Chasles considered segments to be oriented and the cross-ratio to be a signed quantity.

Both Steiner and Chasles used the cross-ratio as a basic tool to characterize the distinguished family of curves of order two, namely conics, and their results were essentially the same. In his *Traité de sections coniques* (1865), Chasles considers the cross-ratio of four lines passing through four given points on a conic and a fifth point. He finds that the cross-ratio has a fixed value as long as the fifth point also lies on the conic, which enableds him to give a projective
definition of conics in terms of the cross-ratio. As a consequence, a homography must map a conic to another conic.

The calculation of the (signed) cross-ratio \( (\lambda) \) involves measurement of (directed) segments along a fixed line. The crucial property is that a projective transformation of the line does not affect the cross-ratio. However, its value \( \lambda \) obviously depends on the order of the four given points; in fact, there are up to six different values and they are \( \neq 0, 1 \), namely

\[
\lambda, 1 - \lambda, \frac{1}{\lambda}, \frac{\lambda - 1}{\lambda}, \frac{\lambda}{\lambda - 1}, \frac{1}{\lambda - 1}.
\]

(69)

The set \( \{P, Q, R, S\} \) is called a *harmonic* quadruple, or the points are said to be in harmonic position, if the cross-ratio is \(-1, 2, \frac{1}{2}\). More specifically and fixing the ratio to be \(-1\), the pairs \((P, Q)\) and \((R, S)\) are said to be harmonically related to each other, or \(S\) is said to be the *harmonic conjugate* of \(R\) with respect to \((P, Q)\).

Poncelet developed his theory of harmonic separation at great length, although the invariance of the cross-ratio itself had somehow escaped him, at least he did not use it. For given points \(P, Q, R\), there are, indeed, simple coordinate free constructions of the 4th harmonic point \(S\), making the cross-ratio equal \(-1\), see for example Gray [2007: 26]. Using analytic geometry, on the other hand, with \(0, q, r, s\) as the coordinates of the points \(P, Q, R, S\) on the real line, a simple calculation yields

\[
(P, Q; R, S) = -1 \implies s = \frac{qr}{2r - q}.
\]

(70)

Von Staudt also made extensive use of harmonic sets; for example, he simply defined projectivity between two lines as a map preserving harmonicity. However, this was a daring step, and critical questions about his approach came up in the 1870's, involving Zeuthen, Klein, Darboux and others.

The invariance of the cross-ratio under projective transformations was a useful fact for geometers who developed projective geometry. Chasles introduced the term homography to describe a transformation between planes which carries points into points and lines into lines. To ensure that such a transformation is also projective, in the sense of Poncelet, he added the extra condition that the cross-ratio must be preserved. Rather surprisingly, however, this was later shown to be a superfluous condition.

By definition \( (\lambda) \), the cross-ratio involves the concept of length of segments of which the ratio is compounded, whereas projective geometry was supposed to be more fundamental than Euclidean geometry and hence should not involve the concept of distance at all. The fundamental criticism of the work of Möbius and Chasles also referred to this fact. However, some geometers realized that direct usage of the general cross-ratio \( (\lambda) \), which involves measurements, could be circumvented by successive construction of 4th harmonic points. Still, geometric purists such as von Staudt found the dependence of the cross-ratio on metric concepts intolerable, and he was the first to advance the study of projective
geometry in a way independent of metric considerations. After his death in 1867, it turned out that his pioneering work had taken him close to a final solution.

3.3 The analyst and the synthesist

Since the beginning of analytic geometry in the 17th century, geometric objects could be studied using tools from algebra and analysis, as an alternative to the synthetic approach in the tradition of Euclid’s *Elements*. The two approaches—analytic contra synthetic—persisted side by side in the development of projective geometry in the 19th century, and thus a distinction was made between analytic or algebraic geometry on the one side, and synthetic geometry on the other side. The *analysts* would gladly use analytic or algebraic techniques from other areas of mathematics, and they would express geometric relations in terms of coordinates and equations. The *synthetics*, on the other hand, inspired by the ancient Appolonius, were the advocates of the purely geometric methods, with intuition as a guide and logic as the instrument for a strict formal reasoning, avoiding measurements and algebra.

Some of the leading geometers were, in fact, clearly favoring one of the two major trends. Prominent synthetics were Carnot, Poncelet, Steiner, von Staudt and Cremona, whereas Möbius, Chasles, Plücker, Cayley and Salmon were heading the analytic trend. To the latter we may also include Grassmann, Hesse and Clebsch, who pioneered the use of algebra and analysis in a modern sense. Gergonne attributed the modern rise of coordinate geometry to Monge, but the latter was also well acquainted with pure geometry, for example through his treatise on descriptive geometry (1799), and wisely he chose to remain neutral with regard to the ensuing controversy between the purists and the analysts.

However, the controversy also caused some bitter rivalry and priority quarrels, and there were upcoming rumours, even with undertones flavored by nationalism. But apparently, Möbius and Chasles were among the more generous and diplomatic ones, and Möbius and von Staudt would stay aloof from discussions not of a purely scientific nature. We should add that Chasles, allegedly an analyst, also defended pure geometry. According to Kline [1972: 850], Chasles thought analytically but presented his results geometrically. He calls this approach the “mixed method”, and says it was used later by others. Presumably, this also includes eclecticists such as Sophus Lie, who thought synthetically but wanted to present his results analytically.

3.3.1 Poncelet, Gergonne, and Chasles

As we have seen, Poncelet himself initiated the purely geometric approach, having returned from the Russian battlefields and soon became convinced that analytical methods are inferior to the synthetic ones. He set himself to undo everything the successors of Descartes had done, and then he would reprove or improve everything. But his dubious continuity principle actually promoted the
analytic trend as well, and in the later years of his career he seemed to be more interested in mechanics and became more of an analyst.

There was also the saying that Poncelet and Steiner occasionally concealed in synthesis what they had discovered by analysis. In fact, for some reason or other, 50 years after they were composed, Poncelet decided to publish his Saratov notes, and they appeared in two volumes in 1862–64, in their original analytic form. In his address given at the Congress of Science and Arts in St. Louis (1904), Darboux "let the analytic cat out of the synthetic bag" by referring to the "unfortunate publication of the Saratov manuscripts", which showed that the principles which served as a foundation for the Traité (1822) were established by the aid of the Cartesian Analysis (cf. Darboux [1904], Bell [1945: 339]).

Gergonne’s mathematical career had been delayed due to his previous military life, but after 1810 his journal and editorial position made him very influential. He was advocating the use of coordinates, and to demonstrate the power of the analytic approach he asked for proofs of classical problems of synthetic geometry, such as the famous Appolonius problem: the construction of a circle tangent to three given ones. His own solution in 1816 became known as Gergonne’s construction, and later many new elegant solutions were also also found, with or without analysis, by Poisson, Plücker, Chasles, Poncelet, Steiner, and others. Gergonne himself wrote about 200 articles, not necessarily on geometry.

Chasles was the only follower of Poncelet of major importance in France, and he devoted his entire life to projective geometry, following up the works of Poncelet and Steiner. He was only four years younger than Poncelet, but from his analytic approach to geometry he seemed to belong to a different scientific epoch. Chasles was also a judicious historian of geometry, but in his first major work Aperçu historique (1837), which is a classic mathematical historiography, he admits that he had neglected the German writers since he did not know the language. His interest in the past also made him the first who fully appreciated the forgotten works of Desargues and Pascal, but surprisingly, Poncelet disliked his appraisal of those 17th century scholars. For some reason or other, the relationship between Poncelet and Chasles was rather hostile.

Like many others from Monge’s school, Chasles was also a professor at École Polytechnique, for about ten years, and in 1846 a chair of higher geometry at Sorbonne was specially created for him. He wrote a very important text showing the power of synthetic geometry, while his general study of geometry, with all the concepts he had introduced, such as cross-ratio, pencils, and involution, appeared in his second major work Traité de géométrie supérieure (1852). It seems that he rediscovered or superseded many of Steiner’s results, with his own analytic approach, but unintentionally since allegedly he did not know Steiner’s papers.

Many classical counting problems are concerned with the enumeration of conics, and some even date back to Appolonius. In 1848 Steiner posed, but wrongly solved the problem of enumerating all conics tangent to five given conics. His answer was 7776, but Chasles developed his theory of characteristics to solve the problem, and he gave the correct answer 3264 in 1864. At this time,
a Danish student named H.G. Zeuthen was in Paris for two years and studied
gometry with Chasles, having received a scholarship in 1863 after graduation
from the University of Copenhagen. His doctoral dissertation (in Copenhagen)
in 1865 was, in fact, on a new method to determine characteristics, and for
the next ten years he worked mostly on enumerative geometry. In 1868 Sophus
Lie met Zeuthen for the first time, a meeting which was very decisive for his
choice of future career as a mathematician. Both Lie and Klein developed a
lasting friendship with Zeuthen, who became Denmark’s first internationally
acknowledged mathematician.

3.3.2 Möbius, Plücker, and Steiner

In Germany, Möbius was the first pioneer of projective geometry. After finishing
his studies in Leipzig he moved to Göttingen in 1813 to study astronomy under
Gauss. In fact, throughout his career he taught mechanics and had the title of
astronomer. In the 1840’s Möbius became a professor in astronomy as well as
the director of the Observatory in Leipzig. But he is best known for his results
in pure mathematics. Möbius had heard about the geometric works of the
French pioneers Poncelet, Gergonne and Chasles and he also acknowledged his
indebtedness to them. But Möbius rather followed his own approach, which was
also largely adopted by Chasles. These two mathematicians, of very different
personality, were regarded as scientific equals and they headed the analytic trend
in projective geometry for many years.

Möbius’s basic work Der barycentrische Calcul (1827) on analytic geometry
became a classic. Here he presents many of his results on affine and projective
geometry, and he discusses projective mappings and other geometric transforma-
ations. But most importantly, he introduces his barycentric coordinates,
which are clearly inspired by the notion of center of mass in classical mechanics.
The key idea is to assign to each point $p$ in the plane a triple of numbers
$(m_0, m_1, m_2)$, depending on a fixed triangle with vertices $p_0, p_1, p_2$, such that
$p$ will be the center of mass of the triangle when the mass $m_i$ is assigned to
the vertex $p_i$. Clearly, the numbers are unique up to scaling and, moreover,
some of them must be negative if $p$ lies outside the triangle. In fact, the $m_i$’s
are homogeneous coordinates applied to projective geometry for the first time.
For analysts such as Plücker, homogeneous coordinates became a flexible tool
in many types of coordinate descriptions, and with Plücker the leadership of
analytic geometry was definitely in Germany.

Like Felix Klein, Plücker also had his basic education in Düsseldorf, but
contrary to Klein’s experience he was inspired by his gymnasium teacher to
study mathematics. The young student moved around to various major univer-
sities such as in Bonn, Heidelberg and Berlin, and after completion of his
doctoral thesis in Marburg in 1823, at the age of 21, he went to Paris. Here
he attended courses in geometry and came under the influence of the great
school of French geometers, in the spirit of its founder Monge, with Poncelet,
Gergonne and Chasles as the leading figures in the development of projective
geometry. Returning to Germany, Plücker submitted in 1824 his habilitation
thesis to the University of Bonn, where he continued his research and further qualified to become a university teacher (extraordinary professor) in 1829. One of his great achievements dates back to this time, namely he proposed the revolutionary idea that the straight line rather than the point may be used as the fundamental geometric element. Thereby he initiated the new discipline called line geometry, a 4-dimensional geometry which represents a new way to study the various geometric configurations in 3-space. His first memoir on the subject was published in the Philosophical Transactions of the Royal Society in London, where many of his later publications also appeared.

Plücker’s first paper, in Gergonne’s Annales (1826), was in fact a synthetic approach to the tangents of conics, a favorite topic at the time. But this also drew him into the crossfire between Gergonne and Poncelet, in particular their dispute over the discovery of duality, a topic which also appealed to Plücker. However, at this time he switched completely to the analytic approach and soon he became the leading expert on both analytic and algebraic geometry. He did not aim at collecting existing results, exploiting existing principles, but modestly building analytic geometry anew and along the lines suggested by Monge. In addition to several published books, Plücker contributed many important papers to various periodicals in Germany, France, England, and Italy, and terms such as "new method" or "new geometry" typically appear in the titles, subtitles, or prefaces. Undoubtedly, no single person contributed more to analytic geometry than Plücker, with regard to both volume and power.

In Germany, however, the relations between Plücker and Steiner were far from being friendly, and they became engaged in an endless feud. Having such an extraordinary geometric intuition, Steiner was said to be the greatest geometer since the legendary Appolonius, and he became the obvious leader of the German school of synthetic geometry. In 1834 a chair of geometry was established for him in at the University of Berlin, which he kept until his death in 1863. His teaching was so influential that courses in projective geometry at many universities have up to present time been based on his outlines, and even some of his "old-fashioned" terminology has survived. On the other hand, in 1834 Plücker had just spent one year as an extraordinary professor in Berlin, and it was perhaps expected that he too would try to make his career here. However, now he quickly decided to leave Berlin and was offered the position as an ordinary professor at Halle, where he stayed for two years before he finally returned to Bonn in 1836.

The story about Jakob Steiner is rather peculiar, indeed. It starts with a Swiss shepherd and farmer son who first went to school at the age of 18, where his great talent for geometry was discovered, and later as a student at German universities he managed to support himself precariously as a tutor. In Berlin he became acquainted with the prosperous engineer and largely self-taught mathematician Leopold Crelle, who had a special ability to spot exceptionally talented young mathematicians and generously offering them his friendship and support. Another of these young men was the Norwegian Niels H. Abel, and together with Steiner they strongly encouraged Crelle in the founding of his mathematical journal in 1826, the first journal in Germany devoted exclusively to mathematics. The first volume came out in 1827, filled with many original
works of Abel and Steiner. Here we find, for example, Steiner’s proof of the formula \((n^3 + 5n)/6 + 1\) for the number of pieces space can be divided into by \(n\) planes. Steiner published altogether 62 papers in the journal.

Crelle’s Journal became an important publication channel for mathematics in general, and also the analyst geometer Plücker submitted papers to this Berliner journal. In Bonn, Plücker was applying his analytic methods and thus promoted an independent development of modern geometry. His success was in a way comparable with that of his great contemporaries Poncelet and Steiner, who continued to cultivate geometry in its purely synthetic form. However, at his chair of geometry in Berlin, Steiner became increasingly more influential and authoritative, and Crelle seemed to favour Steiner over Plücker with regard to their personal conflict. Perhaps somewhat dubitable, but the story goes that Steiner would no longer write in the journal if Plücker did so, and for many years Crelle was in reality forced to deny Plücker access to the journal.

Plücker felt disappointed with the receptions his geometric works were judged in Germany, where the impact of the synthetic approach of Poncelet and Steiner was regarded to be more useful. Perhaps this was the reason why Plücker switched over to natural science when he accepted the chair of physics at Bonn in 1847. For another explanation of the switch, perhaps filling the chair of physics with a mathematician would have been untenable.

It is likely that Plücker’s accomplishments both as a mathematician and physicist were rather unacknowledged in Germany during his lifetime, and certainly English scientists appreciated his work more than his compatriots did. In England, his reputation as a profound geometer flourished, and he was encouraged by Cayley and Sylvester who dominated British pure mathematics in the second half of the 19th century. For example, at a meeting of the British Association for the Advancement of Science in 1848, Sylvester hailed Plücker as “the master” of English mathematicians and, moreover, there was none between Plücker and Descartes when it came to the relation of geometry to analysis (cf. Gray [2007]). In 1863 Steiner died, and a couple of years later Plücker again turned his attention to mathematics. We shall return to some of his mathematical accomplishments later.

### 3.3.3 Steiner, von Staudt, and Cremona

Steiner was the first of the German school of geometry who took over the French ideas, following the strict synthetic approach of Poncelet. He was undoubtedly the most extreme synthesist at all, he was said to be hating analysis and he would teach geometry without using figures. But he came to impressive geometric results, especially in his younger days, and with his first book *Systematische Entwickelungen* (1832) he made a grand effort to unify classical geometry, based on a new conception of projective geometry and a new approach to conic sections. Thus he would seek the common roots and uncover the fundamental properties of the classical geometry. Much of his later works aimed at encompassing more recent results as well, due to himself or others, into his synthetic geometric framework. Quite often, therefore, Steiner was able to con-
struct purely geometric proofs of results first discovered in analytic or algebraic geometry.

A principal idea of Steiner is to use the simple projective concepts such as points, lines, planes, pencils of lines or planes etc. to build up more complicated structures, using the principle of duality and the cross-ratio \[68\] as basic tools. Steiner's approach to duality is in the tradition of Monge, Brianchon and Poncelet. First of all, he establishes a powerful theorem on conic sections, which in fact yields a new definition of these curves. Namely, starting from two projectively related lines, the lines between corresponding points on the two lines envelop a "conic" which is tangent to the two lines. Moreover, from the proof of the theorem it is clear that the "conics" are projections of a circle and hence are actually conics. Now, from the pole and polar theory he can construct the polar map and realize the duality principle. In particular, starting with two projectively related pencils of lines, the locus of all intersections of corresponding lines will be a conic passing through the centers of the pencils. Steiner's synthetic theory of conic sections was one of his chief accomplishment in projective geometry.

Steiner investigated also algebraic curves and surfaces with his synthetic approach. Let us first recall some basic properties of quadrics. A quadric has the property that all its plane sections are conics, and moreover, if it contains a line, then it is a ruled surface and hence contains infinitely many lines. In fact, we know which quadrics are ruled (see Section 2.3.1). Now, Steiner was looking for a surface of degree \(> 2\) with the property that its plane sections are still conics, for example two conics. On a trip to Rome in 1844 Steiner discovered his Roman surface, which is such a surface and it has degree 4. In fact, it has the peculiar property that each tangent plane cuts the surface in two conics and therefore a double infinity of conics is lying on it (see Gray [2007: 270]).

Turning to (real) cubic surfaces, Cayley first showed that the number of straight lines on the surface must be finite, which enabled Salmon to prove there are exactly 27 lines in general, some of which may be complex. This is the content of the Cayley-Salmon theorem, published in 1849. Even 20 years later Cayley wrote on the topic of cubic surfaces, see his memoir [1869b], and this may also have inspired Klein to his own studies in 1872-73. Both Sylvester and Salmon had given a pure geometric construction of a nonsingular cubic surface and its 27 lines, but without any indication of how to make a model (thread or plaster), which was, in fact, constructed for the first time by L.C. Wiener in 1869.

In 1856 Steiner wrote an important paper which was the basis for his purely geometric approach to cubic surfaces and, for example, now he gave a synthetic proof of the Cayley-Salmon theorem. However, he also stated many other results without proofs, which were in fact supplied 7-10 years later by Cremona and Sturm. They were highly regarded geometers in the generation after Steiner and Staudt, working in the same spirit of pure geometry. Sturm studied the cubic surface problems in his dissertation (Breslau, 1863) and continued with this work, whereas Cremona during his years 1860-67 at Bologna did important work on transformations of plane curves, birational transformations, and wrote
a long memoir on cubic surfaces which appeared in Crelle’s Journal in 1868. As was perhaps expected, in 1866 Cremona and Sturm were jointly awarded the Steiner-Preis of the Berlin Academy. Cremona’s transformation theory had many links to Lie’s work in the 1870’s.

Von Staudt was the second major synthesist in Germany, a contemporary of Steiner but his opposite in many respects. He studied under Gauss in Göttingen during 1818-22, and based on his work on the determination of the orbit of a comet he received his doctorate from the University of Erlangen in 1822. In 1835 he was appointed to this university and stayed there to the end of his life in 1867. His lifestyle was modest, he communicated hardly with people but continued peacefully with his own rigorous solitary research.

In his 1822 treatise, Poncelet had pointed out the distinction between projective and metric properties of figures, namely the projective properties are logically more fundamental, but it was von Staudt, rather than Poncelet himself or Steiner, who began to build up projective geometry as a subject independent of distance and hence without reference to length or angle size. With his book Geometrie der Lage (1847), supplemented by three booklets (1856-60), von Staudt presented the essence of his project, aimed at introducing an analogue of length on a projective basis. For information on the publication history of this peculiar book, see Hartshorne [2008].

A major problem von Staudt was facing was how to define the cross-ratio, the fundamental invariant of projective geometry, in an intrinsic way. Its importance was also made clear in the works of Möbius, Steiner and Chasles, but the cross-ratio involves four lengths and length is a metric concept. Consequently, numbers must be associated to the points of a line, but the numbers and their algebraic operations must be defined purely geometrically. In the opposite direction, this would also pave the way for the idea of having non-metric geometry on which a notion of distance can be defined. In fact, the idea of a projective theory of metric geometry was first elaborated by Cayley in his Sixth Memoir [1858], see below.

From a modern viewpoint, it is a general coordinatization problem to represent a geometric lattice by “closed subsets” of some algebraic structure, and von Staudt was the first to realize the possibility of such a coordinatization in projective geometry (cf. Lashkhi [1995]). It was for this purpose he created his “Wurf algebra” or the “algebra of throws”, where a “throw” is the geometric construction which attaches a symbol (number) to any point. Starting from three points on a line $l$, von Staudt constructs a harmonic chain of points on $l$, by first constructing the 4th harmonic point and then iterating the construction indefinitely by choosing at each step three points from the previously constructed points. Similarly, starting from four coplanar points, with no three on the same line, he constructs a harmonic net (or Möbius net) of points in the plane. It is clear, however, that the successive 4th harmonic construction was essentially the same as the ancient construction of commensurable sets, so that the transition from the harmonic chain (or net) to the whole line (or plane) would be similar to proving that things valid for commensurable ratios hold as well for all ratios.
The coordinates (symbols) of the harmonic chain on $l$ can be taken to be rational numbers, with their algebraic operations given by the appropriate geometric constructions. Thus by identifying the points $P, Q, R, S$ in (68) with their coordinates $\overline{p}, \overline{q}, \overline{r}, \overline{s}$, their cross-ratio can be calculated in the usual way as

$$\frac{(\overline{p} - \overline{q})(\overline{s} - \overline{r})}{(\overline{r} - \overline{q})(\overline{s} - \overline{p})}$$

We refer to Kline [1972: 850], which also illustrates the construction of the point labelled 2 on the line $l$, starting from chosen points labelled $0, 1, \infty$, where $\infty$ is the point at infinity. Since $(0, 1; \infty, 2) = 1/2$, this amounts, indeed, to a construction of a 4th harmonic point, see (69).

Möbius and von Staudt defined a general collineation to be a one-to-one transformation which takes points to points, lines to lines (and planes to planes, in dimension 3). Then they asked the question whether these are necessarily "projective" transformations in the appropriate sense. In the plane, Möbius assumed continuity, which enabled him to show the transformation is a composition of perspectivities and hence is projective in the sense of Poncelet; in particular it preserves the cross-ratio. On the other hand, von Staudt had no continuity assumption, but he showed harmonicity is preserved, which would enable him to conclude the cross-ratio is preserved as well. But how could this be true? What about the lowest dimensional case, namely transformations on a single line?

Most likely, the work of von Staudt was poorly understood and little appreciated during his lifetime, although Reye lectured on his approach and even published the lectures in 1866. It seems to be Klein, with his interest in the foundations of geometry, who first focused attention on von Staudt and saw his work in a new light in the early 1870’s. Klein learned from his friend Otto Stolz, who was close to von Staudt’s spirit and was well acquainted with the new geometric ideas of both von Staudt and Lobachevsky. Soon it became clear that von Staudt had, in fact, made implicit use of the Euclidean parallel postulate, which is a blemish since parallelism is not a projective invariant. But the problem was not deeply rooted and Klein was able to remove it, as explained in Klein’s second paper [1872c] on non-Euclidean geometry.

However, in [1872c] Klein was also alerted by a more serious gap in von Staudt’s work, namely he did not really show that a harmonic chain fills the line “densely”, penetrating any small interval. This was needed to ensure that a projective mapping of a line was uniquely determined by its restriction to the harmonic chain. More precisely, the key result asserted by von Staudt amounts to a statement nowadays referred to as the Fundamental Theorem of projective geometry:

*A projective transformation of a line to itself is uniquely determined by its values at three different points. Alternatively, if the transformation fixes three points, then it is the identity transformation.*

The transformations of the line considered by von Staudt were those preserving harmonicity. Therefore, starting from three given points, the image of
the 4th harmonic point is already determined, and by repeated usage of the 4th harmonic point construction the transformation is determined on the whole harmonic chain. From this he simply concluded the transformation was determined on the whole line.

We recall that Eudoxus had tacitly used a basic property of line segments, referred to as the Archimedian axiom since the 19th century, which granted the possibility of fencing an incommensurable ratio by commensurable ones, and the validity of this step was in fact clarified in 1872 by the Cantor–Dedekind axiom. Now Klein insisted in his article that the "density" property of the harmonic chains might as well be postulated as a similar continuity axiom. Klein’s article drew immediate responses from Cantor, Lüroth, and Zeuthen, which Klein described and commented in his subsequent paper [1874a].

Of particular interest was the answer from Zeuthen, which included a proof of the fact that a harmonic chain is, indeed, a dense set of points on the projective line. Still, however, Klein insisted that a continuity assumption was needed in von Staudt’s definition of a collineation, for the same reason that a function defined on the rational numbers may not be extendible to a continuous function on the whole real axis. But the topological concept of continuity was not so well understood in the 1870’s, and earlier it had only been handled in a confusing manner.

Finally, in 1880 Klein received a letter from Darboux with his proof that no extra condition of continuity was, after all, needed in von Staudt’s definition of collineations. The situation was similar to other theorems which were known to be true without an explicit continuity condition, such as for example transformations of the plane which send all circles and lines to themselves. Möbius had assumed a continuity condition to conclude that these transformations are the same as inversions ([63] and their compositions, but the extra continuity assumption is in fact superfluous. We refer to Klein [1880] for Darboux’s continuity argument, and to Gray [2007: 341–42]) for a readable account of the above events, including the arguments of Zeuthen and Darboux.

Briefly, it may be said that the success of Steiner and von Staudt was due to their unification and purification of geometry, respectively. In the 19th century Steiner was usually ranked higher than his successor von Staudt. But since the early 20th century Steiner seems to have fallen below his successor, whose originality and depth have been hailed by many writers starting with Klein [1928] and Coolidge [1934]. However, some new viewpoints are presented in the recent paper Blåsjö [2009].

Cremona studied mathematics under Brioschi and others in his hometown Pavia, where he took a doctorate in civil engineering in 1853. But due to his previous military activity against the ruling Austrian government he was prevented from obtaining a position, so for the following three years he made his living as a private tutor of mathematics. The situation improved in 1856, with his second paper published, and in early 1857 he was secured a full teaching position at the scientific high school in the capital Cremona of the Lombardy region. He wrote a number of papers in the following years, examining curves with projective methods which later became characteristic for his more impor-
tant mathematical works. In 1860 he was appointed professor at Bologna, in fact, by Victor Emmanuel II who was proclaimed king of the recently united Italy in 1861.

Cremona left Bologna in 1867, when he was appointed to the Polytechnic Institute of Milan, on Brioschi’s recommendation, and received the title of Professor in 1872. But the next year he moved to Rome, as director as well as professor of graphic statics at the newly established Polytechnic School of Engineering. The years he spent in Milan were, however, the most creative time of his mathematical career. Projective geometry was a theme which was present in almost all his works, and it culminated with his monumental book [1873] on the Elements of projective geometry. From now on his administrative and teaching duties began to put an effective end to his research career. Although he was appointed to the chair of higher mathematics in Rome in 1877, the political pressure on him finally persuaded him to serve the new Italian State henceforth at the political level.

The method of graphic statics, founded by C. Cullman at Zürich around 1860, applies projective geometry, rather than analytic methods, to the design and analysis of stationary mechanical frames or equilibrium of systems of forces. Cremona gave important contributions to this topic, and improved previous work of Maxwell. For example, Maxwell’s notion of reciprocal figures, which appeared in an engineering journal (1867), was interpreted by Cremona (1872) as duality in projective 3-space. To begin with, Cremona’s geometric views were largely influenced by Chasles, but later he became more attracted to the synthetic approach of Steiner and von Staudt.

Cremona [1873] first reviews the developments of projective geometry and the contributions of the various pioneers. Then, based on his understanding of its present state and the previous efforts to establish projective geometry as the most fundamental geometry, Cremona aims at pulling it all together anew, in purely projective terms without any resort to Euclidean geometry. Much of the basic work was, after all, already provided by Chasles, Steiner, and von Staudt, but the great merit of Cremona is that he finally succeeds in presenting projective geometry as an independent geometric system, released from the embarrassment of its Euclidean origins (cf. Gray [2007: 245]).

3.4 Some basic analytic developments

In the classical description of projective spaces \( P^n \), say in dimensions \( n = 1, 2, 3 \), there is always the distinction between ordinary and ideal points, sometimes called finite points and points at infinity, respectively. This is because \( P^n \) was regarded as an extension of the Euclidean \( n \)-space \( E^n \), or rather the affine \( n \)-space \( A^n \) (since metric properties are non-projective). The set \( A^n \) consists of the ordinary points, whereas the points at infinity, as it turned out, actually constitute a projective space of one dimension lower. For example, the ordinary line \( A^1 \) extended by an ideal point \( \infty \) becomes the projective line \( P^1 \), so by
writing \( P^0 = \{\infty\} \),
\[
P^1 = A^1 \cup P^0, \quad P^2 = A^2 \cup P^1, \quad P^3 = A^3 \cup P^2
\] (71)

While the synthesists were developing projective geometry, the analysts treated the same subject as coordinate geometry and pursued their own methods. To be more specific, let us consider the projective plane \( P^2 \), where \( xy \)-coordinates are introduced in the ordinary plane \( A^2 \) and thus we identify it with the Cartesian plane \( \mathbb{R}^2 \). In this \( xy \)-plane model, the affine transformations have the simple algebraic expressions (65), whereas a general projective transformation, the joint effect of both parallel and central projections, expresses as
\[
(x, y) \mapsto \left( \frac{a_{21} + a_{22}x + a_{23}y}{a_{11} + a_{12}x + a_{13}y}, \frac{a_{31} + a_{32}x + a_{33}y}{a_{11} + a_{12}x + a_{13}y} \right), \quad \det(a_{ij}) \neq 0 \] (72)

Therefore, unless the transformation is affine there are two lines in the \( xy \)-plane
\[
l_A : a_{11} + a_{12}x + a_{13}y = 0, \quad l_B : b_{11} + b_{12}x + b_{13}y = 0
\] (73)
such that the mapping (72) is undefined along \( l_A \) and its image is the complement of \( l_B \), which remains to be determined. This also illustrates the role of the extended plane \( P^2 \) and its ideal line \( l_\infty = P^1 \) in (71). Namely, the mapping (72) extends to a transformation which maps the line \( l_A \) to \( l_\infty \) and \( l_\infty \) to \( l_B \). In effect, we shall obtain an invertible mapping of \( P^2 \), whose expression in terms of homogeneous coordinates becomes, indeed, very simple, see (81) below.

This classical picture is somewhat misleading, however, because the homogeneous structure of \( P^n \) gives no preference to any specific hyperplane \( P^{n-1} \); in fact, any hyperplane can be mapped to any other one by a suitable projective transformation. Therefore, one may choose a hyperplane \( \simeq P^{n-1} \) in \( P^n \) and refer to it as the points at infinity. Then the complementary set identifies with an affine subspace \( A^{n-1} \), and we recover the classical picture (71). This aspect of projective spaces became better understood with the introduction of homogeneous coordinates. Moreover, parallel with the developments of linear algebra and group theory came the modern vector space model of a projective space, which has largely reduced projective geometry to the setting of algebraic geometry. Below we shall explain the vector space model, see (78)-(81).

Projective geometry in higher dimensions \( n > 3 \) was first introduced around 1850, in particular in its analytic form as created by Möbius and Plücker. They also introduced complex coordinates, extending \( P^n \) to the complex projective \( n \)-space \( \mathbb{C}P^n \), an extension of the Cartesian space \( \mathbb{C}^n \) which is completely similar to the real case extension of \( \mathbb{R}^n = A^n \) to \( P^n \).

3.4.1 Algebraization and homogeneous coordinates

The modern rise of coordinate geometry was due to Monge, according to Gergonne, and a major difficulty was the search for the "best" coordinate system,
with regard to a specific type of problems. To demonstrate the analytic approach Gergonne asked for proofs of classical problems in synthetic geometry, such as the famous Appolonius problem: the construction of a circle tangent to three given ones. His own solution became known as Gergonne’s construction, and later many new elegant solutions, analytic or purely synthetic, were also found by Poisson, Plücker, Chasles, Poncelet, Steiner, and others.

The analytic approach to projective geometry became, in fact, a major geometric discipline during the first decades of the 19th century. Up to 1830 or so, the coordinates being used had typical geometric interpretations such as length, angle, area, or volume, but this pattern was broken and a new era began with the introduction of homogeneous coordinates. This also opened the gate to algebraic geometry and was a major step towards the complete arithmetization of the geometry, as seen from a modern viewpoint. It is remarkable that the idea of homogeneous coordinates was discovered almost simultaneously around 1827 and independently by four geometers, namely Bobillier, Möbius, Feuerbach, and Plücker. However, Bobillier and Feuerbach really did not have the coordinate concept in mind. Let us have a closer look at some of these early developments.

The early stage of the algebraization was closely related to the use of abridged notation, a method which, in fact, seeks to avoid the use of coordinates, elimination of variables, and related messy calculations. The idea is to represent algebraic equations or their loci by single symbols \( C, C' \) etc. or \( C = 0, C' = 0 \) etc., and apply algebraic operations to the symbols themselves rather than the coordinates. For example, it follows from the identity

\[
(C_1 - C_2) + (C_2 - C_3) + (C_3 - C_1) = 0
\]

that the three angular bisectors of a triangle are concurrent, and similarly the three chords each of which is common to two of three given circles, are concurrent. One verifies this by writing the lines (resp. circles) on the normal form \( C = px + qy + r, \) with \( p^2 + q^2 = 1 \) (resp. \( C = x^2 + y^2 + ax + by + c = 0 \)). This simple proof is also found in many textbooks nowadays.

Applications of this kind, pioneered by Gergonne and Lamé in 1816, began to appear in Gergonne’s Annales and other journals, as systems of curves or surfaces became a subject of study. Consider, for example, the widely used Gergonne’s ”lambdalization”, which for given curves (resp. surfaces) \( E = 0, E' = 0 \) of degree \( n \) constructs the one-parameter family (called pencil)

\[
E + \lambda E' = 0
\]

which represents curves (resp. surfaces) of degree \( n \) passing through the intersection of the two given loci.

The foremost early user of abridged notation was E. Bobillier (1798-1840), who explained the method and published extensive applications in 1827-28. As an illustrating example, given the three edges \( C_i = 0 \) of a triangle, he considered the two families of equations

\[
\begin{align*}
(i) \quad aC_1 + bC_2 + cC_3 &= 0, \\
(ii) \quad aC_1C_2 + bC_2C_3 + cC_3C_1 &= 0
\end{align*}
\]
with variable parameters $a, b, c$, which in case (i) yield all lines in the plane, and in case (ii) yield all conics circumscribed about the triangle. In particular, in case (i) the parameters $a, b, c$ serve as homogeneous line coordinates in the plane since their mutual ratios determine a unique line. Recall that Möbius also studied figures by relating them to a given triangle, namely he introduced barycentric coordinates relative to the triangle, but these are homogeneous point coordinates rather than line coordinates.

Next, let us see how K.W. Feuerbach (1800-1834) came up with a system of homogeneous coordinates, not by using the abridged notation but by the elegant analytic methods of Lagrange in solid geometry. Feuerbach’s investigations of tetrahedrons (triangular pyramids) in 1827 are similar to what Möbius did in the plane the same year, but his approach was purely geometrical rather than mechanical. He considered a fixed plane $\Pi$ and five generic points $p_j$ in 3-space, the five tetrahedra $T_i$ with the vertices $p_j$ for $j \neq i$, and the five distances $d_i$ from $p_i$ to $\Pi$. Then he observed the relation

$$\sum_{i=1}^{5} d_i \text{Vol}(T_i) = 0$$

where the distances and volumes are signed quantities. Next, by letting the point $p_5$ be a variable point $p$ he introduced the quadruple $(\tau_1, \tau_2, \tau_3, \tau_4)$ depending on $p$, where $\tau_i = \text{Vol}(T_i)$, which yield a kind of homogeneous coordinates for $p$, closely related to the barycentric coordinates of Möbius. However, Feuerbach was not concerned with new coordinates but rather with new theorems in “tetranometry”; he aimed at expressing all geometric invariants (around 45, say) of a tetrahedron in terms of six basic ones, such as its edges.

Finally, we turn to Plücker, whose first major work was the two volumes [1828-1831] on the development of analytic geometry, based on his lectures at Bonn. Apparently unaware of the works of Lamé, Gergonne and Bobillier, Plücker had himself discovered several important aspects of the abridged notation, which he presented in the first volume and elevated the method to the status of a principle. Now he became the real expert and made the widest and most effective use of these ideas. One also speaks of Plücker’s abridged notation, namely when Gergonne’s letter $\lambda$ is replaced by Plücker’s $\mu$ in the pencil (75). For example, an elegant proof of Pascal’s hexagon theorem follows by analysis of the pencil (75) with cubic polynomials $E = pqr, E' = p'q'r'$, where $p, q, ..., r'$ are the six lines of the hexagon suitably partitioned into two triples as indicated.

The first homogeneous point coordinates proposed by Plücker are the so-called trilinear coordinates. Namely, for a fixed reference triangle, a point $p$ is assigned the ordered triple $(d_1, d_2, d_3)$ of signed distances from $p$ to the side lines. Fixed multiples $k_i d_i$ of the functions $d_i$ are still referred to as trilinear coordinates, and the barycentric coordinates of Möbius are recovered as a special case. Henceforth, we shall use the notation $(d_1 : d_2 : d_3)$ to stress the homogeneity of the coordinates, that is, they are determined modulo a common multiple. But Plücker also took the crucial step towards the complete algebraization of projective geometry by introducing homogeneous coordinates $(x_1 : x_2 : x_3)$ without
any geometric interpretation, namely as the result of applying any invertible linear substitution to the above coordinates $d_i$.

The projective lines $\simeq P^1$ in $P^2$ are given by linear equations $ax_1 + bx_2 + cx_3 = 0$, and any of them may serve as the line at infinity $l_\infty$. Henceforth, we shall assume homogeneous coordinates in the plane chosen so that $l_\infty$ is the line $x_3 = 0$:

$$P^1 = l_\infty : x_3 = 0, \quad (x_1 : x_2 : 0) \leftrightarrow (x_1 : x_2) \quad (76)$$

$$A_2 : x_3 \neq 0, \quad (x_1 : x_2 : x_3) = \left(\frac{x_1}{x_3} : \frac{x_2}{x_3} : 1\right) \leftrightarrow \left(\frac{x_1}{x_3}, \frac{x_2}{x_3}\right) = (x, y) \quad (77)$$

and thus the (ordinary) plane $A_2$ is identified with the $xy$-plane. For example, in Möbius’s barycentric coordinates $(m_1 : m_2 : m_3)$, the equation of $l_\infty$ is $m_1 + m_2 + m_3 = 0$, so by setting $x_1 = m_1, x_2 = m_2, x_3 = m_1 + m_2 + m_3$, we obtain new coordinates in terms of which $P^1$ and $A_2$ are characterized as in (76) and (77).

Projective coordinates are just homogeneous coordinates naturally rising from the vector space model of the projective space, as follows. To a given $(n+1)$-dimensional vector space $V \simeq \mathbb{R}^{n+1}$ we can associate a projective space $P^n = P(V)$ whose points (resp. lines) are defined to be the 1-dimensional (resp. 2-dimensional) subspaces of $V$, and more generally a $k$-plane $\simeq P^k$ in $P^n$ is a $(k+1)$-dimensional subspace of $V$. For example, the statement that two points $p_1, p_2$ in $P^n$ span a unique line $l \simeq P^1$ is the statement that two distinct central lines in $V$ span a unique 2-dimensional subspace $\Lambda$. In particular, a vector $v \neq 0$ and any of its nonzero multiples represent a single point $p = [v]$ in $P^n$.

By choosing a basis $v_1, v_2, \ldots, v_{n+1}$ for $V$, every nonzero vector $v = \sum x_i v_i$ is assigned a nonzero coordinate (row) vector

$$x = (x_1, x_2, \ldots, x_{n+1}). \quad (78)$$

This procedure provides $P^n$ with projective coordinate $x_i$, and we shall write

$$p = (x_1 : x_2 : x_3 : \ldots : x_{n+1}) \in P^n \quad (79)$$

Conversely, a coordinate polyhedron in $P^n$ consists of $n+1$ vertices $p_i = [v_i], i = 1, \ldots, n+1$, which are the image of some basis of $V$, and thus the polyhedron gives rise to a projective coordinate system as above.

For example, in projective 3-space, four points $p_1, \ldots, p_4$ in general position span a coordinate tetrahedron. With respect to the associated coordinate system the vertices $p_i$ of the tetrahedron play the role of the "standard basis"

$$p_1 = (1 : 0 : 0 : 0), \quad p_2 = (0 : 1 : 0 : 0), \quad p_3 = (0 : 0 : 1 : 0), \quad p_4 = (0 : 0 : 0 : 1) \quad (80)$$

and the opposite face of $p_i$ is the plane $(x_i = 0)$. Then, if we return to the classical picture (71) and identifies $A^3 \simeq \mathbb{R}^3$ using Cartesian coordinates $(x, y, z)$ as indicated in (77), the four points in (80) have the following interpretation:
$p_4$ is the origin of $\mathbb{R}^3$, and $p_1, p_2, p_3$ are the points at infinity representing the directions of the three coordinate axes of $\mathbb{R}^3$.

In homogeneous coordinates, projective transformations of $P^n$ are simply linear substitutions, namely induced from linear transformations $V \rightarrow V$ in the above vector space model of $P^n$. Therefore, each transformation $\varphi$ has an invertible matrix $A = (a_{ij})$ of dimension $n+1$, unique up to a non-zero multiple, so that

$$\varphi = [A] : (x_1 : x_2 : \ldots : x_{n+1}) \mapsto (x'_1 : x'_2 : \ldots : x'_{n+1}), \quad x'_i = \sum_{i,j=1}^{n+1} a_{ij} x_j$$

(81)

In the language of groups, the set of these matrices $A$ is the matrix group usually denoted by $GL(n+1)$, and the non-zero multiples $kI$ of the identity $I$ is the subgroup $Z$ of all matrices $A$ which yield the identity transformation $[A] = Id$ of $P^n$. Therefore, the full group of projective transformations can be expressed as the quotient group

$$G(P^n) = PGL(n+1) = GL(n+1)/Z$$

(82)

In the case $n = 2$, by restricting the transformation (81) to the affine sub-space $A^2 \simeq \mathbb{R}^2$ with coordinates as in (77), we calculate

$$(x, y) \rightarrow (x : y : 1) \rightarrow (x'_1 : x'_2 : x'_3) = \left(\frac{x'}{x_3} : \frac{x'}{x_3} : 1\right) \rightarrow (x', y')$$

(83)

where $(x', y')$ are exactly the rational expressions in (72). Moreover, the line $l_A$ in (73) expresses as $x'_3 = 0$ and is therefore mapped to the line $l_\infty$ at infinity, whereas the line $l_B$ in (73) is mapped to $l_\infty$ by the inverse transformation, whose matrix is $B = A^{-1}$.

In the projective plane $P^2$ with homogeneous coordinates $x_i$, a second order curve (or conic) is the zero set of a quadratic form

$$F(x, x) = x A x^T = \sum_{i,j=1}^{3} a_{ij} x_i x_j = 0$$

(84)

where $A = (a_{ij})$ is a symmetric matrix of dimension 3. Since all matrices of this kind constitute a 6-dimensional vector space $W$, and matrices differing by a nonzero scalar represent the same conic, the set of conics naturally identifies with the projective 5-space $P^5 = P(W)$.

### 3.4.2 Möbius and his approach to duality

Duality in the projective plane say, is realized by constructing a polar transformation $\pi$ which is a one-to-one correspondence between points (pole) and lines (polar) respecting the incidence relation. For this purpose, pioneers such as Poncelet made use of a chosen conic (84), and their construction is purely geometric. For example, let us choose a circle and determine the polar of a point...
There are exactly two tangents of the circle passing through \( p \), say tangents at the points \( p_1 \) and \( p_2 \), and the polar \( \pi(p) \) is the line joining \( p_1 \) and \( p_2 \).

The approach discovered independently by Möbius (1827) and Plücker (1829), establishes the analytic counterpart of the geometric principle of duality, in a new and simple algebraic way, as follows. When a line \( l \) in \( \mathbb{P}^2 \) is given by a homogeneous linear equation

\[
\mathbf{a} \cdot \mathbf{x} = a_1 x_1 + a_2 x_2 + a_3 x_3 = 0,
\]

the three coefficients \( a_i \) may be regarded as homogeneous line coordinates for \( l \), and we shall write \([a_1 : a_2 : a_3]\) to distinguish them from point coordinates. Then the equation (85) also provides a one-to-one correspondence, namely a duality between lines and points

\[
[a_1 : a_2 : a_3] \leftrightarrow (a_1 : a_2 : a_3).
\]

On the one hand, the equation (85) determines all points \( p = (x_1 : x_2 : x_3) \) on the line \( l \), but on the other hand, it also determines the pencil of lines \([a_1 : a_2 : a_3]\) passing through a fixed point \( p \). We mention that modern mathematicians rather regard \( a_1, a_2, a_3 \) as homogeneous coordinates of another projective plane, namely the dual \( \mathbb{P}^2^* \) of our plane \( \mathbb{P}^2 \). Anyhow, we shall regard \( \mathbf{a} = (a_1, a_2, a_3) \) as the associated coordinate vector (78) of a point in a projective plane.

More generally, let us choose a nondegenerate conic (84), namely the matrix \( A \) is invertible. If points and lines are represented by column vectors \( \mathbf{x}, \mathbf{a}, \) respectively, the corresponding polar transformation can be expressed neatly by matrix multiplication as

\[
\pi: \mathbf{x} \rightarrow A \mathbf{x} = \mathbf{a}, \quad \text{or} \quad \mathbf{a} \rightarrow A^{-1} \mathbf{a} = \mathbf{x}
\]

(87)

Two points \( p, q \) are said to be conjugate if one point lies on the polar of the other, and this symmetric relation expresses as \( A \mathbf{x} \cdot \mathbf{y} = 0 \) in terms of the coordinate vectors \( \mathbf{x}, \mathbf{y} \) of the points. In particular, the self-conjugate points \( p \) are those lying on the conic (84). In view of this, however, the polar map (86) is rather remarkable, since by (85) and (86) the conjugacy between \( p \) and \( q \) simply expresses as \( \mathbf{x} \cdot \mathbf{y} = 0 \). This amounts to using the matrix \( A = I \) associated with the conic

\[
x_1^2 + x_2^2 + x_3^2 = 0
\]

(88)

whose real locus is the empty set. Thus there are no self-conjugate points at all, but years later the analyst Möbius identified the locus with an imaginary circle in the complex extension \( \mathbb{C}\mathbb{P}^2 \) of the projective plane.

We have seen how the duality principle can be realized by constructing polar maps using symmetric invertible matrices \( A \), as in (87), and this works also in higher dimensions. The geometric construction makes use of the associated quadric hypersurface \( A \mathbf{x} \cdot \mathbf{x} = 0 \) in \( \mathbb{P}^n \). In his study of geometric mechanics Möbius made another remarkable discovery, namely that a skew-symmetric matrix \( A \) can also be used to construct a duality, via the equation \( A \mathbf{x} \cdot \mathbf{y} = 0 \) as
before. However, in this case there is no underlying quadric, since $Ax \cdot x = 0$ holds for all $x$ and so all points are self-conjugate. On the other hand, invertible skew-symmetric matrices exist only in even dimensions, $n + 1 = 2k$, so there will be a duality of this kind in $P^3$ but not in $P^2$. Thus, Möbius also settled a dispute between Gergonne and Poncelet on the nature of dualities in the projective plane; now it turned out that all of them are actually associated with conics and symmetric matrices as in [54] (cf. also Gray [2007: 151]).

Plücker presented his work on reciprocity and homogeneous (line) coordinates in the second volume of [1828-31]. Around 1830 he was maybe not aware of the work of Möbius, but proper credit must be given to Möbius, whose geometric results were largely overtaken and extended by Plücker in the 1830’s. This is also the decade that Plücker devoted primarily to an indepth study of algebraic curves in the plane, the major topic of his books [1835], [1839].

3.4.3 Classical algebraic geometry and Plücker’s formulae for plane curves

Modern algebraic geometry arose from the classical studies of curves and surfaces in the Cartesian plane or space, defined by algebraic (polynomial) equations, and besides classification of these objects a central topic has been the behavior of their intersections. For this purpose many concepts and numerical invariants have been gradually introduced to describe and distinguish the various types and their possible singularities. The following review of this topic is largely centered around the achievements of Plücker during the 1830’s, with Boyer [1956] and Gray [2007] as our major references.

Important properties common to curves in the plane were discovered by Descartes and Newton, but the scientific foundations of the theory of plane curves are due to Euler and G. Cramer (1704–1752) around 1750. A classification of the algebraic curves was attempted by Euler, who distinguished them from the transcendental ones, and Cramer’s initial study of their singularities was continued in a modern geometric sense by Poncelet. A very basic result from the 18th century is the celebrated Bézout’s theorem, claiming that the number of common points to two plane algebraic curves with no common component is equal to the product of their degrees. In fact, special cases such as the intersection of lines, conics, and cubics, were known already in the 17th century.

E. Bézout (1730–1783) published his theorem in 1776, based upon heuristic reasoning and cumbersome calculations, but precise conditions for the theorem to hold were not formulated. In general, for the validity of Bézout’s theorem and its generalization to higher dimensions, imaginary points and points at infinity must also be considered. But the most delicate part is the assigning of proper intersection multiplicities. We shall not describe this procedure, but we mention that a common tangent point of two plane algebraic curves has intersection multiplicity at least two.

In the sequel we shall recall some of the classical theory of plane curves, due to Plücker and others. Assuming projective coordinates $(x : y : z)$, say $z = 0$ is the line at infinity, an algebraic curve $C_n$ of degree (or order) $n$ is the locus of
a polynomial equation

\[ C_n : f(x, y, z) = \sum_{i+j+k=n} q_{i,j,k} x^i y^j z^k = 0 \]  \quad (89)

and so its equation in the Cartesian \( xy \)-plane is found by setting \( z = 1 \). Clearly, the latter equation still has degree \( n \), unless \( f(x, y, z) \) is reducible with \( z \) as a factor. The abridged notation \( C_n \) for \( n \)th order curves is rather typical in the classical literature, and it is also used in Klein’s letters to Lie. Similarly, \( F_n \) is used to denote a surface (Fläche) of degree \( n \).

The number of coefficients in (89) is \( 1 + n(n + 3)/2 \), consequently a given curve \( C_n \) is uniquely determined by

\[ \mu_n = \frac{n(n + 3)}{2} \]  \quad (90)

suitably chosen points. Let \( C_n \) and \( C'_n \) be distinct curves having \( \mu_n - 1 \) of these points, but missing the last point \( \bar{p} \), say. Then a principle due to Lamé states that the equations

\[ C_n + \mu C'_n = 0 \]  \quad (91)

represent all curves of degree \( n \) passing through the \( \mu_n - 1 \) given points. Clearly, for a specific value of \( \mu \) we recover the given curve \( C_n \).

On the other hand, Cramer and Euler had observed the paradoxical fact, that although a cubic curve \( C_3 \) in general is uniquely determined by \( \mu_3 = 9 \) points, two cubic curves still have \( n^3 = 9 \) points in common, by Bézout’s theorem. They both realized that somehow, interdependence of points was involved, and this puzzle became known as Cramer’s paradox. It was first resolved by Plücker, who gave a clearer answer which, in fact, follows from the simple observation that the curves \( C_n, C'_n \) in (91) with \( \mu_n - 1 \) given points in common, actually have

\[ \delta_n = n^2 - (\mu_n - 1) = \frac{1}{2}(n - 1)(n - 2) \]  \quad (92)

additional points in common, by Bézout’s theorem. Therefore, the whole curve family (91) have the \( n^2 \) points in common. For example, in the case of quartic curves, they will have 3 extra common points which are determined by the 13 common given points. Similar explanations of Cramer’s paradox were also given by Gergonne, Jacobi, and Lamé.

In Plücker’s analysis of plane curves the principle of duality plays a crucial role, since any curve \( C = C_n \) has a dual curve \( C^* = C'_m \) of some degree \( m \), called the class of \( C \). An alternative definition of the class \( m \) was given by Gergonne in 1826, namely it is the number of tangents to \( C \) passing through a fixed (generic) point. The two definitions would seem to be equivalent, since lines and points are dual to each other and the fixed point becomes a line cutting \( C^* \) in \( m \) points. So, by Bézout’s theorem \( m \) is also the degree of \( C^* \).

Now, Gergonne mistakenly assumed \( n = m \), despite the fact that Monge had earlier estimated the number of tangents to be \( n(n - 1) \), in a theorem which had been overlooked. Thus, Monge’s result seemed to imply \( m = n(n - 1) \).
for all \( n \); in fact, Poncelet confirmed that a curve of degree \( n \) is generally of class \( m = n(n-1) \). But on the other hand, the dual of \( C^* \) is \( C^{**} = C \), which clearly leads to a contradiction when \( n \neq 2 \). This explains the so-called Duality paradox, which was still unsettled in the early 1830’s. But Plücker and others must have realized that the correct value of \( m \) is also sensitive to the behavior of \( C \) and \( C^* \) at their singular points. Since he was using line coordinates as well as point coordinates, Plücker was in a better position to resolve the paradox by simultaneously analyzing the curve and its dual curve.

The advantage of developing point and line conceptions simultaneously had also been noted by Brianchon and Poncelet. In fact, the notion of a curve as the envelope of its tangent lines had been proposed already in the 17th century, and Leibniz (1692) gave rules for calculating envelopes. However, it was Möbius (1827) who determined the condition

\[
f^*(u, v, w) = 0
\]

for a line \( ux + vy + wz = 0 \) to be tangent to the curve \( C_n \). But he did not express the idea that \([u : v : w]\) are coordinates of the line, nor did he associate \([93]\) with the dual curve and hence overlooked the possibility of finding the class \( m \) of \( C_n \) by calculation of the degree of the curve \([93]\). This discovery is essentially due to Plücker, who clearly understood that all curves, except points and lines, have both point equations and line equations. In a paper published in Crelle’s Journal (1830) he made the prescient remark that the general theory of curves should be developed together with the idea of singular points and singular tangents.

With his new insight Plücker clarified the correspondence between tangent singularities of a curve \( C \) and point singularities of its dual curve \( C^* \), and vice versa. This enabled him to set forth the celebrated formulae \([95]\), from which a resolution of the duality paradox follows as a simple consequence. He communicated the formulae in the first place to Crelle’s Journal (1834); a sketchy version of his theory appeared in [1835] and a complete account and further extension in [1839]. Plücker considered two types of point singularities and two types of tangent singularities, namely

\[
\text{double point (δ) } \leftrightarrow \text{ bitangent (τ)} \quad (94)
\]

\[
\text{cusp (κ) } \leftrightarrow \text{ stationary tangent (ι)}
\]

where the letters \( δ, κ, τ, ι \) count the number of each kind. Alternative terms used in the literature are node, double tangent, triple point, and inflectional tangent, respectively. For example, a double point (i) and a cusp (ii) at the origin is illustrated by the cubic curves:

\[
(i) \quad x^3 + y^3 - 5xy = 0, \quad (ii) \quad x^2 - y^3 = 0.
\]

As indicated in [94], double points and bitangents are dual to each other, and similarly cusps and stationary tangents are dual. That is, via the duality
map \( C \rightarrow C^* \) a double point of \( C \) becomes a bitangent of the dual curve, and conversely. Hence, if the symbols \( \delta^*, \tau^* \) etc. count the singularities of \( C^* \), then \( \delta^* = \tau, \tau^* = \delta, \kappa^* = \iota, \kappa = \iota^* \). The six numerical invariants associated with the plane curve \( C_n \) are constrained by Plücker’s formulae, which can be stated as follows:

\[
m = n(n - 1) - 2\delta - 3\kappa, \quad n = m(m - 1) - 2\tau - 3\iota \tag{95}
\]

\[
i = 3m(n - 2) - 6\delta - 8\kappa, \quad \kappa = 3m(m - 2) - 6\tau - 8\iota
\]

As expected, a conic \((n = 2)\) has no singular points or tangents, and its dual is still a conic. A curve is said to be non-singular if \( \delta = \kappa = 0 \), but we note that such a curve has necessarily singular tangents (if \( n > 2 \)), and consequently its dual curve is singular. Point singularities had been studied at least in the previous 200 years, and it was known that the number of such points is limited by the degree \( n \) of the curve. For example, C. MacLaurin (1698-1746) showed the number of double points is limited by the constant \( (92) \). The finiteness of the number of bitangents was first suggested by Poncelet (1832), but shortly afterwards Plücker established an upper bound \( \tau_n \), namely we have

\[
\delta \leq \delta_n = \frac{1}{2}(n - 1)(n - 2), \quad \tau \leq \tau_n = \frac{1}{2}n(n - 2)(n^2 - 9) \tag{96}
\]

Bézout’s theorem is the basic tool for the estimation of the above numerical invariants, namely the singular points in question should be recovered by intersecting the given curve \( C_n \) with some suitably related curve \( C' \), for example the Hessian of \( C_n \). Its equation is \( H(f) = 0 \), where \( H(f) \) is the Hessian determinant of \( f \) \( (89) \), which has degree \( 3(n - 2) \). Hence, the curve and its Hessian have \( 3n(n - 2) \) common points, and they include the inflection points of \( C_n \), that is, the points where the curvature vanishes. Consequently,

\[
\iota \leq \iota_n = 3n(n - 2)
\]

and moreover, equality holds if \( C_n \) is non-singular. In general, the Hessian also passes through any double point or cusp, in fact, it has 6-fold and 8-fold contact with such points, respectively. So, this reduces the maximal value \( \iota_n \) of \( \iota \) by \( 6\delta + 8\kappa \) when the curve is singular, in agreement with \( (94) \).

Plücker estimated the number of bitangents of a non-singular curve \( C_n \) to be the upper bound \( \tau_n \) in \( (96) \), as follows. Since the dual curve \( C^* \) has degree \( m = n(n - 1) \), and the dual of \( C^* \) has degree \( n \), one of Plücker’s formulae amounts to the equation

\[
n = n(n - 1)(n(n - 1) - 1) - 2\tau - 3\iota_n
\]

and as an equation for \( \tau \) this has the unique solution \( \tau = \tau_n \). However, he did not give any proof of this independent of his formulae \( (95) \). Such a proof was given by Jacobi (1850), which in addition to several other results confirmed the validity of Plücker’s approach.
Plücker’s formulae provided an effective tool for the determination of the possible values of the above numerical invariant, say for \( n, m \leq 10 \). But first of all, they enabled him to progress more deeply into the study of cubic and quartic curves. However, Plücker did not proceed to the more general case of higher orders, where more complicated “non-Plückerian” singularities must also be considered, but we shall leave this topic here and refer to Gray [2007: 169].

In the following decades, however, quite new techniques of algebraic geometry were gradually developed, such as resolution of singularities of plane curves as well as space curves. Cremona transformations were found to be useful for this purpose, in particular, they are effective in the reduction of singularities of curves to double points with distinct tangents.

Around 1860 or so, Clebsch came across the following numerical invariant for plane curves

\[
g = \delta_n - \delta - \kappa = \frac{1}{2}(n-1)(n-2) - (\delta + \kappa)
\]

which takes the same value for both \( C_n \) and its dual curve. The term deficiency was originally used, since it counts the maximal number of double points reduced by the actual number of double points and cusps. The same invariant was in fact studied by Riemann, who also realized its importance as a topological invariant. It was renamed the genus of the curve, which is also the modern term.

3.4.4 “Projective geometry is all geometry” (Cayley 1858)

The work of von Staudt (with some later corrections) made it clear that projective geometry can be built up without dependence on Euclidean metric concepts such as angle and distance. But conversely, it is also rather surprising that it is possible to express these metric quantities on the basis of purely projective concepts. For example, the angle in radians between two intersecting lines \( l_1 \) and \( l_2 \) in the Euclidean plane can be expressed in terms of the cross-ratio of four lines

\[
\psi = \angle(l_1, l_2) = \frac{i}{2} \log(l_1, l_2; \omega_1, \omega_2)
\]

where \( \omega_1, \omega_2 \) are the two (imaginary) lines passing through the vertex of the angle and the two circular points at infinity, \((1 : i : 0)\) and \((1 : -i : 0)\) respectively, see equation (i) in (66). Here we regard (as usual) the Euclidean plane as part of the projective plane, which in turn extends to the complex projective plane, namely \( E^2 \subset P^2 \subset \mathbb{C}P^2 \). The coefficient \( i/2 \) is needed to make the angle real and to ensure a right angle has value \( \pi/2 \). This formula was first discovered by Laguerre in 1853. It does not seem that he was looking for a similar formula for the distance between two points.

Independent of Laguerre, Cayley also wanted to show that angle and distance, namely metric notions in Euclidean geometry, can be formulated in projective terms. For this purpose he introduced a conic (resp. a quadric) in the case of plane (resp. space) geometry, which he referred to as the absolute figure. As pointed out in his Sixth Memoir upon Quantics [1858], the two cases
are similar, so let us follow Cayley’s approach in the projective plane, with homogeneous coordinates \((x_1 : x_2 : x_3)\) and an absolute conic \((83)\) with associated real bilinear form

\[
F(x, y) = \sum_{1}^{3} a_{ij} x_i y_j
\]  

(99)

In terms of line coordinates \([u_1 : u_2 : u_3]\) (see \((85)\)), the equation \(F^*(u, u) = 0\) also describes the above conic, where \(F^*(u, v)\) is the bilinear form whose matrix \(A^*\) is the cofactor matrix (or adjoint) of the real symmetric matrix \(A = (a_{ij})\).

Now, Cayley defined the distance between \(x\) and \(y\) by

\[
\delta(x, y) = \arccos \frac{F(x, y)}{\sqrt{F(x, x)} \sqrt{F(y, y)}}
\]

(100)

and for the angle \(\psi = \angle(u, v)\) between lines \(u\) and \(v\) he defined \(\cos \psi\) in the same way using \(F^*\) instead of \(F\).

Depending on the absolute, \(\delta\) is a generalization of the modern notion of a distance function. The projective space has lines, and for three points \(x, y, z\) on a line the function is additive

\[
\delta(x, y) + \delta(y, z) = \delta(x, z)
\]

(101)

Notice that the value of the expression \((100)\) lies in the interval \([0, \pi]\), and the formula for \(\delta\) only gives the distance modulo \(\pi\). Hence, starting from \(x\) and moving along a line towards \(y\), we see from \((101)\) that multiples of \(\pi\) may accumulate before \(y\) is reached. In fact, in 1870 Klein modified Cayley’s definition \((100)\) of distance and removed the use of the arccos function, by expressing the distance between two points as a line integral.

Familiar expressions for \(\delta\) are obtained, for example when \(A = I\) (identity). But similar to Laguerre, Cayley’s absolute could also be the two circular points at infinity, viewed as a degenerate conic, and then he came out with the expression

\[
\delta(x, y)^2 = \frac{(x_1 y_3 - y_1 x_3)^2 + (x_2 y_3 - y_2 x_3)^2}{x_3^2 y_3^2}
\]

(102)

It follows that \(\delta\) restricted to the affine plane \(A^2 = (x_3 = y_3 = 1)\) is, in fact, the usual Euclidean distance function, so the induced geometry on \(A^2\) is that of the Euclidean plane \(E^2\).

With his paper [1858] Cayley had reduced metric geometry to projective geometry, but Cayley himself only showed how Euclidean geometry can be reinterpreted in terms of projective properties. In 1870 Klein suggested the non-Euclidean geometries are related to the projective metric as well, by an appropriate choice of the absolute figure as a standard of reference, and with the paper [1871g] and its second part the following year he set forth his new ideas. Instead of his previous integral expressions Klein defined the distance between \(x\) and \(y\), and the angle between \(u\) and \(v\), by taking the logarithm of a cross-ratio, namely

\[
\delta(x, y) = c \log(x, y; a, b), \quad \psi = c' \log(u, v; w_1, w_2)
\]

(103)
where $a$, $b$ denote the two intersection points of the absolute conic (or quadric) and the line through $x$ and $y$, $w_1$ and $w_2$ are the tangent lines of the conic through the intersection point, and $c$ and $c'$ are appropriate constants.

In plane geometry, Klein showed the geometry is hyperbolic, spherical, or Euclidean, according to whether the conic is real, imaginary, or degenerate. For example, the imaginary circle (88) yields elliptic geometry. Beltrami’s disk model for hyperbolic geometry (see Section 4.4.3) can be derived from Klein’s procedure by taking the following circle as the absolute conic

$$x_1^2 + x_2^2 - x_3^2 = 0 \quad \text{or} \quad x^2 + y^2 = 1 \quad \text{cf. (77)} \quad (104)$$

and the interior of the circle as the hyperbolic plane $H^2$. Beltrami did not derive a distance function or formula for angles in this disk. However, in general one can calculate the cross-ratio expressions in (103) as

$$\left( x, y; a, b \right) = \frac{F(x, y) + \sqrt{F(x, y)^2 - F(x, x)F(y, y)}}{F(x, y) - \sqrt{F(x, y)^2 - F(x, x)F(y, y)}} \quad (105)$$

and similarly for the cross-ratio of lines using $F^*$, cf. also Kline [1972: 911]. In the case (104) we have, of course, $F = F^*$ and

$$F(x, x) = x_1^2 + x_2^2 - x_3^2 \quad (106)$$

Some writers on the topic (wrongfully) attribute Klein’s formulae (103) to Cayley, and in fact they turn out to be just a reformulation of Cayley’s formulae. In his Collected Math. Papers, Cayley added important comments to his Sixth Memoir article [1858] and its connection with Klein’s work, stating that he regarded Klein’s approach as an improvement of his. However, he expressed disagreement with Klein’s interpretation of non-Euclidean geometry and its relation to the projective metric.

But Cayley neglected exploring his metric to the fullest; in particular, he did not relate it to non-Euclidean geometry, perhaps because of his rather ambivalent attitude toward this geometry. Instead, Klein discovered many different metric subgeometries of projective geometry, depending on the choice of the absolute, a valuable experience which contributed to the formulation of his Erlanger Programm in 1872.

Remarks on groups The Cayley-Klein approach also describes the relationship between motions (isometries) of the metric subgeometry and projective transformations of the ambient space. The general principle, which also appeared in Klein’s Erlanger Programm, is that the motions are the restriction of those projective transformations leaving the absolute figure invariant, not necessarily pointwise. Assuming this, let us determine the isometry group of the hyperbolic plane, by calculating the group of projective transformations of $P^2$ leaving the circle (104) invariant. In the notation of (82), the matrix subgroup of $GL(3)$ leaving the bilinear form $F$ invariant is usually denoted $O(2, 1)$, and
extension of this by the scaling group \( Z \) yields the matrix group leaving the form invariant modulo scaling, consequently

\[
Iso(H^2) = \frac{O(2,1) \cdot Z}{\{\pm I\}} = \frac{O(2,1)}{\{\pm I\}} \cong SO(2,1)
\]  

(107)

3.5 Line geometry

Line geometry is an approach to geometry where geometric objects in projective 3-space \( P^3 \) (or n-space in general) are studied by considering the straight lines, rather than the points, as the basic geometric Elements. Thus the geometric objects are represented by appropriate configurations of lines, and so line geometry becomes a branch of projective geometry. It was initiated in the early 19th century, by mathematicians such as Monge, Malus, Möbius, and Hamilton, who studied families of lines in 3-space and their geometric properties, often motivated by studies and experiments in optics. We shall use the terms "ray" and "line" synonymously.

Although line geometry went out of fashion in the early 20th century, the study was revived with modern techniques at the end of the century; for example, we refer to the survey on low order congruences in Arrondo [2002], and to Pottman-Wallner [2001] on computational line geometry and its modern applications. In the sequel we shall assume the underlying 3-space is the complex projective space \( \mathbb{C}P^3 \), in the tradition of Plücker, Kummer, Klein and Lie. For surveys of the classical line geometry, see Lie-Scheffers [1896], Jessop [1903], Rowe [1989].

3.5.1 Ray systems and focal surfaces

Recall that rays parallel to the axis of a parabolic surface are reflected into rays passing through the focal point. In general, however, light rays will not meet at a single focus after reflection or refraction, so there will be overlapping and the rays envelop and thus create an interesting geometric pattern, such as a caustic curve or caustic surface. An example is provided by a normal congruence, namely the lines perpendicular to a given surface. In these examples the rays constitute a 2-parameter family of rays called a ray system. Caustics were maybe first introduced by E.L. Malus (1775-1812), a student of Monge and Fourier, who began publishing papers on optics at École Polytechnique in 1808. It was Plücker who introduced the modern term line congruence for a ray system. The caustic surface became known as the Brennfläche or focal surface in the German or English literature, respectively. Generally the surface has two components and the line congruence consists of their common tangent lines.

In the 1840’s Plücker was a leading figure in line geometry, and it was after his renewal of the theory in the 1860’s that the discipline became, in fact, a major topic in algebraic geometry. Plücker’s approach provided, in fact, much of the geometric framework for the studies of Klein and Lie during the first years of their career. But their interests were also greatly stimulated by recent results of Kummer on specific surfaces arising as the focal surface of certain line
congruences. These surfaces have been known as *Kummer surfaces* ever since. Algebraically, they are quartic surfaces in projective 3-space with 16 double points, and it seems that they arose from his interests in Dupin’s cyclides and the optical properties of biaxial crystals. Needless to say, the relationship between the Kummer surfaces, the theta-function, and quotients of abelian surfaces, was only perceived much later in the development of algebraic geometry.

As a leading algebraic number theorist in 1855, Kummer became a professor at the University in Berlin, and in 1857 he was awarded the Grand Prix at the Academy of Science in Paris for his fundamental work relating to Fermat’s last theorem. At the end of the decade, however, his interests drifted towards geometry and he became interested in the ray systems examined by Hamilton. In fact, the first fundamental paper on line congruences was written by Kummer in 1859, and published in Crelle’s Journal the next year. He introduced, for example, the notion of a density function, which for a normal congruence (see above) equals the Gaussian curvature of the underlying surface. Kummer’s geometric period lasted (at least) through the 1860’s, and at his seminar in Berlin during the fall 1869, with Klein and Lie among the participants, the topic was in fact line geometry.

### 3.5.2 Basic ideas and definitions

It had been known long before Plücker that a line in 3-space has 4 independent degrees of freedom. In his initial paper [1846] on the subject, Plücker described the family of all lines in xyz-space in terms of four line coordinates \((r, s, \rho, \sigma)\) by the equations

\[
x = rz + \rho, \quad y = sz + \sigma
\]

Although this simple and naive 4-parameter representation has singularities and exceptional lines (which can be avoided), this did not prevent Plücker from developing many of the basic concepts and properties of line geometry. He defined a *line complex* to be a 3-parameter family of lines given by an algebraic equation \(F(r, s, \rho, \sigma) = 0\). However, this approach encountered difficulties since the degree of the defining equation is not invariant under linear transformations of the variables \(x, y, z\). He solved the problem in his "English" paper [1865] by introducing the auxiliary line coordinate \(\eta = r\sigma - s\rho\), which enabled him to define the *order* (or degree) of a line complex \(C\) to be the degree of the defining equation

\[
C : F(r, s, \rho, \sigma, \eta) = 0
\]  

(108)

For example, a linear complex is a complex of order one. A *special* line complex consists of the lines intersecting a given line or curve, called the *directrix*, but the complex of lines tangent to a given (non-planary) surface is also called special. These complexes will be of the same order as the curve or surface. For example, there is a special linear complex associated with each line, and these complexes were, in fact, studied by Möbius, who called them *null systems*. Here the lines through a point \(p\) lie in a fixed plane \(C_p\) and constitute the full pencil of lines, and there is a dual relationship \(p \leftrightarrow C_p\) between points and planes.
called the null polarity. First of all, however, it was Plücker who called attention to the applications of line complexes of degree one and two to mechanics and optics.

A 1-parameter family $R$ of lines represents a ruled surface in 3-space, and a 2-parameter family $K$ is called a line congruence. They can be expressed as an appropriate intersection of two or three line complexes

$$K = C_1 \cap C_2, \quad R = C_1 \cap C_2 \cap C_3$$

(109)

The order of a line congruence $K$ is defined to be the number of lines passing through a general point. In (109) this is the product $n_1n_2$ of the orders of $C_1$ and $C_2$. Dually, the class of the congruence is the number of its lines lying in a general plane. The two numbers may be different, but they are equal if $K$ belongs to a linear complex, say $C_1$ in (109) is linear.

The degree of the ruled surface formed by $R$ in (109) equals $2n_1n_2n_3$. In the lowest degree case it is, in fact, a doubly ruled quadric since through each of its points there pass two distinct lines lying on the surface. So the surface is a hyperbolic paraboloid or a hyperboloid of one sheet, see (45), unless it degenerates to a plane or two planes.

Recall from Section 3.5.1, a line congruence gives rise to a specific surface, namely the focal surface enveloped by its lines. In fact, Plücker associated with a given line complex $C$ infinitely many surfaces of this kind - the complex surfaces - namely the focal surfaces of all congruences $C \cap C_1$, where $C_1$ is the special linear complex with a given line $l$ as directrix. In his book [1868], Part III, he classified these surfaces into seven families, for complexes $C$ of order 2 (cf. letter of 20 July, 1871).

Plücker also used the terminology complex lines and complex curves, respectively, for the lines belonging to $C$ and the curves whose tangent lines belong to $C$. Moreover, at each point $p$ there is the complex cone $C_p$, namely the 1-parameter family of all complex lines passing through $p$. Thus, an alternative description of a complex surface is that it is enveloped by the cones $C_p$ as $p$ runs through a fixed line $l$. We also mention that Lie, in his study of tetrahedral line complexes in 1869-70 (cf. [1869], [1870a]) used his own definition of complex surfaces, and they were expressible as solutions of a certain differential equation. According to Lie, a tetrahedral complex consists of those lines whose four intersection points with the planes of a given (coordinate) tetrahedron have a fixed cross-ratio. However, based on another definition these line complexes had, in fact, also been studied by T. Reye in Zürich a few years earlier.

The complex cones $C_p$ describe the local geometry of a line complex of degree $n$. At a non-singular point $p$ the cone $C_p$ is non-degenerate, and it has degree $n$ in the sense that it cuts any plane $P^2$ along an algebraic curve $\gamma = C_p \cap P^2$ of degree $n$. Namely, $\gamma$ intersects a general line in $P^2$ at $n$ points. Dually, associated with the given line complex are also the non-singular planes $\simeq P^2$ in 3-space, with the property that the complex lines in the plane envelop a curve $\Gamma$ of class $n$. From algebraic geometry, the curve is said to be of class $n$ since it has $n$ tangents passing through an arbitrary point in the plane.
On the other hand, the singular points (or planes) of a line complex constitute the singular locus, whose geometry is generally complicated and hard to visualize. But it has attracted some attention in special cases, such as complexes of order two. These are the quadratic complexes, whose cone \( C_p \) at a singular point \( p \) degenerates into two planes intersecting along a singular line. Thus, the locus of singular points is the singularity surface, enveloped by the congruence of singular lines.

Dually, one also arrives at the singularity surface by considering the singular planes of the complex. In such a plane the class curve \( \Gamma \) degenerates into two points \( p_1 \) and \( p_2 \), and the complex lines in the plane are those passing through either \( p_1 \) or \( p_2 \). The line joining \( p_1 \) and \( p_2 \) is the singular line, and again the singular lines form a congruence whose focal surface is the singularity surface of the quadratic line complex.

With the paper [1866] Kummer gave a classification of linear line congruences (missing one case, see ref. in Arrondo [2002]). But more importantly, he also made a deeper study of the focal surface associated with a line congruence of order 2 and class 2, which generally is a surface of order 4 and class 4. In 1864 he had shown that this is a surface in \( \mathbb{CP}^3 \) with 16 double points and 16 double tangent planes, the maximum number possible for quartic surfaces. These are the Kummer surfaces, which due to their nice properties and dominant role among algebraic surfaces of degree 4, became an interesting study object for geometers and algebraists in the ensuing years. For a detailed exposition of the topic, see Hudson [1990].

The Kummer surface (and its generalization to higher dimensions) developed into an intricate speciality among French, German, British, and Italian geometers well into the 20th century, but the interest declined rapidly in the 1920’s. Today these surfaces are regarded as a 3-parameter family which play an important role in the modern theory of so-called K3 surfaces. For example, the following algebraic equation

\[
x^4 + y^4 + z^4 - (x^2 y^2 + x^2 z^2 + y^2 z^2) - (x^2 + y^2 + z^2) + 1 = 0
\]

represents a typical Kummer surface, symmetric with respect to permutations and change of sign of the coordinates, consequently the surface has the symmetries of an embedded regular octahedron.

### 3.5.3 Plücker’s new approach

Now, let us return to the late 1860’s when Plücker was preparing his 2-volume study *Neue Geometrie des Raumes...* with focus on linear and quadratic line complexes, which appeared in 1868-69. Grassman and Cayley were in fact forerunners of the new approach, based upon the use of homogeneous coordinates and *Elements* from exterior linear algebra.

Consider projective 3-space \( \mathbb{P}^3 \) with homogeneous coordinates \( x_i \) relative to a given coordinate tetradron with vertices \( p_1, \ldots, p_4 \), as in (19). This tetrahedron was also referred to as the fundamental tetrahedron. A basic observation is that
the line $\overline{xy}$ between two points $x$ and $y$ is determined by the 2-minors $p_{ij}$ of the following matrix

$$[x, y] = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{bmatrix}, \quad p_{ij} = \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix} = x_i y_j - x_j y_i$$ (111)

Therefore, one can take the 6-tuple of Plücker coordinates

$$(p) = (p_{12} : p_{13} : p_{14} : p_{23} : p_{42} : p_{34})$$ (112)

as homogeneous coordinates for the line $\overline{xy}$. Note, however, the numbers $p_{ij}$ are constrained by the Plücker relation

$$\mathcal{P} = p_{12} p_{34} + p_{13} p_{42} + p_{14} p_{23} = 0$$ (113)

and conversely, it is not difficult to see that each 6-tuple $(p)$ satisfying the condition $\mathcal{P} = 0$ does, indeed, represent a line $l$ in $\mathbb{P}^3$. Let us also define the dual line of $l$ to be the line $l^*$ with Plücker coordinates

$$(p^*) = (p^*_{12} : \ldots : p^*_{34}) = (p_{34} : p_{42} : p_{23} : p_{14} : p_{13} : p_{12})$$

On the other hand, following Möbius and Plücker (see [85]), in the equation $u \cdot x = 0$ for a plane $U$ in $\mathbb{P}^3$, the plane and point coordinates $u_i$ and $x_i$ appear symmetrically. This suggests a simple realization of the Poncelet-Gergonne duality principle as the following correspondence between planes and points

$$U = [u_1 : u_2 : u_3 : u_4] \leftrightarrow (u_1 : u_2 : u_3 : u_4) = u, \quad \text{cf. (86)}$$ (114)

Let $l = U \cap V$ be the line of intersection of two planes, say $l = \overline{xy}$ has Plücker coordinates $(p)$ as above. However, the two points $u$ and $v$ dual to $U$ and $V$ also define a line $l' = \overline{uv}$ where planes $u$ and $v$, say with Plücker coordinates

$$(q) = (q_{12} : q_{13} : \ldots : q_{34}), \quad q_{ij} = u_i v_j - u_j v_i,$$

and the important fact is that $l'$ is the dual line $l^*$. Namely, we have $(q) = (p^*)$, so the dual Plücker coordinates are the Plücker coordinates of the dual line, and thus the duality map (114) extended to lines is the involutive map $l \to l^*$.

Plücker[1869] also introduced another type of mappings of lines to lines, namely a general polar relationship depending on a given quadratic line complex $C$. Recall from Section 3.5.2 that the complex lines in a non-singular plane $\Pi \simeq \mathbb{P}^2$ envelop a curve of degree 2, which yields a polarity in the plane in the usual sense. Thus, for a given line $l$, if we consider the pencil of planes $\pi$ containing $l$, then $l$ has a pole in each of the (non-singular) planes $\pi$. Plücker showed these poles actually lie on a common line $l'$. However, the correspondence $l \to l'$ is not involutive (or reciprocal) since $l''$ is generally not equal to $l$.

The celebrated Plücker imbedding is the construction which regards $(p)$ in (112) as a point in projective 5-space $\mathbb{P}^5$, whereby the set of lines in $\mathbb{P}^3$ becomes a quadratic hypersurface

$$M^4_2 \subset \mathbb{P}^5 : \mathcal{P} = 0$$ (115)
One also observes that the coordinates $p_{ij}$ in (112) are associated with the coordinate polyhedron of $P^5$ whose six vertices are the edges $p_ip_j$ of the fundamental tetrahedron in $P^3$, simply because the only nonzero coordinate of the line $p_ip_j$ is $p_{ij}$. Moreover, given two lines $\mathbf{xy} = (p)$ and $\mathbf{zw} = (q)$, expansion of the determinant of the $4 \times 4$-matrix $[x, y, z, w]$ into 2-minors yields

$$\det[x, y, z, w] = \sum p_{ij}q_{ij}$$

so the vanishing of this expression is the condition that the lines intersect.

The 4-dimensional variety $M^4_2$ is usually referred to as the Plücker quadric, but sometimes also as the Klein quadric. Indeed, it was Klein who completed Plücker’s volume [1869], and with his continuing works in 1869 he added an extra geometric touch to line geometry, relating it to projective geometry in dimension 5. For example, a line complex of order $n$ is defined by a homogeneous polynomial $X = X(p_{ij})$ of degree $n$, and the line complex identifies with the 3-dimensional variety $\{P = 0, X = 0\}$ in $M^4_2$, which by Bézout’s theorem is of degree $2n$ in general. Similarly, the intersection of three line complexes $X_i = 0$ is generally an algebraic curve in $P^5$ of degree $2n_1n_2n_3$, again by Bézout’s theorem, and the curve represents a ruled surface in 3-space $P^3$.

As an example of a 2nd order special line complex, consider the totality of lines which intersect the spherical circle at infinity (66). It is also the family of lines satisfying Monge’s equation (in Cartesian coordinates)

$$dx^2 + dy^2 + dz^2 = 0 \quad (116)$$

In Plücker coordinates the complex is given by $p_{14}^2 + p_{24}^2 + p_{34}^2 = 0$. In the beginning of his career (1868-1870), Lie jokingly referred to the lines as "verrückten Geraden" (crazy lines), as they seemed to have many paradoxical properties, such as having zero length and being perpendicular to themselves. Later he called them minimal lines, but the term isotropic lines, due to the French geometer Ribeaucour, was commonly used.

4 Non-Euclidean geometry

One of the greatest mathematical discoveries in the 19th century is that of non-Euclidean geometry, which did so profoundly affect our conception of space and the entire foundation of geometry. The historical developments prior to this achievement centered around the ancient problem of parallels and the "truth" of the parallel postulate. From the days of Euclid to the middle of the 19th century, many prominent scholars have taken risky steps towards a possible solution of the problem, only to have their name added to the long list of past failures. So, although there were many actors in this historical drama, the actual discovery
of the new geometry is attributed to Gauss, Bolyai and Lobachevsky. Their revolutionary findings in the first third of the century, based on the counterintuitive assumption that the Euclidean parallel postulate is actually false, were unique in the history of mathematics.

4.1 The discoveries of Bolyai, Lobachevsky, and Gauss

The Hungarian János Bolyai did most of his creative work in the 1820’s, as a military engineer in the Austrian army. He was the son of a mathematics professor, Wolfgang (Farkas) Bolyai (1775-1856), who was familiar with the problem of parallels and strongly warned his son against wasting his life on this problem "as 100 geometers before had done". But already in 1823 the young man wrote back to his father that he had "created a new and different world out of nothing". Unfortunately, he was not able to publish his treatise until 1832, when it appeared as a 28-page Appendix in a two-volume geometry textbook by his father. Wolfgang had been a friend of Gauss since their student days in Göttingen in the 1790’s, and a copy of the book was sent to Gauss. However, the approval from Gauss was not so well received by the Bolyais, and it seemed to have a devastating effect on the young Bolyai. His mathematical career almost ceased, mainly due to discouragements and mental depression and, in fact, during his lifetime he received no public recognition for his work.

Let us also mention that János Bolyai proved the interesting result that in the new geometry it is possible to construct by ruler and compass a square of area equal to that of a circle of radius 1. This is the "quadrature of the circle", one of the most celebrated ancient problems, which already appeared in our oldest mathematical document, the Papyrus Rhind (2000 BC). Its impossibility in Euclidean geometry was finally settled by F. Lindemann in 1882, as a consequence of his proof of the transcendence of $\pi$.

The approach of Nicolai I. Lobachevsky, at the University of Kazan in Russia, was amazingly similar to that of Bolyai, but probably he never knew about him. For various reasons, Lobachevsky firmly believed the foundations of the Euclidean geometry were flawed. The first ideas of his alternative imaginary geometry were apparently set forth in a public lecture in 1826, on the principles of geometry and the theory of parallels, but he failed to obtain a publication out of it. On the other hand, his long paper On the Elements of geometry which appeared in two parts in Kazan Vestnik in 1829-30, is probably the first publication ever on non-Euclidean geometry. As we shall see, however, Elements from this geometry had appeared already in printed books by Saccheri and Taurinus.

Lobachevsky wrote several papers on the topic and therefore went further, but not necessarily deeper, than Bolyai. He made two attempts to convey his ideas outside Russia, namely with two publications in Berlin in 1837 and 1840. The first was an account in French, Géométrie imaginaire, published in the new Crelle’s Journal, heavy with formulae and dependent on his papers from 1829-30, so it was virtually impossible to read. The second was the more readable booklet Geometrische Untersuchungen, written in German, and he sent a copy
to Gauss, without knowing about his interest in this topic. As with Bolyai, however, Lobachevsky’s work was not so appreciated in his lifetime, and in 1846 he was even fired from the university. Indeed, the only acclaim he was to receive during his lifetime was the appointment in 1842, recommended by Gauss, of his membership at the Academy of Science in Göttingen.

Gauss, the foremost mathematician of his time and professor at Göttingen since 1807, had certainly anticipated some of the results of Bolyai and Lobachevsky on non-Euclidean geometry. In letter correspondences he praised their talents and the geometric spirit of their work. However, as to the extent of his own investigations we can only judge from various remarks in private letters, unpublished notes, and his reviews of books relating to the theory of parallels. On the basis of this, it is amazing that despite his great reputation he was afraid of making public his own geometric discoveries on the subject. But these were ideas which undoubtedly would have refuted Kant’s position on the nature of space and the unique role of the Euclidean space.

4.2 Absolute geometry and the Euclidean parallel postulate

For more than 2000 years, Euclidean geometry was the true and real geometry, namely the Geometry which was regarded as the science of the Space we live in. Euclid’s *Elements* had introduced five basic postulates expressing self-evident properties of points, lines, right angles etc., and proceeded to deduce altogether 465 propositions by mathematical reasoning. In Section 1.1 we stated these postulates in the spirit of Euclid and denoted them respectively by $E_1, E_2, E_3, E_4, E_5$, but with modern critical eyes the first four postulates are certainly vague statements. Moreover, they are incomplete as far as rigorous proofs of the propositions are concerned, since many additional “evident” assumptions must have been tacitly used as well. Keeping this in mind, we recall that many scholars, even long before Gauss, Bolyai and Lobachevsky, were led to investigate the restricted geometric content based upon $E_1 – E_4$, namely without assuming $E_5$. In the 1820’s Bolyai referred to this geometry as *absolute geometry* and, for convenience, in the sequel we shall also do so.

The logical status and geometric implications of the parallel postulate $E_5$ had remained a challenge since the days of Euclid. Recall that two lines in the plane are said to be *parallel* if they do not intersect. It is worth noticing that Euclid deduced the first 28 propositions without using $E_5$, in other words, they are statements of absolute geometry. In this geometry it follows, for example, that for every line $l$ and point $p$ outside $l$ there is at least one parallel line passing through $p$.

Now, the last postulate $E_5$, which distinguishes absolute geometry from Euclidean geometry, is in fact equivalent to the statement that the above parallel line through $p$ is unique. This alternative version of $E_5$ dates back to Proclus (411-485), but it is known today as Playfair’s axiom, after John Playfair (1748-1819) in Edinburgh, who published in 1795 a new edition of Book I-VI of the *Elements*.
By assuming the postulate $E5$ one can, for example, prove the following two basic results about triangles, namely the theorem of Pythagoras and the statement that the angle sum is $2R$, where $R$ denotes a right angle. Conversely, by accepting these results as obvious or experimental facts, one is also accepting the "truth" of $E5$. This may explain why the search for a "proof" of $E5$ became such an important issue, and there seemed to be two directions to proceed, either

1. to prove $E5$ as a logical consequence of the postulates $E1 - E4$, or

2. to establish the truth of $E5$ from the laws of nature.

Since antiquity many outstanding scholars have followed the first approach, hoping to show that $E5$ is a superfluous postulate. If the scholar succeeded, he would have proved that absolute geometry is the same as Euclidean geometry. But, as we shall see, the scholar would typically invoke another "self-evident" assumption, and in the misbelief that it was true in absolute geometry he would use it to deduce $E5$. On the other hand, despite the long lasting belief that the geometric truths are encoded into the nature, the second direction was not seriously considered until the 19th century, perhaps as a last effort.

4.3 Attempted proofs of the parallel postulate

The first known attempted proofs date back to the ancient scholars Ptolemy (ca. 85-165 AD) and Proclus. Of course, their arguments were flawed since the decisive assumption was merely a disguised version of $E5$ itself. Later examples of this kind are, besides Playfair's axiom, the axioms named after C. Clavius (1537-1612), R. Simson (1687-1768), J. Wallis (1616-1703), and Clairaut (cf. e.g. Greenberg [2001], Gray [2007]). For example, the axioms of Clavius and Clairaut assert the existence of two equidistant lines or a rectangle, respectively. Moreover, the existence of a triangle with angle sum $2R$ also yields $E5$. Some geometers such as Clairaut or Wallis, perhaps after realizing how hopeless it was to prove $E5$, proposed to replace $E5$ by their own axiom in order to improve or simplify Euclid's geometry.

The Persian scholars Omar Khayyám (1048-1131) and Nasir Eddin al-Tusi (1201-1274) are also known for their noteworthy analysis of the parallel postulate. Their ideas made their way to Europe and may, in fact, have contributed to the development of non-Euclidean geometry many centuries later. For example, the quadrilateral named after the Jesuit priest and logician G. Saccheri (1667-1733), was already introduced by them many centuries earlier. Saccheri was a student of Giovanni Ceva's brother Tommaso (1648-1737), who was a professor of mathematics and rhetoric at a Jesuit college in Milan. Saccheri tried to prove $E5$ by reductio ad absurdum, assuming the negation of $E5$, and so he attempted to deduce a contradiction from the ensuing bulk of non-Euclidean results. His little book *Euclid Freed of Every Flaw*, which appeared a few months before he died, created something of a sensation and was examined by leading
mathematicians of the day. But it fell into oblivion after some years, until it was rediscovered by Beltrami in 1889.

The work of Saccheri was most likely known to the Swiss-German scholar and leading mathematician of the 18th century, J.H. Lambert (1728-1777), who proceeded similarly and explored more deeply the consequences of the negation of the parallel postulate. He was a man of extraordinary insight and published more than 150 works in various areas. In 1764 he was invited to become a colleague of Euler and Lagrange at the Preussian Academy of Science in Berlin. Euler had established in 1737 that $e$ and $e^2$ are irrational numbers, and in an outstanding paper of 1768 Lambert showed that $\pi$ is also irrational. He further conjectured that $e$ and $\pi$ are even transcendental, but it was in the next century that Hermite and Lindemann, respectively, verified this conjecture.

Lambert’s work Theory of Parallels was written in 1766, but perhaps due to his unsatisfaction with the work it was only published posthumously, by Johann Bernoulli III 20 years later. Saccheri and Lambert must have regarded the new geometry as ficticious and without reality. Saccheri even believed, but wrongly of course, that he had established the truth of $E_5$, whereas Lambert admitted that his attempts had failed. Like ancient geometers and Clavius as well, Lambert had wrongly assumed that the curve of points equidistant from a given line and on the same side of the line, is itself a line.

In France, the leading analyst A.M. Legendre, influential in the restructuring of higher education during and after the revolution, was also confronted with the parallel postulate and its role in geometry. He became obsessed with proving its truth, and during the years 1794 to 1833 his 12 different attempts appeared, one after another, in the appendix of the revised editions of his highly successful textbook Éléments de Géométrie.

A famous mistake of Legendre was his assumption that through any interior point of an angle one can always draw a line which cuts both sides of the angle, but Legendre never realized that this is just another disguised version of the postulate $E_5$. On the other hand, 100 years after Saccheri, Legendre rediscovered Saccheri’s results in absolute geometry, for example, that the angle sum is at most $2R$ for any triangle.

Truly, in the 18th century, elementary geometry was rather engulfed in the problems raised by the parallel postulate. The situation was well illustrated by the thesis of G.S. Klügel in Göttingen in 1763, who described the flaws of 28 different attempted proofs of $E_5$. In 1767 the leading French scholar J. L.R. d’Alembert (1717-83) referred to the accumulation of false proofs and lack of progress as the Scandal of geometry. Although Klügel expressed doubt that $E_5$ could ever be proved, he did not scare off but rather inspired scholars like Lambert to try their fortune. Even Gauss had been working on the parallel postulate since 1792, at the age of 15, but having made little progress by 1813 he wrote:

*In the theory of parallels we are even now not further than Euclid.*

*This is a shameful part of mathematics.*
Gradually but slowly, scholars became convinced that Euclid’s parallel postulate cannot be proved, namely it is independent of the other Euclidean postulates. For example, Gauss expressed his conviction in 1817 in a letter to the astronomer H.W.M. Olbers (1758-1840). However, the truth of the parallel postulate continued to be an unsettled question for quite another reason, due to the poorly understood relation between abstract ”mathematical geometry ” and ”physical geometry”. So, let us turn to the second direction of attempted proofs of $E_5$, as pointed out above.

The first ancient cosmological model of the universe bounded by the celestial sphere dates back to Eudoxus, who was also the founder of theoretical astronomy, and this model was refined in Ptolemy’s 13 volume treatise *Almagest*. Even Kepler and Galileo regarded the universe to be limited. But a physical interpretation of Euclid’s postulates would certainly be false if the geometry was spherical. In fact, the viewpoint that we live in infinite, unbounded Euclidean space dates back no further than to Descartes in the 17th century. This model became the geometric frame for Newtonian mechanics and was later adopted by the Kantian philosophy.

But now, with the emerging non-Kantian ideas about geometry and space, the question came up whether we rather live in an infinite and unbounded non-Euclidean space. The real Geometry was supposed to represent physical space, and therefore, as in the case of other laws of nature, the truth could be established from physical experiments such as astronomical observations. As a geometer, Lambert knew that similar triangles would, in fact, be congruent if the postulate $E_5$ was wrong, and moreover, the angle sum of a triangle would be less than $2R$. In that case, as an astronomer he would worry about the countless inconveniences, and astronomy would be ”an evil task”.

Lobachevsky measured in 1829 the parallax of stars, which is almost negligible, so his observations were inaccurate and hence inconclusive. Gauss shared Lobachevsky’s view in his letter correspondences with his many astronomer friends, such as Olbers, W. Bessel (1784-1836) and C.L. Gerling(1788-1864). He discussed with them the possibility that physical space was not necessarily Euclidean. Indeed, while surveying the estates of Hanover he set up theodolites on three mountain peaks to test the non-Euclidean hypothesis experimentally, cf. Coxeter [1998].

Gauss wrote to Olbers that we should not put geometry on a par with arithmetic that exists purely a priori, but rather with mechanics. The viewpoint of geometry as an empirical science is also exemplified by the leading French mathematicians Lagrange and Fourier, who tried to deduce the parallel postulate from the law of the lever in statics. Lobachevsky was even more extreme, and being sceptical to the very foundations of Euclidean geometry, he believed that knowledge about the motion of bodies would help building up the concepts of geometry, such as ideas about the straight line.
4.4 The emergence of non-Euclidean geometry

Let us first have a closer look at Saccheri’s approach. On the basis of absolute geometry, he focused attention on a special quadrilateral $ABCD$, where the opposite sides $AD$ and $BC$ are congruent and perpendicular to the base $AB$. Denoting by $\alpha$ the angles at $C$ and $D$ (which are congruent), Saccheri considered the three possible cases

\begin{align*}
(i) & \quad \alpha = R, \\
(ii) & \quad \alpha > R, \\
(iii) & \quad \alpha < R,
\end{align*}

referred to as the hypothesis of the right, obtuse, and acute angle, respectively. Saccheri rightly argued that case (ii) is impossible, being incompatible with postulate $E2$ on the indefinite extension of lines. Case (i) simply means the quadrilateral is a rectangle, namely its angle sum is $4R$, and consequently the geometry must be Euclidean. Thus the negation of the parallel postulate $E5$ amounts to the acute angle hypothesis, $\alpha < R$, from which Saccheri deduced many ”strange” geometric results. Regrettably, as Lambert must have observed, Saccheri concluded with an obscure argument that he had produced a contradiction, so contrary to his own belief he failed to establish the truth of $E5$. However, with all the ”strange” geometric results Saccheri had actually discovered non-Euclidean geometry, although he did not recognize it as such. For example, he deduced the existence of lines approaching each other infinitely close without having intersection – later these became known as asymptotic parallels.

4.4.1 The trigonometry of Lambert, Schweikart, and Taurinus

Lambert was baffled by the observation that in non-Euclidean geometry there would be an absolute measure of length, analogous to the measure of angles. Therefore, similar triangles would also be congruent. For example, all equilateral triangles with angle $\alpha = 50^\circ$ say, are congruent, so their side $s_0$ would yield a natural unit of length. Lambert further noticed that the area of a triangle with angles $\alpha, \beta, \gamma$ (in radian measure) is proportional to $(\pi - \alpha - \beta - \gamma)$. Knowing that the area of a triangle on a sphere of radius $r$ is

\begin{equation}
A_1 = r^2(\alpha + \beta + \gamma - \pi),
\end{equation}

Lambert expressed the area in the former case as

\begin{equation}
A_2 = k^2(\pi - \alpha - \beta - \gamma) = r^2(\alpha + \beta + \gamma - \pi), \quad \text{where } r = k\sqrt{-1}
\end{equation}

and $k$ is a positive constant depending on $s_0$. Then he proclaimed that the geometry behaves like an imaginary sphere of radius $r$.

Lambert’s ideas were continued by two amateur geometers, the law professor F.K. Schweikart (1780-1859) and his nephew F.A. Taurinus (1794-1874), who progressed further using analysis rather than following the classical approach. From the outset Schweikart accepted with no prejudice the new geometry where the angle sum of triangles is less than two right angles. He was not looking for
a contradiction, but rather speculated if his geometry, which he referred to as *Astral Geometry*, would be appropriate for the study of the physical space.

In a noteworthy memorandum, communicated in 1818 to Gauss via the astronomer Gerling, Schweikart introduced his own constant $\mathcal{C}$, in analogy with Lambert’s constant $k$ in \[(119)\], to be the maximal height of any right-angled isosceles triangle. He could not determine its value, and perhaps regarded it as a parameter for many possible astral geometries, but he pointed out that the geometry would be Euclidean if $\mathcal{C}$ is infinite. Gauss complimented him on his results, and remarked that the area of a triangle would have the upper bound $\pi k^2$ when $\mathcal{C}$ is expressed as

$$\mathcal{C} = k \log(1 + \sqrt{2}).$$  \[(120)\]

In letters to Taurinus and Bessel in the 1820’s, Gauss expressed the view that the non-Euclidean geometry is self-consistent and entirely satisfactory, but the parameter cannot be determined a priori. As an astronomer, Bessel bravely suggested to Gauss in 1829 that physical space is maybe slightly non-Euclidean.

On the other hand, Taurinus firmly believed in the truth of Euclidean geometry, and his work was motivated by his desire to prove the parallel postulate. Therefore, he continued with his uncle’s work to prepare himself and to better understand geometry in general. Up to 1825, when his first booklet appeared, he still believed Euclidean geometry was the unique geometry, but in his second booklet in 1826 he accepted the internal consistency and lack of contradictions in his uncle’s astral geometry, and even vaguely suggested it might be the geometry of some surface. With due regard to Saccheri’s book, the above booklets, published in Cologne (Cöln), seem to be the first printed expositions on the *Elements* of non-Euclidean geometry.

Most noticeable, Taurinus broke with the traditional synthetic approach and introduced trigonometry as a method in non-Euclidean geometry, with the usual trigonometric functions replaced by Lambert’s *hyperbolic* functions

$$\sinh x = \frac{1}{2} (e^x - e^{-x}) = i \sin \frac{x}{i}, \quad \cosh x = \frac{1}{2} (e^x + e^{-x}) = \cos \frac{x}{i}.$$  

Indeed, Lambert himself had missed the connection which the functions provide between analysis and non-Euclidean geometry. Taurinus started with the trigonometric relations for a triangle $ABC$ on a sphere of radius $k$, for example the cosine law

$$\cos \frac{a}{k} = \cos \frac{b}{k} \cos \frac{c}{k} + \sin \frac{b}{k} \sin \frac{c}{k} \cos A$$  \[(121)\]

where $a$ is the side opposite to the vertex $A$ etc., and by recalling Lambert’s obscure idea of an imaginary sphere, he formally substituted $k \to ik$ into the equations. Thus, for example, the law \[(121)\] becomes the hyperbolic cosine law

$$\cosh \frac{a}{k} = \cosh \frac{b}{k} \cosh \frac{c}{k} - \sinh \frac{b}{k} \sinh \frac{c}{k} \cos A$$  \[(122)\]

Similarly he obtained the identity

$$\cos A = \sin B \sin C \cos \frac{a}{k} - \cos B \cos C$$  \[(123)\]
By a simple calculation the limiting case as $k \to \infty$ is readily seen to be the Euclidean geometry; for example, formula (122) yields the (usual) cosine law

$$a^2 = b^2 + c^2 - 2bc \cos A$$

Taurinus called the geometry satisfying the resulting hyperbolic laws for *log-spherical geometry*. He found that it agreed with Schweikart’s Astral Geometry; he also expressed his uncle’s constant $C$ in terms of $k$, and it agreed with the above formula (120) of Gauss.

Still, Taurinus was not convinced by the nice analytic results, which appeared in the second booklet. After all he had only worked out the geometry of an imaginary sphere, perhaps with no real content. He had corresponded with Gauss about the geometric ideas, but now he also insisted that Gauss should publish his own ideas on the subject. As a result, however, the angered Gauss ended his correspondence with Taurinus, and after this, the story goes, Taurinus ceased his geometric studies, bought up his own published booklets and burned them.

### 4.4.2 On the work of Bolyai and Lobachevsky

In the 1820’s there were also mathematicians with the conviction that there is a plane geometry in which Euclid’s parallel postulate is wrong, and Bolyai and Lobachevsky were the first ones with enough confidence in their geometric ideas to publish them. Probably they never knew about Taurinus, but his work would have provided the analytical basis of their geometric study. Instead, they derived in their own way trigonometric formulae including those found by Taurinus. Bolyai also became quite an expert on *absolute* geometry, that is, the aggregate of those propositions of Euclidean geometry which are independent of the parallel postulate and hence common to both Euclidean and non-Euclidean geometry. His unifying sine law for triangles $ABC$

$$\frac{\sin A}{\mathcal{O}_a} = \frac{\sin B}{\mathcal{O}_b} = \frac{\sin C}{\mathcal{O}_c},$$

where $\mathcal{O}_a$ denotes the circumference of a circle of radius $a$, is valid for absolute geometry and spherical geometry as well, see also (130).

Bolyai and Lobachevsky were also the first who worked in more generality on the non-Euclidean geometry in three dimensions, perhaps having in mind that it could be the geometry of physical space. Therefore, they also initiated the task of reformulating the basic laws of classical mechanics in the new geometric setting. For example, in 1835 Lobachevsky considered the gravitational law and defined the Kepler problem with an attractive force inversely proportional to the area of the 2-dimensional sphere of radius equal to the distance between the bodies. Bolyai came up with similar ideas about the same time (see Diacu et al (2008)). But now, let us briefly recall their basic geometric approach.

A *horocycle* is the plane curve perpendicular to a family of asymptotic parallel lines; it is also the limiting curve of an expanding circle whose radius tends
to infinity. In the 3-space there are surfaces defined similarly, called *horospheres* by Lobachevsky. Bolyai and Lobachevsky made use of this type of curves and surfaces in their analysis; in fact, they both attacked plane geometry via the horosphere. They made the remarkable observation that the induced geometry of a horosphere is that of a Euclidean plane, whose straight lines correspond to the horocycles lying on the surface. In fact, this result dates back to F.L. Wachter (1792-1817), a pupil of Gauss in 1816, who inquired about the limiting form of a sphere in non-Euclidean 3-space when its radius becomes infinite.

Let us consider a triangle $PQR$ with vertices $P, Q, R$ and opposite sides of length $p, q, r$, respectively. We also assume $QR$ lies on the line $l$ and $PQ$ is perpendicular to $l$, so $r$ is the distance from $P$ to $l$. Now, keep the vertices $P$ and $Q$ fixed and let $R$ move far away along $l$. Then the angle at $P$ increases towards a limiting value $\Pi(r)$, depending only on $r$, and the angle at $R$ tends to zero. Moreover, the side $PR$ approaches the asymptotic parallel ray of $l$ through $P$, so $\Pi(r)$ is also referred to as the angle of *parallelism* at $P$. Bolyai and Lobachevsky discovered the following nice formula for $\Pi(r)$, measured in radians,

$$\Pi(r) = 2 \arctan e^{-r/k}$$  \hfill (125)

where $k$ is the constant appearing in the area formula (119), namely $\pi k^2$ is the maximal area of any triangle. In particular, we recall that Schweikart’s constant $\mathcal{C}$ is the value of $r$ when $\Pi(r)$ is half of a right angle, so the identity (120) found by Gauss and Taurinus is just a special case of the identity (125).

In fact, Taurinus could also have derived the formula (125) from the identity (123) applied to the above triangle $PQR$.

Next, let us recall Lobachevsky’s crucial experiment in 1856 to test whether his geometry would fit better than the Euclidean one. This test differs from his first effort in 1829 and is described in *Pangéométrie*, which he dictated as a blind man the year he died. With reference to the above triangle $PQR$, imagine a distant star at $R$ which is observed at the two diametrically opposite positions $P$ and $Q$ of the Earth’s orbit. Then the observed parallax should exceed the angle $\pi/2 - \Pi(r)$. However, due to the experimental errors the finiteness of the unknown absolute unit $k$ could not be established, only that $k$ would be several million times larger than the diameter $r$ of the orbit.

Lobachevsky strongly believed that his geometry was consistent, that is, it would not lead to any contradictions. For him, geometry was concerned with measurements and numbers, related by formulae whose validity is rather an algebraic problem. He argued that the geometry is based on formulae for a triangle, and they would yield the familiar formulae for a spherical triangle when the sides $a, b, c$ are replaced by $ia, ib, ic$, or when the parameter $k$ is replaced by $ik$. Moreover, the formulae would still make sense and describe the Euclidean geometry in the limit when $k$ tends to infinity.

The above arguments of Lobachevsky were, in fact, insufficient (see e.g. Rosenfeld [1988:227]). On the other hand, Gauss and Bolyai were also familiar with the above trigonometric relations, but they did not regard them as evidence for the logical consistency of the non-Euclidean geometry. In his letter
to Taurinus in 1824, Gauss had expressed his belief that the new geometry is self-consistent, but on the other hand, Bolyai was quite disturbed by his inability to settle this matter. In fact, it remained an open question until 1868, when Beltrami came across the crucial idea of constructing a concrete model for the geometry, which simply reduced the whole question to the consistency of Euclidean geometry itself.

Certainly, the recognition of the work of Bolyai and Lobachevsky was hampered by the fact that none of them were successful in establishing the truth, or the physical existence, of their geometry. After all, almost everyone believed that the physical space is Euclidean, so how could an alternative geometry also be true? However, after Gauss’s death in 1855, his unpublished notebooks and letter correspondences became known among mathematicians. In particular, the publication of his letters to H.C. Schumacher (1780-1850) in 1864, in which he praised Lobachevsky’s work and expressed his own belief that Space might be non-Euclidean, made a strong impression on European mathematicians (Rosenfeld [1988]). As a consequence, during the next few years the original papers of Lobachevsky and Bolyai’s Appendix from 1832 attracted considerable attention and were translated to German, French, Italian, and Russian. Let us briefly recall what actually happened during these years.

In 1866, when R. Baltzer in Germany was preparing the second edition (1867) of his textbook *Die Elemente der Mathematik*, he also included a favorable mention of the discoveries of Bolyai and Lobachevsky. He informed Hoël in France, who issued the same year a French translation of Lobachevsky’s German memoir from 1840, together with excerpts from the Gauss-Schumacher correspondence. Moreover, in 1867 Hoël also wrote a book in order to explain Lobachevsky’s geometry. But on the other hand, Bolyai’s Appendix was not so easily available, so the translation of that paper was delayed until 1868. An historical note on the life and work of Lobachevsky, by E.P. Yanisevskii in Kazan, appeared also translated to French and Italian in 1868.

4.4.3 Beltrami and his model of non-Euclidean geometry

In Italy, Battaglini, Beltrami and Cremona were active proponents for the new geometric ideas. The journal Giornale di matematiche, which Battaglini founded in 1863, became the major Italian publication channel for papers on non-Euclidean geometry. Battaglini published in 1867 a paper on Lobachevsky’s imaginary geometry and a translation of Lobachevsky’s last paper *Pangéométrie*, as well as a translation of Bolyai’s Appendix in 1868. For geometry in Italy, Cremona was certainly very influential, but his own works were mostly within projective geometry.

Beltrami had learned about non-Euclidean geometry by reading the French translations of Hoël in 1866, and in 1868, shortly after the publication of Riemann’s famous habilitation lecture in 1854, Beltrami quickly developed a deeper understanding of the topic by following Riemann’s approach to geometry. Thus appeared his two seminal papers [1868a,b], which settled for the first time the ancient question of a proof of the parallel postulate, namely he demonstrated...
that no proof is possible. He achieved this by exhibiting a Euclidean model of the geometry of Boliai and Lobachevsky.

By taking a closer look at Beltrami’s paper [1868a], one will find that he gives (at least) two different Euclidean models for the hyperbolic plane, with explicit formulae for the metric in the sense of Riemann. For simplicity, consider the upper hemisphere $z > 0$ of radius 1 in Euclidean 3-space with rectangular coordinates $(x, y, z)$. Beltrami modifies its spherical metric by multiplying it with $z^{-2}$, and therefore, in terms of spherical polar coordinates $(\phi, \theta)$, centered at the north pole so that $z = \cos \phi$, the metric becomes

$$ds^2 = \frac{d\phi^2 + \sin^2 \phi \ d\theta^2}{\cos^2 \phi} \quad (126)$$

This is, in fact, a model of the hyperbolic plane with curvature $K = -1$, whose geodesics (lines) are the semicircles on the sphere lying in planes perpendicular to the $xy$-plane. Moreover, by vertical projection of the hemisphere onto the unit disk $x^2 + y^2 < 1$, Beltrami obtains the so-called disk model of hyperbolic geometry, where the hyperbolic lines are, indeed, the Euclidean line segments. Finally, Beltrami further transforms the metric to hyperbolic polar coordinates $(\rho, \theta)$ centered at the midpoint of the disk, namely $\rho$ measures the radial distance from the center of the disk. The connection between $\phi$ and $\rho$ is found to be $\sinh \rho = \tan \phi$, which yields the very simple expression

$$ds^2 = d\rho^2 + \sinh^2 \rho \ d\theta^2 \quad (127a)$$

In England, the algebraist and geometer Arthur Cayley was a leading figure, and with his numerous papers he treated nearly every subject of pure mathematics. In the paper entitled Notes on Lobachevsky’s Imaginary geometry (1865) he made a comparison of the spherical trigonometric formulae with those of the Lobachevskian geometry, but apparently he overlooked the essence of Lobachevsky’s discovery, although his writings helped promoting the new geometric ideas (cf. Rosenfeld [1988: 220]). On the other hand, Cayley’s important theory of projective metrics in his Sixth Memoir [1859], where he proposed a generalized definition of distance, was grasped by the young Felix Klein in 1871, who caught an important idea which was, in fact, overlooked by both Cayley and Beltrami, see Klein [1871g], [1872c], Stillwell [1996].

In summary, due to the above mentioned publicity, around 1870 the ideas of Bolai and Lobachevsky were known to geometers at the major universities in Europe. Beltrami (1868) and Klein (1871), the latter from a projective geometric viewpoint, had finally completed the last step needed for non-Euclidean geometry to be accepted as part of ordinary mathematics, by their construction of convincing Euclidean and projective models which showed that the new geometry was equally consistent with the ancient geometry. The conformal models of non-Euclidean geometry exhibited in Poincaré [1882] are, in fact, implicit among the models presented in Beltrami [1868b]. But by applying them to his study of automorphic functions Poincaré also contributed largely to the uprise of hyperbolic geometry to a respectable mathematical discipline, which by 1890 was finally taught as a course at major universities.
5 The classical geometries: Euclidean, spherical, and hyperbolic

5.1 A unified view of the three classical geometries

In the 1860’s the geometry of Bolyai and Lobachevsky, which Gauss had referred to as non-Euclidean geometry, became also known as Lobachevskian geometry. In fact, still more geometries with non-Euclidean properties were expounded by Riemann, Klein, and others, and in fact, spherical geometry was also called Riemann’s non-Euclidean geometry. Namely, the latter is the geometry of figures drawn on a round spherical surface in 3-dimensional Euclidean space.

Being so closely related to Euclidean geometry, spherical geometry is really the first example in history of a geometry other than the Euclidean one. The ancients called it Sphaerica, and they used it to describe the heavenly bodies moving around on the celestial sphere. For example, Menelaus of Alexandria and the Arabs (around 1000 AD) studied this geometry. The French Albert Girard (1595-1632), whose treatise on trigonometry in 1626 contained the first use of the abbreviations sin, cos, tan, also gave the formula (118) for the area of a spherical triangle, a formula which was later generalized by Gauss to geodesic triangles on more general surfaces in 3-space. We refer to Rosenfeld [1988] for detailed information on the early history of spherical geometry.

Euclidean, spherical, and Lobachevskian geometry constitute the three classical geometries, and for many good reasons. They have many properties in common, deeply rooted in the human conception of space, and the most basic ones are of a non-metric nature. But they are also, somehow, elaborated within another type of ”classical” geometry, namely the grand unifying theory called projective geometry. However, in this section we shall rather focus on the metric properties of the classical geometries, some of which they have in common, but certainly there are also important differences.

One of Klein’s achievements in [1871g] was his unification of these geometries, subsumed by a common ambient projective space, by utilizing the projective measure construction of Cayley [1859]. Klein proposed the suggestive terms parabolic, elliptic, and hyperbolic geometry, respectively, in accordance with their geometric properties and limiting behavior in resemblance with conic sections. We shall henceforth follow Klein and use the modern term hyperbolic geometry instead of Lobachevskian geometry. However, the equivalence of the terms ”parabolic” and ”Euclidean” space ceased in the early 20th century, when more general parabolic spaces were defined as specific homogeneous spaces constructed in terms of Lie group theory.

With the unifying analytic approach inspired by Riemann, around 1870 the focus on classical geometries had shifted to their space forms, which we shall denote by \( E^n, S^n, \) and \( H^n \). Namely, their differential geometric properties, and to some extent also topological properties, were the subject of study in the lowest dimensions \( n = 2, 3 \). The higher dimensional versions with \( n > 3 \) were gradually accepted since Riemann had extended the classical notion of
curvature for surfaces to spaces of higher dimensions, and the above three types of geometries were found to have constant curvature $K$, namely

$$E^n : K = 0, \quad S^n : K > 0, \quad H^n : K < 0$$  \hspace{1cm} (128)$$

It was Beltrami who discovered the link between hyperbolic geometry and spaces of constant negative curvature. He came across the idea when he compared Lobachevsky’s 1837 paper with Minding’s 1840 paper on the geometry of the pseudosphere, both papers printed in Crelle’s Journal. The latter surface has constant negative curvature, and Beltrami observed in 1868 that the two papers had, in fact, the same trigonometric formulae. With the formula (129) below, Beltrami had expressed in polar coordinates $(\rho, \theta)$ the Riemannian metric of $H^2$ for $K = -1$, so he certainly knew the analogous expressions in all cases (128) for $n = 2$ and any $K$, namely

$$ds^2 = d\rho^2 + f(\rho)^2 d\theta^2, \quad \begin{cases} 
\rho & K = 0 \\
\frac{\rho}{\sqrt{K}} \sin(\sqrt{K} \rho) & K > 0 \\
\frac{\rho}{\sqrt{-K}} \sinh(\sqrt{-K} \rho) & K < 0 
\end{cases} \hspace{1cm} (129)$$

The distance $\rho$ from the chosen center $O$ of the coordinate system ranges over $[0, \infty)$ when $K \leq 0$, whereas $0 \leq \rho \leq \pi/\sqrt{K}$ in the sphere case. Another remarkable property of the three geometries is the unifying Bolyai’s sine law (124), where by (129)

$$\bigcirc_a = 2\pi f(a) \hspace{1cm} (130)$$

is the length of the circle of radius $a$, say the circle with the equation $\rho = a$, which also yields a nice geometric interpretation of the size function $f(\rho)$.

We also remark that the trichotomy (117) of the angle $\alpha$ in Saccheri’s quadrilateral reflects the trichotomy of the classical geometries. Thus case (i) is the Euclidean geometry, based upon the ancient Euclidean postulates $E1, E2, .., E5$, and hyperbolic geometry was discovered in case (iii) by renouncing $E5$. Finally, Riemann gave spherical geometry his full recognition as a kind of non-Euclidean geometry, satisfying Saccheri’s hypothesis (ii) of the obtuse angle, with the great circles on the sphere $S^2$ interpreted as the straight lines. In this case $E5$ is violated since there are no parallel lines at all, but $E2$ is also violated because the circles are of finite length. In addition, $E1$ postulates that two points determine a unique line, so $E1$ is violated as well. For an axiomatic approach, Riemann therefore proposed a modification of these three postulates so that, for example, (i) two points would determine at least one line, and (ii) a line is unbounded. However, such an approach to spherical geometry has never been found useful, since the geometry is best understood as a subgeometry of Euclidean geometry.

However, the incompatibility of $E1$ with spherical geometry remained an unsatisfactory issue for many years. Beltrami [1868a] refers to $E1$ as the postulate of the straight line, and several mathematicians, including Beltrami and Weierstrass, believed that this failure was a characteristic property of geometries of
constant positive curvature. It also seemed that Riemann [1867] had identified, although vaguely, such geometries with spherical geometry. But Klein [1871g: 604], [1874c] removed this misconception by describing a truly elliptic geometry where $E_1$ holds. Let us briefly recall the basic ideas.

Klein’s elliptic model in dimension $n$ was taken to be the space obtained from the $n$-sphere $S^n$ by identifying antipodal points, whereby each great circle is reduced to a closed geodesic of half the original length. In fact, the construction identifies the elliptic $n$-space with the projective $n$-space $P^n$. This is clear from the vector space model of $P^n$ (see Section 3.4.1), since a central line in $\mathbb{R}^{n+1}$ cuts the surrounding sphere $S^n$ in two antipodal points. For example, the elliptic plane identifies with the real projective plane $P^2$; in fact, by leaving out metrical concepts and congruence, elliptic geometry becomes real projective geometry. From the Riemannian viewpoint, $P^n$ and $S^n$ cannot be distinguished locally, that is, they are locally isometric. In particular, they have the same constant curvature. Spherical geometry on $S^n$ became also known as doubly elliptic geometry since the mapping $S^n \to P^n$ is two to one.

Before basic topological concepts and constructions had been developed, the distinction between local and global properties of a geometry was poorly understood. This explains why people generally believed, in the 1870’s and maybe even later, that the three classical spaces (128) and Klein’s elliptic space $P^n$ were the only (proper) space forms of constant curvature. But, in fact, in 1873 the English mathematician Clifford described the construction of a flat torus, which can be embedded in the sphere $S^3$ as a closed surface of zero curvature, cf. note to the letter of 4.11.73.

Hawkins [2000] points out that Riemann’s discussion of manifolds of constant curvature impressed the late 19th century geometers. Already in his paper [1872c] Klein comments upon the work of Riemann and Helmholtz, and as late as in his lectures on hyperbolic geometry in 1889 Klein maintains that the concept of an $n$-dimensional manifold of constant curvature is the most essential result of Riemann’s approach. With modern eyes, however, its real importance is the generality of his approach, allowing metrics (59) far more general than the classical ones. But for many years these ”speculations” were largely ignored or dismissed as useless. A notable exception was Clifford, who with his paper On the space-theory of matter (1870) identified energy and matter with two types of curvature of space. Many decades later his ideas were found to be important for the development of Einstein’s general relativity theory.

5.2 The conception of higher-dimensional geometry

Let us also briefly consider how the conception of dimension has influenced the developments of geometric ideas in the past. Obviously, our geometric intuition is largely based upon our experiences with the physical space we live in, so it can be hard to imagine geometry in more than three dimensions. Before 1870, the term geometry was in fact largely synonymous with the classical geometries (128) and projective geometry in dimensions 2 and 3. Geometry was still considered as ”descriptive” and supposed to describe physical space, or at least
some idealization or mental conception of it. Lützen [1995] argues that the only notable exceptions were the non-Euclidean geometers and certainly also Riemann.

Descartes and Euler knew how to identify the Cartesian $n$-space $\mathbb{R}^n$ in dimension $n = 2$ or $3$ with the Euclidean plane or space, respectively. Therefore they could apply analytic geometry in two or three variables to study geometric problems in three dimensions or less. Conversely, geometry was used to illustrate analytic problems in at most three variables. In those days, however, geometry in higher dimensions did not make sense, although the prominent 17th century scholar Pascal proposed, in fact, that a 4th dimension was allowed in geometry. Making a leap to Möbius in 1827, we find that he introduced a 4-dimensional space in his paper *Der barycentrische Calcul*, but still with the reservation that "it cannot be imagined".

However, the *Elements* of $\mathbb{R}^n$ are just $n$-tuples $x = (x_1, \ldots, x_n)$ of numbers, so gradually it became natural to attempt generalize the ideas of analytic geometry to $n > 3$ variables, for example as generalized coordinates of mechanical systems. Depending on the problem at hand, $\mathbb{R}^n$ would be referred to as the coordinate space or, say, the number space in $n$ dimensions. So, it is not surprising that early geometric ideas in higher dimensions appeared in the study of special subsets or "submanifolds" of $\mathbb{R}^n$. For example, Cauchy used geometric notions to describe submanifolds of $\mathbb{R}^n$ of type $f(x_1, x_2, \ldots, x_n) = 0$.

Foremost scholars in the development of multidimensional geometry are Cayley, Grassmann, Schläfli, and Riemann. Grassmann was the first who attempted, with his *Ausdehnungslehre* (1844 and 1862), a systematic study of $n$-dimensional vector spaces from a geometric viewpoint. But his work was hard to understand and also too philosophical, so it was not appreciated until the end of the century, when modern tensor calculus was developed. The Swiss mathematician Ludwig Schläfli (1814-1895) studied the differential geometry of non-linear submanifolds of $\mathbb{R}^n$, such as ellipsoids and their geodesics, and polytopes in higher dimensions, for example the description of all regular solids in four dimensions. But being ahead of his time, he too had problems when he tried to publish his major work *Theorie der vielfachen Kontinuität* (1852). Its importance was fully appreciated when it was finally published at the turn of the century.

In 1843 Hamilton discovered the quaternions, which initiated a 4-dimensional geometry, and Cayley announced his interest in multidimensional geometry, at least in the title of his paper *Chapters in analytic geometry of (n) dimensions*. Like Grassmann he arrived independently at the notion of an $n$-dimensional space, and joined by Sylvester they became the leading British advocates of $n$-dimensional geometry. But they were widely opposed, until the breakthrough at the end of the 1860’s, mainly due to the gradual acceptance of non-Euclidean geometry in France, Germany and Italy, and moreover, the publication in 1868 of Riemann’s celebrated lecture from 1854. Here Riemann’s space models are referred to as $n$-fold extended quantities or manifolds, whose local coordinate systems make them look locally like $\mathbb{R}^n$. In fact, Riemann had anticipated the idea of an $n$-dimensional differentiable manifold, a concept belonging to the new discipline of the 20th century called topology.
Schläfli and Riemann both extended the geometry of the Euclidean 2-sphere $S^2$ to higher dimensions, thus providing the following model for Riemann’s n-dimensional spherical space imbedded in Euclidean (n+1)-space

$$S^n(r) : x_1^2 + x_2^2 + \ldots + x_{n+1}^2 = r^2$$

Riemann suggested in 1854 that the 3-sphere $S^3(r)$, with its intrinsic geometry of constant positive curvature $K = r^{-2}$, provides a possible model for the universe which is finite but still unbounded. In contrast to this, a hyperbolic space model would yield an infinitely large universe. However, since the curvature could be arbitrary small, it would be almost impossible to distinguish a large 3-dimensional sphere from Euclidean 3-space. The 20th century physicist Max Born has described this sphere model as “one of the greatest ideas about the nature of the world which has ever been conceived”. In fact, as his first attempt at a cosmological model based on the general theory of relativity, Einstein chose the 3-sphere as his model of the universe (cf. Osserman [2005]).

5.3 Foundations of geometry and the geometry of Space

During the three decades prior to 1860 or so, the strictly synthetic approach to geometry with Steiner in the forefront had a strong position in German geometry. But in the late 1860’s differential geometry came more to the foreground, and the philosophically oriented essays of Riemann and Helmholtz were published in 1868. Here we merely point out that Riemann, with his mathematical formulation of the concept of space, paved the way for applying geometry to physical reality, whereas Helmholtz, as an exponent for physical geometry, became identified throughout the century with the philosophical foundations of physical space. On the other hand, from a synthesist’s viewpoint, the foundations of geometry, projective or classical, still needed a thorough revision of its basic concepts and postulates. Indeed, there were also geometers who opposed the idea of reducing geometric thinking to analytic geometry and perhaps relying too much on physical intuition or experience.

Moritz Pasch and David Hilbert were the major figures who resurrected the classical synthetic or “elementary” geometry, to become a rigorous axiomatic system with emphasis on the purely formal character of geometry. First of all, the focus was on real projective geometry, considered to be the most basic geometry, amply demonstrated in works by the French pioneers and by Steiner, von Staudt, Cayley, and Klein. To ensure the desired properties of a line, for example, Pasch was the first who recognized the importance of the relations of “order” and “betweenness”. With his book [1882] he published the first rigorous deductive system of geometry in history. But despite his insistence on pure logical reasoning, Pasch still viewed geometry as the science of physical space, and he wanted to justify the geometric axioms from experience.

In 1895 David Hilbert left Königsberg and joined Klein at Göttingen University. Influenced by Pasch’s ideas he became heavily involved with the foundations of elementary geometry, a subject which had also attracted him while
he was in Königsberg. Hilbert’s seminal book, *Grundlagen der Geometrie*, first published in 1899 and to appear in many editions, is a purely logical approach to geometry. In comparison with Euclid’s five postulates, Hilbert formulated about 20 axioms, grouped into five types and introduced stepwise, in order to clarify what type of theorems can be proved at each step. These five types are nowadays referred to as the axioms of incidence, order, congruence, continuity, and parallels.

In addition to Pasch, Hilbert also benefitted from many others who had initiated techniques which he found useful for his program, as for example Gergonne and his use of implicit definitions, von Staudt who created a calculus of line segments, H. Wiener (1857-1939) and his lectures in 1891 on the role of incidence theorems exemplified by the theorems of Pappus and Desargues, and finally the Italian geometers G. Peano (1858-1932), M. Pieri (1860-1913), and A. Padoa (1868-1937), who focused on the strictly logical approach and replaced the processes of reasoning by symbols and formulas. As a consequence, the meaning of the fundamental concepts, or the sense in which the axioms are true, should be excluded altogether from geometry. The story goes that Hilbert, on a certain occasion (see Bos [1993]), expressed this wisdom by saying "instead of points, lines, planes, one should always be able to say tables, chairs, beer mugs".

The wave caused by Hilbert’s "Grundlagen der Geometrie" pervaded the research and teaching of geometry during the first half of the 20th century, but after the initial excitement it also seemed that Hilbert had "killed" the subject, in a tradition which had little to offer with regard to philosophical or foundational reflection. On the other hand, in the tradition of Riemann, Helmholtz and Poincaré, the term "Grundlagen der Geometrie" had in fact been used several times before, for example in works of Lie and Killing.

Poincaré and Hilbert were the leading figures in mathematics at the turn of the century. Poincaré wrote a favorable review [1903] of Hilbert’s work on the foundations of geometry, and as a great philosopher and scientist he developed his geometrical conventionalism, closely linked to his own mathematical studies. In his book [1902], Poincaré gives his answer to an aged problem by declaring that the question of whether Euclidean geometry is true has no meaning, and moreover, one geometry cannot be more true than another, it can only be more convenient (cf. Torretti [1978], Bos [1993]).

### 5.4 Riemann–Helmholtz–Lie space problem

The geometric interests of Helmholtz arose from his work on physiological optics in Heidelberg during the early 1860’s. He had gained the reputation as a leading world scientist, whose interests embraced all the sciences, as well as philosophy and the fine arts. Around 1866, when he moved more towards physics, he questioned the foundations of geometry, viewed as a science of physical space. His first brief report *On the factual foundations of geometry* (1868) appeared in Heidelberg, but his more detailed essay [1868] was printed in the Göttingen Nachrichten shortly afterwards. In the meantime, Helmholtz had obtained a copy of Riemann’s essay from Ernst Schering, who was Riemann’s successor in
Göttingen. We refer to Nowak [1989: 43] for a plausible reason why the titles of the two essays which appeared in Göttingen in 1868 differ literally by only one word - "facts" versus "hypotheses" - as if Helmholtz aimed at a philosophical reproof of Riemann’s view, see Freudenthal [1964].

Helmholtz renounced the conventional attitude to the basic geometric questions, and like an empiricist he wanted to investigate the nature of space on the basis of experimental facts (Thatsachen). Since he regarded differential (or integral) quantities as derived concepts, Helmholtz could not start from a hypothesis on the line element $ds$ as Riemann did. But he wanted to support Riemann’s assumption concerning the nature of $ds$. For him the most basic observed fact is the free mobility of rigid bodies, and nobody before him seems to have used mathematics to analyze the logical consequences of this.

Helmholtz formulated four axioms $H_1 - H_4$, the first of which is similar to Riemann’s postulate that space is (in modern language) an $n$-dimensional manifold with differentiability properties. For simplicity, we can say the second axiom $H_2$ expresses the idea of a metric (distance function), and the motions of the space are the transformations which preserve both the metric and the orientation. $H_3$ explains the meaning of "free movability" (see (63)). In fact, a body moves due to the motion of the space, so the whole space itself is like a rigid body. Helmholtz assumed that by fixing $n-1$ general points of a body in $n$-dimensional space, the remaining mobility is restricted to a 1-parameter family of motions regarded as "rotations". As a consequence, the position of a rigid body, generally speaking, depends on $n(n+1)/2$ quantities. Finally, $H_4$ is the monodromy axiom, according to which the above 1-parameter family of motions is periodic, so that the body eventually returns to its initial position as the parameter increases to a certain value.

From his axioms Helmholtz deduced, in fact, Riemann’s postulate about the squared line element $ds^2$. On the other hand, like Riemann he readily accepted that free movability implies constant curvature, from which he concluded that physical space is either Euclidean or spherical. Furthermore, assuming the space is infinitely large, as was generally believed, Helmholtz wrongfully concluded that its curvature must be zero and is therefore Euclidean. But Beltrami pointed out to him the omission of the hyperbolic geometry, whose curvature is negative, so Helmholtz issued in 1869 a correction to his paper. Now, from the simple observation that rigid bodies exist in the space we live in, Helmholtz arrived at the final solution of his space problem, which we may state as follows (see also Torretti [1978], Chap. 3): Physical space is a 3-dimensional Riemannian manifold with constant curvature.

In 1870 Helmholtz accepted a chair in physics at the university in Berlin, where he became a highly respected colleague of Weierstrass. During the 1870’s he still gave lectures on the origin and meaning of geometric axioms, some of which were published later. He was not the first scholar who argued that the choice between Euclidean and non-Euclidean geometry cannot be resolved by pure geometry. But as a physicist he pointed out that a change in the geometry would impose a change in the laws of mechanics, so a given geometry can be confirmed or refuted by experience. Like Gauss and Lobachevsky many decades
before him, Helmholtz claimed in 1877 that empirical measurements of triangles will decide this question.

The papers Riemann [1867] and Helmholtz [1868] appealed to many scholars interested in the foundations of geometry, as they raised deep mathematical and philosophical questions about the relationship of geometry and the space we live in. The motions in Helmholtz’s approach are distance preserving transformations, and they form a group, which is a concept Helmholtz never mentioned and presumably did not know about. Still, he used notions and methods with a group theoretic flavour, and in this respect he anticipated Klein and Lie, who were the first to stress the importance of the notion of groups in geometry. It was in the fall 1872 that Klein presented his Erlanger Programm on this topic, whereas Lie started to develop his theory of continuous groups in the fall 1873.

Lie was first informed by Klein in the 1870’s about the geometrical works of Helmholtz, but he was not attracted by philosophical speculations concerning the foundation of geometry or the nature of space. On the other hand, it seemed that nobody had so far really questioned the validity of Helmholtz’s mathematical reasoning and the role of his axioms. So Lie became interested in the geometric problem behind the idea of “free movability”, which he referred to as the Riemann-Helmholtz space problem. Encouraged by Klein he realized the problem was well suited for a demonstration of the power of his group theory.

Lie came to Leipzig in the spring 1886, having accepted the chair of geometry after Klein, who had moved to Göttingen. Now Lie was invited to Berlin where the great Meeting of the German Natural Sciences would take place in late September. It seems, however, that he accepted the invitation primarily because of the opportunity to meet Klein in Berlin. Lie’s lecture on September 21 was entitled “Tatsachen, welche der Geometrie zu Grunde liegen”, or “Facts lying at the foundation of geometry”, and in this address he publicly criticized Helmholtz’s famous 1868 paper, even with sharp comments. Lie claimed the monodromy axiom $H4$ was superfluous, at least if the axiom $H3$ on “free movability” is interpreted in the proper way.

Lie argued that his theory of continuous groups would give a more comprehensive and better solution of the basic geometric problems discussed by Helmholtz. His lecture appeared later in 1886 as a note of 5 pages in the Leipziger Berichte, where also Lie’s more detailed elaboration Über die Grundlagen der Geometrie appeared as two articles in 1890 and one in 1892. Lie’s five papers on the topic are collected in GA II, pp. 374-479. However, in Lie [1893] a complete recasting of his solution of the space problem is presented. It should be mentioned that Poincaré had, in fact, solved the special case of dimension $n = 2$ in 1887, also by the use of continuous groups.

Lie worked with infinitesimal groups (Lie algebras) $G$ rather than the corresponding continuous groups (Lie groups) $G$, so first of all he introduced an infinitesimal version of “free movability”, which enabled him to determine the possible local structure of the group $G$ of motions (isometries). This amounts to the determination of infinitesimal generators $A_i$, that is, a basis of $G$ viewed as a vector space. In Lie’s group theory, the Elements of $G$ can be interpreted as vector fields acting on the underlying space. Lie first studied the case of a
3-dimensional geometry, as Helmholtz did, and he showed the condition of free (or maximal) movability implies \( G \) has dimension 6, and moreover, in terms of suitable coordinates there are just the three cases corresponding to the classical geometries \( E^3, S^3, H^3 \), see (128).

**Remarks on homogeneous spaces**

Let us also describe the above result globally from a modern viewpoint, as a pair \( (G, M) \) where \( G \) is the connected isometry group of a 3-dimensional Riemannian manifold \( M \). "Free movability" means, first of all, that \( G \) is transitive, namely each point \( p \) can be moved to any other point \( p' \). Therefore, \( M \cong G/H \) is a homogeneous space, where \( H \subset G \) is the subgroup keeping \( p \) fixed. Moreover, \( H \) identifies with a subgroup of the group \( \cong O(3) \) of local isometries around \( p \). Now, maximal movability around \( p \) means \( H \) should be transitive on the 2-sphere of points at a fixed small distance from \( p \), consequently \( H \cong SO(3) \) or \( O(3) \) and, in particular, \( G \) has dimension 6. But the possibilities for such a pair \( H \subset G \) of Lie groups is very limited. With a fairly standard notation for the groups involved, the solutions representing the classical geometries (128) and elliptic geometry, may be stated for \( n = 3 \) as the following homogeneous spaces

\[
E^3 = \frac{\mathbb{R}^3 \times SO(3)}{SO(3)}, \quad S^3 = \frac{SO(4)}{SO(3)}, \quad P^3 = \frac{SO(4)}{O(3)}, \quad H^3 = \frac{SO(3,1)^+}{SO(3)}, \quad (132)
\]

and these are, in fact, the only possibilities. They represent Euclidean, spherical, elliptic, and hyperbolic geometry, respectively.

The above spaces, however, are not the only ones having constant curvature, even among "well-behaved" manifolds with no boundary (cf. e.g. Section 5.1). But ideas related to local and global topological properties of a space were poorly understood at the time of Lie. In fact, topology and the modern theory of Lie groups and their homogeneous spaces became theories first in the 20th century. On the other hand, although Lie’s solution of the space problem is close to a satisfactory modern solution, he did not remove the differentiability assumptions which sound rather artificial to modern taste, and this remained a major obstacle at the turn of the century.

In the following years various mathematicians, including David Hilbert and Hermann Weyl (1855-1955), contributed to the new formulation and final solution of the problem, which was not found until 1953, by the Belgian mathematician Jacques Tits (1930-). With rather weak topological assumptions, the solution asserts that if a triple of points can be carried by a motion into any other triple having the same mutual distances, then the space is one of the classical geometries (128) in some dimension \( n \), more precisely, either Euclidean, spherical or elliptic, or hyperbolic \( n \)-space. In these spaces the above property is, in fact, the SSS-congruence property for triangles, which is now seen to characterize the classical geometries uniquely.
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