Direct-dynamical Entanglement-Discord relations

Virginia Feldman · Jonas Maziero · A. Auyuanet

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Abstract In this article, by considering Bell-diagonal two-qubit initial states under local dynamics generated by the Phase Damping, Bit Flip, Phase Flip, Bit-Phase Flip, and Depolarizing channels, we report some elegant direct-dynamical relations between geometric measures of Entanglement and Discord. The complex scenario appearing already in this simplified case study indicates that a similarly simple relation shall hardly be found in more general situations.

Keywords Quantum Correlations · Discord · Entanglement

1 Introduction

Since 1935 [1, 2, 3], the quantumness of correlations has triggered philosophical and scientific debates regarding some unusual qualities of the physical world. From these discussions grew quantum information science (QIS) [4, 5, 6], which is excitingly reverberating on several fields of scientific inquire [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19]. There are some key concepts which are very significant for this quantum revolution, among them are quantum contextuality [20, 21], non-locality [22, 23], Steering [24, 25], Entanglement [26, 27, 28, 29, 30, 31, 32], Discord [33, 34, 35, 36], and Coherence [37, 38]. Understanding them and their interplay is a fundamental current problem in QIS. Although some of these quantumness measures have been indirectly related through
purifications, as they are qualitatively and operationally quite distinct, at first sight there is no reason one should expect to exist a direct relation between them.

In recent years special attention has been given to the geometric definition of quantifiers of correlations [39, 40, 41, 42, 43, 44, 45]. This geometric approach requires to choose a measure of distance, which has also been the subject of wide discussions. One specific property that is required for a measure of distance is its contractivity. This feature is particularly important because it ensures that the distance between two quantum states initially in an uncorrelated state with its environment doesn’t increase with time, which assures that the distinguishability doesn’t grow with time. The contractivity of the Trace norm for trace-preserving quantum evolution was proved by Ruskai [46], but for the case of the Hilbert-Schmidt norm Osawa [47] found one example in which this norm is not contractive. More recently, Wang and Schirmer [48] demonstrated necessary and sufficient conditions for contractivity of the Hilbert-Schmidt norm.

Indirect relations between Entanglement and Discord of different bipartitions of a three-partite system were obtained in the literature [49, 50, 51, 52, 53]. However there is no straightforward reason to assume the existence of a direct relation between Entanglement and Discord for a single bipartite system, although numerical and analytical inequalities can be found [54, 55, 56, 57, 58, 59].

In this article we will consider the possible existence of a direct relation between Entanglement and Discord, two of the most prominent quantum correlations. In particular we calculate for initial Bell-diagonal states under the action of several local trace-preserving quantum-channels (Bit Flip, Bit Phase Flip, Phase Damping and Depolarizing) geometric measures for Entanglement and Discord and we find direct relations between them. In Sec. 2 we present the geometric measures of correlations and the corresponding measures of distance. In Sec. 3 we obtain the Entanglement-Discord relations for Bell-diagonal states under the Phase Damping Channel, using the Hilbert-Schmidt norm and the Trace norm. In particular for the Trace norm we find that our geometric measure of Entanglement is equal to the Concurrence of the entangled state. In Sec. 4 we obtain the Entanglement-Discord relations for Bell-diagonal states under the Depolarizing channel. Finally, in Sec. 5 we summarize and discuss our results.

2 Geometric measures of correlations

Concerning the quantification of Entanglement and Discord, we choose a geometric approach in order to set both quantifiers in equal foot, strategy that allows us to find a direct relation between them. We can measure how entangled or Discordant a general bipartite density operator $\rho$ is by its minimum distance to the corresponding set not possessing that property, i.e.:

$$E(\rho) = \min_{\rho^{sep}} d(\rho, \rho^{sep}),$$  

(1)$$D(\rho) = \min_{\rho^{cc}} d(\rho, \rho^{cc}).$$  

(2)

where $\rho^{sep}$ belongs to the set of separables states (zero Entanglement) and $\rho^{cc}$ to the set of classical states (zero Discord).

In order to quantify Entanglement and Discord from a geometric approach, we
choose two particular cases of the Schatten $p$-norm distance, which is defined using the $p$-norm: $||A||_p^p := \text{Tr} \left( \sqrt[p]{A^p} \right)^p$. When $p = 1$ we have the Trace norm that induces the distance $d_1$,

$$d_1(\rho, \sigma) = ||\rho - \sigma||_1$$

and when $p = 2$ we have the Hilbert-Schmidt norm that induces the distance $d_2$,

$$d_2(\rho, \sigma) = ||\rho - \sigma||_2^2.$$  

As we discussed, the Trace norm is contractive; and using the results mentioned above we found that, for the quantum channels we work with: Phase Damping, Depolarizing, Bit Flip and Bit Phase Flip, the Hilbert-Schmidt norm is also contractive.

3 Entanglement-Discord relations for Bell-diagonal states evolving under local Phase Damping channel

Bell-diagonal states have been the workhorse for analytical computations of most quantum correlation functions. This is helpful, for instance, when investigating the dynamics of these correlations under decoherence [48, 49, 50, 51], for their experimental estimation [52, 53, 54], and to more easily visualize and comprehend the geometrical structure of the quantum state space [55, 56, 57, 58]. The Bell-diagonal class of states is equivalent, under local unitaries, to two-qubit states with maximally mixed marginals that can be written as follows:

$$\rho = 2^{-2} \left( I_4 + r \cdot \Sigma \right),$$

with $I_n$ denoting the $n \times n$ identity operator and we define $\Sigma = \sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3$ and $r = r_1 \hat{1} + r_2 \hat{j} + r_3 \hat{k}$. The last vector is dubbed here as the correlation vector, because $r_j = \text{Tr}(\rho \sigma_j \otimes \sigma_j)$. The set $\{i, j, k\}$ is the usual orthonormal basis for $\mathbb{R}^3$ and $\sigma_j$ are the Pauli matrices.

In the following, we study a two-qubit system $ab$ prepared in the state $\rho$ with each party interacting with its own local-independent environment. Thus, if the overall initial state is $\rho \otimes |E_0^a\rangle \langle E_0^a| \otimes |E_0^b\rangle \langle E_0^b|$, the closed system evolved state is $U_a \otimes U_b (\rho \otimes |E_0^a\rangle \langle E_0^a| \otimes |E_0^b\rangle \langle E_0^b|) U_b^\dagger \otimes U_a^\dagger$, with $U_s$ being an unitary transformation in $\mathcal{H}_s \otimes \mathcal{H}_{E_s}$, for $s = a, b$. The partial trace leads then to the evolved state for the bipartite system $ab$:

$$\rho_f = \sum_{m,n} K_{m}^{a} \otimes K_{n}^{b} \rho K_{n}^{a \dagger} \otimes K_{m}^{b \dagger},$$

with the matrix elements of the Kraus’ operators being $\langle s_j | K_{m}^{a} | s_k \rangle = \langle s_j | \otimes \langle E_{m}^{a} | U_{j}^{a} (| s_k \rangle \otimes | E_{k}^{a} \rangle)$, where $|s_j\rangle$ and $|E_{m}^{a}\rangle$ are orthonormal basis for $\mathcal{H}_s$ and $\mathcal{H}_{E_s}$, respectively. Using this, we obtain direct-dynamical Discord-Entanglement relations considering two qubits prepared in a Bell-diagonal state interacting with the Phase Damping channel.

The Phase Damping Channel represents the main source of decoherence in quantum systems. This kind of environment obtains information about the system’s state without exchange of energy. From the phenomenological map describing this process we can obtain the Kraus operators $K_{0}^{pd} = \sqrt{1-p}I_2$, $K_{1}^{pd} = \sqrt{p} |s_0\rangle \langle s_0|$, where
Fig. 1 (color online) $\mathbb{R}^3$ representation of the Bell-diagonal state space. The $(r_1, r_2, r_3)$ points in the tetrahedron are the physical states. The entangled states are those in the yellow regions. The separable convex subset is the green octahedron. The three black axes are the classical-classical states. The dotted lines represent the “temporal” trajectories for the initial state $r_1 = r_2 = r_3 = -0.7$ evolved under local Phase Damping, Bit Flip, Bit Phase Flip and Depolarizing channels.

and $K_{pd} = \sqrt{p_a |s_1\rangle\langle s_1|}$. Note that $p_a$ is the probability for a change in the environment state, which is conditioned on the system state. From now on we will assume symmetric environments, i.e., $p_a = p_b \equiv p$. Substituting these Kraus operators in Eq. (6), we obtain the evolved correlation vector:

$$r_{pd}(p) = r_1(1 - p)^2\hat{i} + r_2(1 - p)^2\hat{j} + r_3\hat{k}. \quad (7)$$

Following this evolution, the Bell-diagonal form of the evolved state is preserved. In Fig. 1 we show the Bell-diagonal state space [66], its classical-classical and separable sub-sets, and examples of trajectories for the dynamics considered in Sec. 3 and in Sec. 4.

3.1 Entanglement - Discord relation using the Hilbert-Schmidt norm

In order to calculate the Discord we use the geometric measure introduced by [43]:

$$D(\rho) = \min_{\chi} \|\rho - \chi\|_2^2, \quad (8)$$

with the minimum $\chi$ taken over the set of the states with zero Discord. As it was noted by [66], for the Bell-diagonal states the only states with zero Discord lay in the Cartesian axes thus the Discord is determined by the euclidean distance from the point $(|r_1(p)|, |r_2(p)|, |r_3(p)|)$ to its closest axis. Taking this into account we define the Discord:

$$D(p) = \text{Min}\{D_1(p), D_2(p), D_3(p)\}, \quad (9)$$

where

$$D_i(p) = r_j(p)^2 + r_k(p)^2, \quad i, j, k = 1, 2, 3, \quad i \neq j \neq k. \quad (10)$$
As we will discuss later, depending on the initial value of the correlation vector, the Discord could present "sudden changes" during the evolution of the system, due to the fact that $D(p)$ could change between the three different functions: $D_1(p)$, $D_2(p)$ or $D_3(p)$.

For the Entanglement calculation we use the known result \cite{18} that for hermitian matrices which pertain to the same trace class, the Hilbert-Schmidt distance between them equals the Euclidean distance between their corresponding correlation vectors: $\|\rho - \zeta\|_2 = \|r - z\|_2$.

The region of separable states is bounded by the octahedron \cite{66} defined by $|r_1| + |r_2| + |r_3| \leq 1$. Using the symmetry of the separable states region in the Bell-diagonal space, we can calculate the Entanglement for the state with evolved correlation vector $r(p)$ as the euclidean distance from the point $(|r_1(p)|, |r_2(p)|, |r_3(p)|)$ to the plane $|r_1| + |r_2| + |r_3| = 1$, when $(|r_1(p)| + |r_2(p)| + |r_3(p)|) \geq 1$. In order to relate Discord Entanglement, we define:

$$E(p) = \min_{\rho^{SEP}} \|\rho - \rho^{SEP}\|_2^2,$$

which for the case of Phase Damping takes the following expression:

$$E(p) = \frac{(|r_1|(1-p)^2 + |r_2|(1-p)^2 + |r_3| - 1)^2}{3}.$$  \hspace{1cm} (11)

Depending on initial conditions we have three different scenarios: (1) considering initial Bell-diagonal states with $|r_1|, |r_2| < |r_3|$, the Discord takes the form $D_3(p)$ for every $p$.

(2) When $|r_2|, |r_3| < |r_1|$ ($|r_1| \neq 0$), the Discord exhibits a sudden change for $p_{13} = 1 - \sqrt{\frac{|r_2|}{|r_1|}}$. It starts equal to $D_1(p)$ for $p \leq p_{13}$, and then it becomes $D_3(p)$ for $p \geq p_{13}$.

(3) Similarly, if $|r_1|, |r_3| < |r_2|$ ($|r_2| \neq 0$), the Discord starts as $D_2(p)$ and then it becomes $D_3(p)$ for $p \geq p_{23}$, where $p_{23} = 1 - \sqrt{\frac{|r_3|}{|r_2|}}$.

Taking into account that the Entanglement undergoes sudden death for $p_{SD} = 1 - \sqrt{\frac{1-|r_3|}{|r_1| + |r_2|}}$, we can express the Discord for $p \in [0, p_{SD}]$ as a function of the Entanglement in $p$ and the parameter $|r_3|$.

(1) When $D(p) = D_1(p)$ or $D_2(p)$, we have $D(p) = D_i(p)$ with

$$D_i(p) = |r_j|^2 \left(\frac{3E(p) - |r_3| + 1}{|r_1| + |r_2|}\right)^2 + |r_3|^2,$$  \hspace{1cm} (13)

where $i, j = 1, 2$ and $i \neq j$.

(2) In the case that $D(p) = D_3(p)$,

$$D(p) = \left(|r_1|^2 + |r_2|^2\right) \left(\frac{3E(p) - |r_3| + 1}{|r_1| + |r_2|}\right)^2.$$  \hspace{1cm} (14)

In Fig.4, for the case where the initial conditions verify $|r_3| < |r_2| < |r_1|$, we show the curve of Discord as a function of Entanglement exhibiting one of the sudden changes described previously.
3.2 Entanglement - Discord relation using the Trace norm

In the following, we calculate Entanglement and Discord based on the trace norm:

\[ E(\rho) = \min_{\rho^{sep}} \| \rho - \rho^{sep} \|_1, \]
\[ D(\rho) = \min_{\rho^{cc}} \| \rho - \rho^{cc} \|_1. \]

(15)

3.2.1 Entanglement

As far as we know there is no expression for the Entanglement by means of the trace distance. In order to calculate Entanglement we need to find the closest separable state to a Bell diagonal state. There are several proposals to determine the closest separable state using different criteria \cite{68, 69, 70} and different measures of distance. In this section we use a result obtained in the context of multipartite systems \cite{71}, which shows that the closest biseparable state to a general X state is also an X state. We propose the following criterion: to search for the closest separable state to an entangled X state over the X states with the same populations as the entangled one. With this assumption we get a very meaningful result.

Proposition 1 Let \( \rho_X \) be the density matrix associated to a quantum X state and \( \sigma_X \) its closest separable state (with the same populations as \( \rho_X \)) under the trace distance. The trace distance between \( \rho_X \) and \( \sigma_X \) is the Concurrence of \( \rho_X \).

Proof - Let’s consider \( \rho_X \), a density matrix associated to a general X state:

\[ \rho_X = \begin{pmatrix}
  a & 0 & 0 & e \\
  0 & b & f & 0 \\
  0 & f & c & 0 \\
  e & 0 & 0 & d
\end{pmatrix}. \]

To represent a physical state, \( e \) and \( f \) must fulfill the following restrictions: \( |e| \leq \sqrt{ad} \) and \( |f| \leq \sqrt{bc} \). We will search for the closest separable state to \( \rho_X \) over the set of X matrices constrained to having the same diagonal as \( \rho_X \):

\[ \sigma_X = \begin{pmatrix}
  a & 0 & 0 & e' \\
  0 & b & f' & 0 \\
  0 & f' & c & 0 \\
  e' & 0 & 0 & d
\end{pmatrix}. \]

If \( \sigma_X \) represents a physical and separable state, \( e' \) and \( f' \) must fulfill the following condition:

\[ |e'|, |f'| \leq \min\{\sqrt{ad}, \sqrt{bc}\}. \]

(16)

As we are searching for the closest separable X state according to the trace distance, we have to minimize

\[ \| \rho_X - \sigma_X \|_{TD} = 2 \left( |(|e| - |e'|)| + (|f| - |f'|) \right), \]

(17)

where we used the fact that the minimum is attained when the off diagonal elements of \( \rho_X \) and \( \sigma_X \) have the same sign.

The expression above reaches its minimum value when both terms are 0, but this only happens when both matrices are equal. In the general case we can only equate to zero one term, depending on the value of the elements of \( \rho_X \): (1) if \( \sqrt{bc} < \sqrt{ad} \)
it is easy to see that the minimum is reached when \( f' = f \) and \( e' = \sqrt{ad} \). In this case we have
\[
\|\rho_X - \sigma_X\|_1 = 2|e| - \sqrt{bc}.
\] (18)

(2) If \( \sqrt{ad} < \sqrt{bc} \) the minimum is reached when \( e' = e \) and \( f' = \sqrt{ad} \). In this case we have
\[
\|\rho_X - \sigma_X\|_1 = 2|f| - \sqrt{ad}.
\] (19)

We can easily see that for \( \sqrt{ad} = \sqrt{bc} \), the matrix \( \rho_X \) is separable.

As demonstrated by [72], for a general \( X \) state the Concurrence has the following expression:
\[
C(\rho_X) = 2 \max\{0, |e| - \sqrt{bc}, |f| - \sqrt{ad}\}.
\] (20)

This bring us to our result: the trace distance between a general \( X \) state and its closest separable state is the Concurrence of the general \( X \) state.

Taking into account this result we have for the geometric Entanglement:
\[
E(\rho) = \min_{\rho^\text{sep}} d(\rho, \rho^\text{sep}) = C(\rho).
\] (21)

This result allows us to calculate the trace norm based geometric Entanglement, even for evolutions that don’t preserve the Bell-diagonal form, as for example \( X \) states under the action of the Amplitude Damping channel.

At this point it is worth mentioning that for the evolution of the Bell-diagonal states under the Phase Damping channel, the Entanglement is determined by the sign of \( r_3 \): if \( r_3 > 0 \) then \( C_1(\rho_X) = 2|e| - \sqrt{bc} \) and if \( r_3 < 0 \), \( C_2(\rho_X) = 2|f| - \sqrt{ad} \), for all the evolution until become separable. In a pictorial view: the Entanglement of states that belong to the upper part of the tetrahedron is described by \( C_1(\rho_X) \) and the Entanglement of states that belong to the lower part of the tetrahedron is described by \( C_2(\rho_X) \).

3.2.2 Discord

Expressions to calculate the Discord by the Schatten-1 norm for the Bell-diagonal states are known [73],
\[
D = \text{int}\{|c_1|, |c_2|, |c_3|\},
\] (22)

where “int” means the intermediate value.

3.2.3 Discord-Entanglement relations

Having properly defined the geometric quantifiers for Entanglement and Discord using the Schatten-1 norm we can relate them. To calculate the Discord we have to establish the ordering between the absolute values of the coefficients \( |r_i| \) as we did when we used the Hilbert-Schmidt norm.

(1) Considering \( |r_i| < |r_j| < |r_3| \), where \( i, j = 1, 2 \) and \( i \neq j \), it is clear that the Discord \( D \) is expressed by \( D = |r_j(p)| \) for all the evolution.

The Entanglement of the state quantified by the concurrence is \( C(p) = \frac{1}{2} \max\{|r_2 \pm r_1|(1 - p)^2 - |1 \pm r_3|\} \). Combining the two expressions we have:
\[
D(p) = \frac{2C(p) + |1 \pm r_3|}{|r_2 \pm r_1|} |r_j|, \forall p \in [0, p_{SD1}],
\] (23)
Fig. 2 (color online) Discord as a function of Entanglement for the initial state \( r_1 = 0.65, r_2 = 0.59, r_3 = -0.38 \) evolved under local Phase Damping channel. The thick (red) line corresponds to the Trace distance and the dashed (blue) line to the Hilber-Schmidt distance.

where \( p_{SD1} \) is when Entanglement sudden death occurs; for this case \( p_{SD1} = 1 - \sqrt{\frac{1 + r_3}{|r_2|}} \).

(2) When \( |r_1| < |r_3| < |r_2| \), the Discord exhibits a sudden change in \( p_I = 1 - \sqrt{\frac{|r_3|}{|r_2|}} \); it starts being equal to \( |r_3| \) for \( p \leq p_I \) and then becomes equal to \( |r_2(p)| \) for \( p \geq p_I \). We note that \( p_I = p_{23} \), \( p_{23} \) being the time for the sudden change of geometric Discord calculated using the Hilbert-Schmidt norm.

(3) When \( |r_3| < |r_2| < |r_1| \), the Discord exhibits two sudden changes, in \( p_I \) and in \( p_{II} = 1 - \sqrt{\frac{|r_2|}{|r_1|}} \), verifying for this initial conditions that \( p_I < p_{II} \). For \( p < p_I \) the Discord \( D \) is expressed by \( D = |r_3(p)| \), for \( p_I \leq p < p_{II} \) the Discord is expressed by \( D = |r_3| \) and for \( p_{II} \leq p \)

\[
D(p) = |r_1(p)| = \frac{2C(p) + |1 \pm r_3|}{|r_2 \pm r_1|} |r_1|.
\]

(24)

We note that, analogous to case (2), \( p_{II} = p_{13} \), \( p_{13} \) being the time for the sudden change of the Hilbert-Schmidt norm based geometric Discord.

The same results are found for the Bit Flip and the Bit Phase Flip channels as we can see from their correlation vectors:

\[
r_{bf}(p) = r_1 \hat{i} + r_2(1 - p) \hat{j} + r_3(1 - p) \hat{k},
\]

(25)

\[
r_{bpf}(p) = r_1(1 - p) \hat{i} + r_2 \hat{j} + r_3(1 - p) \hat{k},
\]

(26)

which are obtained interchanging \( r_3 \) with \( r_1 \) and \( r_3 \) with \( r_2 \), respectively, in the Phase Damping channel.

In Fig. 2 we also show the curve of Discord as a function of Entanglement, for the Trace distance with initial condition \( |r_3| < |r_2| < |r_1| \), exhibiting two sudden changes.

4 Entanglement-Discord relations for Bell-diagonal states evolving under local Depolarizing channel

The Depolarizing channel describes the interaction of the system with the environment that gets its state mixed up with the maximally entropic state, with a
probability $p$, i.e., $\rho \rightarrow (1-p)\rho + p(2^{-1/2})$. This is equivalent to the possibility of occurrence of all the Pauli errors with equal probability $p/4$. The Kraus operators for this kind of noise channel are $K_0^d = \sqrt{1-3p/4} I_2$, $K_1^d = \sqrt{p/4} \sigma_1$, $K_2 = \sqrt{p/4} \sigma_2$, and $K_3 = \sqrt{p/4} \sigma_3$. And the evolved correlation vector is:

$$r_d(p) = r_1(1-p)^2i + r_2(1-p)^2j + r_3(1-p)^2k. \quad (27)$$

4.1 Entanglement - Discord relation using the Hilbert-Schmidt norm

Applying the same reasoning we used above, we determine the Entanglement calculating the distance of the quantum state to the separable octahedron. For this channel the Entanglement is expressed by

$$E(p) = \frac{1}{3} \left( (1-p)^2 (|r_1| + |r_2| + |r_3|) - 1 \right)^2, \quad (28)$$

the sudden death of Entanglement occurs for $p_{SD} = 1 - \frac{1}{\sqrt{|r_1| + |r_2| + |r_3|}}$.

For the Discord we have to determine the minimum

$$D_i(p) = (1-p)^4 \left( r_j^2 + r_k^2 \right), \quad i, j, k = 1, 2, 3,$n\not= j \not= k. \quad (29)$$

For this channel the Discord dynamics is very simple, never exhibiting a sudden change which is obvious from its mathematical expression.

Once more, Discord and Entanglement can be analytically related:

$$D_i(p) = \left( \frac{\sqrt{3}E(p) + 1}{(|r_1| + |r_2| + |r_3|)} \right)^2 \left( r_j^2 + r_k^2 \right), \quad i \not= j \not= k. \quad (30)$$

4.2 Entanglement - Discord relation using the Trace norm

As we discussed previously, the geometric Entanglement for the Bell-diagonal states, measured by the Trace norm is the concurrence of the state. In this case:

$$C(p) = \frac{1}{2} \max\{0, |r_1 \pm r_2|(1-p)^2 - \left( 1 \pm r_3(1-p)^2 \right) \} \quad (31)$$

The expression of the geometric Discord is

$$D(p) = |r_{int}|(1-p)^2 \quad (32)$$

and the relation between the geometric Discord and the geometric Entanglement is:

$$D(p) = |r_{int}| \left( \frac{2C(p) + 1}{|r_1 \pm r_2| + r_3} \right), \quad (33)$$

where the $\pm$ depends on the maximum of Eq. (31).

In Fig. 3, we show the curves of Discord as a function of Entanglement obtained in this section for the Hilbert-Schmidt and the Trace distance.
Fig. 3 (color online) Discord as a function of Entanglement for the initial state $r_1 = 0.65, r_2 = 0.59, r_3 = -0.38$ evolved under local Depolarizing channel. The thick (red) line corresponds to the Trace distance and the dashed (blue) line to the Hilber-Schmidt distance.

5 Concluding remarks

In this work, using a geometric approach, we obtained dynamical Entanglement-Discord relations for a family of initial Bell-diagonal states under the action of several local trace-preserving quantum-channels: Bit Flip, Bit Phase Flip, Phase Damping and Depolarizing. It is worth pointing out that these channels keep the Bell-diagonal feature of the initial Bell-diagonal states, this made possible to work with closed analytic expressions. A direct dynamical relation as the one presented here can be useful in several instances. As a matter of fact, whenever a result is derived for one of the quantities, we may apply it to simplify the description of the other one. An interesting example related to the obtention of an equation of motion for Discord in terms of the equation of motion for Entanglement can be considered. In the last few years, several works were dedicated to obtain factorization equations for Entanglement \[74, 75, 76, 77, 78\], which can considerably simplify the experimental description of its dynamics. Via our relations, we can use those results in order to simplify the experimental description of the evolution of quantum Discord with time.

In addition, we obtained a very meaningful result working with the geometric measure of Entanglement in terms of the trace norm: we found that the trace distance between a general entangled X state and its closest separable state is the concurrence of the former. This result would allow to extend our study to more general evolutions that don’t preserve the Bell-diagonal shape of the states, such as the Amplitude Damping channel. Some work in that direction is in progress.

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