RATIONAL PERIOD FUNCTIONS ON THE
HECKE GROUPS *

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July 11, 2000

Abstract

We describe rational period functions on the Hecke groups and characterize the ones whose poles satisfy a certain symmetry. This generalizes part of the characterization of rational period functions on the modular group, which is one of the Hecke groups.

1 INTRODUCTION

Marvin Knopp [11] introduced the idea of a rational period function (RPF) for an automorphic integral of weight $2k$ ($k \in \mathbb{Z}$) on $\Gamma$, where $\Gamma$ is any Fuchsian group acting on the upper half-plane.

Knopp [11, 12], Hawkins [6], and Choie and Parson [2, 3] took steps toward an explicit characterization of RPFs on the modular group $\Gamma(1)$. Ash [1] used cohomological techniques to provide a characterization, after which Choie and Zagier [4] and, independently, Parson [14] provided an explicit characterization of the RPFs on $\Gamma(1)$. The explicit characterizations use negative continued fractions to establish a connection between the poles of RPFs and binary quadratic forms.

Schmidt [16] generalized Ash’s work, giving an abstract characterization of RPFs on any finitely generated Fuchsian group of the first kind with parabolic elements, a class of groups which includes the Hecke groups. Schmidt [15] and Schmidt and Sheingorn [17] have taken steps toward an explicit characterization of RPFs on the Hecke groups using generalizations of the classical continued fractions and binary quadratic forms.

In this paper we give an explicit characterization of a class of rational period functions on the Hecke groups. Our characterization is for rational period functions of weight $2k$, with $k$ a positive odd integer, and with irreducible pole

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*1991 Mathematics Subject Classification. Primary 11F67; Secondary 11F12, 11E45.
sets which are Hecke-symmetric, as defined in Section 2.3. We also determine
an explicit expression for all rational period functions on Hecke groups. We use
the correspondence, developed in [5], between poles of rational period functions
and generalized binary quadratic forms.

2 DEFINITIONS AND BACKGROUND IDEAS

2.1 Hecke groups

Let \( \lambda \) be a positive real number and let \( G(\lambda) = \langle S, T \rangle/\{\pm I\} \), where \( S = S_\lambda = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \), \( T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \), and \( I \) is the identity. Erich Hecke [8] showed that the only values of \( \lambda \) between 0 and 2 for which \( G(\lambda) \) is discrete are \( \lambda = \lambda_p = 2\cos(\pi/p) \) for \( p = 3, 4, 5, \ldots \). These are the Hecke groups which we will denote by \( G_p = G(\lambda_p) \) for \( p \geq 3 \). Put \( U = U_{\lambda_p} = S_{\lambda_p}T \in G_p \). The Hecke groups have two relations, \( T^2 = U^p = I \).

Hecke showed that \( G(\lambda) \) is also discrete if \( \lambda \geq 2 \), however these groups have only one relation \( T^2 = I \). In this case the related rational period functions have a simpler structure, which is given in [5].

For \( M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_p \), we have \( a, b, c, d \in \mathbb{Z}[\lambda_p] \) and \( ad - bc = 1 \), so \( G_p \) is a subgroup of \( \text{SL}(2, \mathbb{Z}[\lambda_p]) \). It is well-known that \( \text{G}_1 = \text{SL}(2, \mathbb{Z}[\lambda]) \) (i.e., \( \Gamma(1) = \text{SL}(2, \mathbb{Z}) \)), however for the other Hecke groups \( G_p \subseteq \not\subseteq \text{SL}(2, \mathbb{Z}[\lambda_p]) \).

Members of Hecke groups act on the Riemann sphere as linear fractional transformations. An element \( M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) is hyperbolic if \( |a + d| > 2 \), parabolic if \( |a + d| = 2 \), and elliptic if \( |a + d| < 2 \). We will designate fixed points accordingly.

We will use the following lemma when we study the poles of rational period functions.

Lemma 1. Fix \( p \geq 3 \) and let \( U = U_{\lambda_p} \). The nonzero entries of \( U^nT \in G_p \) are all positive for \( 1 \leq n \leq p - 1 \). The only zero entries occur in \( UT = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \) and \( U^{p-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \).

Proof. Write \( U^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \). By induction on \( n \) we have that \( a_n = \frac{\sin((n+1)\pi/p)}{\sin(\pi/p)} \) for \( n \geq 0 \), from which it is clear that \( a_n > 0 \) for \( 0 \leq n \leq p - 2 \). We write \( U^n = UU^{n-1} = U^{n-1}U \), or

\[
U^n = \begin{pmatrix} \lambda a_{n-1} + b_{n-1} & -a_{n-1} \\ \lambda c_{n-1} + d_{n-1} & -c_{n-1} \end{pmatrix}.
\]

Then

\[
U^n = \begin{pmatrix} a_n & -a_{n-1} \\ a_{n-1} & -a_{n-2} \end{pmatrix},
\]

so

\[
U^nT = \begin{pmatrix} a_{n-1} & a_n \\ a_{n-2} & a_{n-1} \end{pmatrix},
\]

which has positive entries for \( 2 \leq n \leq p - 2 \).
2.2 Rational period functions

For \( M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{R}) \) and \( f(z) \) a complex function, we define the weight 2k slash operator \( f \mid_{2k} M = f \mid M \) by

\[
(f \mid M)(z) = (cz + d)^{-2k} f(Mz).
\]

**Definition 1.** Let \( U = U_\lambda^p \) for \( p \geq 3 \) and let \( k \in \mathbb{Z}^+ \). A rational period function (RPF) of weight 2k for \( G_p \) is a rational function \( q(z) \) which satisfies the relations

\[
q + q \mid T = 0, \quad (1)
\]

and

\[
q + q \mid U + q \mid U^2 + \cdots + q \mid U^{p-1} = 0. \quad (2)
\]

Marvin Knopp [11] first introduced RPFs as period functions for automorphic integrals in the following way.

**Definition 2.** Let \( S = S_\lambda^p \) for \( p \geq 3 \), and let \( k \in \mathbb{Z}^+ \). Suppose that \( F \) is a function meromorphic in \( \mathcal{H} \) and at \( i\infty \) which satisfies

\[
(F \mid S)(z) = F(z),
\]

and

\[
(F \mid T)(z) = F(z) + q(z),
\]

where \( q(z) \) is a rational function. Then \( F \) is an automorphic integral of weight 2k on \( G_p \) with rational period function \( q(z) \).

These definitions are equivalent. It is easy to show that any rational period function satisfying Definition 2 also satisfies Definition 1. On the other hand, Knopp showed [10, Theorem 3] that any function satisfying Definition 1 is the period function for an automorphic integral of weight 2k.

For any rational period function \( q \) on \( G_p \) we let \( P(q) \) denote the set of poles of \( q \). Hawkins [6] introduced the idea of an irreducible system of poles (or irreducible pole set), the minimal set of poles which are forced to occur together by the relations (1) and (2). If \( \alpha \in P^*(q) = P(q) \setminus \{0\} \), we let \( P(q; \alpha) \) denote the irreducible set of poles of \( q \) which contains \( \alpha \).

2.3 Binary quadratic forms

An indefinite binary quadratic form on \( \mathbb{Z}[^p \lambda] \) is an expression of the form

\[
Q(x, y) = Ax^2 + Bxy + Cy^2,
\]
with \( A, B, C \in \mathbb{Z}[\lambda_p] \) and \( D = B^2 - 4AC > 0 \). We denote such a form by \( Q = [A, B, C] \) and refer to it as a \( \lambda_p \)-BQF. If \( \mathbb{Z}[\lambda_p] \) is a principal ideal domain we also require that a \( \lambda_p \)-BQF \([A, B, C]\) be primitive, i.e., that \((A)+(B)+(C) = (1)\), where \( (x) = x\mathbb{Z}[\lambda_p] \) denotes the ideal of \( \mathbb{Z}[\lambda_p] \) generated by \( x \).

We need a \( 1 \rightarrow 1 \) correspondence between \( \lambda_p \)-BQFs and certain algebraic numbers. We map the \( \lambda_p \)-BQF \( Q = [A, B, C] \) to the number \( \alpha_Q = \frac{-B + \sqrt{D}}{2A} \in \mathbb{Q}(\lambda_p, \sqrt{D}) \). In this form the mapping is invertible only for \( p = 3 \), \( G_p = \Gamma(1) \). When \( p > 3 \), ambiguity arises from the presence of nontrivial units in \( \mathbb{Z}[\lambda_p] \) and from the fact that the discriminant \( D \) need not be square-free in \( \mathbb{Z}[\lambda_p] \). In order to recover a \( 1 \rightarrow 1 \) correspondence from this mapping we first restrict the range of the mapping to hyperbolic fixed points of \( G_p \). Then we give an algorithm which produces a unique inverse for any hyperbolic number.

**Lemma 2.** If \( \alpha \) is a hyperbolic fixed point of \( G_p \) it may be associated with a unique indefinite \( \lambda_p \)-BQF \( Q = Q_\alpha \) such that \( \alpha = \alpha_Q \). If \( \mathbb{Z}[\lambda_p] \) is a principal ideal domain, \( Q \) may be chosen to be primitive.

If an indefinite \( \lambda_p \)-BQF \( Q \) is associated with a hyperbolic number as in the Lemma, we say that \( Q \) is hyperbolic.

**Proof.** We present an outline of the more detailed proof in [5]. Suppose that \( \alpha \) is a hyperbolic fixed point of \( G_p \). Let \( M_\alpha = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in G_p \) be a generator of the cyclic group of matrices fixing \( \alpha \). \( M_\alpha \) is determined up to inverses and

\[
\alpha = \frac{a - d \pm \sqrt{(a + d)^2 - 4}}{2c}.
\]

We show in [5] that if \( \mathbb{Z}[\lambda_p] \) is a principal ideal domain, there is a uniquely determined positive number \( g \in \mathbb{Z}[\lambda_p] \) with \((c, d - a, -b) = (g)\). Then \( \frac{1}{g}[c, d - a, -b] \) is primitive and the unique primitive \( \lambda_p \)-BQF for \( \alpha \) is

\[
Q_\alpha = \begin{cases} 
\frac{1}{g}[c, d - a, -b], & \text{if } \alpha = \frac{a - d + \sqrt{(a + d)^2 - 4}}{2c}, \\
\frac{1}{g}[c, d - a, -b], & \text{if } \alpha = \frac{a - d - \sqrt{(a + d)^2 - 4}}{2c}.
\end{cases}
\]

If \( \mathbb{Z}[\lambda_p] \) fails to be a principal ideal domain we put \( g = 1 \). \( \square \)

Suppose that \( \alpha = \frac{-B + \sqrt{D}}{2A} \) is a hyperbolic point associated with the \( \lambda_p \)-BQF \( Q_\alpha = [A, B, C] \). We define the Hecke \( (\lambda_p) \)-conjugate of \( \alpha \) to be \( \alpha' = \frac{-B - \sqrt{D}}{2A} \), the number associated with \(-Q_\alpha = [-A, -B, -C]\), i.e., \( \alpha'Q = \alpha - Q \). A calculation shows that if \( V \in G_p \), then \((\alpha)V = V\alpha' \). In the case of \( G_3 = \Gamma(1) \), Hecke conjugation reduces to algebraic conjugation over \( Q \).

If \( R \) is a set of hyperbolic fixed points of \( G_p \) we write \( R' = \{ x' \mid x \in R \} \). We say that \( R \) has Hecke \( (\lambda_p) \)-symmetry if \( R = R' \).

Elements of a Hecke group act on \( \lambda_p \)-BQFs by \((Q \circ M)(x, y) = Q(ax + by, cx + dy)\) for \( M = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in G_p \). This action preserves the discriminant and maps primitive forms to primitive forms. We say that \( \lambda_p \)-BQFs \( Q_1 \) and
are $G_p$-equivalent, and write $Q_1 \sim Q_2$, if there exists a $V \in G_p$ such that $Q_2 = Q_1 \circ V$. It is easy to check that $G_p$-equivalence is an equivalence relation, so $G_p$ partitions the $\lambda_p$-BQFs into equivalence classes of forms.

Every $G_p$-equivalence class of $\lambda_p$-BQFs contains either all hyperbolic forms or no hyperbolic forms. Consequently, we may label equivalence classes themselves as hyperbolic or non-hyperbolic.

Suppose that $Q_1$ and $Q_2$ are hyperbolic $\lambda_p$-BQFs. A calculation shows that $Q_2 = Q_1 \circ V$ if and only if $\alpha_2 = V^{-1}\alpha_1$, where $\alpha_1$ and $\alpha_2$ are associated with $Q_1$ and $Q_2$, respectively. Thus $G_p$-equivalence of hyperbolic $\lambda_p$-BQFs induces a corresponding $G_p$-equivalence of associated numbers.

If $A$ denotes a $G_p$-equivalence class of $\lambda_p$-BQFs, we let $-A = \{-Q|Q \in A\}$. Then $-A$ is another $G_p$-equivalence class of forms, not necessarily distinct from $A$. If $A$ is hyperbolic, so is $-A$, and the numbers associated with the forms in $-A$ are the Hecke conjugates of the numbers associated with the forms in $A$.

We say that a hyperbolic $\lambda_p$-BQF $Q = [A, B, C]$ is ($G_p$-)simple if $A > 0 > C$. If $Q$ is $G_p$-simple we say that the associated hyperbolic number $x_Q$ is ($G_p$-)simple. We show in [5] that a hyperbolic fixed point $x$ is $G_p$-simple if and only if $x' < 0 < x$.

3 POLES OF RATIONAL PERIOD FUNCTIONS

Throughout this section we fix $p \geq 3$, $\lambda = \lambda_p$, and $U = U_{\lambda_p}$. We assume that $q$ an RPF of weight $2k \in 2\mathbb{Z}^+$ on $G_p$ with pole set $P = P(q)$.

3.1 Sets of Poles

In this section we show that the positive poles of $q$ may be put into cycles, which we use to write the irreducible systems of poles. We also make the connection between these cycles of poles and simple $\lambda_p$-BQFs.

Lemma 3. $P(q) \subset \mathbb{R}$ for any RPF $q$ of weight $2k \in 2\mathbb{Z}^+$ on $G_p$.

Remark. This is given without proof by Meier and Rosenberger [3], who state that the proof is analogous to the one by Knopp [4] for rational period functions on the modular group. The following proof is a generalization of the one by Choie and Zagier [1] for the modular group.

Proof. Suppose, by way of contradiction, that $\alpha_1 \in P$ but $\alpha_1 \notin \mathbb{R}$. By the first relation [1] we have $T\alpha_1 \in P$, then by the second relation [2] we have $U^{j_1} T\alpha_1 \in P$ for some $j_1$, $1 \leq j_1 \leq p - 1$. Repeating this, we produce a sequence of poles $\{\alpha_1, \alpha_2, \ldots\}$, none of which are real, with $\alpha_{\nu+1} = U^{j_{\nu}} T\alpha_{\nu}$ and $1 \leq j_{\nu} \leq p - 1$ for each $\nu \geq 1$. Lemma [3] and a geometric argument show that $|\arg(\alpha_{\nu})| > |\arg(\alpha_{\nu+1})|$ for each $\nu \geq 1$. But this is impossible, since $P$ is a finite set. 

5
Put $\lambda = \lambda_p$ and $U = U_\lambda$ for $p \geq 3$. Define $\Phi_p : [0, \infty) \to [0, \infty)$ by

$$
\Phi_p(x) = \begin{cases} 
TU_{ij}(0) \leq x < U_{i-1}(0) 
\end{cases}
$$

This function is given more explicitly by

$$
\Phi_p(x) = \begin{cases} 
x \leq 0 
\end{cases}
$$

It can be shown that $TU^n(x) > 0$ if and only if $U_{i-1}(0) < x < U_{i}(0)$, so the exponent $n$ in $\Phi_p(x) = TU^n(x)$ is the unique exponent between 1 and $p-1$ for which $TU^n(x) > 0$.

If $\mathcal{A}$ is a hyperbolic $G_p$-equivalence class of $\lambda_p$-$BQFs$ we write $Z_A = \{z \mid Q_z \in A, Q_z \text{ simple}\}$. We show in Lemma 1 that each hyperbolic equivalence class of $\lambda_p$-$BQFs$ contains at least one simple form, i.e., $Z_A \neq \emptyset$ for every hyperbolic $\mathcal{A}$. We also show in Lemma 3 that the finite cyclic orbits of $\Phi_p$ are the sets $\{0\}$ and $Z_A$, where $\mathcal{A}$ runs over all hyperbolic $G_p$-equivalence classes of $\lambda_p$-$BQFs$.

Let $P^+$ denote the set of positive poles in $P$, $P^-$ denote the set of negative poles in $P$, so $P = P^+ \cup P^-$. Write $TZ_A = \{Tz \mid z \in Z_A\}$.

The following Lemma establishes the relationship between irreducible pole sets of RPFs on $G_p$ and equivalence classes of simple $\lambda_p$-$BQFs$.

**Lemma 4.** Suppose that $\alpha$ is a positive pole of an RPF on $G_p$. Then $\alpha$ is hyperbolic and $\alpha \in Z_A$, where $\mathcal{A}$ is the $G_p$-equivalence class of $\lambda_p$-$BQFs$ containing $Q_\alpha$. The irreducible pole set containing $\alpha$ is

$$
P(q;\alpha) = Z_A \cup TZ_A.
$$

**Proof.** Suppose that $\alpha = \alpha_1 \in P^+$. As in the proof of Lemma 3, $\alpha_2 = U^{j_1}T\alpha_1 \in P$ for some $j_1$, $1 \leq j_1 \leq p-1$. By Lemma 1 we may take each entry of $U^{j_1}T$ to be non-negative, which implies that $\alpha_2 = U^{j_1}T\alpha_1 > 0$. Repeating this process gives a sequence of poles $\{\alpha_1, \alpha_2, \ldots\} \subset P^+$, with

$$
\alpha_{\nu+1} = U^{j_\nu}T\alpha_\nu,
$$

and $1 \leq j_\nu \leq p-1$ for each $\nu \geq 1$. Since $P^+$ is finite we must have that

$$
\alpha_1 = \alpha_{r+1} = U^{j_r}T\alpha_r,
$$

for some $r \geq 1$. Thus we have a finite cycle of positive poles, $\{\alpha_1, \alpha_2, \ldots, \alpha_r\}$. Reversing (3) we have that $\alpha_r = TU^{p-j_r}\alpha_1 = \Phi_p(\alpha_{\nu+1})$, since $1 \leq p-j_\nu \leq p-1$.
and $\alpha_\nu > 0$ for each $\nu \geq 1$. Reversing (4) we have that $\alpha_\nu = TU^p - j_\nu \alpha_{\nu+1} = \Phi_p(\alpha_1)$, since $1 \leq p - j_\nu \leq p - 1$ and $\alpha_\nu > 0$. Hence $\{\alpha_1, \alpha_2, \ldots, \alpha_r\} \subseteq P^+$ is a finite cyclic orbit of $\Phi_p$, so $\{\alpha_1, \alpha_2, \ldots, \alpha_r\} = Z_{A}$, where $A$ is some $G_p$-equivalence class of $\lambda_p$-BQFs. Hence $A$ is hyperbolic and contains $Q_{\alpha}$, while $Q_{\alpha}$ and $\alpha$ are both hyperbolic and simple.

Since $\alpha$ is a positive pole, each element of $Z_{A}$ must also be positive pole, so $Z_{A} \subseteq P(q; \alpha)$. This, with the first relation (3), implies that every element of $TZ_{A}$ is a negative pole, so $TZ_{A} \subseteq P(q; \alpha)$. On the other hand, if $\beta \in P(q; \alpha)$, then $\beta = M\alpha$, where $M \in G_p$ is a product of matrices, each one equal to $T$ or to $U^i$, $1 \leq i \leq p - 1$. Thus $\alpha$ and $\beta$ are $G_p$-equivalent numbers, hence $Q_{\alpha}$ and $Q_{\beta}$ are $G_p$-equivalent BQFs, so $Q_{\beta} \in A$. If $\beta > 0$, then $\beta \in Z_{A}$, $A$ is hyperbolic and contains $Q_{\alpha}$, while $Q_{\alpha}$ and $\alpha$ are both hyperbolic and simple.

Corollary. The set of nonzero poles of an RPF on $G_p$ has the form

$$P^* = \bigcup_{\ell=1}^{L} (Z_{A_\ell} \cup TZ_{A_\ell})$$

where the $A_\ell$ are distinct indexed hyperbolic $G_p$-equivalence classes of $\lambda_p$-BQFs.

Remark. This is essentially Lemma 3.2 in [15].

The following Lemma allows us to express the negative poles in $P^*$ using Hecke conjugation instead of the action of $T$.

Lemma 5. Let $A$ be a $G_p$-equivalence class of $\lambda_p$-BQFs. Then $TZ_{A} = Z_{-A}$.

Proof. A routine exercise showing containment in both directions establishes the Lemma.

3.2 Principal Parts

In this section we determine the principle part at any pole of any RPF of weight $2k$ on $G_p$. We use this to give an explicit expression of the form of any RPF.

3.2.1 The Pole at Zero

Lemma 6. Suppose that $q$ is an RPF of weight $2k$ on $G_p$. Then

(i) $q$ is regular at $\infty$,

(ii) if $q$ has a pole at 0, the order of the pole is at most $2k$, and

(iii) $q$ has a pole at 0 of order $2k$ if and only if $q(\infty) \neq 0$.

Proof. Fix $p \geq 3$ and put $\lambda = \lambda_p$ and $U = U_{\lambda_p}$. For (i) we apply $|S|$ to the second relation (4), use the first relation (3) and rearrange to get

$$(q - q|S)(z) = (q|US + q|U^2S + \cdots + q|U^{p-2}S)(z).$$
We claim that the right hand side of this expression approaches 0 as \( z \to \infty \).
To prove our claim it suffices to show that for \( 1 \leq n \leq p - 2 \),
\[
(qU^nS)(z) \to 0 \quad \text{as } z \to \infty.
\]
Write \( U^n = \left( \begin{array}{cc} a_n & b_n \\ c_n & d_n \end{array} \right) \), so \( U^nS = \left( \begin{array}{cc} a_n & a_n+1 \\ a_{n-1} & a_n \end{array} \right) \) and
\[
(qU^nS)(z) = (a_{n-1}z - a_n)^{-2k} q(U^nSz).
\]
Now \( (a_{n-1}z - a_n)^{-2k} \to 0 \) as \( z \to \infty \), since \( a_{n-1} \neq 0 \) for \( 1 \leq n \leq p - 2 \). We calculate that \( 0 < U^nS(\infty) < \infty \) for \( 1 \leq n \leq p - 2 \), and \( U^nS(\infty) \) is parabolic, so \( U^nS(\infty) \) cannot be a pole of \( q \), i.e., \( q(U^nSz) \) is bounded near \( \infty \). Thus \( (a_{n-1}z - a_n)^{-2k} q(U^nSz) \to 0 \) as \( z \to \infty \) (\( 1 \leq n \leq p - 2 \)), and the claim holds. If \( q \) had a pole at \( \infty \) of order \( m > 0 \), then
\[
q(z) = r(z) + cmz^m + cm_{-1}z^{m-1} + \cdots + c_0,
\]
where \( r(\infty) = 0 \) and \( c_m \neq 0 \). Then
\[
(q - q|S)(z) = r(z) - r(z + \lambda) - m\lambda cmz^{m-1} + \cdots,
\]
where “...” denotes a polynomial of degree less than \( m - 1 \). Thus as \( z \to \infty \),
\[
(q - q|S)(z) \to \begin{cases} -\lambda c_1, & \text{if } m = 1, \\ \infty, & \text{if } m > 1, \end{cases}
\]
which contradicts the claim and establishes (i).

Parts (ii) and (iii) follow immediately from (i) and the first relation \([1]\) in the form
\[
q \left( \frac{-1}{z} \right) = -z^{2k}q(z).
\]

\( \square \)

An RPF of weight \( 2k \) on \( G_p \) may have a pole only at zero. Meier and Rosenberger show in [3] that such an RPF is of the form
\[
q_{k,0}(z) = \begin{cases} a_0(1 - z^{-2k}), & \text{if } 2k \neq 2, \\ a_0(1 - z^{-2}) + b_1z^{-1}, & \text{if } 2k = 2. \end{cases} \tag{6}
\]

Given \( q \), an RPF of weight \( 2k \) on \( G_p \), we let \( PP_\alpha[q] \) denote the principal part of \( q(z) \) at \( z = \alpha \). Then by (3) and Lemma 2, \( q \) has the form
\[
q(z) = \sum_{\ell=1}^{L} \sum_{\alpha \in Z_{A_{\ell}} \cup T \mathcal{Z}_{A_{\ell}}} PP_\alpha[q](z) + b_0q_{k,0}(z) + \sum_{n=1}^{2k-1} \frac{c_n}{z^n}, \tag{7}
\]
where \( q_{k,0} \) is an RPF given by (5), \( b_0 \) and each \( a_n \) and \( c_n \) is a constant, and the \( \mathcal{A}_\ell \) are indexed hyperbolic \( G_p \)-equivalence classes of \( \lambda_p \)-BQFs.
3.2.2 Nonzero Poles

**Lemma 7.** Let \( q \) be an RPF of weight \( 2k \) on \( G_p \) with a nonzero pole at \( \alpha \). Then the pole at \( \alpha \) is of order \( k \).

**Remark.** Schmidt [13] first states this Lemma with a note that the proof for RPFs on \( \Gamma(1) \) goes through. The following proof generalizes the one for \( \Gamma(1) \) by Choie and Zagier [14].

**Proof.** Fix \( p \geq 3 \) and put \( U = U_{p^\nu} \). Note that if \( f(z) \) has a pole at \( \beta \) and \( V = \left( \begin{smallmatrix} c & d \\ a & b \end{smallmatrix} \right) \) is a linear fractional transformation, then \( (f \mid V)(z) = (cz + d)^{-2k}f(Vz) \) has a pole at \( z = V^{-1}\beta \). In fact,

\[
PP_{V^{-1}\beta}[f \mid V] = PP_{V^{-1}\beta}[PP_{\beta}f \mid V].
\] (8)

By \( 1 \) and \( 8 \) with \( V = T \), we have

\[
PP_{T\beta}[q] = -PP_{T\beta}[PP_{\beta}q \mid T].
\] (9)

In a similar way, we use \( 2 \) and \( 8 \) with \( V = U^{-t} \), \( t \) an integer, to get

\[
PP_{U^{t}\beta}[q] = -PP_{U^{t}\beta}[PP_{\beta}q \mid U^{-t}].
\] (10)

Suppose that \( \alpha > 0 \). Then by the proof of Lemma 4, \( \alpha \) is fixed by an element of \( G_p \) of the form \( M = U^{j_\nu}TU^{j_{\nu-1}} \ldots U^{j_1}T \), where \( 1 \leq j_\nu \leq p - 1 \) for each \( \nu \geq 1 \). Applying \( 8 \) and \( 10 \) \( r \) times each, we get

\[
PP_{T^\alpha}[q] = PP_{T^\alpha}[PP_{T^\alpha}q \mid M^{-1}],
\]

and, since \( M\alpha = \alpha \),

\[
PP_{\alpha}[q] = PP_{\alpha}[PP_{\alpha}q \mid M^{-1}].
\] (11)

If \( \alpha < 0 \), then \( T\alpha \in P^+ \) is fixed by an element of \( G_p \) of the form \( N = U^{j_\nu}TU^{j_{\nu-1}} \ldots U^{j_1}T \). Thus \( \alpha \) is fixed by \( M = TNT = TU^{j_\nu}TU^{j_{\nu-1}} \ldots U^{j_1}T \). We apply \( 8 \) and \( 10 \) \( r \) times each in this case as well, and \( q \) also satisfies \( 1 \) when \( \alpha < 0 \).

Let \( m \) be the order of the pole of \( q \) at \( \alpha \) and suppose that \( q \) is normalized so that \( PP_{\alpha}[q](z) = (z - \alpha)^{-m}r_{\alpha}(z) \), where \( r_{\alpha}(z) \) is a rational function with a pole at \( \alpha \) of order less than \( m \). Put \( M = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \) so \( M^{-1} = \left( \begin{smallmatrix} \frac{d}{a} & \frac{-b}{a} \\ \frac{-c}{d} & \frac{a}{d} \end{smallmatrix} \right) \). We calculate

\[
(PP_{\alpha}[q] \mid M^{-1})(z) = (-cz + a)^{-2k}((M^{-1}z - \alpha)^{-m} + r_{\alpha}(M^{-1}z))
\]

\[
= ((-cz + a)^{-m} - 2k((ca + d)z - (a\alpha + b))^{-m} + r_{\alpha}(z))
\]

\[
= ((-cz + a)^{-m} - 2k(\alpha - z)^{-m} + r_{\alpha}(z))
\]

\[
= ((-cz + a)^{-m} - 2k(\alpha - z)^{-m} + r_{\alpha}(z))
\]

\[
= (ca + d)^{2k-2m}(z - \alpha)^{-m} + r_{\alpha}(z).
\]

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where \( r_{\alpha} \) and \( \tilde{r}_{\alpha} \) are rational functions, each with a pole at \( \alpha \) of order less than \( m \) and a pole at \( \frac{2}{c\alpha} \). We have used partial fractions, the fact that \( M \) fixes \( \alpha \), and the fact that \((c\alpha + d)^{2k-2m} = (c\alpha + d)^{-m} + \tilde{r}_{\alpha}(z)\). Then by (11) we have

\[
P_{\alpha}[q](z) = P_{\alpha}\left[(c\alpha + d)^{2k-2m}(z - \alpha)^{-m} + \tilde{r}_{\alpha}(z)\right] = (c\alpha + d)^{2k-2m}(z - \alpha)^{-m} + \tilde{r}_{\alpha}(z),
\]

where \( \tilde{r}_{\alpha} \) is a rational function with a pole at \( \alpha \) of order less than \( m \). Thus \((c\alpha + d)^{2k-2m} = 1\), so either \( m = k \) or \( c\alpha + d = 1 \). But if \( c\alpha + d = 1 \), then \( \alpha = \frac{1}{c\alpha} \) is parabolic, a contradiction. We conclude that \( m = k \).

**Lemma 8.** Suppose that \( \alpha \neq 0 \) can occur as a pole of an RPF of weight \( 2k \) on \( G_p \). Then there exists a unique function of the form

\[
q_{k,\alpha}(z) = \frac{1}{(z - \alpha)^k} + \frac{a_1(k,\alpha)}{(z - \alpha)^{k-1}} + \cdots + \frac{a_{k-1}(k,\alpha)}{z - \alpha},
\]

such that for any RPF \( q \) of weight \( 2k \) on \( G_p \), \( P_{\alpha}[q] \) is a constant multiple of \( q_{k,\alpha} \).

**Remark.** The function \( q_{k,\alpha} \) is uniquely determined by \( k \) and \( \alpha \), but is independent of the rational period function \( q \).

**Proof.** Functions of the given form exist, since \( P_{\alpha}[q] \) has a pole of order \( k \) at \( \alpha \) and can be normalized so that the coefficient of \((z - \alpha)^{-k}\) is 1. For uniqueness, suppose that \( q_{k,\alpha} \) and \( r_{k,\alpha} \) are functions satisfying the hypotheses of the Lemma, with \( q_{k,\alpha} \neq r_{k,\alpha} \). Then there exist RPFs \( q \) and \( r \) and nonzero constants \( a \) and \( b \) such that \( P_{\alpha}[q] = aq_{k,\alpha} \) and \( P_{\alpha}[r] = br_{k,\alpha} \). But then \( q - \frac{a}{b}r \) is an RPF with a pole at \( \alpha \) of order less than \( 2k \), a contradiction.

The following Lemma gives a formula for \( q_{k,\alpha} \).

**Lemma 9.** Suppose that \( \alpha \neq 0 \) can occur as a pole of an RPF \( q \) of weight \( 2k \) on \( G_p \). Then

\[
q_{k,\alpha}(z) = P_{\alpha}\left[\frac{D^{k/2}}{Q_\alpha(z,1)^k}\right] = P_{\alpha}\left[\frac{(\alpha - \alpha')^k}{(z - \alpha)^k(z - \alpha')^k}\right],
\]

where \( Q_\alpha \) is the \( \lambda_p \)-BQF associated to \( \alpha \), \( D \) is the discriminant of \( Q_\alpha \), and \( \alpha' \) is the Hecke \( \lambda_p \)-conjugate of \( \alpha \).

**Remark.** Schmidt [15] states this Lemma in a modified form for \( \alpha > 0 \) and asserts that the proof follows from the proof in the classical case. The following proof holds for \( \alpha \) positive or negative.
Proof. Write $Q_\alpha(z) = Q_\alpha(z,1) = Az^2 + Bz + C$. Let $M = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)$ be an element of $G_p$ which fixes both $\alpha$ and $\alpha'$. We calculate

$$\left( Q_\alpha^{-k} | M^{-1} \right)(z)$$

$$= (-cz + a)^{-2k} A^{-k} \left( \frac{dz - b}{-cz + a} - \alpha \right)^{-k} \left( \frac{dz - b}{-cz + a} - \alpha' \right)^{-k}$$

$$= A^{-k}((\alpha + d)z - (a\alpha + b))^{-k}((\alpha' + d)z - (a\alpha' + b))^{-k}$$

$$= A^{-k}(\alpha + d)^{-k}(\alpha' + d)^{-k} \left( z - \frac{a\alpha + b}{\alpha + d} \right)^{-k} \left( z - \frac{a\alpha' + b}{\alpha' + d} \right)^{-k}$$

$$= \frac{A^{-k}(z - \alpha)^{-k}(z - \alpha')^{-k}}{Q_\alpha(z)^{-k}}.$$ 

We have used the fact that $(\alpha + d)(\alpha' + d) = 1$. From this it follows that $Q_\alpha^{-k}$ satisfies (11), as does $q$. By an argument similar to the proof of Lemma 8, $PP_\alpha[Q_\alpha^{-k}]$ is a nonzero multiple of $PP_\alpha[q]$, which is in turn a multiple of $q_{k,\alpha}$. Thus $q_{k,\alpha}$ is a multiple of $PP_\alpha[Q_\alpha^{-k}]$. Finally

$$PP_\alpha[Q_\alpha^{-k}](z) = PP_\alpha[A^{-k}(z - \alpha)^{-k}(z - \alpha')^{-k}]$$

$$= A^{-k}(z - \alpha)^{-k}(\alpha - \alpha')^{-k} + \cdots$$

$$= D^{-k/2}(z - \alpha)^{-k} + \cdots,$$

so the proportionality constants are as given in the Lemma.

We now give a complete description of any RPF of weight $2k$ on $G_p$.

**Theorem 1.** An RPF of weight $2k \in 2\mathbb{Z}$ on $G_p$ is of the form

$$q(z) = \sum_{\ell = 1}^{L} C_{\ell} \sum_{\alpha \in Z_{\mathcal{A}_\ell}} (q_{k,\alpha}(z) - q_{k,T\alpha}(z)) + c_0 q_{k,0}(z) + \sum_{n = 1}^{2k-1} \frac{c_n}{z^n},$$

where each $\mathcal{A}_\ell$ is a $G_p$-equivalence class of $\mathbb{Z}[\lambda_p]$-BQFs, $Z_{\mathcal{A}_\ell}$ is the cycle of positive poles associated with $\mathcal{A}_\ell$, $q_{k,\alpha}$ is given by (12), $q_{k,0}$ is given by (11), and the $C_{\ell}$ and $c_n$ are all constants.

**Proof.** By (9) and Lemma 8 we have that any RPF $q$ of weight $2k$ on $G_p$ has the form

$$q(z) = \sum_{\ell = 1}^{L} \sum_{\alpha \in Z_{\mathcal{A}_\ell} \cup T Z_{\mathcal{A}_\ell}} C_\alpha q_{k,\alpha}(z) + c_0 q_{k,0}(z) + \sum_{n = 1}^{2k-1} \frac{c_n}{z^n},$$

where each $\mathcal{A}_\ell$ is a hyperbolic $G_p$-equivalence class of $\lambda_p$-BQFs and $q_{k,0}$ is given by (9). From (9) and (11) we see that the coefficients $C_\alpha$ alternate in sign as $\alpha$ goes through any cycle $\{\alpha_1, T\alpha_1, U^j T\alpha_1, TU^j T\alpha_1, \ldots \}$. Thus for each irreducible system of poles $Z_{\mathcal{A}_\ell} \cup T Z_{\mathcal{A}_\ell}$ there is a constant $C_{\ell}$ such that $C_\alpha = C_{\ell}$ for each $\alpha \in Z_{\mathcal{A}_\ell}$ and $C_\alpha = -C_{\ell}$ for each $\alpha \in T Z_{\mathcal{A}_\ell}$. As a result any RPF of weight $2k$ on $G_p$ has the given form.

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Lemma 5 allows us to identify negative poles using Hecke conjugation instead of the action of $T$ in the Corollary.

**Corollary.** An RPF of weight $2k \in 2\mathbb{Z}$ on $G_p$ is of the form

$$q(z) = \sum_{\ell=1}^{L} C_{\ell} \left( \sum_{\alpha \in Z_{A_\ell}} q_{k,\alpha}(z) - \sum_{\alpha \in Z_{-A_\ell}} q_{k,\alpha'}(z) \right) + c_0 q_{k,0}(z) + \sum_{n=1}^{2k-1} \frac{c_n}{z^n},$$

(13)

where each $A_\ell$ is a $G_p$-equivalence class of $\mathbb{Z}[\lambda_p]$-BQFs, $Z_{A_\ell}$ is the cycle of positive poles associated with $A_\ell$, $q_{k,\alpha}$ is given by (12), $q_{k,0}$ is given by (6), and the $C_{\ell}$ and $c_n$ are all constants.

### 4 HECKE-SYMMETRIC POLE SETS, $k$ ODD

The following theorem characterizes of RPFs of weight $2k$ on $G_p$, for $k$ odd, with Hecke-symmetric sets of poles.

**Theorem 2.** Suppose that $k$ is odd.

Suppose $q$ is an RPF of weight $2k$ on $G_p$ with Hecke $\lambda_p$-symmetric irreducible systems of poles. Then $q$ is of the form

$$q(z) = \sum_{\ell=1}^{L} d_\ell \sum_{\alpha \in Z_{A_\ell}} Q_\alpha(z,1)^{-k} + c_0 q_{k,0}(z),$$

(14)

where each $A_\ell$ is a $G_p$-equivalence class of $\mathbb{Z}[\lambda_p]$-BQFs satisfying $-A_\ell = A_\ell$, the $d_\ell$ ($1 \leq \ell \leq M$) are constants, and $q_{k,0}(z)$ is given by (8).

Conversely, any rational function of the form (14) is an RPF of weight $2k$ on $G_p$ with Hecke $\lambda_p$-symmetric irreducible systems of poles.

**Proof.** Suppose that $q$ is an RPF of weight $2k$ on $G_p$ with Hecke $\lambda_p$-symmetric irreducible systems of poles. Then $-A = A$ for each $G_p$-equivalence class of $\lambda_p$-BQFs associated with poles of $q$. Thus the expression for $q$ in (13) simplifies to

$$q(z) = \sum_{\ell=1}^{L} C_{\ell} \sum_{\alpha \in Z_{A_\ell}} (q_{k,\alpha}(z) - q_{k,\alpha'}(z)) + c_0 q_{k,0}(z) + \sum_{n=1}^{2k-1} \frac{c_n}{z^n},$$

where $q_{k,0}(z)$ is given by (8) and $C_{\ell}$ ($1 \leq \ell \leq L$) and $c_n$ ($1 \leq n \leq 2k-1$) are all constants. A calculation shows that for any pole $\alpha$,

$$q_{k,\alpha}(z) - q_{k,\alpha'}(z) = PP_\alpha \left[ \frac{D^{k/2}}{Q_\alpha(z,1)^k} \right] - (-1)^k PP_{\alpha'} \left[ \frac{D^{k/2}}{Q_\alpha(z,1)^k} \right],$$

where $D$ is the discriminant of $Q_\alpha$. Since $k$ is odd,

$$q_{k,\alpha}(z) - q_{k,\alpha'}(z) = \frac{D^{k/2}}{Q_\alpha(z,1)^k},$$
so

\[ q(z) = \sum_{\ell=1}^{L} d_{\ell} \sum_{\alpha \in \mathbb{Z}_{A_{\ell}}} Q_{\alpha}(z,1)^{-k} + c_{0} q_{k,0}(z) + \sum_{n=1}^{2k-1} c_{n} z^{n}, \tag{15} \]

where \( d_{\ell} = C_{\ell} D_{\ell}^{k/2} \) and \( D_{\ell} \) is the discriminant of \( \lambda_{p}-\text{BQFs} \) in \( A_{\ell} \).

Schmidt proves in [15, Theorem 3.1] that if \( k \) is odd, then

\[ r(z) = \sum_{\alpha \in \mathbb{Z}_{A} \cup \mathbb{Z}_{-A}} Q_{\alpha}(z,1)^{-k} \]

is an RPF of weight 2\( k \) on \( G_{p} \). Thus the first sum in (15) is an RPF of weight 2\( k \) on \( G_{p} \). Since \( q \) and \( q_{k,0} \) are also RPFs, we must have that \( \sum_{n=1}^{2k-1} c_{n} z^{n} \) is an RPF of weight 2\( k \) on \( G_{p} \). But then we must have \( c_{n} = 0 \) for \( 1 \leq n \leq 2k-1 \) or else we contradict Lemma 6 (ii). Thus \( q \) has the form (14).

For the converse, assume that \( q \) has the form (14). Since \( \sum_{\ell=1}^{L} d_{\ell} \sum_{\alpha \in \mathbb{Z}_{A_{\ell}}} Q_{\alpha}(z,1)^{-k} \) and \( q_{k,0} \) are both RPFs of weight 2\( k \) on \( G_{p} \), \( q \) is also an RPF of weight 2\( k \) on \( G_{p} \). The set of nonzero poles of \( q \) is

\[ P^{*}(q) = \bigcup_{\ell=1}^{L} (Z_{A_{\ell}} \cup Z'_{A_{\ell}}), \]

which is clearly Hecke-symmetric. \( \square \)

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