REPRESENTABILITY FOR SOME MODULI STACKS OF FRAMED SHEAVES

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Introduction. Moduli problems for various kinds of framed sheaves have been studied and used in many settings (see, for example, [Tha94], [Bra91], [Nak94]), and there is a good general theory of moduli for semistable framed sheaves, thanks to the work of Huybrechts and Lehn ([HL95a], [HL95b]). By contrast, there seem to be only a few examples in which the full moduli functor for framed sheaves (without conditions of semistability) is known to be represented by a scheme. In this paper, we prove a representability theorem for the full moduli functors of framed torsion-free sheaves on projective surfaces under certain conditions.

Let $S$ denote a smooth, connected complex projective surface, and let $D \subset S$ denote a smooth connected complete curve in $S$. Fix a vector bundle $E$ on $D$. An $E$-framed torsion-free sheaf on $S$ is a pair $(E, \phi)$ consisting of a torsion-free sheaf $E$ on $S$ and an isomorphism $\phi : E|_D \to E$; the isomorphism $\phi$ is called an $E$-framing of $E$. An isomorphism of $E$-framed torsion-free sheaves on $S$ is an isomorphism of the underlying torsion-free sheaves on $S$ that is compatible with the framings. Let $\text{TF}_S(E)$ denote the moduli functor for isomorphism classes of $E$-framed torsion-free sheaves on $S$. The reader should note that in the work of Huybrechts–Lehn the framing $\phi$ need not be an isomorphism; as a consequence of our more restrictive definition, the moduli functors that we study have no hope of being proper.

Suppose the vector bundle $E$ satisfies

$$H^0 \left( D, \text{End } E \otimes N^{-k}_{D/S} \right) = 0$$

for all $k \geq 1$; here $N_{D/S}$ is the normal bundle of $D$ in $S$. If $D \subset S$ is an arbitrary curve, there may be very few such bundles. However, if $D$ is smooth and has positive self-intersection in $S$, then $N^{-1}_{D/S}$ is a negative line bundle on $D$, and consequently this condition on $E$ is an open condition which is satisfied by all semistable vector bundles on $D$.

Theorem 1. Suppose that $S$ is a smooth, connected complex projective surface and $D \subset S$ is a smooth connected complete curve. Suppose, in addition, that $E$ is a vector bundle on $D$ that satisfies Condition (1) for all $k \geq 1$. Then the functor $\text{TF}_S(E)$ is represented by a scheme.

In the proof of Theorem 1 we work in the slightly more general setting of a family of vector bundles on $D$, parametrized by a scheme $U$, that satisfies Condition (1) for all $k \geq 1$ at every point of $U$. Note also that the reader who is familiar with the language of stacks may restate Theorem 1 in the following form: over the substack of its target that parametrizes vector bundles on $D$ that satisfy Condition (1) for all $k \geq 1$, the fibers of the restriction morphism from the moduli stack of torsion-free

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sheaves on $S$ that are locally free along $D$ to the moduli stack of vector bundles on $D$ are schemes.

Functors of the type we study here arose naturally (in some special cases) in the representation-theoretic constructions of Nakajima; Theorem 1 demonstrates that the existence of the fine moduli schemes used by Nakajima is a much more general phenomenon, one which we hope can be exploited more widely in the study of sheaves on noncompact surfaces. The new ingredient in our proof of Theorem 1 is the use of formal geometry along the curve $D$; in particular, the techniques used here are completely different from those of [HL95a], [HL95b], and make no use of geometric invariant theory (GIT). Although Lehn ([Leh93]) has, under some conditions on the curve $D$ and the bundle $E$ along the curve, proven that the full moduli functors for vector bundles on $S$ with framing along $D$ by $E$ are represented by algebraic spaces, from the point of view of the usual GIT techniques it is perhaps surprising that there is a fine moduli scheme (a much stronger fact) for all framed sheaves: indeed, there can be framed sheaves that are not semistable for any polarization.

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Affine bundles over $\text{Bun}^\times(D)$. In this section we construct the fundamental affine bundles $A_n$ (for $n$ in the range $1 \leq n < \infty$) over $U$ that we will use to embed the functor $\text{TF}_S(E)$ in a scheme. The construction of these bundles and the description of the universal properties they possess must be well known (cf. [Gri66], in which the relevant cohomology groups are discussed), but the author does not know a suitable reference.

Fix a surface $S$, a curve $D$ in $S$, a scheme $U$, and a vector bundle $E$ on $D \times U$ as in Theorem 1. Let $D^{(n)}$ (that is, $D$ with structure sheaf $O_S/I_D^{n+1}$, $0 \leq n < \infty$) denote the $n$th order neighborhood of $D$ in $S$.

Definition 2. Let $A_n$ denote the moduli functor over $U$ of isomorphism classes of triples $(\mathcal{E}, V \xrightarrow{f} U, \phi)$ consisting of
1. a vector bundle $\mathcal{E}$ on $D^{(n)} \times V$,
2. a morphism $f : V \to U$, and
3. an isomorphism $\phi : \mathcal{E} |_{D \times V} \to (1_D \times f)^*E$.

Suppose that $\mathcal{E}$ is a vector bundle over $D^{(n)}$; then $\mathcal{E}$ has a canonical (decreasing) filtration as an $O_{D^{(n)}}$-module with filtered pieces $F_j\mathcal{E} = I_D^{n-j}\mathcal{E}$, where $I_D$ is the ideal of $D \subset D^{(n)}$. By its construction, this filtration is preserved by any endomorphism of the vector bundle $\mathcal{E}$, and moreover $F_j\mathcal{E} / F_{j+1}\mathcal{E} \cong N_{D/S}^{-j} \otimes (F_0\mathcal{E} / F_1\mathcal{E})$ provided $0 \leq j \leq n$. Using these facts together with the exact sequence
$$0 \to \text{Hom}(E, E \otimes N_{D/S}^{-j}) \to \text{Hom}(\mathcal{E}, \mathcal{E}) \to \text{Hom}(\mathcal{E}, \mathcal{E} |_{D^{(n-j)}}) \to 0$$
and condition (1), one may prove by induction on $n$ that $\text{End}(\mathcal{E}) \subseteq \text{End}(\mathcal{E} |_D)$ and consequently that $E$-framed bundles on $D^{(n)}$ are rigid.
Evidently $A_0 \cong U$; moreover, there are maps $\pi_{n+1} : A_{n+1} \to A_n$ for all $n \geq 0$.

**Proposition 3.** Each $A_n$ ($n \geq 1$) is represented by a scheme $A_n$ that is an affine bundle over $A_{n-1}$.

**Proof.** Working inductively, it will suffice to construct an $A_{n-1}$-scheme $A_n$ that represents $A_n$ and is an affine bundle over $A_{n-1}$. Fix a universal bundle $E^{(n-1)}$ on $D^{(n-1)} \times A_{n-1}$. For any scheme $T$, an element of $A_n(T)$ determines a map $f : T \to A_{n-1}$, and, if $(E, \phi)$ is the given element of $A_n(T)$, there is an isomorphism of $E|_{D^{(n-1)} \times T}$ with $(1 \times f)^*E^{(n-1)}$ compatibly with the framings by $E$. But then, because $E$-framed bundles on $D^{(n-1)}$ are rigid, we find that $A_n$ as a functor over $A_{n-1}$ is isomorphic to the functor taking $f : T \to A_{n-1}$ to the set of isomorphism classes of pairs $(\mathcal{E}, \phi)$ consisting of a bundle $\mathcal{E}$ on $D^{(n)} \times T$ together with an isomorphism $\phi$ of $E|_{D^{(n-1)} \times T}$ with $(1 \times f)^*E^{(n-1)}$. We will refer to such a pair as an $E^{(n-1)}$-framed bundle.

Because the statement of the proposition is local on $A_{n-1}$, we may assume that $A_{n-1}$ is an affine scheme that is the spectrum of a local ring $R$. For simplicity, write $\mathcal{O} = O_{D^{(n-1)} \times A_{n-1}}$ and $\mathcal{O}' = O_{D^{(n)} \times A_{n-1}}$. The “change of rings” spectral sequence (see Chap. XVI, Section 5 of [E50])

$$E_2^{p,q} = \text{Ext}^p_{\mathcal{O}}(\text{Tor}^\mathcal{O}_{q}(E^{(n-1)}, \mathcal{O}), E(-nD)) \Rightarrow \text{Ext}^{p+q}_{\mathcal{O}'}(E^{(n-1)}, E(-nD))$$

yields the exact sequence of terms of low degree

$$(2) \quad 0 \to \text{Ext}^1_{\mathcal{O}}(E^{(n-1)}, E(-nD)) \to \text{Ext}^1_{\mathcal{O}'}(E^{(n-1)}, E(-nD))$$

\[ \beta \to \text{Hom}(\text{Tor}_1^{\mathcal{O}'}(E^{(n-1)}, \mathcal{O}), E(-nD)) \to 0. \]

Note that $\beta$ is surjective since the next term in the sequence is $\text{Ext}^{2}_{\mathcal{O}'}(E^{(n-1)}, E(-nD))$, which vanishes because $D$ is one-dimensional. Using $\text{Tor}_1^{\mathcal{O}'}(E^{(n-1)}, \mathcal{O}) \cong E(-nD)$ one may check that there is a canonical element $e$ of $\text{Hom}(\text{Tor}_1^{\mathcal{O}'}(E^{(n-1)}, \mathcal{O}), E(-nD))$ such that $\beta^{-1}(e)$ is exactly the $\text{Ext}^1_{\mathcal{O}'}(E^{(n-1)}, E(-nD))$-subtisor of $\text{Ext}^{1}_{\mathcal{O}'}(E^{(n-1)}, E(-nD))$ that classifies 1-extensions

$$0 \to E(-nD) \to \mathcal{E} \to E^{(n-1)} \to 0$$

for which $\mathcal{E}$ is a locally free $\mathcal{O}'$-module. Now, Condition [4], together with Cohomology and Base Change, implies that the $R$-module $\text{Ext}^1_{\mathcal{O}}(E^{(n-1)}, E(-nD)) \cong H^1(D \times A_{n-1}, \text{End}(E) \otimes N_{D/S}^{-n})$ is projective, hence free. One can easily construct, moreover, a universal 1-extension over $D^{(n)} \times A_{n-1} \times \beta^{-1}(e)$ (using, for example, an affine subspace of the Cech cocycles that maps isomorphically to $\beta^{-1}(e)$ to furnish gluing data). Because the exact sequence (2) and the element $e$ are functorial under pullback along morphisms of affine schemes Spec $R' \xrightarrow{f} \text{Spec } R = A_{n-1}$, this universal 1-extension induces a functorial bijection between the set $\beta_{R'}^{-1}(e)$ (the inverse image of the canonical element under the base-changed map $\beta$) and the set of isomorphism classes of pairs $(\mathcal{E}, \phi)$ consisting of a vector bundle $\mathcal{E}$ on $D^{(n)} \times \text{Spec } R'$ and a framing $\phi : \mathcal{E}|_{D^{(n-1)} \times \text{Spec } R'} \to (1 \times f)^*E^{(n-1)}$.

Consequently $A_n$ is represented as a functor over $A_{n-1}$ by the torsor over $\text{Spec } \text{Sym}^* \text{Ext}^1_{\mathcal{O}}(E^{(n-1)}, E(-nD))$ defined by $\beta^{-1}(e)$, proving the proposition. \[\square\]
Proof of Theorem 1. There is a compatible family of morphisms $F_n : TFS(E) \to \mathbb{A}_n$ given by restriction. Fix a Spec $\mathbb{C}$-valued point of $TFS(E)$, that is, a point $u \in U$ together with an $E_u$-framed pair $(\mathcal{F}, \phi)$ on $S$. We will show that there is an open subfunctor $Z$ of $TFS(E)$ that contains $(\mathcal{F}, \phi)$ and is represented by a scheme.

Fix a polarization $H$ of $S$, and choose $m$ sufficiently large that

1. $\mathcal{F} \otimes H^m$ is globally generated and
2. $H^1(\mathcal{F} \otimes H^m) = H^2(\mathcal{F} \otimes H^m) = 0$.

Further, fix $n$ sufficiently large that the restriction map

$$H^0(\mathcal{F} \otimes H^m) \to H^0(\mathcal{F} \otimes H^m|_{D(n)})$$

is injective; it is possible to choose such an $n$ because $\mathcal{F}$ is torsion-free. Finally, choose $m'$ sufficiently large that $H^1(\mathcal{F} \otimes H^{m+m'}|_{D(n)}) = 0$.

Next, let $Z \subseteq TFS(E)$ denote the open subfunctor parametrizing those triples

$$(W \xrightarrow{\nu} U, \mathcal{E}, \phi)_{|D \times W} \to (1 \times f)^* E$$

for which the family $\mathcal{E}$ satisfies the following conditions:

- $\mathcal{E}_W \otimes H^m$ is globally generated for all $w \in W$,
- $H^1(\mathcal{E}_w \otimes H^m) = H^2(\mathcal{E}_w \otimes H^m) = 0$ for all $w \in W$,
- the map $H^0(\mathcal{E}_w \otimes H^m) \to H^0(\mathcal{E}_w \otimes H^m|_{D(n)})$ is injective for all $w \in W$, and
- $H^1(\mathcal{E}_w \otimes H^{m+m'}|_{D(n)}) = 0$ for all $w \in W$.

In the previous section we showed that there is a universal vector bundle $E^{(n)}$ on $D(n) \times \mathbb{A}_n$. Fix an element of $Z(W)$; then the map $F_n(W) : W \to \mathbb{A}_n$ yields a vector bundle $(1 \times F_n)^* E^{(n)}$ on $D(n) \times W$ together with an isomorphism

$$\mathcal{E}_W|_{D^{(n)} \times W} \xrightarrow{\phi_n} (1 \times F_n)^* E^{(n)};$$

here $\mathcal{E}_W$ denotes the torsion-free sheaf on $S \times W$ determined by the fixed element of $Z(W)$. Let $p_W$ denote the projection $S \times W \to W$. Then by construction the sheaves $(p_W)_* \mathcal{E}_W \otimes H^m$, $(p_W)_* \mathcal{E}_W \otimes H^{m+m'}$, and $(p_W)_* \left(\mathcal{E}_W \otimes H^{m+m'}|_{D(n) \times W}\right)$ are vector bundles on $W$, and, choosing a section $s$ of $H^{m'}$ the zero locus of which has transverse intersection with $D$, there is a commutative diagram

$$
\begin{array}{ccc}
(p_W)_* \mathcal{E}_W \otimes H^m & \xrightarrow{(p_W)_*} & (p_W)_* \left(\mathcal{E}_W \otimes H^m|_{D(n) \times W}\right) \\
\downarrow \otimes s & & \downarrow \otimes s \\
(p_W)_* \mathcal{E}_W \otimes H^{m+m'} & \xrightarrow{(p_W)_*} & (p_W)_* \left(\mathcal{E}_W \otimes H^{m+m'}|_{D(n) \times W}\right)
\end{array}
$$

for which the vertical arrows (given by tensoring with $s$) and the top row are injective. Using $\phi_n$, we may replace this diagram canonically with the diagram

$$
\begin{array}{ccc}
(p_W)_*(\mathcal{E}_W \otimes H^m) & \xrightarrow{(p_W)_*} & (p_W)_* \left((1 \times F_n)^* E^{(n)} \otimes H^m\right) \\
\downarrow \otimes s & & \downarrow \otimes s \\
(p_W)_*(\mathcal{E}_W \otimes H^{m+m'}) & \xrightarrow{(p_W)_*} & (p_W)_* \left((1 \times F_n)^* E^{(n)} \otimes H^{m+m'}\right).
\end{array}
$$
Now, by assumption (d) on $W$, we have

$$(p_W)_* \left( (1 \times F_n)^* E^{(n)} \otimes H^{m+m'} \right) = F_n^* \left( (p_{A_n})_* (E^{(n)} \otimes H^{m+m'}) \right),$$

where $p_{A_n} : D^{(n)} \times A_n \to A_n$ is the projection, and so finally we obtain the diagram of vector bundles

$$
\begin{array}{ccc}
(p_W)_* \mathcal{E}_W \otimes H^m & \xrightarrow{r} & F_n^* \left( (p_{A_n})_* (E^{(n)} \otimes H^{m+m'}) \right) \\
\downarrow \circ s & & \downarrow \\
(p_W)_* \mathcal{E}_W \otimes H^{m+m'} & \rightarrow & F_n^* \left( (p_{A_n})_* (E^{(n)} \otimes H^{m+m'}) \right)
\end{array}
$$

on $W$, where the diagonal map $r$ and the map $\circ s$ are injective. By construction, furthermore, the image of the morphism $r$ is a vector subbundle of $F_n^* \left( (p_{A_n})_* (E^{(n)} \otimes H^{m+m'}) \right)$ and consequently determines a morphism $W \to \text{Gr}$ over $A_n$, where $\text{Gr} \xrightarrow{q} A_n$ denotes the relative Grassmannian for the vector bundle $(p_{A_n})_* (E^{(n)} \otimes H^{m+m'})$ on $A_n$, the fiber of which over $a \in A_n$ parametrizes vector subspaces of $H^0(E^{(n)} \otimes H^{m+m'})$ that are of dimension $h^0(F \otimes H^m)$.

We now construct a Quot-scheme over $\text{Gr}$ that we will use to represent $Z$. We may pull back $(p_{A_n})_* (E^{(n)} \otimes H^{m+m'})$ to $\text{Gr}$ to obtain a vector bundle $q^* (p_{A_n})_* (E^{(n)} \otimes H^{m+m'})$ on (an open subset of) $\text{Gr}$, with universal subbundle $U \subset q^* (p_{A_n})_* (E^{(n)} \otimes H^{m+m'})$

of rank $h^0(F \otimes H^m)$. If $p_{\text{Gr}} : S \times \text{Gr} \to \text{Gr}$ denotes the projection to $\text{Gr}$, we obtain a bundle $p_{\text{Gr}}^* U \subset p_{\text{Gr}}^* q^* (p_{A_n})_* (E^{(n)} \otimes H^{m+m'})$ on $S \times \text{Gr}$, as well as a quotient

$$p_{\text{Gr}}^* q^* (p_{A_n})_* (E^{(n)} \otimes H^{m+m'}) \to (1 \times q)^* (E^{(n)} \otimes H^{m+m'})$$

and subquotient $(1 \times q)^* (E^{(n)} \otimes H^m) \subset (1 \times q)^* (E^{(n)} \otimes H^{m+m'})$ that are sheaves on $S \times \text{Gr}$ supported on $D^{(n)} \times \text{Gr}$.

Consider the relative Quot-scheme $q' : \text{Quot}_{S \times \text{Gr}/S} (p_{\text{Gr}}^* U) \to \text{Gr}$ that parametrizes quotient sheaves for the family $p_{\text{Gr}}^* U$ on $S \times \text{Gr}/S$. There is a universal quotient $(1 \times q')^* p_{\text{Gr}}^* U \to Q$ on $S \times \text{Quot}_{S \times \text{Gr}/S}$, giving a diagram

$$
\begin{array}{ccc}
(1 \times q')^* p_{\text{Gr}}^* U & \rightarrow & (1 \times q')^* p_{\text{Gr}}^* q^* (p_{A_n})_* (E^{(n)} \otimes H^{m+m'}) \\
\downarrow & & \downarrow \\
Q & & (1 \times q q')^* (E^{(n)} \otimes H^m) \subset (1 \times q q')^* (E^{(n)} \otimes H^{m+m'}). 
\end{array}
$$

There is a closed subscheme of $\text{Quot}_{S \times \text{Gr}/S}$ (see the proof of Theorem 1.6 of [Ser80]) that represents the subfunctor of those quotients the kernels of which project to zero in $(1 \times q q')^* (E^{(n)} \otimes H^{m+m'})$, and a closed subscheme $C$ of that closed subscheme that represents the sub-subfunctor that parametrizes those quotients the images of which in $(1 \times q q')^* (E^{(n)} \otimes H^{m+m'})$ actually lie in the subsheaf $(1 \times q q')^* (E^{(n)} \otimes H^m)$. $C$ then represents the functor of quotients of $p_{\text{Gr}}^* U$ that map to $(1 \times q q')^* (E^{(n)} \otimes H^m)$—that
is, it is exactly the closed subscheme over which Diagram (3) extends to

\[
(1 \times q')^* \mathcal{P}_{\mathcal{U}} \rightarrow (1 \times q')^* \mathcal{P}_{\mathcal{P}} q^* (p_{A_n})_*(E^{(n)} \otimes H^{m+m'})
\]

(4)

\[
Q \rightarrow (1 \times qq')^*(E^{(n)} \otimes H^m) \subset (1 \times qq')^*(E^{(n)} \otimes H^{m+m'}).
\]

Restricting further to an open subscheme \(C^o\) of \(C\), we may assume that, over \(C^o\), the map \(Q|_{D^{(n)} \times C^o} \rightarrow (1 \times qq')^*(E^{(n)} \otimes H^m)\) is an isomorphism, that \(Q\) is a family of torsion-free sheaves on \(S\), and that conditions (a) through (d) are satisfied.

By construction the morphism \(W \rightarrow \mathcal{P}\) lifts to a morphism \(W \rightarrow C^o\); this construction thus determines a morphism of functors \(Z \rightarrow C^o\). Similarly, there is a forgetful morphism \(C^o \rightarrow Z\). Finally, it is clear from the construction that these two morphisms of functors are inverses of each other, as desired. \(\square\)

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