Inverse Cognitive Radar – A Revealed Preferences Approach
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Abstract—We consider an adversarial signal processing problem involving “us” versus an “enemy” cognitive radar. The enemy’s cognitive radar observes our state in noise; uses a tracker to update its posterior distribution of our state and then chooses an action based on this posterior. Given knowledge of “our” state and the observed sequence of actions taken by the enemy’s radar, we consider three problems: (i) Are the enemy radar’s actions consistent with optimizing a monotone utility function (i.e., is the cognitive radar behavior rational in an economics sense). If so how can we estimate the adversary’s cognitive radar’s utility function that is consistent with its actions. (ii) How to construct a statistical detection test for utility maximization when we observe the enemy radar’s actions in noise? (iii) How can we optimally probe the enemy’s radar by choosing our state to minimize the Type 2 error of detecting if the enemy radar is deploying an economic rational strategy, subject to a constraint on the Type 1 detection error?

The central theme of this paper involves an adversarial signal processing/inverse reinforcement learning problem comprised of “us” and an “enemy”. Figure 1 displays the schematic setup. “Us” refers to a drone/UA V or electromagnetic signal that probes an “adversary” cognitive radar system. The adversary’s cognitive radar estimates our kinematic coordinates using a Bayesian tracker and then adapts its mode (waveform, aperture, revisit time) dynamically using feedback control based on sensing our kinematic state (e.g. position and velocity of drone). At each time the enemy’s kinematic state can be viewed as a probe vector , and thus the radar’s response is to the radar. We measure the radar’s response , and given the time series of probe vectors and responses, does it exist a utility function that the radar is maximizing to generate its response ? How can we estimate such a utility function to predict the future behavior of the cognitive radar?

Fig. 1. Schematic of Adversarial Inference Problem. Our side is a drone/UA V or electromagtic signal that probes the adversary’s cognitive radar system. denotes a fast time scale and denotes a slow time scale. Our state parameterized by , respectively. Based on the noisy observation , the adversary’s radar responds with action . Our aim is to determine if the radar is economic rational, i.e., is generated by optimizing a utility function?

Inverse reinforcement learning seeks to estimate the utility function of a decision system by observing its input output dataset. The revealed preferences framework considered here is more general since we first detect if the behavior is consistent with a utility function and then estimate a set of utility functions that rationalize the dataset.

1Inverse reinforcement learning seeks to estimate the utility function of a decision system by observing its input output dataset. The revealed preferences framework considered here is more general since we first detect if the behavior is consistent with a utility function and then estimate a set of utility functions that rationalize the dataset.

2In Section we give specific examples of how the kinematic state and radar actions are mapped to probe and response , respectively.

I. INTRODUCTION

Cognitive radars use the perception-action cycle of cognition to sense the environment, learn from it relevant information about the target and the background, then adapt the radar sensor to optimally satisfy the needs of their mission. A crucial element of a cognitive radar is feedback control: based on its tracked estimates, the radar adaptively optimizes the waveform, aperture, dwell time and revisit rate.

This paper is motivated by the next logical step, namely, inverse cognitive radar. From the intercepted emissions of an enemy’s radar: (i) Is the enemy’s radar cognitive? That is, are the enemy radar’s actions consistent with optimizing a monotone utility function (equivalently, is the radar’s behavior rational in an economics sense). If so how can we estimate the cognitive radar’s utility function that is consistent with its actions. (ii) How to construct a statistical detection test for utility maximization when we observe the enemy’s radar’s actions in noise? (iii) How can optimally probe the enemy’s radar by choosing our state to minimize the Type 2 error of detecting if the enemy radar is deploying an economic rational strategy, subject to a constraint on the Type 1 detection error?

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A. Revealed Preferences and Afriat’s Theorem

Nonparametric detection of utility maximization behavior is the central theme in the area of revealed preferences in microeconomics; which dates back to Samuelson in 1938 \[^3\].

**Definition 1** (\[^4\], \[^5\]). A system is a utility maximizer if for every probe \(\alpha_n \in \mathbb{R}^n_+\), the response \(\beta_n \in \mathbb{R}^n\) satisfies

\[
\beta_n \in \arg \max_{\alpha_n \beta \leq 1} U(\beta)
\]

where \(U(\beta)\) is a monotone utility function.

In economics, \(\alpha_n\) denotes the price vector and \(\beta_n\) the consumption vector. Then \(\alpha_n \beta \leq 1\) is a natural budget constraint\[^5\] for a consumer with 1 dollar. Given a dataset of price and consumption vectors, the aim is to determine if the consumer is a utility maximizer (rational) in the sense of (1).

The key result in revealed preferences is the following remarkable theorem due to Afriat; see \[^6\], \[^5\], \[^4\], \[^7\], \[^8\] for extensive expositions.

**Theorem 2** (Afriat’s Theorem \[^4\]). Given a data set

\[
D = \{(\alpha_n, \beta_n), n \in \{1, 2, \ldots, N\}\},
\]

the following statements are equivalent:

1. The system is a utility maximizer and there exists a monotonically increasing\[^7\] continuous, and concave utility function by satisfies (7).
2. For \(u_t, \lambda_t > 0\) the following set of inequalities (called Afriat’s inequalities) has a feasible solution:

\[
u_s - u_t - \lambda_t \alpha_t(\beta_s - \beta_t) \leq 0 \quad \forall t, s \in \{1, 2, \ldots, N\}.
\]

3. Explicit monotone and concave utility functions that rationalize the dataset by satisfying (7) are given by:

\[
U(\beta) = \min_{t \in \{1, 2, \ldots, N\}} \{u_t + \lambda_t \alpha_t(\beta - \beta_t)\}
\]

where \(u_t\) and \(\lambda_t\) satisfy the linear inequalities (7).

4. The data set \(D\) satisfies the Generalized Axiom of Revealed Preference (GARP), namely for any \(t \leq N\),

\[
\alpha_t^{k} \beta_t \geq \alpha_t^{k} \beta_{t+1} \quad \forall t \leq k - 1 \quad \Rightarrow \quad \alpha_t \beta_k \leq \alpha_{k+1} \beta_{k+1}.
\]

Afriat’s theorem tests for economics-based rationality; its remarkable property is that it gives a necessary and sufficient condition for a system to be a utility maximizer based on the system’s input-output response. The feasibility of the set of inequalities (3) can be checked using a linear programming solver; alternatively GARP (4) can be checked using Warshall’s algorithm with \(O(N^3)\) computations \[^9\] \[^10\]. A utility function consistent with the data can be constructed\[^5\] using (4).

The recovered utility using \(4\) is not unique; indeed any positive monotone increasing transformation of \(4\) also satisfies Afriat’s Theorem; that is, the utility function constructed is ordinal. This is the reason why the budget constraint \(\alpha_n \beta \leq 1\) is without generality; it can be scaled by an arbitrary positive constant and Theorem 2 still holds. In signal processing terminology, Afriat’s Theorem can be viewed as set-valued system identification of a set-valued system; set-valued since \(4\) yields a set of utility functions that rationalize the finite dataset \(D\).

B. Objectives

In this paper, our working assumption is that a cognitive radar satisfies economics-based rationality; that is, a cognitive radar is a utility maximizer in the sense of (4) with possibly a nonlinear budget constraint. The main objectives of the paper involve answering:

1. **Test for Utility Maximization – Spectral Revealed Preferences:** The first question is: Does a radar satisfy economics based rationality, i.e., is its action \(\beta_n\) consistent with optimizing a utility function \(U\)? By observing how the enemy’s radar switches ambiguity function and waveforms to track a target, or how the radar schedules its beam between targets, is there a utility function that rationalizes the radar’s behavior? Notice that a key requirement in Afriat’s theorem is a budget constraint. How to formulate a useful budget constraint for a radar? A key idea in this paper is to formulate linear and nonlinear budget constraints for a radar in terms of the tracking error covariance where \(\alpha_n\) and \(\beta_n\) are the spectra of the state and covariance noise matrices (as will be justified in Section II) associated with a Kalman filter tracker. From a practical point of view, such spectral revealed preferences yield constructive estimates of the radar’s utility function, and so we can predict (in a Bayesian sense) its future actions.

2. **Cognition Detection in Noise:** If the radar’s response \(\beta_n\) or probe signal \(\alpha_n\) is observed in noise, then violation of Afriat’s theorem could be either due to measurement noise or the absence of utility maximization. We will construct a statistical detection test to decide if the radar is a utility maximizer. The hypothesis test yields a tight bound for the Type-I errors.

3. **Optimal Probing:** Given the detector in the above objective, what choice of probe signal yields the smallest Type II error in detecting if the radar is a utility maximizer, subject to maintaining the Type I error within a specified bound? We construct a stochastic gradient algorithm that estimates our optimal probe sequence.

C. Context and Literature

The above objectives are fundamentally different to the model-centric theme used in the signal processing literature where one postulates an objective function (typically convex) and then proposes optimization algorithms. In contrast the revealed preference approach is data centric - given a dataset, we wish to determine if is consistent with utility maximization. Specifically, Sections \[^3\] and \[^4\] below discuss how revealed...
preferences can be used as a systematic method to detect utility maximization in cognitive radars.

Regarding the literature, in the context of revealed preferences we already mentioned [2], [11], [8], [6], [4]. A nonlinear budget version extension was developed in [12] which we will exploit in our spectral revealed preferences setup in Section III. A stochastic detector for utility maximization given noisy measurements of the probe or response is studied in [13], [14] and we will use these results in Section V. Our earlier work [15], [16] consider utility estimation in adversarial signal processing and social network applications. As mentioned above, revealed preferences are more general than inverse reinforcement learning [2].

Cognitive radars [17] use stochastic control and resource allocation to adapt their waveform, aperture, and service requests. In the last decade there have been several works in adaptive/cognitive radar and radar resource management; see [18], [19], [22] and references therein. Our main aim in this paper is to detect cognitive radars. Below we will use revealed preferences to detect radars that optimize their waveforms; [20] contains detailed analysis of waveform adaptation based on the seminal book of [21].

II. COGNITIVE RADAR RESPONSE MODEL

The setup involves two time scales. Let \( k = 1, 2, \ldots \) denote discrete time (fast time scale) and \( n = 1, 2, \ldots \) denote epoch (slow time scale). Our probe signal is \( \alpha_n \), the radar’s response action is \( \beta_n \) and our measurement of this action is \( z_n \).

The model of “us” interacting with the cognitive radar has the following dynamics, see Figure 2:

\[
\begin{align*}
\alpha_n & \sim p_{\alpha_n}(x|x_{k-1}), \quad x_0 \sim \pi_0 \\
\beta_n & \in \arg \max_{\beta \leq 1} U(\beta) \\
y_k & \sim p_{y_k}(y|x_k) \\
\pi_k & = T(\pi_{k-1}, y_k) \\
z_n & = \beta_n + \epsilon_n
\end{align*}
\]

Let us explain the notation in (6): \( p(\cdot) \) denotes a generic conditional probability density function (or probability mass function), \( \sim \) denotes distributed according to, and

- \( x_k \in \mathcal{X} \) is our Markovian state with transition kernel \( p_{\alpha_n} \) and prior \( \pi_0 \) where \( \mathcal{X} \) denotes the state space. So our dynamics are determined by the control probe signal \( \alpha_n \) which evolves on the slow time scale.
- Based on optimizing a utility function \( U(\cdot) \) which is some monotone function (unknown to us) of the predicted target statistic (e.g. covariance of the target’s estimate) in epoch \( n \), the radar chooses an action \( \beta_n \). It is here that actual tracker structure determines the response.
- \( y_k \in \mathcal{Y} \) is the radar’s noisy observation of our state \( x_k \); with observation likelihoods \( p_{y_k}(y|x) \). Here \( \mathcal{Y} \) denote the observation space. So the observation at the radar depends on its action \( \beta_n \), which evolves on the slow time scale.
- \( \pi_k = p(x_k|y_{1:k}) \) is the radar tracker’s belief (posterior) of our state \( x_k \) where \( y_{1:k} \) denotes the sequence \( y_1, \ldots, y_k \). The tracking functionality \( T(\cdot) \) in (6) is the classical Bayesian optimal filtering update formula [22].

\[
T(\pi, y)(x) = \frac{p_{\beta}(y|x) \int_X p_{\alpha}(x|\zeta) \pi(\zeta) d\zeta}{\int_X p_{\beta}(y|x) \int_X p_{\alpha}(x|\zeta) \pi(\zeta) d\zeta dx}
\]

Let \( \mathcal{Y} \) denote the space of all such beliefs. When the state space \( \mathcal{X} \) is Euclidean space, then \( \Pi \) is a function space comprising the space of density functions; if \( \mathcal{X} \) is finite, then \( \Pi \) is the unit \( X \times 1 \) dimensional simplex of \( X \)-dimensional probability mass functions.
- \( z_n \) denotes our noisy measurement of the radar’s action

III. WAVEFORM ADAPTATION: SPECTRAL REVEALED PREFERENCES TO TEST FOR COGNITIVE RADAR

Waveform adaptation is perhaps one of the most important functionalities of a cognitive radar. A cognitive radar adapts its waveform by adapting its ambiguity function. Our aim is to detect such cognitive behavior of the enemy’s radar when it deploys a Bayesian filter as a tracker. Below we identify economic rationality of a radar controller that interacts with a physical level tracker. For concreteness, in this section we assume that the enemy’s cognitive radar uses a Kalman filter tracker. Also since the probe and response signal evolve on a slow time scale \( n \) (described below) we assume that both the radar and us (observer) have perfect measurements of probe \( \alpha_n \) and response \( \beta_n \).

Our working assumption is that a cognitive radar satisfies economics-based rationality; that is, it adapts its waveform by maximizing a utility function in the sense of [4] with a possibly nonlinear budget constraint. A key requirement in Afriat’s Theorem [2] is the budget constraint. In economics, such a constraint is obvious since it specifies the total available resources of the decision maker. How to formulate useful budget constraints for waveform adaptation? Our key idea here is to formulate linear and nonlinear budget constraints in terms of the Kalman filter error covariance where \( \alpha_n \) and \( \beta_n \) are the spectra (eigenvalues) of the state and covariance noise matrices of the state space model.

A. Waveform Adaptation by Cognitive Radar

Suppose a radar adapts its waveform while tracking a target (us) using a Kalman filter. By observing the radar’s signals, how can we test the radar for economic rationality?
1) Linear Gaussian Target Model and Radar Tracker: Linear Gaussian dynamics for a target's kinematics and linear Gaussian measurements at the radar are widely assumed as a useful approximation. Accordingly, consider the following special case of model with linear Gaussian dynamics and measurements:
\[
x_{k+1} = A x_k + w_k(\alpha_n), \quad x_0 \sim \pi_0 \\
y_k = C x_k + v_k(\beta_n)
\]
Here \( x_k \in \mathcal{X} = \mathbb{R}^X \) is “our” state with initial density \( \pi_0 \sim N(\tilde{x}_0, \Sigma_0) \), \( y_k \in \mathcal{Y} = \mathbb{R}^Y \) denotes the cognitive radar’s observations, \( w_k \sim N(0, Q(\alpha_n)) \), \( v_k \sim N(0, R(\beta_n)) \) and \( \{w_k\}, \{v_k\} \) are mutually independent i.i.d. processes. When \( x_k \) denotes respectively, the x,y,z position and velocity components of the target (so \( x_k \in \mathbb{R}^3 \)) then
\[
A_{6 \times 6} = \text{diag} \left[ \begin{array}{cccccc} 1 & T & 0 & 0 & 1 & T \\ 0 & 1 & 1 & 0 & 1 & 0 \end{array} \right]
\]
where \( T \) is the sampling interval. Recall \( k \) indexes the fast time scale while \( n \) indexes the slow time scale.

In we explicitly indicate the dependence of the state noise covariance \( Q \) on our probe signal \( \alpha_n \) and the observation noise covariance \( R \) on the radar’s response signal \( \beta_n \). These are justified as follows. When the radar controls its ambiguity function, in effect it controls the measurement noise covariance \( R \). Of course, this come as a cost: reducing the observation noise covariance of a target results in increased visibility of the radar (and therefore higher threat) or increased covariance of other targets. Similarly, when we modify our kinematics, we effectively change the state noise covariance \( Q \). For example, in a classical linear Gaussian state space model used in target tracking , our probe \( \alpha_n \) parametrizes the state noise covariance \( Q(\alpha_n) \) which models acceleration maneuvers of our drone.

Based on observation sequence \( y_1, \ldots, y_k \), the tracking functionality in the radar computes the posterior
\[
\pi_k = N(\tilde{x}_k, \Sigma_k)
\]
where \( \tilde{x}_k \) is the conditional mean state estimate and \( \Sigma_k \) is the covariance. These are computed by the classical Kalman filter:
\[
\begin{align*}
\Sigma_{k+1|k} &= A \Sigma_k A' + Q(\alpha_n) \\
K_{k+1} &= C \Sigma_{k+1|k} C' + R(\beta_n) \\
\hat{x}_{k+1} &= A \hat{x}_k + C \Sigma_{k+1|k} K_{k+1} (y_{k+1} - C A \hat{x}_k) \\
\Sigma_{k+1} &= \Sigma_{k+1|k} - C \Sigma_{k+1|k} K_{k+1} \Sigma_{k+1|k}
\end{align*}
\]
Under the assumption that the model parameters in satisfy \([A, C]\) is detectable and \([A, \sqrt{Q}]\) is stabilizable, the asymptotic predicted covariance \( \Sigma_{k+1|k} \) as \( k \to \infty \) is the unique non-negative definite solution of the algebraic Riccati equation (ARE):
\[
A(\alpha, \beta, \Sigma) \triangleq -\Sigma + A(\Sigma - \Sigma C' [C \Sigma C' + R(\beta)]^{-1} C \Sigma) A' + Q(\alpha) = 0
\]
where \( \alpha_n \) and \( \beta_n \) are the probe and response signals of the radar at epoch \( n \). Note \( A(\alpha, \beta, \Sigma) \) is a symmetric \( \mathbb{R}^{m \times m} \) matrix. Since \( \Sigma \) is parametrized by \( \alpha, \beta \), we write the solution of the ARE at epoch \( n \) as \( \Sigma_n(\alpha, \beta) \).

B. Effect of waveform design on observation noise covariance

To give a precise structure to the radar dynamics, this section summarizes how the observation noise covariance \( R(\beta) \) in depends on the radar waveform. The details involve maximum likelihood estimation involving the radar ambiguity function and can be found in [20]. Below:
- \( c \) denotes the speed of light (in free space),
- \( \omega \) denotes the carrier frequency,
- \( \theta \) is an adjustable parameter in the waveform,
- \( \eta \) is the signal to noise ratio at the radar.
- \( j = \sqrt{-1} \) is the unit imaginary number.
- \( \tilde{s}(t) \) is the complex envelope of the waveform.
- \( \beta \) is the vector of eigenvalues of \( R \) (in Section D below) or \( R^{-1} \) (in Section C below).

We now describe 3 waveforms and their resulting observation noise covariance matrices \( R(\beta) \); see [20] for details.

(i) Triangular Pulse - Continuous Wave
\[
\tilde{s}(t) = \begin{cases} \sqrt{\frac{3}{2\beta}} \left(1 - \frac{|t|}{\beta}\right) & -\beta < t < \beta \\ 0 & \text{otherwise} \end{cases}
\]
\[
R(\beta) = \begin{bmatrix} c^2 \beta^2 t^2 & 0 \\ 0 & 2c^2 \beta^2 \eta \end{bmatrix}
\]

(ii) Gaussian Pulse - Continuous Wave
\[
\tilde{s}(t) = \left(\frac{1}{\pi \beta^2}\right)^{1/4} \exp\left(-\frac{t^2}{2\beta^2}\right)
\]
\[
R(\beta) = \begin{bmatrix} c^2 \beta^2 & 0 \\ 0 & c^2 \frac{2c^2 \beta^2 \eta}{\omega c^2} \end{bmatrix}
\]

(iii) Gaussian Pulse - Linear Frequency Modulation chirp
\[
\tilde{s}(t) = \left(\frac{1}{\pi \beta^2 t^2}\right)^{1/4} \exp\left(-\frac{1}{2\beta^2 - j\theta_2} t^2\right)
\]
\[
R(\beta) = \begin{bmatrix} c^2 \beta^2 t^2 & 0 \\ 0 & c^2 \frac{2c^2 \beta^2 \eta}{\omega c^2} & c^2 \frac{\omega c^2 \beta^2 \eta}{\omega c^2} \end{bmatrix}
\]

To summarize, by adapting its waveform parametrized by \( \beta \) (vector of eigenvalues), the radar can change the noise covariance \( R(\beta) \). Below we will use the response \( \beta_n \) to construct revealed preference tests for cognition.

C. Testing for Cognitive Radar: Spectral Revealed Preferences with Linear Budget

We now show that Afriat’s theorem (Theorem 2) can be used to determine if a radar is cognitive. The assumption here is that the utility function \( U(\beta) \) maximized by the radar is a monotone function (unknown to us) of the predicted covariance of the target. Our main task is to formulate and justify a linear budget constraint \( \alpha_n^\prime, \beta \leq 1 \) in Afriat’s theorem. Specifically, suppose

1) Our probe \( \alpha_n \) is the vector of eigenvalues of the positive definite matrix \( Q \).
2) The radar response \( \beta_n \) is the vector of eigenvalues of the positive definite matrix \( R^{-1} \).

Then the cognitive radar chooses its waveform parameter \( \beta_n \) at each slow time epoch \( n \) to maximize a utility \( U(\cdot) \):

\[
\beta_n \in \arg \max_{\alpha_n^*, \tilde{n} \leq \tilde{n}} U(\beta)
\]  

(15)

where \( U \) is a monotone increasing function of \( \beta \).

Then Afriat’s theorem (Theorem 3) can be used to detect utility maximization and construct a utility function that rationalizes the response of the radar. Recall that the 1 in the right hand side of the budget rationalizes the response of the radar. Recall that the 1 in the right hand side of the budget \( \alpha_n^*, \tilde{n} \leq \tilde{n} \) can be replaced by any non-negative constant.

It only remains to justify the linear budget constraint \( \alpha_n^*, \tilde{n} \leq \tilde{n} \) in (15). The i-th component of \( \alpha \), denoted as \( \alpha(i) \), is the incentive for considering the i-th mode of the target; \( \alpha(i) \) is proportional to the signal power. The i-th component of \( \beta \) is the amount of resources (energy) devoted by the radar to this i-th mode; a higher \( \beta(i) \) (more resources) results in a smaller measurement noise covariance, resulting in higher accuracy of measurement by the radar. So \( \alpha_n^*, \tilde{n} \leq \tilde{n} \) measures the signal to noise ratio (SNR) and the budget constraint \( \alpha_n^*, \tilde{n} \leq \tilde{n} \) is a bound on the SNR. A rational radar maximizes a utility \( U(\beta) \) that is monotone increasing in the accuracy (inverse of noise power) \( \beta \). However, the radar has limited resources and can only expend sufficient resources to ensure that the precision (inverse covariance) of all modes at each epoch \( n \) is smaller than some pre-specified precision \( \tilde{\Sigma}^{-1} \). We can then justify the linear budget constraint as follows:

**Lemma 3.** The linear budget constraint \( \alpha_n^*, \tilde{n} \leq \tilde{n} \) implies that solution of the ARE (17) satisfies \( \Sigma_n^{-1}(\alpha_n, \beta) \leq \tilde{\Sigma}^{-1} \) for some symmetric positive definite matrix \( \tilde{\Sigma}^{-1} \).

The proof of Lemma 3 follows straightforwardly using the information Kalman filter formulation [25], and showing that \( \Sigma_n^{-1} \) is increasing in \( \beta \). Afriat’s theorem requires that the constraint \( \alpha_n^*, \tilde{n} \leq \tilde{n} \) be smaller than some pre-specified covariance \( \Sigma \). Clearly, a sufficient condition is that \( \lambda(\Sigma_n^*(\alpha_n, \beta)) \leq \Lambda \). But for revealed preferences involving nonlinear budgets, we need the following (see Theorem 5 below): The constraint \( \lambda(\Sigma_n^*(\alpha_n, \beta)) \leq \Lambda \) needs to be active at \( \beta_n \). This is straightforwardly ensured by choosing \( \Lambda \) as

\[
\Lambda \in [0, \lambda_L], \quad \text{where } \lambda_L = \lambda(\Sigma_n^*(\alpha_n, \tilde{\beta})).
\]

(17)

That is, \( \lambda_L \) is the largest eigenvalue of the unique solution \( \Sigma_L \) of the ARE \( A(\alpha_n, \tilde{\beta}_n, \Sigma) = 0 \). The constraint (17) says that the enemy’s Bayesian tracker cannot perform worse in covariance than that of the worst case observation noise covariance \( R(\beta) \), i.e., \( \Sigma^* \leq \Sigma_L \) (positive definite ordering).

**Remark.** In the special case when the constraint \( \beta \leq \tilde{\beta} \) is omitted, then \( \Sigma_L \) is the solution of the algebraic Lyapunov equation

\[
\Sigma = A\Sigma A' + Q(\alpha)
\]

(18)

The constraint (17) then says that that the enemy’s Bayesian tracker cannot perform worse than the optimal predictor (which has infinite observation noise). Of course, when \( A \) is specified as in (9), since all the eigenvalues of \( A \) are 1, the solution of the algebraic Lyapunov equation is not finite.

With the above definitions, our aim is to test if the radar’s response \( \beta \) satisfies economics-rationality:

\[
\beta_n = \arg \max_{\beta} U(\beta),
\]

subject to: \( \lambda(\Sigma_n^*(\alpha_n, \beta)) \leq \Lambda, \quad \beta \leq \tilde{\beta}_n \)

Since there is no natural ordering of eigenvalues, our assumption is that \( U(\beta) \) is a symmetric function of \( \beta \). Here \( \Sigma_n^* \) is the solution of the ARE (11) at epoch \( n \), and \( \lambda \in \mathbb{R}_+, \tilde{\beta}_n \in \mathbb{R}_+^m \) are user-specified parameters. Note that the constraint \( \beta \leq \tilde{\beta}_n \) holds elementwise.

1) **Discussion of Utility and Nonlinear Budget Constraint (10):** The economics-based rationale for the utility (16) is as follows: The i-th component of \( \alpha \), denoted as \( \alpha(i) \), is the price the radar pays for devoting resources to the i-th mode of the target. Since \( \alpha(i) \) is inversely proportional to the signal power, so a higher \( \alpha(i) \) implies a more expensive mode to track, implying that the radar needs to allocate more resources to the i-th mode. The radar’s response for the i-th mode is \( \beta(i) \); this reflects the cost incurred by the radar for estimating mode \( i \). A rational radar aims to minimize its total effort \( C(\beta) \) where cost \( C(\beta) \) decreases with \( \beta \) since choosing a waveform that results in a larger observation noise variance requires less effort. Equivalently, the radar seeks to maximize a utility function \( U(\beta) = -C(\beta) \) where \( U(\beta) \) is increasing with \( \beta \).

We now discuss the nonlinear budget constraint \( \lambda(\Sigma_n^*(\alpha_n, \beta)) \leq \lambda \) in (16) together with \( \beta \leq \tilde{\beta}_n \). The radar seeks to minimize total effort \( C(\beta) \) subject to maintaining the inaccuracy of all modes (covariance \( \Sigma_n \)) to be smaller than some pre-specified covariance \( \Sigma \). Clearly, a sufficient condition is that \( \lambda(\Sigma_n^*(\alpha_n, \beta)) \leq \Lambda \). But for revealed preferences involving nonlinear budgets, we need the following (see Theorem 5 below): The constraint \( \lambda(\Sigma_n^*(\alpha_n, \beta)) \leq \Lambda \) needs to be active at \( \beta_n \). This is straightforwardly ensured by choosing \( \Lambda \) as

\[
\Lambda \in [0, \lambda_L], \quad \text{where } \lambda_L = \lambda(\Sigma_n^*(\alpha_n, \tilde{\beta})).
\]

Examples of symmetric functions include trace, determinant, nuclear norm, etc. The assumption of symmetry is only required when we choose \( \beta \) to be the vector of eigenvalues since there is no natural ordering of the eigenvalues in terms of the ordering of the elements of the matrix. Specifically, Theorem 5 does not require \( U(\beta) \) to be a symmetric function of \( \beta \).
We can now justify the nonlinear budget for a cognitive radar equipped with a Kalman filter tracker as follows:

**Lemma 4.** Consider the nonlinear budget constraint $\lambda(\Sigma^*(\alpha_n, \beta)) \leq \bar{\lambda}$ in (16) with user defined parameter $\lambda$ satisfying (17). Then the solution of the ARE (17) satisfies $\Sigma^*(\alpha_n, \beta) \preceq \Sigma$, for any choice of symmetric positive definite matrix $\Sigma \preceq \Sigma_k$.

2) Revealed Preference for Nonlinear Budget: Having formally justified the nonlinear budget constraint $\lambda(\Sigma^*(\alpha_n, \beta)) \leq \bar{\lambda}$ in (16), we now state the main revealed preference test (12) which generalizes Afriat’s theorem to nonlinear budgets. The result below provides an explicit test for a cognitive radar and constructs a set of utility functions that rationalizes the decisions $\{\beta_n\}$ of the cognitive radar.

**Theorem 5** (Test for rationality with nonlinear budget [12]). Let $B_n = \{\beta \in \mathbb{R}_+^n | g_n(\beta) \leq 0\}$ with $g_n : \mathbb{R}_+^m \rightarrow \mathbb{R}$ an increasing, continuous function and $g_n(\beta_n) = 0$ for $n = 1, \ldots, N$. Then the following conditions are equivalent:

1) There exists a monotone continuous utility function $U$ that rationalizes the data set $\{\beta_n, B_n\}, n = 1, \ldots, N$. That is

$$\beta_n = \arg \max_{\beta} U(\beta), \quad g_n(\beta_n) \leq 0$$

2) The data set $\{\beta_n, B_n\}, n = 1, \ldots, N$ satisfies GARP:

$$g(\beta_j) \leq g(\beta_i) \implies g_j(\beta_i) \geq 0$$

3) For $u_t$ and $\lambda_t > 0$ the following set of inequalities has a feasible solution:

$$u_s - u_t - \lambda_t g_t(\beta_s) \leq 0 \quad \forall t, s \in \{1, 2, \ldots, N\}.$$  

(19)

4) With $u_t$ and $\lambda_t$ defined in (19), an explicit monotone continuous utility that rationalizes the data set is given by:

$$U(\beta) = \min_{t \in \{1, 2, \ldots, N\}} \{u_t + \lambda_t g(\beta_t)\}$$  

(20)

Note that the classic Afriat theorem (Theorem 2) is a special case of Theorem 5 where $g_n(\beta) = \alpha^*_{n}(\beta - \beta_n)$. Also unlike Afriat’s theorem, the constructed utility function is not necessarily concave.

We now show that the nonlinear radar budget constraint in (16), (17) satisfies the properties of Theorem 5 with

$$g_n(\beta) = \lambda(\Sigma^*(\alpha_n, \beta)) - \bar{\lambda}$$  

(21)

First, clearly $\Sigma^*(\alpha, \beta)$ is increasing in $\beta$ and is a continuous function of $\beta$, and so is $\lambda(\Sigma^*(\alpha, \beta))$. Second Theorem 5 requires the constraint to be active at $\beta_n$. This follows since $\lambda(\Sigma^*(\alpha_n, \beta))$ is increasing in $\beta$ and due to (17).

**Summary:** By choosing the probe signal $\alpha$ as the spectrum of $Q^{-1}$ and the response signal $\beta$ as the spectrum of $R$, we can use the nonlinear budget Theorem 5 to test a cognitive radar for utility maximization. We can then construct explicit utility functions (20) that rationalize the decisions of the radar in terms of waveform adaptation.

IV. BEAM ALLOCATION: REVEALED PREFERENCE TEST

This section constructs a test for cognition of a radar that switches its beam adaptively between targets. We work at a higher level of abstraction than the previous section and consider multiple targets. Suppose a radar adaptively switches its beam between $m$ targets where these $m$ targets are controlled by us. As in (5), on the fast time scale indexed by $k$, each target $i$ has linear Gaussian dynamics and the enemy radar obtains linear Gaussian measurements:

$$x_{k+1} = Ax_k + w_k, \quad x_0 \sim \pi_0$$

$$y_k = Cx_k + v_k$$

(22)

Here $w_k \sim N(0, Q_n(i))$, $v_k \sim N(0, R_n(i))$. We assume that both $Q_n(i)$ and $R_n(i)$ are known to us and the enemy.

The enemy’s radar tracks our $m$ targets using a Kalman filter tracker. The fraction of time the radar allocates to each target $i$ in epoch $n$ is $\beta_n(i)$. The price of each target $i$ at the beginning of epoch $n$ is its predicted covariance obtained as the trace of the predicted asymptotic covariance at epoch $n$ from the Algebraic Lyapunov equation (18) as

$$\alpha_n(i) = \text{Tr}(\Sigma_n(n - 1)(i)), \quad i = 1, \ldots, m$$

Obviously, $\alpha_n(i)$ depends on the maneuver covariance $Q_n(i)$ of target $i$. So unlike the previous section where the spectrum of the probe matrix was chosen as the probe vector, here we abstract the covariance by the resulting $\alpha_n(i)$. Note also that $R_n(i)$ depends on the enemy’s radar response $\beta_n(i)$, i.e., the fraction of time allocated to target $i$. We assume that each target $i$ is equipped with a radar detector and can estimate the fraction of time $\beta_n(i)$ the enemy’s radar devotes to it.

Given the time series $\alpha_n, \beta_n, n = 1, \ldots, N$, our aim is to detect if the enemy’s radar is cognitive. We assume that a cognitive radar optimizes its beam allocation as follows:

$$\beta_n = \arg \max_{\beta} U(\beta)$$

s.t. $\beta' \alpha_n \leq \sigma_*$, $\sum_i \beta(i) \leq 1$  

(23)

where $U(\cdot)$ is some unknown utility function and $\sigma_* \in \mathbb{R}_+$ is a pre-specified maximum allowed average variance of all targets. Note that the setup is almost identical to that of Afriat’s Theorem 2 except that there is the additional constraint $\sum_i \beta(i) \leq 1$. It can be shown [6] that Afriat’s Theorem 2 with the additional constraint that $\sum_i \beta(i) \leq 1$ in (5) can be used to test if the radar satisfies (23) and also estimate the set of utility functions (4). Furthermore as in Afriat’s theorem $\sigma_*$ can be chosen as 1 without loss of generality (and therefore does not need to be known by us).

V. DETECTING COGNITIVE RADARS IN A NOISY SETTING

Afriat’s theorem (Theorem 2) and its generalization to nonlinear budgets (Theorem 5) assumes perfect observation of the probe and response. However, when the response (e.g. enemy’s radar waveform) is measured in noise by us, or the probe signal (e.g. our maneuver) is measured in noise by

7Stochastic revealed preferences is discussed in Section V.
the enemy, violation of the inequalities in Afriat Theorem could be either due to measurement noise or absence of utility maximization (economic rationality). Below we give two statistical tests for utility maximization and characterize the Type-I and Type-II errors.

A. Detecting Cognitive Radar given Noisy Response

Suppose we measure the response $\beta_n$ of the radar in additive noise $\epsilon_n$ as

$$z_n = \beta_n + \epsilon_n.$$  

(24)

Here $\epsilon_n$ are $m$-dimensional random variables that are possibly correlated but functionally independent of $\beta_n$. As an example, consider the beam allocation problem discussed in Section IV where each target $i$ equipped with a radar detector obtains a noisy estimate of the fraction of time $\beta_n(i)$ the radar devotes to it.

Given the noisy data set

$$D_{\text{obs}} = \{(\alpha_n, z_n) : n \in \{1, \ldots, N\}\},$$  

(25)

we propose the following statistical test for testing utility maximization (1). Let $H_0$ denote the null hypothesis that the data set $D_{\text{obs}}$ in (25) satisfies utility maximization. Let $H_1$ denote the alternative hypothesis that the data set does not satisfy utility maximization. There are two possible sources of error:

**Type-I errors:** Reject $H_0$ when $H_0$ is valid.

**Type-II errors:** Accept $H_0$ when $H_0$ is invalid.  

(26)

The following statistical test can be used to detect if an agent is seeking to maximize a utility function.

$$\int_{\Phi^*(z)}^{+\infty} f_M(\psi) d\psi \gtrless \gamma,$$

(27)

In the statistical test (27):

(i) $\gamma$ is the “significance level” of the test.

(ii) The “test statistic” $\Phi^*(z)$, with $z = [z_1, z_2, \ldots, z_N]$ is the solution of the following constrained optimization problem:

$$\min \quad \Phi$$

s.t.  

$$u_s - u_t - \lambda_t \alpha_t'(z_s - z_t) - \lambda_t \Phi \leq 0$$

$$\lambda_t > 0 \quad \Phi \geq 0 \quad \text{for} \quad t, s \in \{1, 2, \ldots, N\}.$$  

(28)

(iii) $f_M$ is the pdf of the random variable $M$ where

$$M \triangleq \max_{t, s} \{\alpha_t'(\epsilon_t - \epsilon_s)\}.$$  

(29)

The intuition behind (27), (28) is clear: if $\Phi = 0$, then (28) is equivalent to Afriat’s theorem. Due to the presence of noise, it is unlikely that $\Phi = 0$ is feasible; so we seek the minimum perturbation $\Phi^*(z)$ that satisfies (28).

The constrained optimization problem (28) is non-convex due to the bilinear constraints $\lambda_t \Phi$. However, since the objective function depends only on the scalar $\Phi$, a one dimensional line search algorithm can be used. In particular, for any fixed value of $\Phi$, (28) becomes a set of linear inequalities, and so feasibility is straightforwardly determined.

The following theorem is our main result for characterizing the detector (27). It states that the probability of Type I error (false alarm) of the detector is bounded by $\gamma$ and that the optimal solution $\Phi^*(z)$ gives the tightest false alarm bound.

**Theorem 6.** Consider the noisy data set (25) where $z = [z_1, z_2, \ldots, z_N]$ and detector (27).

1. Suppose (28) has a feasible solution. Then $H_0$ is equivalent to the event that $\Phi^*(z) \leq M$ in (28).

2. The probability of Type I error (false alarm) is

$$P_{\Phi^*(z)}(H_1|H_0) \leq \gamma$$  

(30)

3. The optimizer $\Phi^*(z)$ in (27) yields the tightest Type-I error bound, in that for any other $\Phi \in [\Phi^*, M]$,

$$P_{\Phi(z)}(H_1|H_0) \geq P_{\Phi^*(z)}(H_1|H_0).$$  

(31)

Proof. Suppose $H_0$ holds. By Theorem 2, $H_0$ is equivalent to having a feasible solution. Let $(\lambda_t^*, u_t^*, \Phi = M)$ denote a feasible solution for (3). Then substituting $\beta_n = z_n - \epsilon_n$, it is easily seen that $(\lambda_t^*, u_t^*, \Phi = M)$ is a feasible solution for the noisy inequalities (28). Since $(\lambda_t^*, u_t^*, \Phi = M)$ is feasible, clearly the minimizing solution of (28) satisfies $\Phi^*(z) \leq M$. Therefore, (28) feasible and $H_0 \implies \Phi^*(z) \leq M$

Similarly, let $(\lambda_t, u_t)$ denote a feasible solution to the noisy inequalities (28). Then $\Phi^*(z) \leq M$ implies that (3) has a feasible solution, i.e.,

(28) feasible and $\Phi^*(z) \leq M \implies H_0$

Therefore if (28) is feasible, $H_0$ is equivalent to $\Phi^*(z) \leq M$.

Let $F_M$ denote the complementary cdf of $M$. From the statistical test (27), the event $H_1$, given $H_0$, is equivalent to the event $\{F_M(\Phi^*(z)) \leq \gamma\}$ given $\{\Phi^*(z) \leq M\}$ and (28). So $P(H_1|H_0) = P(F_M(\Phi^*(z)) \leq \gamma|\Phi^*(z) \leq M)$. Now if $\Phi^*(z) = M$, then since $F_M(\gamma)$ is uniform[0,1] clearly $P(H_1|H_0) = \gamma$. So if $\Phi^*(z) \leq M$ then $P(F_M(\Phi^*(z)) \leq \gamma|\Phi^*(z) \leq M)$, i.e., (30) holds.

Suppose $\Phi(z) > \Phi^*(z)$. Then clearly, $P(F_M(\Phi(z)) < \gamma|\Phi(z) \leq M) \geq P(F_M(\Phi^*(z)) \leq \gamma|\Phi^*(z) \leq M)$, i.e., (31) holds.

B. Detecting Cognitive Radar given Noisy Probe

Here we consider the case where the radar measures our probe signal $\alpha_n$ in additive noise $\zeta_n$, as

$$\alpha_n = \alpha_n + \zeta_n.$$  

(32)

Here $\zeta_n$ are $m$-dimensional i.i.d. random variables.

Given the noisy data set

$$D_{\text{obs}} = \{(\alpha_n, \beta_n) : n \in \{1, \ldots, N\}\},$$  

(33)

we propose the following statistical test for testing utility maximization (1) of the radar:

$$\int_{\Phi^*(\bar{a})}^{+\infty} f_M(\psi) d\psi \gtrless \gamma.$$  

(34)

Clearly the cdf and complementary cdf of any random variable $X$, namely, $F(X)$ and $F_M(X)$, are uniformly distributed in $[0,1]$
In the statistical test (32),
(i) $\gamma$ is the “significance level” of the test.
(ii) The test statistic $\Phi^*(\tilde{\alpha})$, with $\tilde{\alpha} = [\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_N]$ is the solution of the following constrained optimization problem:
\[
\min \Phi \\
\text{s.t.} \quad u_s - u_t - \lambda_t \tilde{\alpha}_t^*(\beta - \beta_t) - \lambda_s \Phi \leq 0 \\
\lambda_t > 0, \ \Phi \geq 0 \quad \text{for} \quad t, s \in \{1, 2, \ldots, N\}.
\] (35)

(iii) $f_M$ is the pdf of the random variable $M$ where
\[
M \equiv \max_{t, s} \{Q^*(\beta_t - \beta_s)\}.
\] (36)

In complete analogy to Theorem 6 we have

**Theorem 7.** Consider the noisy data set (32) and detector (33). Then the assertions of Theorem 6 hold.

**C. Lower Bound for False Alarm Probability**

Given the significance level of the statistical test in (27), a Monte Carlo simulation is required to compute the threshold. Below, we derive an analytical expression for a lower bound on the false alarm probability of the statistical test in (27) when the additive noise $\epsilon_n$ in (24) are standard normal variables.

**Theorem 8.** Consider the noisy data set $D_{obs}$ in (25). Suppose $\{\epsilon_n\}$ in (24) are i.i.d $N(0, I)$ Gaussian vectors and $\mathbf{z} = [z_1, z_2, \ldots, z_N]$. Then the probability of false alarm in (26) is lower bounded by
\[
1 - \prod_n \left\{ 1 - \frac{1}{\pi} \sqrt{2} \left\| \alpha_n \right\|^2 \exp \left( -\frac{\Phi^*(\mathbf{z})}{\Phi^*(\mathbf{z}) + \left\| \alpha_n \right\|^2} \right) \right\}.
\] (37)

The proof is in [16]. From the analytical expression (37), can obtain an upper bound of the test statistic, denoted by $\Phi^*(\mathbf{z})$. Hence, given a data set $D_{obs}$ in (25), if the solution to the optimization problem (28) is such that $\Phi > \Phi^*(\mathbf{z})$, then the conclusion is that the data set does not satisfy utility maximization, for the desired false alarm probability.

**VI. OPTIMIZING OUR PROBE SIGNAL TO MINIMIZE TYPE-II DETECTION ERROR OF ENEMY’S RADAR**

This section deals with adaptively interrogation the enemy radar to detect if it is cognitive. Specifically, given batches of noisy measurements of the enemy’s radar response $\mathbf{z}_k = [z_{1,k}, \ldots, z_{N,k}]$, $k = 1, 2, \ldots$ how can we adaptively design batches of our probe signals $\alpha_k = [\alpha_{1,k}, \ldots, \alpha_{N,k}]$, $k = 1, \ldots$, so as to detect if the radar is cognitive (economically rational)? In this section we formulate this input design problem of minimizing the Type-II error as a stochastic optimization problem.

Theorem 6 above guarantees that if we observe the radar response in noise, then the probability of Type-I errors (deciding that the radar is not cognitive when it is) is less than $\gamma$ for the decision test (27). In this subsection, the statistical test (27) is enhanced by adaptively optimizing the probe vectors $\alpha_k$ to reduce the probability of Type-II errors (deciding that the radar is cognitive when it is not).

![Fig. 3. Optimizing the probe waveform to detect cognition in adversary’s radar by minimizing the Type II errors subject to constraints in Type I errors.](Image)

\[
\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_N] \quad \text{to reduce the probability of Type-II errors (deciding that the radar is cognitive when it is not).}
\]

The framework is shown in Figure 3. The probe signals $\alpha$ are adapted to estimate
\[
\alpha \in \mathbb{R}^{m \times N}
\]

\[
\arg \min J(\alpha) = \mathbb{P} \left( \int_{\Phi^*(\beta(\alpha)) + \epsilon}^{+\infty} f_M(\psi) d\psi > \gamma \left\{ \{\alpha, \beta(\alpha)\} \in \mathcal{A} \right\} \right)
\] (38)

Here $\mathbb{P}(\cdot | \cdot)$ denotes the conditional probability that the statistical test (27) accepts $H_0$, defined in (26), given that $H_0$ is false. In (38), $\epsilon = [\epsilon_1, \epsilon_2, \ldots, \epsilon_N]$ is a random variable defined in (24), and $\gamma$ is the significance level of (27). The set $\mathcal{A}$ contains all the elements $\{\alpha, \beta(\alpha)\}$, with $\beta(\alpha) = [\beta_1, \beta_2, \ldots, \beta_N]$, where $\{\alpha, \beta\}$ does not satisfy $\delta$.

Since the probability density function $f_M$ defined in (36) is not known explicitly, (38) is a simulation based stochastic optimization problem. To determine a local minimum value of $J(\alpha)$, several types of stochastic optimization algorithms can be used [25]. Below we use the simultaneous perturbation stochastic gradient (SPSA) algorithm:

**Step 1:** Choose initial probe $\alpha_0 = [\alpha_1, \alpha_2, \ldots, \alpha_N] \in \mathbb{R}^{m \times N}

**Step 2:** For iterations $k = 1, 2, 3, \ldots$

a) Estimate cost $J(\alpha_k)$ in (38) using
\[
\hat{J}_k(\alpha_k) = \frac{1}{s} \sum_{s=1}^{S} I \left( F_M(\Phi(\mathbf{z}_s)) \leq 1 - \gamma \right)
\] (39)

where $I(\cdot)$ denotes the indicator function, and $F_M(\cdot)$ is an estimate of the cumulative distribution function of $M$ obtained by generating random samples according to (36). In (39), $\Phi(\mathbf{z}_s)$ is obtained from (28) using a noisy observation $\mathbf{z}_s = \beta(\alpha_k) + \epsilon_s$, with $\epsilon_s$ a fixed realization of $\epsilon$, and the data set $\{\alpha_k, \beta(\alpha_k)\} \in \mathcal{A}$ described below (38). The parameter $S$ controls the accuracy of the empirical probability of Type-II errors.

b) Compute the gradient estimate $\hat{\nabla}_\alpha$
\[
\hat{\nabla}_\alpha \hat{J}_k(\alpha_k) = \frac{\hat{J}_k(\alpha_k + \Delta_k \omega) - \hat{J}_k(\alpha_k - \Delta_k \omega)}{2\omega \Delta_k}
\] (40)

where
\[
\Delta_k(i) = \begin{cases} 
-1 & \text{with probability 0.5} \\
+1 & \text{with probability 0.5}
\end{cases}
\]
with gradient step size $\omega > 0$.

c) Update the probe vector $\alpha_k$ with step size $\mu > 0$:

$$\alpha_{k+1} = \alpha_k - \mu \nabla J_k(\alpha_k).$$

A useful property of the SPSA algorithm is that estimating the gradient $\nabla J_k(\alpha_k)$ in (40) requires only two measurements of the cost function (39) corrupted by noise per iteration, i.e., the number of evaluations is independent of the dimension $m \times N$ of the vector $\alpha$. In comparison, a naive finite difference gradient estimator requires computing $2(\alpha \times N)$ estimates of the cost per iteration; see [26] for a tutorial exposition of the SPSA algorithm. For decreasing step size $\mu = 1/k$, the SPSA algorithm converges with probability one to a local stationary point. For constant step size $\mu$, it converges weakly.

REFERENCES

[1] S. Haykin, “Cognitive radar,” IEEE Signal Processing Magazine, pp. 30–40, Jan. 2006.
[2] A. Ng and S. Russell, “Algorithms for inverse reinforcement learning,” in Proc. 17th International Conf. Machine Learning, 2000, pp. 663–670.
[3] P. Samuelson, “A note on the pure theory of consumer’s behaviour,” Economica, pp. 61–71, 1938.
[4] S. Afriat, “The construction of utility functions from expenditure data,” International economic review, vol. 8, no. 1, pp. 67–77, 1967.
[5] ——, Logic of choice and economic theory. Clarendon Press Oxford, 1987.
[6] W. Diewert, “Afriat’s theorem and some extensions to choice under uncertainty,” The Economic Journal, vol. 122, no. 560, pp. 305–331, 2012.
[7] H. Varian, “Revealed preference and its applications,” The Economic Journal, vol. 122, no. 560, pp. 332–338, 2012.
[8] ——, “Non-parametric tests of consumer behaviour,” The Review of Economic Studies, vol. 50, no. 1, pp. 99–110, 1983.
[9] ——, “Revealed preference,” Samuelsen’s economics and the twenty-first century, pp. 99–115, 2006.
[10] ——, “The nonparametric approach to demand analysis,” Econometrica, vol. 50, no. 1, pp. 945–973, 1982.
[11] ——, “Price discrimination and social welfare,” The American Economic Review, pp. 870–875, 1985.
[12] F. Forges and E. Minelli, “Afriat’s theorem for general budget sets,” Journal of Economic Theory, vol. 144, no. 1, pp. 135–145, 2009.
[13] A. Fleissig and G. Whitney, “Testing for the significance of violations of Afriat’s Inequalities,” Journal of Business & Economic Statistics, vol. 23, no. 3, pp. 355–362, 2005.
[14] B. E. Jones and D. L. Edgerton, “Testing utility maximization with measurement errors in the data,” in Measurement Error: Consequences, Applications and Solutions. Emerald Group Publishing Limited, 2009, pp. 199–236.
[15] V. Krishnamurthy and W. Hoiles, “Afriat’s test for detecting malicious agents,” IEEE Signal Processing Letters, vol. 19, no. 12, pp. 801–804, 2012.
[16] A. Aprem and V. Krishnamurthy, “Utility change point detection in online social media: A revealed preference framework,” IEEE Transactions on Signal Processing, vol. 65, no. 7, April 2017.
[17] S. Haykin, “Cognitive dynamic systems: Radar, control, and radio [point of view],” Proceedings of the IEEE, vol. 100, no. 7, pp. 2095–2103, 2012.
[18] E. K. P. Chong, C. Kreucher, and A. Hero, “Partially observable Markov decision process approximations for adaptive sensing,” Discrete Event Dynamic Systems, vol. 19, no. 3, pp. 377–422, 2009.
[19] V. Krishnamurthy and D. Djonin, “Structured threshold policies for dynamic sensor scheduling—a partially observed Markov decision process approach,” IEEE Transactions on Signal Processing, vol. 55, no. 10, pp. 4938–4957, Oct. 2007.
[20] D. Kershaw and R. Evans, “Optimal waveform design for tracking,” IEEE Transactions on Information Theory, pp. 1536–1551, September 1994.
[21] H. V. Trees, Detection, Estimation and Modulation Theory. John Wiley & Sons, 1968.
[22] V. Krishnamurthy, Partially Observed Markov Decision Processes. From Filtering to Controlled Sensing. Cambridge University Press, 2016.
[23] X. R. Li and V. P. Jilkov, “Survey of maneuvering target tracking. part i. dynamic models,” IEEE Transactions on Aerospace and Electronic Systems, vol. 39, no. 4, pp. 1333–1364, 2003.
[24] Y. Bar-Shalom, X. R. Li, and T. Kirubarajan, Estimation with applications to tracking and navigation. New York: John Wiley, 2008.
[25] B. D. O. Anderson and J. B. Moore, Optimal filtering. Englewood Cliffs, New Jersey: Prentice Hall, 1979.
[26] J. Spall, Introduction to Stochastic Search and Optimization. Wiley, 2003.