CONVEX RATIONALLY CONNECTED VARIETIES

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0. INTRODUCTION

Let $X$ be a nonsingular projective variety over $\mathbb{C}$. A morphism,

$$\mu : \mathbb{P}^1 \to X,$$

is unobstructed if $H^1(\mathbb{P}^1, \mu^*T_X) = 0$. The variety $X$ is convex if all morphisms $\mu : \mathbb{P}^1 \to X$ are unobstructed.

A rational curve in $X$ is the image of a morphism

$$\mu : \mathbb{P}^1 \to X.$$ 

The variety $X$ is rationally connected if all pairs of points of $X$ are connected by rational curves.

Homogeneous spaces $G/P$ for connected linear algebraic groups are convex, rationally connected, nonsingular, projective varieties. Convexity is a consequence of the global generation of the tangent bundle of $G/P$. Rational connectedness is consequence of the rationality of $G/P$.

The following speculation arose at dinner after an algebraic geometry seminar at Princeton in the fall of 2002.

Speculation. If $X$ is convex and rationally connected, then $X$ is a homogeneous space.

The failure of the speculation would perhaps be more interesting than the success.


1. Complete intersections

The only real evidence known to the author is the following result for complete intersections in projective space.

**Theorem.** If $X \subset \mathbb{P}^n$ is a convex, rationally connected, nonsingular complete intersection, then $X$ is a homogeneous space.

**Proof.** We first consider nonsingular complete intersections of dimension at most 1:

(i) in dimension 0, only points are rationally connected,

(ii) in dimension 1, only $\mathbb{P}^1$ is rationally connected.

Hence, the rationally connected complete intersections of dimension at most 1 are simply connected.

Let $X \subset \mathbb{P}^n$ be a generic complete intersection of type $(d_1, \ldots, d_l)$. Let $d = \sum_{i=1}^l d_i$. By the results of [4], $X$ is rationally connected if and only if $d \leq n$. Moreover, if $X$ is rationally connected, then $X$ must be simply connected: if the dimension of $X$ at least 2, $X$ is simply connected by Lefschetz, see [5].

Let $M$ denote the parameter space of lines in $X$. $M$ is a non-empty, nonsingular variety of dimension $2n - 2 - d - l$. Non-emptiness can be seen by several methods. For example, the nonvanishing in degree 1 of the 1-point series of the quantum cohomology of $X$ implies $M$ is non-empty, see [1], [6]. Nonsingularity is a consequence of the genericity of $X$. Let $\pi : U \to M$ denote the universal family of lines over $M$, and let

$$\nu : U \to X$$

denote the universal morphism.

Let $L$ be a line on $X$. If $X$ is convex, the normal bundle $N_L$ of $L$ in $X$ must be semi-positive. If $N_L$ has a negative line summand, then every double cover of $L$,

$$\mu : \mathbb{P}^1 \to \mathbb{P}^1,$$
is obstructed. Since the degree of $N_L$ is $n - d - 1$, we may assume $d \leq n - 1$.

Every semi-positive bundle on $\mathbf{P}^1$ is generated by global sections. Hence, if every line $L$ has semi-positive normal bundle, we easily conclude the morphism $\nu$ is smooth and surjects onto $X$. The fiber of $\nu$ over $x \in X$ is the parameter space of lines passing through $x$.

We now consider the Leray spectral sequence for the fibration $\nu$, see [2]. The Leray spectral sequence degenerates at the $E_2$ term,
\[ E_2^{pq} = H^p(X, R^q\nu_* \mathbb{C}). \]
Since $X$ is simply connected, all local systems on $X$ are constant. Hence,
\[ E_2^{pq} = H^p(X, R^q\nu_* \mathbb{C}) = H^p(X, \mathbb{C}) \otimes H^q(F, \mathbb{C}), \]
where $F$ denotes the fiber of $\nu$.

Let $p_U(t), p_F(t)$, and $p_X(t)$ denote the Poincaré polynomials of the manifolds $U$, $F$, and $X$. We conclude,
\[ p_U = p_F \cdot p_X. \]
On the other hand, since $U$ is a locally trivial fibration over $M$, the polynomial $p_{\mathbf{P}^1}$ must divide $p_U$. Since
\[ p_{\mathbf{P}^1} = 1 + t^2 \]
is irreducible over the integers, we find $1 + t^2$ divides either $p_F$ or $p_X$.

We have proven the following result. Let $X \subset \mathbf{P}^n$ be a generic complete intersection of type $(d_1, \ldots, d_l)$ satisfying $d \leq n - 1$. If every line of $X$ has a semi-positive normal bundle, then either $p_F(i) = 0$ or $p_X(i) = 0$.

Consider the fiber $F$ of $\nu$ over $x$. The dimension of $F$ is $n - 1 - d$. In fact, $F$ is a complete intersection of type
\[ (1, 2, 3, \ldots, d_1, 1, 2, 3, \ldots, d_2, \ldots, 1, 2, 3, \ldots, d_l) \]
in the projective space $\mathbf{P}^{n-1}$ of lines of $\mathbf{P}^n$ passing through $x$.

If $p_F(i) = 0$, then the type of $F$ must be one of the three types allowed by the Lemma below. If $p_X(i) = 0$, then the type of $X$ must be one of the three allowed by the Lemma. Since, one of the two polynomial evaluations must
vanish, we conclude the type of $X$ must be either $(1, \ldots, 1)$ or $(1, \ldots, 1, 2)$. Clearly both are types of homogeneous varieties.

If $X$ is not of one of the two above types, then $X$ must contain a line $L$ for which $N_L$ has a negative line summand. Since $X$ was assumed to be general, every nonsingular complete intersection $Y$ of the type of $X$ must also contain such a line by taking a limit of $L$.

Therefore, if the type of a nonsingular complete intersection $Y$ is not $(1, \ldots, 1)$ or $(1, \ldots, 1, 2)$, then $Y$ is not a convex, rationally connected variety. \[\square\]

The proof of the Theorem also shows homogeneous complete intersections in projective space must be of type $(1, \ldots, 1)$ or $(1, \ldots, 1, 2)$.

**Lemma.** Let $Y \subset \mathbb{P}^n$ be a nonsingular complete intersection of dimension $k$. Let $p_Y(t)$ be the Poincaré polynomial of $Y$. If $p_Y(i) = 0$, then one of the following three possibilities hold:

(i) the type of $Y$ is $(1, \ldots, 1)$ and $k$ is odd,
(ii) the type of $Y$ is $(1, \ldots, 1, 2)$ and $k$ is odd,
(iii) the type of $Y$ is $(1, \ldots, 1, 2)$, and $k = 2 \mod 4$.

**Proof.** Let $Y \subset \mathbb{P}^n$ be a nonsingular complete intersection of dimension $k$. The cohomology of $Y$ is determined by the Lefschetz isomorphism except in the middle (real) dimension $k$. The cohomology determined by Lefschetz is of rank 1 in all even (real) dimensions. If $k$ is odd, then

$$p_Y(t) = \sum_{q=0}^{k} t^{2q} + b_k t^k,$$

where $b_k$ is the $k^{th}$ Betti number. We see $p_Y(i) = 0$ if and only if $b_k = 0$. If $k$ is even,

$$p_Y(t) = \sum_{q=0}^{k} t^{2q} + (b_k - 1)t^k.$$

We see $p_Y(i) = 0$ if and only if $k = 2 \mod 4$ and $b_k - 1 = 1$. 

\[\square\]
Assume \( p_Y(i) = 0 \). Let \((e_1, \ldots , e_k)\) be the type of \( Y \). Let \( e \) be the largest element of the type.

Let \( Z \subset \mathbb{P}^n \) be a nonsingular projective variety of dimension \( r \). Let \( H_d \subset \mathbb{P}^n \) be a general hypersurface of degree \( d \). The dimension,

\[
h^{r-1}(Z \cap H_d; \mathbb{C}),
\]

is a non-decreasing function of \( d \), see [3]. Hence, we can bound \( b_k \) for \( Y \) from below by the middle cohomology \( b'_k \) of the complete intersection \( Y' \subset \mathbb{P}^n \) of type \((e, 1, \ldots , 1)\),

\[
b_k \geq b'_k.
\]
The variety \( Y' \) may then be viewed as a hypersurface of degree \( e \) in the smaller projective space \( \mathbb{P}^{k+1} \).

For a hypersurface \( Y' \subset \mathbb{P}^{k+1} \) of degree \( e \), the middle cohomology \( b'_k \) is given by the following formula:

\[
b'_k - \delta_k = \frac{e - 1}{e}((e - 1)^{k+1} - (-1)^{k+1}),
\]

where \( \delta_k \) is 1 if \( k \) is even and 0 if \( k \) is odd. If \( k \) is odd,

\[
b'_k = \frac{e - 1}{e}((e - 1)^{k+1} - 1).
\]

Then, \( b'_k > 0 \) if \( e > 2 \). If \( k \) is even,

\[
b'_k - 1 = \frac{e - 1}{e}((e - 1)^{k+1} + 1).
\]

Then, \( b_k - 1 > 1 \) if \( e > 2 \). Therefore, we conclude \( e \leq 2 \).

If \( e = 1 \), then case (i) of the Lemma is obtained. It is easy to check \( k \) must be odd for \( p_Y(i) = 0 \) to hold.

Let \( e = 2 \). If \( Y \) is of type \((1, \ldots , 1, 2)\), then either case (ii) or (iii) of the Lemma is obtained. If \( k \) is even,

\[
k = 2 \mod 4
\]

must be satisfied in order for \( p_Y(i) = 0 \) to hold.

If \( Y \) is not of type \((1, \ldots , 1, 2)\), then the next largest type of \( Y \) must be at least 2. As before, we may bound \( b_k \) from below by the middle cohomology
$b'_k$ of the complete intersection of type $(2,2)$ in $\mathbb{P}^{k+2}$. If $k$ is odd, the calculation below shows

$$b'_k = k + 1 > 0.$$ 

If $k$ is even, the calculation below shows

$$b'_k - 1 = k + 3 > 1.$$ 

In fact, the type of $Y$ cannot contain two elements greater than 1 in $p_Y(i) = 0$.

Let $k \geq 0$. The Euler characteristic $\chi_{22}(k)$ of a nonsingular complete intersection of type $(2,2)$ in $\mathbb{P}^{k+2}$ is:

$$\int_{\mathbb{P}^{k+2}} \left( \frac{2H}{1+2H} \right)^2 (1 + H)^{k+3} = \sum_{i=0}^{k} 2^{k+2-i} (-1)^{k-i} (k+1-i) \binom{k+3}{i}. $$

On the other hand, since

$$(k+3)(t-1)^{k+2} - (t-1)^{k+3} + (-1)^{k+3} = \sum_{i=0}^{k+2} t^{k+2-i} (-1)^i (k+3-i-t) \binom{k+3}{i},$$

we find:

$$(-1)^k \chi_{22}(k) = k + 2 + (-1)^k (k + 2).$$

The formulas for $b'_k$ then follow easily. \qed

2. Homogeneous complete intersections

It is interesting to see how the homogeneous complete intersections survive the above analysis.

First, consider a complete intersection $X \subset \mathbb{P}^n$ of type $(1, \ldots, 1, 1)$. Then, $F$ is of dimension $n - 1 - l$, and $X$ is a dimension $n - l$. Both are complete intersections of hyperplanes. Since one of $n - 1 - l$ and $n - l$ is odd, exactly one of the conditions $p_F(i) = 0$ or $p_X(i) = 0$ holds by part (i) of the Lemma.

Next, consider a complete intersection $X \subset \mathbb{P}^n$ of type $(1, \ldots, 1, 2)$ where $l + 1 \leq n - 1$. Then, $F$ is of dimension $n - 1 - l - 1$, and $X$ is of dimension $n - l$. There are two cases:
(i) If \( n - l - 2 \) and \( n - l \) are odd, then both \( p_F(i) = 0 \) and \( p_X(i) = 0 \) by part (ii) of the Lemma.

(ii) If \( n - l - 2 \) and \( n - l \) are even, then one of \( (n - l - 2)/2 \) and \( (n - l)/2 \) is odd. Hence, exactly one of the conditions \( p_F(i) = 0 \) and \( p_X(i) = 0 \) holds by part (iii) of the Lemma.

References

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