The Uniform Order Convergence Structure on $\mathcal{ML}(X)$

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Abstract

The aim of this paper is to set up appropriate uniform convergence spaces in which to reformulate and enrich the Order Completion Method [25] for nonlinear PDEs. In this regard, we consider an appropriate space $\mathcal{ML}(X)$ of normal lower semi-continuous functions. The space $\mathcal{ML}(X)$ appears in the ring theory of $C(X)$ and its various extensions [15], as well as in the theory of nonlinear, PDEs [25] and [28]. We define a uniform convergence structure, in the sense of [11], on $\mathcal{ML}(X)$ such that the induced convergence structure is the order convergence structure, as introduced in [7] and [32]. The uniform convergence space completion of $\mathcal{ML}(X)$ is constructed as the space all normal lower semi-continuous functions on $X$. It is then shown how these ideas may be applied to solve nonlinear PDEs. In particular, we construct generalized solutions to the Navier-Stokes Equations in three spatial dimensions, subject to an initial condition.

Keywords General Topology, Uniform Convergence Structures, Function Spaces, Ordered Spaces

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1 Introduction

It is widely held that, in contradistinction to ODEs, there can be no general, type independent theory for the existence and regularity of the solutions to PDEs [8], [14]. As seen in the sequel, this is in fact a misunderstanding which is often attributed to the more complex geometry of $\mathbb{R}^n$, with $n \geq 2$, as as opposed to that of $\mathbb{R}$ which is relevant to ODEs alone, see [8]. Indeed, the difficulties that are typically encountered when solving PDEs by the usual function analytic methods, which are perceived to arise form the complicated geometry of $\mathbb{R}^n$, are rather due to the inherent limitations of the function analytic methods themselves, and are therefore technical obstacles, rather than conceptual ones.

The above is exemplified by the appearance of not only one, but two general, type independent theories for the solutions of nonlinear PDEs. The Central Theory of PDEs, developed by Neuberger [23], see also [24], is based on a generalized method of steepest descent in suitably constructed Hilbert Spaces. The Order Completion Method, as developed by Oberguggenberger and Rosinger [25], is based on the Dedekind completion of suitable spaces of equivalence classes of functions.

1.1 The Order Completion Method

The method of Order Completion results in the existence of generalized solutions to arbitrary, continuous nonlinear PDEs of the form

$$T(x, D)u(x) = f(x), \ x \in \Omega$$

(1)
with the right hand term $f$ a continuous function of $x \in \Omega$, and the partial differential operator $T(x, D)$ defined through a jointly continuous function
\[ F : \Omega \times \mathbb{R}^M \to \mathbb{R} \]
by
\[ T(x, D) u : x \mapsto F(x, u(x), \ldots, D^\alpha u(x), \ldots) \] (2)

With the PDE (1) one associates a mapping
\[ T : \mathcal{M}^m(\Omega) \to \mathcal{M}^0(\Omega) \]
where $\mathcal{M}^m(\Omega)$ is the space of equivalence classes of functions which are continuously differentiable up to order $m$ everywhere except on some closed nowhere dense set $\{25\}$, under the equivalence relation
\[ u \sim v \iff \exists \Gamma \subseteq \Omega \text{ closed nowhere dense :} \]
\[ \begin{align*}
1) & \quad u, v \in \mathcal{C}^m(\Omega) \\
2) & \quad x \in \Omega \setminus \Gamma \Rightarrow v(x) = u(x)
\end{align*} \] (3)

The mapping $T$ induces an equivalence relation $\sim_T$ on $\mathcal{M}^m(\Omega)$ through
\[ \forall \quad u, v \in \mathcal{M}^m(\Omega) : \]
\[ u \sim_T v \iff Tu = Tv \] (4)

With the mapping $T$ one associates in a canonical way an injective mapping
\[ \widehat{T} : \mathcal{M}_T^m(\Omega) \to \mathcal{M}_T^0(\Omega) \]
where $\mathcal{M}_T^m(\Omega)$ denotes the quotient space $\mathcal{M}^m(\Omega) / \sim_T$. The space $\mathcal{M}_T^m(\Omega)$ is ordered through
\[ \forall \quad U, V \in \mathcal{M}_T^m(\Omega) : \]
\[ U \leq_T V \iff \widehat{T}U \leq \widehat{T}V \] (5)

so that $\widehat{T}$ is an order isomorphic embedding. The mapping $\widehat{T}$ extends uniquely to an order isomorphic embedding
\[ \widehat{T}^\sharp : \mathcal{M}_T^m(\Omega)^\sharp \to \mathcal{M}_T^0(\Omega)^\sharp \] (6)

where $\mathcal{M}_T^m(\Omega)^\sharp$ and $\mathcal{M}_T^0(\Omega)^\sharp$ denote the Dedekind order completions of $\mathcal{M}_T^m(\Omega)$ and $\mathcal{M}_T^0(\Omega)$, respectively. This is summarized in the following commutative diagram:

Subject to a mild assumption on the PDE (1), one has
\[ \forall \quad f \in \mathcal{C}^0(\Omega) : \]
\[ \exists! \quad U^\sharp \in \mathcal{M}_T^m(\Omega)^\sharp : \]
\[ \widehat{T}^\sharp U^\sharp = f \]
where \( U^f \in \mathcal{M}_T\kern-0.5em^m (\Omega)^{\sharp} \) is the unique generalized solution to \((11)\). The unique generalized solution should be interpreted as the \textit{totality} of all super solutions, sub solutions and exact solutions to \((11)\). Recently \([6]\) it was shown that the generalized solutions to a PDE of the form \((11)\) may be assimilated with usual Hausdorff continuous functions, in the sense that there is an order isomorphism between \(\mathcal{M}_T\kern-0.5em^m (\Omega)^{\sharp} \) and the space \(\mathbb{H}_{nf} (\Omega)\) of all nearly finite Hausdorff continuous functions.

Taking into account the universality of the existence and regularity result just described, one may notice that there is a large scope for further enrichment of the basic theory of Order Completion \([25]\). In particular, the following may serve as guidelines for such an enrichment.

\((A)\) The space of generalized solutions to \((11)\) may depend on the PDE operator \(T(x, D)\)

\((B)\) There is no differential structure on the space of generalized solutions

In order to accommodate \((A)\), one may do away with the equivalence relation \((4)\) on \(\mathcal{M}(\Omega)\) and consider a partial order other than \((5)\), which does not depend on the partial differential operator \(T(X, D)\). Indeed, somewhat in the spirit of Sobolev, one may consider the partial order

\[
\forall \ u, v \in \mathcal{M}(\Omega) : u \leq_D v \iff (\forall \ |\alpha| \leq m : D^\alpha u \leq D^\alpha v) \tag{7}
\]

which could also solve \((B)\). However, such an approach presents several difficulties. In particular, the existence of generalized solutions in the Dedekind completion of the partially ordered set \((\mathcal{M}(\Omega), \leq_D)\) is not clear. In fact, the possibly nonlinear mapping \(T\) associated with the PDE \((11)\) cannot be extended to the Dedekind completion in a unique and meaningful way, unless \(T\) satisfies some additional and rather restrictive conditions. We mention that the use of partial orders other than \((5)\) was investigated in \([25, \text{Section 13}]\), but the partial orders that are considered are still tied some additional and rather restrictive conditions. We mention that the use of partial orders other than \((5)\) was investigated in \([25, \text{Section 13}]\), but the partial orders that are considered are still tied conditions. We mention that the use of partial orders other than \((5)\) was investigated in \([25, \text{Section 13}]\), but the partial orders that are considered are still tied conditions.

\subsection{1.2 The Order Convergence Structure}

One possible way of going beyond the basic theory of Order Completion is motivated by the fact that the process of taking the supremum of a subset \(A\) of a partially ordered set \(X\) is essentially a process of approximation. Indeed,

\[
x_0 = \sup A
\]

means that the set \(A\) approximates \(x_0\) arbitrarily close from below. Approximation, however, is essentially a topological process. Hence a topological type model for the process of Dedekind completion of \(\mathcal{M}^0 (\Omega)\) may serve as a starting point for the enrichment of the Order Completion Method.

In this regard, we recall that there are several useful modes of convergence on a partially ordered set, defined in terms of the partial order, see for instance \([12, 20\) and \([27\). In particular, we consider the order convergence of sequences defined on a partially ordered set \(X\) as

\[
(x_n) \ \text{order converges to} \ x \in X \iff \\
\exists \ (\lambda_n), (\mu_n) \subset X : \\
\begin{align*}
1) \ & n \in \mathbb{N} \Rightarrow \lambda_n \leq \lambda_{n+1} \leq x_n \leq \mu_{n+1} \leq \mu_n \\
2) \ & \sup \{\lambda_n : n \in \mathbb{N}\} = x = \inf \{\mu_n : n \in \mathbb{N}\}
\end{align*} \tag{8}
\]

It is well known that the order convergence of sequences is in general not topological, as is demonstrated in \([31]\). That is, for a partially ordered set \(X\) there is no topology \(\tau\) on \(X\) such that the \(\tau\)-convergent sequences are exactly the order convergent sequences. However, see \([7\) and \([32\), for a \(\sigma\)-distributive lattice \(X\) there exists a convergence structure \(\lambda_o\), in the sense of \([11]\), on \(X\) that induces the order convergence of sequences through

\[
\forall \ x \in X : \\
\forall \ (x_n) \subset X : \\
(x_n) \ \text{order converges to} \ x \iff \{\{x_n : n \geq k\} : k \in \mathbb{N}\} \in \lambda_o (x) \tag{9}
\]
In particular, the order convergence structure, defined and studied in [7] and [32] induces the order convergence of sequences through (9), and is defined as

$$\forall x \in X : \forall F \text{ a filter on } X :$$

$$F \in \lambda_{o}(x) \Leftrightarrow \begin{cases} 
\exists (\lambda_n), (\mu_n) \subset X : \\
1) n \in \mathbb{N} \Rightarrow \lambda_n \leq \lambda_{n+1} \leq x \leq \mu_{n+1} \leq \mu_n \\
2) \sup \{\lambda_n : n \in \mathbb{N} \} = x = \inf \{\mu_n : n \in \mathbb{N} \} \\
3) \left[\{\lambda_n, \mu_n : n \in \mathbb{N}\} \right] \subseteq F 
\end{cases} \quad (10)$$

and is Hausdorff, regular and first countable, see [32].

A particular case of the above occurs when $X$ is an Archimedean vector lattice. In this case the convergence structure $\lambda_{o}$ is a vector space convergence structure, and as such it is induced by a uniform convergence structure, in the sense of [17]. Indeed, the Cauchy filters are characterized as

$$\forall F \text{ a filter on } X : F \text{ is Cauchy } \Leftrightarrow F - F \in \lambda_{o}(x)$$

The convergence vector space completion of an Archimedean vector lattice $X$, equipped with the order convergence structure $\lambda_{o}$ may be constructed as the Dedekind $\sigma$-completion $X^\sharp$ of $X$, equipped with the order convergence structure, see [32]. If $X$ is order separable, then the completion of $X$ is in fact its Dedekind completion. In the particular case when $X = C(Y)$, with $Y$ a metric space, then the convergence vector space completion is the set $H_{ft}(X)$ of finite Hausdorff continuous functions on $Y$, which is the Dedekind completion of $C(Y)$.

Let us now consider the possibility of applying the above results to the problem of solving nonlinear PDEs. In this regard, consider a nonlinear PDE of the form (1), and the associated mapping

$$T : M^m(\Omega) \rightarrow M^0(\Omega)$$

The Order Completion Method is based on the abundance of approximate solutions to (1), which are elements of $M^m(\Omega)$, and in general one cannot expect these approximations to be continuous, let alone sufficiently smooth, on the whole of $\Omega$. Moreover, the space $H_{ft}(\Omega)$ does not contain the space $M^0(\Omega)$.

On the other hand, the space $M^0(\Omega)$ is an order separable Archimedean vector lattice [25], and therefore one may equip it with the order convergence structure. The completion of this space will be its Dedekind completion $M^0(\Omega)^\sharp$, as desired. However, there are several obstacles. If one equips $M^m(\Omega)$ with the subspace convergence structure, then the nonlinear mapping $T$ is not necessarily continuous. Moreover, the quotient space $M^0(\Omega)^\sharp$ is not a linear space, so that the completion process for convergence vector spaces does not apply. It is therefore necessary to develop a nonlinear topological model for the Dedekind completion of $M(\Omega)$.

### 2 Spaces of Lower Semi-Continuous Functions

The notion of a normal lower semi-continuous function, respectively normal upper semi-continuous function, was introduced by Dilworth [13] in connection with the Dedekind completion of spaces of continuous functions. Dilworth introduced the concept for bounded, real valued functions. Subsequently the definition was extended to locally bounded functions [3]. The definition extends in a straightforward way to extended real valued functions. In particular, a function $u : X \rightarrow \mathbb{R}$, with $X$ a topological space, is normal lower semi-continuous whenever

$$(I \circ S)(u) (x) = u(x), \ x \in X \quad (11)$$

where $I$ and $S$ are the Lower- and Upper Baire Operators, see [2], [9] and [30], defined as

$$I (u) (x) = \sup \{\inf \{u(y) : y \in V\} : V \in \mathcal{V}_x\}, \ x \in X \quad (12)$$
\[
S(u)(x) = \inf \{ \sup \{ y : y \in V \} : V \in V_x \}, \quad x \in X
\]  
(13)

where \( u \) is any extended real valued function on \( X \). A normal lower semi-continuous function \( u \) is called nearly finite if the set

\[
\{ x \in X : u(x) \in \mathbb{R} \}
\]

is open and dense in \( X \). We denote the space of all nearly finite normal lower semi-continuous functions by \( \mathcal{NL}(X) \). The space \( \mathcal{NL}(X) \) is ordered in a pointwise way through

\[
\forall u, v \in \mathcal{NL}(X) : 
\quad u \leq v \iff \left( \forall x \in X : 
\quad u(x) \leq v(x) \right)
\]

(14)

The space \( \mathcal{NL}(X) \) satisfies the following properties.

**Theorem 1** The space \( \mathcal{NL}(X) \) is Dedekind complete. Moreover, if \( A \subseteq \mathcal{NL}(X) \) is bounded from above, and \( B \subseteq \mathcal{NL}(X) \) is bounded from below, then

\[
\sup A = (I \circ S)(\phi)
\]

\[
\inf B = (I \circ S \circ I)(\varphi)
\]

where

\[
\phi : X \ni x \mapsto \sup \{ u(x) : u \in A \}
\]

and

\[
\varphi : X \ni x \mapsto \inf \{ u(x) : u \in B \}
\]

**Proof.** One may prove the result directly. However, it is straightforward to show that \( \mathcal{NL}(X) \) is order isomorphic to the set \( \mathbb{H}_{nf}(\Omega) \) of nearly finite Hausdorff continuous functions [3]. The result follows immediately from the respective result in [3].

Applying similar arguments, we obtain the following useful result.

**Proposition 2** Consider any \( u \in \mathcal{NL}(X) \). Then there is a set \( U \subseteq X \) such that \( X \setminus U \) is of First Baire Category and \( u \in C(X \setminus U) \). What is more, if \( v \in \mathcal{NL}(X) \) and \( D \subseteq X \) is dense in \( X \), then

\[
\forall x \in D : u(x) \leq v(x) \Rightarrow u \leq v
\]

**Proof.** Again a direct proof of this proposition is available. However the result follows easily by considering the order isomorphism

\[
I : \mathbb{H}_{nf}(X) \to \mathcal{NL}(X)
\]

**Proposition 3** The space \( \mathcal{NL}(X) \) is fully distributive.

**Proof.** Consider a set \( A \subseteq \mathcal{NL}(X) \) such that

\[
\sup A = u_0
\]

For \( v \in \mathcal{NL}(X) \) we must show

\[
u_0 \land v = \sup \{ u \land v : u \in A \}
\]

(15)
Suppose that (15) fails for some \( A \subset \mathcal{NL}(X) \) and some \( v \in \mathcal{NL}(X) \). That is,
\[
\exists \quad w \in \mathcal{NL}(X) : \\
\quad u \in A \Rightarrow u \wedge v \leq w < u_0 \wedge v
\] (16)
Clearly, \( u_0, v \geq w \) so that there is some \( u \in A \) such that \( w \) is not larger than \( u \). In view of Proposition 2
\[
\exists \quad V \subseteq X \text{ nonempty, open} : \\
\quad x \in V \Rightarrow w(x) < u(x)
\] (17)
Upon application of Proposition 1 we find
\[(v \wedge u)(x) > u(x), \quad x \in V\]
since the operators \( I \) and \( S \) are monotone and idempotent [2, Section 2]. Hence (16) cannot hold. This completes the proof. ■

The set \( C_{nd}(X) \) of all functions \( u : X \to \mathbb{R} \) that are continuous everywhere except on some closed nowhere dense subset of \( X \), that is,
\[
u \in C_{nd}(X) \iff \exists \quad \Gamma \subset X \text{ closed nowhere dense} : u \in \mathcal{C}(X \setminus \Gamma)
\] plays a fundamental role in the theory of Order Completion [25], as discussed in the introduction. In particular, one considers the quotient space \( \mathcal{M}(X) = C_{nd}(X) / \sim \), where the equivalence relation \( \sim \) on \( C_{nd}(X) \) is defined by
\[
u \sim v \iff \exists \quad \Gamma \subset X \text{ closed nowhere dense} : x \in X \setminus \Gamma \Rightarrow u(x) = v(x)
\] (18)
An order isomorphic representation of the space \( \mathcal{M}(X) \), consisting of normal lower semi-continuous functions, is obtained by considering the set
\[
\mathcal{ML}(X) = \left\{ u \in \mathcal{NL}(X) \mid \exists \quad \Gamma \subset X \text{ closed nowhere dense} : u \in \mathcal{C}(X \setminus \Gamma) \right\}
\] (19)
The advantage of considering the space \( \mathcal{ML}(X) \) in stead of \( \mathcal{M}(X) \) is that the elements of \( \mathcal{ML}(X) \) are actual point valued functions on \( X \), as opposed to the elements of \( \mathcal{M}(X) \) which are equivalence classes of functions. Hence the value \( u(x) \) of \( u \in \mathcal{ML}(X) \) is completely determined for every \( x \in X \).

**Proposition 4** The mapping
\[
I_S : \mathcal{M}(X) \ni U \mapsto (I \circ S)(u) \in \mathcal{ML}(X)
\] (20)
is a well defined order isomorphism.

**Proof.** First we show that the mapping \( I_S \) is well defined. In this regard, consider some \( U \in \mathcal{M}(X) \) and any \( u, v \in U \). Let \( \Gamma \subset X \) be the closed nowhere dense set associated with \( u \) and \( v \) through (18). Since \( \Gamma \) is closed, it follows by (12) and (13) that
\[
(I \circ S)(u)(x) = (I \circ S)(v)(x), \quad x \in X \setminus \Gamma
\] (21)
Since \( X \setminus \Gamma \) is dense in \( X \), it follows that
\[
\forall \quad x \in X : \\
\forall \quad V_1, V_2 \in V_x : \\
\exists \quad x_0 \in X : \\
x_0 \in (X \setminus \Gamma) \cap (V_1 \cap V_2)
\]
For any $x \in X$ we have
\[
\inf \{(I \circ S)(u)(y) : y \in V_1\} \leq (I \circ S)(u)(x_0)
\]
and
\[
(I \circ S)(v)(x_0) \leq \sup \{(I \circ S)(v)(y) : y \in V_2\}
\]
Hence it follows by (21) that
\[
\inf \{(I \circ S)(u)(y) : y \in V_1\} \leq \sup \{(I \circ S)(v)(y) : y \in V_2\}
\]
so that (12) and (13) yields
\[
I((I \circ S)(u)) \leq S((I \circ S)(v)) \quad (22)
\]
It now follows form the idempotency and monotonicity of the operator $I$ [2, Section 2] that
\[
(I \circ S)(u) \leq (I \circ S)((I \circ S)(v))
\]
Since the operator $(I \circ S)$ is also idempotent, see [6, Section ], one obtains
\[
(I \circ S)(u) \leq (I \circ S)(v)
\]
By similar arguments it follows that
\[
(I \circ S)(v) \leq (I \circ S)(u)
\]
so that $(I \circ S)(u) = (I \circ S)(v)$.
It is obvious that the mapping $I_S$ is surjective. To see that it is injective, consider any $U, V \in \mathcal{M}(X)$. Then we may assume that
\[
\exists \ A \subseteq X \text{ nonempty, open}:
\exists \ \epsilon > 0 : \\
\forall \ u \in U, v \in V : \\
1) \ x \in A \Rightarrow u(x) < v(x) - \epsilon \\
2) \ u, v \in C(A)
\]
so that
\[
I_S(U)(x) < I_S(V)(x) - \epsilon, \ x \in A
\]
It remains to verify
\[
\forall \ U, V \in \mathcal{M}(X) : \\
U \leq V \iff I_S(U) \leq I_S(V)
\]
The implication `$U \leq V \Rightarrow I_S(U) \leq I_S(V)$’ follows by similar arguments as those employed to show that $I_S$ is well defined. Conversely, suppose that $I_S U \leq I_S V$ for some $U, V \in \mathcal{M}(X)$. The result now follows in the same way as the injectivity of $I_S$. This completes the proof.

Corollary 5 The space $\mathcal{ML}(X)$ is a fully distributive lattice.

3 The Uniform Order Convergence Structure on $\mathcal{ML}(X)$

As a consequence of Proposition 3 one may define the order convergence structure $\lambda_\circ$ on the space $\mathcal{ML}(X)$. The order convergence structure induces the order convergence of sequences on $\mathcal{ML}(X)$ and is Hausdorff, regular and first countable. In order to define a uniform convergence structure, in the sense of [11], we introduce the following notation. For any open subset $U$ of $X$, and any subset $F$ of $\mathcal{ML}(X)$, we denote by $F\vert_U$ the restriction of $F$ to $U$. That is,
\[
F\vert_U = \left\{ v \in \mathcal{ML}(U) \mid \exists \ w \in F : x \in U \Rightarrow w(x) = v(x) \right\}
\]
Definition 6 Let $\tau$ be the topology on $X$, and let $\Sigma$ consist of all nonempty order intervals in $\mathcal{ML}(X)$. Let $\mathcal{J}_o$ denote the family of filters on $\mathcal{ML}(X) \times \mathcal{ML}(X)$ that satisfy the following:

There exists $k \in \mathbb{N}$ such that

$$\forall \ i = 1, \ldots, k : \exists \ \Sigma_i = (I^0_n) \subseteq \Sigma :$$

1) $I^0_{n+1} \subseteq I^0_n$, $n \in \mathbb{N}$

2) $(\Sigma_1 \times [\Sigma_1]) \cap \ldots \cap ([\Sigma_k] \times [\Sigma_k]) \subseteq \mathcal{U}$

where $[\Sigma_i] = \{ [F : F \in \Sigma_i] \}$. Moreover, for every $i = 1, \ldots, k$ and $V \in \tau$ one has

$$\exists \ u_i \in \mathcal{ML}(X) : \cap_{n \in N} I^0_n = \{ u_i \}_V \quad \text{or} \quad \cap_{n \in N} I^0_n = \emptyset$$

Theorem 7 The family $\mathcal{J}_o$ of filters on $\mathcal{ML}(X) \times \mathcal{ML}(X)$ constitutes a uniform convergence structure.

Proof. The first four axioms [11, Definition 2.1.2] are clearly fulfilled, so it remains to verify

$$\forall \ U, V \in \mathcal{J}_o : \quad U \circ V \text{ exists } \Rightarrow U \circ V \in \mathcal{J}_o$$

So take any $U, V \in \mathcal{J}_o$ such that $U \circ V$ exists, and let $\Sigma_1, \ldots, \Sigma_k$ and $\Sigma'_1, \ldots, \Sigma'_k$ be the collection of order intervals associated with $U$ and $V$, respectively, through Definition 6. Set

$$\Phi = \{ (i, j) : [\Sigma_i] \circ [\Sigma'_j] \text{ exists} \}$$

Then

$$U \circ V \supseteq \bigcap\{ ([\Sigma_i] \times [\Sigma'_j]) : (i, j) \in \Phi \}$$

by [11, Proposition 2.1.1 (i)]. Now, $(i, j) \in \Phi$ exists if and only if

$$\forall \ m, n \in \mathbb{N} : \quad I^0_m \cap I^0_n \neq \emptyset$$

For any $(i, j) \in \Phi$, set $\Sigma_{i,j} = (I^0_{n,j})$ where, for each $n \in \mathbb{N}$

$$I^0_{n,j} = [\inf (I^0_n) \land \inf (I^0_j), \sup (I^0_n) \lor \sup (I^0_j)]$$

Now, using [8], we find

$$U \circ V \supseteq \bigcap\{ [\Sigma_i] \times [\Sigma_j] : (i, j) \in \Phi \} \supseteq \bigcap\{ [\Sigma_{i,j}] \times [\Sigma_{i,j}] : (i, j) \in \Phi \}$$

Clearly each $\Sigma_{i,j}$ satisfies 1) of (24). Since $\mathcal{ML}(X)$ is fully distributive, see Corollary 5, (25) also holds. This completes the proof. \[ \square \]

An important fact to note is that the uniform order convergence structure $\mathcal{J}_o$ is defined solely in terms of the order on $\mathcal{ML}(X)$, and the topology on $X$. This is unusual for a uniform convergence structure on a function space. Indeed, for a space of functions $F(X,Y)$, defined on some set $X$, and taking values in $Y$, one defines the uniform convergence structure either in terms of the uniform convergence structure on $Y$, or in terms of a convergence structure on $F(X,Y)$ which is suitably compatible with the algebraic structure of the space. Indeed, a convergence vector space carries a natural uniform convergence structure, where the Cauchy filters are determined by the linear structure. That is,

$$\mathcal{F} \text{ a Cauchy filter } \iff \mathcal{F} - \mathcal{F} \rightarrow 0$$

This is also the case for the order convergence structure studied in [21] and [32]. The motivation for introducing a uniform convergence structure that does not depend on the algebraic structure of the set $\mathcal{ML}(X)$ comes from nonlinear PDEs, and in particular the Order Completion Method [25], as explained in the Introduction.

The convergence structure $\lambda_{\mathcal{J}_o}$ induced on $\mathcal{ML}(X)$ by the uniform convergence structure $\mathcal{J}_o$ may be characterized as follows.
Theorem 8 A filter $\mathcal{F}$ on $\mathcal{ML}(X)$ belongs to $\lambda_{\mathcal{J}}(u)$, for some $u \in \mathcal{ML}(X)$, if and only if there exists a family $\Sigma_{\mathcal{F}} = (I_n)$ of nonempty order intervals on $\mathcal{ML}(X)$ such that

1) $I_{n+1} \subseteq I_n$, $n \in \mathbb{N}$
2) $\forall V \in \tau : \cap_{n \in \mathbb{N}} I_n|V = \{u\}|V$

and $[\Sigma_{\mathcal{F}}] \subseteq \mathcal{F}$.

Proof. Let the filter $\mathcal{F}$ converge to $u \in \mathcal{ML}(X)$. Then, by [11, Definition 2.1.3], $[u] \times \mathcal{F} \in \mathcal{J}_u$. Hence by Definition 6 there exist $k \in \mathbb{N}$ and $\Sigma_i \subseteq \Sigma$ for $i = 1, \ldots, k$ such that (24) through (25) are satisfied.

Set $\Psi = \{i : [\Sigma_i] \subset [u]\}$. We claim

$$\mathcal{F} \supseteq \bigcap_{i \in \Psi} I_i$$

(30)

Take a set $A \in \cap_{i \in \Psi} I_i$. Then for each $i \in \Psi$ there is a set $A_i \in I_i$ such that $A \supseteq \cup_{i \in \Psi} A_i$. For each $i \in \{1, \ldots, k\} \setminus \Psi$ choose a set $A_i \in I_i$ with $u \in \mathcal{ML}(X) \setminus A_i$. Then

$$(A_1 \times A_1) \cup \ldots \cup (A_k \times A_k) \in (I_1 \times I_1) \cap \ldots \cap (I_k \times I_k) \subset \mathcal{F} \times [u]$$

and so there is a set $B \in \mathcal{F}$ such that

$$B \times \{u\} \subset (A_1 \times A_1) \cup \ldots \cup (A_k \times A_k)$$

If $w \in B$ then $(u, w) \in A_i \times A_i$ for some $i$. Since $u \in A_i$, we get $i \in \Psi$ and so $w \in \cup_{i \in \Psi} A_i$. This gives $B \subseteq \cup_{i \in \Psi} A_i \subseteq A$ and so $A \in \mathcal{F}$ so that (30) holds.

Clearly, for each $i \in \Psi$, we have

$$\forall V \in \tau : \cap_{n \in \mathbb{N}} I_n^i|V = \{u\}|V$$

(31)

Writing each $I_n^i \in \Sigma_i$ in the form $I_n^i = [\lambda_n^i, \mu_n^i]$, we claim

$$\sup\{\lambda_n^i : n \in \mathbb{N}\} = u = \inf\{\mu_n^i : n \in \mathbb{N}\}$$

(32)

Suppose this were not the case. Then there exists $v, w \in \mathcal{ML}(X)$ such that

$$\lambda_n \leq v < w \leq \mu_n, \ n \in \mathbb{N}$$

Then, in view of Proposition 2, there is some nonempty $V \in \tau$ such that

$$v(x) < w(x), \ x \in V$$

which contradicts (25). Since $\mathcal{ML}(X)$ is fully distributive, the result follows upon setting

$$\Sigma_{\mathcal{F}} = \left\{[\lambda_n, \mu_n] : \begin{array}{ll} 1) \lambda_n = \inf\{\lambda_n^i : i \in \Psi\} \\ 2) \mu_n = \sup\{\mu_n^i : i \in \Psi\} \end{array} \right\}$$

(33)

The converse is trivial. ■

The following is now immediate

Corollary 9 Consider a filter $\mathcal{F}$ on $\mathcal{ML}(X)$. Then $\mathcal{F} \in \lambda_{\mathcal{J}}(u)$ if and only if $\mathcal{F} \in \lambda_{o}(u)$.

Therefore $\mathcal{ML}(X)$ is a uniformly Hausdorff uniform convergence space.

In particular, a sequence $(u_n)$ on $\mathcal{ML}(X)$ converges to $u$ if and only if $(u_n)$ order converges to $u$. 

9
4 The Completion of $\mathcal{ML}(X)$

This section is concerned with constructing the completion of the uniform convergence space $\mathcal{ML}(X)$. In this regard, recall that the completion of the convergence vector space $\mathcal{C}(X)$, equipped with the order convergence structure, is the set of finite Hausdorff continuous functions on $X$ [7]. This space is order isomorphic to the set all finite normal lower semi-continuous functions. Note, however, that functions $u \in \mathcal{ML}(X)$ need not be finite everywhere, but may, in contradistinction to functions in $\mathcal{C}(X)$, assume the values $\pm \infty$ on any closed nowhere dense subset of $X$. Hence we consider the space $\mathcal{NL}(X)$ of nearly finite normal lower semi-continuous functions on $X$. Following the results in Section 3, we introduce the following uniform convergence structure on $\mathcal{NL}(X)$.

Definition 10 Let $\tau$ be the topology on $X$, and let $\Sigma$ consist of all nonempty order intervals in $\mathcal{NL}(X)$. Let $\mathcal{J}_\sigma$ denote the family of filters on $\mathcal{NL}(X) \times \mathcal{NL}(X)$ that satisfy the following: There exists $k \in \mathbb{N}$ such that

$$\forall \ i = 1, \ldots, k : \quad \exists \ \Sigma_i = (I_n^i) \subseteq \Sigma :$$

1) $I_{n+1}^i \subseteq I_n^i, \ n \in \mathbb{N}$

2) $(\Sigma_1 \times [\Sigma_1]) \cap \ldots \cap (\Sigma_k \times [\Sigma_k]) \subseteq \mathcal{U}$

where $[\Sigma_i] = \{[F : F \in \Sigma_i]\}$. Moreover, for every $i = 1, \ldots, k$ and $V \in \tau$ one has

$$\exists \ u_i \in \mathcal{NL}(X) : \quad \cap_{n \in \mathbb{N}} I_n^{i[V]} = \{u_i\}_V \quad \text{or} \quad \cap_{n \in \mathbb{N}} I_n^{i[V]} = \emptyset$$

The following now follows by similar arguments as those employed in Section 3.

Theorem 11 The family $\mathcal{J}_\sigma$ of filters on $\mathcal{NL}(X) \times \mathcal{NL}(X)$ is a Hausdorff uniform convergence structure.

Theorem 12 A filter $\mathcal{F}$ on $\mathcal{ML}(X)$ belongs to $\lambda_{\mathcal{J}_\sigma}$ if and only if $\mathcal{F} \in \lambda_{\mathcal{O}_\sigma}(u)$.

We now proceed to show that $\mathcal{NL}(X)$ is the completion of $\mathcal{ML}(X)$. That is, we show that the following three conditions are satisfied:

- The uniform convergence space $\mathcal{NL}(X)$ is complete
- $\mathcal{ML}(X)$ is uniformly isomorphic to a dense subspace of $\mathcal{NL}(X)$
- Any uniformly continuous mapping $\varphi$ on $\mathcal{ML}(X)$ into a complete, Hausdorff uniform convergence space $Y$ extends uniquely to a uniformly continuous mapping $\varphi^\sharp$ from $\mathcal{NL}(X)$ into $Y$.

Proposition 13 The uniform convergence space $\mathcal{NL}(X)$ is complete.

Proof. Let $\mathcal{F}$ be a Cauchy filter on $\mathcal{NL}(X)$, so that $\mathcal{F} \times \mathcal{F} \in \mathcal{J}_\sigma$. Let $\Sigma_1, \ldots, \Sigma_k$ be the families of order intervals associated with $\mathcal{F} \times \mathcal{F}$ through Definition 10. Since $\mathcal{NL}(X)$ is Dedekind complete it follows by (35) that, for each $i = 1, \ldots, k$

$$\sup\{\lambda_n^i : n \in \mathbb{N}\} = u_i = \inf\{\mu_n^i : n \in \mathbb{N}\}$$

for some $u_i \in \mathcal{NL}(X)$, where $I_n^i = [\lambda_n^i, \mu_n^i]$ for each $n \in \mathbb{N}$. By Theorem 12 each of the filters $\mathcal{F}_i = [\Sigma_i]$ converges to $u_i$. Let $\mathcal{G} \supseteq \mathcal{F}$ be an ultrafilter. Since

$$\mathcal{F} \supseteq \mathcal{F}_1 \cap \ldots \cap \mathcal{F}_k$$

it follows that $\mathcal{G} \supseteq \mathcal{F}_i$ for at least one $i = 1, \ldots, k$, so that $\mathcal{G}$ converges to $u_i$. Therefore [11] Proposition 2.3.2 (iii)] the filter $\mathcal{F}$ converges to $u_i$. This completes the proof. □
Theorem 14 Let $X$ be a metric space. Then the space $\mathcal{NL}(X)$ is the uniform convergence space completion of $\mathcal{ML}(X)$.

Proof. First we show that the identity mapping $\iota : \mathcal{ML}(X) \to \mathcal{NL}(X)$ is a uniformly continuous embedding. In this regard, it is sufficient to consider a filter $[\Sigma_F]$ where $\Sigma_F$ is a family of nonempty order intervals in $\mathcal{ML}(X)$ that satisfies 1) of (24) and (25). Clearly

$$\forall \ I_n = [\lambda_n, \mu_n] \in \Sigma_F : \ \iota(I_n) \subseteq [\iota(\lambda_n), \iota(\mu_n)]$$

(37)

The family

$$\Sigma_{\iota(F)} = (I'_n) = \{[\iota(\lambda_n), \iota(\mu_n)] : n \in \mathbb{N}\}$$

(38)

satisfies 1) of (34). To see that (35) holds, we proceed by contradiction. Assume that for some $W \in \tau$

$$\exists \ u, v \in \mathcal{NL}(X) : \ \cap_{n \in \mathbb{N}} I'_n \supseteq \{u, v\} \text{ if } W \subseteq \cap_{n \in \mathbb{N}} I'_n$$

(39)

where $u|_W \neq v|_W$. We may assume that $u(x) < v(x)$, $x \in W$. Clearly,

$$\lambda_n(x) \leq \varphi(x) \leq u(x) < v(x) \leq \mu_n, \ x \in W$$

(40)

for every $n \in \mathbb{N}$, where

$$\varphi(x) = \sup\{\lambda_n(x) : n \in \mathbb{N}\}$$

which is upper semi-continuous. Applying Hahn’s Theorem twice we find

$$\exists \ \phi, \psi \in C(W) : \ \{\phi, \psi\} \subseteq \cap_{n \in \mathbb{N}} I'_n$$

(41)

which contradicts (25) so that (35) must hold. That $\iota^{-1}$ is uniformly continuous is trivial.

To see that $\iota(\mathcal{ML}(X))$ is dense in $\mathcal{NL}(X)$, consider any $u \in \mathcal{NL}(X)$, and set

$$D_u = \{x \in X : u(x) \in \mathbb{R}\}$$

Since $D_u$ is open, it follows that $u$ restricted to $D_u$ is normal lower semi-continuous. Since $u$ is also finite on $D_u$ it follows, see [7, Proof of Theorem 26] that there exists a sequence $(u_n)$ of continuous functions on $D_u$ such that

$$u(x) = \sup\{u_n(x) : n \in \mathbb{N}\}, \ x \in D_u$$

(42)

Consider now the sequence $(v_n) = (I \circ S)(u_n^0)$ where

$$u_n^0(x) = \begin{cases} \ u_n(x) & \text{if } x \in D_u \\ 0 & \text{if } x \notin D_u \end{cases}$$

(43)

Clearly $v_n(x) = u_n(x)$ for every $x \in D_u$. We claim

$$u = \sup\{v_n : n \in \mathbb{N}\}$$

(44)

If (44) does not hold, then

$$\exists \ v \in \mathcal{NL}(X) : \ n \in \mathbb{N} \Rightarrow v_n \leq v < u$$

But then, in view of Proposition 7 and the fact that $D_u$ is open and dense, there exists an open and nonempty set $W \subseteq D_u$ such that

$$\forall \ x \in W : \ n \in \mathbb{N} \Rightarrow u_n(x) \leq v(x) < u(x)$$

(45)
which contradicts (11). Therefore (43) must hold. The sequence \((v_n)\) is clearly a Cauchy sequence in \(\mathcal{M}\mathcal{L}(X)\) so that \(\mathcal{M}\mathcal{L}(X)\) is dense in \(\mathcal{N}\mathcal{L}(X)\).

The extension property for uniformly continuous mappings on \(\mathcal{M}\mathcal{L}(X)\) follows in the standard way.

Note that in the above proof, we actually showed that \(\mathcal{N}\mathcal{L}(X)\) is the Dedekind completion of \(\mathcal{M}\mathcal{L}(X)\). Hence the uniform order convergence structure provides a nonlinear topological model for the process of taking the Dedekind completion of \(\mathcal{M}\mathcal{L}(X)\). In view of Proposition 4, this extends a previous result of Anguelov [2] on the Dedekind completion of \(\mathcal{M}(X)\).

5 An Application to Nonlinear PDEs

As an illustration of how the results developed in this paper may be applied to the problem of obtaining generalized solutions to nonlinear PDEs, we consider the Navier-Stokes equations in three spatial dimensions given by

\[
\frac{\partial}{\partial t} u_i(x, t) + \sum_{j=1}^{3} u_j(x, t) \frac{\partial}{\partial x_i} u_j(x, t) - \nu \sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2} u_i(x, t) + \frac{\partial p}{\partial x_i}(x, t) = f(x, t)
\]

\[
\sum_{i=1}^{3} \frac{\partial}{\partial x_i} u_i(x, t) = 0
\]

(44)

where \((x, t) \in \Omega = \mathbb{R}^3 \times [0, \infty)\), and \(f \in C^0(\Omega, \mathbb{R}^3)\). We also require the unknown function \(u = (u_1, u_2, u_3)\) to satisfy the initial value

\[
u(x, 0) = u^0(x), \quad x \in \mathbb{R}^3
\]

(45)

where \(u^0 \in C^2(\mathbb{R}^3, \mathbb{R}^3)\) is a given, divergence free vector field. The equations (44) are supposed to model the motion of a fluid through three dimensional space, where \(u\) specifies the velocity, and \(p\) the pressure in the fluid. We write the equation (44) in the compact form

\[
T(x, t, D)v(x, t) = g(x, t), \quad (x, t) \in \Omega
\]

where \(v = (u, p)\), \(g = (f, 0)\) and the nonlinear PDE operator \(T(x, t, D)\) is defined through a continuous mapping \(F: \Omega \times \mathbb{R}^K \to \mathbb{R}^4\) by

\[
T(x, t, D)v(x, t) = F(x, t, v(x, t), ..., D^\alpha v(x, t), ...), \quad |\alpha| \leq 2
\]

(46)

With the system of PDEs (44) we can associate a mapping

\[
T: C^2(\Omega)^4 \ni u \mapsto (T_1 u, T_2 u, T_3 u, T_4 u) \in C^0(\Omega)^4
\]

(47)

In view of (46), one may extend the mappings \(T\) uniquely to

\[
T: C^2_{nd}(\Omega)^4 \to C^0_{nd}(\Omega)
\]

Then, for \(i = 1, ..., 3\)

\[
T_i: X \ni v \mapsto (I \circ S) \left( \frac{\partial}{\partial t} u + \sum_{j=1}^{3} u_j \frac{\partial}{\partial x_i} u_j - \nu \sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2} u_i + \frac{\partial p}{\partial x_i} \right) \in Y
\]

(48)

and

\[
T_4: X \ni v \mapsto (I \circ S) \left( \sum_{i=1}^{4} \frac{\partial}{\partial x_i} u \right) \in Y
\]

(49)

define unique extensions of the components of \(T\) to \(X\), where

\[
X = \mathcal{M}\mathcal{L}_0^2(\Omega)^4,
\]

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\[ Y = \mathcal{ML}^0(\Omega)^4 \]

where, for \( m \in \mathbb{N} \),

\[ \mathcal{ML}^m_0(\Omega) = \left\{ u \in \mathcal{ML}^0(\Omega) \left| \begin{array}{l}
1) \quad u(\cdot , 0) \in C^m(\mathbb{R}^3) \\
2) \quad \exists \Gamma \subset \Omega \text{ closed nowhere dense : } u \in C^m(\Omega \setminus \Gamma)
\end{array} \right. \right\} \tag{50} \]

With the initial value problem \([45]\) we associate the mapping

\[ R_0 : X \ni u \mapsto u_{|t=0} \in Z \tag{51} \]

where

\[ Z = C^2(\mathbb{R}^3, \mathbb{R}^3) \]

That is, \( R_0 \) assigns to \( u \in X \) the restriction of \( u \) to the hyperplane \( \mathbb{R}^3 \times \{0\} \). Note that this amounts to a \textit{separation} of the problem of solving the system of PDEs \([44]\), and the problem of satisfying the initial value. This is a characteristic feature of the Order Completion Method \([25]\), and the pseudo-topological version of the theory developed here and in \([33]\). What is more, and as will be seen in the sequel, this allows for the rather straightforward and easy treatment of boundary and / or boundary value problems, when compared to the usual functional analytic methods.

Define the mapping \( T_0 \) as

\[ T_0 : X \ni v = (u, p) \mapsto (Tv, R_0u) \in Y \times Z \tag{52} \]

The mapping \( T_0 \) induces an equivalence relation \( \sim_{T_0} \) on \( X \) through

\[ \forall \quad v, w \in X : \quad v \sim_{T_0} w \iff T_0v = T_0w \tag{53} \]

The quotient space \( X/\sim_{T_0} \) is denotes \( X_{T_0} \). There is then an \textit{injective} mapping

\[ \hat{T}_0 : X_{T_0} \ni V \mapsto (T_0v, R_0u) \in Y \times Z \tag{54} \]

where \( v = (u, p) \) is any member of the equivalence class \( V \), such that the diagram

\[ \begin{array}{ccc}
X & \xrightarrow{T_0} & Y \times Z \\
\downarrow q_{T_0} & & \downarrow i \\
X_{T_0} & \xrightarrow{\hat{T}_0} & Y \times Z \\
\end{array} \]

commutes, with \( q_{T_0} \) the quotient mapping.

We equip the space \( \mathcal{ML}^0(\Omega) \) with the uniform order convergence structure \( J_o \), and \( Y \) carries the product uniform convergence structure. The space \( Z \) carries the uniform convergence structure \( J_\lambda \), see \([11]\), associated with the convergence structure

\[ \forall \quad u \in Z : \lambda(u) = |u| \tag{55} \]
That is,
\[ \forall \ U \text{ a filter on } Z \times Z : \]
\[ U \in J_\lambda \iff \exists u_1, \ldots, u_k \in Z : (\lfloor u_1 \rfloor \times \lfloor u_1 \rfloor) \cap \ldots \cap (\lfloor u_k \rfloor \times \lfloor u_k \rfloor) \subseteq U \]  
(56)

Note that \( J_\lambda \) induces the convergence structure \( \lambda \), and is uniformly Hausdorff and complete [11]. In particular, the sequences which converge with respect to \( J_\lambda \) are exactly the constant sequences.

The product space \( Y \times Z \) carries the product uniform convergence structure, which we denote by \( J_P \). In view of Theorem 14 and [34, Theorem 3.1] the completion \( (Y \times Z)^\sharp \) of \( Y \times Z \) is \( NL(\Omega)^4 \times Z \), equipped with the product uniform convergence structure with respect to the uniform convergence structure \( J_\lambda^2 \) and the uniform convergence structure \( J_\lambda \). We equip \( X_{T_0} \) with the initial uniform convergence structure \( J_{T_0} \) with respect to the mapping \( \hat{T}_0 \). That is,
\[ \forall \ U \text{ a filter on } X_{T_0} \times X_{T_0} : \]
\[ U \in J_{T_0} \iff (\hat{T}_0 \times \hat{T}_0)(U) \in J_P \]  
(57)

Since \( \hat{T}_0 \) is injective, it is a uniformly continuous embedding so that \( X_{T_0} \) is uniformly isomorphic to a subspace of \( Y \times Z \). Therefore, see [34], the mapping \( \hat{T}_0 \) extends to a uniformly continuous embedding
\[ \hat{T}_0^\sharp : X_{T_0}^\sharp \to (Y \times Z)^\sharp \]  
(58)

so that \( X_{T_0}^\sharp \) is uniformly isomorphic to a subspace of \( (Y \times Z)^\sharp \). This is summarized in the following commutative diagram.

A generalized solution to (44) through (45) is any \( V^\sharp \in X_{T_0}^\sharp \) that satisfies the equation
\[ \hat{T}_0^\sharp V^\sharp = g \]  
(59)

The main result of this section, concerning the existence of generalized solutions to (44) through (45), is based on the existence of approximate solutions, which follows form the following [33]. We include the proof to illustrate the technique.

**Lemma 15** Consider any \( g = (f, 0) \in C^0(\Omega) \) and any \( \epsilon > 0 \). Then
\[ \forall \ (x_0, t_0) \in \Omega : \]
\[ \exists \ v = (u, p) \in C^2(\Omega) : \]
\[ \exists \ \delta > 0 : \]
\[ \forall \ (x, t) \in \Omega : \]
\[ (\|x_0 - x\| < \delta) \implies g(x, t) - \epsilon < T(x, t, D) v(x, t) < g(x, t) \]  
(60)

where the order above is coordinatewise, and \( \epsilon \) represents the 4 dimensional vector that corresponds to the real number \( \epsilon \).
We also assume that the interiors of $C$ exist, see [16].

Let $v = (u, p)$ be the $C^2$-smooth function such that

$$D^\alpha u(x,t) = \xi^\alpha$$

The result now follows from the continuity of $v$, $F$ and $g$. ■

The following is essentially a version of Lemma 15 above which incorporates the initial condition $\nu$.

**Lemma 16** Let $g$ and $\epsilon$ be as in Lemma [15] above. Consider any $u^0 \in C^2(\mathbb{R}^3, \mathbb{R}^3)$. Then

$$\forall \ x_0 \in \mathbb{R}^3 : \exists \ v = (u, p) \in C^2(\Omega) : \exists \ \delta > 0 :$$

$$\forall \ (x, t) \in \Omega :$$

$$1) \ ||x_0 - x|| < \delta, |t| < \delta \Rightarrow g(x, t) - \epsilon < T(x, t, D)v(x, t) < g(x, t)$$

$$2) \ x \in \mathbb{R}^3 \Rightarrow u(x, 0) = u^0(x)$$

**Proof.** The proof follows similar arguments as those employed in the proof of Lemma 15 when one sets

$$u(x, t) = u^0(x) + \varphi(t)$$

where $\varphi \in C^2([0, \infty))$ is an appropriate function such that $\varphi(0) = 0$. ■

The main result of this section is now the following.

**Theorem 17** For any $g = (f, 0) \in Y$ and any $u^0 \in Z$, there exists a unique $V^\nu \in X^2_{u^0}$ such that

$$T_0^\nu V^\nu = g$$

**Proof.** Let

$$\Omega = \bigcup_{\nu \in \mathbb{N}} C_\nu$$

where, for $\nu \in \mathbb{N}$, the compact sets $C_\nu$ are 4-dimensional intervals

$$C_\nu = [a_\nu, b_\nu]$$

with $a_\nu = (a_{\nu, 1}, ..., a_{\nu, n})$, $b_\nu = (b_{\nu, 1}, ..., b_{\nu, n}) \in \mathbb{R}^n$ and $a_{\nu, i} \leq b_{\nu, i}$ for every $i = 1, ..., n$. We also assume that $C_\nu$, with $\nu \in \mathbb{N}$ are locally finite, that is,

$$\forall \ (x, t) \in \Omega : \exists \ V_x \subseteq \Omega \ a \ neighborhood \ of \ x : \{\nu \in \mathbb{N} : C_\nu \cap V_x \neq \emptyset\} \ is \ finite$$

We also assume that the interiors of $C_\nu$, with $\nu \in \mathbb{N}$, are pairwise disjoint. We note that such $C_\nu$ exist, see [16].

Select $\nu \in \mathbb{N}$ and $\epsilon > 0$ arbitrary but fixed. For any $(x, t) \in C_\nu$, let $\delta_{x, t} > 0$ be the positive number and $v_{x, t}^{(x, t)}$ the function associated with $(x, t)$ through Lemma [15], if $t > 0$, and Lemma [10] if $t = 0$. Since $C_\nu$ is compact, it follows that

$$\exists \ \delta > 0 : \forall \ (x_0, t_0) \in C_\nu : \exists \ v = (u, p) \in C^2(\mathbb{R}^4, \mathbb{R}^4) :$$

$$1) \ \left( \begin{array}{l} ||x - x_0|| \leq \delta \\ |t - t_0| \leq \delta \end{array} \right) \Rightarrow g(x, t) - \epsilon \leq T(x, t, D)v(x, t) \leq g(x, t), \ (x, t) \in \Omega$$

$$2) \ t_0 = 0 \Rightarrow u(x) = u^0(x), x \in \mathbb{R}^3$$
Subdivide $C_\nu$ into $n$-dimensional intervals $I_{\nu,1}, \ldots, I_{\nu,\mu_\nu}$ such that their interiors are pairwise disjoint and

\[
\forall \ (x_0, t_0), \ (x, t) \in I_{\nu,i} : \\
1) \ |x_0 - x| < \delta \\
2) \ |t_0 - t| < \delta
\]

If $I_{\nu,i} \cap (\mathbb{R}^3 \times \{0\}) = \emptyset$, take $a_i$ to be the center of the interval $I_{\nu,i}$. Then by (66) there exists $v^{\nu,i} = (u, p) \in C^2 (\mathbb{R}^4 \times \mathbb{R}^4)$ such that

\[
g (x, t) - \epsilon \leq T (x, t, D) v^{\nu,i} (x, t) \leq g (x, t), \ (x, t) \in I_{\nu,i}
\]

If, on the other hand, $I_{\nu,i} \cap (\mathbb{R}^3 \times \{0\}) \neq \emptyset$, let $a_i$ denote the projection of the midpoint of $I_{\nu,i}$ on the hyperplane $\mathbb{R}^3 \times \{0\}$. Then by (66) there exists $v^{\nu,i} = (u, p) \in C^2 (\mathbb{R}^4 \times \mathbb{R}^4)$ such that (67) holds and

\[
u (x, 0) = u^0 (x), \ (x, 0) \in (\mathbb{R}^3 \times \{0\}) \cap I_{\nu,i}
\]

Now set

\[
u = (u_1^*, u_2^*, u_3^*, p^*) = \sum_{\nu \in \mathbb{N}} \left( \sum_{\chi \in I_{\nu,i}} \chi I_{\nu,i} \right)
\]

where $\chi I_{\nu,i}$ is the characteristic function of $I_{\nu,i}$. Clearly, $\nu = (u^*, p^*)$ is $C^2$-smooth everywhere except on a closed nowhere dense set, which has measure 0, and $\nu (x, 0) = u^0 (x)$ everywhere except on a closed nowhere dense subset of $\mathbb{R}^3 \times \{0\}$.

Now set $w^* = (u_1^*, u_2^*, u_3^*, p^*)$ where, for $j = 1, \ldots, 3$

\[
u_j^* (I \circ S) (u_j^*)
\]

and

\[
u = (I \circ S) (p^*)
\]

Clearly the function $w^*$ belongs to $X$. What is more, in view of (67) through (68), it follows that

\[
g - \epsilon \leq T w^* \leq g
\]

and

\[
R_0 w^* = u^0
\]

so that the sequence $\left( T_0 w_n \right) = \left( T_0 w^n \right)$ converges to $(g, u^0)$ in $Y \times Z$. For each $n \in \mathbb{N}$, let $W_n$ denote the $\sim_{T_0}$-equivalence class generated by the function $w^n$. The sequence $(W_n)$ is Cauchy in $X_{T_0}$, and since $T_0$ is uniformly continuous, there exists $V^2 \in X^2_{T_0}$ that satisfies (62). Moreover, $V^2$ is unique, since the mapping $T_0^2$ is a uniformly continuous embedding.

The uniqueness of the generalized solution should not be misinterpreted. Note that the completion of $X_{T_0}$ consists of equivalence classes of Cauchy filters on $X_{T_0}$, under the equivalence relation

\[
\mathcal{F} \sim \mathcal{G} \Leftrightarrow \exists \ H \ a \ Cauchy \ filter : \ H \subseteq \mathcal{F} \cap \mathcal{G}
\]

In view of this, the solution $V^2$ is actually the equivalence class of filters $\mathcal{F}$ on $X_{T_0}$ such that $T_0 (\mathcal{F})$ converges to $(g, u^0)$ in $Y \times Z$. What is more, $V^2$ contains also all classical, or smooth, solutions to (44) through (45), as well as all non-classical solutions $v = (u, p) \in C^2_{nd} (\Omega) \times \mathbb{R}^4$, since each such a solution generates a Cauchy sequence in $X_{T_0}$. Hence our notion of a generalized solution is consistent with the usual classical and nonclassical solutions in $C^2_{nd} (\Omega) \times \mathbb{R}^4$ from (44) through (45). Note that the method presented here for the three dimensional Navier-Stokes equations applies equally well to any dimension $n \geq 2$. 

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6 Conclusion

We have constructed an order isomorphic representation $\mathcal{ML}(X)$ of the quotient space $\mathcal{M}(X)$ consisting of normal lower semi-continuous functions on $X$. A nontrivial uniform convergence structure on $\mathcal{ML}(X)$, which induces the order convergence structure was constructed solely in terms of the order on $\mathcal{ML}(X)$. The completion of the uniform convergence space $\mathcal{ML}(\Omega)$ is obtained as the set $\mathcal{NL}(X)$ of nearly finite normal lower semi-continuous functions on $X$. This result essentially relies on the fact that $\mathcal{NL}(X)$ is the Dedekind completion of $\mathcal{ML}(X)$. Hence we have established a topological type model for the Dedekind completion of the space $\mathcal{ML}(X)$. This includes the case when $X = \Omega$ is a subset of $\mathbb{R}^n$, which is relevant to PDEs. This makes it possible to enrich the Order Completion Method for arbitrary nonlinear PDEs of the form (1), by reformulating it within the framework of uniform convergence spaces. In this regard, we obtained the existence of generalized solutions to the Navier-Stokes equations in three spatial dimensions, subject to an initial condition.

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