A Note on Singular Cardinals in Set Theory without Choice

Denis I. Saveliev

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In this talk, I discuss how singular can cardinals be in absence of AC, the axiom of choice. I shall show that, contrasting with known negative consistency results (of Gitik and others), certain positive results are provable. At the end, I pose some problems.
Preliminaries
**Definition.** Given a set $X$, its *cardinal* number $|X|$ is the class of all sets of *the same size* that $X$, i.e., admitting a one-to-one map onto $X$.

Thus

$$|X| = |Y|$$

means “There is a bijection of $X$ onto $Y$”. 
Cardinals of nonempty sets are proper classes; so, we have a little technical obstacle:

How quantify cardinals?

In some happy cases we can represent them by sets:

If \(|X|\) is a well-ordered cardinal, i.e., meets the class of (von Neumann’s) ordinals, take the least such ordinal (an initial ordinal).

If \(|X|\) is a well-founded cardinal, i.e., meets the class of well-founded sets, take the lower level of the intersection (so-called Scott’s trick).
What is in general? The answer is

No matter

because instead of cardinals, we can say about
sets and bijections.

Thus

\[ \varphi(|X|, |Y|, \ldots) \]

means \( \varphi(X', Y', \ldots) \) whenever \( |X| = |X'|, |Y| = |Y'|, \ldots \)
Notations:

The German letters

\[ l, m, n, \ldots \]

denote arbitrary cardinals. The Greek letters

\[ \lambda, \mu, \nu, \ldots \]

denote well-ordered ones (i.e., initial ordinals), while the Greek letters

\[ \alpha, \beta, \gamma, \ldots \]

denote arbitrary ordinals.
Two basic relations on cardinals (dual in a sense):

\[ |X| \leq |Y| \]

means “\(X\) is empty or there is an injection of \(X\) into \(Y\)”, and

\[ |X| \leq^* |Y| \]

means “\(X\) is empty or there is a surjection of \(Y\) onto \(X\)”. 

Equivalently,

\[ |X| \leq |Y| \] means “There is a subset of \(Y\) of size \(|X|\)”,

\[ |X| \leq^* |Y| \] means “\(X\) is empty or there is a partition of \(Y\) into \(|X|\) pieces”. 
Clearly:

(i) Both $\leq$ and $\leq^*$ are reflexive and transitive.

(ii) $\leq$ is antisymmetric (Dedekind; Bernstein), $\leq^*$ is not necessarily.

(iii) $\leq$ is stronger than $\leq^*$. Both relations coincide on well-ordered cardinals.
Two important functions on cardinals (Hartogs and Lindenbaum resp.):

\[ \aleph(n) = \{ \alpha : |\alpha| \leq n \}, \]

\[ \aleph^*(n) = \{ \alpha : |\alpha| \leq^* n \}. \]

Equivalently,

\[ \aleph(n) \] is the least \( \alpha \) such that on a set of size \( n \) there is no well-ordering of length \( \alpha \),

\[ \aleph^*(n) \] is the least \( \alpha \) such that on a set of size \( n \) there is no pre-well-ordering of length \( \alpha \).

Customarily, \( \nu^+ \) denotes \( \aleph(\nu) \) for \( \nu \) well-ordered.
Clearly:

(i) $\aleph(n)$ and $\aleph^*(n)$ are well-ordered cardinals.

(ii) $\aleph(n) \not< n$ and $\aleph^*(n) \not< n$.

It follows $\nu < \nu^+$ and so

$$\aleph_0 < \aleph_1 < \ldots < \aleph_\omega < \ldots < \aleph_{\omega_1} < \ldots$$

(where $\aleph_\alpha$ is $\alpha$th iteration of $\aleph$ starting from $\aleph_0$).

(iii) $\aleph(n) \leq \aleph^*(n)$, and both operations coincide on well-ordered cardinals. On other cardinals, the gap can be very large:

**Example.** Assume AD. Then $\aleph(2^{\aleph_0}) = \aleph_1$ while $\aleph^*(2^{\aleph_0})$ is a very large cardinal (customarily denoted $\Theta$).
Results on Singularity
Notations:

\[ \text{Cov}(l, m, n) \]
means “A set of size \( n \) can be covered by \( m \) sets of size \( l \)”.

\[ \text{Cov}(< l, m, n) \] and \[ \text{Cov}(\mathcal{L}, m, n) \] (where \( \mathcal{L} \) is a class of cardinals) have the appropriate meanings.

**Definition.** A cardinal \( n \) is *singular* iff \( \text{Cov}(< n, < n, n) \), and *regular* otherwise.
What is under AC?

**Fact.** Assume AC. Then \( \text{Cov}(l, m, n) \) implies \( n \leq l \cdot m \).

**Corollary.** Assume AC. Then all the successor alephs are regular.

Thus \( \neg \text{Cov}(\lambda, \lambda, \lambda^+) \) for all \( \lambda \geq \aleph_0 \).
What happens without AC?

**Theorem** (Feferman Lévy). \( \aleph_1 \) can be singular.

Thus \( \text{Cov}(\aleph_0, \aleph_0, \aleph_1) \) is consistent.

Moreover, under a large cardinal hypothesis, so can be all uncountable alephs:

**Theorem** (Gitik). *All uncountable alephs can be singular.*

Clearly, then \( \text{Cov}(<\lambda, \aleph_0, \lambda) \) for all \( \lambda \geq \aleph_0 \).
Remark. What is the consistency strength?

Without successive singular alephs:
The same as of ZFC.

With $\lambda, \lambda^+$ both singular:
Between 1 Woodin cardinal (Schindler improving Mitchell) and $\omega$ Woodin cardinals (Martin Steel Woodin).

So, in general case:
A proper class of Woodins.
Specker’s problem:

Is \( \text{Cov}(\mathfrak{N}_\alpha, \aleph_0, 2^{\aleph_\alpha}) \) consistent for all \( \alpha \) simultaneously?

Partial answer:

**Theorem** (Apter Gitik). Let \( A \subseteq \text{Ord} \) consist either

(i) of all successor ordinals; or

(ii) of all limit ordinals and all successor ordinals of form \( \alpha = 3n, 3n+1, \gamma+3n, \) or \( \gamma+3n+2, \) where \( \gamma \) is a limit ordinal.

Then

\[(\forall \alpha \in A) \text{Cov}(\mathfrak{N}_\alpha, \aleph_0, 2^{\aleph_\alpha}) \]

is consistent (modulo large cardinals).

(Really, their technique gives slightly more.)

In general, the problem remains open.
Question: *How singular* can cardinals be without AC? in the following sense: How small are \( l \leq n \) and \( m \leq n \) satisfying

(i) \( \text{Cov}(< l, < n, n) \)?

(ii) \( \text{Cov}(< n, < m, n) \)?

(iii) \( \text{Cov}(< l, < m, n) \)?

On (iii):
Specker’s problem is a partial case.

On (ii):
The answer is

As small as possible

since Gitik’s model satisfies \( \text{Cov}(< n, \aleph_0, n) \) for all (not only well-ordered) \( n \).
On (i):
For well-ordered $n$, the answer is

$l < n$ is impossible.

**Theorem 1.** $\text{Cov}(\lambda, m, \nu)$ implies $\nu \leq^* \lambda \cdot m$, and so

$$\nu^+ \leq \aleph^*(\lambda \cdot m).$$

**Corollary.** $\neg \text{Cov}(\lambda, \lambda, \lambda^+)$ for all $\lambda \geq \aleph_0$.

Since $\text{Cov}(\lambda, \lambda, \lambda^+)$ is consistent, the result is exact.

**Remark.** $\neg \text{Cov}(\aleph_0, \aleph_0, \aleph_2)$ is an old result of Jech. (I am indebted to Prof. Blass who informed me.) By Corollary, really $\neg \text{Cov}(\aleph_0, \aleph_1, \aleph_2)$. 
Next question: Let Cov(l, m, n), is n estimated via l and m? (when n is not well-ordered). Without Foundation, the answer is

No

Even in the simplest case l = 2 and m = ℵ₀ such an estimation of n is not provable:

**Theorem 2.** *It is consistent that for any p there exists n ∉ p such that Cov(2, ℵ₀, n).*

The proof uses a generalization of permutation model technique to the case of a proper class of atoms. We use non-well-founded sets instead of atoms.
On the other hand, $\aleph(n)$ and $\aleph^*(n)$ are estimated via $\aleph(l)$, $\aleph^*(l)$, and $\aleph^*(m)$:

**Theorem 3.**
$\text{Cov}(L, m, n)$ implies

$$\aleph(n) \leq \aleph^*(\sup_{l \in L} \aleph(l) \cdot m)$$

and

$$\aleph^*(n) \leq \aleph^*(\sup_{l \in L} \aleph^*(l) \cdot m).$$

**Corollary 1.**
$\neg\text{Cov}(\lambda, \lambda, 2^\lambda)$ and $\neg\text{Cov}(n, 2^n, 2^{2n^2})$.

In particular:

$\neg\text{Cov}(\lambda, 2^\lambda, 2^{2^\lambda})$ and $\neg\text{Cov}(\beth_\alpha, \beth_{\alpha+1}, \beth_{\alpha+2})$.

Since $\text{Cov}(n, n, 2^n)$ is consistent, the result is near optimal.
Another corollary is that Specker’s request, even in a weaker form, gives the least possible evaluation of $\aleph^*(2^\lambda)$ (which is $\lambda^{++}$):

**Corollary 2.** $\text{Cov}(\lambda, \lambda^+, 2^\lambda)$ implies

$$\aleph^*(2^\lambda) = \aleph(2^\lambda) = \lambda^{++}.$$

So, if there exists a model which gives the positive answer to Specker’s problem, then in it, all the cardinals $\aleph^*(2^\lambda)$ have the least possible values.
As the last corollary, we provide a “pathology” when a set admits *neither* well-ordered covering (of arbitrary size) by sets of smaller size, *nor* covering of smaller size by well-orderable sets (of arbitrary size). Moreover, it can be the *real line*:

**Corollary 3.** Assume CH holds and Θ is limit. (E.g., assume AD.) Then for any well-ordered λ

\[ \neg \text{Cov}(< 2^{\aleph_0}, \lambda, 2^{\aleph_0}) \quad \text{and} \quad \neg \text{Cov}(\lambda, < 2^{\aleph_0}, 2^{\aleph_0}). \]

(Here CH means “There is no m such that \( \aleph_0 < m < 2^{\aleph_0} \).”)
Problems
Problem 1. Is $\neg \text{Cov}(n, 2^n, 2^{2^n})$ true for all $n$?

That holds if $n = n^2$ (by Corollary 1 of Theorem 3).

Problem 2. Is $\neg \text{Cov}(\langle \mathcal{I}_\alpha, \mathcal{I}_\alpha, \mathcal{I}_{\alpha+1} \rangle)$ true for all $\alpha$?

That near holds if $\alpha$ is successor (again by Corollary 1 of Theorem 3).

Problem 3. Is $\text{Cov}(n, \aleph_0, 2^{n^2})$ consistent for all $n$ simultaneously?

This sharps Specker’s problem of course.

Problem 4. Can Theorem 2 be proved assuming Foundation? More generally, expand the Transfer Theorem (Jech Sohor) to the case of a proper class of atoms.
Problem 5. Is it true that on successor alephs the cofinality can behave anyhow, in the following sense: Let $F$ be any function such that

$$F : \text{SuccOrd} \rightarrow \text{SuccOrd} \cup \{0\}$$

and $F$ satisfies

(i) $F(\alpha) \leq \alpha$ and 
(ii) $F(F(\alpha)) = F(\alpha)$

for all successor $\alpha$. Is it consistent

$$\text{cf } \aleph_\alpha = \aleph_{F(\alpha)}$$

for all successor $\alpha$?

Perhaps if $F$ makes no successive cardinals singular, it is rather easy; otherwise very hard.
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