Coarse Grained Parallel Selection

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Abstract

We analyze the running time of the Saukas-Song algorithm for selection on a coarse grained multicomputer without expressing the running time in terms of communication rounds. This shows that while in the best case the Saukas-Song algorithm runs in asymptotically optimal time, in general it does not. We propose other algorithms for coarse grained selection that have optimal expected running time.

Key words and phrases: selection problem, coarse grained multicomputer, uniform distribution, Chebyshev’s inequality

1 Introduction

The paper [8], by Saukas and Song, presents an algorithm to solve the Selection Problem on coarse grained parallel computers. Saukas and Song present the analysis of the algorithm in terms of the amount of time spent in local sequential operations and the number of communications rounds. In the current paper, we replace analysis of the number of communications rounds with an analysis of their running times, giving us asymptotic analysis of the running time for the entire algorithm. This lets us show that although the Saukas-Song algorithm is efficient, it is not optimal. We propose other algorithms for coarse grained parallel selection that have asymptotically optimal running time or asymptotically optimal expected running time.

2 Preliminaries

2.1 Model of Computation

Material in this section is quoted or paraphrased from [3].

The coarse grained multicomputer model, or $CGM(n, p)$, considered in this paper, has $p$ processors with $\Omega(n/p)$ local memory apiece - i.e., each processor has $\Omega(n/p)$ memory cells of $\Theta(\log n)$ bits apiece. The processors may be connected to some (arbitrary) interconnection network (such as a mesh, hypercube,

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or fat tree) or may share global memory. A processor may exchange messages of \(O(\log n)\) bits with any immediate neighbor in constant time. In determining time complexities, we consider both local computation time and interprocessor communication time, in standard fashion. The term “coarse grained” means that the size \(\Omega(n/p)\) of each processor’s memory is “considerably larger” than \(\Theta(1)\); by convention, we usually assume \(n/p \geq p\) (equivalently, \(n \geq p^2\)), but will occasionally assume other relations between \(n\) and \(p\), typically such that each processor has at least enough local memory to store the ID number of every other processor. For more information on this model and associated operations, see [4].

### 2.2 Terminology and notation

We say an array list \([1 \ldots n]\) is evenly distributed in a CGM \((n, p)\) if each processor has \(\Theta(n/p)\) members of the array.

### 2.3 Semigroup operations

Let \(X = \{x_j\}_{j=1}^n\) be a set of data values and let \(\circ\) be a binary operation on \(X\). A semigroup operation computes \(x_1 \circ x_2 \circ \ldots \circ x_n\). Examples of such operations include sum, average, min, and max.

**Theorem 2.1.** [2] Let \(X = \{x_j\}_{j=1}^n\) be a set of data values distributed \(\Theta(n/p)\) per processor in a CGM \((n, p)\). Then the semigroup computation of \(x_1 \circ x_2 \circ \ldots \circ x_n\) can be performed in \(\Theta(n/p)\) time. At the end of this algorithm, all processors hold the value of \(x_1 \circ \ldots \circ x_n\).

### 2.4 Data movement operations

In the literature of CGM algorithms, many papers, including [8], analyze an algorithm by combining the running time of sequential computations with the number of communications rounds. A communications round is described in [8] as an operation in which each processor of a CGM \((n, p)\) can exchange \(O(n/p)\) data with other processors.

However, this definition does not lead to a clear understanding of the asymptotic running time of a CGM algorithm. E.g., a communication round could require a processor to send \(\Theta(1)\) data to a neighboring processor, which can be done in \(\Theta(1)\) time; or, a communication round could require communication of \(\Theta(1)\) data between diametrically opposite processors of a linear array, which requires \(\Theta(p)\) time.

Further, there is a sense in which the notion of a communication round is not well defined. Consider again the example of communication of \(\Theta(1)\) data between diametrically opposite processors \(P_i, P_p\) of a linear array in which the processors \(P_i\) are indexed sequentially, i.e., \(P_1\) is adjacent to \(P_2\) and \(P_i\) is adjacent to \(P_{i-1}\) and to \(P_{i+1}\) for \(1 < i < p\). This communication can be regarded as a single communication round according to the description given above; or as
3 Analysis of the Saukas-Song algorithm

The algorithm of [8] is given in Figure 1. Note it is assumed in [8] that \( n > p^2 \log p \). We will refer to the steps of the algorithm as labeled in this figure. For convenience, we will take \( c = 1 \) in step (2).

**Theorem 3.1.** [8] After each performance of the body of the loop in step (2), the number of elements remaining under consideration is reduced by at least one quarter. □

**Corollary 3.2.** The number of performances of the body of the loop in step (2) is \( O(\log p) \).

**Proof.** It follows from Theorem 3.1 that after \( k \) performances of the body of the loop, the number of elements remaining under consideration is at most \((3/4)^k n\). Since step (2) terminates in the worst case when the number of elements remaining under consideration is at most \( n/p \), termination requires, in the worst case, \((3/4)^k n \leq n/p\), or \( p \leq (4/3)^k \). The smallest integer \( k \) satisfying this inequality must therefore satisfy \( k = \Theta(\log p) \). Since the loop could terminate after as little as 1 performance of its body, the assertion follows. □

**Theorem 3.3.** Assume \( n > p^2 \log p \). The Saukas-Song algorithm for the Selection Problem runs in \( \Theta(\frac{n \log p}{p}) \) time on a CGM \((n, p)\) in the worst case. In the best case, the running time is \( \Theta(n/p) \). We may assume that at the end of the algorithm, every processor has the solution to the Selection Problem.

**Proof.** We give the following argument.

- Clearly, step (1) executes in \( \Theta(1) \) time.
We analyze step (2) as follows.

1. Step (2.1) is executed by a linear-time sequential algorithm \([1, 5]\). As Saukas and Song observed, our algorithm does not guarantee that the data being considered are evenly distributed among the processors throughout the repetitions of the loop body. In the worst case, some processor could have \(\Theta(n/p)\) data in each iteration of the loop body. Therefore, this step executes in worst case \(\Theta(n/p)\) time.

2. Step (2.2) is performed by a gather operation. By Theorem \([2, 3]\) this can be done in \(\Theta(p)\) time.

3. Step (2.3) is performed in \(\Theta(p)\) time by a linear-time sequential algorithm.

4. Step (2.4) is performed by a broadcast operation. By Theorem \([2, 2]\) this requires \(O(p)\) time.

5. Step (2.5) is performed by sequential semigroup (counting) operations performed by all processors in parallel, in linear time \([5]\). As noted above, in the worst case a processor could have \(\Theta(n/p)\) data in each iteration of the loop body. Therefore, this step executes in \(O(n/p)\) time.

6. Step (2.6) is performed by a gather operation. As above, this can be done in \(O(p)\) time.
7. Step (2.7) is executed by semigroup operations in $\Theta(p)$ time.
8. Step (2.8) is performed by a broadcast operation in $O(p)$ time.
9. In the worst case, Step (2.9) requires each processor $P_i$ to discard $O(n/p)$ data items. This can be done as follows. In parallel, each processor $P_i$ rearranges its share of the data so that the undiscarded items are at the beginning of the segment of (a copy of) list stored in $P_i$, using a sequential prefix operation in $O(n/p)$ time.

Thus, the worst case time required for one performance of the body of the loop of step (2) is $\Theta(n/p + p) = \Theta(n/p)$. By Corollary 3.2, the loop executes its body $\Theta(\log p)$ times in the worst case. Thus, the loop executes all performances of its body in worst case $\Theta(n \log p)$ time.

- Step (3) is performed by a gather operation. Since we now have $N \leq n/p$, this is done in $O(n/p)$ time.
- Step (4) uses a linear time sequential algorithm to solve the problem in $\Theta(N) = O(n/p)$ time.
- Additionally, processor 1 can broadcast its solution to all other processors. By Theorem 2.2, this requires $O(p)$ time.

Thus, the algorithm uses worst case $\Theta(n \log p)$ time.

In the best case, the loop of step (2) executes its body once, when in step (2.9) it is found that $L < k \leq L + E$. In this case, step (2) executes in $\Theta(n/p)$ time, and the algorithm executes in $\Theta(n/p)$ time.

4 Selection for a finite set of support values

Often, the data values under consideration are known to belong to a finite set, e.g., the integers from 0 to 100. Under such circumstances, we can give a coarse grained parallel algorithm that runs in asymptotically optimal time over a greater range of processors for a given $n$ than the Saukas-Song algorithm permits, i.e., here, $p^2 \leq n$. This algorithm may be primarily of theoretical interest, since in practice it will often occur that the $\log p$ factor in the analysis of the Saukas-Song algorithm is smaller than the contribution of the size of the data range to the constant of proportionality in the following.

**Theorem 4.1.** Let $X = \{x_i\}_{i=1}^n$ be a set of data values from a known set $Y = \{y_m\}_{m=1}^c$ of finite cardinality, where $y_1 < y_2 < \ldots < y_{c-1} < y_c$. If $X$ is evenly distributed among the processors of a CGM($n, p$) and $k$ is a positive integer, $1 \leq k \leq n$, then the $k^{th}$ smallest member of $X$ can be found in optimal $\Theta(n/p)$ time.

**Proof.** Consider the following algorithm.
• In parallel, each of the processors $P_j$ uses a Bucket Sort to sort its members of $X$, keeping track of the number $n_{j,m}$ of members $x_i$ of $X$ that $P_j$ has such that $x_i = y_m$, in $\Theta(n/p)$ time [5].

• For $m = 1$ to $c$, do the following.
  – Gather the values $\{n_{j,m}\}_{j=1}^p$ to one processor, say, $P_1$. By Theorem 2.3 this requires $\Theta(p)$ time.
  – Processor $P_1$ computes $n_m = \sum_{i=1}^p n_{j,m}$, the number of members of $X$ equal to $y_m$. This takes $\Theta(p)$ time.

End For
Since $c$ is constant, this takes $\Theta(p)$ time.

• For $m = 1$ to $c$, let $N_m = \sum_{u=1}^m n_u$, the number of members of $X$ that are less than or equal to $y_m$. Processor $P_1$ performs a parallel prefix operation to find the values $N_1, N_2, \ldots, N_c$. This takes $\Theta(c) = \Theta(1)$ time.

• $P_1$ does a binary search to find the smallest index $v$, $1 \leq v < c$, such that $k \leq N_v$. This takes $O(\log c) = \Theta(1)$ time.

• $P_1$ broadcasts $v$ to all processors as the index of a member of $Y$ that is equal to the $k^{th}$ smallest member of $X$. This takes $O(p)$ time.

• A processor with $y_v$ broadcasts $y_v$ to all processors. This takes $O(p)$ time.

Since $p \leq n/p$, the algorithm runs in $\Theta(n/p)$ time. This is asymptotically optimal, since the optimal sequential time for selection is $\Theta(n)$. $\square$

5 An expected-optimal selection algorithm

In this section, we propose a different algorithm for the selection problem on coarse grained multicomputers. Our algorithm is based on the familiar idea that if we know the distribution of data values, then, with high probability, we can estimate the desired value to within a small interval and, in so doing, eliminate (with high probability) most of the data values from consideration as possible solutions. In the following, we assume the data values are uniformly distributed over some interval.

5.1 Useful formulas from probability and statistics

We use the notation $Pr[H]$ for the probability of the event $H$, and $Pr[D|C]$ for the conditional probability of the event $D$ given that $C$ occurs. We use the notation $\overline{H}$ for “not $H$”.

Let $X_{(k)}$ be a random variable for the $k^{th}$ smallest member of a set $\{x_i\}_{i=1}^n$ of data points uniformly distributed over the interval $[0, 1]$. Then the expected value $E(X_{(k)})$ and variance $V(X_{(k)})$ of $X_{(k)}$ are given [7] by the following.

$$E(X_{(k)}) = k/(n + 1) \quad (1)$$
\[ V(X(k)) = \frac{k(n+1-k)}{(n+1)^2(n+2)} \]  

We recall also Chebyshev’s Inequality:

**Theorem 5.1.** [7] Let \( y \) be a random variable with expected value \( E(y) = \mu \) and variance \( V(y) = v \). Then, for any \( t > 0 \),

\[ Pr[|y - \mu| > t] \leq \frac{v}{t^2}. \]

5.2 Algorithm

The solution to the Selection Problem proposed below makes use of the Saukas-Song algorithm, and therefore is subject to the same restriction on the number of processors, \( p^2 \log p < n \). Its expected running time is asymptotically optimal, \( \Theta(n/p) \), and its worst case running time is that of the Saukas-Song algorithm, \( \Theta(\frac{n \log p}{p}) \).

**Theorem 5.2.** Given a set \( A \) of \( n \) elements distributed \( \Theta(n/p) \) per processor among the processors of a CGM \( (n, p) \) such that \( p^2 \log p < n \), and an integer \( k \) such that \( 1 \leq k \leq n \). Assume the key values of the elements of \( A \) are uniformly distributed over an interval \([u, v] \). Then the \( k^{th} \) smallest member of \( A \) can be found in expected \( \Theta(n/p) \) time and in worst case \( \Theta(\frac{n \log p}{p}) \) time.

**Proof.** Without loss of generality, \([u, v] = [0, 1] \).

If the values of \( n \) and \( p \) are not known, they can be computed and made known to all processors via semigroup (counting) operations. By Theorem 2.1, this can be done in \( \Theta(n/p) \) time.

Let \( c \) be a small positive integer.

If \( k \leq c \) then we can find the desired result by \( k \) performances of a minimum computation. I.e., we do the following.

For \( i = 1 \) to \( k \)

Find an element \( a_j \in A \) such that \( a_j = \min\{a \in A\} \).

If \( i = k \) then the desired result is \( a_j \); else set \( A = A \setminus \{a_j\} \).

End For

By Theorem 2.1 the For loop executes in \( \Theta(n/p) \) time, and by Theorem 2.2 the result can be distributed to all processors in \( O(p) \) time. Thus, the case \( k \leq c \) is solved in \( \Theta(n/p) \) time.

If \( k \geq n - c \), then we do the following.

For \( i = n \) to \( k \) step \(-1\)

Find an element \( a_j \in A \) such that \( a_j = \max\{a \in A\} \).

If \( i = k \) then the desired result is \( a_j \); else set \( A = A \setminus \{a_j\} \).

End For
By Theorem 2.1 the For loop executes in \( \Theta(n/p) \) time, and by Theorem 2.2 the result can be distributed to all processors in \( O(p) \) time. Thus, the case \( k \leq c \) and the case \( k \geq n - c \) are solved in \( \Theta(n/p) \) time.

The remaining case is \( c < k < n - c \). Let \( d \) be a constant such that \( 0.5 < d < 1 \). Note \( p < (p^2 \log p)^{1/2} < n^{1/2} < n^d \), hence \( n/p > n/n^d = n^{1-d} \).

Let \( \varepsilon = 1 - d > 0 \). Proceed as follows.

- In \( \Theta(1) \) time, each processor computes

\[
U = \begin{cases} 
0 & \text{if } k \leq n^{1-d/2}; \\
E(X_{(k-n^{1-d/2})}) = \frac{k-n^{1-d/2}}{n+1} & \text{if } k > n^{1-d/2}
\end{cases}
\]

and

\[
V = \begin{cases} 
E(X_{(k+n^{1-d/2})}) = \frac{k+n^{1-d/2}}{n+1} & \text{if } k < n - n^{1-d/2}; \\
1 & \text{if } k \geq n - n^{1-d/2}.
\end{cases}
\]

- In parallel, each processor \( P_j \) scans its portion of the data set \( X = \{x_i\}_{i=1}^n \) to determine

- \( S_j \), the number of elements of \( X \) in \( P_j \) that are less than \( U \); and
- \( M_j \), the number of elements of \( X \) in \( P_j \) that are in \([U, V]\).

This can be done in \( \Theta(n/p) \) time.

- Gather the \( S_j \) values to one processor and compute their sum, \( S = \sum_{j=1}^p S_j \). Similarly, gather the \( M_j \) values to one processor and compute their sum, \( M = \sum_{j=1}^p M_j \). Broadcast the values \( S, M \) to all processors. From Theorems 2.3 and 2.2 we conclude that all this can be done in \( \Theta(p) \) time.

- If \( S \leq k \leq S + M \), then we have \( X_{(k)} \in [U, V] \); and if further we have \( M \leq n/p \) then the \( k^{th} \) smallest member of \( X \) is the \((k-S)^{th}\) smallest member of the subset \( M' \) of \( X \) consisting of members of \( X \cap [U, V] \). Since \( |M'| = M \), we can gather the elements of \( M' \) to one processor in \( \Theta(M) = O(n/p) \) time according to Theorem 2.3 and have that processor sequentially find the \((k-S)^{th}\) smallest member of \( M' \) in \( \Theta(M) = O(n/p) \) time, and broadcast the result to all processors in \( O(p) \) time.

Thus, if this case is realized, the running time of the algorithm is \( \Theta(n/p) \).

Otherwise, either \( k < S \) or \( k > S + M \).

- If \( k < S \) then apply the Saukas-Song algorithm to \( U' \), the set members of \( X \) that are less than \( U \). In the worst case, \( |U'| = \Theta(n) \), so in this case the running time is \( \Theta(\frac{n \log p}{p}) \).
Otherwise, \( k > S + M \) and the desired value is the \( (k - (S + M))^{th} \) smallest member of the subset \( V' \) consisting of those members of \( X \) that are greater than \( V \). Use the Saukas-Song algorithm accordingly, i.e., to find the \( (k - (S + M))^{th} \) smallest member of \( V' \), in worst-case \( \Theta\left(\frac{n \log p}{p}\right) \) time.

Thus the worst-case running time of the algorithm is \( \Theta\left(\frac{n \log p}{p}\right) \). It remains for us to derive the expected running time of this algorithm.

Let \( C \) be the event
\[
C = [U \leq X_{(k)} \leq V].
\]
Notice the event complementary to \( C \) is
\[
\overline{C} = [X_{(k)} < U] \cup [V < X_{(k)}] .
\]
By (5) and (6), \( \overline{C} \subset \left[ |X_{(k)} - E(X_{(k)})| > \frac{2^{1-d/2}}{n+1} \right] \), so by (1), (2), and Theorem 5.1,
\[
P_r[\overline{C}] \leq \frac{k(n+1-k)}{(n+1)^2(n+2)} = \frac{n(n-1-k)}{n^2} \rightarrow_{n \to \infty} 0.
\]
Also,
\[
P_r[\overline{C}] = O\left(\frac{n^2}{n^{3-d}}\right) = O(n^{-1+d}) = O(n^{-\varepsilon}). \tag{7}
\]
Therefore,
\[
\lim_{n \to \infty} Pr[C] = 1. \tag{8}
\]
Let
\[
D' = [\text{The interval } [U, V] \text{ contains at most } n/p \text{ members of } X] ;
\]
\[
D = [\text{The interval } [U, V] \text{ contains at most } n^{1-d} \text{ members of } X].
\]
By (4), \( D \subset D' \).

• For the case \( k \leq n^{1-d/2} \), we have \( U = 0 \), so

\[
\text{given } C \text{ we have that } \overline{D} \text{ occurs if and only if } X_{(n^{1-d/2}+1)} \leq V. \tag{9}
\]
Now,
\[
V - E(X_{(n^{1-d/2}+1)}) = \frac{k + n^{1-d/2} - (n^{1-d} + 1)}{n + 1} > 0 \tag{10}
\]
so
\[
[X_{(n^{1-d}+1)} \leq V] = \]
\[
\left[ X_{(n_1, d+1)} - E(X_{(n_1, d+1)}) \right] \leq \frac{k + n^{1-d/2} - (n_1^{1-d} + 1)}{n+1}
\]
- hence

\[
Pr[X_{(n_1, d+1)} \leq V] = 1 - Pr[X_{(n_1, d+1)} > V] = 1 - Pr \left[X_{(n_1, d+1)} - E(X_{(n_1, d+1)}) > \frac{k + n^{1-d/2} - (n_1^{1-d} + 1)}{n+1} \right].
\] (11)

Now,

\[
Pr \left[X_{(n_1, d+1)} - E(X_{(n_1, d+1)}) > \frac{k + n^{1-d/2} - (n_1^{1-d} + 1)}{n+1} \right] \leq
\]

\[
Pr \left[|X_{(n_1, d+1)} - E(X_{(n_1, d+1)})| > \frac{k + n^{1-d/2} - (n_1^{1-d} + 1)}{n+1} \right] \leq
\]

(by 13)

\[
\frac{(n_1^{1-d} + 1)(n_1^{1-d} - 1)}{(k + n^{1-d/2} - (n_1^{1-d} + 1))^2 (n+2)} = \Theta \left( \frac{n_1^{2-d}}{n^{3-d}} \right) = \Theta(n^{-1}),
\] (12)

so by (11) and (12),

\[
\lim_{n \to \infty} Pr[X_{(n_1, d+1)} \leq V] = 1 - 0 = 1.
\] (13)

By 9, 11 and 13

\[
Pr[D \mid C] = Pr \left[ |X_{(n_1, d+1)} \leq V \mid C \right] = \frac{Pr \left[ X_{(n_1, d+1)} \leq V \cap C \right]}{Pr[C]} =
\]

(by 8 and 13) = \frac{1}{1} = 1
(14)

and, by statements 8 through 12

\[
Pr[D \mid C] = \Theta(n^{-1}).
\]

- The case \( k \geq n - n^{1-d/2} \) is symmetric with the previous case, and similarly yields that \( \lim_{n \to \infty} Pr[D \mid C] = 1 \) and \( Pr[D \mid C] = \Theta(n^{-1}) \).

- This leaves for our consideration the case \( n^{1-d/2} < k < n - n^{1-d/2} \). In the presence of \( C \), \( D \) occurs if and only if for some \( j \) such that \( j \leq k \leq j + n^{1-d} \), we have \( U \leq X_j \leq X_{(j+n^{1-d})} \leq V \). We therefore have

\[
Pr[D \mid C] = Pr \left[ \bigcup_{j=k-n^{1-d}}^{k} [U \leq X_j] \cap [X_{(j+n^{1-d})} \leq V] \mid C \right].
\] (15)
But
\[
\bigcup_{j=k-n^{-d}}^{k} \left( [U \leq X_{(j)}] \cap [X_{(j+n^{-d})} \leq V] \right) \subset \\
\bigcup_{j=k-n^{-d}}^{k} [U \leq X_{(j)}] \subset [U \leq X_{(k-n^{-d})}]
\]
so (15) implies
\[
Pr[D|C] \leq Pr \left[ [U \leq X_{(k-n^{-d})}] \ | \ C \right]
\]
Now,
\[
E(X_{(k-n^{-d})}) - U = \frac{k - n^{-d} - k - n^{-1-d}/2}{n+1} = \frac{n^{-1-d/2} - n^{-1-d}}{n+1} > 0
\]
so
\[
[U \leq X_{(k-n^{-d})}] \subset \left[ |X_{(k-n^{-d})} - E(X_{(k-n^{-d})})| \geq E(X_{(k-n^{-d})}) - U \right]
\]
\[
= \left[ |X_{(k-n^{-d})} - E(X_{(k-n^{-d})})| \geq \frac{n^{-1-d/2} - n^{-1-d}}{n+1} \right],
\]
and hence
\[
Pr \left[ U \leq X_{(k-n^{-d})} \right] \leq Pr \left[ |X_{(k-n^{-d})} - E(X_{(k-n^{-d})})| \geq \frac{n^{-1-d/2} - n^{-1-d}}{n+1} \right]
\]
\[
\leq \text{by (3)} \quad \frac{(k - n^{-1-d})(n+1 - (k - n^{-1-d}))}{(n+1)^2(n+2)} \leq \frac{n^{1-d/2} - n^{-1-d}}{n+1}
\]
\[
= O \left( \frac{n^2}{n^{2-d}} \right) = O(n^{-1+d}) = O(n^{-\varepsilon}). \quad (17)
\]

Since \( \lim_{n \to \infty} Pr[C] = 1 \), it follows from (16) and (17) that
\[
Pr[D|C] = O(n^{-\varepsilon}) \quad \text{and} \quad \lim_{n \to \infty} Pr[D|C] = 1. \quad (18)
\]

Thus, in all cases,
\[
Pr[D|C] \to_{n \to \infty} 1 \quad \text{and} \quad Pr[D|C] = O(n^{-\varepsilon}). \quad (19)
\]

From (3) and (19),
\[
Pr[C \cap D] = Pr[C] Pr[D|C] \to_{n \to \infty} 1.
\]

Let \( R(A) \) denote the running time of this algorithm for the event \( A \). Then the expected running time \( T(n, p) \) of the algorithm is
\[
T(n, p) = R(C \cap D) Pr[C \cap D] + R(C \cup D') Pr[C \cup D'].
\]
Above, we have shown that

\[ R(C \cap D') = \Theta(n/p), \Pr[C \cap D'] \geq \Pr[C \cap D] \rightarrow_{n \to \infty} 1, \quad R(C \cap D') = O \left( \frac{n \log p}{p} \right), \]

so

\[ T(n, p) = \Theta(n/p) \Theta(1) + O \left( \frac{n \log p}{p} \{ \Pr[C] + \Pr[\tilde{D}' \cap C] \} \right). \]

Using (7) and since \( \tilde{D}' \subset \tilde{D} \),

\[ T(n, p) = \Theta(n/p) + O \left( \frac{n \log p}{p} \{ O(n^{-\varepsilon}) + O(\Pr[\tilde{D} \cap C]) \} \right) = \]

\[ \Theta(n/p) + O \left( \frac{n \log p}{p} \{ O(n^{-\varepsilon}) + O(\Pr[\tilde{D} | C] \Pr[C]) \} \right) = (\text{by } (13)) \]

\[ \Theta(n/p) + O \left( \frac{n \log p}{p} \{ O(n^{-\varepsilon}) + O(n^{-\varepsilon}) \} \right) = \]

\[ \Theta(n/p) + O \left( \frac{n^{1-\varepsilon} \log p}{p} \right) = \Theta(n/p). \]

We remark that the expected running time is asymptotically optimal, since the optimal sequential running time for solving the Selection Problem is \( \Theta(n) \).

6 Further remarks

We have given an asymptotic analysis of the running time of the Saukas-Song selection algorithm for coarse grained parallel computers, showing that this algorithm is efficient but not asymptotically optimal. We have given other algorithms for the selection problem on coarse grained parallel computers with asymptotically optimal average running times.

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