**PRO-p LINK GROUPS AND p-HOMOLOGY GROUPS**

JONATHAN HILLMAN¹, DANIEL MATEI² AND MASANORI MORISHITA³

1. School of Mathematics and Statistics, The University of Sydney, Sydney, NSW 2006, Australia
2. Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba, Meguro, Tokyo 153-8914, Japan
3. Department of Mathematics, Kanazawa University, Kakuma-machi, Kanazawa, Ishikawa, 920-1192, Japan

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**Introduction**

Let \( L \) be a tame link in the 3-sphere \( S^3 \) consisting of \( n \) knots \( K_1, \ldots, K_n \) and let \( G_L \) be the link group \( \pi_1(X_L), X_L = S^3 \setminus L \). For a prime number \( p \), let \( \hat{G}_L \) denote the pro-p completion of the group \( G_L \), \( \hat{G}_L = \varprojlim G_L/N \) where \( N \) runs over normal subgroups of \( G_L \) having \( p \)-power indices. By a theorem of J. Milnor [Mi], it is shown that \( \hat{G}_L \) has the following simple presentation as a pro-p group

\[
\hat{G}_L = \langle x_1, \ldots, x_n \mid [x_1, y_1] = \cdots = [x_n, y_n] = 1 \rangle
\]

where \( x_i \) and \( y_i \) represent the meridian and longitude around \( K_i \) respectively (Theorem 1.2.1). The purpose of this paper is to use the pro-p link group \( \hat{G}_L \) and the associated group-theoretic invariants for the study of the p-homology groups of \( p^m \)-fold cyclic branched covers of \( S^3 \) along \( L \), following the analogies between link theory and number theory [Mo1~4],[Rez1,2]. The invariants we derive from \( \hat{G}_L \) are the \( p \)-adic Milnor invariants and the completed Alexander module over the formal power series ring \( \widehat{\Lambda}_n = \mathbb{Z}_p[[X_1, \cdots, X_n]] \) with coefficients in the ring \( \mathbb{Z}_p \) of \( p \)-adic integers. The tool involved here is the Fox differential calculus on a free pro-p group [Ih]. Although these invariants are simply \( p \)-adic analogues of the usual Milnor invariants and Alexander modules, it is natural to work over \( \widehat{\Lambda}_n \) since the completed Alexander module can be presented over \( \widehat{\Lambda}_n \) by a sort of
universal $p$-adic higher linking matrix $\hat{T}_L$, called the $p$-adic Traldi matrix. This is defined in terms of the $p$-adic Milnor numbers and we can derive from $\hat{T}_L$ systematically the “$p$-primary” information on the homology of $p^n$-fold branched covers of $L$. This is an idea analogous to Iwasawa theory [Iw] which may also be regarded as a $p$-adic strengthening of the method employed by W. Massey [Mas] and L. Traldi [T]. We note that the method using the truncated Traldi matrices was considered in [Mat] to study the homology of unbranched covers.

The homology of cyclic branched covers of a link $L$ is one of the basic invariants of $L$ and has been extensively investigated by many authors. The Betti number and the order have been determined in terms of the Alexander (Hosokawa) polynomial ([HK],[MM],[S1] etc) and further the (Galois) module structure has been studied ([Da],[HS],[S2] etc), however most results are concerned mainly with the part which is prime to the covering degree. In [Rez1,2], A. Reznikov studied the $p$-homology of $p$-fold branched covers after the model of the classical problem on $p$-ideal class groups in number theory (see also [Mo1]). In this paper, we push this line of study in arithmetic topology further and determine the Galois module structure of the $p$-homology of a $p$-fold branched cover along a link completely in terms of the $p$-adic higher linking matrices. To be precise, let $M$ be the $p$-fold cyclic branched cover of $S^3$ along $L$ obtained from the completion of the $p$-fold total linking cover of $X_L$ and let $\sigma$ denote a generator of the Galois group of $M$ over $S^3$. The homology group $H_1(M,\mathbb{Z}_p) = H_1(M,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is then a module over the complete discrete valuation ring $\hat{O} := \mathbb{Z}_p(\langle \sigma \rangle)/(\sigma^{p-1} + \cdots + \sigma + 1) = \mathbb{Z}_p[\zeta]$, $\zeta := \sigma \mod (\sigma^{p-1} + \cdots + \sigma + 1)$. Assume that $H_1(M,\mathbb{Z})$ is finite. Then the $p$-primary part $H_1(M,\mathbb{Z}_p)$ has $p$-rank $n-1$ ([Mo1],[Rez2]) so that it has form

$$H_1(M,\mathbb{Z}_p) = \bigoplus_{i=1}^{n-1} \hat{O}/p^{a_i} \quad (a_i \geq 1)$$

as $\hat{O}$-module where $p := (\zeta - 1)$ is the maximal ideal of $\hat{O}$. Hence the determination of the Galois module structure of $H_1(N,\mathbb{Z}_p)$ is equivalent to that of the $p^k$-rank

$$e_k := \# \{ i \mid a_i \geq k \} \quad (k \geq 1).$$

Our main result is to give formulas for $e_k$'s in terms of the higher linking matrices obtained by specializing the truncated $p$-adic Traldi matrices at $X_1 = \cdots = T_n = \zeta - 1$ (Theorem 4.1.3). For the simplest case of $k = 2$, our formula reads

$$e_2 = n - 1 - \text{rank}_{\mathbb{F}_p}(C \mod p)$$

where $C = (C_{ij})$ is the linking matrix defined by $C_{ii} = -\sum_{j \neq i} \text{lk}(K_i, K_j)$ and $C_{ij} = \text{lk}(K_i, K_j)$ for $i \neq j$. In view of the analogy between the linking number and the power
residue symbols [Mo2,3], this is seen as a link-theoretic analog of L. Rédei’s formula for the
4-rank of the class group of a quadratic field ([Réd1]), and our general result was partly
suggested by the relation between Rédei’s triple symbol and the 8-rank of a class group
[Réd2]. In fact, the whole argument here can be translated into arithmetic [Mo5]. In the
last section, we study the asymptotic behavior of the order \(|H_1(M_m, \mathbb{Z}_p)|\) for the
\(p^m\)-fold cyclic branched cover \(M_m\) as \(m \to \infty\), following Iwasawa theory on \(\mathbb{Z}_p\)-extensions [Iw].
Though our results obtained in this paper are rather elementary, they seem to indicate
further possibilities of our arithmetic approach to link theory.

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Notation. Throughout this paper, we fix a prime number \(p\). We denote by \(\mathbb{F}_p\) the field
with \(p\) elements and by \(\mathbb{Z}_p\) the ring of \(p\)-adic integers. Let \(\text{ord}_p\) denote the additive \(p\)-adic
valuation extended on the algebraic closure \(\overline{\mathbb{Q}}_p\) of the \(p\)-adic field \(\mathbb{Q}_p\) with \(\text{ord}_p(p) = 1\)
and set \(|x|_p = p^{-\text{ord}_p(x)}, x \in \overline{\mathbb{Q}}_p\). We use the letter \(q\) to denote \(p\) or 0. For a topological
(possibly discrete) group \(G\), we denote by \(G(k,q)\) the \(k\)-th term of lower central \(q\)-series
defined by \(G(1,q) = G, G(k+1,q) = (G(k,q))^{q}[G(k,q), G]\) where for closed subgroups \(A, B\) of
\(G, [A, B]\) stands for the closed subgroup of \(G\) generated by \([a, b] = aba^{-1}b^{-1}, a \in A, b \in B\). We simply write \(G(k)\) for \(G(k,0)\), the \(k\)-th term of the lower central series of \(G\). For a
pro-finite group \(G\) and a complete local ring \(R\), we denote by \(R[[G]]\) the completed group
ring of \(G\) over \(R\) [Ko,§7].

1. Pro-\(p\) completion of a link group

1.1. The pro-\(p\) completion of a link group. Let \(L\) be a tame link in the 3-sphere \(S^3\) consisting
of \(n\) component knots \(K_1, \ldots, K_n\) and let \(G_L\) be the link group \(\pi_1(X_L), X_L = S^3 \setminus L\).
After the work of K.T. Chen, J. Milnor [Mi] derived the following information about the
presentation of the nilpotent quotient \(G_L/G_L^{(k,q)}\). Let \(F\) be the free group on the \(n\) words
\(x_1, \ldots, x_n\) where \(x_i\) represents the meridian \(m_i\) around \(K_i\) and let \(\pi : F \to G_L\) be the
meridianal homomorphism defined by \(\pi(x_i) = m_i\ (1 \leq i \leq n)\).

Theorem 1.1.1 ([Mi]). For each \(k \geq 1\) and \(i\) \((1 \leq i \leq n)\), there is a word \(y_i^{(k)}\) in
\(x_1, \ldots, x_n\) representing the image of the \(i\)-th longitude in the quotient \(G_L/G_L^{(k,q)}\) such that

\[
y_i^{(k)} \equiv y_i^{(k+1)} \mod F^{(k,q)}
\]

and such that \(\pi : F \to G_L\) induces the isomorphism

\[
F/N_k F^{(k,q)} \simeq G_L/G_L^{(k,q)}
\]
where $N_k$ is the subgroup of $F$ generated normally by $[x_1, y_1^{(k)}], \ldots, [x_n, y_n^{(k)}]$.

Let $\widehat{G}_L$ be the pro-$p$ completion of $G_L$, namely the inverse limit $\lim \limits_{\leftarrow} G_L/N$ of the tower of quotients $G_L/N$ which are finite $p$-groups. Since the quotients by the lower central $p$-series of $G_L$ are cofinal in this tower, we have

$$\widehat{G}_L = \lim \limits_{\leftarrow} G_L / G_L^{(k,p)}.$$ 

Since $\{y_i^{(k)} F^{(k,p)}\}_{k \geq 1}$ forms an inverse system in the tower $\{F/F^{(k,p)}\}_{k \geq 1}$ by (1.1.2), we define the pro-$p$ word $y_i$ to be $(y_i^{(k)} F^{(k,p)})$ in the free pro-$p$ group $\widehat{F} := \lim \limits_{\leftarrow} F/F^{(k,p)}$ which represents the $i$-th “longitude” in $\widehat{G}_L$ under the map $\widehat{\pi} : \widehat{F} \to \widehat{G}_L$ induced by $\pi$. By taking the inverse limit with respect to $k$ in the isomorphism (1.1.3) of finite $p$-groups for $q = p$, we have the following

**Theorem 1.1.2.** The map $\widehat{\pi}$ induces the isomorphism of pro-$p$ groups

$$\widehat{F}/\widehat{N} \simeq \widehat{G}_L$$

where $\widehat{N}$ is the closed subgroup of $\widehat{F}$ generated normally by $[x_1, y_1], \ldots, [x_n, y_n]$. In particular, we have $\widehat{G}_K \simeq \mathbb{Z}_p$ for a knot $K$.

**Remark 1.1.3.** (1) By the construction above, we note $y_i \equiv y_i^{(k)} \mod \widehat{F}^{(k,q)}$ ($F$ is embedded in $\widehat{F}$).

(2) In view of the analogy between knots and primes, the pro-$p$ link group $\widehat{G}_L$ is regarded as an analog of the maximal pro-$p$ Galois group over the rational number field $\mathbb{Q}$ unramified outside prime numbers $p_1, \ldots, p_n$, $p_i \equiv 1 \mod p$ [Mo2].

Theorem 1.1.2 tells us that from the group-theoretic point of view, any link of $n$ components looks like a pure braid link with $n$ strings after the pro-$p$ completion. In particular, by applying the method of D. Anick [A] to determine the graded quotients of the lower central series of a pure braid link group to our pro-$p$ link group $\widehat{G}_L$, we see that the pro-$p$ analog of Murasugi’s conjecture holds (cf. [L]). We define the mod $p$ linking diagram of $L$ to be the graph with vertices the components of $L$ and an edge joining $K_i$ and $K_j$ if and only if the linking number $\text{lk}(K_i, K_j) \not\equiv 0 \mod p$.

**Theorem 1.1.4.** If the mod $p$ linking diagram of $L$ is connected, we have the isomorphisms

$$\frac{\widehat{G}_L^{(q)}}{\widehat{G}_L^{(q+1)}} \simeq \frac{\widehat{F_1}^{(q)}}{\widehat{F_1}^{(q+1)}} \times \frac{\widehat{F_n}^{(q)}}{\widehat{F_n}^{(q+1)}} \text{ for } q \geq 1,$$
where \( \hat{F}_r \) denotes the free pro-\( p \) group of rank \( r \).

1.2. The \( p \)-goodness of a link group. Let \( G \) be a group and \( \hat{G} \) be the pro-\( p \) completion of \( G \). We then call \( G \) \( p \)-good if the natural map \( G \to \hat{G} \) induces the isomorphisms on cohomology \( H^q(\hat{G}, M) \cong H^q(G, M) \) for all \( q \geq 0 \) for any finite \( p \)-primary \( \hat{G} \)-modules \( M \) (cf. [Se]).

**Theorem 1.2.1.** A link group \( G_L \) is \( p \)-good.

**Proof.** We shall say that a subgroup \( G \) of finite index in \( G_L \) is open if \( [G_L : G] \) is a power of \( p \). Let \( M \) be a finite \( p \)-primary \( \hat{G}_L \)-module. We shall show by induction on the length of \( M \) that if \( G \) is an open subgroup of \( G_L \) then there is a smaller open subgroup \( G_1 \) such that restriction from \( H^2(G, M) \) to \( H^2(G_1, M) \) is trivial.

Suppose first that \( M = F_p \), with trivial \( G_L \)-action, and let \( H^*(G) \) denote \( H^*(G, F_p) \), for ease of reading. Since \([G_L : G]=d\) is finite and \( G_L/G_L^{(2)} \cong \mathbb{Z}^n \) there is an epimorphism \( \tau : G \to C = \mathbb{Z}/p\mathbb{Z} \). Then \( K = \text{Ker}(\tau) \) is another open subgroup of \( G_L \). The Hochschild-Serre spectral sequence for \( G \) as an extension of \( C \) by \( K \) has \( E_2 \) term \( E_2^{p,q} = H^p(C, H^q(K)) \), \( r \)-th differential \( d_r \) of bidegree \((r,1-r)\) and converges to \( H^*(G) \). Since \( H^p(K) = 0 \) for \( p > 2 \) there are only three nonzero rows, and since \( H^*(G) = 0 \) for \( \ast > 2 \) we see that \( d_3^{p,2} \) is an isomorphism for all \( p \geq 1 \). The spectral sequence is an algebra over the ring \( H^*(C) = E_2^{*,0} \). Since \( C \) has cohomological period 2, the cup product with a generator of \( H^2(G, M) \) induces isomorphisms \( \gamma_2^{p,q} : E_2^{p,q} \cong E_2^{p+2,q} \) such that \( d_2^{p+2,q} \gamma_2^{p,q} = \gamma_2^{p+2,q-1} d_2^{p,q} \) for all \( p, q \geq 0 \). Therefore we have the isomorphisms \( \text{Ker}(d_2^{p,q}) \cong \text{Ker}(d_2^{p+2,q}) \), \( \text{Im}(d_2^{p,q}) \cong \text{Im}(d_2^{p+2,q}) \) for any \( p, q \geq 0 \).

In particular we have the isomorphisms \( \gamma_3^{0,2} : E_3^{0,2} = \text{Ker}(d_2^{0,2}) \cong E_3^{2,2} = \text{Ker}(d_2^{2,2}) \) and \( \gamma_3^{3,0} : E_3^{3,0} / \text{Im}(d_2^{3,1}) \cong E_3^{5,0} / \text{Im}(d_2^{5,1}) \) with \( d_3^{2,2} \gamma_3^{0,2} = \gamma_3^{2,1} d_3^{0,2} \). It follows that \( d_3^{0,2} \) is also an isomorphism, and so \( E_3^{0,2} = 0 \). But the edge homomorphism from \( H^2(G) \) to \( H^2(K) \) factors through \( E_3^{0,2} \leq E_2^{0,2} = H^2(K)^C \), and so is 0.

In general, \( M \) has a finite composition series whose factors are copies of the simple module \( F_p \). Suppose that \( M_1 \) is a maximal proper submodule of \( M \), with quotient \( M/M_1 \cong F_p \). Restriction from \( G \) to \( K \) induces a homomorphism from the exact sequences of cohomology corresponding to the coefficient sequence \( 0 \to M_1 \to M \to F_p \to 0 \). The result for \( F_p \) implies that the image of \( H^2(G; M) \) lies in the image of \( H^2(K, M_1) \). By the hypothesis of induction we may assume the result is true for \( M_1 \), and so there is an open subgroup \( K_1 < K \) such that restriction from \( H^2(K, M_1) \) to \( H^2(K_1, M_1) \) is trivial. Hence restriction from \( H^2(G, M) \) to \( H^2(K_1, M) \) is also trivial. This establishes the inductive step.

In particular, restriction from \( H^2(G_L, M) \) to \( H^2(J, M) \) is trivial, for some open subgroup \( J \), and so the result follows, as in Exercise 1 of Chapter I, §2.6 of [Se]. (This exercise is stated in terms of profinite completions, but extends easily to the pro-\( p \) case).

Since the cohomological dimension \( cd(G_L) \leq 2 \), with equality if and only if \( L \) is nontrivial,
Theorem 1.2.4 gives the corresponding bound for the pro-$p$ completion $\hat{G}_L$.

**Corollary 1.2.2.** The cohomological $p$-dimension $\text{cd}_p(\hat{G}_L) \leq 2$.

If the Milnor invariants of $L$ are all 0 mod $p$ (cf. Section 2), then $\hat{G}_L$ is a free pro-$p$ group and so $\text{cd}_p(\hat{G}_L)$ may be strictly less than $\text{cd}(G_L)$. In particular, this is so if $L$ is a nontrivial knot.

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### 2. $p$-adic Milnor invariants

#### 2.1. The pro-$p$ Fox differential calculus

Let $\hat{F}$ be the pro-$p$ completion of the free group $F$ on $n$ generators $x_1, \ldots, x_n$. Y. Ihara [Ih] extended the Fox differential calculus on the abstract free group $F ([F])$ to that on $\hat{F}$. The basic result is stated as the following

**Theorem 2.1.1** ([Ih]). There exists a unique continuous $\mathbb{Z}_p$-homomorphism

$$\partial_i = \frac{\partial}{\partial x_i} : \mathbb{Z}_p[[\hat{F}]] \rightarrow \mathbb{Z}_p[[\hat{F}]]$$

for each $i$ ($1 \leq i \leq n$) such that any element $\alpha \in \mathbb{Z}_p[[\hat{F}]]$ is expressed uniquely in the form

$$\alpha = \epsilon(\alpha)1 + \sum_{i=1}^{n} \partial_i(\alpha)(x_i - 1)$$

where $\epsilon$ is the augmentation map $\mathbb{Z}_p[[\hat{F}]] \rightarrow \mathbb{Z}_p$.

The higher order derivatives are defined inductively by

$$\partial_{i_1} \cdots \partial_{i_r}(\alpha) = \partial_{i_1}(\partial_{i_2} \cdots \partial_{i_r}(\alpha)).$$

Here are some basic rules (cf. [Ih,2]).

1. If one restricts $\partial_i$ to $\mathbb{Z}[F]$ under the natural embedding $\mathbb{Z}[F] \rightarrow \mathbb{Z}_p[[\hat{F}]]$, we get the usual Fox derivative on $\mathbb{Z}[F]$ ($[F]$).
2. $\partial_i(x_j) = \delta_{ij}$ (Kronecker delta).
3. $\partial_i(\alpha\beta) = \partial_i(\alpha)\epsilon(\beta) + \alpha\partial_i(\beta)$ (\(\alpha, \beta \in \mathbb{Z}_p[[\hat{F}]]\)).
4. $\partial_i(f^{-1}) = -f^{-1}\partial_i(f)$ (\(f \in \hat{F}\)).
5. For \( f \in \hat{F} \) and \( a \in \mathbb{Z}_p \), \( \partial_i(f^a) = b \partial_i(f) \), where \( b \) is any element of \( \mathbb{Z}_p[[\hat{F}]] \) such that \( b(f - 1) = f^a - 1 \).

6. Let \( \hat{F}' \) be another free pro-\( p \) group on \( x'_1, \ldots, x'_m \) and let \( \varphi : \hat{F} \to \hat{F}' \) be a continuous surjective homomorphism. Then one has \( \partial_i'(\varphi(\alpha)) = \sum_{j=1}^{n} \varphi(\partial_j(\alpha)) \partial_i(\varphi(x_j)) \), where \( \partial_i' = \frac{\partial}{\partial x'_i} \).

7. For \( f \in \hat{F} \), \( \epsilon(\partial_i'(f)) = (a) \) where \( a = \epsilon(\partial_i(f)) \) and \( (a) = \frac{a(a-1)\cdots(a-r+1)}{r!} \in \mathbb{Z}_p \).

Let \( \mathbb{Z}_p\langle\langle X_1, \ldots, X_n \rangle\rangle \) be the formal power series ring over \( \mathbb{Z}_p \) in non-commuting variables \( X_1, \ldots, X_n \) which is compact in the topology taking the ideals \( I(r) \) of power series with homogeneous components of degree \( \geq r \) as the system of neighborhood of 0. The pro-\( p \) Magnus embedding \( M \) of \( \hat{F} \) into \( \mathbb{Z}_p\langle\langle X_1, \ldots, X_n \rangle\rangle \) is defined by

\[
M(x_i) = 1 + X_i, \quad M(x_i^{-1}) = 1 - X_i + X_i^2 - 
\]

and it is extended to give the isomorphism \( \mathbb{Z}_p[[\hat{F}]] \simeq \mathbb{Z}_p\langle\langle X_1, \ldots, X_n \rangle\rangle \) of compact \( \mathbb{Z}_p \)-algebras. The resulting expansion of \( \alpha \in \mathbb{Z}_p[[\hat{F}]] \) is given by the Fox derivatives:

\[
(2.1.3) \quad M(\alpha) = \epsilon(\alpha) + \sum_{I=(i_1\cdots i_r)} \epsilon_I(\alpha) X_{i_1} \cdots X_{i_r},
\]

\[
\epsilon_I(\alpha) = \epsilon(\partial_{i_1} \cdots \partial_{i_r}(\alpha)), \quad I = (i_1 \cdots i_r).
\]

Finally, we recall the following fact ([Ko],7.14): Let \( \hat{J}_q \) be the two-sided ideal of \( \mathbb{Z}_p[[\hat{F}]] \) generated by \( q \) and the augmentation ideal \( I_{\mathbb{Z}_p[[\hat{F}]}} \). Then for \( f \in \hat{F} \) and \( k \geq 1 \), we have

\[
(2.1.4) \quad f \in \hat{F}^{(k,q)} \iff f - 1 \in \hat{J}_q \iff M(f) = 1 + (\text{term of degree } \geq k) \quad \text{ (for case } q = 0).\]

2.2. \textit{p-adic Milnor invariants}. We keep the same notation as in Section 1. Let \( \hat{F} \) be the free pro-\( p \) group on \( x_1, \ldots, x_n \) where each \( x_i \) represents the \( i \)-th meridian. For a multi-index \( I = (i_1 \cdots i_r), r \geq 1 \), we define the \textit{p-adic Milnor number} \( \hat{\mu}(I) \) by

\[
(2.2.1) \quad \hat{\mu}(I) := \epsilon_I'(y_{i_r}) \quad I' = (i_1 \cdots i_{r-1})
\]

\[
= \epsilon(\partial_{i_1} \cdots \partial_{i_{r-1}}(y_{i_r}))
\]

where \( y_j \in \hat{F} \) represents the \( j \)-th “longitude” in \( \hat{G}_L \) (cf. Section 1.1). By convention, we set \( \hat{\mu}(I) = 0 \) for \( |I| = r = 1 \). We let \( \Delta(I) \) denote the ideal of \( \mathbb{Z}_p \) generated by \( \hat{\mu}(I) \) where
$J$ runs over all cyclic permutations of proper subsequences of $I$. We then define the $p$-adic Milnor invariant $\overline{\mu}(I)$ by

$$
\mu(I) := \hat{\mu}(I) \mod \hat{\Delta}(I).
$$

Since the usual Milnor number $\mu(I), I = (i_1 \cdots i_r)$ is defined by $\epsilon(\partial_{i_1} \cdots \partial_{i_{r-1}}(y_{i_r}^{(r)}))$ by (2.1.1), (1), Remark 1.1.3, (1) and (2.2.1) yield

$$
\hat{\mu}(I) = \mu(I) \quad \text{and} \quad \hat{\Delta}(I) = \Delta(I)
$$

as elements in $\mathbb{Z}_p$. Hence, we have

(2.2.4) $\overline{\mu}(I)$'s are isotopy invariants of $L$ and satisfy the same properties such as the cycle symmetry and Shuffle relations $\overline{\pi}(I)$'s enjoy ([Mi]).

Theorem 1.1.2 and (2.1.4) implies the following:

(2.2.5) All $\overline{\mu}(I) = 0$ for $|I| < r$ if and only if $\overline{\pi} : \widehat{\pi} \rightarrow \widehat{G}_L$ induces an isomorphism $\widehat{\pi} / \pi^{(r,q)} \simeq \widehat{G}_L / \widehat{G}_L^{(r,q)}$. In particular, if all $p$-adic Milnor invariants of $L$ are zero, one has $\widehat{G}_L \simeq \widehat{\pi}$. This is the case for boundary links.

Finally, we remark that (2.1.2), 7 implies

$$
\hat{\mu}(i \cdots j) = \left(\frac{\text{lk}(K_i, K_j)}{r}\right) \quad \text{for} \quad i \neq j.
$$

The Milnor invariant is also given by the Massey products in the cohomology of $\widehat{G}_L$. For the normalized Massey system in profinite group cohomology and sign convention, we refer to [Mo3]. Let $\xi_1, \cdots, \xi_n$ be the $\mathbb{Z}_p$-basis of $H^1(\widehat{G}_L, \mathbb{Z}_p)$ dual to the meridians $m_i$'s, and let $\eta_j \in H_2(\widehat{G}_L, \mathbb{Z}_p)$ be the image of $[x_j, y_j]$ under the transgression $H_1(\widehat{N}, \mathbb{Z}_p) \rightarrow H_2(\widehat{G}_L, \mathbb{Z}_p)$. Then for $I = (i_1 \cdots i_r), r \geq 2$, there is a normalized Massey system $M$ for the product $\langle \xi_{i_1}, \cdots, \xi_{i_r} \rangle \in H_2(\widehat{G}_L, \mathbb{Z}_p / \hat{\Delta}(I))$ so that

$$
\langle \xi_{i_1}, \cdots, \xi_{i_r} \rangle(\eta_j) = \begin{cases} 
(-1)^r \overline{\mu}(I) & j = i_{r+1} \neq i_1, \\
(-1)^{r+1} \overline{\mu}(I) & j = i_1 \neq i_r, \\
0 & \text{otherwise}. 
\end{cases}
$$
3. Completed Alexander modules

3.1. The Alexander module of $\hat{G}_L$. The Alexander module of a finitely presented pro-$p$ group was introduced in several modes in [Mo2]. As a particular case, the Alexander module of the pro-$p$ link group $\hat{G}_L$ is defined using the pro-$p$ Fox free differential calculus as follows. We keep the same notation as in Section 1 and 2. Let $\hat{\psi}$ be the abelianization map $\hat{G}_L \to \hat{H} := \hat{G}_L/\hat{G}_L^{(2)} = \mathbb{Z}_p^\mathbb{N}$ and denote by the same $\hat{\psi}$ the continuous $\mathbb{Z}_p$-homomorphism $\mathbb{Z}_p[[\hat{G}_L]] \to \mathbb{Z}_p[[\hat{H}]]$ on the completed group rings. We identify $\mathbb{Z}_p[[\hat{H}]]$ with the commutative formal power series ring $\hat{\Lambda}_n := \mathbb{Z}_p[[X_1, \cdots, X_n]]$ over $\mathbb{Z}_p$ by setting $t_i := \hat{\psi} \circ \hat{\pi}(x_i) = 1 + X_i$. By Theorem 1.2.1, we call the matrix over $\hat{\Lambda}_n$

\begin{equation}
\hat{P}_L = \left( \hat{\psi} \circ \hat{\pi}(\partial_j([x_i, y_i])) \right)
\end{equation}

the Alexander matrix of $\hat{G}_L$. We then define the Alexander module $\hat{A}_L$ of $\hat{G}_L$ by the compact $\hat{\Lambda}_n$-module presented by $\hat{P}_L$:

\begin{equation}
\hat{A}_L := \text{Coker}(\hat{\Lambda}_n \xrightarrow{\hat{P}_L} \hat{\Lambda}_n)
\end{equation}

and call $\hat{A}_L$ the completed Alexander module of $L$ over $\hat{\Lambda}_n$. Let $\hat{\Lambda} = \mathbb{Z}_p[[X]]$, the Iwasawa algebra and $\tau: \hat{\Lambda}_n \to \hat{\Lambda}$ the reducing homomorphism defined by $\tau(X_i) = X$ ($1 \leq i \leq n$). Then the reduced completed Alexander module $\hat{A}_L^{\text{red}}$ is defined by the compact $\hat{\Lambda}$-module

\begin{equation}
\hat{A}_L^{\text{red}} := \tau(\hat{A}_L) = \hat{A}_L \otimes_{\hat{\Lambda}_n} \hat{\Lambda}
\end{equation}

which is presented by $\tau(\hat{P}_L) = \hat{P}_L(X, \cdots, X)$. For a knot $K$, $\hat{P}_L = O_n$ (zero matrix) and so we have

\begin{equation}
\hat{A}_L = \hat{A}_L^{\text{red}} = \hat{\Lambda}.
\end{equation}

We also define the $i$-th completed Alexander ideal $\hat{E}_i(L)$ of $L$ by the $i$-th elementary ideal $E_i(\hat{A}_L)$ of $\hat{A}_L$ and $i$-th $p$-adic Alexander series $\hat{\Delta}_i(L)$ of $L$ by the greatest common divisor $\Delta_i(E_i(L))$ of generators of the ideal $\hat{E}_i(L)$:

\begin{equation}
\hat{E}_i(L) := E_i(\hat{A}_L), \quad \hat{\Delta}_i(L) := \Delta_i(\hat{E}_i(L))
\end{equation}

The relation with the usual Alexander module is given as follows. Let $\psi: G_L \to H = H_1(X_L, \mathbb{Z})$ be the abelianization map which induces the ring homomorphism $\psi: \mathbb{Z}[G_L] \to \mathbb{Z}_p$.
\[ Z[H] \] on the group rings where \( Z[H] \) is identified with the Laurent polynomial ring \( \Lambda_n = Z[t_1^{\pm 1}, \ldots, t_n^{\pm 1}] \), \( t_i = \psi \circ \pi(x_i) \). Given a presentation \( G_L = \langle x_1, \ldots, x_m \mid r_1 = \cdots r_l = 1 \rangle \) \((m \geq n)\), the Alexander module of \( L \) is given as the \( \Lambda_n \)-module presented by the Alexander matrix \( (\psi \circ \pi(\partial_j(r_i))) \). By Theorem 1.1.1, we can take \( m = n \) and the relations to be \( [x_1, y_i^{(k+1)}] \) \((1 \leq i \leq n)\) and some finite number of generators \( f_i^{(k+1)} \) of \( F^{(k+1,p)} \) \((k \geq 1)\).

We let \( J_p = J_p \cap Z[F] \) and \( J_{\Lambda_n,p} = \psi \circ \pi(J_p) \). Since \( \partial_j(f_i^{(k+1)}) \in J_p \) by (2.1.4), passing to the quotients modulo \( (J_{\Lambda_n,p})^k \), \( A_L/(J_{\Lambda_n,p})^k A_L = A_L \otimes_{\Lambda_n} \Lambda_n/(J_{\Lambda_n,p})^k \) is presented by the matrix \( (\psi \circ \pi(\partial_j([x_i, y_i^{(k+1)}]))) \mod (J_{\Lambda_n,p})^k \). Here we see by Remark 1.1.3, (1) that the elements \( \{\psi \circ \pi(\partial_j([x_i, y_i^{(k+1)}])) \mod (J_{\Lambda_n,p})^k \} \) form an inverse system with respect to \( k \) and its limit is given as \( \psi \circ \pi(\partial_j([x_i, y_i])) \) under the identification \( \lim_k \Lambda_n/(J_{\Lambda_n,p})^k = \tilde{\Lambda}_n \).

Hence by (3.1.2) we have

\[(3.1.6) \quad A_L = \lim_k A_L \otimes_{\Lambda_n} (\Lambda_n/(J_{\Lambda_n,p})^k) = A_L \otimes_{\Lambda_n} \tilde{\Lambda}_n.\]

Similarly, \( \tilde{A}_L^{red} \) is related with the usual reduced Alexander module \( A_L^{red} = \tau(A_L) \) by

\[(3.1.7) \quad \tilde{A}_L^{red} = A_L \otimes_{\Lambda} \tilde{\Lambda} \]

where \( \Lambda = Z[t^{\pm 1}] \) is embedded into \( \tilde{\Lambda} \) by \( t = 1 + X \).

3.2. The \( p \)-adic Traldi matrix. The Alexander matrix \( \tilde{P}_L \) of \( \tilde{G}_L \) (3.1.1) is computed explicitly as a universal \( p \)-adic higher linking matrix in terms of \( p \)-adic Milnor numbers. This is regarded as a \( p \)-adic strengthening of the work by L. Traldi [Tr].

Definition 3.2.1. The \( p \)-adic Traldi matrix \( \tilde{T}_L = (\tilde{T}_L(i,j)) \) of \( L \) over \( \tilde{\Lambda}_n \) is defined by

\[ \tilde{T}_L(i,j) = \begin{cases} -\sum_{r \geq 1 \leq i_1 \cdots i_r \leq n} \hat{\mu}(i_1 \cdots i_r)X_{i_1} \cdots X_{i_r} & i = j \\ \hat{\mu}(ji)X_i + \sum_{r \geq 1 \leq i_1 \cdots i_r \leq n} \hat{\mu}(i_1 \cdots i_r ji)X_{i_1}X_{i_1} \cdots X_{i_r} & i \neq j. \end{cases} \]

and we also define the reduced \( p \)-adic Traldi matrix \( \tilde{T}_L^{red} \) of \( L \) over \( \tilde{\Lambda} \) by

\[ \tilde{T}_L^{red} := \tau(\tilde{T}_L) = \tilde{T}_L(X, \cdots, X). \]

Our theorem is then stated as
Theorem 3.2.2. The p-adic Traldi matrix $\hat{T}_L$ gives a presentation matrix for the completed Alexander module $\hat{A}_L$ over $\hat{\Lambda}_n$, and the reduced p-adic Traldi matrix $\hat{T}_L^{\text{red}}$ gives a presentation matrix for the reduced completed Alexander module $\hat{A}_L^{\text{red}}$ over $\hat{\Lambda}$.

Proof. By (3.1.1), (3.1.2) and (3.1.3), it suffices to show $\hat{\psi} \circ \hat{\pi} (\partial_j([x_i, y_i])) = \hat{T}_L(i, j)$. By the rules 2 ∼ 4 of (2.1.2), we have

$$\partial_j([x_i, y_i]) = (1 - x_i y_i x_i^{-1}) \delta_{ij} + x_i (1 - y_i x_i^{-1} y_i^{-1}) \partial_j(y_i).$$

Here $y_i = 1 + \sum_{r \geq 1} \sum_{1 \leq i_1, \ldots, i_r \leq n} \hat{\mu}(i_1 \cdots i_r i)(x_i^{r-1} - 1) \cdots (x_i - 1)$ by (2.1.3) and (2.2.1). Hence we get

$$(3) \quad \hat{\psi} \circ \hat{\pi} (\partial_j([x_i, y_i])) = \delta_{i,j} - \sum_{r \geq 1} \sum_{1 \leq i_1, \ldots, i_r \leq n} \hat{\mu}(i_1 \cdots i_r i) x_{i_1} \cdots x_{i_r}
+ \hat{\mu}(ji) x_i + \sum_{r \geq 1} \sum_{1 \leq i_1, \ldots, i_r \leq n} \hat{\mu}(i_1 \cdots i_r ji) x_i x_{i_1} \cdots x_{i_r}.$$ 

which yields the assertion. □

The following is an extension of (3.1.4).

Corollary 3.2.3. For a link whose p-adic Milnor invariants are all zero, we have

$$\hat{A}_L \cong \hat{\Lambda}_n^n, \quad \hat{A}_L^{\text{red}} \cong \hat{\Lambda}^n.$$ 

This is the case for boundary links.

Finally, we introduce the truncated p-adic Traldi matrices.

Definition 3.2.4. For $k \geq 2$, the $k$-th truncated p-adic Traldi matrix $\hat{T}_L^{(k)} = (\hat{T}_L^{(k)}(i, j))$ is defined by

$$\hat{T}_L^{(k)}(i, j) = \begin{cases} 
- \sum_{r=1}^{k-1} \sum_{1 \leq i_1, \ldots, i_r \leq n, i_r \neq i} \hat{\mu}(i_1 \cdots i_r i) x_{i_1} \cdots x_{i_r} & i = j \\
\hat{\mu}(ji) x_i + \sum_{r=1}^{k-2} \sum_{1 \leq i_1, \ldots, i_r \leq n} \hat{\mu}(i_1 \cdots i_r ji) x_i x_{i_1} \cdots x_{i_r} & i \neq j
\end{cases}$$
and we also define the \( k \)-th truncated reduced \( p \)-adic Traldi matrix \( \hat{T}_L^{\text{red},(k)} \) by

\[
\hat{T}_L^{\text{red},(k)} := \tau(\hat{T}_L^{(k)}) = \hat{T}_L^{(k)}(X, \ldots, X).
\]

We note that \( \hat{T}_L^{\text{red},(2)} \) is the linking matrix multiplied by \( X \), where the linking matrix \( C = (C_{ij}) \) is defined by \( C_{ii} = -\sum_{j \neq i} \text{lk}(K_i, K_j) \) and \( C_{ij} = \text{lk}(K_i, K_j) \) for \( i \neq j \). Thus the \( p \)-adic Traldi matrix \( \hat{T}_L \) is regarded as a universal higher linking matrix over \( \hat{\Lambda}_n \) which contains all information on the completed Alexander module. In the following section, we derive from \( \hat{T}_L \) the information on the \( p \)-homology groups of \( p^m \)-fold cyclic branched covers along \( L \).

4. Galois module structure for the \( p \)-homology group of a \( p \)-fold cyclic branched cover

4.1. Galois module structure of the \( p \)-homology of a \( p \)-fold cover. Let \( X_\infty \) be the infinite cyclic cover of \( X_L \) associated to the kernel of the homomorphism \( G_L \to \langle t \rangle \) sending each meridian to \( t \). Let \( M \) be the completion of the \( p \)-fold subcover of \( X_\infty \) over \( X_L \) so that \( M \) is a \( p \)-fold cyclic cover of \( S^3 \) branched along \( L \). We set \( \nu_d(t) = t^{d-1} + \cdots + t + 1 \) for \( d \geq 1 \). Let \( \phi : M \to S^3 \) be the covering map and \( \sigma \) a generator of its Galois group. Since \( \nu_p(\sigma) = \text{tr} \circ \phi_* = 0 \) where \( \text{tr} : H_1(S^3, \mathbb{Z}) \to H_1(M, \mathbb{Z}) \) is the transfer, we can regard \( H_1(M, \mathbb{Z}) \) as a module over the Dedekind ring \( \mathcal{O} = \mathbb{Z}[[\sigma]]/(\nu_p(\sigma)) = \mathbb{Z}[\zeta] \), \( \zeta = \sigma \mod (\nu_p(\sigma)) \) is a primitive \( p \)-th root of 1. Hence \( H_1(M, \mathbb{Z}_p) = H_1(M, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_p \) is regarded as a module over the complete discrete valuation ring \( \hat{\mathcal{O}} = \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_p = \mathbb{Z}_p[\zeta] \). Note that \( \hat{\mathcal{O}} \) is the completion of \( \mathcal{O} \) with respect to the maximal ideal \( p \) generated by the prime element \( \pi := \zeta - 1 \) and the residue field \( \hat{\mathcal{O}}/p \) is \( \mathbb{F}_p \). By Theorem 2.2.1, we can derive the following information on a presentation matrix for \( \hat{\mathcal{O}} \)-module \( H_1(M, \mathbb{Z}_p) \). Note that the evaluation of a power series \( F(X) \in \mathbb{Z}_p[[X]] \) at \( s = \pi \) makes sense in the \( p = (\pi) \)-adically complete ring \( \hat{\mathcal{O}} \).

**Theorem 4.1.1.** A presentation matrix for \( H_1(M, \mathbb{Z}_p) \oplus \hat{\mathcal{O}} \) over \( \hat{\mathcal{O}} \) is given by \( \hat{T}_L^{\text{red}}(\pi) \). Further, for any integer \( k \geq 2 \), a presentation matrix for \( (H_1(M, \mathbb{Z}_p) \otimes \hat{\mathcal{O}}/p^k) \oplus \hat{\mathcal{O}}/p^k \) over \( \hat{\mathcal{O}}/p^k \) is given by \( \hat{T}_L^{\text{red},(k)}(\pi) \). Here \( \hat{T}_L^{\text{red}} \) (resp. \( \hat{T}_L^{\text{red},(k)} \)) is the reduced (resp. reduced truncated) Traldi matrix defined in Section 3.2.

**Proof.** Note that the well-known relation ([S1, Theorem 6])

\[
H_1(M, \mathbb{Z}) \cong H_1(X_\infty, \mathbb{Z})/\nu_p(t)H_1(X_\infty, \mathbb{Z})
\]
is an \( \hat{O} \)-isomorphism since \( \sigma \) acts on the r.h.s by \( t \). Hence we have the following isomorphisms over \( \hat{O} \):

\[
\begin{align*}
H_1(M, \mathbb{Z}_p) & \simeq (H_1(X_\infty, \mathbb{Z})/\nu_p(t)H_1(X_\infty, \mathbb{Z})) \otimes \mathbb{Z} \mathbb{Z}_p \\
& \simeq H_1(X_\infty, \mathbb{Z}) \otimes_\Lambda (\Lambda \otimes \mathbb{Z} \mathbb{Z}_p)/(\nu_p(t)) \\
& \simeq H_1(X_\infty, \mathbb{Z}) \otimes_\Lambda (\hat{\Lambda}/(\nu_p(1 + X))) \\
& \simeq H_1(X_\infty, \mathbb{Z}) \otimes_\Lambda \hat{\Lambda}.
\end{align*}
\]

Since \( A_{L}^{\text{red}} \simeq H_1(X_\infty, \mathbb{Z}) \oplus \Lambda \) as \( \Lambda \)-module ([H, 5.4]), tensoring with \( \hat{O} \) over \( \Lambda \), we have an isomorphism of \( \hat{O} \)-modules

\[
A_{L}^{\text{red}} \otimes_\Lambda \hat{O} \simeq H_1(M, \mathbb{Z}_p) \oplus \hat{O}.
\]

Since the l.h.s is same as \( \hat{A}_{L}^{\text{red}} \otimes_\Lambda \hat{O} \), the first assertion follows from Theorem 3.2.4. The second assertion is obtained from the first one by taking modulo \( p^k \). \( \Box \)

Now, we assume that \( H_1(M, \mathbb{Z}) \) is finite so that \( H_1(M, \mathbb{Z}_p) \) is the \( p \)-primary part of \( H_1(M, \mathbb{Z}) \). Using Theorem 3.1.1, we will see the \( \hat{O} \)-module structure of \( H_1(M, \mathbb{Z}_p) \) more precisely. First, we recall the following result on the \( p \)-rank of \( H_1(M, \mathbb{Z}_p) \) (cf. [Mo1],[Rez]).

**Lemma 4.1.2.** \( H_1(M, \mathbb{Z}_p) \otimes \hat{O} \mathbb{F}_p \) has dimension \( n - 1 \) over \( \mathbb{F}_p \).

**Proof.** By [Mo1], the map \( \Phi : H_1(M, \mathbb{Z}) \rightarrow \mathbb{F}_p^n \) defined by \( \Phi(c) := (\text{lk}(\phi_*(c), K_i) \mod p) \) induces an isomorphism

\[
H_1(M, \mathbb{Z})/(\sigma - 1)H_1(M, \mathbb{Z}) \simeq \{(\xi_i) \in \mathbb{F}_p^n \mid \sum_{i=1}^{n} \xi_i = 0\} \simeq \mathbb{F}_p^{n-1}
\]

where the l.h.s is \( H_1(M, \mathbb{Z}) \otimes_\mathcal{O} \mathcal{O}/p = H_1(M, \mathbb{Z}_p) \otimes_\hat{O} \mathbb{F}_p \), and hence we are done. \( \Box \)

By Lemma 4.1.2, \( H_1(M, \mathbb{Z}_p) \) has the form

\[
H_1(M, \mathbb{Z}_p) = \bigoplus_{i=1}^{n-1} \hat{O}/p^{a_i} \quad (a_i \geq 1)
\]

as \( \hat{O} \)-module. Hence, the determination of \( \hat{O} \)-module structure of \( H_1(M, \mathbb{Z}_p) \) is equivalent to the determination of the \( p^k \)-rank

\[
e_k := \#\{i \mid a_i \geq k\} = \dim_{\mathbb{F}_p} H_1(M, \mathbb{F}_p) \otimes_\hat{O} (p^{k-1}/p^k) \quad (k \geq 1).
\]
We describe the $p^k$-rank $e_k$ in terms of $\hat{T}_L^{red,(k)}(\pi)$. For an $n$ by $n$ matrix $A$ over $\hat{O}$, we denote by $A \otimes_{\hat{O}} (p^{k-1}/p^k)$ the $F_p$-linear map on the $F_p$-vector space $(p^{k-1}/p^k)^n$ induced by $A$. Then Theorem 4.1.1 is restated as

**Theorem 4.1.3.** Notation and assumption being as above, we have

$$e_k = n - 1 - \text{rank}_{F_p}(\hat{T}_L^{red,(k)}(\pi) \otimes_{\hat{O}} (p^{k-1}/p^k)), \ k \geq 2.$$  

Here we may call $\hat{T}_L^{red,(k)}(\pi)$ the $k$-th higher linking matrix in view of the following

**Corollary 4.1.4.** For $k = 2$, we have

$$e_2 = n - 1 - \text{rank}_{F_p}(C \mod p)$$

where $C = (C_{ij})$ is the linking matrix of $L$ defined by $C_{ii} = -\sum_{j\neq i} \text{lk}(K_i, K_j)$ and $C_{ij} = \text{lk}(K_i, K_j)$ for $i \neq j$.

**Proof.** In fact, we have, by definition,

$$\hat{T}_L^{red,(2)}(\pi) = \pi C$$

and hence $\text{rank}_{F_p}(\hat{T}_L^{red,(2)} \otimes_{\hat{O}} p/p^2) = \text{rank}_{F_p}(C \mod p)$. \qed

4.2. 2-component case. We suppose $n = 2$ and keep to assume $H_1(M, \mathbb{Z})$ is finite. By Lemma 4.1.2, $H_1(M, \mathbb{Z}_p)$ has the $p$-rank 1 so that we have

$$H_1(M, \mathbb{Z}_p) = \hat{O}/p^a, \ a \geq 1.$$  

Hence $e_k = 0$ or 1 for $k \geq 2$, and by Theorem 4.1.3 we have

$$e_k = 1 \iff \hat{T}_L^{red,(k)}(\pi) \equiv O_2 \mod \pi^k.$$  

Since $\hat{T}_L^{red,(k)}(1,2)(\pi) = -\hat{T}_L^{red,(k)}(1,1)(\pi)$, $\hat{T}_L^{red,(k)}(2,2)(\pi) = -\hat{T}_L^{red,(k)}(2,1)(\pi)$, we have the following

**Theorem 4.2.1.** Suppose $n = 2$. For each $k \geq 1$, assuming $e_k = 1$, we have

$$e_{k+1} = 1 \iff \begin{cases} \sum_{r=1}^{k} \sum_{i_1, \ldots, i_{r-1}=1, 2} \hat{\mu}(i_1 \ldots i_{r-1} 12) \pi^r \equiv 0 \mod \pi^{k+1}, \\ \sum_{r=1}^{k} \sum_{i_1, \ldots, i_{r-1}=1, 2} \hat{\mu}(i_1 \ldots i_{r-1} 12) \pi^r \equiv 0 \mod \pi^{k+1}. \end{cases}$$
We give the condition in Theorem 4.2.1 in more concise forms for lower \( k \). In the following computation, we use simply the usual Milnor number \( \mu(I) \) instead of \( \hat{\mu}(I) \) by (2.2.3).

**Example 4.2.2.** \( e_2 \): Since \( e_1 = 1 \), we have by Theorem 4.2.1

\[
(4.2.2.1) \quad e_2 = 1 \iff \mu(12) \pi \equiv 0 \mod \pi^2 \iff \text{lk}(K_1, K_2) \equiv 0 \mod p.
\]

\( e_3 \): Assume \( \text{lk}(K_1, K_2) \equiv 0 \mod p \). By Theorem 4.2.1, we have

\[
e_3 = 1 \iff \begin{cases} 
\mu(21) \pi + (\mu(121) + \mu(221)) \pi^2 \equiv 0 \mod \pi^3, \\
\mu(12) \pi + (\mu(112) + \mu(212)) \pi^2 \equiv 0 \mod \pi^3.
\end{cases}
\]

By cycle symmetry, \( \mu(121) \equiv \mu(112), \mu(221) \equiv \mu(212) \mod \mu(12) \). Here \( \mu(112) \equiv \mu(221) \equiv \left(\mu^{(12)}\right)_2 \mod \mu(12) \) by (2.2.6). Thus we have \( \mu(121) + \mu(221) \equiv \mu(112) + \mu(212) \equiv \mu(12) \equiv 0 \mod p \). Hence, we have

\[
(4.2.2.2) \quad e_3 = 1 \iff \text{lk}(K_1, K_2) \equiv 0 \mod p^2.
\]

As one easily see, this condition is also equivalent to

\[
(4.2.2.3) \quad \mu(112) \equiv \mu(221) \equiv 0 \mod p.
\]

\( e_4 \): Assume \( \text{lk}(K_1, K_2) \equiv 0 \mod p^2 \). By Theorem 4.2.1, we have

\[
e_4 = 1 \iff \begin{cases} 
\mu(21) \pi + (\mu(121) + \mu(221)) \pi^2 \\
+ (\mu(1121) + \mu(1221) + \mu(2121) + \mu(2221)) \pi^3 \equiv 0 \mod \pi^4, \\
\mu(12) \pi + (\mu(112) + \mu(212)) \pi^2 \\
+ (\mu(1112) + \mu(1212) + \mu(2112) + \mu(2212)) \pi^3 \equiv 0 \mod \pi^4.
\end{cases}
\]

As in case of \( e_3 \), we have \( \mu(121) + \mu(221) \equiv \mu(12) \equiv 0 \mod p^2 \). Similarly, \( \mu(1121) \equiv \mu(1112) \equiv \left(\mu^{(12)}\right)_3 \mod \Delta(1121) \) and \( \mu(2221) \equiv \left(\mu^{(12)}\right)_3 \mod \Delta(2221) \) by (2.2.6). Since \( \Delta(1121) \equiv \Delta(2221) \equiv 0 \mod p, \mu(1121) + \mu(2221) \equiv \mu(12)/3 \equiv 0 \mod p \). Finally, by shuffle relation, \( \mu(1221) + \mu(2121) + \mu(2211) \equiv \mu(2121) + 2\mu(2211) \equiv 0 \mod p \). Thus the first condition is equivalent to \( \mu(21) \pi - \mu(2211) \pi^3 \equiv 0 \mod \pi^4 \). Similarly, we see that the second condition is equivalent to \( \mu(12) \pi - \mu(1122) \pi^3 \equiv 0 \mod \pi^4 \) which is same as the first one. Hence, we obtain

\[
e_4 = 1 \iff \text{lk}(K_1, K_2) - \mu(1122) \pi^2 \equiv 0 \mod \pi^3.
\]
For case $p = 2$, this is equivalent to the following condition:

\[
\begin{cases}
\text{lk}(K_1, K_2) \equiv 0 \mod 8, \; \mu(1122) \equiv 0 \mod 2 \\
or \\
\text{lk}(K_1, K_2) \equiv 0 \mod 4, \text{lk}(K_1, K_2) \not\equiv 0 \mod 8, \mu(1122) \equiv 1 \mod 2.
\end{cases}
\]

(4.2.2.4)

For example, the Whitehead link $L = K_1 \cup K_2$ satisfies $\text{lk}(K_1, K_2) = 0$, $\mu(1122) = 1$ and so $e_3 = 1, e_4 = 0$, hence $H_1(M, \mathbb{Z}_p) = \mathbb{Z}/p^3$. For the 2-bridge link of type (48,37), the latter condition of (4.2.2.4) is satisfied and so $H_1(M, \mathbb{Z}_2) = \mathbb{Z}/2^k\mathbb{Z}$, $k \geq 4$.

5. Iwasawa type formulas for the $p$-homology groups of $p^m$-fold cyclic branched covers

5.1. Asymptotic formula for the $p$-homology of $p^m$-fold covers. For $m \geq 1$, let $M_m$ be the completion of the $p^m$-fold subcover of $X_\infty$ over $X_L$ so that $M_m$ is a $p^m$-fold cyclic cover of $S^3$ branched along $L$. In this last Section, we are concerned with the asymptotic behavior of the order of $H_1(M_m, \mathbb{Z}_p)$ as $m \to \infty$ using the standard argument in Iwasawa theory. As in Section 4, we start again with the following isomorphisms

\[
\hat{A}_L^{\text{red}} \simeq (H_1(X_\infty, \mathbb{Z}) \otimes_{\hat{\Lambda}} \hat{\Lambda}) \oplus \hat{\Lambda}, \tag{5.1.1}
\]

\[
H_1(M_m, \mathbb{Z}_p) \simeq (H_1(X_\infty, \mathbb{Z}) \otimes_{\hat{\Lambda}} \hat{\Lambda})/(\nu_p^{m-1}(1 + X)). \tag{5.1.2}
\]

From these, we get immediately an extension of a theorem of M. Dellomo [D] for a knot.

**Proposition 5.1.3** For a link $L$ whose $p$-adic Milnor invariants are all zero, for example a boundary link, we have $H_1(M_m, \mathbb{Z}_p) = \mathbb{Z}_p^{(p^m-1)(n-1)}$ for $m \geq 1$. In particular, $H_1(M_m, \mathbb{Z}_p) = 0$ for $m \geq 1$ if $L$ is a knot.

**Proof.** In fact, $\hat{A}_L^{\text{red}} = \hat{\Lambda}^n$ for such a link $L$ by Corollary 3.2.5. Hence $H_1(X_\infty, \mathbb{Z}) \otimes_{\hat{\Lambda}} \hat{\Lambda} = \hat{\Lambda}^{n-1}$ by (5.1.1) and so $H_1(M_m, \mathbb{Z}_p) = \mathbb{Z}_p^{(p^m-1)(n-1)}$ by (5.1.2). □

In the following, we assume $n \geq 2$. By (5.1.1), the 0-th elementary ideal $E_0(H_1(X_\infty, \mathbb{Z}) \otimes_{\hat{\Lambda}} \hat{\Lambda})$ over $\hat{\Lambda}$ is same as the 1st completed Alexander ideal $\hat{E}_1(L)$ (3.1.5). Note that the 1st $p$-adic Alexander series $\hat{\Delta}_1(L)$ is given as the greatest common divisors of all $n - 1$ minors of the reduced $p$-adic Traldi matrix and so it is written by the form

\[
\hat{\Delta}_1(L) = X^{n-1} \cdot \hat{\nabla}_L.
\]

*K. Murasugi informed us of this condition and examples which are obtained by the relation between the Alexander polynomial and Milnor invariants [Mu].
where we call $\hat{\nabla}_L$ the \textit{$p$-adic Hosokawa series} of $L$. Then by (5.1.2), we have the following formula on the order $|H_1(M_m, \mathbb{Z}_p)|$ which is seen as the $p$-primary part of the well known formula by S. Kinoshita and F. Hosokawa [KT] (See also [MM]). Here we interpret $|H_1(M_m, \mathbb{Z}_p)| = 0$ to mean $H_1(M_m, \mathbb{Z}_p)$ is infinite.

\textbf{Proposition 5.1.4.} $|H_1(M_m, \mathbb{Z}_p)| = p^{m(n-1)} \prod_{\zeta \neq 1}^{\zeta^p = 1} |\hat{\nabla}_L(\zeta - 1)|_p^{-1}$.

Now, we assume $H_1(M_m, \mathbb{Z}_p)$ is finite for any $m$ and see the asymptotic behaviour of the order $|H_1(M_m, \mathbb{Z}_p)|$ as $m \to \infty$. For this, we recall the following standard facts from Iwasawa theory. We call a polynomial $g(X) \in \mathbb{Z}_p[\mathbb{Z}]$ distinguished if $g(X) = X^d + a_{d-1}X^{d-1} + \cdots + a_0, a_i \equiv 0 \mod p$ for $0 \leq i \leq d-1$.

\textbf{Lemma 5.1.5} ([W, Theorem 7.3]). A non-zero element $f(X) \in \widehat{\Lambda}$ is written uniquely as

$$f(X) = p^\mu g(X)u(X)$$

where $\mu$ is a non-negative integer, $g(X)$ is a distinguished polynomial and $u(X) \in \widehat{\Lambda}^\times$.

\textbf{Lemma 5.1.6} ([W, Theorem 7.14]). Let $f(X) \in \widehat{\Lambda}$ and assume $f(\zeta - 1) \neq 0$ for any primitive $p^m$-th root $\zeta$ of 1 for $k \geq 1$. Write $f(X) = p^\mu g(X)u(X)$ according to Lemma 5.5 and define $\lambda$ by the degree of $g(X)$. Then there is an integer $\nu$ independent of $k$ such that we have the equality

$$\ord_p\left(\prod_{\zeta^p = 1} f(\zeta - 1)\right) = \lambda m + \mu p^m + \nu$$

for sufficiently large $m$.

For the convenience of the reader, we include herewith a proof of Lemma 5.1.6.

\textit{Proof of Lemma 5.1.6.} Since $u(x) \in \widehat{\Lambda}^\times$, we have

$$\ord_p\left(\prod_{\zeta^p = 1} f(\zeta - 1)\right) = (p^m - 1)\mu + \ord_p\left(\prod_{\zeta^p = 1} g(\zeta - 1)\right)$$

Write $g(X) = X^\lambda + a_\lambda \cdots + a_0, a_i \equiv 0 \mod p$ ($0 \leq i \leq \lambda - 1$). For a primitive $p^l$-th root $\zeta$ of 1 ($1 \leq l \leq k$), one has the equality $(p) = (\zeta - 1)\phi(p^l)$ of ideals of $\mathbb{Z}[\zeta]$ and so $\ord_p((\zeta - 1)^\lambda) = \frac{\lambda}{\phi(p^l)}$ where $\phi(x)$ is the Euler function. Therefore, if $l$ is large enough,
\[ \text{ord}_p((\zeta - 1)^{\lambda}) < \text{ord}_p(a_i(\zeta - 1)^i) \text{ for } 0 \leq i \leq \lambda - 1 \text{ and so } \text{ord}_p(g(\zeta - 1)) = \text{ord}_p((\zeta - 1)^{\lambda}). \]

Hence, there is a constant \( C \) independent of \( k \) such that for sufficiently large \( m \), we have

\[ \text{ord}_p\left( \prod_{\zeta \neq 1}^{\zeta^m = 1} g(\zeta - 1) \right) = \text{ord}_p\left( \prod_{\zeta \neq 1}^{\zeta^m = 1} (\zeta - 1)^{\lambda} \right) + C = \text{ord}_p(p^{m\lambda}) + C = \lambda m + C. \]

\[ \square \]

To apply Lemma 5.1.6 to \( \hat{\nabla}_L \), write

\[ \hat{\nabla}_L = p^{\mu(L;p)}g(L;p)u(L;p) \]

where \( \mu(L;p) \) is a nonnegative integer, \( g(L;p) \) is a distinguished polynomial and \( u(L;p) \in \hat{\Lambda}^\times \) according to Lemma 5.1.5 and set \( \lambda(L;p) = \deg(g(L;p)) \). Then Proposition 5.1.4 and Lemma 5.1.6 yield the following

**Theorem 5.1.7.** Notation and assumption being as above, there is a constant \( \nu(L;p) \) depending only on \( L \) and \( p \) such that we have

\[ \text{ord}_p\left( |H_1(M_m,\mathbf{Z}_p)| \right) = (n - 1 + \lambda(L;p))m + \mu(L;p)p^m + \nu(L;p) \]

for sufficiently large \( m \).

We call the invariants \( \lambda(L;p), \mu(L;p) \) the Iwasawa \( \lambda, \mu \)-invariants of \( L \) with respect to \( p \) respectively after the model of the Iwasawa invariants in the theory of \( \mathbf{Z}_p \)-extensions [Iw].

5.2. Examples.

1. Let \( L \) be the Whitehead link. We then have

\[ \hat{T}_L = \begin{pmatrix} X_1 X_2 & -X_1^2 X_2 \\ X_1^2 X_2 & -X_1 X_2^2 \end{pmatrix} \]

and \( \hat{\nabla}_L = X^2 \). Hence, we have \( \lambda(L;p) = 2, \mu(L;p) = 0 \) and \( \text{ord}_p(H_1(M_m,\mathbf{Z}_p)) = 3m \) for \( m \geq 1 \).

2. Let \( L = K_1 \cup K_2 \cup K_3 \) be the Borromean rings so that we can take \( y_1 = [x_3, x_2], y_2 = [x_3, x_1], y_3 = [x_1, x_2] \). Then we can compute all Milnor number needed to get the reduced Traldi matrix

\[ \hat{T}_L^{\text{red}} = \begin{pmatrix} X + X^2 & -X^2 & -X \\ -X & X + X^2 & -X^2 \\ -X^2 & -X & X + X^2 \end{pmatrix} \]

and so \( \hat{\nabla}_L = 1 + X + X^2 \). Hence, we have \( \lambda(L;p) = \mu(L;p) = 0 \) and \( \text{ord}_p(H_1(M_m,\mathbf{Z}_p)) = 2m \) for \( m \geq 1 \).
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jonh@maths.usyd.edu.au; matei@ms.u-tokyo.ac.jp; morisita@kenroku.kanazawa-u.ac.jp