Finite-Time Neural Network Backstepping Control of an Uncertain Fractional-Order Duffing System With Input Saturation

Hui Lv and Xiulan Zhang*

1 Department of Applied Mathematics, Huainan Normal University, Huaihua, China, 2 School of Mathematics and Physics, Guangxi University for Nationalities, Nanning, China

In this paper, neural network (NN) control of the fractional-order Duffing system (FODS) by using a backstepping method within finite time in the presence of input saturation has been investigated. A fractional-order filter with an order lying on the interval (1,2) was used to estimate the virtual input together with its fractional derivative, and this showed that the estimation error tends to a small region in some finite time. Fractional-order law is designed for the parameter of the NN, and an adaptive NN controller was given. The proposed method drives the tracking error, tending to an arbitrary small region within a finite time. The simulation results verify the validity of the proposed method.

Keywords: finite time control, fractional-order system, fractional filter, adaptive neural network control, chaos control

1. INTRODUCTION

It is a well-known fact that classical differential operators are local operators and cannot describe some complex properties. For example, Brownian motion, viscoelastic materials, anomalous diffusion, and irregular fluctuations in the turbulent velocity field have memory problems. Fractional-order differential operators are non-local and can well-characterize memory, genetic, and global correlation in the real world. The physical process is an important tool for describing physical processes and complex mechanics [1, 2]. In fact, fractional derivatives exhibit several advantages over integer derivatives: (1) fractional derivatives have a global correlation and can reflect the historical dependence of function development in the system; (2) the fractional derivative model is more consistent with the experimental results when simulating some complex properties, and the effect is better; and (3) when simulating complex mechanics and physical process problems, the expression of fractional-order model is more concise and the meaning is clearer [3, 4]. In view of these three advantages of fractional derivative, scholars have gradually used fractional differential equations to describe some practical problems. In recent decades, fractional calculus and fractional differential equations have developed rapidly and have gradually matured, and they have also been applied in other disciplines, such as quantum mechanics, economics and finance, turbulence, viscoelasticity theory, and superconductivity. A large number of papers on fractional calculus and fractional differential equations and works and so on have appeared [5–10]. The research contents include the theory and application of fractional calculus, the existence and uniqueness of solutions to the Cauchy problem, stability, controllability, the existence and uniqueness of solutions to boundary value problems, analytical solutions, and numerical algorithms. However, some research methods in integer-order differential equations cannot be directly applied to the study of fractional-order differential equations, and new theories and methods need to be sought. There
are many research fields for integer-order differential equations, and research fields for fractional-order differential equations are limited, and the solution mapping of fractional-order differential equations does not have a semigroup property. Therefore, there are many difficulties in the study of fractional differential equations.

On the other hand, it is well-known that chaos control is a research hot topic and has some successful applications. With the in-depth study of chaotic systems, people began to try to migrate the synchronization method of integer-order chaotic systems to the synchronization of fractional-order chaotic systems (FOCSs). This natural idea is not easy to implement. For this reason, people try to use the Laplace transform method to implement the control method using the Laplace transform method, finite time control was investigated in Tavazoie and Haeri [5]. Up to now, many control methods have been used to control or synchronize FOCSs, for example, adaptive robust control, adaptive fuzzy control (AFC), adaptive neural network control (ANNC), sliding mode control (SMC), command filtered control (CFC), etc. [11-18]. In Pham et al. [19], a three-dimensional FOCS that had no equilibrium was introduced and investigated, and it was shown that the system shows chaotic phenomenon when the order $\alpha < 2.7$. In Zhang et al. [20], the lag projective synchronization of FOCSs with time-varying delays was considered by using a comparison principle of linear fractional equation. In Liu et al. [16], the NN was used to control FOCSs in the presence of input faults. It should be mentioned that above works do not consider the finite time stability is. Up to now, the finite time control of the FOCS has barely been investigated [21-23].

Inspired with above discussion, we will address the finite time NN control of the fractional-order-land Coupled system (FODS) with input saturation. Take some related works, such as Liu et al. [16] and Ha et al. [24, 25], our work has included several features: (1) a fractional-order filter whose order lies on (1,2), designed to evaluate the immediate controller and its fractional derivative within some finite time. However, a fractional filter was also used in Liu et al. [16] and Ha et al. [25] whose order lies on (0, 1), and, in addition, the finite-time stability cannot be guaranteed; (2) to cancel the estimation error of the filter, a fractional-order compensated signal was proposed. Compared with the compensated signals proposed in Ha et al. [24], our method can obtain a more rapid convergence; and (3) in the FOCS's mode, we have considered the case of input saturation.

2. PRELIMINARIES

2.1. Description of the NN

The NN with three layers is expressed as

$$y_{j}(s, \mu_{j}) = \sum_{\eta=1}^{h} \omega_{j\eta}q_{j\eta} \left( \sum_{i=1}^{n} v_{n\eta} + \gamma_{\eta} \right) = \mu_{j}^{T} \chi_{j}(\cdot),$$  (1)

where $n, h, and m \in \mathbb{N}$ denote the amount of neurons three layers (input, middle, and output), $\mu_{j} = \begin{bmatrix} \omega_{j1} \\ \vdots \\ \omega_{j\mu_{j}} \end{bmatrix}$, and $\chi_{j} = \begin{bmatrix} \varphi_{1} \left( \sum_{i=1}^{n} v_{n1} + \gamma_{1} \right) \\ \vdots \\ \varphi_{\mu_{j}} \left( \sum_{i=1}^{n} v_{n\mu_{j}} + \gamma_{\mu_{j}} \right) \end{bmatrix}$. $v_{j}$ denotes a weight whose value is on the interval $[-1, 1]$. Usually, $\varphi(\cdot)$ can be defined by

$$\varphi(h) = \frac{e^{h} - e^{-h}}{e^{h} + e^{-h}}.$$  (2)

Then, the NN is given as

$$y = \theta^{T} \chi(h)$$  (3)

with $\theta = \begin{bmatrix} \mu_{1}^{T} \\ \mu_{2}^{T} \\ \vdots \\ \mu_{m}^{T} \end{bmatrix}$ and $\chi(h) = \begin{bmatrix} x_{1}(h) & 0 & \cdots & 0 \\ 0 & x_{2}(h) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_{m}(h) \end{bmatrix}$.

Suppose that $f(h, h) \in \mathbb{R}^{n}$ is known, then it can be approximated by the NN as

$$f(x) = \theta^{*T} \chi(h) + \varepsilon(h),$$  (4)

with $\varepsilon(h)$ denoting the optimal approximation error, where

$$\theta^{*} = \arg \min_{\theta} \left[ \sup_{h} \hat{f}(h) - f(h) \right],$$  (5)

with $\hat{f}(h) = \theta^{T}(t) \chi(h)$.

2.2. Basic Lemmas

The $q$-th fractional integral for a function $g(t)$ is defined as

$$\mathcal{I}^{q}g(t) = \frac{1}{\Gamma(q)} \int_{0}^{t} g(\tau)(t - \tau)^{-q+1} d\tau,$$  (6)

with $\Gamma(\cdot)$ representing Euler's function, and the $q$-th fractional-order derivative for a function $g(t)$, which has a $k$-th continuous derivative, is

$$\mathcal{D}^{q}g(t) = \frac{1}{\Gamma(k - q)} \int_{0}^{t} g^{(k)}(\tau)(t - \tau)^{-q+1-k} d\tau,$$  (7)

where $k - 1 \leq q < k$ ($k \in \mathbb{N}$). The following always assumes that $0 < q \leq 1$ for convenience. The fractional-order calculus has the following properties.

Lemma 1. [16] For a smooth function $x(t)$, it holds that

$$\frac{1}{2} \mathcal{D}^{2}x^{2}(t) \leq x(t) \mathcal{D}^{1}x(t).$$  (8)

Lemma 2. [22] Let $V(\xi)$ be a smooth function satisfying $\mathcal{D}^{\alpha}_{t}V(\xi) + \alpha_{1}V^{\alpha_{2}}(\xi) \leq 0$, $\xi \in \Omega_{1} \subset \mathbb{R}^{n}$, $\alpha_{1} \in \mathbb{R}^{+}$, and $0 < \alpha_{2} < 1$. Then, one can find $\Omega_{2} \subset \mathbb{R}^{n}$, which holds that $V(\xi)$ begins within $\Omega_{2}$ will reach a sufficient small region in some finite time $T^{*}$. 

Lemma 3. [23] Assume $g_{1}, g_{2} > 0$, $0 < g_{3} < 1$, and

$$\mathcal{D}^{\alpha}_{t}V(\xi) + g_{1}V(\xi) + g_{2}V^{g_{3}}(\xi) \leq 0$$

where $\xi \in \mathbb{R}^{n}$. Then, the system is finite time stable.
Lemma 4. [23] Consider
\[
\begin{align*}
\mathcal{D}^\alpha_0 \pi_1(t) &= F(t), \\
\mathcal{D}^\alpha_0 \pi_2(t) &= -g_2 \text{sign}(\pi_2(t) - F(t)), \\
\mathcal{D}^\alpha_0 \dot{\pi}_2(t) &= -g_3 \text{sign}(\pi_1(t) - \xi(t)) + \pi_2(t),
\end{align*}
\]
where \( \xi(t) \subseteq \mathbb{R} \), \( g_4, g_5 \in \mathbb{R}^+ \). Let
\[
\dot{\pi}_1 = \pi_1 - \xi, \quad \dot{\pi}_2 = F - \mathcal{D}^\alpha_0 \xi.
\]
Then, \( \dot{\pi}_1 \) and \( \dot{\pi}_2 \) are finite time stable.

3. MAIN RESULTS

The integer-order Duffing system is written as
\[
\ddot{x}(t) - y(t) + ax(t) + y^3(t) = b \cos(\omega t), \quad (9)
\]
where \( a, b \) are parameters. Denote \( x_1(t) = y(t), x_2(t) = \dot{y}(t), x(t) = [x_1(t), x_2(t)]^T \) and \( f(x(t)) = x_1(t) - ax_2(t) - x_1^3(t) + b \cos(\omega t) \). By putting the fractional calculus into system (9) and considering the input saturation, the controlled FODS is written as
\[
\begin{align*}
\mathcal{D}^\alpha_0 x_1(t) &= x_2(t), \\
\mathcal{D}^\alpha_0 x_2(t) &= \text{sat}(u(t)) + d(t) + f(x(t)),
\end{align*}
\]
in which \( \text{sat} : u(t) \to \text{sat}(u(t)) \) is called a saturator. It can be expressed as:
\[
\text{sat}(u(t)) = \begin{cases} 
  u_r, & u \geq u_r \\
  u(t), & u_l < u(t) < u_r \\
  u_l, & u \leq u_l,
\end{cases} \quad (11)
\]
with \( u_r > 0, u_l < 0 \). Denoting the term that exceeds the saturation limiter as \( \gamma(t) \):
\[
\gamma(t) = \begin{cases} 
  u_r - u(t), & u(t) \geq u_r \\
  0, & u_l < u(t) < u_r \\
  u_l - u(t), & u(t) \leq u_l.
\end{cases} \quad (12)
\]
For the target, let \( x_1(t) \) track a known smooth signal \( x_d(t) \in \mathbb{R} \) in finite time. In this paper, we have used the backstepping method. Define \( \epsilon_1(t) = x_1(t) - x_d(t) \), and let us construct a virtual input \( \alpha(t) \), giving us
\[
\begin{align*}
\mathcal{D}^\alpha_0 \epsilon_1(t) &= x_2(t) - \mathcal{D}^\alpha_0 x_d(t) \\
&= \alpha(t) + \dot{\alpha}(t) - \alpha(t) + x_2(t) - \dot{\alpha}(t) - \mathcal{D}^\alpha_0 x_d(t) \\
&= \alpha(t) + \dot{\alpha}(t) - \alpha(t) + \epsilon_2(t) - \mathcal{D}^\alpha_0 x_d(t)
\end{align*}
\]
where \( \beta_2(t) \) is given later, \( k_1, c_1 > 0 \), and \( \beta(0) = 0 \). Using Lemma 4, we can estimate \( \alpha(t) \) and \( \mathcal{D}^\alpha_0 \alpha(t) \) as
\[
\begin{align*}
\mathcal{D}^\alpha_0 \pi_1(t) &= F(t), \\
\mathcal{D}^\alpha_0 \pi_2(t) &= -b_1 \text{sign}(\pi_1(t) - \alpha(t)) + \pi_2(t), \\
\mathcal{D}^\alpha_0 \dot{\pi}_2(t) &= -b_2 \text{sign}(\pi_2(t) - F(t)).
\end{align*}
\]
Thus, (15) and Lemma 4 imply that \( \dot{\alpha}(t) = \pi_1(t) \) and \( \mathcal{D}^\alpha_0 \dot{\alpha}(t) = F(t) \) within finite time. Let
\[
\begin{align*}
\dot{\epsilon}_1(t) &= \epsilon_1(t) - \beta_1(t), \\
\dot{\epsilon}_2(t) &= \epsilon_2(t) - \beta_2(t),
\end{align*}
\]
where \( \epsilon_2(t) = x_2(t) - \dot{\alpha}(t) \). Then the virtual signal is designed as
\[
\alpha(t) = -k_1 \epsilon_1(t) + \mathcal{D}^\alpha_0 x_d(t) - a_1 \epsilon_1^2(t), \quad (17)
\]
with \( k_1 \in \mathbb{R}^+ \), \( v \in (0, 1) \). Define \( V_1 = \frac{1}{2} \epsilon_1^2(t) \), according to Lemma 1, and its fractional-order derivative is
\[
\mathcal{D}^\alpha_0 V_1 \leq \dot{\epsilon}_1(t) \mathcal{D}^\alpha_0 \epsilon_1(t) = \epsilon_1(t) \left[ u(t) + \alpha(t) + a_1 \epsilon_1^2(t) + \epsilon_2(t) + k_1 \beta_1(t) - \dot{\alpha}(t) + \alpha(t) - \beta_2(t) + c_1 \text{sign}(\beta_1(t)) \right]
\]
\[
\leq \epsilon_1(t) \left[ -k_1 \epsilon_1(t) - a_1 \epsilon_1^2(t) + \epsilon_2(t) + k_1 \beta_1(t) - \beta_2(t) + c_1 \text{sign}(\beta_1(t)) \right]
\]
\[
= -k_1 \epsilon_1^2(t) + a_1 \epsilon_1^2(t) \epsilon_1^2(t) + \epsilon_1(t) \epsilon_2(t) - \epsilon_1(t) \epsilon_2(t) + c_1 \epsilon_1(t) \text{sign}(\beta_1(t))
\]
\[
= -k_1 \epsilon_1^2(t) + a_1 \epsilon_1^2(t) \epsilon_1^2(t) + \epsilon_1(t) \epsilon_2(t) + c_1 \epsilon_1(t) \text{sign}(\beta_1(t)).
\]
It follows from (10), (11), (12), and (16) that
\[
\mathcal{D}^\alpha_0 \dot{\epsilon}_2(t) \leq \text{sat}(u(t)) + d(t) + f(x(t)) - \mathcal{D}^\alpha_0 \dot{\alpha}(t) - \mathcal{D}^\alpha_0 \beta_2
\]
\[
= u(t) + \gamma(t) + d(t) + f(x(t)) - \mathcal{D}^\alpha_0 \dot{\alpha}(t) - \mathcal{D}^\alpha_0 \beta_2
\]
\[
= u(t) + \Theta(t) - \mathcal{D}^\alpha_0 \dot{\alpha}(t) - \mathcal{D}^\alpha_0 \beta_2
\]
with \( \Theta(t) = \gamma(t) + d(t) + f(x(t)) \), \( \mathcal{D}^\alpha_0 \dot{\alpha}(t) \) being driven from (15). The unknown function \( \Theta(x) \) in (19) can be approximated by the NN as
\[
\dot{\Theta}(t) = \theta^T(t) \chi(x(t)). \quad (20)
\]
Let the optimal parameter of NN be \( \theta^* = \arg \min_{\theta(t)} \left[ \sup_{x(t)} |\dot{\Theta}(t) - \Theta(t)| \right] \). Define \( \tilde{\theta}(t) = \theta(t) - \theta^* \), and \( \epsilon(t) = \dot{\Theta}(t) - \Theta(t) \). In fact, according to universal approximation theorem of the NN, we know that, for any continuous non-linear function defined on a compact set, there is a NN in order for the optimal to be as small as possible [16, 26, 27]. Thus, it is possible
for us to assume the optimal estimation error is bounded, i.e., 
\[ |\epsilon| \leq \tilde{\epsilon}, \text{ where } \tilde{\epsilon} \in \mathbb{R}^+ \] is a constant. We then have
\[
\hat{\theta}(t) - \Theta(t) = \theta(t)^T \chi(x(t)) - \theta(t)^* T \chi(x(t)) + \theta^* T \chi(x(t)) - \Theta(t) = \tilde{\theta}^T(t) \chi(x(t)) - \epsilon(t). \tag{21}
\]

To meet the control objective, we can design the compensated signal as
\[
D^{\gamma}_t \beta_2(t) = -k_2 \beta_2(t) - \beta_1(t) - c_2 \text{sign}(\beta_2(t)) \tag{22}
\]
with \( k_2, c_2 > 0 \). Then, let us construct the final input as
\[
u(t) = -k_2 \hat{e}_2(t) + D^{\gamma}_t \tilde{\theta}(t) - \tilde{\theta}^T(t) \chi(x(t)) - \sigma \text{sign}(\tilde{e}_2(t)) - \tilde{a}_2 \tilde{e}_1(t) \tag{23}
\]
where \( \sigma, a_2 > 0 \), and \( \sigma \geq \tilde{\epsilon} \) can be satisfied. It follows from (22) and (23) into (19) that
\[
D^{\gamma}_t \tilde{e}_2(t) = -k_2 \hat{e}_2(t) - \tilde{\theta}^T(t) \chi(x(t)) - \sigma \text{sign}(\tilde{e}_2(t)) - \tilde{a}_2 \tilde{e}_1(t) \tag{24}
\]
\[
= -k_2 \hat{e}_2(t) - \tilde{\theta}^T(t) \chi(x(t)) + \epsilon(t) - \sigma \text{sign}(\tilde{e}_2(t)) - \tilde{a}_2 \tilde{e}_1(t)
\]
\[
= -k_2 \hat{e}_2(t) + k_2 \beta_2(t)
\]
\[
- \hat{\beta}_1(t) + c_2 \text{sign}(\beta_2(t))
\]
\[
= -k_2 \hat{e}_2(t) - \tilde{\theta}^T(t) \chi(x(t)) + \epsilon(t) - \sigma \text{sign}(\tilde{e}_2(t)) - \tilde{a}_2 \tilde{e}_1(t)
\]
\[
- a_2 \tilde{e}_2(t) - \tilde{\beta}_1(t) + c_2 \text{sign}(\beta_2(t)).
\]

Then, (24) implies
\[
\hat{e}_2(t) D^{\gamma}_t \hat{e}_2(t) = -k_2 \hat{e}_2^2(t) - \hat{e}_2(t) \tilde{\theta}^T(t) \chi(x(t)) + \hat{e}_2(t) \epsilon(t)
\]
\[
- \sigma \text{sign}(\hat{e}_2(t)) - \hat{e}_2(t) \hat{\beta}_1(t) + c_2 \text{sign}(\beta_2(t))
\]
\[
\leq -k_2 \hat{e}_2^2(t) - \hat{e}_2(t) \tilde{\theta}^T(t) \chi(x(t)) + \hat{e}_2(t) \tilde{\epsilon}
\]
\[
- \sigma |\hat{e}_2(t)| - \hat{e}_2(t) \hat{\beta}_1(t) + c_2 \text{sign}(\beta_2(t))
\]
\[
\leq -k_2 \hat{e}_2^2(t) - \hat{e}_2(t) \tilde{\theta}^T(t) \chi(x(t)) - a_2 \hat{e}_2(t) \tilde{e}_1(t)
\]
\[
- \hat{e}_2(t) \hat{\beta}_1(t) + c_2 \text{sign}(\beta_2(t)). \tag{25}
\]

Define
\[
V_2(t) = V_1(t) + \frac{1}{2} \hat{e}_2^2(t). \tag{26}
\]

According to (18), (25), and (26), we have
\[
D^\gamma_t V_2(t) = -k_1 \hat{e}_2^2(t) - a_1 \hat{e}_1(t) \hat{e}_1^2(t) + c_1 \hat{e}_1(t) \text{sign}(\beta_1(t))
\]
\[
- k_2 \hat{e}_2^2(t) - \hat{e}_2(t) \tilde{\theta}^T(t) \chi(x(t))
\]
\[
- \hat{e}_2(t) \text{sign}(\beta_2(t))
\]
\[
\leq -k_1 \hat{e}_2^2(t) - a_1 \hat{e}_1^2(t) + c_1 \hat{e}_1(t) \text{sign}(\beta_1(t))
\]
\[
- k_2 \hat{e}_2^2(t) - \hat{e}_2(t) \tilde{\theta}^T(t) \chi(x(t))
\]
\[
- \hat{e}_2(t) \text{sign}(\beta_2(t)) \tag{27}
\]

The fractional-order adaptation law is
\[
D^\gamma_t \hat{\theta}(t) = \kappa_1 \hat{e}_2(t) \chi(x(t)) - \kappa_2 \hat{\theta}(t) \tag{28}
\]
with \( \kappa_1, \kappa_2 > 0 \).

The following theorem provides a conclusion for the discussion.

**Theorem 1.** Let the immediate controller be (17) with the fractional filter (15). Let the compensated signal be (14) and (22). Then, the NN controller (23) with adaptation law (28) drive \( e_1(t) \) to be arbitrary small in finite time.

**Proof.** Let
\[
V(t) = V_2(t) + \frac{1}{2k_1} \tilde{\theta}^T(t) \hat{\theta}(t). \tag{29}
\]

Then, based on (27), (28), and (29), we obtain
\[
D^\gamma_t V(t) \leq \sum_{j=1}^2 \left[ -k_j \hat{e}_2^2(t) - a_j \hat{e}_1^2(t) + c_j \hat{e}_1(t) \text{sign}(\beta_j(t)) \right]
\]
\[
- e_2 \tilde{\theta}^T(t) \chi(x(t)) + \frac{1}{k_1} \tilde{\theta}^T(t) D^\gamma \hat{\theta}(t)
\]
\[
= \sum_{j=1}^2 \left[ -k_j \hat{e}_2^2(t) - a_j \hat{e}_1^2(t) + c_j \hat{e}_1(t) \text{sign}(\beta_j(t)) \right]
\]
\[
- e_2 \tilde{\theta}^T(t) \chi(x(t)) + \frac{1}{k_1} \tilde{\theta}^T(t) D^\gamma \hat{\theta}(t)
\]
\[
\leq \sum_{j=1}^2 \left[ -2k_j \hat{e}_2^2(t) - a_j \hat{e}_1^2(t) + c_j \hat{e}_1(t) \text{sign}(\beta_j(t)) \right]
\]
\[
+ \sum_{j=1}^{\gamma} \frac{\gamma}{2} \tilde{\epsilon}_j(t) \tilde{\epsilon}_j(t) \tag{30}
\]
Then, (30) implies

\[
\mathcal{D}_t^q V(t) \leq \sum_{j=1}^{2} \left[ -\frac{2k_j - c_j}{2} \hat{\theta}_j^2(t) - a_j \hat{\theta}_j^{v+1}(t) \right] - \frac{k_2}{2} \beta^T(t) \tilde{\theta}(t) \tilde{\theta}(t) + \kappa_2 \theta^* \theta^* + \sum_{j=1}^{2} \frac{c_j}{2} \beta^T(t) \tilde{\theta}(t) \tilde{\theta}(t) \right]^{\frac{1}{2}(v+1)}.
\]

Substituting (34) into (31) yields

\[
\mathcal{D}_t^q V(t) \leq \sum_{j=1}^{2} \left[ -\frac{2k_j - c_j}{2} \hat{\theta}_j^2(t) - a_j \hat{\theta}_j^{v+1}(t) \right] - \frac{k_2}{2} \beta^T(t) \tilde{\theta}(t) \tilde{\theta}(t) + \left[ 1 + \kappa_2 \theta^* \theta^* + \sum_{j=1}^{2} \frac{c_j}{2} \beta^T(t) \tilde{\theta}(t) \tilde{\theta}(t) \right]^{\frac{1}{2}(v+1)}.
\]

with \( \varsigma_1 = \min \left\{ \frac{2k_1 - c_1}{2}, \frac{2k_2 - c_2}{2}, \kappa_{\min} \right\} \), \( \varsigma_2 = \min \left\{ \frac{2}{\nu a_1}, \frac{2}{\nu a_2}, \kappa_{\min} \right\} \), and \( \varsigma_3 = 1 + \kappa_2 \theta^* \theta^* + \sum_{j=1}^{2} \frac{c_j}{2} \beta^T(t) \tilde{\theta}(t) \tilde{\theta}(t) \). As a result, (35) can be arranged as

\[
\mathcal{D}_t^q V(t) \leq - \left( \varsigma_1 - \frac{\varsigma_3}{2} \right) V(t) - \left( \varsigma_2 - \frac{\varsigma_3}{2} \right) V(t)^{\frac{1}{2}(v+1)} + \varsigma_3.
\]

According to (36) and Lemma 3, when \( k > \frac{1}{2} c, e_1(t) \) will tend to the region

\[
|e_1(t)| \leq \max \left\{ \sqrt{\frac{\varsigma_3}{\varsigma_1}}, \frac{\varsigma_3}{3\varsigma_2} \right\}
\]

in some finite time. Since \( e_1(t) = \tilde{e}_1(t) + \beta_1(t), e_2(t) = \tilde{e}_2(t) + \beta_2(t) \), if \( \beta_1(t) \) and \( \beta_2(t) \) are bounded, then all signals are bounded. Let \( V_3(t) = \frac{2}{\nu} \beta_1^2(t) + \frac{1}{2} \beta_2^2(t) \). Then, (14) and (22) imply

\[
\mathcal{D}_t^q V_3(t) \leq \beta_1(t) \mathcal{D}_t^q \beta_1(t) + \beta_2(t) \mathcal{D}_t^q \beta_2(t)
\]

\[
= -k_1 \beta_1^2(t) + \beta_1(t)(\alpha(t) - \alpha(t)) + \beta_1(t) \beta_2(t)
\]

\[
- c_1 \beta_1(t) \max(\beta_1(t)) - k_2 \beta_2^2(t)
\]

\[
- \beta_2(t) \beta_1(t) - c_2 \beta_2(t) \max(\beta_2(t))
\]

\[
\leq - \sum_{j=1}^{2} \frac{k_j \beta_j^2(t) - \sum_{j=1}^{2} \frac{c_j \beta_j(t)}{2}}{2} - \rho_j \beta_j \tilde{\alpha}(t)
\]

\[
- \alpha(t) \leq - \sum_{j=1}^{2} \frac{k_j \beta_j^2(t) - \sum_{j=1}^{2} \frac{c_j \beta_j(t)}{2}}{2} - \rho_j \beta_j \tilde{\alpha}(t),
\]

where \( \tilde{\alpha}(t) = \tilde{\alpha}(t) - \alpha(t) \). Then, it follows from (15) and Lemma 4 that \( \tilde{\alpha}(t) \) is bounded in finite time. As a result, we have

\[
\mathcal{D}_t^q V_3(t) \leq - \sum_{j=1}^{2} \frac{k_j \beta_j^2(t) - \sum_{j=1}^{2} \frac{c_j \beta_j(t)}{2}}{2} + \sum_{j=1}^{2} \rho_j \delta |\beta_j(t)|
\]

\[
\leq k \beta_3(t) - \epsilon \sqrt{V_3(t)} + \rho \delta \sqrt{V_5(t)}
\]

\[
= k V_3(t) - (\epsilon - \rho \delta) \sqrt{V_3(t)},
\]

where \( k = 2 \min(k_1, k_2), \epsilon = \min(c_1, c_2), \rho = \max(r_1, r_2) \) and \( \delta = \max(\delta_1, \delta_2) \). Thus, (38) and Lemma 3 imply that \( \beta_1(t) \) and \( \beta_2(t) \) are finite time bounded. This concludes our proof.
Remark 1. In this paper, the finite-control of fractional-order Duffing system was considered. It can be seen from the system model (10) that the non-linear function $f(x)$ is Lipschitz continuous. In addition, under the proposed controller (23), for any initial condition, the solution to the fractional-order Duffing system exists and is unique. In addition, from Theorem 1, it is obvious that all the signals in the closed loop system keep bounded. Thus, the solution of the controlled system (10) is stable.

Remark 2. It should be emphasized that the proposed fractional-order finite-time filter has very convergence ability compared with some related works, such as Liu et al. [16] and Ha et al. [25], where only the following class of lower filter (the order of the filter lying on $[0,1]$) is used:

$$\mathcal{D}_t^\alpha z(t) = \frac{1}{k} (z(t) - \alpha(t)),$$  

(39)
5. CONCLUSIONS

This paper addressed the finite time control of an unknown disturbed FODS in the presence of input saturation. By using the backstepping technique, a high order fractional filter with the order lying on (1,2) is proposed, and thus, the virtual input and its fractional derivative can be approximated. It is proven that the filter’s approximation error can be enough small and can converge to the small region in some finite time. Then, an adaptive NN controller is given. The stability is proven strictly. In addition, the robustness of the proposed method is shown in simulation results. Our future research directions including how to design sliding mode surface for FODS and how to construct a high-order filter.

DATA AVAILABILITY STATEMENT

All datasets generated for this study are included in the article/supplementary material.

AUTHOR CONTRIBUTIONS

HL and XZ contributed the conception of the study. HL wrote the study and organized the literature. XZ wrote the simulation programs. All authors contributed to the manuscript revision, read, and approved the submitted version.

FUNDING

This work was supported by the Basic Ability Promotion Project for Young and Middle-aged Teachers of Guangxi Colleges and Universities (2020KY04022), the Key Natural Science Research Projects in Anhui Universities (KJ2019A0695), and the Xiangshui Young Scholars Innovative Research Team of Guangxi University for Nationalities (2019RSCXSHQN02).

REFERENCES

1. Monje CA, Chen Y, Vinagre BM, Xue D, Feliz-Valle J. Fractional-Order Systems and Controls: Fundamentals and Applications. London: Springer Science & Business Media Verlag (2010).

2. Luo Y, Chen YQ, Wang CY, Pi YG. Tuning fractional order proportional integral controllers for fractional order systems. J Prog Control. (2010) 20:823–31. doi: 10.1016/j.jprocont.2010.04.011

3. Munoz-Pacheco J, Zambrano-Serrano E, Volos C, Tacha O, Stouboulos I, Pham VT. A fractional order chaotic...
system with a 3D grid of variable attractors. *Chaos Solit Fract.* (2018) **113**:69–78. doi: 10.1016/j.chaos.2018.05.015

4. Liu H, Pan Y, Cao J. Composite learning adaptive dynamic surface control of fractional-order nonlinear systems. *IEEE Trans Cybern.* (2019). doi: 10.1109/TCYB.2019.2938754. [Epub ahead of print].

5. Tavazoei MS, Haeri M. Chaotic attractors in incommensurate fractional order systems. *Phys D Nonlin Phenom.* (2008) **237**:2628–37. doi: 10.1016/j.physd.2008.03.037

6. Liu H, Pan Y, Cao J, Zhou Y, Wang H. Positivity and stability analysis for fractional-order delayed systems: a T-S fuzzy model approach. *IEEE Trans Fuzzy Syst.* (2020). doi: 10.1109/TFUZZ.2020.2966420. [Epub ahead of print].

7. Liu H, Pan Y, Li S, Chen Y. Adaptive fuzzy backstepping control of fractional-order nonlinear systems. *IEEE Trans Syst Man Cybern Syst.* (2017) **47**:2209–17. doi: 10.1109/TSMC.2016.2640950

8. Gallegos JA, Aguila-Camacho N, Duarte-Mermoud M. Vector Lyapunov-like functions for multi-order fractional systems with multiple time-varying delays. *Commun Nonlin Sci Numer Simul.* (2020) **83**:105089. doi: 10.1016/j.cnsns.2019.105089

9. Tlelo-Cuautle E, Pano-Azucena AD, Guillén-Fernández O, Silva-Juárez A. Synchronization and applications of fractional-order chaotic systems. In: Azar AT, Vaidyanathan S, editors. *Analog/Digital Implementation of Fractional Order Chaotic Circuits and Applications.* Cham: Springer (2020). p. 175–201. doi: 10.1007/978-3-030-31250-3_6

10. Yuan L, Yang Q. Parameter identification of fractional-order chaotic systems without or with noise: reply to comments. *Commun Nonlin Sci Numer Simul.* (2019) **67**:506–16. doi: 10.1016/j.cnsns.2018.07.032

11. Marí S, Chadli M. Robust admissibility and stabilization of uncertain singular fractional-order linear time-invariant systems. *IEEE/CAA J Autom Sinica.* (2019) **6**:685–92. doi: 10.1109/JCAS.2019.1191480

12. Wei Y, Wang J, Liu T, Wang Y. Sufficient and necessary conditions for stabilizing singular fractional order systems with partially measurable state. *J Frank Inst.* (2019) **356**:1975–90. doi: 10.1016/j.jfranklin.2019.01.022

13. Boroujeni EA, Momeni HR. Non-fragile nonlinear fractional order observer design for a class of nonlinear fractional order systems. *Signal Process.* (2012) **92**:2365–70. doi: 10.1016/j.sigpro.2012.02.009

14. Efe MÖ. Fractional order systems in industrial automation-a survey. *IEEE Trans Ind Inform.* (2011) **7**:582–91. doi: 10.1109/TII.2011.2166775

15. Boukroune A, Bouzeriba A, Bouden T, Azar AT. Fuzzy adaptive synchronization of uncertain fractional-order chaotic systems. In: Azar AT, Vaidyanathan S, editors. *Advances in Chaos Theory and Intelligent Control.* Cham: Springer (2016). p. 681–97. doi: 10.1007/978-3-319-30340-6_28

16. Liu H, Pan Y, Cao J, Wang H, Zhou Y. Adaptive neural network backstepping control of fractional-order nonlinear systems with actuator faults. *IEEE Trans Neural Netw Learn Syst.* (2020). doi: 10.1109/TNNLS.2020.2964044. [Epub ahead of print].

17. Bouzeriba A, Bouklouane A, Bouden T. Projective synchronization of two different fractional-order chaotic systems via adaptive fuzzy control. *Neural Comput Appl.* (2016) **27**:1349–60. doi: 10.1007/s00521-015-1938-4

18. Mirzajani S, Aghababa MP, Heydari A. Adaptive control of nonlinear fractional-order systems using T-S fuzzy method. *Int J Mach Learn Cybern.* (2019) **10**:527–40. doi: 10.1007/s13042-017-0733-1

19. Pham VT, Ouannas A, Volos C, Kapitaniak T. A simple fractional-order chaotic system without equilibrium and its synchronization. *AEU Int J Electron Commun.* (2018) **86**:69–76. doi: 10.1016/j.aeue.2018.01.023

20. Zhang W, Cao J, Wu R, Aisad FE, Aisad A. Lag projective synchronization of fractional-order delayed chaotic systems. *J Frank Inst.* (2019) **356**:1522–34. doi: 10.1016/j.jfranklin.2018.10.024

21. Thanh NT, Phat VN. Improved approach for finite-time stability of nonlinear fractional-order systems with interval time-varying delay. *IEEE Trans Circuits Syst II Express Briefs.* (2018) **66**:1356–60. doi: 10.1109/TCSII.2018.2880777

22. Yang S, Yu J, Hu C, Jiang H. Finite-time synchronization of memristive neural networks with fractional-order. *IEEE Trans Syst Man Cybern Syst.* (2019). doi: 10.1109/TSMC.2019.2931046. [Epub ahead of print].

23. Liang J, Wu B, Liu L, Wang YE, Li C. Finite-time stability and finite-time boundedness of fractional order switched systems. *Trans Inst Meas Control.* (2019) **41**:3364–71. doi: 10.1177/0142331219826333

24. Ha S, Liu H, Li S. Adaptive fuzzy backstepping control of fractional-order chaotic systems with input saturation. *J Intell Fuzzy Syst.* (2019) **37**:6513–25. doi: 10.3233/JIFS-182623

25. Ha S, Liu H, Li S, Liu A. Backstepping-based adaptive fuzzy synchronization control for a class of fractional-order chaotic systems with input saturation. *Int J Fuzzy Syst.* (2019) **21**:1571–84. doi: 10.1007/s40815-019-00663-5

26. Xu W, Hu G, Ho DW, Feng Z. Distributed secure cooperative control under denial-of-service attacks from multiple adversaries. *IEEE Trans Cybern.* (2019). doi: 10.1109/TCSII.2019.2896160. [Epub ahead of print].

27. Wang P, Wen G, Yu X, Yu W, Yan Y. Synchronization of resilient complex networks under attacks. *IEEE Trans Syst Man Cybern Syst.* (2019). doi: 10.1109/TSMC.2019.2895027. [Epub ahead of print].

Conflict of Interest: The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

Copyright © 2020 Lv and Zhang. This is an open-access article distributed under the terms of the Creative Commons Attribution License (CC BY). The use, distribution or reproduction in other forums is permitted, provided the original author(s) and the copyright owner(s) are credited and that the original publication in this journal is cited, in accordance with accepted academic practice. No use, distribution or reproduction is permitted which does not comply with these terms.