FAST TRACK COMMUNICATION

Comment on star–star relations in statistical mechanics and elliptic gamma-function identities

Vladimir V Bazhanov\textsuperscript{1,2}, Andrew P Kels\textsuperscript{1} and Sergey M Sergeev\textsuperscript{3}

\textsuperscript{1} Department of Theoretical Physics, Research School of Physics and Engineering, Australian National University, Canberra, ACT 0200, Australia
\textsuperscript{2} Mathematical Sciences Institute, Australian National University, Canberra, ACT 0200, Australia
\textsuperscript{3} Faculty of Education Science Technology and Mathematics, University of Canberra, Bruce, ACT 2601, Australia

E-mail: Vladimir.Bazhanov@anu.edu.au

Received 18 February 2013
Published 3 April 2013
Online at stacks.iop.org/JPhysA/46/152001

Abstract

We prove a recently conjectured star–star relation, which plays the role of an integrability condition for a class of 2D Ising-type models with multicomponent continuous spin variables. Namely, we reduce this relation to an identity for elliptic gamma functions, previously obtained by Rains.

PACS numbers: 02.30.Ik, 68.43.De

1. Introduction

Recently, two of us [1, 2] introduced a new class of exactly solvable 2D lattice models of statistical mechanics, which involve continuous spin variables taking values on a circle. The interest in these models is motivated by various applications. In statistical mechanics they serve as rather general ‘master models’, containing many important particular limits, such as the Ising, chiral Potts [3, 4], Kashiwara–Miwa [5], Faddeev–Volkov [6, 7] and other models. Mathematically, the new models are deeply related to the theory of elliptic hypergeometric functions [8]. For instance, the celebrated elliptic beta integral [9], which lies at the basis of this theory, is shown [1] to be a Yang–Baxter (star–triangle) relation, defining perfectly physical integrable lattice models of statistical mechanics. Other interesting connections are discussed in [10–13]. We mention, in particular, that the elliptic gamma functions arise in calculations of superconformal indices connected with electric–magnetic dualities in 4D $\mathcal{N} = 1$ superconformal Yang–Mills theories [14]. Most remarkably, as recently discovered in [15–17], the superconformal indices in 4D superconformal quiver gauge theories precisely coincide with partition functions of the 2D lattice ‘master models’ [1, 2] discussed here. Interestingly, in this correspondence the Seiberg duality for the superconformal indices reduces to Baxter’s $Z$-invariance [18] for the partition function under (generalized) ‘star–triangular moves’ of the 2D lattice.
Here, we resolve an outstanding question for these models concerning the so-called star–star relation, conjectured in [2]. This relation serves as an integrability condition, as it implies the Yang–Baxter equation, the $Z$-invariance of the partition function and the commutativity of row-to-row transfer matrices. In this communication we completely prove this star–star relation by reducing it to a transformation formula for elliptic hypergeometric integrals, previously obtained by Rains [19].

2. Edge-interaction model with continuous spins

In this section, we formulate the star–star relation conjectured in [2] (see (14) below). First, we need to briefly describe the associated two-dimensional edge-interaction model; full details can be found in [2]. Consider the regular square lattice, drawn diagonally as in figure 1. The edges of the lattice are shown with bold lines and the sites are shown with either open or filled circles in a checkerboard order. At the moment we will not distinguish these two types of sites. At each lattice site place a $n$-component continuous spin variable $x = \{x_1, \ldots, x_n\} \in \mathbb{R}^n$, $0 \leq x_j < \pi$, $\sum_{j=1}^n x_j = 0 \pmod{\pi}$.

Note that due to the restriction on the total sum, there are only $(n-1)$ independent variables $x_j$. For further reference define the integration measure

$$\int dx = \int_0^\pi \cdots \int_0^\pi dx_1 \cdots dx_{n-1},$$

over the space of states of a single spin. Figure 1 also shows an auxiliary medial lattice whose sites lie on the edges of the original square lattice. The medial lattice is drawn with alternating thin and dotted lines. The lines are directed as indicated by arrows. To each horizontal (vertical) line on the medial lattice assign a rapidity variable $u (v)$. In general, these variables may be different for different lines. However, a convenient level of generality that we shall use here is

Figure 1. The square lattice shown with bold sites and bold edges drawn diagonally. The associated medial lattice is drawn with thin and dotted horizontal and vertical lines. The lines are oriented and carry rapidity variables $u, u', v$ and $v'$. 
to assign the same rapidity $u$ to all thin horizontal lines and the same variable $u'$ to all dotted horizontal lines. Similarly, assign the variables $v$ and $v'$ to thin and dotted vertical lines as indicated in figure 1.

Two spins interact only if they are connected by an edge. To define the Boltzmann weights we need to introduce the elliptic gamma function [8]. Let $q$, $p$ be two elliptic nomes (they play the role of the temperature-like parameters),

$$p = e^{i\pi \sigma}, \quad q = e^{i\pi \tau}, \quad \text{Im} \sigma > 0, \quad \text{Im} \tau > 0,$$

and

$$\eta = -i\pi (\sigma + \tau)/2$$

denote the 'crossing parameter'. Define the elliptic gamma function

$$\Phi_1(z) = \prod_{j,k=0}^{\infty} \frac{1 - e^{2i\pi q^{j+1}p^{k+1}}}{1 - e^{-2i\pi q^{j+1}p^{k+1}}} = \exp \left\{ \sum_{k \neq 0} \frac{e^{-2i\pi k \eta}}{k(q^k - q^{-k})(p^k - p^{-k})} \right\},$$

where the product formula is valid for all $z$, while the exponential formula is only valid in the strip

$$-\text{Re} \eta < \text{Im} z < \text{Re} \eta.$$

The function (5) possesses simple periodicity and 'reflection' properties

$$\Phi(z + \pi \tau) = \Phi(z), \quad \Phi(z) \Phi(-z) = 1.$$

Each edge is assigned a Boltzmann weight, which depends on spins at the ends of the edge and on two rapidities passing through the edge. There are four types of edges differing by orientations and types of the directed rapidity lines passing through the edge. They are assigned different Boltzmann weights as shown in figure 2. These weights are defined as

$$\mathbb{W}_u(x, y) = \kappa_u(\alpha)^{-1} \prod_{j,k=1}^{n} \Phi(x_j - y_k + i\alpha), \quad \mathbb{W}_{u'}(x, y) = \sqrt{S(x)S(y)} \mathbb{W}_{u'-\alpha}(x, y),$$

where $\kappa_u(\alpha)$ is a normalization factor. The single-spin function $S$ is given by

$$S(x) = \kappa_s^{-1} \prod_{j,k=1}^{n} \left( \Phi(x_j - x_k + i\eta) \right)^{-1}, \quad \kappa_s = n! \left( \frac{\pi}{G(q) G(p)} \right)^{-n-1},$$

4 Our function $\Phi(z)$ coincides with $\Gamma(e^{-2i\pi(-\eta)}; p^2, q^2)$ in the notation of [8].
where the indices \( j, k \) run over the values 1, 2, \ldots, \( n \) and

\[
G(z) = \prod_{k=1}^{\infty} (1 - z^{2k}).
\]

The partition function is defined as

\[
Z = \int \prod_{\langle xy \rangle} \mathbb{W}_{uv}(x,y) \prod_{\langle yx \rangle} \mathbb{W}_{uv'}(y,x) \prod_{\langle xy \rangle} \mathbb{W}_{u'v'}(y,x) \prod_{\text{sites}} dx,
\]

where the four products are taken, respectively, over the four types of edges shown in figure 2. The integral is taken over all configurations of the spin variables on the internal lattice sites. The boundary spins are kept fixed.

Note that the lattice in figure 1 can be formed by periodic translations of a four-edge star, consisting of four edges meeting at the same site. A little inspection shows that there are only two different types of such stars shown in figure 3. They are either centred around ‘white’ sites, shown with open circles, or around ‘black’ sites, shown with filled circles. Applying the rules of figure 2, one can write Boltzmann weights corresponding to these stars

\[
\mathbb{V}^{(1)}_{av}(a \ b \ c \ d) = \int dx \ \mathbb{W}_{u-v}(c,x) \ \mathbb{W}_{u'-v'}(b,x) \ \mathbb{W}_{u'-v}(x,a) \ \mathbb{W}_{u-v}(x,d),
\]

and

\[
\mathbb{V}^{(2)}_{av}(a \ b \ c \ d) = \int dy \ \mathbb{W}_{u-v}(y,b) \ \mathbb{W}_{u'-v'}(y,c) \ \mathbb{W}_{u'-v}(d,y) \ \mathbb{W}_{u-v}(a,y),
\]

where the bold symbols \( u = [u, u'] \) and \( v = [v, v'] \) stand for the rapidity pairs. It turns out that the above two expressions are simply connected to each other,

\[
\mathbb{W}_{v-v'}(d,c) \ \mathbb{W}_{u-u'}(d,b) \ \mathbb{V}^{(1)}_{av}(a \ b \ c \ d) = \mathbb{W}_{v-v'}(b,a) \ \mathbb{W}_{u-u'}(c,a) \ \mathbb{V}^{(2)}_{av}(a \ b \ c \ d).
\]

This is precisely the star–star relation conjectured in [2] (in the same paper it was also verified in a few orders of perturbation theory in the parameters \( p \) and \( q \)). Below we will give a complete proof of (14) by reducing it to a mathematical identity, previously obtained by Rains [19]. Apparently, the above star–star relation is the simplest condition for the Boltzmann weights which ensures the integrability of the considered model. In particular, it implies the commutativity of the row-to-row transfer matrices.

\[5\] For \( n = 2 \), the star–star relation (14) is just a consequence of the star–triangle relation, equation (1.5) of [1], which is equivalent to the elliptic beta integral [10, 9]. However, for \( n \geq 3 \), the corresponding star–triangle relation apparently does not exist (at least it is not known to the authors) and the star–star relation (14) seems to be the simplest relation of this type.
3. Proof of the star–star relation

Here we will use the standard notation for the elliptic gamma function, which is simply related to our definition (5),

$$\Gamma(z; p^2, q^2) = \prod_{j,k=0}^{\infty} \frac{(1 - p^{2j+2}q^{2k+2}/z)}{(1 - p^{2j}q^{2k}z)}, \quad \Phi(x) = \Gamma(pq e^{-2ix}; p^2, q^2), \quad pq = e^{-2\eta}.$$  

(15)

In the following we will omit the nome arguments $p^2$ and $q^2$, assuming that $\Gamma(z) \equiv \Gamma(z; p^2, q^2)$. Following Rains [19], introduce the following elliptic hypergeometric integral$^6$:

$$I_{\alpha s_{n-1}}((t_j), (s_i)) \equiv \frac{1}{n!} \left(\frac{G(p)G(q)}{2\pi i}\right)^{\alpha - 1} \int_{|z_i| = 1} \frac{\prod_{j,k=1}^{n} \Gamma(t_jz_k) \Gamma(s_jz_k)}{\prod_{k \neq s_i} \Gamma(z_k/z_i)} \prod_{k=1}^{\infty} \frac{dz_k}{z_k}, \quad (16)$$

which involves $4n$ independent parameters $\{t_j\} = \{t_1, t_2, \ldots, t_{2n}\}, \quad \{s_i\} = \{s_1, s_2, \ldots, s_{2n}\}, \quad |t_j|, |s_i| < 1, \quad i = 1, \ldots, 2n, \quad (17)$

where $n \geq 2$ and the function $G(z)$ is defined in (10). The indices $k$ and $l$ in the denominator of (16) run over the values $1, \ldots, n$. All integrations are taken over the unit circles $|z_k| = 1$, and the variable $z_n$ is determined by the constraint

$$z_1 z_2 \cdots z_n = 1. \quad (18)$$

The theorem 4.1 of [19] (where we set $m = n$ and $Z = 1$) states the following transformation formula:

$$I_{\alpha s_{n-1}}((t_j), (s_i)) = \left(\prod_{j,k=1}^{n} \Gamma(t_j s_k)\right) I_{\alpha s_{n-1}}((\tilde{t}_j), (\tilde{s}_i)), \quad (19)$$

where the new parameters $\{\tilde{t}_j\}$ and $\{\tilde{s}_i\}$ in the rhs are given by

$$\tilde{t}_j = T^j t_j^{-1}, \quad \tilde{s}_i = U^i s_i^{-1}, \quad T = \prod_{j=1}^{2n} t_j, \quad U = \prod_{j=1}^{2n} s_j, \quad i = 1, 2, \ldots, 2n. \quad (20)$$

Consider now the star weight (12) and make a change of variables $z_j = e^{2ivj}$ and $t_j = e^{-2(u-v) - 2ivj}$, $t_{n+j} = e^{-2(u-v) - 2ivj}$, $s_j = e^{2(u-v) - 2ivj}$, $s_{n+j} = e^{2(u-v) - 2ivj}$.

(21)

where $a_j, b_j, c_j, d_j$ are the components of the spin variables and $j = 1, \ldots, n$. Note that due to (1) the new variables $z_j$ obey the constraint (18). Now taking into account that (9) can be written as

$$S(x) = \kappa_x^{-1} \left(\prod_{j \neq k} \Gamma(z_j/z_k)\right)^{-1}, \quad z_j = e^{2ivj}, \quad (22)$$

it is not difficult to check that

$$\eta^{(1)}_{uv}(a \quad b \quad c \quad d) = \vartheta I_{\alpha s_{n-1}}((t_j), (s_i)), \quad (23)$$

where

$$\vartheta = \sqrt{S(\bar{c}) S(\bar{b})} \kappa_x(\eta - u + v) \kappa_y(\eta - u' + v') \kappa_x(\eta - u - v') \kappa_y(\eta - u - v) \kappa_x(\eta - u' - v') \kappa_y(\eta - u' - v). \quad (24)$$

$^6$ We follow section 4 of [19], where we set $Z = 1$. Moreover, our indices numerating parameters in (17) start from 1 instead of 0 in [19].
Further, using (15) and the reflection property (7) for the elliptic gamma function, one can re-write the ratio of the $W$-factors entering (14) in the form

$$\frac{W_{v'-v}(b, a)W_{u'-u}(d, c)}{W_{v'-v}(d, c)W_{u'-u}(b, a)} = \prod_{j,k=1}^{2n} \Gamma(t_js_k). \quad (25)$$

Next, substituting (21) into (20) one obtains

$$\tilde{t}_j = e^{-2(u'-v')-2ic_j}, \quad \tilde{t}_{k+j} = e^{-2(u-v)+2ib_j}, \quad \tilde{s}_j = e^{2(u-v'-\eta)-2ia_j}, \quad \tilde{s}_{k+j} = e^{2(u'-v-\eta)-2id_j}. \quad (26)$$

Consider now the star weight (13) and make a change of variables $z_j = e^{-2i\nu_j}$ (note the minus sign in the exponent). Using the variables (26) one obtains

$$\psi^{(1,2)}_{uv}(a \ b \ c \ d) = \rho \Gamma^{(n-1)}(\tilde{a}_j, \tilde{a}_j), \quad (27)$$

where $\rho$ is the same as in (24). The relations (23), (25) and (27) immediately imply that the star–star relation (14) is equivalent to the Rains transformation formula (19) for the elliptic hypergeometric integrals, obtained in [19].

Acknowledgments

We are grateful to Hjalmar Rosengren who brought our attention to the work of Rains [19]. The work was partially supported by the Australian Research Council.

References

[1] Bazhanov V V and Sergeev S M 2012 A Master solution of the quantum Yang–Baxter equation and classical discrete integrable equations Adv. Theor. Math. Phys. 19 1–28 (arXiv:1006.0651 [math-ph])

[2] Bazhanov V V and Sergeev S M 2012 Elliptic gamma-function and multi-spin solutions of the Yang–Baxter equation Nucl. Phys. B 856 475–96 (arXiv:1106.5874 [math-ph])

[3] Baxter R J, Perk J H H and Au-Yang H 1988 New solutions of the star triangle relations for the chiral Potts model Phys. Lett. A 128 138–42

[4] Au-Yang H, McCoy B M, Perk J H H, Tang S and Yan M-L 1987 Commuting transfer matrices in the chiral Potts models: solutions of star–triangle equations with genus > 1 Phys. Lett. A 123 219–23

[5] Kashiwara M and Miwa T 1986 A class of elliptic solutions to the star–triangle relation Nucl. Phys. B 275 121–34

[6] Volkov A Y and Faddeev L D 1995 Yang-Baxterization of the quantum dilogarithm Zapiski Nauchn. Semin. POMI 224 146–54

Volkov A Y and Faddeev L D 1998 Yang-Baxterization of the quantum dilogarithm J. Math. Sci. 88 202–7 (Engl. transl.)

[7] Bazhanov V V, Mangazeev V V and Sergeev S M 2007 Faddeev–Volkov solution of the Yang–Baxter equation and discrete conformal symmetry Nucl. Phys. B 784 234–58 (arXiv:hep-th/0703041)

[8] Spiridonov V P 2008 Essays on the theory of elliptic hypergeometric functions Uspekhi Mat. Nauk 63 3–72

[9] Spiridonov V P 2001 On the elliptic beta function Uspekhi Mat. Nauk 56 181–2

[10] Spiridonov V P 2001 On the elliptic beta function Uspekhi Mat. Nauk 56 181–2

[11] Spiridonov V P 2010 Elliptic beta integrals and solvable models of statistical mechanics arXiv:1011.3798 [hep-th]

[12] Kashaev R, Luo F and Vartanov G 2012 A TQFT of Turaev-Viro type on shaped triangulations arXiv:1210.8393 [math.QA]

[13] Chicherin D, Derkachov S and Isaev A 2012 Conformal group: $R$-matrix and star–triangle relation arXiv:1206.4150 [math-ph]

[14] Chicherin D, Derkachov S, Karakhanyan D and Kirschner R 2013 Baxter operators with deformed symmetry Nucl. Phys. B 868 652–83 (arXiv:1211.2965 [math-ph])

[15] Spiridonov V and Vartanov G 2011 Elliptic hypergeometry of supersymmetric dualities: part II. Orthogonal groups, knots, and vortices arXiv:1107.5788 [hep-th]
[15] Yamazaki M 2012 Quivers, YBE and 3-manifolds J. High Energy Phys. JHEP05(2012)147 (arXiv:1203.5784 [hep-th])
[16] Terashima Y and Yamazaki M 2012 Emergent 3-manifolds from 4d superconformal indices Phys. Rev. Lett. 109 091602 (arXiv:1203.5792 [hep-th])
[17] Xie D and Yamazaki M 2012 Network and Seiberg duality J. High Energy Phys. JHEP09(2012)036 (arXiv:1207.0811 [hep-th])
[18] Baxter R J 1978 Solvable eight-vertex model on an arbitrary planar lattice Phil. Trans. R. Soc. A 289 315–46
[19] Rains E M 2010 Transformations of elliptic hypergeometric integrals Ann. Math. 171 169–243