Existence of a martingale solution of the stochastic Hall-MHD equations perturbed by Poisson type random forces on \( \mathbb{R}^3 \)

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November 16, 2022

Abstract

Stochastic Hall-magnetohydrodynamics equations on \( \mathbb{R}^3 \) with random forces expressed in terms of the time homogeneous Poisson random measures are considered. We prove the existence of a global martingale solution. The construction of a solution is based on the Fourier truncation method, stochastic compactness method and a version of the Skorokhod theorem for non-metric spaces adequate for Lévy type random fields.

MSC: primary 35Q35, 35Q30; secondary 60H15, 76M35

Keywords Hall-MHD equations; Poisson random measure, martingale solution, compactness method

1 Introduction

We consider the following stochastic Hall-magnetohydrodynamics system on \([0, T] \times \mathbb{R}^3\) with Poisson type noises

\[
du + \left[ (u \cdot \nabla)u + \nabla p - s (B \cdot \nabla)B + s \nabla \left( \frac{|B|^2}{2} \right) - \nu_1 \Delta u \right] dt = \int_{Y_1} F_1(t, u(t-); y) \, d\tilde{\eta}_1(dt, dy),
\]

\[
dB + \left[ (u \cdot \nabla)B - (B \cdot \nabla)u + \varepsilon \text{curl} \left[ (\text{curl} B) \times B \right] - \nu_2 \Delta B \right] dt = \int_{Y_2} F_2(t, B(t-); y) \, d\tilde{\eta}_2(dt, dy)
\]

with the incompressibility conditions

\[
\text{div } u = 0 \quad \text{and} \quad \text{div } B = 0
\]

and the initial conditions

\[
u(t, x) = u_0 \quad \text{and} \quad B(0) = B_0.
\]

In this problem \( u(t, x) = (u_1, u_2, u_3)(t, x) \), \( B(t, x) = (B_1, B_2, B_3)(t, x) \) for \((t, x) \in [0, T] \times \mathbb{R}^3\), are three-dimensional vector fields representing velocity and magnetic fields, respectively.

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and $p(t, x)$ is a real valued function representing the pressure of the fluid. Random forces are defined as the integrals $\int_{Y_1} F_1(t, u(t^-); y) \, d\tilde{\eta}_1(dt, dy)$ and $\int_{Y_2} F_2(t, B(t^-); y) \, d\tilde{\eta}_2(dt, dy)$ with respect to the compensated time homogeneous Poisson random measures $\tilde{\eta}_1$ and $\tilde{\eta}_2$. The positive constants $\nu_1, \nu_2$ and $s$ stand for kinematic viscosity, resistivity and the Hartmann number, respectively. Let us recall that the curl-operator for a vector field $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $\text{curl} \phi := \nabla \times \phi$. The expression $\varepsilon \text{curl} [(\text{curl} B) \times B]$ represents the Hall-term with the Hall parameter $\varepsilon > 0$. For simplicity we will assume that $s = 1$ and $\varepsilon = 1$.

Mathematical treatment of Hall-MHD equations, which are important in the physics of plasma, were introduced in [1]. Since then this subject receives an increasing attention. Let us mention several papers where deterministic Hall-MHD model is considered, e.g. [11], [12], [13], [14]. Stochastic Hall-MHD equations perturbed by Wiener noise were studied in [40], [41], [42] and [29].

Recently, Yamazaki and Mohan [43] proved the existence and uniqueness of a local in time solution for the stochastic Hall-MHD system forced by the Lévy noise. The solution is global provided that the initial data is sufficiently small.

In the present paper we are interested in the existence of a global martingale solution of problem (1.1)-(1.4) (without limitations on the smallness of the initial data). Proceeding similarly as in [29], where the Gaussian type noise terms are considered, we rewrite problem (1.1)-(1.4) in the form of the equation

$$X(t) + \int_0^t \left[ A X(s) + \tilde{B}(X(s)) + \tilde{H}(X(s)) \right] \, ds = X_0 + \int_0^t \int_Y F(s, X(s^-); y) \tilde{\eta}(ds, dy), \quad t \in [0, T],$$

where $X = (u, B)$ and $X_0 := (u_0, B_0)$. Besides, $A$, $\tilde{B}$ and $\tilde{H}$ are the maps corresponding to the Stokes-type operators, the MHD-term and the Hall-term, respectively, defined in Section 2.1. In fact, the deterministic part of problem (1.5) is the same as in [29]. Analysis of the stochastic part is different.

The main result is stated in Theorem 3.4. We prove the existence of a global martingale solution of problem (1.5) understood as a system $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, \eta, X)$, where $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a filtered probability space, $\eta$ is a time-homogeneous Poisson random measure and $X = \{X(t)\}_{t \in [0, T]}$ is a stochastic process. The trajectories of the process $X$ belong, with probability 1, to space of càdlàg functions $\mathbb{D}([0, T]; \mathbb{H}_w)$, where the subscript $w$ indicates the weak topology in $\mathbb{H}$, see Definition 3.3. Besides, we prove that the stochastic process $X$ satisfies, in particular, the energy inequality

$$\mathbb{E} \left[ \sup_{t \in [0, T]} |X(t)|^2_{\mathbb{H}} + \int_0^T \|X(t)\|^2_{\mathbb{V}} \right] < \infty$$

The spaces $\mathbb{H} \subset \mathbb{L}^2$ and $\mathbb{V} \subset \mathbb{H}^1$, defined in Section 2.1, are products of appropriate spaces of divergence free vector fields corresponding to the incompressibility conditions from (1.3). The problem of uniqueness is still open.

Similarly to [29], the construction of a solution is based on the approximation which uses the Fourier transform. Problem (1.5) is approximated by a sequence of the truncated SPDEs perturbed by appropriate Poisson type random forces, in the infinite dimensional Hilbert
spaces $\mathbb{H}_n$, $n \in \mathbb{N}$. The solutions $X_n$ of the truncated equations generate a tight sequence of probability measures $\{\text{Law}(X_n), n \in \mathbb{N}\}$ on appropriate functional space $Z$ defined by (5.8) in Section 5. In comparison to [29] in the definition of the space $Z$, the spaces of continuous functions are replaced by the spaces of càdlàg functions with the Skorokhod topology, see Section 5 and Appendix A. Besides, the construction of the new stochastic basis and new stochastic process is different from that used in [29]. It is based on the version of the Skorokhod theorem adequate for Poisson type noises. It is closely related to the techniques used in [27] in the context of Navier-Stokes equation and in [28] for other hydrodynamic equations and uses Corollary B.1 stated in Appendix B. Because of the Hall term, the framework considered in [28] does not cover directly the Hall-MHD system.

In Section 7 we consider also the Hall-MHD equation with more general Lévy noise, i.e.

$$X(t) + \int_0^t \left[ AX(s) + \tilde{B}(X(s)) + \tilde{H}(X(s)) \right] \, ds = X_0 + \int_0^t f(s) \, ds + \int_0^t \int Y F(s, X(s^-); y) \tilde{\eta}(ds, dy) + \int_0^t G(s, X(s)) \, dW(s), \quad t \in [0, T],$$

where $f$ represents deterministic forces and $W$ is a Wiener process, and state the theorem about the existence of a martingale solution.

The Fourier truncation method is very useful in the study of differential equations. Let us recall the paper [16], where Fefferman and co-authors use this method in deterministic MHD equations on $\mathbb{R}^d$, $d = 2, 3$. The same idea, referred to as the Friedrichs method, is used by Bahouri, Chemin and Danchin [4, Section 4] to study other class of deterministic differential equations.

In the context of SPDEs, Mohan and Srinathran [26] apply the Fourier truncation method to study the stochastic Euler equation with Lévy noise. This method is also used by Manna, Mohan and Srinathran [24] in the stochastic MHD equations with Lévy noise and by Yamazaki and Mohan in [33]. Moreover, Brzeźniak and Dhariwal [7] apply the truncated approximation in the stochastic tamed Navier-Stokes equations.

The paper is organized as follows. In Section 2 we recall the functional setting of the Hall-magnetohydrodynamics equations. The main result is stated in Section 3. In Section 4 we consider approximate equations and prove a priori estimates. Compactness and tightness theorems used in the proof of the main theorem are presented in Section 5. Section 6 is devoted to the proof of the existence of a global martingale solution. Version of the Skorokhod theorem used in the proof of Theorem 3.4 is recalled in Appendix B. Some auxiliary results related to the Fourier analysis are contained in Appendix C.

## 2 Functional setting of the Hall-magnetohydrodynamics equations

Let $C_c^\infty = C_c^\infty(\mathbb{R}^3, \mathbb{R}^3)$ denote the space of all $\mathbb{R}^3$-valued functions of class $C^\infty$ with compact supports in $\mathbb{R}^3$, and let

\begin{align}
V &:= \{ u \in C_c^\infty(\mathbb{R}^3, \mathbb{R}^3) : \text{div} \, u = 0 \}, \quad (2.1) \\
H &:= \text{the closure of } V \text{ in } L^2(\mathbb{R}^3, \mathbb{R}^3), \quad (2.2) \\
V &:= \text{the closure of } V \text{ in } H^1(\mathbb{R}^3, \mathbb{R}^3). \quad (2.3)
\end{align}
In the space $H$ we consider the inner product and the norm inherited from $L^2(\mathbb{R}^3, \mathbb{R}^3)$ and denote them by $(\cdot|\cdot)_H$ and $|\cdot|_H$, respectively, i.e.

$$(u|v)_H := (uv)_{L^2}, \quad |u|_H := |u|_{L^2}, \quad u, v \in H.$$ 

In the space $V$ we consider the inner product inherited from $H^1(\mathbb{R}^3, \mathbb{R}^3)$, i.e.

$$(u|v)_V := (uv)_{H^1} + (\nabla u|\nabla v)_{L^2}, \quad u, v \in V,$$

where

$$(\nabla u|\nabla v)_{L^2} = \sum_{i=1}^3 \int_{\mathbb{R}^3} \frac{\partial u}{\partial x_i} \cdot \frac{\partial v}{\partial x_i} \, dx,$$

and the norm induced by $(\cdot|\cdot)_V$, i.e.

$$|u|_V := \left( |u|^2_H + |\nabla u|^2_{L^2} \right)^{\frac{1}{2}}.$$ 

For any $m \geq 0$ consider the following of Hilbert space $V_m := \text{the closure of } V$ in $H^m(\mathbb{R}^3, \mathbb{R}^3)$ (2.4) with the inner product inherited from the space $H^m(\mathbb{R}^3, \mathbb{R}^3)$. Note that, $V_0 = H$ and $V_1 = V$.

**Notations.** Let $(X, \| \cdot \|_X), (Y, \| \cdot \|_Y)$ be two normed spaces. By $\mathcal{L}(X, Y)$ we denote the space of all bounded linear operators from $X$ to $Y$. If $Y = \mathbb{R}$, the $X^\prime := \mathcal{L}(X, \mathbb{R})$ is called the dual space of $X$. The standard duality pairing is denoted by $X^\prime \langle \cdot | \cdot \rangle_X$. If no confusion seems likely we omit the subscripts $X^\prime$ and $X$ and write $\langle \cdot | \cdot \rangle$.

### 2.1 Spaces used in the Hall-MHD equations

Spaces used in the theory of Hall-magnetohydrodynamics equations are products of the spaces $H$ and $V$ defined by (2.2) and (2.3), respectively. Namely,

$$\mathbb{H} := H \times H, \quad \mathbb{V} := V \times V, \quad \mathbb{V}^\prime := \text{the dual space of } \mathbb{V}$$

(2.5)

The spaces $\mathbb{H}$ and $\mathbb{V}$ are Hilbert spaces with the following inner products

$$(\phi|\psi)_\mathbb{H} := (u|v)_{L^2} + (B|C)_{L^2}$$

for all $\phi = (u, B), \psi = (v, C) \in \mathbb{H}$, and

$$(\phi|\psi)_\mathbb{V} := (\phi|\psi)_\mathbb{H} + ((\phi|\psi))$$

for all $\phi = (u, B), \psi = (v, C) \in \mathbb{V}$, where

$$((\phi|\psi)) := \nu_1 (\nabla u|\nabla v)_{L^2} + \nu_2 (\nabla B|\nabla C)_{L^2}.$$ 

(2.6)

In the spaces $\mathbb{H}$ and $\mathbb{V}$ we consider the norms induced by the inner products $(\cdot|\cdot)_\mathbb{H}$ and $(\cdot|\cdot)_\mathbb{V}$, respectively, i.e. $|\phi|_\mathbb{H} := (\phi|\phi)_\mathbb{H}$ for $\phi \in \mathbb{H}$, and

$$||\phi||_\mathbb{V}^2 := ||\phi||^2_\mathbb{H} + ||\phi||^2,$$

(2.7)

where

$$||\phi||_\mathbb{V}^2 := ((\phi|\phi)), \quad \phi \in \mathbb{V}.$$ 

Let $\mathcal{A}$ be the operator defined by

$$\langle \mathcal{A}\phi|\psi \rangle = ((\phi|\psi)), \quad \phi, \psi \in \mathbb{V},$$

(2.8)

where $((\cdot|\cdot))$ is given by (2.6).
Remark 2.1. It is clear that $A \in \mathcal{L}(V, V')$ and

$$|A\phi|_{V'} \leq \|\phi\|, \quad \phi \in V. \quad (2.9)$$

For $m_1, m_2 \geq 0$ let us define

$$V_{m_1, m_2} := V_{m_1} \times V_{m_2}, \quad (2.10)$$

where $V_{m_1}, V_{m_2}$ are the spaces defined by (2.4). In $V_{m_1, m_2}$ we consider the product norm

$$\|\phi\|_{m_1, m_2}^2 := \|u\|^2_{V_{m_1}} + \|B\|^2_{V_{m_2}} \quad (2.11)$$

for all $\phi = (u, B) \in V_{m_1, m_2}$. In the case when $m_1 = m_2 =: m$ we denote

$$V_m := V_m \times V_m \quad \text{and} \quad \|\cdot\|_m := \|\cdot\|_{m,m}. \quad (2.12)$$

Note that if $m = 1$, then $V_1 = V$ and $\|\cdot\|_1 = \|\cdot\|_V$.

By $L^2_{\text{loc}}(\mathbb{R}^3, \mathbb{R}^3)$, we denote the space of all Lebesgue measurable functions $v : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\int_K |v(x)|^2 \, dx < \infty$ for every compact subset $K \subset \mathbb{R}^3$, equipped with the Fréchet topology generated by the family of seminorms

$$p_R(v) := \left( \int_{O_R} |v(x)|^2 \, dx \right)^{\frac{1}{2}}, \quad R \in \mathbb{N}, \quad (2.13)$$

where $(O_R)_{R \in \mathbb{N}}$ is an increasing sequence of open bounded subsets of $\mathbb{R}^3$ with smooth boundaries and such that $\bigcup_{R \in \mathbb{N}} O_R = \mathbb{R}^3$.

By $H_{\text{loc}}$, we denote the space $H$ defined by (2.2), endowed with the Fréchet topology inherited from the space $L^2_{\text{loc}}(\mathbb{R}^3, \mathbb{R}^3)$.

Let us fix $T > 0$. By $L^2(0,T; H_{\text{loc}})$ we denote the space of measurable functions $u : [0, T] \to H$ such that for all $R \in \mathbb{N}$

$$p_{T,R}(u) = \left( \int_0^T \int_{O_R} |u(t,x)|^2 \, dx \, dt \right)^{\frac{1}{2}} < \infty. \quad (2.14)$$

In $L^2(0,T; H_{\text{loc}})$ we consider the topology generated by the seminorms $(p_{T,R})_{R \in \mathbb{N}}$ defined by (2.14).

By $L^2(0,T; \mathbb{H}_{\text{loc}})$ we denote the space of measurable functions $\phi : [0, T] \to \mathbb{H}$ such that for all $R \in \mathbb{N}$

$$p_{T,R}(\phi) := \left( \int_0^T \int_{O_R} \left[ |u(t,x)|^2 + |B(t,x)|^2 \right] \, dx \, dt \right)^{\frac{1}{2}} < \infty. \quad (2.15)$$

where $\phi = (u, B)$. In the space $L^2(0,T; \mathbb{H}_{\text{loc}})$ we consider the topology generated by the seminorms $(p_{T,R})_{R \in \mathbb{N}}$ defined by (2.15).

2.2 The form $\tilde{b}$ and the operator $\tilde{B}$

We recall the standard maps used in the theory of magnetohydrodynamic equation, see Sermange and Temam [34] and Sango [33]. Let us consider the following tri-linear form

$$b(u, w, v) = \int_{\mathbb{R}^3} (u \cdot \nabla w) v \, dx. \quad (2.16)$$
For the properties of the form $b$, see Temam [28]. Using the form $b$ defined by (2.16) we will consider the tri-linear form $\tilde{b}$ on $V \times V \times V$, where $V$ is defined by (2.5). Namely,

$$
\tilde{b}(\phi^{(1)}, \phi^{(2)}, \phi^{(3)}) := b(u^{(1)}, u^{(2)}, u^{(3)}) - b(B^{(1)}, B^{(2)}, u^{(3)}) + b(u^{(1)}, B^{(2)}, B^{(3)}) - b(B^{(1)}, u^{(2)}, B^{(3)}),
$$

where $\phi^{(i)} = (u^{(i)}, B^{(i)}) \in V$, $i = 1, 2, 3$. Since the form $b$ is continuous on $V \times V \times V$, the form $\tilde{b}$ is continuous on $V \times V \times V$. Moreover, the form $\tilde{b}$ has the following properties

$$
\tilde{b}(\phi^{(1)}, \phi^{(2)}, \phi^{(3)}) = -\tilde{b}(\phi^{(1)}, \phi^{(3)}, \phi^{(2)}), \quad \phi^{(i)} \in V, \quad i = 1, 2, 3
$$

and in particular

$$
\tilde{b}(\phi^{(1)}, \phi^{(2)}, \phi^{(2)}) = 0, \quad \phi^{(1)}, \phi^{(2)} \in V.
$$

Now, let us define a bilinear map $\tilde{B}$ by

$$
\tilde{B}(\phi, \psi) := \tilde{b}(\phi, \psi, \cdot), \quad \phi, \psi \in V. \quad (2.17)
$$

We will also use the notation $\tilde{B}(\phi) := \tilde{B}(\phi, \psi)$.

Let us recall properties of the map $\tilde{B}$ stated in [28].

**Lemma 2.2.** (See [28, Lemma 6.4])

(i) There exists a constant $c_{\tilde{B}} > 0$ such that

$$
|\tilde{B}(\phi, \psi)|_{V'} \leq c_{\tilde{B}} \|\phi\|_{V} \|\psi\|_{V}, \quad \phi, \psi \in V.
$$

In particular, the map $\tilde{B} : V \times V \to \mathbb{V}'$ is bilinear and continuous. Moreover,

$$
\langle \tilde{B}(\phi, \psi) | \theta \rangle = -\langle \tilde{B}(\phi, \theta) | \psi \rangle, \quad \phi, \psi, \theta \in V,
$$

and, in particular,

$$
\langle \tilde{B}(\phi) | \phi \rangle = 0, \quad \phi \in V. \quad (2.18)
$$

(ii) The mapping $\tilde{B}$ is locally Lipschitz continuous on the space $V$, i.e. for every $r > 0$ there exists a constant $L_r > 0$ such that

$$
|\tilde{B}(\phi) - \tilde{B}(\tilde{\phi})|_{V'} \leq L_r \|\phi - \tilde{\phi}\|_{V}, \quad \phi, \tilde{\phi} \in V, \quad \|\phi\|_{V}, \|\tilde{\phi}\|_{V} \leq r.
$$

(iii) If $m > \frac{3}{2}$, then $\tilde{B}$ can be extended to the bilinear mapping from $H \times H$ to $V'_m$ (denoted still by $\tilde{B}$) such that

$$
|\tilde{B}(\phi, \psi)|_{V'_m} \leq c_{\tilde{B}}(m) \|\phi\|_{H} \|\psi\|_{H}, \quad \phi, \psi \in H, \quad (2.19)
$$

where $c_{\tilde{B}}(m)$ is a positive constant.

We will use also the following convergence result for the nonlinear term $\tilde{B}$.

**Corollary 2.3.** (See [24, Corollary 2.8].) Let $\phi, \psi \in L^2(0, T; \mathbb{H})$ and let $(\phi_n), (\psi_n) \subset L^2(0, T; \mathbb{H})$ be two sequence bounded in $L^2(0, T; \mathbb{H})$ and such that

$$
\phi_n \to \phi \quad \text{and} \quad \psi_n \to \psi \quad \text{in} \quad L^2(0, T; \mathbb{H}_{loc}).
$$

If $m > \frac{3}{2}$, then for all $t \in [0, T]$ and all $\varphi \in V_m$:

$$
\lim_{n \to \infty} \int_0^t \langle \tilde{B}(\phi_n(s), \psi_n(s)) | \varphi \rangle \, ds = \int_0^t \langle \tilde{B}(\phi(s), \psi(s)) | \varphi \rangle \, ds,
$$

where $V_m$ is the space defined by (2.12).
2.3 The form \( \tilde{\mathcal{h}} \) and the map \( \tilde{\mathcal{H}} \)

Now we analyze the Hall term. By the integration by parts formula for the curl-operator, we obtain

\[
\int_{\mathbb{R}^3} \text{curl} [u \times \text{curl} w] \cdot v \, dx = \int_{\mathbb{R}^3} [u \times \text{curl} w] \cdot \text{curl} v \, dx
\]

for \( u, w, v \in \mathcal{V} \). We will use the following tri-linear form defined by

\[
\mathcal{h}(u, w, v) := - \int_{\mathbb{R}^3} [u \times \text{curl} w] \cdot \text{curl} v \, dx
\]  \hspace{1cm} (2.20)

for \( u, w, v \in \mathcal{V} \), see [29, Section 2.3].

Since \((a \times b) \cdot c = -(a \times c) \cdot b\) for \( a, b, c \in \mathbb{R}^3 \), we infer that

\[
\mathcal{h}(u, v, w) = - \mathcal{h}(u, w, v).
\]  \hspace{1cm} (2.21)

In particular,

\[
\mathcal{h}(u, v, v) = 0.
\]  \hspace{1cm} (2.22)

Using the form \( \mathcal{h} \) defined by (2.20) we define the tri-linear form \( \tilde{\mathcal{h}} \) on \( \mathcal{V} \times \mathcal{V} \times \mathcal{V}_{1,2} \) by

\[
\tilde{\mathcal{h}}(\phi^{(1)}, \phi^{(2)}, \phi^{(3)}) := \mathcal{h}(B^{(1)}, B^{(2)}, B^{(3)}),
\]

where \( \phi^{(i)} = (u^{(i)}, B^{(i)}) \in \mathcal{V} \) for \( i = 1, 2 \), and \( \phi^{(3)} = (u^{(3)}, B^{(3)}) \in \mathcal{V}_{1,2} \). Due to (2.10), \( \mathcal{V}_{1,2} := V_1 \times V_2 \). The form \( \tilde{\mathcal{h}} \) is continuous on \( \mathcal{V} \times \mathcal{V} \times \mathcal{V}_{1,2} \). Moreover, by (2.21) and (2.22) the form \( \tilde{\mathcal{h}} \) has the following properties

\[
\tilde{\mathcal{h}}(\phi^{(1)}, \phi^{(2)}, \phi^{(3)}) = - \tilde{\mathcal{h}}(\phi^{(1)}, \phi^{(3)}, \phi^{(2)}), \quad \phi^{(i)} \in \mathcal{V}, \quad \phi^{(i)} \in \mathcal{V}_{1,2}, \quad i = 2, 3,
\]

and in particular

\[
\tilde{\mathcal{h}}(\phi^{(1)}, \phi^{(2)}, \phi^{(2)}) = 0, \quad \phi^{(1)} \in \mathcal{V}, \quad \phi^{(2)} \in \mathcal{V}_{1,2}.
\]

Now, let us define a bilinear map \( \tilde{\mathcal{H}} \) by

\[
\tilde{\mathcal{H}}(\phi, \psi) := \tilde{\mathcal{h}}(\phi, \psi, \cdot), \quad \phi, \psi \in \mathcal{V}.
\]  \hspace{1cm} (2.23)

We will also use the notation \( \tilde{\mathcal{H}}(\phi) := \tilde{\mathcal{H}}(\phi, \phi) \).

In the following lemma we state basic properties of the map \( \tilde{\mathcal{H}} \) proved in [29].

Lemma 2.4. (See [29, Lemma 2.9].)

(i) There exists a constant \( c_{\tilde{\mathcal{H}}} > 0 \) such that

\[
|\tilde{\mathcal{H}}(\phi, \psi)|_{\mathcal{V}_{1,2}'} \leq c_{\tilde{\mathcal{H}}} \|\phi\|_{\mathcal{V}} \|\psi\|_{\mathcal{V}}, \quad \phi, \psi \in \mathcal{V}.
\]  \hspace{1cm} (2.24)

In particular, the map \( \tilde{\mathcal{H}} : \mathcal{V} \times \mathcal{V} \to \mathcal{V}_{1,2}' \) is well-defined bilinear and continuous. Moreover,

\[
\langle \tilde{\mathcal{H}}(\phi, \psi) | \Theta \rangle = -\langle \tilde{\mathcal{H}}(\phi, \theta) | \psi \rangle, \quad \phi \in \mathcal{V}, \quad \psi, \theta \in \mathcal{V}_{1,2},
\]

and, in particular,

\[
\langle \tilde{\mathcal{H}}(\phi) | \phi \rangle = 0, \quad \phi \in \mathcal{V}_{1,2}.
\]  \hspace{1cm} (2.25)
(ii) The map \( \tilde{H} \) is locally Lipschitz continuous on the space \( \mathbb{V} \), i.e., for every \( r > 0 \) there exists a constant \( L_r > 0 \) such that
\[
|\tilde{H}(\phi) - \tilde{H}(\tilde{\phi})|_{\mathbb{V}^2} \leq L_r \|\phi - \tilde{\phi}\|_{\mathbb{V}}, \quad \phi, \tilde{\phi} \in \mathbb{V}, \quad \|\phi\|_{\mathbb{V}}, \|\tilde{\phi}\|_{\mathbb{V}} \leq r.
\]

(iii) If \( s \geq 0 \) and \( m > \frac{5}{2} \), then \( \tilde{H} \) can be extended to the bilinear mapping from \( \mathbb{H} \times \mathbb{V} \) to \( \mathbb{V}_{s,m} \) (denoted still by \( \tilde{H} \)) such that
\[
|\tilde{H}(\phi, \psi)|_{\mathbb{V}_{s,m}} \leq c_{\tilde{H}}(s,m) \|\phi\|_{\mathbb{H}} \|\psi\|_{\mathbb{V}}, \quad \phi \in \mathbb{H}, \quad \psi \in \mathbb{V},
\]
where \( c_{\tilde{H}}(s,m) \) is a positive constant.

In particular, if \( m > \frac{5}{2} \), then \( \tilde{H} \) can be extended to the bilinear mapping from \( \mathbb{H} \times \mathbb{V} \) to \( \mathbb{V}_m \) (denoted still by \( \tilde{H} \)) such that
\[
|\tilde{H}(\phi, \psi)|_{\mathbb{V}_m} \leq c_{\tilde{H}}(m) \|\phi\|_{\mathbb{H}} \|\psi\|_{\mathbb{V}}, \quad \phi \in \mathbb{H}, \quad \psi \in \mathbb{V}, \quad \tag{2.26}
\]
where \( c_{\tilde{H}}(m) := c_{\tilde{H}}(m, m) \) and \( \mathbb{V}_m \) is the space defined by (2.12).

We will use also the following convergence result for the map \( \tilde{H} \) proved in Lemma 2.5 and Corollary 2.10 in [29].

**Corollary 2.5.** (See [29, Corollary 2.10].) Let \( \phi \in L^2(0,T;\mathbb{H}) \) and \( \psi \in L^2(0,T;\mathbb{V}) \) and let \( (\phi_n) \subset L^2(0,T;\mathbb{H}) \) and \( (\psi_n) \subset L^2(0,T;\mathbb{V}) \) be two sequences such that

- \( (\phi_n) \) is bounded in \( L^2(0,T;\mathbb{H}) \) and \( \psi_n \to \psi \) weakly in \( L^2(0,T;\mathbb{V}) \),
- \( \phi_n \to \phi \) and \( \psi_n \to \psi \) in \( L^2(0,T;\mathbb{H}_{loc}) \).

If \( s \geq 0 \) and \( m > \frac{5}{2} \), then for all \( t \in [0,T] \) and all \( \varphi \in \mathbb{V}_{s,m} \):
\[
\lim_{n \to \infty} \int_0^t (\tilde{H}(\phi_n(s),\psi_n(s))|\varphi) \, ds = \int_0^t (\tilde{H}(\phi(s),\psi(s))|\varphi) \, ds,
\]
where \( \mathbb{V}_{s,m} \) is the the space defined by (2.10) - (2.11).

### 2.4 Probabilistic preliminaries. Time homogeneous Poisson random measure

We follow the approach due to [3], [8], [9], see also [19] and [30]. Let us denote \( \mathbb{N} := \{0,1,2,\ldots\} \), \( \overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\} \), \( \mathbb{R}_+ := [0,\infty) \). Let \( (S,\mathcal{S}) \) be a measurable space and let \( M_{\overline{\mathbb{N}}}(S) \) be the set of all \( \overline{\mathbb{N}} \)-valued measures on \( (S,\mathcal{S}) \). On the set \( M_{\overline{\mathbb{N}}}(S) \) we consider the \( \sigma \)-field \( \mathcal{M}_{\overline{\mathbb{N}}}(S) \) defined as the smallest \( \sigma \)-field such that for all \( B \in \mathcal{S} \) the map \( i_B : M_{\overline{\mathbb{N}}}(S) \ni \mu \mapsto \mu(B) \in \overline{\mathbb{N}} \) is measurable.

Let \( (\Omega,\mathcal{F},\mathbb{P}) \) be a complete probability space with filtration \( \mathbb{F} := (\mathcal{F}_t)_{t \geq 0} \) satisfying the usual conditions.

**Definition 2.6.** (See [3] and [8].) Let \( (Y,\mathcal{Y}) \) be a measurable space. A **time homogeneous Poisson random measure** \( \eta \) on \( (Y,\mathcal{Y}) \) over \( (\Omega,\mathcal{F},\mathbb{F},\mathbb{P}) \) is a measurable function
\[
\eta : (\Omega,\mathcal{F}) \to (M_{\overline{\mathbb{N}}}(\mathbb{R}_+ \times Y),\mathcal{M}_{\overline{\mathbb{N}}}(\mathbb{R}_+ \times Y))
\]
such that
(i) for all $B \in \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{Y}$, $\eta(B) := i_B \circ \eta : \Omega \to \mathbb{N}$ is a Poisson random variable with parameter $\mathbb{E}[\eta(B)]$;

(ii) $\eta$ is independently scattered, i.e. if the sets $B_j \in \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{Y}$, $j = 1, \ldots, n$, are disjoint then the random variables $\eta(B_j)$, $j = 1, \ldots, n$, are independent;

(iii) for all $U \in \mathcal{Y}$ the $\mathbb{N}$-valued process $(N(t, U))_{t \geq 0}$ defined by

$$N(t, U) := \eta((0, t] \times U), \quad t \geq 0$$

is $\mathcal{F}$-adapted and its increments are independent of the past, i.e. if $t > s \geq 0$, then $N(t, U) - N(s, U) = \eta((s, t] \times U)$ is independent of $\mathcal{F}_s$.

If $\eta$ is a time homogeneous Poisson random measure then the formula

$$\mu(A) := \mathbb{E}[\eta((0, 1] \times A)], \quad A \in \mathcal{Y}$$

defines a measure on $(\mathbb{Y}, \mathcal{Y})$ called an intensity measure of $\eta$. Moreover, for all $T < \infty$ and all $A \in \mathcal{Y}$ such that $\mathbb{E}[\eta((0, T] \times A)] < \infty$, the $\mathbb{R}$-valued process $\{\tilde{N}(t, A)\}_{t \in (0, T]}$ defined by

$$\tilde{N}(t, A) := \eta((0, t] \times A) - t\mu(A), \quad t \in (0, T],$$

is an integrable martingale on $(\Omega, \mathcal{F}, \mathbb{P})$. The random measure $\ell \otimes \mu$ on $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{Y}$, where $\mathcal{B}(\mathbb{R}_+)$ denotes the Borel $\sigma$-field and $\ell$ stands for the Lebesgue measure, is called a compensator of $\eta$, and the difference between a time homogeneous Poisson random measure $\eta$ and its compensator, i.e.

$$\tilde{\eta} := \eta - \ell \otimes \mu,$$

is called a compensated time homogeneous Poisson random measure.

Let us also recall basic properties of the stochastic integral with respect to $\tilde{\eta}$, see [8], [19] and [30] for details. Let $\mathcal{P}$ be a separable Hilbert space and let $\mathcal{P}$ be a predictable $\sigma$-field on $[0, T] \times \Omega$. Let $\mathcal{L}^2_{\mu,T}(\mathcal{P} \otimes \mathcal{Y}, l \otimes \mathbb{P} \otimes \mu; E)$ be a space of all $E$-valued, $\mathcal{P} \otimes \mathcal{Y}$-measurable processes such that

$$\mathbb{E}\left[\int_0^T \int_Y ||\xi(s, \cdot, y)||_E^2 dsd\mu(y)\right] < \infty.$$

If $\xi \in \mathcal{L}^2_{\mu,T}(\mathcal{P} \otimes \mathcal{Y}, l \otimes \mathbb{P} \otimes \mu; E)$ then the integral process $\int_0^t \int_Y \xi(s, \cdot, y) \tilde{\eta}(ds, dy)$, $t \in [0, T]$, is a càdlàg $L^2$-integrable martingale. Moreover, the following isometry formula holds

$$\mathbb{E}\left[\left\|\int_0^t \int_Y \xi(s, \cdot, y) \tilde{\eta}(ds, dy)\right\|_E^2\right] = \mathbb{E}\left[\int_0^t \int_Y \xi(s, \cdot, y) ||\xi(s, \cdot, y)||_E^2 dsd\mu(y)\right], \quad t \in [0, T]. \quad (2.27)$$

### 3 Martingale solutions of the Hall-MHD equations.

Let $H$ and $L^2([0, T]; H_{loc})$ be the spaces defined by (2.2) and (2.14), respectively. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with filtration $\mathcal{F} := \{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions and let us denote $\mathcal{A} := (\Omega, \mathcal{F}, \mathbb{P})$.

**Assumption 3.1.**

(F.1) Let $(Y_i, \mathcal{Y}_i)$, $i = 1, 2$, be measurable spaces and let $\mu_i$, $i = 1, 2$, be $\sigma$-finite measures on $(Y_i, \mathcal{Y}_i)$. Assume that $\eta_i$, $i = 1, 2$, are time homogeneous Poisson random measures on $(Y_i, \mathcal{Y}_i)$ over $\mathcal{A}$ with the (jump) intensity measures $\mu_i$. 

9
Assume that $F_i : [0, T] \times H \times Y_i \to H$, where $i = 1, 2$, are two measurable maps such that there exist constants $L_i$, $i = 1, 2$ such that

$$
\int_{Y_i} |F_i(t, \phi; y) - F_i(t, \psi; y)|_H^2 \mu_i(dy) \leq L_i|\phi - \psi|_H^2, \quad \phi, \psi \in H, \ t \in [0, T],
$$

and for each $q \geq 1$ there exists a constant $K_{iq}$ such that

$$
\int_{Y_i} |F_i(t, \phi; y)|_H^q \mu_i(dy) \leq K_{iq}(1 + |\phi|_H^q), \quad \phi \in H, \ t \in [0, T].
$$

Moreover, for all $\varphi \in H$ the maps $\tilde{F}_{i, \varphi}$, $i = 1, 2$, defined for $\phi : [0, T] \to \mathbb{H}$ by

$$
(\tilde{F}_{i, \varphi}(\phi))(t, y) := (F_i(t, \phi(t^-); y)|\varphi)_H, \quad (t, y) \in [0, T] \times Y_i
$$

are continuous from $L^2(0, T; H_{Ioc})$ into $L^2([0, T] \times Y_i, d\ell \otimes \mu_i; \mathbb{R})$. Here $\ell$ denotes the Lebesgue measure on the interval $[0, T]$.

In the context of Assumption 3.1 let

$$Y := Y_1 \times Y_2, \quad \mathcal{Y} := \mathcal{Y}_1 \otimes \mathcal{Y}_2, \quad \mu := \mu_1 \otimes \mu_2, \quad \eta := (\eta_1, \eta_2).$$

Remark 3.2. (The map $F$ and its properties.) Let $F_1$ and $F_2$ be the maps given in Assumption 3.1. Let us define the following map

$$F(t, \Phi; y) := (F_1(t, u; y_1), F_2(t, B; y_2)),
$$

where $t \in [0, T], \ \Phi = (u, B) \in \mathbb{H}, \ y = (y_1, y_2) \in Y$.

(i) Then $F : [0, T] \times \mathbb{H} \times Y \to \mathbb{H}$ is a measurable map and there exists a constant $L > 0$ such that

$$
\int_Y |F(t, \Phi; y) - F(t, \Psi; y)|_\mathbb{H}^2 \mu(dy) \leq L|\Phi - \Psi|_\mathbb{H}^2, \quad \Phi, \Psi \in \mathbb{H}, \ t \in [0, T],
$$

and for each $q \geq 1$ there exists a constant $K_q$ such that

$$
\int_Y |F(t, \Phi; y)|_\mathbb{H}^q \mu(dy) \leq K_q(1 + |\Phi|_\mathbb{H}^q), \quad \Phi \in \mathbb{H}, \ t \in [0, T].
$$

(ii) Moreover, for every $\Psi \in \mathbb{H}$ the map $\tilde{F}_\Psi$ defined for $\Phi : [0, T] \to \mathbb{H}$ by

$$
(\tilde{F}_\Psi(\Phi))(t, y) := (F(t, \Phi(t^-); y)|\Psi)_\mathbb{H}, \quad (t, y) \in [0, T] \times Y
$$

is continuous from $L^2(0, T; \mathbb{H}_{loc})$ into $L^2([0, T] \times Y, d\ell \otimes \mu; \mathbb{R})$.

Let us recall the spaces $\mathbb{H}$ and $L^2(0, T; \mathbb{H}_{loc})$ are defined in Section 2.1.

Taking into account the above notations, and the maps $\mathcal{A}$, $\tilde{B}$, $\tilde{H}$ and $F$ defined respectively by (2.8), (2.17), (2.23) and (3.4), problem (1.1)-(1.4) can be rewritten in the form of the following equation

$$
X(t) + \int_0^t [\mathcal{A}X(s) + \tilde{B}(X(s)) + \tilde{H}(X(s))] \ ds = X_0 + \int_0^t \int_Y F(s, X(s^-); y) \eta(ds, dy), \quad t \in [0, T],
$$

where $X_0 := (u_0, B_0)$. 

\(10\)
Definition 3.3. Let Assumption 3.1 be satisfied and let $X_0 \in \mathbb{H}$. We say that there exists a martingale solution of problem (3.8) iff there exist

- a stochastic basis $\tilde{\mathfrak{A}} := (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \tilde{\mathbb{P}})$ with a filtration $\tilde{\mathbb{F}} = \{\tilde{\mathcal{F}}_t\}_{t \in [0,T]}$ satisfying the usual conditions,
- a time homogeneous Poisson random measure $\tilde{\eta}$ on $(Y, \mathcal{Y})$ over $\tilde{\mathfrak{A}}$ with the intensity measure $\mu$,
- and an $\tilde{\mathbb{F}}$-progressively measurable process $\tilde{X} : [0, T] \times \Omega \rightarrow \mathbb{H}$ with paths satisfying

$$\tilde{X}(\cdot, \omega) \in D([0, T]; \mathbb{H}) \cap L^2(0, T; \mathbb{V}),$$

(3.9)

for $\tilde{\mathbb{P}}$-a.e. $\omega \in \tilde{\Omega}$, and such that for all $t \in [0, T]$ and $\phi \in \mathbb{V}_{1,2}$ the following identity holds $\tilde{\mathbb{P}}$-a.s.

$$\tilde{X}(t|\phi)_{\mathbb{H}} + \int_0^t \langle A \tilde{X}(s)|\phi \rangle ds + \int_0^t \langle B(\tilde{X}(s))|\phi \rangle ds + \int_0^t \langle \tilde{H}(\tilde{X}(s))|\phi \rangle ds$$

$$= (X_0|\phi)_{\mathbb{H}} + \int_0^t \int_Y \langle F(s, \tilde{X}(s^-); y)|\phi \rangle_{\mathbb{H}} \tilde{\eta}(ds, dy),$$

(3.10)

where $\mathbb{V}_{1,2}$ is the space is defined by (2.10).

If all the above conditions are satisfied, then the system $(\tilde{\mathfrak{A}}, \tilde{\eta}, \tilde{X})$ is called a martingale solution of problem (3.8).

Here, $\mathbb{H}_w$ denotes the Hilbert space $\mathbb{H}$ endowed with the weak topology. The fact that $\Psi \in D([0, T]; \mathbb{H}_w)$ means that for every $h \in \mathbb{H}$

$$[0, T] \ni t \mapsto (\Psi(t)|h)_{\mathbb{H}} \in \mathbb{R}$$

is a real-valued càdlàg function, i.e. it is right continuous and has left limits at every $t \in [0, T]$, see also Appendix A.

The main result of this paper asserts the existence of a martingale solution.

Theorem 3.4. Let Assumption 3.1 be satisfied and let $p \in [2, \infty)$. Then for every $X_0 \in \mathbb{H}$ there exists a martingale solution of problem (3.8) such that for every $q \in [2, p]

$$\mathbb{E}\left[ \sup_{t \in [0, T]} \|\tilde{X}(t)\|_{\mathbb{H}}^q + \int_0^T \|\tilde{X}(t)\|_{\mathbb{V}}^2 dt \right] < \infty.$$  

(3.11)

4 Approximate SPDEs

In this section will construct a sequence of stochastic equations approximating problem (3.8) and prove a priori estimates. This construction is based on the Fourier analysis. The deterministic background is the same as in [29]. Since in the present paper we consider the Hall-MHD equation with the Poisson type noise terms, analysis of the stochastic part is different from [29]. Approximation of this type is closely related to the Littlewood-Paley decomposition, see [4], and has also been used, e.g., in [16], [26], [24] and [7].
4.1 The subspaces $\mathbb{H}_n$ and the operators $P_n$

First we recall the construction of the sequence of subspaces $(\mathbb{H}_n)_{n \in \mathbb{N}}$ and associated sequence of operators $(P_n)_{n \in \mathbb{N}}$. Let

$$\bar{B}_n := \{ \xi \in \mathbb{R}^3 : |\xi| \leq n \}, \quad n \in \mathbb{N}$$

and let

$$H_n := \{ v \in H : \text{supp } \hat{v} \in \bar{B}_n \}.$$ 

In the subspace $H_n$ we consider the norm inherited from the space $H$ defined by (2.2). For each $n \in \mathbb{N}$ let us define a map $\pi_n$ by

$$\pi_n v := F^{-1}(\mathbb{1}_{\bar{B}_n} \hat{v}), \quad v \in H,$$

where $F^{-1}$ denotes the inverse of the Fourier transform, see Appendix C. By Remark C.1, the map $\pi_n : H \to H_n$ is the orthogonal projection onto $H_n$.

Let

$$\bar{\mathbb{B}}_n := \bar{B}_n \times \bar{B}_n$$

and

$$\mathbb{H}_n := H_n \times H_n.$$ 

(4.1)

In the subspace $\mathbb{H}_n$ we consider the norm inherited from the space $\mathbb{H}$ defined by (2.5). Let us define the operator $P_n$:

$$P_n := \pi_n \times \pi_n : \mathbb{H} \to \mathbb{H}_n.$$ 

(4.2)

Explicitly, for $\Phi = (u, B) \in \mathbb{H}$

$$P_n(u, B) = (\pi_n u, \pi_n B) = (F^{-1}(\mathbb{1}_{\bar{B}_n} \hat{u}), F^{-1}(\mathbb{1}_{\bar{B}_n} \hat{B})).$$

Since the map $\pi_n : H \to H_n$ is the orthogonal projection onto $H_n$, we infer that

$$P_n : \mathbb{H} \to \mathbb{H}_n$$

is the orthogonal projection onto $\mathbb{H}_n$.

The following lemma states that the subspaces $\mathbb{H}_n$ are embedded in the spaces $\mathbb{V}_{m_1,m_2}$ for $m_1, m_2 \geq 0$, defined by (2.10) with the equivalence of appropriate norms.

**Lemma 4.1.** (See [29, Lemma 4.1].) Let $n \in \mathbb{N}$ and $m_1, m_2 \geq 0$. Then

$$\mathbb{H}_n \hookrightarrow \mathbb{V}_{m_1,m_2},$$

and for all $u \in \mathbb{H}_n$:

$$\|u\|_{m_1,m_2}^2 \leq (1 + n^2)^m \|u\|_{\mathbb{H}_n}^2,$$

where $m = \max\{m_1, m_2\}$.

**Corollary 4.2.** (See [29, Corollary 4.2].) On the subspace $\mathbb{H}_n$ the norm $\| \cdot \|_{\mathbb{H}_n}$ and the norms $\| \cdot \|_{m_1,m_2}$, for $m_1, m_2 \geq 0$, inherited from the spaces $\mathbb{V}_{m_1,m_2}$ are equivalent (with appropriate constants depending on $m_1, m_2$ and $n$).

Let us recall properties of the operators $P_n$ in the spaces $\mathbb{V}_{m_1,m_2}$ defined by (2.10).
Lemma 4.3. (See [29 Lemma 4.3].) Let us fix \( m_1, m_2 \geq 0 \). Then for all \( n \in \mathbb{N} \):

\[
P_n : \mathbb{V}^{m_1, m_2} \to \mathbb{V}^{m_1, m_2}
\]

is well defined linear and bounded. Moreover, for every \( u \in \mathbb{V}^{m_1, m_2} \):

\[
\lim_{n \to \infty} \|P_n u - u\|_{m_1, m_2} = 0.
\]

From Lemma 4.3 we obtain the following corollary.

Corollary 4.4. (See [29 Corollary 4.4].)

(i) \( P_n \in \mathcal{L}(\mathbb{V}, \mathbb{V}) \), and for all \( u \in \mathbb{V} \)

\[
\lim_{n \to \infty} \|P_n u - u\|_{\mathbb{V}} = 0,
\]

(ii) For every \( m \geq 0 \), \( P_n \in \mathcal{L}(\mathbb{V}_m, \mathbb{V}_m) \) and for all \( u \in \mathbb{V}_m \)

\[
\lim_{n \to \infty} \|P_n u - u\|_{\mathbb{V}_m} = 0,
\]

(iii) For every \( m \geq 1 \), \( P_n \in \mathcal{L}(\mathbb{V}_m, \mathbb{V}) \) and for all \( u \in \mathbb{V}_m \)

\[
\lim_{n \to \infty} \|P_n u - u\|_{\mathbb{V}} = 0.
\]

The spaces \( \mathbb{V} \) and \( \mathbb{V}_m \) is defined by (2.5) and (2.12), respectively.

4.2 The truncated SPDEs

Let us consider the following approximation in the space \( \mathbb{H}_n \) defined by (4.1).

Definition 4.5. Let \( X_0 \in \mathbb{H} \). By a truncated approximation of equation (3.8) we mean an \( \mathbb{H}_n \)-valued càdlàg, \( \mathbb{F} \)-adapted process \( \{X_n(t)\}_{t \in [0,T]} \) such that for all \( t \in [0, T] \) and \( \phi \in \mathbb{H}_n \) the following identity holds \( \mathbb{P} \) - a.s.

\[
(X_n(t) | \phi)_{\mathbb{H}} + \int_0^t \langle AX_n(s) | \phi \rangle ds + \int_0^t \langle \tilde{B}(X_n(s)) | \phi \rangle ds + \int_0^t \langle \tilde{H}(X_n(s)) | \phi \rangle ds = (X_0 | \phi)_{\mathbb{H}} + \int_0^t \int_Y (F(s, X_n(s^-); y) | \phi)_{\mathbb{H}} \tilde{\eta}(ds, dy).
\]

(4.4)

Using Lemma 4.4, Corollary 4.2 and the Riesz representation theorem for continuous linear functionals on \( \mathbb{H}_n \), we will rewrite identity (4.3) as a stochastic equation in the space \( \mathbb{H}_n \) (with the \( (\cdot | \cdot)_{\mathbb{H}} \)-inner product).

Remark 4.6. (See [29 Remark 4.7].) For fixed \( v \in \mathbb{V} \), the maps

\[
\mathbb{H}_n \ni \phi \mapsto \mathbb{V} \langle A v | \phi \rangle_{\mathbb{V}} \in \mathbb{R},
\]

(4.5)

\[
\mathbb{H}_n \ni \phi \mapsto \mathbb{V} \langle \tilde{B}(v) | \phi \rangle_{\mathbb{V}} \in \mathbb{R},
\]

(4.6)

\[
\mathbb{H}_n \ni \phi \mapsto \mathbb{V} \langle \tilde{H}(v) | \phi \rangle_{\mathbb{V}} \in \mathbb{R}.
\]

(4.7)
are continuous linear functionals on $\mathbb{H}_n$. Let $A_n(v), \tilde{B}_n(v), \tilde{H}_n(v) \in \mathbb{H}_n$ denote the Riesz representations in $\mathbb{H}_n$ of the functionals (4.5), (4.9), (4.7), respectively. Then we have for every $\varphi \in \mathbb{H}_n$

\begin{align}
\langle A_n(v) | \varphi \rangle_{\mathbb{H}} &= (A_n(v)|\varphi)_{\mathbb{H}}, \quad (4.8) \\
\langle B(v) | \varphi \rangle_{\mathbb{H}} &= (\tilde{B}_n(v)|\varphi)_{\mathbb{H}}, \quad (4.9) \\
\langle H(v) | \varphi \rangle_{\mathbb{H}} &= (\tilde{H}_n(v)|\varphi)_{\mathbb{H}}. \quad (4.10)
\end{align}

Since $A$ is linear, the map $V \ni v \mapsto A_n(v) \in \mathbb{H}_n$ is linear as well.

Since $P_n : \mathbb{H} \to \mathbb{H}_n$ is the $(\cdot | \cdot)_{\mathbb{H}}$-orthogonal projection, we have

\[
\langle v | P_n \varphi \rangle_{\mathbb{H}} = (P_n v | \varphi)_{\mathbb{H}} \quad \text{for all } \varphi \in \mathbb{H}_n.
\]

In particular, for a fixed $v \in \mathbb{H}$ the Riesz representation of the linear functional $\mathbb{H}_n \ni \varphi \mapsto (v|\varphi)_{\mathbb{H}}$ is equal $P_n v$.

Using Remark 4.6 we can write the integral identity (4.4) (tested by the functions from the subspace $\mathbb{H}_n$) in terms of appropriate Riesz representations. In fact, integral identity (4.4) is equivalent to the following stochastic equation in $\mathbb{H}_n$

\[
X_n(t) + \int_0^t \left[ A_n(X_n(s)) + \tilde{B}_n(X_n(s)) + \tilde{H}_n(X_n(s)) \right] ds = P_n X_0 + \int_0^t \int Y F_n(s, X_n(s^-); y) \tilde{\eta}(ds, dy), \quad t \in [0, T],
\]

where $F_n$ is a map defined by

\[
F_n : [0, T] \times \mathbb{H} \times Y \ni (s, X, y) \mapsto P_n F(t, X; y) \in \mathbb{H}_n \subset \mathbb{H}.
\]

**Proposition 4.7.** Let Assumption 3.1 be satisfied and let $X_0 \in \mathbb{H}$. For each $n \in \mathbb{N}$, there exists a unique global solution $(X_n(t))_{t \in [0, T]}$ of equation (4.11) with $\mathbb{H}_n$-valued càdlàg trajectories.

**Proof.** By Lemmas 2.2 and 2.4, for every $n \in \mathbb{N}$ the nonlinear terms $\tilde{B}_n$ and $\tilde{H}_n$ are locally Lipschitz (with the Lipschitz constants dependent on $n$). Moreover, by (4.9), (2.18), (4.10) and (2.25) for all $X_n \in \mathbb{H}_n$

\[
\langle \tilde{B}_n(X_n) + \tilde{H}_n(X_n) | X_n \rangle = 0.
\]

Now, the assertion follows from [2, Theorem 3.1].

**4.3 A priori estimates**

In this section we will prove some uniform estimates for the solutions $\{X_n, n \in \mathbb{N}\}$ of the truncated equation (4.11). These estimates will be used to prove the tightness of the family of laws of $X_n$, $n \in \mathbb{N}$, on the functional space $\mathcal{Z}$ defined by (5.8). In fact, in the proof of tightness the estimates for $q = 2$ from the following Lemma 4.8 are sufficient. Higher order estimates will be used in the proof of the convergence in Section 6.2.
Lemma 4.8. Let Assumption 3.1 be satisfied, let $X_0 \in \mathbb{H}$ and let $(X_n)_{n \in \mathbb{N}}$ be the solutions of equations (4.11). Then for every $q \geq 2$ there exist positive constants $C_1(q)$ and $C_2(q)$ such that

$$
\sup_{n \in \mathbb{N}} \mathbb{E} \left[ \sup_{s \in [0,T]} |X_n(s)|_{H}^q \right] \leq C_1(q) \tag{4.13}
$$

and

$$
\sup_{n \in \mathbb{N}} \mathbb{E} \left[ \int_0^T |X_n(s)|_{H}^{q-2} \|X_n(s)\|^2 \, ds \right] \leq C_2(q). \tag{4.14}
$$

In particular, there exists a positive constant $C_2$ such that

$$
\sup_{n \in \mathbb{N}} \mathbb{E} \left[ \int_0^T \|X_n(s)\|_{\varphi}^2 \, ds \right] \leq C_2. \tag{4.15}
$$

In the proof of Lemma 4.8 we will use the Itô formula and the following version of the Gronwall lemma.

Lemma 4.9. (See Lemma A.1 in [15].) Let $X,Y,I$ and $\varphi$ be non-negative processes and $Z$ be a non-negative integrable random variable. Assume that $I$ is a non-decreasing and there exist non-negative constants $C, \alpha, \beta, \gamma, \delta$ with the following properties

$$
\int_0^T \varphi(s) \, ds \leq C, \quad a.s., \quad 2\beta e^C \leq 1, \quad 2\delta e^C \leq \alpha, \tag{4.16}
$$

and such that for $0 \leq t \leq T$,

$$
X(t) + \alpha Y(t) \leq Z + \int_0^t \varphi(r) X(r) \, dr + I(t), \quad a.s.,
$$

$$
\mathbb{E}[I(t)] \leq \beta \mathbb{E}[X(t)] + \gamma \int_0^t \mathbb{E}[X(s)] \, ds + \delta \mathbb{E}[Y(t)] + \tilde{C},
$$

where $\tilde{C} > 0$ is a constant. If $X \in L^\infty([0,T] \times \Omega)$, then we have

$$
\mathbb{E}[X(t) + \alpha Y(t)] \leq 2 \exp(C + 2t\gamma e^C) \cdot (\mathbb{E}[Z] + \tilde{C}), \quad t \in [0,T]. \tag{4.17}
$$

Proof of Lemma 4.8. We apply the Itô formula to the function $f$ defined by

$$
f(x) := |x|^q_{H}, \quad x \in \mathbb{H}.
$$

In the sequel we will often omit the subscript $\mathbb{H}$ and write $|\cdot| := |\cdot|_{\mathbb{H}}$. The Fréchet derivative of $f$ is given by

$$
f'(x)(h) = dx f(h) = q \cdot |x|^{q-2} \cdot \langle x|h \rangle_{\mathbb{H}}, \quad h \in \mathbb{H}.
$$

By the Itô formula, see [17], we have for every $t \in [0,T]$

$$
|X_n(t)|^q - |P_n X_0|^q = \int_0^t q |X_n(s)|^{q-2} \langle X_n(s) | - A_n X_n(s) - \tilde{B}_n(X_n(s)) - \tilde{H}_n(X_n(s)) \rangle \, ds \\
+ \int_0^t \int_0^t \left\{ |X_n(s^-) + F_n(s,X_n(s^-);y)|^q_{\mathbb{H}} - |X_n(s^-)|^q_{\mathbb{H}} \\
- q |X_n(s^-)|^{q-2}_{\mathbb{H}} (X_n(s^-)|F_n(s,X_n(s^-);y))_{\mathbb{H}} \right\} \eta(ds,dy) \\
+ \int_0^t \int_0^t q |X_n(s^-)|^{q-2}_{\mathbb{H}} (X_n(s^-)|F_n(s,X_n(s^-);y))_{\mathbb{H}} \tilde{\eta}(ds,dy).
$$
Note that by (4.8) and (2.8) we have

\[(A_n X_n|X_n) = \langle AX_n|X_n \rangle = \|X_n\|^2.\]

By (4.9), (2.18), (4.10) and (2.25) we infer that

\[(\tilde{B}_n(X_n)|X_n) = (\tilde{B}(X_n)|X_n) = 0 \quad \text{and} \quad (\tilde{H}_n(X_n)|X_n) = (\tilde{H}(X_n)|X_n) = 0.\]

Thus for every \(t \in [0,T]\)

\[
|X_n(t)|^q + q \int_0^t |X_n(s)|^{q-2} \|X_n(s)\|^2 ds \leq |P_n X_0|^q + \int_0^t \int_Y \left\{ |X_n(s) + F_n(s, X_n(s); y)|^q_H - |X_n(s)|^q_H \\
- q |X_n(s)|^{q-2}(X_n(s)|F_n(s, X_n(s); y))_H \right\} \eta(ds, dy) \]

\[+ \int_0^t \int_Y q |X_n(s^-)|^{q-2}(X_n(s^-)|F_n(s, X_n(s^-); y))_H \tilde{\eta}(ds, dy).\]

(4.18)

For any \(R > 0 \) let us define the stopping time

\[
\tau_R^n := \inf \{ t \in [0,T] : |X_n(t)|_H > R \} \wedge T. \quad (4.19)
\]

Since \(\{X_n(t), t \in [0,T]\}\) is an \(H\)-valued \(\mathbb{F}\)-adapted and right-continuous process, \(\tau_R^n\) is a stopping time.

Let us fix \(t \in [0,T]\). From (4.18), we infer that

\[
\sup_{s \in [0,t \wedge \tau_R^n]} |X_n(s)|^q + \sup_{s \in [0,t \wedge \tau_R^n]} q \int_0^s |X_n(r)|^{q-2} \|X_n(r)\|^2 dr \leq \sup_{s \in [0,t \wedge \tau_R^n]} \left| \int_0^s \int_Y \left\{ |X_n(r) + F_n(r, X_n(r); y)|^q_H - |X_n(r)|^q_H \\
- q |X_n(r)|^{q-2}(X_n(r)|F_n(r, X_n(r); y))_H \right\} \eta(dr, dy) \right| \]

\[+ \int_0^s \int_Y q |X_n(r^-)|^{q-2}(X_n(r^-)|F_n(r, X_n(r^-); y))_H \tilde{\eta}(dr, dy).\]

(4.20)

Let us denote

\[\mathcal{U}_{n,R}(t) := \sup_{s \in [0,t \wedge \tau_R^n]} |X_n(s)|^q_H\]

\[\mathcal{V}_{n,R}(t) := \sup_{s \in [0,t \wedge \tau_R^n]} q \int_0^s |X_n(r)|^{q-2} \|X_n(r)\|^2 dr = q \int_0^{t \wedge \tau_R^n} |X_n(r)|^{q-2} \|X_n(r)\|^2 dr\]

\[\mathcal{I}_{n,R}(t) := \sup_{s \in [0,t \wedge \tau_R^n]} \left| \int_0^s \int_Y \left\{ |X_n(r) + F_n(r, X_n(r); y)|^q_H - |X_n(r)|^q_H \\
- q |X_n(r)|^{q-2}(X_n(r)|F_n(r, X_n(r); y))_H \right\} \eta(dr, dy) \right| \]

\[+ \int_0^s \int_Y q |X_n(r^-)|^{q-2}(X_n(r^-)|F_n(r, X_n(r^-); y))_H \tilde{\eta}(dr, dy),\]

(4.21)

where \(t \in [0,T]\).
Using the notations introduced in (4.21) by (4.20) we have for every \( t \in [0, T] \)
\[
\mathcal{U}_{n,R}(t) + \mathcal{V}_{n,R}(t) \leq |X_0|^q + \mathcal{I}_{n,R}(t). \tag{4.22}
\]
We will estimate the term \( \mathcal{I}_{n,R}(t) \). To this end, let
\[
\mathcal{I}_n(t) := \int_0^t \int_Y \left\{ |X_n(r) + F_n(r, X_n(r); y)|^q_H - |X_n(r)|^q_H - q |X_n(r)|^{q-2}_H (X_n(r)|F_n(r, X_n(r); y))_H \right\} \eta(dr, dy) \tag{4.23}
\]
\[
+ \int_0^t \int_Y q |X_n(r^-)|^{q-2}_H (X_n(r^-)|F_n(r, X_n(r^-); y))_H \tilde{\eta}(dr, dy)
\]
We decompose \( \mathcal{I}_n(t) \) into two terms
\[
\mathcal{I}_n(t) = \mathcal{J}_n(t) + \mathcal{M}_n(t), \quad t \in [0, T], \tag{4.24}
\]
where
\[
\mathcal{J}_n(t) := \int_0^t \int_Y \left\{ |X_n(r) + F_n(r, X_n(r); y)|^q_H - |X_n(r)|^q_H - q |X_n(r)|^{q-2}_H (X_n(r)|F_n(r, X_n(r); y))_H \right\} \eta(dr, dy), \quad t \in [0, T], \tag{4.25}
\]
and
\[
\mathcal{M}_n(t) := \int_0^t \int_Y q |X_n(r^-)|^{q-2}_H (X_n(r^-)|F_n(r, X_n(r^-); y))_H \tilde{\eta}(dr, dy), \quad t \in [0, T]. \tag{4.26}
\]
We will estimate separately the terms \( \mathcal{J}_n \) and \( \mathcal{M}_n \).
Let us consider first the term \( \mathcal{J}_n \) defined by (4.25). From the Taylor formula, it follows that for every \( q \geq 2 \) there exists a positive constant \( c_q > 0 \) such that for all \( x, h \in \mathbb{H} \) the following inequality holds
\[
||x + h||_H^q - |x|^q_H - q|x|^{q-2}_H |h|_H |x|_H \leq c_q(|x|^{q-2}_H + |h|^{q-2}_H) |h|_H^2. \tag{4.27}
\]
By (4.27), the fact that \( P_n : \mathbb{H} \to \mathbb{H}_n \) is the \((\cdot|\cdot)_H\)-projection, and inequality (3.3) we obtain
the following inequalities

\[ \mathbb{E} \left[ \sup_{s \in [0, t \wedge \tau_R^p]} |J_n(s)| \right] \leq \mathbb{E} \left[ \sup_{s \in [0, t \wedge \tau_R^p]} \int_0^s \int_Y |X_n(r) + F_n(r, X_n(r); y)|_H^q - |X_n(r)|_H^q \right. \]
\[ - q \left| X_n(r) \right|_H^{q-2} \left( X_n(r) |F_n(r, X_n(r); y)|_H^2 \right) \eta(dr, dy) \]
\[ \leq \mathbb{E} \left[ \int_0^{t \wedge \tau_R^p} \int_Y \left| X_n(r) + F_n(r, X_n(r); y) \right|_H^{q} - \left| X_n(r) \right|_H^{q} \right. \]
\[ - q \left| X_n(r) \right|_H^{q-2} \left( X_n(r) |F_n(r, X_n(r); y)|_H^2 \right) \eta(dr, dy) \]
\[ = \mathbb{E} \left[ \int_0^{t \wedge \tau_R^p} \int_Y \left| X_n(r) + F_n(r, X_n(r); y) \right|_H^{q} - \left| X_n(r) \right|_H^{q} \right. \]
\[ - q \left| X_n(r) \right|_H^{q-2} \left( X_n(r) |F_n(r, X_n(r); y)|_H^2 \right) \eta(dr, dy) \]
\[ \leq c_q \mathbb{E} \left[ \int_0^{t \wedge \tau_R^p} \int_Y \left| X_n(s, X_n(s); y) \right|_H^{2} \left( |X_n(s)|_H^{q-2} + |F_n(s, X_n(s); y)|_H^{q-2} \right) \eta(dy)ds \right] \]
\[ \leq c_q \mathbb{E} \left[ \int_0^{t \wedge \tau_R^p} \left\{ K_2 \left| X_n(s) \right|_H^{q-2} (1 + |X_n(s)|_H^2) + K_q (1 + |X_n(s)|_H^q) \right\} ds \right] \]
\[ \leq \tilde{c}_q \mathbb{E} \left[ \int_0^{t \wedge \tau_R^p} \left\{ 1 + \left| X_n(s) \right|_H^q \right\} ds \right] \leq \tilde{c}_q t + \tilde{c}_q \mathbb{E} \left[ \int_0^{t \wedge \tau_R^p} \left| X_n(s) \right|_H^q ds \right], \quad t \in [0, T], \]
where \( \tilde{c}_q > 0 \) is a certain constant.

Let us move to the term \( \mathcal{M}_n \) given by (4.26). By (3.6), the fact that \( P_n \) is the orthogonal projection in \( \mathbb{H} \) and (2.27), we infer that the process \( (\mathcal{M}_n(t \wedge \tau_R^p))_{t \in [0, T]} \) is a square integrable martingale. Indeed, this is a consequence of the following estimates. By (2.27), (3.6) and the fact that \( P_n \) is the orthogonal projection in \( \mathbb{H} \) we infer that for every \( t \in [0, T] \),

\[ \mathbb{E} \left[ \int_0^{t \wedge \tau_R^p} \int_Y \left| q \left| X_n(r) \right|_H^{q-2} \left( X_n(r) |F_n(r, X_n(r); y)|_H^2 \right) \right|_H^2 \eta(dy)dr \right] \]
\[ \leq q^2 \mathbb{E} \left[ \int_0^{t \wedge \tau_R^p} \int_Y \left| X_n(r) \right|_H^{2(q-2)} \left( X_n(r) |F_n(r, X_n(r); y)|_H^2 \right) \eta(dy) \right] \]
\[ = q^2 \mathbb{E} \left[ \int_0^{t \wedge \tau_R^p} \left| X_n(r) \right|_H^{2(q-2)} \left( \int_Y |F_n(r, X_n(r); y)|_H^2 \eta(dy) \right) dr \right] \]
\[ \leq q^2 \mathbb{E} \left[ \int_0^{t \wedge \tau_R^p} \left| X_n(r) \right|_H^{2(q-2)} K_2 (1 + \left| X_n(r) \right|_H^2) dr \right] \leq q^2 K_2 T R^{2q-2} (1 + R^2) < \infty. \]

By the maximal inequality we infer that there exists a positive constant \( \tilde{K}_2 \) such that

\[ \mathbb{E} \left[ \sup_{s \in [0, t \wedge \tau_R^p]} |\mathcal{M}_n(s)| \right] \]
\[ = \mathbb{E} \left[ \sup_{s \in [0, t \wedge \tau_R^p]} \left| \int_0^s \int_Y q \left| X_n(r) \right|_H^{q-2} \left( X_n(r) |F_n(r, X_n(r); y)|_H^2 \right) \eta(dr, dy) \right| \right] \]
\[ \leq \tilde{K}_2 q K_2 \mathbb{E}\left[ \left( \int_0^{t \wedge \tau_R^p} \left| X_n(r) \right|_H^{2(q-2)} (1 + \left| X_n(r) \right|_H^2) dr \right)^{\frac{1}{2}} \right] \]
\[ \leq \tilde{K}_2 q K_2 \mathbb{E}\left[ \left( \sup_{r \in [0, t \wedge \tau_R^p]} \left| X_n(r) \right|_H^q \right)^{\frac{1}{2}} \left( \int_0^{t \wedge \tau_R^p} \left| X_n(r) \right|_H^{2(q-2)} (1 + \left| X_n(r) \right|_H^2) dr \right)^{\frac{1}{2}} \right]. \]
Thus there exists a constant \( \tau \) such that
\[ \mathbb{E} \left[ \sup_{s \in [0, t \wedge \tau^R]} |M_n(s)| \right] \leq \varepsilon \mathbb{E} \left[ \sup_{r \in [0, t \wedge \tau^R]} |X_n(r)|_{\mathbb{H}}^q \right] + C_\varepsilon \mathbb{E} \left[ \int_0^{t \wedge \tau^R} |X_n(r)|_{\mathbb{H}}^{q-2} (1 + |X_n(r)|_{\mathbb{H}}^2) dr \right]. \]

Using moreover the Young inequality (for numbers) we infer that for every \( \varepsilon > 0 \) there exists \( C_\varepsilon > 0 \) such that
\[ \mathbb{E} \left[ \sup_{s \in [0, t \wedge \tau^R]} |M_n(s)| \right] \leq \varepsilon \mathbb{E} \left[ \sup_{r \in [0, t \wedge \tau^R]} |X_n(r)|_{\mathbb{H}}^q \right] + C_\varepsilon \mathbb{E} \left[ \int_0^{t \wedge \tau^R} \left\{ \frac{2}{q} + \left( 1 - \frac{1}{q} \right) |X_n(r)|_{\mathbb{H}}^q \right\} dr \right]. \]

Let us choose \( \varepsilon \in (0, \frac{1}{2}] \). By (4.21), (4.31) and Lemma 4.9 we obtain for every \( t \in [0, T] \)
\[ \mathbb{E} \left[ \mathcal{I}_{n,R}(t) \right] \leq \varepsilon \mathbb{E} \left[ \mathcal{J}_{n,R}(t) \right] + \mathbb{E} \left[ \sup_{s \in [0, t \wedge \tau^R]} |M_n(s)| \right] \]
\[ \leq \varepsilon \mathbb{E} \left[ \sup_{r \in [0, t \wedge \tau^R]} |X_n(r)|_{\mathbb{H}}^q \right] + K_1(q, \varepsilon, T) + K_2(q, \varepsilon) \mathbb{E} \left[ \int_0^{t \wedge \tau^R} |X_n(r)|_{\mathbb{H}}^q dr \right]. \]  \hspace{1cm} (4.30)

where \( K_1(q, \varepsilon, T) \) and \( K_2(q, \varepsilon) \) are some positive constants. Note that \( \mathbb{E} \left[ \int_0^{t \wedge \tau^R} |X_n(r)|_{\mathbb{H}}^q dr \right] \leq \mathbb{E} \left[ \int_0^{t \wedge \tau^R} |X_n(r)|_{\mathbb{H}}^q dr \right] \). Using the notation (4.21), by (4.30) we have
\[ \mathbb{E} \left[ \mathcal{I}_{n,R}(t) \right] \leq \varepsilon \mathbb{E} \left[ \mathcal{J}_{n,R}(t) \right] + K_1(q, \varepsilon, T) + K_2(q, \varepsilon) \int_0^t \mathbb{E} \left[ \mathcal{U}_{n,R}(r) \right] dr, \quad t \in [0, T]. \] \hspace{1cm} (4.31)

Let us choose \( \varepsilon \in (0, \frac{1}{2}] \). By (4.21), (4.31) and Lemma 4.9 we obtain
\[ \mathbb{E} \left[ \mathcal{U}_{n,R}(t) + \mathcal{V}_{n,R}(t) \right] \leq 2e^{2K_2(q, \varepsilon)} \left( \mathbb{E}[|X_0|_{\mathbb{H}}^q] + K_1(q, \varepsilon, T) \right), \quad t \in [0, T]. \] \hspace{1cm} (4.32)

Thus there exists a constant \( C(\varepsilon, q, T) > 0 \) such that for every \( t \in [0, T] \)
\[ \mathbb{E} \left[ \mathcal{U}_{n,R}(t) + \mathcal{V}_{n,R}(t) \right] \leq C(\varepsilon, q, T) \left( \mathbb{E}[|X_0|_{\mathbb{H}}^q] + 1 \right), \] \hspace{1cm} (4.33)

or explicitly
\[ \mathbb{E} \left[ \sup_{s \in [0, t \wedge \tau^R]} |X_n(s)|_{\mathbb{H}}^q + q \int_0^{t \wedge \tau^R} |X_n(s)|_{\mathbb{H}}^{q-2} \|X_n(s)\|^2 ds \right] \leq C(\varepsilon, q, T) \left( \mathbb{E}[|X_0|_{\mathbb{H}}^q] + 1 \right). \] \hspace{1cm} (4.34)

Recall that \( \tau^R \uparrow T \) as \( R \to \infty, \) \( \mathbb{P} \)-a.s. and \( \mathbb{P}\{\tau^R < T\} = 0. \) Using the Fatou lemma we infer that
\[ \mathbb{E} \left[ \sup_{s \in [0, T]} |X_n(s)|_{\mathbb{H}}^q + q \int_0^T |X_n(s)|_{\mathbb{H}}^{q-2} \|X_n(s)\|^2 ds \right] \leq C(\varepsilon, q, T) \left( \mathbb{E}[|X_0|_{\mathbb{H}}^q] + 1 \right). \] \hspace{1cm} (4.35)

From the above inequality we obtain estimates (4.13), (4.14) and (4.15). This concludes the proof of Lemma 4.8. \( \square \)
5 Compactness and tightness results.

5.1 The space $U$

It is well known that in the case when the domain is $\mathbb{R}^3$, thus unbounded, the standard Sobolev embedding are not compact. To overcome this problem we introduce auxiliary space $U$. In the functional setting of the Hall-MHD equations we have the following three basic spaces

$$V_{1,2} \subset V \subset H,$$

see Section 2.1 and Definition 3.3. Let us consider the space

$$V_s := V_m,$$ (5.1)

where $V_m$ is defined by (2.12). The choice of the space $V_s$ corresponds to the properties of nonlinear maps $\tilde{B}$ and $\tilde{H}$, see Lemmas 2.2(iii) and 2.4(iii) and Corollaries 2.3 and 2.5.

Since $V_s$ is dense in $H$ and the embedding $V_s \hookrightarrow H$ is continuous, by Lemma 2.5 from [18] (see [10, Lemma C.1]) there exists a separable Hilbert space $U$ such that $U \subset V_s$, $U$ is dense in $V_s$ and

the embedding $\iota : U \hookrightarrow V_s$ is compact. (5.2)

Then we have

$$U \hookrightarrow V_s \hookrightarrow V_{1,2} \hookrightarrow V \hookrightarrow H.$$ (5.3)

5.2 The space $Z$

In this section we define the space $Z$ which plays important role in our approach. By (5.2) and (5.3), in particular, we have

$$U \hookrightarrow V \hookrightarrow H \cong H' \hookrightarrow U',$$ (5.4)

the embedding $U \hookrightarrow V$ being compact. To define the space $Z$ we will need the following four functional spaces being the counterparts in our framework of the spaces used in [27], see also [23]:

- $D([0, T], U') :=$ the space of càdlàg functions $\phi : [0, T] \to U'$ with the topology $T_1$ induced by the Skorohod metric (see Appendix A),

- $L^2_w(0, T; V) :=$ the space $L^2(0, T; V)$ with the weak topology $T_2$,

- $L^2(0, T; H_{loc}) :=$ the space of measurable functions $\phi : [0, T] \to H$ such that for all $R \in \mathbb{N}$

$$\begin{align*}
PT,R(\phi) := \left( \int_0^T \int_{O_R} [\phi_1(t, x)]^2 + [\phi_2(t, x)]^2 \, dx \, dt \right)^{\frac{1}{2}} \leq \infty,
\end{align*}$$

where $\phi = (\phi_1, \phi_2)$, with the topology $T_3$ generated by the seminorms $(PT,R)_{R \in \mathbb{N}}$.

Let $H_w$ denotes the Hilbert space $H$ endowed with the weak topology. Let us consider the fourth space, see [27],
Let us consider the ball $B := \{ x \in \mathbb{H} : |x|_\mathbb{H} \leq r \}$.

Let $B_w$ denote the ball $B$ endowed with the weak topology. It is well-known that $B_w$ is metrizable, see [6]. Let $q_r$ denote the metric compatible with the weak topology on $B$. Let us denote by $D([0, T]; B_w)$ the space of functions $\phi \in D([0, T]; \mathbb{H}_w)$ such that

$$\sup_{t \in [0, T]} |\phi(t)|_{\mathbb{H}_w} \leq r.$$  \hspace{1cm} (5.6)

The space $D([0, T]; B_w)$ is completely metrizable, as well. In fact, $D([0, T]; B_w)$ is metrizable with

$$\delta_{T, r}(\phi, v) = \inf_{\lambda \in \Lambda_T} \left\{ \sup_{t \in [0, T]} q_r(\phi(t), v \circ \lambda(t)) + \sup_{t \in [0, T]} |t - \lambda(t)| + \sup_{s \neq t} \left| \log \frac{\lambda(t) - \lambda(s)}{t - s} \right| \right\}. \hspace{1cm} (5.7)$$

Since by the Banach-Alaoglu Theorem $B_w$ is compact, $(D([0, T]; B_w), \delta_{T, r})$ is a complete metric space.

**Definition 5.1.** Let us put

$$Z := L^2_w(0, T; \mathbb{V}) \cap L^2(0, T; \mathbb{H}_{loc}) \cap D([0, T]; \mathbb{H}_w) \cap D([0, T]; \mathbb{U'})$$  \hspace{1cm} (5.8)

and let $\mathcal{T}$ be the supremum of the corresponding four topologies, i.e. the smallest topology on $Z$ such that the four natural embeddings from $Z$ are continuous. The space $Z$ will also be considered with the Borel $\sigma$-field, denoted by $\sigma(Z)$, i.e. the smallest $\sigma$-field containing the family $\mathcal{T}$.

### 5.3 Deterministic compactness theorem

The following lemma says that any bounded sequence $(\phi_n) \subset L^\infty(0, T; \mathbb{H})$ convergent in $D([0, T]; \mathbb{U'})$ is convergent in the space $D([0, T]; B_w)$, as well. It is closely related to the lemma due to Strauss, see [36], that says:

$$L^\infty(0, T; \mathbb{H}) \cap C([0, T]; \mathbb{U'_w}) \subset C([0, T]; \mathbb{H}_w),$$  \hspace{1cm} (5.9)

where $C([0, T]; \mathbb{U'_w})$ and $C([0, T]; \mathbb{H}_w)$ denote the space of $\mathbb{U'}$ and $\mathbb{H}$-valued, respectively, weakly continuous functions.

**Lemma 5.2.** (See [27] Lemma 2.) Let $\phi_n : [0, T] \to \mathbb{H}$, $n \in \mathbb{N}$, be functions such that

(i) $\sup_{n \in \mathbb{N}} \sup_{s \in [0, T]} |\phi_n(s)|_{\mathbb{H}} \leq r,$
(ii) $\phi_n \to \phi$ in $\mathbb{D}([0,T];U')$.

Then $\phi, \phi_n \in \mathbb{D}([0,T];B_w)$ and $\phi_n \to \phi$ in $\mathbb{D}([0,T];B_w)$ as $n \to \infty$.

The compactness criterion contained in the following Theorem 5.3 can be seen as the generalization of the classical Dubinsky theorem, see [39, Theorem IV.4.1], to the case when the embedding $V \subset H$ is continuous and possibly not compact and the space of continuous functions is replaced by the space of càdlàg functions $\mathbb{D}([0,T];U')$. Together with Lemma 5.2 it is a simple modification of Lemma 4.1 from [28]. See also [27, Theorem 2] for the case of the Navier-Stokes equations.

**Theorem 5.3.** (See [28, Lemma 4.1] ) Let

\[ \mathcal{K} \subset L^\infty(0,T;H) \cap L^2(0,T;V) \cap \mathbb{D}([0,T];U') \]

be a set satisfying the following three conditions

(a) for all $\phi \in \mathcal{K}$ and all $t \in [0,T]$, $\phi(t) \in H$ and $\sup_{\phi \in \mathcal{K}} \sup_{s \in [0,T]} |\phi(s)|_H < \infty$,

(b) $\sup_{\phi \in \mathcal{K}} \int_0^T \|\phi(s)\|_V^2 \, ds < \infty$, i.e. $\mathcal{K}$ is bounded in $L^2(0,T;V)$,

(c) $\lim_{\delta \to 0} \sup_{\phi \in \mathcal{K}} w_{[0,T],U'}(\phi; \delta) = 0$.

Then $\mathcal{K} \subset \mathcal{Z}$ and $\mathcal{K}$ is $T$-relatively compact in $\mathcal{Z}$ defined by (5.8).

### 5.4 Tightness criterion

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with filtration $\mathbb{F} := (\mathcal{F}_t)_{t \in [0,T]}$ satisfying the usual conditions. Using Theorem 5.3 we get the corresponding tightness criterion in the measurable space $(\mathcal{Z}, \sigma(\mathcal{Z}))$.

**Corollary 5.4.** (See [27, Corollary 1].) Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of càdlàg $\mathbb{F}$-adapted $U'$-valued processes such that

(a) there exists a positive constant $C_1$ such that

\[ \sup_{n \in \mathbb{N}} \mathbb{E} \left[ \sup_{s \in [0,T]} |X_n(s)|_H \right] \leq C_1, \]

(b) there exists a positive constant $C_2$ such that

\[ \sup_{n \in \mathbb{N}} \mathbb{E} \left[ \int_0^T \|X_n(s)\|_V^2 \, ds \right] \leq C_2, \]

(c) $(X_n)_{n \in \mathbb{N}}$ satisfies the Aldous condition in $U'$.

Let $\tilde{\mathbb{P}}_n$ be the law of $X_n$ on $\mathcal{Z}$. Then for every $\varepsilon > 0$ there exists a compact subset $K_\varepsilon$ of $\mathcal{Z}$ such that

\[ \tilde{\mathbb{P}}_n(K_\varepsilon) \geq 1 - \varepsilon. \]

For the completeness of presentation we recall the Aldous condition in the form given by Métivier [25].
Definition 5.5. (M. Métivier) A sequence \((X_n)_{n \in \mathbb{N}}\) satisfies the **Aldous condition** in the space \(U'\) iff

\[
\text{[A]} \quad \text{for every } \varepsilon > 0 \text{ and } \eta > 0 \text{ there exists } \delta > 0 \text{ such that for every sequence } (\tau_n)_{n \in \mathbb{N}} \text{ of } \mathbb{F}\text{-stopping times with } \tau_n \leq T \text{ one has }
\]

\[
\sup_{n \in \mathbb{N}} \sup_{0 \leq \theta \leq \delta} \mathbb{P} \{ |X_n(\tau_n + \theta) - X_n(\tau_n)|_{U'} \geq \eta \} \leq \varepsilon.
\]

In the following lemma we recall a certain condition which guarantees that the sequence \((X_n)_{n \in \mathbb{N}}\) satisfies condition \([A]\).

Lemma 5.6. (See [27], Lemma 9) Let \((E, \| \cdot \|_E)\) be a separable Banach space and let \((X_n)_{n \in \mathbb{N}}\) be a sequence of \(E\)-valued random variables such that

\[
\text{[A']} \quad \text{there exist } \alpha, \beta > 0 \text{ and } C > 0 \text{ such that for every sequence } (\tau_n)_{n \in \mathbb{N}} \text{ of } \mathbb{F}\text{-stopping times with } \tau_n \leq T \text{ and for every } n \in \mathbb{N} \text{ and } \theta \geq 0 \text{ the following condition holds :}
\]

\[
\mathbb{E} \left[ \|X_n(\tau_n + \theta) - X_n(\tau_n)\|_{E}^{\alpha} \right] \leq C \theta^{\beta}.
\]

(5.10)

Then the sequence \((X_n)_{n \in \mathbb{N}}\) satisfies condition \([A]\) in the space \(E\).

6 Existence of a martingale solution

We will use the structure introduced in Section 5.1. By (5.3) we have the following spaces \(U \subset V_{*} \subset V_{1,2} \subset V \subset H\), where \(V_{*} = V_{m}\) for fixed \(m > \frac{3}{2}\). Considering the dual spaces, and identifying \(H\) with its dual \(H'\), we have the following system

\[
U \subset V_{*} \subset V_{1,2} \subset V \subset H \cong H' \to \mathbb{V}' \to \mathcal{V}_{*}' \to U'.
\]

(6.1)

In (4.2) we have defined the map \(P_{n} : H \to \mathbb{H}_{n}\) being the \((\cdot\,|\,\cdot)_{\mathbb{H}}\)-orthogonal projection onto \((\mathbb{H}_{n},\,(\cdot\,|\,\cdot)_{\mathbb{H}})\). Using further properties of the map \(P_{n}\) stated in Corollary 4.4 we will consider the adjoint operators \(P'_{n}\).

Remark 6.1. Since \(P_{n} \in \mathcal{L}(V,V)\), its adjoint \(P'_{n} \in \mathcal{L}(V',V')\) by definition satisfies

\[
\mathcal{V}' \langle \xi | P'_{n} \varphi \rangle_{\mathcal{V}} = \mathcal{V}' \langle P'_{n} \xi | \varphi \rangle_{\mathcal{V}}, \quad \text{for all } \xi \in \mathcal{V}', \ \varphi \in \mathcal{V}.
\]

(6.2)

Since \(P_{n} \in \mathcal{L}(V_{*},V)\), its adjoint \(P'_{n} \in \mathcal{L}(V_{*}',V_{*}')\) satisfies

\[
\mathcal{V}_{*}' \langle \xi | P'_{n} \varphi \rangle_{\mathcal{V}_{*}} = \mathcal{V}_{*}' \langle P'_{n} \xi | \varphi \rangle_{\mathcal{V}_{*}}, \quad \text{for all } \xi \in \mathcal{V}_{*}', \ \varphi \in \mathcal{V}_{*}.
\]

(6.3)

Since \(P_{n} \in \mathcal{L}(V_{*},V_{*})\), its adjoint \(P'_{n} \in \mathcal{L}(V_{*}',V_{*}')\) satisfies

\[
\mathcal{V}_{*}' \langle \xi | P'_{n} \varphi \rangle_{\mathcal{V}_{*}} = \mathcal{V}_{*}' \langle P'_{n} \xi | \varphi \rangle_{\mathcal{V}_{*}}, \quad \text{for all } \xi \in \mathcal{V}_{*}', \ \varphi \in \mathcal{V}_{*}.
\]

(6.4)

Remark 6.1 enables us to rewrite the truncated equation as an equation in the space \(V_{*}'\), and by the injection \(V_{*}' \to U'\) - also as an equation in \(U'\).

Remark 6.2.
(i) If the $\mathbb{H}_n$-valued process $X_n$ satisfies identity (4.4), then in particular, for all $t \in [0, T]$ and $\varphi \in \mathbb{V}_s$ we have $P_n \varphi \in \mathbb{H}_n$ and

$$
(X_n(t)|P_n \varphi)_\mathbb{H} + \int_0^t \nu'(AX_n(s)|P_n \varphi)_\mathbb{V} \, ds + \int_0^t \nu'_*(\tilde{B}(X_n(s))|P_n \varphi)_\mathbb{V}_s \, ds
$$

$$
+ \int_0^t \nu'_*(\tilde{H}(X_n(s))|P_n \varphi)_\mathbb{V}_s \, ds
$$

$$
= (X_0|P_n \varphi)_\mathbb{H} + \int_0^t \int_0^s (F(s, X_n(s^-); y)|P_n \varphi)_\mathbb{H} \, \tilde{\eta}(ds, dy).
$$

Since $P_n : \mathbb{H} \to \mathbb{H}_n$ is an $(\cdot|\cdot)_\mathbb{H}$-orthogonal projection, we have $(X_n(t)|P_n \varphi)_\mathbb{H} = (P_nX_n(t)|\varphi)_\mathbb{H} = (X_n(t)|\varphi)_\mathbb{H}$. Using the operators $P'_n$ from Remark 5.6, identity (6.5) can be written in the form

$$
(X_n(t)|\varphi)_\mathbb{H} + \int_0^t \nu'(P'_nAX_n(s)|\varphi)_\mathbb{V} \, ds + \int_0^t \nu'_*(P'_n\tilde{B}(X_n(s))|\varphi)_\mathbb{V}_s \, ds
$$

$$
+ \int_0^t \nu'_*(P'_n\tilde{H}(X_n(s))|\varphi)_\mathbb{V}_s \, ds
$$

$$
= (P_nX_0|\varphi)_\mathbb{H} + \int_0^t \int_0^s (F_n(s, X_n(s^-); y)|\varphi)_\mathbb{H} \, \tilde{\eta}(ds, dy), \quad t \in [0, T],
$$

where $F_n$ is defined by (4.12).

(ii) By the identification $\mathbb{H} \cong \mathbb{H}'$ and the fact that $\mathbb{H}' \hookrightarrow \mathbb{V}' \hookrightarrow \mathbb{V}'_s$, we identify $X_n(t)$ with the functional induced by $X_n(t)$ on the space $\mathbb{V}_s$. From (6.6), we infer that $X_n(t)$ satisfies the following equation

$$
X_n(t) + \int_0^t [P'_nAX_n(s) + P'_n\tilde{B}(X_n(s)) + P'_n\tilde{H}(X_n(s))] \, ds
$$

$$
= P_nX_0 + \int_0^t \int_0^s F_n(s, X_n(s^-); y) \, \tilde{\eta}(ds, dy), \quad t \in [0, T].
$$

6.1 Tightness

Let us consider the sequence $(X_n)_{n \in \mathbb{N}}$ of approximate solutions. We will prove that the sequence of laws is tight in the space $\mathbb{Z}$ defined by (5.8), i.e.

$$
\mathbb{Z} := L^2_{w}(0, T; \mathbb{V}) \cap L^2(0, T; \mathbb{H}_{loc}) \cap D([0, T]; \mathbb{H}_w) \cap D([0, T]; \mathbb{V}'),
$$

equipped with the Borel $\sigma$-field $\sigma(\mathcal{T})$, see Definition 5.1.

**Lemma 6.3.** The set of probability measures $\{\text{Law}(X_n), n \in \mathbb{N}\}$ is tight on the space $(\mathbb{Z}, \sigma(\mathcal{T}))$.

**Proof.** We apply Corollary 5.4. By estimates (4.13) and (4.15), conditions (a) and (b) of Corollary 5.4 are satisfied. Thus, it is sufficient to prove that the sequence $(X_n)_{n \in \mathbb{N}}$ satisfies the Aldous condition [A]. By Lemma 5.6 it is sufficient to prove that $(X_n)_{n \in \mathbb{N}}$ satisfies condition [A*].
Let \((\tau_n)_{n \in \mathbb{N}}\) be a sequence of stopping times taking values in \([0, T]\). By Remark 6.2 (ii), we have

\[
X_n(t) = P_n X_0 - \int_0^t P_n' A X_n(s) \, ds - \int_0^t P_n' B (X_n(s)) \, ds - \int_0^t P_n' \bar{H}(X_n(s)) \, ds \\
+ \int_0^t \int Y F_n(s, X_n(s^-); y) \eta(dy, ds)
\]

\[
= J^1_n + J^2_n(t) + J^3_n(t) + J^4_n(t) + J^5_n(t), \quad t \in [0, T].
\]

Let us choose \(\theta > 0\). It is sufficient to show that each sequence \(J^i_n\) of processes, \(i = 1, \cdots, 5\), satisfies the sufficient condition \([A']\) from Lemma 5.6.

Obviously the term \(J^1_n\) which is constant in time, satisfies this condition. In fact, we will check that the term \(J^2_n\) satisfies condition \([A']\) from Lemma 5.6 in the space \(E = V'\), the terms \(J^3_n, J^4_n\) satisfy this condition in \(E = V'_s\) and the term \(J^5_n\) satisfies this condition in \(E = H\). Since the embeddings \(H \subset U', V' \subset U'\) and \(V'_s \subset U'\) are continuous, we infer that \([A']\) holds in the space \(E = U'\), as well.

Ad \(J^2_n\). By Remark 2.4 the linear operator \(A : V \to V'\) is bounded, and by Corollary 4.3, \(\sup_{n \in \mathbb{N}} |P_n|_{\mathcal{L}(V, V')} < \infty\). Using the Hölder inequality and (4.15), we obtain

\[
\mathbb{E} \left[ |J^2_n(\tau_n + \theta) - J^2_n(\tau_n)|_{V'} \right] \leq |P_n'|_{\mathcal{L}(V', V')} \mathbb{E} \left[ \int_{\tau_n}^{\tau_n + \theta} |AX_n(s)|_{V'}^2 \, ds \right]^{\frac{1}{2}} \leq |P_n|_{\mathcal{L}(V, V')} C_2^{\frac{1}{2}} \cdot \theta^\frac{1}{2} = c_2 \cdot \theta^\frac{1}{2},
\]

where \(c_2 = C_2^{\frac{1}{2}} \cdot \sup_{n \in \mathbb{N}} |P_n|_{\mathcal{L}(V, V')} < \infty\).

Ad \(J^3_n\). By (2.19) in Lemma 2.2, \(\bar{B} : H \times H \to V'_s\) is bilinear and continuous (and hence bounded so that the norm \(\|\bar{B}\|\) of \(\bar{B} : H \times H \to V'_s\) is finite), and by Corollary 4.4, \(\sup_{n \in \mathbb{N}} |P_n|_{\mathcal{L}(V_s, V_s)} < \infty\). Then by (4.13) we have the following estimates

\[
\mathbb{E} \left[ |J^3_n(\tau_n + \theta) - J^3_n(\tau_n)|_{V'_s} \right] = |P'_n|_{\mathcal{L}(V'_s, V'_s)} \mathbb{E} \left[ \left. \int_{\tau_n}^{\tau_n + \theta} \bar{B}(X_n(r)) \, dr \right|_{V'_s} \right] \\
\leq |P_n|_{\mathcal{L}(V_s, V_s)} \mathbb{E} \left[ \int_{\tau_n}^{\tau_n + \theta} \|\bar{B}(X_n(r))\|_H \, dr \right] \leq |P_n|_{\mathcal{L}(V_s, V_s)} \|\bar{B}\| \mathbb{E} \left[ \int_{\tau_n}^{\tau_n + \theta} |X_n(r)|_H^2 \, dr \right] \\
\leq |P_n|_{\mathcal{L}(V_s, V_s)} \|\bar{B}\| \cdot \mathbb{E} \left[ \sup_{r \in [0, T]} X_n(r) \right] \cdot \theta \leq |P_n|_{\mathcal{L}(V_s, V_s)} \|\bar{B}\| C_1(2) \cdot \theta = c_3 \theta,
\]

where \(c_3 = \sup_{n \in \mathbb{N}} |P_n|_{\mathcal{L}(V_s, V_s)} \|\bar{B}\| C_1(2) < \infty\).

Ad \(J^4_n\). By (2.26) in Lemma 2.4, \(\tilde{H} : H \times V \to V'_s\) is bilinear and continuous (and hence bounded so that the norm \(\|\tilde{H}\|\) of \(\tilde{H} : H \times V \to V'_s\) is finite) and by Corollary 4.4,
sup_{n \in \mathbb{N}} |P_n|_{\mathcal{L}(\mathcal{V}_*, \mathcal{V}_*)} < \infty. \text{ Then by (4.13) and (4.15) we have the following estimates}

\[
\mathbb{E} \left[|J^3_n(\tau_n + \theta) - J^3_n(\tau_n)|_{\mathcal{V}_*}^2\right] \leq |P'_n|_{\mathcal{L}(\mathcal{V}_*, \mathcal{V}_*)} \mathbb{E} \left[\int_{\tau_n}^{\tau_n + \theta} |\tilde{\mathcal{H}}(X_n(r))| \|X_n(r)\| \, dr\right]
\leq |P_n|_{\mathcal{L}(\mathcal{V}_*, \mathcal{V}_*)} \|\tilde{\mathcal{H}}\| \mathbb{E} \left[\int_{\tau_n}^{\tau_n + \theta} |X_n(r)| \|X_n(r)\| \, dr\right]
\leq |P_n|_{\mathcal{L}(\mathcal{V}_*, \mathcal{V}_*)} \|\tilde{\mathcal{H}}\| \left(\mathbb{E} \left[\sup_{r \in [\tau_n, \tau_n + \theta]} |X_n(r)|^2 \right]\right)^{\frac{1}{2}} \left(\mathbb{E} \left[\int_{\tau_n}^{\tau_n + \theta} \|X_n(r)\|^2 \, dr\right]\right)^{\frac{1}{2}} \theta^\frac{1}{2}
\leq |P_n|_{\mathcal{L}(\mathcal{V}_*, \mathcal{V}_*)} \|\tilde{\mathcal{H}}\| \left[C_1(2)\right]^{\frac{1}{2}} \mathbb{E} \left[\sup_{r \in [0, T]} |X_n(s)|^2 \right] \leq c_4 \theta^\frac{1}{2},
\]

where \(c_4 = \sup_{n \in \mathbb{N}} |P_n|_{\mathcal{L}(\mathcal{V}_*, \mathcal{V}_*)} \|\tilde{\mathcal{H}}\| \left[C_1(2)\right]^{\frac{1}{2}} < \infty.

\textbf{Ad} \ J^5_n. \text{ Let us consider the noise term } J^5_n. \text{ By (2.27), the fact that } P_n : \mathbb{H} \to \mathbb{H}_n \text{ is the } (\cdot|\cdot)_{\mathbb{H}}\text{-orthogonal projection, inequality (3.6) with } p = 2 \text{ and by (4.13), we obtain}

\[
\mathbb{E} \left[|J^5_n(\tau_n + \theta) - J^5_n(\tau_n)|^2 \right] = \mathbb{E} \left[\int_{\tau_n}^{\tau_n + \theta} \int_Y F_n(s, X_n(s^{-})); y) \eta(ds, dy) \right]^2
\leq K_2 \mathbb{E} \left[\int_{\tau_n}^{\tau_n + \theta} (1 + |X_n(s)|^2) \, ds\right]
\leq K_2 \cdot \theta \cdot \left(1 + \mathbb{E} \left[\sup_{s \in [0, T]} |X_n(s)|^2 \right]\right) \leq K_2 \cdot (1 + C_1(2)) \cdot \theta =: c_5 \cdot \theta,
\]

where \(c_5 = K_2 \cdot (1 + C_1(2)). \text{ Thus } J^5_n \text{ satisfies condition (5.10).}

Thus the proof of Lemma 6.3 is complete. \(\square\)

### 6.2 Proof of Theorem 3.4

By Lemma 6.3 the set of measures \(\{\text{Law}(X_n), n \in \mathbb{N}\} \) is tight on the space \((\mathcal{Z}, \sigma(T))\) defined by (5.8). Let \(\eta_n := \eta, n \in \mathbb{N}. \text{ The set of measures } \{\text{Law}(\eta_n), n \in \mathbb{N}\} \text{ is tight on the space } M_{\mathbb{H}}([0, T] \times Y). \text{ Thus the set } \{\text{Law}(\eta_n, X_n), n \in \mathbb{N}\} \text{ is tight on } M_{\mathbb{H}}([0, T] \times Y) \times \mathcal{Z}. \text{ By Corollary B.1 and Remark B.2 see Appendix B there exists a subsequence } (n_k)_{k \in \mathbb{N}}, \text{ a probability space } (\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}) \text{ and, on this space, } M_{\mathbb{H}}([0, T] \times Y) \times \mathcal{Z}\text{-valued random variables } (\eta_k, X_k), (\bar{\eta}_k, \bar{X}_k), k \in \mathbb{N} \text{ such that}

(i) \(\text{Law}((\bar{\eta}_k, \bar{X}_k)) = \text{Law}((\eta_{n_k}, X_{n_k})) \text{ for all } k \in \mathbb{N};

(ii) \((\bar{\eta}_k, \bar{X}_k) \to (\eta_k, X_k) \text{ in } M_{\mathbb{H}}([0, T] \times Y) \times \mathcal{Z} \text{ with probability 1 on } (\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}) \text{ as } k \to \infty;

(iii) \bar{\eta}_k(\bar{\omega}) = \eta_k(\bar{\omega}) \text{ for all } \bar{\omega} \in \bar{\Omega}.

We will denote these sequences again by \(\{(\eta_n, X_n)\}_{n \in \mathbb{N}} \) and \(\{\eta_n, X_n\}_{n \in \mathbb{N}}\). Moreover, \(\eta_n, n \in \mathbb{N}, \text{ and } \eta_n \text{ are time homogeneous Poisson random measures on } (Y, \mathcal{Y}) \text{ with the intensity measure } \mu, \text{ see [9 Section 9]. Using the definition of the space } \mathcal{Z}, \text{ see (5.8), we have}

\[
X_n \to X_\ast \text{ in } L_\mu(0, T; \mathcal{V}) \cap L^2(0, T; \mathbb{H}_{\text{loc}}) \cap \mathcal{D}([0, T]; \mathcal{Y}) \cap \mathcal{D}([0, T]; \mathbb{H}_w) \text{ } \bar{P}\text{-a.s. (6.12)}
\]
In particular,
\[ X_\tau \in L^2(0,T; \mathcal{V}) \cap \mathcal{D}([0,T]; \mathbb{H}_{w}) \cap \mathcal{D}([0,T]; \mathbb{U}) . \] (6.13)
Since the random variables \( X_n \) and \( X_\tau \) are identically distributed, using Lemma 4.8 we infer that \( X_n \) satisfy the following inequalities. For every \( q \in [2, \infty) \)
\[
\sup_{n \geq 1} \mathbb{E} \left[ \sup_{0 \leq s \leq T} |X_n(s)|^q_{\mathcal{H}} \right] \leq C_1(q). \] (6.14)
and
\[
\sup_{n \geq 1} \mathbb{E} \left[ \int_0^T \| X_n(s) \|^q \| \varphi \| ds \right] \leq C_2. \] (6.15)
Let us fix \( p \in [2, \infty) \). We will show that for every \( q \in [2,p] \)
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} |X_\tau(t)|^p_{\mathcal{H}} \right] < \infty. \] (6.16)
By inequality (6.14) with \( q := p \) we can choose a further subsequence of \( (X_n) \) convergent weak in the space \( L^p(\Omega; L^\infty(0,T; \mathbb{H})) \), and using (6.12), deduce that
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} |X_\tau(t)|^p_{\mathcal{H}} \right] \leq C_1(p). \]
Let us fix \( q \in [1,p] \). Notice that for every \( t \in [0,T] \)
\[
\| X_\tau(t) \|^q_{\mathcal{H}} = (\| X_\tau(t) \|^p_{\mathcal{H}})^{q/p} \leq \left( \sup_{t \in [0,T]} |X_\tau(t)|^p_{\mathcal{H}} \right)^{q/p} .
\]
Thus, \( \sup_{t \in [0,T]} |X_\tau(t)|^q_{\mathcal{H}} \leq \left( \sup_{t \in [0,T]} |X_\tau(t)|^p_{\mathcal{H}} \right)^{q/p} \), and so by the Hölder inequality
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} |X_\tau(t)|^q_{\mathcal{H}} \right] \leq \mathbb{E} \left[ \left( \sup_{t \in [0,T]} |X_\tau(t)|^p_{\mathcal{H}} \right)^{q/p} \right] 
\leq \left( \mathbb{E} \left[ \sup_{t \in [0,T]} |X_\tau(t)|^p_{\mathcal{H}} \right] \right)^{q/p} \leq C_1(p,q),
\]
where \( C_1(p,q) := (C_1(p))^{q/p} \). Thus (6.16) holds.
By inequality (6.15), we infer the sequence \( (X_n) \) contains further subsequence, denoted again by \( (X_n) \), convergent weakly in the space \( L^2([0,T] \times \tilde{\Omega}; \mathcal{V}) \). Since by (6.12), \( X_n \to X_\tau \) in \( \mathcal{Z} \), we infer that \( X_\tau \in L^2([0,T] \times \tilde{\Omega}; \mathcal{V}) \), i.e.
\[
\tilde{\mathbb{E}} \left[ \int_0^T \| X_\tau \|^2_{\mathcal{V}} ds \right] < \infty . \] (6.17)
We will prove that the system \( (\Omega, \mathcal{F}, \tilde{\mathbb{P}}, \tilde{\mathbb{P}}, \eta_{\tau}, X_\tau) \) is a martingale solution of problem (3.8).

**Step 1.** Let us fix \( \varphi \in \mathcal{V}_* \). Analogously to [28], let us denote
\[
\Lambda_n(X_n, \eta_n, \varphi)(t) := (X_n(0)|P_n \varphi \rangle_{\mathcal{H}} - \int_0^t \langle A X_n(s)|P_n \varphi \rangle ds 
- \int_0^t \langle B(X_n(s))|P_n \varphi \rangle ds - \int_0^t \langle \mathcal{H}(X_n(s))|P_n \varphi \rangle ds 
+ \int_0^t \int_Y \langle F(s, X_n(s^-); y)|P_n \varphi \rangle d\tilde{\eta}_n(ds,dy) \quad t \in [0,T] ,
\] (6.18)
and
\[
\Lambda(X_s, \bar{\eta}, \varphi)(t) := (X_s(0)|\varphi)_H - \int_0^t \langle AX_s(s)|\varphi \rangle ds
- \int_0^t \langle \tilde{B}(X_s(s))|\varphi \rangle ds - \int_0^t \langle \tilde{H}(X_s(s))|\varphi \rangle ds
+ \int_0^t \int_{\mathcal{Y}} (F(s, X_s(s^-); y)|\varphi)_H \bar{\eta}_s(ds, dy), \quad t \in [0, T].
\] (6.19)

In the following lemma we will prove that each term on the r.h.s. of (6.18) is convergent to the corresponding term on the r.h.s of (6.19).

**Lemma 6.4.** We have

(a) for every \( \varphi \in H \)
\[
\lim_{n \to \infty} \mathbb{E} \left[ \int_0^T \left| (\bar{X}_n(t)|P_n \varphi)_H - (X_s(t)|\varphi)_H \right|^2 dt \right] = 0,
\]

(b) for every \( \varphi \in H \)
\[
\lim_{n \to \infty} \mathbb{E} \left[ \left| (\bar{X}_n(0)|P_n \varphi)_H - (X_s(0)|\varphi)_H \right|^2 \right] = 0,
\]

(c) for every \( \varphi \in V \)
\[
\lim_{n \to \infty} \mathbb{E} \left[ \int_0^T \left| \int_0^t (\langle AX_n(s)|P_n \varphi \rangle - \langle AX_s(s)|\varphi \rangle) ds \right| dt \right] = 0,
\]

(d) for every \( \varphi \in V_\ast \)
\[
\lim_{n \to \infty} \mathbb{E} \left[ \int_0^T \left| \int_0^t (\langle \tilde{B}(X_n(s))|P_n \varphi \rangle - \langle \tilde{B}(X_s(s))|\varphi \rangle) ds \right| dt \right] = 0,
\]

(e) for every \( \varphi \in V_\ast \)
\[
\lim_{n \to \infty} \mathbb{E} \left[ \int_0^T \left| \int_0^t (\langle \tilde{H}(X_n(s))|P_n \varphi \rangle - \langle \tilde{H}(X_s(s))|\varphi \rangle) ds \right| dt \right] = 0,
\]

(f) for every \( \varphi \in H \)
\[
\lim_{n \to \infty} \mathbb{E} \left[ \int_0^T \left| \int_0^t \int_{\mathcal{Y}} (P_n F(s, \bar{X}_n(s^-); y)|\varphi)_H \bar{\eta}_n(ds, dy) \right| dt \right] = 0,
\]
\[
- \int_0^T \int_{\mathcal{Y}} \langle F(s, X_s(s^-); y)|\varphi \rangle \bar{\eta}_s(ds, dy) \bigg|^2 dt \right] = 0.
\]

**Proof.** Ad. (a). Let us fix \( \varphi \in H \). Since
\[
(X_n(t)|P_n \varphi)_H = (P_n \bar{X}_n(t)|\varphi)_H = (\bar{X}_n(t)|\varphi)_H,
\]
and by (6.12) \( \bar{X}_n \to X_s \) in \( \mathbb{D}([0, T]; H) \), \( \mathbb{P} \)-a.s., we infer that \( (\bar{X}_n(\cdot)|\varphi)_H \to (X_s(\cdot)|\varphi)_H \) in \( \mathbb{D}([0, T]; \mathbb{R}) \), \( \mathbb{P} \)-a.s.. Hence, in particular, for almost all \( t \in [0, T] \)
\[
\lim_{n \to \infty} (\bar{X}_n(t)|\varphi)_H = (X_s(t)|\varphi)_H, \quad \mathbb{P} - a.s..
\]
Since by (6.14) and (6.16), sup_{t\in[0,T]} |\bar{X}_n(t)|^2 < \infty and sup_{t\in[0,T]} |X_*(t)|^2 < \infty, \mathbb{P}\text{-a.s., using the dominated convergence theorem we infer that}

\[
\lim_{n\to\infty} \int_0^T |(\bar{X}_n(t) - X_*(t)|\varphi)_{|\mathbb{H}}|^2 dt = 0 \quad \mathbb{P}\text{-a.s.} \tag{6.20}
\]

Moreover, by the Hölder inequality, for every n \in \mathbb{N} and every r \in (1, \frac{5}{2}]

\[
\mathbb{E}\left[ \left( \int_0^T |\bar{X}_n(t) - X_*(t)|^{2r} dt \right)^{\frac{2}{r}} \right] \leq T^{r-1} 2^{2r-1} \mathbb{E}\left[ \int_0^T (|\bar{X}_n(t)|^{2r} + |X_*(t)|^{2r}) dt \right] 
\leq T^{r-1} 2^{2r-1} T \mathbb{E}\left[ \sup_{t\in[0,T]} |\bar{X}_n(t)|^{2r} + \sup_{t\in[0,T]} |X_*(t)|^{2r} \right].
\]

Thus by (6.14) and (6.16), we infer that for every r \in (1, \frac{5}{2}]

\[
\sup_{n\in\mathbb{N}} \mathbb{E}\left[ \left( \int_0^T |\bar{X}_n(t) - X_*(t)|^{2r} dt \right)^{\frac{2}{r}} \right] < \infty. \tag{6.21}
\]

Finally, by (6.20), (6.21) and the Vitali theorem we infer that

\[
\lim_{n\to\infty} \mathbb{E}\left[ \int_0^T |(\bar{X}_n(t) - X_*(t)|\varphi)_{|\mathbb{H}}|^2 dt \right] = 0.
\]

The proof of assertion (a) is thus complete.

**Ad. (b).** Let us fix \varphi \in \mathbb{H}. Since by (6.12) \bar{X}_n \to X_* \text{ in } \mathbb{D}(0,T;\mathbb{H}_w) \text{ } \mathbb{P}\text{-a.s. and } X_* \text{ is right-continuous at } t = 0, \text{ we infer that } (\bar{X}_n(0)|\varphi)_{|\mathbb{H}} \to (X_*(0)|\varphi)_{|\mathbb{H}}, \mathbb{P}\text{-a.s. By (6.14), (6.16) and the Vitali theorem, we have}

\[
\lim_{n\to\infty} \mathbb{E}\left[ |(\bar{X}_n(0) - X_*(0)|\varphi)_{|\mathbb{H}}|^2 \right] = 0,
\]

which completes the proof of (b).

**Ad. (c).** Let us fix \varphi \in \mathcal{V}. By (2.8), we have

\[
\int_0^t \langle A\bar{X}_n(\sigma)|P_n\varphi \rangle d\sigma = \int_0^t \langle \bar{X}_n(\sigma)|P_n\varphi \rangle d\sigma.
\]

By (6.12), \bar{X}_n \to X_* \text{ in } L^2_{w}(0,T;\mathcal{V}), \mathbb{P}\text{-a.s. Moreover, since } \varphi \in \mathcal{V}, \text{ we infer that } P_n\varphi \to \varphi \text{ in } \mathcal{V}, \text{ (see Corollary 4.4). Thus}

\[
\lim_{n\to\infty} \int_0^t \langle \bar{X}_n(\sigma)|P_n\varphi \rangle d\sigma = \int_0^t \langle X_*(\sigma)|\varphi \rangle d\sigma = \int_0^t \langle AX_*(\sigma)|\varphi \rangle d\sigma, \quad \mathbb{P}\text{-a.s.} \tag{6.22}
\]

Using (2.8), the Hölder inequality we obtain the following inequality for all t \in [0,T] and n \in \mathbb{N}

\[
\mathbb{E}\left[ \left( \int_0^t \langle A\bar{X}_n(s)|P_n\varphi \rangle ds \right)^2 \right] 
\leq \|P_n\varphi\|^2 \cdot \mathbb{E}\left[ \left( \int_0^t \|\bar{X}_n\| ds \right)^2 \right] 
\leq |P_n|^2 L_{(\mathcal{V},\mathcal{V})} \|\varphi\|^2 T \cdot \mathbb{E}\left[ \int_0^T \|\bar{X}_n(s)\|^2_{\mathcal{V}} ds \right].
\]

Thus by (6.15) we infer that for all t \in [0,T]

\[
\sup_{n\in\mathbb{N}} \mathbb{E}\left[ \left( \int_0^t \langle A\bar{X}_n(s)|P_n\varphi \rangle ds \right)^2 \right] \leq T \sup_{n\in\mathbb{N}} |P_n|^2 L_{(\mathcal{V},\mathcal{V})} \|\varphi\|^2_{\mathcal{V}} \cdot C_2 < \infty. \tag{6.23}
\]
Therefore by (6.22), (6.23), (6.17) and the Vitali theorem we infer that for all $t \in [0, T]$

$$
\lim_{n \to \infty} \mathbb{E} \left[ \int_0^t \left| \langle AX_n(s) | P_n \varphi \rangle - \langle AX_*(s) | \varphi \rangle \right| ds \right] = 0.
$$

Hence by the dominated convergence theorem and the Fubini theorem

$$
\lim_{n \to \infty} \mathbb{E} \left[ \int_0^T \int_0^t \left| \langle AX_n(s) | P_n \varphi \rangle - \langle AX_*(s) | \varphi \rangle \right| ds \, dt \right] = 0. \quad (6.24)
$$

The proof of assertion (c) is thus complete.

**Ad. (d).** Let us move to the $\tilde{B}$-term. Let us fix $\varphi \in \mathcal{V}_*$. For every $s \in [0, T]$ we have

$$
\langle \tilde{B}(X_n(s)) | P_n \varphi \rangle - \langle \tilde{B}(X_*(s)) | \varphi \rangle = \langle \tilde{B}(X_n(s)) | P_n \varphi - \varphi \rangle + \langle \tilde{B}(X_n(s)) - \tilde{B}(X_*(s)) | \varphi \rangle.
$$

By (6.12), the sequence $(\tilde{X}_n)$ is $\mathbb{P}$-a.s. convergent to $X_*$ in $L^2(0, T; \mathbb{V})$. Thus $(\tilde{X}_n)$ is bounded in $L^2(0, T; \mathbb{V})$, and in particular, by (2.1), it is bounded in $L^2(0, T; \mathbb{H})$, as well. Moreover, by (6.12), $\tilde{X}_n \to X_*$ in $L^2(0, T; \mathbb{H}_{loc})$, $\mathbb{P}$-a.s. By Corollary 2.3 we infer that $\mathbb{P}$-a.s. for all $t \in [0, T]$

$$
\lim_{n \to \infty} \int_0^t \langle \tilde{B}(X_n(s)) - \tilde{B}(X_*(s)) | \varphi \rangle ds = 0.
$$

By (2.19) we have

$$
\left| \int_0^t \mathcal{V}_* \langle \tilde{B}(X_n(s)) | P_n \varphi - \varphi \rangle_{\mathcal{V}_*} ds \right| \leq \int_0^t \mathcal{V}_* \langle \tilde{B}(X_n(s)) | P_n \varphi - \varphi \rangle_{\mathcal{V}_*} \, ds \\
\leq \| \tilde{B} \| \cdot \int_0^t |X_n(s)|^2_{\mathcal{H}} \, ds \cdot \| P_n \varphi - \varphi \|_{\mathcal{V}_*} \leq \| \tilde{B} \| \cdot \| X_n \|^2_{L^2(0, T; \mathbb{H})} \cdot \| P_n \varphi - \varphi \|_{\mathcal{V}_*}.
$$

Since $\varphi \in \mathcal{V}_*$ then by Corollary 4.3 (ii), $P_n \varphi \to \varphi$ in $\mathcal{V}_*$, and by the boundedness of the sequence $(\tilde{X}_n)$ in $L^2(0, T; \mathbb{H})$, we infer that $\mathbb{P}$-a.s. for all $t \in [0, T]$

$$
\lim_{n \to \infty} \int_0^t \mathcal{V}_* \langle \tilde{B}(X_n(s)) | P_n \varphi - \varphi \rangle_{\mathcal{V}_*} \, ds = 0.
$$

Thus for all $t \in [0, T]$

$$
\lim_{n \to \infty} \int_0^t \langle (\tilde{B}(X_n(s)) | P_n \varphi) - \langle \tilde{B}(X_*(s)) | \varphi \rangle \rangle ds = 0, \quad \mathbb{P}\text{-a.s.} \quad (6.25)
$$

By (2.19) we obtain for all $r \in \left[ 1, \frac{5}{2} \right]$, $t \in [0, T]$ and $n \in \mathbb{N}$

$$
\mathbb{E} \left[ \left( \int_0^t \| \tilde{B}(X_n(s)) | P_n \varphi \|^r_{\mathcal{V}_*} ds \right)^{\frac{1}{r}} \right] \leq \| \tilde{B} \|^r \mathbb{E} \left[ \left( \int_0^t |X_n(s)|^2_{\mathcal{H}} ds \right)^{\frac{r}{2}} \right]^{\frac{r}{2}} \\
\leq \| \tilde{B} \|^r \mathbb{E} \left[ \left( \int_0^t |X_n(s)|^2_{\mathcal{H}} ds \right)^{\frac{r}{2}} \right]^{\frac{r}{2}} = \| \tilde{B} \|^r \mathbb{E} \left[ \left( \int_0^t |X_n(s)|^2_{\mathcal{H}} ds \right)^{\frac{r}{2}} \right]^{\frac{r}{2}}.
$$

Thus by (6.14) we infer that for all $r \in \left[ 1, \frac{5}{2} \right]$, and $t \in [0, T]$

$$
\sup_{n \in \mathbb{N}} \mathbb{E} \left[ \left( \int_0^t \| \tilde{B}(X_n(s)) | P_n \varphi \|^r_{\mathcal{V}_*} ds \right)^{\frac{1}{r}} \right] \leq \| \tilde{B} \|^r \mathbb{E} \left[ \left( \int_0^t |X_n(s)|^2_{\mathcal{H}} ds \right)^{\frac{r}{2}} \right]^{\frac{r}{2}} T^r C_1(2r) < \infty. \quad (6.26)
$$
Indeed, by (2.26), we have for each $n$

$$
\lim_{n \to \infty} \mathbb{E}\left[ \int_0^t \left( \langle \tilde{B}(X_n(s))|P_n \varphi \rangle - \langle \tilde{B}(X_*(s))|\varphi \rangle \right) ds \right] = 0. \quad (6.27)
$$

By (6.27), (6.26) (with $r = 1$), the dominated convergence theorem and the Fubini theorem, we infer that

$$
\lim_{n \to \infty} \mathbb{E}\left[ \int_0^T \int_0^t \left( \langle \tilde{B}(X_n(s))|P_n \varphi \rangle - \langle \tilde{B}(X_*(s))|\varphi \rangle \right) ds \right] dt = 0. \quad (6.28)
$$

The proof of (d) is complete.

**Ad. (e).** Let us move to the $\tilde{H}$-term. Let us fix $\varphi \in V_\ast$. For every $s \in [0, T]$ we have

$$
\langle \tilde{H}(X_n(s))|P_n \varphi \rangle - \langle \tilde{H}(X_*(s))|\varphi \rangle = \langle \tilde{H}(X_n(s))|P_n \varphi - \varphi \rangle + \langle \tilde{H}(X_*(s)) - \tilde{H}(X_*(s))|\varphi \rangle.
$$

By (6.30), the sequence $(X_n)$ is $\tilde{P}$-a.s. convergent to $X_\ast$ in $L^2(0, T; \mathbb{V}) \cap L^2(0, T; \mathbb{H}_{loc})$. By Corollary 2.5 we infer that $\tilde{P}$-a.s. for all $t \in [0, T]$

$$
\lim_{n \to \infty} \int_0^t \langle \tilde{H}(X_n(s)) - \tilde{H}(X_*(s))|\varphi \rangle ds = 0.
$$

By (2.26), we have

$$
\left| \int_0^t \psi_s \langle \tilde{H}(X_n(s))|P_n \varphi - \varphi \rangle \psi_s \right| ds \leq \int_0^t \psi_s \langle \tilde{H}(X_n(s))|P_n \varphi - \varphi \rangle \psi_s \left| ds \right|
$$

$$
\leq \|\tilde{H}\| \cdot \int_0^t \|X_n(s)\|_H \|X_n(s)\|_V ds \cdot \|P_n \varphi - \varphi\|_{V_\ast}
$$

$$
\leq \|\tilde{H}\| \cdot \|X_n\|_{L^2(0, T; H)} \|X_n\|_{L^2(0, T; V)} \cdot \|P_n \varphi - \varphi\|_{V_\ast}.
$$

Since $\varphi \in V_\ast$ then by Corollary 4.4 (ii), $P_n \varphi \to \varphi$ in $V_\ast$. By the boundedness of the sequence $(X_n)$ in $L^2(0, T; \mathbb{V})$ (and hence in $L^2(0, T; \mathbb{H})$, as well), we infer that $\tilde{P}$-a.s. for all $t \in [0, T]$

$$
\lim_{n \to \infty} \int_0^t \psi_s \langle \tilde{H}(X_n(s))|P_n \varphi - \varphi \rangle \psi_s \left| ds \right| = 0.
$$

Thus for all $t \in [0, T]$

$$
\lim_{n \to \infty} \int_0^t \left( \langle \tilde{H}(X_n(s))|P_n \varphi \rangle - \langle \tilde{H}(X_*(s))|\varphi \rangle \right) ds = 0, \quad \tilde{P}\text{-a.s.} \quad (6.29)
$$

We will show that for $r \in [1, \frac{2p}{p+2}]

$$
\sup_{n \in \mathbb{N}} \sup_{t \in [0, T]} \mathbb{E}\left[ \left| \int_0^t \psi_s \langle \tilde{H}(X_n(s))|P_n \varphi \rangle \psi_s \left| ds \right| \right|^r \right] < \infty. \quad (6.30)
$$

Indeed, by (2.26), we have for each $n \in \mathbb{N}$ and $t \in [0, T]$

$$
\left| \int_0^t \psi_s \langle \tilde{H}(X_n(s))|P_n \varphi \rangle \psi_s \left| ds \right| \right| \leq \int_0^t \left| \psi_s \langle \tilde{H}(X_n(s))|P_n \varphi \rangle \psi_s \right| \left| ds \right|
$$

$$
\leq \|\tilde{H}\| \sup_{n \in \mathbb{N}} \|P_n\|_{L(V_\ast, V_\ast)} \|\varphi\|_{V_\ast} \int_0^t \|X_n(s)\|_H \|X_n(s)\|_V \left| ds \right|.
$$
Let \( r \in [1, \frac{2p}{p+2}] \). By the Hölder inequality and the Young inequality (for numbers with \( \frac{1}{\alpha} = 1 - \frac{r}{2} \) and \( \frac{1}{\beta} = \frac{r}{2} \)), and estimates (6.14) and (6.15) we have for each \( n \in \mathbb{N} \) and \( t \in [0, T] \)

\[
\mathbb{E} \left[ \int_0^t |\bar{X}_n(s)|_H^2 \| \bar{X}_n(s) \|_V^2 \, ds \right]^{\frac{1}{2}} \leq \frac{1}{\alpha} T^{\alpha} \mathbb{E} \left[ \sup_{s \in [0, T]} |\bar{X}_n(s)|_{\mathbb{H}}^{2r} \right] + \frac{1}{\beta} T^{\beta-1} \mathbb{E} \left[ \int_0^T \| \bar{X}_n(s) \|_V^2 \, ds \right]^{\frac{1}{2}}
\]

\[
\leq \frac{1}{\alpha} T^{\alpha} C_1 \left( \frac{2r}{2-r} \right) + \frac{1}{\beta} T^{\beta-1} C_2.
\]

Since moreover by Corollary 4.4(ii), \( \sup_{n \in \mathbb{N}} |P_n|_{L(V, V^*)} < \infty \), the proof of (6.30) is complete.

In view of (6.29) and (6.30) (with \( r \in (1, 2) \)), by the Vitali theorem we obtain for all \( t \in [0, T] \)

\[
\lim_{n \to \infty} \mathbb{E} \left[ \int_0^t \left( \langle \bar{H}(\bar{X}_n(s)) | P_n \varphi \rangle - \langle \bar{H}(X_s(s)) | \varphi \rangle \right) ds \right] = 0.
\]

(6.31)

By (6.31), (6.30) (for \( r = 1 \)) the dominated convergence theorem and the Fubini theorem, we infer that

\[
\lim_{n \to \infty} \mathbb{E} \left[ \int_0^T \int_0^t \left( \langle \bar{H}(\bar{X}_n(s)) | P_n \varphi \rangle - \langle \bar{H}(X_s(s)) | \varphi \rangle \right) ds \, dt \right] = 0.
\]

(6.32)

The proof of (e) is complete.

**Ad. (f).** Let us fix \( \varphi \in \mathbb{H} \). By (4.12), inequality (3.6) in Remark 3.2 (6.14) and (6.16) for every \( t \in [0, T] \) and \( n \in \mathbb{N} \)

\[
\mathbb{E} \left[ \int_0^t \int_Y |(F_n(s, \bar{X}_n(s)); y)|_H^2 \, dy \, ds \right] \leq |\varphi|_H^2 \mathbb{E} \left[ \int_0^t \int_Y |F(s, \bar{X}_n(s)); y)|_H^2 \, dy \, ds \right]
\]

\[
\leq |\varphi|_H^2 K_2 \mathbb{E} \left[ \int_0^t \left( 1 + |\bar{X}_n(s)|_H^2 \right) \, ds \right] \leq |\varphi|_H^2 TK_2 \left( 1 + \mathbb{E} \left[ \sup_{s \in [0, T]} |\bar{X}_n(s)|_H^2 \right] \right) < \infty,
\]

and

\[
\mathbb{E} \left[ \int_0^t \int_Y |(F(s, X_s(s)); y)|_H^2 \, dy \, ds \right] \leq |\varphi|_H^2 TK_2 \left( 1 + \mathbb{E} \left[ \sup_{s \in [0, T]} |X_s(s)|_H^2 \right] \right) < \infty.
\]

By the isometry formula (2.27) and the fact that \( \bar{\eta}_n = \eta_s \), we have

\[
\mathbb{E} \left[ \int_0^t \int_Y \langle F_n(s, \bar{X}_n(s)); y); \varphi \rangle \, \bar{\eta}_n(ds, dy) \right.
\]

\[
\left. - \int_0^t \int_Y \langle F(s, X_s(s)); y); \varphi \rangle \, \eta_s(ds, dy) \right| ^2 = 0
\]

(6.33)

We will prove that for every \( t \in [0, T] \) and \( \varphi \in \mathbb{H} \)

\[
\lim_{n \to \infty} \mathbb{E} \left[ \int_0^t \int_Y \left| \langle F_n(s, \bar{X}_n(s)); y); F(s, X_s(s)); y); \varphi \rangle \right|^2 \, dy \, ds \right] = 0.
\]

(6.34)

Indeed, for all \( t \in [0, T] \) we have

\[
\int_0^t \int_Y \left| \langle F(s, \bar{X}_n(s)); y); F(s, X_s(s)); y); \varphi \rangle \right|^2 \, dy \, ds
\]

\[
\leq \int_0^T \int_Y \left| \langle F(s, \bar{X}_n(s)); y); F(s, X_s(s)); y); \varphi \rangle \right| _H^2 \, dy \, ds
\]

\[
= \int_0^T \int_Y \left| \bar{F}_\varphi(\bar{X}_n)(s, y) - \bar{F}_\varphi(X_s)(s, y) \right| _H^2 \, dy \, ds
\]

\[= \| \bar{F}_\varphi(\bar{X}_n) - \bar{F}_\varphi(X_s) \| _{L^2([0,T] \times Y; \mathbb{R})}^2,
\]

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Moreover, by inequality (3.6) in Remark 3.2 and by (6.14), for every Corollary 6.5.

Directly from Lemma 6.4 we get the following corollary where \( \tilde{\varphi} \) and \( \overline{\varphi} \). Moreover, by (4.12) and the fact that \( t \in [0, T] \).

By (6.34), (6.36) (with \( c > 0 \)). Thus by (6.35), (6.36) (for \( r \in (1, \frac{2}{T}] \)) and the Vitali theorem for all \( t \in [0, T] \). Moreover, by (4.12) and the fact that \( P_n : H \rightarrow H_n \) is the (\( \cdot \))\( \cap H \)-projection onto \( H_n \), see Section 4 we infer that also

This completes the proof of (6.34).

By (6.34), (6.36) (with \( r = 1 \)) and the dominated convergence theorem, we have for all \( \varphi \in H \)

By (6.33), (6.38) and the Fubini theorem

This concludes the proof of (f) and of Lemma 6.4

Directly from Lemma 6.3 we get the following corollary

**Corollary 6.5.** For every \( \varphi \in V_s \),

\[
\lim_{n \to \infty} \| (\tilde{X}_n(\cdot)|\varphi_H - (X_s(\cdot)|\varphi_H \|_{L^2([0,T] \times \Omega)} = 0.
\]

and

\[
\lim_{n \to \infty} \| A_n(\tilde{X}_n, \tilde{\eta}_n, \varphi) - A(X_s, \eta_s, \varphi) \|_{L^1([0,T] \times \Omega)} = 0.
\]
Proof. Assertion (6.40) follows from the equality
\[
\| (\bar{X}_n(t)|\varphi|)_{\mathbb{H}} - (X_*(\cdot)|\varphi|)_{\mathbb{H}} \|^2_{L^2([0,T] \times \Omega)} = \mathbb{E} \left[ \int_0^T \| (\bar{X}_n(t) - X_*(t)|\varphi|)_{\mathbb{H}} \|^2 dt \right]
\]
and Lemma 6.4 (a). To prove (6.41) let us note that by the Fubini theorem, we have
\[
\| \Lambda_n(\bar{X}_n, \eta_n) - \Lambda(X_*, \eta_*|\varphi|) \|_{L^1([0,T] \times \Omega)}
\]
\[
= \int_0^T \mathbb{E} \left[ \| \Lambda_n(\bar{X}_n, \eta_n, \varphi) - \Lambda(X_*, \eta_*|\varphi|) \| \right] dt.
\]
To complete the proof of (6.41) it is sufficient to note that by Lemma 6.4 (b)-(f), each term on the right hand side of (6.18) tends at least in $L^1([0,T] \times \Omega)$ to the corresponding term in (6.19).

\begin{proof}
\end{proof}

**Step 2.** Since $X_n$ is a solution of the truncated equation, for all $t \in [0, T]$ and $\varphi \in \mathcal{V}_*$
\[
(X_n(t)|\varphi|)_{\mathbb{H}} = \Lambda_n(X_n, \eta, \varphi)(t), \quad \mathbb{P}\text{-a.s.}
\]
In particular,
\[
\int_0^T \mathbb{E} \left[ \| (X_n(t)|\varphi|)_{\mathbb{H}} - \Lambda_n(X_n, \eta, \varphi)(t) \| \right] dt = 0.
\]
Since $\text{Law}(X_n, \eta) = \text{Law}(\bar{X}_n, \tilde{\eta}_n)$, using (6.40) and (6.41) we infer that
\[
\int_0^T \mathbb{E} \left[ \| (X_*(t)|\varphi|)_{\mathbb{H}} - \Lambda(X_*, \eta_*, \varphi)(t) \| \right] dt = 0.
\]
Hence for $\ell$-almost all $t \in [0, T]$ and $\mathbb{P}$-almost all $\omega \in \bar{\Omega}$
\[
(X_*(t)|\varphi|)_{\mathbb{H}} - \Lambda(X_*, \eta_*, \varphi)(t) = 0,
\]
(6.42)

Since $X_*$ is $\mathcal{F}_T$-valued random variable, in particular $X_* \in \mathbb{D}([0, T]; \mathbb{H}_w)$, we infer that equality (6.42) holds for all $t \in [0, T]$ and all $\varphi \in \mathcal{V}_*$. Since $\mathcal{V}_*$ is dense in $\mathcal{V}_{1,2}$, equality (6.42) holds for all $\varphi \in \mathcal{V}_{1,2}$, as well. Putting $\tilde{\mathfrak{A}} := (\bar{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \tilde{\mathcal{F}})$, $\tilde{\eta} := \eta_*$ and $\bar{X} := X_*$, by (6.42), (6.19), (6.16) and (6.17) we infer that the system $(\tilde{\mathfrak{A}}, \eta_*, X_*)$ is a martingale solution of equation (3.8) satisfying inequality (3.11). The proof of Theorem 3.3 is thus complete. \square

7 Some generalizations

Let us consider the stochastic Hall-magnetohydrodynamics system on $[0, T] \times \mathbb{R}^3$ with the Lévy noise terms
\[
du + \left[ (u \cdot \nabla)u + \nabla p - (B \cdot \nabla)B + \nabla \left( \frac{|B|^2}{2} \right) - \nu_1 \Delta u \right] dt
\]
\[
= f_1(t) + \int_{V_1} F_1(t, u(t^-); y) \, d\tilde{\eta}_1(dt, dy) + G_1(t, u) \, dW_1(t),
\]
\[
dB + \left[ (u \cdot \nabla)B - (B \cdot \nabla)u + \text{curl} \left( (\text{curl} \, B) \times B \right) - \nu_2 \Delta B \right] dt
\]
\[
= f_2(t) + \int_{V_2} F_2(t, B(t^-); y) \, d\tilde{\eta}_2(dt, dy) + G_2(t, B) \, dW_2(t)
\]
with the incompressibility conditions (1.3), i.e.

\[ \text{div } \mathbf{u} = 0 \quad \text{and} \quad \text{div } \mathbf{B} = 0 \]

and the initial conditions (1.4), i.e.

\[ \mathbf{u}(0) = \mathbf{u}_0 \quad \text{and} \quad \mathbf{B}(0) = \mathbf{B}_0. \]

**Assumption 7.1.**

\( (G.1) \) \( K_1, K_2 \) are separable Hilbert spaces, and

\[ G_i : [0,T] \times V \to \mathcal{L}(2)(K_i, H), \quad i = 1, 2, \]

are two measurable map which are Lipschitz continuous, i.e. there exist constants \( L_i, i = 1, 2, \) such that

\[ \| G_i(t, \phi_1) - G_i(t, \phi_2) \|_{\mathcal{L}^2(K_i, H)}^2 \leq L_i \| \phi_1 - \phi_2 \|_V^2, \quad \phi_1, \phi_2 \in V, \ t \in [0,T]. \]  

(7.3)

In addition, there exist \( \lambda_i, g_i \in \mathbb{R} \) and \( a \in (0,2) \) such that

\[ \| G_i(t, \phi) \|_{\mathcal{L}^2(K_i, H)}^2 \leq \nu_i(2-a)|\nabla \phi|_{L^2}^2 + \lambda_i|\phi|_H^2 + g_i, \quad (t, \phi) \in [0,T] \times V. \]  

(7.4)

\( (G.2) \) The maps \( G_i, i = 1, 2, \) can be extended to measurable maps

\[ g_i : [0,T] \times H \to \mathcal{L}(K_i, V') \]

such that for some \( C_i > 0 \)

\[ \sup_{\psi \in V} \sup_{\| \psi \|_V \leq 1} |\langle V(g(t, \phi)(y)|\psi \rangle|_V^2 \leq C_i(1 + |\phi|_H^2), \quad (t, \phi) \in [0,T] \times H. \]  

(7.5)

\( (G.3) \) Moreover, for every \( \psi \in V \) the maps \( \tilde{g}_i \psi \) defined for \( \phi \in L^2(0,T; H) \) by

\[ (\tilde{g}_i \psi(t))(t) := \{ K_i \ni y \mapsto \langle g_i(t, \phi(t))(y)|\psi \rangle \in \mathbb{R} \} \in \mathcal{L}(2)(K_i, \mathbb{R}), \quad t \in [0,T], \]

are continuous maps from \( L^2(0,T; H_{loc}) \) into \( L^2(0,T; \mathcal{L}(2)(K_i, \mathbb{R})). \)

For any Hilbert spaces \( \mathbb{K} \) and \( Y \) by \( \mathcal{L}(2)(\mathbb{K}; Y) \) we denote the space of Hilbert-Schmidt operators from \( \mathbb{K} \) into \( Y. \)

**Assumption 7.2.** We assume also that

\( (H.1) \) \( p \) is a real number such that

\[ p \in [2,2 + \gamma), \]

where

\[ \gamma := \begin{cases} \frac{a}{2-a}, & \text{if } a \in [0,2), \\ \infty, & \text{if } a = 2, \end{cases} \]

(7.8)

and \( a \) is the parameter from inequality (7.4).

\( (H.2) \) \( X_0 := (\mathbf{u}_0, \mathbf{B}_0) \in H \times H \) and \( f := (f_1, f_2), \) where \( f_i \in L^p(0,T; V') \) for \( i = 1, 2. \)

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Let $W(t) := (W_1(t), W_2(t))$. Then $W(t)$ is a cylindrical Wiener process on $\mathbb{K} := \mathbb{K}_1 \times \mathbb{K}_2$, on the stochastic basis $\mathfrak{A}$. Using the maps $G_1$ and $G_2$ given in Assumption 7.4 we define the map $G$ by

$$G(t, \Phi)(y) := (G_1(t, u)(y_1), G_2(t, B)(y_2)), \quad (7.9)$$

where $t \in [0, T]$, $\Phi := (u, B) \in \mathcal{V}$, $y = (y_1, y_2) \in \mathbb{K}$.

**Remark 7.3.**

(i) Then

$$G : [0, T] \times \mathcal{V} \to \mathcal{L}_2(\mathbb{K}, \mathbb{H}).$$

The map $G$ satisfies the Lipschitz condition, i.e there exists a constant $L > 0$ such that

$$\|G(t, \Phi_1) - G(t, \Phi_2)\|^2_{\mathcal{L}_2(\mathbb{K}, \mathbb{H})} \leq L \|\Phi_1 - \Phi_2\|^2_{\mathcal{V}}, \quad \Phi_1, \Phi_2 \in \mathcal{V}, \ t \in [0, T]. \quad (7.10)$$

(ii) The map $G$ satisfies the following inequality

$$\|G(s, \Phi)\|^2_{\mathcal{L}_2(\mathbb{K}, \mathbb{H})} \leq (2 - a)\|\Phi\|^2 + \lambda |\Phi|_{\mathbb{H}}^2 + \rho, \quad (s, \Phi) \in [0, T] \times \mathcal{V}, \quad (7.11)$$

where $\lambda := \lambda_1 + \lambda_2$ and $\rho := \rho_1 + \rho_2$.

(iii) Let $g_1$ and $g_2$ be the maps from Assumption 7.1 and let us define

$$g(t, \Phi)(y) := (g_1(t, u)(y_1), g_2(t, B)(y_2)), \quad (7.12)$$

where $t \in [0, T]$, $\Phi := (u, B) \in \mathcal{V}$, $y = (y_1, y_2) \in \mathbb{K}$. Then the map $g$ is an extension of the map $G$ to a measurable map

$$g : [0, T] \times \mathbb{H} \to \mathcal{L}(\mathbb{K}, \mathcal{V})$$

and by (7.15) we obtain

$$\sup_{\Psi \in \mathcal{V}, \|\Psi\|_{\mathbb{V}} \leq 1} \sup_{y \in \mathbb{K}, \|y\|_{\mathbb{H}} \leq 1} |\mathcal{V}(g(t, \Phi)(y)\Psi)|_{\mathcal{V}} \leq C(1 + |\Phi|_{\mathbb{H}}^2), \quad (t, \Phi) \in [0, T] \times \mathbb{H}. \quad (7.13)$$

Moreover, for every $\Psi \in \mathcal{V}$ the map $\tilde{g}_\Psi$ defined for $\Phi \in L^2(0, T; \mathbb{H})$ by

$$(\tilde{g}_\Psi(\Phi))(t) := \{\mathbb{K} \ni y \mapsto \mathcal{V}(g(t, \Phi(t))(y)\Psi) \in \mathcal{L}(\mathbb{K}, \mathbb{R})\}, \quad t \in [0, T], \quad (7.14)$$

is a continuous map from $L^2(0, T; \mathbb{H}_{loc})$ into $L^2(0, T; \mathcal{L}_2(\mathbb{K}, \mathbb{R}))$.

Using the maps $\mathcal{A}, \tilde{B}, \tilde{H}, F$ and $G$ defined respectively by (2.8), (2.17), (2.23), (3.4) and (7.9), equations (7.1) - (7.7) with the incompressibility conditions (1.3) and the initial conditions (1.4) can be rewritten as the following abstract SPDE

$$\begin{align*}
X(t) + \int_0^t \left[ AX(s) + \tilde{B}(X(s)) + \tilde{H}(X(s)) \right] ds &= X_0 + \int_0^t f(s) ds \\
+ \int_0^t \int_Y F(s, X(s^-); y) \tilde{\eta}(ds, dy) + \int_0^t G(s, X(s)) dW(s), & t \in [0, T]. \quad (7.15)
\end{align*}$$

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Definition 7.4. Let Assumptions 3.1, 7.1 and 7.2 be satisfied. We say that there exists a martingale solution of problem (7.15) iff there exist

- a stochastic basis \( \tilde{\mathfrak{A}} := (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \tilde{\mathbb{P}}) \) with a filtration \( \tilde{\mathcal{F}} = \{ \tilde{\mathcal{F}}_t \}_{t \in [0,T]} \) satisfying the usual conditions,
- a time homogeneous Poisson random measure \( \tilde{\eta} \) on \( (Y, \mathcal{Y}) \) over \( \tilde{\mathfrak{A}} \) with the intensity measure \( \mu \),
- a \( \mathbb{K} \)-cylindrical Wiener process \( \tilde{W} \) over \( \tilde{\mathfrak{A}} \),
- and an \( \mathcal{F} \)-progressively measurable process \( \tilde{X} : [0,T] \times \Omega \rightarrow \mathbb{H} \) with paths satisfying

\[
\tilde{X}(\cdot, \omega) \in \mathbb{D}([0,T]; \mathbb{H}_w) \cap L^2(0,T; \mathbb{V}),
\]

(7.16)

for \( \tilde{\mathbb{P}} \)-a.e. \( \omega \in \tilde{\Omega} \), and such that for all \( t \in [0,T] \) and \( \phi \in \mathbb{V}_{1,2} \) the following identity holds \( \tilde{\mathbb{P}} \)-a.s.

\[
\begin{align*}
(\tilde{X}(t)|\phi)_{\mathbb{H}} + \int_0^t \langle A\tilde{X}(s)|\phi \rangle ds &+ \int_0^t \langle \tilde{B}(\tilde{X}(s))|\phi \rangle ds + \int_0^t \langle \tilde{H}(\tilde{X}(s))|\phi \rangle ds \\
= (X_0|\phi)_{\mathbb{H}} + \int_0^t \langle f(s)|\varphi \rangle ds &+ \int_0^t \int_Y \langle F(s, \tilde{X}(s^-); y)|\phi \rangle_{\mathbb{V}} \tilde{\eta}(ds,dy) \\
+ \left\langle \int_0^t G(s, \tilde{X}(s)) d\tilde{W}(s) \right| \varphi \right). 
\end{align*}
\]

(7.17)

If all the above conditions are satisfied, then the system \( (\tilde{\mathfrak{A}}, \tilde{\eta}, \tilde{W}, \tilde{X}) \) is called a martingale solution of problem (7.15).

Using the ideas from [28] and [29] we can prove the following generalization of Theorem 3.4.

Theorem 7.5. Let Assumptions 3.1, 7.1 and 7.2 be satisfied. In particular, we assume that \( p \) satisfies \( 7.4 \), i.e.

\[
p \in [2, 2 + \gamma),
\]

where \( \gamma \) is given by \( 7.8 \). Then there exists a martingale solution of problem (7.15) such that

(i) for every \( q \in [1,p]\) there exists a constant \( C_1(p, q) \) such that

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} |\tilde{X}(t)|_{\mathbb{H}}^q \right] \leq C_1(p, q),
\]

(7.18)

(ii) there exists a constant \( C_2(p) \) such that

\[
\mathbb{E} \left[ \int_0^T \| \tilde{X}(t) \|_{\mathbb{V}}^2 dt \right] \leq C_2(p).
\]

(7.19)
A The space of càdlàg functions

Let $(\mathcal{S}, \varrho)$ be a separable and complete metric space. Let $\mathbb{D}([0,T]; \mathcal{S})$ the space of all $\mathcal{S}$-valued càdlàg functions defined on $[0,T]$, i.e. the functions which are right continuous and have left limits at every $t \in [0,T]$. The space $\mathbb{D}([0,T]; \mathcal{S})$ is endowed with the Skorokhod topology.

Remark A.1. A sequence $(u_n) \subset \mathbb{D}([0,T]; \mathcal{S})$ converges to $u \in \mathbb{D}([0,T]; \mathcal{S})$ iff there exists a sequence $(\lambda_n)$ of homeomorphisms of $[0,T]$ such that $\lambda_n$ tends to the identity uniformly on $[0,T]$ and $u_n \circ \lambda_n$ tends to $u$ uniformly on $[0,T]$.

This topology is metrizable by the following metric $\delta_T$

$$\delta_T(u,v) := \inf_{\lambda \in \Lambda_T} \left[ \sup_{t \in [0,T]} \varrho(u(t), v \circ \lambda(t)) + \sup_{t \in [0,T]} |t - \lambda(t)| + \sup_{s \neq t} \log \frac{\lambda(t) - \lambda(s)}{t - s} \right],$$

where $\Lambda_T$ is the set of increasing homeomorphisms of $[0,T]$. Moreover, $(\mathbb{D}([0,T]; \mathcal{S}), \delta_T)$ is a complete metric space, see [22] and [20].

Let us recall the notion of a modulus of the function. It plays analogous role in the space $\mathbb{D}([0,T]; \mathcal{S})$ as the modulus of continuity in the space of continuous functions $C([0,T]; \mathcal{S})$.

Definition A.2. (see [25]) Let $u \in \mathbb{D}([0,T]; \mathcal{S})$ and let $\delta > 0$ be given. A modulus of $u$ is defined by

$$w_{[0,T], \mathcal{S}}(u, \delta) := \inf_{\Pi_{\delta}} \max \sup_{t_i \in \omega} \sup_{t_i \leq s < t < t_{i+1} \leq T} \varrho(u(t), u(s)),$$

where $\Pi_{\delta}$ is the set of all increasing sequences $\omega = \{0 = t_0 < t_1 < ... < t_n = T\}$ with the following property

$$t_{i+1} - t_i \geq \delta, \quad i = 0, 1, ..., n - 1.$$ 

If no confusion seems likely, we will denote the modulus by $w_{[0,T]}(u, \delta)$.

We have the following criterion for relative compactness of a subset of the space $\mathbb{D}([0,T]; \mathcal{S})$, see [22,25, Ch.II] and [5, Ch.3], analogous to the Arzelà-Ascoli theorem for the space of continuous functions.

Theorem A.3. A set $A \subset \mathbb{D}([0,T]; \mathcal{S})$ has compact closure iff it satisfies the following two conditions:

(a) There exists a dense subset $J \subset [0,T]$ such that for every $t \in J$ the set $\{u(t), u \in A\}$ has compact closure in $\mathcal{S}$.

(b) $\lim_{\delta \to 0} \sup_{u \in A} w_{[0,T]}(u, \delta) = 0$.

B A version of the Skorohod embedding theorem

In the proof of Theorem 3.4 we use the following version of the Skorohod embedding theorem following from the version due to Jakubowski [21] and the version due to Brzeźniak, Hausenblas and Razafimandimby [9, Theorem C.1].
Corollary B.1. (Corollary 2 in [27]) Let $X$ be a separable complete metric space and let $X'$ be a topological space such that there exists a sequence $\{f_i\}_{i \in \mathbb{N}}$ of continuous functions $f_i : X' \to \mathbb{R}$ separating points of $X'$. Let $X := X_1 \times X_2$ with the Tykhonoff topology induced by the projections

$$\pi_i : X_1 \times X_2 \to X_i, \quad i = 1, 2.$$ 

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\chi_n : \Omega \to X_1 \times X_2$, $n \in \mathbb{N}$, be a family of random variables such that the sequence $\{\text{Law}(\chi_n), n \in \mathbb{N}\}$ is tight on $X_1 \times X_2$. Finally let us assume that there exists a random variable $\rho : \Omega \to X_1$ such that $\text{Law}(\pi_1 \circ \chi_n) = \text{Law}(\rho)$ for all $n \in \mathbb{N}$.

Then there exists a subsequence $(\chi_{n_k})_{k \in \mathbb{N}}$, a probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$, a family of $X_1 \times X_2$-valued random variables $(\bar{\chi}_k, k \in \mathbb{N})$ on $(\Omega, \mathcal{F}, \mathbb{P})$ and a random variable $\chi_* : \bar{\Omega} \to X_1 \times X_2$ such that

(i) $\text{Law}(\bar{\chi}_k) = \text{Law}(\chi_{n_k})$ for all $k \in \mathbb{N}$;

(ii) $\bar{\chi}_k \to \chi_*$ in $X_1 \times X_2$ a.s. as $k \to \infty$;

(iii) $\pi_1 \circ \bar{\chi}_k(\bar{\omega}) = \pi_1 \circ \chi_*(\bar{\omega})$ for all $\bar{\omega} \in \bar{\Omega}$.

In topological spaces we consider the $\sigma$-fields generated by the topology, i.e. the smallest $\sigma$-field containing the topology.

In the proof of Theorem 3.4 we use Corollary B.1 for the space $X_2 := Z$ defined by (5.8), i.e.,

$$X_2 := Z = L_w^2(0, T; V) \cap L^2(0, T; H_{\text{loc}}) \cap D([0, T]; U') \cap D([0, T]; H_w).$$

Remark B.2. Proceeding similarly as in [27], Remark 2), we infer that the space $(Z, \sigma(Z))$ defined in Definition B.1 satisfies assumptions of Corollary B.1.

C The spaces $L^2_n$ and the cut-off operators $S_n$

We recall crucial facts related to the Fourier truncation method, called also the Friedrichs method, see [61, Section 4, p.174]. This approach is also used, e.g., in [7] and [16]. For details see [29, Appendix A] and references given there.

The Fourier transform of a rapidly decreasing function $\psi \in \mathcal{S}(\mathbb{R}^d)$ is defined by (see [32], [37])

$$\hat{\psi}(\xi) := (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-i\xi \cdot x} \psi(x) \, dx, \quad \xi \in \mathbb{R}^d,$$

and the Fourier transform $\hat{f}$ of a tempered distribution is defined via duality.

For $s \geq 0$ the Sobolev space is defined by

$$H^s(\mathbb{R}^d) := \{ u \in L^2(\mathbb{R}^d) : (1 + |\xi|^2)^{\frac{s}{2}} \hat{u} \in L^2(\mathbb{R}^d) \}$$

and

$$\|u\|_{H^s} := \| (1 + |\xi|^2)^{\frac{s}{2}} \hat{u} \|_{L^2(\mathbb{R}^d)} = \left( \int_{\mathbb{R}^d} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}}.$$

(See [37], [35].) The spaces $H^s(\mathbb{R}^d)$ are also called Lebesgue spaces and denoted by $L^2_s$. 

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Let
\[ \hat{B}_n := \{ \xi \in \mathbb{R}^d : |\xi| \leq n \}, \quad n \in \mathbb{N}, \]
and let
\[ L^2_n := \{ u \in L^2(\mathbb{R}^d) : \text{supp} \hat{u} \subset \hat{B}_n \}. \quad (C.1) \]

On the subspace \( L^2_n \) we consider the norm inherited from \( L^2(\mathbb{R}^d) \).

The cut-off operator \( S_n \) is defined by
\[ S_n(u) := F^{-1}(1_{\hat{B}_n} \hat{u}), \quad u \in L^2(\mathbb{R}^d), \quad (C.2) \]
where \( F^{-1} \) denotes the inverse Fourier transform. (See [4, Section 4, p.174].)

**Remark C.1.** (See [4, Section 4, p.174] and [7].) The map
\[ S_n : L^2(\mathbb{R}^d) \to L^2_n \]
is an \((\cdot | \cdot)_{L^2}\)-orthogonal projection.

Basic properties of the operators \( S_n \) are stated in the following two lemmas, see [29, Appendix A] and references given there.

**Lemma C.2.** (See [29, Lemma A.2].) Let \( s \geq 0 \) be fixed. Then for all \( n \in \mathbb{N} \):
\[ S_n : H^s(\mathbb{R}^d) \to H^s(\mathbb{R}^d) \]
is well defined linear and bounded. Moreover, for every \( u \in H^s(\mathbb{R}^d) \)
\[ \| S_n u \|_{H^s} \leq \| u \|_{H^s} \quad (C.3) \]
and
\[ \lim_{n \to \infty} \| S_n u - u \|_{H^s} = 0. \quad (C.4) \]

**Lemma C.3.** (See [29, Lemma A.3].) If \( s \geq 0 \) and \( k > 0 \), then
\[ S_n : H^{s+k}(\mathbb{R}^d) \to H^s(\mathbb{R}^d) \]
is well defined and bounded and \( |S_n|_{L(H^{s+k},H^s)} \leq 1 \). Moreover, for every \( u \in H^{s+k}(\mathbb{R}^d) \):
\[ \| S_n u - u \|_{H^s}^2 \leq \frac{1}{(1 + n^2)^k} \| u \|_{H^{s+k}}^2. \quad (C.5) \]
Thus
\[ \lim_{n \to \infty} |S_n - I|_{L(H^{s+k},H^s)} = 0, \quad (C.6) \]
where \( I \) stands for the identity operator.

Let us also recall also the relation between the spaces \( L^2_n \) and \( H^s(\mathbb{R}^d) \) for \( s \geq 0 \). By definition, on the spaces \( L^2_n \) we consider the norms inherited from the space \( L^2(\mathbb{R}^d) \), see (C.1).

**Lemma C.4.** (See [29, Lemma A.4].) For each \( n \in \mathbb{N} \)
\[ L^2_n \hookrightarrow H^s(\mathbb{R}^d) \quad \text{for all } s \geq 0 \]
and for every \( s \geq 0 \) and \( u \in L^2_n \):
\[ \| u \|_{H^s}^2 \leq (1 + n^2)^s \| u \|_{L^2_n}^2. \quad (C.7) \]
In particular, the norm of the embedding \( L^2_n \subset H^s(\mathbb{R}^d) \) depends on \( n \) and \( s \).

**Corollary C.5.** (See [29, Corollary A.5].) On the subspace \( L^2_n \) the norms \( \| \cdot \|_{L^2_n} \) and \( \| \cdot \|_{H^s} \), for \( s > 0 \), are equivalent (with appropriate constants depending on \( s \) and \( n \)).
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