Quantum homodyne tomography with \textit{a priori} constraints

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I present a novel algorithm for reconstructing the Wigner function from homodyne statistics. The proposed method, based on maximum-likelihood estimation, is capable of compensating for detection losses in a numerically stable way.

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An intriguing aspect of quantum mechanics is the intricate description of the state of a physical system. Therefore, a great deal of interest has been attracted by the recent experimental demonstration that the quantum state of a simple system, namely a single light mode, can be completely characterized in a feasible scheme [1]. The measurement was based on an observation that marginals of the Wigner function, which contains complete information on the quantum state, can be collected with the help of a balanced homodyne detector [2]. The Wigner function was reconstructed from the statistics of homodyne events using a standard filtered back-projection algorithm developed in image processing. This seminal work initiated an extensive research in the field of quantum state measurement, which has brought beautiful demonstrations of quantum phenomena as well as deeper understanding of the foundations of quantum theory [3].

The Wigner function is particularly well suited for visualization of quantum states, as it represents pictorially quantum coherence in the form of nonclassical phase space structures. The purpose of this Communication is to present a novel method for reconstructing the Wigner function from homodyne statistics. Its essential advantage compared to the standard back-projection algorithm is the capability of compensating, in a numerically stable way, for non-unit efficiency of the homodyne detector. Imperfect detection is well known to have a deleterious effect on nonclassical features of the Wigner function, such as negativities and oscillatory interference patterns [4,5]. Furthermore, an attempt to incorporate compensation into the standard linear reconstruction scheme fails due to rapidly exploding statistical errors and numerical instabilities [6,7]. In the present Communication I show that these difficulties can be effectively overcome by taking into account \textit{a priori} constraints imposed by the quantum mechanical form of the Wigner function. This new method is derived using the maximum-likelihood estimation [8], and constraints are formulated in a way which provides a convenient algorithm for reconstructing the Wigner function from realistic, finite and imperfect homodyne data.

Let us start the considerations from tracing the standard route from raw statistics of homodyne events to the Wigner function. The quantum mechanical probability distribution of measuring the quadrature $x$ in a single run of the homodyne setup with the local oscillator phase $\theta$ is given by the expectation value of the following positive operator-valued measure [9]:

$$\hat{H}(x; \theta) = \frac{1}{\sqrt{\pi(1-\eta)}} \exp \left(-\frac{(x - \sqrt{\eta} \hat{x}_\theta)^2}{1-\eta}\right), \quad (1)$$

where $\eta$ is the efficiency of the homodyne detector, and $\hat{x}_\theta$ is the quadrature operator. In the limit of perfect detection, the above expression reduces to

$$\hat{H}(x; \theta) \xrightarrow{\eta \to 1} \delta(x - \hat{x}_\theta), \quad (2)$$

i.e., the measured statistics become distributions of quadrature operators, which are one-dimensional projections of the Wigner function. This projection relation can be analytically inverted, which yields an expression for the Wigner function as the inverse Radon transform of the family of quadrature distributions.

In standard optical homodyne tomography, regularized inverse Radon transform is applied to frequency histograms of homodyne events. In other words, frequency histograms are regarded as estimates for quantum mechanical quadrature distributions, and statistical uncertainty in the reconstruction process is governed by a simple propagation law. This treatment of statistical noise has crucial consequences, when efficiency of the homodyne detector is less than one. In this case, application of the inverse Radon transform to a family of distributions given by Eq. (2) yields a generalized, $s$-ordered quasidistribution function with the ordering parameter $s = -(1-\eta)/\eta$ [10]. This object is related to the Wigner function by a convolution with a Gaussian operator-valued measure [10]:

$$\int \hat{H}(x; \theta) \, d\theta = \frac{1}{\sqrt{\pi(1-\eta)}} \exp \left(-\frac{x^2}{1-\eta}\right), \quad (3)$$

The source of this difficulty lies in the estimation procedure applied in standard optical homodyne tomography. This procedure is performed at the level of experimental frequency histograms, which are inserted in place of quantum mechanical distributions. In such an approach, statistical noise is regarded as necessary evil, whose impact can be quantified, but cannot be reduced.

Fortunately, this conclusion becomes invalid if we adopt a more careful treatment of experimental data. In fact, there is a powerful tool for improving the performance of the reconstruction procedure: it is \textit{a priori} constraints...
knowledge about constraints satisfied by quantities to be estimated. Use of appropriate estimation methodology allows one to incorporate this information in the reconstruction scheme. Such an approach can lead to substantial reduction of the statistical uncertainty, as in this case estimates are picked only from a priori restricted physically sensible region. Specifically, the quantum mechanical definition of the Wigner function results in certain constraints on its values. The practical problem is how to translate this a priori knowledge into an efficient numerical algorithm. In the following, I will show that this goal can be effectively achieved for optical homodyne tomography.

We will start from the observation that the Wigner function at a phase space point \((q,p)\) is given by the expectation value of the operator \(\hat{W}(q,p)\), which can be decomposed into the diagonal form \([12]\):

\[
\hat{W}(q,p) = \frac{1}{\pi} \sum_{n=0}^{\infty} (-1)^n \hat{\rho}_n(q,p)
\]

where \(\hat{\rho}_n(q,p)\) are projections on displaced Fock states:

\[
\hat{\rho}_n(q,p) = \hat{D}(q,p)|n\rangle\langle n|\hat{D}^\dagger(q,p).
\]

Here \(\hat{D}(q,p)\) denotes the displacement operator. Quantum expectation values of \(\hat{\rho}_n(q,p)\) satisfy obvious conditions:

\[
\langle \hat{\rho}_n(q,p) \rangle \geq 0, \quad \sum_{n=0}^{\infty} \langle \hat{\rho}_n(q,p) \rangle = 1
\]

The decomposition defined in Eq. \([3]\) suggests the following two-step reconstruction scheme: for a given phase space point \((q,p)\) find estimates for the family of observables \(\hat{\rho}_n(q,p)\), taking into account constraints given by Eq. \([5]\). Using these estimates, compute the value of the Wigner function according to Eq. \([4]\). What makes this scheme more robust to statistical noise compared to standard optical homodyne tomography, is that estimates of \(\langle \hat{\rho}_n(q,p) \rangle\) are a priori restricted to the physically sensible region. In contrast, estimates for positive definite observables, such as Fock state projections, obtained via standard tomographic technique of pattern functions, may, and often do take unphysical negative values \([13]\). These artifacts become particularly strong when compensation for detection losses is built into the pattern functions.

The positive definite estimates for \(\langle \hat{\rho}_n(q,p) \rangle\) will be found using the maximum-likelihood approach. The estimation procedure is motivated by the one-to-one relation linking the photon distribution with the phase-averaged homodyne statistics \([3]\). This relation can be expressed in the operator form as

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \, \hat{\mathcal{H}}(y;\theta) = \sum_{n=0}^{\infty} A_n(y)|n\rangle\langle n|,
\]

where

\[
A_n(y) = \sum_{k=0}^{n} \binom{n}{k} (1 - \eta) \eta^{n-k} \frac{H_k(y)}{\sqrt{n!}} \exp\left(-\eta^2\right)
\]

describes contribution generated by the occupation of the \(n\)th Fock state. Here \(H_k(y)\) is the \(k\)th Hermite polynomial. The relation given by Eq. \([7]\), along with statistical characterization of experimental data, has been shown to provide an algorithm for reconstructing the photon distribution with positivity constraints \([14]\). This approach will be now generalized to the estimation of projections on arbitrarily displaced Fock states.

For this purpose let us apply the coherent displacement transformation to both the sides of Eq. \([3]\), and denote the transformed left hand side by \(\tilde{\mathcal{T}}(y;\theta)\). Making use of the identity

\[
\hat{D}(q,p)\hat{\mathcal{T}}(y;\theta)\hat{D}^\dagger(q,p) = \hat{\mathcal{T}}(y;\theta) - q \sin \theta - p \sin \theta
\]

yields

\[
\tilde{\mathcal{T}}(y;\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \, \hat{\mathcal{H}}(y + \sqrt{\eta} q \cos \theta + \sqrt{\eta} p \sin \theta;\theta)
\]

\[
= \sum_{n=0}^{\infty} A_n(y)|\hat{\rho}_n(q,p)\rangle.
\]

Thus, \(\tilde{\mathcal{T}}(y;\theta)\) as a function of \(y\) describes a probability distribution obtained by integrating the family of shifted homodyne statistics over the local oscillator phase \(\theta\). Eq. \([5]\) shows that \(\tilde{\mathcal{T}}(y;\theta)\) is uniquely defined by the set of projections on displaced Fock states \(\hat{\rho}_n(q,p)\) and vice versa: all the quantum expectation values \(\langle \hat{\rho}_n(q,p) \rangle\) can be recovered from \(\langle \hat{\rho}_n(q,p) \rangle\). For \(\eta = 1\), the transformation of homodyne statistics in Eq. \([5]\) has a simple pictorial interpretation in phase space: moving the phase space origin to the point \((q,p)\) corresponds to displacing the distribution of the quadrature \(\hat{x}_0\) by \(q \cos \theta + p \sin \theta\).

Let us now look at the derived relation from the perspective of raw experimental results. Data collected in \(N\) runs of the homodyne experiment consist of pairs \((x_i,\theta_i)\) specifying the outcome \(x_i\) and the phase \(\theta_i\) for an \(i\)th run of the setup \([13]\). For a given phase space point \((q,p)\) these data can be converted to the form

\[
y_i = x_i - \sqrt{\eta} q \cos \theta_i - \sqrt{\eta} p \sin \theta_i.
\]

Statistics of these outcomes is governed by the expectation value of \(\langle \hat{T}(y;\theta) \rangle\) over the quantum state that is measured. This probability distribution \(\langle \hat{T}(y;\theta) \rangle\) contains complete information on the set of \(\langle \hat{\rho}_n(q,p) \rangle\). However, in a real experiment the distribution \(\langle \hat{T}(y;\theta) \rangle\) is not known perfectly. The only information we have in hand is a set of outcomes characterized by this distribution, and a priori knowledge that \(\langle \hat{\rho}_n(q,p) \rangle\) generating it are positive definite and sum up to one. From this incomplete information we need to infer estimates for projections on displaced Fock states, which we will denote for short by \(\hat{\rho}_n\). The solution given to this problem by
the maximum-likelihood methodology is to pick the set of \( \{ q_n \} \) for which it was the most likely to obtain the actual result of the series of measurements. Mathematically, this is done by maximization of the log-likelihood function

\[
\mathcal{L}(\{ q_n \}; \{ y_i \}) = \sum_i \ln \left( \sum_n A_n(y_i) q_n \right) - N \sum_n q_n, \tag{11}
\]

where the outcomes \( y_i \) are treated as fixed parameters for a given phase space point \( (q, p) \). In the above formula, the method of Lagrange multipliers has been used to include the constraint \( \sum_n q_n = 1 \).

Thus we have eventually arrived at the following computational recipe for reconstructing the Wigner function from raw homodyne outcomes: for a given phase space point \( (q, p) \), convert experimental data according to Eq. (10) and find the maximum of the corresponding log-likelihood function \( \mathcal{L}(\{ q_n \}; \{ y_i \}) \) over the manifold defined by Eq. (5). Finally, evaluate the Wigner function as

\[
W(q, p) = \frac{1}{\pi} \sum_n (-1)^n q_n. \tag{12}
\]

The nontrivial step in the above scheme is the multidimensional constrained optimization necessary to maximize the log-likelihood function. In solving this problem, it is useful to note its specific form: the statistics of \( \{ y_i \} \) depends linearly on \( \{ q_n \} \) which has all the properties of a probability distribution as a function of \( n \). This makes our task a special case of linear inverse problems with positivity constraints. An effective tool in solving this class of inverse problems is the so-called expectation-maximization (EM) algorithm. Its principle of operation can be understood by considering the necessary condition for the maximum of the log-likelihood function \( \mathcal{L}(\{ q_n \}; \{ y_i \}) \). For each \( m \), the partial derivative \( \partial \mathcal{L}/\partial q_m \) must vanish, unless the maximum is located on the boundary of the allowed region for which \( q_m = 0 \). These two possibilities can be written jointly as

\[
q_m \frac{\partial \mathcal{L}}{\partial q_m} = 0. \tag{13}
\]

This condition can be rearranged to the form

\[
q_m = \frac{1}{N} \sum_i \frac{A_m(y_i) q_m}{\sum_n A_n(y_i) q_n} \tag{14}
\]

for all \( m \), which shows that the maximum likelihood estimate for \( q_n \) is a fixed point of the nonlinear transformation defined by the right hand side of Eq. (14). The idea of the EM algorithm is simply to iterate this transformation. When sufficient mathematical conditions are fulfilled, this procedure converges to the maximum-likelihood solution.

I illustrate the presented method with the reconstruction of the Wigner function from Monte Carlo simulated imperfect homodyne statistics for a Schrödinger cat state

\[
|\Psi\rangle = \frac{1}{\sqrt{2 - 2 \exp(-2|\alpha|^2)}} (|\alpha\rangle - | - \alpha\rangle) \tag{15}
\]

with \( \alpha = 2i \). The computer generated data consisted of \( 10^5 \) events simulated for each of 64 phases uniformly spaced between 0 and \( \pi \). The detector efficiency was assumed to equal \( \eta = 90\% \). The effect of imperfect detection can be observed in the histogram of homodyne events for the phase \( \theta = 0 \), depicted in Fig. 3. The visibility of interference fringes in the homodyne statistics is substantially smaller than in the corresponding quadrature distribution. This blurring is a result of detection losses, and it analogously affects the quasidistribution function reconstructed by means of inverse Radon transform, decreasing the magnitude of the oscillatory pattern in phase space.

Fig. 3 shows the Wigner function reconstructed along the position axis \( q \) via 10 iterations of the EM algorithm at each point, starting from a flat distribution of \( \{ q_n \} \) for \( 0 \leq n \leq 39 \). For a quantitative comparison, the reconstructed values are plotted together with the true Wigner function evaluated for the state \( |\Psi\rangle \). It is clearly seen that virtually full magnitude of the oscillatory pattern is recovered despite detection losses. In contrast, the dashed line in Fig. 3 depicts the quasidistribution function characterized by the ordering parameter \( s = -(1 - \eta)/\eta \). This function would have been obtained using the standard linear back-projection algorithm from homodyne statistics in the limit of infinite number of measurements.

In conclusion, the reconstruction algorithm presented in this Communication demonstrates that application of appropriate estimation methodology can substantially improve performance of quantum homodyne tomography. The maximum-likelihood approach applied in this paper provides an algorithm that is entirely free from singularities appearing in the standard linear reconstruction scheme. Of course, incorporation of a priori constraints does not automatically cancel all the effects of imperfect detection. For a fixed number of measurements, the quality of the maximum-likelihood estimate worsens with decreasing detector efficiency \( \eta \). What is important, however, is that in any case the obtained estimate for \( \{ q_n \} \) remains between physical bounds, which allows one to evaluate safely the sum in Eq. (12). Quantitative discussion of statistical uncertainty in the proposed algorithm will be a subject of a separate publication.

It is noteworthy that the EM algorithm is a well known tool in classical tomography and image restoration. However, these applications make essential use of the fact that the multidimensional object to be reconstructed is positive definite. This is not the case of the Wigner function, whose negativities are a signature of nonclassical properties. Therefore, classical reconstruction algorithms with a priori constraints usually cannot be directly used in quantum state measurement, and there is an apparent need to develop novel tools for quantum tomography, such as that presented in this paper.

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![FIG. 1. Histogram of Monte Carlo simulated homodyne events for the local oscillator phase $\theta = 0$. The modulation depth of the interference pattern is decreased by imperfect detection.](image1)

![FIG. 2. A comparison of the reconstructed Wigner function (●) with its analytical form (solid line). The dashed line represents the quasidistribution function $\eta^{-1}W(\eta^{-1/2}q,0;−(1−\eta)/\eta)$ that is related via inverse Radon transform to blurred quadrature distributions corresponding to efficiency $\eta = 90\%$](image2)