Relativistic effects and quasipotential equations

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We compare the scattering amplitude resulting from the several quasipotential equations for scalar particles. We consider the Blankenbecler-Sugar, Spectator, Thompson, Erkelenz-Holinde and Equal-Time equations, which were solved numerically without decomposition into partial waves. We analyze both negative-energy state components of the propagators and retardation effects. We found that the scattering solutions of the Spectator and the Equal-Time equations are very close to the nonrelativistic solution even at high energies. The overall relativistic effect increases with the energy. The width of the band for the relative uncertainty in the real part of the scattering \(T\) matrix, due to different dynamical equations, is largest for backward-scattering angles where it can be as large as 40%.

I. INTRODUCTION

Although hadrons are small bags of confined quarks, at low and intermediate energies what is observed are mesons and baryons, suggesting that the genuine QCD degrees of freedom may not be adequate for the study of nuclear structure at this energy regime. At low energies the nuclear dynamics is described in terms of nucleons interacting via potentials, within the framework of the Schrödinger or Lippmann-Schwinger equations. At intermediate energies, however, the Non Relativistic (NR) quantum mechanical approach may fail. In particular, for processes with high momentum and energy transfer, we may expect the NR scattering equation to be inadequate to describe the nucleon-nucleon (NN) scattering.

The NR scattering equation, or Lippmann-Schwinger (LS) equation, gives the NN scattering amplitude \(T\), as:

\[
T(p', p; W) = V(p', p; W) - \int \frac{d^3k}{(2\pi)^3} V(p', k; W) \frac{m}{k^2 - p^2 - i\varepsilon} T(k, p; W),
\]

(1)

where \(m\) is the nucleon mass, \(p, p'\) and \(k\) are respectively the initial, final and intermediate relative 3-momenta; \(W\) is the total energy which depends on \(p^2\), and \(V\) is the interaction kernel. This kernel, in a NR framework, describes the instantaneous interaction or potential. The homogeneous term involves also an effective 2-particle scalar propagator.

For a relativistic formulation, we have several alternative equations since there is not a unique description dictated from first principles \([1]\). An implementation of relativity which satisfies covariance explicitly leads to the Bethe-Salpeter (BS) scattering equation \([2]\):

\[
T(p', p; P) = V(p', p; P) + i \int \frac{d^4k}{(2\pi)^4} V(p', k; P) G(k; P) T(k, p; P).
\]

(2)

Here \(p, p'\) and \(k\) are respectively the final, initial and intermediate relative 4-momenta, and \(P\) the total system 4-momentum. As for \(V\), the interaction kernel, it consists of the sum of all irreducible interaction processes derived from a considered Lagrangian. In the scalar case the 2-particle propagator \(G\) is given by

\[
G(k; P) = \frac{1}{m^2 - (P/2 + k)^2 - i\varepsilon} \frac{1}{m^2 - (P/2 - k)^2 - i\varepsilon}.
\]

(3)

Contrarily to the LS equation, which is 3-dimensional and includes only instantaneous interactions, the BS is a 4-dimensional integral equation and its kernel includes retardation and a dependence on the virtual intermediate states energy (off-mass-shell effects). Since the exact solution of the BS equation with the complete kernel is still out of reach of present calculation capabilities, several approximations have been developed. Nevertheless, all these different methods have in common the truncation of the kernel, which keeps only a restrict set of meson exchange diagrams. In Ref. \([3]\), the 4-dimensional BS equations was applied by Tjon and collaborators to the NN problem with a kernel derived from One-Boson-Exchange (OBE) generating the so called Ladder Approximation. It has been...
proved, however, that the Ladder Approximation does not verify the one-body limit. The addition to the OBE kernel of a crossed-box diagram, implementing a particular off-mass shell extrapolation, has been done by Theufl and Desplanques for a scalar bound state problem. Alternatively, for the same problem Phillips and Wallace simulated the cross-box diagram through a specific modification of the 2-particle propagator and the result is known as the so called 4D Equal-Time equation. Calculations considering ladder and crossed-ladder diagrams have been performed by Niemenuhuis and Tjon based on a Feynman-Schwinger formalism and Monte-Carlo techniques. The calculations are, however, still restricted to scalar particles and the bound state cases.

Even with a truncated kernel the BS equation is, beyond Ladder Approximation, very hard to solve. This difficulty has triggered the development of covariant 3-dimensional reduction equations. These can be classified into two classes: the QuasiPotential (QP) type equations and the Klein type equations.

In the QP equations the 3-dimensional reduction is achieved by fixing the relative energy $k_0$ in a covariant way. These equations can formally be written as a BS equation, if we replace $G$ by another propagator $g$ including a covariant energy constraint. In the last years two QP equations have been applied extensively to hadronic physics problems: the Blankenbecler and Sugar (BbS) and the Spectator (Sp), or Gross equations.

The BbS equation, is constructed assuming that, in all intermediate states, the two particles are equally off-mass-shell. The Sp equation, is obtained restricting one particle to its positive-energy on-mass-shell state in all the intermediate states. Although this equation is exact for scalar particles in the one-body limit, when applied to the NN interaction it implies approximations. Furthermore, to account for the identity of the two particles, a symmetrized propagator has to be used. In the BbS equation the interaction is instantaneous because the exchange boson does not carry any energy, since both nucleons are equally off-mass-shell, whereas in the Sp equation energy exchange is allowed and therefore retardation is included. Another QP equation, the 3D Equal-Time (ET) equation, was proposed by Mandelzweig and Wallace to deal with the QED problem of spin 1/2 particles, and with spin zero and spin 1/2 particles with boson exchange cases. In this equation the interaction is also instantaneous and includes the cross-box diagram in the eikonal approximation, or high energy limit, where the forward scattering components are dominant. This 3D ET equation has been applied by Tjon and collaborators to the NN system with an One-Boson-Exchange kernel. Calculations have shown that the 3D ET equation results are the closest to the BS with OBE and crossed-box kernel for the bound state of scalar particles. Henceforth we refer the 3D ET equation as ET equation.

The Klein equation, obtained also from the BS equation, is related with time ordered perturbation theory without anti-particles, and was used by the Nijmegen and Bonn groups in their applications to two-meson-exchange calculations.

In this work we study relativistic effects included in the most well known QP equations - BbS, Sp, Thompson (Th) and Erkelenz-Holinde (EH) - and compare their results with ones obtained with the NR Lippman-Schwinger equation. Since the exact solution of the relativistic equation is not known, there is no exact relativistic answer to be taken as a reference. Therefore, we focus on the analysis of the relativistic content of the different QP equations. Our natural reference, consequently, is the nonrelativistic LS equation. We emphasize, at this point, that the BS equation in ladder approximation, which has been taken as a reference in the past, has been shown to be inadequate for that purpose because, on the one hand, does not reproduce the one-body limit, as mentioned before, and, on the other hand, excludes non negligible contributions of the crossed-box diagrams, as found by the authors of Ref.

As a first step, we consider in all our calculations (corresponding to different equations), only scalar particles interacting through the Malfliet-Tjon potential. Only by using the same interaction in all the dynamical cases we are able to withdraw conclusions about the way the different formalisms introduce relativistic effects. The relativistic effects considered come from kinematics, retardation and negative-energy state components of the propagators. We point out that retardation is automatically excluded from the instantaneous BbS equation. As for the negative-energy state effects, the different equations account for them in different ways, as it will be shown.

In Sec. we present the QP formalism, and give the explicit expression of all the propagators. In Sec., we use the method of Ref., reviewed in references, to distinguish between the equations that include retardation from the others where the interaction is instantaneous. We also introduce the representation of Wallace and Mandelzweig to separate the positive and negative-energy state effects, and discuss the suppression of the negative-energy states contributions. In reference a similar derivation was presented, but none of them discuss explicitly retardation effects, which is one of the main goals of the present work. In Sec. we explain the numerical method used to evaluate the $T$ scattering amplitudes, which does not involve a partial wave decomposition. This method is based on the work of Elster and collaborators in a NR framework. In Sec. we show and discuss the results for the scattering amplitude at two different energies. As an example we also show the phase shifts for the $s$-wave. In Sec. we summarize and draw some final conclusions and remarks.
II. QUASIPOTENTIAL EQUATIONS

The BS equation (3), can be written in a shorthand notation as

\[ T = V + VGT. \]  

(4)

A common approximation consists on restricting \( V \) to the One Boson Exchange (OBE) diagrams, and is usually called the Ladder Approximation. There are two main objections to this approximation: firstly, as mentioned before, the one body limit is not recovered when one of the masses goes to infinity (4); secondly, it has been shown that, in general, crossed-ladder diagrams may have important contributions and hence should not be neglected.

Alternative ways consists on the replacement of BS Eq. (4) by the set of equations:

\[ T = K + KgT \]  
\[ K = V + V(G - g)K, \]  

(5) (6)

which is perfectly equivalent to the BS equation, and where \( g \) is a different 2-particle propagator, containing the 3-dimensional reduction. Since there are no first principle rules to dictate how this reduction should be performed, a large ambiguity gives room to many options. Usually, different approximations compromise different appropriate choices of the constraint on the energy-component of the 4-momentum in the propagator \( K \).

The rate of convergence of the new kernel \( K \) is determined by the difference \( G - g \). If \( g \) is a good approximation to \( G \), then only the first term in Eq. (4) can be kept and Eqs. (5) and (6) reduces to one simpler 3-dimensional, but still covariant, equation. This is the standard procedure in any application of QP equations. If needed and possible, the lowest order truncation of the kernel \( K = V \) can be corrected by higher order terms.

Any finite order truncation of the kernel beyond the OBE approximation generates unphysical singularities in the scattering equation, due to the delta function constraint on the energy \( \delta(k_0) \). For this reason 3-dimensional QP equations with kernels not restricted to the OBE form became controversial \([6,16,25]\). However, we point out that it has been shown that the contribution of the two-pion subtracted [27] and crossed-box diagrams can be simulated by effective vector meson exchanges [27]. This result supports the usefulness of the OBE approximation.

In the present work we apply several QP equations, with an OBE kernel, to the scattering of scalar particles. More specifically, we considered the scalar Malfliet-Tjon potential and took the form of this potential as the kernel of all the equations considered.

We perform the calculations in the c.m. reference frame, where the expressions are simpler. In this frame the Bethe-Salpeter to 2-particle propagator is written as

\[ G(k; P) = \frac{(2m)^2}{[E_k^2 - (W/2 + k_0)^2 - i\varepsilon][E_k^2 - (W/2 - k_0)^2 - i\varepsilon]} \]  

(7)

Here \( k \) represents the magnitude of \( k \), \( E_k = \sqrt{m^2 + k^2} \) and \( W = \sqrt{P^2 - 2E_P} \), if both initial particles are on-mass-shell. This expression differs from the original of Eq. (3) by the factor \((2m)^2\), which is introduced for convenience. In fact, in the scalar case both the kernel and the amplitude \( T \) are dimensionless, whereas the amplitude generated by the LS equation, which refers to fermions, has dimensions of inverse of energy squared. In order to compare the solutions of the two equations, we must redefine \( V \) and \( T \) of the scalar equation by dividing them by \((2m)^2\), meaning that we have to redefine the propagator by multiplying it by the same factor. This procedure implies the use of dimensionless couplings for both the scalar and Dirac cases.

As mentioned in the introduction, in the present work we consider the following QP equations: the Sp, Th, BbS, EH and the ET. All these equations can be obtained replacing the propagator \( G \) of the Eq. (2) by the following \( g \) functions:

\[ g_{Th}(k; W) = i2\pi \frac{m}{E_k} \frac{m}{W} \frac{\delta(k_0)}{E_k - \frac{W}{2} - i\varepsilon} \]  
\[ g_{BbS}(k; W) = i2\pi \frac{m}{E_k} \frac{m\delta(k_0)}{E_k^2 - \frac{W^2}{4} - i\varepsilon} \]  
\[ g_{EH}(k; W) = i2\pi \frac{m}{E_k} \frac{m\delta(k_0 - E_k + W/2)}{E_k^2 - \frac{W^2}{4} - i\varepsilon} \]  
\[ g_{Sp}(k; W) = i2\pi \frac{m}{E_k} \frac{m}{W} \frac{\delta(k_0 - E_k + W/2)}{E_k - \frac{W}{2} - i\varepsilon}. \]  

(8) (9) (10) (11)
The BbS and EH propagators are identical except for the argument of the delta function containing the energy constraint. The same can be said about the Th and the Sp propagators. We also consider the ET scalar equation \[ g_{ET}(k; W) = i2\pi \frac{m}{E_k} \frac{m\delta(k_0)}{E_k^2 - \frac{W^2}{4} - i\varepsilon} \left( 2 - \frac{W^2}{4E_k^2} \right), \] that differs from BbS propagator \( g_{BbS}(k; W) \) by a kinematical factor, which becomes one when the particles are on-mass-shell \( (W = 2E_k) \).

In all cases the energy-fixing condition is included through a \( \delta \)-function. We note here that even the NR equation \[ g_{NR}(k; W) = i2\pi \frac{m\delta(k_0)}{E_k^2 - \frac{W^2}{4} - i\varepsilon}, \] which can also be written as \[ g_{NR}(k; W) = i2\pi \frac{m\delta(k_0)}{E_k^2 - \frac{W^2}{4} - i\varepsilon}, \] but this does not mean that the LS equation is covariant.

All the propagators defined above are for scalar particles. The first five propagators can be organized into two classes, by means of a function \( f(k; W) \), the BbS and Sp types \[ f_{BbS}(k; W) \rightarrow i2\pi \frac{m\delta(k_0)}{E_k^2 - \frac{W^2}{4} - i\varepsilon} \] \[ f_{Sp}(k; W) \rightarrow i2\pi \frac{m\delta(k_0)}{E_k^2 - \frac{W^2}{4} - i\varepsilon} \delta(k_0 - E_k + W/2). \] Any choice of \( f \) is possible providing that \[ f(p; W) = 1, \] which results from the simultaneous constraint of the 2 nucleons in its physical state (positive-energy on-mass-shell state). Taking \( f(k; W) \equiv 1 \) in Eqs. (15) and (16), we obtain respectively the original BbS and EH propagators. Taking \[ f(k; W) = \frac{W + 2E_k}{2W}, \] in Eqs. (17) and (18), we get respectively the Th and Sp propagators. Finally the ET propagator is obtained from Eq. (15) using the following \( f(k; W) \) function:

\[ f(k; W) = \left( 2 - \frac{W^2}{4E_k^2} \right). \]

We point out that the function \( f(k; W) \), to a great extent arbitrary, is related with the weight of the contribution of the negative-energy state components, as we shall see.

For numerical applications we rewrite the propagators \[ g_{BbS}(k; W) \] and \[ g_{Sp}(k; W) \] in a different form. Firstly, we represent \( g \) the propagator without the \( \delta \)-function factor. Secondly, we define the nonsingular function \( \tilde{f}(k; W) \) accordingly to \[ \tilde{f}(k; W) = \frac{m}{E_k} \frac{m\delta(k_0)}{E_k^2 - \frac{W^2}{4} - i\varepsilon} \] and \( \tilde{f}(k; W) \) is expressed in terms of \( f(k; W) \) by \[ \tilde{f}(k; W) = m \frac{m\delta(k_0)}{E_k^2 - \frac{W^2}{4} - i\varepsilon}. \] These \( \tilde{f}(k; W) \) functions verify the relativistic elastic cut condition, expressed in Eq. (22), corresponding to \[ \tilde{f}(p; W) = 2 \left( \frac{m}{W} \right)^2. \]
In order to clarify the interpretation of the positive and negative-energy state effects we follow here the analysis of Ref. [24]. We start with the homogeneous term of the Eq. (2), in the c.m. frame where \( P = (W, 0) \)

\[
\mathcal{R}(k; P) = \int \int d^3kd\!k\!0 V(p'_0, p'0, k; W)G(k; P)T(k_0, k; p_0, p; W);
\]  

(23)

in the above equation we explicitly separate the energy and 3-momentum components of all momenta. Using the representation of Wallace and Mandelzwieg [6,15,16], the propagator of 1-particle with 4-momentum \( P/2 + k \), can be written as

\[
G_1(k; P) = -\left\{ \frac{N_1^+}{k_0 - E_k + W/2 + i\varepsilon} - \frac{N_1^-}{k_0 + E_k + W/2 - i\varepsilon} \right\},
\]

(24)

and the propagator of particle 2 with of momentum \( P/2 - k \) as

\[
G_2(k; P) = \left\{ \frac{N_2^+}{k_0 - E_k + W/2 - i\varepsilon} - \frac{N_2^-}{k_0 + E_k - W/2 + i\varepsilon} \right\}.
\]

(25)

For scalar particles \( N_i^+ = N_i^- = m/E_k \) whereas for fermions \( N_i^+ \) and \( N_i^- \) are respectively the positive and negative-energy projectors [6,15,16].

Defining

\[
\omega_k^+ = E_k - \frac{W}{2} - i\varepsilon
\]

(26)

\[
\omega_k^- = -(E_k + \frac{W}{2}) + i\varepsilon.
\]

(27)

we can rewrite the BS 2-particle propagator \( G \) as

\[
G(k; P) = -\left\{ \frac{N_1^+}{k_0 - \omega_k^+} - \frac{N_1^-}{k_0 + \omega_k^-} \right\}_{E_1>0} \left\{ \frac{N_2^+}{k_0 + \omega_k^+} - \frac{N_2^-}{k_0 - \omega_k^-} \right\}_{E_2>0} \cdot \left( \frac{N_1^+}{k_0 - \omega_k^+} - \frac{N_1^-}{k_0 + \omega_k^-} \right)_{E_1<0} \left( \frac{N_2^+}{k_0 + \omega_k^+} - \frac{N_2^-}{k_0 - \omega_k^-} \right)_{E_2<0}.
\]

(28)

In the expression above we indicate, for the two terms of each particle propagator, the sign of the particle energy at the corresponding pole.

Following [24,21], the QP equations can be obtained in a different way than the one discussed in the previous section, namely by fixing the relative energy \( k_0 \) in \( V \) and, consequently in \( T \). If we choose \( k_0 = 0 \) in \( V \) and \( T \), we can factorize the energy and 3-momentum integrations in the following form

\[
\mathcal{R}(k; P) = \int d^3kV(0, p'; 0, k; W)T(0, k; 0, p; W)\mathcal{G}(k; P),
\]

(29)

where \( \mathcal{G} \) is

\[
\mathcal{G}(k; P) = \int dk_0G(k; P)
\]

(30)

With this choice the interaction between the nucleons is instantaneous. We call these equations the instantaneous-type equations. The integration in the \( k_0 \) variable can be done analytically using the residues theorem. The pole structure of the integrand function is presented in the Fig. [4]. Choosing the lower half contour for the \( k_0 \) integration, we can conclude that

\[
\mathcal{G}(k; P) = I_1(\omega_k^+) + I_2(-\omega_k^-),
\]

(31)
where the residues $I_1$ and $I_2$ are

$$I_1(\omega_k^+) = i2\pi \left\{ \frac{N_1^+ N_2^+}{2E_k - W - i\varepsilon} + \frac{N_1^+ N_2^-}{W} \right\}$$

$$I_2(-\omega_k^-) = i2\pi \left\{ -\frac{N_1^- N_2^+}{W} + \frac{N_1^- N_2^-}{W + 2E_k} \right\}.$$  \hfill (32)  \hfill (33)

By keeping both $I_1(\omega_k^+)$ and $I_2(-\omega_k^-)$ in Eq. (24) we obtain the BbS equation. Neglecting the residue $I_2(-\omega_k^-)$ we get the Th equation.

The other class of equations can be obtained when we fix the $k_0$ in order to correspond to the positive-energy state of the particle 1. This corresponds to the choice $k_0 = \omega_k^+$. So, in that case

$$\mathcal{R}(k; P) = \int d^3k V(\omega_{p'}, p'; \omega_k^+, k; W) T(\omega_k^+, k; \omega_p^+, p; W) G(k; P).$$  \hfill (34)

All equations in this class include retardation, because they include the transferred energy term in the interaction kernel. If in Eq. (34) we include the two residues $I_1(\omega_k^+)$ and $I_2(-\omega_k^-)$ we obtain Erkelenz-Holinde (EH) equation, and if we neglect the last one, we obtain the Sp equation. In that respect Sp equation is more consistent since it considers the same value of $k_0$ in all integrand functions.

We summarize in Table I the comparison among the several QP propagators accordingly to the the discussion above.

Using the decompositions (32) and (33) we may write

$$\bar{g}_{BbS}(k; W) = \bar{g}_{EH}(k; W) = i2\pi \left\{ \frac{N_1^+ N_2^+}{2E_k - W - i\varepsilon} + \frac{N_1^+ N_2^-}{W} - \frac{N_1^- N_2^+}{W} + \frac{N_1^- N_2^-}{2E_k + W} \right\}$$

$$\bar{g}_{Sp}(k; W) = \bar{g}_{Th}(k; W) = i2\pi \left\{ \frac{N_1^+ N_2^+}{2E_k - W - i\varepsilon} + \frac{N_1^- N_2^-}{W} \right\}.$$  \hfill (35)  \hfill (36)

To avoid any possible confusion we call the reader’s attention to the fact that, for spin 1/2 particles, some authors call BbS to a somewhat different 2-particle propagator $\bar{g}_{BbS}$.

For the BbS/EH equations we need to consider the sum of both terms in (31). It is interesting to note that, if the weight functions are equal, $N_1^+ = N_1^-$, as it happens in the scalar case, the residue $I_1$ at the negative-energy pole of particle 2 cancels exactly against the residue $I_2$ at the positive-energy pole of the particle 1. As a result, we can write for the scalar case

$$\bar{g}_{BbS}(k; W) = i2\pi \left( \frac{m}{E_k} \right)^2 \left\{ \frac{1}{2E_k - W - i\varepsilon} + \frac{\eta}{2E_k + W} \right\}.$$  \hfill (37)

where $\eta = 1$ corresponds to the maximal inclusion of negative-energy state term of the propagators, and the case $\eta = 0$ totally excludes them. In the same situation we obtain for the Sp/Th equations the following

$$\bar{g}_{Sp}(k; W) = i2\pi \left( \frac{m}{E_k} \right)^2 \left\{ \frac{1}{2E_k - W - i\varepsilon} + \frac{\eta}{W} \right\}.$$  \hfill (38)

Comparing (37) with (38) in the case of negative-energy suppression ($\eta = 0$) we conclude that all the propagators are equal.

As for the ET propagator, it has been obtained in Refs. [14,13] and reads

$$\bar{g}_{ET}(k; W) = i2\pi \left\{ \frac{N_1^+ N_2^+}{2E_k - W - i\varepsilon} + \frac{N_1^+ N_2^-}{2E_k} + \frac{N_1^- N_2^+}{2E_k} + \frac{N_1^- N_2^-}{2E_k + W} \right\}.$$  \hfill (39)

Compared to the BbS propagator, the $\bar{g}_{ET}$ contains two additional contributions, which the authors justify as an effective inclusion of the crossed-box diagram. Restricting to the scalar case, this expression simplifies to

$$\bar{g}_{ET}(k; W) = i2\pi \left( \frac{m}{E_k} \right)^2 \left\{ \frac{1}{2E_k - W - i\varepsilon} + \frac{\eta}{E_k} + \frac{\eta}{2E_k + W} \right\}.$$  \hfill (40)

If we restrict ourselves to the positive-energy components ($\eta = 0$) then the ET propagator is equivalent to all the other propagators. In particular, for a complete suppression of negative-energy states, BbS and ET equations generate the same results.
FIG. 1. Poles structure of the $G$ propagator.

|                | $V, T$   | $I_1(\omega_k^+)$ | $I_2(-\omega_k^-)$ |
|----------------|----------|--------------------|---------------------|
| Thompson       | $k_0 = 0$| √                  | ×                   |
| Blankenbecler-Sugar | $k_0 = 0$| √                  | √                   |
| Erkelenz-Holinde          | $k_0 = \omega_k^+$| √                  | √                   |
| Spectator      | $k_0 = \omega_k$| √                  | ×                   |

TABLE I. Comparison among QP equations. Here $k_0 = \omega_k^+$ means that retardation is taken into account, and × means suppression of the residue.
IV. SOLVING THE SCATTERING EQUATION WITHOUT PARTIAL WAVE DECOMPOSITION

In the previous section we derived the QP equations in a way to make explicit, in each of the considered equations, the contribution of the positive and negative-energy terms of the BS propagator, as well as the effects of retardation present in the kernel. However, as mentioned before, this method generates exactly the same QP equations as the one described in Sec. II, and we chose to proceed here with the formalism presented in the latter section.

Our starting point is, consequently, Eq. (5) with \( \gamma \) given by one of the functions of Eqs. (8)-(12), which after performing the \( k_0 \) integration can be written as:

\[
T(p', p; W) = V(p', p; W) - \int \frac{d^3k}{(2\pi)^3} V(p', k; W) \bar{g}(k; W) T(k, p; W). \tag{41}
\]

Here \( \bar{g} \) is given by Eq. (21). The value of the relative energy variable, which is not explicit in Eq. (41), is \( k_0 = 0 \) for the instantaneous-type equations, and \( k_0 = E_k - W/2 \) for the retarded-type equations.

The QP equations are 3-dimensional integral equations, corresponding to a radial integration and a 2-dimensional angular integration. Since the physics is independent of the scattering plane, we may choose the incoming 3-momentum \( p \) to have the z-direction and the scattering plane as the \( xz \) plane, meaning that the final momentum is only defined by its magnitude and by the angle \( \theta \), which then gives the scattering angle. If the interaction is scalar (spinless particles), the \( T \)-matrix and the kernel \( V \) are independent of the angle \( \varphi \) that specifies the scattering plane.

Representing the magnitudes of the initial and final 3-momenta by \( p \) and \( p' \) respectively, and defining \( u = \cos \theta \), we can write then

\[
T(p', p; W) = T(p', u; p, 1; W), \tag{42}
\]
\[
V(p', p; W) = V(p', u; p, 1; W). \tag{43}
\]

The intermediate 3-momentum is characterized in terms of a polar angle, \( v = \cos \theta_k \), and an azimuthal angle \( \varphi_k \). With this notation we write

\[
T(k, p; W) = T(k, v, p, 1; W), \tag{44}
\]
\[
V(p', k; W) = V(p', u; k, v, \varphi_k; W). \tag{45}
\]

Using Eqs. (43) and (47) and factorizing the integration in \( \varphi_k \) we can rewrite (41) as

\[
T(p', u; p, 1; W) = V(p', u; p, 1; W) - \int_0^\infty \frac{k^2dk}{(2\pi)^2} \int_{-1}^1 dv \bar{V}(p', u; k, v; W) \frac{\bar{f}(k; W)}{E_k - \frac{W}{2} - i\varepsilon} T(k, v; p, 1; W) \tag{46}
\]

where

\[
\bar{V}(p', u; k, v; W) = \frac{1}{2\pi} \int_0^{2\pi} V(p', u; k, v, \varphi_k; W) d\varphi_k. \tag{47}
\]

Treating the singularity of the integral function in the usual way, we write

\[
T(p', u; p, 1; W) = V(p', u; p, 1; W) - \int_{-1}^1 dv \mathcal{P} \int_0^\infty \frac{k^2dk}{(2\pi)^2} \bar{V}(p', u; k, v; W) \frac{\bar{f}(k; W)}{E_k - \frac{W}{2}} T(k, v; p, 1; W)
\]
\[
- \frac{i\pi}{W} \frac{(2m)^2p}{W} \int_{-1}^1 dv \bar{V}(p', u; p, v; W) T(p, v; p, 1; W) \tag{48}
\]

where we used

\[
\left. \frac{d}{dk} \frac{k^2\bar{f}(k; P)}{E_k - \frac{W}{2}} \right|_{k=p} = \frac{(2m)^2p}{W}, \tag{49}
\]

accordingly to the eq. (22).
By standard discretization we transform the integral equation into an algebraic linear set of equations, as explained in the appendix A and obtain, for a fixed on-mass-shell initial state, the half-off-mass-shell scattering amplitude
\[ T(p', u_j; p, 1; W). \]
One of the numerical checks that has been performed is the verification of the optical theorem. The number of grid points for the magnitudes of the 3-momenta and for the cosine of the polar angles are fixed by imposing an accuracy of the order of 1% in the verification of the optical theorem. Typical values used were 17-25 for the magnitude of the momenta and 20-50 for the angular discretization. We point out however, that the convergence of the on-shell \( T \) matrix does not require so many grid points.

V. RESULTS AND DISCUSSION

We solve the scattering equation [2] with the propagators [3]-[12]. For instantaneous-type equations we considered the kernel to be the Malfliet-Tjon potential [2]:
\[ V(k, p; W) = \frac{V_R}{\mu_R^2 + (k - p)^2} - \frac{V_A}{\mu_A^2 + (k - p)^2}, \]
with the parameters of Ref. [26]:
\[ V_R = 3,1769 \quad \mu_R = 305,86 \text{ MeV} \]
\[ V_A = 7,9210 \quad \mu_A = 613,69 \text{ MeV} \]
The choice of the interaction is due, on one hand, to its simplicity, and, on the other, to the fact that it as has been used recently in the literature [26], within a nonrelativistic framework and without a partial wave decomposition for the T-matrix. This way we could compare and control our numerical results.

In order to study retardation effects however, we have to consider a covariant version of Eq. (50), obtained by replacing the square of the 3-momentum by the negative square of the 4-momentum \((q^2 \to -q^2)\). Therefore we use for retarded-type QP equations
\[ V(k, p; W) = \frac{V_R}{\mu_R^2 + (k - p)^2 - (E_p - E_k)^2} - \frac{V_A}{\mu_A^2 + (k - p)^2 - (E_p - E_k)^2}. \]

To evaluate the scattering amplitude without partial wave decomposition we apply the method presented in the appendix A. We consider throughout our calculations two energies: \( T_{lab} = 300 \text{ MeV} \), where we expect small relativistic effects, and \( T_{lab} = 1 \text{ GeV} \), which corresponds to the intermediate energy range. The results are presented in the Figs. [2-5]. We will focus mainly on the real part of \( T \), where the effects are more significant, but the imaginary part is also presented for completeness.

We organize our study accordingly to the following topics:

- **Effects of the negative-energy components of the propagators** - where we compare the QP amplitudes calculated with positive-energy components of the intermediate propagators (\( \eta = 0 \)), with the corresponding amplitudes including the full propagators (\( \eta = 1 \)). Moreover, by comparing the amplitudes EH/BbS with the Sp/Th ones, we separate the effects originated from \( I_2(-\omega_k) \), present only in the first pair of equations. We note that accordingly to Eqs. (12) and (33) the inclusion of the \( I_2 \) residue means extra terms originated by negative-energy components. In particular, only \( I_2 \) contains a term involving negative-energy components of the propagators of both particles.

- **Effects of retardation** - where we compare the results of Sp/EH equations with and without retardation. We note that, as shown above, the Th equation is the instantaneous version of the Sp equation, and that BbS equation the instantaneous version of the EH equation (Table I).

In Fig. 2 we show the Sp and Th scattering amplitudes calculated with (\( \eta = 1 \)) and without (\( \eta = 0 \)) negative-energy components of the propagators. The + label corresponds to \( \eta = 0 \). As can be seen from the figure, the Sp and NR amplitudes are indistinguishable at 300 MeV and very close at 1 GeV. Since the Sp equation includes the combined effects of retardation and negative-energy state components, we may conclude that the two types of effects tend to add and to approximate the Sp relativistic calculation to the NR one. We can also see that the negative-energy components decrease the real part of the scattering amplitude for Sp and Th formalisms, and, in the former, retardation enhances this effect. These conclusions can be drawn for the two energies considered. We point out that the difference between the Sp and NR amplitudes increase with energy as expected.
In Fig. 3 we compare the EH and BbS scattering amplitudes, which differ only through the inclusion of retardation in the former. It is also shown the ET result. In all cases the inclusion of negative-energy components decreases the real part of the scattering amplitude, and approaches the amplitudes to the NR limit. More significative is the ET result, which has more content of negative-energy components than the EH one, and is closer to the NR result. We remind here that the ET propagator without negative-energy states corresponds to BbS+ (BbS with only positive-energy states). By comparing the retardation effects that are present in EH and absent in BbS, we conclude that retardation also decreases the real part of the amplitude. This is valid for the positive-energy versions and for the full versions. Once again, we may conclude that the combined effects of retardation and negative-energy state components tend to approximate the relativistic calculations to the NR ones. The fact that the ET curve, which does not contain retardation, is the closest to the NR result is explained by the larger contribution of the negative-energy state components in its propagator.

The full scattering amplitude results for the five QP equations along with the NR one can be observed in Fig. 4. The curves shown confirm the conclusion drawn above, namely that the combined effect of retardation and negative-energy state contributions tend to approximate the relativistic calculations to the NR ones. We point out, however, that the full result depends upon a delicate interplay of these two relativistic effects, and, in particular, on the amount of the contribution of negative-energy state components of the propagators. This can clearly be seen by comparing the Sp result (no $I_2(-\omega_k)$) with the EH result (with $I_2(-\omega_k)$), being the first the closest to the NR limit.

All the previous considerations about the real part of the scattering amplitude remain valid for the imaginary part with an opposite direction that is, in general retardation and negative-energy effects increases the imaginary part of the scattering amplitude.

We note that the result from the BbS equation which, accordingly to Eqs. (3) and (4), indeed generates the NR equation when we replace $m/E_k$ by 1, deviates, the most from the NR result.
FIG. 2. Spectator (Sp) and Thompson (Th) scattering amplitudes for 300 MeV and 1 GeV. NR corresponds to Lippmann-Schwinger amplitude.
FIG. 3. Erkelenz-Holinde (EH), Blankenbecler-Sugar (BbS) and Equal-Time (ET) scattering amplitudes for 300 MeV and 1 GeV. NR corresponds to Lippmann-Schwinger amplitude.
Finally, as an example, we show in Fig. 5 the phase shifts for the $l = 0$ partial wave, generated by the different QP equations, as a function of the incoming energy. As can be seen from the figure, at low energies the different equations generate similar results, but at higher energies discrepancies are obvious. In this partial wave the phases generated by the EH and Th equations are the closest to the NR ones, contrarily to what happens with the full T-matrix, where the Sp and the ET are the closest. This fact has a very simple interpretation, stemming from the fact that at high energies the contribution to the transition matrix of the more peripheral partial waves becomes quite important, and therefore looking only to the s-wave channel is a very limited analysis. This conclusion is also achieved in the work of Ref. [26].
VI. CONCLUSIONS

We consider a family of relativistic quasipotential scattering equations, and study the effects of retardation and negative-energy state contributions in the propagators. We restrict our calculation, for the time being, to the scalar case. Comparison with the NR equation is provided. The numerical calculations of the $T$ matrices were done without partial wave decomposition. Instead, we solved the two-dimensional integral equation, and the accuracy of the method was checked through identity provided by the optical theorem. The method implied a discretization of the integral variables with 17-25 grid points for the magnitude of the relative momentum, and 20-50 grid points for the angular variables. The NR solutions were compared with the results of Ref. [26], where also the partial wave decomposition has been avoided. This comparison provided an extra check to our method.

We conclude that, in general, retardation decreases the real part of the scattering amplitude, and increases the imaginary part of the same amplitude. The equations that include the $I_2$ term, i.e. the ones containing additional negative-energy state effects, deviate the most from the NR results. We found also that the Sp and the ET equations generate the scattering amplitudes which are the closest to the NR result. We also point out that the Sp (with retardation but no $I_2$ terms) and the ET equations (without retardation but with $I_2$ terms) consider, in an effective way, some contribution from the crossed-box diagrams. On the other hand, the ET equation, on top of the $I_2$ term, includes further negative-energy state effects, associated with its specific way of describing effectively the crossed-box diagrams. These additional terms compensate partially the absence of retardation. We may therefore conclude that the inclusion of crossed-box diagrams, even in an effective way, approaches the QP results to NR ones.

It is interesting to note that the way the Spectator formalism deals with negative-energy state components, retardation and effective crossed-box diagrams, makes the corresponding amplitudes the closest to the NR limit, even at high energies. Finally we note that the relative uncertainty in the real part of the scattering $T$ matrix due to different dynamical equations depends on the scattering angle and is the narrowest near the forward-scattering region. In the backward-scattering, it can be as large as 40%. This could be expected since in this region the momentum transfer is larger.

In this work we restrict ourselves to spin zero particles. However, calculations to account relativistically for the spin 1/2, such that our formalism can be applied to the nucleon-nucleon system, is on progress. In such a more realistic calculation, the uncertainty band found in this work for the $T$ matrix will most likely imply differences in the nucleon-nucleon interaction, compatible with the scattering data and originated by the different dynamical formalisms.

FIG. 5. $s$-wave phase shifts generated with the different QP equations, as a function of the energy.
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APPENDIX A: SOLVING SCATTERING EQUATION WITHOUT PARTIAL WAVE DECOMPOSITION

In order to solve the 2-dimensional integral Eq. (48) in the variables k and v, we discretize this variables using a gaussian grid with the following properties:

- Linear grid of \( N_u \) points \( u_j \) the variable \( v \in ]-1, 1[ \), with weight \( h_j \).
- Linear grid of \( N_p \) even points \( x_i \) the variable \( x \in ]0, 1[ \), with weight \( w_i \). The correspondent momentum points are obtained through the transformation

\[
p_i = \Lambda \frac{x_1}{1 - x_i}, \quad (A1)
\]

\[
w_i' = \frac{\Lambda}{(1 - x_i)^2} w_i, \quad (A2)
\]

where \( \Lambda \) is a free parameter. We add the point \( x_{N_p+1} = 1/2 \) with \( w_{N_p+1}' = 0 \).

We take \( \Lambda = p \) the on-mass-shell initial momentum. With this choice the evaluation of the principal part in (48) is simplified

\[
P \int_0^{\infty} f(k)dk \approx \sum_{i=1}^{N_p} f(p_i)w_i'. \quad (A3)
\]

The point \( p_{N_p+1} = p \) corresponding to the pole is not included in the sum, hence, the function have no singularities.

We take the outcoming variables \( p' \) and \( u \) from the same grid. In that case the integral equation (48) is transformed into a linear algebraic system of equations with the unknown \( T_{ij} = T(p_i, u_j; p, 1; W) \). The brackets in \( ij \) indicates that this indices can be contracted in only one. The linear system can be written as

\[
C \cdot T = V, \quad (A4)
\]

where

\[
V_{ij} = V(p_i, u_j; p, 1; W). \quad (A5)
\]

The matrix \( C \) are given by

\[
C = I + A + iB, \quad (A6)
\]

where \( I \) in the unity and

\[
A_{ij},(i'j') = \frac{p_{i'}}{(2\pi)^2 E_{p_{i'}} - \frac{W}{2}} \overline{V}(p_i, u_j; p_{i'}, u_{j'}; W) \quad (A7)
\]

\[
B_{ij},(i'j') = \frac{1}{4\pi} \frac{(2m)^2 p}{W} \overline{V}(p_i, u_j; p_{i'}, u_{j'}; W) \delta_{i',N_p+1} \quad (A8)
\]

The factor \( \delta_{i',N_p+1} \) gives the elastic-cut constraint and fixes the momentum \( p_{i'} = p = p_{N_p+1} \). Note that \( A \) is zero if \( i' = N_p + 1 \), and \( B \) is always zero except for \( i' = N_p + 1 \). Accordingly, the matrix elements of \( A + iB \) are purely real or imaginary.
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