Primordial Graviton Production and Decaying Vacuum Energy Density

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Abstract. The problem of cosmological production of (massless) gravitons is discussed in the framework of an expanding, spatially homogeneous and isotropic FRW type Universe with decaying vacuum energy density ($\Lambda \equiv \Lambda(H(t))$) described by general relativity theory. The gravitational wave equation is established and its time-dependent part has analytically been solved for different epochs in the case of a flat geometry. Unlike the standard $\Lambda$CDM cosmology (no interacting vacuum), we show that massless gravitons can be produced during the radiation era. However, high frequency modes are damped out even faster than in the standard cosmology both in the radiation and matter-vacuum dominated epoch. The formation of the stochastic background of gravitons and the remnant power spectrum generated at different cosmological eras are also explicitly evaluated.
1 Introduction

In the context of General Relativity Theory (GRT), the simplest explanation for the present accelerating stage of the observed Universe is the existence of a dark energy (DE) component - in addition to cold dark matter (CDM) - and whose energy density remains constant during the cosmic evolution [1–6]. Observationally the most accepted candidate for dark energy is the cosmological constant $\Lambda$ with energy density $\rho_\Lambda = \frac{\Lambda}{8\pi G}$. At the level of GRT and quantum field theory (QFT) in curved spacetimes, the $\Lambda$-term is usually interpreted as a relativistic simple fluid (vacuum medium) obeying the equation of state (EoS), $p_{\text{vac}} = -\rho_{\text{vac}}$.

The recent Planck2013 results have confirmed that the cosmic concordance model (LCDM), a flat Universe filled with cold dark matter, baryons and a $\Lambda$-term, is in agreement with all currently available cosmological observations [7, 8]. From the theoretical viewpoint, however, the unsettled situation in the particle physics/cosmology interface in which the cosmological upper bound ($\rho_{\text{vac}} \lesssim 10^{-47} \text{GeV}^4$) differs from theoretical quantum field theory expectations ($\rho_{\text{vac}} \sim 10^{71} \text{GeV}^4$) by more than 100 orders of magnitude, originates an extreme fine-tuning problem, the so called cosmological constant problem [9, 10]. Another unsolved mystery from first principles is why the vacuum energy density is so close to the matter energy density which is sometimes referred to as cosmic coincidence problem.

In order to alleviate the cosmological constant and coincidence problems, many authors have proposed decaying vacuum models, or equivalently, a time dependence of the vacuum energy density, $\rho_{\text{vac}}(t) = \frac{\Lambda(t)}{8\pi G}$. Different decay laws for $\Lambda(t)$ have been proposed by many authors and their predictions confronted with the available observational data [11–16]. It is worth mentioning that the most usual critique to these $\Lambda(t)$CDM scenarios is that in order to establish a model and study their observational and theoretical predictions, one needs first to specify a phenomenological time-dependence for $\Lambda$. However, there are some attempts to represent the $\Lambda$ models by a scalar field [17, 18], a Lagrangian description [19], or
even to calculate the $\Lambda(H)$ based on renormalization group techniques from QFT in curved spacetimes [20].

On the other hand, the BICEP2 collaboration has recently reported a positive detection of the B-mode power spectrum in the cosmic background radiation (CMB) which is a specific signature of the primordial gravitational waves (PGWs) [21]. The detected excess is inconsistent with the null hypothesis ($> 5\sigma$) being readily explained by the inclusion of PGWs with tensor/scalar ratio $r \sim 0.2$, being $r = 0$ disfavored at $7\sigma$. Since PGWs were presumably generated during inflation and the determined value of $r$ is too large, it is natural to expect that simplest inflationary models might be ruled out by this result thereby opening (through B-mode cosmology) an observational window to discuss more realistic models of inflation [22–24] (see, however, [25]). So, it is timely to discuss the production of primordial relic gravitons both at low and high frequencies since the minimal post-inflationary $\Lambda$CDM model may not be the end of the story. The high level of tensor models also points to another possibility e.g., the adiabatic amplification of gravitational waves during the radiation era for slightly modified models describing the physics of the early Universe [26].

In this context, we discuss here the production of primordial gravitational waves (tensor modes) in the framework of a simple $\Lambda(H)$ decaying vacuum cosmology. The power and energy density spectrum will be analytically derived. Interestingly, unlike the cosmic concordance model ($\Lambda$CDM), there exists adiabatic amplification of gravitational waves (tensor modes) during the radiation epoch. As we shall see, such a result remains valid for generic decaying vacuum cosmologies.

2 The Model: Basic Equations

Let us now consider that the Universe is well described by a flat Friedmann-Robertson-Walker (FRW) geometry. In the co-moving coordinate system, the background line element reads ($c = 1$):

$$ds^2 = dt^2 - a^2(t) \left( d\eta^2 - dl^2 \right),$$  \hspace{1cm} (2.1)

where the 3-space metric is $dl^2 = dx^2 + dy^2 + dz^2 \equiv \delta_{ij}dx^i dx^j$, $i,j,k=1,2,3$ and $t$, $\eta$ are, respectively, the cosmic and conformal times, related by

$$dt = a d\eta.$$  \hspace{1cm} (2.2)

The Einstein field equations for a two-fluid mixture (matter + vacuum) is given by:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G (T_{\text{mat}}^{\mu\nu} + T_{\text{vac}}^{\mu\nu}),$$  \hspace{1cm} (2.3)

with $R_{\mu\nu}$ and $R$, being the Ricci tensor and Ricci scalar, respectively. The energy-momentum tensors (matter and vacuum) read:

$$T_{\text{mat}}^{\mu\nu} = (\rho + p) U^\mu U^\nu - p g^{\mu\nu}, \quad T_{\text{vac}}^{\mu\nu} = \rho_{\text{vac}} g^{\mu\nu},$$  \hspace{1cm} (2.4)

where $\rho$, $p$, $U^\mu$ are, respectively, the energy density, pressure and four-velocity of the fluid, whereas $\rho_{\text{vac}} \equiv \Lambda(t)/8\pi G$, is the vacuum energy density. The field equations in the above background take the form [12, 13, 15]
\[ 8\pi G \rho + \Lambda(t) = \frac{3\dot{a}^2}{a^2}, \tag{2.5} \]
\[ 8\pi G p - \Lambda(t) = -2\frac{a}{\dot{a}} - \frac{\dot{a}^2}{a^2}, \tag{2.6} \]

where a dot means derivative with respect to the cosmic time \( t \). In order to solve the above equation we need to know both the functional form of \( \Lambda(t) \) and the equation of state (EoS).

Many phenomenological functional forms have been proposed in the literature for describing a time-varying \( \Lambda(t) \) vacuum \[15\]. Based on dimensional arguments, Carvalho et al. \[12\] shown that a natural dependence is \( \Lambda \propto H^2 \). Later on, this functional dependence was derived within a renormalization group approach by Solà and Shapiro \[27\] thereby arriving to \( \Lambda(H) = \Lambda_0 + \beta H^2 \), where \( \beta \) is a dimensionless free parameter. Observational constraints on \( \beta \) depend whether \( \Lambda_0 \) is present or not in the phenomenological law. In the absence of \( \Lambda_0 \), the free parameter was first constrained from big-bang nucleosynthesis to be \( \beta \leq 0.16 \) \[28, 29\], a value not enough to accelerate the late time observed Universe. In the presence of \( \Lambda_0 \), Basilakos and collaborators \[30, 31\] based on a set of complementary observations found \( \beta \sim 10^{-3} \) (see also \[32\]), a very comprehensible value because \( \Lambda_0 \) models work like an attractor solution.

Nevertheless, in order to discuss analytically some subtleties related to the decaying vacuum contribution for the stochastic background of gravitational waves, it will be assumed here that the \( \Lambda(H) \)-term is the one adopted by Carvalho et al. \[12\]:

\[ \Lambda(H) = 3\beta H^2, \tag{2.7} \]

where \( H = \dot{a}/a \) is the Hubble parameter, and the factor 3 was added for mathematical convenience. By assuming the \( \omega \)-law EoS

\[ p = \omega \rho, \tag{2.8} \]

where \( \omega \) is a different constant for each era, and combining Eqs. (5)-(8), we find that the scale factor is driven by the following differential equation:

\[ a\ddot{a} + \Delta \dot{a}^2 = 0, \tag{2.9} \]

where we have introduced the convenient short notation

\[ \Delta = \frac{3(1+\omega)(1-\beta) - 2}{2}. \tag{2.10} \]

In terms of the conformal time (\( \eta \)), the scale factor equation (2.9) can be rewritten as \[33\]

\[ \frac{a''}{a} + (\Delta - 1)\frac{a'^2}{a^2} = 0. \tag{2.11} \]

where primes denote derivatives with respect to \( \eta \). It is easy to check that the general solution for the scale factor is given by:

\[ a(\eta) = c_1(\Delta \eta - c_2)^{1/\Delta}, \tag{2.12} \]
where \( c_1 \) and \( c_2 \) are integration constants. As usual, the solution for each era is specified by the corresponding value of \( \omega \), namely: inflation (\( \omega_{\text{inf}} = -1 \)), radiation-vacuum (\( \omega_{\text{rad}} = 1/3 \)) and matter-vacuum (\( \omega_{\text{mat}} = 0 \)). For each era we have also assumed that the vacuum decays only on the dominant component. Note also that the early exponential inflation is not modified by the decaying vacuum since \( \omega = -1 \) implies \( \Delta = -1 \) so that \( \ddot{a} > 0 \) regardless of the values of \( \beta \) (see Eqs. (2.9) and (2.10)).

In order to find the integration constants for each era we must use the continuity junction conditions for the transition times between each era, \( a_n(\eta_i) = a_{n+1}(\eta_i) \) and \( a_n' (\eta_i) = a_{n+1}' (\eta_i) \).

With this we have:

\[
a(\eta) = \begin{cases} 
-l_i \eta^{-1} & \eta < 0, \quad \eta \leq \eta_1 \\
_l i a_{0r}(\Delta_{\text{rad}} \eta - \eta_{\text{rad}})^{1/\Delta_{\text{rad}}}, & \eta_1 \leq \eta \leq \eta_{\text{eq}} \\
_l i a_{0m}(\Delta_{\text{mat}} \eta - \eta_{\text{mat}})^{1/\Delta_{\text{mat}}}, & \eta \geq \eta_{\text{eq}}
\end{cases}
\]  

(2.13)

where \( l_i \) is a constant and the parameter \( \Delta_\alpha \) corresponds to the value \( \omega_\alpha \). The transition time between inflation and radiation era is \( \eta_1 \), and between radiation and matter is \( \eta_{\text{eq}} \). The values of the integration constants are

\[
\eta_{\text{rad}} = (\Delta_{\text{rad}} + 1) \eta_1, \\
a_{0r} = (-\eta_1)^{-1(1+1/\Delta_{\text{rad}})}, \\
\eta_{\text{mat}} = (\Delta_{\text{mat}} - \Delta_{\text{rad}}) \eta_{\text{eq}} + \eta_{\text{rad}}, \\
a_{0m} = a_{\text{rad}} (\Delta_{\text{rad}} \eta_{\text{eq}} - \eta_{\text{rad}})^{1/\Delta_{\text{rad}}}, \\
\eta_{\text{mat}} = \beta = 0, \end{cases}
\]  

(2.14)

Now we have a complete solution for the scale factor which depends on the parameter \( \beta \). In the special case of no decaying vacuum (\( \beta = 0 \)), the quantity \( \Delta \) as given by (2.10) reduces to the standard definition and all the expressions appearing in ref. [34] are recovered. For future calculations it is important to estimate values of the transition times \( \eta_1 \) and \( \eta_{\text{eq}} \). In order to do that, let us compare the scale factors of different cosmological eras. Comparing the scale factor at the present time \( \eta_0 \) with the scale factor at the end of inflation we have \( a(\eta_0)/a(\eta_1) \approx 10^{21} \) and for the end of the radiation era \( a(\eta_0)/a(\eta_{\text{eq}}) \approx 10^4 \) [34]. Using the solution for the scale factor (2.13) and solving the equations system we obtain that \( \eta_1 \approx -10^{-17} \) and \( \eta_{\text{eq}} \approx 3 \times 10^{-3} \). Although representing a crude approximation since the value of \( \eta_{\text{eq}} \) must depend on the \( \beta \) parameter, throughout this paper we adopt this values.

In the Figure 1 we show schematically the behavior of the scale factor for some selected values of \( \beta \). As expected, for a given value of \( \omega \), the expansion grows faster for higher values of \( \beta \). This happens because the vacuum component contributes with a negative pressure.

3 Cosmological Tensor Perturbations

In the conformal time, a classical tensor metric perturbation in the FRW flat geometry given by Eq.(2.1) can be written as:

\[
ds^2 = a^2(\eta) [d\eta^2 - (\delta_{ij} + h_{ij}) dx^i dx^j],
\]  

(3.1)
Figure 1. Evolution of the scale factor for different regimes and some selected values of $\beta$ as indicated in the figure. The expansion rate is faster for higher values of $\beta$.

where the perturbation $h_{ij}$ is small, $|h_{ij}| \ll 1$, and satisfy the well known [35] symmetry (transverse-traceless) and gauge constraints, namely: $h_{ij0} = 0$, $h_{ij} = 0$, $\nabla^j h_{ij} = 0$.

Let us now assume that the vacuum is a smooth unperturbed component as discussed by many authors [14, 36]. In the context of GRT, it is easy to see that in this case the evolution of the tensor perturbations maintains the standard form [37]:

$$h_{ij}'' + 2a' a h_{ij}' - \nabla^2 h_{ij} = 0,$$

(3.2)

and its general solution can be expressed in terms of Fourier spatial harmonics expansion:

$$h_{ij}(\eta, x) = \sqrt{16\pi G (2\pi)^3/2} \int d^3 n \sum_{r=+,-}^r \epsilon_{ij}^r(n) \times \left[ r_{h_0}(\eta) e^{i n \cdot x} r^n + r_{h_0}^*(\eta) e^{-i n \cdot x} r_n^\dagger \right],$$

(3.3)

where $h_n(\eta)$ are the mode functions, $n$ is the comoving wave vector, $r^n$ and $r_n^\dagger$ are complex numbers. The polarization tensor, $\epsilon_{ij}(n)$, is symmetric ($\epsilon_{ij}(n) = r_{ij}(n)$), traceless ($\epsilon_{ii}(n) = 0$), and transverse ($n_i \epsilon_{ij}(n) = 0$). We also choose a circular-polarization basis in which $\epsilon_{ij}(n) = (\epsilon_{ij}(-n))^*$, and normalize the basis $\sum_{ij} \epsilon_{ij}(n)(\epsilon_{ij}(n))^* = 2\delta^{rs}$.

The comoving wave number $n = |n|$ is related with the physical wave number $k$ by

$$n = |n| = \frac{2\pi a(\eta)}{\lambda} = k a(\eta).$$

(3.4)

Now, by inserting the solution (3.3) in (3.2) it is readily seen that the temporal part decouples thereby giving the evolution equation for the conformal time modes:

$$r_{h_n}(\eta)'' + 2a' a r_{h_n}(\eta)' + n^2 r_{h_n}(\eta) = 0.$$  

(3.5)

Using the auxiliary function $\tilde{\mu}(\eta, n) = r_{h_n}(\eta) a(\eta)$ the above equation assumes the first obtained by Grishchuck [34] which is independently satisfied for each polarization $r = +, \times$: 

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Figure 2. Potential, $V(\eta) = a''/a$ for some values of $\beta$. Note that there is no adiabatic amplification for $\beta = 0$ and $\omega = 1/3$ (standard radiation era) since in this case $a''/a \equiv 0$ (see also Eqs. (2.10)-(2.11)).

Therefore, by assuming that the vacuum component is smooth we see that the standard wave equation is not modified. Given the solutions for the scale factor $a(\eta)$ in different eras, we can solve (3.6) for each mode $n$. It represents an harmonic oscillator with variable frequency determined by the evolution of the Universe and describes different behaviors for the high and low frequency regimes (with and without a vacuum component). Once the solutions for $\mu(\eta)$ for the different cosmic eras has been calculated, it is immediate to obtain the associated physical quantities like the wave amplitude, energy density and power spectrum.

An important quantity driving the behavior of the primordial gravitational waves is the “potential” $V(\eta) = a''/a$ appearing in equation (3.6) (it should be recalled that the name “potential” come from the mathematical analogue with the stationary Schrodinger equation). The relation between the potential and the wave-number determines the behavior of the limit solutions for $\mu(\eta)$.

For the times when $n^2 \gg |V|$ holds, the solution of (3.6) is oscillatory, $\mu \propto e^{\pm in\eta}$, so that the high-frequency waves are diluted by the cosmic expansion $h = e^{\pm in\eta}/a$. In the opposite limit, $n^2 \ll |V|$, we have $\mu \propto a$, and, consequently, the low-frequency waves obey $h = constant$. The effect of the potential is to avoid the damping of the waves due the universe expansion. The net effect is that the perturbations are relatively enhanced, a phenomenon commonly referred to as adiabatic amplification [38]. Note that in the limit case ($\beta = 0$, $\omega = 1/3$), that is, in the standard radiation phase, we see that $\Delta = 1$ and, therefore $a'' \equiv 0$. It thus follows that in the radiation era ($\eta_1 < \eta < \eta_{eq}$), the potential vanishes identically ($V(\eta) \equiv 0$). Physically, this means that there is no adiabatic amplification of gravitational waves during the standard radiation phase.

In Figure 2 we show the behavior of the potential for the different eras. As it will be discussed next, for $\omega = 1/3$ and $\beta \neq 0$ gravitational waves are produced so that low frequency modes can be slightly amplified even during the radiation phase. Let us now discuss the solutions of the wave equation for the different eras.
4 Gravitational Wave solutions for different eras

4.1 Inflation Era

In the particular case of $\omega = -1$ we have an exponential inflation with a potential, $a''/a = -2/\eta^2$, regardless of the values assumed by the $\beta$ parameter. This case has already been studied in [39]. The scale factor $a_{\text{inf}}(\eta) = -l_i/\eta$ gives a positive constant Hubble parameter $H_I = l_i^{-1}$. In the inflation era the gravitational wave equation (3.6) can be written as:

$$\mu_{\text{inf}}'' + \left( n^2 - \frac{2}{\eta} \right) \mu_{\text{inf}} = 0, \quad (4.1)$$

where for simplicity, we have suppressed the polarization index $r$. The general solution of the above equation can be expressed in terms of Bessel’s functions

$$\mu_{\text{inf}}(n, \eta) = \sqrt{\eta} [A_i J_{-3/2}(n\eta) + B_i J_{3/2}(n\eta)]. \quad (4.2)$$

We have to specify some conditions to calculate the integration constants $A_i$ and $B_i$. In the inflation era, the limit for high frequencies must reach the so-called adiabatic vacuum

$$\lim_{n \to \infty} \mu \propto e^{-i n \eta} \quad [40].$$

Using this condition and also in its first derivative the constants reduce to $A_i = i \sqrt{\pi/2}$ and $B_i = -\sqrt{\pi/2}$. After doing some algebra we have the normalized $(\mu_{\text{inf}} \mu_{\text{inf}}' - \mu_{\text{inf}}^2 \mu_{\text{inf}}'' = i)$ solution

$$\mu_{\text{inf}}(n, \eta) = \frac{1}{\sqrt{2n}} \left( 1 - \frac{i}{n\eta} \right) e^{-i n \eta}. \quad (4.3)$$

Knowing the full expressions for $\mu$ and $a$ is easy to calculate the power $P$ and energy spectrum $\Omega_{gw}$. For details see the Appendix B.

$$P_{\text{inf}}(n, \eta) = \frac{16G}{\eta l_i^2} (1 + n^2 \eta^2),$$

$$\Omega_{gw}^{\text{inf}}(n, \eta) = \frac{8G}{3\pi l_i^2} \eta^4 \left( 2 + \frac{1}{n^2 \eta^2} \right). \quad (4.4)$$

These are standard results already studied, for details see [41]. In particular for long wavelengths $\lambda \gg l_i = H_I^{-1}$ the power spectrum is flat and proportional to $H_I$, $P_{\text{inf}} = \frac{16}{3 \pi} H_I^2$. For our purposes the calculation of $\mu_{\text{inf}}$ is important to obtain a complete solution for $\mu_{\text{rad}}$ in the radiation era that will be shown in the next section.

4.2 Radiation Era

In the radiation era we have $\omega = \omega_{\text{rad}} = 1/3$ and $\Delta_{\text{rad}} = 1 - 2\beta$. From the second solution given by Eq. (13), it is readily seen that the Eq. (20) now takes the form:

$$\mu_{\text{rad}}'' + \left( n^2 - \frac{1 - \Delta_{\text{rad}}}{[\Delta_{\text{rad}}(\eta - \eta_1) - \eta_1]^2} \right) \mu_{\text{rad}} = 0. \quad (4.5)$$

At this point, we have to stress the first important result of the present paper: The potential for this case is $V = -2\beta (a'/a)^2 \neq 0$. In the particular case of $\beta = 0$ (no decaying vacuum) we obtain the well-known result $V = 0$ of no GW amplification in the radiation era. In the decaying vacuum models this does not hold anymore, the $V \neq 0$ condition implies that always the radiation will contribute to the primordial GW spectrum today.
Figure 3. The amplitude of the gravitational wave in the radiation era. 

**a)** Modulus of the mode function $|h_{\text{rad}}|$ as a function of the conformal time for some selected values of $\beta$ and a fixed comoving low frequency, $n = 10$. Note that in the low frequency regime decaying vacuum models amplify the perturbations since the amplitude is higher in comparison to the case $\beta = 0$. However, high frequency modes are always damped out regardless of the value of $\beta$ (see main text).

**b)** The same plot of figure a) but now for a fixed high frequency, $n = 10^5$. Surprisingly, we see that the high frequencies modes are damped even faster than in the standard case ($\beta = 0$).

**c)** Modulus of the mode function $|h_{\text{rad}}|$ as a function of $n$ and the same selected values of $\beta$.

The general solution of the last equation is:

$$
\mu_{\text{rad}}(n, \eta) = \sqrt{\Delta_{\text{rad}} \eta - \eta_1 (\Delta_{\text{rad}} + 1)}
\times \left[ A_r J_{\alpha_r} \left( \frac{n}{\Delta_{\text{rad}}} (\Delta_{\text{rad}} \eta - \eta_1 (\Delta_{\text{rad}} + 1)) \right) + B_r J_{\alpha_r} \left( \frac{n}{\Delta_{\text{rad}}} (\Delta_{\text{rad}} \eta - \eta_1 (\Delta_{\text{rad}} + 1)) \right) \right],
$$

where $\alpha_r = \frac{1}{\Delta_{\text{rad}}} - \frac{1}{2}$. The continuity junction conditions, $\mu_{\text{inf}}(\eta_1) = \mu_{\text{rad}}(\eta_1)$ and $\mu'_{\text{inf}}(\eta_1) = \mu'_{\text{rad}}(\eta_1)$, must be used to calculate the integration constants $A_r$ and $B_r$. Making the calculations we have:

$$
A_r = \frac{e^{-in\eta_1} \pi \sec(\pi/\Delta_{\text{rad}})}{\Delta_{\text{rad}} (-2n \eta_1)^{3/2}} \left[ n \eta_1 (i - n \eta_1) J_{-\alpha_r} (\eta^*) - (2i - n \eta_1 (2 + in \eta_1) - \Delta_{\text{rad}} (i - k \eta_1)) J_{-\alpha_r} (\eta^*) \right],
$$

$$
B_r = \frac{e^{-in\eta_1} \pi \sec(\pi/\Delta_{\text{rad}})}{\Delta_{\text{rad}} (-2n \eta_1)^{3/2}} \left[ -in^2 \eta_1^2 J_{\alpha_r} (\eta^*) - (in \eta - n^2 \eta_1^2) J_{\alpha_r + 1} (\eta^*) \right].
$$

where $\eta^* = -\frac{in \eta_1}{\Delta_{\text{rad}}}$. Note that the above solution for the radiation era is also normalized since it satisfies $\mu_{\text{rad}} \mu'_{\text{rad}} - \mu'_{\text{rad}} \mu_{\text{rad}} = i$. These expressions allow us to obtain the amplitude of the perturbations.

Figures 3a and 3b display the evolution of the amplitude $|h_{\text{rad}}|$ as a function of the conformal time $\eta$ and some selected values of $\beta$. As discussed above, the behavior of low frequency (Figure 3a) and high frequency (Figure 3b) modes are quite different since the former are amplified (even during the radiation phase) while the later are damped. Surprisingly, we see that the high frequencies modes are damped even faster than in the standard case ($\beta = 0$).

How can such a result be understood? The basic point here is that in the high frequency limit the term $a''/a$ can be neglected and $h(\eta) = a^{-1}(\eta)\mu(\eta)$. Therefore, since the solution
of $\mu$ is an oscillating function, the amplitudes are damped out even more intensively (in comparison to $\beta = 0$) since the scale factor expands faster for higher values of $\beta \neq 0$ (see Figure 1).

Conversely, in the low frequency regime we find exactly the opposite behavior. Indeed, due to the condition $n^2 \ll \ddot{a}/a$, the solution for low frequencies is $\mu \propto a$ so that the perturbations, $h(\eta)$, remains nearly constant (see Figure 3a). Is exactly in this regime (low frequencies) that the amplification occurs even during the radiation phase. Finally, in figure 3c we show the behavior of $|h_{\text{rad}}|$ as a function of the frequency for a fixed time.

### 4.3 Matter Dominated era

In the matter dominated phase, the equation of state is $\omega \equiv \omega_{\text{mat}} = 0$ which implies that $\Delta_{\text{mat}} = (1 - 3\beta)/2$. In this case, the scale factor reduces to:

$$a_{\text{mat}}(\eta) = a_0 m \left( \frac{1 - 3\beta}{2} \eta - \eta_{\text{mat}} \right)^{2/(1-3\beta)}$$  \hspace{1cm} (4.7)

where $\eta_{\text{mat}} = \frac{1-2\beta}{2} \eta_{\text{eq}} + 2(1-\beta) \eta_1$ and $a_0 m = (-\eta_1)^{-\frac{2(1-\beta)}{2(3\beta-1)}} [(1 - 2\beta) \eta_{\text{eq}} - 2(1 - \beta) \eta_1]^{\frac{1-3\beta}{1-5\beta+6\beta^2}}$. Calculating the potential and substituting in (3.6) we have

$$\mu''_{\text{mat}} + \left(n^2 - \frac{2 + 6\beta}{(3\beta - 1) \eta + 2 \eta_{\text{mat}}}^2 \right) \mu_{\text{mat}} = 0,$$  \hspace{1cm} (4.8)

solving it we obtain the general solution

$$\mu_{\text{mat}} = \sqrt{\Delta_{\text{mat}} \eta - \eta_{\text{mat}}} \left[ A_{m} J_{\alpha_m} \left( \frac{n(\Delta_{\text{mat}} \eta - \eta_{\text{mat}})}{\Delta_{\text{mat}}} \right) \right] + B_{m} J_{-\alpha_m} \left( \frac{n(\Delta_{\text{mat}} \eta - \eta_{\text{mat}})}{\Delta_{\text{mat}}} \right),$$  \hspace{1cm} (4.9)

with the index $\alpha_m = \frac{1}{\Delta_{\text{mat}}} - \frac{1}{2}$. Again the constants, $A_{m}$ and $B_{m}$, are obtained by using the continuity conditions at the transition time $\eta_{\text{eq}}$ between the radiation and matter era, $\mu_{\text{rad}}(\eta_{\text{eq}}) = \mu_{\text{mat}}(\eta_{\text{eq}})$ and $\mu'_{\text{rad}}(\eta_{\text{eq}}) = \mu_{\text{mat}}(\eta_{\text{eq}})'$, the full expressions are cumbersome and will not be presented.

In Figure 4 we plot $|h_{\text{mat}}|$. We can see that a behavior like $|h_{\text{rad}}|$ is also obtained. As one may conclude, this happens because of the same reasons already discussed in the preceding...
section. For low frequencies if we have two perturbations $|h_1(\beta_1)|$ and $|h_2(\beta_2)|$, with $\beta_1 > \beta_2$, then $|h_1| > |h_2|$. This condition inverts for the high frequency regime (for $\beta_1 > \beta_2$ then $|h_1| < |h_2|$).

5 Power and energy density spectra

Let us now discuss the power spectrum and the spectral energy density parameter (per logarithmic wave number interval) which can be written as (see Appendix for details):

$$P(n, \eta) = \frac{32G}{\pi} n^3 |h_n(\eta)|^2,$$

(5.1)

and

$$\Omega_{gw}(n, \eta) = \frac{8\pi G}{3H^2(\eta)} \frac{n^3}{2\pi^2} (|h'_n(\eta)|^2 + n^2 |h_n(\eta)|^2).$$

(5.2)

In comparison to other alternative cosmologies, one advantage of our simple decaying vacuum models is that the above quantities can analytically be calculated for the radiation and matter dominated eras. Actually, this happens because the amplitude $h(n, \eta) = \mu/a$ and the corresponding solutions of $\mu$ were explicitly obtained for each case (see previous section).

In Figure 5 we show the plots of the power spectra, $P_{\text{rad}}$ and $P_{\text{mat}}$, for radiation and matter dominated eras, respectively. From the first plot we see that $P_{\text{rad}}$ is almost flat until some transition frequency when begins to decrease. Similarly to what happens for the amplitudes, the power spectrum in this regime is slightly larger as the parameter $\beta$ increases. As should be expected, the decaying vacuum contributes to the creation of low-frequency gravitons with the corresponding spectrum remaining essentially flat.

Nevertheless, after some transition frequency (corresponding to $n \sim f ew \times 10^3$), the waves are strongly damped and the associated power spectrum is no longer flat. This means that the decaying vacuum in this regime contributes more to increase the scale factor than to the production of gravitons. $P_{\text{rad}}$ decreases exponentially and the effect is even more pronounced for larger values of $\beta$. Note also that at late times, the $P_{\text{mat}}$ for the matter-vacuum dominated phase presents the same high and low frequencies general properties of the radiation era. It starts with an almost flat spectrum and also decreases faster as long as the vacuum contribution is relatively larger (higher values of $\beta$).

In Figure 6 we display the energy density parameter as a function of the comoving wave number for the radiation and matter dominated eras. An interesting feature of the energy
density spectrum in the radiation era, $\Omega_{gw}^{(\text{rad})}$, is that if $\beta_1 > \beta_2$ then $\Omega_{gw}^{(\text{rad})}(\beta_1) < \Omega_{gw}^{(\text{rad})}(\beta_2)$ for all frequencies. Note also that $\Omega_{gw}^{(\text{rad})}$ grows as a power-law being weakly dependent on the value of $\beta$, but a more strong dependence is obtained at the high frequency limit.

The evolution of $\Omega_{gw}^{(\text{mat})}$ follows a similar trends with some peculiarities at the high frequency limit. As in the radiation case if $\beta_1 > \beta_2$ then $\Omega_{gw}^{(\text{mat})}(\beta_1) < \Omega_{gw}^{(\text{mat})}(\beta_2)$ for all frequencies. In addition, for low frequencies the spectrum also grows as a power law linear being slightly lower for bigger values of $\beta$. However, in the high frequency regime, the spectrum always decreases but varies differently as a function of the $\beta$ parameter. In particular, for $\beta \sim 0.2$, the fall is very abrupt. Note also that for $\beta = 0$ it initially decreases and after remains almost flat for all modes. The basic reason for such a behavior is simple: when $|h'|^2 < |h|^2$ the energy density spectrum is $\Omega_{gw} \propto n^5 |h|^2 H^{-2}$, where $H = a'/a$ (see Eq. (B.7)). This means that the contribution of the factor $H^{-2}$ (which is not present in the power spectrum) largely determines the behavior of $\Omega_{gw}^{(\text{mat})}$.

6 Final Comments

In this paper, by assuming a spatially flat geometry, we have investigated the production of gravitational waves for an interacting mixture of matter and vacuum in the context of general relativity. The dynamical $\Lambda$-term was described by a phenomenological law: $\Lambda(H) = 3\beta H^2$, and a three stage description involving inflation, radiation and matter eras was adopted. It was also assumed that the vacuum decays only on the dominant component.

For each cosmic era, we have determined the general expressions for the scale factor (in the conformal time), as well as the analytical solutions for the gravitational wave equation (see Figs. 1, 3 and 4). We notice that the mode function equations of the primordial GW’s were derived and explicitly solved for each era. More interesting, the corresponding power spectra and the energy density parameter for the radiation and matter era were also obtained (see Figs. 5 and 6). Obviously, exact solutions are allowed in this framework due to the simplified phenomenological form adopted for the decaying $\Lambda(H)$-term.

In the vacuum model adopted here, the scale factor expands faster as long as the $\beta$ parameter increases thereby affecting considerably the production and evolution of the gravitational wave modes for both regimes - low and high frequency limits (see Figs. 3 and 4). However, the most prominent feature is that the “potential” in the radiation era never vanishes (see Fig. 2). As a consequence, unlike models with no decaying vacuum ($\beta = 0$), the
GW’s are adiabatically amplified (in the sense of Grishchuk [38]) even during the radiation phase. This result is completely different from the cosmic concordance ΛCDM model. Actually, for the cosmic concordance model, it has been proved that during the radiation phase (negligible Λ) the gravitationally induced quantum production of particles (scalar or gravitons) due the cosmic expansion is forbidden (in this connection see [42–44]).

On the other hand, our expressions for the modulus of the GW amplitude $|h|$ shown us that the adiabatic amplification (graviton production), is a low-frequency phenomenon even for this kind of Λ($H$) cosmologies. This interesting and known feature is also maintained in the present context even considering that GW are produced in the radiation phase. Here as there, the basic problem is that at the high frequency regime the cosmic expansion dominates, and, therefore, the perturbations are dynamically suppressed in the course of the expansion. This behavior is also reproduced in the power spectrum due $P \propto n^3|h|^2$. However, in the case of the energy density parameter spectrum, $\Omega_{gw} \propto H^{-2}P$, the behavior is different due to the contribution of $H^{-2}$. Naturally, a very important point for this kind of cosmology is to calculate all the observables for the present time and compare with the recent BICEP2 results [21].

Finally, we stress that the present work was based on a very simple decaying vacuum cosmology. It played the role of a toy model for obtaining analytically some basic information, like the production of GW’s during the radiation phase. It can be thought as a starting point for the investigation of more complex and rich decaying vacuum cosmologies, like the one proposed in Refs. [13, 45], and, recently, investigated in a more enlarged framework in Ref. [46, 47]. Although less analytical regarding the calculations involving gravitational waves, such models deserve a closer scrutiny since they furnish a complete cosmological history. In this connection, some interesting questions arise naturally here, as for instance, are such decaying vacuum cosmologies capable to explain the results of the BICEP2 experiment? How to compare the graviton production in this kind of model with the ones predicted by inflationary cosmologies? Some of these issues (including a comparison with the present observations) will be discussed in a forthcoming communication.

A Quantized tensor perturbations

The generating mechanism of the primordial tensor perturbations (gravitational waves) is believed to have a quantum mechanical origin. The basic idea is that quantum fluctuations of the vacuum state in the early universe were stretched to macroscopic scales due the cosmic inflation thereby originating the present observable primordial gravitational wave spectrum. The standard quantization procedure is based on a semi-classical approach where the perturbations are quantized but the gravitational background evolves classically (for details see the review of Giovannini [48]).

When the perturbations are quantized on a classical background, the functions $c_n \in r_n$ in (3.3) are promoted to be quantum creation and annihilation operators which satisfy equal time commutation relations ($\hbar = 1$)

$$[r_n^r, c_m^\dagger] = \delta_{nn} \delta^3(n - m), \quad r_n|0\rangle_n = 0. \quad (A.1)$$

The vacuum state $|0\rangle_n$ is defined for a given time $\eta$ and mode $n$. In general does not exist an unique vacuum state because if for some time $\eta_i$ the vacuum state is $r_n(\eta_i)|0\rangle_n = 0$, for another time $\eta_f$ it changes to $c_n(\eta_f)|0\rangle_n \neq 0$. For an expanding background, the
vacuum state at time $\eta_i$ is different from the vacuum state at $\eta_f$. This so-called vacuum state ambiguity has some interesting consequences, the most prominent of them being the gravitationally-induced particle production (‘Grishchuk particles’). To be more precise, by assuming that there is no particles at time $\eta_i$, that is, $N_i|0_i\rangle = 0$, where $N_i$ is the number operator acting in the vacuum state $[N_i \equiv c_i^\dagger c_i$ (the polarization and wave number index were suppressed for simplicity)]. Later on, at a time $\eta_f$ we have $N_f|0_f\rangle = 0$, but the vacuum state ambiguity gives $N_f|0_i\rangle \neq 0$. Consequently the vacuum state $|0_i\rangle$ contains “$f$” particles and vice versa. This is the physical foundation for the creation of ‘Grishchuk gravitons’ induced by the cosmic expansion.

B Power and energy density spectra

With the perturbations already quantized, the important physical observables are readily calculated. The (dimensionless) power spectrum, that is, the quadratic mean value of the amplitude of the perturbations, can be defined as:

$$P(n, \eta) \equiv \frac{d\langle 0|h_{ij}(\eta, x)h^{ij}(\eta, x)|0\rangle}{d \ln n}, \quad (B.1)$$

Now, by inserting (A.1) into (3.3) one finds

$$\langle 0|h_{ij}(\eta, x)h^{ij}(\eta, x)|0\rangle = \frac{32G}{\pi} \int_0^\infty n^3|h_n(\eta)|^2 d\ln n. \quad (B.2)$$

Note that the equality holds because the modulus of the modes for different polarizations are equal, $|\tilde{h}|^2 = |\tilde{\hat{h}}|^2 = |h|^2$. Finally we have:

$$P(n, \eta) = \frac{32G}{\pi} n^3|h_n(\eta)|^2. \quad (B.3)$$

Another important quantity is the energy spectrum defined in the following manner:

$$\Omega_{gw}(n, \eta) \equiv \frac{\rho_{gw}(\eta)}{\rho_{crit}} \frac{d\langle 0|\rho_{gw}(\eta)|0\rangle}{d \ln n}, \quad (B.4)$$

which represents the gravitational wave energy density ($\rho_{gw}$) per logarithmic wave number interval, in units of the critical density $\rho_{crit}(\eta) = 3H^2(\eta)/8\pi G$. The gravitational wave density is

$$\rho_{gw} = \rho_0 = \frac{1}{64\pi G} \frac{(h_{ij}')^2 + (\nabla h_{ij})^2}{a^2}, \quad (B.5)$$

whose vacuum expectation value in the vacuum state reads:

$$\langle 0|\rho_{gw}|0\rangle = \int_0^\infty \frac{n^3 |h_{ij}'(\eta)|^2 + n^2|h_n(\eta)|^2}{2\pi^2} dn/n, \quad (B.6)$$

while the energy spectrum (B.4) becomes

$$\Omega_{gw}(n, \eta) = \frac{8\pi G}{3H^2(\eta) 2\pi^2} \frac{n^3}{2\pi^2}(|h_{ij}'(\eta)|^2 + n^2|h_n(\eta)|^2). \quad (B.7)$$

Both quantities are the most important primordial GWs observables.
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