SYMMETRIC \( q \)-DEFORMED KP HIERARCHY

KELEI TIAN\(^{\dagger} \), JINGSONG HE\(^{\ddagger,*} \) AND YUCAI SU\(^{\S} \)

\(^{\dagger} \) School of Mathematics, Hefei University of technology, Hefei 230009, China
Email: kltian@ustc.edu.cn
\(^{\ddagger} \) Department of Mathematics, Ningbo University, Ningbo, 315211, China
Email: hejingsong@nbu.edu.cn
\(^{\S} \) Department of Mathematics, Tongji University, Shanghai 200092, China
E-mail: ycsu@tongji.edu.cn

ABSTRACT. Based on the analytic property of the symmetric \( q \)-exponent \( e_q(x) \), a new symmetric \( q \)-deformed Kadomtsev-Petviashvili (\( q \)-KP) hierarchy associated with the symmetric \( q \)-derivative operator \( \partial_q \) is constructed. Furthermore, the symmetric \( q \)-CKP hierarchy and symmetric \( q \)-BKP hierarchy are defined. Here we also investigate the additional symmetries of the symmetric \( q \)-KP hierarchy.

Keywords: \( q \)-derivative, symmetric \( q \)-KP hierarchy, additional symmetries
Mathematics Subject Classification(2000): 35Q53, 37K05, 37K10
PACS(2003): 02.30.Ik

1. INTRODUCTION

The origin of \( q \)-calculus (quantum calculus) [12] traces back to the early 20th century. Many mathematicians have important works in the area of \( q \)-calculus, \( q \)-hypergeometric series and quantum group. There are two different forms of \( q \)-derivative operators, which are defined respectively by

\[
D_q(f(x)) = \frac{f(qx) - f(x)}{(q - 1)x}, \quad q \neq 1
\]

and

\[
\partial_q(f(x)) = \frac{f(qx) - f(q^{-1}x)}{(q - q^{-1})x}, \quad q \neq 1.
\]

The so-called \( q \)-deformation of the integrable system (or \( q \)-deformed integrable system) started in 1990’s by means of the first \( q \)-derivative \( D_q \) in eq. (1.1) instead of usual derivative \( \partial \) with respect to \( x \) in the classical system. As we know, the \( q \)-deformed integrable system reduces to a classical integrable system as \( q \) goes to 1. Several \( q \)-deformed integrable systems have been presented, for example, \( q \)-deformation of the KdV hierarchy [3-6], \( q \)-Toda equation [7], \( q \)-Calogero-Moser equation [8] and so on. The \( q \)-deformed Kadomtsev-Petviashvili (\( q \)-KP) hierarchy is also a subject of intensive study in the literature from [9] to [17]. Indeed, it is worth to point out that there exist two variants of the \( q \)-deformed integrable system, one belonging to E.Frenkel [3] and another to D.H.Zhang et al. [4-17].

It has been known for some time that different sub-hierarchies of the KP hierarchy can be obtained by adding different reduction conditions on Lax operator \( L \). Two important sub-hierarchies of the KP hierarchy are CKP hierarchy [18] through a restriction \( L^* = -L \) and BKP
hierarchy \cite{19} through a restriction $L^* = -\partial L \partial^{-1}$. However, to the best of our knowledge, there is no any results on the $q$-deformed CKP hierarchy and $q$-deformed BKP hierarchy so far. The difficulty to define them is the conjugate operation “$*$” of $q$-derivative $D_q$ in eq.(1.1). In fact, $D_q^* \neq -D_q$ but $D_q^* = -D_q \theta^{-1} = -\frac{1}{q} D_q$. This paper shows a quite interesting fact as $\partial^*_q = -\partial_q$, where the symmetric $q$-derivative operator $\partial_q$ is defined by eq.(1.2). In what follows, we shall fill the gap by constructing the new symmetric $q$-deformed KP hierarchy based on the symmetric $q$-derivative operator $\partial_q$.

The paper is organized as follows. Some basic results of symmetric $q$-derivative operator $\partial_q$ are given in Section 2, and one formula for the symmetric $q$-exponent $e_q(x)$ is established. Then a new symmetric $q$-KP hierarchy are stated in Sections 3 similarly to the classical KP hierarchy \cite{20}, and also symmetric $q$-CKP hierarchy and symmetric $q$-BKP hierarchy are given in this section. We further study the additional symmetries for the symmetric $q$-KP hierarchy in Section 4. Section 5 is devoted to conclusions and discussions.

2. Symmetric quantum calculus

We give some useful facts about the symmetric $q$-derivative operator $\partial_q$ in the form of eq.(1.2) based on the literature \cite{2}. We work in an associative ring of functions which includes a $q$-variable $x$ and infinite time variables $t_i \in \mathbb{R}$

$$F = f = f(x; t_1, t_2, t_3, \cdots).$$

The $q$-shift operator is defined by

$$\theta(f(x)) = f(qx). \quad (2.1)$$

Note that $\theta$ does not commute with $\partial_q$. Indeed, the relation

$$(\partial_q \theta^k(f)) = q^k \theta^k(\partial_q f), \quad k \in \mathbb{Z}$$

holds. The limit of $\partial_q(f(x))$ as $q$ approaches to 1 is the ordinary differentiation $\partial_x(f(x))$. We denote the formal inverse of $\partial_q$ as $\partial^{-1}_q$.

**Proposition 1.** The conjugate of $\partial_q$ can be defined as

$$\partial^*_q = -\partial_q.$$

**Proof.** First step is to prove $\theta^* = q^{-1} \theta^{-1}$. According to the definition, we have

$$\begin{align*}
\partial_q(fg) &= (\theta f)(\partial_q g) + (\partial_q f)(\theta^{-1} g) \\
 &= (\theta g)(\partial_q f) + (\partial_q g)(\theta^{-1} f).
\end{align*}$$

Calculating the quantum integration $\int \cdot \, d_q x$ for the above two formulas separately, it follows that

$$\int (\theta f)(\partial_q g) \, d_q x = -\int (\partial_q f)(\theta^{-1} g) \, d_q x, \quad (2.2)$$

$$\int (\theta g)(\partial_q f) \, d_q x = -\int (\partial_q g)(\theta^{-1} f) \, d_q x. \quad (2.3)$$

Let $g \rightarrow \theta^{-2} g$ in eq.(2.3), it now yields

$$\int (\theta^{-1} g)(\partial_q f) \, d_q x = -\int (\partial_q \theta^{-2} g)(\theta^{-1} f) \, d_q x.$$
Comparing it with the eq. (2.2), the above equation becomes
\[ \int (\theta f)(\partial_q g)d_q x = \int (\partial_q \theta^{-2} g)(\theta^{-1} f)d_q x. \]
It can now be written in the form
\[ < \theta f, \partial_q g > = < \theta^{-1} f, q^{-2} \partial_q g >. \]
By letting \( g \rightarrow \theta^{-2} g \) and \( f \rightarrow \theta f \) in the above equation, we find that
\[ < \theta^2 f, g > = < f, q^{-2} \partial_q g >, \]
so one can choose \( \theta^* = q^{-1} \theta^{-1} \).

We will now proceed to prove \( \partial_q^* = -\partial_q \). Let \( f \rightarrow \theta^{-1} f \) and \( g \rightarrow \theta g \) in the eq. (2.2), it now reads
\[ < \partial_q \theta^{-1} f, g > = -< f, \partial_q \theta g >. \]
This implies
\[(\partial_q \theta)^* = -\partial_q \theta^{-1}.\]
According to the equation \( \theta^* = q^{-1} \theta^{-1} \), we get
\[ \partial_q^* = -q \theta \partial_q \theta^{-1} = -\partial_q. \]

The following \( q \)-deformed Leibnitz rule holds
\[ \partial_q^n \circ f = \sum_{k \geq 0} \binom{n}{k}_q \theta^{n-k}(\partial_q^k f)\theta^{-k}\partial_q^{n-k}, \quad n \in \mathbb{Z} \] \hspace{1cm} (2.4)
where the \( q \)-number
\[ (n)_q = \frac{q^n - q^{-n}}{q - q^{-1}} \]
and the \( q \)-binomial is introduced as
\[ \binom{n}{k}_q = \frac{(n)_q(n-1)_q \cdots (n-k+1)_q}{(1)_q(2)_q \cdots (k)_q}, \quad n \in \mathbb{Z}, k \in \mathbb{Z}_+. \]

To illustrate the \( q \)-deformed Leibnitz rule, the following examples are given.
\[ \partial_q f = \theta(f)\partial_q + (\partial_q f)\theta^{-1}, \]
\[ \partial_q^2 f = (q + q^{-1})\theta(\partial_q f)\theta^{-1}\partial_q + \theta^2(f)\partial_q^2 + (\partial_q^2 f)\theta^{-2}, \]
\[ \partial_q^3 f = (q^2 + q^{-2} + 1)\theta(\partial_q^2 f)\theta^{-2}\partial_q + (q^2 + q^{-2} + 1)\theta^2(\partial_q f)\theta^{-1}\partial_q^2 + (\partial_q^3 f)\theta^{-3} + \theta^3(f)\partial_q^3, \]
\[ \partial_q^{-1} f = \theta^{-1}(f)\partial_q^{-1} - \theta^{-2}(\partial_q f)\theta^{-1}\partial_q^{-2} + \cdots + (-1)^k \theta^{-k-1}(\partial_q^k f)\theta^{-k}\partial_q^{-k-1} + \cdots. \]

Using the Taylor’s formula we can get the following proposition for the symmetric \( q \)-exponent \( e_q(x) \), which is crucial to develop the tau function of the symmetric \( q \)-KP hierarchy and to research the interaction of \( q \)-solitons in the future.
Proposition 2. The $q$-exponent $e_q(x)$ is defined as

$$e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n)_q!}, \tag{2.5}$$

where

$$(n)_q! = (n)_q(n-1)_q(n-2)_q \cdots (1)_q,$$

then the formula

$$e_q(x) = \exp(\sum_{k=1}^{\infty} c_k x^k) \tag{2.6}$$

holds, where

$$c_k = \sum_{i=1}^{k} (-1)^{i-1} \frac{1}{i} \sum_{\substack{v_1+v_2+\cdots+v_i=k\ 1}}^{\infty} \frac{1}{(v_1)_q!(v_2)_q!*\cdots(v_i)_q!}. \tag{2.7}$$

Proof. From the definition of $e_q(x)$ and Taylor’s formula, it follows that

$$e_q(x) = 1 + \sum_{n=1}^{\infty} \frac{x^n}{(n)_q!} \tag{2.5},$$

$$= \exp(\ln(1 + \sum_{n=1}^{\infty} \frac{x^n}{(n)_q!}))$$

$$= \exp(\sum_{i=1}^{\infty} (-1)^{i-1} \frac{1}{i} (\sum_{n=1}^{\infty} \frac{x^n}{(n)_q!})^i)$$

$$= \exp(\sum_{k=1}^{\infty} \sum_{i=1}^{k} (-1)^{i-1} \frac{1}{i} \sum_{\substack{v_1+v_2+\cdots+v_i=k\ 1}}^{\infty} \frac{x^k}{(v_1)_q!(v_2)_q!*\cdots(v_i)_q!})$$

$$= \exp(\sum_{k=1}^{\infty} c_k x^k), \tag{2.7}$$

where $c_k$ is given by eq. (2.7).

Several explicit forms of $q$-exponent $e_q(x)$ can be written out as follows.

$c_1 = 1,$

$c_2 = -\frac{(q-1)^2}{2(q^2+1)},$

$c_3 = \frac{(q-1)^2(q^4 - q^3 - q^2 - q + 1)}{3(q^2+1)(q^4 + q^2 + 1)},$

$c_4 = -\frac{(q-1)^4(q^4 - q^3 - 2q^2 - q + 1)}{4(q^2-q+1)(q^6 + q^4 + q^2 + 1)},$

$c_5 = \frac{(q-1)^4(q^{14} - 2q^{13} - 2q^{11} + q^{10} - 2q^9 + 5q^8 + q^7 + 5q^6 - 2q^5 + q^4 - 2q^3 - 2q - 1)}{5(q^2+1)(q^2-q+1)(q^6 + q^4 + q^2 + 1)(q^8 + q^6 + q^4 + q^2 + 1)} + \frac{5q^2+1)(q^2-q+1)(q^6 + q^4 + q^2 + 1)(q^8 + q^6 + q^4 + q^2 + 1)}{(q^2-q+1)(q^6 + q^4 + q^2 + 1)(q^8 + q^6 + q^4 + q^2 + 1)},$

$c_6 = -\frac{(q-1)^6((q^{12}+1)(q^2 - 3q + 1) + q^2(q^{10} + 1)(q + 1) - 4q^5(q^3 - 1)(q - 1) + 2q^7)}{6(q^2-q+1)(q^6 + q^4 + q^2 + 1)(q^4 - q^3 + q^2 - q + 1)(q^8 - q^7 + q^6 + q^2 - q + 1)}.
For the case \( D_q(f(x)) = \frac{f(qx) - f(x)}{(q-1)x} \) and \( (n)_q = \frac{q^{n-1}}{q-1} \), \( q \)-exponent function \( \tilde{e}_q(x) \) is defined as

\[
\tilde{e}_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n)_q!},
\]

then

\[
\tilde{e}_q(x) = \exp(\sum_{k=1}^{\infty} \tilde{c}_k x^k),
\]

where

\[
\tilde{c}_k = \frac{(1 - q)^k}{k(1 - q^k)}.
\]

Recall that the \( q \)-exponent function \( e_q(x) \) is the eigenfunction of operator \( \partial_q \), i.e.

\[
\partial_q e_q(x) = e_q(x).
\]

Furthermore, from

\[
e_q(xz) = \sum_{n=0}^{\infty} \frac{(xz)^n}{(n)_q!}
\]

one obtains immediately that the formula

\[
\partial_q^m e_q(xz) = z^m e_q(xz), m = 1, 2, 3, \ldots,
\]

which is useful to define the \( q \)-wave function of the symmetric \( q \)-KP hierarchy in the following section.

### 3. Symmetric \( q \)-deformed KP hierarchy

Similar to the classical KP hierarchy \[19,20\], we will define a new symmetric \( q \)-deformed KP hierarchy. The Lax operator \( L \) of the symmetric \( q \)-KP hierarchy is given by

\[
L = \partial_q + u_1 + u_2 \partial_q^{-1} + u_3 \partial_q^{-2} + \cdots.
\]

where \( u_i = u_i(x; t_1, t_2, t_3, \ldots), i = 1, 2, 3, \ldots \). The corresponding Lax equation of the symmetric \( q \)-KP hierarchy is defined by

\[
\frac{\partial L}{\partial t_n} = [B_n, L], \quad n = 1, 2, 3, \ldots,
\]

where the differential part \( B_n = (L^n)_+ = \sum_{i=0}^{n} b_i \partial_q^i \) and the integral part \( (L^n)_- = L^n - (L^n)_+ \).

The first few \( B_n \) and flow equations in eq.\((3.2)\) for dynamical variables \( \{u_1, u_2, u_3, \cdots\} \) can be written out as follows.

\[
B_1 = \partial_q + u_1,
\]

\[
B_2 = \partial_q^2 + v_1 \partial_q + v_0,
\]

\[
B_3 = \partial_q^3 + w_2 \partial_q^2 + w_1 \partial_q + w_0,
\]

where \( L^2 = B_2 + v_- \partial_q^{-1} + \cdots \) and

\[
v_1 = \theta(u_1) + u_1,
v_0 = (\partial_q u_1) \theta^{-1} + \theta(u_2) + u_1^2 + u_2,
v_- = (\partial_q u_2) \theta^{-1} + \theta(u_3) + u_1 u_2 + u_2 \theta^{-1}(u_1) + u_3,
\]
The first flow equations are
\[
\begin{align*}
\frac{\partial u_0}{\partial t_1} &= \theta(u_2) - u_2, \\
\frac{\partial u_1}{\partial t_1} &= (\partial_q u_2)\theta^{-1} + \theta(u_3) + u_1 u_2 - u_2 \theta^{-1}(u_1) - u_3, \\
\frac{\partial u_2}{\partial t_1} &= (\partial_q u_3)\theta^{-1} + \theta(u_4) + u_1 u_3 + u_2(\theta^{-3}(\partial_q u_1))\theta^{-1} - u_3 \theta^{-2}(u_1) - u_4, \\
\frac{\partial u_3}{\partial t_1} &= (\partial_q u_4)\theta^{-1} + \theta(u_5) + u_1 u_4 - u_2(\theta^{-2}(\partial_q u_1))\theta^{-2} - u_4 \theta^{-3}(u_1) - u_5 \\
&\quad + (2)\partial_q u_3(\theta^{-3}(\partial_q u_1))\theta^{-1}.
\end{align*}
\]

The Lax operator \( L \) in eq. (3.1) can be generated by a pseudo-difference operator \( S = 1 + \sum_{k=1}^{\infty} s_k \partial_q^{-k} \) in the following way
\[
L = S\partial_q S^{-1}.
\] (3.3)

Here \( S \) is called dressing operator or wave operator of the symmetric \( q \)-KP hierarchy.

**Proposition 3.** Dressing operator \( S \) of the symmetric \( q \)-KP hierarchy satisfies the Sato equation
\[
\frac{\partial S}{\partial t_j} = -(L^j)_- S, \quad j = 1, 2, 3, \cdots .
\] (3.4)

**Proof.** From the Lax equation \( \frac{\partial L}{\partial t_n} = [B_n, L] \), which is followed by
\[
\begin{align*}
\frac{\partial L}{\partial t_j} &= [B_j, L] = (L^j)^+ L - L(L^j)^+ \\
&\quad = (L^j - (L^j)_-) L - L(L^j - (L^j)_-) \\
&\quad = -(L^j)_- L + L(L^j)_-.
\end{align*}
\]

On the other hand,
\[
\begin{align*}
\frac{\partial L}{\partial t_j} &= \frac{\partial}{\partial t_j}(S\partial_q S^{-1}) \\
&= \frac{\partial S}{\partial t_j} \partial_q S^{-1} + S\partial_q \frac{\partial S^{-1}}{\partial t_j} \\
&= \frac{\partial S}{\partial t_j} S^{-1} \partial_q S^{-1} + S\partial_q (-S^{-1} \frac{\partial S}{\partial t_j} S^{-1}) \\
&= \frac{\partial S}{\partial t_j} S^{-1} L - L \frac{\partial S}{\partial t_j} S^{-1},
\end{align*}
\]
then
\[
\frac{\partial L}{\partial t_j} = -(L_j^-)_- L + L(L_j^-) = \frac{\partial S}{\partial t_j} S^{-1} L - L \frac{\partial S}{\partial t_j} S^{-1}.
\]

The above equation implies that
\[
\frac{\partial S}{\partial t_j} S^{-1} = -(L_j^-)_-, \quad j = 1, 2, 3, \ldots,
\]
which ends the proof. \(\square\)

**Definition 1.** The \(q\)-wave function \(w_q(x, t; z)\) for the symmetric \(q\)-KP hierarchy eq. (3.2) with the wave operator \(S\) in eq. (3.3) is given by
\[
w_q(x, t; z) = Se_q(xz) \exp(\sum_{i=1}^{\infty} t_i z^i),
\]
where \(t = (t_1, t_2, t_3, \ldots)\).

**Proposition 4.** The \(q\)-wave function \(w_q(x, t; z)\) of the symmetric \(q\)-KP hierarchy satisfies the following linear \(q\)-differential equations
\[
Lw_q = zw_q, \quad \partial_m w_q = (L^m)_+ w_q,
\]
where \(\partial_m = \frac{\partial}{\partial t_m}\).

**Proof.** Using the equation \(\partial_q e_q(xz) = z e_q(xz)\), then
\[
Lw_q = S \partial_q S^{-1} S e_q(xz) \exp(\sum_{i=1}^{\infty} t_i z^i)
\]
\[
= S \partial_q e_q(xz) \exp(\sum_{i=1}^{\infty} t_i z^i)
\]
\[
= zw_q.
\]

From the Sato equation \(\partial_m S = -(L^m)_- S\), it follows that
\[
\partial_m w_q = \partial_m (Se_q(xz) \exp(\sum_{i=1}^{\infty} t_i z^i))
\]
\[
= (\partial_m S)e_q(xz) \exp(\sum_{i=1}^{\infty} t_i z^i) + Se_q(xz) \exp(\sum_{i=1}^{\infty} t_i z^i)z^m
\]
\[
= -(L^m)_- Se_q(xz) \exp(\sum_{i=1}^{\infty} t_i z^i) + S \partial_q^m e_q(xz) \exp(\sum_{i=1}^{\infty} t_i z^i)
\]
\[
= -(L^m)_- w_q + (L^m)_+ w_q
\]
\[
= (L^m)_+ w_q.
\]

\(\square\)

Furthermore, we would like to give the definitions of the symmetric \(q\)-CKP hierarchy and the symmetric \(q\)-BKP hierarchy respectively to answer the previous question mentioned in the introduction.
Definition 2. Let the operator $L$ in eq.(3.1) be the Lax operator for the symmetric $q$-KP hierarchy associated with eq.(3.2), if $L$ satisfies the reduction condition $L^* = -L$, then we call it the symmetric $q$-CKP hierarchy.

Definition 3. Let the operator $L$ in eq.(3.1) be the Lax operator for the symmetric $q$-KP hierarchy associated with eq.(3.2), if $L$ satisfies the reduction condition $L^* = -\theta^{-\frac{1}{2}} \partial_q L \partial_q^{-1} \theta^\frac{1}{2}$, then it is the symmetric $q$-BKP hierarchy.

4. ADDITIONAL SYMMETRY OF THE SYMMETRIC $q$-KP HIERARCHY

The another main goal of this note is to consider the additional symmetries of the symmetric $q$-KP hierarchy. First, let us define $\Gamma_q$ and Orlov-Shulman’s $M$ operator as

$$\Gamma_q = \sum_{i=1}^{\infty} \left( it_i + ic_i x^i \right) \partial_q^{i-1},$$

$$M = S \Gamma_q S^{-1},$$

where $c_i$ is given by eq.(2.7). Then the additional flows of the symmetric $q$-KP hierarchy for each pair $\{m, n\}$ are defined by

$$\frac{\partial S}{\partial t_{m,n}^*} = -(M^m L^n)_-, \quad (4.1)$$

Proposition 5. The additional flows act on $L$ and $M$ of the symmetric $q$-KP hierarchy as

$$\frac{\partial L}{\partial t_{m,n}^*} = -[(M^m L^n)_-, L], \quad (4.2)$$

$$\frac{\partial M}{\partial t_{m,n}^*} = -[(M^m L^n)_-, M]. \quad (4.3)$$

Proof. By performing the derivative $\frac{\partial}{\partial t_{m,n}^*}$ on $L = S \partial_q S^{-1}$ and using the eq.(4.1), we observe that

$$\frac{\partial L}{\partial t_{m,n}^*} = \frac{\partial S}{\partial t_{m,n}^*} \partial_q S^{-1} + S \partial_q \frac{\partial S^{-1}}{\partial t_{m,n}^*}$$

$$= -(M^m L^n)_- S \partial_q S^{-1} + S \partial_q(-S^{-1} \frac{\partial S}{\partial t_{m,n}^*} S^{-1})$$

$$= -(M^m L^n)_- L + S \partial_q S^{-1} (M^m L^n)_-$$

$$= -[(M^m L^n)_-, L].$$

For the action on $M = S \Gamma_q S^{-1}$, there exists similar derivation as $\frac{\partial L}{\partial t_{m,n}^*}$, and then

$$\frac{\partial M}{\partial t_{m,n}^*} = \frac{\partial S}{\partial t_{m,n}^*} \Gamma_q S^{-1} + S \Gamma_q \frac{\partial S^{-1}}{\partial t_{m,n}^*}$$

$$= -(M^m L^n)_- S \Gamma_q S^{-1} + S \Gamma_q(-S^{-1} \frac{\partial S}{\partial t_{m,n}^*} S^{-1})$$

$$= -(M^m L^n)_- M + S \Gamma_q S^{-1} (M^m L^n)_-.$$
In the above calculation, the fact that $\Gamma_q$ does not depend on the additional flows variables $t_{m,n}^*$ has been used. □

Corollary 1.

$$\frac{\partial L^k}{\partial t_{m,n}^*} = -[\{M^m L^n\}_{-}, L^k], \tag{4.4}$$
$$\frac{\partial M^k}{\partial t_{m,n}^*} = -[\{M^m L^n\}_{-}, M^k], \tag{4.5}$$
$$\frac{\partial M^k L^l}{\partial t_{m,n}^*} = -[\{M^m L^n\}_{-}, M^k L^l], \tag{4.6}$$
$$\frac{\partial M^k L^l}{\partial t_n} = [B_n, M^k L^l]. \tag{4.7}$$

Proof. We present only the proof of the first equation here. The others can be proved in a similar way.

$$\frac{\partial L^k}{\partial t_{m,n}^*} = \frac{\partial L}{\partial t_{m,n}^*}L^{k-1} + L\frac{\partial L}{\partial t_{m,n}^*}L^{k-2} + \cdots + L^{k-2}\frac{\partial L}{\partial t_{m,n}^*}L + L^{k-1}\frac{\partial L}{\partial t_{m,n}^*}$$
$$= \sum_{l=1}^{k} L^{l-1}\frac{\partial L}{\partial t_{m,n}^*}L^{k-l}$$
$$= \sum_{l=1}^{k} L^{l-1}(-[\{M^m L^n\}_{-}, L])L^{k-l}$$
$$= -[\{M^m L^n\}_{-}, L^k],$$

where we have used the formula $\frac{\partial L}{\partial t_{m,n}^*} = -[\{M^m L^n\}_{-}, L]$ in the Proposition 5. □

Proposition 6. The additional flows $\partial_{m,n}^* = \frac{\partial}{\partial t_{m,n}^*}$ commute with the hierarchy $\partial_k = \frac{\partial}{\partial t_k}$, i.e.

$$[\partial_{m,n}^*, \partial_k] = 0,$$

thus we call them additional symmetries of the symmetric $q$-KP hierarchy.

Proof. According to the definition and the Corollary 1, it equals to

$$[\partial_{m,n}^*, \partial_k]S = \partial_{m,n}^*(\partial_k S) - \partial_k(\partial_{m,n}^* S)$$
$$= \partial_{m,n}^*(-(L^k)_{-}S) - \partial_k(-(M^m L^n)_{-}S)$$
$$= -(\partial_{m,n}^* L^k)_{-}S - (L^k)_{-}(\partial_{m,n}^* S) + (\partial_k M^m L^n)_{-}S + (M^m L^n)_{-}(\partial_k S)$$
$$= [(M^m L^n)_{-}, L^k]_{-}S + (L^k)(M^m L^n)_{-}S + [(L^k)_{+}, M^m L^n]_{-}S - (M^m L^n)_{-}(L^k)_{-}S$$
$$= [(M^m L^n)_{-}, (L^k)_-]_{-}S + [(L^k)_+, (M^m L^n)_{-}]S - [(M^m L^n)_{-}, (L^k)_{+}]S + [(L^k)_{-}, (M^m L^n)_{-}]S$$
$$= 0.$$
\[ ([L^k]_+, (M^m L^n) ]_\pm = [(L^k)_+, (M^m L^n) ]_\pm \text{ and } [(M^m L^n)_-, (L^k)_-] = [(M^m L^n)_-, (L^k)_-] \] have been used in the above derivation. \[\square\]

5. Conclusions and discussions

To summarize, we have derived the antisymmetric property of \(\partial_q\) in Proposition 1 and a crucial expression of \(e_q(x)\) by usual exponential in Proposition 2. The analytic property of symmetric \(e_q(x)\) in Proposition 2 is used to define the wave function of the symmetric \(q\)-KP hierarchy. After introducing the dressing operator and the \(q\)-wave function of the symmetric \(q\)-KP hierarchy in Section 3, we also give the definitions of symmetric \(q\)-CKP hierarchy and symmetric \(q\)-BKP hierarchy. The additional symmetries of the symmetric \(q\)-KP hierarchy are obtained in Section 4. The above results of this paper show obviously that the symmetric \(q\)-KP hierarchy is different with the \(q\)-KP hierarchy \[8–17\] based on the \(D_q(f(x))\).

In comparison with the known interesting results of the KP hierarchy \[18–20\] and the \(q\)-KP hierarchy based on the \(D_q(f(x))\) \[8–17\], the symmetric \(q\)-KP hierarchy defined in this paper deserves further study from several aspects including the tau function and its Hirota bilinear identity, the Hamiltonian structure, the gauge transformation, the symmetry analysis and the interaction of \(q\)-solitons. Furthermore, it is highly nontrivial to consider above topics of the symmetric \(q\)-CKP(or \(q\)-BKP) hierarchy because of the reduction condition \(L^* = -\partial_q L / \partial_q^{-1}\) and the complexity of the \(\partial_q\).

Acknowledgments This work was supported by Erasmus Mundus Action 2 EXPERTS, SMSTC grant no. 12XD1405000, Fundamental Research Funds for the Central Universities, and NSF grant no. 11271210, 11201451, 10825101 of China.

References

[1] Klimyk, A. and Schmüdgen, K., Quantum groups and their representations, Springer, Berlin, 1997.
[2] Kac, V. and Cheung, P., Quantum calculus, Springer-Verlag, New York, 2002.
[3] Frenkel, E., Deformations of the KdV hierarchy and related soliton equations, Int. Math. Res. Not., 2, 1996, 55–76.
[4] Zhang, D. H., Quantum deformation of KdV hierarchies and their infinitely many conservation laws, J. Phys. A., 26, 1993, 2389–2407.
[5] Wu, Z. Y., Zhang, D. H. and Zheng, Q. R., Quantum deformation of KdV hierarchies and their exact solutions: \(q\)-deformed solitons, J. Phys. A., 27, 1994, 5307–5312.
[6] Khesin, B., Lyubashenko, V. and Roger, C., Extensions and contractions of the Lie algebra of \(q\)-pseudodifferential symbols on the circle, J. Funct. Anal., 143, 1997, 55–97.
[7] Tsuboi, Z. and Kuniba, A., Solutions of a discretized Toda field equation for \(D_r\) from analytic Bethe ansatz, J. Phys. A., 29, 1996, 7785–7796.
[8] Iliev, P., \(q\)-KP hierarchy, bispectrality and Calogero-Moser systems, J. Geom. Phys., 35, 2000, 157–182.
[9] Mas, J. and Seco, M., The algebra of \(q\)-pseudodifferential symbols and the \(q\)-W\(_{\text{KP}}^{(n)}\) algebra, J. Math. Phys., 37, 1996, 6510–6529.
[10] Iliev, P., Solutions to Frenkel’s deformation of the KP hierarchy, J. Phys. A., 31, 1998, 241–244.
[11] Iliev, P., Tau function solutions to a \(q\)-deformation of the KP hierarchy, Lett. Math. Phys., 44, 1998, 187–200.
[12] Tu, M. H., \(q\)-deformed KP hierarchy: its additional symmetries and infinitesimal Bäcklund transformations, Lett. Math. Phys., 49, 1999, 95–103.
[13] He, J. S., Li, Y. H. and Cheng, Y., $q$-deformed KP hierarchy and its constrained sub-hierarchy. *SIGMA*, **2**, 2006, 060(33pages).

[14] Tian, K. L., He, J. S., Su, Y. C. and Cheng, Y., String equations of the $q$-KP hierarchy, *Chin. Ann. Math.*, **32B**(6), 2011, 895–904.

[15] Tian, K. L., He, J. S. and Cheng, Y., Virasoro and W-constraints for the $q$-KP hierarchy, *AIP Conf. Proc.*, **1212**, 2010, 35–42.

[16] Lin, R. L., Liu, X. J. and Zeng, Y.B., A new extended $q$-deformed KP hierarchy, *J. Nonl. Math. Phys.*, **15**, 2008, 333–347.

[17] Lin, R.L., Peng, H. and Manas, M., The $q$-deformed mKP hierarchy with self-consistent sources, Wronskian solutions and solitons, *J. Phys. A.*, **43**, 2010, 434022.

[18] Date, E., Kashiwara, M., Jimbo, M. and Miwa, T., KP hierarchy of Orthogonal symplectic type—transformation groups for soliton equations VI, *J. Phys. Soc. Jap.*, **50**, 1981, 3813–3818.

[19] Date, E., Kashiwara, M., Jimbo, M. and Miwa, T., Transformation groups for soliton equations, Nonlinear integrable systems-classical and quantum theory(edited by M. Jimbo and T. Miwa), World Scientific, Singapore, 1983, 39-119.

[20] Dickey, L. A., Soliton Equations and Hamiltonian Systems, 2nd Edition, World Scientific, Singapore, 2003.