Economies-of-scale in resource sharing systems: tutorial and partial review of the QED heavy-traffic regime

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Abstract

Resource sharing systems describe situations in which users compete for service from scarce resources. Examples include check-in lines at airports, waiting rooms in hospitals or queues in contact centers, data buffers in wireless networks, and delayed service in cloud data centers. These are all situations with jobs (clients, patients, tasks) and servers (agents, beds, processors) that have large capacity levels, ranging from the order of tens (checkouts) to thousands (processors). This survey investigates how to design such systems to exploit resource pooling and economies-of-scale. In particular, we review the mathematics behind the Quality-and-Efficiency Driven (QED) regime, which lets the system operate close to full utilization, while the number of servers grows simultaneously large and delays remain manageable. We also discuss emerging research directions related to load balancing, overdispersion and model uncertainty.

1 Introduction

Resource sharing systems describe situations in which users compete for service from scarce resources. Classical examples of such systems include call centers [37,119,147,47,20,26,156,16,88] and health care delivery [5,53,154,54], but large-scale systems are particularly present in communication systems [90,2,86,91,141,138]. In all settings, one can think of such resource sharing systems as being composed of jobs and servers. In call centers, jobs are customers’ requests for help from one of the agents (servers). In communication networks, the data packets are the jobs and the communication channels are the servers. The system scale may refer to the size of the client base it caters to, or the magnitude of its capacity, or both.

Next to the central notions of jobs and servers, most resource sharing systems are subject to uncertainty and hence give rise to stochastic systems. Although arrival volumes over a certain planning horizon can be anticipated to some extent, for instance through historical data and forecasting methods, it is challenging to predict with certainty future arrival patterns. Moreover, job sizes are typically random as well, adding more uncertainty. This intrinsic stochastic variability is a predominant cause of delay experienced by jobs in the system, which is why stochastic models have proved instrumental in both quantifying and improving the operational performance of service systems. Queueing theory provides the mathematical tools to analyze such stochastic models, and to evaluate and improve system performance. Queueing theory can also serve to reveal capacity-sizing rules that prescribe how to scale resource sharing systems, in terms of matching capacity with demand, to meet certain performance targets. Often a trade-off exists between high system utilization and short delays. In order to achieve this dual goal, the system should be scaled in a specific way that allows to use the full potential of resource pooling.
Let us first demonstrate the effects of resource pooling for the most basic multi-server queueing model, the $M/M/s$ queue. This model assumes that jobs arrive according to a Poisson process, that their service times form an i.i.d. sequence of exponential random variables, and that jobs are processed in order of arrival by one of the $s$ parallel servers. Delayed jobs are temporarily stored in an infinite-sized buffer. The three parameters that characterize this model are: the arrival rate $\lambda$, the mean processing time $1/\mu$ and the number of servers $s$. We denote the number of jobs in the system at time $t$ by $Q(t)$. The process $(Q(t))_{t \geq 0}$ is a continuous-time Markov chain with state space $\{0, 1, 2, \ldots\}$. The birth rate $\lambda$ is constant and the death rate is $\mu \cdot \min\{k, s\}$ when there are $k$ jobs in the system. Without loss of generality we shall henceforth set $\mu = 1$.

To illustrate the operational benefits of sharing resources, we compare a system of $s$ separate $M/M/1$ queues, each serving a Poisson arrival stream with rate $\lambda < 1$, against one $M/M/s$ queue with arrival rate $\lambda s$. The two systems thus face the same workload $\lambda$ per server. We now fix the value of $\lambda$ and vary $s$. Obviously, the delay and queue length distribution in the first scenario with parallel servers are unaffected by the parameter $s$, since there is no interaction between the single-server queues. This lack of coordination tolerates an event of having an idle server, while the total number of jobs in the system exceeds $s$, therefore wasting resource capacity. Such an event cannot happen in the many-server scenario, due to the central queue. This central coordination improves the Quality-of-Service (QoS). Indeed Figure 1 shows that the reduction in mean delay and delay probability can be substantial.

The Quality-and-Efficiency driven (QED) regime is a form of resource pooling that goes beyond the typical objective of improving performance by joining forces. For the $M/M/s$ queue, the QED regime is best explained in terms of the square-root rule

$$s = \lambda + \beta \sqrt{\lambda}, \quad \beta > 0,$$

which prescribes how to size capacity as a function of the offered load. Notice that the number of servers $s$ is taken equal to the sum of the mean load $\lambda$ and an additional term $\beta \sqrt{\lambda}$ that is of the same order as the natural load fluctuations of the arrival process (so of the order $\sqrt{\lambda}$). Observe that capacity increases with $\beta$, where we note that the free parameter $\beta$ can take any positive value. The QED regime assumes the coupling between $\lambda$ and $s$ as in (1.1) and then lets both $s$ and $\lambda$ become large. This not only increases the scale of operation, but also lets the load per server $\rho = \lambda/s \sim 1 - \beta/\sqrt{\lambda}$ approach 1 as $s$ (and $\lambda$) become(s) large. Now instead of diving immediately into the mathematical details, we shall first demonstrate the QED regime, or the capacity-sizing
Figure 2: Sample paths of the $M/M/s$ queue with $\lambda = 10, 50$ and 100 and $s$ set according to the three scaling rules in (1.2) with $\beta = 0.5$.

We next show similar sample paths for increasing values of $\lambda$. Since $s > \lambda$ is required for stability, the value of $s$ needs to be adjusted accordingly. We show three scaling rules

$$s^{(1)}_\lambda = \lceil \lambda + \beta \rceil, \quad s^{(2)}_\lambda = \lceil \lambda + \beta \sqrt{\lambda} \rceil, \quad s^{(3)}_\lambda = \lceil \lambda + \beta \lambda \rceil,$$

(1.2)

with $\beta > 0$, where $\lceil \cdot \rceil$ denotes the rounding operator. Note that these three rules differ in terms of overcapacity $s_{\lambda} - \lambda$, and $s_{\lambda}^{(2)}$ is the (rounded) square-root rule introduced in (1.1). Figure 2 depicts typical sample paths of the queue length process for increasing values of $\lambda$ for the three
scaling rules with $\beta = 0.5$. Observe that for all scaling rules, the stochastic fluctuations of the queue length processes relative to $s$ decrease with the system size. Moreover, the paths in Figure 2 appear to become smoother with increasing $\lambda$. Of course, the actual sample path always consists of upward and downward jumps of size 1, but we will show how proper centering and scaling of the queue length process indeed gives rise to a diffusion process in the limit as $\lambda \to \infty$ (Section 2). Although the difference in performance of the three regimes is not yet evident for relatively small $\lambda$, clear distinctive behavior occurs for large $\lambda$.

With $s(1)$, most jobs are delayed and server idle time is low, since $\rho = (1 + \beta / \lambda)^{-1} \to 1$ as $\lambda \to \infty$. Systems scaled according to this rule value server efficiency over QoS and therefore this regime is in the literature also known as the Efficiency-Driven (ED) regime [156]. In contrast, the third scaling rule $s(3)$ yields a constant utilization level $\rho = 1 / (1 + \beta)$, which stays away from 1, even for large $\lambda$. Queues operating in this regime exhibit significant server idle times. Moreover, for the particular realization of the queueing processes for $\lambda = 50$ and $\lambda = 100$, none of the jobs is delayed. This is known as the Quality-Driven (QD) regime [156]. The scaling rule $s(2)$ is in some ways a combination of the other two regimes. First, we have $\rho = (1 + \beta / \sqrt{\lambda})^{-1} \to 1$ as $\lambda \to \infty$, which indicates efficient usage of resources as the system grows. The sample paths, however, indicate that only a fraction of all jobs is delayed, and only small queues arise, indications of good QoS. Figure 2 provides visual confirmation that the square-root rule $s(2)$, related to the QED regime, strikes the right balance between the two profound objectives of capacity allocation in resource sharing systems: negligible delay and idling. We shall latter discuss the mathematical foundations of the QED regime and quantify the favorable properties revealed by Figure 2, including the non-degeneracy of the delay probability. To quote Halfin & Whitt [60]: “The balance between service and economy usually dictates that the probability of delay be kept away from both zero and one, so that the number of jobs present fluctuates between the regions above and below the number of servers.” This property will be one of the universal properties that come with the QED regime. So not only the specific $M/M/s$ queue, but a wide range of stochastic models will possess the same qualitative property in the QED regime, a phenomenon referred to as universality.

Universality plays a central role in research within probability theory, stochastic networks and mathematical physics. We say complex systems belong to the same universality class when they demonstrate the same universal statistical behavior in the long-time/large-scale limit. This survey considers the QED universality class and explains how some tractable member models reveal the key QED properties (like the delay probability being strictly between zero and one). We now first discuss briefly Gaussian universality, not only because it is the most central universality class in science, but also because of the intimate connection with QED universality.

Like most results on universality, the specific properties that come with the Gaussian universality class were first discovered through exact analysis of one particular member model, and only later conjectured and proved to hold for a much larger class of models. In this case the particular model was coin tossing, studied by among others De Moivre, Gauss and Laplace in the 18th and 19th century. A fair coin flipped $n$ times gives rise to the random variable $H$ counting the number of heads with

$$P(H = k) = 2^{-n} \binom{n}{k}. \tag{1.3}$$

Clearly, $E[H] = n/2$ and the law of large numbers says that $H/n$ converges to $1/2$ when $n \to \infty$. But the deviations from these expected results are more intriguing. As it turns out,

$$\lim_{n \to \infty} P \left( H < \frac{1}{2} n + \frac{1}{2} x \sqrt{n} \right) = \Phi(x), \tag{1.4}$$
where \( \Phi \) denotes the cumulative distribution function (cdf) of the standard normal distribution.

This is essentially a statement about the natural deviations of the binomial distribution, and the proof of (1.4) follows from applying Stirling’s formula \( n! \sim n^{n+1}e^{-n}\sqrt{2\pi/n} \) to the binomial coefficient in (1.3). Here, we mean by \( a(n) \sim b(n) \) that \( \lim_{n \to \infty} a(n)/b(n) = 1 \). Rather than giving the precise details of this derivation, we now briefly sketch how Stirling’s formula can be derived and apply this formula to obtain a statement similar to (1.4), but then for the Poisson distribution. First, express \( n! \) in terms of the Gamma function:

\[
n! = \Gamma(n+1) = \int_0^\infty z^n e^{-z} \, dz = \int_0^\infty e^{n \ln z - z} \, dz. \tag{1.5}
\]

Changing variables \( z = ny \) gives

\[
n! = e^{n \ln n} \int_0^\infty e^{n(\ln y - y)} \, dy. \tag{1.6}
\]

With Laplace’s method, see e.g. [34], we have

\[
\int_0^\infty e^{n(\ln y - y)} \, dy \sim \frac{\sqrt{2\pi/n}}{e^{-n}} \frac{e^{-x^2/(2\lambda)}}{\sqrt{2\pi \lambda}}, \tag{1.8}
\]

a Gaussian distribution with mean and variance \( \lambda \), where we have used that

\[
(1 + \delta)^{-\lambda(1+\delta)} = \exp\{-\lambda(1+\delta)\ln(1+\delta)\} \approx \exp\{-\lambda(1+\delta)(\delta - \frac{1}{2}\delta^2)\} = \exp\{-\lambda\delta - \frac{1}{2}\delta^2\}. \tag{1.9}
\]

Notice that for \( x = s_\lambda \) and \( s_\lambda = \lambda + \beta \sqrt{\lambda} \) as in (1.1), the approximation (1.8) becomes

\[
\sqrt{s_\lambda} p(s_\lambda) \approx \varphi(\beta), \tag{1.10}
\]

with \( \varphi \) the probability density function of the standard normal distribution.

The above derivation is somewhat elaborate, and a more elegant proof follows from the Central Limit Theorem stated below. However, the route of writing exact formulas for probabilities in terms of integrals, and then performing asymptotic analysis of integrals, will be one of the main approaches in this survey.

So we have seen that both the binomial distribution and the Poisson distribution give rise to Gaussian approximations in the large-mean limit. The universality of the Gaussian distribution only emerged around 1900 (with again involvement of great mathematicians including Markov, Lyapunov, Lévy, Lindeberg and Kolmogorov) and resulted in the Central Limit Theorem (CLT).
Theorem 1.1 (Central Limit Theorem) Let $X_1, X_2, \ldots$ be i.i.d. random variables of finite mean $m$ and variance $v$. Then, for all $x \in \mathbb{R}$,

$$
\lim_{n \to \infty} \mathbb{P}\left( \sum_{i=1}^{n} X_i < mn + vx \sqrt{n} \right) = \Phi(x).
$$

(1.11)

Proofs of this result and its many extensions, for this or even higher level of generality, are typically not based on exact expressions and precise calculations as in (1.4) and (1.8).

Let us illustrate how the CLT provided a simpler explanation of the convergence results for the two examples above. In the coin tossing model, we can regard the number of heads $H$ as the sum of $n$ Bernoulli random variables with mean $m = \frac{1}{2}$ and variance $v = \frac{1}{4}$. The CLT thus immediately implies (1.4). Similarly, a Pois($\lambda$) random variable, with $\lambda$ integer-valued, is equal in distribution to the sum of $\lambda$ independent Poisson random variables with unit mean and variance, i.e.

$$\text{Pois}(\lambda) \overset{d}{=} \sum_{i=1}^{\lambda} \text{Pois}(1).$$

Direct application of the CLT hence implies that

$$\mathbb{P}(\text{Pois}(\lambda) \leq \lambda + x \sqrt{\lambda}) \to \Phi(x), \quad \text{as } \lambda \to \infty.
$$

(1.12)

Some members of a universality class, in this case the binomial and Poisson distribution, possess favorable algebraic structure and admit exact asymptotic analysis, revealing the universal properties of the class at hand. Other members might remain largely intractable, but can be shown to be part of the same universality class using different, more implicit methods.

In this survey, we explain QED universality in a similar manner as Gaussian universality explained above. In Section 2, we reveal universal properties by exact analysis of basic models, and later we discuss more formal models that belong to the same or related universality classes.

Related surveys & organization. In this survey, we further review the analysis of large-scale resource sharing systems operating in the QED regime, with special focus on universal QED behavior under various modeling assumptions. In recent years, many comprehensible surveys have appeared in the literature on topics related to resource sharing systems and their asymptotic analysis. We take the opportunity to mention a couple of them here. Insightful tutorial papers have been written to aid the understanding of specific applications. Telephone call centers are the main focus of survey papers by Gans et al. [47], Brown et al. [26] and Aksin et al. [1]. Armony et al. [5] provide an extensive overview of queueing phenomenon in the health care environments. On the theoretical side, Pang et al. [120] discuss mathematical techniques to prove stochastic-process limits for several queueing settings. Ward [145] gives an excellent review of queueing systems with abandonments in asymptotic regimes, with special focus on derivation of simple performance measure approximations. The survey paper by Dai and He [32] also targets resource sharing systems with abandonments, particularly focusing on the ED and QED regimes.

The remainder of this survey is organized as follows. Section 2 introduces two classical queueing models that serve as a vehicle to convey the ideas behind the QED regime. We discuss in Section 3 key properties that are common to these models under QED scaling, and illustrate how these features stretch beyond these specific model settings. In Section 4, we explain how asymptotic QED
approximations of performance measures can be transformed into easy-to-use and robust capacity allocation principles. Furthermore, we illustrate how to adapt capacity allocation decisions to time-varying demand. Even though QED stochastic-process limits provide good first-order insight into the performance of large-scale systems, care needs to be taken with regard to the finite-ness of the system. Therefore, we review in Section 5 results that attempt to quantify the error made by asymptotic approximations, leading to both refinements and approximation bounds. We also consider the implication of approximation errors for capacity allocation decisions (so-called optimality gaps). In Section 6 we review some other models and variations that have received much attention due to their applicability in real-world resource sharing systems. Finally, we use Section 7 to discuss recent advances in the study of large-scale queueing systems under QED scaling and highlight interesting open problems.

We conclude this section by introducing some notation that will be used throughout the paper. By \( N(\mu, \sigma^2) \) we denote a normally distributed random variable with mean \( \mu \) and variance \( \sigma^2 \). The probability density function (pdf) and cumulative distribution function (cdf) of the standard normal distribution are denoted by \( \varphi \) and \( \Phi \), respectively. By \( d \approx \), we indicate that two random variables are equal in distribution and \( d \Rightarrow \) we mean convergence in distribution. As indicated before, the relation \( u(\lambda) \sim v(\lambda) \) implies that \( \lim_{\lambda \to \infty} u(\lambda)/v(\lambda) = 1 \). Last, by the relation \( u(\lambda) = O(v(\lambda)) \) we mean that \( \limsup_{\lambda \to \infty} u(\lambda)/v(\lambda) < \infty \), and \( u(\lambda) = o(v(\lambda)) \) implies that \( \limsup_{\lambda \to \infty} u(\lambda)/v(\lambda) = 0 \).

2 Example models

This survey uses two running examples that are illustrative for both the model-specific and universal features of the QED regime. The first example is the already introduced \( M/M/s \) queue, a fully Markovian many-server system. The second example is the so-called bulk-service queue, a standard discrete-time model. Through these models, we shall describe in this section several easy ways of establishing QED limits that only require a standard application of the CLT.

2.1 Many exponential servers

Let us first consider an infinite-server system to which jobs arrive according to a Poisson process with rate \( \lambda \). Each job requires an exponentially distributed service time with unit mean. The steady-state number of jobs presents (or equivalently the steady-state number of busy servers) follows a Poisson distribution with mean \( \lambda \). It is known that a Poisson distribution can be well approximated by a normal distribution for sufficiently large \( \lambda \), so that it is approximately normally distributed with mean and variance \( \lambda \). Therefore, the coefficient of variation (standard deviation divided by the mean) decreases as \( 1/\sqrt{\lambda} \), which makes the steady-state queue length become more concentrated around its mean with increasing \( \lambda \).

If we now pretend, for a moment, that this infinite-server system serves as a good approximation for the \( M/M/s \) queue, we could approximate the steady-state delay probability \( \mathbb{P}(\text{delay}) \) in the \( M/M/s \) queue as

\[
\mathbb{P}(\text{delay}) \approx \mathbb{P}(Q \geq s) = \mathbb{P} \left( \frac{Q - \lambda}{\sqrt{\lambda}} \geq \frac{s - \lambda}{\sqrt{\lambda}} \right) \approx 1 - \Phi \left( \frac{s - \lambda}{\sqrt{\lambda}} \right) = 1 - \Phi(\beta). \tag{2.1}
\]

The use of this normal approximation in support of capacity allocation decisions was explored by Kolesar & Green [93]. Of course, the infinite-server system ignores the one thing that makes a
queueing system unique: that a queue is formed when all servers are busy. During these periods of congestion, a system with a finite number of servers will operate at a slower pace than its infinite-server counterpart, so the approximation in (2.1) is likely to underestimate \( P(\text{delay}) \). Nevertheless, the infinite-server heuristic does suggest that, in large systems, the number of servers can be chosen close to the offered load as in (1.1).

We shall now make more precise statements about QED limits, and use the intimate relation between the \( M/M/s/s \) queue (Erlang loss model) and the \( M/M/s \) queue (Erlang delay model).

When \( \rho = \lambda/s < 1 \) the steady-state distribution of the \( M/M/s \) queue exists and is given by

\[
\pi_k = \lim_{t \to \infty} \mathbb{P}(Q(t) = k) = \begin{cases} \frac{\lambda^k}{k!}, & \text{if } k \leq s, \\ \frac{\rho \lambda^k}{1 - \rho s!}, & \text{if } k > s, \end{cases}
\]  

where

\[
\pi_0 = \left( \sum_{k=0}^{s} \frac{\lambda^k}{k!} + \frac{\rho \lambda^s}{1 - \rho s!} \right)^{-1}.
\]

From Little's law and the PASTA (Poisson Arrivals See Time Averages) property \([153]\), it follows that the delay probability, so the probability that an arbitrary job needs to wait before taken into service, is given by the Erlang C formula

\[
C(s, \lambda) = \frac{\lambda^s}{s!} \frac{1}{(1 - \rho) \sum_{k=0}^{s-1} \frac{\lambda^k}{k!} + \frac{\lambda^s}{s!} + \rho \lambda^s}. 
\]  

The mean steady-state delay is given by

\[
\mathbb{E}[\text{delay}] = \frac{C(s, \lambda)}{(1 - \rho)s}. 
\]  

A closely related performance measure is the probability of blocking in the \( M/M/s/s \) queue, also known as the Erlang loss formula, and is given by

\[
B(s, \lambda) = \frac{\lambda^s}{s!} \frac{\mathbb{P}(\text{Pois}(\lambda) = s)}{\mathbb{P}(\text{Pois}(\lambda) < s)}. 
\]  

where the latter probabilistic representation, with \( \text{Pois}(\lambda) \) denoting a Poisson random variable with mean \( \lambda \), is convenient in light of the CLT. Note also that the Erlang B and C formulae are related by

\[
C(s, \lambda) = \left( \rho + \frac{1 - \rho}{B(s, \lambda)} \right)^{-1}.
\]

See \([148]\) for an extensive overview of properties of the Erlang B and C formulae; see also \([68, 75]\).

We now focus on how these formulae scale when \( \lambda \) and \( s \) both grow large.

Halfin & Whitt \([60]\) showed that, just as the tail probability in the infinite-server setting (2.1), the delay probability in the \( M/M/s \) queue converges under scaling (1.1) to a value between 0 and 1. Moreover, they showed that this is in fact the only scaling regime in which such a non-degenerate limit exists and identified its value.

Let \( \rho_\lambda := \lambda/s_\lambda \) denote the server utilization if capacity \( s_\lambda \) is scaled according to (1.1). The following result is obtained in \([60]\).
Figure 3: The delay probability \( C(s_{\lambda}, \lambda) \) with \( s_{\lambda} = [\lambda + \beta \sqrt{\lambda}] \) for \( \beta = 0.1, 0.5, \) and \( 1 \) as a function of \( \lambda \). The limiting values \( g(\beta) \) are plotted as dashed lines.

**Proposition 2.1** There is non-degenerate limit

\[
\lim_{\lambda \to \infty} C(s_{\lambda}, \lambda) = \left(1 + \frac{\beta \Phi(\beta)}{\varphi(\beta)} \right)^{-1} =: g(\beta) \in (0, 1) \tag{2.7}
\]

if and only if

\[
\lim_{\lambda \to \infty} (1 - \rho_{\lambda}) \sqrt{s_{\lambda}} \to \beta, \quad \beta > 0. \tag{2.8}
\]

In this case

\[
\lim_{\lambda \to \infty} \sqrt{\lambda} B(s_{\lambda}, \lambda) = \frac{\varphi(\beta)}{\Phi(\beta)}. \tag{2.9}
\]

**Proof.** Similar to (2.1) we find

\[
\mathbb{P}(\text{Pois}(\lambda) < s_{\lambda}) = \mathbb{P}\left(\frac{\text{Pois}(\lambda) - \lambda}{\sqrt{\lambda}} < \frac{s_{\lambda} - \lambda}{\sqrt{\lambda}}\right) = \mathbb{P}\left(\frac{\text{Pois}(\lambda) - \lambda}{\sqrt{\lambda}} < (1 - \rho_{\lambda}) \frac{s_{\lambda}}{\sqrt{\lambda}}\right) \to \Phi(\beta), \tag{2.10}
\]

for \( \lambda \to \infty \). Stirling’s formula gives

\[
\mathbb{P}(\text{Pois}(\lambda) = s) = e^{-\lambda_{s_{\lambda}}^{\lambda_{s_{\lambda}}} \frac{s_{\lambda}!}{s_{\lambda}!}} \sim e^{-\lambda_{s_{\lambda}}^{\lambda_{s_{\lambda}}} \frac{1}{\sqrt{2\pi s_{\lambda}}} \left(\frac{e}{s_{\lambda}}\right)^s_{\lambda}} = \frac{1}{\sqrt{2\pi s_{\lambda}}} e^{\lambda_{s_{\lambda}}^{\lambda_{s_{\lambda}}} \ln(\rho_{\lambda})}. \tag{2.11}
\]

Since \( \ln(\rho_{\lambda}) = -(1 - \rho_{\lambda}) - \frac{1}{2}(1 - \rho_{\lambda})^2 + o((1 - \rho_{\lambda})^2) \) we find that

\[
\frac{\mathbb{P}(\text{Pois}(\lambda) = s_{\lambda})}{1 - \rho_{\lambda}} = \frac{1}{(1 - \rho_{\lambda}) \sqrt{s_{\lambda}}} \frac{e^{-\frac{1}{2}(1 - \rho_{\lambda})^2 + o((1 - \rho_{\lambda})^2)}}{\sqrt{2\pi}} \to \frac{1}{\beta} e^{-\frac{1}{2} \beta^2} = \varphi(\beta). \tag{2.12}
\]

Substituting (2.10) and (2.12) into (2.6) gives (2.7), and as by-product also (2.9). \( \square \)

Many of the subsequent results in this survey presented for the \( M/M/s_{\lambda} \) queue can also be derived for the \( M/M/s_{\lambda}/s_{\lambda} \) queue; we refer to \cite{75} for a detailed overview of these results. Observe that \( g(\beta) \) is a strictly decreasing function on \((0, \infty)\) with \( g(\beta) \to 1 \) as \( \beta \to 0 \) and \( g(\beta) \to 0 \) for \( \beta \to \infty \). Thus all possible delay probabilities are achievable in the QED regime, which will prove useful for the dimensioning of systems (see Section 4). Although Proposition 2.1 is an
characterizes this scaling limit formally.

This reflects that the system state typically hovers around the full-occupancy level \( s \). For that reason, we consider the centered and scaled process

\[
\bar{Q}(s, \lambda)(t) := \frac{Q(s, \lambda)(t) - s}{\lambda} \quad \text{for all } t \geq 0,
\]

and ask what happens to this process as \( \lambda \to \infty \). First, we consider the mean drift conditioned on \( \bar{Q}(s, \lambda)(t) = x \). When \( x > 0 \), this corresponds to a state in which \( Q(s, \lambda)(t) > s \lambda \) and hence all servers are occupied. Therefore, the mean rate at which jobs leave the system is \( s \lambda \), while the arrival rate remains \( \lambda \), so that the mean drift of \( \bar{Q}(s, \lambda)(t) \) in \( x > 0 \) satisfies

\[
\frac{\lambda - s}{\lambda} \to -\beta, \quad \text{as } \lambda \to \infty,
\]

under scaling \( \sqrt{s} \lambda(1 - \rho) \to \beta \) in (2.13). When \( x \leq 0 \), only \( s \lambda + x \sqrt{s} \lambda \) servers are working, so that the net drift is

\[
\frac{\lambda - (s \lambda + x \sqrt{s} \lambda)}{\sqrt{s} \lambda} \to -\beta - x, \quad \text{as } \lambda \to \infty.
\]

Now, imagine what happens to the sample paths of \( Q(s, \lambda)(t) \) as we increase \( \lambda \). Within a fixed time interval, larger \( \lambda \) and \( s \lambda \) will trigger more and more events, both arrivals and departures. Also, the jump size at each event epoch decreases as \( 1/\sqrt{s} \lambda \) as a consequence of the scaling in (2.14). Hence, there will be more events, each with a smaller impact, and in the limit as \( \lambda \to \infty \), there will be infinitely many events of infinitesimally small impact. This heuristic explanation suggests that the process \( \bar{Q}(s, \lambda)(t) \) converges to a stochastic-process limit, which is continuous, and has infinitesimal drift \(-\beta \) above zero and \(-\beta - x \) below zero. Figure 3 visualizes the emergence of the suggested scaling limit as \( \lambda \) and \( s \lambda \) increase. The following theorem by Halfin & Whitt [60] characterizes this scaling limit formally.
Figure 4: Sample paths of the normalized queue length process $\bar{Q}(s_\lambda)(t)$ with $\lambda = 50$, $\lambda = 100$ and $\lambda = 500$ and $s_\lambda = [\lambda + 0.5\sqrt{\lambda}]$.

**Theorem 2.2** Let $\bar{Q}(s_\lambda)(0) \overset{d}{\Rightarrow} D(0) \in \mathbb{R}$ and $\sqrt{s_\lambda}(1 - \rho_\lambda) \to \beta$. Then for all $t \geq 0$,

$$\bar{Q}(s_\lambda)(t) \overset{d}{\Rightarrow} D(t), \quad \text{as } \lambda \to \infty,$$

where $D(t)$ is the diffusion process with infinitesimal drift $m(x)$ given by

$$m(x) = \begin{cases} -\beta, & \text{if } x > 0, \\ -\beta - x, & \text{if } x \leq 0 \end{cases}$$

and infinitesimal variance $\sigma^2(x) = 2$.

The limiting diffusion process $(D(t))_{t \geq 0}$ in Theorem 2.2 is a combination of a negative-drift Brownian motion in the upper half plane and an Ornstein-Uhlenbeck process in the lower half plane. We refer to this hybrid diffusion process as the Halfin-Whitt diffusion $[143, 42, 27]$. Studying this diffusion process provides valuable information for the system's performance. The fact that the properly centered and scaled occupancy process $(\bar{Q}(s_\lambda)(t))_{t \geq 0}$ has the weak limit $(D(t))_{t \geq 0}$, as stated in Theorem 2.2, has several important consequences. The boundary between the Brownian motion and the Ornstein-Uhlenbeck process can be thought of as the number of servers, and $(D(t))_{t \geq 0}$ will keep fluctuating between these two regions. The process mimics a single-server queue above zero, and an infinite-server queue below zero, for which Brownian motion and the Ornstein–Uhlenbeck process are indeed the respective heavy-traffic limits. As $\beta$ increases towards $+\infty$, capacity grows and the Halfin–Whitt diffusion will spend more time below zero.

The diffusion process $(D(t))_{t \geq 0}$ can thus be employed to obtain simple approximations for the system behavior. Theorem 2.2 supports approximating the occupancy process in the $M/M/s_\lambda$ queue as

$$Q(s_\lambda)(\cdot) \overset{d}{\approx} s_\lambda + \sqrt{s_\lambda}D(\cdot)$$

when $\lambda$ and $s_\lambda$ are large. It is natural to expect that this carries over to approximations for the steady-state distribution of $(D(t))_{t \geq 0}$. Let $D(\infty) := \lim_{t \to \infty} D(t)$ and $Q(s_\lambda)(\infty) := \lim_{t \to \infty} Q(s_\lambda)(t)$ denote the steady-state random variables. Then,

$$Q(s_\lambda)(\infty) \overset{d}{\approx} s_\lambda + \sqrt{s_\lambda}D(\infty).$$

To rigorously justify the approximation (2.20) it is still required to show that the sequence of steady-state distributions associated with the queue-length process, when appropriately scaled,
converge to the steady-state distribution associated with diffusion process,

$$Q^{(s)}(\infty) - s_\lambda \frac{d}{\sqrt{s_\lambda}} D(\infty), \quad \text{as } \lambda \to \infty. \quad (2.21)$$

This has been done in [60].

The steady-state characteristics of the diffusion were studied in [60]. Since the diffusion process $D(t) \geq 0$ has piecewise linear drift, the procedure developed in [27] to find the stationary distribution can be followed. This procedure consists of composing the density function as in (2.18) based on the density function of a Brownian motion with drift $-\beta$ for $x > 0$ and of an Ornstein-Uhlenbeck process with drift $-\beta - x$ for $x < 0$. The density function of the stationary distribution for $D(t)|t \geq 0$ is then proportional to $\phi(x + \beta)/\Phi(\beta)$ for negative levels $x < 0$ and proportional to $\exp(\int_0^x m(u)du)$ for $x \geq 0$. Then, upon normalization, we find that

$$P(D(\infty) > 0) = g(\beta), \quad (2.22)$$

$$P(D(\infty) \geq x|D(\infty) > 0) = e^{-\beta x}, \quad \text{for } x > 0, \quad (2.23)$$

$$P(D(\infty) \leq x|D(\infty) \leq 0) = \frac{\Phi(\beta + x)}{\Phi(\beta)}, \quad \text{for } x \leq 0. \quad (2.24)$$

This confirms the earlier result for the Erlang C formula in (2.7), i.e.

$$C(s_\lambda, \lambda) \to P(D(\infty) > 0) = g(\beta), \quad \text{as } \lambda \to \infty, \quad (2.25)$$

and the scaled limiting mean delay in (2.13)

$$\frac{E[Q^{(s_\lambda)}]}{\sqrt{s_\lambda}} \to E[D(\infty)] = \int_0^\infty g(\beta)e^{-\beta x}dx = \frac{g(\beta)}{\beta}, \quad \text{as } \lambda \to \infty. \quad (2.26)$$

It is also of interest to study time-dependent characteristics like mixing times, time-dependent distributions and first passage times, to enhance our understanding of how the $M/M/s_\lambda$ queue, behaves over various time and space scales. The mixing time is closely related to the spectral gap, which for the Halfin–Whitt diffusion $(D(t))_{t \geq 0}$ has been identified by Gamarnik & Goldberg [43] building on the results of van Doorn [140] on the spectral gap of the $M/M/s_\lambda$ queue. An alternative derivation of this spectral gap was presented in by [142, 143], along with expressions for the Laplace transform over time, and the large-time asymptotics for the time-dependent density. First passage times to large levels corresponding to highly congested states were obtained in [105, 42].

For obvious reasons, the QED regime is also referred to as the Halfin-Whitt regime, and both these names are used interchangeably in the literature.

### 2.2 Bulk-service queue

We next consider the bulk-service queue, a standard model for digital communication [28], but also many more applications among which wireless networks, road traffic, reservation systems, health care; see [141, Chap. 2] for an overview. Although the bulk-service queue gives rise to a plain reflected random walk, and is not a multi-server queue, in the same sense as the $M/M/s$ queue, we explain below how these two models are connected.

Let jobs again arrive according to a Poisson process with rate $\lambda$, but now we discretize time, so the number of new arrivals per time period is given by a Pois($\lambda$) random variable. Let $Q_k^{(s_\lambda)}$
denote the number of delayed jobs at the start of the \( k \)th period and assume that the system is able to process \( s_\lambda \) jobs at the end of each period. The queue length process can then be described by the Lindley-type recursion [96]

\[
Q^{(s_\lambda)}_{k+1} = \max\{0, Q^{(s_\lambda)}_k + \text{Pois}_k(\lambda) - s_\lambda\},
\]

with \( Q^{(s_\lambda)}_0 = 0 \) and \((\text{Pois}_k(\lambda))_{k \geq 0}\) i.i.d. random variables. The queue length process is thus characterized by a random walk with i.i.d. steps of size \((\text{Pois}(\lambda) - s_\lambda)\), with a reflecting barrier at zero. We can iterate the recursion in (2.27) to find

\[
Q^{(s_\lambda)}_k = \max\left\{0, \sum_{i=1}^{j} (\text{Pois}_{k-i}(\lambda) - s_\lambda)\right\} \quad \text{d} \max\left\{0, \sum_{i=1}^{j} (\text{Pois}_i(\lambda) - s_\lambda)\right\},
\]

where the last equality holds in distribution due to the duality principle for random walks, see e.g. [129] Sec. 7.1. Stability requires that the mean step size satisfies \( \mathbb{E}[\text{Pois}(\lambda) - s_\lambda] = \lambda - s_\lambda < 0 \). We use the shorthand notation for the partial sum \( S_k := \sum_{i=1}^{k} (\text{Pois}_i(\lambda) - s_\lambda) \). Let \( Q^{(s_\lambda)} := \lim_{k \to \infty} Q^{(s_\lambda)}_k \) denote the stationary queue length. The probability generating function (pgf) of \( Q^{(s_\lambda)} \) can then be expressed in terms of the pgf of the positive parts of the partial sum:

\[
\mathbb{E}[z^{Q^{(s_\lambda)}}] = \exp\left\{-\sum_{k=1}^{\infty} \frac{1}{k} \left(1 - \mathbb{E}[z^{S_k^+}] \right)\right\}, \quad |z| \leq 1.
\]

From (2.29) we obtain for the mean queue length and empty-queue probability the expressions

\[
\mathbb{E}[Q^{(s_\lambda)}] = \sum_{k=1}^{\infty} \frac{1}{k} \mathbb{E}[S_k^+],
\]

\[
\mathbb{P}(Q^{(s_\lambda)} = 0) = \exp\left\{-\sum_{k=1}^{\infty} \frac{1}{k} \mathbb{P}(S_k^+ > 0)\right\}.
\]

There is a connection between the bulk-service queue and the \( M/D/s \) queue. To see this, consider the number of queued jobs \( Q^{(s_\lambda)}(k) \) at time epochs \( k = 0, 1, 2, \ldots \). The we set the period length equal to one service time. The number of new arrivals per time period is then given by the sequence of i.i.d. random variables \((\text{Pois}_k(\lambda))_{k \geq 1}\). At the start of the \( k \)th period, \( Q^{(s_\lambda)}_k \) customers are waiting. Since the service time of a customer is equal to the period length, all jobs that are in service at the beginning of the period will have left the system by time \( k + 1 \). This implies that \( \min\{Q^{(s_\lambda)}_k, s_\lambda\} \) of the jobs that were queued at time \( k \) are taken into service during period \( k \). These however cannot possibly have departed before the end of the period, due to their deterministic service times. If \( Q^{(s_\lambda)}_k < s_\lambda \), then additionally \( \min\{\text{Pois}_k(\lambda), s_\lambda - Q^{(s_\lambda)}_k\} \) of the new arrivals are taken into service. This yields a total of \( \text{Pois}_k(\lambda) \) arrivals, and \( \min\{Q^{(s_\lambda)}_k + \text{Pois}_k(\lambda), s_\lambda\} \) departures from the queueing system during period \( k \). In total, this adds up to the Lindley recursion (2.27). Hence, although the bulk-service queue is technically not a multi-server queue, it gives rise to a recursive relation that describes the \( M/D/s \) queue.
The reason why we choose to explain the QED regime through the bulk-service queue is that the elementary random walk perspective allows for a rather direct application of the CLT. To see this, let us ask ourselves what happens if the elementary random walk perspective allows for a rather direct application of the CLT. To see

$$\text{Hence by the CLT}$$

Let 

$$\text{it is easily verified that}$$

random walk with normally distributed increments, i.e. a reflected

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the elementary random walk perspective allows for a rather direct application of the CLT . To see

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$$\text{Prop. 19.2}$$

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the QED regime: for the

$$\text{Theorem 2.3}$$

$$M$$

as in

$$\text{(i) } Q_0 \ldots Q_n$$

$$\text{queue this is the Gaussian random walk. Indeed,}$$

it is easily verified that [73]

$$\text{Hence by the CLT}$$

$$Y_k^{(s)} = A_k^{(\lambda)} - s_\lambda = \frac{A_k^{(\lambda)} - \lambda}{\sqrt{\lambda}} - \beta \Rightarrow Y_k \xrightarrow{d} \mathcal{N}(-\beta, 1),$$

for $$\lambda \to \infty$$. So we expect the scaled queue length process to converge in distribution to a reflected random walk with normally distributed increments, i.e. a reflected Gaussian random walk. Indeed, it is easily verified that [73]

$$\text{Let } M_\beta := \lim_{k \to \infty} M_{\beta,k} \text{ denote the all-time maximum of a Gaussian random walk. It can be shown that } M_\beta \text{ almost surely exists and that } \bar{Q}_{(s)} := \lim_{k \to \infty} \bar{Q}_{(s)} \xrightarrow{d} M_\beta \text{ for instance by [134, Prop. 19.2] and [8 Thm. X6.1]. The following theorem can be proved using a similar approach as in [76].}$$

**Theorem 2.3** If $$(1 - \rho_\lambda) \sqrt{\lambda} \to \beta$$ as $$\lambda \to \infty$$, then

(i) $$\bar{Q}_{(s)} \xrightarrow{d} M_\beta$$ as $$\lambda \to \infty$$;

(ii) $$\mathbb{P}(\bar{Q}_{(s)} = 0) \to \mathbb{P}(M_\beta = 0)$$ as $$\lambda \to \infty$$;

(iii) $$\mathbb{E}[\bar{Q}_{(s)}^k] \to \mathbb{E}[M_\beta^k]$$ as $$\lambda \to \infty$$ for any $$k > 0$$.

Hence, Theorem 2.3 is the counterpart of Theorem 2.2, but for the bulk-service (or $$M/D/s_\lambda$$) queue, rather than for the $$M/M/s_\lambda$$ queue. Both theorems identify the stochastic-process limit in the QED regime: for the $$M/M/s$$ queue this is the Halfin-Whitt diffusion, and for the bulk-service queue this is the Gaussian random walk.

The Gaussian random walk is well studied [132, 29, 70, 18, 70] and there is an intimate connection with Brownian motion. The only difference, one could say, is that Brownian motion is a continuous-time process, whereas the Gaussian random walk only changes at discrete points in time. If $$(B(t))_{t \geq 0}$$ is a Brownian motion with drift $$-\beta < 0$$ and infinitesimal variance $$\sigma^2$$ and $$(W(t))_{t \geq 0}$$ is a random walk with $$\mathcal{N}(-\beta, \sigma^2)$$ distributed steps and $$B(0) = W(0)$$, then $$W$$ can be regarded as the process $$B$$ embedded at equidistant time epochs. That is, $$W(t) \xrightarrow{d} B(t)$$ for all $$t \in \mathbb{N}^+$$.

For the maximum of both processes this coupling implies

$$\max_{k \in \mathbb{N}^+} W(k) = \max_{k \in \mathbb{N}^+} B(k) \leq_{st} \max_{t \in \mathbb{R}^+} B(t), \quad (2.33)$$
where $\leq_{st}$ denotes stochastic dominance. This difference in maxima is visualized in Figure 5. It is known that the all-time maximum of Brownian motion with negative drift $-\mu$ and infinitesimal variable $\sigma^2$ has an exponential distribution with mean $\sigma/2\mu$ [61]. Hence, (2.33) implies that $M_\beta$ is stochastically upper bounded by an exponential random variable with mean $1/2\beta$.

Despite this easy bound, precise results for $M_\beta$ are more involved. Let $\zeta$ denote the Riemann zeta function. In [29] and [70] it is shown that for $0 < 2\beta < 2\sqrt{\pi}$,

$$\mathbb{P}(M_\beta = 0) = \sqrt{2}\beta \exp\left(\frac{\beta}{\sqrt{2\pi}} \sum_{l=0}^{\infty} \frac{\zeta(1/2 - l)}{l!(2l + 1)} \left(\frac{-\beta^2}{2}\right)^l\right),$$  \hspace{1cm} (2.34)

and

$$\mathbb{E}[M_\beta] = \frac{1}{2\beta} + \frac{\zeta(1/2)}{\sqrt{2\pi}} + \frac{\beta}{4} + \frac{\beta^2}{\sqrt{2\pi}} \sum_{l=0}^{\infty} \frac{\zeta(-1/2 - l)}{l!(2l + 1)(2l + 2)} \left(\frac{-\beta^2}{2}\right)^l. \hspace{1cm} (2.35)$$

In Figure 6 we have plotted the exact empty-buffer probability and scaled mean delay, together with their asymptotic approximations. We see that the performance measures associated with the Gaussian random walk serve as accurate approximations to performance measures describing the bulk-service queues of small to moderate size as well, just as we saw in Figure 3 for the $M/M/s_\lambda$ queue.
3 Universal QED properties

Now that we have seen how the square-root rule (1.1) yields non-degenerate limiting behavior in classical queueing models, we shall summarize the revealed universal QED properties and argue that these properties should hold for a more general class of models. The first property relates to the efficient usage of resources, expressed as

\[
\text{system load} \sim 1 - \frac{\text{constant}}{\sqrt{\text{system size}}}. \quad \text{(Efficiency)}
\]

This property for the $M/M/s_\lambda$ queue and bulk-service queue is a direct consequence of the square-root rule. The second distinctive property is the balance between QoS and efficiency:

\[
\mathbb{P}(\text{delay}) \to \text{constant}, \quad \text{(Balance)}
\]

as the system size increases indefinitely. Indeed, we have shown that under (1.1) and letting $\lambda, s_\lambda \to \infty$ that both limiting functions $g(\beta)$ in the $M/M/s_\lambda$ queue and $\mathbb{P}(M_\beta > 0)$ in the bulk-service queue can take all values in the interval $(0, 1)$ by tuning the parameter $\beta$. The third property relates to good QoS:

\[
\mathbb{E}[\text{delay}] = O(1/\sqrt{\text{system size}}). \quad \text{(QoS)}
\]

Indeed, we have

\[
\mathbb{E}[W^{(s_\lambda)}] = \frac{h(\beta)}{\sqrt{s_\lambda}} + o(1/\sqrt{s_\lambda}) \quad \text{and} \quad \mathbb{E}[Q^{(s_\lambda)}] = \sqrt{s_\lambda}\mathbb{E}[M_\beta] + o(1/\sqrt{s_\lambda}), \quad (3.1)
\]

in the $M/M/s_\lambda$ queue and bulk-service queue, respectively. Hence the mean delay vanishes at rate $1/\sqrt{s_\lambda}$.

Since the mathematical underpinning of these properties comes from the CLT (as shown in Section 2), we can expect the properties to hold for a much larger class of models. We will illustrate this by discussing several extensions of the basic models discussed in Section 2. The easiest way to do so seems to interpret the bulk-service queue as a many-sources model. Consider a stochastic system in which demand per period is given by some random variable $A$, with mean $\mu_A$ and variance $\sigma^2_A < \infty$. For systems facing large demand we propose to set the capacity according to the more general rule

\[
s = \mu_A + \beta \sigma_A,
\]

which consists of a minimally required part $\mu_A$ and a variability hedge $\beta \sigma_A$. Assume that the demand is generated by $n$ stochastically identical and independent sources. Each source $i$ generates $A_{i,k}$ work in the $k$th period, with $\mathbb{E}[A_{i,k}] = \mu$ and $\text{Var}A_{i,k} = \sigma^2$. Then the total amount of work arriving to the system during one period is $A^{(n)}_k = \sum_{i=1}^n A_{i,k}$ with mean $n\mu$ and variance $n\sigma^2$. Assume that the system is able to process a deterministic amount of work $s_n$ per period and denote by $Q^{(n)}_k$ the amount of work left over at the end of period $k$. Then,

\[
Q^{(n)}_{k+1} = \left(Q^{(n)}_k + A^{(n)}_k - s_n\right)^+.
\]

Given that $s_n > \mathbb{E}[A^{(n)}_1] = n\mu$, the steady-state limit $Q^{(n)} := \lim_{t \to \infty} Q^{(n)}(t)$ exists and satisfies

\[
Q^{(n)} \overset{d}{=} \left(Q^{(n)} + A^{(n)} - s_n\right)^+.
\]

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With this many-sources interpretation [2 69 71], increasing the system size is done by increasing $n$, the number of sources. As we have seen before, it requires a rescaling of the process $Q^{(n)}$ by an increasing sequence $c_n$, to obtain a non-degenerate scaling limit $Q := \lim_{n \to \infty} Q^{(n)}/c_n$. (We omit the technical details needed to justify the interchange of limits.) From (3.3) it becomes clear that the scaled increment

$$\frac{A_k^{(n)} - s_n}{c_n} = \sum_{i=1}^n A_{i,k} - n\mu + \frac{n\mu - s_n}{c_n}$$

only admits a proper limit if $c_n$ is of the form $c_n = O(\sqrt{n})$, by the virtue of the CLT, and $(s_n - n\mu)/c_n \to \beta > 0$ as $n \to \infty$. Especially for $c_n = \sigma \sqrt{n}$, the standard deviation of the demand per period, this reveals that $Q$ has a non-degenerate limit, which is equal in distribution to the maximum of a Gaussian random walk with drift $-\beta$ and variance 1, if

$$s_n = n\mu + \beta \sigma \sqrt{n} + o(\sqrt{n}).$$

Moreover, the results for the Gaussian random walk presented in Section 2.2 are applicable to this model and the key features of the QED scaling carry over to this more general setting as well using the CLT. That is, for the bulk-service queue under the general assumptions above we get the QED approximation

$$\mathbb{E}[Q^{(n)}] \approx \sigma \sqrt{n} \mathbb{E}[M_\beta] \approx \frac{\sigma \sqrt{n}}{2\beta} \quad (3.5)$$

for small $\beta$. Thus, the many-sources framework shows that the QED scaling finds much wider application than just queueing models with Poisson input.

Let us reflect on a key technical difference between the bulk-service queue and the $M/M/s$ queue. The bulk-service queue is and remains a one-dimensional reflected random walk, even under the QED scaling. Therefore, to establish the QED limits for the performance measure, one only needs to apply the CLT to the increments of the random walk, which readily shows that the queue converges to the Gaussian random walk. Analysis of multi-server queues is typically more challenging. Establishing QED limits for the elementary $M/M/s$ queue already contains some technically advanced steps. While we explained the high-level insights to argue the convergence of the birth-death process taking discrete steps to the continuous diffusion process, the formal proof in Halfin & Whitt [60] relies on Stone’s Theorem [137, 57, 94] for the weak convergence of birth-death processes to diffusion processes. However, for multi-server queue that cannot be viewed as a birth-death process, Stone’s Theorem cannot be applied and entirely different techniques are needed.

To explain this, let us turn to the $G/G/s$ queue, the natural extension of the $M/M/s$ queue to generally distributed interarrival times and service times. Establishing QED limits for the $G/G/s$ queue has led to a remarkable research effort of which the majority took place over the last decade. When one moves beyond the exponential and deterministic assumptions, establishing QED behavior becomes mathematically more challenging and most of the analysis has the $G/G/s$ queue in the QED regime has evolved around the characterization of the stochastic-process limit of the centered and scaled process, under various assumptions on the model primitives. We restrict our discussion to developments on the basic $G/G/s$ queue; a more extensive discussion, including work on abandonments, can be found in the surveys [120, 32]. Puhalskii & Reiman [123] analyzed the multi-class queue with phase-type service times in the QED regime. Heavy-traffic limits for queues in which service time distributions are lattice-based and/or have finite support were studied by Mandelbaum & Momčilović [107] and Gamarnik & Momčilović [45]. The most general
class of distributions was considered by Reed [125] and Puhalskii & Reed [122], who imposed no assumptions on the service time distribution except for the existence of the first moment. Both of these papers focus on the queue length process. The paper by Reed [125] utilized an ingenious connection with the infinite server queue, a connection which was developed further by Puhalskii & Reed [122] where results from modern empirical process theory (including the usage of outer measures to avoid measurability problems) are used in their full potential. Equally important steps forward concern the usage of measure-valued processes by Kang, Kaspi & Ramanan [83, 82, 84]. Assuming minor additional regularity conditions on the service-time distribution (like a bounded density, and sufficiently many finite moments) the paper [84] unraveled the structure of the limit process that appears after scaling. A key insight from these works is that the limiting queue length process can be interpreted as a one-dimensional diffusion with a drift that depends on the entire history of the process. This as opposed to the Halfin-Whitt diffusion that comes with exponential service times, where the drift depends on the current scaled queue length only. As a result, even after taking the limit, the resulting limit process for the \( G/G/s \) queue has still a complicated steady-state distribution and it is therefore not surprising that considerably less is known for the corresponding steady-state distribution of the \( G/G/s \) queue in the QED regime. An exception is the work by Gamarnik & Goldberg [49, 44], who performed their analysis under the mild assumption that the service time distribution has finite \((2 + \epsilon)\) moments and revealed suitable analogues of all three structural properties mentioned at the beginning of this section, and in addition, explicit tail bounds for the distribution of the delay have been developed.

### 4 Dimensioning

We adopt the term *dimensioning* used by Borst et al. [20] to say that the capacity of a resource sharing system is adapted to the load in order to reach certain performance levels. In [20] dimensioning refers to the staffing problem in a large-scale call center and key ingredients are the square-root rule in (1.1) and the QED regime. We now revisit the results in [20] and its follow-up works to explain this connection to the QED regime.

#### 4.1 Constraint satisfaction

Consider the \( M/M/s \) queue with arrival rate \( \lambda \) and service rate \( \mu = 1 \). A classical dimensioning problem is to determine the minimum number of servers \( s \) necessary to achieve a certain target level of service, say in terms of delay.

Suppose we want to determine the minimum number of servers such that the fraction of jobs that are delayed in the queue is at most \( \epsilon \in (0, 1) \). Hence we should find

\[
s_\lambda^\epsilon := \min \{ s > \lambda \mid C(s, \lambda) \leq \epsilon \}.
\]

But alternatively, we can use the QED framework, which says that with \( s_3 \) as in (1.1),

\[
\lim_{\lambda \to \infty} C(s_3, \lambda) = g(\beta) \text{ (see Proposition 2.1).} \]

Then (4.1) can be replaced by

\[
s_\lambda^{QED}(\epsilon) = \lceil \lambda + \beta^*(\epsilon) \sqrt{\lambda} \rceil,
\]

where \( \beta^*(\epsilon) \) solves

\[
g(\beta^*) = \epsilon.
\]
In Figure 7, we plot the exact (optimal) capacity level \( s^*_\lambda(\epsilon) \) and the heuristically obtained capacity level \( s_{\lambda}^{\text{QED}}(\epsilon) \) as functions of \( \epsilon \) for several loads \( \lambda \).

Observe that even for very small values of \( \lambda \), the capacity function \( s_{\lambda}^{\text{QED}}(\epsilon) \) coincides with the exact solution for almost all \( \epsilon \in (0, 1) \) and differs no more than by one server for all \( \epsilon \). Borst et al. [20] recognized this in their numerical experiments too, and [74] later confirmed this theoretically (see Section 4). One can easily formulate other constraint satisfaction problems and reformulate them in the QED regime. For instance, constraints on the mean delay or the tail probability of the duration of delay, e.g. \( P(\text{delay} > T) \), which are asymptotically approximated by \( h(\beta) / \sqrt{\lambda} \) and \( g(\beta) e^{-\beta \sqrt{\lambda} T} \), respectively. See [20, 157, 130] for more examples.

### 4.2 Cost minimization

Alternatively, one can consider optimization problems, for instance to strike the right balance between the capacity allocation costs and delay costs incurred. More specifically, assume an allocation cost of \( a \) per server per unit time, and a penalty cost of \( q \) per delayed job per unit time, yielding the total cost function

\[
\tilde{K}(s, \lambda) := as + q \lambda E[\text{delay}] = as + q\lambda \frac{C(s, \lambda)}{s - \lambda},
\]

see (2.4), and then ask for the capacity level \( s \) that minimizes \( \tilde{K}(s, \lambda) \). Since \( s > \lambda \), we have \( \tilde{K}(s, \lambda) > a \lambda \) for all feasible solutions \( s \). Moreover, the minimizing value of \( \tilde{K}(s, \lambda) \) is invariant with respect to scalar multiplication of the objective function. Hence we equivalently seek to optimize

\[
K(s, \lambda) = r (s - \lambda) + \frac{\lambda}{s - \lambda} C(s, \lambda) \quad \text{with} \quad r = a/q. \tag{4.4}
\]

Denote by \( s^*_\lambda(r) := \arg \min_{s > \lambda} K(s, \lambda) \) the true optimal capacity level. With \( s_\lambda = \lambda + \beta \sqrt{\lambda} \) and the QED limit in (2.13), we can replace (4.4) by its asymptotic counterpart:

\[
\frac{K(s_\lambda, \lambda)}{\sqrt{\lambda}} \to r \beta + \frac{g(\beta)}{\beta} =: K_\lambda(\beta), \quad \text{as} \ \lambda \to \infty. \tag{4.5}
\]

We again obtain a limiting objective function that is easier to work with than its exact pre-limit counterpart. Hence, in the spirit of the asymptotic resource allocation procedure in the previous subsection, we propose the following method to determine the capacity level that minimizes overall costs. First, (numerically) compute the value \( \beta^*(r) = \arg \min_{\beta > 0} K_\lambda(\beta) \), which is well-defined,
because the function $K_r(\beta)$ is strictly convex for $\beta > 0$. Then, set $s_{QED}^{\lambda}(r) = [\lambda + \beta^*(r)\sqrt{\lambda}]$. In Figure 8, we compare the outcomes of this asymptotic resource allocation procedure against the true optima as a function of $r \in (0, \infty)$, for several values of $\lambda$. The capacity levels $s_{QED}^{\lambda}(r)$ and $s_{QED}^{\lambda}(r)$ are aligned for almost all $r$, and differ no more than one server for all instances.

### 4.3 Dynamic resource allocation

We next discuss how the QED regime also finds application in systems facing a time-varying load. A time-varying arrival rate $\lambda(t)$ calls for a time-varying capacity rule $s(t)$. Again, we shall explain the main ideas through the $M/M/s$ queue, but now its time-varying extension in which jobs arrive according to a non-homogeneous Poisson process with rate function $\lambda(t)$, a setting typically referred to as the $M_t/M/s_t$ queue.

As in Section 4.1, we want to set the capacity level $s(t)$ such that the delay probability is at most $\epsilon \in (0, 1)$ for all $t$. The analysis of this time-varying many-server queueing systems is cumbersome and several approximative analysis have been proposed such as the pointwise-stationary approximation (PSA) [51], which evaluates the system at time $t$ as if it were in steady-state with instantaneous parameters $\lambda = \lambda(t)$, $\mu$ and $s = s(t)$. PSA performs well in slowly varying environments with relatively short service times [51, 146], but the steady-state approximation becomes less accurate when $\lambda(t)$ displays significant fluctuations; see the numerical experiment at the end of this section. One reason for this lack of accuracy is that PSA does not account for the jobs that are actually present in the system (being in service or queued), an important piece of real-time information that should be taken into account in capacity allocation decisions. Jennings et al. [77] introduced an alternative to PSA that exploits the relation with infinite-server queues, facing a non-homogeneous Poisson process with rate $\lambda(t)$, in which case the number of jobs at time $t$ is Poisson distributed with mean

$$R(t) = \mathbb{E}[\lambda(t - B)] \mathbb{E}[B] = \int_0^\infty \lambda(t - u) P(B > u) \, du = \int_0^\infty \lambda(t - u) e^{-\mu u} \, du, \quad (4.6)$$

where $B$ denotes the processing time of one jobs, in our case an exponentially distributed random variable. We remark that under general service time assumptions, we should replace $\mathbb{E}[\lambda(t - B)]$ in (4.6) with $\mathbb{E}[\lambda(t - B_t)]$, where $B_t$ denotes the excess service time [36]. Recall that the mean delay in the QED regime is negligible; see (QoS). Hence, the total time in the system is roughly equal to its service time. Under these conditions, the many-server system can be approximated by the infinite-server approximation with offered load as in (4.6). Accordingly, we can determine the capacity levels $s(t)$ for each $t$ based on steady-state $M/M/s$ measures with offered load $R = R(t)$. 

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Figure 8: Optimal capacity levels as a function of $r = a/q$. 

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20
Jennings et al. [77] proceed by exploiting the heavy-traffic results of Halfin & Whitt (2.13). In conjunction with the dimensioning scheme in Section 4.1, it is proposed in [77] to set
\[ s(t) = \left[ R(t) + \beta^*(\epsilon) \sqrt{R(t)} \right], \] (4.7)
where \( \beta^*(\epsilon) \) solves \( g(\beta^*(\epsilon)) = \epsilon \). Remark that the number of servers is rounded up to ensure that the achieved delay probability is indeed below \( \epsilon \). The time-dependent dimensioning rule in (4.7) was dubbed in [77, 112] the modified offered load (MOL) approximation. Let us now demonstrate how MOL works for an example with sinusoidal arrival rate function. Figure 9a shows an arrival rate pattern \( \lambda(t) \) and corresponding offered load function \( R(t) \) for \( \mu = 1/2 \). The resulting time-varying capacity levels based on the PSA and MOL approximations with \( \epsilon = 0.3 \) are plotted in Figure 9b. Through simulation, we evaluate the delay probability as a function of time for \( \epsilon = 0.1, 0.3 \) and 0.5. While the PSA approach fails to stabilize the performance of the queue, the MOL method does stabilize around the target performance, see Figure 9c. The slightly erratic nature of the delay probability as a function of time can be explained by rounding effects of the capacity level.

Because time-varying capacity allocation is an issue that recurs in many practical settings, this has been the topic of many works; see e.g. [40, 85, 150, 151, 97, 98, 99, 64]. For an accessible overview of queueing-theoretical methods for determining capacity levels under time-varying demand, see Kolesar et al. [52] and references therein. Whitt [152] provides an extensive bibliography of literature on queueing models with time-varying demand.

5 Convergence rates

By now, it is clear that the QED paradigm is based on limit theorems that apply when systems become infinitely large. In practice, even large systems are finite, which makes it important to quantify the error made in approximating a finite system by a limiting object. As it turns out, QED approximations are in many cases highly accurate, already for relatively small or moderately sized systems. In this section we show how to quantify these errors by determining the rate of convergence of certain performance measures to their asymptotic limits. A first sign of this was seen through the accuracy of the asymptotic dimensioning schemes in Section 4. These convergence rates are typically of order \( 1/\sqrt{s} \) with \( s \) the system size. This again confirms the deep connection with the CLT with a typically error also of order \( 1/\sqrt{s} \) but then with \( s \) the number of random variables in the sum.
5.1 Bounds

A convergence rate can also be interpreted as the (main) error made when using the QED limits as approximations for the real performance measures. Whenever we find ways to obtain explicit and precise descriptions of the convergence rates, this can also be used to correct the limiting expression for the finite size of the system. We will also show how such effective corrections can be obtained and applied directly in the QED framework.

Recall that when \( \lambda \) is a positive integer, \( \text{Pois}(\lambda) \) can be written as the sum of \( \lambda \) \( \text{Pois}(1) \) random variables. A more general version of the CLT in Theorem 1.1 related to the Berry-Esséen bound, see e.g. [41, Sec. XVI.5], implies that

\[
\mathbb{P}(\text{Pois}(\lambda) \leq s_\lambda) = \Phi(\beta) + O(\lambda^{-1/2}),
\]

as \( \lambda \to \infty \) with \( s_\lambda \) as in (1.1). Comparing (5.1) with (1.12), (5.1) not only shows convergence of \( \mathbb{P}(\text{Pois}(\lambda) \leq s_\lambda) \) to \( \Phi(\beta) \), but also quantifies (roughly) the convergence rate as \( O(\lambda^{-1/2}) \). To obtain better estimates for the error of order \( 1/\sqrt{\lambda} \), one can derive asymptotic expansions. There are various general theorems that yield asymptotic expansions for \( \mathbb{P}(A_{\lambda} \leq s) \) in ascending positive powers of \( \lambda^{-1/2} \), see, e.g. [14, 17, 41, 66, 79, 121]. One example would be the Edgeworth expansion, which for the Poisson distribution yields, see [14, Eq. (4.18)],

\[
\mathbb{P}(\text{Pois}(\lambda) \leq s_\lambda) = \Phi(\beta) - \frac{\varphi(\beta)(\beta^2 - 1)}{6\sqrt{\lambda}} + O(1/\lambda).
\]

The technical challenge in determining convergence rates is that we need to establish an asymptotic expansion rather than just the limit theorem. We shall demonstrate this for the \( M/M/s_\lambda \) queue using the asymptotic evaluation of integrals through the Laplace method. The formula \( C(s_\lambda, \lambda) \) in its basic form is only defined for integer values of \( s_\lambda \). An extension of this formula that is well defined for all real \( s_\lambda > \lambda \) is given by (see for example Jagers & Van Doorn (1986))

\[
C(s_\lambda, \lambda)^{-1} = \lambda \int_0^\infty t e^{-\lambda t} (1 + t)^{s_\lambda - 1} dt.
\]

We introduce the following key parameters:

\[
\alpha = \sqrt{-2s_\lambda(1 - \rho_\lambda + \ln \rho_l)}, \quad \beta = (s_\lambda - \lambda)/\sqrt{\lambda}, \quad \gamma = (s_\lambda - \lambda)/\sqrt{s_\lambda} = (1 - \rho_\lambda)/\sqrt{s_\lambda} = \beta \sqrt{\rho_\lambda}.
\]

It has been shown in [75] that \( \alpha < \beta \). By expanding \( \frac{1}{2} \alpha^2 \) in powers of \( 1 - \rho_\lambda \), it easily follows that \( \gamma < \alpha \), so we have \( \gamma < \alpha < \beta \).

**Theorem 5.1** For \( s > \lambda \),

\[
C(s_\lambda, \lambda) \leq \left[ \rho_l + \gamma \left( \frac{\Phi(\alpha)}{\varphi(\alpha)} + \frac{2}{3} \frac{1}{\sqrt{s_\lambda}} \right) \right]^{-1},
\]

and

\[
C(s_\lambda, \lambda) \geq \left[ \rho_\lambda + \gamma \left( \frac{\Phi(\alpha)}{\varphi(\alpha)} + \frac{2}{3} \frac{1}{\sqrt{s_\lambda}} + \frac{1}{\varphi(\alpha)} \frac{1}{12s_\lambda - 1} \right) \right]^{-1}.
\]
Notice that the structure of the bounds (5.7) and (5.8) is quite similar to the Halfin-Whitt approximation $C(s, \lambda) \approx g(\beta)$. Indeed, using $s_\lambda = \lambda + \beta \sqrt{\lambda}$ with $\beta$ fixed and letting $\lambda \to \infty$, one can see that $\alpha$ and $\gamma$ both converge to $\beta$. With the above theorem at hand, convergence of $C(s, \lambda)$ towards the Halfin-Whitt function $g(\beta)$ is follows, which provides an alternative proof and confirmation of Proposition 2.1. More importantly, the bounds (5.7)–(5.8) are sharp in the QED regime for small and moderate-size systems. The difference between the lower and upper bound is only $O(1/s_\lambda)$ in (5.2). In Table 1, we keep $\beta = 1$ fixed and vary $s_\lambda$. The load $\lambda$ is chosen such that $s_\lambda = \lambda + \beta \sqrt{\lambda}$. The quality of the bounds is apparent, even for small systems, and certainly compared to the asymptotic approximation $g(1) = 0.22336$.

\begin{table}[h]
\centering
\begin{tabular}{cccccccc}
\hline
$s_\lambda$ & $\lambda$ & $\alpha$ & (5.8) & $C(s, \lambda)$ & (5.7) & $\frac{(5.7)-\lambdatd}{C(s, \lambda)}$ & (5.9) & $\frac{(5.9)-C(s, \lambda)}{C(s, \lambda)}$ \\
\hline
1 & 0.382 & 0.830 & 0.36571 & 0.38197 & 0.39437 & 7.504 \cdot 10^{-2} & 0.45085 & 1.803 \cdot 10^{-1} \\
2 & 1.000 & 0.879 & 0.32678 & 0.33333 & 0.33936 & 3.772 \cdot 10^{-2} & 0.36935 & 0.918 \cdot 10^{-2} \\
5 & 3.209 & 0.924 & 0.28886 & 0.29097 & 0.29328 & 1.518 \cdot 10^{-2} & 0.30185 & 3.739 \cdot 10^{-2} \\
10 & 7.298 & 0.946 & 0.26937 & 0.27030 & 0.27142 & 7.616 \cdot 10^{-3} & 0.27540 & 1.886 \cdot 10^{-2} \\
20 & 16.000 & 0.962 & 0.25665 & 0.25668 & 0.25663 & 3.818 \cdot 10^{-3} & 0.25851 & 9.495 \cdot 10^{-3} \\
50 & 43.411 & 0.976 & 0.24361 & 0.24377 & 0.24398 & 1.531 \cdot 10^{-3} & 0.24470 & 3.820 \cdot 10^{-3} \\
100 & 90.488 & 0.983 & 0.23761 & 0.23769 & 0.23779 & 7.665 \cdot 10^{-4} & 0.23814 & 1.916 \cdot 10^{-4} \\
200 & 186.349 & 0.988 & 0.23340 & 0.23344 & 0.23349 & 3.836 \cdot 10^{-4} & 0.23366 & 9.602 \cdot 10^{-4} \\
500 & 478.134 & 0.993 & 0.22969 & 0.22970 & 0.22972 & 1.536 \cdot 10^{-4} & 0.22979 & 3.848 \cdot 10^{-4} \\
1000 & 968.873 & 0.995 & 0.22783 & 0.22783 & 0.22784 & 7.683 \cdot 10^{-5} & 0.22788 & 1.926 \cdot 10^{-4} \\
\hline
\end{tabular}
\caption{Results for the bounds on $C(s, \lambda)$ for $\beta = 1$.}
\end{table}

We by now know that $C(s, \lambda) \to g(\beta)$ and D’Auria and Janssen et al. proved that $C(s, \lambda) \geq g(\beta)$ for all $\lambda, \beta > 0$. Using the bounds in (5.7) and (5.8), it was shown by Janssen et al. that as $\lambda \to \infty$,

$$C(s, \lambda) \approx g(\beta) + g^*(\beta) \frac{\beta}{\sqrt{\lambda}},$$

(5.9)

with

$$g^*(\beta) = g(\beta)^2 \left[ \frac{1}{3} + \frac{\beta^2}{6} + \frac{\Phi(\beta)}{\Phi(\beta)} \left( \frac{\beta}{2} + \frac{\beta^3}{6} \right) \right].$$

(5.10)

This result can be interpreted as the counterpart of (5.2), but then not for the Poisson distribution in the CLT regime, but for the delay probability in the QED regime. In Table 1 we see that (5.9) leads to much sharp approximations than the original asymptotic approximation $g(1) = 0.22336$.

### 5.2 Optimality gaps

Given these refinements to the asymptotic delay probability, we revisit the cost minimization problem discussed in Section 4, and ask ourselves what can be said about the associated optimality gaps when dimensioning principles based on the asymptotic approximations are used.

Recall that under linear cost structure, we aim to find the minimizing value $s_\lambda^* = K(s, \lambda)$ as in (4.1) (we omit the argument $r$ in this section for brevity). Since $K(s_\lambda, \lambda) \to K_\beta(\beta)$ as $\lambda \to \infty$, we alternatively considered asymptotic minimizer $s_\lambda^{\text{QED}} = [\lambda + \beta^* \sqrt{\lambda}]$ with $\beta^*$ minimizing $K_\beta(\beta)$, and Figure 8 illustrated the accuracy of this asymptotic dimensioning scheme.
of systems of various sizes. Indeed, Borst et al. [20] showed that $s_{\lambda}^{\text{QED}}$ is asymptotically optimal in the sense that

$$K(s_{\lambda}^{\text{QED}}, \lambda) = K(s_{\lambda}^*, \lambda) + o(\sqrt{\lambda}).$$  (5.11)

The corrected approximation for the delay probability in (5.9), however, provides a means to improve the accuracy of $s_{\lambda}^{\text{QED}}$. Namely, by substituting (5.9) into (4.4), it is clear that we can write

$$\frac{K(s_{\lambda}, \lambda)}{\sqrt{\lambda}} \approx K_*(\beta) + \frac{g_*(\beta)}{\sqrt{\lambda}} =: K_*(\beta),$$  (5.12)

with an error that is of order $O(1/\lambda)$ for uniformly bounded $\beta$ and $g_*(\beta) := g_*(\beta)/\beta$. If we consider the approximated cost function $K_*(\beta)$ in (5.12), and let $\beta_1^*$ be the associated minimizer, then we expect the refined square-root rule $s_{\lambda}^* := [\lambda + \beta_1^* \sqrt{\lambda}]$ to give a better approximation to the true optimizer $s_{\lambda}^*$. It is shown in Janssen et al. [74], by invoking Taylor’s theorem, that $\beta_1^* = \beta_1^* + \beta_1^*/\sqrt{\lambda} + O(1/\lambda)$ with

$$\beta_* = -\frac{\beta^* g_*(\beta^*)}{K_*(\beta^*) + 2r}.$$  (5.13)

The resulting refined square-root rule $s_{\lambda}^* = [s_{\lambda}^{\text{QED}} + \beta_*]$ indeed yields an improvement over the original square-root rule in terms of the optimality gap. Namely, see [74, Thm. 2],

$$K(s_{\lambda}^*, \lambda) = K(s_{\lambda}^*, \lambda) + O(1/\sqrt{\lambda}).$$  (5.14)

Observe that the characterization of $s_{\lambda}^*$ as an $O(1)$ correction to the original square-root rule (1.1) provides a rigorous mathematical underpinning for the exceptionally good performance of the QED dimensioning scheme observed in Figure 9.

In the context of $M/M/s + M$ queues, Zhang et al. [157] obtained similar results on optimality gaps. Motivated by the results in [74,157], Randhawa [124] takes a more abstract approach to quantify optimality gaps of asymptotic optimization problems. He shows under generally assumptions that when the approximation to the objective function is accurate up to $O(1)$, the prescriptions that are derived from this approximation are $o(1)$-optimal. The optimality gap thus asymptotically becomes zero. This general setup is shown in [124] to apply to the $M/M/s$ queues in the QED regime, which confirmed and sharpened the results on the optimality gaps in [74,157]. The abstract framework in [124], however, can only be applied if refined approximations as discussed above are available. Optimality gaps in settings with admission control in the QED regime, based upon a trade-off between revenue, costs and service quality, have further been studied by Sanders et al. [131].

### 6 Extensions

By now we should have developed a good understanding for why the mathematical theory that comes with the QED regime for many-server systems ranks among the most celebrated principles in applied probability. The general idea is a follows: a finite server system is modeled as a system in heavy traffic, where the number of servers $s$ is large, whereas at the same time, the system is critically loaded achieved by setting $s = \lambda + \beta \sqrt{\lambda}$ and letting $\lambda \to \infty$ while keeping $\beta$ fixed. The system then reaches the desirable QED limiting regime, which provides a basis for establishing resource pooling and economies of scale, and also for solving asymptotic dimensioning problems that trade off revenue, costs and service quality.
The QED regime rests on defining conditions in which both customers and system operators benefit from the advantages that come with systems that operate at large scale. Signs of such conditions are that the system utilization is close to maximal, that the delay probability is low, and that the mean delay asymptotically becomes negligible. The QED regime then refers to the mathematically defined conditions under which this desirable behavior can be realized, and characterizes rigorously this behavior by establishing stochastic-process limits. The QED regime also creates a natural environment for solving dimensioning problems that are formulated so as to achieve an acceptable trade-off between service quality and capacity. The notion of quality is usually formulated in terms of some targeted service level. Take for instance the probability that an arriving customer experiences delay. The targeted service level could be to keep the delay probability below some value $\epsilon \in (0, 1)$. The smaller $\epsilon$, the better the offered level of service. Once the targeted service level is set, the objective from the system’s perspective is to determine the highest load $\lambda$ such that the target $\epsilon$ is still met.

6.1 Abandonments

So far, we have surveyed standard systems in which all arriving jobs join the queue and stay until eventually being processed by one of the servers. Additional features could be added, such as multi-class job types \[62, 12, 56, 59, 139, 105, 11, 10\], heterogeneous servers \[4, 7, 109, 136\] and congestion control mechanisms \[130, 92, 9, 10, 11, 19, 59, 72\]. These models are all interesting in their own respect and are well-understood.

One model extension that is featured predominantly in the literature is abandonment caused by customer impatience, in which case customers leave the system without being served \[47, 26, 119\]. The canonical model for abandonments is the $M/M/s + M$ or Erlang A model \[119, 48\], with similar dynamics as the $M/M/s$ queue, with the additional feature that each job is assigned an i.i.d. patience time, which is exponentially distributed with mean $1/\theta$. If a job’s patience time expires before reaching an available server, the job leaves (abandons) the system. As the number of jobs in the Erlang A queue remains a birth-death process, its stationary distribution and associated performance measures are fairly well-understood, also in the QED regime \[48, 156, 158\]. Garnett et al. \[48\] and Zeltyn & Mandelbaum \[156\] showed that in the QED regime, with $s_\lambda = \lambda + \beta \sqrt{\lambda}$ and $\lambda \to \infty$,

$$P(\text{delay}) \to \left(1 + \sqrt{\theta} \frac{k(\beta / \sqrt{\theta})}{k(-\beta)} \right)^{-1},$$

(6.1)

and

$$\sqrt{\lambda} P(\text{abandon}) \to \frac{\sqrt{\theta} k(\beta / \sqrt{\theta}) - \beta}{1 + \sqrt{\theta} k(\beta / \sqrt{\theta})/k(-\beta)},$$

(6.2)

where $k(\beta) = \varphi(\beta)/\Phi(-\beta)$. Hence, the universal QED properties, discussed in Section 3, remain intact when the model includes abandonments. Moreover, the probability that a job abandons vanishes at rate $O(1/\sqrt{\lambda})$ as $\lambda \to \infty$. In \[156\], the stationary QED limits for more generally distributed patience time were derived, for which similar limiting behavior is proved. More surprisingly, it is shown that the limit is insensitive to the patience time distribution as long as its density at 0, i.e. the probability of abandoning immediately upon arrival, is fixed. On the process
level, the appropriately scaled queue length process of the $M/M/s_\lambda + M$ model in the QED regime can be shown to converge to a piecewise-linear Ornstein–Uhlenbeck process with drift terms

$$m(x) = \begin{cases} -\beta - \theta x, & \text{if } x > 0, \\ -\beta - x, & \text{if } x \leq 0 \end{cases}$$

and infinitesimal variance $\sigma^2(x) = 2$, see e.g. [48]. Notice that for $\theta = 0$, we retrieve the Halfin-Whitt diffusion in Theorem 2.2. Under more general assumptions, [105] characterizes the QED limiting process for the $G/GI/s + GI$ queue. More specifically, they find that the QED limit of the $G/M/s + GI$ queue is still a piecewise-linear Ornstein–Uhlenbeck process. The general $G/G/s + G$ queue under various modeling assumptions and its limiting process in the QED regime has been studied in [48, 47, 149, 110, 156, 108, 82, 31, 126, 78, 158]. Surveys on systems with abandonments are Ward [145] and Dai & He [32].

### 6.2 Finite waiting space

We have assumed so far that systems have infinite buffers for storing delayed jobs. Physical resource sharing systems, such as data centers and hospitals, however, are often limited in the number of jobs that can be held in the system simultaneously. Depending on the practical setting and admission policy, if the maximum capacity, say $n$, is reached, newly arriving jobs can either leave the system immediately (blocking), reattempt getting access later (retries) or queue outside the facility (holding). In any case, expectations are that the queueing dynamics within the resource sharing facility are affected considerably in the presence of such additional capacity constraints.

We illustrate these implications through the $M/M/s/n$ queue, that is, the standard $M/M/s$ queue with additional property that a job that finds upon arrival $n$ jobs already present in the system is blocked/lost. To avoid trivialities, let $n \geq s$. Since the mean workload reaching the servers is less than in an finite buffer ($n = \infty$) scenario, one expects less congestion and resource utilization.

Consider the $M/M/s_\lambda/n_\lambda$ in the QED regime. So, let $\lambda$ increase while $s_\lambda$ scales as in (1.1). We then ask how $n_\lambda$ should scale along with $\lambda$ and $s_\lambda$ to maintain the non-degenerate behavior as seen in Section 2.1. We provide a heuristic answer. Let $Q^{(s_\lambda, n_\lambda)}$ and $W^{(s_\lambda, n_\lambda)}$ denote the number of jobs in the system and delay in the $M/M/s_\lambda/n_\lambda$ queue in steady state. If there were no finite-size constraints, then through (2.22)–(2.24), we find as $\lambda \to \infty$

$$\mathbb{P}(Q^{(s_\lambda)} \geq n_\lambda) = \mathbb{P}\left(\frac{Q^{(s_\lambda)} - s_\lambda}{\sqrt{s_\lambda}} \geq \frac{n_\lambda - s_\lambda}{\sqrt{s_\lambda}}\right) \to g(\beta) e^{-\beta \gamma},$$

where $\gamma = \lim_{\lambda \to \infty} (n_\lambda - s_\lambda)/\sqrt{s_\lambda}$. Hence, asymptotically the finite-size effects only play a role if the extra variability hedge of $n_\lambda$ is of order $\sqrt{s_\lambda}$ (or equivalently $o(\sqrt{\lambda})$). Furthermore, if the variability hedge is $o(\sqrt{\lambda})$, then we argue that asymptotically, all jobs that do enter the system have probability of delay equal to zero. More formally, under the two-fold scaling rule

$$\begin{cases} s_\lambda = \lambda + \beta \sqrt{\lambda} + o(\sqrt{\lambda}), \\ n_\lambda = s_\lambda + \gamma \sqrt{s_\lambda} + o(\sqrt{\lambda}), \end{cases}$$

it is not difficult to deduce that, see e.g. [111],

$$\mathbb{P}(\text{delay}) \to \left(1 + \frac{\beta \Phi(\beta)}{(1-e^{-\beta \gamma})\varphi(\beta)}\right)^{-1}, \quad \text{as } \lambda \to \infty,$$
which is strictly smaller than \( g(\beta) \) in (3), but still bounded away from both 0 and 1. Furthermore, the buffer size of the queue is \( n_\lambda - s_\lambda = \gamma \sqrt{s_\lambda} \), so that by Little’s law, the mean delay of an admitted job is \( O(1/\sqrt{s_\lambda}) \). Even though resource utilization in the \( M/M/s_\lambda/n_\lambda \) is less efficient than in the queue with unlimited waiting space, it can be shown that \( \rho_\lambda \to 1 \) as \( \lambda \to \infty \). Hence, all three key characteristics of the QED regime are carried over to the finite-size setting if one uses (6.5).

On a process level, adding a capacity constraint translates to adding a reflection barrier to the normalized queue length process \( \tilde{Q}^{(s_\lambda, n_\lambda)}(t) / \sqrt{s_\lambda} \), at \( \gamma \), as is illustrated by the sample paths of \( \tilde{Q}^{(s_\lambda, n_\lambda)} \) for three values of \( \lambda \) in Figure 10.

Indeed, non-degenerate limiting behavior can be expected when the additional space \( \gamma \sqrt{s_\lambda} \) is of the same order as the natural fluctuations of the arrival process; see [111]. The idea of the two-fold scaling in (6.5) can be extended to networks of queues, rather than the single-station setting discussed here; see [88, 154, 138] for examples of such semi-open queueing networks.

### 6.3 Endogenous arrival rates

In several applications users have the option to join a certain congestion dependent service or not, leading to a game theoretic setting where the provider of a service maximizes profit, and users decide to join a service depending on their utility, possibly involving the mean delay. If the market size is large, the QED capacity allocation rule can emerge endogenously, though it is possible to obtain other scaling rules as well. Examples of such studies include [95, 118]. For illustrative purposes, we briefly describe the model and results of Nair et al. [118] in more detail.

A user needs to decide whether or not to use a congestion-dependent service which is free for the user (and supported by advertisements, think of Google or Facebook). If the total user base that uses the service has magnitude \( \lambda \), the user receives a utility \( V(\lambda) \) (this may be increasing with \( \lambda \) in a social network context), and a congestion-dependent dis-utility \( \xi(s, \lambda) \), chosen according to the mean delay in the \( M/M/s \) queue, i.e. \( \xi(s, \lambda) = C(s, \lambda) / (s - \lambda) \) for \( \lambda < s \) and \( \infty \) otherwise. Given the choice of a number of data processing units of the service provider, an infinitesimal user will join if and only if \( V(\lambda) - \xi(s, \lambda) \) is non-negative. The total market size of the user base is equal to \( \Lambda \), which is assumed to be large. For illustrative purposes, we restrict to the case where the entire user population can cooperate and therefore the total arrival rate becomes

\[
\hat{\lambda}_\Lambda(s) = \max \left\{ \lambda \in [0, \Lambda] : \max \{ \lambda V(\lambda) - \lambda \xi(s, \lambda) \} \right\}
\]  

(6.7)

The firm optimizes its revenue given this user behavior. The cost of each resource is scaled to 1, and the average advertisement revenue per unit of users is set to \( b_1 \). In this case the optimal

![Figure 10: Sample paths of the normalized queue length process \( \tilde{Q}^{(s_\lambda, n_\lambda)}(t) \) with \( \lambda = 50, 100 \) and \( 500 \) under scaling (6.5) with \( \beta = 0.5 \) and \( \gamma = 1 \).](image)
number of services \( k^*(\Lambda) \) becomes
\[
k^*_\Lambda = \max \left\{ \arg \max_{k \geq 0} b_1 \hat{\lambda}_\Lambda(k) - k \right\}.
\] (6.8)

It is possible to determine how \( k^*_\Lambda \) scales with \( \Lambda \). As is shown in Theorem 1 of [118], if \( \alpha = \lim_{\lambda \to \infty} U'(\lambda) \in (0, \infty) \) (which is the case if \( V \) is converging to a constant, corresponding with an online service like Google), then there exists a strictly positive and decreasing function \( \beta \) of \( \alpha \) such that
\[
k^*_\Lambda = \Lambda + p \beta(\alpha) \Lambda (1 + o(1)).
\]

In the case \( V(\lambda) = \lambda^v \) for some \( v > 0 \), then \( \alpha = \infty \) and users are more interested to join a service if other users are present (as is the case in a social network like Facebook). In this case, the firm can give less QoS: the number of spare servers becomes of the order \( p \Lambda \). If users cannot collaborate, the firm only needs two spare servers to maximize its profit: the choice \( k^*_\Lambda = \Lambda + 2 \) makes the entire user population join the network. This is an example of what is called a \textit{tragedy of the commons}. There are many additional opportunities for research in this domain; the recent monograph [63] on the interface of game theory and queueing provides an excellent starting point.

### 6.4 Networks

The models shown so far all are all single-station models. The analysis of networks in the QED regime is more cumbersome. We present two classes of models in the computer-communication domain for which it is possible to derive tractable results, in the sense that a limit process can be derived of which the invariant distribution is computationally tractable. These are (i) loss networks and (ii) bandwidth sharing networks. For work on fork-join networks, we refer to [100, 101, 102].

A \textit{loss network} is an extension of the Erlang B model. Consider a telecommunication network with \( J \) links, and suppose that link \( j, j = 1, \ldots, J \), comprises \( C_j \) circuits (servers). There are \( R \) classes of calls called routes. A call on route \( r \) uses \( A_{jr} \) circuits from link \( j \), where we take \( A_{jr} \) to be either 0 or 1. Calls of route \( r \) arrive according to a Poisson arrival process of rate \( \lambda_r \) and a call is blocked if the appropriate servers are not available. Assuming unit exponential services on each route, it can be shown that the invariant distribution \( \pi(n) \) can be written as a ratio of two Poisson probabilities. Specifically, let \( N \) be an \( R \)-dimensional vector of independent Poisson random variables where the rate of \( N_r \), \( r \in R \), equals \( \nu_r \). Now
\[
\pi(n) = \frac{P(N = n)}{P(AN \leq C)}.
\] (6.9)

Unfortunately the computation of the normalizing constant \( P(AN \leq C) \) is nontrivial. It turns out though, that it is possible to develop a Gaussian approximation using a central limit approach, using an appropriate normalization around the mode of \( N_r \), assuming both \( \lambda \) and \( C \) are large. What is essentially needed is that \( C = An^*(\lambda) + O(\sqrt{An^*(\lambda)}) \) component-wise, with \( n^*(\lambda) \) the mode of \( \pi(n) \) (which can, for large values of \( \lambda \), be characterized by the solution of a convex programming problem). For details on this procedure we refer to [87], which is still a valuable source of information and the more recent [85]. For recent progress on computational procedures we refer to [81, 3].

In a \textit{bandwidth sharing network} with rate constraints, we can keep the same notation. A crucial difference between the two models is that the arrivals of all routes are accepted, but share the load according to a certain function \( \Lambda(n) \), which can be written as the solution of a concave
programming problem of the form

$$\Lambda(n) = \arg \max \sum_r n_r U_r(\Lambda_r/n_r),$$  \hspace{1cm} (6.10)

subject to the network capacity constraints $A\Lambda \leq C$ and the individual user constraints $\Lambda_r \leq d_r/n_r$. Here, the download rate of each user is constrained by $d_r$, and the vector $C$ now comprises the (possibly non-integer) capacities of each link. $U_r(x)$ is the utility for a user of route $r$ if its service rate equals $x$. If the network capacity links (i.e. the elements of $C$) are large w.r.t. the individual user access links $d_r$, one can consider a scaled sequence of systems which is similar to that in the QED regime.

In [127], this is formally developed, and it is shown that, assuming user impatience on all routes, it is possible to derive a tractable diffusion approximation of which the invariant distribution is multivariate normal. The mean vector and covariance matrix of this distribution are characterized by convex, and quadratic programs, respectively.

7 Emerging research directions

7.1 Pre-limit and robust approximations

A downside of heavy-traffic analysis is that the results are of an asymptotic nature, and therefore approximations. Obtaining corrections or refinements is one of the main goals of many research efforts, and the demonstration in Section 5 is only a small part of a richer and active line of research.

In the field of statistics, Siegmund [133] proposed a corrected diffusion approximation for the waiting time in a single-server queue. In heavy traffic, the workload distribution is approximated by an exponential distribution. Siegmund gave a precise estimate of the correction term, nowadays a classical result and textbook material, cf. [8, p. 369]. Siegmund’s first order correction has been extended recently by Blanchet & Glynn [18], who give a full series expansion for the $G/G/1$ waiting time distribution in heavy traffic.

In this survey, we have seen corrected diffusion approximations for the $M/M/s$ queue in Section 5. In addition to the corrected diffusion approximations presented there, a number of other refinements exist in the literature that provide improved (w.r.t. the heavy-traffic limit) approximation of the invariant distribution. One class of such approximations is based on variations of Stein’s method [22, 24]. Another class of approximations is based on the idea to consider the diffusion limit of a Markovian queue, and to replace the drift and diffusion coefficients by terms that depend on the parameters in the prelimit. The goal is not only to improve the convergence rate in the QED regime, but also make the approximations accurate in other scaling regime, hence the term universal approximations. We refer to [57, 55, 65] for a more in-depth discussion, and explain the idea of modifying a diffusion in the context of the Halfin-Whitt diffusion, following an idea of Dai & Braverman [23].

Recall from Theorem 2.2 that the scaled queue length process in the QED regime converges to a diffusion process with infinitesimal drift $m(x) = -\beta - x\mathbb{1}_{x \leq 0}$ and infinitesimal variance $\sigma^2(x) = 2$. $\beta$ can be expressed in terms of the pre-limit characteristics by the expression $\beta = (s-\lambda)/\sqrt{\lambda}$. The idea in [23] is now to replace the diffusion coefficient and consider

$$\sigma^2_\Lambda(x) = 1 + \mathbb{1}_{x \geq \sqrt{\lambda}} \left( 1 - \frac{m(x)}{\sqrt{\lambda}} \right).$$  \hspace{1cm} (7.1)
The resulting approximation for the steady-state density is explicit and it is shown in [23] that the resulting distributional approximation has an error of the order $1/\lambda$, while the Halfin-Whitt approximation has a much larger error of order $1/\sqrt{\lambda}$. Though the associated approximation for the delay probability is worse than the approximations and bounds presented in Section 5 of this paper, the idea of modifying the limiting diffusion appropriately seems to be of high potential.

7.2 Parameter uncertainty & overdispersed arrivals

Models describing resource sharing systems typically assume perfect knowledge on the model primitives, including the mean demand per time period. For large-scale resource sharing systems, the dominant assumption in the literature is that demand arrives according to a non-homogeneous Poisson process, just as in Section 4, which translates to the assumption that arrival rates are known for each basic time period (second, hour or day). In practice, however, estimates for mean demand typically rely on historical data, and are therefore subject to uncertainty. This parameter uncertainty is likely to affect the effectiveness of capacity sizing rules. The number of studies in which resource allocation rules are considered under parameter uncertainty is limited to [106, 80, 58, 15] and more work to understand the quality of the square-root rule is needed.

As an illustration, consider a resource allocation problem with Poisson $\lambda$ arrivals and exponential ($\mu$) servers. Suppose that $\mu = 1$, and $\lambda$ is unknown. For instructive purposes, we make a resource allocation decision $s$ based on the infinite server approximation $P(\text{Pois}(\lambda) > s) \leq \epsilon$. In case $\lambda$ is known and large, the choice $s_\lambda = \lambda + \beta \sqrt{\lambda}$, with $\beta = 1 - \Phi^{-1}(\epsilon)$, would be natural, see (2.1). If $\lambda$ is not known, but needs to be estimated from data, it is instructive to see how the choice of $s$ is affected. Suppose we have an estimator $\hat{\lambda}$ of $\lambda$ which is approximately normally distributed with a standard deviation $\sigma$. When would it be appropriate to simply take $s = \hat{\lambda} + \beta \sigma$? To obtain some insight, we use the approximation $\text{Pois}(\lambda) \sim \lambda + G \sqrt{\lambda}$, and assume $\hat{\lambda} = \lambda + G_0 \sigma$, where $G$ and $G_0$ are independent standard normal variables. Then we see the following: if $\lambda$ is large, we need to pick $s$ such that $P(\hat{\lambda} + \sigma G_0 + G \sqrt{\hat{\lambda}} > s) = \epsilon$, yielding $s = \hat{\lambda} + \beta \sigma_2 + \hat{\lambda}$. If $\sigma^2$ is of the order $\hat{\lambda}$, it follows that the naive rule $s = \hat{\lambda} + \beta \sqrt{\hat{\lambda}}$ lead to poor system performance.

Another difficulty arises when fluctuations in demand are larger than anticipated by the Poisson assumption. Although natural and convenient from a mathematical viewpoint, the Poisson assumption often fails to be confirmed in practice. A deterministic arrival rate implies that the number of jobs entering the system over a period of time is a Poisson random variable, whose variance equals its expectation. A growing number of empirical studies of service systems shows that the variance of demand typically exceeds the mean significantly, see [13, 15, 26, 90, 47, 58, 89, 106, 114, 128, 135, 155]. The feature that variability is higher than one expects from the Poisson assumption is referred to as overdispersion.

Due to its inherent connection with the CLT, the square-root rule relies heavily on the premise that the variance of the number of jobs entering the system over a period of time is of the same order as the mean. Subsequently, when stochastic models do not take into account overdispersion, resulting performance estimates are likely to be overoptimistic. The system then ends up being underprovisioned, which possibly causes severe performance problems, particularly in critical loading. To deal with overdispersion, existing capacity sizing rules like the square-root rule need to be modified in order to incorporate a correct hedge against (increased) variability. Following our findings in Section 3 the following adapted capacity allocation rule may be proposed

$$s = \mu_A + \beta \sigma_A,$$ (7.2)
where \( \mu_A \) and \( \sigma_A \) are the mean and standard deviation of demand per period, respectively, and \( \beta > 0 \). This is similar to (1.1) in which the original variability hedge is replaced by an amount that is proportional to the square-root of the variance of the arrival process. In [113], it is shown that this rule indeed leads to QED-type behavior in bulk-service queues as the system size grows. For the \( M/M/s \) queue, this has been studied in [106] and [15], but much more work in this area is necessary.

### 7.3 Load balancing

The analysis and design of load balancing policies has attracted strong renewed interest in the last several years, mainly motivated by significant challenges involved in assigning tasks (e.g. file transfers, compute jobs, data base look-ups) to servers in large-scale data centers. Load balancing schemes provide an effective mechanism for improving QoS experienced by users while achieving high resource utilization levels, goals that are perfectly aligned with the QED regime. A distinguishing feature of such systems is that there is no centralized queue.

A first example of a load balancing scheme is round robin scheduling. This is a cyclic service discipline in an \( s \)-server queue under which every \( s \)-th job is assigned to the same server. When service requirements are equal to a constant, this cyclic routing achieves “perfect load balancing” among servers and the delay distribution is the same as that of a single server serving every \( s \)th arrival of a Poisson input, or rather, Erlang input, and the delay distribution can be approximated by a Gaussian random walk, and all three structural properties are still justified. If deterministic job sizes are being replaced with general job sizes, the system will still operate in heavy traffic, and the probability of delay converges to a value in the interval \((0, 1)\), but the mean delay will no longer be of the order \( O(1/\sqrt{s}) \) but constant, so that the third structural QED property no longer holds.

A second example concerns Join-the-Shortest-Queue (JSQ) routing and several of its variations. Many questions, such as versions where the shortest of \( d = d(s) \) randomly chosen queues have been investigated in the literature under various assumptions. [115] [144] [21] [103] [104] [50] [25] [39] [46]. In recent years several new results were discovered for JSQ(\( d(s) \)) parallel systems that operate the QED regime \((s - \lambda(s))/\sqrt{s} \to \beta > 0 \) as \( s \to \infty \). In order to describe these results, we let for any \( d(s) \) \((1 \leq d(s) \leq s)\), \( Q_i^{d(s)}(t) := (Q_1^{d(s)}(t), Q_2^{d(s)}(t), \ldots, Q_b^{d(s)}(t)) \) denote the system occupancy state, where \( Q_i^{d(s)}(t) \) is the number of servers under the JSQ(\( d(s) \)) scheme with a queue length of \( i \) or larger, at time \( t \), including the possible task in service, \( i = 1, \ldots, b \). Here \( b \) is the maximal buffer size of each queue.

Eschenfeldt and Gamarnik [38] first considered the JSQ scheme, so JSQ(\( d(s) \)) with \( d(s) = s \) and introduced a properly centered and scaled version of the system occupancy state \( Q_i^{d(s)}(t) \) needs to be centered around \( s \) while \( Q_i^{d(s)}(t) \), \( i = 2, \ldots, b \), are not, as the fraction of servers with a queue length of exactly one tends to one, and the fraction of servers with a queue length of two or more tends to zero as \( \lambda \to \infty \). For \( d(s) = s \), the sequence of processes \((Q_i^{d(s)}(t))_{t \geq 0} \) converges weakly to a system of \( b \) coupled stochastic differential equations.

Although the resulting process differs from the diffusion limit obtained for the fully pooled \( M/M/s \) queue, In particular, both the number of idle servers and the number of queues with exactly one job are of the order \( O(\sqrt{s}) \). This implies a total delay of \( O(\sqrt{s}) \) per time unit during which \( O(s) \) jobs. Hence, the delay per job is \( O(1/\sqrt{s}) \), as in the usual \( M/M/s \) queue in QED. The cause for this vanishing delay is however different. Any arriving job waits with probability \( O(1/\sqrt{s}) \) (which is \( O(1) \) in the \( M/M/s \) queue) and then is delayed an exponential time with rate
1 (which is rate $s$ in the $M/M/s$ queue).

It was recently shown [117, 116] that for $d(s)$ such that $d(s)/(\sqrt{s}\log(s)) \to \infty$ as $s \to \infty$ the diffusion limit of $JSQ(d(s))$ corresponds to that for the JSQ policy. This indicates that the overhead of the JSQ policy can ‘almost’ be reduced to $O(\sqrt{s}\log s)$ while retaining diffusion-level optimality. Similar results have been shown for the load balancing algorithm. In particular, $JIQ$ and $JSQ(d(s))$ with $d(s)$ growing at least as fast as $\sqrt{s}\log s$ have the same diffusion limit. Many exciting problems in this area, which is still in its infancy, are still open.

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