ON A FAMILY OF QUIVERS RELATED TO THE GIBBONS-HERMSEN SYSTEM

ALBERTO TACCHIELLA

Abstract. We introduce a family of quivers $Z_r$ (labeled by a natural number $r \geq 1$) and study the non-commutative symplectic geometry of the corresponding doubles $Q_r$. We show that the group of non-commutative symplectomorphisms of the path algebra $CQ_r$ contains two copies of the group $GL_n$ over a ring of polynomials in one indeterminate, and that a particular subgroup $P_r$ (which contains both of these copies) acts on the completion $C_{n,r}$ of the phase space of the $n$-particles, rank $r$ Gibbons-Hermsen integrable system and connects each pair of points belonging to a certain dense open subset of $C_{n,r}$. This generalizes some known results for the cases $r = 1$ and $r = 2$.

1. Introduction

1.1. Gibbons-Hermsen manifolds. For every $n, r \in \mathbb{N}$ let us denote by $\text{Mat}_{n,r}(\mathbb{C})$ the complex vector space of $n \times r$ matrices with entries in $\mathbb{C}$. We consider the space

$$V_{n,r} := \text{Mat}_{n,n}(\mathbb{C}) \oplus \text{Mat}_{r,n}(\mathbb{C}) \oplus \text{Mat}_{n,r}(\mathbb{C}) \oplus \text{Mat}_{r,n}(\mathbb{C}).$$

Using the identification between $\text{Mat}_{n,r}(\mathbb{C})$ and the dual of $\text{Mat}_{r,n}(\mathbb{C})$ provided by the bilinear form

$$\langle A, B \rangle \mapsto \text{Tr} AB$$

we can view $V_{n,r}$ as the cotangent bundle of the vector space $\text{Mat}_{n,n}(\mathbb{C}) \oplus \text{Mat}_{r,n}(\mathbb{C})$. In other words, denoting by $(X,Y,v,w)$ a point in $V_{n,r}$ we think of the pair $(Y,v)$ as a cotangent vector applied at the point $(X,w)$. It follows that on $V_{n,r}$ a canonical (holomorphic) symplectic form is defined:

$$\omega(X,Y,v,w) = \sum_{i,j=1}^{n} dy_{ji} \wedge dx_{ij} + \sum_{i=1}^{n} \sum_{\alpha=1}^{r} dv_{\alpha i} \wedge dw_{\alpha i}.$$

The group $GL_n(\mathbb{C})$ acts on $V_{n,r}$ by

$$g.(X,Y,v,w) = (gXg^{-1}, gYg^{-1}, gv, wg^{-1}).$$

This action is Hamiltonian, and the corresponding moment map $\mu: V_{n,r} \to \mathfrak{gl}_n(\mathbb{C})$ is

$$\mu(X,Y,v,w) = [X,Y] - vw.$$

For every $\tau \in \mathbb{C}^*$ the action of $GL_n(\mathbb{C})$ on $\mu^{-1}(\tau I)$ is free; the corresponding Marsden-Weinstein quotient,

$$C_{n,r} := \mu^{-1}(\tau I)/GL_n(\mathbb{C}),$$

is a smooth symplectic manifold of dimension $2nr$ that we call the $(n$-particle, rank $r$) Gibbons-Hermsen manifold. Different choices of $\tau$ yield isomorphic symplectic manifolds; from now on we simply suppose that some choice has been made and stick to it for the rest of the paper.

The manifold $C_{n,r}$ can be seen naturally as a completion of the phase space of the $n$-particle, rank $r$ integrable system introduced by Gibbons and Hermsen in [GHS4] as a generalization of the well-known rational Calogero-Moser model. In more detail, let us denote by $C'_{n,r}$ the subset of
\[ \mathcal{C}_{n,r} \] consisting of the orbits of those quadruples in which the matrix \( X \) has \( n \) distinct eigenvalues, to be interpreted as the positions of the \( n \) particles on the complex plane. In each such orbit we can find a point \((X, Y, v, w)\) such that \( X \) is diagonal, say \( X = \text{diag}(x_1, \ldots, x_n) \), and

\[
Y = \begin{pmatrix}
y_1 & \frac{v_{12}w_{21}}{x_1 - x_2} & \cdots & \frac{v_{1n}w_{n1}}{x_1 - x_n} \\
\frac{v_{21}w_{12}}{x_2 - x_1} & y_2 & \cdots & \\
\vdots & \ddots & \ddots & \\
\frac{v_{n1}w_{1n}}{x_n - x_1} & \cdots & \cdots & y_n
\end{pmatrix}
\]

where \( v_{i*} \) denotes the \( r \)-components row vector given by the \( i \)-th row of the matrix \( v \) and similarly \( w_{*i} \), denotes the \( r \)-components column vector given by the \( j \)-th column of the matrix \( w \); the diagonal entries \((y_1, \ldots, y_n) \in \mathbb{C}^n \) are free. If we fix the ordering of the eigenvalues of \( X \), this representative is unique up to the action of a diagonal matrix \( \text{diag}(\lambda_1, \ldots, \lambda_n) \in \text{GL}_n(\mathbb{C}) \); such a matrix fixes the parameters \((x_1, \ldots, x_n) \) and \((y_1, \ldots, y_n) \) and acts as follows on the vectors \( v_{i*} \) and \( w_{*i} \):

\[
\begin{align*}
v_{i*} &\mapsto \lambda_i v_{i*} \\
w_{*i} &\mapsto w_{*i} \lambda_i^{-1}.
\end{align*}
\]

The restriction of the symplectic form \((7)\) on \( \mathcal{C}_{n,r} \) reads

\[
\omega_{\mathcal{C}_{n,r}} = \sum_{i=1}^{n} (dy_i \wedge dx_i + \sum_{\alpha=1}^{r} dv_{i\alpha} \wedge dw_{i\alpha})
\]

and this shows that the coordinates

\[
(x_1, \ldots, x_n, w_{11}, \ldots, w_{rn}, y_1, \ldots, y_n, v_{11}, \ldots, v_{nr})
\]

are canonical. We conclude that \( \mathcal{C}_{n,r} \) can be interpreted as the phase space of a system of \( n \) point particles of equal mass located at the points \( x_i \) with momenta \( y_i \), each particle having some internal degrees of freedom parametrized by a \( r \)-component covector-vector pair \((v_{i*}, w_{*i})\) living in the complex manifold of dimension \( 2r - 2 \) defined by

\[
\mathcal{V}_r := \{(\xi, \eta) \in \text{Mat}_{1,r}(\mathbb{C}) \times \text{Mat}_{r,1}(\mathbb{C}) \mid \xi \eta = -\tau \}/\mathbb{C}^*,
\]

where the action of \( \lambda \in \mathbb{C}^* \) is given by \( \lambda(\xi, \eta) = (\lambda \xi, \eta \lambda^{-1}) \), as in \((7)\). The condition \( v_{i*}w_{*i} = -\tau \) comes from the diagonal part of the moment map equation \([X, Y] - vw = \tau I\); when \( r = 1 \) this completely fixes the internal degrees of freedom, and one recovers the usual description for the (complexified) rational Calogero-Moser system \([\text{Wil98}]\).

The Gibbons-Hermsen hierarchy is defined by the family of \( \text{GL}_n(\mathbb{C}) \)-invariant Hamiltonians

\[
J_{k,m} := \text{Tr} Y^k v m w
\]

with \( k \in \mathbb{N}, m \in \text{Mat}_{r,r}(\mathbb{C}) \). The equations of motion induced by \( J_{k,m} \) are

\[
\begin{align*}
\dot{X} &= \sum_{i=1}^{k} Y^{k-i} v m w Y^{i-1} \\
\dot{Y} &= 0 \\
\dot{v} &= Y^k v m \\
\dot{w} &= -m w Y^k.
\end{align*}
\]

In particular \( J_{2,1} \) is (up to scalar factors) the Hamiltonian considered by Gibbons and Hermsen in \([\text{GHS84}]\).

It is not difficult to see that the flows determined by the evolution equations \((9)\) are complete on \( \mathcal{C}_{n,r} \); this is the reason for calling this manifold a completion of the phase space \( \mathcal{C}_{n,r} \). As in the
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Calogero-Moser case, these extended flows correspond to motions which have been “analytically continued” through the collisions.

1.2. \( C_{n,r} \) as a quiver variety. In order to investigate the geometry of the manifolds \( C_{n,r} \) it is useful to consider them as particular examples of quiver varieties, a notion introduced by Nakajima in [Nak94].

For the reader’s convenience let us recall that a quiver is simply a directed graph, possibly with loops and multiple edges. We will only consider quivers with a finite vertex set, say \( I = \{1, \ldots, n\} \).

A (complex) representation of such a quiver \( Q \) is specified by giving for each vertex \( i \in I \) a finite-dimensional complex vector space \( V_i \) and for each edge \( \xi: i \to j \) a linear map \( V_\xi: V_i \to V_j \). If we denote by \( d_i \) the dimension of the vector space \( V_i \), the vector \( d = (d_1, \ldots, d_n) \in \mathbb{N}^n \) is called the dimension vector of the representation \( V \). Clearly, representations of \( Q \) with dimension vector \( d \) correspond to points in the complex vector space \( \text{Rep}(Q, d) := \bigoplus_{i,j \in I} \bigoplus_{\xi: i \to j} \text{Mat}_{d_j, d_i}(\mathbb{C}) \).

On this space there is an action of the group \( \text{GL}_d(\mathbb{C}) := \prod_{i \in I} \text{GL}_{d_i}(\mathbb{C}) / C^* \), where \( C^* \) is seen as the subgroup of \( n \)-tuples of the form \( (\lambda d_1, \ldots, \lambda d_n) \) for each \( \lambda \in \mathbb{C}^* \). This action is defined as follows: for every arrow \( \xi: i \to j \) the action of \( \prod_{i \in I} \text{GL}_{d_i}(\mathbb{C}) \) sends the matrix \( V_\xi \) to the matrix \( g_j V_\xi g_i^{-1} \). In other words, each factor \( \text{GL}_{d_i}(\mathbb{C}) \) acts on the \( d_i \)-dimensional space \( V_i \) by change of basis.

The quotient
\begin{equation}
\text{Rep}(Q, d) / \text{GL}_d(\mathbb{C})
\end{equation}
parametrizes isomorphism classes of representations of the quiver \( Q \) with dimension vector \( d \). Unfortunately, as a topological space it is usually quite pathological (e.g. not Hausdorff).

To improve the situation, one replaces the quiver \( Q \) with its double \( \overline{Q} \), obtained by keeping the same vertices and adding for each arrow \( \xi: i \to j \) a corresponding arrow \( \xi^*: j \to i \) going in the opposite direction. Using the isomorphisms provided by the bilinear form \( (2) \) we can identify the representation space \( \text{Rep}(\overline{Q}, d) \) with the cotangent bundle to \( \text{Rep}(Q, d) \); then \( \text{Rep}(\overline{Q}, d) \) is naturally a symplectic vector space. Moreover, the action of the group \( \prod_{i \in I} \text{GL}_{d_i}(\mathbb{C}) \) on this space coincides with the cotangent lift of its action on the base. It follows that this action is Hamiltonian and admits an equivariant moment map
\begin{equation}
\mu: \text{Rep}(\overline{Q}, d) \to \mathfrak{gl}_d(\mathbb{C}),
\end{equation}
where \( \mathfrak{gl}_d(\mathbb{C}) \) is the Lie algebra of \( \text{GL}_d(\mathbb{C}) \) (which we identify with its dual using once again the bilinear form \( (2) \)). The idea is now to replace the badly-behaved quotient \( (11) \) with a symplectic quotient
\begin{equation}
M_\nu := \mu^{-1}(\nu) / G_\nu,
\end{equation}
where \( \nu \) is a point in \( \mathfrak{gl}_d(\mathbb{C}) \) and \( G_\nu \) is its stabilizer with respect to the adjoint action of \( \text{GL}_d(\mathbb{C}) \). If the action of \( G_\nu \) on the inverse image \( \mu^{-1}(\nu) \subseteq \text{Rep}(\overline{Q}, d) \) is free and proper then the quotient \( M_\nu \) is itself a smooth manifold, with a symplectic form induced from the one on \( \text{Rep}(\overline{Q}, d) \).

At first sight, the definition of \( C_{n,r} \) above seems to fit precisely this construction by taking as \( Q \) the quiver
\[
\begin{array}{c}
\bullet_1 \\
\rightarrow
\end{array}
\begin{array}{c}
\bullet_2
\end{array}
\]
whose double $\overline{Q}$ is
\begin{align*}
\begin{tikzpicture}
  \node (1) at (0,0) {$\bullet$};
  \node (2) at (1,0) {$\bullet$};
  \draw (1) to [out=up, in=down] (2);
\end{tikzpicture}
\end{align*}
and considering representations with dimension vector $(n, r)$; then $\text{Rep}(\overline{Q}, (n, r)) = V_{n,r}$. However, the group (10) reads in this case
\[ \text{GL}_{(n,r)}(\mathbb{C}) = (\text{GL}_n(\mathbb{C}) \times \text{GL}_r(\mathbb{C}))/\mathbb{C}^*, \]
hence it differs from $\text{GL}_n(\mathbb{C})$ as long as $r > 1$. To remedy this problem, one should take instead a family of quivers $(Q_r)_{r \geq 1}$ such that the corresponding doubles, call them $Q_r$, have $r$ distinct arrows $1 \to 2$ and $r$ distinct arrows $2 \to 1$:
\begin{align*}
Q_r = \begin{tikzpicture}
  \node (1) at (0,0) {$\bullet$};
  \node (2) at (1,0) {$\bullet$};
  \draw (1) to [out=up, in=down] (2);
\end{tikzpicture}
\end{align*}
and consider their representations with dimension vector $(n, 1)$. Then the space $\text{Rep}(Q_r, (n, 1))$ can again be identified with $V_{n,r}$ by “slicing” the matrices $v$ and $w$ in the $r$ column matrices $(v_{*,1}, \ldots, v_{*,r})$ and in the $r$ row matrices $(w_{1,*}, \ldots, w_{r,*})$, respectively.

1.3. Non-commutative symplectic geometry. One of the advantages in adopting this new point of view is that we can now apply to the manifolds $C_{n,r}$ the powerful tools of non-commutative symplectic geometry. This theory was introduced by Kontsevich in [Kon93] and developed, in particular with applications to quiver varieties, by Ginzburg in [Gin01] and Bocklandt-Le Bruyn in [BLB02].

The basic idea of (algebraic) non-commutative geometry is to generalize the well-known duality between commutative rings and affine schemes to the much larger class of associative, but not necessarily commutative, algebras. In Section 2 below we will briefly review the fundamentals of this approach, and in particular how one can define, starting from an associative algebra $A$, a complex $\text{DR}^\bullet(A)$ of “non-commutative differential forms” on $A$.

The link with quiver varieties comes from the observation that to every quiver $Q$ one can associate in a natural manner an associative algebra $CQ$, the path algebra (over $\mathbb{C}$) of $Q$. This is the complex associative algebra which is generated, as a linear space, by all the (oriented) paths in $Q$ and whose product is given by composition of paths, or zero when two paths do not compose (meaning that the ending point of the first does not coincide with the starting point of the second).

Suppose now that $Q$ is actually the double of some other quiver. Then, as it will be explained in Section 2 on the corresponding path algebra $CQ$ there is a canonical non-commutative 2-form $\omega \in \text{DR}^2(CQ)$ that plays the same rôle of a symplectic form in the usual (i.e., commutative) symplectic geometry. Moreover, the group of “non-commutative symplectomorphisms” of $CQ$, meaning algebra automorphisms of $CQ$ that preserve (in a suitable sense) the above-mentioned symplectic form, naturally acts on every quiver variety derived from $Q$.

Thus we can hope to obtain further information about the family of manifolds $C_{n,r}$ by studying how the group of non-commutative symplectomorphisms of $Q_r$ acts on them. For the case $r = 2$ this was done in the second part of the paper [BP11], where it is proved in particular that this action is transitive. However, in order to endow $Q_r$ with a non-commutative symplectic form one must choose for each $r \geq 1$ a quiver $Q_r$ such that its double coincides with $Q_r$. In other words, one has to decide which of the arrows in (13) are the “unstarred” ones. As pointed out in [BP11], this choice is not irrelevant: different choices will give (in general) different symplectomorphism groups. In Section 4 we define a family of “zigzag” quivers $(Z_r)_{r \geq 1}$ by continuing the pattern...
started in [BP11] for $r = 2$:

\[
\begin{align*}
Z_1 &= a \xymatrix{ 1 \ar[r]^{x_1} & 2 } \\
Z_2 &= a \xymatrix{ 1 \ar[r]_{y_1} & 2 } \\
Z_3 &= a \xymatrix{ 1 \ar[r]^{x_2} & 2 } \\
Z_4 &= a \xymatrix{ 1 \ar[r]_{y_2} & 2 } \\
Z_5 &= a \xymatrix{ 1 \ar[r]^{x_3} & 2 } \\
& \ldots
\end{align*}
\]

In the main body of the paper we view the quiver $Q_r$ as the double of $Z_r$. Other possible choices are briefly discussed in Appendix [3].

1.4. **Aim and organization of the paper.** The aim of this paper is to study the non-commutative symplectic geometry of the family of quivers $(Q_r)_{r \geq 1}$, with particular regard to its group of symplectomorphisms, and to extend some of the results obtained in [BP11, Part 2] and [MT13] for the case $r = 2$ to higher values of $r$, hopefully clarifying their origin in the process.

We start in Section 2 by recalling some basic constructions in non-commutative symplectic geometry that will be used in the sequel.

In Section 3 we study the properties of the path algebra of the quiver $Q_r$ that do not depend on the particular non-commutative symplectic structure chosen.

In Section 4 we consider $Q_r$ as the double of the zigzag quiver $Z_r$ and characterize the corresponding group of (tame) symplectomorphisms $\mathrm{TAut}(\mathbb{C}Q_r; c_r)$.

In Section 5 we define a subgroup $P_r \subset \mathrm{TAut}(\mathbb{C}Q_r; c_r)$ which is the higher-rank version of the group $P = P_2$ considered in [MT13]. As a generalization (and strengthening) of one of the results proved there, we show that this group contains two copies of the classical group $\mathrm{GL}_r$ over the complex polynomial ring in one indeterminate.

In Section 6 we show that one can use the action of $P_r$ on the manifolds $\mathcal{C}_{n,r}$ to connect each pair of points in the open subset

\[
\mathcal{R}_{n,r} := \mathcal{C}'_{n,r} \cup \mathcal{C}''_{n,r}
\]

where $\mathcal{C}''_{n,r}$ is defined, by analogy with $\mathcal{C}'_{n,r}$, as the subset of $\mathcal{C}_{n,r}$ consisting of the orbits of those quadruples in which the matrix $Y$ has $n$ distinct eigenvalues.

Finally, in Section 7 we specialize our machinery to the cases $r \leq 3$, making contact with the previously known results for $r = 1, 2$.

2. **Non-commutative symplectic geometry**

Let us quickly summarize some definitions and results in non-commutative symplectic geometry that will be used in the rest of the paper. The interested reader should look at Ginzburg’s lectures [Gin05] for a broad treatment of the subject.

Let $A$ denote an associative, not necessarily commutative, algebra over the commutative ring $B$. Given an $A$-bimodule $M$, a *derivation* from $A$ to $M$ is a $A$-bimodule map $\theta : A \to M$ such that

\[
\theta(ab) = \theta(a)b + a\theta(b) \quad \text{for all } a, b \in A.
\]

We say that $\theta$ is a derivation relative to $B$ if $\theta(b) = 0$ for every $b \in B$, and write $\mathrm{Der}_B(A, M)$ for the linear space of such derivations. (We will usually write $\mathrm{Der}_B(A)$ instead of $\mathrm{Der}_B(A, A)$.)

The functor

\[
\mathrm{Der}_B(A, -) : A\text{-Bimod} \to \text{Vec}
\]

...
is representable; we denote its representing object by $\Omega_B^1(A)$ and call it the $A$-bimodule of Kähler differentials of $A$ relative to $B$. It comes naturally equipped with a derivation $d: \Omega_B^1(A) \rightarrow \Omega_B^1(A)$ relative to $B$. The tensor algebra of this $A$-bimodule,

$$\Omega_B^n(A) := T_A(\Omega_B^1(A)),$$

turns out to be isomorphic to the universal differential envelope of $A$ relative to $B$ (see [Gin05, Theorem 10.7.1]). As such, it is naturally a differential graded algebra, with a differential $d: \Omega_B^n(A) \rightarrow \Omega_B^{n+1}(A)$ extending the derivation $d: \Omega_B^1(A)$ described above. More concretely, if we denote by $\overline{A}$ the quotient of $A$ by its subalgebra $B$ we have the isomorphism

$$\Omega_B^n(A) \simeq A \otimes_B A \otimes B \cdots \otimes B \overline{A}.$$

Multiplication is given by the rule

$$(a_0 \otimes \cdots \otimes a_n)(a_{n+1} \otimes \cdots \otimes a_m) = \sum_{i=0}^{m} (-1)^{a_{i-1}a_i} a_0 \otimes \cdots \otimes a_{i-1}a_i \otimes a_{i+1} \otimes a_{i+2} \otimes \cdots \otimes a_m$$

and the differential acts as

$$d(a_0 \otimes \cdots \otimes a_n) = 1 \otimes a_0 \otimes \cdots \otimes a_n.$$

In the sequel we will denote a generic element $a_0 \otimes a_1 \otimes \cdots \otimes a_n \in \Omega_B^n(A)$ more concisely as $a_0 a_1 \cdots a_n$.

The non-commutative Karoubi-de Rham complex of $A$ relative to $B$, denoted $\text{DR}_B^\bullet(A)$, is the graded vector space over $\mathbb{C}$ whose degree $n$ part is defined by

$$\text{DR}_B^n(A) := \frac{\Omega_B^n(A)}{\sum_{i=0}^{n}[\Omega_B^i(A), \Omega_B^{n-i}(A)]}$$

where $[\Omega_B^i(A), \Omega_B^{n-i}(A)]$ denotes the linear subspace in $\Omega_B^n(A)$ generated by all the graded commutators $[a, b]$ for $a \in \Omega_B^i(A)$, $b \in \Omega_B^{n-i}(A)$. Elements of the linear space $\text{DR}_B^n(A)$ will be called the Karoubi-de Rham $n$-forms of $A$ relative to $B$. In particular for $n = 0$ we have that

$$\text{DR}_B^0(A) = \frac{A}{[A, A]}$$

does not depend on $B$, hence we will usually denote it simply as $\text{DR}_0^0(A)$. Its elements are to be thought of as the “regular functions” on the non-commutative space determined by $A$. We also define

$$\text{DR}_B^0(A) := \text{DR}_B^0(A)/B \simeq \frac{A}{B + [A, A]}$$

in agreement with our previous notation $\overline{A}$ for the quotient $A/B$.

The differential on $\Omega_B^n(A)$ descends to a well-defined map $d: \text{DR}_B^n(A) \rightarrow \text{DR}_B^{n+1}(A)$, making $\text{DR}_B^n(A)$ a differential graded vector space. In fact, one can introduce a whole “Cartan calculus” on $\text{DR}_B^n(A)$, that is for every derivation $\theta \in \text{Der}_B(A)$ we have a degree $-1$ “interior product” $i_\theta: \text{DR}_B^n(A) \rightarrow \text{DR}_B^{n-1}(A)$ and a degree 0 “Lie derivative” $\mathcal{L}_\theta: \text{DR}_B^n(A) \rightarrow \text{DR}_B^n(A)$ satisfying all the familiar relationships from commutative differential geometry [Gin05, Section 11].

In this framework, a non-commutative symplectic manifold is defined to be a pair $(A, \omega)$ consisting of an associative algebra $A$ and a Karoubi-de Rham 2-form $\omega \in \text{DR}_B^2(A)$ which is closed ($d\omega = 0$ in $\text{DR}_B^2(A)$) and non-degenerate, meaning that the map $\tilde{\omega}: \text{Der}_B(A) \rightarrow \text{DR}_B^2(A)$ defined by $\theta \mapsto i_\theta(\omega)$ is a bijection. In analogy with the commutative case, we say that a derivation $\theta$ is symplectic if $\mathcal{L}_\theta(\omega) = 0$, and denote by $\text{Der}_{B, \omega}(A)$ the Lie subalgebra of $\text{Der}_B(A)$ consisting of symplectic derivations. It follows from the Cartan homotopy formula that a derivation $\theta$ is symplectic if and only if the 1-form $\tilde{\omega}(\theta)$ is closed.
There is also no problem in defining a map $\theta : \text{DR}^0(A) \to \text{Der}_{B,\omega}(A)$ sending each 0-form $f$ to the corresponding “Hamiltonian derivation” $\theta_f := \tilde{\omega}^{-1}(df)$. Just as in the commutative case, this map fits into an exact sequence of Lie algebras
\begin{equation}
0 \to H^0(\text{DR}^*_B(A)) \to \text{DR}^0(A) \xrightarrow{\theta} \text{Der}_{B,\omega}(A) \to H^1(\text{DR}^*_B(A)) \to 0
\end{equation}
where the Lie algebra structure on $\text{DR}^0(A)$ is given by the Poisson brackets determined by $\omega$, defined e.g. by $\{f, g\} := i_{\theta_f} i_{\theta_g}(\omega)$. See [Gin05, Section 14] for the details.

Now let us specialize to the case when $A = \mathbb{C}Q$ is the path algebra of the double of some quiver $Q$. Let us denote again with $I = \{1, \ldots, n\}$ the vertices of $Q$, and by $e_i$ the trivial (i.e. length zero) path at the vertex $i$. Then, as noted in [Gin01, BLB02], $\mathbb{C}Q$ is most naturally viewed as an algebra over the (commutative) ring
\[ \mathbb{C}^n = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_n \]
generated by the complete set of mutually orthogonal idempotents $e_1, \ldots, e_n$. Accordingly, in the sequel we will only consider derivations and differential forms on $\mathbb{C}Q$ relative to this subalgebra, without explicitly displaying this fact in the notation.

A basis for the vector space $\text{DR}^0(\mathbb{C}Q)$ is given by the necklace words in $\mathbb{C}Q$, that is oriented cycles in $Q$ modulo cyclic permutations. A crucial result (see [BLB02, Theorem 3.6]) is that the relative Karoubi-de Rham complex associated to $\mathbb{C}Q$ is acyclic:
\begin{equation}
\text{H}^k(\text{DR}^*(\mathbb{C}Q)) = \begin{cases} \mathbb{C}^n & \text{for } k = 0 \\ 0 & \text{for } k \geq 1. \end{cases}
\end{equation}
The non-commutative symplectic form of the quiver $Q$ is the Karoubi-de Rham 2-form on $\mathbb{C}Q$ represented by
\begin{equation}
\omega := \sum_{\xi \in Q} d\xi d\xi^*\end{equation}
where the sum runs over all the arrows in $Q$. One can easily verify that $\omega$ is both closed and non-degenerate, so that the pair $(\mathbb{C}Q, \omega)$ is in fact a non-commutative symplectic manifold.

The Poisson brackets induced by $\omega$ on $\text{DR}^0(\mathbb{C}Q)$ can be given a very explicit expression, as follows. First, let us introduce for every arrow $\xi \in Q$ a corresponding “necklace derivative” operator
\[ \frac{\partial}{\partial \xi} : \text{DR}^0(\mathbb{C}Q) \to \mathbb{C}Q \]
defined on an arrow $\eta$ of $Q$ as
\[ \frac{\partial \eta}{\partial \xi} = \begin{cases} 1 & \text{if } \eta = \xi \\ 0 & \text{otherwise} \end{cases} \]
and extended on a generic necklace word $w = \eta_1 \ldots \eta_k \in \text{DR}^0(\mathbb{C}Q)$ by
\[ \frac{\partial w}{\partial \xi} = \sum_{k=1}^{k} \frac{\partial \eta_k}{\partial \xi} \eta_{k+1} \ldots \eta_{1} \eta_{1} \ldots \eta_{k-1}. \]
Then the Lie brackets on $\text{DR}^0(\mathbb{C}Q)$ determined by the non-commutative symplectic form are given by
\begin{equation}
\{w_1, w_2\} = \sum_{\xi \in Q} \left( \frac{\partial w_1}{\partial \xi} \frac{\partial w_2}{\partial \xi^*} - \frac{\partial w_1}{\partial \xi^*} \frac{\partial w_2}{\partial \xi} \right) \mod [\mathbb{C}Q, \mathbb{C}Q].
\end{equation}
We also note that, as a consequence of (20), the sequence (19) becomes in this case simply

\[ 0 \rightarrow \mathbb{C}^n \rightarrow \operatorname{DR}^0(\mathbb{C}Q) \rightarrow \operatorname{Der}_\omega(\mathbb{C}Q) \rightarrow 0. \]

Finally, let us clarify what it means for an algebra automorphism of \( \mathbb{C}Q \) to be symplectic. As it is shown in [Gin05, §10.4], for every associative algebra \( A \) there is a well-defined map

\[ b: \operatorname{DR}^1(A) \rightarrow [A, A] \]

given by \( a_0da_1 \mapsto [a_0, a_1] \). Suppose now that \( A = \mathbb{C}Q \) for some quiver \( Q \), and let \( \omega \) be any closed Karoubi-de Rham 2-form on \( \mathbb{C}Q \). By the acyclicity of \( \operatorname{DR}^\bullet(\mathbb{C}Q) \), there exists \( \theta \in \operatorname{DR}^1(\mathbb{C}Q) \) such that \( \omega = d\theta \); then we can define a map\(^1\)

\[ \operatorname{DR}^2(\mathbb{C}Q)_{\text{closed}} \rightarrow [\mathbb{C}Q, \mathbb{C}Q] \]

by sending \( \omega \) to \( b(\theta) \in [\mathbb{C}Q, \mathbb{C}Q] \). In particular, the non-commutative symplectic form (21) is the differential of the “non-commutative Liouville 1-form”

\[ \theta = \sum_{\xi \in Q} \xi d\xi^*. \]

The image of this 1-form under the map (23),

\[ c := \sum_{\xi \in Q} [\xi, \xi^*], \]

will be called, following [BLB02], the moment element of the path algebra \( \mathbb{C}Q \). Then we say that an algebra automorphism \( \psi: \mathbb{C}Q \rightarrow \mathbb{C}Q \) is symplectic if it preserves the moment element: \( \psi(c) = c \).

3. The path algebra of \( Q_r \)

The goal of this section is to study the structure of the path algebra of the quiver (14) and its automorphism group.

We start by introducing some notation. As in section 2 we denote by \( e_1 \) and \( e_2 \) the trivial paths at the two vertices 1 and 2 in \( Q_r \), and consider \( \mathbb{C}Q_r \) as an associative algebra over the ring \( \mathbb{C}^2 = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \) with unit \( e_1 + e_2 \). Let us also define

\[ A_{ij} := e_i \mathbb{C}Q_re_j \quad (i, j = 1, 2) \]

as the linear subspace in \( \mathbb{C}Q_r \) spanned by the paths \( j \rightarrow i \). For the sake of brevity, we put \( A_i := A_{ii} \). Clearly, as a linear space \( \mathbb{C}Q_r \) can be decomposed as the direct sum

\[ \mathbb{C}Q_r = A_1 \oplus A_{12} \oplus A_{21} \oplus A_2. \]

Let us denote by \( a \) and \( a^\ast \) the two loops at 1, by \( b_1, \ldots, b_r \) the \( r \) arrows \( 1 \rightarrow 2 \) and by \( d_1, \ldots, d_r \) the \( r \) arrows \( 1 \leftarrow 2 \) in \( Q_r \). The letters \( b \) and \( d \) are meant to resemble, respectively, something pointing to the right (for arrows \( 1 \rightarrow 2 \)) and something pointing to the left (for arrows \( 1 \leftarrow 2 \)). It is convenient to introduce the matrices

\[ D := \begin{pmatrix} d_1 \\ \vdots \\ d_r \end{pmatrix}, \quad B := (b_1 \ldots b_n) \quad \text{and} \quad E := D \cdot B. \]

We adopt the convention that Greek indices (\( \alpha, \beta, \) etc.) always run from 1 to \( r \). Notice that each entry \( e_{\alpha\beta} = d_{\alpha} b_{\beta} \) of the matrix \( E \) is an element of \( A_1 \), i.e. a cycle at the vertex 1 in \( Q_r \).

\(^1\)In the “absolute” case, that is when the quiver \( Q \) has only one vertex (so that \( \mathbb{C}Q \) is a free algebra over \( \mathbb{C} \)), this map is in fact an isomorphism [Gin05, Proposition 11.5.4(ii)]. We do not know whether this result holds also in the “relative” case, i.e. for a quiver with more than one vertex.
We can now make a number of trivial observations.

**Lemma 1.** The subspace $A_1$ is closed under products and is isomorphic to the free associative algebra on the $r^2 + 2$ generators $a, a^*$ and $(e_{\alpha \beta})_{\alpha, \beta = 1 \ldots r}$, with unit $e_1$.

**Proof.** This follows immediately from the definition of the path algebra. □

Of course it is also true that $A_2$ is a free subalgebra on the $r^2$ generators $(b_\alpha d_\beta)_{\alpha, \beta = 1 \ldots r}$.

**Lemma 2.** The subspace $A_{12}$ is a free left $A_1$-module with basis $(d_1, \ldots, d_r)$.

**Proof.** Suppose we are given a relation

$$\rho_1 d_1 + \cdots + \rho_r d_r = 0$$

for some coefficients $(\rho_\alpha)_{\alpha = 1 \ldots r}$ in $A_1$. By multiplying from the right e.g. by $b_1$, we get the equality

$$\rho_1 e_{11} + \cdots + \rho_r e_{r1} = 0.$$ 

As $A_1$ is a free algebra, this implies $\rho_\alpha = 0$ for every $\alpha = 1 \ldots r$. □

**Lemma 3.** The subspace $A_{21}$ is a free right $A_1$-module with basis $(b_1, \ldots, b_r)$.

**Proof.** Completely analogous to the previous one. □

Let us also note that, as vector spaces,

$$\text{DR}^0(CQ_r) \simeq \text{DR}^0(A_1) \oplus \mathbb{C}e_2$$

because in $Q_r$ every cycle of nonzero length based at 2 can be transformed in a cycle based at 1 by a single cyclic shift. In other words, apart from $e_2$ every “regular function” on the non-commutative manifold $CQ_r$ comes from a “regular function” on the $(r^2 + 2)$-dimensional non-commutative affine space $A_1$.

We proceed to study the group of $C^2$-linear automorphisms of $CQ_r$, denoted simply by $\text{Aut} CQ_r$. Its elements are morphisms of algebras $CQ_r \to CQ_r$ which are invertible and fix both $e_1$ and $e_2$. It follows that each $\psi \in \text{Aut} CQ_r$ preserves the decomposition (26), since for every path $p \in CQ_r$

$$\psi(e_{i}pe_{j}) = e_{i}\psi(p)e_{j}$$

so that $p \in A_{ij}$ implies $\psi(p) \in A_{ij}$. This means in particular that every $\psi \in \text{Aut} CQ_r$ automatically induces, by restriction, three bijections

$$\psi_1 : A_1 \to A_1, \quad \psi_{12} : A_{12} \to A_{12} \quad \text{and} \quad \psi_{21} : A_{21} \to A_{21}.$$ (27)

**Lemma 4.** The map

$$\text{res}_1 : \text{Aut} CQ_r \to \text{Aut} A_1$$

defined by $\psi \mapsto \psi_1$ is a morphism of groups.

**Proof.** Immediate from the definitions. □

To clarify the nature of the other two restriction maps we need the following concept from group theory. Suppose we are given two groups $G$ and $H$ and an action $\varphi : G \to \text{Aut} H$ of $G$ on $H$. A map $f : G \to H$ is then called a crossed (homo)morphism of groups if

$$f(ab) = f(a)\varphi_a(f(b))$$

for all $a, b \in G$.

When the action $\varphi$ is trivial, a crossed morphism is just an ordinary morphism of groups.
In the present situation, let us take $G = \text{Aut} \mathbb{C}Q_r$ and $H = \text{GL}_r(A_1)$, the group of invertible $r \times r$ matrices with entries in the free algebra $A_1$. The morphism $\psi$ induces an action of $G$ on $H$ obtained by letting $\psi_1 = \text{res}_1(\psi)$ act on each matrix element:

\[ \psi(M) \coloneqq (\psi_1(M_{\alpha\beta}))_{\alpha,\beta=1\ldots r} \quad \text{for all } M \in \text{GL}_r(A_1). \]

We claim that the restriction maps $\psi \mapsto \psi_{12}$ and $\psi \mapsto \psi_{21}$ induce two crossed morphisms $G \to H$ with respect to the above action.

To see this, notice first that by lemma 2 the map $\psi_{12}$ is completely specified by its values on the basis $(d_1, \ldots, d_r)$, and these values in turn can be expressed on this basis in a unique way; in other words, there exists a unique $r \times r$ matrix $M^\psi$ with entries in $A_1$ such that

\[ \psi_{12}(d_\alpha) = \sum_{\beta=1}^r M_{\alpha\beta}^\psi d_\beta. \]

Similarly, by lemma 3 the map $\psi_{21}$ is also completely specified by a unique $r \times r$ matrix $N^\psi$ with entries in $A_1$ such that

\[ \psi_{21}(b_\alpha) = \sum_{\beta=1}^r b_\beta N_{\beta\alpha}^\psi. \]

**Theorem 1.** The maps

\[ M : \text{Aut} \mathbb{C}Q_r \to \text{GL}_r(A_1)^{op} \quad \text{and} \quad N : \text{Aut} \mathbb{C}Q_r \to \text{GL}_r(A_1) \]

defined, respectively, by $\psi \mapsto M^\psi$ and $\psi \mapsto N^\psi$ are crossed morphisms of groups with respect to the action \((29)\).

**Proof.** Take $\psi, \sigma \in \text{Aut} \mathbb{C}Q_r$. The action of $\sigma$ followed by $\psi$ on $d_\alpha$ is

\[ \sigma(\psi(d_\alpha)) = \sigma(\sum_{\beta=1}^r M_{\alpha\beta}^\psi d_\beta) = \sum_{\beta=1}^r \sigma(M_{\alpha\beta}^\psi) \sigma_{12}(d_\beta) = \sum_{\beta=1}^r \sum_{\gamma=1}^r \sigma(M_{\alpha\beta}^\psi) M_{\beta\gamma}^\sigma d_\gamma, \]

where $\sigma(M^\psi)$ is given exactly by \((29)\). In other words,

\[ \sigma_{12}(\psi_{12}(d_\alpha)) = \sum_{\gamma=1}^r (\sigma(M^\psi) M^\sigma)_{\alpha\gamma} d_\gamma \]

hence the matrix corresponding to $(\sigma \circ \psi)_{12}$ is $\sigma(M^\psi) M^\sigma$. This would be the same as saying that the map defined by $\psi \mapsto M^\psi$ is a crossed morphism from $\text{Aut} \mathbb{C}Q_r$ to the opposite of $\text{GL}_r(A_1)$, except that we did not prove yet that $M^\psi$ is always invertible. But applying the above result to the equalities $\psi^{-1} \circ \psi = \psi \circ \psi^{-1} = \text{id}_{\mathbb{C}Q_r}$ we get that

\[ \psi^{-1}(M^\psi) M^{\psi^{-1}} = I_r \quad \text{and} \quad \psi(M^{\psi^{-1}}) M^\psi = I_r. \]

From the second equality it follows that $\psi(M^{\psi^{-1}})$ is a left inverse for $M^\psi$, and acting with $\psi$ on the first equality (notice that $\psi(I_r) = I_r$ since $\psi$ fixes $e_1$) we get that $M^\psi \psi(M^{\psi^{-1}}) = I_r$, hence $\psi(M^{\psi^{-1}})$ is also a right inverse for $M^\psi$. This concludes the proof for the map $M$.

With regard to $N$, we only need to perform the analogous computation for $\sigma(\psi(b_\alpha))$ to find out that the matrix corresponding to $(\sigma \circ \psi)_{21}$ is given by $N^\sigma \sigma(N^\psi)$. By applying this result to the equalities $\psi^{-1} \circ \psi = \psi \circ \psi^{-1} = \text{id}_{\mathbb{C}Q_r}$ we can again deduce that $N^\psi$ is always invertible, and its inverse coincides with $\psi(N^{\psi^{-1}})$. \(\square\)

Let us study the structure of the group $\text{Aut} \mathbb{C}Q_r$ in more detail, following [BPT11, Section 6.1]. Let $I_r$ denote the two-sided ideal in $\mathbb{C}Q_r$, generated by $d_1, \ldots, d_r$ and $b_1, \ldots, b_r$. Clearly

\[ \mathbb{C}Q_r / I_r \simeq \mathbb{C}(a, a^*) = \mathbb{C}Q_0 \]
where $Q_0$ is the quiver with a single vertex and two loops $a$ and $a^*$ on it.

Every $\mathbb{C}^2$-linear map $\mathbb{C}Q_r \to \mathbb{C}Q_r$ preserves the ideal $I_r$, hence every $\psi \in \text{Aut} \mathbb{C}Q_r$ descends to an automorphism of $\mathbb{C}Q_0$ and this gives a map

$$\pi: \text{Aut} \mathbb{C}Q_r \to \text{Aut} \mathbb{C}Q_0$$

which is a morphism of groups. On the other hand, every automorphism of $\mathbb{C}Q_0$ extends to an automorphism of $\mathbb{C}Q_r$ acting as the identity on the additional arrows, hence we have a split extension of groups, i.e. a semidirect product

$$\text{Aut} \mathbb{C}Q_r = K \rtimes \text{Aut} \mathbb{C}Q_0$$

where $K$ is the kernel of $\pi$, i.e. the (normal) subgroup of automorphisms of $\mathbb{C}Q_r$ which fix $a$ and $a^*$ after killing all the other arrows. For lack of a better word, we shall call such an automorphism reduced.

**Definition 1.** The subgroup $K$ of $\text{Aut} \mathbb{C}Q_r$ defined by the splitting (32) is called the group of reduced automorphisms of $\mathbb{C}Q_r$.

In the sequel we will be particularly interested in symplectic automorphisms, that is automorphisms of $\mathbb{C}Q_r$ fixing its moment element. Quite generally, given a quiver $Q$ and a path $p \in \mathbb{C}Q$ we can define

$$\text{Aut}(\mathbb{C}Q; p) := \{ \psi \in \text{Aut} \mathbb{C}Q \mid \psi(p) = p \}$$

as the subgroup of automorphisms of $\mathbb{C}Q$ fixing $p$. Now suppose we take some path $c_r \in \mathbb{C}Q_r$ and let $c_0$ be the image of $c_r$ in $\mathbb{C}Q_0$ modulo $I_r$. Then we can repeat the above argument to show that the group $\text{Aut}(\mathbb{C}Q_r; c_r)$ of $c_r$-preserving automorphisms of $\mathbb{C}Q_r$ is again a semidirect product

$$\text{Aut}(\mathbb{C}Q_r; c_r) = K_{c_r} \rtimes \text{Aut}(\mathbb{C}Q_0; c_0)$$

where now $K_{c_r}$ is the subgroup of reduced automorphisms fixing $c_r$.

**4. The non-commutative symplectic geometry of zigzag quivers**

Let us denote by $(Z_r)_{r \geq 1}$ the family of quivers defined in the following way. Each quiver has two vertices, call them 1 and 2, and a loop $a: 1 \to 1$. Moreover, the quiver $Z_r$ has:

- an arrow $x_i: 2 \to 1$ for each $i = 1, \ldots, \lceil r/2 \rceil$, and
- an arrow $y_j: 1 \to 2$ for each $j = 1, \ldots, \lfloor r/2 \rfloor$.

Here $\lfloor z \rfloor$ and $\lceil z \rceil$ denote, respectively, the largest integer not greater than $z$ and the smallest integer not less than $z$, as usual. It follows that the double $\overline{Z}_r$ has an additional loop $a^*: 1 \to 1$, $\lceil r \rceil$ additional arrows $x_i^*: 1 \to 2$ and $\lfloor r \rfloor$ additional arrows $y_j^*: 2 \to 1$. It is immediate to check that $\overline{Z}_r$ coincides with the quiver (14). We will identify the generators of $\mathbb{C}Q_r$ (as labeled in the previous section) with the generators in $\mathbb{C}\overline{Z}_r$ in the following way:

$$d_\alpha = \begin{cases} 
-x_{(\alpha+1)/2} & \text{for } \alpha \text{ odd} \\
y_{\alpha/2} & \text{for } \alpha \text{ even}
\end{cases}$$

$$b_\alpha = \begin{cases} 
x_{(\alpha+1)/2}^* & \text{for } \alpha \text{ odd} \\
y_{\alpha/2} & \text{for } \alpha \text{ even}.
\end{cases}$$
In terms of this new notation, the matrices $D$, $B$ and $E$ read

\begin{equation}
D = \begin{pmatrix}
-x_1 \\
y_1 \\
-x_2 \\
y_2 \\
\vdots
\end{pmatrix}, \quad B = \begin{pmatrix}
x_1^* \\
y_1 \\
x_2^* \\
y_2 \\
\vdots
\end{pmatrix},
\end{equation}

\begin{equation}
E = \begin{pmatrix}
-x_1 x_1^* & -x_1 y_1 & -x_1 x_2^* & -x_1 y_2 & \cdots \\
y_1 x_1^* & y_1 y_1 & y_1 x_2^* & y_1 y_2 & \cdots \\
y_2 x_1^* & -x_2 y_1 & -x_2 x_2^* & -x_2 y_2 & \cdots \\
y_2 x_2^* & y_2 y_1 & y_2 x_2^* & y_2 y_2 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\end{equation}

For later use, we define the sequence $(q_r)_{r \geq 1}$ by $q_r := \lfloor r/2 \rfloor \lceil r/2 \rceil$; these are usually known as \textit{quarter-square numbers}. Clearly, $q_r$ coincides with the number of different cycles at 1 that may be obtained by composing one of the arrows $x_1, \ldots , x_{\lfloor r/2 \rfloor}$ with one of the arrows $y_1, \ldots , y_{\lceil r/2 \rceil}$ in the quiver $Z_r$.

\textbf{Remark 1.} Although it will not play any rôle in the sequel, we note that the Poisson brackets (22) naturally defined on $\text{DR}^0(CQ_r)$ induce, by restriction to the subalgebra $A_1$, a Lie algebra structure on the linear space $\text{DR}^0(A_1)$ whose nonzero part reads

$$\{a, a^\ast\} = 1 \quad \text{and} \quad \{e_{\alpha \beta}, e_{\gamma \delta}\} = \delta_{\beta \gamma} e_{\alpha \delta} - \delta_{\alpha \delta} e_{\gamma \beta}.$$  

In particular the $e_{\alpha \beta}$ obey the same relations that hold in the Lie algebra $\mathfrak{gl}_r(C)$, as emphasized in [BP11] §5.1 for the case $r = 2$.

According to the definition (21), the non-commutative symplectic form of $Z_r$ is given by

$$\omega = da da^\ast + \sum_{i=1}^{\lfloor r/2 \rfloor} dx_i dx_i^\ast + \sum_{j=1}^{\lceil r/2 \rceil} dy_j dy_j^\ast$$

and the corresponding moment element in $CZ_r = CQ_r$ is

\begin{equation}
c_r := [a, a^\ast] + \sum_{i=1}^{\lfloor r/2 \rfloor} [x_i, x_i^*] + \sum_{j=1}^{\lceil r/2 \rceil} [y_j, y_j^*].
\end{equation}

An element of $\text{Aut} CQ_r$ fixing $c_r$ will be called a \textit{symplectic automorphism} of $CQ_r$, or a \textit{symplectomorphism} for short. Using the notation introduced in (33), the group of symplectic automorphisms of $Q_r$ will be denoted by $\text{Aut}(CQ_r; c_r)$.

Notice that the quotient of $c_r$ by the ideal $I_r$,

$$c_0 := [a, a^\ast],$$

is exactly the moment element corresponding to the non-commutative symplectic form on the path algebra $\mathbb{C}Q_0$ (with $Q_0$ seen as a double in the obvious way). We conclude by (34) that every symplectic automorphism of $CQ_r$ can be factorized in a unique way as the composition of a reduced symplectic automorphism followed by a symplectic automorphism of $\mathbb{C}Q_0$ (acting only on $a$ and $a^\ast$). As the group $\text{Aut}(CQ_0; c_0)$ is “well-known” (it is isomorphic to the automorphism group of the first Weyl algebra, see the proof of lemma 7 below), the problem of understanding the group $\text{Aut}(CQ_r; c_r)$ largely reduces to understanding its subgroup $K_{c_r}$ of reduced automorphisms.
Next we note that \( c_r \) can be decomposed as \( c_r^1 + c_r^2 \), where
\[
c_r^1 := [a, a^*] + \sum_{i=1}^{\lfloor r/2 \rfloor} x_i x_i^* - \sum_{j=1}^{\lfloor r/2 \rfloor} y_j^* y_j = [a, a^*] - \sum_{\alpha=1}^{r} d_{\alpha} b_{\alpha}
\]
is a cycle based at 1 and
\[
c_r^2 := -\sum_{i=1}^{\lfloor r/2 \rfloor} x_i^* x_i + \sum_{j=1}^{\lfloor r/2 \rfloor} y_j y_j^* = \sum_{\alpha=1}^{r} b_{\alpha} d_{\alpha}
\]
is a cycle based at 2. Being \( C_2 \)-linear, every automorphism in \( \text{Aut}(\mathbb{C}Q_r; c_r) \) must preserve separately the two cycles \( c_r^1 \) and \( c_r^2 \); this means that
\[
\text{Aut}(\mathbb{C}Q_r; c_r) = \text{Aut}(\mathbb{C}Q_r; c_r^1) \cap \text{Aut}(\mathbb{C}Q_r; c_r^2).
\]
The second component of this intersection can be characterized quite satisfactorily.

**Theorem 2.** An automorphism \( \psi \in \text{Aut}(\mathbb{C}Q_r) \) fixes \( c_r^2 \) if and only if \( (N^\psi)_{-1} = M^\psi \) in \( \text{GL}_r(\mathbb{A}_1) \).

**Proof.** We have that
\[
\psi(c_r^2) = \psi(\sum_{\alpha=1}^{r} b_{\alpha} d_{\alpha}) = \sum_{\alpha, \beta, \gamma=1}^{r} b_{\gamma} N_{\gamma \alpha}^\psi M_{\alpha \beta}^\psi d_{\beta}.
\]
Then \( \psi \) fixes \( c_r^2 \) if and only if
\[
\sum_{\beta, \gamma=1}^{r} b_{\gamma} (N^\psi M^\psi)_{\gamma \beta} d_{\beta} = \sum_{\alpha=1}^{r} b_{\alpha} d_{\alpha},
\]
or in other words
\[
\sum_{\beta, \gamma=1}^{r} b_{\gamma} (N^\psi M^\psi - I)_{\gamma \beta} d_{\beta} = 0.
\]
Set \( C := N^\psi M^\psi - I \). Multiplying the previous relation by \( d_1 \) from the left and \( b_1 \) from the right we get
\[
\sum_{\beta, \gamma=1}^{r} d_1 b_{\gamma} C_{\gamma \beta} d_{\beta} b_1 = \sum_{\gamma=1}^{r} e_{1\gamma} \sum_{\beta=1}^{r} C_{\gamma \beta} e_{\beta 1} = 0.
\]
As \( \mathbb{A}_1 \) is a free algebra, this implies \( \sum_{\beta} C_{\gamma \beta} e_{\beta 1} = 0 \) for every \( \gamma \), which in turn implies that \( C_{\gamma \beta} = 0 \) for every \( \gamma \) and \( \beta \). Hence \( C = 0 \), i.e. \( N^\psi = (M^\psi)^{-1} \), as claimed. \( \square \)

It follows that for \( c_r^2 \)-preserving automorphisms the two crossed morphisms (30) are related by the isomorphism \((\cdot)^{-1}: \text{GL}_r(\mathbb{A}_1) \text{op} \to \text{GL}_r(\mathbb{A}_1)\), i.e. the following diagram commutes:

\[
\begin{array}{ccc}
\text{Aut}(\mathbb{C}Q_r; c_r^2) & \xrightarrow{M} & \text{GL}_r(\mathbb{A}_1) \text{op} \\
N \downarrow & & \downarrow (\cdot)^{-1} \\
\text{GL}_r(\mathbb{A}_1)
\end{array}
\]

Unfortunately, the condition of preserving \( c_r^1 \) does not seem to admit such a straightforward description (even considering only reduced automorphisms). For this reason the structure of the group \( \text{Aut}(\mathbb{C}Q_r; c_r) \) remains somewhat unclear.

To remedy this situation we introduce some classes of “nice” automorphisms.

**Definition 2.** An automorphism \( \psi \in \text{Aut}(\mathbb{C}Q_r) \) will be called:
• **triangular** if it fixes each of the arrows \((a, x_1, \ldots, x_{[r/2]}, y_1, \ldots, y_{[r/2]})\),
• **op-triangular** if it fixes each of the arrows \((a^*, x_1^*, \ldots, x_{[r/2]}^*, y_1^*, \ldots, y_{[r/2]}^*)\),
• **affine** if it sends each arrow in \(Q_r\) to a polynomial which is at most linear in each arrow.

Notice that the definition of (op-)triangularity depends on the choice of the unstarred arrows in \(Q_r\).

Our next goal is to characterize the symplectic automorphisms in each of these classes.

4.1. **Triangular symplectomorphisms.** Let \(\psi\) be a triangular automorphism of \(CQ_r\). Without loss of generality, we can write the action of \(\psi\) as

\[
\begin{align*}
  a &\mapsto a \\
x_i &\mapsto x_i \\
y_j &\mapsto y_j \\
a^* &\mapsto a^* + h \\
x_i^* &\mapsto x_i^* + s_i \\
y_j^* &\mapsto y_j^* + t_j
\end{align*}
\]

for some \(h \in A_1, s_1, \ldots, s_{[r/2]} \in A_{21}\) and \(t_1, \ldots, t_{[r/2]} \in A_{12}\). Now \(\psi\) will preserve \(c_r^2\) if and only if

\[
\sum_i s_ix_i = \sum_j y_jt_j.
\]

This equality can only hold when each \(s_i\) can be written as \(\sum_j y_ju_{ij}\) for some coefficients \(u_{ij} \in A_1\); then equation (39) becomes

\[
\sum_j y_j(\sum_i u_{ij}x_i - t_j) = 0,
\]

which implies that \(t_j = \sum_i u_{ij}x_i\) for every \(j = 1, \ldots, [r/2]\). Hence \(\psi\) acts as

\[
\begin{align*}
  a^* &\mapsto a^* + h \\
x_i^* &\mapsto x_i^* + \sum_j y_ju_{ij} \\
y_j^* &\mapsto y_j^* + \sum_i u_{ij}x_i
\end{align*}
\]

for some \(u_{ij} \in A_1\).

Such an automorphism preserves \(c_r^1\) if and only if

\[
[a, h] + \sum_{i,j} x_iy_ju_{ij} - \sum_{i,j} u_{ij}x_iy_j = 0
\]

or, using \(e_{2i-1,2j} = -x_iy_j\),

\[
[a, h] - \sum_{i,j} (e_{2i-1,2j}, u_{ij}) = 0.
\]

Thus we are led to study equations in \(A_1\) of the form

\[
\sum_{k=0}^n [g_k, u_k] = 0
\]

where \(g_0, \ldots, g_n\) are (distinct) generators of \(A_1\) and \(u_0, \ldots, u_n\) are unknowns. This problem is completely solved by the following result, whose proof is deferred to Appendix A.

**Theorem 3.** The \((n + 1)\)-tuple \((u_0, \ldots, u_n)\) satisfies equation (42) if and only if there exists \(f \in \text{DR}^n(C(g_0, \ldots, g_n))\) such that \(u_k = \frac{\partial f}{\partial g_k}\) for all \(k = 0, \ldots, n\).  

\(^2\)In [BPT11] and [MT13] automorphisms of this kind are called strictly triangular, reserving the word triangular for the larger class of automorphisms preserving the subalgebra \(CZ_r\). As this weaker notion will play no rôle in the sequel, we decided to drop the “strictly” and use the simplest name for the most useful concept.
It is now easy to explicitly describe all the triangular symplectomorphisms of $\mathbb{C}Q_r$. Let us denote by $\text{Tri}_c$, the subgroup of $\text{Aut}(\mathbb{C}Q_r; e_i)$ consisting of such automorphisms. Let also $F_r$ denote the (free) subalgebra of $A_1$ generated by the $q_r+1$ elements $a$ and $b_{ij} := -c_{2i-1,2j} = x_i y_j$ for $i = 1 \ldots \lfloor r/2 \rfloor$ and $j = 1 \ldots \lceil r/2 \rceil$. We define a map

\begin{equation}
\Lambda: \mathbb{DR}^0(F_r) \to \text{Aut} \mathbb{C}Q_r
\end{equation}

by letting a necklace word $f \in \mathbb{DR}^0(F_r)$ correspond to the triangular automorphism

\begin{align}
&\begin{cases}
a \mapsto a \\
x_i \mapsto x_i \\
y_j \mapsto y_j
\end{cases}
&\begin{cases}
a^* \mapsto a^* + \frac{\partial f}{\partial a} \\
x_i^* \mapsto x_i^* + \sum_{j=1}^{\lfloor r/2 \rfloor} y_j \frac{\partial f}{\partial b_{ij}} \\
y_j^* \mapsto y_j^* + \sum_{i=1}^{\lfloor r/2 \rfloor} x_i \frac{\partial f}{\partial b_{ij}}
\end{cases}
\end{align}

**Theorem 4.** The map $\Lambda$ is an isomorphism between the abelian group $(\mathbb{DR}^0(F_r), +)$ and $\text{Tri}_c$.

**Proof.** We saw that every element of $\text{Tri}_c$ has the form \((10)\) and is determined by a $(q_r+1)$-tuple $(h, u_{ij})$ of elements of $A_1$ that satisfies equation \((11)\). By theorem 3, all those tuples are obtained by taking $h = \frac{\partial f}{\partial h}$ and $u_{ij} = \frac{\partial f}{\partial b_{ij}}$ for some $f \in \mathbb{DR}^0(F_r)$, which gives exactly $\Lambda(f)$ as defined above. It follows that $\Lambda$ is a surjective map $\mathbb{DR}^0(F_r) \to \text{Tri}_c$. It is also injective, since necklace derivatives have no kernel on $\mathbb{DR}^0(F_r)$. Finally, it is a morphism of (abelian) groups:

$$\Lambda(f_1 + f_2) = \Lambda(f_1) \circ \Lambda(f_2).$$

This follows easily by noting that necklace derivatives are $\mathbb{C}$-linear and $\Lambda(f)$ translates each starred arrow by a non-commutative polynomial in $a$ and $b_{ij} = x_i y_j$, and every automorphism in $\text{Tri}_c$ fixes such an element. \hfill \Box

When $r$ is even, it is easy to verify that the matrix in $\text{GL}_r(A_1)$ associated to a generic element $\Lambda(f) \in \text{Tri}_c$ via the crossed morphism $N$ is given by

\begin{equation}
N^{\Lambda(f)} = I_r + \left( \begin{array}{cccc}
\frac{\partial f}{\partial b_{11}} & \cdots & \frac{\partial f}{\partial b_{1,\lfloor r/2 \rfloor}} \\
\vdots & \ddots & \vdots \\
\frac{\partial f}{\partial b_{r1}} & \cdots & \frac{\partial f}{\partial b_{r,\lfloor r/2 \rfloor}}
\end{array} \right) \otimes \left( \begin{array}{cc}
0 & 0 \\
1 & 0
\end{array} \right)
\end{equation}

When $r$ is odd one has to take the above matrix for $r+1$ and remove the last row and the last column.

By theorem 2 the matrix $M^{\Lambda(f)}$ is just the inverse of $N^{\Lambda(f)}$; as $\text{Tri}_c$ is abelian, to obtain it we simply need to replace $f$ with $-f$ in \((45)\).

We also note that the triangular symplectomorphism $\Lambda(f)$ is reduced if and only if every monomial in $\frac{\partial f}{\partial h}$ depends on at least one of the variables $b_{ij}$, since it is exactly in this case that the translation affecting $a^*$ gets killed after quotienting out by the ideal $\mathcal{I}_r$.

**4.2. Op-triangular symplectomorphisms.** To describe op-triangular symplectomorphisms we only need to adapt the above analysis in the obvious way. Writing the action of a generic op-triangular automorphism as

\begin{align}
&\begin{cases}
a \mapsto a + h' \\
x_i \mapsto x_i + s_i' \\
y_j \mapsto y_j + t_j
\end{cases}
&\begin{cases}
a^* \mapsto a^* \\
x_i^* \mapsto x_i^* \\
y_j^* \mapsto y_j^*
\end{cases}
\end{align}
for some \( h' \in \mathcal{A}_1 \), \( s'_1, \ldots, s'_{r/2} \in \mathcal{A}_{12} \) and \( t'_1, \ldots, t'_{r/2} \in \mathcal{A}_{21} \) and requiring \( c^2_r \) to be fixed we deduce that

\[
    s'_j = \sum_i v_{ij} y^*_j \quad \text{and} \quad t'_j = \sum_i x^*_i v_{ij}
\]

for some coefficients \( v_{ij} \in \mathcal{A}_1 \). An automorphism of this form will fix \( c^1_r \) if and only if

\[
    [a^*, h'] + \sum_{i,j} [y^*_j x^*_i, v_{ij}] = 0.
\]

This is again an equation of the form (42), as \( y^*_j x^*_i = e_{2j,2i-1} \). By theorem 3, its solutions are parametrized by elements of \( \overline{\text{DR}}(F^*_r) \), where \( F^*_r \) is the free subalgebra of \( \mathcal{A}_1 \) generated by the \( q_r + 1 \) elements \( a^* \) and \( b^*_i = e_{2i,2i-1} = y^*_j x^*_i \), by taking

\[
    h' = \frac{\partial f}{\partial a^*} \quad \text{and} \quad v_{ij} = \frac{\partial f}{\partial b^*_{ij}}.
\]

By mimicking the proof of theorem 3 it is straightforward to show that the map

\[
    \Lambda': \overline{\text{DR}}^0(F^*_r) \to \text{Aut} \mathbb{C}Q_r
\]

defined by sending \( f \in \overline{\text{DR}}^0(F^*_r) \) to the automorphism

\[
(46) \quad \begin{cases}
    a \mapsto a + \frac{\partial f}{\partial a^*} \\
    x_i \mapsto x_i + \sum_j \frac{\partial f}{\partial y^*_j} y^*_j \\
    y_j \mapsto y_j + \sum_i x^*_i \frac{\partial f}{\partial x^*_i}
\end{cases}
\]

is an isomorphism between \( (\overline{\text{DR}}^0(F^*_r),+) \) and the group of op-triangular symplectic automorphisms of \( \mathbb{C}Q_r \), that we will denote by \( \text{opTri}_c \). The matrix associated to \( \Lambda'(f) \) by the crossed morphism \( N \) is

\[
(47) \quad N\Lambda'(f) = \begin{pmatrix}
    1 & \frac{\partial f}{\partial a^*} & 0 & \frac{\partial f}{\partial y^*_1} & \cdots \\
    0 & 1 & 0 & 0 & \cdots \\
    0 & \frac{\partial f}{\partial a^*} & 1 & \frac{\partial f}{\partial x^*_1} & \cdots \\
    \vdots & \vdots & \vdots & \vdots & \ddots 
\end{pmatrix}.
\]

Again, the automorphism \( \Lambda'(f) \) will be reduced exactly when each of the monomials in \( \frac{\partial f}{\partial a^*} \) depends on at least one of the variables \( b^*_i \).

From the above results it is clear that the two groups \( \text{Tri}_c \) and \( \text{opTri}_c \) are isomorphic. For the sequel it will be useful to define the following explicit identification. Consider the isomorphism of linear spaces \( \overline{\text{DR}}(F_r) \to \overline{\text{DR}}(F^*_r) \) obtained by mapping a necklace word \( f(a,b_j) \) in \( \overline{\text{DR}}(F_r) \) to the necklace word \( \hat{f} := -f(a^*,b^*_j) \) in \( \overline{\text{DR}}(F^*_r) \). We denote by \( \circ \): \( \text{Tri}_c \to \text{opTri}_c \) the unique isomorphism of abelian groups making the diagram

\[
(48) \quad \begin{array}{ccc}
    \overline{\text{DR}}(F_r) & \xrightarrow{f \mapsto \hat{f}} & \overline{\text{DR}}(F^*_r) \\
    \Lambda \downarrow & & \downarrow \Lambda' \\
    \text{Tri}_c & \xrightarrow{\circ} & \text{opTri}_c
\end{array}
\]

commute, i.e. such that \( \circ(\Lambda(f)) = \Lambda'(\hat{f}) \).
4.3. Affine symplectomorphisms. Let \( \varphi \) be an affine automorphism of \( \mathbb{C}Q_r \). Being \( \mathbb{C}^2 \)-linear, it must act as

\[
a \mapsto A_{11}a + A_{12}a^* + B_1 \quad a^* \mapsto A_{21}a + A_{22}a^* + B_2
\]

for some \( \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in \text{GL}_2(\mathbb{C}) \), \( \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \in \mathbb{C}^2 \)
on on the linear subspace spanned by \( a \) and \( a^* \) in \( \mathbb{C}Q_r \), whereas its action on the arrows \( 1 \to 2 \) and \( 2 \to 1 \) will be described by the two complex matrices \( M^\varphi \) and \( N^\varphi \in \text{GL}_r(\mathbb{C}) \). Now, by theorem 2 the automorphism \( \varphi \) fixes \( c^2 \) if and only if \( M^\varphi = (N^\varphi)^{-1} \). Granted that, we can compute

\[
\varphi(c^i_j) = \varphi(\{a, a^*\} - \sum_{\alpha=1}^r d_\alpha b_\alpha) = \varphi(\{a, a^*\}) - \sum_{\alpha,\beta,\gamma=1}^r M^\varphi_{\alpha\beta} d_\beta b_\gamma N^\varphi_{\gamma\alpha}.
\]

Since the entries of \( M^\varphi \) and \( N^\varphi \) are complex numbers they commute with everything, so that

\[
\sum_{\alpha,\beta,\gamma=1}^r M^\varphi_{\alpha\beta} d_\beta b_\gamma N^\varphi_{\gamma\alpha} = \sum_{\beta,\gamma=1}^r N^\varphi_{\gamma\alpha} M^\varphi_{\alpha\beta} d_\beta b_\gamma = \sum_{\beta,\gamma=1}^r \delta_{\gamma\alpha} d_\beta b_\gamma = \sum_{\beta} d_\beta b_\beta.
\]

Hence \( \varphi \) preserves \( c^i_j \) if and only if it preserves \( \{a, a^*\} \), and this happens if and only if the matrix \( A \) in the transformation (49) has determinant 1. Denoting by \( \text{ASL}_2(\mathbb{C}) \) the subgroup of affine symplectomorphisms of this form, we conclude that the group of affine symplectic automorphisms of \( \mathbb{C}Q_r \) is the direct product

\[
\text{Aff}_r := \text{ASL}_2(\mathbb{C}) \times \text{GL}_r(\mathbb{C}).
\]

Lemma 5. An affine symplectic automorphism is reduced if and only if it fixes \( a \) and \( a^* \).

Indeed every transformation of the form (49) with \( \det A = 1 \) comes from an automorphism of \( \mathbb{C}Q_0 \). Thus the subgroup of reduced affine symplectic automorphisms coincides with the subgroup \( \text{GL}_r(\mathbb{C}) \) of scalar invertible matrices inside \( \text{GL}_r(\mathbb{A}_1) \).

4.4. Tame symplectomorphisms. Having defined some classes of “nice” symplectic automorphisms, we proceed to consider the subgroup of \( \text{Aut}(\mathbb{C}Q_r; c_r) \) which is generated by them.

Definition 3. A symplectic automorphism of \( \mathbb{C}Q_r \) is called tame if it belongs to the subgroup generated by \( \text{Tri}_r \) and \( \text{Aff}_r \).

We denote by \( \text{TAut}(\mathbb{C}Q_r; c_r) \) the subgroup of tame symplectic automorphisms. It is possible that every symplectic automorphism of \( \mathbb{C}Q_r \) is tame (this is unknown even for \( r = 2 \)); anyway, this issue will be irrelevant for what follows.

It is natural to ask if one can replace \( \text{Tri}_r \) with \( \text{opTri}_r \) in the definition 3. When \( r \) is even (in which case \( \lfloor r/2 \rfloor = \lfloor r/2 \rfloor \)), the answer is positive: let \( \mathcal{F}_r \) denote the affine symplectomorphism of \( \mathbb{C}Q_r \) defined by

\[
a \mapsto -a^* \\
\quad a^* \mapsto a
\]

\[
x_i \mapsto -y_i^* \\
\quad y_i^* \mapsto x_i
\]

We then have the following:

Theorem 5. The map \( \text{Aut}(\mathbb{C}Q_r; c_r) \to \text{Aut}(\mathbb{C}Q_r; c_r) \) defined by \( \psi \mapsto \mathcal{F}_r^{-1} \circ \psi \circ \mathcal{F}_r \) restricts to the isomorphism \( \circ : \text{Tri}_r \to \text{opTri}_r \) defined by the diagram (15).

Proof. Let \( f \in \mathcal{D} \mathcal{R}_0^d (F_r) \); we compute the action of \( \psi := \mathcal{F}_r^{-1} \circ \Lambda(f) \circ \mathcal{F}_r \) on the arrows of \( Q_r \).

It is immediate to check that \( \psi \) fixes the starred arrows. Now let \( p_0 := \frac{\partial f}{\partial a} \), \( p_{ij} := \frac{\partial f}{\partial b_{ij}} \); they
are non-commutative polynomials in the indeterminates \( a \) and \( (x_i, y_i)_{i,j=1,...,r/2} \). Then, by direct computation,
\[
\psi(a) = F_r^{-1}(-a^* - p_0(a, x_1 y_1, \ldots, x_{r/2} y_{r/2})) = a - p_0(a^*, y_1^* x_1^*, \ldots, y_{r/2}^* x_{r/2}^*)
\]
\[
\psi(x_i) = F_r^{-1}(-y_i^* - \sum_k p_k(a, x_1 y_1, \ldots, x_{r/2} y_{r/2})x_k) = x_i - \sum_k p_k(a^*, y_1^* x_1^*, \ldots, y_{r/2}^* x_{r/2}^*)y_k
\]
\[
\psi(y_i) = F_r^{-1}(-x_i^* - \sum_k y_k p_k(a, x_1 y_1, \ldots, x_{r/2} y_{r/2})) = y_i - \sum_k x_k^* y_k(a^*, y_1^* x_1^*, \ldots, y_{r/2}^* x_{r/2}^*).
\]
To conclude, it remains to observe that \(-p_0(a^*, y_1^* x_1^*, \ldots, y_{r/2}^* x_{r/2}^*)\) coincides with the non-commutative polynomial \( \partial_{a^*} f \), and similarly \(-p_{ij}(a^*, y_1^* x_1^*, \ldots, y_{r/2}^* x_{r/2}^*)\) coincides with \( \partial_{a^*} f \), so that \( \psi = \Lambda^*(f) = o(\Lambda(f)) \).

It follows that every op-triangular symplectomorphism is tame and, vice versa, every triangular symplectomorphism is generated by the affine and op-triangular ones.

When \( r \) is odd it is not clear if one can obtain the map \( o \) in a similar way. As a matter of fact, in this case we do not know any general recipe to express an op-triangular symplectomorphism as a composition of affine and triangular ones. Fortunately, we will see in the next section that when we restrict to the subgroup \( P_r \) inside \( \text{TAut}(\mathbb{C}Q_r; c_r) \) this difficulty disappears.

5. The group \( P_r \)

Let us denote by \( \text{Tri}^c_r \) the subgroup of \( \text{Tri}_r \) given by the image under \( \Lambda \) of the subgroup of \( \text{DTT}(F_r) \) consisting of necklace words of the form \( f = p(a)b_{11} \) for some \( p \in \mathbb{C}[a] \). Clearly, \( \text{Tri}^c_r \) is isomorphic to the abelian group \( (\mathbb{C}[a], +) \).

**Definition 4.** We denote by \( P_r \) the subgroup of \( \text{TAut}(\mathbb{C}Q_r; c_r) \) generated by \( \text{Tri}^c_r \) and the subgroup \( \text{Aff}^\text{red}_c \subseteq \text{Aff}_c \) of reduced affine symplectic automorphisms.

Since every automorphism in \( \text{Tri}^c_r \) is itself reduced (as every term in \( \partial_{a^*} \) will contain an occurrence of \( b_{11} \)), we observe that \( P_r \) is in fact a subgroup of \( K_{c_r} \). Our next aim is to prove that this group is isomorphic to \( \text{GL}_r(\mathbb{C}[a]) \).

**Lemma 6.** The restriction of the action \( \Psi \) to \( P_r \) is trivial on the subsets \( N(\text{Tri}^c_r) \) and \( N(\text{Aff}^\text{red}_c) \subseteq \text{GL}_r(A_1) \).

**Proof.** It suffices to show that the generators of \( P_r \) act trivially. It is clear that any \( \mathbb{C}^2 \)-linear automorphism of \( \mathbb{C}Q_r \) acting entry-wise on a matrix in \( \text{GL}_r(A_1) \) fixes every scalar matrix, hence fixes \( N(\text{Aff}^\text{red}_c) \). Now let us take \( \psi \in \text{Tri}^c_r \); since \( \psi = \Lambda(p(a)b_{11}) \) for some \( p \in \mathbb{C}[a] \), the matrix \( N^\psi \) will be of the form

\[
N^\psi = \begin{pmatrix}
1 & 0 & 0 & \ldots & 0 \\
p(a) & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 \\
\end{pmatrix}
\]

and hence only depends on \( a \). But then \( N^\psi \) is fixed both by elements of \( \text{Aff}^\text{red}_c \) (by lemma 5) and by elements of \( \text{Tri}^c_r \) (because every triangular automorphism fixes \( a \)).

**Theorem 6.** The restriction of the map \( N \) to the subgroup \( P_r \) induces an isomorphism of groups \( P_r \to \text{GL}_r(\mathbb{C}[a]) \).
Proof. By a straightforward induction, the previous lemma implies that $P_r$ acts trivially on the image $N(P_r) \subseteq \text{GL}_r(A_1)$. This means that

$$N^\psi N^\psi = N^\psi \psi_2(N^\psi_1) = N^\psi_2 N^\psi_1$$

for each $\psi_1, \psi_2 \in P_r$, or in other words that the crossed morphism $N$ becomes a genuine morphism of groups when restricted to $P_r$. We claim that the kernel of this morphism is trivial. Indeed, suppose $\psi \in P_r$ is such that $N^\psi = I_r$; then by definition $\psi$ fixes the arrows $x_i, x'_i, y_j$ and $y'_j$. Every automorphism in $P_r$ fixes $a$, hence $\psi$ can only act nontrivially on $a^*$, sending it to $a^* + h$ for some $h \in A_1$. But $\psi$ is symplectic, and in particular $\psi(c_i^1) = c_i^1$, hence

$$[a, a^*] + [a, h] + \sum \alpha d_{\alpha} b_{\alpha} = [a, a^*] + \sum \alpha d_{\alpha} b_{\alpha}$$

from which $[a, h] = 0$ follows. This implies that $h$ is a polynomial in $a$; then it must be zero, as $\psi$ is reduced.

It remains to prove that the image of $N|_{P_r}$ coincides with $\text{GL}_r(\mathbb{C}[a]) \subseteq \text{GL}_r(A_1)$. To do this, it is sufficient to show that it contains a family of generators for $\text{GL}_r(\mathbb{C}[a])$. Such a family is given (see e.g. [HO89]) by the invertible diagonal matrices and by the (elementary) transvections, i.e. matrices of the form $I_r + pe_{\alpha,\beta}$ where $p$ is a polynomial and $e_{\alpha,\beta}$ is the $r \times r$ matrix with 1 at the position $(\alpha, \beta)$ and 0 elsewhere. Clearly, invertible diagonal matrices are contained in $\text{GL}_r(\mathbb{C}) = N(\text{Aff}_r^e)$. Also, the matrix $[\alpha \beta]$, associated to $\Lambda(p(a)B_1)$ is exactly the transvection by $p$ in the $(2, 1)$ plane of $\mathbb{C}[a]^r$. But the symmetric group $S_r$, embedded in $\text{GL}_r(\mathbb{C})$ as the subgroup of permutation matrices, acts transitively on the $r$ axes of the $\mathbb{C}[a]^r$, so that by composing the transvection $N^{\Lambda(p(a)B_1)}$ with suitable permutation matrices in $\text{GL}_r(\mathbb{C})$ we obtain every possible transvection in $\mathbb{C}[a]^r$, as we needed.

Together with theorem[2] this implies that that the restriction of the crossed morphism $M$ to $P_r$ induces an isomorphism $P_r \to \text{GL}_r(\mathbb{C}[a])^{op}$.

Let us note another interesting consequence of theorem[3] Denote by

$$\Psi : \text{GL}_r(\mathbb{C}[a]) \rightarrow P_r$$

the inverse morphism to $N$ on $\text{GL}_r(\mathbb{C}[a])$. By definition, given a matrix $A \in \text{GL}_r(\mathbb{C}[a])$ the automorphism $\Psi(A)$ will act on the arrows $1 \rightarrow 2$ and $2 \rightarrow 1$ in $Q_r$ as $B \mapsto BA$ and $D \mapsto A^{-1}D$ respectively, where $B$ and $D$ are the matrices given by (51). Let us write again $a^* \mapsto a^* + h$ for the action of $\Psi(A)$ on $a^*$. As $\Psi(A)$ fixes $c_i^1$, the following equality in $A_1$ must hold:

$$(52) \quad [a, h] = \sum_{\alpha, \beta, \gamma} (A^{-1}_{\alpha, \beta} A_{\beta, \gamma} + e_{\alpha, \beta}).$$

Then theorem[3] can be rephrased by saying that, for every $A \in \text{GL}_r(\mathbb{C}[a])$, equation (52) can be solved uniquely for $h$. This appears to be quite non-trivial to prove directly.

Definition 5. We denote by $P_r$ the subgroup of $\text{Aut}(\mathbb{C}Q_6; c_r)$ obtained by replacing, in the semidirect product (41), the group $K_{\psi_0}$ with its subgroup $P_r$.

It follows that every element in $P_r$ can be written in a unique way as the composition $\psi_0 \circ \psi$, with $\psi_0 \in \text{Aut}(\mathbb{C}Q_6; c_0)$ and $\psi \in P_r$.

Lemma 7. Every automorphism of $\text{Aut}(\mathbb{C}Q_6; c_0)$ is tame.

Proof. It follows from results of Makar-Limanov [ML70, ML84] that the group $\text{Aut}(\mathbb{C}Q_6; c_0)$ is isomorphic to the group of automorphisms of the first Weyl algebra. Moreover, in [Dix68] Dixmier proved that the latter group is generated by the family of triangular automorphisms $(a, a^*) \mapsto (a, a^* + \frac{1}{2\alpha \beta} p(a))$ indexed by a polynomial $p \in a\mathbb{C}[a]$ and by the single automorphism
$F_0$ defined by $(a, a^*) \mapsto (-a^*, a)$. It is immediate to check that all these automorphisms, when extended from $\mathbb{C}Q_0$ to $\mathbb{C}Q_r$, are tame. \hfill \Box

From this lemma it follows that $\mathcal{P}_r$ is a subgroup of $\text{TAut}(\mathbb{C}Q_r; c_r)$. Notice that the automorphism $F_0$ considered in the previous proof is just the affine symplectomorphism $F_r$ defined by (50) modulo the ideal $I_r$.

We conclude this section by clarifying the rôle of op-triangular symplectomorphisms inside $\mathcal{P}_r$. By analogy with $\text{Tri}^r$, let us denote by $\text{opTri}^r_{\mathcal{L}}$ the subgroup of $\text{opTri}^r$ generated by the image under $\Lambda'$ of necklace words of the form $p(a^*)b_{11}^-$. As

$$N^{\Lambda'}(p(a^*)b_{11}^-) = \begin{pmatrix} 1 & p(a^*) & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}$$

it is clear that $\text{opTri}^r_{\mathcal{L}}$ is isomorphic to the abelian group $(\mathbb{C}[a^*], +)$.

The reader can easily convince himself or herself that all the steps leading to theorem 6 can be carried out equally well for the subgroup $P'_r$ of $\text{TAut}(\mathbb{C}Q_r; c_r)$ generated by $\text{opTri}^r_{\mathcal{L}}$ and $\text{Aff}_r^\text{red}$, replacing the group $\text{GL}_r(\mathbb{C}[a])$ with $\text{GL}_r(\mathbb{C}[a^*]) \subseteq \text{GL}_r(A_1)$. In particular, the restriction of $N$ to $P'_r$ induces an isomorphism $P'_r \rightarrow \text{GL}_r(\mathbb{C}[a^*])$.

Suppose now $r \geq 2$. Let $\varphi$ stand for the affine symplectomorphism acting as $F_2$ on the arrows $a, x_1, y_1$ and their starred version:

$$\begin{cases} a \mapsto -a^* \\ a^* \mapsto a \\ x_1 \mapsto -y_1^* \\ x_1^* \mapsto y_1 \\ y_1 \mapsto -x_1^* \\ y_1^* \mapsto x_1 \end{cases}$$

and fixing all the other arrows in $\mathbb{C}Q_r$. Then the same calculations used in the proof of theorem 5 show that, independently of the parity of $r$, the following equality holds:

$$\varphi^{-1} \circ \Lambda(p(a)b_{11}) \circ \varphi = \Lambda'(-p(a^*)b_{11}^-).$$

As $\varphi \in \mathcal{P}_r$ we conclude that every generator of $\text{opTri}^r_{\mathcal{L}}$ belongs to $\mathcal{P}_r$, hence $P'_r \subseteq \mathcal{P}_r$. Of course, equation (55) also shows that one can equivalently define the group $\mathcal{P}_r$ as the semidirect product $P'_r \rtimes \text{Aut}(\mathbb{C}Q_0; c_0)$. In other words, once we include the semidirect factor $\text{Aut}(\mathbb{C}Q_0; c_0)$, the two groups $P_r$ and $P'_r$ are completely interchangeable.

6. The action of $\mathcal{P}_r$ on Gibbons-Hermsen manifolds

In this section we are concerned with the action of (tame) symplectic automorphisms of $\mathbb{C}Q_r = \mathbb{C}Z_r$ on the manifolds $\mathcal{C}_{n,r}$. Let us start by recalling how this action is defined.

To begin with, we should make explicit the embedding of $\mathcal{C}_{n,r}$ inside the moduli space of representations of the quiver $Z_r$. To do this we need a bijection between $\text{Rep}(\mathbb{Z}_r, (n, 1))$ and the linear space $V_{n,r}$ defined by (11). Let us denote a point in $\text{Rep}(\mathbb{Z}_r, (n, 1))$ by a $2(r + 1)$-tuple of the form

$$\left( A, A, X_1, \ldots, X_{r/2}, X_1, \ldots, X_{r/2}, Y_1, \ldots, Y_{r/2}, \bar{Y}_1, \ldots, \bar{Y}_{r/2} \right)$$

where the arrow $a$ is represented by the matrix $A$, the arrow $a^*$ by the matrix $\bar{A}$ and so on. We identify this point with a quadruple $(X, Y, \bar{v}, \bar{w}) \in V_{n,r}$ using the following correspondence:

$$\begin{cases} A \leftrightarrow X \\ \bar{A} \leftrightarrow Y \\ X_i \leftrightarrow -v_{i,2i-1} \\ \bar{X}_i \leftrightarrow w_{2i-1} \\ Y_j \leftrightarrow w_{2j} \\ \bar{Y}_j \leftrightarrow v_{2j} \end{cases} \quad i = 1 \ldots \lfloor r/2 \rfloor \quad j = 1 \ldots \lfloor r/2 \rfloor.$$
In other words, we build the $n \times r$ matrix $v$ using the $r$ columns $-X_1, Y_1, -X_2, Y_2$ and so on; similarly, we build the $r \times n$ matrix $w$ using the $r$ rows $X_1, Y_1, X_2, Y_2$ and so on.

We can sum up the correspondence between the various arrows in $Q_r$ and their representative matrices as follows:

| arrow in $\mathbb{Z}_r = Q_r$ | $a$ $a^*$ $x_i = -d_{2i-1}$ $x_i^* = b_{2i-1}$ $y_j = b_{2j}$ $y_j^* = d_{2j}$ |
|-----------------------------|------------------------------------------------------------------|
| matrix in $\text{Rep}(\mathbb{Z}_r, (n, 1))$ | $A$ $\bar{A}$ $X_i$ $\bar{X}_i$ $Y_j$ $\bar{Y}_j$ |
| matrix in $C_{n,r}$ | $X$ $Y$ $-v_{2i-1}^*$ $w_{2i-1}^*$ $w_{2j}^*$ $v_{2j}$ |

Notice that for every $\alpha = 1 \ldots r$ the arrow $d_\alpha$ is represented on $V_{n,r}$ by the column matrix $v_{\alpha\bullet}$, and the arrow $b_\alpha$ by the row matrix $w_{\bullet\alpha}$.

As explained in the introduction, on $\text{Rep}(\mathbb{Z}_r, (n, 1))$ there is a natural action of the group $\text{GL}_{n,1}(\mathbb{C}) \simeq \text{GL}_{n}(\mathbb{C})$; the corresponding moment map is

$$ J(A, \bar{A}, X_i, \bar{X}_i, Y_j, \bar{Y}_j) = [A, \bar{A}] + \sum_i X_i \bar{X}_i - \sum_j Y_j \bar{Y}_j. $$

Under the correspondence (57), this becomes

$$ J(X, Y, v, w) = [X, Y] - \sum_{\alpha=1}^r v_{\alpha\bullet} w_{\bullet\alpha}, $$

which is exactly the map (5). We conclude that the symplectic quotient

$$ J^{-1}(\tau I_n)/\text{GL}_n(\mathbb{C}) $$

inside $\text{Rep}(\mathbb{Z}_r, (n, 1))$ is isomorphic to the manifold $C_{n,r}$, as defined by the quotient (5).

We can now explain how the group of symplectomorphisms of the path algebra $\mathbb{C}Q_r$ acts on $C_{n,r}$ for each $n \geq 1$. Given $\psi \in \text{Aut}(\mathbb{C}Q_r; c_r)$, we consider for each arrow $\xi$ in $Q_r$ the non-commutative polynomial $\psi(\xi)$. Given a point $p = (A, \bar{A}, X_i, Y_j, \bar{Y}_j) \in \text{Rep}(\mathbb{Q}_r, (n, 1))$, we can evaluate the polynomial $\psi(\xi)$ at $p$ (by substituting each matrix in $p$ for the arrow it represents) to obtain another matrix $\psi(\xi)(p)$. We define our action by declaring that $\psi$ sends the point $p$ to the point

$$ (\psi(a)(p), \psi(a^*)(p), \psi(x_i)(p), \psi(x_i^*)(p), \psi(y_j)(p), \psi(y_j^*)(p)) $$

in $\text{Rep}(\mathbb{Q}_r, (n, 1))$. One can easily verify that, when $\psi$ is symplectic, this action restricts to an action on each fiber of the moment map $J$ which is constant along the orbits of $\text{GL}_n(\mathbb{C})$. The action on $C_{n,r}$ is just the induced action on the quotient. As this is naturally a right action, we will write the image of the point $p$ under the action of $\psi$ as $p \psi$.

**Example 1.** Let $\psi$ denote the triangular automorphism of $\mathbb{C}Q_3$ given by $\Lambda(a^2 b_{21})$. As

$$ \psi(a, a^*, x_1, x_2, x_1^*, x_2^*, y, y^*) = (a, a^* + ax_2y + x_2ya, x_1, x_2, x_1^*, x_2^* + ya^2, y, y^* + a^2x_2) $$

we have that

$$ (A, \bar{A}, X_1, X_2, \bar{X}_1, \bar{X}_2, Y, \bar{Y}), \psi = (A, \bar{A} + AX_2Y + X_2YA, X_1, X_2, \bar{X}_1, \bar{X}_2 + YA^2, Y, \bar{Y} + A^2X_2) $$

or equivalently, in terms of the coordinates on $V_{n,r}$,

$$ (X, Y, v, w), \psi = (X, Y - Xv_{e_{32}}w - ve_{32}wX, v - X^2v_{e_{32}}w + e_{32}wX^2) $$

where again we are denoting by $e_{\alpha\beta}$ the matrix with 1 at the position $(\alpha, \beta)$ and 0 elsewhere.

It is not difficult to guess how a reduced affine symplectomorphism acts on $C_{n,r}$. In what follows, we will denote by $^t m$ the transpose of a matrix $m$.

**Lemma 8.** Let $\varphi \in \text{Aff}_c^{\text{red}}$ and $T$ be the corresponding matrix in $\text{GL}_r(\mathbb{C})$. Then $(X, Y, v, w), \varphi = (X, Y, v, Tw)$. 

Proof. By definition, $T := N^\circ$ is the only matrix such that
\begin{equation}
\varphi(b_\alpha) = b_1 T_{1\alpha} + \cdots + b_r T_{r\alpha}.
\end{equation}
As the arrow $b_\alpha$ is represented on $V_{n,r}$ by the row matrix $w_{\alpha^*}$, a simple computation shows that the row matrix representing $\varphi(b_\alpha)$, as expressed by (59), coincides with the $\alpha$-th row of the matrix $\hat T w$. Similarly, $T^{-1} = M^\circ$ is the only matrix such that
\begin{equation}
\varphi(d_\alpha) = (T^{-1})_{\alpha 1} d_1 + \cdots + (T^{-1})_{\alpha r} d_r.
\end{equation}
As the arrow $d_\alpha$ is represented on $V_{n,r}$ by the column matrix $v_{\alpha^*}$, it is again easy to verify that the column matrix representing $\varphi(d_\alpha)$, as expressed by (60), coincides with the $\alpha$-th column of the matrix $v^\top T^{-1}$.

It would be tempting to conjecture, by analogy with the result proved in [BP11], that the group $T\text{Aut}({\mathbb C}Q_r; c_r)$ acts transitively on (the disjoint union over $n$ of) the manifolds $C_{n,r}$ for every $r$. However, it does not seem feasible to generalize the proof given in [BP11] for $r = 2$ to the new setting. Instead, following [MT13], we will study this problem on the “large” open subset $R_{n,r} \subset C_{n,r}$ which was defined in the introduction (cf. (15)) as the set of points $[X,Y,v,w] \in C_{n,r}$ such that at least one of the matrices $X$ and $Y$ is regular semisimple, i.e., diagonalizable with distinct eigenvalues. This has the advantage of allowing explicit computations. Our result is that, as in the case $r = 2$, to connect each pair of points in $R_{n,r}$ it suffices to take the (much smaller) subgroup $P_r$ inside $T\text{Aut}({\mathbb C}Q_r; c_r)$.

**Theorem 7.** For every pair of points $p, p' \in R_{n,r}$ there exists $\psi \in P_r$ such that $p \psi = p'$.

**Proof.** The basic strategy is the same as the one used in [BP11] Lemma 8.4 and [MT13], i.e. a reduction to the case $r = 1$. Inside $R_{n,r}$ there is the submanifold
\begin{equation}
N_{n,r} := \{ [X,Y,v,w] \in R_{n,r} \mid v_{\bullet^*} = 0 \text{ and } w_{\bullet^*} = 0 \}
\end{equation}
which is isomorphic to $R_{n,r-1}$. Now suppose that, for every $r > 1$, we are able to move every point $p \in R_{n,r}$ to $N_{n,r}$ using a symplectomorphism in $P_r$. Notice that $P_{r-1}$ naturally embeds into $P_r$ as the subgroup of automorphisms fixing the two arrows in $CQ_r$, that do not appear in $CQ_{r-1}$. Then a straightforward induction on $r$ shows that every point in $R_{n,r}$ can be moved inside (an isomorphic copy of) $R_{n,1}$. As the subgroup $\text{Aut}({\mathbb C}Q_0; c_0) \subseteq P_r$ acts transitively on $C_{n,1} \supseteq R_{n,1}$ [BW00], the theorem follows.

So let us take $r > 1$, $p \in R_{n,r}$ and let $(X,Y,v,w)$ be a representative of the point $p$. We can assume $p$ to be in $C_{n,r}$ (otherwise we only need to act with $F_0$, which exchanges $C_{n,r}$ and $C_{n,r}''$) so that $X = \text{diag}(\lambda_1, \ldots, \lambda_n)$ with $\lambda_i \neq \lambda_j$ for $i \neq j$. Also, recall from the introduction that for a point $p \in C_{n,r}$, we have that $v_{\bullet^*} w_{\bullet^*} = -\tau$ for every $k = 1 \ldots n$. As $\tau \neq 0$, this implies that each row of the matrix $v$ and each column of the matrix $w$ have at least one nonzero entry. Clearly, this result also holds when $p \in C_{n,r}''$.

Let us suppose that $r$ is odd, say $r = 2s + 1$. Then the last column of $v$ represents the arrow $x_{s+1}$ and the last row of $w$ represents the arrow $x_{s+1}^*$. As each column of $w$ has a nonzero entry, there exists a matrix $T \in \text{GL}_r({\mathbb C})$ such that each entry in the $2s$-th row of $T w$ is nonzero. The matrix $T$ corresponds, via lemma 5 to some reduced affine symplectomorphism $\varphi$ acting with it, we arrive at a point $(X,Y,v,w')$ such that $X = \text{diag}(\lambda_1, \ldots, \lambda_n)$ and $w'_{2s,k} \neq 0$ for every $k = 1 \ldots n$. We claim that there exists a unique polynomial $p$ of degree $n - 1$ such that
\begin{equation}
w_{2s+1} = w_{2s} p(X) = 0.
\end{equation}
Indeed, as $X$ is diagonal and the row vector $w_{2s}$ has nonzero entries, to solve equation (61) means to find a polynomial whose value at $\lambda_k$ is given by $-w_{2s+1,k}/w_{2s,k}$ for every $k = 1 \ldots n$, Cool...
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and it is well known that such an interpolation problem always has a unique solution of the stated degree. Then, acting with the triangular symplectomorphism

\[
\Lambda(p(a)b_{s+1,s}) = \begin{cases}
    a^* \mapsto a^* + \frac{\partial}{\partial a}(p(a)x_{s+1}y_s) \\
x_{s+1} \mapsto x_{s+1} + y_sp(a) \\
y_s \mapsto y_s + p(a)x_{s+1}
\end{cases}
\]

we get to a point \((X, Y', v', w')\) such that \(w_{2s+1,} = 0\). Now we exchange \(X\) and \(Y'\) using \(F_0\) and we repeat the same algorithm to kill the last row of \(w'\). Namely, as each row of \(v\) has a nonzero entry there exists a matrix \(T \in \text{GL}_r(C)\) such that every entry in the \(2\)-th column of \(vT\) is nonzero. Let us act with the affine symplectic automorphism corresponding to \(v^T\) and acting with the op-triangular symplectomorphisms

\[
\Lambda'(q(a^*)b_{s+1,s}) = \begin{cases}
a \mapsto a + \frac{\partial}{\partial a}(q(a^*)y_s^*x_{s+1}^*) \\
x_{s+1}^* \mapsto x_{s+1}^* + q(a^*)y_s^* \\
y_s \mapsto y_s + x_{s+1}^*q(a^*)
\end{cases}
\]

(which fixes \(x_{s+1}^*\), hence the last row of \(w\)) we finally arrive at a point in \(N_{n,r}\), as we wanted.

When \(r\) is even, say \(r = 2s\), the proof proceeds along completely analogous lines. In this case the last column of \(v\) represents the arrow \(y_s^*\) and the last row of \(w\) represents the arrow \(y_s\). The reduction from \(R_{n,r}\) to \(N_{n,r}\) is again accomplished in two steps. In the first step we make every entry of \(v_{2s}\) nonzero by acting with a suitable element of \(\text{Aff}^\text{red}\), and then act with

\[
\Lambda(p(a)b_{ss}) = \begin{cases}
a^* \mapsto a^* + \frac{\partial}{\partial a}(p(a)x_sy_s) \\
x_s^* \mapsto x_s^* + y_sp(a) \\
y_s \mapsto y_s + p(a)x_s
\end{cases}
\]

where the polynomial \(p\) of degree \(n-1\) is determined by solving the interpolation problem posed by the system

\[
v_{2s} - p(X)v_{2s-1} = 0.
\]

With this step we reach a point \((X, Y', v', w')\) such that \(v_{2s}' = 0\). In the second step we use \(F_0\) to get into \(C_{n,r}'\); make all the entries of \(w_{2s-1,}\) different from zero in the usual manner, determine the polynomial \(q\) of degree \(n-1\) such that

\[
w_{2s,} + w_{2s-1,}q(Y) = 0,
\]

and act with

\[
\Lambda'(q(a^*)b_{ss}) = \begin{cases}
a \mapsto a + \frac{\partial}{\partial a}(q(a^*)y_s^*x_s^*) \\
x_s \mapsto x_s + q(a^*)y_s^* \\
y_s \mapsto y_s + x_s^*q(a^*)
\end{cases}
\]

to finally land into \(N_{n,r}\).

\(\square\)

Remark 2. As in [MT13], we should emphasize that the subset \(R_{n,r}\) is not closed under the action of \(P_r\). This is clear from the fact that the subgroup \(\text{Aut}(\mathbb{C}Q_0; c_0)\) can send a point in \(R_{n,1}\) to a point such that both \(X\) and \(Y\) are nilpotent (cf. [BW00]).
Before concluding this section, let us briefly explain the relation between the group $\mathcal{P}_r$ and the flows determined by the Hamiltonians $\mathfrak{H}$ of the Gibbons-Hermsen hierarchy. First, notice that

$$J_{k,1} = \text{Tr} Y^k w = \text{Tr} Y^k ([X,Y] - \tau I) = -\tau \text{Tr} Y^k,$$

so that the flow of a Hamiltonian of this form coincides with the action of the op-triangular symplectomorphism $\Lambda'(-\tau a^{k})$.

Consider now the functions $J_{k,m}$ for $m = e_{\alpha\beta}$. When $\alpha = \beta$ the corresponding flow is not polynomial, hence it cannot be realized as the action of an element of $\mathcal{P}_r$. When $\alpha \neq \beta$, the flow of $J_{k,e_{\alpha\beta}}$ is given by

$$
X(t) = X + t \sum_{i=1}^{k} Y^{k-i} v_{\alpha} w_{\beta} Y^{i-1} \\
v_{\alpha}(t) = v_{\alpha} - tY^{k} v_{\alpha} \\
w_{\beta}(t) = w_{\beta} + tw_{\beta} Y^{k}.
$$

(62)

In particular, when $\alpha$ is even and $\beta$ is odd it coincides with the action of the op-triangular automorphism

$$
\begin{align*}
a &\mapsto a + t^{b_{\alpha\beta}} (a^{k} y_{\alpha/2} x_{(\beta+1)/2}) \\
x_{(\alpha+1)/2} &\mapsto x_{(\beta+1)/2} + t a^{k} y_{\alpha/2} \\
y_{\alpha/2} &\mapsto y_{\alpha/2} + t a^{k} x_{(\beta+1)/2} Y^{k}.
\end{align*}
$$

For different parities of $\alpha$ and $\beta$ the corresponding automorphism will no longer be op-triangular, but it will still belong to $\mathcal{P}_r$. Indeed, as $\alpha \neq \beta$ we can always find a permutation matrix $P$ such that $e_{\alpha\beta} = P^{-1} e_{21} P$, so that

$$J_{k,e_{\alpha\beta}} = \text{Tr} Y^k v P^{-1} e_{21} P w = \text{Tr} Y^k \bar{v} e_{21} \bar{w}$$

with $\bar{v} = v P^{-1}$ and $\bar{w} = P w$. By lemma 8 the transformation $(v, w) \mapsto (\bar{v}, \bar{w})$ corresponds to the action of the unique reduced affine symplectomorphism $\varphi$ such that $N^\varphi = t^P$.

Completely analogous results hold for the flow of Hamiltonian functions of the form $\text{Tr} X^k$ or $\text{Tr} X^k v e_{\alpha\beta} w$ (again with $\alpha \neq \beta$). In particular when $\alpha$ is odd and $\beta$ is even such a flow will coincide with the action of the triangular automorphism $\Lambda(t a^{k} b_{\alpha\beta})$.

7. Some examples

Let us illustrate some of the results obtained above for the first few values of $r$. The case $r = 1$ is somewhat degenerate, but we will treat it for completeness. The first zigzag quiver is

$$Z_1 = a \xrightarrow{x} 1 \xrightarrow{2} .$$

As $q_1 = 1 \cdot 0 = 0$, triangular symplectomorphisms of $\mathbb{C}Q_1$ are labeled by an element of $\overline{\text{DR}}^1(\mathbb{C}[a],)$, i.e. a polynomial without constant term. The only reduced symplectomorphisms are the affine ones, acting as $(x, x^*) \mapsto (\lambda x, \lambda^{-1} x^*)$ for some $\lambda \in \mathbb{C}^*$; this action is trivial on representation spaces, so that the group acting on $C_{n,1}$ effectively reduces to $\text{Aut}(\mathbb{C}Q_1; c_0)$ and the action is the same as the one first defined by Berest and Wilson in [BW09]. All this is well known, and related to the fact (mentioned in the introduction) that the rank 1 Gibbons-Hermsen system coincides with the rational Calogero-Moser system. The action of the op-triangular automorphism

$$(a, a^*, x) \mapsto (a + p'(a^*), a^*, x)$$

determined by a polynomial $p \in \overline{\text{DR}}^1(\mathbb{C}[a^*])$ corresponds to a finite linear combinations of Calogero-Moser flows.
Let us proceed to the more interesting case $r = 2$. The quiver
\[
Z_2 = a \begin{array}{c}
\circ \rightarrow \\
\circ \leftarrow \\
\end{array} 2
\]
coinsides with the one studied in [BPT11] and [MT13]. As $q_2 = 1$, triangular symplectomorphisms of $\mathbb{C}Q_2$ are labeled by a necklace word in the two generators $a$ and $b := b_{11} = xy$. One can easily verify that the isomorphism $P_2 \simeq GL_2(\mathbb{C}[a])$ described in section 5 identifies the lower unitriangular matrix $(\begin{smallmatrix} 1 & 0 \\ p(a) & 1 \end{smallmatrix})$ with the triangular automorphism $\Lambda(p(a)b)$ acting as
\[
(a^*, x^*, y^*) \mapsto (a^* + \frac{\partial}{\partial a}(p(a)xy), x^* + yp(a), y^* + p(a)x),
\]
and the upper unitriangular matrix $(\begin{smallmatrix} 1 & p(a) \\ 0 & 1 \end{smallmatrix})$ with the automorphism
\[
(a^*, x, y) \mapsto (a^* + \frac{\partial}{\partial a}(p(a)xy), x + p(a)y^*, y + x^*p(a)).
\]
Similarly, the isomorphism $P'_2 \simeq GL_2(\mathbb{C}[a^*])$ identifies the upper unitriangular matrix $(\begin{smallmatrix} 1 & p(a^*) \\ 0 & 1 \end{smallmatrix})$ with the op-triangular automorphism $\Lambda'(p(a^*)b^*)$, and the lower unitriangular matrix $(\begin{smallmatrix} 1 & 0 \\ p(a^*) & 1 \end{smallmatrix})$ with the automorphism
\[
(a, x^*, y^*) \mapsto (a + \frac{\partial}{\partial a^*}(p(a)y^*x^*), x^* + yp(a^*), y^* + p(a^*)x).
\]
The map $k: GL_2(\mathbb{C}[a^*]) \to T\text{Aut}(\mathbb{C}Q_2; c_2)$ defined in the main proof of [MT13] is essentially the inverse of $M|_{P'_2}$; in particular it is injective (as it was conjectured there). The qualifier “essentially” in the previous sentence is due to the unfortunate choice we made of inverting the map $M$ (which is naturally an anti-morphism of groups, at least using the standard ordering for the composition of maps) instead of $N$. As a consequence, the matrices used in [MT13] are actually the transposes of the ones used here. The notation adopted in the present paper is (hopefully) more consistent.

It is also easy to check that the group $P$ defined in [MT13] coincides (apart from the quotienting out of the subgroup of scalar affine symplectomorphisms) with $P_r$ for $r = 2$. Indeed, in [MT13] we denoted by $P$ the subgroup of $T\text{Aut}(\mathbb{C}Q_2; c_2)$ generated by the reduced affine symplectomorphisms, the op-triangular symplectomorphisms of the form $\Lambda'(p(a^*))$ and $\Lambda'(p(a^*)b^*)$ for some polynomial $p$, and the single affine symplectomorphism $P_2$ defined as in [20]. All these automorphisms belong to $P_2$, as it is immediate to verify. Vice versa, every element of $P_2$ can be written as $\psi_0 \circ \psi$ with $\psi_0 \in \text{Aut}(\mathbb{C}Q_2; c_0)$ and $\psi \in P_2'$; both of these groups are contained in $P$, as defined above.

Remark 3. In [MT13] we were actually interested in embedding into $T\text{Aut}(\mathbb{C}Q_2; c_2)$ the larger group $\Gamma^{\text{alg}}(2)$, where
\[
\Gamma^{\text{alg}}(r) := \{e^pI_r \}_{p \in \mathbb{C}[z]} \times \text{PGL}_r(\mathbb{C}[z])
\]
is the group introduced by Wilson in [Wil09] (see [MT13] for its precise definition). From the results proved in Section 5 it follows that also for every $r > 2$ the group $P_r$ contains (roughly speaking) “two copies” of $\Gamma^{\text{alg}}(r)$, one for $z = a$ and the other for $z = a^*$. Clearly, this fact has some bearing on the general picture described in [MT13, Section 4]. However, we will not explore this connection further in this paper.

Finally let us consider the first “new” case, $r = 3$. The third zigzag quiver is
\[
Z_3 = a \begin{array}{c}
\circ \rightarrow \\
\circ \leftarrow \\
\circ \rightarrow \\
\circ \leftarrow \\
\end{array} 2
\]
As $g_2 = 2$, triangular symplectomorphisms of $\mathbb{C}Q_3$ are indexed by a necklace word $f$ in the linear space $\mathfrak{dr}^\ast(\mathbb{C}(a,b_{11},b_{21}))$, where $b_{11} = x_1 y$ and $b_{21} = x_2 y$. The matrix in $GL_3(A_1)$ corresponding to $\Lambda(f)$ is

$$N^\Lambda(f) = \begin{pmatrix} 1 & 0 & 0 \\ \frac{\partial f}{\partial b_{11}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$  

When restricted to $P_3 \subseteq \text{TAut}(\mathbb{C}Q_3;c_3)$, the map $N$ becomes an isomorphism with $GL_3(\mathbb{C}[a])$. For instance one can compute that the generic upper unitriangular matrix

$$\begin{pmatrix} 1 & p_{12}(a) & p_{13}(a) \\ 0 & 1 & p_{23}(a) \\ 0 & 0 & 1 \end{pmatrix}$$

in $GL_3(\mathbb{C}[a])$ corresponds to the symplectomorphic of $\mathbb{C}Q_3$ defined by

$$\begin{aligned}
  a^* &\mapsto a^* + \frac{\partial}{\partial a}(p_{23}(a)x_2 y) - \frac{\partial}{\partial a}(p_{12}(a)y^* x_1) - p_{23}(a)\frac{\partial}{\partial a}(p_{12}(a)x_2 x_1^*) + \frac{\partial}{\partial a}(p_{13}(a)x_2 x_1^*) \\
y &\mapsto y + x_1 p_{12}(a) \\
x_2^* &\mapsto x_2^* + y p_{23}(a) + x_1^* p_{13}(a) \\
x_1 &\mapsto x_1 + p_{12}(a)y^* + p_{12}(a)p_{23}(a)x_2 - p_{13}(a)x_2 \\
y^* &\mapsto y^* + p_{23}(a)x_2
\end{aligned}$$

One also has similar expressions for automorphisms corresponding to matrices in $GL_3(\mathbb{C}[a^*])$.

**Appendix A. Proof of theorem 3**

Let $G$ stand for the subset $\{g_0, \ldots, g_n\}$ of generators of $A_1$ which appear in equation $42$. We start by proving some general results about an $(n+1)$-tuple $\vec{u} = (u_0, \ldots, u_n)$ of elements of $A_1$ that satisfies equation $42$.

**Lemma 9.** For each $i = 0 \ldots n$, each word of nonzero length in the support of $u_i$ begins and ends with a letter in $G$.

**Proof.** Without loss of generality, let us take $i = 0$ and write $u_0 = \ell m + \chi$, where $\ell$ is a generator, $m$ is a word (possibly of length zero) and $\chi \in A_1$ does not contain $-\ell m$. Equation $42$ becomes

$$g_0 \ell m + g_0 \chi - \ell m g_0 - \chi g_0 + g_1 u_1 - u_1 g_1 + \cdots + g_n u_n - u_n g_n = 0.$$  

Suppose $\ell \notin G$. Then the term $-\ell m g_0$ cannot be canceled by the first two terms (because $\ell \neq g_0$), nor by the fourth (by the hypothesis on $\chi$), nor by a term $g_i u_i$ (because $\ell \neq g_i$), nor by a term $-u_i g_i$ (because $g_i \neq g_0$ for every $i \neq 0$). This contradicts equation $42$, hence it must be that $\ell \in G$. Similarly, writing $u_0 = m \ell + \chi$ the same reasoning implies that $\ell \in G$. \qed

**Lemma 10.** For each $i = 0 \ldots n$, $u_i$ contains a word of the form $m g_i$ if and only if it contains a word of the form $g_i m$.

**Proof.** Again let us fix $i = 0$. Notice that the lemma trivially holds if $m = g_0^s$ for some $s \in \mathbb{N}$, hence we can suppose that $m$ contains a letter different from $g_0$. Write $u_0 = m g_0 + \chi$ for some $\chi \in A_1$ not containing $-m g_0$. Equation $42$ becomes

$$g_0 m g_0 + g_0 \chi - m g_0 g_0 - \chi g_0 + g_1 u_1 - u_1 g_1 + \cdots + g_n u_n - u_n g_n = 0.$$  

Consider the term $g_0 m g_0$. It cannot be canceled by the second term (by hypothesis) and it cannot be canceled by terms coming from other commutators (because $g_i \neq g_0$ for every $i \neq 0$).

\footnote{Recall that the support of a non-commutative polynomial $w \in \mathbb{C}(G)$, denoted supp $w$, is simply the set of monomials (or words) in $G$ which appear in $w$.}
Neither can it be canceled by $-m_0g_0$, because as soon as $m$ contains a generator different from $g_0$ we have that $m_0g_0-g_0m \neq 0$. Then it must be canceled by a term in $-\chi g_0$, i.e. $g_0m \in \text{supp} \chi$. The opposite implication is shown in the same way. 

From these results it follows that each $u_i$ is a word in $\mathbb{C}(G)$, the free algebra generated by $G$.

**Lemma 11.** For each $i = 0 \ldots n$ and for each $j = 0 \ldots n$ with $j \neq i$, $u_i$ contains a word of the form $mg_j$, if and only if $u_j$ contains a word of the form $g_jm$.

**Proof.** Fix $i = 0, j = 1$ and suppose $u_0 = mg_1 + \chi$; equation (12) becomes

$$g_0mg_1 + g_0\chi - mg_1g_0 - \chi g_0 + g_1u_1 - u_1g_1 + \sum_{k=2}^{n} (g_ku_k - u_kg_k) = 0.$$ 

Then, reasoning as in the previous proofs, we see that the term $g_0mg_1$ can be canceled only by $-u_1g_1$, hence $g_0m$ appears in $u_1$. The converse is proved in the same way. \hfill \Box

Now let $S_{\tilde{u}}$ denote the (finite) set consisting of pairs $(i, w)$ where $i = 0 \ldots n$ and $w \in \text{supp} u_i$. By the previous lemmas, given an element $(i, w) \in S_{\tilde{u}}$ with $w = mg_j$ the pair $(j, g,m)$ is again in $S_{\tilde{u}}$. By repeating this operation a sufficient number of times we must eventually return to the original pair $(i, w)$, which means that on $S_{\tilde{u}}$ there is an action of a cyclic group of some finite order. As this argument also holds for every other pair, we conclude that the set $S_{\tilde{u}}$ is partitioned into a finite number of orbits of cyclic groups $C_{k_1}, \ldots, C_{k_s}$ for some integers $k_1, \ldots, k_s \in \mathbb{N}$.

**Example 2.** Let $n = 1$, $G = \langle a, b \rangle$ and consider the pair $\tilde{u} = (bab + bb, aba + ab + ba)$ satisfying $[a, u_0] + [b, u_1] = 0$. Then

$$S_{\tilde{u}} = \{(0, bab), (0, bb), (1, aba), (1, ab), (1, ba)\}.$$ 

There are two orbits, the first given by

$$(0, bab) \rightarrow (1, aba) \rightarrow (0, bab)$$

and the second by

$$(0, bb) \rightarrow (1, ab) \rightarrow (1, ba) \rightarrow (0, bb).$$

**Proof of theorem 3.** One direction is immediate: if $u_i = \partial f / \partial g_i$ then by [Gin05 Prop. 11.5.4] we have that

$$\sum_{i=0}^{n} \left[ \frac{\partial f}{\partial g_i} : g_i \right] = 0$$

which is equation (12). For the converse, let $S_{\tilde{u}}$ be as above. Take $(k, w) \in S_{\tilde{u}}$ and let $O_w$ be the orbit of $(k, u)$ under the action of the corresponding cyclic factor. For each $i = 0 \ldots n$, define $u_i^w$ to be the sum of monomials in $u_i$ that belongs to $O_w$. Then the necklace word $f^w := g_kw$ is such that

$$\frac{\partial f^w}{\partial g_i} = c_wu_i^w$$

for every $i$, where $c_w \in \mathbb{N}$ is some combinatorial factor which depends only on the structure of the particular monomial $w$ chosen. By repeating this construction for every orbit and summing all the necklace words so obtained (divided by $c_w$ if necessary) we get the required primitive for $\tilde{u}$. \hfill \Box

In example 2 above, we can take for instance the pair $(0, bab)$ in the first orbit, which produces the necklace word $f_1 = abab$ with

$$\frac{\partial}{\partial a} f_1 = 2bab \quad \text{and} \quad \frac{\partial}{\partial b} f_1 = 2aba,$$
and the pair $(1, ba)$ in the second orbit, which gives the necklace word $f_2 = bba$ with
\[
\frac{\partial}{\partial a} f_2 = bb \quad \text{and} \quad \frac{\partial}{\partial b} f_2 = ba + ab.
\]

Then it is immediate to check that $f = \frac{1}{2} abab + bba$ is in fact a primitive for the pair $\overrightarrow{u} = (bab + bb, aba + ab + ba)$.

**Appendix B. Other possible choices of the non-commutative symplectic form**

In the introduction we pointed out that for each $r > 1$ there are many quivers whose double coincides with $Q_r$; choosing a quiver different from $Z_r$ will alter the non-commutative symplectic form on $\mathbb{C}Q_r$ and consequently its group of (tame) symplectomorphisms.

For an extreme example of this phenomenon, suppose that we make the (apparently very natural) choice of taking the unstarred arrows in $Q_r$ all oriented in the same direction, e.g. by selecting the arrows $d_1, \ldots, d_r$ and putting $d_\alpha^* = b_\alpha$ for every $\alpha = 1 \ldots r$. The moment element in $\mathbb{C}Q_r$ determined by this choice is
\[
\tilde{c}_r = [a, a^*] + \sum_{\alpha = 1}^r [d_\alpha, d_\alpha^*],
\]
which can be decomposed as $\tilde{c}_1^r + \tilde{c}_2^r$ with $\tilde{c}_1^r = [a, a^*] + \sum_{\alpha} d_\alpha d_\alpha^*$ and $\tilde{c}_2^r = -\sum_{\alpha} d_\alpha^* d_\alpha$.

Now let $\psi$ be an automorphism of $\mathbb{C}Q_r$, fixing the arrows $(a, d_1, \ldots, d_r)$. Let us write as usual $\psi(a^*) = a^* + h$ for some $h \in A_1$ and $\psi(d_\alpha^*) = d_\alpha^* + s_\alpha$ for some $s_1, \ldots, s_r \in A_{21}$. Then $\psi$ fixes $\tilde{c}_2^r$ if and only if
\[
(64) \quad \sum_{\alpha = 1}^r s_\alpha d_\alpha = 0.
\]

As $A_{21}$ is a free right $A_1$-module we can write $s_\alpha = \sum_\beta b_\beta u_{\beta \alpha}$ for some unique coefficients $u_{\beta \alpha} \in A_1$, so that equation (64) becomes
\[
\sum_{\alpha, \beta = 1}^r b_\beta u_{\beta \alpha} d_\alpha = 0.
\]

Multiplying by $b_1$ from the right and by $d_1$ from the left we get
\[
\sum_{\alpha, \beta = 1}^r c_{1, \beta} u_{\beta \alpha} e_{\alpha 1} = 0.
\]

As the $c_{\alpha, \beta}$ are independent in $A_1$, this implies that $\sum_\beta c_{1, \beta} u_{\beta \alpha} = 0$ for every $\alpha = 1 \ldots r$, which in turn implies $u_{\beta \alpha} = 0$ for every $\alpha, \beta = 1 \ldots r$. We conclude that with this choice of the non-commutative symplectic form every triangular symplectomorphism fixes all the arrows $d_1^*, \ldots, d_r^*$. Clearly, this makes the resulting group of tame symplectomorphisms hardly useful.

In general, reasoning as in Section 3 one can show that for any given choice of a quiver $Q_r$ such that $\overline{Q}_r = Q_r$ the corresponding triangular symplectomorphisms of $\mathbb{C}Q_r$ will be labeled by a (non-scalar) necklace word in the free algebra generated by all the possible cycles $1 \to 1$ of length at most 2 that one can make in $Q_r$ (i.e. using only the unstarred arrows in $Q_r$). In this sense choosing the zigzag quiver $Z_r$ makes the group $\text{TAut}(\mathbb{C}Q_r; c_r)$ the largest possible.

Another interesting choice, which gives the minimal (non-trivial) result, is to take a single arrow going in one direction, e.g. $x := d_1$, and all the other arrows going in the opposite one, say $y_k := b_{k+1}$ for every $k = 1 \ldots r - 1$. Let us call $Q'_r$ the quiver obtained in this way (notice
that $Q'_r$ is again the Bielawski-Pidstrygach quiver). Then we have the $r - 1$ cycles $\ell_k := xy_k$ in $Q'_r$, so that triangular symplectomorphisms are indexed by elements of the linear space $\mathcal{D}R^0(\mathbb{C}(a, \ell_1, \ldots, \ell_{r-1}))$, the automorphism corresponding to a necklace word $f$ being given by

$$(a^*, x^*, y^*) \mapsto (a^* + \frac{\partial f}{\partial a} x^* + \sum_{k=1}^{r-1} y_k \frac{\partial f}{\partial \ell_k}, y^* + \frac{\partial f}{\partial \ell_k} x^*).$$

It is not difficult to see that the resulting group of tame symplectomorphisms of $\mathbb{C}Q'_r = \mathbb{C}Q_r$ contains the group $P_r$ defined in Section 5. In fact, it seems that most (if not all) of the results of the present paper do not actually depend on the particular choice of the family of quivers $(Z_r)_{r \geq 1}$, as long as one rules out the trivial case shown at the beginning of this appendix.

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ICMC - Universidade de São Paulo, Avenida Trabalhador São-carlense, 400, 13566-590 São Carlos - SP, Brasil.

E-mail address: tacchella@icmc.usp.br