Closed EP and hypo-EP operators on Hilbert spaces

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Abstract
A bounded linear operator $A$ on a Hilbert space $\mathcal{H}$ is said to be an EP (hypo-EP) operator if ranges of $A$ and $A^*$ are equal (range of $A$ is contained in range of $A^*$) and $A$ has a closed range. In this paper, we define EP and hypo-EP operators for densely defined closed linear operators on Hilbert spaces and extend results from bounded linear operator settings to (possibly unbounded) closed linear operator settings.

Keywords Moore-Penrose inverse · EP operator · Hypo-EP operator

Mathematics Subject Classification 47A05 · 47A55 · 47A65

1 Introduction

A square matrix $A$ over the complex field $\mathbb{C}$ is said to be an EP matrix (EP stands for Equal Projections) if ranges of $A$ and $A^*$ are equal. Nevertheless, the EP matrix was defined by Schwerdtfeger [21] in 1950; it could not get any greater attention until Pearl [17] gave an interesting characterization of EP matrix through Moore-Penrose inverse: $A$ is an EP matrix if and only if $A^\dagger A = AA^\dagger$. Campbell and Meyer [4] extended the notion of EP matrix to a bounded linear operator with a closed range defined on a Hilbert space, using the Pearl’s characterization. Itoh [11] introduced hypo-EP operator by weakening the Pearl’s characterization as $A^\dagger A - AA^\dagger$ is a positive operator.
In what follows, we will use the term operator to mean a linear operator with domain and range in (real or complex) Hilbert spaces. EP matrices, bounded EP operators and bounded hypo-EP operators have been studied by many authors [3, 4, 6, 9, 12, 13, 18, 23]. In this paper, we extend characterizations and results of bounded EP and hypo-EP operators to (possibly unbounded) closed operators on Hilbert spaces. We recall that an operator on a Hilbert space $\mathcal{H}$ is bounded if and only if it is continuous. It follows that unbounded operators are discontinuous (everywhere). Although the existence of unbounded operators on the whole of the Hilbert space $\mathcal{H}$ can be shown (cf. [14]), unbounded operators of theoretical and practical interests are closed operators defined on subspaces of $\mathcal{H}$. Consequently, the specification of the subspace $\mathcal{D}$ on which $A$ is defined, called the domain of $A$, denoted by $\mathcal{D}(A)$, is to be given, when $A$ is an unbounded operator. The null space and range of $A$ are denoted by $\mathcal{N}(A)$ and $\mathcal{R}(A)$ respectively, $M^\perp$ denotes the orthogonal complement of a set $M$ whereas $M \oplus N$ denotes the orthogonal direct sum of the subspaces $M$ and $N$. For the sake of completeness of exposition, we first begin with the definition of a closed operator.

**Definition 1.1** Let $A$ be an operator from a Hilbert space $\mathcal{H}$ with domain $\mathcal{D}(A)$ to a Hilbert space $\mathcal{K}$. If the graph of $A$ defined by

$$\mathcal{G}(A) = \{ (x, Ax) : x \in \mathcal{D}(A) \}$$

is closed in $\mathcal{H} \times \mathcal{K}$, then $A$ is called a closed operator. Equivalently, $A$ is a closed operator if $x_n \in \mathcal{D}(A)$ such that $x_n \to x$ and $Ax_n \to y$ for some $x \in \mathcal{H}, y \in \mathcal{K}$, then $x \in \mathcal{D}(A)$ and $Ax = y$.

By the closed graph theorem, a closed operator with domain $\mathcal{D}(A)$ is a bounded operator if and only if $\mathcal{D}(A)$ is closed (cf. [14], Theorem 7.1.2, page 300). In other words, a closed operator defined on all of $\mathcal{H}$ is necessarily bounded. Given an operator $A$ from $\mathcal{H}$ to $\mathcal{K}$ with dense domain $\mathcal{D}(A)$, there exists a unique operator $A^*$ such that

$$\langle Ax, y \rangle = \langle x, A^* y \rangle \quad \text{for } x \in \mathcal{D}(A), y \in \mathcal{D}(A^*),$$

(1) where

$$\mathcal{D}(A^*) = \{ y \in \mathcal{K} : x \to \langle Ax, y \rangle \text{ for all } x \in \mathcal{D}(A) \text{ is continuous} \}.$$ This operator $A^*$ is known as the adjoint of $A$. An operator having a dense domain is called a densely defined operator. The adjoint of any densely defined (not necessarily closed) operator is always closed. The set of all densely defined closed operators from $\mathcal{H}$ to $\mathcal{K}$ is denoted by $\mathcal{C}(\mathcal{H}, \mathcal{K})$ and we write $\mathcal{C}(\mathcal{H}) = \mathcal{C}(\mathcal{H}, \mathcal{H})$. We denote the set of bounded operators from $\mathcal{H}$ into $\mathcal{K}$ by $\mathcal{B}(\mathcal{H}, \mathcal{K})$ and $\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H}, \mathcal{H})$. We call $\mathcal{D}(A) \cap \mathcal{N}(A)^\perp$, the carrier of $A$ and it is denoted by $\mathcal{C}(A)$. We note that, for any $A \in \mathcal{C}(\mathcal{H})$, the closure of $\mathcal{C}(A)$, that is, $\overline{\mathcal{C}(A)}$ is $\mathcal{N}(A)^\perp$.

If $A$ and $B$ are in $\mathcal{C}(\mathcal{H})$ with $\mathcal{D}(A) \subseteq \mathcal{D}(B)$ and $Ax = Bx$ for all $x \in \mathcal{D}(A)$, then $B$ is called an extension of $A$ and is denoted by $A \subset B$. Also, we write $A = B$ if $A \subset B$ and $B \subset A$. Among the unbounded operators, densely defined closed operators are...
certainly the nicest ones. For instance, the adjoint $A^*$ of a densely defined closed operator $A$ exists and it is also densely defined with $A = A^{**}$. However, there are densely defined unbounded operators $A$ with $\mathcal{D}(A^*)$ is zero.

In the sequel, we will need the following definitions and results.

**Definition 1.2** [19, 20] Let $A \in \mathcal{C}(\mathcal{H})$. The operator $A$ is said to be

1. **normal** if $AA^* = A^*A$, where either side is defined on its standard domain.
2. **symmetric** if $A \subset A^*$.
3. **self-adjoint** if $A = A^*$.
4. **positive** if $A$ is self-adjoint and $\langle Ax, x \rangle \geq 0$ for all $x \in \mathcal{D}(A)$, and denote this fact by $A \geq 0$.

The number $\gamma(A) = \inf \{\|Ax\| : x \in C(A), \|x\| = 1\}$ is called the **reduced minimum modulus** of $A$. Moreover, $\gamma(A) = \gamma(A^*)$. The operator $|A| := (A^*A)^{1/2}$ is called the **modulus of $A$**. It can be verified that $\mathcal{D}(|A|) = \mathcal{D}(A)$ and $\mathcal{N}(|A|) = \mathcal{N}(A)$ and $\mathcal{R}(|A|) = \mathcal{R}(A^*)$.

In the above, the square-root of a positive self-adjoint operator $A$ may be defined using spectral theorem (cf. [24]). A simple proof for the existence of a square-root of a positive unbounded operator was given by Bernau [2].

It is well-known that the Moore-Penrose inverse $A^\dagger$ of a bounded operator on $\mathcal{H}$ is a closed densely defined operator with $\mathcal{D}(A^\dagger) := \mathcal{R}(A) \oplus \mathcal{R}(A)^\perp$ (cf. [14], Theorem 14.3.7). Analogously, the Moore-Penrose inverse $A^\dagger$ of a closed densely defined operator $A$ can be defined with $\mathcal{D}(A^\dagger) := \mathcal{R}(A) \oplus \mathcal{R}(A)^\perp$ and taking values in $C(A)$ by associating each $y \in \mathcal{D}(A^\dagger)$ to the unique $A^\dagger y$ such that

$$AA^\dagger y = Qy,$$

where $Q$ is the orthogonal projection of $\mathcal{H}$ onto $\overline{\mathcal{R}(A)}$. It can be seen that $\mathcal{N}(A^\dagger) = \mathcal{R}(A)^\perp$ and

$$A^\dagger Ax = Px \text{ for } x \in \mathcal{D}(A),$$

where $P$ is the orthogonal projection of $\mathcal{H}$ onto $\overline{C(A)}$. Again, $A^\dagger$ is a closed densely defined operator (cf. [15, 16]).

In the following result, we give some characterizations of densely defined closed operators to have closed ranges.

**Theorem 1.3** [1, 22] Let $A \in \mathcal{C}(\mathcal{H})$. Then the following statements are equivalent:

1. $\mathcal{R}(A)$ is closed;
2. $\mathcal{R}(A^*)$ is closed;
3. $\mathcal{R}(AA^*)$ is closed;
4. $\mathcal{R}(A^*A)$ is closed;
5. $\mathcal{R}(A) = \mathcal{R}(AA^*)$;
6. $\mathcal{R}(A^*) = \mathcal{R}(A^*A)$;

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In the following, we list some more properties of operators in $\mathcal{C}(\mathcal{H})$.

**Theorem 1.4** [1, 22] Let $A \in \mathcal{C}(\mathcal{H})$. Then the following statements hold good.

1. If $\mathcal{D}(AA^*) = \mathcal{D}(A^2)$, then $\mathcal{D}(A) = \mathcal{D}(A^2)$.
2. If $\mathcal{D}(A) \subset \mathcal{D}(A^*)$, then $A$ is bounded if and only if $\mathcal{D}(A) = \mathcal{D}(A^2)$.
3. $A$ is bounded if and only if $\mathcal{R}(A^\dagger)$ is closed.
4. $A$ is bounded if and only if $\mathcal{D}(AA^*) = \mathcal{D}(A^*)$.
5. $A^*$ is bounded if and only if $\mathcal{D}(A^*A) = \mathcal{D}(A)$.

In the following result, we list some relationships among the concepts, majorization, range inclusion, and factorization, which are studied in a general setting for densely defined closed operators by Douglas in 1966 [7]. Note that (11) and (12) in Theorem 1.3 give majorization and factorization type characterizations of closed range operators.

**Theorem 1.5** [7] Let $A$ and $B$ be in $\mathcal{C}(\mathcal{H})$. Then the following statements are true.

1. If $AA^* \leq BB^*$, then there exists a contraction $C$ so that $A \subset BC$.
2. If $C$ is an operator so that $A \subset BC$, then $\mathcal{R}(A) \subset \mathcal{R}(B)$.
3. If $\mathcal{R}(A) \subset \mathcal{R}(B)$, then there exists a densely defined operator $C$ so that $A = BC$ and a number $k > 0$ so that $\|Cx\|^2 \leq k\left\{\|x\|^2 + \|Ax\|^2\right\}$ for all $x \in \mathcal{D}(C)$.

Moreover, if $A$ is bounded, then $C$ is bounded; if $B$ is bounded, then $C$ is closed.

Bounded $EP$ and hypo-$EP$ operators have been studied by many authors. However, $EP$ and hypo-$EP$ operators, in the setting of unbounded operators, did not get much attention in the literature. Motivated by the results in [3, 4, 6] for bounded $EP$ operators and in [12, 18, 23] for bounded hypo-$EP$ operators, we have made an effort in the paper to glean the results for the class of (possibly unbounded) closed densely defined operators on Hilbert spaces.

### 2 Closed $EP$ operators on Hilbert spaces

We begin the section with the definition of bounded $EP$ operator and some of its characterizations. The notion of $EP$ operator was introduced by Campbell and Meyer [4] in 1975. Brock [3] gave a few more characterizations of $EP$ operators.
Definition 2.1 [4] An operator \( A \in \mathcal{B}(\mathcal{H}) \) is called an \textbf{EP operator} if \( A \) has a closed range and \( \mathcal{R}(A) = \mathcal{R}(A^*) \).

As already mentioned in the Introduction, \( EP \) stands for \textit{Equal Projections}. Note that the set of all bounded \( EP \) operators contains all bounded normal operators with closed ranges.

Theorem 2.2 [3] Let \( A \in \mathcal{B}(\mathcal{H}) \) with a closed range. Then the following statements are equivalent:

1. \( A \) is an \( EP \) operator;
2. \( AA^\dagger = A^\dagger A \);
3. \( \mathcal{N}(A) = \mathcal{N}(A^\dagger) \);
4. \( \mathcal{N}(A) = \mathcal{N}(A^*) \);
5. \( A^* = PA \), where \( P \) is a bijective bounded operator on \( \mathcal{H} \).

The following result characterizes densely defined closed operators with closed ranges to have the same ranges of ‘operators’ and ‘their adjoints’.

Theorem 2.3 Let \( A \in \mathcal{C}(\mathcal{H}) \) with a closed range. Then the following statements are equivalent:

1. \( \mathcal{R}(A) = \mathcal{R}(A^*) \);
2. \( AA^\dagger = A^\dagger A \) on \( \mathcal{D}(A) \);
3. \( \mathcal{N}(A) = \mathcal{N}(A^\dagger) \);
4. \( \mathcal{N}(A) = \mathcal{N}(A^*) \);
5. \( \mathcal{N}(A) = \mathcal{N}(A^*/C3) \);
6. \( \mathcal{C}(A) = \mathcal{R}(A) \);
7. \( \mathcal{H} = \mathcal{R}(A) \oplus \mathcal{N}(A) \).

\textbf{Proof} Suppose that \( \mathcal{R}(A) = \mathcal{R}(A^*) \). Let \( x \in \mathcal{D}(A) = \mathcal{N}(A) \oplus C(A) \). Then \( x = x_1 + x_2, x_1 \in \mathcal{N}(A), x_2 \in C(A) = \mathcal{R}(A^\dagger) \). Hence \( A^\dagger A x = A^\dagger A(x_1 + x_2) = A^\dagger A x_2 = x_2 \). As \( C(A) = \mathcal{R}(A^\dagger) \subseteq \mathcal{R}(A^*) = \mathcal{R}(A) \) and \( \mathcal{N}(A) = \mathcal{R}(A^*) = \mathcal{R}(A) \) and \( \mathcal{N}(A) = \mathcal{N}(A^*) = \mathcal{N}(A) \) and \( \mathcal{N}(A^*) = \mathcal{N}(A) \). Hence the implication (1 \( \iff \) 2) is proved.

In view of the equivalent statements

\[
\mathcal{R}(A) = \mathcal{R}(A^*),
\]
\[
\mathcal{N}(A) = \mathcal{N}(A^\dagger),
\]
\[
\mathcal{N}(A) = \mathcal{N}(A^*),
\]

we obtain the equivalence (1 \( \iff \) 3 \( \iff \) 4).

Taking the orthogonal complements, we obtain the implication (4 \( \iff \) 5). We have already observed that \( \mathcal{C}(A) = \mathcal{N}(A^*) \). Hence, we obtain (5 \( \iff \) 6).
Next we assume that \( \mathcal{H} = \mathcal{R}(A) \oplus \mathcal{N}(A) \). Since \( \mathcal{H} = \mathcal{R}(A^*) \oplus \mathcal{R}(A^*)^\perp = \mathcal{R}(A^*) \oplus \mathcal{N}(A) \), we get \( \mathcal{R}(A) = \mathcal{R}(A^*) \). Then
\[
\mathcal{H} = \mathcal{R}(A) \oplus \mathcal{R}(A)^\perp = \mathcal{R}(A) \oplus \mathcal{R}(A^*)^\perp = \mathcal{R}(A) \oplus \mathcal{N}(A)
\]
so that (1) and (7) are equivalent. \( \square \)

**Remark 2.4** It is proved in Theorem 2.3 that \( AA^\dagger = A^\dagger A \) on \( \mathcal{D}(A) \) if and only if \( \mathcal{N}(A) = \mathcal{N}(A^*) \). If we drop the assumption that \( \mathcal{R}(A) \) is closed, we get that \( AA^\dagger \subseteq A^\dagger A \) if and only if \( \mathcal{N}(A) = \mathcal{N}(A^*) \) and \( \mathcal{D}(A^\dagger) \subseteq \mathcal{D}(A) \) [8]. Indeed, if \( AA^\dagger \subseteq A^\dagger A \), then \( \mathcal{D}(A^\dagger) \subseteq \mathcal{D}(A) \) and hence \( AA^\dagger = A^\dagger A \) on \( \mathcal{D}(A^\dagger) \). So \( \mathcal{R}(A) = \mathcal{R}(A^\dagger) \), and thus \( \mathcal{N}(A) = \mathcal{N}(A^*) \). Conversely, if \( \mathcal{N}(A) = \mathcal{N}(A^*) \) and \( \mathcal{D}(A^\dagger) \subseteq \mathcal{D}(A) \), then \( \mathcal{N}(A^\dagger) = \mathcal{N}(A^*) \), and hence \( \mathcal{R}(A^\dagger) = \mathcal{R}(A^*) \). Therefore we have \( AA^\dagger x = A^\dagger Ax \) for all \( x \in \mathcal{D}(A^\dagger) \). Hence \( AA^\dagger \subseteq A^\dagger A \). Similarly, we can prove that \( A^\dagger A \subseteq AA^\dagger \) if and only if \( \mathcal{N}(A) = \mathcal{N}(A^*) \) and \( \mathcal{D}(A) \subseteq \mathcal{D}(A^\dagger) \).

To have an analogous result of Theorem 2.2 for closed operators, we give the following definition of \( EP \) for closed operators so that Theorem 2.3 will give necessary and sufficient conditions for an operator in \( \mathcal{C}(\mathcal{H}) \) to be \( EP \).

**Definition 2.5** Let \( A \) be a densely defined closed operator on a Hilbert space \( \mathcal{H} \). The operator \( A \) is said to be an **\( EP \) operator** if \( A \) has a closed range and \( \mathcal{R}(A) = \mathcal{R}(A^*) \).

Clearly, the class of unbounded \( EP \) operators includes all densely defined self-adjoint operators with closed ranges. Also, every densely defined normal operator with a closed range is \( EP \). Indeed, let \( A \in \mathcal{C}(\mathcal{H}) \) be normal with a closed range. Then \( AA^* = A^*A \) and by Theorem 1.4 (1), we have \( \mathcal{D}(A) = \mathcal{D}(A^*) \). Since \( \mathcal{R}(A) \) is closed, \( \mathcal{R}(A) = \mathcal{R}(AA^*) \) and \( \mathcal{R}(A^*) = \mathcal{R}(A^*A) \). Thus \( A \) is \( EP \).

We now see some examples in the class of \( EP \) operators.

**Example 2.6** Define \( A \) on \( \ell_2 \) by
\[
A(x_1, x_2, x_3, \ldots) = (x_1, 2x_2, 3x_3, \ldots)
\]
with domain \( \mathcal{D}(A) = \{ (x_1, x_2, x_3, \ldots) \in \ell_2 : \sum_{n=1}^{\infty} |nx_n|^2 < \infty \} \). As the sequence \( (1/n)_{n=1}^{\infty} \) belongs to \( \ell_2 \) but not in \( \mathcal{D}(A) \) and \( \mathcal{D}(A) \) contains the space \( c_{00} \) of all finitely non-zero sequences, \( \mathcal{D}(A) \) is a proper dense subspace of \( \ell_2 \). Also, \( \mathcal{R}(A) \) is closed because \( A \) is surjective. Moreover, \( A \) is self-adjoint and unbounded. Thus \( A \in \mathcal{C}(\mathcal{H}) \) and it is an \( EP \) operator.

We have now seen an example of an unbounded densely defined closed operator, which is \( EP \). It is known that for \( A \in \mathcal{B}(\mathcal{H}) \), \( A \) is \( EP \) if and only if \( A^\dagger \) is \( EP \), which is not true in general for \( A \in \mathcal{C}(\mathcal{H}) \). Even if \( A \in \mathcal{C}(\mathcal{H}) \) has a closed range, \( A^\dagger \) may not have a closed range by Theorem 1.4 (3). In the following example, we give a densely defined closed operator satisfying the ‘range condition’, but it is not \( EP \).
Example 2.7  Define $A$ on $\ell_2$ by

$$A(x_1, x_2, x_3, \ldots) = \left(x_1, 2x_2, \frac{x_3}{3}, 4x_4, \frac{x_5}{5}, \ldots\right)$$

with domain $\mathcal{D}(A) = \left\{(x_1, x_2, x_3, \ldots) \in \ell_2 : (x_1, 2x_2, \frac{x_3}{3}, 4x_4, \frac{x_5}{5}, \ldots) \in \ell_2\right\}$. As the sequence $(1, \frac{1}{2}, 0, \frac{1}{4}, 0, \ldots) \in \ell_2 \setminus \mathcal{D}(A)$ and $\mathcal{D}(A)$ contains the space $c_{00}$, $\mathcal{D}(A)$ is a proper dense subspace of $\ell_2$. Since $A$ is self-adjoint, $A$ is a closed operator and $\mathcal{R}(A) = \mathcal{R}(A^\ast)$. Moreover, the sequence $(1/n)_{n=1}^\infty \in \ell_2 \setminus \mathcal{R}(A)$ and $\mathcal{R}(A)$ is dense in $\ell^2$ as $c_{00} \subseteq \mathcal{R}(A)$. Hence $\mathcal{R}(A)$ is a proper dense subspace of $\mathcal{H}$, so it cannot be closed. Thus $A$ is not an $EP$ operator.

Example 2.8  [10] Let $\varphi : [0, 1] \to \mathbb{C}$ by

$$\varphi(t) = \begin{cases} 
1 & \text{if } t = 0 \\
\frac{1}{\sqrt{t}} & \text{if } 0 < t \leq 1.
\end{cases}$$

Define

$$Af = \varphi f, \quad f \in \mathcal{D}(A),$$

where $\mathcal{D}(A) = \left\{ f \in L^2[0, 1] : \varphi f \in L^2[0, 1]\right\}$. Then $A$ is a densely defined closed operator. As $|\varphi(t)| \geq 1$ for all $t \in [0, 1]$, we have $\mathcal{R}(A) = L^2[0, 1]$ and $A$ has bounded inverse $A^{-1} : L^2[0, 1] \to L^2[0, 1]$ defined by $A^{-1}g = \psi g$ for all $g \in L^2[0, 1]$ where

$$\psi(t) = \begin{cases} 
1 & \text{if } t = 0 \\
\frac{1}{\sqrt{t}} & \text{if } 0 < t \leq 1.
\end{cases}$$

Hence $A$ is an $EP$ operator on $L^2[0, 1]$.

Example 2.9  Let $\mathcal{H} = L^2[0, 1]$. Let

$$\mathcal{AC}[0, 1] = \left\{ f \in \mathcal{H} : f : [0, 1] \to \mathbb{C} \text{ is absolutely continuous and } f' \in \mathcal{H}\right\}.$$ 

Let $\mathcal{D}(A) = \left\{ f \in \mathcal{AC}[0, 1] : f(0) = f(1)\right\}$. Define $A : \mathcal{D}(A) \to \mathcal{H}$ by

$$Af = if' \quad \text{for all } f \in \mathcal{D}(A).$$

It can be shown that $A$ is a densely defined self-adjoint operator and

$$\mathcal{R}(A) = \left\{ u \in \mathcal{H} : \int_0^1 u(t) dt = 0 \right\} = \text{span}\{1\}^\perp.$$ 

Hence $A$ is an $EP$ operator.

Now, we prove some results related to unbounded $EP$ operators.
**Proposition 2.10** Let $A \in \mathcal{C(H)}$ with a closed range. If $A$ is EP, then the following statements are true.

1. $A^*$ is EP.
2. $AA^*$ is EP.
3. $A^*A$ is EP.
4. $|A|$ is EP.

**Proof** The proof of (1) is obvious from the definition of EP operator and Theorem 1.3. Since $AA^*$ and $A^*A$ are self-adjoint and by Theorem 1.3, $AA^*$ and $A^*A$ are EP. Since $\mathcal{R}(A)$ is closed, by Theorem 1.3, we have $\mathcal{R}(A^*) = \mathcal{R}(A^*A)$. Also, we have $\overline{\mathcal{R}(A^*)} = \mathcal{R}(A^*)$. Now $\mathcal{R}(A^*A) \subseteq \mathcal{R}(A) \subseteq \mathcal{R}(|A|) = \mathcal{R}(A^*) = \mathcal{R}(A^*A)$. Hence $\mathcal{R}(A^*) = \mathcal{R}(|A|)$. Therefore $\mathcal{R}(|A|)$ is closed. Since $|A|$ is self-adjoint and $\mathcal{R}(|A|)$ is closed, $|A|$ is EP. 

**Theorem 2.11** Let $A \in \mathcal{C(H)}$ with a closed range. Then the following statements are equivalent:

1. $A$ is EP;
2. $AA^\dagger = A^\dagger A$ on $\mathcal{D}(A)$;
3. $\mathcal{N}(A) = \mathcal{N}(A^\dagger)$;
4. $\mathcal{N}(A) = \mathcal{N}(A^*)$;
5. $\mathcal{N}(A)^\perp = \mathcal{R}(A)$;
6. $\overline{\mathcal{C}(A)} = \mathcal{R}(A)$;
7. $\mathcal{H} = \mathcal{R}(A) \oplus \mathcal{N}(A)$.

**Proof** Follows from Theorem 2.3. 

**Remark 2.12** If $A \in \mathcal{C(H)}$ is an unbounded EP operator, then $\mathcal{R}(A^\dagger)$ is a proper dense subspace of $\mathcal{R}(A)$ because $\overline{\mathcal{R}(A^\dagger)} = \overline{\mathcal{R}(A^*)} = \mathcal{R}(A^*) = \mathcal{R}(A)$ and by Theorem 1.4 (3).

**Theorem 2.13** Let $A \in \mathcal{C(H)}$ be EP. Then there exists a densely defined bijective linear operator $C : \mathcal{D}(A^*) \to \mathcal{D}(A)$ such that $A^* \subset AC$.

**Proof** Assume that $A$ is EP. Let $y \in \mathcal{D}(A^*)$. Then $y = y_1 + y_2, y_1 \in \mathcal{C}(A^*)$, $y_2 \in \mathcal{N}(A^*)$. Since $\mathcal{R}(A^*) = \mathcal{R}(A)$, $A^*y = Ax$, for some $x \in \mathcal{D}(A)$ with $x = x_1 + x_2, x_1 \in \mathcal{C}(A), x_2 \in \mathcal{N}(A)$. Define $Cy = C(y_1 + y_2) = y_1 + x_2$. It is easily seen that $C$ is well-defined, linear and $A^*y = ACy$, for all $y \in \mathcal{D}(A^*)$.

We shall now prove $C$ is injective. Suppose that $C(y_1 + y_2) = 0$, so $y_1 + x_2 = 0$, hence $y_1 = x_2 = 0$ because $\mathcal{N}(A^*) = \mathcal{N}(A)$. Also, $A^*y_2 = 0$. Therefore $y_2 = 0$, thus $y_1 + y_2 = 0$. To prove surjective, let us take $x = x_1 + x_2 \in \mathcal{D}(A)$. Then there exists $y = y_1 + y_2 \in \mathcal{D}(A^*)$ such that $A^*y = Ax$. Let $z = x_1 + y_2$. Then $Cz = x$, hence $C$ is bijective. 

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3 Closed hypo-EP operators on Hilbert spaces

Weakening the range-space condition by inclusion relation between range spaces, the notion of hypo-EP was defined by Itoh for bounded operators in [11]. Algebraic and analytic characterizations, sum, product, factorization of bounded hypo-EP operators are discussed in [12, 18, 23]. In the collection of bounded operators, the class of all hypo-EP operators contains the class of all EP operators and hyponormal operators with closed ranges. Hence it contains all normal and invertible operators with closed ranges. In the case of finite dimensional, EP and hypo-EP are the same. In this section, we define and discuss hypo-EP operators for densely defined closed operators.

**Definition 3.1** Let $A$ be a densely defined closed operator on a Hilbert space $H$. The operator $A$ is said to be a **hypo-EP operator** if $A$ has a closed range and $\mathcal{R}(A) \subseteq \mathcal{R}(A^*)$.

In the above definition, the inclusion relation for range spaces “$\mathcal{R}(A) \subseteq \mathcal{R}(A^*)$” can equivalently be replaced by the inclusion relation for null spaces “$\mathcal{N}(A) \subseteq \mathcal{N}(A^*)$”.

**Example 3.2** Define $A$ on $\ell_2$ by
\[
A(x_1, x_2, x_3, \ldots) = (0, x_1, 2x_2, 3x_3, \ldots)
\]
with $\mathcal{D}(A) = \{(x_1, x_2, x_3, \ldots) \in H : \sum_{n=1}^{\infty} |nx_n|^2 < \infty\}$. As the sequence $(1/n)_{n=1}^\infty$ belongs to $\ell_2$ but not in $\mathcal{D}(A)$ and $\mathcal{D}(A)$ contains the space $c_0$, $\mathcal{D}(A)$ is a proper dense subspace of $\ell_2$. Also, $A$ is closed and $\mathcal{R}(A) = \ell_2 \setminus \text{span}\{e_1\}$. The adjoint of $A$ is defined by $A^*(x_1, x_2, x_3, \ldots) = (x_2, 2x_3, 3x_4, \ldots)$ with $\mathcal{D}(A^*) = \{(x_1, x_2, x_3, \ldots) \in H : \sum_{n=2}^{\infty} |(n-1)x_n|^2 < \infty\}$. Moreover, $\mathcal{N}(A^*) = \text{span}\{e_1\}$ and $\mathcal{N}(A) = \{0\}$, so $\mathcal{R}(A) \not\subseteq \mathcal{R}(A^*)$. Thus $A$ is hypo-EP but not EP.

**Theorem 3.3** Let $A \in \mathcal{C}(H)$ with a closed range. Then $A$ is hypo-EP if and only if $A^\dagger A A^\dagger = AA^\dagger$.

**Proof** Suppose $\mathcal{R}(A) \subseteq \mathcal{R}(A^*)$ and $\mathcal{R}(A)$ is closed. Then $AA^\dagger x \in \mathcal{R}(A)$ for each $x \in H$ and hence $AA^\dagger x \in \mathcal{R}(A^\dagger) = \mathcal{R}(A^*)$. As $A^\dagger A$ is a projection onto $\mathcal{R}(A^\dagger)$, we have $A^\dagger A (AA^\dagger x) = AA^\dagger x$. Hence $A^\dagger A A^\dagger = AA^\dagger$. Conversely, suppose $A^\dagger A A^\dagger = AA^\dagger$. Then $\mathcal{R}(A) = \overline{\mathcal{R}(A)} = \mathcal{R}(AA^\dagger) = \mathcal{R}(A^\dagger A A^\dagger) \subseteq \mathcal{R}(A^\dagger) = C(A) \subseteq \mathcal{N}(A)^\perp = \overline{\mathcal{R}(A^*)} = \mathcal{R}(A^*)$. Hence $A$ is hypo-EP.

**Theorem 3.4** Let $A \in \mathcal{C}(H)$. Then each of the following statements implies the next statement:

1. $A$ is hypo-EP;
2. $(A^\dagger) A = AA^\dagger$ on $\mathcal{D}(A)$;
3. $AA^\dagger \leq A^\dagger A$ on $\mathcal{D}(A)$;
(4) $\|AA^\dagger x\| \leq \|A^\dagger Ax\|$ for all $x \in D(A)$.

**Proof** Assume that $A$ is hypo-EP and $x \in D(A)$. Then

\[
\langle AA^\dagger A^\dagger Ax, x \rangle = \langle (AA^\dagger)^* A^\dagger Ax, x \rangle \\
= \langle A^\dagger Ax, AA^\dagger x \rangle \\
= \langle (A^\dagger A)^* x, AA^\dagger x \rangle \\
= \langle x, A^\dagger A^2 A^\dagger x \rangle \\
= \langle x, AA^\dagger x \rangle, \text{ by Theorem 3.3} \\
= \langle AA^\dagger x, x \rangle.
\]

Hence $A(A^\dagger)^2 A = AA^\dagger$ on $D(A)$.

Assume that $A(A^\dagger)^2 A = AA^\dagger$ on $D(A)$. Let $x \in D(A)$. Then

\[
\langle AA^\dagger x, x \rangle = \langle AA^\dagger AA^\dagger x, x \rangle \\
= \langle (AA^\dagger)^* AA^\dagger x, x \rangle \\
= \langle AA^\dagger x, (AA^\dagger)x \rangle \\
= \|AA^\dagger x\|^2 \\
= \|A(A^\dagger)^2 Ax\|^2 \\
\leq \|AA^\dagger\|^2 \|A^\dagger Ax\|^2 \\
= \|A^\dagger Ax\|^2 \\
= \langle A^\dagger Ax, A^\dagger Ax \rangle \\
= \langle A^\dagger Ax, x \rangle.
\]

Hence $AA^\dagger \leq A^\dagger A$ on $D(A)$.

Assume that $AA^\dagger \leq A^\dagger A$ on $D(A)$. Let $x \in D(A)$. Then

\[
\langle AA^\dagger x, x \rangle \leq \langle A^\dagger Ax, x \rangle \\
\Rightarrow \langle AA^\dagger AA^\dagger x, x \rangle \leq \langle A^\dagger AA^\dagger Ax, x \rangle \\
\Rightarrow \langle AA^\dagger x, AA^\dagger x \rangle \leq \langle A^\dagger Ax, A^\dagger Ax \rangle \\
\Rightarrow \|AA^\dagger x\|^2 \leq \|A^\dagger Ax\|^2.
\]

Thus $\|AA^\dagger x\| \leq \|A^\dagger Ax\|$ for all $x \in D(A)$. \qed

**Remark 3.5** If $R(A) \subseteq D(A)$, the implication “(4) $\Rightarrow$ (1)” is true in Theorem 3.4. That is, all the necessary conditions for hypo-EP in Theorem 3.4 become sufficient.
conditions as well. Note that the inclusion relation “\( \mathcal{R}(A) \subseteq \mathcal{D}(A) \)” is redundant in the case of bounded operators with full domain \( \mathcal{H} \).

**Theorem 3.6** Let \( A \in \mathcal{C}(\mathcal{H}) \) be hypo-EP. Then for each \( x \in \mathcal{D}(A) \), there exists \( k > 0 \) such that \( |\langle Ax, y \rangle| \leq k \|Ay\| \), for all \( y \in \mathcal{D}(A) \).

**Proof** Suppose \( A \) is hypo-EP. If \( x \in \mathcal{N}(A) \), then the result is trivial. Let \( x \in \mathcal{D}(A) \) such that \( Ax \neq 0 \). Then \( Ax \in \mathcal{R}(A) \subseteq \mathcal{R}(A^*) \). Therefore there exists \( z \in \mathcal{D}(A^*) \) such that \( A^*z = Ax \). Then for all \( y \in \mathcal{D}(A) \),

\[
|\langle Ax, y \rangle| = |\langle A^*z, y \rangle| = |\langle z, Ay \rangle| \leq \|z\||Ay||. 
\]

Taking \( k = \|z\| \), we get

\[
|\langle Ax, y \rangle| \leq k \|Ay\|,
\]

for all \( y \in \mathcal{D}(A) \). \( \square \)

The converse of Theorem 3.6 has been proved for bounded hypo-EP operators on Hilbert spaces in [23] where Douglas’ theorem for bounded operators was used. Unlike the bounded operators, Douglas’ theorem for densely defined closed operators does not guarantee the equivalence of the notions of majorization, range inclusion and factorization.

### 4 A perturbation result

Ding and Huang in [5] considered a perturbation problem associated with injective or surjective bounded operators on Hilbert spaces. Motivated by the work of Ding and Huang, stability of EP and hypo-EP operators under perturbation by bounded operators is investigated in the following result.

**Theorem 4.1** Let \( A \in \mathcal{C}(\mathcal{H}) \) and \( B \in \mathcal{B}(\mathcal{H}) \) be such that \( \|B\|\|A^\dagger\| < 1 \), \( AA^\dagger B = B \) and \( BA^\dagger A = B|_{\mathcal{D}(A)} \).

1. If \( A \) is EP, then \( A + B \) is EP.
2. If \( A \) is hypo-EP, then \( A + B \) is hypo-EP.

**Proof** Let \( x \in \mathcal{N}(A) \). Then \( Bx = BA^\dagger Ax = 0 \). Hence \( (A + B)x = Ax + Bx = 0 \) and \( x \in \mathcal{N}(A + B) \). Therefore \( \mathcal{N}(A) \subseteq \mathcal{N}(A + B) \). It is given that \( \|B\|\|A^\dagger\| < 1 \), hence \( \|BA^\dagger\| < 1 \) which implies \( I + BA^\dagger \) is invertible. Let \( x \in \mathcal{N}(A + B) \). Then

\[
\begin{align*}
(A + B)x &= 0 \\
Ax + BA^\dagger Ax &= 0 \\
(I + BA^\dagger)Ax &= 0 \\
Ax &= 0.
\end{align*}
\]

Therefore \( x \in \mathcal{N}(A) \) and \( \mathcal{N}(A + B) \subseteq \mathcal{N}(A) \). Hence \( \mathcal{N}(A + B) = \mathcal{N}(A) \).
Since $\mathcal{R}(A)$ is closed, $\|Ax\| \geq \gamma(A)\|x\|$, for all $x \in C(A)$. Since $\mathcal{N}(A + B) = \mathcal{N}(A)$, we have $C(A + B) = C(A)$. For any $x \in C(A + B)$, $\|(A + B)x\| = \|Ax + Bx\| \geq \|\|Ax\| - \|Bx\|\| \geq \gamma(A)\|x\| - \|B\|\|x\| = (\gamma(A) - \|B\|)\|x\|$. Since $\gamma(A) = \|A^\dagger\|^{-1}$ and $\|B\|\|A^\dagger\| < 1$, we have $\gamma(A) - \|B\| > 0$. Hence $\mathcal{R}(A + B)$ is closed.

Let $y \in \mathcal{R}(A + B)$. Then there exists $x \in \mathcal{D}(A)$ such that $y = (A + B)x = Ax + Bx = Ax + A\dagger Bx = A(I + A\dagger B)x$. Hence $y \in \mathcal{R}(A(I + A\dagger B)) \subseteq \mathcal{R}(A)$ and $\mathcal{R}(A + B) \subseteq \mathcal{R}(A)$. Let $y \in \mathcal{R}(A)$. Then there exists $x \in \mathcal{D}(A)$ such that $Ax = y$. Since $\|B\|\|A^\dagger\| < 1$, $\|A\dagger B\| < 1$ implies $(I + A\dagger B)^{-1} \in B(\mathcal{H})$. Since $(I + A\dagger B)$ is surjective, for $x \in \mathcal{H}$, there exists $x' \in \mathcal{D}(A)$ such that $(I + A\dagger B)x' = x$. Hence $y = Ax = A(I + A\dagger B)x' = Ax' + A\dagger Bx' = Ax' + Bx' = (A + B)x'$. Therefore $y \in \mathcal{R}(A + B)$ and hence $\mathcal{R}(A) = \mathcal{R}(A + B)$.

Remark 4.2 For $A \in \mathcal{C}(\mathcal{H})$ with a closed range, $AA\dagger$ and $A\dagger A$ are orthogonal projections onto $\mathcal{R}(A)$ and $\overline{C(A)}$ respectively, hence we have the following implications:

1. $AA\dagger B = B$ iff $\mathcal{R}(B) \subseteq \mathcal{R}(A)$.
2. $BA\dagger A = B|_{\mathcal{D}(A)}$ iff $\mathcal{N}(A) \subseteq \mathcal{N}(A + B)$.

It is noted from the proof of Theorem 4.1 that the norm condition $\|B\|\|A^\dagger\| < 1$ is useful to conclude that the perturbed operator $A + B$ will also have the same range and null spaces as that of the operator $A$.

1. If $\|B\|\|A^\dagger\| < 1$ and $AA\dagger B = B$, then $\mathcal{R}(A + B) = \mathcal{R}(A)$.
2. If $\|B\|\|A^\dagger\| < 1$ and $BA\dagger A = B|_{\mathcal{D}(A)}$, then $\mathcal{N}(A + B) = \mathcal{N}(A)$.

The following example illustrates the above theorem.

Example 4.3

1. Define $A$ and $B$ on $\ell_2$ by

$$A(x_1, x_2, x_3, \ldots) = (0, 2x_2, 3x_3, 4x_4, \ldots)$$

$$B(x_1, x_2, x_3, \ldots) = \left(0, \frac{x_2}{2}, \frac{x_3}{3}, \frac{x_4}{4}, \ldots\right)$$

with $\mathcal{D}(A) = \{(x_1, x_2, x_3, \ldots) \in \mathcal{H} : \sum_{n=2}^{\infty} |nx_n|^2 < \infty\}$ and $\mathcal{D}(B) = \mathcal{H}$. Then $A$ is a densely defined closed EP operator and $B$ is a bounded operator with...
Moreover, $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ and $\mathcal{N}(A) \subseteq \mathcal{N}(A + B)$. Hence $A + B$ is EP.

(2) Consider $A$ given in the Example 3.2 and define $B$ on $\ell_2$ by

$$B(x_1, x_2, x_3, \ldots) = \left(0, \frac{x_1}{2}, \frac{x_2}{3}, \frac{x_3}{4}, \ldots\right).$$

Then $A$ is a densely defined closed hypo-EP operator and $B$ is a bounded operator with $\|B\|\|A\| < 1$. Also, we have $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ and $\mathcal{N}(A) \subseteq \mathcal{N}(A + B)$. Hence $A + B$ is hypo-EP.

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