On the flag curvature of invariant Randers metrics

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Abstract. In the present paper, the flag curvature of invariant Randers metrics on homogeneous spaces and Lie groups is studied. We first give an explicit formula for the flag curvature of invariant Randers metrics arising from invariant Riemannian metrics on homogeneous spaces and, in special case, Lie groups. We then study Randers metrics of constant positive flag curvature and complete underlying Riemannian metric on Lie groups. Finally we give some properties of those Lie groups which admit a left invariant non-Riemannian Randers metric of Berwald type arising from a left invariant Riemannian metric and a left invariant vector field.

Keywords: invariant metric, flag curvature, Randers space, homogeneous space, Lie group

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1. Introduction

The geometry of invariant structures on homogeneous spaces is one of the interesting subjects in differential geometry. Invariant metrics are of these invariant structures. K. Nomizu studied many interesting properties of invariant Riemannian metrics and the existence and properties of invariant affine connections on reductive homogeneous spaces (see (Kobayashi, Nomizu, 1969; Nomizu, 1954)). Also some curvature properties of invariant Riemannian metrics on Lie groups has studied by J. Milnor (Milnor, 1976). So it is important to study invariant Finsler metrics which are a generalization of invariant Riemannian metrics.

S. Deng and Z. Hou studied invariant Finsler metrics on reductive homogeneous spaces and gave an algebraic description of these metrics (Deng, Hou, 2004-1; Deng, Hou, 2004-2). Also, in (Esrafilian, Salimi Moghaddam, 2006-1; Esrafilian, Salimi Moghaddam, 2006-2), we have studied the existence of invariant Finsler metrics on quotient groups and the flag curvature of invariant Randers metrics on naturally reductive homogeneous spaces. In this paper we study the flag curvature of invariant Randers metrics on homogeneous spaces and Lie groups. Flag curvature, which is a generalization of the concept of sectional curvature in Riemannian geometry, is one of the fundamental quantities which associate with a Finsler space. In general, the computation of the flag curvature
curvature of Finsler metrics is very difficult, therefore it is important to find an explicit and applicable formula for the flag curvature. One of important Finsler metrics which have found many applications in physics are Randers metrics (see (Antonelli, Ingarden, Matsumoto, 1993; Asanov, 1985).). In this article, by using Püttmann’s formula (Püttmann, 1999), we give an explicit formula for the flag curvature of invariant Randers metrics arising from invariant Riemannian metrics on homogeneous spaces and Lie groups. Then the Randers metrics of constant positive flag curvature and complete underlying Riemannian metric on Lie groups are studied. Finally we give some properties of those Lie groups which admit a left invariant non-Riemannian Randers metric of Berwald type arising from a left invariant Riemannian metric and a left invariant vector field.

2. Flag curvature of invariant Randers metrics on homogeneous spaces

The aim of this section is to give an explicit formula for the flag curvature of invariant Randers metrics of Berwald type, arising from invariant Riemannian metrics, on homogeneous spaces. For this purpose we need the Püttmann’s formula for the curvature tensor of invariant Riemannian metrics on homogeneous spaces (see (Püttmann, 1999).).

Let $G$ be a compact Lie group, $H$ a closed subgroup, and $g_0$ a bi-invariant Riemannian metric on $G$. Assume that $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras of $G$ and $H$ respectively. The tangent space of the homogeneous space $G/H$ is given by the orthogonal compliment $\mathfrak{m}$ of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to $g_0$. Each invariant metric $g$ on $G/H$ is determined by its restriction to $\mathfrak{m}$. The arising $\text{Ad}_H$-invariant inner product from $g$ on $\mathfrak{m}$ can extend to an $\text{Ad}_H$-invariant inner product on $\mathfrak{g}$ by taking $g_0$ for the components in $\mathfrak{h}$. In this way the invariant metric $g$ on $G/H$ determines a unique left invariant metric on $G$ that we also denote by $g$. The values of $g_0$ and $g$ at the identity are inner products on $\mathfrak{g}$ which we denote as $<\ldots,\ldots>_0$ and $<\ldots,\ldots>$. The inner product $<\ldots,\ldots>$ determines a positive definite endomorphism $\phi$ of $\mathfrak{g}$ such that $<X,Y>=\phi X, Y>_0$ for all $X,Y \in \mathfrak{g}$.

Now we give the following Lemma which was proved by T. Püttmann (see (Püttmann, 1999).).

**Lemma 1.** The curvature tensor of the invariant metric $<\ldots,\ldots>$ on the compact homogeneous space $G/H$ is given by

$$< R(X,Y)Z,W > = \frac{1}{2}(< B_-(X,Y), [Z,W]>_0 + < [X,Y], B_-(Z,W)>_0)$$
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\[ + \frac{1}{4} < [X, W], [Y, Z]_m > - < [X, Z], [Y, W]_m > \]
\[ - 2 < [X, Y], [Z, W]_m > + ( < B_+(X, W), \phi^{-1} B_+(Y, Z) > 0 \]
\[- < B_+(X, Z), \phi^{-1} B_+(Y, W) > 0 ), \]

where the symmetric resp. skew symmetric bilinear maps \( B_+ \) and \( B_- \) are defined by

\[ B_+(X, Y) = \frac{1}{2} ([X, \phi Y] + [Y, \phi X]), \]
\[ B_-(X, Y) = \frac{1}{2} ([\phi X, Y] + [X, \phi Y]), \]

and \([., .]_m \) is the projection of \([., .] \) to \( m \).

Let \( \tilde{X} \) be an invariant vector field on the homogeneous space \( G/H \) such that \( \| \tilde{X} \| = \sqrt{g(\tilde{X}, \tilde{X})} < 1 \). A case happen when \( G/H \) is reductive with \( g = m \oplus h \) and \( \tilde{X} \) is the corresponding left invariant vector field to a vector \( X \in m \) such that \( < X, X > < 1 \) and \( Ad(h)X = X \) for all \( h \in H \) (see (Deng, Hou, 2004-2) and (Esrafilian, Salimi Moghaddam, 2006-1)). By using \( \tilde{X} \) we can construct an invariant Randers metric on the homogeneous space \( G/H \) in the following way:

\[ F(xH, Y) = \sqrt{g(xH)(Y, Y)} + g(xH)(\tilde{X}_x, Y) \quad \forall Y \in T_xH(G/H). \quad (2) \]

Now we give an explicit formula for the flag curvature of these invariant Randers metrics.

**THEOREM 1.** Let \( G \) be a compact Lie group, \( H \) a closed subgroup, \( g_0 \) a bi-invariant metric on \( G \), and \( g \) and \( h \) the Lie algebras of \( G \) and \( H \) respectively. Also let \( g \) be any invariant Riemannian metric on the homogeneous space \( G/H \) such that \( < Y, Z > = < \phi Y, Z > \) for all \( Y, Z \in g \). Assume that \( \tilde{X} \) is an invariant vector field on \( G/H \) which is parallel with respect to \( g \) and \( g(\tilde{X}, \tilde{X}) < 1 \) and \( \tilde{X}_H = X \). Suppose that \( F \) is the Randers metric arising from \( g \) and \( \tilde{X} \), and \( (P, Y) \) is a flag in \( T_H(G/H) \) such that \( \{ Y, U \} \) is an orthonormal basis of \( P \) with respect to \( < ., . > \). Then the flag curvature of the flag \( (P, Y) \) in \( T_H(G/H) \) is given by

\[ K(P, Y) = \frac{A}{(1+ < X, Y >)^2(1- < X, Y >)}, \quad (3) \]

where \( A = \alpha. < X, U > + \gamma(1+ < X, Y >) \), and for \( A \) we have:

\[ \alpha = \frac{1}{4}(< [\phi U, Y] + [U, \phi Y], [Y, X] > + < [U, Y], [\phi Y, X] + [Y, \phi X] >) \]
\[ + \frac{3}{4} < [Y, U], [Y, X]_m > + \frac{1}{2} < [U, \phi X] + [X, \phi U], \phi^{-1} ([Y, \phi Y]) > \]
\[ - \frac{1}{4} < [U, \phi Y] + [Y, \phi U], \phi^{-1} ([Y, \phi X] + [X, \phi Y]) >, \quad (4) \]
and

\[ \gamma = \frac{1}{2} < [\phi U, Y] + [U, \phi Y], [Y, X] >_0 + \frac{3}{4} < [Y, U], [Y, U]_m > + < [U, \phi U], \phi^{-1}([Y, \phi Y]) >_0 \]

(5)

\[-\frac{1}{4} < [U, \phi Y] + [Y, \phi U], \phi^{-1}([Y, \phi U] + [U, \phi Y]) >_0 .\]

Proof. \( \bar{X} \) is parallel with respect to \( g \), therefore \( F \) is of Berwald type and the Chern connection of \( F \) and the Riemannian connection of \( g \) coincide (see (Bao, Chern, Shen, 2000), page 305.), so we have \( R^F(U, V)W = R^g(U, V)W \), where \( R^F \) and \( R^g \) are the curvature tensors of \( F \) and \( g \), respectively. Let \( R := R^g = R^F \) be the curvature tensor of \( F \) (or \( g \)). Also for the flag curvature we have ((Shen, 2001)):

\[ K(P, Y) = \frac{g_Y(R(U, Y)Y, U)}{g_Y(Y, Y) \cdot g_Y(U, U) - g_Y^2(Y, U)}, \]

(6)

where \( g_Y(U, V) = \frac{1}{2} \frac{\partial^2}{\partial s \partial t}(F^2(Y + sU + tV))|_{s=t=0} \).

By a direct computation for \( F \) we get

\[ g_Y(Y, Y) \cdot g_Y(U, U) - g_Y(Y, U) = (1 + < X, Y >)^2(1 - < X, Y >). \]

(8)

Also we have:

\[ g_Y(R(U, Y)Y, U) = < R(U, Y)Y, U > + < X, R(U, Y)Y > . < X, U > + < X, Y > . < R(U, Y)Y, U > + < X, U > . < Y, R(U, Y)Y >, \]

(9)

now let \( \alpha = < X, R(U, Y)Y >, \theta = < Y, R(U, Y)Y > \) and \( \gamma = < R(U, Y)Y, U >. \)

By using Pützmann’s formula (see Lemma 1.) and some computations we have:

\[ \alpha = \frac{1}{4}(< [\phi U, Y] + [U, \phi Y], [Y, X] >_0 + < [U, Y], [\phi Y, X] + [Y, \phi X] >_0) \]
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\begin{align*}
+ \frac{3}{4} & \left< [Y, U], [Y, X] \right>_m + \frac{1}{2} \left< [U, \phi X] + [X, \phi U], \phi^{-1}([Y, \phi Y]) \right>_0 > 0 \\
- \frac{1}{4} & \left< [U, \phi Y] + [Y, \phi U], \phi^{-1}([Y, \phi X] + [X, \phi Y]) \right>_0 > 0, \quad (10)
\end{align*}

and

\begin{align*}
\theta = 0, \\
\gamma = \frac{1}{2} & \left< [\phi U, Y] + [U, \phi Y], [Y, U] \right>_0 + \frac{3}{4} \left< [Y, U], [Y, U] \right>_m > 0 \\
& + \left< [U, \phi U], \phi^{-1}([Y, \phi Y]) \right>_0 > 0 \\
& - \frac{1}{4} \left< [U, \phi Y] + [Y, \phi U], \phi^{-1}([Y, \phi U] + [U, \phi Y]) \right>_0 > 0. \quad (12)
\end{align*}

Substituting the equations (7), (8), (9), (10), (11) and (12) in the equation (6) completes the proof. \qed

Remark. In the previous theorem, If we let \( H = \{ e \} \) and \( m = g \) then we can obtain a formula for the flag curvature of the left invariant Randers metrics of Berwald types arising from a left invariant Riemannian metric \( g \) and a left invariant vector field \( \tilde{X} \) on Lie group \( G \).

If the invariant Randers metric arises from a bi-invariant Riemannian metric on a Lie group then we can obtain a simpler formula for the flag curvature, we give this formula in the following theorem.

**THEOREM 2.** Suppose that \( g_0 \) is a bi-invariant Riemannian metric on a Lie group \( G \) and \( \tilde{X} \) is a left invariant vector field on \( G \) such that \( g_0(\tilde{X}, \tilde{X}) < 1 \) and \( \tilde{X} \) is parallel with respect to \( g_0 \). Then we can define a left invariant Randers metric \( F \) as follows:

\[ F(x, Y) = \sqrt{g_0(x)(Y, Y)} + g_0(x)(\tilde{X}_x, Y). \]

Assume that \( (P, Y) \) is a flag in \( T_e G \) such that \( \{ Y, U \} \) is an orthonormal basis of \( P \) with respect to \( < ., . >_0 \). Then the flag curvature of the flag \( (P, Y) \) in \( T_e G \) is given by

\[ K(P, Y) = \frac{< [Y, [U, Y]], X >_0 \cdot < X, U >_0 + < [Y, [U, Y]], U >_0 (1 + < X, Y >_0)}{4(1 + < X, Y >_0)^2(1 - < X, Y >_0)}. \]

Proof. Since \( \tilde{X} \) is parallel with respect to \( g_0 \) the curvature tensors of \( g_0 \) and \( F \) coincide. On the other hand for \( g_0 \) we have \( R(X, Y)Z = \frac{1}{4}[Z, [X, Y]] \), therefore by substituting \( R \) in the equation (6) and using equation (7) the proof is completed. \qed
3. Invariant Randers metrics on Lie groups

In this section we study the left invariant Randers metrics on Lie groups and, in some special cases, find some results about the dimension of Lie groups which can admit invariant Randers metrics. These conclusions are obtained by using Yasuda-Shimada theorem. The Yasuda-Shimada theorem is one of important theorems which characterize the Randers spaces. In the year 2001, Shen’s examples of Randers manifolds with constant flag curvature motivated Bao and Robles to determine necessary and sufficient conditions for a Randers manifold to have constant flag curvature. Shen’s examples showed that the original version of Yasuda-Shimada theorem (1977) is wrong. Then Bao and Robles corrected the Yasuda-Shimada theorem (1977) and gave the correct version of this theorem, Yasuda-Shimada theorem (2001) (see (Bao, Robles, 2003).). (For a comprehensive history of Yasuda-Shimada theorem see (Bao, 2004).)

Suppose that $M$ is an $n$-dimensional manifold endowed with a Riemannian metric $g = (g_{ij}(x))$ and a nowhere zero 1-form $b = (b_i(x))$ such that $\|b\| = b_i(x)b_j(x)g^{ij}(x) < 1$. We can define a Randers metric on $M$ as follows

$$F(x, Y) = \sqrt{g_{ij}(x)Y^i Y^j} + b_i(x)Y^i. \quad (13)$$

Next, we consider the 1-form $\beta = b^i(b^j_{\mid j} - b^j_{\mid i})dx^i$, where the covariant derivative is taken with respect to Levi-Civita connection to $M$. Now we give the Yasuda-Shimada theorem from (Bao, 2004).

**THEOREM 3.** (Yasuda-Shimada) (see (Bao, 2004).) Let $F$ be a strongly convex non-Riemannian Randers metric on a smooth manifold $M$ of dimension $n \geq 2$. Let $g_{ij}$ be the underlying Riemannian metric and $b_i$ the drift 1-form. Then:

$(\pm)$ $F$ satisfies $\beta = 0$ and has constant positive flag curvature $K$ if and only if:

- $b$ is a non-parallel Killing field of $g$ with constant length;
- the Riemann curvature tensor of $g$ is given by

$$R_{hijk} = K(1 - \|b\|^2)(g_{hk}g_{ij} - g_{hj}g_{ik}) + K(g_{ij}b_h b_k - g_{kj}b_i b_h) - K(g_{hk}b_i b_j - g_{hj}b_i b_k) - b_{i[j}b_{h]k} + b_{i[k}b_{h]j} + 2b_{h[i}b_{j]k}$$
(0) $F$ satisfies $\beta = 0$ and has zero flag curvature $\iff$ it is locally Minkowskian.

(-) $F$ satisfies $\beta = 0$ and has constant negative flag curvature if and only if:
- $b$ is a closed 1-form;
- $b_{ijk} = \frac{1}{2} \sigma (g_{ik} - b_i b_k)$, with $\sigma^2 = -16K$;
- $g$ has constant negative sectional curvature $4K$, that is, $R_{ijkl} = 4K(g_{ij}g_{hk} - g_{ik}g_{hj})$.

Since any Randers manifold of dimension $n = 1$ is a Riemannian manifold from now on we consider $n > 1$.

An immediate conclusion of Yasuda-Shimada theorem is the following corollary.

**COROLLARY 1.** There is no non-Riemannian Randers metric of Berwald type with $\beta = 0$ and constant positive flag curvature.

Now by using the results of (Bejancu, Farran, 2003) we obtain the following conclusions.

**THEOREM 4.** Let $F^n = (M, F, g_{ij}, b_i)$ be an $n$-dimensional parallelizable Randers manifold of constant positive flag curvature with $\beta = 0$ on $M$ and complete Riemannian metric $g = (g_{ij})$. Then the dimension of $M$ must be $3$ or $7$.

**Proof.** By using theorem 2.2 of (Bejancu, Farran, 2003) $M$ is diffeomorphic with a sphere of dimension $n = 2k + 1$. But a sphere $S^m$ is parallelizable if and only if $m = 1, 3$ or $7$ (see (Adams, 1960)). Therefore $n = 3$ or $7$. \qed

A family of Randers metrics of constant positive flag curvature on Lie group $S^3$ was studied by D. Bao and Z. Shen (see (Bao, Shen, 2003)). They produced, for each $K > 1$, an explicit example of a compact boundaryless (non-Riemannian) Randers spaces that has constant positive flag curvature $K$, and which is not projectively flat, on Lie group $S^3$. In the following we give some results about the dimension of Lie groups which can admit Randers metrics of constant positive flag curvature. These results show that the dimension $3$ is important.

**COROLLARY 2.** There is no Randers Lie group of constant positive flag curvature with $\beta = 0$, complete Riemannian metric $g = (g_{ij})$ and $n \neq 3$. 


Proof. Any Lie group is parallelizable, so by attention to theorem 4 and the condition \( n \neq 3, n \) must be 7. Since \( G \) is diffeomorphic to \( S^7 \) and \( S^7 \) can not admit any Lie group structure, hence the proof is completed. \( \square \)

Similar to the (Milnor, 1976) for the sectional curvature of the left invariant Riemannian metrics on Lie groups, we compute the flag curvature of the left invariant Randers metrics on Lie groups in the following theorem.

**THEOREM 5.** Let \( G \) be a compact Lie group with Lie algebra \( \mathfrak{g} \), \( g_0 \) a bi-invariant Riemannian metric on \( G \), and \( g \) any left invariant Riemannian metric on \( G \) such that \(< X, Y > = < \phi X, Y >\) for a positive definite endomorphism \( \phi : \mathfrak{g} \rightarrow \mathfrak{g} \). Assume that \( X \in \mathfrak{g} \) is a vector such that \(< X, X > < 1 \) and \( \bar{F} \) is the Randers metric arising from \( \bar{X} \) and \( g \) as follows:

\[ F(x, Y) = \sqrt{g(x)(Y, Y)} + g(x)(\bar{X}_x, Y), \]

where \( \bar{X} \) is the left invariant vector field corresponding to \( X \), and we have assumed \( \bar{X} \) is parallel with respect to \( g \). Let \( \{e_1, \ldots, e_n\} \subset \mathfrak{g} \) be a \( g \)-orthonormal basis for \( \mathfrak{g} \). Then the flag curvature of \( F \) for the flag \( P = \text{span}\{e_i, e_j\}(i \neq j) \) at the point \((e, e_i)\), where \( e \) is the unit element of \( G \), is given by the following formula:

\[ K(P = \text{span}\{e_i, e_j\}, e_i) = \frac{X_j: < R(e_j, e_i) e_i, X > + (1 + X_i)< R(e_j, e_i) e_i, e_j >}{(1 + X_i)^2(1 - X_i)}, \]

where \( X = X^k e_k \),

\[ < R(e_j, e_i) e_i, X > = -\frac{1}{4}( < [\phi e_j, e_i], [e_i, X] > + < [e_j, \phi e_i], [e_i, X] > + < [e_j, e_i], [\phi e_i, X] > + < [e_j, e_i], [e_i, \phi X] > ) \]

\[ + \frac{3}{4} < [e_j, e_i], [e_i, X] > \]

\[ - \frac{1}{2} < [e_j, \phi X] + [X, \phi e_j], \phi^{-1}([e_i, \phi e_i]) > 0 \]

\[ + \frac{1}{4} < [e_j, \phi e_i] + [e_i, \phi e_j], \phi^{-1}([e_i, \phi X] + [X, \phi e_i]) > 0 \]

and

\[ < R(e_j, e_i) e_i, e_j > = -\frac{1}{2}( < [\phi e_j, e_i], [e_i, e_j] > + < [e_j, \phi e_i], [e_i, e_j] > ) \]

\[ + \frac{3}{4} < [e_j, e_i], [e_i, e_j] > - < [e_j, \phi e_j], \phi^{-1}([e_i, \phi e_i]) > 0 \]

\[ + \frac{1}{4} < [e_j, \phi e_i] + [e_i, \phi e_j], \phi^{-1}([e_i, \phi e_j] + [e_j, \phi e_i]) > 0. \]
Proof. By using theorem 1, the proof is clear. \hfill \Box

Now we give some properties of those Lie groups which admit a left invariant non-Riemannian Randers metric of Berwald type arising from a left invariant Riemannian metric and a left invariant vector field.

**THEOREM 6.** There is no left invariant non-Riemannian Randers metric of Berwald type arising from a left invariant Riemannian metric and a left invariant vector field on connected Lie groups with a perfect Lie algebra, that is, a Lie algebra $\mathfrak{g}$ for which the equation $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ holds.

**Proof.** If a left invariant vector field $X$ is parallel with respect to a left invariant Riemannian metric $g$ then, by using Lemma 4.3 of (Brown, Frinck, Spencer, Tapp, Wu , 2007), $g(X, [\mathfrak{g}, \mathfrak{g}]) = 0$. Since $\mathfrak{g}$ is perfect therefore $X$ must be zero. \hfill \Box

**COROLLARY 3.** There is not any left invariant non-Riemannian Randers metric of Berwald type arising from a left invariant Riemannian metric and a left invariant vector field on semisimple connected Lie groups.

**COROLLARY 4.** If a Lie group $G$ admits a left invariant non-Riemannian Randers metric of Berwald type $F$ arising from a left invariant Riemannian metric $g$ and a left invariant vector field $X$ then for sectional curvature of the Riemannian metric $g$ we have $K(X, u) \geq 0$ for all $u$, where equality holds if and only if $u$ is orthogonal to the image $[X, \mathfrak{g}]$.

**Proof.** Since $F$ is of Berwald type, $X$ is parallel with respect to $g$. By using Lemma 4.3 of (Brown, Frinck, Spencer, Tapp, Wu , 2007), $\text{ad}(X)$ is skew-adjoint, therefore by Lemma 1.2 of (Milnor, 1976) we have $K(X, u) \geq 0$. \hfill \Box

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