Study of the neutron star structure in strong magnetic fields including the anomalous magnetic moments

Guangjun Mao\textsuperscript{1,2}, Akira Iwamoto\textsuperscript{1} and Zhuxia Li\textsuperscript{3}

\textsuperscript{1) Japan Atomic Energy Research Institute} \\
Tokai, Naka, Ibaraki 319-1195, Japan

\textsuperscript{2) Institute of High Energy Physics, Chinese Academy of Science} \\
P.O. Box 918(4), Beijing 100039, P.R. China\textsuperscript{1}

\textsuperscript{3) China Institute of Atomic Energy} \\
P.O. Box 275(18), Beijing 102413, P.R. China

Abstract

We study the effects of strong magnetic fields on the neutron star structure. If the interior field of a star is on the same order of the surface field currently observed, the influences of the magnetic field on the star mass and radius are negligible. If one assumes that the internal magnetic field can be as large as that estimated from the scalar virial theorem, considerable effects can be induced. The maximum mass of stars is arisen substantially while the central density is largely suppressed. For two equal-mass stars the radius of the magnetic star can be larger by about $10\% \sim 20\%$ than the nonmagnetic star.

Key words. stars: neutron stars — stars: magnetic fields — equation of state

\textsuperscript{1) Permanent address} \\
e-mail: maogj@mail.ihep.ac.cn
I. INTRODUCTION

It is well known that the structure of neutron stars is mainly determined by the nuclear equation of state (EOS) built on the strong interactions. Recent observations have indicated that large magnetic fields are presented at the surface of neutron stars (Michel 1991; Rothschild, Kulkarni & Lingenfelter 1994; Kouveliotou et al. 1998; Woods et al. 1999). The dipole fields inferred from the spin-down rates of neutron star rotations can be up to $10^{15}$ G (for a recent review, see Reisenegger (2001)). The strength of magnetic fields in the interiority of stars remains unknown. According to the scalar virial theorem (Shapiro & Teukolsky 1983; Lai & Shapiro 1991) the maximum interior field strength could reach $\sim 10^{18}$ G for a star with $R \approx 10$ km and $M \approx 1.4 M_\odot$. Even larger fields may be expected in the core of neutron stars. Such high fields will certainly play a role when one evaluates the EOS of neutron-star matter. Consequently, it may causes considerable effects on the structure of neutron stars.

The same problem has been addressed on the magnetic white dwarfs where a surface magnetic field on the order of $10^5 \sim 10^9$ G and an interior field of $10^9 \sim 10^{13}$ G are estimated. Both the earlier work of Ostriker & Hartwick (1968) and recent calculations of Suh & Mathews (2000) predicted an increase of white dwarf radii in the presence of internal magnetic fields. In the case of white-dwarf binary system of LB11146 (PG 0945+245), if one assumes an equal mass of $M = 0.9 M_\odot$ for the magnetic one and nonmagnetic one, an interior field of $B \approx 0.5 B_c^e$ (here $B_c^e = 4.414 \times 10^{13}$ G is the electron critical field) would cause the radius of the magnetic white dwarf be larger by about 10% than that of the normal star (Suh & Mathews 2000).

Theoretical investigation of ideal noninteracting neutron-proton-electron ($n$-p-e) gas and interacting pure neutron matter under large magnetic fields has been carried out by Suh & Mathews (2001) and Vshivtsev & Serebryakova (1994). Recently, Brückner-Hartree-Fock calculations of spin polarized asymmetric nuclear matter have been performed (Vidana & Bombaci 2002). Based on the meson field theory several authors have incorporated strong magnetic fields into the equation of state of a dense $n$-p-e system.
under the beta equilibrium and the charge neutrality conditions (Chakrabarty, Bandyopadhyay & Pal 1997, Paper I; Broderick, Prakash & Lattimer 2000, Paper II). Evident changes on the EOS have been found. It was demonstrated that the nucleon anomalous magnetic moments (AMM) play a significant role (Paper II) which may overwhelm the softening of the EOS caused by Landau quantization (Paper I) to the stiffening. In the mean time, a dramatic increase of the proton fraction with the increase of the magnetic field was exhibited. However, our previous calculations (Mao et al. 2002) showed that with the AMM of nucleons and electrons taken into account the proton fraction was found to never exceed the field free case. Extremely strong fields would lead to a pure neutron matter rather than a proton-rich matter. In this work we will study the effects of large magnetic fields on the neutron star structure. We will examine the EOS of a dense $n$-$p$-$e$ system with the AMM of both nucleons and electrons taken into account. One may argue that the electron self-energy may not change substantially in magnetic fields when high-order terms are taken into account. However, a systematic incorporation of high-order contributions beyond the AMM term is not yet clear, which will be the topic of our forthcoming works. Here the effects of magnetic fields on different particles within the considered system are treated on an equal footing. The developed EOS will be subsequently applied to investigate the structure of neutron stars with strong internal fields. The paper is organized as follows: a relativistic mean-field theory approach for dense neutron star matter is described in Section 2. In Section 3 we present the numerical results. A brief summary and outlook will be finally given in Section 4.

II. THEORETICAL FRAMEWORK

We consider a neutron-star matter consisting of neutrons, protons and electrons interacting through the exchange of $\sigma$, $\omega$ and $\rho$ mesons in the presence of a uniform magnetic field $B$ along the $z$ axis. The Lagrangian density can be written as (Serot & Walecka 1986)

$$\mathcal{L} = \bar{\psi}[i\gamma_{\mu}\partial^\mu - e\frac{1 + \tau_0}{2}\gamma_{\mu}A^\mu - \frac{1}{4}\kappa_{b}\mu N\sigma_{\mu\nu}F^{\mu\nu} - M_N + g_{\sigma}\sigma - g_{\omega}\gamma_{\mu}\omega^\mu - \frac{1}{2}g_{\rho}\gamma_{\mu}\tau\cdot R^\mu]\psi]$$
\[ \psi_e^\dagger \gamma_\mu \partial^\mu - e \gamma_\mu A^\mu - \frac{1}{4} \kappa_e \mu B \sigma_{\mu \nu} F^{\mu \nu} - m_e \psi_e + \frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma - U(\sigma) + \frac{1}{4} \omega_{\mu \nu} \omega^{\mu \nu} + \frac{1}{2} m_\omega^2 \omega_\mu \omega^\mu - \frac{1}{4} \mathbf{R}_\mu \cdot \mathbf{R}^{\mu} + \frac{1}{2} m_\rho^2 \mathbf{R}_\mu \cdot \mathbf{R}^\mu, \]

and \( U(\sigma) \) is the self-interaction part of the scalar field (Boguta & Bodmer 1977)

\[ U(\sigma) = \frac{1}{2} m_\sigma^2 \sigma^2 + \frac{1}{3} b(g_\sigma \sigma)^3 + \frac{1}{4} c(g_\sigma \sigma)^4. \]

In the above expressions \( \psi \) and \( \psi_e \) are the Dirac spinors of the nucleon and electron; \( \sigma, \omega_\mu, \mathbf{R}_\mu \) represent the scalar meson, vector meson and vector-isovector meson field, respectively. \( A^\mu \equiv (0, 0, Bx, 0) \) refers to a constant external magnetic field. Here the field tensors for the omega, rho and magnetic field are given in terms of their potentials by

\[ \omega_{\mu \nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu, \]

\[ \mathbf{R}_{\mu \nu} = \partial_\mu \mathbf{R}_\nu - \partial_\nu \mathbf{R}_\mu, \]

\[ F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \]

and \( \sigma_{\mu \nu} = \frac{i}{2} [\gamma_\mu, \gamma_\nu] \). \( \tau \) is the isospin operator of the nucleon and \( \tau_0 \) is its third component. \( M_N \) and \( m_e \) are the free nucleon mass and electron mass and \( m_\sigma, m_\omega, m_\rho \) are the masses of the \( \sigma -, \omega - \) and \( \rho - \) meson. \( \mu_N \) and \( \mu_B \) are the nuclear magneton of nucleons and Bohr magneton of electrons; \( \kappa_p = 3.5856, \kappa_n = -3.8263 \) and \( \kappa_e = \alpha/\pi \) are the coefficients of the AMM for protons, neutrons and electrons (Greiner & Reinhardt 1994), respectively. The third set of parameters presented by Glendenning & Moszkowski (1991) is used as the nucleon coupling strengths. It gives \( g_\sigma = 8.7818, g_\omega = 8.7116, g_\rho = 8.4635, b g_\sigma^3 = 27.9060, c g_\sigma^4 = -14.3989 \). This yields a binding energy \( B/A = -16.3 \) MeV, saturation density \( \rho_0 = 0.153 \) fm\(^{-3} \) and bulk symmetry energy \( a_{sym} = 32.5 \) MeV.

Let us first consider nucleons. The Dirac equation for the nucleons in a uniform magnetic field can be written as

\[ \left[ i \gamma_\mu \partial^\mu - \frac{1}{2} \tau_0 \gamma_\mu A^\mu - \frac{1}{4} \kappa_B \mu_B N \sigma_{\mu \nu} F^{\mu \nu} - M_N + g_\sigma \sigma - g_\omega \gamma_\mu \omega^\mu - \frac{1}{2} g_\rho \gamma_\mu \tau_0 R_0^\mu \right] \psi = 0. \]
ones. The positive energy of the protons in the Fermi sea \( (E^p_{\nu,S})_+ \) and the negative energy of the protons in the Dirac sea \( (E^p_{\nu,S})_- \) read as

\[
(E^p_{\nu,S})_+ = \left\{ \left[ \left( \sqrt{m^*_{\nu,S}^2 + 2eB\nu + S\Delta} \right)^2 + p_z^2 \right]^{1/2} + g_\omega\omega_0 + \frac{1}{2}g_{\rho\omega}R_{0,0} \right\},
\]
\[
(E^p_{\nu,S})_- = -\left\{ \left[ \left( \sqrt{m^*_{\nu,S}^2 + 2eB\nu + S\Delta} \right)^2 + p_z^2 \right]^{1/2} - g_\omega\omega_0 + \frac{1}{2}g_{\rho\omega}R_{0,0} \right\}.
\]

The positive-energy and negative-energy spectra of the neutrons are

\[
(E^n_{\nu,S})_+ = \left\{ \left[ \left( \sqrt{p_x^2 + p_y^2 + m^*_{\nu,S}^2 + S\Delta} \right)^2 + p_z^2 \right]^{1/2} + g_\omega\omega_0 - \frac{1}{2}g_{\rho\omega}R_{0,0} \right\},
\]
\[
(E^n_{\nu,S})_- = -\left\{ \left[ \left( \sqrt{p_x^2 + p_y^2 + m^*_{\nu,S}^2 + S\Delta} \right)^2 + p_z^2 \right]^{1/2} - g_\omega\omega_0 - \frac{1}{2}g_{\rho\omega}R_{0,0} \right\}.
\]

Here \( \Delta = -\frac{1}{2}\kappa_b\mu_N B \); \( S = \pm 1 \) for spin-up and spin-down particles. \( \nu \) is the quantum number of Landau levels for charged particles (Landau & Lifshitz 1977). The positive energy of the anti-particle is just the negative of the negative energy of the particle, i.e.,

\( (E^p_{\nu,S})_+ = - (E^p_{\nu,S})_- \), \( (E^n_{\nu,S})_+ = - (E^n_{\nu,S})_- \) (Mao, Stöcker & Greiner 1999).

Since neutron stars are cold dense matter, we shall perform numerical calculations at zero temperature. The general chemical equilibrium is realized for valence particles. The chemical potentials of protons and neutrons are defined as

\[
\mu_p = \epsilon^p_f + g_\omega\omega_0 + \frac{1}{2}g_{\rho\omega}R_{0,0},
\]
\[
\mu_n = \epsilon^n_f + g_\omega\omega_0 - \frac{1}{2}g_{\rho\omega}R_{0,0}.
\]

They are related to the respective Fermi momenta via following equations:

\[
\left( k^p_{\nu,S} \right)^2 = \left( k^p_f \right)^2 - \left( \sqrt{m^*_{\nu,S}^2 + 2eB\nu + S\Delta} \right)^2,
\]
\[
\left( k^n_{\nu,S} \right)^2 = \left( k^n_f \right)^2 - \left( m^* + S\Delta \right)^2.
\]

In the above equations, the effective nucleon mass \( m^* = M_N - g_\sigma\sigma \). \( \sigma, \omega_0 \) and \( R_{0,0} \) are the mean values of the scalar field, the time-like component of the vector field and the time-like isospin 3-component of the vector-isovector field in neutron-star matter, respectively. They are obtained by solving the non-linear equations of the meson fields

\[
m^2_\sigma + bg_\sigma^3\sigma^2 + cg_\sigma^4\sigma^3 = g_\rho\rho_S,
\]
\[ m_{\omega}^2 \omega_0 = g_{\omega} \rho, \]  
\[ m_{\rho}^2 R_{0,0} = \frac{1}{2} g_{\rho} \rho_{0,0}. \]  

Here \( \rho_s, \rho \) and \( \rho_{0,0} \) are the scalar density, the time-like component of the vector density and the time-like isospin 3-component of the vector-isovector density contributed from the valence nucleons, that is, from the Fermi sea. In principle, there exist additional contributions stemming from the Dirac sea. Here they are neglected according to the \( n\)-\( s\)-sea approximation since the renormalization of the system under the external magnetic field is a problem to be solved. Thus, the contributed densities are \( \rho_s = \rho_s^p + \rho_s^n, \rho = \rho_p^p + \rho_0 \) and \( \rho_{0,0} = \rho_0^p - \rho_0^n \), with

\[
\rho_s^p = \frac{eBm^*}{2\pi^2} \sum_s \sum_{\nu} \frac{\sqrt{m^* + 2eB\nu} + S\Delta}{\sqrt{m^* + 2eB\nu}} \ln \left| \frac{k_{f,\nu,s}^p + \epsilon_s^p}{\sqrt{m^* + 2eB\nu} + S\Delta} \right|, 
\]
\[
\rho_s^n = \frac{m^*}{4\pi^2} \sum_s \left\{ \frac{\epsilon_s k_{f,s}^n}{m^* + S\Delta} - (m^* + S\Delta)^2 \ln \left| \frac{k_{f,s}^n + \epsilon_s^n}{m^* + S\Delta} \right| \right\}, 
\]
\[
\rho_0^p = \frac{eB}{2\pi^2} \sum_s \sum_{\nu} k_{f,\nu,s}^p, 
\]
\[
\rho_0^n = \frac{1}{2\pi^2} \sum_s \left[ \frac{1}{3} \left( k_{f,s}^n \right)^3 + \frac{S\Delta}{2} \left( (m^* + S\Delta) k_{f,s}^n + \left( \frac{\epsilon_s^n}{m^* + S\Delta} - \frac{\pi}{2} \right) \right) \right]. 
\]

The summation of \( \nu \) runs up to the largest integer for which \( \left( k_{f,\nu,s}^p \right)^2 \) is positive. For spin-up protons \( \nu \) starts from 1 while for spin-down protons 0. It should be pointed out that here the so-called spin up and spin down are just relative notes since the wave functions are no more eigenfunctions of 3-component spin operator (see Appendix A), mainly attributed to the coupling of the spin to the magnetic field. The contributions of the protons and neutrons to the energy density read as

\[
\varepsilon_p = \frac{eB}{4\pi^2} \sum_s \sum_{\nu} \left[ k_{f,\nu,s}^p \epsilon_s^p + \left( \sqrt{m^* + 2eB\nu} + S\Delta \right)^2 \ln \left| \frac{k_{f,\nu,s}^p + \epsilon_s^p}{\sqrt{m^* + 2eB\nu} + S\Delta} \right| \right], 
\]
\[
\varepsilon_n = \frac{1}{4\pi^2} \sum_s \left\{ \frac{1}{2} \left( \epsilon_s^n \right)^3 k_{f,s}^n + \frac{2}{3} S\Delta \left( \epsilon_s^n \right)^3 \left[ \arcsin \left( \frac{m^* + S\Delta}{\epsilon_s^n} \right) - \frac{\pi}{2} \right] \right. 
\]
\[
+ \left. \left[ \frac{1}{3} S\Delta - \frac{1}{4} (m^* + S\Delta) \right] \left( (m^* + S\Delta) k_{f,s}^n \epsilon_s^n + (m^* + S\Delta)^3 \ln \left| \frac{k_{f,s}^n + \epsilon_s^n}{m^* + S\Delta} \right| \right) \right\}. 
\]

In the \( \beta \)-equilibrium system, the electron is assumed to move freely in the strong magnetic fields. The wave functions of electrons are the same as that of protons in the
free space except that the corresponding quantities are replaced by the electron ones. The energy spectrum of the electrons can be expressed as

\[ (E_{\nu,S}^e)_+ = \left[ \sqrt{m_e^2 + 2eB\nu + S\Delta} \right]^2 + p_z^2 \right]^{1/2}, \]  

(24)

here \( \Delta = -\frac{1}{2}\kappa e\mu_B B \) and \( (E_{\nu,S}^e)_- = - (E_{\nu,S}^e)_+ \). The chemical potential of electrons \( \mu_e = \epsilon_f^e \). Its relation to the electron Fermi momentum is

\[ \left( k_{f,\nu,S}^e \right)^2 = \left( \epsilon_f^e \right)^2 - \left( \sqrt{m_e^2 + 2eB\nu + S\Delta} \right)^2. \]  

(25)

The electron density is defined as

\[ \rho_0^e = \frac{eB}{2\pi^2} \sum_S \sum_\nu k_{f,\nu,S}^e. \]  

(26)

The contribution of the electrons to the energy density reads as

\[ \epsilon_e = \frac{eB}{4\pi^2} \sum_S \sum_\nu \left[ k_{f,\nu,S}^e \epsilon_f^e + \left( \sqrt{m_e^2 + 2eB\nu + S\Delta} \right)^2 \ln \left| \frac{k_{f,\nu,S}^e + \epsilon_f^e}{\sqrt{m_e^2 + 2eB\nu + S\Delta}} \right| \right]. \]  

(27)

Finally, we obtain the energy density contributed from the neutron-star matter

\[ \epsilon_m = \frac{1}{2}m^2\sigma^2 + \frac{1}{3}g_{\sigma\sigma}^3 + \frac{1}{4}g_{\sigma\sigma}^4 
+ \frac{1}{2}m^2\omega_0^2 + \frac{1}{2}m^2\sigma_0^3 + \epsilon_p + \epsilon_n + \epsilon_e. \]  

(28)

The total energy density of the system is given by

\[ \epsilon = \epsilon_m + \frac{B^2}{8\pi}, \]  

(29)

where the last term is the contribution from the external electromagnetic field (Landau, Lifshitz & Pitaevskii 1984). In the charge neutral beta-equilibrated matter, the pressure of the system can be expressed as (Paper II)

\[ p = \mu_n \rho - \epsilon_m + \frac{B^2}{8\pi}. \]  

(30)

The inclusion of the \( B^2/8\pi \) term in Eqs. (29) and (30) is equivalent to introducing the electromagnetic-field tensor in the source term of the gravitational equation of general
relativity (Bocquet et al. 1995; Bonazzola & Gourgoulhon 1996; Cardall, Prakash & Lattimer 2001). Numerical calculations are performed under the constraints of the charge neutrality $\rho_p^0 = \rho_e^0$ and the $\beta$-equilibrium $\mu_n = \mu_p + \mu_e$ (Glendenning 1997). These two constraint equations together with three meson equations are solved self-consistently in an iteration procedure.

**III. NUMERICAL RESULTS**

Since there is no information directly available for the interior magnetic field of a star, we assume that the field varies from the surface to the center and adopt the following parametrization (Bandyopadhyay, Chakrabarty & Pal 1997)

$$B(\rho/\rho_0) = B_{\text{surf}} + B_{\text{cent}} \left[1 - \exp\left(-\beta(\rho/\rho_0)^\gamma\right)\right],$$  \hspace{1cm} (31)

where the parameters are chosen to be $\beta = 0.01$ and $\gamma = 3$. We further set $B_{\text{surf}} = \alpha B_{e}^c$, $B_{\text{cent}} = \alpha \times 10^4 B_{e}^c$ and take $\alpha$ (which should not be confused with the fine structure constant) as a free parameter to check the effects of different fields. In this configuration the magnetic field decreases from the center to the surface of a star. We further take the surface magnetic field in the range of $10^{13} \sim 10^{15}$ G which is in agreement with the values inferred from observations. Therefore, the applied magnetic field does not influence the stationary configuration of a spherical neutron star considered here (note that the stability of an axially symmetric neutron star in large average magnetic fields has been discussed recently (Broderick, Prakash & Lattimer 2002; A.P. Martinez, H.P. Rojas & H.J.M. Cuesta 2003)). The EOS of neutron-star matter under strong magnetic fields is depicted in Fig. 1. It can be found that the equation of state becomes stiffer than the field free case at the main region of density. Stronger field leads to a stiffer EOS. For the very low density part the situation is just inverse. A softer EOS is obtained when the magnetic field is incorporated which can be more clearly seen from the upper panel of the figure.

In Fig. 2 we separate the cases with and without the inclusion of the AMM effects. Here we have dropped the contributions from the electromagnetic field to the pressure.
and energy density and presented the EOS for the matter part. A uniform magnetic field from the surface to the center of neutron stars is assumed. Dashed line denotes the case of \( B = 50 B_c \) and neglecting the AMM effects. One can see that the EOS turns out to be softer than the field free case at small energy densities and approaches to it in the main domain. The inclusion of the AMM effects here practically leads to undistinguishable results since at such low field the anomalous magnetic moments do not play any role. The whole situation changes substantially when a strong magnetic field is considered, as depicted in the figure by the dash-dotted line and dash-dot-dotted line for \( B = 5 \times 10^5 B_c \). In accordance with the findings in Paper I & Paper II the EOS becomes softer compared to the case of \( B = 0 \) when the AMM is neglected and stiffer when it is included. Therefore, the softening of the EOS at small energy densities exhibited in Fig. 1 is mainly due to the fact that at the surface region we have taken a relatively small magnetic field where the effects of the AMM are negligible. Under the employed parametrization the field increases with the increase of the density. The AMM terms gradually get into the game. This leads to a stiffening of the EOS as shown in the region of large energy densities.

The structure of neutron stars can be obtained by applying the developed EOS to solve the Tolman-Oppenheimer-Volkoff (TOV) equation for a relativistic, spherical and static star (Shapiro & Teukolsky 1983; Glendenning 1997). We obtain stable solutions for the TOV equation based on the EOS with the strong magnetic fields applied. Fig. 3 displays the gravitational masses of neutron stars as a function of central densities, i.e., the sequences of stars obtained under different circumstances of interior magnetic fields as indicated by different curves in the figure. Rather evident effects are induced by strong magnetic fields. The maximum mass of stars is increased drastically in the presence of strong fields. For the case of \( \alpha = 50 \) the \( M_{\text{max}} \) is enhanced by about 40% compared to the field free case, which can nearly balance the effects of decreasing the maximum mass if the hyperon degrees of freedom are taken into account (Glendenning & Moszkowski 1991). In the mean time the central density is largely suppressed with the increase of fields. This can be seen from Fig. 1 since the equation of state becomes stiff when a strong magnetic field is buried in a star. Note that an enhancement of the maximum mass around 13% –
30% has been reported (Bocquet et al. 1995; Bonazzola & Gourgoulhon 1996; Cardall, Prakash & Lattimer 2001) where the authors accomplished calculations for the structure of axisymmetric relativistic stars. The effects of strong magnetic fields were included in the stress-energy tensor of the gravitational equation while neglected in the input equations of state.

We show in Fig. 4 the radii of neutron stars as a function of interior magnetic field strength. Three cases of fixed star masses with $M/M_\odot = 1.4, 1.6, \text{ and } 1.8$ are investigated. If the field is weak, as in the normal case, the heavier stars have smaller radii due to the effects of the gravitational force. However, the radius increases with the increase of the magnetic field strength to a large extent. When strong enough fields are presented, the heavier stars can even have larger radii since neutron stars with different internal magnetic fields may belong to different star sequences. Generally speaking, the radius of a magnetic star can be enhanced by about $10\% \sim 20\%$ depending on the star mass compared to the nonmagnetic star of equal mass.

We have also investigated the neutron star structure under a nonvaried field strength from the surface to the center of neutron stars. For $B = 50B_c$ the relationship between the star mass and the central density is undistinguishable from that of the field free case. The enhancement of the star radii is found to be less than 2%. It seems to be difficult to observe any effects of the magnetic field by measuring the star mass and radius if the magnetitude of the interior field is on the same order of the surface field, though the influence of the surface field itself on the star properties should be pursued more closely in a model for matter below neutron drip (Baym, Pethick & Sutherland 1971; Lai & Shapiro 1991). Finally, in Fig. 5 we depict the mass-radius relations of different neutron-star sequences with the magnetic fields considered. In order to describe the surface region a model for non-uniform matter at low densities (Shen 2002) should be utilized, which will be taken into account in our future investigations.

**IV. SUMMARY AND OUTLOOK**

Within a relativistic field theory approach we have studied the effects of strong mag-
netic fields on the equation of state of beta-equilibrium and charge neutrality matter relevant to neutron stars. The anomalous magnetic moments of both nucleons and electrons are incorporated in the model covariantly. We present the analytical expressions of the Dirac spinors under a uniform magnetic field. Numerical results show that if the magnetitude of the magnetic field is on the order of the surface field of neutron stars, the effects of the AMM are negligible. The EOS deviates from the field free case only at very low density region. If a much larger field is considered, the AMM plays a significant role so that the EOS becomes quite stiffer compared to the nonmagnetic case. Consequently, if the interior magnetic field of a neutron star is in the same level as the surface field inferred from pulsars, it may not cause evident influences on the star mass and radius as well as the maximum mass of star sequence. If ultra-strong fields do exist in neutron stars, considerable effects can be observed. We assume the magnetic field varies from the surface to the center of a neutron star. For the surface field we take the values inferred from pulsars. Strong fields up to $10^{18} \sim 10^{19}$ G have been considered for the center of the neutron star. With the employed parametrization of the field changing with the density, the maximum mass could be enhanced by about 40%. The central density of the star is in turn suppressed dramatically. For two equal-mass magnetic and nonmagnetic stars, the radius of magnetic one can be larger by about 10% $\sim$ 20% than the nonmagnetic one.

It would be interesting to check whether the enhancement of the maximum mass induced by strong internal fields can be balanced by the hyperon degrees of freedom which was known to decrease the maximum mass. Since the effects of star rotations increase the star radii, it is necessary to study the effects of rotations and magnetic fields simultaneously. Theoretically one should solve the coupled Einstein-Maxwell equations for axisymmetric configuration with the effects of magnetic fields taken into account both in the source term of the gravitational equation of general relativity and the nuclear equation of state. If in future astrophysical observations one can catch two pulsars with similar rotation periods and masses but quite different radii, a possible explanation is that the star having larger radius may contain a strong interior magnetic field.

Now let us discuss several issues involved for the description of an electron in intense
magnetic fields. It has long been recognized that the ground-state energy of an electron may be shifted after taking into account the AMM term. In previous calculations (O’Connell 1968; Chiu & Canuto 1968) one thought that the Landau levels always start from $\nu = 0$. Thus, the ground-state energy of an electron in the presence of the magnetic field can be written as

$$E_0 = m_e - \frac{\alpha}{2\pi} \mu_B B.$$  \hspace{1cm} (32)

The vacuum becomes unstable with respect to the electron-positron pair creations when $B > 7.6 \times 10^{16}$ G, which may have dramatic astrophysical consequences. As pointed out in the appendix, the Landau levels start from $\nu = 1$ for spin-up particles since the wave functions vanish at $\nu = 0$. It causes the ground-state energy to the form of

$$E_0 = \sqrt{m_e^2 + 2eB} - \frac{\alpha}{2\pi} \mu_B B.$$ \hspace{1cm} (33)

The critical field for pair creations now turns out to be $2.6 \times 10^{20}$ G. However, Eq. (33) is valid at the level of the anomalous magnetic moment term (Jancovici 1969). Higher-order terms, e.g. radiative correction due to vacuum polarization effect (Schwinger 1951; 1973), may become effective at such large field. The incorporation of these terms in the present model is a problem to be addressed in future studies.

**APPENDIX A**

In this appendix we derive the Dirac spinors and energy spectra of nucleons in the presence of a constant uniform magnetic field along the $z$ axis. The Dirac equation for a free nucleon which has an anomalous magnetic moment $\mu_N$ in an external magnetic field can be written as

$$\left[ i\gamma^\mu \partial_\mu - e \frac{1 + \tau_0}{2} \gamma^\mu A^\mu - \frac{1}{4} \kappa_{\mu \nu} \sigma_{\mu \nu} F^{\mu \nu} - M_N \right] \psi = 0.$$  \hspace{1cm} (A1)

If one drops the term concerning the AMM, the solutions for neutrons are just the conventional Dirac spinors (Greiner 1990; Weinberg 1995). The Dirac theory for free electrons in a homogeneous magnetic field was first investigated by Rabi (1928). The wave functions
for charged particles without the inclusion of the AMM have been studied by several authors (Kobayashi & Sakamoto 1983; Das & Hott 1996). Johnson and Lippmann (1950) considered the inclusion of the AMM in the Dirac equation within a noncovariant description. A covariant energy spectrum was discussed by Vshivtsev and Serebryakova (1994). Recently, Broderick, Prakash and Lattimer (Paper II) derived the spinors and energy spectra for baryons in the Dirac representation. However, their formulae are quite complicated which leads to a lack of analytical expressions for the wave functions. Here we resolve the problem in the chiral representation. The employed $\gamma$-matrices then become (Itzykson & Zuber 1980)

$$\gamma^0 = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}, \quad \alpha = \begin{pmatrix} \sigma & 0 \\ 0 & -\sigma \end{pmatrix},$$

$$\gamma = \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

(A2)

where $\sigma$ is the Pauli matrix. Equation (A1) can be rewritten as

$$i\frac{\partial}{\partial t}\psi = \left[ -i\alpha \cdot \nabla - e\frac{1+\tau_0}{2}\alpha_2 B x + \beta M_N + \frac{1}{4}\beta\kappa_{\mu N}\sigma_{\mu\nu} F^{\mu\nu} \right] \psi.$$  

(A4)

In the following we define $\Delta = -\frac{1}{2}\kappa_{\mu N} B$ and consider the cases of protons and neutrons individually.

Protons

Let us specify the wave functions of protons as

$$\psi(X) = e^{-iEt+ip_y y+ip_z z}\phi_p(p_y, p_z, x).$$

(A5)

In the static system we obtain the eigenequation

$$\begin{pmatrix} p_z - E & \xi_+ & -M_N + \Delta & 0 \\ \xi_- & -p_z - E & 0 & -M_N - \Delta \\ -M_N + \Delta & 0 & -p_z - E & -\xi_+ \\ 0 & -M_N - \Delta & -\xi_- & p_z - E \end{pmatrix} \begin{pmatrix} \phi_p^{(1)} \\ \phi_p^{(2)} \\ \phi_p^{(3)} \\ \phi_p^{(4)} \end{pmatrix} = 0.$$  

(A6)
Here we have defined \( \xi_\pm \equiv -i \partial_x \mp i (p_y - eBx) \). Provided \( eB > 0 \), one can introduce the eigenfunction of \( \xi_+ \), \( \xi_- \)

\[
I_{\nu,p_y}(x) = \left( \frac{eB}{\pi} \right)^{1/4} \exp \left[ -\frac{1}{2} eB \left( x - \frac{p_y}{eB} \right)^2 \right] \frac{1}{\sqrt{\nu!}} H_\nu \left[ \sqrt{2eB} \left( x - \frac{p_y}{eB} \right) \right] , \\
(\nu = 0, 1, 2, ...)
\]

(A7)

where \( H_\nu(x) \) is the Hermite polynomial defined by

\[
H_\nu(x) = (-1)^\nu \exp \left( \frac{x^2}{2} \right) \frac{d^n}{dx^n} \exp \left( -\frac{x^2}{2} \right) .
\]

(A8)

\( I_{\nu,p_y}(x) \) is normalized as

\[
\int dx I_{\nu,p_y}(x) I_{\mu,p_y}(x) = \delta_{\nu\mu},
\]

(A9)

\[
\sum_{\nu=0}^{\infty} I_{\nu,p_y}(x) I_{\nu,p_y}(x') = \delta(x-x').
\]

(A10)

It satisfies the following relations:

\[
\xi_- I_{\nu,p_y}(x) = -i \sqrt{2eB\nu} I_{\nu-1,p_y}(x), \quad (I_{-1,p_y}(x) = 0)
\]

(A11)

\[
\xi_+ I_{\nu,p_y}(x) = i \sqrt{2eB(\nu+1)} I_{\nu+1,p_y}(x).
\]

(A12)

The eigenvalues and eigenfunctions of Eq. (A6) can be deduced in a standard way by performing matrix calculations. The energy spectra of protons are

\[
(E^p_{\nu,S})_+ = E^\nu_S, \quad (E^p_{\nu,S})_- = -E^\nu_S,
\]

(A13)

with

\[
E^\nu_S = \left[ \left( \sqrt{M_N^2 + 2eB\nu + S\Delta} \right)^2 + p_z^2 \right]^{1/2} ,
\]

(A14)

here \( S = \pm 1 \) for the spin-up and spin-down particles. The respective eigenfunctions are as follows:

\[
\psi_1(X) = e^{-iE^\nu_{+1}t + ip_y y + ip_z z} \sqrt{\frac{E^\nu_{+1} + p_z}{2E^\nu_{+1}}} \left[ 1 + \frac{2eB\nu}{\left( \sqrt{M_N^2 + 2eB\nu + M_N} \right)^2} \right]^{-1/2}
\]
\[
\psi_2(X) = e^{-iE_{-1}^\nu t + ip_y y + ip_z z} \sqrt{\frac{E_{-1}^\nu + p_z}{2E_{-1}^\nu}} \left[ 1 + \frac{2eB\nu}{\left(\sqrt{M_N^2 + 2eB\nu - M_N}\right)^2} \right]^{-1/2}
\]

\[
\times \left( \begin{array}{c}
- \frac{i\sqrt{2eB\nu}}{\sqrt{M_N^2 + 2eB\nu + M_N}} I_{\nu, p_y}(x) \\
- \frac{\sqrt{M_N^2 + 2eB\nu + \Delta}}{E_{-1}^\nu + p_z} I_{\nu-1, p_y}(x) \\
\end{array} \right) I_{\nu-1, p_y}(x)
\]

\[
\psi_3(X) = e^{-i(-E_{+1}^\nu)t + ip_y y + ip_z z} \sqrt{\frac{E_{+1}^\nu + p_z}{2E_{+1}^\nu}} \left[ 1 + \frac{2eB\nu}{\left(\sqrt{M_N^2 + 2eB\nu + M_N}\right)^2} \right]^{-1/2}
\]

\[
\times \left( \begin{array}{c}
\frac{i\sqrt{2eB\nu}}{\sqrt{M_N^2 + 2eB\nu - M_N}} I_{\nu, p_y}(x) \\
\frac{\sqrt{M_N^2 + 2eB\nu - \Delta}}{E_{+1}^\nu + p_z} I_{\nu-1, p_y}(x) \\
\end{array} \right) I_{\nu-1, p_y}(x)
\]
One can easily check that $\psi_i (i = 1, 4)$ forms a complete orthogonal set. Note that the Landau levels start at $\nu = 0$ for spin-down particles and $\nu = 1$ for spin-up particles since $\psi_1$ and $\psi_3$ vanish at $\nu = 0$. As pointed out in Sect. II, here the so-called spin up and spin down are just relative notes for convenience of description.

Neutrons

The wave functions of neutrons can be specified as

$$\psi(X) = e^{-iE_1t + i p_y y + i p_z z} \left[ \frac{E'_- + p_z}{2E'_-} \left( 1 + \frac{2eB\nu}{\sqrt{M_N^2 + 2eB\nu - M_N}} \right)^{-1/2} \right]$$

One can easily check that $\psi_i (i = 1, 4)$ forms a complete orthogonal set. Note that the Landau levels start at $\nu = 0$ for spin-down particles and $\nu = 1$ for spin-up particles since $\psi_1$ and $\psi_3$ vanish at $\nu = 0$. As pointed out in Sect. II, here the so-called spin up and spin down are just relative notes for convenience of description.

Neutrons

The wave functions of neutrons can be specified as

$$\psi(X) = e^{-iE_1t + i p_y y + i p_z z} \phi_n(p).$$  \hspace{1cm} (A19)
Inserting it into Eq. (A4) we have the following eigenequation:

\[
\begin{pmatrix}
  p_z - E & p_x - ip_y & -M_N + \Delta & 0 \\
  p_x + ip_y & -p_z - E & 0 & -M_N - \Delta \\
  -M_N + \Delta & 0 & -p_z - E & -(p_x - ip_y) \\
  0 & -M_N - \Delta & -(p_x + ip_y) & p_z - E
\end{pmatrix}
\begin{pmatrix}
  \phi^{(1)}_n \\
  \phi^{(2)}_n \\
  \phi^{(3)}_n \\
  \phi^{(4)}_n
\end{pmatrix} = 0. \tag{A20}
\]

Through solving the above matrix equation we obtain the energy spectra of neutrons as

\[
(E^n_S)_+ = E_S, \quad (E^n_S)_- = -E_S, \tag{A21}
\]

with

\[
E_S = \left[ \left( p_x^2 + p_y^2 + M_N^2 + S\Delta \right)^2 + p_z^2 \right]^{1/2}. \tag{A22}
\]

The corresponding eigenfunctions read as

\[
\psi_1(X) = e^{-iE_{+1}t + ip \cdot x} \sqrt{\frac{E_{+1} + p_z}{2E_{+1}}} \left[ 1 + \frac{p_x^2 + p_y^2}{\left( \sqrt{M_N^2 + p_x^2 + p_y^2} + M_N \right)^2} \right]^{-1/2} \times \left( \begin{array}{c}
  \frac{p_x - ip_y}{\sqrt{M_N^2 + p_x^2 + p_y^2 + M_N}} \\
  -\frac{\sqrt{M_N^2 + p_x^2 + p_y^2 + \Delta}}{E_{+1} + p_z} \\
  \frac{p_x - ip_y}{\sqrt{M_N^2 + p_x^2 + p_y^2 + M_N}} \\
  \frac{E_{+1} + p_z}{\sqrt{M_N^2 + p_x^2 + p_y^2 + M_N}}
\end{array} \right), \tag{A23}
\]

\[
\psi_2(X) = e^{-iE_{-1}t + ip \cdot x} \sqrt{\frac{E_{-1} + p_z}{2E_{-1}}} \left[ 1 + \frac{p_x^2 + p_y^2}{\left( \sqrt{M_N^2 + p_x^2 + p_y^2} - M_N \right)^2} \right]^{-1/2}.
\]
\[
\psi_3(X) = e^{-i(E_+1)t + i\mathbf{p} \cdot \mathbf{x}} \left( \frac{p_x - ip_y}{\sqrt{M_N^2 + p_x^2 + p_y^2 + M_N}} \right) \times \left( \frac{\sqrt{M_N^2 + p_x^2 + p_y^2 - \Delta}}{E_+ + p_z} \right) \times \left( -\frac{(p_x - ip_y)\left(\sqrt{M_N^2 + p_x^2 + p_y^2 + \Delta}\right)}{(E_+ + p_z)\left(\sqrt{M_N^2 + p_x^2 + p_y^2 + M_N}\right)} \right) \frac{1}{1 - \frac{p_x - ip_y}{\sqrt{M_N^2 + p_x^2 + p_y^2 + M_N}}} \times \frac{\sqrt{M_N^2 + p_x^2 + p_y^2 + \Delta}}{E_+ + p_z}
\]

\[
\psi_4(X) = e^{-i(E_-1)t + i\mathbf{p} \cdot \mathbf{x}} \left( \frac{p_x - ip_y}{\sqrt{M_N^2 + p_x^2 + p_y^2 + M_N}} \right) \times \left( \frac{\sqrt{M_N^2 + p_x^2 + p_y^2 + \Delta}}{E_- + p_z} \right) \times \left( -\frac{(p_x - ip_y)\left(\sqrt{M_N^2 + p_x^2 + p_y^2 - \Delta}\right)}{(E_- + p_z)\left(\sqrt{M_N^2 + p_x^2 + p_y^2 - M_N}\right)} \right) \frac{-1}{\sqrt{\frac{M_N^2 + p_x^2 + p_y^2 - \Delta}{E_- + p_z}}}
\]

Again, \(\psi_i (i = 1, 4)\) is orthonormalized.

**Acknowledgments:** G. Mao was financially supported by the STA foundation and in part supported by the National Natural Science Foundation of China under the grant 10275072. He thanks the Japan Atomic Energy Research Institute and the China Institute.
of Atomic Energy for local hospitality during visit.

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Figure 1: The equation of state of neutron-star matter with strong magnetic fields. The upper panel gives the relationship between the pressure and total energy density while the lower panel displays the pressure as a function of density. Different curves correspond to different magnetic field strength as indicated in the figure and explained in the text.
Figure 2: The equation of state of neutron-star matter with a uniform magnetic field. The contributions of the magnetic field to the pressure and energy density are neglected. Different curves are related to different cases of magnetic fields $B$ and with or without the inclusion of the anomalous magnetic moments $\kappa$ as indicated in the figure.
Figure 3: The gravitational masses of neutron stars as a function of central densities, i.e., the sequences of stars under different interior magnetic fields as depicted by different curves in the figure.
Figure 4: The radii of neutron stars with equal gravitational masses as a function of interior magnetic field strength. Different curves correspond to different groups of star masses.
Figure 5: The mass-radius relations of neutron-star sequences. The different curves are related to the different interior magnetic fields as depicted in the legend of Fig. 3.