Basic Theorem, Gauge Algebra, \( \theta \)-superfield QED in the Lagrangian Formulation of General Superfield Theory of Fields

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Abstract

The basic theorem of the Lagrangian formulation for general superfield theory of fields (GSTF) is proved. The gauge transformations of general type (GTGT) and gauge algebra of generators of GTGT (GGTGT) as the consequences of the above theorem are studied. It is established the gauge algebra of GGTGT contains the one of generators of gauge transformations of special type (GGTST) as one’s subalgebra. In the framework of Lagrangian formulation for GSTF the nontrivial superfield model generalizing the model of Quantum Electrodynamics and belonging to the class of gauge theory of general type (GThGT) with Abelian gauge algebra of GGTGT is constructed.

PACS codes: 11.10.Ef, 11.15.-q, 12.20.-m, 03.50.-z
Keywords: Lagrangian quantization, Gauge theory, Superfields.

1 Introduction

The Lagrangian and Hamiltonian formulations for GSTF of the superfield (with respect to odd time \( \theta \)) models description, suggested in the papers [1,2], had permitted to solve the problem of constructing the superfield Lagrangian (in usual sense) quantization method for general gauge theories in the framework of general superfield quantization method (GSQM) in the Lagrangian formalism [3].

GSQM permits by means of the path integral method to quantize the ordinary gauge models of the quantum field theory extended, in a natural way, to the superfield GSTF models. GSQM contains the BV quantization method for gauge theories [4] as the particular case under special choice of the generating equation [3]. By the main resulting GSQM feature it appears the Ward identities form for generating functionals (superfunctions) of Green’s functions, including the effective action, which reflect the fact of these superfunctions invariance under their translation with respect to variable \( \theta \) along integral curve of solvable [1,2] Hamiltonian system constructed with respect to quantum gauge fixed action superfunction \( S_H^w(\theta, h) \). It is the above Hamiltonian system with which the standard BRST symmetry transformations are associated [3] under corresponding notations providing the \( \theta \)-superfield realization of that symmetry.

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Theorem 1, formulated in Ref.[1], on reduction of the 1st order with respect to differentiation on \( \theta \) system of \( N \) ordinary differential equations (ODE) to generalized normal form (GNF) in the case of its linear (functional) dependence appears by the key one in GSTF [1,2] and in GSQM [3] construction on the whole. The paper is devoted to its proof, to the investigation of the GTGT and a gauge algebra of the GGTGT, to the connection of the latters with GTST and a gauge algebra of the GGTST, to the demonstration of the efficiency of these results on the example of superfield (on \( \theta \)) quantum electrodynamics model.

In work the definitions, conventions and notations introduced in Ref.[1] are made use unless otherwise stated.

2 Proof of the Basic Theorem

Consider the 2nd order with respect to derivatives on \( \theta \) system of \( N \) ODE in normal form (NF)

\[
\overset{\text{odd}}{g} i(\theta) = f^i(g(\theta), \overset{\text{odd}}{g}(\theta), \theta) , \quad f^i(\theta) \in C^1(T_{\text{odd}}N \times \{\theta\}) ,
\]  
(2.1)

in a some domain of the supermanifold \( N \) parametrized by local coordinates \( g^i(\theta), i = 1, \ldots, N = (N_+, N_-) \) \( (g^i(\theta) = g^i_0 + g^i_1 \theta) \) being by unknown superfunctions\(^1\). Grassmann parities \( \varepsilon_p, \varepsilon_j, \varepsilon \) of quantities \( g^i(\theta), g^i_0, g^i, f^i(\theta) \) are given by the formula \((\varepsilon = \varepsilon_p + \varepsilon_j [1])\)

\[
(\varepsilon_p, \varepsilon_j, \varepsilon)b(\theta) = (\varepsilon_p(g^i_1) + 1, \varepsilon_j(g^i_1), \varepsilon(g^i_1) + 1) = (\varepsilon_p(g^i(\theta)), \varepsilon_j(g^i(\theta)), \varepsilon(g^i(\theta))) ,
\]
(2.2)

Eqs.(2.1) are equivalent to the following system of \( 2N \) ODE at most of the 2nd order with respect to \( \theta \)

\[
\overset{\text{odd}}{g} i(\theta) \equiv \frac{d^2 g^i(\theta)}{d \theta^2} \equiv \frac{d}{d \theta} \frac{d g^i(\theta)}{d \theta} = 0 ,
\]
(2.3)

\[
f^i(g(\theta), \overset{\text{odd}}{g}(\theta), \theta) = 0 ,
\]
(2.4)

so that the Cauchy problem setting for (2.1) is controlled by differential constraints which are the subsystem of the 1st order on \( \theta \) \( N \) ODE (2.4) [1]. In a general case the Eqs.(2.4) appear by (functionally) dependent. Singling from (2.4) the independent subsystem of the 1st order on \( \theta \) ODE is effectively realized in fulfilling of the following assumptions [1]:

1) \( (\overset{\text{odd}}{g} i(\theta), \overset{\text{odd}}{g} i(\theta)) = (0, 0) \in T_{\text{odd}} \Phi = \left\{ (g^i(\theta), \overset{\text{odd}}{g} i(\theta)) \middle| f^i(g(\theta), \overset{\text{odd}}{g}(\theta), \theta) \equiv 0 \right\} \),

(2.5)

2) \( f^i(g(\theta), \overset{\text{odd}}{g}(\theta), \theta) = 0 \) determines the 1st order smooth surface \( T_{\text{odd}} \Phi \) in \( T_{\text{odd}}N \) which the condition holds on

\[
\text{rank}_{\varepsilon_j} \left\| \frac{\delta f^i(g(\theta), \overset{\text{odd}}{g}(\theta), \theta)}{\delta g^j(\theta_1)} \right\|_{T_{\text{odd}} \Phi} \leq N \equiv [f^i] .
\]
(2.6)

Notion of the rank for supermatrix of the form (2.6) with respect to \( \varepsilon_j \) grading for \( f^i(\theta), g^i(\theta) \) was defined in [1,3] and \( \frac{\delta}{\delta g^i(\theta_1)} \) denotes the left superfield variational derivative with respect to superfunction \( g^i(\theta_1) (\theta_1 \neq \theta) \).

\(^1\)Because the index \( i \) possesses by the complicated condensed contents then Eqs.(2.1) are, in general, the system of partial differential equations [1]. The only differential operator \( \overset{\text{odd}}{g} \) is specially singled out here.
Theorem 1
System of the 1st order on \(\theta\) N ODE with respect to \(g^i(\theta)\) (2.4) subject to conditions (2.5), (2.6) being unsolvable with respect to \(\hat{g}^i(\theta)\) is reduced to equivalent system of independent equations in GNF under following nondegenerate parametrization for \(g^i(\theta) = (\alpha^i(\theta), \beta^i(\theta), \gamma^i(\theta)), i = (i, j, \sigma)\)

\[
\hat{\gamma}^i(\theta) = \varphi^i(\alpha(\theta), \gamma(\theta), \hat{\gamma}(\theta), \theta), \quad \beta^i(\theta) = \kappa^i(\alpha(\theta), \gamma(\theta), \theta),
\]

with arbitrary superfunctions \(\gamma^\sigma(\theta)\) and \(\varphi^j(\theta), \kappa^\delta(\theta) \in C^1(T_{odd}\mathcal{N} \times \{\theta\})\). The number of \([\gamma^\sigma]\) coincides with one of differential identities among Eqs.(2.4)

\[
\int d\theta f^i(g(\theta), \hat{g}(\theta), \theta) \bar{R}_{i\sigma}(g(\theta), \hat{g}(\theta), \theta; \theta') = 0,
\]

where operators \(\bar{R}_{i\sigma}(g(\theta), \hat{g}(\theta), \theta; \theta')\) are a) local on \(\theta\) and b) functionally independent ones

a) \(\bar{R}_{i\sigma}(g(\theta), \hat{g}(\theta), \theta; \theta') \equiv \bar{R}_{i\sigma}(\theta; \theta') = \sum_{k=0}^{1} \left( \frac{d}{d\theta} \right)^k \delta(\theta - \theta') \bar{R}_{i\sigma}^k(g(\theta), \hat{g}(\theta), \theta),
\]

b) functional equation

\[
\int d\theta' \bar{R}_{i\sigma}(\theta; \theta') u^\sigma(\theta', \hat{g}(\theta', \theta')) = 0, \quad u^\sigma(\theta) \in C^1(T_{odd}\mathcal{N} \times \{\theta\})
\]

has unique trivial solution.

Proof includes the investigation scheme of the corresponding system of the 1st and 2nd orders with respect to even derivatives on \(t \in \mathbb{R}\) [5] because one can regard that \(t \in i^{[1]}\).

1) In correspondence with (2.6) let us assume that

\[
\text{rank}_{\mathbb{C}} \left\| \frac{\delta_i f^i(\theta)}{\delta \hat{g}^i(\theta_1)} \right\|_{T_{odd}\Phi} = N - M \iff \text{corank}_{\mathbb{C}} \left\| \frac{\delta_i f^i(\theta)}{\delta \hat{g}^i(\theta_1)} \right\|_{T_{odd}\Phi} = M = (M_+, M_-).
\]

Then \(f^i(\theta)\) as the functions of \(\hat{g}^i(\theta)\) are dependent ones and from (2.11) it follows the possibility of the representation

\[
f^i(\theta) = (P^a_1(\theta), p^A_1(\theta), a = 1, \ldots, M; A = M + 1, \ldots, N,
\]

\[
\text{rank}_{\mathbb{C}} \left\| \frac{\partial p^a_1(g(\theta), \hat{g}(\theta), \theta)}{\partial \hat{g}^i(\theta)} \right\|_{T_{odd}\Phi} = N - M \iff \text{corank}_{\mathbb{C}} \left\| \frac{\partial p^a_1(\theta)}{\partial \hat{g}^i(\theta)} \right\|_{T_{odd}\Phi} = M,
\]

\[
P^a_1(g(\theta), \hat{g}(\theta), \theta) = p^A_1(g(\theta), \hat{g}(\theta), \theta) \alpha^a_1 A(g(\theta), \hat{g}(\theta), \theta) + \Delta^a(g(\theta), \theta).
\]

The superfunctions \(\Delta^a(\theta)\) may be dependent ones, i.e.

\[
\text{rank}_{\mathbb{C}} \left\| \frac{\partial \Delta^a(g(\theta), \theta)}{\partial \hat{g}^j(\theta)} \right\|_{\Phi} = M - K \leq M, \quad 0 \leq K = (K_+, K_-).
\]

It means the superfunctions \(\delta^{(1)a_1}(g(\theta), \theta), a_1 = K + 1, \ldots, M\) exist that the condition holds

\[
\text{rank}_{\mathbb{C}} \left\| \frac{\partial \delta^{(1)a_1}(\theta)}{\partial \hat{g}^j(\theta)} \right\|_{\Phi} = M - K.
\]
In (2.11), (2.13)–(2.16) the left partial superfield derivatives with respect to $\tilde{g}^j(\theta), g^j(\theta)$ are denoted as $\frac{\partial}{\partial \tilde{g}^j(\theta)}, \frac{\partial}{\partial g^j(\theta)}$ respectively [1]. By virtue of the assumption (2.5) for $\Delta^a(\theta), \mathcal{P}^a_1(\theta)$ the following representation is valid

$$\Delta^a(g(\theta), \theta) = \delta^{(1)a_1}(g(\theta), \theta)\beta_1^{a_1}(g(\theta), \theta), \quad \text{rank}_{\epsilon_j} \|\beta_1^{a_1}(\theta)\|_{\Phi} = M - K,$$

$$\mathcal{P}^a_1(g(\theta), \tilde{g}(\theta), \theta) = p^A_1(\theta)A_1^{a_A}(\theta) + \delta^{(1)a_1}(\theta)\beta_1^{a_1}(\theta). \quad (2.17)$$

Divide the all $\mathcal{P}^a_1(\theta)$ onto 2 groups: $\mathcal{P}^a_1(\theta) = (\mathcal{P}^{A_1}_1(\theta), \mathcal{P}^{a_1}_1(\theta)), A_1 = 1, \ldots, K; a_1 = K + 1, \ldots, M, a = (A_1, a_1)$

$$\mathcal{P}^{A_1}_1(g(\theta), \tilde{g}(\theta), \theta) = p^A_1(\theta)A_1^{a_A}(g(\theta), \tilde{g}(\theta), \theta) + \delta^{(1)a_1}(\theta)\beta_1^{a_1}(g(\theta), \theta), \text{sdet}\|\beta_1^{a_1}(\theta)\| \neq 0. \quad (2.18)$$

Then from (2.11)–(2.18) it follows that $f^i(\theta) = (\mathcal{P}^{A_1}_1(\theta), \mathcal{P}^{a_1}_1(\theta), p^A_1(\theta))$ are connected with $f^j(1)(\theta, \tilde{g}(\theta), \theta, \theta') = (\mathcal{P}^{B_1}_1(\theta), \delta^{(1)a_1}(\theta), p^B_1(\theta))$ by means of the nondegenerate supermatrix $K^{0,1}(\theta) = \|K^{0,1}_{j}(g(\theta), \tilde{g}(\theta), \theta)\|$ in $T_{odd}V \supset T_{odd}\Phi$ for some $V \subset N$

$$f^i(\theta) = f^j(1)(\theta)K^{0,1}_{j}(\theta), \quad (K^{0,1}(\theta))^{-1} = K^{1,0}(\theta), \quad (2.19)$$

providing the equivalence of Eqs.(2.4) and $f^j(1)(\theta) = 0$. Consider the superfunctions $\mathcal{P}^{A_1}_1(\theta)$ among $\mathcal{P}^a_1(\theta)$ in (2.17). They are the identities for superfunctions $f^j(1)(\theta)$ or $f^i(\theta)$ and can be written by means of two equivalent expressions

$$\int d\theta f^j(1)(g(\theta), \tilde{g}(\theta), \theta)_{(1)}^{(1)}(g(\theta), \tilde{g}(\theta), \theta, \theta') = 0; \quad (2.20)$$

$$\mathcal{P}^{A_1}_1(\theta) = \int d\theta f^{(a_1, A)}(g(\theta'), \tilde{g}(\theta'), \theta')A_{(a_1, A)}^{(a_1, A)}(g(\theta'), \tilde{g}(\theta'), \theta')_{(a_1, A)}; \quad (2.21a)$$

$$f^j(1)(\theta) \equiv \left(\mathcal{P}^{B_1}_1(\theta), f^{(a_1, B)}(g(\theta), \tilde{g}(\theta), \theta')\right). \quad (2.21b)$$

Therefore, among $f^j(1)(\theta) = 0$ the only $f^{(a_1, A)}(\theta) = 0$ are essential. It is the latter equations are equivalent to (2.4).

2) In its turn the superfunctions $\tilde{g}^{(1)a_1}(\theta), p^A_1(\theta)$ may be dependent ones with respect to $\tilde{g}^j(\theta)$

$$\text{rank}_{\epsilon_j} \|\frac{\partial g^{(1)a_1}(\theta)}{\partial \tilde{g}^j(\theta)}p^A_1(\theta)\| |_{T_{odd}\Phi} = N - K - K_1 < [\delta^{(1)}(\theta)] + [p_1(\theta)],$$

$$N - K - K_1 \geq [\delta^{(1)}(\theta)], K_1 = (K_{1+}, K_{1-}). \quad (2.22)$$

It follows from (2.22) the representability of $p^A_1(\theta)$ in the form

$$p^A_1(\theta) = (\mathcal{P}^{a_1, a_1}_2(\theta), a_{11} = M + 1, \ldots, M + K_1; A_{11} = M + K_1 + 1, \ldots, N), \quad (2.23)$$

$$\mathcal{P}^{a_1, a_1}_2(\theta, \tilde{g}(\theta), \theta) = p^{a_1, a_1}_1(\theta)g(\theta), \tilde{g}(\theta), \theta)_{a_{22}}(A_{11}^{a_1}(\theta), \tilde{g}(\theta), \theta) +$$

$$\tilde{g}^{(1)a_1}(\theta)\nu_{11}^{a_1}(a_{11}^{a_1}(\theta), \tilde{g}(\theta), \theta) + \delta^{(1)a_1}(\theta)_{M_11}^{a_1}(\theta)_{a_{21}}^{a_1}(\theta) + \delta^{(2)a_2}(g(\theta), \theta)_{a_{22}}^{a_1}(\theta), \theta), \quad (2.24)$$

$$\text{rank}_{\epsilon_j} \|\beta_{a_{11}}(\theta)\| = [\delta^{(2)}(\theta)] = M_1 = (M_{1+}, M_{1-}),$$

$$a_2 = M + K_1 - M_1 + 1, \ldots, M + K_1, \quad (2.25)$$

$$\text{rank}_{\epsilon_j} \|\frac{\partial \tilde{g}^{(1)a_1}(\theta)}{\partial \tilde{g}^j(\theta)}p^{a_1, a_1}_2(\theta)\| |_{T_{odd}\Phi} = [\delta^{(1)}(\theta)] + [p_2(\theta)] = N - K - K_1, \quad (2.26)$$
where $\delta^{(2)a_2}(\theta)$ are the superfunctions being independent ones on $\delta^{(1)a_1}(\theta)$

$$\operatorname{rank}_{\varepsilon_j} \left| \frac{\partial_l (\delta^{(1)a_1}(\theta), \delta^{(2)a_2}(\theta))}{\partial g^j(\theta)} \right|_{\Phi} = \left[ \delta^{(1)}(\theta) \right] + \left[ \delta^{(2)}(\theta) \right] = M + M_1 - K. \quad (2.27)$$

According to (2.24)–(2.27) divide $\mathcal{P}_{2}^{a_1}(\theta)$ onto 2 groups

$$\mathcal{P}_{2}^{a_1}(\theta) = (\mathcal{P}_{2}^{A_2}, \mathcal{P}_{2}^{A_2})(g(\theta), \tilde{g}(\theta), \theta), A_2 = M + 1, \ldots, M + K_1 - M_1, \quad (2.28)$$

$$\mathcal{P}_{2}^{a_2}(\theta) = p_{2}^{A_1}(\theta) \mu_{a_2}^{a_1}(\theta) + \delta^{(1)a_1}(\theta) \mu_{a_2}^{a_1}(\theta) + \delta^{(1)a_2}(\theta) \mu_{a_2}^{a_2}(\theta), \quad (2.29)$$

and define the set of superfunctions $f^j_{(1)}(g(\theta), \tilde{g}(\theta), \theta) = (\mathcal{P}_{1}^{A_1}(\theta), p_{2}^{A_2}(\theta), \delta^{(1)a_1}(\theta), \delta^{(2)a_2}(\theta), p_{2}^{A_1}(\theta))$ connected with $f^j_{(1)}(g(\theta), \tilde{g}(\theta), \theta) = (\mathcal{P}_{1}^{B_1}(\theta), \delta^{(1)b_1}(\theta), \mathcal{P}_{2}^{B_2}(\theta), \mathcal{P}_{2}^{b_2}(\theta), \theta)$ (and therefore with $f^j(\theta)$ (2.4)) by the nondegenerate supermatrix $K^{1.2}(\theta) = \left| K^{1.2}_{1} (g(\theta), \tilde{g}(\theta), \theta) \right|$ in $T_{odd}V$

$$f^j_{(1)}(\theta) = f^j_{(2)}(\theta) K^{1.2}_{1} j(\theta), \quad K^{1.2}_{1}(\theta) = \begin{vmatrix}
\delta^{B_{1}A_{i}} & 0 & 0 & 0 & 0 \\
0 & \delta^{B_{2}A_{j}} & 0 & 0 & 0 \\
0 & \delta^{b_{1}a_{1}} & A^{b_{2}a_{1}}(\theta) & 0 & 0 \\
0 & 0 & 0 & B^{b_{2}a_{2}}(\theta) & 0 \\
0 & 0 & 0 & C^{b_{2}a_{2}}(\theta) & \delta^{B_{1}A_{1}}
\end{vmatrix}, \quad (2.30)$$

Its inverse supermatrix has the form

$$K^{2.1}\left(\theta\right) = \left( K^{1.2}(\theta) \right)^{-1} = \begin{vmatrix}
\delta^{B_{1}C_{1}} & 0 & 0 & 0 & 0 \\
0 & \delta^{b_{1}c_{1}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{vmatrix}. \quad (2.31)$$

From (2.30), (2.31) it follows both $K^{1.2}(\theta)$ and $K^{2.1}(\theta)$ appear by the local differentiation operators with respect to $\theta$. In its turn from (2.22)–(2.24) it implies the additional, already differential on $\theta$, identities exist among $f^j_{(2)}(\theta) \equiv (\mathcal{P}_{1}^{A_1}(\theta), f^{(a_1,A_1)}_{(1)}(\theta))$ besides of ones (2.20)

$$\int d\theta f^{(a_1,A)}_{(1)}(g(\theta), \tilde{g}(\theta), \theta) \mathcal{R}^{(2)}_{(a_1,A_2)}(g(\theta), \tilde{g}(\theta), \theta; \theta') = 0. \quad (2.32)$$

One can equivalently represent them in the form

$$\mathcal{P}_{2}^{a_2}(\theta) = \int d\theta' f^{(a_1,a_2,A_{11})}_{(2)}(g(\theta'), \tilde{g}(\theta'), \theta') \mathcal{A}_{(a_1,a_2,A_{11})}^{(2)}(g(\theta'), \tilde{g}(\theta'), \theta; \theta'), \quad (2.33a)$$

$$f^j_{(2)}(\theta) \equiv (\mathcal{P}_{1}^{A_1}(\theta), \mathcal{P}_{2}^{a_2}(\theta), f^{(a_1,a_2,A_{11})}_{(2)}(\theta)). \quad (2.33b)$$

Thus the only $f^{(a_1,a_2,A_{11})}_{(2)}(\theta) = 0$ are the essential equations from $f^{(a_1,A_1)}_{(1)}(\theta) = 0$. It is the former equations are completely equivalent to the initial ones (2.4).
3) Let us assume the set of superfunctions \( p_{2A11}(\theta), \delta^{(1)a_1}(\theta), \delta^{(2)a_2}(\theta) \) are dependent with respect to variables \( \hat{g}^i(\theta) \). Then, analogously to above case the new constraints \( \delta^{(3)}(g(\theta), \theta) \) arise being by independent on \( \delta^{(1)}(\theta), \delta^{(2)}(\theta) \). In view of the finiteness of the discrete part of index \( i \) and in ignoring of the covariance requirement with respect to \( i \) it follows from induction principle the existence of the \( l \)th step \((l \leq N)\) of the iterative procedure the such that the equations

\[
\begin{align*}
\mathcal{F}^{(a_1),A_{l-11}}_{(l)}(g(\theta), \hat{g}(\theta), \theta) &= \mathcal{F}^{(a_{11},A_{l-11})}_{(l)}(g(\theta), \hat{g}(\theta), \theta) = 0, \\
\mathcal{F}^{(a_1),A_{l-11}}_{(l)}(\theta) &= \left( \delta^{(1)a_1}(g(\theta), \theta), \ldots, \delta^{(l)a_1}(g(\theta), \theta), p_{l_{A_{l-11}}}(g(\theta), \hat{g}(\theta), \theta) \right)
\end{align*}
\]

are equivalent to Eqs.(2.4) and

\[
\begin{align*}
\text{rank}_{j} \left\| \frac{\partial}{\partial g^j(\theta)} \delta^{(1)a_1}(\theta), \ldots, \delta^{(l)a_1}(\theta), p_{l_{A_{l-11}}}(\theta) \right\|_{T_{a=\Phi}} &= N - K + \sum_{s=1}^{l} (M_s - K_s) = [p_l] + \sum_{s=1}^{l} [\delta^{(s)}], \quad (2.35) \\
\text{rank}_{j} \left\| \frac{\partial}{\partial g^j(\theta)} \delta^{(1)a_1}(\theta), \ldots, \delta^{(l)a_1}(\theta) \right\|_{\Phi} &= \frac{\sum_{s=1}^{l} [\delta^{(s)}]}{M_s}, \quad s = 1, \ldots, l, \quad (2.36) \\
[p_l] &= N - M - \sum_{k=1}^{l-1} K_{k}^{}, [\delta^{(s)}] = M_s^{}, s = 1, \ldots, l^{}
\end{align*}
\]

Formally, the constructed algorithm of system (2.4) reduction to GNF can be written as follows

\[
f^i(\theta) = f^i_{(1)}(\theta)K^01^1j(\theta),
\]

\[
\begin{align*}
f^i_{(1)}(\theta) &= (\mathcal{F}^{A_1}_{(1)}(\theta), \mathcal{F}^{(a_1,A)}_{(1)}(\theta)), \\
\mathcal{F}^{A_1}_{(1)}(\theta) &= \int d\theta^{} \mathcal{F}^{(a_1,A)}_{(2)}(\theta')A_{(1)(a_1,A)}A_1(\theta'; \theta) \Leftrightarrow \int d\theta^{} f^i_{(1)}(\theta)R^A_{j_{A_1}}(\theta; \theta') = 0; \\
f^i_{(2)}(\theta) &= f^i_{(2)}(\theta)K^{12j}(\theta), \\
f^i_{(3)}(\theta) &= f^i_{(3)}(\theta)K^{23j}(\theta), \\
\mathcal{F}^{(a_1,A)}_{(1)}(\theta) &= \int d\theta^{} \mathcal{F}^{(a_1,A)}_{(2)}(\theta')A_{(2)(a_2,A_{11})}A_2(\theta'; \theta) \Leftrightarrow \int d\theta^{} f^i_{(1)}(\theta)R^A_{j_{A_1}}(\theta; \theta') = 0; \\
\mathcal{F}^{(a_1,A)}_{(2)}(\theta) &= \int d\theta^{} \mathcal{F}^{(a_1,A)}_{(3)}(\theta')A_{(3)(a_3,A_{11})}A_3(\theta'; \theta) \Leftrightarrow \int d\theta^{} f^i_{(1)}(\theta)R^A_{j_{A_1}}(\theta; \theta') = 0.
\end{align*}
\]

Thus, the Eqs.(2.4) have been reduced to equivalent ones in GNF (2.34) being by functionally independent. Comparison of (2.34) with (2.7) means by virtue of (2.35), (2.36) that

\[
\begin{align*}
\hat{\delta}^{(s)}(\theta) - \varphi^{(s)}(\alpha(\theta), \gamma(\theta), \hat{\gamma}(\theta), \theta) = 0 \Leftrightarrow p_{l_{A_{l-11}}}(g(\theta), \hat{g}(\theta), \theta) = 0, \quad \hat{i} = A_{l-11}, \\
\hat{\beta}^{(k)}(\theta) - \kappa^{(k)}(\alpha(\theta), \gamma(\theta), \hat{\gamma}(\theta), \theta) = 0 \Leftrightarrow \delta^{(k)\delta_1}(g(\theta), \theta) = 0, \quad k = 1, \ldots, l, \quad \hat{i} = A_{l-11}, \\
\end{align*}
\]

(2.42)
The number of arbitrary superfunctions $\gamma(\theta)$ in (2.7) is equal to

$$[\gamma(\theta)] = [g^i(\theta)] - [\alpha^\gamma(\theta)] - [\beta^\lambda(\theta)] = [f^i(\theta)] - \left[ \sum_{l=0}^{l-1} (K_{s} - M_s) \right] = K + \sum_{s=1}^{l-1} (K_s - M_s)$$

(2.44)

and coincides with one of the identities in (2.39b), (2.40b), . . . , (2.41b).

As far as the supermatrices $K^s,s^{-1}(\theta) = (K^{s-1,s}(\theta))^{-1}$ are the local ones on $\theta$, $s = 1, \ldots, l$, then one can write the identities in terms of the initial equations (2.4) which have the form (2.8) with local on $\theta$ and functionally independent operators $\mathcal{R}_{\sigma}(\theta; \theta')$ being by polynomials with respect to $K^{s,s^{-1}}(\theta)$.

**Remark:** The above proof have not concerned the possible complicated structure of index $i$ (see footnote 1). The locality of operators $\mathcal{R}_{\sigma}(\theta; \theta')$ with respect to other continual parts of the indices $i, \sigma$ may be shown in the analogous way. However, the requirement of functional independence of $\mathcal{R}_{\sigma}(\theta; \theta')$ leads, in general case, to the loss of covariance for these quantities.

From the Theorem 1 proof the validity of its consequence [1] easily follows (with use of the integer-valued functions of degree and least degree: $\deg_{\gamma(\theta)}, \min \deg_{\gamma(\theta)}$, $c(\theta) \in \{g^i(\theta), \hat{g}^i(\theta), g^i(\theta)\hat{g}^i(\theta), \ldots\}$ [1]).

**Corollary 1**

If $f^i(\theta)$ (2.4) are the holonomic constraints

$$\deg_{\gamma(\theta)} f^i(\theta) = 0,$$

(2.45)

then for $f^i(g(\theta), \theta)$ under following parametrization of superfunctions $g^i(\theta) \mapsto g'^i(g(\theta))$

$$g'^i(\theta) = (\alpha^A(\theta), \gamma^\sigma(\theta)), \quad i = (A, \sigma), \quad \sigma = 1, \ldots, [\gamma], \quad A = 1, \ldots, [\alpha]$$

(2.46)

there exists the equivalent system of the holonomic constraints

$$\Phi^A(\alpha(\theta), \gamma(\theta), \theta) = 0.$$  

(2.47)

The number $[\gamma]$ coincides with one of algebraic (in the sense of differentiation with respect to $\theta$) identities among $f^i(\theta)$

$$f^i(g(\theta), \theta)\mathcal{R}_{\sigma}^{(0)}(g(\theta), \theta) = 0$$

(2.48)

being obtained from (2.8) by means of integration on $\theta$ with allowance made for validity of the type (2.9) connection of $\mathcal{R}_{\sigma}(\theta, \theta')$ with algebraic (on $\theta$) operators $\mathcal{R}_{\sigma}^{(0)}(\theta)$

$$\mathcal{R}_{\sigma}(g(\theta), \theta; \theta') = \delta(\theta - \theta')\mathcal{R}_{\sigma}^{(0)}(g(\theta), \theta)(-1)^c(g(\theta)).$$

(2.49)

A dependence upon $\hat{g}^i(\theta)$ in (2.49) may be only parametric one.

### 3 Application to GSTF in Lagrangian Formulation

Consider as $\mathcal{N}$ the supermanifold $\mathcal{M}_{cl}$ parametrized by the classical superfields $\mathcal{A}^i(\theta)$

$$\mathcal{A}^i(\theta) = A^i + \lambda^\theta \varepsilon_p, (\varepsilon_p, \varepsilon_j, \varepsilon) \mathcal{A}^i(\theta) = ((\varepsilon_p)_i, (\varepsilon_j)_i, \varepsilon_i), \quad i = 1, \ldots, n = (n_+, n_-),$$

being by superfunctions defined on $\mathcal{M} = \hat{\mathcal{M}} \times \hat{P}$, in its turn to be the quotient space of the supergroup $J = J \times P = (\hat{\mathcal{M}} \otimes J_{\hat{A}}) \times P$.

$\mathcal{M} = J/J_{\hat{A}}$ with one-parametric subsupergroup $P$.
generated by the Grassmann nilpotent variable $\theta$ [1]. Superspace $\mathcal{M}$ may be parametrized by sets of supernumbers $(z^a, \theta) = (x^\mu, \theta^{A_k}, \theta)$, if the representation for $\tilde{\mathcal{M}}$ is valid [1]

$$\tilde{\mathcal{M}} = \mathbb{R}^{1,D-1|L_c}, \; \mu = 0, 1, \ldots, D - 1, \; A = 1, \ldots, c = 2^{[D/2]}, \; k = 1, \ldots,$$

meaning that $\tilde{\mathcal{M}}$ appears by the real $D$-dimensional Minkowski superspace with $L$ supersymmetries (if $\hat{J}$ is the corresponding group of the usual $L$-extended supersymmetry). Superfields $\mathcal{A}(\theta)$ are considered by belonging to the special Berezin superalgebra $\hat{\Lambda}_{D|L_c+1}(z^a, \theta; K), K = (\mathbb{R}$ or $\mathbb{C})$ [1–3].

The $\Lambda_1(\theta, R)$-valued superfunction $S_L(\theta) \equiv S_L(\mathcal{A}(\theta), \mathcal{A}(\theta), \theta) \in C^k(T_{odd} \mathcal{M}_{cl} \times \{\theta\}), k \leq \infty, (\varepsilon_p, \varepsilon_j, \varepsilon) S_L(\theta) = (0, 0, 0), T_{odd} \mathcal{M}_{cl} = \{(\mathcal{A}(\theta), \mathcal{A}(\theta)) | \mathcal{A}(\theta) \in \mathcal{M}_{cl}\}$ and superfunctional $Z[\mathcal{A}] = \int d\theta S_L(\theta), Z[\mathcal{A}] \in C_{F_1}(\varepsilon_p, \varepsilon_j, \varepsilon) Z[\mathcal{A}] = (1, 0, 1)$ are the central objects in the Lagrangian formulation for GSTF characterizing the superfield model on this stage [1] of investigation. Dynamics of the model follows from a variational principle for $Z[\mathcal{A}]$ and is described by Euler-Lagrange equations [1]

$$\mathcal{L}_i^l(\theta) S_L(\theta) \equiv \left( \frac{\partial_i}{\partial \mathcal{A}^i(\theta)} - (-1)^{\varepsilon_i} \frac{d}{d\theta} \frac{\partial_i}{\partial \mathcal{A}^i(\theta)} \right) S_L(\theta) = \frac{\delta_i Z[\mathcal{A}]}{\delta \mathcal{A}^i(\theta)} = 0, \quad (3.1)$$

being represented equivalently by virtue of (2.1), (2.3), (2.4) by the Lagrangian system (LS)

$$\mathcal{L}_i^l(\theta) \mathcal{A}^i(\theta) \frac{\partial^2 S_L(\theta)}{\partial \mathcal{A}^i(\theta) \partial \mathcal{A}^l(\theta)} = 0, \quad (3.2)$$

$$\Theta_i(\mathcal{A}(\theta), \mathcal{A}(\theta), \theta) \equiv \frac{\partial_i S_L(\theta)}{\partial \mathcal{A}^i(\theta)} - (-1)^{\varepsilon_i} \left( \frac{\partial_i S_L(\theta)}{\partial \mathcal{A}^i(\theta)} + \mathcal{A}^i(\theta) \frac{\partial_i}{\partial \mathcal{A}^i(\theta)} \right) = 0. \quad (3.3)$$

Subsystem (3.3), for deg $z_A(\theta) \neq 0$ called the differential constraints in Lagrangian formalism (DCLF) (for deg $z_A(\theta) = 0$ the holonomic constraints in Lagrangian formalism (HCLF)), restricts an arbitrariness in a choice of $2n$ initial conditions $\left(\mathcal{A}(0), \mathcal{A}(0)^i(0)\right)$ for $\theta = 0$ in setting of Cauchy problem. Subsystem (3.2) are not written in NF with respect to $\mathcal{A}^i(\theta)$ and possibility to pass to NF depends on the nondegeneracy of the supermatrix $K(\theta) = \left\| \frac{\partial^2 S_L(\theta)}{\partial \mathcal{A}^i(\theta) \partial \mathcal{A}^l(\theta)} \right\|$.

DCLF themselves appear, in general, by dependent system of the 1st order on $\theta$ $n$ ODE with respect to unknowns $\mathcal{A}^i(\theta)$. Reduction of $\Theta_i(\theta) \to GN$ is realized independently on subsystem of the 2nd order on $\theta$ $n$ ODE (3.2) in the result of Theorem 1 application directly to (3.3). To this end let us adapt the assumption (2.5), (2.6) to the case of the Lagrangian GSTF [1,3] only in terms of $Z[\mathcal{A}]$:

1) \hfill \exists \left(\mathcal{A}_0^i(\theta), \mathcal{A}_0^i(\theta)^i\right) \in T_{odd} \mathcal{M}_{cl} : \Theta_i(\theta) \left| \left(\mathcal{A}_0(\theta), \mathcal{A}_0(\theta)\right) = \left(\mathcal{A}_0(\theta), \mathcal{A}_0^i(\theta)\right) \right. = 0; \quad (3.4)

2) \hfill \exists \Sigma \subset \mathcal{M}_{cl} (\Sigma \text{ smooth supersurface}) : \left(\mathcal{A}_0(\theta), \mathcal{A}_0^i(\theta)\right) \in T_{odd} \Sigma, \Theta_i(\theta)|_{T_{odd} \Sigma} = 0, \quad (3.5)

\quad \dim \Sigma = m = (m_+, m_-), \dim \Sigma_{T_{odd} \Sigma} \equiv \dim T_{odd} \Sigma = (m_+ + m_-, m_- + m_+); \quad (3.6)

index $\iota$ can be divided $\iota = (A, \alpha), A = 1, \ldots, n - m, \alpha = n - m + 1, \ldots, n$ in a such way that the condition holds

$$\text{rank}_{\Sigma} \left\| \frac{\delta_i}{\delta \mathcal{A}^i(\theta_1)} \frac{\delta_i Z[\mathcal{A}]}{\delta \mathcal{A}^i(\theta)} \right\| = \text{rank}_{\Sigma} \left\| \frac{\delta_i}{\delta \mathcal{A}^i(\theta_1)} \frac{\delta_i Z[\mathcal{A}]}{\delta \mathcal{A}^i(\theta)} \right\| = n - m. \quad (3.7)$$
Remind [1], in the first place, that Σ is considered as local supersurface and, in the second, the following representation is true for DCLF in terms of the superfields \( \hat{A}_i(\theta) = A^i(\theta) - A_0^i(\theta) \): 
\[
\Theta_i(A(\theta), \hat{A}(\theta), \theta) = \Theta_{i, \text{lin}}(\hat{A}(\theta), \hat{A}(\theta), \theta) + \Theta_{i, \text{nl}}(\hat{A}(\theta), \hat{A}(\theta), \theta),
\]

(3.8)

Whereas the assumption 2 gives the possibility to present \( \Theta_i(\theta) \) in the form of two special subsystems in formal ignoring of the requirements of locality and covariance with respect to index \( \iota \) relative to restriction of the superfield representation \( T \) onto subsupergroup \( \hat{J} : T_{J,F} \).

Reduction of DCLF to equivalent system of the 1st order on \( \theta \) ODE in GNF immediately follows from the Theorem 1 application [1] in the form of

**Theorem 2**

A nondegenerate parametrization for \( A^i(\theta) \) exists

\[
A^i(\theta) = (\delta^i(\theta), \beta^i(\theta), \xi^i(\theta)) \equiv (\varphi^i(\theta), \xi^i(\theta)), \quad i = (i_1, \ldots, n - m),
\]

(3.9)

the such that \( \Theta_i(\theta) \) (3.3) are equivalent to the system of independent ODE in GNF

\[
\dot{\delta}^i(\theta) = \phi^i(\delta^i(\theta), \xi^i(\theta), \theta), \quad \beta^i(\theta) = \kappa^i(\delta^i(\theta), \xi^i(\theta), \theta),
\]

(3.10)

with \( \phi^i(\theta), \kappa^i(\theta) \in C^k(T_{\text{odd}} \text{M}_{cl} \times \{\theta\}) \) and arbitrary superfields \( \xi^i(\theta) \): \( [\xi^i(\theta)] = m < n \). Their \( (\xi^i(\theta)) \) number coincides with one of differential identities among \( \Theta_i(\theta) \)

\[
\int d\theta \frac{\delta Z[A]}{\delta A^i(\theta)} \hat{K}_a^i(\theta; \theta') = 0, \quad \varepsilon_p, \varepsilon_j, \varepsilon \hat{R}_a^i(\theta; \theta') = (1 + \varepsilon_p)_{i_1}(\varepsilon_j)_{i_2} + \varepsilon_{i_3} + \varepsilon_{i_4} + 1)
\]

(3.11)

with a) local and b) functionally independent operators \( \hat{K}_a^i(\theta; \theta') \equiv \hat{K}_a^i(\theta; \theta') \):

a) 
\[
\hat{R}_a^i(\theta; \theta') = \sum_{k=0}^{1} \left( \frac{d}{d\theta} \right)^k (\theta - \theta') \hat{K}_a^i(\theta; \theta'),
\]

(3.12)

b) functional equation

\[
\int d\theta' \hat{R}_a^i(\theta; \theta') u^i(\theta') = 0, \quad u^i(\theta) \in C^k(T_{\text{odd}} \text{M}_{cl} \times \{\theta\})
\]

(3.14)

has the unique vanishing solution.

It literally follows from Theorem 2, after change of corresponding symbols, the consequence being analogous to Corollary 1 for HCLF \( \Theta_i(A(\theta), \theta) \) with \( R_{a}^i(\theta; \theta') = \hat{R}_{a}^i(\theta; \theta') \) [1].

The interpretation for operators \( R_a^i(\theta; \theta') \), \( \hat{R}_a(\theta; \theta) \) as the GGTG, GGTST respectively had been given in Ref.[1]. It had been shown the complete sets of the GGTGT, GGTST appear by the bases in affine \( C^k(T_{\text{odd}} \text{M}_{cl} \times \{\theta\}) \)-module \( Q(Z) = \text{Ker}\left\{ \frac{\delta Z[A]}{\delta A^i(\theta)} \right\} \) and affine \( C^k(\text{M}_{cl} \times \{\theta\}) \)-module \( Q(S_L) = \text{Ker}\left\{ \Theta_i(A(\theta), \theta) \right\} \) respectively. By realization of the mentioned consequence for Theorem 2 it appears the Corollary 2.2 from Ref.[1] in the framework
of which a GSTF model is the almost natural system
\[
S_L(\mathbf{A}(\theta), \mathbf{\hat{A}}(\theta), \theta) = T(\mathbf{A}(\theta), \mathbf{\hat{A}}(\theta)) - S(\mathbf{A}(\theta), \theta), \quad \min \deg_{\mathbf{A}(\theta)} S(\theta) = 2, \tag{3.15}
\]
\[
T(\mathbf{A}(\theta), \mathbf{\hat{A}}(\theta)) = T_1(\mathbf{\hat{A}}(\theta)) + \mathbf{\hat{A}}(\theta)T_0(\mathbf{A}(\theta)), \quad T_j(\mathbf{A}(\theta)) = g_{jl}(\theta)\mathbf{A}^l(\theta),
\]
\[
g_{jl}(\theta) = (-1)^{\varepsilon_{jl}} g_{\varepsilon_{jl}}(\theta), \quad g_{jl}(\theta) = P_0(\theta) g_{jl}(\theta), \quad \min \deg_{\mathbf{A}(\theta)} T_1(\theta) = 2, \tag{3.16}
\]
so that the HCLF and condition (3.7) have the form
\[
\Theta_i(\mathbf{A}(\theta), \theta) = -S_{i\alpha}(\mathbf{A}(\theta), \theta)(-1)^{\varepsilon_i} = 0, \quad \text{rank}_{\varepsilon_j}(S_{i\varepsilon_j}(\mathbf{A}(\theta), \theta)) = \sum = n - m. \tag{3.17}
\]
Corresponding GTGT, GTST have the form [1]
\[
\mathcal{A}^l(\theta) \mapsto \mathcal{A}^l(\theta) = \mathcal{A}^l(\theta) + \delta_\varepsilon \mathcal{A}^l(\theta); \quad \delta_\varepsilon \mathcal{A}^l(\theta) = \int d\theta' \mathcal{R}_\alpha^l(\theta; \theta')\xi^\alpha(\theta'),
\]
\[
\mathcal{A}^l(\theta) \mapsto \mathcal{A}^l(\theta) = \mathcal{A}^l(\theta) + \delta \mathcal{A}^l(\theta); \quad \delta \mathcal{A}^l(\theta) = \mathcal{R}_{0, \alpha}^l(\mathbf{A}(\theta), \theta)\xi^\alpha(\theta) \tag{3.18}
\]
and appear by infinitesimal invariance transformations with arbitrary superfunctions \(\xi^\alpha(\theta), \xi^\alpha_0(\theta)\) \([\varepsilon_p, \varepsilon_j, \varepsilon] \xi^\alpha(\theta) = (0, \varepsilon_\alpha, \varepsilon_\alpha)\) for \(Z[\mathbf{A}], S(\mathbf{A}(\theta), \theta)\) respectively.

In addition for local with respect to \(z^a\) models the GGTGT, GGST can be represented by the local differential operators with respect to \(z^a\). At least in ignoring of the requirement of covariance on index \(x\) the GGTGT, GGST, as it follows from Theorem 2, appear by independent and hence define the irreducible GSTF models in the Lagrangian formulation, called the irreducible GThGT and GThST (gauge theory of the special type) respectively [1]. In general the conservation of mentioned conditions on locality and covariance for \(\mathcal{R}_\alpha^l(\theta; \theta'), \mathcal{R}_{0, \alpha}^l(\theta)\) leads to modification of the Theorem 2 conclusion concerning of the solution for Eq.(3.14) and its analog for \(\mathcal{R}_{0, \alpha}^l(\theta)\). Namely, for the last equations the nonvanishing solutions may exist as well. By definition the GThGT (GThST) with the such property are called the reducible GThGT (GThST) with functionally dependent GSTF (GGTST).

4 Gauge Algebra of GGTGT

Following to Refs.[6,7] let us investigate the GTGT (3.18) and algebraic structures connected with them. From GTGT in a more than one-valued form one can construct the finite transformations of invariance for \(Z[\mathbf{A}]\)
\[
\mathcal{A}^l(\theta) \mapsto \mathcal{A}^l(\theta) = G^l(\mathbf{A}(\theta)|\xi(\theta)) , \quad G^l(\mathbf{A}(\theta)|0) = \mathcal{A}^l(\theta), \tag{4.1}
\]
\[
\frac{\delta G^l(\mathbf{A}(\theta)|\xi(\theta))}{\delta \xi^\alpha(\theta)}|_{\xi^\alpha(\theta)=0} = \mathcal{R}_\alpha^l(\theta; \theta'); \quad Z[\mathbf{A}_f] = Z[\mathbf{A}], \quad (\varepsilon_p, \varepsilon_j, \varepsilon) \frac{\delta}{\delta \xi^\alpha(\theta)}(1, \varepsilon_\alpha, \varepsilon_\alpha + 1). \tag{4.2}
\]
As \(G^l(\theta)\) one can make use, for instance, the superfields satisfying to the \(\theta\)-superfield condition [6]
\[
\frac{\partial}{\partial \tau} G^l(\mathbf{A}(\theta)|\tau \xi(\theta)) = \int d\theta' \xi^\alpha(\theta')\mathcal{R}_\alpha^l(\theta; \theta')|_{\mathcal{A}^l(\theta) = G^l(\mathbf{A}(\theta)|\tau \xi(\theta))}, \quad \tau \in \mathbb{R}. \tag{4.3}
\]
Really, having denoted \(Z_{\tau} \equiv Z[\mathbf{A}|_{\mathcal{A}=G^l(\mathbf{A}|\tau \xi)}\), obtain from (3.11), (4.3) the relationships
\[
\frac{\partial}{\partial \tau} Z_{\tau} = \int d\theta' \frac{\delta Z[\mathbf{A}]}{\delta \mathcal{A}^l(\theta)} \int d\theta' \xi^\alpha(\theta')\mathcal{R}_\alpha^l(\theta; \theta')|_{\mathcal{A}^l(\theta) = G^l(\mathbf{A}(\theta)|\tau \xi(\theta))} = 0, \tag{4.4}
\]
from which it follows
\[ Z_{\tau=0} = Z_{\tau=1} \iff Z[A] = Z[A_f]. \] (4.5)

Formal solution for Eq.(4.3) with initial condition from (4.1) has the form
\[ G^i(A(\theta)|\xi(\theta)) = \exp\{\int d\theta' \xi^\alpha(\theta')\hat{\Gamma}_\alpha(\theta')\}A^i(\theta), \] (4.6)
\[ \hat{\Gamma}_\alpha(\theta')F[A] = \int d\theta \frac{\delta F[A]}{\delta A^i(\theta')} \hat{R}_\alpha^i(\theta; \theta')\], \( (\varepsilon, \varepsilon, \varepsilon, \varepsilon)\hat{\Gamma}_\alpha(\theta) = (1, \varepsilon^\alpha, \varepsilon^\alpha + 1), \) \( F[A] \in C_F. \) (4.7)

Really, for an arbitrary superfunctional \( F[A] \), having the polynomial series expansion with respect to \( A^i(\theta) \), the operatorial formula is valid
\[ F[G] \equiv F[A_f] = \exp\{\int d\theta' \xi^\alpha(\theta')\hat{\Gamma}_\alpha(\theta')\}F[A]. \] (4.8)

Choosing \( F[G] \equiv G^i(A(\theta)|\xi(\theta)) \) obtain the solution of Eqs.(4.2) in the form (4.6).

From differential consequences of the identities (3.11)
\[ \left( \int d\theta' \left[ \frac{\delta Z[A]}{\delta A^i(\theta')} \hat{R}_\alpha^i(\theta; \theta') \right] \right) (-1)^{\xi^\alpha} \cdot \left( \frac{\delta Z[A]}{\delta A^i(\theta')} \frac{\delta \hat{R}_\alpha^i(\theta; \theta')}{\delta A^j(\theta')} \right) \] (4.9)

it follows the transformation rule for \( \frac{\delta Z[A]}{\delta A^i(\theta')} \) under finite GTGT (4.1), (4.6)
\[ \frac{\delta Z[A]}{\delta A^i(\theta')} \bigg|_{A^i = G^i(A|\xi)} = \int d\theta' Q^i_A(\theta', \hat{A}(\theta), \theta; \theta') \frac{\delta Z[A]}{\delta A^j(\theta')} \] (4.10)

with nondegenerate supermatrix \( Q^i_A(\theta', \theta) \in C^k(T_{odde}M_{cl} \times \{\theta, \theta\}) \) in some neighbourhood of \( \xi^\alpha(\theta) = 0: Q^i_A(\theta', \theta)|_{\xi(\theta)=0} = \delta(\theta' - \theta)\delta^i. \)

Investigation of the gauge algebra of GTGT properties is based on the study of properties of the supercommutator of the 1st order differential operators \( \hat{\Gamma}_\alpha(\theta) \), with respect to \( A^i(\theta) \), having \( Z[A] \) as the eigensuperfunction with zero eigenvalue. By definition, the supercommutator \( [\hat{\Gamma}_\alpha(\theta_1), \hat{\Gamma}_\beta(\theta_2)]_s \) possesses by the same property as well. Its value in calculating on arbitrary \( F[A] \in C_F \) is equal to
\[ [\hat{\Gamma}_\alpha(\theta_1), \hat{\Gamma}_\beta(\theta_2)]_s F[A] = \hat{\Gamma}_\alpha(\theta_1)\left( \hat{\Gamma}_\beta(\theta_2) F[A] \right) - \left( -1 \right)^{(\varepsilon^\alpha + 1)(\varepsilon^\beta + 1)}\left( (\alpha, \theta_1) \leftrightarrow (\beta, \theta_2) \right) = \] \[ \int d\theta_2' \frac{\delta F[A]}{\delta A^i(\theta_2')} \int d\theta_1' \left( \frac{\delta \hat{R}_\beta^i(\theta_2'; \theta_2)}{\delta A^j(\theta_1')} \hat{R}_\alpha^i(\theta_1'; \theta_1) \right) - \left( -1 \right)^{(\varepsilon^\alpha + 1)(\varepsilon^\beta + 1)}\left( (\alpha, \theta_1) \leftrightarrow (\beta, \theta_2) \right) \]
\[ = \int d\theta_2' \frac{\delta F[A]}{\delta A^i(\theta_2')} \hat{y}_{\beta\alpha}(A(\theta_1'), \hat{A}(\theta_2), \theta_2; \theta_2, \theta_1), \] (4.11)
\[ \hat{y}_{\beta\alpha}(\theta_1', \theta_2, \theta_1) = \left( -1 \right)^{\varepsilon^\alpha(\varepsilon^\beta + 1)}\hat{y}_{\beta\alpha}(\theta_1', \theta_2), \] (4.12)
\[ \hat{y}_{\alpha\beta}(A(\theta_1'), \hat{A}(\theta_2'), \theta_1', \theta_2) \equiv \hat{y}_{\alpha\beta}(\theta_1', \theta_2) \in C^k(T_{odde}M_{cl} \times \{\theta_1', \theta_2\}), \theta_k \equiv \theta_1, \ldots, \theta_k \]

Superfunctions \( \hat{y}_{\beta\alpha}(\theta_1', \theta_2) \) appear by local operators of differentiation on \( \theta \) (and with respect to \( z^\alpha \), if \( \hat{R}_\alpha^i(\theta; \theta') \) are the same). By virtue of completeness of the GTGT \( \hat{R}_\alpha^i(\theta; \theta') \) the quantities \( \hat{y}_{\alpha\beta} \) must be expressed through GTGT and the trivial GTGT \( \hat{\tau}_{\alpha\beta}(\theta_1', \theta_2) \) [1]
\[ \hat{y}_{\alpha\beta}(\theta_1', \theta_2) \equiv \left( -1 \right)^{\varepsilon^\alpha} \int d\theta_3 \hat{R}_\gamma(\theta_1', \theta_3) \hat{F}_{\alpha\beta}^\gamma(\theta_3; \theta_2) + \int d\theta_2' \frac{\delta Z[A]}{\delta A^i(\theta_2')} \hat{M}_{\alpha\beta}(\theta_2; \theta_2)(-1)^{\varepsilon^i}. \] (4.13)
Superfunctions $\mathcal{F}_{\alpha \beta}^*(\theta_3; \tilde{\theta}_2)$, $\mathcal{M}_{\alpha \beta}^{\gamma} (\tilde{\theta}_2; \tilde{\theta}_2) \in C^k (T_{\text{odd}}\mathcal{M}_{\text{cl}} \times \{\tilde{\theta}_3, \tilde{\theta}_2\})$ possess by the properties

\[
\begin{align*}
\mathcal{F}_{\alpha \beta}^*(\theta_3; \tilde{\theta}_2) & | \quad \varepsilon_p & | \quad \varepsilon_j & | \quad \varepsilon \\
\mathcal{M}_{\alpha \beta}^{\gamma} (\tilde{\theta}_2; \tilde{\theta}_2) & | \quad 0 & | \quad \varepsilon_\gamma + \varepsilon_\alpha + \varepsilon_\beta & | \quad \varepsilon_\gamma + \varepsilon_\alpha + \varepsilon_\beta \quad , \quad (4.14) \\
\end{align*}
\]

and can be chosen by $\theta$-local ones.

The explicit form of $\mathcal{F}_{\alpha \beta}^*$, $\mathcal{M}_{\alpha \beta}^{\gamma}$ and their properties are based on the analysis of the general solution for equation

\[
\int d\theta \delta Z[A] \delta A^*(\theta) \hat{y}^*(A(\theta), A(\theta), \theta) = 0. \tag{4.16}
\]

**Lemma 1:**

General solution of the Eq.(4.16) for irreducible GGTGT satisfying to completeness condition has the form in the superalgebra $C^k (T_{\text{odd}}\mathcal{M}_{\text{cl}} \times \{\theta\})$

\[
\begin{align*}
\hat{y}^*(\theta) & \equiv \hat{y}^*(A(\theta), A(\theta), \theta) = \int d\theta \left[ \hat{R}^A_\alpha(\theta; \theta') \hat{y}^\alpha(\theta')(-1)^{\varepsilon_1} + \frac{\delta Z[A]}{\delta A^*(\theta')} \hat{E}^{\gamma}(\theta, \theta')(-1)^{\varepsilon_j} \right], \tag{4.17} \\
\hat{y}^\alpha(\theta) & \equiv \hat{y}^\alpha(A(\theta), A(\theta), \theta), \quad \hat{E}^{\alpha}(\theta, \theta') \equiv \hat{E}^{\alpha}(A(\theta), A(\theta), \theta, \theta') = -(-1)^{(\varepsilon_1 + 1)(\varepsilon_1 + 1)} \hat{E}^{\alpha}(\theta', \theta), \quad (4.18) \\
\end{align*}
\]

**Proof:** Assumption (3.7) permits to represent Eq.(4.16) and identities (3.11) in the form respectively

\[
\begin{align*}
\int d\theta \delta Z[A] \delta A^*(\theta) \hat{y}^\alpha(\theta) + \frac{\delta Z[A]}{\delta A^*(\theta)} \hat{y}^\alpha(\theta) & = 0 , \tag{4.19} \\
\int d\theta \delta Z[A] \delta A^*(\theta) \hat{R}^A_\alpha(\theta; \theta') + \frac{\delta Z[A]}{\delta A^*(\theta)} \hat{R}^\alpha_\beta(\theta; \theta') & = 0 , \tag{4.20} \\
\hat{R}^\alpha_\beta(\theta; \theta') \equiv \left( \hat{R}^A_\alpha(\theta; \theta'), \hat{R}^\alpha_\beta(\theta; \theta') \right), \quad \text{rank}_{\|_{\Sigma}} \| \hat{R}^\alpha_\beta(\theta; \theta') \|_{\Sigma} & = \| \hat{R}^{-1} \|_{\Sigma}, \tag{4.21} \\
\exists (\hat{R}^{-1})^\alpha_\beta(\theta; \theta_1) : \int d\theta' \hat{R}^\alpha_\beta(\theta; \theta')(\hat{R}^{-1})^\alpha_\beta(\theta'; \theta_1) & = \delta^\alpha_\beta \delta(\theta - \theta_1). \tag{4.22} \\
\end{align*}
\]

From (4.19)–(4.22) it follows the equivalent representation for (3.11, 4.16)

\[
\begin{align*}
\frac{\delta Z[A]}{\delta A^*(\theta_1)} & = \int d\theta d\theta' \frac{\delta Z[A]}{\delta A^*(\theta')} \hat{R}^\alpha_\beta(\theta; \theta')(\hat{R}^{-1})^\alpha_\beta(\theta'; \theta_1), \tag{4.23} \\
\int d\theta d\theta' \frac{\delta Z[A]}{\delta A^*(\theta_1)} \hat{z}^\alpha(\theta, \theta_1)(-1)^{\varepsilon_\alpha} & = 0 , \\
\hat{z}^\alpha(\theta, \theta_1) & = \delta(\theta_1 - \theta) \hat{y}^\alpha(\theta_1) - \int d\theta' \hat{R}^\alpha_\beta(\theta_1; \theta')(\hat{R}^{-1})^\alpha_\beta(\theta'; \theta) \hat{y}^\gamma(\theta). \tag{4.24}
\end{align*}
\]
Condition (3.7) guarantees the existence of the special parametrization for $A'(\theta)$

$$A'(\theta) \mapsto \tilde{A}'(\theta) = \left( \frac{\delta Z[A]}{\delta A^A(\theta)} \right) = (F_A(\theta), A^\alpha(\theta)).$$

in terms of which Eq.(4.24) is written in the form

$$\int d\theta d\bar{\theta}_1F_A(\theta_1)\bar{z}'(\tilde{A}(\theta), \tilde{A}(\theta), \theta, \theta_1)(-1)^{\epsilon_A} = 0.$$  

(4.26)

Calculating the variational superfield derivative of expression (4.26) with respect to $F_B(\theta_2)$ we obtain

$$\bar{z}'(\tilde{A}(\theta), \tilde{A}(\theta), \theta, \theta_1) = \frac{\delta \bar{z}'(\tilde{A}(\theta), \tilde{A}(\theta), \theta, \theta_1)}{\delta F'_A(\theta)} = \left( \frac{\delta \bar{z}'(\tilde{A}(\theta), \tilde{A}(\theta), \theta, \theta_1)}{\delta F'_A(\theta)} \right) = \left( \frac{\delta \bar{z}'(\tilde{A}(\theta), \tilde{A}(\theta), \theta, \theta_1)}{\delta F'_A(\theta)} \right) \left( \frac{\delta \bar{z}'(\tilde{A}(\theta), \tilde{A}(\theta), \theta, \theta_1)}{\delta F'_A(\theta)} \right).$$

(4.27)

After scaled transformation $F_A(\theta) \mapsto \tau F_A(\theta)$, $\tau \in [0,1] \subset \mathbb{R}$ in (4.27) this equation will pass into system of the 1st order on $\tau$ ODE

$$\frac{d}{d\tau} \left( \tau \bar{z}'(\tau F_A(\theta), \bar{A}(\theta), \theta, \theta_1) \right) = \int d\theta_2 \tau F_A(\theta_2) \times \left( \frac{\delta \bar{z}'(\tilde{A}(\theta), \tilde{A}(\theta), \theta, \theta_1)}{\delta F'_A(\theta)} \right).$$

(4.29)

By direct integration of Eq.(4.29) with respect to $\tau$ along the segment $[0,1]$ we obtain (the integral is regarded as improper one)

$$\bar{z}'(\tilde{A}(\theta), \tilde{A}(\theta), \theta, \theta_1) = \lim_{\tau \to 0} \tau \bar{z}'(\tau F_A(\theta), \bar{A}(\theta), \theta, \theta_1) = \int d\theta_2 F_A(\theta_2) \left[ \int_0^1 d\tau F_A(\theta_2) \times \left( \frac{\delta \bar{z}'(\tilde{A}(\theta), \tilde{A}(\theta), \theta, \theta_1)}{\delta F'_A(\theta)} \right) \right] \times (4.30)$$

The boundedness of the solution for Eq.(4.16) near $F_A(\theta) = 0$ and existence of the integral from right-hand side by hypothesis of the Lemma mean the limit in the left of (4.30) is equal to 0 and the general solution for (4.16) taking account of (4.24) has the form

$$\bar{z}'(\tilde{A}(\theta), \tilde{A}(\theta), \theta, \theta_1) = \int d\theta_1 \left[ \int d\theta' \hat{R}^A(\theta; \theta') (\hat{R}^{-1})^\alpha(\theta; \theta_1) \delta \bar{z}'(\tilde{A}(\theta), \theta_1) - \hat{z}'(\tilde{A}(\theta), \theta_1) \right] =$$

$$\int d\theta_1 \left[ \hat{R}^A(\theta; \theta_1) \delta \bar{z}'(\tilde{A}(\theta), \theta_1)(-1)^{\epsilon_A} + \left( \frac{\delta Z[A]}{\delta A^B(\theta)} \right) \hat{E}^{AB}(\theta, \tilde{A}(\theta), \theta, \theta_1)(-1)^{\epsilon_B} \right], (4.31)

$$\hat{E}^{AB}(\theta, \tilde{A}(\theta), \theta, \theta_1) = \int d\theta_2 \int_0^1 d\tau P^{AB}(\tau F_A(\theta_2), \bar{A}(\theta_2), \theta_1) \times (4.32)$$

$$\hat{\Phi}_A(\theta, \tilde{A}(\theta), \theta) = (-1)^{\epsilon_A} \int d\theta' \hat{R}^{-1}\gamma(\theta; \theta') \delta \hat{z}'(\theta).$$

(4.33)
with arbitrary superfunctions $\hat{y}^\gamma(\theta)$.

Setting

$$\hat{E}^{\alpha B}(\theta_2) = \hat{E}^{A\beta}(\theta_2) = \hat{E}^{\alpha\beta}(\theta_2) = 0$$

we arrive to validity of the formula (4.17) with the properties (4.18).

**Definitions:**

1) Let us call the identities (3.11), expressions for supercommutator GGTGT (4.13) the \textit{structural} equations of the 1st and 2nd orders respectively of the gauge algebra of GTGT. Call the superfunctional $Z[A]$; superfunctions $\hat{R}_\alpha(\theta; \theta')$; $\hat{F}_{\alpha\beta}(\theta; \theta'_2)$, $\hat{M}_{\alpha\beta}(\theta'_2 : \theta_2)$ by the structural superfunctions of zero; 1st; 2nd orders respectively. The set of quantities $Z[A]$, $\hat{R}_\alpha$, $\hat{F}_{\alpha\beta}$, $\hat{M}_{\alpha\beta}$ and so on together with structural equations let us call the gauge algebra of GTGT on $Q(Z)$.

Thus, the order of the structural superfunction and equation is equal to the number of free lower indices in the nonzero function and equation respectively.

2) The rank $R$ of the gauge algebra of GTGT, by definition, is given by the maximal number of free upper indices for the such structural superfunction from the set of all structural superfunctions that corresponding numbers for other elements of this set not greater than given one ($R \in \mathbb{Z}$, $R \leq \infty$). For $R = 0$ the GSTF model appears by nondegenerate theory of general type (ThGT) [1].

Further structural equations and superfunctions of the gauge algebra are deduced from systematic use of the definitions of GThGT, Lemma 1 and all preceding structural equations and functions including their differential consequences in analyzing of supercommutators of the form $[\hat{\Gamma}_\alpha(\theta_1), [\hat{\Gamma}_\alpha(\theta_2), \ldots [\hat{\Gamma}_\alpha(\theta_{k-1}), \hat{\Gamma}_\alpha(\theta_k)] \ldots]]$, $k \geq 3$. This investigation remains out the paper’s scope. Let us only point out the maximal numbers of different with respect to set of upper indices on the structural functions and equations in the fixed kth order of the gauge algebra of GTGT are equal to $[\frac{k}{2}] + 1$ and $[\frac{k+1}{2}]$ respectively.

Note the non-invariance of the definition for structural superfunctions and rank of gauge algebra because of GGTGT are defined by ambiguously [1] with accuracy up to equivalence transformations and in view of the fact that the form of structural functions and equations depends on a choice of parametrization for superfields $A^i(\theta)$.

5 \hspace{1em} \textbf{Connection with Gauge Algebra of Irreducible GThST}

It had been shown in [1] the $Q(S_L)$ appears by the $C^k(M_{cl} \times \{\theta, \theta'\})$-submodule of the affine $C^k(T_{odd}M_{cl} \times \{\theta, \theta'\})$-module $Q(Z)$. By analogy with Sec.4 one can deduce the basic relationships by means of the literal change of corresponding symbols and operations and find the quantities defining a gauge algebra for irreducible GThST on $Q(S_L) \equiv Q(S) \equiv \text{Ker}\{S, \theta(\theta')\}$ with $S(\theta)$ (3.15) satisfying to (3.17). Namely, the identities (3.11), the finite invariance transformations for $S(A(\theta), \theta)$ constructed from infinitesimal GTST (3.19) analogously to scheme of Sec.4 (relationships (4.1)–(4.8)) and transformation rule for HCLF $\Theta_i(A(\theta), \theta)$ under finite GTST have the form respectively

$$S_{\gamma}(\theta)R_{\alpha}^{\gamma}(A(\theta), \theta) = 0 \text{,}$$

$$A^i(\theta) \mapsto A^i_{fin}(\theta) = G^0_\gamma(A(\theta)\xi(\theta)), \text{ } G^0_{\alpha}(A(\theta)|0) = A^i(\theta) \text{,}$$

$$\frac{\partial G^0_\alpha(A(\theta)|\xi(\theta))}{\partial \xi^\alpha(\theta)} \bigg|_{\xi^\alpha(\theta) = 0} = R_{\alpha}^{\gamma}(A(\theta), \theta) \text{,}$$

$$S(A_{fin}(\theta), \theta) = S(A(\theta), \theta) \text{,}$$
\[ G_0(A(\theta)|\xi(\theta)) = \exp\{\xi(\theta)\Gamma_0(\theta)\}A(\theta), \]  
\[ \Gamma_0(\theta)F(A(\theta), \theta) = \mathcal{F}_\gamma(A(\theta), \theta)R_{\gamma\alpha}(A(\theta), \theta), \]  
\[ \varepsilon, \varepsilon \] \[ \mathcal{F}(G(\theta), \theta) \equiv \mathcal{F}(A_{\text{fin}}(\theta), \theta) = \exp\{\Gamma_0(\theta)|\Gamma_0(\theta)\}F(A(\theta), \theta), \]  
\[ \mathcal{F}(\theta) \in C^k(M_{G\theta} \times \{\theta\}), \]  
\[ S_{\gamma}(\theta)_{|A(\theta)|=G(t(A(\theta)|\xi(\theta))} = Q_{0,\gamma}(A(\theta), \theta)S_{\gamma}(\theta), \]  
\[ \text{sdt} \|Q_{0,\gamma}(\theta)\| \not= 0, Q_{0,\gamma}(\theta)|_{\xi(\theta)=0} = \delta_{\gamma}. \]

Vector fields \( \Gamma_0(\theta) \) on \( M_{G\theta} \times \{\theta\} \), amniling \( S(A(\theta), \theta) \), play the role of operators \( \hat{\Gamma}_0(\theta) \) in \( GThGT \) on \( T_{odd}M_{G\theta} \times \{\theta\} \). Their supercommutator possesses by the same property that a c-

\[ \left[ \Gamma_0(\theta), \Gamma_0(\theta) \right]_{\varepsilon}(\theta) = \mathcal{F}(\varepsilon, \varepsilon) = (\mathcal{R}_{\gamma\alpha}(\theta)R_{\gamma\alpha}(\theta) - (-1)^{\varepsilon_\alpha\varepsilon_\beta}(\alpha \leftrightarrow \beta)), \]  
\[ \mathcal{R}_{\gamma\alpha}(\theta)R_{\gamma\alpha}(\theta) - (-1)^{\varepsilon_\alpha\varepsilon_\beta}(\alpha \leftrightarrow \beta) = -\mathcal{R}_{\gamma\alpha}(\theta)\mathcal{A}(\theta, \theta) - S_{\gamma\alpha}(\theta, \theta)\mathcal{M}_{\gamma\alpha}(\theta, \theta), \]  
\[ \mathcal{F}_{\gamma\alpha}(\theta, \theta) \equiv \mathcal{F}(\gamma, \alpha)(\theta, \theta), \mathcal{M}_{\gamma\alpha}(\theta, \theta) \equiv \mathcal{M}_{\gamma\alpha}(\theta, \theta) \in C^k(M_{G\theta} \times \{\theta\}), \]

The explicit form of the superfunctions \( \mathcal{F}_{\alpha\beta}(\theta, \theta), \mathcal{M}_{\alpha\beta}(\theta, \theta) \) and their properties are based on the being easily proved analog of Lemma 1 in question.

**Lemma 2:**
General solution of the equation

\[ S_{\gamma}(\theta)y_0(A(\theta), \theta) = 0 \]  
for irreducible GGTST satisfying to condition of completeness has the form in \( C^k(M_{G\theta} \times \{\theta\}) \)

\[ y_0(\theta) \equiv y_0(A(\theta), \theta) = \mathcal{R}_0(\theta)\mathcal{A}(\theta, \theta)y_0(\theta, \theta) + S_{\gamma}(\theta)E_0(\theta), \]  
\[ E_0(\mathcal{A}(\theta, \theta) \equiv E_0(\theta) = (-1)^{\varepsilon_\alpha\varepsilon_\beta}E_\theta(\theta, \theta), \]  
\[ \Phi_0(\theta) \equiv \Phi_0(\theta, \theta) \equiv \Phi_0(\theta), \]  
\[ \mathcal{E}_0(\theta) \equiv \mathcal{E}_0(\theta, \theta) \equiv \mathcal{E}_0(\theta), \]

Identities (5.1), expression (5.10) are called in correspondence with Sec.4 and Ref.[7] by the structural equations of the 1st and 2nd orders of a gauge algebra of GTST respectively. Superfunctions \( S(\theta); \mathcal{R}_0(\theta); \mathcal{F}_{\gamma\alpha}(\theta, \theta), \mathcal{M}_{\gamma\alpha}(\theta, \theta) \) are called the structural functions of zero; 1st; 2nd orders respectively. The set of \( S(\theta), \mathcal{R}_0(\theta), \mathcal{F}_{\alpha\beta}(\theta, \theta), \mathcal{M}_{\alpha\beta}(\theta, \theta) \) and so on together with corresponding structural equations is called the gauge algebra of GTST on \( Q(S) \).

All other concepts and remarks in the end of Sec.4 are literally transferred onto \( Q(S) \).

A dependence upon \( \hat{\mathcal{A}}(\theta) \) in the structural functions and equations may be only by parametric one. The results of Sec.5 on the gauge algebra of GTST can be obtained from gauge algebra of ordinary (not superfield on \( \theta \) irreducible gauge theory [4,6,7] by continuation of the component fields \( \mathcal{A}^i \) to the superfields \( \mathcal{A}^i(\theta) \) and simultaneously by deformation on \( \theta \) of the all structural functions and equations (in the sense of their explicit dependence on \( \theta \).

Gauge algebra of GTST for GTHST on the whole can be efficiently described by means of generating equations for superfunction

\[ S(\Gamma_{min}(\theta, \theta)) \in C^k(T_{odd}M_{min} \times \{\theta\}, k \leq \infty, T_{odd}^*M_{min} = \{\Phi_{min}(\theta, \theta) \} \Phi_{min}(\theta, \theta) = (A(\theta), C(\alpha(\theta)) \in M_{min} = M_{G\theta} \times M_C, \Phi_{Amin}(\theta) = (A^i(\theta), \]
\[ C_\alpha^*(\theta), \ A = (\iota, \alpha) \] which in contrast to its analog \( S_{H_{\text{min}}}(\Gamma_{\text{min}}(\theta)) \) in [3] depends upon \( \theta \) explicitly and is not restricted by requirement of ordinary ghost number vanishing.

Not any GThST appears by part of a given GThGT just as not arbitrary GThGT contains a nontrivial GThST (see corollary 2.1, 2.2 for Theorem 2 from Ref.[1]). However if the GThST with \( S(\theta) \) is embedded into the GThGT with \( Z[A] \) (in representing of \( S_L(\theta) \) in the form (3.15), (3.16)) then the corresponding gauge algebra for GThST is the gauge subalgebra in the corresponding gauge algebra for GThGT. Really, the vector fields \( \Gamma^\alpha_0(\theta) \) (5.6) are connected with ones \( \hat{\Gamma}^\alpha_0(\theta) \) of the type (4.7) given and acting on \( C^k(M_\text{cl} \times \{ \theta \}) \) by the formul\a\n
\[
(\Gamma^\alpha_0(\theta')^2 F(\theta')) \delta(\theta' - \theta_1) = \hat{\Gamma}^\alpha_0(\theta_1) F(\theta_1), \ \ F(\theta) \in C^k(M_\text{cl} \times \{ \theta \}),
\]

\[
\hat{\Gamma}^\alpha_0(\theta_1) F[A] = \int d\theta \frac{\delta F[A]}{\delta A^\alpha(\theta)} \hat{\mathcal{R}}^\alpha_0(A(\theta), \theta, \theta_1), \ \ \hat{\mathcal{R}}^\alpha_0(A(\theta), \theta, \theta_1) = \mathcal{R}^\alpha_0(A(\theta), \theta) \delta(\theta - \theta_1), (5.16)
\]

The structural functions \( S(A(\theta), \theta); \mathcal{R}^\alpha_0(A(\theta), \theta); \mathcal{F}^\alpha_0(\theta, \theta); \mathcal{M}^0_{\alpha\beta}(A(\theta), \theta) \) of zero, 1st, 2nd orders of the gauge algebra of GTST are connected with corresponding ones \( Z_0[A]; \hat{\mathcal{R}}^\alpha_0(\theta, \theta'); \hat{\mathcal{F}}^\alpha_0(\theta, \theta'; \theta''), \hat{\mathcal{M}}^0_{\alpha\beta}(\theta, \theta'; \theta'', \theta''') \) of zero, 1st, 2nd orders of the gauge algebra of GTGT by the relationships in addition to the 2nd expression in (5.17)

\[
Z_0[A] = - \int d\theta S(A(\theta), \theta), \ (5.18)
\]

\[
\hat{\mathcal{F}}^\alpha_{\alpha\beta}(A(\theta), \theta; \theta') = (-1)^{\epsilon_\gamma + \epsilon_\delta} \mathcal{F}^\alpha_{\alpha\beta}(A(\theta), \theta) \delta(\theta - \theta_1) \delta(\theta - \theta_2), \ (5.19)
\]

\[
\hat{\mathcal{M}}^{0}_{\alpha\beta}(A(\theta), \theta; \theta') = (-1)^{\epsilon_\gamma + \epsilon_\delta} \mathcal{M}^{0}_{\alpha\beta}(A(\theta), \theta) \delta(\theta - \theta_1) \delta(\theta - \theta_2) \times \frac{1}{2} \delta(\theta_2 - \theta_1) \delta(\theta_1 - \theta_2) + \delta(\theta'_1 - \theta_1) \delta(\theta'_2 - \theta_2), \ (5.20)
\]

in a such way that in fulfilling of the corresponding structural equations of the 1st, 2nd orders for GTST (5.1), (5.10) taking the properties (5.11), (5.12) for \( \mathcal{F}^\alpha_{\alpha\beta}(A(\theta), \theta), \mathcal{M}^{0}_{\alpha\beta}(A(\theta), \theta) \) into account the quantities \( Z_0[A], \hat{\mathcal{R}}^\alpha_0(\theta, \theta'); \hat{\mathcal{F}}^\alpha_0(\theta, \theta'; \theta'') \hat{\mathcal{M}}^{0}_{\alpha\beta}(\theta, \theta'; \theta'', \theta''') \) satisfy exactly to the 1st and 2nd orders structural equations for the gauge algebra of GTGT with \( Z_0[A] = \int d\theta S(A(\theta), \theta) \) (3.11), (4.13) with properties (4.14), (4.15) for \( \hat{\mathcal{F}}, \hat{\mathcal{M}}' \). This embedding of the gauge algebra for GTST on \( Q(S) \) can be established in the all orders \( k \geq 2 \) of the gauge algebra.

Derivation of the formul\a\n
\[
\frac{\delta}{\delta A^\alpha(\theta)}, \ \ \frac{\partial}{\partial A^\alpha(\theta)} \]

(5.16), (5.19), (5.20) are based on the rules of connection for superfield derivatives \( \frac{\delta}{\delta A^\alpha(\theta)} \), \( \frac{\partial}{\partial A^\alpha(\theta)} \) obtained in [1].

### 6 \( \theta \)-Superfield Quantum Electrodynamics

As the initial GSTF model in the Lagrangian formulation consider the superfield model of free spinor superfield of spin \( \frac{1}{2} \) being by the singular theory of special type [1]. The model is described by Dirac bispinor superfield \( \Psi(x, \theta) = (\psi_\gamma(x, \theta), \chi^{\bar{\gamma}}(x, \theta))^T = \psi(x) + \psi_1(x) \theta \) and by its Dirac conjugate one \( \overline{\Psi}(x, \theta) = \Psi^+(x, \theta) \Gamma^0 = (\overline{\chi}^{\bar{\gamma}}(x, \theta), \overline{\psi}_\beta(x, \theta)) = \overline{\psi}(x) + \overline{\psi}_1(x) \theta, \ \gamma, \beta = 1, 2, \ \bar{\gamma}, \bar{\beta} = 1, 2 \) being by elements of \((\frac{1}{2}, 0) \oplus (0, \frac{1}{2}) \) reducible massive superfield (on \( \theta \) representation \( T \) of supergroup \( J = \Pi(1, 3)^\dagger \times \mathcal{P}, \ (\Pi(1, 3)^\dagger \otimes T(1, 3)) \) defined on superspace \( \mathcal{M} = R^{1, 3} \times \mathcal{P} = \{(x^\mu, \theta)\}, \eta_{\mu
u} = \text{diag}(1, -1, -1, -1) \).

Let us point out briefly the only condensed contents of index \( \iota \) for \( A^{\dagger}(\theta) \mapsto (\Psi(x, \theta), \overline{\Psi}(x, \theta)) \), the Grassmann parities table, the transformation laws of \( \Psi(x, \theta), \overline{\Psi}(x, \theta) \) with respect to \( T_L \) representation, the superfunction \( S_L(\theta) \) defining the GSTF model in question and Euler-Lagrange
equations (3.1) in the form of HCLF (3.17) respectively in Ref.[1] notations

\[\begin{array}{cccc}
\varepsilon_P & 0 & 1 & 0 \\
\varepsilon_{\Pi} & 1 & 1 & 1 \\
\varepsilon & 1 & 0 & 1
\end{array}\]

\[K(x, \theta) \in \tilde{A}_{40+1}(x^\mu, \theta; \mathbb{C}), \quad K \in \{\Psi, \overline{\Psi}\},\]

\[\delta \Psi(x, \theta) = \Psi'(x, \theta) - \Psi(x, \theta) = -\mu \tilde{\Psi}(x, \theta) = -\mu \psi_1(x),\]

\[\delta \overline{\Psi}(x, \theta) = \overline{\Psi}'(x, \theta) - \overline{\Psi}(x, \theta) = -\mu \tilde{\overline{\Psi}}(x, \theta) = -\mu \overline{\psi}_1(x),\]

\[S_{L}^{(1)}(\theta) \equiv S_L\left(\Psi(\theta), \overline{\Psi}(\theta), \tilde{\Psi}(\theta), \tilde{\overline{\Psi}}(\theta)\right) = T\left(\tilde{\Psi}(\theta), \tilde{\overline{\Psi}}(\theta)\right) - S_0(\Psi(\theta), \overline{\Psi}(\theta)),\]

\[T^{(1)}(\theta) = T\left(\tilde{\Psi}(\theta), \tilde{\overline{\Psi}}(\theta)\right) = \int d^4x \tilde{\Psi}(x, \theta) \tilde{\overline{\Psi}}(x, \theta) = \int d^4x \mathcal{L}_{kin}^{(1)}(x, \theta),\]

\[
\frac{\delta Z[\Psi, \overline{\Psi}]}{\delta \Psi(x, \theta)} = -\frac{\delta S_{L}^{(1)}(\theta)}{\delta \Psi(x, \theta)} + \frac{d}{d\theta} \frac{\delta T^{(1)}(\theta)}{\delta \overline{\Psi}(x, \theta)} = -(i\mu \overline{\Psi}(x, \theta) \Gamma^\mu + m \overline{\Psi}(x, \theta)) = 0,
\]

\[
\frac{\delta Z[\Psi, \overline{\Psi}]}{\delta \overline{\Psi}(x, \theta)} = -\frac{\delta S_{L}^{(1)}(\theta)}{\delta \overline{\Psi}(x, \theta)} + \frac{d}{d\theta} \frac{\delta T^{(1)}(\theta)}{\delta \Psi(x, \theta)} = -(i\mu \Psi(x, \theta) \Gamma^\mu - m) \Psi(x, \theta) = 0,
\]

\[
\frac{\partial S_0(\Psi(\theta), \overline{\Psi}(\theta))}{\partial \Psi(x, \theta)} - \frac{\partial S_0(\Psi(\theta), \overline{\Psi}(\theta))}{\partial \overline{\Psi}(x, \theta)} - \frac{\partial \mathcal{L}_{kin}^{(1)}(x, \theta)}{\partial \overline{\Psi}(x, \theta)} = \frac{\partial \mathcal{L}_{kin}^{(1)}(x, \theta)}{\partial \Psi(x, \theta)}.
\]

Given model appears by nongauge one and is invariant with respect to global $U(1)$ (phase) transformations with constant parameter $\xi$ and elementary electric charge $e$

\[\Psi(x, \theta) \mapsto \Psi'(x, \theta) = \exp(-ie\xi)\Psi(x, \theta), \quad (\varepsilon_P, \varepsilon_{\Pi}, \varepsilon) = (0, 0, 0), \quad \xi \in \mathbb{R},\]

\[\overline{\Psi}(x, \theta) \mapsto \overline{\Psi}'(x, \theta) = \exp(ie\xi)\overline{\Psi}(x, \theta)\]

Realizing the Yang-Mills type gauge principle [8] let us change the parameter onto arbitrary superfield $\xi(x, \theta)$. In this connection Eqs.(6.6), (6.7) are changed onto $\theta$-superfield generalization of Dirac equations in presence, at least, of external electromagnetic superfield $A^\mu(x, \theta)$ and corresponding superfunction $S_{LQ}^{(1)}(\theta)$ must be invariant with respect to following from (6.10) GTGT

\[A^\mu(x, \theta) = A^\mu_{(1)}(x) + A^\mu_{(2)}(x, \theta) \mapsto A^\mu(x, \theta) = A^\mu(x, \theta) + \partial^\mu \xi(x, \theta),\]

\[C(x, \theta) = C(\theta) + C_{(1)}(x, \theta) \mapsto C'(x, \theta) = C(x, \theta) + \tilde{\xi}(\theta),\]

\[\Psi(x, \theta) \mapsto \Psi'(x, \theta) = \exp(-ie(\xi(x, \theta) + \theta))\Psi(x, \theta),\]

\[\overline{\Psi}(x, \theta) \mapsto \overline{\Psi}'(x, \theta) = \exp(i(e\xi(x, \theta) + \theta))\overline{\Psi}(x, \theta),\]

\[A^\mu(x, \theta) = A^\mu_{(1)}(x) A^\mu_{(2)}(x, \theta) C(x, \theta) C(x) C_{(1)}(x), \quad K(x, \theta) \in \tilde{A}_{0+1}(x^\mu, \theta; \mathbb{R}), \quad K \in \{A^\mu, C, \xi\}.\]

\[
\begin{array}{cccccc}
\varepsilon_P & 0 & 0 & 1 & 1 & 0 \\
\varepsilon_{\Pi} & 0 & 0 & 0 & 1 & 0 \\
\varepsilon & 0 & 0 & 1 & 1 & 1
\end{array}\]
Note the $\varepsilon_\sigma$ Grassmann parity value of superfield $A^i(\theta)$ is not trivial in contrast to corresponding one in Ref. [1] because of the ghost superfield $C(x, \theta)$ inclusion into multiplet $A^i(\theta)$ already on the initial level of the model formulation.

Written in the infinitesimal form (3.18) with parameter $\delta \xi(x, \theta)$ the GTGT and GGTGT have the representation respectively under change of superfield $A^i(\theta)$ and index $i$ (6.11)–(6.14)

$$\delta_A A^i(\theta) = \int d\theta' \hat{R}^i(A(\theta), \theta, \theta') \delta \xi(\theta') = \int d\theta'dy \hat{R}^i(A(x, \theta), x, \theta; y, \theta') \delta \xi(y, \theta'), \alpha = ([\xi], y),$$

$$A^i(\theta) = (\mathbf{\nabla}(x, \theta), \Psi(x, \theta), A^i(x, \theta), C(x, \theta)), i = (\gamma, \gamma', \beta, \mu, [C], x) = (i, x),$$

$$\hat{R}^i(A(x, \theta), x, \theta; y, \theta') = \sum_{k \geq 0} \left( \left( \frac{d}{d\theta} \right)^k \delta(\theta - \theta') \right) \hat{R}_k^i(A(x, \theta), x, \theta) \delta(x - y),$$

$$\hat{R}_0^i(A(x, \theta), x, \theta) = \begin{cases} 
\partial^{\mu} - i e \Psi(x, \theta), & i = (\mu, x) \\
- i e \mathbf{\nabla}(x, \theta), & i = (\beta, \gamma', \beta, \mu, [C], x) = (i, x) \\
- i e \Psi(x, \theta), & i = (\gamma, \gamma', \beta, \mu, [C], x) = (i, x) 
\end{cases}$$

GGTGT (6.17), (6.18) forms the Abelian gauge algebra of GTGT in terminology of Sec. 4. To construct zero order structural superfunction $S_{LQ}^{(1)}(\theta)$ for given algebra let us introduce according to Ref. [8] the prolonged covariant derivatives in $\theta$-superfield form with respect to representation $T$ of supergroup $J$ (not Lorentz type)

$$D_A \equiv \partial_A - i e A_A(x, \theta), \partial_A = (\partial_{\mu}, \frac{d}{d\theta}), D_A = (D_{\mu}, D_\theta), A_A(x, \theta) = (A_{\mu}, C)(x, \theta).$$

The supercommutator of above derivatives leads to expression for superfield $A_A(x, \theta)$ strength being invariant with respect to GTGT (6.11)–(6.14)

$$F_{AB}(x, \theta) = \frac{1}{e} [D_A, D_B]_s = \partial_A A_B(x, \theta) - (-1)^{\varepsilon(A_A)\varepsilon(A_B)} \partial_B A_A(x, \theta),$$

$$F_{AB}(x, \theta) = \left[ F_{\mu\nu} \frac{F_{\mu\nu}}{F_{[C]}} \right] \left( x, \theta \right) = \left[ \partial_{[\mu} A_{\nu]} - \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \right] \left( x, \theta \right) =$$

$$- (-1)^{\varepsilon(A_A)\varepsilon(A_B)} F_{BA}(x, \theta), (A, B) = ((\mu, [C]), (\nu, [C])), \varepsilon(A_A) = (0 \delta_{\mu\nu}, 1 \delta_{\mu[A]}).$$

The following superfunctions being quadratic on $F_{AB}(x, \theta)$ appear by the Poincare and gauge (with respect to GTGT) invariant objects

$$F_{AB}(x, \theta) F^{AB}(x, \theta) = \left( F_{\mu\nu} F^{\mu\nu} + 2 F_{\mu[C]} F^{\mu[C]} + 4 \tilde{C} \tilde{C} \right) (x, \theta) \equiv$$

$$- 4 \left( \mathcal{L}_1^{(0)}(\partial_\mu A_\nu(x, \theta)) + \mathcal{L}_1^{(1)}(\partial_\mu C(x, \theta), \tilde{A_\nu}(x, \theta)) + \mathcal{L}_1^{(2)}(\tilde{C}(x, \theta)) \right),$$

$$\mathcal{L}_1^{(1)}(\partial_\mu C(x, \theta), \tilde{A_\nu}(x, \theta)) \equiv 0; \quad \mathcal{L}_1^{(2)}(\tilde{C}(x, \theta)) = - \frac{d}{d\theta} \left( C(x, \theta) \tilde{C}(x, \theta) \right);$$

$$\varepsilon_{ABCD} F_{AB}(x, \theta) F^{CD}(x, \theta) = \left( \varepsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} + 4 \varepsilon_{\mu\nu\rho[C]} F^{\mu\nu} F^{\rho[C]} + 4 \varepsilon_{\mu\nu[C][C]} F^{\mu\nu} \tilde{C} \right. +$$

$$\left. 4 \varepsilon_{\mu[C][C]} F^{\mu[C]} F^{\mu[C]} + 8 \varepsilon_{\mu[C][C][C]} F^{\mu[C]} \tilde{C} \right) (x, \theta) - 4 \varepsilon_{[C][C][C][C]} \mathcal{L}_1^{(2)}(\tilde{C}(x, \theta)),$$

$$\varepsilon_{ABCD} = - (-1)^{\varepsilon(A_A)\varepsilon(A_B)} \varepsilon_{BACD} = - (-1)^{\varepsilon(A_C)\varepsilon(A_B)} \varepsilon_{ACBD} = - (-1)^{\varepsilon(A_C)\varepsilon(A_B)} \varepsilon_{ABCD}.$$
Choosing the elements of superantisymmetric constant tensor $\varepsilon_{ABCD}$ in the form being compatible with even values of its $(\varepsilon_{ABCD})$, $\varepsilon_P$, $\varepsilon_\Pi$, $\varepsilon$ gradings and with properties (6.24)

$$
\varepsilon_{0123} = \varepsilon[c][c][c][c] = 1, \varepsilon_{\mu\nu[c]} = \varepsilon_{\mu[c][c][c]} = 0, \varepsilon_{\mu[c][\nu[c]} = -\varepsilon_{\mu[c][\nu[c]} = \varepsilon_{\mu\nu}^{(1)} = -\varepsilon_{\nu\mu}^{(1)},
$$

we obtain for (6.23) the result

$$
(\varepsilon_{ABCD}F^{AB}F^{CD})(x,\theta) = (\varepsilon_{\mu\rho\sigma}F_{\mu\nu}F^{\rho\sigma} - 4\varepsilon_{\mu\nu}^{(1)}(F_{\mu\nu}C - F^{\mu}[c]F^{\nu[c]}) + 4C^2)(x,\theta).
$$

In the first place, note the superfunction $L_1^{(2)}(\hat{C}(x,\theta))$ is the self-dual one and with accuracy up to total derivatives with respect to $x^\mu$, $\theta$ the sum of the 2nd and 3rd summands in (6.26) is reduced to the form

$$
4\varepsilon_{\mu\nu}^{(1)}(F_{\mu\nu}C + F_{\mu}[c]F^{\nu[c]})(x,\theta) = 4\varepsilon_{\mu\nu}^{(1)}(\hat{A}^\nu \hat{A}_\mu + 2F_{\mu\nu}C)(x,\theta).
$$

With regard of the last representation the superfunction $S_{LQ}^{(1)}(\theta)$ being invariant with respect to GTGT (6.11)–(6.14), defining the GThGT with nontrivial inclusion of the ghost superfield $C(x,\theta)$ into superfield $(\theta)$ quantum electrodynamics and addition of the "$\theta$-term" (vacuum angle), leading by means of relationships (6.23)–(6.27) to application in the electromagnetic duality theory (see for instance Ref. [9]), has the resultant form

$$
S_{LQ}^{(1)}(\theta) = S_{LQ}^{(1)}(A_\theta, \hat{A}_\theta, \Psi(x,\theta), \bar{\Psi}(x,\theta), \theta, \bar{\theta}) =
$$

$$
T_{inv}(D_\theta\Psi(x,\theta), D_\bar{\theta}\bar{\Psi}(x,\theta)) - S_{inv}^{(1)}(\Psi(x,\theta), \bar{\Psi}(x,\theta), A_\mu(x,\theta)) = 0,
$$

$$
T_{inv}(\theta) \equiv T_{inv}(D_\theta\Psi(x,\theta), D_\bar{\theta}\Psi(x,\theta)) = \int d^4x L_\theta^{(1)}(\Psi(x,\bar{\theta}), \bar{\Psi}(x,\theta), D_\mu\Psi(x,\theta)) =
$$

$$
\int d^4x (D_\theta\bar{\Psi}(x,\theta)) (D_\bar{\theta}\Psi(x,\theta)) = \left(\frac{d}{d\theta} + i\epsilon C(x,\theta)\right) \bar{\Psi}(x,\theta);
$$

$$
S_{inv}^{(1)}(\Psi(x,\theta), \bar{\Psi}(x,\theta), A_\mu(x,\theta)) = \int d^4x L_\theta^{(1)}(\Psi(x,\theta), \bar{\Psi}(x,\theta), D_\mu\Psi(x,\theta)) =
$$

$$
L_\theta^{(1)}(x,\theta) = -\frac{1}{4F_{AB}F^{AB}}(x,\theta) = L_\theta (x,\theta) + L_0^{(0)}(\partial_\mu A_\nu(x,\theta)) + L_1^{(2)}(\hat{C}(x,\theta)),
$$

$$
L_\theta^{(1)}(x,\theta) = -\frac{\tilde{\theta}e^2}{32\pi^2} \varepsilon_{ABCD}F^{AB}(x,\theta)F^{CD}(x,\theta) =
$$

$$
-\frac{\tilde{\theta}e^2}{32\pi^2} (\varepsilon_{\mu\rho\sigma}F^{\mu\nu}(x,\theta)F^{\rho\sigma}(x,\theta) + 4\varepsilon_{\mu\nu}^{(1)}(\hat{A}^\nu \hat{A}_\mu + 2F_{\mu\nu}C)(x,\theta) - 4L_1^{(2)}(\hat{C}(x,\theta))).
$$

Superfunctions $T_{inv}(\theta)$, $S_{inv}^{(1)}(\theta)$, $S_{0}^{(1)}(\theta)$ in (6.28) are invariant with respect to GTGT. Euler-Lagrange equations (3.1) for $Z^{(1)}[\Psi, \bar{\Psi}, A_\theta] = \int d\theta S_{LQ}^{(1)}(\theta)$ read as follows

$$
\frac{\delta Z^{(1)}}{\delta \Psi(x,\theta)} = \frac{\partial Z_{LQ}^{(1)}(\theta)}{\partial \Psi(x,\theta)} + \frac{d}{d\theta}T_{inv}(\theta) = -(iD_\mu \bar{\Psi} + m\Psi)(x,\theta) + i\epsilon \hat{C}(x,\theta) = 0,
$$

$$
\frac{\delta Z^{(1)}}{\delta \bar{\Psi}(x,\theta)} = \frac{\partial Z_{LQ}^{(1)}(\theta)}{\partial \bar{\Psi}(x,\theta)} + \frac{d}{d\theta}T_{inv}(\theta) = -(i\Gamma^\mu D_\mu - m + i\epsilon \hat{C}(x,\theta))\Psi(x,\theta) = 0.
$$
\[
\frac{\delta_l Z^{(1)}}{\delta A_\mu(x, \theta)} = - \frac{\partial_l (S^{(1)}(\theta) + S^{(1)}_0(\theta))}{\partial A_\mu(x, \theta)} + \frac{d}{d\theta} \frac{\partial_l S^{(1)}(\theta)}{\partial A_\mu(x, \theta)} = -\left( \partial_\nu \left[ F^{\nu\mu} - \frac{\hat{\theta} e^2}{2\pi^2} \epsilon^{(1)\nu\mu} \tilde{C} \right] \right) + \epsilon \hat{\Psi} \Gamma^\mu \hat{\Psi}(x, \theta) = 0, \quad (6.35)
\]
\[
\frac{\delta_l Z^{(1)}}{\delta C(x, \theta)} = \frac{\partial_l T_{\text{inv}}(\theta)}{\partial C(x, \theta)} - \frac{d}{d\theta} \frac{\partial_l S^{(1)}(\theta)}{\partial C(x, \theta)} = -\frac{d}{d\theta} \left( \epsilon \Psi \Psi + \frac{\hat{\theta} e^2}{4\pi^2} \epsilon^{(1)\mu\nu} F^{\mu\nu} \right) (x, \theta) = 0, \quad (6.36)
\]
appear by DCLF and represent by themselves the 1st (2nd) order with respect to derivatives on \( \theta \) and \( x^\mu \) nonlinear partial differential equations (6.33), (6.34) for spinor superfields ((6.35), (6.36) for electromagnetic and ghost superfields). In view of degeneracy of the model (6.28) the Cauchy problem setting is not trivial in question and remains out the paper’s scope.

From (6.28–6.36) it follows, in particular, the \( \theta \)-superfield free electrodynamics is described in terms of superfield \( A_\theta(x, \theta) \) by means of the superfunctional with “\( \theta \)-term”

\[
Z_{SED}[A_\theta] \equiv Z^{(1)}[\Psi, \Psi', A_\theta]_{\Psi=\bar{\Psi}=0} = -\int d\theta S^{(1)}(\theta)(A_\theta^0, A_\theta^0(\theta)) \quad (6.37)
\]
the such that for \( C(x, \theta) = 0 \) and in absence of the topological summand \( \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}(x, \theta) \) in (6.37) it is obtained the GThST described with accuracy up to nonessential number multipliers in \( D = 4 \) by means of the free massless vector superfield \( A_\mu(x, \theta) \) model [1]. The model with \( Z_{SED}[A_\theta] \) (6.37) itself belongs to the class of GThST as well as it follows from (6.35), (6.36).

To construct the corresponding to Sec.5 Abelian gauge algebra of GTST being by the gauge subalgebra of the gauge algebra of GTG with structural functions in (6.17), (6.18), (6.28) it is necessary to restrict the model onto hypersurface \( \tilde{\Psi}(x, \theta) = \tilde{\bar{\Psi}}(x, \theta) = A_\theta^0(x, \theta) = 0 \) and next to determine the structural functions of 0 and 1st orders in correspondence with (6.28)–(6.32)

\[
S(\Psi(\theta), \bar{\Psi}(\theta), A_\mu(\theta), C(\theta)) \equiv S(\Psi(\theta), \bar{\Psi}(\theta), A_\mu(\theta), C(\theta)) = S^{(1)}(\theta) + S^{(1)}_0(\theta) \quad (6.38)
\]
and with \( R^0_\mu(\theta, A_\theta(x, \theta), x, \theta) \) coinciding with \( \hat{R}^0_\mu(\theta, A_\theta(x, \theta), x, \theta) \) (6.18) with the exception of superfield \( C(x, \theta) \) not entering in (6.38). Then the formulae (5.17), (5.18) completely establish the embedding of the gauge algebra of GTST into one of GTG for given models.

At last, in setting \( \theta = 0 \) in (6.11)–(6.18), (6.38) or equivalently using the special involution * [1] for \( \theta = 0 \) we obtain the ordinary component quantum electrodynamics formulation on classical level being described by \( A^0(\mu), \Psi(x), \bar{\Psi}(x) \) [10].

Let us note the unification possibility of the gauge \( A_\mu(x) \) and ghost \( C(x) \) \( P_0(\theta) \)-component fields into uniform \( P_0(\theta) \)-component \( A_\theta^0(x) \) of the uniform superfield \( A_\theta(x, \theta) \) (6.19) in order to realize the BRST symmetry in the superfield form and to construct the action functional(!) by means of the gauge strength of the form (6.20) for the Yang-Mills type theories had been considered in the paper [11] (see the references therein). However the form of Abelian superfield \( A_\theta(x, \theta) \) (6.19) and strength \( F_{AB}(x, \theta) \) (6.20) have exhausted the coincidence of the superfield models from the present paper and from Ref.[11]. Their difference is traced not only through the whole corresponding formulae spectrum, among them leading to construction of the actions, but is based on the functionally distinct conceptual formulations of the models.

### 7 Conclusion

The basic Theorem announced in Ref.[1] from which it follows the many properties of GTh-GTs, GThSTs in the framework of the Lagrangian formulation for GSTF is completely proved
and their consequences are studied. Nontrivial differential-algebraic structures, i.e. the gauge algebras of GTGT, GTST have been investigated.

The general results of the Secs.2–5 have obtained the final confirmation on the example of the \( \theta \)-superfield quantum electrodynamics, being on the classical level by ordinary \( \theta \)-superfield spinor electrodynamics, realized on the gauge principle basis [8] (the so-called minimal inclusion of interaction). The cases of \( \theta \)-superfield scalar or vector electrodynamics may be deduced from the free massive complex scalar superfields \( \varphi(x, \theta), \varphi^*(x, \theta) \) model and free massive complex(!) vector superfield \( \mathcal{A}^\mu(x, \theta) \) in \( D = 4 \) model, realized in fact in the Lagrangian formulation of GSTF in Ref.[1], by means of the algorithm from Sec.6. All these models representing the GThGTs with Abelian gauge algebra can be generalized in their constructing, in an obvious way, starting from the case of the initial interacting \( \theta \)-superfield massive spinor, scalar, (complex) vector models.

Specially note the prolongation of the derivative with respect to odd time \( \theta \) have led to necessity already on the classical level of the nontrivial inclusion of the ghost superfield \( C(x, \theta) \) playing the same role for \( \frac{d}{d\theta} \) as the electromagnetic one \( \mathcal{A}^\mu(x, \theta) \) for \( \partial_\mu \).

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