EXISTENCE AND ASYMPTOTIC BEHAVIOR OF SOLUTION TO A SINGULAR ELLIPTIC PROBLEM

to appear in Surveys in Mathematics and its Applications

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Abstract

In this paper we obtain existence results for the positive solution of a singular elliptic boundary value problem. To prove the main results we use comparison arguments and the method of sub-super solutions combined with a procedure which truncates the singularity.

2000 Mathematics Subject Classification: 35J60;35J15;35J05.

Key words: nonlinear elliptic equation; singularity; existence; regularity.

1 Introduction

This paper contains a contribution of a technical nature to the study of positive solutions of the equations

$$-\Delta u + c(x)u^{-1}|\nabla u|^2 = a(x) \text{ for } x \in \mathbb{R}^N, \ u > 0 \text{ in } \mathbb{R}^N, \ u(x) \to 0 \text{ as } |x| \to \infty$$ (1.1)

where $N > 2$, $a : \mathbb{R}^N \to \mathbb{R}$ is a function satisfying the following conditions

AC1) $a, c \in C_{\text{loc}}^{0,\alpha}(\mathbb{R}^N)$ for some $\alpha \in (0, 1)$;

AC2) $a(x) > 0, c(x) > 0$ for all $x \in \mathbb{R}^N$;

A3) for $\varphi(r) = \max_{|x|=r} a(x)$ we have

$$\int_0^\infty r\varphi(r)dr < \infty.$$ (1.2)

Problems like (1.1) have been intensively studied. Our study is motivated by the works of Shu [17], Arcoya, Carmona, Leonori, Aparicio, Orsina and Petitta [2], Arcoya, Barile and Aparicio [3] where the existence, non-existence and uniqueness of solution for the problem like (1.1) are solved.

In this article we present a new argument in the study of the problem (1.1) more simple that used in [2], [3], [17] and where the problem is considered just in the case when $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary.

The above equation contains different quantities, such as: singular nonlinear term (like $u^{-1}$), convection nonlinearity (denoted by $|\nabla u|^2$), as well as potentials ($c$ and $a$). The principal difficulty in the treatment of (1.1) is due to the singular character of the equation combined with the nonlinear gradient term.
The importance of the problem (1.1) is given considering the well known problem

\[ \Delta u = a(x)h(u), \quad u > 0 \text{ in } \Omega, \quad u(x) = \infty \text{ as } x \to \partial \Omega, \tag{1.2} \]

because we can easily deduce the following two remarks:

**Remark 1.1.** When \( h(u) = e^u \), by a transformation of the form \( w = e^{-u} \) the problem (1.2) becomes

\[ -\Delta w + \frac{\|\nabla w\|^2}{w} = a(x), \quad w > 0 \text{ in } \Omega, \quad w(x) \to 0 \text{ as } x \to \partial \Omega, \tag{1.3} \]

but this is the problem (1.1) when \( c(x) = 1 \).

**Remark 1.2.** For \( h(u) = u^\delta \) (\( \delta > 1 \)) and \( w = C[u]^{-(C^{-1})} \), \( C := 1/(\delta - 1) \) in (1.2) we have

\[ -\Delta w + \delta C \frac{\|\nabla w\|^2}{w} = a(x), \quad w > 0, \text{ in } \Omega, \quad w \to 0 \text{ as } x \to \partial \Omega, \tag{1.4} \]

which is the problem (1.1) when \( c(x) = \delta C \).

This finish the motivation of our work.

The main results of the article are:

**Theorem 1.1.** If \( \Omega \subset \mathbb{R}^N \) is a bounded domain with boundary \( \partial \Omega \) of class \( C^{2,\alpha} \) for some \( \alpha \in (0,1) \) and \( a, c \in C^{0,\alpha}(\Omega) \), \( a(x) > 0 \), \( c(x) > 0 \) for any \( x \in \overline{\Omega} \), then the problem

\[ -\Delta u + c(x)u^{-1}\|\nabla u\|^2 = a(x) \text{ in } \Omega, \quad u|_{\partial \Omega} = 0, \tag{1.5} \]

has at least a positive solution \( u \in C(\overline{\Omega}) \cap C^{2,\alpha}(\Omega) \).

In the next result we establish sufficient condition for the existence of solution to the problem (1.1) in the case when \( \Omega = \mathbb{R}^N \).

**Theorem 1.2.** We suppose that hypotheses AC1), AC2), A3) are satisfied. Then, the problem (1.1) has a \( C_{\text{loc}}^2(\mathbb{R}^N) \) positive solution vanishing at infinity. If, in addition,

\[ \lim_{|x| \to \infty} |x|^\mu \varphi(|x|) < \infty, \tag{1.6} \]

for some \( \mu \in (2, N) \), then

\[ u(x) = O(|x|^{2-\mu}) \text{ as } |x| \to \infty. \tag{1.7} \]

To prove the existence of such a solution to (1.1) we establish some preliminary results.

# Preliminary results

Since we apply sub and super solution method due to Amann [Π], we recall the following definition of sub and super solution which are our main tools in the proof of the solvability of problem (1.1).

For \( f_1(x, \eta, \xi) : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R} \) and \( g_1 : \partial \Omega \to \mathbb{R} \), Amann introduce the following definitions:
Definition 2.1. A function \( u \in C^{2,\alpha}(\overline{\Omega}) \) is called a sub solution for the problem
\[
-\Delta u = f_1(x, u, \nabla u) \quad \text{in} \quad \Omega, \quad u = g \quad \text{on} \quad \partial \Omega,
\]
if
\[
-\Delta u \leq f_1(x, u, \nabla u) \quad \text{in} \quad \Omega, \quad u = g \quad \text{on} \quad \partial \Omega.
\]

Definition 2.2. A function \( \overline{u} \in C^{2,\alpha}(\overline{\Omega}) \) is called a super solution of the problem (2.1) if
\[
-\Delta \overline{u} \geq f_1(x, u, \nabla u) \quad \text{in} \quad \Omega, \quad \overline{u} = g \quad \text{on} \quad \partial \Omega.
\]

One of the important results from [1] is:

Lemma 2.1. Let \( \Omega \) be a bounded domain from \( \mathbb{R}^N \), with boundary \( \partial \Omega \) of class \( C^{2,\alpha} \) for some \( \alpha \in (0, 1) \), \( g \in C^{2,\alpha}(\partial \Omega) \) and \( f_1 \) be a continuous function with the property that \( \partial f_1/\partial \eta, \partial f_1/\partial \xi_i, i = 1, N \) exists and are continuous on \( \Omega \times \mathbb{R}^{N+1} \) and such that

(AM1) \( f_1(\cdot, \eta, \xi) \in C^{\alpha}(\overline{\Omega}) \), uniformly for \( (\eta, \xi) \) in bounded subsets of \( \mathbb{R} \times \mathbb{R}^N \);

(AM2) there exists a function \( f_2 : \mathbb{R}^+ \to \mathbb{R}^+ := [0, \infty) \) such that
\[
|f_1(x, \eta, \xi)| \leq f_2(\rho)(1 + |\xi|^2),
\]
for every \( \rho \geq 0 \) and \( (x, \eta, \xi) \in \overline{\Omega} \times [-\rho, \rho] \times \mathbb{R}^N \).

Under these assumption, if the problem (2.1) has a sub solution \( u \) and a super solution \( \overline{u} \) such that \( u(x) \leq \overline{u}(x), \forall x \in \overline{\Omega} \) then there exists at least a function \( u(x) \in C^{2+\alpha}(\overline{\Omega}) \) which satisfies \( u(x) \leq \overline{u}(x) \) for all \( x \in \overline{\Omega} \) and satisfying (2.1) pointwise. More precisely, there exist a minimal solution
\( \tilde{u}(x) \in [u(x), \overline{u}(x)] \) and a maximal solution \( \approx u(x) \in [u(x), \overline{u}(x)] \), in the sense that every solution \( u(x) \in [u(x), \overline{u}(x)] \) satisfies \( \tilde{u}(x) \leq u(x) \leq \approx u(x) \).

We will need the following variant of the maximum principle:

Lemma 2.2. Assume that \( \Omega \) is a bounded open set in \( \mathbb{R}^N \). If \( u : \overline{\Omega} \to \mathbb{R} \) is a smooth function such that
\[
\begin{aligned}
-\Delta u &\geq 0 \quad \text{in} \quad \Omega, \\
u &\geq 0 \quad \text{on} \quad \partial \Omega,
\end{aligned}
\]
then \( u \geq 0 \) in \( \Omega \).

This finishes the auxiliary results. Now we prove the announced Theorems.

3 Proof of the Theorem 1.1

In the following will we use similarly argument that were used by Crandall, Rabinowitz and Tartar [7], Noussair [15] and the author [6].

Let \( \varepsilon \in (0, 1) \). The existence will be established by solving the approximate problems
\[
\begin{aligned}
-\Delta u + c(x)u^{-1} |\nabla u|^2 = a(x), &\quad \text{in} \quad \Omega, \quad u > \varepsilon \quad \text{in} \quad \Omega, \\
u = \varepsilon, &\quad \text{on} \quad \partial \Omega.
\end{aligned}
\]
For this, let \( \varphi_1 \) be the first positive eigenfunction corresponding to the first eigenvalue \( \lambda_1 \) of the problem

\[
-\Delta u(x) = \lambda u(x), \quad \text{in} \quad \Omega, \quad u|_{\partial \Omega} = 0.
\]  

(3.2)

It is well known that \( \varphi_1 \in C^{2+\alpha}(\Omega) \). We note by \( m_2 := \min_{x \in \Omega} a(x) \) and \( M_1 := \max_{x \in \Omega} c(x) \) to prove that the function \( u(x) = \sigma_1 \varphi_1^2 + \varepsilon, \) where

\[
0 < \sigma_1 \leq \min \left\{ \frac{m_2}{2\lambda_1 \max_{x \in \Omega} \varphi_1^2 + 4M_1 \max_{x \in \Omega} |\nabla \varphi_1|^2}, 1 \right\}
\]

(3.3)
is a sub solution of (3.1) in the sense of Lemma 2.1. Indeed, by (3.3) we have

\[
-\Delta u + c(x)u^{-1}|\nabla u|^2 - a(x) \leq -\Delta u + M_1 u^{-1} |\nabla u|^2 - m_2
\]

\[
\leq -2\sigma_1 \varphi_1 \Delta \varphi_1 - 2\sigma_1 |\nabla \varphi_1|^2 + 4M_1 \sigma_1 |\nabla \varphi_1|^2 - m_2
\]

\[
= 2\sigma_1 \lambda_1 \varphi_1^2 - 2\sigma_1 |\nabla \varphi_1|^2 + 4M_1 \sigma_1 |\nabla \varphi_1|^2 - m_2
\]

\[
\leq 2\sigma_1 \lambda_1 \varphi_1^2 + 4M_1 \sigma_1 |\nabla \varphi_1|^2 - m_2 \leq 0.
\]

In the next step we prove the existence of a super solution to the problem (3.1). For this, let \( v \in C^{2+\alpha}(\Omega) \) be the unique solution of the problem

\[
-\Delta y = a(x) \quad \text{in} \quad \Omega, \quad y(x) = 0 \quad \text{for} \quad x \in \partial \Omega.
\]

(3.4)

We observe that, \( \overline{v} = v + \varepsilon \in C^{2+\alpha}(\Omega) \), fulfils

\[
-\Delta \overline{v}(x) + c(x)\overline{v}^{-1}(x) |\nabla \overline{v}(x)|^2 = a(x) + c(x)\overline{v}^{-1}(x) |\nabla \overline{v}(x)|^2 \geq a(x) \quad \text{for} \quad x \in \Omega.
\]

(3.5)

Clearly, \( \overline{v} \) is a super solution to (3.1). Now, since

\[
\left\{ \begin{array}{c}
-\Delta (\overline{v} - u) \geq a(x) + c(x)\overline{v}^{-1}(x) |\nabla \overline{v}(x)|^2 - a(x) \geq 0, \\
\overline{v} - u = 0,
\end{array} \right.
\]

(3.6)

follows from the maximum principle, Lemma 2.1, that \( \underline{u}(x) \leq \overline{v}(x), \quad x \in \Omega \).

We have obtained a sub solution \( \underline{u} \in C^{2,\alpha}(\Omega) \) and a super solution \( \overline{v} \in C^{2,\alpha}(\Omega) \) for the problem (3.1) such that \( \underline{u} \leq \overline{v} \) in \( \Omega \) with the property from Lemma 2.1. Then, there exists \( u_\varepsilon \in C^{2,\alpha}(\Omega) \) such that

\[
\underline{u}(x) \leq u_\varepsilon(x) \leq \overline{v}(x), \quad x \in \Omega.
\]

(3.7)

and satisfying (pointwisely) the problem (3.1).

The relation (3.6) shows that \( u > 0 \) in \( \Omega \). We remark that \( \underline{u} = \sigma_1 v^2 + \varepsilon \), where \( \sigma_1 \) is a positive constant such that

\[
0 < \sigma_1 \leq \min \left\{ \frac{m_2}{\max_{x \in \Omega} [2v + 4M_1 |\nabla v|^2]}, 1 \right\},
\]

(3.7)
In this time we have obtained a function $u_\varepsilon \in C^{2,\alpha}(\Omega)$ that satisfies pointwisely the equivalently form of (3.1): 

$$
\begin{cases}
-\Delta u + c(x) (u + \varepsilon)^{-1} |\nabla u|^2 = a(x), & \text{in } \Omega, \\
u > 0, & \text{in } \Omega, \\
u = 0, & \text{on } \partial\Omega.
\end{cases}
$$

(3.8)

Moreover $u_\varepsilon \in C^{2,\alpha}(\Omega)$ is unique. Indeed, assume that the problem (3.8) has more than one solution and let $v_\varepsilon$ the second solution. Let us show that $u_\varepsilon \leq v_\varepsilon$ or, equivalently, $u_\varepsilon (x) + \varepsilon \leq v_\varepsilon (x) + \varepsilon$ for any $x \in \overline{\Omega}$. Assume the contrary. Set 

$$
\alpha(x) := \frac{u_\varepsilon (x) + \varepsilon}{v_\varepsilon (x) + \varepsilon} - 1.
$$

Since we have $[\alpha (x)]_{\partial\Omega} = 0$ we deduce that $\max_{\overline{\Omega}} \alpha (x)$, exists and is positive. At that point, say $x_0$, we have $\nabla \alpha(x_0) = 0$ and $\Delta \alpha(x_0) \leq 0$, which implies 

$$
\left( - (\varepsilon + \varepsilon) \Delta u_\varepsilon + (u_\varepsilon + \varepsilon) \Delta v_\varepsilon \right)(x_0) \geq 0,
$$

(3.9) and 

$$
\frac{|\nabla u_\varepsilon(x_0)|^2}{(u_\varepsilon(x_0) + \varepsilon)^2} = \frac{|\nabla v_\varepsilon|^2}{(v_\varepsilon(x_0) + \varepsilon)^2}.
$$

(3.10)

By (3.9) and (3.11) we have 

$$
\frac{a(x_0)}{u_\varepsilon(x_0) + \varepsilon} - \frac{a(x_0)}{v_\varepsilon(x_0) + \varepsilon} + c(x_0) \left( \frac{(v_\varepsilon + \varepsilon)^{-1} |\nabla v_\varepsilon|^2}{v_\varepsilon + \varepsilon} - \frac{(u_\varepsilon + \varepsilon)^{-1} |\nabla u_\varepsilon|^2}{u_\varepsilon + \varepsilon} \right)(x_0) \geq 0,
$$

(3.11)

or, equivalently 

$$
a(x_0) \frac{v_\varepsilon(x_0) - u_\varepsilon(x_0)}{(u_\varepsilon(x_0) + \varepsilon)(v_\varepsilon(x_0) + \varepsilon)} \geq 0.
$$

(3.12)

which is a contradiction with $u_\varepsilon(x_0) > v_\varepsilon(x_0)$. So $u_\varepsilon(x) \leq v_\varepsilon(x)$ in $\overline{\Omega}$. A similar argument can be made to produce $v_\varepsilon(x) \leq u_\varepsilon(x)$ forcing $u_\varepsilon(x) = v_\varepsilon(x)$.

We will show that, for any smooth bounded subdomain $\Omega'$ of $\mathbb{R}^N$ there exists a constant $C_4 > 0$ such that 

$$
\|u_\varepsilon\|_{C^{2,\alpha}(\Omega')} \leq C_4.
$$

(3.13)

For any bounded $C^{2,\alpha}$-smooth domain $\Omega' \subset \mathbb{R}^N$, take $\Omega_1$, $\Omega_2$ and $\Omega_3$ with $C^{2,\alpha}$-smooth boundaries, such that $\Omega' \subset \subset \Omega_1 \subset \subset \Omega_2 \subset \subset \Omega_3 \subset \subset \Omega$. Note that 

$$
u_\varepsilon(x) \geq \frac{a(x) - c(x) (u_\varepsilon(x) + \varepsilon)^{-1} |\nabla u_\varepsilon(x)|^2}{x \in \Omega_3}.
$$

(3.14)

Let $h_\varepsilon(x) = a(x) - c(x) (u_\varepsilon(x) + \varepsilon)^{-1} |\nabla u_\varepsilon(x)|^2$, $x \in \overline{\Omega_3}$. Following, we use $C_{i=1,3}$, to denote positive constants which are independent of $\varepsilon$.

Since $-\Delta u_\varepsilon(x) = h_\varepsilon(x)$, $x \in \overline{\Omega_3}$, we see by the interior gradient estimate theorem of Ladyzenskaya and Ural’tseva [13] Theorem 3.1, p. 266] that there exists a positive constant $C_1$ independent of $\varepsilon$ such that 

$$
\max_{x \in \overline{\Omega_2}} |\nabla u_\varepsilon(x)| \leq C_1 \max_{x \in \overline{\Omega_3}} u_\varepsilon(x).
$$

(3.15)
Using (3.6) and (3.15) we obtain that \( \| \nabla u_\varepsilon \| \) is uniformly bounded on \( \overline{\Omega}_2 \). This final result, the property of \( a \) and \( c \) shows that \( |h_\varepsilon| \) is uniformly bounded on \( \overline{\Omega}_2 \) and so \( h_\varepsilon \in L^p(\Omega_2) \) for any \( p > 1 \).

Since \( -\Delta u_\varepsilon(x) = h_\varepsilon(x) \) for \( x \in \Omega_2 \), we see from (6), that there exists a positive constant \( C_2 \) independent of \( \varepsilon \) such that

\[
\| u_\varepsilon \|_{W^{2,p}(\Omega_1)} \leq C_2 (\| h_\varepsilon(x) \|_{L^p(\Omega_2)} + \| u_\varepsilon \|_{L^p(\Omega_2)}),
\]
i.e. \( \| u_\varepsilon \|_{W^{2,p}(\Omega_1)} \) is uniformly bounded.

Choose \( p \) such that \( p > N \) and \( p > N (1 - \alpha)^{-1} \). Then by Sobolev’s imbedding theorem, it follows that \( \| u_\varepsilon \|_{C^{1,\alpha}(\overline{\Omega}_1)} \) is uniformly bounded by a constant independent of \( \varepsilon \).

Moreover, this say that \( h_\varepsilon \in C^{0,\alpha}(\overline{\Omega}_1) \) and \( \| h_\varepsilon \|_{C^{0,\alpha}(\overline{\Omega}_1)} \), is uniformly bounded. Using this and the interior Schauder estimates (see \( g \)), for solutions of elliptic equations (4.1), we have that there exists a positive constant \( C_3 \) independent of \( \varepsilon \) with the property

\[
\| u_\varepsilon \|_{C^{2,\alpha}(\overline{\Omega})} \leq C_3 \left( \| h_\varepsilon \|_{C^{0,\alpha}(\overline{\Omega}_1)} + \sup_{\overline{\Omega}_1} u_\varepsilon \right).
\]

Because \( \| h_\varepsilon \|_{C^{0,\alpha}(\overline{\Omega}_1)} \) is uniformly bounded, we see from (3.14) that

\[
\| u_\varepsilon \|_{C^{2,\alpha}(\overline{\Omega})} \leq C_4.
\]

Thus (3.13) is proved.

Set \( \varepsilon := 1/n \) and \( u_\varepsilon := u^n \). Since the sequence \( u^n \) is bounded in \( C^{2,\alpha}(\overline{\Omega}) \) for any bounded domain \( \Omega' \subset \subset \Omega \) by (3.17), using the Ascoli-Arzela theorem and the standard diagonal process, we can find a subsequence of \( u^n \), denote again by \( u^n \) and a function \( u \in C^2(\overline{\Omega}) \) such that \( \| u^n - u \|_{C^2(\overline{\Omega})} \to 0 \) for \( n \to \infty \). In particular

\[
\Delta u^n \text{ respectively } a(x) - c(x)(u^n(x) + 1/n)^{-1} |\nabla u^n(x)|^2
\]

converge for \( n \to \infty \) in \( \overline{\Omega} \) to

\[
\Delta u \text{ respectively } a(x) - c(x)u(x)^{-1} |\nabla u(x)|^2.
\]

It follows that \( u \) is a solution of

\[
- \Delta u = a(x) - c(x)u^{-1}(x) |\nabla u(x)|^2, \text{ in } \overline{\Omega},
\]

of class \( C^2(\overline{\Omega}) \), and hence of class \( C^{2,\alpha}(\overline{\Omega}) \) by a standard regularity arguments based on Schauder estimates.

Since \( \Omega' \) is arbitrary, we also see that \( u \in C^{2,\alpha}(\Omega) \). We have obtained \( u^n \nrightarrow u \) (pointwisely) in \( C^{2,\alpha}(\Omega) \).

For \( \varepsilon := 1/n \nrightarrow 0 \) in (3.6) we have

\[
u_\varepsilon(x) := \sigma_1 \varphi_1 \leq u(x) \leq \tilde{u}(x) := v(x), \quad x \in \overline{\Omega}.
\]

Moreover, by (3.18) and (3.19), we obtain

\[
- \Delta u = a(x) - c(x)u^{-1} |\nabla u|^2 \text{ a.e. in } \Omega, \ u > 0 \text{ in } \Omega, \ u|_{\partial\Omega} = 0.
\]

Thus \( u \in C(\overline{\Omega}) \cap C^{2,\alpha}(\Omega) \) is the solution of the problem (1.5).
4 Proof of the Theorem 1.2

To prove the existence of solution to (1.1) we consider the following boundary value problem

\[-\Delta u + c(x)w^{-1} |\nabla u|^2 = a(x), \quad u > 0 \text{ in } B_k, \quad u = 0 \text{ on } \partial B_k, \quad (4.1)\]

where \( B_k := \{ x \in \mathbb{R}^N | |x| < k \} \) is the ball of center 0 and radius \( k = 1, 2, ... \) Put \( \Omega = B_k \) in Theorem 1.1. Then the problem (4.1) has at least one solution \( u_k \in C(\overline{B}_k) \cap C^{2,\alpha}(B_k) \), which satisfies

\[ u_2 \leq u_k \leq \overline{u}^2 \text{ in } B_k, \quad (4.2) \]

for \( \underline{u} \) (resp. \( \overline{u}^2 \)) the corresponding functions from Theorem 1.1 when \( \Omega = B_k \). In outside of \( B_k \) we put \( u_k = 0 \). The resulting function is in \( \mathbb{R}^N \). Now, we observe that

\[ w(r) := \int_r^\infty \xi^{1-N} \int_0^\xi \sigma^{N-1} \varphi(\sigma) d\sigma d\xi, \quad r := |x| \quad (4.3) \]

is the unique radial solution of the problem \(-\Delta w = \varphi(|x|) \) in \( \mathbb{R}^N, w > 0 \) in \( \mathbb{R}^N, w \uparrow |x| \to \infty \). We prove that \( w \) is bounded. Using integration by parts and L' Hôpital rule, we have

\[ \int_r^\infty \xi^{1-N} \int_0^\xi \sigma^{N-1} \varphi(\sigma) d\sigma d\xi = \frac{1}{N-2} \left( \int_0^\infty d\xi \left[ \int_0^\xi \sigma^{N-2} \varphi(\sigma) d\sigma \right] \right) \]

\[ = \lim_{R \to \infty} \frac{1}{N-2} \left\{ \int_r^R \xi \varphi(\xi) d\xi - R^{2-N} \int_0^R \sigma^{N-1} \varphi(\sigma) d\sigma + r^{2-N} \int_0^r \sigma^{N-1} \varphi(\sigma) d\sigma \right\} \]

\[ = \lim_{R \to \infty} \frac{R^{N-2} \left[ \int_r^R \xi \varphi(\xi) d\xi + r^{2-N} \int_0^r \xi^{N-1} \varphi(\xi) d\xi \right] - \int_0^R \xi^{N-1} \varphi(\xi) d\xi}{R^{N-2}} \]

\[ = \frac{1}{N-2} \int_0^r \xi \varphi(\xi) d\xi + \int_0^r \xi^{N-1} \varphi(\xi) d\xi, \quad R > r. \quad (4.4) \]

Now, by the second mean value theorem for integrals follows that there exists \( r_1 \in (0, r) \) such that

\[ \int_0^r \xi^{N-1} \varphi(\xi) d\xi = \int_0^r \xi^{N-2} \xi \varphi(\xi) d\xi = \int_0^r \xi^{N-2} \xi \varphi(\xi) d\xi \]

\[ = r^{N-2} \int_r^{r_1} \xi \varphi(\xi) d\xi \leq r^{N-2} \int_0^r \xi \varphi(\xi) d\xi \quad (4.5) \]

for \( N > 2 \). By (4.4)-(4.5) we obtain

\[ w(r) \leq K := \frac{1}{N-2} \int_0^\infty \xi \varphi(\xi) d\xi. \]

We observe, in addition, that \( w \) satisfies \(-\Delta w(|x|) + c(x)w^{-1}(|x|) |\nabla w(|x|)|^2 \geq a(x), \quad x \in \mathbb{R}^N, 0 < w \leq K \) and \( w(r) \to 0 \) as \( r \to \infty \).

We prove that

\[ u_k \leq w(|x|), \quad x \in \mathbb{R}^N, k = 1, 2, 3, ... \quad (4.6) \]

Since \( w(|x|) > 0 \) in \( \mathbb{R}^N \) and \( u_k = 0 \) in \( \mathbb{R}^N \setminus B_k \) it is enough to prove that \( u_k \leq w \) in \( B_k, k = 1, 2, 3, ... \). To prove this we observe that \( w \in C^2(\overline{B}_k) \) and

\[ \begin{cases} -\Delta [w(x) - u_k(x)] \geq c(x)u_k^{-1}(x)|\nabla u_k(x)|^2 - a(x) + a(x) \geq 0, \quad \text{in } B_k, \\ w(x) - u_k(x) > 0, \quad \text{on } \partial B_k. \end{cases} \]

As a consequence of the maximum principle, Lemma 2.2 we have that \( u_k \leq w \) in \( B_k \). So (4.6) holds.
To finish the proof, use the standard convergence procedure (see [6] or [14]) and so $u_k$ has a subsequence, denoted again by $u_k$, such that $u_k \to u$ (pointwise) in $C^{2,\alpha}_{loc}(\mathbb{R}^N)$ and that $u$ is a solution for the problem (1.5) that vanishing at infinity.

In order to show (1.7), from the above arguments we have

$$u \leq w \text{ in } \mathbb{R}^N. \quad (4.7)$$

On the other hand, using (4.3) we have

$$\lim_{|x| \to \infty} \frac{w(|x|)}{|x|^{2-\mu}} = \frac{1}{2 - \mu} \lim_{|x| \to \infty} \frac{w'(x)}{|x|^{1-\mu}} = \frac{1}{\mu - 2} \lim_{|x| \to \infty} \left[ \int_0^{1/|x|} \sigma^{N-1-\mu} \varphi(\sigma) d\sigma / |x|^{N-\mu} \right].$$

$$= \frac{1}{\mu - 2} \lim_{|x| \to \infty} |x|^{\mu} \varphi(|x|) < \infty.$$

The above relation imply

$$w(x) = O(|x|^{2-\mu}) \text{ as } |x| \to \infty. \quad (4.8)$$

Now, (1.7) follows from (4.8) and (4.7). The proof of Theorem 1.2 is completed.

References

[1] Herbert Amann, *Existence and multiplicity theorems for semi-linear elliptic boundary value problems*, Mathematische Zeitschrift, Springer-Verlag, 150, Pages 281-295, (1976).

[2] David Arcoya, Jose Carmona, Tommaso Leonori, Pedro J. Martinez-Aparicio, Luigi Orsina, Francesco Petitta, *Existence and non-existence of solutions for singular quadratic quasilinear equations*, Journal of Differential Equations, 246 (2009) 4006-4042.

[3] David Arcoya, Sara Barile, Pedro J. Martinez-Aparicio, *Singular quasilinear equations with quadratic growth in the gradient without sign condition*, Journal of Mathematical Analysis and Applications, 350 (2009) 401-408.

[4] Catherine Bandle and Moshe Marcus, *Large solutions of semilinear elliptic equations: existence, uniqueness and asymptotic behavior*, Journal d’Analyse Mathématique 58, (1992), 9-24.

[5] Kuo-Shung Cheng and Wei-Ming Ni, *On the structure of the conformal scalar curvature equation on $\mathbb{R}^N$*, Indiana University Mathematics Journal, Vol. 41, No. 1, Pages 261–278, (1992).

[6] Dragoș-Pătru Covei, *A Lane-Emden-Fowler Type Problem With Singular Nonlinearity*, Journal of Mathematics of Kyoto University, Volume 49, No. 2, Pages 325–338, 2009.

[7] Michael G. Crandall, Paul H. Rabinowitz and Luc Charles Tartar, *On a Dirichlet problem with a singular nonlinearity*, Communications in Partial Differential Equations, Volume 2, Issue 2, 1977, Pages 193-222.

[8] David Gilbarg and Neil Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin, Reprint of the 1998 Edition, 1998.

[9] George Dincă, *Metode variaționale și aplicații*, Editura Tehnică, București, 1980 (in Romanian).
[10] Joseph B. Keller, *Electrohydrodynamics I. The Equilibrium of a Charged Gas in a Container*, Journal of Rational Mechanics and Analysis, Vol. 5, No. 4 (1956).

[11] Olga Alexandrovna Ladyzhenskaya and Nina Nikolaevna Uraltseva, *Linear and Quasilinear Elliptic Equations*, Translated by Scripta Technica (Translation Editor Leon Ehrenpreis), Elsevier (Volume 46), Academic Press New York and London, 1968.

[12] Alan C. Lazer and P. Joseph McKenna, *On a problem of Bieberbach and Rademacher*, Nonlinear Analysis 21 (1993) 327–335.

[13] Charles Loewner and Louis Nirenberg, *Partial differential equations invariant under conformal or projective transformations*, Contributions to Analysis, Academic Press, New York, 1974, pp. 245-272.

[14] Wei-Ming Ni, *On the elliptic equation $\Delta u + K(x)u^{(N+2)/(N-2)} = 0$, its generalizations, and applications in geometry*, Indiana University Mathematics Journal, Vol. 31, No. 4, Pages 343–352, (1982).

[15] Ezzat S. Noussair, *On the existence of solutions of nonlinear elliptic boundary value problems*, Journal of Differential Equations 34, 482-495 (1979).

[16] Stanislav Ivanovich Pohozaev (Pokhozhaev), *The Dirichlet problem for the equation $\Delta u = u^2$*, Doklady Acad Sci. USSR, 136, (1960), no. 3, 769-772. English translation: Soviet. Mathematics Doklady, 1 (1961), 1143-1146.

[17] Zhou Wen-Shu, *Existence and multiplicity of weak solutions for singular semilinear elliptic equation*, Journal of Mathematical Analysis and Applications, 346 (2008) 107-119.

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