**q-deformed supersymmetric t-J model with a boundary**

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**Abstract**

The q-deformed supersymmetric t-J model on a semi-infinite lattice is diagonalized by using the level-one vertex operators of the quantum affine superalgebra $U_q[\widehat{sl}(2|1)]$. We give the bosonization of the boundary states. We give an integral expression of the correlation functions of the boundary model, and derive the difference equations which they satisfy.
1 Introduction

Integrable models with quantum superalgebra symmetries have been the focus of recent studies [1, 2, 3, 4, 5, 6] in the context of strongly correlated fermion systems, a subject of high-profile international research activity because of their relevance to high-\(T_c\) superconductivity. The investigations to these models have largely been carried out within the framework of QISM and Bethe ansatz method. The exception are the works in [7], where the algebraic analysis method, developed in [8, 9] and generalized in [11, 12, 13, 14, 15], was used to diagonalize the supersymmetric \(t-J\) model and its multi-component version directly on an infinite lattice.

The algebraic analysis method [8, 9], which we will call the vertex operator method, was formulated with the help of the level-one q-vertex operators [10] and highest weight representations of quantum affine algebras. The vertex operator method was later extended in [16] to treat integrable models with boundary interactions [17, 18]. It was shown in [16] how the space of states of the boundary XXZ spin-\(\frac{1}{2}\) chain on a semi-infinite lattice can be described in terms of level-one q-vertex operators of \(U_q(\hat{sl}_2)\), and how the correlation functions can be computed by the vertex operators. Several other models have been analysed by means of this approach [19, 20, 21, 22].

In this paper, we study the q-deformed supersymmetric \(t-J\) model with an integrable boundary. We will work directly on a semi-infinite lattice. As is known, the q-deformed supersymmetric \(t-J\) model on an infinite lattice (i.e. without a boundary) has as its symmetry algebra the quantum affine superalgebra \(U_q[\hat{sl}(2|1)]\) [17]. On a finite lattice with diagonal boundary reflection K-matrices this model was solved in [3] by the Bethe ansatz method. Here we adapt the vertex operator method. We will diagonalize the boundary model Hamiltonian directly on the semi-infinite lattice, and moreover compute the correlation functions of the boundary model.

This paper is organized as follows. In section 2, we describe the vertex operator approach to the q-deformed supersymmetric \(t-J\) model on a semi-infinite lattice. In section 3, we study the bosonic realization of the boundary states associated with the level-one highest weight representation of \(U_q[sl(2|1)]\). In section 4, we compute the correlation functions of the local operators (including the spin operator \(S_z\)) and derive the difference equations which they satisfy. In appendix A, we review the bosonization of \(U_q[sl(2|1)]\) at level-one and the associated vertex operators.

2 Boundary q-deformed supersymmetric \(t-J\) model

2.1 q-deformed supersymmetric \(t-J\) model on a finite lattice

In this section, we recall some facts about the q-deformed supersymmetric \(t-J\) model on a finite lattice. Throughout this paper, we fix \(q\) such that \(|q| < 1\).

Let \(V\) be the 3-dimensional graded vector space and \(E_{ij}\) be the \(3 \times 3\) matrix whose \((i,j)\)-element is unity and zero otherwise. The grading of the basis vectors \(v_1, v_2, v_3\) of \(V\) is chosen to be \([v_1] = [v_2] = 1, [v_3] = 0\). Let \(V^*\) be the dual space and \(\{v_1^*, v_0^*, v_{-1}^*\}\) the dual basis vectors. Denote by \(V_z\) (resp. \(V_z^{S}\)) the 3-dimensional level-0 representation (resp. dual representation) of \(U_q[sl(2|1)]\) associated with \(V\). Let \(R(z) \in \text{End}(V \otimes V)\) be the R-matrix of \(U_q[sl(2|1)]\) with matrix elements defined by

\[
R(z)(v_i \otimes v_j) = \sum_{k,l} R_{kl}^{ij}(z) v_k \otimes v_l, \quad \forall v_i, v_j, v_k, v_l \in V,
\]
where
\[
R_{33}^{33}(\frac{z_1}{z_2}) = -\frac{z_1 q^{-1} - z_2 q}{z_1 q - z_2 q^{-1}}, \quad R_{23}^{23}(\frac{z_1}{z_2}) = -\frac{z_1 - z_2}{z_1 q - z_2 q^{-1}}, \quad R_{23}^{32}(\frac{z_1}{z_2}) = \frac{(q - q^{-1})z_2}{z_1 q - z_2 q^{-1}}, \quad R_{32}^{32}(\frac{z_1}{z_2}) = -1,
\]
\[
R_{33}^{32}(\frac{z_1}{z_2}) = -\frac{z_1 - z_2}{z_1 q - z_2 q^{-1}}, \quad R_{33}^{31}(\frac{z_1}{z_2}) = \frac{(q - q^{-1})z_1}{z_1 q - z_2 q^{-1}}, \quad R_{33}^{31}(\frac{z_1}{z_2}) = -\frac{z_1 - z_2}{z_1 q - z_2 q^{-1}}, \quad R_{31}^{31}(\frac{z_1}{z_2}) = \frac{(q - q^{-1})z_1}{z_1 q - z_2 q^{-1}},
\]
\[
R_{21}^{21}(\frac{z_1}{z_2}) = -\frac{z_1 - z_2}{z_1 q - z_2 q^{-1}}, \quad R_{21}^{12}(\frac{z_1}{z_2}) = \frac{(q - q^{-1})z_1}{z_1 q - z_2 q^{-1}}, \quad R_{21}^{12}(\frac{z_1}{z_2}) = -1, \quad R_{11}^{11}(\frac{z_1}{z_2}) = -1, \quad R_{ij}^{ij} = 0, \quad \text{otherwise}.
\]

The R-matrix satisfies the graded Yang-Baxter equation (YBE) on \( V \otimes V \)
\[
R_{12}(z)R_{13}(zw)R_{23}(w) = R_{23}(w)R_{13}(zw)R_{12}(z),
\]
and moreover enjoys: (i) initial condition, \( R(1) = P \) with \( P \) being the graded permutation operator; (ii) unitarity condition, \( R_{12}(\frac{z}{w})R_{21}(\frac{w}{z}) = 1 \), where \( R_{21}(z) = PR_{12}(z)P \); and (iii) crossing-unitarity,
\[
R^{-1, st_1}(z) \left( (M \otimes 1)R(zq^{-2})(M \otimes 1) \right)^{st_1} = 1 \otimes 1,
\]
where
\[
M \equiv q^{\alpha \beta} \equiv \begin{pmatrix} q^{2\rho_1} & q^{2\rho_2} \\ q^{2\rho_3} & q^{-2} \end{pmatrix}.
\]

The various supertranspositions of the R-matrix are given by
\[
(R^{st_1}(z))_{ij}^{kl} = R(z)^{il}_{kj}(-1)^{|i||[l]+[k]|}, \quad (R^{st_2}(z))_{ij}^{kl} = R(z)^{kj}_{il}(-1)^{|j||[j]+[l]|}, \quad (R^{st_12}(z))_{ij}^{kl} = R(z)^{ij}_{kl}(-1)^{|[i]+[j]|(|i|+|j|)+|k|}
\]

Following Sklyanin [17], we construct the transfer matrix of an integrable finite chain, with an open boundary condition described by a reflection K-matrix \( K(z) \). Here \( K(z) \) is a solution of the graded reflection equation
\[
K_2(z_2)R_{21}(z_1/z_2)K_1(z_1)R_{12}(z_1/z_2) = R_{21}(z_1/z_2)K_1(z_1)R_{12}(z_1/z_2)K_2(z_2).
\]

With appropriate normalization, we can show that this \( K(z) \) obeys the relations
\[
K(1) = 1, \quad \text{(Boundary initial condition)},
K(z)K(z^{-1}) = 1, \quad \text{(Boundary unitarity)},
\overline{K}(z)\overline{K}(z^{-1}) = 1, \quad \text{(Boundary crossing – unitarity)},
\]
where \( \overline{K}(z) \) is defined by
\[
\overline{K}(z) = -\sum_{\alpha, \beta} R(z^2)^{\alpha}_i^{\alpha j}(-1)^{|i|+[\beta]+|\alpha|}[\alpha]_j^{\beta}K_\alpha^{\beta}(z^{-1}q^{-1})q^{2\rho_\alpha}.
\]

The third relation is the graded extension of the boundary crossing-unitarity \[ [18, 16, 23]. \]

The transfer matrix of the q-deformed supersymmetric t-J model on a finite chain with the open boundary condition is constructed from \( R(z) \) and \( K(z) \) via \([7, 24]\):

\[
T_B^{\text{fin}}(z) = \text{str}_{V_0}(K^+(z)T(z^{-1})K(z)T(z)),
\]

where \( K^+(z) = K(-z^{-1}q^{-3})^*M \) and

\[
T(z) = R_{01}(z)\cdots R_{0N}(z) \in \text{End}(V_0 \otimes V_1 \otimes \cdots \otimes V_N)
\]

is the double-row monodromy matrix. The supertrace is defined as \( \text{str}(A) = \sum(-1)^{|i|}A_{ii} \).

It can be verified that \( T_B^{\text{fin}}(z) \) form a commuting family, \([T_B^{\text{fin}}(z), T_B^{\text{fin}}(w)] = 0 \). The Hamiltonian of the boundary q-deformed supersymmetric t-J model is given by \([7, 8]\):

\[
H_B^{\text{fin}} = \frac{d}{dz} T_B^{\text{fin}}(z)|_{z=1} = \sum_{j=1}^{N-1} h_{j,j+1} + \frac{1}{2} \frac{d}{dz} K(z)|_{z=1}^2 + \frac{\text{str}_{V_0}(K^+(1)h_{0,N})}{K^+(1)},
\]

where \( h_{j,j+1} = P_{j,j+1} \frac{d}{dz} R_{j,j+1}(z)|_{z=1} \).

The transfer matrix \((2.5)\) with diagonal reflection K-matrices was diagonalized by the Bethe ansatz method in \([3]\).

### 2.2 q-deformed supersymmetric t-J model on a semi-infinite lattice

In this paper, we restrict ourselves to the diagonal reflection K-matrix of the form

\[
K(z) = f(z)
\begin{pmatrix}
\frac{1-rr_{zz}}{z-r} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

where \( r \) is an arbitrary parameter which is related with the boundary interaction \([18, 16]\).

One can check that such a K-matrix satisfies the boundary unitarity and crossing-unitarity \((2.3)\).

We now consider Hamiltonian \((2.6)\) in the semi-infinite limit:

\[
H_B^{\text{fin}}|_{N \to \infty} = \sum_{j=1}^{\infty} h_{j,j+1} + \Delta,
\]

where \( \Delta = \frac{1}{2} \frac{d}{dz} f(z)|_{z=1} \) acts formally on the left-infinite tensor product space

\[
\cdots \otimes V \otimes V.
\]

As mentioned in the introduction, the q-deformed supersymmetric t-J model on an infinite lattice has \( U_q[sl(2|1)] \) as its symmetry algebra. Let \( V(\mu_\alpha) \) be the level-one irreducible highest weight \( U_q[sl(2|1)] \)-modules with highest weight \( \mu_\alpha, \alpha \in \mathbb{Z} \) (see \([14, 17]\)). Consider the level-one vertex operators which are intertwining operators between \( V(\mu_\alpha) \) and \( V(\mu_\beta) \). It has been shown in \([7]\) that the following type I vertex operators \( \Phi(z) \) exist, \( \Phi^*(z) \) which intertwine the level-one irreducible highest weight \( U_q[sl(2|1)] \)-modules \( V(\mu_\alpha) \)

\[
\Phi(z) : V(\mu_\alpha) \longrightarrow V(\mu_{\alpha-1}) \otimes V_2, \quad \Phi^*(z) : V(\mu_\alpha) \longrightarrow V(\mu_{\alpha+1}) \otimes V^*_S.
\]

\[
\text{(2.10)}
\]
In this section we construct the bosonic boundary state $|\alpha; r >_B$ and its dual state $B < r; \alpha|$, which satisfy

$$T_B(z) |\alpha; r >_B = |\alpha; r >_B, \quad B < r; \alpha|T_B(z) = B < r; \alpha|.$$

(See Appendix A for more details about $V(\mu_\alpha)$ and its associated vertex operators.) Therefore following [8, 7, 16, 21], we can write the transfer matrix of the q-deformed supersymmetric $t$-$J$ model on the semi-infinite lattice as

$$T_B(z) = - \sum_{i,j=1}^{3} \Phi_i^s(z^{-1})K^j_i(z)\Phi_j(z)(-1)^{|i|} = \sum_{i,j=1}^{3} q^{-2\rho} \Phi_j(z)K^{-j}_i(z^{-1}q^{-1})\Phi_i^s(z^{-1}), \quad (2.11)$$

where $\Phi_i(z)$ and $\Phi_j^s(z)$ are the components of the $U_q[sl(2|1)]$ vertex operators of type I (see (A.6)). We have used the exchange relations of vertex operators (A.14) and the definition of $K(z)$ (2.4) in the above equation.

We remark that the transfer matrix $T_B(z)$ given by (2.11) is an operator with the property

$$T(z) : V(\mu_\alpha) \longrightarrow V(\mu_\alpha), \quad \alpha \in \mathbb{Z}.$$ 

The commutativity of the transfer matrix (2.11), $[T_B(z), T_B(w)] = 0$, then follows from (A.13) and (2.2). Moreover by (A.12), (A.14) and (A.16), one can show

$$T_B(1) = id, \quad T_B(z)T_B(z^{-1}) = id, \quad T_B(z)T_B(z^{-1}q^{-2}) = id. \quad (2.12)$$

These relations correspond to the boundary initial condition, boundary unitarity and boundary crossing-unitarity (2.3) of the K-matrix, respectively. In terms of the transfer matrix, the q-deformed supersymmetric $t$-$J$ model Hamiltonian on the semi-infinite lattice is given by

$$H = \frac{d}{dz}T_B(z)|_{z=1}. \quad (2.14)$$

Following [7], we define the local operators acting on the n-th site:

$$E_{i,j}^{(1)} = -\Phi_i^s(1)\Phi_j(1)(-1)^{|i|}, \quad (2.15)$$

$$E_{i,j}^{(n)} = \sum_{m} (-1)^{|i|+|j|+|m|+|m|} \Phi_m^*(1)E_{i,j}^{(n-1)}\Phi_m(1), \quad n = 2, 3, \cdots. \quad (2.16)$$

In particular, we have the spin operator $S_i^z$

$$S_i^z = \frac{1}{2} (E_{11}^{(1)} - E_{22}^{(1)}) = \frac{1}{2} \{ \Phi_1^s(1)\Phi_1(1) - \Phi_2^s(1)\Phi_2(1) \}.$$

3 The boundary states

In this section we construct the bosonic boundary state $|\alpha; r >_B$ and its dual state $B < r; \alpha|$, which satisfy

$$T_B(z)|\alpha; r >_B = |\alpha; r >_B, \quad B < r; \alpha|T_B(z) = B < r; \alpha|.$$

By (A.15) and (2.11), the above eigenvalue problem is equivalent to

$$\Phi_i(z^{-1})|\alpha; r >_B = \sum_j K^j_i(z)\Phi_j(z)|\alpha; r >_B, \quad (3.18)$$

$$B < r; \alpha|\Phi_j^s(z)(-1)^{|j|} = \sum_i B < r; \alpha|\Phi_i^s(z^{-1})K^j_i(z)(-1)^{|i|}. \quad (3.19)$$
3.1 The boundary state in $V(\Lambda_0)$

Firstly, we consider the boundary state $|0>_B \in V(\mu_0)$ (or $V(\Lambda_0)$). As is shown in Appendix A, $V(\mu_0) = \eta_0 \xi_0 F_{0;\beta}$ and the highest weight vector $|\Lambda_0> = |\beta, \beta, \beta, 0>$ satisfies

$$\eta_0 |\Lambda_0> = 0.$$ 

So we make the following ansatz [16]:

$$|0; r>_B = e^{F_0(r)} |\Lambda_0>,$$  \hspace{1cm} (3.20)

$$F_0(r) = \frac{1}{2} \sum_{m=1}^{\infty} \frac{m}{|m|^2} \alpha_m \{h^1_m h^1_m + h^2_m h^2_m + c_{m+} - c_{m-}\} + \sum_{m=1}^{\infty} \{\beta^1_m h^1_m + \beta^2_m h^2_m + \beta^3_m c_{m-}\},$$  \hspace{1cm} (3.21)

where $\alpha_m, \beta^1_m, \beta^2_m, \beta^3_m$ are functions of the boundary parameter $r$.

We can check that $e^{F_0(r)}$ plays a role of the Bogoliubov transformation

$$e^{-F_0(r)} h^1_m e^{F_0(r)} = h^1_m + \alpha_m h^1_m + \frac{|m|^2}{m} \beta^1_m,$$

$$e^{-F_0(r)} h^2_m e^{F_0(r)} = h^2_m + \alpha_m h^2_m + \frac{|m|^2}{m} \beta^2_m,$$

$$e^{-F_0(r)} c_m e^{F_0(r)} = c_m + \alpha_m c_m + \frac{|m|^2}{m} \beta^3_m,$$

$$e^{-F_0(r)} h^1_m e^{F_0(r)} = h^1_m + \alpha_m h^1_m + \frac{2|m||m|}{m} \beta^1_m - \beta^2_m \frac{|m|^2}{m}.$$ 

Keeping (3.18) in mind and following [16, 19, 21], we find that the coefficients $\alpha_m, \beta^1_m, \beta^2_m, \beta^3_m$ are

$$\alpha_m = -q^{4m}, \hspace{1cm} \beta^1_m = 0,$$  \hspace{1cm} (3.22)

$$\beta^2_m = \frac{r^m}{|m|^2} + \theta_m q^{2m} - q^{2m},$$  \hspace{1cm} (3.23)

$$\beta^3_m = \theta_m q^{2m} |m|,$$  \hspace{1cm} (3.24)

where the function $\theta_m$ is defined by

$$\theta_m = \begin{cases} 1 & \text{if } m \text{ is even} \\ 0 & \text{if } m \text{ is odd} \end{cases}.$$ 

Moreover following [16] one can check that $\eta_0 |0>_B = 0$, namely, the boundary state $|0>_B \in V(\mu_0)$, as required. In the derivation, the following relation are useful

$$e^{h^1_0 + (-q^{4}; -\frac{1}{2})} |0; r>_B = e^{h^1_0 + q^{4}; -\frac{1}{2}} |0; r>_B,$$

$$e^{-h^1_0 + (\omega q^2; -\frac{1}{2})} |0; r>_B = (1 - \omega^{-1})(1 - r\omega^{-1})e^{-h^1_0 + (\omega q^2; -\frac{1}{2})} |0; r>_B,$$

$$e^{c + (\omega q^2; 0)} |0>_B = (1 - \omega^{-2})e^{c + (\omega q^2; 0)} |0; r>_B,$$

$$e^{h^2_0 + (\omega q^2; -\frac{1}{2})} |0>_B = (1 - \omega r)^{-1}e^{h^2_0 + (\omega q^2; -\frac{1}{2})} |0; r>_B.$$
Similarly, the dual state \( B < r; 0 | \in V^*(\mu_0) \) can be constructed

\[
B < r; 0 | = < 0 | e^{G_0(r)}, \quad G_0(r) = -\frac{1}{2} \sum_{m=1}^{\infty} q^{-2m} \left\{ \frac{m}{[m]} \left( h_m^1 h_m^{1*} + h_m^2 h_m^{2*} + c_m c_m \right) \right. \\
+ \left. \sum_{m=1}^{\infty} \left\{ \delta_m^1 h_m^1 + \delta_m^2 h_m^2 + \delta_m^3 c_m \right\} \right\},
\]

where

\[
\delta_m^1 = 0, \\
\delta_m^2 = -\frac{r^{-m} q^{-m}}{[m]} + \theta_m \left( \frac{q^{-m} + q^{-2m}}{[m]} \right), \\
\delta_m^3 = \theta_m \left( \frac{q^{-m}}{[m]} \right).
\]

### 3.2 The general boundary states

Noting that the boundary K-matrix \( K(z) \) have the following properties

\[
K(z)|_{z=r} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},
\]

we may define \( |-1; r>_B = \Phi_1(r^{-1})|0; r>_B |_{r \to rq^{-2}} \). One can check that such \( |-1; r>_B \) satisfies \((3.18)\) with \( \alpha = -1 \). Recursively, we can construct the general boundary state \( |\alpha; r>_B \) from \( |0; r>_B \) by the following recursive relations,

\[
|\alpha; rq^2>_B = \Phi_1(r^{-1})|\alpha + 1; r>_B, \quad |\alpha; r>_B = q^{-2\rho \iota} \Phi_1(r^{-1}q^2)|\alpha - 1; rq^2>_B.
\]

We have used the second invertibility relation \((A.10)\). Similarly, we can obtain the dual boundary states \( B < r; \alpha| \) from \( B < r; 0| \) by the recursive relations,

\[
B < r; \alpha|\Phi_1^*(r) = B < rq^2; \alpha - 1|, \quad B < r; \alpha| = B < rq^2; \alpha - 1|\Phi_1(qr^{-2})q^{-2\rho \iota}.
\]

### 4 Correlation functions

The aim of this section is to calculate the one-point functions \( < E^{(1)}_{i,j}>_\alpha \):

\[
<E^{(1)}_{i,j}>_\alpha = \frac{B < r; \alpha|E^{(1)}_{i,j}|\alpha; r>_B}{B < r; \alpha|\alpha; r>_B}.
\]

The generalization to the calculation of multi-point functions is straightforward. Thanks to the recursive relations \((3.28)\) and \((3.29)\), it is sufficient to calculate \( < E^{(1)}_{i,j}>_0 \). Thus in the following we restrict ourselves to the calculation of \( < E^{(1)}_{i,j}>_0 \).

Define

\[
\oint dz \ f(z) = f_{-1}, \ \text{for formal series function} \ f(z) = \sum_{n \in \mathbb{Z}} f_n z^n.
\]
By the bosonic realization of Drinfeld currents of $U_q[sl(2|1)]$, (A.7)-(A.10) and the normal ordering relations in the appendix A, we obtain the integral expression of the vertex operators [7]

\[
\phi_3(z) = e^{-h_3^z(q^2 z; -\frac{1}{2}) + c(q^2 z; 0)} : e^{-i\pi \sigma_3^2} :,
\]

\[
\phi_2(z) = \left\{ \frac{e^{-c(wq; 0)}}{wq(1 - \frac{wz}{w_1})} + \frac{e^{-c(w^{-1}; 0)}}{zq^2(1 - \frac{w}{wq})} \right\} e^{-h_2^z(q^2 z; -\frac{1}{2}) - h_2(w; -\frac{1}{2}) + c(q^2 z; 0)} e^{-i\pi \sigma_2^2} ;,
\]

\[
\phi_1(z) = \frac{q^2 - 1}{w(1 - \frac{wz}{w_1})(1 - \frac{wq}{w_1})} \times \left\{ \frac{e^{-c(wq; 0)}}{wq(1 - \frac{wz}{w_1})} + \frac{e^{-c(w^{-1}; 0)}}{zq^2(1 - \frac{w}{wq})} \right\} e^{-h_1^z(q^2 z; -\frac{1}{2}) - h_1(w; -\frac{1}{2}) + c(q^2 z; 0)} e^{-i\pi \sigma_1^2} ;,
\]

\[
\phi^*_1(z) = : e^{h^*_1(qz; -\frac{1}{2})} : e^{i\pi \sigma_1^2} ;,
\]

\[
\phi^*_2(z) = \oint dw \frac{1 - q^{-2}}{z(1 - \frac{wz^2}{w_1})} : e^{h^*_1(qz; -\frac{1}{2}) - h_1(w; -\frac{1}{2})} e^{-i\pi \sigma_1^2} ;,
\]

\[
\phi^*_3(z) = \oint dw_1 \oint dw \frac{1 - q^{-2}}{z(1 - \frac{wz^2}{w_1})} \times \frac{e^{-c(w_1q; 0)}}{w_1w(1 - \frac{wz}{w_1})} e^{h^*_1(qz; -\frac{1}{2}) - h_1(w; -\frac{1}{2}) - h_2(w; -\frac{1}{2})} e^{i\pi \sigma_1^2} ;.
\]

Since $\eta_0|0, r > 0$, one may set

\[
P_{i,j}(z_1, z_2) = \frac{B < r; 0|\Phi_i^*(z_1)\Phi_j(z_2)|0; r > B}{< r; 0|0; r > B} \equiv \frac{B < r; 0|\phi_i^*(z_1)\phi_j(z_2)|0; r > B}{< r; 0|0; r > B},
\]

(4.1)

then $E_{i,j}^{(1)} > 0 = -(-1)^{[i]} P_{i,j}(1, 1)$.

The bosonization formulae (A.7)-(A.10) of the vertex operators immediately imply

\[
P_{i,j}(z_1, z_2) = \delta_{ij} F_i(z_1, z_2) \overset{\text{def}}{=} \delta_{ij} \frac{B < r; 0|\phi_i^*(z_1)\phi_i(z_2)|0; r > B}{B < r; 0|0; r > B}.
\]

Using the technique in [10, 24] (see equation (C.4)), after tedious calculation, we get

\[
B(r; 0|0; r)_B = \prod_{n=1}^{\infty} \frac{1}{1 - \alpha_n \gamma_n} \prod_{n=1}^{\infty} \frac{1}{(\alpha_n \gamma_n - 1)^{\frac{1}{2}}} \exp \left[ \frac{1}{2} \sum_{n=1}^{\infty} \frac{[n]^2}{n} \frac{1}{1 - \alpha_n \gamma_n} \left( \gamma_n \left( \beta^3_n \right)^2 + 2 \beta^3_n \delta_n^2 + \alpha_n \left( \delta^3_n \right)^2 \right) \right],
\]

(4.2)

\[
F_1(z_1, z_2) = \frac{1}{B(r; 0|0; r)_B} \oint d\omega_1 \oint d\omega \frac{(q^2 - 1)g_1}{q\omega^2(1 - \frac{\omega_1}{\omega})(1 - \frac{\omega}{\omega_1})(1 - \frac{\omega z^2}{\omega_1})} \times \prod_{n=1}^{\infty} (-\alpha_n \gamma_n - 1)^{-1} \prod_{n=1}^{\infty} (\alpha_n \gamma_n - 1)^{\frac{1}{2}} \exp \left( \sum_{n=1}^{\infty} \frac{[n]^2}{n} \frac{1}{(\alpha_n \gamma_n - 1)} \left( (B_1 - C_1)^2 [2n] \frac{\gamma_n}{\alpha_n \gamma_n - 1} + \gamma_n (B_1 - C_1) \beta_n^2 - \gamma_n (B_1 - C_1) A_1 + (B_1 - C_1) \delta_n^2 \right) \right)
\]
\[
\begin{align*}
&\times \exp(\sum_\frac{[n]^2}{n} \frac{1}{1 - \alpha_n \gamma_n} \left\{ \frac{1}{2} (\beta_n^3)^2 D\gamma_n + \frac{1}{2} \beta_n^3 D_1^2 \gamma_n + \beta_n^3 D_1 \gamma_n + \beta_n^3 \delta_n^3 \\
&\quad + D_1 \delta_n^3 + \frac{1}{2} \alpha_n \left( \delta_n^3 \right)^2 \right\}) \\
&+ \int d\omega \int d\omega^* \frac{(q^2 - 1) g'_1}{q^2 \omega^2 (1 - \frac{\omega^2}{\omega^2_1})(1 - \frac{\omega}{\omega_1}) (1 - \frac{\omega}{\omega^*})} \\
&\times \prod_{n=1}^\infty (-(\alpha_n \gamma_n - 1)^{-1}) \prod_{n=1}^\infty (\alpha_n \gamma_n - 1)^{- \frac{3}{2}} \\
&\times \exp(\sum_\frac{[n]^2}{n} \frac{1}{\alpha_n \gamma_n - 1} \left\{ (B_1 - C_1)^2 \left[ \frac{2n}{[n]} \gamma_n \right] \frac{\gamma_n}{\alpha_n \gamma_n - 1} + \gamma_n (B_1 - C_1) \beta_n^2 \\
&\quad - \gamma_n (B_1 - C_1) A_1 + (B_1 - C_1) \delta_n^2 \right\}) \\
&\times \exp\left( \sum_\frac{[n]^2}{n} \frac{1}{1 - \alpha_n \gamma_n} \left\{ \frac{1}{2} (\beta_n^3)^2 D'_1 \gamma_n + \frac{1}{2} \beta_n^3 (D'_1)^2 \gamma_n + \beta_n^3 D'_1 \gamma_n + \beta_n^3 \delta_n^3 \\
&\quad + D'_1 \delta_n^3 + \frac{1}{2} \alpha_n \left( \delta_n^3 \right)^2 \right\} \right),
\end{align*}
\]

where

\[
\begin{align*}
g_1 &= \exp \left( - \sum_\frac{q^{3n} z_2^{-1} \omega^{-n}}{n} \right) \exp \left( \sum_\frac{q^{n} z_2^{-1} \omega^{-n}}{n} \right) \exp \left( \sum_\frac{q^{n} z_2^{-1} \omega^{-n}}{n} \right) \\
&\quad \exp \left( - \sum_\frac{q^{-n} z_2^{-1} \omega^{-n}}{n} \right) \exp \left( \sum_\frac{r^{n} z_2^{-1} \omega^{-n}}{n} \right) \exp \left( \sum_\frac{t^{n} \omega^{-n} \omega^{-n}}{n} \right) \\
&\quad \exp \left( \sum_\frac{q^{4n} z_2^{-1} \omega^{-n}}{n} \right) \exp \left( \sum_\frac{q^{n} \omega^{-n} \omega^{-n}}{n} \right) \exp \left( - \sum_\frac{q^{n} \omega^{-2n}}{n} \right) \\
&\quad \exp \left( - \sum_\frac{r^{n} q^{2n} \omega^{-1}}{n} \right) \exp \left( - \sum_\frac{q^{n} z_2^{-1} \omega^{-n} \omega^{-n}}{n} \right) \exp \left( - \sum_\frac{q^{n} z_2^{-1} \omega^{-n}}{n} \right) \\
&\quad \exp \left( \sum_\frac{z_2^{-1} \omega^{-n}}{n} \right),
\end{align*}
\]

\[
\begin{align*}
g'_1 &= \exp \left( \sum_\frac{q^{n} z_2^{-1} \omega^{-n}}{n} \right) \exp \left( - \sum_\frac{q^{-n} z_2^{-1} \omega^{-n}}{n} \right) \exp \left( \sum_\frac{q^{n} z_2^{-1} \omega^{-n}}{n} \right) \\
&\quad \times \exp \left( \sum_\frac{q^{5n} \omega^{-n} \omega^{-1}}{n} \right) \exp \left( \sum_\frac{q^{4n} z_2^{-1} \omega^{-n}}{n} \right) \exp \left( \sum_\frac{q^{n} \omega^{-n} \omega^{-1}}{n} \right) \\
&\quad \times \exp \left( - \sum_\frac{q^{4n} \omega^{-2n}}{n} \right) \exp \left( - \sum_\frac{r^{n} q^{2n} \omega^{-1}}{n} \right) \exp \left( - \sum_\frac{q^{n} z_2^{-1} \omega^{-n} \omega^{-n}}{n} \right) \\
&\quad \times \exp \left( - \sum_\frac{q^{2n} z_2^{-1} \omega^{-n} \omega^{-n}}{n} \right) \exp \left( \sum_\frac{z_2^{-1} \omega^{-1}}{n} \right),
\end{align*}
\]

and

\[
\begin{align*}
A_1 &= \sum_\frac{q^{n} z_2^{-n}}{[n]} - \alpha_n \sum_\frac{q^{-n} z_2^{-n}}{[n]} + \sum_\frac{q^{n} \omega^{-n}}{[n]} - \alpha_n \sum_\frac{q^{n} \omega^{-n}}{[n]}, \\
B_1 &= \sum_\frac{q^{n} z_2^{-n}}{[n]} - \alpha_n \sum_\frac{q^{-n} z_2^{-n}}{[n]}, \\
D_1 &= \sum_\frac{q^{2n} z_2^{-n}}{[n]} - \alpha_n \sum_\frac{q^{2n} z_2^{-n}}{[n]} - \sum_\frac{q^{n} \omega^{-n}}{[n]} + \alpha_n \sum_\frac{q^{n} \omega^{-n}}{[n]},
\end{align*}
\]
\[ C_1 = \sum_{\frac{n}{[n]}} q^{2n} \omega_1^{n} - \alpha_n \sum_{\frac{n}{[n]}} q^{2n} \omega_1^{n}, \]
\[ D'_1 = \sum_{\frac{n}{[n]}} q^{2n} \omega_2^{n} - \alpha_n \sum_{\frac{n}{[n]}} q^{2n} \omega_2^{n} - \sum_{\frac{n}{[n]}} q^{-n} \omega_n^{n} + \alpha_n \sum_{\frac{n}{[n]}} q^{n} \omega_n^{n}, \]
\[ F_2(z_1, z_2) = \frac{1}{B(\eta)^2} \int_{z_1} \int_{z_2} \frac{(1 - q^{-2}) g_2}{\omega_1 (1 - \frac{z_1 q^2}{\omega})} \frac{(1 - q^{-2}) g_2'}{(1 - \frac{z_2 q^2}{\omega})} \omega_2 (1 - \frac{z_2 q^2}{\omega_2}) \frac{(1 - q^{-2}) g_2}{\omega_1 (1 - \frac{z_1 q^2}{\omega})} \]
\[ \times \prod_{n=1}^{\infty} \left( - (\alpha_n \gamma_n - 1)^{-1} \right) \prod_{n=1}^{\infty} \left( - (\alpha_n \gamma_n - 1)^{-1} \right) \]
\[ \times \exp \left( \sum_{\frac{n}{[n]}} \frac{1}{\alpha_n \gamma_n - 1} \left[ (B_2 - C_2) \omega_1 q \left( 1 - \frac{z_1 q^2}{\omega} \right) \omega_1 q \left( 1 - \frac{z_2 q^2}{\omega} \right) \right] \gamma_n (B_2 - C_2) \beta_n^2 \right) \]
\[ - \gamma_n (B_2 - C_2) A_2 + (B_2 - C_2) \delta_n^2 \right) \}
\[ \times \exp \left( \sum_{\frac{n}{[n]}} \frac{1}{\alpha_n \gamma_n} \left[ \frac{1}{2} \frac{\beta_n^3}{\gamma_n} D_2^2 \gamma_n + \frac{1}{2} \frac{\beta_n^3}{\gamma_n} D_2^2 \gamma_n + \gamma_n (B_2 - C_2) \beta_n^2 \right) \]
\[ \times \prod_{n=1}^{\infty} \left( - (\alpha_n \gamma_n - 1)^{-1} \right) \prod_{n=1}^{\infty} \left( - (\alpha_n \gamma_n - 1)^{-1} \right) \]
\[ \times \exp \left( \sum_{\frac{n}{[n]}} \frac{1}{\alpha_n \gamma_n} \left[ \frac{1}{2} \frac{\beta_n^3}{\gamma_n} D_2^2 \gamma_n + \frac{1}{2} \frac{\beta_n^3}{\gamma_n} D_2^2 \gamma_n + \gamma_n (B_2 - C_2) \beta_n^2 \right) \]
\[ \times \prod_{n=1}^{\infty} \left( - (\alpha_n \gamma_n - 1)^{-1} \right) \prod_{n=1}^{\infty} \left( - (\alpha_n \gamma_n - 1)^{-1} \right) \]
\[ \times \exp \left( \sum_{\frac{n}{[n]}} \frac{1}{\alpha_n \gamma_n} \left[ \frac{1}{2} \frac{\beta_n^3}{\gamma_n} D_2^2 \gamma_n + \frac{1}{2} \frac{\beta_n^3}{\gamma_n} D_2^2 \gamma_n + \gamma_n (B_2 - C_2) \beta_n^2 \right) \right), \]
\[ \text{(4.4)} \]

where
\[ g_2 = \exp \left( \sum_{\frac{n}{[n]}} \frac{\omega_n z_1^{-n}}{n} \right) \exp \left( - \sum_{\frac{n}{[n]}} \frac{q^{2n} z_1^{-n} z_2^{-n}}{n} \right) \exp \left( \sum_{\frac{n}{[n]}} \frac{q^n \omega_n z_1^{-n}}{n} \right) \]
\[ \times \exp \left( - \sum_{\frac{n}{[n]}} \frac{q^{-n} \omega_1 z_2^{-n}}{n} \right) \exp \left( \sum_{\frac{n}{[n]}} \frac{q^n \omega_1 z_2^{-n}}{n} \right) \exp \left( \sum_{\frac{n}{[n]}} \frac{q^{2n} \omega_n z_1^{-n}}{n} \right) \]
\[ \times \exp \left( - \sum_{\frac{n}{[n]}} \frac{q^{-n} \omega_1 z_2^{-n}}{n} \right) \exp \left( \sum_{\frac{n}{[n]}} \frac{q^n \omega_1 z_2^{-n}}{n} \right) \exp \left( \sum_{\frac{n}{[n]}} \frac{q^{2n} \omega_n z_1^{-n}}{n} \right) \]
\[ \times \exp \left( - \sum_{\frac{n}{[n]}} \frac{q^{2n} z_2^{-n} z_1^{-n}}{n} \right), \]
\[ g'_2 = \exp \left( \sum_{\frac{n}{[n]}} \frac{\omega_n z_1^{-n}}{n} \right) \exp \left( - \sum_{\frac{n}{[n]}} \frac{q^{2n} z_2^{-n} z_1^{-n}}{n} \right) \exp \left( \sum_{\frac{n}{[n]}} \frac{q^n \omega_n z_1^{-n}}{n} \right) \]
\[ g_3 = \exp \left( -\sum \frac{\omega^n z_1^{-n}}{n} \right) \left( -\sum \frac{q^{2n} z_1^{-n} z_2^{-n}}{n} \right) \exp \left( \sum \frac{q^n \omega^n z_2^{-n}}{n} \right) \]
\[ g'_3 = \exp \left( -\sum_{\omega^n z_1^{-n}} n \right) \exp \left( \sum_{\omega^n \omega_1^{-n}} n \right) \exp \left( -\sum_{\omega^{-2n} q^{4n}} n \right) \exp \left( -\sum_{\omega^{-n} q^{2n, n}} n \right) \exp \left( -\sum_{\omega^{-n} q^{2n, n}} n \right), \]

and

\[ A_3 = \sum_{\omega^n z_1^{-n}} n - \alpha_n \sum_{\omega^n z_1^{-n}} n + \sum_{\omega^n \omega_1^{-n}} n - \alpha_n \sum_{\omega^n \omega_1^{-n}} n, \]

\[ B_3 = \sum_{\omega^n z_2^{-n}} n - \alpha_n \sum_{\omega^n z_2^{-n}} n, \]

\[ D_3 = \sum_{\omega^n z_2^{-n}} n - \alpha_n \sum_{\omega^{-2n} z_2^{-n}} n - \sum_{\omega^n \omega_1^{-n}} n + \alpha_n \sum_{\omega^n \omega_1^{-n}} n, \]

\[ C_3 = \sum_{\omega^n \omega_1^{-n}} n - \alpha_n \sum_{\omega^n \omega_1^{-n}} n, \]

\[ D'_3 = \sum_{\omega^n z_2^{-n}} n - \alpha_n \sum_{\omega^{-2n} z_2^{-n}} n - \sum_{\omega^n \omega_1^{-n}} n + \alpha_n \sum_{\omega^n \omega_1^{-n}} n. \]

We now derive the difference equations satisfied by the one-point functions. By (2.11) and (A.13)-(A.16), one obtains

\[ \Phi_i^j(z^{-1}) | \alpha; r \rangle_B = \sum_j \widetilde{K}_j^i(zq) \Phi_j^z(zq^2) | \alpha; r \rangle_B, \quad (4.6) \]

\[ B < r; \alpha | \Phi_i(z)(-1)[i] = \sum_B < r; \alpha | \Phi_j(z^{-1} q^{-2}) \widetilde{R}^j_i(zq)(-1)[j]. \quad (4.7) \]

By (A.14), one derives the exchange relations

\[ \Phi_i^j(z_1) \Phi_j^z(z_2) = \sum_{k,l} \widetilde{R}(z_1, z_2) \Phi_l(z_2) \Phi_k^z(z_1)(-1)[k][l], \quad (4.8) \]

where \( \widetilde{R}(z) = R^{-1, s_{t_1}, t_{n_1}}(z) \).

Using (B.18)-(B.19), (4.6)-(4.8), (A.14) and (A.7)-(A.10), we get the difference equations

\[ F_i(z_1 q^{-2}, z_2) = \sum_{j,k,l,m,n} (-1)^{|k|[l]|j|[j] + [j][n]} K_j^i(z_1 q^{-2}) \widetilde{R}(z_1, z_2) R(z_1, z_2) F_n(z_1, z_2), \quad (4.9) \]
\[ F_i(z_1, z_2 q^2) = \sum_{j,k,l,m,n} (-1)^{[k][l]+[l]+[m]+[n]} K_i^j(z_2^{-1} q^{-2}) \tilde{R} (z_1 z_2 q^2)^{ij}_{kl} \times \tilde{K}_1^m(z_2^{-1} q^{-1}) \tilde{R}(\frac{z_1}{z_2})^{n}_{m} F_n(z_1, z_2). \] (4.10)

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A Appendix A

A.1 Bosonization of \( U_q[sl(2|1)] \)

In this section, we briefly review the bosonization of \( U_q[sl(2|1)] \) at level-one and the corresponding vertex operators [24, 6].

The Cartan matrix of \( U_q[sl(2|1)] \) is

\[
(a_{ij}) = \begin{pmatrix}
0 & -1 & 1 \\
-1 & 2 & -1 \\
1 & -1 & 0
\end{pmatrix}
\]

where \( i, j = 0, 1, 2 \).

In terms of the Drinfeld generators: \( \{ d, X^\pm_m, h^i_n, (K^i)^{\pm 1}, \gamma^{\pm 1/2} \} \), the defining relations of \( U_q[sl(2|1)] \) read

\[
\gamma \text{ is central}, \quad [K^i, h^j_m] = 0, \quad [d, K^i] = 0, \quad [d, h^j_m] = mh^j_m, \\
[K^i, h^j_m] = \delta_{m+n,0} \frac{[a_{ij} m]}{m} (\gamma^m - \gamma^{-m}), \\
[K^i, X^\pm_m] = q^{\pm a_{ij}} X^\pm_m K^i, \quad [d, X^\pm_m] = m X^\pm_m, \\
[h^i_m, X^\pm_m] = \pm \frac{[a_{ij} m]}{m} \gamma^{\pm m/2} X^{\pm}_{m+n}, \\
[X^+, X^-_n] = \frac{\delta_{i,j}}{q - q^{-1}} (\gamma^{(m-n)/2} \psi^{+, j}_{m+n} - \gamma^{-(m-n)/2} \psi^{-, j}_{m+n}), \\
[X^+, X^+_n] = 0, \\
[X^+, X^+_{n+1}]_{q^{a_{ij}}} + [X^-, X^-_{n+1}]_{q^{a_{ij}}} = 0, \quad \text{for } a_{ij} \neq 0,
\]

where \([m] = \frac{q^m - q^{-m}}{q - q^{-1}}, [X, Y]_\xi = XY - (-1)^{[X][Y]} \xi Y X \) and \([X, Y]_1 \equiv [X, Y]; \) the \( \mathbb{Z}_2 \)-grading of Drinfeld generators are: \( [X^\pm_n] = 1 \) for \( m \in \mathbb{Z} \) and zero otherwise.

Introduce the bosonic \( q \)-oscillators [25] \( \{ a^i_n, a^j_n, b_n, c_n, Q_a, Q_{a^2}, Q_b, Q_c | n \in \mathbb{Z} \} \), which satisfy the commutation relations

\[
[a^i_m, a^j_n] = \delta_{i,j} \delta_{m+n,0} \frac{[m]^2}{m}, \quad [a^i_0, Q_{a^j}] = \delta_{i,j},
\]
\[ \{b_m, b_n\} = -\delta_{m+n,0} \frac{m^2}{m}, \quad \{b_0, Q_b\} = -1, \]
\[ \{c_m, c_n\} = \delta_{m+n,0} \frac{m^2}{m}, \quad \{c_0, Q_c\} = 1. \]

Define the generating functions for the Drinfeld basis by \( X_i^\pm(z) = \sum_{m \in \mathbb{Z}} X_m^\pm z^{-m-1} \), and introduce \( h_0^i \) by setting \( K^i = q^{h_0^i} \). Define \( Q_h^i = Q_{a^i} - Q_{a^2} \), \( Q_h^2 = Q_{a^2} + Q_b \) and \( h_i(z; \beta) \) by
\[
h_i(z; \beta) = - \sum_{n \neq 0} \frac{h_i^n}{n} q^{-\beta n} z^{-n} + Q_{h^i} + h_0^i \ln z, \tag{A.1}
\]
where \( \beta \) is a parameter. Other bosonic fields are defined similarly.

The Drinfeld generators at level-one are realized by the free boson fields as [25]
\[
\begin{align*}
&h_1^1 = a_1^1 q^{-|m|/2} - a_1^2 q^{+|m|/2}, \quad h_1^2 = a_1^2 q^{-|m|/2} + b_m q^{+|m|/2}, \quad m \in \mathbb{Z}, \\
&X_1^\pm(z) = \pm e^{\pm h_1(z; \pm \frac{1}{2})}, \quad X_2^\pm(z) = e^{h_2(z; \pm \frac{1}{2})} e^{c(z; 0)} = e^{\pm i \pi a_1^1}, \\
&X_2^z(z) = e^{-h_2(z; -\frac{1}{2})} [\partial_z e^{-c(z; 0)}] = e^{i \pi a_1^1}, \quad \gamma = q,
\end{align*}
\]
where \( \partial_z f(z) = \frac{f(qz) - f(q^{-1}z)}{(q-q^{-1})z} : O : \) stands for the usual normal ordering of \( O \).

Consider the bosonic Fock spaces \( F_{\lambda_1, \lambda_2, \lambda_3, \lambda_4} \), generated by \( a_{-m}^i, b_{-m}, c_{-m} (m > 0) \) over the vacuum vectors \( |\lambda_1, \lambda_2, \lambda_3, \lambda_4> \),
\[
F_{\lambda_1, \lambda_2, \lambda_3, \lambda_4} = C[a_{-1, \lambda_1}^i, a_{-2, \lambda_2}^i, \ldots, b_{-1, \lambda_3}, c_{-1, \lambda_4}, \ldots]|\lambda_1, \lambda_2, \lambda_3, \lambda_4>, \tag{A.2}
\]
where
\[
a_m^i |\lambda_1, \lambda_2, \lambda_3, \lambda_4> = 0, \quad b_m |\lambda_1, \lambda_2, \lambda_3, \lambda_4> = 0, \quad c_m |\lambda_1, \lambda_2, \lambda_3, \lambda_4> = 0, \quad \text{for } m > 0,
\]
\[
|\lambda_1, \lambda_2, \lambda_3, \lambda_4> = e^{\lambda_1 Q_{a^1} + \lambda_2 Q_{a^2} + \lambda_3 Q_b + \lambda_4 Q_c} |0, 0, 0, 0>. 
\]

Introduce the following spaces
\[
F_{(\alpha; \beta)} = \bigoplus_{i, j \in \mathbb{Z}} F_{\beta+i, \beta-i+j, \beta-\alpha+j, -\alpha+j}. \tag{A.3}
\]

It can be shown that the bosonized action of \( U_q[sl(2|1)] \) on \( F_{(\alpha; \beta)} \) is closed. To obtain the irreducible subspace in \( F_{(\alpha; \beta)} \), it convenient to introduce a pair of fermionic currents [26, 25]
\[
\eta(z) = \sum_{n \in \mathbb{Z}} \eta_n z^{-n-1} = e^{c(z; 0)}, \quad \xi(z) = \sum_{n \in \mathbb{Z}} \xi_n z^{-n} = e^{-c(z; 0)},
\]
The mode expansion of \( \eta(z), \xi(z) \) is well defined on \( F_{(\alpha; \beta)} \) for \( \alpha \in \mathbb{Z} \), and it satisfies the following relation
\[
\xi_n \xi_m + \eta_n \eta_m = \eta_n \eta_m + \eta_m \eta_n = 0, \quad \xi_n \eta_m + \eta_n \xi_m = \delta_{m,n}.
\]

Since \( \eta_0 \) commutes (or anticommutes) with \( U_q[sl(2|1)] \), \( \eta_0 \) plays the role of screening charge and \( \eta_0 \xi_0 \) qualify as the projector from \( F_{(\alpha; \beta)} \) to the Kernel of \( \eta_0 \). Set \( \lambda_0 = (1 - \alpha) \Lambda_0 + \alpha \Lambda_2, \ \alpha \in \mathbb{Z} \), where \( \Lambda_i (i = 0, 1, 2) \) are the fundamental weights of \( U_q[sl(2|1)] \), and
\[
\mu_\alpha = \begin{cases} 
\Lambda_\alpha, & \alpha = 0, 1, 2 \\
\lambda_{\alpha-1} & \text{for } \alpha > 2 \\
\lambda_\alpha & \text{for } \alpha < 0.
\end{cases} \tag{A.4}
\]

Define \( V(\mu_\alpha) = \eta_0 \xi_0 F_{(\alpha; \beta-\alpha)} \). Following [25, 1], \( V(\mu_\alpha) \) \((\alpha \in \mathbb{Z})\) are the irreducible highest weight \( U_q[sl(2|1)] \)-modules with the highest weight \( \mu_\alpha \).
A.2 Level-one Vertex operators

Let $V(\lambda)$ be the highest weight $U_q[sl(2|1)]$-module with the highest weight $\lambda$. Consider the following intertwiners of $U_q[sl(2|1)]$-modules:

$$\Phi^V(z) : V(\lambda) \rightarrow V(\mu) \otimes V_z, \quad \Phi^{V^*}(z) : V(\lambda) \rightarrow V(\mu) \otimes V_{z^*}.$$  

They are intertwiners in the sense that for any $x \in U_q[sl(2|1)]$,

$$\Theta(z) \cdot x = \Delta(x) \cdot \Theta(z), \quad \Theta(z) = \Phi(z) \Phi^*(z), \quad (A.5)$$

the grading of these operators is $\Theta(\cdot) = 0$. $\Phi(z)$ is called type I (dual) vertex operator [5]. We expend the vertex operator as

$$\Phi(z) = \sum_{j=1,2,3} \Phi(z_j) \otimes v^j, \quad \Phi^*(z) = \sum_{j=1,2,3} \Phi^*(z_j) \otimes v^j*.$$  

Define the operators $\phi_j(z), \phi_j^*(z), \psi_j(z)$ and $\psi_j^*(z)$ ($j = 1, 2, 3$) by

$$\phi_3(z) = e^{-h_3^2(q^2z^2;-\frac{i}{2})+c(q^2z;0)} e^{i\pi a_0^2}, \quad (A.7)$$

$$\phi_2(z) = -[\phi_3(z), X_0^{-2}]_{q^{-1}}, \quad \phi_1(z) = [\phi_2(z), X_0^{-1}]_{q}, \quad (A.8)$$

$$\phi_1^*(z) = e^{h_1^2(q^2z^{-2};\frac{i}{2})} e^{i\pi a_0^2}, \quad (A.9)$$

$$\phi_2^*(z) = -q^{-1}[\phi_1^*(z), X_0^{-1}]_{q^{-1}}, \quad \phi_3^*(z) = q^{-1}[\phi_2^*(z), X_0^{-2}]_{q^{-1}}, \quad (A.10)$$

where $h_m^* = -h_m^2$, $h_m^* = -h_m^2 - \frac{2|m|}{|m|} h_m^2$ and $Q_{h^2} = -Q_{h^2} = -Q_{h^2} = -2Q_{h^2}$. Since the operator $\phi_i(z), \phi_i^*(z)$ commute (or anti-commute) with $\eta_0$, we define

$$\Phi_i(z) = \eta_0 \xi_0 \phi_i(z) \eta_0 \xi_0, \quad \Phi_i^*(z) = \eta_0 \xi_0 \phi_i^*(z) \eta_0 \xi_0. \quad (A.11)$$

According [23, 7], the vertex operators $\Phi(z)$ and $\Phi^*(z)$ (A.6) given by (A.11) are the only type I vertex operators of $U_q[sl(2|1)]$ which intertwine the level-one irreducible highest weight $U_q[sl(2|1)]$-modules $V(\mu_\alpha)$ ($\alpha \in Z$)

$$\Phi(z) : V(\mu_\alpha) \rightarrow V(\mu_{\alpha-1}) \otimes V_z, \quad \Phi^*(z) : V(\mu_\alpha) \rightarrow V(\mu_{\alpha+1}) \otimes V_{z^*}. \quad (A.12)$$

It is shown [7] that the above vertex operators satisfy the graded Faddeev-Zamolodchikov algebra

$$\Phi_j(z_2)\Phi_i(z_1) = \sum_{kl} R(\frac{z_1}{z_2})_{ij,k} \Phi_k(z_1)\Phi_l(z_2)(-1)^{|i||j|}, \quad (A.12)$$

$$\Phi^*_j(z_2)\Phi^*_i(z_1) = \sum_{kl} R(\frac{z_1}{z_2})_{ij,k} \Phi^*_k(z_1)\Phi^*_l(z_2)(-1)^{|i||j|}, \quad (A.13)$$

$$\Phi_j(z_2)\Phi^*_i(z_1) = \sum_{kl} R(\frac{z_1}{z_2})_{ij,k} \Phi^*_k(z_1)\Phi_l(z_2)(-1)^{|k||l|}, \quad (A.14)$$

where $R(z) = R^{-1,sl_1}(z)$. Moreover, the vertex operators having the following invertibility relations

$$\Phi_i(z)\Phi_j^*|_{V(\Lambda_\alpha)} = -(-1)^{|ij|} \delta_{ij} id|_{V(\Lambda_\alpha)}, \quad (A.15)$$

$$\Phi^*_i(zq^2)\Phi_j(z)|_{V(\Lambda_\alpha)} = \delta_{ij} q^{2\rho_j} id|_{V(\Lambda_\alpha)}, \quad (A.16)$$

$$\sum_k q^{-2\rho_k} \Phi^*_k(zq^2)|_{V(\Lambda_\alpha)} = id|_{V(\Lambda_\alpha)}. \quad (A.17)$$
Appendix B

In this appendix, we give the normal ordering relations of fundamental bosonic fields:

\[
\begin{align*}
\cosh(z_1 + q^{-(\beta_1 + \beta_2)z_2}) &= \cosh(z_1) \cosh(z_2) + \cosh(z_1 - q^{-(\beta_1 + \beta_2)z_2}) \\
\sinh(z_1 + q^{-(\beta_1 + \beta_2)z_2}) &= \cosh(z_1) \sinh(z_2) - \cosh(z_1 - q^{-(\beta_1 + \beta_2)z_2}) \\
\cosh(z_1 - q^{-(\beta_1 + \beta_2)z_2}) &= \cosh(z_1) \cosh(z_2) - \cosh(z_1 + q^{-(\beta_1 + \beta_2)z_2}) \\
\sinh(z_1 - q^{-(\beta_1 + \beta_2)z_2}) &= \cosh(z_1) \sinh(z_2) + \cosh(z_1 + q^{-(\beta_1 + \beta_2)z_2}) \\
\end{align*}
\]

Appendix C

We here summarize the formulas concerning coherent states of bosons which have been used in section 5.

The coherent states \(|\zeta^1, \zeta^2, \zeta^3 >\) and \(< \bar{\zeta}^1, \bar{\zeta}^2, \bar{\zeta}^3|\) in the Fock space \(F_{(0; \beta)}\) and its dual space \(F^*_{(0; \beta)}\) are defined by

\[
|\zeta^1, \zeta^2, \zeta^3 > = \exp \left\{ \sum_{m=1} \frac{2 m}{|m|^2} \zeta^i_m h^i_m + \sum_{m=1} \frac{2 m}{|m|^2} \bar{\zeta}^i_m c_m \right\} |\beta, \beta, 0 >, \quad (C.1)
\]

\[
< \bar{\zeta}^1, \bar{\zeta}^2, \bar{\zeta}^3| = < \beta, \beta, 0 | \exp \left\{ \sum_{m=1} \frac{2 m}{|m|^2} \bar{\zeta}^i_m h^i_m + \sum_{m=1} \frac{2 m}{|m|^2} \zeta^i_m c_m \right\} (C.2)
\]

where \(\zeta^i_m \quad \text{and} \quad \bar{\zeta}^j_m \quad (l = 1, 2, 3, \quad m = 1, 2, \cdots)\) are complex conjugate parameters.

Noting that

\[
\begin{align*}
\cosh(z_1 + q^{-(\beta_1 + \beta_2)z_2}) &= \cosh(z_1) \cosh(z_2) + \cosh(z_1 - q^{-(\beta_1 + \beta_2)z_2}) \\
\sinh(z_1 + q^{-(\beta_1 + \beta_2)z_2}) &= \cosh(z_1) \sinh(z_2) - \cosh(z_1 - q^{-(\beta_1 + \beta_2)z_2}) \\
\cosh(z_1 - q^{-(\beta_1 + \beta_2)z_2}) &= \cosh(z_1) \cosh(z_2) - \cosh(z_1 + q^{-(\beta_1 + \beta_2)z_2}) \\
\sinh(z_1 - q^{-(\beta_1 + \beta_2)z_2}) &= \cosh(z_1) \sinh(z_2) + \cosh(z_1 + q^{-(\beta_1 + \beta_2)z_2}) \\
\end{align*}
\]

one can easily verify

\[
\begin{align*}
\cosh(z_1 + q^{-(\beta_1 + \beta_2)z_2}) &= \cosh(z_1) \cosh(z_2) + \cosh(z_1 - q^{-(\beta_1 + \beta_2)z_2}) \\
\sinh(z_1 + q^{-(\beta_1 + \beta_2)z_2}) &= \cosh(z_1) \sinh(z_2) - \cosh(z_1 - q^{-(\beta_1 + \beta_2)z_2}) \\
\cosh(z_1 - q^{-(\beta_1 + \beta_2)z_2}) &= \cosh(z_1) \cosh(z_2) - \cosh(z_1 + q^{-(\beta_1 + \beta_2)z_2}) \\
\sinh(z_1 - q^{-(\beta_1 + \beta_2)z_2}) &= \cosh(z_1) \sinh(z_2) + \cosh(z_1 + q^{-(\beta_1 + \beta_2)z_2}) \\
\end{align*}
\]

One can also show that the coherent states \(|\zeta^1, \zeta^2, \zeta^3 >\) (resp. \(< \bar{\zeta}^1, \bar{\zeta}^2, \bar{\zeta}^3|\)) form a complete basis in Fock space \(F_{(0; \beta)}\) (resp. \(F^*_{(0; \beta)}\)). Namely, one can verify the completeness
relation

\[ id_{F(0,\beta)} = \int \prod_{m=1}^{\infty} \frac{d\zeta_m^1 d\zeta_m^2 d\zeta_m^3 d\zeta_m^3}{m^2 m} \exp \left\{ -\sum_{m=1}^{\infty} \left( \frac{K_{ij}(m)m}{m} \zeta_m^i \bar{\zeta}_m^j \right) \right\} \times |\zeta^1, \zeta^2, \zeta^3 > < \bar{\zeta}_1, \bar{\zeta}_2, \bar{\zeta}_3|, \] (C.3)

where \( K_{ij}(n) \) is a \( 2 \times 2 \) matrix satisfying

\[ \sum_{l=1}^{2} K_{il}(n) [a_{lj} n] = \delta_{ij}. \]

One may also derive the following identity

\[
\int \prod_{m=1}^{\infty} \frac{d\zeta_m^1 d\zeta_m^2 d\zeta_m^3 d\zeta_m^3}{m^2 m} \exp \left\{ \frac{1}{2} \sum_{m=1}^{\infty} \lambda_m \left( \zeta_m^1, \bar{\zeta}_m^1, \zeta_m^2, \bar{\zeta}_m^2, \zeta_m^3, \bar{\zeta}_m^3 \right) A_m \right\} \\
+ \sum_{m=1}^{\infty} \left( \zeta_m^1, \bar{\zeta}_m^1, \zeta_m^2, \bar{\zeta}_m^2, \zeta_m^3, \bar{\zeta}_m^3 \right) B_m \right\} \\
= \prod_{m=1}^{\infty} \left( -det A_m \right)^{-\frac{1}{2}} \exp \left\{ \frac{1}{2} \sum_{m=1}^{\infty} \frac{m^2}{m} \det \left( \frac{[a_{ij} m]}{m} \right) B_m^t A_m^{-1} B_m \right\}, \] (C.4)

where \( A_m \) are invertible constant \( 6 \times 6 \) matrices and \( B_m \) are constant 6 component vectors.

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