The dispersion relation of GWs in vacuum, GWs propagate at the speed of light and obey the dispersion relation of gravitons. According to general relativity, the time of writing. Nonetheless, no conclusive evidence of lensed gravitational waves has been found using this method. These methods are capable of extracting the graviton mass from unlensed GW signals. On the other hand, lensed GW signals are expected to be detected in the future. To thoroughly understand the nature of GWs in different astrophysical scenarios, developing a test of the dispersion of lensed GWs becomes increasingly pressing. Moreover, since the amplitude of lensed GWs shows more variations across the frequency than the amplitude and phase of unlensed waves, the wave-form morphology of lensed dispersive GWs may depend on the graviton mass more sensitively than the unlensed GWs. Besides, the amplification introduced by lensing may contribute to an improved measurement accuracy of the graviton mass compared to the unlensed case. Furthermore, the dispersion relation of GWs corresponding to the massive graviton also changes the time delay of waves of different frequencies in different directions, leading to additional features of the resultant lensing pattern. These considerations prompt us to explore measuring the graviton mass from lensed GW signals. Measuring the graviton mass by lensing also makes relevant tests more complete in at least two ways. First, lensing involves the strength of gravity intermediately between the near and far strengths of gravity intermediately between the near and far.
II. LENSING PATTERN OF GRAVITATIONAL WAVES WITH DISPERSION

A. Assumptions and approximations

This work makes a few assumptions:

A.1 Following [39, 42, 47], we assume perfect screening of gravity due to the mass of graviton [57]. In other words, general-relativistic limits are recovered at a length scale shorter than the Compton wavelength of the graviton. This assumption implies that we will ignore the effects on the dynamics of binary black hole mergers due to the graviton mass. In the context of lensing, this assumption implies that at a sufficiently far distance $r$ the Newtonian gravitational potential due to a black hole (point-mass lens) of mass $M$ is given by $-\frac{M}{r}$.

A.2 We focus only on the effects on GW lensing due to the graviton mass. Other consequences of lensing, such as modifications on polarization [40] and phase shift [45, 46], will be omitted. These are acceptable approximations because including these effects will include more contrasting features to graviton-mass approximations because including these effects will shift [45, 46], will be omitted. These are acceptable approximations because including these effects will shift [45, 46],

B. Method

If gravitons have mass, phenomenologically, the dispersion relation of GWs will be altered to [47]

$$\omega^2 = k^2 + m_g^2,$$

(1)

where $m_g$ is the mass of graviton. If $m_g \ll k$, the propagation speed of dispersive GWs that obey this dispersion relation can be approximated by the following equation:

$$v_g(f) \approx 1 - \frac{1}{8\pi^2} \frac{m_g^2}{f^2}.$$  (2)

When propagating in a flat space-time, the dispersive GWs obeying Eq. (1) will acquire a dephasing due to the difference in propagation speeds among different frequencies [47],

$$\Psi_{\text{disp}}(f; m_g) = -\frac{\pi D_0}{\lambda_g} \frac{1}{(1 + z)f},$$  (3)

where $\lambda_g = 1/m_g$ is Compton’s wavelength of the graviton, $D_0$ is the propagation distance from the source to the detector and $z$ is the redshift of the source binary. Thus, in the frequency domain, the waveform of unlensed dispersive GWs is

$$\tilde{h}_{\text{disp}}(f) = \tilde{h}(f) e^{i\Psi_{\text{disp}}(f)},$$  (4)

where $\tilde{h}(f)$ is the original (unlensed) GR waveform (see, e.g., [3, 25, 55] for GR waveform approximants).

When encountering a massive compact object, such as an intermediate-mass black hole, GWs will be lensed. The lensing effect is characterized by the amplification function (or transmission factor) [32, 59], $F$, which is the ratio of lensed-wave amplitude to unlensed-wave amplitude,

$$\tilde{h}_L(f) = F(f)\tilde{h}(f),$$  (5)

where $\tilde{h}_L(f)$ is the lensed waveform and $\tilde{h}(f)$ is the unlensed waveform. Given a lensing geometry, $F(f)$ can be computed by [52, 61, 63]

$$F(f; \bar{\delta}_s) = \frac{D_L D_S}{D_{LS}} \frac{c_s^2}{c_0} \frac{(1 + z_L)}{f} \frac{i}{v_g} \times \int d^2 \bar{\theta}_L \exp \left[ 2\pi i f t_d(\bar{\theta}_L, \bar{\theta}_s) \right],$$  (6)

where $v_g$ is GW propagation speed; $D_L, D_S,$ and $D_{LS}$ are, respectively, the lens-to-observer distance, the source-to-observer distance, and the source-to-lens distance; $z_L$ is the redshift of lens; $\bar{\theta}_s$ is the displacement from optical axis to the source on source plane; $\bar{\theta}_L$ is the displacement from optical axis to lens on lens plane; and $t_d$ is the time delay between the lensed ray and unlensed ray,

$$t_d(\bar{\theta}, \bar{\theta}_s) = \frac{(1 + z_L)}{v_g} \left[ \frac{D_L D_S}{2D_{LS}} |\bar{\theta}_s - \bar{\theta}_l|^2 - \psi(\bar{\theta}_s) \right],$$  (7)

where $\psi(\bar{\theta}_s)$ is the lensing potential. Overall, $t_d$ also depends on $v_g$, $\bar{\theta}_s$, and lens $\bar{\theta}_L$, and $\xi_0$ is a length scale.

We note that the amplification function Eq. (6) depends on $\frac{L}{v_g}$ as a whole. Thus, the amplification function

1 Alternatively, this equation can be interpreted as a definition of the massive graviton which leads to the dispersion of gravitational perturbations. In this work, we refer "the mass of graviton" to $m_g$ defined by Eq. (1).
of GWs of the massive graviton is just that of GWs without dispersion with the following replacement:

\[ f \rightarrow \beta(f) f, \]

where

\[ \beta(f) = \frac{c}{v_g(f)} \approx 1 + \frac{1}{2} \frac{m_g^2}{f^2}. \]

From Eq. (9), we expect that the modifications to the lensing pattern due to the dispersion relation Eq. (1) are manifest for \( m_g \geq 10^{-14}\text{eV} \), corresponding to the energy scale of \( hf \) at \( f = 10\text{Hz} \).

As a proof of principle, in this work we focus on the case of a point-mass lens, such as a black hole. For a point-mass lens, the amplification function can be analytically evaluated as \[ F(f; M_{\text{len}}, y, m_g) \]

\[ = \exp \left( \frac{\pi}{4} w \beta \right) \left( \frac{w}{2} \beta \right)^{\frac{1}{2}} \Gamma \left( 1 - i \frac{w}{2} \beta \right) \nonumber \]

\[ \times F_1 \left( i \frac{w}{2} \beta, 1; i \frac{w}{2} \beta y^2 \right), \]

where \( M_{\text{len}} \) is the redshifted mass of the lens, \( y \) is the impact parameter of lensing, \( \Gamma \) is the (complex) Gamma function, \( F_1 \) is confluent hypergeometric function, and \( w = 8\pi M_{\text{len}} f \) is the dimensionless frequency. The resulting lensed waveform of GWs corresponding to the massive gravitons can be written as

\[ \tilde{h}_L(f; m_g) = F(f; M_{\text{len}}, y, m_g) \tilde{h}(f)e^{i \varphi_{\text{disp}}(f)}. \]

Note that, according to Eq. (2), GWs of different frequencies travel at different speeds. The only constant achromatic speed is the speed of light. Therefore, the effects described by Eq. (11) are not degenerate with a constant change of propagation speed of GWs. Thus, the effects of the massive gravitons can be distinguished upon gravitational-wave detection.

Fig. 1 plots the \( F(f) \) corresponding to the lensing by an intermediate-mass black hole of redshifted lens mass \( M_{\text{len}} \) of 400 \( M_\odot \) and \( y = 0.9 \) for \( m_g = 0 \) (solid blue), \( m_g = 10^{-14}\text{eV} \) (dashed red) and \( m_g = 10^{-22}\text{eV} \) (dotted green) as a function of \( f \). For \( m_g = 10^{-14}\text{eV} \), we find that the modifications of the amplification function is manifest for the low-frequency regime \( f \leq 10^2\text{Hz} \), in which \( \beta \) changes significantly with \( m_g \). As GW frequency increases, the changes of the amplification function due to the alternative dispersion become increasingly less manifest because the ultralativistic limit \( E \approx p \) has been attained. For \( m_g = 10^{-22}\text{eV} \), over \( f \in [10, 10^3]\text{Hz} \), the modifications due to the graviton mass are not visible, as expected because \( |\beta(f) - 1| \sim 10^{-15} \ll 1 \) for \( m_g = 10^{-22}\text{eV} \).

Eq. (11) suggests that GW lensing may help to improve the measurement of \( m_g \) in at least three ways.

R.1 Lensing changes the waveform morphology of the signal. Specifically, because of the modulation by the amplification function [Fig. 1], the amplitude and phase of lensed GWs show more variations across the frequencies than the unlensed GWs. This beating pattern may make the waveform morphology of lensed dispersive GWs depend on the graviton mass more sensitively than unlensed waves.

R.2 Lensing increases the signal-to-noise ratio (SNR).

R.3 The graviton mass modifies the amplification function, making the waveform morphology of lensed dispersive GWs depend on the graviton mass even more sensitively.

However, judging from Fig. 1 for \( m_g \) close to the existing constraints of the graviton mass \( \sim 10^{-22}\text{eV} \) \[8, 15, 20\], the changes of the amplification function due to \( m_g \) are not significant. Therefore, R.3 is unlikely to contribute to any significant improvement. In what follows, we focus on investigating the roles of R.1 and R.2.

As a first step, we compare the similarity of the waveform of both lensed and unlensed dispersive GWs of a given \( m_g \) to the dispersive waves of other \( m_g \). In general, the similarity between two waveforms, \( \tilde{h}_1(f) \) and \( \tilde{h}_2(f) \), can be gauged by the match between \( \tilde{h}_1 \) and \( \tilde{h}_2 \),
defined as
\[ M = \frac{\langle \hat{h}_1 | \hat{h}_2 \rangle}{\sqrt{\langle \hat{h}_1 | \hat{h}_1 \rangle \langle \hat{h}_2 | \hat{h}_2 \rangle}}, \tag{12} \]
where the braket notation denotes the noise-weighted inner product \[^3\].

\[ \langle \hat{h}_1 | \hat{h}_2 \rangle = 4 \Re \int_0^{+\infty} df \frac{\hat{h}_1(f) \hat{h}_2^*(f)}{S_n(f)}, \tag{13} \]
and \( S_n(f) \) is the one-sided power-spectral density of the detector. Throughout this work, we assume GW signals are detected by the Advanced LIGO and Virgo detectors operating at their design sensitivity \[^1\][^2\]. To investigate how sensitive lensed dispersive GWs depend on \( m_g \), we chose
\[ \hat{h}_1(f) = \hat{h}_L(f; m_g = m_g^{\text{inj}}), \]
\[ \hat{h}_2(f) = \hat{h}_L(f; m_g), \tag{14} \]
where \( m_g^{\text{inj}} \) is a given value of \( m_g \) and \( \hat{h}_L \) is the lensed waveform defined by Eq. (11) and Eq. (10). Using this waveform, we have defined a match as a function of \( m_g \) for lensed dispersive GWs. Similarly, we can define a match for unlensed dispersive GWs by replacing \( h_L(f; m_g = m_g^{\text{inj}}) \) to \( h(f; m_g = m_g^{\text{inj}}) \) and \( h_L(f; m_g) \) to \( h(f; m_g) \), where \( h(f; m_g) \) is defined by Eq. (4). As \( \sqrt{\langle \hat{h}_1 | \hat{h}_1 \rangle} \) and \( \sqrt{\langle \hat{h}_2 | \hat{h}_2 \rangle} \) are, respectively, the SNRs of \( \hat{h}_1 \) and \( \hat{h}_2 \), \( M \) does not depend on the SNR of the waveform considered. Alternatively, \( M \) can be viewed as a normalized inner product between the two waveforms, and its magnitude is always smaller than 1. If \( \hat{h}_1 \) and \( \hat{h}_2 \) have more similarity, \( M \) is closer to unity. In particular, if \( \hat{h}_1(f) \propto \hat{h}_2(f) \), meaning that \( \hat{h}_1 \) and \( \hat{h}_2 \) have the same morphology, \( M = 1 \).

For the explicit calculations of \( M \), we consider:

\begin{itemize}
  \item U.1 an unlensed waveform due to a GW150914-like source binary black hole \[^7\] at a luminosity distance of 400 Mpc, whose SNR is 46,
  \item L.1 a lensed waveform of the unlensed waveform by an IMBH of reshifted mass of 400\( M_\odot \) and impact parameter \( y = 0.9 \), whose SNR is 57.
\end{itemize}

This mass of lens is chosen because IMBHs of similar masses are hoped to be discovered by GW lensing \[^4\]. This value of \( y \) is chosen because IMBH lensing is more likely to occur at a larger \( y \) (see the subsequent discussion of the prior of \( y \)). The existing constraints on \( m_g \) by GW detection \[^10\][^13][^20\] suggest that we can probe the existence of massive gravitons of \( \sim 10^{-22} \) eV via GW detection. Thus, we consider \( m_g^{\text{inj}} = 10^{-22} \) eV.

\[ \text{FIG. 2. The match, a function gauging the similarity between waveforms, of lensed GWs and unlensed GWs as a function of the graviton mass. In particular, we compare the similarity between lensed dispersive GWs of \( m_g = 10^{-22} \) eV to unlensed dispersive GWs of other \( m_g \) (solid blue) and the similarity between unlensed dispersive GWs of \( m_g = 10^{-22} \) eV to unlensed dispersive GWs of other \( m_g \). The match of lensed dispersive GWs shows a narrower peak, suggesting that the waveform morphology of lensed dispersive GWs vary more sensitively with \( m_g \). As we shall see, this character of lensed GWs can lead to a better measurement accuracy of \( m_g \) over unlensed GWs.} \]

\[^2\] Note that, throughout this work, SNR is defined with respect to the Advanced LIGO and Virgo detectors at the design sensitivity.
III. PARAMETER ESTIMATION

A. Mock signals

To further investigate how lensing modifications of waveform morphology and SNR may help to improve the measurement of $m_g$, we analyze a mock signal of $L.1$ and $U.1$ that is injected into simulated Gaussian noises assuming the design sensitivity of the Advanced LIGO and Virgo detectors. We also inject $L.2$ a lensed signal which is identical to $L.1$ except the source binary is at 500 Mpc.

The luminosity distance of the source binary of $L.2$ is increased so that the SNR of $L.2$ is the same as that of $U.1$. On the other hand, it is estimated that the Advanced LIGO and Virgo detectors will detect 0.05 IMBH-lensed events per year (or 1 IMBH-lensed event per ~20 years) [43]. At its design sensitivity, the Advanced LIGO and Virgo are expected to detect $\leq 360$ unlensed events per year [22]. Thus, a more fair comparison will be with the posterior of $m_g$ combined across $20 \times 360 \sim 7000$ unlensed signals. In practice, the combined measurement accuracy of $m_g$ will be dominated by the signal with the best measurement accuracy, which depends on the SNR of the signal [65]. Thus, we first simulated a population of ~7000 binary black-hole mergers according to [19], each of which has an SNR of $\geq 10$, approximately the minimum SNR for an event to be detectable by the Advanced LIGO and Virgo detectors [18, 20, 48]. Then, we inject the fourth signal, which is $P.1$ the unlensed signal that has the largest SNR (130) among the simulated 7000 unlensed events.

We represent the measurement of $m_g$ combined across these 7000 simulated signals by the posterior of $m_g$ of $P.1$.

B. Bayesian inference

We denote parameters describing the source binary by $\hat{\theta}_{BBH}$ and parameters describing lensing by $\hat{\theta}_{lens} = (M_{lens}, y)$. By Bayes’ theorem, the posterior of $m_g$, $\hat{\theta}_{lens}$ and $\hat{\theta}_{BBH}$ is given by

$$p(\hat{\theta}_{BBH}, \hat{\theta}_{lens}, m_g | \hat{d}, H, I) \propto p_{BBH}(\hat{\theta}_{BBH} | H, I) p_{lens}(\hat{\theta}_{lens} | H, I) \rho(m_g | H, I)$$

$$\times p(\hat{d} | \hat{\theta}_{BBH}, \hat{\theta}_{lens}, m_g, H, I),$$

where $p_{BBH}(\hat{\theta}_{BBH} | H, I)$, $p_{lens}(\hat{\theta}_{lens} | H, I)$ and $\rho(m_g | H, I)$ are, respectively, the prior of $\hat{\theta}_{BBH}$, $\hat{\theta}_{lens}$ and $m_g$, given the hypothesis $H$ that GWs may exhibit dispersion relation due to the massive gravitons and background information $I$, such as that the signal is lensed, the amplification function [Eq. (10)] and lensing geometry etc. Since $\hat{\theta}_{BBH}$, $\hat{\theta}_{lens}$ and $m_g$ should be independent, we have assumed that their priors are factorized. $p(\hat{d} | \hat{\theta}_{BBH}, \hat{\theta}_{lens}, m_g, H, I)$ is the likelihood that a binary black hole of $\hat{\theta}_{BBH}$ and lens of $\hat{\theta}_{lens}$ will generate detected strain data $\hat{d}$,

$$p(\hat{d} | m_g, \hat{\theta}_{lens}, \hat{\theta}_{BBH}, \hat{\theta}, H, I) \propto \exp \left( -\frac{1}{2} \langle \tilde{n}(f) | \tilde{n}(f) \rangle \right)$$,

$$\tilde{n}(f) = \hat{h}_D(f; m_g, \hat{\theta}_{lens}, \hat{\theta}_{BBH}) \sim \hat{d}_D,$$

(17)

where $\hat{h}_D(m_g, \hat{\theta}_{lens}, and \hat{\theta}_{BBH})$ is the frequency-domain responses corresponding to detector $D$ by the waveform equation Eq. (11).

Following [43], we place a uniform prior for $M_{lens}$. For $y$, we place a prior which is uniform for $y^2 \in (0,1)$ instead of $y$. For $m_g$, we place a prior which is uniform for $\log_{10} m_g \in [-26, -20]$, covering the magnitude of the most updated constraints on $m_g$ [20] by GWs and for us to explore tighter constraints. At last, the marginalized posterior of $m_g$ can be obtained by marginalizing Eq. (16) over $\hat{\theta}_{BBH}$ and $\hat{\theta}_{lens}$.

$$p(m_g | \hat{d}, H, I) = \int d\hat{\theta}_{BBH} \int d\hat{\theta}_{lens} p(\hat{\theta}_{BBH}, \hat{\theta}_{lens}, m_g | \hat{d}, H, I).$$

(18)

C. Mock signals of $m_g = 0$

We first analyze $U.1$, $L.1$, $L.2$ and $P.1$ that are generated by assuming $m_g = 0$. The frequency-domain strains of $U.1$ and $P.1$ are generated using the IMRPhenomPv2 template [31, 53], a phenomenological waveform template calibrated against numerical-relativity simulations, using the LALSimulation library [43]. The simulated unlensed signals contain the inspiral, merger, and ringdown phase. We then map $U.1$ into $L.1$ by multiplying the frequency-domain waveform of $U.1$ by the amplification function Eq. (6). $L.2$ is also generated according to these procedures. When inferring $L.1$ and $L.2$, we use the waveform model of Eq. (11) with the dephasing due to the massive gravitons included and infer $m_g$ along with $\hat{\theta}_{BBH}$ and $\hat{\theta}_{lens}$. For $U.1$ and $P.1$, we infer with the waveform model with $F(f; m_g)$ in Eq. (11) set to be 1 for all frequencies and $M_{lens}$ and $y$ are removed from inference.

The diagonal of Fig. 3 shows the posterior of redshifted lens mass $M_{lens}$, $y$ and $\log m_g$ obtained from $L.1$. The off-diagonal plots show the two-dimensional posterior distributions among the variables. The green vertical lines mark the injected values. The red vertical line marks the $3\sigma$ interval of the marginalized posterior of $\log_{10} m_g$ from $m_g = 10^{-26}$eV. From Fig. 3, we find that the posterior of $\log_{10} m_g$ has no support for $\log_{10} m_g > -23.2$ because our measurement of GWs rules out the possibility of an excessive large $m_g$. From the posterior of $M_{lens}$ and $y$, we conclude that we can accurately estimate
FIG. 3. The corner plot shows the marginalized posterior of the redshifted lens mass $M_{\text{len}}$, $y$ and $\log m_g$ and their correlations, although we also infer the parameters of the source binary together as free parameters. The posteriors are estimated from a mock lensed signal due to a GW150914-like binary lensed by a black hole of redshifted mass $M_{\text{len}}$ of 400$M_{\odot}$ at $y = 0.9$. The green lines denote the injected values for $M_{\text{len}}$ and $y$. The red line on the marginalized posterior of $\log_{10} m_g$ denotes the 3σ confidence interval (CI) from the lower limit of the prior of $\log_{10} m_g$. We conclude that we can bound the graviton mass while accurately measuring the lensing-related parameters.

the lensing-related parameters while testing the graviton mass with lensing. Moreover, judging from Fig. 3 there are no strong correlations between the lensing-related parameters and $m_g$.

We now compare the constraints on $m_g$ obtained from different nondispersive GW signals. Fig. 4 shows the posterior of $\log_{10} m_g$ of $L.1$ (solid blue line), its unlensed counterpart $U.1$ (dashed red line), $L.2$ (dashed dotted black line) and $P.1$ (dashed dotted green). We notice that all posteriors are in step-function shape because the measurement rules out large values for $m_g$, which would produce discernible effects on the waveform. All posteriors correspond to a similar 3σ confidence interval (CI), ranging from $3.3 \times 10^{-24}$ eV to $1.3 \times 10^{-23}$ eV. In particular, $L.1$ yields a constraint (3σ CI) on $m_g$ of $5.5 \times 10^{-24}$ eV, slightly better than the constraint on $m_g$ by $U.1$ corresponding to $1.3 \times 10^{-23}$ eV. At the same SNR, we find that the 3σ CI of $L.2$ is $1.3 \times 10^{-23}$ eV, almost the same as that of $U.1$. Even with large SNR, $P.1$ yields a constraint on $m_g$ of $\sim 3.3 \times 10^{-24}$ eV, slightly better than the constraint by all the other signals. These results conclude that lensing and increasing the SNR do not significantly improve the constraints.

D. Mock signals of $m_g = 10^{-22}$ eV

On the other hand, we find that lensing can help to improve the measurement of $m_g$ from dispersive GWs. Fig. 5 shows the posterior of $\log_{10} m_g$ obtained from $U.1$ (dashed red line), $L.1$ (solid blue line), $L.2$ (dot-dot-dashed line) and $P.1$ (dot-dashed green line) that are generated by assuming $m_g = 10^{-22}$ eV (solid vertical black line). The shaded region illustrates the 3σ CI of the posterior of $L.1$. The embed figure shows the zoomed-in comparison of the posterior of $L.1$ and $P.1$. We notice that, at the same SNR, lensing still improves the measurement accuracy of $m_g$ over its unlensed counterpart. The posterior of $L.2$ shows more support for $m_g$ close to the injected $m_g$ than $U.1$, indicating that the posterior of $L.2$ is more accurate than that of $U.1$. This is because lensing modulates the amplitude and phase of GWs, so that lensed GWs depend on $m_g$ more sensitively, increasing the detectability of dispersive GWs, as indicated by Fig. 2. The posterior of $L.1$ and $P.1$ peaks
FIG. 5. The marginalized posterior of \( m_g \) obtained from L.1 (solid blue), U.1 (dashed red), L.2 (dashed-dotted-dotted black line) and P.1 (dashed-dotted green). The embedded figure shows the zoomed-in comparison of the posterior of L.1 and U.1. The shaped region denotes the 3\( \sigma \) confidence interval of the posterior obtained from the lensed signal. For all signals, we assume \( m_g = 10^{-22} \text{eV} \) (vertical line in black). The lensing geometry is the same as that considered in Section III C. By comparing the posterior of L.2 and U.1, we find that, even at the same SNR lensing modifications of waveform morphology contribute to improving measurement accuracy of \( m_g \).

at a \( m_g \) closer to the injected \( m_g \) because of larger SNR. Nevertheless, L.1 still leads to significant improvement of the measurement accuracy of \( m_g \) to an extent comparable to P.1. From the results of Fig. 5, we find that both the lensing modifications of waveform morphology and amplification can contribute to the improved measurement accuracy of the graviton mass.

IV. CONCLUSIONS

This paper studies the lensing pattern by a point-mass lens of GWs with an isotropic dispersion relation due to the massive gravitons. Although the graviton mass close to the existing constraints leads to no significant effects on the lensing amplification, we find that lensing modifies the waveform morphology of dispersive GWs, making the morphology changes more sensitively with the graviton mass, which helps to improve the measurement of the graviton mass. The improvement can also be further enhanced by the increase of signal-to-noise ratio due to lensing. By detecting a lensed gravitational-wave signal, we can measure the graviton mass with an accuracy comparable with the combined measurement across \( O(10^3) \) unlensed signals. Our work lays the foundation for measuring the graviton mass in the era of detectable lensed GWs, making the existing analyses that focus primarily on unlensed signals more complete.

Other than the improvement of measurement accuracy of the graviton mass, our method enjoys several advantages. First, compared to other proposed methods of testing the speed of GWs by observing lensing \cite{27, 33}, our approach requires no observation of the electromagnetic counterpart(s) of a given event. Therefore, our method is more stand-alone and is easier to be performed. Second, our method is independent of the nature of the source binaries. Although in this paper, we focused on GWs generated by binary black holes, our method can be straightforwardly applied to other types of coalescence, such as binary neutron star coalescence \cite{50}. This flexibility greatly extends the scope of graviton-mass measurement. Lastly, our method makes the test of graviton mass more complete. While the far-field propagation of GWs \cite{12, 47} and near-field behavior of black holes \cite{24} have been proposed to constrain the mass of graviton, our test bridges the intermediate region between these two tests. Along with other tests of general relativity via observing the lensing of GWs (such as \cite{33}), our test demonstrates the strong potential to understand the nature of space-time via observing gravitational-wave lensing.

In this work, we ignore the effects of (i) the change of polarization of GWs due to lensing \cite{10}; (ii) the change of the behavior of the source compact binary due to massive graviton, as is the case in \cite{8, 15}; and (iii) the change of the gravitational field around the lens by the graviton mass. Also, our study focusing on the case of point-mass lens. These ignored effects and the lensing of dispersive GWs of other lens types remain fully explored. If we include these effects in our measurement, the accuracy can be further enhanced.

In the future, we plan to extend our studies to other types of lenses, which may help further improve the measurement accuracy. Our study has thus far focused on the point-mass lens, such as intermediate-mass black holes, which leads to microlensing. In reality, it may be very rare for gravitational waves to be lensed by an intermediate-mass black hole of \( \sim 400 M_\odot \). Moreover, the population properties and lensing rates of intermediate-mass black holes are uncertain. On the other hand, strong lensing due to different types of lenses, such as galaxies or galaxy clusters \cite{50}, are expected to be more common, roughly 1 per \( \sim 600 \) unlensed events at the design sensitivity of LIGO and Virgo \cite{41}. Upon strong lensing, a GW signal may split into multiple images whose properties, such as image position and the arrival time differences, may depend on the graviton mass even more sensitively than the diffraction pattern considered in this work \cite{24, 30}. To extend our test to strong lensing, we need to study the strong lensing of dispersive GWs by lenses with structures, such as galactic lenses, singular isothermal sphere, and other possible extended mass distribution \cite{64}. We would also like to investigate the performance of our test for the detection by proposed space-based detectors, such as the Laser Interferometer Space Antenna \cite{58}, which are capable of exquisite phase measurement and much better constraints. Therefore, in the future, we can measure the graviton mass with
unparalleled accuracy by observing lensed gravitational-wave signals.

ACKNOWLEDGEMENTS

The authors are indebted to valuable discussion among the lensing working group of LIGO. A.K.-W.C. would like to acknowledge Patrick C.K. Cheong, Srashiti Goyal and Shasvath Kapadia for stimulating discussions, Jose Maria Ezquiaga Bravo, Mark H.Y. Cheung, Otto A. Hanuksela, Alvin K.Y. Li and Ignacio Magana for their comments on the manuscript and relevant presentations and Robin S.H. Yuen for his advice about computer programming. A.K.-W.C. was supported by the Hong Kong Scholarship for Excellence Scheme (HKSES). The work described in this paper was partially supported by grants from the Research Grants Council of the Hong Kong (Project No. CUHK 24304317 and CUHK 14306218), The Croucher Foundation of Hong Kong, and the Research Committee of the Chinese University of Hong Kong. This manuscript carries a report number of KCL-PH-TH 2021/41 and LIGO Document number of P2100192-v2.

This research has made use of data, software and/or web tools obtained from the GW Open Science Center (https://www.gwopenscience.org), a service of LIGO Laboratory, the LIGO Scientific Collaboration and the Virgo Collaboration. LIGO is funded by the U.S. National Science Foundation. Virgo is funded by the French Centre National de Recherche Scientifique (CNRS), the Italian Istituto Nazionale della Fisica Nucleare (INFN) and the Dutch Nikhef, with contributions by Polish and Hungarian institutes.

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