ON DEGENERATIONS OF MODULI OF HITCHIN PAIRS

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Abstract. The purpose of this note is to announce certain basic results on the construction of a degeneration of \( \mathcal{M}^H_{X_k}(n, d) \) as the smooth curve \( X_k \) degenerates to an irreducible nodal curve with a single node.

Let \( X_k \) be a smooth projective curve of genus \( g \geq 2 \) over an algebraically closed field \( k \) of characteristic zero and let \( L \) be a line bundle on \( X_k \). A Hitchin pair \((E, \theta)\) is comprised of a torsion-free \( \mathcal{O}_{X_k} \)-module \( E \) together with a \( \mathcal{O}_{X_k} \)-morphism \( \theta : E \to E \otimes L \) called the Higgs structure. Let \( \mathcal{M}^H_{X_k}(n, d) \) denote the moduli space of semistable Hitchin pairs on \( X_k \) with Higgs structure given by the line bundle \( L \). The geometry of Hitchin pairs or Higgs bundles has been extensively studied for over twenty-five years beginning with Hitchin ([4], [5]), Nitsure ([8]), and Simpson ([11], [12], [13]).

More precisely, let \( R \) be a discrete valuation ring with quotient field \( K \) and residue field an algebraically closed field \( k \), for instance \( R = k[[t]] \). Let \( S = \text{Spec} \, R \), and \( \text{Spec} \, K \) the generic point and let \( s \) be the closed point of \( S \). Let \( X \to S \) be a proper, flat family with generic fibre \( X_K \) a smooth projective curve of genus \( g \geq 2 \) and with closed fibre \( X_s \) a irreducible nodal curve \( C \) with a single node \( p \in C \). Assume that \( X \) is regular as a scheme over \( k \). Let \( \mathcal{L} \) be a relative line bundle on \( X \) and assume that \( \deg(\mathcal{L}|_C) > \deg(\omega_C) \), where \( \omega_C \) is the dualizing sheaf on \( C \). Let \((n, d)\) be a pair of integers such that \( \gcd(n, d) = 1 \).
We now make the key definitions (motivated by the constructions in Gieseker [3] and Nagaraj-Seshadri [7]) before we state our principal results. Let $\tilde{C}$ be its normalization and let $\nu : \tilde{C} \to C$ be the normalization map and let $\nu^{-1}(p) = \{p_1, p_2\}$.

**Definition 1.** A scheme $R(m)$ is called a chain of projective lines if $R(m) = \bigcup_{i=1}^{m} R_i$, with $R_i \simeq \mathbb{P}^1$, and if $i \neq j$,

$$R_i \cap R_j = \begin{cases} \text{singleton} & \text{if } |i - j| = 1 \\ \emptyset & \text{otherwise} \end{cases}$$  \quad (1)

**Definition 2.** Let $E$ be a vector bundle of rank $n$ on a chain $R(m)$. Let $E|_{R_i} = \bigoplus_{j=1}^{n} \mathcal{O}(a_{ij})$. Say that $E$ is standard if $0 \leq a_{ij} \leq 1$, $\forall i, j$. Say that $E$ is strictly standard if moreover, for every $i$ there is an index $j$ such that $a_{ij} = 1$.

**Definition 3.** Let $C(m)$ denote the semi-stable curve which is semistably equivalent to $C$, which is obtained as follows: the normalization $\tilde{C}$ is a component of $C(m)$ and further, if $\nu : C(m) \to C$ is the canonical morphism, the fibre $\nu^{-1}(p)$ is a chain $R(m)$ of projective lines of length $m$ cutting $\tilde{C}$ in $p_1$ and $p_2$.

Let $p : X \to S$ be as before a family of smooth curves degenerating to the singular curve $C$. For an $S$-scheme $T$, let $X_T := X \times_S T$.

**Definition 4.** (cf. [6, Definition 3.8]) For every $S$-scheme $T$, a modification is a diagram:

$$\begin{array}{ccc}
X_T^{(mod)} & \xrightarrow{\nu} & X_T \\
p_T \downarrow & & \downarrow p \\
T & &
\end{array}$$

(1) $p_T : X_T^{(mod)} \to T$ is flat,
(2) the $T$-morphism $\nu$ is finitely presented which is an isomorphism when $(X_T)_t$ is smooth,
(3) over each closed point $t \in T$ over $s \in S$, we have $(X_T^{(mod)})_t = C(m)$ for some $m$ and $\nu$ restricts to the morphism which contracts the $\mathbb{P}^1$’s on $C(m)$.

**Definition 5.** (see [7] and [10]) A vector bundle $V$ on $C(m)$ of rank $n$ is called a Gieseker vector bundle if it satisfies the following conditions:

(1) for $m \geq 1$, the restriction $V|_{R(m)}$ is strictly standard,
(2) the direct image $\nu_*(V)$ to be a torsion-free $\mathcal{O}_C$-module.
A Gieseker vector bundle on a modification \( X_T^{(\text{mod})} \) is a vector bundle such that its restriction to each \( C^{(m)} \) in it is a Gieseker vector bundle.

Let \( \mathcal{L}_{\text{mod}} \) be the line bundle on \( X_T^{(\text{mod})} \) defined by \( \mathcal{L}_{\text{mod}} := \nu^*(\mathcal{L}) \). In particular, \( \mathcal{L}_{\text{mod}}|_{R^{(m)}} = O_{R^{(m)}} \) on the chain \( R^{(m)} \) in \( C^{(m)} \).

**Definition 6.** A Gieseker-Hitchin pair on \( X_T^{(\text{mod})} \) is a locally free Hitchin pair \( (V_T, \phi_T) \), with an element

\[ \phi_T \in H^0(T, (p_T)_*(\mathcal{L}_{\text{mod}} \otimes \text{End}(V_T))) , \]

i.e., a morphism \( \phi_T : V_T \to V_T \otimes \mathcal{L}_{\text{mod}} \) satisfying the following:

1. \( V_T \) is a Gieseker vector bundle on \( X_T^{(\text{mod})} \) (Definition 5).
2. For each closed point \( t \in T \) over \( s \in S \), the direct image \( \nu_*(V_t, \phi_t) \) is a torsion-free Hitchin pair on \( X_t = C \).

A Gieseker-Hitchin pair \( (V_T, \phi_T) \) is called stable if the direct image \( \nu_*(V_T, \phi_T) \) is a family of stable Hitchin pairs on \( X_T \) over \( T \) (for the notion of (semi)stability of torsion-free Hitchin pairs, see [12], [13] and [1]).

**Definition 7.** Two families \( (V_T, \phi_T) \) and \( (V'_T, \phi'_T) \) parametrized by \( T \) are called equivalent if there exists a \( X_T \)-automorphism \( \sigma \), i.e.,

\[ \begin{array}{ccc}
X_T^{(\text{mod})} & \xrightarrow{\sigma} & X_T^{(\text{mod})} \\
\downarrow & & \downarrow \\
X_T & \xrightarrow{\nu} & X_T
\end{array} \]

and a line bundle \( \mathcal{D}_T \) on the parameter space \( T \) such that

\[ \sigma^*((V_T, \phi_T) \otimes \mathcal{D}_T) \simeq (V'_T, \phi'_T) . \]

Equivalently, for each closed point \( t \in T \) over \( s \in S \), there exists an automorphism \( g \) of \( C^{(m)} \) which is the identity automorphism on the normalization \( \tilde{C} \), with the property that \( g^*(V_t, \phi_t) \simeq (V'_t, \phi'_t) \).

Let \( \mathcal{M}_S^H(n, d) \) be the functor which associates to every \( S \)-scheme \( T \), the set \( \mathcal{M}_S^H(n, d)(T) \) of the equivalence classes of families of p-semistable torsion-free Hitchin pairs \( (E, \theta) \) on \( X_T := X \times_S T \) with Hilbert polynomial \( P \) given by \( n \) and \( d \), where \( (E_T, \theta_T) \sim (E'_T, \theta'_T) \) if there exists a line bundle \( L_T \) on \( T \) such that \( E_T \simeq E'_T \otimes p_T^*(L_T) \) which sends \( \theta_T \) to \( \theta'_T \otimes \text{id} \).

**Definition 8.** The Gieseker-Hitchin functor \( \mathcal{G}_S^H(n, d)(T) \) is defined as follows: for every \( S \)-scheme \( T \),

\[ \mathcal{G}_S^H(n, d)(T) := \left[ X_T^{(\text{mod})}, (V_T, \phi_T) \right] . \]
i.e., equivalence classes such that \((V_T, \phi_T)\) is a stable Gieseker-Hitchin pair on \(X_T^{(mod)}\) and \(\nu_s(V_T, \phi_T) \in \mathcal{M}_S^H(n, d)(T)\).

Our principal results are the following:

**Theorem 1.**

1. There is a quasi-projective \(S\)-scheme \(\mathcal{G}_S^H(n, d)\) of Gieseker-Hitchin pairs which coarsely represents the functor \(\mathcal{G}_S^H(n, d)\); the \(S\)-scheme \(\mathcal{G}_S^H(n, d)\) is flat over \(S\) and regular over \(k\), with the closed fibre a divisor with (analytic) normal crossing singularities.
2. The generic fibre is isomorphic to the classical Hitchin space \(\mathcal{M}_{X_K}^H(n, d)\).

**Theorem 2.** We have a Hitchin morphism of \(S\)-schemes

\[ g_S : \mathcal{G}_S^H(n, d) \to \mathcal{A}_S \]  

(6)

to an affine space \(\mathcal{A}_S\) over \(S\) which extends the classical Hitchin map on \(\mathcal{M}_{X_K}^H(n, d)\). Furthermore, \(g_S\) is proper and has the following properties:

1. To a general section \(\xi : S \to \mathcal{A}_S\) we can associate a spectral fibered surface \(Y_\xi\) over \(S\) with smooth projective generic fibre \(Y_{\xi, K}\) and whose closed fibre \(Y_{\xi, s}\) is an irreducible vine curve with \(n\)-nodes (cf. [2]).
2. Let \(\delta = d + \deg(L) \frac{n(n-1)}{2}\) and let \(P_{\delta, Y_\xi}\) denote the compactified relative Picard \(S\)-scheme of the spectral fibered surface \(Y_\xi\) over \(S\) (see [2]). Then we have a proper birational morphism

\[ \nu_* : g_S^{-1}(\xi) \to P_{\delta, Y_\xi} \]  

(7)

which is an isomorphism over the generic fibre and this map coincides with the classical Hitchin isomorphism of the Hitchin fibre with the Jacobian of \(Y_{\xi, K}\).
3. The \(S\)-scheme \(g_S^{-1}(\xi)\) gives a new compactification of the Picard variety, whose fibre over \(s\) is a divisor with analytic normal crossing singularities.

The compactified Picard variety \(P_{\delta, Y_{\xi, s}}\) of the irreducible vine curve \(Y_{\xi, s}\) with \(n\)-nodes, has a stratification in terms of the complexity of the torsion-freeness of the sheaves. This can be given as follows:

\[ P_{\delta, Y_{\xi, s}} = \bigsqcup P_{\delta, Y_{\xi, s}}(j), \]  

(8)

where

\[ P_{\delta, Y_{\xi, s}}(j) := \{\eta | \eta \text{ is non-free at exactly } j \text{ nodes}\}. \]  

(9)
In this description the stratum $P_{\delta,Y_{\xi,s}}(0)$ corresponds to the open subset of line bundles on $Y_{\xi,s}$ of degree $\delta$. The fibres of the morphism $\nu_*$ to the compactified Picard variety of the vine curve $Y_{\xi,s}$ gets the following description:

**Theorem 3.** The morphism $\nu_*$ is an isomorphism over the subscheme of locally free sheaves of rank 1 and for each $j$, over the stratum $P_{\delta,Y_{\xi,s}}(j)$ the fibres are canonical toric subvarieties of the wonderful compactification $PGL(j)$ obtained from the closures of the maximal tori of $PGL(j)$. These are toric varieties associated to the Weyl chamber of $PGL(j)$ (see [9]).

For the details of this announcement see [1].

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