MASS AND POLYHEDRA IN ASYMPTOTICALLY HYPERBOLIC MANIFOLDS

XIAOXIANG CHAI

Abstract. Using the upper half space model, we evaluate a component of the hyperbolic mass functional evaluated on a special family of polyhedra extending a formula of Miao-Piubello.

1. Introduction

We say that a three dimensional manifold \((M, g)\) is an asymptotically hyperbolic manifold if outside a compact set \(M\) is diffeomorphic to the standard hyperbolic space \((\mathbb{H}^3, \bar{g})\) minus a geodesic ball and

\[ |e|_{\bar{g}} + |\nabla e|_{\bar{g}} + |\nabla \nabla e|_{\bar{g}} = O(e^{-\tau r}), \]

where \(\nabla\) is the connection on \(\mathbb{H}^3\), \(r\) is the distance function to a fixed point \(o\) and \(\tau > \frac{3}{2}\). We fix \(o\) to be the point \((0, 0, 1)\) in the upper half space model

\[ \bar{g} = \frac{1}{(x_3)^2}((dx_1)^2 + (dx_2)^2 + (dx_3)^2). \]

Then

\[ 2 \cosh r = \frac{1}{\tau^2}((x_1)^2 + (x_2)^2 + (x_3)^2 + 1). \]

See [BP92, Chapter A]. We assume in this article that \(M\) is diffeomorphic to \(\mathbb{H}^3\).

Using the upper half space coordinates, we say that \(\Delta\) is a polyhedron if each of its faces is a plane in the Euclidean sense whatever the metric the upper half space carries.

Let \(V = \frac{1}{3}\) and \(F\) be a face of the polyhedron, let \(\bar{\nu}\) be the \(\bar{g}\)-normal pointing outward of \(\Delta := \Delta_q\), we see that

\[ \bar{\nu} = x^n a^i \partial_i \]

where \(a^i\) are constants and \(a\) is a vector of length one under the Euclidean metric. Due to conformality of \(\bar{g}\) to the Euclidean metric \(\delta\), faces of \(\Delta\) meets at constant angles. Easily,

\[ \partial_i V = x^3 a^i \partial_i \left( \frac{1}{x_3} \right) = -a^3 \frac{1}{x_3} = -a^3 V. \]

Note that \(a^3 \in [-1, 1]\) and the case of \(a^3 = \pm 1\) is used in the works of Jang-Miao [JM21] and the author [Cha21]. When \(|a^3| < 1\), this faces lies in a so-called equidistant hypersurface. Suppose that the face lies in a plane which intersect the \(x^3\) axis at \(x_3 = z_0\), then following [Cha21], the vector \(X = x - z_0 \partial_n\) is tangent to the face and \(\text{div}(x - z_0 \partial_n) = z_0 V\). We calculate the second fundamental form of each face under the metric \(b\). Pick a local coordinate \(y^\alpha\) where \(\alpha = 1, 2\). Using the
conformality to $\delta$,

\[
\bar{A}_{\alpha\beta} = \langle \partial_{\alpha}, \partial_{\beta} \rangle \delta x^3 \partial_3 \frac{1}{(x^3)^2} \\
= -a^3 \frac{1}{|\partial_3|^2} \langle \partial_{\alpha}, \partial_{\beta} \rangle \delta \\
= -a^3 \langle \partial_{\alpha}, \partial_{\beta} \rangle b.
\]

(2)

Each face $F$ is umbilic and the mean curvature is then $\bar{H} = -2a^3$. Euclidean spheres are also umbilic in the hyperbolic metric, however, they do not satisfy the condition (1). Obviously, these discussions work for higher dimension.

The mass integrand for an asymptotically hyperbolic manifold (See [CH03]) is

\[
U = V \text{div} e - V d(tr_b e) + tr_b edV - e(\bar{\nabla} V, \cdot).
\]

We assume that for a family of polyhedra indexed by $q$ the mass satisfies the following

\[
M(V) = \int_{\partial \Delta_q} \bar{\nabla} \bar{\nu}_i d\bar{\sigma} + o(1),
\]

where $\bar{\nu}$ is the $b$-normal to the face of $\Delta_q$ and $d\bar{\sigma}$ is the two dimensional volume element.

Such a family is easy to find. For example, according to [CH03] or [Mic11], if each polyhedron of the family is enclosed by a geodesic sphere and enclose another geodesic sphere, radius of each sphere goes to infinity as $\Delta_q$ exhaust the manifold $M$, then such a family provides an example. Let $E_q$ be the all edges of $\Delta_q$, $\alpha$ be the dihedral angle for by neighboring faces along $E_q$. We denote by $dv, d\sigma$ and $d\lambda$ respectively the three, two and one dimensional volume element. We put a bar over a letter to indicate the quantity is calculated with respect to the background metric $\bar{g}$.

**Theorem 1.** Assuming each dihedral angle satisfies the bound $\sin \bar{\alpha} \geq c > 0$, then the mass $M(V)$ is

\[
M(V) = -\int_{\partial \Delta_q} 2V(H - \bar{H}) d\bar{\sigma} + 2 \int_{E_q} V(\alpha - \bar{\alpha}) d\bar{\lambda} \\
+ \int_{\partial \Delta_q} O(\cosh^{-2r+1} r) d\bar{\sigma} + \int_{E_q} O(\cosh^{-2r+1} r) d\bar{\lambda} + o(1).
\]

As suggested by Miao [Min20], this type of formula may be used to promote the Gromov dihedral rigidity [Gro18] to an integrated form. However, in the asymptotically flat case, one can perturb graphically of a face and use Taylor expansion to the third order to see a counterexample.

Also, the Miao-Piubello type formula (3) leads us to do comparison between a Riemannian polyhedron with the polyhedron whose faces are realized as Euclidean planes in the upper half space model. In particular, one could consider the Gromov type dihedral rigidity. The case when the model is a cone type polyhedron with base faces lying on a horosphere is possible using the method developed by [Li20]. Instead of minimal surface with capillary angle condition, one uses constant mean curvature two surfaces with capillary angle condition.
2. Example and proof

Before we show the proof of Theorem 1, we calculate an easy example other than the parabolic cylinder [JM21] to illustrate that the terms \( \int_{\partial \Delta_q} \cosh^{-2\tau+1} r d\sigma \) and \( \int_{E_q} \cosh^{-2\tau+1} r d\bar{\lambda} \) can be \( o(1) \). We assume the base face \( B \) is a regular \( n \)-side polygon lies at the horosphere \( \{ x^3 = \varepsilon \} \) and centered at \((0, 0, \varepsilon)\) for \( \varepsilon \) small, the apex is \((0, 0, \varepsilon^{-1})\). The distance between the vertex of the polygon to the \( x^3 \)-axis is \( \rho(\varepsilon) \). We assume that

\[
\rho(\varepsilon) = o(\varepsilon^{-2\tau})
\]

and \( \rho(\varepsilon) \to \infty \) as \( \varepsilon \to 0 \). We use aliases \( x = x^1, y = x^2, z = x^3 \) for the coordinates.

First, we deal with integrals on the edges. There are two type edges. By rotational symmetry, we can assume one edge is \( E_1 = \{(0, \frac{1-\varepsilon z}{1-\varepsilon^2 \rho}, z) : z \in [\varepsilon, \varepsilon^{-1}]\} \).

The length element \( d\bar{\lambda} \) on \( E_1 \) is

\[
z^{-1} \sqrt{1 + \frac{\varepsilon^2 \rho^2}{(1-\varepsilon^2)^2}} d\varepsilon \leq \sqrt{2z}^{-1} \max\{C\varepsilon \rho, 1\}
\]

since we are take \( \varepsilon \to 0 \). So

\[
\int_{E_1} \cosh^{-2\tau+1} r d\bar{\lambda} \leq C \max\{\varepsilon \rho, 1\} \int_{\varepsilon}^{\varepsilon^{-1}} z^{-2+2\tau} (z^2 + 1 + (\frac{1-\varepsilon z}{1-\varepsilon^2 \rho})^2)^{-2\tau+1} dz.
\]

We divide the integral into three parts. On \([\varepsilon, 1]\), we have

\[
\max\{C\varepsilon \rho, 1\} \int_{\varepsilon}^{1} z^{-2+2\tau} (z^2 + 1 + (\frac{1-\varepsilon z}{1-\varepsilon^2 \rho})^2)^{-2\tau+1} dz
\]

\[
\leq \max\{C\varepsilon \rho, 1\} (\frac{1}{1-\varepsilon^2 \rho})^{-2\tau+2} \int_{\varepsilon}^{1} z^{-2+2\tau} dz
\]

\[
= o(1)
\]

(4)

since \( \tau > \frac{3}{2} \). On \([1, \frac{1}{2}^{-1}]\), for \( 1 < a < 2\tau - 1 \),

\[
\max\{C\varepsilon \rho, 1\} \int_{1}^{1/2} z^{-2+2\tau} (z^2 + 1 + (\frac{1-\varepsilon z}{1-\varepsilon^2 \rho})^2)^{-2\tau+1} dz
\]

\[
= \max\{C\varepsilon \rho, 1\} \int_{1}^{1/2} z^{-2+2\tau-(2\tau-2+a)} (z^2 + 1 + (\frac{1-\varepsilon z}{1-\varepsilon^2 \rho})^2)^{-2\tau+1+\frac{1}{2}(2\tau-2+a)} dz
\]

\[
\leq C \max\{C\varepsilon \rho, 1\} \rho^{-4\tau+2+2\tau-2+a} \int_{1}^{1/2} z^{-a} dz
\]

(5)\leq C \max\{C\varepsilon \rho, 1\} \rho^{-2\tau+a}
which is $o(1)$ as $\rho(\varepsilon) \to \infty$. On $[\varepsilon^{-1}/2, \varepsilon^{-1}]$, we have that
\[
\max\{C\varepsilon\rho, 1\} \int_{\varepsilon^{-1}/2}^{\varepsilon^{-1}} z^{-2+2\tau}(z^2 + 1 + (1 - \varepsilon \rho)^2)^{-2\tau+1} dz \\
\leq \max\{C\varepsilon\rho, 1\} \int_{\varepsilon^{-1}/2}^{\varepsilon^{-1}} z^{-2+2\tau-4\tau+2} dz \\
\leq \max\{C\varepsilon\rho, 1\} \varepsilon^{2\tau-1}.
\]
Requiring that $\rho(\varepsilon) = o(\varepsilon^{-2\tau})$ this term is $o(1)$. By rotational symmetry again, the other type edges we can just consider
\[
E_2 = \{(\rho \cos \frac{\pi}{n}, y, \varepsilon) : y \in [-\rho \sin \frac{\pi}{n}, \rho \sin \frac{\pi}{n}]\}.
\]
The integral of $\cosh^{-2\tau+1} r$ on $E_2$ is
\[
\int_{E_2} \cosh^{-2\tau+1} r d\lambda \\
\leq C \int \rho \sin \frac{\pi}{n} \varepsilon^{-1} \cdot \varepsilon^{-1+2\tau}(y^2 + \rho^2 \cos^2 \frac{\pi}{n} + \varepsilon^2 + 1)^{-2\tau+1} dy \\
\leq C\varepsilon^{-2+2\tau} \int_R (x^2 + 1)^{-2\tau+1} dy = o(1)
\]
as $\varepsilon \to 0$.

For the face $F$ of $\Delta$ lying on $\{z = \varepsilon\}$, we have
\[
\int_F \cosh^{-2\tau+1} r d\tilde{v} \\
\leq C \int F \varepsilon^{-2-1+2\tau}(x^2 + y^2 + \varepsilon^2 + 1)^{-2\tau+1} dxdy \\
\leq C\varepsilon^{2\tau-3} \int_0^{\rho(\varepsilon)} (s^2 + 1)^{-2\tau+1} sds = o(1)
\]
as $\varepsilon \to 0$. Now we consider a side face, we use the symbol $S$. We use the Euclidean distance $\xi$ of a point in side edges to the $z$-axis and $z$ to parametrized $S$. First, $\xi = \frac{1 - \varepsilon \rho}{\varepsilon^{2\tau}}$, then
\[
S = \{ (\xi \cos \frac{\pi}{n}, y, z) : y \in [-\xi \sin \frac{\pi}{n}, \xi \sin \frac{\pi}{n}], z \in [\varepsilon, \varepsilon^{-1}] \}.
\]
It is easy to show that
\[
\int_S \cosh^{-2\tau+1} r d\tilde{\sigma} \\
= \sqrt{1 + \frac{\varepsilon^2 \rho^2}{(1-\varepsilon^2)^2}} 2^{2\tau-1} \int_{\xi}^{\varepsilon^{-1}} \int_{-\xi \sin \frac{\pi}{n}}^{\xi \sin \frac{\pi}{n}} \xi^{2\tau-3}(\xi^2 \cos^2 \frac{\pi}{n} + y^2 + 1 + z^2)^{-2\tau+1} dy dz \\
\leq \max\{C\varepsilon\rho, C_1\} \int_{\xi}^{\varepsilon^{-1}} \xi^{2\tau-3}(\xi^2 \cos^2 \frac{\pi}{n} + 1 + z^2)^{-2\tau+1} dz
\]
Similar to (4) and (5), we have that
\[
\max\{C\varepsilon\rho, C_1\}(\int_{\varepsilon}^{1} + \int_{1}^{\varepsilon^{-1}/2}) = o(1).
\]
For the integral over \([\varepsilon^{-1}/2, \varepsilon^{-1}]\), we absorb \(\xi \sin \frac{\pi}{n}\) by \(\xi^2 \cos^2 \frac{\pi}{n} + z^2 + 1\) and

\[
\max\{C\varepsilon \rho, C_1\} \int_{\varepsilon^{-1}/2}^{\varepsilon^{-1}} \xi \sin \frac{\pi}{n} z^{2\tau-3}(\xi^2 \cos^2 \frac{\pi}{n} + 1 + z^2)^{-2\tau+1} dz
\]

\[
\leq \max\{C\varepsilon \rho, C_1\} \int_{\varepsilon^{-1}/2}^{\varepsilon^{-1}} z^{2\tau-3}(\xi^2 \cos^2 \frac{\pi}{n} + 1 + z^2)^{-2\tau+1+\frac{1}{2}} dz
\]

\[
\leq \max\{C\varepsilon \rho, C_1\} \int_{\varepsilon^{-1}/2}^{\varepsilon^{-1}} z^{2\tau-3-4\tau+2+1} dz
\]

\[
\leq \max\{C\varepsilon \rho, C_1\} 2^{2\tau-1} = o(1).
\]

where we have required that \(\rho(\varepsilon) = o(\varepsilon^{-2\tau})\).

**Proof of Theorem 4.** We have on each face \(F\) of \(\Delta_q\),

\[
2V(H - \bar{H}) = -U^i \bar{\nu}_i - \text{div}_F(VX) + O(e^{-2\tau+r})
\]

where \(X\) is the vector field dual to the 1-form \(e(\bar{\nu}, \cdot)\) with respect to the metric \(b|_F\).

This is an easy consequence of [JM21, (2.5)] that

\[
\mathbb{U}(\bar{\nu}) = 2V(\bar{H} - H) - \text{div}_F(VX) + [(\text{tr}_b e - e(\bar{\nu}, \nu))dV, \bar{\nu}] - V(\bar{A}, e|_b) + O(e^{-2\tau+r}).
\]

From (1) and (2), we obtain the desired formula (3). From (4), we have that

\[
\int_{\partial\Delta_q} U^i \bar{\nu}_i d\tilde{\sigma} = \int_{\partial\Delta_q} [-2V(H - \bar{H}) - \text{div}_{\partial\Delta_q}(VX)]d\tilde{\sigma} + o(1)
\]

On each face \(F\), using divergence theorem

\[
\int_F \text{div}_F(VX) d\tilde{\sigma} = \int_{\partial F} V(e(\bar{\nu}, \bar{n}))d\tilde{\lambda},
\]

where \(\bar{n}\) is the \(b\)-normal to \(\partial F\) in \(F\). On the edge \(F_A \cap F_B\), the contribution is

\[
\int_{F_A \cap F_B} V[e(\bar{\nu}_A, \bar{n}_A) + e(\bar{\nu}_B, \bar{n}_B)]d\tilde{\lambda}.
\]

Let \(g_{ij} = g(\partial_i, \partial_j)\), we have that

\[
\varepsilon_{ij} := (x^n)^2 e_{ij} = (x^n)^2 g_{ij} - (x^n)^2 \bar{g}_{ij} = (x^n)^2 \bar{g}_{ij} - \delta_{ij} = O(e^{-\tau})
\]

The \(g\)-normal to the face \(F_A\) is then expressed as

\[
\nu_A = \frac{g^{ij} a_i a_j}{\sqrt{g^{kl} a_k a_l}}.
\]

We write

\[
\cos \theta = g(\nu_A, \nu_B) = g_{ij} \bar{g}_{ij} = \frac{1}{2} (g^{kl} a_k a_l)^{-1/2} \frac{1}{2} (g^{kl} a_k a_l)^{-1/2}
\]

\[
= a_i b_j \left(\frac{x^n}{2}\right)\left(\frac{x^n}{2}\right) a_k a_l^{-1/2} (x^n)^2 a_k a_l^{-1/2}.
\]
Up to here, it follows from the same lines as in [MP21, (3.9)-(3.30)] to show that
\[
\int_{F_A \cap F_B} V[e(\bar{\nu}_A, \bar{n}_A) + e(\bar{\nu}_B, \bar{n}_B)]d\tilde{\lambda}
= \int_{F_A \cap F_B} V(\bar{\alpha} - \alpha) + \int_{F_A \cap F_B} V \cosh^{-2\tau} r d\tilde{\lambda}.
\]
Therefore,
\[
M(V) = -\int_{\partial \Delta_q} 2V(H - \bar{H})d\tilde{\sigma} + 2\int_{E_q} V(\alpha - \bar{\alpha})d\tilde{\lambda}
+ \int_{\partial \Delta_q} \cosh^{-2\tau+1} r d\tilde{\sigma} + \int_{E_q} \cosh^{-2\tau+1} r d\tilde{\lambda} + o(1),
\]

obtaining the theorem. 

**References**

[BP92] Riccardo Benedetti and Carlo Petronio. *Lectures on Hyperbolic Geometry*. Universitext. Springer Berlin Heidelberg, Berlin, Heidelberg, 1992.

[CH03] Piotr T. Chruściel and Marc Herzlich. The mass of asymptotically hyperbolic Riemannian manifolds. *Pacific journal of mathematics*, 212(2):231–264, 2003.

[Cha21] Xiaoxiang Chai. Asymptotically hyperbolic manifold with a horospherical boundary. *ArXiv:2102.08889 [gr-qc]*, 2021.

[Gro18] Misha Gromov. Dirac and Plateau Billiards in Domains with Corners. *ArXiv:1811.04318 [math]*, 2018.

[JM21] Hyun Chul Jang and Pengzi Miao. Hyperbolic mass via horospheres. *ArXiv:2102.01036 [gr-qc]*, 2021.

[Li20] Chao Li. A polyhedron comparison theorem for 3-manifolds with positive scalar curvature. *Invent. Math.*, 219(1):1–37, 2020.

[Mia20] Pengzi Miao. Measuring mass via coordinate cubes. *Comm. Math. Phys.*, 379(2):773–783, 2020.

[Mic11] B. Michel. Geometric invariance of mass-like asymptotic invariants. *Journal of Mathematical Physics*, 52(5):52504, 2011.

[MP21] Pengzi Miao and Annachiara Piubello. Mass and Riemannian Polyhedra. *ArXiv:2101.02693 [gr-qc]*, 2021.

Korea Institute for Advanced Study, Seoul 02455, South Korea

*Email address: xxchai@kias.re.kr*