THE CONCEPT OF A SET

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ABSTRACT. Metaphysical interpretations of set theory are either inconsistent or incoherent. The uses of sets in mathematics actually involve three distinct kinds of collections (surveyable, definite, and heuristic), which are governed by three different kinds of logic (classical, intuitionistic, and minimal). A foundational system incorporating this analysis and based on the principles of mathematical conceptualism [8] accords better with actual mathematical practice than Zermelo-Fraenkel set theory does.

1. Metaphysical interpretations

1.1. Sets in ordinary language. Elementary introductions to set theory tend to give the impression that the concept of a set is trivial, something with which we are already thoroughly familiar from everyday life. We may be told that such things as a flock of birds, a deck of cards, or a pair of apples are examples of sets.

This immediately seems strange because sets in the mathematical sense are supposed to be abstract objects not existing in space and time, whereas it is hard to believe that a simple assertion about, say, a flock of birds carries any significant metaphysical content.

Slater ([7], Section II) analyzed the way collective expressions are actually used in ordinary language and showed that they indeed express no metaphysical content. A deck of cards exists in space and time; it is a physical object composed of cards in the same way that a house is made out of bricks, something philosophers call a “mereological sum”. It is not a set of cards in the mathematical sense, with cards as elements. The same is true of such things as a bunch of grapes, a herd of cows, etc. (You don’t eat the elements of a bunch of grapes, you eat the bunch of grapes. And “the fact that such collections are mereological sums is also shown by the fact that shoals, herds, packs, tribes, and the like, are located and can move around in physical space, just like their members” ([7], p. 61).)

In contrast, the word “pair” in the phrase “pair of apples” is a numerical measure word; to say that you ate a pair of apples merely tells how many apples you ate. It is analogous to the word “half” in the phrase “half a loaf of bread”. And “in ‘There is a half of a loaf’ there is obviously no reference to something other than bread; there is not, in addition, reference to one of a range of mysterious, further objective entities, ‘halves’, ‘quarters’, ‘parts’, etc. There is merely a specification of how much of a loaf there is” ([7], p. 62).

The idea that the mathematical concept of a set is obvious and in no need of any special explanation is not correct. This is sometimes noted in the philosophical literature on mathematical foundations, but it seems the conclusion that is usually
drawn is not that the set concept is bankrupt but rather that it merely needs some
more sophisticated interpretation.

1.2. Inconsistency versus incoherence. But however we might interpret set
language, it is clear that a straightforward belief in the actual existence of a well-
defined objective world of sets immediately gives rise to the classical paradoxes
of naive set theory. Thus, naive metaphysical interpretations of set language are
inconsistent. Is there any cogent alternative metaphysical interpretation which
evades the paradoxes?

The standard answer is that this is to be done in terms of the iterative conception
of sets, according to which sets are not to be thought of as arbitrary collections, but
rather as being hierarchically built up from the empty set via the two operations of
(1) forming a subset of a given set and (2) collecting together all subsets of a given
set, i.e., forming its power set. This suggestion can be interpreted in two ways. On
one reading, it marks a distinction between two concepts, “set” and “collection”,
with sets effectively being just those collections which appear in the cumulative
hierarchy. But this is nothing more than a change in terminology, so that the
classical paradoxes about sets are not defused but simply become paradoxes about
collections. Although the paradoxes no longer directly invalidate set theory, they
now do so indirectly by showing that the notion of a collection, in terms of which
sets are defined, is itself inconsistent. Thus this interpretation of the set concept is
incoherent.

The iterative conception can also be understood in a different way, not as differ-
entiating sets out of a background universe of collections, but rather as clarifying
the concept of a collection as something which must be in some sense “formed” out
of elements that in some sense exist “before” it does. Exactly what this means for
abstract objects not existing in space and time is hard to pin down, so its status
as a clarification is questionable, to say the least. The point is apparently that the
existence of a set has something to do with its being constructible, in some obscure
sense. What makes this interpretation really incoherent is the fact that each stage
of the process by which sets are supposed to be built up involves forming a power
set, which in the case of infinite sets is an absolutely non-constructive operation. In
other words, the set-theoretic universe is thought of as being built up in an iterative
process, yet we pass from each stage of this process to the next in a completely
non-constructive way. The power set of the preceding level is not constructed in
any sense whatever, it simply appears.

The source of this incoherence is obviously the fact that axiomatic set theory was
developed not in a philosophically principled way, but rather in an opportunistic
attempt to preserve as much naive set theory as possible without admitting any
obvious inconsistencies. This was accomplished by, in effect, adding just enough
constructivity to the naive picture to avoid the standard paradoxes. The result is
a nonsensical jumble of constructive and non-constructive ideas.

If the temporal (“in stages”) and constructive metaphors, which are highly du-
bious anyway in light of the purported abstract nature of sets, are left out of the
iterative conception, then it loses its force as a resolution of the paradoxes. One
is simply left with a bald assertion that sets are layered in a hierarchy with no
explanation as to why this is so. And we still have the problem of justifying the
power set axiom.
It seems that the fundamental error in all metaphysical interpretations of set theory is the reification of a collection as a separate object, and that this is done as a result of a series of grammatical confusions [7]. This reification is the ultimate source of the paradoxes, and once one accepts it there is no cogent way to avoid them.

1.3. The philosophical dilemma. The basic dilemma faced by philosophers of mathematics was summarized well by Hazen: “On the one hand, the notion of a set is central to modern mathematics . . . On the other hand, we seem unable to obtain for set theory the kind of metaphysical or epistemological legitimacy that would come from a characterization of the sort of entity a set is or an account of how we become acquainted with them” ([2], p. 173).

Because the task of legitimizing axiomatic set theory has seemed so crucial, a great deal of effort has been expended towards this goal. Various authors have either proposed to defend set theory on its own (metaphysical) terms, or tried to justify it in some weaker sense by more roundabout methods. However, it seems fair to say that none of these attempts has met with general approval.

But the dilemma rests on a faulty assumption. Although the language of set theory is used throughout mathematics in an elementary way, the actual discipline of axiomatic set theory is not central to modern mathematics. In fact it is quite peripheral, making only occasional and relatively minor contact with mainstream areas. Indeed, virtually all modern mathematics outside set theory itself can be carried out in formal systems which are far weaker than Zermelo-Fraenkel set theory and which can be justified in very concrete terms without invoking any supernatural universe of sets (see [9] or [12], and also [10] for borderline cases). Thus, axiomatic set theory is not indispensable to mathematical practice, as most philosophers of mathematics have apparently assumed it to be. It is one arena in which mathematics can be formalized, but it is not the only one, nor even necessarily the best one (see Section 3.3 below).

The point of view taken here could be expressed by moving the word “seem” in the passage quoted above: we are unable to obtain metaphysical or epistemological legitimacy for set theory, but it only seems that sets are central to modern mathematics. I discuss this issue further in [11].

2. Collections

2.1. Proxies for sets. We must reject as nonsensical the idea that sets literally exist as some sort of mysterious non-physical entities. But this does not mean that we cannot make sense of any kind of set language. For example, we do not need to be set-theoretic platonists to grasp the meaning of the statement that every nonempty set of natural numbers has a least element. The same idea could be expressed by saying that if either a 0 or a 1 appears in each cell of a one-way infinite tape, and at least one cell contains 1, then there is a first cell that contains 1. This formulation is clumsier than the first one and it introduces extraneous notions (the symbols 0 and 1, the image of an infinite tape), but it does show us that we can understand what is expressed by the first statement in a way that does not require us to think of sets of natural numbers as actual objects.

This is somewhat analogous to the difference between “two plus three is five” and “two apples plus three apples is five apples”. Here too the second formulation is clumsier and unnecessarily specific. But again, it shows that we do not need to
assume the literal existence of a platonic world of numbers in order to make sense of basic arithmetic. In other words, we do not gain any substantive mathematical content by supposing that number words literally refer to some sort of abstract objects.

Thus, the idea is that we can find meaning in at least some language involving fictional “abstract objects” like numbers and sets by using actual physical objects (apples) or possible physical objects (infinite tapes) as proxies for them. Of course philosophical questions can be raised about the idea of an infinite tape, but the point is that we avoid the far more serious difficulties attaching to set theory. The naive notion of an infinite tape is not obviously paradoxical, and Hazen’s questions (What sort of entities are they? How do we become acquainted with them?) hardly have the same force they have against abstract sets.

The moderately substantial set-theoretic system used in [9] could be interpreted in a similar way, in this case using formal expressions as proxies for sets. In some ways this example is even less problematic because the structure in question, $J_\alpha$, is countable and the formal expressions are finite. $J_2$ can be thought of as a toy model for the Cantorian universe, and despite its small size it is, surprisingly, rich enough to permit the development of ordinary mainstream mathematics, even up to subjects like measure theory and functional analysis. This was shown in some detail in [9].

Because it is so concrete, if we agree to practice mathematics only within $J_2$ then philosophical concerns about sets largely disappear. However, although $J_2$ is an attractive structure and is quite adequate for ordinary mathematics, we can easily extend its construction to obtain richer toy universes. Reasoning about $J_\alpha$ for arbitrary $\alpha$ then reintroduces philosophical issues because we cannot model the class of ordinals with a single physically instantiated structure. This forces us to come to grips with the general notion of a collection outside of any particular system of proxies.

2.2. Types of collections. There are three essentially different kinds of collections. First, we have collections that could be instantiated in some possible world by a physical structure in which the elements of the collection appear as discrete components. I call this kind of collection surveyable. In principle we could exhaustively search through all the individuals in such a collection.

What counts as a possible physical structure is obviously open to debate. According to finitism it is precisely the finite collections that are surveyable. In classical set theory, on the other hand, a collection is surveyable if and only if it is a set. It might take a transfinite amount of space to contain the elements of a set, or a transfinite amount of time to survey them, but this would be seen as falling within the realm of logical possibility.

The next kind of collection is definite. This is a much broader category. We no longer require that we be able to search through all the individuals in the collection; instead, we ask only that the statement that an individual belongs to the collection have a fixed well-defined meaning. For example, the prime numbers are finitistically definite, since we can finitistically test any natural number for primality. Thus the statement that a number is prime has, finitistically, a fixed well-defined meaning. In classical set theory “definite collection” is synonymous with “class”.
Definite collections generally cannot be physically modelled in the way that surveyable collections can. So the most natural proxies for definite collections will be the predicates that define them.

We do not assume that membership in a definite collection is decidable. For example, consider the values of $k$ such that every graph with chromatic number $k$ has a $k$-clique minor. On the face of it we cannot finitistically test whether a given natural number has the stated property, because determining the truth of this condition for any given value of $k$ apparently requires us to quantify over infinitely many graphs. But the condition still has a fixed well-defined meaning for each value of $k$. As we build up a successively larger repertoire of natural numbers our understanding of what the condition means does not change. Thus, this collection is finitistically definite but (on its face) not decidable. In classical set theory we are in the same position with regard to classes that are defined by means of conditions that involve quantification over the entire set-theoretic universe (or over any proper class).

Our notion of definite collections may be contrasted with Dummett’s notion of “indefinitely extensible” concepts [1]. One trivial difference is that we include surveyable collections among definite collections, whereas Dummett’s indefinitely extensible concepts are supposed to be complementary to definite concepts. But any collection that is, in the above sense, definite but not surveyable would have to have the property that it is not exhausted by any surveyable subcollection. That is, given any surveyable collection that is contained in a definite collection which is not surveyable, there would have to be a new individual that belongs to the latter but not the former. This looks similar to Dummett’s condition that “if we can form a definite conception of a totality all of whose members fall under the concept, we can, by reference to that totality, characterize a larger totality all of whose members fall under it” ([1], p. 441). However, Dummett seems to require that there be an explicit prescription for producing these new elements and we do not assume this. Also, Dummett apparently takes all concepts to be decidable, which, as I just indicated, I do not do.

In the case of a collection that is definite but not surveyable, we can go beyond any surveyable subcollection, but the collection as a whole is still well-defined. The last kind of collection, which I call heuristic, goes one step further: any definite subcollection can be extended. For example, the collection of all well-defined predicates is heuristic because the property of being a well-defined predicate does not itself have a fixed well-defined meaning. To see this let $P$ be any well-defined predicate with the property that $P(x)$ implies that $x$ is a well-defined predicate. Then let $Q$ be the predicate “$P(x)$ and $x$ does not hold of itself”. Since $P$ is well-defined so is $Q$, but if $P$ holds of $Q$ then we immediately get a contradiction: if $Q$ holds of itself then it does not hold of itself, and vice versa. We cannot imagine a state of affairs in which the assertion that $Q$ holds of itself has a truth value, so $Q$ cannot be a well-defined predicate, contradicting the assumption that $P$ holds of $Q$. So $P$ does not exhaust all well-defined predicates. This shows that any definite collection of well-defined predicates can be extended. Other basic examples of heuristic collections include the collection of all valid proofs and the collection of all valid definitions.

By definition, we cannot give a precise meaning to the notion of a heuristic collection. But we would still like to be able to reason about — in particular, to quantify
over — all well-defined predicates, or all valid proofs, or all valid definitions. (For instance, we want to be able to say: if there is a valid proof of $A$ and a valid proof of $B$ then there is a valid proof of $A \land B$.) The classical semantic paradoxes might be taken to show that this is an unreasonable wish. But in fact these paradoxes only arise when heuristic concepts are treated as if they were definite. Once the distinction between definite and heuristic concepts is recognized the semantic paradoxes are easily resolved.

I will say that a variable is surveyable, definite, or heuristic according to whether it is supposed to range over the individuals belonging to a surveyable, definite, or heuristic collection.

In Section 3 I will make a case that there is a reasonable version of the power set operation for surveyable and definite collections, but that the result of this operation when it is applied to a surveyable collection in general is merely definite, and the result when it is applied to a definite collection in general is merely heuristic. There is no meaningful version of the power set operation for heuristic collections.

2.3. Systems of logic. Legitimate forms of logical reasoning vary depending on whether the language one is using expresses surveyable, definite, or heuristic properties. I will argue that the appropriate corresponding forms of logical reasoning are respectively classical, intuitionistic, and minimal. Similar suggestions have been made before; most notably, Dummett has proposed that intuitionistic logic should be used when reasoning about the individuals falling under an indefinitely extensible concept.

The differences between the three logics are most elegantly expressed in terms of natural deduction. To present the logical rules as simply as possible we introduce a symbol $\bot$ for “falsehood” and regard $\neg A$ as an abbreviation of $A \rightarrow \bot$. Informally, the rules for classical logic are:

1. Given $A$ and $B$ deduce $A \land B$; given $A \land B$ deduce $A$ and $B$.
2. Given either $A$ or $B$ deduce $A \lor B$; given $A \lor B$, a proof of $C$ from $A$, and a proof of $C$ from $B$, deduce $C$.
3. Given a proof of $B$ from $A$ deduce $A \rightarrow B$; given $A$ and $A \rightarrow B$ deduce $B$.
4. Given $A(x)$ deduce $(\forall x)A(x)$; if the term $t$ is free for $x$, given $(\forall x)A(x)$ deduce $A(t)$.
5. If the term $t$ is free for $x$, given $A(t)$ deduce $(\exists x)A(x)$; if $y$ does not occur freely in $B$, given $(\exists x)A(x)$ and a proof of $B$ from $A(y)$ deduce $B$.
6. Given $(A \rightarrow \bot) \rightarrow \bot$ (i.e., $\neg \neg A$) deduce $A$.

In intuitionistic logic the final rule is weakened to the ex falso law “given $\bot$ deduce $A$” (for any formula $A$), and in minimal logic it is dropped altogether. (For a more precise exposition of this system of reasoning see Chapter 1 of [5].)

2.4. Classical versus constructive truth. Formal assertions can be interpreted either classically or constructively. The classical interpretation is formulated in terms of an assumed underlying reality: we understand the statement that $A$ is true just to mean that what $A$ asserts is the case. Thus, for example, we interpret $A \rightarrow B$ to mean that any evaluation of the free variables which makes $A$ the case also makes $B$ the case. In contrast, the constructive interpretation of formal assertions has to do with what can be proven. Under this interpretation, to say that $A$ is true is to say that we are entitled to assert $A$, i.e., that we can prove $A$. For example, here we are entitled to assert $A \rightarrow B$ — that is, we can prove $A \rightarrow B$ —
precisely if we have a construction that will convert any proof of $A$ into a proof of $B$.

Although the classical interpretation is arguably more straightforward, I claim that it cannot be used in the heuristic setting where definitions are not fixed and hence there is no well-defined underlying reality; that in the surveyable setting, on the other hand, there is no effective difference between classical and constructive truth; and thus that the only case where a real choice has to be made is in the definite setting.

First, for statements that involve only surveyable variables and predicates, there is operationally no difference between the classical and constructive interpretations. In principle, we could determine the truth value of any such statement simply by performing an exhaustive search (if there are multiple quantifiers, a finite series of exhaustive searches). In other words, any sentence that is classically true could in principle be proven and hence would also be constructively true. So in this kind of setting classical logic is appropriate. In particular, the law of excluded middle (the statement $A \lor \neg A$, for all formulas $A$, which is generally absent from intuitionistic logic) is constructively valid here.

For statements which quantify over variables that are definite but not surveyable, the situation is more subtle. We are still entitled to assume an underlying notion of being the case which would support a classical interpretation of truth. However, in general we should not expect that every classically true sentence will be provable. That is, we may, even in principle, have no way to determine the truth value of a statement that quantifies over definite variables. Consequently, either classical or intuitionistic logic can be used. But adopting a classical interpretation in a definite but not surveyable context does not seem very helpful. All we gain is the existence of truth values that in principle could never be known, and this cannot have any substantive consequences.

The fact that classical logic is an option does, however, imply that under the constructive interpretation of truth the ex falso law is valid. Even if the classical truth values of some statements might not be knowable, we can still incorporate their existence into our notion of a valid proof. That is, we can insist that a proof cannot be valid if it yields a conclusion that is classically false. Given this restriction, even though we cannot access all classical truth values, we can still affirm that $\bot$ is not provable. Thus the ex falso law $\bot \rightarrow A$ is constructively valid because it is vacuously the case that we can convert any proof of $\bot$ into a proof of $A$.

The preceding argument no longer holds in the heuristic case. In this setting it does not make sense to suppose an underlying classical reality. (If we could do this, we would be dealing with definite concepts.) So we have to interpret truth constructively. Furthermore, we have no sharp global distinction between valid and invalid reasoning. Thus we can no longer argue that $\bot$ has no proof, and hence that the ex falso law is valid, because this assumes that constructive reasoning is sound (as otherwise a false statement might indeed have a proof). Now one may find the validity of constructive reasoning perfectly compelling, but the point is that this must be decided before the justification of ex falso can proceed. The problem is that this leads to an infinite regress if one is arguing that ex falso itself ought to be included among the axioms of constructive reasoning.
This difficulty is genuine in the heuristic setting where we are not dealing with fixed concepts with well-defined meanings. Here any axioms we adopt will to some extent play a definitional role, in contrast to the definite case where we need only to choose axioms which accurately embody concepts that are already well-defined. In other words, formal systems modelling definite concepts are merely descriptive, whereas systems modelling heuristic concepts may be at least partly prescriptive. In the latter context we lack the guarantee of consistency that comes from correctly modelling an already well-defined concept, and there is a real danger that adopting ex falso could introduce an inconsistency.

To summarize: if we adopt a classical interpretation of truth then we can use classical logic in both the surveyable and definite settings, but we cannot reason about heuristic collections. If we adopt a constructive interpretation of truth then we must use classical logic in the surveyable case, intuitionistic logic in the definite case, and minimal logic in the heuristic case.

3. Conceptualist mathematics

3.1. Supertasks. The basic tenets of mathematical conceptualism [8] can be summarized as follows: mathematics is the study of logical possibility; conceivability is a sufficient condition for logical possibility; and constructions of length $\omega$ are conceivable.

The first point can be contrasted with the view that mathematics is about physical possibility. This view is expressed, for example, when philosophical conclusions about mathematics are drawn from purported upper bounds on the number of elementary particles in the universe. The conceptualist objection to arguments like these would be that any computational constraints we face as a result of special features of our universe are not relevant to the concept of mathematical truth. They may be interesting and important in many other ways but they have no bearing on this issue.

Some sort of physicalist attitude seems to underlie constructivist ideas generally, in particular the idea that only those constructions which terminate after a finite number of steps are legitimate because it is only these which could actually be performed. This suggestion draws a very clean line between what is acceptable and what is not, but it is suspect for that very reason. What “could actually be performed” is not a sharp concept, and putting all finite computations into this category already involves a substantial idealization. This leads to the question why we should not allow the further idealization to computations of length $\omega$.

Various commentators have felt that computations of length $\omega$ — “supertasks” — clearly fall within the realm of logical possibility [4]. If this point is granted, then in the terminology of Section 2 we should conclude that the natural numbers are surveyable. Moreover, the truth value of any statement of first order number theory could be mechanically decided using infinite truth tables by a computation of length $n \cdot \omega$ for some $n$ (i.e., a finite series of supertasks), so that classical logic is indeed appropriate to this setting, in line with the suggestion made in Section 2.4.

More generally, any countable structure familiar from ordinary mathematics ought to be surveyable on the preceding analysis. Of course we need to specify how such a structure is to be physically modelled, but in worlds that permit supertasks there should be no serious difficulty in finding appropriate models. At this level logic is classical and abstract set theory is wholly absent.
3.2. **Uncountable sets and proper classes.** Thus, at the countable level conceptualist mathematics is practically indistinguishable from classical mathematics. We are “platonists about number theory”. However, this is no longer true at the level of (what are classically thought of as) uncountable sets. We cannot regard uncountable structures, even familiar ones like the real line, the power set of \( \omega \), or \( \aleph_1 \), to be surveyable. This is because we have no clear conception of what it would be like to look through all real numbers, or all countable ordinals, even if supertasks were allowed.

The surveyability of collections like these is widely taken to be self-evident. But this is just a consequence of failing to draw a clear distinction between surveyability and definiteness. If one lacks this conceptual distinction then one is bound to conclude that \( \mathcal{P}(\omega) \) is *the same kind of thing* as \( \omega \) since, after all, both are collections. This line of thought leads straight into the bizarre theology of Cantorian set theory, with its remote cardinals that have no connection to mainstream mathematics.

In fact the surveyability of \( \mathcal{P}(\omega) \) is an extraordinary claim, completely different in character from the idea that \( \omega \) is surveyable. It is inherent in our conception of natural numbers that we can imagine searching through them one at a time. But we have no clear intuition for how one would go about exhaustively searching through all sets of natural numbers. It is only to be expected that mathematics that requires the assumption that \( \mathcal{P}(\omega) \) is surveyable would be highly pathological, have no meaningful scientific applications, and make little connection to mainstream mathematics generally.

The point can be made in terms of possible worlds. Can we clearly conceive of a world in which (representatives of) all real numbers are present and surveyable? The Löwenheim-Skolem theorem suggests that we cannot. Any description we could give of such a world as an abstract structure would already be satisfied by some countable substructure, which evidently would not contain all real numbers.

However, classically uncountable sets like \( \mathcal{P}(\omega) \) and \( \aleph_1 \) are at least conceptually definite. If we accept that \( \omega \) is surveyable then we should also accept that any infinite sequence of 0’s and 1’s is surveyable, leading to the conclusion that the predicate “is an infinite sequence of 0’s and 1’s” is in principle decidable true or false. This shows that the power set of \( \omega \) can be conceptually regarded as a definite collection. The real line can be handled in essentially the same way, either identifying real numbers with their binary expansions, or in terms of Dedekind cuts.

Once we accept that the real line is definite, it is a short step to the same conclusion for all of the standard spaces that one encounters in ordinary mathematics. In some cases this will require fairly straightforward coding. For instance, the real Banach space \( C[0,1] \) may be regarded as definite because elements of this space can be identified with uniformly continuous functions from \( [0,1] \cap \mathbb{Q} \) to \( \mathbb{R} \); such a function is specified by a countable family of real numbers, which is easily seen to be definite, or it can be encoded as a single real number, allowing a reduction to the fact that \( \mathbb{R} \) is definite. For details see [12].

In general, any classical set whose elements can be reasonably encoded as real numbers will be conceptually definite. For instance, the set of all separable manifolds and the set of all separable Banach spaces fall in this category via encodings familiar from reverse mathematics [6].
Broadly speaking, the proper classes that appear in ordinary mathematics (e.g., the class of all manifolds, or the class of all Banach spaces) are conceptualistically realized as heuristic collections.

3.3. **Comparison with Zermelo-Fraenkel set theory.** In classical set theory every set has a power set. Conceptualistically, the situation is not so simple. The example of \( \omega \) and \( \mathcal{P}(\omega) \) illustrates the principle that for every surveyable collection there is a definite collection which plays the role of its power set. “Subsets” of \( \omega \) can be represented by sequences of 0’s and 1’s, which are surveyable just like \( \omega \), so that the collection of infinite sequences of 0’s and 1’s is definite. But it is not surveyable.

The point is that sequences of 0’s and 1’s play the role of proxies for surveyable subcollections of \( \omega \). In general, given any surveyable collection we should be able to set up a natural system of proxies for surveyable subcollections, such that the collection of all surveyable subcollections is definite. The surveyable subcollections of any surveyable collection constitute a definite collection.

(We may note here that there exist non-surveyable but definite subcollections of \( \omega \), for instance the Church-Kleene ordinal notations, as well as non-definite but heuristic subcollections of \( \omega \), for instance the Gödel numbers of the well-defined predicates.)

Similarly, the definite subcollections of any definite collection constitute a heuristic collection. Since the notion of a well-defined predicate is merely heuristic, so is the notion of a definite subcollection of a given collection, in general.

The process stops here. There is no meaningful sense in which one can talk about arbitrary heuristic subpredicates of a given heuristic predicate, because the general concept of a heuristic predicate is not even well-defined.

The accord between conceptualist philosophy and mainstream mathematics is striking. In general, discrete objects in ordinary mathematics are conceptualistically represented as surveyable; continuous objects are represented as definite; and proper classes are represented as heuristic. In Zermelo-Fraenkel set theory the sets \( \omega, \mathcal{P}(\omega), \text{ and } \mathcal{P}(\mathcal{P}(\omega)) \) are just the first three terms of an infinite sequence. But all mainstream mathematics takes place, or can be construed as taking place, in these sets. To fully appreciate this point, consider that classically the power set operation can be iterated not just any finite number of times, but any transfinite number of times (taking unions at limit stages) — even uncountably many times. Even, say, a measurable cardinal number of times, if one believes in measurable cardinals. Thus ordinary mathematics is grossly discordant with the Cantorian universe, while it fits the conceptualist picture perfectly.

The conceptualist universe is much smaller than the Cantorian universe, but what is lost is a vast realm of set-theoretic pathology that plays no essential role in mainstream mathematics. This is particularly seen in fields where the basic definitions allow objects of arbitrary cardinality. For instance, Banach spaces can be arbitrarily large. Yet nearly all Banach spaces of real mainstream interest are either separable or the duals of separable spaces, and hence fall into the conceptualist framework. There are a few important spaces that are neither separable nor the duals of separable spaces (such as spaces of almost periodic functions, or CCR algebras, or the Calkin algebra), but these rare exceptions invariably still turn out to lie within the conceptualist framework. Genuinely extra-conceptualist spaces...
like the dual of $L^\infty[0,1]$ are seen as pathological and do not attract substantial mainstream interest.

The naive concept of a set is not only grounded in an untenable philosophical stance, it also yields a picture of a universe which bears little relation to mainstream mathematics. When we pay closer attention to the kinds of collections that we actually use, we find that they stratify into three levels, and this stratification fits with mathematical practice in a way that Cantorian set theory does not. For more on this point see [12].

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1See [http://www.math.wustl.edu/~nweaver/conceptualism.html](http://www.math.wustl.edu/~nweaver/conceptualism.html)