Marginally trapped submanifolds in Lorentzian space forms and in the Lorentzian product of a space form by the real line

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Abstract

We give local, explicit representation formulas for \( n \)-dimensional spacelike submanifolds which are marginally trapped in the Minkowski space \( \mathbb{R}^{n+2}_1 \), the de Sitter space \( dS^{n+2} \), the anti de Sitter space \( AdS^{n+2} \) and the Lorentzian products \( S^{n+1} \times \mathbb{R} \) and \( H^{n+1} \times \mathbb{R} \) of the sphere and the hyperbolic space by the real line.

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Introduction

Let \( S \) be a submanifold of a pseudo-Riemannian manifold \( (\mathcal{N},g) \). In the induced metric on \( S \) is non-degenerate, one may define its mean curvature \( \vec{H} \), a normal vector field along \( S \). We shall say that \( S \) is marginally trapped if \( \vec{H} \) is a null vector, i.e. \( g(\vec{H},\vec{H}) \) vanishes identically. Of course this may happen only if \( S \) has codimension greater than two and if the induced metric on the normal bundle is indefinite.

The case of a spacelike surface \( S \) of a four-dimensional Lorentzian manifold \( (\mathcal{N},g) \) is the most interesting because of its physical interpretation in the setting of General Relativity: marginally outer trapped surfaces (MOTS) play a fundamental role in the study of black holes and spacetime singularities ([Pe],[CGP]). Despite their physical relevance and the fact that marginally trapped is the most natural curvature condition which is purely pseudo-Riemannian, these submanifolds are still not very well understood.

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1Some authors call such submanifolds quasi-minimal or pseudo-minimal.
Recently, the classification of marginally trapped surfaces satisfying several additional properties, such as being Lagrangian ([CD]), isotropic ([CFG]), having flat normal bundle ([AGM]), constant curvature ([Ch]) or positive relative nullity ([CVdV], [VdV]), has been done. On the other hand, in [Pa] (see also [APa]) a very interesting minimization property has been discovered concerning marginally trapped surfaces: a minimal spacelike surface of Minkowski space $\mathbb{R}^4$, although it is unstable, minimizes the area among marginally trapped surfaces satisfying a natural boundary data.

The purpose of this paper is to give local, explicit representation formulas for $n$-dimensional marginally trapped submanifolds in some of the simplest Lorentzian manifolds: the Minkowski space $\mathbb{R}^{n+2}$, the non-flat Lorentzian spaces forms, i.e. the de Sitter space $d\mathbb{S}^{n+2}$ and the anti de Sitter space $Ad\mathbb{S}^{n+2}$, and finally the Lorentzian products $S^{n+1} \times \mathbb{R}$ and $\mathbb{H}^{n+1} \times \mathbb{R}$ of the sphere and the hyperbolic space by the real line.

Our construction is inspired by [An1] and is based, although not explicitly, on the contact structure enjoyed by the space of null geodesics of a pseudo-Riemannian manifold ([GS],[KT]). For example, in the case of Minkowski space $\mathbb{R}^{n+2}$, a spacelike, $n$-dimensional submanifold $\tilde{S}$ is locally described in terms of its height function (i.e. its timelike coordinate) and a hypersurface $S$ of $\mathbb{R}^{n+1}$. Then the marginally trapped condition amounts to a simple algebraic relation between the height function of $\tilde{S}$ and the second fundamental form of $S$. The interpretation in terms of contact geometry is the following: the set of null geodesics normal to $S$ is a Legendrian submanifold in the set of null geodesics of $\mathbb{R}^{n+2}$, which is contactomorphic to the unit tangent bundle of $\mathbb{R}^{n+1}$; then the hypersurface $S$ is nothing but the projection on the basis $\mathbb{R}^{n+1}$ of this Legendrian submanifold.

This idea works in the same way in the other simple Lorentzian spaces $d\mathbb{S}^{n+2}$, $Ad\mathbb{S}^{n+1}$, $S^{n+1} \times \mathbb{R}$ and $\mathbb{H}^{n+1} \times \mathbb{R}$. The construction can be performed in Robertson-Walker spaces as well, but the analysis becomes quite more involved since the equation relating the height function of $\tilde{S}$ and the second fundamental form of $S$ is not any more algebraic, but differential. This case will be discussed in a forthcoming paper ([An2]).

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1 Statement of results

Let $(\mathbb{R}^{n+1}, \langle ., . \rangle_0)$ be the Euclidean space endowed with its canonical Riemannian metric

$$\langle ., . \rangle_0 := dx_1^2 + ... + dx_{n+1}^2,$$
and denote by
\[ \langle \cdot, \cdot \rangle_1 := \langle \cdot, \cdot \rangle_0 - dx_{n+2}^2 \]
the flat Lorentzian metric of the Cartesian product \( \mathbb{R}^{n+2}_1 = \mathbb{R}^{n+1} \times \mathbb{R} \).

We denote by \( \iota : \mathbb{R}^n \rightarrow \mathbb{R}^{n+1} \) the canonical totally geodesic embedding \( \iota(x) = (x,0) \) and denote by \( \nu_0 = (0, ..., 0, 1) \) its (constant) unit normal vector.

We recall that the second fundamental form \( h \) of an immersion \( M \rightarrow (N, g) \) with non-degenerate first fundamental form is the symmetric tensor \( h : T M \times T M \rightarrow N M \) defined by
\[ h(X,Y) := (D_X Y)^\perp, \]
where \( (\cdot)^\perp \) denotes the projection onto the normal space \( N M \) and \( D \) is the Levi-Civita connection of \( g \). If \( \nu \) is a normal vector field along \( M \), we have the following important relation:
\[ g(h(X,Y),\nu) = -g(D_X \nu,Y). \]
The mean curvature vector of the immersion is the trace of \( h \) with respect to the induced metric.

**Theorem 1.** Let \( \Omega \) be an open domain of \( \mathbb{R}^n \) and \( \tau \in C^2(\Omega) \). Then the immersion \( \bar{\varphi} : \Omega \rightarrow \mathbb{R}^{n+2}_1 \) defined by
\[ \bar{\varphi}(x) = (\iota(x),0) + \tau(x)(\nu_0,1) \]
is flat and its second fundamental form is given by
\[ \bar{h}(X,Y) = \text{Hess}_\tau(X,Y)(\nu_0,1). \]
In particular \( \bar{\varphi} \) has null second fundamental form and is therefore marginally trapped. Conversely, any \( n \)-dimensional spacelike submanifold with null second fundamental form is locally congruent to the image of such an immersion.

Let \( \varphi \) be an immersion of class \( C^4 \) of an \( n \)-dimensional manifold \( M \) into \( \mathbb{R}^{n+1} \) and denote by \( \nu \) the Gauss map of \( \varphi \), which is therefore \( S^n \)-valued. Assume that \( \varphi \) admits \( p \) distinct, non-vanishing principal curvatures \( \kappa_1, ..., \kappa_p, p \geq 2 \) with multiplicity \( m_i \) and denote by \( \tau_i \) the \( p-1 \) roots of the polynomial
\[ P(\tau) := \sum_{i=1}^p m_i \prod_{j \neq i} (\kappa_j^{-1} - \tau). \]
Then the \( p-1 \) immersions \( \bar{\varphi}_i : M \rightarrow \mathbb{R}^{n+2}_1 \) defined by
\[ \bar{\varphi}_i = (\varphi + \tau_i \nu, \tau_i) \]
are marginally trapped.

Conversely, any \( n \)-dimensional marginally trapped submanifold of \( \mathbb{R}^{n+2}_1 \) whose second fundamental form is not null is locally congruent to the image of such an immersion.
In particular, in the \(n = 2\) case, let \(\varphi\) be a non-flat \(C^4\)-immersion of a surface \(\mathcal{M}\) into \(\mathbb{R}^3\) which is free of umbilic points. Denote by \(\nu\) its Gauss map, by \(H\) and \(K\) the mean curvature with respect to \(\nu\) and the Gaussian curvature of \(\varphi\). Then immersion \(\bar{\varphi} : \mathcal{M} \to \mathbb{R}^4_1\) defined by

\[
\bar{\varphi} := \left( \varphi + \frac{H}{K} \nu, \frac{H}{K} \right)
\]

is marginally trapped. As a corollary, an immersed surface which is contained in a time slice \(\{x_4 = \text{const.}\}\) is marginally trapped if and only if \(\frac{H}{K}\) is constant, i.e. \(\varphi\) is linear Weingarten.

It may be interesting to relate the latter formula to one found by B. Palmer: it is proved in [Pa] that, given \(\Omega\) an open subset of \(\mathbb{S}^2\) and \(f \in C^4(\Omega)\), the immersion \(\varphi : \Omega \to \mathbb{R}^4_1\) defined by

\[
\bar{\varphi}(x) = \left( \nabla f(x) + f(x)x, -f(x) - \frac{1}{2} \Delta f(x) \right),
\]

where \(\nabla\) and \(\Delta\) denote the gradient and Laplace operators w.r.t. the round metric, is marginally trapped.

We first observe that a convex surface of Euclidean space may be constructed from its support function: let \(\varphi : \Omega \to \mathbb{R}^3\) parametrized by its unit normal vector \(\nu\) and introduce \(f(\nu) = \langle \varphi(\nu), \nu \rangle_0\), i.e. \(f\) is the support function of \(\varphi\). Hence we have (see [AG]):

\[
\varphi(\nu) = f(\nu) \nu + \nabla f(\nu).
\] (2)

It will be seen along the proof of Theorem 1 that a spacelike immersion \(\bar{\varphi} : \Omega \to \mathbb{R}^4_1\) such that the null geodesic \(\{(\varphi(\nu), 0) + t(\nu, 1) | t \in \mathbb{R}\}\) crosses orthogonally the surface \(\bar{\varphi}(\Omega)\) at \(\bar{\varphi}(\nu)\) takes the form

\[
\bar{\varphi}(\nu) = (\varphi(\nu) + \tau(\nu) \nu, \tau(\nu)),
\]

where \(\tau\) is a smooth real map on \(\Omega\). Moreover, the mean curvature vector of \(\bar{\varphi}\) is collinear to the null vector field \(\bar{\nu} = (\nu, 1)\) if and only if \(\tau = \frac{H}{K}\). On the other hand, the following formula holds (see [AG]):

\[
\frac{H}{K} = -f - \frac{1}{2} \Delta f,
\] (3)

so using Formulas (2) and (3) we recover Palmer’s formula (Formula (1)).
The same construction may be applied in the case of the Lorentzian space forms. In order to state the result we set

$$S^{n+1} := \{ x \in \mathbb{R}^{n+2} \mid \langle x, x \rangle_0 = 1 \} \quad \text{and} \quad \mathbb{H}^{n+1} := \{ x \in \mathbb{R}^{n+2} \mid \langle x, x \rangle_1 = -1 \}.$$

These are the hyperquadric models of the $n+1$-dimensional Riemannian space forms. Similarly the $(n+2)$-dimensional Lorentzian space forms are defined as follows:

$$dS^{n+2} := \{ x \in \mathbb{R}^{n+3} \mid \langle x, x \rangle_1 = 1 \} \quad \text{and} \quad AdS^{n+2} := \{ x \in \mathbb{R}^{n+3} \mid \langle x, x \rangle_2 = -1 \},$$

where

$$\langle \cdot, \cdot \rangle_2 := dx_1^2 + \ldots + dx_{n+1}^2 - dx_{n+2}^2 - dx_{n+3}^2.$$

**Theorem 2.** Let $\Omega$ be an open domain of $S^n$ (resp. $\mathbb{H}^n$) and $\tau \in C^2(\Omega)$. Denote by $\iota : S^n \to S^{n+1} \subset \mathbb{R}^{n+2}$ (resp. $\mathbb{H}^n \to \mathbb{H}^{n+1} \subset \mathbb{R}^{n+2}$) the canonical totally geodesic embedding $\iota(x) = (x,0)$ and denote by $\nu_0 = (0,\ldots,0,1)$ the corresponding (constant) unit normal vector. Then the immersion $\bar{\varphi} : S^n \to dS^{n+2}$ (resp. $\mathbb{H}^n \to AdS^{n+2}$) defined by

$$\bar{\varphi}(x) = (\iota(x),0) + \tau(x)(\nu_0,1)$$

is flat and its second fundamental form is given by

$$\bar{h}(X,Y) = \text{Hess}_\tau(X,Y)(\nu_0,1).$$

In particular $\bar{\varphi}$ has null second fundamental form and is therefore marginally trapped. Conversely, any $n$-dimensional spacelike submanifold with null second fundamental form is locally congruent to the image of such an immersion.

Let $\varphi$ be an immersion of class $C^4$ of an $n$-dimensional manifold $\mathcal{M}$ into $S^{n+1}$ (resp. $\mathbb{H}^{n+1}$) and denote by $\nu$ the Gauss map of $\varphi$, which is therefore $S^{n+1}$-valued (resp. $dS^{n+1}$-valued). Assume that $\varphi$ admits $p$ distinct, non-vanishing principal curvatures $\kappa_1, \ldots, \kappa_p$, $p \geq 2$ with multiplicity $m_i$ and denote by $\tau_i$ the $p-1$ roots of the polynomial

$$P(\tau) := \sum_{i=1}^{p} m_i \prod_{j \neq i} (\kappa_j^{-1} - \tau).$$

Then the $p-1$ immersions $\bar{\varphi}_i : \mathcal{M} \to dS^{n+2}$ defined by

$$\bar{\varphi}_i = (\varphi + \tau_i \nu, \tau_i)$$

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are marginally trapped.

Conversely, any \( n \)-dimensional marginally trapped submanifold of \( d\mathbb{S}^{n+2} \) whose second fundamental form is not null is locally congruent to the image of such an immersion.

We observe that all the examples found in \([CVdV]\) and \([CFG]\) have null fundamental form (see Section 5).

The construction works as well in the case of the Lorentzian product of a space form by the real line. We endow \( \mathbb{S}^{n+1} \times \mathbb{R} \) with the Lorentzian metric \( \langle ., . \rangle_0 - dx_{n+3}^2 \), where \( \langle ., . \rangle_0 \) is the round metric of \( \mathbb{S}^{n+1} \) and \( x_{n+3} \) denotes the canonical coordinate of the real line \( \mathbb{R} \).

**Theorem 3.** There is no non-totally geodesic \( n \)-dimensional submanifold of \( \mathbb{S}^{n+1} \times \mathbb{R} \) with null second fundamental form.

Let \( \varphi \) be an immersion of class \( C^4 \) of an \( n \)-dimensional manifold \( M \) into \( \mathbb{S}^{n+1} \). Denote by \( \nu \) the Gauss map of \( \varphi \) and by \( \kappa_1, \ldots, \kappa_p \) its \( p \) distinct curvatures with multiplicity \( m_i \). Then the polynomial

\[
P(s) := \sum_{i=1}^{p} m_i (\kappa_i s + 1) \prod_{j \neq i} (s - \kappa_j)
\]

admits exactly \( p-1 \) roots \( s_i \) if \( \varphi \) is minimal and \( p \) roots otherwise. Moreover, the \( p-1 \) or \( p \) immersions \( \bar{\varphi}_i : M \to \mathbb{S}^{n+1} \times \mathbb{R} \) defined by

\[
\bar{\varphi}_i := \left( \frac{s_i \varphi + \nu}{\sqrt{1 + s_i^2}}, \cot^{-1} s_i \right), \quad 1 \leq i \leq p-1 \text{ or } p,
\]

are marginally trapped.

Conversely, any \( n \)-dimensional marginally trapped submanifold of \( \mathbb{S}^{n+1} \times \mathbb{R} \) is locally congruent to the image of such an immersion.

In particular, in the \( n = 2 \) case, given a non minimal \( C^4 \)-immersion \( \varphi \) of a surface into \( \mathbb{S}^3 \) and

\[
a := \frac{\kappa_1 \kappa_2 - 1}{\kappa_1 + \kappa_2},
\]

the two immersions into \( \mathbb{S}^3 \times \mathbb{R} \) defined by

\[
\bar{\varphi}_\pm := \left( \frac{(a \pm \sqrt{a^2 + 1}) \varphi + \nu}{\sqrt{2} \sqrt{a^2 + 1} \pm a \sqrt{a^2 + 1}}, \cot^{-1}(a \pm \sqrt{a^2 + 1}) \right)
\]

are marginally trapped.
Analogously, \( \mathbb{H}^{n+1} \times \mathbb{R} \) is endowed with the metric \( \langle \cdot, \cdot \rangle_1 - dx_{n+3}^2 \), where \( \langle \cdot, \cdot \rangle_1 \) is the standard metric of \( \mathbb{H}^{n+1} \) and \( x_{n+3} \) denotes the canonical coordinate of the real line \( \mathbb{R} \).

**Theorem 4.** There is no non-totally geodesic \( n \)-dimensional submanifold of \( \mathbb{H}^{n+1} \times \mathbb{R} \) with null second fundamental form.

Let \( \varphi \) be an immersion of class \( C^4 \) of an \( n \)-dimensional manifold \( \mathcal{M} \) into \( \mathbb{H}^{n+1} \). Denote by \( \nu \) the Gauss map of \( \varphi \) and by \( \kappa_1, \ldots, \kappa_p \) its \( p \) distinct curvatures with multiplicity \( m_i \). Denote by \( s_i \), \( 1 \leq i \leq q \leq p \), the \( q \) roots of the polynomial

\[
P(s) := \sum_{i=1}^{p} m_i (\kappa_i s - 1) \prod_{j \neq i} (s - \kappa_j)
\]

satisfying \( |s_i| > 1 \). Then the \( q \) immersions \( \bar{\varphi}_i : \mathcal{M} \to \mathbb{H}^{n+1} \times \mathbb{R} \) defined by

\[
\bar{\varphi}_i := \left( \frac{s_i \varphi + \nu}{\sqrt{s_i^2 - 1}}, \coth^{-1}(s_i) \right), \quad 1 \leq i \leq q,
\]

are marginally trapped.

Conversely, any \( n \)-dimensional marginally trapped submanifold of \( \mathbb{H}^{n+1} \times \mathbb{R} \) is locally congruent to the image of such an immersion.

In particular, in the \( n = 2 \) case, given a non minimal \( C^4 \)-immersion \( \varphi \) of a surface into \( \mathbb{H}^3 \), such that \( a := \frac{\kappa_1 \kappa_2 + 1}{\kappa_1 + \kappa_2} \in (1, \infty) \), the immersion into \( \mathbb{H}^3 \times \mathbb{R} \) defined by

\[
\bar{\varphi} := \left( \frac{(a + \sqrt{a^2 - 1}) \varphi + \nu}{\sqrt{2 \sqrt{a^2 - 1} + a \sqrt{a^2 - 1}}}, \frac{1}{2} \coth^{-1}(a) \right)
\]

is marginally trapped.

### 2 The Minkowski case: proof of Theorem 1

Let \( \bar{\varphi} = (\psi, \tau) \) be an immersion of a \( n \)-dimensional manifold \( \mathcal{M} \) into \( \mathbb{R}^{n+2} \) which is spacelike, i.e. the induced metric \( \bar{g} := \bar{\varphi}^* \langle \cdot, \cdot \rangle_1 \) is definite positive. In particular the induced metric on the normal space of \( \bar{\varphi} \) is Lorentzian and we may define locally two null, non-vanishing normal vector fields. Moreover a null vector field \( \bar{\nu} \) may be normalized on the following form: \( \bar{\nu} = (\nu, 1) \), with \( \nu : \mathcal{M} \to \mathbb{S}^n \). From now on we consider a null normal vector field \( \bar{\nu} := (\nu, 1) \) and we set \( \varphi := \psi - \tau \nu \).
Lemma 1. The map $(\varphi, \nu) : \mathcal{M} \to \mathbb{R}^{n+1} \times S^n$ is an immersion.

Proof. Suppose $(\varphi, \nu)$ is not an immersion, so that there exists a non-vanishing vector $v \in T\mathcal{M}$ such that $(d\varphi(v), d\nu(v)) = (0, 0)$. Since we have $d\psi = d\varphi + \tau d\nu + d\tau \nu$, it follows that

$$d\tilde{\varphi}(v) = (d\psi(v), d\tau(v)) = (d\tau(v)\nu, d\tau(v)) = d\tau(v)\nu,$$

which is a null vector. This contradicts the assumption that $\tilde{\varphi}$ is spacelike.

Lemma 2. We have the following relation$^2$:

$$\langle d\varphi, \nu \rangle_0 = 0.$$

Proof. Using again that $d\psi = d\varphi + \tau d\nu + d\tau \nu$ and observing that $\langle \nu, d\nu \rangle_0 = 0$, we have

$$0 = \langle d\tilde{\varphi}, \tilde{\nu} \rangle_1 = \langle (d\psi, d\tau), (\nu, 1) \rangle_1 = \langle d\psi, \nu \rangle_0 - d\tau = \langle d\varphi, \nu \rangle_0.$$

Lemma 3. Given $\epsilon > 0$, there exists $t_0 \in (-\epsilon, \epsilon)$ such that $\varphi + t_0\nu$ is an immersion, and $\nu$ is its Gauss map.

Proof. The claim follows from the fact that the set $\{t \in \mathbb{R} | \varphi + t\nu$ is not an immersion$\}$ contains at most $n$ elements. To see this, observe that given a pair of distinct real numbers $(t, t')$, we have

$$Ker(d\varphi + td\nu) \cap Ker(d\varphi + t'd\nu) = \{0\}$$

(otherwise we would have a contradiction with the fact that $(\varphi, \nu)$ is an immersion). Hence there cannot be more than $n$ distinct values $t$ such that $Ker(d\varphi + td\nu) \neq \{0\}$. The fact that $\nu$ is the Gauss map of $\varphi + t_0\nu$ comes from Lemma 2:

$$\langle d(\varphi + t_0\nu), \nu \rangle_0 = \langle d\varphi, \nu \rangle_0 + t_0\langle d\nu, \nu \rangle_0 = 0.$$

---

$^2$This corresponds to the fact that the immersion $(\varphi, \nu)$ is Legendrian with respect to the canonical contact structure of the unit bundle of $\mathbb{R}^{n+1}$.
The last lemma shows that there is no loss of generality in assuming that \( \varphi \) is an immersion: if it is not the case, we may translate the immersion \( \varphi \) along the vertical direction, setting \( \varphi_{t_0} := \varphi - (0, t_0) \). Of course \( \varphi \) is marginally trapped if and only if \( \varphi_{t_0} \) is so, and moreover the vector field \( \bar{\nu} \) is still normal to \( \varphi_{t_0} \). Finally, observe that the map \( \varphi_{t_0} : \mathcal{M} \to \mathbb{R}^{n+1} \) associated to \( \varphi_{t_0} \) is
\[
\varphi_{t_0} = \psi - (\tau - t_0)\nu = \psi - \tau \nu + t_0 \nu = \varphi + t_0 \nu,
\]
hence an immersion.

We now describe the first fundamental form of \( \bar{\varphi} \) and its second fundamental form with respect to \( \bar{\nu} \), both in terms of the geometry of the immersion \( \varphi \):

**Lemma 4.** Denote by \( g := \varphi^*(\langle ., . \rangle)_0 \) the metric induced on \( \mathcal{M} \) by \( \varphi \) and \( A \) the shape operator associated to \( \nu \), i.e. \( A(v) := -dv(v), \forall v \in T\mathcal{M} \). Then the metric \( \bar{g} := \varphi^*(\langle ., . \rangle)_1 \) induced on \( \mathcal{M} \) by \( \bar{\varphi} \) is given by the formula
\[
\bar{g} = g(., .) - 2\tau g(A., .) + \tau^2 g(A., A.).
\]
In particular, the non-degeneracy assumption on \( \bar{g} \) implies that \( \tau^{-1} \) is not equal to any principal curvature of \( \varphi \). Moreover, the second fundamental form of \( \bar{\varphi} \) with respect to \( \bar{\nu} \) is given by
\[
\bar{h}_\bar{\nu} := \langle \bar{H}(., \bar{\nu}) \rangle_1 = g(., A.) - \tau g(A., A.).
\]
and
\[
\langle \bar{H}_\varphi, \bar{\nu} \rangle_1 = \frac{1}{n} \sum_{i=1}^{n} \frac{\kappa_i}{(1 - \tau \kappa_i)}, \tag{4}
\]
where the \( \kappa_i \) are the principal curvatures of \( \varphi \).

**Proof.** Since \( \langle d\varphi, \nu \rangle_0 = \langle dv, \nu \rangle_0 = 0 \), we have, given \( v_1, v_2 \in T_x\mathcal{M}^n \),
\[
\bar{g}(v_1, v_2) = \langle d\bar{\varphi}(v_1), d\bar{\varphi}(v_2) \rangle_1
= \langle d\varphi(v_1), d\varphi(v_2) \rangle_0 + \tau \langle d\varphi(v_1), dv(v_2) \rangle_0 + \tau \langle dv(v_1), d\varphi(v_2) \rangle_0 + \tau^2 \langle dv(v_1), dv(v_2) \rangle_0 + d\tau(v_1)d\tau(v_2) \langle \nu, \nu \rangle_0 - d\tau(v_1)d\tau(v_2)
= g(v_1, v_2) - \tau (g(v_1, Av_2) + g(Av_1, v_2)) + \tau^2 g(Av_1, Av_2)
= g(v_1, v_2) - 2\tau g(Av_1, v_2) + \tau^2 g(Av_1, Av_2).
\]
We calculate the second fundamental form of \( \bar{\varphi} \) with respect to \( \bar{\nu} := (\nu, 1) \):
\[
\bar{h}_\bar{\nu} = -\langle d\bar{\varphi}, d\bar{\nu} \rangle_1
= -\langle d\varphi + \tau dv + d\tau \nu, dv \rangle_0
= -\langle d\varphi, dv \rangle_0 - \tau \langle dv, dv \rangle_0
= g(., A.) - \tau g(A., A.).
\]
To complete the proof, observe that in the totally umbilic case $A = \kappa \text{Id}$, we obviously have
\[
\langle \vec{H}_\varphi, \vec{\nu} \rangle_1 = \frac{\kappa}{1 - \tau \kappa}.
\]

If $\varphi$ is not totally umbilic, we introduce, away from isolated umbilic points, a principal orthonormal frame $(e_1, \ldots, e_n)$ along $\mathcal{M}$, i.e. such that \( g(e_i, e_j) = \delta_{ij} \) and $Ae_i = \kappa_i e_i$. Hence
\[
\bar{g}(e_i, e_j) = (1 - 2\tau \kappa_i + \tau^2 \kappa_i^2) \delta_{ij}
\]
and
\[
\bar{h}(e_i, e_j) = \kappa_i(1 - \tau \kappa_i) \delta_{ij}
\]
and the proof follows.

From Equation 4 it is immediately seen that if $\varphi$ is totally geodesic, i.e. $A$ vanishes, then $\bar{h}_\varphi$ vanishes as well, so the second fundamental form $\bar{h}$ of $\varphi$ takes value in the null line directed by $\vec{\nu}$. There is no loss of generality in assuming that $\nu_0 = (0, \ldots, 0, 1)$, i.e. the image of $\varphi$ is contained in the hyperplane $\{x_{n+1} = 0\}$. It is then straightforward to check that $\bar{h}(X, Y) = \text{Hess}_e(X, Y)(\nu_0, 1)$. We therefore recover the first part of Theorem 1.

In order to complete the proof we order the non-vanishing principal curvatures $\kappa_i$, taking into account their multiplicity $m_i$, in such a way that the corresponding radii of curvature are increasing $r_1 := \kappa_1^{-1} < \ldots < r_p := \kappa_p^{-1}$. Hence
\[
\langle \vec{H}_\varphi, \vec{\nu} \rangle_1 = 0
\]

\[
\iff \sum_{i=1}^{p} \frac{m_i \kappa_i}{(1 - \tau \kappa_i)} = 0
\]

\[
\iff \sum_{i=1}^{p} \frac{m_i}{r_i - \tau} = 0
\]

\[
\iff P(\tau) := \sum_{i=1}^{p} m_i \prod_{j \neq i} (r_j - \tau) = 0.
\]

We have
\[
P(r_i) = \sum_{k=1}^{p-1} m_k \prod_{j \neq k} (r_j - r_i) = m_i \prod_{j \neq i} (r_j - r_i).
\]

It follows that $P(r_p) > 0$, $P(r_{p-1}) < 0$ and that more generally the signs of $P(r_i)$, $i = 1, \ldots, p$ are alternate. We deduce that $P(\tau)$ admits at least $p - 1$
distinct roots $\tau_i, i = 1, \ldots, p-1$, satisfying $r_i < \tau_i < r_{i+1}$. Since $P(\tau)$ has degree $p-1$, it has no other roots.

Remark 1. If $\varphi$ is minimal, $\tau = 0$ is a root of $P(\tau)$. The corresponding immersion $\tilde{\varphi} = (\varphi, 0)$ is not only marginally trapped, but minimal.

3 The de Sitter and anti de Sitter cases: proof of Theorem 2

3.1 The de Sitter case

Let $\tilde{\varphi} = (\psi, \tau) : \mathcal{M} \to d\mathbb{S}^{n+2}$ an immersion such that the induced metric $\tilde{g} := \tilde{\varphi}^*\langle \cdot, \cdot \rangle_1$ is spacelike. Let $\tilde{\nu} = (\nu, 1)$ be one of the two normalized, null normal field to $\tilde{\varphi}$. We define the null projection of $\tilde{\varphi}$ to be $\varphi := \psi - \tau \nu$. The fact that $(\nu, 1) \in T_{\tilde{\varphi}}d\mathbb{S}^{n+2}$, i.e. $0 = \langle (\psi, \tau), (\nu, 1) \rangle_1 = \langle \psi, \nu \rangle_0 - \tau$, implies that $\langle \psi, \nu \rangle_0 = \tau$. Hence

$$\langle \varphi, \varphi \rangle_0 = \langle \psi, \psi \rangle_0 - 2\tau \langle \psi, \nu \rangle_0 + \tau^2 \langle \nu, \nu \rangle_0 = \langle \varphi, \varphi \rangle_1 = 1,$$

which shows that $\varphi$ is $\mathbb{S}^{n+1}$-valued. The proofs of the next two lemmas are omitted, since they are similar to the Minkowski case:

Lemma 5. The map $(\varphi, \nu) : \mathcal{M} \to \mathbb{S}^{n+1} \times \mathbb{S}^{n+1}$ is an immersion:

Lemma 6. We have the following relations:

$$\langle \varphi, \nu \rangle_0 = 0 \quad \text{and} \quad \langle d\varphi, \nu \rangle_0 = 0.$$

Unlike in the Minkowski case, there is no vertical translation in $d\mathbb{S}^{n+2}$. We may however, up to an arbitrarily small, linear perturbation, assume that $\varphi$ is an immersion.

Lemma 7. Given $\epsilon > 0$, there exists $\alpha \in (-\epsilon, \epsilon)$ and a hyperbolic rotation $R^\alpha$ of angle $\alpha$ such that the null projection $\varphi^\alpha$ of $\tilde{\varphi} := R^\alpha \tilde{\varphi}$ is an immersion.
Proof. Set

\[ R^\alpha = \begin{pmatrix} \cosh \alpha & \sinh \alpha \\ -\sinh \alpha & \cosh \alpha \end{pmatrix} \in SO(n + 2, 1) \]

and \( \varphi^\alpha := R^\alpha \bar{\varphi}, \bar{\nu}^\alpha := R^\alpha \bar{\nu} \). Observe that \( \bar{\nu}^\alpha := (\nu^\alpha, \sigma^\alpha) \) is not anymore normalized, a priori, since its last component \( \sigma^\alpha := \bar{\nu}^\alpha_{n+3} \) is equal to \( \cosh(\alpha) + \sinh(\alpha) \nu_1 \), where \( \nu_1 \) is the first component of the vector \( \nu \).

Nevertheless, the null geodesic passing through the point \( \varphi^\alpha \) and directed by the vector \( \bar{\nu}^\alpha \) crosses the slice \( dS_{n+2} \cap \{x_{n+3} = 0\} \) at the point \( (\varphi^\alpha, 0) := (\psi^\alpha - \frac{\tau^\alpha}{\sigma^\alpha} \nu^\alpha, 0) \).

Clearly, \( \varphi^\alpha \) is an immersion if and only if \( R^{-\alpha} \varphi^\alpha = \psi - \frac{\tau^\alpha}{\sigma^\alpha} \nu = \varphi + (\tau - \frac{\tau^\alpha}{\sigma^\alpha}) \nu \) is so. Observe that

\[ \tau - \frac{\tau^\alpha}{\sigma^\alpha} = \tau - \frac{\cosh(\alpha) \tau + \sinh(\alpha) \psi_1}{\cosh(\alpha) + \sinh(\alpha) \nu_1} \]
\[ = \tanh(\alpha)(\psi_1 - \tau \nu_1) + o(\alpha) \]
\[ = \varphi_1 \alpha + o(\alpha). \]

Now, assume that \( R^{-\alpha} \varphi^\alpha \) is not an immersion for \( \alpha \in (-\varepsilon, \varepsilon) \). Hence there exists a one-parameter family of unit tangent vectors \( v^\alpha \) such that

\[ 0 = d(R^{-\alpha} \varphi^\alpha)(v^\alpha) = d\varphi(v^\alpha) + (d\varphi_1(v^\alpha) + \varphi_1 dv(v^\alpha)) \alpha + o(\alpha), \forall \alpha \in (-\varepsilon, \varepsilon). \]

Thus

\[ \begin{cases} d\varphi(v^\alpha) = 0 \\ d\varphi_1(v^\alpha) \nu + \varphi_1 dv(v^\alpha) = \varphi_1 dv(v^\alpha) = 0. \end{cases} \]

By Lemma 5, \( dv(v^\alpha) \) and \( d\varphi(v^\alpha) \) cannot vanish simultaneously, therefore \( \varphi_1 \) vanishes. Repeating the argument with suitable rotations yields that all the other coordinates of \( \varphi \) vanish, a contradiction since \( \varphi \in S^{n+1} \).

By the previous lemma we may assume that \( \varphi \) is an immersion. The remainder of the proof of Theorem 2 follows the lines of Theorem 1, in particular Lemma 4 still holds.
3.2 The anti de Sitter case

Let $\bar{\varphi} = (\psi, \tau) : M \to \text{AdS}^{n+2}$ an immersion such that the induced metric $\bar{g} := \bar{\varphi}^* <.,.>_2$ is spacelike. Let $\bar{\nu} = (\nu, 1)$, be a normalized, null vector field which is normal to $\bar{\varphi}$. We define the null projection of $\bar{\varphi}$ to be $\varphi := \psi - \tau \nu$.

The fact that $\langle \nu, 1 \rangle \in T_{\bar{\varphi}} \text{AdS}^{n+2}$, i.e. $0 = \langle (\psi, \tau), (\nu, 1) \rangle_2 = \langle \psi, \nu \rangle_1 - \tau$ implies that $\langle \psi, \nu \rangle_1 = \tau$.

Hence

$$\langle \varphi, \varphi \rangle_1 = \langle \psi, \psi \rangle_1 - 2\tau \langle \psi, \nu \rangle_1 + \tau^2 \langle \nu, \nu \rangle_1 = \langle \varphi, \varphi \rangle_2 - \tau^2 = -1,$$

which shows that $\varphi$ is $\mathbb{H}^{n+1}$-valued.

The remainder of the proof is similar to the previous case (de Sitter case) and is therefore omitted.

4 The case of the product of a space form by the real line: proof of Theorems 3 and 4

4.1 The $S^{n+1} \times \mathbb{R}$ case

Let $\bar{\varphi} = (\psi, \tau) : M \to S^{n+1} \times \mathbb{R}$ an immersion such that the induced metric $\bar{g} := \bar{\varphi}^* <.,.>_1$ is spacelike. Let $\bar{\nu} = (\nu, 1)$, where $\nu \in S^{n+1}$, be a normalized, null normal field along $\bar{\varphi}$. We set $\varphi := \cos(\tau)\psi - \sin(\tau)\nu$ and $\nu_\varphi = \sin(\tau)\psi + \cos(\tau)\nu$.

Lemma 8. The map $(\varphi, \nu_\varphi) : M \to S^{n+1} \times S^{n+1}$ is an immersion.

Proof. Suppose $(\varphi, \nu_\varphi)$ is not an immersion, so that there exists a non-vanishing vector $v \in TM$ such that $(d\varphi(v), d\nu_\varphi(v)) = (0, 0)$. Since $\psi = \cos(\tau)\varphi + \sin(\tau)\nu_\varphi$ we have

$$d\psi = \cos(\tau)d\varphi + \sin(\tau)d\nu_\varphi + d\tau(-\sin(\tau)\varphi + \cos(\tau)\nu_\varphi).$$

Then

$$d\bar{\varphi}(v) = (d\psi(v), d\tau(v)\nu) = d\tau(v)(\nu, 1)$$

is a null vector, which contradicts the assumption that $\bar{\varphi}$ is spacelike. \qed

Lemma 9.

$$\langle d\varphi, \nu_\varphi \rangle_0 = 0.$$
Proof. Using that $d\psi = \cos(\tau)d\varphi + \sin(\tau)d\nu_\varphi + d\tau \nu$ and observing that $\langle \nu, d\nu \rangle_0 = 0$, we have

$$0 = \langle d\bar{\varphi}, \bar{\nu} \rangle_1 = \langle (d\psi, d\tau), (\nu, 1) \rangle_1 = \langle d\psi, \nu \rangle_0 - d\tau = \langle d\varphi, \varphi_\varphi \rangle_0.$$ 

□

**Lemma 10.** Given $\epsilon > 0$, there exists $t_0 \in (-\epsilon, \epsilon)$ such that $\cos(t_0)\varphi + \sin(t_0)\nu_\varphi$ is an immersion, and $\cos(t_0)\nu_\varphi - \sin(t_0)\varphi$ is its Gauss map.

**Proof.** We prove that the set

$$\{t \in \mathbb{R} | \cos(t)\varphi + \sin(t)\nu_\varphi \text{ is not an immersion}\}$$

contains at most $n$ elements: For a pair of real numbers $(t, t')$, such that $t - t'/\pi \in \pi \mathbb{Z}$, we have

$${\text{Ker}}(\cos(t)d\varphi + \sin(t)d\nu_\varphi) \cap {\text{Ker}}(\cos(t')d\varphi + \sin(t')d\nu_\varphi) = \{0\}$$

(otherwise we would have a contradiction with the fact that $(\varphi, \nu_\varphi)$ is an immersion). Hence there cannot be more than $n$ distinct values $t$ such that $\text{Ker}(\cos(t)d\varphi + \sin(t)d\nu_\varphi) \neq \{0\}$. The fact that $\cos(t_0)\nu_\varphi - \sin(t_0)\varphi$ is the Gauss map of $\cos(t_0)\varphi + \sin(t_0)\nu_\varphi$ comes from Lemma 9.

□

The last lemma proves that, up to a vertical translation, we may assume that $\varphi$ is an immersion.

**Lemma 11.** Denote by $g = \varphi^*(\ldots)_0$ the metric induced on $\mathcal{M}$ by $\varphi$ and $A$ the shaped operator associated to $\nu$. Then the metric $\bar{g} = \bar{\varphi}^*(\ldots)_1$ induced on $\mathcal{M}$ by $\bar{\varphi}$ is given by the formula

$$\bar{g} = \cos^2(\tau)g(\ldots) - 2\sin(\tau)\cos(\tau)g(A.,.) + \sin^2(\tau)g(A.,A.).$$

In particular, the non-degeneracy assumption on $\bar{g}$ implies to $\cot(\tau)$ is not equal to a principal curvature of $\varphi$. Moreover,

$$\bar{h}_\varphi := \langle \bar{h}(\ldots), \bar{\nu} \rangle_1 = (\cos^2(\tau) - \sin^2(\tau))g(A.,.) + \sin(\tau)\cos(\tau)(g(\ldots) - g(A.,A.)),$$

and

$$\langle \bar{H}_\varphi, \bar{\nu} \rangle_1 = \frac{1}{n} \sum_{i=1}^{n} \frac{\kappa_i + \tan(\tau)}{1 - \tan(\tau)\kappa_i},$$

where the $\kappa_i$ are the principal curvatures of $\varphi$. 

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We first claim that if $\bar{h}_\varphi$ vanishes, it must be totally geodesic: according to the previous lemma, this implies $A = \pm \kappa \text{Id}$, (hence $\kappa$ is constant) and $\kappa = \cot(\tau + \pi/2)$. Then a routine calculation shows that the shape operator of $\psi = \cos(\tau)\varphi + \sin(\tau)\nu$ vanishes, i.e. $\psi$ is totally geodesic. Since the height function $\tau$ is constant, $\bar{\varphi}$ is totally geodesic itself.

We now label the principal curvatures $\kappa_1 < ... < \kappa_p$, taking into account their multiplicity $m_i$. Hence $\langle \bar{H}\bar{\varphi}, \bar{\nu}\rangle_1$ vanishes if and only if
\[
\sum_{i=1}^{p} m_i \frac{\kappa_i \tan(\tau)}{1 - \tan(\tau)\kappa_i} = 0.
\]
Introducing $s := \cot(\tau)$, we see that $\bar{\varphi}$ is marginally trapped with respect to $\bar{\nu}$ if and only if the following polynomial vanishes:
\[
P(s) := \sum_{i=1}^{p} m_i (\kappa_i s + 1) \prod_{j \neq i} (s - \kappa_j) = 0.
\]
It is easy to check that signs of $P(\kappa_i)$ are alternate. Therefore, the polynomial $P(s)$ admits at least $p - 1$ distinct roots $s_i$ such that $\kappa_i < s_i < \kappa_{i+1}$. In particular, the degree of $P(s)$ is at least $p - 1$. Since the term of degree $p$ of $P(s)$ is $\sum_{i=1}^{p} m_i \kappa_i = nH$, it has degree $p - 1$ when $\varphi$ is minimal and degree $p$ otherwise. In the first case, since we already found $p - 1$ roots, we conclude that there are exactly $p - 1$ roots. Observe moreover that if $\varphi$ is minimal and $\tau = 0$ (which corresponds to $s = \pm \infty$), the immersion $\bar{\varphi} = (\varphi, 0)$ is not only marginally trapped, but also minimal. In the non-minimal case, by looking at $\lim_{s \to \pm \infty} \frac{P(s)}{s^p}$, we check that there exists one more root $s_p$ in $(-\infty, \kappa_1)$ or in $(\kappa_p, \infty)$, depending on whether $p$ is even or odd and $H$ is positive or negative.

The conclusion of Theorem 3 comes from the formula
\[
\left( \cos(\cot^{-1} s), \sin(\cot^{-1} s) \right) = \left( \frac{s}{\sqrt{1 + s^2}}, \frac{1}{\sqrt{1 + s^2}} \right).
\]
Finally, if $n = 2$, and $\varphi$ is not minimal, setting $a := \frac{\kappa_1 \kappa_2 - 1}{\kappa_1 + \kappa_2}$, the polynomial $P(s)$ is equivalent to $s^2 - 2as - 1 = 0$, whose two distinct roots are $s_\pm = a \pm \sqrt{a^2 + 1}$. Hence
\[
\tau_\pm = (\cot)^{-1} \left( a \pm \sqrt{a^2 + 1} \right),
\]
so we get the required formula. Observe however that if $a = 0$, then $\tau$ is constant and moreover $\psi = \frac{1}{\sqrt{2}}(\pm \varphi + \nu)$ is minimal, so again $\bar{\varphi}$ is minimal.
4.2 The $\mathbb{H}^{n+1} \times \mathbb{R}$ case

Let $\varphi = (\psi, \tau) : M \to \mathbb{H}^{n+1} \times \mathbb{R}$ an immersion whose induced metric is spacelike. Let $\bar{\nu} = (\nu, 1)$, where $\nu \in d\mathbb{S}^{n+1}$, be a normalized, null normal field along $\varphi$. We set $\varphi := \cosh(\tau)\psi + \sinh(\tau)\nu$. Reasoning like in the previous cases, we easily prove that, up to a vertical translation, we may assume that $\varphi$ is an immersion and that $\nu_{\varphi} := \sinh(\tau)\psi + \cosh(\tau)\nu$ is its Gauss map. Moreover, the non-degeneracy assumption on the induced metric on $\bar{\nu}$ implies that coth($\tau$) is not equal to a principal curvature $\kappa_i$ of $\varphi$. Finally, counting the principal curvatures with their multiplicity $m_i$, we have that $\langle \bar{H}_{\varphi}, \bar{\nu} \rangle_1$ vanishes if and only if $\sum_{i=1}^{p} m_i\kappa_i - \tanh(\tau)\kappa_i$ vanishes as well. Hence, if $s_i$ is a root of the polynomial

$$P(s) := \sum_{i=1}^{p} m_i(\kappa_i s - 1) \prod_{j \neq i} (s - \kappa_j)$$

satisfying in addition $|s_i| > 1$, the immersion

$$\varphi_i := (\cosh(\text{coth}^{-1}(s_i))\varphi + \sinh(\text{coth}^{-1}(s_i))\nu, \text{coth}^{-1}(s_i))$$

$$= \left( \frac{s_i\varphi + \nu}{\sqrt{s_i^2 - 1}}, \text{coth}^{-1}(s_i) \right)$$

is marginally trapped.

It seems difficult to determine exactly the number $q$ of roots of $P(s)$ such that $|s| > 1$. However, observe that given a monotone sequence of $\kappa_i$ such that $\kappa_i < -1$, $|\kappa_i| > 1$ or $\kappa_i > 1$, the signs of $P(\kappa_i) = m_i(\kappa_i^2 - 1) \prod_{j \neq i} (\kappa_i - \kappa_j)$ are alternate. Hence introducing

$$\alpha := \# \{ \kappa_i, 1 \leq i \leq p, \kappa_i < -1 \},$$

$$\beta := \# \{ \kappa_i, 1 \leq i \leq p, |\kappa_i| < 1 \},$$

$$\gamma := \# \{ \kappa_i, 1 \leq i \leq p, \kappa_i > 1 \},$$

$$\delta := \# \{ \kappa_i, 1 \leq i \leq p, |\kappa_i| = 1 \},$$

We deduce that there exist $\alpha - 1$ roots $s_i$ satisfying $\kappa_i < s_i < \kappa_{i+1} < -1$, giving rise to $\alpha - 1$ marginally trapped immersions $\varphi_i$. Analogously there exist $\gamma - 1$ solutions satisfying $1 < \kappa_i < s_i < \kappa_{i+1}$. Analysing the signs of $\lim_{s \to \pm \infty} \frac{P(s)}{s^{p}}$ as in the $\mathbb{S}^{n+1} \times \mathbb{R}$ case, we see that if $\sum_{i=1}^{p} m_i\kappa_i = nH$ does not vanish, the existence of one more solution $s \in (-\infty, \inf_{i} \kappa_i) \cup
(sup \{ \kappa_i \}, +\infty) is granted. On the other hand, if \( \beta \neq 0 \), the \( \beta - 1 \) roots satisfying \(-1 < \kappa_i < s_i < \kappa_{i+1} < 1 \) lead to no marginally trapped immersion. Finally, if 1 or \(-1 \) is a principal curvature, it is also a root of \( P(s) \), which again corresponds to no marginally trapped immersion. Finally, if \( \varphi \) is not minimal, we obtain the following inequalities:

\[ \alpha + \gamma - 1 \leq q \leq p - (\beta - 1) - \delta = \alpha + \gamma + 1. \]

If \( n = 2 \) and \( \varphi \) is not minimal, the polynomial \( P(s) \) is equivalent to \( s^2 - 2as + 1 \), where we set \( a := \frac{\kappa_1 \kappa_2 + 1}{\kappa_1 + \kappa_2} \). Without loss of generality we assume that \( a > 0 \). If \( a < 1 \), \( P(s) \) has no real solution and if \( a = 1 \), the unique solution is \( s = 1 \). Finally, if \( a > 1 \), the two distinct roots of \( P(s) \) are \( a \pm \sqrt{a^2 - 1} \), one of which is less than one and the other greater than one. Hence we get \( \tau := (\cot)^{-1} \left( a + \sqrt{a^2 - 1} \right) \), so we get the required formula. Observe that if \( |\kappa_1| > 1 \) and \( |\kappa_2| > 1 \), we have \( \alpha + \gamma - 1 = q = 1 \), i.e. the left hand side inequality above is sharp.

### 5 Examples

Here we briefly discuss how some of the examples of [CVdV] can be recovered from our construction.

The two families of marginally trapped surfaces found by B.-Y. Chen and J. Van der Veken in \( \mathbb{R}^4_1 \) are:

\[ L_1(x, y) := (x, y, f(x), f(x)), \]

where \( f \) is an arbitrary differentiable function with \( f''(x) \) being nowhere zero, and

\[ L_2(x, y) := \left( y \cos x - \int_0^x r(x) \sin x \, dx, y \sin x + \int_0^x r(x) \cos x \, dx, \right. \]

\[ \left. \int_0^x r(x)q'(x) \, dx + q(x)y, \int_0^x r(x)q'(x) \, dx + q(x)y \right), \]

where \( q \) and \( r \) are defined on an open interval \( I \supseteq 0 \) satisfying \( q''(x) + q(x) \neq 0 \) for each \( x \in I \).

Thus \( L_1(x, y) = (i(x, y), 0) + f(x)(\nu_0, 1), \) where \( \nu_0 = (0, 0, 1) \), so we are in the first case (null second fundamental form) of Theorem 1. Moreover, since

\[ T(x, y) := \left( y \cos x - \int_0^x r(x) \sin x \, dx, y \sin x + \int_0^x r(x) \cos x \, dx \right) \]
is simply a reparametrization of an open subset of the plane, setting
\[ \tau(x, y) := \int_0^x r(x)q'(x)dx + q(x)y, \]
the second family takes the form \( L_2(x, y) = (\iota \circ T(x, y) + \tau(x, y)\nu_0, \tau(x, y)) \), so we are again in the case of null second fundamental form.

Next, consider the immersion in \( d\mathbb{S}^4 \) given by
\[ L_3(x, y) = (\sin x \cos y, \sin y, \cos x \cos y, f(x) \cos y, f(x) \cos y) \]
where \( f \) is an arbitrary differentiable function defined on an open interval \( I \) satisfying \( f'' + f \neq 0 \) at each point in \( I \). It takes the form
\[ L_3(x, y) = (\iota(x, y), 0) + \tau(x, y)(\nu, 1), \]
where \( \tau(x, y) := f(x) \sin y, \nu := (0, 0, 0, 1) \) and \( \iota(x, y) : \mathbb{S}^2 \to \mathbb{S}^3 \) is the totally geodesic embedding given in coordinates by \( \iota(x, y) = (\sin x \cos y, \sin y, \cos x \cos y, 0) \).

Hence \( L_3 \) has null second fundamental form (first case of Theorem 2).

Finally, consider the immersion in \( Ad\mathbb{S}^4 \) given by
\[ L_4(x, y) := \left( e^y - 2 \sinh y, xe^y, x^2 e^y - \frac{1}{2}e^y, \frac{3}{2}e^y - 2 \sinh y, x^2 e^y \right). \]
A straightforward calculation shows that a normalized, null normal vector along \( L_4 \) is \( \bar{\nu} = (-1, 0, 1, -1, 1) = (\nu, 1) \). Since \( \bar{\nu} \) is constant, \( \langle \bar{h}(.,.), \bar{\nu} \rangle_2 \) vanishes, so is in particular the second fundamental form \( \bar{h}(.,.) \) is null and \( L_4 \) is marginally trapped. Observe moreover that \( \varphi := \psi - \tau \nu = \psi - x^2 e^y(-1, 0, 1, -1) \) is an immersion whose normal unit vector \( \nu = (-1, 0, 1, -1) \) is constant, therefore \( \varphi \) is totally geodesic.

We leave to the reader the easy task to check that all other examples of \[CVdV\] have null second fundamental form.

References

[AGM] J. Aledo, J. Galvez, P. Mira, Marginally trapped surfaces in \( L^4 \) and an extended Weierstrass-Bryant representation, Ann. Global Anal. Geom. 28 (2005), no. 4, 395–415

[APa] L. Alías, B. Palmer, Deformations of stationary surfaces Class. Quantum Gravity 14 (1997) 2107–2111
[An1] H. Anciaux, *Spaces of geodesics of pseudo-Riemannian space forms and normal congruences of hypersurfaces*, to appear in Trans. of the AMS

[An2] H. Anciaux, *Marginally trapped submanifolds in Robertson-Walker spaces*, in preparation

[AG] H. Anciaux, B. Guilfoyle, *On the three-dimensional Blaschke-Lebesgue problem*, Proc. Amer. Math. Soc. 139 (2011), no. 5, 1831–1839

[CFG] J.-L. Cabrerizo, M. Fernández, J.-S. Gómez, *Isotropy and marginally trapped surfaces in a spacetime*, Classical Quantum Gravity 27 (2010), no. 13, 135005 (12pp)

[Ch] B.-Y. Chen, *Classification of marginally trapped surfaces of constant curvature in Lorentzian complex plane*, Hokkaido Math. J. 38 (2009), no. 2, 361–408

[CD] B.-Y. Chen, F. Dillen, *Classification of marginally trapped Lagrangian surfaces in Lorentzian complex space forms*, J. Math. Phys. 48 (2007), no. 1, 013509, 23 pp.

[CvdV] B.-Y. Chen, J. Van der Veken, *Marginally trapped surfaces in Lorentzian space forms with positive relative nullity*, Class. Quantum Grav. 24 (2007) 551–563

[CGP] P. Chruściel, G. Galloway, D. Pollack, *Mathematical general relativity: a sampler*, Bull. Amer. Math. Soc. (N.S.) 47 (2010), no. 4, 567–638

[GS] Y. Godoy, M. Salvai, *The canonical contact structure on the space of oriented null geodesics of pseudospheres and products*, to appear in Adv. Geom.

[KT] B. Khesin, S. Tabachnikov, *Pseudo-Riemannian geodesics and billiards*, Adv. Math. 221 (2009), 1364–1396

[Pa] B. Palmer, *Area minimization among marginally trapped surfaces in Lorentz-Minkowski space*, Calc. Var. and Partial Diff. Eq. 41 (2011), no. 3-4, 387–395

[Pe] R. Penrose, *Gravitational collapse and space-time singularities*, Phys. Rev. Lett. 14 (1965), 57–59
[VdV] J. Van der Veken, *Marginally trapped surfaces in Lorentzian space forms*, Pure and applied differential geometry, PADGE 2007, 261–266, Ber. Math., Shaker Verlag, Aachen, 2007

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