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1. INTRODUCTION

Given a positive integer \( m \), let \( \mathbb{N}_{>0}^m := \{(a_1, a_2, \ldots, a_m) \mid a_i \in \mathbb{N}_{>0} \forall 1 \leq i \leq m \} \) be the set of \( m \)-tuples of positive integers. We now fix an \( n \in \mathbb{N}_{>0} \).

**Definition 1.1.** An \( m \)-tuple \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathbb{N}_{>0}^m \) is a composition of \( n \) if \( \sum_{i=1}^{m} \lambda_i = n \); if moreover, \( \lambda_i \geq \lambda_{i+1} \forall 1 \leq i \leq m - 1 \), we say that \( \lambda \) is a partition of \( n \). We refer to \( \lambda_i \) as the parts of \( \lambda \) and denote the number of parts of \( \lambda \) by \( l(\lambda) := m \).

We denote the set of all partitions of \( n \) by \( \mathcal{P}(n) \).

**Definition 1.2.** Let \( e \geq 2 \) be an integer. We call a partition \( \lambda \) \( e \)-singular if there is an integer \( 1 \leq i \leq l(\lambda) - e + 1 \) such that \( \lambda_i = \lambda_{i+1} = \cdots = \lambda_{i+e-1} \neq 0 \); we call \( \lambda \) \( e \)-regular otherwise. We denote the set of all \( e \)-singular partitions of \( n \) by \( \mathcal{P}^{e-\text{sing}}(n) \) and the set of all \( e \)-regular partitions of \( n \) by \( \mathcal{P}^{e-\text{reg}}(n) \).

Suppose that \( q \) is a non-zero element of a field \( \mathbb{F} \). Unless mentioned explicitly, we assume that \( \mathbb{F} \) can have any characteristic (including zero). Let \( e \) be the least positive integer such that \( 1 + q + \cdots + q^{e-1} = 0 \), assuming throughout the paper that it exists. Let \( n \) be a positive integer. Denote by \( \mathcal{S}_n = \mathcal{S}_q(n, n) \) the \( q \)-Schur algebra (over \( \mathbb{F} \)) defined in [5]. When \( q = 1 \), this is just the classical Schur algebra over \( \mathbb{F} \). To each partition \( \lambda \in \mathcal{P}(n) \), we associate a Weyl module \( W^\lambda \). Each \( W^\lambda \) has a simple head \( L^\lambda \) which is self-dual with respect to the contravariant duality induced by the anti-automorphism of \( \mathcal{S}_n \). Moreover, the set \( \{ L^\lambda \mid \lambda \in \mathcal{P}(n) \} \) is a complete set of mutually non-isomorphic simple modules of \( \mathcal{S}_n \). Given any two partitions \( \lambda, \mu \in \mathcal{P}(n) \), the composition multiplicities \([W^\lambda : L^\mu]\) are called the decomposition numbers of \( \mathcal{S}_n \). We record these in a decomposition matrix with rows and columns indexed by \( \mathcal{P}(n) \) and whose \((\lambda, \mu)\)-entry is \([W^\lambda : L^\mu]\).

Let \( \mathcal{H}_n = \mathcal{H}_{\mathbb{F}, q}(\mathfrak{S}_n) \) denote the Iwahori-Hecke algebra of the symmetric group \( \mathfrak{S}_n \). This is a ‘deformation’ of the group algebra \( \mathbb{F}\mathfrak{S}_n \); we refer the reader to [17] for its definition. When \( q = 1 \), this is simply the group algebra \( \mathbb{F}\mathfrak{S}_n \). The representation theory of \( \mathcal{H}_n \) is very similar to that of \( \mathcal{S}_n \). To each partition \( \lambda \in \mathcal{P}(n) \), we associate a Specht Module \( S^\lambda \) for \( \mathcal{H}_n \). If \( \lambda \in \mathcal{P}^{e-\text{reg}}(n) \), then \( S^\lambda \) has a simple head \( D^\lambda \). The set \( \{ D^\lambda \mid \lambda \in \mathcal{P}^{e-\text{reg}}(n) \} \) is a complete set of mutually non-isomorphic simple \( \mathcal{H}_n \)-modules. The decomposition numbers of \( \mathcal{H}_n \) are the composition multiplicities \([S^\lambda : D^\mu]\); the decomposition matrix for \( \mathcal{H}_n \) has rows indexed by \( \mathcal{P}(n) \) and columns indexed by \( \mathcal{P}^{e-\text{reg}}(n) \), with \((\lambda, \mu)\)-entry \([S^\lambda : D^\mu]\).
**Warning:** The Specht and Weyl modules, $S^\lambda$ and $W^\lambda$ defined in [17] are isomorphic to the dual of the Specht and Weyl modules defined by Dipper and James in [4] indexed by $\lambda$. We adopt the Specht and Weyl modules defined by Dipper and James in [4] rather than those in [17].

A central problem in the study of the representation theory of $S_n$ and $H_n$ is to determine their decomposition matrices. These two problems are closely related. In fact, there is an exact functor called the Schur functor from the category of $S_n$-modules to the category of $H_n$-modules. Given $\mu \in \mathcal{P}(n)$, the Schur functor sends the Weyl module $W^\mu$ to the Specht module $S^\mu$. If $\mu \in \mathcal{P}^{e-\text{reg}}(n)$, the simple module $L^\mu$ is sent to the simple module $D^\mu$; otherwise if $\mu \in \mathcal{P}^{e-\text{sing}}(n)$, $L^\mu$ is sent to zero. Using the Schur functor, one can deduce the following:

**Theorem 1.3.** [17, Theorem 4.18] Suppose that $\lambda$ and $\mu$ are partitions of $n$ and that $\mu$ is $e$-regular. Then, 

$$[W^\lambda : L^\mu] = [S^\lambda : D^\mu].$$

In other words, the decomposition matrix of $H_n$ is a submatrix of the decomposition matrix of $S_n$; obtained by deleting the columns indexed by $\mathcal{P}^{e-\text{sing}}(n)$. Lascoux, Leclerc and Thibon conjectured that their recursive LLT algorithm [15] calculates the decomposition matrices of $H_n$ over $\mathbb{C}$. This was later proved by Ariki in [1]. It is known that decomposition matrices over fields of prime characteristic may be obtained from decomposition matrices over $\mathbb{C}$ by post-multiplying by an ‘adjustment matrix’. In view of this, we often study the adjustment matrices instead of the decomposition matrices directly when working over fields of prime characteristic. The complexity of the representation theory of a block of $H_n$ or $S_n$ is roughly captured by a measure called the weight of that block. James conjectured [11] that when the weight of a block is less than the characteristic $p > 0$ of the underlying field, the adjustment matrix is the identity matrix; therefore the decomposition matrix is computable using the LLT algorithm in principle. In the case of $H_n$, the conjecture has been proved for weights up to 4 by the works of Richards [19] and Fayers [8, 9]. The author [16] has also proved that the adjustment matrix for the principal block of $H_n$ of weight 5 is the identity matrix when $\text{char}(\mathbb{F}) \geq 5$ and $e \neq 4$. For the $q$-Schur algebras, the weight 2 case was proved by Schroll and Tan [21]. However, Williamson found a counter-example [25] to James’s Conjecture. Nevertheless, the smallest counter-example produced in his paper occurs in the symmetric group $S_n$ where $n = 1744860$. There is considerable interest in finding smaller counter-examples.

This paper is structured in the following way. In section 2 we mention some fundamental results in the modular representation theory of $H_n$ and $S_n$. Our main object of interest, the adjustment matrices are introduced in section 3. We give an overview of the existing results on adjustment matrices and provide some new tools for studying them. In sections 4 and 5 we apply the results from sections 2 and 3 to prove James’s Conjecture for blocks of $S_n$ of weights 3 and 4. We also show that the decomposition numbers for weight 3 blocks of $S_n$ are bounded above by one.

2. Background

2.1. **Partitions and abacus displays.** Take an abacus with $e$ vertical runners, numbered $0, \ldots, e-1$ from left to right, marking positions $0, 1, \ldots$ on the runners increasing from left to right along successive ‘rows’. Given $\lambda \in \mathcal{P}(n)$, take an integer $r \geq l(\lambda)$. Define

$$\beta_i = \begin{cases} 
\lambda_i + r - i, & \text{if } 1 \leq i \leq l(\lambda), \\
 r - i, & \text{if } r > l(\lambda).
\end{cases}$$

Now, place a bead at position $\beta_i$ for each $i$. The resulting configuration is an abacus display for $\lambda$ with $r$ beads. If a bead has been placed at position $j$, we say that the position $j$ is occupied. Otherwise, we say that the position $j$ is unoccupied or vacant. We remark that moving a bead from position $a$ to an unoccupied position $b < a$ is akin to removing a rim hook of length $h = b - a$ from the Young diagram of $\lambda$. The leg-length of the hook is given by the number of occupied positions.
between $a$ and $b$. By moving all the beads as high as possible on their runners, the resulting configuration is an abacus display for the $e$-core of $\lambda$. The relative $e$-sign of $\lambda$, denoted by $\sigma_e(\lambda)$ is $(-1)^t$, where $t$ is the total leg lengths of $e$-hooks removed to obtain the $e$-core (See \cite[\S 2]{15}). Note that although $t$ depends on the choice of $e$-hooks removed, its parity is constant. We define the $e$-weight of a bead to be the number of vacant positions above it on the same runner. The $e$-weight of $\lambda$ is the sum of $e$-weights of all the beads in an abacus display for $\lambda$. Thus, if $\lambda$ has $e$-weight $w$, then its $e$-core is a partition of $n - ew$. If there is no ambiguity, we often just refer to $e$-weight as weight and $e$-core as core.

It is easy to read addable and removable nodes from an abacus display. Display $\lambda$ on an abacus with $e$ runners and $r$ beads, where $r \geq t(\lambda)$. Let $i$ be the residue class of $(j + r)$ modulo $e$. Then, the removable nodes of $\lambda$ with $e$-residue $j$ correspond to the beads on runner $i$ with unoccupied preceding positions, while the addable nodes of $e$-residue $j$ correspond to the vacant positions on runner $i$ with occupied preceding positions. We call the removable (resp. addable) nodes with $e$-residue $j$ $j$-removable (resp. $j$-addable). Removing a removable node corresponds to moving the corresponding bead to its preceding unoccupied position, while adding an addable node corresponds to moving the corresponding bead to its succeeding unoccupied position. We adopt the convention that position 0 has an occupied preceding position.

2.2. Blocks of $S_n$ and $\mathcal{H}_n$.

**Theorem 2.1.** \cite[Theorem 5.37]{17} Let $\lambda$ and $\mu$ be partitions of $n$. Then, $W^\lambda$ and $W^\mu$ lie in the same block of $S_n$ if and only if $\lambda$ and $\mu$ have the same $e$-core.

Applying the Schur functor to the theorem above, we get the following corollary.

**Corollary 2.2.** (Nakayama Conjecture) \cite[Corollary 5.38]{17} Let $\lambda$ and $\mu$ be partitions of $n$. Then, $S^\lambda$ and $S^\mu$ lie in the same block of $\mathcal{H}_n$ if and only if $\lambda$ and $\mu$ have the same $e$-core.

Given a block $B$ of $S_n$ or $\mathcal{H}_n$, we say that a partition $\lambda \in \mathcal{P}(n)$ lies in $B$ if $W^\lambda$ or $S^\lambda$ lies in $B$. Given the Nakayama Conjecture, we may define the $e$-weight and $e$-core of a block $B$ of $S_n$ or $\mathcal{H}_n$ simply to be the $e$-weight and $e$-core of a partition lying in $B$.

Let $\lambda(i)$ be the partition corresponding to the abacus display containing only a single runner, the $i^{th}$ runner. Denote the number of beads on the $i^{th}$ runner as $b_i$. Then, we may write $\lambda$ as

$$\lambda = \langle 0_{\lambda(0)}, \ldots, (e - 1)_{\lambda(e-1)} \mid b_0, \ldots, b_{e-1} \rangle;$$

we omit $i_{\lambda(i)}$ if $\lambda(i) = \emptyset$ and write $i_{\lambda(i)}$ simply as $i$ if $\lambda(i) = (1)$. Additionally, we may omit $b_0, \ldots, b_{e-1}$ if there is no ambiguity. If $\lambda$ has $e$-weight $w$ and lies in a block $B$ of $S_n$ or $\mathcal{H}_n$, we say that $B$ is the block of $e$-weight $w$ with the $\langle b_0, \ldots, b_{e-1} \rangle$ notation.

2.3. Jantzen-Schaper formula and the product order. Let $\succeq$ denote the usual dominance order for partitions. Due to the fact that $S_n$ is a cellular algebra \cite{10}, we have the following result.

**Theorem 2.3.** \cite[Corollary 4.13]{5} Suppose that $\lambda$ and $\mu$ are partitions of $n$. We have

- $[W^\mu : L^\mu] = 1$,
- $[W^\lambda : L^\mu] = 0$ unless $\mu \succeq \lambda$.

Let $\lambda$ be a partition and consider its abacus display, say with $r$ beads. Suppose that after moving a bead at position $a$ up its runner to a vacant position $a - ie$, we obtain the partition $\mu$. Denote $l_{\lambda\mu}$ for the number of occupied positions between $a$ and $a - ie$, and let $h_{\lambda\mu} = i$.

Further, write $\lambda \overset{\lambda}{\rightarrow} \tau$ if the abacus display of $\tau$ with $r$ beads is obtained from that of $\mu$ by moving a bead at position $b - ie$ to a vacant position $b$, and $a < b$.

**Definition 2.4.** Jantzen-Schaper bound

Let $p = \text{char}(\mathbb{F})$. For any ordered pair $\langle \lambda, \tau \rangle$, we define the Jantzen-Schaper coefficient to be the
integer
\[ J_\mathbb{F}(\lambda, \tau) := \sum_{\lambda \to \tau} (-1)^{\lambda_0 + \lambda_\omega} (1 + v_p(h_{\lambda_0})), \]
where \( v_p \) denotes the standard \( p \)-valuation if \( p > 0 \) and \( v_0(x) = 0 \ \forall x \). The Jantzen-Schaper bound is defined as the integer
\[ B_\mathbb{F}(\lambda, \mu) = \sum_{\tau} J_\mathbb{F}(\lambda, \tau)[W_\mathbb{F}^\tau : L_\mathbb{F}^\mu]. \]

**Remark 2.5.** If \( \lambda \) has \( e \)-weight \( w \) and \( p > w \), then \( v_p(h_{\lambda_0}) \) is always zero. In this case, \( J_\mathbb{F}(\lambda, \tau) \) is independent of \( \text{char}(\mathbb{F}) \) and we may just refer to it as \( J(\lambda, \tau) \). Similarly, if \( B_\mathbb{F}(\lambda, \tau) \) turns out to be independent of \( \text{char}(\mathbb{F}) \), we just refer to it as \( B(\lambda, \tau) \).

**Theorem 2.6.** Jantzen-Schaper formula(\[13\] Theorem 4.7])
\[ [W_\mathbb{F}^\lambda : L_\mathbb{F}^\mu] \leq B_\mathbb{F}(\lambda, \mu). \]
Moreover, the left-hand side is zero if and only if the right-hand side is zero.

**Corollary 2.7.** If \( B_\mathbb{F}(\lambda, \mu) \leq 1 \), then
\[ [W_\mathbb{F}^\lambda : L_\mathbb{F}^\mu] = B_\mathbb{F}(\lambda, \mu). \]

We write \( \lambda \rightarrow \tau \) if there exists some \( \mu \) such that \( \lambda \nu \rightarrow \tau \). Further, write \( \lambda <_J \sigma \) if there exist partitions \( \tau_0, \tau_1, \ldots, \tau_r \) such that \( \tau_0 = \lambda, \tau_r = \sigma \) and \( \tau_{i-1} \rightarrow \tau_i \) \( \forall i \in \{1, 2, \ldots, r\} \). We call \( \leq_J \) the Jantzen order and it is clear that this defines a partial order on the set of all partitions. Only partitions in the same block are comparable under this partial order. Moreover, the dominance order extends the Jantzen order in the following sense: \( \mu \geq_J \lambda \) implies \( \mu \geq \lambda \). Theorem 2.6 can be used to refine Theorem 2.3 in the following way:

**Theorem 2.8.** Suppose that \( \lambda \) and \( \mu \) are partitions of \( n \). Then,
- \( [W_\mathbb{F}^\mu : L_\mathbb{F}^\mu] = 1; \)
- \( [W_\mathbb{F}^\lambda : L_\mathbb{F}^\mu] \geq 0 \Rightarrow \mu \geq_J \lambda. \)

It is difficult to check that \( \mu \not\geq_J \lambda \) by inspection. To this end, we introduce the product order on partitions which was first defined by Tan in \[23\]. Let \( \lambda \) be a partition, displayed on an abacus with \( e \) runners and \( r \) beads. Suppose that the beads having positive \( e \)-weights are at positions \( a_1, a_2, \ldots, a_s \) with weights \( w_1, w_2, \ldots, w_s \) respectively. The induced \( e \)-sequence of \( \lambda \), denoted \( s(\lambda)_r \), is defined as
\[ \bigcup_{i=1}^{s} (a_i, a_i - e, \ldots, a_i - (w_i - 1)e), \]
where \( (b_1, b_2, \ldots, b_t) \cup (c_1, c_2, \ldots, c_u) \) denotes the weakly decreasing sequence obtained by rearranging terms in the sequence \( (b_1, \ldots, b_t, c_1, \ldots, c_u) \). Note that \( s(\lambda)_r \in \mathbb{N}_{>0}^w \), where \( w \) is the \( e \)-weight of \( \lambda \).

We define a partial order \( \geq_p \) on the set of partitions by: \( \mu \geq_p \lambda \) if and only if \( \mu \) and \( \lambda \) have the same \( e \)-core and \( e \)-weight, and \( s(\mu)_r \geq s(\lambda)_r \) (for sufficiently large \( r \)) in the standard product order of \( \mathbb{N}_{>0}^w \).

**Lemma 2.9.** (\[23\] Lemma 2.9])
\[ \lambda \leq_J \mu \Rightarrow \lambda \leq_p \mu. \]
Therefore, \( \mu \not\geq_p \lambda \Rightarrow [W_\mathbb{F}^\lambda : L_\mathbb{F}^\mu] = 0. \)
Example 2.10. Suppose that \( e = 5, r = 10, \lambda = (2, 4, 2) \mid 2^5 \) = \( (10, 6, 5, 2, 1^2) \) and \( \mu = (0, 1, 4, 3) \mid 2^5 \) = \( (15, 3^2, 2^2) \).

![Abacus Display]

Then, \( s(\lambda)_{10} = (19, 14, 14, 12, 9) \), \( s(\mu)_{10} = (24, 19, 14, 11, 10) \). Hence, \( \mu \triangleright \lambda \) but \( \mu \not\succ_P \lambda \).

2.4. \( v \)-decomposition numbers. For a brief introduction to \( v \)-decomposition numbers, the reader may refer to [21] §2.7. For our purposes, all we need to know is that given 2 partitions \( \lambda \) and \( \mu \) of \( n \), we may define a polynomial \( d^e_{\lambda \mu}(v) \in \mathbb{N}[v] \) with the following crucial property (which explains its name):

**Theorem 2.11.** Let \( \lambda, \mu \in \mathcal{P}(n) \). Then,

1. \( d^e_{\lambda \mu}(1) = [W^\lambda C : L^\mu C] \),
2. \( \frac{d}{dv}(d^e_{\lambda \mu}(v))|_{v=1} = B_C(\lambda, \mu) \).

**Proof.** (1) has been proven by Varagnolo and Vasserot in [24]. (2) was proved by Schroll and Tan in [21] Theorem 2.13. \( \square \)

The following result tells us that \( d^e_{\lambda \mu}(v) \) is either an even or an odd polynomial, depending on the relative \( e \)-signs of \( \lambda \) and \( \mu \).

**Theorem 2.12.** [24] Theorem 2.4 If \( d^e_{\lambda \mu}(v) \neq 0 \), then

\[
d^e_{\lambda \mu}(v) \in \begin{cases} \mathbb{N}[v^2] & \text{if } \sigma_e(\lambda) = \sigma_e(\mu), \\ v\mathbb{N}[v^2] & \text{otherwise.} \end{cases}
\]

**Remark 2.13.** In section 3 we will sometimes calculate that \( B_C(\lambda, \mu) = 2 \) for some pair of partitions \( (\lambda, \mu) \). In order to verify whether \( [W^\lambda C : L^\mu C] \) is 1 or 2, we would calculate \( \sigma_e(\lambda) \) and \( \sigma_e(\mu) \). If they turn out to be the same, then \( d^e_{\lambda \mu}(v) = v^2 \) and \( [W^\lambda C : L^\mu C] = 1 \) by Theorem 2.11 and Theorem 2.12. Otherwise, \( d^e_{\lambda \mu}(v) = 2v \) and \( [W^\lambda C : L^\mu C] = 2 \).

The following two theorems commonly known as the Runner Removal Theorems allow us to relate \( v \)-decomposition numbers for different values of \( e \).

**Theorem 2.14.** [14] Theorem 4.5] Suppose that \( e \geq 3 \). Let \( \lambda \) and \( \mu \) be partitions lying in the same block, and display them on an abacus with \( e \) runners and \( r \) beads, for some large enough \( r \). Suppose that there exists some \( i \) such that in both abacus displays, the last bead on runner \( i \) occurs before every unoccupied space on the abacus. Define two abacus displays with \( e - 1 \) runners by deleting runner \( i \) from each display, and let \( \lambda^- \) and \( \mu^- \) be the partitions defined by these displays. Then,

\[
d^e_{\lambda \mu}(v) = d^{e-1}_{\lambda^- \mu^-}(v).
\]

**Theorem 2.15.** [7] Suppose that \( e \geq 3 \). Let \( \lambda \) and \( \mu \) be partitions lying in the same block, and display them on an abacus with \( e \) runners and \( r \) beads, for some large enough \( r \). Suppose that there exists some \( i \) such that in both abacus displays, the first unoccupied space on runner \( i \) occurs after every bead on the abacus. Define two abacus displays with \( e - 1 \) runners by deleting runner \( i \) from each display, and let \( \lambda \) and \( \mu \) be the partitions defined by these displays. Then,

\[
d^e_{\lambda \mu}(v) = d^{e-1}_{\lambda \mu}(v).
\]
Remark 2.16. Calculating υ-decomposition numbers in practice. When μ ∈ ℙ^c-reg(n), we have a relatively fast recursive algorithm for calculating d^c_μ(υ) called the LLT algorithm [15] (developed by Lascoux, Leclerc and Thibon). The author uses the GAP package hecke (https://www.gap-system.org/Packages/hecke.html) which was first written by Mathas for running the LLT algorithm.

If μ ∈ ℙ^c-sing(n), we may calculate d^c_μ(υ) by adding an empty runner to λ and μ and using Theorem 2.14. By Theorem 2.11, the decomposition matrix for Sn can be calculated in principle when the underlying field is C.

Example 2.17. Suppose that υ ≥ 4, μ = (0,1,2,3 | 2^4), λ = (0_1,2_1,2), μ = (0,1,2,3 | 2^4), λ = (0_1,2_1,2), μ = (0,1,2,3 | 2^4,0) and λ = (0_1,2_1,2 | 2^4,0).

\[
\begin{array}{c}
\mu & \ldots & \hat{\mu} & \mu^+ \\
\lambda & \ldots & \hat{\lambda} & \lambda^+ \\
\end{array}
\]

By Theorem 2.15 d^c_λ(μ) = d^4_λ(υ). By Theorem 2.14 and the LLT algorithm, d^4_λ(υ) = d^5_λ(μ^+) = υ.

2.5. The modular branching rules. We use some notational conventions for modules. We write

\[ M \sim M_1^a_1 + M_2^a_2 + \cdots + M_t^a_t \]

to indicate that M has a filtration in which the factors are M_1, \ldots, M_t appearing a_1, \ldots, a_t times respectively. Additionally, we write M^{\oplus a} to indicate the direct sum of a isomorphic copies of M.

There are restriction and induction functors which are exact functors between Sn−1 and Sn. If M is a module for Sn, the restriction of M to Sn−1 is denoted by M↓Sn−1. Similarly, the induction of M to Sn+1 is denoted by M↑Sn+1. If B is a block of Sn−1, we write M↓B to indicate the projection of M↓Sn−1 onto B. Similarly, if C is a block of Sn+1, we write M↑B to indicate the projection of M↑Sn+1 onto C. In this section, we describe the restriction and induction of Weyl modules and simple modules.

Suppose that A, B and C are blocks of Sn−k, Sn and Sn+k respectively, and that there is an e-residue j such that a partition lying in A may be obtained from a partition lying in B by removing exactly k j-removable nodes, while a partition lying in C may be obtained from a partition lying in B by adding exactly k j-addable nodes.

Suppose that λ is a partition in B, and that λ−1, λ−2, \ldots, λ−t are the partitions in A that may be obtained from λ by removing k j-removable nodes. Similarly, let λ+1, λ+2, \ldots, λ+s be the partitions in C that may be obtained from λ by adding k j-addable nodes. We have the following result.

Theorem 2.18. (The Branching Rule 2) Suppose that A, B, C and λ are as above. Then,

\[ W^λ↓A \sim (W^λ^{-1})^{k!} + (W^λ^{-2})^{k!} + \cdots + (W^λ^{-t})^{k!} \]

and

\[ W^λ↑C \sim (W^λ^{+1})^{k!} + (W^λ^{+2})^{k!} + \cdots + (W^λ^{+s})^{k!} \]

We now discuss the restriction and induction of simple modules. Suppose that the nodes with residue j are on runner i. The j-signature of λ is the sequence of signs defined as follows. Starting from the top row of the abacus display for λ and working downwards, write a − if there is a bead on runner i but no bead on runner i − 1; write a + if there is a bead on runner i − 1 but no bead on runner i; write nothing for that row otherwise. Given the j-signature of λ, successively delete all neighbouring pairs of the form −+ to obtain the reduced j-signature of λ. If there are any −
(resp. +) signs in the reduced \( j \)-signature of \( \lambda \), we call the corresponding nodes on runner \( i \) normal (resp. conormal). Normal (resp. conormal) nodes with residue \( j \) are also called \( j \)-normal (resp. \( j \)-conormal).

**Definition 2.19.** Let \( \lambda \) be a partition. We denote the number of \( j \)-normal nodes of \( \lambda \) by \( \epsilon_j(\lambda) \) and the number of \( j \)-conormal nodes of \( \lambda \) by \( \varphi_j(\lambda) \). For \( t \leq \epsilon_j(\lambda) \), we define \( \tilde{E}_j^t \lambda \) to be the partition obtained from \( \lambda \) by removing the \( t \) highest (in an abacus display for \( \lambda \)) \( j \)-normal nodes. For \( t \leq \varphi_j(\lambda) \), we define \( \tilde{F}_j^t \lambda \) to be the partition obtained from \( \lambda \) by adding the \( t \) lowest (in an abacus display for \( \lambda \)) \( j \)-conormal nodes.

**Theorem 2.20.** \( [2] \) Suppose that \( A, B, C \) and \( \lambda \) are as above.

- If \( \epsilon_j(\lambda) < k \), then \( L^A \downarrow \lambda = 0 \).
- If \( \epsilon_j(\lambda) > k \), then \( \text{soc}(L^A \downarrow \lambda) \cong (L^\tilde{E}_j^k \lambda)^{\oplus k!} \).
- If \( \epsilon_j(\lambda) = k \), then \( L^A \downarrow \lambda \cong (L^\tilde{E}_j^k \lambda)^{\oplus k!} \).
- If \( \varphi_j(\lambda) < k \), then \( L^A \uparrow \lambda = 0 \).
- If \( \varphi_j(\lambda) > k \), then \( \text{soc}(L^A \uparrow \lambda) \cong (L^\tilde{F}_j^k \lambda)^{\oplus k!} \).
- If \( \varphi_j(\lambda) = k \), then \( L^A \uparrow \lambda \cong (L^\tilde{F}_j^k \lambda)^{\oplus k!} \).

The following lemma guarantees that the weight of a partition will not increase if we remove all of its \( j \)-normal nodes (or add all of its \( j \)-conormal nodes) for some \( e \)-residue \( j \).

**Lemma 2.21.** Suppose that \( \mu \) lies in a block of \( S_n \) of weight \( w \). Let \( k = \epsilon_j(\mu) \) and \( l = \varphi_j(\mu) \). Then, \( \tilde{E}_j^k \mu \) and \( \tilde{F}_j^l \mu \) have weight \( w - kl \).

Before we proceed with the proof of this lemma, it may be helpful to first look at an example.

**Example 2.22.** In the diagrams below, we only display two runners of the abacus displays for the partitions; the runner on the right corresponds to the nodes with \( e \)-residue equal to \( j \). We highlight the reduced \( j \)-signature in red. In this example, \( k = 3 \) and \( l = 2 \).

Assuming for simplicity that the other runners which are not displayed in the diagram have no weight, we may count that \( \mu \) has weight 43 while \( \tilde{E}_j^k \mu \) and \( \tilde{F}_j^l \mu \) have weight 37.

**Proof.** We only show the proof for \( \tilde{E}_j^k \mu \) here as the other case is similar. Let \( \mu \) be a partition, \( k = \epsilon_j(\mu) \) and \( l = \varphi_j(\mu) \). We focus our attention on the two adjacent runners, with the runner on the right corresponding to the nodes with \( e \)-residue equal to \( j \). Our task is to keep track of the change of weight of each bead in these two runners when \( \tilde{E}_j^k \) is applied to \( \mu \). We may categorize the beads in these two runners into four categories:

1. Normal; in which case there are no conormal beads in the rows below it.
2. Conormal; in which case there are no normal beads in the rows above it.
3. Two beads in the same row.

Two beads in two distinct rows forming a \((-\, +)\) pair that was deleted from the \(j\)-signature to form the reduced \(j\)-signature; by definition, there are no normal or conormal beads in between these two rows.

The change of weight of each of these types of beads when \(\tilde{E}_j^k\) is applied to \(\mu\) is summarised below:

1. Each normal bead loses weight \(l\) when moved to the left.
2. The conormal beads do not experience any change in weight since there are no normal beads above them.
3. If two beads are in the same row and that there are \(\alpha\) normal beads above this row, then the bead on the right gains weight \(\alpha\) while the bead on the left loses weight \(\alpha\). Therefore, there is no net change in weight contributed by beads of this type.
4. Suppose that we have two beads in two distinct rows forming a \((-\, +)\) pair that was deleted from the \(j\)-signature to form the reduced \(j\)-signature. If there are \(\beta\) normal beads in the rows above this \((-\, +)\) pair (this is well defined), then the bead on the right corresponding to the \(-\) gains weight \(\beta\), while the bead on the left corresponding to the \(+\) gains weight \(\beta\). Therefore, there is no net change in weight contributed by beads of this type.

Since there are \(k\) normal beads, \(\tilde{E}_j^k\mu\) must have weight \(kl\) less than \(\mu\). □

3. Adjustment Matrices

Let the \(q\)-Schur algebra over an arbitrary field \(F\) be denoted by \(S_n\), and denote the \(q\)-Schur algebra over \(\mathbb{C}\) by \(\mathcal{S}_n^0\). Let \(\zeta\) be a primitive \(e^{th}\)-root of unity in \(\mathbb{C}\). We write \(\mathcal{H}_n^0\) for \(\mathcal{H}_{\mathbb{C},\zeta}(\mathcal{S}_n)\) and \(\mathcal{H}_n\) for \(\mathcal{H}_{\mathbb{F},q}(\mathcal{S}_n)\). By Theorem 2.1, the Weyl modules corresponding to two partitions lie in the same block of \(\mathcal{S}_n\) if and only if they lie in the same block of \(\mathcal{S}_n^0\). Similarly, the Specht modules corresponding to two partitions lie in the same block of \(\mathcal{H}_n\) if and only if they lie in the same block of \(\mathcal{H}_n^0\) by Corollary 2.2. Therefore, given a block \(B\) of \(\mathcal{S}_n\) or \(\mathcal{H}_n\), we may denote \(B^0\) to be its corresponding block in \(\mathcal{S}_n^0\) or \(\mathcal{H}_n^0\) respectively.

The Grothendieck group \(\mathcal{G}(\mathcal{S}_n)\) of \(\mathcal{S}_n\) is the additive abelian group (with complex coefficients) generated by the symbols \([E]\), where \(E\) runs over the isomorphism classes of finite dimensional \(\mathcal{S}_n\)-modules, together with the relations \([F] = [E] + [G]\) whenever there exists a short exact sequence \(0 \to E \to F \to G \to 0\). Thus, as a complex vector space, \(\mathcal{G}(\mathcal{S}_n)\) has a basis given by \(\{[L^\lambda_\mathbb{F}] : \lambda \in \mathcal{P}(n)\}\). We denote the decomposition matrix for \(\mathcal{S}_n\) by \(D_S\). Since \(D_S\) is unitriangular, \(\{[W^\lambda_\mathbb{F}] : \lambda \in \mathcal{P}(n)\}\) must be another basis for \(\mathcal{G}(\mathcal{S}_n)\) with \(D_S\) being the transition matrix between these two bases. In other words, given any \(\lambda \in \mathcal{P}(n)\),

\[
[W^\lambda_\mathbb{F}] = \sum_{\mu \in \mathcal{P}(n)} [W^\lambda_\mathbb{F} : L^\mu_\mathbb{F}][L^\mu_\mathbb{F}].
\]

There is a well-defined homomorphism \(d_S : \mathcal{G}(\mathcal{S}_n^0) \to \mathcal{G}(\mathcal{S}_n)\) which fixes the Weyl modules, \(d_S([W^\lambda_\mathbb{F}]) = [W^\lambda_\mathbb{F}] \forall \lambda \in \mathcal{P}(n)\). In the literature [17], this homomorphism is known as the decomposition map.

**Theorem 3.1.** [17, Theorem 6.35] Let \(D_S\) and \(D_S^0\) be the decomposition matrices for \(\mathcal{S}_n\) and \(\mathcal{S}_n^0\) respectively. Let \(A_S\) be the matrix \((a^S_{\mu\nu})_{\mu,\nu \in \mathcal{P}(n)}\), where \(a^S_{\mu\nu}\)'s satisfy \(d_S([L^\mu_\mathbb{F}]) = \sum_{\nu \in \mathcal{P}(n)} a^S_{\mu\nu}[L^\mu_\mathbb{F}]\).

Then, \(a^S_{\mu\nu} \in \mathbb{N}\) for all \(\mu, \nu \in \mathcal{P}(n)\) and

\[
D_S = D^0_S A_S.
\]

We call the matrix \(A_S\) in Theorem 3.1 the adjustment matrix for \(\mathcal{S}_n\). Replacing \(\mathcal{S}_n\) by \(\mathcal{H}_n\) and \(\mathcal{S}_n^0\) by \(\mathcal{H}_n^0\) in Theorem 3.1 yields the following theorem.
Theorem 3.2. Let $D_{\mathcal{H}}$ and $D_{\mathcal{H}}^{0}$ be the decomposition matrices for $\mathcal{H}_{n}$ and $\mathcal{H}_{n}^{0}$ respectively. Let $A_{\mathcal{H}}$ be the matrix $(a_{\mu\nu}^{\mathcal{H}})_{\mu,\nu\in P^{e-reg}(n)}$, where $a_{\mu\nu}^{\mathcal{H}}$’s satisfy $d_{\mathcal{H}}([D_{\mathcal{C}}^{\mathcal{H}}]) = \sum_{\nu\in P^{e-reg}(n)} a_{\mu\nu}^{\mathcal{H}}[D_{\mathcal{F}}^{\mathcal{H}}]$. Then, $a_{\mu\nu}^{\mathcal{H}} \in \mathbb{N}$ for all $\mu, \nu \in P^{e-reg}(n)$ and

$$D_{\mathcal{H}} = D_{\mathcal{H}}^{0}A_{\mathcal{H}}.$$  

We call the matrix $A_{\mathcal{H}}$ in Theorem 3.2 the adjustment matrix for $\mathcal{H}_{n}$. The matrix $A_{S}$ has rows and columns indexed by $P(n)$, whereas $A_{\mathcal{H}}$ has rows and columns indexed by $P^{e-reg}(n)$. One may argue using the Schur functor that $A_{\mathcal{H}}$ is the submatrix of $A_{S}$ obtained by removing the rows and columns of $A_{S}$ indexed by $P^{e-sing}(n)$. Therefore, we refer to adjustment matrices as $A$ when it is clear whether we are dealing with $S_{n}$ or $\mathcal{H}_{n}$. Moreover, given any two partitions $\lambda, \mu \in P(n)$, we may refer to the $(\lambda, \mu)$-entry of $A$ as $\adj_{\lambda\mu}$ without any ambiguity.

We highlight the unitriangular property of adjustment matrices inherited from the unitriangularity of the decomposition matrices in the following corollary.

Corollary 3.3. Suppose that $\lambda$ and $\mu$ are partitions lying in a block $B$ of $S_{n}$. Then,

- $\adj_{\mu\mu} = 1$,
- $\adj_{\lambda\mu} = 0$ unless $\lambda \geq_{J} \mu$.

It follows from Lemma 2.9 and Corollary 3.3 that $\mu \not\lhd_{J} \lambda \Rightarrow \adj_{\lambda\mu} = 0$. As mentioned before, it is difficult to check that $\mu \not\lhd_{J} \lambda$, whereas $\mu \not\lhd_{J} \lambda$ can be verified by inspection. In terms of adjustment matrices, we have the following corollary of Theorem 2.11.

Corollary 3.4. Suppose that $\lambda$ and $\mu$ are partitions lying in a block $B$ of $S_{n}$ of weight $w < \text{char}(F)$. Additionally, suppose that $\adj_{\nu\mu} = 0$ for all partitions $\nu$ such that $\lambda <_{J} \nu <_{J} \mu$, and that $d_{\mathcal{C}^{\mu}}^{\mathcal{C}^{\mu}}(v) \in \{0, v\}$. Then, $\adj_{\lambda\mu} = 0$.

Proof. The proof of this corollary is essentially the same as the proof of [9, Corollary 2.12] by replacing $S^{\nu}$ with $W^{\nu}$ and $D^{\mu}$ with $L^{\mu}$. \hfill \square

In section 3, we will use the following easy fact several times.

Lemma 3.5. Let $\lambda$ and $\mu$ be two distinct partitions lying in some block $B$ of $S_{n}$. If $[W_{\mathcal{F}}^{\lambda} : L_{\mathcal{F}}^{\mu}] = [W_{\mathcal{C}}^{\lambda} : L_{\mathcal{C}}^{\mu}]$, then $\adj_{\lambda\mu} = 0$.

Proof. By Theorem 3.1, Theorem 2.8 and Corollary 3.3,

$$[W_{\mathcal{F}}^{\lambda} : L_{\mathcal{F}}^{\mu}] = [W_{\mathcal{C}}^{\lambda} : L_{\mathcal{C}}^{\mu}] + \adj_{\lambda\mu} + \sum_{\lambda <_{J} \nu <_{J} \mu} [W_{\mathcal{C}}^{\lambda} : L_{\mathcal{C}}^{\mu}] \adj_{\nu\mu}.$$  

The terms in the sum are all non-negative, so $\adj_{\lambda\mu} = 0$. \hfill \square

3.1. James’s Conjecture. Throughout the rest of this paper, we shall adopt the Kronecker delta. In view of the LLT algorithm (see Remark 2.16), the decomposition matrix $D^{0}$ of $S_{n}^{0}$ or $H_{n}^{0}$ can be calculated in principle. Therefore, we focus our attention on studying the adjustment matrices $A$. The following is the famous James’s Conjecture for adjustment matrices.

Conjecture 3.6. (James’s Conjecture [11, §4]) Suppose that $\lambda$ and $\mu$ are partitions lying in a block $B$ of $S_{n}$ or $\mathcal{H}_{n}$ with $e$-weight $w$. If $w < \text{char}(F)$, then $\adj_{\lambda\mu} = \delta_{\lambda\mu}$.

James’s Conjecture is easy to verify for blocks of weight 0 or 1. A block of weight 0 contains its core $\kappa$ as the only partition, so $W^{\kappa} = L^{\kappa}$ regardless of the underlying field. Blocks of weight 1 each contain $e$ partitions which can be totally ordered by the dominance order, $\lambda^{1} \triangleright \cdots \triangleright \lambda^{e}$. The decomposition numbers are independent of the underlying field; $[W^{\lambda^{i}} : L^{\lambda^{i}}]$ is equal to 1 if $i \in \{j, j+1\}$, and is equal to 0 otherwise. We summarize the progress on James’s Conjecture made so far by the works of Fayers, Richards, Schroll and Tan in the following two theorems.
Theorem 3.7. (James’s Conjecture for blocks of Iwahori-Hecke algebras of weight at most 4) \[ \text{[8, Theorem 4.1]} \ [9, Theorem 2.6] \ [19] \]
Suppose that \( \text{char}(\mathbb{F}) \geq 5 \). Let \( \lambda \) and \( \mu \) be \( e \)-regular partitions lying in \( B \), a block of \( H_n \) of weight at most 4. Then, \( \text{adj}_{\lambda \mu} = \delta_{\lambda \mu} \).

Theorem 3.8. (James’s Conjecture for blocks of \( \text{q-Schur} \) algebras of weight 2) \[ \text{[21, Corollary 3.6]} \]
Suppose that \( \text{char}(\mathbb{F}) \geq 3 \). Let \( \lambda \) and \( \mu \) be partitions lying in \( B \), a block of \( S_n \) of weight 2. Then, \( \text{adj}_{\lambda \mu} = \delta_{\lambda \mu} \).

In section 4 we prove James’s Conjecture for blocks of \( S_n \) of weight 3.

Theorem 3.9. (James’s Conjecture for blocks of \( \text{q-Schur} \) algebras of weight 3) Suppose that \( \text{char}(\mathbb{F}) \geq 5 \). Let \( \lambda \) and \( \mu \) be partitions lying in \( B \), a block of \( S_n \) of weight 3. Then, \( \text{adj}_{\lambda \mu} = \delta_{\lambda \mu} \).

In section 5 we prove James’s Conjecture for blocks of \( S_n \) of weight 4.

Theorem 3.10. (James’s Conjecture for blocks of \( \text{q-Schur} \) algebras of weight 4) Suppose that \( \text{char}(\mathbb{F}) \geq 5 \). Let \( \lambda \) and \( \mu \) be partitions lying in \( B \), a block of \( S_n \) of weight 4. Then, \( \text{adj}_{\lambda \mu} = \delta_{\lambda \mu} \).

In the context of Theorem 3.9 and Theorem 3.10 we already know that \( \text{adj}_{\lambda \mu} = \delta_{\lambda \mu} \) when \( \lambda \) and \( \mu \) are both \( e \)-regular due to Theorem 3.7. In fact, this knowledge can be strengthened for \( \text{q-Schur} \) algebras:

Proposition 3.11. Let \( B \) be a block of \( S_n \) and let \( \bar{B} \) be its corresponding block of \( H_n \) with the same \( e \)-core and \( e \)-weight. Let \( \lambda \) and \( \mu \) be partitions lying in \( B \) with \( \mu \) being \( e \)-regular. If the adjustment matrix for \( B \) is the identity matrix, then \( \text{adj}_{\lambda \mu} = \delta_{\lambda \mu} \).

Proof. Suppose that \( \lambda \neq \mu \). Then,
\[
[\lambda : \mu] = [\lambda : \mu] + \text{adj}_{\lambda \mu} + \sum_{\lambda \mu}^\text{adj}_{\lambda \mu}.
\]
Since \( \mu \) is \( e \)-regular, we can apply Theorem 1.3 to get \( [\lambda : \mu] = [\lambda : \mu] \) and \( [\lambda : \mu] = [\lambda : \mu] \). On the other hand, \( [\lambda : \mu] = [\lambda : \mu] \) since the adjustment matrix for \( B \) is the identity matrix by assumption. Moreover, the terms in the sum are non-negative, so \( \text{adj}_{\lambda \mu} \) must be zero. \( \square \)

3.2. The row and column removal theorems.

Definition 3.12. We define the row removal function \( \mathcal{R} \) as
\[
\mathcal{R} : \bigcup_{n>0} \mathcal{P}(n) \longrightarrow \bigcup_{n>0} \mathcal{P}(n)
\]
\[
\nu = (\nu_1, \nu_2, \ldots, \nu_{l(\nu)}) \mapsto (\nu_2, \nu_3, \ldots, \nu_{l(\nu)}).
\]
We define the column removal function \( \mathcal{C} \) as
\[
\mathcal{C} : \bigcup_{n>0} \mathcal{P}(n) \longrightarrow \bigcup_{n>0} \mathcal{P}(n)
\]
\[
\nu = (\nu_1, \nu_2, \ldots, \nu_{l(\nu)}) \mapsto (\nu_1 - 1, \nu_2 - 1, \ldots, \nu_{l(\nu)} - 1).
\]

Theorem 3.13. (\[6, \S 4.2 (9)]\) Suppose that \( \lambda \) and \( \mu \) are partitions of \( n \) with \( \lambda_1 = \mu_1 \). Then,
\[
[\lambda : \mu] = [\lambda : \mu].
\]

Corollary 3.14. (\[6, Corollary 2.19\] Suppose that \( \lambda \) and \( \mu \) are partitions of \( n \) with \( \lambda_1 = \mu_1 \). Then,
\[
\text{adj}_{\lambda \mu} = \text{adj}_{\mathcal{R}(\lambda), \mathcal{R}(\mu)}.
\]
Theorem 3.15. (§4.2 (15)) Suppose that \( \lambda \) and \( \mu \) are partitions of \( n \) with \( l(\lambda) = l(\mu) \). Then,
\[
[W^\lambda : L^\mu] = [W^{C(\lambda)} : L^{C(\mu)}].
\]

Corollary 3.16. Suppose that \( \lambda \) and \( \mu \) are partitions of \( n \) with \( l(\lambda) = l(\mu) \). Then,
\[
\adj_{\lambda\mu} = \adj_{C(\lambda)C(\mu)}.
\]

Proof. This is similar to the proof of Corollary 3.14 in [16, Corollary 2.19]; we use Theorem 3.15 instead of Theorem 3.13.

Suppose that \( \nu \) is a partition of weight \( w \). Let us examine the weights of \( \mathcal{R}(\nu) \) and \( \mathcal{C}(\nu) \). An abacus display for \( \mathcal{R}(\nu) \) is obtained from that of \( \nu \) by replacing the bead corresponding to \( \nu_1 \) (the maximal occupied position) with an empty space. Therefore, if this bead has weight \( s \geq 0 \), then \( \mathcal{R}(\nu) \) would have weight \( w - s \leq w \). On the other hand, an abacus display for \( \mathcal{C}(\nu) \) is obtained from that of \( \nu \) by replacing the first unoccupied space with a bead. If there are \( r \geq 0 \) beads in the same runner under this position, then \( \mathcal{C}(\nu) \) would have weight \( w - r \leq w \). For our purposes, the crucial point is that the weights of \( \mathcal{R}(\nu) \) and \( \mathcal{C}(\nu) \) are at most \( w \).

Remark 3.17. In sections 4 and 5, we will want to prove that \( \adj_{\lambda\mu} = \delta_{\lambda\mu} \) for all pairs of partitions \( (\lambda, \mu) \) lying in a block of \( \mathcal{S}_n \) of weight \( w \), assuming that James’s conjecture holds for all blocks of \( \mathcal{S}_m \) of weight at most \( w \), where \( m < n \). Note that \( \mu > \lambda \) implies that \( \mu_1 > \lambda_1 \) and \( l(\mu) \leq l(\lambda) \). When \( \mu_1 = \lambda_1 \) or \( l(\mu) = l(\lambda) \), we may apply Corollary 3.14 or Corollary 3.16 respectively to conclude that \( \adj_{\lambda\mu} = \delta_{\lambda\mu} \). Thus, in this setting, we have \( \adj_{\lambda\mu} = 0 \) unless \( \mu > \lambda, \mu_1 > \lambda_1 \) and \( l(\mu) < l(\lambda) \). We write \( \mu \gg \lambda \) when \( \mu > \lambda, \mu_1 > \lambda_1 \) and \( l(\mu) < l(\lambda) \).

3.3. Lowerable partitions. Recall the decomposition map \( d_S \) between the Grothendieck groups \( \mathcal{G}(\mathcal{S}_0) \) and \( \mathcal{G}(\mathcal{S}_n) \). Suppose that \( A, B \) and \( C \) are blocks of \( \mathcal{S}_{n-1}, \mathcal{S}_n \) and \( \mathcal{S}_{n+1} \) respectively, and that there is an \( e \)-residue \( j \) such that a partition lying in \( A \) may be obtained from a partition lying in \( B \) by removing exactly one \( j \)-removable node, while a partition lying in \( C \) may be obtained from a partition lying in \( B \) by adding exactly one \( j \)-addable node. Let \( \mu \) be an arbitrary partition in \( B \). We define \( E_j \) to be the \( j \)-restriction functor from \( \mathcal{G}(\mathcal{S}_n) \) to \( \mathcal{G}(\mathcal{S}_{n-1}) \) and \( F_j \) to be the \( j \)-induction functor from \( \mathcal{G}(\mathcal{S}_n) \) to \( \mathcal{G}(\mathcal{S}_{n+1}) \) in the following way:
\[
E_j([M]) := [M_{\downarrow A}],
\]
\[
F_j([M]) := [M_{\uparrow C}].
\]

Similarly, we define \( \bar{E}_j \) to be the \( j \)-restriction functor from \( \mathcal{G}(\mathcal{S}_0) \) to \( \mathcal{G}(\mathcal{S}_{n-1}) \) and \( \bar{F}_j \) to be the \( j \)-induction functor from \( \mathcal{G}(\mathcal{S}_0) \) to \( \mathcal{G}(\mathcal{S}_{n+1}) \) in the following way:
\[
\bar{E}_j([M]) := [M_{\downarrow A}],
\]
\[
\bar{F}_j([M]) := [M_{\uparrow C}].
\]

It is easy to check using Theorem 2.18 that
\[
d_S \bar{E}_j([W^\mu_C]) = E_j d_S([W^\mu_C]),
\]
\[
d_S \bar{F}_j([W^\mu_C]) = F_j d_S([W^\mu_C]).
\]

Since \( \{[W^\mu_C] \mid \mu \in \mathcal{P}(n) \} \) is a basis for \( \mathcal{G}(\mathcal{S}_0) \), the following diagrams commute.
\[
\begin{array}{ccc}
\mathcal{G}(\mathcal{S}_0) & \xrightarrow{d_S} & \mathcal{G}(\mathcal{S}_n) \\
\downarrow E_j & & \downarrow E_j \\
\mathcal{G}(\mathcal{S}_{n-1}) & \xrightarrow{d_S} & \mathcal{G}(\mathcal{S}_{n+1}) \\
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{G}(\mathcal{S}_0) & \xrightarrow{d_S} & \mathcal{G}(\mathcal{S}_n) \\
\downarrow F_j & & \downarrow F_j \\
\mathcal{G}(\mathcal{S}_{n-1}) & \xrightarrow{d_S} & \mathcal{G}(\mathcal{S}_{n+1}) \\
\end{array}
\]
Let \( w \) be a weight less than \( B \) of \( \mathcal{S}_n \). Under this identification, we get \( \tilde{E}_j = E_j \) and \( \tilde{F}_j = F_j \) from the commutative diagrams. In particular,

\[
\begin{align*}
E_j(B_{\lambda\mu}^\uparrow) &= [L_{C_j}^\uparrow \downarrow], \\
F_j(B_{\lambda\mu}^\uparrow) &= [L_{C_j}^\uparrow \downarrow A], \\
E_j(B_{\lambda\mu}^\uparrow C) &= [L_{C_j}^\uparrow \downarrow A], \\
F_j(B_{\lambda\mu}^\uparrow C) &= [L_{C_j}^\uparrow \downarrow A].
\end{align*}
\]

Let \( t \) be a positive integer. We denote the divided power \( j \)-restriction functor and divided power \( j \)-induction functor as \( \tilde{E}_j^{(t)} := \frac{1}{t} E_j^t \) and \( \tilde{F}_j^{(t)} := \frac{1}{t} F_j^t \) respectively.

**Proposition 3.18.** Let \( \lambda \) and \( \mu \) be two distinct partitions lying in some block \( B \) of \( S_n \). If \( \epsilon_j(\lambda) < \epsilon_j(\mu) \) (resp. \( \varphi_j(\lambda) < \varphi_j(\mu) \)) for some \( e \)-residue \( j \), then \( \text{adj} \lambda \mu = 0 \).

If \( \epsilon_j(\lambda) = \epsilon_j(\mu) \) (resp. \( \varphi_j(\lambda) = \varphi_j(\mu) \)), for some \( e \)-residue \( j \), then \( \text{adj} \lambda \mu = \text{adj} \tilde{E}_j^k \lambda \tilde{E}_j^k \mu \) (resp. \( \text{adj} \lambda \mu = \text{adj} \tilde{E}_j^k \lambda \tilde{E}_j^k \mu \)), where \( k := \epsilon_j(\lambda) \) (resp. \( k := \varphi_j(\lambda) \)).

**Proof.** Let \( l := \epsilon_j(\mu) \) and \( k := \epsilon_j(\lambda) \). If \( l > k \), then

\[
\tilde{E}_j^{(t)}([L_{C_j}^\uparrow]) = \sum_{\nu \in \mathcal{P}(n)} \text{adj} \lambda \nu \tilde{E}_j^{(t)}[L_{F_j}^\nu].
\]

By Theorem 2.20 \( \tilde{E}_j^{(t)}([L_{C_j}^\uparrow]) = 0 \) and \( \tilde{E}_j^{(t)}[L_{F_j}^\nu] \neq 0 \), so \( \text{adj} \lambda \mu \) must be zero.

If \( l = k \), then

\[
[L_{C_j}^\uparrow \lambda \nu] = \tilde{E}_j^{(k)}([L_{C_j}^\uparrow]) = \sum_{\nu \in \mathcal{P}(n), \epsilon_j(\nu) \geq k} \text{adj} \lambda \nu \tilde{E}_j^{(k)}[L_{F_j}^\nu] = \sum_{\nu \in \mathcal{P}(n), \epsilon_j(\nu) = k} \text{adj} \lambda \nu \tilde{E}_j^{(k)}[L_{F_j}^\nu],
\]

where the third equality is due to Theorem 2.20 and the last equality is due to the case \( l > k \) that we just proved above. On the other hand,

\[
[L_{C_j}^\uparrow \lambda \sigma] = \sum_{\sigma \in \mathcal{P}(n-k)} \text{adj} \tilde{E}_j^{k} \lambda \tilde{E}_j^{k} \mu
\]

Comparing the coefficients of \([L_{F_j}^\nu] \tilde{E}_j^{k} \lambda \tilde{E}_j^{k} \mu \) in equations (3.1) and (3.2), we conclude that \( \text{adj} \lambda \mu = \text{adj} \tilde{E}_j^{k} \lambda \tilde{E}_j^{k} \mu \).

The proof of the other case considering conormal nodes is similar. \( \square \)

**Remark 3.19.** In sections 4 and 5 we will want to prove that \( \text{adj} \lambda \mu = \delta \lambda \mu \) for all pairs of partitions \( (\lambda, \mu) \) lying in a block \( \mathcal{S}_n \) of weight \( w \), assuming that James’s conjecture holds for all blocks of \( \mathcal{S}_n \) of weight at most \( w \), where \( m < n \). If \( \epsilon_j(\mu) > 0 \) and \( \epsilon_j(\lambda) \leq \epsilon_j(\mu) \) for some \( e \)-residue \( j \), then we may apply Proposition 3.18 to conclude that \( \text{adj} \lambda \mu = \delta \lambda \mu \).

**Definition 3.20.** Let \( \lambda \) and \( \mu \) be two distinct partitions lying in some weight \( w \) block \( B \) of \( \mathcal{S}_n \). We say that the pair \( (\lambda, \mu) \) is **lowerable** if there is some \( e \)-residue \( j \) such that \( \epsilon_j(\mu) > 0 \), \( \varphi_j(\mu) > 0 \), and \( \epsilon_j(\lambda) \leq \epsilon_j(\mu) \).

**Corollary 3.21.** Suppose that \( \lambda \) and \( \mu \) are two distinct partitions lying in some weight \( w \) block \( B \) of \( \mathcal{S}_n \) and that \( (\lambda, \mu) \) is lowerable. Moreover, suppose that the adjustment matrix is the identity matrix for blocks of weight less than \( w \). Then \( \text{adj} \lambda \mu = 0 \).

**Proof.** Let \( k := \epsilon_j(\mu) \) and \( l := \varphi_j(\mu) \). If \( \epsilon_j(\lambda) < k \), this follows directly from Proposition 3.18. If \( \epsilon_j(\lambda) = k \), then we have \( \text{adj} \lambda \mu = \text{adj} \tilde{E}_j^{k} \lambda \tilde{E}_j^{k} \mu \) by Proposition 3.18. By Lemma 2.21 \( \tilde{E}_j^{k} \lambda \) has weight \( w - kl < w \), so the result follows. \( \square \)
Note that our notion of lowerable partitions here is inspired by and generalises Fayers’s definition of lowerable partitions in [9 Proposition 2.17].

**Example 3.22.** Suppose that $e = 8$ and $w = 5$. Let $\lambda = \langle 1, 2, 3, 4, 5 \rangle$ and $\mu = \langle 2, 3, 4, 5, 6 \rangle$ be the partitions lying in the block $B$ of $\mathcal{H}_n$ with the $\langle 2^e \rangle$ notation.

![Diagram of partitions 

Observe that $\varepsilon_2(\mu) = 1$, $\varphi_2(\mu) = 1$ and $\varepsilon_2(\lambda) = 0$. Hence, $(\lambda, \mu)$ is lowerable by Theorem 3.7.

3.4. [**[w:k]-pairs**](#).

Let $B$ be a weight $w$ block of $\mathcal{S}_n$ whose core $\kappa_B$ has exactly $k$ removable nodes on a given runner $i$ with residue $j$ (if $0 < i < e$, runner $i$ has exactly $k$ more beads than runner $i-1$. If $i = 0$, runner 0 has exactly $k+1$ more beads than runner $e-1$). Suppose that $A$ is the weight $w$ block of $\mathcal{S}_{n-k}$ whose core $\kappa_A$ is obtained from $\kappa_B$ by removing all of the $k$ $j$-removable nodes. We say that the blocks $A$ and $B$ form a $[w : k]$-pair. We note that for each partition $\mu$ in $A$ (resp. $B$), we have $\varphi_j(\mu) - \varepsilon_j(\mu) = k$ (resp. $\varepsilon_j(\mu) - \varphi_j(\mu) = k$).

Given a partition $\mu$ in $A$, recall that $\tilde{F}_j^k \mu$ is the partition in $B$ obtained from $\mu$ by adding the $k$ lowest $j$-conormal nodes. Then, $\tilde{F}_j^k$ is a bijection from the set of partitions in $A$ to the set of partitions in $B$. Moreover, we have the following:

**Theorem 3.23.** [2] Let $\mu$ be a partition in $A$. Then, $\varphi_j(\mu) = k$ (equivalently $\varepsilon_j(\mu) = 0$) if and only if $\varepsilon_j(\tilde{F}_j^k \mu) = k$, in which case, $L^\mu_{\downarrow B} \cong (L^{\tilde{F}_j^k \mu})^{\oplus k!}$ and $L^{\tilde{F}_j^k \mu}_{\downarrow A} \cong (L^{\mu})^{\oplus k!}$. If this happens, we say that $\mu$ and $\tilde{F}_j^k \mu$ are non-exceptional for the $[w : k]$-pair $(A, B)$. We say that $\mu$ and $\tilde{F}_j^k \mu$ are exceptional otherwise.

When $w \leq k$, every partition is non-exceptional for the $[w : k]$-pair $(A, B)$, and we say that $A$ and $B$ are Scopes equivalent; they are in fact Morita equivalent [20].

**Remark 3.24.** Let $\lambda$ be a partition in $A$.

- If $\lambda$ were exceptional, then $\varphi_j(\lambda) > k$ (equivalently $\varepsilon_j(\lambda) > 0$).
- If $\lambda$ were non-exceptional, then $\varphi_j(\lambda) = k$ (equivalently $\varepsilon_j(\lambda) = 0$).

Let $\sigma$ be a partition in $B$.

- If $\sigma$ were exceptional, then $\varepsilon_j(\sigma) > k$ (equivalently $\varphi_j(\sigma) > 0$).
- If $\sigma$ were non-exceptional, then $\varepsilon_j(\sigma) = k$ (equivalently $\varphi_j(\sigma) = 0$).

**Definition 3.25.** Let $\mu \in \mathcal{P}(n)$ and let $j$ be some $e$-residue. If $\varphi_j(\mu) - \varepsilon_j(\mu) = k > 0$, we define $\hat{F}_j^k \mu$ to be $\tilde{F}_j^k \mu$ ($\hat{F}_j^k \mu$ is defined only when $\varphi_j(\mu) - \varepsilon_j(\mu) > 0$). Given a positive integer $m$, we define $\hat{F}_j^{m_1} \mu$ and $\hat{F}_j^{m_2} \mu$ recursively in the following way:

- If $\varphi_j(\mu) - \varepsilon_j(\mu) > 0$, $\hat{F}_j^{m_1} \mu := \hat{F}_j^k \mu$ ($\hat{F}_j^{m_1} \mu$ is not defined when $\varphi_j(\mu) - \varepsilon_j(\mu) \leq 0$).
- If $m > 2$ and $\varphi_{j-m+1}(\hat{F}_j^{m_1} \mu) - \varepsilon_{j-m+1}(\hat{F}_j^{m_1} \mu) > 0$, $\hat{F}_j^{m_1} \mu := \hat{F}_{j-m+1} \hat{F}_j^{m_1} \mu$.

**Proposition 3.26.** Suppose that $A$ and $B$ are blocks forming a $[w : k]$-pair as above with $k < w$. Let $\lambda$ and $\mu$ be two distinct partitions in $A$.

1. If $\lambda$ and $\mu$ are both non-exceptional, then $\text{adj}_{\lambda \mu} = \text{adj}_{\hat{F}_j^k \lambda, \hat{F}_j^k \mu}$.

2. If $\lambda$ is non-exceptional but $\mu$ is exceptional, then $\text{adj}_{\lambda \mu} = 0$.

**Proof.**

1. By Remark 3.24 $\varphi_j(\lambda) = \varphi_j(\mu) = k$, so $\text{adj}_{\lambda \mu} = \text{adj}_{\hat{F}_j^k \lambda, \hat{F}_j^k \mu}$ by Proposition 3.18.
Suppose that we have a series of \( t + 1 \) blocks \( B_m \) of weight \( w \) with cores \( \kappa_m \), where \( 0 \leq m \leq t \). Moreover, for \( 1 \leq m \leq t \), \( B_m \) and \( B_{m-1} \) forms a \([w : k_m]\)-pair with \( \kappa_{m-1} \) being obtained from \( \kappa_m \) by adding \( k_m > 0 \) addable nodes of residue \( j_m \). Given any partition \( \lambda \) in \( B_t \), \( f(\lambda) := \hat{F}_{j_1} \hat{F}_{j_2} \cdots \hat{F}_{j_{t-1}} \hat{F}_{j_t} \lambda \) is well-defined. We say that \( f(\lambda) \) is \textit{semisimply induced} if \( \lambda, \hat{F}_{j_1} \lambda, \hat{F}_{j_{t-1}} \hat{F}_{j_t} \lambda, \ldots, \hat{F}_{j_2} \cdots \hat{F}_{j_{t-2}} \hat{F}_{j_{t-1}} \hat{F}_{j_t} \lambda \) and \( f(\lambda) \) are all non-exceptional. We say that \( f(\lambda) \) is \textit{not semisimply induced} otherwise. Using this notation, we may restate our working version of Proposition 3.26 in the following corollary.

**Corollary 3.27.** We adopt the notation above. Let \( \lambda \) and \( \mu \) be two distinct partitions in \( B_t \).

1. If \( f(\lambda) \) and \( f(\mu) \) are both semisimply induced, then \( \text{adj}_{\lambda\mu} = \text{adj}_{f(\lambda)f(\mu)} \).
2. If \( f(\lambda) \) is semisimply induced but \( f(\mu) \) is not semisimply induced, then \( \text{adj}_{\lambda\mu} = 0 \).

**Proof.** We just apply Proposition 3.26 repeatedly. □

When we want to show that \( \text{adj}_{\lambda\mu} = 0 \) for some pair of partitions \((\lambda, \mu)\) in section 5, we sometimes do this by finding a sequence of \( e \)-residues \( j_1, \ldots, j_t \) such that \( f(\lambda) := \hat{F}_{j_1} \cdots \hat{F}_{j_t} \lambda \) is semisimply induced and \( f(\lambda), f(\mu) \) is lowerable. We may also use this formalism without making \( f \) explicit by writing \( \lambda \sim \nu \) (and say that \( \lambda \) induces semisimply to \( \nu \)) to indicate that there exists a sequence of \( e \)-residues \( j_1, \ldots, j_t \) such that \( f(\lambda) := \hat{F}_{j_1} \cdots \hat{F}_{j_t} \lambda \) is semisimply induced and equals \( \nu \). When we do so, it is hoped that it will not be too difficult for the reader to construct an appropriate sequence \( j_1, \ldots, j_t \).

**Example 3.28.** Let \( \lambda = \langle 0_{12}, 3_{12} \rangle \), \( \mu = \langle 0_{32}, 4 \rangle \) and \( \nu = \langle 0_{2}, 3_{12} \rangle \) be the partitions lying in the weight 4 block \( B \) of \( S_n \) with the \((4^3, 5^2, 4^3)\) notation (the beads on runner 0 have \( e \)-residue 5). For any partition \( \sigma \) in \( B \), define \( a(\sigma) := \hat{F}_0 \hat{F}_0 \hat{F}_0 \hat{F}_0 \hat{F}_0 \sigma \).

![Diagram](image)

Observe that \( a(\lambda) \) and \( a(\mu) \) are semisimply induced and that \((a(\lambda), a(\mu))\) is lowerable. However, \( a(\nu) \) is not semisimply induced.

We state the following observation from the list of exceptional partitions of weights 3 and 4 in the appendix (Figure 1, Figure 2 and Figure 3).

**Remark 3.29.** Adopting the same notation as the beginning of this section, let \( A \) and \( B \) be blocks forming a \([w : k]\)-pair with \( k < w \) and \( w \in \{3, 4\} \). Let \( \lambda \) be a partition in \( A \).

- If \( \lambda \) were exceptional, then \( \varphi_j(\lambda) = k + 1 \) (equivalently \( \epsilon_j(\lambda) = 1 \)).
- If \( \lambda \) were non-exceptional, then \( \varphi_j(\lambda) = k \) (equivalently \( \epsilon_j(\lambda) = 0 \)).

Let \( \sigma \) be a partition in \( B \).
• If $\sigma$ were exceptional, then $\epsilon_j(\sigma) = k + 1$ (equivalently $\varphi_j(\sigma) = 1$).
• If $\sigma$ were non-exceptional, then $\epsilon_j(\sigma) = k$ (equivalently $\varphi_j(\sigma) = 0$).

Therefore, we have the following result specific to blocks of $S_n$ of weights 3 and 4.

**Proposition 3.30.** Suppose that $A$ and $B$ are blocks forming a $[w : k]$-pair as above, with $k < w$ and $w \in \{3, 4\}$. Let $\lambda$ and $\mu$ be two distinct partitions in $A$. If $\mu$ is exceptional, then both pairs of partitions $(\lambda, \mu)$ and $(\hat{\lambda}, \hat{\mu})$ are lowerable.

**Proof.** This follows directly from Remark 3.29 and Definition 3.20.

3.5. Rouquier blocks. A weight $w$ block $B$ of $S_n$ with the $\langle b_0, \ldots, b_{e-1} \rangle$ notation is **Rouquier** if for every $0 \leq i < j \leq e - 1$, either $b_i - b_j \geq w$ or $b_j - b_i \geq w - 1$. The Rouquier blocks are Scopes equivalent to each other and are now well understood. In particular, we know that James's Conjecture holds for Rouquier blocks.

**Theorem 3.31.** [12 Corollary 3.15] If $B$ is a Rouquier block of weight $w < \text{char}(\mathbb{F})$, then $\text{adj}_{\lambda\mu} = \delta_{\lambda\mu}$ for all $\lambda$ and $\mu$ in $B$.

We say that a partition $\lambda$ induces semi-simply to a Rouquier block if $\lambda \sim \nu$ for some $\nu$ lying in a Rouquier block. As a consequence of Corollary 3.27 and Theorem 3.31, we have the following result.

**Proposition 3.32.** Suppose that $\lambda$ and $\mu$ are partitions lying in a block $B$ of $S_n$ of weight $w < \text{char}(\mathbb{F})$. If $\lambda$ induces semi-simply to a Rouquier block, then $\text{adj}_{\lambda\mu} = \delta_{\lambda\mu}$.

In sections 4 and 5, we will sometimes state a partition $\nu$ lying in a Rouquier block and invite the reader to verify that $\lambda \sim \nu$ for some partition $\lambda$ for which we want to show that $\text{adj}_{\lambda\mu} = \delta_{\lambda\mu}$.

3.6. Outline of the proof of James’s Conjecture for the blocks of $q$-Schur algebras of weights 3 and 4. From now on, we assume that $\text{char}(\mathbb{F}) \geq 5$ and $3 \leq w \leq 4$. We prove Theorem 3.9 and Theorem 3.10 by induction on $n$, with the base case being the unique weight $w$ block of $S_{we}$. We will deal with the base case at the beginning of sections 4 and 5.

For the inductive step, we use $[w : k]$-pairs. If $B$ is a weight $w$ block of $S_n$ and $n > we$, then there is at least one block $A$ forming a $[w : k]$-pair with $B$. Suppose that $A_1, \ldots, A_t$ are all the blocks with $A_m$ forming a $[w : k_m]$-pair with $B$, for each $m$. If $\lambda$ is a partition in $B$ which is non-exceptional for some pair $(A_m, B)$, then $\text{adj}_{\lambda\mu} = \delta_{\lambda\mu}$ for every partition $\mu$ lying in $B$ by Proposition 3.26 and induction.

Therefore, we may assume that $\lambda$ is exceptional for every pair $(A_m, B)$. Suppose that an abacus display for the core of $A_m$ is obtained from that of $B$ by removing $k_m$ removable nodes on runner $i_m$, for each $m$. If $\lambda$ were exceptional for $(A_m, B)$, then there must be at least $k_m + 1$ normal beads on runner $i_m$ in the abacus display for $\lambda$. Hence, $|\lambda(i_m)| \geq k_m + 1$ for each $m$ (see Figure 1 and Figure 2), so we have $(k_1 + 1) + \cdots + (k_t + 1) \leq |\lambda(i_1)| + \cdots + |\lambda(i_t)| \leq w$. When $w = 3$, this implies that $t = 1$ and $k_1 \leq 2$. When $w = 4$, this implies that either $t = 1$ and $k_1 \leq 3$ or $t = 2$ and $k_1 = k_2 = 1$. The blocks $B$ satisfying these conditions are dealt with in the rest of the paper.

By Remark 3.17, we may always assume that $\mu \gg \lambda$; that is $\mu \gg_p \lambda$, $\mu_1 > \lambda_1$ and $l(\mu) < l(\lambda)$. Additionally, for each $e$-residue $j$ such that $\epsilon_j(\mu) > 0$, we may assume that $\epsilon_j(\lambda) < \epsilon_j(\mu)$ by Remark 3.19.

Finally, we may assume that $\mu$ is $e$-singular by Proposition 3.11 and Theorem 3.7.

4. Proof of James’s Conjecture for weight 3 blocks of $S_n$

In this section, we will first prove Theorem 3.9 and then use it to show that the decomposition numbers for weight 3 blocks of $S_n$ are bounded above by one. Whenever $\lambda$ and $\mu$ are partitions with weight less than 3, $\text{adj}_{\lambda\mu} = \delta_{\lambda\mu}$ by Theorem 3.8. We will use this fact repeatedly without further comment.
4.1. The principal block of $S_{3e}$. Let $B$ be the principal block of $S_{3e}$; that is the weight $3^e$ block which we display on an abacus with the $\langle 3^e \rangle$ notation.

**Proposition 4.1.** Suppose that $\lambda$ and $\mu$ are partitions lying in $B$. Then, $\text{adj}_{\lambda\mu} = \delta_{\lambda\mu}$.

*Proof.* Let $\lambda$ be an arbitrary partition lying in $B$. Observe that each runner of $\lambda$ has at most one normal bead. By Remark 3.19, we may assume that $\mu$ has no normal beads on any runner. By Remark 3.17, we may also assume that $\mu \gg \lambda$. Table 1 (in the appendix) lists all the possible pairs of partitions $(\lambda, \mu)$. We invite the reader to check that every partition in column $\lambda$ of Table 1 induces semi-simply to a Rouquier block.

- $\langle 0_{13} \rangle \sim \langle 0_{1} | 3, 5, \ldots, 2e + 1 \rangle$,
- $\langle 1_{13} \rangle \sim \langle 0_{12}, 1 | 3, 5, \ldots, 2e + 1 \rangle$,
- $\langle 0_{12}, 1 \rangle \sim \langle 0_{2}, 1 | 3, 5, \ldots, 2e + 1 \rangle$,
- $\langle 0, 1_{12} \rangle \sim \langle 0_{2}, 1 | 3, 5, \ldots, 2e + 1 \rangle$.

Proposition 3.32 completes the proof of Proposition 4.1. □

4.2. Blocks forming exactly one [3 : 1]-pair. In this section, we prove the following proposition:

**Proposition 4.2.** Suppose that $A$ and $B$ are weight $3$ blocks of $S_{n-1}$ and $S_{n}$ respectively, forming a $[3 : 1]$-pair. Moreover, suppose that there is no block other than $A$ forming a $[3 : k]$-pair with $B$ for any $k$. Additionally, suppose that the adjustment matrix for every weight $3$ block of $S_{m}$ is the identity matrix whenever $m < n$. Then, the adjustment matrix for $B$ is the identity matrix.

The conditions on $A$ and $B$ mean that we may $B$ on an abacus with the $(3^a, 4^{b-a}, 3^{e-b})$ notation, where $0 < a < b \leq e$. Suppose that $\lambda$ and $\mu$ are two distinct partitions lying in $B$, so we want to prove that $\text{adj}_{\lambda\mu} = 0$. By Proposition 3.26 and Proposition 3.30, we may assume that $\lambda$ is exceptional and that $\mu$ is non-exceptional. We list all the exceptional partitions in $B$ below:

- $\langle a_{2,1} \rangle$,
- $\langle a_{12}, i \rangle$, $a < i < e$,
- $\langle i, a_{12} \rangle$, $0 \leq i \leq a - 2$,
- $\langle a_{13} \rangle$.

We invite the reader to check that $\langle a_{13} \rangle$ induces semi-simply to a Rouquier block, so we may assume that $\lambda \neq \langle a_{13} \rangle$ by Proposition 3.32.

$$\langle a_{13} \rangle \sim \begin{cases} 
\langle 0, a, a + e - b | 3, 5, \ldots, 2e + 1 \rangle & \text{if } e - b > 0, \\
\langle 0, a_{12} | 3, 5, \ldots, 2e + 1 \rangle & \text{if } e - b = 0.
\end{cases}$$

Let $\nu := \langle 0, a_{12} \rangle$. Note that $l(\nu) \geq l(\lambda)$ for all the remaining possible $\lambda$, so we may assume that $l(\mu) < l(\nu)$ by Remark 3.17. Since the first unoccupied position in the abacus display for $\lambda$ occurs at position 0, the first unoccupied position in an abacus display for $\mu$ must occur strictly after position 0. Consequently, all beads in runner 0 of an abacus display for $\mu$ have zero weight, therefore $\mu$ must be $e$-regular and $\text{adj}_{\lambda\mu} = \delta_{\lambda\mu}$ by Proposition 3.11 and Theorem 3.7. This completes the proof of Proposition 4.2.
4.3. Blocks forming exactly one $[3 : 2]$-pair.

**Proposition 4.3.** Suppose that $A$ and $B$ are weight 3 blocks of $S_{n-2}$ and $S_n$ respectively, forming a $[3 : 2]$-pair. Moreover, suppose that there is no block other than $A$ forming a $[3 : k]$-pair with $B$ for any $k$. Additionally, suppose that the adjustment matrix for every weight 3 block of $S_m$ is the identity matrix whenever $m < n$. Then, the adjustment matrix for $B$ is the identity matrix.

The conditions on $B$ mean that we may represent $B$ on an abacus with the $\langle 3^a, 5^{b-a}, 4^{c-b}, 3^{e-c} \rangle$ notation, where $0 < a < b \leq c \leq e$. If $\lambda$ and $\mu$ are two distinct partitions in $B$, then we have $\text{adj}_{\lambda\mu} = 0$ by Proposition 3.26 unless $\lambda$ is the unique exceptional partition for $(A,B)$, namely $\lambda = \langle a_1 \rangle$.

Observe that the first unoccupied position in the abacus display for $\lambda$ occurs at position $a$. By Remark 3.17 we may assume that $l(\mu) < l(\lambda)$, so the first unoccupied position in an abacus display for $\mu$ must occur after position $a$. Consequently, all beads in runner 0 of an abacus display for $\mu$ have zero weight, therefore $\mu$ must be $e$-regular and $\text{adj}_{\lambda\mu} = \delta_{\lambda\mu}$ by Proposition 3.11 and Theorem 3.7.

As discussed in section 3.6, the combination of Proposition 4.1, Proposition 4.2 and Proposition 4.3 completes the proof of Theorem 3.9.

In his weight 3 paper, Fayers proved an upper bound for the decomposition numbers of $H_n$.

**Theorem 4.4.** [8, Theorem 1.1] Suppose that $\text{char}(F) \geq 5$ and that $B$ is a block of $H_n$ of weight 3. Let $\lambda$ and $\mu$ be partitions in $B$, with $\mu$ being $e$-regular. Then,

$$[S^\lambda : D^\mu] \leq 1.$$ 

An easy consequence of Theorem 3.9 is that we may extend this upper bound to the case of the $q$-Schur algebras.

**Corollary 4.5.** Suppose that $\text{char}(F) \geq 5$ and that $B$ is a block of $S_n$ of weight 3. Let $\lambda$ and $\mu$ be partitions in $B$. Then,

$$[W^\lambda : L^\mu] \leq 1.$$ 

**Proof.** If $\mu$ is $e$-regular, then $[W^\lambda : L^\mu] = [S^\lambda : D^\mu]$ by Theorem 1.3 so we are done by Theorem 4.4. If $\mu$ is $e$-singular, we display $\lambda$ and $\mu$ on an abacus with $e$ runners and $r$ beads, for some $r$ large enough. Then, we define two abacus displays with $e + 1$ runners each by adding a runner with every space unoccupied to the right of all the existing runners in the abacus displays for $\lambda$ and $\mu$ (see example 2.17). Let $\lambda^+$ and $\mu^+$ be the partitions corresponding to these two new abacus displays. Theorem 2.14 applies and so we have $d^e_{\lambda\mu}(v) = d^{e+1}_{\lambda^+\mu^+}(v)$. Moreover, $\mu^+$ is $(e+1)$-regular, so $[W^\lambda : L^\mu] = [W^\lambda_{C^+} : L^\mu_{C^+}] = [S^\lambda_{C^+} : D^\mu_{C^+}] \leq 1$ by Theorem 1.3, Theorem 2.11 and Theorem 4.4 (note that $\lambda^+$ and $\mu^+$ have weight 3). By Theorem 3.9 $[W^\lambda_{C^+} : L^\mu_{C^+}] = [W^\lambda_{C'} : L^\mu_{C'}]. \square$

5. Proof of James’s Conjecture for weight 4 blocks of $S_n$

We shall prove Theorem 3.10 in this section. Whenever $\lambda$ and $\mu$ are partitions with weight less than 4, $\text{adj}_{\lambda\mu} = \delta_{\lambda\mu}$ by Theorem 3.8 and Theorem 3.9. We will use this fact repeatedly without further comment.
5.1. The principal block of $S_{4e}$. Let $B$ be the principal block of $S_{4e}$; that is the weight 4 block which we display on an abacus with the $\langle 4^e \rangle$ notation.

**Proposition 5.1.** Suppose that $\lambda$ and $\mu$ are partitions lying in $B$. Then, $\text{adj}_{\lambda \mu} = \delta_{\lambda \mu}$.

**Proof.** Observe that when $\lambda = \langle i_{22} \rangle$ and $0 < i < e$, $\lambda$ has two normal beads on runner $i$ and no normal beads on every other runner. By Remark 3.19 we may assume that $\mu$ has at most one normal bead on runner $i$ and no normal beads on every other runner. By Remark 3.17 we may also assume that $\mu \gg \lambda$. We list all the possibilities for $\mu$ below:

- $\langle i_4 \rangle$,
- $\langle i_3, i + 1 \rangle$, $i + 1 < e$,
- $\langle 0, i_3 \rangle$,
- $\langle i_2, (i + 1) \rangle$, $i + 1 < e$.

Notice that every partition $\mu$ in the list above are $e$-regular, therefore $\text{adj}_{\lambda \mu} = \delta_{\lambda \mu}$ by Proposition 3.11 and Theorem 3.7.

Hence, we may assume that $\lambda \neq \langle i_{22} \rangle$ for $0 < i < e$. In this case, observe that in the abacus displays for the remaining possibilities for $\lambda$, no runner has more than one normal bead. By Remark 3.19 we may assume that $\mu$ has no normal beads on any runner. By Remark 3.17 we may also assume that $\mu \gg \lambda$. Table 2 and Table 3 (in the appendix) list all the possible pairs of partitions $\langle \lambda, \mu \rangle$. Note that for each $\mu$, we list the partitions $\lambda$ in descending lexicographic order.

We invite the reader to check that the following partitions induce semisimply to a Rouquier block, thus $\text{adj}_{\lambda \mu} = \delta_{\lambda \mu}$ if $\lambda$ is one of those partitions by Proposition 3.32:

- $\langle 0, 2, 1 \rangle \sim \langle 0, 3, 1 \rangle | 4, 7 \rangle$ when $e = 2$,
- $\langle 0, 1, 2 \rangle \sim \langle 0, 3, 1 \rangle | 4, 7 \rangle$ when $e = 2$,
- $\langle 0, 1, 2 \rangle \sim \langle 0, 3, 1 \rangle | 4, 7, \ldots, 3e + 1 \rangle$,
- $\langle 0, 1, 2 \rangle \sim \langle 0, 3, 1 \rangle | 4, 7, \ldots, 3e + 1 \rangle$,
- $\langle 1, 2 \rangle \sim \langle 0, 2, 1 \rangle | 4, 7, \ldots, 3e + 1 \rangle$,
- $\langle 0, 2 \rangle \sim \langle 0, 2, 1, 2 \rangle | 4, 7, \ldots, 3e + 1 \rangle$.

Let $\lambda^1, \lambda^2, \lambda^3, \mu^1, \mu^2$ and $\mu^3$ (see Table 2 and Table 3) be the following partitions:

- $\lambda^1 := \langle 0, 3, 2 \rangle$, $e = 3$,
- $\mu^1 := \langle 0, 2, 1 \rangle$, $e = 3$,
- $\lambda^2 := \langle 0, 3, 2 \rangle$, $e \geq 4$,
- $\mu^2 := \langle 0, 1, 2, 3 \rangle$, $e \geq 4$,
- $\lambda^3 := \langle 0, 2 \rangle$, $e = 2$,
- $\mu^3 := \langle 0, 2, 1 \rangle$, $e = 2$.

If we managed to show that $\text{adj}_{\lambda^1 \mu^1} = \text{adj}_{\lambda^2 \mu^2} = \text{adj}_{\lambda^3 \mu^3} = 0$, the proof of Proposition 5.1 would follow from Proposition 3.4.

We now relax the definition of $\lambda^3$ and $\mu^3$ slightly, so that $\lambda^3 := \langle 0, 2 \rangle | 4^2, 5^{e-2} \rangle$ and $\mu^3 := \langle 0, 1, 2 \rangle | 4^2, 5^{e-2} \rangle$, where $e \geq 2$. We also define $\lambda^0 := \langle 0, 1, e \rangle | 4^2, 5^{e-2} \rangle$, where $e \geq 2$. Recall from Definition 3.25 that $\hat{F}_{1}^{-\lambda^0} \hat{F}_{0}^{-\lambda^3} \hat{F}_{0}^{-\mu^3} \hat{F}_{0}^{-\mu^3} \hat{F}_{0}^{-\mu^3} \hat{F}_{0}^{-\mu^3} \hat{F}_{0}^{-\mu^3}$ is well-defined for partitions $\nu$ in the block $B$. When $i \in \{1, 2\}$, we invite the reader to check (in an abacus display with the $\langle 4^e \rangle$ notation, the beads on runner 0 have $e$-residue 0) that $\hat{F}_{1}^{-\lambda^i} \hat{F}_{0}^{-\mu^i} \hat{F}_{0}^{-\mu^i} \hat{F}_{0}^{-\mu^i} \hat{F}_{0}^{-\mu^i} \hat{F}_{0}^{-\mu^i}$ are semisimply
induced and moreover, $\tilde{F}_1 \setminus_{\omega \equiv 2} \tilde{F}_0 \setminus_{\omega \equiv 2} \lambda^i = \lambda^0$ and $\tilde{F}_1 \setminus_{\omega \equiv 2} \tilde{F}_0 \setminus_{\omega \equiv 2} \mu^i = \mu^3$. Therefore, we are left to show that $\text{adj}_{\lambda^0, \mu^3} = 0$ and $\text{adj}_{\lambda^3, \mu^3} = 0$ by Corollary 3.27.

**Proposition 5.2.** $\text{adj}_{\lambda^3, \mu^3} = 0$.

**Proof.** Let $f(\lambda^3) := \tilde{F}_2 \setminus_{\omega \equiv 2} \lambda^3$ and $f(\mu^3) := \tilde{F}_2 \setminus_{\omega \equiv 2} \mu^3$ (in an abacus display with the $\langle 4^2, 5^{e-2} \rangle$ notation, the beads on runner 0 have $\omega$-residue 2. When $\omega = 2$, $f$ is the identity map). We observe that $f(\lambda^3)$ and $f(\mu^3)$ are both semisimply induced and moreover, $f(\lambda^3) = \langle (e-1)_2 | 6^{e-2}, 4, 3 \rangle$ and $f(\mu^3) = \langle (e-2)_2, (e-1)_2 | 6^{e-2}, 4, 3 \rangle$. Hence, $\text{adj}_{\lambda^3, \mu^3} = \text{adj}_{f(\lambda^3), f(\mu^3)}$ by Corollary 3.27. We also note that $d_{f(\lambda^3)}(v) = d_{f(\mu^3)}(v) = v^3 + v$ by Theorem 2.15. Hence, $W_{\text{C}}^{f(\lambda^3)} : L_{\text{C}}^{f(\mu^3)} = 2$ and it suffices to prove that $W_{\text{F}}^{f(\lambda^3)} : L_{\text{F}}^{f(\mu^3)} = 2$ by Lemma 3.5.

Let $B^i$ be the weight 4 block with the $\langle 6^{e-1-i}, 3, 6^i, 4 \rangle$ notation for $0 \leq i \leq e - 2$. We define $\lambda^y$, $\lambda^x$ and $\mu^x$ to be the partitions lying in $B^0$ by their abacus displays below. We may check using the modular branching rules (Theorem 2.18 and Theorem 2.20) that

\[
W_{\text{F}}^{f(\lambda^3)} \uparrow_{B^0} \sim W_{\text{F}}^{\lambda^y} + W_{\text{F}}^{\lambda^x},
\]

\[
L_{\text{F}}^{f(\mu^3)} \uparrow_{B^0} \cong L_{\text{F}}^{\mu^x}.
\]

We define $\lambda^{y,x}$, $\lambda^{x,x}$ and $\mu^{x,x}$ to be the partitions lying in $B^{e-2}$ by their abacus displays below. We may check using the modular branching rules that

\[
W_{\text{F}}^{\lambda^{y,x} \uparrow_{B^0} B^1 \uparrow_{B^2 \cdots \uparrow_{B^{e-2}}} \sim W_{\text{F}}^{\lambda^{y,x}}},
\]

\[
W_{\text{F}}^{\lambda^{x,x} \uparrow_{B^0} B^1 \uparrow_{B^2 \cdots \uparrow_{B^{e-2}}} \sim W_{\text{F}}^{\lambda^{x,x}}},
\]

\[
L_{\text{F}}^{\mu^{x,x} \uparrow_{B^0} B^1 \uparrow_{B^2 \cdots \uparrow_{B^{e-2}}} \cong L_{\text{F}}^{\mu^{x,x}}},
\]

Let $C$ be the weight 4 block with the $\langle 5, 6^{e-2}, 2 \rangle$ notation. We define $\lambda^{y,x}, \lambda^{x,x}, \lambda^{x,y}, \lambda^{x,z}$ and $\mu^{x,x}$ to be the partitions lying in $C$ by their abacus displays below. We may check using the modular branching rules that

\[
W_{\text{F}}^{\lambda^{y,x} \uparrow_{B^{e-2}} \sim (W_{\text{F}}^{\lambda^{y,x}})^2},
\]
\[ W_F^{\lambda} \uparrow_{B^e-2} \sim (W_F^{\lambda})^2 + (W_F^{\lambda})^2 + (W_F^{\lambda})^2, \]

\[ L_{\mu}^{\mu} \uparrow_{B^e-2} \cong L_{\mu}^{\mu} \oplus L_{\mu}^{\mu} \]

Hence, we have the upper bound

\[ 2|W_F^{f(\lambda)} : L_{\mu}^{f(\mu)}| = |W_F^{f(\lambda)} : L_{\mu}^{f(\mu)}| \leq \sum_{\nu} |W_F^{f(\lambda)} : L_{\nu}^{f(\mu)}| \]

\[ = 2|W_{\mu}^{\lambda} : L_{\mu}^{\mu} | + |W_{\mu}^{\lambda} : L_{\mu}^{\mu} | + |W_{\mu}^{\lambda} : L_{\mu}^{\mu} | + |W_{\mu}^{\lambda} : L_{\mu}^{\mu} | + |W_{\mu}^{\lambda} : L_{\mu}^{\mu} |. \]

Using the LLT algorithm and Theorem 2.15, we have \( d_{\lambda}^{\mu} (v) = v^2 \), \( d_{\lambda}^{\mu} (v) = v^3 \), and \( d_{\lambda}^{\mu} (v) = v^4 \). Additionally, observe that \( \lambda_{i,x} (v) = \lambda_{i,x} (v) = \lambda_{i,x} (v) = \mu_{i,x} \) for \( 1 \leq i \leq e-1 \), so we may combine Corollary 3.14 (applied \( e - 1 \) times) and Theorem 3.9 to conclude that

\[ |W_{\mu}^{\lambda} : L_{\mu}^{\mu} | = |W_{\mu}^{\lambda} : L_{\mu}^{\mu} | = 1, |W_{\mu}^{\lambda} : L_{\mu}^{\mu} | = |W_{\mu}^{\lambda} : L_{\mu}^{\mu} | = 1 \]

and \( |W_{\mu}^{\lambda} : L_{\mu}^{\mu} | = |W_{\mu}^{\lambda} : L_{\mu}^{\mu} | = 0. \)

By Theorem 3.1 and Corollary 3.3, we have

\[ |W_{\mu}^{\lambda} : L_{\mu}^{\mu} | = |W_{\mu}^{\lambda} : L_{\mu}^{\mu} | + \text{adj}_{\lambda \mu} + \sum_{\lambda \mu \mu} |W_{\mu}^{\lambda} : L_{\mu}^{\mu} |. \]

From the abacus display of \( \mu^{x,x} \), we observe that \( R_{\mu}^{-1}(\mu^{x,x}) \) has weight 3. Therefore, the terms in the sum above are non-zero only if \( \lambda^{x,x} <_{J} \nu <_{J} \mu^{x,x} \) and \( \nu_{e-1} < \mu_{e-1} \) by Corollary 3.14 Corollary 3.16 and Theorem 3.9. It is easy to check that the set \( \{ \nu : \lambda^{x,x} <_{J} \nu <_{J} \mu^{x,x} \} \) induces semisimply to a Rouquier block, so \( \text{adj}_{\lambda \mu} = 0 \). Additionally, \( |W_{\mu}^{\lambda} : L_{\mu}^{\mu} | = d_{\lambda}^{\mu} (1) = 0, \) therefore \( |W_{\mu}^{\lambda} : L_{\mu}^{\mu} | = 0 \) and \( |W_{\mu}^{f(\lambda)} : L_{\mu}^{f(\mu)} | = 2. \)

To prove Proposition 5.1, all that remains is to show that \( \text{adj}_{\lambda \mu} = 0 \). Since \( d_{\lambda}^{\mu} (v) = 0 \) (calculated using the LLT algorithm and runner removal theorems 2.14 and 2.15), we just need to check that \( \text{adj}_{\lambda \mu} = 0 \) for every partition \( \nu \) satisfying \( \lambda \nu < \nu < \mu \) in order to apply Proposition 3.4.

We list all the partitions \( \nu \) such that \( \lambda \nu < \nu < \mu \):

- \( \nu = (0_2, 1_1, 1 | 2, 5 \nu - 2) \sim (0_3, 1 | 4, 7, \ldots, 3e + 1), \) so \( \text{adj}_{\lambda \mu} = 0 \) by Proposition 3.32
- \( \nu = (0_2, 1, 1 | 2, 5 \nu - 2) \sim (0_3, 1 | 4, 7, \ldots, 3e + 1), \) so \( \text{adj}_{\lambda \mu} = 0 \) by Proposition 3.32
- \( \nu = \lambda^3, \) so \( \text{adj}_{\lambda \mu} = 0 \) by Proposition 5.2.

This concludes the proof of Proposition 5.1.

5.2. Blocks forming exactly two \([4 : 1]\)-pairs.\) In the next two sections, we prove the following proposition.
Proposition 5.3. Let $B$ be a weight 4 block of $S_n$. Suppose that there are exactly two blocks $A_1$ and $A_2$ forming a $[4 : 1]$-pair with $B$, and that there are no other blocks $C$ forming a $[4 : k]$-pair with $B$ for any $k$. Additionally, suppose that the adjustment matrix for every weight 4 block of $S_m$ is the identity matrix whenever $m < n$. Then, the adjustment matrix for $B$ is the identity matrix.

The conditions above give two distinct types of block $B$.

5.2.1. Blocks with the $\langle 4^a, 5^{b-a}, 4^{c-b}, 5^{d-c}, 4^{e-d} \rangle$ notation. We first consider the case where $B$ has the $\langle 4^a, 5^{b-a}, 4^{c-b}, 5^{d-c}, 4^{e-d} \rangle$ notation, for some $0 < a < b < c < d \leq e$. Let $\lambda$ and $\mu$ be two distinct partitions in $B$ for which we want to prove that $\text{adj}_{\lambda\mu} = 0$. By Proposition 3.26, if $\lambda$ were non-exceptional for any one of $(A_i, B)$, then $\text{adj}_{\lambda\mu} = \delta_{\lambda\mu}$. Therefore, we may assume that $\lambda$ is exceptional for both pairs $(A_1, B)$ and $(A_2, B)$; that is $\lambda = \langle a_1, c_1 \rangle$.

Observe that the first unoccupied position in the abacus display for $\lambda$ above occurs at position $a$. By Remark 3.17, we may assume that $l(\mu) < l(\lambda)$, so the first unoccupied position in an abacus display for $\mu$ must occur after position $a$. Consequently, all beads in runner 0 of an abacus display for $\mu$ have zero weight, therefore $\mu$ must be $e$-regular and $\text{adj}_{\lambda\mu} = \delta_{\lambda\mu}$ by Proposition 3.11 and Theorem 3.7.

5.2.2. Blocks with the $\langle 4^a, 5^{b-a}, 6^{c-b}, 5^{d-c}, 4^{e-d} \rangle$ notation. We complete the proof of Proposition 5.3 by considering blocks $B$ with the $\langle 4^a, 5^{b-a}, 6^{c-b}, 5^{d-c}, 4^{e-d} \rangle$ notation, for some $0 < a < b < c \leq d \leq e$. Let $\lambda$ and $\mu$ be two distinct partitions in $B$ for which we want to prove that $\text{adj}_{\lambda\mu} = 0$. By Proposition 3.26, we may assume that $\lambda$ is exceptional for both pairs $(A_1, B)$ and $(A_2, B)$; that is $\lambda = \langle a_1, b_1 \rangle$ with $b - a \geq 2$.

Observe that the first unoccupied position in the abacus display for $\lambda$ above occurs at position $a$. By Remark 3.17, we may assume that $l(\mu) < l(\lambda)$, so the first unoccupied position in an abacus display for $\mu$ must occur after position $a$. Consequently, all beads in runner 0 of an abacus display for $\mu$ have zero weight, therefore $\mu$ must be $e$-regular and $\text{adj}_{\lambda\mu} = \delta_{\lambda\mu}$ by Proposition 3.11 and Theorem 3.7.

This completes the proof of Proposition 5.3.

5.3. Blocks forming exactly one $[4 : 3]$-pair.

Proposition 5.4. Suppose that $A$ and $B$ are weight 4 blocks of $S_{n-1}$ and $S_n$ respectively, forming a $[4 : 3]$-pair. Moreover, suppose that there is no block other than $A$ forming a $[4 : k]$-pair with $B$ for any $k$. Additionally, suppose that the adjustment matrix for every weight 4 block of $S_m$ is the identity matrix whenever $m < n$. Then, the adjustment matrix for $B$ is the identity matrix.

Proof. The conditions on $B$ mean that we may represent $B$ on an abacus with the $\langle 4^a, 7^{b-a}, 6^{c-b}, 5^{d-c}, 4^{e-d} \rangle$ notation, where $0 < a < b \leq c \leq d \leq e$. If $\lambda$ and $\mu$ are two distinct partitions in $B$, then we have
\[ \text{adj}_\lambda \mu = 0 \] by Proposition 3.26 unless \( \lambda \) is the unique exceptional partition for \((A, B)\), namely \( \lambda = \langle a_{14} \rangle \).

Observe that the first unoccupied position in the abacus display for \( \lambda \) above occurs at position \( a \). By Remark 3.17, we may assume that \( l(\mu) < l(\lambda) \), so the first unoccupied position in an abacus display for \( \mu \) must occur after position \( a \). Consequently, all beads in runner 0 of an abacus display for \( \mu \) have zero weight, therefore \( \mu \) must be \( e \)-regular and \( \text{adj}_\lambda \mu = \delta_{\lambda} \mu \) by Proposition 3.11 and Theorem 3.7.

5.4. Blocks forming exactly one \([4 : 2]\)-pair.

**Proposition 5.5.** Suppose that \( A \) and \( B \) are weight 4 blocks of \( S_{n-1} \) and \( S_n \) respectively, forming a \([4 : 2]\)-pair. Moreover, suppose that there is no block other than \( A \) forming a \([4 : k]\)-pair with \( B \) for any \( k \). Additionally, suppose that the adjustment matrix for every weight 4 block of \( S_m \) is the identity matrix whenever \( m < n \). Then, the adjustment matrix for \( B \) is the identity matrix.

The conditions on \( B \) mean that we may represent \( B \) on an abacus with the \((4^a, 6^{b-a}, 5^{c-b}, 4^{e-c})\) notation, where \( 0 < a < b \leq c \leq e \). Suppose that \( \lambda \) and \( \mu \) are two distinct partitions lying in \( B \) and we want to prove that \( \text{adj}_\lambda \mu = 0 \). If \( \lambda \) were non-exceptional for \((A, B)\), then \( \text{adj}_\lambda \mu = \delta_{\lambda} \mu \) by Proposition 3.26. Therefore we may assume that \( \lambda \) is exceptional; that is \( \lambda \) must be one of the following partitions:

- \( \langle a_{2,12} \rangle \),
- \( \langle a_{13}, i \rangle, i \notin \{a-1, a\} \),
- \( \langle a_{14} \rangle \).

We invite the reader to verify that \( \langle a_{14} \rangle \) induces semi-simply to a Rouquier block:

\[
\langle a_{14} \rangle \sim \begin{cases} 
(0, a, a + e - c, a + e - b | 4, 7, \ldots, 3e + 1) & \text{if } e - c > 0, c - b > 0, \\
(0, a_{12}, a + e - c | 4, 7, \ldots, 3e + 1) & \text{if } e - c = 0, c - b > 0, \\
(0, a_{13}, a + e - c | 4, 7, \ldots, 3e + 1) & \text{if } e - c > 0, c - b = 0, \\
(0, a_{13} | 4, 7, \ldots, 3e + 1) & \text{if } e - c = 0, c - b = 0. 
\end{cases}
\]

Therefore, we may assume that \( \lambda \neq \langle a_{14} \rangle \) by Proposition 3.32. We are left to consider the cases \( \lambda = \langle a_{2,12} \rangle \) or \( \lambda = \langle a_{13}, i \rangle, i \notin \{a-1, a\} \).

Let \( \nu := \langle 0, a_{13} \rangle \). Note that \( l(\nu) \geq l(\lambda) \) for all the remaining possible \( \lambda \), so we may assume that \( l(\mu) < l(\nu) \) by Remark 3.17. Since the first unoccupied position in the abacus display for \( \lambda \) occurs at position 0, the first unoccupied position in an abacus display for \( \mu \) must occur strictly after position 0. Consequently, all beads in runner 0 of an abacus display for \( \mu \) have zero weight, therefore \( \mu \) must be \( e \)-regular and \( \text{adj}_\lambda \mu = \delta_{\lambda} \mu \) by Proposition 3.11 and Theorem 3.7. This completes the proof of Proposition 5.5.
5.5. Blocks forming exactly one $[4 : 1]$-pair. In this section, we prove the following proposition:

**Proposition 5.6.** Suppose that $A$ and $B$ are weight 4 blocks of $S_{n-1}$ and $S_n$ respectively, forming a $[4 : 1]$-pair. Moreover, suppose that there is no block other than $A$ forming a $[4 : k]$-pair with $B$ for any $k$. Additionally, suppose that the adjustment matrix for every weight 4 block of $S_m$ is the identity matrix whenever $m < n$. Then, the adjustment matrix for $B$ is the identity matrix.

The conditions on $B$ mean that we may represent $B$ on an abacus with the $(4^a, 5^{b-a}, 4^{e-b})$ notation, where $0 < a < b \leq e$. Suppose that $\lambda$ and $\mu$ are two distinct partitions lying in $B$ and we want to prove that $\text{adj}_{\lambda\mu} = 0$. By Proposition 3.26 and Proposition 3.30 we may assume that $\lambda$ is exceptional, so that $\lambda$ is one of the following partitions:

- $\langle a_3, 1 \rangle$,
- $\langle a_2^2 \rangle$,
- $\langle a_2, 1_2 \rangle$,
- $\langle a_{2,1}, i \rangle$, $i \notin \{a - 1, a\}$, $e \geq 3$,
- $\langle a_{12}, i_2 \rangle$, $i \notin \{a - 1, a\}$, $e \geq 3$,
- $\langle a_{12}, i, j \rangle$, $i, j \notin \{a - 1, a\}$ and $i < j$, $e \geq 4$,
- $\langle a_{11}, i \rangle$, $i \neq a$,
- $\langle a_{12, 1_2} \rangle$, $i \notin \{a - 1, a\}$, $e \geq 3$,
- $\langle a_4 \rangle$.

We invite the reader to check that $\langle a_{14} \rangle$ induces semi-simply to a Rouquier block:

$$\langle a_{14} \rangle \sim \begin{cases} \langle 0_{12}, a, a + e - b \mid 4, 7, \ldots, 3e + 1 \rangle & \text{if } e - b > 0, \\
\langle 0_{12}, a_{12} \mid 4, 7, \ldots, 3e + 1 \rangle & \text{if } e - b = 0. \end{cases}$$

Therefore, we may assume that $\lambda \neq \langle a_{14} \rangle$ by Proposition 3.32. In the remaining possibilities for $\lambda$, observe that $\lambda$ has exactly two normal beads on runner $a$ and at most one normal bead on each of the other runners. By Remark 3.19 we may assume that $\mu$ has at most one normal bead on runner $a$ (non-exceptional) and no normal beads on every other runner. Note that $\nu := \langle 0_{12}, a_{12} \rangle \leq P \lambda$ for all the remaining possibilities for $\lambda$. By Remark 3.17 we may further assume that $\mu \gg \lambda$ and $\lambda \geq P \nu$ imply that $\mu \gg \nu$. By Corollary 3.11 we may also assume that $\mu$ is $e$-singular.

We list every possibility for $\mu$ satisfying all the conditions stated above:

- (A1) $\langle a_2, b_{12} \rangle$, $e - b > 0$,
- (A2) $\langle a, b_{2,1} \rangle$, $b - a = 1$, $e - b > 0$,
- (A3) $\langle a, a + 1, b_{2,2} \rangle$, $b - a \geq 2$, $e - b > 0$,
- (B1) $\langle 0_{13}, a \rangle$, $a = 1$, $b - a = 1$, $e - b = 0$,
- (B2) $\langle 0_{2, a, a + 1} \rangle$, $a = 1$, $e - b = 0$, $b - a \geq 2$,
- (C1) $\langle 0, a_3 \rangle$,
- (C2) $\langle 0, a_2, a + 1 \rangle$, $b - a \geq 2$,
- (C3) $\langle 0, a_{2,1} \rangle$, $a = 1$,
- (C4) $\langle 0, 1, a_2 \rangle$, $a \geq 2$,
- (C5) $\langle 0, a, a + 1, a + 2 \rangle$, $b - a \geq 3$,
- (C6) $\langle 0, a_{12}, a + 1 \rangle$, $a = 1$, $b - a \geq 2$,
- (C7) $\langle 0, 1, a, a + 1 \rangle$, $a \geq 2$, $b - a \geq 2$.

We now consider 2 separate cases.

5.5.1. The case $e - b > 0$. For any partition $\nu$ lying in $B$, we define $a(\nu) := \tilde{\tilde{E}}_{a-b}^{a} \nu$ (in an abacus display with the $(4^a, 5^{b-a}, 4^{e-b})$ notation, the beads on runner 0 have $e$-residue $a - b$). We may check that $a(\lambda)$ and $a(\mu)$ are both semisimply induced and that $(a(\lambda), a(\mu))$ is lowerable except when one of the following happens:

- $\lambda = \langle a_{22} \rangle$ or $\lambda = \langle i_{2}, a_{12} \rangle$ for some $0 \leq i \leq a - 2$, 


the Jantzen-Schaper formula. We denote their induced counterparts as \(\hat{\lambda}, \hat{\mu}\), for partitions in block cases C1, C2 and C5. Suppose that \(\langle \lambda, \mu \rangle\) are:
- \(\langle a_{13}, i \rangle, 0 \leq i \leq a - 1\) or \(i = b\),
- \(\langle i_{12}, a_{12} \rangle, 0 \leq i \leq a - 2\).

For any partition \(\nu\) lying in \(B\), we define \(b(\nu) := \hat{F}_a \backslash b_{-a} \nu\) (in an abacus display with the \(\langle 4^a, 5^{e-a} \rangle\) notation, the beads on runner \(b\) have \(e\)-residue \(a\)). We observe that for the remaining pairs \((\lambda, \mu), b(\lambda)\) and \(b(\mu)\) are both semisimply induced and that \((b(\lambda), b(\mu))\) is lowerable, so the proof of Proposition 5.6 when \(e - b > 0\) is complete by Corollary 3.21 and Corollary 3.27.

5.5.2. The case \(e - b = 0\). The case \(e = b\) is much more difficult to deal with. From now on, we assume that \(e = b\). We will first refine the list of possible \(\mu\). We note that \(\langle 0, a_{13} \rangle \sim \langle 0, a_{12} | 4, 7, \ldots, 3e + 1\rangle\), so we may assume that \(\lambda \neq \langle 0, a_{13} \rangle\) by Proposition 3.32. In view of this, we have \(\tau := \langle a_{13}, a + 1 \rangle \leq \mu\lambda\) for the remaining possibilities for \(\lambda\) when \(a = 1\) and \(e - a \geq 2\). When \(a = 1\) and \(e - a = 1\), we have \(\gamma := \langle a_{212} \rangle \leq \mu\lambda\) for the remaining possibilities for \(\lambda\).

By Remark 3.17, we may assume that \(\lambda \gg \tau\) when \(a = 1\) and \(e - a \geq 2\), since \(\lambda \gg \lambda\) and \(\lambda \geq \tau\) implies that \(\mu \gg \gamma\); when \(a = 1\) and \(e - a = 1\), we may assume that \(\mu \gg \gamma\). These two additional restrictions produce the following refined list of possibilities for \(\mu\):

- \((C1) \langle 0, a_{3} \rangle,\)
- \((C2) \langle 0, a_{2}, a + 1 \rangle, e - a \geq 2\),
- \((C3) \langle 0, a_{21} \rangle, a = 1, e - a \geq 2\),
- \((C4) \langle 0, 1, a_{2} \rangle, a \geq 2\),
- \((C5) \langle 0, a, a + 1, a + 2 \rangle, e - a \geq 3\),
- \((C7) \langle 0, 1, a, a + 1 \rangle, a \geq 2, e - a \geq 2\).

For any partition \(\sigma\) in \(B\), define \(a(\sigma) := \hat{F}_a \backslash a \sigma\) (in an abacus display with the \(\langle 4^a, 5^{e-a} \rangle\) notation, the beads on runner 0 have \(e\)-residue \(a\)). When \(\mu\) is in cases C1, C3 and C4 with \(e - a \geq 2\), \(a(\lambda)\) and \(a(\mu)\) are both semisimply induced and \(a(\lambda), a(\mu)\) is lowerable. When \(e - a \geq 3\) and \(\mu\) is in case C2, \(a(\lambda)\) and \(a(\mu)\) are both semisimply induced and \(a(\lambda), a(\mu)\) is lowerable. In view of Corollary 3.21 and Corollary 3.27, we are left with the following possibilities for \(\mu\):

- \((C1) \langle 0, a_{3} \rangle, e - a = 1\),
- \((C2) \langle 0, a_{2}, a + 1 \rangle, e - a = 2\),
- \((C4) \langle 0, 1, a_{21} \rangle, a \geq 2, e - a = 1\),
- \((C5) \langle 0, a, a + 1, a + 2 \rangle, e - a \geq 3\),
- \((C7) \langle 0, 1, a, a + 1 \rangle, a \geq 2, e - a \geq 2\).

**Cases C1, C2 and C5.** Suppose that \(\mu\) is in either case C1, C2 or C5. When \(e - a = 1\), \(\langle a_{212} \rangle \sim \langle 0, a_{21} | 4, 7, \ldots, 3e + 1\rangle\), so we may assume that \(\lambda \neq \langle a_{212} \rangle\) by Proposition 3.32. By Remark 3.17, we may also assume that \(\lambda \ll \mu\). We find that the only remaining possibilities for \(\lambda\) are:
- \(\langle i_{12}, a_{12} \rangle\) for some \(0 \leq i \leq a - 2\),
- \(\langle i, a_{13} \rangle\) for some \(1 \leq i \leq a - 1\).

Our strategy to deal with the remaining pairs \((\lambda, \mu)\) is to first induce them up semisimply via the same sequence of inductions (for example using \(f := \hat{F}_{a+1} \backslash a \sigma_{e-a} \ldots \hat{F}_{2a-1} \backslash a \sigma_{e-a-1} \hat{F}_{2a-2} \backslash a \sigma_{e-a-2} \ldots \hat{F}_{a+1} \backslash a \sigma_{e-a} \hat{F}_{a} \backslash a \sigma_{e-a}\) which is well-defined for partitions in block \(B\)) to the block with the \(\langle 4^a, 6, 9^{e-a-1} \rangle\) notation, followed by an application of the Jantzen-Schaper formula. We denote their induced counterparts as \((\hat{\lambda}, \hat{\mu})\); that is \((\lambda, \mu) \sim (\hat{\lambda}, \hat{\mu})\).
Using Theorem 2.6, we would show that \( W^\lambda : L^\hat{\mu} \) is independent of \( \text{char}(F) \), so that \( \text{adj}_\lambda \hat{\mu} = \delta_\lambda \hat{\mu} \) and hence \( \text{adj}_{\lambda \mu} = \delta_{\lambda \mu} \) by Corollary 3.27 and Lemma 3.5. We may check that

- \( (i_2, a_2) \sim (i_{2,1}, a \mid 4^a, 6, 9^{e-a-1}) \) when \( i \leq a - 2 \),
- \( (i, a_{2,1}) \sim (0, i_2, a \mid 4^a, 6, 9^{e-a-1}) \) when \( 1 \leq i \leq a - 1 \),
- \( \mu \sim (0, a_3 \mid 4^a, 6, 9^{e-a-1}) \).

Note that \( \langle 0_{2,1}, a \mid 4^a, 6, 9^{e-a-1} \rangle \leq J \hat{\lambda} \) for all remaining possible \( \lambda \). Table 4, Table 5 and Table 6 (in the appendix) illustrates how we use Theorem 2.6. The entries of the tables are the Jantzen-Scherer coefficients (Definition 2.4) \( J(\nu, \sigma) \), for partitions \( \langle 0_{2,1}, a \mid 4^a, 6, 9^{e-a-1} \rangle \leq \nu < J \sigma \leq J \hat{\mu} = (0, a_3 \mid 4^a, 6, 9^{e-a-1}) \). Note that \( J_F(\nu, \sigma) = J_C(\nu, \sigma) \) since \( \text{char}(F) > w = 4 \). We also omit the columns indexed by \( \nu \) if \( W^\nu : L^\hat{\mu} = 0 \) as these do not contribute to our calculations. If we are able to justify the last column of Table 4, Table 5 and Table 6, it would imply that \( \text{adj}_{\lambda \mu} = \delta_{\lambda \mu} \) for every remaining possible \( \lambda \) by Corollary 3.27 and Lemma 3.5.

We now proceed to justify the last column of Table 4, Table 5 and Table 6. When \( B(\nu, \hat{\mu}) \leq 1 \), we have \( W^\nu : L^\hat{\mu} = B(\nu, \hat{\mu}) \) by Corollary 2.7 so \( \text{adj}_{\nu \hat{\mu}} = \delta_{\nu \hat{\mu}} \) by Lemma 3.5. When \( B(\nu, \hat{\mu}) > 1 \), we have to do more work.

Let \( \nu^0 := \langle (a - 1)_{2,1}, a \mid 4^a, 6, 9^{e-a-1} \rangle \). Observe that \( B(\nu^0, \hat{\mu}) = 2 \) and that this is the only partition \( \nu \) in Table 4, Table 5 and Table 6 with \( B(\nu, \hat{\mu}) > 1 \) (shaded the tables). Following Remark 2.13, we calculate \( \sigma_\nu(\nu^0) \) and \( \sigma_\nu(\hat{\mu}) \), and find that they are both \((-1)^a\), hence \( W^{\nu^0} : L^\hat{\mu}_C = 1 \). A priori, we only know that \( W^{\nu^0} : L^\hat{\mu}_F \leq 2 \).

**Lemma 5.7.** We have \( W^{\nu^0} : L^\hat{\mu}_F = 1 \).

**Proof.** We may check that \( \nu^0 \) induces semi-simply to a Rouquier block:

\[
\nu^0 \sim \begin{cases} 
\langle 0_{2,1}, a \mid 4, 7, \ldots, 3e + 1 \rangle & \text{if } a = 1, \\
\langle 0, (a - 1)_{2,1}, a \mid 4, 7, \ldots, 3e + 1 \rangle & \text{if } a \geq 2.
\end{cases}
\]

Hence, \( \text{adj}_{\nu \hat{\mu}} = 0 \) by Proposition 3.32. Moreover, we deduce from the rows above \( \nu^0 \) in Table 4, Table 5 and Table 6 that \( W^\nu : L^\hat{\mu} \) is independent of \( F \) whenever \( \nu^0 < J \nu < J \hat{\mu} \), therefore \( \text{adj}_{\nu \hat{\mu}} = 0 \) by Lemma 3.5. Finally,

\[
[W^{\nu^0}_F : L^\hat{\mu}_F] = \sum_{\nu^0 \leq \nu \leq J \hat{\mu}} [W^{\nu^0}_C : L^\nu_C] \text{adj}_{\nu \hat{\mu}} = [W^{\nu^0}_C : L^\hat{\mu}_C] = 1.
\]

\( \square \)

**Cases C4 and C7.** Suppose that \( \mu \) is in either case C4 or C7. When \( e - a = 1, \langle a_{2,1} \rangle \sim \langle 0, a_{2,1} \mid 4, 7, \ldots, 3e + 1 \rangle \), so we may assume that \( \lambda \neq \langle a_{2,1} \rangle \) by Proposition 3.32. By Remark 3.17, we may also assume that \( \lambda \ll \mu \). We find that the only remaining possibilities for \( \lambda \) are:

- \( \langle 1, a_{13} \rangle \),
- \( \langle 1_{12}, a_{12} \rangle \),
- \( \langle 0_{12}, a_{13} \rangle \).

We use the Jantzen Schaper formula and Corollary 3.27 in the same fashion as in section 5.5.2 to deal with the remaining pairs of \( (\lambda, \mu) \). We may check that (for example using \( f := F_{6a+2} \setminus e_{a-e-1} \cdots e_{a-e-1} F_{2a} \setminus e_{a-e-1} F_{2a} \setminus e_{a-e-1} F_{2a} \setminus e_{a-e-1} \cdots F_{a+1} \setminus e_{a-e-1} F_a \setminus e_{a-e-1} ) \):

- \( \langle 1, a_{13} \rangle \sim \langle 0, 1_{12}, a \mid 4^a, 6, 9^{e-a-1} \rangle \),
- \( \langle 1_{12}, a_{12} \rangle \sim \langle 0_{2,1}, a \mid 4^a, 6, 9^{e-a-1} \rangle \),
- \( \langle 0_{12}, a_{12} \rangle \sim \langle 0_{2,1}, a \mid 4^a, 6, 9^{e-a-1} \rangle \),
- \( \mu \sim \langle 0, 1, a_{2} \mid 4^a, 6, 9^{e-a-1} \rangle \).
Note that $\langle 0_{2,1}, a \mid 4^a, 6, 9^{e-a} - 1 \rangle <_J \langle 1_{2,1}, a \mid 4^a, 6, 9^{e-a} - 1 \rangle <_J \langle 0, 1_2, a \mid 4^a, 6, 9^{e-a} - 1 \rangle$. The entries of Table 7 (in the appendix) are the Jantzen-Schaper coefficients (Definition 2.4) $J(\nu, \sigma)$, for partitions $(0_{2,1}, a \mid 4^a, 6, 9^{e-a} - 1) <_J \nu <_J \sigma \leq _J \mu := \langle 0, 1_2, a \mid 4^a, 6, 9^{e-a} - 1 \rangle$. Note that $J_p(\nu, \sigma) = J_C(\nu, \sigma)$ since $\text{char}(\mathbb{F}) > w = 4$. We also omit the columns indexed by $\nu$ if $[W^\nu : L^\mu] = 0$ as these do not contribute to our calculations. If we are able to justify the last column of Table 7 it would imply that $[W^\nu : L^\mu]$ is independent of $\mathbb{F}$, hence $\text{adj}_{\nu, \sigma} = \delta_{\nu, \sigma}$ for all $(0_{2,1}, a \mid 4^a, 6, 9^{e-a} - 1) \leq _J \nu <_J \sigma \leq _J \mu$ by Lemma 3.5. This shows that $\text{adj}_{\nu, \mu} = \delta_{\nu, \mu}$ for every remaining possible $\lambda$ by Corollary 3.27.

We now proceed to justify the last column of Table 7. When $B(\nu, \mu) \leq 1$, we have $[W^\nu : L^\mu] = B(\nu, \mu)$ by Corollary 2.7, so $\text{adj}_{\nu, \mu} = \delta_{\nu, \mu}$ by Lemma 3.5. When $B(\nu, \mu) > 1$, we have to do more work.

Let $\nu^0 := \langle 1_{2,1}, a \mid 4^a, 6, 9^{e-a} - 1 \rangle$. Observe that $B(\nu^0, \mu) = 2$ (shaded in Table 7) and that this is the only partition $\nu$ in Table 7 with $B(\nu, \mu) > 1$. Following Remark 2.13, we calculate $\sigma_{e}(\nu^0)$ and $\sigma_{\nu}(\mu)$, and find that they are both +1, hence $[W_{C}^{\nu^0} : L_{C}^{\mu}] = 1$. A priori, we only know that $[W_{F}^{\nu^0} : L_{F}^{\mu}] \leq 2$.

Lemma 5.8. We have $[W_{F}^{\nu^0} : L_{F}^{\mu}] = 1$.

Proof. We may check that $\nu^0 \sim \tilde{\nu} := \langle 0, 1, 2 \mid 7, 3, 4^a - 1, 10^{e-a} - 1 \rangle$ and $\mu \sim \tilde{\mu} := \langle 0, 1_2, 7 \mid 3, 4^a - 1, 10^{e-a} - 1 \rangle$ (see Figure 4 in the appendix). Observe that $l(\tilde{\nu}) = l(\tilde{\mu})$ and that $C(\tilde{\nu})$ and $C(\tilde{\mu})$ have weight 3. Hence $\text{adj}_{\nu^0, \mu} = 0$ by Theorem 3.9 and Corollary 3.16. Moreover, we deduce from the rows above $\nu^0$ in Table 7 that $[W^\nu : L^\mu]$ is independent of $\mathbb{F}$ whenever $\nu^0 <_J \nu <_J \mu$, therefore $\text{adj}_{\nu^0, \mu} = 0$ by Lemma 3.5. Finally,

$$[W_{F}^{\nu^0} : L_{F}^{\mu}] = \sum_{\nu^0 \leq \nu \leq _J \mu} [W_{C}^{\nu^0} : L_{C}^{\nu}] \text{adj}_{\nu^0, \mu} = [W_{C}^{\nu^0} : L_{C}^{\mu}] = 1.$$ 

This completes the proof of Proposition 5.6.

As discussed in section 3.6, the combination of Proposition 5.1, Proposition 5.6, Proposition 5.5, Proposition 5.4 and Proposition 5.3 completes the proof of Theorem 3.10.

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Appendix A. Figures and tables

Figure 1 Exceptional partitions for [3 : 1]-pairs.

Figure 2 Exceptional partitions for [3 : 2]-pairs.
Figure 3 Exceptional partitions for $[4 : k]$-pairs, $0 < k < 4$.

Table 1 Some cases $(\lambda, \mu)$ in section 4.1

| $\mu$ | $\lambda$ |
|-------|-----------|
| $\langle 0_1 \rangle$ | NIL |
| $\langle 0_2, e - 1 \rangle$, $e = 2$ | $\langle 0_1 \rangle$ $\langle 1_1 \rangle$ $\langle 0_2, 1 \rangle$ $\langle 0, 1_2 \rangle$ |
| $\langle 0_1, 1 \rangle$ | $\langle 0_1 \rangle$ |
| $\langle 0, 1, 2 \rangle$, $e \geq 3$ | $\langle 0_1 \rangle$ $\langle 1_1 \rangle$ $\langle 0_2, 1 \rangle$ $\langle 0, 1_2 \rangle$ |

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Figure 4 $\nu^0 = \langle 1_{2,1}, a \mid 4^a, 6, 9^{e-a-1} \rangle$, $\hat{\mu} = \langle 0, 1, a_2 \mid 4^a, 6, 9^{e-a-1} \rangle$
Table 2 Some cases $(\lambda,\mu)$ in section 5.1. For each $\mu$, the possibilities for $\lambda$ are listed in descending lexicographic order.

| $\mu$                  | $\lambda$   | $d_{\lambda \mu}^k(v)$ |
|------------------------|-------------|------------------------|
| $(0_{14})$             | NIL         |                        |
| $(0_{13}, 1)$          | $0_{14}$    |                        |
| $(0_{12}, 1_{12})$     | $0_{14}$    |                        |
| $(0_{12}, 1, 2)$, $e \geq 3$ | $0, 1_{13}$ |                        |
|                        | $0_{13}, 1$ |                        |
|                        | $1_{14}$    |                        |
|                        | $0_{14}$    |                        |
| $(0_{2,1}, 1)$, $e = 2$ | $0, 1_{13}$ |                        |
|                        | $0_{13}, 1$ |                        |
|                        | $1_{14}$    |                        |
|                        | $0_{14}$    |                        |
| $\mu^1 := (0_{2,1}, 2)$, $e = 3$ | $0, 1, 2_{12}$ |                        |
|                        | $0, 1_{12}, 2$ |                        |
|                        | $0_{12}, 1, 2$ |                        |
|                        | $1_{12}, 2_{12}$ |                        |
|                        | $0_{12}, 2_{12}$ |                        |
|                        | $1, 2_{13}$ |                        |
|                        | $1_{13}, 2$ |                        |
|                        | $0, 2_{13}$ |                        |
| $\lambda^1 := (0_{13}, 2)$ | $2_{14}$    |                        |
|                        | $0_{12}, 1_{12}$ |                        |
|                        | $0, 1_{13}$ |                        |
|                        | $0_{13}, 1$ |                        |
|                        | $1_{14}$    |                        |
|                        | $0_{14}$    |                        |
| $\mu^2 := (0_{1,2,3}), e \geq 4$ | $(0, 1, 2_{12})$ | $d_{\lambda \mu}^4(v) = v$ |
|                        | $(0, 1_{12}, 2)$ | $d_{\lambda \mu}^4(v) = v$ |
|                        | $(0_{12}, 1, 2)$ | $d_{\lambda \mu}^4(v) = v^3$ |
Table 3 Some cases \((\lambda, \mu)\) in section 5.1. For each \(\mu\), the possibilities for \(\lambda\) are listed in descending lexicographic order.

| \(\lambda\)             | \(d_{\lambda\mu}^{\mu}(v)\) |
|-------------------------|-------------------------------|
| \(\langle 0, l_{12}\rangle\) | \(v^3 + v\)                   |
| \(\langle 0, l_{12}\rangle\) | \(0\)                          |
| \(\langle 0, 1_{13}\rangle\) | \(0\)                          |
| \(\langle 1_{14}\rangle\) | \(0\)                          |

Table 4 We have \(a > 3\), \(\nu\) and \(\hat{\mu} = (0, a_3)\) lie in the block with the \(\langle 4^a, 6, 9^{e-a-1}\rangle\) notation. See section 5.5.2 for an explanation of how we obtained the last column.

| \(\langle 0, a_3\rangle\) | \(\langle 0, a_2\rangle\) | \(\langle (a - 1)_{12}, a_2\rangle\) | \(\langle 0, (a - 1)_{2}, a\rangle\) | \(\langle (a - 1)_{2,1, a}\rangle\) | \(\langle (a - 2)_{1,2, a}\rangle\) | \(B(\nu, \hat{\mu})\) | \([W^\nu : L^\hat{\mu}]\) |
|--------------------------|--------------------------|----------------------------------|-------------------------------|--------------------------|--------------------------|------------------|---------------------|
| \(\langle 0, a_3\rangle\) | \(\langle 0, a_2\rangle\) | \(\langle (a - 1)_{12}, a_2\rangle\) | \(\langle 0, (a - 1)_{2}, a\rangle\) | \(\langle (a - 1)_{2,1, a}\rangle\) | \(\langle (a - 2)_{1,2, a}\rangle\) | \(B(\nu, \hat{\mu})\) | \([W^\nu : L^\hat{\mu}]\) |
| \(\langle 0, a - 1, a_2\rangle\) | \(\langle 0, a - 1, a_{12}\rangle\) | \(\langle 0, 1, a_{12}\rangle\) | \(\langle (0, a - 1)_{12}, a_2\rangle\) | \(\langle (0, a - 1)_{2}, a\rangle\) | \(\langle (0, a - 2)_{1,2, a}\rangle\) | \(B(\nu, \hat{\mu})\) | \([W^\nu : L^\hat{\mu}]\) |
Table 5 We have $a = 3$, $\nu$ and $\hat{\mu} = \langle 0, 3 \rangle$ lie in the block with the $\langle 4^a, 6, 9^{e-a-1} \rangle$ notation. See section 5.5.2 for an explanation of how we obtained the last column.

|       | (0, 3a) | (02, 32) | (212, 32) | (0, 22, 3) | (221, 3) | (121, a) | $B(\nu, \hat{\mu})$ | $[W^\nu : L^\hat{\mu}]$ |
|-------|---------|---------|-----------|-----------|---------|---------|-----------------|-----------------|
| (0, a3) | 0       |         |           |           |         |         |                 |                 |
| (0, a2, 1) | 0       | 0       |           |           |         |         |                 |                 |
| (02, 32) | 1       |         |           |           |         |         |                 |                 |
| (0, 32) | −1      | 1       |           |           |         |         |                 |                 |
| (0, a1, a) | 1       |         |           |           |         |         |                 |                 |
| (0, a1, a) | 0       | 0       |           |           |         |         |                 |                 |
| (0, 22, 3) | −1      | 1       |           |           |         |         |                 |                 |
| (0, 1, 3) | 1 −1    |         |           |           |         |         |                 |                 |
| (0, 2, 3) | 1 0 −1  |         |           |           |         |         |                 |                 |
| (0, 1, 3) |         |         |           |           |         |         |                 |                 |
| (0, 2, 3) |         |         |           |           |         |         |                 |                 |
| (02, 32) | 0       |         |           |           |         |         |                 |                 |
| (0, 1, 3, 2) |         |         |           |           |         |         |                 |                 |
| (0, 2, 3, 2) |         |         |           |           |         |         |                 |                 |
| (0, 1, 3, 2) |         |         |           |           |         |         |                 |                 |
| (0, 2, 3, 2) |         |         |           |           |         |         |                 |                 |
| (0, 2, 3) |         |         |           |           |         |         |                 |                 |
| (0, 1, 3) |         |         |           |           |         |         |                 |                 |

Table 6 We have $a = 2$, $\nu$ and $\hat{\mu} = \langle 0, 2 \rangle$ lie in the block with the $\langle 4^a, 6, 9^{e-a-1} \rangle$ notation. See section 5.5.2 for an explanation of how we obtained the last column.

|       | (0, 3a) | (02, 2a) | (121, 2a) | (0, 12, 2) | (121, 2) | (121, 2) | $B(\nu, \hat{\mu})$ | $[W^\nu : L^\hat{\mu}]$ |
|-------|---------|---------|-----------|-----------|---------|---------|-----------------|-----------------|
| (0, 2a) | 0       |         |           |           |         |         |                 |                 |
| (0, 22, 1) | 0       | 0       |           |           |         |         |                 |                 |
| (02, 2a) | 1       |         |           |           |         |         |                 |                 |
| (0, 2a) | −1      | 1       |           |           |         |         |                 |                 |
| (0, 2a) |         |         |           |           |         |         |                 |                 |
| (0, 1, 2) | 0       |         |           |           |         |         |                 |                 |
| (0, 2a) |         |         |           |           |         |         |                 |                 |
| (0, 1, 2) |         |         |           |           |         |         |                 |                 |
| (0, 2a) |         |         |           |           |         |         |                 |                 |
| (0, 1, 2) |         |         |           |           |         |         |                 |                 |
| (0, 2a) |         |         |           |           |         |         |                 |                 |
| (0, 1, 2) |         |         |           |           |         |         |                 |                 |
| (0, 2a) |         |         |           |           |         |         |                 |                 |
| (0, 2a) |         |         |           |           |         |         |                 |                 |
| (0, 1, 2) |         |         |           |           |         |         |                 |                 |
| (0, 2a) |         |         |           |           |         |         |                 |                 |
| (0, 1, 2) |         |         |           |           |         |         |                 |                 |
| (0, 2a) |         |         |           |           |         |         |                 |                 |
| (0, 1, 2) |         |         |           |           |         |         |                 |                 |
Table 7 We have $a \geq 2$, $e - a \geq 1$, $\nu$ and $\hat{\mu} = \langle 0, 1, a_2 \rangle$ lie in the block with the $\langle 4^a, 6, 9e-a-1 \rangle$ notation. See section 5.5.2 for an explanation of how we obtained the last column.

|          | $\langle 0, 1, a_2 \rangle$ | $\langle 0, 1_2, a \rangle$ | $\langle 1_2, a_2 \rangle$ | $\langle 1_2, 1, a \rangle$ | $B(\nu, \hat{\mu})$ | $[W^\nu : L^\mu]$ |
|----------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|
| $\langle 0, 1, a_2 \rangle$ |                             |                             |                             |                             |                             | 1                           |
| $\langle 0, 1_2, a \rangle$ | 0                           |                             |                             |                             |                             | 0                           |
| $\langle 0, 1_2, a \rangle$ |                             | 1                           |                             |                             |                             | 0                           |
| $\langle 1_2, a_2 \rangle$ |                             |                             | 0                           |                             |                             | 1                           |
| $\langle 1_2, 1_2 \rangle$ |                             |                             |                             | 1                           |                             | 1                           |
| $\langle 1_2, 1, a \rangle$ |                             |                             |                             |                             | 2                           | 1                           |
| $\langle 0_2, 1, a \rangle$ | $-1$                        |                             |                             |                             | 0                           | 0                           |
| $\langle 0_2, 1_2 \rangle$ | $-1$                        | 1                           |                             |                             | 0                           | 0                           |
| $\langle 0_2, 1_2 \rangle$ |                             |                             |                             |                             | 0                           | 0                           |
| $\langle 0_2, 1, a \rangle$ |                             |                             |                             |                             | 1                           | 1                           |

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