**Maximum turnaround radius in \( f(R) \) gravity**

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I. INTRODUCTION

The present cosmological picture, the so-called concordance model, shows that the universe is fueled by some unknown fluid, which drives the cosmic dynamics at different epochs \(^1\). In particular, the accelerated expansion is today described by some sort of dark energy, whose standard origin is presumed to be the cosmological constant, \( \Lambda \). Although appealing and straightforward, the concordance model, dubbed the \( \Lambda \)CDM model, fails to be predictive at quantum regime, i.e. as standard gravity breaks down \(^4\). In turn, the fact that at quantum scales gravity cannot be described by general relativity (GR) suggests more complicated time-dependent dark energy contributions, different from the \( \Lambda \)CDM model.

Hence, the concordance paradigm may represent only a first effective explanation of present universe speed up \(^8\). Moreover, structure formation and small perturbations require non-baryonic dark matter contributions, which dominate over the luminous matter throughout the universe expansion history. To assess the issues of dark energy and dark matter, one wonders whether the net fluid responsible for them is due to geometric corrections, in particular curvature or torsion corrections \(^9,12\). In this picture additional geometric terms drive the universe dynamics both at ultra-violet and infra-red scales \(^12\). In the case of \( f(R) \) gravity, as curvature corrections propagate, one can imagine that dark energy and dark matter are nothing else but effects of geometry at infrared scales \(^10\). This prescription generalizes GR without introducing barotropic fluids by hand, proposing a way to interpret the dark side of the universe by a geometrical point of view. In the teleparallel equivalent representation of GR, the so called TEGR, the generalization is achieved considering functions of the torsion scalar \( T \), and then one deals with the so called \( f(T) \) gravity \(^{14}\).

Assuming the \( f(R) \) gravity, if on the one hand, curvature effects are sub-dominant terms that at certain point lead to a transition at which the expansion starts, on the other hand, a curvature radius related to the scale at which structures form can be defined for \( f(R) \) models. In such a way, understanding how curvature corrections affect the scale of structure formation is essential to confront dynamics with a given background\(^1\). For these reasons, one invokes the existence of a maximum turnaround radius, say \( R_{\text{TA,max}} \), which plays the aforementioned role. The turnaround radius can be defined as the exact point at which the attraction of gravity and the repulsion of dark energy cancel each other. For example, considering spherical structures with mass \( M \), in the concordance paradigm, the maximum turnaround radius is: \( R_{\text{TA,max}} = (3G_NM/\Lambda c^2)^{1/3} \). It denotes the maximum size that any spherically-symmetric structures can assume in the framework of \( \Lambda \)CDM model. A test parti-

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\(^1\) At small redshifts, dark energy models degenerate to \( \Lambda \)CDM, so that higher redshift regimes are essential to discriminate among viable extensions of concordance model.
structure formation. In the last few years, this approach has been extensively adopted to constrain dark energy models and alternative theories of gravity. In particular Brans-Dicke theory, cubic galileon model, phantom braneworld approach, quintessence, and so forth have been tested against turnaround radius (see for example e.g. [18 28]). A covariant definition of the maximum turnaround radius, together with the turnaround equation in metric theories of gravity, can be found in [29].

In this paper, we discuss the turnaround radius in the framework of \(f(R)\) gravity. In this perspective, we assume \(f(R)\) models which, at certain point, assume a constant curvature, \(R = R_{\text{AS}}\). We thus compute the \(R_{\text{TA,max}}\) in two different pictures. In the first approach, we assume spherically symmetric space-times. In the second, we discuss the cosmological case considering perturbations. As expected, curvature terms account for repulsive effects, mimicking dark energy even at the level of turnaround radius. In this way, we are able to put constraints on generic \(f(R)\) models. We find that the observed bounds on \(R_{\text{TA,max}}\) are analogous if one considers spherically symmetric space-times or cosmological perturbations\(^2\). We therefore derive limits over the form of \(f(R)\) which should guarantee structure formation for any model even in the approximation of quasi-constant curvature.

The paper is organized as follows. In Sec. II we introduce the role of turnaround radius in cosmology, highlighting its physical properties and the main consequences it may have in structure formation. In Sec. III we summarize \(f(R)\) gravity with particular attention to cosmological implications. In addition, we evaluate the maximum bound of the turnaround radius in two distinct cases. In the first one, we consider a static and spherically symmetric space-time and we find a bound for the ratio \(G_{\text{eff}}/R_{\text{AS}}\), where \(G_{\text{eff}}\) is the effective gravitational coupling. In the second case, we take into account cosmological perturbations and we find a constraint for viable \(f(R)\) models. Both the results agree with each other, as well as with results in literature. Conclusions and perspectives are reported in Sec. IV

II. THE TURNAROUND RADIUS

In the concordance \(\Lambda\)CDM model, the cosmological constant is associated to the vacuum energy and plays a significant role both at large and small scales. Indeed, in the \(\Lambda\)-dominated universe, the normalized dark energy abundance is: \(\Omega_{\Lambda,0} \simeq 0.73\), affecting structure formation by repulsive interaction\(^3\). In this picture, there exists a point, or more precisely a surface, where the attractive force of gravity cancels out with repulsive effects. This point is the **turnaround radius**. A test particle inside it falls into the gravitational well, while outside the test particle follows the Hubble flow. Clearly, the point of turnaround radius has an unstable equilibrium.

Alternatively, we can define the turnaround radius as the *scale at which, initially-expanding and gravitationally-bound structures halt their expansion, turn around, and collapse*. This prerogative is a property of structure formation and its evolution. Thus, it does not depend on the adopted metric and gravitational theory. On the contrary, the maximum value of the turnaround radius for a structure of given mass \(M\) turns out to be independent of the cosmic epoch. So that, it is a bound to the maximum size that a structure may have.

In the rest of this section, we sum up the turnaround derivation following the recipe presented in [29]. We first calculate the turnaround radius in the \(\Lambda\)CDM model using static coordinates; afterwards, we generalize it to arbitrary theories with static solutions. Furthermore, we compute it using the McVittie metric and, finally, we combine the two results, i.e. static and McVittie, using scalar cosmological perturbations.

Let us consider a spherical mass \(M\) in a universe with a cosmological constant, i.e. the Schwarzschild-de Sitter (SdS) space-time

\[
 ds^2 = -A(r)dt^2 + B^{-1}(r)dr^2 + r^2d\Omega^2, \tag{1}
\]

with:

\[
 B(r) = A(r) = 1 - \frac{2G_NM}{rc^2} - \frac{\Lambda r^2}{3}, \tag{2}
\]

and \(d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2\). The 4-velocity of a stationary observer is given by \(u^\mu = (A(r)^{-1/2}, 0, 0, 0)\). The turnaround radius is defined as the point where, the 4-acceleration of the observer vanishes, i.e.

\[
 R_{\text{TA,max}} \equiv \left(\frac{3G_NM}{\Lambda c^2}\right)^{1/3}. \tag{3}
\]

Since the SdS-metric is not necessarily a solution of any theory of gravity, this result does not hold in general. In theories that obey the weak equivalence principle and have solutions of the form (1), the turnaround radius, following the same rationale as before, i.e. the vanishing

\(^2\) Throughout the paper we use the Lorentzian signature \((-+,+,+,-+)\) for the metric. \(G_N\) is the Newton gravitational constant and we set \(c = 1\) unless otherwise specified.

\(^3\) For alternative effects of repulsive gravity which emerge without the need of *any* dark energy contribution see [30 31].
4-acceleration of a stationary observer, can be computed from

$$A'(r)|_{r = R_{TA,\text{max}}} = 0,$$  \hspace{1cm} (4)$$
which is an algebraic equation in terms of $r$. Note that, even if the solutions are not static, they can always be diagonalized in the form (11) and thus the formula (4) is still valid. In that case, however, the turnaround radius becomes time-dependent.

Eq. (4) only holds for this specific coordinate system. Thus, it is more convenient if we find a covariant version of it. So that, for a stationary observer in a spherically symmetric space-time, the point at which $u^\mu \nabla_\mu u^\nu = 0$ is identified with the turnaround radius.

Starting from the fact that we want to extend the result to cosmology, we consider a McVittie space-time, which describes a spherical mass $M$ in an expanding universe. Its line element is given by

$$ds^2 = -\left(1 - \frac{\mu}{1 + \mu}\right)^2 dt^2 + (1 + \mu)^4 a^2 \left(dr^2 + r^2 d\Omega^2\right),$$  \hspace{1cm} (5)$$
where $\mu = G_NM/(2ar)$. If we consider a pressureless fluid with $8\pi G_N \rho = \Lambda$, adopting the coordinate transformations

$$\tilde{t}(t, r) = t + Q(\tilde{r}),$$

$$\tilde{r}(t, r) = (1 + \mu)^2 ar,$$  \hspace{1cm} (6a), (6b)$$
with $Q(\tilde{r})$ satisfying the condition

$$\frac{\partial Q}{\partial \tilde{r}} = \frac{\sqrt{\Lambda/3\tilde{r}}}{\left(1 - \frac{2G_NM}{\tilde{r}} - \frac{\Lambda}{3}\tilde{r}^2\right)^{1/2}}$$  \hspace{1cm} (7)$$
we recover the SdS-metric given by (11) and (2).

In order to see how the turnaround radius looks like in this space-time, we need to transform results in SdS-space-time adopting the inverse transformations (6a), (6b). The observer 4-velocity becomes now

$$u^\mu = \frac{1 + \mu}{\sqrt{(1 - \mu)^2 - H^2(1 - \mu)^6 a^2 r^2}} (1, -rH, 0, 0),$$  \hspace{1cm} (8)$$
and using $u^\mu \nabla_\mu u^\nu = 0$ we find the turnaround radius in McVittie space-time

$$2\mu = (1 + \mu)^6 H^2 a^2 r^2,$$  \hspace{1cm} (9)$$
which takes the form (8) if we use (6a), (6b).

In general, the SdS-metric and static, spherically symmetric metrics that can be transformed into McVittie metric, cannot be solutions in arbitrary modified theories of gravity. Nonetheless, a perturbed Friedmann-Robertson-Walker (FRW) can be used for characterizing small perturbations.

Indeed, the McVittie metric, for $\mu \ll 1$, can be seen as a perturbed FRW metric. Identifying $\Phi = \Psi = -2\mu = -G_NM/(ar)$, Eq. (11) can be expressed as the perturbed FRW metric in the Newtonian gauge

$$ds^2 = -\left(1 + 2\Phi(t, r^i)\right) dt^2 + a^2 \left(1 - 2\Psi(t, r^i)\right) \gamma_{ij} dx^i dx^j.$$  \hspace{1cm} (10)$$
In Cartesian coordinates $\gamma_{ij} dx^i dx^j = dx^2 + dy^2 + dz^2$ and $r^i = \{x, y, z\}$, assuming spherical symmetry, it is $R = r^i = (x^2 + y^2 + z^2)^{1/2}$. In spherical coordinates, it is $r^i = r$ and $\gamma_{ij} dx^i dx^j = dr^2 + r^2 d\Omega^2$.

Taking advantage of this fact, we find the 4-velocity (8) in the limit $\mu \ll 1$ and $aHr < 1$. Using the geodesic equation $u^\mu \nabla_\mu u^\nu = 0$, we finally get the turnaround for spherically symmetric space-times in cosmological framework, that is

$$a^2 \left(H^2 + \dot{H}\right) = \frac{\partial \Phi}{\partial r}.$$  \hspace{1cm} (11)$$
Specifically, given a Hubble parameter and a potential $\Phi$, Eq. (11) gives the maximum turnaround radius $R_{TA,\text{max}} = ar$. For the general turnaround equation, including non-spherical space-times, we refer to [22].

A quick look at the previous results shows that the two outcomes, i.e. Eq. (11) and (3), are equivalent. In fact, in the $\Lambda$CDM model, the potential $\Phi$ is given by $\Phi = -G_NM/(ar)$ and the Hubble constant is $H = \sqrt{\Lambda/3}$.

Before proceeding with our considerations, let us see how the turnaround radius can be used to constrain different cosmological models. To do so, we can compare the masses and the radii of known cosmic structures. Observationally the turnaround radius is the zero-velocity surface of a structure and can be determined by observations of the Hubble flow in structures that consist of, at least, several galaxies. The concept of mass needs some more investigation. The usual procedure for the mass estimation of a structure is the combination of turnaround radius measurements and predictions of the spherical collapse models in the given cosmology. The straightforward application of this approach can be very complicated. Alternatives are provided by summing up the masses of the constituent galaxies, strong or weak gravitational lensing studies, or X-ray spectroscopy studies. Even a single observation of a structure that exceeds the theoretically predicted bound (assuming that the confidence of the measurement is really high) would be enough to constrain a cosmological model.

In the $\Lambda$CDM model, the obtained formula for the turnaround radius (3), even for structures as big as $10^{15}M_\odot$ (Virgo cluster), is 10% greater than its actual observed size. A detailed analysis about the structures used as well as relevant plots can be found in [17].
means that the turnaround radius in any theory can be at most 10% smaller than in ΛCDM model. The observed size of structures is, in general, at low redshift, i.e. when the structures have almost reached their maximum size. This means that we do not need data from higher redshifts or from the early stages of the universe to evaluate the turnaround radius.

III. TURNAROUND RADIUS IN f(R) GRAVITY

We have reviewed the definition of the turnaround radius. In particular, we proved that Eq. (11) can give the maximum turnaround radius in any theory of gravity where the Einstein Equivalence Principle holds. In this section, we compute the turnaround radius in f(R) gravity taking into account two cases. First we consider a static and spherically symmetric space-time, after, we investigate the turnaround radius in the context of cosmological perturbations.

Briefly, f(R) gravity is obtained by substituting the Hilbert-Einstein action, linear in the Ricci scalar R, with an arbitrary function of R, i.e.

$$S = \int d^4x \sqrt{-g} \left( \frac{1}{16\pi G_N} f(R) + \mathcal{L}_m \right). \tag{12}$$

By varying the above action with respect to the metric, we get the field equations

$$f'(R)R_{\mu\nu} - \frac{1}{2} f(R)g_{\mu\nu} + \left( g_{\mu\nu} \Box - \nabla_{\mu} \nabla_{\nu} \right) f'(R) = 8\pi G_N T^{\mathcal{M}}_{\mu\nu}, \tag{13}$$

where $T^{\mathcal{M}}_{\mu\nu}$ is the energy-momentum tensor of matter. Another interesting equation, is the trace of Eq. (13),

$$f'(R)R - 2f(R) + \Box f'(R) = 8\pi G_N T^{\mathcal{M}}, \tag{14}$$

which relates the Ricci scalar to f(R) and its derivative in R, that is f′(R).

Several efforts have been spent to get constraints on viable f(R) models (see for example [1, 13, 52, 33]). In particular, we can assume $f''(R) > 0$, in order to avoid tachyonic instabilities, and $f'(R) > 0$, to make the theory ghost-free.

A. The spherically symmetric case

As mentioned in the Introduction, we can restrict our attention to f(R) models with constant curvature solutions, i.e. $R = R_{\text{ds}} = \text{const}$. This $R_{\text{ds}}$ is the solution of the trace Eq. (14), which, for constant curvature, takes the form

$$R f'(R) - 2 f(R) = 0. \tag{15}$$

Solutions⁵ to this equation are known as de Sitter points [34, 35]. Moreover, following [36, 37], nonlinear models should have stable de Sitter points at late times. From the Eq. (13), using Eq. (15), we get

$$R_{\mu\nu} = \frac{f(R_{\text{ds}})}{2f'(R_{\text{ds}})} g_{\mu\nu} = \frac{R_{\text{ds}}}{4} g_{\mu\nu}, \tag{16}$$

which gives rise to an effective cosmological constant of the form

$$\Lambda_{\text{eff}} = \frac{f(R_{\text{ds}})}{2f'(R_{\text{ds}})} = \frac{R_{\text{ds}}}{4}. \tag{17}$$

The stability of these points is discussed in [34, 38]. They asymptotically approach the (anti)-de Sitter space with $\Lambda = \Lambda_{\text{eff}}$.

If we consider a static and spherically symmetric metric of the form (11), the equations of motion (14), for both the (0,0) and (1,1) components, together with the trace equation, are respectively

$$\left( \frac{1}{2} A f(R) + \frac{A'}{2B} \right) \mid_{R=R_{\text{ds}}} = 0, \tag{18a}$$

$$\left( -\frac{1}{2} B f(R) + \frac{A'^2}{4A^2} + \frac{B' f'(R)}{4AB} - \frac{A' B' f'(R)}{4B} \right) \mid_{R=R_{\text{ds}}} = 0, \tag{18b}$$

$$\left( R f'(R) - 2 f(R) \right) \mid_{R=R_{\text{ds}}} = 0. \tag{18c}$$

We can easily see, from the first two equations, (18a) and (18b), that

$$B(r) = \frac{c}{A(r)} \tag{19},$$

where c is an integration constant. The constant can be set equal to 1 to recover the Minkowski space-time as limiting case. Hence, Eq. (18b) provides:

$$A(r) = a_1 - \frac{a_2}{r} - \frac{r^2 f(R)}{6 f'(R)} \mid_{R_{\text{ds}}} = a_1 - \frac{a_2}{r} - \frac{R_{\text{ds}}}{12} r^2, \tag{20}$$

where, in the second line, we used Eq. (18c), and $a_1, a_2$ are constants. Without losing generality, we choose

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⁵ Such solutions exist in the constant curvature f(R) models. They imply the validity of the Birkhoff theorem that does not always hold in f(R) gravity [34, 35].
\[ a_1 = 1 \text{ and } a_2 = 2G_{\text{eff}}M/c^2, \] where \( G_{\text{eff}} \) is the effective gravitational coupling. This allows to recover a Schwarzschild-like solution in the \( R_{\text{ds}} \to 0 \) limit.

It is worth-mentioning that in the Hilbert-Einstein limit, i.e. \( f(R) = R - 2\Lambda \), we can set \( R_{\text{ds}} = 4\Lambda > 0 \), since \( \Lambda \) is positive-definite. As a consequence, we recover the known Schwarzschild-de Sitter solution \[^4\].

Finally, the maximum turnaround radius for any \( f(R) \) model with \( R = R_{\text{ds}} \) is given by Eq. \[^4\]

\[ A'(R_{\text{TA,max}}) = 0 \Rightarrow R_{\text{TA,max}} = \left( \frac{12G_{\text{eff}}M}{R_{\text{ds}}} \right)^{1/3}. \] (21)

As we already mentioned in Sec. \[^1\] the maximum turnaround radius in any alternative theory of gravity can be, at most, 10\% smaller than the corresponding one in GR. Thus, by comparing (21) with (3), we get the following constraint

\[ \frac{G_{\text{eff}}}{R_{\text{ds}}} \geq \frac{0.18G_N}{\Lambda}. \] (22)

At this point, a short comment on Eq. \[^1\] is needed. Solutions \( R_{\text{ds}} = 0 \), are not excluded \textit{a priori}. However, these are trivial Minkowski solutions, instead of de Sitter ones, leading to neither expanding nor contracting universes. In this case, the maximum turnaround radius cannot be defined. In the next section, we will see that scalar cosmological perturbations give the same result as Eq. \[^22\], together with a specific form for the gravitational coupling.

### B. The cosmological case

In the previous section, we studied the turnaround radius derived from static and spherically symmetric space-times in \( f(R) \) gravity. However, we want to find a more general formula for the turnaround radius and thus we turn to cosmology. Specifically, in this section, we are going to use Eq. \[^1\] in order to see, whether we can extend the result \[^22\].

To this end, let us consider a spherical cosmic structure described by a perfect fluid with non-relativistic matter, i.e. with pressure \( P = 0 \), and all its mass is assumed to be at the center, \( r = 0 \). We perturb this structure by a test fluid and we study its dynamics. The whole configuration can be described by a perturbed FRW metric, which in conformal Newtonian gauge, can be expressed in the form \[^10\].

The homogeneous background Eqs. \[^13\] in a FRW flat space-time \( g_{\mu\nu} = \text{diag}(-1, a^2(t), a^2(t), a^2(t)) \), and with non-relativistic matter, i.e. \( T_{\mu\nu} = \text{diag}(-\rho, 0, 0, 0) \), are given by

\[ 3f'f''H^2 = 8\pi\rho + \frac{1}{2}(Rf' - f) - 3f''H\dot{R}, \] (23a)

\[ f'\left(2H + 3H^2\right) = \frac{1}{2}(Rf' - f) - f''\dot{R} - f''\dot{R} \] (23b)

together with the continuity equation

\[ \dot{\rho} + 3H\rho = 0. \] (24)

The prime denotes differentiation with respect to \( R \), the dot with respect to time \( t \), and \( H = \dot{a}/a \) is the Hubble parameter. In this case, the Ricci scalar is

\[ R = 6\left(2H^2 + \dot{H}\right), \] (25)

and Eq. \[^21\] gives \( \rho = \rho_0/a^3 \), where \( \rho_0 \) is the constant rest-mass density.

The perturbed energy momentum tensor is given by \( T_{\mu\nu} = -\rho - \delta\rho, \ T_{0i} = -\rho\delta v_i \) and \( T_{ij} = 0 \). Thus, the perturbed equations are \[^34\] \[^41\] \[^42\]

\[ -\frac{\Delta \Psi}{a^2} + 3H \left( H\Phi + \dot{\Psi} \right) = -\frac{1}{2f'}\left[8\pi G_N\delta\rho + \left(3H^2 + 3\dot{H} + \frac{\Delta}{a^2}\right)\delta f' - 3H \left(\delta f' - 2f'\Phi\right) + 3f'\dot{\Psi}\right], \] (26a)

\[ H\dot{\Phi} + \dot{\Psi} = \frac{1}{2f'} \left[8\pi G_N\rho\delta v + \delta f' - H\delta f' - f'\Phi\right], \] (26b)

\[ \Psi - \Phi = \frac{\delta f'}{f'}, \] (26c)

\[ 3 \left( H\dot{\Phi} + H\dot{\Psi} + \dot{\Psi} \right) + 6H \left( H\dot{\Phi} + \dot{\Psi} \right) + \left(3H + \frac{\Delta}{a^2}\right)\Phi = \frac{1}{2f'} \left[8\pi G_N\delta\rho + 3\delta f' + 3H\delta f' - \left(6H^2 + \frac{\Delta}{a^2}\right)\delta f' - 3\dot{f}' \left(\dot{\Phi} + 2H\Phi + \dot{\Psi}\right) - 6\dot{f}'\Phi\right], \] (26d)

\[ \delta f' + 3H\delta f' - \left(\frac{\Delta}{a^2} + \frac{R}{3}\right)\delta f' = \frac{8\pi G_N}{3}\delta\rho + \dot{f}' \left(\dot{\Phi} + 6H\Phi + 3\dot{\Psi}\right) + 2\dot{f}'\Phi - \frac{1}{3} f'\delta R, \] (26e)
where
\[
\delta R = -2 \left[ 3 \left( \dot{H} \Phi + H \dot{\Phi} + \ddot{\Psi} \right) + 12 H \left( H \Phi + \dot{\Psi} \right) + \frac{\Delta}{a^2} \left( \Phi - 2 \Psi \right) + 3 \dot{H} \Phi \right].
\] (27)

As before, \( f' = df/dR \), dot denotes differentiation with respect to cosmic time \( t \) and \( \Delta \) is the Laplacian in comoving coordinates.

We can safely use the quasi-static approximation \( \frac{f}{f' \rho} \) according to which the inhomogeneities \( \Phi \) and \( \Psi \) are primarily produced by the spatial distribution of matter. This means that the spatial derivatives of the fields are dominant in the equations (see \( 29) \). In this way, Eq. (26a), (26d), (26e) yield\(^6\)
\[
\frac{\Delta \Psi}{a^2} = \frac{1}{2 f} \left[ 8 \pi G_N \delta \rho + \frac{\Delta f'}{a^2} \right],
\] (28a)
\[
\frac{\Delta \Phi}{a^2} = \frac{1}{2 f} \left[ 8 \pi G_N \delta \rho - \frac{\Delta f'}{a^2} \right],
\] (28b)
\[
-\frac{\Delta \delta f'}{a^2} = \frac{8 \pi G_N \delta \rho - \Delta f'}{3} - \frac{1}{3} \frac{f'}{a^2} \delta R,
\] (28c)

where \( \delta R = \frac{2 a}{3} (\Phi - 2 \Psi) \). By using \( \delta f' = f'' \delta R \), (28a) and (28b) immediately give
\[
\Psi = \frac{f''}{2 f} \delta R + \phi, \quad \Phi = -\frac{f''}{2 f} \delta R + \phi,
\] (29)

where \( \phi \) satisfies the Poisson-like equation
\[
\frac{\Delta \phi}{a^2} = \frac{4 \pi G_N a^2}{f'} \delta \rho.
\] (30)

In addition, (28c) gives
\[
(\Delta - a^2 M^2) \delta R = -\frac{8 \pi G_N a^2}{3 f''} \delta \rho.
\] (31)

where we defined the mass term (of the scalar degree of freedom) as
\[
M^2 = \frac{1}{3} \left( \frac{f'(R_{\text{ds}})}{f''(R_{\text{ds}})} - R_{\text{ds}} \right).
\] (32)

We notice that Eq. (31) is a modified-Helmholtz equation and, if we set \( \delta \rho(r) \sim \mathcal{M} \delta(r)/a^3 \), where \( \mathcal{M} \) is the mass of the structure, Eq. (31) and Eq. (30) give respectively
\[
\delta R = \frac{2 G_N \mathcal{M}}{3 f'' ar} e^{-a M r}, \quad \phi = -\frac{G_N \mathcal{M}}{f'' ar},
\] (33)

where we assumed that \( \delta R \to 0 \) when \( r \) is large. Finally, the gravitational potentials from (29) become
\[
\Phi = \frac{G_N \mathcal{M}}{f''(R_{\text{ds}})} \left( 1 + \frac{e^{-a M r}}{3} \right), \quad \Psi = \frac{G_N \mathcal{M}}{f''(R_{\text{ds}})} \left( 1 - \frac{e^{-a M r}}{3} \right).
\] (34)

Before proceeding to the calculation of the turnaround radius through Eq. (11), let us comment on these results. First of all, Eq. (34) shows that the effective gravitational coupling is
\[
G_{\text{eff}} = \frac{G_N}{f'(R_{\text{ds}})} \left( 1 + \frac{e^{-a M r}}{3} \right),
\] (35)

where \( r_{\text{ph}} = a r \) is the physical distance. So, even in the weak field limit, deviations from GR, i.e. \( f'(R) \neq 1 \), become evident. Apart from this, we see that the inhomogeneities caused by the test fluid, together with the non-linearities of the theory, contribute as a Yukawa-like correction to the gravitational fields. This has been observed in different contexts \[43,44\]. Such corrections can effectively explain the flat rotation curves of galaxies, without invoking any exotic form of matter \[44\]. Moreover, it is worth noticing that, with growing \( r \), the Yukawa corrections vanish, while \( \Phi \) and \( \Psi \) evolve towards a McVittie-like form, which is expected in scalar cosmological perturbations.

Let us proceed now with the calculation of the turnaround radius. Eq. (11) gives
\[
r^3_{\text{ph}} = \frac{12 G_N \mathcal{M}}{R_{\text{ds}} f'(R_{\text{ds}})} \left[ 1 + \frac{e^{-a M r}}{3} \left( 1 + \frac{1}{3} M r_{\text{ph}} \right) \right],
\] (36)

As already mentioned, \( M \) is the effective mass related to the further degree of freedom of \( f(R) \) gravity. Thus, \( M r \ll 1 \) means that the mass of the scalar field is small and the related effective length is very large with respect to the the Solar System scale. On the other hand, if the mass \( M \) is large, it cannot have observable effects at late times\(^7\). Hence, Eq. (30) becomes of zeroth order in \( M r \)
\[
R_{\text{TA, max}} \simeq \left( \frac{12 G_N \mathcal{M}}{R_{\text{ds}} f'(R_{\text{ds}})} \right)^{1/3}.
\] (37)

Clearly, Eq. (37) is the same of Eq. (24) for \( G_{\text{eff}} = G_N/f'(R_{\text{ds}}) \), which is \[35\] for \( M r \ll 1 \). Thus the con-

\(^6\) Eq. (26b) means that, in the considered order of approximation, the peculiar velocities of the test fluid are ignored and Eq. (26c) is in the correct order and it is satisfied, \textit{a posteriori}, by Eq. (29).

\(^7\) One can claim that the mass \( M \) is a function of the curvature, and then of the energy density, and thus it is small at cosmological scale and large at Solar System scales like in the so-called Chameleon Mechanism \[33,49\].
where $n$ constant. For $n > 3$, it is $\alpha \lesssim 1/(20\Lambda^2)$, as shown in Fig.1.

**FIG. 1:** In the picture, the observational values of the turnaround radius for some astrophysical structures are reported. Details about the used data can be found in [17, 18] and references therein. The theoretical bound of the maximum turnaround radius in $\Lambda$CDM model (blue line), as well as in the power-law $f(R)$ model with $n = 3$ (orange line) are reported.

The constraint (22) becomes

$$R_{\text{dS}} f'(R_{\text{dS}}) \leq 5.48\Lambda.$$  \hspace{1cm} (38)

This is the main result of the paper; the maximum turnaround radius can be used to set a further criterion for the viability of $f(R)$ models, through the stability of cosmic structures. As we already mentioned in Sec. III.A for late times, when the matter density becomes negligible compared to $\Lambda$, the Ricci curvature scalar takes the value $R_{\text{dS}} = 4\Lambda$; thus the upper bound for the first derivative of the model is $f'(R_{\text{dS}}) \leq 1.37$ (which, of course, in GR is $f'(R_{\text{dS}}) = 1$). Summarizing, viable $f(R)$ models are those which obey the following criteria

$$0 < f'(R) \leq 1.37, \quad f''(R) > 0 \quad R \geq R_{\text{dS}} \geq 0.$$  \hspace{1cm} (39)

As an example we can consider a power-law model

$$f(R) = R + \alpha R^n, \quad n > 0, \hspace{1cm} (40)$$

where $n$ is a positive real number and $\alpha$ is a dimensional constant. For $n = 0$, $\alpha$ plays the role of cosmological constant. In the literature, there are several constraints on $n$ (e.g. [50] and references therein) and thus we will consider only $n > 2$. From the constraint (39), we find, for example, that for $n = 3$, it is $\alpha \lesssim 1/(20\Lambda^2)$, as shown in Fig.1.

**IV. FINAL OUTLOOKS AND PERSPECTIVES**

In this work, we considered $f(R)$ gravity to find out a turnaround radius for cosmological structures. Standard requirements of such theories are that $f''(R) > 0$ to avoid tachyonic instabilities and $f'(R) > 0$ to let the theory be ghost-free. Several bounds and conditions may be put on the $f(R)$ derivatives to check whether it is possible to circumscribe the functional form of $f(R)$. This is a consequence of the fact that the form of $f(R)$ is not known a priori and deserves reconstruction techniques to be determined. Hence, we studied whether it is possible to constrain $f(R)$ invoking a maximum turnaround radius, i.e. a radius that fixes the stability of large scale structures with respect to the background cosmic evolution. To do so, we considered two approaches: the first concerning a spherically symmetric metric and the second adopting cosmic perturbations. In both cases, we got analogous outcomes which allow the existence of stable structures according to a stability criterion which is $f'(R_{\text{dS}}) \leq 1.37$. To achieve this result, we first focused on $f(R)$ models with constant curvature solutions. In this case, the result is analytical. The same result is valid if curvature is approximately constant as in the case of several astrophysical cases. In a forthcoming paper, considering also the study reported in [57], the present criterion will be confronted with galaxy clusters.

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