Number Fields in Fibers: the Geometrically Abelian Case with Rational Critical Values

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Abstract

Let \( X \) be an algebraic curve over \( \mathbb{Q} \) and \( t \in \mathbb{Q}(X) \) a non-constant rational function such that \( \mathbb{Q}(X) \neq \mathbb{Q}(t) \). For every \( n \in \mathbb{Z} \) pick \( P_n \in X(\overline{\mathbb{Q}}) \) such that \( t(P_n) = n \). We conjecture that, for large \( N \), among the number fields \( \mathbb{Q}(P_1), \ldots, \mathbb{Q}(P_N) \) there are at least \( cN \) distinct. We prove this conjecture in the special case when \( \overline{\mathbb{Q}}(X)/\overline{\mathbb{Q}}(t) \) is an abelian field extension and the critical values of \( t \) are all rational. This implies, in particular, that our conjecture follows from a more famous conjecture of Schinzel.

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1 Introduction

Everywhere in this paper “curve” means “smooth geometrically irreducible projective algebraic curve”.

Let \( X \) be a curve over \( \mathbb{Q} \) and \( t \in \mathbb{Q}(X) \) a non-constant rational function such that \( \mathbb{Q}(X) \neq \mathbb{Q}(t) \). We fix, once and for all, an algebraic closure \( \overline{\mathbb{Q}} \). All number fields occurring in this article are subfields of this \( \overline{\mathbb{Q}} \).

Dvornicich and Zannier [2, Theorem 2(a)] proved the following theorem.

Theorem 1.1 (Dvornicich, Zannier) For every \( n \in \mathbb{Z} \) pick \( P_n \in X(\overline{\mathbb{Q}}) \) such that \( t(P_n) = n \). There exists a real number \( c > 0 \) (depending on \( X \) and \( t \), but not on the particular selection of every \( P_n \)) such that for every sufficiently large integer \( N \) the number field \( \mathbb{Q}(P_1, \ldots, P_N) \) is of degree at least \( e^{cN/\log N} \) over \( \overline{\mathbb{Q}} \).

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An immediate consequence is the following result.

**Corollary 1.2** In the above set-up, there exists a real number \( c > 0 \) such that for every sufficiently large integer \( N \), among the number fields \( \mathbb{Q}(P_1), \ldots, \mathbb{Q}(P_N) \) there are at least \( cN/\log N \) distinct.

Theorem 1.1 is, in general, best possible, but Corollary 1.2 is, probably, not; see the discussion in the introduction of [1]. In particular, in [1] we suggest the following conjecture.

**Conjecture 1.3** Let \( X \) be a curve over \( \mathbb{Q} \) and \( t \in \mathbb{Q}(X) \) a non-constant \( \mathbb{Q} \)-rational function such that \( \mathbb{Q}(X) \neq \mathbb{Q}(t) \). Then there exists a real number \( c > 0 \) such that for every sufficiently large integer \( N \), among the number fields \( \mathbb{Q}(P_1), \ldots, \mathbb{Q}(P_N) \) there are at least \( cN/\log N \) distinct.

There is also a more famous conjecture (attributed in [2, 3] to Schinzel), which relates to Theorem 1.1 in the same way as Conjecture 1.3 relates to Corollary 1.2. To state it, recall that \( \alpha \in \overline{\mathbb{Q}} \cup \{\infty\} \) is called a critical value (or a branch point) of \( t \in \overline{\mathbb{Q}}(X) \) if the rational function \( 1/t - \alpha \) has at least one multiple zero in \( X(\overline{\mathbb{Q}}) \). It is well-known that any rational function \( t \in \overline{\mathbb{Q}}(X) \) has at most finitely many critical values, and that \( t \) has at least 2 distinct critical values if \( \overline{\mathbb{Q}}(X) \neq \overline{\mathbb{Q}}(t) \) (a consequence of the Riemann-Hurwitz formula). In particular, in this case \( t \) admits at least one finite critical value.

**Conjecture 1.4 (Schinzel)** In the set-up of Conjecture 1.3, assume that either \( t \) has at least one finite critical value not belonging to \( \mathbb{Q} \), or the field extension \( \overline{\mathbb{Q}}(X)/\overline{\mathbb{Q}}(t) \) is not abelian. Then there exists a real number \( c > 0 \) such that for every sufficiently large integer \( N \) the number field \( \mathbb{Q}(P_1), \ldots, \mathbb{Q}(P_N) \) is of degree at least \( e^{cN/\log N} \) over \( \mathbb{Q} \).

As Dvornichich and Zannier remark in [2, 3], the hypothesis in Conjecture 1.4 is necessary. Indeed, when all finite critical values of \( t \) belong to \( \mathbb{Q} \) and the field extension \( \overline{\mathbb{Q}}(X)/\overline{\mathbb{Q}}(t) \) is abelian, it follows from Kummer’s Theory that \( \mathbb{Q}(X) \) is contained in the field of the form \( L(t, (t - \gamma_1)^{1/e_1}, \ldots, (t - \gamma_s)^{1/e_s}) \), where \( L \) is a number field, \( \gamma_1, \ldots, \gamma_s \) are rational numbers and \( e_1, \ldots, e_s \) are positive integers. Now if we denote by \( A \) the maximal absolute value of the denominators and the numerators of the rational numbers \( \gamma_1, \ldots, \gamma_s \), and set \( E = \text{lcm}(e_1, \ldots, e_s) \), then the number field \( \mathbb{Q}(P_1, \ldots, P_N) \) is contained in the field, generated over \( L \) by the \( E \)th roots of prime numbers not exceeding \( AN + A \); by the Prime Number Theorem, the degree of this field cannot exceed \( e^{cN/\log N} \) for some \( c > 0 \).

Dvornichich and Zannier [2, 3] obtain several results in favor of Schinzel’s Conjecture. In particular, they show [2, Theorem 2(b)] that it holds true if \( t \) admits a critical value of degree 2 or 3 over \( \mathbb{Q} \).

In [1] we improve on Corollary 1.2, showing that \( cN/\log N \) can be replaced by \( N/(\log N)^{1-\eta} \) with some \( \eta > 0 \). See the introduction of [1] for further relevant references.

\(^1\)We use the standard convention \( t - \infty = t^{-1} \).
The purpose of the present note is to show that Conjecture 1.3 holds true in the case excluded in Schinzel’s conjecture. The following theorem is proved in Section 3.

**Theorem 1.5** Conjecture 1.3 holds true when all finite critical values of $t$ belong to $\mathbb{Q}$ and the field extension $\overline{\mathbb{Q}}(X)/\overline{\mathbb{Q}}(t)$ is abelian.

An immediate consequence of Theorem 1.5 is that Conjecture 1.4 implies Conjecture 1.3.

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## 2 Abundance of Almost Square-Free Values of Polynomials with Rational Roots

Let $S$ be a finite set of prime numbers and $\ell$ a positive integer. We say that $a \in \mathbb{Z}$ is $S$-square-free if $\nu_p(a) \in \{0,1\}$ for every prime $p \not\in S$. If, in addition to this, $\nu_p(a) \leq \ell$ for all $p \in S$, then we say that $a$ is $(S, \ell)$-square-free.

We say that integers $a$ and $b$ are $S$-distinct if there exists a prime $p \not\in S$ such that $\nu_p(a) \neq \nu_p(b)$, and $S$-equal otherwise.

In the following lemma we collect some elementary properties of the notions just introduced.

**Lemma 2.1** Let $S$ and $\ell$ be as above.

1. Let $a_1, \ldots, a_k$ be distinct $(S, \ell)$-square-free integers which are, however, all $S$-equal. Then $k \leq 2(\ell + 1)^{|S|}$.

2. Let $L$ be a number field and $S$ a finite set of (rational) prime numbers containing all the primes ramified in $L$. Let $a, b$ be $S$-distinct $S$-square-free integers. Let $e > 1$ be an integer whose all prime divisors belong to $S$, and let $A, B$ be integers satisfying

   $$a \mid A, \quad A \mid a^{e-1}, \quad b \mid B, \quad B \mid b^{e-1}. $$

   Then the number fields $L(A^{1/e})$ and $L(B^{1/e})$ are not isomorphic.

3. Let $L$ and $S$ be as in part 2. Let $a_1, \ldots, a_N$ be distinct $(S, \ell)$-square-free integers. Let $e > 1$ be an integer whose all prime divisors belong to $S$, and let $A_1, \ldots, A_n$ be positive integers satisfying

   $$a_i \mid A_i, \quad A_i \mid a_i^{e-1} \quad (i = 1, \ldots, N).$$
Then among the number fields $L(A_1^{1/\varepsilon})$ there are at least $N/2(\ell + 1)^{|S|}$ distinct.

**Proof** Part 1 is obvious. To prove 2, observe that, by the hypothesis, there exists a prime $p \notin S$ such that one of the numbers $\nu_p(a)$, $\nu_p(b)$ is 1 and the other is 0; say, $\nu_p(a) = 1$ and $\nu_p(b) = 0$. Then $1 \leq \nu_p(A) \leq e - 1$ and $\nu_p(B) = 0$, which implies that $p$ ramifies in the field $L(A_1^{1/\varepsilon})$ but not in $L(B_1^{1/\varepsilon})$. This proves 2. Finally, 3 follows from 1 and 2. 

In the sequel

$$f(T) = \alpha_d T^d + \cdots + \alpha_0 = \alpha_d (T - \gamma_1) \cdots (T - \gamma_d) \in \mathbb{Z}[T]$$

is a separable polynomial whose all roots $\gamma_1, \ldots, \gamma_d$ belong to $\mathbb{Q}$. For every prime number $p$ set

$$\lambda_i(p) = \nu_p(f'(\gamma_i)) \quad (i = 1, \ldots, d), \quad \lambda(p) = \max_{1 \leq i \leq d} \lambda_i(p).$$

Note that, while individual $\lambda_i(p)$ may be negative, we always have $\lambda(p) \geq 0$, and, moreover,

$$\lambda(p) \geq \delta(p), \quad (1)$$

where $\delta(p) = \min_{1 \leq i \leq d} \nu_p(\alpha_i)$. Indeed, it follows from the Gauss Lemma that

$$\delta(p) = \nu_p(\alpha_d) + \sum_{i=1}^d \min\{0, \nu_p(\alpha_i)\}. $$

Now, if, say, $\nu_p(\gamma_1) \geq \nu_p(\gamma_i)$ for $i \geq 2$ then

$$\lambda_1(p) = \nu_p(\alpha_d) + \sum_{i=2}^d \nu_p(\gamma_1 - \gamma_i) \geq \nu_p(\alpha_d) + \sum_{i=2}^d \min\{0, \nu_p(\gamma_i)\} \geq \delta(p),$$

proving $(1)$.

We will use the following variation of Hensel's lemma.

**Lemma 2.2** Let $n$ be an integer such that $\nu_p(f(n)) > 2\lambda(p)$. Then there exists a unique $j \in \{1, \ldots, d\}$ such that $\nu_p(n - \gamma_j) = \nu_p(f(n)) - \lambda_j$.

**Proof** We will write $\nu(\cdot)$, $\lambda_j$, $\lambda$ and $\delta$ instead of $\nu_p(\cdot)$, $\lambda_j(p)$, $\lambda(p)$ and $\delta(p)$.

Choose $j$ such that $\nu(n - \gamma_j) \geq \nu(n - \gamma_i)$ for all $i \neq j$. (A priori this $j$ is not uniquely defined, but in the course of the proof we will see that it actually is.) First of all, we claim that

$$\nu(\gamma_j) \geq 0. \quad (2)$$

Indeed, if $\nu(\gamma_j) < 0$ then $\nu(n - \gamma_i) = \nu(\gamma_i) < 0$ for all $i = 1, \ldots, n$, which implies that

$$\nu(f(n)) = \nu(\alpha_d) + \sum_{i=1}^d \nu(\gamma_i) = \nu(\alpha_d) + \sum_{i=1}^d \min\{0, \nu(\gamma_i)\} = \delta.$$
Since \( \nu(f(n)) > 2\lambda \), this contradicts (1). This proves (2).

We claim further that
\[
\nu(n - \gamma_j) > \lambda_j. \tag{3}
\]
Indeed, our definition of \( j \) implies that
\[
\nu(n - \gamma_i) \leq \nu(\gamma_j - \gamma_i) \quad (i \neq j).
\]
Hence
\[
\nu(f(n)) = \nu(\alpha_d) + \sum_{i=1}^{d} \nu(n - \gamma_i) \\
\leq \nu(\alpha_d) + \sum_{i \neq j} \nu(\gamma_j - \gamma_i) + \nu(n - \gamma_j) \\
= \lambda_j + \nu(n - \gamma_j).
\]
Therefore
\[
\nu(n - \gamma_j) \geq \nu(f(n)) - \lambda_j > 2\lambda - \lambda_j \geq \lambda_j,
\]
which proves (3).

Since \( \nu(f'(\gamma_j)) = \lambda_j \), inequality (3) implies that
\[
\nu(f'(n)) = \lambda_j. \tag{4}
\]
Thus, we have \( \nu(f(n)) > 2\lambda \geq 2\nu(f'(n)) \). Hensel’s lemma implies that \( f \) has a unique root \( \gamma \in \mathbb{Q}_p \) with the property
\[
\nu(n - \gamma) \geq \nu(f(n)) - \nu(f'(n)) > 2\lambda - \lambda_j \geq \lambda_j.
\]
Since the root \( \gamma_j \) has this property, we must have \( \gamma = \gamma_j \).

To conclude the proof of the lemma, observe that the Taylor expansion
\[
f(n) = f(\gamma_j) + f'(\gamma_j)(x - \gamma_j) + \cdots
\]
implies the congruence
\[
f(n) \equiv f'(\gamma_j)(n - \gamma_j) \mod p^{2\nu(n - \gamma_j)},
\]
which, together with (3), proves that \( \nu(n - \gamma_j) = \nu(f(n)) - \lambda_j \).

For all primes \( p \) with finitely many exceptions we have
\[
\lambda_i(p) = \nu_p(\gamma_i) = 0 \quad (i = 1, \ldots, d). \tag{5}
\]
In particular, \( \lambda(p) = 0 \) for all but finitely many \( p \). We denote by \( S_0 \) the finite set of primes for which (5) does not hold, and we set \( \ell_0 = \max_p \lambda(p) \). We also denote by \( U \), respectively, \( V \), the maximum of absolute values of the numerators, respectively, denominators, of rational numbers \( \gamma_i \): if \( \gamma_i = u_i/v_i \) with coprime \( u_i, v_i \in \mathbb{Z} \) then
\[
U = \max_{1 \leq i \leq d} |u_i|, \quad V = \max_{1 \leq i \leq d} |v_i|.
\]

The following is a version of Lemma 2 from [4].
Lemma 2.3 Let $S$ be a finite set of primes containing $S_0$ and let $\ell$ be an integer satisfying $\ell \geq 2\ell_0$. Let $P$ be the smallest prime not belonging to $S$. Then, given an integer $N \geq 1$, there are at most
\[
d \left( \zeta(\ell + 1 - \ell_0) + \frac{1}{P - 1} \right) N + d(VN + U)^{1/2} + d|S| \tag{6}
\]
positive integers $n \leq N$ with the property
\[
f(n) \text{ is not } (S, \ell)-\text{square-free}. \tag{7}
\]
Here $\zeta(\cdot)$ is the Riemann $\zeta$-function.

**Proof** Let $n \in \{1, \ldots, N\}$ satisfy (7). Then we have one of the following options:
\[
\begin{align*}
\nu_p(f(n)) &> \ell & \text{for some } p \in S, & \tag{8} \\
\nu_p(f(n)) &> 1 & \text{for some } p \notin S. & \tag{9}
\end{align*}
\]
In the case (8) we have $\nu_p(f(n)) > 2\ell_0 \geq 2\lambda(p)$. Lemma 2.2 implies that for some root $\gamma_i$ we have $n \equiv \gamma_i \mod p^{\nu_p(f(n)) - \lambda_i(p)}$. Since $\nu_p(f(n)) \geq \ell + 1$ and $\lambda_i(p) \leq \ell_0$, this implies
\[
n \equiv \gamma_i \mod p^{\ell + 1 - \ell_0}. \tag{10}
\]
When $p$ and $i$ are fixed, the number of $n \in \{1, \ldots, N\}$ satisfying (10) is bounded by $N/p^{\ell + 1 - \ell_0} + 1$. Summing up over all $p \in S$ and $i \in \{1, \ldots, d\}$, we estimate the total number of $n$ satisfying (8) as
\[
d \sum_{p \in S} \left( \frac{N}{p^{\ell + 1 - \ell_0}} + 1 \right) \leq dN \sum_{p} \frac{1}{p^{\ell + 1 - \ell_0}} + d|S| = d(\zeta(\ell + 1 - \ell_0)N + d|S|. \tag{11}
\]
In the case (9) we have $\lambda(p) = 0$ and $\nu_p(f(n)) \geq 2$. Lemma 2.2 implies that for some root $\gamma_i$ we have
\[
n \equiv \gamma_i \mod p^2. \tag{12}
\]
Since $1 \leq n \leq N$, this implies $n = \gamma_i$ or $p \leq (VN + U)^{1/2}$.

When $p$ and $i$ are fixed, the number of $n \in \{1, \ldots, N\}$ satisfying (12) is bounded by $N/p^2 + 1$. Summing up over all $p$ satisfying $P \leq p \leq (VN + U)^{1/2}$ and all $i \in \{1, \ldots, d\}$, we estimate the total number of $n$ satisfying (9) as
\[
d \sum_{P \leq p \leq (VN + U)^{1/2}} \left( \frac{N}{p^2} + 1 \right) \leq dN \sum_{p \geq P} \frac{1}{p^2} + d(VN + U)^{1/2} \leq \frac{dN}{P - 1} + d(VN + U)^{1/2}. \tag{13}
\]
Summing (11) and (13), we obtain (6). $\square$

An immediate consequence is that, with suitably chosen $S$ and $\ell$, “most” of the values $f(n)$ are $(S, \ell)$-square-free. Here is the precise statement.
Corollary 2.4 There exist a finite set of primes $S_1$ and a positive integer $\ell_1$ (both depending only on $f$) such that the following holds. For every $S \supseteq S_1$ and every $\ell \geq \ell_1$ there exists $N_0 = N_0(f, S)$ such that for $N \geq N_0$, at most $N/2$ positive integers $n \leq N$ satisfy (7).

Proof Let $\ell_1$ be a positive integer and $P_1$ a prime number satisfying

$$d \zeta(\ell_1 + 1 - \ell_0) < \frac{1}{6}, \quad \frac{d}{P_1 - 1} < \frac{1}{6}.$$

Setting $S_1 = S_0 \cup \{\text{primes } p < P_1\}$, the result follows. □.

3 Proof of Theorem 1.5

We start with the special case of a superelliptic curve.

Theorem 3.1 Let $F(T) \in \mathbb{Q}[T]$ be a non-constant polynomial whole all roots are rational numbers, $L$ a number field and $e$ a positive integer. Assume that $F(T)$ is not an $e$th power in $\bar{\mathbb{Q}}[T]$. Then there exists a positive number $c$ such that, for large $N$, among the number fields

$$L(F(1)^{1/e}, \ldots, L(F(N)^{1/e})$$

there is at least $cN$ distinct.

Proof We may assume that the roots of $F$ are all of multiplicity not exceeding $e - 1$. Furthermore, multiplying $F$ by $a^e$ with a suitable non-zero integer $a$, we may assume that $F(T) \in \mathbb{Z}[T]$. Then there exists a separable polynomial $f(T) \in \mathbb{Z}[T]$ such that $f(T) | F(T)$ and $F(T) | f(T)^{e-1}$ in the ring $\mathbb{Z}[T]$.

Corollary 2.4 implies that, with suitably chosen $S$ and $\ell$ the following holds: for large $N$, at least half of the numbers

$$f(1), \ldots, f(N)$$

are $(S, \ell)$-square-free. The polynomial $f$ takes every value at most $d$ times, where $d = \deg f$. Hence among (15) there are at least $N/2d$ distinct $(S, \ell)$-square-free numbers. We complete the proof applying Lemma 2.1:3. □.

Now we can prove Theorem 1.5 in full generality. Note first of all that, if $P, Q \in X(\bar{\mathbb{Q}})$ and $L$ is a number field, then $L(P) \neq L(Q)$ implies $\mathbb{Q}(P) \neq \mathbb{Q}(Q)$. Hence, it suffices to show that, for some number field $L$, among the fields

$$L(P_1), \ldots, L(P_N)$$

there are at least $cN$ distinct.

Now we use Kummer’s theory. Since $\bar{\mathbb{Q}}(X)/\bar{\mathbb{Q}}(t)$ is a abelian extension, for some number field $L$ we have $L(X) = L(t, F_1(t)^{1/e_1}, \ldots, F_s(t)^{1/e_s})$, where $F_i(t) \in L[t]$, $e_i \geq 2$ and $F_i(t)$ is not a $e_i$th power in $\bar{\mathbb{Q}}[t]$. 7
Moreover, the roots of every $F_i$ are finite critical values of $t$, which, by the hypothesis, belong to $\mathbb{Q}$. In particular, we may assume that $F_i(t) \in \mathbb{Q}[t]$.

Pick some $F_i$ and $e_i$ and call them $F$, $e$ in the sequel. Theorem 3.1 implies that, for large $N$, among the fields (14) there are at least $c'N$ distinct. But $L(F(n)^{1/e})$ is a subfield of $L(P_n)$ (provided $L$ contains the $e$th roots of unity, which can be always achieved by extending $L$). It remains to note that the fields $L(P_n)$ are of degree over $\mathbb{Q}$ bounded independently of $n$:

$$[L(P_n) : \mathbb{Q}] \leq [L(X) : \mathbb{Q}(t)].$$

A field of degree $r$ over $\mathbb{Q}$ may have at most $c(r)$ distinct subfields. Hence, producing $c'N$ distinct subfields of the fields (16) implies that among the fields (16) there are at least $cN$ distinct.

□

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