An application of harmonic measure in complex analysis about estimation of the absolute value of holomorphic function provided boundary data

Liangchen Zou

Departments of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui, 230026, China

*Corresponding author’s e-mail: zlc0601@mail.ustc.edu.cn

Abstract: This paper introduces some basic concepts of harmonic measure—a special sort of harmonic function and gives a proof to the existence of harmonic measure, which is also unique, in some simple cases using Poisson’s formula and Riemann mapping theorem. A solution to a problem in complex analysis is given using conformal mapping and harmonic measure. This problem is about estimation of the absolute value of holomorphic functions with its boundary data provided.

1. Introduction

The following is a problem in the exercise part of [1]:

Let $D$ denote a square region in the complex plain with center 0 and vectors $z_i, 1 \leq i \leq 4$. Function $f \in H(D) \cap C(\overline{D})$. Assume that $M = \max_{z \in D} |f(z)|, m = \max_{z \in [z_1, z_2]} |f(z)|$

Prove:

(i) $|f(0)| \leq m^{1/3}M^{2/3}$

(ii) $|f(z)| \leq m^{1/3}M^{2}, \text{ for } z \in \triangle z_10z_2$.

Generally speaking, only using complex analysis tools, it’s not easy to estimate the absolute value of a holomorphic function within a region with its boundary data prescribed. However, there is basic observation: If $f \in H(D)$ and $f(z) \neq 0$ for $z \in D$, then $\log |f|$ is harmonic in $D$. Since harmonic functions have many nice properties, such as the maximal value principle, one can give an estimate on $\log |f|$ and thus $|f|$. This is how this study derives from the problem above.

We will first summarize some of the properties of harmonic functions, which act as an important role in the following discussion. We will then introduce some basic concepts about harmonic measure and then prove the existence of harmonic measure of the union of finite many disjoint arcs on the unit disc and use Riemann mapping theorem to generalize the result to any simply connected region. Then, we will use harmonic measure along with some complex analysis tools to solve the original problem as an application. Finally, we will generalize the methods used to solve this problem to a general scale and give more details.

2. Material and Methods

The well-known Dirichlet principle states that for any bounded connected open set $\Omega \in \mathbb{R}^n$ with its
boundary in \( C^1 \) class and function \( f \in C^1(\partial \Omega) \), there is a unique function \( u \) which is harmonic in \( \Omega \) and continuous on \( \overline{\Omega} \). When \( n = 2 \) and \( \Omega \) is unit disc, Poisson’s formula gives an explicit expression of \( u \). One may ask what if \( f \) is not continuous and in more general cases, such as \( f \in L^1(\partial \Omega) \). In fact, when \( f \) is not too "intricate", the Poisson’s formula still works in some ways, and the function \( u \) given by Poisson’s formula is still harmonic in \( \Omega \). Although \( u \) is no longer continuous on the whole \( \partial \Omega \), \( u \) is continuous at the points where \( f \) is continuous and still maintains some nice properties of harmonic functions which is continuous on boundary. Since conformal map maintains "harmonic", one can use Riemann mapping theorem to generalize this result to any simply connected Jordan region. With these observations, we can verify the existence of harmonic measure on simply connected region by choosing specific function \( f \).

We will see later in this paper that the estimation of absolute value of a holomorphic function is associated with the harmonic measure of the region where \( f \) is defined. Therefore, we will calculate harmonic measure of two simple regions, which will help to solve the problem we mentioned at the beginning of introduction.

The following are basic tools and concepts we need to work with the problem mentioned above.

2.1. Properties of harmonic function

In this section, \( \Omega \) is always a region in \( \mathbb{R}^n \).

**Definition 2.1.** Function \( u \in C^2(\Omega) \) is harmonic, if \( \Delta u = 0 \) in \( \Omega \).

**Theorem 2.2.** \( u \) is harmonic in \( \Omega \) if and only if for any sphere \( B(a, r) \subseteq \Omega \),

\[
    u(a) = \frac{1}{|B(a, r)|} \int_{B(a, r)} u(y) dy
\]

This theorem is called the mean-value property of harmonic measure, which is included in many textbooks (For example [2]), but the proofs are usually long. Here we give a simple proof for the first part of this theorem. For the converse property, one can refer to [2].

**Proof** When \( u \) is harmonic, we prove the property in a more general sense using Stokes formula. Let \( \alpha(n) \) denote the volume of the unit ball in \( \mathbb{R}^n \). \( \Sigma \) denote an arbitrary simply connected region with smooth boundary included in \( \Omega \) containing \( a \). Let \( \vec{p} = \mathbf{y} - a \) where \( \mathbf{y} \in \partial \Sigma \) and \( p = |\vec{p}h| \), \( \nu \) denote the unit outer vector of \( \partial \Sigma \). The following equations are true:

\[
    \frac{1}{(n\alpha(n))} \int_{\partial \Sigma} \left( \frac{u\vec{p}}{p^n} + \frac{\nabla u}{(n-2)p^{n-2}} \right) \cdot dS = u(a), n \geq 3
\]

\[
    \frac{1}{(2\alpha(2))} \int_{\partial \Sigma} \left( \frac{u\vec{p}}{p^2} - \ln p \nabla u \right) \cdot dS = u(a), n = 2
\]

We only prove the case \( n \geq 3 \), the case \( n = 2 \) is similar. Note that when \( \Sigma = B(a, r) \),

\[
    \int_{\partial \Sigma} \left( \frac{u\vec{p}}{p^n} + \frac{\nabla u}{(n-2)p^{n-2}} \right) \cdot dS = \int_{\partial B(a,r)} \left( \frac{u\vec{p}}{r^n} + \frac{\nabla u}{(n-2)r^{n-2}} \right) \cdot dS
\]

\[
    = \frac{1}{r^{n-1}} \int_{\partial B(a,r)} u \cdot dS + \frac{1}{(n-2)r^{n-2}} \int_{\partial B(a,r)} \nabla u \cdot dS
\]

\[
    = l_1 + l_2
\]

By Stokes formula, \( l_2 = 0 \), since \( \nabla \cdot \nabla u = \Delta u = 0 \). Meanwhile, \( l_1 \) tends to \( n\alpha(n)u(a) \) as \( r \) tends to zero. Note that

\[
    \nabla \cdot \left( \frac{u\vec{p}}{p^n} + \frac{\nabla u}{(n-2)p^{n-2}} \right)
\]
\begin{equation}
\n\Delta u = \frac{\Delta u}{(n-2)p^{n-2}}
\end{equation}

Hence for any simply connected region \( \Sigma' \) with smooth boundary included in \( \Sigma \) containing \( a \),

\[
\int_{\partial \Sigma} \frac{u^p}{p^n + (n-2)p^{n-2}} \nu \cdot \mathbf{v} dS = \int_{\partial \Sigma'} \frac{u^p}{p^n + (n-2)p^{n-2}} \cdot \mathbf{v} dS
\tag{3}
\]

By taking \( \Sigma' = B(a,s) \), and letting \( s \) tends to zero, according to (1), we obtain

\[
\frac{1}{(n\alpha(n))} \int_{\partial \Sigma} \frac{u^p}{p^n + (n-2)p^{n-2}} \nu \cdot \mathbf{v} dS = u(a)
\tag{4}
\]

When \( \Sigma = B(a,r) \),

\[
uu(a) = \frac{1}{(n\alpha(n))} \int_{\partial \Sigma} \frac{u^p}{p^n + (n-2)p^{n-2}} \nu \cdot \mathbf{v} dS = \frac{r^{1-n}}{(n\alpha(n))} \int_{\partial B(a,r)} u \, dS
\]

which implies the original equation. \( \blacksquare \)

**Corollary 2.3.** If \( u \) is harmonic in \( \Omega \) and reaches a local maximal value at \( a \in \Omega \), then \( u \) is a constant.

**Corollary 2.4.** If \( \Omega \) is bounded, \( u \in C^2(\Omega) \cap C(\overline{\Omega}) \) is harmonic in \( \Omega \), then

\[
\sup_{x \in \partial \Omega} u(x) = \sup_{x \in \partial \Omega} u(x)
\]

**Remark 2.5.** When \( \Omega \) is not bounded and \( \overline{\Omega} \neq \mathbb{R}^n \), if we additionally assume \( \max_{x \in \overline{\Omega}} u(x) \) exists, the same result in Corollary 2.4 can be obtained.

**Theorem 2.6.** If \( \Omega \) is bounded, \( u \) is harmonic in \( \Omega \), continuous on \( \overline{\Omega} \setminus F \) and bounded, where \( F \) is a finite subset of \( \partial \Omega \), then

\[
\sup_{x \in \overline{\Omega} \setminus F} u(x) = \sup_{x \in \partial \Omega \setminus F} u(x)
\]

**Proof** Let \( u_\epsilon(x) = u(x) - \epsilon \sum_{y \in F} |x - y|^{2-n} \), when \( n \leq 3 \).

\[
uu_\epsilon(x) = u(x) - \epsilon \sum_{y \in F} \log \frac{d}{|x - y|} \text{ when } n = 2, \text{ where } d = \max_{y \in \overline{\Omega}} |x - y|\] Then \( u_\epsilon \) is harmonic in \( \Omega \), continuous in \( \overline{\Omega} \setminus F \) and tends to negative infinity as \( x \) tends to \( F \). For any \( y \in \Omega \), pick \( \delta_1 \) small enough such that \( z \in \Omega \setminus \cup_{y \in F} B(y, \delta_1) \). Pick \( a \in \partial \Omega \setminus \cup_{y \in F} B(y, \delta_1) \), and take \( \delta_2 \) small enough such that \( u_\epsilon(x) \leq u_\epsilon(a) \) for any \( x \in \cup_{y \in F} B(y, \delta_2) \). Let \( \delta = \min \{ \delta_1, \delta_2 \} \). By Corollary 2.4,

\[
uu_\epsilon(z) \leq \max_{x \in \partial \Omega \setminus \cup_{y \in F} B(y, \delta)} u_\epsilon(x)
\]

Since \( u_\epsilon(x) \leq u(a) \) for any \( x \in \cup_{y \in F} B(y, \delta) \),

\[
uu_\epsilon(z) \leq \max_{x \in \partial \Omega \setminus \cup_{y \in F} B(y, \delta)} u_\epsilon(x) = \max_{x \in \partial \Omega \setminus \cup_{y \in F} B(y, \delta)} u_\epsilon(x) \leq \sup_{x \in \partial \Omega} u(x)
\]

Since \( \epsilon \) can be chosen arbitrarily small, by letting \( \epsilon \) tend to zero, we obtain

\[
uu(z) \leq \sup_{x \in \partial \Omega} u(x)
\]

Since \( z \) is an arbitrary point in \( \Omega \), the proof is completed. \( \blacksquare \)

**Remark 2.7.** When \( n \leq 2 \), the precondition \( \Omega \) is bounded can be replaced by \( \overline{\Omega} = \mathbb{R}^2 \).

**Proof** View \( \mathbb{R}^2 \) as \( \mathbb{C} \). Take \( a \in \mathbb{C} \setminus \overline{\Omega} \). Let \( \varphi(z) = \frac{1}{z-a} \). Then \( \varphi(\Omega) \) is bounded and \( u(\varphi^{-1}(z)) \)

\[
\text{is bounded, harmonic in } \varphi(\Omega) \text{ and continuous in } \varphi(\Omega) \setminus \varphi(F)
\]

(if necessarily \( \varphi(\Omega) \setminus (\varphi(F) \cup \{0\}) \)). Then apply Theorem 2.6 to \( u(\varphi^{-1}(z)) \). \( \blacksquare \)

**Remark 2.8.** Since we only use the fact that \( u \) has upper bound in the proof of theorem 2.6, the condition "\( u \) is bounded" can be replaced by "\( \sup u < \infty \)".

2.2. Introduction to harmonic measure

In this section, \( \Omega \) denotes a region in \( \mathbb{C} \).
Definition 2.9. Suppose $\Omega \neq \emptyset$, $E \subseteq \partial \Omega$ is the union of finite many arcs on $\partial \Omega$. The harmonic measure of $E$ on $\Omega$, denoted by $w(z, E, \Omega)$, is defined as a function satisfying the following conditions:

(a) $\Delta w(z) = 0, z \in \Omega$

(b) $w(z) = 1, z \in E$, $w(z) = 0, z \in \partial \Omega \setminus E$

(c) $w$ is continuous on $\Omega$ except for the endpoints of $E$.

Example

(a) $\Omega = \{z \in \mathbb{C} | 0 < z < a\}, E = \{z \in \mathbb{C} | Re(z) = a\}$

$$w(z, E, \Omega) = \frac{1}{a} Re(z)$$

(b) $\Omega = \{z \in \mathbb{C} | r < |z| < R\}, E = \{z \in \mathbb{C} | |z| = R\}$

$$w(z, E, \Omega) = (\log \frac{R}{r})^{-1} \log \frac{|z|}{r}$$

Theorem 2.10. (Uniqueness) If $w_1$ and $w_2$ both satisfy the three conditions in Definition 2.9, then $w_1 = w_2$ on $\Omega$ except for the endpoints of $E$.

Proof Let $\nu = w_1 - w_2$, then $\nu$ is harmonic in $\Omega$, continuous on $\partial \Omega \setminus F$, where $F$ is the set of finite endpoints of $E$. $\nu = 0$ on $\partial \Omega \setminus F$. Applying Theorem 2.6 and Remark 2.7 to $\nu$ and $-\nu$, one can conclude $w_1 - w_2 = \nu = 0$ on $\Omega \setminus F$. ■

Theorem 2.11. (Conformal Invariance) Suppose $\Omega_1$ and $\Omega_2$ are two regions in $\mathbb{C}$, $\Omega_1 \neq \emptyset$. $E_1 \subset \partial \Omega_1, E_2 \subset \partial \Omega_2$ are unions of finite many arcs. $f$ is a biholomorphic function from $\Omega_1$ to $\Omega_2$ such that $f$ maps one to one from $\partial \Omega_1$ to $\partial \Omega_2$ and $f(E_1) = E_2$. Then $w(f(z), E_2, \Omega_2) = w(z, E_1, \Omega_1)$.

Proof Write $f = (f^1, f^2)$. Let $u(z)$ denote $w(z, E_2, \Omega_2)$

$$\frac{\partial^2}{\partial x^2} (u \circ f) = u_x f^1_{xx} + u_{xx} (f_x^1)^2 + u_{xy} f^1_x f^2_x + u_{yy} (f_y^2)^2 + u_{xy} f^1_x f^2_y + u_{yy} f^1_x f^2_x$$

$$\frac{\partial^2}{\partial y^2} (u \circ f) = u_x f^1_y + u_{xx} (f_y^1)^2 + u_{xy} f^1_x f^2_y + u_{yy} (f_y^2)^2 + u_{xy} f^1_x f^2_y + u_{yy} f^1_y f^2_y$$

$$\Delta (u \circ f) = u_x (f_{xx}^1 + f_{yy}^1) + (u_{xx} (f_{xx}^2)^2 + u_{yy} (f_{yy}^2)^2)$$

$$+ u_{xy} (f_{xx}^1 f_{yy}^1 + f_{yy}^1 f_{yy}^1)$$

Since $f(E_1) = E_2$, we also have $u \circ f |_{E_1} = u |_{E_2} = 1, u \circ f |_{\partial \Omega_1 \setminus E_1} = u |_{\partial \Omega_2 \setminus E_2} = 0$.

By theorem 3.2, $u \circ f(z) = w(z, E_1, \Omega_1), z \in \Omega_1 \setminus F$. ■

Riemann mapping theorem tells that any simply connected region in $\mathbb{C}$ except for $\mathbb{C}$ itself has a biholomorphic function which maps it to the unit ball. Therefore, we only need to consider the existence of harmonic measure on the unit ball in order to prove the existence of harmonic measure on general simply connected regions.

Recall the well-known Poisson’s formula for Dirichlet problem on the unit ball:

$$u(z) = \frac{1}{2\pi} \int_{[0,2\pi]} f(e^{i\theta}) \frac{1-|z|^2}{|e^{i\theta} - z|^2} d\theta$$

It is the solution for the equation:

$$\Delta u = 0, z \in D$$

$$u(z) = f(z), z \in \partial D$$

where $f \in C(\partial D), D$ denotes the unit ball in $\mathbb{C}$.

However, in order to obtain the harmonic measure of an arc $E$ in the unit ball, one needs to take 

4
$f = \chi_E$, which is not continuous. But we still hope that Poisson’s formula can give the harmonic measure. Hence, it’s necessary to check whether the function given by (5) through replacing $f$ by $\chi_E$ satisfies the conditions in definition 2.9.

**Theorem 2.12.** $u(z)$ given by (5) is harmonic in $D$, if $f \in L^1(\partial D)$.

**Proof.** By Theorem 2.2, we only need to check $u$ satisfies the mean-value property. Suppose $B(a, r) \subseteq D$. We should check

$$u(a) = \frac{1}{2\pi} \int_{[0,2\pi]} u(a + re^{i\theta}) d\theta$$

$$\frac{1}{2\pi} \int_{[0,2\pi]} u(a + re^{i\theta}) d\theta$$

$$= \frac{1}{4\pi^2} \int_{[0,2\pi]} \int_{[0,2\pi]} f(a) \frac{1 - |a + re^{i\theta}|^2}{|e^{ia} - a - re^{i\theta}|^2} d\alpha d\theta$$

$$= \frac{1}{4\pi^2} \int_{[0,2\pi]} f(a) \left( \int_{[0,2\pi]} \frac{1 - |a + re^{i\theta}|^2}{|e^{ia} - a - re^{i\theta}|^2} d\theta \right) d\alpha$$

Apply residue theorem,

$$\int_{[0,2\pi]} \frac{1 - |a + re^{i\theta}|^2}{|e^{ia} - a - re^{i\theta}|^2} d\theta$$

$$= \int_{\partial D} \frac{1 - (a + rz)(\bar{a} + r\frac{1}{z})}{(e^{ia} - a - rz)(e^{-ia} - \bar{a} - r\frac{1}{z})} dz$$

$$= \frac{1}{i} \int_{\partial D} \frac{arz^2 + (a^2 + r^2 - 1)z + ar}{z(rz - e^{ia} + a)((e^{-ia} - \bar{a})z - r)} dz$$

$$= 2\pi \left( \frac{e^{-ia} - a}{e^{-ia} - \bar{a}} + \frac{e^{-ia}}{1 - |a|^2} \right)$$

$$= 2\pi \frac{|e^{-ia} - a|^2}{|e^{-ia} - \bar{a}|^2}$$

Taking (7) into (6), we obtain

$$\frac{1}{2\pi} \int_{[0,2\pi]} u(a + re^{i\theta}) d\theta = \frac{1}{2\pi} \int_{[0,2\pi]} f(a) \frac{1 - |a|^2}{|e^{ia} - a|^2} d\alpha = u(a)$$

**Theorem 2.13.** $u$ is defined as (5). If $f \in L^1(\partial D)$ and is continuous at $e^{i\theta_0}$, then

$$\lim_{z \to e^{i\theta_0}} u(z) = f(e^{i\theta_0})$$

**Proof.** Since $f$ is continuous at $e^{i\theta_0}$, for any positive $\varepsilon$, there exists an interval $I$ containing $\theta_0$ such that $|f(e^{i\theta}) - f(e^{i\theta_0})| < \varepsilon$

$$u(z) - f(e^{i\theta_0}) = \frac{1}{2\pi} \int_{[0,2\pi]} f(e^{i\theta}) \frac{1 - |z|^2}{|e^{i\theta} - z|^2} d\theta - f(e^{i\theta}) = I_1 + I_2$$

where

$$I_1 = \frac{1}{2\pi} \int_{[0,2\pi]\setminus I} (f(e^{i\theta}) - f(e^{i\theta_0})) \frac{1 - |z|^2}{|e^{i\theta} - z|^2} d\theta$$

$$I_2 = \frac{1}{2\pi} \int_{I} (f(e^{i\theta}) - f(e^{i\theta_0})) \frac{1 - |z|^2}{|e^{i\theta} - z|^2} d\theta$$

By dominated convergence theorem

$$\lim_{z \to e^{i\theta_0}} I_1 = 0$$
\[ |I_2| \leq \frac{1}{2\pi} \int_{1} |f(e^{i\theta}) - f(e^{i\theta_0})| \frac{1 - |z|^2}{|e^{i\theta} - z|^2} d\theta \leq \varepsilon \]

Therefore, we conclude that for any positive \( \varepsilon \)
\[
\limsup_{z \to e^{i\theta_0}} |u(z) - f(e^{i\theta_0})| \leq \varepsilon
\]

Hence, \( u \) is continuous at \( e^{i\theta_0} \). \( \blacksquare \)

**Corollary 2.14.** For any simply connected region \( \Omega \subseteq \mathbb{C} \), \( \overline{\Omega} \neq \mathbb{C} \) and any union of finite many arcs in \( \partial\Omega \), denoted by \( E \). There is a function \( w \) satisfying conditions (a)(b)(c) in Definition 2.9.

**Proof** by Riemann mapping theorem, there is a biholomorphic map \( \varphi \) from \( \Omega \) to \( D \), the unit ball. Let \( w(z) = w(\varphi(z), \varphi(E), D) \). Then by Theorem 2.11, 2.12 and 2.13, \( w \) satisfies the conditions we require. \( \blacksquare \)

### 3. Results

Now we have prepared to work on the original problem:

Let \( D \) denote a square region in the complex plain with center 0 and vectors \( z_i, 1 \leq i \leq 4 \). Function \( f \in H(D) \cap C(\overline{D}) \). Given that

\[
M = \max_{z \in D} |f(z)|, \quad m = \max_{z \in [z_1, z_2]} |f(z)|
\]

prove:

(i) \( |f(0)| \leq \frac{1}{2} M^2 \)

(ii) \( |f(z)| \leq \frac{1}{2} M^2 \), for any \( z \in \Delta z_1 \Omega z_2 \).

As mentioned in the introduction, \( \log |f| \) is harmonic near \( z_0 \) if \( z_0 \) is not a zero of \( f \). Define \( g \) as following:

\[
g(z) = \log |f(z)| + w(z, (z_1, z_2), D) \log \frac{M}{m} - \log M
\]

Note that if we "ignore" the zeros of \( f \) in \( \overline{D} \), then \( g(z) \leq 0 \) on \( \partial D \), and \( \Delta g = 0 \), for \( z \in D \). In this case, we can apply theorem 2.6 and remark 2.8 to demonstrate \( g(z) \leq 0 \) for any \( z \in \overline{D} \). However, \( \log |f| \) may fail to be defined globally on \( \overline{D} \), for \( f \) may have zeros on \( \overline{D} \). Therefore, we have to deal with the zeros of \( f \) in order to apply "maximal principle" on \( g \). In the following discussion, we will formally define \( \log |f| = -\infty \) when \( f(z) = 0 \).

**Theorem 3.1.** In the sense that view \( g(z) = -\infty \) when \( f(z) = 0 \) and view \( -\infty < r \) for any real \( r \), we have

\[
\sup_{z \in \mathbb{D} \setminus F} g(z) = \sup_{z \in \partial \mathbb{D} \setminus F} g(z)
\]

where \( F = \{z_1, z_2\} \).

**Proof.** Let \( N \) denote the set of zeros of \( f \). Since \( D \) is bounded and \( N \) is closed, \( N \) is compact. Without loss of generality, we assume \( f \) is not a constant. In this case, \( N \cap D \) has no accumulation point according to the uniqueness theorem for holomorphic function. Let \( z_0 \in D \setminus N \). For any \( w \in N \), there exists a ball \( B_w \) of radial \( r_w \) and center \( w \) which doesn’t contain \( z_0 \), such that \( r_w < 1 \) and \( |g(z)| < |g(z_0)| \) for any \( z \in B_w \). Moreover, if \( w \in D \cap N \), we can assume \( B_w \cap N = \{w\} \). Let \( \Delta = \{B'_w | w \in N\} \), where \( B'_w \) is the ball with center \( w \), radial \( \frac{r_w}{4} \). Then \( \Delta \) is an open cover of \( N \), and thus has an affine sub-cover \( \Delta' \). By shrinking \( B'_w \) and adding new balls into \( \Delta' \) properly, we can assume \( D \setminus \bigcup B'_{w \in \Delta'} B'_w \) is connected. Furthermore, Since \( f \) has no zero in \( D \setminus \bigcup B'_{w \in \Delta'} B'_w \), \( g \) is harmonic in \( D \setminus \bigcup B'_{w \in \Delta'} B'_w \) and continuous on \( D \setminus \bigcup B'_{w \in \Delta'} B'_w \setminus \{z_1, z_2\} \). Applying theorem 2.6 to \( g \) on \( D \setminus \bigcup B'_{w \in \Delta''} B'_w \setminus \{z_1, z_2\} \) and noting that \( |g(z)| < |g(z_0)| \) for \( z \in B_w \), we obtain

\[
g(z_0) \leq \sup_{z \in \partial \mathbb{D} \setminus \{z_1, z_2\}} g(z)
\]

Since \( z_0 \) can be piked arbitrarily and \( f \) is continuous on \( \overline{D} \setminus F \),
Here we have not indicated explicitly how to shrink and add new balls into $\Delta''$ for the reason that this procedure is prolix and not instructive, and we will prove this theorem again in a more general case later in theorem 4.1.

Applying the theorem above, one can see $|f(z)| \leq M^{1-w(z)} m^{w(z)}$. Therefore, it’s sufficient to show $w(z) \geq \frac{1}{4}$ for $z \in \Delta z_1 0 z_2$. A natural thought is to calculate $w(z, (z_1, z_2), D)$ using the definition of harmonic measure. In fact, we can indeed express $w(z, (z_1, z_2), D)$ as

$$
\sum_{k=1}^{\infty} \frac{4}{(2k-1)} \sin (2k-1) \pi x \frac{e^{(2k-1) \pi y} - e^{-(2k-1) \pi y}}{e^{(2k-1) \pi} - e^{-(2k-1) \pi}}
$$

(8)

if we take $z_1 = 1 + i, z_2 = i, z_3 = 0, z_4 = 1$. However, it’s not easy to estimate its value, even only at the center of the square. Nevertheless, using some geometry methods, considering the symmetry of D, the estimation of $w$ above can be completed.

**Proof of (i):**

Let $\rho$ denote the rotation $\rho(z) = iz$. By theorem 3.3, $w(z, (z_1, z_2), D) = w(\rho(z), (z_2, z_3), D) = w(\rho^2(z), (z_3, z_4), D) = w(\rho^3(z), (z_4, z_1), D)$. Take $z = 0$,

$$
\begin{align*}
\psi(0, (z_1, z_2), D) &= \psi(0, (z_2, z_3), D) = \psi(0, (z_3, z_4), D) = \psi(0, (z_4, z_1), D) \\
\end{align*}
$$

Noting that

$$
\begin{align*}
w(z, (z_1, z_2), D) + w(z, (z_2, z_3), D) + w(z, (z_3, z_4), D) + w(z, (z_4, z_1), D) &= 1 \\
\end{align*}
$$

Therefore, we obtain

$$
\begin{align*}
w(0, (z_1, z_2), D) &= w(0, (z_2, z_3), D) = w(0, (z_3, z_4), D) \\
&= w(0, (z_4, z_1), D) = \frac{1}{4}
\end{align*}
$$

(9)

which complete the proof.

**Proof of (ii):**

We will assume $D$ center at 0 with vectors $\pm 1, \pm i$.

**Step 1:** Calculate the harmonic measure of an arc on the unit disc.

Suppose $0 \leq \theta_1 < \theta_2 < 2\pi$. Let $E$ denote the arc with endpoints $e^{\theta_1}, e^{\theta_2}$

$$
w(z, E, B(0,1)) = \frac{1}{2\pi} \int_{[\theta_1, \theta_2]} |e^{i\theta} - z|^2 d\theta
$$

Let $z = |z| e^{i\theta_0}$, by basic calculus, we obtain

$$
w(z, E, B(0,1)) = \frac{1}{\pi} \arctan \left( \frac{1 - |z|^2}{(1 + |z|^2) \cos \frac{\theta_2 - \theta_1}{2} - 2|z| \cos \left( \frac{\theta_1 + \theta_2}{2} - \theta_0 \right)} \right)
$$

(10)

When $\theta_2 - \theta_1 = \frac{\pi}{2}$ and $\theta_1 \leq \theta_0 \leq \theta_2$,

$$
w(z, E, B(0,1)) = \frac{1}{\pi} \arctan \left( \frac{\frac{\sqrt{2}}{2} (1 - |z|^2)}{\frac{\sqrt{2}}{2} (1 + |z|^2) - 2|z| \cos \left( \theta_0 - \theta_1 - \frac{\pi}{4} \right)} \right)
$$

Since $|\theta_0 - \theta_1 - \frac{\pi}{4}| < \frac{\pi}{4}$.

$$
w(z, E, B(0,1)) \geq \frac{1}{\pi} \arctan \left( \frac{\frac{\sqrt{2}}{2} (1 - |z|^2)}{\frac{\sqrt{2}}{2} (1 + |z|^2) - \sqrt{2} |z|} \right) \geq \frac{1}{4}
$$

**Step 2:** Construct a biholomorphic function mapping disc to square and mapping $\frac{1}{4}$ circle to right
defined on \( \mathcal{N} \).

Here we are going to generalize the results obtained in theorem 3.1.

### 4. Discussion

Here we are going to generalize the results obtained in theorem 3.1.

**Theorem 4.1.** Let \( \Omega \subseteq \mathbb{C} \) be a connected region and \( \overline{\Omega} \neq \mathbb{C} \). Suppose \( N \) is a subset of \( \overline{\Omega} \) and \( N \cap \Omega \) has no accumulation in \( \Omega \). Suppose also \( F \) is a finite subset of \( \partial \Omega \). Let \( g \) be a function defined on \( \overline{\Omega} \setminus (N \cup F) \) satisfies the following conditions:

(a) \( g \) is harmonic in \( \Omega \setminus N \).
(b) \( g \) is continuous in its domain.
(c) \( \sup_{z \in \Omega \setminus F} g(z) < \infty \).
(d) \( \lim_{z \to \xi} g(z) = -\infty \) for any \( \xi \in N \).
(e) \( \limsup_{x \to \xi} g(z) \leq \limsup_{\partial \Omega \cap x \to \xi} g(z) \) for any \( \xi \in F \).
For simpleness, we define \( g(z) = -\infty \) for \( z \in \mathbb{N} \) and say \( g \) is continuous in \( \Omega \setminus F \). Then \( g \) satisfies maximal principle, in other words
\[
\sup_{z \in \Omega \setminus F} g(z) = \sup_{z \in \partial \Omega \setminus F} g(z)
\]

**Proof** Considering Riemann mapping theorem, we can assume \( \Omega \) is bounded. Let \( \Omega_\varepsilon := \{ z \in \Omega | dist(z, \partial \Omega) > \varepsilon \} \) for any \( \varepsilon > 0 \). Note that \( N \cap \Omega_\varepsilon \) is finite since \( \Omega_\varepsilon \) is compact. Let \( z_0 \) be an arbitrary point in \( \Omega \).

Claim: \( g(z_0) \leq \sup_{z \in \partial \Omega_\varepsilon} g(z) \) for any \( \varepsilon \) such that \( \Omega_\varepsilon \) contains \( z_0 \).

Since \( \Omega_\varepsilon \cap N \) is finite, one can pick finite many and small enough balls \( B_j = B(\xi_j, r_j) \), \( \xi_j \in \Omega_\varepsilon \cap N \) such that \( \Omega_\varepsilon \setminus \bigcup B_j \) is connected and \( g(z) < g(z_0) \) for any \( z \in \bigcup B_j \). Applying theorem 2.6 and remark 2.8 to \( g \) on \( \Omega_\varepsilon \setminus \bigcup B_j \) and noting that \( g(z) < g(z_0) \) for any \( z \in \bigcup B_j \), this claim is verified. Now, assume otherwise \( g(z_0) > \delta + \sup_{z \in \partial \Omega_\varepsilon} g(z) \) for some \( \delta > 0 \). In that case, since \( g \) is continuous, for any \( w \in \partial \Omega \setminus F \), there is a ball \( B_w \) with center \( w \) such that \( g(z) < g(z_0) - \delta, z \in B_w \). Meanwhile, for \( w \in F \), since we have condition (e), there is also a ball \( B_w \) with center \( w \) such that \( g(z) < g(z_0) - \delta, z \in B_w \). Hence, these \( B_w \) form an open cover of \( \partial \Omega \). Therefore, we can choose finite many \( B_{w_j} \) so that \( \partial \Omega \subseteq \bigcup B_{w_j} \) and \( g(z) < g(z_0) - \delta \), for \( z \in \bigcup B_{w_j} \). However, since \( \bigcup B_{w_j} \) is finite, we have \( \partial \Omega \subseteq \bigcup B_{w_j} \) for some small \( \varepsilon \). Consequently, \( \sup_{z \in \partial \Omega} g(z) \leq g(z_0) - \delta \), which is a contradiction with the claim above.

**Proposition 4.2.** Let \( \Omega \subseteq \mathbb{C} \) be a connected region and \( \Omega \neq \mathbb{C} \). Let \( \partial \Omega = \alpha \cup \beta \) where \( \alpha \) is the union of finite many arcs on \( \partial \Omega \) and \( \beta = \partial \Omega \setminus \alpha \). Suppose \( f \in H(\Omega) \cap C(\overline{\Omega}) \). Let \( g(z) = \log | f(z) | + C w(z, \alpha, \Omega) \), where \( C \) is a constant and \( w(z, \alpha, \Omega) \) is the harmonic measure of \( \alpha \) in \( \Omega \). Then \( g \) satisfies the conditions required in the theorem above.

**Proof** Only need to check the condition (e). Let \( \xi \) be an endpoint of \( \alpha \).
\[
\lim \sup_{z \to \xi} g(z) = \lim \sup_{z \to \xi} \log | f(z) | + C w(z, \alpha, \Omega)
\]
\[
\leq \lim \sup_{z \to \xi} \log | f(z) | + C
\]
\[
= \lim \sup_{\partial \Omega \ni z \to \xi} \log | f(z) | + C w(z, \alpha, \Omega)
\]
\[
= \lim \sup_{\partial \Omega \ni z \to \xi} g(z)
\]

\[\blacksquare\]

5. **Conclusion**

**Theorem 5.1.** Let \( \Omega \subseteq \mathbb{C} \) be a connected region and \( \Omega \neq \mathbb{C} \). Let \( \partial \Omega = \alpha \cup \beta \) where \( \alpha \) is the union of finite many arcs on \( \partial \Omega \) and \( \beta = \partial \Omega \setminus \alpha \). Suppose \( f \in H(\Omega) \cap C(\overline{\Omega}) \) and provided that
\[
M = \sup_{z \in \partial \Omega} | f(z) |, m = \sup_{z \in \partial \Omega} | f(z) |
\]
Then \( | f(z) | \leq M^{1 - w(z, \alpha, \Omega)} \), where \( w(z, \alpha, \Omega) \) is the harmonic measure of \( \alpha \) in \( \Omega \).

**Proof** Let \( g(z) = \log | f(z) | + w(z, \alpha, \Omega) \log M \). According to the boundary data provided, we obtain \( \sup_{z \in \partial \Omega} g(z) \leq 0 \), where \( F \) is the set of endpoints of \( \alpha \). Then apply theorem 4.1 and proposition 4.2.

In the third section, we constructed a biholomorphic function \( \alpha \) which maps the unit ball \( B(0,1) \) to the square \( D \) and maintains the line segments connecting the center and vectors of the square. In fact, we can generalize this result to any regular polygons.

**Theorem 5.2.** Assume \( D \) is a regular polygon with \( n \) vectors. For simpleness, suppose \( D \) has center \( 0 \), and vectors \( \exp \left( \frac{2k\pi}{n} \right) \), \( k \in [0, n - 1] \). Let \( S_k = \left\{ z \in B(0,1) \mid \frac{2k\pi}{n} < \arg(z) < \frac{(2k+1)\pi}{n} \right\} \)
and \( T_k = S_k \cap D \). Then there exists a biholomorphic function \( \alpha \) mapping \( B(0,1) \) to \( D \) such that 
\[ \alpha(S_k) = T_k, \quad k \in [0, n-1]. \]

**Proof** Let \( S_0 = \{ z \in B(0,1) | 0 < \arg(z) < \frac{\pi}{n} \} \) and \( T_0 \) be the triangle with vectors \( 0, e^{i\pi/n}, \cos \frac{\pi}{n} \).

By almost repeating the Step 2 in the previous section, one can construct a biholomorphic function \( \varphi \) mapping \( S_0 \) to \( T_0 \) such that \( \varphi(0) = 0, \varphi(1) = \cos \frac{\pi}{n}, \varphi(e^{i\pi/n}) = e^{i\pi/n} \). Applying theorem 3.2 to \( \varphi, \varphi \) extends to \( \{ z \in B(0,1) | -\frac{\pi}{n} < \arg(z) < \frac{\pi}{n} \} \). Let \( \psi(z) = e^{i\pi/n} \varphi(e^{i\pi/n}z) \). It’s easy to see \( \psi \) is biholomorphic from \( S_0 \) to \( T_0 \) and maps \( [0,1), \{ e^{i\theta} | 0 \leq \theta \leq \frac{2\pi}{n} \} \) to \([0,1], [1, e^{i\pi/n}], [0, e^{i\pi/n}] \).

Define \( \alpha \) as following:
\[ \alpha(z) = \exp \left\{ \frac{2k\pi i}{n} \right\} \psi(\exp \left\{ \frac{-2k\pi i}{n} \right\} z) \]
where \( z \in S_k, k \in [0, n-1] \). Using theorem 3.2, one can check \( \alpha \) is biholomorphic from \( B(0,1) \) to \( D \) such that \( \alpha(S_k) = T_k \) for \( k \in [0, n-1] \).

**Theorem 5.3.** Using Schwarz-Christoffel formula (for its statement and proof, see [4]), we can give an explicit expression of \( \psi \) in theorem 5.2.

\[ \psi(z) = C \int_{z_0}^z \left( -\frac{z^n}{2} + \frac{z^n}{2} + 1 \right) \frac{1}{\pi \sqrt{z}} \left( -\frac{z^n}{2} + \frac{z^n}{2} - 1 \right) \frac{1}{\pi \sqrt{z}} dz + C_1 \]

where \( C, C_1, z_0 \) only depend on \( n \). The value of \( C, C_1, z_0 \) can be obtained using the equations
\( \psi(0) = 0, \psi(1) = 1, \psi(e^{i\pi/n}) = e^{i\pi/n} \).

**Proof** Let \( \varphi_1(z) = z^{\frac{1}{n}}, \varphi_2(z) = -\frac{z^{1+i}z}{2} \). One can easily see \( \varphi_1 \) maps \( S_0 \) to the upper half-disc and maps \( 1, e^{i\pi/n}, 0 \) to \( 1, -1, 0 \) in order. Then \( \varphi_2 \) maps the upper half-disc to the upper half-plane and maps \( 1, -1, 0 \) to \( -1, 1, \infty \). By Schwarz-Christoffel formula, for some constants \( C, C_1, z_0 \),
\[ \varphi_3(z) = C \int_{z_0}^z (z + 1) \frac{1}{\pi \sqrt{z}} dz + C_1 \]
maps the upper half-plane to \( T_0 \). Therefore, \( \psi \) and \( \varphi_3 \circ \varphi_2 \circ \varphi_1 \) both maps \( S_0 \) to \( T_0 \) and maintain three points \( 1, 0, -1 \). Hence \( \psi = \varphi_3 \circ \varphi_2 \circ \varphi_1 \).

**Proposition 5.4.**

\[ 4 \sum_{k=1}^{\infty} \frac{(1)^{k-1}}{(2k-1)} \left( e^{\frac{(2k-1)\pi}{2}} + e^{\frac{-\pi(2k-1)}{2}} \right) = \frac{1}{4} \]

**Proof** By equation (8), the left side of the formula above equals to \( w(0, (z_1, z_2), D) \), where \( z_1 = \frac{1}{2} + \frac{i}{2}, z_2 = -\frac{1}{2} + \frac{i}{2}, z_3 = \frac{1}{2} - \frac{i}{2}, z_4 = \frac{1}{2} - \frac{i}{2} \). Then by equation (9), \( w(0, (z_1, z_2), D) = \frac{1}{4} \).

**Reference**

[1] Shi J, Liu T. (1998) Complex function. University of Science and Technology of China. Press, 176-178, Hefei. (in Chinese)

[2] Stein, E., Shakarchi R. (2005) Real Analysis Measure Theory, Integration, and Hilbert. Spaces, Princeton University Press, 234-243, Princeton.

[3] Stein, E., Shakarchi R. (2003) Complex Analysis, Princeton University Press, 218-223, Princeton.

[4] Stein, E., Shakarchi R. (2003) Complex Analysis, Princeton University Press, 235-237, Princeton.