Sandwiches for promise constraint satisfaction

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Abstract. Promise Constraint Satisfaction Problems (PCSP) were proposed recently by Brakensiek and Guruswami as a framework to study approximations for Constraint Satisfaction Problems (CSP). Informally a PCSP asks to distinguish between whether a given instance of a CSP has a solution or not even a specified relaxation can be satisfied. All currently known tractable PCSPs can be reduced in a natural way to tractable CSPs. In 2019 Barto presented an example of a PCSP over Boolean structures for which this reduction requires solving a CSP over an infinite structure. We give a first example of a PCSP over Boolean structures which reduces to a tractable CSP over a structure of size 3 but not smaller. Further we investigate properties of PCSPs that reduce to systems of linear equations or to CSPs over structures with semilattice or majority polymorphism.

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1. Introduction

The Constraint Satisfaction Problem (CSP) for a fixed relational structure $\mathbf{A}$ can be formulated as the following decision problem:

$$\text{CSP}(\mathbf{A})$$

Input: a relational structure $\mathbf{X}$ of the same type as $\mathbf{A}$

Output: yes, if there exists a homomorphism $\mathbf{X} \rightarrow \mathbf{A}$, no, otherwise

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For example, the question of whether a given graph is $r$-colorable is a CSP where $A$ is the complete graph $K_r$ on $r$ vertices.

Bulatov [3] and Zhuk [9] showed that CSP$(A)$ for finite $A$ is either in P or NP-complete. In [4] Brakensiek and Guruswami introduced Promise Satisfaction Problems (PCSP) as relaxations and generalizations of CSP. For relational structures $A, B$ of the same type, let

$$\text{PCSP}(A, B)$$

Input: a relational structure $X$ of the same type as $A$
Output: yes, if there exists a homomorphism $X \to A$, no, if there exists no homomorphism $X \to B$

Here the promise is that for the input $X$ exactly one of the two alternatives $\exists X \to A$ or $\nexists X \to B$ holds. A typical example of a PCSP is to distinguish graphs that are $r$-colorable from those that are not even $s$-colorable for $r \leq s$.

Let $A, B, C$ be relational structures of the same type with homomorphisms $A \to C \to B$. Then we say $C$ is sandwiched by $A$ and $B$; imagine $C$ as the cheese between two slices of bread $A$ and $B$. In this case PCSP$(A, B)$ has a straightforward (constant time) reduction to CSP$(C)$: a structure $X$ is a yes-instance for PCSP$(A, B)$ iff $X$ is a yes-instance for CSP$(C)$. PCSP has been studied in a series of papers by Brakensiek and Guruswami [4,5,6], Brakensiek, Guruswami, Wrochna and Živný [7], Barto, Bulín, Krokhin and Opšal [2] among others. Still the complexity of PCSP in general is wide open. All currently known tractable PCSP$(A, B)$ can be reduced to tractable CSP$(C)$ for some $C$ sandwiched by $A$ and $B$.

In a research project for undergraduate students (REU) organised by P. Mayr and A. Sparks at CU Boulder in Summer 2019, we considered the following meta question on PCSP:

Given finite $A, B$, does there exists some $C$ sandwiched by $A$ and $B$ such that CSP$(C)$ is tractable?

If the answer is yes, then clearly PCSP$(A, B)$ is tractable. However a negative answer may not necessarily yield hardness of PCSP$(A, B)$.

In any case it is not known whether the meta question is decidable. The main obstacle is that tractable sandwiched structures may grow in size. Barto gave an example of Boolean structures $A, B$, that is, structures with two element domains, for which all tractable sandwiched $C$ are infinite [1]. Moreover, it is open whether the size of the smallest tractable sandwiched $C$, that is, the function

$$c(A, B):= \min\{|C| : A \to C \to B, \text{CSP}(C) \text{ tractable}\}$$

is computable. If all such $C$ are infinite, set $c(A, B):=\infty$. If no such $C$ exists, let $c(A, B)$ be undefined. One outcome of the REU is a first example of Boolean $A, B$ for which the smallest sandwiched $C$ with tractable CSP$(C)$ has size 3 (see Theorem 2.1). In particular the finite values of $c(A, B)$ are not bounded above by max($|A|, |B|$).
In Section 3 we show that the existence of certain symmetries can already be observed in small sandwiches. Recall that for a structure $C$ 

$$\text{Pol}(C) := \{ f : C^k \rightarrow C \mid k \in \mathbb{N} \}$$

denotes the set of polymorphisms of $C$. We show that if $A, B$ sandwich some $C$ with a conservative polymorphism $f$ (or a majority polymorphism in case $|A| = 2$), then they already sandwich some substructure $D$ of $C$ with size $|D| \leq |A|$ and restricted polymorphism $f|_D$.

2. Affine sandwiches

A relational structure $C$ is affine if there exists an abelian group $C := (C, +, -, 0)$ on the domain of $C$ such that $x - y + z$ is a polymorphism of $C$. In other words, $C$ is affine if every $n$-ary relation $R^C$ of $C$ is a coset of a subgroup of $C^n$. Then CSP($C$) encodes a system of linear equations and can be solved in polynomial time.

We present an example of Boolean $A, B$ with sandwiched affine $C$ of size 3 but without any sandwiched tractable Boolean structure.

**Theorem 2.1.** Let $A = (\{0, 1\}, R^A), B = (\{0, 1\}, R^B), C = (\{0, 1, 2\}, R^C)$ with

$$R^A := \{100011, 010101, 001110\},$$

$$R^B := \{0, 1\}^6 \setminus \{000000, 000111, 111000, 111111\},$$

$$R^C := \text{the closure of } R^A \text{ under } x - y + z \mod 3.$$ 

Then

1. the affine $C$ is sandwiched by $A$ and $B$ via homomorphisms $A \xrightarrow{\text{id}} C \xrightarrow{g} B$ where $g : \{0, 1, 2\} \rightarrow \{0, 1\}$ is defined by $g(0) = g(2) = 0$ and $g(1) = 1$,

2. but there exists no Boolean $D$ such that $A \rightarrow D \rightarrow B$ and CSP($D$) is tractable (assuming $P \neq \text{NP}$).

**Proof.** For (1) note first that $A$ is a substructure of $C$ by definition, that is, the identity map $A \rightarrow C$ is a homomorphism. More explicitly $R^A$ spans the affine subspace

$$R^C = \{(x_1, \ldots, x_6) \in \mathbb{Z}_3^6 : x_1 + x_2 + x_3 = 1, \quad x_1 + x_4 = 1, \quad x_2 + x_5 = 1, \quad x_3 + x_6 = 1\}$$

$$= \{100011, 010101, 001110, 000000, 000111, 111000, 111111\},$$

$$= \{220221, 202212, 022122, 211200, 121020, 112002\}.$$ 

Applying $g$ we get

$$g(R^C) = \{100011, 010101, 001110, 000001, 000010, 000100, 011000, 101000, 110000\} \subseteq R^B.$$
Hence $g : C \to B$ is a homomorphism, and (1) is proved.

For (2) suppose there exists Boolean $D$ and homomorphisms $A \xrightarrow{f} D \xrightarrow{h} B$ such that $\text{CSP}(D)$ is tractable. Since $R^B$ contains no constant tuple, both $f$ and $h$ are bijections.

**Case, $f$ and $h$ both are the identity:** Then $A \leq D \leq B$ is a chain of substructures. By Schaefer’s Dichotomy Theorem for Boolean CSP [8], $D$ has one of the following polymorphisms: $0, 1, \wedge, \vee$, minority or majority. However, closing $R^A$ under any of the first 4 clearly yields a constant tuple, e.g.,

$$100011 \wedge 010101 \wedge 001110 = 000000,$$

which is not in $R^B$. Hence $R^D$ is not preserved by $0, 1, \wedge, \vee$. Next applying the minority operation $d$ to the elements of $R^A \subseteq R^D$ yields, e.g.,

$$d(100011, 010101, 001110) = 111000$$

which is not in $R^B$, hence not in $R^D$. Similarly applying the majority operation $m$ yields

$$m(100011, 010101, 001110) = 000111 \notin R^D.$$

Hence no substructure of $B$ that contains $A$ has tractable CSP.

**Case, $f$ is the identity and $h$ is negation:** As in the previous case, closing $R^A$ under one of the six polymorphisms of Schaefer’s Theorem and then applying $h$ to the results yields a constant tuple, $111000$, or $000111$. Since neither is in $R^B$, we have a contradiction.

The remaining cases that $f$ is negation, $h$ is the identity or that both $f$ and $h$ are negation follow similarly. Thus (2) is proved. $\square$

There is no known characterization of structures $A, B$ that sandwich an affine $C$. But we have some necessary conditions in terms of the polymorphisms from $A$ to $B$,

$$\text{Pol}(A, B) := \{A^k \to B : k \in \mathbb{N}\}.$$  

First recall that for $k \in \mathbb{N}$ a function $f : A^k \to B$ is symmetric if $f$ is invariant under permutation of its arguments $x_1, \ldots, x_k \in A$. x More generally Brakensiek and Guruswami [6] call $f : A^k \to B$ block-symmetric for a partition of $\{1, \ldots, k\}$ into blocks $B_1, \ldots, B_\ell$ if $f$ is invariant under permutation of arguments $x_{i_1}, \ldots, x_{i_m}$ for any block $B_j = \{i_1, \ldots, i_m\}$. Note that every function $f$ is block-symmetric for the partition into singletons. Further there exists a unique coarsest partition for which $f$ is block-symmetric, that is, a partition with maximal blocks $B_1, \ldots, B_\ell$. The width of $f$ is the size of the smallest block of this coarsest partition for which $f$ is block symmetric, that is,

$$\max\{\min\{|B_1|, \ldots, |B_\ell|\} : f \text{ is block-symmetric for the partition } B_1, \ldots, B_\ell\}.$$  

We can now formulate some weak necessary conditions for structures to have an affine sandwich. Clearly they are not sufficient. Note that e.g. for $A = B = (\{0, 1\}, \leq, \{0\}, \{1\})$, the $k$-ary minimum operation is in $\text{Pol}(A, A)$ for every $k \in \mathbb{N}$. In particular $A$ has symmetric polymorphisms of all arities. However
there is no affine $C$ with $A \rightarrow C \rightarrow A$ because otherwise $A$ would be the core of $C$ and would inherit the polymorphism $x - y + z \mod 2$, which it does not have.

**Theorem 2.2.** Let $A, B, C$ be relational structures of the same type with homomorphisms $A \rightarrow C \rightarrow B$ and $C$ affine.

(1) Then $\text{Pol}(A, B)$ contains block-symmetric polymorphisms of arbitrarily large width.

(2) If $C$ is finite, then $\text{Pol}(A, B)$ contains symmetric polymorphisms of arbitrarily large arity.

**Proof.** Let $f: A \rightarrow C$ and $g: C \rightarrow B$. Since $C$ is affine, we have for each $k \in \mathbb{N}$ and $a_1, \ldots, a_k \in \mathbb{Z}$ such that $\sum_{i=1}^k a_i x = x$ for all $x \in C$, that

$$C^k \rightarrow C, (x_1, \ldots, x_k) \mapsto \sum_{i=1}^k a_i x_i ,$$

is in $\text{Pol}(C)$. Composing this polymorphism with the homomorphisms $f$ and $g$, we obtain that

$$A^k \rightarrow B, (x_1, \ldots, x_k) \mapsto g \left( \sum_{i=1}^k a_i f(x_i) \right) ,$$

is in $\text{Pol}(A, B)$.

For (1), it follows that for $k \in \mathbb{N}$

$$A^{2k+1} \rightarrow B, (x_1, \ldots, x_{2k+1}) \mapsto g \left( \sum_{i=1}^{2k+1} (-1)^{i-1} f(x_i) \right) ,$$

is a block-symmetric polymorphism with partition into two blocks $B_1 = \{1, 3, \ldots, 2k+1\}$, $B_2 = \{2, 4, \ldots, 2k\}$, hence width $\geq k$.

For (2) assume $C$ is finite of size $n$. Then for $k \in \mathbb{N}$

$$A^{nk+1} \rightarrow B, (x_1, \ldots, x_{nk+1}) \mapsto g \left( \sum_{i=1}^{nk+1} f(x_i) \right) ,$$

is a symmetric polymorphism of arity $nk + 1$. □

Assume $A$ and $B$ sandwich an affine $C$. Then $\text{PCSP}(A, B)$ reduces to $\text{CSP}(C)$. Every instance of the latter can be expressed as system of linear equations and hence solved in polynomial time. More generally, Brakensiek and Guruswami showed that if $\text{Pol}(A, B)$ contains (block)-symmetric polymorphisms of arbitrarily large arity (width), then $\text{PCSP}(A, B)$ can be solved in polynomial time via the so-called basic linear programming relaxation over the non-negative rationals and over the integers [6, Theorem 3.1, 4.1].
3. Conservative and majority sandwiches

We add some straightforward observations on non-affine sandwiches. Recall that \( f: A^3 \to A \) is a majority operation if it satisfies
\[
f(y, x, x) = f(x, y, x) = f(x, x, y) = x
\]
for all \( x, y \in A \). More generally, let \( k \geq 2 \) and \( f: A^k \to A \). Then \( f \) is a weak near-unanimity (WNU) operation if
\[
f(y, x_1, x_2, \ldots, x_k) = f(x_1, y, x_2, \ldots, x_k) = \cdots = f(x_1, x_2, \ldots, y)
\]
for all \( x, y \in A \). Further \( f \) is conservative if \( f(x_1, \ldots, x_k) \in \{x_1, \ldots, x_k\} \) for all \( x_1, \ldots, x_k \in A \). For example, the binary operation min on \( \{0, 1\} \) is a conservative WNU operation. These operations play a crucial role for CSPs: CSP(\( C \)) for finite \( C \) is tractable iff \( C \) has a WNU polymorphism \([3,9]\).

For a structure \( C \) and \( D \subseteq C \) the induced substructure \( C|_D \) on \( D \) has domain \( D \) and relations \( R^D := R^C \cap (D \times \cdots \times D) \) for every \( R \) in the type of \( C \).

Lemma 3.1. Let \( A, B, C \) be relational structures of the same type with homomorphisms \( A \xrightarrow{f} C \xrightarrow{g} B \), and let \( D := C|_{f(A)} \). Then we have homomorphisms \( A \xrightarrow{f} D \xrightarrow{g|_{f(A)}} B \), and every conservative polymorphism of \( C \) restricts to a polymorphism of \( D \).

Proof. Let \( p \) be a conservative polymorphism of \( C \). Then \( p(D, \ldots, D) \subseteq D \). Hence \( p \) preserves \( R^D \) for every relation \( R \) in the type of \( C \). \(\square\)

As a consequence, if \( A \) and \( B \) sandwich a structure with conservative WNU polymorphism, then they sandwich such a structure of size \( \leq |A| \). Hence given finite \( A, B \) it is decidable whether they sandwich some structure with conservative WNU polymorphism.

Lemma 3.2. Let \( A, B, C \) be relational structures of the same type with homomorphisms \( A \xrightarrow{f} C \xrightarrow{g} B \). Assume that \( A \) is Boolean and \( C \) has a majority polymorphism \( m \). Then \( D := C|_{f(A)} \) is sandwiched by \( A \) and \( B \), has a majority polymorphism \( m|_{f(A)} \), and has size \( \leq 2 \).

Proof. Since \( m \) preserves every 2-element subset of \( C \), in particular, \( f(A) \), this follows as in Lemma 3.1. \(\square\)

As a consequence, if some Boolean \( A \) and finite \( B \) sandwich a structure with majority polymorphism, then they sandwich such a structure of size \( \leq 2 \). In particular, given Boolean \( A, B \) it is decidable whether they sandwich some structure with majority polymorphism.

4. Summary

We gave some necessary conditions for finite structures \( A, B \) to sandwich a finite affine \( C \) and showed that the smallest such \( C \) can be strictly larger than \( A, B \). The following remain open:
Problem 1. Are there some finite graphs $A, B$ with $\max(|A|, |B|) < c(A, B) < \infty$?

Problem 2. Given finite $A, B$, is it decidable whether they sandwich some (finite) affine $C$?

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