Multivariate box spline wavelets in higher-dimensional Sobolev spaces

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Abstract
We construct wavelets and derive a density condition of MRA in a higher-dimensional Sobolev space. We give necessary and sufficient conditions for orthonormality of wavelets in $H^s(\mathbb{R}^d)$. We construct nonseparable orthonormal wavelets in a higher-dimensional Sobolev space by using multivariate box spline.

Keywords: Wavelets; Box Splines; Multiresolution analysis; Sobolev space

1 Introduction
Box splines are refinable functions, and we can easily choose various directions to have a box spline function with a desired order of smoothness. Naturally, they have been used to construct various wavelet functions. Mathematically box splines offer an elegant toolbox for constructing a class of multidimensional elements with flexible shape and support. In multivariate setting, box splines are often considered as a generalization of B-splines [1]. Theoretically, the computational complexity of a box spline is lower than that of an equivalent B-spline, since its support is more compact and its total polynomial degree is lower. To investigate this potential in practice, several attempts were made. Recurrence relation [1, 2] is the most commonly used technique for evaluating box splines at an arbitrary position. There are many papers on multivariate spline wavelet theory, in particular, on orthogonal spline wavelets [3], compactly spline prewavelets [4–6], bivariate and trivariate compactly supported biorthogonal box spline wavelets [7, 8], and multivariate compactly supported tight wavelet frames [9].

Wavelets in a Sobolev space and their properties were instigated by Bastin et al. [10, 11], Dayong and Dengfeng [12], and Pathak [13]. Regular compactly supported wavelets and compactly supported wavelets of integer order in a Sobolev space by B-spline are given in [10, 11]. Further, bivariate box splines in a Sobolev space were introduced in [14].

Inspired by the works mentioned, in this paper, we study nonseparable wavelets in a higher-dimensional Sobolev space by using a multivariate box spline. To the best of our knowledge, no previous studies of multivariate box spline wavelets exist in higher-dimensional Sobolev spaces. This paper is organized as follows. In Sect. 2, we hereby present construction of wavelets and density conditions of MRA in a higher-dimensional Sobolev space. Also, we give necessary and sufficient conditions for the orthonormality of wavelets in $H^s(\mathbb{R}^d)$. In Sect. 3, we construct nonseparable wavelets in a higher-dimensional Sobolev space by using a multivariate box spline.

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1.1 Sobolev space $H^s(\mathbb{R}^d)$

For any real number $s$, the Sobolev space $H^s(\mathbb{R}^d)$ consists of tempered distributions in $S'((\mathbb{R}^d))$ such that

$$
\|f\|_s^2 := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (1 + \|\xi\|^2)^s |\hat{f}(\xi)|^2 d\xi,
$$

where $\| \cdot \|$ denotes the Euclidean norm in $\mathbb{R}^d$, and the corresponding inner product is

$$
(f, g)_s := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (1 + \|\xi\|^2)^s \hat{f}(\xi) \overline{\hat{g}(\xi)} d\xi.
$$

The Fourier transform $\hat{f}$ of $f \in L^1(\mathbb{R}^d)$ is defined as

$$
\hat{f}(\xi) := \int_{\mathbb{R}^d} e^{-i(x,\xi)} f(x) dx,
$$

where $(x,\xi)$ is the inner product of two vectors $x$ and $\xi$ in $\mathbb{R}^d$.

2 Multiresolution analysis

To adapt classical theory of MRA over $H^s(\mathbb{R}^d)$, we first derive an orthonormality and density condition. The main problem is that $H^s$-norm is not dilation invariant. We also don’t achieve orthonormality at each level of dilation by a single scaling function. This lead us to a more general construction of MRA, where the scaling function depends on the level of dilation. Throughout this paper, the superscript $j$ of a function $\psi^{(j)}$ represents level $j$.

**Proposition 2.1** If $s$ is a real number, $\psi^{(j)} \in H^s(\mathbb{R}^d)$, and $j$ is an integer, then the distributions $\psi^{(j)}(x) = 2^{jd/2} \psi^{(j)}(2^j x - k), k \in \mathbb{Z}^d$, are orthonormal in $H^s(\mathbb{R}^d)$ iff

$$
\sum_{k \in \mathbb{Z}^d} (1 + 2^j \|\xi + 2^j k\|^2)^s |\hat{\psi}^{(j)}(\xi + 2^j k)|^2 = 1
$$

almost everywhere. It follows that we have the bound

$$
|\hat{\psi}^{(j)}(2^j \xi)| \leq (1 + \|\xi\|^2)^{-s/2}.
$$

**Proof** Since $\psi^{(j)}(t) \in H^s(\mathbb{R}^d)$, the series

$$
M(\xi) = \sum_{r \in \mathbb{Z}^d} |\hat{\psi}^{(j)}(\xi + 2\pi r)|^2 (1 + 2^j \|\xi + 2\pi r\|^2)^s
$$

converges almost everywhere, belongs to $\mathcal{L}_{\text{loc}}^1(\mathbb{T}^d)$, and is $2\pi \mathbb{Z}^d$-periodic, that is, $M(\xi) \in L^1(\mathbb{T}^d)$, where $\mathbb{T}^d = [0,2\pi]^d$ is the $d$-dimensional torus. Moreover, for every $l \in \mathbb{Z}^d$, we have

$$
\int_{\mathbb{T}^d} M(\xi) e^{-i\langle \xi, (k - l) \rangle} d\xi
$$

$$
= \sum_{r \in \mathbb{Z}^d} \int_{\mathbb{T}^d} |\hat{\psi}^{(j)}(\xi + 2\pi r)|^2 (1 + 2^j \|\xi + 2\pi r\|^2)^s e^{-i\langle \xi, (k - l) \rangle} d\xi
$$

$$
= \int_{\mathbb{R}^d} |\hat{\psi}^{(j)}(v)|^2 (1 + 2^j \|v\|^2)^s e^{-i\langle v, (k - l) \rangle} dv
$$
Proof

Let us prove the first part with Proposition 2.2.

Moreover, which implies the distributions for every \( j \)

\[
M = \left\{ (2\pi)^d e^{i(\xi, k - l)} : k, l \in \mathbb{Z}^d \right\}
\]

is an orthonormal basis for \( L^2(\mathbb{T}^d) \), we have

\[
\frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} M(\xi) e^{i(\xi, k - l)} d\xi = \{ \phi_{j,k}(t), \phi_{j,l}(t) \}_s = \delta_{k,l}
\]

if \( M(\xi) = 1 \).

From (1) we get

\[
(1 + 2^j \| \xi \|^2)^{\alpha} |\hat{\phi}(\xi)|^2 \leq 1,
\]

which implies

\[
|\hat{\phi}(\xi)| \leq (1 + 2^j \| \xi \|^2)^{-\alpha/2}.
\]

\[\square\]

**Proposition 2.2** Let \( \psi^{(j)} \in \mathbb{Z}_j \) be a sequence of elements of \( H^s(\mathbb{R}^d) \) such that, for every \( j \), the distributions \( \psi^{(j)}(x) = 2^jd^2 \psi^{(j)}(2x - k) \), \( k \in \mathbb{Z}^d \), are orthonormal in \( H^s(\mathbb{R}^d) \). If \( P_j \) is the orthogonal projection from \( H^s(\mathbb{R}^d) \) onto \( V_j := \text{span} \{ \psi^{(j)} : k \in \mathbb{Z}^d \} \), then, for every \( h \in H^s(\mathbb{R}^d) \), we have

\[
\lim_{j \to +\infty} \left( \|P_j h\|^2 - \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} (1 + \| \xi \|^2)^{2\alpha} \hat{h}(\xi) |\hat{\psi}^{(j)}(2^{-j} \xi)|^2 d\xi \right) = 0.
\]

Moreover, if there are \( A, \alpha > 0 \) such that

\[
\int_{\mathbb{T}^d} (1 + \| \xi \|^2)^{\alpha} |\hat{\phi}(\xi)|^2 d\xi \leq A
\]

for every \( j \leq 0 \), then \( \bigcap_{j=\infty}^{+\infty} V_j = \{0\}^d \).

**Proof** Let us prove the first part with \( h \in C^\infty_c(\mathbb{R}^d) \). By the definition of \( P_j \), we get

\[
\|P_j h\|^2 = \sum_{k \in \mathbb{Z}^d} \|h, \phi^{(j)}_{j,k}\|^2 = \frac{2^{-jd}}{(2\pi)^{2d}} \sum_{k \in \mathbb{Z}^d} \left( \int_{\mathbb{T}^d} (1 + \| \xi \|^2)^{2\alpha} \hat{h}(\xi) \hat{\phi}^{(j)}(2^{-j} \xi) e^{2\pi i (k, \xi)} d\xi \right)^2.
\]

Moreover, since \( h \) and \( \psi^{(j)} \) belong to \( H^s(\mathbb{R}^d) \),

\[
\int_{\mathbb{T}^d} (1 + \| \xi \|^2)^{2\alpha} \hat{h}(\xi) \hat{\phi}^{(j)}(2^{-j} \xi) e^{2\pi i (k, \xi)} d\xi = \int_{[0,2\pi]^d} \sum_{p \in \mathbb{Z}^d} (1 + \| \xi + 2\pi p \|^2)^{2\alpha} \hat{h}(\xi + 2\pi p) \hat{\phi}^{(j)}(2^{-j} \xi + 2\pi p) d\xi.
\]
Hence, using the Parseval formula in $L^2([0, 2\pi]^d)$, we get

\[
\|P_j h\|^2 = \frac{1}{(2\pi)^d} \int_{[0, 2\pi]^d} \left| \sum_{p \in \mathbb{Z}^d} (1 + \|\xi + 2\pi p\|^2)^{\frac{1}{2}} \hat{h}(\xi + 2\pi p) \hat{\psi}(2^{-j}\xi + 2\pi p) \right|^2 d\xi
\]

\[
= \frac{1}{(2\pi)^d} \int_{[0, 2\pi]^d} \left( 1 + \|\xi\|^2 \right)^{\frac{1}{2}} (1 + \|\xi + 2\pi q\|^2)^{\frac{1}{2}} \hat{h}(\xi) \hat{\psi}(2^{-j}\xi) d\xi
\]

\[
\times \hat{h}(\xi + 2\pi q) \hat{\psi}(2^{-j}\xi + 2\pi q) d\xi
\]

\[
= \frac{1}{(2\pi)^d} \int_{[0, 2\pi]^d} (1 + \|\xi\|^2) \hat{h}(\xi)^2 |\hat{\psi}(2^{-j}\xi)|^2 d\xi
\]

\[
+ \frac{1}{(2\pi)^d} \sum_{q \in \mathbb{Z}^d \setminus \{0\}^d} \int_{[0, 2\pi]^d} (1 + \|\xi\|^2)^{\frac{1}{2}} (1 + \|\xi + 2\pi q\|^2)^{\frac{1}{2}} \hat{h}(\xi) \hat{\psi}(2^{-j}\xi) d\xi
\]

\[
\times \hat{h}(\xi + 2\pi q) \hat{\psi}(2^{-j}\xi + 2\pi q) d\xi.
\]

The term associated with $q = \{0\}^d, \{0\}^d = (0, 0, \ldots, 0) \in \mathbb{Z}^d$ is used as an approximation for $\|P_j h\|^2$. Using Proposition 2.1, the inequality $|\hat{\psi}(2^{-j}\xi)| \leq (1 + \|\xi\|^2)^{-\alpha/2}$, and the fact that $\hat{h}$ belongs to the Schwartz space $S(\mathbb{R}^d)$ (i.e., $|\hat{h}(\xi)| \leq C(1 + \|\xi\|^2)^{-\alpha}$ for any $\alpha > 0$), we obtain that the sum of the other ones is bounded by

\[
\sum_{q \in \mathbb{Z}^d \setminus \{0\}^d} \int_{[0, 2\pi]^d} (1 + \|\xi\|^2)^{\frac{1}{2}} (1 + \|\xi + 2\pi q\|^2)^{\frac{1}{2}} |\hat{h}(\xi) \hat{\psi}(2^{-j}\xi)| \ d\xi
\]

\[
\leq C \sum_{q \in \mathbb{Z}^d \setminus \{0\}^d} \frac{1}{(1 + \|\xi\|^2)^{\frac{1}{2}}} \int_{[0, 2\pi]^d} \frac{1}{(1 + \|\xi\|^2)^{\frac{1}{2}}} \ d\xi
\]

\[
\leq C \sum_{q \in \mathbb{Z}^d \setminus \{0\}^d} \frac{1}{\|2\pi q\|^2 \ (1 + \|\xi\|^2)^2} \ d\xi
\]

\[
\leq C 2^{2d(j+1)} \left( \sum_{q \in \mathbb{Z}^d \setminus \{0\}^d} \frac{1}{|q|^2} \right) \int_{[0, 2\pi]^d} \frac{1}{(1 + \|\xi\|^2)^{\frac{1}{2}}} \ d\xi,
\]

where $|q| = (\sum_{r=1}^d |q_r|^2)^{\frac{1}{2}}, q = (q_1, q_2, \ldots, q_d) \in \mathbb{Z}^d$. This expression converges to 0 as $j \to +\infty$.

Now let $h \in H^s(\mathbb{R}^d)$. Recall the inequality

\[
\|f + g\|^2 \leq (1 + \varepsilon)\|f\|^2 + \left(1 + \frac{1}{\varepsilon}\right)\|g\|^2,
\]

which is valid for every $\varepsilon > 0$ and any seminorm. For any $\chi \in C^\infty_0(\mathbb{R}^d)$, we have

\[
\|P_j h\|^2 - \frac{1}{(2\pi)^d} \int_{[0, 2\pi]^d} (1 + \|\xi\|^2)^{\alpha} |\hat{h}(\xi)|^2 |\hat{\psi}(2^{-j}\xi)|^2 d\xi
\]

\[
\leq (1 + \varepsilon)\|P_j h\|^2 + \left(1 + \frac{1}{\varepsilon}\right)\|P_j (h - \chi)\|^2
\]

\[
- \frac{1}{(2\pi)^d(1 + \varepsilon)} \int_{[0, 2\pi]^d} (1 + \|\xi\|^2)^{\alpha} |\hat{\chi}(\xi)|^2 |\hat{\psi}(2^{-j}\xi)|^2 d\xi
\]
We know that

\[ \text{now we construct wavelets in } H^s \]

The last expression converges to zero as \( s \) large. This follows immediately from the fact that the Fourier transform of \( 2^{-j} \chi \) is

\[ \sum_{q \in \mathbb{Z}^d} \left( 1 + \| \xi + 2^j \alpha \|_2^2 \right)^{i/2} |\hat{\chi}(\xi + 2^j \alpha)|^2 \]

\[ \leq \left( \sum_{q \in \mathbb{Z}^d} \left( 1 + \| \xi + 2^j \alpha \|_2^2 \right)^{i/2} |\hat{\chi}(\xi + 2^j \alpha)|^2 \right)^{1/2}. \]

We know that

\[ \sum_{q \in \mathbb{Z}^d} \left( 1 + \| \xi + 2^j \alpha \|_2^2 \right)^{i/2} |\hat{\chi}(\xi + 2^j \alpha)|^2 (2^{j+1} \pi)^d \rightarrow \int_{\mathbb{R}^d} \left( 1 + \| \xi \|_2^2 \right)^{i/2} |\hat{\chi}(\xi)|^2 d\xi \]

if \( j \leq -1 \). It follows that

\[ \| P_j h \|_2^2 \leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left( 1 + \| \xi \|_2^2 \right)^{i/2} |\hat{\chi}(\xi)\hat{\phi}^{(j)}(2^{-j} \xi)| (2^j \pi)^d C \| h \|_2 d\xi \]

\[ \leq \frac{2^{-j} C \| h \|_2}{(2\pi)^d} \left( \int_{\mathbb{R}^d} \left( 1 + 2^j \| \xi \|_2^2 \right)^{i/2} |\hat{\phi}^{(j)}(2^{-j} \xi)|^2 d\xi \right)^{1/2} \]

\[ \times \left( \int_{\mathbb{R}^d} \left( 1 + 2^j \| \xi \|_2^2 \right)^{i/2} |\hat{\chi}(\xi)|^2 d\xi \right)^{1/2} \]

\[ \leq \frac{C \sqrt{A \| h \|_2}}{(2\pi)^d} \left( \int_{\mathbb{R}^d} \left( 1 + 2^j \| \xi \|_2^2 \right)^{i/2} |\hat{\chi}(\xi)|^2 d\xi \right)^{1/2}. \]

The last expression converges to zero as \( j \) converges to \(-\infty\). \( \square \)

Now we construct wavelets in \( H^s(\mathbb{R}^d) \) with the help of previous propositions.

By definition, \( V_j \) is the set of all \( f \in H^s(\mathbb{R}^d) \) such that

\[ \hat{f}(\xi) = m(2^{-j} \xi) \hat{\phi}^{(j)}(2^{-j} \xi), \]

where \( m \in L^2_{\text{loc}}(\mathbb{R}^d) \) is \( 2\pi \mathbb{Z}^d \)-periodic. This follows immediately from the fact that the Fourier transform of \( 2^{jd/2} \phi^{(j)}(2x - k) \) is

\[ 2^{-jd/2} e^{-2\pi i (k \cdot \xi)} \hat{\phi}^{(j)}(2^{-j} \xi). \]
We have $V_j \subset V_{j+1}$ for every $j \in \mathbb{Z}^d$ iff there are $2\pi \mathbb{Z}^d$-periodic functions $m_0^{(j)} \in L^2_{\text{loc}}(\mathbb{R}^d)$ such that the following scale relation holds:

$$
\hat{\phi}^{(j)}(2\xi) = m_0^{(j+1)}(\xi)\hat{\phi}^{(j+1)}(\xi);
$$

(2)

moreover, $\phi^{(j)}$ and $\phi^{(j+1)}$ satisfy the hypothesis of Proposition 2.1. Now, using our theorems and propositions, we develop the definition of MRA in $H^s(\mathbb{R}^d)$.

**Definition 2.3** Let $s$ be a real number. The MRA of $H^s(\mathbb{R}^d)$ is a sequence $V_j, j \in \mathbb{Z}$, of closed linear subspaces of $H^s(\mathbb{R}^d)$ such that

(a) $V_j \subset V_{j+1}$,

(b) $\bigcap_{j=-\infty}^{+\infty} V_j = H^s(\mathbb{R}^d)$,

(c) $\bigcap_{j=-\infty}^{+\infty} V_j = \{0\}^d$, and

(d) for every $j$, there is a function $\phi^{(j)}$ such that the distributions $2^{jd} \phi^{(j)}(2^j x - k), k \in \mathbb{Z}^d$, form an orthonormal basis for $V_j$.

Before giving a necessary condition for the orthonormality, we define $E_d := (0, 1)^d$ as the unit cube in the $d$-dimensional Euclidean space.

**Theorem 2.4** If $\phi^{(j)}$ and $\phi^{(j+1)}$ satisfy the hypothesis of Proposition 2.1, then

$$
\sum_{q=0}^{2^d-1} \left| m_0^{(j+1)}(\xi + \gamma_q \pi) \right| = 1, \quad \gamma_q \in E_d, q = 1, 2, \ldots, 2^d - 1.
$$

**Proof** We know from Proposition 2.1 that if the system is orthonormal, then

$$
\delta_{k,l} = \langle \phi^{(j)}(x), \phi^{(j+1)}(x) \rangle_{L^2} = \frac{2^{-jd}}{(2\pi)^d} \int_{\mathbb{R}^d} |\hat{\phi}^{(j)}(2^{-j}\xi)|^2 e^{-i2^j \langle \xi, (k-l) \rangle} (1 + \|\xi\|^2)^s d\xi
$$

$$
= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\hat{\phi}^{(j)}(u)|^2 e^{-i2^j \langle u, (k-l) \rangle} (1 + \|u\|^2)^s du
$$

$$
= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |m_0^{(j+1)}(u/2)|^2 |\hat{\phi}^{(j+1)}(u/2)|^2 e^{-i2^j \langle u, (k-l) \rangle} (1 + 2^j \|u\|^2)^s du
$$

$$
= \frac{2^d}{(2\pi)^d} \int_{\mathbb{R}^d} |m_0^{(j+1)}(v)|^2 |\hat{\phi}^{(j+1)}(v)|^2 e^{-i2^j \langle v, (k-l) \rangle} (1 + 2^j \|v\|^2)^s dv
$$

$$
= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} m_0^{(j+1)}(v) \sum_{r \in \mathbb{Z}^d} \left| \hat{\phi}^{(j+1)}(v + 2\pi r) \right|^2
$$

$$
\times (1 + 2^j \|v + 2\pi r\|^2)^s e^{-i2^j \langle v, (k-l) \rangle} dv
$$

$$
= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} m_0^{(j+1)}(v) \left| \hat{\phi}^{(j+1)}(v) \right|^2 e^{-i2^j \langle v, (k-l) \rangle} dv
$$

$$
= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \sum_{q=0}^{2^d-1} \left| m_0^{(j+1)}(v + \gamma_q \pi) \right|^2 e^{-i2^j \langle v, (k-l) \rangle} dv.
$$
which implies that
\[
\sum_{q=0}^{2^d-1} |m_0^{(j+1)}(\xi + \gamma_q \pi)|^2 = 1, \quad \gamma_q \in E_d,
\]
if \( k = l. \) □

With the help of (2) and Theorem 2.4, we may define \( \psi^{(j)} \) by
\[
\hat{\psi}^{(j)}(\xi) = m_0^{(j+1)}(\xi/2) \hat{\psi}^{(j+1)}(\xi/2) = \prod_{l=1}^{j} m_0^{(i_l)}(\xi/2^l) \hat{\psi}^{(i_l)}(\xi/2^l) = \cdots = \frac{1}{(1 + \|\xi\|^2)^{d/2}}\prod_{l=1}^{\infty} m_0^{(i_l)}(\xi/2^l) \tag{3}
\]
for \( j \in \mathbb{Z} \). For \( V_j \), let \( W_j \) be the orthogonal complement of \( V_j \) in \( V_{j+1} \). We have
\[
\psi^{(j)}_{j,k} := 2^{jd/2} \psi^{(j)}(2^j x - k) \in V_{j+1} \tag{4}
\]
if there are \( 2\pi \mathbb{Z}^d \)-periodic functions \( m_1^{(j)}, m_2^{(j)}, \ldots, m_{2^d-1}^{(j)} \in L^2_{\text{loc}}(\mathbb{R}^d) \) such that
\[
\hat{\psi}^{(j)}(2^j \xi) = m_0^{(j+1)}(2^{-j-1} \xi) \hat{\psi}^{(j+1)}(2^{-j-1} \xi), \quad p = 1, 2, \ldots, 2^d - 1.
\]

Theorem 2.5 The distributions \( \psi^{(j)}_{j,k,p}(x) = 2^{jd/2} \psi^{(j)}(2^j x - k) \) are orthonormal if
\[
\sum_{q=0}^{2^d-1} |m_0^{(j)}(\xi + \gamma_q \pi)|^2 = 1, \quad \gamma_q \in E_d, \forall p = 1, 2, \ldots, 2^d - 1,
\]
and they are orthogonal to \( V_j \) if
\[
\sum_{q=0}^{2^d-1} m_0^{(j+1)}(\xi + \gamma_q \pi) m_0^{(j+1)}(\xi + \gamma_q \pi) = 0, \quad \gamma_q \in E_d, \forall p = 1, 2, \ldots, 2^d - 1. \tag{5}
\]

Proof
\[
\left< \psi^{(j)}_{j,k,p}, \psi^{(j)}_{j,l,p} \right> = \frac{2^{-jd}}{(2\pi)^d} \int_{\mathbb{R}^d} |\hat{\psi}^{(j)}_p(2^j \xi)|^2 e^{-i2^j(u,k-l)} (1 + \|\xi\|^2)^d \, d\xi
\]
\[
= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\hat{\psi}^{(j)}_p(u)|^2 e^{-i(u,l-k)} (1 + 2^d\|u\|^2)^d \, du
\]
\[
= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |m_{p}^{(j+1)}(u/2)|^2 |\hat{\psi}^{(j+1)}_p(u/2)|^2 e^{-i(u,l-k)} (1 + 2^d\|u\|^2)^d \, du
\]
\[
= \frac{2^d}{(2\pi)^d} \int_{\mathbb{R}^d} |m_{p}^{(j+1)}(v)|^2 |\hat{\psi}^{(j+1)}_p(v)|^2 e^{-i2^j(v,l-k)} (1 + 2^{2(j+1)}\|v\|^2)^d \, dv
\]
\[
\frac{1}{(2\pi)^d} \int_{\mathbb{S}^d} |m_p^{(j+1)}(v)|^2 \sum_{r=2}^{d} \left| \hat{\psi}^{(j+1)}(v + 2\pi r) \right|^2 \\
\times (1 + 2^{2(j+1)} \|v + 2\pi r\|^2) e^{-i2\langle v, (k-l) \rangle} dv
\]
\[
= \frac{1}{(2\pi)^d} \int_{\mathbb{S}^d} |m_p^{(j+1)}(v)|^2 e^{-i2\langle v, (k-l) \rangle} dv
\]
\[
= \frac{1}{(2\pi)^d} \int_{[0,\pi]^d} \sum_{q=1}^{2^d-1} |m_p^{(j+1)}(v + y_q\pi)|^2 e^{-i2\langle v, (k-l) \rangle} dv
\]
\[
= \frac{1}{(2\pi)^d} \int_{[0,\pi]^d} e^{-i2\langle v, (k-l) \rangle} dv.
\]

Therefore

\[
\{\psi_{j,k,p}^{(j)}, \psi_{j,k,p}^{(j)}\}_s = 1
\]

if \(\sum_{q=0}^{2^d-1} |m_p^{(j+1)}(\xi + y_q\pi)| = 1, y_q \in E_d, \) and \(k = l.\)

Now we prove second part of the theorem:

\[
0 = \{\psi_{j,k,p}^{(j)}, \psi_{j,k,p}^{(j)}\}_s
\]

\[
= \frac{2^{-jd}}{(2\pi)^d} \int_{\mathbb{S}^d} (1 + \|\xi\|^2)^k \overline{\psi_j^{(j)}(2^{-j}\xi)} \overline{\hat{\psi}^{(j+1)}(2^{-j}\xi)} e^{-i2\langle \xi, (k-l) \rangle} d\xi
\]

\[
= \frac{1}{(2\pi)^d} \int_{\mathbb{S}^d} (1 + 2^{2j} \|\xi\|^2)^k m_p^{(j+1)}(\xi/2)m_0^{(j+1)}(\xi/2) \overline{\hat{\psi}^{(j+1)}(\xi/2)} e^{-i2\langle \xi, (k-l) \rangle} d\xi
\]

\[
= \frac{2^d}{(2\pi)^d} \int_{\mathbb{S}^d} (1 + 2^{2(j+1)} \|\xi\|^2)^k m_p^{(j+1)}(\xi)m_0^{(j+1)}(\xi) \overline{\hat{\psi}^{(j+1)}(\xi)} e^{-i2\langle \xi, (k-l) \rangle} d\xi
\]

\[
= \frac{1}{(2\pi)^d} \int_{\mathbb{S}^d} m_p^{(j+1)}(\xi)m_0^{(j+1)}(\xi)
\]

\[
\times \sum_{r=2}^{d} (1 + 2^{2(j+1)} \|\xi+2\pi r\|^2)^k \overline{\hat{\psi}^{(j+1)}(\xi+2\pi r)} e^{-i2\langle \xi, (k-l) \rangle} d\xi
\]

\[
= \frac{1}{(2\pi)^d} \int_{[0,\pi]^d} \sum_{q=1}^{2^d-1} m_p^{(j+1)}(\xi + y_q\pi)m_0^{(j+1)}(\xi + y_q\pi) e^{-i2\langle \xi, (k-l) \rangle} d\xi,
\]

which implies

\[
\sum_{q=0}^{2^d-1} m_p^{(j+1)}(\xi + y_q\pi)m_0^{(j+1)}(\xi + y_q\pi) = 0, \quad y_q \in E_d, \forall p = 1, 2, \ldots, 2^d - 1.
\]

Now we define unitary matrix with the help of our theorems,

\[
\begin{bmatrix}
  m_0^{(j)}(\xi + y_0\pi) & m_0^{(j)}(\xi + y_1\pi) & \cdots & m_0^{(j)}(\xi + y_{2^d-1}\pi) \\
  m_1^{(j)}(\xi + y_0\pi) & m_1^{(j)}(\xi + y_1\pi) & \cdots & m_1^{(j)}(\xi + y_{2^d-1}\pi) \\
  \vdots & \vdots & \ddots & \vdots \\
  m_{2^d-1}^{(j)}(\xi + y_0\pi) & m_{2^d-1}^{(j)}(\xi + y_1\pi) & \cdots & m_{2^d-1}^{(j)}(\xi + y_{2^d-1}\pi)
\end{bmatrix}
\]

(6)
Theorem 2.6  Suppose that the scaling function $\psi^{(0)}$, $j \in \mathbb{Z}$, generate an MRA $\{V_j\}$ of $H^s(\mathbb{R}^d)$ and $\psi^{(0)}_{jk}$, $k \in \mathbb{Z}^d$, form an orthonormal basis for $V_j$, $j \in \mathbb{Z}$. Suppose that, for each $j \in \mathbb{Z}$, $m_p^{(0)}$ for $p = 1, 2, \ldots, 2^d - 1$ are such that matrix (6) is unitary. Define $\psi^{(0)}_{jk,p}$ by (4) for $p = 1, 2, \ldots, 2^d - 1$ and $j \in \mathbb{Z}$. Then $W_j = W_{j,1} \oplus W_{j,2} \oplus \cdots \oplus W_{j,2^d-1}$ with $W_{j,p} = \text{span}(2^{jd/2} \psi^{(0)}_{jk}(2^j x - k) : k \in \mathbb{Z})$, $p = 1, 2, \ldots, 2^d - 1$, is perpendicular to $V_j$ in $V_{j+1}$, and $V_{j+1} = V_j \oplus W_j$. Therefore

$$2^{jd/2} \psi^{(0)}_{jk}(2^j x - k), \quad k \in \mathbb{Z}, p = 1, 2, \ldots, 2^d - 1,$$

is an orthonormal basis for $H^s(\mathbb{R}^d)$.

Proof  First, we show that $\psi^{(0)}_{jk,p} \perp V_j$ for all $k \in \mathbb{Z}^d$ and $p = 1, 2, \ldots, 2^d - 1$. Indeed,

$$(2\pi)^d \langle \psi^{(0)}_{jk,p}(x), \psi^{(0)}_{jk}(x) \rangle_s = (2\pi)^d \langle 2^{jd/2} \psi^{(0)}_{jk}(2^j x - k_1), 2^{jd/2} \psi^{(0)}_{jk}(2^j x - k_2) \rangle_s = \int_{\mathbb{R}^d} (1 + \|\xi\|^2)^{1/2} \mathcal{F}(2^{jd/2} \psi^{(0)}_{jk}(2^j \xi - k_1)) \mathcal{F}(2^{jd/2} \psi^{(0)}_{jk}(2^j \xi - k_2)) d\xi = 2^{-jd} \int_{\mathbb{R}^d} (1 + \|\xi\|^2)^{1/2} \hat{\psi}^{(0)}_{jk}(2^{-j} \xi) \hat{\psi}^{(0)}_{jk}(2^{-j} \xi) e^{2\pi i \langle \xi, k - k_1 \rangle} d\xi = \int_{\mathbb{R}^d} (1 + 2^d \|\xi\|^2)^{1/2} m_p^{(j+1)}(\xi/2) \hat{\psi}^{(j+1)}_{jk}(\xi/2) \times m_0^{(j+1)}(\xi/2) \hat{\psi}^{(j+1)}_{jk}(\xi/2) e^{2\pi i \langle \xi, k - k_1 \rangle} d\xi = \int_{\mathbb{R}^d} \sum_{l \in \mathbb{Z}^d} (1 + 2^d \|\xi\|^2 + 2\pi l \|\xi\|^2) m_p^{(j+1)}(\xi/2 + \pi l) \hat{\psi}^{(j+1)}_{jk}(\xi/2 + \pi l) \times m_0^{(j+1)}(\xi/2 + \pi l) e^{2\pi i \langle \xi, k - k_1 \rangle} d\xi = \int_{\mathbb{R}^d} \sum_{q=0}^{2^d-1} m_p^{(j+1)}(\xi/2 + \gamma_q \pi) m_0^{(j+1)}(\xi/2 + \gamma_q \pi) e^{2\pi i \langle \xi, k - k_1 \rangle} d\xi$$

by Proposition 2.1. This expression is equal to zero because matrix (6) is unitary. Similarly, we can show that $W_{j,p_1} \perp W_{j,p_2}$ for all $p_1, p_2 \in \{1, 2, \ldots, 2^d - 1\}$.

We know show that $V_{j+1} = V_j \oplus W_{j,1} \oplus W_{j,2} \oplus \cdots \oplus W_{j,2^d-1}$ for any $f \in V_{j+1}$. We write

$$\hat{f}(\xi) = B(2^{-j-1} \xi) \hat{\psi}^{(j+1)}(2^{-j-1} \xi).$$

We will demonstrate that there exist $2\pi \mathbb{Z}^d$-periodic functions $G(2^{-j} \xi)$ and $H_p(2^{-j} \xi)$ such that

$$\hat{f}(\xi) = G(2^{-j} \xi) \hat{\psi}^{(j)}(2^{-j} \xi) + \sum_{p=1}^{2^d-1} H_p(2^{-j} \xi) \hat{\psi}^{(j)}_p(2^{-j} \xi).$$

Now, we have

$$B(\xi/2) \hat{\psi}^{(j+1)}(\xi/2) = G(\xi) \hat{\psi}^{(j)}(\xi) + \sum_{p=1}^{2^d-1} H_p(\xi) \hat{\psi}^{(j)}_p(\xi).$$
\[ = G(\xi) m^{(j+1)}(\xi/2) + \sum_{p=1}^{2d-1} H_p(\xi) m^{(j+1)}_p(\xi/2) + \sum_{p=1}^{2d-1} H_p(\xi) m^{(j+1)}_p(\xi/2) \phi^{(j+1)}(\xi/2). \]

It follows that
\[ B(\xi/2) = G(\xi) m^{(j+1)}(\xi/2) + \sum_{p=1}^{2d-1} H_p(\xi) m^{(j+1)}_p(\xi/2). \]

By the periodicity \((2\pi \mathbb{Z}^d)\)-periodic of \(G\) and \(H_p\) we have
\[ B(\xi/2 + \gamma q \pi) = G(\xi) m^{(j+1)}(\xi/2 + \gamma q \pi) + \sum_{p=1}^{2d-1} H_p(\xi) m^{(j+1)}_p(\xi/2 + \gamma q \pi) \]
for \(q = 0, 1, \ldots, 2^d - 1\). This completes proof. \(\Box\)

3 Multivariate box spline
Now we give an example of multivariate box splines in a Sobolev space. Using them, we construct a wavelet in \(H^s(\mathbb{R}^d)\).

Let \(D\) be the direction matrix of order \(d \times \sum_{i=1}^{d+1} m_i, m_i \in \mathbb{N}_0, \forall i\), whose column vectors consist of \((m_1, m_2, \ldots, m_{d+1})\) copies of the following \(d + 1\) column vectors: \((1, 0, \ldots, 0)^T, (0, 1, 0, \ldots, 0)^T, \ldots, (0, 0, \ldots, 1)^T, (1, 1, \ldots, 1)^T\).

Fix \(s \geq 0\) and the natural numbers \((m_1, m_2, \ldots, m_{d+1})\) such that
\[ \{m[D] := \min\{m_i + m_j : i \neq j \text{ for all } i, j = 1, 2, \ldots, d + 1\} + \frac{1}{2} > s. \]

Let \(M_{m_1, m_2, \ldots, m_{d+1}}\) be a multivariate box spline function defined in terms of the Fourier transform by
\[ \hat{M}_{m_1, m_2, \ldots, m_{d+1}}(\xi) = \prod_{j=1}^{d+1} \left( 1 - e^{-i k_j \xi} \right)^{m_j}, \quad k_j \in D, m_j \in \mathbb{N}_0, \forall j. \]

The multivariate box spline \(M_{m_1, m_2, \ldots, m_{d+1}}\) belongs to \(C^{|m[D]|-1}\), where \(m[D] + 1\) is the minimum number of columns that can be discarded from \(D\) to obtain a matrix of rank \(< d\) (see [15]).

For
\[ W_{m_1, m_2, \ldots, m_{d+1}}^{(j)}(\xi, l) := \sum_{l' \in \mathbb{Z}^d} \left( 1 + 2^j \|\xi + 2\pi l\|^2 \right)^{j} \left| \hat{M}_{m_1, m_2, \ldots, m_{d+1}}(\xi + 2\pi l) \right|^2, \]
it is known that there exist \(c, C \geq 0\) such that
\[ 0 \leq c \leq \sum_{l' \in \mathbb{Z}^d} \left| \hat{M}_{m_1, m_2, \ldots, m_{d+1}}(\xi + 2\pi l) \right|^2 \leq C < \infty. \]

Considering \(\xi := (\xi_1, \xi_2, \ldots, \xi_d)\) and \(l := (l_1, l_2, \ldots, l_d)\), we have
\[ \sum_{l' \in \mathbb{Z}^d} \left( 1 + 2^j \|\xi + 2\pi l\|^2 \right)^{j} \left| \hat{M}_{m_1, m_2, \ldots, m_{d+1}}(\xi + 2\pi l) \right|^2 \]
By mathematical induction we know that, for positive real numbers \(x_i, i = 1, \ldots, d\),

\[
\left(\sum_{i=1}^{d} x_i\right)^{m} \leq d^{m} \left(\sum_{i=1}^{d} x_i^{m}\right), \quad x_i \in \mathbb{R}. \tag{8}
\]

From (7) and (8) we have

\[
\sum_{(l_1,l_2,\ldots,l_d) \in \mathbb{Z}^d} \left(1 + 2^{2j} \sum_{i=1}^{d} |\xi_i + 2\pi l_i|^2\right)^{s} |\hat{M}_{m_1,m_2,\ldots,m_d,1}(\xi + 2\pi l)|^2 \\
\leq (d + 1)^j \left[c + 2^{2j} \sum_{(l_1,l_2,\ldots,l_d) \in \mathbb{Z}^d} \left(\sum_{i=1}^{d} |\xi_i + 2\pi l_i|^2\right)^{\frac{s}{2}} |\hat{M}_{m_1,m_2,\ldots,m_d,1}(\xi + 2\pi l)|^2\right] \\
\leq (d + 1)^j \left[c + 2^{2j} C \sum_{(l_1,l_2,\ldots,l_d) \in \mathbb{Z}^d} \left(\sum_{i=1}^{d} |\hat{M}_{m_i,m_d+1}(\xi + 2\pi l)|^2\right)^{\frac{s}{2}}\right] \\
\leq C_j < +\infty,
\]

where \(C', C_j > 0\), and \(m_i - s, m_{d+1}\) is the \(i\)th term subtracted by \(s\). Hence we have the following:

**Lemma 3.1** There exist two constants \(c_j\) and \(C_j\) such that

\[
0 < c_j \leq W_{m_1,m_2,\ldots,m_{d+1}}^{(j)}(\xi) \leq C_j < +\infty.
\]

Now, for every \(j \in \mathbb{Z}\), we define

\[
\hat{\varphi}^{(j)}(\xi) = \frac{\hat{M}_{m_1,m_2,\ldots,m_{d+1}}(\xi)}{\sqrt{W_{m_1,m_2,\ldots,m_{d+1}}^{(j)}(\xi)}}. \tag{9}
\]

Now we find a \(2\pi \mathbb{Z}^d\)-periodic function \(m_0^{(j)} \in L^2(\mathbb{Z}^d)\) for which the scaling relation (5) holds:

\[
\varphi^{(j)}(2\xi) = m_0^{(j+1)}(\xi)\varphi^{(j+1)}(\xi).
\]

From (9) we get

\[
m_0^{(j+1)}(\xi) = \frac{\hat{M}_{m_1,m_2,\ldots,m_{d+1}}(2\xi)}{\hat{M}_{m_1,m_2,\ldots,m_{d+1}}(\xi)} \frac{W_{m_1,m_2,\ldots,m_{d+1}}^{(j+1)}(\xi)}{W_{m_1,m_2,\ldots,m_{d+1}}^{(j)}(\xi)} \frac{W_{m_1,m_2,\ldots,m_{d+1}}^{(j+1)}(2\xi)}{W_{m_1,m_2,\ldots,m_{d+1}}^{(j)}(2\xi)} \\
= \prod_{i=1}^{d+1} \left(1 + e^{-i\langle k_i, \xi \rangle} \right) m_i \frac{W_{m_1,m_2,\ldots,m_{d+1}}^{(j+1)}(\xi)}{W_{m_1,m_2,\ldots,m_{d+1}}^{(j)}(2\xi)}.
\]
Finally, let us construct wavelets associated with the scaling function \( \phi_0 \), \( j \in \mathbb{Z} \). We define the \( 2\pi \mathbb{Z}^d \)-periodic functions \( m_p^j, p = 1, 2, \ldots, 2^{d-1}, \) by

\[
m_p^j(\xi) = e^{-i(\gamma_p \xi) L_p^j(2\xi)m_0^{p+1}(\xi + \gamma_p \pi)},
\]

where the trigonometric polynomial \( L_p^j \) is to be chosen such that \( m_p^j \) satisfies (6) for all \( p \).

**Proposition 3.2** Suppose \( \phi_0 \) is a scaling function for an MRA \( V_j, j \in \mathbb{Z} \), of \( H^s(\mathbb{R}^d) \) and \( m_0^j \) is the associated low pass filter. Then the distributions \( 2^{j/2} \phi_0(2^j \xi - k), j \in \mathbb{Z}, k \in \mathbb{Z}^d \), are an orthonormal basis for \( H^s(\mathbb{R}^d) \) if and only if

\[
\hat{\phi}_p^j(2\xi) = e^{-i(\gamma_p \xi) L_p^j(2\xi)m_0^{p+1}(\xi + \gamma_p \pi)}\phi_0^{(p+1)}(\xi), \quad \forall p = 1, 2, \ldots, 2^{d-1},
\]

a.e. on \( \mathbb{R}^d \) for some \( 2\pi \mathbb{Z}^d \)-periodic function \( L_p^j \) such that

\[
\left| L_p^j(\xi) \right| = 1, \quad \forall p = 1, 2, \ldots, 2^{d-1}, \text{a.e. } \xi \in \mathbb{T}^d.
\]

**Proof** From Proposition 2.1 we get

\[
\sum_{k \in \mathbb{Z}^d} \left( 1 + 2^{2d} \left| \xi + 2k \pi \right|^2 \right)^{1/2} \left| \hat{\phi}_p^j(\xi + 2k \pi) \right|^2 = 1, \quad \forall p = 1, 2, \ldots, 2^{d-1}.
\]

Now, we have only to verify the density condition

\[
\lim_{j \to +\infty} \left| \phi_0^j(2^{-j} \xi) \right| = \left( 1 + \left| \xi \right|^2 \right)^{-s/2}.
\]

By definition,

\[
W_{m_1,m_2,\ldots,m_d}^j(2^{-j} \xi) = \sum_{k \in \mathbb{Z}^d} \left( 1 + \left| \xi + 2^{-j} \pi k \right|^2 \right)^{1/2} \left| \hat{M}_{m_1,m_2,\ldots,m_d}^j(2^{-j} \xi + 2\pi k) \right|^2.
\]

The term associated with \( k = 0 \) converges to \( (1 + \left| \xi \right|^2)^s \). Using the estimates

\[
\left| \hat{M}_{m_1,m_2,\ldots,m_d}^j(2^{-j} \xi + 2\pi k) \right| = \left| \prod_{j=1}^{d+1} \frac{1 - e^{-i(k_j 2^{-j} \xi)}}{i(k_j 2^{-j} \xi + 2\pi k)} \right|^{m_j}
\]

for \( \xi = (\xi_1, \xi_2, \ldots, \xi_d) \) and

\[
\left| \prod_{j=1}^{d+1} \frac{1 - e^{-i(k_j 2^{-j} \xi)}}{i(k_j 2^{-j} \xi + 2\pi k)} \right|^{m_j} \leq \left( \prod_{j=1}^{d+1} \left| \sin(2^{-j+1} \xi_j) \right|^{m_j} \right) \left( \frac{\sin(2^{-j+1} \sum_{j=1}^d \xi_j)}{2^{-j+1} \sum_{j=1}^d \xi_j + d\pi k} \right)^{d+1} \left( \sum_{j=1}^d |\xi_j|^{m_j} \right)^{d+1}.
\]

Finally, let us construct wavelets associated with the scaling function \( \phi_0^j, j \in \mathbb{Z} \). We define the \( 2\pi \mathbb{Z}^d \)-periodic functions \( m_p^j, p = 1, 2, \ldots, 2^{d-1}, \) by

\[
m_p^j(\xi) = e^{-i(\gamma_p \xi) L_p^j(2\xi)m_0^{p+1}(\xi + \gamma_p \pi)},
\]
for $2^{j}(|\prod_{r=1}^{d}|\xi_{r}|^{m_{r}})(|\sum_{r=1}^{d}|\xi_{r}|^{m_{d+1}}) < 1$ and $k = 0$, we see that, as $j \to +\infty$, the sum of the other terms converges to 0. The conclusion follows easily.

If $\psi_{j}^{[p]}$ is an orthonormal wavelet, then the orthonormality of $(2^{j/2}\psi_{j}^{[p]}(2^{j} \cdot k) : j \in \mathbb{Z}, k \in \mathbb{Z}^{d}, p = 1, 2, \ldots, 2^{d-1})$ gives us

$$
1 = \sum_{k \in \mathbb{Z}^{d}} \left(1 + 2^{j} \|\xi + 2k\pi\|^{2}\right)^{2} |\hat{\psi}_{p}^{[j]}(\xi + 2k\pi)|^{2} \\
= \sum_{k \in \mathbb{Z}^{d}} \left(1 + 2^{j} \|\xi + 2k\pi\|^{2}\right)^{2} |\hat{L}_{p}^{[j]}(\xi)|^{2} |\hat{\phi}_{p}^{[j+1]}(\xi/2 + k\pi)|^{2} \\
\times |m_{0}^{[j+1]}(\xi/2 + k\pi + \gamma_{p}\pi)|^{2} \\
= |L_{p}^{[j]}(\xi)|^{2} \left(\sum_{k \in \mathbb{Z}^{d}} \left(1 + 2^{j} \|\xi + 2k\pi\|^{2}\right)^{2} |\hat{\phi}_{p}^{[j+1]}(\xi/2 + 2k\pi)|^{2} \\
\times \sum_{q=1}^{2^{d-1}} |m_{0}^{[j+1]}(\xi/2 + \gamma_{q}\pi)|^{2} + \sum_{l \in \mathbb{Z}^{d}} (1 + 2^{j} \|\xi + 2l\pi + \gamma_{q}\pi\|^{2})^{2} \\
\times |\hat{\phi}_{p}^{[j+1]}(\xi/2 + 2l\pi + \gamma_{q}\pi)|^{2} |m_{0}^{[j+1]}(\xi/2)|^{2}\right) \\
= |L_{p}^{[j]}(\xi)|^{2} \left(\sum_{q=0}^{2^{d-1}} |m_{0}^{[j+1]}(\xi/2 + \gamma_{q}\pi)|^{2}\right) = |L_{p}^{[j]}(\xi)|^{2}, \quad p = 1, 2, \ldots, 2^{d-1},
$$

for a.e. $\xi \in \mathbb{T}^{d}$ and $\gamma_{q}, \gamma_{p} \in E_{d}$, which finishes our proof. □

### 4 Conclusion

In this paper, we have successfully generalized MRA over higher-dimensional Sobolev spaces by giving orthonormality and density conditions. Further, we constructed nonseparable orthonormal wavelets in a higher-dimensional Sobolev space by using multivariate box splines. The main obstacle in constructing wavelets is constructing low-pass and high-pass filters with the help of multivariate box splines, which satisfy the condition of orthonormality in $H^{s}(\mathbb{R}^{d})$ for every scale $j$ (because the $H^{s}$-norm is not dilation invariant).

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Both authors contributed equally to the manuscript. Both authors read and approved the final manuscript.

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