REPRESENTATIONS OF LOW COPOLARITY
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Abstract. We classify irreducible representations of compact connected Lie groups whose orbit space is isometric to the orbit space of a representation of a compact Lie group of dimension 7, 8 or 9. They turn out to be closely related to symmetric spaces, with one exception only.

1. Introduction

A representation \( \rho : G \to O(V) \) of a compact Lie group \( G \) is called polar if there exists a subspace \( \Sigma \), called a section, that meets all \( G \)-orbits, and meets them always orthogonally. This notion was made explicit over 35 years ago and has many connections and ramifications, among them, with symmetric spaces, invariant theory and submanifold geometry (see e.g. [Dad85, DK85, BCO03]). For a polar representation as above, the space of orbits \( V/G \) can be recovered, as a metric space, as the space of orbits of a finite group action, namely, \( \Sigma/W \), where \( W \) is the largest subquotient group of \( G \) acting on the section \( \Sigma \). Indeed, as an easy consequence of O’Neill’s formula for Riemannian submersions, one sees that the later property characterizes polar representations (cf. [GL15], introd.).

Polar representations were classified by Dadok [Dad85]; it follows from this classification that a polar representation of a connected group has the same orbits as the isotropy representation of a symmetric space. The above considerations and the general importance of polar representations led the authors of [GL14] to seek other classes of orthogonal representations whose orbit spaces can be presented in different ways, and to investigate the mysterious interplay between geometric and algebraic aspects of representations. Namely, one says that two representations are quotient-equivalent if they have isometric orbit spaces; in addition, if the underlying group of one representation has dimension strictly less than the dimension of the underlying group of the other one, then one says the former representation is a reduction of the latter one. It follows that the polar representations of connected groups are precisely those that admit a reduction to a finite group action.

A minimal reduction of a given representation is a reduction whose underlying group has minimal dimension in that quotient-equivalence class; in this case, the dimension of this group is called the abstract copolarity of the given representation. For instance, polar representations of connected groups are precisely those representations of abstract copolarity zero. As one of the main results of [GL14], it was shown that if a non-polar irreducible representation of a compact connected
Lie group admitting a reduction has abstract copolarity at most six, then the representation is toric, that is, it has a minimal reduction to a representation of a finite extension of a torus. In the same paper, it was found a counterexample for abstract copolarity 7: \((U(3) \times \text{Sp}(2), \mathbb{C}^3 \otimes \mathbb{C}^4)\) reduces to a \(\mathbb{Z}_2\)-extension of \((\text{O}(3) \times \text{U}(2), \mathbb{R}^3 \otimes \mathbb{R}^4)\), where \(\text{O}(3) \times \text{U}(2)\) sits inside \(U(3) \times \text{Sp}(2)\) as a symmetric subgroup (see also [Gom21]). Toric irreducible representations were later classified in [GL15] (they are mostly related to Hermitean symmetric spaces; see also [Pan17]).

In the present work, we wanted to understand the extent of the above counterexample. As it turns out, we can show it is the only counterexample in abstract copolarity 7, and we go a little further:

**Theorem 1.1.** Let \(\rho : G \to \text{O}(V)\) be a non-polar, non-reduced, irreducible representation of a compact connected Lie group \(G\) on the Euclidean space \(V\). Assume the abstract copolarity of \(\rho\) is 7, 8 or 9. Then \(\rho\) is either toric, quaternion-toric, or equivalent to \((U(3) \times \text{Sp}(2), \mathbb{C}^3 \otimes \mathbb{C}^4)\).

A representation is called non-reduced if it admits a reduction; it is called quaternion-toric (or q-toric, for short) if it is non-polar and admits a reduction to a representation of a group whose identity component is isomorphic to \(\text{Sp}(1)^k\) for some \(k > 0\). Q-toric representations were classified in the irreducible case in [GG18] (they are related to quaternion-Kähler symmetric spaces); in particular, it was shown that \(k = 3\) always holds, so they have abstract copolarity 9. In particular, an irreducible representation of a connected Lie group with abstract copolarity 8 must be toric.

Another consequence of Theorem 1.1 is that, for the representations in the theorem, the abstract copolarity coincides with the copolarity. The copolarity of a representation \(\rho : G \to \text{O}(V)\) is the minimal possible dimension of \(H\), where \(\tau : H \to \text{O}(W)\) is an arbitrary reduction of \(\rho\) such that \(H\) is a subgroup of \(G\), \(W\) is a subspace of \(V\), and \(\tau\) is the restriction of \(\rho\). Thus the copolarity considers only reductions “embedded” in the given representation; in particular, the abstract copolarity is bounded above by the copolarity. It is an open problem to decide whether they always coincide.

In view of the results and calculations in the present paper, one is tempted to formulate the following conjecture: Suppose \(G\) is connected and \(\rho : G \to \text{O}(V)\) is a non-polar, non-reduced, irreducible representation that reduces to \(\tau : H \to \text{O}(W)\). Then \(H^0\) is not simple.

The results in this paper are part of the PhD thesis of the first author.

2. Preliminaries

Let \(\rho : G \to \text{O}(V)\) be a representation of a compact Lie group \(G\). Then \(V\) inherits a \(G\)-invariant stratification by orbit types, namely, two points in \(V\) are in the same stratum if and only if they have conjugate isotropy groups. This stratification projects to a stratification of the orbit space \(X = V/G\). The strata in \(X\) are locally convex Riemannian manifolds, and their connected components can be recognized metrically as points in \(X\) with isometric tangent spaces. The maximal dimensional stratum \(X_{\text{reg}}\) is unique, convex, open and dense, and consists of the principal orbits, namely, those orbits with minimal isotropy groups, called principal isotropy groups. The cohomogeneity of \(\rho\) is the codimension of the principal orbits,
which also coincides with the topological dimension of $X$. The closure of the union of strata of codimension one in $X$ is called the boundary of $X$, and it is denoted by $\partial X$. We say that $p \in V$ is $G$-principal if it lies in a $G$-principal orbit, and it is $G$-important if it projects to a stratum of codimension one in $X$. Locally at $p \in V$, the orbit decomposition of $V$ is completely determined by the slice representation of the isotropy group $G_p$ on the normal space $\nu_p(Gp)$ to the orbit $Gp$: namely, the tangent cone $T_p X$ is isometric to the orbit space of the slice representation at $p$, where $x$ is the projection of $p$; in particular, the fixed point set of the slice representation is tangent to the stratum of $p$ in $V$, and the cohomogeneity of the slice representation modulo the fixed point set is equal to the codimension of the stratum of $x$ in $X$.

The strategy to investigate reductions of representations introduced in [GL14] is based on the following dichotomy. Suppose $\rho : G \to O(V)$ is a minimal reduction of $\tau : H \to O(W)$, where $H$ is connected. Then the principal isotropy groups of $\rho$ are trivial, and: (i) either $G$ is connected, and $V/G = W/H$ has non-empty boundary [GL14 Proposition 5.2]; (ii) or $G$ is disconnected, and $G/G^0$ is generated by nice involutions, that is, elements of $G$ of order 2 that fix a $G^2$-principal and $G$-important point, and act on $V/G^0$ as a reflection [GL14 Proposition 1.2 and § 4.3].

In case (i), the boundary components arise as $S^a$-isotropy types, where $a = 1$ or $a = 3$. More precisely, if $p \in V$ is a $G$-important point, then $G_p$ is a sphere $S^a$ and the slice representation, modulo the fixed point set, is $(S^1, \mathbb{C})$ or $(S^3, \mathbb{R})$. In case (ii), the boundary components arise as $S^0$-isotropy types, where $S^0 = \mathbb{Z}_2$, and the slice representation at a corresponding important point, modulo the fixed point set, is $(\mathbb{Z}_2, \mathbb{R})$. In any case, we have the dimension formula

$$\text{(2.1)} \quad \dim V - a - 1 = \dim G - \dim N_G(G_p) + \dim V^{G_p},$$

where $N_G(G_p)$ denotes the normalizer of $G_p$ in $G$, and $\dim V^{G_p}$ denotes the fixed point set of $G_p$ in $V$.

3. Abstract copolarity 7

We proceed to prove Theorem 1.1 using the method outlined in section 2. In this section, we address the case of abstract copolarity 7. So assume $\tau : H \to O(W)$ is a non-polar irreducible representation of a compact connected Lie group $H$ that reduces to $\rho : G \to O(V)$, where $\dim G = 7$ and $G^0$ is not Abelian, and $\rho$ is a minimal reduction.

Recall that $\rho$ is irreducible, since $\tau$ is assumed irreducible, and the irreducibility of a representation can be read off its quotient [GL14, § 5.2]. If the restriction of $\rho$ to the identity component $G^0$ were reducible, then $\tau$ would be toric [GL14 Theorem 1.7], so we know that also $\rho|_{G^0}$ is irreducible. Now the center of $G^0$ can be at most one-dimensional, so $G^0$ is locally isomorphic to $\text{U}(1) \times \text{SU}(2) \times \text{SU}(2)$.

3.1. Connected case. Herein we assume $G$ is connected. Then $V/G$ has non-empty boundary. Let $p \in V$ be a $G$-important point. We first rule out the case $G_p = S^3$. Indeed a $S^3 = \text{SU}(2)$-subgroup of $G$ must correspond to one of the SU(2)-factors in $G$, or the diagonal in $\text{SU}(2) \times \text{SU}(2)$. In any case, the nontrivial element $z$ in the center of $G_p$ is a central element of $G$. The fixed point set of $z$ is not zero, as it contains $p$, and it is preserved by $G$; but this contradicts the irreducibility of $G$. 

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Now $G_p = S^1$. Since the rank of $G$ is 3, $\dim N_G(G_p) \geq 3$. By the dimension formula (2.1),

$$\dim V \leq 6 + \dim V^{G_p}.$$  

Since $G$ contains a central circle, $\rho$ is a representation of complex type. For a maximal torus of $G$ containing $G_p$, we have that $V^{G_p}$ is a sum of weight spaces; more precisely, those weight spaces whose weights lie in the annihilator of a generator of the Lie algebra of $G_p$. Since the dimension of the annihilator is two, there are at most two linearly independent weights appearing in $V^{G_p}$. Next, note that the restriction of any weight to the central circle in $G$ is independent of the weight. It follows that there are no linearly dependent weights appearing in $V^{G_p}$. We deduce that there are at most two weights appearing in $V^{G_p}$; hence

$$\dim V \leq 10.$$

Now the cohomogeneity of $\rho$ is at most 3. It is known that such representations must be either polar or of abstract copolarity 1 ([Str94, Theorem 5.1], [GOT04, Theorem 1.1] and [GL14, Corollary 1.6 and Example 1.9]). Therefore this case is not possible.

3.2. **Disconnected case.** Now we assume $G$ is disconnected. Then there exists a nice involution $w \in G$, namely, $w^2 = I$ and

$$\dim V = 8 - \dim Z_{G^0}(w) + \dim V^w,$$

where $Z_{G^0}(w)$ denotes the centralizer of $w$ in $G^0$ and $V^w$ denotes the fixed point set of $w$ in $V$.

The conjugation of $G^0$ by $w$ defines an involutive automorphism $\varphi$ of $G^0$, namely,

$$\rho(\varphi(g)) = w\rho(g)w^{-1}$$

for all $g \in G^0$. Since $w \not\in \rho(G^0)$, and the centralizer of $\rho(G^0)$ in $\mathcal{O}(V)$ coincides with its center, $\varphi$ cannot be an inner automorphism of $G^0$.

We may lift $\rho|_{G^0}$ to the universal covering and assume $G^0 = U(1) \times SU(2) \times SU(2)$. Any automorphism of this group must preserve the central circle and permute the $SU(2)$-factors. Since $SU(2)$ has no outer automorphisms, the group of outer automorphisms $\text{Aut}(G^0)/\text{Inn}(G^0)$ is generated by the commuting involutions $\sigma_1$, $\sigma_2 : G^0 \to G^0$, where $\sigma_1(t, g_1, g_2) = (\bar{t}, g_1, g_2)$ and $\sigma_2(t, g_1, g_2) = (z, g_2, g_1)$. We may replace $\sigma_1$ by an appropriate $G^0$-conjugate and assume $\sigma_1(t, g_1, g_2) = (t, \bar{g}_1, g_2)$.

Note that $V = \mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}^m \otimes_{\mathbb{C}} \mathbb{C}^n$ as a $U(1) \times SU(2) \times SU(2)$-representation. We have $\sigma_1$ is induced by conjugation by $\iota \in \mathcal{O}(V)$, where $\iota$ is complex conjugation of $V$ over the real form $\mathbb{R} \otimes_{\mathbb{R}} \mathbb{R}^m \otimes_{\mathbb{R}} \mathbb{R}^n$, namely,

$$\rho(\sigma_1(g)) = \iota \rho(g) \iota^{-1}$$

for all $g \in G^0$.

In case $m = n$, also $\sigma_2$ is induced by conjugation by $\iota \in \mathcal{O}(V)$, where $\iota(z, z_1, z_2) = (z, \bar{z}_2, z_1)$, namely,

$$\rho(\sigma_2(g)) = \iota \rho(g) \iota^{-1}$$

for all $g \in G^0$.

Write $\varphi = \text{Inn}_h \circ \sigma$, where $1 \neq \sigma \in \langle \sigma_1, \sigma_2 \rangle$ and $h \in G^0$, and $\text{Inn}_h$ denotes the inner automorphism defined by $h$. We may assume $h$ has no component in the central circle. Now $\varphi^2 = I$ implies $h \sigma(h)$ lies in the center $Z(G^0)$ of $G^0$. Since
either $(1, -1, 1)$ or $(-1, -1, 1)$ (resp. $(1, 1, -1)$ or $(-1, 1, -1)$) lies in the kernel of $\rho$, we can write

\[ \sigma(h) = h^{-1}. \]

If $\varphi$ permutes the SU(2)-factors of $G^0$, i.e. $\sigma$ involves $\sigma_2$, then $m = n$. In particular, if $\sigma = \sigma_2$, then (3.6) implies that $h = (1, h_1, h_1^{-1}) \in G_0$ for some $h_1 \in \text{SU}(2)$. Replacing $w$ by $\rho(1, 1, h_1)w\rho(1, 1, h_1)^{-1}$, we may assume that $\varphi = \sigma_2$. It follows from (3.3) and (3.5) that $w$ centralizes $\rho(G_0)$. By Schur’s lemma, $w = \lambda i$ for some $\lambda \in \text{U}(1)$, and from $w^2 = I$ we deduce that $w = \pm i$. In any case, $\dim Z_{G^0}(w) = 4$, so (3.2) gives $2n^2 = 4 + \dim V^w$. Note that $V^w$ is the space of symmetric (resp. skew-symmetric) tensors, so $\dim V^w = n(n + 1)$ (resp. $n(n - 1)$). It follows that $n^2 \pm n = 4$, but these equations have no integer solutions, so $\sigma = \sigma_2$ is impossible.

Similarly, if $\sigma = \sigma_1\sigma_2$, then (3.6) implies that $h = (1, h_1, h_1^{-1}) \in G_0$ for some $h_1 \in \text{SU}(2)$. Replacing $w$ by $\rho(1, 1, h_1)w\rho(1, 1, h_1)^{-1}$, we may assume that $\varphi = \sigma_1\sigma_2$. It follows from (3.4) and (3.5), together with Schur’s lemma and $w^2 = I$, that $w = \pm \epsilon i$. In any case, $\dim Z_{G^0}(w) = 3$ so (3.2) gives $2n^2 = 5 + \dim V^w$. Note that $V^w$ is the space of Hermitian (resp. skew-Hermitian) tensors, so $\dim V^w = n^2$. It follows that $n^2 = 5$, but this equation has no integer solutions, so $\sigma = \sigma_1\sigma_2$ is impossible.

It remains the case $\sigma = \sigma_1$. Using (3.6), we obtain $h = (1, h_1, h_2)$, where $h_i \in \text{SU}(2)$ is a symmetric matrix, for $i = 1, 2$. Taking the imaginary part of $h_1h_1^* = h_2h_2^* = 1$ shows that $\text{Re} h_1$ and $\text{Re} h_2$ are commuting (real symmetric) matrices. Therefore there is $k_i \in \text{SO}(2)$ such that $k_1h_1k_1^{-1}$ is a diagonal matrix $d_i$. Now let $d_i^{1/2}$ be a square root of $d_i$ and put $a_i = d_i^{-1/2}k_i$. Then $a_1a_2 = 1$. Let $a = (a_1, a_2)$. Then $\varphi(a) = ha$. Using this, we see that by replacing $w$ by $\rho(a)w\rho(a)^{-1}$, $\varphi$ gets replaced by $\sigma_1$. Therefore we may assume $\varphi = \sigma_1$. Now $w$ centralizes $\rho(G^0)$, so Schur’s lemma and $w^2 = I$ yield that $w = \pm \epsilon i$. In any case, $\dim Z_{G^0}(w) = 2$ and $\dim V^w = \frac{1}{2} \dim V$, so (3.2) yields that $\dim V = 12$. This is the representation $(\text{U}(2) \times \text{SU}(2), \mathbb{C}^2 \otimes \mathbb{C}^3)$ or, equivalently, $(\text{U}(2) \times \text{SO}(3), \mathbb{R}^4 \otimes \mathbb{R} \mathbb{R}^3)$, which is the representation in the statement of Theorem 1.1.

4. Abstract copolarity 8

In this section we assume $\tau : H \to \text{O}(W)$ is a non-polar irreducible representation of a compact connected Lie group $H$ that reduces to $\rho : G \to \text{O}(V)$, where $\dim G = 8$ and $G^0$ is not Abelian, and $\rho$ is a minimal reduction. As in the previous section, $\rho|_{G^0}$ must be irreducible, and hence $G^0$ must be covered by $\text{SU}(3)$.

4.1. Connected case. Herein we assume $G$ is connected. We will show that $V/G$ must have empty boundary and hence $\rho$ cannot be a reduction of $\tau$. This follows immediately from the following result from [GKW21]: Every irreducible representation of a compact connected simple Lie group with non-empty boundary in the orbit-space must be polar, toric, q-toric or a half-spin representation of $\text{Spin}(11)$. In the sequel we give an alternative argument.

We first note that there are no $S^2$-boundary components. Indeed it follows from [Sch80, Corollary 13.4] that a necessary condition for the existence of boundary is that the Dynkin index of the complexification $\rho^C$ is less than one. But [AVÉ67, Table 1] implies that such a representation would be polar.
Now, let $p$ be a $G$-important point projecting to a $S^1$-boundary component. Since the rank of $G^0$ is 2, the dimension formula (2.1) says
\begin{equation}
\dim V \leq 8 + \dim V^{G_p}.
\end{equation}
We next estimate $\dim V^{G_p}$. Consider a maximal torus $T$ of $G^0$ that contains $G_p$, so that $V^{G_p}$ is a sum of weight spaces. Recall that the diagram of weights of $V$ look like a sequence of concentric hexagons followed by a sequence of concentric triangles [FH91, p. 183-184]. Since the Lie algebra of $G_p$ has codimension one in the Lie algebra of $T$, the weights associated to weight spaces appearing in $V^{G_p}$ all lie in a line through the origin. This line can meet at most two weights in each hexagon (resp. triangle).

Consider first the case $\rho$ is the realification of $\pi_{a,b}$, where $a > b$, $\pi_{a,b}$ is the complex irreducible representation of $SU(3)$ of highest weight $a\lambda_1 + b\lambda_2$, and $\lambda_1$, $\lambda_2$ denote the fundamental weights. Then the diagram of weights consists of hexagons $H_0, \ldots, H_{b-1},$ and triangles $T_0, \ldots, T_{\sigma_b}$, the multiplicities of weights on $H_i$ being $i + 1$, and the multiplicities of weights on any triangle being $b + 1$. We obtain the rough estimate
\[ \dim V^{G_p} \leq 2b(b + 1) + \frac{4}{3}(b + 1)(a - b). \]
The Weyl dimension formula gives $\dim V = (a + 1)(b + 1)(a + b + 2)$, and one easily sees that the only solution to (4.7) is $(a, b) = (1, 0)$, which gives the vector representation, a polar representation.

Consider next the case $\rho$ is a real form of $\pi_{a,a}$. Then the diagram of weights consists of hexagons $H_0, \ldots, H_{a-1},$ and triangles $T_0, \ldots, T_{\sigma_a}$, the multiplicities of weights on $H_i$ being $i + 1$, and a point $H_a$ with multiplicity $a + 1$. We obtain
\[ \dim V^{G_p} \leq (a + 1)^2. \]
Since $\dim V = (a + 1)^3$, the only solution to (4.7) is $a = 1$, the adjoint representation, which is polar.

4.2. Disconnected case. Herein we assume $G$ is disconnected. We will show that $V/G$ must have empty boundary and hence $\rho$ cannot be a reduction of $\tau$.

There is a nice involution $w \in G$, so that $w^2 = I$ and
\begin{equation}
\dim V = 9 - \dim Z_{G^0}(w) + \dim V^w.
\end{equation}
As in subsection 3.2, the conjugation of $G^0$ by $w$ defines an automorphism $\varphi$ of $G^0$. We will deal separately with the cases in which $\varphi$ is of inner or outer type.

4.2.1. Outer type. It follows from [Wol11] that, replacing $w$ by a $\rho(G^0)$-conjugate, we may assume $\varphi = \sigma$, where $\sigma$ is the Weyl involution, given by $\sigma(g) = \bar{g}$ for all $g \in G^0$.

We first check $\rho|_{G^0}$ cannot be absolutely irreducible. Indeed, if this is the case, the complexification $(\rho|_{G^0})^c$ is an irreducible. Denote by $\epsilon$ the complex conjugation of $V^c$ over $V$. Then $\epsilon \circ \pi \circ \epsilon$ and $\pi \circ \sigma$ are equivalent representations. So there is a unitary transformation $A$ of $V^c$ such that
\[ \pi \circ (g) \pi A^{-1} = \pi \circ (\sigma(g)) = w \pi(g) w^{-1}, \]
for all $g \in G^0$, where we have denoted the complex linear extension of $w$ to $V^c$ by the same letter. This says that $(A \rho w)$ centralizes $\pi(G^0)$, so Schur’s lemma yields that $\varphi$ is a complex multiple of $A\epsilon$. We reach a contradiction, as $\epsilon A$ is not a complex transformation of $V^c$. 

Now $\rho|_{G^0}$ is not absolutely irreducible. Then it is the realification of a representation $\pi$ of $G^0$ on a complex vector space $U$. Let $\epsilon$ be the conjugation of $U$ over any real form. Then $\epsilon \circ \pi \circ \epsilon$ and $\pi \circ \sigma$ are equivalent representations, so as above we can write $w = \lambda \epsilon A$, for some $\lambda \in S^1$, and some unitary transformation $A$ of $U$. We can replace $w$ by $\lambda^{-1/2}w\lambda^{1/2} = \epsilon A$. Next, since $\epsilon$ is conjugate linear, we have
\[ I = w^2 = (\lambda \epsilon A)^2 = (\epsilon A \epsilon)A = AA, \]
that is, $A$ is unitary and symmetric. Write $A = BB^t$, where $B$ is a unitary matrix, similar what was done in the end of section 3.2. Then $B^t w(B^t)^{-1} = B^{-1}\epsilon A(B^{-1}) = \epsilon B^{-1}(BB^t)(B^t)^{-1} = \epsilon$, so we may indeed assume $w = \epsilon$. Since $w$ is conjugate linear, we have $\dim V^w = \frac{1}{2} \dim V$. Further, $\dim Z_{G^0}(w) = \dim SO(3) = 3$, so the dimension formula (4.8) yields $\dim V = 12$. This implies that the cohomogeneity of $\rho$, and hence also of $\tau$, is 4, but all non-polar, non-reduced, irreducible representations of cohomogeneity 4 of compact connected Lie groups have abstract copolarity 2 [GL14 Theorem 1.11].

4.2.2. Inner type. Write $w = \rho(h)z$, where $h \in G^0$, and $I \neq z \in O(V)$ centralizes $\rho(G^0)$. From $w^2 = I$ we deduce that $h^2$ lies in the center $Z(G^0)$. If $h \in Z(G^0)$ then $\rho(h)$ and $z$ are scalar maps, and so $w = \pm I$, which cannot be. Now $h \notin Z(G^0)$, and we can conjugate $h$ to a diagonal matrix and assume $h = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}$, where $\theta$ is a cubic root of 1. It follows that $\dim Z_{G^0}(w) = \dim Z_{G^0}(h) = 4$, and the dimension formula gives $\dim V - \dim V^w = 5$ is odd. This already implies that $\rho$ cannot leave invariant a complex structure on $V$. Now Schur’s lemma yields $z = -I$, so $w = -\rho(h) = -\rho(\theta \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}) = \pm \rho(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix})$, and we get $\dim V^h = 5$.

We know that $\rho$ is a real form of $\pi_{a,a}$ (notation as in subsection 4.1). Consider the maximal torus of $G^0$ consisting of diagonal matrices, so $(V^c)^h = (V^h)^c$ is a sum of weight spaces. It is clear that the set of weights associated to weight spaces in $(V^c)^h$ is invariant under multiplication by $-1$. Since the multiplicity of the zero weight in $\pi_{a,a}$ is $a + 1$, we deduce that $5 = \dim V^h = \dim C(V^c)^h \geq a + 1$, and $a$ cannot be an odd number. We are only left to examine the cases $a = 2$ and $a = 4$.

Let $e_1, e_2, e_3$ denote the canonical basis of $C^3$ and let $e'_1, e'_2, e'_3$ denote the dual basis of $C^3^*$. The highest weight vector of $\pi_{a,a}$ is $e''_a \otimes e''_a$, which is fixed by $h$ for all $a$. Using the action of the Weyl group, the vectors $e''_j \otimes e''_j$ lie in $V^c$ for $j = 1, 2, 3$, and they are clearly also fixed by $h$, together with the corresponding weight vectors with the opposite weights, and the zero weight space. This gives the estimate $\dim V^h \geq 3 \cdot 2 + (2 + 1) = 9$ for $a \geq 2$, which contradicts $\dim V^h = 5$.

5. Abstract copolarity 9

In this section we assume $\tau : H \to O(W)$ is a non-polar irreducible representation of a compact connected Lie group $H$ that reduces to $\rho : G \to O(V)$, where $\dim G = 9$ and $G^0$ is not Abelian, and $\rho$ is a minimal reduction. As in the previous section, $\rho|_{G^0}$ must be irreducible, and hence $G^0$ must be locally isomorphic to $U(3)$ or $Sp(1)^3$, but the latter case is q-toric, so we need not consider it. Now $\rho$ is a representation of complex type.

5.1. Connected case. Herein we assume $G$ is connected, so $G$ is covered by $U(1) \times SU(3)$. We will show that $V/G$ must have empty boundary and hence $\rho$ cannot be a reduction of $\tau$. 

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Note that if \( p \) is a \( G \)-important point that projects to a \( S^3 \)-boundary component, then the restriction of \( p \) to the simple subgroup \( SU(3) \) is a (non-orbit equivalent) representation which is either irreducible of complex type, or with two equivalent irreducible components of real type, and which also has \( p \) as an important point projecting to a \( S^3 \)-boundary component. By \([\text{Sch}80, \S13]\) and \([\text{AV} \acute{E}67, \text{Table 1}]\), there are no possibilities for \( \rho \).

Now, let \( p \) be a \( G \)-important point projecting to a \( S^1 \)-boundary component. Since the rank of \( G \) is 3, the dimension formula (2.1) says
\[
\dim V \leq 8 + \dim V^G \rho.
\]
Again, since \( G \) has rank 3 and contains a central circle, as in subsection 3.1 we see that \( V^G \rho \) is a sum of at most two weight spaces. Suppose \( \rho \) is the realification of \( \pi_{a,b} \). Then the highest multiplicity of a weight is \( b+1 \), so \( \dim V^G \rho \leq 4b+4 \), and (5.9) implies
\[
(a+1)(b+1)(a+b+2) \leq 12 + 4b.
\]
The only solution is \((a, b) = (1, 0)\), which corresponds to a polar representation.

5.2. **Disconnected case.** Herein we assume \( G \) is disconnected. We will show that \( V/G \) must have empty boundary and hence \( \rho \) cannot be a reduction of \( \tau \).

There is a nice involution \( w \in G \), so that \( w^2 = I \) and
\[
\dim V = 10 - \dim Z_{G^0}(w) + \dim V^w.
\]
As in subsection 3.2, the conjugation of \( G^0 \) by \( w \) defines an automorphism \( \varphi \) of \( G_0 \). Since \( w \notin \rho(G^0) \), and the centralizer of \( \rho(G^0) \) in \( O(V) \) coincides with its center, \( \varphi \) must be an outer automorphism of \( G^0 \).

Since the center of \( \rho(G^0) \) acts by scalars and is normalized by \( w \), we may assume that \( G^0 = U(3) \), up to orbit-equivalence. Now the group of outer automorphisms of \( G^0 \) is generated by \( \sigma \), where \( \sigma(g) = \bar{g} \) for \( g \in U(3) \). Similar to the situation in subsection 4.2.1 we may replace \( w \) by a conjugate and assume \( \varphi = \sigma \). Here we already know \( \rho|_{G^0} \) is not absolutely irreducible. Continuing with the argument in subsection 4.2.1 we may assume \( w = \epsilon \). Now \( \dim Z_{G^0}(w) = 3 \), so the dimension formula (5.10) yields \( \dim V = 14 \). However, there exist no irreducible representations of dimension 14 of \( U(3) \). This finishes the proof of Theorem 1.1.

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