Derivation of the probability distribution function for the local density of states of a disordered quantum wire via the replica trick and supersymmetry

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Abstract

We consider the statistical properties of the local density of states of a one-dimensional Dirac equation in the presence of various types of disorder with Gaussian white-noise distribution. It is shown how either the replica trick or supersymmetry can be used to calculate exactly all the moments of the local density of states. Careful attention is paid to how the results change if the local density of states is averaged over atomic length scales. For both the replica trick and supersymmetry the problem is reduced to finding the ground state of a zero-dimensional Hamiltonian which is written solely in terms of a pair of coupled “spins” which are elements of $u(1,1)$. This ground state is explicitly found for the particular case of the Dirac equation corresponding to an infinite metallic quantum wire with a single conduction channel. The calculated moments of the local density of states agree with those found previously by Al’tshuler and Prigodin [Sov. Phys. JETP 68 (1989) 198] using a technique based on recursion relations for Feynman diagrams.

Key words: disordered systems, replica trick, supersymmetry, path integral, localization, mesoscopics
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1 Introduction

In the presence of disorder mesoscopic systems such as quantum wires and quantum dots have electronic properties which are characterised by large sta-
istical fluctuations between different samples [1]. For example, the conductance of a typical sample can be quite different from the ensemble averaged conductance. A widely studied phenomena, both experimentally and theoretically, is that of universal conductance fluctuations: the root mean square of the conductance is of order $e^2/h$ (where $e$ is the electronic charge and $h$ is Planck's constant) and is independent of the strength of the disorder or the sample size [2].

Field theoretical techniques based on supersymmetry and the replica trick have proven to be extremely useful in studying non-perturbative effects associated with disorder in non-interacting electron systems. Supersymmetry has been used to evaluate exactly the disorder averaged density of states for a one-dimensional Schrödinger equation with a random white-noise potential [3], various random one-dimensional Dirac equations [4–7], and a number of models associated with the lowest Landau level of a two-dimensional electron gas in a high magnetic field [8]. The replica trick has been used to evaluate exactly the disorder averaged density of states of the same one-dimensional models involving the Schrödinger equation [9], and Dirac equation [10,7]. Non-linear sigma models based on either the replica trick [11] or supersymmetry [12–15] have also proven to be extremely powerful tools for studying Anderson localisation in non-interacting electron systems. Supersymmetric spin chains describe the delocalisation transition associated with the quantum Hall effect [16].

In a disordered electronic system static fluctuations in the local electric potential will produce statistical fluctuations in the local density of states. The corresponding probability distribution function was first studied by Al'tshuler, Kravtsov, and Lerner for the case of two and $2 + \epsilon$ dimensions [17]. Their results were based on a perturbative renormalisation group analysis of the non-linear sigma model introduced by Wegner [11]. They found that near the mean value of the density of states the distribution function was Gaussian but slowly decayed with tails of logarithmically normal form. Near the metal-insulator transition due to Anderson localisation the distribution approached a logarithmically normal form for all values of the local density of states. Mirlin and Fyodorov [14,15] have developed a supersymmetric formalism to calculate all the moments of the local density of states for a graded non-linear sigma model relevant to disordered systems in one, two, and three dimensions and a quasi-one-dimensional wire. A novel path-integral approach was used by Kolokolov to calculate the probability distribution function for the inverse participation ratio of both a finite and an infinite disordered one-dimensional system [18].

Al'tshuler and Prigodin [19] evaluated exactly all of the moments of the local density of states for a one-dimensional metal. They applied the Berezinskii diagrammatic technique [20] to the microscopic Hamiltonian. From the moments it is straight-forward to construct the complete probability distribution
function. They also showed that due to the resulting distribution in Knight shifts the distribution function could be related to the line shape of a nuclear magnetic resonance (NMR) line. With decreasing temperature and increasing disorder the line shape becomes increasingly asymmetric, with large differences between the average and typical values. Recently, the results of Reference [19] were generalized to the case of a finite-sized ring threaded by a magnetic flux [21]. Efetov and Prigodin have used the supersymmetry method to calculate the NMR line shape for a small metallic particle which corresponds to the case of zero dimensions [22,23].

Due to advances in nanotechnology it is now possible to experimentally test some of the theoretical predictions concerning fluctuations in the local density of states. Recent nuclear magnetic resonance experiments on monodisperse molecular metal clusters of platinum atoms have tested the theoretical predictions for zero-dimensions and find a broad asymmetric line but that the density of states fluctuations are obscured by additional contributions from surface effects [24]. Experiments have imaged the local density of states in a three-dimensional disordered metal (heavily doped GaAs) by measuring the differential conductance associated with resonant tunneling through impurity states in asymmetric double-barrier heterostructures [25]. Fluctuations in the local density of states were detected and found to be enhanced in a magnetic field [26].

In this paper we show in detail how both the supersymmetry and replica trick methods can be used to derive all the moments of local density of states for a random one-dimensional Dirac equation. Balents and Fisher [6] recently used supersymmetry to evaluate the disorder-average of the one-particle Green function for a one-dimensional Dirac equation with a random mass. Bouquet [7] used group theory to simplify their analysis and also treated the problem using the replica trick. Our analysis extends their formalism to allow for the evaluation of the disorder average of products of Greens functions. This is explicitly evaluated for the case of a quantum wire. Our results agree with those of Reference [19]. However, in our view the field theoretic method used here is more transparent and less cumbersome than the Berezinskii technique.

In section 2 we introduce the model Hamiltonian, the one-dimensional random Dirac equation and briefly mention the different physical systems it is relevant to, ranging from quantum wires to random spin chains. In section 3 we introduce different forms of the local density of states and define the generalized participation ratios that are a measure of the spatial extent of localised wave functions. We show how the the moments of the density of states and the participation ratios are related to one another. These quantities can be expressed in terms of the disorder average of products of retarded and advanced Green’s functions. Section 4 shows how these products of Green’s functions may be written in path integral form. Either the replica trick or
supersymmetry can then be used to evaluate exactly the average over disorder. One is then left with a one-dimensional interacting field theory. Section 5 shows how a transfer matrix approach can be used to reduce this to studying a zero-dimensional transfer Hamiltonian. For both the replica trick and supersymmetry this Hamiltonian involves a pair of “spin” operators from the non-compact algebra $su(1, 1)$. Section 6 shows how the moments of the local density of states can be written in terms of the expectation value of components of these operators in the ground state of the transfer Hamiltonian. In section 7 we find the ground state for the particular case of the random Dirac equation corresponding to a quantum wire. This is then used to evaluate all the moments of the density of states. In section 8 we make some corrections to the moments of the density of states. In section 9 we use the moments of the density of states to calculate the distribution of the density of states.

2 The model

The one dimensional random Dirac equation Hamiltonian, $h$, for a relativistic particle with mass $M$ in a magnetic field $\Phi$ and in a scalar potential $V$ is a $2 \times 2$ matrix

$$h = -i \sigma_z \partial_x + \Phi(x) \sigma_x + M(x) \sigma_y + V(x). \quad (1)$$

The $\sigma_j$’s are Pauli spin matrices. For constant $\Phi$, $M$ and $V$ this model describes a one dimensional semiconductor when particles are near the Fermi energy with a narrow forbidden band. The Hamiltonian is a $2 \times 2$ matrix because it describes particles moving in both directions in one dimension. The wave function is of the form $\Psi = (\psi_\uparrow, \psi_\downarrow)$ where $\psi_\uparrow$ and $\psi_\downarrow$ correspond to right and left moving electrons, respectively.

The variables $\Phi$, $M$ and $V$ are random variables with a one dimensional spatial dependence. We can separate each of these random variables into an average part and a random part

$$\Phi(x) = \phi + \tilde{\phi}(x), \quad M(x) = m + \tilde{m}(x), \quad V(x) = v + \tilde{v}(x). \quad (2)$$

The random part of each of the three variables is assumed to have a Gaussian white noise distribution centred about zero. If $D_\phi$ is the disorder strength of $\tilde{\phi}(x)$

$$\langle \tilde{\phi}(x) \tilde{\phi}(x') \rangle = 2D_\phi \delta(x - x'), \quad \langle \tilde{\phi}(x) \rangle = 0, \quad (3)$$

and similarly for $\tilde{m}(x)$ and $\tilde{v}(x)$. The disorder strength of $\tilde{m}(x)$ and $\tilde{v}(x)$ is $D_m$ and $D_v$ respectively. We shall use the general form of the Dirac equation shown in equation (1) up until section 7.
The random mass Dirac equation in a random magnetic field is when there is no scalar potential, $V = 0$. It also known as an incommensurate Dirac equation. The Green's function and localised density of states and has been calculated for this model [27,28]. The incommensurate Dirac equation is equivalent to the random $XX$ spin chain in a random magnetic field [29,30]. In [6] Ballents and Fisher consider a random mass Dirac equation where $\Phi = V = 0$ so only $M$ is non-zero. This is a commensurate Dirac equation [31,4]. The commensurate Dirac equation is equivalent to the $XY$ spin chain in a random transverse magnetic [29,30]. Bocquet [7] considered a number different cases, the random mass model (like [6]), the massless particle in a scalar potential ($M = \Phi = 0$) and a multiple disorder model ($m = \phi = v$).

In section 7 we shall consider the special case of the incommensurate Dirac equation. The average parts of $M$ and $\Phi$ vanish, $m = \phi = 0$, and there is no scalar potential, $V = 0$. We make the further assumption that the disorder strength of the mass and the magnetic field are equal, $D_\Phi = D_m$. This corresponds to a disordered quantum wire with a single channel.

3 The probability distribution function

3.1 The different density of states

The local density of states at the position $x$ for a general one-dimensional system with Hamiltonian $\hat{h}$ is defined by

$$\rho(E, x) = \langle x | \delta(E - \hat{h}) | x \rangle_Q = \sum_j |\psi_j(x)|^2 \delta(E - e_j), \quad (4)$$

with eigenfunction $\psi_j$ and eigenvalue $e_j$. The subscript $Q$ indicates a quantum average (which uses the eigenfunctions as weights) rather than a disorder average. For a finite closed sample the electron states make up a discrete spectrum. However, the function $\rho$ can only take on two values, zero or infinity. We can define a new and more appropriate density of states from [19]

$$\rho_f(E, x) = \int_{-\infty}^{\infty} \rho(E', x) f(E - E') dE', \quad (5)$$

where $f$ is some arbitrary weight. We define the density of states $\rho_1(\eta, E, x)$ as the density derived from the weight

$$f_\eta(x) = \frac{1}{\pi} \frac{\eta}{\eta^2 + x^2}, \quad (6)$$

where $\eta$ is small, positive and real. This weight is equivalent to the Dirac delta function, $\delta(x)$, in the limit $\eta \to 0$. To obtain the thermodynamic limit of $\rho_1$,
As $L \to \infty$, we take the limit $\eta \to 0$. In the thermodynamic limit $\rho(E, x) = \rho_1(0, E, x)$.

A full one-dimensional wave function of the general one-dimensional system for states near the Fermi energy is related to the spinor $\Psi = (\psi_\up, \psi_\down)$ in the Dirac equation by

$$\psi(x) = \psi_\up(x)e^{ik_F x} + \psi_\down(x)e^{-ik_F x},$$

(7)

where $\psi_\up$ and $\psi_\down$ denote the amplitudes for right and left moving electrons, respectively. These amplitudes are slowly varying on atomic length scales of order $k_F^{-1}$ but the exponential terms vary rapidly over this length scale. The intensity of the total wave function is

$$|\psi(x)|^2 = |\psi_\up(x)|^2 + |\psi_\down(x)|^2 + \psi_\up^*(x)\psi_\down(x)e^{-2ik_F x} + \psi_\down^*\psi_\up(x)e^{2ik_F x}.$$  

(8)

Thus, the local density of states defined in equation (4) contains terms which oscillate rapidly over atomic length scales. We can define an alternative density of states that does not include these rapid oscillations

$$\tilde{\rho}(E, x) = \sum_j \left[|\psi_{j\up}(x)|^2 + |\psi_{j\down}(x)|^2\right] \delta(E - e_j).$$

(9)

We can write an equation similar to (5) but with $\tilde{\rho}$ substituted for $\rho$

$$\tilde{\rho}_f(E, x) = \int_{-\infty}^{\infty} \tilde{\rho}(E', x)f(E - E')dE'.$$

(10)

We define $\tilde{\rho}_1(\eta, E, x)$ as we defined $\rho_1(\eta, E, x)$, using the weight $f_\eta$ in equation (6), but the former is obtained from equation (10) rather than equation (5). We shall find, as did Al'tshuler and Prigodin [19], that these two different local density of states have quite different probability distribution functions.

For the Dirac equation the Green’s function is a $2 \times 2$ matrix. A component of the retarded and advanced Green’s function matrix is defined as

$$G_{\alpha,\beta}^{R,A}(E, x, x') = \langle x, \alpha \left| \frac{1}{E \pm i\eta - \hbar} \right| x, \beta \rangle_Q,$$

(11)

where $\alpha$ and $\beta$ are spin indices. In this paper we are interested in the trace of this matrix, $G^{R,A}(E, x, x) = \text{Tr}_{\text{spin}}G_{\alpha,\beta}^{R,A}(E, x, x)$. We shall refer to $G^{R,A}$ as the retarded or advanced Green’s function. In terms of Green’s functions the density of states is

$$\tilde{\rho}_1(\eta, E, x) = \frac{1}{2i\pi} \left( G^A(E, x, x) - G^R(E, x, x) \right).$$

(12)
3.2 Participation ratio

There are a number of different criteria to distinguish energy regions of extended and localized states of a particle in a random potential. The inverse participation ratio, $P_j$, is the sum over all positions of the fourth power of the $j$th eigenfunction

$$P_j = \sum_x |\psi_j(x)|^4. \quad (13)$$

The eigenfunction at position $x$ is $\psi_j(x)$ with a corresponding eigenvalue $e_j$ and Hamiltonian $\hat{h}$ which acts on the wave function $|x\rangle$ as follows

$$\hat{h}|x\rangle = \hat{h} \sum_j \psi_j(x) = \sum_j e_j \psi_j(x). \quad (14)$$

The inverse participation ratio is positive for localized states but vanishes for extended states in the thermodynamic limit. The participation ratio averaged over the ensemble (i.e. the random potential), $\langle P_j \rangle$, measures those sites which make a significant contribution to the eigenfunction normalization. The participation ratio is related to how much the wave functions extend through space. For localized states $\langle P_j \rangle$ is nonzero. If the eigenstates are completely delocalized then $\langle P_j \rangle$ vanishes for an infinite system.

Instead of considering $P_j$ which is for individual eigenstates of individual systems we consider the ensemble averaged quantity in a narrow energy window (spectral average) centred about the energy $E$

$$P^{(2)}(E) = \left\langle \sum_{j,x} |\psi_j(x)|^4 \delta(E - e_j) \right\rangle \left\langle \sum_j \delta(E - e_j) \right\rangle^{-1}, \quad (15)$$

where the angled brackets refer to the average over the ensemble, or the disorder average. Once we take the disorder average of some quantity it will no longer depend on the position $x$. Hence, after summing equation (4) over all $x$ and taking the disorder average we obtain

$$2L \langle \rho(E) \rangle = \sum_j \delta(E - e_j), \quad (16)$$

where $2L$ is the length of the system. Similarly, the average in the numerator of equation (15) does not depend on the position, $x$, so

$$P^{(2)}(E) \langle \rho(E) \rangle = \left\langle \sum_j |\psi_j(x)|^4 \delta(E - e_j) \right\rangle. \quad (17)$$
We can generalise equation (17) to

\[ P^{(k)}(E)\langle \rho(E) \rangle = \left\langle \sum_j |\psi_j(x)|^{2k} \delta(E - e_j) \right\rangle, \]

where \( P^{(k)} \) is the \( k \)th moment of the participation ratio. The first moment \((k = 1)\) of the participation ratio is unity.

### 3.3 Moments of the local density of states

The \( k \)th moment of the local density of states, \( \rho_1 \), is the disorder average of the \( k \)th power of the density of states, \( \langle \rho_1(\eta, E)^k \rangle \). We can write the \( k \)th moment of the density of states in terms of the density of states,

\[ \langle \rho_1(\eta, E)^k \rangle = \int_{-\infty}^{\infty} dE \cdots f_\eta(E - E_1) \cdots f_\eta(E - E_k) \times \langle \rho(E_1)\rho(E_2)\cdots\rho(E_k) \rangle. \]

Wegner [11] showed that in the limit \( \eta \to 0 \)

\[ \langle \rho_1(\eta, E)^k \rangle = \left\langle \sum_j |\psi_j|^2 \delta(E - e_j) \right\rangle \int_{-\infty}^{\infty} dE \cdots f_\eta(E - E_1)^k. \]

The disorder average part is the \( k \)th moment of the participation ratio defined in equation (18) so the \( k \)th moment of the density of states is related to the \( k \)th moment of the participation ratio by

\[ \langle \rho_1(\eta, E)^k \rangle = P^{(k)}\langle \rho(E) \rangle \int_{-\infty}^{\infty} dE \cdots f_\eta(E - E_1)^k. \]

We can evaluate the integral over the weight function

\[ \int_{-\infty}^{\infty} dx f_\eta(x)^k = (4\pi\eta)^{1-k} \frac{\Gamma(2k - 1)}{\Gamma(k)^2}, \]

so that

\[ \langle \rho_1(\eta, E)^k \rangle = (4\pi\eta)^{1-k} \frac{\Gamma(2k - 1)}{\Gamma(k)^2} P^{(k)}\langle \rho(E) \rangle. \]

From equation (18) \( P^{(1)} = 1 \), since the left hand side is \( \langle \rho(E) \rangle \). This agrees with the above equation in the thermodynamic limit since in this limit \( \rho_1 = \rho \).

The general equation (5) for the density of states \( \rho_f \) can be generalised to resemble equation (21) [19]

\[ \langle \rho_f(E)^k \rangle = P^{(k)}\langle \rho(E) \rangle \int_{-\infty}^{\infty} dE f(E - E_1)^k. \]
The moments of the participation ratio are independent of the weight $f$. We can write a similar equation for the moments of $\tilde{\rho}_f$ by replacing $\rho$ with $\tilde{\rho}$ in the above equation. The $k$th moment of $\tilde{\rho}_1$ can be written in terms of Green’s functions by using equation (12)

$$
\langle \tilde{\rho}_1(\eta,E)^k \rangle = \frac{k!}{(2i\pi)^k} \sum_{j=0}^{k} \frac{(-1)^j}{j!(k-j)!} \langle G^A(E,x,x)^{k-j}G^R(E,x,x)^j \rangle. \quad (25)
$$

In this paper we calculate the Green’s function correlator

$$
\langle G^A(E,x,x)^{k-j}G^R(E,x,x)^j \rangle. \quad (26)
$$

This correlator enable us to calculate the moments of the density of states, $\langle \tilde{\rho}_1^k \rangle$. By extending this method a little we can also calculate $\langle \rho_1^k \rangle$. This allows us to calculate the moments of the participation ratio via equation (23).

In section 7 we will calculate the moments of the local density of states, $\langle \tilde{\rho}_1(\eta,E)^k \rangle$, for a random mass Dirac equation. In section 8 we find $\langle \rho_1(\eta,E)^k \rangle$ for the same model which gives the moments of the participation ratio, $P^{(k)}$. Knowing the moments of the participation ratio we can solve equation (24) for $\langle \rho_f^k \rangle$ using a general weight, $f$. In section 9 we take $f$ to be the derivative of the Fermi-Dirac distribution function.

## 4 Path integral form of products of Green’s functions

A component of the advanced and retarded Green’s function matrix at some position $x$ and energy $E$ is defined in equation (11). Note that the retarded Green’s function is the complex conjugate of the advanced Green’s function. A solution for the Green’s function can often be obtained by writing it in terms of a path integral. There are two standard ways to construct the path integral, the replica trick and supersymmetry. Both techniques allow the average over the disorder to be performed analytically.

### 4.1 Replica trick

To construct the replica trick path integral form of the advanced Green’s function we firstly write it as a path integral over two complex variable, $s$ and
\[ G^A(E, x, x) = i \left[ \int Ds^* Ds^* (x) s(x) \right. \]
\[ \times \exp \left( -i \int_{-L}^L dy s^*(y) [E - i\eta - h] s(y) \right) \]
\[ \times \left[ \int Ds^* Ds D\exp \left( -i \int_{-L}^L dy s^*(y) [E - i\eta - h] s(y) \right) \right]^{-1} \quad (27) \]

The region of integration \((-L, L)\) is over the length of the system. We shall later take the limit \(L \to \infty\). The retarded Green’s function is the complex conjugate of the trace of the advanced Green’s function

\[ G^R(E, x, x) = -i \left[ \int Ds^* Ds^* (x) s(x) \right. \]
\[ \times \exp \left( i \int_{-L}^L dy s^*(y) [E + i\eta - h] s(y) \right) \]
\[ \times \left[ \int Ds^* Ds D\exp \left( i \int_{-L}^L dy s^*(y) [E + i\eta - h] s(y) \right) \right]^{-1}. \quad (28) \]

The dominator is \(Z\), the partition function.

If the variables \(s\) and \(s^*\) are replicated \(n\) times the advanced Green’s function becomes

\[ G^A(E, x, x) = \frac{i}{n} \left[ \int Ds^* Ds \sum_{l=1}^n s^*_l (x) s_l (x) \right. \]
\[ \times \exp \left( -i \int_{-L}^L dy \sum_{l=1}^n s^*_l [E - i\eta - h] s_l \right) \]
\[ \times \left[ \int Ds^* Ds D\exp \left( -i \int_{-L}^L dy \sum_{l=1}^n s^*_l [E - i\eta - h] s_l \right) \right]^{-1}. \quad (29) \]

The dominator is now \(Z^n\). Thus, if we let \(n \to 0\) the denominator is unity and

\[ G^A(E, x, x) = \lim_{n \to 0} \frac{i}{n} \left[ \int Ds^* Ds \sum_{l=1}^n s^*_l (x) s_l (x) \right. \]
\[ \times \exp \left( -i \int_{-L}^L dy \sum_{l=1}^n s^*_l [E - i\eta - h] s_l \right) \]
\[ \left. \times \left[ \int Ds^* Ds D\exp \left( -i \int_{-L}^L dy \sum_{l=1}^n s^*_l [E - i\eta - h] s_l \right) \right]^{-1} \right]. \quad (30) \]

This is one form of the replica trick path integral. A similar expression may be constructed for the retarded Green’s function. There are other forms of the replica trick path integral which only differ in the placement of the complex number \(i\), such as the form used by Bocquet [7].
We shall now construct a replica trick form of a multiple of Green’s functions. The \( j \)th power of the advanced Green’s function is

\[
G^A(E, x, x)^j = \frac{i^j}{j!} \times \left[ \int \mathcal{D}s^* \mathcal{D}s (s^*(x)s(x))^j \exp \left( -i \int_L^L ds^*(y) [E - i\eta - h]s(y) \right) \right] \times \left[ \int \mathcal{D}s^* \mathcal{D}s \exp \left( -i \int_L^L ds^*(y) [E - i\eta - h]s(y) \right) \right]^{-1}, \tag{31}
\]

and the \( j \)th multiple of the retarded Green’s function is the complex conjugate of this. The product of the \((k-j)\)th power of the advanced Green’s function and the \( j \)th power of the retarded Green’s function may be written as a path integral over four complex variables, \( s, s^*, s' \) and \( s'^* \),

\[
G^A(E, x, x)^{k-j}G^R(E, x, x)^j = \frac{i^k \Gamma(n)}{(k-j)!j!} \times \left[ \int \mathcal{D}s^* \mathcal{D}s \mathcal{D}s' \mathcal{D}s'' \left( s^*(x)s(x) \right)^{k-j} \left( s'^*(x)s'(x) \right)^j \times \exp \left( -i \int_L^L dy \left[ s^*[E - i\eta - H]s - s'^*[E + i\eta - h]s' \right] \right) \right] \times \left[ \int \mathcal{D}s^* \mathcal{D}s \mathcal{D}s' \mathcal{D}s'' \left( s^*(x)s(x) \right)^{k-j} \left( s'^*(x)s'(x) \right)^j \times \exp \left( -i \int_L^L dy \left[ s^*[E - i\eta - H]s - s'^*[E + i\eta - h]s' \right] \right) \right]^{-1}. \tag{32}
\]

If we replicate \( n \) times and \( s' \) \( n' \) times then

\[
G^A(E, x, x)^{k-j}G^R(E, x, x)^j = \frac{i^k \Gamma(n)\Gamma(n')}{\Gamma(k-j+n)\Gamma(j+n')} \times \left[ \int \mathcal{D}s^* \mathcal{D}s \mathcal{D}s' \mathcal{D}s'' \left( \sum_{l=1}^n s_l^*(x)s_l(x) \right)^{k-j} \left( \sum_{l=1}^{n'} s_l'^*(x)s_l'(x) \right)^j \times \exp \left( -i \int_L^L dy \left[ \sum_{l=1}^n s_l^*[E - i\eta - h]s_l - \sum_{l=1}^{n'} s_l'^*[E + i\eta - h]s_l' \right] \right) \right] \times \left[ \int \mathcal{D}s^* \mathcal{D}s \mathcal{D}s' \mathcal{D}s'' \left( \sum_{l=1}^n s_l^*(x)s_l(x) \right)^{k-j} \left( \sum_{l=1}^{n'} s_l'^*(x)s_l'(x) \right)^j \times \exp \left( -i \int_L^L dy \left[ \sum_{l=1}^n s_l^*[E - i\eta - h]s_l - \sum_{l=1}^{n'} s_l'^*[E + i\eta - h]s_l' \right] \right) \right]^{-1}. \tag{33}
\]

The prefactor \( \Gamma(n)\Gamma(n')/\Gamma(k-j+n)\Gamma(j+n') \) is derived in appendix A. When
taking the limit $n, n' \rightarrow 0$, as before, the denominator is unity and

$$G^A(E, x, x)^{k-j}G^R(E, x, x)^j = \lim_{n,n' \rightarrow 0} \frac{i^k(-1)^j\Gamma(n)\Gamma(n')}{\Gamma(k-j+n)\Gamma(j+n')}$$

$$\times \int \mathcal{D}s^*\mathcal{D}s\mathcal{D}s'^*\mathcal{D}s' \left( \sum_{l=1}^{n} s^*_l(x)s_l(x) \right)^{k-j} \left( \sum_{l=1}^{n'} s^*_l(x)s'_l(x) \right)^j$$

$$\times \exp \left( -i \int_{-L}^{L} dy \left[ \sum_{l=1}^{n} s^*_l[E-i\eta-h]s_l - \sum_{l=1}^{n'} s^*_l[E+i\eta-h]s'_l \right] \right).$$

(34)

4.2 Supersymmetry

The supersymmetric path integral form of the advanced Green’s function is an integral over two complex variables, $s$ and $s^*$, and two Grassman variables, $\xi$ and $\xi^*$ [32,12],

$$G^A(E, x, x) = i \int \mathcal{D}\xi^*\mathcal{D}\xi\mathcal{D}s^*\mathcal{D}s s(x)$$

$$\times \exp \left( -i \int_{-L}^{L} dy \left[ s^*[E-i\eta-h]\xi + s^*[E+i\eta-h]s \right] \right).$$

(35)

Unlike the replica trick no denominator corresponding to a normalization (or partition function) appears because the unwanted contributions from the complex and Grassman variables cancel. The Grassman variables are introduced for this reason. As stated before, the retarded Green’s function is the complex conjugate of the advanced Green’s function. The $j$th multiple of the advanced Green’s function is

$$G^A(E, x, x)^j = \frac{i^j}{j!} \int \mathcal{D}\xi^*\mathcal{D}\xi\mathcal{D}s^*\mathcal{D}s (s^*(x)s(x))^j$$

$$\times \exp \left( -i \int_{-L}^{L} dy \left[ s^*[E-i\eta-h]\xi + s^*[E+i\eta-h]s \right] \right).$$

(36)

The product $G^A(E, x, x)^{k-j}G^R(E, x, x)^j$ can be written in terms of a supersymmetric path integral over eight variables [12]

$$G^A(E, x, x)^{k-j}G^R(E, x, x)^j = \frac{i^k(-1)^j}{(k-j)!j!}$$

$$\times \int \mathcal{D}\xi^*\mathcal{D}\xi\mathcal{D}s^*\mathcal{D}s\mathcal{D}\xi^*\mathcal{D}\xi'\mathcal{D}s'^*\mathcal{D}s' (s^*(x)s(x))^{k-j}(s'^*(x)s'(x))^j$$

$$\times \exp \left( -i \int_{-L}^{L} dy \left[ s^*[E-i\eta-h]\xi + s^*[E+i\eta-h]s 
- \xi'^*[E+i\eta-h]\xi' - s'^*[E+i\eta-h]s' \right] \right).$$

(37)
5 The transfer Hamiltonian

We now briefly review the transfer matrix method [33,3–5] which can be used to reduce a one-dimensional field theory into a zero-dimensional Schrödinger-type equation. We wish to calculate the disorder average of a product of Green’s functions, \( \langle G^A(E, x, x) G^R(E, x, x) \rangle \). We have shown in the previous section that this average is of the form

\[
\langle \phi(x) \rangle_S = \left\langle \int \mathcal{D}X \phi(x) e^{-S(X)} \right\rangle,
\]

where \( S \) is some action and \( x \) is a spatial variable. For the replica trick

\[
\phi(x) = \left( \sum_{l=1}^{n} s^*_l(x) s_l(x) \right)^{k-j} \left( \sum_{l=1}^{n'} s^*_l(x) s'_l(x) \right)^j,
\]

and for supersymmetry

\[
\phi(x) = (s^*(x) s(x))^{k-j} (s'^*(x) s'(x))^j.
\]

The action is defined as the integral of the Lagrangian

\[
S = \int_{-L}^{L} \mathcal{L} dy.
\]

All the disorder is contained in the Hamiltonian, \( h \), which is contained in the exponential term \( e^{-S(X)} \). If we have a random Gaussian variable, \( V \), which has an average value \( v \) and disorder strength \( D_v \) and \( g \) is some non-random function then the average may be taken by using [4,5]

\[
\left\langle \exp \left( i \int_{-L}^{L} dy V(y) g(y) \right) \right\rangle = \exp \left( -D_v \int_{-L}^{L} dy g(y)^2 \right).
\]

Once we have taken the disorder average we can find a Hamiltonian, known as the transfer Hamiltonian [3–5], from the disorder averaged Lagrangian. If we have a coordinate \( q \) and a momentum \( p \) the transfer Hamiltonian is

\[
H = \mathcal{L} - pq.
\]

A path integral is a sum over all possible configurations. We can rewrite the path integral of \( \langle \phi \rangle_S \) as such a sum. The continuous interval \((-L, L)\) can be written as a regular lattice with spacing \( \alpha \) so that there are \( 2L/\alpha + 1 \) sites. The value of the variable \( X \) at site \( x \) is \( X_x \). Because of periodic boundary conditions \( X_{-L} = X_L \). The path integral is now an integral over the variables
$X_x$ with $x = -L, -L + \alpha, \ldots, L$. Or more compactly

$$\langle \phi(x) \rangle_S = \int dX_L dX_x K(-L, x, X_L, X_x) \phi K(x, L, X_x, X_L),$$

(44)

where most of the integrals are hidden inside the functions $K$ which may be thought of as propagators. Note that the $\phi$ inside the integral longer has any spatial dependence because we have already taken the disorder average.

From a propagator such as $K$ we can define a set of eigenfunctions, $\Psi_{mL}$ and $\Psi_{mR}$. The two eigenfunctions, $\Psi_{mL}$ and $\Psi_{mR}$, are left and right eigenfunctions respectively. We need to distinguish between the two because the transfer Hamiltonian need not be Hermitian. The right eigenfunctions and propagator should satisfy an equation of the form

$$\Psi_{mR}(X_x) = \int dX_y K(y, x, X_y, X_x) \Psi_{mR}(X_y).$$

(45)

Using Feynman’s transfer matrix technique [33] we take $x = y + \alpha$ and find a Schrödinger-type equation for $\Psi_{mR}$:

$$i \frac{\partial \Psi_{mR}}{\partial x} = H \Psi_{mR},$$

(46)

where $H$ is the transfer Hamiltonian defined in equation (43). The left eigenfunction satisfies a similar equation. We can define some eigenstates, $E_m$, such that $H \Psi_{mR} = E_n \Psi_{mR}$ and $\Psi_{mL} H = E_n \Psi_{mL}$. We define the eigenfunctions to be orthonormal so that

$$\int dX \Psi_{mL}^*(X) \Psi_{nR}(X) = \delta_{mn}.$$  

(47)

A propagator can be written in terms of its eigenfunctions and eigenstates

$$K(y, x, X_y, X_x) = \sum_m \Psi_{mR}(X_y) \Psi_{mL}^*(X_x) \exp[-E_m(x - y)].$$

(48)

We substitute this into equation (44) and using the orthonormality of the eigenfunctions obtain

$$\langle \phi(x) \rangle_S = \sum_m \int |\Psi_m(X)|^2 \phi dX \exp[-2LE_m],$$

(49)

where we have defined $|\Psi_m|^2 = \Psi_{mL}^* \Psi_{mR}$.

We shall now derive an equation of the above form for the Green’s function correlator $\langle G^A(E, x, x)^{k-j} G^R(E, x, x)^j \rangle$. We obtain both a replica and a supersymmetric version. The equation can be further simplified in the limit $L \to \infty$. 

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5.1 Replica trick

Since the Hamiltonian, $h$, for the random Dirac equation (1) is a $2 \times 2$ matrix
the path integral variables in equation (34) are two component vectors. For example
\[ s = (s^\uparrow, s^\downarrow). \]  
(50)

By substituting the Dirac Hamiltonian (1) into equation (34) we can obtain
the action
\[ S = -\int_{-L}^{L} dy \left[ \sum_{l=1}^{n} \left[ s^*_l \sigma_z \partial s_l + i \Phi s^*_l \sigma_x s_l + i M s^*_l \sigma_y s_l + (iV - \epsilon_-) s^*_l s_l \right] 
- \sum_{l=1}^{n'} \left[ s^*_l \sigma_z \partial s_l + i \Phi s^*_l \sigma_x s_l + i M s^*_l \sigma_y s_l + (iV - \epsilon_+) s^*_l s_l \right] \right]. \]  
(51)

The coordinates are
\[ q_l = s_l = (s^\uparrow_l, s^\downarrow_l), \quad q'_l = s'_l = (s'^\uparrow_l, s'^\downarrow_l), \]  
(52)
and so we can calculate the momenta
\[ p_l = (s^\uparrow_l, s^\downarrow_l), \quad p'_l = (s'^\uparrow_l, s'^\downarrow_l). \]  
(53)

To obtain the transfer Hamiltonian the action needs to be transformed into
the form of a coherent path integral [6,7]
\[ s^\uparrow_l \rightarrow b^\uparrow_l, \quad s^\downarrow_l \rightarrow b^\downarrow_l, \]
\[ s'^\uparrow_l \rightarrow b'^\uparrow_l, \quad s'^\downarrow_l \rightarrow b'^\downarrow_l, \]  
(54)
and $b_l = (b^\uparrow_l, b^\downarrow_l)$. The transformation of the primed terms is identical. The
coordinates and momenta in terms of these new variables are
\[ q_l = (b^\uparrow_l, b^\downarrow_l), \quad p_l = (-b'^\uparrow_l, b'^\downarrow_l), \]
\[ q'_l = (b'^\uparrow_l, b'^\downarrow_l), \quad p'_l = (b'^\uparrow_l, -b'^\downarrow_l). \]  
(55)

The transformed action is
\[ S = -\int_{-L}^{L} dy \left[ \sum_{l=1}^{n} \left[ b'^\uparrow_l \partial b^\uparrow_l - b'^\downarrow_l \partial b^\downarrow_l + i \Phi (b'^\uparrow_l b'^\downarrow_l + b^\downarrow_l b^\uparrow_l) 
+ M (b'^\uparrow_l b^\downarrow_l - b^\uparrow_l b'^\downarrow_l) + (iV - \epsilon_-) (b'^\uparrow_l b^\uparrow_l + b^\downarrow_l b'^\downarrow_l) \right] 
- \sum_{l=1}^{n'} \left[ b'^\uparrow_l \partial b'^\uparrow_l - b'^\downarrow_l \partial b'^\downarrow_l + i \Phi (b'^\uparrow_l b'^\downarrow_l + b'^\downarrow_l b'^\uparrow_l) 
+ M (b'^\uparrow_l b'^\downarrow_l - b'^\downarrow_l b'^\uparrow_l) + (iV - \epsilon_+) (b'^\uparrow_l b'^\uparrow_l + b'^\downarrow_l b'^\downarrow_l) \right] \right]. \]  
(56)
At this stage we take the ensemble average of the action using equation (42). After averaging over the disorder we can find the transfer Hamiltonian [3–5]. From equation (43) the transfer Hamiltonian is

$$H = -2i\phi(Z_y - Z'_y) + 2im(Z_x - Z'_x) - 2i(v - E)(Z_z - Z'_z) + 2\eta(Z_z + Z'_z) + 4D_\phi(Z_y - Z'_y)^2 + 4D_m(Z_x - Z'_x)^2 + 4D_e(Z_z - Z'_z)^2,$$  \hspace{1cm} (57)

where

$$Z = -\frac{1}{2} \sum_{l=1}^{n} p_l(i\sigma_x, i\sigma_y, \sigma_z) q_l, \quad Z' = \frac{1}{2} \sum_{l=1}^{n} p'_l(i\sigma_x, i\sigma_y, \sigma_z) q'_l$$

$$Z_x = \frac{i}{2} \sum_{l=1}^{n} (b_{l\uparrow} b_{l\downarrow}^\dagger - b_{l\downarrow} b_{l\uparrow}^\dagger), \quad Z'_x = \frac{i}{2} \sum_{l=1}^{n'} (b'_{l\uparrow} b'_{l\downarrow}^\dagger - b'_{l\downarrow} b'_{l\uparrow}^\dagger),$$

$$Z_y = \frac{i}{2} \sum_{l=1}^{n} (b_{l\uparrow} b_{l\downarrow}^\dagger + b_{l\downarrow} b_{l\uparrow}^\dagger), \quad Z'_y = \frac{i}{2} \sum_{l=1}^{n'} (b'_{l\uparrow} b'_{l\downarrow}^\dagger + b'_{l\downarrow} b'_{l\uparrow}^\dagger),$$

$$Z_z = \frac{i}{2} \sum_{l=1}^{n} (b_{l\uparrow} b_{l\downarrow}^\dagger + b_{l\downarrow} b_{l\uparrow}^\dagger), \quad Z'_z = \frac{i}{2} \sum_{l=1}^{n'} (b'_{l\uparrow} b'_{l\downarrow}^\dagger + b'_{l\downarrow} b'_{l\uparrow}^\dagger).$$ \hspace{1cm} (58)

Note that the transfer Hamiltonian does not depend on the spatial coordinate \(x\) making it a zero-dimensional equation. We can define some ladder operators

$$Z_{\pm} = Z_y \mp iZ_x,$$  \hspace{1cm} (59)

and similarly for \(Z_{\pm}'\). The vector \(Z\) satisfies \(su(1,1)\) commutation rules [7]

$$[Z_+, Z_-] = -2Z_z, \quad [Z_z, Z_+] = Z_+, \quad [Z_z, Z_-] = -Z_-,$$ \hspace{1cm} (60)

and also \((Z'_z)^\dagger = Z'_z\). Identical rules govern \(Z\).

We define the \(m\)th eigenstate of the transfer Hamiltonian as \(E_m\) with corresponding eigenfunction \(\Psi_m\). Since the transfer Hamiltonian need not be Hermitian the left and right eigenstates may be different. After undergoing the transformation in (54) the disorder average of equation (34) in terms of \(E_m\) and \(\Psi_m\) is [3–5]

$$\langle G^A(E, x, x)^{k-j} G^B(E, x, x)^j \rangle = \lim_{n,n' \to 0} \frac{i^k(-1)^j \Gamma(n) \Gamma(n')}{\Gamma(k-j+n) \Gamma(j+n')} \times \int dB^\dagger dB dB^\dagger dB^\dagger \left( \sum_{l=1}^{n} (b_{l\uparrow} b_{l\downarrow}^\dagger + b_{l\downarrow} b_{l\uparrow}^\dagger) \right)^{k-j} \left( \sum_{l=1}^{n'} (b'_{l\uparrow} b'_{l\downarrow}^\dagger + b'_{l\downarrow} b'_{l\uparrow}^\dagger) \right)^j \times \sum_m |\Psi_m(b, \sigma^\dagger, b', \sigma^\dagger)|^2 e^{-2LE_m}.$$ \hspace{1cm} (61)

The partition function for \(n,n' \to 0\) is defined as

$$Z_0 = \lim_{n,n' \to 0} \int Ds^a Ds Ds'^a Ds'^a e^{-S},$$ \hspace{1cm} (62)
and when deriving equation (34) we showed that $Z_0 = 1$. After transforming as shown in equation (54) the disorder averaged partition function in terms of $E_m$ and $\Psi_m$ is

$$
\langle Z_0 \rangle = \lim_{n, n' \to 0} \int db db' db'' db''' \sum_m |\Psi_m(b, b', b'', b''')|^2 e^{-2LE_m}. \quad (63)
$$

As the system becomes infinite in length, that is $L \to \infty$, the exponential term, $e^{-2LE_m}$ will vanish for all positive energy levels. In the the disorder averaged partition function the dominant term is the ground state energy term

$$
\langle Z_0 \rangle = \lim_{n, n' \to 0} \int db db' db'' db''' |\Psi_0(b, b', b', b'')|^2 e^{-2LE_0} \quad (64)
$$

which may alternatively be be written as

$$
\langle Z_0 \rangle = \lim_{n, n' \to 0} L \langle \Psi | e^{-2LE_0} | \Psi \rangle_R \quad (65)
$$

where $|\Psi\rangle_L$ and $|\Psi\rangle_R$ are the left and right ground state wave functions respectively with ground state energy $E_0$. As stated above this average partition function must be unity so $E_0 = 0$ and

$$
\lim_{n, n' \to 0} L \langle \Psi | \Psi \rangle_R = 1. \quad (66)
$$

Hence, to calculate the Green’s function correlator in the limit of an infinite system we need only calculate the ground state of the transfer Hamiltonian, $H$. After taking the $L \to \infty$ limit of (61) and using equation (58) to write it in terms of $Z$ and $Z'$

$$
\langle G^A(E, x, x)^k - G^R(E, x, x)^j \rangle = \lim_{n, n' \to 0} \frac{(2i)^k(-1)^j\Gamma(n)\Gamma(n')}{\Gamma(k - j + n)\Gamma(j + n')} \times L \langle \Psi | Z_z^{k-j} Z_z' | \Psi \rangle_R. \quad (67)
$$

From equation (25) the $k$th moment of the local density of states, $\tilde{\rho}_1$, is

$$
\langle \tilde{\rho}_1(\eta, E)^k \rangle = \lim_{n, n' \to 0} \frac{k!\Gamma(n)\Gamma(n')}{\pi^k} \sum_{j=0}^k \frac{L \langle \Psi | Z_z^{k-j} Z_z' | \Psi \rangle_R}{j!(k - j)!\Gamma(k - j + n)\Gamma(j + n')}. \quad (68)
$$
5.2 Supersymmetry

We substitute the Dirac Hamiltonian (1) into equation (37) to get the supersymmetric action

\[
S = -\int_{-L}^{L} dy \left[ \xi^{*}\sigma_{x}\partial\xi + s^{*}\sigma_{x}\partial s - \xi^{*}\sigma_{z}\partial\xi' - s^{*}\sigma_{z}\partial s' \right] \\
+ i\Phi(\xi^{*}\sigma_{x}\xi + s^{*}\sigma_{x}s - \xi^{*}\sigma_{x}\xi' - s^{*}\sigma_{x}s') + iM(\xi^{*}\sigma_{y}\xi + s^{*}\sigma_{y}s - \xi^{*}\sigma_{y}\xi' - s^{*}\sigma_{y}s') \\
+ i(V - (E - i\eta))(\xi^{*}\xi + s^{*}s) - i(V - (E + i\eta))(\xi^{*}\xi' + s^{*}s').
\]  
(69)

If we define the coordinates as

\[
q_{1} = s = (s_{\uparrow}, s_{\downarrow}), \quad q_{2} = \xi = (\xi_{\uparrow}, \xi_{\downarrow}), \\
q'_{1} = s' = (s'_{\uparrow}, s'_{\downarrow}), \quad q'_{2} = \xi' = (\xi'_{\uparrow}, \xi'_{\downarrow}),
\]  
(70)

the momenta are

\[
p_{1} = (-s^{*}_{\uparrow}, s^{*}_{\downarrow}), \quad p_{2} = (-\xi^{*}_{\uparrow}, \xi^{*}_{\downarrow}), \\
p'_{1} = (s'^{*}_{\uparrow}, -s'^{*}_{\downarrow}), \quad p'_{2} = (\xi'^{*}_{\uparrow}, -\xi'^{*}_{\downarrow}).
\]  
(71)

As in the replica case we need to transform this action. We shall use the same transformation as Bocquet [7] rather than the transformation used by Balents and Fisher [6]

\[
s_{\uparrow} \rightarrow b_{\uparrow}, \quad \xi_{\uparrow} \rightarrow f_{\uparrow}, \\
s_{\downarrow} \rightarrow b'_{\downarrow}, \quad \xi_{\downarrow} \rightarrow f'_{\downarrow}, \\
s^{*}_{\uparrow} \rightarrow b^{*}_{\uparrow}, \quad \xi^{*}_{\uparrow} \rightarrow f^{*}_{\uparrow}, \\
s^{*}_{\downarrow} \rightarrow b^{*}_{\downarrow}, \quad \xi^{*}_{\downarrow} \rightarrow -f^{*}_{\downarrow}.
\]  
(72)

The negative sign in the transformation of the Grassman variables is because for any Grassman variable \((\xi^{*})^{*} = -\xi\). In terms of the new variables the coordinates and momenta are

\[
q_{1} = (b_{\uparrow}, b^{*}_{\downarrow}), \quad q_{2} = (f_{\uparrow}, f^{*}_{\downarrow}), \\
q'_{1} = (b'_{\uparrow}, b'^{*}_{\downarrow}), \quad q'_{2} = (f'_{\uparrow}, f'^{*}_{\downarrow}), \\
p_{1} = (-b^{*}_{\uparrow}, b_{\downarrow}), \quad p_{2} = -(f^{*}_{\uparrow}, f_{\downarrow}), \\
p'_{1} = (b'^{*}_{\uparrow}, -b'_{\downarrow}), \quad p'_{2} = (f'^{*}_{\uparrow}, f'_{\downarrow}).
\]  
(73)
The transformed action is

\[
S = -\int_{-L}^{L} dy \left[ f_{\uparrow}^{\dagger} \partial f_{\uparrow} + f_{\downarrow} \partial f_{\downarrow}^{\dagger} - f_{\uparrow}^{\dagger} \partial f_{\uparrow} - f_{\downarrow} \partial f_{\downarrow}^{\dagger} + b_{\uparrow} \partial b_{\uparrow} - b_{\downarrow} \partial b_{\downarrow}^{\dagger} - b_{\uparrow}^{\dagger} \partial b_{\uparrow}^{\dagger} + b_{\downarrow}^{\dagger} \partial b_{\downarrow}^{\dagger} \\
+ i \Phi(f_{\uparrow}^{\dagger} f_{\downarrow} + f_{\downarrow}^{\dagger} f_{\uparrow} - f_{\uparrow}^{\dagger} f_{\downarrow}' - f_{\downarrow}^{\dagger} f_{\uparrow}' - f_{\uparrow}^{\dagger} B_{\uparrow} + b_{\downarrow}^{\dagger} B_{\downarrow} + b_{\uparrow}^{\dagger} B_{\uparrow}' - b_{\downarrow}^{\dagger} B_{\downarrow}' - b_{\uparrow}^{\dagger} B_{\uparrow} + b_{\downarrow}^{\dagger} B_{\downarrow}
\right]
\]

We take the average over the disorder as described in equation (42). Following Bocquet [7] we define two superspins with three components

\[
\mathcal{J} = -\frac{1}{2} p_1(i \sigma^x, i \sigma^y, \sigma^z) q_1 - \frac{1}{2} p_2(i \sigma^x, i \sigma^y, \sigma^z) q_2,
\]

\[
\mathcal{J}' = \frac{1}{2} p_1'(i \sigma^x, i \sigma^y, \sigma^z) q_1' + \frac{1}{2} p_2'(i \sigma^x, i \sigma^y, \sigma^z) q_2'.
\]

Both these superspins satisfy \(su(1,1)\) algebra like \(\mathbf{Z}'\) and \(\mathbf{Z}\) in equations (60). We can write the transfer Hamiltonian in terms of components of these two superspins

\[
H = -2i \phi(\mathcal{J}_y - \mathcal{J}'_y) + 2im(\mathcal{J}_x - \mathcal{J}'_x) - 2i(v - E)(\mathcal{J}_z - \mathcal{J}'_z) + 2\eta(\mathcal{J}_z + \mathcal{J}'_z)
\]

\[
+ 4D_{\phi}(\mathcal{J}_y - \mathcal{J}'_y)^2 + 4D_{m}(\mathcal{J}_x - \mathcal{J}'_x)^2 + 4D_{\eta}(\mathcal{J}_z - \mathcal{J}'_z)^2.
\]

Note that the transfer Hamiltonian, like the transfer Hamiltonian of the replica trick, is independent of the spatial coordinate \(x\). We can decompose the superspins, \(\mathcal{J} = \mathbf{J} + \mathbf{S}\), into fermionic, \(\mathbf{S}\), and bosonic, \(\mathbf{J}\), parts

\[
\mathbf{J} = -\frac{1}{2} p_1(i \sigma^x, i \sigma^y, \sigma^z) q_1, \quad \mathbf{J}' = \frac{1}{2} p_1'(i \sigma^x, i \sigma^y, \sigma^z) q_1',
\]

\[
\mathbf{S} = -\frac{1}{2} p_2(i \sigma^x, i \sigma^y, \sigma^z) q_2, \quad \mathbf{S}' = \frac{1}{2} p_2'(i \sigma^x, i \sigma^y, \sigma^z) q_2'.
\]

The \(\mathbf{J}, \mathbf{J}', \mathbf{S}\) and \(\mathbf{S}'\) all satisfy the \(su(1,1)\) algebra. In terms of \(b, b', f\) and \(f'\)

\[
\mathbf{J}_x = \frac{1}{2}(b_{\uparrow}^{\dagger} b_{\downarrow}' - b_{\downarrow}^{\dagger} b_{\uparrow}'), \quad \mathbf{J}_x' = \frac{1}{2}(b_{\uparrow}'^{\dagger} b_{\downarrow} - b_{\downarrow}'^{\dagger} b_{\uparrow}),
\]

\[
\mathbf{J}_y = \frac{1}{2}(b_{\uparrow}^{\dagger} b_{\downarrow}' + b_{\downarrow}^{\dagger} b_{\uparrow}'), \quad \mathbf{J}_y' = \frac{1}{2}(b_{\uparrow}'^{\dagger} b_{\downarrow}' + b_{\downarrow}^{\dagger} b_{\uparrow}),
\]

\[
\mathbf{J}_z = \frac{1}{2}(b_{\uparrow} b_{\uparrow} + b_{\downarrow} b_{\downarrow}), \quad \mathbf{J}_z' = \frac{1}{2}(b_{\uparrow}' b_{\uparrow}' + b_{\downarrow}' b_{\downarrow}'),
\]

\[
\mathbf{S}_x = \frac{1}{2}(f_{\uparrow}^{\dagger} f_{\downarrow}' - f_{\downarrow}^{\dagger} f_{\uparrow}), \quad \mathbf{S}_x' = \frac{1}{2}(f_{\uparrow}'^{\dagger} f_{\downarrow}' - f_{\downarrow}'^{\dagger} f_{\uparrow})
\]

\[
\mathbf{S}_y = \frac{1}{2}(f_{\uparrow}^{\dagger} f_{\downarrow}' + f_{\downarrow}^{\dagger} f_{\uparrow}), \quad \mathbf{S}_y' = \frac{1}{2}(f_{\uparrow}'^{\dagger} f_{\downarrow}' + f_{\downarrow}'^{\dagger} f_{\uparrow}),
\]

\[
\mathbf{S}_z = \frac{1}{2}(f_{\uparrow}^{\dagger} f_{\uparrow}' + f_{\downarrow}^{\dagger} f_{\downarrow}'), \quad \mathbf{S}_z' = \frac{1}{2}(f_{\uparrow}'^{\dagger} f_{\uparrow}' + f_{\downarrow}'^{\dagger} f_{\downarrow}').
\]
\( f, f', f^\dagger \) and \( f'^\dagger \). The \( m \)th eigenstate of the transfer Hamiltonian, \( \Psi_m \), has a dependence on the fermionic variables as well as the bosonic variable \( s \). Like the replica trick we can define a disorder averaged partition function which is set to unity. When we take the limit \( L \to \infty \) we obtain the supersymmetric equivalent of equation (67)

\[
\langle G^A(E, x, x)^{k-j}G^R(E, x, x)^j \rangle = \frac{(2i)^k(-1)^j}{(k-j)!j!} L \langle \Psi | J_z^{k-j} J_z^j | \Psi \rangle_R, \tag{79}
\]

where, like the replica trick, \( | \Psi \rangle_R \) and \( | \Psi \rangle_L \) are the right and left ground state wave functions of the transfer Hamiltonian with ground state energy \( E_0 = 0 \).

From equation (25) the \( k \)th moment of the local density of states, \( \tilde{\rho}_1 \), is

\[
\langle \tilde{\rho}_1(\eta, E)^k \rangle = \frac{k!}{\pi^k} \sum_{j=0}^{k} \frac{L \langle \Psi | J_z^{k-j} J_z^j | \Psi \rangle_R}{(j!(k-j)!)^2}. \tag{80}
\]

The first moment of this local density of states is

\[
\langle \tilde{\rho}_1(\eta, E) \rangle = \frac{1}{\pi} L \langle \Psi | J_z + J_z' | \Psi \rangle_R. \tag{81}
\]

If we take the limit \( \eta \to 0 \) this equation should be equivalent to equation (9). Using equation (78) we can relate \( \psi^\uparrow \) and \( \psi^\downarrow \) to the bosonic variables \( b \) and \( b' \).

To find the moments of the full local density of state, \( \rho_1 \), rather than \( \tilde{\rho}_1 \), we must include the rapidly oscillating terms in (8). This means that instead of \( J_z \) in equation (80) we have \( J_z + \frac{1}{2} J_+ e^{-2ik_F x} + \frac{1}{2} J_- e^{2ik_F x} \), and similarly for \( J_z' \). So,

\[
\langle \rho_1(\eta, E)^k \rangle = \frac{k!}{\pi^k} \sum_{j=0}^{k} \frac{1}{(j!(k-j)!)^2} L \langle \Psi | (J_z + \frac{1}{2} J_+ e^{-2ik_F x} + \frac{1}{2} J_- e^{2ik_F x})^{k-j} \times (J_z' + \frac{1}{2} J_+ e^{-2ik_F x} + \frac{1}{2} J_- e^{2ik_F x})^j | \Psi \rangle_R. \tag{82}
\]

Note that when this is expanded out terms involving \( J_+ J_- \) and \( J_- J_+ \) can produce contributions to the moment that are slowly varying in space. This argument to find \( \langle \rho_1^k \rangle \) is also valid for the replica trick. We used supersymmetry rather than the replica trick because the notation is simpler.

6 The ground state

6.1 Replica trick

Bocquet [7] has calculated the ground state for the transfer Hamiltonian associated with the one-particle retarded Green’s function. This ground state is
a linear combination of the basis states \{\ket{p}\}\n
\[ |\Psi\rangle_R = \sum_{p=0}^{\infty} \zeta_p \ket{p}. \quad (83) \]

This basis set is infinite due to the non-compactness of \(u(1,1)\). The basis states are generated by a raising operator \(Z_+\)

\[ |p\rangle = \frac{(Z_+)^k}{k!} |\Omega\rangle, \quad (84) \]

where \(|\Omega\rangle\) is the lowest basis state. The basis states are orthogonal with norm

\[ \langle p|p' \rangle = \frac{\Gamma(p+n)}{\Gamma(n)\Gamma(p+1)}. \quad (85) \]

As \(n \to 0\), \(\langle p|p \rangle = \delta_{0p}\). Bocquet’s \(Z\) is identical to the \(Z'\) in this paper.

We use the same general form for the ground state of our transfer Hamiltonian but summed over an additional variable

\[ |\Psi\rangle_R = \sum_{p,p'=0}^{\infty} \zeta_{pp'} |p,p'\rangle. \quad (86) \]

The non-primed terms are related to the advanced Green’s function and the primed terms are related to the retarded Green’s function. The primed term should act in the same way as Bocquet’s variables and the non-primed should act like complex conjugates.

We construct some raising and lowering operators, \(Z_{\pm} = Z_y \mp iZ_x\) and \(Z'_{\pm} = Z'_y \mp iZ'_x\) (as was done in section 5). The basis states, \(|p,p'\rangle\), are generated by the action of \(Z'_+\)

\[ |p,p'\rangle = \frac{(Z'_+)^{p'}}{p'!} |p,\Omega'\rangle, \quad (87) \]

and, since the non-primed terms act like complex conjugates of the primed terms, the action of \(Z_-\)

\[ |p,p'\rangle = \frac{(Z_-)^{p}}{p!} |\Omega,p'\rangle. \quad (88) \]

The state \(|\Omega,\Omega'\rangle\) is the lowest vector in this representation. Alternatively, we could have \(Z_+\) generating the basis states since \((Z_-)^\dagger = Z_+\)

\[ \langle p,p' | = \langle \Omega,p' | \frac{(Z_+)^p}{p!}. \quad (89) \]
The vector $Z'$ operates on the $|p, p'\rangle$ basis state in the following way

$$Z'_z|p, p\rangle = \left(p' + \frac{n}{2}\right)|p, p\rangle$$
$$Z'_+|p, p\rangle = (p' + 1)|p, p' + 1\rangle$$
$$Z'_-|p, p\rangle = (p' + n' - 1)|p, p' - 1\rangle,$$  \hspace{1cm} (90)

and for $Z$ operating on $\langle p, p'|$

$$\langle p, p'|Z_z = \left(p + \frac{n}{2}\right)\langle p, p'|$$
$$\langle p, p'|Z_+ = (p + 1)\langle p + 1, p'|$$
$$\langle p, p'|Z_- = (p + n - 1)\langle p - 1, p'|.$$  \hspace{1cm} (91)

The scalar products of these states can be calculated by using, for example,

$$(Z'_-|p, p\rangle)^\dagger = \langle p, p'|Z'_+$$
$$= \langle p, p' - 1|(p' + n' - 1).$$  \hspace{1cm} (92)

The norm is

$$\langle p, p'|p, p'\rangle = \left(\frac{1}{p!p!'(Z'_-)^p(Z'_+)^{p'}}\right)\langle \Omega, \Omega'|\Omega, \Omega'$$
$$= \frac{\Gamma(p + n)\Gamma(p' + n')}{\Gamma(n)\Gamma(n')\Gamma(p + 1)\Gamma(p' + 1)}\langle \Omega, \Omega'|\Omega, \Omega'$$
$$= \frac{\Gamma(p + n)\Gamma(p' + n')}{\Gamma(n)\Gamma(n')\Gamma(p + 1)\Gamma(p' + 1)}.$$  \hspace{1cm} (93)

Note that $Z$ and $Z'$ commute. In the limit $n, n' \to 0$

$$\langle p, p'|p, p'\rangle = \delta_{0p}\delta_{0p'}.$$  \hspace{1cm} (94)

Similarly, it can also be shown that the scalar product of $|p, p\rangle$ and $|q, q\rangle$ with $p \neq q$ or $p' \neq q'$ vanishes, assuming $\langle p, p'\Omega, \Omega' = 0$ if $p \neq \Omega$ and $p' \neq \Omega'$. Hence, the basis states are orthogonal.

As discussed previously, because the transfer Hamiltonian isn’t Hermitian the left wave function is not the Hermitian conjugate of the right wave function. The Hamiltonian and its Hermitian conjugate are related by the transform $R = \exp(i\pi(Z_z + Z'_z))$ [7]. The left and right ground state wave functions are related by this same transform

$$|\Psi\rangle_L = R|\Psi\rangle_R.$$  \hspace{1cm} (95)

Using equation (93) and the orthogonality of the basis states

$$L\langle\Psi|\rangle_R = \frac{1}{\Gamma(n)\Gamma(n')} \sum_{p,p'=0}^{\infty} (-1)^{p+p'+s+n+s} \frac{\Gamma(p + n)\Gamma(p' + n')}{\Gamma(p + 1)\Gamma(p' + 1)} |\zeta_{pp'}|^2.$$  \hspace{1cm} (96)
As \( n, n' \to 0 \) this reduces to

\[
L \langle \Psi | \Psi \rangle_R = |\zeta_{00}|^2
\]

(97)

so we can make our normalization condition \( |\zeta_{00}|^2 = 1 \), in agreement with equation (66).

Using equations (90), (91) and (96)

\[
L \langle \Psi | Z_x^{k-j} Z_{x'}^{j} | \Psi \rangle_R = \frac{1}{\Gamma(n)\Gamma(n')} \sum_{p,p'=0}^{\infty} (-1)^{p+p'+\frac{n+n'}{2}} \left( p + \frac{n}{2} \right)^{k-j} \\
\times \frac{\Gamma(p+n)\Gamma(p'+n')}{\Gamma(p+1)\Gamma(p'+1)} |\zeta_{pp'}|^2.
\]

(98)

Thus, by substitution into equation (67),

\[
\langle G^A(E, x, x)^{k-j} G^R(E, x, x)^{j} \rangle = (2i)^k \langle \frac{(-1)^j}{\Gamma(k-j)\Gamma(j)} \sum_{p,p'=0}^{\infty} (-1)^{p+p'} \times p^{k-j-1} p'^{j-1} |\zeta_{pp'}|^2. \rangle
\]

(99)

If we can find the ground state of the transfer Hamiltonian we can calculate all the \( \zeta_{pp'} \) and hence we have calculated the Green's function correlator. We will calculate \( \zeta_{pp'} \) for a special case of the Dirac equation corresponding to a single channel quantum wire in section 7.1.

### 6.2 Supersymmetry

The bosonic parts of the superspins, \( J \) and \( J' \), are defined in exactly the same way as the \( Z \) and \( Z' \) used in the replica trick calculation when \( n = n' = 1 \). So, \( J \) and \( J' \) satisfy all the same equations as \( Z \) and \( Z' \) when we set \( n = n' = 1 \) in all the replica trick calculations. Like the replica trick we have a bosonic wave function \(|p,p'\rangle\) but here we will use the notation \(|p\rangle|p'\rangle\). This bosonic wave function is not affected by the fermionic operators \( S \) and \( S' \). From equations (90) and (91)

\[
J_{z}^'|p\rangle' = \left( p + \frac{z}{2} \right) |p\rangle', \quad \langle p|J_{z} = \left( p + \frac{z}{2} \right) \langle p|,
\]

\[
J_{+}^'|p\rangle' = (p+1) |p+1\rangle', \quad \langle p|J_{+} = (p+1) \langle p+1|,
\]

\[
J_{-}^'|p\rangle' = p |p-1\rangle', \quad \langle p|J_{-} = p \langle p-1|,
\]

(100)

where \( J_{\pm} = J_{y} \mp i J_{x} \) and \( J_{\pm}^' = J_{y}^' \mp i J_{x}^' \). From equation (93)

\[
\langle p|p \rangle = \langle p|p\rangle' = 1.
\]

(101)
The fermionic operators, $S$ and $S'$, are like spin operators. There are two spin up states and two spin down states and the fermionic operators operate on them as follows

$$S'_\downarrow \downarrow' = \frac{1}{2} \downarrow' \downarrow,$$

$$S'_\uparrow \uparrow' = 0,$$

$$S'\downarrow \uparrow' = - \downarrow \downarrow',$$

$$S'\uparrow \downarrow' = -\frac{1}{2} \downarrow \downarrow',$$

$$S'_\downarrow \downarrow' = \uparrow \uparrow',$$

$$S'_\uparrow \uparrow' = 0,$$

where $S_{\pm} = S_y \mp iS_x$ and $S'_\pm = S'_y \mp iS'_x$. Note that the operation of $S_-$ and $S'_-$ on the spin up has the opposite sign from usual. This is due to the Grassman variables which make up the spin operators. For the same reason it can be shown that $(S_z^3) = (S'_z^3) = 0$.

Balents and Fisher [6] show that the basis state for the non-primed bosons and fermions is

$$|p\rangle_R = \begin{cases} |p\rangle \downarrow + |p - 1\rangle \uparrow, & p > 0, \\ |0\rangle \downarrow, & p = 0, \end{cases}$$

and similarly for $|p\rangle'_R$. The total basis state combines primed and non-primed parts and is written as $|p\rangle_R |p\rangle'_R$. The left basis state for $|p\rangle_R$ is

$$|p\rangle_L = (-1)^{J_z} |p\rangle_R$$

$$= \begin{cases} (-1)^p (|p\rangle \downarrow + |p - 1\rangle \uparrow), & p > 0, \\ |0\rangle \downarrow, & p = 0, \end{cases}$$

and similarly for $|p\rangle'_L$.

As discussed in Bocquet [7] the norm of the lowest basis state, $|0\rangle_R$, can be taken to be unity

$$L \langle 0 | 0 \rangle_R = \langle 0 | 0 \rangle \langle \downarrow | \downarrow \rangle = 1.$$  

(105)

Since the $|p\rangle$'s are orthonormal the norm for the down spins must be unity. The up spin can be defined in terms of the down spin:

$$\langle \uparrow | = \langle \downarrow | S_+.$$  

(106)

This allows us to write the norm of the up spin as

$$\langle \uparrow | \uparrow \rangle = \langle \downarrow | S_+ S_- \uparrow \rangle,$$  

(107)
which can be evaluated using the spin rules of equation (102). We can calculate the norms of the primed spins in the same way. The norms for all the spin states are

\[ -'\langle \uparrow | \uparrow \rangle' = '\langle \downarrow | \downarrow \rangle' = -\langle \uparrow | \uparrow \rangle = \langle \downarrow | \downarrow \rangle = 1. \]  

(108)

The total left and right wave functions are created by summing over all the basis states

\[
|\Psi\rangle_L = \sum_{p,p'=0}^{\infty} \zeta_{pp'} |p\rangle_L |p'\rangle'_L, \\
|\Psi\rangle_R = \sum_{p,p'=0}^{\infty} \zeta_{pp'} |p\rangle_R |p'\rangle'_R. 
\]  

(109)

Using the rules given above and the normalization condition on the wave function it can be shown that

\[ L\langle \Psi|\Psi\rangle_R = |\zeta_{00}|^2 = 1. \]  

(110)

Combining equations (100) and (102) we obtain the following equations for \( J \) and \( J' \)

\[
J'_z|p\rangle'_R = p|p\rangle'_R, \quad L\langle p|J_z = L\langle p|p, \\
J'_+|p\rangle'_R = (p + 1)|p + 1\rangle'_R, \quad L\langle p|J_+ = L\langle p + 1|(p + 1), \\
J'_-|p\rangle'_R = (p - 1)|p - 1\rangle'_R, \quad L\langle p|J_ - = L\langle p - 1|(p - 1). 
\]  

(111)

These are exactly the same as \( Z \) and \( Z' \) for the replica trick for \( n = n' = 0 \).

To calculate (79) we require

\[ L\langle \Psi|J_z^{k-j}J_z^{j}|\Psi\rangle_R = L\langle \Psi|k-j|S_z|S_z^j|\Psi\rangle_R. \]

(112)

This expression can be expanded using equations (102) and (111) and by using
the form of the ground state in equation (109)

\[ J_{z}^{k-j} J_{z}^{j} |\Psi\rangle_R = (J_z - S_z) (J_z' - S_z') |\Psi\rangle_R \]

\[ = (J_z^{k-j} - (k-j)J_z^{k-j-1}S_z + \frac{1}{2}(k-j)(k-j-1)J_z^{k-j-2}S_z^2) \]

\[ \times (J_z' - jJ_z'^{-1}S_z' + \frac{1}{2}j(j-1)J_z'^{-2}S_z'^2) |\Psi\rangle_R \]

\[ = \sum_{p,p'} \left( \left( p^{k-j} - (k-j)p^{k-j-1}S_z + \frac{1}{2}(k-j)(k-j-1)p^{k-j-2}S_z^2 \right) \right) \]

\[ \times \left( p'^j - jp'^{-1}S_z' + \frac{1}{2}j(j-1)p'^{-2}S_z'^2 \right) \zeta_{pp'} |p\rangle_R |p'\rangle'_R \]

\[ = \sum_{p,p'} \left[ \left( p^{j} + \frac{1}{2}j p^{j-1} + \frac{1}{8}j(j-1)p^{j-2} \right) \right] \]

\[ + \left( p^{j} - \frac{1}{2}j p^{j-1} + \frac{1}{8}j(j-1)p^{j-2} \right) |p'\rangle_R \]

\[ \times \left[ \left( -j p^{j-1} + \frac{1}{8}(k-j)(k-j-1)p^{j-2} \right) |\uparrow\rangle \right] \zeta_{pp'} , \quad (113) \]

where, in order to simplify the above formula, we have set |−1⟩_R = |−1⟩'_R = 0.

Using equations (108) and (104) and the orthogonality of the basis states

\[ L(\Psi | J_z^{k-j} J_z^{j} |\Psi\rangle_R) = \sum_{p,p'} \left( -1 \right)^{p+p'} |\zeta_{pp'}|^2 \]

\[ \times \left( k-j \right) jp^{k-j-1} p'^{-1} \]

\[ + \sum_{p=1}^{\infty} \left( -1 \right)^{p} |\zeta_{p0}|^2 \left( k-j \right) p^{k-j-1} \left( p'^j + \frac{1}{2}j p'^{j-1} + \frac{1}{8}j(j-1)p'^{j-2} \right) \]

\[ \times \sum_{p'=0}^{\infty} \left( -1 \right)^{p'} |\zeta_{0p'}|^2 \left( p'^j + \frac{1}{2}j p'^{j-1} + \frac{1}{8}j(j-1)p'^{j-2} \right) \]

\[ \times \left( p^{j} + \frac{1}{2}j p^{j-1} + \frac{1}{8}j(j-1)p^{j-2} \right) \zeta_{pp'} , \quad (114) \]

The terms arising from \( p = 0 \) will vanish for \( k-j \geq 2 \). The terms due to \( p' = 0 \)
will vanish for \( j > 2 \). The above equation may be substituted into equation (79). When \( k-j \geq 2 \) and \( j > 2 \) this equation will resemble equation (99) but with the \( Z_z \) and \( Z_z' \) replaced with \( J_z \) and \( J_z' \). For \( k-j \leq 2 \) or \( j \leq 2 \) the supersymmetric form will have some additional constant terms. However, as will be shown in the following section, the density of states calculated from supersymmetry does reduce to the density of states calculated from the replica trick. As for replica trick we need to solve for \( \zeta_{pp'} \) in order to calculate the Green’s function correlator.
7 Evaluation of the moments of the local density of states

We shall now evaluate the moments of the local density of states for a specific case corresponding to a disordered quantum wire with a single channel. We set \( \phi = m = v = D_v = 0 \) and \( D_\phi = D_m = D \).

7.1 Solution of the ground state wave function of the transfer Hamiltonian

The replica trick transfer Hamiltonian (57) with \( \phi = m = v = D_v = 0 \) and \( D_\phi = D_m = D \) is

\[
H = 2iE(Z_z - Z'_z) + 2\eta(Z_z + Z'_z) + 2D\left((Z_+ - Z'_+)(Z_- - Z'_-) + (Z_- - Z'_-)(Z_+ - Z'_+)\right). \quad (115)
\]

The supersymmetric transfer Hamiltonian (76) has a similar form but with the vectors \( Z \) and \( Z' \) replaced with the superspins \( J \) and \( J' \). Also, as can be seen by comparing equations (90) and (91) with equation (111) the operation of \( Z \) on the replica ground state with \( n = n' = 0 \) is identical to the operation of the \( J \) on the supersymmetric ground state. Hence, the ground state eigenvalue equations must be identical when we take \( n \) and \( n' \) to zeroth order in the replica ground state. We shall calculate the zeroth order replica ground state here and use the same ground state in the supersymmetry calculations.

If substitute the transfer Hamiltonian into

\[
\langle p, p' | H | \Psi \rangle = 0, \quad (116)
\]

using, for example,

\[
\begin{align*}
\langle p, p' | | \Psi \rangle &= \zeta_{pp'} \\
\langle p, p' | Z_z | \Psi \rangle &= p\zeta_{pp'} \\
\langle p, p' | Z_+ Z_- | \Psi \rangle &= (p + 1)p\zeta_{pp'} \\
\langle p, p' | Z_+ Z'_- | \Psi \rangle &= pp'\zeta_{p+1p'+1}.
\end{align*}
\]

we obtain

\[
iE(p - p')\zeta_{pp'} + \eta(p + p')\zeta_{pp'} \\
+ 2D(p^2\zeta_{pp'} + p'^2\zeta_{pp'} - pp'\zeta_{p-1p'-1} - pp'\zeta_{p+1p'+1}) = 0. \quad (118)
\]

This equation implies that \( \zeta_{pp'}^* = \zeta_{p'p} \) which implies that \( \zeta_{pp} \) is real. If we assume that \( \zeta_{pp'} = \delta_{pp'}\zeta_{pp} \) then

\[
\eta p\zeta_{pp} + Dp^2(2\zeta_{pp} - \zeta_{p-1p-1} - \zeta_{p+1p+1}) = 0. \quad (119)
\]
Note that this equation is the same as equation (47) in Reference [19] which was obtained using the Berezinskii technique. We define a new variable \( x = \eta p / D \). As \( p \) increases to \( p + 1 \) \( x \) only increases by \( \eta / D \). It is reasonable to assume that \( \eta \ll D \) so we can take \( x \) to be a continuous variable. If \( \zeta_{pp} = \zeta(x) \) we obtain the following differential equation of \( \zeta \) in terms of \( x \)

\[
\frac{d^2 \zeta(x)}{dx^2} - \frac{1}{x} \zeta(x) = 0,
\]

which has the general solution

\[
\zeta(x) = x^{1/2} \left[ a' K_1(2x^{1/2}) + b I_1(2x^{1/2}) \right],
\]

where \( a' \) and \( b' \) are constants. In terms of \( p \)

\[
\zeta_{pp} = p^{1/2} \left[ a K_1 \left( 2 \sqrt{\frac{\eta p}{D}} \right) + b I_1 \left( 2 \sqrt{\frac{\eta p}{D}} \right) \right],
\]

where \( a \) and \( b \) are constants.

Boundary conditions require that \( \zeta \) remains finite for all \( p \) so we must have \( b = 0 \). We can find the other constant of integration, \( a \), by the normalization condition on the ground state. The normalizing condition is \( |\zeta_{00}|^2 = 1 \). We also know that \( \zeta_{00} \) must be real so

\[
\zeta_{00} = 1 = a \lim_{p \to 0} p^{1/2} K_1 \left( 2 \sqrt{\frac{\eta p}{D}} \right),
\]

which gives

\[
a = 2 \sqrt{\frac{\eta}{D}}.
\]

Hence

\[
|\Psi\rangle_R = \sum_{p=0}^{\infty} 2 \sqrt{\frac{\eta p}{D}} K_1 \left( 2 \sqrt{\frac{\eta p}{D}} \right) |p, p\rangle
\]

and

\[
\zeta_{pp'} = \delta_{pp'} 2 \sqrt{\frac{\eta p}{D}} K_1 \left( 2 \sqrt{\frac{\eta p}{D}} \right).
\]
7.2 Replica trick

By substituting equation (126) into (99)

\[
\langle G^A(E,x,x)G^R(E,x,x) \rangle_k - \langle G^R(E,x,x) \rangle_j = \left( 2i \right)^k \left( -1 \right)^j \sum_{p=0}^{\infty} p^{k-1} K_1 \left( 2 \sqrt{\eta p / D} \right)^2
\]

\[
= \frac{(2i)^k (-1)^j}{\Gamma(k-j) \Gamma(j)} \left( \frac{D}{\eta} \right)^{k-1} \frac{k}{k-1} \frac{\Gamma(k)^4}{\Gamma(2k)}. \tag{127}
\]

We have used the fact that in the limit \( \eta \to 0 \) the sum over \( p \) can be converted into an integral by defining the variable \( x = 2 \sqrt{\eta p / D} \) which may be approximated to a continuous variable since \( \eta \ll D \)

\[
\frac{4\eta}{D} \sum_{p=0}^{\infty} p^{k-1} K_1 \left( 2 \sqrt{\eta p / D} \right)^2 = 2 \left( \frac{D}{4\eta} \right)^{k-1} \int_0^{\infty} dx x^{2k-1} K_1(x)^2
\]

\[
= \left( \frac{D}{\eta} \right)^{k-1} \frac{k}{k-1} \frac{\Gamma(k)^4}{\Gamma(2k)}. \tag{128}
\]

and [34]

\[
\int_0^{\infty} x^\lambda K_1(x)^2 dx = 2^{\lambda-2} \frac{\lambda + 1}{\lambda - 1} \frac{\Gamma \left( \frac{\lambda+1}{2} \right)^4}{\Gamma(\lambda+1)}, \tag{129}
\]

if \( \lambda > 1 \). Note that as \( \eta \to 0 \) the sum in equation (127) is dominated by the terms with large \( p \). Hence, the non-compactness of \( u(1,1) \) is playing an essential role here.

By substituting (127) into equation (25) we obtain the moments of the density of states

\[
\langle \tilde{\rho}_k \rangle = \frac{1}{\pi^k} \left( \frac{D}{\eta} \right)^{k-1} \frac{\Gamma(k+1)}{(2k-1)} \tag{130}
\]

by using the relation (calculated by Maple 5.1)

\[
k! \sum_{j=0}^{k} \frac{(k-j)j}{((k-j)!)^2} = \frac{(k-1)\Gamma(2k-1)}{\Gamma(k)^3}. \tag{131}
\]

When \( k = 1 \) we find that \( \langle \tilde{\rho}_1 \rangle = 1/\pi \). This equation for \( \langle \tilde{\rho}_1 \rangle \) agrees with equation (69) in Al’tshuler and Prigodin [19] and, more recently [21], where \( s_1 = 8\eta \tau = \eta / D \). The two results differ by a factor of \( 1/\pi^k \) because what Al’tshuler and Prigodin define as \( \langle \rho^k \rangle \) we define as \( \langle \rho^k \rangle / \langle \rho \rangle^k \), and similarly for all the density of states.
7.3 Supersymmetry

To calculate the moments of the density of states from supersymmetry we substitute equation (126) into equation (114). The terms from $p = 0$ and $p' = 0$ will be negligible for small $\eta$ for all cases except $k = 1$. From equation (114) and for $k > 1$ we obtain the same result as for the replica trick, shown in equation (127). The moments of the density of states are given in equation (130).

The calculations for $k = 1$ are straightforward. It is easy to show

$$L\langle \Psi | J_z | \Psi \rangle_R = L\langle \Psi | J'_z | \Psi \rangle_R = \frac{1}{2}$$

which gives $\langle \tilde{\rho}_1 \rangle = \frac{1}{\pi}$ which agrees with equation (130).

8 Moments of the full density of states

We showed at the end of section 5 how to calculate the moments of the full density of states, $\langle \rho^k_1 \rangle$, described by equation (82). We require these moments rather than the moments of the density of states which removes the rapidly varying terms, $\langle \tilde{\rho}^k_1 \rangle$, described by equation (80). By comparing these two equations we see that to obtain the moments of the full density of states instead of $J^j_z$ we need the term

$$\sum_{m=0}^{j} \sum_{l=0}^{m} \sum_{r=0}^{m} \sum_{q=0}^{r} (\frac{1}{2})^{j-m+l} J^j_z J^m_z J^l_z J^r_z \frac{j! e^{2ikF(l+m-j)}}{(j-m)!(m-l)!!!(j-m-r)!}.$$  

A similar expression may be written for the additional non-primed terms. We have ignored that fact that the components of $J'$ do not commute since, as will be shown, the terms arising from interchanging the components are negligible. Instead of $J^k_z - j J^j_z$ we use

$$\sum_{m=0}^{j} \sum_{r=0}^{m} \sum_{l=0}^{m} \sum_{q=0}^{r} (\frac{1}{2})^{k-m+r+l+q} J^j_z J^m_z J^l_z J^r_z$$

$$\times \frac{j!(k-j)! e^{2ikF(m+r+l+q-k)}}{(j-m)!(m-l)!!!(k-j-r)!(r-q)!!}.$$  

Now we can neglect those terms which contain rapidly varying exponentials. We retain those terms for which $q = k - l - m - r$. However $q$ must be in the range $[0, r]$. This places a restriction on $l$

$$\max(0, k - m - 2r) \leq l \leq (m, k - m - r).$$  

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The above sum is now
\[
\sum_{j} \sum_{k-j \text{ min}(m,k-m-r)} \sum_{m=0}^{r=0} \sum_{l=\max(0,k-m-2r)} \left( \frac{1}{2} \right)^{2(k-m-r)} J_+^{j-m} J_-^{m-l} J_+^{k-j-r} J_-^{k-m-r-l} J_{2r+m+l-k}^r \times \frac{j! (k-j)!}{(j-m)!(m-l)! (k-j-r)!(2r+m+l-k)!(k-m-r-l)!}. \tag{136}
\]

We need to calculate something of the form \( L\langle \Psi | J_+^a J_-^b J_z^c | \Psi \rangle_R \). To get insight into how this is done we first evaluate
\[
L\langle \Psi | J_+^a J_-^b J_z^c | \Psi \rangle_R = \sum_{p=0}^{p=\infty} |\zeta_{pp'}|^2 (p-1) \ldots (p-b+1)(p-b+1) \ldots (p-b+a-1) \\
\times [p(p-b+a)(p^c + \frac{1}{2} cp^{c-1} + \frac{1}{8} (c-1)(c-2)p^{c-2}) \\
- (p-b)^2(p^c - \frac{1}{2} cp^{c-1} + \frac{1}{8} (c-1)(c-2)p^{c-2})], \tag{137}
\]
where we have used \( \zeta_{pp'} = \delta_{pp'} \zeta_{pp} \). As \( \eta \to 0 \) the sum is dominated by terms with large \( p \) (since \( \zeta_{pp} \) is proportional to \( \eta \)) and so we retain just those terms with the highest power of \( p \) which is \( p^{a+b+c-1} \). This is why in equation (133) we can swap the components of \( J' \) around, because the extra terms are a smaller power of \( p \) than \( p^{a+b+c-1} \) which become negligible as \( \eta \to 0 \). The simplified result is
\[
L\langle \Psi | J_+^a J_-^b J_z^c | \Psi \rangle_R = \sum_{p=0}^{p=\infty} |\zeta_{pp}|^2 (a+b+c)p^{a+b+c-1}. \tag{138}
\]
When we include both the primed and non-primed terms we obtain
\[
L\langle \Psi | J_+^a J_-^b J_z^c J_+^d J_-^e J_z^f | \Psi \rangle_R = \sum_{p=0}^{p=\infty} |\zeta_{pp}|^2 (a+b+c)(d+e+f)p^{a+b+c+d+e+f-2}. \tag{139}
\]
Substituting these identities into equation (82) gives
\[
\langle p_1(\eta, E)^k \rangle = \frac{k!}{\pi^k} \sum_{p=0}^{p=\infty} |\zeta_{pp}|^2 \left( \sum_{j=0}^{j=k} \frac{j! (k-j)!}{k-j} \sum_{m=0}^{m=0} \sum_{r=0}^{r=0} \sum_{l=\max(0,k-m-2r)}^{l=\max(0,k-m-r)} \left( \frac{1}{2} \right)^{2(k-m-r)} \times [(j-m)!(m-l)!(k-j-r)!(2r+m+l-k)!(k-m-r-l)!]^{-1}. \tag{140}
\]
The sum over \( p \) can be evaluated as in the previous section using equation (126) and (128).
The sum over \( j, m, r \) and \( l \) can be rewritten as [23]

\[
S = \sum_{j=0}^{k} \frac{j(k-j)}{j!(k-j)!} \sum_{m=0}^{k-j} \sum_{r=0}^{m} \sum_{l=r}^{m-r} \delta_{w} \left( \frac{1}{\pi} \right)^{2(k-m-r)}
\times [(j-m)!(m-l)!(k-j-r)!(2r+m+l'-k)!(k-m-r-l')!]^{-1}.
\]

(141)

If we write the kronecker delta as

\[
\delta_{w} = \frac{1}{2\pi} \int_{0}^{2\pi} e^{i(l-l')t} dt
\]

(142)

and rewrite the sum over \( l' \) as a sum over \( q = l' - k - m + 2r \) it can be shown that

\[
S = \frac{(k-1)\Gamma(2k-1)}{k\Gamma(k)^{4}} \frac{1}{2\pi} \int_{0}^{2\pi} dt \left( \frac{1}{4} e^{-it} + 1 + e^{it} \right)^{k},
\]

(143)

by making use of equation (131). The integral can be evaluated after a binomial expansion. After integration the resulting sum may be simplified using Maple 5.1 so that

\[
S = \frac{(k-1)(2k-1)\Gamma(2k-1)^{2}}{2^{k-1}k^{2}\Gamma(k)^{6}}.
\]

(144)

The moments of the total density of states is thus

\[
\langle \rho_{1}(\eta, E) \rangle^{k} = \frac{1}{\pi^{k}} \left( \frac{D}{2\eta} \right)^{k-1} \frac{\Gamma(2k-1)}{\Gamma(k)}. \]

(145)

This agrees with equation (68) in Al’tshuler and Prigodin [19] although we have an additional factor of \( 1/\pi^{k} \), as discussed previously, due to different scaling of the density of states. Using equation (23) we can use the moments of the density of states in equation (145) to calculate the \( k \)th moment of the participation ratio

\[
P^{(k)} = (2D)^{k-1}\Gamma(k),
\]

(146)

where we have used \( \langle \rho \rangle = \lim_{\eta \to 0} \langle \rho_{1} \rangle = 1/\pi \).

9 Probability distribution function

The moments of the participation ratio may be used to find moments of density of states, \( \langle \rho_{f}^{k} \rangle \), which are derived from some arbitrary weight \( f \) as defined in

32
The probability distribution function of any local density of states, \( W(\rho_f) \), may be obtained from these moments of the density of states by the following \([19]\)

\[
\int_0^\infty W(\rho) \exp(-p\rho) d\rho = \exp \left[ \sum_{k=1}^\infty \frac{(-p)^k \langle \rho_f^k \rangle_c}{k!(\rho_f)^k} \right]. \tag{147}
\]

The cumulants are defined by

\[
\langle \rho_f^k \rangle_c = \langle \rho_f^k \rangle - \langle \rho_f^{k-1} \rangle. \tag{148}
\]

Solving equation (147) for \( W \) requires an inverse Laplace transform.

Following Al’tshuler and Prigodin \([19]\) we now summarise how one can construct a density of states which is related to the NMR line shape. In a metal the NMR line undergoes a frequency shift, known as the Knight shift, which is proportional to the local density of states derived using (5) with weight

\[
f_2(E) = -\frac{n_F(E + g\mu_e H) - n_F(E - g\mu_e H)}{2g\mu_e H}. \tag{149}
\]

This weight is the derivative of the Fermi distribution function, \( n_F \), about the Zeeman splitting \( g\mu_e H \). The Fermi distribution function is

\[
n_F(E) = \left[ \exp \left( \frac{E - \epsilon_F}{T} \right) + 1 \right]^{-1}, \tag{150}
\]

where \( \epsilon_F \) is the Fermi energy. The density of states derived from this weight within the region \( T \gg g\mu_e H \) will be defined as \( \rho_2(T, \epsilon_F) \). In this high temperature limit

\[
f_2(E) = \frac{1}{4T} \cosh^{-2} \left( \frac{E - \epsilon_F}{2T} \right). \tag{151}
\]

The local density of states, \( \rho_2 \), fluctuates throughout the sample. Hence, the Knight shift also fluctuates. The net result of these various Knight shifts is a broadening of the NMR line shape. If the relaxation rate is of the sample is less than the Knight shift the NMR line shape is fully determined by the superposition of all the local Knight shifts. This is proportional to the distribution of the density of states, \( \rho_2 \).

By substituting equation (146) and (151) into equation (24) we find the moments of this new density of states are

\[
\langle \rho_2(T, \epsilon_F)^k \rangle = \left( \frac{D}{2T} \right)^{k-1} \frac{\Gamma(k)^2 \Gamma\left(\frac{3}{2}\right)}{\pi \Gamma(k + \frac{1}{2})}. \tag{152}
\]
where we have used [34]

$$\int_{-\infty}^{\infty} dy \cosh^{-2k} y = 2 \frac{\Gamma(k) \Gamma\left(\frac{3}{2}\right)}{\Gamma(k + \frac{1}{2})}. \quad (153)$$

The sum in equation (147) is dominated by the large $k$ terms. If $k$ is large we can assume that the $k$th cumulant is approximately equal to the $k$th moment.

---

Fig. 1. The probability distribution function, equation (155), for a number of different values of $s = 2T/\pi D$. For large $s$ (high temperature and low disorder) the distribution is symmetric so that the typical value of the density of states is the same as the average value. As $s$ decreases (low temperature and high disorder) the distribution becomes skewed and there is a marked difference between the typical and the average value of the density of states. This probability distribution can be related to the NMR line shape.
of the density of states

\[ \langle \rho_k^2 \rangle_c = \langle \rho_k^2 \rangle. \]  

(154)

After substituting the moments shown in equation (152) into equation (147) and taking the inverse Laplace transform Al’tshuler and Prigodin [19] obtained

\[ W(\rho_2) = \frac{s}{2\pi} \int_{-\infty}^{\infty} dx \sinh 2x \sin(\pi sx) \exp \left( -s \left[ \frac{\rho_2}{\langle \rho_2 \rangle} \cosh^2 x + x^2 - \frac{\pi}{4} \right] \right), \]

(155)

where \( s = 2T/\pi D \) in our notation but \( s = 16T\tau/\pi \) in Al’tshuler and Prigodin’s notation. The relationship between the disorder strength, \( D \), and the scattering time, \( \tau \), is \( 8\tau = D^{-1} \).

The probability distribution function at large temperature of the density of states, \( \rho_2 \), is plotted in figure 1 for a number of values of \( s \). Note that for \( s \leq 1 \) the distribution is quite asymmetric and the typical (most probable) value is significantly less than the average value. We have set various constants to unity, such as the Boltzmann constant, \( k_B \), the Fermi velocity, \( v_F \), and \( \hbar \). If all these constants are included we find that near the Fermi energy \( s = 2k_B T \hbar v_F / \pi D \) or, in terms of the mean free path, \( l = v_F \tau = (\hbar v_F)^2 / 8D \), we have \( s = 16k_B Tl / \pi \hbar v_F = 6.67 \times 10^{11} Tl / v_F \). If we have a sample with a mean free path of 1 \( \mu \)m and a Fermi velocity of \( v_F = 1 \times 10^6 \) ms\(^{-1} \) then some skewness in the NMR line shape should be observable when the temperature is 1.5 K or less (\( s \leq 1 \)).

10 Conclusion

We have presented a general formalism to calculate exactly the disorder average of a product of Green’s functions, \( \langle G^A(E, x, x) k^{-j} G^R(E, x, x) \rangle \), for the random Dirac equation using both the replica trick and supersymmetry. This extends the formalism developed recently to calculate the disorder average of one Green’s function using the replica trick [7] and supersymmetry [6,7]. The problem is reduced to that of finding the ground state of a zero-dimensional Hamiltonian involving a pair of “spins” that are elements of \( u(1, 1) \). The moments of the different density of states can be written in terms of the expectation values of various “spin” operators in this ground state. We explicitly found this ground state wave function and the associated expectation values for the case of the Dirac equation corresponding to a disordered quantum wire with a single channel. The non-compactness of \( u(1, 1) \) played an essential role in the results. We showed how the moments of the local density of states changed when it was averaged over atomic length scales. Our results for the
moments agree with those obtained previously by Al’tshuler and Prigodin [19] using the Berezinskii diagram method. In our view, our derivation is more transparent from a field-theoretic point of view. Future work should focus on finding the ground state of the transfer Hamiltonian corresponding to more general random Dirac equations, particularly those associated with quantum phase transitions in random spin chains.

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A APPENDIX

In this appendix we derive the prefactor in the replica trick expansion of the $k$th power of the Green’s function shown in equation (33). Consider an $N$ dimensional integral of the form

$$I_N = \int d^N x F(x_i^2)$$  \hspace{1cm} (A.1)

where $F(x_i^2)$ is a function which depends only on the length $r = \sum_{i=1}^N x_i^2$. By making a change of variables into polar coordinates:

$$(x_1, x_2, \ldots, x_N) \rightarrow (r, \phi, \theta_1, \ldots, \theta_{N-2})$$  \hspace{1cm} (A.2)

we can evaluate the integrals over the $\theta$'s and the $\phi$ (since $F(x_i^2) = F(r)$ has no dependence on the $\theta$'s or the $\phi$) and we are left with [35]

$$I_N = \frac{\pi^{N/2}}{\Gamma(N/2)} \int_0^\infty dr r^{N-2} F(r).$$  \hspace{1cm} (A.3)

For example, if $s$ and $s^*$ are one dimensional variables such that $s = x_1 + ix_2$ and $A$ is a constant

$$\frac{1}{A^k} = \frac{1}{k!} \left[ \int ds^* ds (s^* s)^k e^{-s^* A s} \right] \left[ \int ds^* ds e^{-s^* A s} \right]^{-1}$$  \hspace{1cm} (A.4)

then the integral has dimension $N = 2$ and $F(r) = r^k e^{-Ar}$ where $r = s^* s = x_1^2 + x_2^2$. Using equation (A.3) it can be shown that in terms of $r$ the integral
\[
\frac{1}{A^k} = \frac{1}{k!} \left[ \int_0^\infty dr r^k e^{-Ar} \right] \left[ \int_0^\infty dr e^{-Ar} \right]^{-1} \quad (A.5)
\]

We now consider a replicated version of equation (A.4). If this equation is replicated \(n\) times

\[
\frac{1}{A^k} = \frac{X}{k!} \left[ \prod_{l=1}^n ds_l^* ds_l \left( \sum_{l=1}^n s_l^* s_l \right)^k \exp \left( -\sum_{l=1}^n s_l^* A s_l \right) \right] \times \left[ \int \prod_{l=1}^n ds_l^* ds_l \exp \left( -\sum_{l=1}^n s_l^* A s_l \right) \right]^{-1} \quad (A.6)
\]

where \(X\) is a factor which must be calculated. We can write

\[
s_l = x_{2l-1} + ix_{2l} \quad \text{so now we are dealing with a } N = 2n \text{ dimensional integral and } r = \sum_{l=1}^n s_l^* s_l = \sum_{l=1}^{2n} x_l^2.
\]

The integral becomes

\[
\frac{1}{A^k} = \frac{X}{k!} \frac{\pi^n/\Gamma(n)}{\pi^n/\Gamma(n)} \left[ \int_0^\infty dr r^{n-1} r^k e^{-Ar} \right] \left[ \int_0^\infty dr r^{n-1} r^{-A} \right]^{-1}. \quad (A.7)
\]

The integrals over the \(r\) variable may be evaluated to show that

\[
\frac{1}{A^k} = \frac{X}{k!} \frac{1}{A^k} \frac{\Gamma(n+k)}{\Gamma(n)} \quad (A.8)
\]

so

\[
X = \frac{k! \Gamma(n)}{\Gamma(n+k)} \quad (A.9)
\]

and hence the replicated form of equation (A.4) is

\[
\frac{1}{A^k} = \frac{\Gamma(n)}{\Gamma(n+k)} \left[ \prod_{l=1}^n ds_l^* ds_l \left( \sum_{l=1}^n s_l^* s_l \right)^k \exp \left( -\sum_{l=1}^n s_l^* A s_l \right) \right] \times \left[ \int \prod_{l=1}^n ds_l^* ds_l \exp \left( -\sum_{l=1}^n s_l^* A s_l \right) \right]^{-1} \quad (A.10)
\]

Note that when \(n = 1\) we get back to equation (A.4). If \(k = 1\) then the factor before the integral is \(1/n\) which agrees with Bocquet [7].

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