Equilateral Sets in Banach Spaces of the form $C(K)$

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Abstract

We show that for "most" compact non-metrizable spaces, the unit ball of the Banach space $C(K)$ contains an uncountable 2-equilateral set. We also give examples of compact non-metrizable spaces $K$ such that the minimum cardinality of a maximal equilateral set in $C(K)$ is countable.

Introduction

A subset $S$ of a metric space $(M, d)$ is said to be equilateral if there is a constant $\lambda > 0$ such that $d(x, y) = \lambda$, for $x, y \in S, x \neq y$; we also call such a set a $\lambda$-equilateral set. An equilateral set $S \subseteq M$ is said to be maximal if there is no equilateral set $B \subseteq M$ with $A \subsetneq B$.

Equilateral sets have been studied mainly in finite dimensional spaces, see [13], [15] and [14] for a survey on equilateral sets. More recently there are also results on infinite dimensions, see [11], [7] and also on maximal equilateral sets, see [16].

In this paper we study equilateral sets in Banach spaces of the form $C(K)$, where $K$ is a compact space. The paper is divided into two sections. In the first section we introduce the combinatorial concept of a linked family of pairs of a set $\Gamma$; using this concept we characterize those compact spaces $K$, such that the unit ball of $C(K)$ contains a $(1 + \varepsilon)$-separated (equivalently: a 2-equilateral) set of a given cardinality (Theorem 1). Then we show that in "most" cases a compact non-metrizable space $K$ admits an uncountable linked family of closed pairs and hence its unit ball contains an uncountable 2-equilateral set (Theorem 2).

In the second section we focus on maximal equilateral sets on the space $C(K)$. Following [16] (Definition 2), given a normed space $E$, we denote by $m(E)$ the minimum cardinality of a maximal equilateral set in $E$. The main results here are the following: for every infinite locally compact space $K$ we have $m(C_0(K)) \geq \omega$ (Theorem 3) (thus in particular, $m(C(K)) = \omega$, for any infinite compact metric space $K$). For every infinite product $K = \prod_{\gamma \in \Gamma} K_\gamma$ of nontrivial compact metric

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spaces, \( m(C(K)) = |\Gamma| \) (Theorem 4). Then using proper linked families of pairs of \( \mathbb{N} \), we give a variety of examples of compact non metrizable spaces \( K \) (including scattered compact and the Stone-Čech compactification \( \beta \Gamma \) of any infinite discrete set \( \Gamma \)) such that \( m(C(K)) = \omega \) (Theorems 6, 7 and Corol.5).

If \( E \) is any (real) Banach space then \( B_X \) denotes its closed unit ball. If \( K \) is any compact Hausdorff space, then \( C(K) \) is the Banach space of all continuous real functions on \( K \) endowed with the supremum norm \( \| \cdot \|_\infty \).

Linked families and equilateral sets in Banach spaces of the form \( C(K) \)

In this section we introduce the concept of a linked family of pairs of a set \( \Gamma \) and then use this, in order to investigate the existence of equilateral sets in \( C(K) \), where \( K \) is any compact (non metrizable) space.

**Definition 1.** Let \( \mathcal{F} = \{(A_\alpha, B_\alpha) : \alpha \in A\} \) be a family of pairs of a nonempty set \( \Gamma \). We say that this family is **linked** (or **intersecting**) if

1. \( A_\alpha \cap B_\alpha = \emptyset \), for \( \alpha \in A \)
2. \( A_\alpha \cup B_\alpha \neq \emptyset \), for \( \alpha \in A \) and
3. for \( \alpha, \beta \in A \) with \( \alpha \neq \beta \) we have, either \( A_\alpha \cap B_\beta \neq \emptyset \) or \( A_\beta \cap B_\alpha \neq \emptyset \).

If we replace condition (2) with the stronger one: \( (2') A_\alpha \neq \emptyset \neq B_\alpha \), for \( \alpha \in A \) we shall say that \( \mathcal{F} \) is a linked family of nonempty pairs.

We note the following easily verified facts:

1. If \( \alpha \neq \beta \in A \) then \( A_\alpha \neq A_\beta \) and \( B_\alpha \neq B_\beta \) (and hence)
2. there is at most one \( \alpha \in A \) such that \( A_\alpha = \emptyset \) and at most one \( \beta \in A \) such that \( B_\beta = \emptyset \).
3. If \( \mathcal{F} \) is a linked family of nonempty pairs of the set \( \Gamma \), then the family \( \mathcal{F} \cup \{(\Gamma, \emptyset), (\emptyset, \Gamma)\} \) is a linked family of pairs of \( \Gamma \).

**Examples 1**

1. Let \( \{A_\alpha : \alpha \in A\} \) be a family of distinct subsets of the set \( \Gamma \). Then the family \( \{(A_\alpha, \Gamma \setminus A_\alpha) : \alpha \in A\} \) is linked. Assuming furthermore that \( \emptyset \neq A_\alpha \neq \Gamma, \) for \( \alpha \in A \), we get that \( \mathcal{F} \) is a linked family of nonempty pairs. It follows in particular that the family \( \{(A, \Gamma \setminus A) : A \subseteq \Gamma \} \) (resp. the family \( \{(A, \Gamma \setminus A) : \emptyset \neq A \subseteq \Gamma \} \) ) is linked (resp. a linked family of nonempty pairs).

2. Let \( K \) be a compact totally disconnected space, then the family \( \{(V, K \setminus V) : V \text{ is a clopen subset of } K\} \) is linked.
3. The family \( F = \{(\{1\}, \{2\}), (\{2\}, \{3\}), (\{3\}, \{1\})\} \) is a linked family of nonempty pairs of the set \( \Gamma = \{1, 2, 3\} \).

In section 2 we shall present much more examples of linked families. Now we are going to examine the interrelation between the concepts of linked families and equilateral sets in Banach spaces of the form \( C(K) \), where \( K \) is a compact Hausdorff space.

**Lemma 1.** Let \( K \) be a compact Hausdorff space and \( S \subseteq [0,1]^K \cap C(K) \). Set \( A_f = f^{-1}(\{0\}) \) and \( B_f = f^{-1}(\{1\}) \), for \( f \in S \). Then the following are equivalent:

(a) The family \( F = \{(A_f, B_f) : f \in S\} \) of (closed) pairs of \( K \) is linked.

(b) The set \( S \) is 1-equilateral in \( C(K) \).

**Proof.** (a) \( \Rightarrow \) (b) Let \( f, g \in S \) with \( f \neq g \); clearly \( 0 < \|f - g\|_\infty \leq 1 \). Since we either have \( A_f \cap B_g \neq \emptyset \) or \( A_g \cap B_f \neq \emptyset \), there is a \( t_0 \in K \) such that \( |f(t_0) - g(t_0)| = 1 \), hence \( \|f - g\|_\infty = 1 \).

(b) \( \Rightarrow \) (a) If \( f \neq g \in S \), then \( \|f - g\|_\infty = 1 \); so by the compactness of \( K \), there is a \( t_0 \in K \) such that \( |f(t_0) - g(t_0)| = \|f - g\|_\infty = 1 \). Since \( 0 \leq f(t_0), g(t_0) \leq 1 \) we get that \( \{f(t_0), g(t_0)\} = \{0, 1\} \). Therefore, either \( t_0 \in A_f \cap B_g \) or \( t_0 \in A_g \cap B_f \) and \( F \) is as required.

**Note:** Since there is at most one \( f_0 \in S \) with \( A_{f_0} = \emptyset \) (\( \iff \inf f(f_0) > 0 \)) and at most one \( g_0 \in S \) with \( B_{g_0} = \emptyset \) (\( \iff \|g_0\|_\infty < 1 \)), we get that the family \( \{(A_f, B_f) : f \in S \setminus \{f_0, g_0\}\} \) is a linked family of nonempty closed pairs of \( K \) and the set \( S \setminus \{g_0\} \) is a subset of the positive part \( S^+_{C(K)} \) of the unit sphere \( S_{C(K)} \) of the space \( C(K) \).

**Lemma 2.** Let \( F = \{(A_\alpha, B_\alpha) : \alpha \in \mathcal{A} \} \) be a linked family of closed pairs of the compact space \( K \). Then we can associate with \( F \) a 1-equilateral subset \( S \) of \( C(K) \) with \( |S| = |\mathcal{A}| \) and \( S \subseteq [0,1]^K \cap C(K) \).

**Proof.** Let \( \alpha \in \mathcal{A} \); we distinguish the following cases for the pair \( (A_\alpha, B_\alpha) \):

(I) \( A_\alpha \neq \emptyset \neq B_\alpha \). We consider a Urysohn function \( f_\alpha : K \rightarrow [0,1] \) so that \( f_\alpha(x) = 0 \) for \( x \in A_\alpha \) and \( f_\alpha(x) = 1 \) for \( x \in B_\alpha \); clearly \( \inf f_\alpha = 0 < \|f_\alpha\|_\infty = 1 \).

(II) Assume that \( A_\alpha = \emptyset \), thus \( B_\alpha \neq \emptyset \). If \( B_\alpha \neq K \), pick \( t_0 \in K \setminus B_\alpha \) and consider a Urysohn function \( f_\alpha : K \rightarrow [0,1] \) so that \( f_\alpha|_{B_\alpha} = 1 \) and \( f_\alpha(t_0) = 0 \). In case when \( B_\alpha = K \), we let \( f_\alpha = 1 \) on \( K \).

(III) Assume that \( B_\alpha = \emptyset \), thus \( A_\alpha \neq \emptyset \). This case is similar to case (II). So we consider a Urysohn function \( f_\alpha : K \rightarrow [0,1] \) so that \( f_\alpha|_{A_\alpha} = 0 \) and \( f_\alpha(t_0) = 1 \) for some \( t_0 \in K \setminus A_\alpha \), if \( A_\alpha \neq K \) and define \( f_\alpha \) to be the constant zero function in case when \( A_\alpha = K \).

Now set \( A'_\alpha = f_\alpha^{-1}(\{0\}) \) and \( B'_\alpha = f_\alpha^{-1}(\{1\}) \), for \( \alpha \in \mathcal{A} \). Since \( A'_\alpha \cap B'_\alpha = \emptyset \), \( A_\alpha \subseteq A'_\alpha \) and \( B_\alpha \subseteq B'_\alpha \) for \( \alpha \in \mathcal{A} \), we have that the family \( \{(A'_\alpha, B'_\alpha) : \alpha \in \mathcal{A} \} \) is a linked family of closed pairs of the space \( K \), hence by Lemma 1 the set \( S = \{f_\alpha : \alpha \in \mathcal{A}\} \) is a 1-equilateral subset of \( [0,1]^K \cap C(K) \).
Lemma 1. If in the proof of Lemma 2 we consider (as we may) continuous functions \( f_\alpha : K \to [-1, 1] \) such that \( f_\alpha / A_\alpha = 1 \) and \( f_\alpha / B_\alpha = -1 \), then the set \( \{ f_\alpha : \alpha \in \mathcal{A} \} \) is a 2-equilateral subset of the unit ball of \( C(K) \).

Proof. (2) Let \( F = \{ (A_\alpha, B_\alpha) : \alpha \in \mathcal{A} \} \) be a family of disjoint pairs of a set \( \Gamma \). Set \( \mathcal{F} = \{ (A_\alpha, B_\alpha) : \alpha \in \mathcal{A} \} \) where \( A_\alpha = cl_{\beta \Gamma} A_\alpha, B_\alpha = cl_{\beta \Gamma} B_\alpha \) and \( \beta \Gamma \) is the Stone–Čech compactification of the discrete set \( \Gamma \). Then it is easy to see that \( \mathcal{F} \) is a linked family of (nonempty) pairs of \( \Gamma \) iff \( \mathcal{F} \) is a linked family of (nonempty) pairs of \( \beta \Gamma \).

Remarks 1
(1) If in the proof of Lemma 2 we consider (as we may) continuous functions \( f_\alpha : K \to [-1, 1] \) such that \( f_\alpha / A_\alpha = 1 \) and \( f_\alpha / B_\alpha = -1 \), then the set \( \{ f_\alpha : \alpha \in \mathcal{A} \} \) is a 2-equilateral subset of the unit ball of \( C(K) \).

(2) Let \( \mathcal{F} = \{ (A_\alpha, B_\alpha) : \alpha \in \mathcal{A} \} \) be a family of disjoint pairs of a set \( \Gamma \). Set \( \mathcal{F} = \{ (A_\alpha, B_\alpha) : \alpha \in \mathcal{A} \} \) where \( A_\alpha = cl_{\beta \Gamma} A_\alpha, B_\alpha = cl_{\beta \Gamma} B_\alpha \) and \( \beta \Gamma \) is the Stone–Čech compactification of the discrete set \( \Gamma \). Then it is easy to see that \( \mathcal{F} \) is a linked family of (nonempty) pairs of \( \Gamma \) iff \( \mathcal{F} \) is a linked family of (nonempty) pairs of \( \beta \Gamma \).

(3) Let \( K \) be a compact space and \( S \subseteq [0, 1]^K \cap C(K) \) be a 1-equilateral set. We consider the linked family \( \mathcal{F} = \{ (A_f, B_f) : f \in S \} \) given by Lemma 1. Then it is not difficult to verify that \( \mathcal{F} \) is a maximal linked family of closed pairs of \( K \) if the set \( S \) is a maximal (with respect to inclusion) 1-equilateral subset of \( [0, 1]^K \cap C(K) \), endowed with the norm metric (the equilateral set \( S \) is not necessarily maximal in the space \( C(K) \)), see Remark 5(3).

Theorem 1. Let \( K \) be a compact Hausdorff space and \( \alpha \) be an infinite cardinal. The following are equivalent:

1. The unit ball \( B_{C(K)} \) of \( C(K) \) contains a \( \lambda \)-equilateral set with \( \lambda > 1 \), of size \( \alpha \).

2. The unit sphere \( S_{C(K)} \) (resp. the positive part of the unit sphere \( S_{C(K)}^{+} \)) of \( C(K) \) admits a 2-equilateral (resp. a 1-equilateral) set of size \( \alpha \).

3. The unit ball \( B_{C(K)} \) of \( C(K) \) contains a \((1+\varepsilon)\)-separated set, for some \( \varepsilon > 0 \), of size equal to \( \alpha \).

4. There exists a linked family of closed (nonempty) pairs in \( K \) of size equal to \( \alpha \).

Proof. (2) \( \Rightarrow \) (1) Let \( S \) be a 1-equilateral subset of \( S_{C(K)}^{+} \) with \( |S| = \alpha \). Then by Lemma 1, \( \mathcal{F} = \{ (A_f, B_f) : f \in S \} \) is a linked family of closed pairs of \( K \) with \( |\mathcal{F}| = \alpha \). Therefore, by Lemma 2 and Remark 1(1) \( \mathcal{F} \) defines a 2-equilateral set contained in the unit ball of \( C(K) \).

(1) \( \Rightarrow \) (3) is obvious.

(3) \( \Rightarrow \) (4) Let \( D \subseteq B_{C(K)} \) be a \((1+\varepsilon)\)-separated set \((\varepsilon > 0)\), with \( |D| = \alpha \). We may assume that \( ||f||_\infty = 1 \), for \( f \in D \). We define \( A_f = f^{-1}([-1, -\frac{\varepsilon}{2}]) \) and \( B_f = f^{-1}([\frac{\varepsilon}{2}, 1]) \), for \( f \in D \); clearly \( A_f \cup B_f \neq \emptyset \). Let \( f, g \in D \) with \( f \neq g \), so there is a \( t_0 \in K \) such that \( ||f-g||_\infty = |f(t_0) - g(t_0)| \geq 1 + \varepsilon \). Assume without loss of generality that \( f(t_0) < g(t_0) \); then we have, \( f(t_0) \leq -\frac{\varepsilon}{2} \) and \( g(t_0) \geq \frac{\varepsilon}{2} \), that is, \( A_f \cap B_g \neq \emptyset \). For, suppose otherwise, then we would either have \( f(t_0) > -\frac{\varepsilon}{2} \) or \( g(t_0) < \frac{\varepsilon}{2} \). Assuming that \( f(t_0) > -\frac{\varepsilon}{2} \) we get that \( -\frac{\varepsilon}{2} < f(t_0) < g(t_0) \leq 1 \), hence \( g(t_0) - f(t_0) < 1 + \frac{\varepsilon}{2} \), a contradiction.

In a similar way we get a contradiction assuming that \( g(t_0) < \frac{\varepsilon}{2} \). It follows that the family \( \mathcal{F} = \{ (A_f, B_f) : f \in D \} \) defined above is a linked family of closed pairs in \( K \) of size \( \alpha \).
(4) ⇒ (2) This implication is a direct consequence of Lemma 2. The proof of the Theorem is complete.

Let $K$ be an infinite compact space; as is well known the Banach space $c_0$ is isometrically embedded in $C(K)$, hence the assertions of Theorem 1 hold true for $\alpha = \omega$. The following questions are open for us:

Questions. Let $K$ be a compact Hausdorff non metrizable space.

1. Does there exist an uncountable $(1 + \varepsilon)$-separated $D \subseteq B_{C(K)}$? Does there exist (at least) an uncountable $D \subseteq B_{C(K)}$ such that $f \neq g \in D \Rightarrow \|f - g\|_\infty > 1$?

Note that the unit ball of every infinite dimensional Banach space contains an infinite $(1 + \varepsilon)$-separated set, see [5].

2. Does the space $C(K)$ contain an uncountable equilateral set?

We note that, regarding question (1), by transfinite induction it can be shown that there is an uncountable $D \subseteq S^+_{C(K)}$ such that $f \neq g \in D \Rightarrow \|f - g\|_\infty \geq 1$.

However we can show that in “most” cases the answer to the above questions is positive. For this purpose we recall that a (Hausdorff and completely regular) topological space $X$ is said to be:

(i) hereditarily Lindelöf (HL) if every subspace $Y$ of $X$ is Lindelöf. It is well known that a space $X$ is HL iff there is no uncountable right separated family in $X$; that is, a family $\{t_\alpha : \alpha < \omega_1\} \subseteq X$ such that $t_\alpha \notin \text{cl}_X \{t_\beta : \alpha < \beta < \omega_1\}$ for $\alpha < \omega_1$ and

(ii) hereditarily separable (HS) if every subspace $Y$ of $X$ is separable. It is also well known that a space $X$ is HS iff there is no uncountable left separated family in $X$; that is, a family $\{t_\alpha : \alpha < \omega_1\} \subseteq X$ such that $t_\alpha \notin \text{cl}_X \{t_\beta : \beta < \alpha\}$ for $1 \leq \alpha < \omega_1$ (see [8] p. 151).

We are going to use the following standard

Fact. A compact space $K$ is HL if and only if it is perfectly normal (i.e. each closed subset of $K$ is $G_\delta$).

Theorem 2. Let $K$ be a compact space. If $K$ satisfies one of the following conditions, then $K$ admits an uncountable linked family of closed pairs (and hence by Theorem 1 the unit ball of $C(K)$ contains an uncountable 2-equilateral set).

1. There exists a closed subset $\Omega$ of $K$ admitting uncountably many relatively clopen sets (in particular $\Omega$ is non metrizable and totally disconnected).

2. $K$ is non hereditarily Lindelöf.

3. $K$ is non hereditarily separable.

4. $|K| > c = 2^\omega$ (=the cardinality of continuum).

5. $K$ admits a Radon probability measure of uncountable type.
Proof. (1) Let $\mathcal{B}$ be any uncountable family of clopen sets in $\Omega \subseteq K$. Then clearly the family $\mathcal{F} = \{(V, \Omega \setminus V) : V \in \mathcal{B}\}$ is an uncountable linked family of closed pairs in $K$. (It is clear that condition (1) can be stated as follows: there is a closed subset $\Omega \subseteq K$ such that the unit ball of $C(\Omega)$ has uncountably many extreme points).

(2) Let $\{t_\alpha : \alpha < \omega_1\} \subseteq K$ be an uncountable right separated family. Set $A_\alpha = \{t_\alpha\}$ and $B_\alpha = cl_K\{t_\beta : \alpha < \beta < \omega_1\}$ for $\alpha < \omega_1$. Then it is easy to see that the family $\{(A_\alpha, B_\alpha) : \alpha < \omega_1\}$ is a linked family of closed (nonempty) pairs of $K$.

(3) Since $K$ is non HS, there exists an uncountable left separated family in $K$ and the proof is similar to that of the previous case.

(4) This follows from (2), since if $|K| > c$ then $K$ is not HL. Indeed, any compact HL space is first countable (each point set of $K$ is $G_\delta$ by the Fact preceding the Theorem). By a classical result of Archangel’skii each compact first countable space has cardinality $\leq c$.

(5) Let $\mu \in P(K)$ be a Radon probability measure on $K$ of uncountable type (i.e., $dim L_1(\mu) \geq \omega_1$). We consider (as we may) an uncountable stochastically independent family $\{\Gamma_\alpha : \alpha < \omega_1\}$ of $\mu$-measurable subsets of $K$. This means that

$$\mu\left(\bigcap_{k=1}^n \varepsilon_k \Gamma_{\alpha_k}\right) = \frac{1}{2^n}, \quad \alpha_1 < \cdots < \alpha_n, \quad and \quad \varepsilon_1, \cdots, \varepsilon_n \in \{-1, 1\}$$

where, if $A \subseteq K$ we let $1 \cdot A = A$ and $(-1) \cdot A = K \setminus A$. By the regularity of the measure $\mu$ we can find compact sets $A_\alpha \subseteq \Gamma_\alpha$ and $B_\alpha \subseteq K \setminus \Gamma_\alpha$, for each $\alpha < \omega_1$ such that $\mu((\Gamma_\alpha \setminus A_\alpha) < \frac{3}{8}$ and $\mu((K \setminus \Gamma_\alpha) \setminus \varepsilon_\alpha) < \frac{3}{8}$ (1).

Claim. The family of closed pairs $\{(A_\alpha, B_\alpha) : \alpha < \omega_1\}$ is linked.

Proof of the Claim: Let $\alpha < \beta < \omega_1$; we are going to show the stronger property $\mu(A_\alpha \cap B_\beta) > 0$ and $\mu(A_\beta \cap B_\alpha) > 0$. Assume that $\mu(A_\alpha \cap B_\beta) = 0$; it then follows from (1) that $\mu(\Gamma_\alpha \setminus A_\alpha) = \mu(\Gamma_\alpha) - \mu(A_\alpha) = \frac{1}{2} - \mu(A_\alpha) < \frac{1}{4}$, thus $\mu(A_\alpha) > \frac{1}{4} - \frac{3}{8} = \frac{3}{8}$. Also $\mu((K \setminus \Gamma_\alpha) \setminus B_\beta) = \mu(K \setminus \Gamma_\beta) = \mu(\Gamma_\beta) - \mu(B_\beta) = \frac{1}{2} - \mu(B_\beta) < \frac{1}{8}$, thus $\mu(B_\beta) > \frac{1}{8} - \frac{3}{8} = \frac{3}{8}$.

Therefore $\mu(\Gamma_\alpha \cup (K \setminus \Gamma_\beta)) \geq \mu(A_\alpha \cup B_\beta) = \mu(A_\alpha) + \mu(B_\beta) > \frac{3}{8} + \frac{3}{8} = \frac{3}{4}$, a contradiction because $\mu(\Gamma_\alpha \cup (K \setminus \Gamma_\beta)) = \frac{3}{4}$.

In a similar way we get that $\mu(A_\beta \cap B_\alpha) > 0$ and the proof of the Claim is complete.

The above Theorem has some interesting consequences. If $K$ is any compact space, then $P(K)$ denotes the set of Radon probability measures on $K$. Recall that both spaces $P(K)$ and $B_{C(K)'}$ are compact with the weak* topology.

**Corollary 1.** Let $K$ be any compact non metrizable space. Denote by $\Omega$ any of the compact spaces $K \times K$, $P(K)$ and $B_{C(K)'}$. Then the unit ball of $C(\Omega)$ contains an uncountable 2-equilateral set.
Proof. The compact space $\Omega$ is not HL. Indeed, if $\Omega = K \times K$, then since $K$ is not metrizable, its diagonal $\Delta = \{(x, x) : x \in K\}$ is closed but not $G_\delta$ subset of $\Omega$ (by a classical result if the diagonal of a compact space $K$ is a $G_\delta$ subset of $K \times K$ then $K$ is metrizable). So by the Fact before Th.1 the space $K \times K$ is not HL.

Let $\Omega = P(K)$. We consider the continuous map $\Phi : K \times K \to P(K)$ : $\Phi(x, y) = \frac{1}{2}\delta_x + \frac{1}{2}\delta_y$ ($\delta_x$ is the Dirac measure at $x \in K$). Then $\Delta = \Phi^{-1}(\{(\delta_x : x \in K\})$. If the space $P(K)$ were HL, then by the Fact above $K$ would be a closed $G_\delta$ subset of $P(K)$, therefore $\Delta$ would be a $G_\delta$ subset of $K \times K$, a contradiction.

If $\Omega = B_{C(K)^*}$ then since $P(K)$ is a weak* closed subset of $\Omega$ we get that $\Omega$ is not HL.

**Corollary 2.** Let $X$ be a nonseparable Banach space. Denote by $\Omega$ its closed dual unit ball $B_X^*$ with the weak* topology. Then the unit ball of $C(\Omega)$ contains an uncountable 2-equilateral set.

Proof. Using transfinite induction and Hahn-Banach Theorem we may construct for each $\varepsilon > 0$ two long sequences $\{x_\alpha : \alpha < \omega_1\} \subseteq B_X$ and $\{f_\alpha : \alpha < \omega_1\} \subseteq (1 + \varepsilon)B_X^*$ satisfying $f_\beta(x_\alpha) = 0$ for $\beta > \alpha$ and $f_\alpha(x_\alpha) = 1$ for $\alpha < \omega_1$ (see the Fact 4.27 of §). It is easy to see that the sequence $\{f_\alpha : \alpha < \omega_1\}$ is right separated in the compact space $(1 + \varepsilon)B_X^*$. So the space $(1 + \varepsilon)B_X^*$ is not HL and the same is valid for $\Omega = B_X^*$.

In the sequel we relate the concept of a linked family of closed pairs with the known concept of a weakly separated subspace of some topological space ($\mathbb{Z}$, $\mathbb{W}$).

A subspace $Y$ of a topological space $X$ is said to be weakly separated if there are open sets $U_y, y \in Y$ in $X$ such that $y \in U_y \forall y \in Y$ and whenever $y_1 \neq y_2 \in Y$ we either have $y_1 \notin U_{y_2}$ or $y_2 \notin U_{y_1}$.

We note the following easily verified facts:

(i) If $Y = \{t_\alpha : \alpha < \omega_1\}$ is any right (resp. left) separated family in the topological space $X$, then $Y$ is an uncountable weakly separated subspace of $X$; we say in this case that $Y$ is an uncountable right (resp. left) separated subspace of $X$.

(ii) Let $Y$ be any weakly separated subspace of $X$ by the family of open sets $U_y, y \in Y$. Then the family $\{(\{y\}, X \setminus U_y) : y \in Y\}$ is a linked family of closed pairs in $X$.

As we shall see, linked families of closed pairs in a topological space $X$ can be interpreted as a special kind of weakly separated subspaces in $expX$, the hyperspace of closed nonempty subsets of $X$ endowed with the Vietoris topology. If $G_1, \ldots, G_n$ are subsets of $X$ we denote $<G_1, \ldots, G_n> = \{F \in expX : F \subseteq \bigcup_{k=1}^{n} G_k$ and $F \cap G_k \neq \emptyset \forall k = 1, 2, \ldots, n\}$. The Vietoris topology on $expX$ has as base the sets of the form $<G_1, \ldots, G_n>$, where $G_1, \ldots, G_n$ are open subsets of $X$.

We shall say that a weakly separated subspace $Y$ of $expX$ is separated by open subsets of $X$, if the sets $U_y, y \in Y$ of the definition above are of the form
$y \nsubseteq V_y$ for $y \in Y$ and if $y_1 \neq y_2 \in Y$, then either $y_1 \nsubseteq V_{y_2}$ or $y_2 \nsubseteq V_{y_1}$.

More exactly we have the following (easy) Proposition, the proof of which is left to the reader.

**Proposition 1.** Let $X$ be a topological space and $\kappa$ any cardinal. The following are equivalent:

1. $X$ admits a linked family of closed pairs of cardinality $\kappa$.
2. $expX$ admits a weakly separated subspace by open subsets of $X$ of cardinality $\kappa$.

**Remarks 2** (1) As was shown by Todorcevic assuming Martin’s Axiom and the negation of the continuum hypothesis, if $K$ is compact and non metrizable then the space $C(K)$ admits an uncountable (bounded) biorthogonal system ([17], Th.11). So by using Theorem 3 of [11], the space $C(K)$ can be given an equivalent norm that admits an uncountable equilateral set.

(2) It is consistent with ZFC to assume that there exists a compact non metrizable space $K$ having no uncountable weakly separated subspace (see [2]). The space $K$ constructed there, among its many interesting properties, is totally disconnected and hence admits an uncountable linked family of closed (and open) pairs.

(3) Let $K$ be a compact non metrizable space. Then the hyperspace $expK$ of $K$ is not HL. Actually its closed subspace $[K]^{\leq 2} = \{ A \subseteq K : |A| \leq 2 \}$ is not HL. (The proof is similar to the proof that the space $(P(K), w^*)$ is not HL; we consider the continuous map $\Phi : K \times K \to expK : \Phi(x, y) = \{ x, y \}$ and note that $\Phi(K \times K) = [K]^{\leq 2}$). It follows that there is an uncountable right separated subspace $Y = \{ F_\alpha : \alpha < \omega_1 \}$ of $expK$, but it is not clear whether $Y$ is (or another uncountable weakly separated subspace of $expK$ can be chosen so as to be) separated by open subsets of $K$.

**Maximal equilateral sets in Banach spaces of the form $C(K)$**

Our goal here is the study of maximal equilateral sets of minimum cardinality, mainly in Banach spaces of the form $C(K)$. As we shall see, proper linked families of pairs of $\mathbb{N}$ play a key role.

**Definition 2.** Let $(M, d)$ be a metric space. We define, for $x \in M$, $m(M, x) = \min\{|A| : x \in A$ and $A$ is a maximal equilateral set in $M\}$. We also define $m(M) = \min\{|A| : A$ is a maximal equilateral set in $M\}$

It is clear that $m(M) = \min\{m(M, x) : x \in M\}$.

**Lemma 3.** Let $(X, ||\cdot||)$ be a normed space. Then we have $m(X) = m(S_X \cup \{0\}, 0)$. 
Proof. Let $A \subseteq X$ be any maximal equilateral set in $X$. Assume that $A$ is $\lambda$-equilateral. Let $x_0 \in A$; then the set $B = \{1/(x - x_0) : x \in A\}$ is a 1-equilateral subset of $S_X \cup \{0\}$ containing the point 0. Note that $|B| = |A|$ and that $B$ is a maximal equilateral set (in $X$ and hence) in the metric space $S_X \cup \{0\}$.

In the converse direction, consider any maximal equilateral subset $B$ of the metric space $S_X \cup \{0\}$ with $0 \in B$. Then clearly $B$ is 1-equilateral. We claim that $B$ is a maximal equilateral subset of $X$; indeed, if $x \in X$ with $x \notin B$ such that $B \cup \{x\}$ is equilateral then $1 = ||x - 0|| = ||x||$, so $x \in S_X$ which contradicts the maximality of $B$ in the metric space $S_X \cup \{0\}$. □

Lemma 4. Let $(X, || \cdot ||)$ be a normed space. Then we have $m(B_X) \leq m(X) (= m(S_X \cup \{0\}, 0))$.

Proof. By the (method of proof of) the previous Lemma any maximal equilateral set in $X$ gives rise to a maximal equilateral set in $X$ of the same cardinality, contained in $S_X \cup \{0\} \subseteq B_X$, so we are done. □

Remarks 3 (1) Swanepoel and Villa have shown in [16] the following result, generalizing an example of Petty [13]:
If $X$ is any Banach space with $dim X \geq 2$ having a norm which is Gâteaux differentiable at some point and $Y = (X \oplus \mathbb{R})_1$, then we have $m(Y) = 4$.
(Their proof is based on the following simple result: Let $X$ be any normed space with $dim X \geq 2$ and also let $x, u \in S_X$ such that $||u - x|| = ||u + x|| = 2$. Then the unit ball of the subspace $Z = < u, x >$ of $X$ is the parallelogram with vertices $\pm u, \pm x$.) One can easily check that the result of Swanepoel and Villa can be generalized (by the method of its proof) as follows:
If $dim X \geq 2$ and the norm of $X$ is either strictly convex or Gâteaux differentiable at some point, then we have $m(Y) = 4$ and $m(B_X) = 2 (= m(S_X))$.
(2) For the Hilbert space $X = \ell_2$ clearly we have $m(X) = \omega$ and since the norm of $X$ is strictly convex, we get from the preceding remark that $m(B_X) = 2$. So the inequality in Lemma 4 can be strict.
(3) Let $\Gamma$ be any set with $|\Gamma| \geq 2$ and let $|| \cdot ||$ be an equivalent strictly convex norm on the Banach space $\ell_1(\Gamma)$, see [3]. Now set $X = (\ell_1(\Gamma), || \cdot ||)$, then we get from Remark 3(1) that $m(X \oplus \mathbb{R})_1 = 4$.

Now we are going to generalize the following result of Swanepoel and Villa in [16]: If $d \in \mathbb{N}$ then $m(\ell^d_c) = d + 1$.
Let $X$ be a locally compact Hausdorff space and $C_0(X)$ be the Banach space (endowed with supremum norm) of all continuous functions $f : X \to \mathbb{R}$ with the property that, $\forall \varepsilon > 0 \exists K \subseteq X$ compact: $|f(x)| < \varepsilon$, for all $x \in X \setminus K$. As we know $C_0(X)$ is the completion of the space $C_c(X)$ of continuous functions $f : X \to \mathbb{R}$ with compact support. We note the following facts:
(1) If $f, g \in C_0(X)$, then $\max \{f, g\}$ and $\min \{f, g\}$ belong to $C_0(X)$.
(2) If $A$ is any finite nonempty subset of $X$ and $g : A \to \mathbb{R}$ (resp. $g : A \to [0, 1]$) is any function, then there is a continuous extension $f : X \to \mathbb{R}$ (resp. $f : X \to [0, 1]$) of $g$, which has compact support (the proof of this fact uses Urysohn’s Lemma).
Theorem 3. Let \( X \) be any infinite locally compact Hausdorff space. Then we have \( m(C_0(X)) \geq \omega \).

Proof. We shall show that each finite equilateral subset of \( C_0(X) \) can be extended. So let \( S = \{f_1, \ldots, f_n\}, n \geq 2 \) be any 1-equilateral set in \( C_0(X) \). Since \( |f_k(x) - f_l(x)| \leq 1 \) for all \( k \neq l \leq n \) and \( x \in X \), we may assume that \( S \subseteq [0,1]^X \cap C_0(X) \).

(Indeed, set \( f(x) = \min\{f_k(x) : 1 \leq k \leq n\} \) for \( x \in X \), then the function \( f \in C_0(X) \) and \( 0 \leq f_k(x) - f(x) \leq 1 \) for \( k \leq n \) and \( x \in X \). So the set \( \{g_k = f_k - f : 1 \leq k \leq n\} \) is a 1-equilateral subset of \([0,1]^X \cap C_0(X)\).)

We consider any finite subset \( A \) of \( X \) with \( |A| \geq n \), such that:

(1) \( \forall k \leq n \exists t \in A \) with \( |f_k(t_k)| = ||f_k||_\infty \) and

(2) \( \forall k \neq l \leq n \exists t = (k, l) \in A : ||f_k - f_l||_\infty = |(f_k - f_l)(t)| = 1. \)

Then the set \( \{h_k = f_k/A : 1 \leq k \leq n\} \) is 1-equilateral in the space \( \ell_\infty(A) \) and since \( n \leq |A| < \omega \), it can be extended on \( \ell_\infty(A) \) to a 1-equilateral set with at least \( |A| + 1 \geq n + 1 \) elements (see Prop.12 in [10]). Let \( h \in \ell_\infty(A) \) taking values in \([0,1] \) such that \( ||h - h_k||_\infty = 1 \) for \( k \leq n \). Then by using Fact (2) mentioned above, we can find a continuous extension \( f : X \to [0,1] \) of \( h \) on \( X \) having compact support. It is obvious that the set \( S \cup \{f\} \) is a 1-equilateral set in \( C_0(X) \), so we are done.

Let \( \Gamma \) be an infinite set endowed with discrete topology. Then \( c_0(\Gamma) \) is the space of all functions \( f : \Gamma \to \mathbb{R} \) that vanish at infinity. We shall show that the number \( m(c_0(\Gamma)) \) is as big as possible.

Proposition 2. Let \( \Gamma \) be an infinite set, then \( m(c_0(\Gamma)) = |\Gamma| \).

Proof. We claim that each equilateral subset \( S \) in \( c_0(\Gamma) \) with \( |S| < |\Gamma| \) can be extended. If \( \Gamma \) is countable, then \( S \) is finite and can be extended by the previous theorem. So assume \( \Gamma \) is uncountable. It is also clear by Lemma 3 that we may assume \( S \subseteq B_{c_0(\Gamma)} \) and that it is 1-equilateral. Set \( \Delta = \bigcup \{\text{supp}x : x \in S\} \); since \( |S| < |\Gamma| \geq \omega_1 \) and each element of \( c_0(\Gamma) \) has at most countable support, we get that \( |\Delta| < |\Gamma| \). Let \( \gamma_0 \in \Gamma \setminus \Delta \), then it is easy to see that the set \( S \cup \{e_{\gamma_0}\} \) (\( e_{\gamma_0} \) is the \( \gamma_0 \)-member of the usual basis of \( c_0(\Gamma) \)) is 1-equilateral. Now we can proceed by transfinite induction, using the above claim to show that \( m(c_0(\Gamma)) = |\Gamma| \). We omit the details of this (easy) proof.

Let \( K \) be any infinite compact metric space, then the Banach space \( C(K) \) is separable and by Theorem 3 we get that \( m(C(K)) = \omega \). This result can be generalized as follows:

Theorem 4. Let \( \{X_\gamma : \gamma \in \Gamma\} \) be an infinite family of compact metric spaces, so that \( |X_\gamma| \geq 2 \), for all \( \gamma \in \Gamma \). Set \( X = \prod_{\gamma \in \Gamma} X_\gamma \), then we have \( m(C(K)) = |\Gamma| \).

The main tool for proving Theorem 4 is the following
**Lemma 5.** Let $X, Y$ be compact spaces such that $X$ is metrizable and $|Y| \geq 2$. Let $S$ be any equilateral set in $C(X)$. Then there is a linear isometry extension operator $T : C(X) \to C(X \times Y)$, so that the set $T(S)$ can be extended to an equilateral set in $C(X \times Y)$.

**Proof.** Assume without loss of generality that $S$ is a 1-equilateral set such that $S \subseteq S_{C(X)} \cup \{0\}$. Consider two distinct elements, say $a, b$ of the space $Y$ and set $E = C(X \times \{a, b\})$. Let $T_1 : C(X) \to E$ be the isometric embedding defined by $T_1(f)(x, a) = f(x)$ and $T_1(f)(x, b) = 0$.

We now consider a linear isometry extension operator $T_2 : E \to C(X \times Y)$ (with $T_2(1) = 1$) given by Borsuk’s Theorem, see [3] p. 250 and [1]. Set $T = T_2 \circ T_1$; then $T$ is the desired operator. Indeed, it is clear that $T$ is an isometry so that $T(f) / X \times \{a\} = T_1(f)$. Let $g \in E : g / X \times \{a\} = 0$ and $||g|| = 1$; then it is easy to see that the set $T(S) \cup \{T_2(g)\}$ is a 1-equilateral subset of $C(X \times Y)$. 

**Remark 4** A compact Hausdorff space $X$ is said to be a Dugundji space, if for any compact space $Y$ with $X \subseteq Y$ there is a linear extension operator $T : C(X) \to C(Y)$; that is, a linear operator $T$ such that:

(i) $T(1) = 1$, (ii) $||T|| = 1$ and (iii) $T(f) / X = f$, for $f \in C(X)$ (see [1]).

It is clear that such an operator is an isometry. We note that: (a) Every compact metric space is Dugundji and (b) the class of Dugundji spaces is closed under cartesian products.

It follows in particular from the above remark that Lemma 5 remains true if we assume that $X$ is any Dugundji space and $Y$ any compact space with at least two points.

In order to prove Theorem 4 we need to introduce some notation and to remind the reader of some concepts. Let $\{X_\gamma : \gamma \in \Gamma\}$ be a family of topological spaces. Set $X = \prod_{\gamma \in \Gamma} X_\gamma$, endowed with the product topology. Fix a point, say $o = (o)_\gamma \in \Gamma$ in $X$. For any $\emptyset \neq A \subseteq \Gamma$ we set $X_A = \prod_{\gamma \in A} X_{\gamma} \times \{o\}_{\gamma \in \Gamma \setminus A}$ and define a map $\pi_A : X \to X$ by $\pi_A(x)(\gamma) = \begin{cases} x(\gamma) & , \gamma \in A \\ o & , \gamma \in \Gamma \setminus A \end{cases}$. The map $\pi_A$ is a continuous retraction with $\pi_A(X) = X_A$, which in its turn induces a norm one projection $P_A : C(X) \to C(X)$ by the rule, $P_A(f)(x) = f(\pi_A(x))$, for $f \in C(X)$ and $x \in X$. Note that the range of $P_A$ is identified with the range of the isometry $T_A : C(X_A) \to C(X)$ defined by $T_A(g) = g \circ \pi_A$, for $g \in C(X_A)$.

We remind the reader that a map $f : Y \subseteq X \to \mathbb{R}$ depends on a set $A \subseteq \Gamma$, if whenever $x, y \in Y$ and $\pi_A(x) = \pi_A(y)$ then $f(x) = f(y)$. It is well known that if each $X_\gamma$ is separable, then every continuous map $f : X \to \mathbb{R}$ depends on countably many coordinates. In fact there is a countable set $A \subseteq \Gamma$ and a continuous map $h : X_A \to \mathbb{R}$ such that $f = h \circ \pi_A$ (see [4] pp.157-9).

**Proof. (of Theorem 4)**

We may (and shall) assume that $|\Gamma| \geq \omega_1$ (if $|\Gamma| \leq \omega$, then $X$ is compact metric
and the result holds true). So let $S$ be any equilateral set in $C(X)$ with $|S| < |\Gamma|$. Since by Theorem 3, $m(C(K)) \geq \omega$ for any infinite compact space $K$, we may also assume that $S$ is infinite. We are going to prove that $S$ can be extended (cf. the proof of Prop. 2). Since each continuous function on $X$ depends on countably many coordinates and $|S| < |\Gamma|$, it follows that there is an $A \subseteq \Gamma$ with $|A| = |S| < |\Gamma|$ such that each member of $S$ depends on $A$. So if $A \subseteq B \subseteq \Gamma$, then the set $P_B(S) = S$ is an equilateral subset of $C(X_B)$. Let $\gamma_0 \in \Gamma \setminus A$; set $B = A \cup \{\gamma_0\}$. Then $P_A(S)$ is an equilateral set in $C(X_A)$, $X_A$ is a Dugundji space and $X_B = X_A \times X_{\gamma_0}$, therefore by Lemma 5 and Remark 4 we can extend $S$ to an equilateral subset $S \cup \{f\}$ of $C(X_B) \subseteq C(X)$. The proof of the theorem is complete.

Let $T$ be the dyadic tree, i.e. $T = \bigcup_{n=0}^{\infty} \{0,1\}^n$ ordered by the relation "$s$ is an initial segment of $t$", denoted by $s \leq t$. By the term chain (resp. antichain) of $T$ we mean a set of pairwise comparable (resp. incomparable) elements of $T$. A branch of $T$ is any maximal chain of $T$.

Let $A = \{s_1, s_2, \ldots, s_n, \ldots\}$ be any infinite antichain of $T$. We let for any $n \in \mathbb{N}$, $A_n = \{k \leq |s_n| : s_n(k) = 1\}$ and $B_n = \{k \leq |s_n| : s_n(k) = 0\}$ (where $|s|$ denotes the length of $s \in T$, that is, if $s = (s(1), s(2), \ldots, s(m))$, then $|s| = m$).

**Lemma 6.** The family $\mathcal{F}(A) = \{(A_n, B_n) : n \in \mathbb{N}\}$ is a linked family of (finite) pairs of $\mathbb{N}$.

**Proof.** Let $n, m \in \mathbb{N}$ with $n < m$; we then have that $s_n$ and $s_m$ are incomparable. Let $k \leq \min\{|s_n|, |s_m|\}$ such that $s_n(k) \neq s_m(k)$. If $s_n(k) = 1$ then $s_m(k) = 0$, thus $k \in A_n \cap B_m$. If $s_n(k) = 0$ then $s_m(k) = 1$ and thus $k \in A_m \cap B_n$. So we are done.

**Example 2** Let $\Delta = \{0,1\}^\mathbb{N}$ be the Cantor set. If $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n, \ldots) \in \Delta$, then the sequence

$$C(\sigma) = \{(\sigma_1), (\sigma_1, \sigma_2), \ldots, (\sigma_1, \sigma_2, \ldots, \sigma_n), \ldots\}$$

is a maximal chain (that is, a branch) of $T$ and the sequence

$$A(\sigma) = \{ (\varphi(\sigma_1), (\sigma_1, \varphi(\sigma_2)), \ldots, (\sigma_1, \ldots, \sigma_{n-1}, \varphi(\sigma_n)), \ldots \},$$

where $\varphi(0) = 1$ and $\varphi(1) = 0$, is a maximal antichain of $T$.

It follows immediately from Lemma 6 that the family $\mathcal{F}(A(\sigma))$ is a linked family of pairs of $\mathbb{N}$. Furthermore, the family $\mathcal{F}(A(\sigma)) \cup \{(A_\omega, B_\omega)\}$, where $A_\omega = \{n \in \mathbb{N} : \sigma_n = 1\}$ and $B_\omega = \{n \in \mathbb{N} : \sigma_n = 0\}$ is linked. Indeed, let $(t_n)$ be an arbitrary sequence in the interval $(0,1)$ and let

$$p_1 = (\varphi(\sigma_1), t_1, \ldots, t_1, \ldots)$$

$$p_2 = (\sigma_1, \varphi(\sigma_2), t_2, \ldots, t_2, \ldots)$$

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Let $c$ be a lateral subset of $\mathbb{C}$. More exactly:

Lemma 7. For any $N$ the discrete set $|\alpha|$ is a maximal equilateral subset of $\ell_\infty$ and that

$|\alpha| = \{\sigma_n: n \geq 1\}$. From the equations $|\alpha - p_1| = |\alpha - p_2| = 1$, we get that $|\alpha_1 - \varphi(\sigma_1)| \leq 1$ and $|\alpha_1 - \varphi(\sigma_1)| = \{0, 1\}$ we conclude that $\alpha_1 \in \{0, 1\}$. From the equations $|\alpha - p_2| = |\alpha - p_3| = 1$, we get that $|\alpha_2 - \varphi(\sigma_2)| \leq 1$ and $|\alpha_2 - \varphi(\sigma_2)| = \{0, 1\}$ we conclude that $\alpha_2 \in \{0, 1\}$. Similarly we conclude that $\alpha_n \in \{0, 1\}$, for all $n \geq 1$.

Since we have $|t_1 - \alpha_k| < 1$, for all $k \geq 2$, we get that $1 = |p_1 - \alpha| = |\alpha - \varphi(\sigma_1)|$, which clearly implies that $\alpha_1 = \sigma_1$. We observe that $|t_2 - \alpha_k| < 1$, for all $k \geq 3$, therefore $1 = |p_2 - \alpha| = \max\{|\alpha_1 - \varphi(\sigma_1)|, |\alpha_2 - \varphi(\sigma_2)|\} = |\alpha_2 - \varphi(\sigma_2)|$, which implies that $\alpha_2 = \sigma_2$. In the same way we get that $\alpha_n = \sigma_n$, for all $n \in \mathbb{N}$. So we are done.

Corollary 3. The family $\mathcal{F}(A(\sigma)) \cup \{(A_{\omega}, B_{\omega})\}$ is a maximal linked family of pairs of $\mathbb{N}$, for any $\sigma \in \Delta$.

Proof. It follows immediately from Lemma 7 and Remarks 1 (2) and (3).

Theorem 5. $m(\ell_\infty) = \omega(= m(c))$.

Proof. It is an immediate consequence of Theorem 3 and Lemma 7 (The fact that $m(c) = \omega$ also follows from Theorem 3, this is so because $c$ is isometric to the space $C(\mathbb{N})$, where $\mathbb{N}$ is the one point compactification of the discrete set $\mathbb{N}$).

Theorem 6. Let $K$ be a compactification of the discrete set $\mathbb{N}$. Then we have that $m(C(K)) = \omega$.
Proof. Let $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n, \ldots) \in \Delta$ be an eventually constant sequence, that is, there are $N \in \mathbb{N}$ and $i \in \{0, 1\}$ such that $\sigma_{N+\lambda} = i$, for $\lambda \geq 1$. Also let $(t_n) \subseteq (0, 1)$ be an arbitrary sequence. We define a 1-equilateral subset of the space $C(K)$ as follows:

$$f_1 = \varphi(\sigma_1) \cdot \chi_{\{1\}} + t_1 \cdot \chi_{K \setminus \{1\}}$$
$$f_2 = \sigma_1 \cdot \chi_{\{1\}} + \varphi(\sigma_2) \cdot \chi_{\{2\}} + t_2 \cdot \chi_{K \setminus \{1, 2\}}$$
$$\vdots$$
$$f_n = \sigma_1 \cdot \chi_{\{1\}} + \ldots + \sigma_{n-1} \cdot \chi_{\{n-1\}} + \varphi(\sigma_n) \cdot \chi_{\{n\}} + t_n \cdot \chi_{K \setminus \{1, 2, \ldots, n\}}$$
$$\vdots$$
$$f_\omega = \sigma_1 \cdot \chi_{\{1\}} + \ldots + \sigma_N \cdot \chi_{\{N\}} + i \cdot \chi_{K \setminus \{1, 2, \ldots, N\}}$$

Let $\pi : \beta \mathbb{N} \to K$ be the continuous surjective map so that $\pi(n) = n$, for $n \in \mathbb{N}$. Then the operator $\Phi : C(K) \to C(\beta \mathbb{N}) \cong \ell_\infty$ defined by $\Phi(f) = f \circ \pi$, for $f \in C(K)$, is an isometric embedding. Since clearly $\Phi(f_n) = p_n$, for $n \leq \omega$, where $\{p_n : n \leq \omega\}$ is the sequence defined in Lemma 7, we get the conclusion. \hfill \Box

The aforementioned results (Theorems 5 and 6) culminate in the following more general result.

**Theorem 7.** Let $\Gamma$ be an infinite set and $K$ be a compact space containing an infinite set of isolated points having a unique limit point. Then $m(\ell_\infty(\Gamma)) = m(C(K)) = \omega$.

**Proof.** Let $D = \{x_n : n \in \mathbb{N}\}$ be a sequence of distinct isolated points of $K$. We consider a non eventually constant sequence $\sigma = (\sigma_1, \ldots, \sigma_n, \ldots) \in \Delta$ and a dense sequence $(t_n) \subseteq (0, 1)$. We define a 1-equilateral subset of $\ell_\infty(K)$, as follows:

$$f_n(x) = \begin{cases}
\sigma_k, & x = x_k, k \leq n - 1 \\
\varphi(\sigma_n), & x = x_n \\
t_n, & x \in K \setminus \{x_1, x_2, \ldots, x_n\}
\end{cases}$$

$$f_\omega(x) = \begin{cases}
\sigma_n, & x = x_n, n \in \mathbb{N} \\
t_1, & x \in K \setminus D
\end{cases}$$

It is clear that the set $S = \{f_n : n \in \mathbb{N}\} \subseteq C(K)$.

**Claim.** The set $S$ is a maximal equilateral subset of $C(K)$ and the set $S \cup \{f_\omega\}$ is a maximal equilateral subset of $\ell_\infty(K)$.

**Proof of the Claim:** Let $f \in \ell_\infty(K)$ such that

$$||f - f_n||_\infty = 1, \text{ for } n \in \mathbb{N}. \quad (1)$$

It then follows from (1) that $|f(x) - f_n(x)| = |f(x) - t_n| \leq 1$ for all $x \in K \setminus D$ and $n \in \mathbb{N}$. Since $t_n$ is dense in $(0, 1)$ we get that $0 \leq f(x) \leq 1$, for $x \in K \setminus D$. 

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We also have from (1) that \(|f(x_n) - \sigma_n| \leq 1\) and \(|f(x_n) - \phi(\sigma_n)| \leq 1\), for \(n \in \mathbb{N}\); so we get that \(0 \leq f(x_n) \leq 1\), for \(n \in \mathbb{N}\). Therefore,

\[ 0 \leq f(x) \leq 1, \text{ for } x \in K. \] (2)

Let now \(N \in \mathbb{N}\); we then have \(f_N(x) = t_N\), for \(x \in K \setminus D\). It follows from (2) that

\[ |f(x) - f_N(x)| = |f(x) - t_N| \leq \max\{t_N, 1 - t_N\} < 1 \text{ for } x \in K \setminus D. \]

So we get that

\[ \|f - f_n\|_\infty = \sup_{x \in D} |f(x) - f_n(x)| = 1, \text{ for } n \in \mathbb{N}. \]

The method of proof of Lemma 7 then yields that \(f(x_n) = \sigma_n\), for \(n \in \mathbb{N}\).

Assume for the moment that \(D\) has a unique limit point, say \(x\), thus \(x_n \to x\). Since \(f(x_n) = \sigma_n = 1\) for infinite \(n \in \mathbb{N}\) and \(f(x_n) = \sigma_n = 0\) for infinite \(n \in \mathbb{N}\), we conclude that \(f\) cannot be continuous on \(K\). So the set \(S\) is a maximal equilateral subset of \(C(K)\).

Also \(S \cup \{f_\omega\}\) is a maximal equilateral subset of \(\ell_\infty(K)\), since (for an arbitrary infinite set \(K\))

\[ \|f - f_\omega\|_\infty = \sup_{x \in K \setminus D} |f(x) - f_\omega(x)| = \sup_{x \in K \setminus D} |f(x) - t_1| \leq \max\{t_1, 1 - t_1\} < 1. \]

The proof of the Claim and hence of the Theorem is complete. \(\square\)

**Corollary 4.** Let \(K\) be an infinite compact scattered space. Then \(m(C(K)) = \omega\).

**Proof.** Since \(K\) is scattered, the set of isolated points of \(K\) is dense in \(K\); moreover \(K\) is sequentially compact. So let \((x_n)\) be a sequence of distinct isolated points of \(K\), then \((x_n)\) has a convergent subsequence, say \(y_n = x_{m_n}\), \(n \in \mathbb{N}\). Therefore the set \(D = \{y_n : n \in \mathbb{N}\}\) has a unique limit point and the previous theorem can be applied. \(\square\)

**Remarks 5** (1) Let \(X, Y\) be compact spaces, \(\pi : X \to Y\) a continuous surjective map which is non irreducible (i.e., there is \(\Omega \subseteq X\) compact such that \(\pi(\Omega) = Y\)) and \(T : C(Y) \to C(X)\) be the linear isometry given by \(T(f) = f \circ \pi\), for \(f \in C(Y)\). We consider a 1-equilateral subset \(S\) of \(S^+(Y)\), then it is rather easy to prove that there is \(g \in S^+_p\), such that the set \(T(S) \cup \{g\}\) is equilateral.

Given this result, it can be shown by transfinite induction, that if a compact non metrizable space \(K\) is roughly the ”limit” of a long system of ”smaller” compact spaces connected by non irreducible maps, then \(m(S^+_p \cup \{0\}, 0) \geq \omega_1\). This is the case for instance when:

(a) \(K\) is an Eberlein, or more general a Corson compact space, because then \(K\) admits a long sequence of compatible retractions, see [12], [3] and
(b) $K$ is a compact group, since then $K$ is a projective limit of compact metrizable groups, see [9].

(2) Note that, given any infinite set $\Gamma$, the Banach spaces $c_0(\Gamma)$ and $c(\tilde{\Gamma})$ are isomorphic, where $\tilde{\Gamma}$ is the one point compactification of the discrete set $\Gamma$. But if $\Gamma$ is uncountable, then by Prop. 2, Cor. 4 and the above remark we have

$$m(C(\tilde{\Gamma})) = \omega < m(S^+_{C(\tilde{\Gamma})} \cup \{0\}, 0) = m(c_0(\Gamma)) = |\Gamma|.$$ 

(3) The following example is related with Remark 1(3):

Let $K$ be a compact nonempty space. We denote by $\Omega$ the disjoint union of the compact spaces $K$ and $\tilde{\mathbb{N}}$, where $\tilde{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ is the one point compactification of $\mathbb{N}$. Let $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n, \ldots, \in \Delta$ be an eventually constant sequence, so that $\sigma_n = i \in \{0, 1\}$ for $n \geq N$. We define a 1-equilateral set $S = \{f_n : n \leq \omega\} \subseteq [0, 1]^\Omega \cap C(\Omega)$ as follows:

$$f_n(x) = \begin{cases} 
\sigma_k, & x = k, k \leq n - 1 \\
0, & x = n \\
\frac{1}{2}, & x \in \Omega \setminus \{1, 2, \ldots, n\}
\end{cases}, \quad n \in \mathbb{N}$$

and $f_\omega(x) = \begin{cases} 
\sigma_n, & x = n \\
0, & x = \infty \\
\frac{1}{2}, & x \in K
\end{cases}$.

It is easy to verify that the linked family $F$ corresponding to this equilateral set according to Lemma 1 is maximal (cf. Lemma 7) and also that $S$ can be extended to an equilateral set in $C(\Omega)$.

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