A Model for Adversarial Wiretap Channel

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Abstract—In wiretap model of secure communication, the goal is to provide (asymptotic) perfect secrecy and reliability over a noisy channel that is eavesdropped by an adversary with unlimited computational power. This goal is achieved by taking advantage of the channel noise and without requiring a shared key. The model has attracted considerable attention in recent years because it captures eavesdropping attack in wireless communication. The wiretap adversary is a passive eavesdropping adversary at the physical layer of communication.

In this paper we propose a model for adversarial wiretap (AWTP) channel to capture active attacks at this layer. We consider a \( (\rho_r, \rho_w) \) wiretap adversary who can see a fraction \( \rho_r \), and modify a fraction \( \rho_w \), of the sent codeword. The code components that are read and/or modified can be chosen adaptively, and the subsets of read and modified components in general, can be different. AWTP codes provide secrecy and reliability definitions for these codes, and define secrecy capacity of an AWTP channel. The paper has two main contributions. First, we prove a tight upper bound on the rate of AWTP codes for \( (\rho_r, \rho_w) \) AWTP channels, and use the bound to derive the secrecy capacity of the channel. Second, we give an explicit construction for a perfectly secure capacity achieving AWTP code family. We show that our AWTP model is a natural generalization of Wyner’s wiretap models, and somewhat surprisingly, also provides a generalization of a seemingly unrelated cryptographic primitive, Secure Message Transmission (SMT). This relation is used to derive a new (and the only known) bound on the secrecy capacity of a Wyner wiretap model with observed noise over the main channel in Wyner’s wiretap model with specified limits. The model effectively replaces probabilistic variations of the basic model including extending the goal of communication to key agreement also. There have also been numerous implementations of the system [4], [5].

Considering active adversaries in wiretap model is well motivated by real life application scenarios. In wireless communication it is relatively easy for an attacker to inject signals in the channel resulting in the transmitted symbols to be erased, or selectively modified [48]. Studying active adversaries that model channel tampering is also important for bringing wiretap model in line with cryptographic models used for confidentiality (e.g. authenticated encryption). Recent proposals [1], [9], [44] for physical layer active adversaries in wiretap setting consider a general adversary modelled as an arbitrary varying channel, but fall short on one or more of the following, (i) considering adaptive adversaries that uses its current knowledge to perform its next action, (ii) using a strong definition of security, (iii) deriving an expression or a tight upper-bound for secrecy capacity, and (iv) providing an efficient explicit construction.

In this paper we propose an Adversarial Wiretap Channel (AWTP Channel) model in which the adversary can adaptively choose his/her view of the channel, and tamper with the transmission over the channel using this view. Adversary’s observation and tampering strategies are arbitrary as long as the total number of observed and tampered symbols stay within specified limits. The model effectively replaces probabilistic noise over the main channel in Wyner’s wiretap model with adversarial noise, and for the wiretapper channel adopts the Wyner’s wiretap II model, allowing the adversary to adaptively select the symbols that they want to observe. The model thus can be seen as extensions of Wyner’s wiretap and wiretap II models both. In Section VII we will also show that this model also generalizes the seemingly unrelated cryptographic model

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of Secure Message Transmission \cite{26, 27, 34, 47}, used for networks. There are however subtle differences between the two that will be further discussed in Section VI. We use a definition of security and reliability that is in-line with related cryptographic primitives such as SMT and also the security definition of wiretap adversaries in [4].

We will derive an expression for upper bound on the rate of AWTP codes and give an explicit construction for a code that achieves the bound.

A. Our Results

1) AWTP Channel: An AWTP channel is specified by a pair of parameters \((\rho_r, \rho_w)\): for a codeword of length \(N\), the adversary can choose a subset \(S_r\) of \(\rho_r N\) components of the codeword to read, and a subset \(S_w\) of \(\rho_w N\) components to write, and writing is by adding an error vector to the codeword. The goal is to provide reliability and secrecy for communication against the above adversary. Secrecy is defined as the indistinguishability of the adversary’s view of the communication for two messages chosen by the adversary, and indistinguishability is measured by the statistical distance between the two views. Reliability is given by the receiver’s probability of correctly decoding a sent message, possibly chosen by the adversary (see Definition 5). Perfect secrecy and perfect reliability correspond to zero information leakage and always correct recovery of the message by the receiver, respectively. The \(\epsilon\)-secrecy capacity \(C^\epsilon\) of a \((\rho_r, \rho_w)\)-AWTP channel achieves the highest possible information rate (number of message bits divided by the number of communicated bits) with \(\epsilon\)-secrecy and the guarantee that the decoder error probability asymptotically approaches zero. Secrecy capacity of a channel gives the potential of the channel for secure communication and achieving this capacity with efficient construction, is the ultimate goal of the system designer.

2) Adversarial Wiretap Codes (AWTP Code): An AWTP code provides security and reliability for message transmission over \((\rho_r, \rho_w)\)-AWTP channels. An AWTP code is specified by a triple \((M, N, \Sigma)\), denoting the message space, code length, and alphabet set respectively, and a pair of algorithms \((\text{AWTPenc}(-), \text{AWTPdec}(-))\) that are used for encoding and decoding, respectively. Encoding is probabilistic and maps a message \(m \in M\) to a codeword \(c \in C\). Decoding is deterministic and outputs a message that could be incorrect. Decoding error is worst case and assumes that the adversary uses their best strategy for choosing the message (message distribution) and tampering with communication to make the decoder output in error. An \((\epsilon, \delta)\)-AWTP code guarantees that the information leaked about the message (measured using statistical distance) and the probability of decoding error are upper bounded by \(\epsilon\) and \(\delta\), respectively. The rate of an AWTP code \(C^N\) of length \(N\), denoted by \(R(C^N)\), is defined as \(R(C^N) = \frac{\log |\Sigma|}{\log |C^N|}\). An AWTP code family \(C\) is a family \(\{C^N\}_{N \in \mathbb{N}}\) of AWTP codes indexed by the code length \(N\). The rate \(R(C)\) is achievable by a code family \(C\), if for any \(\xi > 0\) there exists an \(N_0\) such that for all \(N \geq N_0\), we have \(\frac{1}{N} \log |\Sigma| |M| \geq R(C) - \xi\), and decoding error probability satisfies \(\delta \leq \xi\). The \(\epsilon\)-secrecy capacity of AWTP channel, denoted by \(C^\epsilon\), is the largest achievable rate of all AWTP code families that provide \(\epsilon\)-secrecy for the channel. \(C^0\) for perfect secrecy.

3) Rate Upper Bound of AWTP channel: We prove the bound \(H(M) = \log |\Sigma| \cdot (1 - \rho_r + \rho_w) / N\), for an arbitrary message distribution with entropy \(H(M)\). Using uniform message distribution, we obtain an upper bound on the rate of an \((\epsilon, \delta)\)-AWTP code for a \((\rho_r, \rho_w)\)-AWTP channel, that leads to the following upper bound on the secrecy capacity of a \((\rho_r, \rho_w)\)-AWTP channel,

\[ C^\epsilon \leq 1 - \rho_r - \rho_w + 2\epsilon \rho_r (1 + \log |\Sigma|)^{\frac{1}{\epsilon}} \]  \hspace{1cm} (1)

For \(\epsilon = 0\), we obtain the upper bound \(C^0 \leq 1 - \rho_r - \rho_w\), on the secrecy capacity of perfectly secure AWTP code families (Corollary 1), which is achieved by the construction in Section VI (Theorem 2), and we obtain the perfect secrecy capacity of a \((\rho_r, \rho_w)\)-AWTP channel,

\[ C^0 = 1 - \rho_r - \rho_w \]  \hspace{1cm} (2)

The bound on \(C^0\) implies that perfect security for \((\rho_r, \rho_w)\)-AWTP channels is possible only if, \(\rho_r + \rho_w < 1\), indicating a trade-off between read and write (adding noise) capabilities of an AWTP adversary. In particular, when the adversary is almost oblivious (\(\rho_r\) is small), the rate stays positive even when the adversary writes over the large fraction \(1 - \rho_r\) of the codeword, and on the other extreme when \(\rho_r\) is close to 1, few \(1 - \rho_r\) corrupted symbols can be tolerated. In general, the subset of components of a codeword that is either read, or written to, cannot contribute to secure and reliable transmission of information. Since the capacity result must hold for all adversaries, that is all choices of \(S_r\) and \(S_w\) (subject to the bound on the size), considering an adversary that uses \(S_r \cap S_w = \emptyset\), results in the capacity to be less than \(1 - \rho_r - \rho_w\). Oblivious writing adversaries have been previously considered in Algebraic Manipulation Detection (AMD) codes \cite{11} where the goal is to detect algebraic tampering. In AWTP codes however, the aim is to correct errors and recover the sent message.

4) A Capacity Achieving AWTP Code Family: We construct a \((0, \delta)\)-AWTP code family \(C = \{C^N : N \in \mathbb{N}\}\), where \(N\) is the code length, for a \((\rho_r, \rho_w)\)-AWTP channel. For any small \(\xi > 0\), the code \(C^N\) has rate \(R(C^N) = 1 - \rho_r - \rho_w - \xi\); the code alphabet size is \(|\Sigma| = O(q^{1/2})\), and decoding error probability is \(\delta \leq q^{2^{-N}}\). The construction gives a construction for a code family that achieves the capacity \(C^0\).

The construction uses three building blocks: Algebraic Manipulation Detection Code (AMD code), Subspace Evasive Sets, and Folded Reed-Solomon code (FRS code). An AMD code \cite{11} (Definition 1) is a randomized code that protects against an oblivious adversary that “algebraically manipulates” (adds error to) the AMD codeword; A \((v, \ell)\)-subspace evasive set is a subset \(S\) of a vector space \(F_q^v\) with the property that, any subset of dimension \(v\) has at most \(\ell\) common elements with \(S\); An FRS code \cite{31} is a special type of Reed-Solomon
code that has efficient list decoding algorithm for errors up to
the list decoding capacity.

AWTP encoding algorithm of a message uses these three
building blocks as follows: the message is first encoded into an
AMD codeword; the resulting AMD codeword is then encoded
into a vector in a subspace evasive sets with appropriate
parameters; the resulting vector is finally encoded into a
codeword of the FRS code. AWTP decoding algorithm uses
inverse steps: it first uses the FRS decoder to output a list of
possible codewords which contains the correct codeword.
Using the decoding algorithm of FRS code \[\text{[31]},\] the list
size will be exponential in the code length \(N\). Using the
intersection algorithm of the subspace evasive sets, the list
is pruned to a list of size at most \(\ell\) that contains the correct
codeword. The decoder combines the above two steps and so
effectively avoids the generation of the exponential size list
in the FRS decode, which would result in inefficient decoder.
The final step uses the AMD code to find the correct coded
message, and outputs the correct message with probability
at least \(1 - \delta\). We prove that with appropriate choices of
parameters, the rate of the code family meets the rate upper
bound of the \((\rho, \rho_w)\)-AWTP channels with equality, and
so the family is capacity achieving.

5) Relations with SMT and AMD codes: AWTP model of
secure and reliable communication over adversarially con-
trolled channels, is closely related to 1-round SMT \[\text{[21]},\]
a model proposed for secure and reliable communication in
networks. In SMT setting Alice is connected to Bob through
a set of \(N\) node disjoint paths (wires) in a network, \(t\) of
which are controlled by a Byzantine adversary. The goal of
an SMT protocol is to provide reliability and privacy for
communication: an \((\epsilon, \delta)\)-SMT ensures that the privacy loss
(indistinguishability based) is bounded by \(\epsilon\), and the proba-
nility of failing to decode the message is bounded by \(\delta\) \[\text{[34]}\].
A notable difference between an AWTP channel adversary
and an SMT adversary is that, in the former error is \emph{added
to the codeword} while in the latter, it is a \emph{replacement error}
and allows the adversary to replace what is sent over a wire
with its own adversarial choices. In Section \[\text{VI}\] we consider
the relationship between the two primitives, and in particular
show how the results in one, can give results for the other.
The relationship also suggests a new efficiency measure for
SMT systems using information rate of a family.

B. Related Work

Wiretap model and its extensions has attracted much atten-
tion in recent years. There is a large body of excellent works
on extensions of wiretap model \[\text{[10], [13], [15], [14], [31],
[43], [46]},\] construction of capacity achieving codes \[\text{[4], [32]},\]
and implementation of codes in practice \[\text{[6], [8]}\]. We only consider
the works that are directly related to this work. Considering
adversarial control in wiretap channel dates back to Wyner
wiretap II \[\text{[46]}\] model in which the adversary can select their
view of the communication. The adversary however, does
not modify the transmission over the main channel which
is assumed noise-free. Physical layer active adversaries for
wiretap channels that tamper with the transmission, have
been considered more recently \[\text{[1], [2], [44]}\]. These works
model wiretap channels under active attack, as an arbitrarily
varying channel. An \emph{arbitrarily varying channel} (AVC) \[\text{[2],
[3], [7], [13]}\] is specified by two finite sets \(X\) and \(Y\) of
input and output alphabets, a finite set \(\mathcal{A}\) of channel states,
and a set of channels specified by transition probabilities
\(\Pr(y|x,a), x \in X, y \in Y, a \in \mathcal{A}\). The channel state in general
varies with each channel use (possibly with memory) and
\[
\Pr(y^n|x^n,a^n) = \prod_{i=1}^n \Pr(y_i|x_i,a_i)
\]
where \(a^n = (a_1 \cdots a_n), a^n \in \mathcal{A}^n\), is the sequence of channel
states. An \emph{arbitrarily varying wiretap channels} (AVWC) is
specified by an input alphabet set \(X\), two sets of output
alphabets, \(Y\) and \(Z\), representing the legitimate receiver’s
and the wiretapper’s input values, respectively, and a family
of channels, each specified with a transition probability
\(\Pr(y,z|x,a), x \in X, y \in Y, z \in Z, a \in \mathcal{A}\) indexed by
the channel state \(a\). In \[\text{[44]}\], a jammer chooses the state \(a_i\) (jam-
ming signal) independent of the eavesdroppers’ observation \(z\).
Transmitter and receiver know the state space but not the state
chosen by the adversary.

The message is chosen randomly, with uniform distribution,
from the message space. \emph{Encoding and decoding is random-
ized}; that is the system uses a family of encoder and decoder
pairs, and the pair used by the sender and receiver is specified
by a random value (also called key) that is known to the
eavesdropper but \emph{not the jammer}. The family of the codes is
known to the jammer. Security is measured by the \emph{rate of the mutual
information between the message and the adversary’s observation},
and reliability for a single encoder and decoder is in terms of the expected error probability over all messages.
For randomized codes security and reliability are averaged
over all realizations of the code.

Authors defined randomized-code secrecy capacity of AVWP
channels and derived an upper bound for that. They also
obtained the capacity for the special case of \emph{strongly degraded
with independent states} where certain Markov chains hold
among \(X, Y\) and \(Z\), and the set of states \(\mathcal{A}\) is decomposable
as \(\mathcal{A} = \mathcal{A}_y \times \mathcal{A}_z\) and states of the receiver’s channel and the
eavesdroppers channels are selected independently.

In summary, the model (i) assumes common randomness
between the sender and the receiver that is unknown to the
jammer, but is known to the eavesdropper, (ii) uses weak
definition of secrecy, and (iii) the jammer’s corruption does
not depend on the eavesdropper view. In \[\text{[5], [2]}\] these results
are strengthened. These works both use a strong definition of
secrecy using total mutual information (instead of rate), and
\[\text{[9]}\] uses the AVWC model and analyzes active adversaries that
exploit common randomness.

Our adversarial channel model can be seen as a special
class of arbitrarily varying channel. For this class we can
remove some of the restrictions of the above line of work.
More specifically, (i) we do not assume shared randomness
between the sender and the receiver, (ii) we consider an
integrated adaptive eavesdropping and jamming adversary
and assume that to corrupt the next symbol, the adversary uses
all its knowledge up to that point; (iii) we allow adversary to
choose the message distribution and so the error is worst case;

\[
\begin{align*}
\Pr(y^n|x^n,a^n) &= \prod_{i=1}^n \Pr(y_i|x_i,a_i) \\
n &= a^n = (a_1 \cdots a_n), a^n \in \mathcal{A}^n, a^n \in \mathcal{A}\n\end{align*}
\]
and finally (iv) the secrecy measure in our case is in terms of statistical distance between the adversary’s views of two adversarially chosen messages. We note that in [9], secrecy is measured as the mutual information of a random message (uniform distribution on messages) and the adversary’s view. Our security definition using statistical distance is equivalent to the mutual information security when the message distribution is adversarially chosen [4].

In [1] wiretap II model is extended to include an active adversary, using two types of corruption. In the first one the adversary erases symbols that are observed, and in the second, corrupts them. Authors give constructions that achieve good rates. The adversary’s corruption capability in our work is more general (adversary is more powerful). No other comparison can be made because of insufficient details.

Wyner [54] quantifies security of the system using the adversary’s equivocation defined as the average (per message symbol) uncertainty about the message, given the adversary’s view of the sent codeword. Strengthening this security definition has been considered in [4], [10], [32], [42]. In [4], the relationship among security notions used for wiretap channels is studied, and it is also shown that distinguishability based definition using statistical distance is equivalent to a security notion that is called mutual information security. This is a stronger notion compared to the strong security notion used in [42], with the difference being that the adversary chooses the message distribution.

Adversarial channels have been widely studied in literature [18], [30], [57], with [56] providing a comprehensive survey. An adversarial channel closely related to this work is limited view (LV) adversary channel [30], [51]. An LV adversary power is identical to the adversary in AWTP channel but the goal of communication in the former is reliability, while transmission over AWTP channels requires reliability and secrecy both.

Computationally limited active adversary at network layer of communication, has been considered in [20], [33], [41], [43], [59]. This adversary can tamper with the whole message, and to provide protection, access to resources such as shared randomness [28], [52], close secrets [33], or extra channels [45], is required. The adversary in AWTP setting is at physical layer of communication, and the only advantage of communicants over the adversary is limited access of the adversary to the channel.

In all above models, the reliability goal is to correctly recover the sent message. A less demanding reliability goal in detection of errors. Adversarial tampering by an adversary that cannot “see” the encoded message, has been considered in the context of AMD codes [11]. AMD codes with strong security, are randomized codes that detect algebraic manipulation resulting from the addition of an arbitrary error vector. Weak AMD codes are deterministic and use the message randomness to provide protection. AMD codes with leakage [3] allow the adversary to “see” a fraction of the codeword. The writing ability of the adversary however is unrestricted. This is possible because the goal of the encoding is to detect tampering, while in AWTP model, the goal is message recovery and so the corruption must be limited.

C. Discussion and Future Work

Providing security against a computationally unlimited adversary and without assuming a shared key, requires limiting adversary’s physical access to the information. The bound on the write ability of the adversary captures limitations of the adversary’s transmitting power, and the complexity of effecting symbol change [43] in real systems. The bound on the read ability captures limited physical access to sent symbols due to the inadequacy of the adversary’s receiver to perfectly decode all the sent symbols.

A number of results in this paper can be improved. Construction of capacity achieving AWTP codes for $\epsilon > 0$, and construction of AWTP code for constant size alphabets, and in particular $\mathbb{F}_2$, remain open problems.

Organization: Section 2 provides background. In Section 3, we introduce AWTP channels and codes. Section 4 is on bounds and Section 5 gives the construction. Section 6 concludes the paper.

II. Preliminaries

We use calligraphic symbols $X$ to denote sets, $\Pr(X)$ to denote a probability distribution over $X$, and $X$ to denote a random variable that takes values from $X$ with probability $\Pr(X)$. The conditional probability given an events $E$ is $\Pr(X|E)$. Conditional entropy of a random variable $X$ is, $H(X) = -\sum_x \Pr(x) \log \Pr(x)$, and conditional entropy of a random variable $X$ given $Y$ is defined by, $H(X|Y) = -\sum_{x,y} \Pr(x,y) \log \Pr(x|y)$. Statistical distance between two random variables $X_1$, $X_2$ defined over the same set is given by $SD(X_1, X_2) = \frac{1}{2} \sum_x |\Pr[X_1 = x] - \Pr[X_2 = x]|$. Mutual information between random variables $X$ and $Y$ is given by, $I(X, Y) = H(X) - H(X|Y)$. For a vector $e$, Hamming weight of a vector $e$ is denoted by $wt(e)$.

1) Algebraic Manipulation Detection Code (AMD code): Consider a storage device $\Sigma(G)$ that holds an element $x$ from a group $G$. The storage $\Sigma(G)$ is private but can be manipulated by the adversary by adding $\Delta \in G$. AMD codes allow the manipulation to be detected.

Definition 1 (AMD Code [11]): An $(X, G, \delta)$-Algebraic Manipulation Detection code code $((\lambda', G, \delta), G)$-AMD code consists of two algorithms (AMDenc, AMDdec). Encoding, AMDenc : $X \rightarrow G$, is probabilistic and maps an element of a set $X$ to an element of an additive group $G$. Decoding, AMDdec : $G \rightarrow X \cup \{\bot\}$, is deterministic and for any $x \in X$, we have AMDdec$(\text{AMDenc}(x)) = x$. Security of AMD codes is defined by requiring,

$$\Pr[\text{AMDdec}(\text{AMDenc}(x) + \Delta) \in \{x, \bot\}] \leq \delta,$$

for all $x \in X$, $\Delta \in G$.

An AMD code is systematic if the encoding has the form AMDenc : $X \rightarrow X \times G_1 \times G_2$, and $x \rightarrow (x, r, t = f(x, r))$, for some function $f$ and $r \leftarrow G_1$. The decoding function results in AMDdec$(x, r, t) = x$, if and only if $t = f(x, r)$, and $\bot$ otherwise.

We use the systematic AMD code in [11] over an extension field. Let $\phi$ be a bijection between vectors $v$ of length $N$ over $\mathbb{F}_q$, and elements of $\mathbb{F}_{q\cdot \ell}$, and let $\delta$ be an integer such that
Let $s + 2$ be not divisible by $q$. Define the function $\text{AMDenc} : \mathbb{F}_q^n \rightarrow \mathbb{F}_q^n \times \mathbb{F}_q^m \times \mathbb{F}_q^N$ as, $\text{AMDenc}(x) = (x, r, f(x,r))$ where,

$$f(x, r) = \phi^{-1}\left(\phi(r)^{d+2} + \sum_{i=1}^{d} \phi(x_i) \phi(r)^i \right) \mod q^N$$

**Lemma 1:** For the AMD code above, the success chance of an adversary, that has no access to the codeword $(x, r, t)$, in constructing a new codeword $(x', r', t') = (x' = x + \Delta x, r' = r + \Delta r, t' = t + \Delta t)$, that satisfies $t' = f(x', r')$, is no more than $\frac{d+2}{q^n}$. Proof of this Lemma is the direct application of Theorem 2 in [11], when the underlying field is $\mathbb{F}_q^n$.

2) **Subspace Evasive Sets:** Subspace evasive sets are used to reduce the list size of list decodable code [22].

**Definition 2 (Subspace Evasive Sets [22, 27]):** Let $S \subseteq \mathbb{F}^n_q$. We say $S$ is a $(v, \ell)$-subspace evasive if for all $v$-dimensional affine subspaces $\mathcal{H} \subseteq \mathbb{F}^n_q$, we have $|S \cap \mathcal{H}| \leq \ell$.

Dvir et al. [22] gave an efficient explicit construction of subspace evasive sets $S \subseteq \mathbb{F}^n_q$, with an efficient intersection algorithm that computes $S \cap \mathcal{H}$ for any $v$-dimensional subspace $\mathcal{H} \subseteq \mathbb{F}^n_q$.

A $v \times w$ matrix is called strongly-regular if all its $r \times r$ minors are regular (have non-zero determinant) for all $1 \leq r \leq v$.

**Lemma 2:** (Theorem 3.2 [22]) Let $v, \epsilon > 0$ and $\mathbb{F}$ be a finite field. Let $w = v/\epsilon$ and, assume $w$ divides $n$. Let $A$ be a $v \times w$ matrix with coefficients in $\mathbb{F}$ which is strongly-regular. Let $d_1 > \cdots > d_w$ be integers. For $i \in [v]$ let,

$$f_i(x_1, \ldots , x_w) = \sum_{j=1}^w A_{i,j} x_j^{d_j}$$

and define the subspace evasive set $S \subseteq \mathbb{F}^n$ to be $(n/w)$ times cartesian product of $V_{\mathbb{F}}(f_1, \ldots , f_v) \subseteq \mathbb{F}^w$. That is,

$$S = V_{\mathbb{F}}(f_1, \ldots , f_v) \times \cdots \times V_{\mathbb{F}}(f_1, \ldots , f_v)$$

$$= \{ x \in \mathbb{F}^n : f_i(x_{tw+i}, \ldots , x_{tw+w}) = 0, \forall 0 \leq t < n/w, 1 \leq i \leq v \}$$

Then $S$ is $(v, v^{D-v}\log\log v)$-subspace evasive set, and $|S| = \|\mathbb{F}\|^{(1-\epsilon)n}$.

To use a subspace evasive sets for efficient list decoding, two efficient algorithms are needed: (i) a bijection mapping that maps messages that are elements of a space $\mathbb{F}^{n_1}$ into the subspace evasive set $S$, and an intersection algorithm that computes the intersection between $S$ and any subspace $\mathcal{H}$ with dimension at most $v$. The lemmas below show the existence of these two algorithms for the subspace evasive set above.

**Lemma 3:** [22] Let $v, w, n_1 \in \mathbb{N}$, $b = \frac{\log n}{w-1}$, $n = bw$, and $\mathbb{F}_q$ be a finite field. For any vector $v \in \mathbb{F}_q^n$, there is a bijection which maps $v$ into an element of the subspace evasive set $S \subseteq \mathbb{F}_q^n$. That is, $\mathcal{S} : v \rightarrow s \in S$. The encoding algorithm is $\text{Pol}(n)$.

**Lemma 4:** [22] Let $S \subseteq \mathbb{F}_q^n$ be the $(v, \ell)$-subspace evasive sets (described above). Then there exists an algorithm that, given a basis for any $\mathcal{H}$, outputs $S \cap \mathcal{H}$ in time $\text{Pol}(v^\ell\log\log v)$.

3) **Folded Reed-Solomon Code (FRS code):** An error correcting code $C$ over $\mathbb{F}_q$ is a subspace of $\mathbb{F}_q^N$. The rate of the code is $\log_2 |C|/N$. A code $C$ of length $N$ and rate $R$ is $(\rho, \ell, \text{List})$-list decodable if the number of codewords within distance $\rho N$ from any received word is at most $\ell \text{List}$. List decodable codes can correct up to $1 - R$ fraction of errors in a codeword, which is twice that of unique decoding. This is however at the cost of outputting a list of possible sent codewords (messages). Construction of good codes with efficient list decoding algorithms has been an active research area. An explicit construction of a list decodable code that achieves the list decoding capacity $\rho = 1 - R$, is given by Guruswami et al. [31]. The code is called Folded Reed-Solomon code (FRS code), and can be seen as a Reed-Solomon code with extra structure. The code has polynomial time encoding and list decoding algorithms.

**Definition 3:** [31] A $u$-Folded Reed-Solomon code is an error correcting code with block length $N$ over $\mathbb{F}_q^u$ where $q > Nu$. The message of an FRS code is written as a polynomial $f(x)$ with degree $k$ over $\mathbb{F}_q$. The FRS codeword corresponding to the message is a vector over $\mathbb{F}_q^u$ where each component is a $u$-tuple $(f(\gamma^j u), f(\gamma^{j+1} u), \ldots , f(\gamma^{j+k-1} u))$, $0 \leq j < N$, and $\gamma$ is a generator of $\mathbb{F}_q^u$, the multiplicative group of $\mathbb{F}_q$. A codeword of a $u$-folded Reed-Solomon code of length $N$ is in one-to-one correspondence with a codeword $c$ of a Reed-Solomon code of length $uN$, and is obtained by grouping together $u$ consecutive components of $c$. We use FRSenc to denote the encoding algorithm of the FRS code. $u$ is called the folding parameter of the FRS code.

We will use the linear algebraic FRS decoding algorithm of these codes [31].

**Lemma 5:** [31] For a Folded Reed-Solomon code of block length $N$ and rate $R = \frac{\log k}{3}$, the following holds for all integers $1 \leq v \leq u$. Given a received word $y \in (\mathbb{F}_q^u)^N$ agreeing with $c$ in at least a fraction,

$$(N - \rho N) > N\left(\frac{1}{v+1} + \frac{v}{v+1} uR\right)$$

one can compute a matrix $M \in \mathbb{F}_q^{u^2(v-1)}$ and a vector $z \in \mathbb{F}_q^u$, such that the message polynomials $f \in \mathbb{F}_q[X]$ in the decoded list are contained in the affine space $M b + z$ for $b \in \mathbb{F}_q^{u^2}$ in $O(\left(\frac{Nu \log q}{v}\right)^2)$ time.

III. MODEL AND DEFINITIONS

We consider the following scenario. Alice wants to send messages $m \in M$, reliably and securely, to Bob, over a communication channel that is partially controlled by an adversary, Eve. Let $\Sigma$ denote the channel alphabet, and $C$ be a code, $C \subseteq \Sigma^N$, together with a probabilistic encoder, $\text{AWTPenc} : \Sigma \times \mathcal{R} \rightarrow C$, and a deterministic decoder, $\text{AWTPdec} : \Sigma^N \rightarrow \mathcal{M}$. The encoder takes a message $m$ and a random string $r_S \overset{\$}{\leftarrow} \mathcal{R}$ and outputs a codeword $c = \text{AWTPenc}(m, r_S)$. The codewords associated with a message $m$ and different $r_S$, define a random variable over $C$. Alice will use the encoding algorithm and for a message $m$ (also referred to as the information), generates a codeword $\text{AWTPenc}(m)$. The adversary interacts with the codeword as
described below, resulting Bob to receive a corrupted word $y \neq c$. Bob uses the decoding algorithm to recover the message.

A. Adversarial Wiretap: Channel and Code

Let $[N] = \{1, \ldots, N\}$, and $S_r = \{i_1, \ldots, i_{\rho_r N}\} \subseteq [N]$ and $S_w = \{j_1, \ldots, j_{\rho_w N}\} \subseteq [N]$, denote two subsets of the $N$ coordinates, and for a vector $x \in \Sigma^N$, $\text{SUPP}(x)$ denote the set of coordinates where $x_i$ is non-zero.

**Definition 4:** A $(\rho_r, \rho_w)$-Adversarial Wiretap channel (or a $(\rho_r, \rho_w)$-AWTP channel), is an adversarially corrupted communication channel between Alice and Bob, such that it is (partially) controlled by the adversary Eve with two capabilities: Reading and Writing. For a codeword of length $N$, Eve can do the following.

- **Reading** (also called Eavesdropping): Eve can select a subset $S_r \subseteq [N]$ of size at most $\rho_r N$ and read the components of the sent codeword $c$, on positions associated with $S_r$. Eve’s view of the codeword is given by, \[ \text{View}_A(\text{AWTPenc}(m, r), r_A) = \{c_{i_1}, \ldots, c_{i_{\rho_r N}}\}, \] and consists of all the components that are read (observed).

- **Writing** (also called Jamming): Eve can choose a subset $S_w \subseteq [N]$ of size at most $\rho_w N$, for “writing”. This is by adding an error vector $e$ to vector $c$, where the addition is component-wise over $\Sigma$. It holds that $\text{SUPP}(e) \subseteq S_w$. The corrupted components of $c$ are $\{y_{j_1}, \ldots, y_{j_{\rho_w N}}\}$ and $y_{ji} = c_{ji} + e_{ji}$. The error $e$ is generated according to the Eve’s best strategy for making Bob’s decoder to output in error.

We assume the adversary is adaptive and can select components of the sent codeword for reading and writing one by one, at each step using their knowledge of the codeword at that time.

Let $S = S_r \cup S_w$ denote the set of codeword components that the adversary either reads, or writes to. We have $|S| = \rho N$, and $\rho \leq \rho_r + \rho_w$.

An AWTP channel is called restricted if, $S_r = S_w$. Restricted AWTP channel are a special type of AWTP channel where the adversary is limited in its selection of $S_r$ and $S_w$, and so is a weaker type of AWTP channel.

Alice and Bob will use an Adversarial Wiretap Code to provide secure and reliable communication over AWTP channel.

**Definition 5:** An $(\epsilon, \delta)$-Adversarial Wiretap Code $(\epsilon, \delta)$-AWTP code) over $(\rho_r, \rho_w)$-AWTP channel, consists of a randomized encoding $\text{AWTPenc} : \mathcal{M} \times \mathcal{R} \rightarrow \mathcal{C}$, from the message space $\mathcal{M}$ to a code $\mathcal{C}$, and a deterministic decoding algorithm $\text{AWTPdec} : \Sigma^N \rightarrow \mathcal{M}$. The code guarantees the following two properties:

- **Secrecy:** For any two messages $m_1, m_2 \in \mathcal{M}$, the statistical distance between the adversary’s views, when the same randomness $r_A$ is used by the adversary, is bounded by $\epsilon$. That is, \[ \text{Adv}_s(\text{AWTPenc}, \text{View}_A) \triangleq \max_{m_0, m_1} \text{SD}((\text{View}_A(\text{AWTPenc}(m_1), r_A), \text{View}_A(\text{AWTPenc}(m_2), r_A)) \leq \epsilon \]

- **Reliability:** For any message $m$ that is encoded to $c$ by the sender, and corrupted to $y = c + e$ by the $(\rho_r, \rho_w)$-AWTP channel, the probability that the receiver outputs the correct information $m$ is at least $1 - \delta$. That is, \[ \Pr(M_S \neq M_R) \leq \delta \]

where the probability is over the choice of the message, randomness of the communicants and the adversary.

The AWTP code is perfectly secure if $\epsilon = 0$.

For $\epsilon > 0$, an $\epsilon$-secure AWTP code family $\mathcal{C}_\epsilon$, is a family $\{C_N\}_{N \in \mathbb{N}}$ of $(\epsilon, \delta)$-AWTP codes, indexed by $N \in \mathbb{N},$ for a $(\rho_r, \rho_w)$-AWTP channel. When $\epsilon = 0$, the family is called a perfectly secure AWTP code family.

In the following, when $\epsilon \neq 0$, we omit $\epsilon$ and simply write $\mathcal{C}$ to denote, $\mathcal{C}_\epsilon$.

**Definition 6:** For a family $\mathcal{C}$ of $(\epsilon, \delta)$-AWTP codes the rate $R(\mathcal{C})$ is achievable if for any $\xi$, there exists $N_0$ such that for any $N \geq N_0$, we have, $\frac{1}{N} \log |\mathcal{C}| \geq R(\mathcal{C}) - \xi$, and the encoding error probability satisfies, $\Pr(M_S \neq M_R) \leq \xi$.

To define secrecy capacity of an AWTP channel, we will use achievable rate of a code family for the channel.

**Definition 7:** The $\epsilon$-secrecy capacity of a $(\rho_r, \rho_w)$-AWTP channel denoted by $C^\epsilon$, is the largest achievable rate of all AWTP-code families $\mathcal{C}$ that provide $\epsilon$-secrecy for the channel.

The perfect secrecy capacity of a $(\rho_r, \rho_w)$-AWTP channel is denoted by $C^0$, and is the highest achievable rate of perfectly secure AWTP-code families for the channel.

IV. A Bound on the Rate of $(\epsilon, \delta)$-AWTP Codes

We derive an upper bound on the rate of AWTP codes, and use it to find the secrecy capacity of AWTP channels.

The bound is derived on the rate of an arbitrary code when the adversary uses a special strategy, given below. Since the strategy can always be used, it follows that the code rate cannot be higher than the bound. The adversary’s strategy is a probabilistic strategy.

1) Before the start of the transmission, the adversary selects two pairs of read and write sets, $\{S_{r1}^i, S_{w1}^i\}, i = 1, 2,$ satisfying $S_{r1}^i \cap S_{w1}^i = \emptyset$.

The adversary then selects one of the two pairs with probability $1/2$; that is, $\Pr(S_{r1}^i, S_{w1}^i) = \Pr(S_{r2}^i, S_{w2}^i) = \frac{1}{2}$.

The set sizes satisfy the following: for $i = 1, 2$, we have $|S_{r1}^i| = \rho_r N$, $|S_{w1}^i| = \rho_w N$, and $|S_{r1}^i \cup S_{w1}^i| = \rho N$, where $0 \leq \rho \leq 1$.

2) For the chosen read and write pair, $\{S_{r2}^i, S_{w2}^i\}$, the adversary, (i) reads the $\rho_r N$ components of the codeword corresponding to the subset $S_{r2}^i$, (ii) chooses an error vector $e_i \in \Sigma^{\rho_w N}$ randomly with uniform distribution, and adds it component-wise to the codeword components corresponding to $S_{w2}^i$.

3) The adversary chooses the uniform distribution on the message space.

We associate a random variable $C_i$ to the $i^{th}$ component of an $(\epsilon, \delta)$-AWTP code. For $i = 1, 2$, let $C_{S_{r1}^i}$ and $C_{S_{w1}^i}$ be the components of a codeword on the sets $S_{r1}^i, S_{w1}^i$, respectively.

Let $Y$ denote the word that Bob receives.
In the following we will derive the secrecy capacity, \( C_s \), of a \((\rho_r, \rho_w)\) AWTP channel.

**Theorem 1:** The upper bound on the secrecy capacity of AWTP code family over \((\rho_r, \rho_w)\)-AWTP channel is,

\[
C_s \leq 1 - \rho_r - \rho_w + 2\epsilon \rho_r (1 + \log_2 \frac{1}{\epsilon})
\]

We first prove an upper-bound on the rate of an \((\epsilon, \delta)\)-AWTP code (Lemma 3), and then extend the bound to the achievable rate of a code family, and so the secrecy capacity of the channel. To prove the bound on the rate of a code, we prove two lemmas that use the secrecy and reliability guarantees of the code, respectively, and use them to prove the bound on the rate of the code.

**Lemma 6:** An \((\epsilon, \delta)\)-AWTP code for a \((\rho_r, \rho_w)\)-AWTP channel satisfies,

\[
H(M) - H(M|C_{s1}) \leq 2\epsilon \rho_r N \log \frac{\Sigma}{\epsilon}
\]

**Proof:** From the definition of \(\epsilon\)-secrecy we have,

\[
\begin{align*}
\text{Adv}_{\epsilon}^s(\text{AWTPenc, View}_A) &= \frac{1}{2} \sum_{c_{s1}} |\Pr(c_{s1}|m_o) - \Pr(c_{s1}|m_1)| \\
&+ \frac{1}{2} \sum_{c_{s2}} |\Pr(c_{s2}|m_0) - \Pr(c_{s2}|m_1)| \leq \epsilon
\end{align*}
\]

This implies that for any pair of messages, \(m_0, m_1 \in \mathcal{M}\), we have,

\[
2 \sum_{c_{s1}} |\Pr(c_{s1}|m_0) - \Pr(c_{s1}|m_1)| \leq \epsilon
\]

and so it follows that,

\[
\begin{align*}
\text{SD}(P_{c_{s1}m}, P_{c_{s1}} P_M) &= \frac{1}{2} \sum_{c_{s1}} |\Pr(c_{s1}|m) - \Pr(c_{s1})| \\
&= \frac{1}{2} \sum_{c_{s1}} |\Pr(c_{s1}|m) - \sum_{m' \in \mathcal{M}} \Pr(c_{s1}|m')\Pr(m')| \\
&= \frac{1}{2} \sum_{c_{s1}} |\sum_{m' \in \mathcal{M}} \Pr(c_{s1}|m') - \Pr(c_{s1})| \\
&\leq \sum_{m \in \mathcal{M}} \Pr(m) \frac{1}{2} \sum_{c_{s1}} |\Pr(c_{s1}|m) - \sum_{m' \in \mathcal{M}} \Pr(c_{s1}|m')| \\
&\leq \sum_{m \in \mathcal{M}} \Pr(m) \epsilon
\end{align*}
\]

By Theorem 17.3.3 (Page 370, [17]), for sufficiently small \(\epsilon\)

\[
I(M, C_{s1}) 
\leq 2 \text{SD}(P_{c_{s1}m}, P_{c_{s1}} P_M) \log \frac{\Sigma|^{\rho_r N}}{\text{SD}(P_{c_{s1}m}, P_{c_{s1}} P_M)}
\]

\[
\leq 2\epsilon \rho_r N \log \frac{\Sigma}{\epsilon}
\]

**Lemma 7:** An \((\epsilon, \delta)\)-AWTP code for a \((\rho_r, \rho_w)\)-AWTP channel satisfies,

\[
H(M|Y \bar{S}_r^2 \bar{S}_w^2) \leq H(\delta) + \delta N \log |\Sigma|
\]

**Proof:** From Fano’s inequality (Theorem 2.10.1, Page 38, [17]), the decoding error probability \(\delta\), implies,

\[
H(M|Y) \leq H(\delta) + \delta N \log |\Sigma|
\]

We have \(\Pr(M) = \Pr(M|S_{r1}^1 S_{w1}^1) + \Pr(M|S_{r2}^2 S_{w2}^2)\) and so,

\[
\begin{align*}
H(M|Y) &= H(M|S_{r1}^1 S_{w1}^1 Y) + H(M|S_{r2}^2 S_{w2}^2 Y) \\
&= H(S_{r1}^1 S_{w1}^1 Y) H(M|Y S_{r1}^1 S_{w1}^1) + H(S_{r2}^2 S_{w2}^2 Y) H(M|Y S_{r2}^2 S_{w2}^2) \\
&= \frac{1}{2} H(S_{r1}^1 S_{w1}^1) H(M|Y S_{r1}^1 S_{w1}^1) + \frac{1}{2} H(S_{r2}^2 S_{w2}^2) H(M|Y S_{r2}^2 S_{w2}^2)
\end{align*}
\]

(1) is because \(S_{r1}^1, S_{w1}^1\) are selected before transmission starts, and independent of \(Y\); (2) is from \(H(S_{r1}^1 S_{w1}^1) = \frac{1}{2}\). Since \(H(M|Y S_{r1}^1 S_{w1}^1) \geq 0\), we have,

\[
H(M|Y S_{r}^2 S_{w}^2) \leq 2H(\delta) + 2\delta N \log |\Sigma|
\]

We denote the \((\epsilon, \delta)\)-AWTP code with length \(N\) as \(C^N\), and the rate of \((\epsilon, \delta)\)-AWTP code as \(R(C^N)\).

**Lemma 8:** The upper bound rate of \((\epsilon, \delta)\) AWTP code \(C^N\) over \((\rho_r, \rho_w)\) AWTP channel is,

\[
R(C^N) \leq 1 - \rho_r - \rho_w + 4H(\delta) + 2\epsilon \rho_r (1 + \log_2 \frac{1}{\epsilon})
\]

**Proof:** We have,

\[
\begin{align*}
H(M) &= I(M; Y S_{r}^2 S_{w}^2) + H(M|Y S_{r}^2 S_{w}^2) \\
&= I(M; C_{s1}) + I(M; C_{s1}) \\
&\leq I(M; Y C_{s1} S_{r}^2 S_{w}^2) + I(M; C_{s1}) \\
&+ H(M|Y S_{r}^2 S_{w}^2) + I(M; C_{s1}) \\
&= I(M; Y C_{s1} S_{r}^2 S_{w}^2) + I(M; C_{s1}) \\
&+ H(M|Y S_{r}^2 S_{w}^2) + I(M; C_{s1}) \\
&= H(Y|C_{s1} S_{r}^2 S_{w}^2) + I(M; C_{s1}) \\
&+ H(M|Y S_{r}^2 S_{w}^2) + I(M; C_{s1}) \\
&\leq H(Y|C_{s1} S_{r}^2 S_{w}^2) + H(E_2|MC_{s1} S_{r}^2 S_{w}^2) \\
&+ H(M|Y S_{r}^2 S_{w}^2) + I(M; C_{s1}) \\
&\leq H(Y|C_{s1} S_{r}^2 S_{w}^2) + H(E_2 S_{r}^2 S_{w}^2) \\
&+ H(M|Y S_{r}^2 S_{w}^2) + I(M; C_{s1}) \\
&\leq H(Y|C_{s1} S_{r}^2 S_{w}^2) + H(E_2 S_{r}^2 S_{w}^2) \\
&+ H(M|Y S_{r}^2 S_{w}^2) + I(M; C_{s1})
\end{align*}
\]

(5) (1) is from \(\{S_{r}^2, S_{w}^2\} \rightarrow C_{s1} \rightarrow M\) from which it follows that \(\Pr(M|C_{s1} S_{r}^2 S_{w}^2) = \Pr(M|C_{s1})\). The Markov chain holds because knowledge of \(C_{s1}\) implies that subset pair \(\{S_{r}^2, S_{w}^2\}\) is used, and so \(\{S_{r}^2, S_{w}^2\}\) does not provide extra information; (2) is by noting that \(E_2 = Y - C\) if the adversary selects \(\{S_{r}^2, S_{w}^2\}\). (3) is from the Markov chain \(MC_{s1} C\rightarrow\)
\{S_r^2, S_w^2\} \rightarrow E_2$, which implies $I(MC_1; C; E_2|S_r^2S_w^2) = H(E_2|S_r^2S_w^2) - H(E_2|MC_1S_r^2S_w^2) = 0$.

So we have
\[
H(M) \leq H(Y|S_r^2S_w^2) - H(E_2|S_r^2S_w^2) + H(M|Y S_r^2S_w^2)
\]
\[
+ I(M; C S_1^2)
\]

We can upper bound $H(M)$ by bounding the four terms on the right hand side of the inequality.

First, we have the bound, $H(Y|S_r^2S_w^2) \leq (1 - \rho_r)N \log |\Sigma|$. Let $[N]\backslash S_1^2$ be the subset of $[N]$ that is not in $S_1^2$, and $Y_{[N]\backslash S_1^2}$ be the components of $Y$ on the set $[N]\backslash S_1^2$. Since $S_r^2 \cap S_w^2 = \emptyset$, if the adversary selects the set pair $\{S_r^2, S_w^2\}$, the components of $Y$ on the set $S_1^2$ will not have error and will be equal to the components of $C$ on $S_1^2$. That is,
\[
H(Y|S_1^2) = 0
\]
So we have,
\[
H(Y|S_r^2S_w^2) = H(Y|S_1^2) + H(Y_{[N]\backslash S_1^2}|S_r^2S_w^2)
\]
\[
= H(Y_{[N]\backslash S_1^2}|S_r^2S_w^2) \leq (1 - \rho_r)N \log |\Sigma|
\]

To bound the second item notice that if the adversary selects $\{S_r^2, S_w^2\}$, $E_2$ is uniformly distributed and so,
\[
H(E_2|S_r^2S_w^2) = \rho_wN \log |\Sigma|
\]

From Lemma 4 and 7 we also have the bounds $H(M|Y S_r^2S_w^2) \leq 2H(\delta) + 2\delta N \log |\Sigma|$ and $H(M) - H(M|S_1^2) \leq 2r \rho_r N \log \frac{\log 1}{\epsilon}$. So the upper bound on $H(M)$ is,
\[
H(M) \leq (1 - \rho_r - \rho_w)N \log |\Sigma| + 2H(\delta) + 2\delta N \log |\Sigma|
\]
\[
+ 2\epsilon \rho_r N \log \frac{\log 1}{\epsilon}
\]

Since the message is uniformly distributed, we have $H(M) = \log |\mathcal{M}|$. Since for $0 < \delta < \frac{1}{2}$ it holds $\delta < H(\delta)$, the upper bound on the rate of an AWTP code of length $N$ is obtained by using $H(\delta) + \delta N \log |\Sigma| \leq 2H(\delta)N \log |\Sigma|$. That is,
\[
R(C^N) = \frac{\log |\mathcal{M}|}{N \log |\Sigma|}
\]
\[
\leq 1 - \rho_r - \rho_w + 2\epsilon \rho_r (1 + \log |\Sigma|) \frac{1}{\epsilon} + 4H(\delta)
\]

The following is the proof of Theorem 1.

Proof: (Theorem 1) Proof by contradiction. Suppose there is a code family $\mathcal{C}$ with achievable rate $C^* = 1 - \rho_r - \rho_w + 2\epsilon \rho_r (1 + \log |\Sigma|) \frac{1}{\epsilon}$, for some small constant $0 < \xi < \frac{1}{2}$.

Let $H(p_0) = \frac{\xi}{8}$. For any $\xi' \leq p_0$, we have $4H(\xi') \leq \frac{\xi}{2}$ and $\hat{\xi}' \leq H(\xi') \leq \frac{\xi}{2}$. From Definition 8 for any $0 < \xi' \leq p_0$, there is an $N_0$ such that for any $N > N_0$, we have $\delta < \xi'$ and,
\[
R(C^N) \geq C' - \xi'
\]
\[
= 1 - \rho_r - \rho_w + 2\epsilon \rho_r (1 + \log |\Sigma|) \frac{1}{\epsilon} + 4H(\delta) + \frac{\xi}{2} - \xi'
\]
\[
(1) \geq 1 - \rho_r - \rho_w + 2\epsilon \rho_r (1 + \log |\Sigma|) \frac{1}{\epsilon} + 4H(\delta)
\]

(1) is from $H(\delta) \leq H(\xi') < \frac{\xi}{2}$; (2) is from $\xi' < \frac{\xi}{2}$.

This contradicts the bound on $R(C^N)$ in Lemma 8 and so,
\[
C^* \leq 1 - \rho_r - \rho_w + 2\epsilon \rho_r (1 + \log |\Sigma|) \frac{1}{\epsilon}
\]

For $\epsilon = 0$, we have the upper bound on the achievable rate of an AWTP code family with perfect secrecy.

Corollary 1: The upper bound on the achievable rate of a perfectly secure AWTP code family for a $(\rho_r, \rho_w)$-AWTP channel is,
\[
C^0 \leq 1 - \rho_r - \rho_w
\]

A. Restricted AWTP channels

Note that the above proof is general in the sense that the sets $S_r$ and $S_w$ can have nonempty intersection. Restricted channels limit the adversary to the case that $S_r = S_w$. Using the same approach, we can derive the following bounds on $C^0$ and $C^*$.

Corollary 2: The upper bound of rate of restricted-AWTP code family with over $(\rho_r, \rho_w)$-AWTP channel is, for a perfectly secure code family,
\[
C^0 \leq 1 - \rho_r - \rho_w
\]

and for an $\epsilon$-secure code family,
\[
C^\epsilon \leq 1 - \rho_r - \rho_w + 2\epsilon \rho_r (1 + \log |\Sigma|) \frac{1}{\epsilon}
\]

We note that a more direct proof of Theorem 1 is to use an adversary strategy in which $S_r \cap S_w = \emptyset$. However this proof cannot be used for the subcase of restricted AWTP because, for this subclass this is not a valid adversarial strategy. The above proof with randomized adversarial strategy removes this restriction and allows us to apply the same proof method for restricted AWTP channels.

V. AWTP Code Construction

Let $q$ be a prime satisfying, $q > Nu$. A message is an element of $\mathcal{M} = \mathbb{F}_q^{\rho R N}$, given by $m = \{m_1, \cdots, m_{\rho R N}\} \in \mathcal{M}$, and $m_i \in \mathbb{F}_q$.

We construct a $(0, \delta)$-AWTP code family $C^0 = \{C^N\}_{N \in \mathbb{N}}$, for a $(\rho_r, \rho_w)$-AWTP channel. The construction uses, (i) an FRS code, (ii) an AMD code and, (iii) a subset evasive sets, with the following parameters.

1) A $u$-Folded RS-codes of length $N$ over $\mathbb{F}_q$, with a linear algebraic decoder $[31]$ using the decoding parameter $u$.

Let $\xi_1 = \xi/13$. Parameters of the FRS code are chosen as, (i) folding parameter $u = \xi_1^{-2}$, (i) decoding parameter $v = \xi_1^{-1}$, (ii) the length $N \geq (1/\xi_1)^{D/\xi_1 \log \log 1/\xi_1}$,
and (iv) the field size satisfying $q > Nu$, and condition 2 of Theorem 3.2 in [22]. This latter condition on $q$ is required for efficient injective mapping into the subset evasive set, and hence efficient encoding and decoding.

2) Assume, for simplicity, that $uR$ is an integer. (The argument can straightforwardly be extended to the case that this condition does not hold.) The AMD code will be the code in Section II-1, and will have message space $X = \mathbb{F}_q^{uR}$, codeword space $G = \mathbb{F}_q^{uR + 2}$, and $\delta \leq \frac{uR + 1}{q^2}$.

3) We will use a $(v, vD-v\log v)$-subspace evasive sets $S$, that is a subset of size $q^n$, in $\mathbb{F}_q$, using the construction in Theorem 3.2 in [22]. Here $D$ is a constant (See Claim 4.3 in [22]), and the bound $vD-v\log v$ on the intersection list size of a $v$-dimensional affine subspace with $S$, follows from Claim 4.3 and 3.3 in [22].

The parameters $n$ and $n_1$ are chosen as shown below, to achieve a rate $R(C^N) = 1 - \rho_r - \rho_w - \xi_l$.

Let $w = v^2$ and $b = \lceil uR + 2N \rceil$. Then we choose $n_1 = (w - v)b$, $n = wb$.

Note that $n_1$ is almost the same as the codeword length of the AMD code, i.e. $(uR + 2)N$. Also, $v = \xi_l^{-1}$ and,

$$n_1 = \frac{w - v}{w} = \frac{v^2 - v}{v^2} = (1 - 1/v)n = (1 - \xi_l)n$$

and so the size of $S$ satisfies condition 2 in Theorem 3.2 in [22].

We use $\gamma$ to denote a primitive element of $\mathbb{F}_q$.

Let $AWTPenc_N$ and $AWTPdec_N$, be the encoding and decoding algorithms of the code, respectively.

The constructions of the encoder and the decoder of $C^N$ are given in Figure [V].

| Encoding | Alice does the following: |
|----------|--------------------------|
| 1)      | Interpret an information block $m$ of length $uRN$, as a vector $x \in \mathbb{F}_q^{uR}$, Generate a random vector $r \in \mathbb{F}_q^N$ and use it to find the codeword associated with $x$ using the AMD construction in section II-1. $AMDenc(x) = (x, r, t)$. The AMD codeword is of length $uRN + 2N$ over $\mathbb{F}_q$. |
| 2)      | Extend the AMD codeword to length $n_1$ by appending zeros from $\mathbb{F}_q$, Encode the AMD codeword to an element $s \in S$, using the bijection mapping of the subspace evasive sets, $s = SE(x, r, t||0, \cdots, 0)$ |
| 3)      | Append a random vector $a = (a_1, \cdots, a_{u\rho_r}N) \in \mathbb{F}_q^{u\rho_r}N$ to $s$ and form the vector that will be the message of the FRS code. Use $s$ and $a$ as the coefficients of the FRS codeword polynomial, $f(x)$, over $\mathbb{F}_q$. That is $(f_0, \cdots, f_{k-1}) = (s||a)$. We have $k = deg(f) + 1 = n + u\rho_rN$. |

4) Use $FRSenc$ to construct the FRS codeword $c = FRSenc(f(X)) = (c_1, \cdots, c_N)$, with $c_i = (f(\gamma^{iu1-1})), f(\gamma^{iu1-1})) \in \mathbb{F}_{q}^{r}$, for $i = 1, \cdots, N$.

| Decoding | Bob does the following: |
|----------|--------------------------|
| 1)      | Let $y = c + e$, and $w_H(e) \leq \rho_w$. Let, $y = (y_1, \cdots, y_N)$ and $y_i = (y_i, 1, \cdots, y_i, u)$ for $i = 1, \cdots, N$. |

Use the FRS (linear algebraic) decoding algorithm $FRSdec(y)$ to output a matrix $M \in \mathbb{F}_{q}^{k \times \langle w \rho_rN \rangle}$, and a vector $z \in \mathbb{F}_q$, such that the decoder output list is of the form, $L_{list} = Mb + z$. $M$ has $k = n + u\rho_rN$ rows, each giving a component of the output vector as a linear combination of $(b_1, \cdots, b_v)$, and the corresponding component of $z$. |
| 2)      | Let $H$ denote the vector space spanned by the first $n$ equations. That is $H = M_n \times b + z_n$, $b \in \mathbb{F}_q^n$, where $M_n \times v$ is the first $n$ rows of the submatrix of $M$ and $z_n$ is the first $n$ elements of $z$. |

We prove secrecy and reliability of the above code, and derive the rate of the AWTP code family.

**Lemma 9 (Secrecy):** The AWTP code $C$ above provides perfect security for $(\rho_r, \rho_w)$-AWTP channels.

**Proof:** It is sufficient to show that an AWTP codeword sent over a $(\rho_r, \rho_w)$-AWTP channel does not leak any information about the encoded element in subspace evasive sets, which includes the message sent by Alice.

The codeword polynomial is of degree $n + u\rho_rN - 1$ and so has $n + u\rho_rN$ coefficients, $u\rho_rN$ of which are randomly chosen. The adversary sees $r\rho_rN$ elements of $\mathbb{F}_q$, each corresponding to a linear equation on the coefficients. This means that the adversary has no information about the remaining $n$ coefficients corresponding to the message. The FRS coding can be seen as coset coding in [46], and so inheriting the security of these codes. Hence for an arbitrary observation $View_A = \{c_{j_1}, \cdots, c_{j_{\rho_rN}}\}$,

$$H(S|View_A) = H(S)$$

where $S$ is the element of the subspace evasive sets which is the encoding from the message $M$. 

---

**Figure [V]**

Theorem 3.2 in [22].
Lemma 10 (Reliability): i) Given \( N \geq v^2 \), the AWTP code \( C^N \) described above provides reliability for a \((\rho_r, \rho_w)\)-AWTP channel if the following holds:
\[
\rho_w < \frac{v}{v+1 - \frac{u}{v+1}(uR + 3) + u\rho_r}.
\]  
(8)

ii) The decoding error probability of AWTPdec is bounded by \( \delta \leq \frac{v^{D'}v \log \log v}{vD'v \log \log v} \).  

Proof: i) FRS decoding algorithm FRSdec \([31]\) requires,
\[
N - \rho_w N > N\left(\frac{1}{v+1} + \frac{u \rho_{FRS}}{v} \right)
\]  
(9)
The dimension of the FRS code is bounded by,
\[
k = u \rho_{FRS} N + u \rho_r N + n
\]
\[
= u \rho_r N + w\left[\frac{uRN + 2N}{w-v} \right]
\]  
(1)
is from \( N \geq v^2 \).

Thus, we have, \( u \rho_{FRS} \leq u \rho_r + \frac{w}{w-v}(uR + 3) \), and replacing \( R_{FRS} \) in the decoding condition for FRS code \([9]\) gives,
\[
\rho_w < \frac{v}{v+1 - \frac{u}{v+1}(uR + 3) + u\rho_r},
\]
(10)
ii) There is a decoding error if there are at least codewords in the FRS decoder output list, that are AMD encodings of two messages \( \mathbf{m} \neq \mathbf{m}' \).

First note that the correct message is always in the decoder list. This is because the FRS decoder output, combined with the subspace evasive sets intersection algorithm, gives all codewords that are at distance at most \( \rho_w N \) from the received word \( y \), and, have messages that are elements of the subset evasive set. This list includes the sent codeword as the message \( \mathbf{m} \) had been mapped to \( S \), that is the information is first, encoded using AMD coding \((x, r, t) = AMDenc(m)\), and then mapped to the subspace evasive sets,

\[
SE(x, r, t)[0, \cdots, 0] \in L = S \cap H \quad \text{and} \quad t = f(x, r),
\]
and finally, encoded using FRS code. Hence the sent codeword is in the output list of the FRS decoder.

Next, we show that the probability that the message associated with any other codeword in the decoder list is a valid AMD codeword, is small. That is,
\[
Pr([SE(x', r', t') \in S \cap H] \land [t' = f(x', r')]) \leq \frac{uR + 1}{q^N}
\]
From Lemma \([5]\) the adversary has no information about the encoding of any message \( s \), and so the AMD codeword \( SE(x, r, t) = s \). This means that the adversary’s error, \((\Delta x_i = x' - x, \Delta r_i = r' - r, \Delta t_i = t' - t)\), is independent of \((x, r, t)\). According to Lemma \([1]\) the probability that a tampered AMD codeword, \((x', r', t')\), passes the verification is no more than \( \frac{v^2}{p^v} \).

Finally, we show the probability of decoding error is at most \( \delta \leq \frac{v^{D'}v \log \log v}{p^v} \). The list size is at most \(|S \cap H| \leq v^D v^w \log \log v\) and, \( uR + 1 \leq u + 1 = v^2 + 1 \). So by using the union bound and letting \( D = D' + 3 \), the probability that some \((x', r', t') \neq (x, r, t)\) in the decoded list passes the verification \( t' = f(x', r')\), is no more than \( \frac{v^{D'}v \log \log v}{p^v} \).

A. Secrecy Capacity

The achievable rate of the code family \( C = \{C^N\}_{N \in \mathbb{N}} \) is given by the following Lemma.

Lemma 11 (Achievable Rate of \( C \)): The information rate of the AWTP code family \( C = \{C^N\}_{N \in \mathbb{N}} \) for a \((\rho_r, \rho_w)\)-AWTP channel is \( R(C) = 1 - \rho_r - \rho_w \).

Proof: For a given small \( 0 < \xi < \frac{1}{2} \), let the code parameters be chosen as, \( \xi_1 = \frac{\xi}{\sqrt{2}}, v = 1/\xi_1 \) and \( u = 1/\xi_2^2 \). Finally let, \( N_0 > (1/\xi_1)^D/\xi_1 \log \log \xi_1/\xi \) where \( D > 0 \) is a constant.

We have, starting from the right hand side of \((8)\),
\[
\frac{v}{v+1} - \frac{v}{v+1} \frac{uR + 3 + u\rho_r}{uR + 3 + u\rho_r}
\]
\[
= 1 - \frac{1}{\xi_1 + 1} - \frac{1}{\xi_1 + 1} - \frac{1}{\xi_1 + 1} - \frac{1}{\xi_1 + 1}
\]
\[
\geq 1 - \xi_1 - \frac{1}{\xi_1 + 1} - \frac{1}{\xi_1 + 1} - \frac{1}{\xi_1 + 1} - \frac{1}{\xi_1 + 1}
\]
\[
\geq 1 - \xi_1 - \frac{1}{\xi_1 + 1} - \frac{1}{\xi_1 + 1} - \frac{1}{\xi_1 + 1} - \frac{1}{\xi_1 + 1}
\]
\[
= 1 - \xi_1 - \frac{1}{\xi_1 + 1} - \frac{1}{\xi_1 + 1} - \frac{1}{\xi_1 + 1} - \frac{1}{\xi_1 + 1}
\]
\[
= 1 - \rho_r - \rho_w - 2\xi_1.
\]  
(11)
and so, \( R(C^N) = 1 - \rho_r - \rho_w - 2\xi_1 \).

Now since \( \xi_1 = \frac{\xi}{\sqrt{2}} \), for any \( N > N_0 \), the rate of the AWTP code \( C^N \) is
\[
\frac{1}{\sqrt{N}} \log \{M\} = R(C^N) = 1 - \rho_r - \rho_w - 2\xi_1
\]
\[
> 1 - \rho_r - \rho_w - \xi = R(C) - \xi
\]
and the probability of the decoding error is bounded as,
\[
\delta \leq (1/\xi_1)^D/\xi_1 \log \log 1/\xi_1 \leq Nq^{-N} \leq \xi
\]
This concludes that the achievable rate of AWTP code family \( C \) is \( R(C) = 1 - \rho_r - \rho_w \).

The computational complexity of encoding is \( O((N \log q)^2) \). The combined computational complexity of the FRS decoding algorithm and subspace evasive sets intersection algorithm is, \( Poly((1/\xi)^D/\xi \log \log 1/\xi) \). An AMD verification costs \( O((N \log q)^2) \), and so the total complexity of the AWTP decoding is \( Poly(N) \).

Theorem 2: For any small \( \xi > 0 \), there is a \((0, \delta)\)-AWTP code \( C^N \) of length \( N \) for \((\rho_r, \rho_w)\)-AWTP channel, such that the information rate is \( R(C^N) = 1 - \rho_r - \rho_w - \xi \), the alphabet
proved [26] that probabilistic SMT protocol satisfies the following two properties:

1. **Reliability:** Receiver $R$ outputs the wrong message with probability no more than $\delta_{\text{SMT}}$.

\[
\Pr(M_S \neq M_R) \leq \delta_{\text{SMT}}
\]

2. **Adversary’s view:** When it is clear from the context, we omit the subscript “SMT” and simply use $(\epsilon, \delta)$-SMT.

A perfect SMT protocol has $\delta_{\text{SMT}} = 0$ and $\delta_{\text{SMT}} = 0$. It was proved [26] that $(\epsilon_{\text{SMT}}, \delta_{\text{SMT}})$-SMT for $\delta_{\text{SMT}} < \frac{1}{2} (1 - \frac{1}{|S|})$, is possible only if $N \geq 2t + 1$ and 1-round $(0,0)$-SMT is possible only if $N \geq 3t + 1$ [21]. Let $V_i$ denote the set of possible transmissions (also called transcripts) of each wire. Transmission rate of an SMT protocol is defined as $r(S) = \frac{\text{Total Length of transcripts}}{\text{Length of message}} = \frac{\sum_{i=1}^{N} \log |V_i|}{\log |S|}$.

For 1-round $(0,0)$-SMT protocols, the lower bound on transmission rate is $\frac{N}{N-3t}$ [21], and for $(0, \delta_{\text{SMT}})$-SMT, the bound is $\frac{N}{N-2t}$ [47]. 1-round $(0,0)$-SMT and $(0, \delta_{\text{SMT}})$-SMT protocols whose transmission rates asymptotically reach $O(\frac{N}{N-3t})$ and $O(\frac{N}{N-2t})$, respectively, are called transmission optimal.

### B. Relation between AWTP Code and SMT

$(\epsilon, \delta)$-AWTP codes are closely related to 1-round $(\epsilon_{\text{SMT}}, \delta_{\text{SMT}})$-SMT protocols. In the following we show the relationship between these two primitives.

**Definition 9 (Symmetric SMT):** An SMT protocol is called symmetric SMT if the protocol remains invariant under any permutation of the wires.

Let $(W_1^r, W_2^r, \ldots, W_N^r)$ denote the set of possible transmissions on the $N$ wires in an $r$-round SMT protocol. In a symmetric protocol, for each round $i$, we have $W_j^r = W_j = 1 \cdots N$. That is the set of possible transmissions on a wire, is independent of the wire. All known constructions of SMT protocols are symmetric.

**Theorem 3:** There is a one-to-one correspondence between an $(\epsilon, \delta)$-AWTP code $C_N$ of length $N$ that provides security for a restricted AWTP channel with $S = S_r = S_w$, and a 1-round $(\epsilon_{\text{SMT}}, \delta_{\text{SMT}})$ symmetric SMT protocol for $N$ wires with security against a $(t, N)$ threshold adversary.

Furthermore, an $(\epsilon, \delta)$-AWTP code for a $(\rho_r, \rho_w)$-AWTP channel can be used to construct a code for a restricted AWTP channel, resulting in a 1-round $(\epsilon_{\text{SMT}}, \delta_{\text{SMT}})$ symmetric SMT.

**Proof:** Consider an $(\epsilon, \delta)$-AWTP code $C_N$ over a restricted AWTP channel with $S = S_r = S_w$. By associating each component of the code with a distinct wire, one can construct a 1-round $(\epsilon_{\text{SMT}}, \delta_{\text{SMT}})$ symmetric SMT protocol for $N$ wires. The protocol security is against a threshold $(t, N)$ adversary with $t = \rho N$. The SMT encoding and decoding are obtained from the corresponding functions in the $(\epsilon, \delta)$-AWTP code; that is, $\text{SMTenc}(m, r_s) = \text{AWTPenc}(m, r_s)$ and $\text{SMTdec}(y) = \text{AWTPdec}(y)$. To relate the security and reliability of the SMT protocol to those of the AWTP code, we note the following:

1. Definitions of privacy in the two primitives are both in terms of the statistical distance of the adversary’s view for two messages chosen by the adversary (Compare definition 8 and definition [5]).

2. Definitions of error in decoding for the two primitives both require the decoder to output the correct message with probability at least $1 - \rho$

3. Adversary’s capabilities in the two models are the same. The corruption of the codeword in $(\epsilon, \delta)$-AWTP code is by an additive error, while in SMT the adversary can arbitrarily modify the $|S| = t$ wires. However for restricted AWTP channels with $S = S_r = S_w$, modifying $t$ components $(c_1, \ldots, c_t)$ to $(c'_1, \ldots, c'_t)$ is equivalent to “adding” the error $\epsilon$ with SUPP$(\epsilon) = S$ and $(c_1, \ldots, c_t) = ((c'_1 - c_1), \ldots, (c'_t - c_t))$ and so for these channels additive errors cover all possible adversarial tampering.

The theorem follows by constructing a restricted $(\epsilon, \delta)$-AWTP code for a restricted AWTP channel with $S = S_r = S_w$ from a 1-round $(\epsilon_{\text{SMT}}, \delta_{\text{SMT}})$ symmetric SMT, using the same association of the code components and the wires. We will have $\epsilon = \epsilon_{\text{SMT}}$ and $\delta = \delta_{\text{SMT}}$.

**Corollary 5** below follows from the one-to-one correspondence established in Theorem 8.
Corollary 3: Let \( R(C^N) \) be the rate of an \((\epsilon, \delta)\)-AWTP code \( C^N \) for a restricted AWTP channel. The transmission rate of the associated 1-round \((\epsilon_{SMT}, \delta_{SMT})\) symmetric SMT is given by, \( \tau_R(SMT) = \frac{N \log |M|}{N - 2t + 2\epsilon(1 + \log |V|)} \).

The upper bound on the secrecy rate (Lemma 2) of \((0, \delta)\)-AWTP codes for restricted AWTP channels, gives a lower bound on the transmission rate of 1-round \((0, \delta_{SMT})\) symmetric SMT protocols.

Theorem 4: For a 1-round \((\epsilon_{SMT}, \delta_{SMT})\) symmetric SMT protocol, transmission rate is lower bounded as,

\[
\tau_R(SMT) \geq \frac{N}{N - 2t + 2\epsilon(1 + \log |V|)} \frac{1}{\epsilon}
\]

For \( \epsilon_{SMT} = 0 \), the bounded reduces to the known bound,

\[
\tau_R(SMT) \geq \frac{N}{N - 2\epsilon(1 + \log |V|)} \frac{1}{\epsilon}.
\]

Proof: Using Theorem 2 for a 1-round \((\epsilon, \delta)\) symmetric SMT over \( N \) wires with \( t = \rho N \), there is a corresponding \((\epsilon, \delta)\)-AWTP code for a restricted AWTP channel with \( S = S_r = S_w \) whose information rate is upper bounded by,

\[
R(C^N) \leq 1 - 2\rho + 2\epsilon(1 + \log |V|) \frac{1}{\epsilon}
\]

Since the transmission rate of an \((\epsilon, \delta)\) symmetric SMT protocol is the inverse of the information rate of the corresponding \((\epsilon, \delta)\)-AWTP code, we have

\[
\tau_R(SMT) = \frac{1}{R(C^N)} \geq \frac{1}{1 - 2\rho + 2\epsilon(1 + \log |V|) \frac{1}{\epsilon}} = \frac{N}{N - 2t + 2\epsilon(1 + \log |V|) \frac{1}{\epsilon}}
\]

It has been proved [26] that for 1-round \((\epsilon, \delta)\)-SMT protocols for \( \delta \leq \frac{1}{2}(1 - \frac{1}{2t+1}) \) can be constructed only if, \( N \geq 2t + 1 \).

Corollary 4: For \( N = 2t + 1 \), we will have,

\[
\tau_R(SMT) = \frac{1}{R(C^N)} \geq \frac{2t + 1}{1 + 2t\epsilon(1 + \log |V|) \frac{1}{\epsilon}}
\]

This is the first and the only known lower bound on the transmission rate of \((\epsilon, \delta)\) symmetric SMT protocols. Using a similar approach one can obtain an alternative proof for the known lower bound on the transmission rate of 1-round \((0, \delta)\) symmetric SMT protocols (Theorem 10, [47]).

VII. CONCLUDING REMARKS

We proposed a model for active adversaries in wiretap channels, derived secrecy capacity and gave an explicit construction for a family of capacity achieving codes. The model is a natural extension of Wyner wiretap models when the adversary is a powerful active adversary that uses its partial observation of the communication channel to introduce adversarial noise in the channel. The adversary’s view of communication is the same as wiretap II model. However unlike noiseless main channel in wiretap II, we allow the main channel to be corrupted by the adversary inline with the corruption introduced by the adversary in Hamming’s model of reliable communication. This is the first model of adversarial wiretap where the adversary’s view is used by the adversary for forming its adversarial noise. All previous work (See Section [47]) assume the view of the eavesdropper does not affect the noise added to the channel.

AWTP model provides a framework for studying SMT which so far has been studied independent of wiretap model. The fruitfulness of this was demonstrated by deriving a new lower bound on the transmission rate of 1-round \((\epsilon, \delta)\) symmetric SMT protocols. It is an interesting question if this bound applies to all SMT protocols. That is if allowing different transcript set would increase the bound.

In the discussion part of Section 1, we listed a number of open questions that will improve our results. More general settings such as allowing interaction between the sender and the receiver, as well as variations in the adversarial power including considering adversarial and probabilistic noise both, will be interesting directions for future work. Another important direction for future work is the study of key agreement problem over an AWTP channel.

APPENDIX A

SUBSPACE Evasive SETS

A. Encoding Algorithm

We show the encoding map \( SE : v \to s \). Assuming there is a vector \( v \) of length \( n_1 \) and \((w - v)|n_1 \). First we divide the vector into \( \frac{n_1}{w-v} \) blocks. Then for each block \( v_i \) for \( i = 1, \cdots, \frac{n_1}{w-v} \), we encode into a block \( s_i \) using bijection \( \varphi \). Then we concatenate each block \( s_i \) for \( i = 1, \cdots, \frac{n_1}{w-v} \) and generate \( s \) in \( S \). We give the function \( \varphi \) in the following.

Lemma 12: (Claim 4.1) Assume that at least \( v \) of the degree \( d_1, \cdots, d_w \) are co-prime to \(|P| - 1\). Then there is an easy to compute bijection \( \varphi : \mathbb{F}^{w - v} \to \mathbb{V}_P \subseteq \mathbb{F}^w \). Moreover, there are \( w - v \) coordinates in the output of \( \varphi \) that can be obtained from the identity mapping \( \text{Id} : \mathbb{F}^{w - v} \to \mathbb{F}^{w - v} \).

Let \( d_{j_1}, \cdots, d_{j_v} \) be the degree among \( d_1, \cdots, d_w \) co-prime to \(|P| - 1\) and let \( J = \{j_1, \cdots, j_v\} \) and \( x_{d_{j_i}} = y_i \). On the positions \([w]\)\text{\textbackslash}J, the map \( \varphi \) takes the elements from \( \mathbb{F}^{w - v} \) to \( \mathbb{F}^{[w]\text{\textbackslash}J} \). For the elements on \( J \), there is

\[
\sum_{j \in J} A_{i,j}x_{d_{j_i}} = -\sum_{j \notin J} A_{i,j}x_{d_{j_i}}
\]

Let \( A' = v \times v \) minor of \( A \) given by restricting \( A \) to columns in \( J \) and \( b_i = -\sum_{j \notin J} A_{i,j}x_{d_{j_i}} \). Then

\[
A' y = b
\]

and for each \( y_i \), there is unique solution of \( x_{d_{j_i}} = y_i \mod q \) because \( d_{j_i} \) is co-prime to \( q - 1 \).

The computational complexity of mapping each vector \( v_i \) into \( s_i \) is \( \text{Poly}(v) \). Since \( v \) is consisted by \( b = \frac{n}{w} \) vectors, the total computational complexity of encoding a vector into an element in subspace evasive sets is \( \text{Poly}(n) \).

B. Intersection Algorithm

We show how to compute the intersection \( S \cap H \) given \((v, \ell)\) subspace evasive sets \( S \) and \( v\)-dimensional subspace \( H \). The subspace evasive sets \( S \) filter out the elements in \( H \) and output a set of elements \( S \cap H \) with size no more than \( \ell \).
Because $H$ is $v$-dimensional subspace and $H \subset \mathbb{F}^w$, there exists a set of affine maps $\{\ell_1, \ldots, \ell_n\}$ such that for any elements $x = \{x_1, \ldots, x_m\} \in H$, there is $x_i = \ell_i(s_1, \ldots, s_v)$. We show the result by induction of the number of blocks $i = 1, \ldots, n/w$. If $i = 1$, let $H_1 := \{(x_1, \ldots, x_w) : (x_1, \ldots, x_n) \in H\}$, the dimension of $H_1$ is $r = v$ and $H_{x_1,\ldots,x_w} = \{(x_1, \ldots, x_n) \in H : (x_1, \ldots, x_w)\}$ such that $H = \bigcup_{(x_1, \ldots, x_w) \in H_1} H_{x_1,\ldots,x_w}$, and the dimension of $H_{x_1,\ldots,x_w}$ is $v - r$. There is

$$V(f_1, \ldots, f_v) \cap H_1 = \{(x_1, \ldots, x_w) = (\ell_1(s_1, \ldots, s_v), \ldots, \ell_w(s_1, \ldots, s_v)) : f_i(\ell_1(s_1, \ldots, s_v), \ldots, \ell_w(s_1, \ldots, s_v)) = 0, \ldots, f_v(\ell_1(s_1, \ldots, s_v), \ldots, \ell_w(s_1, \ldots, s_v)) = 0\}$$

We can solve the $v$ equations to get $(s_1, \ldots, s_v)$ and then obtain $(x_1, \ldots, x_w)$. Since $H_1 \subset \mathbb{F}^w$, $V(f_1, \ldots, f_v) \cap H_1 = V(f_1, \ldots, f_v) \subset H_1$.

By Bezout’s theorem, there is $|V(f_1, \ldots, f_v) \cap H_1| \leq (d_1)^r$. So there are at most $(d_1)^r$ solutions for $(x_1, \ldots, x_w) \in H_1$. The computational time of solving the equation system follows from powerful algorithms that can solve a system of polynomial equations (over finite fields) in time polynomial in the size of the output, provided that the number of solutions is finite in the algebraic closure (i.e., the zero-dimensional case). So for $i = 1$, there are $(d_1)^r$ solutions for $(x_1, \ldots, x_w)$. The computational time is at most $\text{Poly}(d_1)^r$.

For every fixed of the first $w$ coordinates, we reduce the dimension of $H$ and obtained a new subspace on the remaining coordinates. By induction, we have $|V(f_1, \ldots, f_v) \cap H_{x_1,\ldots,x_w}| \leq (d_1)^{v-r}$ for all $(x_1, \ldots, x_w) \in H_1$. Hence there is $|V(f_1, \ldots, f_v) \cap H_1| \leq (d_1)^r$.

Similarly, we can compute all the solutions in times $\text{Poly}(d_1 ^ {r_1}) \cdot \text{Poly}(d_1^{r_2}) \cdot \cdots \cdot \text{Poly}(d_1^{v/w})$, where $r_1 + r_2 + \cdots + r_{n/w} = v$. So the running time of decoding algorithm is $\text{Poly}(d_1)^{v/w}$. Since $d_1$ can be bounded by $d_1 \leq v^{D \log \log v}$ (Claim 4.3 [22]), with constant $D$, the total running time for the intersection algorithm is $\text{Poly}(v^{D \log \log v})$.

APPENDIX B
List Decodable Code

A. Decoding algorithm of FRS code

Linear algebraic list decoding [31] has two main steps: interpolation and message finding as outlined below.

- Find a polynomial, $Q(X, Y_1, \ldots, Y_v) = A_0(X) + A_1(X)Y_1 + \cdots + A_v(X)Y_v$, over $\mathbb{F}_q$ such that $\deg(A_i(X)) \leq D$, for $i = 1, \ldots, v$, and $\deg(A_0(X)) \leq D + k - 1$, satisfying $Q(\alpha_i, y_i, y_1, \ldots, y_v) = 0$ for $1 \leq i \leq n_0$, where $n_0 = (u - v + 1)N$.

- Find all polynomials $f(X) \in \mathbb{F}_q[X]$ of degree at most $k - 1$, with coefficients $f_0, f_1, \ldots, f_{k-1}$, that satisfy, $A_0(X) + A_1(X)f(X) + A_2(X)f(\gamma X) + \cdots + A_v(X)f(\gamma^{v-1}X) = 0$, by solving linear equation system.

The two above requirements are satisfied if $f \in \mathbb{F}_q[X]$ is a polynomial of degree at most $k - 1$ whose FRS encoding agrees with the received word $y$ in at least $t$ components:

$$t > N(\frac{1}{v + 1} + \frac{v}{v + 1} \frac{uR}{u + 1})$$

This means we need to find all polynomials $f(X) \in \mathbb{F}_q[X]$ of degree at most $k - 1$, with coefficients $f_0, f_1, \ldots, f_{k-1}$, that satisfy,

$$A_0(X) + A_1(X)f(X) + A_2(X)f(\gamma X) + \cdots + A_v(X)f(\gamma^{v-1}X) = 0$$

Let us denote $A_i(X) = \sum_{j=0}^{D+k-1} a_{i,j}X^j$ for $0 \leq i \leq v$. $(a_{i,j} = 0$ when $i \geq 1$ and $j \geq D)$. Define the polynomials,

$$B_0(X) = a_{1,0} + a_{2,0}X + a_{3,0}X^2 + \cdots + a_{v,0}X^{v-1}$$

$$B_{k-1}(X) = a_{1,k-1} + a_{2,k-1}X + a_{3,k-1}X^2 + \cdots + a_{v,k-1}X^{v-1}$$

We examine the condition that the coefficients of $X^i$ of the polynomial $Q(X) = A_0(X) + A_1(X)f(X) + A_2(X)f(\gamma X) + \cdots + A_v(X)f(\gamma^{v-1}X) = 0$ equals 0, for $i = 0 \cdots k - 1$. This is equivalent to the following system of linear equations for $f_0 \cdots f_{k-1}$.

$$\begin{bmatrix}
B_0(\gamma^0) & 0 & 0 & \cdots & 0 \\
B_1(\gamma^0) & B_0(\gamma^1) & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
B_{k-1}(\gamma^0) & B_{k-2}(\gamma^1) & B_{k-3}(\gamma^2) & \cdots & B_0(\gamma^{k-1})
\end{bmatrix} \times
\begin{bmatrix}
f_0 \\
f_1 \\
\vdots \\
f_{k-1}
\end{bmatrix} = \begin{bmatrix}
-\alpha_{0,0} \\
-\alpha_{0,1} \\
\vdots \\
-\alpha_{0,k-1}
\end{bmatrix}$$

(12)

The rank of the matrix of (Eqs. 12) is at least $k-v+1$ because there are at most $v-1$ solutions of equation $B_0(X) = 0$ so at most $v - 1$ of $\gamma^i$ that makes $B_0(\gamma^i) = 0$. The dimension of solution space is at most $v - 1$ because the rank of matrix of (Eqs. 12) is at least $k - v + 1$. So there are at most $q^{v-1}$ solutions to (Eqs. 12) and this determines the size of the list which is equal to $q^{v-1}$.

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