Research Article

Second-Order Neutral Differential Equations: Improved Criteria for Testing the Oscillation

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The main purpose of this study is to establish new improved conditions for testing the oscillation of solutions of second-order neutral differential equation \( (r(l)(u'(l)))' + q(l)x^\beta (\sigma(l)) = 0 \), where \( l \geq l_0 \) and \( u(l) = x(l) + px(q(l)) \). By optimizing the commonly used relationship \( x > (1 - p)u \), we obtain new criteria that give sharper results for oscillation than the previous related results. Moreover, we obtain criteria of an iterative nature. Our new results are illustrated by an example.

1. Introduction

In this study, we consider the neutral delay differential equation (NDDE) with second-order of the form

\[
(r(l)(u'(l)))' + q(l)x^\beta (\sigma(l)) = 0,
\]

(1)

where \( l \geq l_0 \) and \( u(l) = x(l) + px(q(l)) \). Throughout the results, we always suppose \( \gamma, \beta \in Q_\text{odd} = \{a/b: a, b \in \mathbb{Z}^+ \text{ are odd} \} \), \( p \) is a nonnegative constant, \( r \in C([l_0, \infty), (0, \infty)), q \in C([l_0, \infty), [0, \infty)), \) and \( \sigma \in C([l_0, \infty), \mathbb{R}) \). By optimizing the commonly used relationship \( x > (1 - p)u \), we obtain new criteria that give sharper results for oscillation than the previous related results. Moreover, we obtain criteria of an iterative nature. Our new results are illustrated by an example.

then it is called an oscillatory solution; otherwise, it is called nonoscillatory. The equation itself is called oscillatory if all its solutions oscillate.

In real-world life problems, the NDDEs have interesting applications. The NDDEs appear in the modeling of the networks containing lossless transmission lines, in the study of vibrating masses attached to an elastic bar, in the theory of automatic control, and others, see [1]. It is easy—in recent times—to observe the great development in the theory of oscillation for differential equations of different orders.

In the following, we review some of the works that contributed to the development of the oscillation theory of second-order NDDEs were the motivation for this work.

At studying the oscillatory behavior of NDDEs with canonical Case (2), the relationship between the solution and the corresponding function

\[
x(l) > (1 - p)u(l)
\]

(3)

has been commonly used in the literature. For canonical case (2), by using the Riccati technique, Xu and Meng [2]
presented some oscillation criteria for (1) when $\gamma = \beta$. In the case, where $0 < p(t) \leq p_0 < \infty$, Baculikova and Dzurina [3] established the oscillation criteria for (1).

**Theorem 1** (see Corollary 2 in [3]). Let $0 < \beta \leq 1$, $\beta \leq \gamma$, \begin{equation}
\sigma(l) \leq q(l) \leq l, q'(l) \geq q_0 > 0, q \circ \sigma = \sigma \circ q.
\end{equation}

If
\begin{equation}
\liminf_{l \to \infty} \frac{1}{\psi(l)} \int_{\sigma(l)}^{l} G(s) \left( v^*(\sigma(s)) \right)^{q} \, ds \geq 1,
\end{equation}

then (1) is oscillatory, where $G(l) = \min \{q(l), q(q(l))\}$.

Using relationship (3), Grace et al. [4] studied the oscillatory behavior of solutions of (1) when $\gamma = \beta$ and $p < 1$. Moreover, they improved previous results in the literature.

**Theorem 2** (see Theorems 3 and 6 in [4]). If
\begin{equation}
\limsup_{l \to \infty} \int_{0}^{l} \left( \phi(s) \exp \left( -\int_{s}^{l} \frac{d\xi}{r(\xi)\phi^p(\xi)} \right) \right) \, ds = \infty,
\end{equation}

then (1) is oscillatory, where $v(l) = \int_{0}^{l} r^{-1/(1+p)}(s) \, ds$, $\phi \in C([l_0, \infty), (0, \infty))$, $\phi(l) = \max \{\phi(0), l\}$, and
\begin{equation}
G(l) = q(l) \left( 1 - p(\sigma(l)) \right)^{q}.
\end{equation}

Recently, Moaaz et al. [5, 6] generalized and complemented the results in [4]. They established the following criteria for oscillation of (1) with $p < 1$.

**Theorem 3** (see Theorem 2 in [6]). Let $\beta = \gamma$. If
\begin{equation}
\liminf_{l \to \infty} \frac{1}{\psi(l)} \int_{l}^{\infty} r^{-1/(1+p)}(\xi) \psi^{1/q}(\xi) d\xi > \frac{1}{(p+1)^{(1+p)/p}}.
\end{equation}

then (1) is oscillatory, where
\begin{align*}
\varphi(l) &= \exp \left( -\int_{l}^{\infty} \frac{1}{\psi(s) r^{-1/(1+p)}(s)} \, ds \right),
\psi(l) &= \int_{l}^{\infty} \varphi(\xi) G(\xi) d\xi,
\end{align*}

and $G(l)$ and $v^*(l)$ are defined as in (8).

The similar results as those above have been extended for even-order NDDEs in [7–11]. For the works that dealt with the noncanonical case, that is,
\begin{equation}
\int_{l}^{\infty} r^{-1/(1+p)}(\xi) d\xi < \infty,
\end{equation}

see, for example, [12–14].

The objective of this paper is to establish new oscillation criteria for the NDDE (1) by improving (3). The new relationship enables us to
\begin{enumerate}
\item Create more effective criteria for studying neutral equations in both cases $p < 1$ and $p > 1$
\item Essentially take into account the influence of the delay argument $q(l)$ that has been careless in all related results
\item Exclude some restrictions that are usually imposed on the coefficients of the neutral equations in the case where $p > 1$
\end{enumerate}

Moreover, we use an iterative technique to establish new oscillation criteria for the NDDE (1) when $\beta = \gamma$ and $p < 1$. One purpose of this paper is to further improve Theorems 2 and 1. The results reported in this paper generalize, complement, and improve those in [3–6]. To show the importance of our results, we provide an example.

### 2. Main Results I: New Relationship between $x$ and $u$

For simplicity, we just write the functions without the independent variable, such as $f(l) = f$ and $f(g(l)) = f(g)$. Moreover, assuming
\begin{align*}
\varrho^0 &= l, \\
\varrho^m &= \varrho \circ \varrho^{m-1}, \\
\varrho^{-m-1} &= \varrho^{-1} \circ \varrho^{-m}, \quad \text{for } m = 1, 2, \ldots, \\
\eta_1(l) &= \int_{l_0}^{l} r^{-1/(1+p)}(s) \, ds, \\
B &= \begin{cases} 
\varrho^{-\gamma}, & \text{if } \gamma \leq \beta, \\
\varrho^{-\gamma} \eta_1(l), & \text{if } \gamma > \beta,
\end{cases}
\end{align*}

where $c_1$ and $c_2$ are positive constants, the set of all eventually positive solutions of (1) is denoted by $X^+$. 

**Lemma 1** (see Lemma 3 in [3]). Let $x \in X^+$. Then,
\begin{equation}
\begin{cases}
\quad u > 0, \\
\quad u' > 0, \\
\quad (r(u')')' \leq 0,
\end{cases}
\end{equation}

for $l \geq l_1$, where $l_1$ is sufficiently large.
The following lemma is a direct observation from the proof of 2.1 in [5].

**Lemma 2.** If \( x \in X^+ \), then \( u^{\beta - \gamma} (l) \geq B(l) \), eventually.

**Lemma 3.** Let \( x \in X^+ \) and \( p > 0 \), and there exists an even positive integer \( n \) such that

\[
\bar{p} := \sum_{m=1}^{n/2} \frac{1}{p^{2m-1}} \left( 1 - \frac{\eta_1 (\theta^{-2m})}{\eta_1 (\theta^{-(2m-1)})} \right) > 0. 
\tag{14}
\]

Then,

\[
x(l) \geq \bar{p}(l) u(l). 
\tag{15}
\]

**Proof.** Suppose that \( x \in X^+ \). Thus, \( x(l) \), \( x(\varrho(l)) \), and \( x(\sigma(l)) \) are positive for all \( l \geq l_1 \), where \( l_1 \) is sufficiently large. From Lemma 1, we see that (13) holds. Since \((r^{1/2} u')^l \leq 0\), we have that

\[
u(l) \geq \int_{l_1}^{l} \frac{1}{r^{1/2} (\xi)} r^{1/2} (\xi) u' (\xi) d\xi > r^{1/2} (l) u'(l) \eta_1 (l), \tag{16}
\]

for all \( l \geq l_1 \). Using the definition of \( u(l) \), we obtain

\[
x = \frac{1}{p} (u^{\varpi}(\xi) - x^{\varpi}(\xi)) = \frac{1}{p} (u(\varpi) - 1) u^{\varpi}(\xi) + \frac{1}{p} x^{\varpi}(\xi). 
\tag{17}
\]

Repeating this procedure, we obtain

\[
x = \sum_{m=1}^{n} (-1)^{m+1} p_m x^{\varpi}(\xi) + \frac{1}{p} x^{\varpi}(\xi) \geq \sum_{m=1}^{n/2} \frac{1}{p^{2m-1}} \left( u^{\varpi}(\xi) - \frac{1}{p} u(\xi) \right),
\tag{18}
\]

for \( l \geq l_2 \geq l_1 \), where \( l_2 \) is sufficiently large, and any even positive integer \( n \). Taking (16) and \( \varpi^{2m} \geq \varpi^{-(2m-1)} \) into account, we obtain

\[
u^{\varpi}(\xi) < \nu^{\varpi}(\xi) \frac{\eta_1 (\theta^{-2m})}{\eta_1 (\theta^{-(2m-1)})} \tag{19}
\]

for \( m = 1, 2, \ldots, (n/2) \). Combining (18) and (19), we obtain

\[
x > \sum_{m=1}^{n/2} \frac{1}{p^{2m-1}} \left( 1 - \frac{\eta_1 (\theta^{-2m})}{\eta_1 (\theta^{-(2m-1)})} \right) u^{\varpi}(\xi), 
\tag{20}
\]

This completes the proof. \( \square \)

**Lemma 4.** Let \( x \in X^+ \) and \( p_0 < 1 \). Then,

\[
x(l) \geq \bar{p}(l) u(l), 
\tag{21}
\]

for any odd positive integer \( n \), where

\[
\bar{p} := (1 - p) \sum_{m=0}^{(n-1)/2} \frac{p^{2m} \eta_1 (\theta^{2m+1})}{\eta_1 (\theta^{2m+1})}. 
\tag{22}
\]

**Proof.** Proceeding as in the proof of Lemma 3, we arrive at (16). From the definition of \( u(l) \), we have that

\[
x = u - px(\xi) = u - pu(\xi) + p^2 x(\xi^2). 
\tag{23}
\]

Repeating this procedure, we obtain

\[
x = \sum_{m=0}^{n} (-1)^m p_m u(\xi^m) + p^{n+1} x(\xi^{n+1}) \geq \sum_{m=0}^{(n-1)/2} \left( p^{2m} u(\xi^{2m}) - p^{2m+1} u(\xi^{2m+1}) \right). \tag{24}
\]

for \( l \geq l_2 \geq l_1 \), where \( l_2 \) is sufficiently large, and any odd \( n \in \mathbb{Z}+ \). Since \( \varpi^{2m+1}(l) \leq \varpi^{2m}(l) \), we see that

\[
u^{\varpi}(\xi) \leq \cdots \leq u^{\varpi}(\xi^{2m+1}) \leq u^{\varpi}(\xi^{2m}) \leq \cdots \leq u, \tag{25}
\]

for \( m = 0, 2, \ldots, (n-1)/2 \), which with (24) gives

\[
x \geq \sum_{m=0}^{(n-1)/2} p^{2m} (1 - p) u(\xi^{2m+1}). \tag{26}
\]

From (16), we find

\[
u^{\varpi}(\xi) > u \frac{\eta_1 (\theta^{2m+1})}{\eta_1 (\theta^{2m+1})}, \tag{27}
\]

which with (26) gives

\[
x \geq (1 - p) u \sum_{m=0}^{(n-1)/2} p^{2m} \eta_1 (\theta^{2m+1}). \tag{28}
\]

This completes the proof. \( \square \)

**Theorem 4.** Assume that \( p_0 < 1 \). If there exists a function \( \theta \in C^1 ([l_0, \infty), (0, \infty)) \) such that

\[
\limsup_{l \to \infty} \int_{l_1}^{l} \left( \theta(\xi) q(\xi) \bar{p}(\sigma(\xi)) \delta(\xi) B(\sigma(\xi)) - \frac{1}{(\gamma + 1)^{\gamma+1}} r(\xi) (\theta(\xi))^{\gamma+1} \right) d\xi = \infty, 
\tag{29}
\]
then (1) is oscillatory, where
\[
\hat{\eta}_l(l) = \eta_l(l) + \frac{1}{y} \int_l^l \eta_l(\xi)q(\xi)\tilde{p}^\beta(\sigma(\xi))B(\sigma(\xi))d\xi.
\]
\[
\delta(l) = \exp \left( -y \int_l^l \frac{1}{\eta_l(\xi)} d\xi \right).
\]
(30)

Proof. Assume the contrary that \( x \) is a nonoscillatory solution of (1). Without loss of generality, we suppose that \( x \in X^+ \). Thus, \( x(l), x(\sigma(l)) \), and \( x(\sigma(l)) \) are positive for all \( l \geq l_1 \), where \( l_1 \) is sufficiently large. Using Lemma 4, we have that (21) holds. Using (1) and (21), we obtain
\[
\left( r(u')^y \right)' \leq -q\tilde{p}^\beta(\sigma)u^\beta(\sigma).
\]
(31)

Using the chain rule and simple computation, we find
\[
y \left( r^\gamma u' \right)^{y-1} \frac{d}{dl}(u - \eta_l r^\gamma u') = -y \left( r^\gamma u' \right)^{y-1} \eta_l(r^\gamma u')' = -\eta_l(r^\gamma u')',
\]
(32)

which with (31) gives
\[
\frac{d}{dl}(u - \eta_l r^\gamma u') \geq \frac{1}{y} \left( r^\gamma u' \right)^{1-y} \eta_l q\tilde{p}^\beta(\sigma)u^\beta(\sigma)
\geq \frac{1}{y} \left( r^\gamma u' \right)^{1-y} \eta_l q\tilde{p}^\beta(\sigma)B(\sigma)u^\gamma(\sigma).
\]

Integrating (33) from \( l_1 \) to \( l \), we obtain
\[
u \geq \eta_l r^\gamma u' + \frac{1}{y} \int_{l_1}^l \left( r^\gamma (\xi)u'(\xi) \right)^{1-y} \eta_l(\xi)q(\xi)\tilde{p}^\beta(\sigma(\xi))B(\sigma(\xi))d\xi.
\]
(34)

Since \( (r^\gamma (l)u'(l))^y \leq 0 \), we have
\[
u(\sigma) \geq \eta_l(\sigma)r^\gamma(\sigma)u'(\sigma) \geq \eta_l(\sigma)r^\gamma u'.
\]
(35)

Thus, (24) becomes
\[
u \geq \left( \eta_l(l) + \frac{1}{y} \int_l^l \eta_l(\xi)(\sigma(\xi))q(\xi)\tilde{p}^\beta(\sigma(\xi))B(\sigma(\xi))d\xi \right) r^\gamma u',
\]
that is,
\[
u \geq \eta_l r^\gamma u'.
\]
(36)

Integrating \( (u'/u) \leq (1/(r^\gamma \eta_l)) \) from \( \sigma(l) \) to \( l \), we find
\[
\ln \frac{u(l)}{u(\sigma(l))} \leq \int_{\sigma(l)}^l \frac{1}{r^\gamma(\xi)\eta_l(\xi)} d\xi,
\]
that is,
\[
u(\sigma(l)) \geq \exp \left( -\int_{\sigma(l)}^l \frac{1}{r^\gamma(\xi)\eta_l(\xi)} d\xi \right) u(l).
\]
(39)

Next, we define the function
\[
\Theta = \theta r^\gamma u'.
\]
(40)

Clearly, \( \Theta(l) > 0 \) for all \( l \geq l_1 \) and
\[
\Theta' = \frac{\theta'}{\theta} \Theta + \theta \left( \frac{r(u')^y}{u^y} - \gamma \theta r(u')^y \frac{u'}{u^y-1} \right).
\]
(41)

It follows from (31) and (39) that
\[
\Theta' \leq \frac{\theta'}{\theta} \Theta - \theta \delta q\tilde{p}^\beta(\sigma)B(\sigma) - \gamma \theta r(u')^y \frac{u'}{u^y-1} \Theta.
\]
(42)

from definition \( \Theta \), we have
\[
\Theta' \leq \frac{\theta'}{\theta} \Theta - \theta \delta q\tilde{p}^\beta(\sigma)B(\sigma) - \gamma \frac{y}{r^\gamma \theta^y} \Theta^{1+y/(y-1)}.
\]
(43)

Using the inequality (see Lemma 1.2 in [5]),
\[
A\phi - B\phi^{(r+1)/y} \leq \frac{y}{(y+1)^{(r+1)/y}} A^{(r+1)} B^y, \quad B > 0,
\]
with \( A = (\theta'/\theta), B = (\gamma/(r^\gamma \theta^{(r-1)/y})) \), and \( \phi = \Theta \), we obtain
\[
\Theta' \leq -\theta \delta q\tilde{p}^\beta(\sigma)B(\sigma) + \frac{1}{(y+1)^{(r+1)/y}} \frac{r(\theta')^{r+1}}{\theta^r}.
\]
(45)

Integrating this inequality from \( l_1 \) to \( l \), we obtain
\[
\int_{l_1}^l \left( \theta(\xi)q(\xi)\tilde{p}^\beta(\sigma(\xi))\delta(\xi)B(\sigma(\xi)) \right) \frac{1}{(y+1)^{(r+1)/y}} \frac{r(\theta')^{r+1}}{\theta^r} d\xi \leq \Theta(l_1),
\]
(46)
which contradicts (29). This completes the proof.

\[ \limsup_{l \to \infty} \int_{l}^{1} \left( \frac{1}{\eta_1(l)} \right) d\xi = \infty, \]

then (1) is oscillatory, where

\[ \tilde{\eta}_1(l) = \eta_1(l) + \frac{1}{p} \int_{\eta_1(l)}^{1} \eta_1' \phi(l) \eta_1^\gamma \left( \sigma(l) \right) \eta_1 \left( \sigma(l) \right) d\xi, \]

\[ \tilde{\delta}(l) = \exp \left( -\gamma \int_{\eta_1(l)}^{1} \phi(l) \eta_1 \left( \sigma(l) \right) d\xi \right). \]

(48)

**Proof.** To prove this theorem, it suffices to use (15) instead of (21) in the proof of Theorem 4.

### 3. Main Results II: Iterative Technique

**Lemma 5.** Assume that \( x \in X^+ \), \( \gamma = \beta \), and \( p_0 < 1 \). Then,

\[ u(l) \geq \phi_k(l)r^{1/y}(l)u(l), \]

for \( k = 0, 1, \ldots \), where \( \phi_0(l) = \tilde{\eta}_1(l) \) and

\[ \phi_{k+1}(l) = \int_{l}^{1} \left( \frac{1}{\sigma(l)} \right) \exp \left( \int_{l}^{1} q(l) \phi^{\gamma}(\sigma(l))\phi_k(l) \right) \right) ds. \]

(50)

**Proof.** Suppose that \( x \in X^+ \). Thus, \( x(l) \), \( x(q(l)) \), and \( x(\sigma(l)) \) are positive for all \( l \geq l_1 \), where \( l_1 \) is sufficiently large. From Lemma 1, we see that (13) holds. Now, we will prove (49) using induction.

For \( k = 1 \), proceeding as in proof of Theorem 4, we obtain that (31) and (37) hold. From (37), we obtain

\[ u \geq \tilde{\eta}_1(l)r^{1/y}u. \]

(51)

Next, we assume that (49) holds at \( k = n \), that is,

\[ u \geq \phi_n r^{1/y}u. \]

Thus, since \( (r^{1/y}u)^{1/y} \leq 0 \), we find

\[ u(\sigma) \geq \phi_n(\sigma)r^{1/y}(l)u(\sigma) \geq \phi_n(\sigma)r^{1/y}u. \]

(52)

which with (31) gives

\[ \left( \frac{r \phi(l)}{\psi(l)} \right)^{1/y} + q \psi(\sigma)\phi_k(\sigma)r(l)u \leq 0. \]

(53)

If we set \( H = (r \phi(l))^{1/y} \), (53) becomes

\[ H'(l) + q \psi(\sigma)\phi_k^\gamma(\sigma)H(l) \leq 0. \]

(54)

Applying the Grönwall inequality in (54), we obtain

\[ H(s) \geq H(l)\exp \left( \int_{l}^{s} q(l) \phi^{\gamma}(\sigma(l))\phi_k(\sigma(l))d\xi \right). \]

(55)

**Theorem 5.** Assume that (14) holds for some even positive integer \( n \). If there exists a function \( \theta \in C^1([l_0, \infty), (0, \infty)) \) such that

\[ \limsup_{l \to \infty} \int_{l}^{1} \left( \frac{1}{\gamma + 1} \phi(l) \eta_1 \right) \right) d\xi = \infty, \]

for \( l \geq s \geq l_1 \), so

\[ u'(s) \geq r^{1/y}(l)u'(l) \left( \frac{1}{r(s)} \exp \left( \int_{l}^{s} q(l) \phi^\gamma(\sigma(l))\phi_k(\sigma(l))d\xi \right) \right). \]

(56)

Integrating this inequality from \( l_1 \) to \( l \), we obtain

\[ u(l) \geq r^{1/y}(l)u'(l) \left( \int_{l}^{s} \frac{1}{r(s)} \exp \left( \int_{l}^{s} q(l) \phi^\gamma(\sigma(l))\phi_k(\sigma(l))d\xi \right) \right]^{1/y} \]

\[ = \phi_{m+1}(l)r^{1/y}(l)u(l). \]

(57)

This completes the proof.

**Theorem 6.** Assume that \( \gamma = \beta \) and \( p_0 < 1 \). If

\[ \liminf_{l \to \infty} \int_{\sigma(l)}^{1} q(l) \phi^\gamma(\sigma(l))\phi_k(\sigma(l))d\xi > \frac{1}{c}, \]

for some integers \( k \geq 0 \), then (1) is oscillatory, where \( \tilde{\phi} \) and \( \phi_k \) are defined as in (22) and (50), respectively.

**Proof.** Assume the contrary that \( x \) is a nonoscillatory solution of (1). Without loss of generality, we suppose that \( x \in X^+ \). Thus, \( x(l) \), \( x(q(l)) \), and \( x(\sigma(l)) \) are positive for all \( l \geq l_1 \), where \( l_1 \) is sufficiently large. From Lemma 5, we have that (49) holds. Proceeding as in the proof of Theorem 4, we arrive at (31). Combining (53) and (49), we obtain

\[ \left( \frac{r \phi(l)}{\psi(l)} \right)^{1/y} + q \psi(\sigma)\phi_k(\sigma)r(l)u(\sigma) \leq 0. \]

(59)

If we set \( w = r \phi(l) \), we have that \( w \) is a positive solution of the delay differential inequality:

\[ w'(l) + q(l)\psi(\sigma)\phi_k(\sigma)w(\sigma) \leq 0. \]

(60)

Using Theorem 1 in [15], the associated DDE

\[ w'(l) + q(l)\psi(\sigma)\phi_k(\sigma)w(\sigma) = 0 \]

(61)

has also a positive solution. However, condition (58) ensures oscillation of (61), which is a contradiction. This completes the proof.

**Theorem 7.** Assume that \( \gamma = \beta \) and \( p_0 < 1 \). If there exists a function \( \rho \in C^1([l_0, \infty), (0, \infty)) \) such that
for some integers \(k \geq 0\), then (1) is oscillatory, where
\[
\tilde{\delta}_k (l) = \exp \left( - \gamma \int_{l_i}^l \frac{1}{r(\xi)} \, d\xi \right),
\]
and \(\tilde{\rho}\) and \(\varphi_k\) are defined as in (22) and (50), respectively.

\[\text{Proof.} \] Assume the contrary that \(x\) is a nonoscillatory solution of (1). Without loss of generality, we suppose that \(x \in X^+\). Thus, \(x(l), x'(l), \) and \(x(\sigma(l))\) are positive for all \(l \geq l_1\), where \(l_1\) is sufficiently large. Now, we define the function \(\psi = \rho (u'/u)^\gamma\). Thus, \(\psi(l) > 0\) and
\[
\psi' = \frac{\rho'}{\rho} \psi + \rho \left( \frac{r(u')^\gamma}{u^\gamma} \right)' - \gamma \rho \varphi_k \left( \frac{u'}{u^{\gamma+1}} \right)^{\gamma+1}. \tag{64}\]

From Lemma 5, we have that (49) holds. By replacing (37) with (49) in the proof of Theorem 4, this part of proof is similar to that of Theorem 4, so we omit it. \(\square\)

Now, we give an example to illustrate our main results.

Example 1. Consider the NDDE:
\[
\left( ((x(l) + px(\mu l))^\gamma)' + \frac{q_0}{p^\gamma} x^\gamma (\lambda l) \right) = 0, \tag{65}\]
where \(q_0 > 0\) and \(p, \mu, \lambda \in (0, 1)\). It is easy to verify that \(\eta_\zeta (l) = l, \ \varphi_m(l) = \mu^m l, \ \tilde{\eta}_\zeta (l) = \lambda l, \ \tilde{\delta}_\zeta (l) = \lambda l\), where
\[
\tilde{\rho}(l) = (1 - p) \sum_{m=0}^{(n+1)/2} \mu^{2m+1} = \tilde{\rho}_0,
\]
\[
\tilde{\gamma} = \left( 1 + (1/\gamma) \tilde{\rho}_0 q_0 \lambda \right),
\]
\[
\tilde{p}(l) = \sum_{m=1}^{n/2} \frac{1}{p^{2m-1}} \left( 1 - \frac{1 - \mu p}{\mu p} \right) = \tilde{p}_0,
\]
\[
\tilde{\gamma} = \left( 1 + (1/\gamma) \tilde{p}_0 q_0 \lambda \right).
\]
Using Theorem 4, we see that (65) is oscillatory if \(p < 1\) and
\[
\tilde{\rho}_0^{\gamma+1} q_0 > \left( \frac{\gamma}{\gamma + 1} \right)^{\gamma+1}. \tag{67}\]

Using Theorem 5, we see that (65) is oscillatory if
\[
\tilde{\rho}_0^{\gamma+1} q_0 > \left( \frac{\gamma}{\gamma + 1} \right)^{\gamma+1}. \tag{68}\]

Remark 1. The best-known criteria for oscillation of NDDE (65) are
\[
q_0 (1 - p)^\gamma \lambda \left( \frac{\gamma (1 + (1/\gamma) (1 - p)^\gamma q_0 \lambda)}{\gamma + 1} \right) > \left( \frac{\gamma}{\gamma + 1} \right)^{\gamma+1} \tag{69}\]

\[
q_0 \lambda^\gamma \ln \left( \frac{\mu}{\lambda} \right) > \frac{\mu + p^\gamma}{\mu} \tag{70}\]

for \(p < 1\) and \(p > 1\), respectively.

Giving values for the parameters \(p, \mu, \) and \(\lambda,\) we can determine the lower bound of the parameter \(q_0\) to ensure that every solution of (65) is oscillatory. Table 1 shows the lower boundaries of the parameter \(q_0\) in different special cases of (65) when \(\delta = 1\) by using conditions (69) and (67).

| \((p, \lambda, \mu)\)          | \(n = 5\)   | \(n = 9\)   | \(n = 49\)   |   |
|-------------------------------|-------------|-------------|-------------|---|
| \((2/3, 0.1, 0.755)\)          | 5.3342      | 5.2529      | 5.2474      | 5.30610 |
| \((0.5, 0.5, 0.830)\)          | 0.8844      | 0.8799      | 0.88227     |   |
| \((0.9, 0.5, 0.900)\)          | 3.2491      | 1.9189      | 1.6857      | 4.41130 |

\[\text{Table 1: The lower boundaries of the parameter } q_0 \text{ in oscillation criteria.}\]

\[\text{Remark 2. Using the boundedness condition } p_1 \leq p(l) \leq p_2, \text{ it will be easy to infer results similar to ours if } p \text{ is a function in } l.\]

4. Conclusion

This article is concerned with oscillatory behavior of a class of the neutral delay differential equation (NDDE) with second-order. By optimizing the commonly used relationship \(x > (1 - p)u,\) we obtained new criteria that give sharper results for oscillation than the previous related results. Moreover, we obtained criteria of an iterative nature. Furthermore, in the future work, we can try to get some oscillation criteria of (1) under case of \(x(l) = x(l) + c(l)x'(\sigma(l)), \ 0 < c < 1.\)
Data Availability
No data were used to support the findings of the study.

Conflicts of Interest
The authors declare that they have no conflicts of interest.

Authors’ Contributions
The authors contributed equally to this article.

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