The countable existentially closed pseudocomplemented semilattice

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Abstract

As the class \( \mathcal{PCSL} \) of pseudocomplemented semilattices is a universal Horn class generated by a single finite structure it has a \( \aleph_0 \)-categorical model companion \( \mathcal{PCSL}^* \). As \( \mathcal{PCSL} \) is inductive the models of \( \mathcal{PCSL}^* \) are exactly the existentially closed models of \( \mathcal{PCSL} \). We will construct the unique existentially closed countable model of \( \mathcal{PCSL} \) as a direct limit of algebraically closed pseudocomplemented semilattices.

1 Introduction

For a first-order language \( \mathcal{L} \) and an \( \mathcal{L} \)-structure \( M \) with universe \( M \) the language \( \mathcal{L}(M) \) is obtained by adding a constant symbol for every \( m \in M \).

To define the notion of model companion we first have to define the notion of model completeness. An \( \mathcal{L} \)-theory \( T \) is said to be model complete if for every model \( M \) of \( T \) the set of \( \mathcal{L} \)-sentences \( T \cup \text{diag}(M) \) is complete, where \( \text{diag}(M) \) is the set of atomic and negated atomic \( \mathcal{L}(M) \)-sentences that hold in \( M \). \( T^* \) is said to be a model companion of \( T \) if (i) every model of \( T^* \) is embeddable in a model of \( T \) and vice versa and (ii) \( T^* \) is model complete.

A theory \( T \) need not have a model companion as is the case for the theory of groups and theory of commutative rings, see Wheeler [9]. However, if \( T \) is a set of Horn sentences and the class \( \text{Mod}(T) \) of its models is finitely generated then \( T \) has a model companion \( T^* \) as was shown by Burris and Werner [4].

If \( T \) is additionally inductive—that is \( \text{Mod}(T) \) is closed under the union of chains—then we have the characterization \( \text{Mod}(T^*) = \text{Mod}(T)^{ec} \), that is the models of \( T^* \) are the existentially closed models of \( T \), see Macintyre [7]. A definition of the notions of algebraically and existentially closed can be found in [1]. Finally, if \( \text{Mod}(T) \) is generated by single finite structure then \( \text{Mod}(T^*) \) is \( \aleph_0 \)-categorical, see Burris [3].

Horn and Balbes [2] proved that \( \mathcal{PCSL} \) is equational, Jones [6] showed that it is generated by a single finite structure. Thus \( \mathcal{PCSL}^* \) is \( \aleph_0 \)-categorical and its only countable model is the countable existentially closed pseudocomplemented semilattice.

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In Section 2 we provide the basic properties and algebraic notions concerning pseudocomplemented semilattices—p-semilattices for short—while in Section 3 the countable existentially closed p-semilattice is constructed.

2 Basic properties of pseudocomplemented semilattices and notation

A p-semilattice \( \langle P; \wedge, \ast, 0 \rangle \) is an algebra where \( \langle P; \wedge \rangle \) is a meet-semilattice with least element 0, and for all \( x, y \in P \), \( x \wedge a = 0 \) if and only if \( x \leq a^* \).

\( 1 := 0^* \) is obviously the greatest element of \( P \). \( x \parallel y \) is defined to hold if neither \( x \leq y \) nor \( y \leq x \) holds. An element \( d \) of \( P \) satisfying \( d^* = 0 \) is called dense, and if additionally \( d \neq 1 \) holds, then \( d \) is called a proper dense element. For \( P \in \mathcal{PCS} \) the set \( \text{D}(P) \) denotes the subset of dense elements of \( P \), \( \langle \text{D}(P); \wedge \rangle \) being a filter of \( \langle P; \wedge \rangle \). An element \( s \) is called skeletal if \( s^{**} = s \). The subset of skeletal elements of \( P \) is denoted by \( \text{Sk}(P) \). The abuse of notation \( \text{Sk}(x) \) for \( x \in \text{Sk}(P) \) and \( \text{D}(x) \) for \( x \in \text{D}(P) \) should not cause ambiguities. Obviously, \( \text{Sk}(P) = \{ x^* : x \in P \} \).

For any p-semilattice \( P \) the p-semilattice \( \hat{P} \) is obtained from \( P \) by adding a new top element. The maximal dense element of \( \hat{P} \) different from 1 is denoted by \( e \). Furthermore, the p-semilattices \( \mathcal{B} \) with \( \mathcal{B} \) being a boolean algebra are exactly the subdirectly irreducible p-semilattices. Moreover, let \( 2 \) denote the two-element boolean algebra and \( A \) the countable atomfree boolean algebra interpreted as p-semilattices.

An equational set \( \Sigma \) of axioms for \( \mathcal{PCS} \) can be found in [1], for more background on p-semilattices in general consult Frink [5] and [6].

In Schmid [8] the following characterization of algebraically closed p-semilattices is established:

**Theorem 2.1.** A p-semilattice \( P \) is algebraically closed if and only if for any finite subalgebra \( F \leq P \) there exists \( r, s \in \omega \) and a p-semilattice \( F' \) isomorphic to \( 2^r \times (\hat{A})^s \) such that \( F \leq F' \leq P \).

In [1] the following (syntactic) description of existentially closed p-semilattices is given:

**Theorem 2.2.** A p-semilattice \( P \) is existentially closed if and only if \( P \) is algebraically closed and satisfies the following list of axioms:

\((EC1)\) if
\[
(\forall b_1, b_2 \in \text{Sk}(P)) (\exists b_3 \in \text{Sk}(P)) (b_1 < b_2 \rightarrow b_1 < b_3 < b_2),
\]

\((EC2)\) if
\[
(\forall b_1, b_2 \in \text{Sk}(P), d \in \text{D}(P)) (\exists b_3 \in \text{Sk}(P)) (\begin{array}{l}
(b_1 \leq b_2 < d < 1 \& b_1^* \parallel d) \\
\rightarrow (b_2 < b_3 < 1 \& b_1^* \land b_3 \parallel d \& b_1 \lor b_3 < d)
\end{array}),
\]

\((EC3)\) if
\[
(\exists d \in \text{D}(P)) (d < 1),
\]

\((EC4)\) if
\[
(\forall d_1, d_2 \in \text{D}(P)) (\exists d_3 \in P) (d_1 < d_2 \rightarrow (d_1 < d_3 < d_2)),
\]

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if

\[(\forall b \in \text{Sk}(P), d_1 \in D(P)) ((\exists d_2 \in D(P))(0 < b < d_1
\rightarrow (d_2 < d_1 \& b \parallel d_2 \& d_1 \& b^* = d_2 \& b^*))).\]

Constructing the unique countable model of the model companion $\Sigma^*$ of $\mathcal{P}_{\mathcal{C}}$ thus amounts to constructing a countable algebraically closed p-semilattice that satisfies (EC1)–(EC5).

3 The construction

As the objects of the direct limit we are going to construct we take $\{G_n : n \in \mathbb{N} \setminus \{0\}\}$, where $G_n := (\hat{A})^n$. In view of Theorem 2.1 $G_n$ is algebraically closed for all $n \in \mathbb{N} \setminus \{0\}$. We have to define embeddings $f_n : G_n \to G_{n+1}$, $n \geq 1$, such that the direct limit of the directed family $\{(G_n, g_{m,n}) : m, n \in \mathbb{N}, 1 \leq m \leq n\}$ where $g_{i,j} := f_{j-1} \circ \cdots \circ f_i$ for $i < j$ and $g_{i,i} := id_{G_i}$ additionally satisfies (EC1)–(EC5) of Theorem 2.2. We will see that this is obtained if all $f_n : G_n \to G_{n+1}$ have the following properties:

(P1) For every anti-atom $d$ of $D(G_n)$ there is a $k \in \mathbb{N}$ such that $g_{n,n+k}(d)$ is not an anti-atom of $G_{n+k}$ anymore. This will imply that the order restricted to the dense elements of the direct limit is dense.

(P2) For every $a \in \text{Sk}(G_n) \setminus \{0, 1\}$ and $d \in D(G_n)$ with $a < d$ there is an $m \in \mathbb{N}$ such that $\pi_{n+m}(g_{n,n+m}(a)) = \pi_{n+m}(g_{n,n+m}(d)) = 1$.

Let $d_1, \ldots, d_n$ be an enumeration of the anti-atoms of $D(G_n)$, where $D(G_n) = \{e, 1\}^n$. For every anti-atom $d_i \in D(G_n)$ let $\sigma_n(i) \in \{1, \ldots, n\}$ be such that $\pi_n(d_i) = e$ if and only if $k = \sigma_n(i)$, $1 \leq k \leq n$. That is, $\sigma_n$ is the permutation that assigns $i$ the place of the component of $d_i$ that is $e$. Furthermore, let $a_{\sigma_n(i),1}, a_{\sigma_n(i),2}, \ldots$ be an enumeration of the elements of $\pi_{n}(\text{Sk}(G_n)) \setminus \{0, 1\}$. To define $f_n$ we use the following notation: For $x = (x_1, \ldots, x_n) \in G_n$ and $u \in \hat{A}$ we put $(x^u, u) = (x_1, \ldots, x_n, u) \in G_{n+1}$. We distinguish between $n$ being even and $n$ being odd.

- For $n$ even we put $f_n(x) = (x^u, x_{\sigma_n(1)})$ for $x = (x_1, \ldots, x_n) \in G_n$. $f_n$ obviously is an embedding. $f_n(d_i) = (\hat{d}_i, \pi_{\sigma_n(1)}(d_i)) = (\hat{d}_i, e)$ is not an anti-atom of $G_{n+1}$ anymore, whereas $f_n(d_i) = (\hat{d}_i, \pi_{\sigma_n(1)}(d_i)) = (\hat{d}_i, 1)$ still is an anti-atom of $G_{n+1}$ for $i = 2, \ldots, n$. These anti-atoms of $G_{n+1}$ are numbered 1 to $n - 1$ whereas the two anti-atoms of $D(G_{n+1} \setminus f_n(G_n))$ are numbered $n$ and $n + 1$ according to the place of the $e$-component. This guarantees that for every anti-atom $d$ of $G_n$ there is $k \in \mathbb{N}$ such that $g_{n,n+k}(d)$ is not an anti-atom of $G_{n+k}$ anymore. Thus (P1) is satisfied.

The enumeration of $\pi_i(\text{Sk}(G_{n+1})) \setminus \{0, 1\}$ is the same as the enumeration of $\pi_i(\text{Sk}(G_n)) \setminus \{0, 1\}$ for $1 \leq i \leq n+1$. $\pi_{n+1}(\text{Sk}(G_{n+1})) \setminus \{0, 1\}$ can be enumerated arbitrarily.

- For $n$ odd we define

\[f_n(x) = \begin{cases} (\hat{x}, 1), & \pi_{\sigma_n(1)}(x) \in U_{\sigma_n(1)} \cup \{e\}, \\ (\hat{x}, 0), & \text{otherwise}, \end{cases}\]

(1)
where $U_{\sigma_n(i)}$ is an ultrafilter on $\pi_{\sigma_n(i)}(\text{Sk}(G_n))$ containing $a_{\sigma_n(i)}$. $a_{\sigma_n(1)}$ is denoted as the distinguished element for $n$.

Then $f_{n}(d) = \{ \bar{d}, 1 \}$ for all $d \in D(G_n)$: $\pi_{\sigma_n(1)}(d) \in \{ e, 1 \} \cup \{ \bar{0} \}$, thus $f_{n}(d_{i}), i = 1, \ldots, n,$ are still anti-atoms of $G_{n+1}$. They are numbered 1 to $n$, the anti-atom $(1, \ldots, 1, e)$ is numbered $n + 1$. We obtain $f_{n}(\text{min}(D(G_n))) \neq \text{min}(D(G_{n+1}))$. The enumeration of $\pi_{1}(\text{Sk}(G_{n+1})) \setminus \{ 0, 1 \}$, $1 \leq i \leq n + 1$, is as follows: For $j \notin \{ \pi_{\sigma_n(1)}, n + 1 \}$ the enumeration is the same as for $\pi_{1}(\text{Sk}(G_n)) \setminus \{ 0, 1 \}$.

$\pi_{n+1}(\text{Sk}(G_{n+1})) \setminus \{ 0, 1 \}$ can be enumerated arbitrarily. Let now $x$ be an element of $\pi_{\sigma_n(1)}(\text{Sk}(G_{n+1})) \setminus \{ 0, 1 \} = \pi_{\sigma_n(1)}(\text{Sk}(G_{n+1})) \setminus \{ 0, 1 \}$. Therefore, $x = a_{\sigma_n(1)}$ is the $j$th element of $\pi_{\sigma_n(1)}(\text{Sk}(G_n)) \setminus \{ 0, 1 \}$ for a $j \in \mathbb{N}$. We distinguish three cases depending on the value of $j$.

If $j = 1$ then $x$ receives the number $2^n$. If $2 \leq j \leq 2^n$, $x$ receives the number $j - 1$. Finally, if $2^n < j$, then $x$ receives the number $j$. This guarantees that every $x \in \pi_{1}(\text{Sk}(G_n)) \setminus \{ 0, 1 \}$, $1 \leq i \leq n$, becomes the distinguished element for some $n' \geq n$. Thus $[\text{P2}]$ is satisfied in $G_{n'}$.

Claim. The direct limit $G$ of the directed family $\{ (G_m, g_{m,n}) : m, n \in \mathbb{N}, 1 \leq m \leq n \}$ of p-semilattices is countable and existentially closed.

Proof. $G$ is countable since a countable union of countable sets is countable. That $G$ is algebraically closed follows from Theorem 2.1. Let $S$ be a finite subalgebra of $G$. By the construction of $G$, there is an $n \in \mathbb{N}$ such that the carrier $S$ of $S$ is a subset of $G_n$. Therefore, there is a subalgebra $S'$ of $G$ isomorphic to $G_n = (\hat{A})^n$ containing $S$.

By Theorem 2.2 it remains to show that $G$ satisfies $[\text{EC1}]$ $[\text{EC5}]$. $[\text{EC5}]$ is satisfied as it is satisfied in $A$. $[\text{EC3}]$ is obviously satisfied. To prove the remaining three axioms we denote for $x \in \bigcup_{n=1}^{\infty} G_n$ with $[x] \in G$ the equivalence class of $x$.

For $[\text{EC2}]$ consider arbitrary $b_1, b_2 \in \text{Sk}(G)$ and $d \in D(G)$ such that $b_1 \leq b_2 < d$ and $b_1^n \parallel d$. There is $n \in \mathbb{N}$ and $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, $w = (w_1, \ldots, w_n) \in G_n$ such that $b_1 = [x]$, $b_2 = [y]$, $d = [w]$, $\text{Sk}(x)$, $\text{Sk}(y)$, $D(w)$, $x \leq y < w$ and $x^* \parallel w$. We first assume that $w$ is not an anti-atom of $G_n$. Then without loss of generality we can assume $x_1 = 0$, $w_1 = w_2 = e$. Then put $z = (1, z_2, 1, \ldots, 1)$ with $y_2 < z_2 < e$ to obtain $y < z$, $z \parallel w$, $x^* \land z \parallel w$ and $x \lor z^* < w$. The last inequality follows from $x_2 < y_2 < z_2$, which implies $0 < x^*_2 \land z_2$, thus $e > (x^*_2 \land z_2)^* = x_2 \lor z_2^*$. Putting $b_3 = [z]$ yields what is requested in $[\text{EC2}]$.

If $w$ is an anti-atom there is by $[\text{PT}]$ an $l \in \mathbb{N}$ such that $g_{n,n+l}(w)$ is not an anti-atom of $G_{n+l}$ anymore. For $x' := g_{n,n+l}(x)$, $y' := g_{n,n+l}(y)$ and $w' := g_{n,n+l}(w)$ we find as above $y' < z$, $x'^* \land z \parallel w'$ and $x' \lor z^* < w'$. Putting $b_3 = [z]$ yields what is requested in $[\text{EC2}]$ because $[x] = [x']$, $[y] = [y']$, $[w] = [w']$.

For $[\text{EC4}]$ consider arbitrary $d_1, d_2 \in D(G)$ such that $d_1 < d_2$. There is $n \in \mathbb{N}$ and $x, y \in G_n$ such that $d_1 = [x]$, $d_2 = [y]$. There are $l \in \mathbb{N}$ and $z \in D(G_{n+l})$ such that $g_{n,n+l}(z) < z < g_{n,n+l}(y)$. We have $x = \bigvee_{j \in J_x} x_j$, $y = \bigvee_{j \in J_y} x_j$ for subsets $J_x \subseteq J_y \subseteq \{ 1, \ldots, n \}$, $x_j$ being an anti-atom of $D(G_n)$ for $j \in J_x$. For $j_0 \in J_x \setminus J_y$ there is by $[\text{PT}]$ a least $l \in \mathbb{N}$ such that $g_{n+l}(x_{j_0})$ is not an anti-atom of $G_{n+l}$ anymore, that is, there is an anti-atom $u_1, u_2 \in G_{n+l}$ with $g_{n,n+l}(x_{j_0}) < u_1 \land u_2$ and $g_{n,n+l}(x_{j_0}) \parallel u_i$ for all $i \in J_y$, $i = 1, 2$. Because $D(G_{n+l})$ is boolean, that is, a $\land$-reduct of
a boolean algebra, we obtain

\[ g_{n,n+l}(x) = \bigwedge_{j \in J \setminus \{j_0\}} g_{n,n+l}(x_j) \land u_1 \land u_2 < \bigwedge_{j \in J \setminus \{j_0\}} g_{n,n+l}(x_j) \land u_1 \]

\[ \leq \bigwedge_{j \in J} g_{n,n+l}(x_j) \land u_1 < \bigwedge_{j \in J} g_{n,n+l}(x_j), \]

which implies

\[ g_{n,n+l}(x) < g_{n,n+l}(y) \land u_1 < g_{n,n+l}(y). \]

We have

\[ d_1 = [x] = [g_{n,n+l}(x)] < [g_{n,n+l}(y) \land u_1] < [g_{n,n+l}(y)] = [y] = d_2, \]

and we can choose \( d_3 = [g_{n,n+l}(y) \land u_1], \)

For (EC5) consider an arbitrary \( b \in \text{Sk}(G) \) and \( d_1 \in D(G) \) such that \( 0 < b < d_1 \). There is a \( n \in \mathbb{N} \) and \( x = (x_1, \ldots, x_n) \), \( y = (y_1, \ldots, y_n) \) \( \in G_n \) such that \( b = [x] \), \( d_1 = [y] \), \( \text{Sk}(x) \), \( D(y) \), \( 0 < x < y \). Let us assume that there is no \( z \in D(G_n) \) such that \( z < y \), \( x||z \) and \( x^* \land y = x^* \land z \), since otherwise we put \( d_2 = [z] \).

By (P2) there is an \( l \in \mathbb{N} \) such that \( \pi_{n+l}(g_{n,n+l}(x)) = \pi_{n+l}(g_{n,n+l}(y)) = 1 \). Defining \( z \in G_{n+l} \) by putting \( \pi_j(z) = \pi_j(g_{n,n+l}(y)) \) for \( 1 \leq j \leq n+l-1 \) and \( \pi_{n+l}(z) = e \) we can then choose \( d_2 = [z] \).

\[ \square \]

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