Isotropic LQC and LQC–inspired Models

with a massless scalar field

as Generalised Brans–Dicke theories

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ABSTRACT

We explore whether generalised Brans–Dicke theories, which have a scalar field $\Phi$ and a function $\omega(\Phi)$, can be the effective actions leading to the effective equations of motion of the LQC and the LQC–inspired models, which have a massless scalar field $\sigma$ and a function $f(m)$. We find that this is possible for isotropic cosmology. We relate the pairs $(\sigma, f)$ and $(\Phi, \omega)$ and, using examples, illustrate these relations. We find that near the bounce of the LQC evolutions for which $f(m) = \sin m$, the corresponding field $\Phi \to 0$ and the function $\omega(\Phi) \propto \Phi^2$. We also find that the class of generalised Brans–Dicke theories, which we had found earlier to lead to non-singular isotropic evolutions, may be written as an LQC–inspired model. The relations found here in the isotropic cases do not apply to the anisotropic cases, which perhaps require more general effective actions.
1. Introduction

In Loop quantum cosmology (LQC), the big bang singularities do not arise [1] – [8]. Starting with a large (3 + 1) dimensional universe and going back in time, one finds that its physical volume decreases, reaches a non vanishing minimum because of the quantum effects in LQC, and starts increasing again. The densities and the temperatures of the constituents of the universe do not diverge and remain finite throughout. As explained in detail in the review [8], these non singular evolutions arising due to quantum effects can be described very well by effective equations of motion. In the classical limit, these equations reduce to Einstein’s equations.

In a recent work [9], we constructed the ‘LQC–inspired models’ by generalising empirically these effective LQC equations to higher dimensions and to functions other than the trigonometric ones. In [10], we studied the cosmological evolutions in these models using several examples.

It is natural now to search for an effective action which will lead to the effective equations of the LQC–inspired models. Besides being of interest for its own sake, such an action may also be used practically: for example, one may use it to obtain the generalisation of Schwarzschild solutions, or to study spherically symmetric stars and their collapses. In the context of LQC, $F(R)$ theories [11, 12, 13] have been used to construct an effective action which leads to the isotropic effective equations of LQC [14] – [19]. It is shown in [19] that the anisotropic case requires more general $F(R, Q)$ theories where $Q = R_{\mu\nu}R^{\mu\nu}$.

It is very likely that the effective actions for the LQC–inspired models may also be obtained using $F(R)$ theories in the isotropic case or $F(R, Q)$ theories in the anisotropic case. However, in this paper, we explore a different possibility: the possibility of obtaining the effective actions of the LQC and the LQC–inspired models using the $(d + 1)$ dimensional scalar tensor theories [20]. Our motivations for this exploration are the following.

• $F(R)$ theories can also be written as special cases of scalar tensor theories with a particular value for the Brans – Dicke parameter $\omega$ and with a particular scalar field potential which depends on the function $F$ [11, 12].

• The effective actions will, clearly, be generalisations of the standard Einstein’s action and will reduce to it in the classical limit. Now,
the generalisations of Einstein’s action generically involve a scalar field which often appears as a modulus field with no potential. For example: The Brans – Dicke theory arose as a theory which violates the strong, but obeys the weak, equivalence principle of general relativity and, in this theory, the gravitational constant was elevated to a dynamical scalar field [21]. In Kaluza – Klein theories, a scalar field appears upon dimensional reduction and it describes the size of an extra dimension. Moduli fields are ubiquitous in supergravity theories. In string theory effective actions, a dilaton field is always present which is related to the string coupling constant, see [22] in this context, or equivalently to the size of the eleventh dimension in M theory.

- A massless scalar field is often used in LQC to define internal time, with respect to which the quantum evolution of the universe has a bounce, see [8]. If this had been the only way to define a time variable then one may say that a scalar field with no potential appears in LQC also and plays the role of internal time. However, as pointed out by the referee, a time variable can be defined using pressureless dust or radiation also [23, 24].

- Long ago, in [25, 26], we had found that non singular evolution of a homogeneous isotropic universe is possible in generalised Brans – Dicke theories when a scalar field with no potential is non minimally coupled with the coupling function obeying certain conditions. It is possible that such theories are related to some LQC–inspired models.

With these motivations in mind, we explore in this paper whether \((d + 1)\) dimensional scalar tensor theories can be the effective actions which will lead to the effective equations of the LQC and the LQC–inspired models. We restrict ourselves to models where there is only one scalar field and its potential vanishes. And, on the other side, we restrict to a subclass of scalar tensor theories, referred to as the generalised Brans – Dicke theories, which contain only one scalar field, one coupling function, and no scalar field potential.

The LQC and the LQC–inspired models have one scalar field \(\sigma\) with no potential and one function \(f(m)\) where \(m\) is a certain Hubble-parameter-like variable in the gravitational sector. The function \(f(m) = \sin m\) for LQC, \(f(m) = m\) gives Einstein’s equations, and the classical limit corresponds to
taking the limit \( m \to 0 \). One requires that \( f(m) \to m \) in the limit \( m \to 0 \) to ensure that the classical limit leads to Einstein’s equations.

The scalar field and the coupling function in the effective action may be defined in several equivalent ways – for example, as the Brans – Dicke field \( \Phi \) and the function \( \omega(\Phi) \). In the original Brans – Dicke theory, \( \omega \) is constant. Einstein theory follows in the limit \( \omega(\Phi) \to \infty \). Determining the pair \((\Phi, \omega(\Phi))\) for a given \((\sigma, f(m))\) will then specify the effective action corresponding to the effective equations of motion of the LQC–inspired models.

In this paper, we consider isotropic effective equations and find that one can indeed relate the pairs \((\sigma, f(m))\) and \((\Phi, \omega(\Phi))\) – actually, an equivalent pair \((\phi, \Psi(\phi))\). We describe the relations between these pairs and they involve some integrations and functional inversions. We can not carry out these operations explicitly for any non trivial case. However, by studying a few explicit examples and some limiting cases, we can illustrate several important features of these relations.

The main results of this paper are the following.

We find that the LQC function \( f(m) = \sin m \) corresponds to an effective action where the function \( \omega(\Phi) \), equivalently \( \Psi(\phi) \), has the following properties: In the limit \( m \to 0 \) or \( \pi \), the function \( \omega \to \infty \), equivalently \( \phi \to \infty \) and \( \Psi \to const \). In the limit \( m \to \frac{\pi}{2} \) from below, the field \( \Phi \propto (\frac{\pi}{2} - m) \) and the function \( \omega(\Phi) \propto \Phi^2 \), equivalently \( \phi \to -\infty \) and \( \Psi \to \frac{\kappa \phi}{\sqrt{d(d-1)}} \). Another copy of \((\phi, \Psi(\phi))\) is needed when \( m \) evolves further from \( \frac{\pi}{2} \) to \( \pi \).

Also, we show that an LQC–inspired model which gives a non singular evolution similar to that in LQC, but now with a much flatter bounce, corresponds to a similar function \( \omega(\Phi) \) as above, but now with \( \omega(\Phi) \propto \Phi^{2n} \) near \( \Phi \to 0 \) where the positive integer \( n \) indicates how flat the minimum is.

We also find that an LQC–inspired model with \( f(m) \to m \) in the limit \( m \to 0 \) and \( f(m) \propto (m_s - m)^q \) in the limit \( m \to m_s \) from below, and with \( \frac{1}{2} \leq q < \frac{d}{2d-1} \), corresponds to the generalised Brans – Dicke theories which led to non singular isotropic evolutions in our earlier works [25, 26].

Lastly, by studying an example, we find that the relations obtained here
between the pairs \((\sigma, f)\) and \((\phi, \Psi)\) apply only to the homogeneous isotropic cases, and not to the anisotropic ones. This means that anisotropic evolutions of the LQC–inspired models can not be described by the scalar tensor theories of the type considered here. A similar situation arises in the LQC case also where the anisotropic evolution can not be described by \(F(R)\) theories and requires a further generalisation to \(F(R, Q)\) theories with \(Q = R_{\mu\nu}R^{\mu\nu}\) [19]. Perhaps a further generalisation is needed here also, but its nature is not clear to us.

The organisation of this paper is as follows. In section 2, we write down the effective equations of motion of the LQC–inspired models for the general anisotropic case. In section 3, we specialise these equations to the isotropic case and to the scalar field with no potential. In section 4, we briefly describe the relevant aspects of the generalised Brans – Dicke theory, write the action in several forms, write the resulting equations of motion, and specialise them to the anisotropic and the isotropic cases of interest here. In section 5, we relate the pairs \((\sigma, f)\) and \((\phi, \Psi)\) and work out various examples and limiting cases. We derive several results and also present a simple model for the function \(\Psi(\phi)\) which can capture several interesting features of the evolution. In section 6, we study an example and find that the relations obtained in the isotropic cases are not applicable for the anisotropic ones. In section 7, we conclude with a brief summary and some discussions.

2. Effective equations in the LQC–inspired models

Consider a \((d+1)\) dimensional homogeneous, anisotropic spacetime where \(d \geq 3\). Let the \(d\) dimensional space be toroidal, let \(x^i\) be the coordinate and \(L^i\) the coordinate length of the \(i^{th}\) direction, and let the line element \(ds\) be given by\(^1\)

\[
ds^2 = -dt^2 + \sum_i a_i^2 \left(dx^i\right)^2
\]

where \(i = 1, 2, \cdots, d\) and the scale factors \(a_i\) are functions of \(t\) only. We will write down the LQC–inspired equations of motion for \(a_i\) which are straightforward and natural generalisations of the effective equations in LQC.

\(^1\)In the following, the convention of summing over repeated indices is not always applicable. Hence we will write explicitly the indices to be summed over.
For the LQC case, $d = 3$. In the Loop quantum gravity (LQG) formalism, the canonical pairs of phase space variables consist of an $SU(2)$ connection $A^i_a = \Gamma^i_a + \gamma K^i_a$ and a triad $E^a_i$ of density weight one.\(^2\) In these expressions, $\Gamma^i_a$ is the spin connection defined by the triad $e^a_i$, $K^i_a$ is related to the extrinsic curvature, and $\gamma > 0$ and $\approx 0.2375$ is the Barbero-Immirzi parameter of LQG, its numerical value being suggested by the black hole entropy calculations. For the anisotropic universe, whose line element $ds$ is given in equation (1), one has $A^i_a \propto \hat{c}_i$ and $E^a_i \propto \hat{p}_i$ where $\hat{c}_i$ will turn out to be related to the time derivative of $a_i$, and $\hat{p}_i$ is given by

$$\hat{p}_i = \frac{V}{a_iL_i} , \quad V = \prod_j a_jL_j$$

with $V$ being the physical volume. The full expressions for $A^i_a$ and $E^a_i$ contain various fiducial triads, cotriads, and other elements, and are given in [6, 8]. The non-vanishing Poisson brackets among $\hat{c}_i$ and $\hat{p}_j$ are given by

$$\{\hat{c}_i, \hat{p}_j\} = \gamma \kappa^2 \delta_{ij}$$

where $\kappa^2 = 8\pi G_{d+1}$ is the gravitational constant. The effective equations of motion are given by the ‘Hamiltonian constraint’ $C_H = 0$ and by the Poisson brackets of $\hat{p}_i$ and $\hat{c}_i$ with $C_H$ which give the time evolutions of $\hat{c}_i$ and $\hat{p}_i$: namely, by

$$C_H = 0 , \quad (\hat{p}_i)_t = \{\hat{p}_i, C_H\} , \quad (\hat{c}_i)_t = \{\hat{c}_i, C_H\}$$

where the $t$–subscripts denote derivatives with respect to $t$. As reviewed in detail in [8], there exists an effective $C_H$ which leads to the equations of motion which describe very well the quantum dynamics of LQC. In a suitable limit, this effective $C_H$ reduces to the classical one which leads to Einstein’s equations.

The expression for the $C_H$ is of the form

$$C_H = H_{grav}(\hat{p}_i, \hat{c}_i) + H_{mat}(\hat{p}_i; \{\phi_{mat}\}, \{\pi_{mat}\})$$

\(^2\)See the review [8] for a complete description of the various LQG/C terms and concepts mentioned here and in the following.
where $H_{\text{grav}}$ denotes the effective gravitational Hamiltonian and $H_{\text{mat}}$ denotes a generalised matter Hamiltonian. In the matter sector, the density $\tilde{\rho}$ and the pressure $\tilde{p}_i$ in the $i^{th}$ direction are defined by

$$
\tilde{\rho} = \frac{H_{\text{mat}}}{V}, \quad \tilde{p}_i = -\frac{a_i L_i}{V} \frac{\partial H_{\text{mat}}}{\partial (a_i L_i)}.
$$

(6)

The pressure $\tilde{p}_i$ is thus, as to be physically expected, proportional to the change in energy per fractional change in the physical length in the $i^{th}$ direction. As indicated in equation (5), $H_{\text{mat}}$ is assumed to be independent of $\hat{c}_i$. Since $\hat{c}_i$ will turn out to be related to $(a_i)_t$, this assumption is equivalent to assuming that matter fields couple to the metric fields but not to the curvatures. It can also be shown [9] that, irrespective of what $H_{\text{grav}}$ is, this assumption leads to the standard conservation equation

$$
\tilde{\rho}_t = (\frac{H_{\text{mat}}}{V})_t = -\sum_i (\tilde{\rho} + \tilde{p}_i) \frac{(a_i)_t}{a_i}.
$$

(7)

In the gravitational sector, the effective $H_{\text{grav}}$ is given by

$$
H_{\text{grav}} = -\frac{V}{\gamma^2 \lambda_{qm}^2 \kappa^2} \left( \sin(\tilde{\mu}^1 \hat{c}_1) \sin(\tilde{\mu}^2 \hat{c}_2) + \text{cyclic terms} \right)
$$

(8)

where $V = \sqrt{\hat{p}_1 \hat{p}_2 \hat{p}_3}$ is the physical volume, $\lambda_{qm}^2 = \sqrt{\frac{3}{4}} \gamma \kappa^2$ is the quantum of area, and $\tilde{\mu}^i = \frac{\lambda_{qm} \hat{p}_i}{V}$ in what is referred to as the $\tilde{\mu}$–scheme. Classical $H_{\text{grav}}$ follows in the limit $\tilde{\mu}^i \hat{c}_i \to 0$ where $\sin (\tilde{\mu}^i \hat{c}_i) \to \tilde{\mu}^i \hat{c}_i$.

**LQC–inspired models**

In recent papers [9, 10], we have generalised the effective LQC equations and studied their solutions. Our generalisations are empirical but simple, straightforward, and natural. We generalised from $(3 + 1)$ to $(d + 1)$ dimensions, and generalised the trigonometric and the $\tilde{\mu}$ functions appearing in the effective $H_{\text{grav}}$ in equation (8). In this paper, we will consider only the generalisation of the trigonometric function, keeping the $\tilde{\mu}$ function the same as in the $\tilde{\mu}$–scheme. Upon generalisation:

- The index $i$ takes the values $i = 1, 2, \cdots, d$ and $d \geq 3$ now.
The canonical pairs of phase space variables are given by $\hat{c}_i$ which will be related to $(a_i)_i$, and $\hat{p}_i$ which is given by equation (2). The non-vanishing Poisson brackets among $\hat{c}_i$ and $\hat{p}_j$ are given by equation (3) where $\gamma$ now characterises the quantum of the $(d-1)$ dimensional area given by $\lambda_{qm}^{d-1} \sim \gamma \kappa^2 [27, 28, 29, 30]$.

The effective equations of motion are given by equation (4) where $C_H$ is of the form given in equation (5). In the matter sector, the density $\hat{\rho}$ and the pressures $\hat{p}_i$ are given by equations (6), and they satisfy the standard conservation equation (7).

In the gravitational sector, the effective $H_{grav}$ in equation (8) is now generalised to

$$
H_{grav} = - \frac{V}{\gamma^2 \lambda_{qm}^2 \kappa^2} \sum_{i<j} f^i f^j
$$

where $V = (\prod_i \hat{p}_i)^{\frac{1}{d-1}}$ is the $d$ dimensional physical volume and

$$
f^i = f(m^i) \ , \quad m^i = \bar{\mu}^i \hat{c}_i \ , \quad \bar{\mu}^i = \frac{\lambda_{qm} \hat{p}_i}{V} \ .
$$

The function $f(x)$ which appears in equation (10) is arbitrary, but with the only requirement that $f(x) \to x$ as $x \to 0$ so that classical $H_{grav}$ is obtained in the limit $m^i \to 0$. It is easy to see that the LQC case follows upon setting $d = 3$ and $f(x) = \sin x$.

The equations of motion may now be obtained using the generalised $H_{grav}$ given in equation (9). These equations will describe the evolution of a $(d+1)$ dimensional homogeneous anisotropic universe in our LQC–inspired models. The required algebra is straightforward but involved, and we present only the final equations in a convenient form. Defining $\lambda^i$, $\Lambda$, $G_{ij}$, and $G^{ij}$ by

$$
a_i = e^{\lambda^i} \ , \quad \Lambda = \sum_i \lambda^i \ , \quad G_{ij} = 1 - \delta_{ij} \ , \quad G^{ij} = \frac{1}{d-1} - \delta^{ij} \ ,
$$

the resulting equations of motion may be written as

$$
\sum_{ij} G_{ij} f^i f^j = 2 \gamma^2 \lambda_{qm}^2 \kappa^2 \hat{\rho}
$$

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$$

the resulting equations of motion may be written as

$$
\sum_{ij} G_{ij} f^i f^j = 2 \gamma^2 \lambda_{qm}^2 \kappa^2 \hat{\rho}
$$
\[ (m^i)_t + \sum_j (m^i - m^j) X_j = -\gamma\lambda qm\kappa^2 \sum_j G^{ij} (\tilde{\rho} + \tilde{\rho}_j) \]  \hspace{1cm} (12)

\[ (\gamma\lambda qm) \lambda^i_t = \sum_j G^{ij} X_j \]  \hspace{1cm} (13)

\[ \tilde{\rho}_t = -\sum_i (\tilde{\rho} + \tilde{\rho}_i) \lambda^i_t \]  \hspace{1cm} (14)

where \( X_i \) is given by

\[ X_i = g_i \sum_j G_{ij} f^j \]

Einstein’s equations follow when \( f(m^i) = m^i \). Then \( g_i = 1 \), equation (13) gives \( (\gamma\lambda qm) \lambda^i_t = m^i \), and, after a little algebra, equations (11) and (12) give the Einstein’s equations for a \((d + 1)\) dimensional homogeneous anisotropic universe:

\[ \sum_{ij} G_{ij} \lambda^i_t \lambda^j_t = 2\kappa^2 \tilde{\rho} \]  \hspace{1cm} (15)

\[ \lambda^i_{tt} + \Lambda_t \lambda^i_t = \kappa^2 \sum_j G^{ij} (\tilde{\rho} - \tilde{\rho}_j) \]  \hspace{1cm} (16)

3. Isotropic case with a massless scalar field

For the isotropic case, one has \((\tilde{\rho}_i, m^i, f^i, a_i) = (\tilde{\rho}, m, f, a)\). Then

\[ \lambda^i_t = \frac{a_t}{a} \equiv h \hspace{0.5cm}, \hspace{0.5cm} X_i = (d - 1) gf \hspace{0.5cm}, \hspace{0.5cm} g_i = g = \frac{df(m^i)}{dm^i} \]  \hspace{1cm} (17)

Also \( e^\Lambda = a^d \) and \( \Lambda_t = d h \). Equations of motion (11) – (14) now give

\[ f^2 = \frac{2\gamma^2\lambda qm\kappa^2 \tilde{\rho}}{d(d - 1)} \]  \hspace{1cm} (18)

\[ m_t = -\gamma\lambda qm\kappa^2 \frac{\tilde{\rho} + \tilde{\rho}}{d - 1} \]  \hspace{1cm} (19)
\[ h = \frac{a_t}{a} = \frac{g f}{\gamma \lambda_{qm}} \quad (20) \]

\[ \tilde{\rho}_t = -d h (\tilde{\rho} + \tilde{p}) . \quad (21) \]

Consider a massless scalar field \( \sigma \) with no potential. It is straightforward to show that its density, pressures, and the equation of motion in a homogeneous, isotropic universe are given by

\[ \tilde{\rho} = \tilde{p} = \frac{(\sigma_t)^2}{2} , \quad \sigma_t + d h \sigma_t = 0 . \quad (22) \]

Equation (21) is then satisfied. Equations (18) and (19) now become

\[ f^2 = \frac{\gamma^2 \lambda_{qm}^2 \kappa^2}{d (d-1)} (\sigma_t)^2 \quad (23) \]

\[ m_t = -\frac{\gamma \lambda_{qm}^2 \kappa}{d-1} (\sigma_t)^2 . \quad (24) \]

With no loss of generality, let \( \sigma_t = \sqrt{2\tilde{\rho}} = \sqrt{\frac{d (d-1)}{\gamma \lambda_{qm} \kappa}} f \) where the square roots are always taken with a positive sign. Then equations (22) – (24) give

\[ \frac{\sigma_t}{\sigma_{t0}} = \sqrt{\frac{\tilde{\rho}}{\tilde{\rho}_0}} = \frac{f}{f_0} = \left( \frac{a_0}{a} \right)^d \quad (25) \]

where the 0–subscripts denote the values at an initial time \( t_0 \) and

\[ \frac{dm}{f^2} = -c_{qm} dt , \quad \frac{dm}{f} = -c_1 d\sigma \quad (26) \]

where \( c_{qm} = \frac{d}{\gamma \lambda_{qm}} \) and \( c_1 = \sqrt{\frac{\sigma^2}{d-1}} \). Thus, for a given function \( f(m) \), equations (25) and (26) give \( a(m) \), \( t(m) \), and \( \sigma(m) \) which, in principle, then give the solutions \( m(t) \), \( a(t) \), and \( \sigma(t) \).

For example, let \( f(m) = m \) which leads to Einstein’s equations. Then, using \( dh_0 = c_{qm} m_0 \) and after a little algebra, one obtains that

\[ \frac{f_0}{f} = \frac{m_0}{m} = \left( \frac{a}{a_0} \right)^d = e^{c_1 (\sigma - \sigma_0)} = dh_0 \left( t - t_0 + \frac{1}{dh_0} \right) . \quad (27) \]
It can also be shown that the evolution near any simple zero of \( f \) is same as that given by Einstein’s equations upto a constant scaling of \( t \).

As another example, consider the evolution near a maximum of \( f \). Let \( f \) reach a maximum at \( m = m_b \) and, near \( m_b \), let

\[
f(m) \simeq f_b \left(1 - f_1(m_b - m)^{2n}\right)
\]  

(28)

where \( f_b \) and \( f_1 \) are positive constants and \( n \) is a positive integer which indicates how flat the maximum is. Let \( t_b \) be the time when \( f \) reaches its maximum. Then, as \( t \to t_b \), it follows from equations (26) that

\[
m_b - m \simeq f_b^2 c_{qm}(t - t_b)
\]

\[
c_1(\sigma - \sigma_b) \simeq f_b c_{qm}(t - t_b)
\]

\[
a \simeq a_{mn} \left(1 + a_1(t - t_b)^{2n}\right)
\]  

(29)

where \( \sigma_b \) is a constant, \( a_{mn} = a_0 \left(\frac{f_0}{f_b}\right)^{\frac{1}{d}} \) and \( a_1 = \frac{f_0}{f_b} (f_b^2 c_{qm})^{2n} \). If \( f(m) = \sin m \) then \( m_b = \frac{\pi}{2} \) and \( f_b = 2f_1 = n = 1 \) in equation (28). It can then be checked that equations (29) are consistent with the explicit solution given, for example, in [10].

### 4. Generalised Brans–Dicke theory

The effective equations (22) – (24) or, equivalently, (25) and (26) describe the evolution of a \((d+1)\) dimensional homogeneous isotropic universe with a massless scalar field in the LQC–inspired models. For such an universe, the line element \( ds \) and the metric \( g_{\mu\nu} \), \( \mu, \nu = 0, 1, 2, \cdots, d \), are given by

\[
ds^2 = \sum_{\mu\nu} g_{\mu\nu} \, dx^\mu dx^\nu = -dt^2 + a^2 \sum_i (dx^i)^2
\]  

(30)

where the scale factor \( a \) is a function of \( t \) only. It is natural to search for an effective action which, if exists, will lead to covariant equations that generalise the standard Einstein’s equations and, for the above line element, lead to the effective equations (25) and (26).
We assume that such an effective action exists and proceed to construct it invoking the following line of reasoning. The action should contain the metric field $g_{\mu\nu}$ and a scalar field $\phi$ which, in general, may be different from $\sigma$. The construction should also involve an arbitrary function to act as an equivalent for the function $f$ of the LQC–inspired models. The required action cannot be the canonical minimally coupled action for $\phi$ with a potential $V(\phi)$ since it cannot lead to a non singular evolution of $a(t)$ with a bounce.

However, the scalar field $\phi$ may couple non minimally. Then, irrespective of whether there is a potential for the scalar field, the general non minimal coupling will involve a function which may act as an equivalent for the function $f$. Non minimally coupled scalar field appears naturally, and generically with no potential, in several contexts. For example, it appears as a dilaton field in string theories and as a moduli field in supergravity theories. It also appears in Brans–Dicke theories where the gravitational constant is elevated to a dynamical scalar field. Also, in our past works in this context [25, 26], we had found that non singular evolution of a homogeneous isotropic universe is possible for appropriate choices of the non minimal coupling function. For these reasons, and also due to the motivations listed in the Introduction, we consider in this paper the generalised Brans–Dicke theories where a scalar field with no potential is non minimally coupled with a non trivial coupling function.  

**Action : different forms**

The $(d + 1)$ dimensional action for a scalar field $\phi$ with no potential and with a non minimal coupling function may be written in several equivalent forms. Consider the action $S_{st}$ for a scalar – tensor theory given by

$$S_{st} = \frac{1}{2\kappa^2} \int d^{d+1}x \sqrt{-g} \left( \Omega R - B (\nabla \phi)^2 \right)$$

(31)

3 In LQC, $(3 + 1)$ dimensional effective actions have been constructed in [14] – [19] by generalising Einstein’s term $R$ to $F(R)$ and finding an appropriate function $F$ which will give the isotropic LQC evolution. Any $F(R)$–theory, including that for LQC, can be written as a scalar – tensor theory with the Brans–Dicke constant $\omega = 0$ and with a potential that depends on $F$ [11, 12]. Similar approach may also work for the present $(d + 1)$ dimensional LQC–inspired models for any arbitrary function $f$, but we will not pursue it in this paper.
where $\Omega$ and $B$ are functions of $\phi$ and the scalar potential $V(\phi) = 0$. There is a freedom in defining the scalar field which may be used to specify the functions $\Omega$ and $B$ in a convenient form. For example, setting $\phi = \Omega = \Phi$ and $B = \omega(\Phi)$ gives the generalised Brans–Dicke action

$$S_{bd} = \frac{1}{2\kappa^2} \int d^{d+1}x \sqrt{-g} \left( \Phi R - \frac{\omega(\Phi)}{\Phi} (\nabla \Phi)^2 \right)$$

(32)

which elevates the gravitational constant to a spacetime dependent dynamical scalar field $\Phi$ and the Brans–Dicke constant $\omega$ to a function of $\Phi$ now. Or, the action $S_{st}$ may instead be written as

$$S_{\Psi} = \int d^{d+1}x \sqrt{-g} \ e^{(d-1)\Psi} \left( \frac{R}{2\kappa^2} - \frac{A}{2} (\nabla \phi)^2 \right)$$

(33)

where we have set

$$B = \kappa^2 \Omega \ A \ , \ \Omega = e^{(d-1)\Psi} \ , \ A = 1 - \frac{d(d-1)}{2\kappa^2} (\Psi_{\phi})^2$$

(34)

and the $\phi$–subscripts denote derivatives with respect to $\phi$. The function $\Psi(\phi)$ is now the non minimal coupling function. The action $S_{\Psi}$ follows upon setting $g_{\ast\mu\nu} = e^{2\Psi} g_{\mu\nu}$ in the ‘Einstein frame’ action $S_{\ast}$ given by

$$S_{\ast} = \int d^{d+1}x \sqrt{-g_{\ast}} \left( \frac{R_{\ast}}{2\kappa^2} - \frac{(\nabla_{\ast} \phi)^2}{2} \right)$$

(35)

where the action for $g_{\ast\mu\nu}$ is as in Einstein’s theory, the scalar field $\phi$ has a canonical kinetic term, and $\phi$ is coupled minimally to $g_{\ast\mu\nu}$. However, other fields and the probes in the theory are assumed to couple minimally, not to $g_{\ast\mu\nu}$, but to $g_{\mu\nu}$. Hence, in the Einstein frame, they experience a force due to scalar field $\phi$ and will not fall freely along the geodesics of $g_{\ast\mu\nu}$.

Note that $(\Phi, \omega(\Phi))$ in the action $S_{bd}$ and $(\phi, \Psi(\phi))$ in the action $S_{\Psi}$ may be related to each other easily. If given $\phi$ and $\Psi(\phi)$ then it follows from equations (32) and (33) that

$$\Phi = e^{(d-1)\Psi} \ , \ \omega = \frac{\kappa^2 A}{(d-1)^2 (\Psi_{\phi})^2} = \frac{\kappa^2}{(d-1)^2 (\Psi_{\phi})^2} - \frac{d}{d-1}.$$  

(36)

Inverting the first expression gives, in principle, $\phi(\Phi)$ which then gives $\omega(\Phi)$.

Thus, the limit $\Psi(\phi) \rightarrow \text{const}$ which leads to Einstein’s theory can now be
seen easily to correspond to the limit $\omega \to \infty$. If given $\Phi$ and $\omega(\Phi)$ instead, it then follows from equation (36) that

$$(d - 1) \Psi = \ln \Phi, \quad \kappa \ d\phi = \frac{d\Phi}{\Phi} \left( \omega + \frac{d}{d - 1} \right)^{\frac{1}{2}} . \quad (37)$$

The function $\phi(\Phi)$ now follows upon an integration. Inverting this function then gives, in principle, $\Phi(\phi)$ and thereby $\Psi(\phi)$.

**Equations of motion from $S_{st}$**

**General:** Consider the action $S_{st}$ given in equation (31). Defining $t_{\mu\nu}$ by

$$\kappa^2 t_{\mu\nu} = B \left( \nabla_\mu \phi \nabla_\nu \phi - \frac{g_{\mu\nu}}{2} (\nabla \phi)^2 \right) ,$$

the equations of motion following from the action $S_{st}$ may be written as

$$\Omega \left( R_{\mu\nu} - \frac{g_{\mu\nu}}{2} R \right) = \kappa^2 t_{\mu\nu} + \nabla_\mu \nabla_\nu \Omega - g_{\mu\nu} \nabla^2 \Omega \quad (38)$$

and

$$2B \nabla^2 \phi + B_\phi (\nabla \phi)^2 + \Omega_\phi R = 0 \quad . \quad (39)$$

It follows from equation (38) that

$$\Omega R = \frac{2d}{d - 1} \nabla^2 \Omega + B (\nabla \phi)^2$$

and then from equation (39) that

$$\left( B + \frac{d (\Omega_\phi)^2}{(d - 1) \Omega} \right) \nabla^2 \phi + \left( \frac{B_\phi}{2} + \frac{B \Omega_\phi}{2 \Omega} + \frac{d \Omega_\phi \Omega_{\phi\phi}}{(d - 1) \Omega} \right) (\nabla \phi)^2 = 0 \quad . \quad (40)$$

For the generalised Brans – Dicke action $S_{bd}$ given in equation (32), we have $\phi = \Omega = \Phi$ and $B = \frac{\omega(\Phi)}{\Phi}$ . Equation (40) then simplifies to

$$2 \left( \omega + \frac{d}{d - 1} \right) \nabla^2 \Phi + \omega_\Phi (\nabla \Phi)^2 = 0 \quad . \quad (41)$$
Anisotropic case: Let the metric $g_{\mu\nu}$ be given by

$$ds^2 = \sum_{\mu\nu} g_{\mu\nu} \, dx^\mu \, dx^\nu = -dt^2 + \sum_i a_i^2 \,(dx^i)^2$$

(42)

where the scale factors $a_i$ and the scalar field $\phi$ depend on $t$ only. Defining $a_i = e^{\lambda_i}$ and $\Lambda = \sum_i \lambda_i^i$, the non vanishing components of $R_{\mu\nu}$ and $\nabla^\mu \nabla_\nu \Omega$, and thereby $R$ and $\nabla^2 \Omega$, are given by

$$R^{t\,t} = \Lambda_{tt} + \sum_i \left(\lambda_i^t\right)^2 , \quad R^{i\,i} = \lambda_i^{tt} + \Lambda_t \lambda_i^t$$

$$\nabla^i \nabla_t \Omega = -\Omega_{tt} , \quad \nabla^i \nabla_i \Omega = -\Omega_t \lambda_i^t$$

$$R = 2 \Lambda_{tt} + (\Lambda_t)^2 + \sum_i \left(\lambda_i^t\right)^2 , \quad \nabla^2 \Omega = -\Omega_{tt} - \Omega_t \Lambda_t .$$

It then follows from equations (38) and (39) that

$$\Omega \left((\Lambda_t)^2 - \sum_i (\lambda_i^t)^2\right) = B \left(\phi_t\right)^2 - 2\Lambda_t \Omega_t$$

(43)

$$\Omega \left(\lambda_{tt}^i + \Lambda_t \lambda_i^t\right) + \Omega_t \lambda_i^t = -\frac{\Omega_{tt} + \Lambda_t \Omega_t}{d-1}$$

(44)

$$2B \left(\phi_{tt} + \Lambda_t \phi_t\right) = -B_\phi \left(\phi_t\right)^2 + \Omega_{tt} R .$$

(45)

Isotropic case: For the isotropic case, $a_i = a$, $\lambda_i^i = \frac{a}{a} = h$, and $\Lambda_t = dh$. Equations (43) and (44) then give

$$d(d-1) \Omega \, h^2 = B \left(\phi_t\right)^2 - 2d \, h \, \Omega_t$$

(46)

$$- (d-1) \Omega \, h_t = B \left(\phi_t\right)^2 + \Omega_{tt} - h \, \Omega_t .$$

(47)

Equations of motion from $S_*$ and their solutions

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Anisotropic case : Let the metric $g_{\mu \nu \nu}$ in the Einstein frame be given by

$$ds^2_\ast = \sum_{\mu \nu} g_{\mu \nu} \, dx^\mu \, dx^\nu = -dT^2 + \sum_i A_i^2 \, (dx^i)^2$$  \hspace{1cm} (48)$$

where the scale factors $A_i$ and the scalar field $\phi$ depend on $T$ only. The equations of motion resulting from $S_\ast$ follow from equations (43) – (45) upon setting $\Omega = 1$ and $B = \kappa^2$. Defining $A_i = e^{l_i}$ and $L = \sum_i l_i$, we obtain

$$\left( L_T \right)^2 - \sum_i (l_T^i)^2 = \kappa^2 \left( \phi_T \right)^2$$ \hspace{1cm} (49)$$

These equations can be solved explicitly. Equations (50) give

$$l_T^i = l_T^0 \, e^{L_T^0 - L}, \quad \phi_T = \phi_T^0 \, e^{L_T^0 - L}$$ \hspace{1cm} (51)$$

where the 0—subscripts denote the values at an initial time $T_0$. We then have $L_T = L_T^0 \, e^{L_T^0 - L}$ where $L_T^0 = \sum_i l_T^0$ and, hence,

$$e^{L_T^0 - L} = L_T^0 \, \tilde{T}, \quad \tilde{T} = T - T_0 + \frac{1}{L_T^0}.$$ \hspace{1cm} (52)$$

Writing $l_T^0 = \alpha^i \, L_T^0$ and $\kappa \, \phi_T^0 = \beta \, L_T^0$, we obtain $\sum_i \alpha^i = 1$,

$$e^{l_T^0} = \left( L_T^0 \, \tilde{T} \right)^{\alpha^i}, \quad e^{\kappa (\phi - \phi_0)} = \left( L_T^0 \, \tilde{T} \right)^{\beta},$$ \hspace{1cm} (53)$$

and then, from equation (49), the constraint $\sum_i (\alpha^i)^2 + \beta^2 = 1$.

Isotropic case : For the isotropic case, we have $e^{l_T} = A$, $l_T = \frac{A_T}{A} = H$, and $L_T = dH$. Hence,

$$d(d - 1) \, H^2 = - (d - 1) \, H_T = \kappa^2 \left( \phi_T \right)^2.$$ \hspace{1cm} (54)$$

Also, $\alpha^i = \frac{1}{d}$ and $\beta = \sqrt{\frac{d-1}{d}}$ in equations (53). Hence $\frac{\kappa}{\beta} = \sqrt{\frac{\kappa^2 \, d}{d-1}} = c_1$ and the isotropic solutions may be written as

$$\left( \frac{A}{A_0} \right)^d = e^{\epsilon_i (\phi - \phi_0)} = L_T^0 \, \tilde{T} = \left( \frac{\phi_T^0}{\phi_T} \right).$$ \hspace{1cm} (55)$$
Equations of motion from $S_{\Psi}$

The equations of motion resulting from $S_{\Psi}$ follow similarly from equations (43) – (45) upon using equation (34) for the functions $\Omega$ and $B$. They may also be obtained by noting that $S_{\Psi}$ gives $S_{\Psi}$ upon setting $g_{\mu\nu} = e^{2\Psi}g_{\mu\nu}$. Equations (42) and (48) then give

\[ dT = e^\Psi dt, \quad A_i = e^\Psi a_i \quad \rightarrow \quad l^i = \Psi + \lambda^i. \]  

(56)

Substituting these expressions in equations (49) and (50) must then give the equations obtained by substituting equations (34) for $\Omega$ and $B$ in equations (43) – (45). We have verified that this is indeed the case.

5. Relating the functions $f$ and $\Psi$

Consider a $(d+1)$ dimensional homogeneous isotropic universe with a massless scalar field. In an LQC or LQC–inspired model which is specified by a function $f$, the evolution is described by equations (25) and (26), rewritten below for ease of reference:

\[ \frac{f}{f_0} = \left( \frac{a_0}{a} \right)^d, \quad \frac{dm}{f^2} = -c_{qm} dt, \quad \frac{dm}{f} = -c_1 d\sigma \]  

(57)

where $c_{qm} = \frac{d}{\gamma \lambda_m}$ and $c_1 = \sqrt{\frac{\kappa}{d-1}}$. These equations give $a(m)$, $t(m)$, and $\sigma(m)$ for a given function $f(m)$. In generalised Brans – Dicke theory, the evolution is described by equations (55) and (56) with $A_i = A$ and $a_i = a$, rewritten below for ease of reference:

\[ \left( \frac{\phi T}{\phi T_0} \right) = \left( \frac{A_0}{A} \right)^d = e^{-c_1(\phi - \phi_0)} \]  

(58)

which give $T(\phi)$ and $A(\phi)$, and

\[ dT = e^\Psi dt, \quad A = e^\Psi a \]  

(59)

which then give $t(\phi)$ and $a(\phi)$ for a given function $\Psi(\phi)$. The functions $f$ and $\Psi$ can then be related to each other. In the following, we take the initial
values \( f_0, \sigma_{t0}, \) and \( \phi_{T0} \) to be positive for the sake of definiteness, and set \( \sigma_0 = \phi_0 = 0 \) with no loss of generality.

We now study the relation between the functions \( f \) and \( \Psi \). The scalar field \( \sigma \) may be different from \( \phi \) and, hence, the relation between them also needs to be studied. Now, after some algebra and with \( c_m = c_1 c_\sigma f_0 \) and \( c_\sigma = \frac{a_0}{\psi_0 \phi_{T0}} \), it follows from equations (57) – (59) that

\[
\begin{align*}
    f &= f_0 e^{d (\Psi - \Psi_0)} - c_1 \phi \\
    dm &= -c_m e^{(2d-1) (\Psi - \Psi_0) - c_1 \phi} \, d\phi \\
    d\sigma &= c_\sigma e^{(d-1) (\Psi - \Psi_0)} \, d\phi
\end{align*}
\]

It then follows from these equations, or from \( \frac{dt}{a} = \frac{dA}{A} \), that

\[
\left( \frac{f_0}{f} \right)^{2 - \frac{d}{2}} dm = -c_m e^{(1-\frac{d}{2}) c_1 \phi} \, d\phi.
\]

Consider equations (60) – (63). If a function \( \Psi(\phi) \) is given then equation (60) gives \( f(\phi) \); equation (61) gives \( m(\phi) \); and equation (62) gives \( \sigma(\phi) \). The function \( f(m) \) follows now, in principle, from \( f(\phi) \) and \( m(\phi) \). Thus, a given function \( \Psi(\phi) \) determines the functions \( f(m) \) and \( \sigma(\phi) \).

If a function \( f(m) \) is given then \( \sigma(m) \) follows from the last equation in (57) or from equations (60) – (62). Equation (63) gives \( \phi(m) \) and, then, equation (60) gives \( \Psi(m) \). The function \( \Psi(\phi) \) follows now, in principle, from \( \phi(m) \) and \( \Psi(m) \); and the function \( \phi(\sigma) \) from \( \sigma(m) \) and \( \phi(m) \). Thus, a given function \( f(m) \) determines the functions \( \Psi(\phi) \) and \( \phi(\sigma) \).

**Examples and Limiting cases**

The functions \( f(m) \) and \( \Psi(\phi) \) can be related to each other as described above. This will then relate the corresponding LQC–inspired model and the generalised Brans–Dicke theory. However, we are not able to carry out explicitly all the required integrations and functional inversions for any non trivial
function, including the LQC function \( f(m) = \sin m \). Nevertheless, several important features of the relation between these models can be understood by studying a few explicit examples and some limiting cases.

**Example (1) :** \( f(m) = m \)

Consider the example \( f(m) = m \). Then the last equation in (57) gives \( m = m_0 e^{-c_1 \sigma} \). Equation (63) then gives \( m = m_0 c_{c_1}^{d^d} e^{-c_1 \phi} \). Equation (60) then gives \( c_\sigma e^{(d-1)(\Psi - \Psi_0)} = 1 \) which implies that \( \Psi = \text{const} \) and, hence, that \( \Psi = \Psi_0 \) and \( c_\sigma = 1 \) which, in turn, gives \( \phi = \sigma \). Note that equations (60) and (61) give \( g = \frac{df}{dm} = \left( \frac{c_1 - d \Psi_0}{c_1 c_\sigma} \right) e^{-(d-1)(\Psi - \Psi_0)} \) from which also it follows that \( c_\sigma = 1 \) when \( f(m) = m \) and \( \Psi = \Psi_0 \).

Thus, when \( f(m) = m \) one has \( \Psi = \text{const} \) and \( \phi = \sigma \). This, of course, corresponds to Einstein’s theory. It is easy to see that a similar result follows for any linear function \( f = \alpha_1 m + \alpha_2 \) where \( \alpha_1 \) and \( \alpha_2 \) are constants. Indeed, now start with \( \Psi(\phi) = \text{const} = \Psi_0 \). Then equation (60) gives \( f = f_0 e^{-c_1 \phi} \); equation (62) gives \( \sigma = c_\sigma \phi \); and, equation (61) gives \( m = m_0 + c_\sigma (f - f_0) \) which is of the form \( f = \alpha_1 m + \alpha_2 \).

**Example (2) :** \( \Psi(\phi) = k \, c_1 \phi + \Psi_0 \)

Consider the example where \( \Psi(\phi) = k \, c_1 \phi + \Psi_0 \), and \( k \) and \( \Psi_0 \) are constants. It follows from equation (36) that this corresponds to a Brans – Dicke theory with

\[
\Phi = \Phi_0 \, e^{k(d-1) \, c_1 \phi} \quad \omega = \frac{1 - k^2 d^2}{d(d - 1) \, k^2}
\]

and \( \Phi_0 = e^{(d-1) \Psi_0} \). For this example, equations (60) – (61) give

\[
f = f_0 \, e^{(kd-1) \, c_1 \phi} \quad (65)
\]

\[
c_1 \, \sigma = \frac{c_\sigma}{k(d-1)} \left( e^{k(d-1) \, c_1 \phi} - 1 \right)
\]

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\[ m - m_0 = m_1 \left( 1 - e^{D c_1 \phi} \right) \]  
\[ \Rightarrow f(m) = f_0 \left( \frac{m_1 + m_0 - m}{m_1} \right)^q \]

where

\[ m_1 = \frac{c_m}{c_1 D}, \quad D = k(2d - 1) - 1, \quad q = \frac{kd - 1}{D} . \]

Let \( k > \frac{1}{2} \) so that \( D \) and \( q \) are positive. Then, in the limit \( m \to m_s \) where \( m_s = m_0 + m_1 \), the scalar field \( \phi \to -\infty \), the function \( f \to 0 \), and the scale factor \( a \propto f^{-\frac{1}{d}} \to \infty \). Also, equations (58) and (59) give \( t(\phi) \):

\[ c_t (t - t_0) = e^{(1-k) c_1 \phi} - 1 , \quad c_t = (1-k) c_1 \phi_{t_0} \]  

for \( k \neq 1 \), and \( \phi_{t_0} (t - t_0) = \phi \) for \( k = 1 \). Hence, in the limit \( \phi \to -\infty \), the time \( t \to t_s = t_0 - \frac{1}{c_t} \) if \( k < 1 \) and \( t \to -\infty \) if \( k \geq 1 \).

It now follows that if \( k \geq 1 \) then \( q \geq \frac{1}{2} \) and, in the limit \( m \to m_s \), the evolution is non singular because the scale factor \( a \to \infty \) and the time \( t \to -\infty \). This is the same result obtained in [10] in the context of LQC–inspired models, and in [25, 26] in the context of generalised Brans–Dicke theories. Also, if \( k = 1 \) then it is easy to see that the entire evolution is non singular and the scale factor \( a(t) \) evolves exponentially in time: As \( \phi \) varies from \( -\infty \) to \( +\infty \), the time \( t \propto \phi \) also varies the same way, \( m \) varies from \( m_s \) to \( -\infty \), the function \( f(t) \propto e^{(\star) t} \) varies from 0 to \( \infty \), and the scale factor \( a(t) \propto e^{-(\star) t} \) varies from \( \infty \) to 0, the \( (\star) \)'s being some positive constants.

Note that if \( \frac{1}{d} < k < \infty \) then \( 0 < q < \frac{d}{2d-1} \). This upper bound on \( q \) may also be understood as follows. Integrating equation (63), which follows from \( \frac{dt}{a} = \frac{dT}{A} \), gives

\[ - \int_{m_0}^{m} dm' f^{\frac{1}{2} - 2} = (\star) \int_{\phi_0}^{\phi} d\phi' e^{\frac{d-1}{d} c_1 \phi'} \]

where \( (\star) \) is an unimportant constant. Now, the \( \phi \)-integral in the above equation is finite in the limit \( \phi \to -\infty \). In the corresponding limit, \( m \to m_s \) and \( f \sim (m_s - m)^q \). Therefore the \( m \)-integral in the above equation will also be finite in this limit only if \( q < \frac{d}{2d-1} \).
Example (3): \( f(m) \simeq f_b \left( 1 - f_1(m_b - m)^{2n} \right) \)

Consider the example where \( f(m) \) has a maximum at \( m_b \) and, near its maximum, is given by \( f(m) \simeq f_b \left( 1 - f_1(m_b - m)^{2n} \right) \) where \( f_b \) and \( f_1 \) are positive constants, \( \tilde{m} = (m_b - m) \), and \( n \) is a positive integer which indicates how flat the maximum is. As \( f \) reaches its maximum and then decreases, the scale factor \( a \) reaches a minimum and increases again – it has a ‘bounce’. The scalar field \( \phi \) decreases as \( m \) increases, see equation (61). In the limit \( m \to m_b \) from below then two cases are possible: (i) \( \phi \to \phi_b \), a finite value, or (ii) \( \phi \to -\infty \).

Case (i) \( \phi \to \phi_b \): Let \( \phi \) have a finite value \( \phi_b \) at the bounce. Let \( \tilde{\phi} = \phi - \phi_b \) and \( \tilde{\Psi} = \Psi - \Psi_b \). Here and in the following, we encounter various unimportant constants. We will denote them all by \( \ast \), keeping only their signs. It then follows from equations (63) and (60) that, near the bounce,

\[
\tilde{m} \simeq \ast \tilde{\phi} \quad (70)
\]

\[
e^{\tilde{\phi}} \simeq e^{c_1 \phi} \left( 1 - \ast \tilde{m}^{2n} \right) \quad (71)
\]

\[
\Rightarrow \quad \Psi_\phi \simeq \frac{c_1}{d} - \ast \tilde{m}^{2n-1} . \quad (72)
\]

It is now illuminating to change over from the present \( \phi \) and \( \Psi(\phi) \) to the field \( \Phi \) and the function \( \omega(\Phi) \) of the generalised Brans – Dicke theory, given by equations (36). Let \( \Phi_b = e^{(d-1)} \Psi_b \) and \( \tilde{\Phi} = \Phi - \Phi_b \). Using the above expressions, it then follows that

\[
\Phi = e^{(d-1)} \Psi \simeq \Phi_b \left( 1 + \ast \tilde{m} \right) ,
\]

\[
\omega(\Phi) \simeq \ast \tilde{m}^{2n-1} \simeq \ast \left( \Phi - \Phi_b \right)^{2n-1} . \quad (73)
\]

Case (ii) \( \phi \to -\infty \): Let \( \phi \) diverge to \(-\infty\) at the bounce. It then follows from equations (63) and (60) that

\[
\tilde{m} \simeq \ast e^{\left( 1 - \frac{1}{d} \right) c_1 \phi} \quad (74)
\]
\[ e^\Psi \simeq (\ast) e^{\frac{c_1}{d} \phi} (1 - (\ast) \tilde{m}^{2n}) \]  \hspace{1cm} (75)

\[ \Rightarrow \quad \Psi_\phi \simeq \frac{c_1}{d} - (\ast) \tilde{m}^{2n} . \]  \hspace{1cm} (76)

Changing over to the field \( \Phi \) and the function \( \omega(\Phi) \) using equations (36) and the above expressions, it then follows that

\[ \Phi = e^{(d-1) \Psi} \simeq (\ast) \tilde{m} , \]

\[ \omega(\Phi) \simeq (\ast) \tilde{m}^{2n} \simeq (\ast) \Phi^{2n} . \]  \hspace{1cm} (77)

These expressions for \( \Phi \) and \( \omega(\Phi) \) illuminate nicely the salient features of the relation between the LQC–inspired models and the corresponding generalised Brans – Dicke theories. When the function \( f \) reaches its maximum, equivalently the scale factor \( a \) reaches its minimum, the field \( \Phi \) may reach a finite value or may vanish; and the function \( \omega(\Phi) \) will have a zero of odd or even order respectively. The positive integer \( n \), which indicates how flat the maximum is, corresponds to the order of the zero of \( \omega : \omega(\Phi) \propto (\Phi - \Phi_b)^{2n-1} \) has a zero of odd order if \( \Phi \) reaches a finite value at the bounce; and \( \omega(\Phi) \propto \Phi^{2n} \) has a zero of even order if \( \Phi \) vanishes at the bounce.

Also, it follows that \( \tilde{m}(t) \simeq (\ast) (t - t_b) \) where \( t_b \) is the time when \( m = m_b \), see the second equation in (57). Now, as \( m \) approaches \( m_b \) from below and evolves past it, \( \tilde{m} \) crosses zero and becomes negative. The corresponding evolution in case (i) is straightforward. Consider case (ii). The scalar field \( \phi \), being \( \propto (\ln \tilde{m}) \), is not well defined when \( \tilde{m} \) becomes negative. Another copy of \( \phi \) and the function \( \Psi \) seems to be needed to dictate further evolution. However, the Brans – Dicke field \( \Phi \propto \tilde{m} \) and can smoothly cross zero and become negative. Further evolution will depend on the function \( \omega \). It can be seen from the action \( S_{bd} \) given in equation (32) that, during this crossing, the kinetic terms for both the metric and the scalar field change signs. The significance of these changes of signs is not clear to us, perhaps they are unphysical. However, the equations of motion (41) and (43) – (47) remain the same, and we have verified that the corresponding solutions continue smoothly across \( t_b \) where \( \Phi(t) \) crosses zero and \( \omega(\Phi) \propto \Phi^{2n} \).

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Example (4) : \( f(m) = m \) near 0 ; \( f(2) \) or \( f(3) \) near \( m_r \)

Consider the example where \( f(m) = m \) near \( m = 0 \); remains positive with all its derivatives finite in the interval \( 0 < m < m_r \); and is given by \( f(2) \propto (m_r - m)^q \) as in Example (2), or by \( f(3) \propto (1 - (\ast) (m_r - m)^{2n}) \) as in Example (3), in the limit \( m \to m_r \).

The evolution corresponding to such a function \( f \) is as follows, see [10]. As \( m \) increases from 0, the evolution is initially as in Einstein’s theory: the time \( t \) decreases from \( \infty \) and the scale factor \( a \) decreases from \( \infty \). The evolution proceeds smoothly until \( m \to m_r \). In the limit \( m \to m_r \) if the function \( f \to f(2) \to 0 \) then the scale factor \( a \to \infty \), the time \( t \to t_s \) or \( -\infty \) depending on whether \( 0 < 2q < 1 \) or \( 2q \geq 1 \), and the evolution will be singular or non singular respectively. If the function \( f \to f(3) \) in the limit \( m \to m_r \) then the scale factor \( a \) will reach a minimum and will have a bounce. Further evolution requires specification of \( f(m) \) beyond \( m_r \) which will then be related to another copy of \( \phi \) and another function \( \Psi \). Choosing an \( f \) symmetric around its maximum, e.g. \( f(m) = \sin m \), will require the same \( \Psi \) and will lead to a symmetric bounce.

Consider now the function \( \Psi(\phi) \) which may correspond to the function \( f(m) \) of the present Example (4). We assume that this function \( \Psi(\phi) \) will be such that \( \Psi \to const \) as \( \phi \to \infty \); \( \Psi \to k c_1 \phi \) as \( \phi \to -\infty \); and all the derivatives of \( \Psi \) are finite for all \( \phi \). A simple model for such a function \( \Psi(\phi) \) is given by

\[
e^{-s\Psi} = c + b e^{-sk c_1 \phi} \quad \longleftrightarrow \quad e^{s\Psi} = \frac{e^{sk c_1 \phi}}{b + c e^{sk c_1 \phi}}
\]  

(78)

where \( s, k, b, \) and \( c \) are positive constants. Using

\[
\Psi_\phi = k c_1 \left(1 - c e^{s\Psi} \right) = \frac{k c_1 b}{b + c e^{sk c_1 \phi}}
\]

and equations (36), it follows that the Brans – Dicke field \( \Phi \) and the function \( \omega \) corresponding to the present model are given by

\[
\Phi^s = \frac{e^{(d-1)sk c_1 \phi}}{(b + c e^{sk c_1 \phi})^{d-1}}
\]

\[
\omega = \frac{(b + c e^{sk c_1 \phi})^2 - b^2 k^2 d^2}{d(d - 1) b^2 k^2}.
\]  

(79)
The scalar field $\phi$, and then the function $\omega$, can be expressed in terms of $\Phi$ but this is not necessary here. By construction, we have $\Psi \to k \, c_1 \phi$ as $\phi \to -\infty$. It therefore follows that if $k > \frac{1}{d}$ then, in the limit $\phi \to -\infty$, one obtains the function $f(m) \propto (m_r - m)^q$ in the limit $m \to m_r$ and with $q > 0$. Note that $q$ and $k$ are related by $q = \frac{k d - 1}{k (2d - 1) - 1}$ and, hence, that if $\frac{1}{d} < k < 1$ then $0 < 2q < 1$ and if $1 \leq k < \infty$ then $1 \leq 2q < \frac{2d}{d-1}$.

On the other hand, the choice $k = \frac{1}{d}$ and $s = 2n(d-1)$ will give the function $f(m)$ as in Example (3). With $k = \frac{1}{d}$ and in the limit $\phi \to -\infty$, one has

$$
\Phi \simeq (\ast) \, e^{(d-1)k \, c_1 \phi} , \quad \omega \simeq (\ast) \, e^{sk \, c_1 \phi} \simeq (\ast) \, \Phi^{\frac{1}{d-1}} .
$$

Choosing $s = 2n(d-1)$ will now give a $\omega(\Phi)$ having a zero of order $2n$ at $\Phi = 0$ and, hence, will lead to a function $f(m)$ which has a maximum of the type considered in Example (3). This can be seen more explicitly also. Note that equation (60) gives

$$
\left( \frac{f}{f_0} \right)^s = \left( \frac{a}{a_0} \right)^{-sd} = \left( \frac{b + c}{b + c \, e^{sk \, c_1 \phi}} \right)^d e^{(kd-1)s \, c_1 \phi} .
$$

(80)

We are not able to do the integrations needed to obtain $t(\phi)$ explicitly. Hence, consider the limit $\phi \to -\infty$ where $\Psi \to k \, c_1 \phi$. For $k = \frac{1}{d}$, equations (57) and (61) now give

$$(m_r - m) \simeq (\ast) \, (t - t_r) \simeq (\ast) \, e^{\frac{d}{d-1} \, c_1 \phi}$$

where $t_r$ is a constant. If $s = 2n \, (d-1)$ then $e^{sk \, c_1 \phi} \propto (t - t_r)^{2n}$ and equations (80) give $f(m)$ and $a(t)$ as in Example (3):

$$
f(m) \propto \left( 1 - (\ast) \, (m_r - m)^{2n} \right) , \quad a(t) \propto \left( 1 + (\ast) \, (t - t_r)^{2n} \right) .
$$

Further evolution beyond $m_r$ requires specification of $f(m)$ beyond $m_r$. For example, let the function be symmetric around $m_r$, namely let $f(m) = f(2m_r - m)$ for $m_r \leq m \leq 2m_r$. It is then easy to see that as $m$ varies from $m_r$ to $2m_r$, the evolution will be described by the same function $\Psi(\phi)$, now with $\phi$ varying from $-\infty$ to $+\infty$.

Consider now the LQC function $f(m) = \sin m$ which is symmetric around $\pi$ and for which $n = 1$. Let $\Psi_{lqc}(\phi)$ be the corresponding function in
the generalised Brans–Dicke theory. We are not able to obtain $\Psi_{\text{lqc}}(\phi)$ in an explicit form. However, it follows from Examples (1) and (3) that $\phi \to +\infty$ and $\Psi_{\text{lqc}}(\phi) \to \text{const}$ in the limit $m \to 0$; that $\phi \to -\infty$ and $\Psi_{\text{lqc}}(\phi) \to \frac{\epsilon_m}{d}$ in the limit $m \to \frac{\pi}{2}$ from below; and that the field $\Phi \propto (\frac{\pi}{2} - m)$ and the function $\omega(\Phi) \propto \Phi^2$ in the limit $m \to \frac{\pi}{2}$. Furthermore, as $m$ varies from $\frac{\pi}{2}$ to $\pi$, it also follows that the evolution will be described by the same function $\Psi_{\text{lqc}}(\phi)$, now with $\phi$ varying from $-\infty$ to $+\infty$.

6. Anisotropic case

Consider the example of the LQC–inspired models where $f(m) = m$ near $m = 0$; remains positive with all its derivatives finite in the interval $0 < m < m_r$; and $f(m) \propto (m_r - m)^q$ in the limit $m \to m_r$. Consider the anisotropic case. The line element $ds$ is now given by equation (1) and the equations of motion by (11) – (14) with $\dot{\rho} = \ddot{p}_i = (\sigma_i)^2$. In a recent work [10], we have analysed the cosmological evolution in such LQC–inspired models and have shown that if $2q > 1$ then the corresponding anisotropic evolution is non singular. The analysis is straightforward but involved and, hence, will not be presented here.

The function $\Psi(\phi)$ corresponding to such an $f(m)$ is given in Example (4) in the isotropic case. If the LQC–inspired model for a given function $f(m)$ and the generalised Brans–Dicke theory with the corresponding function $\Psi(\phi)$ are equivalent to each other then this equivalence may be obtained by studying the isotropic case, and it should be applicable for the anisotropic case also. It turns out that this is not the case.

To see the absence of this equivalence in the anisotropic case, consider now the generalised Brans–Dicke theories in the limit $\phi \to -\infty$. In this limit, let $\Psi \to \tilde{k} (\kappa\phi)$ where $\tilde{k} > 0$ is a constant. The solutions for time $t$ and the scale factors $a_i$ can now be obtained in this limit: Equations (53) give the solutions for the scale factors $A_i$ and the field $\phi$ in Einstein frame; equations (56) then give the time $t$ and the scale factors $a_i$. We rewrite these expressions below for ease of reference:

$$A_i = A_{i0} \left( L_{T0} \tilde{T} \right)^{\alpha_i}, \quad e^{\kappa\phi} = \left( L_{T0} \tilde{T} \right)^{\beta}$$
\[ dt = e^{-\Psi} \, dT , \quad a_i = e^{-\Psi} \, A_i \]  

(81)

where \( \sum_i \alpha^i = \sum_i (\alpha^i)^2 + \beta^2 = 1 \) and \( \bar{T} = T - T_0 + \frac{1}{e^T_0} \). In the following, we take \( \beta > 0 \) with no loss of generality. This implies, since \( \beta \) is non vanishing, that \( |\alpha^i| < 1 \).

Consider now the limit \( \bar{T} \to 0 \) where \( \phi \to -\infty \). In this limit, let \( \Psi \to \tilde{k} (\kappa \phi) \) where \( \tilde{k} > 0 \) is a constant. Then \( e^\Psi \propto \bar{T}^{\tilde{k} \beta} \to 0 \) and equations (81) give

\[ t \simeq (\ast) \frac{\bar{T}^{1-\tilde{k} \beta}}{1-\tilde{k} \beta} , \quad a_i \simeq (\ast) \bar{T}^{\alpha^i-\tilde{k} \beta} \]  

(82)

if \( \tilde{k} \beta \neq 1 \), and \( t \simeq (ln \bar{T}) \) if \( \tilde{k} \beta = 1 \). Since \( |\alpha^i| < 1 \), it now follows that if \( \tilde{k} \beta > 1 \) then \( (\alpha^i - \tilde{k} \beta) < 0 \) for all \( i \) and, therefore, the evolution is non singular because all the scale factors \( a^i \to \infty \) and the time \( t \to -\infty \). Also, it follows from equations (36) that \( \omega \) is given in this limit by

\[ \omega = \frac{1}{(d-1)^2 \kappa^2} - \frac{d}{d-1} . \]  

(83)

Note that \( \beta \) is assumed to be positive and non vanishing, but it can be very small : \( 0 < \beta \ll 1 \). This means that if the anisotropic evolution must be non singular for all such values of \( \beta \) also then \( \tilde{k} \beta > 1 \) for all \( \beta \ll 1 \) also and, hence, it follows that \( \tilde{k} \gg 1 \). In turn, this implies that the corresponding \( \omega \) must be close to \(-\frac{d}{\delta-1} \). Conversely, if \( \tilde{k} \) is large but finite then the anisotropic evolution of \( a^i(t) \) will not always be non singular. There will be set of values \( \{\alpha^i\} \) of non zero measure for which \( \beta = \sqrt{1-\sum_i (\alpha^i)^2} < \frac{1}{\tilde{k}} \) and, hence, the corresponding anisotropic evolution will be singular.

In the isotropic case, \( \omega = -\frac{d}{\delta-1} \) corresponds to the function \( f(m) \propto (m_s - m)^q \) where \( q = \frac{d}{2d-1} \). However, in the LQC–inspired models, a non singular anisotropic evolution is possible for any value of \( q > \frac{1}{2} \), see [10]. It therefore follows that the relations between the functions \( \Psi \) and \( f \) studied here apply only to the homogeneous isotropic cases, and not to the anisotropic ones.

In this context, note that a similar situation occurs also in the case of \( F(R) \) theories which are constructed to give the isotropic LQC evolution. As shown in [19], to be able to describe the anisotropic evolution also, one needs to generalise \( F(R) \) theories to \( F(R, Q) \) theories where \( Q = R_{\mu \nu} R^{\mu \nu} \). Perhaps
then it is not surprising that the generalised Brans – Dicke theories, which are constructed here to give the isotropic evolution of the LQC–inspired models, are not able to describe the anisotropic evolution also. Moreover, it is likely that some further generalisation is needed but the nature of such a generalisation is not clear to us.

7. Conclusion

We first give a brief summary. In this paper, we explore whether a subclass of scalar tensor theories can be the effective actions which will lead to the effective equations of motion of the LQC and the LQC–inspired models. We consider models where there is only one scalar field with no potential, and consider the generalised Brans – Dicke theories which contain only one scalar field, one coupling function, and no scalar field potential. Thus, a scalar field $\sigma$ and a function $f(m)$ of the LQC–inspired models need to be related to the field $\Phi$ and a function $\omega(\Phi)$ of the corresponding action.

We consider the isotropic case and find the relation between these two pairs. We cannot do explicit calculations for non trivial cases. Hence, we study a few explicit examples and some limiting cases and, using them, illustrate several important features of these relations. For example, we find that near the bounce of the LQC evolutions for which $f(m) = \sin m$, the corresponding field $\Phi \rightarrow 0$ and the function $\omega(\Phi) \propto \Phi^2$. Also, we find in this paper that the class of generalised Brans – Dicke theories, which was found in our earlier works to lead to non singular isotropic evolutions, may be written as an LQC–inspired model with an appropriate function $f(m)$.

We further find that the relations between the LQC-inspired models and the generalised Brans – Dicke theories do not apply to the anisotropic cases. A similar situation arises in LQC where also the anisotropic cases cannot be described by $F(R)$ theories, and a further generalisation to $F(R, R_{\mu\nu}R^{\mu\nu})$ theories is needed. Perhaps a further generalisation is needed here also, but the nature of such a generalisation is not clear to us.

We now conclude by mentioning a few topics for further studies. It is desirable to have an unique effective action which will give the effective equations of motion of the LQC and the LQC–inspired models. Such an action should give the equations of motion, for example, of the LQC
of Bianchi type I, II, IX models. We are not aware of a physical principle that guarantees the existence of such an action and, furthermore, its uniqueness. It seems miraculous in this context that even the effective equations of motion exist, and that they describe well the quantum effects of the LQC in various Bianchi type models.

The effective equations of motion themselves are very useful. They can be generalised empirically, as we have done recently to obtain the LQC–inspired models, and can be used efficiently as a laboratory to model and study a variety of cosmological evolutions – four or higher dimensional, isotropic or anisotropic, with or without compactifications, et cetera.

An effective action, if it exists and can be found, will be even more useful. It may be used, for example, to study spherical stars and their collapses, or to study how the perturbations of a homogeneous universe evolve when the universe undergoes a bounce. It is, of course, too much to expect that such an action will describe all the quantum effects of LQG itself in these new situations. Nevertheless, it may be expected to lead to new effects. We note here that the class of generalised Brans–Dicke theories, which was found in our earlier works to lead to non singular isotropic evolutions, also leads to interesting new effects when applied to stars [31, 26].

A class of theories, called mimetic gravity and degenerate higher order scalar tensor theories, have been proposed which contain higher order derivative terms but without the associated pathologies. Also, in [32], a class of mimetic theories has been constructed which resolve the cosmological singularities. These theories also lead to the effective isotropic equations of LQC, as shown in [33, 34]. See [35] also where mimetic theories are used to construct non singular black hole solutions. It is thus of interest to explore the connection, if any, between such theories and generalised Brans–Dicke theories considered here.

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