CANONICAL SELF-AFFINE TILINGS BY ITERATED FUNCTION SYSTEMS.

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Abstract. An iterated function system $\Phi$ consisting of contractive affine mappings has a unique attractor $F \subseteq \mathbb{R}^d$ which is invariant under the action of the system, as was shown by Hutchinson [Hut]. This paper shows how the action of the function system naturally produces a tiling $T$ of the convex hull of the attractor. These tiles form a collection of sets whose geometry is typically much simpler than that of $F$, yet retains key information about both $F$ and $\Phi$. In particular, the tiles encode all the scaling data of $\Phi$. We give the construction, along with some examples and applications. The tiling $T$ is the foundation for the higher-dimensional extension of the theory of complex dimensions which was developed in [La-vF1] for the case $d = 1$.

1. Introduction

This paper presents the construction of a self-affine tiling which is canonically associated to a given self-affine system $\Phi$ (as defined in Def. 3.1). The term “self-affine tiling” is used here in a sense quite different from the one often encountered in the literature. In particular, the region being tiled is the complement of the self-affine set $F$ within its convex hull, rather than all of $\mathbb{R}^d$. Moreover, the tiles themselves are neither self-affine nor are they all of the same size; in fact, the tiles may even be simple polyhedra. However, the name “self-affine tiling” is appropriate because we will have a tiling of the convex hull: the union of the closures of the tiles is the entire convex hull, and the interiors of the tiles intersect neither each other, nor the attractor $F$. While the tiles themselves are not self-affine, the overall structure of the tiling is.

The construction of the tiling begins with the definition of the generators, a collection of open sets obtained from the convex hull of $F$. The rest of the tiles will be seen to be images of these generators under the action of the original self-affine system. Thus, the tiling $T$ essentially arises as a spray on the generators, in the sense of [LaPo] and [La-vF1]. The tiles thus obtained form a collection of sets whose geometry is typically much simpler than that of $F$, yet retains key information about both $F$ and $\Phi$. In particular, the tiles encode all the scaling data of $\Phi$.

Section 2 describes the intended context of the present results, and indicates how they may be used to develop a tube formula for the tiling. Section 3 gives the...
tiling construction. Section 4 illustrates the method with several familiar examples, including the Koch snowflake curve, Sierpinski gasket and pentagasket. Section 5 describes the basic properties of the tiling and contains the main results of the paper.

2. Motivation and primary application

The motivation behind the self-affine tiling is to find a means of extending the work of [La-vF1] to higher dimensions. Preliminary investigations for this project began with [LaPe1], and the present paper, together with [LaPe2], has made significant inroads. An outline of these results is given in the survey paper [LaPe3]. The self-affine tiling provides a natural way to define the geometric zeta function of a self-affine subset of general Euclidean space $\mathbb{R}^d$, and thus obtain the complex dimensions of such a set. These terms require some discussion.

The research monograph [La-vF1] is an investigation of the theory of fractal subsets of $\mathbb{R}$. The complement of a fractal within the interval containing it is called a fractal string and may be represented by a sequence of bounded open intervals $L = \{L_n\}_{n=1}^{\infty}$, where the interval $L_n$ has length $\ell_n$. For technical reason, the fractal string may be considered as

$$\mathcal{L} := \{\ell_n\}_{n=1}^{\infty}, \quad \text{with} \quad \sum_{n=1}^{\infty} \ell_n < \infty. \quad (2.1)$$

The authors are able to relate geometric and spectral properties of such objects through the use of zeta functions which encode this data. The most important such function is the geometric zeta function $\zeta_{\mathcal{L}}(s) = \sum_{n=1}^{\infty} \ell_n^s$, and the complex dimensions are defined to be the poles of this function. Prior to the construction of the self-affine tiling, it was not known how to define these objects in the higher-dimensional case, that is, to fractal subsets of $\mathbb{R}^d$ (for $d > 1$).

The poles of the geometric zeta function contain much geometric information about the underlying fractal, including the dimension and measurability of the fractal under consideration. Another main result of [La-vF1] is an explicit tube formula, that is, a formula for the volume $V_{\mathcal{L}}(\varepsilon)$ of the $\varepsilon$-neighbourhood of a fractal string $\mathcal{L}$. This formula is roughly just the sum of the residues of the geometric zeta function at each of the complex dimensions.

The basis for extending these (and other) results to higher dimensional self-affine sets is a suitable higher-dimensional analogue of fractal strings: the self-affine tiling developed in this paper. In [LaPe2], we show how a self-affine tiling $T$ allows one to define a geometric zeta function for self-affine subsets of $\mathbb{R}^d$. Further, we compute an explicit inner tube formula for $V_T(\varepsilon)$ analogous to [La-vF1, Thm. 8.1], using tools from geometric measure theory and convexity theory. That is, for $A \subseteq \mathbb{R}^d$, we obtain an explicit expression for

$$V_A(\varepsilon) := \text{vol}_d \{x \in A \mid d(x, \partial A) < \varepsilon\}, \quad (2.2)$$

where $\text{vol}_d$ is $d$-dimensional Lebesgue measure. Each tile in the self-affine tiling contributes much data to the final formula, including curvature information for each (topological) dimension $0, 1, \ldots, d$, of the tile. Thus, the present construction is used in an essential manner in [LaPe2] to obtain the following result.
Theorem 2.1. The $d$-dimensional volume of the inner tubular neighbourhood of a tiling $T$ is given by the following distributional explicit formula:

$$V_T(\varepsilon) = \sum_{\omega \in \mathcal{D}_T} \text{res}(\zeta_T(\varepsilon, s); \omega) = \sum_{\omega \in \mathcal{D}_T} c_\omega \varepsilon^{d-\omega}. \quad (2.3)$$

In this formula, $\zeta_T$ is the geometric zeta function of the self-affine tiling whose residues define the coefficients $c_\omega$. The meromorphic distribution-valued function $\zeta_T$ is defined by the combinatorics of the scaling ratios of $\Phi$, and various geometric properties of the generators. In particular, $\zeta_T$ incorporates the 0-dimensional through $d$-dimensional curvatures of the generators of the self-affine tiling. The sum in (2.3) is taken over the set of complex dimensions $\mathcal{D}_T$, that is, the set of poles of $\zeta_T$. Further discussion of these topics, however, is beyond the scope of the current paper; please see [LaPe2].

Additionally, under certain conditions, Theorem 2.1 can also be used to obtain the volume of the exterior tubular neighbourhood of a fractal set, that is, an explicit expression for

$$V_F(\varepsilon) := \text{vol}_d \left\{ x \in F \mid d(x, \partial F) < \varepsilon \right\}. \quad (2.4)$$

(The key point here is that $V_F$ is valid for the attractor $F$, not the tiling as in the previous theorem.) The precise conditions under which this may be accomplished are given in [PeWi]. The forthcoming survey paper [LaPe3] gives a detailed discussion of the role of the present construction in applications related to tube formulas.

3. The self-affine tiling

3.1. Basic terms.

Definition 3.1. A self-affine system is a family $\Phi := \{\Phi_j\}_{j=1}^J$ (with $2 \leq J < \infty$) of affine mappings whose eigenvalues $\lambda$ all satisfy $0 < \lambda < 1$. A self-similar system is the special case where each mapping is a similitude, i.e.,

$$\Phi_j(x) := r_j A_j x + a_j,$$

where for $j = 1, \ldots, J$, we have $0 < r_j < 1$, $a_j \in \mathbb{R}^d$, and $A_j \in O(d)$, the orthogonal group of $d$-dimensional Euclidean space $\mathbb{R}^d$. The numbers $r_j$ are referred to as the scaling ratios of $\Phi$. Thus, a similarity is a composition of an (affine) isometry and a homothety (scaling). All the examples presented here are self-similar, with the exception of the harmonic gasket, Example 4.3.

Remark 3.2. Different self-affine systems may give rise to the same self-affine set. In this paper, the emphasis is placed on the self-affine system and its corresponding dynamical system, rather than on the self-affine set.

Definition 3.3. A self-affine system is thus just a particular type of iterated function system (IFS). It is well known\(^2\) that for such a family of maps, there is a unique and self-affine set $F$ satisfying the fixed-point equation

$$F = \Phi(F) := \bigcup_{j=1}^J \Phi_j(F). \quad (3.1)$$

\(^2\)See [Hut], as described in [Fal] or [Kig], for example.
We call \( F \) the attractor of \( \Phi \), or the self-affine set associated with \( \Phi \). The action of \( \Phi \) is the set map defined by (3.1). Thus, one says that \( F \) is invariant under the action of \( \Phi \).

**Definition 3.4.** We fix some notation for later use. Let
\[
C := [F] \tag{3.2}
\]
be the convex hull of the attractor \( F \), that is, the collection of all convex combinations of points in \( F \). (Equivalently, \([F]\) is the intersection of all convex sets containing \( F \).) Since \( F \) is a compact set, it follows that \( C \) is also compact, by [Sch, Thm. 1.1.10]. Further, let
\[
C^\circ := \text{int}(C) = C \setminus \partial C. \tag{3.3}
\]
Here, \( \partial A := \overline{A} \cap \overline{A}^c \), where \( A^c \) is the complement of \( A \) and \( \overline{A} \) denotes the (topological) closure of \( A \).

**Remark 3.5.** For this paper, it will suffice to work with the ambient dimension
\[
d = \dim C, \tag{3.4}
\]
restricting the maps \( \Phi_j \) as appropriate. In (3.4), \( \dim C \) is defined to be the usual topological dimension of the smallest affine space containing \( C \). An appropriate change of coordinates allows one to think of this convention as using a minimal space \( \mathbb{R}^d \) in which to embed \( F \); e.g., if \( F \) is a Cantor set in \( \mathbb{R}^3 \), we study it as if the ambient space were the line containing it, rather than \( \mathbb{R}^3 \). Note that this means \( C^\circ \) is open in the standard topology; and so we have \( C^\circ \neq \emptyset \). This remark is intended to allay any fears about possibly needing to use relative interior instead of interior (see [KIro] or [Sch]) and other unnecessary complications.

**Definition 3.6.** A self-affine system satisfies the tileset condition iff for \( j \neq \ell \),
\[
\text{int } \Phi_j(C) \cap \text{int } \Phi_\ell(C) = \emptyset. \tag{3.5}
\]
It is shown in Cor. 5.10 that because \( C = \text{int} C \), (3.5) implies that the images \( \Phi_j(C) \) and \( \Phi_\ell(C) \) can intersect only on their boundaries:
\[
\Phi_j(C) \cap \Phi_\ell(C) = \partial \Phi_j(C) \cap \partial \Phi_\ell(C).
\]
To avoid trivialities, we also require
\[
C^\circ \nsubseteq \Phi(C). \tag{3.6}
\]
The nontriviality condition (3.6) disallows the case \( C^\circ \setminus \Phi(C) = \emptyset \), and hence guarantees the existence of the tiles in \S \S 3.2. It is shown in [PeWi] that (3.6) fails iff \( F \) is convex.

**Remark 3.7.** The tileset condition is a restriction on the overlap of the images of the mappings, comparable to the open set condition (OSC). The OSC requires a nonempty bounded open set \( U \) such that the sets \( \Phi_j(U) \) are disjoint but \( \Phi(U) \subseteq U \). See, e.g., [Fal, Chap. 9]. It is clear from Cor. 5.3 of \S 5 (with \( k = 0 \) and \( U = \text{int} C \)) that the OSC follows from (3.5). However, the following example shows that the converse is false.
Example 3.8. Consider a system of three similarity mappings, each with scaling ratio $1/\sqrt{3}$ and a clockwise rotation of $\pi/2$. The mappings are illustrated in Figure 1 and form a system which satisfies the open set condition (simply take the interior of the attractor) but not the tileset condition. On the right, the attractor has been shaded for clarity; the dark overlay indicates the intersection of the convex hulls of the lower two images of the attractor.

Definition 3.9. Denote the words of length $k$ (of $\{1, 2, \ldots, J\}$) by

$$W_k = W^J_k : = \{w = w_1w_2\ldotsw_k \mid w_j \in \{1, 2, \ldots, J\}\}, \quad (3.7)$$

and the set of all (finite) words by $W := \bigcup_k W_k$. Generally, the dependence of $W_k^J$ on $J$ is suppressed. For $w$ as in (3.7), we use the standard IFS notation

$$\Phi_w(x) := \Phi_{w_1} \circ \Phi_{w_2} \circ \ldots \circ \Phi_{w_k}(x) \quad (3.8)$$

to describe compositions of maps from the self-affine system.

Definition 3.10. Let $A \subseteq \mathbb{R}^d$ be a set which is the closure of its interior. A tiling of $A$ is a collection of nonempty connected $d$-dimensional sets $\{A_n\}_{n=1}^{\infty}$ such that

(i) $A = \bigcup_{n=1}^{\infty} A_n$, and

(ii) $A_n \cap A_m \subseteq \partial A_n \cap \partial A_m$ for $n \neq m$.

We then say that the sets $A_n$ tile $A$. Further, define

Definition 3.11. A tiling of $A$ by open sets (or open tiling) is a collection of nonempty connected open sets $\{A_n\}_{n=1}^{\infty}$ such that

(i) $\overline{\bigcup_{n=1}^{\infty} A_n}$, and

(ii) $A_n \cap A_m = \emptyset$ for $n \neq m$.

Figure 2 shows an example of a tiling by open sets, each of which is an equilateral triangle.

3.2. The construction. In this section, we construct a self-affine tiling, that is, a tiling which is constructed via the action of a self-affine system (as defined in Def. 3.1). Such a tiling will consequently have a self-affine structure, and is defined precisely in Def. 3.15 below. The reader is invited to look ahead at Figure 3, where
Figure 2. Tiling the complement of the Koch curve $K$. The equilateral triangles form an open tiling of the convex hull $C = [K]$, in the sense of Def. 3.10.

the construction is illustrated step-by-step for the illuminative example of the Koch curve.

For the system $\{\Phi_j\}$ with attractor $F$, denote the convex hull of the attractor by

$$C_0 = C := [F].$$

(3.9)

Denote the image of $C$ under the action of $\Phi$ (in accordance with (3.1)) by

$$C_k := \Phi^k(C) = \bigcup_{w \in W_k} \Phi_w(C), \quad k = 1, 2, \ldots$$

(3.10)

Note that this is equivalent to the inductive definition

$$C_k := \Phi(C_{k-1}), \quad k = 1, 2, \ldots$$

(3.11)

Definition 3.12. The \textit{tilesets} are the sets

$$T_k := C_{k-1} \setminus C_k, \quad k = 1, 2, \ldots$$

(3.12)

Definition 3.13. The \textit{generators} $G_q$ are the connected components of the open set

$$\text{int}(C \setminus \Phi(C)) = G_1 \sqcup G_2 \sqcup \cdots \sqcup G_Q.$$ 

(3.13)

Remark 3.14. The symbol $\sqcup$ is used in place of $\cup$ to emphasize that a given union of sets is disjoint. This should not be confused with the operation of disjoint union, i.e., the coproduct in the category of sets.

As will be shown in Theorem 5.11, it follows from the tileset conditions (3.5)–(3.6) (and some other facts) that the tilesets and tiles are nonempty, and that each tileset is the closure of its interior. Also, Theorem 5.15 will justify the terminology “generators” by showing

$$T_k = \bigcup_{q=1}^Q \Phi^{k-1}(G_q),$$

(3.14)

that is, that any difference $C_{k-1} \setminus C_k$ is (modulo some boundary points) the image of the generators under the action of $\Phi$. The number $Q$ of generators depends on the specific geometry of $C$ and on the self-affine system $\Phi$. It is conceivable that $Q = \infty$ for some systems $\Phi$, but no such examples are known. This possibility will be investigated further in [PeWi].

Definition 3.15. The self-affine tiling of $F$ is

$$\mathcal{T} := \left( \{\Phi_j\}_{j=1}^J, \{G_q\}_{q=1}^Q \right).$$

(3.15)
Figure 3. The left column shows images of the convex hull $C$ under successive applications of $\Phi$. The right column shows how the components of the $T_k$ tile the complement; they are overlaid in Figure 2. This tiling has one generator $G_1 = \text{int} T_1$.

We may also abuse the notation a little, and use $\mathcal{T}$ to denote the set of corresponding tiles:

$$\mathcal{T} = \{ R_n \}_{n=1}^{\infty} = \{ \Phi_w(G_q) \mid w \in W, q = 1, \ldots, Q \},$$

where the sequence $\{ R_n \}$ is an enumeration of the tiles. Clearly, each tile is nonempty and $d$-dimensional. Furthermore, Theorem 5.16 will confirm that (3.16) is a tiling by open sets, as in Def. 3.10.
4. Examples

All the examples discussed in this section have polyhedral generators, but this is not the general case. In fact, it is possible to have generators with boundary that is continuously differentiable, although it is not possible that they be twice continuously differentiable. This was observed to be true for the convex hull of an attractor in [StWa], and it immediately carries over to the generators as well. We will study this eventuality further in [PeWi] and [LaPeWi2]. See also Remark 5.18.

4.1. The Koch curve. Figure 2 shows the self-affine tiling of the Koch curve; the steps of the construction are illustrated in Figure 3. The tiling is \( \mathcal{K} = (\{\Phi_j\}_{j=1}^2, \{G\}) \), and it is easiest to write down the maps as \( \Phi_j : \mathbb{C} \to \mathbb{C} \), with the natural identification of \( \mathbb{C} \) and \( \mathbb{R}^2 \). In this case, \( \Phi_1(z) := \xi z \) and \( \Phi_2(z) := (1 - \xi)(z - 1) + 1 \). Figure 3 depicts the case when \( \xi = (1 + \sqrt{-1/3})/2 \), so that \( r_1 = r_2 = 1/\sqrt{3} \) and the single generator \( G \) is the equilateral triangle of side length \( \frac{2}{3} \).

In general, \( \xi \) may be any complex number satisfying \( |\xi|^2 + |1 - \xi|^2 < 1 \), i.e., lying within the circle of radius 1/2 centered at \( z = 1/2 \). Basic geometric considerations show that this inequality must be satisfied in order for the tileset condition (3.5) to be met. Any member of this family will have one isosceles triangle \( G = G_1 = T_1 \) for a generator. A key point of interest in this example is that, in the language of [La-vF1], curves from this family will generically be nonlattice, i.e., the logarithms of the scaling ratios will not be rationally dependent. Figure 3 thus shows a very exceptional case.

4.2. The Sierpinski gasket. The Sierpinski gasket system consists of the three maps \( \Phi_j(x) = (x + p_j)/2 \), where the \( p_j \) are the vertices of an equilateral triangle; the standard example is \( p_1 = (0,0), p_2 = (1,0) \), and \( p_3 = (\frac{1}{2}, \frac{\sqrt{3}}{2}) \). The convex hull of the gasket is the triangle with vertices \( p_1, p_2, p_3 \). The generator \( G \) is the ‘middle fourth’ of the hull (see \( T_1 \) in Figure 4).

4.3. The harmonic Sierpinski gasket. Recent studies of PDE on post-critically finite fractals (following the analytic methods of [Kig]) have led to interest in the Sierpinski gasket as it is embedded into \( \mathbb{R}^2 \) via harmonic coordinates. The resulting figure is a self-affine homeomorphic image of the usual gasket which is not self-similar. The eigenvalues of the affine maps are \( \frac{3}{5}, \frac{1}{5} \). See Figure 5.

Figure 4. Self-similar tiling of the Sierpinski gasket.
4.4. **The pentagasket.** The pentagasket is constructed via five maps \( \Phi_j(x) = \phi^{-2}x + p_j \), where the \( p_j \) form the vertices of a pentagon of side length 1, and \( \phi = \frac{1 + \sqrt{5}}{2} \) is the golden ratio, so that the scaling ratio of each mapping is

\[
r_j = \phi^{-2} = \frac{3 - \sqrt{5}}{2}, \quad j = 1, \ldots, 5.
\]

The pentagasket is a self-similar (not just self-affine) example of multiple generators \( G_q \) with \( q = 1, \ldots, 6 \). In fact, \( T_1 = G_1 \cup \cdots \cup G_6 \) where \( G_1 \) is a pentagon and \( G_2, \ldots, G_6 \) are triangles. See Figure 6.

4.5. **The Sierpinski carpet.** The Sierpinski carpet is constructed via eight maps \( \Phi_j(x) = \frac{x + p_j}{3} \), where \( p_j = (a_j, b_j) \) for \( a_j, b_j \in \{0, 1, 2\} \), excluding the single case \((1,1)\). The Sierpinski carpet is an example which is not finitely ramified; indeed, it is not even post-critically finite (see [Kig]). See Figure 7.

4.6. **The Menger sponge.** The Menger sponge is constructed via twenty maps \( \Phi_j(x) = \frac{x + p_j}{3} \), where \( p_j = (a_j, b_j, c_j) \) for \( a_j, b_j, c_j \in \{0, 1, 2\} \), except for the six cases when exactly two coordinate are 1, and the single case when all three coordinates are 1. The Menger sponge system provides an example with an generator of dimension greater than 2, and is also an example with a nonconvex generator. See Figure 8.
5. Properties of the tiling

The results of this section indicate that a self-affine tiling may be constructed for any self-affine system satisfying the tileset condition (and nontriviality condition) of Def. 3.6. Throughout, we will use the fact that $\overline{A} = \text{int } A \cup \partial A$, where we denote the closure of $A$ by $\overline{A}$, the interior of $A$ by $\text{int } A$, and the boundary of $A$ by $\partial A = \overline{A} \cap \overline{A^c}$, where $A^c = \mathbb{R}^d \setminus A$. Recall from Rem. 3.14 that $\cup$ indicates a union of disjoint sets.

The first part of this section, Thm. 5.1–Cor. 5.5, establishes the nested structure of the attractor and the images of the hull $C$, $C_{k+1} \subseteq C_k$, and $\bigcap C_k = F$.

These results are reminiscent of [Hut, 5.2(3)], but are developed in terms of the convex hull of the attractor, rather than a set satisfying the open set condition.

**Theorem 5.1.** For each $k \in \mathbb{N}$, one has $C_{k+1} \subseteq C_k \subseteq C$.

*Proof.* Any point $x \in C$ is a convex combination of points in $F$. Since affine transformations preserve convexity, $\Phi_j(x)$ will be a convex combination of points in $\Phi_j(F) \subseteq F$. Hence $\Phi_j(C) \subseteq [F] = C$ for each $j$, so $\Phi(C) \subseteq C$. By iteration of this argument, we immediately have $\Phi^k(C) \subseteq C$ for any $k \in \mathbb{N}$. From (3.11), it is
clear that
\[ C_{k+1} = \Phi(C_k) = \Phi^{k+1}(C) = \Phi^k(\Phi(C)) \subseteq \Phi^k(C) = C_k, \quad (5.1) \]
where the inclusion follows by \( \Phi(C) \subseteq C \), as established initially. 

\[ \square \]

**Corollary 5.2.** The tileset condition is preserved under the action of \( \Phi \), i.e.,
\[ \text{int } \Phi_j(C_k) \cap \text{int } \Phi_\ell(C_k) = \emptyset, \quad j \neq \ell, \quad \forall k \in \mathbb{N}. \quad (5.2) \]

**Proof.** From Theorem 5.1 we have \( \text{int } \Phi_j(C_k) \subseteq \text{int } \Phi_j(C_k) \), and similarly for \( \Phi_\ell \). Then (5.2) follows immediately from the tileset condition (3.5). \( \square \)

**Corollary 5.3.** For \( A \subseteq C_k \), we have \( \Phi_w(A) \subseteq C_k \), for all \( w \in \mathbb{W} \). In particular, \( F \subseteq C_k, \forall k \).

**Proof.** By iteration of (5.1), it is immediate that \( C_m \subseteq C_k \) for any \( m \geq k \). Since \( \Phi(A) \subseteq \Phi(C_k) = C_{k+1} \subseteq C_k \) by Theorem 5.1, the first conclusion follows. The special case follows by induction on \( k \) with basis case \( A = F \subseteq C = C_0 \). The inductive step is
\[ F \subseteq C_k \implies F = \Phi(F) \subseteq \Phi(C_k) = C_{k+1}. \quad \square \]

**Remark 5.4.** Cor. 5.3 is a modified form of [Hut, 3.1(8)].

**Corollary 5.5.** The decreasing sequence of sets \( \{C_k\} \) converges to \( F \).

**Proof.** Cor. 5.3 shows \( F \subseteq C_k \) for every \( k \), so it is clear that \( F \subseteq \bigcap C_k \). For the reverse inclusion, suppose \( x \notin F \), so that \( x \) must be some positive distance \( \varepsilon \) from \( F \). Let \( \lambda \) be the largest eigenvalue of the maps \( \{\Phi_j\} \), and recall that \( 0 < \lambda < 1 \). For \( w \in \mathbb{W}_k \), we have \( \text{diam} (\Phi_w(C)) \leq \lambda^k \text{diam}(C) \), which clearly tends to 0 as \( k \to \infty \). Therefore, we can find \( k \) for which all points of \( C_k = \Phi^k(C) \) lie within \( \varepsilon/2 \) of \( F \). Thus \( x \) cannot lie in \( C_k \) and hence \( x \notin \bigcap C_k \). \( \square \)

**Remark 5.6.** The convergence \( C_k \to F \) also holds in the sense of Hausdorff metric, by [Hut, 3.2(1)]; see also [Fal] or [Kig] for a nice discussion. Hutchinson showed that \( \Phi \) is a contraction mapping on the metric space of compact subsets of \( \mathbb{R}^d \), which is complete when endowed with the Hausdorff metric. (The hull \( C \) is shown to be compact in Cor. 5.9.) An application of the contraction mapping principle then shows that \( \Phi \) has a unique attracting fixed point (as stated in Def. 3.3). This phenomenon is apparent in several of the figures. The main observation in the current situation is that \( C_k \) decreases to \( F \), by the nestedness described in Theorem 5.1.

We now use the conditions on the mapping system \( \Phi \) to obtain some technical nondegeneracy results in Lemma 5.7–Cor. 5.12. There are two main ideas:

1. All the sets we work with \( (C_k, T_k, \text{etc.}) \) are nondegenerate in the sense of being the closure of their interior.
2. The action of \( \Phi \) preserves several key properties, including closure, the tileset conditions, and the nondegeneracy condition mentioned just above.

In the latter part of this section, Thm. 5.14–Cor. 5.17, these allow us to connect properties of the hull differences \( C_k \setminus C_{k+1} \) to properties of the tilesets \( T_k \). Heuristically, we are leveraging the relation
\[ C_k \setminus C_{k+1} \approx \Phi^k(G_1 \cup \cdots \cup G_q) \]
to show that the sets $\Phi_n(G_q)$ give a tiling of $C$ by open sets, i.e., the construction can always be carried out for an iterated function system satisfying the tileset condition. The displayed equation above is only approximate because the sets may differ at the boundary; unfortunately, this necessitates several technicalities in the proofs of the main results.

**Lemma 5.7.** The action of $\Phi$ commutes with set closure, i.e., $\Phi(\overline{A}) = \overline{\Phi(A)}$

**Proof.** It is well known that closure commutes with finite unions, i.e., for any sets $A, B$, one has $\overline{A \cup B} = \overline{A} \cup \overline{B}$. See, e.g., [Mu, Chap. 2, §17]. Also, each $\Phi_j$ is a homeomorphism, and is thus a closed, continuous map. Therefore,

$$\Phi(\overline{A}) = \bigcup_{j=1}^{J'} \Phi_j(\overline{A}) = \overline{\bigcup_{j=1}^{J'} \Phi_j(A)} = \overline{\bigcup_{j=1}^{J'} \Phi_j(A)} = \overline{\Phi(A)}.$$  \hfill \Box

**Theorem 5.8.** If $A$ is the closure of its interior, then so is $\Phi(A)$.

**Proof.** Since $\Phi_j$ is a homeomorphism, this is a simple exercise in basic topology. \hfill \Box

**Corollary 5.9.** Each set $C_k$ is the closure of its interior.

**Proof.** The set $C = [F]$ is convex by definition, and compact by [Sch, Thm. 1.1.10]. Therefore, $C$ is the closure of its interior by [Sch, Thm. 1.1.14]. The conclusion follows by iteration of Theorem 5.8. \hfill \Box

**Corollary 5.10.** The tileset condition implies that images of the hull can only overlap on their boundaries:

$$\Phi_j(C) \cap \Phi_\ell(C) = \partial \Phi_j(C) \cap \partial \Phi_\ell(C), \quad \text{for } j \neq \ell. \quad (5.3)$$

**Proof.** Let $x \in \Phi_j(C) \cap \Phi_\ell(C)$. Suppose, by way of contradiction, that $x \in \text{int} \Phi_j(C)$. Then we can find an open neighbourhood $U$ of $x$ which is contained in $\text{int} \Phi_j(C)$. Since $x \in \Phi_\ell(C)$, there must be some $z \in U \cap \text{int} \Phi_\ell(C)$, by Cor. 5.9. But then $z \in \text{int} \Phi_j(C) \cap \text{int} \Phi_\ell(C)$, in contradiction to the tileset condition. For the reverse inclusion, note that $\partial A \cap \partial B \subseteq A \cap B$ whenever $A, B$ are closed sets. \hfill \Box

**Theorem 5.11** (Nondegeneracy of tilesets). Each tileset is the closure of its interior.

**Proof.** We need only show $T_k \subseteq \overline{\text{int} T_k}$, since the reverse containment is clear by the closedness of $T_k$. Since $\overline{A} = \text{int} A \cup \partial A$, take $x \in \text{int}(C_{k-1} \setminus C_k)$ to begin. Using Cor. 5.9, we have equality in the first step of the following derivation:

$$C_{k-1} \setminus C_k = \overline{\text{int} (C_{k-1} \setminus C_k)} \subseteq \overline{\text{int} (C_{k-1}) \setminus C_k} \subseteq \overline{\text{int} (C_{k-1} \setminus C_k)}, \quad (5.4)$$

because $\text{int} (C_{k-1}) \setminus C_k = \text{int} (C_{k-1}) \setminus C_k \subseteq \text{int} (C_{k-1} \setminus C_k)$.

Now consider the case when $x \in \partial (C_{k-1} \setminus C_k)$. Pick an open neighbourhood $U$ of $x$. By the same argument as above, choose $z \in U \cap (C_{k-1} \setminus C_k)$ to see that $x$ is a limit point (and hence an element) of $\overline{\text{int} T_k}$. \hfill \Box

The following two corollaries will be useful in the proof of Theorem 5.15.

**Corollary 5.12.** For $j = 1, \ldots, J$, $\overline{\Phi_j(C_{k-1}) \setminus \Phi_j(C_k)}$ is the closure of its interior.
Proof. Because each $\Phi_j$ is a homeomorphism, the set $\Phi_j(C_{k-1} \setminus C_k)$ will be the closure of its interior by Theorem 5.11. However, we have

$$\Phi_j(C_{k-1} \setminus C_k) = \Phi_j(C_{k-1} \setminus C_k) = \Phi_j(C_{k-1}) \setminus \Phi_j(C_k),$$

(5.5)
since $\Phi_j$ is closed and injective. \hfill $\square$

Corollary 5.13. It is always the case that one has $C \setminus \Phi(C) = \text{int}(C \setminus \Phi(C))$ and $\partial(C \setminus \Phi(C)) = \partial(\text{int}(C \setminus \Phi(C)))$.

Proof. For the first statement, $\text{int}(C \setminus \Phi(C)) \subseteq C \setminus \Phi(C)$ immediately shows one containment. For the other, choose $x \in C \setminus \Phi(C)$ so that there is a sequence $x_k \rightarrow x$ with $x_k \in C \setminus \Phi(C)$. Since $\Phi(C) = \Phi(C)$, we have $x_k \in \Phi(C) = \text{int}(\Phi(C))$. Further, since $x_j \in C = \text{int}(C)$, one can always find

$$y_k \in B(x_k, \frac{1}{10}) \cap \text{int}(C) \cap \text{int}(\Phi(C)).$$

Then we have a sequence $y_k \rightarrow x$, and since $\text{int}(\Phi(C)) = \Phi(C) = \Phi(C)$, one has

$$y_k \in \text{int}(C) \cap \text{int}(\Phi(C)) = (\text{int} C \setminus \Phi(C))$$

$$\subseteq \text{int}(C \setminus \Phi(C)).$$

Thus, $x \in \text{int}(C \setminus \Phi(C))$. To see the second statement holds, it suffices to show

$$(C \setminus \Phi(C))^c = \text{int}(C \setminus \Phi(C))^c.$$  However, this is clear by applying the identity

$$(\text{int} A = A) = \text{int}(\text{int}(C \setminus \Phi(C))) = \text{int}(C \setminus \Phi(C)).$$  \hfill $\square$

We are now ready to prove the main results of this paper. The reader may find Figure 2 and Figure 3 helpful, as illustrations of Theorems 5.14–5.17. The next result shows that each tileset is the image under $\Phi$ of its predecessor; speaking very roughly, the hulls $C_k$ form a sequence of “neighbourhoods” of the attractor $F$. In the sense of dynamical systems, the tiles describe the orbits of $\Phi$.

Theorem 5.14. Each tileset is the image under $\Phi$ of its predecessor, i.e.,

$$\Phi(T_k) = T_{k+1}, \quad \text{for} \ k \in \mathbb{N}. \quad (5.6)$$

Proof. One would like to say simply that

$$\Phi(T_k) = \Phi \left( C_{k-1} \setminus C_k \right) = \Phi(C_{k-1} \setminus C_k) = \Phi(C_{k-1}) \setminus \Phi(C_k) = C_k \setminus C_{k+1} = T_{k+1}.$$

Unfortunately, the central equality is not immediately justifiable. Using using Def. 3.12 and (5.5), we have the identities

$$\Phi(T_k) = \bigcup_{j=1}^j \Phi_j(C_{k-1} \setminus C_k), \quad \text{and} \quad (5.7)$$

$$T_{k+1} = C_k \setminus C_{k+1}. \quad (5.8)$$

(\subseteq) To see that (5.7) is a subset of (5.8), pick $x \in \Phi(T_k)$ and proceed by cases.

(i) For any $x \in \text{int}(\Phi_j(C_{k-1}) \setminus \Phi_j(C_k))$, it must be that $x \in \text{int} \Phi_j(C_{k-1}) \subseteq C_k$.

It suffices to show $x \in C_k \setminus C_{k+1}$, so by way of contradiction, suppose that $x \in C_{k+1}$. Then $x \in \Phi_\ell(C_k)$ for some $\ell \neq j$. Inasmuch as Theorem 5.1 gives $x \in \Phi_\ell(C_k)$, Cor. 5.10 implies $x \in \partial \Phi_j(C_{k-1}) \cap \partial \Phi_\ell(C_{k-1})$, contradicting the fact that $x \in \text{int} \Phi_j(C_{k-1})$. 

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(ii) Now consider \( x \in \partial(\Phi_j(C_{k-1}) \setminus \Phi_j(C_k)) \). Let \( U \) be an open neighbourhood of \( x \). By Cor. 5.12, we can find \( w \in U \cap \text{int}(\Phi_j(C_{k-1}) \setminus \Phi_j(C_k)) \). Repeating the arguments of part (i), we obtain \( w \in C_k \setminus C_{k+1} \). Thus \( x \) is a limit point (and hence a member) of \( C_k \setminus C_{k+1} \).

\[ \square \] Now we show that (5.8) is a subset of (5.7). This direction is easier.

Let \( x \in C_k \setminus C_{k+1} \) so that \( x = \Phi_j(y) \) for some \( y \in C_{k-1} \). We know \( y \notin C_k \), because otherwise \( x = \Phi_j(y) \in \Phi_j(C_{k-1}) \subseteq C_{k+1} \). Thus \( y \in C_{k-1} \setminus C_k \), which implies \( x \in \Phi(C_{k-1} \setminus C_k) \). This establishes \( C_k \setminus C_{k+1} \subseteq \Phi(C_{k-1} \setminus C_k) \); taking closures completes the proof of the equality (5.6).

Consider the set \( C_k \setminus C_{k+1} \) and the set \( \Phi^k(\bigcup G_q) \). The following result states that, modulo some boundary points, these two sets are the same. In terms of Figure 3, for example, this means that the difference between two successive stages in the left column is roughly equal to the corresponding stage in the right column.

**Theorem 5.15.** The tilesets can be recovered as the closure of the images of the generators under the action of \( \Phi \), that is,

\[ T_k = \bigcup_{q=1}^{Q} \Phi^{k-1}(G_q). \] (5.9)

**Proof.** First, observe that Corollary 5.13 gives

\[ \bigcup_{q=1}^{Q} G_q = \bigcup_{q=1}^{Q} \text{int}(C \setminus \Phi(C)) = C \setminus \Phi(C) = T_1. \] (5.10)

Now take \( \Phi^{k-1} \) of both sides, using Lemma 5.7 on the left and Theorem 5.14 on the right, to obtain the conclusion:

\[ \bigcup_{q=1}^{Q} \Phi^{k-1}(G_q) = \Phi^{k-1} \left( \bigcup_{q=1}^{Q} G_q \right) = \Phi^{k-1}(T_1) = T_k. \] (5.11)

The union at left is disjoint because each \( \Phi_j \) is injective, \( G_q \subseteq \text{int} C \), and the tileset condition (3.5) prohibits overlaps of interiors. \( \square \)

**Theorem 5.16.** The collection \( T = \{ \Phi_w(G_q) \} \) is a tiling of \( C \setminus F \) by open sets:

\[ C = \bigcup R_n = \bigcup \Phi_w(G_q). \] (5.12)

**Proof.** (i) To see the forward inclusion of (5.12), let \( x \in C \). If \( x \notin F \), we can find \( k \) such that \( x \in C_{k-1} \setminus C_k \subseteq T_k \), by Cor. 5.5. By Thm. 5.15, it follows that \( x \in \bigcup_{q=1}^{Q} \Phi^{k-1}(G_q) \) and we are done. Now suppose that \( x \in F \), and let \( B_i \) be the open ball around \( x \) of radius \( 1/i \). By Cor. 5.5 again, we can find \( x_i \in B_i \cap (C \setminus F) \). The previous argument shows \( x_i \in \bigcup \Phi_w(G_q) \), and hence the same holds for \( x = \lim x_i \). The reverse inclusion is obvious from Theorem 5.1 and the definition of the tiles as subsets of the \( C_k \), in (3.16).

(ii) To see that the tiles are disjoint, note first that the generators are disjoint by definition. Suppose \( R_n \) and \( R_m \) are both in the same tileset \( T_k \). Then (5.11) shows that they are disjoint. Now suppose \( R_n \subseteq T_k \) and \( R_m \subseteq T_\ell \), where \( k < \ell \). Then \( R_n \) is disjoint from \( C_k \) by definition of \( T_k \), and it follows from Theorem 5.1 that \( R_n \) is disjoint from \( C_\ell \) for all \( \ell \geq k \). (See, e.g., Figure 3.)

It is also clear that \( R_n \cap C_k = \emptyset \) implies that \( R_n \cap F = \emptyset \), so no tiles intersect the attractor \( F \). Thus, \( T \) is an open tiling of \( C \setminus F \). \( \square \)
Corollary 5.17. The tiling $T$ is subselfaffine in that $\Phi(T) = T \setminus \bigcup_q G_q$.

Remark 5.18. What kinds of generators are possible? In general, this is a difficult question to answer; it is explored in detail in [LaPeWi2]. The generators inherit many geometric properties from the convex hull $C = [F]$ and may therefore have a finite or infinite number of nonregular boundary points. In fact, by an observation of [StWa], it is possible (even generic) for the boundary of a 2-dimensional generator to be a piecewise $C^1$ curve. However, it is impossible for it to be a piecewise $C^2$ curve, unless it is polyhedral.

Remark 5.19. One might ask why the convex hull plays such a unique role in the construction of the tiling. There may exist other sets which are suitable for initiating the construction; however, some properties seem to make the convex hull the natural choice:

1. Any convex set (which is not a singleton set) is the closure of its relative interior (as shown in the proof of Cor. 5.9).
2. The affine image of a convex set is convex. Consequently, $\Phi(C) \subseteq C$ as in Theorem 5.1.
3. The convex hull of $F$ obviously contains $F$.

Note that nonlinear maps do not preserve convexity, and so the convex hull would likely not be appropriate for constructing a tiling in the case that the mappings are not affine.

Bandt et al. introduce the notion of central open set in [BaHuRa]. This provides a more natural (but less intuitive) feasible open set which is nonempty precisely when the open set condition is satisfied, and its closure may provide a substitute for the convex hull of the attractor. This possibility is considered in [PeWi].

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References

[BaHuRa] C. Bandt, N. Hung and H. Rao, On the open set condition for self-similar fractals, Proc. Amer. Math. Soc. (5) 134 (2005), 1369–1374.
[Fal] K. J. Falconer, Fractal Geometry — Mathematical Foundations and Applications, John Wiley, Chichester, 1990.
[Hut] J. E. Hutchinson, Fractals and self-similarity, Indiana Univ. Math. J. 30 (1981), 713–747.
[Kig] J. Kigami, Analysis on Fractals, Cambridge Univ. Press, Cambridge, 1999.
[KlRo] D. A. Klain, G.-C. Rota, Introduction to Geometric Probability, Accademia Nazionale dei Lincei, Cambridge University Press, Cambridge, 1999.
[LaPe1] M. L. Lapidus and E. P. J. Pearse, A tube formula for the Koch snowflake curve, with applications to complex dimensions, *J. London Math. Soc.* (2) No. 2, 74 (2006), 397–414. (arXiv: math-ph/0412029, 2005).

[LaPe2] M. L. Lapidus and E. P. J. Pearse, Tube formulas and complex dimensions of self-similar tilings. *preprint.* 41 pages. arXiv: math.DS/0605527

[LaPe3] M. L. Lapidus and E. P. J. Pearse, Tube formulas of self-similar fractals. *preprint.* 20 pages. submitted to Proc. of Symposia in Pure Mathematics, eds. P. Kuchment et al., Amer. Math. Soc., Providence, RI, 2008.

[LaPeWi1] M. L. Lapidus, E. P. J. Pearse and S. Winter, Pointwise and distributional tube formulas for fractal sprays with Steiner-like generators, *in preparation.*

[LaPeWi2] M. L. Lapidus, E. P. J. Pearse and S. Winter, Tube formulas for generators of self-similar tilings, *in preparation.*

[LaPeWi3] M. L. Lapidus, E. P. J. Pearse and S. Winter, Fractal curvature measures and local tube formulas, *in preparation.*

[LaPo] M. L. Lapidus and C. Pomerance, Counterexamples to the modified Weyl-Berry conjecture on fractal drums, *Math. Proc. Cambridge Philos. Soc.* 119 (1996), 167–178.

[La-vF1] M. L. Lapidus and M. van Frankenhuijsen, *Fractal Geometry, Complex Dimensions and Zeta Functions: geometry and spectra of fractal strings,* Springer Monographs in Mathematics, Springer-Verlag, New York, 2006.

[Mu] J. R. Munkres, *Topology,* 2nd ed., Prentice Hall, Upper Saddle River, NJ, 2000.

[Pe] E. P. J. Pearse, *Complex Dimensions of Self-Similar Systems,* Ph.D. Dissertation, Univ. of California, Riverside, June 2006.

[PeWi] E. P. J. Pearse and S. Winter, Geometry of self-similar tilings, work in progress.

[Sch] R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory,* Encyclopedia of Mathematics and its Applications, vol. 44, Cambridge Univ. Press, Cambridge, 1993.

[Ste] E. M. Stein, *Singular integrals and differentiability properties of functions,* Princeton University Press, Princeton, N.J., 1970.

[StWa] R. S. Strichartz and Y. Wang, Geometry of self-affine tiles I, *Indiana Univ. Math. J.* 48 (1999), 1–23.