Quantum properties of a superposition of squeezed displaced two-mode vacuum and single-photon states

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Abstract

In this paper, we study some quantum properties of a superposition of displaced squeezed two-mode vacuum and single-photon states, such as the second-order correlation function, the Cauchy–Schwarz inequality, quadrature squeezing, quasiprobability distribution functions and purity. These type of states include two mechanisms, namely interference in phase space and entanglement. We show that these states can exhibit sub-Poissonian statistics, squeezing and deviate from the classical Cauchy–Schwarz inequality. Moreover, the amount of entanglement in the system can be increased by increasing the squeezing mechanism. In the framework of the quasiprobability distribution functions, we show that the single-mode state can tend to the thermal state based on the correlation mechanism. A generation scheme for such states is given.

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(Some figures in this article are in colour only in the electronic version.)

1. Introduction

Developing new states besides the traditional ones is an important topic in quantum optics and quantum information theories. The Fock state $|n\rangle$ and the coherent state $|\alpha\rangle$ are the most commonly used states in these theories. The single-mode squeezed states of electromagnetic field are purely quantum states since they have less uncertainty in one quadrature than the vacuum noise level. Additionally, these states exhibit a variety of nonclassical effects, e.g. sub-Poissonian statistics [1] and oscillatory behavior in the photon-number distribution [2]. These states can be generated via a degenerate parametric amplifier [3]. The third type of state given in the literature is the two-mode squeezed state [4], which contains quantum correlations between two different modes of the field. The importance of these states comes from their connection with the two-photon nonlinear processes, e.g. the non-degenerate parametric amplifier [5]. These states have been used in continuous-variable teleportation [6], quantum key distribution [7], verification of EPR correlations [8], etc. The single-mode state—obtained from the two-mode squeezed state by tracing out the other mode—cannot exhibit squeezing [9, 10]. Precisely because of the correlation between modes in the two-mode squeezed operator, the squeezing of the quantum fluctuations does not occur in the individual modes but it occurs in the superposition of the two modes.

Great attention has been devoted to producing mesoscopic superposition states. These states have more interesting, distinct characteristics than the classical ones,
such as interference in phase space, squeezing and quantum entanglement [11]. These remarkable properties present the mesoscopic superposition states as powerful tools in quantum information processing, metrology [12, 13] and experimental studies of decoherence [14]. The most famous superimposed state in the literature is the Schrödinger-cat state [15]. There are several proposals for generating superposition of optical coherent states in the literature. For a recent review, the reader can consult [13] and the references cited therein. Besides the Schrödinger-cat state, various types of superposition have been developed, e.g., the superposition of squeezed and displaced number states without [16] and with thermal noise [17]. Moreover, the superposition of multiple mesoscopic states is given in [18], and has been generated using resonant interaction between atoms and the field in a high-quality cavity. The superposition of the two-mode states is discussed in [19]. These states, under certain conditions, become very close to the well-known Bell states [11] and they can be generated by a resonant bichromatic excitation of $N$ trapped ions [20]. The entanglement of a superposition of two bipartite states in terms of the correlation of the two states constituting the superposition has been discussed in [21].

Developing new states is an important topic for understanding the boundary between classical and quantum mechanics as well as for covering the needs of progress in quantum information theory. Moreover, the investigation of the nonclassical effects of the quantum states is of considerable and continuing interest, since it plays an important role both fundamentally and practically in the quantum information theory. Throughout this paper, we study the quantum properties of the superposition of squeezed displaced two-mode number states (SDTNS), in particular, the vacuum and single-photon states. In these states, the squeezing mechanism is involved via nondegenerate squeezed operators. These states are different from the superposition of the single-mode states [16, 17] in the following sense: they include two mechanisms: (i) entanglement and/or correlation between the two modes and (ii) two-mode interferences in phase space. These states can be generated via two-mode trapped ions [22], as we will show in section 6. For SDTNS, we study the single-mode second-order correlation, Cauchy–Schwarz inequality, quadrature squeezing, quasiprobability functions and purity. We show that the nonclassical effects are remarkable in the different quantities. Also the single-mode state tends to the thermal state based on the correlation mechanism and the amount of entanglement can be increased by increasing the squeezing mechanism.

We perform this investigation in the following order. In section 2, we introduce the state formalism and comment on its photon-number distribution. In section 3, we discuss the second-order correlation function and the Cauchy–Schwarz inequality. In section 4, the quadrature squeezing in the framework of principal squeezing is investigated. In section 5, quasiprobability distribution functions and purity are investigated. The generation of the SDTNS is discussed in section 6; however, the conclusions are summarized in section 7.

### 2. State formalism

The correlated two-mode squeezed states are connected with the two-mode squeezed operator, which has the form

$$\hat{S}(r) = \exp \left[\frac{r}{2} (\hat{a} \hat{b} - \hat{a}^\dagger \hat{b}^\dagger)\right], \quad (1)$$

where $\hat{a}$ ($\hat{a}^\dagger$) and $\hat{b}$ ($\hat{b}^\dagger$) denote the annihilation (creation) operators of the first (signal) and second (idler) mode, respectively. By means of this operator and the superposition principle, we develop a new class of states, namely SDTNS, as

$$|\psi\rangle = |r, \alpha, \beta\rangle_c \exp[\frac{r}{2} (\alpha \hat{a}-\alpha^* \hat{a}^\dagger)] \exp[\frac{r}{2} (\beta \hat{b}-\beta^* \hat{b}^\dagger)], \quad (2)$$

where $\epsilon = |\epsilon| \exp(i\phi)$, $\hat{S}(r)$ is given by (1) and $\hat{D}(\alpha_1, \alpha_2)$ is the two-mode displaced operator defined as

$$\hat{D}(\alpha_1, \alpha_2) = \exp[\alpha_1 \hat{a} - \alpha_1^* \hat{a}^\dagger] \exp[\alpha_2 \hat{b} - \alpha_2^* \hat{b}^\dagger], \quad (3)$$

and $\alpha_1$ is generally a complex parameter (a field amplitude); however, throughout the investigation in this paper it will be considered real. Also the prefactor $\lambda_\epsilon$ is the normalization constant, which can be easily evaluated as

$$|\lambda_\epsilon|^2 = 1 + |\epsilon|^2 + 2|\epsilon| \mu \lambda_n (4\epsilon_1^2 \lambda_m (4\epsilon_2^2) \cos \phi), \quad (4)$$

where

$$t_1 = \alpha_1 C_\epsilon + \alpha_2 S_\epsilon, \quad t_2 = \alpha_2 C_\epsilon + \alpha_1 S_\epsilon, \quad S_\epsilon = \sinh r, \quad C_\epsilon = \cosh r, \quad \mu = \exp[-2(t_1^2 + t_2^2)], \quad (5)$$

and $L_n(\cdot)$ is the Laguerre polynomial of order $n$ (see (8) below). Throughout the paper, we study only two choices for the parameter $|\epsilon|$, namely 1 and 0; however, for the parameter $\phi$ we take the values 0, $\pi$ and $\pi/2$. Precisely, when $|\epsilon| = 1$ and $\phi = 0, \pi, \pi/2$, the states (2) are called even-type, odd-type and Yurke-type states, respectively.

When $|n, m\rangle = |0, 0\rangle$ the states (2) can be expressed in a closed form in terms of the Fock states [23] as

$$|\psi\rangle = \sum_{n_1, n_2=0}^{\infty} C(n_1, n_2)|n_1, n_2\rangle, \quad (6)$$

where

$$C(n_1, n_2) = \lambda_\epsilon \left[1 + (-1)^{n_1+n_2} \frac{1}{\cos r} \exp \left[\frac{1}{2} (\alpha_1 C_\epsilon + \alpha_2 S_\epsilon)\right] \exp \left[\frac{1}{2} (\alpha_1 C_\epsilon + \alpha_2 S_\epsilon)\right]\right. \left. \frac{M^M}{N! (\mu_1)^{n_1-M} (\mu_2)^{n_2-M}} \times \frac{(\tanh r)^M}{M^M} L_M^n \left(\frac{1}{\tanh r}\right) \mu_1 = \alpha_1 - \alpha_2 \tanh r, \quad \mu_2 = \alpha_1 - \alpha_2 \tanh r, \quad M = \min(n_1, n_2), \quad N = \max(n_1, n_2) \right), \quad (7)$$

and $L_M^n(\cdot)$ is the associated Laguerre polynomial having the form

$$L_M^n(x) = \sum_{l=0}^{k} \frac{(x+y)!}{(x+y-k)!k!} \frac{(x+y)!}{(y+l)!(k-l)!k!}, \quad (8)$$
The photon-number distribution of (6) can be evaluated as
\[ P(m_1, m_2) = |C(m_1, m_2)|^2, \]
where \( C(m_1, m_2) \) is given by (7). It is obvious that \( P(m_1, m_2) \) can exhibit pairwise oscillations based on the values of the sum \( m_1 + m_2 \), even if \( r = 0 \). We have to remark that the components of the SDTNS can exhibit oscillatory behavior in \( P(m_1, m_2) \) [23], apart from the superposition mechanism, which can make this behavior more or less pronounced. Moreover, the single-mode photon-number distribution can be obtained via the relation
\[ P(m_1) = \sum_{m_2=0}^{\infty} |C(m_1, m_2)|^2. \]
In \( P(m_1) \) the occurrence of the oscillatory behavior results from the interference mechanism. We can explain this fact for the simplest case \( r = 0, n = m = 0, \epsilon = \exp(i\phi) \) and hence (10) reduces to
\[ P(m_1) = 2\lambda_1^2 \exp(-\alpha_1^2) \frac{\alpha_1^{2m_1}}{m_1!} \times \left[ 1 + (-1)^m \exp(-2\alpha_1^2) \cos \phi \right]. \]
The oscillatory behavior in \( P(m_1) \) depends on the values of \( \alpha_2 \) and \( \phi \), i.e. for large values of \( \alpha_2 \), \( P(m_1) \) tends to that of the coherent state. This means that one can use the second mode to control the nonclassical effects in the first mode and vice versa.

In the following sections, we investigate the properties of the state (2). For the sake of simplicity we treat the second-order correlation function, the Cauchy–Schwarz inequality, squeezing and the purity using the form (6) (i.e. \( |n, m| = (0, 0) \)); however, the quasiprobability functions are given for the case \( |n, m| = (0, 1) \). This is to estimate global information on the generic form.

3. Second-order correlation function and Cauchy–Schwarz inequality

In this section, we investigate the behavior of the second-order correlation function and Cauchy–Schwarz inequality for the state (6). These two quantities can give information on the correlation between the modes in the quantum system. The second-order correlation function for the first mode, e.g. \( \hat{a} \), is defined by
\[ g^{(2)}(0) = \frac{\langle \hat{a}^\dagger \hat{a}^2 \rangle}{\langle \hat{a}^\dagger \hat{a} \rangle^2} - 1, \]
where \( g^{(2)}(0) = 0 \) for Poissonian statistics (standard case), \( g^{(2)}(0) < 0 \) for sub-Poissonian statistics (nonclassical effects) and \( g^{(2)}(0) > 0 \) for super-Poissonian statistics (classical effects). The second-order correlation function can be measured by a set of two detectors [24], e.g. the standard Hanbury Brown–Twiss coincidence arrangement. For this quantity, we restrict the discussion to the first mode only. For this mode, one can easily obtain
\[ \langle \hat{a}^\dagger \hat{a} \rangle = |\lambda_1|^2 \left\{ \left( \alpha_1^2 + \alpha_2^2 \right) \left( 1 + |e|^2 \right) + 2|e| \mu \cos \phi \right\} \times \left[ \left( \alpha_1^2 - 4\alpha_1 \alpha_2 \right) C_r - 2|e| (\alpha_1 C_r + e^2) \right], \]
(13)
\[ \langle \hat{a}^2 \rangle = |\lambda_1|^2 \left\{ \left( \alpha_1^2 + 2\alpha_2^2 \right) \left( 1 + |e|^2 \right) + 2|e| \mu \cos \phi \right\} + \frac{16 \alpha_1 \alpha_2 \alpha_1^2}{\mu^2} + 16 \alpha_1 \alpha_2 \alpha_2^2 + 2 \alpha_2^4 \]
\[ + 8 \alpha_1 \alpha_2 \alpha_1^2 \left( C_r - 2 \alpha_1^2 \right) - 8 \alpha_1 \alpha_2 \alpha_2^2 - 4 \alpha_1^2 C_r, \]
\[ - 16 \alpha_1 \alpha_2 \alpha_1^2 (C_r t_1^2 - 16 \alpha_1^2 C_r t_1) - 16 \alpha_1 \alpha_2 \alpha_2^2 (C_r t_1), \]
\[ - 8 \alpha_1 \alpha_2 \alpha_2^2 (C_r t_1), \]
\[ \right\}. \]
(14)
It is worth mentioning that the most general cases for equations (13) and (14) have been given in [25] for the multimode squeezed cat states but with different parameterizations.
Substituting (13) and (14) into (12) and taking \( r = 0 \), we obtain
\[ g^{(2)}(0) = \frac{1 + |e|^2 + 2|e| \exp \left[ -2(\alpha_1^2 + \alpha_2^2) \right] \cos \phi \alpha_1^2}{1 + |e|^2 - 2|e| \exp \left[ -2(\alpha_1^2 + \alpha_2^2) \right] \cos \phi} = 1. \]
(15)
From equation (15), it is obvious that the sub-Poissonian statistics can occur only for \( \phi = \pi \) and \( 2(\alpha_1^2 + \alpha_2^2) \) small. This means that the odd-type state can exhibit nonclassical effects in the framework of \( g^{(2)}(0) \). In this case, the mode under consideration reduces to the standard odd-coherent state with the components \( \pm \sqrt{\alpha_1^2 + \alpha_2^2} \). The obvious remark is: when the mode \( \hat{a} \) is prepared in the vacuum state \( |0\rangle \), its \( g^{(2)}(0) \) can exhibit sub-Poissonian statistics based on the values of \( \alpha_2 \) of the second mode. A similar argument can be given for the second mode. This reflects the role of the correlation between the modes in the system, which leads to the possibilities of controlling one mode by the other one. Now we draw the attention of the reader to the general case when the squeezing mechanism is involved. We have noted that the even-type and the Yurke-type states cannot exhibit sub-Poissonian statistics. Information about \( g^{(2)}(0) \) of the odd-type states is depicted in figures 1(a)–(c) for given values of the parameters \( \alpha_1 \) and \( \alpha_2 \). From figure 1(a) one can observe the occurrence of the sub-Poissonian statistics, in particular, for small values of \( \alpha_1 \). When the squeezing mechanism is involved, the amounts of the nonclassical effects in \( g^{(2)}(0) \) decrease and eventually vanish for a large value of \( r \) (see figure 1(b)). For \( \alpha_1 = \alpha_2 = 0 \) and \( r \neq 0 \), state (6) reduces to the two-mode squeezed vacuum state. In this case we have \( g^{(2)}(0) = 1 \), which is independent of \( r \). This is remarkable in figure 1(b), which shows sub-Poissonian statistics only when one or both of \( \alpha_j > 0 \). Figure 1(c) gives the range of the parameter \( r \) (for certain values of \( \alpha_j \), for which the sub-Poissonian statistics occur. It is obvious that the smaller the values of \( \alpha_j \), the greater this range.
We conclude this section by investigating the intermodal correlations in terms of the deviation from the classical Cauchy–Schwarz inequality. Classically, Cauchy–Schwarz inequality has the form [26]
\[ \langle I_j I_j \rangle \leq \langle I_j \rangle \langle I_j \rangle, \]
(16)
where \( I_j, j = 1, 2 \), are classical intensities of light measured by different detectors in a double-beam experiment. In quantum theory, the deviation from this classical inequality

\[ \langle I_j I_j \rangle \leq \langle I_j \rangle \langle I_j \rangle. \]
can be represented as $V < 0$, where the factor $V$ takes the form [27]

$$V = \frac{\sqrt{\langle a^2 b^2 \rangle \langle a^* b^* \rangle}}{\langle a^* a b^* b \rangle} - 1. \tag{17}$$

Occurrence of negative values in $V$ means that the intermodal correlation is larger than the correlation between photons in the same mode [28] and this indicates a strong deviation from the classical Cauchy–Schwarz inequality. The origin in this deviation is that in the quantum mechanical treatment, we involve pseudodistributions instead of the true ones. This implies that the Glauber–Sudarshan $P$ function possesses strong quantum properties [27]. Moreover, the deviation from the Cauchy–Schwarz inequality can be observed in a two-photon interference experiment [29]. For completeness, the expectation value $\langle a^* a b^* b \rangle$ for state (6) can easily be evaluated as

$$\langle a^* a b^* b \rangle = |\lambda|^{-2} \left[ S_4 + (S_2 C_r - \alpha_1 \alpha_2)^2 + (\alpha_1^2 + \alpha_2^2) S_2 \right] \times (1 + |\epsilon|^2) + 2|\epsilon| \mu \cos \phi \left( S_2 \cosh(2r) \right. \right.$$}

$$- \alpha_1 \alpha_2 \sinh(2r) + (\alpha_1^2 + \alpha_2^2) S_2$$

$$- 2t_1 t_2 \left[ \sinh(2r)(2S_2^2 + \alpha_1^2 + \alpha_2^2 + \cosh(2r)) \right]$$

$$- 2\alpha_1 \alpha_2 \cosh(2r) \right] + 4 \sinh^2(2r) t_1^2 t_2^2$$

$$+ \alpha_1 \alpha_2 \sinh(2r) \alpha_2 + \alpha_1 S_r \cosh(2r)$$

$$- \alpha_1 \alpha_2 (\alpha_1 C_r - \alpha_2 S_r) - 2t_1 \left[ (\alpha_1 C_r - \alpha_2 S_r) S_2^2 \right.$$

$$+ (\alpha_1 S_r - \alpha_2 C_r)(S_r - \alpha_1 \alpha_2)$$

$$- 4 \sinh(2r)(\alpha_1 C_r - \alpha_2 S_r) t_1 t_2$$

$$- 2\alpha_1 \alpha_2 \cosh(2r) t_1 t_2 \left. \right] . \tag{18}$$

The expectation value $\langle \hat{a}^* \hat{a} \hat{b}^* \hat{b} \rangle$ can be obtained from (14) using the interchange $\alpha_1 \leftrightarrow \alpha_2$. One can easily find $V = 0$ for $r = 0$. Generally, we have noted that $V < 0$ only when $\alpha_1$ are small (see figure 2). Figure 2(a) is given for the Yurke-type state, which is identical to that of the two-mode squeezed displaced states (c.f. (14) and (18) for $\phi = \pi/2$). From these figures the strongest deviation from the classical inequality occurs for $\alpha_j(\neq 0)$ and $r$ small, i.e. the photons are more strongly correlated than is classically possible, and then the curve monotonically increases as $r$ increases. As the values of $\alpha_2$ increase, the negative values in the factor $V$ decrease and eventually disappear (compare different curves in these figures). Also a comparison between figures 2(a) and (b) shows that the nonclassical effects occurring in the factor $V$ for the odd-type state are greater than those for the Yurke-type state.

4. Quadrature squeezing

In this section, we discuss quadrature squeezing for the state under consideration. As is well known, quadrature squeezing can be measured by a homodyne detector in which the signal is superimposed on a strong coherent beam of the local oscillator [30]. Here we use the notion of principal squeezing [31], which can give one form for the single-mode and two-mode cases. In this respect, we define the two quadratures in the following forms:

$$\hat{X} = \hat{X}_1 \cos \nu + \hat{X}_2 \sin \nu, \quad \hat{Y} = \hat{Y}_1 \cos \nu + \hat{Y}_2 \sin \nu, \tag{19}$$

where the subscripts 1 and 2 stand for the first and second modes, respectively, and $\nu$ is a rotation angle. When $\nu = 0, \pi/2, \pi/4$, the quadratures (19) yield those of the first mode, second mode and compound modes, respectively. For the first mode, the quadrature operators can be defined as

$$\hat{X}_1 = \frac{1}{2} (\hat{a} + \hat{a}^\dagger), \quad \hat{Y}_1 = \frac{1}{2\nu} (\hat{a} - \hat{a}^\dagger). \tag{20}$$

A similar definition can be quoted for the second mode via the interchange $\hat{a} \leftrightarrow \hat{b}$. The quadratures (19) satisfy the following commutation rule:

$$[\hat{X}, \hat{Y}] = \frac{i}{2}. \tag{21}$$

Therefore, the squeezing factors associated with $\hat{X}$ and $\hat{Y}$ can be expressed as

$$F = \left( \langle (\Delta \hat{X})^2 \rangle - 1 \right.$$

$$= F_1 \cos^2 \nu + F_2 \sin^2 \nu + F_3 \sin(2\nu), \tag{22}$$

$$S = \left( \langle (\Delta \hat{Y})^2 \rangle - 1 \right.$$}

$$= S_1 \cos^2 \nu + S_2 \sin^2 \nu + S_3 \sin(2\nu). \tag{22}$$

\[ \text{Figure 1. The second-order correlation function of the first mode for } (|\epsilon|, \phi) = (1, \pi) \text{ against } (\alpha_1, \alpha_2) \text{ (a) and (b) for different values of the parameters as indicated.} \]
The expressions for \( F \) be squeezed in \( x \) and \( y \) quadratures. Additionally, the behavior of the quadratures in the range \( \pi/2 < v < \pi \) is just a mirror image of that in \( 0 < v < \pi/2 \). In figure 3(a), we take \( \alpha_1 = \alpha_2 = \alpha \) and \( r = 0 \). From this figure—regardless of the values of \( v \)—squeezing occurs within the range \( 0 < \alpha \leq 1.5 \), otherwise \( S \geq 0 \). Furthermore, the minimum value in \( S \) is observable around \( v = \pi/4 \) and \( \alpha \approx 0.6 \). On the other hand, we have noted that the squeezing mechanism decreases the amount of squeezing involved in the system. This is obvious in figure 3(b), which shows the range of \( r \) over which squeezing is available, i.e. \( 0 \leq r \leq 0.35 \). In other words, \( r = 0.35 \) is the critical value for \( \alpha_1 = \alpha_2 = 0.6 \). This critical value is \( \alpha \) dependent, however, we have found it difficult to obtain an analytical form for it. Comparison between figures 3(a) and (b) shows that involving the two mechanisms (i.e. squeezing and superposition) in the system destroys the nonclassical effects contributed by each one independently.

5. Quasiprobability distribution function

Quasiprobability distribution functions, namely the Husimi function \((Q)\), Glauber \(P\) functions [33], are important tools in quantum optics. Knowing these functions, all nonclassical effects can be predicted and the different moments of the operators.
can be evaluated. Most importantly, these functions can be measured by various means, e.g. photon counting experiments [34], using simple experiments similar to that used in the cavity (QED) and ion traps [35, 36], and homodyne tomography [37]. In this section, we investigate the single-mode quasiprobability distribution functions, in particular, the $W$ and $Q$ functions as well as the purity. We start with the symmetric characteristic function $C_w(\beta)$ of the first mode, which is defined as

$$C_w(\beta) = \text{Tr}[\hat{\rho} \exp(\beta \hat{a}^\dagger - \beta^* \hat{a})],$$

where $\hat{\rho}$ is the density matrix of the system under consideration. It is worth mentioning that the moments of the bosonic operators in symmetric form can be evaluated from $C_w(\beta)$ by differentiation. From equations (2) and (27), one can easily obtain

$$C_w(\beta) = |\lambda_1|^2 \left\{ \exp \left( -\frac{|\beta|^2}{2} \cosh(2r) \right) \right.$$

$$\times \left[ \exp ((\beta - \beta^*) \alpha) + |\epsilon|^2 \exp ((\beta^* - \beta) \alpha) \right]$$

$$\times \text{L}_m(S_0 |\beta|^2)L_m(C_1^2 |\beta|^2)$$

$$+ |\epsilon| \left[ \exp \left( -i\phi - \frac{k_1 + k'_1}{2} \right) \text{L}_m(k_1)\text{L}_n(k_1) \right.$$ 

$$+ \left. \exp \left( i\phi - \frac{k_1 + k'_1}{2} \right) \text{L}_m(k'_1)\text{L}_n(k_1) \right\}. \tag{28}$$

where

$$k_1 = |\beta| c \pm 2t_1|^2,$$

$$k'_1 = |\beta^*| s \pm 2t_2|^2$$

and $\text{L}_m(\cdot)$ is the Laguerre polynomial, which can be obtained from (8) by simply setting $v = 0$. The $W$ and $Q$ functions can be evaluated, respectively, through the following relations:

$$W(z) = \pi^{-2} \int d^2 \beta C_w(\beta) \exp(z \beta^* - z^* \beta^*), \tag{29}$$

$$Q(z) = \pi^{-2} \int d^2 \beta C_w(\beta) \exp(z \beta^* - z^* \beta^* - \frac{1}{2} |\beta|^2), \tag{30}$$

Figure 3. Squeezing factor $S$ for $(|\epsilon|, \phi) = (1, 0)$ with $(r, \alpha_1, \alpha_2) = (0, \alpha, \alpha)$ (a) and $\alpha_1 = \alpha_2 = 0.6$ (b).

where $z = x + iy$. Generally, it is difficult to obtain closed forms for these functions for $m \neq 0, n \neq 0$; however, the integration can be numerically treated. Therefore, we restrict ourselves to the case $n = 0, m \neq 0$, which is sufficient to obtain information on the system. On substituting equation (28) into (30) and carrying out the integration, we arrive at

$$W(z) = \frac{2|\lambda_1|^2}{\pi \cosh^{m+1}(2r)} \left\{ \exp \left[ -\frac{2|z - \alpha|^2}{\cosh(2r)} \right] \right.$$

$$\times \text{L}_m \left[ -\frac{4S_2^2}{\cosh(2r)} |z - \alpha_1|^2 + |\epsilon|^2 \exp \left[ -\frac{2|z + \alpha|^2}{\cosh(2r)} \right] \right]$$

$$\times \text{L}_m \left[ -\frac{4S_2^2}{\cosh(2r)} |z + \alpha|^2 \right]$$

$$+ 2|\epsilon| \exp \left[ -\frac{2}{\cosh(2r)} (\alpha_2^2 + x^2 + y^2) \right]$$

$$\times \text{Re} \left[ \exp \left( -i\phi + i\frac{4\alpha_1}{\cosh(2r)} \right) \text{L}_m(h) \right] \tag{31}$$

$$Q(z) = \frac{|\lambda_1|^2}{\pi C_1^{2m+2}} \left\{ \exp \left[ -\frac{|z - \alpha|^2}{C_1^2} \right] \right.$$ 

$$\times \text{L}_m \left[ -|z - \alpha|^2 \tanh^2 r \right] + |\epsilon|^2 \exp \left[ -\frac{|z + \alpha|^2}{C_1^2} \right]$$

$$\times \text{L}_m \left[ -|z + \alpha|^2 \tanh^2 r \right]$$

$$+ 2|\epsilon| \exp \left[ -\frac{1}{C_1^2} (\alpha_1^2 + 2t_2^2 + x^2 + y^2) \right]$$

$$\times \text{Re} \left[ \exp \left( -i\phi - i\frac{2\alpha_1}{C_1^2} \right) \text{L}_m(h') \right],$$

where

$$\Lambda = t_1 C_r + t_2 S_1,$$

$$h = \frac{4}{\cosh(2r)} \left[ \alpha_2^2 C_2^2 + iy \alpha_2 \sinh(2r) - S_2^2 |z|^2 \right], \tag{32}$$

$$h' = \frac{1}{C_1^2} \left[ (\alpha_2 C_r + t_2)^2 + 2iy S_1 (2\alpha_2 C_r + \alpha_1 S_1) - |z|^2 S_2^2 \right].$$

$\Lambda = t_1 C_r + t_2 S_1,$

$h = \frac{4}{\cosh(2r)} \left[ \alpha_2^2 C_2^2 + iy \alpha_2 \sinh(2r) - S_2^2 |z|^2 \right], \tag{32}$$

$h' = \frac{1}{C_1^2} \left[ (\alpha_2 C_r + t_2)^2 + 2iy S_1 (2\alpha_2 C_r + \alpha_1 S_1) - |z|^2 S_2^2 \right].$
In the derivation of equation (31), we have used the generating function of the Laguerre polynomial [38], namely
\[
\exp\left(-\frac{\mu}{1-t}\right) = \sum_{n=0}^{\infty} t^n L_n(y),
\]
and the following identity [39]:
\[
\int \exp\left[-B|\beta|^2 + (c/2)\beta^2 + (c_1/2)\beta^2 + \gamma \beta + \gamma^* \beta^* \right] d^2\beta
= \frac{\pi}{\sqrt{K}} \exp\left\{ \frac{1}{K} \left[ y \gamma_1 B + \gamma^2 (c_1/2) + \gamma^2 (c/2) \right] \right\},
\]
where \( K = B^2 - cc_1 \) if \( \text{Re}[B + \frac{1}{2}(c + c_1)] \) and \( \text{Re} \ K > 0 \). In the appendix, we show that the quasiprobability functions (31) are normalized. It is worth mentioning that the explicit analytical expressions for the \( W \) and \( Q \) functions of the even/odd superpositions of two-mode squeezed coherent states, which are special cases of (31) by simply setting \( m = 0 \), were obtained earlier in [40]. We start the investigation with the \( W \) function. The \( W \) function has received considerable attention in the literature since it can be implemented by various means, e.g., [34–37], and it is sensitive to the interference in phase space, as we shall show below. From (31) we can extract several analytical facts. For instance, when \((r, m) = (0, 0)\) and the value of \( \alpha_2 \) is very small, the \( W \) function of the first mode exhibits the well-known shape of the cat-state function, i.e. two Gaussian bell and interference fringes in-between (we have checked this fact). This can be easily understood, where—in this case—the second mode is very close to the vacuum state and hence (2) reduces to \( |\psi| \simeq \lambda_1 |\alpha_1 + \epsilon - \alpha_1| \otimes |0\rangle \), i.e. the first mode evolves in the Schrödinger-cat state. Additionally, when \( \alpha_2 \) increases, the negative values in the \( W \) function gradually decrease and eventually vanish, showing the \( W \) function of the statistical mixture of coherent states. This is related to the fact that the interference term in the \( W \) function includes the factor \( \exp(-2\alpha_2^2) \), which tends to zero for large values of \( \alpha_2 \). On the other hand, when \( |\epsilon| = 0, (r, m) \neq (0, 0) \), the \( W \) function cannot exhibit either negative values or stretching contour in phase space since \( L_0(-\kappa) > 0 \) as \( \kappa \geq 0 \). In this case, the behavior of the \( W \) function is close to that of the thermal state for which the peak occurring in the \( W \) function is greater than that of the coherent light.

Now we draw the attention of the reader to the general case (see figure 4 for the even-type state). For \( m = n = 0 \), we have numerically noted that the \( W \) function exhibits negative values only when \( \alpha_1 \geq \alpha_2 \). This condition can be analytically realized as follows. From (31) the interference term in the \( W \) function includes \( \cos(\phi - 4y \Lambda / \cosh(2r)) \), which is responsible for the occurrence of the negative values in the \( W \) function. Assume that we choose the values of
the interaction parameters to verify $\cos(\cdot) = -1$ and take $(|\epsilon|, r) = (1, 0)$ for simplifying the problem. Thus, the $W$ function reduces to
\begin{equation}
W(0, y) = \frac{4|\lambda_1|^2}{\pi \cosh(2r)} \left\{ \exp \left[ \frac{-2(y^2 + x_1^2)}{\cosh(2r)} \right] - \exp \left[ \frac{-2(2x_1^2 + y^2)}{\cosh(2r)} \right] \right\}.
\end{equation}

The $W$ function involves negative values when $W(0, y) < 0$. Solving this inequality yields the above-mentioned condition. From figure 4(a), one can observe that the $W$ function has two-Gaussian bell and interference fringes in between but with negative values smaller than those of the standard cat states, e.g. [41]. These fringes can be amplified for certain values of the squeezing parameter (compare figures 4(a) and (b)). It is worth mentioning that the amplification of the cat states in the parametric down conversion has been discussed in [42]. Now we draw the attention to the case where the second mode includes the Fock state $|1\rangle$ (see figure 4(c)). From this figure, the $W$ function exhibits two-Gaussian bell around $(x, y) = (\pm x_1, 0)$ and inverted peak in between with maximum negative value. Comparison between figures 4(a) and 4(c) shows that the existence of the Fock state in the second mode increases the amounts of the nonclassical effects in the first one. Precisely, the interference in phase space in the first mode can be controlled by the information involved in the second mode. In this respect, the nonclassical effects can be transferred from one of the modes to the other through the entanglement process. Furthermore, involving the squeezing mechanism in the system smooths out the negative values in the $W$ function (compare figures 4(c) and (d)), which vanish for large values of $r$, we get back to this point shortly. In this case, we found that the $W$ behaves quite similarly to the thermal light. This is related to the correlation mechanism in the system. Comparison between figures 4(b) and 4(d) shows that the squeezing mechanism makes the interference fringes more or less pronounced based on the value of $m$. Now we draw the attention of the reader to figures 5 given for the $W$ and $Q$ functions, as indicated, for the odd-type states. From figure 5(a), one can observe that the $Q$ function exhibits a symmetric two-peak structure, which is representative of the cat states as well.
as the statistical mixture states [41]. Moreover, when the value of $r$ increases, i.e. the entanglement between the two modes becomes stronger, the $Q$ function exhibits a quite similar shape to that of the thermal light (see figure 5(d)), which is a single peak localized in the phase space origin with contour greater than that of the vacuum state. In the framework of the $W$ ($Q$) function, the first mode exhibits (nonclassical) super-classical light (compare figures 5(b) and (d)). This confirms the fact that the $W$ function is more informative than the $Q$ function. Similar conclusions have been noticed for the case $\{\mid \varepsilon \rangle, \phi \rangle = (1, \pi / 2)$.

We conclude this part by investigating the relation between the occurrence of negative values in the $W$ function and the value of the squeezing parameter $r$. To do so, we plot figure 6 for the $W$ function in terms of $r$ for the same values of the parameters as in figure 4(c) (dashed curve) and figure 5(b) (solid curve). The values of $x$ and $y$ have been chosen as they give maximum negative values in figure 4(c) and 5(b). From figure 6, we can obtain rough information about the minimal value of $r$ for which the negative values in the $W$ function vanish, e.g. for even-type and odd-type states, it is $r = 1$ and 2.3, respectively.

We have checked the behavior of the $W$ function for these values and found that the negative values are negligible. Furthermore, after plotting the $W$ function for various values of $r$ (not detailed here), we observed that the exact minimal value for the even-type state is $r = 1.5$. Nevertheless, for the odd-type state we found that the negative values—even they are very small—are still observed for all values of $r$.

Entanglement is a global property of a system. For a bipartite pure state it has been proved that there is a unique measure of the entanglement, which is the von Neumann entropy of the reduced state of either of the parties [43]. On the other hand, the purity, which gives information on the mixedness in the system, can be used to estimate some information on the entanglement in the system. In this respect, we can mention that the purity and the von Neumann entropy can give quite similar behavior for the quantum system [44].

Also for the Jaynes–Cummings model, it has been shown that the von Neumann entropy and purity are equivalent [45]. As the purity is easy to calculate and can provide some exact information about the system, we use it here to study the mixedness and/or the entanglement in the state under consideration. The single-mode purity can be evaluated via the characteristic function through the relation

$$\text{Tr} \hat{\rho}_j^2 = \frac{1}{\pi} \int |C_{\omega}(\beta)|^2 d^2 \beta,$$

where $\hat{\rho}_j$ is the density matrix for the mode under consideration. For the pure (mixed) state, we have $\text{Tr} \hat{\rho}_j^2 = 1 (< 1)$. From equations (28) and (36) and setting $\{\varepsilon, m, n\} = (1, 0, 0)$, we obtain

$$\text{Tr} \hat{\rho}_j^2 = \frac{2|\lambda_\varepsilon|^4}{\cosh(2r)} \left\{ 1 + \mu^2 \cos(2\phi) + \exp \left( -\frac{4\mu^2}{\cosh(2r)} \right) \right.$$

$$+ 4\mu \cos \phi \exp \left[ -\frac{\varepsilon^2}{\cosh(2r)} \right]$$

$$\left. + \mu^2 \exp \left( \frac{4\lambda_\varepsilon^2}{\cosh(2r)} \right) \right\}. \quad (37)$$

In figure 7, we have plotted $\text{Tr} \hat{\rho}_j^2$ for the even-type states. From figure 7(a) it is obvious that for $\alpha_1 = 0$ or $\alpha_2 = 0$ the two modes are disentangled, where $\text{Tr} \hat{\rho}_j^2 = 1$. When $\alpha_1$ increases, the first mode abruptly tends to the partial mixed state (i.e. $\text{Tr} \hat{\rho}_j^2 = 0.5$), which indicates a strong entanglement between the two modes. In this case the behavior is quite similar to that of the thermal light with mean photon number $\bar{n} = 1$, which satisfies the inequality $1/(1 + \bar{n}) \leq \text{Tr} \hat{\rho}_j^2 < 1$. This behavior can be analytically realized by evaluating the limiting case $(\alpha_1, \alpha_2) = (\infty, \infty)$ for the purity (37), which gives

$$\text{Tr} \hat{\rho}_j^2 = \frac{1}{2 \cosh(2r)}. \quad (38)$$

It is evident that $\text{Tr} \hat{\rho}_j^2 = 0.5$ for $r = 0$. When the squeezing mechanism is involved in the system, the degree of mixedness and/or the amount of entanglement is increased and the purity becomes more structured (compare figures 7(a) and (b)). The purity tends to the steady state for large values of $\alpha_1$. The value of the purity at $(\alpha_1, \alpha_2) = (0, 0)$ can be easily obtained from (37) as:

$$\text{Tr} \hat{\rho}_j^2 = \frac{1}{\cosh(2r)}. \quad (39)$$

It is evident that when the value of $r$ increases, the amount of mixedness increases, too. From expressions (38) and (39), one can realize that the minimal value of the purity can be achieved for large $r$; e.g. for $(\alpha_1, \alpha_2, r) = (2, 2, 5)$ we have $\text{Tr} \hat{\rho}_j^2 \approx 0.006$.

### 6. State generation

In this section, we give a generation scheme for the states (2) in the framework of the trapped ions. To do so, we consider a two-level ion of mass $M$ moving in a 2D harmonic potential of frequency $\omega_x$ in the $x$-direction and $\omega_y$ in the $y$-direction. Also $\hat{a}$ ($\hat{a}^\dagger$) and $\hat{b}$ ($\hat{b}^\dagger$) represent
the annihilation (creation) operators for the vibronic quanta in the x- and y-directions, respectively. Then the position operators are given by $\hat{x} = \Delta \omega_0 (\hat{a} + \hat{a}^\dagger)$, $\hat{y} = \Delta \omega_0 (\hat{b} + \hat{b}^\dagger)$, where $\Delta \omega_0 = (2\omega_0 M)^{-1/2}$, $\Delta \omega_0 = (2\omega_0 M)^{1/2}$ are the width of the harmonic ground state. Six beams are used to drive the interaction with the ion in the cavity; two are propagating in the x-direction detuned by $\pm \omega_0$, from the transition frequency of the ion. Two are propagating in the y-direction detuned by $\pm \omega_0$ and two are propagating in the x-y plane detuned by $\pm (\omega_0 + \omega_0)$. Thus the interaction Hamiltonian can be written in the form

$$\hat{H}_\text{int} = - (\hat{\mu} \cdot \hat{E}^- \hat{\sigma}_- + \text{h.c.}),$$

where

$$\hat{E}^- = E_1 \exp[i((\omega_0 - \omega_c) t - k_1 x + \vartheta_1)]$$

$$+ E_2 \exp[i((\omega_0 + \omega_c) t - k_2 x + \vartheta_1)]$$

$$+ E_3 \exp[i((\omega_0 - \omega_c) t - k_3 y + \vartheta_3)]$$

$$+ E_4 \exp[i((\omega_0 + \omega_c) t - k_4 y + \vartheta_3)]$$

$$+ E_5 \exp[i((\omega_0 - \omega_c) t - k_5 y + \vartheta_3)]$$

$$+ E_6 \exp[i((\omega_0 + \omega_c) t - k_6 y + \vartheta_3)]$$

(41)

with $\hat{\sigma}_\pm$ being the Pauli spin operators, $E_j$, $k_j$ and $\vartheta_j$ are the amplitudes, wave vectors and phases of the driving modes, and $\omega_0$ is the ionic transition frequency. Using the operator forms for $x, y$ and provided that the field is resonant with one of the vibronic side bands, the ion-field interaction can be described by a nonlinear Jaynes–Cummings model [46]. In the interaction picture and in the Lamb–Dicke limit, it is sufficient to keep the first few terms. Thus we have the following effective Hamiltonian:

$$\hat{H}_\text{int} = \hat{H}_1 + \hat{H}_2,$$

$$\hat{H}_1 = - \left( g_1 \hat{a}^\dagger + g_2 \hat{a} + g_3 \hat{b}^\dagger + g_4 \hat{b} \right) \hat{\sigma}_- + \text{h.c.},$$

$$\hat{H}_2 = - \left( g_5 \hat{a} \hat{b}^\dagger + g_6 \hat{a}^\dagger \hat{b} \right) \hat{\sigma}_+ + \text{h.c.},$$

(42)

where

$$g_j = i\Omega_j \eta_j \exp(i(\vartheta_j - \frac{1}{2} \eta_j^2)), \quad j = 1, 2, 3, 4,$$

$$g_j = - \Omega_j \eta_j \exp(i(\vartheta_j - \frac{1}{2} (\eta_j^2 + \eta_j'^2))), \quad j = 5, 6,$$

(43)

Under this Hamiltonian any particle prepared in the state $|\psi_0\rangle = \exp(-i\hat{H}_1\tau_1)|\psi_0\rangle$ will stay in this state and the dynamics is reduced to that of the motional degrees of freedom only. Now, assume that the system is initially prepared in the following state:

$$|\Psi(0)\rangle = (|e\rangle + |g\rangle)|n, m\rangle,$$

(45)

where $|e\rangle$, $|g\rangle$ are the excited and the ground state of the ion. Also we have dropped the normalization constant in (45) since it has no effect on the following calculations. It is worth mentioning that the Fock state $|n\rangle$ can be prepared with very high efficiency according to the recent experiments [48]. We proceed, and by applying the Hamiltonian $\hat{H}_2$ on the state (45) for a duration of time $\tau_1$, we obtain

$$|\Psi_1\rangle = \exp(-i\hat{H}_2\tau_2)|\Psi(0)\rangle$$

$$= \hat{S}(\tau_2)|n, m\rangle(|e\rangle + |g\rangle).$$

(46)

Then we apply $\hat{H}_1$ for a duration of time $\tau_2$ to obtain

$$|\Psi_2\rangle = \exp(-i\hat{H}_1\tau_2)|\Psi_1\rangle$$

$$= D(\omega_1, \omega_2)\hat{S}(\tau_2)|n, m\rangle(|e\rangle + |g\rangle),$$

(47)

Figure 7. The purity of the first mode for $(|e\rangle, \phi) = (1, 0)$ with $r = 0$ (a) and 0.5 (b).
where \( r = (g_1 + g_2^\dagger) \tau_1 \), \( \alpha_1 = -i(g_1 + g_2^\dagger) \tau_2 \), and \( \alpha_2 = -i(g_3 + g_4^\dagger) \tau_2 \). We choose the polarization in the quantized field so that it affects the excited state only \([48]\), and applying the Hamiltonian \( \hat{H}_1 \) for a duration of time \( \tau_3 \), we arrive at

\[
|\Psi_3\rangle = \exp(-i\hat{H}_1 \tau_3)|\Psi_2\rangle
\]

\[
= [(D(\beta_1, \beta_2)|e\rangle + D(\alpha_1, \alpha_2)|g\rangle)] S(r)|n, m\rangle, \quad (48)
\]

where \( \beta_1 = \alpha_1 - i(g_1 + g_2^\dagger) \tau_3 \) and \( \beta_2 = \alpha_2 - i(g_3 + g_4^\dagger) \tau_3 \). After that we apply a carrier pulse of Rabi frequency \( \Omega_0 \), whose evolution operator is

\[
\hat{U}(t) = \cos(\Omega_0 t)(|e\rangle\langle e| + |g\rangle\langle g|) - i \sin(\Omega_0 t) \times (exp(i\theta)|e\rangle\langle g| + exp(-i\theta)|g\rangle\langle e|), \quad (49)
\]

to the state \( |\Psi_3\rangle \) to obtain

\[
|\Psi_4\rangle = \hat{U}(\tau_4)|\Psi_3\rangle
\]

\[
= [D(\beta_1, \beta_2) \cos(\Omega_0 \tau_4) - i \exp(i\theta) \sin(\Omega_0 \tau_4)] \times D(\alpha_1, \alpha_2) \tau_4 |e\rangle\langle e| + [D(\alpha_1, \alpha_2) \cos(\Omega_0 \tau_4) - i \exp(-i\theta) \sin(\Omega_0 \tau_4)] D(\beta_1, \beta_2) \tau_4 |g\rangle\langle g|.
\]

(50)

Then detecting the particle in either of its states gives the states \( (2) \), where we can choose \( \beta_1 = -\alpha_1, \beta_2 = -\alpha_2 \). Throughout the investigation of this paper, we have considered that the parameters \( \alpha_j \) and \( r \) are real. This can be achieved in the above equations by simply setting, e.g. \( \vartheta_j = 0 \) or \( \pi, j = 1, \ldots, 4 \), and \( \vartheta_5 = \vartheta_6 = \pi/2 \).

7. Conclusion

The superposition principle is at the heart of quantum mechanics, which can produce new states having nonclassical effects greater than those attributed to the components. In this paper, we have studied the quantum properties for a new class of states, namely superposition of squeezed displaced two-mode number states. Particular attention has been given to the two-mode vacuum and single-photon states of this class. These states include two mechanisms: interference in phase space and entanglement between the two modes of the system. We have studied the second-order correlation function, the Cauchy–Schwarz inequality, quadrature squeezing, quasiprobability distribution functions and purity. We have shown that the system can exhibit sub-Poissonian statistics even if the mode under consideration is in the vacuum state. This reflects the role of entanglement in the system. The deviation from the classical Cauchy–Schwarz inequality has been investigated, showing that the photons are more strongly correlated than is allowed classically. For certain values of \( \epsilon \), the system can exhibit squeezing provided that the values of \( \alpha_1 \) and \( \alpha_2 \) are small. From the \( W \) function, it has been shown that the single-mode state resulting from this class can behave as a thermal state as a result of the correlation process. Also for \( m = n = 0 \), the \( W \) function of the first mode provides negative values only when \( \alpha_1, \alpha_2 \) are small. The interference in phase space of one of the subsystems can be controlled by the information involved in the other subsystem. Additionally, the squeezing mechanism can make the interference fringes more or less pronounced. The \( W \) function is more informative than the \( Q \) function in the description of the quantum systems. For the purity it has been shown when the values of \( \alpha \)'s increase, the mode under consideration abruptly tends to the partially mixed state. The amount of entanglement in the system is increased when the value of \( r \) is increased, too. Also, we have discussed how this class of states can be generated by means of trapped ions and pulses for appropriate durations.

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Appendix

In this appendix, we prove that the \( W \) and \( Q \) functions (31) are normalized. Precisely, we would like to prove the following:

\[
\int W(z) \, d^2z = 1,
\]

\[
\int Q(z) \, d^2z = 1.
\]

To do so, we use the generating function technique. We focus our attention on the \( Q \) function only, where the \( W \) function can be similarly treated. Moreover, we evaluate the integration for one of the interference terms in the \( Q \) function, which we denote \( I_m \) and has the form

\[
I_m = \frac{1}{\pi C^2 t^2} \int \exp \left[ -\frac{1}{C^2} \left( \alpha^2 + 2t^2 + |z|^2 \right) \right] \times \exp \left[ -i\phi - \frac{(z - z^*)A}{C^2} \right] L_m(h) \, d^2z. \quad (52)
\]

Multiplying both sides of (52) by \( t^m \), hence suming over index \( m \) and using the identity (33), we obtain

\[
\sum_{m=0}^{\infty} t^m I_m = \frac{\exp(-i\phi)}{\pi(1 - t^2) C^2} \int \exp \left[ -\frac{(\alpha^2 + 2t^2)}{C^2} \right] \times \int \exp \left[ -\frac{|z|^2}{C^2} - \frac{A(z - z^*)}{C^2} + \frac{h'h'}{t' - 1} \right] \, d^2z,
\]

(53)

where \( t' = t/C^2 \). Invoking the value of \( h' \) from equation (32) into (53) and applying the identity (34), we arrive at

\[
\sum_{m=0}^{\infty} t^m I_m = \frac{\exp(-i\phi)}{1 - t'} \exp \left[ -\frac{(\alpha_1^2 + 2t^2) + (t')^2 C^2}{C^2} \right] \times \exp \left\{ -\left[ S_1 (2\alpha_2 C_1 + \alpha_1 S_1) t' - A(t' - 1)^2 \right] / (t' - 1)(t' - 1)C^2 \right\}.
\]

(54)
The exponent in the above equation can be rewritten in terms of the parameter \( t \) as

\[
\sum_{m=0}^{\infty} t^m I_m = \frac{\exp(-i\phi)}{1-t} \exp\left[\frac{-(\alpha_1^2 + 2t_1^2 + \Lambda^2)}{C_r^2}\right]
= \exp(-i\phi) \exp\left[\frac{-(\alpha_1^2 + 2t_1^2 + \Lambda^2)}{C_r^2}\right]
\times \sum_{m=0}^{\infty} t^m I_m \left[\frac{(2\alpha_2 C_r + \alpha_1 S_r + S_r \Lambda)^2}{C_r^4}\right].
\]  
(55)

The transition from the first part to the second one has been done by means of the identity (33). Now the value of the required integral is

\[
I_m = \exp(-i\phi) \exp\left[\frac{-(\alpha_1^2 + 2t_1^2 + \Lambda^2)}{C_r^2}\right]
\times L_m \left[\frac{(2\alpha_2 C_r + \alpha_1 S_r + S_r \Lambda)^2}{C_r^4}\right].
\]  
(56)

Through minor treatments one can easily prove that

\[
\frac{(\alpha_1^2 + 2t_1^2 + \Lambda^2)}{C_r^2} = 2(t_1^2 + t_2^2),
\frac{(2\alpha_2 C_r + \alpha_1 S_r + S_r \Lambda)^2}{C_r^4} = 4t_2^2.
\]  
(57)

Therefore, the quantity \( I_m \) takes the form

\[
I_m = \exp(-i\phi) \exp\left[-2(t_1^2 + t_2^2)\right] L_m(4t_2^2).
\]  
(58)

The value of the integration of the second interference term is just the complex conjugate of (58). Similar procedures lead to the following:

\[
\frac{\pi}{C_r^{2m+2}} \int \exp\left[-\frac{|z-\alpha_1|^2}{C_r^2}\right] L_m[-|z-\alpha_1|^2 \tanh^2 r] r^2 dz
= \frac{\pi}{C_r^{2m+2}} \int \exp\left[-\frac{|z+\alpha_1|^2}{C_r^2}\right] L_m[-|z+\alpha_1|^2 \tanh^2 r] r^2 dz
= 1.
\]  
(59)

From these results, we conclude that

\[
\int Q(z) d^2z = |\lambda_e|^{-2}|\lambda_e|^2 = 1.
\]  
(60)

Using procedures similar to those given above, one can prove that the \( W \) function is normalized (we have checked it).

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