A SUPERCritical Elliptic Problem In A CYLindrical SHEll

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Abstract. We consider the problem
\[-\Delta u = |u|^{p-2} u \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega,
\]
where \(\Omega := \{(y, z) \in \mathbb{R}^{m+1} \times \mathbb{R}^{N-m-1} : 0 < a < |y| < b < \infty\},\)
0 \(\leq m \leq N - 1\) and \(N \geq 2\). Let \(2^*_{N,m} := 2(N-m)/(N-m-2)\) if \(m < N-2\) and \(2^*_{N,m} := \infty\) if \(m = N-2\) or \(N-1\). We show that \(2^*_{N,m}\) is the true critical exponent for this problem, and that there exist nontrivial solutions if \(2 < p < 2^*_{N,m}\) but there are no such solutions if \(p \geq 2^*_{N,m}\).

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To Bernhard Ruf on his birthday, with our friendship and great esteem.

1. Introduction

Consider the Lane-Emden-Fowler problem
\[(1.1) \quad -\Delta u = |u|^{p-2} u \text{ in } \mathcal{D}, \quad u = 0 \text{ on } \partial \mathcal{D}, \]
where \(\mathcal{D}\) is a smooth domain in \(\mathbb{R}^N\) and \(p > 2\).

If \(\mathcal{D}\) is bounded it is well-known that this problem has at least one positive solution and infinitely many sign changing solutions when \(p\) is smaller than the critical Sobolev exponent \(2^*\), defined as \(2^* := \frac{2N}{N-2}\) if \(N \geq 3\) and as \(2^* := \infty\) if \(N = 1\) or \(2\). In contrast, the existence of solutions for \(p \geq 2^*\) is a delicate issue. Pohozaev’s identity \[12\] implies that problem \[(1.1)\] has no nontrivial solution if the domain \(\mathcal{D}\) is strictly starshaped. On the other hand, Bahri and Coron \[2\] proved that a positive solution to \[(1.1)\] exists if \(p = 2^*\) and \(\mathcal{D}\) is bounded and has nontrivial reduced homology with \(\mathbb{Z}/2\) coefficients.

One may ask whether this last statement is also true for \(p > 2^*\). Pasese showed in \[10\] \[11\] that this is not so: for each \(1 \leq m < N-2\) he exhibited a bounded smooth domain \(\mathcal{D}\) which is homotopy equivalent to the \(m\)-dimensional sphere, in which problem \[(1.1)\] has infinitely many solutions.
if $p < 2^*_N,m := \frac{2(N-m)}{N-m-2}$ and does not have a nontrivial solution if $p \geq 2^*_N,m$.

Examples of domains with richer homology were recently given by Clapp, Faya and Pistoia in [3]. Wei and Yan established in [17] the existence of infinitely many positive solutions for $p = 2^*_N,m$ in some bounded domains.

For $p$ slightly below $2^*_N,m$ solutions concentrating along an $m$-dimensional manifold were recently obtained in [1, 4]. Note that $2^*_N,m$ is the critical Sobolev exponent in dimension $N-m$. It is called the $(m+1)$-st critical exponent for problem (1.1).

The purpose of this note is to exhibit unbounded domains in which this problem has the behavior described by Passaseo.

We consider the problem
\begin{equation}
\begin{cases}
-\Delta u = |u|^{p-2} u & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega, \\
|\nabla u|^2, |u|^p & \in L^1(\Omega),
\end{cases}
\end{equation}
in a cylindrical shell
\[ \Omega := \{ x = (y, z) \in \mathbb{R}^{m+1} \times \mathbb{R}^{N-m-1} : a < |y| < b \}, \quad 0 < a < b < \infty, \]
for $p > 2$.

If $m = N-1$ or $N-2$, we set $2^*_N,m := \infty$. First note that if $m = N-1$ then $\Omega = \{ x \in \mathbb{R}^N : a < |x| < b \}$, and a well-known result by Kazdan and Warner [9] asserts that (1.2) has infinitely many radial solutions for any $p > 2$. In the other extreme case, where $m = 0$, the domain $\Omega$ is the union of two disjoint strips $(a, b) \times \mathbb{R}^{N-1}$ and $(-b, -a) \times \mathbb{R}^{N-1}$. Each of them is starshaped, so there are no solutions for $p \geq 2^*_N,0 = 2^*$. Esteban showed in [5] that there are infinitely many solutions in $(a, b) \times \mathbb{R}^{N-1}$ if $N \geq 3$ and $p < 2^*$, and one positive solution if $N = 2$ (in fact, she considered a more general problem). These solutions are axially symmetric, i.e. $u(y, z) = u(y, |z|)$ for all $(y, z) \in \Omega$.

Here we study the remaining cases, i.e., $1 \leq m \leq N-2$. Our first result states the nonexistence of solutions other than $u = 0$, if $p \geq 2^*_N,m$.

**Theorem 1.1.** If $1 \leq m < N-2$ and $p \geq 2^*_N,m$, then problem (1.2) does not have any nontrivial solution $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$.

Our next result shows that solutions $u \neq 0$ do exist if $2 < p < 2^*_N,m$.

As usual, we write $O(k)$ for the group of linear isometries of $\mathbb{R}^k$ (represented by orthogonal $k \times k$-matrices). Recall that if $G$ is a closed subgroup of $O(N)$ then a subset $X$ of $\mathbb{R}^N$ is $G$-invariant if $gX = X$ for every $g \in G$, and a function $u : X \to \mathbb{R}$ is called $G$-invariant provided $u(gx) = u(x)$ for all $g \in G$, $x \in X$. 
Note that $\Omega$ is $[O(m+1) \times O(N - m - 1)]$-invariant for the obvious action given by $(g, h)(y, z) := (gy, hz)$ for all $g \in O(m+1), h \in O(N - m - 1), y \in \mathbb{R}^{m+1}, z \in \mathbb{R}^{N-m-1}$.

**Theorem 1.2.** (i) If $1 \leq m < N - 2$ and $2 < p < 2^*_N$, then problem (1.2) has infinitely many $[O(m+1) \times O(N - m - 1)]$-invariant solutions and one of these solutions is positive.

(ii) If $1 \leq m = N - 2$ and $2 < p < \infty$, then problem (1.2) has a positive $[O(N - 1) \times O(1)]$-invariant solution.

In Section 2 we prove Theorem 1.1. Theorem 1.2 is proved in Section 3.

We conclude the paper with a multiplicity result and an open question in Section 4.

2. A Pohožaev identity and the proof of Theorem 1.1

We prove Theorem 1.1 by adapting Passaseo’s argument in [10, 11], see also [3]. The proof relies on the following special case of a Pohožaev type identity due to Pucci and Serrin [13].

For $(u, v) \in \mathbb{R} \times \mathbb{R}^N$ we set

$$
\phi(u, v) := \frac{1}{2} |v|^2 - \frac{1}{p} |u|^p.
$$

**Lemma 2.1.** If $u \in C^2(\Omega)$ satisfies $-\Delta u = |u|^{p-2} u$ in $\Omega$ then, for every $\chi \in C^1(\overline{\Omega}, \mathbb{R}^N)$, the equality

$$
(\text{div } \chi) \phi(u, \nabla u) - D\chi [\nabla u] \cdot \nabla u = \text{div} [\phi(u, \nabla u) \chi - (\chi \cdot \nabla u) \nabla u]
$$

holds true.

**Proof.** Put $\chi = (\chi_1, \ldots, \chi_N)$, denote the partial derivative with respect to $x_k$ by $\partial_k$ and let LHS and RHS denote the left- and the right-hand side of (2.1). Then

$$
\text{LHS} = (\text{div } \chi) \phi(u, \nabla u) - \sum_{j,k} \partial_k \chi_j \partial_j u \partial_k u
$$

and

$$
\text{RHS} = (\text{div } \chi) \phi(u, \nabla u) + \sum_{j,k} \chi_k \partial_j u \partial^2_{jk} u - |u|^{p-2} u \nabla u \cdot \chi
$$

$$
- (\nabla u \cdot \chi) \Delta u - \sum_{j,k} \partial_k \chi_j \partial_j u \partial_k u - \sum_{j,k} \chi_j \partial_k u \partial^2_{jk} u
$$

$$
= (\text{div } \chi) \phi(u, \nabla u) - (\nabla u \cdot \chi)(\Delta u + |u|^{p-2} u) - \sum_{j,k} \partial_k \chi_j \partial_j u \partial_k u.
$$

Since $-\Delta u = |u|^{p-2} u$, the conclusion follows. \qed
Using a well-known truncation argument, we can now prove the following result.

**Proposition 2.2.** Assume that $\chi \in C^1(\overline{\Omega}, \mathbb{R}^N)$ has the following properties:

(a) $\chi \cdot \nu$ is bounded on $\partial \Omega$, where $\nu(s)$ is the outer unit normal at $s \in \partial \Omega$,
(b) $|\chi(x)| \leq |x|$ for every $x \in \Omega$,
(c) $\text{div} \chi$ is bounded in $\Omega$,
(d) $|D\chi(x)\xi \cdot \xi| \leq |\xi|^2$ for all $x \in \Omega$, $\xi \in \mathbb{R}^N$.

Then every solution $u \in C^2(\Omega) \cap C^1(\Omega)$ of (1.2) satisfies

$$
\text{(2.2)} \quad \frac{1}{2} \int_{\partial \Omega} |\nabla u|^2 \chi \cdot \nu = - \int_{\Omega} (\text{div} \chi) \phi(u, \nabla u) + \int_{\Omega} D\chi [\nabla u] \cdot \nabla u.
$$

**Proof.** Choose $\psi \in C^\infty(\mathbb{R})$ such that $0 \leq \psi(t) \leq 1$, $\psi(t) = 1$ if $|t| \leq 1$ and $\psi(t) = 0$ if $|t| \geq 2$. For each $k \in \mathbb{N}$ define

$$
\psi_k(x) := \psi \left( \frac{|x|^2}{k^2} \right) \quad \text{and} \quad \chi^k(x) := \psi_k(x) \chi(x).
$$

Note that there is a constant $c_0 > 0$ such that

$$
\text{(2.3)} \quad |x| |\nabla \psi_k(x)| \leq c_0 \quad \text{for all} \quad x \in \mathbb{R}^N, \ k \in \mathbb{N}.
$$

Next, choose a sequence of bounded smooth domains $\Omega_k \subset \Omega$ such that

$$
\text{(2.4)} \quad \Omega_k \supset \Omega \cap B_{2k}(0).
$$

Integrating (2.1) with $\chi := \chi^k$ in $\Omega_k$ and using the divergence theorem and Lemma 2.1 we obtain

$$
\int_{\Omega_k} \left( \text{div} \chi^k \right) \phi(u, \nabla u) - \int_{\Omega_k} D\chi^k [\nabla u] \cdot \nabla u =
\int_{\partial \Omega_k} \left[ \phi(u, \nabla u) \left( \chi^k \cdot \nu^k \right) - \left( \chi^k \cdot \nabla u \right) \left( \nabla u \cdot \nu^k \right) \right],
$$

where $\nu^k$ is the outer unit normal to $\Omega_k$. Property (2.3) implies that $\chi^k = 0$ in $\Omega \setminus \Omega_k$, so we may replace $\Omega_k$ by $\Omega$, $\partial \Omega_k$ by $\partial \Omega$ and $\nu^k$ by $\nu$ in the previous identity. Moreover, since $u = 0$ on $\partial \Omega$, we have that

$$
\nabla u = (\nabla u \cdot \nu) \nu \quad \text{on} \quad \partial \Omega.
$$

Therefore,

$$
\int_{\Omega} \left( \text{div} \chi^k \right) \phi(u, \nabla u) - \int_{\Omega} D\chi^k [\nabla u] \cdot \nabla u =
\int_{\partial \Omega} \left[ \phi(u, \nabla u) \left( \chi^k \cdot \nu \right) - \left( \chi^k \cdot \nabla u \right) \left( \nabla u \cdot \nu \right) \right] =
\int_{\partial \Omega} \left[ \phi(u, \nabla u) - |\nabla u|^2 \right] \left( \chi^k \cdot \nu \right) = - \frac{1}{2} \int_{\partial \Omega} |\nabla u|^2 \psi_k(x) (\chi \cdot \nu).
$$
Since $\text{div} \chi^k = \psi_k \text{div} \chi + \nabla \psi_k \cdot \chi$, using (2.3) and properties (b) and (c) we obtain

\[(2.6) \quad |\text{div} \chi^k| \leq |\text{div} \chi| + |\nabla \psi_k| |\chi| \leq |\text{div} \chi| + c_0 \leq c_1 \quad \text{in } \Omega.\]

Similarly, since

\[D\chi^k(x)\xi \cdot \xi = \psi_k(x)D\chi(x)\xi \cdot \xi + (\nabla \psi_k \cdot \xi)(\chi \cdot \xi),\]

property (d) yields

\[(2.7) \quad |D\chi^k(x)\xi \cdot \xi| \leq (1 + c_0) |\xi|^2 \quad \text{for all } x \in \Omega, \xi \in \mathbb{R}^N.\]

Inequalities (2.6), (2.7) and property (a) allow us to apply Lebesgue’s dominated convergence theorem to the left- and the right-hand sides of (2.5) to obtain

\[\int_\Omega (\text{div} \chi) \phi(u, \nabla u) - \int_\Omega D\chi [\nabla u] \cdot \nabla u = -\frac{1}{2} \int_{\partial \Omega} |\nabla u|^2 (\chi \cdot \nu),\]

as claimed. □

**Proof of Theorem 1.1.** Let $\varphi(t) = \frac{1}{m+1} \left[1 - \left(\frac{a}{t}\right)^{m+1}\right]$ be the solution to the boundary value problem

\[\begin{cases} 
\varphi'(t) + (m+1) \varphi(t) = 1, & t \in (0, \infty), \\
\varphi(a) = 0.
\end{cases}\]

Define

\[(2.8) \quad \chi(y, z) := (\varphi(|y|) y, z).\]

Then, if $\nu$ denotes the outer unit normal on $\partial \Omega$,

\[(2.9) \quad (\chi \cdot \nu)(y, z) = \begin{cases} 
0 & \text{if } |y| = a, \\
\frac{1}{m+1} \left[1 - \left(\frac{a}{b}\right)^{m+1}\right] b & \text{if } |y| = b.
\end{cases}\]

So property (a) of Proposition 2.2 holds. Clearly, (b) holds. Now,

\[(2.10) \quad \text{div} \chi(y, z) = [\varphi'(|y|)|y| + (m+1) \varphi(|y|)] + N - m - 1 = N - m.\]

In particular, (c) holds. To prove (d) notice that $\chi$ is $O(m+1)$-equivariant, i.e.

\[\chi(gy, z) = g\chi(y, z) \quad \text{for every } g \in O(m+1).\]

Therefore, $g \circ D\chi(y, z) = D\chi(gy, z) \circ g$ and, hence,

\[\langle D\chi(y, z) [\xi], \xi \rangle = \langle g(D\chi(gy, z) [\xi]), g\xi \rangle = \langle D\chi(gy, z) [g\xi], g\xi \rangle\]

for all $\xi \in \mathbb{R}^N$. Thus, it suffices to show that the inequality (d) holds for $y = (t, 0, \ldots, 0)$ with $t \in (a, b)$. A straightforward computation shows that, for such $y$, $D\chi(y)$ is a diagonal matrix whose diagonal entries are $a_{11} =
\[1 - m \varphi(t), \ a_{jj} = \varphi(t) \text{ for } j = 2, \ldots, m + 1, \text{ and } a_{jj} = 1 \text{ for } j = m + 2, \ldots, N.\]

Since \(a_{jj} \in (0, 1],\)
\[
(2.11) \quad 0 < \langle D\chi(y, z)[\xi], \xi \rangle \leq |\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^N \setminus \{0\}
\]
and (d) follows. From (2.3), (2.2), (2.11) and (2.10) we obtain
\[
0 < \frac{1}{2} \int_{\partial\Omega} |\nabla u|^2 \chi \cdot \nu - \int_{\Omega} (\text{div } \chi) \phi(u, \nabla u) + \int_{\Omega} D\chi(\nabla u) \cdot \nabla u
\]
\[
\leq (N - m) \left( \frac{1}{p} |u|^p - \frac{1}{2} |\nabla u|^2 \right) + \int_{\Omega} |\nabla u|^2
\]
\[
= (N - m) \left( \frac{1}{p} - \frac{1}{2} \frac{1}{N - m} \right) \int_{\Omega} |\nabla u|^2.
\]

The first (strict) inequality follows from the unique continuation property \([8, 7].\) This immediately implies that \(p < 2^*_{N,m}.\)

\[\square\]

3. THE PROOF OF THEOREM 1.2

An \(O(m + 1)-\text{invariant function} u(y, z) = v(|y|, z)\) solves problem (1.2) if and only if \(v = v(r, z)\) solves
\[
(3.1) \quad \begin{cases} -\Delta v - \frac{m}{r} \frac{\partial v}{\partial r} = |v|^{p-2}v & \text{in } (a, b) \times \mathbb{R}^{N-m-1} =: \mathcal{S}, \\ \text{on } \{a, b\} \times \mathbb{R}^{N-m-1} = \partial\mathcal{S}, \end{cases}
\]
and \(|\nabla v|^2, |v|^p \in L^1(\mathcal{S}).\) Problem (3.1) can be rewritten as
\[
(3.2) \quad -\text{div}(r^m \nabla v) = r^m |v|^{p-2}v \quad \text{in } \mathcal{S}, \quad v = 0 \quad \text{on } \partial\mathcal{S}.
\]

By Poincaré’s inequality (see Lemma 3 in [5]) and since \(a < r < b,\) the norms
\[
(3.3) \quad \|v\|_m := \left( \int_{\mathcal{S}} r^m |\nabla v|^2 \right)^{1/2} \quad \text{and} \quad |v|_{m,p} := \left( \int_{\mathcal{S}} r^m |v|^p \right)^{1/p}
\]
are equivalent to those of \(H^1_0(\mathcal{S})\) and \(L^p(\mathcal{S})\) respectively.

Consider the functional \(I(v) := \|v\|_m^2\) restricted to
\[M := \{v \in H^1_0(\mathcal{S}) : |v|_{m,p} = 1\}.
\]

Then \(M\) is a \(C^2\)-manifold, and \(v\) is a critical point of \(I|_M\) if and only if \(v \in H^1_0(\mathcal{S})\) and \(\|v\|_m^{2(p-2)}v\) is a nontrivial solution to (3.2). Note that \(I|_M\) is bounded below by a positive constant.

**Proof of Theorem 1.2 (i).** Assume that \(1 \leq m < N - 2\) and \(2 < p < 2^*_{N,m}.\) Set \(G := O(N - m - 1)\) and denote by \(H^1_0(\mathcal{S})^G\) and \(L^p(\mathcal{S})^G\) the subspaces of \(H^1_0(\mathcal{S})\) and \(L^p(\mathcal{S})\) respectively, consisting of functions \(v\) such that \(v(r, gz) = v(r, z)\) for all \(g \in G.\) Esteban and Lions showed in [6] that, for these values of \(m\) and \(p, H^1_0(\mathcal{S})^G\) is compactly embedded in \(L^p(\mathcal{S})^G\) (see also Theorem
1.24 in [18]. So $H^1_0(S)^G$ is compactly embedded in $L^p(S)^G$ for the norms (3.3) as well.

Let

$$M^G := \{ v \in H^1_0(S)^G : |v|_{m,p} = 1 \}.$$

It follows from the principle of symmetric criticality [18, Theorem 1.28] that the critical points of $I|_{M^G}$ are also critical points of $I|_M$. The manifold $M^G$ is radially diffeomorphic to the unit sphere in $H^1_0(S)^G$, so its Krasnoselskii genus is infinite. A standard argument, using the compactness of the embedding $H^1_0(S)^G \hookrightarrow L^p(S)^G$ for the norms (3.3), shows that $I|_{M^G}$ satisfies the Palais-Smale condition. Hence $I|_{M^G}$ has infinitely many critical points (see e.g. Theorem II.5.7 in [15]). It can also be shown by a well-known argument that the critical values of $I|_{M^G}$ tend to infinity (see e.g. Proposition 9.33 in [14]).

It remains to show that (3.2) has a positive solution. The argument is again standard: since $I|_{M^G}$ satisfies the Palais-Smale condition,

$$c_0^G := \inf \{ I(v) : v \in M^G \}$$

is attained at some $v_0$. Since $I(v) = I(|v|)$ and $|v| \in M^G$ if $v \in M^G$, we have that $I(|v_0|) = c_0^G$ and we may assume $v_0 \geq 0$. The maximum principle applied to the corresponding solution $u_0$ of (1.2) implies $u_0 > 0$. \qed

If $m = N - 2$, then $G = O(1)$ and it is easy to see that the space $H^1_0(S)^G$ is not compactly embedded in $L^p(S)^G$. So part (ii) of Theorem 1.2 requires a different argument.

**Proof of Theorem 1.2 (ii).** Assume that $1 \leq m = N - 2$ and $2 < p < \infty$. We shall show that

$$c_0 := \inf \{ I(v) : v \in M \}$$

is attained. Clearly, a minimizing sequence $(v_n)$ is bounded, so we may assume that $v_n \rightharpoonup v$ weakly in $H^1_0(S)$. According to P.-L. Lions’ lemma [18, Lemma 1.21] either $v_n \to 0$ strongly in $L^p(S)$, which is impossible because $v_n \in M$, or there exist $\delta > 0$ and $(r_n, z_n) \in [a, b] \times \mathbb{R}$ such that, after passing to a subsequence if necessary,

$$\int_{B_1(r_n, z_n)} v_n^2 \geq \delta. \tag{3.4}$$

Here $B_1(r_n, z_n)$ denotes the ball of radius 1 and center at $(r_n, z_n)$. Since the problem is invariant with respect to translations along the z-axis, replacing $v_n(r, z)$ by $v_n(r + z_n)$, we may assume the center of the ball above is $(r_n, 0)$. It follows that for this - translated - sequence the weak limit $v$ cannot be zero due to (3.4) and the compactness of the embedding of $H^1_0(S)$
in $L^2_{loc}(\mathcal{S})$. Passing to a subsequence once more, we have that $v_n(x) \to v(x)$ a.e. It follows from the Brezis-Lieb lemma [18, Lemma 1.32] that

$$1 = |v_n|_{m,p}^p = \lim_{n \to \infty} |v_n - v|_{m,p}^p + |v|_{m,p}^p.$$  

Using this identity and the definition of $c_0$ we obtain

$$c_0 = \lim_{n \to \infty} ||v_n||^2_m = \lim_{n \to \infty} ||v_n - v||^2_m + ||v||^2_m \geq c_0 \left( \lim_{n \to \infty} |v_n - v|_{m,p}^2 + |v|_{m,p}^2 \right)$$

$$= c_0 \left( (1 - |v|_{m,p}^2)^{2/p} + (|v|_{m,p}^2)^{2/p} \right) \geq c_0 (1 - |v|_{m,p}^p + |v|_{m,p}^p)^{2/p} = c_0.$$  

Since $v \neq 0$, it follows that $|v_n - v|_{m,p} \to 0$ and $|v|_{m,p} = 1$. So $v \in M$ and, as $c_0 = \lim_{n \to \infty} I(v_n) \geq I(v)$, we must have $I(v) = c_0$.

So the infimum is attained at $v$ and using the moving plane method [18, Appendix C], we may assume, after translation, that $v(r, -z) = v(r, z)$, i.e. $v \in H^1_0(\mathcal{S})^{O(1)}$. As in the preceding proof, replacing $v$ by $|v|$, we obtain a positive solution. 

\[\square\]

4. Further solutions and an open question

If $1 \leq m = N - 2$ and $p \in (2, 2N/m)$, the method we have used to prove Theorem 1.2 only guarantees the existence of two solutions to problem (1.2), one positive and one negative, up to translations along the $z$-axis. However, if $p \in (2, 2^*)$, then it is possible to show that there are infinitely many solutions, which are not radial in $y$, but have other prescribed symmetry properties.

Write $y = (y^1, y^2) \in \mathbb{R}^2 \times \mathbb{R}^{m-1} \equiv \mathbb{R}^{m+1}$ and identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$. Following [16], we denote by $G_k$, $k \geq 3$, the subgroup of $O(2)$ generated by two elements $\alpha, \beta$ which act on $\mathbb{C}$ by

$$\alpha y^1 := e^{2\pi i/k} y^1, \quad \beta y^1 := e^{2\pi i/k} \overline{y^1},$$

i.e. $\alpha$ is the rotation in $\mathbb{C}$ by the angle $2\pi/k$ and $\beta$ is the reflection in the line $y^1_2 = \tan(\pi/k)y^1_1$, where $y^1 = y^1_1 + iy^1_2 \in \mathbb{C}$. Observe that $\alpha, \beta$ satisfy the relations $\alpha^k = \beta^2 = e$, $\alpha \beta \alpha = \beta$. Let $G_k$ act on $\mathbb{R}^N$ by $g x = (gy^1, y^2, z)$.

**Theorem 4.1.** If $1 \leq m \leq N - 2$ and $2 < p < 2^*$ then, for each $k \geq 3$, problem (1.2) has a solution $u_k$ which satisfies

\[4.1\]

$$u_k(x) = \det(g)u_k(g^{-1} x) \quad \text{for all } g \in G_k,$$

and $u_k \neq u_j$ if $k \neq j$.

**Proof.** Since the approach is taken from [16], we give only a brief sketch of the proof here and refer to Section 2 of [16] for more details.
The group $G_k$ acts on $H^1_0(\Omega)$ by
\[(gu)(x) := \det(g)u(g^{-1}x),\]
where $\det(g)$ is the determinant of $g$. Let
\[H^1_0(\Omega)^{G_k} := \{u \in H^1_0(\Omega) : u(gx) = \det(g)u(g^{-1}x) \text{ for all } g \in G_k\}\]
be the fixed point space of this action, and define $I(u) := \int_\Omega |\nabla u|^2$ and
\[M^{G_k} := \{u \in H^1_0(\Omega)^{G_k} : |u|_p = 1\}.
By the principle of symmetric criticality the critical points of $I|_{M^{G_k}}$ are nontrivial solutions to problem \((1.2)\) which satisfy \((1.1)\). Now we can see as in the proof of part (ii) of Theorem \(1.2\) that there exists a minimizer $u_k$ for $I$ on the manifold $M^{G_k}$. Moreover, we may assume that $u_k$ has exactly $2k$ nodal domains, see Corollary 2.7 in \[16\]. So in particular, $u_k \neq u_j$ if $k \neq j$.

The question whether problem \((1.2)\) has infinitely many solutions when $1 \leq m = N-2$ and $p \in [2^*, 2^*_{N,m})$ remains open. We believe that the answer is yes, but the proof would require different methods.

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