Pair-distribution functions of correlated composite fermions

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Pair-distribution functions \( g(r) \) of Laughlin quasielectrons (composite fermions in their second Landau level) are calculated in the fractional quantum Hall states at electron filling factors \( \nu_e = 4/11 \) and \( 3/8 \). A shoulder in \( g(r) \) is found, supporting the idea of cluster formation. The intra- and inter-cluster contributions to \( g(r) \) are identified, largely independent of \( \nu_e \). The average cluster sizes are estimated; pairs and triplets of quasielectrons are suggested at \( \nu_e = 4/11 \) and \( 3/8 \), respectively.

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I. INTRODUCTION

Pan et al.\(^1\) have recently observed the fractional quantum Hall (FQH) effect\(^2\) in a spin-polarized two-dimensional electron gas (2DEG) at the \( \nu = \frac{4}{11}, \frac{3}{8} \), and \( \frac{5}{13} \) fillings of the lowest Landau level (LL). In the composite fermion (CF) model\(^3\),\(^4\),\(^5\) these values correspond to the fractional fillings \( \nu = \frac{1}{2}, \frac{1}{3}, \) and \( \frac{2}{5} \) of the second CF LL, respectively. In Haldane’s hierarchy picture\(^6\), these states, Laughlin quasielectrons (QE’s) fill (the same) fraction \( \nu \) of their LL. The most striking conclusion from Pan’s discovery is that the CF’s (or QE’s) can also form incompressible states when partially filling a LL. This could not be predicted by a simple analogy with known fractional electron liquids (Laughlin, Jain, or Moore–Read states), because of a different form of QE–QE interaction\(^6\),\(^7\),\(^8\),\(^9\) therefore yielding qualitatively different QE–QE correlations.

Although several numerical studies of interacting QE’s have been reported\(^10\),\(^11\),\(^12\),\(^13\) and ideas such as CF flavor-mixing\(^14\), QE pairing\(^15\),\(^16\) or stripes\(^17\),\(^18\) were invoked, the correlations responsible for the FQHE at \( \nu_e = \frac{4}{11} \) and \( \frac{3}{8} \) are not yet understood. It has not even been settled if these QE states are isotropic, and the energies of liquid and solid phases were compared recently\(^19\) (although the Laughlin form was arbitrarily assumed for the liquid).

Sometimes overlooked is a general connection\(^18\),\(^19\) between the form of Haldane pseudopotential\(^20\) occurrence of Laughlin correlations, and the validity of CF transformation. Actually, the form of QE–QE interaction is known from independent calculations\(^8\),\(^9\),\(^10\) and Laughlin correlations among the QE’s have been ruled out using both a general pseudopotential argument\(^21\) and a direct analysis of many-QE wavefunctions\(^22\). In this paper we refer to the following well-established facts:

(i) The QE–QE Haldane pseudopotential\(^22\) is known from exact diagonalization of the Coulomb interaction among electrons in the lowest LL\(^8\),\(^9\),\(^10\). Since there are no unchecked assumptions in such a calculation, it must be regarded a “numerical experiment.” Neither finite-size errors, lowest-LL restriction, finite 2DEG width, nor other details of realistic experimental systems affect the dominant feature of the pseudopotential which is the lack of strong QE–QE repulsion at short range.

(ii) The QE’s do not\(^13\) have Laughlin correlations at \( \nu = \frac{1}{2} \) corresponding to \( \nu_e = \frac{4}{11} \). The Moore–Read halffilled state is not\(^13\) an adequate description of QE–QE correlations at \( \nu = \frac{1}{2} \) corresponding to \( \nu_e = \frac{3}{8} \).

(iii) A sequence of nondegenerate finite-size QE ground states with a gap, extrapolating to \( \nu = \frac{1}{2} \) has been found\(^13\) on a sphere. Although spherical geometry is not adequate for studying crystal or other broken-symmetry phases, the identified states appear incompressible and have the lowest energy of all QE liquids (considerably below the Laughlin state).

To address the problem of correlations at \( \nu_e = \frac{4}{11}, \frac{3}{8}, \) and \( \frac{5}{13} \) we calculate pair-distribution functions \( g(r) \) in the incompressible liquid ground states of up to \( N = 14 \) QE’s. Their comparison with the (known) curves of the Laughlin and Moore–Read states implies a different nature of the QE correlations in these novel FQH states. It shows that their incompressibility cannot be explained by a simple analogy between the QE and electron liquids, and suggests that different wavefunctions need be proposed for the correlated CF’s. Unfortunately, the calculated \( g(r) \) are of little help in a precise definition of these wavefunctions, even though some qualitative statements can be made about the QE correlations.

From our finite-size results we identify and analyze the size-independent features in \( g(r) \): the \( \sim r^2 \) behavior at short range and a shoulder at a medium range, and argue that they are consistent with the idea\(^12\) of QE cluster formation. Short- and long-range contributions to \( g(r) \) are found, describing correlations between the QE’s from the same or different clusters. Both intra- and inter-cluster QE–QE correlations depend rather weakly on \( \nu \). The average size of the clusters is estimated; it seems that the QE’s form pairs at \( \nu = \frac{4}{11} \) and triplets at \( \nu = \frac{3}{8} \).

A similar analysis of \( g(r) \) carried out for the Moore–Read state reveals a qualitatively different behavior.

II. MODEL

A. Haldane sphere

The numerical calculations have been carried out in Haldane’s spherical geometry\(^22\) convenient for the exact
B. Exact diagonalization

Recently, we have identified the series of finite-size spin-polarized states that in the thermodynamic limit describe the FQHE at $\nu_c = \frac{4}{\pi}$ and $\frac{3}{8}$. To do so, we have carried out extensive exact-diagonalization calculations for interacting QE’s (particles in the second CF LL). On Haldane sphere, $N$ fermionic QE’s were confined in a standard way to an angular momentum shell of degeneracy $\Gamma = 2l + 1$, corresponding to the QE filling factor $\nu \sim \sqrt{\frac{N}{\bar{\nu}}}$. The Haldane QE–QE pseudopotential $V(R)$ was taken from earlier calculations.8,9,10

Regardless of the electron layer width $\nu$, magnetic field $B$, or other experimental parameters, the dominant feature of $V(R)$ is strong repulsion at $R = 3$. This feature alone determines the wavefunctions at $\frac{4}{\pi} \leq \nu \leq \frac{3}{8}$ (with the QE–QE correlations consisting of maximum possible avoidance of Haldane pair amplitude $G$ at $R = 3$), which are hence virtually insensitive to the (sample-dependent) details of $V(R)$. This justifies model calculations using $V(R)$ of Refs. 8,9,10. Actually, a model pseudopotential as simple as $V = \delta_{R,3}$ is sufficient to reproduce correct correlations and incompressibility at $\nu_c = \frac{4}{\pi}$ or $\frac{3}{8}$.

III. NUMERICAL RESULTS

A. Energy spectra

The numerical results carried out for $N \leq 14$ (two sample spectra are displayed in Fig. 1) showed a sequence of nondegenerate (i.e., at the total angular momentum $L = 0$) ground states at $2l = N/\nu - \gamma$ with $\nu = \frac{2}{5}$ and $\gamma = 7$. The significant and well-behaved (as a function of $N$) excitation gap along this sequence strongly suggests that it represents the infinite $\nu_c = \frac{4}{\pi}$ FQH state observed in experiment. The value $\gamma \neq 3$ precludes Laughlin correlations among QE’s in this state (earlier ruled out indirectly, based on the form of QE–QE pseudopotentials,).

B. Pair-distribution functions

The QE–QE pair-distribution functions $g(r)$ have been calculated for the incompressible many-QE ground states as expectation values of the appropriate pair interaction,

$$g(r) = \langle \frac{2}{N} \delta(R \theta - r) \rangle.$$  

Here, $\theta$ is the relative angle on a sphere, so that $r$ measures interparticle distance along the surface (rather than
chord distance). More accurately, \( r \) is the distance between the centers of extended QE’s (note that in the calculation of many-QE wavefunctions, the system of QE’s is mapped onto the lowest LL of point charges interacting through an effective pseudopotential). The prefactor in Eq. (1) ensures proper normalization, \( g(\infty) \to 1 \). Denoting infinitesimal area by \( dS = 2\pi R^2 \, d(\cos \theta) \) or (in magnetic units) by \( ds = dS/2\pi \lambda^2 \), we get an equivalent normalization condition,

\[
\int [1 - g(r)] \, ds = \frac{2l}{N} \to \nu^{-1}
\]

in large systems. Since \( ds = l \, d(\cos \theta) \), a “local filling factor” can also be defined as \( \nu(r) = dN/ds = (N/2l) \, g(r) \), and it satisfies \( \nu(\infty) = \nu \) and \( \int \nu(r) \, ds = N - 1 \).

The results for the \( \nu = \frac{1}{3} \) sequence at \( 2l = 3N - 7 \) are shown in Fig. 2(a). Similarity of all four curves is evident, indicating size-independent form of correlations (hence, describing an infinite system), with a well-developed shoulder around \( r \approx 2.5\lambda \). Similar shoulders occur in \( g(r) \) of all incompressible ground states at \( \nu = \frac{2}{3} \) or \( \frac{1}{3} \) (the \( \nu = \frac{2}{3} \) sequence at \( 2l = \frac{3}{2}N + 2 \) is obtained from \( 2l = 3N - 7 \) by replacing \( N \) with \( \Gamma - N \), while at \( \nu = \frac{1}{3} \), there are two particle–hole conjugate sequences at \( 2l = 2N - 3 \) and \( 2N + 1 \), denoted by \( \nu = \frac{1}{3}^{\pm} \). The four curves representative of \( \nu = \frac{1}{3}, \frac{2}{3}, \frac{1}{2} \), and \( \frac{1}{3}^{\pm} \) are shown in Fig. 2(b). They are all clearly different from those marked with thin lines and describing correlations known for other incompressible FQH states (full LL, Laughlin \( \nu = \frac{1}{3} \) state, or Moore–Read half-filled state). This is a direct indication of a different nature of QE–QE correlations responsible for the FQHE at \( \nu e/2l = \frac{1}{3} \) and \( \frac{2}{3} \).

Let us stress that although the QE–QE interactions are not known with great accuracy, the correlation functions in Fig. 2 are rather insensitive to the details of \( V(R) \), as long as the dominant repulsion occurs at \( R = 3 \) (which seems to be universally true in the systems studied experimentally). This insensitivity is reminiscent of the Laughlin wavefunction, which also very accurately describes the actual \( \nu = \frac{1}{3} \) ground state for a wide class of electron–electron pseudopotentials. However, while the avoidance of \( R = 1 \) by the electrons in the lowest LL can be elegantly described by flux attachment in the CF picture, no similar model has been proposed yet for the avoidance of \( R = 3 \) by the QE’s. Therefore, knowing the \( g(r) \) curves of QE’s and understanding their correlations, we still cannot write their wavefunctions.

C. Gaussian deconvolution

The curves of Fig. 2(b) can be accurately deconvoluted using gaussians, \( G(r/\lambda) = A \exp(-r^2/(2\lambda^2)) \). This is shown in Fig. 3 where the symbols mark the exact data of Fig. 2(b) and the lines give the (nearly perfect) fits using three gaussians, \( g = 1 - G_0 - G_1 - G_2 \) (sufficient for \( r \leq 6\lambda \)). The fitted values of \([A_i, \delta_i, \sigma_i] \) for all four curves are listed in Tab. I. Note that \( A_0 = 1, \delta_0 = 0, \) and \( \delta_1 = 3 \) for all curves (the latter value being least obvious, but probably resulting from the avoidance of the same \( R_3 = 3 \) by the QE’s at all values of \( \nu \)). The values of the \( G_2 \) parameters are not very meaningful when the next term in the approximation (\( G_3 \)) is neglected. The clearest difference between the four curves is in \( A_1 \).

D. Short/long-range deconvolution

It appears more physically meaningful to decompose \( g(r) \) into \( g_0 = 1 - \exp(-r^2/2\lambda^2) \) describing a full lowest
We note that: (i) For the pairs of particle–hole conjugate states, the QE–QE pseudopotential nearly vanishes at \( R = 1 \) and is strongly repulsive at \( R = 3 \), causing an increase of \( G(1) \) and a simultaneous decrease of \( G(3) \) compared to the Laughlin-correlated state.\(^{12}\)

The assumption that QE’s form clusters naturally explains a shoulder in \( g(r) \), and allows one to interpret \( g_0 \) and \( g_{\text{diff}} \) as the intra- and inter-cluster QE–QE correlations, i.e. the short- and long-range contributions to \( g \), corresponding to the QE pairs belonging to the same or different clusters, respectively. The vanishing of \( g_{\text{diff}}(r) \) at short-range reflects isolation of QE’s belonging to different clusters. The reason why \( g_{\text{diff}} \) is not positive definite is that intra-cluster correlations are accurately described by \( g_0 \) only within a certain radius. In other words, the actual inter-cluster contribution to \( g \) is not exactly given by \( g_{\text{diff}} \) defined by Eq. (3). Nevertheless, the following two conclusions remain valid: (i) the intra- and inter-cluster QE–QE correlations are similar at \( \nu = \frac{1}{3} \), and \( \frac{5}{3} \), with the respective correlation-hole radii \( g_0 \sim \lambda \) and \( g_1 \sim 4\lambda \); and (ii) the cluster size \( K \) depends on \( \nu \).

Similar form of \( g(r) \) was found\(^{22}\) for broken-symmetry Laughlin states, in which the shoulder results from angular averaging of an anisotropic function \( g(\rho, \phi) \sim r^2 \) or \( r^\phi \), depending on \( \phi \). However, the present case of QE’s is different, because \( g(r) \) is isotropic (wavefunctions have \( L = 0 \)) and the shoulders result from radial averaging of inter- and intra-cluster correlations, (beginning as \( \sim r^2 \) and a higher power of \( r \) at short range, respectively).

**IV. DISCUSSION**

**A. QE clustering**

Some information about the form of QE–QE correlations can be easily deduced from the form of interaction pseudopotential \( V(R) \), which is simply the interaction hamiltonian defined only for those particle states allowed in the lowest LL. In low-energy many-body states the particles generally tend to avoid pair eigenstates with high interaction energy, which means minimization of the corresponding Haldane pair amplitude \( G \). If repulsion \( V \) decreases sufficiently quickly as a function of \( R \) (the exact criterion being that \( V \) decreases sublinearly as a function of \( \sqrt{(r^2)} \)), the smallest value of \( R = 1 \) is avoided. This Laughlin type of correlation is elegantly described by attachment of \( 2p \) = 2 fluxes to each particle in the CF transformation. In a Laughlin-correlated state, each particle avoids being close to any other particle (as much as possible at a given finite \( \nu \)).

When short-range repulsion weakens (\( V \) at \( R = 1 \) decreases compared to \( V \) at \( R \geq 3 \), Laughlin correlations disappear and can be replaced by pairing or formation of larger clusters. Pairs\(^{15,16}\) or clustering\(^{15}\) were suggested by several authors for the QE’s. This idea was justified by an observation that QE–QE pseudopotential nearly vanishes at \( R = 1 \) and is strongly repulsive at \( R = 3 \), causing an increase of \( G(1) \) and a simultaneous decrease of \( G(3) \) compared to the Laughlin-correlated state.\(^{12}\)

**B. Average cluster size**

In a clustered state, the (average) cluster size \( K \) is connected to \( \alpha \), and the form of \( g_{\text{diff}} \) depends on correlations between the clusters. The values of \( K \) at \( \nu = \frac{1}{3} \) or \( \frac{5}{3} \) can be estimated by comparison of the actual parameters \( \alpha \) with those predicted for the hypothetical states of \( N \) particles arranged into \( N/K \) independent \( K \)-clusters. By independence of the clusters we mean that inter-cluster correlations do not affect the local filling factor \( \nu(r) \) at short range. For a single cluster, which on a sphere is the \( K \)-particle state with the maximum total angular momentum \( L = KL - \frac{1}{2}K(K-1) \), the \( \nu_K(r) \) depends on the surface curvature and thus (through \( R/\lambda = \sqrt{L} \)) on \( 2l \).

We have calculated the prefactors \( \beta_K \) of the short-range approximation \( \nu_K(r) \approx \beta_K g_0(r) \) for different values of \( K \) and \( l \) and listed some in Tab. (note that \( \nu_2(r) \) is known exactly). These coefficients are to be compared with \( \beta = (N/2l)\alpha \). Of course, this approach is somewhat questionable as one generally can-
TABLE II: Parameters $\beta_K$ of the short-range approximation $\nu(r) \sim \beta \gamma_0(r)$ obtained for independent clusters of size $K$.

| $2l$ | $\beta_2$ | $\beta_3$ | $\beta_4$ | $\beta_5$ | $\beta_6$ |
|------|----------|----------|----------|----------|----------|
| 25   | 0.2768   | 0.4196   | 0.5110   | 0.5765   | 0.6269   |
| 29   | 0.2730   | 0.4134   | 0.5029   | 0.5669   | 0.6159   |
| 60   | 0.2699   | 0.3938   | 0.4778   | 0.4372   | 0.5821   |
| $\infty$ | 0.2500   | 0.3763   | 0.4555   | 0.5110   | 0.5527   |

FIG. 5: (a) Pair-distribution functions $g(r)$ of lowest $L = 0$ states of finite systems corresponding to $\nu = \frac{1}{3}$ and $\frac{1}{2}$, for pseudopotentials of electrons in the first and second LL, and of CF’s in the second LL. (b) The total $g(r)$ and “remainder” $g_{\text{diff}}(r)$ curves of the Moore–Read $\nu = \frac{1}{2}$ state; circles mark a fitting linear combination of the curves for Laughlin states.

not deduce the precise cluster size from the short-range behavior of $g(r)$ for the following reasons: (i) $K$ is not a well-defined (conserved) quantum number; (ii) $\nu = \frac{1}{2}$ states occur for all $N$ (not only those divisible by two or three) which means that all clusters cannot have the same $K$; (iii) parameters $\alpha$ and $\beta$ are size-dependent and their extrapolation to large systems is not very reliable based on limited number of $N$-QE systems we are able to diagonalize; (iv) inter-cluster exchange of QE’s makes the “independent-cluster” picture only an approximation.

Fortunately, we can use the Moore–Read states (known to be paired\[22\]) as a test. Our calculation (for details see Sec. IV C) for $N = 14$ and $2l = 25$ gives $\beta_{\text{MR}} \approx 0.336$, somewhat larger than $\beta_2$. Hence, we shall assume that $\beta_K$ in general underestimates the actual value of $\beta$ in a many-body $K$-clustered state.

For the QE’s, we got $\beta \approx 0.319 \approx \beta_{\text{MR}}$ for $N = 12$ and $2l = 29$ ($\nu = \frac{1}{3}$), and $\beta = 0.479$ for $N = 14$ and $2l = 25$ ($\nu = \frac{1}{2}$); directly comparable with the Moore–Read state. With appropriate reservation, we can hence risk a hypothesis that QE’s (on the average) form pairs at $\nu = \frac{1}{3}$ and triplets at $\nu = \frac{1}{2}$ (possible triplet formation might turn out especially intriguing in the context of parafermion statistics\[22\]).

C. Comparison with Moore–Read state

The evolution of $g(r)$ when going from the lowest electron LL to the second CF LL (i.e., from LL$_0$ to CF-LL$_1$) is clear when using a model pseudopotential $V_{ij}(R) = \zeta \delta_{R,1} + (1 - \zeta) \delta_{R,3}$. For $\zeta \approx 0$ or 1, the correlations (avoidance of $R = 1$ or 3) are insensitive to $\zeta$, and both Laughlin and QE–QE correlations are accurately reproduced by $V_0$ and $V_1$, respectively. Modeling correlations among electrons in LL$_1$ (the second LL) is more difficult, because they are very sensitive to the exact form of $V(R)$ at the corresponding $\zeta \approx \frac{1}{2}$. As a result, the N-electron Coulomb eigenstates in LL$_1$ are more susceptible to finite-size errors than in LL$_0$ or CF-LL$_1$. In large systems, a good trial state is only known at $\nu = \frac{1}{2}$ (Moore–Read state), and much less is established about the correlations at $\nu = \frac{1}{3}$. Still, the $g(r)$ curves for electrons in LL$_1$ must certainly fall between the two extreme curves for $\zeta = 0$ and 1 (and differ from both of them). This is shown in Fig. 5(a) for both $\nu = \frac{1}{2}$ and $\frac{3}{5}$.

The exact Moore–Read wavefunctions were calculated on a sphere for $N \leq 14$ and $2l = 2N - 5 = 25$ by diagonalizing a short-range three-body repulsion. In Fig. 5(a) we only plotted $g(r)$ for $N = 14$ because the $N = 12$ curve is too close to be easily distinguished. The values of $\alpha = 0.602$ and 0.600 for $N = 12$ and 14. The $g_{\text{diff}}(r)$, also shown, is positive definite, very different from the QE curves in Fig. 5(b), and rather close to $g_2(r)$, where $g_2$ describes a Laughlin $\nu = (2p + 1)^{-1}$ state. Assuming $\alpha_{\text{MR}} = \frac{2}{3}$ and expanding $g_{\text{diff}}$ into $g_1$ and $g_2$ in accordance with Eq. (4) one obtains an approximate formula

$$g_{\text{diff}}(r) \approx \frac{3}{5}g_0(r) + \frac{3}{10}g_1(r) + \frac{1}{10}g_2(r),$$

marked with the circles in Fig. 5(b) that appears to be quite accurate (the largest finite-size error is in $g_2$ calculated for only $N = 8$, while $g_1$ is for $N = 12$).

The fact that $g_{\text{diff}}$ is positive and rather featureless (similar to $g_0$) for the Moore–Read wavefunction is in contrast with the result for QE’s. This difference may indicate that the QE clusters cannot be understood as literally as Moore–Read pairs. Indeed, even the lack of correlation between the occurrence of $L = 0$ ground states (or size of the excitation gap) and the divisibility of $N$ by $K = 2$ or 3 precludes such a simple picture. The fact that $g_{\text{diff}}(r \sim 3\lambda) < 0$ could mean that the average relative (with respect to center of mass) angular momentum $R_K$ of the QE clusters is much larger than $R_K^{\text{min}} = \frac{1}{2}K(K - 1)$. Certainly, $R_K$ is only conserved for an isolated cluster, but it is possible that the QE clusters are more relaxed due to cluster–cluster interaction than the Moore–Read pairs are. This would make $g_0$ underestimate the radius of the actual intra-cluster QE–QE correlation hole, and explain the negative sign of $g_{\text{diff}}$.\[22\]
V. CONCLUSION

From exact numerical diagonalization on Haldane sphere, we obtained the energy spectra and wavefunctions of up to \( N = 14 \) interacting Laughlin QE’s (CF’s in the second LL). We identified the series of finite-size liquid ground states with a gap, which extrapolate to the experimentally observed incompressible FQH states at \( \nu_c = \frac{4}{11}, \frac{3}{8}, \) and \( \frac{5}{13} \). In these states, we calculated QE–QE pair-distribution functions \( g(r) \), and showed that they increase as \( \sim r^2 \) at short range and have a pronounced shoulder at a medium range. This behavior supports the idea of QE cluster formation, suggested earlier from the analysis of QE–QE interaction pseudopotential. The \( g(r) \) is decomposed into short- and long-range contributions, interpreted as correlations between the QE from the same or different clusters. The intra-cluster contribution to \( g(r) \) is that of a full LL, and the remaining term identified with the inter-cluster QE–QE correlations appears to be the same in all three \( \nu = \frac{1}{3}, \frac{1}{2}, \) and \( \frac{2}{3} \) states. The (average) cluster size on the other hand does depend on \( \nu \), and we present arguments which suggest that the QE’s form pairs at \( \nu = \frac{1}{3} \) and triplets at \( \nu = \frac{2}{3} \).

The qualitative difference between the \( g(r) \) curves obtained here for correlated CF’s and those known for the Laughlin and Moore–Read liquids of electrons are another indication that the origin of incompressibility at \( \nu_c = \frac{1}{11}, \frac{3}{8}, \) and \( \frac{5}{13} \) is different. Of other hypotheses invoked in literature and mentioned here in the introduction, the broken-symmetry states cannot be excluded by our calculation in spherical geometry. However, we anticipate that the QE’s form a liquid (studied in this paper) also in experimental samples, because of the whole series of isotropic ground states with a gap occurring in finite systems of different size.

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