Measures Form a Complete Lattice

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April 14, 2021

Abstract

We show that the set of all measures on any measurable space is a complete lattice, i.e. every collection of measures has both a greatest lower bound and a least upper bound.

In a first course on measure theory, one learns that the sum of any collection of measures, as well as any non-negative constant multiple of a measure, is also a measure on the same measurable space. Additionally, the limit of an increasing sequence of measures is also a measure on the same measurable space.

On the other hand, the minimum and maximum of two measures are not measures (in general). This is shown by the following example:

Example. Suppose $X = \{a, b\}$, $A = \mathcal{P}(X)$ and define $\mu, \nu : A \to [0, \infty]$ as follows:

$\mu(\emptyset) = 0 \quad \mu(\{a\}) = 1 \quad \mu(\{b\}) = 0 \quad \mu(\{a, b\}) = 1$

$\nu(\emptyset) = 0 \quad \nu(\{a\}) = 0 \quad \nu(\{b\}) = 1 \quad \nu(\{a, b\}) = 1$

In other words, $\mu$ and $\nu$ are the Dirac measures at $a$ and $b$ respectively. Then we have:

$\min\{\mu, \nu\}(\{a\}) = 0 \neq 1 = \min\{\mu, \nu\}(\{a, b\})$

$\max\{\mu, \nu\}(\{a\}) = 1 \neq 2 = \max\{\mu, \nu\}(\{a, b\})$

Thus $\min\{\mu, \nu\}$ and $\max\{\mu, \nu\}$ are not measures on $(X, A)$.

Thus, if we want to establish a “greatest lower bound” or a “least upper bound” of two measures, we need to be more careful. We first define precisely the partial order in use:

Definition 1. Suppose $(X, A)$ is a measurable space. Then $\mathfrak{M}(X, A)$ denotes the poset of all measures on $(X, A)$, with the following partial order:

$\mu \preceq \nu \iff \mu(A) \leq \nu(A)$ for all $A \in A$

It follows directly from the definition that $\preceq$ is a partial order on $\mathfrak{M}(X, A)$. From now on, we will denote $\mathfrak{M}(X, A)$ by $\mathfrak{M}$, to indicate that the measurable space is arbitrary.

Clearly $\mathfrak{M}$ has a least element, the **zero measure** ($\mu(A) = 0$ for all $A \in A$). It also has a greatest element, the **infinity measure** ($\mu(\emptyset) = 0$ and $\mu(A) = \infty$ for all $A \in A$, $A \neq \emptyset$).

Going back to the question of the greatest lower bound and least upper bound, we will present two ways to motivate the answer:

Method 1. Suppose $\lambda \in \mathfrak{M}$ such that $\lambda \preceq \mu$ and $\lambda \preceq \nu$. Then for all $A \in A$, we must have $\lambda(A) \leq \mu(A)$ and $\lambda(A) \leq \nu(A)$. This alone implies that $\lambda(A) \leq \min\{\mu(A), \nu(A)\}$, which we have already seen is not good enough. To optimize this, we can introduce another set $B \in A$ and do the following:

$\lambda(A) = \lambda(A \cap B) + \lambda(A \cap B^c) \leq \mu(A \cap B) + \nu(A \cap B^c)$

In other words, we must have $\lambda(A) \leq \mu(A \cap B) + \nu(A \cap B^c)$ for every $B \in A$. This motivates the following definition:

$\lambda(A) = \inf_{B \in A} \{\mu(A \cap B) + \nu(A \cap B^c)\}$

In the example of the Dirac measures, this definition works, as it produces the zero measure.

Method 2. If $f : S \to \mathbb{R}$ is a real-valued function (on any set $S$), we can define its **positive part** $f^+ = \max\{f, 0\}$ and **negative part** $f^- = -\min\{f, 0\}$. These form a decomposition of $f$, since $f = f^+ - f^-$. Also $f^+$ and $f^-$ are ‘complementary’, in the sense that at every point $x \in S$, at least one of $f^+(x)$ and $f^−(x) = 0$.

In the case of signed measures, there is an analogous notion, known as **Jordan decomposition**: If $\lambda$ is a signed measure, then we can write $\lambda = \lambda^+ - \lambda^−$, where $\lambda^+$ and $\lambda^−$ are measures and $\lambda^+ \perp \lambda^−$. See [1] Theorem 9.30, Page 269 or [4] Theorem 3.4, Page 87) for a proof.
If $f$ and $g$ are real-valued functions, we can express $\min\{f,g\}$ and $\max\{f,g\}$ using positive and negative parts:

\[
\min\{f,g\} = f + \min\{0, g-f\} = f - (g-f)^- \\
\max\{f,g\} = f + \max\{0, g-f\} = f + (g-f)^+
\]

This motivates the following definitions:

\[
\mu \land \nu = \mu - (\nu - \mu)^- \\
\mu \lor \nu = \mu + (\nu - \mu)^+
\]

However, this is not always valid. In particular, $\nu - \mu$ may not be well-defined (due to $\infty - \infty$). When it is valid, though, it is equivalent to the definition from Method 1, which can be seen using the following formulas:

\[
\lambda^+(A) = \sup\{\lambda(E) \mid E \in \mathcal{A}, E \subseteq A\} \\
\lambda^-(A) = -\inf\{\lambda(E) \mid E \in \mathcal{A}, E \subseteq A\}
\]

We are now ready to answer the question of the greatest lower bound and least upper bound of two measures. From now on, we will use $\cup$ to denote disjoint unions, and $\bigcup$ to denote arbitrary unions.

**Theorem 2.** Suppose $\mu, \nu \in \mathfrak{M}$. Then their greatest lower bound $\mu \land \nu$ and least upper bound $\mu \lor \nu$ are given by:

\[
(\mu \land \nu)(A) = \inf_{B \in \mathcal{A}} \{\mu(A \cap B) + \nu(A \cap B^c)\} \\
(\mu \lor \nu)(A) = \sup_{B \in \mathcal{A}} \{\mu(A \cap B) + \nu(A \cap B^c)\}
\]

**Proof.** We will prove the result for $\mu \land \nu$, the result for $\mu \lor \nu$ can be proved similarly. Define $\lambda = \mu \land \nu$. We first show that $\lambda \in \mathfrak{M}$, i.e. $\lambda$ is a measure on $(X, \mathcal{A})$.

For all $B \in \mathcal{A}$, we have $\mu(\varnothing \cap B) + \nu(\varnothing \cap B^c) = 0 + 0 = 0$, so $\lambda(\varnothing) = 0$.

Suppose $\{A_n\}_{n=1}^{\infty}$ is a countable collection of disjoint sets in $\mathcal{A}$, and define $A = \bigcup_{n=1}^{\infty} A_n$. Then for all $B \in \mathcal{A}$, we have:

\[
A \cap B = \left( \bigcup_{n=1}^{\infty} A_n \right) \cap B = \bigcup_{n=1}^{\infty} (A_n \cap B) \\
A \cap B^c = \left( \bigcup_{n=1}^{\infty} A_n \right) \cap B^c = \bigcup_{n=1}^{\infty} (A_n \cap B^c)
\]

This yields:

\[
\mu(A \cap B) + \nu(A \cap B^c) = \mu \left( \bigcup_{n=1}^{\infty} (A_n \cap B) \right) + \nu \left( \bigcup_{n=1}^{\infty} (A_n \cap B^c) \right) = \sum_{n=1}^{\infty} \mu(A_n \cap B) + \sum_{n=1}^{\infty} \nu(A_n \cap B^c)
\]

Taking the infimum over all $B \in \mathcal{A}$ yields $\lambda(A) \geq \sum_{n=1}^{\infty} \lambda(A_n)$.

If $\sum_{n=1}^{\infty} \lambda(A_n) = \infty$, then we have $\lambda(A) \geq \sum_{n=1}^{\infty} \lambda(A_n)$ as both sides are $\infty$. Suppose $\sum_{n=1}^{\infty} \lambda(A_n) < \infty$. Then for any $n \in \mathbb{N}$ and any $\varepsilon > 0$, there is a set $B_n \in \mathcal{A}$ such that:

\[
\mu(A_n \cap B_n) + \nu(A_n \cap B_n^c) < \lambda(A_n) + \frac{\varepsilon}{2n} \quad (1)
\]

Define $C_n = A_n \cap B_n$ and $C = \bigcup_{n=1}^{\infty} C_n$ (this is a disjoint union as $C_n \subseteq A_n$ and all $A_n$ are disjoint). Since $A_n, B_n \in \mathcal{A}$ for all $n \in \mathbb{N}$, so are all $C_n$ and $C$. Note that:

\[
A_n \cap C = A_n \cap \left( \bigcup_{k=1}^{\infty} C_k \right) = \bigcup_{k=1}^{\infty} (A_n \cap C_k) = A_n \cap C_n \\
= A_n \cap (A_n \cap B_n) = A_n \cap B_n \quad (2)
\]

\[
A_n \cap C^c = A_n \cap \left( \bigcup_{k=1}^{\infty} C_k^c \right) = A_n \cap \left( \bigcap_{k=1}^{\infty} C_k \right) = A_n \cap C_n^c \\
= A_n \cap (A_n \cap B_n^c) = A_n \cap (A_n^c \cup B_n^c) = A_n \cap B_n^c \quad (3)
\]

This yields:

\[
\mu(A \cap C) + \nu(A \cap C^c) = \mu \left( \bigcup_{n=1}^{\infty} (A_n \cap C) \right) + \nu \left( \bigcup_{n=1}^{\infty} (A_n \cap C^c) \right) = \sum_{n=1}^{\infty} \mu(A_n \cap C) + \sum_{n=1}^{\infty} \nu(A_n \cap C^c)
\]

\[
= \sum_{n=1}^{\infty} (\mu(A_n \cap C) + \nu(A_n \cap C^c))
\]

\[
= \sum_{n=1}^{\infty} (\mu(A_n \cap B_n) + \nu(A_n \cap B_n^c)) \quad (\text{by } (2) \text{ and } (3))
\]

\[
< \sum_{n=1}^{\infty} \left( \lambda(A_n) + \frac{\varepsilon}{2n} \right) \quad (\text{by } (1))
\]

\[
= \sum_{n=1}^{\infty} \lambda(A_n) + \sum_{n=1}^{\infty} \frac{\varepsilon}{2n} = \sum_{n=1}^{\infty} \lambda(A_n) + \varepsilon
\]
Thus \( \lambda(A) < \sum_{n=1}^{\infty} \lambda(A_n) + \varepsilon \). Since this holds for all \( \varepsilon > 0 \), we have \( \lambda(A) \leq \sum_{n=1}^{\infty} \lambda(A_n) \), and so:

\[
\lambda(A) = \sum_{n=1}^{\infty} \lambda(A_n)
\]

Thus \( \lambda \) is a measure on \( (X, \mathcal{A}) \).

For all \( A \in \mathcal{A} \), we have \( \lambda(A) \leq \mu(A) \) and \( \lambda(A) \leq \nu(A) \) (these follow by setting \( B = X \) and \( B = \emptyset \) respectively in the definition). Thus \( \lambda \leq \nu \) and \( \lambda \leq \mu \). Suppose \( \rho \in \mathfrak{M} \) such that \( \rho \leq \mu \) and \( \rho \leq \nu \). Then for all \( A, B \in \mathcal{A} \), we have:

\[
\rho(A) = \rho(A \cap B) + \rho(A \cap B^c) \leq \mu(A \cap B) + \nu(A \cap B^c)
\]

Thus the infimum over all \( B \in \mathcal{A} \) yields \( \rho(A) \leq \lambda(A) \). Thus \( \lambda \) is the greatest lower bound of \( \mu \) and \( \nu \). \( \square \)

Theorem 2 appears as an exercise in [1, Exercise 9B.8, Page 278], [4, Exercise 3.1.7a, Page 88] and [5, Exercise 6.76, Page 364]. The formulas also appear in [2, Ch. III, §1, No. 5, Th. 3, Page III.12] and [3, Page 111] (again, without proof), though only for a special class of Borel measures on a locally compact topological space.

This proves that \( \mathfrak{M} \) is a lattice, i.e. a poset in which every pair of elements has a greatest lower bound and a least upper bound. We will show in Theorem 4 that it is a complete lattice, i.e. every (possibly uncountable) subset of \( \mathfrak{M} \) has a greatest lower bound and a least upper bound.

To motivate the answer, let’s go back to the formula \( (\mu \wedge \nu)(A) = \inf_{B \in \mathcal{A}} \{\mu(A \cap B) + \nu(A \cap B^c)\} \). Clearly we can freely switch \( B \) and \( B^c \), since \( \mathcal{A} \) is a \( \sigma \)-algebra and is thus closed under complements. The important bit here is that \( B \) and \( B^c \) partition \( X \). In other words, we split \( X \) into two pieces, give one to \( \mu \) and the other to \( \nu \). Of course, this readily generalizes to arbitrary collections of sets. With this, you might think of the following definition of the greatest lower bound of a collection \( \{\mu_\alpha\}_{\alpha \in I} \subseteq \mathfrak{M} \):

\[
\lambda(A) = \inf \left\{ \sum_{\alpha \in I} \mu_\alpha(A \cap B_\alpha) \right\}
\]

(4)

Where the infimum is over all partitions \( \{B_\alpha\}_{\alpha \in I} \) of \( X \) into measurable sets \( B_\alpha \in \mathcal{A} \). While this does yield a lower bound of \( \{\mu_\alpha\}_{\alpha \in I} \), it may not be the greatest lower bound, as the following example shows.

Example. Suppose \( X = [0,1] \), \( \mathcal{A} = B([0,1]) \) (the Borel \( \sigma \)-algebra on \([0,1]\)) and \( I = [4,5] \). For each \( \alpha \in [4,5] \), define \( \mu_\alpha = \alpha \mu_L \) (\( \alpha \) times the Lebesgue measure). If we partition \([0,1]\) into individual points, say \( B_\alpha = \{\alpha - 4\} \), then we have:

\[
\sum_{\alpha \in [4,5]} \mu_\alpha(\{\alpha - 4\}) = \sum_{\alpha \in [4,5]} 0 = 0
\]

Thus \( \lambda([0,1]) = 0 \), i.e. \( \lambda \) is the zero measure. However, for all \( \alpha \in [4,5] \), we have \( \mu_\alpha \geq 4 \mu_L \), so \( 4 \mu_L \) is a greater lower bound.

Remark. The problem with (1) is not that the sum may be uncountable: any sum of non-negative extended real numbers is well-defined as an extended real number. The problem is that measures need not be additive over uncountable collections of sets, so the formula \( \mu(\bigsqcup_{n \in \mathbb{N}} A_n) = \sum_{n \in \mathbb{N}} \mu(A_n) \) does not apply if we replace \( \mathbb{N} \) with an uncountable set.

The above example shows that (1) is not the correct formula. Before stating what the correct formula is, we make the following definition:

Definition 3. Suppose \( (X, \mathcal{A}) \) is a measurable space and \( \{\mu_\alpha\}_{\alpha \in I} \) is a collection of measures on \( (X, \mathcal{A}) \). Then \( \Sigma \) denotes the set of all pairs \( (\{B_\alpha\}_{\alpha \in \mathbb{N}}, \{\mu_\alpha\}_{\alpha \in \mathbb{N}}) \), where \( \{B_\alpha\}_{\alpha \in \mathbb{N}} \) is a countable collection of disjoint sets in \( \mathcal{A} \) such that \( \bigsqcup_{\alpha \in \mathbb{N}} B_\alpha = X \) (also known as a countable measurable partition of \( X \)), and for each \( n \in \mathbb{N} \), \( \mu_\alpha \) is some measure in the collection \( \{\mu_\alpha\}_{\alpha \in I} \). In other words, we choose a countable collection of measurable sets that partition \( X \), and for each of these sets, we attach one of the measured from the collection \( \{\mu_\alpha\}_{\alpha \in I} \).

In the proof of Theorem 4, we will use the fact that if \( a_{m,n} \geq 0 \) for all \( m, n \in \mathbb{N} \), then:

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{m,n} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{m,n}
\]

(5)

This is simply Tonelli’s theorem applied to the counting measure on \((\mathbb{N}, \mathcal{P}(\mathbb{N}))\). See [1, Theorem 5.31, Page 131] or [3, Example 4.23b, Page 249] for a proof.

Theorem 4. Suppose \( \{\mu_\alpha\}_{\alpha \in I} \) is a collection of measures on \( (X, \mathcal{A}) \). Then their greatest lower bound \( \bigwedge_{\alpha \in I} \mu_\alpha \) and least upper bound \( \bigvee_{\alpha \in I} \mu_\alpha \) are given by:

\[
\left( \bigwedge_{\alpha \in I} \mu_\alpha \right)(A) = \inf \left\{ \sum_{n=1}^{\infty} \mu_{\alpha_n}(A \cap B_n) \right\}
\]

\[
\left( \bigvee_{\alpha \in I} \mu_\alpha \right)(A) = \sup \left\{ \sum_{n=1}^{\infty} \mu_{\alpha_n}(A \cap B_n) \right\}
\]

Footnotes:
1. We will assume without loss of generality that it is countably infinite, since if it is finite, we can extend it by making the rest of the sets empty.
2. We will assume without loss of generality that all of these measures are distinct, since if any of them coincide, we can merge the corresponding sets from the partition.
Where the infimum and supremum are over all \((\{B_n\})_{n=1}^\infty, \{\mu_n\})_{n=1}^\infty \in \Sigma\).

**Proof.** We will prove the result for \(\bigwedge_{n \in I} \mu_n\), the result for \(\bigvee_{n \in I} \mu_n\) can be proved similarly. Define \(\lambda = \bigwedge_{n \in I} \mu_n\). We will use the shorthand \((B_n, \mu_n)\) for a pair \((\{B_n\})_{n=1}^\infty, \{\mu_n\})_{n=1}^\infty \in \Sigma\).

We first show that \(\lambda \in \mathcal{M}\), i.e. \(\lambda\) is a measure on \((X, \mathcal{A})\).

For all measurable partitions \((B_n)_{n=1}^\infty\) of \(X\), we have \(\mu_n(\emptyset) = \sum_{n=1}^\infty \mu_n(\emptyset) = \sum_{n=1}^\infty 0 = 0, \) so \(\lambda(\emptyset) = 0\).

Suppose \((A_m)_{m=1}^\infty\) is a countable collection of disjoint sets in \(\mathcal{A}\), and define \(A = \bigsqcup_{m=1}^\infty A_m\). Then for any \((B_n, \mu_n)\), we have:

\[
A \cap B_n = \left( \bigcup_{m=1}^\infty A_m \right) \cap B_n = \bigcup_{m=1}^\infty (A_m \cap B_n)
\]

This yields:

\[
\sum_{n=1}^\infty \mu_n(A \cap B_n) = \sum_{n=1}^\infty \mu_n \left( \bigcup_{m=1}^\infty (A_m \cap B_n) \right) = \sum_{n=1}^\infty \sum_{m=1}^\infty \mu_n(A_m \cap B_n)
\]

\[
= \sum_{m,n=1}^\infty \mu_n(A_m \cap B_n) \geq \sum_{m=1}^\infty \lambda(A_m) \quad \text{(by \(5\))}
\]

Taking the infimum over all \((B_n, \mu_n)\) yields \(\lambda(A) \geq \sum_{m=1}^\infty \lambda(A_m)\).

If \(\sum_{m=1}^\infty \lambda(A_m) = \infty\), then we have \(\lambda(A) = \sum_{m=1}^\infty \lambda(A_m)\) as both sides are \(\infty\). Suppose \(\sum_{m=1}^\infty \lambda(A_m) < \infty\). Then for any \(m \in \mathbb{N}\) and any \(\varepsilon > 0\), there exists \((B_n, \mu_n)\) such that:

\[
\sum_{n=1}^\infty \mu_n(A_m \cap B_m,n) < \lambda(A_m) + \frac{\varepsilon}{2^m}
\]

Define \(C_{m,n} = A_m \cap B_m,n\) and \(C_n = \bigsqcup_{m=1}^\infty C_{m,n}\) (this is a disjoint union as \(C_{m,n} \subseteq A_m\) and all \(A_m\) are disjoint). Since \(A_m, B_m,n \in \mathcal{A}\) for all \(m, n \in \mathbb{N}\), so are all \(C_m,n\) and all \(C_n\). We also have:

\[
A_m \cap C_n = A_m \cap \left( \bigcup_{k=1}^\infty C_{k,n} \right) = \bigcup_{k=1}^\infty (A_m \cap C_{k,n}) = A_m \cap C_{m,n}
\]

\[
= A_m \cap (A_m \cap B_m,n) = A_m \cap B_m,n
\]

This yields:

\[
\sum_{n=1}^\infty \mu_n(A \cap C_n) = \sum_{n=1}^\infty \mu_n \left( \bigcup_{m=1}^\infty (A_m \cap C_n) \right) = \sum_{n=1}^\infty \sum_{m=1}^\infty \mu_n(A_m \cap C_n)
\]

\[
= \sum_{m,n=1}^\infty \mu_n(A_m \cap C_n) \quad \text{(by \(5\))}
\]

\[
= \sum_{m=1}^\infty \sum_{n=1}^\infty \mu_n(A_m \cap B_m,n) \quad \text{(by \(7\))}
\]

\[
< \sum_{m=1}^\infty \lambda(A_m) + \sum_{m=1}^\infty \frac{\varepsilon}{2^m} \quad \text{(by \(5\))}
\]

\[
= \sum_{m=1}^\infty \lambda(A_m) + \sum_{m=1}^\infty \frac{\varepsilon}{2^m} = \sum_{m=1}^\infty \lambda(A_m) + \varepsilon
\]

Again, we can swap the sums as all summands are non-negative. While the sets \(C_n\) are disjoint, they do not partition \(X\) (since \(\bigsqcup_{n=1}^\infty C_n = A\), not \(X\)). We can fix this by redefining one of the sets \(C_k\) to be \(C_k \cup A^c\). This does not affect the steps above, as we only considered the measures of intersections with \(A_m\) or \(A\) (and so they are subsets of \(A\)). With this, \((C_n)_{n=1}^\infty\) becomes a countable measurable partition of \(X\) such that:

\[
\sum_{n=1}^\infty \mu_n(A \cap C_n) < \sum_{m=1}^\infty \lambda(A_m) + \varepsilon
\]

Thus \(\lambda(A) < \sum_{m=1}^\infty \lambda(A_m) + \varepsilon\). Since this holds for all \(\varepsilon > 0\), we have \(\lambda(A) \leq \sum_{m=1}^\infty \lambda(A_m)\), and so:

\[
\lambda(A) = \sum_{m=1}^\infty \lambda(A_m)
\]

Thus \(\lambda\) is a measure on \((X, \mathcal{A})\).

For all \(\alpha \in I\) and all \(A \in \mathcal{A}\), we have \(\lambda(A) \leq \mu_\alpha(A)\) (this follows by setting \(\alpha_1 = \alpha, B_{\alpha_1} = X\) and \(B_{\alpha_n} = \emptyset\) for all other \(n > 1\) in the definition of \(\lambda\)). Suppose \(\rho \in \mathcal{M}\) such that \(\rho \geq \mu_\alpha\) for all \(\alpha \in I\). Then for all \(A \in \mathcal{A}\) and all \((B_n, \mu_n)\), we
have:

\[ \rho(A) = \sum_{n=1}^{\infty} \rho(A \cap B_n) \leq \sum_{n=1}^{\infty} \mu_{\alpha_n}(A \cap B_n) \]

Taking the infimum over all \( \{B_n, \mu_{\alpha_n}\} \) yields \( \rho(A) \leq \lambda(A) \). Thus \( \lambda \) is the greatest lower bound of \( \{\mu_{\alpha}\}_{\alpha \in I} \). ■

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