BRAID GROUP ACTION AND QUASI-SPLIT AFFINE $\tilde{\mathfrak{u}}$QUANTUM GROUPS I

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Abstract. This is the first of two papers on quasi-split affine quantum symmetric pairs $(\tilde{\mathfrak{u}}(\tilde{\mathfrak{g}}), \tilde{\mathfrak{u}}^\dagger)$, focusing on the real rank one case, i.e., $\mathfrak{g} = \mathfrak{sl}_3$ equipped with a diagram involution. We construct explicitly a relative braid group action of type $A_2^{(2)}$ on the affine quantum group $\tilde{\mathfrak{u}}^\dagger$. Real and imaginary root vectors for $\tilde{\mathfrak{u}}^\dagger$ are constructed, and a Drinfeld type presentation of $\tilde{\mathfrak{u}}^\dagger$ is then established. This provides a new basic ingredient for the Drinfeld type presentation of higher rank quasi-split affine $\tilde{\mathfrak{u}}$quantum groups in the sequel.

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1. INTRODUCTION

1.1. Background. Besides the Drinfeld-Jimbo presentation, affine quantum groups admit a second presentation due to Drinfeld [Dr87, Dr88, Da93, Be94, Da15], which is a remarkable quantum analog of the current presentation of affine Lie algebras. The Drinfeld (current) presentation has played a fundamental role in representation theory and mathematical physics; cf., e.g., [CP91, FR99, FM01], and see the survey [CH10] for extensive references.

The quantum groups $\mathfrak{u}^\dagger$ arising from quantum symmetric pairs $(\mathfrak{u}, \mathfrak{u}^\dagger)$ associated to Satake diagrams [Let99] (see [Ko14]) can be viewed as a vast generalization of Drinfeld-Jimbo quantum groups associated to Dynkin diagrams; see the survey [W22] and references therein. In this paper, we shall work with universal quantum groups $(\tilde{\mathfrak{u}}, \tilde{\mathfrak{u}}^\dagger)$ following [LW22, WZ22], where $\tilde{\mathfrak{u}}$ is the Drinfeld double quantum group, as this allows us to formulate the relative braid group action conceptually. Just as the quantum group $\mathfrak{u}$ is obtained by $\tilde{\mathfrak{u}}$

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by a central reduction, a central reduction from universal quantum groups recovers quantum groups with parameters [Let02, Ko14).

In [LW21a], two authors of this paper obtained a Drinfeld type presentation for affine quantum groups of split ADE type; here “split” means that the underlying Satake diagram contains only white nodes and a trivial diagram involution, and this class of algebras appeared in [BB10] in connection to boundary affine Toda field theories. Subsequently, the third author streamlined some of the main arguments in the split ADE type and succeeded in generalizing to the split BCFG type [Z21].

The affine quantum group of split rank 1 is also known as the q-Onsager algebra in the mathematics and physics literature; cf. [T18, BB10] and references therein. The real and imaginary root vectors for q-Onsager algebra were first constructed and their commutator relations in a somewhat tedious form were also obtained in [BK20]. These relations were transformed and upgraded in [LW21a] into a Drinfeld type relation for the (universal) q-Onsager algebra. These constructions on q-Onsager algebra were instrumental for the Drinfeld presentation of split affine quantum groups of higher rank.

1.2. Goal. In this paper and its sequel, we shall construct the Drinfeld type presentations of quasi-split affine quantum groups, where the Satake diagrams are of affine types \( \hat{A}_n, \hat{D}_n, \hat{E}_6 \) with nontrivial diagram involutions fixing the affine simple root. This class of quantum groups contains 3 distinct affine quantum groups of real rank one:

(i) Drinfeld-Jimbo quantum group \( \hat{U}(\hat{sl}_2) \);
(ii) q-Onsager algebra \( \hat{U}(\hat{sl}_2) \);
(iii) the quantum group \( \hat{U}(\hat{sl}_3, \tau) \) associated to the Satake diagram \( (I, \tau) \) in (2.3), where \( \tau \) is the involution on \( I = \{0, 1, 2\} \) of affine type \( \hat{A}_2 \)
\[
\tau : I \rightarrow I, \quad \tau(0) = 0, \quad 1 \leftrightarrow 2.
\]

The goal of this paper is to give a Drinfeld type presentation for \( \hat{U}(\hat{sl}_3, \tau) \), the case (iii) in the above list of 3 affine rank one types. Toward this goal, we shall also establish the relative braid group action of twisted affine type \( \hat{A}_2^{(2)} \) on \( \hat{U}(\hat{sl}_3, \tau) \), which is another main result of this paper. The constructions in this paper (as well as the known Drinfeld presentations in Cases (i)–(ii)) will play a basic role in the sequel, in which a Drinfeld type presentation for arbitrary quasi-split affine quantum groups will be established.

The Drinfeld type presentations of affine quantum groups are expected to play a foundational role in their representation theory, to which we shall return elsewhere. They may have additional applications to quantum integrable systems (such as XXZ spin chain, Sine-Gordon and Liouville field theories), cf. [BK05].

1.3. Features of new affine rank 1. There are several reasons why the algebra \( \hat{U}(\hat{sl}_3, \tau) \) is substantially more involved than the other 2 affine rank one types and deserves a separate new investigation.

While \( \hat{U}(\hat{sl}_3, \tau) \) is viewed as a new case (iii) of affine real rank one (a building block which cannot be further reduced), it is a subalgebra of the quantum group \( \hat{U}(\hat{sl}_3) \) and behaves with complexity of affine rank two, in contrast to Cases (i)–(ii). As we shall see, the Drinfeld presentation of \( \hat{U}(\hat{sl}_3, \tau) \) requires 2 infinite series of real (respectively, imaginary)
roots and Serre relations, and moreover, $\tilde{U}^\dagger(\tilde{\mathfrak{sl}}_3, \tau)$ admits a braid group action of twisted affine type $A_2^{(2)}$.

To construct Drinfeld presentations of all affine quantum groups including twisted types, besides the usual type $\tilde{A}_1$, one needs to treat the twisted affine rank one type $A_2^{(2)}$ separately; see [Da00, Da15]. In the study of geometry of twisted affine Grassmannians, there are two special parahoric groups for $A_2^{(2)}$, one of which (known as absolutely special or hyperspecial) requires a separate treatment. The quasi-split affine quantum groups of type $\tilde{A}_n$ with nontrivial diagram involution (which is $\tilde{U}^\dagger(\tilde{\mathfrak{sl}}_3, \tau)$ if $n = 2$) has been realized geometrically in [FLLLW], and its canonical basis admits favorable positivity properties (compare [Lus93]).

The Hall algebra constructions based on quivers or weighted projective lines (cf. [LW22, LR21]) can be used to realize all affine quasi-split quantum groups except $\tilde{A}_2^{(2)}$ with nontrivial diagram involution. The simplest affine case not covered by the current $t$Hall algebra approach is exactly $\tilde{U}^\dagger(\tilde{\mathfrak{sl}}_3, \tau)$. So unlike the split ADE types [LW21b, LRW20, LR21], we do not have access to a Hall algebra construction to gain insights into the Drinfeld type presentation for $\tilde{U}^\dagger(\tilde{\mathfrak{sl}}_3, \tau)$.

We shall denote $\tilde{U}^\dagger(\tilde{\mathfrak{sl}}_3, \tau)$ by $\tilde{U}$ in the remainder of this paper.

1.4. Relative braid group symmetries. The algebra $\tilde{U}$ is generated by $B_i, K_i$, for $i = \{0, 1, 2\}$, subject to Serre type relations (2.4)–(2.8). The relative root system for $(\tilde{U}, \tilde{U}^\dagger)$, which contains 2 simple roots $\alpha_0, \alpha_1$, is of twisted affine type $A_2^{(2)}$. Very recently, two of the authors in [WZ22] constructed relative braid group symmetries on $\tilde{U}$ of arbitrary finite type, confirming a longstanding conjecture of Kolb-Pellegrini [KP11] and generalizing braid group symmetries on quantum groups [Lus93]. It is natural to hope for automorphisms $T_0, T_1$ on $\tilde{U}$ which generate a relative braid group action of type $A_2^{(2)}$.

The automorphisms $T_0$ and $T_0^{-1}$ with respect to the simple root $\alpha_0$ are not difficult to construct; see Proposition 2.5. Actually, the general constructions of relative braid group symmetries in [WZ22] can be applied to cover this case (even though our $\tilde{U}$ is not of finite type), and the formulas for the actions on generators of $\tilde{U}$ essentially arise from finite type considerations (compare [KP11, LW21a]).

In contrast, the automorphisms $T_1$ and $T_1^{-1}$ with respect to $\alpha_1$ are difficult to establish.

**Theorem A** (Theorems 2.7–2.8). There exists a $\mathbb{Q}(v)$-algebra automorphism $T_1$ on $\tilde{U}$ such that

$$T_1(K_1) = v^{-1}K_2^{-1}, \quad T_1(K_2) = v^{-1}K_1^{-1}, \quad T_1(K_0) = v^2K_0K_1^2K_2^2,$$

$$T_1(B_1) = -v^{-2}B_1K_2^{-1}, \quad T_1(B_2) = -v^{-2}B_2K_1^{-1},$$

$$T_1(B_0) = v\left([B_0, B_1]_v, [B_2, B_1]_v\right) - [B_0, [B_2, B_1]_v] K_1 + v B_0 K_1 K_2.$$

The inverse automorphism $T_1^{-1}$ is given by

$$T_1^{-1}(K_1) = v^{-1}K_2^{-1}, \quad T_1^{-1}(K_2) = v^{-1}K_1^{-1}, \quad T_1^{-1}(K_0) = v^2K_0K_1^2K_2^2,$$

$$T_1^{-1}(B_1) = -vB_1K_1^{-1}, \quad T_1^{-1}(B_2) = -vB_2K_2^{-1},$$

$$T_1^{-1}(B_0) = v\left([B_1, B_2]_v, [B_2, B_1]_v\right) - [B_1, [B_2]_v] K_2 + v B_0 K_1 K_2.$$
Moreover, $T_1$ and $T_1^{-1}$ are related by the $\sigma$-conjugation: $T_1^{-1} = \sigma_\tau \circ T_1 \circ \sigma$. (For the anti-involution $\sigma$, see Lemma 2.4.)

Let us discuss the obstacles about $T_1$. One has to guess the explicit formulas for the action of $T_1$ on generators of $\tilde{\mathbb{U}}^\prime$, and then to show it is indeed an automorphism. The action of $T_1$ on most generators of $\tilde{\mathbb{U}}^\prime$ can be formulated without much trouble, except $T_1(B_0)$. Earlier on we guessed a formula for $T_1(B_0)$ (as a polynomial in $B_i$ of degree 5), based on the internal consistency and desired properties of root vectors. (Recall [Lus93] a particular braid group operator formula of type $A_2^{(2)}$ is of degree 5 when the relevant Cartan integer is $-4$.) However, in contrast to the q-Onsager algebra case as done in [BK20] (see also [T18]), it turned out to be too difficult for a (super) computer to verify that $T_1$ is an automorphism of $\tilde{\mathbb{U}}^\prime$ (e.g., that the Serre relations for $\tilde{\mathbb{U}}^\prime$ is preserved by $T_1$).

The conceptual and general approach toward relative braid group action on quantum groups developed in [WZ22] is conjectured to be valid for Kac-Moody type. One advantage of this approach is a built-in mechanism for proving $T_1$ is an automorphism of $\tilde{\mathbb{U}}^\prime$. We follow the strategy loc. cit. to define $T_1$ via a certain rescaled braid operator on the Drinfeld double $\tilde{\mathbb{U}}$ and the quasi $K$-matrix of type $A_{II}$ [BW18] (with an explicit formula given in [DK19]). Accordingly, we obtain formulas for $T_1$ on $B_1, B_2$ and $K_i$, for $i \in \mathbb{I}$. It takes substantial computations however to make this approach work to produce a neat closed formula for $T_1(B_0)$ in Theorem A (which in particular asserts $T_1(B_0) \in \tilde{\mathbb{U}}^\prime$). This in particular verifies [WZ22, Conjecture 5.13] (formulated for $\tilde{\mathbb{U}}^\prime$ of Kac-Moody type) in the first new case beyond finite types.

1.5. Drinfeld type presentation of $\tilde{\mathbb{U}}^\prime$. Using the “translation” braid group operator

$$T_v := T_v T_1,$$

we define the real root vectors $B_{i,k}$ in (4.3), for $i \in \{1, 2\}$ and $k \in \mathbb{Z}$ (cf. [Da93, BK20]). On the other hand, we define inductively in (4.6)–(4.8) the imaginary $\nu$-root vectors $\Theta_{i,m}$ via commutators between real root vectors, for $i \in \{1, 2\}$ and $m \geq 1$. The definition of $\Theta_{i,m}$ is by no mean obvious; see Remark 4.1. Sometimes, it is more convenient to work with a new set of imaginary root vectors $H_{i,m}$; see (5.7) for its relation to $\Theta_{i,m}$.

The Drinfeld type presentation of $\tilde{\mathbb{U}}^\prime$ is built on the real and imaginary root vectors; compare [LW21b,Z21] in split affine types for similarities and differences. These earlier works (and also [BK20]) help us to formulate the relations in Theorem B below. However, the proof of this theorem remains challenging due to the affine rank two complexity of $\tilde{\mathbb{U}}^\prime$.

**Theorem B** (Definition 5.1, Theorem 5.5). The $\mathbb{Q}(\nu)$-algebra $\tilde{\mathbb{U}}^\prime$ has a presentation with generators $B_{i,l}, H_{i,m}, K_{i,l}^{\pm 1}, C^{\pm 1}$, where $i \in \{1, 2\}$, $l \in \mathbb{Z}$ and $m \in \mathbb{Z}_{\geq 1}$, subject to the following relations: for $m, n \geq 1, k, l \in \mathbb{Z}$, and $i,j \in \{1, 2\}$,

1. $C$ is central, $K_i K_j = K_j K_i$, $K_i H_{j,m} = H_{j,m} K_i$, $K_i B_{j,l} = \nu^{c_{ij} - c_{ij}} B_{j,l} K_i$,

$$[H_{i,m}, H_{j,n}] = 0,$$

2. $[H_{i,m}, B_{j,l}] = \frac{mc_{ij}}{m} B_{j,l+m} - \frac{mc_{ri,j}}{m} B_{j,l-m} C^m$,

$$[B_{i,k}, B_{i,l+1}]_{\nu - 2} - \nu^{-2} [B_{i,k+1}, B_{i,l}]_{\nu^2} = 0.$$
\[ [B_{i,k}, B_{\tau_i,l+1}]_v - v[B_{i,k+1}, B_{\tau_i,l}]_{v^{-1}} = -\Theta_{\tau_i,l-k-1} C^{k+1} K_i + v \Theta_{\tau_i,l-k-1} C^{k+1} K_i \]

\[ - \Theta_{i,k-l+1} C^{d+1} K_{\tau_i} + v \Theta_{i,k-l+1} C^{d+1} K_{\tau_i}, \quad (1.5) \]

\[ S(k_1, k_2 \mid l; i) = \text{Sym}_{k_1, k_2} \sum_{p \geq 0} [v^{-1} \Theta_{\tau_i,l-k-3p} K_i - \Theta_{\tau_i,l-k-3p-2} K_i C, B_{i,k_1-p}]_v C^{k_2+p} \]

\[ + \text{Sym}_{k_1, k_2} \sum_{p \geq 0} [\Theta_{i,k_2-l-3p} K_{\tau_i} C - v^{-1} \Theta_{i,k_2-l-3p} K_{\tau_i}, B_{i,k_1-p-1}]_v C^{l+2p+1} \]

\[ + \text{Sym}_{k_1, k_2} \sum_{p \geq 0} [B_{i,k_1+p+1}, \Theta_{\tau_i,l-k-3p} K_i C - v^{-1} \Theta_{\tau_i,l-k-3p} K_i C, B_{i,k_1-p-1}]_v C^{k_2+p} \]

\[ + \text{Sym}_{k_1, k_2} \sum_{p \geq 0} [B_{i,k_1-p}, v^{-1} \Theta_{i,k_2-l-3p} K_{\tau_i} - \Theta_{i,k_2-l-3p} K_{\tau_i} C, B_{i,k_1-p-1}]_v C^{l+2p}. \quad (1.6) \]

If we set all the summands involving \( C \) to zero in the above relations, the above presentation is essentially reduced to the Drinfeld presentation for half of \( \hat{U}(\mathfrak{sl}_3) \). Theorem B admits a generating function reformulation in terms of \( B_i(z), \Theta_i(z), H_i(z) \) \((i = 1, 2)\) and \( \Delta(z) \) from (5.9); see Theorem 5.7.

Denoting by \( \hat{\mathcal{U}}_l \) the \( \mathbb{Q}(v) \)-algebra with generators and relations given in Theorem B, we are reduced to establish an algebra isomorphism \( \Phi : \hat{\mathcal{U}}_l \to \hat{\mathcal{U}}_l \), which matches generators in the same notations (for \( \hat{\mathcal{U}}_l \) they stand for the root vectors). One shows that \( \Phi \) is surjective by checking all generators of \( \hat{\mathcal{U}}_l \) lie in the image of \( \Phi \). The injectivity is reduced by some filtration argument to the corresponding isomorphism for Drinfeld presentation of \( \hat{\mathfrak{sl}}_3 \).

It remains to show that \( \Phi \) is a homomorphism, which is the most involved part of the proof, i.e., to verify all the relations stated in Theorem B are satisfied by the root vectors in \( \hat{\mathcal{U}}_l \). As explained in §6.1, the overall strategy of the verification of the relations is an inductive argument which goes like a big spiral. The earlier approaches in [Da93, BK20] for affine quantum group \( \hat{U}(\mathfrak{sl}_2) \) and q-Onsager algebra \( \hat{U}(\mathfrak{sl}_2) \) have provided us a helpful roadmap but we have to deal with additional complexity of affine rank two. It is worth noting that some crucial proofs here follow more closely the approach in [Z21] (instead of [LW21b]), especially in establishing the relation (1.3) and the Serre relation (1.6). Along the way, we establish the \( T_{\omega} \)-invariance of \( \Theta_{i,n} \).

**Theorem C (Theorem 6.17).** We have

\[ T_{\omega}(\Theta_{i,n}) = \Theta_{i,n}, \quad \text{for all } n \geq 1 \text{ and } i = 1, 2. \]

1.6. **The organization.** The paper is organized as follows. In Section 2, we formulate the algebra \( \hat{\mathcal{U}}_l \) and some basic properties. We then formulate the relative braid group symmetries \( T_0^{\pm 1} \) and \( T_1^{\pm 1} \) on \( \hat{\mathcal{U}}_l \). The detailed proof of Theorem A regarding \( T_1^{\pm 1} \) is given in Section 3. Some additional technical computations for an identity used in the proof of Theorem A can be found in Appendix A.1.

In Section 4, we construct the real and imaginary \( v \)-root vectors for \( \hat{\mathcal{U}}_l \). We further identify the classical limits as \( v \to 1 \) of these \( v \)-root vectors and their Drinfeld type relations.

In Section 5, we formulate the Drinfeld type presentation for \( \hat{\mathcal{U}}_l \), and prove Theorem B except the verification of the relations. The lengthy proofs of the relations are given in Section 6. Theorem C is proved along the way.
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2. Relative braid group action

2.1. Quantum groups. Let $\mathbb{I} = \{0, 1, 2\}$ and $(c_{ij})_{i,j \in \mathbb{I}}$ be the Cartan matrix of affine type $\hat{A}_2$. Let $\mathfrak{g} := \mathfrak{s}\mathfrak{l}_3$ be the simple Lie algebra of type $A_2$ corresponding to $\mathbb{I}_0 = \{1, 2\}$, and $\hat{\mathfrak{g}}$ be the affine Lie algebra of affine type $A_2^{(1)}$. Let $\{\alpha_i \mid i \in \mathbb{I}\}$ be the simple roots of the affine Lie algebra $\hat{\mathfrak{g}}$, and

$$\delta = \alpha_0 + \alpha_1 + \alpha_2$$

be the basic imaginary root. Let $\mathbb{Z}\mathbb{I} := \bigoplus_{i \in \mathbb{I}} \mathbb{Z}\alpha_i$ be the root lattice with a symmetric bilinear form

$$(\cdot, \cdot) : \mathbb{Z}\mathbb{I} \times \mathbb{Z}\mathbb{I} \rightarrow \mathbb{Z}, \quad (\alpha_i, \alpha_j) = c_{ij}.$$  \hfill (2.1)

Let $W$ be the affine Weyl group of type $\hat{A}_2$. Let $P = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ be the weight lattice for $\mathfrak{g}$ where $\omega_i, i = 1, 2$ are the fundamental weights. Let $\tilde{W} := S_3 \ltimes P$ be the extended affine Weyl group, where $S_3$ denotes the symmetric group of 3 letters.

Let $v$ be the quantum parameter. For $n \in \mathbb{Z}, r \in \mathbb{N}$, denote the quantum binomial coefficients by

$$[n] = \frac{v^n - v^{-n}}{v - v^{-1}}, \quad [n]_r = \frac{[n][n-1]\ldots[n-r+1]}{[r]!}.$$ 

For $A, B$ in a $\mathbb{Q}(v)$-algebra, we shall denote $[A, B]_v = AB - v^a BA$, and $[A, B] = AB - BA$. The Drinfeld double quantum group $\tilde{U} = \tilde{U}(\hat{\mathfrak{g}})$ is generated by $E_i, F_i, K_i^{\pm 1}, K_i'^{\pm 1}$, for $i \in \mathbb{I}$, subject to the relations $K_i K_j = K_j K_i, K_i' K_j' = K_j' K_i', K_i K'_j = K_j K_i'$, $K_i E_j = v^{c_{ij}} E_j K_i, K_i F_j = v^{-c_{ij}} F_j K_i, K_i' E_j = v^{-c_{ij}} E_j K_i', K_i' F_j = v^{c_{ij}} F_j K_i'$, and standard quantum Serre relations. Note that $K_i K_i'$ are central in $\tilde{U}$. The Drinfeld-Jimbo quantum group $U = U(\hat{\mathfrak{g}})$ is recovered from $\tilde{U}$ by a central reduction:

$$U = \tilde{U}/\langle K_i K_i' - 1 \mid i \in \mathbb{I} \rangle.$$ 

Let $\text{Br}(W)$ be the braid group associated to $W$:

$$\text{Br}(W) = \langle s_0, s_1, s_2 \mid s_i s_j s_i = s_j s_i s_j, \forall i \neq j \rangle.$$ 

Let $T_i$ be the braid group automorphisms on $\tilde{U}$ defined by (see [LW21c])

$$T_i(K_i) = K_i', \quad T_i(E_i) = -F_i K_i, \quad T_i(F_i) = -K_i' E_i \quad T_i(K_j) = K_i K_j, \quad T_i(E_j) = [E_i, E_j] v^{-1} \quad T_i(F_j) = [F_j, F_i] v$$

for $0 \leq i \neq j \leq 2$. These formulas (by setting $K_i' = K_i^{-1}$) recover Lusztig's braid group action on $U$ [Lus93].

We also have

$$T_{\omega_1} = \bigcirc T_2 T_1, \quad T_{\omega_2} = \bigcirc^{-1} T_1 T_2$$ \hfill (2.2)

where $\bigcirc$ is the diagram automorphism given by $\bigcirc(0) = 1, \bigcirc(1) = 2, \bigcirc(2) = 0$. 

2.2. The quantum group. Let $\tau$ be the following diagram automorphism given by swapping vertices 1 and 2 while fixing 0:

\[ \begin{array}{c}
\tau
\end{array} \]

(2.3)

Recall $\mathbb{I} = \{0, 1, 2\}$. Let $\tilde{U} \colon \tilde{U}^q(\mathfrak{g})$ be the universal quantum group associated to the Satake diagram (2.3); see [LW22, CLW21] (also cf. [Let99, Ko14]). By definition, $\tilde{U}$ is generated by $B_i, K_i (i \in \mathbb{I})$, subject to the following relations:

\begin{align*}
& K_i K_j = K_j K_i, \quad K_i B_j = v^{-c_{ij} - c_{ji}} B_j K_i, \quad (2.4) \\
& B_2 B_1 - [2] B_1 B_2 B_1 + B_1^2 B_2 = -[2](v K_1 B_1 + v B_1 K_2), \quad (2.5) \\
& B_1 B_2 - [2] B_2 B_1 B_2 + B_2^2 B_1 = -[2](v K_2 B_2 + v B_2 K_1), \quad (2.6) \\
& B_0 B_1^2 - [2] B_1 B_0 B_1 + B_1^2 B_0 = 0 \quad (i = 1, 2), \quad (2.7) \\
& B_i B_0^2 - [2] B_0 B_i B_0 + B_0^2 B_i = -v^{-1} K_0 B_i \quad (i = 1, 2). \quad (2.8)
\end{align*}

Note that $K_0$ is central. It helps to make (2.4), say for $i = 1$, explicit (the relations remain valid up to a swap of indices $1 \leftrightarrow 2$):

\begin{align*}
& K_1 B_1 = v^{-3} B_1 K_1, \quad K_1 B_2 = v^3 B_2 K_1, \quad K_1 B_0 = B_0 K_1. \quad (2.9)
\end{align*}

There are some flexibility on scaling on the generators $K_i$; our convention on $K_i$ and the relations (2.4)-(2.8) gives us an embedding $\iota : \tilde{U} \hookrightarrow \tilde{U}$ (compare [LW22, §6.1]) by letting

\begin{align*}
& B_i \mapsto F_i + E_{\tau i} K_i', \quad K_0 \mapsto -v^2 K_0 K_0', \quad K_j \mapsto K_j K_j', \quad \forall i \in \mathbb{I}, j \neq 0. \quad (2.10)
\end{align*}

We often identify $\tilde{U}$ with the image of the embedding $\iota$, $B_i = F_i + E_{\tau i} K_i'$, and so on.

For $\mu = \sum_{i \in \mathbb{I}} a_i \alpha_i \in \mathbb{Z} \mathbb{I}$, define $K_\mu = \prod_{i \in \mathbb{I}} K_i^{a_i}$ and

\[ K_\delta = K_0 K_1 K_2. \]

The algebra $\tilde{U}$ is endowed with a filtered algebra structure

\[ \tilde{U}_{\tau,0} \subset \tilde{U}_{\tau,1} \subset \ldots \subset \tilde{U}_{\tau,m} \subset \ldots \]

(2.11)

by setting

\[ \tilde{U}_{\tau,m} = \mathbb{Q}(v) \text{-span} \{ B_{i_1} B_{i_2} \ldots B_{i_s} K_\mu \mid \mu \in \mathbb{Z} \mathbb{I}, i_1, \ldots, i_s \in \mathbb{I}, s \leq m \}. \quad (2.12) \]

Note that

\[ \tilde{U}_{\tau,0} = \bigoplus_{\mu \in \mathbb{Z} \mathbb{I}} \mathbb{Q}(v) K_\mu, \quad (2.13) \]

is the $\mathbb{Q}(v)$-subalgebra generated by $K_i$ for $i \in \mathbb{I}$. The following is a $\tilde{U}$-variant of a basic result of Letzter and Kolb on quantum symmetric pairs $(U, U')$. 


Proposition 2.1. (cf. [Let02, Ko14]) The associated graded $\text{gr} \tilde{U}^i$ with respect to (2.11)–(2.12) admits the following identification:

$$\text{gr} \tilde{U}^i \simeq U^- \otimes \mathbb{Q}(v)[K_i^\pm \mid i \in I],$$

$$B_i \mapsto F_i, \quad K_i \mapsto K_i \quad (i \in I).$$

(2.14)

Lemma 2.2. The $\mathbb{Q}(v)$-algebra $\tilde{U}^i$ is $\mathbb{Z}_I$-graded by letting

$$\text{wt}(B_i) = \alpha_i, \quad \text{wt}(K_i) = \alpha_i + \alpha_{ri}, \quad \text{for } i \in I.$$  

(2.15)

For any $\gamma \in \mathbb{Z}_I$, we denote $\tilde{U}^i_\gamma$ the homogeneous subspace of degree $\gamma$, and then

$$\tilde{U}^i = \bigoplus_{\gamma \in \mathbb{Z}_I} \tilde{U}^i_\gamma.$$  

(2.16)

Recall the bilinear form $(\cdot, \cdot)$ on $\mathbb{Z}_I$.

Lemma 2.3. We have

$$K_i X = v^{-(\alpha_i - \alpha_{ri}, \gamma)} X K_i,$$

for $i \in I$ and $X \in \tilde{U}^i_\gamma$ ($\gamma \in \mathbb{N}_I$). In particular, we have $K_i X = X K_i$, for $X \in \tilde{U}^i_{k \delta}$ ($k \in \mathbb{Z}$).

Proof. We first observe from (2.4) that $K_i X = v^{-(\alpha_i - \alpha_{ri}, \text{wt}(X))} X K_i$, for all generators $X = K_j, B_j$. The formula for general $X$ follows. The special case follows by $(\cdot, \delta) = 0$. \hfill $\square$

(2.17)

The diagram involution $\tau$ gives rise to an involution $\hat{\tau}$ on the algebra $\tilde{U}^i$:

$$\hat{\tau}(B_i) = B_{ri}, \quad \hat{\tau}(K_i) = K_{ri}, \quad \forall i = 0, 1, 2.$$

The following lemma follows by inspection of the defining relations of $\tilde{U}^i$.

Lemma 2.4. There exists a $\mathbb{Q}(v)$-algebra anti-involution $\sigma_\tau : \tilde{U}^i \rightarrow \tilde{U}^i$ such that

$$\sigma_\tau(B_i) = B_i, \quad \sigma_\tau(K_i) = K_{ri}, \quad \forall i \in I.$$  

2.3. Relative braid group operators $T_{01}^+$. Set

$$\alpha_0 := \alpha_0, \quad \alpha_1 := (\alpha_1 + \alpha_2)/2.$$  

The relative root system with simple roots $\{\alpha_0, \alpha_1\}$ is of twisted affine type $A_2^{(2)}$:

$$\circ \quad \quad \quad \quad \quad \circ$$

$$0 \quad \quad \quad \quad \quad 1$$

Let $\text{Br}(W, \theta) = \langle r_0, r_1 \rangle$ be the free group of two generators, which can be regarded as the braid group associated to the Weyl group of twisted affine type $A_2^{(2)}$. We have a group embedding

$$\text{Br}(W, \theta) \longrightarrow \text{Br}(W)$$

$$r_0 \mapsto s_0, \quad r_1 \mapsto s_1 s_2 s_1.$$  

The diagram involution $\tau$ induces an involution on $\text{Br}(W)$, $s_i \mapsto s_{ri}$ for all $i$. Then we can identify $\text{Br}(W, \theta)$ with the $\tau$-fixed subgroup $\text{Br}(W)^\tau$ via the above embedding.

Following the proposal [KP11] as established in [WZ22] for quantum groups of finite types, one expects a relative braid group action of $\text{Br}(W, \theta)$ on $\tilde{U}^i$. Our first main goal is to construct
the braid group operators $T_0, T_1$ (and respectively, $T_0^{-1}, T_1^{-1}$) explicitly corresponding to $r_0, r_1$. We start with the easier one $T_0^{\pm 1}$. The formulas for the braid operator $T_0^{\pm 1}$ essentially coincide with the ones appearing in finite type; cf. [KP11, LW21a].

**Proposition 2.5 ([WZ22])**. There exists a $\mathbb{Q}(v)$-algebra automorphism $T_0 : \tilde{U}^s \rightarrow \tilde{U}^s$ such that

$$T_0(\mathbb{K}_\gamma) = \mathbb{K}_{\gamma_0, \gamma}, \quad T_0(B_0) = B_0 \mathbb{K}_0^{-1}, \quad T_0(B_i) \mapsto [B_i, B_0]_v,$$

for $\gamma \in \mathbb{N}^2$, and $i = 1, 2$. Its inverse is given by

$$T_0^{-1}(\mathbb{K}_\gamma) = \mathbb{K}_{\gamma_0, \gamma}, \quad T_0^{-1}(B_0) = B_0 \mathbb{K}_0^{-1}, \quad T_0^{-1}(B_i) = [B_0, B_i]_v.$$  

Moreover,

$$T_0^{-1} = \sigma_r T_0 \sigma_r.$$  

*Proof.* The formulation of $T_0$ and its proof are covered by the constructions in [WZ22] (see Remark 5.8 therein), even though the formulation *loc. cit.* is focused on finite types. □

### 2.4. Relative braid group operators $T_1^{\pm 1}$.

Formulating and establishing the braid group operator $T_1$ on $\tilde{U}^s$ turns out to require a significant amount of new work. We caution that the process of formulating $T_1$ in this subsection involves some extension of the field $\mathbb{Q}(v)$, but the final formulas for the action of $T_1$ on generators of $\tilde{U}^s$ are valid over $\mathbb{Q}(v)$.

Recall the Satake diagram of quasi-split affine $A_2$ type (2.3). Setting $\varsigma_0 = -v^{-1/2}$, we denote by $\Psi_\varsigma$ the scaling automorphism on $\tilde{U}$ defined by

$$\Psi_\varsigma : \tilde{U} \longrightarrow \tilde{U}, \quad K_i \mapsto \varsigma_0^{1/2} K_i, \quad K'_i \mapsto \varsigma_0^{1/2} K'_i, \quad E_i \mapsto \varsigma_0^{1/2} E_i, \quad F_i \mapsto F_i.$$

Following [WZ22, §2.3], we define the rescaled braid group operators on $\tilde{U}$ as follows:

$$\tilde{T}_j := \Psi_\varsigma^{-1} \circ T_j \circ \Psi_\varsigma \quad (j \in \{1, 2\}),$$

$$(\tilde{T}_1^{-1})^{(1)} = \tilde{T}_1^{-1} \tilde{T}_2^{-1} \tilde{T}_1^{-1} = \tilde{T}_2^{-1} \tilde{T}_1^{-1} \tilde{T}_2^{-1}. \quad (2.21)$$

(The scaling looks a little different from *loc. cit.*, but leads to the same $\tilde{T}_j$.)

Let $\tilde{\Upsilon}_1$ be the quasi $K$-matrix for the universal quantum symmetric pair associated to the rank 1 Satake subdiagram ($\{1, 2\}, \tau$); cf. [BW18]. The following formula is due to [DK19] (who works in the setting of quantum groups with parameters).

**Lemma 2.6 ([DK19]).** We have

$$\tilde{\Upsilon}_1 = \left( \sum_{m \geq 0} \frac{v^{-m(m-1)/2} (-1)^m}{[m]!} [E_1, E_2]^m_{v^{-1}} \right) \left( \sum_{m \geq 0} \frac{v^{-m(m-1)/2} (-1)^m}{[m]!} [E_2, E_1]^m_{v^{-1}} \right). \quad (2.22)$$

The following theorem is an affine analogue of [WZ22, Theorem B], where we treat $T_1^{-1}$ as a symbol for now.

**Theorem 2.7.** There exists an automorphism $T_1^{-1}$ of $\tilde{U}^s$ such that

$$T_1^{-1}(x) \cdot \tilde{\Upsilon}_1 = \tilde{\Upsilon}_1 \cdot \tilde{T}_1^{-1}(x), \quad \text{for all } x \in \tilde{U}^s. \quad (2.23)$$

More explicitly, the action of $T_1^{-1}$ is given by

$$T_1^{-1}(\mathbb{K}_1) = v^{-1} \mathbb{K}_2^{-1}, \quad T_1^{-1}(\mathbb{K}_2) = v^{-1} \mathbb{K}_1^{-1}, \quad T_1^{-1}(\mathbb{K}_0) = v^2 \mathbb{K}_0 \mathbb{K}_1^2 \mathbb{K}_2^2, \quad (2.24)$$
\[ T_1^{-1}(B_1) = -vB_1K_1^{-1}, \quad T_1^{-1}(B_2) = -vB_2K_2^{-1}, \quad (2.25) \]
\[ T_1^{-1}(B_0) = v\left([B_1, B_2]_v, [B_2, [B_1, B_0]_v]\right) - \left([B_1, B_2]_v, B_0\right)K_2 + vB_0K_1K_2. \quad (2.26) \]

The proof of Theorem 2.7 is highly nontrivial and lengthy; it will occupy Section 3 below.

**Theorem 2.8.** There exists a \( Q(v) \)-algebra automorphism \( T_1 : \tilde{U}^v \to \tilde{U}^v \) such that
\[
T_1(K_1) = v^{-1}K_2^{-1}, \quad T_1(K_2) = v^{-1}K_1^{-1}, \quad T_1(K_0) = v^2K_0K_1^2K_2^2, \quad (2.27) \\
T_1(B_1) = -v^{-2}B_1K_2^{-1}, \quad T_1(B_2) = -v^{-2}B_2K_1^{-1}, \quad (2.28) \\
T_1(B_0) = v\left([B_0, B_1]_v, [B_2, B_1]_v\right) - \left([B_0, [B_2, B_1]_v]_v, B_1\right)K_1 + vB_0K_1K_2. \quad (2.29) \\

Moreover, \( T_1^{-1} \) and \( T_1 \) are mutual inverses, and
\[
T_1^{-1} = \sigma_vT_1\sigma_v. \quad (2.30) \\

Proof. Just as the automorphism \( T_1^{-1} \) is defined by the intertwining property (2.23), the automorphism \( T_1 \) can be characterized by the following intertwining property (cf. [WZ22, Theorem 6.1]):
\[
T_1(x) \cdot \tilde{T}_r(\tilde{T}_1^{-1}) = \tilde{T}_r(\tilde{T}_1^{-1}) \cdot \tilde{T}_r(x), \quad \text{for all } x \in \tilde{U}^v. \quad (2.31) \\

The formulas (2.27)–(2.28) were already established in [WZ22, Proposition 6.2-6.3], which are valid over Kac-Moody setting. The proof of the formula (2.29) is a version of the proof for \( T_1^{-1} \) given in Section 3, and will be omitted.

The proof that \( T_1^{-1} \) and \( T_1 \) are mutual inverses is the same as for [WZ22, Theorem 6.7]; it formally boils down to the uniqueness of the intertwining properties (2.23) and (2.31).

The identity (2.30) follows by comparing the formulas (2.24)–(2.26) and (2.27)–(2.29) and using Lemma 2.4. \( \Box \)

**Lemma 2.9.** We have
\[
T_1^{-1}(B_0) = \left[B_2, \left([B_1, B_2]_v, [B_1, B_0]_v\right)\right] - v\left[B_2, [B_1, B_0]_v^2\right]_v - \left[B_1, [B_2, B_0]_v\right]_vK_2 - v^{-1}\left[B_2, [B_1, B_0]_v\right]_vK_1 + vB_0K_1K_2. \quad (2.32) \\
T_1(B_0) = \left[[B_0, B_1]_v, [B_2, B_1]_v\right]_vB_2 - v\left[[B_0, B_1]_v, B_2\right]_vK_1 - \left[[B_0, B_2]_v, B_1\right]_vK_2 - v^{-1}\left[[B_0, B_1]_v, B_2\right]_vK_1 + vB_0K_1K_2. \quad (2.33) \\

Proof. We shall verify the formula (2.32). Indeed, we have
\[
\left[B_2, \left([B_1, B_2]_v, [B_1, B_0]_v\right)\right]_v = v\left[[B_1, B_2]_v, [B_2, [B_1, B_0]_v]\right]_v - v\left[[B_1, B_0]_v, [B_2, [B_1, B_2]_v]\right]_{v-1} \\
= v\left[[B_1, B_2]_v, [B_2, [B_1, B_0]_v]\right]_v - v\left[[B_1, B_0]_v, [B_2, [B_1, B_2]_v]\right]_{v-1} \\
= v\left[[B_1, B_2]_v, [B_2, [B_1, B_0]_v]\right]_v + v^2\left[[B_2, [B_1, B_0]_v]_v, K_2 + v^{-1}\left[[B_2, [B_1, B_0]_v]_v, K_1. \\

The formula (2.33) follows by (2.32) by applying the anti-involution \( \sigma_v \) and (2.30). \( \Box \)
Remark 2.10. If one is willing to work over an extension field $\mathbb{Q}(v^{1/2})$, it is possible to use variants of $K_1, K_2$ rescaled by $v^{1/2}$ to make the formulas for the action of $T_i^{+1}$ on $K_j$, for $j \in I$, look simpler (namely, removing the $v$-powers in the formulas (2.24) and (2.27)). To achieve the same effect, another option is to rescale only $K_i$ by a factor $v$, for a fixed $i \in \{1, 2\}$ (however, there is a broken symmetry between rescaled $K_1, K_2$).

3. Proof of Theorem 2.7

In this section, we shall prove Theorem 2.7 on the relative braid group operator $T_i^{-1}$. The proof follows the basic strategy developed in [WZ22], but the execution requires additional technical long computations.

3.1. Steps for proof of Theorem 2.7. Let us rephrase Theorem 2.7 to facilitate the discussion of the strategy of its proof.

(S1) For any $x \in \tilde{U}^i$, there exists a unique element $x' \in \tilde{U}^i$ such that $x' \cdot \tilde{\gamma}_1 = \tilde{\gamma}_1 \cdot \tilde{T}_r^{-1}(x)$.
(S2) Sending $x \mapsto x'$ defines an injective homomorphism of $\tilde{U}^i$, denoted by $T_i^{-1}$.
(S3) The formulas for $T_i^{-1}(x)$, with $x = K_i (i \in I)$, $x = B_i (i = 1, 2)$, and $x = B_0$ are given by (2.24), (2.25), and (2.26), respectively. That is, the identity $T_i^{-1}(x) \cdot \tilde{\gamma}_1 = \tilde{\gamma}_1 \cdot \tilde{T}_r^{-1}(x)$ holds, for each generator $x \in \{K_i, B_i | i \in I\}$ of $\tilde{U}^i$.
(S4) $T_i^{-1}$ is surjective (and hence an automorphism of $\tilde{U}^i$).

Step (S2) follows clearly from (S1) and the invertibility of $\tilde{\gamma}_1$. Since $\tilde{T}_r^{-1}$ is an automorphism, if $\tilde{\gamma}_1 \cdot \tilde{T}_r^{-1}(x_1) \cdot \tilde{\gamma}_1^{-1} \in \tilde{U}^i$ and $\tilde{\gamma}_1 \cdot \tilde{T}_r^{-1}(x_2) \cdot \tilde{\gamma}_1^{-1} \in \tilde{U}^i$, for $x_1, x_2 \in \tilde{U}^i$, then $\tilde{\gamma}_1 \cdot \tilde{T}_r^{-1}(x_1 x_2) \cdot \tilde{\gamma}_1^{-1} \in \tilde{U}^i$. This reduces the verification of (S1) for arbitrary $x$ to (S1) for $x$ being generators of $\tilde{U}^i$, i.e., (S3). Step (S4) actually follows when the counterparts to (S1)–(S3) above for $T_1$ in Theorem 2.8 are proved (cf. proof of [WZ22, Theorem 6.7]).

Therefore, to complete the proof of Theorem 2.7, it remains to prove (S3) above.

3.2. The identity (2.23) in cases (2.24)–(2.25). The statement that the “rank one” formulas (2.24)–(2.25) satisfy the intertwining property (2.23) were already proved in [WZ22, Proposition 4.11, Theorem 4.15]; as remarked loc. cit., these statements are actually valid for $\tilde{U}^i$ of arbitrary Kac-Moody type. This settles (S3) above, i.e., the intertwining property (2.23), for all generators of $\tilde{U}^i$ except for $x = B_0$.

We shall deal with this exceptional case on $T_i^{-1}(B_0)$ in the remainder of this section.

3.3. Identities (3.1) ⇔ (3.2)–(3.3). The remainder of this section is devoted to the challenging proof that the intertwining relation (2.23) is satisfied by $T_i^{-1}(B_0)$ in (2.26), i.e.,

$$T_i^{-1}(B_0) \cdot \tilde{\gamma}_1 = \tilde{\gamma}_1 \cdot \tilde{T}_r^{-1}(B_0).$$

Write $B_0 = F_0 + E_0 K_0'$, and hence $\tilde{T}_r^{-1}(B_0) = \tilde{T}_r^{-1}(F_0) + \tilde{T}_r^{-1}(E_0 K_0')$. Set

$$\xi = v \left[ [B_1, B_2]_v, [B_2, [B_1, F_0]_v] - [[B_1, B_2]_v^3, F_0]_K + v F_0 K_1 K_2, \right]$$

$$\xi = v \left[ [B_1, B_2]_v, [B_2, [B_1, E_0 K_0']_v] - [[B_1, B_2]_v^3, E_0 K_0 ']_K + v E_0 K_0' K_1 K_2. \right]$$
The desired intertwining relation (3.1) can be reformulated as
\[ T_i^{-1}(B_0) = \mathfrak{F} + \mathfrak{E}, \]
and it will follow once we verify the following 2 identities:
\[ \mathfrak{F} \cdot \tilde{\Upsilon}_1 = \tilde{\Upsilon}_1 \cdot \tilde{T}_{r_i}^{-1}(F_0), \quad (3.2) \]
\[ \mathfrak{E} \cdot \tilde{\Upsilon}_1 = \tilde{\Upsilon}_1 \cdot \tilde{T}_{r_i}^{-1}(E_0 K_0'). \quad (3.3) \]

3.4. Identities (3.2) ⇔ (3.8). The following lemma is an affine counterpart of [WZ22, Lemma 5.1] for $\tilde{\Upsilon}$ of finite type.

**Lemma 3.1.** $\tilde{T}_{r_i}^{-1}(E_0 K_0')$ commutes with $\tilde{\Upsilon}_1$.

**Proof.** We first observe that the following identity holds, for $i = 1, 2$:
\[ \tilde{T}_{r_i}^{-1} \tilde{\Upsilon}_1^{-1}(E_0) \cdot E_{r_i} = v^2 E_{r_i} \cdot \tilde{T}_{r_i}^{-1} \tilde{\Upsilon}_1^{-1}(E_0). \quad (3.4) \]
Indeed, it follows by definition of the rescaled braid operators (2.21) that
\[ \tilde{T}_{r_i}^{-1} \tilde{\Upsilon}_1^{-1}(E_0) = \varsigma_0^{-3/2}[[E_0, E_{r_i}]_{v^{-1}}, [E_i, E_{r_i}]_{v^{-1}}]_{v^{-1}}. \quad (3.5) \]
Hence, the identity (3.4) follows from (3.5) and the following two $v$-commuting relations:
\[ [E_0, E_{r_i}]_{v^{-1}}, E_{r_i} = v E_{r_i}[E_0, E_{r_i}]_{v^{-1}}, \quad [E_i, E_{r_i}]_{v^{-1}}, E_{r_i} = v E_{r_i}[E_i, E_{r_i}]_{v^{-1}}. \]
These 2 $v$-commuting relations are simple reformulations of the Serre relations in $\tilde{\Upsilon}$.
The identity (3.4) admits the following 2 equivalent reformulations:
\[ \tilde{T}_{r_i}^{-1}(E_0) \tilde{\Upsilon}_1^{-1}(E_{r_i}) = v^2 \tilde{T}_{r_i}^{-1}(E_{r_i}) \tilde{T}_{r_i}^{-1}(E_0); \quad (3.6) \]
\[ [[\tilde{T}_{r_i}^{-1}(E_0 K_0'), \tilde{T}_{r_i}^{-1}(E_{r_i})]] = 0. \quad (3.7) \]
Note that $\tilde{T}_{r_i}^{-1}(E_{r_i}) = \varsigma_0^{-1/2}[E_{r_i}, E_i]_{v^{-1}}$ and $\tilde{T}_{r_i}^{-1}(E_i) = \varsigma_0^{-1/2}[E_i, E_{r_i}]_{v^{-1}}$ are terms appearing in the formula (2.22) for $\tilde{\Upsilon}_1$. Hence, the lemma follows by (3.7). □

By [WZ22, Proposition 2.15, Lemma 5.1], we have
\[ B_1 \tilde{\Upsilon}_1 = \tilde{\Upsilon}_1 (F_1 + v^{-1} E_2 K_1), \quad B_2 \tilde{\Upsilon}_1 = \tilde{\Upsilon}_1 (F_2 + v^{-1} E_1 K_2), \quad F_0 \tilde{\Upsilon}_1 = \tilde{\Upsilon}_1 F_0. \]
Hence, we obtain
\[ \mathfrak{F} \cdot \tilde{\Upsilon}_1 = \tilde{\Upsilon}_1 \cdot \mathfrak{F}, \]
where
\[ \mathfrak{F} := v \left[[F_1 + v^{-1} E_2 K_1, F_2 + v^{-1} E_1 K_2]_{v^3}, [F_2 + v^{-1} E_1 K_2, [F_1 + v^{-1} E_2 K_1, F_0]_{v^3}] \right] \]
\[ - \left[[F_1 + v^{-1} E_2 K_1, F_2 + v^{-1} E_1 K_2]_{v^3}, F_0 \right] K_2 K_1' + v F_0 K_1 K_2' K_1'. \]
Thus, the desired identity (3.2) is equivalent to the identity
\[ \mathfrak{F} = \tilde{T}_{r_1}^{-1}(F_0). \quad (3.8) \]
3.5. Identities \((3.3) \Leftrightarrow (3.8)\).

**Lemma 3.2.** There exists an automorphism \(\varphi\) of \(\tilde{U}\) which sends, for \(i = 1, 2\),

\[
F_i \mapsto E_{ri} K'_i, \quad E_i \mapsto v F_i K'_{ri}, \quad K_i \mapsto K'_{ri}, \quad K_i' \mapsto K_{ri}.
\]

\[
F_0 \mapsto E_0 K'_0, \quad E_0 \mapsto v^{-2} F_0 K^{-1}_0, \quad K_0 \mapsto K'_0, \quad K'_0 \mapsto K_0.
\]

**Proof.** By a direct computation, the images of \(\varphi\) satisfy the defining relations of \(\tilde{U}\) and hence \(\varphi\) is an endomorphism of \(\tilde{U}\). One writes down an obvious candidate for the inverse homomorphism \(\varphi^{-1}\) acting on generators, which is indeed a homomorphism by another direct computation. Hence \(\varphi\) is an automorphism of \(\tilde{U}\). \(\square\)

**Lemma 3.3.** We have

1. \(\tilde{T}_{r_1}^{-1}(F_0) = v \left[ [F_1, F_2]_v, [F_2, [F_1, F_0]_v] \right];\)
2. \(\tilde{T}_{r_1}^{-1}(E_0 K'_{0}) = v \left[ [E_1, E_2]_{v^{-1}}, [E_1, [E_0, E_2]_{v^{-1}}] \right] K'_0 (K'_2 K'_1)^2.\)

**Proof.** We prove (1) only, as (2) can be obtained in a similar way. We have

\[
\tilde{T}_1^{-1} \tilde{T}_2^{-1} \tilde{T}_1^{-1}(F_0) = \left[ [F_1, F_2]_v, -E_1 (K'_1)^{-1} \right]_v, \left[ [F_1, F_2]_v, [F_1, F_0]_v \right]_v. \tag{3.9}
\]

Note that

\[
\left[ [F_1, F_2]_v, -E_1 (K'_1)^{-1} \right]_v = \left[ [F_1, F_2]_v, -E_1 \right] (K'_1)^{-1} = \frac{1}{v - v^{-1}} [K_1 - K'_1, F_2]_{v^2} (K'_1)^{-1} = F_2. \tag{3.10}
\]

Hence, rewriting (3.9) with the help of (3.10), we have

\[
\tilde{T}_1^{-1} \tilde{T}_2^{-1} \tilde{T}_1^{-1}(F_0) = \left[ [F_2, [F_1, F_2]_v, [F_1, F_0]_v]_v \right]_v
\]

\[= \left[ [F_2, [F_1, F_2]_v]_v, [F_1, F_0]_v \right]_v + v \left[ [F_1, F_2]_v, [F_2, [F_1, F_0]_v] \right], \]

where the last equality follows from the Serre relations. \(\square\)

In terms of the automorphism \(\varphi\) in Lemma 3.2, we can rephrase Lemma 3.3 as

\[
\varphi(\tilde{T}_{r_1}^{-1}(F_0)) = \tilde{T}_{r_1}^{-1}(E_0 K'_{0}). \tag{3.11}
\]

By Lemma 3.1, the identity \((3.3)\) is equivalent to the identity

\[
\mathcal{E} = \tilde{T}_{r_1}^{-1}(E_0 K'_{0}). \tag{3.12}
\]

The identity \((3.12)\) follows from \((3.8)\) by applying \(\varphi\) thanks to the identities \((3.11)\) and

\[
\varphi(\tilde{S}) = \mathcal{E}. \tag{3.13}
\]

The identity \((3.13)\) holds since by Lemma 3.2, \(\varphi(F_i + v^{-1} E_{ri} K_i) = B_i\) and \(\varphi(K_i K'_{ri}) = K_i K'_{ri}\), for \(i = 1, 2\). Therefore, we have concluded that the identity \((3.3)\) follows from \((3.8)\).

Let us summarize what we have achieved in this section. We reduced in §3.1–3.2 the proof of Theorem 2.7 to the identity \((3.1)\). Furthermore, in §3.3–3.5, we reduced the proof of the identity \((3.1)\), which is equivalent to the identities \((3.2)–(3.3)\), to the identity \((3.8)\).
The proof of the identity (3.8) shall be given in Appendix A.1.

4. Constructions of root vectors

In this section, we shall introduce the root vectors for the quasi-split quantum groups $\tilde{U}^i$.

4.1. Real and imaginary root vectors. Fix signs $o(1), o(2) \in \{\pm 1\}$ such that $o(1) = -o(2)$.

Denote, for $i \in \mathbb{I} = \{0, 1, 2\}$,

$$B_{\alpha_i} := B_i, \quad \mathbb{K}_{\alpha_i} := \mathbb{K}_i, \quad \mathbb{K}_\delta := \mathbb{K}_1 \mathbb{K}_2 \mathbb{K}_0, \quad C := -v\mathbb{K}_\delta. \quad (4.1)$$

Introduce the following “translation” symmetry on $\tilde{U}^i$:

$$T_\omega := T_0 T_1. \quad (4.2)$$

Similar to [Da93, Be94, BK20, LW21b], we define the real $v$-root vectors

$$B_{i,k} = B_{k\delta + \alpha_i} := (o(i)T_\omega)^{-k}(B_i), \quad \text{for } k \in \mathbb{Z}, i \in \{1, 2\}. \quad (4.3)$$

Note that

$$B_{i,-1} = o(i)T_\omega(B_i) = o(i)v^{-1}[B_i, B_0]_v \mathbb{K}_i C^{-1} = -o(i)v^{-2}[B_i, B_0]_v \mathbb{K}_i \mathbb{K}_\delta^{-1}. \quad (4.4)$$

Denote, for $i = 1, 2$ and $k \in \mathbb{Z}$,

$$D_{i,k} := -[B_{ri}, B_{i,k}]_{v^{-1}} - [B_{i,k+1}, B_{ri,-1}]_{v^{-1}}. \quad (4.5)$$

Set $\Theta_{i,0} = \frac{1}{v^{i-1}}$. Define the imaginary $v$-root vectors $\Theta_{i,m}$, for $m \geq 1$, inductively:

$$\Theta_{i,1} = -o(i)[B_i, [B_i, B_0]_v]_{v^2} - vB_0 \mathbb{K}_i, \quad (4.6)$$

$$\Theta_{i,2} = -vD_{i,0} C\mathbb{K}_{\tau_i}^{-1} + v\Theta_{i,0} C - \Theta_{ri,0} C\mathbb{K}_{ri}^{-1} \mathbb{K}_i, \quad (4.7)$$

$$\Theta_{i,m} - v\Theta_{i,m-2} C = -vD_{i,m-2} C\mathbb{K}_{\tau_i}^{-1}, \quad \text{for } m \geq 3. \quad (4.8)$$

For convenience, we set $\Theta_{i,m} = 0$ for $m < 0$.

Remark 4.1. The recursion (4.6)–(4.8) for $\Theta_{i,m}$ become more transparent when they are viewed as special cases of the Drinfeld type relation (5.5). A crucial property satisfied by $\Theta_{i,m}$ is the commutativity relation (5.11).

By definition, we have, for $k \in \mathbb{Z}, m \geq 0$ and $i \in \{1, 2\}$,

$$B_{i,k} = (-1)^{k\tau_i}(B_{ri,k}), \quad \Theta_{i,m} = (-1)^{m\tau_i}(\Theta_{ri,m}), \quad (4.9)$$

where $\tau_i$ is the involution in (2.17).

Recall from (2.16) that the algebra $\tilde{U}^i = \bigoplus_{\alpha \in \mathbb{Z}_i} \tilde{U}^i_\alpha$ is $\mathbb{Z}_i$-graded.

Lemma 4.2. For $i = 0, 1$, we have

$$T_0(\tilde{U}^i_\gamma) \subseteq \tilde{U}^i_{s_0(\gamma)}, \quad T_1(\tilde{U}^i_\gamma) \subseteq \tilde{U}^i_{s_1 s_2 s_1(\gamma)}, \quad \forall \gamma \in \mathbb{Z}_i.$$

Proof. This follows from the formulas for $T_i$ in Proposition 2.5 and Theorem 2.8 since $T_i$ are algebra automorphisms.

Lemma 4.3. We have $B_{i,k} \in \tilde{U}^i_{k\delta + \alpha_i}$, and $\Theta_{i,m} \in \tilde{U}^i_{m\delta}$, for any $l \in \mathbb{Z}, m > 0$.
Proof. Follows from Lemma 4.2, the definition of $B_{i,k}$ in (4.3) and the recursive definition of $\Theta_{i,m}$ in (4.6)–(4.8). \qed

4.2. The classical limit. Recall $U$ is the quantum group of affine type $A_2^{(1)}$. The quantum group $U^e_\zeta$ with parameter $\zeta = (\zeta_i)_{i=0,1,2} \in \mathbb{Q}(v)^{\times,3}$ (cf. [Ko14]) is a subalgebra of $U$ generated by $B_i$, for $i \in \mathbb{I}$, and $k_1$, via the embedding

$$
i : U^e_\zeta \longrightarrow U$$

$$k_1 \mapsto K_1 K_2^{-1}, \quad B_i \mapsto F_i + \zeta_i E_{\tau i} K_i^{-1} \quad (i \in \mathbb{I}).$$

Alternatively, the quantum group $U^e_\zeta$ can be obtained from $\mathcal{U}$ by the central reduction (cf. [LW22, WZ22]):

$$\mathcal{U} / (\mathbb{K}_0 + v^2 \mathbb{S}_0, \mathbb{K}_1 \mathbb{K}_2 - \zeta_1 \zeta_2) \cong U^e_\zeta,$$

$$B_i \mapsto B_i, \quad \mathbb{K}_0 \mapsto -v^2 \mathbb{S}_0, \quad \mathbb{K}_1 \mapsto \zeta_1 k_1, \quad \mathbb{K}_2 \mapsto \zeta_2 k_1^{-1}.$$  

The enveloping algebra $U(\hat{\mathfrak{g}})$ is recovered from $U$ at the $v \sim 1$ limit by letting

$$E_i \sim e_i, \quad F_i \sim f_i, \quad K_i \sim 1, \quad (K_i - K_i^{-1}) / (v - v^{-1}) \sim h_i.$$  

Let $\zeta, \zeta_0 \in \mathbb{Q}^\times$. Assume that the parameters $(\zeta_i)_{i=0,1,2} \in \mathbb{Q}(v)^{\times,3}$ have limits as $v \sim 1$ as follows: $\lim_{v \sim 1} \mathbb{S}_0 = \zeta_0$, and $\lim_{v \sim 1} \zeta_1 = \lim_{v \sim 1} \zeta_2 = \zeta$.

For $\mathfrak{g} = \mathfrak{sl}_3$, we denote by $\hat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{Q}c$ the corresponding affine Lie algebra of type $\hat{A}_2$. Let $\{e_i, h_i, f_i \mid i = 1, 2\}$ (and respectively, $\{e_i, h_i, f_i \mid i \in \mathbb{I}\}$) be the standard Chevalley generators of $\mathfrak{g}$ (and respectively, $\hat{\mathfrak{g}}$). Set

$$b_0 := f_0 + \zeta_0 e_0, \quad b_i := f_i + \zeta e_{\tau i}, \quad \text{for } i \in \{1, 2\}.$$  

Then we obtain the limits of generators of $U^e_\zeta$ as $v \sim 1$:

$$B_i \sim b_i \quad (i \in \mathbb{I}), \quad k_1 \sim 1, \quad (k_1 - k_1^{-1}) / (v - v^{-1}) \sim (h_1 - h_2).$$  

(4.10)

Introduce a shorthand notation

$$\hat{c} := \zeta_0 \zeta^2,$$  

(4.11)

which will be often used in this section. Note that

$$\mathcal{K}_\delta \sim -\hat{c}, \quad C \sim \hat{c}.$$  

Denote $e_\theta = [e_1, e_2], f_\theta = [f_2, f_1]$. Write $x_k := x \otimes t^k \in \hat{\mathfrak{g}}$, for $x \in \mathfrak{g}$ (and so $e_{i,k} = e_i \otimes t^k, e_{\theta,k} = e_\theta \otimes t^k$, and so on). We identify $e_0 = f_\theta \otimes t, f_0 = e_\theta \otimes t^{-1}$, and $h_0 = [e_0, f_0]$. Denote by

$$\omega_r : \hat{\mathfrak{g}} \longrightarrow \hat{\mathfrak{g}}$$

the involution such that

$$f_0 \mapsto \zeta_0 e_0, \quad h_0 \mapsto -h_0, \quad f_i \mapsto \zeta e_{\tau i}, \quad h_i \mapsto -h_{\tau i},$$

for $i \in \{1, 2\}$. It follows that $\omega_r(c) = -c$. For $i = 1, 2$, we denote

$$b_{i,r} = f_{i,-r} + \zeta \hat{c} e_{\tau i+r},$$

$$t_{i,r} = -h_{i,-r} + \hat{c} h_{\tau i+r},$$

$$b_{\theta,r} = f_{\theta,-r} + \zeta^2 \hat{c} e_{\theta,r}. $$

(4.12)
Note that \( t_{i,r} = -\hat{c}^r t_{\tau_i, -r} \). One checks that the Lie subalgebra of \( \hat{g} \) of \( \omega_r \)-fixed points, \( \hat{g}^{\omega_r} \), has a basis
\[
\{ b_{\theta,r}, b_{i,r}, t_{i,m}, h_1 - h_2 \mid r \in \mathbb{Z}, m \in \mathbb{Z}_{\geq 1}, i = 1, 2 \}.
\]
The enveloping algebra \( U(\hat{g}^{\omega_r}) \) is recovered from \( U_c \) at the \( v \to 1 \) limit.

**Proposition 4.4.** The Lie algebra \( \hat{g}^{\omega_r} \) is generated by \( \{ b_{i,r}, t_{i,r} \mid r \in \mathbb{Z}, i = 1, 2 \} \). Moreover, the following relations in \( \hat{g}^{\omega_r} \) are satisfied: for \( r, s, l, k_1, k_2 \in \mathbb{Z} \),
\[
[b_{i,s}, b_{i,r}] = 0,
\]
\[
[b_{i,r}, b_{\tau_i, s+1} - b_{i,r+1}, b_{\tau_i, s}] = -\zeta \hat{c}^r t_{\tau_i, s-r+1} + \hat{c}^{r+1} t_{\tau_i, s-r-1},
\]
\[
t_{i,r} = -\hat{c}^r t_{\tau_i, -r},
\]
\[
[t_{\tau_i, r}, b_{i,s}] = 2b_{i,s+r} + \hat{c}^r b_{i,s-r},
\]
\[
[t_{\tau_i, r}, b_{\tau_i, s}] = -b_{i,s+r} - 2\hat{c}^r b_{i,s-r},
\]
\[
[b_{i,k_1} [b_{i,k_2}, b_{\tau_i, l}]] = -2\hat{c}^r b_{i,k_1+k_2-1} - \hat{c}^{k_1} b_{i,k_2+k_1-1} - \hat{c}^{k_2} b_{i,k_1+k_2-1}.
\]

**Proof.** Let us write down some details on proving (4.15) and (4.19) while skipping the details on the remaining easier identities. Using (4.12)–(4.13), we compute
\[
[b_{\tau_i, s}, b_{i,r}] = \hat{c}^r t_{\tau_i, s-r} + [f_{\tau_i}, f_i]_{-s+r} + \hat{c}^{s+r}[e_i, e_{\tau_i}]_{-s-r}.
\]
Then the identity (4.15) follows by using the above formula twice.

We also compute again using (4.12)–(4.13):
\[
[b_{i,k_2}, b_{\tau_i, l}] = -b_{\theta, k_2+l} + \hat{c}^l t_{i,k_2-l},
\]
\[
[b_{i,k_1}, -b_{\theta, k_2+l}] = -\hat{c}^{k_1} b_{i,l+k_2-k_1},
\]
\[
[b_{i,k_1}, -\hat{c}^l t_{i,k_2-l}] = -2\hat{c}^l b_{i,k_1+k_2-l} - \hat{c}^{k_2} b_{i,l+k_1-k_2}.
\]
Then the identity (4.19) follows from these formulas.

Finally, the generating set statement follows by (4.20). \( \square \)

**Remark 4.5.** One can use a smaller generating set \( \{ b_{i,r}, t_{i,m}, t_{1,0} \mid r \in \mathbb{Z}, m \in \mathbb{Z}_{\geq 1}, i = 1, 2 \} \) for Lie algebra \( \hat{g}^{\omega_r} \), thanks to (4.16). By arguments similar to the proof of Theorem 5.5, one can show the above generators and relations provide a presentation for the Lie algebra \( \hat{g}^{\omega_r} \). This is the classical limit of the Drinfeld type presentation of \( \hat{U}^r \) established in this paper.

**Proposition 4.6.** Set \( o(1) = -o(2) = 1 \). Then the classical limits of the Drinfeld generators \( B_{i,r}, \Theta_{i,n} \) are \( b_{i,r}, t_{i,n} \) respectively, for \( i = 1, 2, r \in \mathbb{Z}, \) and \( n \geq 1 \).

**Proof.** Using the formula (4.4) for \( B_{i,-1} \), (6.1) for \( B_{i,1} \) and (4.6) of \( \Theta_{i,1} \), we see that the classical limits of these root vectors are \( b_{i,-1}, b_{i,1}, t_{1,1} \), respectively:
\[
B_{1,-1} \rightsquigarrow [b_1, b_0](\hat{s}_0 \hat{\zeta})^{-1} = f_{1,1} + (\hat{s}_0 \hat{\zeta})^{-1} e_{2,-1} = b_{1,-1},
\]
\[
B_{2,-1} \rightsquigarrow -[b_2, b_0](\hat{s}_0 \hat{\zeta})^{-1} = f_{2,1} + (\hat{s}_0 \hat{\zeta})^{-1} e_{1,-1} = b_{2,-1},
\]
\[
\Theta_{1,1} \rightsquigarrow [b_1, [b_2, b_0]] + \hat{\zeta} b_0 = -h_{1,-1} + \hat{s}_0 \hat{\zeta} h_{2,1} = t_{1,1},
\]
\[
\Theta_{2,1} \rightsquigarrow [b_2, [b_1, b_0]] - \hat{\zeta} b_0 = -h_{2,-1} + \hat{s}_0 \hat{\zeta} h_{1,1} = t_{2,1},
\]
\[
B_{1,1} \rightsquigarrow [[b_1, b_2], [b_1, b_0]] - 3[\hat{\zeta} b_1, b_0] = f_{1,-1} + \hat{s}_0 \hat{\zeta} e_{2,1} = b_{1,1}.
\]

More generally, using $[\Theta_{i,j}, B_{j,k}] = [c_{ij}] B_{j,k+1} - [c_{r,ij}] B_{j,k-1} C$ from (5.3) and (4.17)-(4.18), one shows inductively that the classical limit of $B_{i,r}$ is $b_{i,r}$, for $r \in \mathbb{Z}$. Similarly, using the recursive definition (4.6)-(4.8) of $\Theta_{i,n}$ and (4.15), one shows inductively that the classical limit of $\Theta_{i,n}$ is $t_{i,n}$, for $n \geq 1$.

\textbf{Remark 4.7.} Define a new Lie algebra $\hat{g}^{\omega r}$, a variant of $\tilde{g}^{\omega r}$, in which $\hat{c}$ is viewed as a central element (instead of being a scalar), with formally the same relations (4.19)-(4.21). Then $\hat{g}^{\omega r}$ is a $\mathbb{Z}$-graded Lie algebra, by assigning deg $b_{i,r} = r$, deg $t_{i,r} = r$, and deg $\hat{c} = 2$, and noting that the relations (4.19)-(4.21) have become homogeneous. Compare Remark 5.6.

4.3. A \textbf{vanishing criterion.} We continue the notations from the previous subsection. Recall $g = \mathfrak{sl}_3$, which is also identified with $g \otimes t^0$ in $\tilde{g}$. Denote by $l = g^{\omega r} \subset \tilde{g}^{\omega r}$, as we note $\omega_r$ preserves the subalgebra $g$ of $\tilde{g}$. Denote by $l$ the (proper) Lie subalgebra of $\tilde{g}^{\omega r}$ generated by $t_{i,0}, b_{i,n}$, for all $i \in \{1, 2\}$ and $n \in \mathbb{Z}$. Denote by $\mathfrak{l}_{ev}$ the Lie subalgebra of $\tilde{g}^{\omega r}$ spanned by $x_{2n} + (x^{\omega r})_{2n}$, for all $x \in g$ and $n \in \mathbb{Z}$.

\textbf{Lemma 4.8.} We have $\mathfrak{l}_{ev} \subset l$.

\textbf{Proof.} We observe by a direct computation on $[b_{i,m}, b_{j,k}]$, with $m + k \in 2\mathbb{Z}$ and $i \in \{1, 2\}$, that $b_{\theta,j}, t_{i,n} \in l$, for $n$ even.

\textbf{Lemma 4.9.} Let $u \in U(\tilde{g}^{\omega r})$. If ad ($\mathfrak{l}_{ev}$)$u = 0$, then $u = 0$.

\textbf{Proof.} To prove the lemma, it is equivalent to prove its counterpart in the symmetric algebra (in place of the enveloping algebra):

\begin{equation}
\text{If ad} (\mathfrak{l}_{ev})u = 0, \text{ for } u \in S(\tilde{g}^{\omega r}) \text{ then } u = 0.
\end{equation}

The Lie algebra $\tilde{g}^{\omega r}$ (and hence its subalgebra $\mathfrak{l}_{ev}$) admits an $\mathbb{N}$-filtered algebra structure, by letting deg$(x_n + (x^{\omega r})_{-n}) = n$, for all nonzero $x \in g$ and $n \in \mathbb{N}$. Then the associated graded algebras can be identified as follows: gr $\tilde{g}^{\omega r} \cong l \oplus t^1 g[t]$ and gr $\mathfrak{l}_{ev} \cong l \oplus t^1 g[t^2]$.

The proof of the claim (4.21) is reduced to the proof of the following counterpart in the associated graded:

\begin{equation}
\text{If ad (gr } \mathfrak{l}_{ev})u = 0, \text{ for } u \in S(\text{gr } \tilde{g}^{\omega r}) \text{ then } u = 0.
\end{equation}

The proof of (4.22) is entirely analogous to the ones of [M07, Lemmas 2.8.1, 1.7.4], by analyzing the action of gr $\mathfrak{l}_{ev,2} \cong g \otimes t^2$ on $u$, where $\mathfrak{l}_{ev,2}$ is the subspace of $\mathfrak{l}_{ev}$ spanned by $x_2 + x^{\omega r}_2$, for all $x \in g$. We omit the detail.

(The proof of the lemma could have proceeded directly as for [M07, Lemmas 2.8.1, 1.7.4] without referring to the filtration and associated graded explicitly, though it would involve somewhat messy notations if one insists on writing out details.)

\textbf{Lemma 4.10.} If $u \in U(\tilde{g}^{\omega r})$ commutes with $b_{i,1}$ and $t_{i,0}$, for all $i \in \{1, 2\}$ and $l \in \mathbb{Z}$, then $u = 0$.

\textbf{Proof.} By assumption of the lemma and the definition of $l$, we have ad ($l$)$u = 0$ in $S(\tilde{g}^{\omega r})$, and hence ad ($\mathfrak{l}_{ev}$)$u = 0$ by Lemma 4.8. Now it follows by Lemma 4.9 that $u = 0$.

The following lemma is the main point of this subsection.

\textbf{Lemma 4.11.} Let $X \in \bar{U}^1$ be a noncommutative polynomial of $B_1, B_2, B_0$ with coefficients in $\mathbb{Q}(v)[K_i^\pm 1, i \in \mathbb{I}]$ without constant term. If $T_\omega(X) = X$ and $[X, B_i] = [X, K_i] = 0$ for $i \in \{1, 2\}$, then $X = 0$. 
5. A Drinfeld Type Presentation

In this section, we shall give a Drinfeld type presentation for the quasi-split affine quantum group \( \hat{U}^v \) of rank 1.

5.1. The definition. Recall \( \mathbb{I} = \{0, 1, 2\} \) and \( (c_{ij})_{i,j \in \mathbb{I}} \) denotes the Cartan matrix of affine type \( A_2^{(1)} \). We denote by \( \text{Sym}_{k_1,k_2} \) the symmetrization with respect to indices \( k_1, k_2 \) in the sense \( \text{Sym}_{k_1,k_2}(f(k_1,k_2)) = f(k_1,k_2) + f(k_2,k_1) \).

\( \text{Definition 5.1.} \) Let \( D^v \hat{U}^v \) be the \( \mathbb{Q}(v) \)-algebra generated by the elements \( B_{i,l}, H_{i,m}, \mathbb{K}_i^{\pm 1}, C^{\pm 1}, \) where \( i \in \{1, 2\}, \ l \in \mathbb{Z} \) and \( m \in \mathbb{Z}_{\geq 1} \), subject to the following relations: for \( m,n \geq 1, k,l \in \mathbb{Z} \), and \( i,j \in \{1, 2\} \),

\[
C \text{ is central, } \mathbb{K}_i \mathbb{K}_j = \mathbb{K}_j \mathbb{K}_i, \quad \mathbb{K}_i H_{j,m} = H_{j,m} \mathbb{K}_i, \quad \mathbb{K}_i B_{j,l} = v^{c_{ri,j} - c_{ij}} B_{j,i} \mathbb{K}_i, \quad (5.1)
\]

\[
[H_{i,m}, H_{j,n}] = 0, \quad (5.2)
\]

\[
[H_{i,m}, B_{j,l}] = \frac{mc_{ij}}{m} B_{j,l+m} - \frac{mc_{ri,j}}{m} B_{j,l-m} C^m, \quad (5.3)
\]

\[
[B_{i,k}, B_{i,l}]_{v^{-2}} - v^2[B_{i,k+1}, B_{i,l}]_{v^2} = 0, \quad (5.4)
\]

\[
[B_{i,k}, B_{ii,l+1}]_v - v[B_{i,k+1}, B_{ii,l}]_{v^{-1}} = -\Theta_{ri,l-k-1} C^k \mathbb{K}_i - v \Theta_{ri,l-k} C^{k+1} \mathbb{K}_i - \Theta_{i,k-l-1} C^l \mathbb{K}_{ri} + v \Theta_{i,k-l} C^{l+1} \mathbb{K}_{ri}, \quad (5.5)
\]

\[
\mathbb{S}(k_1, k_2 | l; i) = \text{Sym}_{k_1,k_2} \left\{ \sum_{p \geq 0} [v^{-1} \Theta_{ri,l-k-3p} \mathbb{K}_i - \Theta_{ri,l-k-2p} \mathbb{K}_i C, B_{i,k_1+p}]_{v} C^{k_2+p} \right\}
\]

\[
+ \text{Sym}_{k_1,k_2} \left\{ \sum_{p \geq 0} [\Theta_{i,k_2-l-3p-3} \mathbb{K}_{ri} C - v^{-1} \Theta_{i,k_2-l-3p-1} \mathbb{K}_{ri}, B_{i,k_1-p-1}]_{v} C^{l+2p+1} \right\}
\]

\[
+ \text{Sym}_{k_1,k_2} \left\{ \sum_{p \geq 0} [B_{i,k_1+p+1}, \Theta_{ri,l-k-3p-3} \mathbb{K}_i C - v^{-1} \Theta_{ri,l-k-3p-1} \mathbb{K}_i]_{v} C^{k_2+p} \right\}
\]

\[
+ \text{Sym}_{k_1,k_2} \left\{ \sum_{p \geq 0} [B_{i,k_1-p}, v^{-1} \Theta_{i,k_2-l-3p} \mathbb{K}_{ri} - \Theta_{i,k_2-l-3p-2} \mathbb{K}_{ri}]_{v} C^{l+2p} \right\}. \quad (5.6)
\]

Here \( H_{i,m} \) are related to \( \Theta_{i,m} \) by the following equation:

\[
1 + \sum_{m \geq 1} (v - v^{-1}) \Theta_{i,m} u^m = \exp \left( (v - v^{-1}) \sum_{m \geq 1} H_{i,m} u^m \right). \quad (5.7)
\]

We have also denoted

\[
\mathbb{S}(k_1, k_2 | l; i) := \text{Sym}_{k_1,k_2} \left\{ B_{i,k_1} B_{i,k_2} B_{ii,l} - [2] B_{i,k_1} B_{ii,l} B_{i,k_2} + B_{ii,l} B_{i,k_1} B_{i,k_2} \right\}. \quad (5.8)
\]
Remark 5.2. Using Proposition 4.6, one can show that the classical limits of relations (5.1)-(5.6) are given by (4.14)-(4.19); see also Remark 4.7.

The \( \mathbb{Q}(v) \)-algebra \( \tilde{\mathcal{U}}^1 \) admits the following translation symmetry.

**Lemma 5.3.** There exists an automorphism \( \omega \) of the algebra \( \tilde{\mathcal{U}}^1 \) given by

\[
\omega(B_{j,k}) = B_{j,k-1}, \quad \omega(H_{j,m}) = H_{j,m}, \quad \omega(K_j) = K_j C^{-1}, \quad \omega(C) = C,
\]

(and hence \( \omega(\Theta_{j,m}) = \Theta_{j,m} \), for all \( k \in \mathbb{Z}, m \geq 1 \), and \( j \in \{1, 2\} \).

**Proof.** Follows by inspection of the defining relations for \( \tilde{\mathcal{U}}^1 \) in Definition 5.1. \( \square \)

We introduce the following generating functions in a variable \( z \):

\[
B_i(z) = \sum_{k \in \mathbb{Z}} B_{i,k} z^k, \quad \Theta_i(z) = 1 + \sum_{m \geq 1} (v - v^{-1}) \Theta_{i,m} z^m, \quad H_i(z) = \sum_{m \geq 1} H_{i,m} z^m, \quad \Delta(z) = \sum_{k \in \mathbb{Z}} C^k z^k,
\]

(5.9)

where \( \Theta_i(z) \) and \( H_i(z) \) are related by

\[
\Theta_i(z) = \exp ( (v - v^{-1}) H_i(z) ).
\]

(5.10)

The following lemma will be used later.

**Lemma 5.4.** The following equivalences hold:

1. The identity (5.2) is equivalent to

\[
[\Theta_{i,m}, \Theta_{j,n}] = 0, \quad \forall i, j = 1, 2, \text{ and } m, n \geq 1.
\]

(5.11)

2. The identity (5.3) is equivalent to

\[
[\Theta_{i,m}, B_{j,k}] + v^{c_{i,j} - c_{r_{i,j}}} [\Theta_{i,m-2}, B_{j,k}] v^{2(c_{r_{i,j}} - c_{i,j})} C
\]

\[
- v^{c_{i,j}} [\Theta_{i,m-1}, B_{j,k+1}] v^{-2c_{i,j}} - v^{-c_{r_{i,j}}} [\Theta_{i,m-1}, B_{j,k-1}] v^{2c_{r_{i,j}}} C = 0,
\]

(5.12)

for any \( m > 0 \) and \( k \in \mathbb{Z} \).

**Proof.** The equivalence in (1) follows directly from (5.7).

The proof of the equivalence in (2) is very similar to [LW21b, Proposition 2.8], via a generating function formalism (5.9). We outline the main steps below.

The identity (5.3) can be equivalently reformulated as a generating function identity

\[
(v - v^{-1}) [H_i(z), B_j(w)] = \ln \left( \frac{(1 - v^{-c_{i,j}} zw^{-1})(1 - v^{c_{r_{i,j}}} zwC)}{(1 - v^{c_{i,j}} zw^{-1})(1 - v^{-c_{r_{i,j}}} zwC)} \right) \cdot B_j(w).
\]

Via integration and (5.10), this identity is equivalent to the identity

\[
\Theta_i(z) B_j(w) \Theta_i(z)^{-1} = \frac{(1 - v^{-c_{i,j}} zw^{-1})(1 - v^{c_{r_{i,j}}} zwC)}{(1 - v^{c_{i,j}} zw^{-1})(1 - v^{-c_{r_{i,j}}} zwC)} B_j(w),
\]

or equivalently,

\[
\Theta_i(z) B_j(w) = \frac{(1 - v^{-c_{i,j}} zw^{-1})(1 - v^{c_{r_{i,j}}} zwC)}{(1 - v^{c_{i,j}} zw^{-1})(1 - v^{-c_{r_{i,j}}} zwC)} B_j(w) \Theta_i(z).
\]

Comparing the coefficients of \( z^m w^k \) of both sides of the last identity, for \( m \geq 1, k \in \mathbb{Z} \), we obtain the equivalent identity (5.12). \( \square \)
5.2. The isomorphism.

Theorem 5.5. There is a \( \mathbb{Q}(v) \)-algebra isomorphism \( \Phi : \mathcal{D}_r \mathcal{U} \rightarrow \mathcal{U}^r \), which sends
\[
B_{i,l} \mapsto B_{i,l}, \quad \Theta_{i,m} \mapsto \Theta_{i,m}, \quad K_i \mapsto K_i, \quad C \mapsto C, \quad \text{for } m \geq 1, \; i \in \mathbb{Z}, \; i \in \{1, 2\}. \tag{5.13}
\]
The inverse \( \Phi^{-1} : \mathcal{U}^r \rightarrow \mathcal{D}_r \mathcal{U} \) sends
\[
K_0 \mapsto -v^{-1} C K_1^{-1} K_2^{-1}, \quad K_i \mapsto K_i, \quad B_i \mapsto B_{i,0}, \quad \text{for } i \in \{1, 2\}, \quad B_0 \mapsto o(1) v^{-1}(\Theta_{1,1} - v[B_1, B_{2,-1}]_{v^{-1} C K_2^{-1}}) K_1^{-1}.
\]

(We shall refer to \( \mathcal{D}_r \mathcal{U} \) the Drinfeld type presentation of \( \mathcal{U}^r \). The proof that \( \Phi \) is a homomorphism requires long computations, and will be carried out in Section 6.)

Proof. To show \( \Phi \) is a homomorphism, we shall verify that all the defining relations in \( \mathcal{D}_r \mathcal{U} \) (see Definition 5.1) are preserved by \( \Phi \) in the next Section 6. More precisely, the relations (5.1)–(5.6) in \( \mathcal{D}_r \mathcal{U} \) hold for the images of the generators of \( \mathcal{D}_r \mathcal{U} \) under \( \Phi : \mathcal{D}_r \mathcal{U} \rightarrow \mathcal{U}^r \), thanks to Lemma 6.1, Propositions 6.16, 6.18, 6.6, 6.19 and 6.20, respectively.

Next we show that \( \Phi \) is surjective. To that end, it suffices to show that the generator \( B_0 \) of \( \mathcal{U}^r \) lies in the image of \( \Phi \) (as other generators clearly do). By definition of \( \Theta_{1,1} \) in (4.6), \( \Phi \) maps \( v^{-1} o(1) (\Theta_{1,1} - v[B_1, B_{2,-1}]_{v^{-1} C K_2^{-1}}) K_1^{-1} \) to \( B_0 \), and hence the surjectivity of \( \Phi \) follows.

The injectivity of \( \Phi \) follows by an analogous argument as for the injectivity in [LW21b, Theorem 3.13]. For the sake of completeness, we sketch below.

We set \( \mathbb{I}_0 = \{1, 2\} \) in this proof. Denote by \( \mathcal{U}^r_\succ \) (respectively, \( \mathcal{D}_r \mathcal{U}^r_\succ \)) the subalgebra of \( \mathcal{U}^r \) (respectively, \( \mathcal{D}_r \mathcal{U}^r \)) generated by \( B_{i,m}, H_{i,m}, K_i \), for \( m \geq 1 \), and \( i \in \mathbb{I}_0 \). Then \( \Phi : \mathcal{D}_r \mathcal{U}^r \rightarrow \mathcal{U}^r \) restricts to a surjective homomorphism \( \Phi : \mathcal{D}_r \mathcal{U}^r_\succ \rightarrow \mathcal{U}^r_\succ \).

The translation symmetries \( \omega \) on \( \mathcal{D}_r \mathcal{U}^r \) (see Lemma 5.3) and \( T_\omega \) in (4.2) are compatible under \( \Phi \), i.e.,
\[
\Phi \circ \omega = T_\omega \circ \Phi.
\]
The injectivity of \( \Phi : \mathcal{D}_r \mathcal{U}^r \rightarrow \mathcal{U}^r \) is then reduced to the injectivity of \( \Phi : \mathcal{D}_r \mathcal{U}^r_\succ \rightarrow \mathcal{U}^r_\succ \), since any element in the kernel of \( \Phi : \mathcal{D}_r \mathcal{U}^r \rightarrow \mathcal{U}^r \) gives rise to (via a translation automorphism \( \omega^{-N} \), for \( N \gg 0 \)) to an element in the kernel of \( \Phi : \mathcal{D}_r \mathcal{U}^r_\succ \rightarrow \mathcal{U}^r_\succ \).

It remains to prove the injectivity of \( \Phi : \mathcal{D}_r \mathcal{U}^r_\succ \rightarrow \mathcal{U}^r_\succ \). We shall accomplish this by examining a certain filtration and its associated graded.

Define a filtration on \( \mathcal{D}_r \mathcal{U}^r_\succ \) by
\[
(\mathcal{D}_r \mathcal{U}^r_\succ)_0 \subset (\mathcal{D}_r \mathcal{U}^r_\succ)_1 \subset \cdots \subset (\mathcal{D}_r \mathcal{U}^r_\succ)_m \subset \cdots \tag{5.14}
\]
by setting
\[
(\mathcal{D}_r \mathcal{U}^r_\succ)_m = \mathbb{Q}(v)\text{-span}\{x = B_{i_1,m_1} B_{i_2,m_2} \cdots B_{i_r,m_r} \Theta_{j_1,n_1} \Theta_{j_2,n_2} \cdots \Theta_{j_s,n_s} K_\mu \mid \mu \in \mathbb{N}_0, i_1, \ldots, i_r, j_1, \ldots, j_s \in \mathbb{I}_0, m_1, \ldots, m_r, n_1, \ldots, n_s \geq 1, \text{ht}^+(x) \leq m\}. \tag{5.15}
\]
Here we have denoted
\[
\text{ht}^+(x) := \sum_{a=1}^r \text{ht}(m_a \delta + \alpha_{i_a}) + \sum_{b=1}^s n_b \text{ht}(\delta), \tag{5.16}
\]
where $\text{ht}(\beta)$ denotes the height of a positive root $\beta$. Recalling $\tilde{U}^{r,0}$ from (2.13), we have $(\text{Dr}\tilde{U})^{r,0} = \tilde{U}^{r,0} = \mathbb{Q}(v)[K_i^{\pm 1} \mid i \in \mathbb{I}]$. The filtration (5.14)–(5.15) on $\text{Dr}\tilde{U}^r$ defined via a height function is compatible with the filtration (2.11)–(2.12) on $\tilde{U}^r$ under $\Phi$, and thus the surjective homomorphism $\Phi : \text{Dr}\tilde{U}^r \rightarrow \tilde{U}^r$ induces a surjective homomorphism

$$
\text{gr}\Phi : \text{gr}\text{Dr}\tilde{U}^r \rightarrow \text{gr}\tilde{U}^r.
$$

(5.17)

The Drinfeld presentation $\text{Dr}U$ of the affine quantum group $U$ has generators $x_{i,k}^{\pm 1}$, $h_i$, $m$, $K_i^{\pm 1}$, $C^{\pm 1/2}$, for $i \in \mathbb{I}$, $k \in \mathbb{Z}$, $m \in \mathbb{Z} \setminus \{0\}$, cf. [Dr87, Be94]; moreover, we have an isomorphism $\phi : \text{Dr}U \rightarrow U$. Denote by $\text{Dr}U_<$ the $\mathbb{Q}(v)$-subalgebra of $U$ generated by $x_{i,-k}^-$, for $i \in \mathbb{I}$, $k > 0$, and denote by $U_\prec = \phi(\text{Dr}U_<)$. Then $\phi$ restricts to an isomorphism [Be94, Da15]

$$
\phi : \text{Dr}U_< \xrightarrow{\cong} U_<.
$$

(5.18)

Recall the following algebra isomorphism from (2.14) with respect to the filtration on $\tilde{U}^r$ (2.11)–(2.12):

$$
\mathcal{G} : U^- \otimes \tilde{U}^{r,0} \rightarrow \text{gr}\tilde{U}^r, \quad F_i \mapsto \overline{B}_i, \quad K_i \mapsto \overline{K}_i,
$$

where $U^- = \langle F_i \mid i \in \mathbb{I} \rangle$. The homomorphism $\mathcal{G}$ above restricts to an isomorphism

$$
\mathcal{G} : U_<^- \otimes \tilde{U}^{r,0} \xrightarrow{\cong} \text{gr}\tilde{U}^r_<.
$$

(5.19)

Finally, by definition (5.15) of the filtration on $\text{Dr}\tilde{U}_<^r$, its associated graded is compatible with setting $C = 0$ in the defining relations (5.1)–(5.6) of $\text{Dr}\tilde{U}^r$, which reproduces the defining relations of half the affine quantum group in its Drinfeld presentation. Thus, we have a surjective homomorphism

$$
\Xi : \text{Dr}U_<^r \otimes \tilde{U}^{r,0} \rightarrow \text{gr}\text{Dr}\tilde{U}^r_<,
$$

(5.20)

which sends $x_{i,-k}^- \mapsto \overline{B}_{i,k}$, for $k > 0$ (note the opposite signs in indices).

Combining (5.17)–(5.20), we have obtained the following commutative diagram

$$
\begin{array}{ccc}
\text{Dr}U_<^r \otimes \tilde{U}^{r,0} & \xrightarrow{\Xi} & \text{gr}\text{Dr}\tilde{U}^r_< \\
\downarrow \phi \circ \cong & & \downarrow \text{gr}\Phi \circ \cong \\
U_<^- \otimes \tilde{U}^{r,0} & \xrightarrow{\mathcal{G} \circ \cong} & \text{gr}\tilde{U}^r_<
\end{array}
$$

Since $\Xi$ and $\text{gr}\Phi$ are surjective while $\phi$ and $\mathcal{G}$ are isomorphisms, we conclude that $\text{gr}\Phi : \text{gr}\text{Dr}\tilde{U}^r_< \rightarrow \text{gr}\tilde{U}^r_<$ is injective (and indeed an isomorphism), and so is $\Xi$.

The proof of Theorem 5.5 is completed. 

□

Remark 5.6. By definition, the algebra $\text{Dr}\tilde{U}^r$ (and hence $\tilde{U}^r$ by Theorem 5.5) is $\mathbb{Z}$-graded by letting

$$
deg C = 2, \quad \deg K_i = 0, \quad \deg B_{i,k} = k, \quad \deg \Theta_{i,n} = n, \quad (i = 1, 2).
$$

Compare Remark 4.7.
5.3. Presentation via generating functions. Recall the generating functions $B_i(z)$, $Θ_i(z)$, $H_i(z)$ and $∆(z)$ from (5.9). We also denote
\[
S(w_1, w_2 | z; i) = \text{Sym}_{w_1, w_2} \left( B_i(w_1)B_i(w_2)B_{ri}(z) - [2]B_i(w_1)B_{ri}(z)B_i(w_2) + B_{ri}(z)B_i(w_1)B_i(w_2) \right).
\]

Theorem 5.5 admits the following reformulation via generating functions.

**Theorem 5.7.** $\tilde{U}^i$ is isomorphic to the $\mathbb{Q}(v)$-algebra generated by the elements $B_d$, $H_{ik}$, $K_i^\pm 1$, $C^\pm 1$ where $i = 1, 2$, $l \in \mathbb{Z}$ and $k > 0$, subject to the following relations, for $i, j \in \{1, 2\}$:
\[
K_iK_j = K_jK_i, \quad K_iH_j(z) = H_j(z)K_i, \quad K_iB_j(z) = v^{ε_{i}j}c_{ij}B_j(z)K_i, \quad C \text{ is central},
\]
\[
[H_i(z), H_j(w)] = 0,
\]
\[
Θ_i(z)B_j(w) = \frac{(1 - v^{-ε_{ij}zw^{-1}})(1 - v^{ε_{ij}zwC})}{(1 - v^{ε_{ij}zw^{-1}})(1 - v^{-ε_{ij}zwC})}B_j(w)Θ_i(z),
\]
\[
(v^2z - w)B_i(z)B_i(w) + (v^2w - z)B_i(w)B_i(z) = 0,
\]
\[
(v^{-1}z - w)B_i(z)B_{ri}(w) + (v^{-1}w - z)B_{ri}(w)B_i(z) = \frac{∆(zw)}{1 - v^2}((z - vw)K_iΘ_{ri}(w) + (w - vz)K_{ri}Θ_i(z)),
\]
\[
S(w_1, w_2 | z; i) = v^{-1} \text{Sym}_{w_1, w_2} \frac{1 - vw_1^{-1}z}{1 - w_1^{-1}w_2^{-1}z^2} \left[ Θ_{ri}(z)K_i, B_i(w_1) \right] v,
\]
\[
+ v^{-1} \text{Sym}_{w_1, w_2} \frac{vw_1w_2z^{-2} - w_1z^{-1}}{1 - w_1w_2z^{-2}} \left[ Θ_i(w_2)K_{ri}, B_i(w_1) \right] v,
\]
\[
+ v^{-1} \text{Sym}_{w_1, w_2} \frac{vw_1^{-1}w_2^{-1}z^2 - w_1^{-1}z}{1 - w_1^{-1}w_2^{-1}z^2} \left[ B_i(w_1), Θ_{ri}(z)K_i \right] v,
\]
\[
+ v^{-1} \text{Sym}_{w_1, w_2} \frac{1 - vw_2z^{-1}}{1 - w_1w_2z^{-2}} \left[ B_i(w_1), Θ_i(w_2)K_{ri} \right] v.
\]

**Proof.** We simply rewrite the relations (5.1)–(5.6) in Definition 5.1 by using the generating functions (5.9). For example, the relation (5.27) is obtained by multiplying both sides of the relation (5.6) by $w_1^{k_1}w_2^{k_2}z^l$ and summing over $k_1, k_2, l \in \mathbb{Z}$.

6. Verification of the current relations

In this section, we verify that all the defining relations in $\text{Dr}^i\tilde{U}^i$ are preserved by the homomorphism $Φ$ in (5.13). This completes the proof of Theorem 5.5.

6.1. Strategy of proofs. Let us explain the general strategy of the proofs before getting to the technical details. The approach of [BK20] for $q$-Onsager algebra (see also [Da93] for quantum affine $\mathfrak{sl}_2$) provides a helpful though rough guideline for the overall inductive arguments. However, our quasi-split affine rank 1 setting behaves with complexity of affine rank two, as there are 2 infinite series of real (and respectively, imaginary) root vectors as Drinfeld type generators; in contrast, there is only one infinite series of real (and respectively, imaginary) root vectors for $q$-Onsager algebra. Accordingly, the approach developed in
[LW21b] and especially in [Z21] dealing with Serre relations in higher ranks also helps to inspire new ways to get around various technical difficulties.

A central relation (5.2) concerns about the commutativity among imaginary root vectors $\Theta_{i,n}$, for $i = 1, 2$ and $n \geq 1$; a closely related property is the $T_\omega$-invariance of $\Theta_{i,n}$. Assuming these properties, several additional relations among root vectors for $\tilde{U}^i$ can be proved. However, $\Theta_{i,n}$ is defined in terms of (a linear combination of) $v$-commutators between real root vectors. To establish the commutativity among $\Theta_{i,n}$, one has to first understand to some extent commutators of $\Theta_{i,n}$ with real root vectors, and so we would run in a vicious circle if we were not very careful.

By direct computations, we shall establish $T_\omega(\Theta_{j,1}) = \Theta_{j,1}$, and some formula for the commutator $[\Theta_{j,1}, B_1]$. This suffices to derive fully the first nontrivial Relation (5.4).

The actual inductive proofs of Relations (5.2), (5.3), and (5.5) run like a spiral. Under the assumption on the partial commutativity between $\Theta_{j,1}$ and $\Theta_{i,m}$, for $m < n$ (for some fixed positive integer $n$), we show that $T_\omega(\Theta_{i,n}) = \Theta_{i,n}$, and then establish part of (bounded by $n$) the relations for commutators between real root vectors; the Serre relations (5.6) are also partially proved along the way, which imply partial relations for commutators between imaginary and real root vectors. We then use these to establish fully the commutativity among $\Theta_{i,m}$, for $m \leq n$, and then all these relations hold unconditionally. The proof of Relation (5.3) by induction independent of (the proof of) the Serre relations (5.6) follows closely the approach developed in [Z21] (instead of [LW21b]). Finally, we fully prove the Serre relations (5.6) for $\tilde{U}^i$, imitating [Z21] again.

### 6.2. Relation (5.1).

**Lemma 6.1.** Relation (5.1) holds in $\tilde{U}^i$.

*Proof.* Recall that $C = -vK_\delta = -vK_0K_1K_2$. Then $C$ is central and $K_iK_j = K_jK_i$ by (2.4).

We have $H_{j,m} \in \tilde{U}_{m\delta}^i$ by using (5.7) and noting that $\Theta_{j,m} \in \tilde{U}_{m\delta}^i$ (see Lemma 4.3). Hence, by Lemma 2.3, we have $K_iH_{j,m} = H_{j,m}K_i$.

The last relation $K_iB_{j,1} = v^{\varepsilon_{j,i} - c_{i,0}}B_{j,1}K_i$ in (5.1) follows by Lemma 2.3 and Lemma 4.3. □

### 6.3. Properties of $B_{i,1}$ and $\Theta_{i,1}$.

**Lemma 6.2.** For $i = 1, 2$, we have

$$B_{i,1} = o(i)[([B_i, B_{\tau i}]v, [B_i, B_0]v)]_v - v[B_i, B_0]v,K_{\tau i} - v^{-1}[2][B_i, B_0]v,K_i].$$  \hspace{1cm} (6.1)

The proof of Lemma 6.2 is long and can be found in Appendix A.2.

**Lemma 6.3.** We have $T_\omega(\Theta_{i,1}) = \Theta_{i,1}$, for $i = 1, 2$.

*Proof.* Recall from (4.6) that $\Theta_{i,1} = -o(i)([B_i, B_{\tau i}, B_0]v^2 - vB_0K_i)$. It suffices to check that

$$T_1^{-1}T_0^{-1}([B_i, B_{\tau i}, B_0]v^2 - vB_0K_1) = [B_i, B_{\tau i}, B_0]v^2 - vB_0K_1.$$  \hspace{1cm} (6.2)

Since $B_{i,1} = o(1)T_1^{-1}T_0^{-1}(B_1)$ by definition, we have by Lemma 6.2 that

$$T_1^{-1}T_0^{-1}(B_1) = \left(\left([B_1, B_{\tau i}, B_0]v^2 - v[B_1, B_0]v^3K_2 - v^{-1}[2][B_1, B_0]v,K_1\right)\right).$$
Then we compute
\[
T^{-1}_1 T^{-1}_0 \left( [B_1, [B_2, B_0]_v]_{v^2} \right)
\]
\[
= T^{-1}_1 \left( T^{-1}_0(B_1), [B_0, B_2]_{v^2}, B_0 \mathbb{K}^{-1}_{0} \right)_{v^2}
\]
\[
= T^{-1}_1 \left( T^{-1}_0(B_1), [B_0, B_2]_{v^2}, B_0 \mathbb{K}^{-1}_{0} \right) \mathbb{K}^{-1}_{0}
\]
\[
= T^{-1}_1 \left( T^{-1}_0(B_1), B_2 \mathbb{K}^{-1}_{0} \right) \mathbb{K}^{-1}_{0}
\]
\[
= T^{-1}_1 \left( \left[ B_1, B_2 \right] \mathbb{K}^{-1}_{0} \right) \mathbb{K}^{-1}_{0}
\]
\[
= \left( [B_1, B_2]_{v}, [B_1, B_0]_{v} \right)_{v^2} - v[B_1, B_0]_{v^3} \mathbb{K}_2 - v^{-1} [2] [B_1, B_0]_{v} \mathbb{K}_1, -vB_2 \mathbb{K}^{-1}_{2}
\]
\[
= -v \left( [B_1, B_2]_{v}, [B_1, B_0]_{v} \right)_{v^2} - v[B_1, B_0]_{v^3} \mathbb{K}_2 - v^{-1} [2] [B_1, B_0]_{v} \mathbb{K}_1, B_2 \right)_{v^{-1}} \mathbb{K}^{-1}_{2}
\]
\[
= [B_2, [B_1, B_2]_{v}, [B_1, B_0]_{v}]_{v^2} - v[B_2, [B_1, B_0]_{v^3} \mathbb{K}_2 - v^{-1} [2] [B_2, [B_1, B_0]_{v} \mathbb{K}_1, \mathbb{K}^{-1}_{2}]
\]
\[
= [B_2, [B_1, B_2]_{v}, [B_1, B_0]_{v}]_{v^2} \mathbb{K}^{-1} - v[B_2, [B_1, B_0]_{v^3}]_{v^{-2}} - v^{-1} [2] [B_2, [B_1, B_0]_{v} \mathbb{K}_1, \mathbb{K}^{-1}_{2}]
\]

Also, by the formula (2.26) for \( T^{-1}_1(B_0) \), we have
\[
-vT^{-1}_1 T^{-1}_0(B_0 \mathbb{K}_1) = -T^{-1}_1(B_0) \mathbb{K}^{-1}_{2}
\]
\[
= -v \left( [B_1, B_2]_{v}, [B_2, [B_1, B_0]_{v}] \right) \mathbb{K}^{-1}_{2} + [B_1, B_2]_{v^3}, B_0] - vB_0 \mathbb{K}_1.
\]

Summing the above 2 computations and using the Serre relation (2.6), we have
\[
T^{-1}_1 T^{-1}_0 \left( [B_1, [B_2, B_0]_{v}]_{v^2} - vB_0 \mathbb{K}_1 \right)
\]
\[
= [B_2, [B_1, B_2]_{v}, [B_1, B_0]_{v}]_{v^2} \mathbb{K}^{-1} - v[B_2, [B_1, B_0]_{v^3}]_{v^{-2}} - v^{-1} [2] [B_2, [B_1, B_0]_{v} \mathbb{K}_1, \mathbb{K}^{-1}_{2}]
\]
\[
- v \left( [B_1, B_2]_{v}, [B_2, [B_1, B_0]_{v}] \right) \mathbb{K}^{-1} + [B_1, B_2]_{v^3}, B_0] - vB_0 \mathbb{K}_1
\]
\[
= -[2] [B_1, B_0]_{v^3} B_2 \mathbb{K}_2 + v^3 B_2 \mathbb{K}_1 \right]_{v^{-1}} \mathbb{K}^{-1} - v \left( [B_2, [B_1, B_0]_{v^3}]_{v^{-2}}
\]
\[
= -[2] [B_2, [B_1, B_0]_{v^3}]_{v^2} \mathbb{K}_0, B_0] - vB_0 \mathbb{K}_1
\]
\[
= [B_1, B_2]_{v^2} - vB_0 \mathbb{K}_1.
\]

The lemma is proved.

We prove a very special \( m = 1 \) case of (5.12) (equivalent to Relation (5.3)) below.

**Lemma 6.4.** For \( i = 1, 2 \), and \( l \in \mathbb{Z} \), we have
\[
\Theta_{i,1}, B_{j,l} = [c_{ij}] B_{j,l+1} - [c_{ri,j}] B_{j,l-1} C.
\]

**Proof.** We only need to consider \( i = 1 \) thanks to the symmetry \( \hat{r} \).
First, assume \( j = 1 \). The identity (6.3) for \( l = 0 \) reads as

\[
[\Theta_{1,1}, B_1] = [2]B_{1,1} + B_{1,-1}C.
\] (6.4)

Then (6.3) for a general \( l \) follows by applying the \((-l)\)th power of \( T_\omega \) to (6.4) since \( \Theta_{i,1} \) is fixed by \( T_\omega \) (see Lemma 6.3). Hence, it remains to prove the identity (6.4).

To that end, we compute

\[
[\Theta_{1,1}, B_1] = -o(1)\left[ [B_1, [B_2, B_0]_v]_{v^2} - vB_0 K_{1,1}, B_1 \right]
= -o(1)\left[ [B_1, [B_2, B_0]_v]_{v^2}, B_1 \right] + v o(1) [B_0 K_{1,1}, B_1]
= -o(1)\left[ [B_1, [B_2, B_0]_v]_{v^2}, B_1 \right] + v^{-2} o(1) [B_0, B_1]_v^3 K_1.
\]

On the other hand, we compute

\[
\left[ [B_1, [B_2, B_0]_v]_{v^2}, B_1 \right]
= -v^2 \left[ B_2, [B_1, [B_1, B_0]_v]_{v^{-1}} \right]_{v^{-1}} + \left[ [B_1, B_0]_v, [B_2, B_1]_v \right]_v
+ v^3 \left[ B_0, [B_1, [B_1, B_2]_v]_{v^{-1}} \right]_{v^{-1}} - v^{-1} \left[ [B_1, B_2]_v, [B_1, B_0]_v \right]_v
= -2 \left[ [B_1, B_2]_v, [B_1, B_0]_v \right] - v [2] [B_0, B_1]_{v^3} K_1 - v^4 [2] [B_0, B_1]_{v^3} K_2.
\]

Combining the above 2 computations, we obtain

\[
[\Theta_{1,1}, B_1] = [2] o(1) \left[ [B_1, B_2]_v, [B_1, B_0]_v \right]_v + [2] v o(1) [B_0, B_1]_{v^{-3}} K_1
+ [2] v^4 o(1) [B_0, B_1]_{v^3} K_2 + v^{-2} o(1) [B_0, B_1]_v^3 K_1.
\]

This formula can then be converted to the desired identity (6.4) by using (4.4) and (6.1).

Now assume \( j = 2 \). As above, the proof of (6.3) for a general \( l \) follows from the case with \( l = 0 \); that is, it remains to prove

\[
[\Theta_{1,1}, B_2] = -B_{2,1} - [2] B_{2,-1} C = -B_{2,1} + v [2] B_{2,-1} K_3.
\] (6.5)

Indeed, we have

\[
[\Theta_{1,1}, B_2] = -o(1) \left[ [B_1, [B_2, B_0]_v]_{v^2} - vB_0 K_{3,1}, B_2 \right]
= -o(1)\left[ [B_1, [B_2, B_0]_v]_{v^2}, B_2 \right] + v o(1) [B_0 K_{3,1}, B_2]
= -o(1)\left[ [B_1, [B_2, B_0]_v]_{v^2}, B_2 \right] + v^4 o(1) [B_0, B_2]_{v^{-3}} K_1
= o(1)\left[ [B_2, B_1]_v, [B_2, B_0]_v \right]_v + v^4 o(1) [B_0, B_2]_{v^{-3}} K_1.
\]

This formula can be converted to (6.5) by using (4.4), (6.1) and \( o(1) = -o(2) \). \( \square \)

6.4. Relation (5.4).

**Lemma 6.5.** For \( i \in \{1, 2\} \) and \( k \in \mathbb{Z} \), we have \([B_{i,k-1}, B_{i,k}]_{v^{-2}} = 0\).
Proof. We note that the formula in the lemma for different $k$ are equivalent by applying suitable powers of $T_\omega$.

It remains to prove the desired formula for $k = 1$. By the formula (4.4) for $B_{i,-1}$, we have

$$-v^2 o(1) \cdot [B_{i,-1}, B_i]_{v-2} = [B_i, B_0]_v K_i B_i K_i^{-1} - v^{-2} B_i [B_i, B_0]_v K_i K_i^{-1}$$

$$= v^{-3} [ [B_i, B_0]_v, B_i ]_v K_i K_i^{-1} = 0,$$

where the last equality follows by the Serre relation (2.7). \hfill $\Box$

Proposition 6.6. Relation (5.4) holds in $\widehat{U}^i$, i.e.,

$$[B_{i,k}, B_{i,l+1}]_{v-2} + [B_{i,l}, B_{i,k+1}]_{v-2} = 0,$$

for $i \in \{1, 2\}$, and $k, l \in \mathbb{Z}$.

Proof. Without loss of generality, we assume that $i = 1$. It follows by (6.3) that

$$[2] \Theta_{1,1} + \Theta_{2,1}, B_{1,k} = [3] B_{1,k+1}. \quad (6.6)$$

Denote $F_k = [B_{1,k}, B_{1,1}]_{v-2} + [B_{1}, B_{1,k+1}]_{v-2}$, for $k \in \mathbb{Z}$. Then we have

$$[2] \Theta_{1,1} + \Theta_{2,1}, F_k = [3] (F_{k+1} + T_\omega^{-1} F_{k-1}). \quad (6.7)$$

In particular, we have $[2] \Theta_{1,1} + \Theta_{2,1}, F_0 = [3] (F_1 + T_\omega^{-1} F_{-1})$. Since $F_0 = 0$ by Lemma 6.5 and $F_1 = T_\omega^{-1} F_{-1}$, we obtain $F_1 = F_{-1} = 0$. It then follows by (6.7) and an induction on $k$ that $F_k = 0$, for all $k \in \mathbb{Z}$.

The desired relation now follows by applying a suitable power of $T_\omega$ to $F_{k-1} = 0$. \hfill $\Box$

6.5. Partial commutativity and $T_\omega$-invariance of $\Theta_{i,n}$. Recall $D_{i,n}$ and $\Theta_{i,n}$ defined in (4.5) and (4.8), respectively.

Lemma 6.7. For $i = 1, 2$, we have

$$D_{i,-1} = - [B_{ri}, B_{i,-1}]_{v-1} - [B_i, B_{ri,-1}]_{v-1} = -v^{-1} (\Theta_{ri,1} K_i + \Theta_{ri,1} K_{ri}) C^{-1}. \quad (6.8)$$

Proof. We assume $i = 1$. By (4.4) and (4.6), we have

$$- [B_i, B_{2,-1}]_{v-1} = -v^{-1} \Theta_{1,1} K_i C^{-1} - o(1) v^{-1} B_0 K_0^{-1}, \quad (6.9)$$

$$- [B_2, B_{1,-1}]_{v-1} = -v^{-1} \Theta_{2,1} K_1 C^{-1} - o(2) v^{-1} B_0 K_0^{-1}.$$

Hence, (6.8) follows. \hfill $\Box$

We prove a very special $m = 2$ case of (5.12) (equivalent to Relation (5.3)) below.

Lemma 6.8. For $i = 1, 2$, we have

$$[\Theta_{i,2}, B_i] + v^4 [\Theta_{i,0}, B_i]_{v-4} C = v^2 [\Theta_{i,1}, B_i]_{v-4} + v [\Theta_{i,1}, B_{i,-1}]_{v-2} C.$$

Proof. We assume $i = 1$. We shall compute $[\Theta_{1,2}, B_1]$. Using $C = -v K_2 K_1 K_0$ from (4.1), recall $\Theta_{1,2}$ from (4.7):

$$\Theta_{1,2} = -v^2 \left( [B_2, B_1]_{v-1} + [B_{1,1}, B_{2,-1}]_{v-1} \right) K_1 K_0 + v \Theta_{1,0} C - \Theta_{2,0} C K_2^{-1} K_1. \quad (6.10)$$

The main part in $[\Theta_{1,2}, B_1]$ is

$$-v^2 [ [B_{1,1}, B_{2,-1}]_{v-1} K_1 K_0, B_1 ] = v^2 [B_i, [B_{1,1}, B_{2,-1}]_{v-1}]_{v-4} K_1 K_0$$

$$= v^2 [ [B_1, B_{1,1}]_{v-2}, B_{2,-1} ]_{v-2} K_1 K_0 + [B_{1,1}, [B_1, B_{2,-1}]_{v-1}]_{v-2} K_1 K_0.$$
where we used $[B_1, B_{1,1}]_v = 0$ (by Lemma 6.5) and (6.9) in the last step.

Next, we compute the term $[B_{1,1}, B_0]_v$ in (6.11). Recall the formula for $B_{i,1}$ from (6.1):

$$B_{i,1} = o(i) \left( [[B_i, B_{ri}]_v, [B_i, B_0]_v]_v - v[B_i, B_0]_v \mathbb{K}_1 - v^{-1}[2][B_i, B_0]_v \mathbb{K}_i \right).$$

Then the main part in $[B_{1,1}, B_0]_v$ is

$$o(1) \left( [[B_1, B_2]_v, [B_1, B_0]_v]_v, B_0 \right)_v$$

$$= o(1) \left( [B_1, B_2]_v, [[B_1, B_0]_v, B_0]_{v^{-1}} \right)_v - vo(1) \left( [B_1, B_0]_v, [[B_1, B_2]_v, B_0]_{v^2} \right)_{v^{-2}}$$

$$= o(1) \left( [B_1, B_2]_v, [[B_1, B_0]_v, B_0]_{v^{-1}} \right)_v$$

$$- vo(1) \left( [B_1, B_0]_v, [B_1, [B_2, B_0]_v]_{v^2} \right)_{v^{-2}} + v^2 o(1) \left( [B_1, B_0]_v, [B_2, [B_1, B_0]_v] \right)_{v^{-2}}$$

$$= -o(1)v^{-1} \left( [B_1, B_2]_v, B_1 \right)_{v \mathbb{K} o}$$

$$+ v^{-1} \left[ \Theta_{1,1} + vo(1)B_0 \mathbb{K}_1, [B_1, B_0]_v \right]_{v^2} + [\Theta_{2,1} - vo(1)B_0 \mathbb{K}_2, [B_1, B_0]_v]$$

$$= -o(1) \left( v^{-1}[[B_1, B_2]_v, B_1]_{v^3} \mathbb{K}_0 - [B_0 \mathbb{K}_1, [B_1, B_0]_v]_{v^2} + v[B_0 \mathbb{K}_2, [B_1, B_0]_v] \right)$$

$$+ o(1)v^2 \left( v^{-1}[\Theta_{1,1}, B_{1,-1}]_{v^2} + [\Theta_{2,1}, B_{1,-1}] \right) \mathbb{K}_2 \mathbb{K}_0.$$

Then we reorganize the last computation as

$$-v^{-1}o(1)[B_{1,1}, B_0]_v \mathbb{K}_1 + v[\Theta_{1,1}, B_{1,-1}]_{v^{-2}} C$$

$$= \left( [2[\Theta_{1,1} + \Theta_{2,1}, B_{1,-1}] C + v^{-2}[[B_1, B_2]_v, B_1]_{v^3} \mathbb{K}_1 \mathbb{K}_0 \right.$$}

$$+ v^{-1}3(B_1B_0^2 - [2]B_0B_1B_0 + B_0^2B_1)\mathbb{K}_2^3$$

$$= 3B_1C + v^{-2}[[B_1, B_2]_v, B_1]_{v^3} \mathbb{K}_1 \mathbb{K}_0 - v^{-2}3B_1 \mathbb{K}_2^3 \mathbb{K}_0,$$

where we have used (6.6) and the Serre relation (2.8) in the last equality. Arranging the pieces (6.10), (6.11) and (6.12) together, we have

$$[\Theta_{1,2}, B_1] + v^3[\Theta_{1,0}, B_1]_{v^{-8}} C - v^2[\Theta_{1,1}, B_{1,1}]_{v^{-4}} - v[\Theta_{1,1}, B_{1,-1}]_{v^{-2}} C$$

$$= -v^2[[B_2, B_1]_{v^{-1}} \mathbb{K}_1 \mathbb{K}_0, B_1] + \frac{v}{v - v^{-1}}[\mathbb{K}_1 \mathbb{K}_0, B_1]$$

$$- v^{-2}[[B_1, B_2]_v, B_1]_{v^3} \mathbb{K}_1 \mathbb{K}_0 + v^{-2}3B_1 \mathbb{K}_2^3 \mathbb{K}_0$$

$$= 0.$$

The lemma is proved.\qed

**Lemma 6.9.** Let $n \geq 1$, and $i, j \in \{1, 2\}$. Assume that

$$\begin{cases}
T_\omega(\Theta_{i,m}) = \Theta_{i,m}, & \text{for } m \leq n, \\
[\Theta_{j,1}, \Theta_{i,m}] = 0, & \text{for } m < n.
\end{cases}$$

Then $[\Theta_{j,1}, \Theta_{i,n}] = [c_{ij}](1 - T_\omega)\Theta_{i,n+1}$. 


Proof. By definition of $D_{i,k}$ and using Lemma 6.4, we have

$$[\Theta_{i,1}, D_{i,n-2}] = [B_{\tau_i,1}, B_{i,n-2}]_{v^{-1}} + 2[B_{\tau_i,1}, B_{i,n-2}]_{v^{-1}}C - 2[B_{\tau_i,1}, B_{i,n-2}]_{v^{-1}} - [B_{\tau_i,1}, B_{i,n-3}]_{v^{-1}}C$$

$$+ [B_{i,n-1}, B_{\tau_i}]_{v^{-1}} + 2[B_{i,n-1}, B_{\tau_i}]_{v^{-1}}C - 2[B_{i,n-1}, B_{\tau_i}]_{v^{-1}} - [B_{i,n-2}, B_{\tau_i}]_{v^{-1}}C$$

$$= 2(1 - C T_\omega) D_{i,n-1} - T_\omega^{-1}(1 - C T_\omega) D_{i,n-3},$$

for any $n \geq 0$. By the assumption $T_\omega(\Theta_{i,m}) = \Theta_{i,m}$ for $m \leq n$, we have

$$(1 - C T_\omega) D_{i,n-1} = (1 - C T_\omega)(-v^{-1}\Theta_{i,n+1}\kappa_{\tau_i}C^{-1} + \Theta_{i,n-1}\kappa_{\tau_i})$$

$$= -v^{-1}(1 - T_\omega)\Theta_{i,n+1}\kappa_{\tau_i}C^{-1},$$

$$(1 - C T_\omega) D_{i,n-3} = (1 - C T_\omega)(-v^{-1}\Theta_{i,n-1}\kappa_{\tau_i}C^{-1} + \Theta_{i,n-3}\kappa_{\tau_i}) = 0.$$

So

$$[\Theta_{i,1}, D_{i,n-2}] = -v^{-1}(1 - T_\omega)\Theta_{i,n+1}\kappa_{\tau_i}C^{-1}.$$

Replace $D_{i,n-2}$ by $-v^{-1}\Theta_{i,n}\kappa_{\tau_i}C^{-1} + \Theta_{i,n-2}\kappa_{\tau_i}$. By the assumption $[\Theta_{i,1}, \Theta_{i,n-2}] = 0$, we have

$$[\Theta_{i,1}, \Theta_{i,n}]\kappa_{\tau_i}C^{-1} = 2(1 - T_\omega)\Theta_{i,n+1}\kappa_{\tau_i}C^{-1},$$

and then the desired formula follows.

For $j = \tau i$, the proof is entirely similar by using now the identity

$$[\Theta_{\tau_i,1}, D_{i,n-2}] = -(1 - C T_\omega) D_{i,n-1} + 2T_\omega^{-1}(1 - C T_\omega) D_{i,n-3}.$$

We omit the details. □

The following corollary will be used repeatedly in the subsequent inductive arguments.

**Corollary 6.10.** Let $n \geq 1$ and $i, j \in \{1, 2\}$. Assume that $[\Theta_{j,1}, \Theta_{i,m}] = 0$, for all $m < n$. Then $\Theta_{i,m}$ are fixed by $T_\omega$, for $1 \leq m \leq n$.

**Proof.** Follows from Lemmas 6.3, 6.9 and an induction on $m$. □

6.6. The commutator $[\Theta_{i,n}, B_i]$. For $i = 1, 2$ and $n \geq 0$, we denote by $H_{i,n}$ the $\mathbb{Q}(v)$-subalgebra of $\tilde{U}^i$ generated by $\{\Theta_{i,m}, \kappa_1, \kappa_2, \kappa_0 \mid 1 \leq m < n\}$.

**Proposition 6.11.** Let $n \geq 1$, and assume that $[\Theta_{j,1}, \Theta_{i,m}] = 0$ for all $i, j \in \{1, 2\}$ and all $m < n$. Then there exist $X_{i,n}^{(i)} \in H_{i,n}$, for $-n \leq k \leq n$, such that

$$[\Theta_{i,n}, B_1] = \sum_{k=-n}^{n} B_{i,k} X_{i,n}^{(i)}.$$ (6.13)

In order to prove Proposition 6.11, we need to prepare some notations and lemmas. For any $k_1, k_2 \in \mathbb{Z}$ and $i = 1, 2$, we denote

$$\mathbb{R}(k_1, k_2|l; i) := \text{Sym}_{k_1, k_2} \left[ -v^{-1}\Theta_{\tau_i,l-k_2+1} C^{k_2} \kappa_i + \Theta_{\tau_i,l-k_2-1} C^{k_2+1} \kappa_i, B_{i,k_1} \right]_v$$

$$+ \text{Sym}_{k_1, k_2} \left[ -v^{-1}\Theta_{i,k_2-l+1} C^{l} \kappa_{\tau_i} + \Theta_{i,k_2-l-1} C^{l+1} \kappa_{\tau_i}, B_{i,k_1} \right]_v,$$ (6.14)

$$\mathbb{P}(k_1, k_2|l; i) := \text{Sym}_{k_1, k_2} \left[ B_{i,k_1}, -v^{-1}\Theta_{\tau_i,l-k_2+1} C^{k_2-1} \kappa_i + \Theta_{\tau_i,l-k_2-1} C^{k_2} \kappa_i \right]_v.$$
\[ S(k_1, k_2; 1) + S(k_1, k_2 + 1|l; i) - 2[S(k_1, k_2|l + 1; i)] = 2[|S(k_1, k_2|l; i)], \] (6.16)

\[ S(k_1 - 1, k_2|l; i) + S(k_1, k_2 - 1|l; i) - 2[S(k_1, k_2|l - 1; i)] = 2[|S(k_1, k_2|l; i)]. \] (6.17)

**Proof.** If \([\Theta_{i,1}, \Theta_{i,m}] = [\Theta_{r,i,1}, \Theta_{i,m}] = 0\) for all \(m < n\), then \(\Theta_{1,m}, \Theta_{2,m}, 1 \leq m \leq n\) are fixed by \(T_\omega\) by Corollary 6.10. Thus, by applying \(T_\omega\) to the identity (4.8) we obtain the relation (5.5) for \(|l - k| \leq n - 1\).

Below we shall work only with \(i = 1\). For \(|k_1 - l| \leq n - 1, |k_2 - l| \leq n - 1\), we compute
\[
S(k_1 + 1, k_2|l; 1) + S(k_1, k_2 + 1|l; 1)
= \text{Sym}_{k_1,k_2} \left( B_{1,k_1+1}B_{1,k_2}B_{2,l} - 2B_{1,k_1+1}B_{2,l}B_{1,k_2} + B_{2,l}B_{1,k_1+1}B_{1,k_2} \right)
+ \text{Sym}_{k_1,k_2} \left( v^2B_{1,k_2}B_{1,k_1+1}B_{2,l} - 2B_{1,k_1+1}B_{2,l}B_{1,k_2} + 2B_{2,l}B_{1,k_1+1}B_{1,k_2} \right)
+ \text{Sym}_{k_1,k_2} \left( vB_{1,k_1}B_{1,k_2+1}B_{2,l} - 2B_{1,k_1}B_{2,l}B_{1,k_2} + v^{-1}B_{2,l}B_{1,k_2+1}B_{1,k_1} \right)
= [2\text{Sym}_{k_1,k_2} \left( vB_{1,k_1}B_{2,l}B_{1,k_2+1}B_{1,k_1} \right)\] (6.18)

where the second equality follows from \(\text{Sym}_{k_1,k_2} [B_{1,k_1}, B_{1,k_2+1}] v^{-2} = 0\); see Proposition 6.6.

Hence, we have
\[
S(k_1 + 1, k_2|l; 1) + S(k_1, k_2 + 1|l; 1) - 2[S(k_1, k_2|l + 1; 1)] = 2[\text{Sym}_{k_1,k_2} \left( vB_{1,k_1}B_{2,l}B_{1,k_2+1}B_{1,k_1} \right)\] (6.18)

Applying the relation (5.5) for \(|l - k| \leq n - 1\) (which holds as shown in the first paragraph of this proof) to sum up each of the 2 columns of the RHS of (6.18), we have
\[
S(k_1 + 1, k_2|l; 1) + S(k_1, k_2 + 1|l; 1) - 2[S(k_1, k_2|l + 1; 1)]
= [2\text{Sym}_{k_1,k_2} \left( -v^{-1}\Theta_{2,l-k_2+1}C^{k_2}K_1 + \Theta_{2,l-k_2+1}C^{k_2+1}K_1, B_{1,k_1} \right)\] (6.19)

This proves the identity (6.16).

The proof of the remaining identity (6.17) is similar and will be omitted here. □
Lemma 6.13. Let $n \geq 3$ and assume that $[\Theta_{i,1}, \Theta_{i,m}] = [\Theta_{r_i,1}, \Theta_{i,m}] = 0$ for any $i = 1, 2$, and all $m < n$. We have, for $|k_1 - l| \leq n - 2, |k_2 - l| \leq n - 2$,

\[
\mathbb{R}(k_1 - 1, k_2|l; i) + \mathbb{R}(k_1, k_2 - 1|l; i) - [2] \mathbb{R}(k_1, k_2|l - 1; i)
\]

\[= \mathbb{P}(k_1 + 1, k_2|l; i) + \mathbb{P}(k_1, k_2 + 1|l; i) - [2] \mathbb{P}(k_1, k_2|l + 1; i).\]

In particular, for $|k - l| \leq n - 2$, we have

\[
\mathbb{R}(k - 1, k|l) + \mathbb{R}(k, k - 1|l) - [2] \mathbb{R}(k, k|l - 1)
\]

\[= \mathbb{P}(k + 1, k|l) + \mathbb{P}(k, k + 1|l) - [2] \mathbb{P}(k, k|l + 1).\]

Proof. Follows by replacing both $\mathbb{R}$-terms and $\mathbb{P}$-terms in (6.20) by the $S$-terms via (6.16)–(6.17).

For $m \geq 0, k \in \mathbb{Z}$, we define

\[
Y_{1,m}^k := [\Theta_{1,m}, B_{1,k}] + v^3[\Theta_{1,m-2}, B_{1,k}]v^{-3}C - v^2[\Theta_{1,m-1}, B_{1,k+1}]v^{-4} - v[\Theta_{1,m-1}, B_{1,k-1}]v^{-2}C,
\]

(6.22)

\[
Y_{2,m}^k := [\Theta_{2,m}, B_{1,k}] + v^{-3}[\Theta_{2,m-2}, B_{1,k}]v^3C - v^{-1}[\Theta_{2,m-1}, B_{1,k+1}]v^2 - v^{-2}[\Theta_{2,m-1}, B_{1,k-1}]v^4C.
\]

(6.23)

(Note that the desired relation (5.3) can be formulated as $Y_{1,m}^k = 0 = Y_{2,m}^k$, by Lemma 5.4.)

For convenience, we set $Y_{1,-m}^k = 0 = Y_{2,-m}^k$ for $m > 0, l \in \mathbb{Z}$. We rewrite (6.21) as

\[
(Y_{2,l-k}^k C - v^{-1}Y_{2,l-k+2}^k)C^{k-1}K_1 + (v^3Y_{1,k-l}^k C - v^2Y_{1,k-l+2}^k)C^{l-1}K_2 = 0,
\]

(6.24)

for $|k - l| \leq n - 2$.

Lemma 6.14. Let $n \geq 1$, and assume that $[\Theta_{j,1}, \Theta_{i,m}] = 0$, for $i, j = 1, 2$ and all $m < n$. Then we have

\[
Y_{1,n}^k = 0, \quad Y_{2,n}^k = 0, \quad \forall k \in \mathbb{Z}.
\]

(6.25)

Proof. Since $\Theta_{i,m}, \Theta_{r_i,m}, 1 \leq m \leq n$ are fixed by $T_\omega$, by Corollary 6.10, $T_\omega$ acts on $Y_{i,m}^k$ by shifting the superscript $k$. Hence, for fixed $1 \leq m \leq n$, if $Y_{i,m}^k = 0$ for some $k$, then $Y_{i,m}^k = 0$ for all $k \in \mathbb{Z}$.

Denote $Y_{i,m} = Y_{i,m}^0$. It remains to show by induction on $n$ that $Y_{i,n} = 0$ for $i = 1, 2$. This holds for $n = 1, 2$ by Lemma 6.4 and Lemma 6.8. Let $n \geq 3$. Applying a suitable power of $T_\omega$ to (6.24) gives us the identity

\[
(Y_{2,l-k}^k C - v^{-1}Y_{2,l-k+2}^k)C^{k-1}K_1 + (v^3Y_{1,k-l}^k C - v^2Y_{1,k-l+2}^k)C^{l-1}K_2 = 0,
\]

(6.26)

for $|k - l| \leq n - 2$. For $n = 3$, set $k - l = 1$ in (6.26). Note $Y_{i,p} = 0$ for $p \leq 0$. Then

\[
Y_{1,2}K_2 = -v^{-3}Y_{2,1}C K_1 + vY_{1,1}C K_2 = 0.
\]

Similarly, we have $Y_{2,3}K_1 = vY_{2,1}C K_1 - v^3Y_{1,1}C K_2 = 0$. Let $n \geq 4$. Setting $k - l = n - 2$ in (6.26) gives us $Y_{1,n} = vY_{1,n-2}C$, while setting $l - k = n - 2$ in (6.26) gives us $Y_{2,n} = vY_{2,n-3}C$. Therefore, by induction on $n$, we have proved $Y_{i,n} = 0$ for $i = 1, 2$.

Proof of Proposition 6.11. For the sake of notational simplicity, we shall work with $i = 1$ in this proof (the case for $i = 2$ is entirely similar).
We proceed by an induction on \( n \). The statement for \( n = 1, 2 \) by Lemma 6.4 and Lemma 6.8. Suppose that the statement holds for \([\Theta_{1,m}, B_1]\), where \( 1 \leq m \leq n - 1 \). By Lemma 6.14 (see (6.22) for notation \( Y_{i,n}^k \), we have

\[
[\Theta_{1,n}, B_1] = -v^3[\Theta_{1,n-2}, B_1]v^{-C} + v^2[\Theta_{1,n-1}, B_1]v^{-4} + v[\Theta_{1,n-1}, B_{1,-1}]v^{-2}C
\]

\[
= -v^3\Theta_{1,n-2}B_1C + v^3\Theta_{1,n-2}B_1C + v^2\Theta_{1,n-1}B_{1,1} - v^{-2}B_{1,1}\Theta_{1,n-1}
\]

\[
+ v\Theta_{1,n-1}B_{1,-1}C - v^{-1}B_{1,-1}\Theta_{1,n-1}C
\]

\[
= -v^3\Theta_{1,n-2}B_1C + v^2\Theta_{1,n-1}B_{1,1} + v\Theta_{1,n-1}B_{1,-1}C + \sum_{k=-n}^{n} B_{1,k}Z_{k,n} \tag{6.27}
\]

for some \( Z_{k,n} \in \mathcal{H}_{1,n} \). Furthermore, by the inductive assumption, we have

\[
\Theta_{1,n-2}B_1C = B_1\Theta_{1,n-2}C + \sum_{k=-n+2}^{n-2} B_{1,k}X'_{k,n-2};
\]

\[
\Theta_{1,n-1}B_1 = B_1\Theta_{1,n-1} + \sum_{k=-n+1}^{n-1} B_{1,k}X''_{k,n-1}, \tag{6.28}
\]

for some \( X'_{k,n-2} \in \mathcal{H}_{1,n-2} \) and \( X''_{k,n-1} \in \mathcal{H}_{1,n-1} \). By applying \( T_{\omega}^{\pm 1} \) to (6.28), we have

\[
\Theta_{1,n-1}B_{1,1} = B_{1,1}\Theta_{1,n-1} + \sum_{k=-n+1}^{n-1} B_{1,k+1}\tilde{X}'_{k,n-1};
\]

\[
\Theta_{1,n-1}B_{1,-1}C = B_{1,-1}\Theta_{1,n-1}C + \sum_{k=-n+1}^{n-1} B_{1,k-1}\tilde{X}''_{k,n-1};
\]

for some \( \tilde{X}'_{k,n-1}, \tilde{X}''_{k,n-1} \in \mathcal{H}_{1,n-1} \), since \( T_{\omega}(\Theta_{1,n-1}) = \Theta_{1,n-1} \) and \( T_{\omega}(\mathcal{H}_{1,n-1}) \subseteq \mathcal{H}_{1,n-1} \) by Corollary 6.10. Thus the desired formula (6.13) follows from (6.27).

\[\square\]

6.7. Relation (5.2). With all the technical preparation in the prior subsections, we are ready to prove the crucial commutativity among the imaginary root vectors \( \Theta_{i,m} \).

**Lemma 6.15.** Let \( n \geq 1 \), and assume that \([\Theta_{j,1}, \Theta_{i,k}] = 0 \) for all \( i, j \in \{1, 2\} \), and all \( k < n \). Then for all \( 1 \leq m_1, m_2 \leq n \) and \( i, j \in \{1, 2\} \), we have

1. \([\Theta_{i,m_1}, [\Theta_{j,m_2}, B_1]] = [\Theta_{j,m_2}, [\Theta_{i,m_1}, B_1]]\);
2. \([\Theta_{i,m_1}, \Theta_{j,m_2}] = 0\).

**Proof.** Without loss of generality, we assume \( m_1 \geq m_2 \). Let “<” denote the lexicographic ordering on \( \mathbb{N}^2 \). We use induction on \((m_1, m_2)\) in lexicographic ordering. Assume that

\[
[\Theta_{i,k}, \Theta_{j,l}] = 0 \tag{6.29}
\]

for all \( i, j \) and all \((k, l) < (m_1, m_2)\). By Corollary 6.10, \( \Theta_{i,k}, \Theta_{j,l} \) are fixed by \( T_{\omega} \). By Proposition 6.11, there exist \( X_{k,m}^{(i)} \in \mathcal{H}_{i,m} \) such that

\[
[\Theta_{i,m}, B_{1,r}] = \sum_{k=-m}^{m} B_{1,r+k}X_{k,m}^{(i)}, \tag{6.30}
\]
for $1 \leq m \leq n, r \in \mathbb{Z}$. Hence,

$$[\Theta_{i,m_1}, [\Theta_{j,m_2}, B_1]] = \sum_{l=-m_2}^{m_2} [\Theta_{i,m_1}, B_{1,l}] X_{l,m_2}^{(j)}$$

$$= \sum_{l=-m_2}^{m_2} [\Theta_{i,m_1}] X_{l,m_2}^{(j)}$$

$$= \sum_{l=-m_2}^{m_2} \sum_{k=-m_1}^{m_1} B_{1,k+l} X_{k,m_1}^{(i)} X_{l,m_2}^{(j)}$$

$$= \sum_{k=-m_1}^{m_1} \sum_{l=-m_2}^{m_2} B_{1,k+l} X_{k,m_1}^{(j)} X_{l,m_2}^{(i)} = [\Theta_{j,m_2}, [\Theta_{i,m_1}, B_1]],$$

where the second equality follows from $[\Theta_{i,m_1}, X_{l,m_2}] = 0$ by the inductive assumption (6.29), and the commutativity of $X_{k,m_1}, X_{l,m_2}$ used in the fourth equality also follows from (6.29) and $X_{k,m_1} \in H_{1,m_1}, X_{l,m_2} \in H_{1,m_2}$. This proves (1).

It follows by the Jacobi identity that $[[\Theta_{i,m_1}, \Theta_{j,m_2}], B_1] = 0$. By symmetry, we also have $[[\Theta_{i,m_1}, \Theta_{j,m_2}], B_2] = 0$. By Lemma 2.3 and Lemma 4.3, we have $[[\Theta_{i,m_1}, \Theta_{j,m_2}], K_a] = 0$, for all $a \in \mathbb{I}$. Therefore, Lemma 4.11 is applicable and implies $[\Theta_{i,m_1}, \Theta_{j,m_2}] = 0$, whence (2).

**Proposition 6.16.** Relation (5.2) holds in $\tilde{U}^i$.

**Proof.** Since $\Theta_{i,0} = \frac{1}{v_0 - v_i - r}$, we have $[\Theta_{i,1}, \Theta_{i,0}] = 0$, for all $i, j$. Then it follows from Lemma 6.15 that $[\Theta_{i,m_1}, \Theta_{j,m_2}] = 0$ by induction. So (5.2) holds in $\tilde{U}^i$ by Lemma 5.4(1). □

Now we can remove the assumption in Corollary 6.10.

**Theorem 6.17.** We have $T_\omega(\Theta_{i,n}) = \Theta_{i,n}$, for all $n \geq 1$ and $i \in \{1, 2\}$.

**Proof.** Follows by Corollary 6.10 and Proposition 6.16. □

6.8. Relation (5.3).

**Proposition 6.18.** Relation (5.3) holds in $\tilde{U}^i$.

**Proof.** We shall prove the identity (5.12), which is equivalent to (5.3) by Lemma 5.4(2).

By Proposition 6.16, the assumption of Lemma 6.14 holds for any $n \geq 1$, and hence

$$Y_{1,n}^k = 0, \quad Y_{2,n}^k = 0, \quad \forall n \geq 1, \, k \in \mathbb{Z}.$$ 

That is, the identity (5.12) holds for $j = 1$; it holds also for $j = 2$ by the symmetry $\hat{\tau}$ in (2.17). □

6.9. Relation (5.5). The proof of Relation (5.5) is rather straightforward as we now have Theorem 6.17 at our disposal.

**Proposition 6.19.** Relation (5.5) holds in $\tilde{U}^i$. 
Proof. In the proof below, we refer to the relation (5.5) as $R(k, l)$.

Relation $R(k, k)$ follows by applying a suitable power of $\Theta_{i,m}$ to the identity (6.8), thanks to the $\Theta_{i,m}$-invariance of $\Theta_{i,m}$ by Theorem 6.17.

Relation $R(k, -l)$ for $k \geq 0$ follows by definition of $\Theta_{i,m}$ in (4.6)-(4.8). Then $R(k, l)$ for $k > l$ follows by applying $\Theta_{i,m}$ to the known case $R(k-l-1, -1)$ and noting that $\Theta_{i,m}$ is $\Theta_{i,m}$-invariant by Theorem 6.17.

For $k < l$, we apply the involution $\hat{\tau}$ in (2.17) and reduce to the case $R(l, k)$ already established above (the second equation below follows by a simple rewriting of the commutators):

\[
\hat{\tau}(\text{LHS}(5.5)) = [B_{\tau i,k}, B_{i,l+1}]_v - v[B_{\tau i,k+1}, B_{i,l}]_v^{-1}
= [B_{i,t}, B_{\tau i,k+1}]_v - v[B_{i,l+1}, B_{\tau i,k}]_v^{-1}
= -\Theta_{\tau i,k-l+1}C^dK_i + v\Theta_{\tau i,k-l-1}C^{d+1}K_i - \Theta_{i,l-k+1}C^{k+1}K_{\tau i} + v\Theta_{i,l-k-1}C^{k+1}K_{\tau i}.
\]

Another application of the involution $\hat{\tau}$ to the RHS above turns it into RHS (5.5). This completes the proof of the proposition.

6.10. Relation (5.6). Finally we shall prove the Drinfeld type Serre relations.

Proposition 6.20. Relation (5.6) holds in $\tilde{U}^\iota$.

Proof. Since (5.6) is equivalent to the relation (5.27) in generating function form, it suffices to prove (5.27).

We can rewrite (6.16) as

\[
(w_1^{-1} + w_2^{-1} - [2]z^{-1})S(w_1, w_2 | z; i) = -v^{-1}[2]\text{Sym}_{w_1, w_2} \Delta(w_2 z)  
\times \left((z^{-1} - vw_2^{-1})[\Theta_{\tau i}(z)K_i, B_i(w_1)]_v + (w_2^{-1} - vz^{-1})[\Theta_i(w_2)K_{\tau i}, B_i(w_1)]_v \right).
\]

Similarly, we rewrite (6.17) as

\[
(w_1 + w_2 - [2]z)S(w_1, w_2 | z; i) = -v^{-1}[2]\text{Sym}_{w_1, w_2} \Delta(w_2 z)  
\times \left((w_2 - vz)[B_i(w_1), \Theta_i(z)K_i]_v + (z - vz)[B_i(w_1), \Theta_i(w_2)K_{\tau i}]_v \right).
\]

Multiplying both sides of the identity (6.31) by $w_1 w_2$ and then subtracting by (6.32), we obtain

\[
(w_1 w_2 z^{-1})S(w_1, w_2 | z; i)  
= -v^{-1}\text{Sym}_{w_1, w_2} \Delta(w_2 z)(w_1 w_2 z^{-1} - vw_1)\left[\Theta_{\tau i}(z)K_i, B_i(w_1)\right]_v  
- v^{-1}\text{Sym}_{w_1, w_2} \Delta(w_2 z)(w_1 - vz)(w_2 - vz)\left[\Theta_i(w_2)K_{\tau i}, B_i(w_1)\right]_v  
+ v^{-1}\text{Sym}_{w_1, w_2} \Delta(w_2 z)(w_2 - vz)\left[\Theta_i(z)K_i, B_i(w_1)\right]_v + (z - vz)\left[\Theta_i(w_2)K_{\tau i}, B_i(w_1)\right]_v.
\]

Note that the constant component of (6.33) is exactly the relations (2.5)-(2.6). Then, dividing both sides of (6.33) by $z - w_1 w_2 z^{-1}$, we obtain the desired relation (5.27). 

Therefore, we have verified that all the relations (5.1)-(5.6) are preserved under $\Phi : Dr\tilde{U}^\iota \to \tilde{U}^\iota$; that is, $\Phi : Dr\tilde{U}^\iota \to \tilde{U}^\iota$ defined in (5.13) is a homomorphism.
Appendix A. Proofs of Identity (3.8) and Lemma 6.2

A.1. Proof of Identity (3.8). In this subsection, we prove the identity (3.8), and hence complete the proof of the identity (3.1). When combining with discussions in §3.1–§3.2, we have completed the proof of Theorem 2.7.

The identity \( \tilde{\mathcal{F}} = \tilde{T}_1^{-1}(F_0) \) in (3.8) is not homogeneous in \( \tilde{\mathcal{U}} \), and it consists of homogeneous summands of 4 different weights: \(- (\alpha_0 + 2\alpha_1 + 2\alpha_2), -(\alpha_0 + \alpha_1 + \alpha_2), -\alpha_0, \) and \(-\alpha_0 + \alpha_1 + \alpha_2, \) respectively. Accordingly, the proof of the identity (3.8) is reduced to the proofs of the 4 identities (A.1)–(A.4) below:

\[
v \left[ [F_1, F_2]_v, [F_2, [F_1, F_0]_v] \right] = \tilde{T}_1^{-1}(F_0), \tag{A.1}
\]

\[
\left[ [F_1, E_1 K_2]_v, [F_2, [F_1, F_0]_v] \right] + v \left[ K_1 K_2 - K_1 K_2', \left[ F_2, [F_1, F_0]_v \right] \right] = [[F_1, F_2]_v, [E_1 K_2, [F_1, F_0]_v]] \tag{A.2}
\]

\[
\left[ [E_2 K_1, E_1 K_2]_v, [F_2, [F_1, F_0]_v] \right] + v \left[ K_1 K_2 - K_1 K_2', \left[ E_1 K_2, [F_1, F_0]_v \right] \right] = [[F_1, E_2 K_1]_v, [E_1 K_2, [F_1, F_0]_v]] \tag{A.3}
\]

\[
\left[ [E_2 K_1, E_1 K_2]_v, [E_1 K_2, [F_1, F_0]_v] \right] = [[E_2 K_1, E_2 K_1]_v, [F_0] K_2 K_1'] \tag{A.4}
\]

The identity (A.1) holds by Lemma 3.3. The proofs for (A.2)–(A.4) require only elementary computations and are given in the following 3 subsections.

A.1.1. Proof of (A.2). Denote \( X := [F_1, F_0]_v \).

Lemma A.1. We have

1. \( X F_1 = vF_1 X \);
2. \( [E_1, X] = F_0 K_1' \);
3. \( X E_1 K_2 = v^{-2}E_1 K_2 X - F_0 K_1' K_2 \).

Note the third term on LHS (A.2) is 0, i.e., \( \left[ \frac{K_1 K_2 - K_1 K_2'}{v - v^{-1}} \right], [F_2, [F_1, F_0]_v] \right] = 0 \). We now compute using Lemma A.1 as follows:

\[
\left[ [F_1, E_1 K_2]_v, [F_2, X] \right] + \left[ [F_1, F_2]_v, [E_1 K_2, X] \right] = \left[ [F_1, E_1 K_2]_v, F_2 X - [F_1, E_1 K_2]_v X F_2 - F_2 X [F_1, E_1 K_2]_v + X F_2 [F_1, E_1 K_2]_v \right] + [F_1, F_2]_v E_1 K_2 X - E_1 K_2 X [F_1, F_2]_v + X E_1 K_2 [F_1, F_2]_v \\
= \left( \left[ [F_1, E_1 K_2]_v, F_2 - v^{-1} F_2 [F_1, E_1 K_2]_v + [F_1, F_2]_v E_1 K_2 - v^{-2} [F_1, F_2]_v E_1 K_2 \right] X \right) + v F_2 F_0 K_1' K_2 - v F_2 F_0 K_1' K_2 + [F_1, F_2]_v F_0 K_1' K_2 \\
+ X \left( F_2 [F_1, E_1 K_2]_v - v [F_1, E_1 K_2]_v F_2 + E_1 K_2 [F_1, F_2]_v - v^2 E_1 K_2 [F_1, F_2]_v \right) - v^{-1} F_1 F_0 K_1' K_2 + v F_0 F_1 K_1' K_2 - v^2 F_0 [F_1, F_2]_v K_1' K_2
\]
\[ -F_2 K_1' K_2 X + F_1 F_2 F_0 K_1' K_2 - v^4 F_2 F_0 K_1' K_2 \]
\[ + v^{-1} X F_2 K_1' K_2 + v^3 F_0 F_2 F_1 K_1' K_2 - v^{-1} F_1 F_2 K_1' K_2 \]
\[ = [(F_1, F_2)_v^3, F_0] K_1' K_2. \]

This proves the identity (A.2).

A.1.2. **Proof of (A.3).** Denote again \( X := [F_1, F_0]_v \). Denote \( \tilde{E}_1 := E_1 K_2, \tilde{E}_2 := E_2 K_1 \).

**Lemma A.2.** We have

1. \( X \tilde{E}_2 = v \tilde{E}_2 X; \)
2. \( [[\tilde{E}_2, \tilde{E}_1], F_2]_{v^{-1}} = -v \tilde{E}_1 K_1 K_2; \)
3. \( [[\tilde{E}_2, \tilde{E}_1], X]_v = v^2 [\tilde{E}_2, F_0]_{v^3} K_1' K_2; \)
4. \( [F_1, \tilde{E}_1]_{v^{-1}} = \frac{K_1 - K_1'}{v - v^{-1}} K_2; \)
5. \( [[F_1, \tilde{E}_1], X]_v = v^2 [F_1, F_0]_{v^3} K_1' K_2. \)

**Proof.** Parts (1)-(2) and (4) follow by a direct computation. Parts (3) and (5) follow from (1), Lemma A.1, and the following two \( v \)-Jacobi identities, respectively:

\[ [[\tilde{E}_2, \tilde{E}_1]_v, X]_v = [\tilde{E}_2, [[\tilde{E}_1], X]_v] - v [[\tilde{E}_1], [\tilde{E}_2, X]_v]_v, \]
\[ [[F_1, \tilde{E}_1]_v, X]_v = [F_1, [[\tilde{E}_1], X]_v] - v [[\tilde{E}_1], [F_1, X]_v]_v. \]

The lemma is proved. \( \square \)

**Proposition A.3.** We have

\[ [[\tilde{E}_2, \tilde{E}_1]_v, [F_2, X]] + v \left[ \frac{K_1 K_2 - K_1' K_2'}{v - v^{-1}}, [\tilde{E}_1, X] \right] \]
\[ = [[\tilde{E}_2, F_2]_{v^3}, F_0] K_1' K_2 - v^2 F_0 K_1 K_2 K_1' K_2'. \quad \text{(A.5)} \]

**Proof.** By Lemma A.2(2)-(3), the first term on LHS (A.5) is equal to

\[ [[\tilde{E}_2, \tilde{E}_1]_v, [F_2, X]] = \left[ [[\tilde{E}_2, \tilde{E}_1]_v, F_2]_{v^{-1}}, [\tilde{E}_2, \tilde{E}_1, X]_v \right] + v^{-1} [F_2, [[\tilde{E}_2, \tilde{E}_1], X]_v]_v \]
\[ = -v^2 [\tilde{E}_1, X] K_1 K_2 + v [F_2, [\tilde{E}_2, F_0]_{v^3}] K_1' K_2. \]

Hence, LHS (A.5) is simplified as

\[ [[\tilde{E}_2, \tilde{E}_1]_v, [F_2, X]] + v \left[ \frac{K_1 K_2 - K_1' K_2'}{v - v^{-1}}, [\tilde{E}_1, X] \right] \]
\[ = -v^2 [\tilde{E}_1, X] K_1 K_2 + v [F_2, [\tilde{E}_2, F_0]_{v^3}] K_1' K_2 + v^2 [\tilde{E}_1, X] K_1 K_2 \]
\[ = v [F_2, [\tilde{E}_2, F_0]_{v^3}] K_1' K_2 \]
\[ = [[\tilde{E}_2, F_2]_{v^3}, F_0] K_1' K_2 - v^2 F_0 K_1 K_2 K_1' K_2', \]

as desired. Here the last equality follows by

\[ v [F_2, [\tilde{E}_2, F_0]_{v^3}] = [(\tilde{E}_2, F_2)_{v^3}, F_0] \]
\[ = (v - v^3) F_2 \tilde{E}_2 F_0 + (v^2 - 1) F_0 \tilde{E}_2 F_2 - \tilde{E}_2 F_2 F_0 + v^3 F_2 \tilde{E}_2 F_0 + F_0 \tilde{E}_2 F_2 - v^3 F_0 F_2 \tilde{E}_2 \]
\[= v[F_2, E_2]K_1F_0 + v^3F_0[E_2, F_2]K_1\]
\[= -v \frac{K_2 - K'_2}{v - v^{-1}} K_1F_0 + v_3F_0 \frac{K_2 - K'_2}{v - v^{-1}}\]
\[= -v^2F_0K_1K'_2.\]

Hence, Proposition A.3 is proved. \(\square\)

**Proposition A.4.** We have

\[[[F_1, E_1], [E_1, X]] = [[F_1, E_1]v^3, F_0]K'_1K_2.\]

**Proof.** By Lemma A.2(4)-(5), we have

\[[[F_1, E_1]v, [E_1, X]] = [[F_1, E_1]v, [E_1, X]]_{v^{-1}} + v^{-1}\]
\[=[[F_1, E_1]v, [F_1, E_1]v, X]_{v^{-1}} + v^{-1}\]
\[=[[K'_1 - K_2, \tilde{E}_1]v, X]_{v^{-1}} + v[F_1, F_0]v^3K'_1K_2\]
\[=(v + v^{-1})[[F_1, F_0]v, \tilde{E}_1]v_2K'_1K_2 + v[F_1, F_0]v^3v^{-2}K'_1K_2\]
\[\equiv -v(v + v^{-1})[[F_0, [F_1, \tilde{E}_1]v]v^{-2}K'_1K_2 - v^{-2}[[F_1, \tilde{E}_1]v, F_0]v^3K'_1K_2\]
\[=[[F_1, E_1]v, F_0]v^3K'_1K_2.\]

where for the equation (*) above we have used the following identities

\[[[F_1, F_0]v, \tilde{E}_1]v^2 = -v[[F_0, [F_1, \tilde{E}_1]v^3]v^{-2}, [[\tilde{E}_1, [F_1, F_0]v^3]v^{-2} = [[\tilde{E}_1, F_1]v^{-3}, F_0]v^4.\]

The proposition is proved. \(\square\)

The identity (A.3) follows by Proposition A.3 and Proposition A.4.

**A.1.3. Proof of (A.4).** Denote \(\tilde{E}_1 := E_1K_2, \tilde{E}_2 := E_2K_1\). Again \(X := [F_1, F_0]v\).

**Lemma A.5.** We have

1. \([\tilde{E}_1, F_1]v = v^{K_1 - K'_1}K_2;\]
2. \([\tilde{E}_1, F_0]v = [\tilde{E}_2, F_0]v = 0;\]
3. \([\tilde{E}_2, F_1]v^{-2} = 0;\]
4. \([\tilde{E}_2, \tilde{E}_1]v^{-1} = 0;\]
5. \([\tilde{E}_2, X]v^{-1} = 0;\]
6. \([\tilde{E}_1, X]v^2 = v^2F_0K'_1K_2;\]
7. \([\tilde{E}_2, \tilde{E}_1]v, X]v = v^2[\tilde{E}_2, F_0]v^3K'_1K_2].\]

**Proof.** Parts (1)-(3) are clear, (4) is a Serre relation, and (5) follows from (1)-(3). Part (6) follows from (1)-(3) and a \(v\)-Jacobi identity \([\tilde{E}_1, [F_1, F_0]v]v^2 = [[\tilde{E}_1, F_1]v, F_0]v^3 + v[F_1, [\tilde{E}_1, F_0]v].\]

Part (7) follows from (5)-(6) and \([[[\tilde{E}_2, \tilde{E}_1], X]v = [	ilde{E}_2, [\tilde{E}_1, X]v^2] - v[\tilde{E}_1, [\tilde{E}_2, X]v^{-1}]v.\] \(\square\)
We now compute LHS (A.4) by applying Lemma A.5:
\[
\left[ [\tilde{E}_2, \tilde{E}_1]_v, [\tilde{E}_1, X] \right] = \left[ \left[ [\tilde{E}_2, \tilde{E}_1]_v, \tilde{E}_1 \right]_{v-1}, X \right]_v + v^{-1} \left[ \tilde{E}_1, [\tilde{E}_2, \tilde{E}_1]_v, X \right]_v
\]
\[
= v \left[ \tilde{E}_1, \left[ \tilde{E}_2, F_0K'_1K_2 \right] \right]_v
\]
\[
= v \left[ \tilde{E}_1, \left[ \tilde{E}_2, F_0 \right]_{v^3} K'_1K_2 \right]
\]
\[
= v \left( \tilde{E}_1 \tilde{E}_2 F_0 - v^3 \tilde{E}_1 F_0 \tilde{E}_2 - v^{-2} \tilde{E}_2 F_0 \tilde{E}_1 + v F_0 \tilde{E}_2 \tilde{E}_1 \right) K'_1K_2
\]
\[
= v(v - v^{-1}) F_0 \left( - v^3 \tilde{E}_1 \tilde{E}_2 + \tilde{E}_2 \tilde{E}_1 \right) K'_1K_2
\]
\[
= (v^2 - 1) F_0 [\tilde{E}_2, \tilde{E}_1]_{v^3} K'_1K_2.
\]
On the other side, by Lemma A.5, we have \([\tilde{E}_2, \tilde{E}_1]_{v^3} F_0 = v^2 F_0 [\tilde{E}_2, \tilde{E}_1]_{v^3}\), and thus
\[
\left[ [\tilde{E}_2, \tilde{E}_1]_{v^3}, F_0 \right] = (v^2 - 1) F_0 [\tilde{E}_2, \tilde{E}_1]_{v^3}.
\]
Hence, (A.4) follows from the above computations.

A.2. Proof of Lemma 6.2. In this subsection, we shall prove Lemma 6.2.

A.2.1. A preparatory lemma. We denote \(B_{i_1i_2\cdots i_r} := B_{i_1}B_{i_2}\cdots B_{i_r}\). First, we prepare the following lemma.

Lemma A.6. We have
\[
\left[ \left[ [B_1, B_2]_v, [B_1, B_0]_v \right], [B_1, B_2]_v \right]_v
\]
\[
= v^3 B_{0112} \mathbb{K}_1 - (v^4 - 2) B_{0112} \mathbb{K}_2 + v^2 B_{0121} \mathbb{K}_1 + v^5 B_{0121} \mathbb{K}_2
\]
\[
+ v B_{0211} \mathbb{K}_1 + v^6 B_{0211} \mathbb{K}_2 - (2v^2 + v^4) B_{1012} \mathbb{K}_1 - (v + 2v^{-1}) B_{1012} \mathbb{K}_2
\]
\[
- (2v + v^{-1}) B_{1021} \mathbb{K}_1 - (2v^4 + v^2) B_{1021} \mathbb{K}_2 + v[3] B_{1102} \mathbb{K}_1 + (v^2 + 2) B_{1102} \mathbb{K}_2
\]
\[
- B_{1120} \mathbb{K}_1 - v B_{1120} \mathbb{K}_2 + (v^2 + 2) B_{1201} \mathbb{K}_1 + (v^5 + 2v^3) B_{1201} \mathbb{K}_2
\]
\[
- v^{-1} B_{1210} \mathbb{K}_1 - v^2 B_{1210} \mathbb{K}_2 - v[2] B_{2011} \mathbb{K}_1 - v^6[2] B_{2011} \mathbb{K}_2
\]
\[
+ v^{-1} B_{2101} \mathbb{K}_1 + v^2 B_{2101} \mathbb{K}_2 - (v^7 - v^5) B_{2110} \mathbb{K}_2.
\]
Denote
\[
X_1 = B_{01112}, X_2 = B_{01221}, X_3 = B_{02211}, X_4 = B_{102211}, X_5 = B_{110221}, X_6 = B_{111022},
\]
\[
X_7 = B_{11120}, X_8 = B_{112012}, X_9 = B_{112021}, X_{10} = B_{120211}, X_{11} = B_{122110},
\]
\[
X_{12} = B_{201112}, X_{13} = B_{202111}, X_{14} = B_{210211}, X_{15} = B_{211102}, X_{16} = B_{221110}.
\]
Then we have
\[
B_{101122} = \frac{[2]}{[3]} X_1 + \frac{1}{[3]} X_6,
\]
\[
B_{110122} = \frac{1}{[3]} X_1 + \frac{[2]}{[3]} X_6,
\]
\[
B_{110212} = \frac{1}{[3]} X_1 + \frac{1}{[2]} X_5 + \frac{1}{[3]} X_6 + v B_{1102} \mathbb{K}_1 + v^{-2} B_{1102} \mathbb{K}_2,
\]
\begin{align*}
B_{021211} &= \frac{1}{[2]} X_2 + \frac{1}{[2]} X_3 + v^4 B_{02111} \mathbb{K}_2 + v^{-5} B_{02111} \mathbb{K}_1, \\
B_{220111} &= [2] X_3 - X_1, \\
B_{221011} &= \frac{[2]^2}{[3]} X_13 - \frac{[2]}{[3]} X_3 + \frac{1}{[3]} X_16, \\
B_{211210} &= X_{11}, \\
B_{211120} &= \frac{[3]}{[2]} X_{11} - \frac{1}{[2]} X_{16} - v^{-3}[3] B_{21101} \mathbb{K}_1 - v^2[3] B_{21101} \mathbb{K}_1, \\
B_{212110} &= \frac{1}{[2]} X_{11} + \frac{1}{[2]} X_{16} + v^{-5} B_{21101} \mathbb{K}_1 + v^4 B_{21101} \mathbb{K}_2, \\
B_{021112} &= \frac{3}{[2]} X_2 - \frac{1}{[2]} X_3 - v^{-3}[3] B_{02111} \mathbb{K}_1 - v^2[3] B_{02111} \mathbb{K}_2, \\
B_{021121} &= X_2, \\
B_{011212} &= \frac{2}{[3]} X_1 + \frac{1}{[2]} X_2 - \frac{1}{[3]} X_3 + [2] B_{01121} \mathbb{K}_1 + v^{-1}[2] B_{01121} \mathbb{K}_2 - v^{-3}[2] B_{01121} \mathbb{K}_1 - v^2[2] B_{01121} \mathbb{K}_2, \\
B_{021121} &= B_{011221} = \frac{1}{[3]} X_1 + X_2 - \frac{1}{[3]} X_3 - v^{-3}[2] B_{02111} \mathbb{K}_1 - v^2[2] B_{02111} \mathbb{K}_2 \\
&\quad + v^{-1}[2] B_{01121} \mathbb{K}_1 + [2] B_{01121} \mathbb{K}_2, \\
B_{012121} &= \frac{2}{[3]} X_1 + \frac{2}{[2]} X_2 - \frac{1}{[3]} X_3 + v^{-1} B_{01121} \mathbb{K}_1 + B_{01121} \mathbb{K}_2 + v^{-2} B_{01121} \mathbb{K}_1 + v B_{01121} \mathbb{K}_2 \\
&\quad - v^{-3} B_{02111} \mathbb{K}_1 - v^2 B_{02111} \mathbb{K}_2, \\
B_{101212} &= \frac{[2]^2}{[3][2]} X_1 + \frac{1}{[2]^2} X_2 - \frac{1}{[3][2]} X_3 + \frac{1}{[2]} X_5 + \frac{1}{[3]} X_6 + B_{01121} \mathbb{K}_1 + v B_{01121} \mathbb{K}_2 \\
&\quad + \frac{v}{[2]} B_{1102} \mathbb{K}_1 + \frac{v^{-2}}{[2]} B_{1102} \mathbb{K}_2 - \frac{v^{-3}}{[2]} B_{02111} \mathbb{K}_1 - \frac{v^2}{[2]} B_{02111} \mathbb{K}_2, \\
B_{102121} &= B_{102112} = \frac{[3]}{[2]} X_1 + \frac{1}{[2]} X_2 - \frac{1}{[3]} X_3 + \frac{1}{[2]} X_5 + B_{01121} \mathbb{K}_1 + v^{-1}[2] B_{01121} \mathbb{K}_2 \\
&\quad - v^{-3} B_{02111} \mathbb{K}_1 - v^2 B_{02111} \mathbb{K}_2 - v[2] B_{10121} \mathbb{K}_1 - v^{-2}[2] B_{10121} \mathbb{K}_2 \\
&\quad + v B_{1102} \mathbb{K}_1 + v^{-2} B_{1102} \mathbb{K}_2, \\
B_{102121} &= \frac{1}{[3][2]} X_1 + \frac{1}{[2]^2} X_2 - \frac{1}{[3][2]} X_3 + \frac{1}{[2]} X_4 + \frac{1}{[2]^2} X_5 + B_{01121} \mathbb{K}_1 + v^{-1} B_{01121} \mathbb{K}_2 \\
&\quad - v^{-3} B_{02111} \mathbb{K}_1 - v^2 B_{02111} \mathbb{K}_2 - v B_{10121} \mathbb{K}_1 - v^{-2} B_{10121} \mathbb{K}_2 + v^2 B_{10211} \mathbb{K}_1 \\
&\quad + v B_{1102} \mathbb{K}_1 + \frac{v}{[2]} B_{1102} \mathbb{K}_1 + \frac{v^{-2}}{[2]} B_{1102} \mathbb{K}_2, \\
B_{121102} &= \frac{2}{[3]} X_1 + \frac{1}{[3]} X_7 + v^{-1}[2] B_{1102} \mathbb{K}_1 + [2] B_{1102} \mathbb{K}_2, 
\end{align*}
\[ B_{112102} = \frac{1}{[3]} X_{15} + \frac{2}{[3]} X_7 + \frac{[2]}{[2]} B_{1102} \mathbb{K}_1 + v^{-1}[2] B_{1102} \mathbb{K}_2, \]
\[ B_{111220} = \frac{[2]}{[2]} X_7 - X_6, \]
\[ B_{112120} = \frac{[2]}{[3]} X_7 - \frac{[2]}{[3]} X_6 + \frac{1}{[2]} X_{11} - \frac{1}{[3]}! X_{16} + \frac{[2]}{[2]} B_{1120} \mathbb{K}_1 + v^{-1}[2] B_{1120} \mathbb{K}_2 \]
\[ \quad - v^{-3} B_{2110} \mathbb{K}_1 - v^2 B_{2110} \mathbb{K}_2, \]
\[ B_{121120} = B_{112210} = \frac{2}{[3]} X_7 - \frac{1}{[3]} X_6 + \frac{1}{[3]} X_{11} - \frac{1}{[3]} X_{16} - \frac{[2]}{[2]} B_{2110} \mathbb{K}_1 - v^{-3}[2] B_{2110} \mathbb{K}_1 - v^2[2] B_{2110} \mathbb{K}_2 \]
\[ \quad + v^{-1}[2] B_{1120} \mathbb{K}_1 + [2] B_{1120} \mathbb{K}_2, \]
\[ B_{121210} = \frac{1}{[3]} X_7 - \frac{1}{[3]}! X_6 + \frac{2}{[2]} X_{11} - \frac{1}{[3]}! X_{16} + v^{-1} B_{1120} \mathbb{K}_1 + B_{1120} \mathbb{K}_2 + v^{-2} B_{1210} \mathbb{K}_1 \]
\[ \quad + v B_{1210} \mathbb{K}_2 - v^{-3} B_{2110} \mathbb{K}_1 - v^2 B_{2110} \mathbb{K}_2, \]
\[ B_{122011} = \frac{2}{[2]} X_{10} - X_4, \]
\[ B_{122101} = B_{211201} = X_{10} - \frac{2}{[2]} X_4 + \frac{1}{[2]} X_{11}, \]
\[ B_{212011} = \frac{2}{[3]} X_{13} - \frac{1}{[3]} X_3 + \frac{1}{[3]} X_{16} + X_{10} - \frac{1}{[2]} X_4 + v^{-5} B_{2011} \mathbb{K}_1 + v^4 B_{2011} \mathbb{K}_2, \]
\[ B_{212101} = \frac{1}{[3]} X_{13} - \frac{1}{[3]}! X_3 + \frac{1}{[3]} X_{16} + \frac{2}{[2]} X_{10} - \frac{1}{[2]}! X_4 + \frac{1}{[2]}! X_{11} + v^{-5} B_{2101} \mathbb{K}_1 + v^4 B_{2101} \mathbb{K}_2, \]
\[ B_{201121} = \frac{1}{[3]} X_{13} + \frac{2}{[3]} X_{12} + v^{-3}[2] B_{2011} \mathbb{K}_1 + v^2[2] B_{2011} \mathbb{K}_2, \]
\[ B_{202111} = \frac{2}{[3]} X_{13} + \frac{1}{[3]} X_{12} + v^{-4}[2] B_{2011} \mathbb{K}_1 + v^3[2] B_{2011} \mathbb{K}_2, \]
\[ B_{210112} = \frac{2}{[3]} X_{12} + \frac{1}{[3]} X_{15}, \]
\[ B_{211012} = \frac{1}{[3]} X_{12} + \frac{2}{[3]} X_{15}, \]
\[ B_{210121} = \frac{1}{[3]} X_{12} + \frac{1}{[3]}! X_{15} + \frac{1}{[2]} X_{14} + v^{-2} B_{2101} \mathbb{K}_1 + v B_{2101} \mathbb{K}_2, \]
\[ B_{211021} = \frac{1}{[3]} X_{15} + X_{14} - \frac{1}{[3]}! X_{13} - v^{-3}[2] B_{2011} \mathbb{K}_1 - v^2[2] B_{2011} \mathbb{K}_2 \]
\[ \quad + v^{-2}[2] B_{2101} \mathbb{K}_1 + v[2] B_{2101} \mathbb{K}_2, \]
\[ B_{112201} = \frac{1}{[2]} X_9 - X_5, \]
\[ B_{121021} = \frac{1}{[2]} X_9 + \frac{1}{[3]}! X_{15} + \frac{1}{[2]} X_{14} - \frac{1}{[3]}! X_{13} + v^{-2} B_{1021} \mathbb{K}_1 + v B_{1021} \mathbb{K}_2 \]
\[ \quad - v^{-3} B_{2011} \mathbb{K}_1 - v^2 B_{2011} \mathbb{K}_2 + v^{-2} B_{2101} \mathbb{K}_1 + v B_{2101} \mathbb{K}_2, \]
\[ B_{121201} = X_9 - \frac{1}{[2]} X_5 + \frac{1}{[2]} X_{10} - \frac{1}{[2]}! X_4 + \frac{1}{[2]}! X_{11} + v^{-2} B_{1201} \mathbb{K}_1 + v B_{1201} \mathbb{K}_2, \]
\[ B_{121012} = \frac{1}{2} X_8 + \frac{1}{[3]} X_{12} + \frac{1}{[3]} X_{15} + v^{-2} B_{1012 K_1} + v B_{1012 K_2} \]

\[ B_{120112} = X_8 + \frac{1}{[3]} X_{12} - \frac{1}{[3]} X_7 + v^{-2} [2] B_{1012 K_1} + v [2] B_{1012 K_2} - v^{-1} [2] B_{1102 K_1} - [2] B_{1102 K_2}. \]

Lemma A.6 follows now by a routine computation using the above formulas.

A.2.2. Completing the proof of Lemma 6.2. By definition, we have

\[ o(1) B_{1,1} K_1 = T^{-1}_1 T^{-1}_0 (B_1) K_1 = -v [T^{-1}_1 (B_0), B_1 K_1^{-1}] v K_1 \]

\[ = -v \left[ B_2, \left[ [B_1, B_2], v, [B_1, B_0], v \right] \right], B_1 \right] + v^2 \left[ B_2, \left[ B_1, B_0 \right], v \right], B_1 \right] \]

\[ + v \left[ B_1, \left[ B_2, B_0, v \right], v, K_2, B_1 \right] + [2] \left[ B_2, \left[ B_1, B_0 \right], v, K_1, B_1 \right] \]

\[ - v^3 \left[ B_0, B_1 \right], v, K_2 K_1 \]

\[ = v^2 \left[ B_1, \left[ B_2, \left[ B_1, B_2 \right], v, B_1, B_0, v \right] \right], B_1 \right], v^{-1} \]

\[ = v^3 \left[ B_2, \left[ B_1, \left[ B_1, B_2 \right], v, B_1, B_0, v \right], v \right] \]

\[ - v \left[ v \left[ \left[ B_1, B_2 \right], v, B_1, B_0, v \right], B_1, B_2 \right] \]

\[ = v^2 \left[ B_2, \left[ B_1, B_0, v \right], B_1 \right], v^{-3} K_1 \]

\[ - v^2 \left[ B_2, \left[ B_1, B_0, v \right], B_1 \right], v^{-3} K_2 \]

\[ = v^2 \left[ B_1, \left[ B_2, \left[ B_1, B_0, v \right], B_1 \right], v^{-3} K_1 \right] + v^2 \left[ B_2, \left[ B_1, B_0, v \right], B_1 \right], v^{-3} K_2 \]

\[ - v^2 \left[ B_1, \left[ B_2, \left[ B_1, B_0, v \right], B_1 \right], v^{-3} K_1 \right]. \]

So it suffices to check that

\[ - v \left[ v \left[ \left[ B_1, B_2 \right], v, B_1, B_0, v \right], B_1, B_2 \right] \]

\[ = -v^2 \left[ B_2, \left[ B_1, B_0, v \right], B_1 \right], v^{-3} \left[ B_2, \left[ B_1, B_0, v \right], v, B_1 \right], v^{-3} K_1 \]

\[ - v^3 \left[ B_2, \left[ B_1, B_0, v \right], B_1 \right], v^{-3} \left[ B_2, \left[ B_1, B_0, v \right], v, B_1 \right], v^{-3} K_2 \]

\[ - v^2 \left[ B_1, \left[ B_2, \left[ B_1, B_0, v \right], B_1 \right], v^{-3} K_1 \right] - v^2 \left[ B_1, \left[ B_2, \left[ B_1, B_0, v \right], B_1 \right], v^{-3} K_2 \right] \]

\[ - v \left[ v \left[ \left[ B_1, B_2 \right], v, B_1, B_0, v \right], B_1, B_2 \right] \].
By using Lemma A.6, this last identity follows by comparing the coefficients with the help of

\[ B_{0121} = \frac{1}{2} B_{0112} + \frac{1}{2} B_{0211} + v^{-2} B_{01} K_1 + v B_{01} K_2, \]

\[ B_{1201} = \frac{1}{2} B_{1120} + \frac{1}{2} B_{2110} + v^{-2} B_{10} K_1 + v B_{10} K_2, \]

\[ B_{1012} = \frac{1}{2} B_{0112} + \frac{1}{2} B_{1102}, \quad B_{2101} = \frac{1}{2} B_{2110} + \frac{1}{2} B_{2011}. \]

This completes the proof of Lemma 6.2.

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