DIAMETER OF REDUCED SPHERICAL CONVEX BODIES

MAREK LASSAK AND MICHAŁ MUSIELAK

Abstract. The intersection $L$ of two different non-opposite hemispheres of the unit sphere $S^2$ is called a lune. By $\Delta(L)$ we denote the distance of the centers of the semicircles bounding $L$. By the thickness $\Delta(C)$ of a convex body $C \subset S^2$ we mean the minimal value of $\Delta(L)$ over all lunes $L \supset C$. We call a convex body $R \subset S^2$ reduced provided $\Delta(Z) < \Delta(R)$ for every convex body $Z$ being a proper subset of $R$. Our aim is to estimate the diameter of $R$, where $\Delta(R) < \frac{\pi}{2}$, in terms of its thickness.

1. Introduction

Let $S^2$ be the unit sphere of the 3-dimensional Euclidean space $E^3$. A great circle of $S^2$ is the intersection of $S^2$ with any two-dimensional linear subspace of $E^3$. By a pair of antipodes of $S^2$ we mean any pair of points being the intersection of $S^2$ with a one-dimensional subspace of $E^3$. Observe that if two different points $a, b \in S^2$ are not antipodes, there is exactly one great circle passing through them. By the arc $ab$ connecting them we understand the shorter part of the great circle through $a$ and $b$. By the distance $|ab|$ of $a$ and $b$ we mean the length of the arc $ab$. The notion of the diameter of any set $A \subset S^2$ not containing antipodes is taken with respect to this distance and denoted by $\text{diam}(A)$.

A subset of $S^2$ is called convex if it does not contain any pair of antipodes of $S^2$ and if together with every two points it contains the arc connecting them. A closed convex set $C \subset S^2$ with non-empty interior is called a convex body. Its boundary is denoted by $\text{bd}(C)$. If in $\text{bd}(C)$ there is no arc, we say that the body is strictly convex. Convexity on $S^2$ is considered in very many papers and monographs. For instance in [1], [2], [3], [4], [5] and [14].

2010 Mathematics Subject Classification. 52A55.

Key words and phrases. spherical convex body, spherical geometry, hemisphere, lune, width, constant width, thickness, diameter.
The set of points of $S^2$ in the distance at most $\rho$, where $0 < \rho \leq \frac{\pi}{2}$, from a point $c \in S^2$ is called a spherical disk, or shorter a disk, of radius $\rho$ and center $c$. Disks of radius $\frac{\pi}{2}$ are called hemispheres. Two hemispheres whose centers are antipodes are called opposite hemispheres. The set of points of a great circle of $S^2$ which are at distance at most $\frac{\pi}{2}$ from a fixed point $p$ of this great circle is called a semicircle. We call $p$ the center of this semicircle.

Let $C \subset S^2$ be a convex body and $p \in \text{bd}(C)$. We say that a hemisphere $K$ supports $C$ at $p$ provided $C \subset K$ and $p$ is in the great circle bounding $K$.

Since the intersection of every family of convex sets is also convex, for every set $A \subset S^2$ contained in an open hemisphere of $S^2$ there is the smallest convex set $\text{conv}(A)$ containing $A$. We call it the convex hull of $A$.

If non-opposite hemispheres $G$ and $H$ are different, then $L = G \cap H$ is called a lune. The semicircles bounding $L$ and contained in $G$ and $H$, respectively, are denoted by $G/H$ and $H/G$. The thickness $\Delta(L)$ of $L$ is defined as the distance of the centers of $G/H$ and $H/G$.

After [8] recall a few notions. For any hemisphere $K$ supporting a convex body $C \subset S^2$ we find a hemisphere $K^*$ supporting $C$ such that the lune $K \cap K^*$ is of the minimum thickness (by compactness arguments at least one such a hemisphere $K^*$ exists). The thickness of the lune $K \cap K^*$ is called the width of $C$ determined by $K$ and it is denoted by $\text{width}_K(C)$. By the thickness $\Delta(C)$ of a convex body $C \subset S^2$ we understand the minimum of $\text{width}_K(C)$ over all supporting hemispheres $K$ of $C$. We say that a convex body $R \subset S^2$ is reduced if for every convex body $Z \subset R$ different from $R$ we have $\Delta(Z) < \Delta(R)$. This definition is analogous to the definition of a reduced body in normed spaces (for a survey of results on reduced bodies see [10]). If for all hemispheres $K$ supporting $C$ the numbers $\text{width}_K(C)$ are equal, we say that $C$ is of constant width. Spherical bodies of constant width are discussed in [12] and applied in [9].

Just bodies of constant width, and in particular disks, are simple examples of reduced bodies on $S^2$. Also each of the four parts of a disk dissected by two orthogonal great circles through the center of this disk is a reduced body. It is called a quarter of a disk. There is a wide class of reduced odd-gons on $S^2$ (see [21]). In particular, the regular odd-gons of thickness at most $\frac{\pi}{2}$ are reduced.
2. Two lemmas

Lemma 2.1. Let $L \subset S^2$ be a lune of thickness at most $\frac{\pi}{2}$ whose bounding semicircles are $Q$ and $Q'$. For every $u, v, z \in Q$ such that $v \in uz$ and for every $q \in L$ we have $|qv| \leq \max\{|qu|, |qz|\}$.

Proof. If $\Delta(L) = \frac{\pi}{2}$ and $q$ is the center of $Q'$, then the distance between $q$ and any point of $Q$ is the same, and thus the assertion is obvious. Consider the opposite case when $\Delta(L) < \frac{\pi}{2}$, or $\Delta(L) = \frac{\pi}{2}$ but $q$ is not the center of $Q'$. Clearly, the closest point $p \in Q$ to $q$ is unique. Observe that for $x \in Q$ the distance $|qx|$ increases as the distance $|px|$ increases. This easily implies the assertion of our lemma. □

In a standard way, an extreme point of a convex body $C \subset S^2$ is defined as a point for which the set $C \setminus \{e\}$ is convex (see [8]). The set of extreme points of $C$ is denoted by $E(C)$.

Lemma 2.2. For every convex body $C \subset S^2$ of diameter at most $\frac{\pi}{2}$ we have \(\text{diam}(E(C)) = \text{diam}(C)\).

Proof. Clearly, $\text{diam}(E(C)) \leq \text{diam}(C)$.

In order to show the opposite inequality $\text{diam}(C) \leq \text{diam}(E(C))$, thanks to $\text{diam}(C) = \text{diam}(\text{bd}(C))$, it is sufficient to show that $|cd| \leq \text{diam}(E(C))$ for every $c, d \in \text{bd}(C)$. If $c, d \in E(C)$, this is trivial. In the opposite case, at least one of these points does not belong to $E(C)$. If, say $d \notin E(C)$, then having in mind that $C$ is a convex body and $d \in \text{bd}(C)$ we see that there are $e, f \in E(C)$ different from $d$ such that $d \in ef$. From $E(C) \subset \text{bd}(C)$, we see that $e, f \in \text{bd}(C)$. Since also $d \in \text{bd}(C)$, the arc $ef$ is a subset of $\text{bd}(C)$.

Recall that by Theorem 3 of [8] we have $\text{width}_K(C) \leq \text{diam}(C)$ for every hemisphere $K$ supporting $C$. In particular, for the hemisphere $K$ supporting $C$ at all points of the arc $ef$. Thus by the assumption that $\text{diam}(C) \leq \frac{\pi}{2}$ we obtain $\text{width}_K(C) \leq \frac{\pi}{2}$ for our particular $K$. Hence we may apply Lemma 2.1 taking this $K/K^*$ in the part of $Q$ there. We obtain $|cd| \leq \max\{|ce|, |cf|\}$.

If $c \notin E(C)$, from $c, e \in E(C)$ we conclude that $|cd| \leq \text{diam}(E(C))$. If $c \notin E(C)$, from $c \in \text{bd}(C)$ we see that there are $g, h \in E(C)$ such that $c \in gh$. Similarly to the preceding paragraph we show that $|ce| \leq \max\{|cg|, |ch|\}$.
and \(|fc| \leq \max\{|fg|, |fh|\}\). By these two inequalities and by the preceding paragraph we get \(|cd| \leq \max\{|eg|, |eh|, |fg|, |fh|\} \leq \text{diam}(E(C))\), which ends the proof.

The assumption that \(\text{diam}(C) \leq \frac{\pi}{2}\) is substantial in Lemma 2.2, as it follows from the example of a regular triangle of any diameter greater than \(\frac{\pi}{2}\). The weaker assumption that \(\Delta(C) \leq \frac{\pi}{2}\) is not sufficient, which we see taking in the part of \(C\) any isosceles triangle \(T\) with \(\Delta(T) \leq \frac{\pi}{2}\) and the arms longer than \(\frac{\pi}{2}\) (so with the base shorter than \(\frac{\pi}{2}\)). The diameter of \(T\) equals to the distance between the midpoint of the base and the opposite vertex of \(T\). Hence \(\text{diam}(T)\) is over the length of each of the sides.

3. Diameter of reduced spherical bodies

The following theorem is analogous to the first part of Theorem 9 from [7] and confirms the conjecture from [9], p. 214. By the way, the much weaker estimate \(2 \arctan\left(\sqrt{2} \tan \frac{\Delta(R)}{2}\right)\) results from Theorem 2 in [13].

**Theorem 3.1.** For every reduced spherical body \(R \subset S^2\) with \(\Delta(R) < \frac{\pi}{2}\) we have \(\text{diam}(R) \leq \arccos(\cos^2 \Delta(R))\). This value is attained if and only if \(R\) is the quarter of disk of radius \(\Delta(R)\). If \(\Delta(R) \geq \frac{\pi}{2}\), then \(\text{diam}(R) = \Delta(R)\).

**Proof.** Assume that \(\Delta(R) < \frac{\pi}{2}\). In order to show the first statement, by Lemma 2.2 it is sufficient to show that the distance between any two different points \(e_1, e_2\) of \(E(R)\) is at most \(\arccos(\cos^2 \Delta(R))\). Since \(R\) is reduced, according to the statement of Theorem 4 in [8] there exist lunes \(L_j \supset R\), where \(j \in \{1, 2\}\), of thickness \(\Delta(R)\) with \(e_j\) as the center of one of the two semicircles bounding \(L_j\) (see Figure). Denote by \(b_j\) the center of the other semicircle bounding \(L_j\).

If \(e_1 = b_2\) or \(e_2 = b_1\), then \(|e_1e_2| = \Delta(R)\), which by \(\Delta(R) \in (0, \frac{\pi}{2})\) is at most \(\arccos(\cos^2 \Delta(R))\). Otherwise \(L_1 \cap L_2\) is a non-degenerate spherical quadrangle with points \(e_1, b_2, b_1, e_2\) in its consecutive sides. Therefore, since \(e_1 \neq e_2\), arcs \(e_1b_1\) and \(e_2b_2\) intersect at exactly one point. Denote it by \(g\). Observe that it may happen \(b_1 = b_2 = g\).

Let \(F\) be the great circle orthogonal to the great circle containing \(e_1b_1\) and passing through \(e_2\). Since \(e_2 \in L_1\), we see that \(F\) intersects \(e_1b_1\). Let \(f\)
be the intersection point of them. Note that we do not exclude the case $f = b_1$. From $|e_2b_2| = \Delta(R)$ we see that $|ge_2| \leq \Delta(R) < \frac{\pi}{2}$. Thus from the right spherical triangle $gf e_2$ we conclude that $|fe_2| \leq \Delta(R)$. Moreover, from $|e_1b_1| = \Delta(R)$ and $f \in e_1b_1$ we obtain $|fe_1| \leq \Delta(R)$. Consequently, from the formula $\cos k = \cos l_1 \cos l_2$ for the right spherical triangle with hypotenuse $k$ and legs $l_1, l_2$ applied to the triangle $e_1fe_2$ (again see Figure) we obtain $|e_1e_2| \leq \arccos(\cos^2 \Delta(R))$.

Observe that thanks to $\Delta(R) < \frac{\pi}{2}$, the shown inequality becomes the equality only if $g = f = b_1 = b_2$. In this case, by Proposition 3.2 of [12], our body $R$ is a quarter of disk of radius $\Delta(R)$.

Finally, we show the last statement of the theorem. Assume that $\Delta(R) \geq \frac{\pi}{2}$. By Theorem 4.3 of [12], the body $R$ is of constant width $\Delta(R)$.

There is an arc $pq \subset R$ of length equal to $\text{diam}(R)$. Clearly, $p \in \text{bd}(R)$. Take a lune $L$ from Theorem 5.2 of [12] such that $p$ is the center of a semicircle bounding $L$. Denote by $s$ the center of the other semicircle $S$ bounding $L$. By the third part of Lemma 3 in [8], we have $|px| < |ps|$ for every $x \in S$ different from $s$. Hence $|px| \leq |ps|$ for every $x \in S$. So also $|pz| \leq |ps|$ for every $z \in L$. In particular, $|pq| \leq |ps|$. Since $\text{diam}(R) = |pq|$ and $\Delta(R) = \Delta(L) = |ps|$, we obtain $\text{diam}(R) \leq \Delta(R)$.

Let us show the opposite inequality.

If $\text{diam}(R) \leq \frac{\pi}{2}$, by Theorem 3 of [8] we get $\Delta(R) = \text{diam}(R)$. If $\text{diam}(R) > \frac{\pi}{2}$, then by Proposition 1 of [8] we get $\Delta(R) \leq \text{diam}(W)$. Consequently, always we have $\Delta(R) \leq \text{diam}(R)$.
From the inequalities obtained in the two preceding paragraphs we get the equality $\text{diam}(R) = \Delta(R)$. Hence the last sentence of the theorem is proved. □

Proposition 3.5 of [12] implies that for every reduced spherical body $R$ with $\Delta(R) \leq \frac{\pi}{2}$ on $S^2$ we have $\text{diam}(R) \leq \frac{\pi}{2}$. Here is a more precise form of this statement.

**Proposition 3.2.** Let $R \subset S^2$ be a reduced body. Then $\text{diam}(R) < \frac{\pi}{2}$ if and only if $\Delta(R) < \frac{\pi}{2}$. Moreover, $\text{diam}(R) = \frac{\pi}{2}$ if and only if $\Delta(R) = \frac{\pi}{2}$.

**Proof.** We start with proving the first statement of our proposition. The function $f(x) = \arccos(\cos^2 x)$ is increasing in the interval $[0, \frac{\pi}{2}]$ as a composition of the decreasing functions $\arccos x$ and $\cos^2 x$. From $f(\frac{\pi}{2}) = \frac{\pi}{2}$ we conclude that in the interval $[0, \frac{\pi}{2})$ the function $f(x)$ accepts only the values below $\frac{\pi}{2}$. Thus by Theorem 3.1, if $\Delta(R) < \frac{\pi}{2}$, then $\text{diam}(R) < \frac{\pi}{2}$. The opposite implication results from the inequality $\Delta(C) \leq \text{diam}(C)$ for every spherical convex body $C$, which in turn follows from Theorem 3 and Proposition 1 of [8].

Let us show the second part of our proposition.

Assume that $\text{diam}(R) = \frac{\pi}{2}$. The inequality $\Delta(R) < \frac{\pi}{2}$ is impossible, by the first statement of our proposition. Also the inequality $\Delta(R) > \frac{\pi}{2}$ is impossible, because by $\Delta(R) \leq \text{diam}(R)$ (see Proposition 1 of [8]), it would imply $\text{diam}(R) > \frac{\pi}{2}$ in contradiction to the assumption at the beginning of this paragraph. So $\Delta(R) = \frac{\pi}{2}$.

Now assume that $\Delta(R) = \frac{\pi}{2}$. By the second statement of Theorem 3.1 we get $\text{diam}(R) = \frac{\pi}{2}$. □

**References**

[1] BESAU F., SCHUSTER F., Binary operations in spherical convex geometry, *Indiana Univ. Math. J.* 65 (2016), no. 4, 1263–1288.

[2] DANZER L., GRÜNBAUM B., KLEE V., Helly’s theorem and its relatives, *Proc. of Symp. in Pure Math.* vol. VII, Convexity, 1963, pp. 99–180.

[3] FERREIRA O. P., IUSEM A. N., NÉMETH S. Z., Projections onto convex sets on the sphere, *J. Global Optim.* 57 (2013), 663–676.

[4] GAO F., HUG D., SCHNEIDER R., Intrinsic volumes and polar sets in spherical space, *Math. Notae* 41 (2003), 159-176.

[5] HADWIGER H., Kleine Studie zur kombinatorischen Geometrie der Sphäre, *Nagoya Math. J.* 8 (1955), 45–48.
[6] HAN H., NISHIMURA T., Self-dual Wulff shapes and spherical convex bodies of constant width $\pi/2$, *J. Math. Soc. Japan* 69 (2017), 1475–1484.

[7] LASSAK M., Reduced convex bodies in the plane, *Israel J. Math.* 70 (1990), 365–379.

[8] LASSAK M., Width of spherical convex bodies, *Aequationes Math.* 89 (2015), 555–567.

[9] LASSAK M., Reduced spherical polygons, *Colloq. Math.* 138 (2015), 205–216.

[10] LASSAK M., MARTINI H., Reduced convex bodies in finite-dimensional normed spaces – a survey, *Results Math.* 66 (2014), No. 3-4, 405–426.

[11] LASSAK M., MUSIEŁAK M., Reduced spherical convex bodies, *Bull. Pol. Ac. Math.* 66 (2018), 87–97.

[12] LASSAK M., MUSIEŁAK M., Spherical bodies of constant width, *Aequationes Math.* 92 (2018), 627–640.

[13] MUSIEŁAK M., Covering a reduced spherical body by a disk, *Ukr. Math. J.*, to appear (see also arXiv:1806.04246).

[14] VAN BRUMMELEN G., *Heavenly mathematics. The forgotten art of spherical trigonometry*. Princeton University Press (Princeton, 2013).

Marek Lassak  
**Instytut Matematyki i Fizyki**  
University of Science and Technology  
85-796 Bydgoszcz, Poland  
*E-mail address*: marek.lassak@utp.edu.pl

Michał Musielak  
**Instytut Matematyki i Fizyki**  
University of Science and Technology  
85-796 Bydgoszcz, Poland  
*E-mail address*: michal.musielak@utp.edu.pl