Disordered Systems, Spanning Trees and SLE
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Abstract

We define a minimization problem for paths on planar graphs that, on the honeycomb lattice, is equivalent to the exploration path of the critical site percolation and than has the same scaling limit of SLE$_6$. We numerically study this model (testing several SLE properties on other lattices and with different boundary conditions) and state it in terms of spanning trees. This statement of the problem allows the definition of a random growth process for trees on two dimensional graphs such that SLE is recovered as a special choice of boundary conditions.

Keywords: Domain Walls, SLE, Combinatorial Optimization, Matching Problem, Spin Glasses, Spanning Trees

1 Introduction

Recently, some efforts have been done to relate minimal paths in two dimensional disordered systems and SLE processes

Schramm-Loewner Evolution (SLE) is a stochastic process that describes the growth of random curves in simply connected two dimensional domains; for a review see [1] and references therein.

2 The model

We are given a planar two dimensional lattice $G$ and a set of real weights $\omega(f)$ on its faces (plaquettes).

Any edge $e \equiv (i, j)$ linking two vertices $i$ and $j$, is adjacent to two plaquettes: $f_{i,j}$ and $f_{j,i}$ (unless $e$ is an edge on the border of $G$). For a given set of $\{\omega(f)\}$, and a fixed threshold $\theta$, we associate to each edge a weight

$$W_{i,j} = [\omega(P_{i,j,1}) - \theta] \cdot [\omega(P_{i,j,2}) - \theta]$$

Let $(G, W)$ denote the graph with so defined weights on the edges.

Given a path $\gamma$ of length $N(\gamma) = |\gamma|$ with endpoints $i_0$ and $i_N$ we associate to this path the ordered list (in decreasing order): $\tilde{W}(\gamma) = \text{sort}(\{W_e\}_{e \in \gamma}) = (W_1(p), \ldots, W_N(\gamma))$
We define the order relation "<" among paths as follows: $\gamma_1 < \gamma_2$ if

- Exists $k$ such that $W_j(\gamma_1) = W_j(\gamma_2) \forall j < k$, $W_k(\gamma_1) < W_k(\gamma_2)$;
- $N(\gamma_1) < N(\gamma_2)$ and $W_j(\gamma_1) = W_j(\gamma_2) \forall j \leq N(\gamma_1)$

with this definition, either $\gamma_1 \equiv \gamma_2$, or $\gamma_1 \not< \gamma_2$, i.e. we have a full order provided $W_e \neq W_{e'}$ for $e \neq e'$.

Notice that for each $\gamma_1 < \gamma_2$ exists a $\beta$ such that $\forall \beta' \geq \beta$, $\sum_{e \in \gamma_1} \exp(\beta' W_e) < \sum_{e \in \gamma_2} \exp(\beta' W_e)$, so that the function $f_\beta(\gamma) := \sum_{e \in \gamma} \exp(\beta W_e)$, in the large $\beta$ limit, is an additive cost function ($f_\beta(\gamma_1 \cup \gamma_2) = f_\beta(\gamma_1) + f_\beta(\gamma_2)$) which reproduce our order relation.

With abuse of language the word cost will be used also for the first entry $W_1(\gamma)$ of the vector $\vec W(\gamma)$.

### 2.1 Some Remarks

First of all notice that the optimal path connecting two vertices is always a simple path. Call $p_{i,j}$ the optimal path connecting $i$ to $j$. Simple reasonings show that $\forall i, j, k, l$ (also coincident), $p_{i,j} \cup p_{k,l}$ cannot contain any cycle (as happens for every additive cost function without negative cost loops). Furthermore, for our cost function, the union $T := \cup_{i,j \in V} p_{i,j}$ of all the optimal paths is a tree. This tree, spanning for definition, is also the one which minimizes the global cost function $H(T) = \sum e \in T W_e$ on the ensemble of the spanning tree; it is the Minimum Spanning Tree of $(G, W)$, as one sees analysing the Prim’s Algorithm (cfr App. A).

All of this holds for a generic $(G, W)$, but crucially relies on our choice of order relation (and thus of optimality for $\gamma_{i,j}$).

Summarizing: given a planar lattice with arbitrary weights on the plaquettes we introduced some weights on the edges of the graph as to obtain a graph with weighted edges $(G, W)$, a cost function associated to each path and an order relation associated to the set of all the paths. We stress the fact that the union of all the optimal paths is the MST for $(G, W)$. Now we can specialize to a set of planar graphs (we will consider only rectangular domains) and to a probability measure for the weights on the plaquettes (we will only consider i.i.d. weights on the plaquettes).

Consider now a simply connected two dimensional domain (e.g. a square) covered with a honeycomb lattice. Extract the weights $\omega(G)$ from the distribution $\chi_{[0,1]}$ and let the threshold be $\theta = 0.5$. Fix two different edges on the boundary: $s$ (start) and $t$ (end). Constrain all the boundary plaquettes on the right of $s$ and $t$ to have a cost larger than the threshold and all the other boundary plaquettes to have a cost smaller than the threshold; then the path starting in $s$ and ending on $t$ has cost less than 0 and is exactly the boundary wall of the percolation process on the lattice with weights $\omega(G)$ and threshold 0.5. Then the measure of the optimal paths $p_{s,t}$ is the same as that of critical site percolation exploration path on the honeycomb lattice, that is (see [7]), in the thermodynamic limit (infinitesimal lattice spacing), SLE measure with $\kappa = 6$. 

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2 The Minimum Spanning Tree of an arbitrary weighted graph is the loopless cover of the graph $G$ that covers every vertex and minimizes the sum of the weights on the edges.
2.2 Motivations

In the study of domain walls in disordered systems it happens that the union of the domain walls is a tree; this is the case for example in Ising Spin Glasses for domain walls constrained to start from a fixed point, it is the case also for the boundary given by the symmetric difference of opportune matchings on planar graph \[4\].

Such ubiquity suggests to search for a disordered model not only such as to reproduce the SLE distribution of probability, but also such that the union of the optimal paths was a tree and, both for his mathematical properties and for numerical reasons, we wanted this tree to be easy to find.

One of the properties of MST that make it easy to find (in computational sense) is a locality property (see \[A\]) similar to the locality property of SLE for \(\kappa = 6\).

3 Results

3.1 The Samples

We simulated numerically rectangular samples of sizes ranging from \(32 \times 32\) to \(1024 \times 1024\).

For clarity, suppose that the four vertices of the rectangle are \((0, 0), (0, a), (b, 0)\) and \((a, b)\). Let \(s = (a/2, 0)\) be the point at one half of the bottom edge of the rectangle and \(e = (a/2, b)\) be the point at one half of the top edge. We will consider four different boundary conditions and a slightly different model:

- **Free**: the weights on the boundary plaquettes are extracted as the bulk ones.
- **SLE-like**: the weights on the boundary plaquettes on the left of \(s\) and \(t\) are constrained to have a weight higher than \(\theta\) and the other boundary plaquettes are constrained to have a weight lower than \(\theta\).
- **SLE/Free**: The boundary plaquettes on the right edge of the rectangle are constrained to be higher than \(\theta\) and the ones on the left edge are constrained to have a weight lower than \(\theta\). The weights on the top edge and on the bottom edge are unconstrained (free).
- **Repulsive**: All the boundary plaquettes are constrained to have a weight higher than the threshold \(\theta\).
- **Random**: the weights on the edges are i.i.d. variables. Remark that it is not a peculiar choice of the boundary conditions for our model. It is another model: the random MST model. We study this random measure on the weights of the edges just for comparison with this well known model.

In order to study systems at criticality we mainly concentrate on a \(\theta\) equal to the percolation threshold for site percolation (0.5 on the honeycomb lattice, 0.5927463 on the square one). On the square models some boundary conditions (\(SLE\) – \(like\), \(SLE\) – \(free\)) break the left-right symmetry because the critical threshold is different from 0.5, so we simulated our model on the square lattice also with \(\theta = 0.5\), we expected a trivial limit for these paths, the fact that we
did not observe it means that the scaling limit was not reached by the numerical simulations.

3.2 Observables

We measured the fractal-dimension of the paths. We know that the fractal-dimension of SLE-Walks is linked to the parameter $\kappa$ by the relation $d = 1 + \kappa/8$.

We measured left-passage probability, which because of dilation invariance, has to be a function only of the radial coordinate on the half plane. Schramm’s formula (see [8]) links the shape of the left-passage probability to the parameter $\kappa$. As we observed (see [4]) it can happen, for disordered systems, that the parameter $\kappa$ found by left-passage probability and the fractal dimension are not compatible, indicating that SLE and minimizing paths are not equal in measure.

As it is usual in literature ([2] [3]) we consider both the path starting in $s$ and ending in $t$ and the optimal path among the ones starting on the bottom and ending on the top. Notice that, for SLE-like boundary conditions, the two paths coincide: the optimal path connecting the bottom to the top is also the one starting in $s$ and ending in $t$.

If we want to compare the left-passage probability of paths on rectangles with Schramm’s formula we have to transform the domain into the half plane. For the path starting in $s$ and ending in $t$ we choose the conformal transformation that maps $(a/2, 0)$ in $(0, 0)$, $(a/2, b)$ in $\infty$, $(0, 0)$ in $(-1, 0)$ and $(a, 0)$ in $(1, 0)$. For the path connecting the top to the bottom we consider the conformal transformation that maps the rectangle to the semi-annulus such that the vertices of the rectangle are sent on the vertices of the rectangles of the half annulus and $(0, b/2)$, $(a, b/2)$ are sent respectively in $(-1, 0)$ $(1, 0)$. This transformation sends the rectangle to the half plane only in the limit $b/a \to \infty$, the limit considered in [3] to study the horizontal displacement. For $b/a < \infty$ boundary effects are observed at top and bottom.

The horizontal displacement is the difference $\Delta(x)$ between the abscissae of starting and ending point for the optimal path connecting the top of the square to the bottom. We measured the average value of $\Delta x^2$ with $\Delta x$ expressed in unit of $a$, the horizontal length of the rectangle, so that $\Delta x \in [-1, 1]$ for every path.

3.2.1 Fractal Dimension

The fractal dimensions of the curves is measured by comparing the number of steps of the paths in lattices of different sizes. Having fixed the boundary conditions, the fractal dimension is independent of the path considered.

|                | Square | Honeycomb |
|----------------|--------|-----------|
| Free           | 1.21 ± 0.01 | 1.75 ± 0.01 |
| SLE − like     | 1.20 ± 0.01 | 1.75 ± 0.01 |
| SLE/Free       | 1.22 ± 0.01 | 1.75 ± 0.01 |
| Repulsive      | 1.22 ± 0.01 | 1.75 ± 0.01 |
| Random         | 1.21 ± 0.01 | 1.22 ± 0.01 |
3.2.2 Left-Passage Probability

Left passage probabilities (the probability for a point in the domain to be at the left or at the right of the path) have been measured in rectangular domains. For the path with ends in $s$ and $t$ we transformed the domain to the half plane to compare the measured probability (over $10^5$ samples) to the Schramm formula:

$$1/2 + \frac{\Gamma\left(\frac{4}{\kappa}\right)}{\sqrt{\pi}\Gamma\left(\frac{8-\kappa^2}{2\kappa}\right)} \tan t \cdot 2F_1\left[\frac{1}{2}, \frac{4}{\kappa}; \frac{3}{2}, -\tan^2(t)\right]$$

where $t$ is the angle subtended between the ray in $s$ and the real axis.

For the paths with free ends we compared measured probabilities with the formula via the identification of $x$ (the coordinate on the rectangle) with the angle $\theta$ on the half plane.

| Path $s \rightarrow t$ | Square | Honeycomb |
|------------------------|--------|-----------|
| Free                   | 2.8 ± 0.1 | 2.7 ± 0.1 |
| SLE-like               | XXX    | 6.0 ± 0.1 |
| SLE/Free               | XXX    | XXX       |
| Repulsive              | 2.8 ± 0.1 | 2.8 ± 0.1 |
| Random                 | 3.2 ± 0.1 | 3.2 ± 0.1 |

| Optimal path           | Square | Honeycomb |
|------------------------|--------|-----------|
| Free                   | 3.2 ± 0.1 | 2.9 ± 0.1 |
| SLE/Free               | XXX    | XXX       |
| Repulsive              | 5.9 ± 0.1 | 3.2 ± 0.1 |
| Random                 | 3.2 ± 0.1 | 3.2 ± 0.1 |

The entries marked with XXX correspond to measured left passage probabilities that do not fit with Schramm’s formula for any value of $\kappa$.

Notice that the critical model (on both the lattices) has compatible values of $\kappa$ for Free and Random boundary conditions, but they are very different for Repulsive boundary conditions.

3.2.3 Horizontal Displacement

We observe that when the height of the rectangle is bigger than the base, the position of the starting point and the position of the ending point are uncorrelated. As a consequence the average value of $\Delta x^2$ is constant for $b \gg a$ and converges to a value $\langle \Delta x^2 \rangle$. For $b \ll a$ we measure the exponent $l$ in $\Delta x(b/a) \sim (b/a)^l$.

|                     | $\langle \Delta x^2 \rangle$ | $l$  |
|---------------------|-------------------------------|-----|
| **Square**          |                               |     |
| Free                | 0.134 ± 0.05                  | 2.07 ± 0.03 |
| SLE/Free            | 0.126 ± 0.002                 | 2.10 ± 0.16 |
| Repulsive           | 0.24 ± 0.01                   | 2.13 ± 0.09 |
| Random              | 0.128 ± 0.005                 | 2.14 ± 0.26 |
| **Honeycomb**       |                               |     |
| Free                | 0.104 ± 0.001                 | 2.22 ± 0.14 |
| SLE/Free            | 0.190 ± 0.01                  | 2.05 ± 0.02 |
| Repulsive           | 0.190 ± 0.01                  | 2.05 ± 0.12 |
| Random              | 0.129 ± 0.005                 | 2.24 ± 0.19 |
3.2.4 Conformal Invariance of the Trees

We have investigated the conformal invariance of the Minimum Spanning Tree. As we know [5], for the Random Spanning Tree the conformal invariance does not hold. To test conformal invariance we measure the distribution of probability of the triple point $T$ on the square. The triple point is defined as the unique site in the tree connected to $(0,0)$, $(0,b)$, $(a,0)$ by three paths with null intersection. We transform conformally the rectangle into a disk so to map the points $(0,0)$, $(0,b)$, $(a,0)$ on the vertices of an equilateral triangle inscribed in the disk. If conformal invariance for the tree holds (as it happens for instance for the uniform spanning trees), the transformed distribution of probability should be invariant under rotations of $2\pi/3$ of the disk. This test has been done for all the models with boundary conditions that do not break conformal invariance (Free and Repulsive) and has shown that conformal invariance does not hold for the trees we defined.

4 Conclusions and Perspectives

Several surprising facts emerged from the numerical simulations. The fractal dimension of paths is irrespective of the boundary conditions, but it depends dramatically on the kind of lattice. The left-passage probability, also on the honeycomb lattice, is not compatible with the fractal dimension for every choice of the boundary conditions different from the standard one, anyway on the square lattice with Repulsive boundary conditions and at the critical percolation threshold, the left-passage probability obtained is consistent with $\kappa = 6$. These facts are not well understood and need more investigations also on different lattices.

Given two vertices $i$, $j$, we say that they are connected if the cost of the minimal path between $i$ and $j$ is less than 0. The behaviour of the connection probability could be studied both numerically and theoretically using CFT’s tools, as in [6] for critical percolation. The structure of the connected domains is better understood in the scheme of the Krushkal’s algorithm (see A). The SLE boundary conditions are peculiar because all the boundary sites but two are disconnected.

It is possible to define a process of growth of trees in the scheme of Prim’s algorithm, in fact one could start to grow the tree from a starting point on the boundary and progressively increasing it with Prim’s algorithm. This is the definition of a process of growth for Spanning Trees. It would be interesting to understand if, using the reparametrization of the time such that the rate of increase of the capacity be constant, the continuum limit of this evolution process makes sense. Notice that SLE$_6$ is recovered as the growth of the tree with opportune boundary conditions.

This method to define growth processes for trees such that, with opportune boundary conditions, SLE$_6$ is recovered could be easily generalized to other spin models. In fact, given a spin configuration extracted with the Gibbs measure and the usual boundary conditions to force a boundary wall to exist starting on $s$ and ending on $t$, we need only to associate a weight bigger than $\theta$ to sites with up spins and smaller than $\theta$ to sites with down spin. Then, the minimum spanning tree on the honeycomb lattice with weights induced by equation (1)
will contain by construction the boundary between up spins and down spins starting in \( s \) and ending in \( t \).

In this draft we studied only the optimal spanning tree; it could be interesting also to study the low temperature behaviour: the almost optimal trees.

One could investigate the stability of walks under perturbations of the instance. It is not a very hard task when working in the scheme of Krushkal’s algorithm, thanks to MST properties.

\section*{A Minimum Spanning Tree}

Let \( G = (V, E) \) be a connected graph with \( N \) vertices, a spanning tree \( T \) is a loopless subgraph with \( N - 1 \) edges. i.e. it is a tree (loopless and connected) and it is spanning (every vertex in \( V \) has at least one incident edge in \( T \)). Given a weighted graph \( (G, W) \) with real weights on the edges, a Minimum Spanning Tree is a Spanning Tree of minimum weight (where the weight of a tree \( T \) is the sum of the weights of the edges in \( T \)).

MST have an important property that we would like to stress: given a subset \( V' \) of the vertices \( V \) of \( G \), let \( B_{V'} \) be the edges on the boundary between \( V' \) and its complement \( \bar{V'} \) then the MST contains the edge \( e \) of minimum weight among the edges in \( B_{V'} \).

This property allows some local algorithms to work. By locality we mean that we do not need to know all the weights over the whole graph to find one edge in \( B_{V'} \) that will be also in \( T \). Notice also that if the MST on \( (G(V, E), W) \) restricted to a subset \( V' \) of \( V \) is connected then it coincides with the MST of \( (G(V', E_{V'}), W) \).

There are two basics strategies for the search of the MST, one consists on the progressive increase of a minimum tree until it is spanning, the other one consists on the progressive coalescence of the trees in a spanning forest (collection of trees) until a single tree is obtained.

In the first one (Prim’s algorithm) one starts with a given vertex \( i \in V' \) and progressively increases this set with the minimal edge in \( B_{V'} \) until every vertex is in \( V' \). In the other one (Krushkal’s algorithm) one starts with a forest consisting of all the vertices and no edges, then one starts to increment the set of edges in the MST with the edge of minimal weight, and goes on adding minimal edges unless a cycle would result.

In our numerical simulations we used the Kruskal algorithm which is polynomial \( (|E| \ln |E|) \) in the number \( |E| \) of edges of the graph.

We remark that it is easy to write polynomial algorithms to study the excited states of the MST (spanning trees with almost minimum cost), so to investigate low temperature properties of the model.

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