The large $N$ limit of $\mathcal{N}=2$ super Yang-Mills, fractional instantons and infrared divergences

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We investigate the large $N$ limit of pure $\mathcal{N}=2$ supersymmetric gauge theory with gauge group SU($N$) by using the exact low energy effective action. Typical one-complex dimensional sections of the moduli space parametrized by a global complex mass scale $v$ display three qualitatively different regions depending on the ratio between $|v|$ and the dynamically generated scale $\Lambda$. At large $|v|/\Lambda$, instantons are exponentially suppressed as $N \to \infty$. When $|v| \sim \Lambda$, singularities due to massless dyons occur. They are densely distributed in rings of calculable thicknesses in the $v$-plane. At small $|v|/\Lambda$, instantons disintegrate into fractional instantons of charge $1/(2N)$. These fractional instantons give non-trivial contributions to all orders of $1/N$, unlike a planar diagrams expansion which generates a series in $1/N^2$, implying the presence of open strings. We have explicitly calculated the fractional instantons series in two representative examples, including the $1/N$ and $1/N^2$ corrections. Our most interesting finding is that the $1/N$ expansion breaks down at singularities on the moduli space due to severe infrared divergencies, a fact that has remarkable consequences.

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1 Introduction

Quantum SU(N) gauge theory has a single free parameter, the number of colours $N$, or equivalently the number of interacting gluons $N^2 - 1$. Our best chance to understand the strongly coupled dynamics of gluons is probably to perform an analysis of the theory at large $N$ \[1\]. In the present work we will use exact results available for theories with eight supercharges ($\mathcal{N} = 2$ supersymmetry) \[2, 3\] in order to try to better understand the large $N$ limit of four dimensional gauge theories. Some basic qualitative features were first discovered by the author while working on simple two dimensional toy models, which can be supersymmetric \[4\], but also non supersymmetric \[5, 6\]. In four dimensions, we can unfortunately provide explicit calculations for $\mathcal{N} = 2$ theories only.

The paper is organized as follows. In Section 2, we give a general qualitative discussion of some of our main results, without entering into technical details. In Sections 3 and 4, we discuss in turn the structure of the moduli space at large $N$, the nature and properties of the large $N$ expansion of the low energy observables, and two full large $N$ calculations of representative Seiberg-Witten period integrals. In Section 5 we discuss some open problems.

2 General discussion

*Large $N$ in SU(N) gauge theories and planar diagrams*

From the point of view of perturbation theory, the large $N$ expansion is a reordering of Feynman diagrams with respect to their topology, with diagrams of genus $h$ being proportional to $1/N^{2h}$. The original perturbative series has zero radius of convergence, but the large $N$ reordering can produce convergent series in the gauge coupling constant at each order in $1/N^2$, thus providing a non-perturbative treatment. The series in $1/N^2$ is reminiscent of the genus expansion of an oriented closed string theory \[7\] of coupling constant $\kappa \sim 1/N$, and one may hope that a full non-perturbative equivalence between string theories and gauge theories could exist. This is supported by the empirical facts that hadrons are found on a Regge trajectory, and that quarks seem to be confined by the collimation of Faraday flux lines in string-like structures. In a supersymmetric context, one can make this idea more precise by using the fact that D-branes, on which gauge theories live, can be viewed as solitons in closed string theories \[8\].

2
Large \( N \) and instantons

A general amplitude in gauge theory can pick up contributions of different types. In addition to the terms having a Feynman diagram representation that are discussed in the context of the ’t Hooft expansion, non-trivial field configurations invisible in Feynman diagrams can in principle contribute to the path integral. The most popular configurations of this type are instantons \[10\]. Instantons are responsible for important semi-classical effects like tunneling. In the case of \( \mathcal{N} = 2 \) supersymmetric gauge theories, on which we will focus in the present paper, a non-renormalization theorem \[11\] implies that the low energy effective action \( S_{\text{eff}} \) up to two derivatives terms picks up only a trivial one-loop term from Feynman diagrams, but has an infinite series of instanton contributions. From the point of view of the large \( N \) limit, however, instantons are exponentially suppressed, and thus do not seem to play an interesting rôle \[12\]. The instanton action is indeed proportional to \( N \) in the ’t Hooft scaling \( g_{\text{YM}}^2 \propto 1/N \). For \( \mathcal{N} = 2 \) super Yang-Mills, this seems to imply that \( S_{\text{eff}} \) at large \( N \) is simply given by the one-loop formula. One would have to look at higher derivative terms in order to obtain a non-trivial large \( N \) expansion.

More precisely, and as was emphasized in \[12\], the effects of instantons of size \( 1/v \) are proportional in the one-loop approximation to

\[
e^{-\frac{8\pi^2}{g_{\text{YM}}^2}} = \left( \frac{\Lambda}{v} \right)^{\beta_0},
\]

where \( g_{\text{YM}} \) is the gauge coupling constant at scale \( v \), \( \Lambda \) the dynamically generated scale of the theory, and \( \beta_0 \propto N \) is given by the one-loop \( \beta \) function. The one-loop formula \( \[11\] \) is exact for \( \mathcal{N} = 2 \) super Yang-Mills, with \( \beta_0 = 2N \). It suggests that the only smooth limit of instanton contributions when \( N \to \infty \) is zero \[12\]. Large instantons (small \( v \)), if relevant, would produce catastrophic exponentially large contributions, and if one is willing to assume that the large \( N \) limit makes sense the only physically sensible conclusion is that instantons are irrelevant variables. In real-world QCD, instantons of all sizes can potentially contribute, and this led Witten to argue that the instanton gas must vanish \[12\]. In Higgs theories, like \( \mathcal{N} = 2 \) super Yang-Mills, the Higgs vevs introduce a natural cutoff \( v \) on the size of instantons, and for \( v \) large enough (“weak coupling”) the instanton gas can exist but is just negligible at large \( N \). At small \( v \) (“strong coupling”), we run into the same difficulties as in QCD, and instantons must somehow disappear.

Large \( N \) and fractional instantons

An interesting puzzle is to try to understand the nature of the physics at strong coupling in the large \( N \) limit. Unlike in non-supersymmetric theories where one may
argue that the physics is dominated by planar diagrams, in $\mathcal{N} = 2$ gauge theories we have already pointed out that some important observables do not have any non-trivial contribution from Feynman diagrams. In their derivation of the low energy effective action $S_{\text{eff}}$, Seiberg and Witten [2] made use of the electric/magnetic duality transformations of the low energy abelian theory. These transformations are very useful to understand the physics near singularities where magnetically charged particles become massless, but they say nothing about the large $N$ limit at strong coupling. The purpose of the present paper is to elucidate this problem. We will see that instantons actually do not literally vanish, but disintegrate into fractional instantons of topological charge $1/(2N)$ (or a multiple of $1/(2N)$). Such fractional instantons give contributions of order

$$\left(\frac{\Lambda}{\nu}\right)^{\beta_0/2N} = \frac{\Lambda}{\nu},$$

and can obviously survive at large $N$. The fact that fractional instantons could play a rôle in gauge theories has been suspected for some time, independently of the large $N$ approximation. In the case of $\mathcal{N} = 1$ supersymmetric gauge theories, chiral symmetry is broken by a gluino condensate $\langle \lambda\lambda \rangle = \Lambda^3$. Since $\beta_0 = 3N$ in this context, we see that the gluino condensate can be interpreted as coming from a fractional instanton of charge $1/N$ (a recent discussion of the gluino condensate can be found in [13]; see also [14] for a discussion of the closely related problem of $\mathcal{N} = 1$ superpotentials in confining vacua). In the following we will see that fractional instantons do play a fundamental rôle in $\mathcal{N} = 2$ gauge theories as well, when the theory is analysed at large $N$.

Fractional instantons is a new type of contribution in the large $N$ limit. Though it is known that Feynman diagrams generate a series in $1/N^2$ and that instantons must be exponentially suppressed, it is not obvious what is the analogous statement for fractional instantons. We will explicitly demonstrate below that in $\mathcal{N} = 2$ gauge theories, fractional instantons generate a series in $1/N$. The leading contribution can be of the same order of magnitude as the one coming from planar graphs. The first correction is then of order $1/N$, and would dominate any subleading $1/N^2$ corrections from Feynman diagrams. This proves directly from the field theory point of view that the string theory dual must contain open strings in addition to the familiar closed strings. This fact is actually consistent with our present knowledge on string duals to $\mathcal{N} = 2$ gauge theories. In the supergravity approximation, the closed string background dual to such theories usually has unphysical naked singularities called “repulsions” [15]. String theory does actually make sense on such backgrounds thanks to the “enhançon” mechanism [13]: the constituent branes expand and form a finite
shell excising the singularity. The open strings responsible for the fractional instantons contributions must be open strings attached to these branes. Such open strings, and the fact that they would produce unusual $1/N$ corrections on the field theory side, do not seem to have been discussed in the literature. It would be extremely interesting if they could be used to match with non-trivial results of the type we are going to derive on the gauge theory side.

*Large $N$ and singularities*

Possibly the most interesting result derived in this paper concerns the behaviour of the large $N$ expansion at singularities on the moduli space. These results were anticipated in our study of two dimensional models [4, 5, 6].

On the moduli space of vacua, the gauge group $SU(N)$ is broken down to $U(1)^{N-1}$, and generically the low energy theory is a simple pure $\mathcal{N} = 2$ abelian gauge theory. For some particular, “critical,” values of the moduli, however, some additional hypermultiplets can become massless [2]. In the pure $SU(N)$ case on which we focus, such critical points can only be observed at strong coupling (the W bosons have then a mass of order $\Lambda$) and correspond to massless magnetically charged particles called dyons. Since those couple non-locally to the original photons, it is a priori not obvious what the low energy theory can be, but we know from [2] that the dyons couple locally to dual photons and generally produce a trivial free abelian theory in the IR. When the light particles are not mutually local with respect to each other, which can be achieved by adjusting several moduli to special values (higher critical point), a genuinely non-trivial CFT develops in the IR [17]. In gauge theories with quark hypermultiplets, very general types of critical behaviour can be obtained, see e. g. [18].

Commonly, trying to find a good approximation scheme to describe a non-trivial critical point can be subtle in field theory. A typical example is $\phi^4$ theory in dimension $D$. The theory has two parameters, the bare mass $m$ (or “temperature”) and the bare coupling constant $g$. By adjusting the temperature, we can go to a point where we have massless degrees of freedom, and a non-trivial Ising CFT in the IR. The difficulty is that the renormalized fixed point coupling $g_*$ is large, and thus ordinary perturbation theory in $g$ fails. It is meaningless to try to calculate universal quantities like critical exponents as power series in $g$, since those are $g$-independent. Either the tree-level, $g$-independent contributions are exact and the corrections vanish (this occurs above the critical dimension, which is $D_c = 4$ for the Ising model, and we have a trivial fixed point $g_* = 0$ well described by mean field theory), or the expansion parameter corresponds to a relevant operator and corrections to mean field theory are plagued by untamable IR divergencies. The way out of this problem is to use
expansion parameters on which the low energy CFT depends (corresponding in some sense to marginal or nearly marginal operators): we can use an $\epsilon$ expansion by going to $4 - \epsilon$ dimensions, or a large $N$ expansion by considering $O(N)$ vector models and $O(N)$ invariant critical points.

The critical points on the moduli space of $\mathcal{N} = 2$ super Yang-Mills are very similar to the Ising critical point below the critical dimension, and $1/N$ is very similar to the $\phi^4$ coupling $g$. On any generic finite dimensional submanifold of the moduli space, one finds critical points, characterized by a set of critical exponents $[17, 19, 18]$, that are independent of the number of colours $N$ of the gauge theory in which the CFT is embedded. These critical exponents cannot consistently be calculated in a $1/N$ expansion. Even the simple monopole critical points that are known to be trivial are not described consistently by the $N \to \infty$ limit of the original gauge theory, because electric-magnetic duality is not implemented naturally in this approximation scheme. We will indeed explicitly demonstrate in Section 4 that, though the leading large $N$ approximation for the critical points is smooth (“mean field theory”), the $1/N$ corrections are IR divergent.

Note that by adjusting a large number $\sim N$ of moduli, one can also study $N$-dependent critical points. For these, the large $N$ expansion can certainly be consistent. The $N$ special vacua where a maximum number $N - 1$ of mutually local dyons become massless and which where studied in [20] because of their relationship with $\mathcal{N} = 1$ vacua are of this type.

3 The structure of moduli space at large $N$

3.1 Brief review of $\mathcal{N} = 2$ super Yang-Mills

The fields of pure SU($N$), $\mathcal{N} = 2$ supersymmetric gauge theory all transform in the adjoint representation of the gauge group and make up an $\mathcal{N} = 2$ vector multiplet which contains the gluons, gluinos and a complex scalar field $\Phi$ with scalar potential

$$V = \frac{1}{g^2} \text{tr}[\Phi, \Phi^\dagger]^2. \quad (3)$$

The $D$-flatness conditions $[\Phi, \Phi^\dagger] = 0$ are solved by

$$\langle \Phi \rangle = \text{diag}(a_1, \ldots, a_N) \quad (4)$$

with

$$\sum_{i=1}^{N} a_i = 0. \quad (5)$$
Classically, the gauge group is generically broken to $U(1)^{N-1}$, with W bosons of masses $m_{ij} = \sqrt{2}|a_i - a_j|$. When some of the $a_i$s coincide, the gauge symmetry is enhanced. Quantum mechanically, the moduli space is parametrized by the vevs of the gauge invariant operators

$$ u_i = \langle \text{tr} \Phi^i \rangle, \quad 2 \leq i \leq N, \quad (6) $$

or equivalently by the $\phi_i$s defined implicitly by the relations

$$ u_k = \sum_{i=1}^{N} \phi_i^k, \quad \sum_{i=1}^{N} \phi_i = 0. \quad (7) $$

The $a_i$s are non-trivial functions of the $\phi_i$s, with $a_i \simeq \phi_i$ only at weak coupling. In particular, it turns out to be impossible to have $a_i = a_j$, which implies that a non-abelian gauge group is never restored in the quantum theory.

The low energy effective action is then generically a pure abelian $U(1)^{N-1}$ theory. The leading terms in a derivative expansion can be written in $\mathcal{N} = 1$ superspace as

$$ S_{\text{eff}} = \frac{1}{4\pi} \text{Im} \int d^4x \left[ \int d^4\theta \partial_i \mathcal{F}(A) \bar{A}^i + \frac{1}{2} \int d^2\theta \partial_i \partial_j \mathcal{F}(A) W^i W^j \right], \quad (8) $$

where the $(A_i, W_i)$s are the massless abelian $\mathcal{N} = 2$ vector multiplets satisfying $\sum_i A_i = 0$ and $\mathcal{F}(A)$ is a holomorphic function called the prepotential.

According to Gauss’s law, the central charge $Z$ of the supersymmetry algebra, and the masses

$$ M_{\text{BPS}} = \sqrt{2} |Z| \quad (9) $$

of the BPS states, only depend on the values of the fields at large distances, and thus they can be expressed in terms of $\mathcal{F}$ only. Introducing the dual variables

$$ a_{Di} = \frac{\partial \mathcal{F}}{\partial a_i} \quad (10) $$

we have $[2]$

$$ Z = \sum_{i=1}^{N} (q_i a_i + h_i a_{Di}) \quad (11) $$

where the integers $q_i$, $\sum_i q_i = 0$, and $h_i$, $\sum_i h_i = 0$, are the electric and magnetic quantum numbers respectively. Low energy electric/magnetic duality is manifest on the formulas $(8)$ and $(11)$ which are $\text{Sp}(2(N-1), \mathbb{Z})$ invariant, and it turns out that the variables $a$ and $a_D$ are most naturally interpreted as sections of an $\text{Sp}(2(N-1), \mathbb{Z})$ bundle.
If one introduces the genus $N - 1$ hyperelliptic curve $\mathcal{C}$ defined by

$$\mathcal{C} : \quad Y^2 = \prod_{i=1}^{N}(X - \phi_i)^2 - \Lambda^{2N} = P(X)^2 - \Lambda^{2N} \quad (12)$$

equipped with a canonical basis of homology cycles $(\alpha_i, \beta_i)$, and the differential form

$$\lambda_{SW} = \frac{X \, dP}{2i\pi Y}, \quad (13)$$

then $a$ and $a_D$ are simply given by

$$a_i = \oint_{\beta_i} \lambda_{SW}, \quad a_{Di} = \oint_{\alpha_i} \lambda_{SW}. \quad (14)$$

It is natural to rewrite the equation for $\mathcal{C}$ as

$$Y^2 = (P(X) - \Lambda^n)(P(X) + \Lambda^n) = P_+(X)P_-(X) = \prod_{i=1}^{N}(X - X_{i,+})(X - X_{i,-}) \quad (15)$$

and to view the curve $\mathcal{C}$ as two copies of the complex plane with cuts joining the branching points $X_{i,+}$ and $X_{i,-}$. The “electric” contour $\beta_i$ defining $a_i$ encircles the cut from $X_{i,+}$ to $X_{i,-}$. At weak coupling (small $\Lambda$), one then recovers $a_i \simeq \phi_i$. For our purposes, however, it is not particularly useful to try to make a distinction between electric and magnetic variables. At strong coupling, the two notions are mixed by the non-trivial monodromies [2]. The important point is that there is a one-to-one correspondence between possible BPS states and the integer homology of the curve $\mathcal{C}$, in such a way that to any cycle $\gamma \in H_1(\mathcal{C}, \mathbb{Z})$ we can associate the central charge

$$Z(\gamma) = \oint_{\gamma} \lambda_{SW}. \quad (16)$$

Singularities on the moduli space are obtained when $H_1(\mathcal{C}, \mathbb{Z})$ degenerates, or equivalently when two of the branching points coincide. Since the polynomials $P_+$ and $P_-$ obviously have no common roots, this can happen only when the discriminants of $P_+$ or $P_-$ independently vanish,

$$\Delta(P_+) = \prod_{i<j}(X_{i,+} - X_{j,+})^2 = 0 \quad \text{or} \quad \Delta(P_-) = \prod_{i<j}(X_{i,-} - X_{j,-})^2 = 0. \quad (17)$$
3.2 The large N limit

3.2.1 Generalities

At large $N$, it is convenient to parametrize the $N - 1$ dimensional moduli space by the density function

$$\rho_N(\phi) = \frac{1}{N} \sum_{i=1}^{N} \delta^{(2)}(\phi - \phi_i)$$

which satisfies

$$\int \rho_N(\phi) \, d^2\phi = 1, \quad \int \phi \rho_N(\phi) \, d^2\phi = 0.$$  \hspace{1cm} (19)

At weak coupling, $\rho_N(\phi)$ simply gives the density of complex eigenvalues of the Higgs field. The differences $\phi_i - \phi_j$ are related to the masses of the W bosons. If the lightest W has a mass of order $m_W$, then the heaviest W will generically have a mass of order $M_W \sim \sqrt{N} m_W$. Particles having a mass growing with $N$ are problematic in the large $N$ expansion. This is due to the fact that the amplitude for the propagation of a state of mass $M$ for a time $t$ involves the factor $\exp(-i M t)$ (this argument was used long ago by Witten for baryons [21]). A consistent large $N$ limit, for which propagators of W bosons have a smooth expansion, can thus be achieved only if the heaviest Ws have a mass of order $N^0$, and thus the lightest Ws have generically a mass of order $1/\sqrt{N}$. For one dimensional distributions corresponding to the case where all the $\phi_i$s are aligned, the lightest Ws have a mass of order $1/N$. This correct large $N$ scaling was already considered in [20]. Under the above conditions, the distribution $\rho_N(\phi)$ will typically have an $N \to \infty$ limit $\rho(\phi)$ which is the sum of a smooth function with compact support plus $\delta$ function terms. We will consider examples below.

A good strategy to study the large $N$, $N - 1$ dimensional, moduli space, is to consider well-chosen finite dimensional subspaces. The simplest subspaces $\mathcal{M}_\rho$ are one-complex dimensional and are parametrized by a complex mass scale $v$. One picks up a particular fixed distribution $\phi_i$ characterized by $\rho(\phi)$ and considers $\phi_i^{(v)} = v \phi_i$ or equivalently

$$\rho_v(\phi) = \frac{1}{v^2} \rho\left(\frac{\phi}{v}\right).$$  \hspace{1cm} (20)

When $\rho(\phi)$ is a smooth function, then the limit $v \to \infty$ is a weak coupling limit since the lightest Ws have a mass proportional to $v$. If $\rho(\phi)$ has some $\delta$ function singularities, which means that a large number (of order $N$) of the $\phi_i$s coincide, then we are at strong coupling for any $v$. The study of the simple subspaces $\mathcal{M}_\rho$ can reveal a great deal of physics, including the transition from weak coupling to strong coupling and the presence of the simplest singularities. To obtain more complicated singularities ($k$th order critical points), one has to consider $k$ dimensional subspaces,
or introduce quark flavors and quark mass parameters. For the purposes of the present paper, it is enough to stick to the one-dimensional subspaces $\mathcal{M}_\rho$ defined by (20). It is then natural to introduce the dimensionless ratio

$$r = \frac{v}{\Lambda}$$

(21)

and the polynomials

$$p(x) = \prod_{i=1}^{N} (x - \phi_i), \quad p_\pm(x) = p(x) \mp 1/r^N,$$

(22)

and to rewrite the fundamental equations (12), (13) and (16) in terms of dimensionless variables,

$$C : \quad y^2 = p(x)^2 - 1/r^{2N} = p_+(x) p_-(x),$$

(23)

$$\lambda = \lambda_{SW} / v = x dp / (2i\pi y),$$

(24)

$$z(\gamma) = Z(\gamma) / v = \oint_\gamma \lambda.$$  

(25)

### 3.2.2 Singularities

The locus of singularities $\mathcal{S}$ corresponds to the points where the discriminants vanish, or equivalently when one of the polynomials $p_+$ or $p_-$ has a multiple root. The discriminants are themselves polynomials of degree $N - 1$ in $r^N$, and thus we will have generically $2N(N - 1)$ singularities where a dyon becomes massless on $\mathcal{M}_\rho$. These singularities are arranged on $2(N - 1)$ concentric circles. To better understand the structure of $\mathcal{S}$, it is useful to study first the large $N$ distribution of the roots of the polynomials $p_\pm$. The equation $p_\pm(x) = 0$ can be rewritten

$$\ln |r^N p(x)| + i \arg \left( \pm r^N p(x) \right) = 0.$$  

(26)

The real part of the equation is of order $N$ while the imaginary part is of order $N^0$. In the leading large $N$ approximation, both $p_+ = 0$ and $p_- = 0$ thus reduce to

$$E_N(x) \overset{\text{def}}{=} |p(x)|^{1/N} = 1/|r|.$$  

(27)

The $N \to \infty$ limit of the “envelope” $E_N$ is easily evaluated,

$$E_\infty(x) = \exp \int d^2 \phi \rho(\phi) \ln |\phi - x|.$$  

(28)

When $x$ is in the support of $\rho$, the above approximate formula only gives an average value of $E_N$, which in fact has very sharp wells around the $\phi_i$s, $E_N(\phi_i) = 0$. In the
Figure 1: Plot of the function $E_{N=25}(x)$ corresponding to the distribution (32) with $\phi_i = 2(i - 1)/N - 1$ ($1 \leq i \leq N$) (thin line) together with the envelope $E_\infty$ (thick line).

large $N$ expansion, the wells are approximated by infinitely thin lines joining the smooth envelope $E_\infty$ (see Figure 1 for an example). Any correction to this picture would be exponentially small at large $N$, an instanton effect. At large $|x|$, 

$$E_\infty(x) = |x| \left(1 + O(1/|x|)\right).$$

Equation (27) is now easily solved. When $1/|r| < \min_{x\in\mathbb{C}} E_\infty(x)$, the plane $z = 1/|r|$ can only intersect $z = E_N$ along the thin wells: the roots of $p_+$ and $p_-$ coincide with the classical roots $\phi_i$. This corresponds to the instanton dominated region, and those are suppressed at large $N$. When $|r|$ decreases and $1/|r| > \min_{x\in\mathbb{C}} E_\infty(x)$, $z = 1/|r|$ and $z = E_N$ continues to meet along some of the thin wells, but the roots nearest to the minimum of $E_\infty$ are now arranged along an inflating curve. This curve gives the general shape of the “enhançon” discussed in [16]. Note that the enhançon exists for all $r$ if the distribution $\rho$ has some $\delta$ function singularities. It has in general several connected components associated with the different connected components of the support of $\rho$. At large $1/|r|$, all the roots are located on the enhançon whose various connected components eventually merge to form asymptotically a circle of radius $1/|r|$ according to (29).

For example, let us consider the simple uniform distribution

$$\rho_0(\phi) = \begin{cases} \frac{1}{\pi} & \text{if } |\phi| < 1 \\ 0 & \text{if } |\phi| > 1. \end{cases}$$

(30)
It yields an envelope
\[ E_{\infty,0}(x) = \begin{cases} 
  e^{(|x|^2-1)/2} & \text{if } |x| < 1 \\
  |x| & \text{if } |x| > 1.
\end{cases} \quad (31) \]

For \(|r| > \sqrt{e}\), all the roots are classical. At \(|r| = \sqrt{e}\), the enhançon begins to inflate. It is a circle of radius \(\sqrt{1 - \ln |r|^2}\). At \(|r| = 1\), the enhançon has eaten up all the classical roots, and for \(|r| < 1\) it is simply a circle of radius \(1/|r|\). For the purpose of illustration, we have depicted in Figure 1 the case of a unidimensional uniform distribution
\[ \rho_1(\phi) = \begin{cases} 
  \delta(\text{Im}(\phi))/2 & \text{if } |\phi| < 1 \\
  0 & \text{if } |\phi| > 1
\end{cases} \quad (32) \]
for which
\[ E_{\infty,1}(x \in \mathbb{R}) = \begin{cases} 
  (1-x)^{(1-x)/2}(-1-x)^{(1+x)/2}/e & \text{if } x < -1 \\
  (1+x)^{(1+x)/2}(1-x)^{(1-x)/2}/e & \text{if } -1 < x < 1. \\
  (1+x)^{(1+x)/2}(-1+x)^{(1-x)/2}/e & \text{if } x > 1.
\end{cases} \quad (33) \]

Now that we understand the location of the roots of \(p_{\pm}\), it is straightforward to deduce the singularity locus. Generically, roots are either equal to the classical values, or are smoothly distributed along the enhançon. Singularities can then occur only when the classical roots are eaten up by the inflating enhançon, or when two connected components of the enhançon merge. This is particularly transparent on Figure 1, where real roots join the enhançon by pairs and then become complex. We thus deduce that the singularities are densely distributed on rings in the \(r\)-plane corresponding to the melting of classical roots with the enhançon (singularities of the first class) or with the melting of several connected components of the enhançon with each other (singularities of the second class). In Figure 2 we have represented the space \(\mathcal{M}_{\rho_0}\) with the associated singularity locus \(S\) (only singularities of the first class are present in this example). We will see in Section 4 that the large \(N\) expansion is afflicted with power-like divergences near singularities of the second class, while logarithmic divergencies also mix up near singularities of the first class.

### 3.2.3 Special cases

We have discussed above how the singularity locus \(S\) can be deduced for a given density \(\rho\). It is amusing to study the inverse question: can we find a density \(\rho\) that would correspond to an a priori given locus \(S\)? This amounts to inverting the functional relation (28) between \(E_{\infty}\) and \(\rho\). If we limit ourselves to the consideration of unidimensional distributions (by constraining all the \(\phi_i\)'s to be real), the mathematical problem is very similar to the one encountered in the study of the leading \(N \to \infty\)
Figure 2: The one-complex dimensional subspace \( M_{\rho_0} \) corresponding to the uniform distribution (30) with the associated ring of singularity \( S \).

As an application, let us try to find a simple unidimensional smooth density \( \rho \) with a connected compact support \([-1, 1]\) for which all the singularities are distributed on a single circle (a singularity ring of zero thickness). From the discussion of Section 3.2.2 it is clear that this can only happen when \( dE_\infty/dx = 0 \) on the support of \( \rho \). The only solution to (36) consistent with \( \omega(x) \sim 1/x \) at infinity is then

\[
\omega(x) = \frac{1}{\sqrt{(x-1)(x+1)}},
\]

and (35) implies

\[
\rho(\phi) = \frac{1}{\pi \sqrt{1 - \phi^2}}.
\]

This is the density distribution of the roots of the Chebyshev polynomials \( T_N \), for which \( \phi_i = \cos(\pi(1/2 + i)/N) \). This very special distribution was studied in [20].
Equation (28) then gives the radius of the singularity circle $|r| = 2$. Clearly, many generalisations could be studied.

### 3.2.4 Corrections to $N = \infty$

The singularity rings of the first class separate an outer purely instanton-dominated region from an inner strongly coupled region which cannot be described by semiclassical physics (the outer region only exists when $\rho$ does not have any $\delta$ function singularities). In the outer region, corrections at large $N$ are exponentially suppressed. To the contrary, we can a priori expect non-trivial corrections in the inner, strongly coupled, regions of moduli space. We will elucidate the nature and the main properties of these corrections in Section 4, but we can have a first instructive look at them by investigating the envelope $E_\infty$ or equivalently the shape of the enhançon on a simple example. Let us choose $\phi_i = 2(i - 1)/N - 1$ as in Figure 1, and let us calculate the corrections to (33) for $x > 1$. By applying Euler’s formula

$$\frac{b - a}{n} \sum_{k=0}^{n-1} f\left(a + k \frac{b - a}{n}\right) = \int_a^b f - \frac{(b - a)(f(b) - f(a))}{2n} + \mathcal{O}(1/n^2)$$

we obtain

$$E_{N,1}(x) = E_{\infty,1}(x) \left(1 - \frac{1}{2N} \ln \frac{x - 1}{x + 1} + \mathcal{O}(1/N^2)\right) \text{ for } x > 1. \quad (40)$$

This formula displays two of the main features of the corrections to $N = \infty$: they are of order $1/N$ (not of order $1/N^2$) and they diverge near the singularity locus which corresponds here to $|r| \to (1/2)e^+$ or equivalently $x \to 1^+$ (see (27) and (33)). A third feature, that the corrections are given by series of fractional instantons, is also to be expected if one consider that (27) depends on $1/r$.

### 4 Calculation of the $1/N$ and $1/N^2$ corrections

This Section is devoted to a detailed study of the corrections to the leading $N \to \infty$ approximation in two particular, but generic, examples, encompassing the cases of both types of singularities (first class and second class). Since we want to consider the purely strongly coupled region, where the physics cannot be described by instantons, we can put most of the singularities at infinity. Yet, we will keep one singular circle at finite distance, so that we can discuss the physics associated with singularities.
Figure 3: Plot of the function \( E_{N=25}(x) \) corresponding to the distribution (41) (thin line) together with the envelope \( E_\infty \) (thick line). The roots \( x_- \) and \( x_1 \) of the polynomial \( p_- \) discussed in the text (equations (44) and (45)) are also indicated for a particular value of \( 1/r \).

4.1 An example with a first class singularity

We choose the distribution \( \rho_N \) to be

\[
\rho_N(\phi) = \frac{N - 1}{N} \delta(\phi + 1/N) + \frac{1}{N} \delta(\phi + 1/N - 1). 
\]

(41)

The associated function \( E_N \) and envelope \( E_\infty \) are depicted in Figure 3. The parameter \( r \) will be real and positive, without loss of generality (the final formulas can be trivially analytically continued). We have a singularity of the first class corresponding to the melting of the classical root \( x_{cl} = 1 - 1/N \) with the enhançon.

4.1.1 Infrared divergences and fractional instantons

A very simple, yet very instructive, calculation one can do is the critical value \( r_c \) of \( r \) for which the singularity occurs. The leading \( N \to \infty \) approximation \( r_c = 1 \) can be read off immediately from Figure 3. The exact value can be found by noting that it corresponds to the point where the two positive real roots of the polynomial \( p_- \) defined by (22) coincide,

\[
r_c = \frac{N}{(N - 1)^{1-1/N}}. 
\]

(42)
The large $N$ expansion of the exact formula (42) has strange looking logarithmic terms,
\[ r_c = 1 + \frac{\ln N + 1}{N} + o(1/N). \]  
(43)
A standard, well-behaved, large $N$ expansion cannot produce such $(\ln N)/N$ corrections. Remarkably, the same kind of $(\ln N)/N$ corrections to the critical parameter were found recently by the author in a study of a two dimensional QFT that was argued to be very similar to gauge theories with Higgs fields [5, 6]. These $(\ln N)/N$ corrections were the consequence of the breakdown of the large $N$ expansion near the critical point due to infrared divergences (a full calculation can be found in Appendix C of [6]). We will see below that the qualitative physics in four dimensions is strictly similar to the physics previously discussed in two dimensions [4, 5, 6].

The next simple calculation one can obviously do is the large $N$ expansion of the roots of the polynomials $p_+$ and $p_-$. These roots also enter explicitly in the Seiberg-Witten period integrals (25). We will focus on the two positive real roots of the polynomial $p_-$ which exists for $r > r_c$ and coincide at $r = r_c$. One of these two roots that we call $x_-$ is simply
\[ x_- = 1 - 1/N - 1/r^N + O(1/r^{2N}). \]  
(44)
The non-trivial corrections to the classical root $x_{cl} = 1 - 1/N$ are exponentially small. The other root $x_1$ has a non-trivial and very interesting large $N$ expansion that can be straightforwardly deduced from (26),
\[ x_1 = \frac{1}{r} - \frac{r + \ln(r - 1)}{N r} + \frac{1}{2N^2 r} \left( (\ln(r - 1))^2 - \frac{2r \ln(r - 1)}{r - 1} \right) + O(1/N^3). \]  
(45)
This very simple formula displays all the features that were advertised in Section 2. In particular, by writing $\ln(r - 1) = \ln r + \ln(1 - 1/r) = \ln r - \sum_{k \geq 1} 1/(kr^k)$, we see that each order in $1/N$ is given by a series of fractional instantons of fractional charge $1/(2N)$, mixed with the one-loop diagram contribution in $\ln r$. At $r = 1$, the corrections are blowing up like the logarithm of the mass of the lightest degrees of freedom, an infrared divergence. Using (44) and (45) to solve $x_- = x_1$, one can recover the expansion (43) with the $(\ln N)/N$ correction.

Of course, roots like $x_1$ are not directly observables. The physics is encoded in the Seiberg-Witten periods (25), and though they depend on the roots, one might argue that cancellations could occur and that the final result could be smooth. It is also conceivable that only corrections in $1/N^2$ could show up in the final results. To answer rigorously these questions, we will compute in the next subsection the Seiberg-Witten period integral $z$ corresponding to the cycle $\gamma$ which vanishes at $r = r_c$. It
turns out that the only potential divergences come from (45) in this case, and that the qualitative features of the expansion for \(x_1\) and for \(z\) are the same.

4.1.2 The calculation of the period

We thus proceed to compute the large \(N\) expansion of

\[
z = \oint \lambda = \frac{1}{i\pi} \int_{x_1}^{x} dx \frac{x p'(x)}{p(x)^2 - 1/r^{2N}},
\]

with

\[
p(x) = (x + 1/N)^{N-1} (x - 1 + 1/N).
\]

It is convenient to trade the variable \(x\) for \(u = 1/(r(x + 1/N))\), in terms of which

\[
z = \frac{N}{i\pi r} \int_{u_-}^{u_1} du \frac{(1 - ru/N)(1 - (1 - 1/N)ru)}{\sqrt{(1 - ru)^2 - u^{2N}}},
\]

with

\[
u_- = \frac{1}{r} \left( 1 + 1/r^N + \mathcal{O}(1/r^{2N}) \right),
\]

\[
u_1 = 1 + \frac{\ln(r - 1)}{N} + \frac{1}{2N^2} \left( (\ln(r - 1))^2 + \frac{2r}{r - 1} \ln(r - 1) \right) + \mathcal{O}(1/N^3).
\]

At large \(N\), it is tempting to neglect the term \(u^{2N}\) in (48), which is exponentially small on most of the integration region. This is correct except for \(u \simeq u_1 \simeq 1\), when \(u^{2N}\) is not negligible, and for \(u \simeq u_- \simeq 1/r\), when \((ru - 1)^2\) can be smaller than \(u^{2N}\). We will thus distinguish three different contributions to \(z\),

\[
z = z_0 + z_< + z_>,
\]

with

\[
z_0 = \frac{N}{i\pi r} \int_{u_-}^{u_1} du \frac{1}{u^2} \left( 1 - \frac{ru}{N} \right) \left( 1 - \frac{(N - 1)ru}{N} \right) \frac{1}{ru - 1},
\]

\[
z_> = \frac{N}{i\pi r} \int_{u}^{U} du \frac{1}{u^2} \left( 1 - \frac{ru}{N} \right) \left( 1 - \frac{(N - 1)ru}{N} \right) \left( \frac{1}{\sqrt{(ru - 1)^2 - u^{2N}}} - \frac{1}{ru - 1} \right),
\]

\[
z_< = \frac{N}{i\pi r} \int_{u_-}^{U} du \frac{1}{u^2} \left( 1 - \frac{ru}{N} \right) \left( 1 - \frac{(N - 1)ru}{N} \right) \left( \frac{1}{\sqrt{(ru - 1)^2 - u^{2N}}} - \frac{1}{ru - 1} \right).
\]
where \( U \in [u_-, u_1] \) is an arbitrary constant, for example \( U = (u_- + u_1)/2 \). The integral \( z_0 \) can be explicitly evaluated, and its expansion is given by

\[
\frac{i\pi z_0}{N} = -\frac{r - 1}{r} + \left(1 - \frac{1}{N}\right) \ln r + \frac{r - 1}{N} \ln(r - 1) + \frac{(\ln(r - 1))^2}{2N^2r^2} + \mathcal{O}(1/N^3). \tag{55}
\]

Finding the asymptotic expansion of \( z_0 \) at large \( N \) is a bit more subtle. The integral would be exponentially small if not for the integration region near \( u_1 \). The term \( u^{2N} \) cannot be neglected in a region of size \( 1/N \) around \( u_1 \). The idea is to use a new variable \( w \) defined by

\[
u = u_1 - w/N. \tag{56}\]

The large \( N \) expansion of the integrand can then be straightforwardly deduced, and one is left with several definite integrals on the variable \( w \in [0, N(u_1 - U)] \). The integration region can be extended to \([0, \infty]\) at the expense of neglecting exponentially suppressed terms. The calculation of the \( 1/N \) and \( 1/N^2 \) corrections involve five non-trivial definite integrals that can be explicitly evaluated. Instead of writing down all these complicated formulas, we have illustrated the procedure on a simpler, but qualitatively similar, example in the Appendix. Fortunately, the final result is simple-looking,

\[
\frac{i\pi z_0}{N} = -\frac{\ln 2}{N} + \frac{1}{N^2} \left(\left(\ln(r - 1) + r + \frac{1}{2} \ln 2\right) \ln 2 - \frac{\pi^2}{24}\right). \tag{57}
\]

The evaluation of \( z_0 \) proceeds along the same lines, except that now the relevant integration region around \( u = u_- \) is exponentially small. The asymptotic expansion to all orders in \( 1/N \) can then be obtained in this case by changing the variable to \( w \) defined by \( u = u_- + w/r^{N+1} \). We get

\[
\frac{i\pi z_0}{N} = \frac{N - 1}{N^2} \ln 2 + \mathcal{O}(1/r^N). \tag{58}
\]

Putting all the contributions together, we finally end up with

\[
\frac{i\pi z}{N} = -\frac{r - 1}{r} + \ln r + \frac{1}{N} \left(- \ln r + \frac{r - 1}{r} \ln 2 + \frac{(r - 1) \ln(r - 1)}{r}\right) \tag{59}
\]

\[+ \frac{1}{N^2} \left(\frac{1}{2} (\ln(r - 1))^2 + (\ln 2) \ln(r - 1) + \frac{1}{2} (\ln 2)^2 - \frac{\pi^2}{24}\right) + \mathcal{O}(1/N^3).
\]

This final formula displays all the remarkable properties of the large \( N \) expansion of \( \mathcal{N} = 2, \) SU\((N)\) super Yang-Mills theory in four dimensions that we have already discussed at length in the previous Sections. We note that genuine divergencies when \( r \to 1 \) show up only at order \( 1/N^2 \), but non-analytic terms are already present at order \( 1/N \).
Figure 4: The enhançon for the distribution (60) at \( r = 1.02, r = 1, r = 0.98 \) and \( r = 0.5 \). The dots correspond to the actual roots of the polynomial \( p(x)^2 - 1/r^{2N} \) for \( N = 30 \). At \( r = r_c = 1 \), two roots coincide, the two components of the enhançon merge, and we have a second class singularity. At small \( r \) the enhançon is a circle of radius \( 1/r \).

### 4.2 An example with a second class singularity

Let us now pick \( N \) to be even for convenience and consider the case of the distribution

\[
\rho_N(\phi) = \frac{1}{2} \left( \delta(\phi - 1) + \delta(\phi + 1) \right)
\]

which corresponds to the polynomial \( p(x) = (x^2 - 1)^{N/2} \) (see (22)). The enhançon has two components at large \( |r| \) which merge at a second class singularity for \( |r| = 1 \). The shape of the enhançon, together with the roots of \( p(x)^2 - 1/r^{2N} \) are depicted in Figure 4.

If we take \( r \) to be real for concreteness, then the critical value of \( r \) is exactly given by

\[
r_c = 1
\]

and corresponds to the merging of the branching points

\[
x_\pm = \pm \sqrt{1 - 1/r^2}.
\]

The formulas (61) and (62) are to be compared with (42, 43, 44, 45). They exhibit one important difference between first class and second class singularities: the large
The large $N$ expansion of the roots (or the shape of the enhançon) is perfectly smooth in the case of second class singularities. However, for the same general reasons as in the case of first class singularities, the large $N$ expansion does break down as well. To demonstrate this, we have to consider the Seiberg-Witten periods which are the physical observables,

$$z = \frac{1}{i\pi} \int_{x_-}^{x_+} dx \frac{xp'(x)}{\sqrt{p(x)^2 - 1/r^{2N}}} = -\frac{N}{i\pi} \int_{1/r^2}^{1} \frac{du}{u} \sqrt{\frac{1 - 1/(ur^2)}{1 - u^N}}, \quad (63)$$

where $u = 1/(r^2(1 - x^2))$. A straightforward calculation along the lines of the preceding subsection or of the Appendix then yields the large $N$ expansion

$$\frac{i\pi z}{N} = 2\sqrt{1 - 1/r^2} - 2\ln \left(1 + \sqrt{1 - 1/r^2}\right) - 2\ln r - \frac{2\sqrt{1 - 1/r^2} \ln 2}{N}$$

$$+ \frac{\pi^2/6 - 2(\ln 2)^2}{2N^2r^2\sqrt{1 - 1/r^2}} + O(1/N^3). \quad (64)$$

We get non-analytic terms at leading order, and genuine divergences at order $1/N^2$.

## 5 Open problems

We have seen that, despite the fact that the low energy effective action of $\mathcal{N} = 2$ super Yang-Mills is, in some sense, generated by instantons only, it has a surprisingly rich and interesting large $N$ expansion.

Low-dimensional sections of the moduli space can be easily analysed. For example, Figure 2 is very reminiscent of the structure of the moduli space for $\mathcal{N} = 2$. This suggests that the spectrum of BPS states and the curves of marginal stability could have a simple description at large $N$, and that the methods of $[23]$ may be useful.

We have elucidated the fate of large instantons at strong coupling and large $N$ in $\mathcal{N} = 2$ super Yang-Mills: they disintegrate into fractional instantons. These fractional instantons are responsible for a new class of contributions in the large $N$ expansion, generating a series in $1/N$ in addition to the standard series in $1/N^2$ from Feynman diagrams. These new contributions were brought to the fore particularly clearly because we have studied observables which pick up a single one-loop term from Feynman diagrams. More general observables will probably have non-trivial contributions from both types of terms. Note that the leading corrections to $N = \infty$ are dominated by fractional instantons. It would be very interesting to find direct general arguments explaining why fractional instantons contribute a well-behaved
asymptotic series in $1/N$. The fate of instantons in real world QCD remains of course an open problem, but it is fascinating to contemplate the possibility that phenomena qualitatively similar to the one studied in the present paper could occur. This is suggested by the fact that the basic difficulty with instantons—IR problems due to the Landau pole for large instantons—are independent of supersymmetry.

The standard ’t Hooft analysis of perturbation theory at large $N$ in SU($N$) gauge theories \cite{1} is consistent with a dual description in terms of closed oriented strings. Our analysis of the fractional instanton contributions shows that open strings must also be present. Though adding open strings to a closed string theory is a non-trivial step, it is satisfying that fractional instanton terms can be interpreted in a string picture at all.

The breakdown of the large $N$ expansion at singularities may come as a surprise (it was anticipated in the recent studies of two-dimensional models by the author \cite{4,5,6}), but we have argued that the physics from the field theory point of view is the same as the one that produces IR divergencies in corrections to mean field theory below the critical dimension. On the other hand, the interpretation from the string theory point of view is much less clear. A naive application of the UV/IR relation seems to indicate that the string dual is not UV renormalizable, but this would be a rather strange property for a string theory. Another possibility, which was pointed out to me by I. Klebanov, is that the divergencies are due to tensionless strings at the singularities. This interpretation requires the appearance of divergences (or at least non-analytic contributions) at leading order, which is not the case in (59). This might nevertheless be made consistent by correctly identifying the terms generated by closed strings or open strings diagrams. This point clearly deserves further investigation.

Another fascinating line of research, suggested by the divergencies of the $1/N$ expansion near singularities, is that it might be possible to extract finite universal string amplitudes by taking $N \to \infty$ and $r \to r_c$ in a correlated way, along the lines of \cite{24}. A preliminary investigation will be presented in a forthcoming paper \cite{25}.

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Appendix: a simple toy integral

Let us consider the integral

\[ I_N = \int_0^1 dx \left( \frac{1}{\sqrt{1 - x^N}} - 1 \right). \] (65)

By changing variable to \( y = x^N \), \( I_N \) is straightforwardly expressed in terms of Euler beta function,

\[ I_N = \frac{1}{N} B(1/N, 1/2) - 1 = \sqrt{\pi} \frac{\Gamma(1 + 1/N)}{\Gamma(1/2 + 1/N)} - 1. \] (66)

The large \( N \) expansion can then be readily found,

\[ I_N = \frac{2 \ln 2}{N} + \frac{2(\ln 2)^2 - \pi^2/6}{N^2} + O(1/N^3). \] (67)

The difficulty one encounters to understand the large \( N \) asymptotic expansion directly on the integral formula (65) is the same as for the integral (53) studied in the main text: the integrand is exponentially small (an instanton effect) except for a region of size \( 1/N \) near \( x = 1 \). One then use the variable \( y \) defined by \( x = 1 - y/N \), and expand the integrand, which yields

\[ I_N = \frac{1}{N} \int_0^N dy \left( \frac{1}{\sqrt{1 - e^{-y}}} - 1 \right) - \frac{1}{4N^2} \int_0^N dy \frac{y^2 e^{-y}}{(1 - e^{-y})^{3/2}} + O(1/N^3). \] (68)

The integration region can be extended to \( y \in [0, \infty[ \) by neglecting terms of order \( e^{-N} \). The integral at order \( 1/N \) is elementary and gives the \( (2 \ln 2)/N \) correction. The integral at order \( 1/N^2 \) is of the type of the integrals encountered in the main text. By going to the variable \( y' = 1/\sqrt{1 - e^{-y}} \) and integrating by part, it can be related to the dilogarithm function \( \text{Li}_2 \) and thus explicitly evaluated.
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