TIME TO MRCA FOR STATIONARY CBI-PROCESSES

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Abstract. Motivated by sample path decomposition of the stationary continuous state branching process with immigration, a general population model is considered using the idea of immortal individual. We compute the joint distribution of the random variables: the time to the most recent common ancestor (MRCA), the size of the current population and the size of the population just before MRCA. We obtain the bottleneck effect as well. The distribution of the number of the oldest families is also established. The results generalize those in the recent paper by Chen and Delmas [8].

1. Introduction

Continuous state branching processes (CB-processes) are non-negative real-valued Markov processes first introduced by Jirina [17] to model the evolution of large populations of small particles. Continuous state branching processes with immigration (CBI-processes) are generalizations of those describing the situation where immigrants may come from outer sources, see e.g. Kawazu and Watanabe [18]. It is shown in Lamperti [22] that a CB-process can be obtained as the scaling limit of a sequence of Galton-Watson processes; see also [5, 6, 23]. A genealogical tree is naturally associated with the Galton-Watson process. This has given birth to the continuum random tree theory first introduced by Aldous [3, 4] to code the genealogy of the CB-process. Duquesne and Le Gall [9] further developed the continuum Lévy tree to give the complete description of the genealogy of the CB-process in (sub)-critical case. Kingman has initiated the study of the coalescent process in 1982 in his famous papers [19, 20]. Then coalescents with multiple collisions, also known as Λ-coalescents, were first introduced and studied independently by Pitman [26] and by Sagitov [27]. Recently some authors have studied the coalescent process associated with branching processes, see e.g. Lambert [21] on coalescent time, Evans and Ralph [11] on the dynamics of the time to the most recent common ancestor (MRCA), Chen and Delmas [8] on MRCA on some special stationary CBI-process, and Berestycki et al. [7] on the coupling between Λ-coalescents and branching processes.

This paper is motivated by Chen and Delmas [8]. The model considered here is a direct extension of [8]. We will use some notations and definitions in that paper and consider the general CBI-process here instead. The fact that the CBI-process may have a non-trivial stationary distribution makes it a more interesting model to be considered here than the CB-process since for the CB-process either the population becomes extinct or blows up with positive probability. We consider a (sub)-critical CBI-process $Y = (Y_t, t \geq 0)$ with branching mechanism $\psi$ given by (2.1) and immigration mechanism $F$ given by (2.7). Our main interest is in presenting a further model of random size varying population and exhibiting some interesting properties. Afterwards we will give some properties of the coalescent tree.

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We consider the stationary CBI-process defined on the real line $Z = (Z_t, t \in \mathbb{R})$. In order for the time to MRCA to be finite, we assume condition (A1):

$$\int_1^\infty \frac{dz}{\psi(z)} < \infty.$$ 

In order for $Z_t$ to be finite, we shall assume condition (A2):

$$\int_0^{\lambda} \frac{F(z)}{\psi(z)} \, dz < \infty, \quad \text{for some } \lambda > 0.$$ 

Using the look-down construction for the population with constant sizes, we represent the process $Z$ by means of the picture of an immortal individual which gives birth to independent populations. We first give some notations. For fixed time $t = 0$ (indeed we can choose any time by stationarity), we denote by $A$ the time to the MRCA of the population living at time 0, $Z^A = Z_{(-A)_-}$ the size of the population just before MRCA, and $Z^I$ the size of the population at time 0 which has been generated by the immortal individual over the time interval $(-A, 0)$ and $Z^O = Z_0 - Z^I$ the size of the population at time 0 generated by the immortal individual at time $-A$. We will see that conditionally on $A$, the random variables $Z^A, Z^I$ and $Z^O$ are independent, and the joint distribution of the random variables is also considered. We also obtain the result that the size of the population just before MRCA is stochastically smaller than that of the population at the current time, that is the bottleneck effect.

Let $N^A + 1$ represent the number of individuals involved in the last coalescent event of the genealogical tree. We present the joint distribution of $A, N^A$ and $Z_0$. Using the measured rooted real tree formulation of the genealogy of the stationary CBI-process developed in [2], we give the asymptotic for the number of ancestors.

We will give the transition probabilities of the MRCA age process $(A_t, t \in \mathbb{R})$, which has been studied by Evans and Ralph in [11] for the CB-process conditioned on non-extinction. We generalize it to the general case with the similar lines as their proof. In the end we study the zero set of the CBI-process as well, which is a stationary regenerative set. Foucart and Bravo [12] have studied the CBI case on the positive half line. The stationary case is a bit different as the subordinator is not naturally associated with the regenerative set; see [28] and [16] for details. For this situation see also [15].

This paper is organized as follows. We first recall some well-known results on the CB-process and CBI-process in Section 2. The family and clan decomposition of the CBI-process are then introduced in Section 3. We will give the condition for the existence of the stationary CBI-process, determine the joint distribution of $A, Z^A, Z^I, Z^O$ and prove the bottleneck effect in Section 4, that is $Z^A$ is stochastically smaller than $Z_0$. In Section 5 the distribution of the number of individuals involved in the last coalescent event $N^A$ is computed. In the latter part of Section 6 we will introduce the genealogy of CB-process using continuum random Lévy trees. Then the asymptotic for the number of ancestors is given. In Section 7 we give the transition probabilities of the MRCA age process and the properties of the zero set.

2. CB-PROCESS AND CBI-PROCESS

We recall some well-known results on CB-process and CBI-process derived from Li [24, 25]. We consider a (sub)-critical branching mechanism $\psi$:

$$\psi(z) = bz + cz^2 + \int_0^\infty \left( e^{-zu} - 1 + zu \right) m(du), \quad z \geq 0,$$ (2.1)
where \( b = \psi'(0+) \geq 0, c \geq 0 \) are constants and \((u \land u^2) m(du)\) is a finite measure on \((0, \infty)\). We will consider the non-trivial case, that is, assumption \((A3)\):

\[
\text{either } c > 0 \text{ or } \int_{(0,1)} u m(du) = \infty.
\]

There exists a \( \mathbb{R}_+ \)-valued strong Markov process \( X = (X_t, t \geq 0) \) called continuous state branching process (CB-process) with branching mechanism \( \psi \) whose distribution is characterized by its Laplace transform

\[
\mathbb{E}_x [e^{-\lambda X_t}] = e^{-x \upsilon_t(\lambda)},
\]

where \( \mathbb{E}_x \) means that \( X_0 = x \) and the function \( \upsilon_t(\lambda) \) is the unique non-negative solution of the backward equation

\[
\begin{cases}
\frac{\partial}{\partial t} \upsilon_t(\lambda) = -\psi(\upsilon_t(\lambda)), & t > 0, \lambda \geq 0, \\
\upsilon_0(\lambda) = \lambda, & \lambda \geq 0,
\end{cases}
\]

The CB-process has a canonical Feller realization. Let \( \mathbb{P}_x \) be the law of such a CB-process started at mass \( x > 0 \). Moreover, \( X \) has no fixed discontinuities. The probability measure \( Q_t(x, \cdot) \) is infinitely divisible and under condition \((A3)\), \( \upsilon_t(\lambda) \) can be expressed canonically as

\[
\upsilon_t(\lambda) = \int_0^{\infty} (1 - e^{-\lambda u}) l_t(du), \quad t > 0, \lambda \geq 0,
\]

where \( u l_t(du) \) is a finite measure on \((0, \infty)\); see Theorem 3.10 in [24]. The Markov property of \( X \) implies that for any \( \lambda, s, t \geq 0 \)

\[
\upsilon_{t+s}(\lambda) = \upsilon_t(\upsilon_s(\lambda)).
\]

We also have the forward differential equation

\[
\frac{\partial}{\partial t} \upsilon_t(\lambda) = -\psi(\lambda) \frac{\partial}{\partial \lambda} \upsilon_t(\lambda).
\]

Let \( \zeta = \inf\{s \geq 0, X_s = 0\} \) be the extinction time of \( X \) and \( c(t) = \lim_{\lambda \to \infty} \upsilon_t(\lambda) \). Under \((A1)\), \( c(t) > 0 \) is finite. We have by \((2.3)\) that

\[
\upsilon_s(c(t)) = c(t+s).
\]

We consider an immigration mechanism \( F \):

\[
F(z) = \beta z + \int_0^{\infty} (1 - e^{-z u}) n(du), \quad z \geq 0,
\]

where \( \beta \geq 0 \) is a constant and \((1 \land u) n(du)\) is a finite measure on \((0, \infty)\). Then there exists a strong Markov process \( Y = (Y_t, t \geq 0) \) called continuous-state branching process with immigration (CBI-process) with branching mechanism \( \psi \) and immigration mechanism \( F \) defined on \( \mathbb{R}_+ \) with Laplace transform given by

\[
\mathbb{E}_x [e^{-\lambda Y_t}] = e^{-x \upsilon_t(\lambda)} - \int_0^t F(\upsilon_s(\lambda)) ds,
\]

where \( \mathbb{E}_x \) means that \( Y_0 = x \). We also denote \( \mathbb{P} \) the corresponding probability measure.
3. Sample path decomposition

In this section we will recall some results from Li [24] [25]. We will give the clan and family decomposition of the CBI-process.

The process \( X \) is infinitely divisible. It is well known that there exists a canonical measure (we also call it excursion law) \( Q_0 \) on the space \( D \) of càdlàg functions on \([0, \infty)\) with Skorokhod topology. Notice that \( Q_0(\{X, X_{0+} \neq 0\}) = 0 \).

We can give a reconstruction of the sample paths of the CB-process by means of the excursion law. Let \( x \geq 0 \) and let \( N(dX) = \sum_{i \in I} \delta_{X_i}(dX) \) be a Poisson random measure on \( D \) with intensity \( xQ_0(dX) \). We define the process \( (X'_t, t \geq 0) \) by

\[
\begin{align*}
X'_0 &= x, \\
X'_t &= \int_D X_t N(dX), \quad t > 0.
\end{align*}
\]

Then \( X' \) is a realization of the CB-process \( X \). We will not distinguish \( X' \) from \( X \). For the proof see Li [24, Theorem 8.24] or [25, Theorem 2.4.2]. As one can see (3.1) is equivalent to the well-known decomposition as follows: If \( N(dx, dX) = \sum_{i \in I} \delta_{(x_i, X_i)}(dx, dX) \) is a Poisson point measure on \( \mathbb{R}_+ \times D \) with intensity \( 1_{[0, \infty)}(x)dxQ_0(dX) \), then \( \sum_{i \in I} 1_{\{x_i \leq x\}}X_i \) is distributed as \( X \) under \( \mathbb{P}_x \). Further we have for \( \lambda \geq 0 \),

\[
Q_0(1 - e^{-\lambda X_t}) = \lim_{x \to 0} \frac{1}{x} E_x [1 - e^{-\lambda x}] = v_t(\lambda),
\]

and

\[
c(t) = Q_0(\zeta > t) = Q_0(X_t > 0).
\]

We will put \( X_t = 0 \) for \( t < 0 \).

Now we will introduce the family decomposition of the CBI-process. We consider the following Poisson point measures.

1. Let \( N_0(dr, dt) = \sum_{i \in I} \delta_{(r_i, t_i)}(dr, dt) \) be a Poisson point measure on \((0, \infty) \times \mathbb{R}\) with intensity \( n(dr)dt \).

2. Conditionally on \( N_0 \), let \( (N_{1,i}, i \in I) \) be independent Poisson point measures with intensity \( r_i \delta_{t_i}(dt)Q_0(dX) \), where \( N_{1,i}(dt, dX) = \sum_{j \in J_{1,i}} \delta_{(X_i, t_i)}(dt, dX) \). Note that for all \( j \in J_{1,i} \), we have \( t_j = t_i \).

3. Let \( N_2(dt, dX) = \sum_{j \in J_2} \delta_{(t_j, X_j)}(dt, dX) \) be a Poisson point measure with intensity \( \beta dt \) independent of \( N_0, N_1 \).

We set \( J = J_1 \cup J_2 \). We shall call \( X^j \) a family and \( t_j \) its birth place for \( j \in J \). We will consider the process \( (Y'_t, t \geq 0) \) and its stationary version \( (Z_t, t \in \mathbb{R}) \). They are usually called family decomposition of the CBI-process defined as follows:

\[
Y'_t = \sum_{j \in J, t_j > 0} X^j_{t-t_j}, \quad Z_t = \sum_{j \in J} X^j_{t-t_j},
\]

Putting this another way we can deduce that it corresponds to a special immigration process shown in Corollary 3.4.2 in [25].

For \( i \in I \) denote \( X^i = \sum_{j \in J_{1,i}} X^j \) and \( I = I \cup J_2 \). The random measure

\[
N_3(dt, dX) = \sum_{i \in I} \delta_{(t_i, X_i)}(dt, dX)
\]
is a Poisson point measure with intensity \( dt\mu(dx) \), where \( \mu \) is given by

\[
(3.4) \quad \mu(dx) = \beta Q_0(dx) + \int_{(0,\infty)} n(dx)P_x(dx).
\]

It corresponds to the entrance law \( (H_t, t > 0) \) in Li [25] given by

\[
H_t = \beta I_t + \int_0^\infty n(dx) Q_t(x, \cdot).
\]

We shall call \( X^i \) with \( i \in I \) a clan and \( t_i \) its birth place. For \( j \in J_2, \) \( X^j \) is a clan and a family. Then we have

\[
(3.5) \quad Y'_t = \sum_{i \in I, t_i > 0} X^i_{t-t_i}, \quad Z_t = \sum_{i \in I} X^i_{t-t_i}.
\]

\( Y' \) with this representation is just the sample path decomposition of the CBI-process. We shall call this the clan decomposition of \( Y \). \( Y' \) is a version of \( Y \) and \( Z \) is the stationary version of \( Y \). Usually the family decomposition is more precise than the clan decomposition.

We give an interpretation of \( Z \) in population terms. At time \( t \), \( Z_t \) corresponds to the size of the population generated by an immortal individual giving birth at rate \( \beta \) with sizes evolving independently as \( X \) under \( Q_0 \) and at rate 1 with intensity \( n(dx) \) with initial size \( x \) which evolve independently as \( X \) under \( P_x \). We first give a lemma on the family representation.

**Lemma 3.1.** Let \( f \) be a non-negative measurable function. We have

\[
(3.6) \quad \mathbb{E}\left[e^{-\sum_{j \in J} f(t_j, X^j)}\right] = e^{-\int_\mathbb{R} F(Q_0(1-e^{-f(t, X)})) dt}.
\]

**Proof.** Due to the independence of the Poisson random measures and the exponential formula, we have

\[
\mathbb{E}\left[e^{-\sum_{j \in J} f(t_j, X^j)}\right] = \mathbb{E}\left[e^{-\sum_{j \in J_1} f(t_j, X^j)}\right] \mathbb{E}\left[e^{-\sum_{j \in J_2} f(t_j, X^j)}\right] = \mathbb{E}\left[e^{-\sum_{i \in I} \sum_{j \in J_1} f(t_j, X^j)}\right] e^{-\beta \int_\mathbb{R} Q_0(1-e^{-f(t, X)}) dt} = \mathbb{E}\left[e^{-\sum_{i \in I} r_i Q_0(1-e^{-f(t, X)})}\right] e^{-\beta \int_\mathbb{R} Q_0(1-e^{-f(t, X)}) dt} = e^{-\int_\mathbb{R} F(Q_0(1-e^{-f(t, X)})) dt}.
\]

\( \square \)

The necessary and sufficient condition for which the CBI-process has a stationary version is that (A2) holds, see Theorem 3.20 in [24]. If (A2) holds, then \( X_t \) converges in distribution to \( X_\infty \) as \( t \to \infty \), with the distribution of \( X_\infty \) characterized by its Laplace transform

\[
(3.7) \quad \mathbb{E}[e^{-\lambda X_\infty}] = e^{-\int_0^\infty F(v_s(\lambda)) ds}.
\]

Then with (A2) in force, \( Z \) defined by (3.5) and (3.7) is the stationary version of \( Y \).

**Corollary 3.1.** Assume that (A2) holds. We have for \( \lambda > 0, t \in \mathbb{R}, \)

\[
(3.8) \quad \mathbb{E}[Z_t \exp(-\lambda Z_t)] = \frac{F(\lambda)}{\psi(\lambda)} \mathbb{E}[e^{-\lambda Z_t}].
\]

In particular, we have

\[
(3.9) \quad \mathbb{E}[Z_t] = \frac{F'(0)}{\psi'(0)}.
\]
Proof. We can see from (3.7) that
\[ \mathbb{E}[Z_t \exp(-\lambda Z_t)] = \mathbb{E}[\exp(-\lambda Z_t)] \int_0^\infty F(v_s(\lambda)) \, ds. \]

Using the forward equation (2.5) we can deduce that
\[ \partial_\lambda \int_0^\infty F(v_s(\lambda)) \, ds = \int_0^\infty \partial_\lambda F(x)|_{x=v_s(\lambda)} \partial_\lambda v_s(\lambda) \, ds \]
\[ = -\frac{1}{\psi(\lambda)} \int_0^\infty F'(v_s(\lambda)) \partial_\lambda v_s(\lambda) \, ds = \frac{F(\lambda)}{\psi(\lambda)}. \]

The second part is obvious. \( \square \)

In the following we will always suppose that (A1), (A2) and (A3) are in force.

4. Time to MRCA and the population sizes

With the decomposition procedure in force, we will follow the steps of Chen and Delmas [8]. We consider the coalescence of the genealogy at a fixed time \( t_0 \). We may as well assume that \( t_0 = 0 \) because of stationarity. There are infinitely many number of clans contributing to the population at time 0. We can further prove that there are only finite number of clans born before time \( a \) and still alive at time 0. Only one oldest clan is expected to be still alive at time 0.

First we will give the notations using the decomposition. \( -A \) is the birth time of the unique oldest clan at time 0 (A is also the time to the most recent common ancestor (MRCA) of the population at time 0) given by \( A := -\inf\{t_i \leq 0, X_{i-t_i} > 0, i \in I\} \); \( Z^O \) is the population size of this clan at time 0, i.e. \( Z^O := X_{i-t_i} \) if \( A = -t_i \). The size of all the clans alive at time 0 with birth time in \( (-A,0) \) is given by \( Z^I := Z_0 - Z^O \), and the size of the population just before the MRCA is given by \( Z^A := Z_{(-A)_-} = \sum_{i \in I} X_{i-A-t_i} 1_{\{t_i < -A\}} \).

Theorem 4.1. Let \( f: \mathbb{R} \to \mathbb{R}_+ \) be a measurable function. For \( \lambda, \gamma, \eta \geq 0 \), we have
\[ \mathbb{E}[(\exp(-\lambda Z^A - \gamma Z^I - \eta Z^O) f(A))] \]
\[ = \int_0^\infty dt \, f(t) (F(c(t)) - F(v_1(\gamma))) \exp \left(-\int_0^t F(v_s(\gamma)) \, ds - \int_0^\infty F(v_s(\lambda + c(t))) \, ds\right). \]

Proof. We have
\[ \mathbb{E}[(\exp(-\lambda Z^A - \gamma Z^I - \eta Z^O) f(A))] \]
\[ = \mathbb{E} \left[ \exp \left(-\lambda \sum_{i \in I} X_{i-t_i} - \gamma \sum_{i \in I, t_i > t_j} X_{i-t_i} - \eta X_{i-t_j} \right) \right] \]
\[ \times f(-t_j) 1_{\{X_{i-t_j} > 0, \sum_{i \in I, t_i < t_j} 1_{\{X_{i-t_i} > 0\}} = 0\}} \]
\[ = \int_0^\infty dt \, f(t) \mu(\exp(-\eta X_t) 1_{\{X_t > 0\}}) \mathbb{E} \left[ \exp \left(-\gamma \sum_{i \in I, t_i > -t} X_{i-t_i} \right) \right] \]
\[ \times \lim_{K \to \infty} \mathbb{E} \left[ \exp \left(-\lambda \sum_{i \in I, t_i < -t} (X_{i-t_i} - K 1_{\{X_{i-t_i} > 0\}}) \right) \right], \]

where the first equality is based on the values of \( A \) and the second one holds since Poisson point measures over disjoint sets are independent. We will calculate the terms respectively.
First we have
\[ \mu(e^{-\eta X_t} 1_{\{X_t > 0\}}) = \mu(1_{\{X_t > 0\}} - (1 - e^{-\eta X_t})). \]
Using the expression of \( \mu \), we have
\[
\begin{align*}
\mu(1_{\{X_t > 0\}}) &= \beta Q_0(X_t > 0) + \int_0^\infty n(dx) P_x(X_t > 0) \\
&= \beta c(t) + \int_0^\infty n(dx) (1 - P_x(X_t = 0)) \\
&= \beta c(t) + \int_0^\infty n(dx) (1 - e^{-x(t)}) = F(c(t)),
\end{align*}
\]
and
\[
\begin{align*}
\mu(1 - e^{-\eta X_t}) &= \beta Q_0(1 - e^{-\eta X_t}) + \int_0^\infty n(dx) P_x(1 - e^{-\eta X_t}) \\
&= \beta v_t(\eta) + \int_0^\infty n(dx) (1 - e^{-xv_t(\eta)}) = F(v_t(\eta)).
\end{align*}
\]
Second, using Lemma 3.1 we get
\[
\mathbb{E} \left[ \exp \left( -\gamma \sum_{i \in \mathcal{I}, t_i > -t} X_{t_i}^i \right) \right] = \exp \left( -\int_0^t F(v_s(\gamma)) \, ds \right).
\]
Finally we see that
\[
\begin{align*}
\lim_{K \to \infty} \mathbb{E} \left[ \exp \left( -\lambda \sum_{i \in \mathcal{I}, t_i < -t} (X_{t_i}^i + K 1_{\{X_{t_i}^i > 0\}}) \right) \right] \\
&= \exp \left( -\int ds 1_{\{s > 0\}} \mu(1 - e^{-\lambda X_s} 1_{\{X_{s+t} = 0\}}) \right) \\
&= \exp \left( -\int ds 1_{\{s > 0\}} \mu(1 - e^{-\lambda X_s} P_{X_s}(X_t = 0)) \right) \\
&= \exp \left( -\int ds 1_{\{s > 0\}} \mu(1 - e^{-(\lambda + c(t))X_s}) \right) \\
&= \exp \left( -\int_0^\infty ds F(v_s(\lambda + c(t))) \right),
\end{align*}
\]
where we use the exponential formula for the Poisson point measure in the first equality and the Markov property of \( X \) in the second one.

Putting all the calculations together we obtain the result.

It is then straightforward to derive the distribution of the TMRCA \( A \).

**Corollary 4.1.** The distribution of \( A \) is given by
\[
\mathbb{P}(A \leq t) = \exp \left( -\int_0^t F(c(s)) \, ds \right) = \mathbb{E}[e^{-c(t)Z_0}],
\]
and \( A \) has density \( f_A \) with respect to the Lebesgue measure given by
\[
f_A(t) = 1_{\{t > 0\}} F(c(t)) \exp \left( -\int_0^t F(c(s)) \, ds \right).
\]
Proof. Using Theorem 4.1 and (2.6), we see that the first equality holds easily. The second one is immediate. □

We see that in this general case the expression of the distribution of $A$ is invariant compared with $[S]$. The next result is also a direct consequence of Theorem 4.1.

Corollary 4.2. Conditionally on $A$, the random variables $Z^I$, $Z^A$, $Z^O$ are independent.

We also derive from Theorem 4.1 the distribution and the mean of the population size just before MRCA. As can be seen from below that the expression for the Laplace transform is the same as that of $[8]$.

Corollary 4.3. Let $t > 0$. Then

$$
\mathbb{E}[e^{-\lambda Z^A} | A = t] = \frac{\mathbb{E}[e^{-(\lambda + c(t))Z_0}]}{\mathbb{E}[e^{-c(t)Z_0}]} \quad \text{and} \quad \mathbb{E}[Z^A | A = t] = \frac{F(c(t))}{\psi(c(t))}.
$$

Proof. This is a direct consequence of Theorems 4.1 and Corollary 4.1. □

We can further deduce that conditionally on $\{A = t\}$, the distribution of $Z^A$ converges to the distribution of $Z_0$ as $t \to \infty$.

Another application of Theorem 4.1 we call the bottleneck effect, is that the size of the population just before MRCA is stochastically smaller than that of the current population. Note that this inequality does not hold in the almost surely sense in general. The proof is the same as that of $[S]$.

Corollary 4.4. For all $z \geq 0$ and $t \geq 0$, we have $\mathbb{P}(Z^A \leq z | A = t) \geq \mathbb{P}(Z_0 \leq z)$. Hence the population size $Z^A$ is stochastically smaller than $Z_0$, that is $\mathbb{P}(Z^A \leq z) \geq \mathbb{P}(Z_0 \leq z)$, for all $z \geq 0$. In particular we have $\mathbb{E}[Z^A | A = t] \leq \mathbb{E}[Z_0]$.

Remark 4.1. Instead of considering the size of the population just before MRCA, we consider the size at MRCA, $Z^A_+$, which is given as $Z^A_+ = Z^A + \sum_{i \in I} X^I_0 1_{\{t_i = -A\}}$. We don’t take into account the contribution of $i \in J_2$ since for those we have $X^I_0 = 0$. Similar calculations as those of Theorem 4.1 show that for $\lambda, t > 0$,

$$
\mathbb{E}[e^{-\lambda Z^A_+} | A = t] = \mathbb{E}[e^{-\lambda Z^A} | A = t] \frac{F(\lambda + c(t)) - F(\lambda)}{F(c(t))}.
$$

If $F'(0) = \infty$, then $\lim_{t \to \infty} \mathbb{E}[e^{-\lambda Z^A_+} | A = t] = 0$, which means that conditionally on $\{A = t\}$, $Z^A_+$ is likely to be very large, as $t \to \infty$. We can interpret it as this: a clan is born at time $-t$ and it survives up to time $0$, if $t$ is large enough, it is likely to have a large initial size. Therefore, $Z^A_+$ is not stochastically smaller than $Z_0$ in general.

5. The number of oldest families

In this section we will consider the number of families in the oldest clan alive at time 0. It is equivalent as that of individuals involved in the last coalescent event of the genealogical tree. We will use the family representation in this section.

The number of oldest families alive at time 0 (excluding the immortal individual) is defined as:

$$
N^A = \sum_{j \in J} 1_{\{A = -t_j, X^A_{-t_j} > 0\}} = \sum_{j \in J} 1_{\{A = -t_j, \zeta_j > -t_j\}}.
$$

Obviously $N^A \geq 1$. In particular when $\beta > 0$ and the measure $n \equiv 0$, we have $N^A = 1$.

The following theorem gives the joint distribution of $A, N^A$ and $Z_0$. 

**Theorem 5.1.** Let $0 \leq a \leq 1$. For any non-negative measurable function $f$, we have

$$
\mathbb{E}[a^{N^A} e^{-\lambda Z_0} f(A)] = \int_0^\infty ds \ f(s) \exp \left( - \int_0^s F(v_r(\lambda)) dr - \int_s^\infty F(c(r)) dr \right) 
\times \left( F(c(s)) - F((1-a)c(s) + av_s(\lambda)) \right).
$$

**Proof.** For $i \in \mathcal{I}$, set

$$
J_i^* = \begin{cases} 
J_{1,i}, & \text{if } i \in I, \\
\{i\}, & \text{if } i \in \mathcal{J}_2.
\end{cases}
$$

For $f$ non-negative measurable, we have

$$
\mathbb{E}[a^{N^A} e^{-\lambda Z_0} f(A)]
= \mathbb{E} \left[ e^{-\lambda \sum_{k \in \mathcal{I}, t_k < s} X^k_{t_k}} \sum_{i \in \mathcal{I}} a^{\sum_{j \in J_i^*}} 1_{\{t_k > t_i\}} \ f(-t_i) 1_{\{X_{t_i} > 0\}} \prod_{j \in \mathcal{I}} 1_{\{t_k < s, X^k_{t_k} > 0\}} \right] 
\times \left( \beta \mathbb{Q}_0[a e^{-\lambda X_s} 1_{\{X_s > 0\}}] + \int_0^\infty n(dx) \mathbb{E}_x \left[ a^{\sum_{j \in J_3} 1_{\{X^j_s > 0\}}} e^{-\lambda \sum_{j \in J_3} X^j_s} 1_{\{\sum_{j \in J_3} X^j_s > 0\}} \right] \right),
$$

where the first equality is based on the decomposition of $A$, the second on splitting $t_k$ into three parts: $t_k > s$, $t_k < s$ and $t_k = s$, and $\sum_{j \in J_3} \delta_{X_j}(dX)$ is a Poisson point measure with intensity $x \mathbb{Q}_0(dX)$ under $\mathbb{P}$. We will calculate the terms separately.

By Lemma [3.1] we have

$$
\mathbb{E} \left[ \exp \left( - \lambda \sum_{k \in \mathcal{I}, t_k < s} X^k_{t_k} 1_{\{t_k > s\}} \right) \right] = \exp \left( - \int_0^s F(v_r(\lambda)) dr \right),
$$

and

$$
\mathbb{P} \left( \sum_{k \in \mathcal{I}} 1_{\{t_k < s, X^k_{t_k} > 0\}} = 0 \right) = \exp \left( - \int_s^\infty F(c(r)) dr \right).
$$

The next equation is obtained by splitting the terms into two parts:

$$
\mathbb{E}_x \left[ a^{\sum_{j \in J_3} 1_{\{X^j_s > 0\}}} \exp \left( - \lambda \sum_{j \in J_3} X^j_s \right) 1_{\{\sum_{j \in J_3} X^j_s > 0\}} \right] 
= \mathbb{E}_x \left[ a^{\sum_{j \in J_3} 1_{\{X^j_s > 0\}}} \exp \left( - \lambda \sum_{j \in J_3} X^j_s \right) \right] - \mathbb{P}_x \left( \sum_{j \in J_3} X^j_s = 0 \right).
$$

The first part is calculated as follows:

$$
\mathbb{E}_x \left[ a^{\sum_{j \in J_3} 1_{\{X^j_s > 0\}}} e^{-\lambda \sum_{j \in J_3} X^j_s} \right] = \exp \left( - x \mathbb{Q}_0[1 - a 1_{\{X_s > 0\}} e^{-\lambda X_s}] \right)
= \exp \left( - x \mathbb{Q}_0[X_s > 0] + xa \mathbb{Q}_0[1_{\{X_s > 0\}} e^{-\lambda X_s}] \right)
= \exp \left( - x [(1-a)c(s) + av_s(\lambda)] \right).
$$
The second part is
\[ \mathbb{P}_x \left( \sum_{j \in J_3} X_j = 0 \right) = \lim_{\lambda \to \infty} e^{-xQ_0(1-e^{-\lambda X_3})} = \lim_{\lambda \to \infty} e^{-xv_3(\lambda)} = e^{-xc(s)}. \]

For the last part, we see that
\[ Q_0[e^{-\lambda X_3} 1_{\{X_3 > 0\}}] = Q_0[X_3 > 0] - Q_0[1 - e^{-\lambda X_3}] = c(s) - v_3(\lambda). \]

Putting all the calculations together we see that
\[ E[a^A e^{-\lambda Z_0} f(A)] = \int_0^\infty ds f(s) \exp \left( - \int_0^s F(v_\tau(\lambda)) d\tau - \int_s^\infty F(c(r)) dr \right) \times \left( F(c(s)) - F((1-a)c(s) + av_3(\lambda)) \right). \]

This finishes the proof. \[\square\]

Using the density of \( A \), the following corollary is immediate.

**Corollary 5.1.** For \( 0 < a < 1, \lambda, t \geq 0 \), we have
\[ E[a^A e^{-\lambda Z_0} 1_{A = t}] = \frac{F(c(t)) - F((1-a)c(t) + av_3(\lambda))}{F(c(t))} \exp \left( - \int_0^t F(v_\tau(\lambda)) d\tau \right) \]
and
\[ E[a^A | A = t] = \frac{F(c(t)) - F((1-a)c(t))}{F(c(t))} = 1 - \frac{F((1-a)c(t))}{F(c(t))}. \]

The next corollary is direct from Corollary 5.1

**Corollary 5.2.** We have for \( n \geq 1 \),
\[ \mathbb{P}[N^A = n | A = t] = (-1)^{n+1} \frac{c(t)^n F(n)(c(t))}{n! F(c(t))}. \]
Then \( E[N^A | A = t] = \frac{F'(0+)(c(t))}{F(c(t))} \in [0, \infty] \). In addition if \( F'(0+) < \infty \), the function \( t \mapsto E[N^A | A = t] \) is non-increasing.

Notice that if we let \( F(t) = ct^\alpha \), \( c > 0 \) and \( 0 < \alpha < 1 \), then the conditional distribution will not depend on the CB-process but only on the immigration structure.

## 6. The number of ancestors at a fixed time

In this section we will consider the number of ancestors \( M_s \) at time \( -s \) of the current population living at time 0 and how fast it tends to infinity. To answer this question we need to introduce the genealogy of the families which is a richer structure studied in [9, 10, 2, 1].

### 6.1. Genealogy of CB-process.**

The construction developed by Duquesne and Le Gall [9, 10] for (sub)-critical CB-process is well known. Results in [10] is restated in the framework of the measured rooted real trees, see [2]. We will follow Section 2 in [1].
6.1.1. **Real tree framework.** A metric space \((T, d)\) is a real tree if the following two properties hold for every \(s, t \in T\),

- (unique geodesic) There is a unique isometric map \(f_{s,t}\) from \([0, d(s, t)]\) into \(T\) such that \(f_{s,t}(0) = s\) and \(f_{s,t}(d(s, t)) = t\).
- (no loop) If \(q\) is a continuous injective map from \([0,1]\) into \(T\) such that \(q(0) = s\) and \(q(1) = t\), we have \(q([0,1]) = f_{s,t}([0, d(s, t)])\).

A rooted real tree is a real tree \((T, d)\) with a distinguished vertex \(\emptyset\) called the root. Denote such a tree by \((T, d, \emptyset)\). If \(s, t \in T\), we will note \([s, t]\) the range of the isometric map \(f_{s,t}\) described above. We also denote \([s, t] = [s, t] \setminus \{t\}\).

If \(x \in T\), the degree of \(x\), \(n(x)\), is the number of connected components of the set \(T \setminus \{x\}\). The set of leaves is defined as \(Lf(T) = \{x \in T \setminus \emptyset \mid n(x) = 1\}\). The skeleton of \(T\) is the set of points in the tree that are not leaves: \(Sk(T) = T \setminus Lf(T)\).

For every \(x \in T\), \([0, x]\) is interpreted as the ancestral line of vertex \(x\) in the tree. If \(x, y \in T\), there exists a unique \(z \in T\), called the Most Recent Common Ancestor (MRCA) of \(x, y\), such that \([0, x] \cap [0, y] = [0, z]\). Then the root can be seen as the ancestor of all the population in the tree. We shall call the height of \(x\), \(h(x)\), the distance \(d(\emptyset, x)\) to the root. The function \(x \mapsto h(x)\) is continuous on \(T\), and we define the height of \(T\) by \(H_{\max}(T) = \sup_{x \in T} h(x)\).

6.1.2. **Measured rooted real trees.** We will denote by \(\mathcal{T}\) the set of the measured rooted real trees \((T, d, \emptyset, m)\) where \((T, d, \emptyset)\) is a locally compact rooted real tree and \(m\) is a locally finite measure on \(T\). We may simply write \(T\) in case of no confusion.

Let \(T \in \mathcal{T}\). For \(a \geq 0\), we set \(T(a) = \{x \in T \mid d(\emptyset, x) = a\}\) for the level set at height \(a\), and \(\pi_a(T) = \{x \in T \mid d(\emptyset, x) \leq a\}\) for the truncated tree \(T\) up to level \(a\). We consider \(\pi_a(T)\) with the root \(\emptyset\), \(d_{\pi_a(T)}\) and \(m_{\pi_a(T)}\) are the restrictions of \(d\) and \(m\) to \(\pi_a(T)\). Let \((T^k, k \in \mathcal{K})\) be the connected components of \(T \setminus \pi_a(T)\). Denote by \(\emptyset_k\) the MRCA of all the vertices of \(T^k\). Set \(T^k = T^k \cup \{\emptyset_k\}\) which is a real tree rooted at point \(\emptyset_k\) with mass measure \(m^T_k\) defined as the restriction of \(m^T\) to \(T^k\). We will consider the point measure on \(T \times T\):

\[
N_a^T = \sum_{k \in \mathcal{K}} \delta_{(\emptyset_k, T^k)}.
\]

6.1.3. **Excursion measure of Lévy tree.** Recall that \(\psi\) is a (sub)-critical branching mechanism. There exists a \(\sigma\)-finite measure (or an excursion measure of Lévy tree) \(N[dT]\) on \(\mathbb{T}\), with the following properties:

(i) (Height). \(\forall a > 0, N[H_{\max}(T) > a] = c(a)\).
(ii) (Mass measure). The mass measure \(m^T\) is supported on \(Lf(T), N[dT]\)-a.e.
(iii) (Local time). There exists a \(T\)-measure valued process \((\ell^a, a \geq 0)\) c\(\ddash\)ad\(\ddash\) for the weak topology on finite measure on \(T\) such that \(N[dT]\)-a.e.:

\[
m^T(dx) = \int_0^\infty \ell^a(dx) da,
\]

\(\ell^0 = 0, \inf\{a > 0; \ell^a = 0\} = \sup\{a \geq 0; \ell^a \neq 0\} = H_{\max}(T)\) and for every fixed \(a > 0\), \(N[dT]\)-a.e.:

- The measure \(\ell^a\) is supported on \(T(a)\).
We have for every bounded continuous function $\phi$ on $T$:
\[
\langle \ell^a, \phi \rangle = \lim_{\epsilon \downarrow 0} \frac{1}{v(\epsilon)} \int \phi(x) \mathbf{1}_{H_{\max}(T') \geq \epsilon} N_{a}^{T'}(dx, dT')
\]
\[
= \lim_{\epsilon \downarrow 0} \frac{1}{v(\epsilon)} \int \phi(x) \mathbf{1}_{H_{\max}(T') \geq \epsilon} N_{a-\epsilon}^{T'}(dx, dT').
\]

Under $\mathbb{N}$, the process $\langle \ell^a, 1 \rangle$, $a \geq 0$ is distributed as $X$ under $Q_0$.

(iv) (Branching property). For every $a > 0$, the conditional distribution of the point measure $N_{a}^{T}(dx, dT')$ under $\mathbb{N}[dT|H_{\max}(T) > a]$, given $\pi_a(T)$, is that of a Poisson point measure on $T(a) \times T$ with intensity $\ell^a(dx)\mathbb{N}[dT']$.

In order to simplify notations, we will identify $X$ with $\langle \ell^a, 1 \rangle$, $a \geq 0$ as well as $Q_0$ with $\mathbb{N}$.

We give a definition for the number of ancestors.

**Definition 6.1.** The number of ancestors at time $a$ of the population living at time $b$ is the number of subtrees above level $a$ which reach level $b > a$:

\[
R_{a,b}(T) = \sum_{k \in K} \mathbf{1}_{H_{\max}(T^k) \geq b-a}.
\]

6.2. **Genealogy of stationary CBI-process.** We use (5.2) to construct the genealogy of $Z_0$.

- Conditionally on $N_0$, let $\tilde{N}_1(dt, dT) = \sum_{j \in J_1} \tilde{\delta}_{(t_j, T_j)}(dt, dT)$ be a Poisson point measure with intensity $v(dt)Q_0(dT)$ with $v(dt) = \sum_{i \in I} r_i \tilde{\delta}_{t_i}(dt)$.
- Let $\tilde{N}_2(dt, dT) = \sum_{j \in J_2} \delta_{(t_j, T_j)}(dt, dT)$ be a Poisson point measure independent of $N_0, \tilde{N}_1$ with intensity $\beta dt Q_0(dT)$.

We will write $X^j$ for $\ell^a(T^j)$ for $j \in J$. Thus notation (5.2) is still consistent with the previous sections. $\sum_{j \in J_2} \delta_{(t_j, T_j)}$ allows to code the genealogy of the family of $Z_0$ in the Lévy tree sense.

Let $s > 0$, we will consider the number of ancestors at time $-s$ of the current population living at time $0$, that is,

\[
M_s = \sum_{j \in J} \mathbf{1}_{t_j < -s} R_{-s-t_j,-t_j}(T^j).
\]

6.3. **Asymptotic for the number of ancestors.** First we present the following theorem.

**Theorem 6.1.** The conditional joint distribution of $M_s$ and $Z_0$ is: for $\eta, \lambda \geq 0$, $s > 0$,

\[
\mathbb{E}[e^{-\eta M_s - \lambda Z_0} | Z_{-s}] = e^{-\int_0^s F(v_r(\lambda)) \, dr} e^{-Z_{-s}[(1-e^{-\eta})c(s)+e^{-\eta} v_s(\lambda)]}.
\]

In particular, conditionally on $Z_{-s}$, $M_s$ is distributed as a Poisson random variable with parameter $c(s)Z_{-s}$.
Proof. For any $\eta, \lambda \geq 0$, we have
\[
\mathbb{E}[e^{-bZ_s-\eta M_s-\lambda Z_0}]
= \mathbb{E}\left[\exp\left(-\lambda \sum_{j \in J} 1_{\{-s < t_j \leq 0\}} X^j_{t_j}\right)\right] \mathbb{E}\left[\exp\left(-bZ_s-\eta M_s-\lambda \sum_{j \in J} 1_{\{t_j \leq -s\}} X^j_{t_j}\right)\right]
= \exp\left(-\int_0^s F(v(r)) \, dr\right)
\times \mathbb{E}\left[\exp\left(-\sum_{j \in J} 1_{\{t_j \leq -s\}}(b X^j_{s-t_j} + \eta R_{s-t_j,-t_j}(T^j) + \lambda X^j_{t_j})\right)\right]
= \exp\left(-\int_0^s F(v(r)) \, dr\right) \exp\left(-\int_0^\infty da F\left(Q_0\left(1-e^{-bX_a-\eta R_{a,a+s}(T)-\lambda X_{a+s}}\right)\right)\right).
\]
where in the first equality we use the fact that the Poisson point measures over disjoint sets are independent, in the second one we use Lemma 3.1 and we use an immediately generalization of Lemma 3.1 to genealogies in the third equality. Using branching property we have
\[
Q_0\left[1-e^{-bX_a-\eta R_{a,a+s}(T)-\lambda X_{a+s}}\right] = Q_0\left[1-e^{-bX_a-\sum_{k \in K} \eta_1(H_{\text{max}}(T^k) \geq s)-\lambda X(T^k)_s}\right]
= Q_0\left[1-e^{-X_a(b+Q_0[1-\exp(\eta_1(H_{\text{max}}(T^k) \geq s)-\lambda X_s])}\right] .
\]

Since $X_s = 0$ on $\{H_{\text{max}}(T) < s\}$, we have $1-e^{-\eta_1(H_{\text{max}}(T) \geq s)-\lambda X_s} = (1-e^{-\eta})1_{(\zeta \geq s)} + e^{-\eta}(1-e^{-\lambda X_s})$. Then we deduce that
\[
Q_0\left[1-e^{-bX_a-\eta R_{a,a+s}(T)-\lambda X_{a+s}}\right] = Q_0\left[1-e^{-\lambda' X_a}\right] = v_a(\lambda')
\]
with $\lambda' = b + (1-e^{-\eta})c(s) + e^{-\eta} v_a(\lambda)$. Then we get the result.

Intuitively $M_s$ counts the number of excursions of the height process at time $-s$ above level $s$. Similar result as that of Duquesne and Le Gall 10 can be deduced here.

Corollary 6.1. The following convergence holds:
\[
\lim_{s \to 0} \frac{M_s}{c(s)} = Z_0, \quad a.s.
\]

7. The MRCA age process and the zero set of the CBI-process

In this section we will deal with the MRCA age process $(A_t, t \in \mathbb{R})$ and the zero set of the CBI-process $Z = \{t \in \mathbb{R}, Z_t = 0\}$ by using the sample path decomposition of the CBI-process.

For the clan decomposition of the CBI-process shown in Section 3, we have used the Poisson point process $N_3(dt, dX) = \sum_{i \in I} \delta_{(t_i, X_i)}(dt, dX)$ with intensity $dt \mu(dX)$, where $\mu(dX) = \beta Q_0(dX) + \int_{(0,\infty)} n(dx)P_\lambda(dx)$. The corresponding Poisson point process which is in charge of the birth time and duration time is $N_4(dt, d\zeta) = \sum_{i \in I} \delta_{(t_i, \zeta_i)}(dt, d\zeta)$ with intensity $dt \mu(\zeta \in \cdot)$, where $\mu(\zeta > t) = F(c(t))$.

7.1. The MRCA age process. Define the left leaning wedge with apex at $(t, r)$ by
\[
\Delta(t, r) := \{(u, v) : u < t \text{ and } u + v > r\},
\]
which is the set of points that give birth before $t$ and is still alive at time $r$. We can thus define the MRCA age process $(A_t, t \in \mathbb{R})$ as follows:
\[
A_t := t - \inf\{s : \exists \zeta > 0, \text{ such that } (s, \zeta) \in \{(t_i, \zeta_i) : i \in I\} \cap \Delta(t, 0)\}.
\]
The strong Markov property of the Poisson point processes \( N_t \) implies that \( (A_t, t \in \mathbb{R}) \) is a time homogeneous Markov process. The transition probabilities of the MRCA age process can be proved along the same lines as that of part (a) of Theorem 1.1 in Evans and Ralph [11].

**Theorem 7.1.** The transition probabilities of the Markov process \( (A_t, t \in \mathbb{R}) \) are separated into two parts:

- for \( 0 < y < x + t \), the continuous part
  
  \[ P(A_{x+t} \in dy | A_x = x) = \left( 1 - \frac{F(c(x+t))}{F(c(x))} \right) e^{-\int_{x}^{x+t} F(c(u)) \, du} \frac{F(c(y))}{F(c(x))} \, dy, \]

- otherwise, a single atom \( P(A_{x+t} = x + t | A_x = x) = \frac{F(c(x+t))}{F(c(x))} \).

7.2. **The zero set of the CBI-process.** From the clan decomposition, we have

- either \( Z_t \neq 0 \), for \( t \in [t_i, t_i + \zeta_i), i \in I \);
- or \( Z_t \neq 0 \), for \( t \in (t_i, t_i + \zeta_i), i \in J_2 \).

Then we have

\[ \{t, Z_t = 0\} = \mathbb{R} \setminus \bigcup_{i \in I} (t_i, t_i + \zeta_i) \cup \bigcup_{i \in J_2} (t_i, t_i + \zeta_i). \]

Then we can derive that the zero set of the CBI-process is a random renewal set with the following equation:

\[ Z = \mathbb{R} \setminus \bigcup_{i \in I} (t_i, t_i + \zeta_i). \]

To see why this is always true, we only need to focus on those points that belong to \( \mathbb{R} \setminus \bigcup_{i \in I} (t_i, t_i + \zeta_i) \) but not to \( \{t, Z_t = 0\} \). These points actually are the left accumulation points of \( \{t, Z_t = 0\} \). Indeed suppose that this is not true. Let \( \{t_i\} \) be such point. Then for any \( t_i \), there exists \( \epsilon_i > 0 \) such that \( [t_i - \epsilon_i, t_i) \cap \{t, Z_t = 0\} = \emptyset \). This is impossible since after taking the closure we obtain that \( [t_i - \epsilon_i, t_i] \cap \mathbb{Z} = \emptyset \); while \( t_i \in \mathbb{Z} \).

We derive the following theorem.

**Theorem 7.2.**

1. \( Z = \emptyset \) if and only if \( \int_0^1 \exp \left( \int_t^\infty F(c(u)) \, du \right) \, dt = \infty \).
2. If \( \int_0^1 \exp \left( \int_t^\infty F(c(u)) \, du \right) \, dt < \infty \), the random set \( Z \) has a positive Lebesgue measure a.s. if and only if \( \int_0^\infty \frac{F(t)}{\psi(t)} \, dt < \infty \); the random set \( Z \) is the union of the closed intervals of positive lengths if and only if \( \beta = 0 \) and \( n(dx) < \infty \).

**Proof.** We have known that \( \mu(\zeta > t) = F(c(t)) \). We can derive directly from Corollary 5 in [15] that the first assertion holds.

With a slightly modification of Proposition 1 in [15] or Proposition 1.22 in [14], we can obtain that \( Z \) has a positive Lesbesgue measure if and only if

\[ \int_0^\infty t \mu(\zeta \in dt) = \int_0^\infty \mu(\zeta > t) \, dt = \int_0^\infty F(c(t)) \, dt < \infty. \]

Letting \( r = c(t) \) in the above equation yields \( \int_0^\infty \frac{F(t)}{\psi(t)} \, dt < \infty \). In order to derive the last statement, we use Corollary 2 in [15], which requires \( \mu(\zeta \in dt) \) to be finite, i.e. \( F(c(t)) \) is finite, as \( t \to 0^+ \). Since we have \( c(t) \to \infty \) as \( t \to 0^+ \), we need \( \beta = 0 \) and \( n((0, \infty)) < \infty \).

Easy calculation gives a simple example that \( \psi(u) = 2\beta u + \beta u^2 \) and \( F(u) = 2\beta u \) satisfying condition (1), and \( Z = \emptyset \). Another example is given in [13] in stable case.
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