Maximum Principles for Laplacian and Fractional Laplacian with Critical Integrability

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Received: 19 October 2022 / Accepted: 20 March 2023 / Published online: 20 April 2023
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Abstract
In this paper, we study the maximum principles for Laplacian and fractional Laplacian with critical integrability. We first consider the critical cases for Laplacian with zero-order term and first-order term. It is well known that for the Laplacian with zero-order term $-\Delta + c(x)$ in $B_1$, $c(x) \in L^p(B_1)(B_1 \subset \mathbb{R}^n)$, the critical case for the maximum principle is $p = \frac{n}{2}$. We show that the critical condition $c(x) \in L^\frac{n}{2}(B_1)$ is not enough to guarantee the strong maximum principle. For the Laplacian with first-order term $-\Delta + \vec{b}(x)\vec{b}(x) \in L^p(B_1)$, the critical case is $p = n$. In this case, we establish the maximum principle and strong maximum principle for Laplacian with first-order term. We also extend some of the maximum principles above to the fractional Laplacian. We replace the classical lower semi-continuous condition on solutions for the fractional Laplacian with some integrability condition. Then we establish a series of maximum principles for fractional Laplacian under some integrability condition on the coefficients. These conditions are weaker than the previous regularity conditions. The weakened conditions on the coefficients and the non-locality of the fractional Laplacian bring in some new difficulties. Some new techniques are developed.

Keywords Maximum principles · Laplacian · Fractional Laplacian · Critical integrability

Mathematics Subject Classification 35B50 · 35D30 · 35J15
1 Introduction

Maximum principles are fundamental tools in the study of partial differential equations. The classical maximum principle for harmonic functions and subharmonic functions can be traced to the work of Gauss [22]. Hopf established the classical strong maximum principle in [24, 25] which is a basic building block for the analysis of the second-order elliptic partial differential equations. Later, various versions of maximum principles have been discussed by many researchers, as in Littman [30], Ancona [1], Brezis and Ponce [9], Pucci and Serrin [34, 35], Vitolo [45], and Cavaliere and Transirico [15].

In the last few decades, the Schrödinger operator \(-\Delta + c(x)\) has attracted a lot of attention from many scientists, where \(c(x)\) is a given potential in an open connected set \(\Omega \subset \mathbb{R}^n\). The classical strong maximum principle states that under the certain condition on \(c(x)\), if \(u\) satisfies \(-\Delta u + cu \geq 0\), \(u \geq 0\), and \(u(x_0) = 0\) for some point \(x_0 \in \Omega\), then \(u \equiv 0\) in \(\Omega\). It is known that the strong maximum principle holds if \(c(x) \in L^p(\Omega)\) for some \(p > \frac{n}{2}\) (See Serrin [38], Stampacchia [40], Trudinger [43]). Later, the assumption for \(c(x)\) is weakened if adding some vanishing condition on \(u\). Ancona [1], Brezis and Ponce [9] showed that \(u \equiv 0\) a.e. in \(\Omega\) if \(c(x) \in L^1(\Omega)\), \(c \geq 0\) a.e. in \(\Omega\), and there exists a quasi-continuous function related to \(u\) vanishes on a set of positive \(H^1\)-capacity in \(\Omega\), respectively. In [32], Orsina and Ponce proved that for \(p > 1\) and \(c(x) \in L^p(\Omega)\), \(u(x) = 0\) a.e. in \(\Omega\) if \(u(x)\) satisfies vanishing condition \(\lim_{r \to 0} \frac{1}{B(x,r)} \int_{B(x,r)} u(x) dx = 0\) for every point \(x\) in a compact subset of \(\Omega\) with positive \(W^{2,p}\) capacity. In [6], Bertsch, Smarrazzo and Tesei gave a necessary and sufficient condition for the validity of the strong maximum principle in one dimension.

A question we focus on is that whether the critical integrability condition for \(c(x)\) can ensure the classical strong maximum principle for the Schrödinger operator. The strong maximum principle for the Schrödinger operator asserted in [38, 40, 43] requires that \(c(x) \in L^p(\Omega)\) for some \(p > \frac{n}{2}\). It is also known that the strong maximum principle fails for \(p < \frac{n}{2}\), and a counterexample is the function \(u(x) = |x|^2\) which satisfies \(-\Delta u + cu = 0\) in \(B_1 \subset \mathbb{R}^n\) with \(c(x) = \frac{2n}{|x|^2}\). Then the remaining case is if the strong maximum principle for the Schrödinger operator still holds when \(p = \frac{n}{2}\).

One of the key results in this paper is to answer this question, and we show that \(c(x) \in L^{\frac{n}{2}}(B_1)\) is not enough to ensure the strong maximum principle for the Schrödinger operator no matter how small \(\|c\|_{L^{\frac{n}{2}}(B_1)}\) is.

More precisely, our first main result can be stated as follows.

**Theorem 1.1** Let \(n \geq 3\). There exist \(c_\epsilon(x) \in L^{\frac{n}{2}}(B_1)\) and \(u_\epsilon(x) \in H^1(B_1) \cap C(\bar{B}_1)\), \(\epsilon > 0\) such that

\[-\Delta u_\epsilon(x) + c_\epsilon(x)u_\epsilon(x) = 0\quad\text{in } B_1,\]

and

(i) \(u_\epsilon(x) \geq 1\) on \(\partial B_1\),
\[(ii) u_\varepsilon(0) = 0, \]
\[(iii) \lim_{\varepsilon \to 0^+} ||c_\varepsilon||_{L^\frac{n}{2}(B_1)} = 0.\]

We consider the maximum principle for Laplacian with both zero-order and first-order terms: \(-\Delta + \tilde{b}(x) \cdot \nabla + c(x)\). The critical integrability conditions are \(\tilde{b} \in L^n(B_1)\) and \(c \in L^\frac{n}{2}(B_1)\). We establish the maximum principle for the critical case in the following theorem.

**Theorem 1.2** Assume that \(\tilde{b}(x) \in L^n(B_1), c(x) \in L^\frac{n}{2}(B_1)\) and \(u(x) \in H^1(B_1)(n \geq 3)\) is a weak solution of

\[
\begin{cases}
-\Delta u(x) + \tilde{b}(x) \cdot \nabla u(x) + c(x) u(x) \geq 0 & \text{in } B_1 \\
u(x) \geq 0 & \text{on } \partial B_1.
\end{cases}
\]

(2)

There exists a positive constant \(k(n)\) such that if \(\|\tilde{b}\|_{L^n(B_1)} + \|c^-\|_{L^\frac{n}{2}(B_1)} \leq k(n)\), then \(u(x) \geq 0\) in \(B_1\). Here \(c^-(x) = -\min[c(x), 0]\).

**Remark 1.1** Note that the maximum principle for the Schrödinger operator (Laplacian with zero-order term) is not true if \(n = 2\). For instance, one can take \(u_\varepsilon(x) = -\ln(\varepsilon |\ln| x ||)\) changing sign in \(B_1, \varepsilon > 0\), with corresponding \(\|c_\varepsilon\|_{L^1(B_1)} \to 0\) as \(\varepsilon \to 0^+\).

**Remark 1.2** Actually, it follows from Remark 1.1 that the strong maximum principle for the Schrödinger operator with critical integrability \(c(x) \in L^\frac{n}{2}(B_1)\) is not true if \(n = 2\).

Next, we give a refined version of the strong maximum principle for the Schrödinger operator which is useful for analysis of PDEs in practice. Various refined versions of the strong maximum principle have been studied, see [28, 29] and references therein. Note that in [29], they required that the corresponding coefficient \(c(x)\) is bounded.

The following theorem is not really new. For convenience and completeness, we present it here.

**Theorem 1.3** Assume that \(c(x) \in L^p(B_1)(p > \frac{n}{2}),\) and \(u(x) \in H^1(B_1)(n \geq 3)\) is a weak solution of

\[
\begin{cases}
-\Delta u(x) + c(x) u(x) \geq 0 & \text{in } B_1 \\
u(x) \geq m > 0 & \text{on } \partial B_1.
\end{cases}
\]

(3)

There exists a positive constant \(k(n, p)\) such that if \(\|c^+\|_{L^p(B_1)} \leq k(n, p)\), then

\[u(x) \geq \gamma m \text{ in } B_1\]

where \(c^+(x) = \max\{c(x), 0\}\), and \(\gamma\) is a positive constant depending only on \(n, p\) and \(\|c^+\|_{L^p(B_1)}\).
As a corollary of Theorem 1.2, we deduce the following strong maximum principle for Laplacian with first-order term. The crucial observation is that compared to Theorem 1.3, the strong maximum principle for Laplacian with first-order term holds in the critical condition \( \vec{b} \in L^n(B_1) \).

**Corollary 1.4** Assume that \( \vec{b}(x) \in L^n(B_1) \), and \( u(x) \in H^1(B_1)(n \geq 3) \) is a weak solution of

\[
\begin{cases}
-\Delta u(x) + \vec{b}(x) \cdot \nabla u(x) \geq 0 & \text{in } B_1 \\
u(x) \geq m > 0 & \text{on } \partial B_1.
\end{cases}
\]  

(4)

There exists a positive constant \( k(n) \) such that if \( \| \vec{b} \|_{L^n(B_1)} \leq k(n) \), then \( u(x) \geq m \) in \( B_1 \).

**Remark 1.3** Note that Theorems 1.2, 1.3 and Corollary 1.4 also hold in a general bounded and open domain with smooth boundary. The proofs on the ball can be carried to the general domain and we omit these here.

In the following, we consider the maximum principle and strong maximum principle for fractional Laplacian. Lots of efforts have been made on this study, see [11, 16, 18, 19, 21, 26, 28, 29, 31, 36, 39, 46–48] and references therein. As is well known, the standard Laplacian in an \( n \)-dimensional domain possesses an explanation with regard to the diffusion and occurs in differential equations that describe many physical phenomena. In recent years, there have been a lot of fruitful works on anomalous diffusion which is extensively observed in physics, chemistry, and biology. To characterize anomalous diffusion phenomena, the fractional Laplacian is introduced and has been widely used to model diverse physical phenomena, such as molecular dynamics, turbulence and water waves, and quasi-geostrophic flows [13, 14, 20, 41]. Furthermore, the fractional Laplacian has a long history and various applications in probability and finance [2, 5]. In particular, the fractional Laplacian can be understood as the infinitesimal generator of stable Lévy process.

In comparison with Laplacian which is a local operator, the fractional Laplacian is non-local and does not act by pointwise differentiation. However, the fractional Laplacian can be defined by a global integration with respect to a singular kernel, taking the form

\[
(-\Delta)^s u(x) = C_{n,s} P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} \, dy,
\]

where \( s \) is any real number between 0 and 1, and \( P.V. \) stands for the Cauchy principle value.

Note that the operator \( (-\Delta)^s \) is well defined if \( u(x) \in L^2_{2s} \cap C^{1,1}_{loc}(\mathbb{R}^n) \), where

\[
L_{2s} = \left\{ u : \mathbb{R}^n \to \mathbb{R} \left| \int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+2s}} \, dx < \infty \right. \right\}.
\]
On the other hand, if we consider \( u(x) \) in the sense of distribution, only the condition \( u(x) \in L_{2s} \) is required. More precisely, for any \( \psi(x) \in C_0^\infty(\mathbb{R}^n) \),

\[
(-\Delta)^su(\psi) = \int_{\mathbb{R}^n} u(x)(-\Delta)^s\psi(x)dx.
\]

Throughout this paper, if a function \( f(x) \) satisfies \( f(x) \geq 0 \) in \( B_1 \) in the sense of distribution, then we write

\[
f(x) \geq 0 \quad \text{in} \quad \mathcal{D}'(B_1).
\]

For a long time, it is difficult to investigate the fractional Laplacian due to its non-locality. Many scientists made strong research efforts to extend the works on Laplacian to fractional Laplacian. Caffarelli and Silvestre [12] introduced the extension method, which transforms a non-local problem to a local one in higher dimensions. This method has been used to show the maximum principle and strong maximum principle for fractional Laplacian by many researchers, see [11, 21, 36] and references therein.

Here, we study the maximum principle and strong maximum principle for fractional Laplacian with critical integrability. Instead of the extension method introduced by Caffarelli and Silvestre, we work directly on the fractional Laplacian. An important issue is to study the maximum principle for fractional superharmonic functions. In [39, Proposition 2.17], Silvestre established the following maximum principle.

**Proposition 1.5** Let \( \Omega \) be an open set in \( \mathbb{R}^n \). Let \( u(x) \in L_{2s} \) be a lower semi-continuous function in \( \bar{\Omega} \) such that

\[
\begin{cases}
(-\Delta)^su(x) \geq 0 & \text{in} \ \Omega \\
u(x) \geq 0 & \text{in} \ \Omega^c,
\end{cases}
\]

(5)

in the sense of distribution. Then \( u(x) \geq 0 \) in \( \mathbb{R}^n \).

Later, Chen et al. [16, Theorem 2.1] provided a simpler proof for Proposition 1.5 by adding the regularity condition \( u(x) \in C^{1,1}_{loc}(\Omega) \). They proved the maximum principle for fractional superharmonic functions by using the integral definition of the fractional Laplacian directly. Note that the maximum principles for fractional superharmonic functions proved in [16, 39] require that the functions are lower semi-continuous on \( \bar{\Omega} \).

In the next theorem, we improve the results in [16, 39] by weakening the assumptions on \( u \).

**Theorem 1.6** Assume that \( u(x) \in L_{2s} \cap L^{\frac{1}{1-s}}(B_1) \) satisfies

\[
\begin{cases}
(-\Delta)^su(x) \geq 0 & \text{in} \ \mathcal{D}'(B_1) \\
u(x) \geq 0 & \text{in} \ B_1^c.
\end{cases}
\]

(6)

Then \( u(x) \geq 0 \) in \( B_1 \).
Remark 1.4 Li and Liu [27] showed the uniqueness of \( u(x) \) if \( u(x) \in L_{2,s} \cap L^{\frac{1}{1-s}}(B_1) \) satisfies \( (-\Delta)^s u(x) = 0 \) in \( D'(B_1) \) and \( u(x) = 0 \) in \( B_1^c \).

The study on maximum principles for fractional Laplacian with zero-order term \( (-\Delta)^s + c(x) \) has attracted a lot of attentions owing to its applications in the analysis of partial differential equations, for instance, the method of moving planes for the fractional Laplacian, see [16, 18, 19, 47] and references therein. However, the condition that \( c(x) \) is bounded below is always required.

In the following, we prove the maximum principle for fractional Laplacian with both zero- and first-order terms which only requires a weaker condition on \( c(x) \).

Theorem 1.7 Assume that \( \vec{b}(x) \in L^{\frac{n}{n-1}}(B_1) \), \( c(x) \in L^{\frac{n}{n-1}}(B_1) \), and \( u(x) \in L_{2,s} \cap L^{\frac{1}{1-s}}(B_1) \) with \( s \in (\frac{1}{2}, 1)(n \geq 3) \) satisfies

\[
\begin{align*}
(-\Delta)^s u(x) + \vec{b}(x) \cdot \nabla u(x) + c(x)u(x) & \geq 0 \quad \text{in } D'(B_1), \\
u(x) & \geq 0 \quad \text{in } B_1^c.
\end{align*}
\]  

There exists a positive constant \( k(n, s) \) such that if \( \|\vec{b}\|_{L^{\frac{n}{n-1}}(B_1)} + \|di \vec{b}\|_{L^{\frac{n}{n-1}}(B_1)} + \|\frac{\nabla}{d} h\|_{L^{\frac{n}{n-1}}(B_1)} + \|c\|_{L^{\frac{n}{n-1}}(B_1)} \leq k(n, s) \), then \( u(x) \geq 0 \) in \( B_1 \). Here \( d(x) = \text{dist}(x, \partial B_1) \).

The result also holds for \( s \in (0, 1) \) if \( \vec{b}(x) = 0 \).

Remark 1.5 It is still an open question if the requirement \( \|\frac{\nabla}{d} h\|_{L^{\frac{n}{n-1}}(B_1)} \leq k(n, s) \) in Theorem 1.7 can be removed.

Remark 1.6 From the estimates on Poisson kernels and Green functions for the fractional Laplacian established in [17], one can deduce that Theorem 1.6, and Theorem 1.7 with \( b(x) = 0 \) also hold in a bounded and open domain with smooth boundary.

Remark 1.7 To the best knowledge of the authors, it is not easy to obtain an effective estimate on the gradient of Green function, so the maximum principle for fractional Laplacian with first-order term in a general domain needs further work.

The refined version of the strong maximum principle for fractional Laplacian with zero-order term was discussed by Li et al. [29] which is closely related to the Bôcher-type theorem. They investigated the strong maximum principle in a punctured ball and required that \( c(x) \) is bounded. With the aid of Theorem 1.6, we weaken the requirement on \( c(x) \).

Theorem 1.8 Assume that \( c(x) \in L^p(B_1)(p > \frac{n}{2s}) \), and \( u(x) \in L_{2,s} \cap L^{\frac{1}{1-s}}(B_1) \) with \( s \in (0, 1) \) \((n \geq 3) \) satisfies

\[
\begin{align*}
(-\Delta)^s u(x) + c(x)u(x) & \geq 0 \quad \text{in } D'(B_1), \\
u(x) & \geq m > 0 \quad \text{in } B_2 \setminus B_1, \\
u(x) & \geq 0 \quad \text{in } B_1^c.
\end{align*}
\]
There exists a positive constant \( k(n, s, p) \) such that if \( \| c^+ \|_{L^p(B_1)} \leq k(n, s, p) \), then

\[
 u(x) \geq \gamma m \quad \text{in } B_1,
\]

where \( \gamma \) is a positive constant depending only on \( n, p \), \( \| c^+ \|_{L^p(B_1)} \) and \( c^+(x) = \max\{c(x), 0\} \).

Finally, we show the strong maximum principle for the fractional Laplacian with first-order term.

**Theorem 1.9** Assume that \( \vec{b}(x) \in L^{\frac{n}{2s-1}}(B_1) \), and \( u(x) \in L^{2s} \cap L^{1/(1-s)}(B_1) \) with \( s \in (\frac{1}{2}, 1)(n \geq 3) \) satisfies

\[
\begin{align*}
 (-\Delta)^s u(x) + \vec{b}(x) \cdot \nabla u(x) &\geq 0 \quad \text{in } D'(B_1) \\
u(x) &\geq m > 0 \quad \text{in } B_2 \setminus B_1 \\
u(x) &\geq 0 \quad \text{in } B_2^c.
\end{align*}
\]

(9)

There exists a positive constant \( k(n, s) \) such that if \( \| \vec{b} \|_{L^{\frac{n}{2s-1}}(B_1)} + \| \text{div } \vec{b} \|_{L^{\frac{n}{s}}(B_1)} + \| \vec{b} \|_{L^{\frac{n}{s}}(B_1)} \leq k(n, s) \), then

\[
 u(x) \geq \gamma m \quad \text{in } B_1
\]

where \( \gamma \) is a positive constant depending only on \( n \) and \( s \).

The paper is organized as follows. In Sect. 2, we present the proofs of different types of maximum principle and strong maximum principle for Laplacian. The maximum principle and strong maximum principle for fractional Laplacian are discussed in Sect. 3.

### 2 Maximum Principles for Laplacian

In this section, we show the maximum principle and strong maximum principle for Laplacian with zero-order term (Schrödinger operator) and Laplacian with first-order term. We first prove Theorem 1.1 that \( c(x) \in L^{\frac{n}{2}}(B_1) \) is not enough to ensure the strong maximum principle for Schrödinger operator.

**Proof of Theorem 1.1** Let \( \alpha > 0 \), we define an auxiliary function

\[
 u(x) = \frac{1}{(-\ln(|x|/e))^{\alpha}},
\]

for \( x \in B_1 \subset \mathbb{R}^n, n \geq 3 \).

Clearly, one has

\[
 u(0) = 0,
\]
and

$$u(x) = 1 \text{ for } x \in \partial B_1.$$ 

A simple calculation shows that

$$-\Delta u(x) + c(x)u(x) = 0 \text{ in } B_1,$$

where

$$c(x) = \frac{\alpha(\alpha+1)}{-\ln(|x|/\epsilon)} + \alpha(n-2) \frac{\ln(|x|/\epsilon)}{|x|^2(-\ln(|x|/\epsilon))}.$$ 

Moreover, one finds that

$$c(x) \in L^\frac{n}{2}(B_1).$$

Now by scaling, we define functions $$u_\epsilon(x)$$ and $$c_\epsilon(x) (0 < \epsilon < \frac{1}{e})$$ as follows:

$$u_\epsilon(x) := u(\epsilon x) = \frac{1}{(-\ln(\epsilon|x|))^{\alpha}}, \quad c_\epsilon(x) := c(\epsilon x) = \frac{\alpha(\alpha+1)}{-\ln(\epsilon|x|)} + \alpha(n-2) \frac{\ln(\epsilon|x|)}{|x|^2(-\ln(\epsilon|x|))}.$$ 

We show that $$u_\epsilon(x)$$ and $$c_\epsilon(x)$$ satisfy the properties (i)-(iii) in Theorem 1.1. Indeed, one has

$$u_\epsilon(0) = 0,$$

$$u_\epsilon(x) = \frac{1}{(-\ln \epsilon)^{\alpha}} > 0, \quad x \in \partial B_1,$$

and

$$-\Delta u_\epsilon(x) + c_\epsilon(x)u_\epsilon(x) = 0, \quad x \in B_1.$$ 

By direct calculations, one has $$c_\epsilon(x) \in L^\frac{2}{n}(B_1)$$ and

$$\lim_{\epsilon \to 0^+} \|c_\epsilon\|_{L^\frac{2}{n}(B_1)} = 0.$$ 

Hence the proof of the theorem is completed. \qed

Next, we present the proof of Theorem 1.2 about the maximum principle for Laplacian with both zero-order and first-order terms in the critical case that \( \vec{b} \in L^n(B_1) \) and \( c \in L^\frac{2}{n}(B_1) \).
**Proof of Theorem 1.2** Define \( u^{-}(x) = -\min\{u(x), 0\} \). It follows from (2) that
\[
\int_{B_{1}} \nabla u^{-}(x) \cdot \nabla u(x) \, dx + \int_{B_{1}} (\vec{b}(x) \cdot \nabla u(x)) u^{-}(x) \, dx + \int_{B_{1}} c(x) u^{-}(x) u(x) \, dx \geq 0.
\]
Then
\[
\int_{B_{1}} |\nabla u^{-}(x)|^{2} \, dx \leq \int_{B_{1}} |\vec{b}(x)||\nabla u^{-}(x)||u^{-}(x)| \, dx + \int_{B_{1}} c^{-}(x)(u^{-}(x))^{2} \, dx.
\]
Clearly, \( u^{-}(x) \in L^{\frac{2n}{n-2}}(B_{1}) \) by Sobolev embedding theorem. It follows from Hölder inequality that if
\[
\| \nabla u^{-} \|_{L^{2}(B_{1})} \neq 0,
\]
then
\[
\int_{B_{1}} |\nabla u^{-}(x)|^{2} \, dx \\
\leq \| \vec{b} \|_{L^{n}(B_{1})} \| \nabla u^{-} \|_{L^{2}(B_{1})} \| u^{-} \|_{L^{\frac{2n}{n-2}}(B_{1})} + \| c^{-} \|_{L^{\frac{n}{2}}(B_{1})} \| u^{-} \|^{2}_{L^{\frac{2n}{n-2}}(B_{1})} \\
< C(n) \left( \| \vec{b} \|_{L^{n}(B_{1})} + \| c^{-} \|_{L^{\frac{n}{2}}(B_{1})} \right) \| \nabla u^{-} \|^{2}_{L^{2}(B_{1})},
\]
where \( C(n) \) is a constant depending only on \( n \).
Thus, one has
\[
\| \nabla u^{-} \|_{L^{2}(B_{1})} < \left\{ C(n) \left( \| \vec{b} \|_{L^{n}(B_{1})} + \| c^{-} \|_{L^{\frac{n}{2}}(B_{1})} \right) \right\}^{\frac{1}{2}} \| \nabla u^{-} \|_{L^{2}(B_{1})}.
\]
This leads to a contradiction if
\[
\| \vec{b} \|_{L^{n}(B_{1})} + \| c^{-} \|_{L^{\frac{n}{2}}(B_{1})} \leq \frac{1}{C(n)}.
\]
Therefore, one has
\[
\| \nabla u^{-} \|_{L^{2}(B_{1})} = 0,
\]
which implies \( u^{-}(x) = 0 \) in \( B_{1} \). Thus, one has \( u(x) \geq 0 \) in \( B_{1} \).
This completes the proof of the theorem. \( \square \)

In the following, with the aid of Theorem 1.2, we give a proof of Theorem 1.3 about a refined version of strong maximum principle for Schrödinger operator if \( p > \frac{n}{2} \).

**Proof of Theorem 1.3** It follows from Theorem 1.2 that
\[
u(x) \geq 0 \quad \text{in} \quad B_{1}.
\]
Then one can rewrite (3) as
\[
\begin{cases}
-\Delta u(x) + c^+(x)u(x) \geq c^-(x)u(x) \geq 0 & \text{in } B_1 \\
u(x) \geq m > 0 & \text{on } \partial B_1.
\end{cases}
\]  
\tag{11}

Clearly, one has $c^+(x) \in L^p(B_1) \ (p > \frac{n}{2})$.

Suppose that $f(x)$ satisfies the following Dirichlet problem
\[
\begin{cases}
-\Delta f(x) = c^+(x) & \text{in } B_1 \\
f(x) = 0 & \text{on } \partial B_1.
\end{cases}
\]  
\tag{12}

By the classical elliptic estimates, one derives that
\[
\|f\|_{L^\infty(B_1)} \leq C(n, p)\|c^+\|_{L^p(B_1)},
\]
where $C(n, p)$ is a positive constant.

Let $w(x) = u(x) - m + mf(x)$. Combining (11) and (12) yields that $w(x)$ satisfies the following equation
\[
\begin{cases}
-\Delta w(x) + c^+(x)w(x) \geq 0 & \text{in } B_1 \\
w(x) \geq 0 & \text{on } \partial B_1.
\end{cases}
\]  
\tag{13}

It follows from Theorem 1.2 that $w(x) \geq 0$ in $B_1$.

Therefore, one has, for $x \in B_1$,
\[
u(x) \geq m(1 - f(x))
\geq m(1 - \|f\|_{L^\infty(B_1)})
\geq m(1 - C(n, p)\|c^+\|_{L^p(B_1)}).
\]

Let
\[
k(n, p) = \frac{1}{2C(n, p)}.
\]

If $\|c^+\|_{L^p(B_1)} \leq k(n, p)$, then one has
\[
u(x) \geq \frac{m}{2}.
\]

Hence the proof of the theorem is completed. \hfill \Box

Finally, we show the strong maximum principle for Laplacian with first-order term.

**Proof of Corollary 1.4** Let $v(x) = u(x) - m$. One can derive that
\[
\begin{cases}
-\Delta v(x) + \vec{b}(x) \cdot \nabla v(x) \geq 0 & \text{in } B_1 \\
v(x) \geq 0 & \text{on } \partial B_1.
\end{cases}
\]  
\tag{14}
It follows from Theorem 1.2 that \( v(x) \geq 0 \) in \( B_1 \). Thus,

\[
    u(x) \geq m \text{ in } B_1.
\]

\[\square\]

3 Maximum Principles for Fractional Laplacian

The aim of this section is to prove the maximum principle and strong maximum principle for fractional Laplacian. These results are useful in the analysis of the fractional Laplace equations and play an important role in understanding the non-local property for the fractional Laplacian.

3.1 Preliminaries

In this subsection, we first introduce the explicit formula for the Poisson kernel and Green function for the fractional Laplacian. Then we present two basic lemmas which are helpful to prove the maximum principles for the fractional Laplacian.

The formula for the Poisson kernel of balls was obtained by Riesz in [37], and the formula for Green function of balls was obtained by Blumenthal, Getoor and Ray in [7]. Specifically, let \( r > 0 \). For any \( x \in B_r \) and \( y \in \overline{B}_r \), the Poisson kernel \( P_r(x, y) \) is defined by

\[
P_r(x, y) = c(n, s) \left( \frac{r^2 - |x|^2}{|y|^2 - r^2} \right)^s \frac{1}{|x - y|^n},
\]

where \( c(n, s) \) is a constant depending only on \( n \) and \( s \). For any \( x, z \in B_r \) and \( x \neq z \), the Green function \( G_r(x, z) \) is given by

\[
    G_r(x, z) = \Phi(x - z) - \int_{B_r^c} \Phi(z - y) P_r(x, y) dy,
\]

where

\[
    \Phi(x - z) = \frac{c(n, s)}{|x - z|^{n-2s}}.
\]

Moreover, the explicit formula for the Green function is that for fixed \( r > 0 \), \( n > 2s \),

\[
    G_r(x, z) = \kappa(n, s) |z - x|^{2s-n} \int_0^{s} \frac{t^{s-1}}{(t + 1)^{n}} \, dt, \quad (15)
\]

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where

\[ \sigma_r(x, z) = \frac{(r^2 - |x|^2)(r^2 - |z|^2)}{r^2|x - z|^2}, \]

and \( \kappa(n, s) \) is a constant depending only on \( n \) and \( s \).

Bucur [10] showed that if \( g(x) \in L^2_2 \cap C(\mathbb{R}^n) \), then

\[ u_g(x) = \begin{cases} \int_{B_r} P_r(x, y)g(y)dy & \text{if } x \in B_r, \\ g(x) & \text{if } x \in B_r^c, \end{cases} \tag{16} \]

is the unique pointwise continuous solution of the equation

\[ \begin{cases} (-\Delta)^s u(x) = 0 & \text{in } B_r, \\ u(x) = g(x) & \text{in } B_r^c. \end{cases} \tag{17} \]

On the other hand, if \( h(x) \in C^{2s+\epsilon}(B_r) \cap C(\bar{B}_r) \), then

\[ u(x) = \begin{cases} \int_{B_r} h(y)G_r(x, y)dy & \text{if } x \in B_r, \\ 0 & \text{if } x \in B_r^c, \end{cases} \tag{18} \]

is the unique pointwise continuous solution of the equation

\[ \begin{cases} (-\Delta)^s u(x) = h(x) & \text{in } B_r, \\ u(x) = 0 & \text{in } B_r^c. \end{cases} \tag{19} \]

Now, we give two basic lemmas for the fractional Laplacian.

**Lemma 3.1** Let \( \eta_{\epsilon}(x) = \frac{1}{\epsilon^n} \eta\left(\frac{x}{\epsilon}\right) \), where \( \eta(x) \) satisfies \( \eta(x) \in C^\infty_0(B_1) \), \( \eta(x) \geq 0 \) and \( \int_{B_1} \eta(x)dx = 1 \). Assume that \( u(x) \in L^2_2 \) satisfies

\[ (-\Delta)^s u(x) \leq 0 \quad \text{in } D'(B_1), \tag{20} \]

and define its mollification \( u_{\epsilon}(x) = \eta_{\epsilon} * u(x) \) in \( B_{1-\epsilon} \). Then \( u_{\epsilon}(x) \) satisfies

\[ (-\Delta)^s u_{\epsilon}(x) \leq 0 \quad \text{in } B_{1-\epsilon}. \tag{21} \]

**Proof** Taking into account the definition of the fractional Laplacian and mollification, it follows that for \( x \in B_{1-\epsilon} \), one has

\[ (-\Delta)^s u_{\epsilon}(x) = C_{n,s} P.V. \int_{\mathbb{R}^n} \frac{u_{\epsilon}(x) - u_{\epsilon}(y)}{|x - y|^{n+2s}}dy \]

\[ = C_{n,s} P.V. \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \eta_{\epsilon}(x - z)u(z)dz \int_{\mathbb{R}^n} \eta_{\epsilon}(y - z)u(y)dz dy \]

\[ = \int_{\mathbb{R}^n} u(z)(-\Delta)^s \eta_{\epsilon}(x - z)dz \]
\[= \int_{\mathbb{R}^n} u(z)(-\Delta)_{\varepsilon}^s \eta_\varepsilon(x - z) \, dz \]
\[\leq 0.\]

This finishes the proof of the lemma. \(\square\)

Inspired by the idea of [28, 29], we give the following lemma. Note that the statement in [28, Theorem 5.4] requires

\[\|\vec{\beta}(x)\|_{C^1(\Omega)} + \|c(x)\|_{L^\infty(\Omega)} < \infty,\] (22)

where \(\Omega\) is a domain in \(\mathbb{R}^n\). Here, we weaken the conditions on \(\vec{\beta}(x)\) and \(c(x)\).

**Lemma 3.2** Assume that \(u(x) \in \mathcal{L}_{2s} \cap L^{\frac{1}{1-2s}}(B_1)\), \(div \vec{\beta} \in L^{\frac{1}{2}}(B_1)\) and \(c(x) \in L^{\frac{1}{2}}(B_1)\) with \(s \in (\frac{1}{2}, 1)\) satisfy

\[-\Delta^s u(x) + \vec{\beta}(x) \cdot \nabla u(x) + c(x)u(x) \geq 0 \text{ in } \mathcal{D}'(B_1).\] (23)

Then for \(v(x) = \min\{u(x), 0\}\), one has

\[-\Delta^s v(x) + \vec{\beta}(x) \cdot \nabla v(x) + c(x)v(x) \geq 0 \text{ in } \mathcal{D}'(B_1).\] (24)

**Proof** The proof is divided into three steps.

**Step 1.** Set \(U = \{x \in B_1 | u(x) < 0\}\). In this step, we assume that \(u(x)\) is a smooth function and \(\partial U\) is \(C^1\). Let \(\varphi(x) \in C^\infty_0(B_1)\) be a nonnegative function and \(n\) be the outward unit normal vector of \(\partial U\). From the definition of the fractional Laplacian, one can derive that

\[
\int_{\mathbb{R}^n} v(x)(-\Delta)^s \varphi(x) \, dx
\]
\[
= \int_{\mathbb{R}^n} u(x) \left( C_{n,s} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{\varphi(x) - \varphi(y)}{|x - y|^{n+2s}} \, dy \right) \, dx
\]
\[
= \int_{U} u(x) \left( C_{n,s} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{\varphi(x) - \varphi(y)}{|x - y|^{n+2s}} \, dy \right) \, dx
\]
\[
= \int_{U} C_{n,s} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{(u(x) - u(y))\varphi(x)}{|x - y|^{n+2s}} \, dy \, dx
\]
\[
+ \int_{U} C_{n,s} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{u(y)\varphi(x) - u(x)\varphi(y)}{|x - y|^{n+2s}} \, dy \, dx
\]
\[
= \int_{U} (-\Delta)^s u(x)\varphi(x) \, dx + \int_{U} C_{n,s} \lim_{\varepsilon \to 0} \int_{U \setminus B_\varepsilon(x)} \frac{u(y)\varphi(x) - u(x)\varphi(y)}{|x - y|^{n+2s}} \, dy \, dx
\]
\[
+ \int_{U} C_{n,s} \lim_{\varepsilon \to 0} \int_{U \setminus B_\varepsilon(x)} \frac{u(y)\varphi(x) - u(x)\varphi(y)}{|x - y|^{n+2s}} \, dy \, dx.
\]
Clearly, one has
\[
\int_U C_{n,s} \lim_{\epsilon \to 0} \int_U \frac{u(y)\varphi(x) - u(x)\varphi(y)}{|x - y|^{n+2s}} dy dx = 0,
\]
and
\[
\int_U C_{n,s} \lim_{\epsilon \to 0} \int_U \frac{u(y)\varphi(x) - u(x)\varphi(y)}{|x - y|^{n+2s}} dy dx \geq 0.
\]
Thus, one has
\[
\int_{\mathbb{R}^n} v(x)(-\Delta)^s \varphi(x) dx \geq \int_U (-\Delta)^s u(x)\varphi(x) dx. \tag{25}
\]
On the other hand, for the lower order term, one has
\[
\int_{B_1} v(x) \left(-d\psi(b(x)\varphi(x)) + c(x)\varphi(x)\right) dx
\]
\[
= \int_U u(x) \left(-d\psi(b(x)\varphi(x)) + c(x)\varphi(x)\right) dx
\]
\[
= \int_U (b(x) \cdot \nabla u(x) + c(x)u(x))\varphi(x) dx - \int_{\partial U} u(x)\varphi(x)b(x) \cdot \mathbf{n} dS
\]
\[
= \int_U (\nabla b(x) + c(x)u(x))\varphi(x) dx. \tag{26}
\]
Combining (25) and (26) leads to
\[
(-\Delta)^s v(x) + b(x) \cdot \nabla v(x) + c(x)v(x) \geq \left((-\Delta)^s u(x) + b(x) \cdot \nabla u(x) + c(x)u(x)\right) \chi_{\{u(x) < 0\}}
\]
in $D'(B_1)$.
Thus, one can obtain that
\[
(-\Delta)^s v(x) + b(x) \cdot \nabla v(x) + c(x)v(x) \geq 0 \text{ in } D'(B_1). \tag{27}
\]

**Step 2.** In this step, we assume that $u(x)$ is a smooth function, however $\partial U$ may not be $C^1$. We show that (27) still holds.

Let $u_{\theta}(x) = u(x) - \theta$ for $\theta < 0$. By Sard’s theorem, one can choose a non-negative monotone increasing sequence $\{\theta_j\}_{j=1}^{\infty}$ satisfying $\theta_j \to 0$, such that the set
\[
\tilde{U} = \{x \in B_1 | u_{\theta_j}(x) < 0\}
\]
satisfies $\partial \tilde{U} \in C^1$ for each $j$.

Denote $v_{\theta_j}(x) = \min\{u_{\theta_j}(x), 0\}$. It follows from the results in Step 1 that
\[
(-\Delta)^s v_{\theta_j}(x) + b(x) \cdot \nabla v_{\theta_j}(x) + c(x)v_{\theta_j}(x)
\]
\[
\geq \left((-\Delta)^s u_{\theta_j}(x) + b(x) \cdot \nabla u_{\theta_j}(x) + c(x)u_{\theta_j}(x)\right) \chi_{\{u(x) < \theta_j\}}
\]
Thus, for \( \varphi(x) \in C_0^\infty(B_1) \), one derives that

\[
\int_{B_1} v_{\theta_j}(x) \left( (-\Delta)^s \varphi(x) - d i v(\tilde{b}(x)\varphi(x)) + c(x)\varphi(x) \right) dx \\
\geq \int_{B_1} -\theta_j c(x)\varphi(x) \chi_{\{u(x) < \theta_j\}} dx.
\]

Clearly, one has

\[
\bigcup_{j=1}^\infty \{ x \in B_1 \mid u(x) < \theta_j \} = \{ x \in B_1 \mid u(x) < 0 \},
\]

where \( \{ x \in B_1 \mid u(x) < \theta_j \} \) is a sequence of monotonically increasing sets. One can obtain that

\[
\int_{B_1} c(x)\varphi(x) \chi_{\{u(x) < \theta_j\}} dx \\
\leq \left( \int_{\{u(x) < \theta_j\}} |c(x)|^{\frac{1}{s}} dx \right)^s \left( \int_{\{u(x) < \theta_j\}} |\varphi(x)|^{\frac{1}{1-s}} dx \right)^{1-s} \\
\leq C(n, s).
\]

Thus, letting \( j \to \infty \) in (29), one has

\[
\int_{B_1} v(x) \left( (-\Delta)^s \varphi(x) - d i v(\tilde{b}(x)\varphi(x)) + c(x)\varphi(x) \right) dx \geq 0 \text{ in } B_1.
\]

This implies

\[
(-\Delta)^s v(x) + \tilde{b}(x) \cdot \nabla v(x) + c(x)v(x) \geq 0 \text{ in } \mathcal{D}'(B_1).
\]

**Step 3.** In the last step, we assert that (27) still holds even if \( u(x) \) may not be a smooth function and \( \partial U \) may not be \( C^1 \). Actually, one can derive that its mollification \( u_\epsilon(x) \) satisfies

\[
(-\Delta)^s u_\epsilon(x) + \tilde{b}(x) \cdot \nabla u_\epsilon(x) + c(x)u_\epsilon(x) \geq P_\epsilon(x) + Q_\epsilon(x) \text{ in } B_{1-\epsilon},
\]

where \( P_\epsilon(x) = \tilde{b}(x) \cdot \nabla u_\epsilon(x) - (\tilde{b} \cdot \nabla u)_\epsilon(x) \) and \( Q_\epsilon(x) = c(x)u_\epsilon(x) - (cu)_\epsilon(x) \).

It follows from the results in Step 2 that

\[
(-\Delta)^s v_\epsilon(x) + \tilde{b}(x) \cdot \nabla v_\epsilon(x) + c(x)v_\epsilon(x) \geq (P_\epsilon(x) + Q_\epsilon(x)) \chi_{\{u_\epsilon(x) < 0\}} \text{ in } \mathcal{D}'(B_{1-\epsilon}).
\]
where \( v_\epsilon(x) = \min\{u_\epsilon(x), 0\} \). Once we know that \( P_\epsilon(x) \to 0 \) and \( Q_\epsilon(x) \to 0 \) in \( L^1_{loc}(B_1) \) as \( \epsilon \to 0^+ \), then by letting \( \epsilon \to 0^+ \) in (30), one can obtain

\[
(-\Delta)^s v(x) + \tilde{b}(x) \cdot \nabla v(x) + c(x)v(x) \geq 0 \text{ in } \mathcal{D}'(B_1)
\]
as desired.

In the following, we prove \( P_\epsilon(x) \to 0 \) and \( Q_\epsilon(x) \to 0 \) in \( L^1_{loc}(B_1) \) as \( \epsilon \to 0^+ \), respectively.

(i) Prove \( P_\epsilon(x) \to 0 \) in \( L^1_{loc}(B_1) \) as \( \epsilon \to 0^+ \).

\[
P_\epsilon(x) = \tilde{b}(x) \cdot \nabla u_\epsilon(x) - (\tilde{b} \cdot \nabla u)_\epsilon(x)
\]

\[
= \int_{B_\epsilon(x)} u(y)\tilde{b}(x) \cdot \nabla x e_\epsilon(x - y)dy + \int_{B_\epsilon(x)} u(y)di v_\epsilon(x)\eta_\epsilon(x - y)dy
\]

\[
= \int_{B_\epsilon(x)} u(y)di v_\epsilon[(\tilde{b}(y) - \tilde{b}(x))\eta_\epsilon(x - y)]dy
\]

\[
= \int_{B_\epsilon(x)} (u(y) - u(x))di v_\epsilon[(\tilde{b}(y) - \tilde{b}(x))\eta_\epsilon(x - y)]dy
\]

\[
+ \int_{B_\epsilon(x)} (u(y) - u(x))(\tilde{b}(y) - \tilde{b}(x)) \cdot \nabla z \eta_\epsilon(x - y)dy
\]

\[
= I_1(x) + I_2(x).
\]

Then, one has

\[
\int_{B_{1-\epsilon}} |I_1(x)|dx \leq \int_{B_{1-\epsilon}} \int_{B_\epsilon(x)} |u(y) - u(x)||di v_\epsilon(x)(\tilde{b}(y))\eta_\epsilon(x - y)dydx
\]

\[
= \int_{B_{1-\epsilon}} \int_{B_\epsilon(0)} |u(x + z) - u(x)||di v_\epsilon(x)(\tilde{b}(x + z))\eta_\epsilon(z)dzdx
\]

\[
= \int_{B_\epsilon(0)} \left( \int_{B_{1-\epsilon}} |u(x + z) - u(x)||di v_\epsilon(x)(\tilde{b}(x + z)|dx \right) \eta_\epsilon(z)dz
\]

\[
\leq \int_{B_\epsilon(0)} \left( \left( \int_{B_{1-\epsilon}} |u(x + z) - u(x)|^{1-s} dx \right)^{1-s} \left( \int_{B_{1-\epsilon}} |di v_\epsilon(x)(\tilde{b}(x + z)|^s dx \right)^s \right) \eta_\epsilon(z)dz,
\]

where we have used Hölder inequality.

It follows from the condition \( di v_\epsilon(x) \in L^1(B_1) \) and the properties of the mollifier that

\[
\lim_{\epsilon \to 0^+} \int_{B_{1-\epsilon}} |I_1(x)|dx
\]
\[
\leq C(n, s) \lim_{\epsilon \to 0^+} \int_{B_\epsilon(0)} \left( \left( \int_{B_1} |u(x + z) - u(x)|^{\frac{1}{1-s}} \, dx \right)^{1-s} \right) \eta_\epsilon(z) \, dz = 0.
\]

(31)

On the other hand, one can deduce that

\[
\int_{B_1} |I_2(x)| \, dx \leq \int_{B_1} \int_{B_\epsilon(x)} |u(y) - u(x)||\vec{b}(y) - \vec{b}(x)||\nabla \eta_\epsilon(x - y)| \, dy \, dx
\]

\[
= \int_{B_1} \int_{B_\epsilon(0)} |u(x + z) - u(x)||\vec{b}(x + z) - \vec{b}(x)||\nabla \eta_\epsilon(z)| \, dz \, dx
\]

\[
= \int_{B_\epsilon(0)} \left( \int_{B_1} |u(x + z) - u(x)||\vec{b}(x + z) - \vec{b}(x)| \, dx \right) |\nabla \eta_\epsilon(z)| \, dz
\]

\[
= \int_{B_\epsilon(0)} \left( \int_{B_1} |u(x + z) - u(x)| \int_0^1 \frac{d}{dt} \vec{b}(x + tz) \, dt \, dx \right) |\nabla \eta_\epsilon(z)| \, dz
\]

\[
\leq \int_0^1 \int_{B_\epsilon(0)} \left( \int_{B_1} |u(x + z) - u(x)||\nabla \vec{b}(x + tz)||z| \, dx \right) |\nabla \eta_\epsilon(z)| \, dz \, dt.
\]

It follows from Hölder inequality and \(di \, v \vec{b}(x) \in L^{\frac{1}{s}}(B_1)\) that

\[
\int_{B_1} |I_2(x)| \, dx
\]

\[
\leq \int_0^1 \epsilon \int_{B_\epsilon(0)} \left( \left( \int_{B_1} |u(x + z) - u(x)|^{\frac{1}{1-s}} \, dx \right)^{1-s} \int_{B_1} |\nabla \cdot \vec{b}(x + tz)|^{\frac{1}{s}} \, dx \right)^s |\nabla \eta_\epsilon(z)| \, dz \, dt
\]

\[
\leq \epsilon C(n, s) \int_{B_\epsilon(0)} \left( \left( \int_{B_1} |u(x + z) - u(x)|^{\frac{1}{1-s}} \, dx \right)^{1-s} \right) |\nabla \eta_\epsilon(z)| \, dz.
\]

Clearly, for \(z \in B_\epsilon(0)\),

\[
\epsilon \int_{B_\epsilon(0)} |\nabla \eta_\epsilon(z)| \, dz \leq \epsilon \int_{B_\epsilon(0)} \frac{C}{\epsilon^{p+1}} \, dz \leq C(n)
\]

Thus, one has

\[
\lim_{\epsilon \to 0^+} \int_{B_1} |I_2(x)| \, dx = 0.
\]

(32)

Combining (31) and (32) leads to

\[
\lim_{\epsilon \to 0^+} \int_{B_1} |P_\epsilon(x)| \, dx = 0.
\]
This implies $P_\epsilon(x) \to 0$ in $L^1_{loc}(B_1)$ as $\epsilon \to 0^+$.

(ii) Prove $Q_\epsilon(x) \to 0$ in $L^1_{loc}(B_1)$ as $\epsilon \to 0^+$.

$$Q_\epsilon(x) = c(x)u_\epsilon(x) - (cu)_\epsilon(x)$$

$$= \int_{B_\epsilon(x)} c(x)u(y)\eta_\epsilon(x-y)dy - \int_{B_\epsilon(x)} c(y)u(y)\eta_\epsilon(x-y)dy$$

$$= \int_{B_\epsilon(x)} u(y)(c(x) - c(y))\eta_\epsilon(x-y)dy.$$ 

It follows that

$$\int_{B_1 - \epsilon} |Q_\epsilon(x)|dx \leq \int_{B_1 - \epsilon} \int_{B_\epsilon(x)} |u(y)||c(x) - c(y)|\eta_\epsilon(x-y)dydx$$

$$= \int_{B_1 - \epsilon} \int_{B_\epsilon(0)} |u(x+z)||c(x+z) - c(x)|\eta_\epsilon(z)dzdx$$

$$= \int_{B_\epsilon(0)} \left( \int_{B_1 - \epsilon} |u(x+z)||c(x+z) - c(x)|dx \right)\eta_\epsilon(z)dz$$

$$\leq \int_{B_\epsilon(0)} \left( \int_{B_1 - \epsilon} |u(x+z)|^{1-s}dx \right)^{1-s}\left( \int_{B_1 - \epsilon} |c(x+z) - c(x)|^{\frac{1}{s}}dx \right)^{s}\eta_\epsilon(z)dz.$$

It follows from the condition $c(x) \in L^{\frac{1}{s}}(B_1)$ and the properties of the mollifier that

$$\lim_{\epsilon \to 0^+} \int_{B_1 - \epsilon} |Q_\epsilon(x)|dx = 0.$$ 

This implies $Q_\epsilon(x) \to 0$ in $L^1_{loc}(B_1)$ as $\epsilon \to 0^+$.

Thus, one has

$$(-\Delta)^s v(x) + \vec{b}(x) \cdot \nabla v(x) + c(x)v(x) \geq 0 \text{ in } \mathcal{D}'(B_1).$$

This completes the proof of the lemma. \qed

### 3.2 Maximum Principle for Fractional Superharmonic Functions

In this subsection, we prove the maximum principle for fractional superharmonic functions. Our strategy is to use the properties for the mollification of the fractional superharmonic function and the representation formula for the fractional Laplacian.

For convenience, we state Theorem 1.6 again here.

**Theorem 3.3** Assume that $u(x) \in \mathcal{L}_{2s} \cap L^{\frac{1}{s}}(B_1)$ satisfies

$$\begin{cases}
(-\Delta)^s u(x) \geq 0 & \text{in } \mathcal{D}'(B_1) \\
u(x) \geq 0 & \text{in } B_1^c,
\end{cases}
$$

(33)
then $u(x) \geq 0$ in $B_1$.

**Proof** Set $u^- = -\min\{u(x), 0\}$. It follows from Lemma 3.2 that

$$\begin{cases}
(-\Delta)^s u^- (x) \leq 0 & \text{in } \mathcal{D}'(B_1) \\
u^- (x) = 0 & \text{in } B_1^c.
\end{cases} \quad (34)$$

Then for $\epsilon > 0$, it follows from Lemma 3.1 that the mollification $u^-_\epsilon (x)$ satisfies

$$\begin{cases}
(-\Delta)^s u^-_\epsilon (x) = f(x) \leq 0 & \text{in } B_{1-\epsilon} \\
u^-_\epsilon (x) = 0 & \text{in } B_{1+\epsilon}^c,
\end{cases} \quad (35)$$

where $f(x)$ is a smooth function.

For any $r \in (0, 1 - \epsilon]$, applying the representation formula of $u^-_\epsilon (x)$, one has

$$u^-_\epsilon (x) = \int_{B^c_r} P_r (x, y) u^-_\epsilon (y) dy + \int_{B_r} f(y) G_r (x, y) dy \leq \int_{B^c_r} P_r (x, y) u^-_\epsilon (y) dy. \quad (36)$$

Now, for fixed $t \in (0, 1)$ and any $x \in B_t$, one can choose $\epsilon \in (0, \frac{1-t}{3})$, take average of the right side of (36) with $r \in [1 - 2\epsilon, 1 - \epsilon]$ and deduce that

$$u^-_\epsilon (x) \leq \frac{1}{\epsilon} \int_{1-2\epsilon}^{1-\epsilon} dr \int_{B^c_r} P_r (x, y) u^-_\epsilon (y) dy$$

$$= \frac{1}{\epsilon} \int_{B^c_{1-2\epsilon}} \int_{1-2\epsilon}^{\min\{1-\epsilon, |y|\}} P_r (x, y) u^-_\epsilon (y) dr dy$$

$$= \frac{1}{\epsilon} \int_{B^c_{1-2\epsilon}} \int_{1-2\epsilon}^{\min\{1-\epsilon, |y|\}} \left( \frac{r^2 - |x|^2}{|y|^2 - r^2} \right)^s \frac{c(n, s) u^-_\epsilon (y)}{|x - y|^n} dr dy.$$

Clearly, for $x \in B_t$ and $y \in B^c_{1-2\epsilon}$,

$$|x - y|^n \geq (|y| - |x|)^n \geq (1 - 2\epsilon - t)^n \geq \left( \frac{1-t}{3} \right)^n, \quad (37)$$

and

$$\frac{1}{(|y|^2 - r^2)^s} = \frac{1}{(|y| - r)^s} \frac{1}{(|y| + r)^s} \leq \frac{C(t)}{(|y| - r)^s}. \quad (38)$$

Combining (37) and (38) yields

$$\int_{1-2\epsilon}^{\min\{1-\epsilon, |y|\}} \left( \frac{r^2 - |x|^2}{|y|^2 - r^2} \right)^s \frac{c(n, s) u^-_\epsilon (y)}{|x - y|^n} dr$$
\[
\begin{align*}
& \leq C(t, n, s) \int_{1-2\epsilon}^{\min\{1-\epsilon, |y|\}} \frac{1}{(|y|-r)^s} dr \\
& \leq \frac{C_1(t, n, s)}{\epsilon^{s-1}}.
\end{align*}
\]

It follows that

\[
\begin{align*}
& u_{\epsilon}^{-}(x) \leq \frac{C_1(t, n, s)}{\epsilon^{s}} \int_{B_{1-2\epsilon}^c} u_{\epsilon}^{-}(y) dy \\
& \leq \frac{C_2(t, n, s)}{\epsilon^{s}} \int_{B_1 \setminus B_{1-3\epsilon}} u^{-}(y) dy \\
& \leq \frac{C_2(t, n, s)}{\epsilon^{s}} |B_1 \setminus B_{1-3\epsilon}| s \left( \int_{B_1 \setminus B_{1-3\epsilon}} (u^{-}(y)) \frac{1}{|y|-r} dy \right)^{1-s} \\
& \leq C_3(t, n, s) \left( \int_{B_1 \setminus B_{1-3\epsilon}} (u^{-}(y)) \frac{1}{|y|-r} dy \right)^{1-s}.
\end{align*}
\]

Clearly, \( u^{-}(x) \in L^{\frac{1}{1-s}}(B_1) \) and

\[
C_3(t, n, s) \left( \int_{B_1 \setminus B_{1-3\epsilon}} (u^{-}(y)) \frac{1}{|y|-r} dy \right)^{1-s} \to 0 \quad \text{as } \epsilon \to 0^+.
\]

Hence, one has

\[
\|u^{-}\|_{L^{\frac{1}{1-s}}(B_t)} = \lim_{\epsilon \to 0^+} \|u_\epsilon^{-}\|_{L^{\frac{1}{1-s}}(B_t)} = 0.
\]

Since \( t \) is arbitrary in \((0, 1)\), letting \( t \to 1^-\), one has

\[
\|u^{-}\|_{L^{\frac{1}{1-s}}(B_1)} = 0.
\]

This implies that \( u(x) \geq 0 \) in \( B_1 \) and completes the proof of the theorem.

\[\square\]

### 3.3 Maximum Principles for Fractional Laplacian

In this subsection, we prove the maximum principle and strong maximum principle for fractional Laplacian with zero-order term and fractional Laplacian with first-order term.

We first give the proof of Theorem 1.7 about the maximum principle for fractional Laplacian with both zero-order and first-order terms.

**Proof of Theorem 1.7** It follows from Lemma 3.2 and (7) that

\[
\begin{align*}
& \left\{ (-\Delta)^s u^{-}(x) + \vec{b}(x) \cdot \nabla u^{-}(x) + c(x) u^{-}(x) \leq 0 \quad \text{in } D'(B_1) \\
& u^{-}(x) = 0 \quad \text{in } B_1^{c}.
\right.
\end{align*}
\]
Let

\[
v(x) = \begin{cases} \int_{B_1} d\nu_y(G_1(x, y)\tilde{b}(y))u^-(y) + G_1(x, y)(-c(y)u^-(y))dy & \text{in } B_1 \\
0 & \text{in } B_1^c. \end{cases}
\]

Thus, for \(x \in B_1\), one has

\[
v(x) = \int_{B_1} \nabla_y G_1(x, y) \cdot \tilde{b}(y)u^-(y)dy + \int_{B_1} (d\nu_y\tilde{b}(y) - c(y))G_1(x, y)u^-(y)dy
\]

\[\vdash T_1(x) + T_2(x).\]

Note that

\[0 < G_1(x, y) < \Phi(x - y) \text{ for any } x, y \in B_1, x \neq y.\]

Then

\[
|T_2(x)| \leq C_1(n, s) \int_{B_1} \frac{(d\nu_y\tilde{b}(y) + |c(y)|)u^-(y)}{|x - y|^{n-2s}}dy. \tag{40}
\]

On the other hand, a simple calculation yields that

\[
\nabla_y G_1(x, y) = \kappa(n, s)
\]

\[
\left(\frac{(2s - n)(y - x)}{|x - y|^{n-2s+2}} \int_0^{\sigma_1(x, y)} t^{s-1} \frac{d\tau}{(t + 1)^{\frac{2}{n}}} + \frac{1}{|x - y|^{n-2s}} \frac{\sigma_1^{s-1}(x, y)}{\sigma_1(x, y) + 1} \nabla_y \sigma_1(x, y)\right).
\]

Note that

\[
\nabla_y \sigma_1(x, y) = (1 - |x|^2) \left(\frac{-2y}{|x - y|^2} + \frac{2(x - y)(1 - |y|^2)}{|x - y|^4}\right)
\]

\[= \left(\frac{-2y}{1 - |y|^2} + \frac{2(x - y)}{|x - y|^2}\right)\sigma_1(x, y).\]

Thus, one has

\[
|\nabla_y G_1(x, y)| \leq C(n, s) \left(\frac{1}{|x - y|^{n-2s+1}} \int_0^{\sigma_1(x, y)} t^{s-1} \frac{d\tau}{(t + 1)^{\frac{2}{n}}} + \frac{1}{|x - y|^{n-2s}} \frac{1}{1 - |y|}\right)
\]

\[
+ \frac{1}{|x - y|^{n-2s}} \left(\frac{1}{1 - |y|} + \frac{1}{|x - y|}\right) \frac{\sigma_1^{s}(x, y)}{\sigma_1(x, y) + 1} \frac{d\tau}{(t + 1)^{\frac{2}{n}}} + \frac{1}{|x - y|^{n-2s}} \frac{1}{1 - |y|}\right).
\]
Then
\[
|T_1(x)| \leq C_2(n, s) \int_{B_1} \left( \frac{1}{|x-y|^{n-2s+1}} + \frac{1}{|x-y|^{n-2s}} \frac{1}{|y|^{1}} \right) |\vec{b}(y)||u^- (y) dy. \tag{41}
\]

Therefore, for \( p > \frac{n}{n-2s} \), combining (40) and (41), together with Hardy–Littlewood–Sobolev and Hölder inequalities, yields
\[
\|v\|_{L^p(B_1)} \\
\leq C_3(n, s) \int_{B_1} \frac{|\vec{b}(y)||u^- (y)|}{|x-y|^{n-2s+1}} + \frac{u^-(y)}{|x-y|^{n-2s}} \left( \frac{|\vec{b}(y)|}{d(y)} + |\text{div}_y \vec{b}(y)| + |c(y)| \right) dy \|_{L^p(B_1)} \\
\leq C(n, s, p) \left( \|\vec{b}u^-\|_{L^{\frac{np}{p+1}}(B_1)} + \|(|\text{div}_y \vec{b}| + \frac{|\vec{b}|}{d} + |c|)u^-\|_{L^{\frac{np}{p+2np}}(B_1)} \right) \\
\leq C_1(n, s, p) \left( \|\vec{b}\|_{L^{\frac{n}{n-1}}(B_1)} + \|\text{div}_y \vec{b}\|_{L^{\frac{n}{n-2}}(B_1)} + \frac{|\vec{b}|}{d} \|_{L^{\frac{n}{n}}(B_1)} + \|c\| \|_{L^{\frac{n}{n}}(B_1)} \right) \|u^-\|_{L^p(B_1)}. \tag{42}
\]

Clearly, \( v(x) \in \mathcal{L}_{2s} \cap L^{\frac{1}{1-\varepsilon}} (B_1) \). On the other hand, \( v(x) \) satisfies the following equation
\[
\begin{cases}
(-\Delta)^s v(x) + \vec{b}(x) \cdot \nabla u^- (x) + c(x)u^- (x) = 0 & \text{in } \mathcal{D}'(B_1) \\
v(x) = 0 & \text{in } B_1^c. \tag{43}
\end{cases}
\]

Combining (39) and (43) yields that
\[
\begin{cases}
(-\Delta)^s (v(x) - u^- (x)) \geq 0 & \text{in } \mathcal{D}'(B_1) \\
v(x) - u^- (x) = 0 & \text{in } B_1^c. \tag{44}
\end{cases}
\]

It follows from Theorem 1.6 that \( v(x) \geq u^- (x) \) in \( B_1 \).

Thus, from (42), one can derive that
\[
\|u^-\|_{L^{\frac{1}{1-\varepsilon}}(B_1)} \\
\leq \left( \|\vec{b}\|_{L^{\frac{n}{n-1}}(B_1)} + \|\text{div}_y \vec{b}\|_{L^{\frac{n}{n-2}}(B_1)} + \frac{|\vec{b}|}{d} \|_{L^{\frac{n}{n}}(B_1)} + \|c\| \|_{L^{\frac{n}{n}}(B_1)} \right) \|u^-\|_{L^{\frac{1}{1-\varepsilon}}(B_1)}.
\]

If
\[
\frac{1}{C(n, s)}
\]
then

\[ \| u^- \|_{L^{\frac{1}{1-\frac{n}{2s}}} (B_1)} = 0. \]

Therefore, one has \( u^- (x) = 0 \) in \( B_1 \). This implies that \( u(x) \geq 0 \) in \( B_1 \).

If \( \vec{b}(x) = 0 \), the argument above also works for \( s \in (0, 1) \). Hence the proof of the theorem is completed. \( \square \)

In the following, we present the proof of Theorem 1.8 about a refined version of the strong maximum principle for fractional Laplacian with zero-order term if \( p > \frac{n}{2} \).

**Proof of Theorem 1.8** It follows from Theorem 1.7 that

\[ u(x) \geq 0 \text{ in } B_1. \]

Then (8) can be written as

\[
\begin{cases}
(-\Delta)^s u(x) + c^+(x)u(x) \geq c^-(x)u(x) \geq 0 \text{ in } \mathcal{D}'(B_1) \\
u(x) \geq m > 0 \text{ in } B_2 \setminus B_1 \\
u(x) \geq 0 \text{ in } B_2^c.
\end{cases}
\]  \hspace{1cm} (45)

Let

\[ v(x) = \begin{cases}
\int_{B_1} G_1(x, y)c^+(y)dy \text{ in } B_1 \\
0 \text{ in } B_1^c.
\end{cases} \]

By direct calculations and classical elliptic estimates, one has

\[
\begin{cases}
(-\Delta)^s v(x) = c^+(x) \text{ in } \mathcal{D}'(B_1) \\
v(x) = 0 \text{ in } B_1^c,
\end{cases}
\]  \hspace{1cm} (46)

and

\[ \| v \|_{L^\infty(B_1)} \leq C(n, s, p) \| c^+ \|_{L^p(B_1)} \text{ for } p > \frac{n}{2s}. \]

Let

\[ h(x) = \begin{cases}
\int_{B_2 \setminus B_1} P_1(x, y)m dy \text{ in } B_1 \\
m \text{ in } B_2 \setminus B_1 \\
0 \text{ in } B_2^c.
\end{cases} \]

Then \( h(x) \) satisfies the following problem

\[
\begin{cases}
(-\Delta)^s h(x) = 0 \text{ in } \mathcal{D}'(B_1) \\
h(x) = m \text{ in } B_2 \setminus B_1 \\
h(x) = 0 \text{ in } B_2^c.
\end{cases}
\]  \hspace{1cm} (48)
Moreover, there exists a positive constant $\beta < 1$ depending on $n, s$ such that
\[ \beta m \leq h(x) \leq m \text{ in } B_1. \quad (49) \]

Define $w(x) = u(x) - h(x) + mv(x)$. It follows from (45)–(49) that $w(x)$ satisfies the following equation
\[
\begin{cases}
(-\Delta)^s w(x) + c^+(x)w(x) \geq 0 & \text{in } \mathcal{D}'(B_1) \\
w(x) \geq 0 & \text{in } B_1^c.
\end{cases}
\quad (50)
\]
Thus, $w(x) \geq 0$ in $B_1$ by Theorem 1.7.

Therefore, one has, for $x \in B_1$,
\[
u(x) \geq h(x) - mv(x) \geq \beta m - m\|v\|_{L^\infty(B_1)} \geq m(\beta - C(n, s, p)\|c^+\|_{L^p(B_1)}).
\]

Let
\[ k(n, s, p) = \frac{\beta}{2C(n, s, p)}. \]
If $\|c^+\|_{L^p(B_1)} \leq k(n, s, p)$, then one has
\[ u(x) \geq \frac{\beta}{2}m. \]

Hence the proof of the theorem is completed. \(\square\)

Finally, we show the strong maximum principle for fractional Laplacian with first-order term.

**Proof of Theorem 1.9** Let
\[ l(x) = \frac{\int_{B_2 \setminus B_1} \frac{1}{|x-y|^{n+2s}} \, dy}{\int_{B_2 \setminus B_1} \frac{1}{|x-y|^{n+2s}} \, dy + \int_{B_2^c} \frac{1}{|x-y|^{n+2s}} \, dy}, \quad x \in B_1. \quad (51)\]

By direct calculations, there exist constants $C_1(n, s)$ and $C_2(n, s)$ such that
\[
\int_{B_2 \setminus B_1} \frac{1}{|x-y|^{n+2s}} \, dy \geq C_1(n, s) \quad \text{and} \quad \int_{B_2^c} \frac{1}{|x-y|^{n+2s}} \, dy \leq C_2(n, s) \text{ for } x \in B_1.
\]

Let $\gamma = \frac{C_1(n, s)}{C_1(n, s) + C_2(n, s)}$. Thus, one has
\[ l(x) \geq \gamma, \quad x \in B_1. \quad (52)\]
Define
\[ h(x) = \begin{cases} 
\gamma m & \text{in } B_1 \\
m & \text{in } B_2 \setminus B_1 \\
0 & \text{in } B_2^c
\end{cases} \] (53)
and
\[ v(x) = u(x) - h(x). \]

One can derive that
\[ \begin{cases} 
(-\Delta)^s v(x) + \vec{b}(x) \cdot \nabla v(x) \geq 0 & \text{in } D'(B_1) \\
v(x) \geq 0 & \text{in } B_1^c.
\end{cases} \] (54)

It follows from Theorem 1.7 that \( v(x) \geq 0 \) in \( B_1 \). Thus,
\[ u(x) \geq \gamma m \text{ in } B_1. \]

Hence the proof of the theorem is completed. \( \Box \)

Acknowledgements The research of Li was partially supported by NSFC Grants 12031012 and 11831003 and the Institute of Modern Analysis-A Frontier Research Center of Shanghai. The authors would like to thank Professor Genggeng Huang and Chenkai Liu for their helpful discussions.

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