Strongly Regular Graphs With The 7-Vertex Condition

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January 28, 2014

Abstract

The $t$-vertex condition, for an integer $t \geq 2$, was introduced by Hestenes and Higman in 1971, providing a combinatorial invariant defined on edges and non-edges of a graph. Finite rank 3 graphs satisfy the condition for all values of $t$. Moreover, a long-standing conjecture of M. Klin asserts the existence of an integer $t_0$ such that a graph satisfies the $t_0$-vertex condition if and only if it is a rank 3 graph.

We construct the first infinite family of non-rank 3 strongly regular graphs satisfying the 7-vertex condition. This implies that the Klin parameter $t_0$ is at least 8. The examples are the point graphs of a certain family of generalised quadrangles.

1 Introduction

Strongly regular graphs (see Section 2) occur naturally as rank 3 representations of finite permutation groups, and there are many other examples. Whereas in principle all finite rank 3 graphs are known (see, e.g., [25], [30]), the more general problem of classifying the finite strongly regular graphs appears completely hopeless. Hence, it is natural to consider properties of graphs which are satisfied by the rank 3 graphs and also some other, but not all, strongly regular graphs.

Hestenes and Higman [19] introduced a family of such properties of this type that are called the $t$-vertex condition for integers $t \geq 2$. These conditions are described in detail in Section 3. For a graph $\Gamma$ and integer $t \geq 3$ the $t$-vertex condition leads to a combinatorial invariant on pairs of distinct vertices of $\Gamma$; if this invariant is constant on the edges, we say that $\Gamma$ satisfies the $t$-vertex condition on edges, and similarly for non-edges.

The 2-vertex condition is equivalent to regularity of graphs, while the 3-vertex condition is equivalent to strong regularity. Also for any $t \geq 3$, if a

*Partially supported by the School of Mathematics and Statistics at the University of Western Australia
A graph satisfies the \( t \)-vertex condition then it also satisfies the \((t - 1)\)-vertex condition (see Section 4). In a rank 3 graph, all edges and all non-edges are equivalent under the automorphism group, and hence cannot be distinguished combinatorially, therefore these graphs satisfy the \( t \)-vertex condition for any \( t \).

There is a long-standing conjecture by M. Klin \[15\] that there exists a number \( t_0 \) such that the only graphs satisfying the \( t_0 \)-vertex condition are the rank 3 graphs. Since for any fixed \( t \) the \( t \)-vertex condition can be checked in polynomial time (see Theorem 4), a proof of Klin’s conjecture would have the remarkable consequence that rank 3 graphs can be recognised combinatorially in polynomial time.

Hence it is interesting to consider graphs which are not rank 3 graphs, but which satisfy the \( t \)-vertex condition for large \( t \).

1. For \( t = 2 \) it is easy to find regular graphs which are not strongly regular, and hence not rank 3 graphs; a small example being the cycle \( C_6 \).

2. The smallest strongly regular graph \((t = 3)\) which is not a rank 3 graph is the well-known Shrikhande graph on 16 vertices, which can be constructed from a Latin square of order 4.

3. In 1971, Higman \[20\] gave the first examples for \( t = 4 \). He showed that the point graphs of generalised quadrangles (see Section 5) satisfy the 4-vertex condition. The smallest example of a generalised quadrangle which does not admit a rank 3 group is the unique \( GQ(5, 3) \) on 96 points.

4. In the late 1980’s Ivanov \[21\] constructed the first graph with the 5-vertex condition that was not a rank 3 graph. It has the parameters of a Latin square graph \( L_8(16) \) on 256 vertices. Its first and second sub-constituents (subgraphs induced by the neighbours and non-neighbours, respectively, of a given vertex) of orders 120 and 135 are strongly regular and satisfy the 4-vertex condition. The graph on 135 vertices was the first example known to satisfy the 4-vertex condition and have an intransitive automorphism group.

5. Later, Brouwer, Ivanov, and Klin \[4\] generalised the construction above, obtaining an infinite series of graphs of order \( 4^k \) with the 4-vertex condition. These graphs in fact satisfy the 5-vertex condition as was shown by the author in \[36\].

For more historical details we refer to Section 9.

Here, we will once more look at the point graphs of generalised quadrangles. We show the following:

\textbf{Theorem 1.} The point graph of a generalised quadrangle satisfies the 5-vertex condition.

\textbf{Theorem 2.} For an integer \( s \), the point graph of a generalised quadrangle of order \((s, s^2)\) satisfies the 7-vertex condition.
Both results are best possible in the sense that there is a generalised quadrangle of order \((5,3)\) which does not satisfy the 6-vertex condition, and there is a generalised quadrangle of order \((5,25)\) which does not satisfy the 8-vertex condition.

This provides us with an infinite family of graphs with intransitive automorphism groups which satisfy the 7-vertex condition.

**Corollary 1.1.** The constant \(t_0\) in Klin’s Conjecture is at least 8.

The author acknowledges a set of notes by A. Pasini [33], communicated by M. Klin, which give the proof of the 5-vertex condition for generalised quadrangles of order \((s,s^2)\). Thanks to M. Klin for attracting my attention to this problem, and for countless proposed improvements. He also contributed essentially to the discussion in Section 9. Thanks also to C. Praeger, A. Niemeyer and C. Pech for helpful discussions, as well as to A. Woldar for helping to polish the text.

## 2 Strongly regular graphs

All graphs considered in this text are finite and simple, i.e., undirected and without loops or multiple edges. Thus, a graph \(\Gamma\) is a finite set \(V = V(\Gamma)\) of vertices together with a binary symmetric and anti-reflexive relation referred to as adjacency. We call \(v = |V|\) the order of \(\Gamma\).

If \(x\) and \(y\) are distinct vertices of \(\Gamma\), we write \(x \sim y\) if they are adjacent, and say that \(y\) is a neighbour of \(x\). Otherwise, we call \(y\) a non-neighbour of \(x\), and write \(x \not\sim y\).

Let \(\Gamma(x) = \{ y \in V | x \sim y \}\) denote the set of neighbours of \(x\) in \(\Gamma\). The valency of \(x\) is defined as \(val(x) = |\Gamma(x)|\).

**Definition 1** (Regular graph). A graph \(\Gamma\) is regular if there is a number \(k\) such that \(val(x) = k\) for all vertices \(x \in V\). In this case, \(k\) is called the valency of \(\Gamma\).

**Definition 2** (Strongly regular graph). A graph \(\Gamma\) is strongly regular if there are numbers \(k, \lambda, \mu\) such that

- \(\Gamma\) is regular of valency \(k\);
- any two adjacent vertices of \(\Gamma\) have exactly \(\lambda\) common neighbours, i.e., \(|\Gamma(x) \cap \Gamma(y)| = \lambda\) whenever \(x \sim y\).
- any two distinct, non-adjacent vertices of \(\Gamma\) have exactly \(\mu\) common neighbours, i.e., \(|\Gamma(x) \cap \Gamma(y)| = \mu\) whenever \(x \not\sim y\).

In this case, the numbers \((v, k, \lambda, \mu)\) are called the parameters of \(\Gamma\), where \(v = |V|\) is its order.

We refer to Section 9 for a brief discussion of numerical restrictions on putative parameters of \(\Gamma\).
3 Isoregular graphs

There are several ways to generalise the concept of strong regularity. One of them is the \( t \)-vertex condition, which we will discuss in Section 4. Another one is isoregularity.

Let \( \Gamma = (V, E) \) be a graph, and let \( S \subset V \) be a set of vertices. We define the valency of \( S \) as

\[
val(S) = \left\lvert \bigcap_{x \in S} \Gamma(x) \right\rvert,
\]

i.e., the number of vertices adjacent to all elements of \( S \). Note that this generalises the notion of valency since for any vertex \( x \), \( val(\{x\}) = val(x) \).

**Definition 3.** Let \( \Gamma = (V, E) \) be a graph, and \( k \geq 1 \) be an integer. Suppose that for each set \( S \) of at most \( k \) vertices, the valency of \( S \) depends only on the isomorphism class of the subgraph of \( \Gamma \) induced by \( S \). Then \( \Gamma \) is called \( k \)-isoregular.

**Proposition 3.1.** For \( k > 1 \), \( k \)-isoregularity implies \((k-1)\)-isoregularity.

**Proof.** Follows directly from the definition. If the valency of any set \( S \) of up to \( k \) elements depends only on the isomorphism class of the induced subgraph, this holds in particular for all sets of up to \( k - 1 \) elements.

**Proposition 3.2.** A graph \( \Gamma \) is \( k \)-isoregular if and only if its complement \( \overline{\Gamma} \) is \( k \)-isoregular.

**Proof.** Let \( \Gamma \) be \( k \)-isoregular. Let \( S \) be a set of \( k \) vertices in \( \Gamma \). Since we have \( i \)-isoregularity for \( 1 \leq i \leq k \), we know the valency of every subset of \( S \). Using the Principle of Inclusion and Exclusion, we can determine the number of vertices not adjacent to any element of \( S \), i.e., the valency of \( S \) in the complement of \( \Gamma \). Since the calculation depends only on the isomorphism class of the subgraph of \( \Gamma \) induced by \( S \), and since this holds for any \( k \)-subset \( S \), \( \overline{\Gamma} \) is \( k \)-isoregular.

**Proposition 3.3.** A graph \( \Gamma \) is \( 1 \)-isoregular if and only if it is regular. It is \( 2 \)-isoregular if and only if it is strongly regular.

**Proof.** Since there is only one isomorphism class of graphs of order 1, and since the valency of a singleton \( \{x\} \) is the same as the valency of the vertex \( x \), a graph is \( 1 \)-isoregular if and only if each vertex has the same valency, in other words, if and only if the graph is regular.

There are two isomorphism classes of graphs of order 2, namely, edges and non-edges. A graph is \( 2 \)-isoregular if and only if it is \( 1 \)-isoregular, i.e., regular, and if the number of common neighbours of two distinct vertices \( x \) and \( y \) depends only on whether \( x \) is adjacent to \( y \). However, this is exactly the definition of strong regularity.

For a graph \( \Gamma \), a vertex \( x \), and an integer \( i \), we define the \( i \)-th subconstituent \( \Gamma_i(x) \) as the subgraph of \( \Gamma \) induced by all vertices at distance \( i \) from \( x \).
Proposition 3.4. For a strongly regular graph $\Gamma$, the following are equivalent:

- $\Gamma$ is 3-isoregular.
- The subconstituents $\Gamma_i(x)$, $i = 1, 2$, are strongly regular, with parameters which do not depend on the choice of $x$.

Proof. There are four isomorphism classes of graphs of order 3, determined uniquely by the number of edges they contain. We will denote these classes by $\Delta_i$, where $0 \leq i \leq 3$ is the number of edges.

Let $\Gamma$ be a 3-isoregular graph. By Proposition 3.1, it is 2-isoregular, and hence strongly regular, with parameters $(v, k, \lambda, \mu)$.

Let $x$ be a vertex of $\Gamma$, and let $\Gamma_1 = \Gamma(x)$, the subgraph induced by the neighbours of $x$. $\Gamma_1$ is a regular graph of order $k$ and valency $\lambda$. Let $y$ and $z$ be vertices of $\Gamma_1$. Then the common neighbours of $y$ and $z$ in $\Gamma_1$ are exactly the common neighbours of $x, y, z$ in $\Gamma$.

If $y \sim z$, then $x, y, z$ induce a complete graph $\Delta_3$ in $\Gamma$. Hence, $y$ and $z$ have $\text{val}(\Delta_3)$ neighbours in $\Gamma_1$. Similarly, if $y \not\sim z$, $x, y, z$ induce the graph $\Delta_2$ in $\Gamma$. Thus $y$ and $z$ have $\text{val}(\Delta_2)$ neighbours in $\Gamma_1$. So we get that $\Gamma_1$ is strongly regular, with parameters:

\[
\begin{align*}
v_1 &= k \\
k_1 &= \lambda \\
\lambda_1 &= \text{val}(\Delta_3) \\
\mu_1 &= \text{val}(\Delta_2).
\end{align*}
\]

We get strong regularity for the second subconstituents by working with the complement of $\Gamma$. More precisely, if $\Gamma$ is 3-isoregular, then so is the complement $\bar{\Gamma}$, by Proposition 3.2. Then, by the argument above, the first subconstituent of $\bar{\Gamma}$ is strongly regular; however, this is precisely the complement of the second subconstituent of $\Gamma$.

Conversely, assume that $\Gamma$ is a strongly regular graph with parameters $(v, k, \lambda, \mu)$, such that the subconstituents $\Gamma_i(x)$, $i = 1, 2$, are strongly regular with parameters $(v_i, k_i, \lambda_i, \mu_i)$. By assumption, $\Gamma$ is 2-isoregular, hence we only need to check the graphs $\Delta_i$ of order 3. Clearly, in $\Gamma$ we have $\text{val}(\Delta_3) = \lambda_1$, and $\text{val}(\Delta_2) = \mu_1$.

Let $x, y, z$ be pairwise non-adjacent vertices in $\Gamma$. $y$ and $z$ have $\mu$ common neighbours in $\Gamma$; of these, $\mu_2$ are not neighbours of $x$. Hence we get that $\text{val}(\Delta_0) = \mu - \mu_2$.

Similarly, if $y \sim z$, and both are non-adjacent to $x$, they have $\lambda$ neighbours in $\Gamma$, and $\lambda_2$ neighbours in $\Gamma_2$. Hence, $\text{val}(\Delta_1) = \lambda - \lambda_2$.

Altogether, we get that $\Gamma$ is 3-isoregular.

4 The $t$-vertex condition

Here, we discuss another way of generalising strong regularity. Let $\Gamma$ be a graph of order $v$. Let $T$ be a graph of order $t$, with two distinguished vertices $x_0$ and
We will denote such a triple \((T, x_0, y_0)\) as a **graph type**. \(x_0\) and \(y_0\) will be called **fixed vertices**, the other vertices of \(T\) will be called **additional vertices**.

Let \(x, y\) be vertices of \(\Gamma\), and let \(\Delta\) be an induced subgraph of \(\Gamma\) containing both \(x\) and \(y\). \(\Delta\) is said to be of **type** \(T\) (with respect to \(x\) and \(y\)) if there is an isomorphism \(\phi : T \rightarrow \Delta\) which maps \(x_0\) to \(x\) and \(y_0\) to \(y\).

**Definition 4.** Let \(\Gamma\) be a graph, and let \(t \geq 2\) be an integer. Suppose that for any graph type \((T, x_0, y_0)\) of order at most \(t\), and for any pair of vertices \((x, y)\) of \(\Gamma\), the number of subgraphs of \(\Gamma\) which are of type \(T\) w.r.t. \(x\) and \(y\) depends only on whether \(x\) and \(y\) are equal, adjacent, or non-adjacent. Then \(\Gamma\) is said to satisfy the **\(t\)-vertex condition**.

In other words, \(\Gamma\) satisfies the \(t\)-vertex condition if we cannot distinguish its edges (vertices, non-edges) by considering subgraphs of order up to \(t\).

**Proposition 4.1.** For \(t > 2\), the \(t\)-vertex condition implies the \((t - 1)\)-vertex condition.

**Proof.** Follows directly from the definition.

**Proposition 4.2.** A graph \(\Gamma\) satisfies the 2-vertex condition if and only if it is regular. It satisfies the 3-vertex condition if and only if it is strongly regular.

**Proposition 4.3.** A rank 3 graph of order \(v\) satisfies the \(v\)-vertex condition.

**Proof.** In a rank 3 graph, the automorphism group acts transitively on vertices, (directed) edges and non-edges. Hence we cannot distinguish edges (resp. non-edges) on a combinatorial level.

For growing \(t\), the number of graph types to check increases very quickly. However, it turns out that many of these checks are redundant.

**Theorem 3.** Let \(\Gamma\) be a graph which is \(k\)-isoregular, and which satisfies the \((t - 1)\)-vertex condition. Then \(\Gamma\) satisfies the \(t\)-vertex condition if and only if \(\Gamma\) contains a vertex of valency at least \(k + 1\). This contradicts the choice of \(T\).

**Corollary 4.1.** Let \(\Gamma\) be a graph which is \(k\)-isoregular, and which satisfies the \((t - 1)\)-vertex condition, but not the \(t\)-vertex condition. Then there is a graph type \((T, x, y)\) such that all vertices other than \(x\) and \(y\) have valency at least \(k + 1\), and such that the number of graphs of type \((T, x, y)\) depends on the choice of \(x\) and \(y\) in \(\Gamma\).
Finally, we present a complexity result related to the $t$-vertex condition:

**Theorem 4.** For a fixed integer $t$ and a graph $\Gamma$ of order $n$, the $t$-vertex condition can be checked in time polynomial in $n$.

**Proof.** We may assume that $t \geq 3$, and that the $(t-1)$-vertex condition has already been checked.

Fix a graph type $T = (\Delta, z_1, z_2)$ of order $t$ and two vertices $z_1', z_2'$ of $\Gamma$. Suppose that either both $(z_1, z_2)$ is an edge in $\Delta$ and $(z_1', z_2')$ is an edge in $\Gamma$, or both pairs are non-edges in their respective graphs. Arbitrarily label the remaining vertices in $\Delta$ by $z_3, \ldots, z_t$.

Now we consider all sequences $(z_3', \ldots, z_t')$ of distinct vertices in $\Gamma$; there are fewer than $n^{t-2}$ of them. For each sequence we may consider the labelled subgraph induced by the $z_i', 1 \leq i \leq t$. We can check in time proportional to $t^2$ whether $\phi : z_i \mapsto z_i'$ is a graph isomorphism. Hence, in time $O(n^{t-2}t^2)$ we can count the graphs of type $T$ with respect to the given vertices $z_1'$ and $z_2'$.

There are fewer than $n^2$ possible pairs $(z_1', z_2')$ of vertices. Repeating the count for all of them gives us the numbers of graphs of type $T$ for all these pairs in time $O(n^{t-2} \cdot n^2) = O(n^t)$, since $t$ is constant. Since there are only finitely many graph types of order $t$, this proves the theorem. $\square$

5 Generalised quadrangles

**Definition 5.** Let $P$ be a finite set of points. Let $L$ be a set of distinguished subsets of $P$, called lines. Suppose there are integers $s$ and $t$ such that

- any two lines intersect in at most one point;
- each line contains exactly $s + 1$ points;
- each point is contained in exactly $t + 1$ lines.

Then $(P, L)$ is a partial linear space of order $(s, t)$.

We use the traditional geometric language: Two points are collinear if they lie on a common line; two lines are concurrent if they intersect.

Given a partial linear space, we can define a graph as follows:

**Definition 6.** Let $(P, L)$ be a partial linear space. Let $\Gamma$ be the graph with vertex set $P$, two points being adjacent if they are collinear. Then $\Gamma$ is called the point graph of $(P, L)$.

Generalised quadrangles are partial linear spaces which satisfy one additional regularity property.

**Definition 7.** Let $(P, L)$ be a partial linear space. Suppose that for each line $l$ and each point $P \notin l$, there is exactly one point on $l$ collinear to $P$. Then $(P, L)$ is a generalised quadrangle of order $(s, t)$, or a GQ$(s, t)$ for short.
Below we survey a few classical results about generalized quadrangles.

**Theorem 5.** The point graph of a GQ(\(s, t\)) is strongly regular, with parameters

\[
\begin{align*}
v &= (s + 1)(st + 1) \\
k &= (s - 1)t \\
\lambda &= s - 1 \\
\mu &= t + 1
\end{align*}
\]

We now look at subgraphs of the point graphs of generalised quadrangles.

**Theorem 6 (Cameron [10]).** The point graph of a generalised quadrangle does not contain \(K_{4} - e\) as an induced subgraph. Here, \(K_{4} - e\) denotes the graph obtained by removing one edge from the complete graph on 4 vertices.

**Definition 8.** A **triad** in a GQ(\(s, t\)) is a triple \(\{x, y, z\}\) of pairwise non-collinear points. A **center** of a triad is a point collinear to all three points of the triad.

**Theorem 7 (see [35]).** For a generalised quadrangle of order \((s, t)\), we have \(s^2 \geq t\). Equality holds if and only if each triad has exactly \(s + 1\) centers.

**Corollary 5.1.** The point graph of a generalised quadrangle of order \((s, s^2)\) is 3-isoregular.

**Proof.** The isomorphism class of a graph of order 3 is uniquely determined by the number of edges. By Theorem 7, each graph with 0 edges has \(s + 1\) common neighbours. By Theorem 6, each graph with 2 edges has no common neighbours.

If the graph has 1 edge, two of the points are connected by a line. The third point has exactly one neighbour on this line, which is the unique common neighbour of all three points. Thus, a graph with 1 edge has one common neighbour.

If the graph has 3 edges, it is complete, and hence contained in a line. The common neighbours of the three points are the remaining \(s - 2\) points on this line.

\[\square\]

6 Proof of Theorem 1

6.1 Goals and strategy

In this section we prove Theorem 1 which states that the point graph of a generalised quadrangle satisfies the 5-vertex condition. For generalised quadrangles of order \((s, s^2)\), this has been previously shown by A. Pasini (unpublished [33]). Although the proof presented here is independent, the result in [33] showed that generalised quadrangles yield a class of strongly regular graphs which should be further investigated. Also, some of the techniques used by Pasini helped to obtain the result presented here.

By Theorem 6 we have to count graphs with minimal valency 3 and order 5. In other words, if \(x, y\) are the fixed vertices, and \(a, b, c\) the additional vertices,
we need to consider the graphs in which each of \( a, b, c \) is non-adjacent to at most one vertex.

If we take the complements of these graphs, we get that the valency of \( a, b, c \) is at most 1. If we discard a possible edge \((x, y)\), that implies that the size of the graph, i.e., the number of its edges, is at most 3. Thus, we use the following strategy:

1. Enumerate all graphs of order 5 and size \( i = 0, 1, 2, 3 \) (Subsection 6.2);
2. Discard those graphs which cannot appear as subgraphs of generalised quadrangles (Subsection 6.3);
3. For the few remaining graph types, check that their numbers do not depend on the choice of the edge or non-edge \((x, y)\) (Subsection 6.4).

### 6.2 Graph types relevant to the 5-vertex condition

We start by determining all graph types which need to be considered in order to check whether a given graph satisfies the 5-vertex condition. As stated above, the size of the complements of these graph types is bounded by 3, in other words, the complements contain at most 3 edges.

In order to be sure not to miss anything, we will completely enumerate these complements. In the following we enumerate the relevant graphs up to isomorphism under the group \( S(\{x, y\}) \times S(\{a, b, c\}) \) of order 12, counting also the cardinalities of the related isomorphism classes. We will not consider the set \( \{x, y\} \) as a possible edge. This leaves \( \binom{5}{2} - 1 = 9 \) possible edges.

We denote graphs by a list of its edges.

#### Graphs of size 0

The empty graph is uniquely determined by its size.

#### Graphs of size 1

a. \( \{x, a\} \), Aut = \( \langle (b, c) \rangle \), \( |\text{Aut}| = 2 \), length of orbit: 6.

b. \( \{a, b\} \), Aut = \( \langle (a, b), (x, y) \rangle \), \( |\text{Aut}| = 4 \), length of orbit: 3.

This accounts for \( 6 + 3 = 9 = \binom{5}{2} \) graphs.

#### Graphs of size 2

a. \( \{x, a\}, \{x, b\} \), Aut = \( \langle (a, b) \rangle , |\text{Aut}| = 2 \), length of orbit: 6.

b. \( \{x, a\}, \{y, b\} \), Aut = \( \langle (a, b)(x, y) \rangle , |\text{Aut}| = 2 \), length of orbit: 6.

c. \( \{x, a\}, \{b, c\} \), Aut = \( \langle (b, c) \rangle , |\text{Aut}| = 2 \), length of orbit: 6.

d. \( \{x, a\}, \{y, a\} \), Aut = \( \langle (x, y), (b, c) \rangle , |\text{Aut}| = 4 \), length of orbit: 3.

e. \( \{x, a\}, \{a, b\} \), Aut = \( \langle e \rangle , |\text{Aut}| = 1 \), length of orbit: 12.

f. \( \{a, b\}, \{b, c\} \), Aut = \( \langle (a, c), (x, y) \rangle , |\text{Aut}| = 4 \), length of orbit: 3.
This accounts for $6 + 6 + 3 + 12 + 6 + 3 = 36 = \binom{3}{2}$ graphs.

**Graphs of size 3**

| Type | Graph Description | Automorphism Group | Length of Orbit |
|------|------------------|--------------------|-----------------|
| a.   | $\{x, a\}, \{x, b\}, \{x, c\}$, $\text{Aut} = S(\{a, b, c\}$, $|\text{Aut}| = 6$, length of orbit: 2. |
| b.   | $\{x, a\}, \{x, b\}, \{y, c\}$, $\text{Aut} = \langle (a, b) \rangle$, $|\text{Aut}| = 2$, length of orbit: 6. |
| c.   | $\{x, a\}, \{x, b\}, \{y, a\}$, $\text{Aut} = \langle e \rangle$, $|\text{Aut}| = 1$, length of orbit: 12. |
| d.   | $\{x, a\}, \{x, b\}, \{a, b\}$, $\text{Aut} = \langle (a, b) \rangle$, $|\text{Aut}| = 2$, length of orbit: 6. |
| e.   | $\{x, a\}, \{x, b\}, \{a, c\}$, $\text{Aut} = \langle e \rangle$, $|\text{Aut}| = 1$, length of orbit: 12. |
| f.   | $\{x, a\}, \{y, a\}, \{a, b\}$, $\text{Aut} = \langle (x, y) \rangle$, $|\text{Aut}| = 2$, length of orbit: 6. |
| g.   | $\{x, a\}, \{y, a\}, \{b, c\}$, $\text{Aut} = \langle (x, y), (b, c) \rangle$, $|\text{Aut}| = 4$, length of orbit: 3. |
| h.   | $\{x, a\}, \{y, b\}, \{a, b\}$, $\text{Aut} = \langle (a, b)(x, y) \rangle$, $|\text{Aut}| = 2$, length of orbit: 6. |
| i.   | $\{x, a\}, \{y, b\}, \{a, c\}$, $\text{Aut} = \langle e \rangle$, $|\text{Aut}| = 1$, length of orbit: 12. |
| j.   | $\{x, a\}, \{a, b\}, \{a, c\}$, $\text{Aut} = \langle (b, c) \rangle$, $|\text{Aut}| = 2$, length of orbit: 6. |
| k.   | $\{x, a\}, \{a, b\}, \{b, c\}$, $\text{Aut} = \langle e \rangle$, $|\text{Aut}| = 1$, length of orbit: 12. |
| l.   | $\{a, b\}, \{a, c\}, \{b, c\}$, $\text{Aut} = \langle (x, y), (a, b), (a, b, c) \rangle$, $|\text{Aut}| = 12$, length of orbit: 1. |

This accounts for $2 + 4 \cdot 12 + 5 \cdot 6 + 3 + 1 = 84 = \binom{3}{2}$ graphs. This check sum confirms that our enumeration is complete.

The following graphs can be discarded because one of the additional vertices has valency greater than 1: 2.d–f, 3.c–l

This can be summarised as follows (recall that above we enumerated the complements of the relevant graphs):

**Theorem 8.** If a graph $\Gamma$ satisfies the 4-vertex condition, then to check the 5-vertex condition it is sufficient to count the graphs of the eight types given in Table 1. Here, a dashed line indicates an optional edge connecting the fixed vertices.

### 6.3 Easily discarded cases

We now proceed to apply the results obtained above to point graphs of generalised quadrangles. In this case, most of these graph types can be discarded since they contain a subgraph $K_4 - e$, which cannot happen in a generalised quadrangle (see Theorem 8).

**Lemma 6.1.** In a generalised quadrangle, the following graph types do not appear: 1a, 1b, 2b, 2c, 3b.
Table 1: Types to check for 5-vertex condition
Proof. In the pictures of Table 1, we denote the fixed vertices left to right by $x, y$, and the additional vertices left to right by $a, b, c$. For each of the graph types, we note a set of vertices which induces a $K_4-e$:

1a: $x, a, b, c$
1b: $x, a, b, c$
2b: $x, a, b, c$
2c: $y, a, b, c$
3b: $y, a, b, c$

So these types do not occur, independent of whether $x$ and $y$ are adjacent. 

6.4 The remaining cases

This leaves us to check the following types: 0, 2a, 3a.

We will now show that the numbers for these subgraphs are uniquely determined by the axioms of generalised quadrangles. We will arrange these verifications in three lemmas, one for each type.

Lemma 6.2. The number of graphs of type 0 w.r.t. $x$ and $y$ is $\binom{s-1}{3}$ if $x \sim y$, 0 otherwise.

Proof. If $x \not\sim y$, then the graph contains a $K_4-e$. Otherwise, each additional vertex is adjacent to both $x$ and $y$ and hence lies on the line connecting them. Each set of three points on this line yields a graph of type 0.

Lemma 6.3. The number of graphs of type 2a is 0 if $x \sim y$. Otherwise, it is $(t+1)^\binom{s-1}{2}$.

Proof. If $x \sim y$, then $\{x, y, b, c\}$ induces a $K_4-e$. Let $x \not\sim y$. The points $\{y, a, b, c\}$ form a clique, hence lie on a line. Thus to find such a graph, we choose a line $l$ through $y$, with $t+1$ possibilities. The point $c$ is the unique neighbour of $x$ on $l$; for $a$ and $b$ we can choose any two of the remaining points on $l$.

Lemma 6.4. The number of graphs of type 3a is $t\binom{s}{3}$ if $x \sim y$, $(t+1)\binom{s-1}{3}$ otherwise.

Proof. The vertices $\{y, a, b, c\}$ form a clique, hence are collinear. If $x \sim y$, we can choose any line through $y$ but not through $x$; there are $t$ such lines. We need to choose 3 other points on this line; this gives us $\binom{3}{3}$ choices.

If $x \not\sim y$, then we can choose any line through $y$ $(t+1$ possibilities) and then choose any three points on that line that are not adjacent to $x$, giving $\binom{s-1}{3}$ choices.

We see that for all the graph types we have to check, the numbers do not depend on the particular choice of $x$ and $y$. Together with Theorem 3, this proves Main Theorem 1.
7 Proof of Theorem 2

7.1 Goals and preliminaries

For the remainder of the text, we concentrate on point graphs of generalised quadrangles of order \((s, s^2)\). Thus, let \(s\) be fixed, and let \(\Gamma\) be the point graph of such a generalised quadrangle.

Suppose that \(\Gamma\) does not satisfy the \(t_0\)-vertex condition for some value of \(t_0\). We may assume that \(t_0\) is minimal with this property. Thus there is a graph type \((T, x_0, y_0)\) of order \(t_0\) such that the number of graphs of type \(T\) with respect to \(x\) and \(y\) does depend on the choice of \(x\) and \(y\). We may further assume that \(T\) is maximal, i.e., adding an edge to \(T\) leads to a graph type such that the corresponding number does not depend on the choice of the fixed vertices.

We will try to find a lower bound on \(t_0\); this will yield the result that \(\Gamma\) satisfies the \((t_0 - 1)\)-vertex condition.

First, let us collect some properties of \(T\).

**Proposition 7.1.** We may assume that the valency of any additional vertex in \(T\) is at least 4.

**Proof.** Since \(\Gamma\) is 3-isoregular by Corollary 5.1, this follows from Corollary 4.1.

**Lemma 7.1.** \(T\) does not contain an induced subgraph isomorphic to \(K_4 - e\).

**Proof.** Follows from Theorem 6 and the fact that \(T\) is an induced subgraph of \(\Gamma\).

**Proposition 7.2.** Let \(X\) be the set of vertices adjacent to both \(x\) and \(y\) in \(T\). If \(x \sim y\), then the subgraph induced by \(X\) is complete. If \(x \not\sim y\), then the subgraph induced by \(X\) is empty.

**Proof.** Let \(w\) and \(z\) be two distinct vertices both adjacent to each of \(x\) and \(y\). If exactly one of \((x, y), (w, z)\) is an edge in \(T\), then these four vertices induce a graph isomorphic to \(K_4 - e\), contradicting the lemma above.

Let \(S\) be the subgraph obtained from \(T\) by deleting \(x\) and \(y\), and all edges incident with either \(x\) or \(y\), see Figure 1. Then we have

- \(S\) has order \(t_0 - 2\);
- \(S\) has minimal valency at least 2;
- The vertices of valency 2 in \(S\) induce either a complete graph or an empty graph.
- For any maximal clique \(C\) in \(S\) and any vertex \(z\) not in \(C\), \(z\) has at most one neighbour in \(C\).

We now look at the possible subgraphs induced by \(S\). In Subsection 7.2 we look at the case that \(S\) induces a complete graph. In Subsection 7.3 we consider the other case.
7.2 \; S \text{ is complete}

We first consider the case that \( S \) induces a complete graph in \( T \). In this case, we can determine the number of graphs of type \( T \), independent of the size of \( S \).

**Theorem 9.** Let \( \Gamma \) be the point graph of a \( GQ(s, s^2) \). Let \((T, x, y)\) be a graph type of arbitrary size \( t_0 \), such that the additional vertices induce a complete graph \( S \) in \( T \). Then the number of graphs of type \((T, x_0, y_0)\) with respect to vertices \( x \) and \( y \) does not depend on the choice of \( x \) and \( y \) in \( \Gamma \).

**Proof.** If \( S \) is a complete graph, then \( S \) is contained in a line, say \( l \). Either \( x \in l \) and thus is collinear to all \( t_0 - 2 \) points in \( S \), or it has at most one neighbour in \( S \). The same holds for \( y \). Let \( d_x = |T(x) \cap S| \) be the number of neighbours of \( x \) in \( S \), and similar \( d_y = |T(y) \cap S| \). We have that \( d_x, d_y \in \{0, 1, t_0 - 2\} \), and without loss of generality we may assume that \( d_x \geq d_y \). We get six possibilities for the pair \((d_x, d_y)\), and in each case we can determine the number of graphs of type \( T \) with respect to any pair \((x, y)\) of distinct vertices in \( \Gamma \).

\[(d_x, d_y) = (t_0 - 2, t_0 - 2) : \]

\[
\begin{array}{c}
\bullet \\
\bullet
\end{array}
\]

In this case, both \( x \) and \( y \) are contained in \( l \); in particular, \( x \sim y \). Thus there are \( \binom{|S| - 1}{1} \) choices for the additional vertices.

\[(d_x, d_y) = (t_0 - 2, 1) : \] Here, \( x \in l \), \( y \notin l \), and the unique neighbour \( z \) of \( y \) on \( l \) lies in \( S \). We have to distinguish the cases \( x = z \) and \( x \neq z \).
If $x \sim y$, we can choose any line $l$ through $x$ but not through $y$, which gives $s^2$ possibilities. Then we need to choose a subset $S \subseteq l \setminus \{x\}$. Hence, altogether there are $s^2 \binom{s-1}{t_0-2}$ such graphs.

If $x \neq z$, then $x \not\sim y$. In order to find such a graph, we have to first choose the line $l$ through $x$, which gives $s^2 + 1$ possibilities. The point $z \in l$ is uniquely determined by $y$, and we need to choose the $t_0 - 3$ remaining vertices of $S$ on $l$. Altogether, there are $(s^2 + 1) \binom{s-1}{t_0-3}$ such graphs.

$(d_x, d_y) = (t_0 - 2, 0)$: Here, $x \in l$, and the unique neighbour $z$ of $y$ on $l$ does not lie in $S$. In particular, $x \not\sim y$.

We can choose any line $l$ through $x$, and then a subset $S \subseteq l$ avoiding both $x$ and the unique neighbour of $y$ on $l$. Thus, the total number of such graphs is $(s^2 + 1) \binom{s-1}{t_0-3}$.

$(d_x, d_y) = (1, 1)$: Neither $x$ nor $y$ lie on $l$, and their unique neighbours $z_x$ and $z_y$ lie in $S$. We need to distinguish four cases, according as $x \sim y$, and $z_x = z_y$.

$x \not\sim y$, $z_x \neq z_y$:

Choose any line $l_1$ through $x$ ($s^2 + 1$ possibilities). On $l_1$ we take any other point $z_x$ which is not adjacent to $y$ ($s - 1$ possibilities). Through $z_x$ we take any line $l \neq l_1$ ($s^2$ choices). On $l$, $z_x$ and $z_y$ are
fixed, so we have to choose $|S| - 2$ additional points. Thus the total number of such graphs is

$$s^2(s^2 + 1)(s - 1)\binom{s - 1}{|S| - 2}.$$  

$x \sim y$, $z_x \neq z_y$:

Take any line $l_1$ through $x$ which does not contain $y$ ($s^2$ choices). Choose $z_x \neq x$ on $l_1$ ($s$ choices). By construction, $z_x \not\sim y$. Take any line $l \neq l_1$ through $z_x$ ($s^2$ choices); this determines $z_y$. Again, we need to choose the remaining $|S| - 2$ points on $l$. The total number is

$$s^5\binom{s - 1}{|S| - 2}.$$  

$x \not\sim y$, $z_x = z_y$:

We take any common neighbour $z$ of $x$ and $y$ ($\mu$ choices). Through $z$ there are $s^2 - 1$ lines $l$ which do not contain $x$ or $y$. On $l$, we need to choose $|S| - 1$ additional points. Hence, we get a total number of

$$\mu(s^2 - 1)\binom{s}{|S| - 1}.$$  

$x \sim y$, $z_x = z_y$:

Here, the points $x$, $y$, and $z_x$ are pairwise adjacent, hence they are collinear. Let $l_1$ be the line through $x$ and $y$. We can choose any third point $z_x$ on $l_1$ ($s - 1$ possibilities) and any line $l \neq l_1$ through $z_x$ ($s^2$ possibilities). Finally, we need to take $|S| - 1$ additional points on $l$. The total number is

$$s^2(s - 1)\binom{s}{|S| - 1}.$$
\((d_x, d_y) = (1, 0) : \) Again, we distinguish the cases \(x \sim y\) and \(x \not\sim y\).

\[
\begin{array}{c}
\bullet \\
\bullet
\end{array}
\]

If \(x \not\sim y\), we can take any line \(l_1\) through \(x\), and choose \(x \neq z \not\sim y\). Through \(z\), we can take any line \(l \neq l_1\), and on \(l\) we need \(|S| - 1\) additional points, avoiding the neighbour of \(y\). The total number of choices is

\[
(s^2 + 1)(s - 1)s^2 \left(\frac{s - 1}{|S| - 1}\right).
\]

\[
\begin{array}{c}
\bullet \\
\bullet
\end{array}
\]

If \(x \sim y\), we choose \(x \in l_x \not\sim y\). On \(l_x\) we can take any point \(z_x \neq x\). Through \(z_x\), we can take any line \(l \neq l_x\), and on \(l\) we need \(|S| - 1\) more points, none of which is adjacent to \(y\). The total number is

\[
s^2 \cdot s \cdot s^2 \left(\frac{s - 1}{|S| - 1}\right).
\]

\[
(d_x, d_y) = (0, 0) :
\]

\[
\begin{array}{c}
\bullet \\
\bullet
\end{array}
\]

If \(x \neq y\), we can divide all lines in the GQ into three different parts. There are \(2(s^2 + 1)\) lines which pass through either \(x\) and \(y\). On each of the remaining lines, both \(x\) and \(y\) have one common neighbour each. Let \(L_1\) be the set of lines where these neighbours coincide, and \(L_2\) the set of lines where the neighbours are distinct.

Now \(x\) and \(y\) have \(\mu\) common neighbours. Through each such neighbour there are \(s^2 - 1\) lines which do not pass through either \(x\) or \(y\). All these lines are distinct, since otherwise, \(x\) would have two neighbours on one line. Hence, we get that \(|L_1| = \mu(s^2 - 1)\), and thus \(|L_2| = l - |L_1| - 2(s^2 + 1)\), where \(l\) is the total number of lines.

In order to obtain a graph depicted above, we take any line from \(L_1 \cup L_2\). On this line, we need to choose \(|S|\) points avoiding the neighbours of \(x\) and \(y\). Thus, the total number of such graphs is

\[
|L_1| \binom{s}{|S|} + |L_2| \binom{s - 1}{|S|}.
\]
If \( x \sim y \), we distinguish four types of lines: The line \( l_0 \) connecting \( x \) and \( y \); the set \( L_1 \) of lines intersecting \( l_0 \) in either \( x \) or \( y \); the set \( L_2 \) of lines intersecting \( l_0 \) in other points; and finally the set \( L_3 \) of lines skew to \( l_0 \).

Clearly, \( |L_1| = 2s^2 \). There are \( s - 1 \) additional points on \( l_0 \); through each of them, there are \( s^2 \) lines distinct from \( l_0 \). All these lines are distinct, since otherwise, two lines would intersect in more than one point. Hence we get that \( |L_2| = s^2(s-1) \), and hence, \( |L_3| = l - 1 - |L_1| - |L_2| \).

Now, by a similar reasoning as above, we get that the total number of graphs is

\[
|L_2| \binom{s}{|S|} + |L_3| \binom{s-1}{|S|}.
\]

This completes the proof.

7.3 \( S \) is not complete

Recall that \( S \) is a set of \( t_0 - 2 \) vertices in the graph \( T \) of order \( t_0 \).

**Proposition 7.3.** \( S \) does not contain a clique of size \( t_0 - 3 \).

**Proof.** If \( S \) contains only one point \( z \) not contained in a maximal clique \( C \), then \( z \) has to have two neighbours in \( C \), which is a contradiction.

**Proposition 7.4.** \( S \) does not contain a clique of size \( t_0 - 4 \).

**Proof.** Suppose that \( S \) contains such a clique \( C' \), and that it contains two more vertices, \( w \) and \( z \). By the condition on the minimal valency, \( w \) and \( z \) are adjacent, and both have exactly one neighbour in \( C' \). Moreover, in \( T \), they are both adjacent to each of \( x \) and \( y \), forcing \( x \sim y \). Thus, in \( T \), we have two cliques, \( C' \) of size \( t_0 - 4 \), and \( C = \{x, y, w, z\} \) of size 4.

Let \( l \) be the line in the generalised quadrangle containing \( C \), and let \( l' \) be the line connecting \( x \) and \( y \). The latter is uniquely determined. Denote the unique neighbours of \( x, y, w, z \) on \( l' \) by \( x', y', w', z' \) respectively.

If \( w' = z' \), then \( l \) and \( l' \) intersect, and \( x' = y' = w' \in C' \). In this case, we can count the graphs as follows: We select \( w \) and \( z \) on the line connecting \( x \) and \( y \). The intersection point \( x' \) has to lie on \( l \setminus C \), which gives \( s-3 \) choices. There are \( s^2 \) other lines through \( x' \); we choose one of them and then select \( t_0 - 5 \) additional points on this line. Altogether, we get \( \binom{s-1}{2}(s-3)s^2\binom{s}{t_0-5} \) such graphs.
If \( w' \neq z' \), then \( l \) and \( l' \) do not intersect, and hence \( x', y', w', z' \) are all distinct. We choose the points \( w \) and \( z \) on \( l \), and a line \( l' \) not intersecting \( l \). The points \( w' \) and \( z' \) are uniquely determined; we can choose the remaining points according to whether or not \( x' \) and \( y' \) are in \( C' \). In each case, we can count easily how many graphs we obtain.

Thus, the number of graphs of type \( T \) with respect to \( x \) and \( y \) does not depend on the choice of \( x \) and \( y \), which is a contradiction to the choice of \( T \).

Collecting what we have so far, we get:

**Corollary 7.1.** \( S \) is a graph of order \( t_0 - 2 \) of minimal valency at least 2 not containing a \( t_0 - 4 \)-clique.

**Corollary 7.2.** \( t_0 \geq 7 \).

**Proof.** By the minimal valency condition, \( S \) contains an edge, i.e., a 2-clique. Thus \( t_0 - 4 > 2 \).

We now consider how many vertices in \( S \) can have valency 2.

**Proposition 7.5.** If \( x \sim y \), then there are at most three vertices in \( S \) of valency 2.

**Proof.** In this case, the vertices of valency 2 induce a complete graph by Proposition 7.2.

**Proposition 7.6.** If \( x \not\sim y \), then there are at least two vertices with valency at least 3.

**Proof.** In this case, the vertices of valency 2 induce an empty graph by Proposition 7.2. Each one has to have two neighbours, which necessarily have a valency greater than 2.

**Proposition 7.7.** Let \( x \not\sim y \). Let \( z \) be a vertex of valency 3 in \( S \). Then \( z \) can be adjacent to at most one vertex of valency 2.

**Proof.** Let \( z \) be a vertex of valency 3. In \( T \) it has to be adjacent to at least one of \( x \) and \( y \); let us assume that \( z \sim x \). Let \( v \) and \( w \) be two vertices of valency 2 adjacent to \( z \); this implies \( v \not\sim w \). Both \( v \) and \( w \) have to be adjacent to \( x \), which implies that \( \{x, z, v, w\} \) induces a \( K_4 - e \).

**Proposition 7.8.** The case \( t_0 = 7 \) is impossible.
Proof. Assume that $t_0 = 7$, i.e., $S$ has $t_0 - 2 = 5$ vertices. Then $S$ does not contain a 3-clique. Let $z$ be a vertex with maximal valency in $S$. Then $z$ has at least 3 neighbours. These neighbours must be mutually non-adjacent to avoid a 3-clique. Since they have valency at least 2, they are all adjacent to one additional vertex. Since this accounts for 5 vertices, there are no additional vertices or edges in $S$.

However, now $z$ has valency 3, and it is adjacent to three non-adjacent vertices of valency 2, which is a contradiction. Thus we have proved Theorem 2.

Corollary 7.3. The constant $t_0$ in Klin’s Conjecture is at least 8.

Proof. There are generalised quadrangles of order $(s, s^2)$ for some prime powers $s \geq 5$ with intransitive automorphism groups, see [34].

8 Towards the 8-vertex condition

For the 8-vertex condition, we have to consider a 6-vertex graph $S$ satisfying all the conditions stated in the previous section. A computer search has been performed, and the result is that there are 5 different graphs to consider: The complete bipartite graphs $K_{3,3}$ and $K_{4,2}$; the graph $K_{3,3} - e$ obtained by removing an edge from the complete bipartite graph; the triangular prism $K_3 \circ K_2$, and the graph obtained from the prism by removing one edge from the prism which is not contained in a 3-clique. These graphs are shown in Figure 2.

It can be shown that the number of graphs of the types related to the prism can be determined in a generalised quadrangle of order $(s, s^2)$. However, for the bipartite graph types this is not the case:
Proposition 8.1. There is a GQ(5, 25) whose point graph does not satisfy the 8-vertex condition.

This result was obtained via computer search using known generalised quadrangles. The quadrangle in question is obtained using the set of matrices

\[
\left\{ \begin{pmatrix} t & 3t^2 \\ 0 & 3t^3 \end{pmatrix} \mid t \in GF(5) \right\},
\]

see [35] and [34] for details of this construction. Its automorphism group has several orbits on edges, which can be distinguished by counting complete bipartite graphs \(K_{4,4}\) containing a given edge.

9 Discussion

In this section we consider a number of topics which are naturally related to the main part of the paper but which were not immediately required for our presentation.

Nevertheless, we believe that a wider picture may be helpful for the reader. Also credits should be given to many other researchers who were dealing with the \(t\)-vertex condition, either explicitly or implicitly.

It is also worth mentioning that a first draft of the result given above appeared in the authors’ thesis [37], prepared at the University of Delaware.

9.1 More about strongly regular graphs

The classical concept of a strongly regular graph (SRG) goes back to the investigations of R.C. Bose in relation to the design of statistical experiments. For more than two decades it was considered in terms of partially balanced incomplete block designs, while the term itself was coined in [3].

There are a number of necessary conditions for the parameters \((v, k, \lambda, \mu)\) of a putative SRG, which are formulated by means of standard (or slightly more sophisticated) tools from spectral graph theory (cf. [6]) and which are called feasibility conditions.

There are many known infinite classes and series of SRG’s, in particular those coming from finite permutation groups and finite geometries.

There is a continuous interest to know all SRG’s (up to isomorphism) for a given feasible parameter set. A lot of such information is accumulated on the home page of Andries Brouwer [5].

On the other hand there exist so-called prolific constructions of SRG’s (in the sense of [9]); this usually means that for a given infinite series of feasible parameter sets the number of SRG’s grows exponentially. Techniques to present such prolific constructions were suggested by W.D. Wallis and developed quite essentially by D. Fon-Der-Flaass and M. Muzychuk, see [31].

The discovery of prolific constructions put a final dot in the understanding of the fact that the full classification of SRG’s is a hopeless problem.
One who is interested in “nice” SRG’s is forced to rigorously formulate additional requirements to the considered objects, using suitable group-theoretical or combinatorial language.

9.2 \( k \)-isoregular graphs

The concept of a \( k \)-isoregular graph has two independent origins, both related to the investigation of rank 3 permutation groups. The first origin is the Ph.D. thesis of J.M.J. Buczak (1980), fulfilled at Oxford. The main results of this thesis were briefly mentioned in Section 8 of [8], while the text itself was not available to M. Klin and his colleagues for three decades.

The other origin is the paper [17], where all absolutely homogeneous graphs were classified and first steps were taken towards the investigation of \( k \)-homogeneous graphs for \( k \geq 2 \); note that 2-homogeneous graphs are exactly rank 3 graphs.

It should be mentioned that the investigation of absolutely homogeneous graphs was an attractive goal for many other researchers. More or less at the same time the full classification was suggested in publications by G. Cherlin, A.D. Gardiner, H. Enomoto, Ch. Ronse, J. Sheehan and others, see, e.g., the detailed bibliography in the book [12].

It was Ja. Gol’fand who first realized that the concept of a \( k \)-homogeneous graph may be approximated in purely combinatorial terms by \( k \)-isoregular graphs. In particular, absolutely regular graphs coincide with the absolutely homogeneous graphs, and the proof of this fact may be achieved without any use of group-theoretical arguments. An outline of the proof (due to Gol’fand) is presented in the Section 4.3 of [28].

In fact, the history of this proof is just a small visible part of the related drama of ideas. Around 1979-80 Gol’fand prepared a manuscript with the claim that each 5-regular graph is absolutely regular. Unfortunately a fatal mistake (discovered by M. Klin) appeared on the very last page of this detailed text. However, all previous results in this ingenious text were correct, thus the real claim obtained by Gol’fand was the description of two putative infinite classes of parameter sets for 4-regular graphs. One class corresponds to absolutely regular graphs, while the other class, denoted \( M(r) \) by Gol’fand, was presenting an infinite series of possible parameters. The graph \( M(1) \) in this series corresponds to the famous Schläfi graph on 27 vertices, while \( M(2) \) is the unique McLaughlin graph with the parameters \((275, 112, 30, 56)\). The existence of the graphs \( M(r) \) for \( r \geq 3 \) remained open. (A brief introduction about this series is also available in [28].)

Klin and his colleagues made a lot of efforts to convince Gol’fand to publish his brilliant result (without the final page), however they never succeeded. Gol’fand spent many years in attempts to reach the full desired result. After the collapse of the USSR he lived in relative poverty. At the end of the century he was killed in his apartment under unclear circumstances.

Around 1995, when he was still alive, Klin and Woldar established an attempt to publish a series of papers, starting from Gol’fand’s text as Part I and
with [27] as Part II. Unfortunately after Gol’fand’s death the fate of his heritage still remains indefinite, this is one of the reasons why the work over the unfinished text [27] was conserved.

It should be mentioned that the original term $k$-regular suggested by Gol’fand lead to possible confusion. Indeed, for many researchers in graph theory, $k$-regular means regular of valency $k$. This is why Klin and Woldar in [27] decided to use the new term $k$-isoregular, which does not imply any confusion.

We refer to [36] for all necessary precise formulations related to the concept of $k$-isoregularity.

One more approach, influenced by the consideration of graphs with the $t$-vertex condition is developed in publications by J. Wojdylo, see [40] for example.

### 9.3 Smith graphs

In 1975 a few papers were published by M. Smith, see for example [38], devoted to rank 3 permutation groups with the property that each of the two transitive subconstituents is also a rank 3 group. All possible feasible parameters for the corresponding SRG’s were presented in evident form. At that time the result was regarded as a program for further attacks toward a better understanding of such graphs. A few years later, with the announcement of the classification of finite simple groups, this research program became obsolete.

Since that time any SRG with the parameters described by Smith is called a Smith graph, be it rank 3 or not. A great significance of Smith graphs in the current context follows from the the paper [11], in which those SRG’s were investigated for which for each vertex $x$ both induced subgraphs $\Gamma_1(x)$ and $\Gamma_2(x)$ are also SRG’s. It is easy to check that this class of graphs strictly coincides with 3-isoregular graphs. (Note that the complete graph and the empty graph may be regarded as degenerate SRG’s.)

The main result in [11] claims that each 3-isoregular graph is either the pentagon, a pseudo Latin square or negative Latin square graph, or (up to complement) a Smith graph. Another significant ingredient of [11] is the established link of the class of 3-isoregular graphs with extremal properties of the Krein parameters of SRG’s and spherical $t$-designs (this link will be briefly mentioned in Section 9.6).

We should also mention Section 8 of [11] which deals with generalised quadrangles of order $(q, q^2)$. It was proved there that such geometrical structures coexist with orthogonal arrays of strength 3 and order $q$, as well as with certain codes over an alphabet with $q$ letters. This coexistence creates an additional challenge for the reader to try to obtain our main result in absolutely different context, relying only on the geometrical analysis of the related codes.

We wish also to mention that 3-isoregular graphs are frequently called triply regular graphs, which from time to time are subject of ongoing research, see [18]. Hopefully this section of our paper will help the modern investigators to better comprehend all the facets of these striking combinatorial objects.

The concept of a 3-isoregular graph can be generalised to an arbitrary symmetric association scheme. In this more general approach links with spherical
9.4 The 4-vertex condition

Here we provide more details regarding the consideration of the 4-vertex condition.

The paper [22] (the last research input by Andrei Ivanov before his transfer to the software industry) introduces two infinite families of graphs on $2^{2n}$ vertices via the use of suitable finite geometries. It also contains a rigorous proof of a folklore claim that in the definition of the $t$-vertex condition for a regular graph it is enough to check its fulfillment just on edges and non-edges (that is, the inspection of loops is redundant).

Note that [22] contains interesting conjectures about the induces subgraphs of his graphs which still remain unnoticed by modern researchers (although one of the conjectures was confirmed in [36]).

Besides the well-known paper [19], Higman himself was considering graphs with the 4-vertex condition in (at least) one more paper [20]. In particular, he considered the block graphs of Steiner triple systems with $m$ vertices (briefly $STS(m)$), which are known to be SRG’s. Higman proved that the block graph satisfies the 4-vertex condition if the $STS(m)$ consists of the points and lines in projective space $PG(n, 2)$ over the field of two elements, here, $m = 2^n - 1$. Besides this, the 4-condition may be satisfied for $STS(m)$, for $m \in \{9, 13, 25\}$.

While the cases $m = 9$ and $m = 13$ can be easily settled, the case $m = 25$ remained open for four decades. Following the advice of M. Klin, P. Kaski et al obtained the negative answer (see [26]), essentially relying on the use of computer and some extra clever ad hoc tricks (the complete enumeration of all $STS(25)$ looks hopeless at the moment).

There are also known a few sporadic proper (non rank 3) SRG’s which satisfy the 4-condition. The smallest such graphs have 36 vertices, they were carefully investigated in [29].

9.5 Generalisations for association schemes

Every SRG $\Gamma$ together with the complement $\overline{\Gamma}$ forms a symmetric association scheme with two classes (or of rank 3). Therefore it is natural to consider the concept of the $t$-vertex condition for arbitrary association schemes.

Non-symmetric schemes with two classes are equivalent to pairs of complementary doubly regular tournaments. Each such (skew-symmetric) scheme satisfies the 4-vertex condition [32].

The situation for higher ranks is much more sophisticated and goes beyond the scope of this survey. The first results in this direction were presented in [16] and [7], though without the evident use of association schemes. Further steps stem from [14].
9.6 Interactions with other problems

During the last two decades highly symmetrical association schemes, and in particular SRG’s, were used quite essentially in a number of significant research approaches in diverse parts of mathematics. For each such case it is not so easy to recognize immediately which properties of the related graphs or schemes are exactly requested. The reason of difficulties is as a rule a serious discrepancy in the languages adopted in the corresponding parts of mathematics.

A few examples are briefly mentioned below, each together with at least one striking reference.

Two decades ago F. Jaeger demonstrated in [24] how one could produce spin models for the applications in the theory of link invariants (in the sense of V.R.F. Jones), using suitable self-dual association schemes. The requested SRG’s should be 3-isoregular.

The problem of description of finite point systems in Euclidean space which satisfy some energy minimization criteria was attacked by H. Cohn et al in a number of publications throughout the last decade. Surprisingly, they established links of putative optimal systems with some famous SRG’s, among them there are a few 3-isoregular graphs, see [13].

Deterministic polynomial factoring is considered in the framework of m-schemes as they are called in [23]. In fact, this class of objects is strictly related to diverse kinds of highly symmetrical association schemes (as they were mentioned in Section 9.5). The recent thesis [1] is a good source to digest sophisticated interactions, which appear in this new line of research.

Spherical designs are intimately related to the existence of some classes of SRG’s, in particular to putative 3- and 4-isoregular graphs. The survey [13] may be helpful as an initial introduction. In this context, the negative results presented in [2] imply the non-existence of infinitely many 4-isoregular graphs. This is, in a sense, a partial fulfilment of the foremost dream of the late Ja. Gol’fand. It should be mentioned that such an implication is not immediately visible for a non-perplexed reader.

10 Conclusion

We have shown that the point graphs of generalised quadrangles of order $(s, s^2)$ satisfy the 7-vertex condition. This provides us with an infinite family of strongly regular graphs satisfying that condition which have intransitive automorphism groups.

Hence, if Klin’s Conjecture holds (i.e., if there is a number $t_0$ such that the $t_0$-vertex condition implies a rank 3 automorphism group) then $t_0 \geq 8$.

Still, the proof of this conjecture in its full generality appears intractable. If we restrict ourselves to point graphs of generalised quadrangles, then it looks reasonable to prove a similar statement, in particular since many characterization of the “classical” generalised quadrangles (which include those with a rank 3 group) are known.
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