Efficiency of Optimized Pulse Position Modulation with Noisy Direct Detection

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Abstract—We analyze theoretically the impact of background counts on the efficiency of optical communication in the photon-starved regime using the pulse position modulation (PPM) format with direct detection. Degradation of the photon information efficiency is studied in the case when the background count rate is at most comparable with the rate of photodetection events generated by the incoming optical signal. The PPM symbol length is optimized under the constraint of a fixed average spectral power density using an analytical approximation. The results are compared with generalized on-off keying (OOK) optimized over the a priori probability distribution for the input binary alphabet. The generalized OOK scheme can be viewed as a relaxation of the PPM scheme by removing the requirement that a light pulse must occur exactly once in a fixed-length frame of time bins that constitute the PPM symbol. It is shown that the asymptotic scaling of the photon information efficiency does not change qualitatively in this scenario.

I. INTRODUCTION

Pulse position modulation (PPM) is the format of choice for deep-space optical communication, when the average signal power spectral density corresponds to much less than the energy of one photon at the carrier frequency [1]. In the PPM format the information is encoded in the position of a light pulse in a frame of otherwise empty time bins. It can be recovered on the receiver side from the timing of a detection event on a photon counting detector. A very weak light pulse may produce no photocount, resulting in the erasure of photodetection events [6]. The purpose of this contribution is to discuss the efficiency of the PPM scheme in the presence of background noise. Specifically, we will study photon information efficiency when the background count probability is at most comparable with the average received number of photons in one time bin. We will show that in this regime one can obtain closed approximate formulas for the quantities of interest. These approximations are compared with the results of numerical optimization. Our analysis of the PPM scheme is restricted to the case of a simple-decision receiver, which interprets as erasures instances when counts have been registered in two or more time bins within one PPM frame. In order to gain an insight into possible enhancement achievable through more sophisticated encoding and decoding strategies, we consider also generalized on-off keying (OOK) which uses a binary alphabet consisting of an empty time bin and a light pulse used with an arbitrary a priori probability distribution. The relative frequency with which pulses occur in the PPM format, given by the inverse of the PPM order, can be related directly to the probability of sending a pulse in the generalized OOK scheme, but in the latter case it is no longer required that exactly one pulse must be sent in each frame of a constant length. Furthermore, Shannon mutual information for the generalized OOK scheme provides an upper bound on more intricate decoding schemes beyond the simple-decision model.

This paper is organized as follows. In Sec. II we review optimization of PPM with direct detection and no background noise. Results for generalized OOK without and with background noise are presented in Sec. III. Optimization of PPM in the presence of background counts is discussed in Sec. IV. Sec. V concludes the paper. The main results of the paper are analytical expressions (14) and (19) for the photon information efficiency respectively in the case of noisy generalized OOK and noisy PPM with a simple-decision receiver. In the presented expressions \( n_b \) is the average number of received photons per time bin and \( n_e \) is the average number of background counts in one bin. Auxiliary functions used in these formulas are defined in Eqs. (6) and (13).

II. PPM WITHOUT BACKGROUND NOISE

Before addressing the impact of background counts, let us first review the efficiency of PPM communication in a background-free scenario. As depicted schematically in Fig. 1 the \( M \)-ary PPM format uses \( M \) equiprobable symbols defined...
by the position of a single light pulse in a frame that consists of $M$ time bins. Direct detection of the incoming signal allows one to identify unambiguously the received symbol unless the detector does not click at all, which corresponds to an erasure outcome. For the average photon number per time bin in the received signal equal to $n_a$, the light pulse contains on average $Mn_a$ photons which corresponds to the optical energy available in the entire frame under the average signal power constraint. Assuming Poissonian statistics for the photon number distribution in the received light pulse, the probability that it will generate at least one photocount on the detector is $p_p = 1 - \exp(-Mn_a)$. We will refer to such an event as a detector click. To simplify subsequent formulas, the non-unit detection efficiency has been included in the overall transmission of the optical channel, lowering the effective value of $n_a$. Shannon mutual information expressed in bits per time bin for an $M$-ary erasure channel with non-erasure probability $p_p$ reads

$$I_{PPM} = \frac{p_p}{M} \log_2 M. \tag{1}$$

Anticipating that the optimal PPM order $M$ is high for $n_a \ll 1$, in the following we will treat $M$ as a continuous variable. As seen in Fig. 2(a), for a fixed $n_a$ the mutual information $I_{PPM}$, or equivalently the photon information efficiency PIE $I_{PPM}/n_a$, has a well defined maximum in $M$. Its approximate location can be found by assuming that the received pulse contains on average much less than one photon, $Mn_a \ll 1$, and expanding the photocount probability $p_p$ up to the second order in $Mn_a$,

$$p_p \approx Mn_a - \frac{1}{2} (Mn_a)^2. \tag{2}$$

Inserting this expansion into Eq. (1) and equating to zero its first derivative with respect to $M$ yields a transcendental equation for the optimal $M^*$, which can be solved in terms of the Lambert $W$ function [5]:

$$M^* = \frac{2}{n_a} \left[ W \left( \frac{2e}{n_a} \right) \right]^{-1}. \tag{3}$$

The above approximation is compared with the results of numerical optimization of the exact expression for $I_{PPM}$ in Fig. 2(b). It is seen that the optimal PPM order is indeed high in the considered range of $n_a$ and that it grows monotonically with decreasing $n_a$. Fig. 3 depicts the mean photon number $M^*n_a$ contained in the pulse for the optimal PPM order. Notably, over the range $10^{-7} < n_a < 10^{-2}$ covering five orders of magnitude, $M^*n_a$ changes only by a factor of 4. The behavior in the limit $n_a \to 0$ can be obtained from the asymptotic form [7] of the Lambert function $W(x)$ for large arguments $x \gg 1$,

$$W(x) = \log x - \log \log x + \ldots, \tag{4}$$

where the omitted terms are of the order in $x$ lower than a constant. Using the leading term of the above expansion in
Eq. (5) yields
\[ M^* n_a \approx 2 \left( \log \left( \frac{2e}{n_a} \right) \right)^{-1}, \]
which implies that the mean photon number in the pulse decreases very slowly when \( n_a \) tends to zero. In this regime Eq. (5) implies the hierarchy \( n_a \ll M^* n_a \ll 1 \), which justifies expansion (2).

Inserting the quadratic expansion (2) and the approximate expression for the optimal PPM order (3) into Eq. (1) yields a closed formula for the maximum photon information which can be cast as \( I_{PPM} = n_a \Pi(n_a) \), where \( \Pi(n_a) \) is the photon information efficiency specifying how much information can be on average transmitted per one received photon for optimized PPM with direct detection and no background noise. This quantity can be written using the Lambert \( W \) function in a compact form (8)
\[ \Pi(n_a) = \left\{ W \left( \frac{2e}{n_a} \right) - 2 + \left[ W \left( \frac{2e}{n_a} \right) \right]^{-1} \right\} \log_2 e. \] (6)

In Fig. 4 the above approximate formula is compared with results of numerical optimization of the complete expression given in Eq. (1). The agreement is remarkably good. The specific form of the photon information efficiency given in Eq. (6) is convenient for analyzing the limit \( n_a \to 0 \). Using the asymptotics of the Lambert function (4) the expression given in Eq. (6) can be expanded up to the term constant in \( n_a \) as
\[ \Pi(n_a) = \log_2 \frac{1}{n_a} - \log_2 \left( \frac{2e}{n_a} \right) - \log_2 \frac{e}{2} + \ldots. \] (7)

This matches the results of the asymptotic study presented in (4). It is instructive to compare this expression with the upper bound on the photon information efficiency implied by the ultimate capacity \( C \) of a lossy bosonic channel under the constraint of a fixed mean photon number per channel use (9), which in the limit \( n_a \ll 1 \) admits the expansion
\[ \frac{1}{n_a} C = \log_2 \frac{1}{n_a} + \log_2 e + \ldots. \] (8)

Both expressions (7) and (8) have the same form in the leading order of \( n_a \), growing unbounded when \( n_a \to 0 \). A difference appears in the second to the leading order. The photon information efficiency for PPM includes a negative double logarithmic term \( - \log_2 \log(2e/n_a) \) which produces a gap with respect to the ultimate capacity limit seen in Fig. 4.

III. GENERALIZED ON-OFF KEYING

Let us now relax the requirement of the PPM format that exactly one pulse needs to be positioned in each frame of \( M \) time bins. This is equivalent to generalized on-off keying, when light pulses are sent in arbitrarily chosen bins with the overall probability \( q \) and remaining bins are left empty. To satisfy the constraint on the average signal power spectral density, the mean photon number in a single pulse is \( n_a/q \). When noiseless direct detection is used at the output, from the information theoretic viewpoint the scheme is described by the \( Z \) channel shown in Fig. 5(a). An empty bin always produces an unambiguous outcome in the form of a no-count event, whereas a light pulse generates a detector click with a probability \( p_p = 1 - \exp(-n_a/q) \), corresponding to registering at least one photocount. As before, Poissonian photon number statistics is assumed. Shannon mutual information for this scenario reads
\[ I_{OOK} = H(qp_p) - qH(p_p), \] (9)

where \( H(x) = -x \log_2 x - (1 - x) \log_2 (1 - x) \) is the binary entropy function. Let us now assume that both \( q \) and the mean
In the presence of background noise an empty bin is identified unambiguously via direct detection while a light pulse is detected with a probability \( p_b \). When the background noise is present, an empty bin generates a click with a probability \( p_b \) and a light pulse is detected with an overall probability \( p_c \).

The background noise occurs with a probability of one. For an empty input bin a detector click generated by the background noise is identified unambiguously via direct detection while a light pulse is detected with a probability \( p_b \). Consequently it should be sufficient to retain only the first term in Eq. (10) and one can expect that the approximate formula in Eq. (10) with \( q = 1/M \), i.e. the relative frequency with which pulses occur in the PPM signal. Maximization of the first term alone yields the same result as in the PPM scenario with the optimal \( q^* = 1/M^* \). For this value, the combined second and third terms in Eq. (11) are of the order of \( M^*n_a^2 \), hence their contribution to the photon number efficiency scales as \( M^*n_a^2 \) which is lower order in \( n_a \) than a constant according to Eq. (5).

Let us now consider generalized OOK with direct detection in the presence of background noise. The corresponding channel model is shown schematically in Fig. 5(b). We will assume that the background noise generates Poissonian count statistics with an average \( n_b \). We will restrict our attention to the regime when the noise power does not exceed substantially the average signal power, i.e. the ratio \( n_b/n_a \) is of the order of one. For an empty input bin a detector click generated by background noise occurs with a probability \( p_b = 1 - e^{-n_b} \), while the overall probability \( p_c \) that an incoming light pulse results in a click is given by

\[
p_c = p_p + p_b - p_p p_b = 1 - \exp(-n_b - n_a/q).
\]

In this setting Eq. (5) needs to be replaced by a more general expression for the Shannon mutual information which can be rearranged as

\[
I_{\text{OOK}} = \frac{q p_c \log_2 \frac{1}{q} + q(1 - p_p) \log_2(1 - p_p)}{1 - (1 - q p_p) \log_2(1 - q p_p)}.
\]

The first term in the second line has the form familiar from Eq. (12) for numerical optimization and on Eq. (14) as the analytical approximation. The shaded area represents the physically inaccessible region beyond the ultimate capacity of a lossy bosonic channel [9].

In the absence of background noise an empty bin is identified unambiguously via direct detection while a light pulse is detected with an overall probability \( p_c \). Let us now consider generalized OOK with direct detection in the presence of background noise. The corresponding channel model is shown schematically in Fig. 5(b). We will assume that the background noise generates Poissonian count statistics with an average \( n_b \). We will restrict our attention to the regime when the noise power does not exceed substantially the average signal power, i.e. the ratio \( n_b/n_a \) is of the order of one. For an empty input bin a detector click generated by background noise occurs with a probability \( p_b = 1 - e^{-n_b} \), while the overall probability \( p_c \) that an incoming light pulse results in a click is given by

\[
p_c = p_p + p_b - p_p p_b = 1 - \exp(-n_b - n_a/q).
\]
information efficiency is the subtraction of a constant term equal to $g(n_b/n_a)$.

IV. PPM WITH NOISY DIRECT DETECTION

Let us now turn our attention to the case of PPM with noisy direct detection. The response of the detector in an individual time bin is given by the same set of conditional probabilities as for the generalized OOK scheme shown in Fig. 5(b). An additional layer of complexity stems from the fact that in the presence of background noise clicks may occur in two or more distinct time bins within a single PPM frame. We will treat such instances as erasures and consider only outcomes when for a given PPM symbol the detector clicks at most in one time bin. The probability of the detector clicking exclusively in the bin corresponding to the position of the light pulse in the input symbol is

$$p_e = p_c(1 - p_b)^{M-1} = e^{-(M-1)n_b} - M(n_a + n_b).$$

(15)

On the other hand, the probability that a click occurs in a specific time bin that differs from the position of the light pulse is $p_d = (1 - p_c)p_b(1 - p_b)^{M-2}$. The full expression for the Shannon mutual information, expressed in bits per time bin, in such a scenario reads

$$I_{\text{PPM}} = p_e \frac{1}{M} \log_2 M + p_d \frac{M-1}{M} \log_2 \frac{M p_d}{p_c} - p_c + (M-1) p_d \log_2 \left( 1 + \frac{(M-1)p_d}{p_c} \right).$$

(16)

The right hand side has structure similar to that of Eq. (15). However, care needs to be taken when optimizing analytically the first term with respect to the PPM order $M$. This is because the quadratic expansion of the probability $p_c$ has the form

$$p_c \approx M n_a + \frac{1}{2} \frac{M}{M n_a}^2$$

(17)

which compared to Eq. (2) differs in the second order by an additional factor $\gamma = 1 + 2n_b/n_a$. An analogous factor appears in the analysis of PPM efficiency in the absence of background noise, when the light pulse has non-Poissonian photon number statistics. In the latter case, $\gamma$ has the physical interpretation of the normalized second-order intensity correlation function of the light source [5]. Optimization of the first term in Eq. (16) using the expansion (17) can be carried out similarly as in Sec. II with the maximum value equal to $n_a \Pi(\gamma n_a)$ and attained at

$$M^* = \frac{2}{\gamma n_a} \left[ W \left( \frac{2e}{\gamma n_a} \right) \right]^{-1}.$$

(18)

This expression is compared with results of numerical optimization for several values of the ratio $n_b/n_a$ in Fig. 4(b). The remaining two terms in Eq. (16) can be simplified using approximations $p_c \approx M n_a$ and $(M-1)p_d \approx M p_d \approx M n_b$. The final result for the photon information efficiency is

$$\text{PIE}_{\text{PPM}} = \Pi(n_a + 2n_b) - g(n_b/n_a).$$

(19)

The difference with the generalized OOK expression (14) is in the first term. Because the function $\Pi$ is monotonically decreasing, adding a term $2n_b$ to its argument lowers its value. This effect is clearly seen in Fig. 4 when compared to the generalized OOK case with the same ratio $n_b/n_a$ depicted in Fig. 6. The difference between $\Pi(n_a)$ and $\Pi(n_a + 2n_b)$ can be estimated using the leading order of the expansion given in Eq. (18). A simple calculation yields the value of the gap equal to $\log_2(1 + 2n_b/n_a)$. Consequently, for a fixed ratio $n_b/n_a$ the asymptotic scaling of the photon information efficiency is the same for both generalized OOK and PPM with a simple-decision receiver.

V. CONCLUSIONS

We have derived approximate analytical formulas for the photon information efficiency attainable using PPM and generalized OOK with noisy direct detection under the constraint of a given average power spectral density, expressed in terms of the average photon number per bin. The parameters of the input symbol ensemble have been optimized assuming that the ratio of the noise power to the average signal power is of the order of one. For the fixed ratio, with diminishing signal power the photon information efficiency in both scenarios follows in the leading order the ultimate capacity limit of a lossy bosonic channel. Remarkably, the performance of optimized PPM and generalized OOK is not significantly affected as long as the background noise power remains comparable to the average signal power.

The presented formulas can serve as a reference for benchmarking the performance of practical deep-space communication systems. It should be noted that with decreasing average power spectral density the optimal PPM order grows unbounded, which in the case of generalized OOK is reflected by the increasing imbalance between a priori probabilities for the input symbols. This results in escalating requirements for the peak-to-average power ratio, which may be difficult to meet with a satisfactory electrical-to-optical conversion efficiency.

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