A NOVEL PREDICTOR-CORRECTOR SCHEME FOR SOLVING VARIABLE-ORDER FRACTIONAL DELAY DIFFERENTIAL EQUATIONS INVOLVING OPERATORS WITH MITTAG-LEFFLER KERNEL

ANTONIO CORONEL-ESCAMILLA
Tecnológico Nacional de México/CENIDET
Interior Internado Palmira S/N, Col. Palmira
C.P. 62490, Cuernavaca Morelos, México

JOSÉ FRANCISCO GÓMEZ-AGUILAR
CONACyT-Tecnológico Nacional de México/CENIDET
Interior Internado Palmira S/N, Col. Palmira
C.P. 62490, Cuernavaca Morelos, México

Abstract. In this work we present a numerical method based on the Adams-Bashforth-Moulton scheme to solve numerically fractional delay differential equations. We focus in the fractional derivative with Mittag-Leffler kernel of type Liouville-Caputo with variable-order and the Liouville-Caputo fractional derivative with variable-order. Numerical examples are presented to show the applicability and efficiency of this novel method.

1. Introduction. Recent studies in science and engineering demonstrated that the dynamics of many systems may be described more accurately by means of differential equations of non-integer order. The concept of fractional differentiation has been singled out as outstanding mathematical tools to portray more accurately many real world problems in the passes decades. Mathematical and physical considerations in favor of the use of models based on derivatives of non-integer order are given in [35]-[40] and the references therein.

In the mathematical description of a physical process, one generally assumes that the behavior of the process considered depends only on the present state, an assumption which is verified for a large class of dynamical systems. However, there exist situations where this assumption is not satisfied. In the such cases, it is better to consider that the system’s behavior includes also information on the former state. These systems are called time-delay systems [31]. Inclusion of delay in the fractional differential equations finding applications in all disciplines including chemistry, physics, and finance [7]-[42]. Several works have extending standard numerical methods such as Adams-Bashforth method to solve non-linear fractional differential equations [18]-[30]. Recently Daftardar-Gejji introduced a
new predictor-corrector method to numerically solve fractional differential equations [13]- [14]. The authors shown the accurate and time efficient compared with other methods. Also, in [29], the authors studied the system of first order delay differential equations, using spline functions, and studied the stability and the error analysis.

Several studies [32]- [39] have shown that, many complex physical problems can be described with great success via variable-order (VO) derivatives. A novel study underlining the advantages of using these derivatives rather than constant order fractional derivative had been presented in [34]. Some applications include processing of geographical data in [10], diffusion processes in [38]- [33] and groundwater flow equation [4]. Since the equations described by the VO derivatives are highly complex, difficult to handle analytically, it is therefore advisable to investigate their solutions numerically. Possible numerical implementations of VO fractional derivatives are given in [5]- [23]. In this paper, we considered the fractional derivative with Mittag-Leffler kernel of type Liouville-Caputo and the Liouville-Caputo definition.

The paper is organized as follows. Notations and basic definitions of fractional operators with variable-order are given in Section 2. In Section 3 fractional Adams-Bashforth-Moulton method for fractional delay differential equations with Mittag-Leffler kernel is outlined. In Section 4 some illustrative examples are given and conclusions are summarized in Section 5.

2. Fractional operators with variable-order. The Liouville-Caputo (C) fractional operator with variable-order \( \alpha(t) \) is defined as [23]

\[
C_0^\alpha D_t^\alpha f(t) = \frac{1}{\Gamma(1 - \alpha(t))} \int_0^t (t - \tau)^{-\alpha(t)} \frac{d}{dt} f(\tau) d\tau, \quad 0 < \alpha(t) \leq 1, \tag{1}
\]

where, \( C_0^\alpha D_t^\alpha \) is a Liouville-Caputo fractional operator of order \( \alpha(t) \) with respect to \( t \).

Replaced the power law with exponential decay law, we obtain the Caputo-Fabrizio operator in Liouville-Caputo sense (CFC) with variable variable-order \( \alpha(t) \) defined as follows

\[
CFC_0^\alpha D_t^\alpha f(t) = \frac{B(\alpha(t))}{1 - \alpha(t)} \int_0^t \frac{d}{dt} f(\tau) \exp \left[ -\frac{\alpha(t)(t - \tau)}{1 - \alpha(t)} \right] d\tau, \quad 0 < \alpha(t) < 1, \tag{2}
\]

where, \( B(\alpha(t)) \) is a normalization function, when the fractional order \( \alpha \) is variable with respect to the time, this normalization take the form

\[
B(\alpha(t)) = 1 - \alpha(t) + \frac{\alpha(t)}{\Gamma(\alpha(t))}. \tag{3}
\]

The Caputo-Fabrizio operator in Riemann-Liouville sense (CFR) with variable-order \( \alpha(t) \) is defined as follows

\[
CFR_0^\alpha D_t^\alpha f(t) = \frac{B(\alpha(t))}{1 - \alpha(t)} \frac{d}{dt} \int_0^t f(\tau) \exp \left[ -\frac{\alpha(t)(t - \tau)}{1 - \alpha(t)} \right] d\tau, \quad 0 < \alpha(t) < 1, \tag{4}
\]

where, \( B(\alpha(t)) \) is a normalization function defined in Eq. (3).

Atangana and Baleanu, considering the generalized Mittag-Leffler function as kernel of differentiation, this kernel is non-singular and nonlocal and preserve the benefits of the above fractional operators [6]. Replaced the exponential kernel with generalized Mittag-Leffler function, we obtain the fractional operator of type
Atangana-Baleanu in Liouville-Caputo sense (ABC) of variable-order \( \alpha(t) \) defined as follows

\[
\textcolor{blue}{0} \text{ABC} \mathcal{D}_t^{\alpha(t)} f(t) = \frac{B(\alpha(t))}{1-\alpha(t)} \int_0^t f(\tau) E_{\alpha(t)} \left[ - \alpha(t) \frac{(t-\tau)^{\alpha(t)}}{1-\alpha(t)} \right] d\tau, \quad 0 < \alpha(t) \leq 1,
\]

(5)

where, \( B(\alpha(t)) \) is a normalization function defined in Eq. (3).

Atangana and Baleanu also suggest another fractional operator with Mittag-Leffler kernel in Riemann-Liouville sense (ABR) of variable-order \( \alpha(t) \) defined as follows

\[
\textcolor{blue}{0} \text{ABR} \mathcal{D}_t^{\alpha(t)} f(t) = \frac{1-\alpha(t)}{B(\alpha(t))} \frac{d}{dt} \int_0^t f(\tau) E_{\alpha(t)} \left[ - \alpha(t) \frac{(t-\tau)^{\alpha(t)}}{1-\alpha(t)} \right] d\tau, \quad 0 < \alpha(t) \leq 1,
\]

(6)

where, \( B(\alpha(t)) \) is a normalization function defined in Eq. (3).

The Atangana-Baleanu (AB) fractional integral of variable-order \( \alpha(t) \) is defined as

\[
\textcolor{blue}{0} \text{AB} \mathcal{I}_t^{\alpha(t)} f(t) = \frac{1-\alpha(t)}{B(\alpha(t))} f(t) + \frac{\alpha(t)}{B(\alpha(t)) \Gamma(\alpha(t))} \int_0^t f(\tau)(t-\tau)^{\alpha(t)-1} d\tau.
\]

(7)

3. Fractional delay differential equations. In this paper, we provide a novel algorithm for solving fractional delay differential equations using the ABC derivative. First, let us consider the following time delayed fractional system

\[
\textcolor{blue}{0} \text{ABC} \mathcal{D}_t^{\alpha(t)} y(t) = f(t, y(t), y(t-\delta)), \quad 0 \leq t \leq T, \quad 0 < \alpha(t) \leq 1,
\]

(8)

\[
y(t) = g(t), \quad -\delta \leq t \leq 0,
\]

(9)

where, \( T \in \mathbb{R}^+ \), \( g(t) \) and the coefficients of \( y(t) \) and \( y(t-\delta) \) represent smooth functions, \( \delta \in \mathbb{R}^+ \) denotes the delay.

If the function \( f \) is continuous, the solution of Eq. (8) can be rewritten using the fractional Atangana-Baleanu (AB) integral as follows

\[
y(t) = y_0 + \frac{1-\alpha(t)}{B(\alpha(t))} f(t, y(t), y(t-\delta)) + \frac{\alpha(t)}{B(\alpha(t)) \Gamma(\alpha(t))} \int_0^t f(\tau, y(\tau), y(\tau-\delta))(t-\tau)^{\alpha(t)-1} d\tau.
\]

(10)

The solution of Eq. (10) can be discretized using the predictor-corrector algorithm, where, \( y(t) \) should be consistent in the interval \([0, T]\) and differential consistent in the interval \((0, T)\).

Consider a uniform grid \( \{t_n = nh : n = -m, -m + 1, ..., -1, 0, 1, ..., N\} \), \( m \) and \( N \) are integers such that, \( m = \frac{\delta}{h} \) and \( N = \frac{T}{h} \). Let

\[
y(t_n) = g(t_n), \quad n = -m, -m + 1, ..., -1, 0,
\]

(11)

and note that

\[
y(t_n - \delta) = y(nh - mh) = y(t_{n-m}), \quad n = 0, 1, 2, ..., N.
\]

(12)

Now the integral (10) is evaluated using the trapezoidal quadrature formula and we have the corrector formula as

\[
y(t_{n+1}) = y_0 + \frac{1-\alpha(t_{n+1})}{B(\alpha(t_{n+1}))} f(t_{n+1}, y_{n+1}, y_{n+1-m}) +
\]
where, 
\[ b \]

the predicted value \( y \)

\[ a \]

\[ n \]

\[ \{ \]

\[ \alpha \]

\[ \frac{B(a(t_{n+1}))}{B(a(t_{n+1}))} \left[ \frac{h_{\alpha(t_{n+1})}}{\Gamma(a(t_{n+1}) + 2)} f(t_{n+1}, y_{n+1}, y_{n+1-m}) + \right. \]

\[ + \frac{h_{\alpha(t_{n+1})}}{\Gamma(a(t_{n+1}) + 2)} \sum_{j=0}^{n} a_{j,n+1} f(t_{j}, y_{j}, y_{j-m}) \right], \] (13)

where,

\[ a_{j,n+1} = \]

\[ \begin{cases} 
    n^{\alpha(t_{n+1})} - (n - \alpha(t_{n+1}))(n + 1)^{\alpha(t_{n+1})} & j = 0, \\
    (n - j + 2)^{\alpha(t_{n+1})} + (n - j)^{\alpha(t_{n+1})} - 2(n - j + 1)^{\alpha(t_{n+1})} & 1 \leq j \leq n,
\end{cases} \]

the predicted value \( y^p(t_{n+1}) \) is determined by

\[ y^p(t_{n+1}) = y_0 + \frac{1 - \alpha(t_{n+1})}{B(a(t_{n+1}))} f(t_{n+1}, y_{n+1}, y_{n+1-m}) + \]

\[ + \frac{\alpha(t_{n+1})}{\Gamma(a(t_{n+1})) B(a(t_{n+1}))} \sum_{j=0}^{n} b_{j,n+1} f(t_{j}, y_{j}, y_{j-m}), \] (14)

where, \( b_{j,n+1} = \frac{h_{\alpha(t_{n+1})}}{\alpha(t_{n+1})} ((n + 1 - j)^{\alpha(t_{n+1})} - (n - j)^{\alpha(t_{n+1})}) \), \( j = 0, 1, 2, ..., n \).

The Eqs. (13) and (14) constitute a novel algorithm for the numerical approximation of Eq. (8) considering the fractional ABC derivative with variable-order \( \alpha(t) \).

3.1. Existence and uniqueness of the solution. Let us consider the following space [15]

\[ M(T_{max}, y_{max}) = A_{T_{max}}(t_0) \times D_{y_{max}}(t_0), \] (15)

where, we defined

\[ A_{T_{max}} = [t_0 - t_{max}, t_0 + t_{max}], \] (16)

and

\[ D_{y_{max}} = [y_0 - y_{max}, y_0 + y_{max}], \] (17)

of course, \( M(T_{max}, y_{max}) \) is a compact cylinder of defined function \( f \) in Eq. (8) and Eq. (10). In addition let us consider \( K = sup\|f\| \) to be the maximum gradient \( M(T_{max}, y_{max}) \) of the function in modulus, moreover let us take \( \delta \) to be the Lipschitz constant of the function \( f \) with respect to time. The Banach fixed-point theorem will be applied using the metric on own constructed space \( M(T_{max}, y_{max}) \) which will induce the uniform \( m \) norm

\[ ||\Phi||_{\infty} = \sup_{t \in A_{max}} |\Phi(t)|. \] (18)

Moreover, we construct the following Picard’s operator

\[ \Phi : M(T_{max}, y_{max}) \rightarrow M(T_{max}, y_{max}), \] (19)

defined by

\[ \Phi \lambda(t) = y_0 + \frac{1 - \alpha(t)}{B(a(t))} f(t, \lambda(t), \lambda(t - \delta)) + \]

\[ + \frac{\alpha(t)}{B(a(t)) \Gamma(a(t))} \int_{0}^{t} f(\tau, \lambda(\tau), \lambda(\tau - \delta))(t - \tau)^{(a(t)-1)} d\tau, \] (20)
first we construct the condition for well-posedness that is we find the condition for
which the norm $f\Phi \lambda (t) - y_0$ is less than $y_{\text{max}}$, to do this we evaluate

$$
||\Phi \lambda (t) - y_0|| = \left|\left|1 - \frac{\alpha(t)}{B(\alpha(t))} f(t, \lambda(t), \lambda(t - \delta)) + \frac{\alpha(t)}{B(\alpha(t))\Gamma(\alpha(t))} \int_0^t f(\tau, \lambda(\tau), \lambda(\tau - \delta))(t - \tau)^{\alpha(t)-1}d\tau \right|\right|.
$$

(21)

Using the triangular inequality, we get

$$
||\Phi \lambda (t) - y_0|| \leq \left|\left|1 - \frac{\alpha(t)}{B(\alpha(t))} \right|\right| f(t, \lambda(t), \lambda(t - \delta)) \left|\left| + \frac{\alpha(t)}{B(\alpha(t))\Gamma(\alpha(t))} \int_0^t (t - \tau)^{\alpha(t)} \left|\left| f(t, \lambda(t), \lambda(t - \tau)) \right|\right| d\tau \right|\right|
$$

$$
\leq \left|\left|1 - \frac{\alpha(t)}{B(\alpha(t))} \right|\right| K + \frac{\alpha(t)}{B(\alpha(t))\Gamma(\alpha(t))} K T_{\text{max}}^{\alpha(t)} \leq y_{\text{max}},
$$

(22)

for the above, we need to have

$$
K < \frac{y_{\text{max}}}{1 - \alpha(t) + \frac{\alpha(t) T_{\text{max}}^{\alpha(t)}}{B(\alpha(t))\Gamma(\alpha(t) + 1)}},
$$

(23)

Now, let us impose the Picard’s operator to be a contraction under certain con-
dition on the value $T_{\text{max}}$.

Let consider two different functions $\lambda_1$ and $\lambda_2$ in $\mathbb{C}[A_{T_{\text{max}}(t_0), D_{y_{\text{max}}}(y_0)}]$, then let us evaluate the following

$$
\left|\left|\Phi(\lambda_1) - \Phi(\lambda_2)\right|\right|_{\infty} = \left|\left|1 - \frac{\alpha(t)}{B(\alpha(t))} f(t, \lambda_1(t), \lambda_1(t - \delta)) + \frac{\alpha(t)}{B(\alpha(t))\Gamma(\alpha(t))} \int_0^t f(\tau, \lambda_1(\tau), \lambda_1(\tau - \delta))(t - \tau)^{\alpha(t)-1}d\tau - \frac{\alpha(t)}{B(\alpha(t))} f(t, \lambda_2(t), \lambda_2(t - \delta)) - \frac{\alpha(t)}{B(\alpha(t))\Gamma(\alpha(t))} \int_0^t f(\tau, \lambda_2(\tau), \lambda_2(\tau - \delta))(t - \tau)^{\alpha(t)-1}d\tau \right|\right|
$$

$$
\leq \left|\left|1 - \frac{\alpha(t)}{B(\alpha(t))} \right|\right| \left|\left| f(t, \lambda_1(t), \lambda_1(t - \delta)) - f(t, \lambda_2(t), \lambda_2(t - \delta)) \right|\right|_{\infty} + \frac{\alpha(t)}{B(\alpha(t))\Gamma(\alpha(t))} \int_0^t \left|\left| f(\tau, \lambda_1(\tau), \lambda_1(\tau - \delta)) - f(\tau, \lambda_2(\tau), \lambda_2(\tau - \delta)) \right|\right|_{\infty} (t - \tau)^{\alpha(t)-1}d\tau
$$

$$
\leq \left|\left|1 - \frac{\alpha(t)}{B(\alpha(t))} \right|\right| L \left|\left| \lambda_1(t) - \lambda_2(t) \right|\right|_{\infty} + \frac{\alpha(t)}{B(\alpha(t))\Gamma(\alpha(t) + 1)} L T_{\text{max}}^{\alpha(t)} \left|\left| \lambda_1(t) - \lambda_2(t) \right|\right|_{\infty}
$$

(24)

The condition for contraction is

$$
1 - \frac{\alpha(t)}{B(\alpha(t))} + \frac{\alpha(t)}{B(\alpha(t))\Gamma(\alpha(t) + 1)} T_{\text{max}}^{\alpha(t)} < 1.
$$

(25)

Under the above condition the constructed Picard’s operation is a contraction on
a Banach space with the metric induced by uniform norm, thus $\Phi$ has the property
that there exist a unique function $\lambda$ such that $\Phi \lambda = \lambda$, which in the unique solution
of Eq. (10).
4. Illustrative examples. In this section, fourth examples are considered and solved by means of the proposed method.

Example 1. Consider a fractional order model of type [41]

\[ ABC_{0}D_{t}^{\alpha}y(t) = \frac{2y(t - 2)}{1 + y(t - 2)^{0.65}}, \]
\[ y(t) = 0.5, \quad t \leq 0. \]  

(26)

In this example, we take the step size of \( h = 0.01 \) and \( t = 100 \) seg. The delay is \( \delta = 2 \). Using ABC fractional derivative, the Figure 1a show the behavior of \( y(t) \) with an order \( \alpha = 1 \); Figure 1b, shows the phase portrait \( y(t) \) vs. \( y(t - 2) \) of the system. In this paper, for the Liouville-Caputo fractional derivative with variable-order, we used the numerical scheme developed in [23]. Considering the same conditions above mentioned, the Figure 1c, show the behavior of \( y(t) \) and Figure 1d, shows the phase portrait \( y(t) \) vs. \( y(t - 2) \) of the system.

![Figure 1](image-url)

**Figure 1.** Numerical solution of Eq. (26); using ABC derivative, in (a) we show the evolution of \( y(t) \) when \( \alpha = 1 \), in (b) we obtain the phase diagram when \( \alpha = 1 \). Using Liouville-Caputo derivative, in (c) we show the evolution of \( y(t) \) when \( \alpha = 1 \) and in (d) we obtain the phase diagram when \( \alpha = 1 \).
Due to the limitation faced by the power law, Atangana and Baleanu suggested fractional operators based on generalized Mittag-Leffler law, these operators could be used in the limit of power law. The principal difference between ABC and Liouville-Caputo derivative is the kernel of the derivative, in the first case, this kernel is nonlocal and nonsingular, in the second case, the kernel is singular. Now we can see the difference between one and another derivative by proving with a noninteger order of the system, we use $\alpha = 0.85$ and the same conditions and parameters as in the previous test.

Using ABC fractional derivative, the Figure 2a show the behavior of $y(t)$ with an order $\alpha = 0.85$. Figure 2b shows the phase portrait $y(t)$ vs. $y(t-2)$ of the system. As we can see this behavior results different in comparison when we use the Liouville-Caputo derivative [23], for this case, the Figure 2c shows the behavior of $y(t)$ and the Figure 2d shows the phase portrait $y(t)$ vs. $y(t-2)$ of the system.

The numerical simulations showed that the decreased the value of $\alpha$ the system becomes periodic for $\alpha < 0.85$. 

**Figure 2.** Numerical solution of Eq. (26); using ABC derivative, in (a) we show the evolution of $y(t)$ when $\alpha = 0.85$, in (b) we obtain the phase diagram when $\alpha = 0.85$. Using Liouville-Caputo derivative, in (c) we show the evolution of $y(t)$ when $\alpha = 0.85$ and in (d) we obtain the phase diagram when $\alpha = 0.85$. 

The numerical simulations showed that the decreased the value of $\alpha$ the system becomes periodic for $\alpha < 0.85$. 


Example 2. The second example considers the fractional model of the four year life cycle of a population of lemmings [36], the model is given by

\[
\begin{align*}
ABC_0 D_t^\alpha y(t) &= 3.5y(t) \left(1 - \frac{y(t - 0.74)}{19}\right), \\
y(t) &= 19, \quad t \leq 0.
\end{align*}
\]

(27)

In this example, the step size was of \(h = 0.01\) and \(t = 60\) seg. The delay is \(\delta = 0.74\). Figures 3a-3f, shows the behavior of \(y(t)\) using fractional derivatives of type Atangana-Baleanu-Caputo and Liouville-Caputo. Figures 4a-4f, shows the phase portrait \(y(t)\) vs. \(y(t - 2)\) of the system for different values of \(\alpha\) using Atangana-Baleanu-Caputo and Liouville-Caputo approach.

In this case we observed that the phase portrait \(y(t)\) vs. \(y(t - 2)\) gets stretched as the value of \(\alpha\) decreases. This stretching is towards positive side of the axes.

Example 3. In this example we prove the algorithm for a variable-order \(\alpha(t)\) in the derivative. The fractional model describes the growth of a population and it is called the Verhulst-Pearl model [37], the equations are given by

\[
\begin{align*}
ABC_0 D_t^{\alpha(t)} y(t) &= 0.3y(t) - 0.3y(t)y(t - \delta), \\
y(t) &= 0.1, \quad t \leq 0.
\end{align*}
\]

(28)

For this example, the sample time was chosen of \(h = 1/64\), the variable-order is of the form \(\alpha(t) = \frac{1 - \cos(2t)}{3}\) and \(\delta = 1\). Figure 5a, shows the behavior of \(y(t)\) using ABC approach and Figure 5b, presents the behavior of \(y(t)\) using Liouville-Caputo approach. The phase portrait \(y(t)\) vs. \(y(t - 2)\) of the system is showed in Figure 5c and Figure 5d via ABC and Liouville-Caputo derivatives, respectively.

Example 4. The last example consider a model called Kalecki’s business cycle system [16] and is formulates as follow

\[
\begin{align*}
ABC_0 D_t^{\alpha(t)} K(t) &= aK(t) - bK(t - \delta), \\
K(t) &= 1, \quad t \leq 0,
\end{align*}
\]

(29)

where, \(a = \frac{\lambda v}{3(1-c)}\), \(b = \lambda(1 + \frac{v}{3(1-c)})\), \(\lambda = 2/5\), \(\delta = 1\), \(c = 3/4\) and \(v = 0.5\).

In this example, the sample time was chosen of \(h = 1/64\), the variable-order is of the form \(\alpha(t) = \frac{1 - \cos(2t)}{3}\). Figure 6a, shows the behavior of \(y(t)\) using ABC approach and Figure 6b, presents the behavior of \(y(t)\) using Liouville-Caputo approach. The phase portrait \(y(t)\) vs. \(y(t - 2)\) of the system is showed in Figure 6c and Figure 6d via ABC and Liouville-Caputo derivatives, respectively.

5. Conclusions. Within the framework of the Atangana-Baleanu fractional differentiation, a modification of the Adams-Bashforth-Moulton method was suggested to solve nonlinear variable order fractional delay differential equations with power law and Mittag-Leffler kernel of type Liouville-Caputo. The complex dynamics of the system are analyzed with the change of fractional order \(\alpha\) or delay. From the Figures obtained we conclude that the Liouville-Caputo fractional derivative is more affected by the past compared to the Atangana-Baleanu-Caputo fractional derivative, which shows a rapid stabilization. It is observed that even one dimensional delayed systems of fractional variable order show chaotic behavior, and below some critical order, the system changes its nature and becomes periodic. In some cases it is observed that the phase portrait \(y(t)\) vs. \(y(t - 2)\) gets stretched as the order of
Figure 3. Numerical solution of Eq. (27). In (a)-(c)-(e) we show the evolution of $y(t)$ using ABC derivative. In (b)-(d)-(f) we show the evolution of $y(t)$ using Liouville-Caputo derivative.

the derivative $\alpha$ is reduced. We showed that, for certain values of parameters the systems are chaotic and for others the systems tends to a stable periodic orbits. We observe also that the algorithm proposed provides accurate, efficiency and stable numerical results. The reported results illustrate that the fractional approach with
Figure 4. Numerical solution of Eq. (27). In (a)-(c)-(e) we show the phase diagram $y(t)$ vs. $y(t - 0.74)$ using ABC derivative. In (b)-(d)-(f) we show phase diagram $y(t)$ vs. $y(t - 2)$ using Liouville-Caputo derivative.

variable order is more suitable to describe the complex dynamics of the investigated models. The numerical simulations show that Atangana-Baleanu fractional derivative has some filter properties. Finally we observe novel behaviors that cannot be
Figure 5. Numerical solution of Eq. (28): using ABC derivative, in (a)-(c) we show the evolution of $y(t)$ and the phase diagram $y(t)$ vs. $y(t-2)$, when $\alpha(t) = \frac{1 - \cos(2t)}{3}$, respectively; using Liouville-Caputo derivative, in (b)-(d) we show the evolution of $y(t)$ and the phase diagram $y(t)$ vs. $y(t-2)$, when $\alpha(t) = \frac{1 - \cos(2t)}{3}$, respectively.

obtained with standard models. All computed results obtained by using Matlab programme.

Acknowledgments. The authors appreciate the constructive remarks and suggestions of the anonymous referees that helped to improve the paper. Antonio Coronel Escamilla acknowledges the support provided by CONACyT through the assignment doctoral fellowship. José Francisco Gómez Aguilar acknowledges the support provided by CONACyT: cátedras CONACyT para jóvenes investigadores 2014. José Francisco Gómez Aguilar acknowledges the support provided by SNI-CONACyT.
Figure 6. Numerical solution of Eq. (29): using ABC derivative, in (a)-(c) we show the evolution of $y(t)$ and the phase diagram $y(t)$ vs. $y(t-2)$, when $\alpha(t) = \frac{1 - \cos(2t)}{3}$, respectively; using Liouville-Caputo derivative, in (b)-(d) we show the evolution of $y(t)$ and the phase diagram $y(t)$ vs. $y(t-2)$, when $\alpha(t) = \frac{1 - \cos(2t)}{3}$, respectively.

REFERENCES

[1] A. Atangana, Non-validity of index law in fractional calculus: A fractional differential operator with Markovian and non-Markovian properties, Physica A: Statistical Mechanics and its Applications, 505 (2018), 688–706.
[2] A. Atangana and J. F. Gómez Aguilar, Decolonisation of fractional calculus rules: Breaking commutativity and associativity to capture more natural phenomena, The European Physical Journal Plus, 133 (2018), 166.
[3] A. Atangana, On the stability and convergence of the time-fractional variable order telegraph equation, Journal of Computational Physics, 293 (2015), 104–114.
[4] A. Atangana and J. F. Botha, A generalized groundwater flow equation using the concept of variable-order derivative, Boundary Value Problems, 2013 (2013), 1–11.
[5] A. Atangana and D. Baleanu, Numerical solution of a kind of fractional parabolic equations via two difference schemes, Abstr. Appl. Anal., 2013 (2013), Art. ID 828764, 8 pp.
[6] A. Atangana and D. Baleanu, New fractional derivatives with non-local and non-singular kernel: Theory and application to heat transfer model, *Thermal Science, 20* (2016), 763–769.

[7] S. Bhalekar, V. Daftardar-Gejji, D. Baleanu and R. Magin, Generalized fractional order bloch equation with extended delay, *International Journal of Bifurcation and Chaos, 22* (2012), 1250071.

[8] W. C. Chen, Nonlinear dynamics and chaos in a fractional-order financial system, *Chaos, Solitons and Fractals, 36* (2008), 1305–1314.

[9] C. Coimbra, Mechanics with variable-order differential operators, *Ann. Phys.*, 12 (2003), 692–703.

[10] G. R. J. Cooper and D. R. Cowan, Filtering using variable order vertical derivatives, *Computers and Geosciences, 30* (2004), 455–459.

[11] J. Dabas and A. Chauhan, Existence and uniqueness of mild solution for an impulsive neutral fractional integro-differential equation with infinite delay, *Mathematical and Computer Modelling, 57* (2013), 754–763.

[12] V. Daftardar-Gejji, Y. Sukale and S. Bhalekar, Solving fractional delay differential equations: A new approach, *Fractional Calculus and Applied Analysis, 18* (2015), 400–418.

[13] V. Daftardar-Gejji, Y. Sukale and S. Bhalekar, A new predictor-corrector method for fractional differential equations, *Appl. Math. Comput., 244* (2014), 158–182.

[14] V. Daftardar-Gejji and H. Jafari, Analysis of a system of non autonomous fractional differential equations involving Caputo derivatives, *J. Math. Anal. Appl., 328* (2007), 1026–1033.

[15] J. F. Gómez-Aguilar, Analytical and Numerical solutions of a nonlinear alcoholism model via variable-order fractional differential equations, *Physica A: Statistical Mechanics and its Applications, 494* (2018), 52–75.

[16] M. Kalecki, A macroeconomic theory of business cycle, *Econom., 3* (1935), 327–344.

[17] M. M. Khader and A. S. Hendy, The approximate and exact solutions of the fractional-order delay differential equations using Legendre seudospectral Method, *International Journal of Pure and Applied Mathematics, 74* (2012), 287–297.

[18] J. A. Len and S. Tindel, Malliavin calculus for fractional delay equations, *Journal of Theoretical Probability, 25* (2012), 854–889.

[19] Y. Luchko, A New Fractional Calculus Model for the Two-dimensional Anomalous Diffusion and its Analysis, *Mathematical Modelling of Natural Phenomena, 11* (2016), 1–17.

[20] M. M. Meerschaert and C. Tadjeran, Finite difference approximations for fractional advection dispersion equations, *J. Comput. Appl. Math., 172* (2004), 65–77.

[21] B. P. Moghaddam and Z. S. Mostaghim, A numerical method based on finite difference for solving fractional delay differential equations, *Journal of Taibah University for Science, 7* (2013), 120–127.

[22] B. P. Moghaddam and J. A. T. Machado, A stable three-level explicit spline finite difference scheme for a class of nonlinear time variable order fractional partial differential equations, *Computers and Mathematics with Applications, 73* (2017), 1262–1269.

[23] B. P. Moghaddam, S. Yaghoobi and J. T. Machado, An extended predictor-corrector algorithm for variable-order fractional delay differential equations, *Journal of Computational and Nonlinear Dynamics, 11* (2016), 061001, 7pp.

[24] M. L. Morgado, N. J. Ford and P. M. Lima, Analysis and numerical methods for fractional differential equations with delay, *Journal of Computational and Applied Mathematics, 252* (2013), 159–168.

[25] T. A. Nadzharyan, V. V. Sorokin, G. V. Stepanov, A. N. Bogolyubov and E. Y. Kramarenko, A fractional calculus approach to modeling rheological behavior of soft magnetic elastomers, *Polymer, 92* (2016), 179–188.

[26] K. M. Owolabi, Mathematical modelling and analysis of two-component system with Caputo fractional derivative order, *Chaos, Solitons and Fractals, 103* (2017), 544–554.

[27] K. M. Owolabi, Robust and adaptive techniques for numerical simulation of nonlinear partial differential equations of fractional order, *Communications in Nonlinear Science and Numerical Simulation, 44* (2017), 304–317.

[28] K. M. Owolabi and A. Atangana, Numerical simulation of noninteger order system in subdiffusive, diffusive, and superdiffusive scenarios, *Journal of Computational and Nonlinear Dynamics, 12* (2016), 031010, 7pp.

[29] M. A. Ramadan and M. N. Shrif, Numerical solution of system of first order delay differential equations using spline functions, *International Journal of Computer Mathematics, 83* (2006), 925–937.
[30] U. Saeed, Hermite wavelet method for fractional delay differential equations, *Journal of Difference Equations*, 2014 (2014), Article ID 359093, 8 pages.

[31] F. Shakeri and M. Dehghan, Solution of delay differential equations via a homotopy perturbation method, *Mathematical and Computer Modelling*, 48 (2008), 486–498.

[32] J.-J. Shyu, S.-C. Pei and C.-H. Chan, An iterative method for the design of variable fractional-order FIR differintegrators, *Signal Process*, 89 (2009), 320–327.

[33] H. G. Sun, W. Chen, C. Li and Y. Q. Chen, Fractional differential models for anomalous diffusion, *Physica A*, 389 (2010), 2719–2724.

[34] H. G. Sun, W. Chen, H. Wei and Y. Q. Chen, A comparative study of constant-order and variable-order fractional models in characterizing memory property of systems, *Eur. Phys. J. Spec. Top.*, 193 (2011), 185–192.

[35] A. A. Tateishi, H. V. Ribeiro and E. K. Lenzi, The role of fractional time-derivative operators on anomalous diffusion, *Frontiers in Physics*, 5 (2017), 1–9.

[36] L. Tavernini, *Continuous-Time Modeling and Simulation*, Gordon and Breach, Amsterdam, 1996.

[37] A. Tsoularis and J. Wallace, Analysis of logistic growth models, *Mathematical Biosciences*, 179 (2002), 21–55.

[38] S. Umarov and S. Steinberg, Variable order differential equations and diffusion with changing modes, *Z. Anal. Anwend.*, 28 (2009), 431–450.

[39] D. Valério and J. S. Da Costa, Variable-order fractional derivatives and their numerical approximations, *Signal Processing*, 91 (2011), 470–483.

[40] Z. B. Vosika, G. M. Lazovic, G. N. Misevic and J. B. Simic-Krstic, Fractional calculus model of electrical impedance applied to human skin, *PloS one*, 8 (2013), e59483.

[41] D. R. Will and C. T. Baker, DELSOL-A numerical code for the solution of systems of delay-differential equations, *Applied Numerical Mathematics*, 9 (1992), 209–222.

[42] W. Zhen, H. Xia and S. Guodong, Analysis of nonlinear dynamics and chaos in a fractional order financial system with time delay, *Computers and Mathematics with Applications*, 62 (2011), 1531–1539.

Received April 2018; revised May 2018.

E-mail address: antoniocelie@cenidet.edu.mx

E-mail address: jgomez@cenidet.edu.mx