ABSTRACT

We study nonlinear wave phenomena in self-gravitating fluid systems, with a particular emphasis on applications to molecular clouds. This paper presents analytical results for one spatial dimension. We show that a large class of physical systems can be described by theories with a “charge density” $q(\rho)$; this quantity replaces the density on the right hand side of the Poisson equation for the gravitational potential. We use this formulation to prove general results about nonlinear wave motions in self-gravitating systems. We show that in order for stationary waves to exist, the total charge (the integral of the charge density over the wave profile) must vanish. This “no-charge” property for solitary waves is related to the capability of a system to be stable to gravitational perturbations for arbitrarily long wavelengths. We find necessary and sufficient conditions on the charge density for the existence of solitary waves and stationary waves. We study nonlinear wave motions for Jeans-type theories [where $q(\rho) = \rho - \rho_0$] and find that nonlinear waves of large amplitude are confined to a rather narrow range of wavelengths. We also study wave motions for molecular clouds threaded by magnetic fields and show how the allowed range of wavelengths is affected by the field strength. Since the gravitational force in one spatial dimension does not fall off with distance, we consider two classes of models with more realistic gravity: Yukawa potentials and a pseudo-two-dimensional treatment. We study the allowed types of wave behavior for these models. Finally, we discuss the implications of this work for molecular cloud structure. We argue that molecular clouds can support a wide variety of wave motions and suggest that stationary waves (such as those considered in this paper) may have already been observed.

Subject headings: hydromagnetics – wave motions – interstellar: molecules – stars: formation

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1. INTRODUCTION

Molecular clouds are self-gravitating fluid systems and are capable of supporting a wide variety of oscillatory motions. The overall goal of this work is to understand the dynamics of volume density waves in molecular clouds and to explore the extent to which waves can explain the observed cloud structure. These clouds comprise a substantial fraction of the gas in the galaxy, they provide the initial conditions for star formation, and they are interesting astrophysical objects in their own right. Molecular clouds are now fairly well-observed and exhibit very complicated highly nonlinear structures (see, e.g., the recent review of Blitz 1993). Thus far, however, no definitive theory exists to explain or predict the observed large scale structure of these clouds. Our working hypothesis is that volume density wave motions can produce (at least in part) this complicated structure.

The study of nonlinear waves and solitons began over a century ago (see the classic papers by Russell 1844 and Riemann 1858). Similarly, the study of gravitational instability in astrophysical fluids has had a long and distinguished history (see Jeans 1928). In the meantime, however, remarkably little work has been done on the intersection of these two subjects, i.e., the study of nonlinear waves and solitons in systems where self-gravity is important. Liang (1979) has studied nonlinear waves in the cosmological fluid and has shown that soliton solutions are not allowed for ordinary pressure laws (see also Ray 1983). Götz (1988) has searched for solitons in Newtonian gravity, although the solutions found there are not relevant for our present discussion. Other related previous work does not directly address the problem of nonlinear wave motions in a self-gravitating fluid. Nonlinear Alfvén waves have been studied and have been proposed as a mechanism for producing clumpy structure in clouds (Elmegreen 1990), although this study does not include self-gravity. Much of the previous work has been devoted to the linear stability of clouds (see, e.g., Langer 1978; Pudritz 1990; Dewar 1970; see also the review of Shu, Adams, & Lizano 1987). On a smaller size scale, the formation of molecular cloud cores has been considered through the process of ambipolar diffusion (Mouschovias 1976, 1978; Shu 1983; Nakano 1985; Lizano & Shu 1989; Shu et al. 1987). On the size scale of molecular clouds, the study of wave motions including self-gravity remains largely untouched.

Self-gravity is an important ingredient for the existence of stationary nonlinear waves in a neutral fluid. In the absence of self-gravity, acoustic waves in fluids are known to steepen and shock (see, e.g., Whitham 1974; Shu 1992). On the other hand, self-gravity provides dispersion (Jeans 1928) which tends to spread out wave packets. Thus, fluids with self-gravity can, in principle, reach a balance between nonlinear steepening and gravitational dispersion. This balance leads to the possibility of nonlinear stationary waves and solitary waves, which we study in this paper.

In an earlier paper (Adams & Fatuzzo 1993; hereafter Paper I), we began a study of nonlinear waves and solitons in molecular clouds. We found one class of nonlinear waves for molecular clouds with no magnetic fields; we also showed that no soliton solutions exist for these systems. The one dimensional system of Paper I (with no magnetic fields) is highly idealized and hence unphysical in the following ways: (1) The usual Poisson
equation for gravity in one dimension produces a gravitational force that does not fall off at large distances, i.e., there is no asymptotic regime where the gravitational force goes to zero. (2) The one-dimensional formulation of the problem does not allow for any two (or three) dimensional effects. (3) Magnetic fields are ignored, but are known to be dynamically important. (In Paper I, we briefly considered clouds with magnetic fields and derived a model equation which allows a wide variety of solutions, including nonlinear waves, solitons, and topological solitons – see §2.3 below). (4) Real (observed) molecular clouds are not collapsing as a whole, whereas the one dimensional model does collapse. In other words, the one dimensional model does not allow for a static equilibrium state. We note that items (1) and (2) are coupled in the sense that any physical system will have a finite spatial extent in the direction perpendicular to the wave motion. This finite extent implies a finite mass and hence allows gravity to fall off with distance. Similarly, items (3) and (4) are also coupled. Magnetic fields are (at least in part) responsible for the fact that clouds are not collapsing as a whole.

This present paper is a generalization of Paper I. Since the full magnetohydrodynamic problem is rather complicated and the chances of finding meaningful analytic solutions seem remote, we use the full physical problem as motivation to derive a collection of model equations which approximate the true behavior of the system. In order to obtain analytic results, some of these approximations are by necessity rather severe. However, our goal is to obtain a qualitative physical understanding of the behavior of nonlinear waves in self-gravitating fluids. This approach has the advantage of allowing for analytic results which in turn provide us with a clear picture of “what depends on what”. Since we must sacrifice quantitative accuracy in our model equations, this work should be complemented by both numerical studies and by studies in higher spatial dimensions. We are currently pursuing these avenues of research.

This paper is organized as follows. We begin with a general formulation of the problem in §2. We develop the concept of a “charge density” where we replace the right hand side of Poisson’s equation for the gravitational field with a more complicated function (this concept was introduced in Adams, Fatuzzo, & Watkins 1993; hereafter AFW). We show that many types of physical systems can be modeled in this manner. In §3 we prove general results which hold for all theories with a charge density. In particular, we prove that stationary waves must have zero total charge when integrated over one wavelength. We also derive necessary and sufficient conditions on the charge density for the existence of stationary waves and solitary waves. In §4 we consider the special case where the charge density \( q = \rho - \rho_0 \); this case corresponds to the traditional Jeans analysis. In §5, we include the effects of a magnetic field which is perpendicular to the wave propagation direction. In §6, we study nonlinear waves for theories where gravity is modeled with a Yukawa potential. This formulation of the problem allows the gravitational force to fall off at large distances. In §7, we show how stationary waves in two spatial dimensions can produce an effective charge density theory in one spatial dimension. We conclude in §8 with a discussion and summary of our results.

2. GENERAL FORMULATION FOR ONE-DIMENSIONAL WAVES

In this section, we introduce a class of model equations for the study of nonlinear
waves in molecular clouds. We begin with the equations of motion for a molecular cloud fluid. In one spatial dimension, these equations take the form:

\[
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho u) = 0, \tag{2.1}
\]

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{\partial \psi}{\partial x} = 0, \tag{2.2}
\]

\[
\frac{\partial^2 \psi}{\partial x^2} = 4\pi G \rho. \tag{2.3}
\]

For most of this paper, we take the pressure of the molecular cloud fluid to have a general barotropic form,

\[
p = p(\rho), \tag{2.4a}
\]

which includes most equations of state of interest. In order to obtain explicit results, we sometimes must specify the form of the pressure law. In these cases, we adopt the form

\[
p = a_s^2 \rho + p_0 \log(\rho/\rho_R), \tag{2.4b}
\]

where the first term corresponds to an isothermal equation of state ($a_s$ is the isothermal sound speed). The second term arises from a “turbulent” contribution. This term is motivated by the observational finding that linewidths in molecular clouds vary with density according to $\Delta v \sim \rho^{-1/2}$ (see, e.g., Larson 1981; Myers 1983; Dame et al. 1986; Myers 1987; and Myers & Fuller 1992). If we interpret the observed linewidth as the effective transport speed in the fluid, we obtain $(\Delta v)^2 = v_{\text{turb}}^2 = \partial p_{\text{turb}}/\partial \rho = p_0/\rho$, where the final equality follows from the observed relation. We then obtain the form given in equation [2.4b] by integration (see also Lizano & Shu 1989; Myers & Fuller 1992).

We find it convenient to work in dimensionless units. We let $\rho_R$ denote a reference density and $a_s$ denote the sound speed. We can non-dimensionalize all quantities according to

\[
u \rightarrow u/a_s, \tag{2.5a}
\]

\[
\rho \rightarrow \rho/\rho_R, \tag{2.5b}
\]

\[
x \rightarrow kx \quad \text{where} \quad k^2 = 4\pi G \rho_R/a_s^2, \tag{2.5c}
\]

\[
t \rightarrow k a_s t, \tag{2.5d}
\]

\[
p \rightarrow \frac{p}{\rho_R a_s^2}, \tag{2.5e}
\]

\[
\psi \rightarrow \psi/a_s^2, \tag{2.5f}
\]

\[
\kappa \equiv \frac{p_0}{a_s^2 \rho_R}. \tag{2.5g}
\]

For applications to molecular clouds, we are primarily interested in spatial size scales of $1 – 30$ pc where number densities are typically $\sim 100 – 1000$ cm$^{-3}$. The temperature
is expected to be in the range \( T = 10 - 35 \text{ K} \) and hence the sound speed \( a_s = 0.20 - 0.35 \text{ km/s} \). The wavenumber \( k = 2\pi/\lambda_J \) where \( \lambda_J \) is the (usual) Jean’s length that would result from thermal pressure (with sound speed \( a_s \) and fiducial density \( \rho_R \)). For example, if we take \( a_s = 0.2 \text{ km/s} \) and \( \rho_R = 2 \times 10^{-22} \text{ g cm}^{-3} \) (the conditions roughly appropriate for the Taurus molecular cloud), we get \( x = 1 \) at a physical length scale of \( \sim 0.5 \text{ pc} \). The parameter \( \kappa \) determines the relative size of the “turbulent” contribution to the pressure; the aforementioned observations indicate that \( \kappa \) lies in the range \( 6 - 50 \).

2.1 The Concept of Charge Density

In this section we generalize the theory described in equations [2.1–2.4] by introducing the concept of a “charge density” (see AFW). We can include a variety of additional behavior simply by modifying the Poisson equation to take the form

\[
\frac{\partial^2 \psi}{\partial x^2} = q(\rho),
\]

where \( q(\rho) \) is a function of the density \( \rho \). We denote the quantity \( q(\rho) \) as the charge density. We use this terminology because, in general, whatever appears on the right hand side of a Poisson equation is often called “the charge density”. Notice that \( q(\rho) \) defined here has absolutely nothing to do with the electric charge density (which of course is completely negligible in this problem). Notice also that the choice \( q = \rho \) gives us back the usual system. In the following subsections (§2.2–2.4), we present physically motivated examples which can be written as charge density theories.

One general motivation for introducing a charge density is to consider modifications of one-dimensional gravity (see below) and/or to include additional long range forces in the problem. We can model such systems by assuming the forces are conservative (\( \sim \nabla \psi_{\text{ex}} \)) and thus can be written in terms of a new potential \( \psi_{\text{ex}} \). We then consider the potential \( \psi \) to be the total potential, i.e., the sum of the usual gravitational potential and the new potential \( \psi_{\text{ex}} \). The dimensionless Poisson equation must have the form

\[
\frac{\partial^2 \psi}{\partial x^2} = \rho + \text{something} \equiv q,
\]

where we have defined the right hand side of the equation to be the charge density. Here we consider those special cases where the charge density is a function of \( \rho \) only.

2.2 Jeans Theory

The simplest possible nontrivial extension of the theory arises from subtracting out the contribution to the gravitational potential due to the background fluid (see, e.g., Jeans 1928; Binney & Tremaine 1987). This approximation can be written in the form of a charge density theory with

\[
q(\rho) = \rho - \rho_0,
\]

where \( \rho_0 \) is the dimensionless background density of the fluid. Thus, the traditional Jeans analysis results in a charge density theory.
One physically motivated way to obtain a theory with the same mathematical form as in equation [2.7] is to posit a physical system which rotates at a uniform rate \( \Omega \) (see, e.g., Shu 1992). In the direction perpendicular to the rotation axis, Gauss’s law for a uniform density cylinder with density \( \rho_0 \) implies

\[
-\varpi \Omega^2 = -2\pi G \rho_0 \varpi ,
\]

where \( \varpi \) is the radial coordinate (note that all quantities in equation [2.8] have their usual dimensions). Thus, the uniform density state will be in mechanical balance provided that

\[
\rho_0 = \frac{\Omega^2}{2\pi G} .
\]

It is straightforward to show (a homework problem in Shu 1992) that for waves propagating along the axis of rotation in this system, we obtain equations of motion with a “charge density” of the form \( q = \rho - \rho_0 \), where \( \rho_0 \) is given by equation [2.9].

We now consider “typical” values for the rotationally induced fiducial density \( \rho_0 \). At the solar circle, the rotation rate around the center of the galaxy is of order \( \Omega \sim 10^{-15} \) rad/s. With this value of \( \Omega \), the density implied by equation [2.9] corresponds to a number density of \( n_0 \sim 1 \) cm\(^{-3} \). In molecular clouds, the observed rotation rates are larger than this fiducial value by factors of 10 – 100 (see, e.g., Goldsmith & Arquilla 1985) and hence the implied number density becomes \( n_0 \sim 10^2 – 10^4 \) cm\(^{-3} \). These values thus lie in an interesting range for molecular clouds.

### 2.3 Charge Density from Magnetic Field Effects

Another way to obtain a charge density is to simulate the effects of an embedded magnetic field. In Paper I, we derived a model equation on this basis. In this model, we assumed that the magnetic field points in a direction perpendicular to that of the wave motion and that the neutral component of the fluid is coupled to the magnetic field through its frictional interaction with the ionized fluid component (which is well-coupled to the field). We also ignored any chemical effects so that the ionized component of the fluid obeys a continuity equation. The resulting model equation can be derived from a theory with the charge density written in the form

\[
q(\rho) = \rho + \Gamma \left( \frac{1}{\rho} - \frac{1}{\rho_F} \right) ,
\]

where \( \Gamma \) represents the coupling strength between the neutral and ionized components and where \( \rho_F \) determines the ion density (assumed to be constant in Paper I). We note that the model equation corresponding to the charge density [2.10] is idealized in two ways. First, a dissipative term has been dropped to obtain this form. Second, the ion density is not calculated self-consistently. The net result of these approximations is to make the magnetic force on the neutral component into a long-range force that balances the long-range force of gravity. This treatment is unphysical in the sense that these magnetic forces, which arise from the frictional force exerted on the neutral components
by the ions, are intrinsically local. On the other hand, this approximation allows for something to cancel the long-range force of gravity.

2.4 Charge Density from Yukawa Potentials

We can also derive a charge density for theories in which gravity is modeled by a Yukawa potential. The motivation for this approximation is to include the effects of a decreasing gravitational field strength while retaining the one-dimensional treatment of the problem. As we show here, Yukawa potentials provide one means of realizing this behavior. In particular, we write the Poisson equation in the generalized form

\[ \frac{\partial^2 \psi}{\partial x^2} = m^2 \psi + \rho. \]  (2.11)

The Green’s function for the operator \( \frac{\partial^2}{\partial x^2} - m^2 \) has an exponential fall off and hence produces an exponential fall off in the gravitational force between point masses. The value of the parameter \( m \) determines the effective range of the force. †

The right hand side of equation [2.11] defines an effective charge density \( \rho \), although it is not written explicitly as a function of density only. We note, however, that we can integrate the stationary version of the force equation [2.2] to obtain

\[ \psi + h(\rho) + \frac{1}{2}v^2 = E, \]  (2.12)

where \( E \) is the constant of integration (a discussion of the stationary wave approximation is given in §2.5). The quantity \( h(\rho) \) is the enthalpy and is defined by

\[ h(\rho) = \int_0^\rho \frac{dp}{\rho}. \]  (2.13)

The speed \( v \) can be eliminated by using the solution of the continuity equation for stationary waves, i.e., \( v = A/\rho \) (see equation [2.17] below). If we now solve equation [2.12] for \( \psi \) and use the result in equation [2.11], we can read off the appropriate form of the charge density, i.e.,

\[ q(\rho) = \rho + m^2 \left[ E - h(\rho) - \frac{A^2}{2\rho^2} \right]. \]  (2.14)

As we show in §7, another way to obtain a one dimensional theory where gravity falls off with distance is to begin with a two dimensional theory and then reduce it to a one dimensional theory through the use of rather severe approximations. We can thus obtain yet another charge density theory (compare equations [7.9] and [2.14]).

† For completeness, we note that a more complicated additional term of the form \( m^2 \psi^n \) could be used; this term represents nonlinear interactions in the gravitational field and is not of interest here. Notice, however, that this nonlinear theory can also be written as a charge density theory.
2.5 Stationary Waves and Solitary Waves

In this paper, we are interested in stationary nonlinear waves and their limiting forms known as solitary waves and solitons. Since the definitions of these wave entities vary greatly in the literature (e.g., Whitham 1974; Coleman 1985; Rajaraman 1987; Drazin & Johnson 1989; and Infeld & Rowlands 1990), we present the following working definitions from Paper I: We use the term stationary wave to mean any wave structure that is a function of the variable $\xi = x - v_0t$ only. We use the term solitary wave to refer to any solution of a nonlinear wave equation (or system of equations) which (1) represents a stationary wave, and (2) is localized in space so that the wave form either decays or approaches a constant at spatial infinity. We use the term soliton to be synonymous with solitary wave, although the term soliton is often reserved for waves which also satisfy the additional requirement: (3) the wave form can interact strongly with other solitons and retain its identity. Solutions which satisfy requirement (3) must have extraordinary stability in order to pass through each other and, after emerging from the collision, retain their initial forms. Such solitons are very rare; wave entities which satisfy the first two requirements and not (3) occur much more frequently.

We first consider stationary waves, which correspond to traveling waves of permanent form. For these waves, the fluid fields are functions of the quantity

$$\xi = x - v_0t,$$  \hspace{1cm} (2.15)

where $v_0$ is the (nondimensional) speed of the wave. Next, we introduce a new velocity variable

$$v = u - v_0,$$  \hspace{1cm} (2.16)

which is simply the speed of the fluid relative to the speed $v_0$ of the wave. Using the above definitions in the continuity equation [2.1], we find the relation

$$\rho v = A = \text{constant},$$  \hspace{1cm} (2.17)

where the constant of integration $A$ is the “Mach number” of the wave.

We want to combine the equations of motion to obtain a single nonlinear differential equation for the density $\rho$. If we differentiate the force equation [2.2] with respect to $x$ and use the generalized Poisson equation [2.6] to eliminate the potential, we obtain

$$\rho \rho_{\xi \xi} \left[ \rho^2 \frac{\partial p}{\partial \rho} - A^2 \right] + \rho \rho_{\xi} \left[ 3A^2 - \rho^2 \frac{\partial p}{\partial \rho} + \rho^3 \frac{\partial^2 p}{\partial \rho^2} \right] + \rho^4 q(\rho) = 0,$$  \hspace{1cm} (2.18)

where subscripts denote differentiation and where we have eliminated the velocity dependence by using relation [2.17]. The equation of motion [2.18] is the fundamental equation of this paper. Notice its highly nonlinear nature. Fortunately and somewhat surprisingly, however, we can integrate this equation to obtain

$$\frac{1}{2} \rho_{\xi}^2 = \rho^6 \left[ \rho^2 \frac{\partial p}{\partial \rho} - A^2 \right]^{-2} f(\rho) \equiv F(\rho),$$  \hspace{1cm} (2.19)
where we have defined \( f(\rho) \) to be an integral that depends on the form of the charge density \( q(\rho) \), i.e.,

\[
 f(\rho) = \int \frac{q(\rho)}{\rho} \left[ \frac{A^2}{\rho^2} - \frac{\partial p}{\partial \rho} \right].
\]

(2.20)

The existence or non-existence of stationary waves for a particular physical system can be understood through the methods of phase plane analysis (see, e.g., Infeld & Rowlands 1990; Drazin & Johnson 1989). As illustrated above, in this method we reduce the system of equations to a single equation of the form

\[
 \frac{1}{2} \rho_\xi^2 = F(\rho, C_j),
\]

(2.21)

where the \( C_j \) are constants (see equations [2.19] and [2.20]). Notice that equation [2.21] is just the virial theorem for an analogue particle moving in a potential \(-F\) (see, e.g., Rajaraman 1987); when viewed in this manner, equation [2.21] shows that the properties of the “potential” \( F \) determine the allowed behavior of the analogue particle and hence the properties of the wave solutions \( \rho(\xi) \) in a fairly simple manner. In particular, the form of \( F(\rho) \) determines whether or not solitary wave solutions can exist. We also note that if dissipative terms are present in the original equation of motion, then the solution cannot be written in the form of equation [2.21] and hence stationary wave solutions do not exist.

We first note that physically meaningful solutions must have \( F \geq 0 \) (so that the solutions are real). For the systems considered here, the field \( \rho \) is a mass density and must always be positive. Thus, physically relevant solutions exist when \( F(\rho) \) is positive over a range of positive densities. Wave solutions exist when \( F(\rho) \) is positive between two zeroes of the function \( F \), where the zeroes of \( F \) correspond to maximum and minimum densities of the wave profile. The nature of the zeroes of \( F \) determines the nature of the wavelike solutions. For example, if \( F \) is positive between two simple zeroes, then (ordinary) nonlinear waves result. However, if \( F \) is positive between a simple zero and a double zero or higher order zero (i.e., any point where both \( F \) and \( \partial F/\partial \rho \) vanish), then a new type of solution – a solitary wave – can arise. Suppose we expand equation [2.21] about the double zero, which we denote as \( \rho_\infty \); we obtain

\[
 \rho_\xi^2 = (\rho - \rho_\infty)^2 F''(\rho_\infty) + O[(\rho - \rho_\infty)^3],
\]

(2.22)

where \( F''(\rho_\infty) > 0 \) because we are considering the case in which \( F \) is positive. Thus, as \( \rho \to \rho_\infty \), the wave profile has the form

\[
 \rho - \rho_\infty \sim \delta \exp \left[ \mp \sqrt{F''(\rho_\infty)} \xi \right],
\]

(2.23)

where \( \delta \) is a constant. We see that a soliton consists of a single large hump of material and that the density smoothly approaches its asymptotic value \( \rho_\infty \) as \( \xi \to \pm \infty \). Formally, the wavelength of the solution diverges for a solitary wave.

Another interesting type of behavior can arise when the function \( F \) is positive between two double zeroes of \( F \), say \( \rho_A \) and \( \rho_B \). In this case, the solution \( \rho(\xi) \) can approach
one value (e.g., $\rho_A$) in the limit $\xi \to -\infty$ and the other value ($\rho_B$) in the limit $\xi \to \infty$. Solutions of this type are known as kinks or topological solitons and are well studied in the context of quantum field theory (e.g., Coleman 1985; Rajaraman 1987).

3. GENERAL RESULTS

In this section we present general analytic results that apply to all theories of the form described in §2. These results are applicable for arbitrary forms of the charge density $q(\rho)$. In particular, we state and prove four elementary “Theorems” which greatly constrain the allowed types of wave behavior for charge density theories. In the subsequent sections, we use these results to study the behavior of theories which contain specific forms for the charge density.

3.1 The No-Charge Property for Stationary Waves

To begin, we present an argument which shows that for any theory with a charge density $q(\rho)$, a strong constraint must be met in order for physically relevant stationary waves to exist. This constraint arises from the fact that a stationary wave must have local extrema where the pressure and velocity gradients go to zero. Therefore, in order for the wave to remain stationary, the force of gravity must also vanish at these points. This behavior can only occur if the integral of the charge density $q(\rho)$ between two extrema is zero. This argument can be stated as follows:

**Result 1.** If a stationary wave solution $\rho(\xi)$ exists for the one-dimensional theory, then the integral of the charge density over one wavelength must vanish, i.e.,

$$Q \equiv \int_{-\lambda/2}^{\lambda/2} d\xi q[\rho(\xi)] = 0,$$

where the wavelength of a soliton is taken to be infinite. This claim is limited to the case of nonsingular solutions; for solitons we also require that the density does not vanish at spatial infinity.

In order to show that Result 1 is true, we first define the total charge $Q$ contained in one wavelength to be

$$Q \equiv \int_{-\lambda/2}^{\lambda/2} q \, d\xi = 2 \int_{0}^{\lambda/2} q \, d\xi,$$

(3.1)

where $\lambda \to \infty$ for soliton solutions. The second equality holds because of the symmetry of the problem about $\xi = 0$ (this result is valid for all classes of solutions except topological solitons, for which the formula for $Q$ will be different by a factor of two). The first integral of the equation of motion for a stationary wave can be written in the form

$$\rho_\xi = \pm \sqrt{2} \rho \left[ \frac{\partial p}{\partial \rho} - \frac{A^2}{\rho^2} \right]^{-1} f^{1/2},$$

(3.2)
where the function \( f \) (defined in equation [2.20]) must be positive definite for valid solutions. Using the relation

\[
d(f^{1/2}) = \frac{1}{2} f^{-1/2} \frac{q}{\rho} \left[ \frac{A^2}{\rho^2} - \frac{\partial p}{\partial \rho} \right] d\rho,
\]

along with equation [3.2], we find the identity

\[
q d\xi = \pm \sqrt{2} d(f^{1/2}).
\]

Equation [3.4] may be substituted into equation [3.1], where care must be taken to ensure that the integrand has the proper sign. Using this result, we find

\[
Q = 2\sqrt{2} \int_{\rho_1}^{\rho_2} d(f^{1/2}),
\]

which can be integrated to obtain

\[
Q = 2\sqrt{2} \left[ f(\rho_2)^{1/2} - f(\rho_1)^{1/2} \right] = 0.
\]

This result holds provided that we do not integrate over the singularity at the sonic point (see below). The second equality in equation [3.6] follows because the first integral of the equation of motion (and therefore \( f \)) must vanish at the values of density \( \rho_1 \neq 0 \) and \( \rho_2 \) (by definition). Thus, the total charge \( Q \) must vanish.

Result 1 greatly limits the allowed types of wave behavior in systems with self-gravity. For example, one important consequence of this “No-Charge Property” is that the charge density \( q(\rho) \) must vanish in the asymptotic limit \( \xi \to \infty \) for a solitary wave or soliton. For these waves, the mass density \( \rho(\xi) \) approaches a constant as \( \xi \to \infty \).

In Paper I, we considered the case of molecular clouds with no magnetic fields and found solutions corresponding to stationary nonlinear waves. For this case, the charge density is given by \( q(\rho) = \rho \) and thus the total “charge” \( Q \) of the stationary wave is simply the total mass contained in one wavelength; this mass must be positive. Therefore, by Result 1, neither soliton nor stationary wave solutions can arise for this physical system.

The apparent contradiction between Result 1 and the stationary wave solutions found in Paper I can be resolved by examination of their singularity. The stationary wave solutions of Paper I contain a singularity, \( \rho_\xi \to \infty \) as \( \rho \to \rho_C \), where \( \rho_C \) satisfies \( \partial p/\partial \rho = A^2/\rho^2 \). Since \( v = A/\rho \), the singularity is associated with the sonic point of the fluid, i.e., where the “effective” sound speed of the fluid is given by \( a_{\text{eff}}^2 \equiv \partial p/\partial \rho \).

In the presence of the singularity, \( d(f^{1/2}) \) changes sign relative to \( d\rho \) across \( \rho_C \), and we must rewrite equation [3.5] as

\[
Q = 2\sqrt{2} \left[ \int_{\rho_1}^{\rho_C} d(f^{1/2}) - \int_{\rho_C}^{\rho_2} d(f^{1/2}) \right].
\]
The total charge is then given by

\[ Q = 2\sqrt{2} \left[ 2f(\rho_C)^{1/2} - f(\rho_2)^{1/2} - f(\rho_1)^{1/2} \right] \neq 0. \]  

(3.8)

The discontinuity that allows these solutions to avoid the No-Charge Property necessarily produces a discontinuity in the gravitational force \( g(\xi) \), although the density profiles of the waves are continuous. As a result, the solutions of Paper I for the \( q(\rho) = \rho \) theory are unphysical. In this present work, we shall therefore require that realistic solutions of our equations be nonsingular.

The most effective way to ensure nonsingular equations is to limit the discussion to a mass density range that excludes the sonic point \( \rho_C \). However, a nonsingular class of solutions can be found by choosing the constant of integration in equation [2.20] such that \( f(\rho_C) = 0 \). For the theories considered in this paper, this class of solutions are characterized by a lower density bound of \( \rho_1 = 0 \) and a non-zero charge. We will consider this class of solutions briefly for the case of Jeans theory (see §4 and Appendix B).

3.2 Relationship Between Solitary Waves and Jeans Stability

In this section, we discuss the relationship between the existence of solitary waves and the absence of Jeans instability. This relationship arises because a soliton is in some sense an infinite wavelength perturbation on a uniform medium. On the other hand, the existence of a Jeans length implies that all perturbations with a sufficiently large wavelength will collapse. Thus, the existence of a stationary perturbation of infinite wavelength (a soliton) is inconsistent with the presence of a Jeans length. This relationship can be stated more precisely as follows:

**Result 2.** Suppose a physical system obeys a generalized model equation of motion of the form [2.18] and this model equation has solitary wave solutions. Then the system can have a uniform density state that is Jeans stable for arbitrarily large length scales.

Suppose we have such a system. We can write the equations of motion in the form [2.1], [2.2], and [2.6]. We now consider a standard Jeans-type stability analysis, i.e., we take the unperturbed state to be \( \rho = \rho_0 = constant \) and \( u = 0 \). (The existence of such a uniform density state requires \( q(\rho_0) = 0 \); systems which have solitary wave solutions always have such a zero – see the discussion of Result 3 below.) We expand the first order quantities according to

\[ f = f_0 + f_1 e^{i(kx - \omega t)}, \]  

(3.9)

where \( f_1 \) is a constant. After some algebra, the dispersion relation can be written in the form

\[ \omega^2 \left( \frac{\partial p}{\partial \rho} \right)_0 k^2 - \rho_0 \left( \frac{dq}{d\rho} \right)_0, \]  

(3.10)

where the subscript denotes that the quantities are evaluated at the unperturbed density. Thus, the system can be stable to perturbations of all lengthscales provided that

\[ \left( \frac{dq}{d\rho} \right)_0 \leq 0. \]  

(3.11)
for some density $\rho = \rho_0 > 0$ such that $q(\rho_0) = 0$.

We now show that the condition [3.11] can be realized for systems which have solitary wave solutions. The generalized model equation of motion can be solved for $q(\rho)$ to obtain

$$q(\rho) = -e^{-\mu/2} \frac{d}{d\rho} \left\{ e^{\mu} F(\rho) \right\},$$

(3.12)

where $F$ is the first integral of the equation of motion (see equation [2.19]) and where $\mu$ is an integrating factor,

$$\mu \equiv -6 \log \rho + 2 \log \left[ \rho^2 \frac{\partial p}{\partial \rho} - A^2 \right].$$

(3.13)

We thus obtain an expression for $dq/d\rho$:

$$\frac{dq}{d\rho} = -e^{-\mu/2} \frac{d^2}{d\rho^2} \left\{ e^{\mu} F(\rho) \right\} + e^{-\mu/2} \frac{d}{d\rho} \left\{ e^{\mu} F(\rho) \right\} \frac{\mu}{2} \frac{d\mu}{d\rho}.$$  

(3.14)

Since the system has solitary wave solutions (by hypothesis), we know that for those solutions the quantity $e^{\mu} F$ has a double zero with a positive second derivative at some density $\rho = \rho_0$. Equation [3.14] shows that for $\rho = \rho_0$, the quantity $dq/d\rho$ is negative. Thus, by equation [3.11], a Jeans stable configuration can arise for this physical system.

We note that the converse of Result 2 is not true. Physical systems of this type can be Jeans stable to perturbations of all wavelengths and still not have soliton solutions. Thus, the requirement of Jeans stable configurations represents a necessary condition (and not a sufficient condition) for the existence of solitons. We consider other related conditions in the following section.

3.3 Necessary and Sufficient Conditions for the Existence of Stationary Waves and Solitons

In this section, we consider the conditions required for one dimensional systems to exhibit solitary waves and stationary waves. The question of whether or not solitary wave solutions exist is fundamental to the study of nonlinear dynamics and cannot, in general, be answered in a definitive manner. In this paper, we have shown that a fairly large class of physical systems can be modeled using a theory with a charge density $q(\rho)$. The question can then be posed as follows: What properties must the charge density $q(\rho)$ have in order for solitary wave solutions to exist and what properties are required for stationary waves to exist? Fortunately, we can provide a partial answer to this question.

We first find a necessary condition on the charge density. In order for solitary wave solutions to exist, the first integral $F$ of the equation of motion (see equation [2.21]) must be positive between a double zero and an ordinary zero (see §2). Using the solution in the form of equation [2.19], we can write

$$F(\rho) = \rho^6 \left[ \rho^2 \frac{\partial p}{\partial \rho} - A^2 \right]^{-2} f(\rho),$$

(3.15)
where \( f(\rho) \) is defined by equation [2.20] and depends on the form of the charge density. We restrict our discussion to waves which avoid the singularity at the sonic point. We also restrict our discussion to waves with finite velocity; this second restriction eliminates the point \( \rho = 0 \) as a candidate for the double zero (if we let \( \rho \to 0 \), we get \( v = A/\rho \to \infty \) by the continuity equation). Thus, the required double zero of \( \mathcal{F} \) must occur at a double zero of \( f(\rho) \). Furthermore, \( f(\rho) \) must be a local minimum at the double zero. In addition, in order for the second (ordinary) zero of \( \mathcal{F} \) to exist, the function \( f(\rho) \) must also have a local maximum (at some other density). By definition, the derivative of \( f(\rho) \) is given by

\[
\frac{df}{d\rho} = \frac{q(\rho)}{\rho} \left[ \frac{A^2}{\rho^2} - \frac{\partial p}{\partial \rho} \right]. \tag{3.16}
\]

Since we must avoid the singularity at the sonic point, both the required minimum and the maximum of \( f(\rho) \) must occur where the charge density \( q(\rho) \) vanishes. We denote the location of the minimum as \( \rho_1 \) and the maximum as \( \rho_M \). In order for the point \( \rho_1 \) to be a minimum, \( dq/d\rho \) must have the correct sign, which depends on the sign of the quantity in brackets in equation [3.16]. In other words, the sign depends on whether we are considering purely subsonic or purely supersonic waves. For subsonic waves we require \( dq/d\rho < 0 \) at \( \rho_1 \) and \( dq/d\rho > 0 \) at \( \rho_M \). For supersonic waves, the sign requirements are reversed. These arguments can be summarized as the following necessary condition on the charge density \( q(\rho) \) for the existence of solitary waves:

**Result 3.** In order for solitary wave solutions to exist for the class of theories considered in this paper, the charge density \( q(\rho) \) must have (at least) two zeroes \( \rho_1 \) and \( \rho_M \). For subsonic waves, \( q(\rho) \) must be negative between the two zeroes; for supersonic waves, \( q(\rho) \) must be positive between the zeroes. (This result applies only to nonsingular solutions where \( \rho_C \) does not lie between \( \rho_1 \) and \( \rho_M \)).

Result 3 is potentially very powerful as a test to see if solitary wave solutions exist. For example, many possible charge density functions do not have two zeroes and thus solitary wave behavior can be easily ruled out. We note, however, that Result 3 is not sufficient to guarantee the existence of solitary waves. We must place an additional constraint on the charge density to make sure that the second (ordinary) zero of the function \( f(\rho) \) exists. Given the two zeroes of \( q(\rho) \), the possible behavior of the function \( f(\rho) \) is sketched in Figure 1. The first possibility is that \( f \) has a second ordinary zero (solid curve in Figure 1) and thus solitary wave solutions exist. If the function \( f \) has another minimum at some density \( \rho_3 > \rho_M \) and hence turns up (as shown by the short dashed curve in Figure 1), then we can always choose the constant of integration \( \beta \) to make \( \rho_3 \) a double zero of \( f \). We thus obtain a depression soliton or void solution. Only for the special case where \( f(\rho) \) approaches a constant asymptotically (long dashed curve in Figure 1) does the system fail to have solitary wave solutions.

For the usual case of purely subsonic waves, the existence of an ordinary second zero of \( f \) is guaranteed provided that the function \( f \to -\infty \) as \( \rho \to \infty \). In the limit \( \rho \to \infty \),

\[
\frac{df}{d\rho} = -\frac{1}{\rho} \frac{\partial p}{\partial \rho} q(\rho), \tag{3.17}
\]
where we have assumed that the equation of state is well behaved (i.e., the pressure increases with increasing density). Thus, the requirement that

$$\lim_{\rho \to \infty} \frac{1}{\rho} \frac{\partial p}{\partial \rho} q(\rho) > 0 \quad (3.18)$$

is sufficient, but not necessary, to ensure the existence of the second zero of \(f(\rho)\) for the case of subsonic waves. For most of the examples considered in this paper, \(q \to \rho\) and \(\partial p/\partial \rho \to 1\) in the limit \(\rho \to \infty\) (for the case of a magnetic pressure with \(p \sim \rho^2\), \(\partial p/\partial \rho \to \infty\)); thus, the constraint \([3.18]\) is always satisfied for these theories.

We now consider the simpler case of ordinary stationary waves. The required conditions on the charge density \(q(\rho)\) for the existence of these waves can be stated as follows:

**Result 4.** In order for stationary wave solutions to exist for the class of theories considered in this paper, the charge density \(q(\rho)\) must have (at least) one zero \(\rho_M\). For subsonic waves, \(dq/d\rho > 0\) at \(\rho_M\) and this zero must occur at a density larger than the sonic density \(\rho_C\) (the density of singularity). Similarly, for supersonic waves, we must have \(dq/d\rho < 0\) at \(\rho_M < \rho_C\).

The conditions outlined in Result 4 are both necessary and sufficient for the existence of stationary waves. This result is straightforward to understand. In order for such waves to exist, the function \(f(\rho)\) must have a maximum and two zeroes \(\rho_1 < \rho_2\) such that \(\rho_C < \rho_1\) for subsonic waves and \(\rho_C > \rho_2\) for supersonic waves. The zero of \(q(\rho)\) is required to provide a critical point for \(f(\rho)\) – see equation \([3.16]\). The sign requirement on \(dq/d\rho\) ensures that the critical point is a maximum. The third condition, that the critical point \(\rho_M\) be larger than the sonic point for subsonic waves and smaller than the sonic point for supersonic waves, is required so that the singularity does not lie in the range of densities in the wave profile.

4. **NONLINEAR WAVES AND SOLITONS IN JEANS THEORY**

In this section, we explicitly consider the case of fluid systems with a charge density of the form \(q = \rho - \rho_0\). As discussed above, this choice for the charge density is equivalent to adopting the original approximation of Jeans or to considering a uniformly rotating cloud (see §2.2). This system is also the simplest modification of the basic fluid equations that allows for the existence of stationary waves. Application of the No-Charge Property to this system requires \(\rho_0\) to be the average density for stationary wave solutions.

We can analyze the possible behavior of this system using the results of §3. The charge density \(q(\rho)\) has only a single zero (at \(\rho = \rho_0\)) and hence Result 3 shows that no solitary wave solutions exist (except for solutions with vanishing density as \(\xi \to \infty\); see Appendix B). In addition, \(dq/d\rho = 1 > 0\) everywhere and Result 4 implies that subsonic stationary nonlinear waves are possible provided that the singularity at the sonic point can be removed from the wave profile. This requirement implies that \(\rho_0 > \rho_C\) for subsonic waves and can be written as a constraint on the Mach number \(A\):

$$A^2 < \rho_0^2 \left[ \frac{\partial p}{\partial \rho} \right]_{\rho_0} \quad (4.1)$$
This condition thus defines a maximum value of the Mach number $A$ for nonlinear stationary waves.

In order to study the properties of the stationary waves, we must specify the equation of state and find the function $f(\rho)$ defined by equation [2.20]. For the sake of definiteness, we take the pressure to have the form given by equation [2.4b]. For this equation of state, the integral in equation [2.20] can be evaluated to obtain

$$f(\rho) = \beta - \rho - (\kappa - \rho_0) \log(\rho) - \frac{A^2 + \kappa \rho_0}{\rho} + \frac{A^2 \rho_0}{2 \rho^2},$$  \hspace{1cm} (4.2)

where $\beta$ is an integration constant. The function $f(\rho)$ does allow for wave solutions; a schematic is shown in Figure 2.

Without loss of generality, we can choose $\rho_0 = 1$ (see Appendix A). This choice leaves us with a two parameter family of wave solutions. The parameter $A$ sets the velocity of the wave and also determines the value of $\rho_C$. The parameter $A$ can vary from zero up to a maximum given by the condition [4.1]. For a given value of $A$, the constant of integration $\beta$ merely slides $f(\rho)$ in the vertical direction (see Figure 2) and thus determines the wave amplitude, which can vary from zero (when $\rho_1 = \rho_2 = \rho_0$) to a maximum value (when $\rho_1 = \rho_C$). In general, the range of possible amplitudes decreases with increasing $A$. A typical example of a nonlinear stationary wave in Jeans theory is shown in Figure 3. Note that the wavelength is $\lambda \sim 20$ for this example; in physical units (scaled to the Taurus cloud – see the discussion following equation [2.5]), the wavelength is $\sim 10$ pc.

Given the solutions for nonlinear stationary waves discussed above, we can determine the allowed wavelengths for nonlinear waves as a function of amplitude. For specified values of the constants $A$ and $\beta$, the wavelength of a stationary wave is given by

$$\lambda = 2 \int_{\rho_1}^{\rho_2} \left( \frac{d\rho}{d\xi} \right)^{-1} d\rho,$$  \hspace{1cm} (4.3)

an integral which unfortunately must be done numerically. In Figure 4 we plot the wavelength as a function of wave amplitude $\delta\rho/\rho \equiv (\rho_2 - \rho_0)/\rho_0$ for two extreme cases. The first case has $A = 0$ and corresponds to the “Jeans” length generalized for nonlinear waves; note that we recover the linear Jeans length for small amplitude. In fact, the Jeans length is shown to change very little over a large range of amplitudes. The opposite limit corresponds to the “shock” limit; the constraint that all velocities must remain below the effective sound speed $a_{\text{eff}}$ gives a minimum possible wavelength for a wave of a given amplitude. This minimum wavelength approaches the “Jeans” length rapidly with increasing amplitude, so that for $\delta\rho/\rho > 1$ only a narrow range of wavelengths is possible.

Nonlinear waves are characterized by relatively narrow, high density peaks. While the wavelength corresponds to the distance between peaks, the width of the peaks is also an interesting quantity. In order to obtain a measure of the width of the wave crests, we define the quantity $\lambda_p$ to be the length of the region of the wave profile where $\rho > \rho_0$, i.e.,

$$\lambda_p \equiv 2 \int_{\rho_0}^{\rho_2} \left( \frac{d\rho}{d\xi} \right)^{-1} d\rho.$$  \hspace{1cm} (4.4)
In Figure 5, the width $\lambda_p$ is plotted as a function of wave amplitude for both the “Jeans” and the “shock” limits described above. Figure 5 clearly shows that for $\delta \rho/\rho \gtrsim 1$, the range of allowed values of $\lambda_p$ is very tightly constrained. In fact, the width $\lambda_p$ has almost a single value as a function of $\delta \rho/\rho$.

These results have a relatively simple interpretation. For stationary waves, a balance must exist between dispersion, which acts to spread the wave, and nonlinear effects, which act to steepen the wave and cause it to shock. For the case at hand, the dispersion is supplied entirely by the self-gravitational interaction. (Recall that in the linear Jeans analysis the dispersion relation is given by $\omega^2 = a_{\text{eff}}^2 k^2 - 4\pi G \rho_0$.) For waves with $\delta \rho/\rho \gtrsim 1$, nonlinear effects will be large. To balance these large nonlinear effects, the effects of self-gravity must also be large, hence the observation that the natural length scale for nonlinear stationary waves is of order the Jeans length. The narrow range of allowed stationary waves for large amplitudes $\delta \rho/\rho \gtrsim 1$ is thus a consequence of the fact that the amount of self-gravity needed to balance the nonlinearity is comparable to that required to cause the wave to collapse.

For completeness we note that a class of solitary wave solutions exists for Jeans theory. In these solutions, the singularity is removed by choosing the integration constant $\beta$ appropriately (see Appendix B for further discussion). An example of such a solitary wave is shown in Figure 6.

5. NONLINEAR WAVES AND SOLITONS IN CLOUDS WITH MAGNETIC FIELDS

In this section we consider the effects of magnetic fields, which are expected to play an important role in the dynamics of molecular clouds (see, e.g., Shu et al. 1987). In Paper I, we derived a model equation which incorporates the effects of magnetic fields in the equations of motion and produces a charge density theory (see the review in §2.3 and equation [2.10]). This model equation assumes that the largely neutral fluid is coupled to the magnetic fields through the ionic component as an intermediary. In order to obtain a simple form for the equation of motion, a dissipative term was dropped from consideration. Since this approximation has been discussed at some length in Paper I, it will not be considered here.

Here we adopt an alternate approach; we assume that the magnetic field is strongly coupled to the neutral fluid. This assumption is valid for large lengthscales in molecular clouds (scales larger than those associated with dense cores), since the magnetic diffusion time is long compared to the magnetic sound crossing time (see Shu 1992). In fact, the existence of stationary wave solutions requires the absence of diffusive effects (recall that in Paper I we also ignored a diffusion term).

With the neutrals strongly coupled to the magnetic field, the ions do not explicitly enter in the dynamics (the ions act only as an intermediate coupling agent). The equation of motion for the neutral component becomes

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{\partial \psi}{\partial x} = \frac{1}{\rho} \left[ (\nabla \times \mathbf{B}) \times \mathbf{B} \right] \cdot \hat{x}, \quad (5.1)$$
where we have considered only one spatial dimension and where \( B \) is written in dimensionless form \( B \to B/(4\pi a^2 \rho_R)^{1/2} \). Observations of magnetic field strengths in molecular clouds (e.g., Goodman et al. 1989; Myers & Goodman 1988) indicate that \( B = 10 - 30 \mu G \) (with these values of field strength, the dimensionless version of \( B \) will be of order unity). The equation of motion \([5.1]\) must be supplemented by an equation of motion for the magnetic field itself,

\[
\frac{\partial B}{\partial t} = \nabla \times (u \hat{x} \times B).
\]  

(5.2)

Given the form of the magnetic force term, we see that the one-dimensional formalism is consistent as long as the magnetic tension does not result in a \( y \)-directed force (since no other forces exist to balance such a term). The magnetic field must also satisfy the usual divergenceless (no-monopole) condition

\[
\nabla \cdot B = 0.
\]  

(5.3)

These two requirements imply that the magnetic field cannot have components both parallel and perpendicular to \( \hat{x} \). Since a parallel component has no effect on the neutral dynamics, we consider a magnetic field of the form

\[
B = B \hat{y}.
\]  

(5.4)

Equations \([5.1]\) and \([5.2]\) may therefore be expressed as

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{1}{\rho} \frac{\partial}{\partial x} \left( p + \frac{B^2}{2} \right) + \frac{\partial \psi}{\partial x} = 0,
\]  

(5.5)

and

\[
\frac{\partial B}{\partial t} + \frac{\partial}{\partial x} (uB) = 0.
\]  

(5.6)

In the stationary wave approximation, the equation of motion \([5.6]\) for the magnetic field can be integrated to obtain the relation

\[
B v = constant.
\]  

(5.7)

This relation, along with the solution \([2.17]\) to the continuity equation, gives the expected flux-freezing condition for a one-dimensional system,

\[
\frac{B}{\rho} = \alpha = constant.
\]  

(5.8)

In typical molecular clouds, the magnetic contribution to the pressure is comparable to the “turbulent” contribution and hence we expect \( \alpha^2 \sim \kappa \). We can use the flux freezing condition \([5.8]\) in the equation of motion \([5.5]\) to eliminate the magnetic field. The resulting equation of motion (including magnetic effects) is related to the previous one (equation \([2.19]\)) through the transformation \( p \to p + \alpha^2 \rho^2 / 2 \). In other words, we have
simply added a magnetic pressure term to the nonmagnetic case. This new pressure is given by

\[ p(\rho) = \rho + \kappa \log(\rho) + \alpha^2 \rho^2 / 2, \]  

and we can proceed as in §4. For the values expected in molecular clouds, the second and third terms in equation [5.9] are comparable in size. In other words, the “turbulent” contribution to the pressure is comparable to the magnetic pressure.

Here, we take the charge density \( q(\rho) \) to be the same as in the previous section. The results of §3 thus imply that the system cannot have solitary wave solutions, but can have subsonic stationary wave solutions. To study these waves, we find the function \( f(\rho) \) which becomes

\[ f(\rho) = \beta - \frac{\alpha^2}{2} \rho^2 + (\alpha^2 \rho_0 - 1) \rho - (\kappa - \rho_0) \log(\rho) - \frac{A^2 + \kappa \rho_0}{\rho} + \frac{A^2 \rho_0}{2 \rho^2}, \]  

(5.10)

where \( \beta \) is an integration constant. As we found in §4, the function \( f(\rho) \) is positive for small \( \rho \) and becomes negative as \( \rho \to \infty \). For this form of \( f \), the wave solutions are analogous to those found in §4 and hence the discussion given there also applies to this present case.

Figure 7 presents the density profile found by integrating equation [2.19] for the pressure given by equation [5.9]. The parameters used are representative of cloud environments. Comparison of Figure 7 with Figure 3 shows that a strong magnetic field is capable of sustaining a broader wave structure. This result is expected since the added magnetic pressure provides additional support against self-gravity and because the equation of state [5.9] has a stiffer component. As in the case of pure Jeans theory (§4), nonlinear waves can only exist in a fairly narrow range of wavelengths. The allowed range of wavelengths for molecular clouds with magnetic pressure is shown in Figure 8. Notice that the maximum allowed wavelength varies more rapidly with amplitude than in the case of Figure 4.

6. NONLINEAR WAVES AND SOLITONS WITH YUKAWA POTENTIALS

In this section, we present a class of model equations with charge densities arising from modifications of the long range forces of gravity. In particular, we consider theories in which gravity is modeled with a Yukawa potential. As derived in §2.4, this approximation leads to a charge density theory with \( q(\rho) \) of the form

\[ q(\rho) = \rho - m^2 [h(\rho) + \frac{A^2}{2 \rho^2} - E]. \]  

(6.1)

With this form for the charge density, the function \( f(\rho) \) can be easily found:

\[ f(\rho) = \beta - p(\rho) - \frac{A^2}{\rho} + \frac{1}{2} m^2 [h(\rho) + \frac{A^2}{2 \rho^2} - E]^2. \]  

(6.2)

Notice that for the equation of state [2.4b], the charge density \( q(\rho) \) is of cubic order and has either one or three real zeroes; thus, solitary wave behavior is possible.
We now show that all stationary waves in this theory must propagate at subsonic speeds. Result 4 shows that supersonic stationary waves must have \( q = 0 \) and \( dq/d\rho < 0 \) at some density \( \rho_M < \rho_C \). The first of these conditions \( (dq/d\rho < 0) \) implies that

\[
\frac{\partial p}{\partial \rho} - \frac{A^2}{\rho^2} > \frac{\rho}{m^2}.
\]

(6.3a)

On the other hand, the requirement that the wave is supersonic \( (\rho_M < \rho_C) \) implies that

\[
\frac{\partial p}{\partial \rho} - \frac{A^2}{\rho^2} < 0,
\]

(6.3b)

where both conditions [6.3a] and [6.3b] are to be evaluated at the zero of \( q \). Since both of these conditions cannot be simultaneously satisfied, no supersonic wave solutions exist for this theory. In other words, all physical stationary wave solutions must propagate at subsonic velocities.

We now consider solitons in this theory. Result 3 shows that soliton solutions require the charge density \( q(\rho) \) to have at least two zeroes; for subsonic waves, \( q \) must be negative between two zeroes. Suppose \( q(\rho) \) has a zero at \( \rho_S \). We can always choose the constant of integration \( \beta \) to make the point \( \rho_S \) a double zero of \( F \). In this case, \( \rho_S \) represents the fluid density far from the soliton. Without loss of generality, we can rescale the variables so that \( \rho_S = 1 \) (see Appendix A). In order for the charge density to be negative between \( \rho_S \) and a second zero of \( q \), we must have

\[
\frac{dq}{d\rho} = 1 - \frac{m^2}{\rho_S} \left[ \frac{\partial p}{\partial \rho} - \frac{A^2}{\rho^2} \right]_{\rho_S} < 0,
\]

(6.4)

where the term in square brackets is to be evaluated at \( \rho = \rho_S = 1 \). Thus, a necessary condition for the existence of soliton solutions is that

\[
m^2 > \left[ \frac{\partial p}{\partial \rho} \right]_1^{-1} - A^2.
\]

(6.5)

This condition has two important consequences. First, we note that for the theory to have solitons at all, the quantity \( m^2 \) must satisfy the condition

\[
m^2 > \left[ \frac{\partial p}{\partial \rho} \right]_1^{-1}.
\]

(6.6)

This constraint is equivalent to the condition that the theory has no Jeans length, as obtained from the linear dispersion relation (see also Result 2). Second, if the value of \( m^2 \) satisfies equation [6.6], then the Mach number \( A \) must satisfy the constraint

\[
A^2 < \left[ \frac{\partial p}{\partial \rho} \right]_1 - \frac{1}{m^2}.
\]

(6.7)
Since \( A/\rho_S \) is the velocity of the soliton in the rest frame of the fluid, this constraint implies a maximum soliton velocity for a given value of \( m \).

The conditions outlined above are necessary for the existence of solitons. For equations of state for which \( f \rightarrow -\infty \) as \( \rho \rightarrow \infty \), these conditions are also sufficient. Notice that this applies for our usual choice of equation of state [2.4b].

Thus far, our discussion has been limited to compression solitons, i.e., waves in which the density of the soliton is enhanced over that of the background. However, under certain conditions, depression solitons (voids) can exist as well. This type of wave behavior can occur if the function \( f \) drops below zero for \( \rho_C < \rho < 1 \). For the equation of state [2.4b], we can show that a necessary and sufficient condition for the existence of depression solitons is that \( f(\rho_C) < 0 \). After some straightforward algebra, this requirement can be written as a condition on the parameter \( m \),

\[
m^2 < \frac{2 \left[ p(\rho_C) - p(1) - A^2(1 - \rho_C)^2/2\rho_C^2 - h(\rho_C) + h(1) \right]}{[h(\rho_C) - h(1) + A^2(1 - \rho_C^2)/2\rho_C^2]^2}.
\]

(6.8)

In Figure 9, we show wave profiles for both a standard (compression) soliton and a depression (or void) soliton.

The constraints derived above define regions of allowed parameter space in the \((m, A)\) plane. In Figure 10, we plot these constraints for the equation of state given in equation [2.4b] with \( \kappa = 10 \). The vertical dashed line on the right of the figure corresponds to the sonic point; all wave solutions in this theory must be subsonic and hence \( A \) must lie to the left of this line. The constraints of equations [6.5] and [6.8] are shown as solid curves. In region I, ordinary (compression) soliton solutions are allowed; in region II, both compression and depression (void) solitons are allowed. If the constraint [6.5] is not satisfied, then no solitary waves exist (this case corresponds to region III of Figure 10). However, the theory does allow nonlinear waves of various amplitudes, which are determined by varying \( \beta \). In analogy with the compression and depression solitons described above, these nonlinear waves can have profiles with narrow valleys as well as the usual profiles which are narrowly peaked.

The inclusion of a magnetic pressure term \( \alpha^2 \rho^2/2 \) (see §5) can significantly change the results described above. For this case, \( h \rightarrow \alpha^2 \rho \) in the limit \( \rho \rightarrow \infty \) and thus \( f \rightarrow (m^2 \alpha^4 - \alpha^2)\rho^2/2 \). For \( m^2 \alpha^2 > 1 \), the function \( f \rightarrow \infty \) as \( \rho \rightarrow \infty \) in contrast to the behavior discussed above (see the discussion following equation [6.2]). For this pressure law, compression solitary wave solutions do not exist when \( m^2 \alpha^2 > 1 \).

7. APPROXIMATION OF TWO-DIMENSIONAL THEORY

In this section we derive one dimensional model equations by approximating two dimensional fluid systems. The goal of this procedure is to obtain a qualitative understanding of wave motions in higher dimensions while retaining the mathematical simplicity of the one dimensional theory.

We begin with the equations of motion for a two dimensional fluid. We consider stationary wave solutions, so we define the variable \( \xi = x - v_0 t \) and let \( v = u - v_0 \) be
the velocity along the \( \hat{x} \) direction as before. We let \( w \) denote the velocity along the \( \hat{y} \) direction. The continuity equation in this stationary wave approximation becomes

\[
\partial_x (\rho v) + \partial_y (\rho w) = 0. \tag{7.1}
\]

The two components of the force equation become

\[
vv_x + vw_y + h_x + \psi_x = 0, \tag{7.2}
\]

\[
vw_x + ww_y + h_y + \psi_y = 0, \tag{7.3}
\]

and the Poisson equation for the gravitational potential is

\[
\psi_{xx} + \psi_{yy} = q_0(\rho), \tag{7.4}
\]

where we have left open the possibility of a nontrivial charge density \( q_0(\rho) \).

We now consider a plane wave traveling in the \( \hat{x} \) direction. However, we want the plane wave to have finite extent in the perpendicular (\( \hat{y} \)) direction. As a first approximation, we assume that the flow velocity in the transverse direction is much smaller than that in the direction of propagation and set \( w = 0 \). We are thus considering a wave structure with a more or less fixed profile in the transverse direction. In this case, the \( \hat{y} \) component of the force equation [7.3] reduces to the hydrostatic condition \( h_y = -\psi_y \), and the Poisson equation can be written in the suggestive form

\[
\psi_{xx} = q_0(\rho) + h_{yy}. \tag{7.5}
\]

To make further progress, we must specify the equation of state (in order to specify the form of the enthalpy), the charge density \( q_0 \), and the transverse profile of the wave. We note that the equation of state in the transverse direction need not be the same as that in the direction of wave propagation and set \( w = 0 \). We are thus considering a wave structure with a more or less fixed profile in the transverse direction. For example, magnetic fields can provide support in the transverse direction. For simplicity, however, we consider an equation of state of the form [2.4b] and the enthalpy becomes

\[
h(\rho) = \log \rho - \kappa/\rho. \tag{7.6}
\]

In order to isolate any possible pseudo-two-dimensional effects, we adopt a trivial form \( q_0 = \rho \) for the original charge density. If we assume that the wave structure is a localized lump in the transverse direction, we can take the density profile to have the separable form

\[
\rho = \rho(\xi) \exp[-\mu^2 y^2]. \tag{7.7}
\]

This form is chosen because it is symmetric about the \( y = 0 \) plane and decreases with \( y \). The parameter \( \mu \) is like a wavenumber that sets the scale of the structure in the transverse direction. Using this ansatz for the density profile, we obtain a particularly simple form for the term \( h_{yy} \), namely,

\[
h_{yy} = -2\mu^2 (1 + \kappa/\rho) - 4\mu^4 y^2 \kappa/\rho. \tag{7.8}
\]
Because the equation of motion in nonlinear, we cannot self-consistently assume a profile of the form [7.7] for all values of $y$. However, near the center of the wave profile (i.e., near $y = 0$), we can neglect the $y$ dependence everywhere except for keeping the first term in equation [7.8]. We thus obtain the Poisson equation in the form

$$\psi_{\xi\xi} = \rho - 2\mu^2(1 + \kappa/\rho) \equiv q(\rho),$$

(7.9)

where in the second equality we have defined the (total) effective charge density including two dimensional effects. We note that the form [7.9] is not sensitive to the particular form of the wave profile in the transverse direction; we simply require that the profile is even in $y$ and decreases on a lengthscale given by $\mu^{-1}$. We stress that we have not solved the full two dimensional problem; we have simply assumed a reasonable structure in the transverse direction. The physical interpretation of this approximation is that if the wave structure has finite spatial extent in the transverse direction, then the gravitational field falls off with increasing distance (in the $\hat{x}$ direction). For the particular simple form of the transverse wave structure used here, we obtain yet another charge density theory, which can be solved using the methods developed in this paper.

Given the form of the charge density (equation [7.9]), we see immediately that this theory does not allow for solitary wave solutions (because the charge density $q(\rho)$ has only one zero with positive density – see Result 3). We also see immediately that $dq/d\rho$ is always positive; thus, by Result 4, only subsonic nonlinear stationary waves are allowed. In order to ensure that physically relevant wave solutions exist, we still must show that the sonic point can be removed from the range of densities of the wave profile. This final requirement can be met if the zero of $q(\rho)$ occurs at a density larger than that of the sonic point, i.e., provided that

$$\mu^2 + \mu[\mu^2 + 2\kappa]^{1/2} > -\frac{\kappa}{2} + \frac{1}{2}[\kappa^2 + 4A^2]^{1/2}. \quad \text{(7.10)}$$

This condition can clearly be satisfied if the parameter $\mu$ is large enough, i.e., if the density profile in the transverse direction falls off sufficiently rapidly. In order to obtain a better feeling for how restrictive this condition actually is, we consider the limit $A \ll \kappa$ (typically, $A \sim 1$ and $\kappa \sim 10$, so this limit is quite reasonable for molecular clouds). In this case, the constraint [7.10] reduces to the form

$$\mu^2 > \frac{A^4}{2\kappa^3}, \quad \text{(7.11)}$$

where we have kept terms to leading order in $A^2/\kappa^2$. Using the values $A \sim 1$ and $\kappa \sim 10$, we find that $\mu > 0.022$ is sufficient to allow for stationary wave solutions to exist. In other words, the length scale of the density fall-off in the transverse direction can be 45 times the thermal Jeans length and still allow for stationary waves (see equation [7.7]).

Before leaving this section, we note that the Yukawa theory of the previous section can be recovered by using a pseudo-two-dimensional argument similar to that given above. We write the Poisson equation as

$$\psi_{\xi\xi} = \rho - \psi_{yy}$$

(7.12)
and then expand the equation about the plane \( y = 0 \). If we assume that the potential \( \psi \) is separable near \( y = 0 \), we can write

\[
\psi_{yy} = -\mu^2 \psi(\xi).
\]  
(7.13)

Combining equations [7.12] and [7.13], we obtain the Poisson equation for a Yukawa theory (see equation [2.11]).

### 8. DISCUSSION

#### 8.1 Summary of Results

In this paper we have developed further the theory of wave motions in self-gravitating astrophysical fluids. Although many of the results are general, the application to the fluid dynamics of molecular clouds is our primary motivation. Our results can be summarized as follows:

1. We have introduced the concept of “charge density” for the study of self-gravitating fluid systems (see also AFW). In this formalism, the density is replaced by a “charge density” \( q(\rho) \) on the right hand side of the Poisson equation (the continuity equation and the force equation remain as usual). We have shown that a large class of physical systems can be modeled with a charge density; we can thus include a wide range of physical effects while retaining a simple semi-analytic theory.

2. We have proven a “No-Charge Property” (Result 1) which shows that no solitons or stationary waves can exist in one-dimensional self-gravitating fluids unless the total charge vanishes (where the total charge is the integral of the charge density over one wavelength). This requirement greatly constrains the types of model equations that allow for stationary wave behavior.

3. We have shown that in order for a physical system to exhibit solitary wave behavior, the system must also be capable of having a configuration which is stable to gravitational perturbations of arbitrarily large wavelengths (see Result 2). We have thus discovered a fundamental relationship between gravitational stability and the existence of solitary waves.

4. We have found constraints on the form of the charge density \( q(\rho) \) for the existence of solitary waves and ordinary stationary waves (see §3.3, Results 3 and 4). These conditions allow us to determine (or at least constrain) the possible types of wave behavior for any charge density theory without having to solve the equations of motion.

5. The original Jeans analysis provides the simplest nontrivial theory with a charge density and has \( q(\rho) = \rho - \rho_0 \). For this theory, we have studied the propagation of nonlinear waves in one dimension and found a class of wave solutions. In spite of this theoretical idealization, these waves are probably more physical than the nonlinear wave solutions found in Paper I.

6. Using the charge density formulation of the problem, we have performed a nonlinear Jeans analysis for molecular clouds. We find that the Jeans length is a slowly increasing function of the wave amplitude \( \delta \rho/\rho \). The wavelengths for stationary waves
must be less than this Jeans length. In addition, we find that waves with sufficiently small wavelengths tend to shock and dissipate. As the wave amplitude $\delta \rho / \rho$ becomes larger, nonlinear waves become confined to a rather narrow range of wavelengths. These clouds thus select out a particular lengthscale as a function of density contrast (see Figures 4 and 5).

[7] The effects of magnetic fields can be incorporated into this theory in two conceptually different ways. In Paper I, we derived a model equation which takes into account the relative drift between the neutral and ionic species; this model equation, which omits a dissipative term, allows for a wide variety of nonlinear wave behavior including solitary waves and topological solitons. In this paper, we have studied the other extreme — flux freezing. In this case, the magnetic field simply adds another component to the pressure. As in the case of Jeans theory (see item [6]), nonlinear waves are confined to a narrow range of allowed wavelengths (see Figure 8).

[8] We have studied theories with Yukawa potentials which allow the gravitational force to fall off with distance while retaining a one-dimensional formulation of the problem. This theory contains both stationary waves and solitary waves. In addition, depression solitons or voids are allowed.

[9] Using a two-dimensional theory as a starting point, we have derived a model equation which takes the form of a one-dimensional charge density theory. We can thus heuristically take into account two-dimensional effects while retaining a one-dimensional formulation. This particular model does not allow solitary wave solutions, but does allow stationary waves.

We have thus introduced a new class of “charge density theories” for the study of wave motions in self-gravitating fluids. We have proven general results (Results 1 – 4) which provide powerful constraints on the allowed types of wave behavior in these systems. Finally, we have argued that many physical systems can be modeled with charge density theories and we have studied several examples relevant to molecular clouds.

8.2 Applications and Comparison with Observations

Many of the results of this paper are general and can be applied to any self-gravitating fluid system (and other similar systems with long range forces; see AFW). However, one primary motivation for this work is to understand the formation of substructure in molecular clouds. These cloud systems are self-gravitating (see, e.g., the reviews of Blitz 1993; Shu et al. 1987) and exhibit highly nonlinear structures (e.g., Myers 1991; Houlahan & Scalo 1992; Wood, Myers, & Daugherty 1993; Wiseman & Adams 1993; Blitz 1993; Adams 1992; de Geus, Bronfman, & Thaddeus 1990). In addition, these clouds exhibit a wide range of velocity structure as measured by line of sight velocity variations and by varying line-widths. These observations indicate cloud motions which are faster than the thermal sound speed but generally slower than the Alfvén speed. In the models of this paper, the fluid speeds are determined by the Mach number $A$ which is normalized to the thermal sound speed. Thus, for applications to molecular clouds, $A$ should lie in the approximate range $1 < A < (1 + \kappa)^{1/2} \sim 3$.

The first obvious signature of wave motions in molecular clouds is periodic or nearly
periodic structures. Several examples can be found in the existing literature. For example, the star forming region NGC 6334 contains five regularly spaced clumps, as deduced from the 69 μm continuum map of the region (see McBreen et al. 1979); the inferred wavelength is ∼3 pc. Such behavior is not rare. A survey of 23 globular filaments (defined to be filamentary dark clouds which exhibit condensations) shows that “the most striking similarity among all of the globular filaments is the regularity of their segmentation” (Schneider & Elmegreen 1979). This survey also shows that the characteristic spacing of the condensations along the filaments is about three times the width of the filament. Another example of possible wave motion is provided by the Taurus Molecular Cloud complex, which shows evidence for a periodicity in the velocity of 21cm observations of self-absorption (Shuter, Dickman, & Klatt 1987). These authors interpret their data as velocity waves with a peak to peak amplitude of ∼3 km/s and a wavelength of 32 pc (see also Gomez de Castro & Pudritz 1992). To summarize, periodic structures with wavelengths in the range 3 – 30 pc are often found in molecular clouds.

The solutions presented in this paper represent stationary waves in the molecular cloud fluid. Fortunately, a characteristic observable signature of these stationary waves is the relation given by the continuity equation: ρv = A. If we observe a candidate wave train in a molecular cloud, then the observed line-center velocity (along the propagation direction of the wave) should vary inversely with the density. Notice also that the linewidth is known to vary with density (see the discussion following equation [2.4]). Since the linewidths decrease with increasing density, the observed linewidths, when measured along the direction of a wavetrain, should also be anticorrelated with the density. Keep in mind, however, that this linewidth variation is a signature of the equation of state, whereas the aforementioned line-center velocity variation is a signature of stationary waves.

One example of stationary wave behavior in an observed molecular cloud may be provided by the Lynds 204 complex (McCutcheon et al. 1986). This cloud is highly elongated and shows periodic structure in column density with a wavelength of ∼1° (∼3 pc for an assumed distance to the cloud of 160 pc). In addition, these authors find that the line of sight velocities (of CO line observations) along the filament are very well correlated with the density enhancements in the sense that the most massive parts of the filament have the smallest velocity displacement from the mean v0 (see especially their Figures 4 and 5). Finally, the authors derive the relation $M_j(v_j - v_0) = constant$, where $M_j$ and $v_j$ are the mass and velocity of the $j$th segment of the filament. This relation looks suspiciously like the solution to the one dimensional continuity equation for a stationary wave (see equation [2.17]).

We have also found that nonlinear stationary waves select out particular length-scales for their wavelengths (see, e.g., Figures 4, 5, and 8). The observed wavelength should thus be a calculable function of the various physical parameters involved (wave amplitude, sound speed, magnetic field strength, etc.). Although several parameters are necessary to completely specify the preferred wavelength (or range of wavelengths), all of the parameters are physical quantities and can be determined, in principle, independently of the the observations of the wavelengths.

Another observable quantity is the relative widths of the waves. The candidate
wavetrains in molecular cloud regions appear as chains of “blobs” which are (more or less) lined up and regularly spaced. The aspect ratio of these blobs (at a chosen density contour) is thus a well-defined observable quantity (this aspect ratio has a value of $3 \pm 1$ for the sample of Schneider & Elmegreen 1979). We are currently studying stationary waves on filaments and can obtain a prediction for the relative widths of the wavetrains (Gehman et al. 1993) as a function of the physical parameters of the problem.

In addition to forming substructure within molecular clouds, wave motions might be partly responsible for the formation of molecular clouds themselves. Consider, for example, a large scale nonlinear wavetrain (with a wavelength $\lambda$ of several pc) traveling through low density ($n \sim 1$) atomic gas. Atomic regions with sufficiently large column densities can become self-shielding and transform into regions containing molecular gas. After the wave has passed by, a wake of molecular material can remain. This process would thus leave behind a permanent record of the passage of the wave. We note that somewhat similar behavior forms clouds in the Earth’s atmosphere, where waves set up regions of both low and high density; in the regions of high density, moisture condenses to form clouds and the so-called herring-bone cloud structure is produced (see, e.g., the introduction of Infeld & Rowlands 1990). We note that molecular cloud formation probably involves many different physical processes (see, e.g., the review of Elmegreen 1987). However, the large scale waves discussed here may play an important role and should be studied further in this context.

8.3 Discussion and Directions for Future Work

This paper (see also Paper I and AFW) provides a first step toward an understanding of wave motions and structure formation in molecular clouds. However, much more work remains to be done. Perhaps the key point that one should keep in mind is that complex fluid systems, such as molecular clouds, can wiggle around in many different ways. In this paper, we have concentrated on the study of stationary waves (travelling waves of permanent form) in one spatial dimension. Within this class of waves, a great variety of solutions exists. However, an even greater variety of wave motions can arise from the time dependent problem. These more general types of wave solutions should be studied.

Another related topic is the actual generation of the wave motions. As discussed in Paper I, we expect that self-gravitating clouds will tend to collapse and excite a wide spectrum of wave motions (see, e.g., Arons & Max 1975). The self-gravity of the cloud can provide a more than adequate energy source for the waves. Any wave motions which get excited will generally either grow or disperse. However, the waves which live the longest will be the waves of permanent form – the stationary waves studied in this paper. One classic example of this type of scenario was studied by Zabusky & Kruskal (1965). They considered the time evolution of the Korteweg-deVries equation (a model equation which describes, among other things, surface waves on water; see Korteweg & de Vries 1895) and showed that initially simple cosine waves develop into a train of highly nonlinear pulses of permanent form; these pulses are the classic example of true (stable) solitons. Analogous calculations should be performed for the self-gravitating fluid systems of this paper and for more detailed models which more closely resemble real molecular clouds.

Wave stability provides another avenue for future research. Even though the waves
Considered here are stationary and thus possess a balance between gravitational dispersion and nonlinear steepening, they may be subject to instabilities (see, e.g., Pego & Weinstein 1992). In the likely event that the waves are unstable, we must find the wave configurations with the longest lifetimes; these waves will be the ones which provide, in part, the observed structure of molecular clouds.

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APPENDIX A: SCALING TRANSFORMATIONS

In this Appendix, we consider a class of transformations which leave the fundamental equation of motion of this theory invariant. We begin with the equation itself:

\[
\rho \rho \xi \left[ \rho^2 \frac{\partial p}{\partial \rho} - A^2 \right] + \rho \xi \rho \left[ 3A^2 - \rho^2 \frac{\partial p}{\partial \rho} + \rho^3 \frac{\partial^2 p}{\partial \rho^2} \right] + \rho^4 q(\rho) = 0 , \tag{A1}
\]

which is written with an arbitrary charge density. We now consider transformations of the density \( \rho \) and the variable \( \xi \) of the form

\[
\rho \rightarrow \Lambda \rho , \tag{A2}
\]
\[\xi \rightarrow \gamma \xi . \tag{A3}\]

The equation of motion \([A1]\) remains invariant under this transformation provided that

\[
\gamma = \Lambda^{-1/2} , \tag{A4}
\]

and the constituents of the problem scale according to

\[
A \rightarrow \Lambda A , \tag{A5}
\]
\[p \rightarrow \Lambda p , \tag{A6}\]
\[q \rightarrow \Lambda q . \tag{A7}\]

The Mach number \( A \) and the parameters in the pressure \( p \) can generally be rescaled as required. In order for the scaling of the charge density (equation \([A7]\)) to be satisfied, the parameters of the charge density must also be rescaled appropriately, as we discuss below.
For Jeans theory, \( q = \rho - \rho_0 \), and the rescaling of equation [A7] can be met if \( \rho_0 \to \Lambda \rho_0 \). In practice, we choose \( \Lambda \) to make \( \rho_0 = 1 \) and then rescale the remaining parameters as described above.

Both the Yukawa theory and the pseudo-two-dimensional theory of §7 can also be rescaled. We require

\[ m^2 \to \Lambda m^2, \quad \text{(A8)} \]

for the former and

\[ \mu^2 \to \Lambda \mu^2, \quad \text{(A9)} \]

for the latter. These scalings are not unexpected since both \( m \) and \( \mu \) represent inverse lengthscales (very roughly they are the wavenumbers in the direction perpendicular to the propagation direction of the wave) and since lengthscales transform as in equations [A2] and [A3].

**APPENDIX B: SOLITARY WAVE SOLUTIONS IN JEANS THEORY**

In this Appendix, we consider a class of solitary wave solutions in Jeans theory (see §4 in the text). For these solutions, the singularity is removed by choosing \( \beta \) such that \( f(\rho_C) = 0 \). We first expand \( f \) (see equation [2.20]) about \( \rho_C \) to obtain

\[
f = \frac{(\rho - \rho_C)^2 (\rho_0 - \rho_C)}{\rho_C^3} \left\{ \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial p}{\partial \rho} \right) \right\}_C + O[(\rho - \rho_C)^3], \quad \text{(B1)}
\]

where the \( C \) subscript represents evaluation of the term in brackets at \( \rho_C \). We now expand equation [B1] about the point \( \rho_C \) and obtain

\[
\frac{1}{2} \rho_C^2 = \frac{\rho_C^3}{2} (\rho_0 - \rho_C) \left[ \left\{ \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial p}{\partial \rho} \right) \right\}_C \right]^{-1} + O[(\rho - \rho_C)]. \quad \text{(B2)}
\]

For the particular equation of state used here (see equation [2.4b]), the expansion reduces to a simpler form

\[
\frac{1}{2} \rho_C^2 = \frac{\rho_C^3}{2} (\rho_0 - \rho_C) [2\rho_C + \kappa]^{-1} + O[(\rho - \rho_C)]. \quad \text{(B3)}
\]

Since the term in square brackets is nonzero, and since \( f \geq 0 \) near \( \rho_C \), the quantity \( \rho_C^2 \) must be positive and nonsingular.

For physically valid solutions, the density must lie in the range between \( \rho_1 = 0 \) and \( \rho_2 \) (where the peak density obeys the ordering \( \rho_2 > \rho_0 > \rho_C \)). The first integral of the equation of motion (see equation [2.19]) may be expanded about \( \rho = \rho_1 = 0 \) to obtain

\[
\rho_C^2 = \frac{\rho_0}{A^2} \rho^4 + O(\rho^5). \quad \text{(B4)}
\]
Thus, in the limit $\rho \to \rho_1 = 0$, the wave profile has the form

$$\rho \approx \left[ \frac{A^2}{\rho_0} \right]^{1/2} \frac{1}{|\xi|}.$$  \hspace{1cm} (B5)

It is evident from equation [B5] that this class of solutions represents solitary waves (see the discussion in §2.6). The density profile $\rho(\xi)$ for this case is shown in Figure 6. The dashed lines correspond to the sonic points. This solution corresponds to an unshocked flow with both supersonic and subsonic regimes and with densities both above and below the “background” density $\rho_0$.

One complication of this solution is that the boundary conditions in the limits $\xi \to \pm \infty$ are unphysical because the velocity $v \sim \rho^{-1} \sim \xi$ diverges. Furthermore, the total charge $Q$ also diverges, as shown by substituting the limiting form of $f(\rho \to 0) \to A^2 \rho_0/2\rho^2$ into equation [3.8]. In order for this solution to be physical, appropriate boundary conditions must be applied at some finite distance. Suppose we interpret the initial unperturbed fluid to be at uniform density $\rho_0$ and to have zero initial charge. The nonlinear structures that form out of this configuration must also have zero total charge. Thus, one “natural” choice of boundary conditions is to assume a symmetric density profile that also has zero total charge, i.e., we cut off the density profile at a finite lengthscale such that the total charge inside vanishes. As shown by equation [3.8], the $Q = 0$ solitary wave is bounded by the sonic points (since $f(\rho_C) = 0$).

It is worth noting that charge densities $q(\rho)$ can exist for which these solutions (i.e., solutions in which the singularity is removed in this manner) produce wave profiles which are bounded by a lower density $\rho_1 \neq 0$, thereby allowing physical boundary conditions at spatial infinity.

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FIGURE CAPTIONS

Figure 1. Schematic of the function $f(\rho)$ showing possible behavior of a charge density theory in which $q(\rho)$ has at least two zeroes. The solid curve shows the usual case in which a second ordinary zero of $f$ exists. The long dashed curve shows the case in which $f(\rho)$ asymptotically approaches a constant and hence no soliton solutions are allowed. For the remaining case shown by the short dashed curve, depression solitons are allowed.

Figure 2. Schematic of the function $f(\rho)$ for Jeans theory.

Figure 3. Wave profile $\rho(\xi)$ for nonlinear waves in Jeans theory. This example has $A = 1$ and $\kappa = 10$.

Figure 4. Allowed range of wavelengths for nonlinear waves in the Jeans theory. The upper curve shows the (nonlinear) Jeans wavelength $\lambda_J$ plotted as a function of amplitude $\delta \rho / \rho$ of the wave. The lower curve shows the “shock limit”, i.e., waves with wavelengths smaller than this critical value will shock the dissipate. The wavelengths are expressed...
in units of the linear Jeans length for the pressure given by equation [2.4b] with $\kappa = 10$. Nonlinear waves in this theory must have wavelengths which lie between the two curves.

Figure 5. Allowed range of wave profile widths $\lambda_p$ for nonlinear waves in the Jeans theory. The upper curve shows the width for the longest possible wavelength (the Jeans wavelength) plotted as a function of amplitude $\delta \rho / \rho$ of the wave. The lower curve shows the width for the “shock limit”. All quantities are the same as in Figure 4. The width of the wave profiles for nonlinear waves in this theory are thus confined to a rather narrow range.

Figure 6. Solitary wave solution for Jeans theory for $A = 3$ and $\kappa = 10$. Vertical dashed lines show the location of the sonic points (see §4 and Appendix B).

Figure 7. Wave profile $\rho(\xi)$ for nonlinear waves in Jeans theory including the effects of magnetic fields as a pressure term. This example has $A = 1$, $\kappa = 10$, and $\alpha = 3$; the constant of integration $\beta$ has been adjusted to make the amplitude comparable to the field-free case shown in Figure 3. Notice the enhanced width of the waves relative to those of Figure 3.

Figure 8. Allowed range of wavelengths for nonlinear waves in the Jeans theory with magnetic fields included in the flux freezing approximation. The upper curve shows the (nonlinear) Jeans wavelength $\lambda_J$ plotted as a function of amplitude $\delta \rho / \rho$ of the wave. The lower curve shows the “shock limit”, i.e., waves with wavelengths smaller than this critical value will shock and dissipate. Here, $\kappa = 10$, $\alpha = 3$, and the wavelengths are normalized as in Figure 4. Nonlinear waves in this theory must have wavelengths which lie between the two curves.

Figure 9. Wave profiles $\rho(\xi)$ for a solitary wave (upper curve) and a “depression soliton” (lower curve) in Yukawa theory. Both examples have $A = 1$ and $\kappa = 10$. Notice that the depression soliton has a much lower density constraint than the ordinary solitary wave.

Figure 10. Allowed regions of the $A - m$ plane for various types of stationary waves (for Yukawa theory and $\kappa = 10$). Solitary wave solutions are allowed in both regions I and II. Depression solitons are limited to region II. Ordinary nonlinear stationary waves are allowed in region III. The vertical dashed curve on the right side of the figure corresponds to the sonic point for this equation of state; all allowed Mach numbers $A$ must be less than this value.
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