Low memory implementation of Orthogonal Matching Pursuit like greedy algorithms: Analysis and Applications

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Abstract

The convergence analysis of a low memory implementation of the Orthogonal Matching Pursuit method, which is termed Self Projected Matching Pursuit, is presented. The approach is extended to improve the sparsity ratio of a signal representation when approximating the signal by partitioning. A backward strategy, for reducing terms in a signal decomposition, is discussed. The suitability of the methods, to be applied on cases where standard implementations of Orthogonal Matching Pursuit are not feasible due to memory requirements, is illustrated by producing high quality approximation of melodic music and X-Ray medical images.

1 Introduction

The process by which a signal is transformed, in order to significantly reduce is dimensionality, is refereed to as sparse representation of the signal. For the class of signals known as compressible, such as images and audio signals, this process can be realized without much loss of information content. The degree of the achieved sparsity depends on the suitability of the transformation for representing the particular signal. Traditional methods implement the transformation using fast orthogonal transforms. However, much higher levels of sparsity are attained, in many cases, if the transformation is carried out using a large redundant set called dictionary. The gain comes at expenses of increment in complexity. However, advances in computational facilities, including multiprocessors for personal computers, have encouraged the developments of techniques for signal representation using dictionaries. For the most part these techniques comprise strategies based on minimization of the $l_1$-norm \cite{1,3} and the so called ‘greedy strategies’. The latter consist in adaptively constructing a signal representation as a linear superposition of elements.
taken from the dictionary. In this Communication we focus on a low memory implementation of a particular method within this category.

Greedy strategies have been the subject of much research in the last two decades [1–19]. The simplest, yet very effective greedy algorithm for the sparse representation of large signals, was introduced to the signal processing community in [4] with the name of Matching Pursuit (MP). It had previously appeared as a regression technique in statistics though [20], where the convergence property was established [21]. While MP converges asymptotically to a signal in the linear span of the dictionary, or to its orthogonal projection if the signal is out of that space, the approach is not stepwise optimal because it does not yield an orthogonal projection at each step. Consequently, in addition to failing to minimize the norm of the approximation error at each step, it may select linearly dependent elements. As will be illustrated by the example of Sec. 2, this feature significantly compromises sparsity in some cases. The performance of the seminal MP approach has also been shown to be poor for the recovery of some exactly sparse signals [1,22].

A refinement to MP, which does yield an orthogonal projection at each iteration, is refereed to as Orthogonal Matching Pursuit (OMP) [5]. This is a very effective technique up to some dimensionality. When processing large signals, however, the storage requirements of its standard implementations frequently exceed the memory capacity of a personal computer used for research purposes. Some techniques addressing this matter are known as Gradient Pursuits [16]. The methods in this line produce a satisfactory approximation of the OMP criterion in many practical situations. However, as shown by the example of Sec. 2, such an approximation may not be satisfactory in cases where the basic MP method performs badly. An alternative implementation of OMP, which requires much less memory than standard implementations, is considered in [23]. The approach is termed Self Projected Matching Pursuit (SPMP), because it produces the orthogonal projection of the signal, at each iteration, by applying MP using a dictionary consisting only of the already selected elements.

Real world signals are too large to be approximated using dictionaries. Often the approximation is carried out by dividing the signal into small units (or ‘blocks’), constructing what we call a ‘signal partition’. Even so, the memory requirements of standard implementations of
OMP limit the size of the blocks in the partition. Since the memory demands of SPMP are of the same order as MP, this implementation extends the limit of applicability of the OMP refinement as far as memory is concerned. The SPMP implementation has been shown to be affective for sparse approximation of astronomical images [23]. The convenient feature in that context being that SPMP fully exploits the separability of dictionaries. In the case of one dimensional signals such as melodic music, which are well approximated by trigonometric dictionaries, the SPMP approach allows to implement OMP without having to save the dictionary and performs the calculation via the Fast Fourier Transform [24].

Whilst the approximation of a signal partition is often realized by approximating every element of the partition completely independently of the others, in some cases of interest much higher levels of sparsity can be achieved if the approximation of all the blocks in the partition are related by a global constraint on sparsity. This gives rise to what is termed the Hierarchized Block Wise (HBW) implementation of greedy strategies for approximating by partitioning [25,26]. In particular, the HBW versions of the MP/OMP algorithms is refereed to as HBW-MP/OMP.

Even when the HBW-OMP approach is implemented on partition units of relatively small size, for which the standard OMP implementations would not introduce any memory issue, its storage requirements are much more demanding. This limits the size of the signals that can be approximated by the HBW-OMP strategy without segmentation. Contrarily, the HBW-SPMP implementation of the same approach provides us with a suitable alternative for dealing with large signals.

Summary of contributions

The main contributions of the paper are listed below:

- The paper presents the convergence analysis of the SPMP approach, which deals with those cases where the standard implementations of the OMP approach is not feasible due to storage requirements.

- The HBW-SPMP implementation of the HBW-OMP method is introduced. Its advantage upon the standard implementation which would approximate each element of the partition
at once, and totally independently of the other elements, is illustrated by a) producing high quality sparse approximation of music signals using simple trigonometric dictionaries and b) producing high quality sparse approximation of X-ray medical images.

- The HBW Backward SPMP strategy, for downgrading the decomposition of a signal, is considered.

The paper is organized as follows: Sec. 2 revises the SPMP method and proofs the exponential convergence of the self projection step. Sec. 3 extends the SPMP method by introducing its HBW version for approximating a signal partition subjected to a global constraint on sparsity. The suitability of the HBW-SPMP strategy to produce sparse high quality approximation of music signals and X-Ray medical images is illustrated in section Sec. 4. Sec. 5 introduces the HBW manner for downgrading a signal approximation. The conclusions are summarized in Sec. 6.

2 Self Projected Matching Pursuit (SPMP)

Before reviewing the general SPMP technique let’s define some basic notation: $\mathbb{R}, \mathbb{C}$ and $\mathbb{N}$ represent the sets of real, complex and natural numbers, respectively. Boldface fonts are used to indicate Euclidean vectors or matrices and standard mathematical fonts to indicate components, e.g., $\mathbf{d} \in \mathbb{C}^N$ is a vector of $N$-components $d(i) \in \mathbb{C}^N$, $i = 1, \ldots, N$ and $\mathbf{I} \in \mathbb{R}^{N_x \times N_y}$ a matrix of elements $I(i,j)$, $i = 1, \ldots, N_x$, $j = 1, \ldots, N_y$. The operation $\langle \cdot, \cdot \rangle$ indicates the Euclidean inner product and $\| \cdot \|$ the induced norm, i.e. $\| \mathbf{d} \|^2 = \langle \mathbf{d}, \mathbf{d} \rangle$, with the usual inner product definition:

For $\mathbf{d} \in \mathbb{C}^N$ and $\mathbf{f} \in \mathbb{C}^N$

$$\langle \mathbf{f}, \mathbf{d} \rangle = \sum_{i=1}^{N} f(i)d^*(i), \tag{1}$$

where $d^*(i)$ stands for the complex conjugate of $d(i)$.

The operation $\langle \cdot, \cdot \rangle_F$ indicates the Frobenious inner product and $\| \cdot \|_F$ the corresponding norm, i.e., for $\mathbf{D}_1 \in \mathbb{C}^{N_x \times N_y}$ and $\mathbf{D}_2 \in \mathbb{C}^{N_x \times N_y}$

$$\langle \mathbf{D}_1, \mathbf{D}_2 \rangle_F = \sum_{i,j=1}^{N_x, N_y} D_1(i,j)D_2^*(i,j). \tag{2}$$

Let’s consider a finite set $\mathcal{D}$ of $M$ normalized to unity vectors $\mathcal{D} = \{ \mathbf{d}_n \in \mathbb{C}^N : \| \mathbf{d}_n \| = 1 \}_{n=1}^{M}$ spanning $\mathbb{C}^N$. For $M > N$ the over-complete set $\mathcal{D}$ is called a dictionary and the elements
are called *atoms*. Given a signal, as a vector $f \in \mathbb{C}^N$, the $k$-term *atomic decomposition* for its approximation takes the form

$$f^k = \sum_{n=1}^{k} c(n) d_{\ell_n}.$$  
(3)

The problem of how to select from $\mathcal{D}$ the smallest number of $k$ atoms $d_{\ell_n}, n = 1 \ldots, k$, such that $\|f^k - f\| < \rho$, for a given tolerance parameter $\rho$, is an NP-hard problem [6]. In practical applications one looks for ‘tractable sparse’ solutions. This is to say a representation involving a number of $k$-terms, with $k$ acceptable small in relation to $N$. The simple MP approach to tackle this problem evolves by successive approximations as follows [4]: Setting $k = 0$ and starting with an initial approximation $f^0 = 0$ and residual $r^0 = f$ the algorithm iterates by sub-decomposing the $k$-th order residue into

$$r^k = \langle d_{\ell_{k+1}}, r^k \rangle d_{\ell_{k+1}} + r^{k+1},$$  
(4)

with the atom $d_{\ell_{k+1}}$ selected as

$$\ell_{k+1} = \arg \max_{n=1, \ldots, M} |\langle d_n, r^k \rangle|.$$  
(5)

From (4) it follows that $\|r^{k+1}\|^2 \leq \|r^k\|^2$, since

$$\|r^k\|^2 = |\langle d_{\ell_{k+1}}, r^k \rangle|^2 + \|r^{k+1}\|^2.$$  
(6)

**Lemma 1.** *In the limit* $k \to \infty$ *the sequence* $f^k$ *given in (3) converges to* $f$, *or to* $\hat{P}_V f$, *the orthogonal projection of* $f$ *onto* $\mathbb{C}^N = \text{span}\{d_{\ell_n}\}_{n=1}^{M}$ *if* $f$ *were not in* $\mathbb{C}^N$.

This lemma is just a particular case of the well established and more general convergence properties of MP [4, 8, 21]. However, for pedagogical reasons, due to its crucial importance for this work, we present here a particular proof holding for finite dimension spaces which, for this reason, is very simple and transparent.

**Proof.** We notice, from (6), that $\|r^k\|^2$ is a decreasing sequence which, since $\|r^k\|^2 \geq 0$ for all $k$, is bounded. It is a classic results of analysis that a decreasing and bounded sequence converges to the infimum [28], i.e., $\lim_{k \to \infty} \|r^k\|^2 = b$. We prove next that $b = 0$. Since

$$\|r^{k+1}\|^2 = \|r^k\|^2 - |\langle d_{\ell_{k+1}}, r^k \rangle|^2,$$
taking \( \lim_{k \to \infty} \) both sizes, we have:

\[
b^2 = b^2 - \lim_{k \to \infty} |\langle d_{\ell_{k+1}}, r^k \rangle|^2.
\]

Thus, \( \lim_{k \to \infty} |\langle d_{\ell_{k+1}}, r^k \rangle| = 0 \), which, from (5), implies \( \lim_{k \to \infty} |\langle d_n, r^k \rangle| = 0 \), \( n = 1, \ldots, M \). Consequently, either \( \lim_{k \to \infty} r^k = 0 \) or, if the dictionary is incomplete, \( \lim_{k \to \infty} r^k \) is orthogonal to all the elements in \( D \). This result is readily obtainable here, because of the finite dimension framework. Indeed, in finite dimension any set of vectors is a frame for its linear span. Hence the existence of a reciprocal frame \( \tilde{D} = \{ \tilde{d}_n \in \mathbb{C}^N \}_{n=1}^M \) of the dictionary \( D \) is guaranteed. Furthermore, every vector in \( \mathbb{C}^N \), and in particular \( r^k \), can be decomposed in the form \( r^k = \hat{P}_{V_M} r^k + \hat{P}_{V_M} r^k \), where \( \hat{P}_{V_M} r^k \) is the orthogonal projection onto \( V_M \) and \( \hat{P}_{V_M} r^k \) the orthogonal complement in \( \mathbb{C}^N \). From the relation

\[
\hat{P}_{V_M} r^k = \sum_{n=1}^M d_n \langle \tilde{d}_n, r^k \rangle = \sum_{n=1}^M \tilde{d}_n \langle d_n, r^k \rangle
\]

we conclude that \( \lim_{k \to \infty} |\langle d_n, r^k \rangle| = 0 \), \( n = 1, \ldots, M \) \( \implies \lim_{k \to \infty} \hat{P}_{V_M} r^k = 0 \). Then, either \( \lim_{k \to \infty} r^k = 0 \) or \( \lim_{k \to \infty} r^k \in \mathbb{V}_M^\perp \).

### 2.1 Adding Self Projections

The SPMP method uses Lemma 1 to achieve, at each iteration, the orthogonal projection of a given signal on the selected subspace. In other words, SPMP relies on Lemma 1 to produce an alternative implementation of the OMP approach. Given a signal and a dictionary it proceeds as follows [23]: Set \( S = \{ \emptyset \} \), \( f^0 = 0 \) and \( r^0 = f \). Starting from \( k = 1 \), at each iteration \( k \) implement the steps below.

1. Apply the MP criterion for selecting one atom from \( D \), i.e., select \( \ell_k \) such that

\[
\ell_k = \arg \max_{n=1, \ldots, M} |\langle d_n, r^{k-1} \rangle|,
\]

and assign \( S_k = S_{k-1} \cup d_{\ell_k} \). Set \( c(k) = \langle d_{\ell_k}, r^{k-1} \rangle \), update the approximation of \( f \) as \( f^k = f^{k-1} + c(k)d_{\ell_k} \) and evaluate the new residue \( r^k = f - f^k \).

2. Approximate via MP the residue \( r^k \) using only the selected set \( S_k \) as the dictionary, which guarantees the asymptotic convergence to the approximation \( \hat{P}_{V_k} r^k \) of \( r^k \), where
\[ \mathbb{V}_k = \text{span}\{S_k\}, \]
and a residue \( r^\perp = r^k - \hat{P}_{\mathbb{V}_k} r^k \) having no component in \( \mathbb{V}_k \). The outputs of the MP algorithm for the approximation of \( r^k \) in \( \mathbb{V}_k \) give the scalers \( \{t(n)\}_{n=1}^k \), which satisfies:

\[ \hat{P}_{\mathbb{V}_k} r^k = \sum_{n=1}^k t(n)d_{\ell_n}. \]

iii) Set \( f^k \leftarrow f^k + \hat{P}_{\mathbb{V}_k} r^k \) and \( r^k \leftarrow r^\perp \). Update the set of coefficients \( \{c(n)\}_{n=1}^k \leftarrow \{c(n) + t(n)\}_{n=1}^k \) and increment the number of iterations \( k \leftarrow k + 1 \).

Repeat steps i) - iii) until, for a required numerical tolerance \( \rho \), the condition \( ||r^k|| < \rho \) is reached.

Notice that, by means of the self-projections, at iteration \( k \) the SPMP algorithm renders an approximation \( f^k = \hat{P}_{\mathbb{V}_k} f \) and a residue \( r^k = f - \hat{P}_{\mathbb{V}_k} f \). This implies that the outputs of the algorithm are theoretically equivalent to those of OMP.

**Lemma 2.** If the atoms \( d_{\ell_i}, i = 1, \ldots, k \) are selected by criterion [5], and the residue refined by self projections at each iteration, the selected atoms constitutes a linearly independent set.

**Proof.** If the set consists of a single atom the lemma is triviality true. Assuming that it is true for the first \( k \) atoms we prove that it is true for \( k + 1 \) atoms.

Suppose, on the contrary, that \( |\langle d_{\ell_{k+1}}, r^k \rangle| > 0 \) and \( d_{\ell_{k+1}} = \sum_{i=1}^k a_i d_{\ell_i} \), where \( a_i, i = 1, \ldots, k \) are scales such that \( \sum_{i=1}^k |a_i|^2 > 0 \). At iteration \( k \) the SPMP algorithm renders a residue that satisfies \( r^k = f - \hat{P}_{\mathbb{V}_k} f \). Hence,

\[ \langle d_{\ell_{k+1}}, r^k \rangle = \left\langle \sum_{i=1}^k a_i d_{\ell_i}, f - \hat{P}_{\mathbb{V}_k} f \right\rangle = 0, \]

which contradicts the assumption that \( |\langle d_{\ell_{k+1}}, r^k \rangle| > 0 \). It is concluded then that \( d_{\ell_{k+1}} \) cannot be expressed as a linear combination of the previously selected atoms. \( \square \)

### 2.2 Convergence rate of the self projection steps

We start by recalling some properties of hermitian matrices, which will be used for the analysis.

Let’s the atoms in \( S_k \), as defined in the previous section, be the columns of the matrix \( D_k \).

According to Lemma 2, the corresponding Gram matrix \( G_k = D_k^\dagger D_k \), with \( D_k^\dagger \) indicating the transpose conjugate of \( D_k^\dagger \), is full rank. In terms of its \( k \) eigenvalues and eigenvectors it can be
expressed as
\[ G_k = U_k \Lambda_k U_k^\top, \]
where \( \Lambda_k \) is a diagonal matrix, containing in the diagonal its eigenvalues \( \lambda_i^k > 0, i = 1, \ldots, k \) in descending order. The columns of matrix \( U_k \) are its normalized to unity eigenvectors \( u_i^k, i = 1, \ldots, k \). Since \( G_k \) is hermitian its eigenvectors are orthogonal. Moreover, as it is easy to check, the vectors \( w_i^k = \frac{D_k u_i^k}{\sqrt{\lambda_i^k}}, i = 1, \ldots, k \), called singular vectors, constitute an orthonormal basis for \( \mathbb{V}_k = \text{Range}(D_k) \). Accordingly, the orthogonal projector \( \hat{P}_{\mathbb{V}_k} \) admits a representation of the form:
\[ \hat{P}_{\mathbb{V}_k} = W_k W_k^\top, \]
where the columns of matrix \( W_k \) are the vectors \( w_i^k, i = 1, \ldots, k \), while the matrix \( D_k D_k^\top \) admits a representation of the form:
\[ D_k D_k^\top = W_k \Lambda_k W_k^\top. \]

The following inequality, arising from (9) and (10),
\[ ||D_k^\top g||^2 = \langle g, D_k D_k^\top g \rangle \geq \lambda_k^k \| \hat{P}_{\mathbb{V}_k} g \|^2, \quad \forall g \in \mathbb{C}^N \]
will be used for the analysis of the convergence rate of the self-projection step.

Let us recall that the self-projection consists in approximating via MP the residue \( r_k \), using as dictionary the set \( S_k \) with the atoms selected up to iteration \( k \). Thus, setting \( r_k^{0} = r_k \), at the \( j \)-th iteration of the self-projection procedure the residue \( r_k^{j} \) of the residue’s approximation fulfills the equation
\[ r_k^{j} = \langle d_{\ell_j}, r_k^{j} \rangle d_{\ell_j} + r_k^{j+1}, \]
where
\[ \ell_j = \arg \max_{i=1, \ldots, k} | \langle d_{\ell_j}, r_k^{j} \rangle |. \]
Since \( \hat{P}_{\mathbb{V}_k} d_{\ell_j} = d_{\ell_j}, j = 1, \ldots, k \), applying the operator \( \hat{P}_{\mathbb{V}_k} \) on both sides of (12) we have,
\[ \hat{P}_{\mathbb{V}_k} r_k^{j+1} = \hat{P}_{\mathbb{V}_k} r_k^{j} - \langle d_{\ell_j}, r_k^{j} \rangle d_{\ell_j}, \]
and consequently, since \( \hat{P}_{\mathbb{V}_k} \) is hermitian,
\[ \| \hat{P}_{\mathbb{V}_k} r_k^{j+1} \|^2 = \| \hat{P}_{\mathbb{V}_k} r_k^{j} \|^2 - | \langle r_k^{j}, d_{\ell_j} \rangle |^2 \]

(14)
By definition of the index $\ell_j$ (cf.\((13)\)), and using \((11)\), we assert that
\[
|\langle d_{\ell_j}, r^{k,j} \rangle|^2 \geq \frac{1}{k} \sum_{i=1}^{k} |\langle d_{\ell_i}, r^{k,j} \rangle| = \frac{1}{k} \|D_k^T r^{k,j}\|^2 \geq \frac{\lambda_k}{k} \|\hat{P}_{V_k} r^{k,j}\|^2.
\]

Then, we finally obtain:
\[
\|\hat{P}_{V_k} r^{k,j+1}\|^2 \leq \left(1 - \frac{\lambda_k}{k}\right) \|\hat{P}_{V_k} r^{k,j}\|^2,
\]
and applying the inequality back $j$-times:
\[
\|\hat{P}_{V_k} r^{k,j+1}\|^2 \leq \left(1 - \frac{\lambda_k}{k}\right)^j \|\hat{P}_{V_k} r^{k,0}\|^2 \leq \left(1 - \frac{\lambda_k}{k}\right)^j \|r^{k,0}\|^2.
\]

The above bound indicates the exponential convergence to a residue having no component in $V_k$. It also shows the dependence of the convergence rate on the smallest eigenvalue of the Gram matrix $G_k$ of the selected atoms up to iteration $k$. According to the Cauchy’ interlace theorem [29] it is true that $\lambda_{k+1} < \lambda_k$. Hence, in general one could expect the convergence rate of the self projection to slow down as the iterative selection of atoms progresses.

**Remark 1:** The exponential convergence of MP in terms of the dictionary’s coherence is derived in [14] for the case of quasi incoherent dictionaries. That condition is too stringent for signals of practical interest, which are far more compressible when using a highly coherent dictionaries than when using an orthogonal or quasi orthogonal basis. Contrarily, the expression \((16)\) gives a realistic appreciation of the broad range of effective applicability of the SPMP approach. Regardless of the dictionary coherence, SPMP can be an effective low memory implementation of OMP as long as the least square problem, for the determination of the coefficients in the atomic decomposition of each $f_q$, $q = 1, \ldots, Q$, is not an ill well posed problem.

**Numerical example motivating the SPMP approach**

Let’s consider the example introduced in Sec. B of [23], which is a hard test for MP. The dictionary is the Redundant Discrete Cosine (RDC) dictionary $D^c$ given below:
\[
D^c = \left\{ \frac{1}{w^c(n)} \cos\left(\frac{\pi(2i-1)(n-1)}{2M}\right), i = 1, \ldots, N \right\}_{n=1}^{M}.
\]

where
\[
w^c(n) = \begin{cases} 
\sqrt{N} & \text{if } n = 1, \\
\sqrt{\frac{N}{2}} + \frac{\sin\left(\frac{\pi(n-1)}{M}\right) \sin\left(\frac{\pi(n-1)N}{M}\right)}{2(1-\cos(\frac{\pi(n-1)}{M}))} & \text{if } n \neq 1.
\end{cases}
\]
To represent the chirp signal \( \cos(2\pi t^2) \) depicted in Fig. 1 we take \( N = 2000 \) equidistant points in the interval \([0, 8]\) and sample the chirp at those points \( f(i), i = 1, \ldots, N \). The goal is to find an approximation of these points, up to precision \( \rho = 0.001\|f\| \), using the dictionary \( D^c \). Taking \( M = N = 2000 \) in the above definition of \( D^c \) we have an orthonormal basis and therefore both MP and OMP methods give the sparsest approximation of the signal in orthogonal components. For an approximation up to the given precision (coinciding visually with the theoretical chirp in Fig. 1) it is necessary to use \( K = 683 \) orthonormal elements from (17). Now, setting \( M = 2N = 4000 \) the dictionary \( D^c \) is no longer an orthonormal basis but a redundant tight frame and the algorithms MP and OMP produce very different decompositions. While OMP improves the sparsity of the representation requiring only \( K = 286 \) components, MP needs \( K = 1638 \) different atoms, i.e. significantly more than with the orthonormal basis. The reason for the poor performance of MP is that in the redundant dictionary the atoms are highly correlated and the method picks linearly dependent atoms. When applying the proposed refinement, SPMP, the number of required components is the same as with OMP, i.e. \( K = 286 \).

For this particular example the low memory approximation of OMP called Gradient Pursuit (GP) requires \( K = 433 \) atoms, and the Approximated Conjugate Gradient Pursuit (ACGP) method \( K = 385 \) atoms. The results for GP and ACGP were obtained with the MATLAB implementation of those methods available on [30].

This example illustrates the capability of SPMP for reproducing OMP outcomes in a situation
where low memory approximations to OMP fail to produce a similar result. We consider next a particular strategy for approximating a signal partition which benefits from a low memory implementation of OMP.

3 Hierarchized Block Wise SPMP

The HBW version of pursuit strategies is an implementation of those techniques dedicated to approximating by partitioning. It approximates each element of the partition independently of each other but links all the partition’s elements by a global constraint on sparsity [25][26]. The approach simply ranks the partition units for their sequential stepwise approximation.

Let’s suppose that a given signal $f$ is split into $Q$ disjoint ‘blocks’ $f_q$, $q = 1, \ldots, Q$, where each $f_q$ is an element of $\mathbb{R}^{N_b}$. Denoting by $\tilde{J}$ the concatenation operator, the signal $f \in \mathbb{R}^N$ is ‘assembled’ from the blocks as $f = \tilde{J}_q f_q$. This operation implies that the first $N_1$ components of the vector $f$ are given by the vector $f_1$, the next $N_2$ components by the vector $f_2$ and so on.

The HBW version of SPMP, for approximating the signal’s partition is implemented by the following steps.

1) For $q = 1, \ldots, Q$ set $r^0_q = f_q$, $f_0^q = 0$, and $k_q = 1$. Initialize the algorithm by selecting the ‘potential’ first atom for the atomic decomposition of every block $q$, according to the MP criterion:

$$\ell^q_{k^q_q} = \arg \max_{n = 1, \ldots, M} | \langle d_n, r^{k^q_q - 1}_q \rangle |, \; q = 1, \ldots, Q.$$ 

2) Select the block $q^*$ such that

$$q^* = \arg \max_{q = 1, \ldots, Q} | \langle d_{\ell^q_{k^q_q}}, r^{k^q_q - 1}_q \rangle |$$

and upgrade the atomic decomposition of the block $q^*$ by incorporating the atom corresponding to the index $\ell^q_{k^q_q}$, i.e., use $c^q(k^q_q) = \langle d_{\ell^q_{k^q_q}}, r^{k^q_q - 1}_q \rangle$ to compute

$$f^{k^q_q}_{q^*} = f^{k^q_q - 1}_{q^*} + c^q(k^q_q) d_{\ell^q_{k^q_q}},$$

$$r^{k^q_q}_{q^*} = f_{q^*} - f^{k^q_q}_{q^*}.$$ 

If $k^q_{q^*} > 1$ apply the self projection step of Sec. 2 to obtain $\hat{P}_{\tilde{J}^q_{k^q_q}}$, $r^{k^q_q}_{q^*} = \sum_{n=1}^{k^q_{q^*}} t(n) d_{\ell^q_{n}}$ and update:
the coefficients \( \{c^q(n)\}_{n=1}^{k_q} \leftarrow \{c^q(n) + t(n)\}_{n=1}^{k_q} \),

- the signal approximation \( f_q^{k_q} \leftarrow f_q^{k_q} + \hat{P}_{v_{k_q}} \mathbf{r}_{k_q} \),

- the residue \( \mathbf{r}_{k_q} \leftarrow \mathbf{r}_{k_q} - f_q^{k_q} \).

3) Check if, for the given numbers \( K \) and \( \rho \) either the condition \( \sum_{q=1}^{Q} k_q = K \) or \( \| \mathbf{f} - \mathbf{f}^K \|^2 < \rho \) has been met. Otherwise:

- Increase \( k_q \leftarrow k_q + 1 \).

- Select a new potential atom for the atomic decomposition of block \( q^* \) using the same criterion as in 1).

- Repeat steps 2) and 3).

Remark 2: Notice that the memory requirements of the SPMP approach and its HBW version are equivalent to that of the MP approach and its HBW version, respectively. This implies a significant saving in memory with respect of the standard implementations of OMP and the HBW version of it. Certainly, for the implementation of the orthogonal projection step, through Gram Smith orthogonalization for instance, the OMP approach would require to constructs \( k_q \) vectors, each of dimension \( N_b \). Thus, its HBW version would need to save these vectors for each of the \( Q \) blocks, only for the calculation of the projections for the selection of atoms. On the contrary, the HBW-SPMP implementation saves directly the coefficients, reducing the storage demand for the projection step from \( O(N_bK) \) to \( O(K) \).

4 Sparse approximation of real world signals

We construct now the atomic decomposition of the Pop Piano and Classic Guitar clips shown in Fig. 2. Both clips consists of \( N = 262144 \) samples at 44100Hz each (5.94 secs length). The approximation aims at producing high quality reconstruction. The quality is assessed by the Signal to Noise Ratio (SNR), which is defined as

\[
\text{SNR} = 10 \log_{10} \frac{\| \mathbf{f} \|^2}{\| \mathbf{f} - \mathbf{f}^K \|^2} = 10 \log_{10} \frac{\sum_{i=1}^{N_bQ} |f_q(i)|^2}{\sum_{q=1}^{N_bQ} \sum_{i=1}^{|f_q(i) - f_q^{k_q}(i)|^2}}.
\]
The global sparsity is measured by the Sparsity Ratio (SR) which is defined as
\[
\text{SR} = \frac{N}{K},
\]
where \(K\) is the total number of coefficients in the signal representation.

Figure 2: Pop Piano (top graph) and Classic Guitar music signals. Both clips consist of \(N = 262144\) samples at 44100Hz each (5.94 secs length).

While the HBW-SPMP approach can be applied with any dictionary we use the trigonometric dictionaries \(\mathcal{D}^{cs} = \mathcal{D}^c \cup \mathcal{D}^s\), with \(\mathcal{D}^c\) as in (17) and \(\mathcal{D}^s\) as given below

\[
\mathcal{D}^s = \left\{ \frac{1}{w^s(n)} \sin\left(\frac{\pi(2i-1)n}{2M}\right), \quad i = 1, \ldots, N \right\}_{n=1}^{M}, \quad (18)
\]

where

\[
w^s(n) = \begin{cases} 
\sqrt{N} & \text{if } n = 1, \\
\sqrt{\frac{N}{2}} - \frac{\sin\left(\frac{\pi n}{M}\right)\sin\left(\frac{2\pi nN}{M^2}\right)}{2(1-\cos\left(\frac{\pi n}{M}\right))} & \text{if } n \neq 1.
\end{cases}
\]

The dictionary \(\mathcal{D}^{cs}\) with redundancy four has been shown to produce high quality approximation of melodic music involving much less terms than what are needed when using an orthonormal trigonometric basis [26], [24]. It has also been shown that the sparsity produced by this dictionaries leads to significant compression of the signal file [27]. An additional advantage of using the dictionary \(\mathcal{D}^{cs}\) is that, because the inner products with the dictionary’s atoms can be computed via Fast Fourier Transform (FFT), the calculations are fast and there is no need to store the dictionary. The convenience of applying the HBW version of a greedy approach for approximating by partitioning using this dictionary was also previously discussed in [26]. Here
we are simply extending the applicability of the HBW-OMP approach by implementing it via SPMP.

The sparsity results of the clips in Fig. 2 are shown in Fig. 3 for the MP, HBW-MP, SPMP, HBW-SPMP approaches and partitions of unit size $N_b$ equal to 1024, 2048, 4096, and 8192 samples. For larger values of $N_b$ the sparsity does not improve significantly. The quality of the approximation is fixed to yield a SNR of 35dB. This value produces a high quality approximation of the signals. It corresponds to a residual error with very small mean ($O(10^{-6})$) and very small variance ($O(10^{-7})$). As observed in Fig. 3 for the two clips in Fig. 2, the gain in sparsity achieved by implementing the SPMP approach in the HBW manner is significant.

Figure 3: SR vs partition unity size $N_b = 1024, 2048, 4096$ and 8192 samples for the music clips of Fig. 2. The graph on left corresponds to the Pop Piano and the other to the Classic Guitar.

**Two dimensional (2D) case**

While there is a number of techniques for learning suitable dictionaries for approximating particular images [31][35], in consistency with the low memory demands of the present context, we illustrate the approach using a simple separable dictionary, which has been shown to produce large SRs with respect to both the Discreet Cosine and the Wavelet Transforms [25]. For approximating the X-ray medical images in Fig. 4, we simply include the Euclidean basis,

$$\mathcal{B}^e = \{e_i \in \mathbb{R}^{N_b}, e_i(j) = 0, \text{if } i \neq j \text{ and } e_j(j) = 1, j = 1, \ldots, N_b \} \cup_{i=1}^{N_b},$$

to form a separable dictionary $\mathcal{D}^x \otimes \mathcal{D}^y$, where $\mathcal{D}^e = \mathcal{D}^x \cup \mathcal{D}^y \cup \mathcal{B}^e$ and $\mathcal{D}^y = \mathcal{D}^x$. Because the dictionary is separable it is only necessary to store $\mathcal{D}^x$ and $\mathcal{D}^y$ if different from $\mathcal{D}^x$. 
The HBW-SPMP approach in 2D is applied as in one dimension, but the image is partitioned into \( Q \) square blocks \( I_q \in \mathbb{R}^{N_b \times N_b}, q = 1, \ldots, Q \) and the inner products are the Frobenious inner products. Given the 2D arrays \( I_q \in \mathbb{R}^{N_b \times N_b} \), for separable atoms \( d^x_n \otimes d^y_m \in \mathbb{R}^{N_b \times N_b} \) the Frobenious inner products \( \langle d^x_n \otimes d^y_m, I_q \rangle_F \) are calculated as:

\[
\langle d^x_n \otimes d^y_m, I_q \rangle_F = \sum_{i=1}^{N_b} \sum_{j=1}^{N_b} d^x_n(i)I_q(i,j)d^y_m(j), \quad q = 1, \ldots, Q.
\]  

(19)

The approximation of X-ray medical images is more effective when carried out in the wavelet domain. Hence, we first apply the Cohen-Daubechies-Feauveau 9/7 (CDF97) wavelet transform to convert the images in Fig. 4 to the transformed arrays, say \( W_1 \) and \( W_2 \) both in \( \mathbb{R}^{N_x \times N_y} \), and approximate these arrays. The transformation was carried out using the \texttt{waveletcdf97} MATLAB function available at [37]. The quality of the approximation is assessed after inverting the approximated wavelet transform. The quality of the approximated image \( I^K \) is measured using two criteria: a) the PSNR, which for a 8-bit \( N_x \times N_y \) image is defined as

\[
\text{PSNR} = 10 \log_{10} \left( \frac{255^2}{\text{MSE}} \right), \quad \text{with} \quad \text{MSE} = \frac{\|I - I^K\|_F^2}{N_xN_y},
\]
and b) the Mean Structure Similarity index (MSSIM) \textsuperscript{38}. For medical images the quality has to be ensured very high. Thus, the approximation is realized to achieve a PSNR=47dB which produces a MSSIM greater than 0.997. This is a very good indication of high quality, in light of the fact that the MSSIM for any image and itself is 1.

We consider uniform partitions into blocks of size $8 \times 8$, $16 \times 16$, $24 \times 24$, and $32 \times 32$. Fig.\textsuperscript{5} shows the corresponding SRs obtained with MP and SPMP methods, as well as with their HBW versions. As observed in the figures, for both images the MP and SPMP approaches applied to approximate each block in the partition at once, and totally independently of the others, are more sensitive to the block size than the corresponding HBW versions. This represents an advantage of the latter, because the complexity for calculating inner products increases with the size of the block. However, from a computational perspective, the advantage of approximating a signal partition by approximating every element of the partition completely independently of the others is the possibility of straightforward parallelization using multiprocessors. Then, in the next section we consider the HBW manner of just reducing the coefficients of a given approximation by means of self projections.

![SR values for the Knee X-Ray Image](image1)

![SR values for the Foot X-Ray Image](image2)

Figure 5: SR vs partition unit size $N_b \times N_b = 8 \times 8, 16 \times 16, 24 \times 24$, and $32 \times 32$ yielded by MP, SPMP, HBH-MP and HBW-SPMP for the X-Ray images in Fig.\textsuperscript{4} The graph on the left corresponds to the knee and the one on the right to the foot.

5 HBW Backwards SPMP (HBW-BSPMP)

Let’s consider now that the approximation of a signal partition is given as $f^K = \hat{J}_Q^{\hat{q}} f_{kq}^q$, with

$$f_{kq}^q = \sum_{n=1}^{k_q} c^q(n) d_{kq}^q,$$

(20)
In order to downgrade the approximation by disregarding some coefficients in a HBW fashion we proceed as follows.

1) For \( q = 1, \ldots, Q \) select the ‘potential’ coefficient \( c^q(j^q) \) to be eliminated from the atomic decomposition of every block \( q \), according to the criterion:

\[
j^q = \arg \min_{n=1,\ldots,k_q} |c^q(n)|^2.
\]

(21)

2) Select the block \( q^* \) such that

\[
q^* = \arg \min_{q=1,\ldots,Q} |c^q(j^q)|^2
\]

and downgrade the atomic decomposition of the block \( q^* \) by removing the atom corresponding to the index \( j^q \).

3) Project the removed term \( c^{q^*}(j^{q^*})d_{j^{q^*}} \), using as dictionary only the atoms \( \{d_n\}_{n=1}^{k_{q^*}} \), to obtain the set \( \{t(n)\}_{n=1}^{k_{q^*}} \) giving rise to the orthogonal projection of the removed term.

4) Update:

- the approximation

\[
\mathbf{r}^{k_{q^*}-1} = \mathbf{r}^{k_{q^*}} - c^{q^*}(j^{q^*})d_{j^{q^*}} + \sum_{n=1 \atop n \neq j^{q^*}}^{k_{q^*}} t(n)d_n
\]

- and the coefficients

\[
\{c^q(n)\}_{n=1}^{k_q} \quad \leftarrow \quad \{c^{q^*}(n) + t(n)\}_{n=1}^{k_{q^*}} \quad \leftarrow \quad \{c^q(n)\}_{n=1}^{k_q} \quad \leftarrow \quad \{c^{q^*}(n) + t(n)\}_{n=1}^{k_{q^*}}
\]

5) Appropriately shift the indices of the coefficients and atoms corresponding to the atomic decomposition of block \( q^* \), to allow for the removal of the coefficient \( c^{q^*}(j^{q^*}) \) and the atom \( d_{j^{q^*}} \). Set \( k_{q^*} \leftarrow k_{q^*} - 1 \) and check if the stopping criterion has been met. Otherwise:

- Select a new potential coefficient to be removed from the atomic decomposition of block \( q^* \) using the same criterion as in 1), i.e.,

\[
\mathbf{j}^{q^*} = \arg \min_{n=1,\ldots,k_{q^*}} |c^{q^*}(n)|^2.
\]
Repeat steps 2) - 5).

**Remark 3:** The selection criterion (21) is not optimal in the sense of minimizing the norm of the residual error. The criterion which does fulfills this condition is called Backward Optimized Orthogonal Matching (BOOMP) [11]. The corresponding HBW version is discussed in [26]. Nevertheless, if applied by self-projections the BOOMP approach would became computationally very demanding.

**Numerical Example**

The purpose of this example is to illustrate the improvement to the SR of a given approximation, when applying the HBW-BSPMP approach to slightly downgrade the signal’s atomic decomposition. For this we first approximate the music clips of Fig. 2 up to SNR=36dB with the SPMP method, approximating each block in the partition up to the same SNR. Afterwards we reduce the coefficients of the approximation by ranking the blocks for the downgrading through the HBW-BSPMP approach, so as to attain a global SNR of 35dB. As observed in Fig. 6 the SR results are similar to those obtained by applying the HBW-SPMP method up to a SNR=35dB. Fig. 7 shows the SR obtained by the same strategy as in Fig. 6 but in this case the

![Figure 6: SR vs partition unity size](image)

Figure 6: SR vs partition unity size $N_b = 1024, 2048, 4096$ and $8192$ samples for the music clips of Fig. 2. The dotted lines correspond to the SR obtained by the SPMP approach to approximate the signal partition, block by block independently of each other, up to PSNR=36dB, followed by a downgrading of the approximation up to PSNR=35dB by means of the the HBW-BSPMP strategy.

SPMP approach is applied to approximate each of the images in Fig. 4 up to PSNR=48dB. The HBW-BSPMP method is next applied to downgrade that approximation up to PSNR=47dB.

These numerical examples show that, even within a mild reduction of quality, downgrading
in a HBW fashion can significantly improve upon a forward independent approximation of the partition units (c.f. Figs. 3 and 5).

Figure 7: Same details as in Fig. 6 but with partition units $N_b \times N_b = 8 \times 8, 16 \times 16, 24 \times 24, 32 \times 32$, for the X-Ray images in Fig. 4 and the downgrading from PSNR=48dB to PSNR=47dB.

Note: The MATLAB functions implementing the methods, as well as the signals used in the numerical examples, have been made available on [39].

6 Conclusions

The exponential convergence of the SPMP algorithm, which implements the OMP approach by means of the MP one, was derived. The orthogonal projection step, intrinsic to the OMP method, is realized within the SPMP framework by approximating the residual error using the MP algorithm and a dictionary consisting only of the already selected atoms, up to the particular step. Thus, the memory requirements are kept within the same scale as for MP. The bound for the self projection convergence rate (c.f. (16)) clearly highlights the broad range of cases for which implementing OMP through SPMP is effective. Since the convergence rate depends on the minimum eigenvalue of the Gram matrix constructed with the selected atoms, the cases for which the convergence could become very slow fall within the rage of ill posed problems.

A particularly convenient feature of SPMP, when applied for approximating two dimensional images, is that it fully exploits the separability of a dictionaries. The HBW extension of a pursuit strategy for approximating a signal partition also benefits from the reduction in memory
requirements. The HBW version of SPMP was introduced and illustrated by approximating two clips of music and two X-Ray medical images. In these cases, the convenience of the HBW manner for approximating a partition is highlighted by the numerical results.

The advantage of the popular way of approximating a signal partition (i.e. realizing the approximation of each unit totally independent of each other) is recognized with regards to the possibility of straightforward parallelization using multi-processors. Hence, the HBW-BSPMP technique for downgrading a given atomic decomposition was introduced. The relevance of this approach is strengthen by the fact that it can refine, in an effective manner, any given approximation, regardless of the method by which that approximation was obtained.

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