Realizations of the Lie superalgebra $q(2)$ and applications

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Abstract

The Lie superalgebra $q(2)$ and its class of irreducible representations $V_p$ of dimension $2p$ ($p$ being a positive integer) are considered. The action of the $q(2)$ generators on a basis of $V_p$ is given explicitly, and from here two realizations of $q(2)$ are determined. The $q(2)$ generators are realized as differential operators in one variable $x$, and the basis vectors of $V_p$ as 2-arrays of polynomials in $x$. Following such realizations, it is observed that the Hamiltonian of certain physical models can be written in terms of the $q(2)$ generators. In particular, the models given here as an example are the sphaleron model, the Moszkowski model and the Jaynes-Cummings model. For each of these, it is shown how the $q(2)$ realization of the Hamiltonian is helpful in determining the spectrum.

1 Introduction

Since their introduction in supersymmetry [1, 2, 3], Lie superalgebras and their irreducible representations (simple modules) have been the subject of much attention in both the mathematical [4, 5, 6] and the physics literature, where both finite dimensional [7, 8, 9] and infinite dimensional representations [10, 11, 12, 13, 14] have been studied. When Kac obtained his classification [4] of simple Lie superalgebras, he subdivided them into the classical Lie superalgebras and the Lie superalgebras of Cartan type. The classical Lie superalgebras consist of the basic Lie superalgebras – $A(m, n)$, $B(m, n)$, $C(n)$, $D(m, n)$ and the exceptional ones $D(2, 1; \alpha)$, $G(3)$ and $F(4)$ – and the strange series $P(n)$ and $Q(n)$. The basic Lie superalgebras have made their appearance in various physical models. As far as we know, the strange Lie superalgebras have not been used in relation to any physical model or example. In this paper, we shall discuss the strange Lie superalgebra $Q(1)$ of rank 1; more precisely we shall be dealing with its central extension which is usually denoted by $q(2)$ [13]. It will be shown that $q(2)$ has a class of interesting representations $V_p$ labelled by a positive integer $p$. These representations allow for certain realizations of $q(2)$, and it will be shown that these realizations in turn are appropriate for the study of

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certain physical models: the so-called sphaleron model, the Moszkowski model, and the Jaynes-Cummings model.

The strange Lie superalgebras $q(n)$ can be considered as a super-analogue of $gl(n)$. Representations of $q(n)$ have been studied from the mathematical point of view. In [15, 16, 17], the finite dimensional irreducible graded representations of $q(n)$ have been determined together with their characters, both in the so-called typical and atypical case. These representations possess the strange property that the multiplicity of the highest weight is in general greater than 1 [16]. More recently, a new class of finite dimensional irreducible representations of $q(n)$ was determined [18]. These representations are not graded and thus they are not among the ones classified by Penkov and Serganova [16]. However, they possess many other interesting properties: the highest weight has multiplicity 1, they can be equipped with an inner product, and in an appropriate context they can be considered as Fock spaces.

In the present paper we shall concentrate on these representations for the Lie superalgebra $q(2)$. The representations $V_p$ are of dimension $2p$ ($p$ is a positive integer). When decomposed to the even subalgebra $gl(2)$ of $q(2)$, $V_p$ consists of the direct sum of two $gl(2)$ irreps: one of dimension $p + 1$ and one of dimension $p - 1$. Having two $gl(2)$ irreps of such dimension as part of an irreducible representation of another algebra (namely $q(2)$), will help in determining physical applications for the representations $V_p$.

The structure of the paper is as follows. In section 2 the algebra $q(2)$ and its class of representations $V_p$ are defined. In section 3 we shall discuss a relation between these representations and certain representations of $so(4)$. Two realizations of $q(2)$ and of the corresponding representations $V_p$ will be given in section 4. The appearance and usefulness of these realizations in physical models will then be illustrated in the following sections: the sphaleron model in section 5, the Moszkowski model in section 6 and the Jaynes-Cummings model in section 7.

## 2 The Lie superalgebra $q(2)$ and the representations $V_p$

For the definition of $q(n)$ and a corresponding class of representations, we refer to [18]. Here we shall deal only with the case $n = 2$. The Lie superalgebra $q(2)$ has a basis consisting of 4 even elements $e_0^{ij}$ ($i, j = 0, 1$) and 4 odd elements $e_1^{ij}$ ($i, j = 0, 1$), satisfying the bracket relation

$$[e_0^{ij}, e_0^{kl}] = \delta_{jk}e_0^{i+l} + (-1)^{\sigma\theta} \delta_{il}e_0^{i+j},$$

where $\sigma, \theta \in Z_2 = \{0, 1\}$, and $i, j, k, l \in \{0, 1\}$. Here, $[\cdot, \cdot]$ stands for the Lie superalgebra bracket, which could be a commutator or an anti-commutator, depending on the grading of the elements considered. We write explicitly $[\cdot, \cdot]$ (resp. $\{\cdot, \cdot\}$) if this stands for a commutator (resp. anti-commutator).

It is clear that the even part of $q(2)$ (i.e. the 4 elements with upper index equal to 0) is the Lie algebra $gl(2)$. For convenience, a different notation will be introduced for the root vectors, i.e. the elements $e_0^{ij}$ with $i \neq j$, since these elements can be interpreted as “creation and annihilation operators” for $q(2)$ [18]. So we put:

$$b^+ = e_0^{10}, \quad b^- = e_0^{01}.$$
\[ f^+ = e_{10}^1, \quad f^- = e_{01}^1. \]  

These operators satisfy certain triple relations (see [18, (8)–(11)]), and together with their supercommutators they form a basis of \( q(2) \).

The algebra \( q(2) \) has finite dimensional representations labelled by a positive integer \( p \). The representation space \( V_p \) arises as a quotient module \( V_p = \bar{V}_p/M_p \) of an infinite dimensional \( q(2) \) module \( \bar{V}_p \) by its maximal submodule \( M_p \) [18]. The space \( \bar{V}_p \) is spanned by the vectors

\[ v_k = (b^+)^k v_0, \quad k = 0, 1, \ldots; \]
\[ w_k = (b^+)^{k-1} f^+ v_0, \quad k = 1, 2, \ldots, \]

where \( v_0 \) is a vacuum (or highest weight vector) satisfying:

\[ e_{00} v_0 = p v_0, \quad e_{10} v_0 = \sqrt{p} v_0, \]
\[ e_{11} v_0 = 0, \quad e_{01} v_0 = 0, \]
\[ b^- v_0 = f^- v_0 = 0. \]

The following actions in \( \bar{V}_p \) of the creation and annihilation operators on \( v_k \) and \( w_k \) can be computed:

\[ b^+ v_k = v_{k+1}, \quad b^+ w_k = w_{k+1}, \]
\[ f^+ v_k = w_{k+1}, \quad f^+ w_k = 0, \]
\[ b^- v_k = k(p - k + 1) v_{k-1}, \]
\[ f^- v_k = k \sqrt{p} v_{k-1} - k(p - k - 1) w_{k-1}, \]
\[ b^- w_k = \sqrt{p} v_{k-1} + (k - 1)(p - k) w_{k-1}, \]
\[ f^- w_k = p v_{k-1} - (k - 1) \sqrt{p} w_{k-1}. \]

In \( \bar{V}_p \), \( v_p - \sqrt{p} w_p \) is a primitive vector (the action of \( b^- \) and \( f^- \) on it are zero) generating the submodule \( M_p \). The quotient module \( V_p = \bar{V}_p/M_p \) is therefore a finite dimensional module. A set of basis vectors of \( V_p \), together with the corresponding weight in the natural basis \( (\epsilon_0, \epsilon_1) \) of the \( gl(2) \) weight space, is given by

\[ \begin{align*}
v_0 & \quad p \epsilon_0 \\
v_1, w_1 & \quad (p - 1) \epsilon_0 + \epsilon_1 \\
v_2, w_2 & \quad (p - 2) \epsilon_0 + 2 \epsilon_1 \\
\vdots & \\
v_{p-1}, w_{p-1} & \quad \epsilon_0 + (p - 1) \epsilon_1 \\
v_p + \sqrt{p} w_p & \quad p \epsilon_1.
\end{align*} \]

The top and bottom weight appear with multiplicity 1, the other weights have multiplicity 2. Observe that we use the same notation for vectors in \( V_p \) and in \( \bar{V}_p \).

From the above weight structure one can determine the decomposition of this finite dimensional \( q(2) \) module with respect to the even subalgebra \( gl(2) \subset q(2) \):

\[ V_p \to (p, 0) \oplus (p - 1, 1), \quad (p > 1). \]
So $V_p$ splits into two irreducible $gl(2)$ modules, both of which have been labelled by their highest weight (in the $(e_0, e_1)$-basis). In other words, the two components of the $gl(2)$ representations have dimension $p + 1$ and $p - 1$; often this $gl(2)$ representation would be denoted by $D(\frac{p}{2}) \oplus D(\frac{p}{2} - 1)$.

The actions of the remaining $g(2)$ basis elements on the representation space $V_p$ can easily be determined:

\[
\begin{align*}
    e_{00}^0 v_k &= (p - k)v_k, & e_{00}^0 w_k &= (p - k)w_k, \\
    e_{11}^0 v_k &= k v_k, & e_{11}^0 w_k &= k w_k, \\
    e_{00}^1 v_k &= \sqrt{p} w_k - k w_k, & e_{00}^1 w_k &= v_k - \sqrt{p} w_k, \\
    e_{11}^1 v_k &= k w_k, & e_{11}^1 w_k &= w_k.
\end{align*}
\]  

(9)

On the representation space $V_p$, a positive-definite metric can be introduced by requiring

\[
\langle v_0 | v_0 \rangle = 1, \quad \langle b^+ v | v' \rangle = \langle v | b^- v' \rangle, \quad \langle f^+ v | v' \rangle = \langle v | f^- v' \rangle, \quad \forall v, v' \in V_p.
\]  

(10)

Then

\[
\begin{align*}
    \langle v_k | v_l \rangle &= \delta_{kl} \frac{k! p!}{(p - k)!}, & \langle w_k | w_l \rangle &= \delta_{kl} \frac{(k - 1)! p!}{(p - k)!}, & \langle v_k | w_l \rangle &= \delta_{kl} \frac{k! p!}{(p - k)! \sqrt{p}}.
\end{align*}
\]  

(11)

Because of the last relation, the basis (9) is not orthogonal with respect to this metric, so it will be convenient to introduce another (and more convenient) orthogonal basis of $V_p$ as follows:

\[
\begin{align*}
    \Lambda_k &= \frac{(p - k)!}{p!} v_k, & (k = 0, 1, \ldots, p - 1), \\
    \Lambda_p &= \frac{1}{2 p!} (v_p + \sqrt{p} w_p), \\
    \chi_l &= \frac{(p - l - 1)!}{p!} (v_l - \sqrt{p} w_l), & (l = 1, 2, \ldots, p - 1).
\end{align*}
\]  

(12) \hspace{1cm} (13) \hspace{1cm} (14)

The action of the creation and annihilation operators on this basis reads (in the following equations, $k = 0, 1, \ldots, p$ and $l = 1, 2, \ldots, p - 1$):

\[
\begin{align*}
    b^- \Lambda_k &= k \Lambda_{k-1}, & b^- \chi_l &= (l - 1) \chi_{l-1}, \\
    b^+ \Lambda_k &= (p - k) \Lambda_{k+1}, & b^+ \chi_l &= (p - l - 1) \chi_{l+1}, \\
    f^- \Lambda_k &= (k \Lambda_{k-1} + k(k - 1) \chi_{k-1}) / \sqrt{p}, & f^- \chi_l &= -(\Lambda_{l-1} + (l - 1) \chi_{l-1}) / \sqrt{p}, \\
    f^+ \Lambda_k &= ((p - k) \Lambda_{k+1} - (p - k)(p - k - 1) \chi_{k+1}) / \sqrt{p}, & f^+ \chi_l &= (\Lambda_{l+1} - (p - l - 1) \chi_{l+1}) / \sqrt{p}.
\end{align*}
\]  

(15)

Note that in all computations, one has to remember to work in the quotient module $V_p = \bar{V}_p / M_p$, where $M_p$ is generated by the primitive vector $v_p - \sqrt{p} w_p$ of $V_p$. This often requires a separate calculation for the cases $k = p$ or $k = p - 1$. For example,

\[
\frac{b^+}{p!} \Lambda_{p-1} = \frac{1}{p!} v_p = \frac{1}{p!} \left( v_p - \frac{1}{2} (v_p - \sqrt{p} w_p) \right) = \frac{1}{2 p!} (v_p + \sqrt{p} w_p) = \Lambda_p.
\]  

(16)
The actions of the remaining \(q(2)\) elements in this basis are given by

\[
\begin{align*}
\hat{e}_{00}^0 \Lambda_k &= (p-k)\Lambda_k, \\
\hat{e}_{00}^l \chi_l &= (p-l)\chi_l, \\
\hat{e}_{11}^0 \Lambda_k &= k\Lambda_k, \\
\hat{e}_{11}^l \chi_l &= l\chi_l, \\
\hat{e}_{00}^\dagger \Lambda_k &= ((p-k)\Lambda_k + k(p-k)\chi_k)/\sqrt{p}, \\
\hat{e}_{00}^\dagger \chi_l &= (\Lambda_l - (p-l)\chi_l)/\sqrt{p}, \\
\hat{e}_{11}^\dagger \Lambda_k &= (k\Lambda_k - k(p-k)\chi_k)/\sqrt{p}, \\
\hat{e}_{11}^\dagger \chi_l &= -(\Lambda_l + l\chi_l)/\sqrt{p},
\end{align*}
\]

where again \(k = 0, 1, \ldots, p\) and \(l = 1, 2, \ldots, p-1\). Observe that the subalgebra \(gl(2)\) with basis \(\{b^+, b^-, \hat{e}_{00}^0, \hat{e}_{11}^0\}\) acts irreducibly on the vectors \(\Lambda_k (k = 0, 1, \ldots, p)\) and \(\chi_l (l = 1, 2, \ldots, p-1)\); so from here the decomposition of \(V_p\) into two irreducible \(gl(2)\) irreps is obvious.

### 3 A relation with \(so(4)\) representations

Consider the Lie algebra \(so(4) \equiv sl(2) \oplus sl(2)\) with generators \(J_i\) and \(K_i\) \((i = 0, \pm)\) and commutation relations:

\[
\begin{align*}
[J_0, J_\pm] &= \pm J_\pm, & [J_+, J_-] &= 2J_0, \\
[K_0, K_\pm] &= \pm K_\pm, & [K_+, K_-] &= K_0, \\
[J_i, K_j] &= 0.
\end{align*}
\]  

(17)

Rather than dealing with the abstract generators of \(so(4)\), we shall consider these generators in a particular representation. The operators \(J_i\) \((i = 0, \pm)\) are realized in the representation \(D^{(\frac{p+1}{2})}\) of \(sl(2)\) \((p\) a positive integer), and the operators \(K_i\) \((i = 0, \pm)\) are realized in the representation \(D^{(\frac{q+1}{2})}\) of \(sl(2)\). We shall continue to denote the representatives of the abstract operators \([17]\) by the same names, \(J_i\) and \(K_i\). Thus the \(K_i\) satisfy

\[
(K_\pm)^2 = 0, \quad K_0^2 = \frac{1}{4}I, \quad \{K_+, K_-\} = I, \quad \{K_0, K_{\pm}\} = 0,
\]

(18)

where \(I\) is the identity operator.

The Lie algebra \(so(4) = sl(2) \oplus sl(2)\) has the subalgebra \(sl(2)\) with generators \(J_i + K_i\) \((i = 0, \pm)\). Since in the present realization the tensor product \(D^{(\frac{p+1}{2})} \otimes D^{(\frac{q+1}{2})}\) decomposes as \(D^{(\frac{q}{2})} \oplus D^{(\frac{q}{2}-1)}\), the representation of \(so(4)\) considered here decomposes as \(D^{(\frac{q}{2})} \oplus D^{(\frac{q}{2}-1)}\) with respect to this \(sl(2)\) subalgebra. This implies that the \(so(4)\) representation space is isomorphic to the space \(V_p\), with the same \(sl(2)\) action. Denoting the representatives of \(q(2)\) in \(V_p\) again by \(b^\pm, f^\pm, e_{\sigma ii}^\alpha\) \((\sigma = 0, 1, i = 0, 1)\), the following identification holds:

\[
\begin{align*}
b^- &= J_+ + K_+, & b^+ &= J_- + K_-, & e_{00}^0 - e_{11}^0 &= 2J_0 + 2K_0, \\
f^- &= \sqrt{p}K_+, & f^+ &= \sqrt{p}K_-, & e_{00}^1 - e_{11}^1 &= 2\sqrt{p}K_0, \\
e_{00}^0 + e_{11}^0 &= pI, & e_{00}^1 + e_{11}^1 &= \frac{2}{\sqrt{p}}(2J_0K_0 + J_+K_- + J_-K_+ + \frac{1}{2}).
\end{align*}
\]  

(19)
These relations can be verified by considering the representations of the \( so(4) \) generators in a standard basis of \( D(p-1,2) = D(p-1,2) \otimes D(1,2) \), and comparing with (15)-(16). Indeed, let the standard basis of \( D(p-1,2) \) be given by

\[
|\frac{p-1}{2}, m\rangle \otimes |\frac{1}{2}, \mu\rangle,
\]

where \( m = -\frac{p-1}{2}, -\frac{p-1}{2} + 1, \ldots, \frac{p-1}{2} \) and \( \mu = \pm \frac{1}{2} \), then the standard action of the \( so(4) \) basis elements reads

\[
\begin{align*}
J_0 |\frac{p-1}{2}, m\rangle \otimes |\frac{1}{2}, \mu\rangle &= m |\frac{p-1}{2}, m\rangle \otimes |\frac{1}{2}, \mu\rangle, \\
J_\pm |\frac{p-1}{2}, m\rangle \otimes |\frac{1}{2}, \mu\rangle &= \left(\frac{p-1}{2} \pm m\right) |\frac{p-1}{2}, m \pm 1\rangle \otimes |\frac{1}{2}, \mu\rangle, \\
K_0 |\frac{p-1}{2}, m\rangle \otimes |\frac{1}{2}, \mu\rangle &= \mu |\frac{p-1}{2}, m\rangle \otimes |\frac{1}{2}, \mu\rangle, \\
K_\pm |\frac{p-1}{2}, m\rangle \otimes |\frac{1}{2}, \mu\rangle &= \left(\frac{1}{2} \pm \mu\right) |\frac{1}{2}, \mu \pm 1\rangle \otimes |\frac{1}{2}, \mu\rangle.
\end{align*}
\]

(20)

Using the following relation between the \((\Lambda_k, \chi_l)\)-basis and the present one,

\[
\Lambda_k = \sqrt{\frac{(p-k)!k!}{p!}} \left( \sqrt{\frac{p-k}{p}} |\frac{p-1}{2}, \frac{p-1}{2} - k\rangle \otimes |\frac{1}{2}, \frac{1}{2}\rangle + \sqrt{\frac{k}{p}} |\frac{p-1}{2}, \frac{p+1}{2} - k\rangle \otimes |\frac{1}{2}, -\frac{1}{2}\rangle \right),
\]

(21)

\[
\chi_l = \sqrt{\frac{(p-l-1)!(l-1)!}{p!}} \left( \sqrt{\frac{l}{p}} |\frac{p-1}{2}, \frac{p-1}{2} - l\rangle \otimes |\frac{1}{2}, \frac{1}{2}\rangle - \sqrt{\frac{l}{p}} |\frac{p-1}{2}, \frac{p+1}{2} - l\rangle \otimes |\frac{1}{2}, -\frac{1}{2}\rangle \right),
\]

(22)

(23)

(24)

it is straightforward to verify that (19) holds, using the actions (15)-(16) and (20).

Observe that \( so(4) \) has 2 Casimir operators \( C_1 \) and \( C_2 \), which are independent in general :

\[
C_1 = J_0^2 + K_0^2 + \frac{1}{2} \{J_+, J_-\} + \frac{1}{2} \{K_+, K_-\},
\]

(25)

\[
C_2 = J_0^2 - K_0^2 + \frac{1}{2} \{J_+, J_-\} - \frac{1}{2} \{K_+, K_-\}.
\]

(26)

In the present representation, however, these operators are not independent. They can be rewritten in terms of the \( q(2) \) operators, in which case \( C_1 \) and \( C_2 \) coincide apart from a multiple of the operator \( e_{00}^0 + e_{11}^0 \) (with eigenvalue \( p \) in the representation). The Casimirs \( C_1 \) and \( C_2 \) have the value \( 2p^2 - 1 \) and \( 2p^2 - 4 \) respectively.
4 Two realizations of $q(2)$ and its representation $V_p$

In order to find applications of the algebra $q(2)$ and its representations $V_p$, it will be useful to construct certain differential realizations of $q(2)$. Here we shall give two different differential realizations. The main difference comes from the distinction between the spaces of polynomials that the $q(2)$ elements act upon.

A simple realization of $q(2)$ is found by realizing the basis elements $\Lambda_k, \chi_l$ as follows:

\[
\Lambda_k = \left( x^k \atop 0 \right), \quad k = 0, 1, \ldots, p, \\
\chi_l = \left( 0 \atop x^{l-1} \right), \quad l = 1, 2, \ldots, p-1.
\]  

(27)

Thus the basis elements are $(2 \times 1)$-arrays of polynomials in a variable $x$. The representation space can then be identified with

\[
\left( \begin{array}{c} \mathcal{P}(p) \\ \mathcal{P}(p-2) \end{array} \right),
\]

(28)

where $\mathcal{P}(m)$ stands for the space of polynomials in $x$ of degree at most $m$, thus $\mathcal{P}(m)$ has a basis $\{1, x, \ldots, x^m\}$. The Lie superalgebra $q(2)$ will have a realization preserving the space (28).

With this realization of the basis vectors $\Lambda_k$ and $\chi_l$, a differential realization for $q(2)$ is easily derived from (15)-(16). There comes:

\[
b^- = \frac{d}{dx}, \quad b^+ = -x^2 \frac{d}{dx} + (p - 1)x + x\sigma_3, \\
\tilde{e}^{00}_0 - \tilde{e}^{01}_1 = -2x \frac{d}{dx} + p - 1 + \sigma_3, \quad \tilde{e}^{00}_0 + \tilde{e}^{01}_1 = p, \\
f^- = \frac{1}{\sqrt{p}} \left( \frac{d}{dx} \sigma_3 - \sigma_+ + \frac{d^2}{dx^2} \sigma_- \right), \\
f^+ = \frac{1}{\sqrt{p}} \left( -x^2 \frac{d}{dx} + (p - 1)x \right) \sigma_3 + \frac{1}{\sqrt{p}} x + \frac{1}{\sqrt{p}} x^2 \sigma_+ \\
- \frac{1}{\sqrt{p}} \left( x^2 \frac{d^2}{dx^2} + 2(1-p)x \frac{d}{dx} + p(p-1) \right) \sigma_- , \\
\tilde{e}^{00}_0 - \tilde{e}^{11}_1 = \frac{1}{\sqrt{p}} \left( -2x \frac{d}{dx} + p - 1 \right) \sigma_3 + \frac{1}{\sqrt{p}} + \frac{2}{\sqrt{p}} x \sigma_+ + \frac{2}{\sqrt{p}} \left( -x \frac{d^2}{dx^2} + (p - 1) \frac{d}{dx} \right) \sigma_- , \\
\tilde{e}^{00}_0 + \tilde{e}^{11}_1 = \sqrt{p} \sigma_3.
\]

(29)

Herein, $\sigma_\pm$ and $\sigma_3$ are the common notations for the Pauli matrices. We shall refer to (29) as the first differential realization of $q(2)$.

A second useful realization of $q(2)$ will be found by considering a different basis for $V_p$. Let, for $k = 0, 1, \ldots, p - 1$,

\[
\mu_k = \Lambda_{p-k} - k\chi_{p-k}, \\
\mu_{p+k} = \Lambda_{p-k-1} + (p - k - 1)\chi_{p-k-1}.
\]

(30)  (31)
Then the action of the \( q(2) \) operators on this new basis reads:

\[
b^+ \mu_k = k \mu_{k-1}, \quad b^+ \mu_{p+k} = \mu_k + k \mu_{p+k-1},
\]

\[
f^+ \mu_k = 0, \quad f^+ \mu_{p+k} = \sqrt{p} \mu_k,
\]

\[
b^- \mu_k = (p - k - 1) \mu_{k+1} + \mu_{p+k}, \quad b^- \mu_{p+k} = (p - k - 1) \mu_{p+k+1},
\]

\[
f^- \mu_k = \sqrt{p} \mu_{p+k}, \quad f^- \mu_{p+k} = 0,
\]

\[
(e^0_0 + e^0_1) \mu_k = p \mu_k, \quad (e^0_0 + e^0_1) \mu_{p+k} = p \mu_{p+k},
\]

\[
(e^0_0 - e^0_1) \mu_k = (2k - p) \mu_k, \quad (e^0_0 - e^0_1) \mu_{p+k} = (2k + 2 - p) \mu_{p+k},
\]

\[
(e^1_0 + e^1_1) \mu_k = \frac{1}{\sqrt{p}} (p - 2k) \mu_k + \frac{1}{\sqrt{p}} (2k) \mu_{p+k-1},
\]

\[
(e^1_0 + e^1_1) \mu_{p+k} = \frac{1}{\sqrt{p}} (2k + 2 - p) \mu_{p+k} + \frac{2}{\sqrt{p}} (p - k - 1) \mu_{k+1},
\]

\[
(e^1_0 - e^1_1) \mu_k = -\sqrt{p} \mu_k, \quad (e^1_0 - e^1_1) \mu_{p+k} = \sqrt{p} \mu_{p+k}.
\]

Just as the basis \( \Lambda_k, \chi_k \) could be represented by \((2 \times 1)\)-arrays of polynomials in a variable, the same holds for the present basis. Let us consider

\[
\mu_k = \begin{pmatrix} x^k \\ 0 \end{pmatrix}, \quad \mu_{p+k} = \begin{pmatrix} 0 \\ x^k \end{pmatrix}, \quad k = 0, 1, \ldots, p - 1.
\]

When expressed in this basis, the Lie superalgebra will have a realization preserving the space

\[
\left( \mathcal{P}(p-1) \mathcal{P}(p-1) \right).
\]

Following from the action given in \((32)\), this realization reads:

\[
b^- = -x^2 \frac{d}{dx} + (p - 1)x + \sigma_-, \quad b^+ = \frac{d}{dx} + \sigma_+,
\]

\[
e^0_0 - e^0_1 = 2x \frac{d}{dx} + 1 - p - \sigma_3, \quad e^0_0 + e^0_1 = p,
\]

\[
f^- = \sqrt{p} \sigma_-, \quad f^+ = \sqrt{p} \sigma_+, \quad e^1_0 - e^1_1 = -\sqrt{p} \sigma_3,
\]

\[
e^1_0 + e^1_1 = \frac{1}{\sqrt{p}} \left( -2x \frac{d}{dx} \sigma_3 + 1 + (p - 1) \sigma_3 + 2 \frac{d}{dx} \sigma_- + 2(p - 1)x \sigma_+ - 2x^2 \frac{d}{dx} \sigma_+ \right).
\]

and will be referred to as the second differential realization of \( q(2) \).

## 5 The sphaleron model

In this section, we discuss a (physical) system of two coupled equations. In particular, this system will have algebraic solutions in the representation spaces \((28)\) and \((34)\). Such a system arises in the study of the stability of sphalerons \([13]\) (i.e. unstable classical solutions) in the Abelian gauge-Higgs model in \(1+1\) dimensions. The relevant equations read \([20]\):

\[
\frac{d^2}{dy^2} + \lambda - \theta^2 k^2 s n^2) f(y) - 2 \theta k \ cn \ dn \ g(y) = 0,
\]

\[
\frac{d^2}{dy^2} + \lambda + 1 + k^2 - (\theta^2 + 2) k^2 s n^2) g(y) - 2 \theta k \ cn \ dn \ f(y) = 0,
\]
and are considered on the Hilbert space of periodic functions over $[0, 4K(k)]$ ($K(k)$ is the complete elliptic integral of the second type). The three elliptic functions \[ sn = sn(y, k), \quad cn = cn(y, k) \text{ and } dn = dn(y, k) \] are periodic with respective periods $4K(k)$, $4K(k)$ and $2K(k)$. The spectral parameter $\lambda$ is the mode eigenvalue of the system while $\theta$ stands for the mass ratio $2M_H/M_W$, $M_H$ and $M_W$ being respectively the masses of the Higgs and gauge bosons.

Introducing the following new function

\[
W(y) \equiv \frac{df(y)}{dy} - \theta k \, sn \, g(y)
\]

as well as of the change of variables

\[
x = sn^2(y, k),
\]

the system (36)-(37) becomes

\[
\begin{align*}
(4x(1-x)(1-k^2x) \frac{d^2}{dx^2} + 2(1-2(1+k^2)x+3k^2x^2) \frac{d}{dx} + \lambda - k^2\theta^2x)W(x) &= 0, \\
(4x(1-x)(1-k^2x) \frac{d^2}{dx^2} + 2(-1+k^2x^2) \frac{d}{dx} + \lambda - k^2\theta^2x)f(x) &= -2\sqrt{\frac{(1-x)(1-k^2x)}{x}}W(x).
\end{align*}
\]

It has been proved \[20\] that this system has algebraic solutions in a $2p$-dimensional space if

\[
\theta^2 = 2p(2p + 1) \quad \text{or} \quad \theta^2 = 2p(2p - 1).
\]

This result suggests a connection between this sphaleron model and the $q(2)$-representations we are dealing with. More precisely, if $\theta^2 = 2p(2p + 1)$, we can put either

\[
W(x) = P_{p-1}(x) + xQ_{p-1}(x), \quad f(x) = \sqrt{x(1-x)(1-k^2x)}P_{p-1}(x),
\]

where $P_m(x)$ and $Q_m(x)$ stand for polynomials of degree $m$ in $x$, or else

\[
W(x) = \sqrt{(1-x)(1-k^2x)}P_{p-1}(x), \quad f(x) = \sqrt{x}(P_{p-1}(x) + xQ_{p-1}(x)).
\]

Under one of these two substitutions, the system of equations (40)-(41) has polynomial solutions for $P_{p-1}(x)$ and $Q_{p-1}(x)$. Indeed, in the case (43), the system of equations becomes

\[
\begin{align*}
(4x(1-x)(1-k^2x) \frac{d^2}{dx^2} + 2(1-4(1+k^2)x+7k^2x^2) \frac{d}{dx} + \lambda - k^2\theta^2x)P_{p-1}(x) &= -2Q_{p-1}(x), \\
(4x(1-x)(1-k^2x) \frac{d^2}{dx^2} + 2(5-6(1+k^2)x+7k^2x^2) \frac{d}{dx} + \lambda - 4(1+k^2) - k^2(4p^2 + 2p - 6)x)Q_{p-1}(x) &= \left((8k^2x - 4(1+k^2)) \frac{d}{dx} - 6k^2\right)P_{p-1}(x).
\end{align*}
\]
The differential operators of (43)-(46) map any element \( \begin{pmatrix} P_p^{-1}(x) \\ Q_{p-1}(x) \end{pmatrix} \) of the space (34) into an element of the same space. Thus (43)-(46) reduces to an algebraic eigenvalue system for \( \lambda \). The differential operator can be written as

\[
\Delta_{(43)} + \lambda = 4x \frac{d^2}{dx^2} - 4(1 + k^2)x^2 \frac{d^2}{dx^2} + 4k^2 x^3 \frac{d^2}{dx^2} + (6 - 10(1 + k^2)x + 14k^2 x^2) \frac{d}{dx} \\
+ (-4 + 2(1 + k^2)x) \frac{d}{dx} \sigma_3 + (-4p^2 - 2p + 6)k^2 x - 2(1 + k^2) + 2(1 + k^2) \sigma_3 \\
+ 2 \sigma_+ - 6k^2 \sigma_- + (4(1 + k^2) - 8k^2 x) \frac{d}{dx} \sigma_- + \lambda.
\]

(47)

Since this operator leaves the space of polynomials (34) invariant, we might expect that it can be expressed in terms of the \( q(2) \)-generators realized as in the so-called second realization (i.e. as in \( (33) \)). We actually have

\[
\Delta_{(43)} + \lambda = 2(e_0^0 - e_1^0)b^+ - \frac{2}{\sqrt{p}}(e_0^0 - e_1^0)f^+ - 2k^2(e_0^0 - e_1^0)b^- \\
+ \frac{2k^2}{\sqrt{p}}(e_0^0 - e_1^0)b^- - (1 + k^2)(e_0^0 - e_1^0)^2 + \frac{2}{\sqrt{p}}(e_0^0 - e_1^0) \\
- 6p f^+(e_0^0 - e_1^0) + (1 + k^2) \frac{1}{\sqrt{p}}(e_0^0 - e_1^0)(e_0^0 - e_1^0) - \frac{2k^2}{\sqrt{p}}(e_0^0 - e_1^0)f^- \\
+ \frac{2k^2}{p}(e_0^0 - e_1^0)f^- + 4(1 + k^2) \frac{1}{\sqrt{p}}b^+ f^- - 4(1 + k^2) \frac{1}{p}f^+ f^- - 2k^2(1 - p) \frac{1}{\sqrt{p}}f^- \\
+ 2(p + 2)b^+ - 2(p - 1) \frac{1}{\sqrt{p}}f^- + 6k^2 pb^- - (1 + k^2)(2p + 1)(e_0^0 - e_1^0) \\
+ (1 + k^2)\frac{1}{p}(e_0^0 - e_1^0) - p(p + 1)(1 + k^2) + \lambda.
\]

(48)

The same result holds for the case (44) where we obtain

\[
\Delta_{(44)} + \lambda = 2(e_0^0 - e_1^0)b^+ - \frac{2}{\sqrt{p}}(e_0^0 - e_1^0)f^+ - 2k^2(e_0^0 - e_1^0)b^- \\
+ \frac{2k^2}{\sqrt{p}}(e_0^0 - e_1^0)b^- - (1 + k^2)(e_0^0 - e_1^0)^2 + \frac{2}{\sqrt{p}}(e_0^0 - e_1^0) \\
- 6p f^+(e_0^0 - e_1^0) + (1 + k^2) \frac{1}{\sqrt{p}}(e_0^0 - e_1^0)(e_0^0 - e_1^0) - \frac{2k^2}{\sqrt{p}}(e_0^0 - e_1^0)f^- \\
+ \frac{2k^2}{p}(e_0^0 - e_1^0)f^- + 4(1 + k^2) \frac{1}{\sqrt{p}}b^+ f^- - 4(1 + k^2) \frac{1}{p}f^+ f^- + 2k^2\sqrt{p} f^- \\
+ 2(p + 2)b^+ - 2\sqrt{p} f^+ - 6k^2 pb^- - (1 + k^2)(2p + 1)(e_0^0 - e_1^0) \\
+ (1 + k^2) \frac{1}{p}(p + 1)(e_0^1 - e_1^1) - p(p + 1)(1 + k^2) + \lambda.
\]

(49)

In the case that \( b^2 = 2p(2p - 1) \), we can consider either

\[
W(x) = \sqrt{x}Q_{p-1}(x), \quad f(x) = \sqrt{(1 - x)(1 - k^2 x)}P_{p-1}(x),
\]

(50)

or else

\[
W(x) = \sqrt{x(1 - x)(1 - k^2 x)}Q_{p-2}(x), \quad f(x) = P_p(x).
\]

(51)
With the substitution (50), the space preserved by the differential operator is still (34). Acting on an array of polynomials \( \left( \frac{P_p(x)}{Q_{p-1}(x)} \right) \), the equation reduces to an algebraic eigenvalue equation; using the second realization (33) one is again able to express the differential operator subtended by this physical model in terms of the \( q(2) \)-generators. Explicitly this reads:

\[
\Delta_{(50)} + \lambda = 2(e^{0}_{00} - e^{0}_{11})b^+ - \frac{2}{\sqrt{P}}(e^{0}_{00} - e^{0}_{11})f^+ - 2k^2(e^{0}_{00} - e^{0}_{11})b^-
+ \frac{2k^2}{\sqrt{P}}(e^{0}_{00} - e^{0}_{11})f^+ + \frac{2k^2}{\sqrt{P}}(e^{0}_{00} - e^{0}_{11})b^- - \frac{2k^2}{P}(e^{0}_{00} - e^{0}_{11})f^-
- (1 + k^2)(e^{0}_{00} - e^{0}_{11})^2 + \frac{2}{\sqrt{P}}b^+(e^{0}_{00} - e^{0}_{11})
- \frac{6}{P}f^+(e^{0}_{00} - e^{0}_{11}) + (1 + k^2)\frac{1}{\sqrt{P}}(e^{0}_{00} - e^{0}_{11})(e^{0}_{00} - e^{0}_{11})
+ 2pb^+ + \frac{2}{\sqrt{P}}(3 - p)f^+ - 2k^2(3p - 2)b^- + \frac{2k^2(3p - 2)}{1}\frac{1}{\sqrt{P}}f^-
+ (1 + k^2)(-2p + 1)(e^{0}_{00} - e^{0}_{11}) - (1 + k^2)(1 - p)\frac{1}{\sqrt{P}}(e^{0}_{00} - e^{0}_{11})
- p(p - 1)(1 + k^2) + \lambda.
\]

(52)

The context for the substitution (51) is slightly different, so it deserves more attention. This time, the differential operator coming from the system (40)-(41) acts on an element \( \left( \frac{P_p(x)}{Q_{p-2}(x)} \right) \) from the space (28). Since also this space is a representation space for \( q(2) \), as we have proved in the previous section, we can again expect that the differential operator can be written in terms of the \( q(2) \)-generators. This is indeed the case when using the first differential realization of \( q(2) \) as given in (29). There comes

\[
\Delta_{(51)} + \lambda = 2k^2b^+(e^{0}_{00} - e^{0}_{11}) - k^2f^+(e^{1}_{00} + e^{1}_{11}) - \frac{1}{\sqrt{P}}b^-(e^{1}_{00} + e^{1}_{11})
- 2(e^{0}_{00} - e^{0}_{11})b^- + \frac{1}{\sqrt{P}}k^2b^+(e^{1}_{00} + e^{1}_{11}) + 4(1 + k^2)b^+b^-
+ f^-(e^{0}_{00} + e^{0}_{11}) + \frac{1}{2}(1 + k^2)(e^{0}_{00} - e^{0}_{11})(e^{0}_{00} + e^{0}_{11}) - \frac{1}{2\sqrt{P}}(1 + k^2)(e^{0}_{00} - e^{0}_{11})(e^{1}_{00} + e^{1}_{11})
+ 2p - 1)b^- + k^2\sqrt{P}f^+ + k^2(-6p + 1)b^+ - \sqrt{P}f^- + (1 + k^2)(2p + \frac{1}{2})(e^{0}_{00} - e^{0}_{11})
- \sqrt{P}(1 + k^2)(e^{1}_{00} + e^{1}_{11}) - \frac{1}{2}\sqrt{P}(1 + k^2)(e^{0}_{00} - e^{1}_{11}) + (-2p^2 + p)(1 + k^2) + \lambda.
\]

(53)

We have thus written each of the differential operators \( \Delta_{(40)}, \Delta_{(41)}, \Delta_{(50)} \) and \( \Delta_{(51)} \) associated with the sphaleron model in terms of the \( q(2) \) generators. The Lie superalgebra \( q(2) \) acts as a “spectrum generating superalgebra” for this physical model. More precisely both the sets of linear differential operators playing a role in the sphaleron model, those preserving the vector space of 2-arrays of polynomials of degrees \( p - 1 \) and \( p - 1 \) on the one hand and those preserving the vector space of 2-arrays of polynomials of degrees \( p \) and \( p - 2 \) on the other hand, correspond to realizations of \( q(2) \) and make the determination
of \( \lambda \) possible. Such a determination is relatively straightforward due to the fact that the (linear) Lie superalgebra \( q(2) \) has a particularly simple structure, much simpler than the algebras used in previous papers \([20, 22]\) devoted to the calculation of \( \lambda \). Indeed in these papers, the algebra \( so(4) \) (for \( \Delta(4)\), \( \Delta(4) \) and \( \Delta(5) \)) as well as an associative (non-linear) graded algebra denoted by \( A(2) \) (for \( \Delta(5) \)) have been used for such a task and this required heavy techniques in connection with the study \([22]\) of the irreps of this \( A(2) \). Such a simplification obtained by considering \( q(2) \) instead of \( A(2) \) leads to the hope of a more direct diagonalization of the operators connected with \( A(n) \) \([22]\) by using \( q(n) \).

6 The Moszkowski model

We now turn to the Moszkowski model \([23]\). This is a two-level model, each of the levels being \( N \)-fold degenerate with \( N_a \) particles of type \( a \) and \( N_b \) particles of type \( b \). The state of each particle is specified by the quantum numbers \( \sigma = \pm \frac{1}{2} \) (taking the value \( \frac{1}{2} \) in the upper level and \( -\frac{1}{2} \) in the lower level) and \( q \) which refers to the particular degenerate state within a given level. The corresponding Hamiltonian associated to the model reads \([23]\)

\[
H_M = c (J_0(a) - J_0(b)) + V \{ \hat{J}_+, \hat{J}_- \},
\]

where \( c \) is the energy difference between the two levels and \( V \) denotes the interaction strength. In \((54)\), the operators \( J_0(a), J_\pm(a) \) are defined according to

\[
J_0(a) = \frac{1}{2} \sum_q (a_{q,\frac{1}{2}}^a a_{q,-\frac{1}{2}}^a - a_{q,-\frac{1}{2}}^a a_{q,\frac{1}{2}}^a),
\]

\[
J_+(a) = \sum_q a_{q,\frac{1}{2}}^a a_{q,-\frac{1}{2}}^a,
\]

\[
J_-(a) = \sum_q a_{q,-\frac{1}{2}}^a a_{q,\frac{1}{2}}^a,
\]

where \( a_{q,\pm \frac{1}{2}}^a \) (\( a_{q,\pm \frac{1}{2}}^a \)) denotes the creation (annihilation) operator of a particle of type \( a \) in the state \( q \) with \( \sigma = \pm \frac{1}{2} \). Similar definitions hold for \( J_0(b), J_\pm(b) \) and we also have

\[
\hat{J}_i = J_i(a) + J_i(b), \quad i = 0, \pm.
\]

The operators \( J_0(i), J_\pm(i) \) (\( i = a, b \)) satisfy the \( so(4) \) \( \equiv sl(2) \oplus sl(2) \) commutation relations

\[
[J_0(i), J_\pm(j)] = \pm \delta_{ij} J_\pm(i),
\]

\[
[J_+(i), J_-(j)] = 2 \delta_{ij} J_0(i), \quad (i, j = a, b).
\]

Because of this \( sl(2) \oplus sl(2) \) symmetry of the Moszkowski Hamiltonian, we can also expect the \( q(2) \) Lie superalgebra to play a role within this model. Associating the operators \( J_i \) and \( K_i \) (\( i = 0, \pm \)) of \([17]\) with the current operators \( J_i(b) \) and \( J_i(a) \) respectively, we can rewrite \( H_M \) as

\[
H_M = c (K_0 - J_0) + V \{ K_+, K_- \} + \{ J_+, J_- \} + 2J_+ K_- + 2J_- K_+.
\]

12
According to (19), this can be rewritten as

\[ H_M = c \left( \frac{1}{\sqrt{p}} (e_{10}^1 - e_{11}^1) - \frac{1}{2} (e_{00}^0 - e_{11}^0) \right) \\
+ V \left( \frac{1}{2} p^2 - \frac{1}{2} (e_{00}^0 - e_{11}^0)^2 + \sqrt{p} (e_{00}^1 + e_{11}^1) \right). \]  

(62)

Although in principle the Hamiltonian \( H_M \) can be diagonalized using the expression (61) in terms of \( so(4) \)-generators, it turns out to be much simpler using the expression (62) in terms of \( q(2) \)-generators together with the second differential realization (35) of \( q(2) \). Then the Hamiltonian becomes

\[ H_M = c \left( -x \frac{d}{dx} + \frac{1}{2} (p - 1) - \frac{1}{2} \sigma_3 \right) + V \left( -2x^2 \frac{d^2}{dx^2} + (2p - 4) x \frac{d}{dx} \right) \\
+ p + 2 \frac{d}{dx} \sigma_- + 2(p - 1)x \sigma_+ - 2x^2 \frac{d}{dx} \sigma_+ \].  

(63)

Considering the action of this on the representation space (34), or equivalently, the action (32) of (62) on the basis vectors (30)-(31), leads to an eigenvalue system that is almost trivial to solve, i.e. :

\[ E_0^+ = pV - (1 - \frac{p}{2})c, \]
\[ E_k^+ = -2V k(p - k) + c(\frac{p}{2} - k) \pm \sqrt{V^2 p^2 + c^2 - 2(p - 2k)V c}, \quad (k = 1, 2, \ldots, p - 1), \]
\[ E_p^+ = pV + (1 - \frac{p}{2})c. \]

Thus we have recovered the well-known diagonalization of the Moszkowski Hamiltonian but by using one of the differential realizations of the Lie superalgebra \( q(2) \). The latter can then be considered as a “spectrum generating superalgebra” of the Moszkowski model.

### 7 The Jaynes-Cummings model

The well-known Jaynes-Cummings Hamiltonian [24] is one of the diagonalizable Hamiltonians of quantum optics. It describes a two-level atom interacting with a single-mode radiation. Under the so-called rotating wave approximation for which only real transitions (e.g. a photon is absorbed while the electron jumps from level 1 to level 2) are taken into account, the Jaynes-Cummings Hamiltonian is

\[ H_{JC} = \omega (a^+ a^- + \frac{1}{2}) - \frac{1}{2} \omega_0 \sigma_3 + g (a^- \sigma_- + a^+ \sigma_+). \]  

(64)

Here \( \omega \) is the field mode frequency, \( \omega_0 \) the atomic frequency while \( g \) is a real coupling constant and, as usual, \( a^- \) and \( a^+ \) denote the photon annihilation and creation operators, respectively.

In order to determine the spectrum of \( H_{JC} \), one can use the irreducible representations of the Lie superalgebra \( u(1, 1) \) as shown in [25]. We will prove in this section that the Lie superalgebra \( q(2) \) can play a similar role and thus be considered as a “spectrum generating superalgebra” for the Jaynes-Cummings Hamiltonian. For this purpose, we
shall use the basis vectors (12)-(14) consisting of the states $\Lambda_k$ ($k = 0, 1, \ldots, p$) and $\chi_l$ ($l = 1, 2, \ldots, p - 1$). This time, however, we shall consider the following realization of these basis vectors:

$$
\Lambda_k = \begin{pmatrix} p_{x^p-k} \\ (p-k)x^{p-k-1} \end{pmatrix}, \quad (k = 0, 1, \ldots, p),
$$

$\chi_l = \begin{pmatrix} 0 \\ x^{p-l-1} \end{pmatrix}, \quad (l = 1, 2, \ldots, p - 1),
$$

(65)
as opposed to [27]. This new realization of the basis states leads to a third differential realization of the $q(2)$-generators given by

$$
b^- = -x^2 \frac{d}{dx} + (p-1)x + x\sigma_3 + \sigma_-, \quad b^+ = \frac{d}{dx},
$$

$$
e_{00}^0 - e_{11}^0 = 2x \frac{d}{dx} + 1 - p - \sigma_3, \quad e_{00}^0 + e_{11}^0 = p,
$$

$$
f^- = \sqrt{p}(x\sigma_3 + \sigma_- - x^2\sigma_+), \quad f^+ = \sqrt{p}\sigma_+,
$$

$$
e_{10}^1 - e_{11}^1 = \sqrt{p}(-\sigma_3 + 2x\sigma_+), \quad e_{00}^1 + e_{11}^1 = \frac{2}{\sqrt{p}} \frac{p}{2} \sigma_3 + \frac{d}{dx} \sigma_-.
$$

(66)

It has to be noticed that the realization of the $sl(2)$ subalgebra generated by $b^-, b^+$ and $e_{00}^0 - e_{11}^0$ as defined in (65) coincides with the one performed in [26], but with other arguments. Taking in the Hamiltonian (64) the realization

$$
a^+ = x, \quad a^- = \frac{d}{dx},
$$

(67)

we can express $H_{JC}$ as

$$
H_{JC} = \frac{\omega}{2}(e_{00}^0 - e_{11}^0) + \frac{1}{2} p\omega + \frac{g}{2} \sqrt{p}(e_{00}^1 + e_{11}^1)
$$

$$
+ \frac{1}{2} \frac{g}{\sqrt{p}}(e_{00}^1 - e_{11}^1) + \frac{1}{2}(\omega_0 - \omega + g(p-1))\sigma_3.
$$

(68)

From this equation it is clear that the $q(2)$ superalgebra is a “spectrum generating superalgebra” of the Jaynes-Cummings Hamiltonian provided the detuning $\Delta(\equiv \omega - \omega_0)$ satisfies

$$
\Delta = g(p-1).
$$

(69)

Suppose this is the case. Then the action of (68) on the basis elements $\Lambda_k$ and $\chi_l$ follows from (15) and (16). In fact, $\Lambda_0$ and $\Lambda_p$ are directly eigenvectors of $H_{JC}$ (with the eigenvalues $E_0^+$ and $E_p^+$ respectively), whereas the other eigenvectors are simple linear combinations of $\Lambda_k$ and $\chi_k$ ($k = 1, 2, \ldots, p - 1$). Thus it is straightforward to recover the Jaynes-Cummings spectrum i.e.

$$
E_0^+ = \omega p + \frac{1}{2}(p+1)g,
$$

$$
E_k^+ = \omega(p-k) \pm g\sqrt{\frac{1}{4}p^2 + \frac{1}{2}p + \frac{1}{4} - k}, \quad (k = 1, 2, \ldots, p - 1),
$$

$$
E_p^+ = \frac{1}{2}(p-1)g,
$$

where the positive integer $p$ is arbitrary.
Acknowledgements

The authors would like to thank Y. Brihaye for some useful comments and for pointing out Reference [22].

References

[1] Gol'fand Yu A and Lihtman E P 1971 JETP Lett. 13 323
[2] Wess J and Zumino B 1974 Nucl. Phys. B78 1
[3] Salam A and Strathdee J 1974 Nucl. Phys. B80 499
[4] Kac V G 1977 Adv. Math. 26 8
[5] Kac V G 1978 Lecture Notes in Math. 676 597
[6] Scheunert M 1979 The theory of Lie superalgebras (Springer, Berlin)
[7] Corwin L, Ne’eman Y and Sternberg S 1975 Rev. Mod. Phys. 47 573
[8] Balantekin A B 1984 J. Math. Phys. 25 2028
[9] Hurni J P 1987 J. Phys. A 20 5755
[10] Heidenreich W 1982 Phys. Lett. B110 461
[11] Freedman D Z and Nicolai H 1984 Nucl. Phys. B237 342
[12] Flato M and Fronsdal C 1984 Lett. Math. Phys. 8 159
[13] Dobrev V K and Petkova V B 1985 Phys. Lett. B162 127
[14] Van der Jeugt J 1987 J. Math. Phys. 28 758
[15] Penkov I 1986 Funct. Anal. Appl. 20 30
[16] Penkov I and Serganova V 1997 Lett. Math. Phys. 40 147
[17] Penkov I and Serganova V 1997 J. Math. Sci. 84 1382
[18] Palev T D and Van der Jeugt J 2000 J. Phys. A 33 2527
[19] Klinkhamer F R and Manton N S 1984 Phys. Rev. D 30 2212
[20] Brihaye Y, Kosinski P, Giller S and Kunz J 1992 Phys. Lett. B 293 383
   Brihaye Y and Kosinski P 1994 J. Math. Phys. 35 3089
[21] Arscott F M 1964 Periodic Differential Equations (Pergamon, Oxford)
[22] Brihaye Y, Giller S, Kosinski P and Nuyts J 1997 Comm. Math. Phys. 187 201
[23] Moszkowski S A 1958 Phys. Rev. 110 403
   Bonatsos D, Brito L and Menezes D 1993 J. Phys. A 26 895
[24] Jaynes E T and Cummings F W 1963 Proc. IEEE 51 89

[25] Buzano C, Rasetti M G and Rastello M L 1989 Phys. Rev. Lett. 62 137

[26] Zhdanov R Z 1997 Phys. Lett. B 405 253