Minimal Surface Linear Combination Theorem

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Abstract

Given two univalent harmonic mappings $f_1$ and $f_2$ on $D$, which lift to minimal surfaces via the Weierstrass-Enneper representation theorem, we give necessary and sufficient conditions for $f_3 = (1 - s)f_1 + sf_2$ to lift to a minimal surface for $s \in [0, 1]$. We then construct such mappings from Enneper’s surface to Scherk’s singularly periodic surface, Scherk’s doubly periodic surface to the catenoid, and the 4-Enneper surface to the 4-noid.

1 Background

Complex-valued harmonic mappings can be regarded as generalizations of analytic functions. In particular, a harmonic mapping is a complex-valued function $f = u + iv$, where the $C^2$ functions $u$ and $v$ satisfy Laplace’s equation. The Jacobian of such a function is given by $J_f = u_x v_y - u_y v_x$. On a simply connected domain $D \subset \mathbb{C}$, a harmonic mapping $f$ has a canonical decomposition $f = h + g$, where $h$ and $g$ are analytic in $D$, unique up to a constant [2]. We will only consider harmonic mappings that are univalent with positive Jacobian on $D = \{z : |z| < 1\}$. The dilatation $\omega$ of a harmonic map $f$ is defined by $\omega \equiv g'/h'$. A result by Lewy [10] states that $|h'(z)| > |g'(z)|$ if and only if $f = h + \bar{g}$ is sense-preserving and locally univalent. The reader is referred to [6] for many interesting results on harmonic mappings.

One area of study is the construction of families of harmonic mappings [7] and their corresponding minimal surfaces [11, 14, 15]. We now present some necessary background concerning minimal surfaces. Let $M$ be an orientable surface that arises from a differentiable mapping $\mathbf{x}$ from a domain $V \subset \mathbb{R}^2$ (or $\mathbb{C}$) into $\mathbb{R}^3$, so that $\mathbf{x}(u, v) = (x^1(u, v), x^2(u, v), x^3(u, v))$. The parametrization $\mathbf{x}$ is isothermal (or conformal) if and only if $\mathbf{x}_u \cdot \mathbf{x}_v = \mathbf{x}_v \cdot \mathbf{x}_u = \lambda > 0$. Note that there exists an isothermal parametrization on any regular minimal surface (see [8]). Fix a point $p$ on $M$. Let $t$ denote a
vector tangent to \( M \) at \( p \) and \( n \) the unit normal vector to \( M \) at \( p \). Then \( t \) and \( n \) determine a plane that intersects \( M \) in a curve \( \gamma \). The normal curvature \( \kappa_t \) at \( p \) is defined to have the same magnitude as the curvature of \( \gamma \) at \( p \) with the sign of \( \kappa_t \) chosen to be consistent with the choice of orientation of \( M \). The principal curvatures, \( \kappa_1 \) and \( \kappa_2 \), of \( M \) at \( p \) are the maximum and minimum of the normal curvatures \( \kappa_t \) as \( t \) ranges over all directions in the tangent space. The mean curvature of \( M \) at \( p \) is the average value \( H = \frac{1}{2}(\kappa_1 + \kappa_2) \).

**Definition 1.** A minimal surface in \( \mathbb{R}^3 \) is a regular surface for which the mean curvature is zero at every point.

The following standard theorem provides the link between harmonic univalent mappings and minimal surfaces:

**Theorem 2.** (Weierstrass-Enneper Representation). Every regular minimal surface has locally an isothermal parametric representation of the form

\[
X = \left( \text{Re} \left\{ \int_0^z p(1 + q^2)dw \right\}, \right.
\left. \text{Re} \left\{ \int_0^z -ip(1 - q^2)dw \right\}, \right.
\left. \text{Re} \left\{ \int_0^z -2ipqd\zeta \right\} \right)
\]

(1)

in some domain \( D \subset \mathbb{C} \), where \( p \) is analytic and \( q \) is meromorphic in \( D \), with \( p \) vanishing only at the poles (if any) of \( q \) and having a zero of precise order \( 2m \) wherever \( q \) has a pole of order \( m \). Conversely, each such pair of functions \( p \) and \( q \) analytic and meromorphic, respectively, in a simply connected domain \( D \) generate through the formulas (1) an isothermal parametric representation of a regular minimal surface.

We will use (1) in the following form:

**Corollary 1.** For a harmonic function \( f = h + \overline{g} \), define the analytic functions \( h \) and \( g \) by \( h = \int z p\,d\zeta \) and \( g = \int z pq^2d\zeta \). Then the minimal surface representation (1) becomes

\[
\left( \text{Re}\{h + g\}, \text{Im}\{h - g\}, 2\text{Im}\left\{ \int_0^z \sqrt{h'g'}d\zeta \right\} \right)
\]

(2)
2 Harmonic Linear Combinations

The main consideration of this work is the study of harmonic mappings of the form \( f_3 = tf_1 + (1 - t)f_2 \), where \( t \in [0, 1] \) and \( f_1, f_2 \) are both harmonic mappings. We will provide conditions for \( f_3 \) to lift to a minimal surface via (2), and demonstrate several examples which further the work of [4] and relate seemingly disconnected minimal surfaces. Let \( f_1 = h_1 + \bar{g}_1 \) and \( f_2 = h_2 + \bar{g}_2 \) be two univalent harmonic mappings on \( \mathbb{D} \), which lift to minimal surfaces, with dilatations \( q_1^2 = g_1'/h_1' \) and \( q_2^2 = g_2'/h_2' \) respectively, where \( q_1, q_2 \) are analytic. Construct a third harmonic mapping

\[
    f_3 = tf_1(z) + (1 - t)f_2 = [th_1(z) + (1 - t)h_2(z)] + [t\overline{g_1(z)} + (1 - t)\overline{g_2(z)}]
    = h_3 + \bar{g}_3
\]

and define its dilatation to be \( \omega_3 = g_3'/h_3' \).

**Lemma 1.** If \( \omega_1 = \omega_2 \), then \( \omega_3 \) is a perfect square of an analytic function and hence \( f_3 \) is locally univalent.

**Proof.** Suppose that \( \omega_1 = \omega_2 \). Then we have

\[
    \omega_3 = \frac{th'_1\omega_1 + (1 - t)h'_2\omega_1}{th'_1 + (1 - t)h'_2} = \omega_1,
\]

which shows \( \omega_3 \) is a perfect square of an analytic function. Since \( f_1 \) is univalent, \( |\omega_3| = |q_1^2| > 0 \) and so \( f_3 \) is locally univalent.

We now seek to study conditions under which \( f_3 \) is globally univalent and thus lifts to a minimal surface. To do his, we need a few definitions and theorems.

**Definiton 3.** A domain \( D \subset \mathbb{C} \) is said to be convex in the \( e^{i\beta} \) direction if for all \( a \in \mathbb{C} \) the set

\[
    D \cap \{a + te^{i\beta} : t \in \mathbb{R}\}
\]

is either connected or empty. Specifically, a domain is convex in the direction of the imaginary axis if all lines parallel to the imaginary axis have a connected intersection with the domain.
Theorem 4 ([9], [11]). Given a harmonic function $f = h + g$, let $\phi = h - g$. $\phi$ is convex in the $e^{i\beta}$ direction if

$$\text{Re}\{\phi'(1 + ze^{i(\alpha + \beta)})(1 + ze^{-i(\alpha - \beta)})\} > 0$$

for some $\alpha \in \mathbb{R}$ and for all $z \in \mathbb{D}$.

The following theorem will allow us to prove global univalence of a class of harmonic mappings.

Theorem 5 (Clunie and Sheil-Small, [2]). A harmonic function $f = h + g$ locally univalent in $U$ is a univalent mapping of $U$ onto a domain convex in the $e^{i\beta}$ direction if and only if $\phi = h - e^{i2\beta}g$ is a conformal univalent mapping of $U$ onto a domain convex in the $e^{i\beta}$ direction.

The following theorem allows us to determine if a function maps onto a domain convex in the direction of the imaginary axis:

Theorem 6 (Hengartner and Schober, [8]). Suppose $f$ is analytic and non-constant in $\mathbb{D}$. Then

$$\text{Re}\{(1 - z^2)f'(z)\} \geq 0, z \in \mathbb{D}$$

if and only if

1. $f$ is univalent in $\mathbb{D}$,
2. $f$ is convex in the imaginary direction, and
3. there exists points $z'_n, z''_n$ converging to $z = 1, z = -1$, respectively, such that

$$\lim_{n \to \infty} \text{Re}\{f(z'_n)\} = \sup_{|z|<1} \text{Re}\{f(z)\}$$

$$\lim_{n \to \infty} \text{Re}\{f(z''_n)\} = \sup_{|z|<1} \text{Re}\{f(z)\}.$$  \hspace{1cm} (3)

Note that the the normalization in (3) can be thought of in some sense as if $f(1)$ and $f(-1)$ are the right and left extremes in the image domain in the extended complex plane.

Using the above results, we derive the following two theorems.

Theorem 7. Let $f_1 = h_1 + g_1$, $f_2 = h_2 + g_2$ be harmonic mappings convex in the imaginary direction. Suppose $\omega_1 = \omega_2$ and $\phi_i = h_i - g_i$ is univalent, convex in the imaginary direction, and satisfies the normalization given in (3) for $i = 1, 2$. Then $f_3 = tf_1 + (1 - t)f_2$ is convex in the imaginary direction ($0 \leq t \leq 1$).
Proof. We want to show that \( \phi_3 = t\phi_1 + (1-t)\phi_2 \) is convex in the imaginary direction. Then by Theorem 5, \( f_3 \) is convex in the imaginary direction. By the hypotheses, Theorem 6 applies to \( \phi_1, \phi_2 \). That is,

\[
\text{Re}\{(1 - z^2)\phi_i'(z)\} \geq 0, \forall i = 1, 2.
\]

Consider

\[
\text{Re}\{(1 - z^2)\phi_3'(z)\} = \text{Re}\{(1 - z^2)(t\phi_1'(z) + (1-t)\phi_2'(z))\}
\]

\[
= t\text{Re}\{(1 - z^2)\phi_1'(z)\} + (1-t)\text{Re}\{(1 - z^2)\phi_2'(z)\} \geq 0.
\]

Hence, by applying Theorem 6 again, \( \phi_3 \) is convex in the imaginary direction.

We need not only restrict to surfaces convex in the imaginary direction. The following gives a condition for a function to be convex in an arbitrary direction:

**Theorem 8.** For a harmonic function \( f = h+ig \), define \( h-g = \phi = \phi_R+i\phi_I \). Then \( \phi \) is convex in the \( e^{i\beta} \) direction if

\[
[\cos \alpha + \cos(\beta + \gamma)] [\phi_R \cos(\beta + \gamma) - \phi_I \sin(\beta + \gamma)] > 0
\]

for some \( \alpha \in \mathbb{R} \) and for all \( z = re^{i\gamma} \in \mathbb{D} \).

Proof. This theorem follows by applying Theorem 4 to \( \phi \) to get

\[
\text{Re}\{(\phi_R' + i\phi_I')(1 + re^{i(\alpha+\beta+\gamma)})(1 + re^{i(\gamma-\alpha+\beta)})\}
\]

\[
= \phi_R' + 2r \cos \alpha (\phi_R' \cos \theta - \phi_I' \sin \theta) + r^2 (\phi_R' \cos 2\theta - \phi_I' \sin 2\theta)
\]

\[
= 2(\cos \alpha + \cos(\beta + \gamma))(\phi_R' \cos(\beta + \gamma) - \phi_I' \sin(\beta + \gamma)) > 0,
\]

where \( \theta = \beta + \gamma \).

3 Examples

We now proceed to give two interesting examples resulting from Theorems 7 and 8.

**Example 1 (Ennepers to Scherks singly-periodic).**
Consider the harmonic maps

\[ f_E = z + \frac{1}{3}z^3 \]

\[ f_S = \left[ \frac{1}{4} \ln \left( \frac{1 + z}{1 - z} \right) + \frac{i}{4} \ln \left( \frac{i - z}{i + z} \right) \right] + \left[ \frac{1}{4} \ln \left( \frac{1 + z}{1 - z} \right) - \frac{i}{4} \ln \left( \frac{i - z}{i + z} \right) \right] \]

It is straightforward to show that their dilatations are \( \omega = z^2 \) and both harmonic maps satisfy the hypotheses of Theorem \( \text{7} \). Hence

\[ f_t = (1 - t)f_E + tf_S \]

is globally univalent on \( z \in \mathbb{D} \) and \( \forall t \in [0, 1] \). By Corollary \( \text{2} \), \( f_t \) lifts to a family of minimal surfaces. Note that \( f_0 \) lifts to Ennepers surface parametrized by:

\[ X_0 = \left( \text{Re}\left\{ z + \frac{1}{3}z^3 \right\}, \text{Im}\left\{ z - \frac{1}{3}z^3 \right\}, \text{Im}\left\{ z^2 \right\} \right) \]

and \( f_1 \) lifts to Scherks singly-periodic surface parametrized by

\[ X_1 = \left( \text{Re}\left\{ \frac{1}{2} \ln \left( \frac{1 + z}{1 - z} \right) \right\}, \text{Im}\left\{ \frac{i}{2} \ln \left( \frac{i - z}{i + z} \right) \right\}, \text{Im}\left\{ \frac{1}{2} \ln \left( \frac{1 + z^2}{1 - z^2} \right) \right\} \right). \]

So for \( t \in [0, 1] \) we get a continuous family of minimal surfaces transforming from Ennepers to Scherks singly-periodic. In Figure \( \text{2} \) we have shown six equal increments in this transformation.

**Example 2 (Scherks doubly-periodic to catenoid).**

Consider the harmonic maps \( f_D = h_D + g_D \), where

\[ h_D(z) = \frac{1}{4} \ln \left( \frac{1 + z}{1 - z} \right) - \frac{i}{4} \ln \left( \frac{1 + iz}{1 - iz} \right) \]

\[ g_D(z) = -\frac{1}{4} \ln \left( \frac{1 + z}{1 - z} \right) - \frac{i}{4} \ln \left( \frac{1 + iz}{1 - iz} \right) \]
and $f_C = h_C + \overline{g_C}$, where

\[ h_C(z) = \frac{1}{4} \ln \left( \frac{1 + z}{1 - z} \right) + \frac{z}{2(1 - z^2)} \]
\[ g_C(z) = \frac{1}{4} \ln \left( \frac{1 + z}{1 - z} \right) - \frac{z}{2(1 - z^2)}. \]

Notice that both $f_D$ and $f_C$ are convex in the direction of the imaginary axis, satisfy the hypotheses of Theorem 7, and for each $\omega = -z^2$. Hence

\[ f_t = (1 - t)f_D + tf_C \]

is globally univalent on $z \in \mathbb{D}$ and $\forall t \in [0, 1]$.

Note that $f_1$ lifts to a family of minimal surfaces, where $f_0$ lifts to Scherks doubly-periodic surface and $f_1$ lifts to a catenoid. So for $t \in [0, 1]$ we get a continuous family of minimal surfaces transforming from Scherks doubly-periodic surface to a catenoid. In Figure ??, we have shown six equal increments in this transformation.

### 4 Linear combinations that are not convex in one direction

**Example 3 (The 4-noid to 4-Enneper).** The harmonic function that lifts to the 4-ennepers surfaces is given by

\[ h_{4E} + g_{4E} = z - \frac{z^7}{7} \quad h_{4E} - g_{4E} = z + \frac{z^7}{7} \]

and that of the 4-noid is given by

\[ h_{4N} + g_{4N} = \frac{1}{8} \left( \frac{2z}{1 + z^2} - 3 \log \left( \frac{z + 1}{z - 1} \right) \right) \quad h_{4N} - g_{4N} = \frac{1}{4} \left( \frac{z}{1 - z^2} + \frac{3i}{2} \log \left( \frac{1 - iz}{1 + iz} \right) \right) \]

We restrict the domain of these surfaces to $B(0, .95)$ to avoid self intersections. Neither of these surfaces is convex, thus we need to pursue other means then the above for showing that the combination $f = sf_{4E} + (1-s)f_{4N}$ is minimal for all $s \in (0, 1)$. The following lemma will prove univalence:

**Lemma 2.** Let $s$ be fixed such that $0 \leq s < 1$. For any $n \geq 2$, $f$ is univalent in $\mathbb{D}$. 7
Proof. Fix $r_0$ such that $0 < r_0 < 1$ and consider $\Omega \subset \mathbb{D}$ the region bounded by $\sigma_1 \cup \{0\}, \sigma_2, \sigma_3$, and $\sigma_4$, where $\sigma_1 = \{r : 0 < r \leq r_0\}, \sigma_2 = \{re^{i\pi/4} : 0 < r \leq 1\}, \sigma_3 = \{e^{i\pi(1-r)/4} : 0 \leq r \leq r_0\}$, and $\sigma_4 = \{z = tr_0 + (1-t)e^{i\pi(1-r_0)/4} : 0 \leq t \leq 1\}$. We will prove this claim in three steps. First, we will show that $f$ is univalent in $\Omega$ for $r_0$ arbitrarily close to 1, and that $0 \leq \text{Arg}(f(\Omega)) \leq \frac{\pi}{4}$. Second, we verify that $f$ is univalent in the sector $\Omega \cup \Omega'$, where $\Omega'$ is the reflection of $\Omega$ across the real axis, and $\frac{-\pi}{4} \leq \text{Arg}(f(\Omega \cup \Omega')) \leq \frac{\pi}{4}$. Finally, we will verify that $f$ is univalent in $\mathbb{D}$.

Step One: The argument principle for harmonic functions [?] is valid if $f$ is continuous on $\overline{D}$, $f(z) \neq 0$ on $\partial D$, and $f$ has no singular zeros in $D$, where $D$ is a Jordan domain. Note $z_0$ is a singular point if $f$ is neither sense-preserving nor sense-reversing at $z_0$. We will show that for arbitrary $M > 0$, we may choose $r_0 < 1$ so that each value in the region bounded by $|w| < M$ and $0 < \text{Arg}(w) < \frac{\pi}{4}$ is assumed exactly once in the sector bounded by $|z| < 1$ and $0 < \text{Arg}(z) < \frac{\pi}{4}$, while no value in the region bounded by $|w| < M$ and $\frac{-\pi}{4} < \text{Arg}(w) < 2\pi$ is assumed in this sector.

Observe that $f_1(z) = 0$ only if $z$ is an 4th root of -1. Thus, on $\sigma_1$, $f_1$ is an increasing function of $r$ with $\text{Arg}(f_1) = 0$. Also, as $|z|$ increases on $\sigma_2$ and $\text{Arg}(z)$ decreases on $\sigma_3$, $|f_1(z)|$ increases. Note that $\text{Arg}(f_1(\sigma_2 \cup \sigma_3)) = \frac{\pi}{4}$. For $f_2$, if we let $z = \rho e^{i\theta}$ and use the fact that $f_2 = h_2 + e^{i\text{Arg}(z)}$, we get

Note that $f_2(0) = 0$. For $z \in \sigma_1$, $\frac{d}{d\rho}(f_2(\rho)) > 0$, and so $f_2$ increases on $\sigma_1$ as $r$ increases. Also $f_2(\rho) > 0$; hence $\text{Arg}(f_2(\sigma_1)) = 0$. For $z \in \sigma_2$, $\frac{d}{d\rho}(f_2(\rho e^{i\pi/4})) \neq 0$, and so $f_2(\sigma_2)$ does not reverse its direction. Finally we note $\text{Arg}(f_2(\sigma_2)) = \frac{\pi}{4}$. Recall $f_2$ is constant on $\sigma_3$. Therefore, we see that for $j = 1, 2, 3$, $f(\sigma_j)$ is a simple curve with $\text{Arg}(f(\sigma_j)) = 0$ while $\text{Arg}(f(\sigma_2 \cup \sigma_3)) = \frac{\pi}{4}$. To complete the proof that $f$ is univalent on $\Omega$, it suffices to show that given any $M > 0$ there exists an $r_0$ such that $|f(z)| > M$ for all $z \in \sigma_4$. To see this note that $|f_2(z)|$ is bounded for all $z \in \mathbb{D}$ while for $s$ fixed ($0 \leq s < 1$) and for $z \in \sigma_4$, $(1-s)f_1(z) \rightarrow \infty$ as $r \rightarrow 1$. Hence for a given $M$ the inequality will hold if we take $r_0$ sufficiently close to 1.

The proof is now complete since we have shown that every point outside the wedge is not assumed while every point inside the wedge is assumed exactly once by $f$.

Step Two: Since $f$ is univalent in $\Omega$, we can use reflection across the real axis to establish that $f$ is univalent in the sector $\Omega'$. In particular, suppose $z_1, z_2 \in \Omega'$ with $f(z_1) = f(z_2)$. Then by symmetry $\overline{f(z_1)} = f(z_1) = f(z_2) = \overline{f(z_2)}$. Hence, $\overline{f(z_1)} = \overline{f(z_2)}$, or $\overline{f(z_1)} = \overline{f(z_2)}$. Arguing in the same manner as in Step One, we can show that $0 \geq \text{Arg}(f(\Omega')) \geq \frac{-\pi}{4}$. Therefore, $f$ is univalent in $\Omega \cup \Omega'$ and its image is in the wedge between the angles $\frac{-\pi}{4}$ and $\frac{\pi}{4}$.
Step Three: First, it is true that $e^{i\pi j/2}f(ze^{-i\pi j/2}) = f(z)$, for all $z \in \mathbb{D}$ where $j = 0, 1, \ldots, 4$. To see this note that

Now, using this fact that $e^{i\pi j/2}f(ze^{-i\pi j/2}) = f(z)$, we see that if $z$ is any point in $\mathbb{D}$, it can be rotated so that it is in the sector $\Omega'$, in which $f$ is univalent, and then rotated back by multiplying by the constant $e^{i\pi j/2}$ and hence preserving univalency.

Example 4 (The 4-noid to 4-Enneper). The harmonic function that lifts to the 4-ennepers surfaces is given by

$$h_{4E} + g_{4E} = z - \frac{z^7}{7} \quad h_{4E} - g_{4E} = z + \frac{z^7}{7}$$

and that of the 4-noid is given by

$$h_{4N} + g_{4N} = \frac{1}{8} \left( \frac{2z}{1 + z^2} - 3\log \left( \frac{z + 1}{z - 1} \right) \right) \quad h_{4N} - g_{4N} = \frac{1}{4} \left( \frac{z}{1 - z^2} + \frac{3i}{2} \log \left( \frac{1 - iz}{1 + iz} \right) \right)$$

Neither of these surfaces is convex, thus we need to pursue other means then the above for showing that the combination $f = sf_{4E} + (1-s)f_{4N}$ is minimal for all $s \in (0,1)$.

We plot in (3) four instances of this transformation.

References

[1] R. Berry, M. Dorff, and W. L. Petersen, Lie symmetries of minimal surfaces, preprint.

[2] J. Clunie and T. Sheil-Small, Harmonic univalent functions, *Ann. Acad. Sci. Fenn. Ser. A.I 074 Math.* 9 (1984), 3-25.

[3] U. Dierkes, S. Hildebrandt, A. Kster, and O. Wohlrab, *Minimal surfaces I*, Grundlehren der Mathematischen Wissenschaften, 295, Springer-Verlag, Berlin, 1992.

[4] M. Dorff and J. Szynal, Harmonic shears of elliptic integrals, *Rocky Mountain Journal of Mathematics*, 35 (2005), no. 2, 485-499.

[5] K. Driver and P. Duren, Harmonic shears of regular polygons by hypergeometric functions, *J. Math. Anal. App.* 239 (1999), 72-84.
[6] P. Duren, *Harmonic mappings in the plane*, Cambridge Tracts in Mathematics, 156, Cambridge University Press, Cambridge, 2004.

[7] P. Duren, J. McDougall, and L. Schaubroeck, Harmonic mappings onto stars, *J. Math. Anal. Appl.* 307 (2005), no. 1, 312-320.

[8] W. Hengartner and G. Schober, On schlicht mappings to domains convex in one direction, *Comment. Math. Helv.* 45 (1970), 303-314.

[9] W. Koepf, Parallel accessible domains and domains that are convex in some direction, *Partial Differential Equations with Complex Analysis*, Pitman Res. Notes Math. Ser., 262, Longman Sci. Tech., Harlow, 1992, 93-105.

[10] H. Lewy, On the non-vanishing of the Jacobian in certain one-to-one mappings, *Bull. Amer. Math. Soc.* 42 (1936), 689-692.

[11] W. C. Royster and M. Ziegler, Univalent functions convex in one direction, *Publ. Math. Debrecen.*, 23, no. 3-4, 339-345.
Figure 1: Ennepers to Scherks singly-periodic transformation for $t = i/5$ for $i = 0, \ldots, 5$. 
Figure 2: Ennepers to Scherks singly-periodic transformation for $t = i/5$ for $i = 0, \ldots, 5$. 
Figure 3: 4 Enneper to 4-noid