Global Existence and Uniqueness Theorem for 3D – Navier-Stokes System on $\mathbb{T}^3$ for Small Initial Conditions in the Spaces $\Phi(\alpha)$.

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Dedicated to G.A. Margulis on the occasion of his sixtieth birthday.

Abstract: We consider Cauchy problem for three-dimensional Navier-Stokes system with periodic boundary conditions with initial data from the space of pseudo-measures $\Phi(\alpha)$. We provide global existence and uniqueness of the solution for sufficiently small initial data.

1. Introduction

Three-dimensional Navier-Stokes system with periodic boundary conditions after Fourier transform can be written in the form:

\[(1) \quad v(t,k) = \exp{-t|k|^2}v_0(k) + 2\pi i \int_0^t \exp{-(t-s)|k|^2} \sum_{l \in \mathbb{Z}^3} \langle k, v(s, k-l) \rangle P_k v(s, l) ds\]

Here $k \in \mathbb{Z}^3$, $t \in \mathbb{R}_+$, $v(t,k) \in \mathbb{C}^3$, $v(t,k) \perp k$ for any $k \neq 0$ and $v(t,0) = 0$ for all $t > 0$. $v_0(k)$ is the initial condition and $P_k$ denotes Leray projector to the subspace orthogonal to $k$ and has the form $P_k = \text{Id} - \frac{\langle k, \cdot \rangle}{|k|^2} k$. Also (1) assumes that the viscosity $\nu = 1$ and that the external forcing is absent.

T. Kato in [?] proved the local existence theorem for the 3D-Navier-Stokes system on $\mathbb{R}^3$ and global existence and uniqueness theorem in the space $L^3(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$ for small initial conditions.

In this paper we consider Cauchy problem for the system (1) with initial data from the space $\Phi(\alpha)$ which is analogous to the subspace $\Phi(\alpha, \alpha)$ introduced in [?], [?] and consists of functions of the form
\[ \Phi(\alpha) = \left\{ f(k) = \frac{c(k)}{|k|^\alpha}, k \neq 0 \mid \sup_k |c(k)| < \infty \right\}, \quad \|f(k)\|_\alpha = \sup_{k \in \mathbb{Z}^3} |k|^{\alpha} |f(k)| \]

We assume \( \alpha > 2 \) and shall write \( \alpha = 2 + \varepsilon \). V. Kaloshin and Yu. Sannikov announced the global existence theorem in the spaces \( \Phi(\alpha), \alpha \geq 2 \) for small initial data (see [?]). In this paper we give a detailed proof of this result which shows also the character of decay of solutions in this case. It is worthwhile to mention that according to our point of view a similar result is not valid in the continuous case of \( k \in \mathbb{R}^3 \).

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2. MAIN RESULT

The purpose of this paper is to prove the following theorem.

**Theorem 1.** Let \( 0 < 3\varepsilon < 1 \) and \( \|v_0\|_\alpha \leq \delta \) where \( v_0 = \frac{c_0(k)}{|k|^\alpha} \) is the initial condition and \( \delta = \delta(\alpha) \) is sufficiently small. Then the equation (??) has a global solution \( v(t, k) = \frac{c(t, k)}{|k|^\alpha} \) such that \( c(t, k) \) is a continuous mapping of \([0, \infty)\) into \( L^\infty(\mathbb{Z}^3 \setminus \{0\}) \), \( t > 0 \).

The proof of the Theorem ?? goes by induction. Put \( H_0^{(0)}(k) = \frac{c_0(k)}{|k|^\alpha}, k \neq 0, H_0^{(1)}(k) = G_0(k) = 0 \) and assume that for some integer \( m \) we constructed the solution \( v(t, k), 0 \leq t \leq m \), such that

\[ v(m, k) = H_m^{(0)}(k) + H_m^{(1)}(k) + G_m(k) \]

where

\[ H_m^{(0)}(k) = \exp\{-m|k|^2\}c_0(k) \]

\[ H_m^{(1)}(k) = \sum_{j=1}^{m} \exp\{-m-j|k|^2\}h_j^{(1)}(k) \]

and

\[ G_m(k) = \sum_{j=1}^{m} \exp\{-m-j|k|^2\}g_j(k) \]
Suppose that for all $j \leq m$ functions $h_j^{(1)}(k)$ satisfy the inequalities:

$$|h_j^{(1)}(k)| \leq \frac{D_1 \delta^2 \exp\left\{-\frac{2}{3}|k|^2\right\}}{|k|^{2\varepsilon}},$$

while the functions $g_j(k)$ satisfy the inequalities:

$$|g_j(k)| \leq \frac{D_2 \delta^2 \exp\left\{-d_1|k|\sqrt{m}\right\}}{|k|^\beta}$$

Here $\beta > 3$ is a constant and $d, D$ with indices denote various absolute constants which appear during the proof, but their exact values play no role in the arguments.

Consider $0 \leq t \leq 1$ and write down the solution of (??) in the form:

$$v(t + m, k) = \exp\left\{-m|k|^2\right\}c_0(k) + \sum_{j=1}^{m} \exp\left\{-m - j + t\right\}h_j^{(1)}(k) + \sum_{j=1}^{m} \exp\left\{-m - j + t\right\}g_j(k) + g_{m+1}(t, k)$$

We show that the inequalities (??), (??) holds for $h_{m+1}^{(1)}(1, k)$ and $g_{m+1}(1, k)$ respectively.

### 3. Proof of the main result.

Denote

$$H_{m+1}^{(0)}(t, k) = \exp\left\{-m|k|^2\right\}c_0(k),$$

$$H_{m+1}^{(1)}(t, k) = \sum_{j=1}^{m} \exp\left\{-m - j + t\right\}h_j^{(1)}(t, k) + \frac{h_{m+1}^{(1)}(t, k)}{|k|^{2\varepsilon}},$$

$$G_{m+1}(t, k) = \sum_{j=1}^{m} \exp\left\{-m - j + t\right\}g_j(t, k)$$
and
\[
(H' \otimes H'') (t, k) = i \int_0^t \exp\{-(t-s)|k|^2\} \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \frac{\langle k, H'(s, k-l) \rangle P_k H''(s,l)}{|k-l|^\alpha |l|^\alpha}
\]

If we substitute (9) into (8) we can write the expression for \( h_{m+1}^{(1)} (t, k) \):
\[
h_{m+1}^{(1)} (t, k) = |k|^2 \varepsilon \left( H_{m+1}^{(0)} \otimes H_{m+1}^{(0)} \right) (t, k)
\]

and the expression for \( g_{m+1} (t, k) \):
\[
g_{m+1} (t, k) = \sum_{j_1=1}^8 I_{m+1}^{(1,j_1)} (t, k) + \sum_{j_2=1}^3 I_{m+1}^{(2,j_2)} (t, k) + I_{m+1}^{(3)} (t, k)
\]
where
\[
I_{m+1}^{(1,j_1)} (t, k) = (H' \otimes H'') (t, k),
I_{m+1}^{(2,j_2)} (t, k) = (H' \otimes g_{m+1}) (t, k) + (g_{m+1} \otimes H') (t, k)
\]

and \( H', H'' \) are either \( H_{m+1}^{(0)} (t, k) \), or \( H_{m+1}^{(1)} (t, k) \) or \( G_{m+1} (t, k) \) except the case \( H' = H'' = H_{m+1}^{(0)} \) which corresponds to the \( h_{m+1}^{(1)} \) according to (9). Therefore \( j_1 \) changes from 1 to 8 and \( j_2 \) changes from 1 to 3. Also
\[
I_{m+1}^{(3)} (t, k) = g_{m+1} \otimes g_{m+1}.
\]

We see that \( I_{m+1}^{(1)} \) does not depend on \( g_{m+1} (t, k) \), \( I_{m+1}^{(2)} \) is a linear function of \( g_{m+1} (t, k) \) and \( I_{m+1}^{(3)} \) is a quadratic function of \( g_{m+1} (t, k) \). Therefore (9) is a typical equation which can be solved by iterations if the coefficients are small enough. Below we provide necessary estimates and later we come back to the analysis of (8).

### 3.1. First estimates

Here we show that all functions \( h_{m+1}^{(1)} \) behaves like gaussian functions of \( t|k| \) and then provide necessary estimates for coefficients in (9).

As in [?, ?], we use the identity:
\[
a_1|k-l|^2 + a_2|l|^2 = \frac{a_1a_2}{a_1 + a_2} |k|^2 + (a_1 + a_2) \left| l - \frac{a_1}{a_1 + a_2} k \right|^2
\]

**An estimate of \( H_{m+1}^{(1)} \).** At first we estimate \( h_{m+1}^{(1)} (t, k) \). From (9) it follows that
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\[ h_{m+1}^{(1)}(t, k) = (H_{m+1}^{(0)} \circledast H_{m+1}^{(0)})(t, k) = 2\pi i \int_0^t \exp\{-(t-s)|k|^2\} \cdot \sum_{l \in \mathbb{Z} \setminus \{0\}} \frac{\langle k, c_0(k-l) \rangle P_k c_0(l)}{|k-l|^\alpha |l|^\alpha} \exp\{-|k-l|^2 - (m+s)|l|^2\} ds \]

Using \((??)\) we can write

\[ |h_{m+1}^{(1)}(t, k)| \leq \exp\left\{-\frac{m|k|^2}{2}\right\} \int_0^t \exp\{-(t-s)|k|^2\} \cdot \sum_{l \in \mathbb{Z} \setminus \{0\}} \frac{\langle k, c_0(k-l) \rangle P_k c_0(l)}{|k-l|^\alpha |l|^\alpha} \exp\{-2m|l-\frac{1}{2}k|^2\} ds \leq \delta^2 \exp\left\{-\frac{m|k|^2}{2}\right\} \frac{\exp\{-\frac{t}{2}|k|^2\} - \exp\{-t|k|^2\}}{|k|^2} \cdot \sum_{l \in \mathbb{Z} \setminus \{0\}} \frac{\exp\{-2m|l-\frac{1}{2}k|^2\}}{|k-l|^\alpha |l|^\alpha} \leq D_3 \delta^2 \exp\left\{-\frac{(m+t)|k|^2}{2}\right\} \frac{1 - \exp\{-\frac{t}{2}|k|^2\}}{|k|^2} \]

Substituting this inequality to \((??)\) we conclude

\[ |H_{m+1}^{(1)}(t, k)| \leq \frac{D_3 \delta^2}{|k|^{2\varepsilon}} \left(1 - \exp\{-\frac{t}{2}|k|^2\}\right) \sum_{j=1}^{m+1} \exp\left\{-\frac{1}{2}j|k|^2\right\} \leq \frac{D_4 \delta^2}{|k|^{2\varepsilon}} \left(1 - \exp\{-\frac{t}{2}|k|^2\}\right) \exp\left\{-\frac{(m+1)|k|^2}{2}\right\} \]

**Estimates for** \(H_{m+1}^{(j_1)} \circledast H_{m+1}^{(j_2)}\). We present detailed estimate only for \(H_{m+1}^{(0)} \circledast H_{m+1}^{(1)}\) since all other terms can be estimated in the same manner. From \((??)\) and
we have
\[
\left| (H_{m+1}^{(0)} \otimes H_{m+1}^{(1)})(t, k) \right| \leq |k| \delta^3 \int_0^t \exp\{-|t-s||k|^2\} \cdot \sum_{\substack{l \in \mathbb{Z}^3 \setminus \{0\} \\
k \not\in |l|^2 \epsilon |l|^\alpha}} \exp\{- (m+s)|l|^2 - \frac{m+2}{2}|k-l|^2\} ds \leq
\]
\[
\leq |k| \delta^3 \exp\left\{ - \frac{m+1}{3}|k|^2 \right\} \int_0^t \exp\{-|t-s||k|^2\} \sum_{\substack{l \in \mathbb{Z}^3 \setminus \{0\} \\
k \not\in |l|^2 \epsilon |l|^\alpha}} \exp\{-\frac{3}{2}(m+s)|l - \frac{m+s}{3}|k|^2\} ds
\]
Since $\alpha + 2\epsilon > 2$ the last sum is not more than some constant $D_5$. We get

\[\left| (H_{m+1}^{(0)} \otimes H_{m+1}^{(1)})(t, k) \right| \leq |k| D_5 \exp\left\{ - \frac{m+1}{3}|k|^2 \right\} \frac{1 - \exp\{-|t||k|^2\}}{|k|^2} \]  

Similarly, for $(H_{m+1}^{(1)} \otimes H_{m+1}^{(1)})(t, k)$ we can write

\[\left| (H_{m+1}^{(1)} \otimes H_{m+1}^{(1)})(t, k) \right| \leq D_6 \exp\left\{ - \frac{m+1}{4}|k|^2 \right\} \frac{1 - \exp\{-|t||k|^2\}}{|k|} \]

3.2. Spaces $\mathcal{F}_m(c)$. Fix positive constant $\beta > 0$ and introduce functional space $\mathcal{F}_m(c)$

\[\mathcal{F}_m(c) = \left\{ f(k) \mid |f(k)| \leq \frac{D_7}{|k|^\beta} \exp\{-c\sqrt{m}|k|\}, k \neq 0 \right\}, \|f\|_{m,c} = \inf D_7 \]

We show that functions $g_{m+1}(t, k)$ belong to the spaces $\mathcal{F}_m$ with uniform constant if only all coefficients in (??) are sufficiently small. First of all we show that $H^{(j_1)} \otimes H^{(j_2)}$, $j_1 + j_2 > 0$ belongs to the space $\mathcal{F}_m(d_2)$ for some constant $d_2$. It follows from previous estimates, that all of these functions decay as a Gaussian functions. For our purpose it is convenient to consider them as functions from the space $\mathcal{F}_m(d_2)$. Since $m|k| \geq 1$ we can write

\[\exp\left\{ - \frac{m|k|^2}{3} \right\} \leq \frac{D_8}{|k|^\beta} \exp\left\{ - \frac{\sqrt{m}|k|}{\sqrt{3}} \right\} \]

for some constant $D_8$. We see that $(H_{m+1}^{(0)} \otimes H_{m+1}^{(1)})(t, k) \in \mathcal{F}_{m+1}(\frac{1}{\sqrt{3}})$ and

\[\left\| H_{m+1}^{(0)} \otimes H_{m+1}^{(1)} \right\|_{m+1, \frac{1}{\sqrt{3}}} \leq D_9 \]

for some constant $D_9$, which does not depend on $t$. 
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Assuming that \( G_{m+1} \in \mathcal{F}_{m+1}(d_2) \) we can write for \( G_{m+1} \otimes H_{m+1}^{(0)} \)

\[
|(G_{m+1} \otimes H_{m+1}^{(0)})(t, k)| \leq \|G_{m+1}\|_{m+1,d_2} \delta |k| \int_{0}^{t} \exp\{-(t-s)|k|^2\} \cdot \sum_{l \in \mathbb{Z} \setminus \{0\}} \frac{\exp\{-d_2 \sqrt{m}|k-l|-m|l|^2\}}{|l|^\alpha |k-l|^{\beta_1}} ds
\]

For the last expression we get

\[
\exp\{-d_2 \sqrt{m}|k-l|-m|l|^2\} \leq \exp\{-d_2 \sqrt{m}|k|\} \exp\{d_2 \sqrt{m}|l| - m|l|^2\} \leq \exp\{-d_2 \sqrt{m}|k|\} \exp\{|l| - \frac{d_2}{2\sqrt{m}}|l|^2\}
\]

So for \( (G_{m+1} \otimes H_{m+1}^{(0)})(t, k) \) we obtain

\[
|(G_{m+1} \otimes H_{m+1}^{(0)})(t, k)| \leq D_{11} \|G_{m+1}\|_{m+1,d_2} \delta \frac{\exp\{-d_2 |k| \sqrt{m}\} \{1 - \exp\{-t|k|^2\}\}}{|k|^{\beta_1}}
\]

All other terms in \( I_{m+1}^{(1)}(t, k) \) can be similarly estimated.

Thus we embed the first term in the representation of \( g_{m+1}(t, k) \) given by (??) into the space \( \mathcal{F}_{m+1}(d_2) \). Now we provide the necessary estimates for the terms \( I_{m+1}^{(3)} \) and \( I_{m+1}^{(2)} \).

**Estimate for \( I_{m+1}^{(3)} \).** We show, that for given functions \( f_1, f_2 \in \mathcal{F}_{m+1}(d_2) \) \( f_1 \otimes f_2 \) also belongs to the space \( \mathcal{F}_{m+1}(d_2) \) and \( \|f_1, f_2\|_{m+1,d_2} \leq D_{12} \|f_1\|_{m+1,d_2} \|f_2\|_m + \) for some constant \( D_{12} \).

Write down the estimate

\[
|f_1 \otimes f_2| \leq \|f_1\|_{m+1,d_2} \|f_2\|_{m+1,d_2} |k| \int_{0}^{t} \exp\{-(t-s)|k|^2\} \cdot \sum_{l \in \mathbb{Z} \setminus \{0\}} \frac{\exp\{-d_2 \sqrt{m+1}(|l| - |k-l|)\}}{|l|^{\beta_1} |k-l|^{\beta_1-3}} ds \leq \frac{D_{13} \|f_1\|_{m+1,d_2} \|f_2\|_{m+1,d_2}}{|k|^{2\beta - 3}} \exp\{-d_2 |k| \sqrt{m+1}\} \frac{1 - \exp\{-t|k|^2\}}{|k|}
\]

Since \( \beta > 3 \) and the last expression is not more than 1, we get

\[
(16) \quad |f_1 \otimes f_2| \leq \frac{D_{14} \|f_1\|_{m+1,d_2} \|f_2\|_{m+1,d_2}}{|k|^{\beta}} \exp\{-d_2 \sqrt{m+1}|k|\}
\]
In particular, for \( g_{m+1}(t, k) \in \mathcal{F}_{m+1}(d_2) \) it follows that
\[
\|I_{m+1}^{(3)}(t, k)\|_{m+1,d_2} \leq D_{14}\|g_{m+1}(t, k)\|^2_{m+1,d_2}
\]

Estimates for \( I_{m+1}^{(2)} \). Here we produce the upper bound for \( \|I_{m+1}^{(2)}(t, k)\|_{m+1,D_{14}} = \sum_{j_2=1}^{3} I_{m+1}^{(2,j_2)}(t, k) \) assuming, that \( g_{m+1}(t, k) \in \mathcal{F}_{m+1}(d_2) \).

\[
|(g_{m+1}(t, k) \otimes H_{m+1}^{(0)}(t, k))| \leq \|g_{m+1}\|_{m+1,d_2}\delta|k|\int_0^t \exp\{-|t-s|k^2\} \cdot \sum_{l \in \mathbb{Z} \setminus \{0\} \atop k-l \neq 0} \exp\{-d_2\sqrt{m}|k-l|-m|l|^2\} \frac{|k-l|^\alpha}{|l|^\beta} ds
\]

Again for the last expression holds
\[
\exp\{-d_2\sqrt{m}|k-l|-m|l|^2\} \leq D_{15} \exp\{-d_2\sqrt{m}|k|\} \exp\{|l-\frac{d_2}{2\sqrt{m}}|^2\}
\]

So for \( (g_{m+1}(t, k) \otimes H_{m+1}^{(0)}(t, k)) \) we can write
\[
|(g_{m+1}(t, k) \otimes H_{m+1}^{(0)}(t, k))| \leq D_{16}\|g_{m+1}\|_{m+1,d_2}\delta\frac{1-\exp\{-|t|k^2\}}{|k|} \exp\{-d_2\sqrt{m}|k|\}
\]

For the terms \( g_{m+1} \otimes G_{m+1} \) we can produce an appropriate estimate using (??).

All other terms in \( I_{m+1}^{(2)}(t, k) \) can be estimated in a similar way.

Collecting all present estimates we see that for some constant newcon
\[
\|g_{m+1}(t, k)\|_{m+1,d_2} \leq D_{17}\delta^2 + D_{18}\delta\|g_{m+1}\|_{m+1,d_2} + D_{19}\|g_{m+1}\|^2_{m+1,d_2}
\]

So for sufficiently small \( \delta \) all coefficients in (??) are small and the equation (??) can be solved by iterations. The solution \( g_{m+1}(t, k) \) belongs to \( \mathcal{F}_{m+1}(d_2) \) and unique in this class of functions. Each function \( g_{m+1}(t, k) \) provides the unique solution \( v(m + t, k) \) of (??). The Theorem ?? is proven.

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