 Canonical quantization of Plebanski gravity

Eyo Eyo Ita III

May 4, 2010

Department of Applied Mathematics and Theoretical Physics
Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road
Cambridge CB3 0WA, United Kingdom
eei20@cam.ac.uk

Abstract

In this paper we show that the Plebanski theory of gravity implies a theory dual to the Ashtekar variables where the antiself-dual Weyl curvature is the fundamental momentum space variable. The dual theory implies the Einstein equations modulo the initial value constraints, and appears to be consistent in the Dirac sense. Using the dual theory we have obtained a reduced phase space for gravity through implementation of the initial value constraints. Additionally we have computed the classical dynamics and have performed a quantization on this space, constructing a Hilbert space of states for vanishing cosmological constant. Finally, we have clarified the canonical structure of the dual theory in relation to the Ashtekar theory.
1 Introduction

The canonical formulation of the metric representation of general relativity produces a totally constrained system as a consequence of diffeomorphism invariance. The Hamiltonian consists of a linear combination of first class constraints $H_\mu = (H, H_i)$, respectively the Hamiltonian and diffeomorphism constraints. These constraints $H_\mu$ have thus far turned out to be intractable in the metric representation due to their nonpolynomial structure in the basic variables. A major development occurred in 1988 with the introduction of the Ashtekar variables (see e.g. [1],[2],[3]), which led to the simplification of the initial value constraints into polynomial form. The Ashtekar variables can be seen as a result of enlarging the metric phase space $\Omega$, essentially by embedding it into the phase space of a $SO(3)$ Yang–Mills theory. A remnant of this embedding is the inclusion of the Gauss’ law constraint $G_a$ in the list of constraints $H_\mu \rightarrow (H_\mu, G_a)$. The projection to the constraint shell has been problematic in the full theory also in the Ashtekar variables due to the presence of this additional constraint $G_a$. In the attempt to express the Ashtekar theory in covariant form it was shown by Jacobson et. al. in [4], [5], and [6] that the Ashtekar formulation of GR is essentially the 3+1 decomposition of an action where self-dual two forms are the basic variables. One also has the Ashtekar connection as well as the antiself-dual part of the Weyl curvature tensor as variables, the latter regarded as an auxiliary field. This covariant form of the Ashtekar action as noted was actually discovered earlier by Plebanski (See e.g. [7]), and the equations of motion of the Plebanski theory imply the Einstein equations.

In this paper we will show that there are actually two theories of gravity which can arise from the starting Plebanski theory. One of these theories is the Ashtekar theory, which has been well studied in the literature. The second theory to the best of the author’s knowledge appears to be unknown, and will be referred to in this paper as the ‘dual’ theory. This theory is dual to the Ashtekar theory in a sense that we will make precise in this paper. The organization of this paper is as follows. Section 2 provides a review of the Plebanski theory of gravity, and shows the manner in which the Ashtekar formulation is obtained by elimination of the CDJ matrix. The CDJ matrix $\Psi_{ae}$ is the antiself-dual part of the Weyl curvature expressed as a 3 by 3 matrix, and in the Plebanski formulation is regarded as an auxiliary field. In section 3 we derive the dual theory, which eliminates the Ashtekar

\footnote{The spin network states of loop quantum gravity solve the Gauss’ law constraint by construction, and provide a kinematic Hilbert $H_{Kin}$ space for GR. However, they have not yet to the author’s knowledge been shown to solve the Hamiltonian constraint, which encodes the dynamics of the theory. Still, many insights have resulted from the application of the Ashtekar variables at the classical and at the quantum level.}
densitized triad $\tilde{\sigma}_a^i$ from the starting Plebanski theory in favor of the CDJ matrix $\Psi_{ae}$. In the dual theory it is $\Psi_{ae}$ which is now the momentum space variable, with the remaining Ashtekar variables along with their physical interpretations left intact. Having changed the canonical structure of the theory, we verify closure of the algebra of constraints on the phase space of the dual theory. The final results of this algebra are provided, though the details of the calculation and their physical interpretation have been relegated to a separate paper [8] for considerations of brevity. A main result is that the Hamiltonian constraint forms a subalgebra. This is different from the case in the Ashtekar theory, and allows for the possibility to eliminate the kinematic constraints $(H_i, G_a)$ by Dirac brackets while preserving Dirac consistency. The implication is that one may now obtain a reduced phase space constrained by the dynamics only of the Hamiltonian constraint (e.g. the kinematic phase space) and perform a quantization of this space.

In section 4 we perform a reduction to the kinematic phase space of the dual theory, defined as the phase space after implementation of the diffeomorphism and Gauss’ constraints and prior to the Hamiltonian constraint. We compute the Lagrangian and Hamiltonian dynamics on this space, including the verification of the closure of the Hamiltonian constraint algebra. It is clear that the momentum space of the dual theory is naturally adapted to the implementation of the initial value constraints. In section 5 we reproduce the Einstein equations, using the dual theory as the starting point. More precisely, we show that the Einstein equations follow in the same sense that the original Plebanski theory implies the Einstein equations, but now modulo the initial value constraints. The implication for the dual theory is that the solution to the Einstein equations is directly linked to the solution of the initial value constraints. In this section we show that the canonical structure of the dual theory is different from that of the Ashtekar variables for generic phase space configurations. For the remainder of the sections to follow we specialize to the case of a vanishing cosmological constant. Section 6 computes the Lagrangian and Hamiltonian dynamics on the kinematic phase space for $\Lambda = 0$, and as well the dynamics of the spacetime metric. The metric in the dual theory is a derived quantity, constructable from a CDJ matrix $\Psi_{ae}$ solving the initial value constraints.

Section 7 performs a quantization of the kinematic phase space for $\Lambda = 0$, constructing a Hilbert space of normalizable states solving the Hamiltonian constraint. We also provide a prescription for explicitly computing expectation values and observables. The measure of normalization for the states

\[2\text{The usual method to obtain the reduced phase space of a constrained system is to quotient the constraint surface by the gauge orbits generated by the constraints. But the initial value constraints of the dual theory constrain only the momentum part of the phase space. This leaves considerable freedom in the selection of the reduced configuration space } \Gamma_{\text{Kin}}. \text{ A judicious selection of } \Gamma_{\text{Kin}} \text{ can be based on the requirement of globally holonomic coordinates, which yields quantizable configurations, as shown in } [9].\]
is Gaussian, which is reminiscent of the Bargmann representation on holomorphic functions. Since the wavefunctions have been constructed on the kinematic phase space, then they solve the kinematic constraints by construction, in addition to the Hamiltonian constraint. The Hilbert space thus constructed is the Hilbert space of the dual theory, which arises from the Plebanski action and has been shown to imply the Einstein equations. The form of the wavefunctions constructed mimic that of a Hamilton–Jacobi functional on the reduced phase space, obtained by holographic projection. Section 8 establishes the conditions under which the dual theory is canonically equivalent to the Ashtekar theory. Recall that on the unconstrained phase space the two theories are not canonically related. Canonical equivalence is established precisely upon solution of the initial value constraints and projection to the reduced phase space, which we demonstrate in two ways. Essentially, the canonical commutation relations of the Ashtekar variables transform into affine commutation relations, which in turn transform into canonical commutation relations on the kinematic phase space of the dual theory upon the re-designation of variables. Since this space is readily accessible and quantizable in the dual theory, this provides a recourse to addressing the aforementioned issues for the Ashtekar variables. In essence, it is now possible to obtain a quantization of gravity subject to initial value constraints. This result extends from the dual to the Ashtekar theory precisely on nondegenerate configurations. The remainder of section 8 shows the manner in which the initial value constraints correspondingly map between the unconstrained and the physical phase spaces.
2 Plebanski theory of gravity

It has been shown by Plebanski in [7] that general relativity may be written using two forms in lieu of the metric as the basic variables. We adapt the starting action to the language of the $SO(3,\mathbb{C})$ gauge algebra as

$$I_{Pleb} = \frac{1}{G} \int_M \delta_{ae} \Sigma^a \wedge F^e - \frac{1}{2} (\delta_{ae} \varphi + \psi_{ae}) \Sigma^a \wedge \Sigma^e,$$

(1)

where $\varphi$ is a numerical constant. We have defined $SO(3,\mathbb{C})$-valued two forms $\Sigma^a$ and curvature two forms $F^a$, given by

$$\Sigma^a = \frac{1}{2} \Sigma^a_{\mu \nu} dx^\mu \wedge dx^\nu; \quad F^a = \frac{1}{2} F^a_{\mu \nu} dx^\mu \wedge dx^\nu.$$

(2)

The quantity $F^a$ is the curvature two form of an $SO(3,\mathbb{C})$-valued connection one form $A^a = A^a_\mu dx^\mu$, written in component form as

$$F^a_{\mu \nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu$$

(3)

with structure constants $f^{abc} = \epsilon^{abc}$. There are three equations of motion resulting from (1). The first equation

$$\frac{\delta I}{\delta \psi_{ae}} = \Sigma^a \wedge \Sigma^e - \frac{1}{3} \delta_{ae} \Sigma^g \wedge \Sigma_g = 0$$

(4)

implies that the two forms $\Sigma^a$ can be derived from a set of tetrad one forms $e^I = e^I_\mu dx^\mu$ occurring in a self dual combination

$$\Sigma^a = i e^0 \wedge e^a - \frac{1}{2} \epsilon_{a fg} e^f \wedge e^g,$$

(5)

which enforces the equivalence of (1) to general relativity. The volume form for the spacetime corresponding to (5) is given by

$$\frac{i}{2} \Sigma^a \wedge \Sigma^e = \delta^{ae} \sqrt{-g} d^4 x,$$

(6)

which fixes the conformal class of the spacetime metric $g_{\mu \nu} = \eta_{IJ} e^I_\mu \otimes e^J_\nu$.

The second equation of motion

$$\frac{\delta I}{\delta A^g} = D \Sigma^g = d \Sigma^g + \epsilon^g_{fh} A^f \wedge \Sigma^h = 0,$$

(7)

The quantity $F^a$ is the curvature two form of an $SO(3,\mathbb{C})$-valued connection one form $A^a = A^a_\mu dx^\mu$, written in component form as

$$F^a_{\mu \nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu$$

(3)

with structure constants $f^{abc} = \epsilon^{abc}$. There are three equations of motion resulting from (1). The first equation

$$\frac{\delta I}{\delta \psi_{ae}} = \Sigma^a \wedge \Sigma^e - \frac{1}{3} \delta_{ae} \Sigma^g \wedge \Sigma_g = 0$$

(4)

implies that the two forms $\Sigma^a$ can be derived from a set of tetrad one forms $e^I = e^I_\mu dx^\mu$ occurring in a self dual combination

$$\Sigma^a = i e^0 \wedge e^a - \frac{1}{2} \epsilon_{a fg} e^f \wedge e^g,$$

(5)

which enforces the equivalence of (1) to general relativity. The volume form for the spacetime corresponding to (5) is given by

$$\frac{i}{2} \Sigma^a \wedge \Sigma^e = \delta^{ae} \sqrt{-g} d^4 x,$$

(6)

which fixes the conformal class of the spacetime metric $g_{\mu \nu} = \eta_{IJ} e^I_\mu \otimes e^J_\nu$.

The second equation of motion

$$\frac{\delta I}{\delta A^g} = D \Sigma^g = d \Sigma^g + \epsilon^g_{fh} A^f \wedge \Sigma^h = 0,$$

(7)
\[ D \text{ is the exterior covariant derivative with respect to } A^a, \text{ states that the connection } A^a \text{ is the self-dual part of the spin connection compatible with the tetrad implicit in } \Sigma^a \text{ through (5). Moreover, } A^a \text{ is uniquely fixed by } \Sigma^a. \] The third equation of motion is given by

\[ \frac{\delta I}{\delta \Sigma^a} = F^a - \Psi^{-1}\Sigma^e = 0 \quad \rightarrow \quad F^a_{\mu
u} = \Psi^{-1}\Sigma^e_{\mu\nu} \]

where we have defined (for this article we will assume that \( \Psi_{ae} \) is nondegenerate so that its inverse exists)

\[ \Psi^{-1}_{ae} = \delta_{ae}\varphi + \psi_{ae}. \]  

Equation (8) states that the curvature of \( A^a \) is self-dual as a two form, which implies that the metric derived from the tetrad one-forms \( \epsilon^\ell \) satisfies the vacuum Einstein equations. The starting action (1) is equivalent to metric general relativity when the equations of motion (4), (7) and (8) are satisfied. However, a canonical analysis shows that the theory (1) in its present form is a second class constrained system (see e.g. [10] and [11]), whereas the metric theory of gravity is first class due to closure of the hypersurface deformation algebra. Note that (1) is expressed in terms of three different fields \( A^a, \Sigma^a \) and \( \psi_{ae} \), written in component form as

\[ I_{\text{Pleb}}[\Sigma^a, A^a, \Psi] = \frac{1}{4} \int_M d^4x \left( \Sigma^a_{\mu\nu} F^a_{\rho\sigma} - \frac{1}{2} \Psi^{-1}_{ae} \Sigma^e_{\mu\nu} \Sigma^e_{\rho\sigma} \right) \epsilon^{\mu\nu\rho\sigma} \]  

where \( \epsilon^{\text{123}} = 1 \), whereas in metric general relativity there is only one field, namely the spacetime metric \( g_{\mu\nu} \). This implies that to re-establish the link to metric GR, some fields must be eliminated from (10).

### 2.1 Derivation of the Ashtekar theory of gravity

The 3+1 decomposition of (10) is given by

\[ \frac{1}{2} \int dt \int_\Sigma d^3x \epsilon^{ijk} \Sigma^a_{jk} \dot{A}^a_i + A^a_0 D_i (\epsilon^{ijk} \Sigma^a_{jk}) + \Sigma^a_0 \epsilon^{ijk} (F^a_{jk} - \Psi^{-1}_{ae} \Sigma^e_{jk}), \]

where we have integrated by parts, using \( F^a_{0i} = \dot{A}^a_i - D_i A^a_0 \) from the temporal component of (3).\(^3\) Let us rename the variables, defining the spatial parts as

---

\(^3\)As with the convention of this paper, lowercase symbols from the Latin alphabet \( a, b, c, \ldots \) will denote internal \( SO(3, C) \) indices, and those from the middle \( i, j, k, \ldots \) will denote spatial indices.
\[ e^{ijk} \Sigma^a_{jk} = 2 \tilde{\sigma}^i_a, \quad e^{ijk} F^a_{jk} = 2 B^i_a, \]  
(12)

where \( B^i_a \) is the \( SO(3, C) \) magnetic field. Then (11) becomes\(^4\)

\[ I_{Pl} = \int dt \int d^3 x \tilde{\sigma}^i_a \dot{A}^a_i + A^a_0 D_i \tilde{\sigma}^i_a + \Sigma^a_{0i} (B^i_a - \Psi^{-1} \tilde{\sigma}^i_e). \]  
(13)

Rather than repeat the canonical analysis of [10] and [11], we will use (4) and (5) to redefine the two form components in (13). Making the identification

\[ e^a_i = \frac{1}{2} \epsilon^{ijk} \epsilon^{abc} \tilde{\sigma}^j_b \tilde{\sigma}^k_c (\det \tilde{\sigma})^{-1/2} = \sqrt{\det \tilde{\sigma}} (\tilde{\sigma}^{-1})^a_i, \]  
(14)

we see that \( \tilde{\sigma}^i_a \) in (12) takes on the interpretation of a densitized spatial triad. In a special gauge \( e^0_i = 0 \), known as the time gauge, the temporal components of the two forms (5) are given by

\[ \Sigma^a_{0i} = \frac{i}{2} N \epsilon^{ijk} \epsilon^{abc} \tilde{\sigma}^j_b \tilde{\sigma}^k_c + \epsilon^{ijk} N^j \tilde{\sigma}^k_a, \]  
(15)

where \( N = N (\det \tilde{\sigma})^{-1/2} \) and \( N^i \) are a set of four nondynamical fields (See e.g. [12],[13]).

Substituting (15) into (13), we obtain the action

\[ I = \int dt \int d^3 x \tilde{\sigma}^i_a \dot{A}^a_i + A^a_0 \tilde{G}_a - N^\mu H_\mu [\tilde{\sigma}, A, \Psi]. \]  
(16)

The fields \( A^a_0 \) and \( N^\mu = (N, N^i) \) are auxiliary fields whose variations yield respectively the following constraints. First we have the constraint \( \tilde{G}_a \), given by

\[ \tilde{G}_a = D_i \tilde{\sigma}^i_a = 0, \]  
(17)

and constraints \( H_\mu = (H, H_i) \) given by

\[ H_i = \epsilon_{ijk} N^j \tilde{\sigma}^i_a B^k_a + \epsilon_{ijk} \tilde{\sigma}^j_b \tilde{\sigma}^k_e \Psi^{-1}. \]  
(18)

\(^4\)We have omitted a factor of \( \frac{1}{2} \) from the action. We will insert this factor when we are ready to proceed to the quantum theory.
\[ H = (\det \tilde{\sigma})^{-1/2} \left( \frac{1}{2} \epsilon_{ijk} \epsilon^{abc} \tilde{\sigma}^i_a \tilde{\sigma}^j_b B^k_c - \frac{1}{6} \text{tr} \Psi^{-1} \epsilon_{ijk} \epsilon_{abc} \tilde{\sigma}^i_a \tilde{\sigma}^j_b \tilde{\sigma}^k_c \right). \]  \hspace{1cm} (19)

Having redefined the auxiliary fields, (16) is still expressed in terms of three variables \( \tilde{\sigma}^i_a, A^a_i, \) and \( \Psi^{-1}_{ae} \). This situation, which implies the existence of second class constraints, can be rectified by eliminating \( \Psi^{-1}_{ae} \), seen as an auxiliary field. The result is an action in terms of the variables \( (\tilde{\sigma}^i_a, A^a_i) \) while preserving the equivalence of (16) to general relativity. This can be accomplished by imposition of the following conditions on \( \Psi^{-1}_{ae} \)

\[ \epsilon^{bae} \Psi^{-1}_{ae} = 0; \quad \text{tr} \Psi^{-1} = -\Lambda \]  \hspace{1cm} (20)

where \( \Lambda \) is the cosmological constant. Equation (20) eliminates the antisymmetric part of \( \Psi_{ae} \) and fixes its trace. The physical interpretation of (20) arises from the following decomposition

\[ \Psi^{-1}_{ae} = -\frac{\Lambda}{3} \delta_{ae} + \psi_{ae}, \]  \hspace{1cm} (21)

where \( \psi_{ae} \) is the self-dual part of the Weyl curvature tensor expressed in \( SO(3,C) \) language. The consequence of (20) is that \( \psi_{ae} \) has five D.O.F. at the level prior to implementation of the constraint \( G_a \).

When (20) holds, then \( \Psi^{-1}_{ae} \) becomes eliminated and equation (16) reduces to the action for general relativity in the Ashtekar variables ([1],[2],[3])

\[ I_{\text{Ash}} = \frac{1}{G} \int dt \int d^3x \tilde{\sigma}^i_a \dot{A}^a_i + A^a_0 D_i \tilde{\sigma}^i_a - \epsilon_{ijk} N^i \tilde{\sigma}^j_a B^k_a + \frac{i}{2} N \epsilon_{ijk} \epsilon_{abc} \tilde{\sigma}^i_a \tilde{\sigma}^j_b (B^k_c + \frac{\Lambda}{3} \delta^k_c), \]  \hspace{1cm} (22)

where \( N = N(\det \tilde{\sigma})^{-1/2} \) is the lapse density function. The action (22) is expressed in terms of two canonically conjugate dynamical variables with Poisson bracket relations

\[ \{ A^a_i(x,t), \tilde{\sigma}^j_b(y,t) \} = G \delta^a_b \delta^j_i \delta^{(3)}(x,y), \]  \hspace{1cm} (23)

and the offending variable \( \Psi^{-1}_{ae} \) has been eliminated. The auxiliary fields \( A^0_a, N \) and \( N^i \) respectively are the \( SO(3,C) \) rotation angle, the lapse function and the shift vector. Not only is (22) equivalent to metric general relativity, modulo reality conditions on the phase space variables, by complex canonical transformation, but also the algebra of constraints \( G_a, H \) and \( H_i \) is first class as shown in [2],[3]. So we will not requote it here.
3 The dual theory to Ashtekar’s theory

By imposing the simplicity constraint and eliminating $\Psi^{-1}_{ae}$, equation (1) has been transformed from a second class into a canonically consistent first class-constrained theory of general relativity in the Ashtekar variables. Note also that $\tilde{\sigma}^i_a$ in the original Plebanski theory was part of an auxiliary field $\Sigma^a$, but upon elimination of $\Psi^{-1}_{ae}$ has now become promoted to the status of a dynamical momentum space variable. But this is not the only way to eliminate variables. We will show that there exists a theory of gravity based on the field $\Psi_{ae}$, which is dual to the Ashtekar formulation of gravity, which can also be derived directly from (1). Let us, instead of eliminating $\Psi^{-1}_{ae}$, eliminate the densitized triad $\tilde{\sigma}^i_a$ from (16) by enforcing the initial value constraints in the Ashtekar variables. Hence returning to the level of (18) and (19), re-quoted here for completeness,

$$H_i = \epsilon_{ijk} N^j \tilde{\sigma}^k_a B^k_a + \epsilon_{ijk} \tilde{\sigma}^j_a \tilde{\sigma}^k_e \Psi^{-1}_{ae}$$  \hspace{1cm} (24)$$

and

$$H = (\text{det} \tilde{\sigma})^{-1/2} \left( \frac{1}{2} \epsilon_{ijk} \epsilon^{abc} \tilde{\sigma}^i_a \tilde{\sigma}^j_b B^k_c - \frac{1}{6} (\text{tr} \Psi^{-1}) \epsilon_{ijk} \epsilon^{abc} \tilde{\sigma}^i_a \tilde{\sigma}^j_b \tilde{\sigma}^k_c \right),$$  \hspace{1cm} (25)$$

we will impose the Hamiltonian and diffeomorphism constraints from the theory based on the Ashtekar variables

$$\epsilon_{ijk} \epsilon^{abc} \tilde{\sigma}^i_a \tilde{\sigma}^j_b B^k_c = -\frac{\Lambda}{3} \epsilon_{ijk} \epsilon^{abc} \tilde{\sigma}^i_a \tilde{\sigma}^j_b \tilde{\sigma}^k_c, \quad \epsilon_{ijk} \tilde{\sigma}^j_a B^k_a = 0.$$  \hspace{1cm} (26)$$

Substitution of the second equation of (26) into (24) yields

$$H_i = \epsilon_{ijk} \tilde{\sigma}^j_a \tilde{\sigma}^k_e \Psi^{-1}_{ae},$$  \hspace{1cm} (27)$$

while substitution of the first equation of (26) into (25) yields

$$H = (\text{det} \tilde{\sigma})^{-1/2} \left( -\frac{\Lambda}{6} \epsilon_{ijk} \epsilon^{abc} \tilde{\sigma}^i_a \tilde{\sigma}^j_b \tilde{\sigma}^k_c - \frac{1}{6} (\text{tr} \Psi^{-1}) \epsilon_{ijk} \epsilon^{abc} \tilde{\sigma}^i_a \tilde{\sigma}^j_b \tilde{\sigma}^k_c \right) = -\sqrt{\text{det} \tilde{\sigma}} (\Lambda + \text{tr} \Psi^{-1}).$$  \hspace{1cm} (28)$$

Hence substituting (27) and (28), into (16), we obtain an action given by

$$I = \int dt \int_{\Sigma} \sqrt{\text{det} \tilde{\sigma}} \left( A^a_d A^d_a + A^a_0 D^i_a \tilde{\sigma}^i_a \right) + \epsilon_{ijk} N_i \tilde{\sigma}^j_a \tilde{\sigma}^k_e \Psi^{-1}_{ae} - i N \sqrt{\text{det} \tilde{\sigma}} (\Lambda + \text{tr} \Psi^{-1}).$$  \hspace{1cm} (29)$$
But (29) still contains $\tilde{\sigma}_a^i$, therefore we will completely eliminate $\tilde{\sigma}_a^i$ by substituting the spatial restriction of the equation of motion (8)

$$\tilde{\sigma}_a^i = \Psi_{ae}B_e^i,$$

into (29). This substitution, known as the CDJ Ansatz, yields the action

$$I_{Dual} = \int dt \int \Sigma d^3x \Psi_{ae}B_e^i \dot{A}_a^i + A_0^a B_e^i D_i \Psi_{ae} + \epsilon_{ijk} N^j B_e^j B_e^k \Psi_{ae} - iN(\text{det}B)^{1/2} \sqrt{\text{det}\Psi (\Lambda + \text{tr}\Psi^{-1})},$$

which depends on the CDJ matrix $\Psi_{ae}$ and the Ashtekar connection $A_0^a$, with no appearance of $\tilde{\sigma}_a^i$. In the original Plebanski theory $\Psi_{ae}$ was an auxiliary field which could be eliminated. But in (31) $\Psi_{ae}$ is clearly more than just an auxiliary field. As shown in [15], [16] and [17] it is inappropriate to go through the Dirac procedure to define primary and secondary constraints for variables which are already part of the canonical structure in a first order phase space action in canonical form. Therefore we will regard the dynamical variables as $\Psi_{ae}$ and $A_0^a$, which satisfy elementary Poisson brackets

$$\{A_b^i(x, t), \Psi_{ae}(y, t)\} = \delta^b_a (B^{-1})^e_j \delta^{(3)}(x, y).$$

We refer to (32) as a phase space with globally holonomic coordinates with an inhomogeneous symplectic structure, since the right hand side contains field dependence. The (true) constraints in (31) arise from the fact that the time derivatives of the fields $(A_0^a, N^\mu)$ neither appear in the starting action nor multiply fields whose velocities appear in the action. These fields have vanishing conjugate momenta and according to the Dirac procedure for constrained systems [18] yield primary constraints

$$\pi_i = \frac{\delta I_{Dual}}{\delta \dot{N}} = 0; \quad \pi = \frac{\delta I_{Dual}}{\delta \dot{N}} = 0; \quad \pi_a = \frac{\delta I_{Dual}}{\delta A_0^a} = 0.$$

The preservation of the primary constraints (33) in time yields the secondary constraints $(G_a, H, H_i)$. The Gauss’ law constraint $G_a$ is given by

---

5 The CDJ Ansatz is valid when $B_e^i$ and $\Psi_{ae}$ are nondegenerate as three by three matrices. Hence all results of this paper will be confined to configurations where this is the case.

6 One may attempt to treat the velocity of $\Psi_{ae}$ as a primary constraint, but one finds that the corresponding secondary constraint is the Hamilton’s equation of motion for $\Psi_{ae}$, which could alternatively have been derived from the canonical structure on the same footing as $A_0^a$. 

9
\[ G_a = -\dot{\pi}_a = \frac{\delta I_{\text{Dual}}}{\delta \Lambda^a \psi} = v_e \{ \Psi_{ae} \} + C^f_a \Psi_{fg} = 0, \tag{34} \]

which is distinguished by two structures. First there is a triple of vector fields \( v_a = B_a \partial_i \) constructed from the \( SO(3, \mathbb{C}) \) magnetic field \( B^i_a \) which contracts one of the indices on \( \Psi_{ae} \) as a kind of internal divergence operator. The second structure is an object

\[ C^f_a = (f_{abf} \delta_{ge} + f_{ebg} \delta_{af}) C_{be}, \tag{35} \]

where \( C_{be} = A^b_i B^i_e \) is defined as the ‘magnetic helicity density matrix’. The effect of (35) in (34) is to act on \( \Psi_{ae} \), seen as a \( SO(3, \mathbb{C}) \)-valued second-rank tensor, in the tensor representation of the gauge group. The diffeomorphism constraint is given by

\[ H_i = -\dot{\pi}_i = \frac{\delta I_{\text{Dual}}}{\delta N^i} = \epsilon_{ijk} B^j_a B^k_e \Psi_{ae}, \tag{36} \]

which is distinguished by the fact that it is linear in the antisymmetric part of \( \Psi_{ae} \). Lastly, the Hamiltonian constraint is given by

\[ H = -\dot{\pi} = \frac{\delta I_{\text{Dual}}}{\delta N} = i (\det B)^{1/2} \sqrt{\det \psi} (\Lambda + \text{tr} \Psi^{-1}). \tag{37} \]

As a note of caution, the solutions of the initial value constraints can only be used subsequent to, and not before, computing the constraints algebra and the equations of motion.

We argue that (31) is equivalent to general relativity, since it is dual to a theory which is equivalent to GR, which was derived from a theory which is on-shell implies GR. But we will rigorously prove this through the equations of motion, and perform a quantization of (31). But first let us verify the preservation of the secondary constraints of the theory.

3.1 Algebra of secondary constraints

We will now compute the algebra of secondary constraints for the dual theory, by smearing the constraints with their respective smearing functions. The smeared initial value constraints of the dual theory will retain the same names as their counterparts in the Ashtekar variables. The smeared Gauss’ law constraint is given by

\[ \bar{G}[\theta] = \int_{\Sigma} d^3 x \theta^a w_e \{ \Psi_{ae} \} \equiv \Psi_{ae} [W^{ae}(\theta)], \tag{38} \]
the notation being designed to clearly depict its relationship to the smearing function \( \theta^a \). We have made the definition

\[
w_e \{ \Psi_{ae} \} = v_e \{ \Psi_{ae} \} + C_{fg}^a \Psi_{fg}
\]

(39)

for the unsmeared Gauss’ law constraint. The diffeomorphism constraint is given by

\[
\vec{H}[\vec{N}] = \int_\Sigma d^3 x \epsilon_{ijk} N^i B^j_a B^k_e \Psi_{ae} \equiv \Psi_{[ae]}[V^{ae}(\vec{N})],
\]

(40)

and the Hamiltonian constraint is given by

\[
H[N] = \int_\Sigma d^3 x N (\det B)^{1/2} \sqrt{\det \Psi} (\Lambda + \text{tr} \Psi^{-1}).
\]

(41)

In verification of the closure of the secondary constraints, we will compute the Poisson algebra of (34), (36) and (37) using the Poisson brackets

\[
\{ f, g \}_{NH} = \int_\Sigma d^3 x \left( \frac{\delta f}{\delta \Psi_{bf}} (B^{-1})^j_f \frac{\delta g}{\delta A^b_j} - \frac{\delta g}{\delta \Psi_{bf}} (B^{-1})^j_f \frac{\delta f}{\delta A^b_j} \right).
\]

(42)

The details of the calculations can be found in [8], and thus will not be displayed here. The final result of the algebra is

\[
\{ \Psi_{[ae]}[V^{ae}(\vec{N})], \Psi_{[bf]}[V^{bf}(\vec{M})] \} = \Psi_{[ae]}[V^{ae}(\vec{N}, \vec{M})];
\]

\[
\{ \Psi_{[ae]}[V^{ae}(\vec{N})], \Psi_{bf}[W^{bf}(\vec{\theta})] \} = \Psi_{[ae]}[V^{ae}(\vec{N}, \vec{\theta})] + \Psi_{bf}[W^{bf}(\vec{\theta})];
\]

\[
\{ \Psi_{ae}[W^{ae}(\vec{\theta})], \Psi_{bf}[W^{bf}(\vec{\lambda})] \} = \Psi_{ae}[W^{ae}(\vec{\theta}, \vec{\lambda})];
\]

\[
\{ H[N], \Psi_{[ae]}[V^{ae}(\vec{N})] \} = V^{ae}(\vec{N}, N) \Psi_{[ae]} + H[N, \vec{N}];
\]

\[
\{ H[N], \Psi_{ae}[W^{ae}(\vec{\theta})] \} = W^{ae}(N, \vec{\theta}) \Psi_{ae};
\]

\[
\{ H[M], H[N] \} = \vec{m} \cdot \vec{V}(M, N) H.
\]

(43)

According to the notation in (43), the bold quantities signify the corresponding constraints in (38), (40) and (41) but containing momentum-dependent structure functions. The physical interpretation of the initial value constraints for the theory dual to the Ashtekar theory is also provided in [8]. The main point that we wanted to illustrate is that the algebra of constraints closes. This algebra can be summarized in the following notation
\[
\{\bar{H}[\vec{N}], \bar{H}[\vec{M}]\} \sim \bar{H}[\vec{N}, \vec{M}]; \\
\{\bar{H}[\vec{N}], \bar{G}[\vec{\theta}]\} \sim \bar{H}[\vec{N}, \vec{\theta}] + \bar{G}[\vec{N}, \vec{\theta}]; \\
\{\bar{G}[\vec{\theta}], \bar{G}[\vec{\lambda}]\} \sim \bar{G}[\vec{\theta}, \vec{\lambda}]; \\
\{H[N], \bar{H}[\vec{N}]\} \sim \bar{H}[\vec{N}, N] + H[\vec{N}, N]; \\
\{H[N], \bar{G}[\vec{\theta}]\} \sim \bar{G}[N, \vec{\theta}]; \\
\{H[M], H[N]\} \sim H[M, N], \\
\]

which means that the dual theory to the Ashtekar theory contains no second class constraints and is first class in the Dirac sense. To find the number of physical degrees of freedom per point, we implement a first class system of seven constraints\(^7\) on an eighteen (complex) dimensional phase space, which leaves \(2 \times 9 - 2 \times 7 = 4\) complex phase space degrees of freedom. Note that the initial value constraints are interpreted as constraints on the CDJ matrix \(\Psi_{ae}\), which constitutes the momentum space of the dual theory. To obtain the physical degrees of freedom of the configuration space, one must select an equivalence class from the orbits generated by each constraint, which amounts to the judicious imposition of restrictions on \(A^a_i\).\(^8\) Hence, with a cotangent bundle structure on the reduced phase space, we should have two momentum and two configuration degrees of freedom per point, which matches the D.O.F. of complex GR. For this reason, the dual theory is not a topological field theory.

Another main feature of the algebra (44) is that the Hamiltonian constraint forms a subalgebra. This means that the dual theory allows for the possibility to eliminate the Gauss’ law and diffeomorphism constraints, leaving behind a physical system on which the dynamics solely of the Hamiltonian constraint can be implemented. Note that this feature does not appear in the Ashtekar variables since in the latter, the Poisson bracket of two Hamiltonian constraints does not yield a Hamiltonian constraint.

\(^7\)This is the Hamiltonian constraint (1), the diffeomorphism constraint (3) and the Gauss’ law constraint (3).

\(^8\)Reference [9] demonstrates the restriction of \(A^a_i\) to quantizable configurations of the dual theory with respect to globally holonomic coordinates on configuration space \(\Gamma_{\text{Dual}}\).
4 Reduction to the kinematic phase space

Next we will reduce the dual theory to the kinematical level, defined as the level where the diffeomorphism and the Gauss’ law constraint have been implemented, leaving remaining the Hamiltonian constraint. The starting action is given by

\[
I_{Dual} = \int dt \int_{\Sigma} d^3x \Psi_{ae} B^i_e A_i^a + A^a_0 w_e \{\Psi_{ae}\} - N^i H_i - iN H. \tag{45}
\]

Variation of (45) with respect to the shift vector \(N^i\) yields the diffeomorphism constraint

\[
\frac{\delta I_{Dual}}{\delta N^i} = (\det B)(B^{-1})^j_i \psi_d = 0 \rightarrow \psi_d = 0, \tag{46}
\]

where \(\psi_d \equiv \epsilon_{dae} \Psi_{ae}\) parametrizes the antisymmetric part of the CDJ matrix. Since \(\psi_d\) vanishes, then \(\Psi_{ae}\) on-shell must be symmetric. Accompanied with the imposition of the diffeomorphism constraint we will gauge fix the shift vector \(N^i\), using the equation of motion for \(\psi_d\)

\[
\frac{\delta I_{Dual}}{\delta \psi_d} = \epsilon_{dae} B^i_e F^a_{0i} + 2 N^i (B^{-1})^d_i (\det B) = 0. \tag{47}
\]

Using the property of the determinant of nondegenerate 3 by 3 matrices \(B^i_a\), this yields the solution

\[
N^j = \frac{1}{2} \epsilon^{jik} F^a_{0i} (B^{-1})^i_k. \tag{48}
\]

Upon substitution of the solutions (46) and (48) into (45), we obtain an action given by

\[
I_{Dual} = \int dt \int_{\Sigma} d^3x \Psi_{(ae)} B^i_e A_i^a + A^a_0 w_e \{\Psi_{(ae)}\} - iN H
= \int dt \int_{\Sigma} d^3x \Psi_{(ae)} B^i_e F^a_{0i} - iN H, \tag{49}
\]

where we have integrated by parts to transform the Gauss’ law constraint and canonical term into the temporal component of the curvature. Since \(\Psi_{(ae)}\) is symmetric, then (49) can be written as

\[
I_{Dual} = \frac{1}{8} \int_M d^4x \Psi_{ae} F^a_{\mu\nu} F^e_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma} - iN (\det B)^{1/2} \sqrt{\det \Psi} (\Lambda + tr\Psi^{-1}). \tag{50}
\]
Equation (50) is the dual action at the level of implementation of the diffeomorphism constraint, with the Hamiltonian constraint effectively appended to a term of the form $\Psi F \wedge F$. But this is not a topological field theory since as we have shown, there are four physical degrees of freedom per point on the reduced phase space. The first term of (50) by itself possesses three symmetries. It is invariant under spacetime diffeomorphisms due to contraction of its world indices $\mu$. It is invariant under simultaneous $SO(3,C)$ rotations of $\Psi$ and the two curvatures. It is also invariant under the translations

$$\Psi_{ae} \rightarrow \epsilon_{acd}m_d; \quad \Psi_{ae} \rightarrow \Psi_{ae} + \phi_{ae}, \quad (51)$$

where $D\phi_{ae} = 0$. Note, of the three aforementioned invariances there is only one invariance not broken by the Hamiltonian constraint term of (50), namely the invariance under $SO(3,C)$ rotations. This can be seen from the fact that the Hamiltonian constraint depends completely on $SO(3,C)$ invariants, which will bring us to consider the Gauss' law constraint.

Since $\Psi_{ae}$ is symmetric we can write it as a polar decomposition

$$\Psi_{ae} = (e^{\theta \cdot T})_{af} \lambda_f (e^{-\theta \cdot T})_{fe}, \quad (52)$$

using a $SO(3,C)$ transformation $(e^{\theta \cdot T})_{ae}$ parametrized by three complex angles $\theta = (\theta^1, \theta^2, \theta^3)$. This corresponds to a rotation of the diagonal matrix of eigenvalues $\lambda_e = (\lambda_1, \lambda_2, \lambda_3)$ from the intrinsic frame, where $\Psi_{ae}$ is diagonal, into an arbitrary $SO(3,C)$ frame. The Hamiltonian constraint is given by

$$H = (\det B)^{1/2} \sqrt{\det \Psi} (\Lambda + \text{tr} \Psi^{-1}), \quad (53)$$

which is invariant under $SO(3,C)$ since it depends only on the $SO(3,C)$ invariants. Hence (53) can equally be written explicitly in terms of the eigenvalues

$$H = (\det B)^{1/2} \sqrt{\lambda_1 \lambda_2 \lambda_3} \left( \Lambda + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} \right), \quad (54)$$

which is the same for each $\tilde{\theta}$ in (52). Upon substitution of (52) into the first term of (50) we have

$$I_1 = \frac{1}{8} \int_M d^4 x \lambda_f ((e^{-\theta \cdot T})_{fa} F_{\mu \nu}^a [A]) ((e^{-\theta \cdot T})_{fe} F_{\rho \sigma}^e [A]) \epsilon^{\mu \nu \rho \sigma}, \quad (55)$$

\footnote{We assume that $\Psi_{ae}$ is diagonalizable, which requires the existence of three linearly independent eigenvectors \cite{19}.}
where $A^a_\mu$ is a four dimensional connection with curvature $F^a_{\mu\nu}[A]$ given by

$$F^a_{\mu\nu}[A] = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_{\mu} A^c_{\nu}. \quad (56)$$

The internal index on each curvature in (55) is rotated by $e^{-\theta \cdot T}$, which corresponds to a $SO(3, C)$ gauge transformation. Therefore there exists a gauge transformed version of $F^a_{\mu\nu}$, given by curvature $f^a_{\mu\nu}$ such that

$$I_1 = \int_M d^4x \left( \frac{1}{8} \lambda_f f^f_{\mu\nu}[a] f^f_{\rho\sigma}[a] \epsilon^{\mu\nu\rho\sigma} - iNH \right) \quad (57)$$

for some four dimensional connection $a^a_\mu$. The relation between $a^a_\mu$ and $f^a_{\mu\nu}$, which contains no explicit reference to the $SO(3, C)$ angles $\bar{\theta}$, is given by

$$f^a_{\mu\nu}[a] = \partial_\mu a^a_\nu - \partial_\nu a^a_\mu + f^{abc} a^b_\mu a^c_\nu. \quad (58)$$

It then follows that the connection $a^a_\mu$ is a $SO(3, C)$ gauge transformed version of $A^a_\mu$ related by

$$a^a_\mu = (e^{-\theta \cdot T})_{ae} A^e_\mu - \frac{1}{2} \epsilon^{abc} (\partial_\mu (e^{-\theta \cdot T})_{bf}) (e^{-\theta \cdot T})_{cf}, \quad (59)$$

which corresponds to the adjoint representation of the gauge group [20]. Next, perform a 3+1 decomposition of (57), which yields

$$I_{Dual} = \int_M d^4x \left( \frac{1}{8} \lambda_f f^f_{\mu\nu}[a] f^f_{\rho\sigma}[a] \epsilon^{\mu\nu\rho\sigma} - iNH \right)$$

$$= \int dt \int_{\Sigma} d^3x \left( \lambda_f b^i j^i \dot{a}^f_j - \lambda_f w_f \{ a^f_0 \} \right)$$

$$- N (\text{det} b)^{1/2} \sqrt{\lambda_1 \lambda_2 \lambda_3} \left( \Lambda + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} \right). \quad (60)$$

We have defined $b^i_a = \frac{1}{2} \epsilon^{ijk} f^a_{jk}$ as the spatial part of (58). Additionally, the following identifications have been made

$$\text{det} B = \text{det} b; \quad b^i_a = (e^{-\theta \cdot T})_{ae} B^i_e. \quad (61)$$

The first equation of (61) is a result of the special orthogonal property that $\text{det}(e^{\theta T}) = 1$, and the second equation corresponds to a $SO(3, C)$ rotation of the internal index. Integration of (60) by parts with discarding of boundary terms yields
\[ I_{Dual} = \int dt \int_\Sigma d^3x \left( \lambda_f b_j^f \dot{a}_i^f + a_0^f w_f(\lambda_f) \right) - iN(\det b)^{1/2}/\sqrt{\lambda_1 \lambda_2 \lambda_3} \left( \Lambda + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} \right). \] (62)

Variation of (62) with respect to \( a_0^f \) would yield

\[ \frac{\delta I_{Dual}}{\delta a_0^f} = w_f(\lambda_f) = 0 \] (63)

with no summation over \( f \) which is unsatisfactory, since this would constitute a premature restriction on \( \lambda_f \) which we would like to use for the physical degrees of freedom. To preserve three D.O.F. in \( \lambda_f \) at the kinematical level we must instead set \( a_0^f = 0 \), which corresponds to the choice of a gauge. For Yang–Mills theory \( a_0^f = 0 \) is known as the temporal gauge \([20]\).

The temporal gauge in Yang–Mills theory admits the residual freedom to perform time independent gauge transformations. For gravity the infinitesimal \( SO(3,C) \) gauge transformation of \( a_0^f \) would be given by

\[ \delta \xi a_0^f = \dot{\xi}_f + f^{gh} a_0^g \theta^h \bigg|_{a_0^h=0} = \dot{\xi}_f. \] (64)

From (64), one sees that the gauge choice \( a_0^f = 0 \) is preserved only for \( \dot{\xi}_f = 0 \), or \( \xi_f = \xi_f(x) \), namely gauge transformations which are independent of time.

Note that the \( SO(3,C) \) angles \( \theta \) can still be chosen arbitrarily. Imposition of \( a_0 = 0 \) yields the action

\[ I_{Dual} = \int dt \int_\Sigma d^3x \left( \lambda_f b_j^f \dot{a}_i^f - iN(\det b)^{1/2}/\sqrt{\lambda_1 \lambda_2 \lambda_3} \left( \Lambda + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} \right) \right). \] (65)

Equation (65) seems a feasible starting point for describing the dynamics of the physical D.O.F. for the dual theory to the Ashtekar theory, since \( \text{Dim}(\Omega_{Kin}) = 6 \),\(^{10}\) but in the process of setting \( a_0 \) to zero we have eliminated the ability to impose the Gauss\(^\ast \) law constraint \( G_a \). This will bring us to the alternate sequence of 3+1 decomposition.

Performing the 3+1 decomposition of (50) prior to the polar decomposition leads to a first order action

---

\(^{10}\)This refers to both the classical and the quantum dynamics. Additionally, by eliminating three D.O.F. from the Ashtekar connection we have also eliminated three unphysical degrees of freedom, which should bring us a step closer toward metric general relativity.
\[
I_{\text{Dual}} = \int dt \int d^3x \left( \Psi_{ae} B^i_e A_i^a + A_0^a w_e \left\{ \Psi_{ae} \right\} \right.
- iN (\det B)^{1/2} \sqrt{\det \Psi} \left( \Lambda + \text{tr} \Psi^{-1} \right) \), \tag{66}
\]

which essentially is (62) with \( f_{\mu \nu}^a \) replaced by \( F_{\mu \nu}^a \). In contrast to (62), there is no restriction on the angles \( \vec{\theta} \) and the Gauss’ law constraint upon variation of \( A_0^a \)

\[
\frac{\delta I_{\text{Dual}}}{\delta A_0^a} = w_e \left\{ \lambda_f (e^{-\theta T} f_o (e^{-\theta T} f_e) \right\} = 0 \tag{67}
\]

allows us to restrict the \( SO(3, C) \) angles \( \theta^a \) in lieu of prematurely restricting \( \lambda_f \). Namely, for each configuration \( A_i^a \) and triple of eigenvalues \( \lambda_f \), one must invert (67) to solve for \( \vec{\theta} = \vec{\theta} [\lambda^a; A] \), which defines a functional. The main result is that the eigenvalues \( \lambda_f \) are preserved as the physical degrees of freedom. The consistency of (66) with (62) requires that \( a_0 = 0 \) and that \( \vec{\theta} \) be fixed by (67).\textsuperscript{11} Hence, the implementation of the kinematic constraints is equivalent to transforming from (45) directly to

\[
I_{\text{Dual}} = \int_M d^4x \left( \frac{1}{8} \Psi_{ae} F_{\mu \nu}^a F_{\rho \sigma}^{e} \epsilon^{\mu \nu \rho \sigma} - iN (\det B)^{1/2} \sqrt{\det \Psi} \left( \Lambda + \text{tr} \Psi^{-1} \right) \right) \), \tag{68}
\]

which resembles a kind of ‘generalized’ \( F \wedge F \) term modulo a Hamiltonian constraint. In the manner of this section one may proceed to obtain the physical degrees of freedom of the dual theory, which we will expand upon in a later section.

\textsuperscript{11}The procedure for solving (67) is treated in [21],[22] and [23] and therefore will not be covered here.
5 Verification of the Einstein equations of motion

To verify that the Einstein equations follow from the dual theory let us take a step back to the level prior to implementation of the kinematic constraints. The starting action of the dual theory is

$$I_{\text{Dual}} = \int dt \int_{\Sigma} d^3x \Psi ae B^i_a F_{0i}^a + \epsilon_{ijk} N^i B^j_a B^k_e \Psi ae$$

$$-iN(\det B)\sqrt{\det \Psi}(\Lambda + \text{tr} \Psi^{-1}).$$

(69)

The Hamiltonian constraint is given by the equation of motion for the lapse function $N$

$$\frac{\delta I}{\delta N} = H = (\det B)^{1/2} \sqrt{\det \Psi}(\Lambda + \text{tr} \Psi^{-1}) = 0.$$  

(70)

Since $B^i_a$ and $\Psi ae$ are nondegenerate by assumption, then the requirement that the Hamiltonian constraint be satisfied is equivalent to the vanishing of the term in brackets

$$\Lambda + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} = 0.$$  

(71)

Equation (71) leads to the following relation

$$\lambda_3 = -\frac{\lambda_1 \lambda_2}{\Lambda \lambda_1 \lambda_2 + \lambda_1 + \lambda_2},$$  

(72)

which expresses $\lambda_3$ explicitly as a function of $\lambda_1$ and $\lambda_2$, which in the dual theory will be regarded as the physical degrees of freedom.

Since we have already examined the equations involving the antisymmetric part of $\Psi ae$, we will now focus on the symmetric part. In what follows, the reader should keep in mind that the symmetric part of the $\Psi ae B^i_a F_{0i}^a$ term of (69) is the same as the first term of (68). We will now show that (69) implies the same Einstein equations of motion arising from the original Plebanski action (1). More precisely, we will verify consistency with equations (4), (5) and (7) and (8). Using

$$\sqrt{-g} = N \sqrt{\det \sigma} = N \sqrt{h} = N(\det B)^{1/2} \sqrt{\det \Psi},$$

(73)

which writes the determinant of $g_{\mu\nu}$ in terms of its 3+1 decomposition and uses the determinant of (30), we have
\[
\delta I_{\text{fast}} = \frac{1}{8} F^b_{\mu\nu} F^f_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma} + i \sqrt{-g} (\Psi^{-1} \Psi^{-1})^{bf} = 0.
\]  
\hfill (74)

Left and right multiplying (74) by \(\Psi\), we obtain

\[
\frac{1}{4} (\Psi^{bl'} F^l_{\mu\nu}) (\Psi^{fj'} F^j_{\rho\sigma}) \epsilon^{\mu\nu\rho\sigma} = -2i \sqrt{-g} \delta^{bf}.
\]  
\hfill (75)

Note that this step and the steps that follow require that \(\Psi_{ae}\) be nondegenerate as a 3 by 3 matrix. Let us make the definition

\[
\Sigma^a_{\mu\nu} = (\Psi^{-1})^{ae} F^e_{\mu\nu} = \Sigma^a_{\mu\nu}[\Psi, A],
\]  
\hfill (76)

which retains \(\Psi_{ae}\) and \(A^a_{\mu}\) as fundamental, with the two form being derived quantities. Upon using (76) as a re-definition of variables, which amounts to using the curvature and the CDJ matrix to construct a two form, (75) reduces to

\[
\frac{1}{4} \Sigma^b_{\mu\nu} \Sigma^f_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma} dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma = \Sigma^b \wedge \Sigma^f = -2i \sqrt{-g} \delta^{bf} d^4x.
\]  
\hfill (77)

Using (8) as a change of variables, namely that a CDJ matrix combined with a curvature determines a two form, enables one to re-write (75) as

\[
\Sigma^b \wedge \Sigma^f = -2i \sqrt{-g} \delta^{bf} d^4x.
\]  
\hfill (78)

One recognizes (78) as the condition that the two forms thus constructed, which are now derived quantities as in (8), be derivable from tetrads, which is the analogue of (4) and (5). To complete the demonstration that the dual theory to the Ashtekar theory yields the Einstein equations, it remains to show that the connection \(A^a_{\mu}\) is compatible with the two forms \(\Sigma^a\) as constructed in (76).

The equation of motion for the connection \(A^a_{\mu}\) from (69) can be seen as arising from the relevant covariant part encoded in (50), which is given by

\[
\frac{\delta I_{\text{Dual}}}{\delta A^a_{\mu}} = \epsilon^{\mu\sigma\nu\rho} D_{\sigma} (\Psi_{ae} F^e_{\nu\rho}) - \frac{\delta}{\delta A^a_{\mu}} \int_M d^4x \left( \epsilon_{mnlf} N^n B_b^l B_f \Psi_{bf} ight.
\]
\[
- i N \sqrt{\det B} \sqrt{\det \Psi} (\Lambda + \text{tr} \Psi^{-1}) \Big) = 0.
\]  
\hfill (79)

Since there is no occurrence of \(A^a_{0}\) in the \(N^\nu H_\mu\) terms, then the equation of motion for the temporal component is given by
\[
\frac{\delta I_{\text{Dual}}}{\delta A^a_i} = \epsilon^{ijk} D_i(\Psi_{ae} F^e_{jk}) = D_i(\Psi_{ae} B^e_i) = 0,
\]
which is the Gauss’ law constraint \(G_a\) upon use of the spatial restriction of (76). The equations of motion for the spatial components \(A^a_i\) are given by

\[
\frac{\delta I_{\text{Dual}}}{\delta A^a_i} = \epsilon^{ijk} D_i(\Psi_{ae} F^e_{jk}) - \frac{\delta I_{\text{Dual}}}{\delta A^a_i} \int_M d^4x \epsilon_{mnl} N^m B^n_i B^j_f \Psi_{bf} \\
+ \frac{\delta}{\delta A^a_i} \int_M d^4x N \sqrt{\det B} \sqrt{\det \Psi} (\Lambda + \text{tr} \Psi^{-1}) = 0.
\]

Let us consider the contributions to (81) due to the Hamiltonian and diffeomorphism constraints \(H_\mu = (H_i, H_\delta).\) Defining

\[
D_{ea}^{ji}(x, y) \equiv \frac{\delta}{\delta A^a_i(x)} B^i_j(y) = \epsilon^{jki} (-\delta_{ae} \partial_k + f_{eda} A^d_k) \delta^{(3)}(x, y),
\]
the contribution due to the diffeomorphism constraint is given by

\[
\frac{\delta H_i[N]}{\delta A^a_i} = \frac{\delta}{\delta A^a_i} \int_M d^4x \epsilon_{mnl} N^m B^n_i B^j_f \Psi_{bf} \\
= 2D_{ba}^{ji}(\epsilon_{mnl} N^m B^n_i \Psi_{[bf]}) + 2D_{ja}^{ji}(\epsilon_{mnl} N^m B^n_i \Psi_{[bf]}) \\
= 4D_{ba}^{ji}(\epsilon_{mnl} N^m B^n_i \Psi_{[bf]}),
\]
and the contribution due to the Hamiltonian constraint is given by

\[
\frac{\delta H[N]}{\delta A^a_i} = \frac{\delta}{\delta A^a_i} \int_M d^4x N (\det B)^{1/2} \sqrt{\det \Psi} (\Lambda + \text{tr} \Psi^{-1}) \\
= iD_{da}^{ki} \left( \frac{N}{2} (\det B)^{1/2} (B^{-1})^d_k \sqrt{\det \Psi} (\Lambda + \text{tr} \Psi^{-1}) \right) \\
= iD_{ba}^{ki} \left( \frac{N}{2} (B^{-1})^b_k H \right).
\]

Hence the equation of motion for \(A^a_i\) is given by

\[
\epsilon^{\mu
u\rho\sigma} D_\nu(\Psi_{ae} F^e_{\rho\sigma}) + \frac{1}{2} \delta^{\mu
u\rho\sigma} D_{\nu\rho}^{\delta\omega} (i(B^{-1})^b_k N H + 4\epsilon_{\rho\sigma\mu} N^m B^j_f \Psi_{[bf]}) = 0,
\]
where we have used that \(B^i_k\) is nondegenerate. The first term of (85) when zero implies (7) upon use of (76) to construct \(\Sigma_{\mu\nu}\). The obstruction to this equality, namely the compatibility of \(A^a_i\) with \(\Sigma_{\mu\nu}\) thus constructed,
arises due to the second and third terms of (85). These latter terms contain spatial gradients acting on the diffeomorphism and Hamiltonian constraints $H_\mu$. In order that $A_\mu^a$ be compatible with the two form $\Sigma_\mu^a = \Psi_{ae} F_{\mu\nu}^e$, we must require that these terms of the form $\partial_i H_\mu$ must vanish, which can be seen from the following argument. Since $H_\mu = 0$ when the equations of motion are satisfied, then the spatial gradients from $D_{\dot{e}a}^i$ acting on terms proportional to $H_\mu$ in (85) must vanish.

According to Dirac the constraints must be evaluated only subsequent to taking derivatives, and not prior. Our interpretation is that this refers to functional derivatives and time derivatives but not spatial gradients, which are nondynamical. The vanishing of the spatial gradients can be seen if one discretizes 3-space $\Sigma$ onto a lattice of spacing $\epsilon$ and computes the spatial gradients of the constraints $\Phi$ as $\partial\Phi = \frac{1}{2\epsilon} \lim_{\epsilon \to 0} (\Phi(x_{n+1}) - \Phi(x_{n-1}))$, and uses the vanishing of the constraints $\Phi(x_n) = 0 \forall n$ at each lattice point $x_n$. For another argument, smear the gradient of the Hamiltonian constraint with a test function $f$

$$S = \int_{\Sigma} d^3x f \partial_i H = - \int_{\Sigma} d^3x (\partial_i f) H_\mu \sim 0, \quad (86)$$

where we have integrated by parts. The result is that (86) vanishes on the constraint shell $\forall f$ which vanish on the boundary of 3-space $\Sigma$. This is tantamount to the condition that the spatial gradients of a constraint must vanish when the constraint is satisfied.\(^{12}\) Of course, the constraints $H_\mu$ follow from the equations of motion for $N^\mu = (N, N^i)$.

This completes the demonstration of the Einstein equations. The Einstein equations have arisen in the same sense as from (1) using (69) as the starting point, which is defined on the phase space $\Omega_{Dual} = (\Psi_{ae}, A_\mu^a)$. These equations are modulo the initial value constraints (46) and (71) and their spatial gradients, which also have arisen from (69).

### 5.1 Verification of the canonical structure

Next, we will compare the canonical and the symplectic structures of the dual in relation to the Ashtekar theory, in preparation for a quantization. It is not hard to verify that substitution of

$$\Psi_{ae}^{-1} = B_i^e (\hat{\sigma}^{-1})_i^a \rightarrow \hat{\sigma}_a^i = \Psi_{ae} B_i^e, \quad (87)$$

the spatial restriction of (8), into (31) yields the Ashtekar action (22) for nondegenerate $B_i^a$ and $\hat{\sigma}_a^i$. Additionally, the canonical structure can be

\(^{12}\)The author is grateful to Chopin Soo for pointing out this latter argument.
verified as follows. The Ashtekar phase space variables $\Omega_{Ash} = (\tilde{\sigma}_i^a, A_i^e)$ form a canonical pair with relations

$$\{A_i^a(x,t), \tilde{\sigma}_j^b(y,t)\} = \delta_b^a \delta_i^j \delta^{(3)}(x,y)$$  \hspace{1cm} (88)$$

along with

$$\{A_i^a(x,t), A_j^b(y,t), \} = \{\tilde{\sigma}_i^a(x,t), \tilde{\sigma}_j^b(y,t), \} = 0.$$  \hspace{1cm} (89)$$

Substituting (87) into (88) in the form $\tilde{\sigma}_i^a = \Psi_{ae} B_i^e$, we have

$$\{A_i^a(x,t), \Psi_{ae}(y,t) B_i^e(y,t)\} = \{A_i^a(x,t), \Psi_{ae}(y,t) B_i^e(y,t)\} = \delta_b^a \delta_i^j \delta^{(3)}(x,y)$$  \hspace{1cm} (90)$$

from the Liebniz rule. The second term on the right hand side of (90) vanishes on account of (89). Transferring the magnetic field to the right hand side, since it is nondegenerate by assumption, leads to (32). Note, in the quantization of the full unreduced theory, that (90) corresponds to a Schrödinger representation

$$\tilde{\sigma}_i^a \equiv \frac{\delta}{\delta A_i^a} \rightarrow \Psi_{ae} \equiv (B^{-1})_i^a \frac{\delta}{\delta A_i^a}. \hspace{1cm} (91)$$

The result is that the canonical relations of the dual theory transform directly into the those of the Ashtekar variables. While this may be the case, there is one subtle difference. If (87) were a canonical transformation, then the phase space structure of (29) would imply that the variable canonically conjugate to $\Psi_{ae}$ is an object $X_{ae}$ whose time derivative is $B_i^e A_i^a$. However, (87) is not a canonical transformation, which can be seen as follows. The symplectic two form on the phase space $\Omega_{Ash}$ is given by

$$\Omega_{Ash} = \int_\Sigma d^3 x \delta \tilde{\sigma}_i^a(x) \wedge \delta A_i^a(x) = \delta \left( \int_\Sigma d^3 x \tilde{\sigma}_i^a(x) \delta A_i^a(x) \right) = \delta \theta_{Ash}. \hspace{1cm} (92)$$

which is the exterior derivative of its canonical one form $\theta_{Ash}$. Using the functional Liebniz rule in conjunction with the variation of (87) we have $\delta \tilde{\sigma}_i^a = B_i^e \delta \Psi_{ae} + \Psi_{ae} \delta B_i^e$, which transforms the left hand side of (92) into

$$\Omega_{Dual} = \int_\Sigma d^3 x \delta \Psi_{ae} \wedge B_i^e \delta A_i^a + \int_\Sigma \epsilon_{ijk} \Psi_{ae} \delta (D_j A_i^e) \wedge \delta A_i^a. \hspace{1cm} (93)$$

Due to the second term on the right hand side of (93), the symplectic two form in the dual theory is not in general exact. Therefore there is no variable.
canonically conjugate to $\Psi_{ac}$ such that the fundamental Poisson brackets do not contain field dependence. If there are configurations where the second term of (93) vanishes, then such a canonical relation may be established.\textsuperscript{13}

\textsuperscript{13}This corresponds to a restriction of the degrees of freedom of the theory. However, in a few sections we will show that the remaining degrees of freedom are precisely the physical degrees of freedom upon implementation of the initial value constraints of GR.
6 Dynamics on the kinematic phase space

We will now compute the classical dynamics of the reduced theory, on the
kinematic phase space $\Omega_{Kin}$. Setting $a_f^0 = 0$ in (62), we have

$$I_{Dual} = \int dt \int_\Sigma d^3 x \left( \lambda_f b_f^j a'_j - i N (\det b)^{1/2} \sqrt{\lambda_1 \lambda_2 \lambda_3} \left( \Lambda + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} \right) \right).$$  (94)

Note, since $B_a^i$ and $\Psi_{ae}$ are nondegenerate, that the Hamiltonian constraint can equivalently be written as

$$\Phi = \Lambda + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} = 0.$$  (95)

From now on we will limit consideration in this paper to the $\Lambda = 0$ case. The canonical one form of (94) allows globally holonomic coordinates in the full theory for six distinct configurations $A^a_i$, which is proved in [9]. For the purposes of this paper we will treat the diagonal case $A^a_i = \delta^a_i a_a$. This corresponds to a canonical one form

$$\theta = \int_\Sigma d^3 x \left( \lambda_1 a_2 a_3 \delta a_1 + \lambda_2 a_3 a_1 \delta a_2 + \lambda_3 a_1 a_2 \delta a_3 \right)$$  (96)

where $a_f = a_f(x, t)$ contain three independent degrees of freedom per point (and therefore corresponds to the full theory). There is no globally holonomic coordinate in (96), but we can transform it into a theory with globally holonomic coordinates via the transformation

$$\Pi_f = (a_1 a_2 a_3) \lambda_f; \quad X^f = \ln \left( \frac{a_f}{a_0} \right); \quad T = X^1 + X^2 + X^3$$  (97)

where $(\det A) = a_1 a_2 a_3 \neq 0$, which imposes the following ranges on the configuration space

$$-\infty < |X^f| < \infty \longrightarrow 0 < |a_f| < \infty.$$  (98)

---

14The kinematic phase space is defined as the phase space at the level where the Gauss’ law and the diffeomorphism constraints have been implemented on the original full phase space $\Omega_{Dual}$, leaving remaining the Hamiltonian constraint.

15The theory for $\Lambda \neq 0$ is treated classically in [24] at the quantum level in [25].

16The special feature of these configurations is that the canonical one form (96) is free of any terms containing spatial gradients, even though the variables are in general not spatially homogeneous. Hence one is free to quantize the full theory on these configurations with all the advantages of the simplicity of minisuperspace.
Regarding \( \Pi \) and \( X \) in (97) as the fundamental variables implies a symplectic two form

\[
\Omega = \int_\Sigma d^3x \delta \Pi \wedge \delta X = \delta \left( \int_\Sigma d^3x \Pi \delta X \right) = \delta \theta,
\]

(99)

which is the exact variation of the canonical one form \( \theta \). The starting action (94) in terms of the new variables is given by

\[
I_{\text{Dual}} = \int dt \int_\Sigma d^3x \left( \Pi \dot{X}^f - iNa^3/2 e^{T/2} U \sqrt{\Pi_1 \Pi_2 \Pi_3} \left( \frac{1}{\Pi_1} + \frac{1}{\Pi_2} + \frac{1}{\Pi_3} \right) \right),
\]

(100)

where \( U \), which depends entirely on spatial gradients of \( X^f \), is as defined in Appendix A. Equation (100) is canonically well-defined and will form the basis of the reduced classical theory and its quantization. For one example of what can follow, consider the Hamiltonian constraint, which for \( \Lambda = 0 \) reduces to

\[
\frac{1}{\Pi_1} + \frac{1}{\Pi_2} + \frac{1}{\Pi_3} = 0.
\]

(101)

Substitution of (101) into (100) and interchanging the order of spatial with time integration implies the following Hamilton–Jacobi functional \( S_{HJ} \), where

\[
\delta S_{HJ} = \int_\Sigma d^3x \left[ \Pi_1 \delta X^1 + \Pi_2 \delta X^2 - \left( \frac{\Pi_1 \Pi_2}{\Pi_1 + \Pi_2} \right) \delta X^3 \right],
\]

(102)

which is essentially (94) evaluated on the solution to the Hamiltonian constraint.\(^{17}\) Equation (102) preserves the form of the Hamiltonian constraint solution on each spatial hypersurface \( \Sigma \), and reflects two physical degrees of freedom. We will show in this paper that the integrated form of (102) resembles in form the argument of the exponential of the quantum states of the dual theory to Ashtekar’s theory for \( \Lambda = 0 \).

6.1 Classical dynamics for \( \Lambda = 0 \)

We will now formulate the classical dynamics of the kinematic phase space of the dual theory for \( \Lambda = 0 \). For our starting action we will take the first order action given by

\(^{17}\)Since (94) is at the kinematical level, it then follows that (102) is also a Hamilton–Jacobi functional evaluated on the solution of all of the initial value constraints, and contains two degrees of freedom. Note that these D.O.F. have been projected holographically to the final spatial hypersurface \( \Sigma \) forming the boundary of spacetime \( M = \Sigma \times R \).
\[ S_{\text{Kin}} = \frac{1}{G} \int \sigma d^3 x \left( \Pi_f \dot{X}^f - iNa_0^{3/2} e^{T/2} U \sqrt{\Pi_1 \Pi_2 \Pi_3} \Phi \right), \]  

(103)

where \( U \), which contains spatial gradients of the configuration variables \( X^f \), is as defined in Appendix A. Also we have defined

\[ \Phi = \frac{1}{\Pi_1} + \frac{1}{\Pi_2} + \frac{1}{\Pi_3}. \]  

(104)

There are seven fields, \( \Pi_f = (\Pi_1, \Pi_2, \Pi_3) \) which we require to be nonvanishing, \( X^f = (X^1, X^2, X^3) \), and \( N \) and we have defined \( T = X^1 + X^2 + X^3 \). The Euler–Lagrange equations of motion from (103) are given by

\[ \frac{d}{dt} \left( \frac{\delta L}{\delta \dot{N}} \right) = \frac{\delta S_{\text{Kin}}}{\delta N}. \]  

(105)

It is clear from the starting action (103) that the velocity \( \dot{N} \) is absent. Additionally, \( N \) does not multiply a velocity, therefore it is an auxiliary field and (105) yields

\[ a_0^{3/2} e^{T/2} U \sqrt{\Pi_1 \Pi_2 \Pi_3} \Phi = 0. \]  

(106)

We require that \( e^{T/2} U \sqrt{\Pi_1 \Pi_2 \Pi_3} \) be nonzero, hence (106) reduces to

\[ \Phi = \frac{1}{\Pi_1} + \frac{1}{\Pi_2} + \frac{1}{\Pi_3} = 0, \]  

(107)

which is a constraint on the variables \( \Pi_f \). Note that this constraint is independent of the other variables \( X^f \) and \( N \).

The equation of motion for \( X^f \) is given by

\[ \frac{d}{dt} \left( \frac{\delta L}{\delta \dot{X}^f} \right) = \frac{\delta L}{\delta X^f}, \]  

(108)

which is

\[ \Pi_f = -Na_0^{3/2} e^{T/2} U \sqrt{\Pi_1 \Pi_2 \Pi_3} \Phi \{ \sqrt{\Pi_1 \Pi_2 \Pi_3} \Phi \}. \]  

(109)

There are spatial gradients from \( U \) which act on the terms in curly brackets. But since these terms are proportional to \( \Phi \), by the same argument as in (86) they vanish on solutions to (107). This leads to
\[ \Pi_f(x, t) = \Pi_f(x), \]  
\[ \text{(110)} \]

which are arbitrary functions of position, independent of time.

To find the equations of motion for \( \Pi_f \), we subtract a total time derivative \( \frac{d}{dt}(\Pi_f X^f) \) from the starting action (103) and obtain

\[ \frac{d}{dt} \left( \frac{\delta L}{\delta \dot{\Pi}_f} \right) = \frac{\delta L}{\delta \Pi_f}, \]

\[ \text{(111)} \]

which is

\[ -\dot{X}^f = -N a_0^{3/2} e^{T/2} \frac{\delta (\Pi_1 \Pi_2 \Pi_3)^{1/2}}{\delta \Pi_f} \Phi - N a_0^{3/2} e^{T/2} U \sqrt{\Pi_1 \Pi_2 \Pi_3} \frac{\delta \Phi}{\delta \Pi_f}. \]

\[ \text{(112)} \]

The first term on the right hand side of (112) vanishes on account of (107), and we are left with the following equations

\[ \dot{X^1} = -N a_0^{3/2} e^{T/2} U \sqrt{\Pi_1 \Pi_2 \Pi_3} \left( \frac{1}{\Pi_1} \right)^2; \]
\[ \dot{X^2} = -N a_0^{3/2} e^{T/2} U \sqrt{\Pi_1 \Pi_2 \Pi_3} \left( \frac{1}{\Pi_2} \right)^2; \]
\[ \dot{X^3} = -N a_0^{3/2} e^{T/2} U \sqrt{\Pi_1 \Pi_2 \Pi_3} \left( \frac{1}{\Pi_3} \right)^2. \]

\[ \text{(113)} \]

It will be convenient to make the following definitions

\[ \eta = a_0^{3/2} \sqrt{\Pi_1 \Pi_2 \Pi_3} \left( \left( \frac{1}{\Pi_1} \right)^2 + \left( \frac{1}{\Pi_2} \right)^2 + \left( \frac{1}{\Pi_3} \right)^2 \right); \]
\[ \eta_f = a_0^{3/2} \sqrt{\Pi_1 \Pi_2 \Pi_3} \left( \left( \frac{1}{\Pi_f} \right)^2 \right); \]
\[ \eta = \eta_1 + \eta_2 + \eta_3. \]

\[ \text{(114)} \]

Then defining \( T = X^1 + X^2 + X^3 \), then (113) is given by

\[ \dot{X}^f = \left( \frac{\eta_f}{\eta} \right) \dot{T} \quad \dot{T} = -NU e^{T/2} \eta. \]

\[ \text{(115)} \]

We have to integrate the equation for \( T \)

\[ -e^{-T/2} \dot{T} = 2 \frac{d}{dt} e^{-T/2} = NU \eta \]

\[ \text{(116)} \]

\[ ^{18} \text{In analogy to the arguments of Jackiw as applied to the case in the unreduced dual theory, it would be inappropriate to treat } \Pi_f \text{ as a primary constraint.} \]
which yields

\[ e^{-T/2} = e^{-T_0/2} + \frac{\eta(x)}{2} \int_0^T N(x, t') U(x, t'; T) dt', \quad (117) \]

where we have defined \( T_0 = T(x, 0) \). Equation (117) is a nonlinear relation between \( T \) and itself. This can be written as

\[ T = \ln \left( e^{-T_0/2} + \frac{\eta(x)}{2} \int_0^T N(x, t') U(x, t'; T) dt' \right)^{-2}. \quad (118) \]

One may proceed from (119) to perform a fixed point iteration procedure. Define a sequence \( T_n(x, t) \) where \( T_0(x, t) = T_0 \), and the following recursion relation holds

\[ T_{n+1}(x, t) = \ln \left( e^{-T_n/2} + \frac{\eta(x)}{2} \int_0^T N(x, t') U(x, t'; T_n(x, t')) dt' \right)^{-2}. \quad (119) \]

For given initial data \( T(x, 0) \) on a 3 dimensional spatial hypersurface \( \Sigma \) and a choice of the lapse function \( N(x, t) \) through spacetime, if the iteration converges to a fixed point, then one has that

\[ \lim_{n \to \infty} T_n(x, t) = T(x, t). \quad (120) \]

The motion of \( X^f \) is given by

\[ X^f(x, t) = X^f(x, 0) + \left( \frac{\eta_f}{\eta} \right) T(x, t), \quad (121) \]

with \( T(x, t) \) given by (119). The variables \( X^f \) evolve linearly with respect to \( T \), seen as a time variable on configuration space \( \Gamma \). The solutions for \( X^f(x, t) \) in principle are directly constructible from (119) and (120), combined with the specification of boundary data \( X^f(x, 0) \).

### 6.2 Hamiltonian formalism for \( \Lambda = 0 \)

From (103), we will now perform a Legendre transformation into the Hamiltonian formalism. The momentum conjugate to \( X^f \) is given by

\[ \Pi_f(x, t) = \frac{\delta S}{\delta X^f(x, t)}. \quad (122) \]

\[ ^{19} \text{This seems to be the nearest gravitational analogy to the motion of a free particle in ordinary classical mechanics.} \]
Since (103) already appears in first order form, we can directly read off from the canonical structure the following elementary Poisson brackets

\[ \{X^f(x,t), \Pi_g(y,t)\} = G\delta^f_g \delta^{(3)}(x,y). \] (123)

The momentum conjugate to \( N \) is given by

\[ P_N = \frac{\delta L}{\delta \dot{N}} = 0, \] (124)

which leads to the primary constraint \( \Pi_N = 0 \). Conservation of this constraint under time evolution leads to the secondary constraint

\[ \dot{\Pi}_N = -\frac{\delta L}{\delta N} = a_0^{3/2} e^{T/2} U \sqrt{\Pi_1 \Pi_2 \Pi_3} \Phi = 0. \] (125)

We must now check for preservation of (125). The smeared constraint is given by

\[ H[N] = \int_\Sigma d^3x N a_0^{3/2} e^{T/2} U \sqrt{\Pi_1 \Pi_2 \Pi_3} \left( \frac{1}{\Pi_1} + \frac{1}{\Pi_2} + \frac{1}{\Pi_3} \right). \] (126)

The functional derivatives of (126) are of the form

\[ \frac{\delta H[N]}{\delta \Pi_f} = N \left( q_f H + q \left( \frac{1}{\Pi_f} \right)^2 \right) \] (127)

where \( q \) and \( q_f \) are functions on phase space, whose specific forms are not important for what follows. The variational derivative with respect to \( X^f \) is of the form

\[ \frac{\delta H[N]}{\delta X^f} = Q_f N \Phi + Q_{fi} \partial_i (QN \Phi) \] (128)

for some \( Q, Q_f \) and \( Q_{fi} \), which are phase space functions.

We will now compute the algebra of the constraint \( H \) using Poisson brackets

\[ \{H[M], H[N]\} = \int_\Sigma d^3x \left( \frac{\delta H[M]}{\delta X^f} \frac{\delta H[N]}{\delta \Pi_f} - \frac{\delta H[N]}{\delta X^f} \frac{\delta H[M]}{\delta \Pi_f} \right) \]

\[ = \int_\Sigma d^3x M \left( q_f \Phi + q \left( \frac{1}{\Pi_f} \right)^2 \right) \left( Q_f N \Phi + Q_{fi} \partial_i (QN \Phi) \right) - N \leftrightarrow M. \] (129)
All terms which are proportional to $\Phi$ vanish on-shell on account of (107), so we need only consider terms of the form

$$
\int_{\Sigma} d^3x M q \left( \frac{1}{\Pi_f} \right)^2 Q_{fi} (QN \Phi) - N \leftrightarrow M.
$$

(130)

Integrating by parts and discarding boundary terms, one sees that the only nontrivial contributions to (130) are due to the spatial gradients acting on the smearing functions $M$ and $N$. This yields

$$
\int_{\Sigma} d^3x Q \left( \frac{1}{\Pi_f} \right)^2 Q_{fi} (M \partial_i N - N \partial_i M) \Phi.
$$

(131)

The result is that

$$
\{H[M], H[N]\} = \{H[Q^i (M \partial_i N - N \partial_i M)]\},
$$

(132)

where $Q^i = Q^i (X^f, \Pi_f)$ are phase space dependent structure functions. The Poisson bracket of two Hamiltonian constraints $H$ on the phase space $\Omega_0 = (X^f, \Pi_f)$ is proportional to a Hamiltonian constraint. Therefore $H$ is first class and there are no second class constraints. Since we started with a phase space of $2 \times 3 = 6$ degrees of freedom, the degrees of freedom per point subsequent to implementation of the Hamiltonian constraint are

$$
D.O.F. = 2 \times 3 - 2 \times 1 = 4.
$$

(133)

With four phase space degrees of freedom per point, this shows that the reduced dual theory is not a topological field theory. This was also the case in the unreduced dual theory.

### 6.3 The spacetime metric

The spacetime metric in the dual theory to the Ashtekar theory is not a fundamental object and must be derived. The fundamental objects are $X^f$, or alternatively the corresponding connection components which are given by exponentiation of (121)

$$
a_f(x, t) = a_0 \left( (\text{det}(x, 0)/a_0^3)^{-1/2} + \frac{\eta(x)}{2} \int_0^t N(x, t') U(x, t'; T) dt' \right)^{-2\eta_f/\eta}.
$$

(134)

Equation (134) provides the explicit time variation for the diagonal connection in the reduced full theory. Taking the product over $i = 1, 2, 3$ one finds
that for $t = 0$ the condition $\det a = \det a(x, 0)$ is satisfied, which can be chosen arbitrarily on the initial spatial hypersurface $\Sigma_0$. One must then choose the lapse function $N(x, t)$ to specify the manner in which the boundary data becomes evolved for $t > 0$. The solutions are labelled by the conjugate momenta $\Pi_f$ as encoded in $\eta_f/\eta$. Equation (134) can also be written as

$$a_f(x, t) = \left(\frac{\det a(x, t)}{\det a(x, 0)}\right)^{\eta_f/\eta} = a_0 e^{(\eta_f/\eta)t},$$

whence the variables evolve with respect to $\det a$, seen as a time variable on configuration space. We will illustrate the construction of the metric for a simple example where the spatial gradients are zero. Recall in the original Ashtekar variables that the contravariant 3-metric $h^{ij}$ is given by

$$hh^{ij} = \tilde{\sigma}_i^{\alpha} \tilde{\sigma}_j^{\alpha} \rightarrow h^{ij} = (\det \tilde{\sigma})^{-1} \tilde{\sigma}_i^{\alpha} \tilde{\sigma}_j^{\alpha}. \quad (136)$$

The covariant form in the variables of the dual theory is given by

$$h_{ij} = (\det \Psi)^{-1} \Psi^{-1}_{ae} (B^{-1})^e_i (B^{-1})^f_j (\det B). \quad (137)$$

Restricted to the subspace of diagonal connection variables, which in the dual theory admit the proper canonical relation to the densitized eigenvalues of the CDJ matrix $\lambda_f$, this is given by

$$h_{ij} = (\lambda_1 \lambda_2 \lambda_3) \begin{pmatrix} (a_1/\lambda_1)^2 & 0 & 0 \\ 0 & (a_2/\lambda_2)^2 & 0 \\ 0 & 0 & (a_3/\lambda_3)^2 \end{pmatrix}$$

which upon the substitution $\lambda_i = \Pi_i (\det a)^{-1}$ yields

$$h_{ij} = \delta_{ij} (\Pi_1 \Pi_2 \Pi_3) \left(\frac{a_i^3}{\det a}\right) \left(\frac{a_j}{\Pi_j}\right)^2 \quad (138)$$

with $a_j$ given by (134). For simplicity consider the case where the variables are independent of spatial position and depend only on time. Then $\Pi_i$ are numerical constants, $a_i(x, t) = a_i(t)$, and moreover $U = 1$. As a special case, take $a_i(x, 0) = a_0$, and take $N(x, t) = 2$, namely a constant lapse. Then the metric evolves in time via

$$ds^2 = dt^2 + \delta_{ij} \left(\frac{\Pi_1 \Pi_2 \Pi_3}{\Pi_j^2}\right) (1 + \eta t)^{2(1-\eta/\eta)} dx^i dx^j, \quad (139)$$

which has the same form as the Kasner solution, with a re-definition of variables. One may compute the initial volume of the universe

31
Vol(Σ₀) = \int_Σ d^3x\sqrt{\gamma} = \int_Σ d^3x\sqrt{h} = l^3\bigg(\frac{\Pi_1 \Pi_2 \Pi_3}{\text{det}a(0)}\bigg) = l^3\bigg(\text{det}a(0)\bigg)^{-1}\bigg(\frac{(\Pi_1 \Pi_2)^2}{\Pi_1 + \Pi_2}\bigg) \quad (140)

at t = 0, where l is a characteristic length scale of the universe from integration over minisuperspace. Note that this volume is labelled by two arbitrary constants Π₁ and Π₂ which determine the algebraic classification of the spacetime, as well as deta(0). This provides a physical interpretation for deta in terms of metric variables. A more in-depth analysis of minisuperspace, as well as a generalization of the above procedure to the full theory, is reserved for a separate paper.
7 Quantization and Hilbert space structure for vanishing cosmological constant

We now proceed to the quantum theory on the kinematic phase space. We have already eliminated the Gauss’ law and diffeomorphism constraints, leaving behind a Dirac consistent phase space which admits a canonical formulation and classical dynamics. This implies that we may proceed to the quantum theory by promoting the dynamical variables to quantum operators $X^f \rightarrow \hat{X}^f$ and $\Pi_f \rightarrow \hat{\Pi}_f$, and Possion brackets (123) to commutators

$$\left[\hat{X}^f(x,t),\hat{\Pi}_g(y,t)\right] = (\hbar G)\delta^f_g \delta^{(3)}(x,y). \quad (141)$$

The operators in the functional Schrödinger representation act respectively by multiplication and by functional differentiation of a wave functional

$$\hat{X}^f(x,t)\psi = X^f(x,t)\psi; \quad \hat{\Pi}_f(x,t)\psi = (\hbar G)\frac{\delta}{\delta X^f(x,t)}\psi. \quad (142)$$

Note that the following wavefunctionals are eigenstates of $\hat{\Pi}_f$

$$\psi_{\lambda}[X] = \exp\left[(hG)^{-1}\int_{\Sigma} d^3x \bar{\lambda}_f(x)X^f(x,t)\right], \quad (143)$$

where $\bar{\lambda}_f(x)$ are arbitrary continuous functions of position, which do not contain any functional dependence on $X^f(x,t)$. We will see that these play the role of labels for the state. The following action ensues for the momentum operator

$$\hat{\Pi}_f(x,t)\psi_{\lambda}[X] = \bar{\lambda}(x)\psi_{\lambda}[X]. \quad (144)$$

We will now search for states $\psi \in Ker\{\hat{H}\}$. But prior to quantization let us put the smeared constraint into polynomial form

$$H[N] = \int_{\Sigma} d^3x Na_0^{3/2} e^{T/2}U(\Pi_1\Pi_2\Pi_3)^{-1/2}(\Pi_1\Pi_2 + \Pi_2\Pi_3 + \Pi_3\Pi_1). \quad (145)$$

To obtain a nontrivial solution it suffices for the operator in brackets in (145) upon quantization to annihilate the state for each $x$. Hence

$$(\hat{\Pi}_1(x)\hat{\Pi}_2(x) + \hat{\Pi}_2(x)\hat{\Pi}_3(x) + \hat{\Pi}_3(x)\hat{\Pi}_1(x))\psi_{\lambda}[X] = 0 \quad \forall x$$

$$\rightarrow \left(\bar{\lambda}_1(x)\bar{\lambda}_2(x) + \bar{\lambda}_2(x)\bar{\lambda}_3(x) + \bar{\lambda}_3(x)\bar{\lambda}_1(x)\right)\psi_{\lambda}[X] = 0 \quad \forall x. \quad (146)$$
This leads to the dispersion relation

$$\tilde{\lambda}_3 = -\left(\frac{\tilde{\lambda}_1 \tilde{\lambda}_2}{\lambda_1 + \lambda_2}\right) \forall x. \quad (147)$$

Conventionally in quantum field theory, when there are products of momenta evaluated at the same point a regularization procedure is needed to obtain a well-defined action on states. However, there exist states for which the action of (128), is already well-defined without the need for regularization, namely plane wave-type states annihilated by $\hat{\Phi}$. These are states for which the momenta are functionally independent of the configuration variables and act as labels. The solution is given by\(^{20}\)

$$\psi_{\lambda_1, \lambda_2}[X(x)] = \exp\left[ (hG)^{-1} \sum_f \tilde{\lambda}_f(x) X^f(x) \right] \bigg|_{\lambda_3 = -\lambda_1 \lambda_2 / (\lambda_1 + \lambda_2)} \quad (148)$$

for each $x \in \Sigma$. Hence $|\lambda\rangle = |\lambda_1, \lambda_2\rangle \in \text{Ker}\{\hat{\Phi}\}$ defines a Hilbert space of states annihilated by the Hamiltonian constraint, labelled by $\lambda_1$ and $\lambda_2$, once the measure of normalization has been defined. The full Hilbert space consists of a direct product of the Hilbert spaces $\forall x \in \Sigma$, since (147) must be satisfied independently at each point $x$. If one regards each spatial hypersurface $\Sigma$ as a lattice of finite lattice spacing $x_{n+1} - x_n = \Delta x$, then

$$H = \bigotimes_{x_n} H(x_n) \longrightarrow \psi_{\lambda_1, \lambda_2} \sim \prod_{x_n} \psi_{\lambda_1, \lambda_2}(x_n). \quad (149)$$

In the continuum limit $\Delta x \to 0$, the product in (149) goes to a Riemannian integral

$$\psi_{\lambda_1, \lambda_2}[X] = \exp\left[ (hG)^{-1} \int_\Sigma d^3x \left( \tilde{\lambda}_1 X^1 + \tilde{\lambda}_2 X^2 - \left(\frac{\tilde{\lambda}_1 \tilde{\lambda}_2}{\lambda_1 + \lambda_2}\right) X^3 \right) \right]. \quad (150)$$

Equation (150) solves the quantum Hamiltonian constraint by construction. The momentum labels $(\lambda_1, \lambda_2)$ correspond to two functions of spatial position $x \in \Sigma$, which is consistent with the classical solution and also with the Hamilton–Jacobi functional (102).

---

\(^{20}\)We use the tilde notation to distinguish $\tilde{\lambda}_f$, the eigenvalue of $\hat{\Pi}_f$ on $\psi$, from the (undensitized) eigenvalues $\lambda_f$ of $\Psi_{(ae)}$. Since $\Pi_f = \lambda_f (\text{det} a)$ at the classical level, then $\tilde{\lambda}_f$ can be seen as a 'densitized' version of $\lambda_f$. We do not include the tilde in the specification of the state $|\lambda_1, \lambda_2\rangle$, since it would be redundant owing to the invariance of $\Phi$ under rescaling of $\lambda_f$ for $\Lambda = 0$. 

34
7.1 Measure on the Hilbert space

To formalize the Hilbert space structure we need square integrable wavefunctions for solutions to the constraints, which requires the specification of a measure for normalization. If all variables were real, as for spacetimes of Euclidean signature, one would be able to use delta-functional normalizable wavefunctions.

\[ D_{\text{Eucl}}(X) = \prod_x \delta X^1(x) \delta X^2(x) \delta X^3(x). \] (151)

In (151) \( X^f \) is real and on the replacement \( \lambda_f \rightarrow i\lambda_f \), we have

\[
\langle \psi_{\lambda} | \psi_{\zeta} \rangle_{\text{Eucl}} = D_{\mu_{\text{Eucl}}} \exp \left[ -i(hG)^{-1} \int_{\Sigma} d^3x \tilde{\lambda} f(x) X^f(x) \right] \\
\exp \left[ i(hG)^{-1} \int_{\Sigma} d^3x \tilde{\zeta} f(x) X^f(x) \right] = \prod_x \prod_f \delta(\tilde{\lambda}_f(x) - \tilde{\zeta}_f(x)),
\] (152)

or that two states are orthogonal unless their CDJ matrix eigenvalues are identical at each point \( x \in \Sigma \). This can be written more compactly as

\[
\langle \psi_{\lambda} | \psi_{\zeta} \rangle_{\text{Eucl}} = \int_{\Gamma} D_{\mu_{\text{Eucl}}} \exp \left[ -\nu(hG)^{-1} \tilde{X} \cdot X + i(hG)^{-1} \tilde{\zeta} \cdot X \right] = \delta_{\lambda \zeta}.
\] (153)

For spacetimes of Lorentzian signature, the variables are in general complex and a Euclidean measure does not produce normalizable wavefunctions. One may then rather use a Gaussian measure to ensure square integrability for the basis wavefunctions in this case. This Gaussian measure is given by

\[
D_{\mu_{\text{Lor}}} = \bigotimes_{x} \nu^{-1} \delta_X e^{-\nu^{-1} \tilde{X} \cdot X} \\
= \prod_x \delta X^f \exp \left[ -\nu^{-1} \int_{\Sigma} d^3x \tilde{X} f(x) X^f(x) \right],
\] (154)

where \( \nu \) is a numerical constant with mass dimensions \([\nu] = -3\), needed to make the argument of the exponential dimensionless. The inner product of two un-normalized states is now given by

\[
\langle \lambda | \zeta \rangle_{\text{Lor}} = \prod_{x,i} \int_{\Gamma} \nu(0) \delta X^f \exp \left[ -\nu^{-1} \int_{\Sigma} d^3x \tilde{X} f(x) X^f(x) \right] \\
\exp \left[ (hG)^{-1} \int_{\Sigma} d^3x \tilde{\lambda}_f(x) \tilde{\zeta}_f(x) \right] = \exp \left[ \nu(hG)^{-2} \int_{\Sigma} d^3x \tilde{X} f(x) \tilde{X} f(x) \right].
\] (155)
A necessary condition for the wavefunction to be normalizable, as for the inner product to exist, is that the functions \( \tilde{\lambda}_i(x) \) and \( \tilde{\zeta}_i(x) \) be square integrable. In shorthand notation, (155) can be written as

\[
\langle \lambda | \zeta \rangle_{\text{Lor}} = \int \mathcal{D} \mu_{\text{Lor}}(\mathbf{X}, X) e^{(\hbar G)^{-1} \tilde{\lambda}^* \cdot \mathbf{X}} e^{(\hbar G)^{-1} \tilde{\zeta} \cdot X} = e^{\nu(hG)^{-2} \tilde{\lambda} \cdot \tilde{\lambda}}. \tag{156}
\]

Note how the balance of the mass dimensions is ensured in spite of the existence of infinite dimensional spaces. \(^{21}\) The norm of a state is given by

\[
\langle \lambda | \lambda \rangle_{\text{Lor}} = \int \mathcal{D} \mu_{\text{Lor}}(\xi, \xi) e^{(\hbar G)^{-1} \tilde{\lambda}^* \cdot \xi} e^{(\hbar G)^{-1} \tilde{\lambda} \cdot \xi} = e^{\nu(hG)^{-2} \tilde{\lambda} \cdot \tilde{\lambda}}, \tag{157}
\]

and we define the normalized wavefunction by

\[
| \Psi_\lambda \rangle = e^{-\nu(hG)^{-2} \tilde{\lambda} \cdot \tilde{\lambda}} | \lambda \rangle. \tag{158}
\]

The overlap of two states in the Lorentzian measure is given by

\[
| \langle \psi_\lambda | \psi_\zeta \rangle_{\text{Lor}} | = \exp \left[ -\nu(hG)^{-2} \int_{\Sigma} d^3 x |\tilde{\lambda}_i(x) - \tilde{\zeta}_i(x)|^2 \right]. \tag{159}
\]

where

\[
\tilde{\lambda}_3 = -\left( \frac{\tilde{\lambda}_1 \tilde{\lambda}_2}{\lambda_1 + \lambda_2} \right); \quad \tilde{\zeta}_3 = -\left( \frac{\tilde{\zeta}_1 \tilde{\zeta}_2}{\zeta_1 + \zeta_2} \right). \tag{160}
\]

There is always a nontrivial overlap between any two states corresponding to different functions for the eigenvalues. \(^{22}\)

### 7.2 Expectation values and observables

The expectation value of the configuration variable \( X^f \) is given by

\[
\langle \psi_\lambda | \hat{X}^f(x) | \psi_\zeta \rangle_{\text{Lor}} = \prod_{x,i} \int_{\Gamma} d^3 \mathbf{X} f(x) X^f(x) \exp \left[ -\nu^{-1} \int_{\Sigma} d^3 x \mathbf{X} f(x) X^f(x) \right] \exp \left[ (hG)^{-1} \int_{\Sigma} d^3 x \tilde{\lambda}_i(x) \mathbf{X} f(x) (X^f(x) \exp \left[ (hG)^{-1} \int_{\Sigma} d^3 x \tilde{\zeta}_i(x) X^f(x) \right] \right].
\tag{161}
\]

\(^{21}\)The dimensionful constant \( \nu \) remains a parameter of the theory. One may think that such a measure cannot exist on infinite dimensional spaces unless \( \nu = 1 \) with \( [\nu] = 0 \). But we have rescaled the measure by the same factor of \( \nu^{(0)} \) to cancel out these factors arising from the Gaussian integral.

\(^{22}\)It is shown in [9] that the eigenvalues of \( \Psi_{ae} \) encode the Petrov classification of spacetime, since \( \Psi_{ae} \) is the antiself-dual part of the Weyl curvature tensor. This classification is independent of coordinates and of tetrad frames.

36
By replacing multiplication by $X^f$ with functional differentiation with respect to $\tilde{\zeta}_f$, one may simplify the matrix element to

$$\langle \psi_\lambda | \hat{X}^f(x) | \psi_\zeta \rangle_{Lor} = \prod_{x,i} \nu^{(0)} \delta_{\zeta_i(x)} \exp \left[ -\nu^{-1} \int_\Sigma d^3 x \bar{X}_f(x) X^f(x) \right]$$

$$\exp \left[ (hG)^{-1} \int_\Sigma d^3 x \bar{\lambda}_f^* (x) \bar{X}_f (x) \right] \left( \exp \left[ hG^{-1} \int_\Sigma d^3 x \tilde{\zeta}_f(x) X^f(x) \right] \right) \tag{162}$$

whereupon commuting the functional derivative outside the integral we obtain

$$\langle \psi_\lambda | \hat{X}^f(x) | \psi_\zeta \rangle_{Lor} = \frac{\delta}{\delta \tilde{\zeta}_f(x)} \left( \exp \left[ \nu(hG)^{-2} \int_\Sigma d^3 x \bar{\lambda}_f^* (x) \tilde{\zeta}_f(x) \right] \right)$$

$$\exp \left[ \nu(hG)^{-2} \bar{\lambda}_f^* (x) \exp \left[ \nu(hG)^{-2} \int_\Sigma d^3 x \bar{\lambda}_f^* (x) \tilde{\zeta}_f(x) \right] \right]$$

$$= (\nu(hG)^{-2} \bar{\lambda}_f^* (x)) \langle \psi_\lambda | \psi_\zeta \rangle_{Lor}. \tag{163}$$

Going through a similar analysis for various operators, one obtains

$$\langle \psi_\lambda | \tilde{\Pi}_f(x) | \psi_\zeta \rangle_{Lor} = \langle \psi_\lambda | (hG)^{-2} \tilde{\zeta}_f(x) \rangle \langle \psi_\lambda | \psi_\zeta \rangle_{Lor} \tag{164}$$

$$\langle \psi_\lambda | \hat{\Pi}_f(x) | \psi_\zeta \rangle_{Lor} = \langle \psi_\lambda | (hG)^{-2} \tilde{\lambda}_f^* (x) \rangle | \psi_\zeta \rangle_{Lor} = \tilde{\zeta}_f(x) \langle \psi_\lambda | \psi_\zeta \rangle_{Lor} \tag{165}$$

as well as

$$\langle \psi_\lambda | \frac{\delta}{\delta \tilde{X}_f(x)} | \psi_\zeta \rangle_{Lor} = \langle \psi_\lambda | (hG)^{-1} \bar{\lambda}_f^* (x) \rangle | \psi_\zeta \rangle_{Lor}. \tag{166}$$

Hence, with respect to the Lorentzian measure one has, schematically,

$$\frac{\delta}{\delta \tilde{X}_f} \sim hG \nu^{-1} \tilde{X}_f; \quad \frac{\delta}{\delta \bar{X}_f} \sim hG \nu^{-1} X^f. \tag{167}$$

This property of the infinite generalization of a Bargmann-like representation, combined with generating functional techniques, enables an explicit calculation of the matrix element of any observable $O$

$$\langle \psi_\lambda | \hat{O} [\tilde{X}_f; \bar{\lambda}_f] | \psi_\zeta \rangle_{Lor} = O[\nu(hG)^2 \bar{\lambda}_f^*; \nu \bar{\lambda}_f] \langle \psi_\lambda | \psi_\zeta \rangle_{Lor}. \tag{168}$$

Hence, the existence of a function $O$ signifies the existence the expectation value or matrix element corresponding to $O$. 

37
8 Canonical equivalence to the Ashtekar variables

We have provided a direct map from the full phase space $\Omega_{Dual} = (\Psi_{ae}, A^a_i)$ of the dual theory to the nondegenerate sector of the full phase space of the Ashtekar variables $\Omega_{Ash} = (\tilde{\sigma}_a^i, A^a_i)$ via the CDJ Ansatz $\tilde{\sigma}_a^i = \Psi_{ae} B^i_e$. By implementation of the Gauss’ law and diffeomorphism constraints we have reduced the dual theory to $\Omega_0$, its kinematic phase space where we have computed the Hamiltonian dynamics. Subsequently, we have performed a quantization of $\Omega_0$, obtaining a Hilbert space of states solving the quantum Hamiltonian constraint for $\Lambda = 0$. In this section we will demonstrate canonical equivalence of the dual theory to the Ashtekar theory via various routes, which should imply that the results obtained in the dual theory extend to certain regimes of the Ashtekar theory. Note that both theories at the unconstrained level share in common the Ashtekar connection $A^a_i$ as the configuration space variable. In what follows we will exploit the preservation of this property at all levels of reduction sequence.

8.1 Map from $\Omega_0$ to the reduced phase space of the Ashtekar variables

First we will provide the map from $\Omega_0$ to the Ashtekar theory. Since the dimension of the kinematic phase space is $\text{Dim}(\Omega_0) = 6$ per point, then these degrees of freedom must map to six corresponding D.O.F. of the Ashtekar variables. We choose these D.O.F. on $\Omega_A$, the reduced Ashtekar phase space, as the diagonal degrees of freedom. Starting from the commutation relations on $\Omega_0$ (using units where $G = 1$)

$$[X^f(x,t), \Pi^g(y,t)] = \delta^f_g \delta^{(3)}(x,t)$$

(169)

with vanishing relations

$$[X^f(x,t), X^g(y,t)] = [\Pi^f(x,t), \Pi^g(y,t)] = 0,$$

(170)

perform the following change of variables

$$X^1 = \ln\left(\frac{A^1_0}{a_0}\right); \quad X^2 = \ln\left(\frac{A^2_0}{a_0}\right); \quad X^3 = \ln\left(\frac{A^3_0}{a_0}\right),$$

(171)

where $a_0$ is a numerical constant with $[a_0] = 1$. Then (169) is given by

$$\left[\ln\left(\frac{A^f_0(x,t)}{a_0}\right), \Pi^g(y,t)\right] = (A^{-1})^f_y(x,t) \left[A^f_0(x,t), \Pi^g(y,t)\right] = \delta^f_y \delta^{(3)}(x,y).$$

(172)
where we restrict to nonvanishing $A_f^f$. Multiplying (172) by $A_f^f$ we have the following equivalent relations

$$[A_f^f(x,t), \Pi_g(y,t)] = \delta_g^f A_f^f(x,t) \delta^{(3)}(x,y) = \delta_g^f A_g^g(y,t) \delta^{(3)}(x,y). \quad (173)$$

In (173) we have used that the only nontrivial contribution comes from $f = g$ and $x = y$. We can now transfer $A_g^g$ to the left hand side in the form

$$[A_f^f(x,t), \Pi_g(y,t)(A^{-1})_g^g(y,t)] = \delta_g^f \delta^{(3)}(x,y). \quad (174)$$

Equation (174) is justified by application of the Liebniz rule to the commutator, using $[A_f^f, A_g^g] = 0$, to obtain (173). Equation (174) will constitute the starting point for mapping $\Omega_0$ into $\Omega_A$.

Starting from the full Ashtekar phase space $\Omega_{Ash} = (A_a^i, \tilde{\sigma}_a^i)$, where $A_a^i$ is the self-dual Ashtekar connection and $\tilde{\sigma}_a^i$ is the densitized triad, satisfying commutation relations

$$[A_a^i(x,t), \tilde{\sigma}_b^j(y,t)] = \delta_a^b \delta^i_j \delta^{(3)}(x,y), \quad (175)$$

let us perform the substitution

$$\tilde{\sigma}_a^i = \Psi_{ae} B_e^i, \quad (176)$$

where $\Psi_{ae} \in SO(3,C) \otimes SO(3,C)$ is in unreduced form. We will now decompose $\Psi_{ae}$ into its symmetric and its antisymmetric parts

$$\tilde{\sigma}_a^i = \Psi_{(ae)} B_e^i + \Psi_{[ae]} B_e^i. \quad (177)$$

When diagonalizable, the symmetric part of $\Psi_{ae}$ can be written as the $SO(3,C)$ rotation of its eigenvalues in a polar decomposition, which brings (176) into the form

$$\tilde{\sigma}_a^i = (e^{\theta - T})_{ae} \Lambda_g (e^{-\theta - T})_{ge} B_e^i + \epsilon_{aed} B_e^i \psi_d'. \quad (178)$$

We have parametrized the antisymmetric part of $\Psi_{ae}$ as a $SO(3,C)$-valued 3-vector $\psi_d'$. Performing a $SO(3,C)$ rotation of both sides of (178), we have

---

23The concept of multiplication of canonical commutation relations (C.C.R.) by canonical variables is not new. This yields to affine commutation relations, which have also been used by Klauder in [26]. In the present paper, the affine commutation relations (173) are an intermediate stage in the re-establishment of the link from the C.C.R. (169) of the dual theory back into to Ashtekar variables, whose commutation relations are canonical.
\[(e^{-\theta \cdot T})_{ga} \tilde{\sigma}^i_a = \lambda_g (e^{-\theta \cdot T})_{ge} B^i_e + (e^{-\theta \cdot T})_{ga} \epsilon_{aed} B^i_e \psi'_d. \tag{179}\]

Let us now redefine the Ashtekar variables, adapted to the intrinsic \(SO(3, C)\) frame, defined as the frame where \(\Psi_{(ae)}\) is diagonal. Hence we have

\[
\tilde{P}^i_a \equiv (e^{-\theta \cdot T})_{ga} \tilde{\sigma}^i_a; \quad (e^{-\theta \cdot T})_{ge} B^i_e \equiv b^i_e; \quad \psi_d = (e^{-\theta \cdot T})_{df} \psi'_f. \tag{180}\]

Then the following relation holds

\[
(e^{-\theta \cdot T})_{ga} \epsilon_{aed} B^i_e \psi_d = (e^{-\theta \cdot T})_{ga} (e^{-\theta \cdot T})_{fe} (e^{-\theta \cdot T})_{hd} \epsilon_{aed} b^j_f \psi_h = \epsilon_{gfh} b^j_f \psi_h. \tag{181}\]

In (181) we have used the special orthogonal property that \(\det(e^{-\theta \cdot T}) = 1\). Hence substituting (180) and (181) into (179), we have that

\[
\tilde{P}^i_g = \lambda_g b^i_g + \epsilon_{gfh} b^j_f \psi_h. \tag{182}\]

The result is that we have expressed (176) with respect to the intrinsic \(SO(3, C)\) frame. On the diagonal subspace of the Ashtekar variables, the commutation relations (175) should be given by

\[
[A^f_j(x, t), \tilde{P}^g_k(y, t)] = \delta^f_g \delta^{(3)}(x, y). \tag{183}\]

We will now perform the following map from \(\Omega_0\)

\[
A^a_i = \delta^a_i \ A^a_i; \quad \lambda_f = (A^1_1 A^2_2 A^3_3)^{-1} \Pi_f \tag{184}\]

for \(f = 1, 2, 3\). Then the diagonal part of (182) is given by

\[
\tilde{P}^a_g = \lambda_g b^a_g + \epsilon_{gfh} b^j_f \psi_h. \tag{185}\]

Recalling the definition of the magnetic field (for \(\det a \neq 0\))

\[
b^i_g = \epsilon^{ijk} \delta_j a^g_k + (\det a^{-1})^i_g \tag{186}\]

one sees that for diagonal connections \(A^a_i = \delta^a_i A^a_i\), the diagonal terms \(b^a_g\) do not contain any spatial gradients. Take the 1 component of (185) without loss of generality, where the remaining components follow by cyclic permutation of indices.

40
\[ \tilde{P}_1^1 = \lambda_1 b_1^1 + b_1^2 \psi_3 - b_1^3 \psi_2 = \lambda_1 (\partial_2 A_3^1 - \partial_3 A_2^1 + A_2^2 A_3^2 - A_3^2 A_2^3) + \psi_3 (\partial_2 A_3^2 - \partial_3 A_2^2 + A_2^3 A_3^3 - A_3^3 A_2^3) - \psi_2 (\partial_2 A_3^3 - \partial_3 A_2^3 + A_2^1 A_3^1 - A_3^1 A_2^2) \] (187)

The phase space \( \Omega_0 = (\lambda_f, \Pi_f) \) already has a cotangent bundle structure, with 6 degrees of freedom per point. The configuration space \( \Gamma_0 \) maps directly to the three diagonal components of the Ashtekar connection, which in turn can be canonically conjugate only to the three diagonal components of the densitized triad in the intrinsic \( SO(3, C) \) frame. Hence we will set the off-diagonal components of \( A_a^i \) in (187) to zero, yielding

\[ \tilde{P}_1^1 = \lambda_1 A_2^2 A_3^3 - \psi_3 (\partial_3 A_2^2) - \psi_2 (\partial_2 A_3^3). \] (188)

Equation (188) violates the commutation relations (183) even on the diagonal subspace, due to the spatial gradient terms. To avoid this contradiction a necessary and sufficient condition that (188) is that \( \psi_2 = \psi_3 = 0 \). Setting \( \psi_d = 0 \) for \( d = 1, 2, 3 \), we can now write (188) as

\[ \tilde{P}_1^1 \bigg|_{\tilde{\psi}=0} = \lambda_1 A_2^2 A_3^3 = \Pi_1 \left( \frac{1}{A_1^1} \right). \] (189)

Hence we have that

\[ [A_f^f(x, t), \tilde{P}_g^g(y, t)] = \delta_f^g \delta^{(3)}(x, y) = [A_f^f(x, t), \Pi_g(y, t)(A^{-1})_g^g(y, t)] \] (190)

which is the same as (174). The result is that \( \Omega_0 \), the reduced phase space of the dual theory, is canonically equivalent to \( \Omega_{Ash} \) restricted to the diagonal subspace.25

### 8.2 Map from the full Ashtekar variables to the kinematic phase space of the dual theory

We will now prove, using the unconstrained Ashtekar theory as a starting point, that the map to \( \Omega_0 \) requires as a necessary and sufficient condition the implementation of the kinematic initial value constraints. The canonical commutation relations for the Ashtekar variables are given by

\[ \psi_d = 0 \] is precisely the condition that the diffeomorphism constraint be satisfied. Hence we have shown that this is consistent with the configuration space reduction to diagonal \( A_a^i \).

25Since the corresponding unreduced theories are also equivalent, then this implies that the dual theory could serve as a mechanism for obtaining the reduced phase space for Ashtekar’s gravity on the constraint shell.
\[ [A_i^a(x), \tilde{\sigma}_b^j(y)] = \delta^a_b \delta^j_i \delta^{(3)}(x, y), \tag{191} \]

where we have omitted the time dependence to avoid cluttering up the notation. Let us now substitute the CDJ Ansatz \( \tilde{\sigma}_a^j = \Psi_{ae}B_e^j \) into (191)

\[ [A_i^a(x), \Psi_{be}(y)B_e^j(y)] = \delta^a_b \delta^j_i \delta^{(3)}(x, y). \tag{192} \]

We will now multiply (192) by \( A_c^j(y) \) in the following form

\[ [A_i^a(x), \Psi_{be}(y)B_e^j(y)A_c^j(y)] = \delta^a_c \delta^j_i \delta^{(3)}(x, y), \tag{193} \]

which is allowed since \( [A_i^a, A_c^j] = 0 \) for the Ashtekar connection. Define the magnetic helicity density matrix \( C_{ce} = A_b^jB_e^j \), written in component form as

\[ C_{ce} = \epsilon_{ijk} A_c^i \partial_j A_e^k + \delta_{ce}(\det A), \tag{194} \]

which has a diagonal part free of spatial gradients and an off-diagonal part containing spatial gradients. Then the commutation relations read

\[ [A_i^a(x), \Psi_{be}(y)C_{ce}(y)] = \delta^a_b \delta^j_i \delta^{(3)}(x, y). \tag{195} \]

The kinematic configuration space \( \Gamma_{K\text{in}} \) must have three degrees of freedom per point.\(^\text{26}\) Let us choose, without loss of generality, for these D.O.F. to be the three diagonal elements \( A_i^a = \delta^a_i A_i^a \). Then we can set \( a = i \) in (195) to obtain

\[ [A_i^a(x), \Psi_{be}(y)C_{ce}(y)] = \delta^a_b \delta^j_i \delta^{(3)}(x, y). \tag{196} \]

Since \( A_i^a \) is diagonal by supposition, then the only nontrivial contribution to (196) occurs for \( a = c \). Since \( a = b \) also is the only nontrivial contribution, it follows that \( b = c \) as well. Hence the commutation relations for diagonal connection are given by

\[ [A_i^a(x), \Psi_{be}(y)C_{be}(y)] = \delta^a_b \delta^j_i \delta^{(3)}(x, y). \tag{197} \]

Substituting (194) subject to a diagonal connection into (197) we have

\(^\text{26}\)This is nine total degrees of freedom, minus three corresponding to \( G_a \), and minus three corresponding to \( H_i \).
\[
\sum_{e=1}^{3} \left[ A^a_e(x), \Psi_{be}(y) \delta_{be}(\det A) \right] \\
+ \sum_{e=1}^{3} \left[ A^a_e(x), \Psi_{be}(y) \epsilon^{bje} A^b_j A^c_e \right] = \delta^a_b A^b_c(y) \delta^{(3)}(x,y),
\]

which has split up into two terms. We have been explicit in putting in the summation symbol to indicate that \( e \) is a dummy index, while \( a \) and \( b \) are not. There are two cases to consider, \( e = b \) and \( e \neq b \). For \( e = b \) the first term of (198) vanishes, leaving remaining the second term. Since the right hand side stays the same, then this would correspond to the commutation relations for a CDJ matrix whose diagonal components are zero. For the second possibility \( e = b \) the second term of (198) vanishes while the first term survives, with the right hand side the same as before. This case occurs only if the CDJ matrix \( \Psi_{ae} \) is diagonal. Let us choose \( \Psi_{ae} = \text{Diag}(\lambda_1, \lambda_2, \lambda_3) \) as the diagonal matrix of eigenvalues, then (198) reduces to

\[
\left[ A^a_e(x), \lambda_b(y)(\det A(y)) \right] = \delta^a_b A^a_c(y) \delta^{(3)}(x,y).
\]

The conclusion is that in order for (199) to have arisen from (191), that: (i) The antisymmetric part of \( \Psi_{ae} \) must be zero, namely, the diffeomorphism constraint must be satisfied. (ii) The symmetric off-diagonal part of \( \Psi_{ae} \) is not part of the commutation relations on the diffeomorphism invariant phase space \( \Omega_{\text{diff}} \). Given the eigenvalues \( \lambda_f \) on this space, the Gauss’ law constraint can be solved separately from the quantization process. The choice of diagonal \( A^a_a \) is consistent with the implementation of the kinematic constraints, which means that only the Hamiltonian constraint is necessary to obtain the physical phase space \( \Omega_{\text{phys}} \).

Equation (199) are not canonical commutation relations owing to the field-dependence on the right hand side. However, they can be transformed into canonical commutation relations using the following change of variables \( A^a_a = a_0 e^{X^a} \) for \( a = 1, 2, 3 \). This yields

\[
\left[ e^{X^a(x)}, \lambda_b(y)(\det A(y)) \right] = e^{X^a(x)} \left[ X^a(x), \lambda_b(y)(\det A(y)) \right] = \delta^a_b e^{X^a(y)} \delta^{(3)}(x,y).
\]
Since the only nontrivial contribution to (200) comes from \( x = y \), we can cancel the pre-factor of \( e^{X^a} \) from both sides. Defining densitized eigenvalues \( \Pi_b = \lambda_b (\text{det} A) \) as the fundamental momentum space variables, we have that the canonical version of (199) is given by

\[
[X^a(x), \Pi_b(y)] = \delta^a_b \delta^{(3)}(x, y),
\]  

(201)

The coordinate ranges are \( \infty < |X^f| < \infty \), which corresponds to \( 0 < |A_{f}^a| < \infty \), which is a subset of the latter. To utilize the full range of \( A_{f}^a \), which includes the degenerate cases, one may instead use (199). We have shown that \( \Omega_{Kn} \) of the instanton representation admits a cotangent bundle structure with diagonal connection \( A_{f}^a(x) \). It happens from (191) that \( A_{f}^a(x) \) is canonically conjugate to \( \tilde{\sigma}_a^i(x) \). Since the instanton representation maps to the Ashtekar formalism and vice versa on the unreduced phase space for nondegenerate \( B_{a}^{i} \), it follows that (201) corresponds as well to the kinematic phase space of the Ashtekar variables for \( (\text{det} A) \neq 0 \), six phase space degrees of freedom per point, where the variables are diagonal. The bonus is that all the kinematic constraints have been implemented, leaving behind the Hamiltonian constraint which in the instanton representation is easy to solve.

We have shown that a nondegenerate and diagonal \( A_{f}^a \) admits globally holonomic coordinates in the reduced theory. Since \( A_{f}^a \) serves also as the configuration variable for the Ashtekar phase space \( \Omega_{Ash} \), it follows that on this subspace the densitized triad must also be nondegenerate. Hence

\[
[A_{f}^a(x, t), \tilde{\sigma}_g^i(y, t)] = \delta^a_f \delta^{(3)}(x, y).
\]  

(202)

The conclusion is that the kinematic phase space of the dual theory must correspond the reduced phase under \((G_a, H_i)\) of the Ashtekar theory, restricted to nondegenerate triads. Note in both phase spaces that the cotangent bundle structure has been preserved, and the two theories are equivalent when restricted to these configurations. The bonus is that we have now implemented the initial value constraints, computed the dynamics performed a quantization, and have constructed a Hilbert space using the dual theory.

### 8.3 Verification of the initial value constraints

We will now prove that the Hamiltonian constraint on \( \Omega_0 \) maps directly into the Hamiltonian constraint on \( \Omega_{Ash} \).\(^{29}\) The Hamiltonian constraint on the kinematic phase space of the dual theory is given by

\(^{29}\)Note that this is the full Hamiltonian constraint on the diffeomorphism constraint shell.
Note that there are spatial gradients of the diagonal variables in the magnetic field through the $U$ dependence of (203). Putting the components $a_i^a$ into a diagonal matrix

$$a_i^a = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} ; \quad b_i^b \equiv \varepsilon^{ijk} \partial_j a_k^a + \frac{1}{2} \varepsilon^{ijk} f^{abc} a_j^b a_k^c,$$

where $a = 1, 2, 3$ are internal indices and $i = 1, 2, 3$ are spatial with $a_f \neq 0$, and $b_i^a$ is the magnetic field for $a_i^a$, seen as a gauge connection (note that $a_i^a$ is not spatially constant and contains three degrees of freedom per point. Therefore we are dealing with the full theory and not minisuperspace. Hence (203) is given by

$$NH = NU(a_1 a_2 a_3)^{1/2} \sqrt{\Pi_1 \Pi_2 \Pi_3} \left( \frac{1}{\Pi_1} + \frac{1}{\Pi_2} + \frac{1}{\Pi_3} \right). \quad (203)$$

The notation in (204) signifies that it is restricted to diagonal connections $A_f^a = a_i^a$. Note that this restriction does not affect the momentum-dependent terms, which are the objects directly constrained by the Hamiltonian constraint. Substituting $\Pi_f = \lambda_f (a_1 a_2 a_3)$ from (184) into (204), we have

$$NH = N (\text{det} b)^{1/2} \left[ \text{Diag}(a) \right]^{-3/2} \sqrt{\lambda_1 \lambda_2 \lambda_3} \left( \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} \right). \quad (204)$$

Note that (205) is composed of the determinant and the trace of $\Psi_{(ae)}$, and is therefore $SO(3, C)$ invariant. So we can perform the following change of variables

$$\Psi_{(ae)} = (e^{\theta \cdot T})_a f \lambda_f (e^{-\theta \cdot T})_f e; \quad B_i^a = (e^{\theta \cdot T})_{ae} b_i^a \quad (206)$$

where $\theta$ can be chosen arbitrarily. The Hamiltonian constraint at this level, where the diffeomorphism constraint has already been implemented, is transparent to the D.O.F. in $\theta$. Using the cyclic property of the trace, (205) is the same as
\[
N H = N \sqrt{\det B} \sqrt{\det \Psi \text{tr} \Psi^{-1}} \bigg|_{\text{Sym}(\Psi) ; \text{diag}(a)}.
\]  

(207)

The notation in (207) signifies that \( \Psi_{ae} \) is restricted to symmetric matrices and that \( B^i_a \) is restricted to those magnetic field obtainable from the diagonal \( a^i_a \) by \( SO(3, C) \) transformation. So we can define a connection \( A^a_i \) corresponding to \( B^i_a = B^i_a[A] \), which is the gauge transformed version of \( a^i_a \) into the \( SO(3, C) \) frame \( \bar{\Theta} \)

\[
A^a_i \equiv (e^{-\bar{\Theta}T})_{ae} a^e_i + \frac{1}{2} \epsilon^{abc} (e^{-\bar{\Theta}T})_{bf} \partial_i (e^{-\bar{\Theta}T})_{cf}.
\]  

(208)

Note at the level of (206) that both \( \Psi_{ae} \) and \( A^a_i \) contain six degrees of freedom per point. Three of those D.O.F. are due to the angles \( \bar{\Theta} \), which are unphysical. These three D.O.F. can be eliminated by imposition of the Gauss’ law constraint, reducing us back to the kinematic phase space \( \Omega_9 \). The restrictions on \( \Psi_{ae} \) and \( A^a_i \) can be lifted to bring us from 6 to 9 D.O.F. each by appending to \( \Psi_{ae} \) an antisymmetric part

\[
\Psi_{ae} = (e^{-\bar{\Theta}T})_{af} \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} (e^{-\bar{\Theta}T})_{ge} + \epsilon_{aed} \psi^d \\
\end{pmatrix}
\]

and replacing \( a^i_a \) by an arbitrary symmetric matrix

\[
a^i_a = \begin{pmatrix} a_{11} & a_{12} & a_{31} \\ a_{12} & a_{22} & a_{23} \\ a_{31} & a_{23} & a_{33} \end{pmatrix}; \quad A^a_i \equiv (e^{-\bar{\Theta}T})_{ae} a^e_i + \frac{1}{2} \epsilon^{abc} (e^{-\bar{\Theta}T})_{bf} \partial_i (e^{-\bar{\Theta}T})_{cf}.
\]

This is a polar decomposition of \( \Psi_{ae} \), where \( \psi^d \) is a \( SO(3, C) \) 3-vector parametrizing the antisymmetric part, as well as a polar decomposition of \( A^a_i \), which is the gauge transformation of \( a^i_a \). Having increased the phase space to 18 D.O.F., we must now impose constrain it in order to obtain the kinematic phase space. This can be accomplished by requiring that the antisymmetric part of \( \Psi_{ae} \) vanish using a constraint

\[
H_i[N^i] = \int_{\Sigma} d^3 x \epsilon_{ijk} N^i B^j_a B^k_e \Psi_{ae} = 0.
\]  

(209)

The to obtain the reduced phase space under just diffeomorphisms, one in conjunction with (209) must remember to set \( a_{12} = a_{23} = a_{31} = 0 \) and then one is left with the Gauss’ law and Hamiltonian constraints. The Gauss’ law constraint is given by
\[ w_e \{ \Psi_{ae} \} = 0, \] 

which is used to eliminate the degrees of freedom in \( \vec{\theta} \). This simultaneously reduces \( \Psi_{ae} \) and \( A_i^a \) by another three D.O.F. to obtain the kinematic phase space.

We can now go back and rephrase the above in the language of the Ashtekar variables. Using the CDJ Ansatz (177) subject to a diagonal CDJ matrix and a diagonal baseline connection, then (207) becomes

\[
NH = N \sqrt{\det(\tilde{\sigma}^{-1})} B_i^a = \frac{N}{\sqrt{\det(\tilde{\sigma})}}(\det(\tilde{\sigma}))^{1/2} B_i^a. \tag{211}
\]

Using the properties of the determinant of nondegenerate three by three matrices, we have that

\[
HN = \frac{1}{2} \epsilon_{ijk} \epsilon^{abc} \tilde{\sigma}_a \tilde{\sigma}_b \tilde{\sigma}_c B^k_i. \tag{212}
\]

where we have defined the ‘densitized’ lapse density function \( \sqrt{\det(\tilde{\sigma})} = N(\det(\tilde{\sigma}))^{-1/2} \).

Equation (212) is the Hamiltonian constraint in the Ashtekar variables, restricted to the set of connections obtainable from the diagonal connections by \( SO(3, C) \) rotation, and restricted to the set of symmetric CDJ matrices \( \Psi_{ae} \). We will now prove that this corresponds to the diffeomorphism invariant phase space.

The diffeomorphism constraint in the Ashtekar variables is given by

\[
H_i = \epsilon_{ijk} \tilde{\sigma}_j B^k_i. \tag{213}
\]

Now substitute (177) into (213). Due to antisymmetry, the symmetric part of \( \Psi_{ae} \) drops out and we are left with

\[
H_i = \epsilon_{ijk} N^j B^k_d \epsilon_{aed} \Psi_{ae} = 0. \tag{214}
\]

which arises from varying \( N^i \) in (209). The final result is that there is a well-defined map between the kinematic phase space of the dual theory and the reduced phase space of the Ashtekar variables, as well as from the former to the full unconstrained theory both in the dual and in the Ashtekar case. Moreover, the kinematic phase space is Dirac consistent, admits a quantization and classical dynamics.
8.4 Map to the unreduced dual theory

To obtain the unreduced dual theory from its kinematic version (100) it suffices to augment the variables, appending terms corresponding to the Gauss’ law and diffeomorphism constraints. The canonical one form on \( \Omega_0 \) is given by

\[
\theta_0 = \int_{\Sigma} d^3 x \Pi_f (x) \delta X^f (x) = \int_{\Sigma} d^3 x \left( \lambda_1 a_2 a_3 \delta a_1 + \lambda_2 a_3 a_1 \delta a_2 + \lambda_3 a_1 a_2 \delta a_3 \right). \tag{215}
\]

To see where (215) could potentially have originated from, consider the canonical one form on the unreduced phase space \( \Omega_{\text{Dual}} \)

\[
\theta_{\text{Inst}} = \int_{\Sigma} d^3 x \Psi_{ae} B^i_e \delta A^a_i = \int_{\Sigma} d^3 x \tilde{\sigma}^i_a \delta A^a_i. \tag{216}
\]

Perform the decomposition (178) on the left hand side of (216), yielding an integrand

\[
\Psi_{ae} B^i_e \delta A^a_i = \lambda_f ((e^{-\theta \cdot T}) f _e B^i_e)((e^{-\theta \cdot T}) f _a \delta A^a_i) + \epsilon_{dae} \psi_d B^i_e \delta A^a_i. \tag{217}
\]

Next, write (217) in the intrinsic \( SO(3, C) \) frame

\[
\lambda_f b^j _i \delta a^f _i + \epsilon_{dae} \psi_d b^i_e \delta a^a_i. \tag{218}
\]

Using \( \lambda_f = (a_1 a_2 a_3)^{-1} \Pi_f \), we have

\[
\Pi_f (a_1 a_2 a_3)^{-1} \left( \epsilon^{ijk} (\delta a^f _i) \partial_j a^f _k + (\text{det} a)(a^{-1})^j_f \delta a^f _i \right) + \epsilon_{dae} \psi_d b^i_e \delta a^a_i. \tag{219}
\]

Application of the diffeomorphism constraint implies \( \psi_d = 0 \), which makes the second term of (219) vanish. Let us now expand the part of the term in brackets involving spatial gradients. Taking the \( f = 1 \) component without loss of generality, this is given by

\[
(\delta a^1 _i (\partial_2 a^1 _3 - \partial_3 a^1 _2) + \delta a^2 _i (\partial_3 a^1 _1 - \partial_1 a^1 _3) + \delta a^3 _i (\partial_1 a^1 _2 - \partial_2 a^1 _3)). \tag{220}
\]

Note that for diagonal \( a^a_i \), (220) vanishes. The result is that restricted to a diagonal connection in the intrinsic diffeomorphism invariant \( SO(3, C) \) frame, (216) reduces to (215) which in turn admits a quantization and a Hilbert space structure. All that remains is to extend the Hamiltonian constraint (207) to include and antisymmetric \( \Psi_{ae} \), which is actually what we started from in (45). The result is thus a map between the full Ashtekar
variables and its corresponding reduced phase space via the dual theory. The unreduced dual theory is given by

\[ I_{\text{Dual}} = \int dt \int_{\Sigma} d^3x \left( \Psi_{ae} B_i^a \dot{A}_i^a - A_0^a \omega_e \{ \Psi_{ae} \} ight) - \epsilon_{ijk} N^i B_a^j D_e^k \Psi_{ae} - iN(\det B)^{1/2} \sqrt{\det \Psi} (\Lambda + \text{tr} \Psi^{-1}), \] (221)

which as we have shown follows from the Pleanski action. Using (73) and the symmetric properties of the four dimensional epsilon tensor this can be written as

\[ I_{\text{Dual}} = \int_M d^4x \left( \frac{1}{8} \Psi_{ae} F^{a}_{\mu \nu} F^{e}_{\rho \sigma} \epsilon^{\mu \nu \rho \sigma} + (B_i^a \dot{A}_i^a - \epsilon_{ijk} N^i B_a^j D_e^k) \Psi_{ae} - \sqrt{-g}(\Lambda + \text{tr} \Psi^{-1}) \right), \] (222)

where we have absorbed the Gauss’ law constraint into the definition of the covariant curvature.
9 Summary

The this paper we have shown the following things. From the starting Plebanski action which implies the Einstein equations, there are two theories that can result. There is the Ashtekar theory based on the phase space $\Omega_{Ash} = (\tilde{\sigma}_i^a, A_i^a)$ and there is a dual theory based on the phase space $(\Psi_{ae}, A_i^a)$. The momentum space variables for both theories originated at the Plebanski level as auxiliary fields. We have shown that the dual theory is consistent in the Dirac sense since its constraints algebra closes. Next we performed a reduction to the kinematical phase space of the reduced theory by implementing the Gauss’ law and the diffeomorphism constraints. Since the initial value constraints in the reduced theory constrain only the momentum space, we were free to choose diagonal configuration space variables canonically conjugate to the eigenvalues of $\Psi_{ae}$. Next, we demonstrated that the dual theory implies the Einstein field equations provided that the initial value constraints are satisfied. Implementation of the kinematic parts of these constraints led us to the kinematic phase space where we computed the corresponding Lagrangian and Hamiltonian dynamics of the theory. One result is that we were able to construct a Hamilton–Jacobi functional on the spatial boundary of spacetime by holographic projection of the physical degrees of freedom. Additionally, we verified the Dirac consistency of the constraints algebra even after projection to this kinematic phase space.

We then performed a quantization of the kinematic phase space, constructing a Hilbert space of normalizable constraints annihilated by the quantum Hamiltonian constraint. The states are labelled by two eigenvalues of $\Psi_{ae}$, and have the same form implied by the Hamilton–Jacobi functional. Lastly, we clarified the relationship between of the canonical structure of the reduced dual theory to its counterpart in the Ashtekar variables, as well as its relation to the unreduced theories. According to our analysis the dual theory provides a direct route from the full Ashtekar theory to a reduced phase space on the nondegenerate sector via implementation of the initial value constraints.
10 Appendix A: Expansion of the determinant on diagonal configurations

It is convenient to factor out the leading order behaviour of the determinant of the connection from the Ashtekar magnetic field as

\[(\det B) = (U \det A)^2, \tag{223}\]

where \(U\) will be determined. The Ashtekar magnetic field is given by

\[B_a^i = \epsilon^{ijk} \partial_j A^a_k + \frac{1}{2} \epsilon^{ijk} f_{abc} A^b_j A^c_k \equiv f_a^i + (\det A)(A^{-1})^i_a. \tag{224}\]

In (224), \(f_a^i = \epsilon^{ijk} \partial_j A^a_k\) refers to the ‘abelian’ part and the second term is a correction due to nonabeliantity. We have used the fact that the \(SU(2)\) structure constants \(f_{abc} = \epsilon_{abc}\) are numerically the same as the Cartesian epsilon symbol in order to write the determinant, which also assumes that \(A^a_i\) is nondegenerate. Putting (224) into the expansion of the determinant, we have

\[
\begin{align*}
\det B &= \frac{1}{6} \epsilon_{ijk} \epsilon^{abc} (f_a^i + (\det A)(A^{-1})^i_a) (f_b^j + (\det A)(A^{-1})^j_b) (f_c^k + (\det A)(A^{-1})^k_c) \\
&= \det f + (\det A)^2 + \frac{1}{2} \epsilon_{ijk} \epsilon^{abc} [f_a^i f_b^j (A^{-1})^k_c (\det A) + f_a^k A^a_i (\det A)^{-1}] \tag{225}
\end{align*}
\]

On diagonal connections the second term in (230) in square brackets vanishes, since

\[
A^a_i f_a^i = \epsilon^{ijk} A^a_i \partial_j A^a_k = \epsilon^{ijk} (\delta^a_i a_i) \partial_j (\delta^a_k a_k) = \epsilon^{aj} a_a \partial_j a_a = 0 \tag{226}
\]

on account of the antisymmetry of the epsilon symbol. We must now expand the first term in square brackets, evaluated on diagonal connections. Hence we have

\[
\begin{align*}
\frac{1}{2} \epsilon^{ijk} \epsilon^{abc} f_a^i f_b^j (A^{-1})^k_c (\det A) &= \frac{1}{4} \epsilon^{ijk} \epsilon^{klm} \epsilon_{cd} \epsilon^{e} f_a^i f_b^j A^d_l A^e_m \\
&= \frac{1}{4} (\delta^l_i \delta^m_j - \delta^l_j \delta^m_i) (\delta^a_d \delta^b_e - \delta^a_e \delta^b_d) f_a^i f_b^j A^d_l A^e_m \\
&= \frac{1}{4} (f^l_a f^m_b - f^m_a f^l_b) (A^a_i A^b_m - A^a_m A^b_i) \\
&= \frac{1}{2} ((f^l_a A^a_i)^2 - f^m_a A^a_i f^l_b A^b_m). \tag{227}
\end{align*}
\]
The first term on the right hand side of (227) vanishes on diagonal connections as proven in (226). The second term is given by

$$f^l_a A^a_m f^m_b A^b_l = \epsilon^{ij} \partial_i (\delta_{aj} a_a) (\delta_{ac} a_c) \epsilon^{b'i'} \partial_{b'} (\delta_{b'j} a_{b'}) (\delta_{b'c} a_{b'}) = \epsilon^{ija} \epsilon^{b'i'} a_i \partial_{b'} a_{b'} - \frac{1}{4} \epsilon^{iab} \epsilon^{ja'b'} (\partial_{a_i} a_{a_i}^2) (\partial_{b'} a_{b'}^2)$$

(228)

where we have relabelled indices $i' \rightarrow j$ on the last term. The only nontrivial contribution to (228) occurs for $i = j$, which yields

$$r = -\frac{1}{8} \sum_{i=1}^{3} I_{iab}(\partial_i a_a^2)(\partial_i a_b^2).$$

(229)

The determinant of the Ashtekar magnetic field for a diagonal connection, which constitutes the kinematic configuration space, is given by

$$(\det B) = (A_1^a A_2^a A_3^a)^2 + (\partial_2 A_1^a)(\partial_3 A_1^a)(\partial_2 A_3^a) - (\partial_3 A_2^a)(\partial_1 A_2^a)(\partial_3 A_1^a)
+ (A_2^a A_3^a)(\partial_1 A_2^a)(\partial_1 A_3^a) + (A_3^a A_1^a)(\partial_2 A_3^a)(\partial_2 A_1^a) + (A_1^a A_2^a)(\partial_3 A_1^a)(\partial_3 A_2^a)
= a_0^6 e^{2T} \left[ 1 + a_0^{-3} e^{-T} \left( (\partial_2 X^3)(\partial_1 X^1)(\partial_1 X^2) - (\partial_3 X^2)(\partial_1 X^3)(\partial_3 X^1) \right) + a_0^{-2} \left( e^{-2X^1}(\partial_1 X^2)(\partial_1 X^3) + e^{-2X^2}(\partial_2 X^3)(\partial_2 X^1) + e^{-2X^3}(\partial_3 X^1)(\partial_3 X^2) \right) \right] \equiv a_0^6 e^{2T} U^2$$

(230)

where we have defined $T = X^1 + X^2 + X^3$. The end result in the full theory is that

$$\det B = (\det a)^2 + r(\partial a),$$

(231)

where we have defined

$$r = (\det f)^2 - \frac{1}{8} \sum_{i=1}^{3} I_{iab}(\partial_i a_a^2)(\partial_i a_b^2).$$

(232)

This fixes the definition of $U$ as

$$U = \sqrt{1 + r(\det A)^{-2}}.$$
References

[1] Ahbay Ashtekar. ‘New perspectives in canonical gravity’, (Bibliopolis, Napoli, 1988).

[2] Ahbay Ashtekar ‘New Hamiltonian formulation of general relativity’ Phys. Rev. D36(1987)1587

[3] Ahbay Ashtekar ‘New variables for classical and quantum gravity’ Phys. Rev. Lett. Volume 57, number 18 (1986)

[4] Riccardo Capovilla, John Dell, Ted Jacobson ‘Self-dual 2-forms and gravity’ Class. Quantum Grav. 8 (1991) 41-57

[5] Richard Capovilla, Ted Jacobson, John Dell ‘General Relativity without the Metric’ Class. Quant. Grav. Vol 63, Number 21 (1989) 2325-2328

[6] Capovilla, Dell and Jacobson ‘A pure spin-connection formulation of gravity’ Class. Quantum Grav. 8 (1991)59-73

[7] Jerzy Plebanski ‘On the separation of Einsteinian substructures’ J. Math. Phys. Vol. 18, No. 2 (1977)

[8] Eyo Eyo Ita III ‘Instanton representation of Plebanski gravity III. Classical constraints algebra’ arXiv: gr-qc/0704.0367

[9] Eyo Eyo Ita III ‘Instanton representation of Plebanski gravity XIII. Canonical structure of the Petrov classification of nondegenerate space-times’ arXiv: gr-qc/0805.1892

[10] E. Buffenoir, M. Henneaux, K. Noir and Ph. Roche ‘Hamiltonian Analysis of Plebanski theory’ Class. Quantum Grav. 21 (2004) 5203-5220

[11] S. Alexandrov, E. Buffenoir and Ph. Roche ‘Plebanski Theory and Covariant Canonical Formulation’ Class. Quantum Grav. 24: 2809-2824, 2007

[12] Micheal Reisenberger ‘New constraints for canonical general relativity’ Nucl. Phys. B457 (1995) 643-687

[13] Richard Capovilla, John Dell, Ted Jacobson and Lionel Mason ‘Self-dual 2-forms and gravity’ Class. Quant. Grav. 8(1991)41-57

[14] Ingemar Bengtsson ‘Note on non-metric gravity.’ arXiV:gr-qc/0703114

[15] R. Jackiw ‘(Constrained) Quantization Without Tears’ Abstract: No publication information available
[16] L. Faddeev and R. Jackiw ‘Hamiltonian Reduction of Unconstrained and Constrained systems’ Phys. Rev. Lett. Number 17 (1988) 1692-1694

[17] Werner M. Seiler ‘Involution and constrained dynamics: II. The Faddeev–Jackiw approach’ J. Phys. A: Math. Gen. 28 (1995) 7315-7331

[18] Paul Dirac ‘Lectures on quantum mechanics’ Yeshiva University Press, New York, 1964

[19] Asher Peres ‘Diagonalization of the Weyl tensor’ Phys. Rev. D18, Number 2 (1978)

[20] Michael Creutz, I.J. Muzinich, and Thomas N. Tudron ‘Gauge fixing and canonical quantization’ Phys. Rev. D, Vol. 19 Number 2, 531-539 (1979)

[21] Eyo Eyo Ita III ‘Instanton representation of Plebanski gravity IV. Induced geometric structures’ arXiv: gr-qc/0705.0927

[22] Eyo Eyo Ita III ‘Instanton representation of Plebanski gravity VII. Initial value and Gauss’ law constraints in rectangular form’ arXiv: gr-qc/0706.2699

[23] Eyo Eyo Ita III ‘Instanton representation of Plebanski gravity VIII. Initial value and Gauss’ law constraints in polar form’ arXiv: gr-qc/0706.2702

[24] Eyo Eyo Ita III ‘Instanton representation of Plebanski gravity XVIII. Hamiltonian and Hamilton–Jacobi dynamics on superspace’ arXiv: gr-qc/0805.3760

[25] Eyo Eyo Ita III ‘Instanton representation of Plebanski gravity XVIII. Quantization and proposed resolution of the Kodama state’ arXiv: gr-qc/0806.3959

[26] J. R. Klauder ‘The affine quantum gravity programme’ Class. Quantum Grav. 19, 817-826 (2002)