Risk-averse estimation, an axiomatic approach to inference, and Wallace-Freeman without MML

Michael Brand
Faculty of IT (Clayton), Monash University, Clayton, VIC 3800, Australia.
E-mail: michael.brand@monash.edu

Summary. We define a new class of Bayesian point estimators, which we refer to as risk averse. Using this definition, we formulate axioms that provide natural requirements for inference, e.g. in a scientific setting, and show that for two classes of estimation problems the axioms uniquely characterise an estimator. Namely, for estimation problems with a discrete hypothesis space, the axioms characterise Maximum A Posteriori (MAP) estimation, whereas for well-behaved continuous estimation problems they characterise the Wallace-Freeman estimator (WF). Elsewhere, risk-averse estimators are not unique, and describe a new class of estimation methods.

Our results provide a novel justification for WF, which previously was derived only as an approximation to the information-theoretic Strict Minimum Message Length estimator. By contrast, our derivation requires neither approximations nor coding.

Keywords: axiomatic approach, Bayes estimation, inference, MML, risk-averse, Wallace-Freeman

1. Introduction

One of the fundamental statistical problems is point estimation. In a Bayesian setting, this can be described as follows. Let \((x, \theta) \in X \times \Theta\) be a pair of random variables with a known joint distribution that assigns positive probability / probability density to any \((x, \theta) \in X \times \Theta\). Here, \(x\) is known as the observation, \(X \subseteq \mathbb{R}^N\) as observation space, \(\theta\) as the parameter and \(\Theta \subseteq \mathbb{R}^M\) as parameter space. We aim to describe a function \(\hat{\theta} : X \rightarrow \mathbb{R}^M\) such that \(\hat{\theta}(x)\) is our “best guess” for \(\theta\) given \(x = x\).

Such a problem appears frequently for example in scientific inference, where we aim to decide on a theory that best fits the known set of experimental results.

The optimal choice of a “best guess” \(\hat{\theta}(x)\) naturally depends on our definition of “best”. The most common Bayesian approach regarding this is that used by Bayes estimators (Berger, 2013), which define “best” explicitly, by means of a loss function. This allows estimators to optimally trade off different types of errors, based on their projected costs.

In this paper, we examine the situation where errors of all forms are extremely costly and should therefore be minimised and if possible avoided, rather than factored in. The scientific scenario, where one aims to decide on a single theory, rather than a convenient trade-off between multiple hypotheses, is an example. We define this scenario rigorously under the name risk-averse estimation.
We show that for discrete \( \theta \) inference problems, i.e. problems where \( \Theta \) is countable, the assumption of risk-averse estimation is enough to uniquely characterise Maximum A Posteriori (MAP),

\[
\hat{\theta}_{\text{MAP}}(x) \overset{\text{def}}{=} \arg\max_{\theta} P(\theta = x = x) = \arg\max_{\theta} P(\theta|x)
\]

Risk-averse estimation does not suffice alone, however, to uniquely characterise a solution for continuous problems, i.e. problems where the joint distribution of \( (x, \theta) \) can be described by a probability density function \( f = f(x, \theta) \). To do so, we introduce three additional axioms, two of which relate to invariance to representation and the last to invariance to irrelevant alternatives, which reflect natural requirements for a good inference procedure, all of which are also met by MAP.

(Notably, the estimator maximising the posterior probability density \( f(\theta|x) \), which in the literature is usually also named MAP, does not satisfy invariance to representation. To avoid confusion, we refer to it as \( f\text{-MAP} \).

We prove regarding our risk-aversion assumption and three additional axioms that together (and only together) they do uniquely characterise a single estimation function in the continuous case, namely the Wallace-Freeman estimator (WF) (Wallace and Freeman, 1987),

\[
\hat{\theta}_{\text{WF}}(x) \overset{\text{def}}{=} \arg\max_{\theta} \frac{f(\theta|x)}{\sqrt{|I_{\theta}|}},
\]

where \( I_{\theta} \) is the Fisher information matrix (Lehmann and Casella, 2006), whose \((i, j)\) element is the conditional expectation

\[
I_{\theta}(i, j) \overset{\text{def}}{=} E \left( \left( \frac{\partial \log f(x|\theta)}{\partial \theta(i)} \right) \left( \frac{\partial \log f(x|\theta)}{\partial \theta(j)} \right) \bigg| \theta = \theta \right).
\]

While this proves that in a wide variety of scenarios the axioms uniquely characterise either MAP or Wallace-Freeman estimation, for completeness we demonstrate that for other notable cases, such as for problems with a continuous \( \theta \) but a discrete \( x \), the axioms do not provide a unique solution, but rather define a new class of estimation functions.

The fact that our axioms uniquely characterise the Wallace-Freeman estimator is in itself of interest, because this estimator exists almost exclusively as part of Minimum Message Length (MML) theory (Wallace, 2005), and even there is defined merely as a computationally convenient approximation to Strict MML (SMML) (Wallace and Boulton, 1975), which MML theory considers to be the optimal estimator, for information-theoretical reasons.

Importantly, because SMML is computationally intractable in all but the simplest cases (Farr and Wallace, 2002), it is generally not used directly, and MML practitioners are encouraged instead to approximate it by MAP in the discrete case and by WF in the continuous (See, e.g., Comley and Dowe, 2005, p. 268). Thus, MML’s standard practice coincides with what risk-averse estimation advocates in those cases where our characterisation is unique. However, in the case of risk-averse
estimation, neither MAP nor WF is an approximation. Rather, they are both optimal estimators in their own rights (within their respective domains), and the justifications given for them are purely Bayesian and involve no coding theory.

Thus, risk-averse estimation provides a new theoretical foundation, unrelated to MML, that explains the empirical success of the MML recipe, for which recent examples include Sumanaweera et al. (2018); Schmidt and Makalic (2016); Saikrishna et al. (2016); Jin and Rumantir (2015); Kasarapu et al. (2014).

2. Background

2.1. Bayes estimation

The most commonly used class of Bayesian estimators is Bayes estimators. A Bayes estimator, $\hat{\theta}_L$, is defined over a loss function,

$$L = L(x, \theta) : \Theta \times \Theta \rightarrow \mathbb{R}_{\geq 0},$$

where $L(\theta_1, \theta_2)$ represents the cost of choosing $\theta_2$ when the true value of $\theta$ is $\theta_1$. The estimator chooses an estimate that minimises the expected loss given the observation, $x$:

$$\hat{\theta}_L(x) \overset{\text{def}}{=} \arg\min_{\theta \in \Theta} \mathbb{E}(L(\theta, \theta) | x = x).$$

We denote by $f_{\theta} = f(x, \theta)$ the likelihood function $f_{\theta}(x, \theta) = f(x, \theta)(x | \theta)$, and assume for all estimation problems and loss functions

$$L(\theta_1, \theta_2) = 0 \Leftrightarrow \theta_1 = \theta_2 \Leftrightarrow f_{\theta_1} = f_{\theta_2}.$$ (1)

We say that $L$ is discriminative for an estimation problem $(x, \theta)$ if for every $\theta \in \Theta$ and every neighbourhood $B$ of $\theta$, the infima over $\theta' \in \Theta \setminus B$ of both $L(\theta, \theta')$ and $L(\theta', \theta)$ are positive.

Notably, Bayes estimators are invariant to a linear monotone increasing transform in $L$. They may also be defined over a gain function, $G$, where $G$ is the result of a monotone decreasing affine transform on a loss function.

Examples of Bayes estimators are posterior expectation, which minimises quadratic loss, and MAP, which minimises loss over the discrete metric. In general, Bayes estimators such as posterior expectation may return a $\hat{\theta}$ value that is not in $\Theta$. This demonstrates how their trade-off of errors may make them unsuitable for a high-stakes “risk-averse” scenario.

2.2. Set-valued estimators

Before defining risk-averse estimation, we must make a note regarding set-valued estimators.

Typically, estimators are considered as functions from the observation space to (extended) parameter space, $\theta : X \rightarrow \mathbb{R}^M$. However, all standard point estimators are defined by means of an argmin or an argmax. Such functions intrinsically allow the result to be a subset of $\mathbb{R}^M$, rather than an element of $\mathbb{R}^M$. 

---

*Risk-averse estimation*
We say that an estimator is a well-defined point estimator for \((x, \theta)\) if it returns a single-element set for every \(x \in X\), in which case we take this element to be its estimate. Otherwise, we say it is a set estimator. The set estimator, in turn, is well-defined on \((x, \theta)\) if it does not return an empty set as its estimate for any \(x \in X\).

All estimators discussed will therefore be taken to be set estimators, and the use of point-estimator notation should be considered solely as notational convenience.

We also define set limit and use the notation

\[
\lim_{k \to \infty} B_k,
\]

where \((B_k)_{k \in \mathbb{N}}\) is a sequence of sets with an eventually bounded union (i.e., there exists a \(k\), such that \(\bigcup_{i \geq k} B_i\) is bounded), to mean the set \(\Omega\) of elements \(\omega\) for which there exists a monotone increasing sequence of naturals \(k_1, k_2, \ldots\) and a sequence \(\omega_1, \omega_2, \ldots\), such that for each \(i\), \(\omega_i \in B_{k_i}\) and \(\lim_{i \to \infty} \omega_i = \omega\).

3. Risk-averse estimation

The idea behind MAP is to maximise the posterior probability that the estimated value is the correct \(\theta\) value. In the continuous domain this cannot hold verbatim, because all \(\theta \in \Theta\) have probability zero. Instead, we translate the notion into the continuous domain by maximising the probability that the estimated value is essentially the correct value. The way to do this is as follows.

**Definition 1.** A continuously differentiable, monotone decreasing function, \(A: \mathbb{R}^\geq 0 \to \mathbb{R}^\geq 0\), satisfying

\[(a)\; A(0) > 0,\]
\[(b)\; \exists a_0 \forall a \geq a_0, A(a_0) = 0,\]

will be called an attenuation function, and the minimal \(a_0\) will be called its threshold value.

**Definition 2.** Let \(L\) be the loss function of a Bayes estimator and \(A\) an attenuation function.

We define a risk-averse estimator over \(L\) and \(A\) to be the estimator satisfying

\[
\hat{\theta}_{L,A}(x) = \lim_{k \to \infty} \hat{\theta}_k(x),
\]

where \(\hat{\theta}_k\) is the Bayes estimator whose gain function is

\[
G_k(\theta_1, \theta_2) = A(kL(\theta_1, \theta_2)).
\]

By convention we will assume \(A(0) = 1\), noting that this value can be set by applying a positive multiple to the gain function, which does not affect the definition of the estimator.
The rationale behind this definition is that we use a loss function, $L$, to determine how similar or different $\theta_2$ is to $\theta_1$, and then use an attenuation function, $A$, to translate this divergence into a gain function, where a 1 indicates an exact match and a 0 that $\theta_2$ is not materially similar to $\theta_1$. (Such a gain function is often referred to as a similarity measure.) The parameter $k$ is then used to contract the neighbourhood of partial similarity, to the point that anything that is not “essentially identical” to $\theta_1$ according to the loss function is considered a 0. Note that this is done without distorting the loss function, as $k$ merely introduces a linear multiplication over it, a transformation that preserves not only the closeness ordering of pairs but also the Bayes estimator defined on the scaled function.

In this way, the risk-averse estimator maximises the probability that $\theta_2$ is essentially identical to $\theta_1$, while preserving our notion, codified in $L$, of how various $\theta$ values interrelate.

4. Discrete $\theta$ problems

We begin by characterising risk-averse estimation in the discrete $\theta$ case, i.e. where $\Theta$ is countable.

**Theorem 1.** Any risk-averse estimator, $\hat{\theta}_{L,A}$, regardless of its loss function $L$ or its attenuation function $A$, satisfies for any $x$ in any estimation problem $(x, \theta)$ with discrete $\theta$ that

$$\hat{\theta}_{L,A}(x) \subseteq \hat{\theta}_{\text{MAP}}(x)$$

and is a nonempty set, provided $L$ is discriminative for the estimation problem. In particular, $\hat{\theta}_{L,A}$ is in all such cases a well-defined set estimator, and where $\text{MAP}$ is a well-defined point estimator, so is $\hat{\theta}_{L,A}$, and

$$\hat{\theta}_{L,A} = \hat{\theta}_{\text{MAP}}.$$

**Proof.** The risk-averse estimator for a discrete $\theta$ problem is defined by

$$\hat{\theta}_{L,A}(x) = \lim_{k \to \infty} \arg\max_{\theta} \sum_{\theta' \in \Theta} P(\theta'|x) A(kL(\theta', \theta)). \quad (2)$$

Fix $x$, and let $V_k(\theta) = \sum_{\theta' \in \Theta} P(\theta'|x) A(kL(\theta', \theta))$.

Let $N_k(\theta) = \{ \theta' \in \Theta : A(kL(\theta', \theta)) > 0 \}$. The value of $V_k(\theta)$ is bounded from both sides by

$$P(\theta|x) \leq V_k(\theta) \leq P(\theta \in N_k(\theta)|x) = \sum_{\theta' \in N_k(\theta)} P(\theta'|x). \quad (3)$$

Because $\sum_{\theta' \in \Theta} P(\theta'|x)$ is finite (in fact, it equals 1) and because for each element $\theta' \neq \theta$ in the sum there is a $k$ value from which it is excluded from the sum, as $k$ goes to infinity both bounds converge to $P(\theta|x)$. So, this is the limit for $V_k(\theta)$. Also, $V_k(\theta)$ is a monotone decreasing function of $k$. 
The above proves that
\[ \arg\max_{\theta} \lim_{k \to \infty} \sum_{\theta' \in \Theta} P(\theta'|x) A(kL(\theta', \theta)) \]
is the MAP solution. To show that it is also the limit of the argmax (i.e., when switching back to the order of the quantifiers in (2)), we need to show certain uniformity properties on the speed of convergence, which is what the remainder of this proof is devoted to.

Define an enumeration \((\theta_i)_i\) over the values in \(\Theta\), where the \(\theta_i\) values are sorted by descending \(P(\theta_i|x)\). (Such an enumeration is not necessarily unique.) If \(\Theta\) is countably infinite, the values of \(i\) range in \(N\). Otherwise, it is a finite enumeration, with \(i\) in \(\{1, \ldots, |\Theta|\}\).

We now partition \(\Theta\) into three subsets \(S, T\) and \(U\), each holding values of \(\theta\) that are consecutive under the enumeration, defined as follows.

Let \(S\) be the set \(\{\theta_i: 1 \leq i \leq n\}\) for which \(P(\theta|x)\) attains its maximum value, \(v_0\).

Let \(v_1 = P(\theta_{n+1}|x)\).

Let \(U\) be a set \(\{\theta_i: i > m\}\), where \(m \geq n\) is such that
\[ P(\theta \in U|x) < v_0 - v_1. \] (4)

Again, the choice of \(m\) is not unique. However, such an \(m\) always exists because \(\sum_{\theta' \in \Theta} P(\theta'|x) = 1\), so for a large enough \(m\), \(\sum_{\theta' \in U} P(\theta'|x)\) is arbitrarily close to zero.

Define \(T = \Theta \setminus (S \cup U)\).

Importantly, sets \(S\) and \(T\) are necessarily finite, regardless of whether \(|\Theta|\) is finite or not.

Let \(\{B_i\}_{i=1}^{m}\) be a set of neighbourhoods of \(\{\theta_i\}_{i=1}^{m}\), respectively, such that no two neighbourhoods intersect. Because this set of \(\theta\) values is finite, there is a minimum distance between any two \(\theta\) and therefore such neighbourhoods exist.

For each \(i \in 1, \ldots, m\), let \(\delta_i = \inf_{\theta' \in \Theta \setminus B_i} L(\theta_i, \theta')\).

Because \(L\) is discriminative, all \(\delta_i\) are positive. Because this is a finite set, \(\delta_{\min} = \min_i \delta_i\) is also positive.

Consider now values of \(k\) which are larger than \(a_0/\delta_{\min}\), where \(a_0\) is the attenuation function’s threshold value.

Because we chose all \(B_i\) to be without intersection, any \(\theta \in \Theta\) can be in at most one \(B_i\). For \(k\) values as described, only \(\theta\) values in \(B_i\) can have \(\theta_i \in N_k(\theta)\). In particular, each \(N_k(\theta)\) can contain at most one of \(\theta_1, \ldots, \theta_m\).

By (4), any such neighbourhood that does not contain one of \(\theta_1, \ldots, \theta_n\), i.e. set \(S\), the MAP solutions, has a \(V_k(\theta)\) value lower than \(v_0\), the lower bound for \(V_k(\theta_1)\) given in (3). This is because the total value from all elements in \(U\) can contribute, by construction, less than \(v_0 - v_1\), whereas the one element from \(T\) that may be in the same neighbourhood can contribute no more than \(v_1\).

Therefore, any \(\hat{\theta}_k \in \arg\max_{\theta} V_k(\theta)\) must have an \(N_k(\hat{\theta}_k)\) containing exactly one of \(\theta_1, \ldots, \theta_n\).
Let us partition any sequence \( (\hat{\theta}_k)_{k \in \mathbb{N}} \) of such elements \( \hat{\theta}_k \) according to the element of \( S \) contained in \( N_k(\hat{\theta}_k) \), discarding any subsequence that is finite.

Consider now only the subsequence \( k_1, k_2, \ldots \) such that \( \theta_i \in N_{k_j}(\hat{\theta}_{k_j}) \) for some fixed \( i \leq n \).

By the same logic as before, because \( L \) is discriminative, for any neighbourhood \( B \) of \( \theta_i \) \( \inf_{\theta' \in \Theta \setminus B} L(\theta_i, \theta') > 0 \), and therefore there is a \( K \) value such that for all \( k_j \geq K \), if \( \theta_i \in N_{k_j}(\theta') \) then \( \theta' \in B \).

We conclude, therefore, that \( \hat{\theta}_{k_j} \in B \) for all sufficiently large \( j \). By definition, the \( \hat{\theta}_{k_j} \) sequence therefore converges to \( \theta_i \), and the set limit of the entire sequence is the subset of the MAP solution, \( \{\theta_1, \ldots, \theta_n\} \), for which such infinite subsequences \( k_1, k_2, \ldots \) exist.

Because the entire sequence is infinite, at least one of the subsequences will be infinite, hence the risk-averse solution is never the empty set.

5. The axioms

We now describe additional good properties satisfied by the MAP estimator which make it suitable for scenarios such as scientific inference. These natural desiderata will form axioms of inference, which we will then investigate outside the discrete setting.

Our interest is in investigating inference and estimation in situations where all errors are highly costly, and hence we begin with an implicit “Axiom 0” that all estimators investigated are risk averse.

Our remaining axioms are not regarding the estimators themselves, but rather regarding what constitutes a reasonable loss function for such estimators. We maintain that these axioms can be applied equally in all situations in which loss functions are used, such as with Bayes estimators.

In all three axioms, our requirement is that the loss function \( L \) satisfies the specified conditions for every estimation problem \( (x, \theta) \), and every pair of parameters \( \theta_1 \) and \( \theta_2 \) in parameter space.

As always, we take the parameter space to be \( \Theta \subseteq \mathbb{R}^M \) and the observation space to be \( X \subseteq \mathbb{R}^N \).

**Axiom 1: Invariance to Representation of Parameter Space (IRP)**

A loss function \( L \) is said to satisfy IRP if for every invertible, continuous, differentiable function \( F : \mathbb{R}^M \rightarrow \mathbb{R}^M \), whose Jacobian is defined and non-zero everywhere,

\[
L_{(x, \theta)}(\theta_1, \theta_2) = L_{(x, F(\theta))}(F(\theta_1), F(\theta_2)).
\]

**Axiom 2: Invariance to Representation of Observation Space (IRO)**

A loss function \( L \) is said to satisfy IRO if for every invertible, piecewise continuous, differentiable function \( G : \mathbb{R}^N \rightarrow \mathbb{R}^N \), whose Jacobian is defined and non-zero everywhere,

\[
L_{(x, \theta)}(\theta_1, \theta_2) = L_{(G(x), \theta)}(\theta_1, \theta_2).
\]
Axiom 3: Invariance to Irrelevant Alternatives (IIA)

A loss function $L$ is said to satisfy IIA if $L(x, \theta_1, \theta_2)$ does not depend on any $f(x, \theta)$ [or, in a discrete setting, $P(x = x$ and $\theta = \theta\}$] for which $\theta \notin \{\theta_1, \theta_2\}$.

A loss function that satisfies both IRP and IRO is said to be representation invariant.

The conditions of representation invariance follow Wallace and Boulton (1975), whereas IIA was first introduced in a game-theoretic context by Nash (1950).

6. The continuous case

6.1. Well-behaved problems

We now move to the harder, continuous case, which is the case where both $X$ and $\Theta$ are uncountable, and their joint distribution is given by a probability density function $f = f(x, \theta)$. In this case, we show that for any well-behaved estimation problem and well-behaved loss function $L$, if $L$ satisfies the three invariance axioms of Section 5, any risk-averse estimator over $L$ equals the Wallace-Freeman estimator, regardless of its attenuation function. Note, however, that unlike the discrete $\theta$ case, in the continuous case we restrict our analysis to “well-behaved” problems. The reason for this is mathematical convenience and simplicity of presentation.

In this section we define well-behavedness.

We refer to a continuous estimation problem as well-behaved if it satisfies the following criteria.

(a) The function $f(x, \theta)$ is piecewise continuous in $x$ and three-times continuously differentiable in $\theta$.

(b) The set $\Theta$ is a compact closure of an open set.

Additionally, we say that a loss function $L$ is well-behaved if it satisfies the following conditions.

Smooth: If $(x, \theta)$ is a well-behaved continuous estimation problem, then the function $L(x, \theta)(\theta_1, \theta_2)$ is three times differentiable in $\theta_1$ and these derivatives are continuous in $\theta_1$ and $\theta_2$.

Sensitive: There exists at least one well-behaved continuous estimation problem $(x, \theta)$ and at least one choice of $\theta_0$, $i$ and $j$ such that

$$\left. \frac{\partial^2 L(x, \theta)(\theta, \theta_0)}{\partial \theta(i) \partial \theta(j)} \right|_{\theta = \theta_0} \neq 0.$$ 

Problem-continuous: $L$ is problem-continuous (or "$M$-continuous"), in the sense that if $((x_i, \theta))_{i \in \mathbb{N}}$ is a sequence of well-behaved continuous estimation problems, such that for every $\theta \in \Theta$, $(f_\theta(x_i, \theta)) \xrightarrow{M} f_\theta(x, \theta)$, then for every $\theta_1, \theta_2 \in \Theta$, $\lim_{i \to \infty} L(x_i, \theta)(\theta_1, \theta_2) = L(x, \theta)(\theta_1, \theta_2)$. 


In the last criterion, the symbol \( \overset{M}{\to} \) indicates convergence in measure (Halmos, 2013). This is defined as follows. Let \( \mathcal{M} \) be the space of normalisable, non-atomic measures over some \( \mathbb{R}^s \), let \( f \) be a function \( f : \mathbb{R}^s \to \mathbb{R}_{\geq 0} \) and let \((f_i)_{i \in \mathbb{N}}\) be a sequence of such functions. Then \((f_i) \overset{M}{\to} f\) if
\[
\forall \epsilon > 0, \lim_{i \to \infty} \mu(\{x \in \mathbb{R}^s : |f(x) - f_i(x)| \geq \epsilon\}) = 0, \tag{5}
\]
where \( \mu \) can be, equivalently, any measure in \( \mathcal{M} \) whose support is at least the union of the support of \( f \) and all \( f_i \).

We will usually take \( f \) and all \( f_i \) to be probability density functions (pdfs). When this is the case, \( \mu \)'s support only needs to equal the support of \( f \). Furthermore, because \( \mu \) is normalisable, one can always choose values \( a \) and \( b \) such that \( \mu(\{x : 0 < f(x) < a\}) \) and \( \mu(\{x : f(x) > b\}) \) are both arbitrarily small, for which reason one can substitute the absolute difference \( \geq \epsilon \) in (5) with a relative difference \( \geq \epsilon f(x) \), and reformulate it in the case that \( f \) and all \( f_i \) are pdfs as
\[
\forall \epsilon > 0, \lim_{i \to \infty} P_{x \sim f}(|f(x) - f_i(x)| \geq \epsilon f(x)) = 0, \tag{6}
\]
where \( "x \sim f" \) is shorthand for \( "x" \) is a random variable whose distribution has \( f \) as its pdf.

This reformulation makes it clear that convergence in measure over pdfs is a condition independent of representation: it is invariant to transformations of the sort we allow on the observation space.

### 6.2. The main theorem

Our main theorem for continuous problems is as follows.

**Theorem 2.** If \((x, \theta)\) is a well-behaved continuous estimation problem for which \( \hat{\theta}_{WF} \) is a well-defined set estimator, and if \( L \) is a well-behaved loss function, discriminative for \((x, \theta)\), that satisfies all of IIA, IRP and IRO, then any risk-averse estimator \( \hat{\theta}_{L,A} \) over \( L \), regardless of its attenuation function \( A \), is a well-defined set estimator, and for every \( x \),
\[
\hat{\theta}_{L,A}(x) \subseteq \hat{\theta}_{WF}(x).
\]

In particular, if \( \hat{\theta}_{WF} \) is a well-defined point estimator, then so is \( \hat{\theta}_{L,A} \), and
\[
\hat{\theta}_{L,A} = \hat{\theta}_{WF}.
\]

We prove this through a progression of lemmas. For the purpose of this derivation, the dimension of the parameter space, \( M \), and the dimension of the observation space, \( N \), are throughout taken to be fixed, so as to simplify notation.

**Lemma 1.** For continuous estimation problems \((x, \theta)\), if \( L \) satisfies both IIA and IRP then \( L_{(x, \theta)}(\theta_1, \theta_2) \) is a function only of \( f_{\theta_1} \) and \( f_{\theta_2} \).

**Proof.** The IIA axiom is tantamount to stating that \( L_{(x, \theta)}(\theta_1, \theta_2) \) is dependent only on the following:
(a) the function’s inputs $\theta_1$ and $\theta_2$,  
(b) the likelihoods $f_{\theta_1}$ and $f_{\theta_2}$, and  
(c) the priors $f(\theta_1)$ and $f(\theta_2)$.

This is because knowledge of $f(x, \theta)$ for $\theta \in \{\theta_1, \theta_2\}$ is equivalent to knowledge of the last two items, and IIA restricts $L$ from utilising any other value of $f$, or, equivalently, any other part of the description of the estimation problem.

We can assume without loss of generality that $\theta_1 \neq \theta_2$, or else the value of $L(\theta_1, \theta_2)$ can be determined to be zero by (1).

Our first claim is that, due to IRP, $L$ can also not depend on the problem’s prior probability densities $f(\theta_1)$ and $f(\theta_2)$. To show this, construct an invertible, continuous, differentiable function $F : \mathbb{R}^M \rightarrow \mathbb{R}^M$, whose Jacobian is defined and non-zero everywhere, in the following way.

Let $(\vec{v}_1, \ldots, \vec{v}_M)$ be an orthonormal basis for $\mathbb{R}^M$ wherein $\vec{v}_1 = \theta_2 - \theta_1$. We design $F$ as

$$F \left( \theta_1 + \sum_{i=1}^{M} b_i \vec{v}_i \right) = \theta_1 + F_1(b_1) \vec{v}_1 + \sum_{i=2}^{M} b_i \vec{v}_i,$$

where $F_1 : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous, differentiable function onto $\mathbb{R}$, with a derivative that is positive everywhere, satisfying

(a) $F_1(0) = 0$ and $F_1(1) = 1$, and  
(b) $F_1'(0) = d_0$ and $F_1'(1) = d_1$, for some arbitrary positive values $d_0$ and $d_1$.

Such a function is straightforward to construct for any values of $d_0$ and $d_1$, and by an appropriate choice of these values, it is possible to map $(x, \theta)$ into $(x, F(\theta))$ in a way that does not change $f_{\theta_1}$ or $f_{\theta_2}$, but adjusts $f(\theta_1)$ and $f(\theta_2)$ to any desired positive values.

Lastly, we show that $L(\theta_1, \theta_2)$ can also not depend on the values of $\theta_1$ and $\theta_2$ other than through $f_{\theta_1}$ and $f_{\theta_2}$. For this we once again invoke IRP: by applying a similarity transform on $\Theta$, we can map any $\theta_1$ and $\theta_2$ values into arbitrary new values, again without this affecting their respective likelihoods.

In light of Lemma 1, we will henceforth write $L(f_{\theta_1}, f_{\theta_2})$ instead of $L(x, \theta)(\theta_1, \theta_2)$. A loss function that can be written in this way is referred to as a *likelihood-based* loss function.

For pdfs $f$ and $g$ with a common support $X \subseteq \mathbb{R}^N$, let

$$r_{f,g}(x) \overset{\text{def}}{=} \frac{f(x)}{g(x)}$$

within the common support, and define the function $c[f, g] : [0, 1] \rightarrow \mathbb{R}_{\geq 0}$ by

$$c[f, g](t) \overset{\text{def}}{=} \inf \{ r \in \mathbb{R}_{\geq 0} : t \leq \mathbb{P}_{x \sim g}(r_{f,g}(x) \leq r) \}.$$

**Lemma 2.** The function $c$ is $\mathcal{M}$-continuous, in the sense that if $(f_i)_{i \in \mathbb{N}} \overset{\mathcal{M}}{\rightarrow} f$ and $(g_i)_{i \in \mathbb{N}} \overset{\mathcal{M}}{\rightarrow} g$, then $(c[f_i, g_i])_{i \in \mathbb{N}} \overset{\mathcal{M}}{\rightarrow} c[f, g]$.  


Proof. For any $t_0 \in [0, 1]$, let $r_0 = c[f, g](t_0)$, and let $t_{\text{min}}$ and $t_{\text{max}}$ be the infimum $t$ and the supremum $t$, respectively, for which $c[f, g](t) = r_0$.

Because $c[f, g]$ is a monotone increasing function, $t_{\text{min}}$ is also the supremum $t$ for which $c[f, g](t) < r_0$ (unless no such $t$ exists, in which case $t_{\text{min}} = 0$), so by definition $t_{\text{min}}$ is the supremum of $\text{P}_{x \sim g}(r_{f,g}(x) \leq r)$, for all $r < r_0$, from which we conclude

$$t_{\text{min}} = \text{P}_{x \sim g}(r_{f,g}(x) < r_0).$$

Because $(f_i) \xrightarrow{\mathcal{M}} f$ and $(g_i) \xrightarrow{\mathcal{M}} g$, we can use (6) to determine that using a large enough $i$ both $f_i(x)/f(x)$ and $g_i(x)/g(x)$ are arbitrarily close to 1 in all but a diminishing measure of $X$. Hence,

$$\lim_{i \to \infty} \text{P}_{x \sim g_i}(r_{f_i,g_i}(x) < r_0) = \lim_{i \to \infty} \text{P}_{x \sim g}(r_{f_i,g_i}(x) < r_0) = \text{P}_{x \sim g}(r_{f,g}(x) < r_0) = t_{\text{min}}.$$

We conclude that for any $t^+ > t_{\text{min}}$ a large enough $i$ will satisfy $\text{P}_{x \sim g_i}(r_{f_i,g_i}(x) < r_0) < t^+$, and hence $c[f_i, g_i](t^+) \geq r_0$. For all such $t^+$, and in particular for all $t^+ > t_0$,

$$\liminf_{i \to \infty} c[f_i, g_i](t^+) \geq r_0 = c[f, g](t_0). \quad (7)$$

A symmetrical analysis on $t_{\text{max}}$ yields that for all $t^- < t_0$,

$$\limsup_{i \to \infty} c[f_i, g_i](t^-) \leq c[f, g](t_0). \quad (8)$$

Consider, now, the functions

$$c_{\text{sup}}(t) \overset{\text{def}}{=} \limsup_{i \to \infty} c[f_i, g_i](t)$$

and

$$c_{\text{inf}}(t) \overset{\text{def}}{=} \liminf_{i \to \infty} c[f_i, g_i](t).$$

Because each $c[f_i, g_i]$ is monotone increasing, so are $c_{\text{sup}}$ and $c_{\text{inf}}$. Monotone functions can only have countably many discontinuity points (for a total of measure zero). For any $t_0$ that is not a discontinuity point of either function, we have from (7) and (8) that $\lim_{i \to \infty} c[f_i, g_i](t_0)$ exists and equals $c[f, g](t_0)$, so the conditions of convergence in measure hold.

Lemma 3. If $L$ satisfies IRO and is a well-behaved likelihood-based loss function and $p$ and $q$ are piecewise-continuous probability density functions over $X \subseteq \mathbb{R}^N$, then $L(p, q)$ depends only on $c[p, q]$.

Proof. The following conditions are equivalent.

(a) $c[p, q]$ equals the indicator function on $(0, 1]$ in all but a measure zero of values,

(b) $p$ equals $q$ in all but a measure zero of $X$,

(c) $p$ and $q$ are $\mathcal{M}$-equivalent, in the sense that a sequence of elements all equal to $p$ nevertheless satisfies the condition of $\mathcal{M}$-convergence to $q$, and

(d) $L(p, q) = 0$, 

Risk-averse estimation 11
where the equivalence of the last condition follows from the previous one by problem continuity, together with (1). Hence, if the second condition is met, we are done. We can therefore assume that \( p \) and \( q \) differ in a positive measure of \( X \), and (because both integrate to 1) that they are consequently not linearly dependent.

Because \( L \) is known to be likelihood-based, the value of \( L(p, q) \) is not dependent on the full details of the estimation problem: it will be the same in any estimation problem of the same dimensions that contains the likelihoods \( p \) and \( q \). Let us therefore design an estimation problem that is easy to analyse but contains these two likelihoods.

Let \((x, \theta)\) be an estimation problem with \( \Theta = [0,1]^M \) and a uniform prior on \( \theta \). Its likelihood at \( \theta_0 = (0, \ldots, 0) \) will be \( p \), at \( \theta_1 = (1, 0, \ldots, 0) \) will be \( q \), and we will choose piecewise continuous likelihoods, \( f_\theta \), over the rest of the \( \theta \in [0,1]^M \) so all \( 2^M \) are linearly independent, share the same support, and differ from each other over a positive measure of \( X \), and so that their respective \( r_{p,q} \) values are all monotone weakly increasing with \( r_{p,q} \) and with each other.

If \( M > 1 \), we further choose \( f_{\theta_2} \) at \( \theta_2 = (0, 1, 0, \ldots, 0) \) to satisfy that \( c[f_{\theta_2}, q] \) is monotone strictly increasing. If \( M = 1 \), this is not necessary and we, instead, choose \( \theta_2 = \theta_0 \).

We then extend this description of \( f_\theta^{(x,\theta)} \) at \( [0,1]^M \) into a full characterisation of all the problem’s likelihoods by setting these to be multilinear functions of the coordinates of \( \theta \).

We now create a sequence of estimation problems, \((x_i, \theta)\) to satisfy the conditions of \( L \)’s problem-continuity assumption. We do this by constructing a sequence \((S_i)_{i \in \mathbb{N}}\) of subsets of \( \mathbb{R}^N \) such that for all \( \theta \in [0,1]^M \), \( P(x \in S_i | \theta = \theta) \) tends to 1, and for every \( x \in S_i \),

\[
\left| f_\theta^{(x,\theta)}(x) - f_\theta^{(x,\theta)}(x) \right| < \epsilon_i, \quad (9)
\]

for an arbitrarily-chosen sequence \((\epsilon_i)_{i \in \mathbb{N}}\) tending to zero. By setting the remaining likelihood values as multilinear functions of the coordinates of \( \theta \), as above, the sequence \((x_i, \theta)\) will satisfy the problem-continuity condition and will guarantee \( \lim_{i \to \infty} L(x, \theta)(\theta_0, \theta_1) = L(x, \theta)(\theta_0, \theta_1) = L(p, q) \).

Each \( S_i \) will be describable by the positive parameters \((a, b, d, r)\) as follows. Let \( C^N_d = \{ x \in \mathbb{R}^N : |x|_\infty \leq d/2 \} \), i.e. the axis parallel, origin-centred, \( N \)-dimensional cube of side length \( d \). \( S_i \) will be chosen to contain all \( x \in C^N_d \) such that for all \( \theta \in [0,1]^M \), \( a \leq f_\theta^{(x,\theta)}(x) \leq b \) and \( x \) is at least a distance of \( r \) away from the nearest discontinuity point of \( f_\theta^{(x,\theta)} \), as well as from the origin. By choosing small enough \( a \) and \( r \) and large enough \( b \) and \( d \), it is always possible to make \( P(x \in S_i | \theta = \theta) \) arbitrarily close to 1, so the sequence can be made to satisfy its requirements.

We will choose \( d \) to be a natural.

We now describe how to construct each \( f_\theta^{(x,\theta)} \) from its respective \( f_\theta^{(x,\theta)} \). We first describe for each \( \theta \in [0,1]^M \) a new function \( g_\theta^i : \mathbb{R}^N \to \mathbb{R}^\geq 0 \) as follows. Begin by setting \( g_\theta^i(x) = f_\theta^{(x,\theta)}(x) \) for all \( x \in S_i \). If \( x \notin C^N_d \), set \( g_\theta^i(x) \) to zero. Otherwise, complete the \( g_\theta^i \) functions so that all are linearly independent and so that each is positive and continuous inside \( C^N_d \), and integrates to 1. Note that because a
Risk-averse estimation

neighbourhood around the origin is known to not be in \( S_i \), it is never the case that
\( S_i = C_d^N \). This allows enough degrees of freedom in completing the functions \( g \) in
order to meet all their requirements.

As all \( g_\theta \) are continuous functions over the compact domain \( C_d^N \), by the Heine-
Cantor Theorem (Rudin, 1964) they are uniformly continuous. There must therefore
exist a natural \( n \), such that we can tile \( C_d^N \) into sub-cubes of side length \( 1/n \) such
that by setting each \( f_\theta^{(x,\theta)} \) value in each sub-cube to a constant for the sub-cube
equal to the mean over the entire sub-cube tile of \( g_\theta \), the result will satisfy for all
\( x \in C_d^N \) and all \( \theta \in \{0,1\}^M \),
\[
| f_\theta^{(x,\theta)}(x) - g_\theta(x) | < \epsilon_i. 
\]
Because \( f^{(x,\theta)} \) is by design multi-linear in \( \theta \), this implies that for all \( \theta \in \{0,1\}^M \) and all \( x \in S_i \), condition (9) is attained. Furthermore, by choosing a large enough \( n \), we can always ensure,
because the \( g \) functions are continuous and linearly independent, that also the
\( f_\theta^{(x,\theta)} \) functions, for \( \theta \in \{0,1\}^M \) are linearly independent and differ in more than
a measure zero of \( \mathbb{R}^N \). Together, these properties ensure that the new problems
constructed are both well defined and well behaved.

We have therefore constructed \((x_i, \theta)\) as a sequence of well-behaved estimation
problems that \( M \)-approximate \((x, \theta)\) arbitrarily well, while being entirely composed
of \( f_\theta^{(x,\theta)} \) functions whose support is \( C_d^N \), for some natural \( d_i \) and whose values
within their support are piecewise-constant inside cubic tiles of side-length \( 1/n_i \); for some
natural \( n_i \).

We now use IRO to reshape the observation space of the estimation problems in
the constructed sequence by a piecewise-continuous transform.

Namely, we take each constant-valued cube of side length \( 1/n_i \) and transform it using a scaling transformation in each coordinate, as follows. Consider a single
cubic tile, and let the value of \( f_\theta^{(x,\theta)}(x) \) at points \( x \) that are within it be \( G_i \). We
scale the first coordinate of the tile to be of length \( G_i/n_i^N \), and all other coordinates
to be of length 1. Notably, this transformation increases the volume of the cube by
a factor of \( G_i \), so the probability density inside the cube, for each \( f_\theta \), will drop by
a corresponding factor of \( G_i \).

We now place the transformed cubes by stacking them along the first coordinate,
sorted by increasing \( f_\theta^{(x,\theta)}(x) \).

Notably, because the probability density \( f_\theta \) in all transformed cubes is \( G_i/G_i = 1 \), it is possible to arrange all transformed cubes in this way so that, together, they
fill exactly the unit cube in \( \mathbb{R}^N \). Let the new estimation problems created in this
way be \((x'_i, \theta)\), let \( t_i : \mathbb{R}^N \to \mathbb{R}^N \) be the transformation, \( t_i(x_i) = x'_i \), applied on the
observation space and let \( t_i(x) \) be the first coordinate value of \( t_i(x) \).

By IRO, \( L(f_\theta^{(x'_i,\theta)}, f_\theta^{(x,\theta)}) = L(f_\theta^{(x,\theta)}, f_\theta^{(x,\theta)}) \), which we know tends to \( L(p,q) \).

Consider the probability density of each \( f_\theta^{(x'_i,\theta)} \) over its support \([0,1]^N \). This is
a probability density that is uniform along all axes except the first, but has some
marginal, \( s = s_i^{(1)} \), along the first axis. We denote such a distribution by \( D_N(s) \).
Specifically, for \( \theta = \theta_2 \), because of our choice of sorting order, we have
\( s_i^{(1)} = c[f_\theta^{(x,\theta)}, f_\theta^{(x,\theta)}] \), so by Lemma 2, this is known to \( M \)-converge to
\( c[f_\theta^{(x,\theta)}, f_\theta^{(x,\theta)}] \).
If $M = 1$, the above is enough to show that the $\mathcal{M}$-limit problem of $(x', \theta)$ exists. If $M > 1$, consider the following.

Let $t : X \rightarrow [0, 1]$ be the transformation mapping each $x \in X$ to the supremum $\tilde{t}$ for which $c[f_{\theta_2}^{(x, \theta)}(\tilde{t})] \leq f_{\theta_1}^{(x, \theta)}(x)$. This will be satisfied with equality wherever $c[f_{\theta_2}^{(x, \theta)}, f_{\theta_1}^{(x, \theta)}]$ is continuous, which (because it is monotone) is in all but a measure zero of the $\tilde{t}$, and therefore of the $x$.

Thus, in all but a diminishing measure of $x$ we have that $f_{\tilde{t}_2}^{(x, \theta)}(x)/f_{\tilde{t}_1}^{(x, \theta)}(x)$ approaches $f_{\theta_2}^{(x, \theta)}(x)/f_{\theta_1}^{(x, \theta)}(x)$, which in turn equals $c[f_{\theta_2}^{(x, \theta)}, f_{\theta_1}^{(x, \theta)}](t(x))$. On the other hand, we have that $s_{\tilde{t}_2} = c[f_{\theta_2}^{(x, \theta)}, f_{\theta_1}^{(x, \theta)}]$ also $\mathcal{M}$-approaches $c[f_{\theta_2}^{(x, \theta)}, f_{\theta_1}^{(x, \theta)}]$ by Lemma 2, and satisfies $f_{\tilde{t}_2}^{(x, \theta)}(x)/f_{\tilde{t}_1}^{(x, \theta)}(x) = s_{\tilde{t}_2}^{(t_1(x))}$. Together, this indicates that $c[f_{\theta_2}^{(x, \theta)}, f_{\theta_1}^{(x, \theta)}](t_1(x)) \mathcal{M}$-converges to $c[f_{\theta_2}^{(x, \theta)}, f_{\theta_1}^{(x, \theta)}](t(x))$.

Because $c[f_{\theta_2}^{(x, \theta)}, f_{\theta_1}^{(x, \theta)}]$ is monotone strictly increasing, it follows that $t_1(x)$ $\mathcal{M}$-converges to $t(x)$.

For all other $\theta \in [0, 1]^M$ this then implies that $s_{\tilde{t}} \mathcal{M}$-converges to $c[f_q^{(x, \theta)}, f_{\theta_1}^{(x, \theta)}]$, by construction, all the problem’s $r_{f_\theta, q}$ are monotone increasing with each other.

All $f_{\tilde{t}}^{(x, \theta)}$ therefore have a limit, that limit being $D_N(c[f_{\theta_2}^{(x, \theta)}, f_{\theta_1}^{(x, \theta)}])$.

In particular, the limit at $\theta = \theta_0$ is $D_N(c[p, q])$ and the limit at $\theta = \theta_1$ is $U([0, 1]^N)$, the uniform distribution over the unit cube.

By problem-continuity of $L$, $L(D_N(c[p, q]), U([0, 1]^N)) = L(p, q)$. Hence, $L(p, q)$ is a function of only $c[p, q]$.

For a well-behaved continuous estimation problem $(x, \theta)$, if for every $\theta \in \Theta$ the function $L_{\theta}(\theta') = L((x, \theta)(\theta', \theta)$ has all its second derivatives at $\theta' = \theta$ (a condition that is true for every well-behaved $L$), denote its Hessian matrix $H_{\theta}^L = H_{\theta}^L[(x, \theta)]$.

**Lemma 4.** Let $(x, \theta)$ be a well-behaved continuous estimation problem, and let $L$ be a well-behaved likelihood-based loss function satisfying IRO.

There exists a nonzero constant $\gamma$, dependent only on the choice of $L$, such that for every $\theta \in \Theta$ the Hessian matrix $H_{\theta}^L$ equals $\gamma$ times the Fisher information matrix $I_{\theta}$.

**Proof.** We wish to calculate the derivatives of $L(f_{\theta_2}, f_{\theta_1})$ according to $\theta_1$. For convenience, let us describe $L$ in a one-parameter form. Let $p_{r,q}(x) = r(x)q(x)$.

This is the inverse operation to our definition of $r_{p,q}$. We now define

$L_q(r) \overset{\text{def}}{=} L(p_{r,q}, q)$

so

$L_q(r_{p,q}) = L(p, q)$,

where the domain of $r$ is taken as $X$ to avoid divisions by zero.

We differentiate $L_q(r_{p,q})$ as we would any composition of functions. The derivatives of $r_{f_\theta, f_{\theta_2}}$ in $\theta_1$ are straightforward to compute, so we concentrate on the question of how minute perturbations of $r$ affect $L_q(r)$.
For this, we first extend the domain of \( L_q(r) \). Natively, \( L_q(r) \) is only defined when \( E_{x \sim q}(r(x)) = 1 \). However, to be able to perturb \( r \) more freely, we define, for finite expectation \( r \), \( L_q(r) = L_q(r/E_{x \sim q}(r(x))) \).

Let \( Y \subseteq X \) be a set with \( P_{x \sim q}(x \in Y) = \epsilon > 0 \) such that for all \( x \in Y \), \( r(x) \) is a constant, \( r_0 \). The derivative of \( L_q(r) \) in \( Y \) is defined as

\[
\frac{\partial L_q(r)}{\partial \theta_1(i)} = \int_X \Delta_x(\epsilon q_{f_{\theta_1}, f_{\theta_2}}(x) q_{f_{\theta_1}, f_{\theta_2}}(x) \Delta_{\theta_1}(i)) f_{\theta_2}(x) dx.
\]

The same reasoning can be used to describe the second derivative of \( L \) (this time in the directions \( i \) and \( j \) of \( \theta_1 \)). The second derivative of \( L_q(r) \) when perturbing \( r \) relative to two subsets \( Y_1 \) and \( Y_2 \) is defined as

\[
\lim_{\Delta \to 0} \frac{\nabla_{Y_1}(L_q)(r + \Delta \cdot \chi_{Y_2}) - \nabla_{Y_1}(L_q)(r)}{\Delta},
\]

and once again using Lemma 3 and the same symmetry, we can see that if \( Y_1 \) and \( Y_2 \) are disjoint, if the measures of \( Y_1 \) and \( Y_2 \) in \( q \) are, respectively, \( \epsilon_1 \) and \( \epsilon_2 \), both positive, and if the value of \( r(x) \) for \( x \) values in each subset is a constant, respectively \( r_1 \) and \( r_2 \), then any such \( Y_1 \) and \( Y_2 \) will perturb \( L_q(r) \) in exactly the same amount. We name the second derivative coefficient in this case \( \Delta_{xy}(r_1, r_2 | r) \).

The caveat that \( Y_1 \) and \( Y_2 \) must be disjoint is important, because if \( Y = Y_1 = Y_2 \) the symmetry no longer holds. This is a second case, and for it we must define a different coefficient \( \Delta_{xy}(r_0 | r) \), where \( r_0 = r_1 = r_2 \).
In the case where $c[f_{\theta_1}, f_{\theta_2}]$ is a piecewise-constant function, the second derivative, $\frac{\partial^2 L(\theta_1, \theta_2)}{\partial \theta_1(i) \partial \theta_1(j)}$, can therefore be written as

$$
\begin{align*}
\int_X \Delta_x(r_{f_{\theta_1}, f_{\theta_2}}(x)) \frac{\partial^2 r_{f_{\theta_1}, f_{\theta_2}}(x)}{\partial \theta_1(i) \partial \theta_1(j)} f_{\theta_2}(x) dx \\
+ \int_X \int_X \Delta_{xy}(r_{f_{\theta_1}, f_{\theta_2}}(x_1), r_{f_{\theta_1}, f_{\theta_2}}(x_2)) \frac{\partial r_{f_{\theta_1}, f_{\theta_2}}(x_1)}{\partial \theta_1(i)} \frac{\partial r_{f_{\theta_1}, f_{\theta_2}}(x_2)}{\partial \theta_1(j)} f_{\theta_2}(x_2) dx_2 f_{\theta_2}(x_1) dx_1 \\
+ \int_X \Delta_{xx}(r_{f_{\theta_1}, f_{\theta_2}}(x)) \frac{\partial r_{f_{\theta_1}, f_{\theta_2}}(x)}{\partial \theta_1(i)} \frac{\partial r_{f_{\theta_1}, f_{\theta_2}}(x)}{\partial \theta_1(j)} f_{\theta_2}(x) dx.
\end{align*}
$$

(10)

In order to calculate $H_L^0(i, j)$, consider $H_L^0(i, j)$. This equals $\frac{\partial^2 L(\theta_1, \theta_2)}{\partial \theta_1(i) \partial \theta_1(j)}$ where $\theta_1 = \theta_2 = \theta$. In particular, $c[f_{\theta_1}, f_{\theta_2}]$ is $\chi_{(0, 1]}$ and $r_{f_{\theta_1}, f_{\theta_2}}$ is $\chi_X$.

The value of (10) in this case becomes

$$
\begin{align*}
\Delta_x(1|XX) \int_X \frac{\partial^2 r_{f_{\theta_1}, f_{\theta_2}}(x)}{\partial \theta_1(i) \partial \theta_1(j)} f_{\theta}(x) dx \\
+ \Delta_{xy}(1, 1|XX) \left( \int_X \left. \frac{\partial r_{f_{\theta_1}, f_{\theta_2}}(x)}{\partial \theta_1(i)} \right|_{\theta_1 = \theta} f_{\theta}(x) dx \left( \int_X \left. \frac{\partial r_{f_{\theta_1}, f_{\theta_2}}(x)}{\partial \theta_1(j)} \right|_{\theta_1 = \theta} f_{\theta}(x) dx \right) \\
+ \Delta_{xx}(1|XX) \int_X \left. \frac{\partial r_{f_{\theta_1}, f_{\theta_2}}(x)}{\partial \theta_1(i)} \right|_{\theta_1 = \theta} \left( \frac{\partial r_{f_{\theta_1}, f_{\theta_2}}(x)}{\partial \theta_1(j)} \right|_{\theta_1 = \theta} f_{\theta}(x) dx.
\end{align*}
$$

(11)

Note, however, that because $(x, \theta)$ is an estimation problem, i.e. all its likelihoods are probability measures, not general measures, it is the case that

$$
\int_X r_{f_{\theta_1}, f_{\theta}}(x) f_{\theta}(x) dx = \int_X f_{\theta_1}(x) dx = 1,
$$

and is therefore a constant independent of either $\theta_1$ or $\theta$. Its various derivatives in $\theta_1$ are accordingly all zero. This makes the first two summands in (11) zero. What is left, when setting $\gamma = \Delta_{xx}(1|XX)$, is

$$
H_L^0(i, j) = \gamma \int_X \left( \frac{\partial r_{f_{\theta_1}, f_{\theta}}(x)}{\partial \theta_1(i)} \right|_{\theta_1 = \theta} \left( \frac{\partial r_{f_{\theta_1}, f_{\theta}}(x)}{\partial \theta_1(j)} \right|_{\theta_1 = \theta} f_{\theta}(x) dx
$$

$$
= \gamma \int_X \left( \frac{\partial f_{\theta_1}(x)}{\partial \theta_1(i)} \right|_{\theta_1 = \theta} \left( \frac{\partial f_{\theta_1}(x)}{\partial \theta_1(j)} \right|_{\theta_1 = \theta} f_{\theta}(x) dx
$$

$$
= \gamma \int_X \left( \frac{\partial \log f_{\theta_1}(x)}{\partial \theta_1(i)} \right) \left( \frac{\partial \log f_{\theta_1}(x)}{\partial \theta_1(j)} \right) f_{\theta}(x) dx
$$

$$
= \gamma \mathbb{E}_{x \sim f_{\theta}} \left( \left( \frac{\partial \log f_{\theta}(x)}{\partial \theta(i)} \right) \left( \frac{\partial \log f_{\theta}(x)}{\partial \theta(j)} \right) \right)
$$

$$
= \gamma I_{\theta}(i, j).
$$
Hence, $H^\theta_L = \gamma I_\theta$.

As a final point in the proof, we remark that $\gamma$ must be nonzero, because if it had been zero, $H^\theta_L$ would have been zero for every $\theta$ in every well-behaved continuous estimation problem, contrary to our sensitivity assumption on well-behaved loss functions.

**Lemma 5.** If $(x, \theta)$ is a well-behaved continuous estimation problem, $L$ is a well-behaved loss function that is discriminative for it, and $A$ is an attenuation function, and if, further, for every $\theta \in \Theta$, $H^\theta_L$ is a positive definite matrix, define

$$\hat{\theta}(x) \overset{\text{def}}{=} \arg\max_{\theta} \frac{f(\theta|x)}{\sqrt{|H^\theta_L|}}.$$ 

For every $x$, $\hat{\theta}_{L,A}(x)$ is a non-empty subset of $\hat{\theta}(x)$, where $\hat{\theta}_{L,A}$ is the risk-averse estimator defined over $L$ and $A$. In particular, if $\hat{\theta}$ is a well-defined point estimator for the problem, then $\hat{\theta}_{L,A} = \hat{\theta}$.

**Proof.** When calculating

$$\int_{\Theta} f(\theta'|x) A(kL(\theta', \theta)) d\theta'$$

for asymptotically large $k$ values one only needs to consider the integral over $B(\theta, \epsilon)$, the ball of radius $\epsilon$ around $\theta$, for any $\epsilon > 0$, as for a sufficiently large $k$, the rest of the integral values will be zero by the discriminativity assumption.

By definition of $H^\theta_L$, the second order Taylor approximation for $L(\theta', \theta)$ around $\theta$ is

$$L(\theta', \theta) = \frac{1}{2}(\theta' - \theta)^T H^\theta_L (\theta' - \theta) \pm \frac{m}{6} |\theta' - \theta|^3,$$

where $m$ is the maximum absolute third derivative of $L$ over all $\Theta \times \Theta$ and all possible differentiation directions. This maximum exists because the third derivative is continuous and $\Theta \times \Theta$ is compact.

As $k$ grows to infinity, the value of (12) therefore tends to

$$f(\theta|x) \int_{B(\theta, \epsilon)} A\left( \frac{k}{2}(\theta' - \theta)^T H^\theta_L (\theta' - \theta) \right) d\theta',$$

which can be computed via a Jacobian transformation as

$$\frac{f(\theta|x)}{\sqrt{|H^\theta_L|}} \left( \frac{2^{M/2} \int_{\mathbb{R}^M} A(|\omega|^2) d\omega}{\sqrt{k^M}} \right),$$

where the parenthesised expression on the right is a multiplicative factor independent of $\theta$.

We have therefore shown for $F_x(\theta) \overset{\text{def}}{=} f(\theta|x) / \sqrt{|H^\theta_L|}$ that

$$\arg\max_{\theta \in \Theta} \lim_{k \to \infty} \sqrt{k^M} \int_{\Theta} f(\theta'|x) A(kL(\theta', \theta)) d\theta' = \arg\max_{\theta \in \Theta} F_x(\theta) = \hat{\theta}(x).$$

(15)
What we are trying to compute, however, is
\[ \hat{\theta}_{L,A}(x) = \lim_{k \to \infty} \arg\max_{\theta \in \Theta} \int_{\Theta} f(\theta'|x) A(kL(\theta', \theta))d\theta'. \] (16)

The multiplicative difference of \( \sqrt{k^M} \) between (16) and (15) is immaterial, as its addition into (16) would not have changed the argmax value, but the reversal in the order of the quantifiers can, at least potentially, change the result.

In order to show that no value other than a maximiser of \( F_x \) can be part of \( \hat{\theta}_{L,A}(x) \), we need to prove that outside of any neighbourhood of a maximiser of \( F_x \), \( F_x \) is bounded from above away from its maximum, and that the rates of convergence in \( k \) over all \( \theta \) are uniformly bounded.

The fact that \( \theta \) values outside of a neighbourhood of the maximisers of \( F_x \) are bounded away from the maximum is due to \( F_x \) being continuous over a compact space. Let us therefore bound the convergence rates of (12) as \( k \) goes to infinity.

Equation (13) allows us to bound (17), by means of \( m \) and of \( A_{\max}' \), the maximum absolute derivative of \( A \), while the bound on \( |\theta' - \theta| \) allows us to bound \( f(\theta'|x) \) similarly, as \( f(\theta|x) \pm f_{\max}'(\theta' - \theta) \), where \( f_{\max}' \) is the global maximum absolute derivative of \( f(\theta|x) \) over \( \theta' \) at \( x \), for any \( \theta' \in \Theta \) and in any direction.

The four elements that globally bound the speed of convergence are therefore \( m, H_{\min}, A_{\max}' \) and \( f_{\max}' \), all of which are finite, positive numbers, because they are extreme values of continuous, positive functions over a compact space, so the convergence rates are all uniformly bounded, as required.

As a last point, we remark that the setlim of (16) is calculated entirely on subsets of the compact set \( \Theta \), so by the Bolzano-Weierstrass theorem (Bartle and Sherbert, 2011) \( \hat{\theta}_{L,A} \) cannot be the empty set.

We now turn to prove our main claim.

**Proof (of Theorem 2).** By Lemma 1 we know \( L \) to be a likelihood-based loss function, and by Lemma 4 we know that up to a nonzero constant multiple \( \gamma \) its \( H_L^\theta \) is the Fisher information matrix \( I_\theta \). This, in turn, we’ve assumed to be positive definite by requiring \( \hat{\theta}_{WF} \) to be well-defined.
Furthermore, $\gamma$ cannot be negative, as in combination with a positive definite Fisher information matrix this would indicate that $H^\theta_L$ is not positive semidefinite, causing $L$ to attain negative values in the neighbourhood of $\theta$.

It is therefore the case that $H^\theta_L$ must be positive definite, and Lemma 5 can be used to conclude the correctness of the theorem.

7. Feasibility and necessity

By convention, when using the axiomatic approach one also shows that the assumptions taken are all feasible, and that all axioms are necessary. We do so in this section.

7.1. Feasibility

An $f$-divergence (Ali and Silvey, 1966) is a loss function $L$ that can be computed as

$$L(p, q) = \int_X F(p(x)/q(x))q(x)dx.$$ 

We call $F$ the $F$-function of the $f$-divergence (refraining from using the more common term “$f$-function” so as to avoid unnecessary confusion with our probability density function). It should be convex and satisfy $F(1) = 0$.

Any $f$-divergence whose $F$-function has 3 continuous derivatives and satisfies $F''(1) > 0$ meets most of our requirements regarding a well-behaved $L$ function. The one outstanding requirement is that of $M$-continuity. In the very general case discussed here, where, for example, $f(x, \theta)$ can diverge to infinity over $x$, there is a risk that for some estimation problems under some $L$ functions a subset of $X$ of diminishing measure can have a non-negligible impact on the value of $L(p, q)$, for some $p$ and $q$. In order to guarantee $M$-continuity for all estimation problem sequences, we further require our chosen $f$-divergence’s $F$-function to not be in absolute value super-linear in its input, i.e. for its absolute value, $|F(r)|$, to be upper-bounded by some linear function $Ar + B$. When this is the case,

$$|L(p, q)| = \left| \int_X F(p(x)/q(x))q(x)dx \right| \leq \int_X |F(p(x)/q(x))|q(x)dx$$

$$\leq \int_X \left(\frac{Ap(x)}{q(x)} + B\right)q(x)dx = A + B < \infty,$$

so $M$-continuity holds.

An example of a commonly-used $L$ function satisfying all criteria for well-behavedness is squared Hellinger distance (Pollard, 2002),

$$H^2(p, q) = \frac{1}{2} \int_X \left(\sqrt{p(x)} - \sqrt{q(x)}\right)^2 dx,$$

which is the $f$-divergence whose $F$-function is $F(r) = 1 - \sqrt{r}$.
7.2. **Necessity of the axioms**

We now demonstrate that removing any one of our axioms allows us to reach in the continuous case an estimate that differs from the WF estimate.

7.2.1. **Risk aversion**

As presently worded, IRP, IRO and IIA all describe the loss function, \( L \), and by this assume the use of a risk-averse estimator. It is therefore not the case that one can simply omit this one assumption.

However, all three of these assumptions have weaker versions that relate to the representation invariance and to the invariance to irrelevant alternatives of the estimates themselves, rather than the loss function.

For example, IIA can be reworded to state that if for estimation problem \((x, \theta)\) the estimate for any \( x \) is some \( \hat{\theta}(x) \), then for every \( \Omega \subseteq \Theta \) that contains a neighbourhood of \( \theta \) of positive measure in the prior of \( \theta \), if \((x', \theta')\) is distributed as \(\{x, \theta\} \) given \( \theta \in \Omega \) then for this new problem we must also have \( \hat{\theta}(x) \). Such an interpretation would have been very much in the spirit of the original formulation of IIA by Nash (1950).

However, even so, IRP, IRO and IIA are not enough to characterise \( \hat{\theta}_{WF} \). For example, one can equally switch the argmax in the definition of \( \hat{\theta}_{WF} \) for an argmin without this altering any of the estimator’s invariance properties.

7.2.2. **IRP**

A well-known loss function that satisfies all axioms except IRP is quadratic loss, \( L(\theta_1, \theta_2) = |\theta_1 - \theta_2|^2 \). A risk-averse estimator with this loss function yields the \( f \)-MAP estimate.

7.2.3. **IRO**

For a loss function satisfying all axioms except IRO, let \( L(p, q) = \int_X q(x)(p(x) - q(x))^2 dx \), which is the expected square difference between the probability densities at \( x \sim q \). The risk-averse estimator over \( L \) is described by Lemma 5.

Calculating \( H_{L}^\theta \) we get

\[
H_{L}^\theta(i, j) = E_{x \sim f_\theta} \left( 2 \left( \frac{\partial f_\theta(x)}{\partial \theta(i)} \right) \left( \frac{\partial f_\theta(x)}{\partial \theta(j)} \right) \right),
\]

which is different to the Fisher information matrix, and defines an estimator that is not WF.

7.2.4. **IIA**

We construct a loss function \( L \) that satisfies all axioms except IIA as follows. Let \( L_1 \) and \( L_2 \) be two well-behaved loss functions satisfying all of IIA, IRP and IRO (such as, for example, two well-behaved \( f \)-divergences) and let \( t \) be a threshold value.
Consider the function
\[ P(\theta) = P(L_1(\theta, \theta) \leq t). \]

By construction, this function is independent of representation.

Define \( L(\theta_1, \theta_2) = P(\theta_2)^2L_2(\theta_1, \theta_2). \)

By Lemma 5, the resulting risk-averse estimator will equal
\[
\arg\max_{\theta \in \Theta} \frac{f(\theta|x)}{\sqrt{|H^\theta_L|}} = \arg\max_{\theta \in \Theta} \frac{f(\theta|x)}{P(\theta)\sqrt{|H^\theta_{L_2}|}},
\]

because \( P(\theta_2)^2 \) is independent of \( \theta_1 \) and therefore acts as a constant multiplier in the calculation of the Hessian.

This new estimator is different to the Wallace-Freeman estimator in the fact that it adds a weighing factor \( P(\theta) \). Any other weighing factor can similarly be added, given that it is a function of \( \theta \) that is independent of representation and satisfies the well-behavedness criteria of a loss function.

8. Estimating the binomial

In this section we consider the classic example of estimating the \( p \) parameter in a binomial distribution, for a known \( n \). For simplicity of presentation, instead of observing \( \text{Bin}(n, p) \), we will assume our observations are \( n \) Bernoulli trials. The observation space is therefore \( X = \{0, 1\}^n \).

In addition, we modify the classic problem slightly by limiting our parameter space to only \( \Theta = [\epsilon, 1/2] \) for some small \( \epsilon > 0 \), rather than its full possible range. We assume that the prior for \( p \) is uniform within \( \Theta \).

The purpose of this analysis is to demonstrate that in the case of continuous parameters and discrete observations, our axioms do not uniquely characterise a solution, but rather define a family of risk-averse estimators.

Specifically, we will show regarding estimators of the form
\[
\hat{\theta}(x) = \arg\max_{\theta \in \Theta} \frac{f(\theta|x)}{F(\theta)}, \tag{18}
\]

where \( F \) can be any twice continuously differentiable function into \( \mathbb{R}^+ \), that they are members of this family.

To construct a loss function that will satisfy (18), let \( G(\theta) = \int_\theta^1 F(\theta')d\theta' \) and for any \( p \in \Theta \) let \( P_p \) be the distribution of \( x \) given \( p \), i.e. the likelihood, with
\[
P_p(x) \overset{\text{def}}{=} \prod_{i=1}^n p^{x(i)}(1-p)^{1-x(i)}. \]

Consider that for all \( p \), \( \prod_{x' \in X} P_p(x') = (p(1-p))^{n2^{n-1}} \). This allows us to retrieve the original value of \( p \) from the distribution as
\[
p = \tilde{p}(P_p) \overset{\text{def}}{=} \frac{1}{2} - \sqrt{\frac{1}{4} - \left( \prod_{x' \in X} P_p(x') \right)^{\frac{1}{n2^{n-1}}}}.
\]
We can now use as a likelihood-based loss function

\[ L(P_p, P_q) = \frac{1}{2} (G(\hat{p}(P_p)) - G(\hat{p}(P_q)))^2. \]

This loss function leads to (18) because \( R_L^\theta \) equals \( F(\theta)^2 \).

9. Future work

The framework defined in this paper opens up many new avenues for research, including the potential to characterise explicitly the family of estimators satisfying our four axioms in the case of discrete \( X \) and continuous \( \Theta \), as well as the potential to explore other axiomatic setups, some of which may perhaps allow unique characterisation also in this remaining case.

Perhaps the most interesting follow-up question, however, is the one presented in Section 7.2.1: if one poses our four axioms not as one axiom (risk aversion) to which three other axioms are subordinate, but rather as four independent axioms, all of which describe the estimates directly rather than the inner workings of the estimator, would these four axioms still uniquely characterise the Wallace-Freeman estimator?

While we conjecture this to be the case, the question remains open.

References

Ali, S. and Silvey, S. (1966) A general class of coefficients of divergence of one distribution from another. *Journal of the Royal Statistical Society. Series B (Methodological)*, 131–142.

Bartle, R. and Sherbert, D. (2011) *Introduction to real analysis*. Hoboken, NJ: Wiley.

Berger, J. (2013) *Statistical decision theory and Bayesian analysis*. Springer Science & Business Media.

Comley, J. and Dowe, D. (2005) Minimum Message Length and generalized Bayesian nets with asymmetric languages. In *Advances in Minimum Description Length: Theory and applications*, 265–294. MIT Press.

Farr, G. and Wallace, C. (2002) The complexity of Strict Minimum Message Length inference. *The Computer Journal*, **45**, 285–292.

Halmos, P. (2013) *Measure theory*, vol. 18. Springer.

Jin, Y. and Rumantir, G. (2015) A two tiered finite mixture modelling framework to cluster customers on EFTPOS network. In *Australasian Joint Conference on Artificial Intelligence*, 276–284. Springer.

Kasarapu, P., Garcia de la Banda, M. and Konagurthu, A. (2014) On representing protein folding patterns using non-linear parametric curves. *IEEE/ACM Transactions on Computational Biology and Bioinformatics (TCBB)*, **11**, 1218–1228.
Lehmann, E. and Casella, G. (2006) *Theory of point estimation*. Springer Science & Business Media.

Nash, Jr., J. (1950) The bargaining problem. *Econometrica: Journal of the Econometric Society*, 155–162.

Pollard, D. (2002) *A user’s guide to measure theoretic probability*, vol. 8. Cambridge University Press.

Rudin, W. (1964) *Principles of mathematical analysis*, vol. 3. McGraw-Hill New York.

Saikrishna, V., Dowe, D. and Ray, S. (2016) Statistical compression-based models for text classification. In *Eco-friendly Computing and Communication Systems (ICECCS), 2016 Fifth International Conference on*, 1–6. IEEE.

Schmidt, D. and Makalic, E. (2016) Minimum Message Length analysis of multiple short time series. *Statistics & Probability Letters*, 110, 318–328.

Sumanaweera, D., Allison, L. and Konagurthu, A. (2018) The bits between proteins. In *2018 Data Compression Conference*, 177–186. IEEE.

Wallace, C. (2005) *Statistical and Inductive Inference by Minimum Message Length*. Information Science and Statistics. Springer Verlag.

Wallace, C. and Boulton, D. (1975) An invariant Bayes method for point estimation. *Classification Society Bulletin*, 3, 11–34.

Wallace, C. and Freeman, P. (1987) Estimation and inference by compact coding. *Journal of the Royal Statistical Society series B*, 49, 240–252. See also Discussion on pp. 252–265.